INDUCED CHARACTER IN EQUIVARIANT K-THEORY, 
WREATH PRODUCTS AND PULLBACK OF GROUPS

GERMAN COMBARIZA, JUAN RODRIGUEZ, AND MARIO VELÁSQUEZ

Abstract. Let $G$ be a finite group, $X$ be a compact $G$-space. In this note we study the $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graded algebra

$$F^G_\mathbb{Q}(X) = \bigoplus_{n \geq 0} q^n \cdot K_{G \wr \mathbb{S}_n}(X^n) \otimes \mathbb{C},$$

defined in terms of equivariant K-theory with respect to wreath products as a symmetric algebra. More specifically, let $H$ be another finite group and $Y$ be a compact $H$-space, we give a decomposition of $F^G_\mathbb{Q} \times_\mathbb{Q} H(X \times Y)$ in terms of $F^G_\mathbb{Q}(X)$ and $F^H_\mathbb{Q}(Y)$. For this, we need to study the representation theory of pullbacks of groups. We discuss also some applications of the above result to equivariant connective K-homology.

Notation

In this note we denote by $\mathbb{S}_n$ the symmetric group in $n$ letters. Let $G$ be a finite group, let $g, g' \in G$, we say that $g$ and $g'$ are conjugated in $G$ (denoted by $g \sim_G g'$) if there is $s \in G$ such that $g = sg's^{-1}$. We denote by

$$[g]_G = \{g' \in G \mid g \sim_G g'\}$$

to the conjugacy class of $g$ in $G$ (or simply by $[g]$ when $G$ is clear from the context). We denote by $G_s$ the set of conjugacy classes of $G$. We denote by $C_G(g)$ the centralizer of $g$ in $G$. Also $R(G)$ will be the complex representation ring of $G$, with operations given by direct sum and tensor product, and generated as abelian group by the isomorphism classes of irreducible representations of $G$. The class function ring of $G$ is the set

$$\text{Class}(G) = \{f : G \to \mathbb{C} \mid f \text{ is constant in conjugacy classes}\}$$

with the usual operations.

1. Introduction

Let $X$ be a finite CW-complex. In Segal studied the vector spaces

$$\mathcal{F}(X) = \bigoplus_{n \geq 0} K_{\mathbb{S}_n}(X^n) \otimes \mathbb{C},$$

these spaces carries several interesting structures, for example they admit a Hopf algebra structure with the product defined using induction on vector bundles and the coproduct defined using restriction.

Date: November 21, 2019.
Later in [Wan00], Wang generalizes Segal’s work to an equivariant context. Let $G$ be a finite group and $X$ be a finite $G$-CW-complex, Wang defines the vector space
\[ F_G(X) = \bigoplus_{n \geq 0} K_G \wr \mathfrak{S}_n(X^n) \otimes \mathbb{C}, \]
where $G \wr \mathfrak{S}_n$ denotes the wreath product acting naturally over $X^n$. Wang proves that $F_G(X)$ admits similar structures as $F(X)$. In particular $F_G(X)$ has a description as a supersymmetric algebra in terms of $K_G(X) \otimes \mathbb{C}$. In this paper we present a proof of the above fact using an explicit description of the character of an induced vector bundle. This description indicates that $F_G(X)$ has the size of the Fock space of a Heisenberg superalgebra.

Following ideas of [Seg96], in [Vel15] appears another reason to study $F_G(X)$. When $X$ is a $G$-spin$^c$-manifold of even dimension, $F_G(X)$ is isomorphic to the homology with complex coefficients of the $G$-fixed point set of a based configuration space $\mathcal{C}(X, x_0, G)$ whose $G$-equivariant homotopy groups correspond to the reduced $G$-equivariant connective K-homology groups of $X$. This description allows to relate generators of $F_G(X)$ with some homological versions of the Chern classes.

Let $G$ and $H$ be finite groups, $X$ be a finite $G$-CW-complex and $Y$ be a finite $H$-CW-complex, we also prove a Künneth formula for $F_{G \times H}(X \times Y)$, obtaining an isomorphism
\[ F_{G \times H}(X \times Y) \cong F_G(X) \otimes F_H(Y) \]
that is compatible with the decomposition as a supersymmetric algebra. In order to do this, we need to study the representation theory of pullbacks of groups.

Let
\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p_2} & G \\
p_1 \downarrow & & \downarrow \pi_1 \\
H & \xrightarrow{\pi_2} & K
\end{array}
\]
be a pullback diagram of finite groups, with $\pi_1$ and $\pi_2$ surjective, in this case $\Gamma$ can be realized as a subgroup of $G \times H$. We prove that when $\Gamma$ is conjugacy-closed (see Definition 5.2) in $G \times H$ then we have a ring isomorphism
\[ \text{Class}(\Gamma) \cong \text{Class}(H) \otimes \text{Class}(K) \otimes \text{Class}(G). \]
Moreover we use that some conjugacy classes in $(G \times H) \wr \mathfrak{S}_n$ are closed in $(G \wr \mathfrak{S}_n) \times (H \wr \mathfrak{S}_n)$ to prove the Künneth formula for the algebra $F_{G \times H}(X \times Y)$.

This paper is organized as follows:

In Section 2 we recall basic facts about equivariant K-theory, in particular we recall the construction for the character. Following ideas of [Ser77], we give an explicit definition of the induced bundle and recall a formula (proved in [HKR00 Thm. D]) for a character of the induced bundle. In Section 3 we recall basic facts about wreath products and its action over $X^n$. In Section 4 we recall the definition of $F_G(X)$ and give another way to obtain the description as a supersymmetric algebra using the formula of the induced character. In Section 5 we study the representation theory of pullbacks. In Section 6 we recall some basic properties of semidirect products of direct products. In Section 7 we use results in Section 5 to give a Künneth formula for the Hopf algebra $F_{G \times H}(X \times Y)$. In Section 8 we do some final remarks about the relation of $F(X)$ and homological versions of Chern classes.
2. Induced character in equivariant K-theory

In this section we recall a decomposition theorem for equivariant K-theory with complex coefficients obtained by Atiyah and Segal in [AS89]. In the next section we use that result to give a simple description of $F^G_q(X)$. In this paper all CW-complexes (and G-CW-complexes) that we consider are finite.

**Definition 2.1.** Let $X$ be a $G$-space. A $G$-vector bundle over $X$ is a map $p : E \to X$, where $E$ is a $G$-space satisfying the following conditions

(i) $p : E \to X$ is a vector bundle.

(ii) $p$ is a $G$-map

(iii) For every $g \in G$ the left translation $E \to E$ by $g$ is bundle map.

If $p : E \to X$ is a $G$-vector bundle we define the fiber over $x \in X$ to the set $p^{-1}(x) = \{ v \in E | p(v) = x \}$, when $p$ is clear from the context we also denote this set by $E_x$. Also if $H \subseteq G$ is a subgroup, we can consider $E$ as a $H$-vector bundle over $X$, we denote it by $\text{res}^G_H(E)$.

**Definition 2.2.** Let $X$ and $Y$ be $G$-spaces. If $p : E \to Y$ is a $G$-vector bundle and $f : X \to Y$ is a $G$-map, then the pullback $p^*E \to X$ is a $G$-vector bundle over $X$ defined as $p^*E = \{(e, x) \in E \times X | p(e) = f(x)\}$.

When $i : X \to Y$ is an inclusion we usually denote $i^*(E)$ by $E | Y$.

Details about $G$-vector bundles can be found in [Ati89].

**Definition 2.3.** Let $G$ be a group, let $X$ be a finite $G$-CW-complex (see [tD87]), the equivariant K-theory group of $X$, denoted by $K_G(X)$ is defined as the Grothendieck group of the monoid of isomorphism classes of $G$-equivariant vector bundles over $X$ with the operation of direct sum. The functor $K_G(\_)$ could be extended to an equivariant cohomology theory $K^*(\_)$, defining for $n > 0$:

$$K^{-n}_G(X) = \ker \left( K_G(X \times S^n) \xrightarrow{i_n} K_G(X) \right).$$

And for any $G$-CW-pair $(X, A)$, set

$$K^{-n}_G(X, A) = \ker \left( K^{-n}_G(X \cup_A X) \xrightarrow{i_n} K^{-n}_G(X) \right).$$

Finally for $n < 0$

$$K^{-n}_G(X) = K^n_G(X) \text{ and } K^{-n}_G(X, A) = K^n_G(X, A).$$

For more details about equivariant K-theory the reader can consult [Seg68].

**Example 2.4.** If the action of $G$ over $X$ is free, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong K(X/G).$$

**Example 2.5.** If the action of $G$ over $X$ is trivial, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong R(G) \otimes \mathbb{Z} K(X),$$
when \( R(G) \) denotes the (complex) representation ring of \( G \). In particular when \( X = \{ \bullet \} \) we obtain
\[
K_G(\{ \bullet \}) \cong R(G).
\]

If \( Y \) is a finite \( G\text{-}CW \)-complex, we can define a \( G \)-action on \( K(Y) \). Let \( g \in G \), the pullback
\[
g^*: K(Y) \to K(Y),
\]
defines a \( G \)-action over \( K(Y) \). We will need the following lemma.

