Asymptotic properties of a stochastic Gilpin-Ayala model under regime switching

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Abstract: In this paper, we consider a stochastic Gilpin-Ayala population model with Markovian switching and white noise. All parameters are influenced by stochastic perturbations. We analyze the existence of global positive solution, asymptotic stability in probability, pth moment exponential stability, extinction, weak persistence, stochastic permanence and stationary distribution of the model, which generalize the results in the literatures. Moreover, the conditions presented for the stochastic permanence and the existence of stationary distribution improve the previous results.

Keywords: Global positive solution; Weak persistence; Extinction; Stationary distribution

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1 Introduction

In order to describe the nonlinear rate change of the population size, Gilpin and Ayala (1973) [2] proposed a general logistic model, called GA model:

$$dx(t) = x(t)[a - bx^\theta(t)]dt,$$

where $\theta > 0$ denotes the parameter to modify the classical deterministic logistic model.

But in the real ecosystem, population systems are always influenced by stochastic environmental noise, which cannot be neglected for all population sizes. May (1973) [12] has revealed the fact that due to environmental noise, the birth rate, carrying capacity, competition coefficient and all other parameters involved in the system exhibit stochastic fluctuation to a greater or lesser extent. After that stochastic systems become more and more popular, and many authors have done some excellent works in this field. The pioneer work was due to Khasminskii (1980) [3], who established and studied an unstable system by using two white noise sources, his work opened a new chapter in the study of stochastic stabilisation. Mao et. al (2002) [10] presented an important claim that the environmental noise can suppress explosions in a finite time in population dynamics.

Consider the environmental noise in the birth rate and competition coefficient in GA model, one can obtain the following stochastic GA model:

$$dx(t) = x(t)[a - bx^\theta(t)]dt + \sigma_1 x(t)dB_1(t) + \sigma_2 x^{1+\theta}(t)dB_2(t).$$

(1)

Liu et. al (2012) [9] and Li (2013) [5] studied the stationary distribution, ergodicity and extinction of the model.

It was known that besides the white noise there is another type of environment noise in the real ecosystem, that is the telegraph noise, which can be demonstrated as a switching between two or more regimes of environment. The regime switching can be modeled by a continuous time Markov chain $(r_t)_{t\geq 0}$ taking values in a finite state space $S = \{1, 2, ..., m\}$ and with infinitesimal generator $Q = (q_{ij}) \in R^{m\times m}$. That is $r_t$ satisfies

$$P(r_{t+\delta} = j|r_t = i) = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + q_{ij}\delta + o(\delta), & \text{if } i = j, \end{cases} \quad \text{as } \delta \to 0^+.$$
where \( q_{ij} \geq 0 \) is the transition rate from \( i \) to \( j \) for \( i \neq j \), and \( q_{ii} = -\sum_{i \neq j} q_{ij} \) for each \( i \in S \), see Khasminskii et. al (2007) [4] and Zhu et. al (2007, 2009) [14, 15]. Inspired by this, Liu et. al (2011, 2012) [7, 8] consider the following stochastic GA model under regime switching:

\[
\dot{x}(t) = x(t)\left[a(r_t) - b(r_t)x^\theta(t)\right]dt + \sigma_1(r_t)x(t)dB_1(t) + \sigma_2(r_t)x^{1+\gamma(t)}dB_2(t),
\]

(2)

where \( \theta > 0, \gamma > 0 \). They studied the existence of global positive solution, persistence, extinction and non-persistence of the species, and obtained the stochastic permanence of the species under the condition that \( \gamma \in (0, 1] \) and \( \theta \in (0, 1 + \gamma] \). Settati et. al (2015) [13] considered a GA model with the GA parameter \( \theta \) under regime switching:

\[
\dot{x}(t) = x(t)\left[a(r_t) - b(r_t)x^{\theta(r_t)}(t)\right]dt + \sigma(r_t)x(t)dB(t),
\]

(3)

and presented global stability of the trivial solution, and sufficient conditions for the extinction, persistence and existence of stationary distribution. Under the assumption that \( \theta(i) \in (0, 1] \) for all \( i \in S \), Liu et. al (2015) [6] investigate the asymptotical stability in probability and the existence of stationary distribution of the following GA model with regime switching:

\[
\dot{x}(t) = x(t)\left[a(r_t) - b(r_t)x^{\theta(r_t)}(t)\right]dt + \sigma_1(r_t)x(t)dB_1(t) + \sigma_2(r_t)x^{1+\theta(r_t)}(t)dB_2(t).
\]

(4)

Motivated by the claim of May (1973) [12] that all parameters involved in ecosystems exhibit stochastic fluctuation, in this paper we consider telegraph noise in the GA parameters \( \theta \) and \( \gamma \) in model (2), and get a more general stochastic GA model under regime switching in the form:

\[
\dot{x}(t) = x(t)\left[a(r_t) - b(r_t)x^{\theta(r_t)}(t)\right]dt + \sigma_1(r_t)x(t)dB_1(t) + \sigma_2(r_t)x^{1+\gamma(r_t)}(t)dB_2(t),
\]

(5)

with initial value \((x_0, r_0) \in \mathbb{R}_+ \times S\), and for each \( i \in S \), \( \theta(i) > 0 \) and \( \gamma(i) > 0 \).

