Irreducibility of stochastic complex Ginzburg-Landau equations driven by pure jump noise and its applications

Hao Yang\textsuperscript{1}\textsuperscript{*} Jian Wang\textsuperscript{2}\textsuperscript{†} Jianliang Zhai\textsuperscript{2}\textsuperscript{‡}

1. School of Mathematics, Hefei University of Technology, Hefei, Anhui 230009, China.
2. School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China.

Abstract: Considering irreducibility is fundamental for studying the ergodicity of stochastic dynamical systems. In this paper, we establish the irreducibility of stochastic complex Ginzburg-Landau equations driven by pure jump noise. Our results are dimension free and the conditions placed on the driving noises are very mild. A crucial role is played by criteria developed by the authors of this paper and T. Zhang for the irreducibility of stochastic equations driven by pure jump noise. As an application, we obtain the ergodicity of stochastic complex Ginzburg-Landau equations. We remark that our ergodicity result covers the weakly dissipative case with pure jump degenerate noise.

Keywords: Irreducibility; Pure jump noise; Complex Ginzburg-Landau equation; Ergodicity.

AMS Subject Classification (2020): 60H15; 60G51; 37A25.
1 Introduction and motivation

Let $H$ be a topological space with Borel $\sigma$-field $\mathcal{B}(H)$, and let $X := \{X^x(t), t \geq 0; x \in H\}$ be an $H$-valued Markov process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $X$ is said to be strongly irreducible in $H$ if for each $t > 0$ and $x \in H$

$$\mathbb{P}(X^x(t) \in B) > 0$$

for any nonempty open set $B$.

$X$ is said to be weakly irreducible (also called accessible) to $x_0 \in H$ if the resolvent $R_\lambda, \lambda > 0$ satisfies

$$R_\lambda(y, U) = \lambda \int_0^\infty e^{-\lambda t} \mathbb{P}(X^x(t) \in U) dt > 0$$

for all $x \in H$ and all neighborhoods $U$ of $x_0$, where $\lambda > 0$ is arbitrary. It is clear that strong irreducibility implies accessibility.

Irreducibility is a fundamental property of stochastic dynamic systems. The importance of the study of this irreducibility lies in its relevance in the analysis of the ergodicity of Markov processes. The uniqueness of the invariant measures is usually obtained by proving irreducibility and the strong Feller property, or the asymptotic strong Feller property, or the $e$-property; see [4, 8, 9, 11, 14, 15, 18, 26]. The main aim of the current paper is to study the irreducibility of stochastic complex Ginzburg-Landau equations driven by pure jump noise.

The Ginzburg-Landau equation was proposed by physicists Ginzburg and Landau in the 1950s as a low-temperature superconducting model [10]. The model, for which Ginzburg and Landau won the Nobel prize in physics in 2003, is widely used in fields such as superconductivity, superfluidity, Bose-Einstein condensation, and physical phase transition processes [1], in particular, the model is used in the description of spatial pattern formation and the onset of instabilities in nonequilibrium fluid dynamical systems [3, 6, 12].

We now survey previous results concerning the irreducibility of stochastic Ginzburg-Landau equations driven by pure jump noise. To do this, we first introduce the so-called cylindrical pure jump Lévy processes defined by the orthogonal expansion

$$L(t) = \sum_i \beta_i L_i(t)e_i, \quad t \geq 0,$$

(1.1)

where $\{e_i\}$ is an orthonormal basis of a separable Hilbert space $H$, $\{L_i\}$ are real-valued i.i.d. pure jump Lévy processes, and $\{\beta_i\}$ is a given sequence of nonzero real numbers.

In 2013, the authors in [25] obtained the accessibility to zero of stochastic real-valued Ginzburg-Landau equations on torus $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$ in $H := \{h \in L^2(\mathbb{T}) : \int_{\mathbb{T}} h(y) dy = 0\}$; see the proof of [25 Theorem 2.4]. The driving noises they considered are the so-called cylindrical symmetric $\alpha$-stable processes with $\alpha \in (1, 2)$, which have the form (1.1) with $\{L_i\}$ replaced
by real-valued i.i.d. symmetric $\alpha$-stable processes. A key point in their analysis is to prove
the claim that the stochastic convolutions with respect to $L$ stay, with positive probability,
in an arbitrary small ball with zero centre on some path spaces. Subsequently, the authors
solved a control problem to obtain the accessibility. Some technical restrictions are placed
on the driving noises. For example, (ii) on page 3713 of [25], i.e.,

$$\alpha \in (1, 2)$$

and

$$C_1\gamma_i^{-\beta} \leq |\beta_i| \leq C_2\gamma_i^{-\beta} \text{ with } \beta > \frac{1}{2} + \frac{1}{2\alpha}$$

for some positive constants $C_1$ and $C_2$, (1.3)

here $\{\gamma_i = 4\pi^2|i|^2\}$ are the eigenvalues of the Laplace operator on $H$. Under the same
assumptions of [25], in 2017, the authors in [24] established the strong irreducibility of
stochastic real-valued Ginzburg-Landau equations; see [24, Theorem 2.3]. To do this, they
established a support result for stochastic convolutions with respect to $L$ on some suitable
path space; see [24, Lemma 3.2]. They then solved a new control problem with polynomial
term to obtain the irreducibility. By improving the methods in [24], in 2018, the authors in
[23] obtained the strong irreducibility of stochastic real-valued Ginzburg-Landau equations
driven by subordinated cylindrical Wiener process with a $\alpha/2$-stable subordinator, $\alpha \in (1, 2)$;
see [23, Theorem 2.2]. Since the main ideas of [23] are similar to that of [24], the technical
restrictions (1.2) and (1.3) on the driving noises are required in [23].

We remark that all the prior results on the irreducibility of stochastic Ginzburg-Landau
equations driven by pure jump noise concern the real-valued and one-dimensional case. The
previous methods regarding the irreducibility of stochastic real-valued Ginzburg-Landau
equations driven by pure jump noise are basically along the same lines as that of the Gaussian
case; that is, two ingredients play a very important role: the (approximate) controllability
of the associated PDEs and the support of stochastic convolutions on path spaces. These
methods always have some very restrictive assumptions on the driving noises, such as that the
driving noises are additive type and in the class of stable processes, and technical assumptions
such as (1.2) and (1.3) are required. The use of those methods to deal with the case of
other types of pure jump noises is unclear. Moreover, using these methods to study the
irreducibility of stochastic complex Ginzburg-Landau equations would be very hard, if not
impossible.

To the best of our knowledge, there are no results on the irreducibility of stochastic com-
plex Ginzburg-Landau equations driven by pure jump noise. This strongly motivates the
current paper. In this paper, on the one hand, we obtain the strong irreducibility of stochastic
complex Ginzburg-Landau equations driven by multiplicative pure jump nondegenerate
noise. The conditions placed on the driving noises are very mild, including a large class of compound Poisson processes and Lévy processes with heavy tails such as cylindrical symmetric and non-symmetric \( \alpha \)-stable processes with \( \alpha \in (0, 2) \) and subordinated cylindrical Wiener processes with a \( \alpha /2 \)-stable subordinator, \( \alpha \in (0, 2) \), etc. Therefore, our results not only cover all of the previous results but also remove certain technical restrictions required in those results, which we have mentioned above. See Theorems 3.1 and 3.2 in this paper. On the other hand, we establish the accessibility of stochastic complex Ginzburg-Landau equations driven by additive pure jump noise and the driving noises could be degenerate. See Theorem 4.1 in this paper. To prove these main results in this paper, a crucial role is played by the criteria for the irreducibility of stochastic equations driven by pure jump noise developed in [20, 21] by the authors of this paper and T. Zhang. As an application, we obtain the ergodicity of stochastic complex Ginzburg-Laudau equations. We remark that our ergodicity results cover the weakly dissipative case with pure jump degenerate noise. See Section 5 in this paper. Finally, we point out that all of our results are dimension free.

