A MIRROR CONJECTURE FOR PROJECTIVE BUNDLES

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Abstract. We propose, motivate and give evidence for a relation between the \( D \)-modules of the quantum cohomology of a smooth complex projective manifold \( X \) and a projective bundle \( \mathbb{P}(\oplus L_i) \) over \( X \).

1. Introduction and results

Let \( Y \) be a projective manifold. Denote by \( Y_{k,\beta} \) the moduli stack of rational stable maps of class \( \beta \in H_2(Y,\mathbb{Z}) \) with \( k \)-markings [FP] and \([Y_{k,\beta}]\) its virtual fundamental class [BP,LT]. Throughout this paper we will be interested mainly in \( k = 1 \). Recall the following features

- \( e : Y_{1,\beta} \to Y \) - the evaluation map.
- \( \psi \) - the first chern class of the cotangent line bundle on \( Y_{1,\beta} \).
- \( ft : Y_{1,\beta} \to Y_{0,\beta} \) - the forgetful morphism.

Let \( h \) be a formal variable and

\[
J_\beta(Y) := e_* \left( \frac{[Y_{1,\beta}]}{h(h - \psi)} \right) = \sum_{k=0}^{\infty} \frac{1}{h^{2+k}} e_* (\psi^k \cap [Y_{1,\beta}]).
\]

The sum is finite for dimension reasons. Let \( p = \{p_1, p_2, ..., p_k\} \) be a nef basis of \( H^2(Y,\mathbb{Q}) \). For \( t = (t_0, t_1, ..., t_k) \) let

\[
kp := t_0 + \sum_{i=1}^{k} t_i p_i.
\]

The \( D \)-module for the quantum cohomology of \( Y \) is generated by \([G3]\)

\[
J_Y = \exp \left( \frac{tp}{h} \right) \sum_{\beta \in H_2(Y,\mathbb{Z})} q^\beta J_\beta(Y)
\]

where we use the convention \( J_0 = 1 \). The generator \( J_Y \) encodes all of the one marking Gromov-Witten invariants and gravitational descendents of \( Y \).

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For any ring $A$, the formal completion of $A$ along the semigroup $MY$ of the rational curves of $Y$ is defined to be

$$\mathcal{A}[[q^\beta]] := \{ \sum_{\beta \in MY} a_\beta q^\beta, \; a_\beta \in A, \; \beta - \text{effective} \}.$$  

where $\beta \in H_2(Y, \mathbb{Z})$ is effective if it is a positive linear combination of rational curves. This new ring behaves like a power series since for each $\beta$, the set of $\alpha$ such that $\alpha$ and $\beta - \alpha$ are both effective is finite. We may identify $q^\beta$ with $q^{d_1} \cdots q^{d_k} = \exp(t_1 d_1 + \cdots + t_k d_k)$ where $\{d_1, d_2, \ldots, d_k\}$ are the coordinates of $\beta$ relative to the dual basis of $\{p_1, \ldots, p_k\}$. We view $J_Y$ as an element of $H^*(Y, \mathbb{Q})[[q^\beta]]$.

For toric varieties $J_Y$ is related to an explicit hypergeometric series $I_Y$ via a change of variables ([G2], [LLY3]). Furthermore, if $Y$ is Fano then the change of variables is trivial, i.e. $J_Y = I_Y$ thus completely determining the one point Gromov-Witten invariants and gravitational descendants of $Y$. This fact was known even earlier for $Y = \mathbb{P}^n$ (see for example [G1]), which is the case of relevance for us. An elementary proof of its equivariant version has been presented in part 3 of section 3. Its nonequivariant limit yields $J_{\mathbb{P}^n} = I_{\mathbb{P}^n}$. (An alternative proof in terms of related torus actions and equivariant considerations follows as a trivial application from [B], [G3], [LLY1].) We seek to extend this result in the case of a projective bundle.

Let $X$ be a projective manifold. Following Grothendieck’s notation, let $\pi: \mathbb{P}(V) = \mathbb{P}(\bigoplus_{i=0}^n L_i) \to X$ be the projective bundle of hyperplanes of a vector bundle $V$. Assume that $L_0 = \mathcal{O}_X$. The $H^*X$-module $H^*\mathbb{P}(V)$ is generated by $z := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ with the relation

$$z \prod_{i=1}^n (z - c_1(L_i)) = 0.$$  

Let $s_i: X \to \mathbb{P}(V)$ be the section of $\pi$ determined by the $i$-th summand of $V$ and $X_i := s_i(X)$. Then $\mathcal{O}_{\mathbb{P}(V)}(1)|_{X_i} \simeq L_i$. Let $\{p_1, \ldots, p_k\}$ be a nef basis of $H^2(X, \mathbb{Q})$.