**Remark 1.1.** If \( \sigma_2(r_t) = 0 \), then it is model [8], and transforms to model [11] with \( \gamma(i) = \theta(i) \) for each \( i \in S \), and reduces to model [2] while there is no switching in the GA parameters \( \theta \) and \( \gamma \). Thus model [11] generalizes the previous models.
The contribution of this paper is that. Compared with the models in the literatures (e.g., [2, 5–9, 13]), our model (5) provides a more realistic modeling of the population dynamics. The results on the existence of global positive solution, asymptotic stability in probability, $p$th moment exponential stability, weak persistence, extinction, stochastic permanence and stationary distribution of the model generalize the results in previous works. Moreover, the conditions imposed on the positive recurrent, stochastic permanence and the existence of a unique ergodic asymptotically invariant distribution improve those of Liu et. al (2011, 2012, 2015)(e.g., [6–8]).

Throughout this paper, we assume that there is a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions in which the one dimensional Brownian motions $B_1(t)$ and $B_2(t)$ are defined. For convenience and simplicity, in this paper we using the following notations:

$$
\mu(t) = a(r_t) - 0.5 \sigma_1^2(r_t); \quad f_1(y) = \sum_{i \in S} \pi_i \left[ \mu(i) - b(i)y^{\theta(i)} \right]; \\
\tau_\varepsilon = \inf \{ t \geq 0 \mid x(t) \leq \varepsilon \}; \\
\tau_K = \inf \{ t \geq 0 \mid x(t) \geq K \}; \quad \tau_0 = \inf \{ t \geq 0 \mid x(t) = 0 \}; \quad \hat{a} = \max_{i \in S} \{ a(i) \}; \quad \hat{a} = \min_{i \in S} \{ a(i) \};
$$

$$
\mathbb{R}_+ = (0, \infty), \text{ where } \varepsilon > 0 \text{ and } K > 1 \text{ are constants.}
$$

**Assumption:** The Markov chain $(r_t)_{t \geq 0}$ is irreducible, and $B_1(t), B_2(t)$ and $r_t$ are independent.

**Definition 1.1.** (See Liu et. al (2011) [7] for definitions 1-3, Khasminskii et. al (2007) [4] for definition 4, and Mao et. al (2006) for definitions 5-6)

1. The species $x(t)$ is said to be extinctive if $\lim_{t \to \infty} x(t) = 0$;
2. The species $x(t)$ is said to be weak persistent if $\limsup_{t \to \infty} x(t) > 0$;
3. The species $x(t)$ is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there is a pair of positive constants $\alpha, \beta$ such that $\liminf_{t \to \infty} \mathbb{P}\{ x(t) \geq \beta \} \geq 1 - \varepsilon$ and $\liminf_{t \to \infty} \mathbb{P}\{ x(t) \leq \alpha \} \geq 1 - \varepsilon$;
4. The trivial solution is said to be asymptotically stable in probability if it is stable in probability, that is, for any $\varepsilon \in (0, 1)$ and any $r_0 \in S$, $\lim_{x_0 \to 0} \mathbb{P}\{ \sup_{t \geq 0} |x^{x_0,r_0}(t)| > \varepsilon \} = 0$, and satisfying $\lim_{x_0 \to 0} \mathbb{P}\{ \lim_{t \to \infty} x^{x_0,r_0}(t) = 0 \} = 1$ for any $r_0 \in S$;
5. For $p > 0$, the trivial solution is said to be $p$th moment exponentially stable if for all $(x_0, r_0) \in \mathbb{R}_+ \times S$, $\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E}[x^p(t)]) < 0$.

6. A square matrix $A = (a_{ij})_{n \times n}$ is called a nonsingular M-matrix if $A$ can be expressed in the form $A = sI - G$ with a nonnegative square matrix $G$ (i.e., each element of $G$ is nonnegative) and $s > \rho(G)$, where $I$ is the identity $n \times n$ matrix and $\rho(G)$ the spectral radius of $G$.

The organization of this paper is as follows. In section 2, we prove the existence of global positive solution of model (5). In section 3, we investigate the asymptotic stability in probability and $p$th moment exponential stability of the trivial solution to model (5). In section 4, we present some sufficient conditions for weak persistence and extinction of the species described by model (5). In section 5, we prove that there exists a stationary distribution of the solution $x(t)$ to model (5). In section 5, we analyze two examples by computer simulating to verify the results of previous sections. We give some conclusions in the last section.

2 Global Positive Solution

Theorem 2.1. For any $(x_0, r_0) \in \mathbb{R}_+ \times S$, the solution $x(t)$ of model (5) satisfying $\mathbb{P}_{x_0, r_0}(x(t) > 0, \forall t \geq 0) = 1$.

Proof. Define $V(x, i) = x^{-\beta}$ with $\beta > 0$, for $(x, i) \in \mathbb{R}_+ \times S$ we have
\[
LV(x, i) = -\beta x^{-\beta} \left[ \mu(i) + \frac{\beta}{2} \sigma_1^2(i) - b(i)x^{\theta(i)} + \frac{1}{2}(\beta + 1)\sigma^2_2(i)x^{2\gamma(i)} \right] \leq \beta b x^{\theta(i)-\beta}.
\]
Here and in the following sections we will drop $(t)$ from $x(t)$ for simplicity sometimes. Then by Itô’s formula, we obtain
\[
\mathbb{E}[x^{-\beta}(D(t))] = x_0^{-\beta} + \mathbb{E} \int_0^{D(t)} LV(x(s), r_s)ds \leq x_0^{-\beta} + \beta b K^{\hat{\theta}} \mathbb{E} \int_0^{D(t)} x^{-\beta}(s)ds,
\]
where $D(t) = t \wedge \tau_e \wedge \tau_K$. It follows from Gronwall’s inequality that
\[
\mathbb{E}[x^{-\beta}(D(t))] \leq x_0^{-\beta} e^{\beta b K^{\hat{\theta}} t}.
\]
If \( P(\tau_0 < \infty) > 0 \), then we can choose \( t \) and \( K \) large enough such that \( P(\tau_0 < t \wedge \tau_K) > 0 \). So by Chebeshev’s inequality, we get