The organization of the paper is as follows. Section 2 presents the well-posedness of stochastic complex Ginzburg-Landau equations driven by pure jump noise. In Sections 3 and 4, we prove the strong irreducibility and accessibility of stochastic complex Ginzburg-Landau equations, respectively. In Section 5, we apply our main results to obtain the ergodicity of stochastic complex Ginzburg-Landau equations.

\section{Preliminaries and Well-posedness}

In this section, we introduce stochastic complex Ginzburg-Landau equations driven by pure jump noise and present its well-posedness.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions. Let \(d \in \mathbb{N}\) and \(D\) be a bounded open domain in \(\mathbb{R}^d\) with smooth boundary \(\partial D\). \(L^q = L^q(D)\) stands for the space of complex-valued measurable functions \(u\) satisfying the Dirichlet boundary condition, i.e., \(u(x) = 0, x \in \partial D\), such that

\[\|u\|_{L^q} = \left( \int_D |u(x)|^q dx \right)^{\frac{1}{q}} < \infty.\]

We regard \(H = L^2\) as a complex Hilbert space with the scalar product

\[\langle u, v \rangle = \text{Re} \int_D u(x) \overline{v(x)} dx, \; u, v \in H,\]

and denote by \(\| \cdot \|\) the corresponding norm. \(V = H^1_0\) is the space of functions that belong to the complex-valued Sobolev space \(H^1(D)\) and satisfy the Dirichlet boundary condition.
We denote by \( \tilde{\nu} \) associated to \( N := N \cup \{0, \infty\} \). We provide a condition driven by pure jump noise: \( \nu \) constants, and for any \( m \in \mathbb{N}, Z_m^c \) denotes the complement of \( Z_m \) relative to \( Z \).

Consider the following stochastic Ginzburg-Landau equations with Dirichlet boundary condition driven by pure jump noise:

\[
\begin{align*}
\frac{du_t}{dt} &= ((\alpha_1 + i\beta_1)\Delta u_t + (\alpha_2 + i\beta_2)|u_t|^{2\theta}u_t + \lambda u_t)dt + \int_{Z^c_t} \sigma(u_{t-}, z)\tilde{\nu}(dt, dz) \\
&\quad + \int_{Z_t} \sigma(u_{t-}, z)\tilde{N}(dt, dz), \\
u_t &= 0 \text{ on } \partial D \text{ for any } t \geq 0,
\end{align*}
\]

where \( \sigma : H \times Z \to H \) is a measurable mapping, \( \theta > 0, \beta_1, \beta_2, \lambda \in \mathbb{R}, \alpha_1 > 0, \alpha_2 < 0 \) are constants, and for any \( m \in \mathbb{N}, Z_m^c \) denotes the complement of \( Z_m \) relative to \( Z \).

Here is the definition of a solution to (2.1).

**Definition 2.1** An \( H \)-valued càdlàg \( \mathcal{F}_t \)-adapted process \( u = (u_t)_{t \in [0, \infty)} \) is called a solution of (2.1), if there exists a \( dt \times \mathbb{P} \)-equivalent class \( \hat{u} \) of \( u \) such that

1. \( \hat{u} \in L^2_{\text{loc}}([0, \infty); V) \cap L^{2\theta+2}_{\text{loc}}([0, \infty); L^{2\theta+2}(D)), \mathbb{P} \)-a.s.
2. The following equality holds, \( \mathbb{P} \)-a.s., for any \( t \geq 0, 
\begin{align*}
u_t &= u_0 + \int_0^t ((\alpha_1 + i\beta_1)\Delta \hat{u}_s + (\alpha_2 + i\beta_2)|\hat{u}_s|^{2\theta}|\hat{u}_s + \lambda \hat{u}_s)ds + \int_0^t \int_{Z^c_s} \sigma(u_{s-}, z)\tilde{\nu}(ds, dz) \\
&\quad + \int_0^t \int_{Z_s} \sigma(u_{s-}, z)\tilde{N}(ds, dz).
\end{align*}

The above equation is interpreted as an equation in \( V^* \), the dual space of \( V \).

We introduce the following conditions.

(C1): \( \alpha_2 + \frac{\beta_1^2}{2\theta+1} < 0. \)

(C2): There exists a positive constant \( k_0 \) such that

\[
\int_{Z_t^c} \|\sigma(0, z)\|^2\nu(dz) \leq k_0.
\]

(C3): There exist positive constants \( k_1 \) and \( 2 \leq p < 2\theta + 2 \) such that, for all \( v_1, v_2 \in H, 
\int_{Z_t^c} \|\sigma(v_1, z) - \sigma(v_2, z)\|^2\nu(dz) \leq k_1(\|v_1 - v_2\|^2 V \|v_1 - v_2\|^p).

(C4): For any \( z \in Z_1, \sigma(\cdot, z) : H \to H \) is continuous.
Remark 2.1 Conditions (C2) and (C3) implies that for any \( v \in H \)
\[
\int_{Z} \| \sigma(v, z) \|^{2} \nu(dz) \leq 2(k_{0} + k_{1} + k_{1} \| v \|^{p}). \tag{2.2}
\]

Now we state the result on the existence and uniqueness of the solution to (2.1).

**Theorem 2.1** Under Conditions (C1)–(C4), for any \( u_{0} \in H \), there exists a unique solution
\[
u_{0} = (u_{t}^{0})_{t \geq 0}
\]
with the initial data \( u_{0} \). Moreover, \( \{u^{x}\}_{x \in H} \) forms a strong Markov process.

Combining the weak convergence argument and monotonicity argument as in [2] and the idea of proving [16, Theorem 2.5], under Conditions (C1)–(C3), one can establish the well-posedness of (2.1). The strong Markov property of \( \{u^{x}\}_{x \in H} \) follows from the Feller property, the fact \( u^{x} \in D([0, \infty); H) \), \( \mathbb{P} \)-a.s., and Condition (C4). Since our primary concern in this paper is the irreducibility of \( \{u^{x}\}_{x \in H} \), the proof of Theorem 2.1 is omitted here.

In the sequel, the symbol \( C \) will denote a positive generic constant whose value may change from line to line. For any \( x \in H \), let \( u_{t}^{x} = (u_{t}^{x})_{t \geq 0} \) be the unique solution to (2.1) with the initial data \( x \).

### 3 Strong Irreducibility

In this section, we study the strong irreducibility of (2.1).

To study the strong irreducibility of (2.1), we introduce a nondegenerate condition on the intensity measure \( \nu \), which basically says that for any \( h, y \in H \), one can reach the neighborhoods of \( y \) from \( h \) through a finite number of choosing jumps.

