The problems of classifying pairs of forms and local algebras with zero cube radical are wild

Genrich Belitskii*
Dept. of Mathematics, Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel, genrich@cs.bgu.ac.il

Vitalij M. Bondarenko
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine
vit-bond@imath.kiev.ua

Ruvim Lipyanski
Dept. of Mathematics, Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel, lipyansk@cs.bgu.ac.il

Vladimir V. Plachotnik
Mech.-Math. Faculty, Kiev National University,
Vladimirskaja 64, Kiev, Ukraine

Vladimir V. Sergeichuk†
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine
sergeich@imath.kiev.ua

Dedicated to C. M. Ringel on the occasion of his 60th birthday

Abstract

This is the authors’ version of a work that was published in Linear Algebra Appl. 402 (2005) 135–142.

*Partially supported by Israel Science Foundation, Grant 186/01.
†Corresponding author. Partially supported by Ukrainian SFFR, Grant 01.07/00132.
We prove that over an algebraically closed field of characteristic not two the problems of classifying pairs of sesquilinear forms in which the second is Hermitian, pairs of bilinear forms in which the second is symmetric (skew-symmetric), and local algebras with zero cube radical and square radical of dimension 2 are hopeless since each of them reduces to the problem of classifying pairs of \(n\)-by-\(n\) matrices up to simultaneous similarity.

AMS classification: 17B30; 15A21; 16G60

Keywords: Local algebras; Pairs of bilinear forms; Wild problems

1 Introduction

All matrices, vector spaces, and algebras are considered over an algebraically closed field \(\mathbb{F}\) of characteristic not two.

The problem of classifying pairs of \(n \times n\) matrices up to similarity transformations

\[
(A, B) \mapsto (S^{-1}AS, S^{-1}BS), \quad S \text{ is nonsingular,}
\]

is hopeless since it contains the problem of classifying any system of linear operators and the problem of classifying representations of any finite-dimensional algebra; see [1]. Classification problems that contain the problem of classifying pairs of matrices up to similarity are called wild and the others are called tame; see strict definitions in [3].

We prove that the problem of classifying local algebras \(\Lambda\) with \((\text{Rad } \Lambda)^3 = 0\) and \(\dim(\text{Rad } \Lambda)^2 = 2\) is wild (Theorem 3). Recall that an algebra \(\Lambda\) over \(\mathbb{F}\) is a finite dimensional vector space being also a ring with respect to the same addition and some multiplication such that

\[
a(uv) = (au)v = u(av) \quad \text{for all } a \in \mathbb{F} \text{ and } u, v \in \Lambda.
\]

An algebra \(\Lambda\) is local if the set \(R\) of its noninvertible elements is closed under addition (then \(R\) is the radical of \(\Lambda\) and is denoted by \(\text{Rad } \Lambda\)).

We prove in passing the wildness of the problems of classifying

(i) pairs of sesquilinear forms, in which the second is Hermitian (with respect to a nonidentity involution on \(\mathbb{F}\)),

(ii) pairs of bilinear forms, in which the second is symmetric, and
pairs of bilinear forms, in which the second is skew-symmetric.

The hopeless of the problems of classifying tuples (i)–(iii) was also proved in [4] by another method (which was used in [3] too): each of them reduces to the problem of classifying representations of a wild quiver.

Belitskii, Lipyanski, and Sergeichuk worked on these problems when Sergeichuk was visiting the Ben-Gurion University of the Negev in November and December 2003. They discussed applications of [1, Theorem 4.5] stating that the problem of classifying tensors \( T \in U^* \otimes U^* \otimes U \) on a vector space \( U \) is wild (such a tensor determines a bilinear binary operation on \( U \)). Then these authors knew that the wildness of the problem of classifying algebras was also proved by Bondarenko and Plachotnik using another reduction to a matrix problem. So we decided to write this paper jointly.