\[
0 < P(\tau_0 < t \wedge \tau_K) \leq P(\tau_\varepsilon < t \wedge \tau_K) \leq P(x(D(t)) \leq \varepsilon)
\]

where \( \varepsilon \) is increasing as \( n \to \infty \). Let \( K \) be so large that \( x \in \varepsilon \) when \( n \) is large enough such that \( \varepsilon \) is the explosion time. Therefore, there is an integer \( N \geq n \) such that \( P\{\tau_n \leq N\} > \varepsilon \). Then by Itô’s formula, we have

\[
V(x_0, i) = x^p + \int_0^{\tau_n \wedge N} \sigma_1(r_s) x^{p+\gamma(i)} dB_1(s) + \int_0^{\tau_n \wedge N} \sigma_2(r_s) x^{p+\gamma(r_s)} dB_2(s).
\]

Taking expectation on both sides of it gives

\[
E[x^p(N \wedge \tau_n)] = x_0^p + K E[N \wedge \tau_n] \leq x_0^p + KN.
\]

Let \( \Omega_n = \{\tau_n \leq N\} \), then \( P(\Omega_n) \geq \varepsilon \). In view of that for every \( \omega \in \Omega_n \), \( x(\tau_n, \omega) \) equals to \( n \), and we get a contradiction,

\[
\infty > x_0^p + KN \geq E[1_{\Omega_n}(\omega) x^p(\tau_n)] \geq \varepsilon n^p \to \infty \text{ as } n \to \infty.
\]

\[\square\]

**Theorem 2.2.** For any \((x_0, r_0) \in \mathbb{R}_+ \times S\), there is a unique global solution \( x(t) \) to model (5).

**Proof.** All the coefficients of model (5) are locally Lipschitz continuous, thus there is a unique local solution \( x(t) \) to model (5) with initial value \((x_0, r_0) \in \mathbb{R}_+ \times S\) on \( t \in [0, \tau) \), where \( \tau \) is the explosion time.

Next we show that the solution is globally existent, that is \( \tau = \infty \). Let \( n_0 > 0 \) be so large that \( x_0 \in (0, n_0) \). For each \( n > n_0 \), define stopping times \( \tau_n = \inf\{t \in [0, \tau] | x \geq n\} \), then \( \tau_n \) is increasing as \( n \to \infty \). Let \( \tau_\infty = \lim_{n \to \infty} \tau_n \), whence \( \tau_\infty \leq \tau \) a.s.

Now we claim \( \tau_\infty = \infty \). Otherwise, there must exist a pair of constants \( N > 0 \) and \( \varepsilon \in (0, 1) \) such that \( P\{\tau_\infty \leq N\} > \varepsilon \). Therefore, there is an integer \( N_1 \geq n_0 \) such that

\[
P\{\tau_n \leq N\} > \varepsilon \text{ for } n \geq N_1.
\]

Define Lyapunov function \( V(x, i) = x^p \) with \( p \in (0, 1) \), we have

\[
LV(x, i) = p x^p \left[ a(i) + \frac{1}{2} b(p - 1) \sigma_1^2(i) - b(i) x^\theta(i) \right] + \frac{1}{2} p(p - 1) \sigma_2^2(i) x^{p+2\gamma(i)} \leq K,
\]

for \((x, i) \in \mathbb{R}_+ \times S\). Then by Itô’s formula, we have

\[
V(x(\tau_n \wedge N), r_{\tau_n \wedge N}) = V(x(\tau_n), i) + \int_0^{\tau_n \wedge N} LW(x_s, r_s) ds
+ p \int_0^{\tau_n \wedge N} \sigma_1(r_s) x^{p+\gamma(i)} dB_1(s) + p \int_0^{\tau_n \wedge N} \sigma_2(r_s) x^{p+\gamma(r_s)} dB_2(s).
\]

Taking expectation on both sides of it gives

\[
E[x^p(N \wedge \tau_n)] \leq x_0^p + K E[N \wedge \tau_n] \leq x_0^p + KN.
\]
Thus \( \tau_\infty = \infty \).

From above two theorems, one can easily get the following result.

**Corollary 2.1.** For any \((x_0, r_0) \in \mathbb{R}_+ \times \mathcal{S}\), the solution \(x(t)\) to model (5) is global existence and remains in \(\mathbb{R}_+\) for \(t \geq 0\).

### 3 Stability of Trivial Solution

**Theorem 3.1.** For any \((x_0, r_0) \in \mathbb{R}_+ \times \mathcal{S}\). If \(\sum_{i \in \mathcal{S}} \pi_i \mu(i) < 0\), then the trivial solution to model (5) is asymptotically stable in probability.