(C5): For any \( h, y \in H \) with \( h \neq y \) and any \( \bar{\eta} > 0 \), there exist \( n, m \in \mathbb{N} \), and \( \{l_{i}, i = 1, 2, ..., n\} \subset Z_{m} \) such that, for any \( \eta \in (0, \bar{\eta}) \), there exist \( \{\epsilon_{i}, i = 1, 2, ..., n\} \subset (0, \infty) \) and \( \{\eta_{i}, i = 0, 1, ..., n\} \subset (0, \infty) \) such that the following hold, denoting
\[
q_{0} = h, \ q_{i} = q_{i-1} + \sigma(q_{i-1}, l_{i}), \ i = 1, 2, ..., n,
\]

- \( 0 < \eta_{0} \leq \eta_{1} \leq ... \leq \eta_{n-1} \leq \eta_{n} \leq \eta \);
- for any \( i = 0, 1, ..., n - 1 \), \( \{\tilde{q} + \sigma(\tilde{q}, l) : \tilde{q} \in B(q_{i}, \eta_{i}), l \in B(l_{i+1}, \epsilon_{i+1})\} \subset B(q_{i+1}, \eta_{i+1}) \);
- \( B(q_{n}, \eta_{n}) \subset B(y, \bar{\eta}) \);
- for any \( i = 1, 2, ..., n \), \( \nu(B(l_{i}, \epsilon_{i})) > 0 \);
- there exists \( m_{0} \geq m \) such that \( \bigcup_{i=1}^{m} B(l_{i}, \epsilon_{i}) \subset Z_{m_{0}} \).
We have the following main result in this paper.

**Theorem 3.1** Under Conditions (C1)–(C5), the solution \( \{u^x(x)\}_{x \in H} \) of (2.1) is strongly irreducible in \( H \).

To prove Theorem 3.1, the following result taken from [17, Lemmas 2.1 and 2.2] will be used.

**Lemma 3.1** Let \( U \) be a complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \|_U \). Then

- for any \( \gamma > 1 \) and any nonzero \( a, b \in U \) with \( a \neq b \),
  \[
  \frac{|\text{Im}(\|a\|^{\gamma-2}a - \|b\|^{\gamma-2}b, a - b)|}{\text{Re}(\|a\|^{\gamma-2}a - \|b\|^{\gamma-2}b, a - b)} \leq \frac{|\gamma - 2|}{2\sqrt{\gamma - 1}}. \tag{3.1}
  \]

- for any \( p \geq 2 \) and any \( a, b \in U \),
  \[
  \text{Re}(\|a\|^{p-2}a - \|b\|^{p-2}b, a - b) \geq 2^{2-p}\|a - b\|^p. \tag{3.2}
  \]

**Proof** (Proof of Theorem 3.1)

Applying [20, Theorem 2.1] and Theorem 2.1 in this paper, we see that the proof of this theorem will be complete once we prove the following claim.

**Claim 1:** For any \( x, y \in H \), \( \eta > 0 \), define the stopping time \( \tau_{\eta x,y} = \inf\{t \geq 0; u^x_t \notin B(y, \eta)\} \). It is easy to see that \( \tau_{\eta x,x} > 0 \), \( \mathbb{P} \)-a.s.

Denote by \( \tilde{u}^h_t \) and \( \tilde{u}^\tilde{h}_t \) the solutions of (3.3) with the initial data \( h, \tilde{h} \in H \), respectively. Using the Itô formula, we have

\[
\begin{align*}
\|\tilde{u}^h_t - \tilde{u}^\tilde{h}_t\|^2 &= \|\tilde{h} - \tilde{h}\|^2 - 2\alpha_1 \int_0^t \|\tilde{u}^h_s - \tilde{u}^\tilde{h}_s\|^2 ds + 2\lambda \int_0^t \|\tilde{u}^h_s - \tilde{u}^\tilde{h}_s\|^2 ds \\
&\quad + \frac{2\alpha_1}{\gamma} \int_0^t \langle \tilde{u}^h_s - \tilde{u}^\tilde{h}_s, \alpha_2 + i\beta_2 |\tilde{u}^h_s|^{2\gamma} \tilde{u}^h_s - |\tilde{u}^\tilde{h}_s|^{2\gamma} \tilde{u}^\tilde{h}_s \rangle ds
\end{align*}
\]
\[ + 2 \text{Re} \int_0^t \int_{Z_t^-} \langle \hat{u}_s^h - \hat{u}_s^h, \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \rangle \hat{N}(ds, dz) \]
\[ + \int_0^t \int_{Z_t^-} \| \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \|^2 \hat{N}(ds, dz). \quad (3.4) \]

For the fourth term on the right side of the above equality, we apply Lemma 3.1 with \( \gamma = p = 2 \theta + 2 \) and Condition (C1) to obtain
\[ 2 \text{Re}(\hat{u}_s^h - \hat{u}_s^h, (\alpha_2 + i \beta_2)(|\hat{u}_s^h|^{2}\hat{u}_s^h - |\hat{u}_s^h|^{29} \hat{u}_s^h)) \]
\[ = 2 \alpha_2 \text{Re}(\hat{u}_s^h - \hat{u}_s^h, |\hat{u}_s^h|^{29} \hat{u}_s^h - |\hat{u}_s^h|^{29} \hat{u}_s^h) \]
\[ - 2 \beta_2 \text{Im}(\hat{u}_s^h - \hat{u}_s^h, |\hat{u}_s^h|^{29} \hat{u}_s^h - |\hat{u}_s^h|^{29} \hat{u}_s^h) \]
\[ \leq 2(\alpha_2 + \frac{\theta |\beta_2|}{\sqrt{2\theta + 1}}) \mu(\hat{u}_s^h - \hat{u}_s^h, |\hat{u}_s^h|^{29} \hat{u}_s^h - |\hat{u}_s^h|^{29} \hat{u}_s^h) \]
\[ \leq 2^{1-2\theta}(\alpha_2 + \frac{\theta |\beta_2|}{\sqrt{2\theta + 1}}) \| \hat{u}_s^h - \hat{u}_s^h \|^{29+2}_{L^{29+2}} \leq 0. \quad (3.5) \]

Since \( \alpha_1 > 0 \), (3.4) and (3.5) imply that, for any \( t \geq 0 \),
\[ \| \hat{u}_t^h - \hat{u}_t^h \|^2 \]
\[ \leq \| \tilde{h} - h \|^2 + 2 \lambda \int_0^t \| \hat{u}_s^h - \hat{u}_s^h \|^2 ds \]
\[ + 2^{1-2\theta}(\alpha_2 + \frac{\theta |\beta_2|}{\sqrt{2\theta + 1}}) \int_0^t \| \hat{u}_s^h - \hat{u}_s^h \|^{29+2}_{L^{29+2}} ds \]
\[ + 2 \text{Re} \int_0^t \int_{Z_t^-} \langle \hat{u}_s^h - \hat{u}_s^h, \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \rangle \hat{N}(ds, dz) \]
\[ + \int_0^t \int_{Z_t^-} \| \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \|^2 \hat{N}(ds, dz) \]
\[ + \int_0^t \int_{Z_t^-} \| \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \|^2 \nu(dz) ds \]
\[ \leq \| \tilde{h} - h \|^2 + 2 \lambda \int_0^t \| \hat{u}_s^h - \hat{u}_s^h \|^2 ds \]
\[ + 2^{1-2\theta}(\alpha_2 + \frac{\theta |\beta_2|}{\sqrt{2\theta + 1}}) \int_0^t \| \hat{u}_s^h - \hat{u}_s^h \|^{29+2}_{L^{29+2}} ds \]
\[ + 2 \text{Re} \int_0^t \int_{Z_t^-} \langle \hat{u}_s^h - \hat{u}_s^h, \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \rangle \hat{N}(ds, dz) \]
\[ + \int_0^t \int_{Z_t^-} \| \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \|^2 \hat{N}(ds, dz) \]
\[ + k_1 \int_0^t \| \hat{u}_s^h - \hat{u}_s^h \|^2 \hat{N}(ds, dz) \]
\[ \leq \| \tilde{h} - h \|^2 + Ct + 2Re \int_0^t \int_{Z_t^-} \langle \hat{u}_s^h - \hat{u}_s^h, \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \rangle \hat{N}(ds, dz) \]
\[ + \int_0^t \int_{Z_t^-} \| \sigma(\hat{u}_s^h, z) - \sigma(\tilde{u}_s^- , z) \|^2 \hat{N}(ds, dz) \]. \quad (3.6)
To get the second and last inequalities of (3.6), Conditions (C1) and (C3) and the following fact have been used. Condition (C1) and $2 \leq p < 2\theta + 2$ imply that there exists a constant $C$ such that