2 Pairs of forms

Let \( a \mapsto \bar{a} \) be any involution on \( \mathbb{F} \), that is, a bijection \( \mathbb{F} \to \mathbb{F} \) such that
\[
\bar{a} + \bar{b} = \bar{a + b}, \quad \bar{a b} = \bar{a} \bar{b}, \quad \bar{a} = a.
\]

For a matrix \( A = [a_{ij}] \) we define \( A^* := \bar{A}^T = [\bar{a}_{ji}] \). If \( S^* AS = B \) for a nonsingular matrix \( S \), then \( A \) and \( B \) are said to be *congruent* (the involution \( a \mapsto \bar{a} \) can be the identity; we consider congruence of matrices as a special case of *congruence*).

Each matrix tuple in this paper is formed by matrices of the same size, which is called the size of the tuple. Denote
\[
R(A_1, \ldots, A_t) := (RA_1, \ldots, RA_t), \quad (A_1, \ldots, A_t)S := (A_1S, \ldots, A_tS).
\]

We say that matrix tuples \( (A_1, \ldots, A_t) \) and \( (B_1, \ldots, B_t) \) are equivalent and write
\[
(A_1, \ldots, A_t) \sim (B_1, \ldots, B_t)
\]
if there are nonsingular \( R \) and \( S \) such that
\[
R(A_1, \ldots, A_t)S = (B_1, \ldots, B_t).
\]

These tuples are *congruent* if \( R = S^* \).
For each \( \varepsilon \in \mathbb{F} \), define the pair

\[
T_\varepsilon(x, y) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & 0 \\ \varepsilon x^* & 0 & 0 \\ 0 & \varepsilon y^* & 0 \end{pmatrix}
\] (2)

of polynomial matrices in \( x, y, x^* \), and \( y^* \). Then

\[
T_\varepsilon(A, B) := \begin{pmatrix} 0 & I_n & 0 \\ 2I_n & I_n & 0 \\ 0 & 2I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & A & 0 \\ \varepsilon A^* & 0 & 0 \\ 0 & \varepsilon B^* & 0 \end{pmatrix}
\] (3)

for each pair \((A, B)\) of \( n \times n \) matrices.

The statement (a) of the following theorem is used in the next section.

**Theorem 1.** (a) For each \( \varepsilon \in \mathbb{F} \), matrix pairs \((A, B)\) and \((C, D)\) over \( \mathbb{F} \) are similar if and only if \( T_\varepsilon(A, B) \) and \( T_\varepsilon(C, D) \) are *congruent.

(b) The problems of classifying tuples (i)–(iii) from Section \( \mathbb{F} \) are wild.

Define the direct sum of matrix tuples:

\((A_1, \ldots, A_t) \oplus (B_1, \ldots, B_t) := (A_1 \oplus B_1, \ldots, A_t \oplus B_t)\).

A matrix tuple is said to be indecomposable with respect to equivalence if it is not equivalent to a direct sum of matrix tuples of smaller sizes. A tuple of square matrices is indecomposable with respect to *congruence if it is not *congruent to a direct sum of tuples of square matrices of smaller sizes.

**Lemma 2.** (a) Each tuple of \( m \times n \) matrices is equivalent to a direct sum of tuples that are indecomposable with respect to equivalence. This sum is determined uniquely up to permutation of summands and replacement of summands by equivalent tuples.

(b) Each tuple of \( n \times n \) matrices is *congruent to a direct sum of indecomposable tuples. This sum is determined uniquely up to permutation of summands and replacement of summands by *congruent tuples.

**Proof.** (a) Each \( t \)-tuple of \( m \times n \) matrices determines the \( t \)-tuple of linear mappings \( \mathbb{F}^n \to \mathbb{F}^m \); that is, the representation of the quiver consisting of two vertices 1 and 2 and \( t \) arrows \( 1 \to 2 \). By the Krull–Schmidt theorem
Section 8.2], every representation of a quiver is isomorphic to a direct sum of indecomposable representations, which are determined uniquely up to permutation and replacement by isomorphic representations.