**Lemma 1.0.1.** If the line bundles $L_i, i = 1, \ldots, n$ are nef then

(a) $\{p_1, \ldots, p_k, z\}$ is a nef basis of $H^2(\mathbb{P}(V), \mathbb{Q})$.

(b) The Mori cones of $X$ and $\mathbb{P}(V)$ are related via

$$M\mathbb{P}(V) = MX \oplus \mathbb{Z}_{\geq 0} \cdot [l]$$

where $[l]$ is the class of a line in the fiber of $\pi$.  

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Proof. Consider the fibration
\[ \mathbb{P}^n \longrightarrow \mathbb{P}(V) \]
\[ \pi \downarrow \]
\[ X. \]
There is a first quadrant homology Leray-Serre spectral sequence with
\[ E^2_{p,q} = H_p(X, H_q(\mathbb{P}^n, \mathbb{Z})) \]
which abuts to \( H_{p+q}(\mathbb{P}(V), \mathbb{Z}) \). The diagonal \( p + q = 2 \) and the fact that \( H_1(\mathbb{P}^n, \mathbb{Z}) = 0 \) yield a short exact sequence
\[ 0 \rightarrow H_2\mathbb{P}^n \rightarrow H_2\mathbb{P}(V) \overset{\pi_*}{\rightarrow} H_2X \rightarrow 0. \]
The pushforward
\[ s_0* : H_2(X) \rightarrow H_2(\mathbb{P}(V)) \]
_splits the above sequence. Note that for any \( \beta \in H_2(X, \mathbb{Q}) \)
\[ (3) \quad s_0*(\beta) \cdot z = 0. \]
Let \( C \) be a curve in \( \mathbb{P}(V) \) and \( f : C \rightarrow \mathbb{P}(V) \) the inclusion map. There is a surjection
\[ (4) \quad \pi^*(V) \rightarrow O_{\mathbb{P}(V)}(1) \rightarrow 0. \]
Restricting this sequence to \( C \) we obtain
\[ (5) \quad \oplus_i f^*(L_i) \rightarrow O_C(z \cdot C) \rightarrow 0. \]
Since \( \text{deg} f^*(L_i) \geq 0 \) for all \( i \), we obtain that \( z \cdot C \geq 0 \). Hence \( z \) is nef. Now if we let \( \beta = \pi_*(C) \) then \( \pi_*(C - s_0*(\beta)) = 0 \). Therefore \( C - s_0*(\beta) = \nu \cdot [l] \). But
\[ \nu \cdot (C \cdot [l]) = z \cdot (C - s_0*(\beta)) = z \cdot C. \]
Since \( z \) is nef we get \( \nu \geq 0 \). \( \square \)

As seen in the above lemma, if \( C \in \mathbb{P}(V) \) is a curve, there exists a unique pair \( (\nu \geq 0, \beta \in MX) \) such that \([C] = \nu[l] + \beta\). We will identify the homology class \([C]\) with \((\nu, \beta)\). Note that the generator \( J_{\mathbb{P}(V)} \) is an element of \( H^*(\mathbb{P}(V), \mathbb{Q})[t, t_{k+1}][[q_1^\nu, q_2^\beta]]. \)
To understand the the relation between the generators \( J_{\mathbb{P}(V)} \) and \( J_X \) we first consider the case when the projective bundle is trivial \( \mathbb{P}(V) = \mathbb{P}^n \times X \). From now on we will suppress \( X \) and use \( J_\beta \) instead of \( J_{\beta}(X) \). Denote the homology class of a rational curve in \( \mathbb{P}^n \) by its degree \( \nu \) and the hyperplane class by \( H \). Then
\[ J_{\mathbb{P}^n \times X} = J_{\mathbb{P}^n} \times J_X = \exp \left( \frac{tp + t_{k+1}H}{h} \right) \sum_{\nu \geq 0, \beta} q_1^\nu q_2^\beta \frac{1}{\prod_{m=1}^{\nu}(H + mh)^{\nu+1}} J_\beta. \]
For a general vector bundle $V$ we propose twisting the coefficient in front of $J$ by the Chern classes of $V$. More precisely, let $v_\beta := \beta \cdot c_1(L_i)$ and define the “twisting” factor

$$\mathcal{T}_{\nu, \beta} := \prod_{i=0}^{n} \prod_{m=-\infty}^{0} (z - c_1(L_i) + m\hbar).$$