$$2\lambda \|\tilde{u}_s^h - \tilde{u}_s^h\|^2 + 2^{1-2\theta}(\alpha_2 + \theta|\beta_2|/\sqrt{2\theta + 1})\|\tilde{u}_s^h - \tilde{u}_s^h\|^{2\theta + 2} + k_1[\|\tilde{u}_s^h - \tilde{u}_s^h\|^2 \vee \|\tilde{u}_s^h - \tilde{u}_s^h\|^p] \leq C < \infty.$$ 

Applying stochastic Gronwall’s inequality (see [22, Lemma 3.7]), we deduce from (3.6) that for any $0 < q < p < 1$ and any $T > 0$,

$$\mathbb{E}\left[\left(\sup_{0 \leq t \leq T} \|\tilde{u}_t^h - \tilde{u}_t^h\|^2\right)^q\right] \leq \left(\frac{p}{p-q}\right)\left(\|\tilde{h} - h\|^2 + CT\right)^q. \quad (3.7)$$

Therefore, applying Chebysev’s inequality, for any $\eta > 0$, there exists $\tilde{c}, \tilde{t} > 0$ such that, for any $\tilde{h} \in B(h, \tilde{c})$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq \tilde{t}} \|\tilde{u}_s^h - \tilde{u}_s^h\| \leq \frac{\eta}{4}\right) = 1 - \mathbb{P}\left(\sup_{0 \leq s \leq \tilde{t}} \|\tilde{u}_s^h - \tilde{u}_s^h\| > \frac{\eta}{4}\right) \geq \frac{1}{2}. \quad (3.8)$$

**Step 2.** Denote

$$N(Z_1, t) := \int_0^t \int_{Z_1} N(dz, ds), \quad t \geq 0.$$ 

Let $\tau_1$ be the first jumping time of the Poisson process $N(Z_1, t)$, $t \geq 0$, i.e.,

$$\tau_1 = \inf\{t \geq 0; N(Z_1, t) = 1\}. \quad (3.9)$$

$\tau_1$ has an exponential distribution with parameter $\nu(Z_1) < \infty$, that is,

$$\mathbb{P}(\tau_1 > s) = e^{-\nu(Z_1)s}, \quad \mathbb{P}(\tau_1 \leq s) = 1 - e^{-\nu(Z_1)s}.$$ 

We can choose $t_1$ small enough such that

$$0 < t_1 \leq \tilde{t} \quad \text{and} \quad \mathbb{P}(\tau_1 > t_1) > \frac{1}{2}. \quad (3.10)$$

It is easy to see that, for any $x \in H$, $\{u_s^x, t \in [0, \tau_1]\}$ is the unique solution to (3.3) with the initial data $x$ on $t \in [0, \tau_1)$. Then, by (3.8) and (3.10), for any $\tilde{h} \in B(h, \tilde{c})$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq t_1} \|u_s^h - u_s^h\| \leq \frac{\eta}{4}\right) = \mathbb{P}\left(\sup_{0 \leq s \leq t_1} \|u_s^h - u_s^h\| \leq \frac{\eta}{4}, \tau_1 > t_1\right) + \mathbb{P}\left(\sup_{0 \leq s \leq t_1} \|u_s^h - u_s^h\| \leq \frac{\eta}{4}, \tau_1 \leq t_1\right) \geq \mathbb{P}\left(\sup_{0 \leq s \leq t_1} \|u_s^h - u_s^h\| \leq \frac{\eta}{4}, \tau_1 > t_1\right) \geq \mathbb{P}\left(\sup_{0 \leq s \leq t_1} \|\tilde{u}_s^h - \tilde{u}_s^h\| \leq \frac{\eta}{4}, \tau_1 > t_1\right).$$
\[
\mathbb{P}\left( \sup_{0 \leq s \leq t_1} \| \tilde{u}^h_x - u^h_x \| \leq \frac{\eta}{4} \right) \times \mathbb{P}(\tau_1 > t_1) \\
\geq \frac{1}{4},
\]
(3.11)

For the last equality of (3.11), we have used the fact that, for any \( x \in H \), \( \sigma\{ \tilde{u}^x_t, t \geq 0 \} \) and \( \sigma\{\tau_1\} \) are independent.

Since \( u^h \in D([0, \infty); H) \) \( \mathbb{P}\)-a.s., we know that there exists \( t_2 > 0 \) small enough such that
\[
\mathbb{P}( \sup_{0 \leq s \leq t_2} \| u^h_s - h \| > \frac{\eta}{4} ) < \frac{1}{8}.
\]

(3.12)

Set
\[
\epsilon = \frac{\eta}{8} \wedge \tilde{c}, \quad t = t_1 \wedge t_2.
\]

By (3.11) and (3.12),
\[
\inf_{h \in B(h, \epsilon)} \mathbb{P}(\tau^\eta_{h,h} \geq t) \\
\geq \inf_{h \in B(h, \epsilon)} \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - h \| \leq \frac{\eta}{2} \right) \\
\geq \inf_{h \in B(h, \epsilon)} \mathbb{P}\left( \left\{ \sup_{0 \leq s \leq t} \| u^h_s - u^h_s \| \leq \frac{\eta}{4} \right\} \cap \left\{ \sup_{0 \leq s \leq t} \| u^h_s - h \| \leq \frac{\eta}{4} \right\} \right) \\
\geq \inf_{h \in B(h, \epsilon)} \left( 1 - \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - u^h_s \| > \frac{\eta}{4} \right) \right) - \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - h \| > \frac{\eta}{4} \right) \\
= \inf_{h \in B(h, \epsilon)} \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - u^h_s \| \leq \frac{\eta}{4} \right) - \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - h \| > \frac{\eta}{4} \right) \\
\geq \inf_{h \in B(h, \epsilon)} \mathbb{P}\left( \sup_{0 \leq s \leq t} \| u^h_s - u^h_s \| \leq \frac{\eta}{4} \right) - \frac{1}{8} \\
\geq \frac{1}{8},
\]
(3.13)

This completes the proof of Claim 1, so the proof of Theorem 3.1 is also complete.

Let us now consider the particular case of additive noise. Let \( Z = H \), and let \( \nu \) denote a given \( \sigma \)-finite intensity measure of a Lévy process on \( H \). Recall that \( \nu(\{0\}) = 0 \) and \( \int_H (\|z\|^2_H + 1) \nu(dz) < \infty \). Let \( \tilde{N} : \mathcal{B}(H \times \mathbb{R}^+) \times \Omega \rightarrow \tilde{\Omega} \) be the time homogeneous Poisson random measure with intensity measure \( \nu \). Again \( \tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt \) denotes the compensated Poisson random measure associated to \( N \).