**Lemma 2.6.** Let \( Y \) be a finite \( G\text{-}CW \)-complex, then
\[
K(Y/G) \otimes \mathbb{C} \cong K(Y)^G \otimes \mathbb{C}
\]

**Proof.** It is a consequence of the Chern character and the analogous fact for singular cohomology. \( \Box \)

In [ASS99] a character for equivariant K-theory is constructed, that generalizes the character of representations. We will recall this construction briefly. Let \( E \) be a \( G \)-vector bundle over \( X \) and \( g \in G \). Note that \( X^g \) is a \( C_G(g) \)-space, then if \( E \) is a \( G \)-vector bundle, \( E|X^g \) is canonically a \( C_G(g) \)-vector bundle over \( X^g \). Considering the action given by pullback we have that the isomorphism class \([E|X^g] \in K(X^g)\) is a \( C_G(g) \)-fixed point. Then \([E|X^g] \in K(X^g)^{C_G(g)} \). Finally for every element \( \lambda \in S^1 \), we can form the vector bundle of \( \lambda \)-eigenvectors considering the action of the element \( g \) over \( \pi(E|X^g) \) denoted by \( \pi(E|X^g)\lambda \). Then we can define a map
\[
\text{char}_G : K_G(X) \otimes \mathbb{C} \to \bigoplus_{[g]} K(X^g)^{C_G(g)} \otimes \mathbb{C}
\]
[\[
[E] \mapsto \left( \bigoplus_{\lambda \in S^1} [\pi(E|X^g)\lambda] \otimes \lambda \right)_{[g]}
\]
Using the above Lemma we identify \( K(X^g)^{C_G(g)} \) with \( K(X^g/C_G(g)) \).

**Theorem 2.7.** The map \( \text{char}_G \) is an isomorphism of complex vector spaces.

For a proof of the theorem see [ASS99].

2.1. The induced bundle. Now we will give an explicit construction of the induced vector bundle. It is a direct generalization of the induced representation defined for example in Section 3.3 in [Ser77].

Let \( H \subseteq G \) be a subgroup of \( G \) and \( E \to X \) an \( H \)-vector bundle over a \( G \)-space \( X \). If we choose an element from each left coset of \( H \), we obtain a subset \( R \) of \( G \) called a system of representatives of \( H \setminus G \); each \( g \in G \) can be written uniquely as \( g = sr \), with \( r \in R = \{ r_1, \ldots, r_n \} \) and \( s \in H \), \( G = \bigsqcup_{i=1}^{n} Hr_i \), we suppose that \( r_1 = e \) the identity of the group \( G \). Consider the vector bundle \( F = \bigsqcup_{i=1}^{n} (r_i)^*E \), with projection \( \pi_F : F \to X \) and consider the following \( G \)-action defined over \( F \):

Let \( f \in F \), then
\[
f = f_{r_1} + \cdots + f_{r_n},
\]
where \( f_{r_i} \in (r_i)^*E \). If \( \pi_F(f) = x \) then \( f_{r_i} = (x, e) \), where \( e \in E_{r_i} \). Let \( g \in G \), note that \( r_ig^{-1} \) is in the same left coset of some \( r_j \), i.e. \( r_ig^{-1} = sr_j \), for some \( s \in H \). Define
\[
g(f_{r_i}) = (gx, s^{-1}e) \in (r_j)^*E,
\]
and define the action of $g$ on $f$ by linearly.

Now we will see that $F$ does not depend on the set of representatives up to isomorphism. Let \( \{ r'_1, \ldots, r'_n \} \) be another set of representatives of $H\setminus G$ and let $F' = \bigoplus_{i=1}^n (r'_i)^*E$. By reordering we can assume that $r_i$ and $r'_i$ are in the same left coset, then $r'^{-1}_ir^{-1}_i \in H$.

We have an isomorphism of vector bundles over $X$

\[
(r'_i r^{-1}_i)(x,e) \rightarrow (x, r'^{-1}_i r^{-1}_i e)
\]

inducing an isomorphism of $G$-vector bundles

\[
(r'_i r^{-1}_i)^{-1} \oplus \ldots \oplus (r'_n r_n^{-1})^{-1} : F \rightarrow F'.
\]

We only need to verify that this map commutes with the action of $G$. To see this, let $g \in G$ and $f_{r_i} = (x,e) \in (r_i)^*E$, there exist $s,s' \in H$ such that

\[
(2.8) \quad r_i^*g^{-1} = sr_j \quad \text{and} \quad r'^{-1}_i g^{-1} = s'r'_j.
\]

Note that $g f_{r_i} \in (r_j)^*E$, then

\[
(r'_j r^{-1}_j)g (f_{r_i}) = (gx, r'_j r^{-1}_j s^{-1}e).
\]

On the other hand

\[
g (r'_i r^{-1}_i f_{r_i}) = (gx, (s')^{-1}(r'^{-1}_i r^{-1}_i e)),
\]

but we know from (2.8)

\[
(s')^{-1} r'_i r^{-1}_i = r'_j r^{-1}_j s^{-1}.
\]

Then the map $r'^{-1}_i r^{-1}_i \oplus \ldots \oplus r'^{-1}_n r_n^{-1}$ commutes with the $G$-action and then $F$ and $F'$ are isomorphic as $G$-vector bundles. We will denote the $G$-vector bundle $F$ defined above by $\text{Ind}^G_H(E)$. Summarizing we have.

**Theorem 2.9.** Let $G$ be a finite group, let $H \subseteq G$ be a subgroup. Let $X$ be a $G$-CW-complex, and let $E$ be a $H$-vector bundle over $X$, there is a unique $G$-vector bundle $\text{Ind}^G_H(E)$ over $X$, up to isomorphism of $G$-vector bundles such that for every $G$-vector bundle $F$ over $X$ we have a natural identification

\[
\text{Hom}_G(\text{Ind}^G_H(E), F) \cong \text{Hom}_H(E, \text{res}^G_H(F)).
\]

**Proof.** Only remains to prove the identification. Let $\xi \in \text{Hom}_G(\text{Ind}^G_H(E), F)$, recall that we have an inclusion of $H$-vector bundles

\[
E \rightarrow \text{Ind}^G_H(E)
\]

\[
v \in E_x \mapsto (x,v).
\]

Define $r(\xi) \in \text{Hom}_H(E, \text{res}^G_H(F))$ as follows, if $v \in E_x$

\[
(r(\xi))(v) = \xi(x,v).
\]

It is clear that $r(\xi) \in \text{Hom}_H(E, \text{res}^G_H(F))$. On the other hand if $\eta \in \text{Hom}_H(E, \text{res}^G_H(F))$, define $I(\eta) : \text{Ind}^G_H(E) \rightarrow F$ as follows, if $v_i \in r_i^*E$, then $v_i = (x,v)$ with $x \in X$ and $v \in E_{r_i x}$, then we define

\[
I(\eta)(v_i) = r_i^{-1}(\eta(v)).
\]

Extending linearly $I(\eta)$ to $\text{Ind}^G_H(E)$.

Now we will see that $I(\eta)$ is $G$-equivariant. Let $g \in G$, let $s \in H$ such that

\[
r_g^{-1} = sr,
\]
then \[ g \cdot v_i = (gx, s^{-1}v). \]

Now,
\[
I(\eta)(g \cdot v_i) = I(\eta)(gx, s^{-1}v) \\
= r_j^{-1}(\eta(s^{-1}v)) \\
= r_j^{-1}s^{-1}\eta(v) \\
= gr_i^{-1}\eta(v) \\
= gI(\eta)(v_i).
\]

Then \( I(\eta) \in \text{Hom}_G(\text{Ind}_H^G(E), F) \). Now we will see that \( r \) and \( I \) are inverse of each other. It is clear that \( r(I(\eta)) = \eta \). On the other hand,
\[
I(r(\xi))(v_i) = r_i^{-1}(r(\xi)(v)) \\
= r_i^{-1}\xi(rx, v) \\
= \xi(x, v) \\
= \xi(v_i).
\]

We have a formula for the character of an induced \( H \)-vector bundle, it is a particular case of a formula for induced character of generalized cohomology theories in [HKR00] and [Kuh89]. We include a proof for completeness.