**Proof.** Let \(\zeta = (\zeta_1, \ldots, \zeta_m)^T\) be a solution of the Poisson system:

\[
Q\zeta = -\mu + \sum_{i \in \mathcal{S}} \pi_i \mu(i) 1,
\]

where \(\mu = (\mu(1), \mu(2), \ldots, \mu(m))^T\). Choose sufficient large positive constant \(p\) such that

\[
\frac{p}{p + \zeta_i} \sum_{k \in \mathcal{S}} \pi_i \mu(i) + \frac{\sigma_1^2(i)}{2p} + \frac{\zeta_i \mu(i)}{p + \zeta_i} < 0 \quad \text{and} \quad p > -\min_{i \in \mathcal{S}} \{\zeta_i\} \lor 1.
\]

Define \(V(x, i) = (p + \zeta_i)x^{1/p}\), \(i \in \mathcal{S}\), we obtain

\[
LV(x, i) = -\frac{1}{p} V(x, i) \left[ \mu(i) + \sum_{k \in \mathcal{S}} q_{ik} \zeta_k - b(i)x^{\theta(i)} + \frac{1-p}{2p} \sigma_2^2(i)x^{2\gamma(i)} + \frac{\sigma_1^2(i)}{2p} - \frac{\zeta_i}{p + \zeta_i} \sum_{k \in \mathcal{S}} q_{ik} \zeta_k \right]
\]

\[
= -\frac{1}{p} V(x, i) \left[ \sum_{k \in \mathcal{S}} \pi_i \mu(i) - b(i)x^{\theta(i)} - \frac{p-1}{2p} \sigma_2^2(i)x^{2\gamma(i)} + \frac{\sigma_1^2(i)}{2p} + \frac{\zeta_i \mu(i)}{p + \zeta_i} - \frac{\zeta_i}{p + \zeta_i} \sum_{k \in \mathcal{S}} \pi_i \mu(i) \right]
\]

\[
\leq -Kx^{1/p},
\]

where \(K = K(p)\) is a positive constant. Then, for any sufficient small \(\varepsilon \in (0, r)\) we have

\[
LV(x, i) \leq -K\varepsilon^{1/p} \quad \text{for any} \quad x \in (\varepsilon, r) \quad \text{and} \quad i \in \mathcal{S}.
\]

Thus according to Lemma 3.3 and Remark 3.5–(i) in Khasminskii et. al (2007) \([4]\), we get that the trivial solution of model (5) is asymptotically stable in probability.
Theorem 3.2. For $p \in (0, 1)$, if $A(p) := \text{diag}(h_1(p), \cdots, h_m(p)) - Q$ is a nonsingular M-matrix, then the trivial solution of model (4) is $p$th moment exponentially stable, where $h_i(p) = \frac{1}{2}p(1-p)\sigma^2_1(i) - p a(i)$.

Remark 3.1. If $h_i(p) > 0$ for all $i \in S$, then all the row sums of $A(p)$ are positive, then according to Minkovski Lemma, $\det(A(p))$ is positive. Furthermore, by the properties of generator $Q$ we know that all the principle minors of $A(p)$ are positive, thus according to Theorem 2.10-(2) of Mao et. al (2006) [11] that $A(p)$ is a nonsingular M-matrix. Then it follows from Theorem 3.2 that the trivial solution of model (4) is $p$th moment exponentially stable, which also implies the asymptotic stability in probability.

On the other hand, $h_i(p) > 0$ for all $i \in S$ imply $\mu(i) < 0$ for all $i \in S$, thus according to Theorem 3.1, the trivial solution of model (4) is asymptotic stability in probability.

Proof. It follows from Theorem 2.10-(9) of Mao et. al (2006) [11] that there is a vector $\beta = (\beta_1, \cdots, \beta_m)^T > 0$, i.e., $\beta_i > 0$ for all $1 \leq i \leq m$, such that $(\bar{\beta}_1, \cdots, \bar{\beta}_m)^T := A(p)\beta > 0$. Then

$$h_i(p)\beta_i - \sum_{j \in S} q_{ij}\beta_j = \bar{\beta}_i > 0 \text{ for } 1 \leq i \leq m.$$  

Define the Lyapunov in the form $V(x, i) = \beta_i x^p$, then

$$LV(x, i) = \left[ p\beta_i(a(i) - b(i)x^{\theta(i)}) + \frac{1}{2}p(p-1)\beta_i\sigma^2_1(i) + \frac{1}{2}p(p-1)\beta_i\sigma^2_2(i)x^{2\gamma(i)} + \sum_{k \in S} q_{ik}\beta_k \right] x^p$$

$$\leq \left[ (p a(i) + \frac{1}{2}p(p-1)\sigma^2_1(i))\beta_i + \sum_{k \in S} q_{ik}\beta_k \right] x^p$$

$$= \left[ -h_i(p)\beta_i + \sum_{k \in S} q_{ik}\beta_k \right] x^p \leq -\lambda x^p,$$

where $\lambda = \min_{1 \leq i \leq m} \bar{\beta}_i$. Thus according to Theorem 5.8 in Mao et. al (2006) [11] we get the result. \qed

4 Weak Persistence and Extinction

Theorem 4.1. If $\sum_{i \in S} \pi_i\mu(i) > 0$, then $\liminf_{t \to \infty} x(t) \leq x_* \leq x^* \leq \limsup_{t \to \infty} x(t)$, where $x^*$ and $x_*$ are the unique positive solutions of $f_1(y) = 0$ and $f_2(y) = 0$, respectively.
Remark 4.1. \( x^* \geq x_0 > 0 \) can be obtained from the fact that \( f_1 \geq f_2 \) and \( f_2(0) > 0 \). Theorem 4.1 shows that solutions of model (5) will oscillate infinitely often about interval \([x_*, x^*] \), and the amplitude is no less than \( A = x^* - x_* \), which may be decreasing with the decreasing of \( \sigma^2_2 \). If \( \sigma_2 \neq 0 \), then \( A > 0 \), and then \( \liminf_{t \to \infty} x(t) \leq x_* < x^* \leq \limsup_{t \to \infty} x(t) \). Moreover, this theorem implies the weak persistence of the species because of \( \limsup_{t \to \infty} x(t) \geq x^* > 0 \).