Let $I_{\nu, \beta} := \mathcal{T}_{\nu, \beta} \cdot \pi^* J_\beta$ where $\pi^*$ is the flat pull back and define a “twisted” hypergeometric series for the projective bundle $\mathbb{P}(V)$:

$$I_{\mathbb{P}(V)} := \exp \left( \frac{tp + t_{k+1}z}{\hbar} \right) \sum_{\nu, \beta} q_1^\nu q_2^\beta I_{\nu, \beta} \in H^*(\mathbb{P}(V), \mathbb{Q})[t, t_{k+1}][[q_1^\nu, q_2^\beta]].$$

In the next section we motivate and propose the following

**Conjecture 1.0.1.** Let $L_i$, $i = 1, ..., n$ be nef line bundles such that $-K_X - c_1(V)$ is ample. Then $J_{\mathbb{P}(V)} = I_{\mathbb{P}(V)}$.

There is much evidence for this conjecture. First the extremal cases: on one end, the pure fiber case, i.e. the equality of the $\beta = 0$ terms is known to be true; on the other end we prove the equality of the $d = 0$ terms when all the line bundles $L_i$ are ample. We also show that this conjecture is consistent with the Quantum Lefshetz Principle applied to all the complete intersections $\prod_{j \neq i} (z - c_1(L_i))$. Finally, when $X$ is a toric variety and $L_i$ are toric line bundles the projective bundle $\mathbb{P}(V)$ is also a toric variety and the validity of the conjecture follows from the toric mirror theorem ([G2], [LLY3]).

### 2. Motivation

In this section we motivate the conjecture by comparing $J_{\mathbb{P}(V)}$ and $I_{\mathbb{P}(V)}$. Recall first that there is a natural grading in the quantum cohomology of $\mathbb{P}(V)$ ([G2], [G3])

- $\deg q_1 := n + 1$
- $\deg q_2^\beta := \beta \cdot (-K_X - c_1(V))$
- $\deg(\alpha) = \text{codim}_C (\alpha)$ for any pure cycle $\alpha \in H_4(\mathbb{P}(V), \mathbb{Z})$.

**Lemma 2.0.2.** Both $J_{\mathbb{P}(V)}$ and $I_{\mathbb{P}(V)}$ are homogeneous of degree zero.

**Proof.** From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \pi^* V^*(1) \to T_{\mathbb{P}(V)} \to \pi^* T_X \to 0$$

we find that

$$K_{\mathbb{P}(V)} = \pi^* K_X + \pi^* c_1(V) - (n + 1)z.$$
Multiplying by a curve class \((\nu, \beta)\) in \(H_2\mathbb{P}(V)\) we obtain
\[-(\nu, \beta) \cdot K_{\mathbb{P}(V)} = (n + 1)\nu - \beta \cdot K_X - \beta \cdot c_1(V) = \deg(q_1^\nu) \cdot \deg(q^\beta).\]

Hence
\[\dim e_*([\mathbb{P}(V)_{1,(\nu, \beta)}]) = \deg(q_1^\nu) \cdot \deg(q^\beta) + \dim \mathbb{P}(V) - 2\]

Therefore
\[\deg e_*(\psi^k \cap [\mathbb{P}(V)_{1,(\nu, \beta)}]) = 2 + k - \deg(q_1^\nu) \cdot \deg(q^\beta).\]

It follows from the presentation (1) that the degree of \(J_{\nu, \beta}\) is
\[-\deg(q_1^\nu) \cdot \deg(q^\beta),\]

hence \(\sum q_1^\nu q^\beta J_{\nu, \beta}(\mathbb{P}(V))\) is homogeneous of degree zero.