**Theorem 2.10** (Formula for the induced character). Let \( X \) be a \( G \)-CW-complex, let \( H \) be a subgroup of \( G \), let \( h \) be the order of \( H \) and \( E \) be a \( H \)-vector bundle, consider the map
\[
\text{char}_G \circ \text{Ind}^G_H : K_H(X) \otimes \mathbb{C} \to \bigoplus_{[g]} K(X^g)^{C_G(g)} \otimes \mathbb{C},
\]
let \( R \) be a system of representatives of \( H \setminus G \). For each \( g \in G \), we have
\[
\text{char}_G(g) \circ \text{Ind}^G_H([E]) = \bigoplus_{r \in R, r^{-1}gr \in H} r^* \left( \text{char}^H(r^{-1}gr)([E]) \right)
= \frac{1}{h} \bigoplus_{r \in G, r^{-1}gr \in H} r^* \left( \text{char}^H(r^{-1}gr)([E]) \right).
\]

**Proof.** Our explicit definition of the induced bundle allows us to prove this result just by adapting the proof for representations contained in [Ser77]. The vector bundle \( F = \text{Ind}^G_H(E) \) is the direct sum \( \bigoplus_{i=1}^n r_i^* E \), with \( R = \{r_1, \ldots, r_n\} \).

\( H \setminus G = \{Hr_1, \ldots, Hr_n\} \).

We know from the definition of the induced bundle that if we write \( r_i g^{-1} \) in the form \( sr_j \) with \( r_j \in R \) and \( s \in H \), then \( g \) sends \( r_i^* E \) to \( r_j^* E \). Considering the action of \( g \) in \( H \setminus G \), we have that
\[
\text{char}_G(g) \left( \text{Ind}^G_H(E) \right) = \text{char}_G(g) \left( \bigoplus_{Hr_i = Hr_j g^{-1}} r_i^* E \oplus \bigoplus_{Hr_i \neq Hr_j g^{-1}} r_i^* E \right)
\]
A matrix $\rho \in G$ if $\rho$ acts in $G$.

This section is dedicated to describe the conjugacy classes and centralizers of elements in $G$. We will see that $\text{char}(r_i E) = \text{char}(s_i E)$. Finally as the character commutes with pullbacks we have that,

$$\text{char}_G(g)(\text{Ind}_H^G(E)) = \bigoplus_{r \in R, rgr^{-1} \in H} r_i^*(\text{char}_H(rgr^{-1})(E)) = \frac{1}{h} \bigoplus_{s \in G, sg^{-1} \in H} r_i^*(\text{char}_H(s^{-1}hs)(E)) .$$

\square

3. Wreath product and its action on $X^n$

Let $C$ be a set. There is a natural action of $\mathfrak{S}_n$ on $C^n$ defined as

$$\sigma \bullet (c_1, \ldots, c_n) = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})$$

if $G$ is a group, we define the wreath product as the semidirect product

$$G_n = G \wr \mathfrak{S}_n = G^n \rtimes \mathfrak{S}_n .$$

This section is dedicated to describe the conjugacy classes and centralizers of elements in $G_n$, we follow [Mac15, Chapter 1, App. B] and [Van00].

First, we must recall the conjugacy classes in $\mathfrak{S}_n$. Two elements $s_1, s_2 \in \mathfrak{S}_n$ are conjugate if their cycle factorization correspond to the same partition of $n$. For example the elements $(1, 2)(3, 4, 5)$ and $(1, 4)(2, 3, 5)$ are conjugated and correspond to the partition $5 = 2 + 3$. Note that every partition of $n$ can be view as a function $m : \{1, 2, \cdots, n\} \to \mathbb{N}$ as follows. If $s \in \mathfrak{S}_n$ then $m_s(r)$ is the number of $r$-cyces in $s$. Now in the general case, if $x = (g, s) \in G_n$, then $s$ can be decomposed as a product of disjoint cycles, if $z = (i_{112} \cdots i_r)$ is one of these cycles, the element $g_{i_1} g_{i_{r-1}} \cdots g_{i_1}$ is called the cycle product of $x$ corresponding to $z$.

Recall that $G_\ast$ denotes the set of conjugacy classes of $G$. If $x = (g, s) \in G_n$, let $\rho(x) = m_x(r, c)$ denote the number of $r$-cycles in $s$ whose cycle product belongs to $c$, where $c \in G_\ast$ and $r \in \{1, 2, \cdots n\}$. In this way every element $x \in G_n$ determines a matrix $\rho(x) = m_x(r, c)$ of non-negative integers such that $\sum_{r,c} m_x(r, c) = n$. The matrix $\rho(x)$ is called the cycle product of $x$. Note that $x \in G_\ast$ is one of these cycles, the element $g_{i_1} g_{i_{r-1}} \cdots g_{i_1}$ is called the cycle product of $x$ corresponding to $z$. If $x = (g, s) \in G_n$, let $\rho(x) = m_x(r, c)$ denote the number of $r$-cycles in $s$ whose cycle product belongs to $c$, where $c \in G_\ast$ and $r \in \{1, 2, \cdots n\}$. In this way every element $x \in G_n$ determines a matrix $\rho(x) = m_x(r, c)$ of non-negative integers such that $\sum_{r,c} m_x(r, c) = n$. The matrix $\rho(x)$ is called the cycle product of $x$. Note that $x \in G_\ast$ is one of these cycles, the element $g_{i_1} g_{i_{r-1}} \cdots g_{i_1}$ is called the cycle product of $x$ corresponding to $z$.
For example let $G$ be the cyclic group, \{\(g^0, g^1, g^2, g^3\)\}, of 4 elements generated by $g$ and $s = (1, 2)(3, 4, 5) \in S_5$. If $x = (g, g, g, g, g, s)$ then $\rho(x) = m_x(r, c)$ looks like

\[
\begin{array}{cccc}
  r = 1 & g^0 & g^1 & g^2 & g^3 \\
  r = 2 & 0 & 0 & 1 & 0 \\
  r = 3 & 0 & 0 & 0 & 1 \\
  r = 4 & 0 & 0 & 0 & 0 \\
  r = 5 & 0 & 0 & 0 & 0 \\
\end{array}
\]

the map $\rho : G \to M_{n,v}(\mathbb{Z})$ is called the type of $x \in G_n$, where $v = |G|$.

**Proposition 3.1.** Two elements in $G_n$ are conjugate iff they have the same type.

**Proof.** See [Mac15, Appendix 1.B] □

By the above proposition we can assume that every element $x \in G_n$ is conjugated to a product of elements of the form

\[((g, 1, \ldots, 1), (i_{u_1}, \ldots, i_{u_r})).\]

Denote by $g_r(c) = ((g, 1, \ldots, 1), (1, \ldots, r))$.

**Proposition 3.2.** The elements in the centralizer $C_{G_n}(g_n(c))$ are of the form

\[((g z, \ldots, z, g z), (1, \ldots, n)^k), \text{ with } z \in C_G(g). \text{ Moreover } C_{G_n}(g_n(c)) \cong C_G(g) \times \langle (1, \ldots, n) \rangle.\]

**Proof.** It follows from a direct computation. □

We have described centralizers of elements in $G_n$ and in the next section we will use this description to write $\text{char}_{G_n}$ in terms of $\text{char}_G$.

Let $X$ be a $G$-space, there is canonical $G_n$-action over $X^n$ defined from the $G$-action over $X$

\[
G_n \times X^n \to X^n
\]

\[
((\tilde{g}, \sigma), \tilde{x}) \mapsto \tilde{g}(\sigma \cdot \tilde{x})
\]

where $\tilde{g}$ acts component-wise.

In order to relate $\text{char}_{G_n}$ with $\text{char}_G$ we need to describe the fixed point set of a representative of each conjugacy class of $G_n$. Let us start with the conjugacy classes of elements $(\tilde{g}, \sigma)$ where $\sigma$ is an $n$-cycle. To this end we will need the following result.

**Proposition 3.3.** Let $\zeta = ((g, 1, \ldots, 1), \sigma)$ with $g \in G$ and $\sigma$ is a $n$-cycle. There is a canonical homeomorphism

\[
(X^n)^{\zeta} / C_{G_n}(\zeta) \cong X^g / C_G(g).
\]

**Proof.** We can assume $\sigma = (1, \ldots, n)$. Let $(x_1, \ldots, x_n) \in (X^n)^{\zeta}$, then

$\zeta(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$

it implies

$\tilde{g}(x_n, x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_n)$. 