Remark 4.2. If \( \sigma_2 = 0 \), then \( f_1(y) = f_2(y) \), \( x^* = x_* \) and the equalities hold, which is the case of Theorem 3.1 in Settati et al. (2015) \([3]\). Thus our result is an extension of it.

Proof. One can see that \( f_i \), \( i = 1, 2 \) are continuous and strictly decreasing functions on \( \mathbb{R}_+ \) and
\[
f_i(0^+) = \sum_{i \in S} \pi_i \mu(i) > 0, \quad f_i(\infty) = -\infty.
\]

So there exist unique positive solutions \( x^*, x_* \), \( x_* \leq x^* \), such that \( f_1(x^*) = 0 \), \( f_2(x_*) = 0 \), respectively.

**Step 1.** Assume \( \mathbb{P}(\omega \in \Omega, \limsup_{t \to \infty} x(t, \omega) < x^*) > 0 \), then there exists a positive constant \( \alpha \in (1/2, 1) \) such that \( \mathbb{P}(\Omega_1) > 0 \), where \( \Omega_1 = \{ \omega \in \Omega, \limsup_{t \to \infty} x(t, \omega) < (2\alpha - 1)x^* \} \). So for every \( \omega \in \Omega_1 \), there is a \( T(\omega) > 0 \) such that
\[
x(t) \leq (2\alpha - 1)x^* + (1 - \alpha)x^* = \alpha x^* \quad \text{for all} \quad t \geq T(\omega).
\]

Then it follows from model (5) that
\[
\log x(t) \geq \log x_0 + \int_0^t \mu(r_s)dt - \int_0^T \left[ b(r_s)x^{\theta(r_s)} + \frac{1}{2}\sigma_2^2(r_s)x^{2\gamma(r_s)} \right] dt
- \alpha^{\delta \lambda_2} \int_T^t \left[ b(r_s)(x^*)^{\theta(r_s)} + \frac{1}{2}\sigma_2^2(r_s)(x^*)^{2\gamma(r_s)} \right] dt + M_1(t) + M_2(t),
\]
where \( M_1(t) = \int_0^t \sigma_1(r_s)dB_1(t), \quad M_2(t) = \int_0^t \sigma_2(r_s)x^{\gamma(r_s)}dB_2(t) \). Notice that \( M_i(t) \) are real valued continuous local martingale with the quadratic variations:
\[
\langle M_1(t), M_1(t) \rangle = \int_0^t \sigma_1^2(r_s)ds \leq \tilde{\sigma}_1^2 t,
\]

\[
\langle M_2(t), M_2(t) \rangle = \int_0^t \sigma_2^2(r_s)x^{\gamma(r_s)}ds \leq \tilde{\sigma}_2^2 t.
\]
and
\[ \langle M_2(t), M_2(t) \rangle \leq \alpha^{2\gamma}(x^*)^{2\gamma} + \alpha^{2\gamma} \int_0^t \sigma_2^2(r_s) ds \leq \sigma_2^{2\gamma}(x^*)^{2\gamma} + (x^*)^{2\gamma} t. \]

Thus by the large number theorem for martingales and the ergodic theory of the Markov chain, we obtain from inequality (7) that there is \( \Omega' \subset \Omega \) such that \( \mathbb{P}(\Omega') = 1 \), and for every \( \omega \in \Omega' \),
\[
\liminf_{r \to \infty} \frac{1}{t} \log x(t) \geq \sum_{i \in S} \pi_i \left\{ \mu(i) - \alpha^{\theta \wedge 2\gamma} \left[ b(i)(x^*)^{\theta(i)} + \frac{\sigma_2^2(i)}{2}(x^*)^{2\gamma(i)} \right] \right\} \\
\geq (1 - \alpha^{\theta \wedge 2\gamma}) \sum_{i \in S} \pi_i \mu(i) > 0,
\]
which implies \( \lim_{t \to \infty} x(t) = \infty \). But this contradicts assumption. Thus \( x^* \leq \limsup_{t \to \infty} x(t) \).

**Step 2.** Assume \( \mathbb{P}(\omega \in \Omega, \liminf_{t \to \infty} x(t, \omega) > x) > 0 \), then there exists a constant \( \beta > 1 \) such that \( \mathbb{P}(\Omega_2) > 0 \), where \( \Omega_2 = \{ \omega \in \Omega, \liminf_{t \to \infty} x(t, \omega) \geq (2\beta - 1)x \} \). Thus, for every \( \omega \in \Omega_2 \), there exists a \( T_1(\omega) > 0 \) such that
\[
x(t) \geq (2\beta - 1)x - (\beta - 1)x = \beta x \text{ for } t \geq T_1(\omega).
\]

Then we have
\[
\log x(t) = \int_0^t \mu(r_s) dt - \int_0^t \left[ b(r_s)x^{\theta(r_s)} + \frac{1}{2}\sigma_2^2(r_s)x^{2\gamma(r_s)} \right] dt + M_1(t) + M_2(t) + \log x_0.
\]

(8)

It follows from the exponential martingale inequality, see Applebaum (2009) [11], that for any positive numbers \( T, \varepsilon \) and \( \delta \),
\[
\mathbb{P}\left\{ \sup_{t \in [0,T]} [M_2(t) - \frac{\varepsilon}{2} \langle M_2(t), M_2(t) \rangle] > \delta \right\} \leq e^{-\varepsilon \delta}.
\]

Let \( T = T_1, \varepsilon = 1 \) and \( \delta = 2 \log T_1 \), then we have
\[
\mathbb{P}\left\{ \sup_{t \in [0,T_1]} [M_2(t) - \frac{\varepsilon}{2} \langle M_2(t), M_2(t) \rangle] > 2 \log T_1 \right\} \leq \frac{1}{T_1^2}.
\]