We turn our attention to the generator \(I_{\mathbb{P}(V)}\). Notice that
\[\deg e_*(\psi^k \cap [X_{1,\beta}] ) = \dim X - (\dim [X_{1,\beta}] - k) = \beta \cdot K_X + 2 + k\]

and so
\[\deg (\pi^*(J_\beta)) = \deg J_\beta = \beta \cdot K_X.\]

Now
\[\deg \bar{I}_{\nu, \beta} = -\beta \cdot c_1(V) - (n + 1)\nu.\]

It follows that
\[\deg I_{\nu, \beta} = \beta \cdot c_1(V) - (n + 1)\nu + \beta \cdot K_X = -\deg(q_1^\nu) \cdot \deg(q^\beta)\]

and the lemma is proved. \(\square\)

For a trivial bundle \(V\), we showed that \(J_{\mathbb{P}(V)} = I_{\mathbb{P}(V)}\) using the corresponding identity for projective spaces. From the general framework of mirror theorems, it is reasonable to expect this identity to hold for general \(V\), if the \(h\)-asymptotic expansions of \(J_{\mathbb{P}(V)}\) and \(I_{\mathbb{P}(V)}\) coincide in orders \(h^0\) and \(h^{-1}\) (3). By the definition
\[\sum_{\nu, \beta} q_1^\nu q^\beta J_{\nu, \beta} = 1 + o(h^{-1}).\]

We now show that, under the assumptions of the conjecture, \(I_{\mathbb{P}(V)}\) has the same asymptotic expansion. For \(0 \neq (\nu, \beta) \in M\mathbb{P}(V)\), let \(l_{\nu, \beta}\) be the cardinality of the set
\[S_{\nu, \beta} = \{i : \nu - v_i^\beta < 0\}.\]

A simple algebra shows that
\[\bar{I}_{\nu, \beta} = \frac{a_{\nu, \beta}}{h^\nu + (n + 1)\nu - c_1(V)\beta} \cdot P(h^{-1}).\]
Here \( a_{\nu, \beta} \in H_* (\mathbb{P} (V)) \) is homogeneous of degree \( l_{\nu, \beta} \) and \( P (h^{-1}) \) is a power series of \( h^{-1} \). On the other hand, it follows from (1) that
\[
\pi^* (J_\beta) = \frac{a_{\beta}}{h_{\nu, \beta}} \cdot Q (h^{-1})
\]
where \( n_{\beta} = \max (2, -\beta \cdot K_X) \) and \( Q (h^{-1}) \) is also a power series of \( h^{-1} \).

It follows that \( I_{\nu, \beta} = \tilde{I}_{\nu, \beta} \cdot \pi^* (J_\beta) \) is a power series of \( h^{-1} \) with leading term

\[
A_{\nu, \beta} \cdot \frac{1}{h_{\nu, \beta} + (n + 1) \nu - \beta \cdot c_1 (V) + n_{\beta}}
\]

for every \( 0 \neq (\nu, \beta) \in M \mathbb{P} (V) \). Let \( l_{\nu, \beta} := l_{\nu, \beta} + (n + 1) \nu - \beta \cdot c_1 (V) + n_{\beta} \).

If \( \nu > 0 \) then \( n_{\nu, \beta} > (n + 1) \nu > 1 \). If \( \beta \cdot c_1 (V) = 0 \) then \( n_{\nu, \beta} = 0 \), \( i = 0, 1, \ldots, n \) hence \( l_{0, \beta} = 0 \) and \( n_{0, \beta} = n_{\beta} \geq 2 \). Finally if \( \beta \cdot c_1 (V) > 0 \) then \( l_{0, \beta} > 0 \) hence \( n_{0, \beta} > (-K_X - c_1 (V) \cdot \beta \geq 1 \). It follows that \( I_{\nu, \beta} = o (h^{-1}) \) for any \( 0 \neq (\nu, \beta) \in M \mathbb{P} (V) \).