Therefore

\[ x_n = x_{n-1} = \cdots = x_1, \quad gx_n = x_1, \]

and then \( (x_1, \ldots, x_n) = (y, \ldots, y) \) lies in the diagonal and \( y \in X^g \). This proves that \( (X^n)^\mathbb{Z} \cong X^g \). On the other hand, if \( \bar{b} \in C_{G_n}(\zeta) \) then by Proposition 3.2

\[ \bar{b} = ((gz, \ldots, z_{k+1}, \ldots, g), \sigma^k) \]

where \( z \in C_G(g) \). Then we obtain

\[ \bar{b}(y, \ldots, y) = ((gz, \ldots, z_{k+1}, \ldots, g), \sigma^k) \cdot (y, \ldots, y) = (gzy, \ldots, zy, \ldots, gzy), \]

showing that the orbit of \( (y, \ldots, y) \) by \( C_{G_n}(\zeta) \) is

\[ \{(gzy, \ldots, zy, \ldots, gzy) : z \in C_G(g)\}. \]

This proves the result. \( \square \)

4. Fock space

Let \( X \) be a \( G \)-space, by the equivariant Bott periodicity theorem we know that \( K^*_G(X) = K^0_G(X) \oplus K^1_G(X) \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded group. Denote by

\[
\mathcal{F}_G(X) = \bigoplus_{n \geq 0} K^*_G(X^n) \otimes \mathbb{C}, \quad \mathcal{F}_G^q(X) = \bigoplus_{n \geq 0} q^n K^*_G(X^n) \otimes \mathbb{C}
\]

where \( q \) is formal variable giving a \( \mathbb{Z}_+ \)-grading in \( \mathcal{F}_G^q(X) \). They both have a natural structure of abelian groups, we endow them with a product \( \cdot \), defined as the composition of the induced bundle and the Künneth isomorphism \( \boxtimes \) (see [Min69])

\[
K^*_G(X^n) \times K^*_G(X^m) \xrightarrow{\boxtimes} K^*_G(X^{n+m}) \xrightarrow{\text{Ind}} K^*_G(X^{n+m})
\]

**Proposition 4.1.** With the above operations \( \mathcal{F}_G^q(X) \) is a commutative \( (\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z}) \)-graded ring.

Proof. The associativity follows from the following fact. Let \([E_1] \in K^*_G(X^n), [E_2] \in K^*_G(X^m)\) and \([E_3] \in K^*_G(X^k)\), then

\[
(E_1 \cdot E_2) \cdot E_3 \cong \text{Ind}_{G_{n+m+k}}^{G_{m+k}} \left( \text{Ind}_{G_n \times G_m}^{G_{n+m}} (E_1 \boxtimes E_2 \boxtimes E_3) \right)
\]

\[
\cong \text{Ind}_{G_{n+m+k}}^{G_{m+k}} \left( E_1 \boxtimes \text{Ind}_{G_m \times G_m}^{G_{m+k}} (E_2 \boxtimes E_3) \right) \cong \text{Ind}_{G_{n+m+k}}^{G_{m+k}} \left( E_1 \boxtimes \text{Ind}_{G_{n+m+k}}^{G_{m+k}} (E_2 \boxtimes E_3) \right).
\]

For the graded commutativity, let \([E_1] \in K^*_G(X^n)\) and \([E_2] \in K^*_G(X^m)\), we will prove that \( E_1 \cdot E_2 \) and \( E_2 \cdot E_1 \) has the same character as \( G_{n+m} \)-vector bundles over \( X^{n+m} \).

Consider two inclusions of \( \mathfrak{S}_n \) into \( \mathfrak{S}_{n+m} \). The first one is the inclusion by the first \( n \) letters denoted by \( S_n \xrightarrow{i_1} S_{n+m} \), the second one is the inclusion by the last \( n \) letters denoted by \( S_n \xrightarrow{i_2} S_{n+m} \). Let \( x = (\bar{g}, \sigma) \in G_{n+m} \) and let \( r = (\bar{h}, \tau) \in G_{n+m} \), such that \( r^{-1}x\tau = (\bar{g}, \sigma) \), then there is \( \eta_1 \in \mathfrak{S}_n \) and \( \eta_2 \in \mathfrak{S}_m \) such that \( \tau^{-1}\sigma\tau = i_1(\eta_1)i_2(\eta_2) \).
to \( i_1(\eta_2)i_2(\eta_1) \), then there is \( \gamma \in \mathfrak{S}_{n+m} \) such that \( \gamma^{-1}(\tau^{-1}\sigma\tau)\gamma = i_1(\eta_2)i_2(\eta_1) \).

Then

\[
((e, \ldots, e), \gamma^{-1}(r^{-1}xr)((e, \ldots, e), \gamma) \in G_m \times G_n.
\]

It implies that we have bijective correspondence between elements \( r \in G_{n+m} \) such that \( r^{-1}xr \in G_n \times G_m \) and elements \( s \in G_{n+m} \) such that \( s^{-1}xs \in G_m \times G_n \). Then if we apply Theorem 2.10, the number of terms in each direct sum computing

\[
\text{Ind}_{G_m \times G_n}^{G_{n+m}}(E_1 \boxtimes E_2) \text{ and } \text{Ind}_{G_m \times G_n}^{G_{n+m}}(E_2 \boxtimes E_2)
\]

are the same. Moreover, for every \((\alpha, \beta) \in G_n \times G_m\),

\[
\text{char}_{G_m \times G_n}((\alpha, \beta))(E_1 \boxtimes E_2) \cong \text{char}_{G_m \times G_n}((\beta, \alpha))(E_2 \boxtimes E_1),
\]

then we have \( E_1 \cdot E_2 \) and \( E_2 \cdot E_1 \) have the same character, then the product is commutative. The other properties follows directly.

\[\square\]

**Definition 4.2.** Let \( R \) be a commutative ring, the graded-symmetric algebra of a \( \mathbb{Z} \)-graded \( R \)-module \( M \) (denoted by \( \mathcal{S}(M) \)) is the quotient of the tensor algebra of \( M \) by the ideal \( I \) generated by elements of the form

1. \( x \otimes y - (-1)^{\deg(x)\deg(y)}(y \otimes x) \)
2. \( x \otimes x \), when \( \deg(x) \) is even.

Now we will give another proof of the description of \( \mathcal{F}_G^{I}(X) \) as a graded-symmetric algebra given in [Seg96] or Theorem 3 in [Van00].

**Theorem 4.3.** There is an isomorphism of \((\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})\)-graded algebras

\[
\Phi : \mathcal{F}_G^{I}(X) \rightarrow \mathcal{S}(\oplus_{n \geq 1}q^n K_G^*(X) \otimes \mathbb{C}).
\]

**Proof.** First note that using char\(_n\) we can define an injective group homomorphism in the following way. Consider the following sequence of maps

\[
K_G^*(X) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{g \in G_n} K^*(X)^{C_G(g)} \otimes \mathbb{C} \xrightarrow{\lambda} \bigoplus_{x \in G_{n^i}} K^*((X^n)^x)^{C_G(x)} \otimes \mathbb{C} \xrightarrow{\cong} K_G^*(X^n) \otimes \mathbb{C}
\]

where the map \( \lambda \) is given by the assigning \( [((g, 1, \ldots, 1), (1, \ldots, n))_G] \) to the conjugacy class \([g]_G\) and using the identification in Proposition 3.3. This map is certainly injective. Define

\[
\phi : K_G^*(X) \otimes \mathbb{C} \rightarrow K_G^*(X^n) \otimes \mathbb{C}
\]

by the composition of the above sequence so that \( \phi \) is injective and by the universal property of the graded-symmetric algebra we have a unique map

\[
\Phi : \mathcal{S} \left( \bigoplus_{n \geq 1} K_G^*(X) \otimes \mathbb{C} \right) \rightarrow \mathcal{F}_G^{I}(X)
\]

extending \( \phi \).

Suppose inductively that \( \text{im}(\Phi) \) contains \( K_G^*(X^k) \otimes \mathbb{C} \) for \( k < n \). Then by induction we know that the image of the following composition

\[
\mathcal{S} \left( \oplus_{n \geq 1}q^n K_G^*(X) \otimes \mathbb{C} \right) \times \mathcal{S} \left( \oplus_{n \geq 1}q^n K_G^*(X) \otimes \mathbb{C} \right) \xrightarrow{\Phi \times \Phi} \mathcal{F}_G^{I}(X) \times \mathcal{F}_G^{I}(X) \rightarrow \mathcal{F}_G^{I}(X)
\]

contains

\[
K_G^*(X^k) \otimes \mathbb{C} \cdot K_G^*(X^{n-k}) \otimes \mathbb{C} \subseteq K_G^*(X) \otimes \mathbb{C}.
\]

Now we have that the image under char\(_n\) of
\[
\bigoplus_{k=1}^{n-1} (K^*_{G_k}(X^k) \otimes \mathbb{C}) \cdot (K^*_{G_{n-k}}(X^{n-k}) \otimes \mathbb{C})
\]
cointides with
\[
\bigoplus_{x \in J} K^*((X^n)^x) / C_{G_n}(x) \otimes \mathbb{C},
\]
where \( J \) is the set of conjugacy classes in \( G_n \) such that for every \( c \), \( m_n(n, c) = 0 \), in other words \( J \) is the set of conjugacy classes whose components in \( S_n \) are not an \( n \)-cycle.

On the other hand if \( x = ((g, 1, \ldots, 1), (1, \ldots, n)) \) for some \( g \in G \), then Proposition \( \ref{prop:conj_classes} \) gives us that \( \text{im}(\text{char}_{G_n} \circ \Phi) \) contains \( K^*((X^n)^x) / C_{G_n}(x) \). Finally, since \( \text{char}_{G_n} \) is an isomorphism we can conclude that \( \Phi \) is surjective.