Let us point out that (as shown by e.g., [19] Theorems 4.23 and 6.8]) in this case
\[
L(t) = \int_0^t \int_{0 < \|z\|_H \leq 1} z\tilde{N}(dz, ds) + \int_0^t \int_{\|z\|_H > 1} zN(dz, ds), t \geq 0
\]
defines an \( H \)-valued Lévy process.

Consider (2.1) driven by \( L(t), t \geq 0; \) that is, \( \sigma(\cdot, z) \equiv z \) and
\[
du_t = ((\alpha_1 + i\beta_1)\Delta u_t + (\alpha_2 + i\beta_2)|u_t|^{2\theta}u_t + \lambda u_t)dt + dL(t),
\]
where
\[ u_t = 0, \text{ on } \partial D \text{ for any } t \geq 0. \] (3.14)

In this setting, we introduce the following hypothesis regarding the intensity measure \( \nu \):

\textbf{(C5')} For any \( h \in H \) and \( \eta > 0 \), there exist \( n \in \mathbb{N} \), a sequence of strict positive numbers \( \eta_1, \eta_2, \ldots, \eta_n \), and \( a_1, a_2, \ldots, a_n \in H \setminus \{0\} \), such that
\[ 0 \notin B(a_i, \eta_i), \nu(B(a_i, \eta_i)) > 0, \]
\[ 1 \leq i \leq n \] and \( \sum_{i=1}^{n} h_i : h_i \in B(a_i, \eta_i), 1 \leq i \leq n \} \subset B(h, \eta) \).

As an application of Theorem 3.1 (see also [20, Theorem 2.2]), we have

**Theorem 3.2** Under Conditions (C1) and (C5’), the solution \( \{u^x\}_{x \in H} \) to (3.14) is strongly irreducible in \( H \).

For any measure \( \rho \) on \( H \), its support \( S_\rho = S(\rho) \) is defined to be the set of \( x \in H \) such that \( \rho(G) > 0 \) for any open set \( G \) containing \( x \). Set
\[ H_0 := \left\{ \sum_{i=1}^{n} m_i a_i, n, m_1, \ldots, m_n \in \mathbb{N}, a_i \in S_\nu \right\}. \] (3.15)

Finally, we point out the fact based on [20, Subsection 4.1] that Condition (C5’) holds if and only if \( H_0 \) is dense in \( H \).

**Remark 3.1** The conditions placed on the driving noises, i.e., conditions (C5) and (C5’), are very mild, and many examples satisfying these conditions can be found in [20, Subsections 4.1 and 4.2]. In particular, for the case of additive noise, the driving noises include a large class of Lévy processes with heavy tails, i.e., the intensity measures \( \nu \) satisfy for some \( \alpha \in (0, 2], \int_{\|z\| > 1} \|z\|^{2\alpha} \nu(dz) = \infty \), e.g., the cylindrical symmetric and nonsymmetric \( \alpha \)-stable processes with \( \alpha \in (0, 2) \), and subordinated cylindrical Wiener processes with a \( \alpha/2 \)-stable subordinator, \( \alpha \in (0, 2) \), etc. The driving noises even include processes whose intensity measures \( \nu \) satisfy for any small \( \alpha > 0 \), \( \int_{\|z\| > 1} \|z\|^{\alpha} \nu(dz) = \infty \). Also included is a large class of compound Poisson processes, which is somewhat surprising. For more details, see [20, Section 4].

### 4 Accessibility

In this section, we study the accessibility of (3.14). The driving noises could be degenerate.

We need to consider
\[ dU_t^x = \left( (\alpha_1 + i \beta_1) \Delta U_t^x + (\alpha_2 + i \beta_2) |U_t^x|^{2\alpha} U_t^x + \lambda U_t^x \right) dt + \int_{0 < \|z\| \leq \epsilon} z \tilde{N}(dt, dz), \quad \epsilon \in (0, 1) \]
\[ U_t^x = 0 \text{ on } \partial D, \forall t \geq 0, \] (4.1)
and
\[ dU_t = \left( (\alpha_1 + i \beta_1) \Delta U_t + (\alpha_2 + i \beta_2) |U_t|^{2\alpha} U_t + \lambda U_t \right) dt, \]
\[ U_t = 0 \text{ on } \partial D, \forall t \geq 0. \]  

(4.2)

By Theorem 2.1, for any \( x \in H \), there exist unique global solutions \( u^x \), \( U^{x,x} \) and \( U^x \) to (3.14), (4.1) and (4.2) starting from \( x \), respectively. The following condition is required to study the accessibility.

(C6): \( \lambda \leq \alpha_1 \lambda_1 \). Here and in the following, \( \lambda_1 \) is the first eigenvalue of \( -\Delta \).

Theorem 4.1 Under Conditions (C1) and (C6), assume that \( \nu \) is symmetric, the solution \( \{ u^x \}_{x \in H} \) of (3.14) is accessible to zero.

Proof According to [21, Theorem 2.1], to complete the proof of this theorem, we only need to verify the following claims:

(A1) For any \( x \in H \), \( \lim_{t \to \infty} \| U^x_t \| = 0 \).

(A2) For any \( t > 0 \) and \( x \in H \), \( \lim_{\epsilon \to 0} \| U^{x,x}_{\epsilon,x} - U^x_x \| = 0 \) in probability.

(A3) For any \( \eta > 0 \), there exist \((\zeta, t) = (\zeta(\eta), t(\eta)) \in (0, \frac{\eta}{2}) \times (0, \infty)\) such that

\[ \inf_{y \in B(0, \zeta)} \mathbb{P}(\sup_{s \in [0, t]} \| u^y_s \| \leq \eta) > 0. \]

Here \( B(0, \zeta) = \{ h \in H : \| h \| < \zeta \} \).

Now, we verify Claim (A1).

Applying the chain rule gives

\[ \| U^x_t \|^2 = \| x \|^2 - 2\alpha_1 \int_0^t \| U^x_s \|_V^2 ds + 2\lambda \int_0^t \| U^x_s \|^2 ds + 2\alpha_2 \int_0^t \| U^x_s \|^2_{L^{2\theta+2}} ds. \]

By (C6), \( \alpha_2 < 0 \), the Poincaré inequality and Hölder inequality,

\[ \frac{d\| U^x_t \|^2}{dt} = -2\alpha_1 \| U^x_t \|_V^2 + 2\lambda \| U^x_t \|^2 + 2\alpha_2 \| U^x_t \|^{2\theta+2}_{L^{2\theta+2}} \leq 2\alpha_2 \| U^x_t \|^{2\theta+2}_{L^{2\theta+2}} \leq \frac{2\alpha_2}{m(D)^{2\theta}} \| U^x_t \|^{2\theta+2}, \]  

(4.3)

where \( m(D) \) is the Lebesgue measure of \( D \). Applying \( \alpha_2 < 0 \) again, (4.3) implies that the map \( t \to \| U^x_t \|^2 \) is decreasing. Then by a contradiction argument, it is easy to see that

\[ \lim_{t \to +\infty} \| U^x_t \|^2 = 0. \]  

(4.4)

Using an argument similar to that used in the proof of Theorem 3.1 for any \( T > 0 \),

\[ \lim_{\epsilon \to 0} \mathbb{E}\| U^{x,x}_{\epsilon,T} - U^x_T \|^2 = 0, \]  

(4.5)

which implies Claim (A2). Claim (A3) follows from Claim 1 in the proof of Theorem 3.1.

The proof of Theorem 4.1 is complete. □
5 Applications

In this section, we apply the irreducibility obtained above to study stochastic complex Ginzburg-Landau equations driven by pure jump noise.