3. Evidence for the conjecture

1. The conjecture holds for toric varieties. Indeed, assume \( X \) is a toric manifold and let \( \Sigma \subset \mathbb{Z}^m \) be its fan. Denote by \( b_1, \ldots, b_r \) its one dimensional cones. If \( L_i, i = 0, 1, \ldots, n \) are toric line bundles then the projective bundle \( \mathbb{P} (\bigoplus_{i=0}^n L_i) \) \( X \) is also a toric variety and there is a canonical way to obtain its fan [Q]. Let \( \mathbb{Z}^n \) be a new lattice with basis \( \{ f_1, \ldots, f_n \} \). The edges \( b_1, \ldots, b_r \) of \( \Sigma \) are lifted to new edges \( B_1, B_2, \ldots, B_r \) in \( \mathbb{Z}^m \oplus \mathbb{Z}^n \) and subsequently \( \Sigma \) is lifted in a new fan \( \Sigma_1 \) in the obvious way. Let \( \Sigma_2 \subset 0 \oplus \mathbb{Z}^n \) be the fan of \( \mathbb{P}^n \) with edges \( F_0 = - \sum_{i=1}^n f_i, F_1 = f_1, \ldots, F_n = f_n \). The canonical fan associated to \( \mathbb{P} (V) \) consists of the cones \( \sigma_1 + \sigma_2 \) where \( \sigma_1, \sigma_2 \) are cones in \( \Sigma_1, \Sigma_2 \). Let \( N = r + n + 1 \). The edges \( F_i \) in the canonical fan of the projective bundle, correspond to the divisors \( z - c_1 (L_i) \) where \( z := c_1 (\mathcal{O}_{\mathbb{P}^n} (1)) \) while \( B_i \) correspond to the pullback of the base divisors [M]. Since \( L_i, i = 0, 1, \ldots, n \) are nef and \( -K_X - \sum_i L_i \) is ample it follows that 

\[
J_{F(V)} = \exp \left( \frac{tp + t_{k+1}z}{h} \right) \cdot \sum_{(\nu, \beta) \in M \mathbb{P} (V)} q_1^{\nu} q_2^{\beta} \tau_{\nu, \beta} \prod_{i} \frac{\prod_{m=-\infty}^{0} (b_i + mh)}{\prod_{m=-\infty}^{0} (b_i + mh)}
\]

and

\[
J_X = \exp \left( \frac{tp}{h} \right) \cdot \sum_{\beta \in M X} q_2^{\beta} \prod_{i} \frac{\prod_{m=-\infty}^{0} (b_i + mh)}{\prod_{m=-\infty}^{0} (b_i + mh)}.
\]

The equality \( J_{F(V)} = I_{F(V)} \) follows readily.
2. Consistency with the Quantum Hyperplane Section Principle. Let \( W = \bigoplus_{i=1}^{r} \mathcal{L}_i \to Y \) be a vector bundle over a projective manifold \( Y \). Let \( Z = Z(s) \) be the zero locus of a generic section \( s \) of \( W \) and \( \mu \) the inclusion map. For any \( 0 \neq \beta \in H_2(Y, \mathbb{Z}) \), denote by \( W_\beta \) the bundle on \( M_{0,1}(Y, \beta) \) whose fiber over a stable map \( (C, x_1, f) \) is the vector space of the sections of \( H^0(C, f^*(W)) \) that vanish at \( x_1 \). The following \( H^*(Y)[[q^\beta]] \)-valued generator

\[
(7) \quad J_W = \exp \left( \frac{tp}{\hbar} \right) c_*(W) \left( 1 + \sum_{\beta \neq 0} q^\beta e_* \left( \frac{c_{top}(W_\beta)}{\hbar(h - \psi)} \right) \right).
\]

is intrinsically related to the quantum \( \mathcal{D} \)-module of \( Z \). Assume that \( \mu^*: H^2(Y, \mathbb{Q}) \to H^2(Z, \mathbb{Q}) \) is surjective. The Gysin map in cohomology extends to a map of completions

\[
\mu^*: H^*(Z, \mathbb{Q})[[t_Z]][[q^\beta]] \to H^*(Y, \mathbb{Q})[[t_Y]][[q^\alpha]]
\]

via \( \mu^*(q^\beta) = q^{\mu^*(\beta)} \). It is shown in \( \text{AE} \) that

\[
\mu^*(J_Z) = J_W.
\]