To see that \( \Phi \) is injective we can use the formula for the induced character, because this formula implies that if \( A \in S(\bigoplus_{k \geq 1} \Phi^* K^*_G(X) \otimes \mathbb{C}) \) is not zero then there exists \( n \) and \( x \in G_n \) such that \( \text{char}_{G_n}(x)(\Phi(A)) \neq 0 \), then \( \Phi(A) \neq 0 \).

\[\square\]

5. Pullback of groups

Let \( \Gamma \) be a group fitting into the following pullback diagram

\[
\begin{array}{ccc}
\Gamma & \to & G \\
p_2 & \downarrow & \downarrow \pi_2 \\
\downarrow p_1 & & \downarrow \pi_1 \\
H & \to & K
\end{array}
\]

If the group \( \Gamma \) comes from a diagram \( \ref{diag:pullback} \) then it is isomorphic to a subgroup of \( G \times H \), namely \( \Gamma \cong \{(g, h) \in G \times H \mid \pi_1(g) = \pi_2(h)\} \). When maps \( \pi_1 \) and \( \pi_2 \) are clear from the context we denote \( \Gamma \) by \( G \times_K H \). We suppose that \( \pi_1 \) and \( \pi_2 \) are surjective.

In this section we describe the class function ring of \( \Gamma \) in terms of the class function rings of \( G \), \( H \) and \( K \). In order to obtain this description we need that \( \Gamma \) satisfies the following condition.

**Definition 5.2.** Let \( G \) be a finite group and let \( H \subseteq G \) be a subgroup, let \([h]_H \in H_+\), we say that \([h]_H \) is closed in \( G \) if,

\[ [h]_H = [h]_G \cap H. \]

We say that \( H \) is conjugacy-closed in \( G \) if, for every \( h \in H \), \([h]_H \) is closed in \( G \).

**Example 5.3.** The following are examples of conjugacy-closed subgroups:

- The general linear groups over subfields are conjugacy-closed.
- The symmetric group is conjugacy-closed in the general linear group.
- The symmetric group on subsets are conjugacy-closed.
- The orthogonal group is conjugacy-closed in the general linear group over real numbers.
- The unitary group is conjugacy-closed in the general linear group.

**Remark 5.4.** Let \( G \) and \( H \) be groups
• If $H \subseteq G$ is conjugacy-closed in $G$, then the pullback of the inclusion
  
  $$i^* : \text{Class}(G) \to \text{Class}(H)$$
  
  is surjective.

• If $H$ is a retract in $G$, the pullback of the inclusion
  
  $$i^* : R(G) \to R(H)$$
  
  is surjective.

When $\Gamma$ is conjugacy-closed in $G \times H$, we have a way to express the class function ring of $\Gamma$ in terms of the class function rings of $G$, $H$ and $K$. The same is true for the representations ring when $\Gamma$ is a retract of $G \times H$.

5.1. The class function ring of a pullback. Consider a pullback diagram of finite groups such as (5.1). If we apply the representation ring functor we obtain the following diagram

$$(5.5) \quad \begin{array}{ccc}
R(\Gamma) & \xrightarrow{p_2^*} & R(G) \\
\downarrow{p_1^*} & & \downarrow{p_2^*} \\
R(H) & \xrightarrow{\pi_1^*} & R(K)
\end{array}$$

This diagram endows the rings $R(G)$ and $R(H)$ with a $R(K)$-module structure. A similar statement is true changing the representation ring by the class function ring. We will prove that if $\Gamma$ is a retract of $G \times H$, then the diagram (5.5) is a pushout. In fact, we have the following theorem.

**Theorem 5.6.** Let $\Gamma$, $G$, $H$ and $K$ be finite groups such as in the diagram (5.1). If $\Gamma$ is conjugacy-closed in $G \times H$, there is an isomorphism

$$m : \text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H) \to \text{Class}(\Gamma)$$

of $\text{Class}(K)$-modules

Moreover, if $\Gamma$ is a retract of $G \times H$, we have an isomorphism

$$f : R(G) \otimes_{R(K)} R(H) \to R(\Gamma)$$

of $R(K)$-modules.

**Proof.** In order to avoid confusion, in this proof we denote the product on $\text{Class}(\Gamma)$, $\text{Class}(G)$ and $\text{Class}(H)$ by $\cdot$ and the generators of the tensor product by $\rho \otimes \gamma$.

The map $f$ is defined as

$$f : \text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H) \to \text{Class}(\Gamma)$$

$$\rho \otimes \gamma \mapsto p_1^* (\rho) \cdot p_2^* (\gamma)$$

First we prove that the map $f$ is well defined. Let $\xi \in \text{Class}(K)$, $\rho \in \text{Class}(G)$ and $\gamma \in \text{Class}(H)$. Let $(g, h) \in \Gamma$

$$f (\pi_1^* (\xi) \cdot \rho \otimes \gamma) (g, h) = (p_1^* (\pi_1^* (\xi) \cdot \rho) \cdot p_2^* (\gamma)) (g, h)$$

$$= (\pi_1^* (\xi) \cdot \rho) (g) \gamma (h)$$

$$= \xi (\pi_1 (g)) \rho (g) \gamma (h)$$

$$= \rho (g) \xi (\pi_2 (h)) \gamma (h)$$

$$= f (\rho \otimes \pi_2^* (\xi) \cdot \gamma) (g, h).$$
Now we will prove that $f$ is an isomorphism. Consider the following diagram with exact rows

$$
\begin{array}{cccccccc}
0 & \rightarrow & \ker(\pi) & \rightarrow & \text{Class}(G) \otimes_\mathbb{C} \text{Class}(H) & \rightarrow & \pi(\text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H)) & \rightarrow & 0 \\
& & \downarrow f_2 & & \downarrow f_1 & & \downarrow f & & \\
0 & \rightarrow & \ker(i^* \pi) & \rightarrow & \text{Class}(G \times H) & \rightarrow & i^* \rightarrow \text{Class}(\Gamma) & \rightarrow & 0.
\end{array}
$$

Where map $\pi$ is the quotient by the relations defining tensor product over $\text{Class}(K)$, map $i^*$ is the pullback of the inclusion $i : \Gamma \rightarrow G \times H$, map $f_1$ is the natural isomorphism given by tensor product over $\mathbb{C}$ and the map $f_2$ is the restriction of $f_1$ to $\ker(\pi)$. Note that as $\Gamma$ is closed conjugacy in $G \times H$ the map $i^*$ is surjective. We will prove that above diagram is commutative and that $f_2$ is an isomorphism.

First we need to verify that $f_1(\ker(\pi)) \subseteq \ker(i^*)$. Let $(g, h) \in \Gamma$,

$$i^*[f_1(\pi_1(\xi) \cdot \rho \otimes \gamma - \rho \otimes \pi_2(\xi) \cdot \gamma)](g, h) = [p_2^*(\pi_1(\xi)) \cdot p_1^*(\rho) \cdot p_2^*(\gamma) - p_1^*(\rho) \cdot p_2^*(\pi_2(\xi)) \cdot p_2^*(\gamma)](g, h) = 0$$

Now we prove that $\ker(i^*) = f_1(\ker(\pi))$. For this we will prove that if $f$ is a class function in $G \times H$ such that $i^*(f) \equiv 0$ and $f$ is orthogonal to every element in $f_1(\ker(\pi))$, then $f$ has to be zero.

Suppose that for every $\xi \in \text{Class}(K)$, $\rho \in \text{Class}(G)$ and $\gamma \in \text{Class}(H)$ we have

$$\sum_{(g, h) \in G \times H} \bar{f}(g, h) \rho(g) \gamma(h) [\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0.$$

Let us fix $\rho \in \text{Class}(G)$ and let

$$\eta(g) = \sum_{h \in H} \bar{f}(g, h) \rho(g) \gamma(h) [\xi(\pi_2(h)) - \xi(\pi_1(g))].$$

We observe that $\eta$ is a class function on $G$ that is orthogonal to every $\rho$ in $\text{Class}(G)$, then $\eta \equiv 0$.

By a similar argument we conclude that for every $(g, h) \in G \times H$ and $\xi \in \text{Class}(K)$

$$\bar{f}(g, h)[\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0.$$  \hfill (5.7)

We already know that $f(g, h) = 0$ if $(g, h) \not\in \Gamma$, then let $(g, h) \not\in \Gamma$, we have two cases. First suppose that $\pi_1(g)$ is conjugate to $\pi_2(h)$ in $K$, in this case there is $h \in H$ such that $(g, h, h^{-1}) \not\in \Gamma$ and then $f(g, h) = f(g, h, h^{-1}) = 0$.

Suppose now that $\pi_1(g)$ is not conjugate to $\pi_2(h)$ in $K$, in this case there is $\xi \in \text{Class}(K)$ such that $\xi(\pi_1(g)) \neq \xi(\pi_2(g))$ and equation (5.7) gives us that $f(g, h) = 0$. Then we conclude that $\ker(i^*) = f_1(\ker(\pi))$. The map $f_2$ is an isomorphism because it is the restriction of $f_1$ and as the diagram is commutative we conclude that $f$ is $\text{Class}(K)$-module isomorphism.