We first investigate the existence of the invariant measure.

5.1 Existence of the invariant measure

To prove the existence of the invariant measure, we need the following condition on \( \sigma \):

(C7): Assume that there exists \( \hat{\theta} \in (0, 1) \), \( r \in [0, 2\theta + \hat{\theta}) \) and positive constants \( k_4, k_5 \) such that for all \( v \in H \)

\[
\int_{Z_1} \|u(z)\|^2 \nu(dz) + \int_{Z_1} \|v(z)\|^2 \nu(dz) \leq k_4\|u\|^r + k_5.
\]

**Theorem 5.1** Under Conditions (C1)-C4) and (C7), there exists at least one invariant measure for \( (2.7) \).

**Proof** The proof is in the spirit of [7]. Define a function \( f \) on \( H \) by

\[ f(u) = (\|u\|^2 + 1)^{\frac{\theta}{2}}. \]

By simple calculations, for any \( u, v \in H \)

\[ |f(u) - f(v)| \leq (\|u\|^2 + 1)^{\frac{\theta}{2}} - (\|v\|^2 + 1)^{\frac{\theta}{2}} \leq \|u - v\|^\theta, \tag{5.1} \]

and

\[ \nabla f(u) = \frac{\hat{\theta}u}{(\|u\|^2 + 1)^{1 - \frac{\theta}{2}}}, \quad \nabla^2 f(u) = \frac{\hat{\theta}N_{i=1}^\infty e_i \otimes e_i}{(\|u\|^2 + 1)^{1 - \frac{\theta}{2}}} - \frac{\hat{\theta}(2 - \hat{\theta})u \otimes u}{(\|u\|^2 + 1)^{1 - \frac{\theta}{2}}}. \tag{5.2} \]

Here \( \{e_i, i \in \mathbb{N}\} \) is an orthonormal basis of \( H \).

Fix \( x \in H \). Let \( u^x \) be the unique solution to \( (2.7) \) with initial data \( x \). Applying the Itô formula gives

\[ f(u^x_t) = f(x) - \int_0^t \int_{Z_1} \frac{\alpha_1 \hat{\theta}u_x^2}{(\|u_x^2 + 1)^{1 - \frac{\theta}{2}}}ds + \int_0^t \int_{Z_1} H_{u_x}^2 N(dz, dz) + \int_0^t \int_{Z_1} N(\|u_x^2 + 1)^{1 - \frac{\theta}{2}}ds \]

\[ + \int_0^t \int_{Z_1} \frac{\hat{\theta}N_{i=1}^\infty e_i \otimes e_i}{(\|u_x|^2 + 1)^{1 - \frac{\theta}{2}}}ds + \int_0^t \int_{Z_1} N(\|u_x^2 + 1)^{1 - \frac{\theta}{2}}ds \]

\[ + \int_0^t \int_{Z_1} \frac{\hat{\theta}(2 - \hat{\theta})u_x \otimes u_x}{(\|u_x|^2 + 1)^{1 - \frac{\theta}{2}}}ds + \int_0^t \int_{Z_1} N(\|u_x^2 + 1)^{1 - \frac{\theta}{2}}ds \]

\[ \tag{5.3} \]

\[ + \int_0^t \int_{Z_1} \frac{\hat{\theta}(u_x^2 \otimes u_x)(\sigma(u_x^2 + 1)^{1 - \frac{\theta}{2}} - \sigma(u_x^2 + 1)^{1 - \frac{\theta}{2}})}{(\|u_x^2 + 1)^{1 - \frac{\theta}{2}}}ds. \]

13
Using a standard stopping time argument, by Condition (C7), (5.1), (5.2), and Taylor’s expansion,

\[
\mathbb{E} f(u_t^x) + \mathbb{E} \int_0^t \frac{\alpha_1 \hat{\theta} \|u_s^x\|_V^2}{\|u_s^x\|^2 + 1} ds \\
\leq f(x) + \mathbb{E} \int_0^t \frac{\alpha_2 \hat{\theta} \|u_s^x\|_{L^{2\theta + 2}}^2 + \hat{\theta} \lambda \|u_s^x\|^2}{\|u_s^x\|^2 + 1} ds \\
+ \mathbb{E} \int_0^t \int_{Z_1} \|\sigma(u_s^x, z)\| \nu(dz) ds + \mathbb{E} \int_0^t \|\sigma(u_s^x, z)\|^2 \nu(dz) ds \\
\leq f(x) + \mathbb{E} \int_0^t \frac{\alpha_2 \hat{\theta} \|u_s^x\|_{L^{2\theta + 2}}^2 + \hat{\theta} \lambda \|u_s^x\|^2}{\|u_s^x\|^2 + 1} ds \\
+ \mathbb{E} \int_0^t \int_{Z_1} \|\sigma(u_s^x, z)\|^2 \nu(dz) ds + \mathbb{E} \int_0^t (k_4 \|u_s^x\|)^r + k_5 ds \\
\leq f(x) + \mathbb{E} \int_0^t \frac{\alpha_2 \hat{\theta} \|u_s^x\|_{L^{2\theta + 2}}^2 + \hat{\theta} \lambda \|u_s^x\|^2 + k_6 \|u_s^x\|^{r + 2 - \theta} + k_7}{\|u_s^x\|^2 + 1} ds.
\]

Since \(\alpha_2 < 0\) and \((r + 2 - \hat{\theta}) < 2\theta + 2\), there exists a constant \(C\) such that

\[
\alpha_2 \hat{\theta} \|u_s^x\|_{L^{2\theta + 2}}^2 + \hat{\theta} \lambda \|u_s^x\|^2 + k_6 \|u_s^x\|^{r + 2 - \theta} + k_7 \leq C.
\]

Hence (5.4) implies that

\[
\mathbb{E} f(u_t^x) + \mathbb{E} \int_0^t \frac{\alpha_1 \hat{\theta} \|u_s^x\|_V^2}{\|u_s^x\|^2 + 1} ds \leq f(x) + Ct.
\]

Therefore,

\[
\mathbb{E} \int_0^t \|u_s^x\|_{V}^{\hat{\theta}} ds \leq \mathbb{E} \int_0^t \frac{\|u_s^x\|^{\hat{\theta}}_V (\|u_s^x\|^2 - \theta + 1)}{\|u_s^x\|^2 + 1} ds \leq C\mathbb{E} \int_0^t \frac{\|u_s^x\|_{V}^2 + 1}{\|u_s^x\|^2 + 1} ds \leq C(1 + f(x) + t).
\]

By the classical Bogoliubov-Krylov argument (cf. [5]), the above inequality implies the existence of the invariant measure. \(\square\)

### 5.2 Ergodicity: The case of additive noise

In this subsection, we will consider the uniqueness of the invariant measure of (3.14) with the so-called weakly dissipative condition, i.e., \(\lambda = \alpha_1 \lambda_1\). The driving noises could be degenerate. The following theorem is the main result of this subsection.
Theorem 5.2 Assume that Condition (C1) and \( \lambda = \alpha_1 \lambda_1 \) hold. If one of the following conditions is satisfied

- Condition (C5') holds;
- \( \nu \) is symmetric;

then there exists at most one invariant measure for (3.14). If, moreover,

\[
\exists \, \hat{\theta} \in (0, 1], \quad \int_{Z_1} \|z\|^\hat{\theta} \nu(dz) < \infty,
\]

then there exists a unique invariant measure for (3.14).