Consider now

\[
(8) \quad I_W = \exp \left( \frac{tp}{\hbar} \right) c_r(W) \left( 1 + \sum_{\beta \neq 0} q^\beta L_W^\beta e_* \left( \frac{[Y_{1,\beta}]}{\hbar(h - \psi)} \right) \right)
\]

where the modifying Lefschetz factor \( L_W^\beta \) is defined to be

\[
L_W^\beta := \prod_{i=1}^{r} \prod_{m=-\infty}^{b_i} \frac{c_1(\mathcal{L}_i) + mh}{\prod_{m=-\infty}^{0} (c_1(\mathcal{L}_i) + mh)}
\]

with \( b_i := \beta \cdot c_1(\mathcal{L}_i) \). The Quantum Hyperplane Section Principle asserts that, under suitable conditions, \( J_W \) equals \( I_W \) up to a change of variables of the form

\[
t_0' = t_0 + f_0(q)\hbar + f(q) \\
t_i' = t_i + f_i(q), \quad i = 1, 2, ..., n
\]

where \( f_i \)'s are homogeneous power series of degree 0 and \( f \) is homogeneous power series of degree 1 (\( \text{[K2, K3, YPL]} \), and for the most general form \( \text{[CC]} \)). Each section \( s_i: X \to X_i \subset \mathbb{P}(V) \) is the zero locus of a section of the line bundle \( \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}(V)}(1) \otimes L_j^* \) in \( \mathbb{P}(V) \).

**Proposition 3.0.1.** Quantum Hyperplane Section Principle applied appropriately to the generator \( I_{\mathbb{P}(V)} \) produces the (Gysin image of the) generator \( J_{X_i} \) of the quantum \( \mathcal{D} \)-module of \( X_i \).
Proof. For notational simplicity we focus in the case \( i = 0 \) and let \( \mu_0 : X_0 \rightarrow \mathbb{P}(V) \) be the inclusion map. Consider the following cohomology-valued function

\[
I = \exp \left( \frac{tp + t_{k+1}z}{\hbar} \right) \prod_{i=1}^{n}(z - c_1(L_i)) \sum_{d,\beta} q_1^d q_2^\beta \cdot \prod_{d=1}^{n} \prod_{m=-\infty}^{0}(z - c_1(L_i) + m\hbar) \prod_{m=-\infty}^{0}(z - c_1(L_i) + m\hbar) \pi^* J_\beta = \exp \left( \frac{tp + t_{k+1}z}{\hbar} \right) \prod_{i=1}^{n}(z - c_1(L_i)) \sum_{d,\beta} q_1^d q_2^\beta \cdot \prod_{d=1}^{n} \prod_{m=-\infty}^{0}(z - c_1(L_i) + m\hbar) \pi^* J_\beta.
\]

Expanding

\[
I = \prod_{i=1}^{n}(z - c_1(L_i)) \exp \left( \frac{tp}{\hbar} \right) \exp \left( \frac{q_1}{\hbar} \right) \sum_{\beta} q_2^\beta \pi^* (J_\beta).
\]

Let \( t'_0 = t_0 + q_1 \). With this new variables we have \( \mu_0_*(J_{X_0}) = I \). The proposition is proven. \( \square \)

3. Extremal Case I: Pure fiber. In the case where \( \beta = 0 \) our conjecture reads

\[
e_*(\left[ \mathbb{P}(V)_{I,\nu,0} \right] \big/ \hbar(h - \psi)) = \prod_{i=0}^{n} \prod_{m=1}^{n}(z - c_1(L_i) + m\hbar).\]

This is true. In fact the restriction of \( J_{\mathbb{P}(V)} \) to \( \beta = 0 \) is obtained from the relative Gromov-Witten theory of \( \mathbb{P}(V) \) over \( X \), which can be defined via intersection theory in the spaces of holomorphic curves to the fibers of \( \pi \). If we let \( \mathbb{T} = (\mathbb{C}^*)^{n+1} \) act diagonally on \( \mathbb{P}^n \) then the relative GW theory of \( \mathbb{P}^n \) bundle over \( B\mathbb{T} \) associated with the universal bundle \( E\mathbb{T} \rightarrow B\mathbb{T} \) is precisely the \( \mathbb{T} \)-equivariant GW theory of \( \mathbb{P}^n \). Furthermore, if \( f : X \rightarrow B\mathbb{T} \) induces the principal bundle associated with \( \mathbb{P}(V) \), then the \( \mathbb{T} \)-equivariant theory of \( \mathbb{P}^n \) pulls back via \( f \) to the relative GW theory of \( \mathbb{P}(V) \) over \( X \). It follows that the restriction of \( J_{\mathbb{P}(V)} \) to \( \beta = 0 \) is obtained by substituting \( c_1(L_i) \) for the generators.
λ of \( H^*(BT) = \mathbb{Q}[\lambda_0, \ldots, \lambda_n] \) in the \( T \)-equivariant generator \( J_T^{\mathbb{P}^n} \). Now recall that \( J_T^{\mathbb{P}^n} \) and its derivatives form a fundamental solution matrix to the quantum differential equation