By the Borel-Cantelli’s Lemma, for almost all \( \omega \in \Omega \), there is a random integer \( n_0 = n_0(\omega) \) such that
\[
\sup_{t \in [0,T_1]} [M_2(t) - \frac{1}{2} \langle M_2(t), M_2(t) \rangle] \leq 2 \log T_1
\]

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for all $T_1 \geq n_0$ a.s., which yields
\[ M_2(t) \leq 2 \log T_1 + \frac{1}{2} \int_0^t \sigma_2^2(r_s)x^{2r(r_s)} ds \]
for all $t \in [0, T_1]$, $T_1 \geq n_0$ a.s. Inserting the inequality into (8) gives
\[ \log x(t) \leq \log x_0 + \int_0^t \mu(r_s) dt - \beta \hat{\theta} \int_0^t b(r_s)(x^\theta(r_s)) dt - \int_0^{T_1} b(r_s)x^{\theta(r_s)} dt + M_1(t) + 2 \log T_1. \]
Similarly, by using the large number theorem for martingale $M_1(t)$ and the ergodic theory of the Markov chain, we obtain
\[ \limsup_{t \to \infty} \frac{1}{t} \log x(t) \leq \sum_{i \in S} \pi_i [\mu(i) - \beta \hat{\theta} b(i)(x^\theta(i))] = (1 - \beta \hat{\theta}) \sum_{i \in S} \pi_i \mu(i) < 0, \]
which leads to a contradiction. The proof is now completed. \[ \square \]

**Corollary 4.1.** For any $(x_0, r_0) \in \mathbb{R}_+ \times S$, the solution $x(t)$ of model (5) has the following property: $\limsup_{t \to \infty} \frac{1}{t} \log x(t) \leq \sum_{i \in S} \pi_i \mu(i)$ a.s. Moreover, if $\sum_{i \in S} \pi_i \mu(i) < 0$, then the species $x(t)$ will be extinctive a.s.

## 5 Stationary Distribution

**Theorem 5.1.** Assume $\theta(i) + 1 \geq 2\gamma(i)$ for each $i \in S$ and $\sum_{i \in S} \pi_i \mu(i) > 0$, then, for any $(x_0, r_0) \in \mathbb{R}_+ \times S$, the solution $x(t)$ of model (5) is positive recurrent and admits a unique ergodic asymptotically invariant distribution $\nu$. Moreover, the species described by model (5) is stochastically permanent.

**Remark 5.1.** One can see that our condition $\theta(i) + 1 \geq 2\gamma(i)$ for each $i \in S$ holds for large $\theta(i)$, and $\gamma(i)$ is allowed to be bigger than 1 while $\theta(i)$ is larger than 1. Meanwhile, if for $i \in S$, $\gamma(i) = \theta(i)$ and $\theta(i) \in (0, 1]$ (which is assumed to guarantee the positive recurrent and the existence of a unique ergodic asymptotically invariant distribution in Liu et al (2015) [2]), then $\theta(i) + 1 \geq 2\gamma(i)$ holds. For the case of no switching in the GA parameters $\theta$ and $\gamma$, under the conditions that $\gamma \in (0, 1]$ and $\theta \in (0, 1 + \gamma]$, the stochastic permanence is obtained in Liu et al (2011, 2012) [7,8]. Therefore, our conditions improve the corresponding ones in these literatures.
Remark 5.2. From Eq.(9) in the following proof we find that the condition $\theta(i) + 1 \geq 2\gamma(i)$ for each $i \in S$ can be removed from Theorem 5.7 in three cases:

i) For all $i \in S$, $\sigma_2(i) = 0$, then model (5) reduces to model (3), and the condition $\theta(i) + 1 \geq 2\gamma(i)$ for each $i \in S$ can be removed, then we get the same result as that in Settati et. al (2015) [13].

ii) For all $i \in S$, $\sigma_1(i) = 0$, and there is no switching in the GA parameters, that is for all $i \in S$, $\theta(i) = \theta = \text{const.}$, and $\gamma(i) = \gamma = \text{const.}$, then $\mu(i) = a(i)$. In this case we can choose sufficient large $p$ such that $\theta + 1 > 2\gamma - p > 0$, and thus the condition $\theta(i) + 1 \geq 2\gamma(i)$ for each $i \in S$ can also be removed;

iii) For all $i \in S$, $\gamma(i) = \theta(i)$ and $\theta(i) \in (0, 1]$, which are presented in Liu et. al (2005) [6]. In this case, the condition $\theta(i) + 1 \geq 2\gamma(i)$ holds for each $i \in S$.

Proof. Define a Lyapunov function in the form: $V(x, i) = (1 - p\zeta_i)x^{-p} + x$, where $p$ is positive number satisfying $1 > p\max_{i \in S}\{\zeta_i\}$, and $\zeta = (\zeta_1, ..., \zeta_m)^T$ is a solution of the Poisson system (3). Then we have

$$LV(x, i) = -p(1 - p\zeta_i)x^{-p}[a(i) - b(i)x^{\theta(i)}] + \frac{1}{2}p(p + 1)(1 - p\zeta_i)^2(i)x^{-p} + \frac{1}{2}p(p + 1)(1 - p\zeta_i)\sigma_2(i)x^{2\gamma(i) - p} - px^{-p}\sum_{k \in S} q_{ik}\zeta_k + x[a(i) - b(i)x^{\theta(i)}]$$