Proof For any \( x, y \in H \), denote by \( u^x, u^y \) the unique solutions to (3.14) with initial data \( x, y \), respectively. Applying the Itô formula gives

\[
\|u^x_t - u^y_t\|^2 = \|x - y\|^2 + 2\lambda \int_0^t \|u^x_s - u^y_s\|^2 ds - 2\alpha_1 \int_0^t \|u^x_s - u^y_s\|^2 \nu(ds)
\]

\[
+ 2\text{Re} \int_0^t \langle u^x_s - u^y_s, (\alpha_2 + i\beta_2)(|u^x_s|^{2\theta} u^x_s - |u^y_s|^{2\theta} u^y_s) \rangle ds
\]

\[
\leq \|x - y\|^2.
\]

(5.6)

To obtain the last inequality of (5.6), we have used \( \|u^x_s - u^y_s\|^2 \nu \geq \lambda_1 \|u^x_s - u^y_s\|^2 \), \( \lambda = \alpha_1 \lambda_1 \) and the following estimate

\[
2\text{Re} \int_0^t \langle u^x_s - u^y_s, (\alpha_2 + i\beta_2)(|u^x_s|^{2\theta} u^x_s - |u^y_s|^{2\theta} u^y_s) \rangle ds \leq 0.
\]

(5.7)

The proof of (5.7) follows that of (5.5).

(5.6) implies that \( \{u^x\}_{x \in H} \) satisfies the so-called \( e \)-property, i.e., for any \( x \in H \) and Lipschitz bounded function \( \phi : H \to \mathbb{R} \),

\[
\limsup_{y \to x} \sup_{t \geq 0} |\mathbb{E}\phi(u^x_t) - \mathbb{E}\phi(u^y_t)| = 0.
\]

[14, Theorem 2] implies that if a Markov process has the \( e \)-property and irreducibility, then the Markov process has at most one invariant measure. Therefore, combining Theorem 3.2 and Theorem 4.1, if either Condition (C5') holds or \( \nu \) is symmetric, then there exists at most one invariant measure for (3.14).

Moreover, if (5.5) holds, then Theorem 5.1 implies the existence of invariant measure for (3.14).

The proof of Theorem 5.2 is complete. \( \square \)

In the following, we present examples of additive driving noises satisfying (5.5).
Example 5.1 Cylindrical Lévy process

Let \( \{e_i, i \in \mathbb{N}\} \) be an orthonormal basis of \( H \). Let \( \{(L_i(t))_{t \geq 0}, i \in \mathbb{N}\} \) be a sequence of independent one-dimensional pure jump Lévy processes with the intensity measure \( \mu_i \).

Choose \( \beta_i \in \mathbb{R}, i \in \mathbb{N} \) such that
\[
\sum_{i=1}^{\infty} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \mu_i(dx_i) < \infty, \tag{5.8}
\]
and there exists a constant \( \hat{\theta} \in (0, 1] \) such that
\[
\sum_{i=1}^{\infty} \int_{|\beta_i x_i| > 1} |\beta_i x_i|^\hat{\theta} \mu_i(dx_i) < \infty. \tag{5.9}
\]

Then, the intensity measure \( \nu \) of \( L(t) = \sum_{i=1}^{\infty} \beta_i L_i(t)e_i, t \geq 0, \) satisfies (5.5). Indeed,
\[
\int_H \|z\|^2 \wedge 1 \nu(dz) = \sum_{i=1}^{\infty} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \mu_i(dx_i) < \infty,
\]
and
\[
\int_{Z_1} \|z\|^\hat{\theta} \nu(dz) = \sum_{i=1}^{\infty} \int_{|\beta_i x_i| > 1} |\beta_i x_i|^\hat{\theta} \mu_i(dx_i) < \infty.
\]

There are many concrete examples such that \( \sum_{i=1}^{\infty} 1_{\mathbb{R}\setminus\{0\}}(\beta_i)\mu_i(\mathbb{R}) < \infty \), and then, the driving Lévy process \( L \) is a compound Poisson process on \( H \). Another concrete example is the so-called cylindrical \( \alpha \)-stable processes with \( \alpha \in (0, 2) \).

The following example is concerned with the subordination of Lévy processes, which is an important idea to obtain new Lévy processes. Here we just introduce a concrete example, the so-called subordinated cylindrical Wiener process. We refer the reader to [20, Example 4.3] for more details and examples.

Example 5.2 Subordinated cylindrical Wiener process

Let \( \{e_i, i \in \mathbb{N}\} \) be an orthonormal basis of \( H \) and \( \{W_i^t, t \geq 0; i \in \mathbb{N}\} \) be a sequence of i.i.d. one-dimensional Brownian motions. Let \( \beta_i \in \mathbb{R}, i \in \mathbb{N} \) be given constants satisfying \( \sum_{i=1}^{\infty} \beta_i^2 < \infty \). Let \( W_t, t \geq 0 \) be the Wiener process on \( H \) given by
\[
W_t = \sum_{i=1}^{\infty} \beta_i W_t^i e_i.
\]

For \( \alpha \in (0, 2) \), let \( S_t, t \geq 0 \) be an \( \alpha/2 \)-stable subordinator, i.e., an increasing one-dimensional Lévy process with Laplace transform
\[
\mathbb{E}e^{-\eta S_t} = e^{-\eta^{\alpha/2}}, \quad \eta > 0.
\]

\( W_t, t \geq 0 \) and \( S_t, t \geq 0 \) are independent. The subordinated cylindrical Wiener process \( L_t, t \geq 0 \) on \( H \) is defined by
\[
L_t := W_{S_t} = \sum_{i=1}^{\infty} \beta_i W_{S_t}^i e_i.
\]

Then the intensity measure \( \nu \) of \( L_t, t \geq 0 \) is symmetric and satisfies (5.5).
We stress that the driving Lévy process $L$ introduced in the above two examples could be degenerate; that is, there could exist some $I \subset \mathbb{N}$ such that $\beta_i = 0$ for any $i \in I$.

### 5.3 Ergodicity: The case of multiplicative noise

To derive the uniqueness of the invariant measure in the case of multiplicative noise, we require the following conditions:

(C8): Assume that $\sigma$ satisfies for all $v_1, v_2 \in H$

\[
\|\sigma(v_1, z) - \sigma(v_2, z)\| \leq L(z)\|v_1 - v_2\|
\]

with the Lipschitz coefficient satisfying $\int_H L^2(z)\nu(dz) < \infty$ and $\int_{0<\|z\|<1} L(z)\nu(dz) < \infty$.

(C9) (Weak dissipation):

\[
2 \int_{\mathbb{Z}_1} L(z)\nu(dz) + \int_H L^2(z)\nu(dz) + 2\lambda \leq 2\alpha_1\lambda_1.
\]

We now state the ergodicity of (2.1) as follows:

**Theorem 5.3** Under Conditions (C1)-(C5), (C8), and (C9), there exists at most one invariant measure for (2.1), and if that invariant measure exists, then the support of that measure is $H$. If, moreover, (C7) holds, then there exists a unique invariant measure for (2.1).