\[
hq \frac{dS}{dq} = H \ast S,
\]

where we use the same letter \( H \) to denote the equivariant hyperplane class and \( \ast \) is the product in the small equivariant quantum cohomology algebra of \( \mathbb{P}^n \) (this is essentially due to Dijkgraaf-Witten and Dubrovin [D]). For dimension reasons, if \( k < n \) we have \( H \ast H^k = H^{k+1} \), where the powers of \( H \) are the classical ones. The product \( H \ast H^n \) is found from the only possible graded deformation

\[
\prod_{i=0}^n (H - \lambda_i) = xq
\]

of the classical relation (where \( x = 0 \)). The coefficient \( x \) is equal to 1 (two points in \( \mathbb{P}^n \) determine a unique line). The small equivariant quantum cohomology of \( \mathbb{P}^n \) is

\[
\mathbb{Q}[H, \lambda_0, \ldots, \lambda_n, q]/(\prod_{i=0}^n (H - \lambda_i) - q)
\]

(see also [AS], [K1].) The relations imply that \( J_T^{\mathbb{P}^n} \) is annihilated by the operator

\[
D = \prod_{i=0}^n (hq \frac{d}{dq} - \lambda_i) - q.
\]

One can see that \( J_T^{\mathbb{P}^n} \) is the only solution of the above equation that has the form

\[
\exp \left( \frac{t_0 + H \ln q}{\hbar} \right) (1 + q \sum_{\nu \geq 0} a_\nu q^\nu),
\]

where \( a_\nu \in H^*_T(\mathbb{P}^n)[h^{-1}] \). But

\[
I_T^{\mathbb{P}^n} = \exp \left( \frac{t_0 + H \ln q}{\hbar} \right) \sum_{\nu \geq 0} q^\nu \prod_{i=0}^n \prod_{m=1}^\nu (H - \lambda_i + mh)
\]

has also the above form and a direct calculation shows that it is annihilated by the operator \( D \). Hence

(9) \( J_T^{\mathbb{P}^n} = I_T^{\mathbb{P}^n} \)

and the pure fiber case follows. (For another approach see [H].) We note that the nonequivariant limit (i.e. \( \lambda_i \to 0 \), \( i = 0, 1, \ldots, n \)) of (9) yields \( J_{\mathbb{P}^n} = I_{\mathbb{P}^n} \).
4. Extremal Case II: No fiber. This is the case where \( \nu = 0 \).

**Proposition 3.0.2.** If the line bundles \( L_i, i = 1, 2, \ldots, n \) are ample then the conjecture holds for \( d = 0 \) i.e.

\[
J_{0,\beta}(\mathbb{P}(V)) = \prod_{i=1}^{n} \prod_{m=0}^{\beta c_1(L_i) - 1} (z - c_1(L_i) - mh)^\pi^*(J_\beta).
\]

**Proof.** The trivial action of the torus \( \mathbb{T} = (\mathbb{C}^* )^{n+1} \) on \( X \) lifts to a diagonal action on the fibers of \( \mathbb{P}(V) \). Let \( \mathcal{O}(1)_i \) be the pull back of the \( i \)-th \( \mathcal{O}_{\mathbb{P}^{\infty}}(1) \) to \( X_\mathbb{T} = X \times \mathbb{P}^{\infty} \times \ldots \times \mathbb{P}^{\infty} \) and \( \mathcal{L}_i := L_i \otimes \mathcal{O}(1)_i \). Then \( \mathbb{P}(V)_\mathbb{T} = \mathbb{P}(\bigoplus_{i=0}^{n}(\mathcal{L}_i)) \) and

\[
H_{\mathbb{T}}^*(\mathbb{P}(V)) := H^*(\mathbb{P}(V)_\mathbb{T}) = H^*X[z_\mathbb{T}, u_0, \ldots, u_n] / \prod_{i=0}^{n}(z_\mathbb{T} - c_1(L_i) - u_i),
\]

where \( z_\mathbb{T} := c_1(\mathcal{O}(\mathbb{P}(V)_\mathbb{T})(1)) \) and \( u_i := c_1(\mathcal{O}(1)_i) \). The connected components of the fixed point loci are the \( n+1 \) sections \( s_i : X \simeq X_i \subset \mathbb{P}(V) \) that correspond to the quotients \( V \to L_i \). Note that

\[
s_i^*(z_\mathbb{T}) = c_1(L_i) + u_i.
\]