When $\Gamma$ is a retract in $G \times H$, the same argument works changing characters by representations, in particular the map $i^* : R(G \times H) \rightarrow R(\Gamma)$ is surjective. \hfill \square

Observe that the pullback is not always conjugacy-closed in the product as the following example shows.
Example 5.8. Consider the pullback of the symmetric groups $\mathfrak{S}_3$ over the cyclic group $C_2$

$$\Gamma \rightarrow \mathfrak{S}_3 \rightarrow C_2$$

In this case the pullback $\Gamma$ has 6 conjugacy classes, \{\gamma_1, \cdots, \gamma_6\}. The product $\mathfrak{S}_3 \times \mathfrak{S}_3$ has 9 conjugacy classes, \{\chi_1 \cdots, \chi_9\}. Observe that the elements $((1, 2, 3), (1, 2, 3))$ and $((1, 2, 3), (1, 3, 2))$ are conjugate in the group $\mathfrak{S}_3 \times \mathfrak{S}_3$ by the element $(e, (1, 2))$ but they are not conjugate in $\Gamma$.

The pullback of the inclusion can be described in the class function ring as follows:

$$\text{Class}(\mathfrak{S}_3 \times \mathfrak{S}_3) \rightarrow \text{Class}(\Gamma)$$

$$\begin{align*}
\chi_1 &\mapsto \gamma_1 \\
\chi_2 &\mapsto \gamma_1 \\
\chi_3 &\mapsto \gamma_2 \\
\chi_4 &\mapsto \gamma_2 \\
\chi_5 &\mapsto \gamma_3 \\
\chi_6 &\mapsto \gamma_3 \\
\chi_7 &\mapsto \gamma_4 \\
\chi_8 &\mapsto \gamma_4 \\
\chi_9 &\mapsto \gamma_5 + \gamma_6.
\end{align*}$$

This map is not surjective.

Example 5.9. Consider the following pullback

$$\Gamma \rightarrow D_{12} \rightarrow \mathfrak{S}_3$$

In this case the pullback $\Gamma$ is isomorphic to the group $C_2 \times (C_3 \ltimes C_4)$ and it is conjugacy closed in the group $D_{12} \times (C_3 \ltimes C_4)$. According with GAP\cite{GAP2018} the group $D_{12}$ has group id (12,4) and generators $d_1, d_2$ and $d_3$ of orders 2, 2 and 3 respectively. The homomorphism $\psi_1$ is given by

$$\psi_1 : D_{12} \rightarrow \mathfrak{S}_3$$

$$d_1 \mapsto (2, 3)$$
$$d_2 \mapsto (1)$$
$$d_3 \mapsto (1, 2, 3).$$
The group $C_3 \times C_4$ has id $(12,1)$ and it is generated by three elements $g_1, g_2, g_3$ of orders 4, 2 and 3 respectively. The homomorphism $\psi_2$ is given by

$$\psi_2 : C_3 \times C_4 \rightarrow S_3$$

- $g_1 \mapsto (2,3)$
- $g_2 \mapsto (1)$
- $g_3 \mapsto (1,2,3)$.

For these groups the pullback $\Gamma$ is isomorphic to the group $C_2 \times (C_3 \times C_4)$ with group id $(24,7)$ and four generators $f_1, f_2, f_3, f_4$ of orders 4, 6, 2 and 2 respectively.

Applying Theorem 5.6 we obtain an isomorphism

$$\text{Class}(\Gamma) \cong \text{Class}(D_{12}) \otimes_{\text{Class}(S_3)} \text{Class}(C_3 \times C_4).$$

For more examples please see https://sites.google.com/site/combariza/research/pullbacks-with-kernels/.

### 6. Semidirect Product of a Direct Product

Let $A_1, A_2$ be groups with an action of a group $G$ by automorphisms noted by $a_2^g = g \cdot a_1$ for $a_1 \in A_1$ and $g \in G$. Note that $G$ acts also on the direct product $A_1 \times A_2$ by acting on each component, i.e. $(a_1, a_2)^g := (a_1^g, a_2^g)$. In this section we describe the semidirect product of a direct product as a pullback of two semidirect products and then, we apply this for the wreath product of a direct product which will allow us to compute the Fock ring of a product.

Consider the projections $\pi_i : A_i \times G \rightarrow G$ and the pullback $\Gamma$ associated

$$\begin{array}{ccc}
\Gamma & \longrightarrow & A_1 \times G \\
\downarrow & & \downarrow \\
A_2 \times G & \longrightarrow & G
\end{array}$$

**Proposition 6.1.** The pullback $\Gamma$ is isomorphic to the semidirect product $(A_1 \times A_2) \rtimes G$.

**Proof.** Note that the pullback is the subgroup of $(A_1 \times G) \times (A_2 \times G)$ given by

$$\Gamma = \{(a_1, g_1, a_2, g_2) \in (A_1 \times G) \times (A_2 \times G) : \pi_1(a_1, g_1) = \pi_2(a_2, g_2), a_1 \in A_1, g_1 \in G\}$$

that is, $g_1 = g_2$. Consider the bijective function $\phi : \Gamma \rightarrow (A_1 \times A_2) \rtimes G$ given by $\phi(a_1, g, a_2, g) = (a_1, a_2, g)$. On one hand

$$\phi((a_1, g, a_2, g) \cdot (b_1, h, b_2, h)) = \phi(a_1 b_1^g, gh, a_2 b_2^g, gh) = (a_1 b_1^g, a_2 b_2^g, gh).$$

On the other hand $(a_1, a_2, g) \cdot (b_1, b_2, h) = (a_1 b_1^g, a_2 b_2^g, gh)$ which shows that $\phi$ is a homomorphism of groups.

**Corollary 6.2.** Let $A, B$ be groups, there is an isomorphism

$$(A \times B)_n \cong A_n \times_{S_n} B_n.$$

Now we proof that certain conjugacy classes in $(A \times B)_n$ are closed in $A_n \times B_n$.

**Proposition 6.3.** Let $(\tilde{g}, \tilde{h}, \sigma) \in (A \times B)_n$, where $\sigma$ is an $n$-cycle. Then its conjugacy class in $A_n \times B_n$ is closed.
That means, we have a commutative diagram

\[ \begin{array}{c}
K \\
\downarrow d \\
S(X \times Y) \\
\end{array} \quad \begin{array}{c}
\pi_G \otimes \pi_H \\
\downarrow & d \otimes d \\
S(X) \otimes \mathcal{F}(\bullet) S(Y) \\
\end{array} \]

\[ \mathcal{F}_{G \times H}(X \times Y) \xrightarrow{\mathcal{K}} \mathcal{F}_G(X) \otimes \mathcal{F}(\bullet), \mathcal{F}_H(Y). \]

The map \( \mathcal{K} \) is compatible with the symmetric algebra decomposition in Thm. 4.3. That means, we have a commutative diagram

\[ \mathcal{F}_{G \times H}(X \times Y) \xrightarrow{\mathcal{K}} \mathcal{F}_G(X) \otimes \mathcal{F}(\bullet), \mathcal{F}_H(Y). \]

\[ \begin{array}{c}
S(X \times Y) \\
\downarrow d \\
S(X) \otimes \mathcal{F}(\bullet) S(Y) \\
\end{array} \]

Where \( S(X) \) stands for \( S(\bigoplus_{n \geq 0} K_G(X) \otimes \mathbb{C}) \), and similarly for \( S(Y) \) and \( S(X \times Y) \).

**Proof.** Let \( x = (\bar{g}_1, \bar{h}_1, \sigma_1) \) and \( y = (\bar{g}_2, \bar{h}_2, \sigma_2) \) be elements in \( (A \times B)_n \) that are conjugated in \( A_n \times B_n \), where \( \sigma_1 \) and \( \sigma_2 \) are \( n \)-cycles. We can suppose that

\[ \sigma_1 = \sigma_2 = (1 \cdots n). \]

Note that

\[ \begin{cases} 
1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{1,i}, \prod_{i=1}^n h_{1,i}) \in c \\
0 & \text{in any other case}
\end{cases} \]

and the type of \( y \) is given by

\[ \begin{cases} 
1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{2,i}, \prod_{i=1}^n h_{2,i}) \in c \\
0 & \text{in any other case}
\end{cases} \]

Then the types of \( x \) and \( y \) are equal, hence \( x \) and \( y \) are conjugated in \( (A \times B)_n \). \( \square \)

7. The Fock space of a product of spaces

In this section we apply results of Section 5 in order to obtain a decomposition of \( \mathcal{F}_{G \times H}(X \times Y) \) in terms of \( \mathcal{F}_G(X) \) and \( \mathcal{F}_H(Y) \). Let \( X \) be a \( G \)-space, we can endow to \( \mathcal{F}_G(X) \) with natural module structures as follows:

- Consider the trivial \( G_n \)-space \( \{ \bullet \} \), and the unique \( G_n \)-map \( \pi : X^n \rightarrow \{ \bullet \} \), then the pullback

\[ \pi^* : \text{Class}(G_n) \rightarrow K^*_G(X^n) \otimes \mathbb{C} \]

induces a \( \text{Class}(G_n) \)-module structure over \( K^*_G(X^n) \otimes \mathbb{C} \), hence we have a \( \mathcal{F}_G(\{ \bullet \}) \)-module structure over \( \mathcal{F}_G(X) \) defined componentwise.