**Proof** Applying the Itô formula gives

\[
\|u_t^x - u_t^y\|^2
\]

\[
= \|x - y\|^2 + 2\lambda \int_0^t \|u_s^x - u_s^y\|^2 ds - 2\alpha_1 \int_0^t \|u_s^x - u_s^y\|_V^2 ds
\]

\[
+ 2\Re \int_0^t \langle u_s^x - u_s^y, (\alpha_1 + i\beta_2)(|u_s^x|^{2\theta} u_s^x - |u_s^y|^{2\theta} u_s^y) \rangle ds
\]

\[
+ \int_0^t \int_{\mathbb{Z}_1} (\|u_{s-}^x - u_{s-}^y + \sigma(u_{s-}^x, z) - \sigma(u_{s-}^y, z)\|^2 - \|u_{s-}^x - u_{s-}^y\|^2)\nu(ds, dz)
\]

\[
+ \int_0^t \int_{\mathbb{Z}_1} (\|u_{s-}^x - u_{s-}^y + \sigma(u_{s-}^y, z) - \sigma(u_{s-}^x, z)\|^2 - \|u_{s-}^x - u_{s-}^y\|^2)\tilde{\nu}(ds, dz)
\]

\[
+ \int_0^t \int_{\mathbb{Z}_1} \|\sigma(u_s^x, z) - \sigma(u_s^y, z)\|^2 \nu(dz)ds.
\] (5.10)

Using a standard stopping time argument, by (5.7), Conditions (C8) and (C9), and the Poincaré inequality, we have

\[
\mathbb{E}\|u_t^x - u_t^y\|^2
\]

\[
\leq \|x - y\|^2 + 2\lambda \int_0^t \mathbb{E}\|u_s^x - u_s^y\|^2 ds - 2\alpha_1 \int_0^t \mathbb{E}\|u_s^x - u_s^y\|_V^2 ds
\] (5.11)
\[
\begin{align*}
&+ \left( 2 \int_{Z_1} L(z) \nu(dz) + \int_H L^2(z) \nu(dz) \right) \mathbb{E} \int_0^t \| u^x_s - u^y_s \|_2^2 ds \\
\leq & \| x - y \|_2^2 + \left( 2 \int_{Z_1} L(z) \nu(dz) + \int_H L^2(z) \nu(dz) + 2\lambda - 2\alpha_1 \lambda_1 \right) \int_0^t \mathbb{E} \| u^x_s - u^y_s \|_2^2 ds.
\end{align*}
\]

If \( 2 \int_{Z_1} L(z) \nu(dz) + \int_H L^2(z) \nu(dz) + 2\lambda < 2\alpha_1 \lambda_1 \), then by (5.11) there exists a constant \( c > 0 \) such that
\[
\mathbb{E} \| u^x_t - u^y_t \|_2^2 \leq \| x - y \|_2^2 e^{-ct},
\]
which implies the uniqueness of the invariant measure. Theorem 3.1 implies that if invariant measures exist, then the support of any invariant measure is \( H \). If \( 2 \int_{Z_1} L(z) \nu(dz) + \int_H L^2(z) \nu(dz) + 2\lambda = 2\alpha_1 \lambda_1 \), then (5.11) implies that \( \{ u^x; x \in H \} \) satisfies the \( e \)-property. Combining Theorem 3.1 and [14, Theorem 2], there exists at most one invariant measure for (2.1), and if invariant measures exist, then the support of any invariant measure is \( H \). We remark that the novelty of our result in this subsection concerns the case
\[
2 \int_{Z_1} L(z) \nu(dz) + \int_H L^2(z) \nu(dz) + 2\lambda = 2\alpha_1 \lambda_1.
\]

If, moreover, (C7) holds, then Theorem 5.1 implies the existence of invariant measure for (2.1).

The proof of Theorem 5.3 is complete.

Acknowledgement. This work is partially supported by NSFC (No. 12131019, 11971456, 11721101).

References

[1] I. S. Aranson and L. Kramer, The world of the complex Ginzburg-Landau equation, Rev. Mod. Phys. 74 (2002), 99-143.

[2] Z. Brzeźniak, W. Liu, J. Zhu, Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise, Nonlinear Anal. Real World Appl. 17 (2014):283-310.

[3] M.C. Cross, P.C. Hohenberg, Pattern formation outside of equilibrium, Rev. Modern Phys. 65 (1993) 851-1089.

[4] G. Da Prato and J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, London Math.Soc. Lecture Note Ser. 229, Cambridge University Press, Cambridge, UK, 1996.

[5] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions, Encycloedia of Mathematics and its Applications, vol.44, Cambridge University Press, Cambridge, 1992.
[6] J.D. Doering, C.R. Gibbon, C.D. Levermore, Weak and strong solutions of the complex Ginzburg-Landau equation, Phys. D. 71 (1994) 285-318.

[7] Z. Dong, L. Xu, X. Zhang, Invariance measures of stochastic 2D Navier-Stokes equations driven by α-stable processes, Electron. Comm. Probab. 16 (2011), 678-688.

[8] J. L. Doob, Asymptotic property of markov transtion probability. Trans. Amer. Math. Soc., 64 (1948) 393-421.

[9] D. Down, S.P. Meyn, R.L. Tweedie, Exponential and uniform ergodicity of Markov processes. Ann. Probab. 23 (1995) 1671-1691.

[10] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity, Springer, Berlin, Heidelberg, 2009, 113-137.

[11] M. Hairer, J.C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. Ann. of Math. (2006)(2) 164 993-1032.

[12] V. Ginzburg, L. Landau, On the theory of superconductivity, Zh. Eksp. Teor. Fiz. 20 (1950) 1064; English transl.: I. Ter Haar (Ed.), Men of Physics: L.D. Landau, vol. I, Pergamon Press, New York, 1965, pp. 546-568.

[13] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd ed. North-Holland Mathematical Library 24. North-Holland, Amsterdam (1989).

[14] R. Kapica, T. ´Szarek, M. leczka. On a unique ergodicity of some Markov processes. Potential Anal. 2012, 36 (4) 589-606.

[15] T. Komorowski, S. Peszat, T. Szerek, On ergodicity of some Markov processes, Ann. Probab. 38 (4) (2010), 1401-1443.

[16] L. Lin, H. Gao, A stochastic generalized Ginzburg-Landau equation driven by jump noise, J. Theoret. Probab. 32 (1) (2019) 460-483.

[17] N. Okazawa, T. Yokota, Global existence and smoothing effect for the complex Ginzburg-Landau equation with p-Laplacian, J. Differ. Equ. 182 (2002), 514-576.

[18] S. Peszat, J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert spaces. Ann. Probab. 1995, 157-172.

[19] S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy Noise: Evolution Equations Approach, Cambridge University Press, Cambridge, 2007.
[20] J. Wang, H. Yang, J. Zhai, T. Zhang, Irreducibility of SPDEs driven by pure jump noise. http://arxiv.org/abs/2207.11488.

[21] J. Wang, H. Yang, J. Zhai, T. Zhang, Accessibility of SPDEs driven by pure jump noise and its applications. https://doi.org/10.48550/arXiv.2209.04875.

[22] L. Xie, X. Zhang, Ergodicity of stochastic differential equations with jumps and singular coefficients. Ann. Inst. Henri Poincaré Probab. Stat., 56(1) 2020, 175-229.

[23] R. Wang, L. Xu Asymptotics for stochastic reaction-diffusion equation driven by subordinate Brownian motion. Stochastic Process. Appl. 128 (5) 2018, 1772-1796.

[24] R. Wang, J. Xiong, L. Xu, Irreducibility of stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises and applications. Bernoulli, 23 (2) (2017), 1179-1201.

[25] L. Xu, Ergodicity of the stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises, Stochastic Process. Appl. 123 (2013) 3710-3736

[26] X. Zhang, Exponential ergodicity of non-Lipschitz stochastic differential equations. Proc. Amer. Math. Soc. 137 (1) (2009), 329-337.