The proposition will follow as a nonequivariant limit (i.e. \( u_i = 0 \)) of the equivariant identity

\[
e_\ast \left( \frac{[\mathbb{P}(V)_{1,0,\beta}]}{h(h - \psi)} \right) = \prod_{i=1}^{n} \prod_{m=0}^{\beta c_1(L_i) - 1} (z^\mathbb{T} - c_1(L_i) - u_i - mh)^\pi^*(J_\beta)
\]

where the left hand side is the equivariant pushforward.

By the localization theorem (\[(AB)\]), it suffices to show that the restrictions to the fixed point components \( X_i \) agree. For \( i = 1, 2, \ldots, n \) the restriction of the right hand side obviously vanishes. The \( \mathbb{T} \)-action on \( \mathbb{P}(V) \) induces an action on \( \mathbb{P}(V)_{1,0,\beta} \). If \( i \neq 0 \), none of the connected components of \( \mathbb{P}(V)_{1,0,\beta}^\mathbb{T} \) is mapped to \( X_i \) by the evaluation map \( e \). Hence the restriction of the left side to \( X_i \) vanishes for \( i = 1, 2, \ldots, n \).

We notice that \( N_{X_0/\mathbb{P}(V)} = \bigoplus_{i=1}^{n} L_0 \otimes L_i^* \simeq \bigoplus_{i=1}^{n} L_i^* \). The induced \( \mathbb{T} \) action is such that

\[
e^\mathbb{T}(N_{X_0/\mathbb{P}(V)}) = \prod_{i=1}^{n} (u_0 - u_i - c_1(L_i)) = \prod_{i=1}^{n} (-c_1^\mathbb{T}(L_i)).
\]

The restriction of the right hand side to \( X_0 \) is

\[
e^\mathbb{T}(N_{X_0/\mathbb{P}(V)}) \prod_{i=1}^{n} \prod_{m=1}^{\beta c_1(L_i) - 1} (-c_1^\mathbb{T}(L_i) - mh)J_\beta.
\]


Only one fixed point component of $\mathbb{P}(V)_{1,0,\beta}$ is mapped to $X_0$. It is isomorphic to $X_{1,\beta}$. Since $f^*(L_i)$ is positive for every nonconstant map $f: \mathbb{P}^1 \to X$, 

$$H^1 := R^1ft_**e^*(N_{X_0/\mathbb{P}(V)}) = R^1ft_**e^*(\bigoplus_{i=1}^n L_i^*)$$

is a vector bundle on this fixed point component. A simple analysis of the obstruction theory shows that the equivariant Euler class of the normal bundle to this component is inverse to the equivariant Euler class of $H^1$. We apply the localization theorem for the evaluation map $e: \mathbb{P}(V)_{1,0,\beta} \to \mathbb{P}(V)$ and the fixed point component $X_0 \subset \mathbb{P}(V)$ \cite{B}, \cite{LLY2} to obtain

$$e^* \left( \frac{c_{\text{top}}(H^1) \cap [X_{1,\beta}]}{h(h - \psi)} \right) = \frac{1}{e^*(N_{X_0/\mathbb{P}(V)})} \cdot s^*_0 \left( \frac{[\mathbb{P}(V)_{1,0,\beta}]}{h(h - \psi)} \right).$$

Suffices to show that

\begin{equation}
\tag{10}
e^* \left( \frac{c_{\text{top}}(H^1) \cap [X_{1,\beta}]}{h(h - \psi)} \right) = \prod_{i=1}^n \prod_{m=1}^{-\beta c_1(L_i) - 1} (-c_1^T(L_i) - m\hbar)J_{\beta}.
\end{equation}

This is an equivariant form of the concave Quantum Lefshetz Principle for the diagonal fiber action. The nonequivariant proofs of \cite{CG} or \cite{YPL} can be trivially modified for this equivariant version. \qed

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