- Note that we have a quotient map \( s : G_n \rightarrow \mathfrak{S}_n \), then the pullback

\[ (\pi \circ s)^* : \text{Class}(\mathfrak{S}_n) \rightarrow K_{\mathfrak{G}_n}(X^n) \otimes \mathbb{C} \]

induce a \( \text{Class}(\mathfrak{S}_n) \)-module structure over \( K_{\mathfrak{G}_n}(X^n) \otimes \mathbb{C} \), hence we have a \( \mathcal{F}(\{ \bullet \}) \)-module structure over \( \mathcal{F}_G(X) \) defined componentwise.

As we observe in Section 5, \( (G \times H)_n \) is not closed conjugacy in \( G_n \times H_n \), then we cannot expect a decomposition of \( \text{Class}((G \times H)_n) \) in terms of \( \text{Class}(G_n) \), \( \text{Class}(H_n) \) and \( \text{Class}(\mathfrak{S}_n) \), but as the conjugacy classes with an \( n \)-cycle as component in \( \mathfrak{S}_n \) are closed we have a decomposition of \( \mathcal{F}_{G \times H}(X \times Y) \) in terms of \( \mathcal{F}_G(X) \), \( \mathcal{F}_H(Y) \) and \( \mathcal{F}(\{ \bullet \}) \), with \( X \) a \( G \)-space and \( Y \) a \( H \)-space.

**Theorem 7.1.** There is an isomorphism of \( \mathcal{F}(\{ \bullet \}) \)-modules

\[ \mathcal{F}_{G \times H}(X \times Y) \xrightarrow{\mathcal{K}} \mathcal{F}_G(X) \otimes \mathcal{F}(\bullet), \mathcal{F}_H(Y). \]

The map \( \mathcal{K} \) is compatible with the symmetric algebra decomposition in Thm. 4.3. That means, we have a commutative diagram

\[ \begin{array}{c}
\mathcal{F}_{G \times H}(X \times Y) \\
\downarrow d \\
S(X \times Y) \\
\end{array} \quad \begin{array}{c}
\mathcal{F}_G(X) \otimes \mathcal{F}(\bullet) \mathcal{F}_H(Y). \\
\downarrow \pi_G \otimes \pi_H \\
S(X) \otimes \mathcal{F}(\bullet) S(Y) \\
\end{array} \]

Where \( S(X) \) stands for \( S(\bigoplus_{n \geq 0} K_G(X) \otimes \mathbb{C}) \), and similarly for \( S(Y) \) and \( S(X \times Y) \).
Proof. Let \( \{E_1, \ldots, E_m\} \) be a basis of \( K^*_G(X) \otimes \mathbb{C} \) as complex vector space and let \( \{F_1, \ldots, F_s\} \) be a basis of \( K^*_H(Y) \otimes \mathbb{C} \) as complex vector space. From the proof of Theorem \ref{thm:induced_character} we can conclude that

\[
\{ \Delta_{G,n,c,k} \in K^*_{G_n}(X^n) \otimes \mathbb{C} \mid n \geq 0, c \in G_*, 1 \leq k \leq m \}
\]
is a basis of \( \mathcal{F}_G(X) \) as \( \mathbb{C} \)-algebra, where

\[
\operatorname{char}_{G_n}(\Delta_{G,n,c,k})(\{g_1, \ldots, g_n\}, \sigma) = \begin{cases} E_k & \text{if } \prod_{i=1}^n g_{\sigma(i)} \in c \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case} \end{cases}
\]

In a similar way we define

\[
\{ \Delta_{H,n,d,l} \in K^*_H(Y^n) \otimes \mathbb{C} \mid n \geq 0, d \in H_*, 1 \leq l \leq s \}
\]
a basis of \( \mathcal{F}_H(Y) \) as \( \mathbb{C} \)-algebra, where

\[
\operatorname{char}_{H_n}(\Delta_{H,n,d,l})(\{h_1, \ldots, h_n\}, \sigma) = \begin{cases} F_l & \text{if } \prod_{i=1}^n h_{\sigma(i)} \in d \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case} \end{cases}
\]

Recall that we have an isomorphism

\[
K_{G \times H}(X \times Y) \otimes \mathbb{C} \xrightarrow{\cong} (K_G(X) \otimes K_H(Y)) \otimes \mathbb{C},
\]
given by the external tensor product. It is proved in [Min69] or can be obtained (for complex coefficients) directly from the character.

Using the above identification we have that

\[
\{ \Delta_{G \times H,n,c \times d,(k,l)} \mid n \geq 0, c \in G_*, d \in H_*, 1 \leq k \leq m, 1 \leq l \leq s \}
\]
is a basis as \( \mathbb{C} \)-algebra of \( \mathcal{F}_{G \times H}(X \times Y) \), where

\[
\operatorname{char}_{(G \times H)_n}(\Delta_{G \times H,n,c \times d,(k,l)})((\bar{g}, \bar{h}), \sigma) =
\begin{cases}
E_k \otimes F_l & \text{if } \prod_{i=1}^n g_{\sigma(i)} \in c, \prod_{i=1}^n h_{\sigma(i)} \in d \text{ and } \sigma \text{ is an } n\text{-cycle} \\
0 & \text{in any other case}
\end{cases}
\]

As the conjugacy classes when the character of the above elements is not zero is closed in \( G_n \times H_n \) we have

\[
\Delta_{G \times H,n,c \times d,(k,l)} = \Delta_{G,n,c,k} \cdot \Delta_{H,n,d,l}
\]
hence the map defined on generators as

\[
\Delta_{G \times H,n,c \times d,(k,l)} \mapsto \Delta_{G,n,c,k} \otimes \Delta_{H,n,d,l}
\]
is an isomorphism of \( \mathcal{F}(\bullet) \)-modules satisfying the required conditions. \( \square \)

8. Final remarks

In [Vel15] and [Vel19] a configuration space representing equivariant connective K-homology for finite groups was constructed. We recall the construction briefly.

**Definition 8.1.** Let \( G \) be a finite group and \((X, x_0)\) be a based \( G \)-connected, \( G \)-CW-complex. Let \( \mathcal{C}(X, x_0, G) \) be the \( G \)-space of configurations of complex vector spaces over \((X, x_0)\), defined as the increasing union, with respect to the inclusions \( M_n(\mathbb{C}[G]) \to M_{n+1}(\mathbb{C}[G]) \)

\[
\mathcal{C}(X, x_0, G) = \bigcup_{n \geq 0} \operatorname{Hom}^*(C_0(X), M_n(\mathbb{C}[G]))
\]
with the compact open topology. Notice that \( * \) refers to \(*\)-homomorphism, \( C_0(X) \) denotes the \( C^*\)-algebra of complex valued continuous maps vanishing at \( x_0 \) and \( \mathbb{C}[G] \) denotes the complex group ring.

We endow \( \mathcal{C}(X, x_0, G) \) with a continuous \( G \)-action as follows. If \( F \in \mathcal{C}(X, x_0, G) \), we define

\[
g \cdot F : C_0(X) \to M_n(\mathbb{C}[G])
\]

\[
f \mapsto g \cdot F(g^{-1} \cdot f).
\]

The space \( \mathcal{C}(X, x_0, G) \) can be described as the configuration space whose elements are formal sums

\[
\sum_{i=1}^n (x_i, V_i),
\]

when \( x_i \in X - \{x_0\} \) and \( V_i \subseteq C[X]^\infty \) such that if \( x_i \neq x_j \) then \( V_i \perp V_j \), subject to some relations, for details see [Vel19, Sec. 2.1]. We call the elements \( x_i \) the points and to the vector spaces \( V_i \) the labels.

**Remark 8.2.** When the based \( G \)-CW-complex \((X, x_0)\) is not supposed to be \( G \)-connected, we define the configuration space \( \mathcal{C}(X, x_0, G) = \Omega_0 \mathcal{C}(\Sigma X, x_0, G) \), where \( \Omega_0 \) denotes the based loop space and \( \Sigma \) denotes the reduced suspension.

That description allow us to define a Hopf space structure on \( \mathcal{C}(X, x_0, G) \) by putting together two configurations when labels in both of them are mutually orthogonal.