$$= -p(1 - p\zeta_i)x^{-p}\left[a(i) - \frac{1}{2}(p + 1)\sigma_1(i) + \frac{1}{1 - p\zeta_i}\sum_{k \in S} q_{ik}\zeta_k\right] + x[a(i) - b(i)x^{\theta(i)}] + p(1 - p\zeta_i)b(i)x^{\sigma_1(i) - p} + \frac{1}{2}p(p + 1)(1 - p\zeta_i)\sigma_2(i)x^{2\gamma(i) - p}$$

$$= -p(1 - p\zeta_i)x^{-p}\left[\sum_{i \in S} \pi_i \mu(i) - \frac{1}{2}p\sigma_1(i) + \frac{p\zeta_i}{1 - p\zeta_i}\sum_{k \in S} q_{ik}\zeta_k\right] + x[a(i) - b(i)x^{\theta(i)}] + p(1 - p\zeta_i)b(i)x^{\theta(i) - p} + \frac{1}{2}p(p + 1)(1 - p\zeta_i)\sigma_2(i)x^{2\gamma(i) - p}.$$ (9)

Let $U(N) = (1/N, N) \subset R_+$. In view of $\sum_{i \in S} \pi_i \mu(i) > 0$, we can choose $p$ sufficient small such that

$$\sum_{i \in S} \pi_i \mu(i) - \frac{1}{2}p\sigma_1(i) + \frac{p\zeta_i}{1 - p\zeta_i}\sum_{k \in S} q_{ik}\zeta_k > 0,$$
and note that \( \theta(i) + 1 \geq 2\gamma(i) \) for each \( i \in S \), then we have \( LV(x, i) \to -\infty \), as \( N \to \infty \).
Thus, for any given positive constant \( K \), there exists a sufficient large \( N_0 \) such that \( LV(x, i) \leq -K \) for all \( x \in U^c(N_0) \). Then according to the Theorems 3.13 and 4.3 in Zhu et. al (2007) [14] or Theorem 4.1 in Settati et. al (2015) [13] we obtain the first part of the theorem.

Now we prove the second part. By the ergodicity of \( x(t) \), we have
\[
\frac{1}{t} \int_0^t I_{\{x(s) \in \mathbb{R}_+\}} ds \to \int_0^\infty I_{\{x \in \mathbb{R}_+\}}(x)\pi(dx) = \nu(\mathbb{R}_+),
\]
which together with \( x(t) \in \mathbb{R}_+ \) yields \( \nu(\mathbb{R}_+) = 1 \). It follows from the asymptotically invariant distribution of \( x(t) \) that, for positive constants \( \alpha, \beta \),
\[
\liminf_{t \to \infty} \mathbb{P}(x(t) \geq \alpha) = \nu([\alpha, \infty)) \quad \text{and} \quad \liminf_{t \to \infty} \mathbb{P}(x(t) \leq \beta) = \nu((0, \beta)).
\]
Thus \( \lim_{\alpha \to 0^+} \nu([\alpha, \infty)) = \lim_{\beta \to \infty} \nu((0, \beta)) = \nu(\mathbb{R}_+) = 1 \). Then for any \( \epsilon \in (0, 1) \) there exists a sufficient small positive constant \( k \) such that
\[
\liminf_{t \to \infty} \mathbb{P}(x(t) \geq k) \geq 1 - \epsilon \quad \text{and} \quad \liminf_{t \to \infty} \mathbb{P}(x(t) \leq 1/k) \geq 1 - \epsilon.
\]
This proof is now completed. \( \square \)

6 Examples

In order to verify the theorems obtained in previous sections, we give the following two examples. The numerical method used here is Milsteins Higher Order Method, see Higham (2001) [16] for more details.

The states space of Markov chain \( r_t \) is \( S = \{1, 2, 3, 4\} \), and its generator \( Q \) and one step transition probability matrix \( P \) are as follows,
\[
Q = \begin{pmatrix}
-10 & 3 & 2 & 5 \\
6 & -9 & 2 & 1 \\
3 & 3 & -8 & 2 \\
1 & 5 & 3 & -9 \\
\end{pmatrix}; \quad P = \exp(\Delta \cdot Q) = \begin{pmatrix}
0.9990 & 0.0003 & 0.0002 & 0.0005 \\
0.0006 & 0.9991 & 0.0002 & 0.0001 \\
0.0003 & 0.0003 & 0.9992 & 0.0002 \\
0.0001 & 0.0005 & 0.0003 & 0.9991 \\
\end{pmatrix},
\]
where \( \Delta \) is the step. Then the stationary distribution is \( \pi = (0.2622, 0.2879, 0.2227, 0.2272) \).
The computer simulation of the Markov chain is given by Figure 1 with the step \( \Delta = 1e^{-4} \).
Example 1. Let

\[ a = (0.4, 0.3, 0.6, 0.55), \quad b = (0.15, 0.2, 0.13, 0.4), \quad \sigma_1 = (0.3, 0.2, 1.4, 0.5), \]

\[ \sigma_2 = (0.13, 0.21, 0.11, 0.24), \quad \theta = (1.5, 0.5, 1, 0.7), \quad \gamma = (1.2, 0.6, 1, 0.8). \]

Then \( \mu = (0.3550, 0.2800, -0.3800, 0.4250) \) and \( \sum_{i \in S} \pi_i \mu(i) = 0.1856 > 0 \), and \( x^* = 0.7681, \ x^* = 0.8280 \).

Thus according to Theorems 4.1, 5.1 and Definition 4.1, we get the weak persistence and the stochastic permanence of the species \( x \) of model (5). The evolution of \( x(t) \) along with the Markov chain \( r_t \) and \( r_t \equiv i \ (i \in S) \) are simulated in Figures 2-3, respectively. From Figure 3, we find that the solution \( x(t) \) of model (5) with \( r_t \equiv 3 \) goes to zero, that is the trivial solution of model (5) with \( r_t = 3 \) is stable; but the other three states of it are unstable. However, the solution \( x(t) \) along with the Markov chain \( r_t \) is unstable finally because of \( \sum_{i \in S} \pi_i \mu(i) > 0 \). The density and distribution of \( x(t) \) along with the Markov chain \( r_t \) are shown in Figure 4.

Example 2. Replace \( \sigma_1(4) = 0.5 \) by \( \sigma_1(4) = 1.8 \) in Example 1 above. Then we have

\[ \mu = (0.3550, 0.2800, -0.3800, -1.0700) \] and \( \sum_{i \in S} \pi_i \mu(i) = -0.1540 < 0. \)

Hence, it follows from Theorem 3.1 and Corollary 4.1 that the trivial solution to model (5) is asymptotically stable in probability, and the species \( x \) will go to be extinctive finally. The computer simulation results of this example are given by Figures 5-6.

Moreover, let \( p = 0.5 \), we have \( h(p) = (-0.1888, -0.1450, -0.0550, 0.1300) \) and

\[
\mathcal{A}(p) = \begin{pmatrix}
9.8112 & -3.0000 & -2.0000 & -5.0000 \\
-6.0000 & 8.8550 & -2.0000 & -1.0000 \\
-3.0000 & -3.0000 & 7.9450 & -2.0000 \\
-1.0000 & -5.0000 & -3.0000 & 9.1300
\end{pmatrix}.
\]

Its eigenvalues are \( \lambda = (-0.0751, 12.7683 + 3.0485i, 12.7683 - 3.0485i, 10.2797) \), which yields \( \mathcal{A}(p) \) is not a nonsingular \( M \)-matrix. Thus Theorem 3.2 is invalid, but the computer simulation shows that the trivial solution of model (5) is 0.5th moment exponentially stable, see Figure 7 for more details. This shows that the assumption of \( \mathcal{A}(p) \) is a nonsingular \( M \)-matrix is not a necessary condition for the \( p \)th moment exponential stability of \( x \) to model (5).
7 Conclusion

In this paper, we study a stochastic GA model perturbed by regime switching and white noise. Especially, we allow the GA parameters $\theta$ and $\gamma$ to vary according to a continuous-time Markov chain, reflecting the fact that the GA parameters may change in different environments. Compared with the models in the literatures (e.g., [2, 5–9, 13]), our formulation provides a more realistic modeling of the population dynamics. However, our model does introduce extra difficulty in the analysis because of the regime switching mechanism. We overcome these difficulties by constructing suitable Lyapunov functions and using some analysis technics. We show the existence of global positive solutions, asymptotic stability in probability and extinction of model (5) under the condition $\sum_{i \in S} \pi_i \mu(i) < 0$, and obtain the weak persistence of the species described by the model under the condition that $\sum_{i \in S} \pi_i \mu(i) > 0$, also we get that the amplitude of the solution $x(t)$ to the model is at least $[x_*, x^*]$, which generalize the previous results. Meanwhile, we prove that its trivial solution is $p$th moment exponential stable by using the properties of nonsingular $M$-matrix $A(p)$, but from the computer simulation, we know that this condition is not necessary. Under the assumptions $\theta(i) + 1 \geq 2\gamma(i) > 0$ for each $i \in S$ and $\sum_{i \in S} \pi_i \mu(i) > 0$, we prove that, for any $(x_0, r_0) \in \mathbb{R}_+ \times S$, the solution $x(t)$ of model (5) is positive recurrent and admits a unique ergodic asymptotically invariant distribution $\nu$. Moreover, the species $x(t)$ described by model (5) is stochastically permanent. Even for the case of no regime switching in the GA parameters, our condition imposed on GA parameters $\theta$ and $\gamma$ is $\theta + 1 \geq 2\gamma > 0$, which improves those in the previous works.

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Figure 1. The simulation of Markov chain $r_t$ with initial state $r_0 = 3$ and step $\Delta = 10^{-4}$. 
Figure 2. The evolution of $x(t)$ along with Markov chain $r_t$, with initial value $x(0) = 1$ and step $\Delta = 1e^{-3}$. And the unique positive solutions of $f_i(y) = 0 (i = 1, 2)$ are given by $x_*$ and $x^*$, respectively.
Figure 3.  The evolution of $x(t)$ for $r_t \equiv i (i \in S)$, respectively, with initial value $x(0) = 1$ and step $\Delta = 1e - 3$. These figures show that $x(t)$ with $r_t \equiv 3$ is stable and vanish finally (left down figure), but the other states are unstable.
Figure 4. The density and distribution of $x(t)$ along with Markov chain $r_t$, where $x(t)$ with initial value $x(0) = 1$ and computed by step $\Delta = 1e^{-3}$. 
Figure 5. The evolution of $x(t)$ along with Markov chain $r_t$, where $x(t)$ with initial value $x(0) = 1$ and computed by step $\Delta = 1e - 3$. 
Figure 6. The evolution of $x(t)$ for $r_t = i (i \in S)$, respectively, with initial value $x(0) = 1$ and step $\Delta = 1e - 3$. These figures show that $x(t)$ with $r_t = 3$, $r_t = 4$ are stable and vanish finally (down figures), but the other two states are unstable.
Figure 7. The 0.5th moment of $x(t)$ along with Markov chain $r_t$ is exponential stable, where $x(t)$ with initial value $x(0) = 1$ and computed by step $\Delta = 1e - 3$. 

\[ E[x^{0.5}] \]