We have the following result:

**Theorem 8.3** (Thm. 5.2 in [Vel13]). Let \((X, x_0)\) be a based finite \( G \)-connected \( G \)-CW-complex. If we denote by \( k_n^G(X, x_0) \) the \( n \)-th \( G \)-equivariant connective \( K \)-homology groups of the pair \((X, x_0)\), then there is a natural isomorphism

\[
\pi_n(\mathcal{C}(X, x_0, G)^G) \xrightarrow{\cong} k_n^G(X, x_0).
\]

When a Hopf space \( \mathcal{H} \) is path-connected, consider the Hurewicz morphism

\[
\lambda : \pi_*(\mathcal{H}; \mathbb{C}) = \bigoplus_{n \geq 0} \pi_n(\mathcal{H}) \otimes \mathbb{C} \to H_*(\mathcal{H}; \mathbb{C}).
\]

We have the following result

**Theorem 8.4** (Thm. of the Appendix in [MM65]). If \( \mathcal{H} \) is a pathwise connected homotopy associative Hopf space with unit, and \( \lambda : \pi_*(\mathcal{H}; \mathbb{C}) \to H_*(\mathcal{H}; \mathbb{C}) \) is the Hurewicz morphism viewed as a morphism of \( \mathbb{Z} \)-graded Lie algebras, then it induces an isomorphism of Hopf algebras

\[
\tilde{\lambda} : S(\pi_*(\mathcal{H}; \mathbb{C})) \to H_*(\mathcal{H}; \mathbb{C}).
\]

Applying the above theorem to \( \mathcal{C}(X, x_0, G) \) we obtain.

**Corollary 8.5.** Let \( X \) be a finite \( G \)-CW-complex, if \( X \) is \( G \)-connected we have an isomorphism

\[
S(k_*^G(X, x_0) \otimes \mathbb{C}) \cong H_*(\mathcal{C}(X, x_0, G)^G; \mathbb{C}).
\]
In order to relate \( H_\ast(\mathcal{C}(X, x_0, G); \mathbb{C}) \) with \( \mathcal{F}_G(X) \) we need to recall the following result proved in Theorem 6.13 in [Vel15] using the equivariant Chern character obtained in [Luc02].

**Theorem 8.6.** Let \( X \) be a \( G \)-CW-complex. There is a natural isomorphism of \( \mathbb{Z} \)-graded complex vector spaces (here the graduation is given by \( q \))

\[
\bigoplus_{n \geq 0} k^G_n(X) \otimes \mathbb{C} \cong \bigoplus_{n \geq 0} K^G_n(X) \otimes \mathbb{C}[q].
\]

Finally we can relate \( H_\ast(\mathcal{C}(X, x_0, G); \mathbb{C}) \) with \( \mathcal{F}^G_q(X) \) when \( X \) is an even dimensional \( G \)-connected, \( G \)-\( \text{Spin}^c \)-manifold. First we recall Poincaré duality for equivariant K-theory.

**Theorem 8.7.** [BHS07] Let \( M \) be an \( n \)-dimensional \( G \)-\( \text{Spin}^c \)-manifold. Then there exists an isomorphism

\[
D : K^G_\ast(M_+) \rightarrow K^G_{n-\ast}(M).
\]

Applying Theorem 8.7 and Theorem 8.4 we can obtain the main result of the section.

**Theorem 8.8.** Let \( (M, m_0) \) be an even dimensional \( G \)-connected, \( G \)-\( \text{Spin}^c \)-manifold. We have an isomorphism of \( \mathbb{Z} \)-graded Hopf algebras

\[
H_\ast(\mathcal{C}(M, m_0, G); \mathbb{C}) \cong \mathfrak{g}_G^\ast(M).
\]

**Proof.** Since \( M \) is a \( G \)-\( \text{Spin}^c \) manifold we can use Theorem 8.7 and obtain the following isomorphism of \( \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-graded Hopf algebras

\[
S(k^G_\ast(M, m_0) \otimes \mathbb{C}) \cong S\left( \bigoplus_{n \geq 1} q^n K^G_\ast(M, m_0) \otimes \mathbb{C} \right)
\cong S\left( \bigoplus_{n \geq 1} q^n K^G_\ast(M_+, +) \otimes \mathbb{C} \right)
\cong S\left( \bigoplus_{n \geq 1} q^n K^G_\ast(M) \otimes \mathbb{C} \right).
\]

Combining Corollary 8.5, Theorem 8.4 and Theorem 4.3 we obtain

\[
H_\ast(\mathcal{C}(M, m_0, G); \mathbb{C}) \cong \mathfrak{g}_G^\ast(M).
\]

For the case when \( M \) is not necessarily \( G \)-connected, we can obtain also a similar result. For details consult [Vel15, Proposition 6.11].

**Proposition 8.9.** Let \( X \) be a finite \( G \)-CW-complex, we have an isomorphism

\[
H_\ast(\Omega \mathcal{C}(\Sigma X, G); \mathbb{C}) \cong S(k^G_\ast(X, x_0) \otimes \mathbb{C}).
\]

In particular we have.

**Example 8.10.** For \( X = S^0 \) we have

\[
\Omega \left( \mathcal{C}(\Sigma (S^0), G) \right) \simeq BU_G.
\]
Where $BU_G$ can be taken as the Grassmannian of finite dimensional vector subspaces of a complete $G$-universe. A complete $G$-universe is a countably infinite-dimensional representation of $G$ with an inner product such that contains a copy of every irreducible representation of $G$, contains countably many copies of each finite-dimensional subrepresentation. Applying the above discussion to this Hopf space we conclude that

$$H_*((BU_G)^G; \mathbb{C}) \cong R(G) \otimes S \left( \bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C} \right) \cong R(G) \otimes S \left( \bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C} \right).$$

Summarizing, we have an isomorphism

$$H_*((BU_G)^G; \mathbb{C}) \cong F^G_H(\{\bullet\}) = S \left( \bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C} \right).$$

We also have

$$H_*((BU_G)^G; \mathbb{C}) \cong S \left( \bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C} \right) \cong \mathbb{C}[\sigma_1^k, \ldots, \sigma^k_1, \sigma_2^k, \ldots].$$

where $\{\sigma_1^k, \ldots, \sigma^k_1\}$ is a complete set of non isomorphic irreducible representations of $G_i$. We expect that the elements $\sigma_i^k$ correspond in some sense with duals of $G$-equivariant Chern classes.

Now suppose that $M$ is a $G$-connected $G$-Spin$^c$-manifold and $N$ is a $H$-connected $H$-Spin$^c$-manifold, then we have an isomorphism of $\mathbb{Z}$-graded Hopf algebras

$$H_*((C(M \times N, (m_0, n_0), G \times H)^{G \times H}; \mathbb{C}) \cong F^G_M(\{\bullet\}) \otimes F^H_N(\{\bullet\}).$$

In the case that $M = N = S^0$ with trivial action we obtain

$$H_*((BU_{G \times H})^{G \times H}; \mathbb{C}) \cong F^G_M(\{\bullet\}) \otimes F^H_N(\{\bullet\}).$$

References

[AS89] M. F. Atiyah and Graeme Segal. On equivariant Euler characteristics. J. Geom. Phys., 6(4):671–677, 1989.

[Ati89] M. F. Atiyah. K-theory. Advanced Book Classics. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989. Notes by D. W. Anderson.

[BHS07] Paul Baum, Nigel Higson, and Thomas Schick. On the equivalence of geometric and analytic K-homology. Pure Appl. Math. Q., 3(1, part 3):1–24, 2007.

[GAP18] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.8.10, 2018.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. J. Amer. Math. Soc., 13(3):553–594, 2000.

[Kuh89] Nicholas J. Kuhn. Character rings in algebraic topology. In Advances in homotopy theory (Cortona, 1988), volume 139 of London Math. Soc. Lecture Note Ser., pages 111–126. Cambridge Univ. Press, Cambridge, 1989.
[Lüc02] Wolfgang Lück. Chern characters for proper equivariant homology theories and applications to $K$- and $L$-theory. *J. Reine Angew. Math.*, 543:193–234, 2002.

[Mac15] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144].

[Min69] Haruo Minami. A Künneth formula for equivariant $K$-theory. *Osaka J. Math.*, 6:143–146, 1969.

[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math.* (2), 81:211–264, 1965.

[Seg68] Graeme Segal. Equivariant $K$-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[Seg96] Graeme Segal. Equivariant $K$-theory and symmetric products. Preprint, 1996.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[tD87] Tammo tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

[Vel15] Mario Velásquez. A configuration space for equivariant connective $K$-homology. *J. Noncommut. Geom.*, 9(4):1343–1382, 2015.

[Vel19] Mario Velásquez. A description of the assembly map for the Baum-Connes conjecture with coefficients. *New York Journal of Mathematics*, 29:668–686, 2019.

[Wan00] Weiqiang Wang. Equivariant $K$-theory, wreath products, and Heisenberg algebra. *Duke Math. J.*, 103(1):1–23, 2000.