(b) This statement is a special case of the following generalization of the law of inertia for quadratic forms [4, Theorem 2 and §2]: each system of linear mappings and sesquilinear forms on vector spaces over $\mathbb{F}$ decomposes into a direct sum of indecomposable systems uniquely up to isomorphisms of summands.

\[
\text{Proof of Theorem } (\text{a}) \text{ If } (A, B) \text{ is similar to } (C, D), \text{ then } T_\varepsilon(A, B) \text{ is congruent to } T_\varepsilon(C, D) \text{ since } S^{-1}(A, B)S = (C, D) \text{ implies }
\]

\[
R^* T_\varepsilon(A, B) R = T_\varepsilon(C, D), \quad R := \text{diag}((S^*)^{-1}, (S^*)^{-1}, S, S). \quad (4)
\]

Conversely, let $T_\varepsilon(A, B)$ be congruent to $T_\varepsilon(C, D)$, this means that

\[
R^* T_\varepsilon(A, B) R = T_\varepsilon(C, D)
\]

for some nonsingular $R$. Then also

\[
\mathcal{P}_\varepsilon(x, y) := \left(\begin{array}{ccc} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right), \left(\begin{array}{ccc} 0 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right), \left(\begin{array}{ccc} 0 & x & 0 \\ \varepsilon x^* & 0 & \varepsilon y^* \\ 0 & 0 & 0 \end{array}\right) \right).
\]

Hence, $\mathcal{P}_\varepsilon(A, B) \sim \mathcal{P}_\varepsilon(C, D)$ (in the notation (11)), and so

\[
\mathcal{F}(A, B) \oplus \mathcal{G}_\varepsilon(A, B) \sim \mathcal{F}(C, D) \oplus \mathcal{G}_\varepsilon(C, D), \quad (5)
\]

\[
\mathcal{F}(x, y) := \left(\begin{array}{ccc} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{array}\right), \left(\begin{array}{ccc} 2 & 0 \\ 1 & 2 \\ 0 & y \end{array}\right), \left(\begin{array}{ccc} \varepsilon x^* & 0 \\ 0 & \varepsilon y^* \\ 0 & 0 \end{array}\right) \right).
\]

The equivalence

\[
\mathcal{G}_\varepsilon(C, D) \sim \left[\begin{array}{cc} 2I_n & I_n \\ 0 & 2I_n \end{array}\right]^{-1} \mathcal{G}_\varepsilon(C, D) = \left(\begin{array}{ccc} I_n & 0 \\ 0 & I_n \end{array}\right), \left(\begin{array}{ccc} I_n/2 & -I_n/4 \\ 0 & I_n/2 \end{array}\right), \ldots \right).
\]
ensures that there are no triples $\mathcal{H}$ (with matrices of size not $0 \times 0$), $\mathcal{H}_1$, and $\mathcal{H}_2$ such that

$$\mathcal{F}(A, B) \sim \mathcal{H} \oplus \mathcal{H}_1 \quad \text{and} \quad \mathcal{G}_\varepsilon(C, D) \sim \mathcal{H} \oplus \mathcal{H}_2.$$  

The same holds for $\mathcal{F}(C, D)$ and $\mathcal{G}_\varepsilon(A, B)$. By (5) and Lemma 2(a), $\mathcal{F}(A, B) \sim \mathcal{F}(C, D)$; that is, $R\mathcal{F}(A, B) = \mathcal{F}(C, D)S$ for some nonsingular $R$ and $S$. Equating the corresponding matrices of these triples gives

$$RI_{2n} = I_{2n}S, \quad R\begin{bmatrix} 2I_n & 0 \\ I_n & 2I_n \end{bmatrix} S, \quad R\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} S.$$ 

By the first equality, $R = S$. By the second equality,

$$S = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$$

for some $P$ and $Q$. By the last equality, $P(A, B) = (C, D)P$; that is, $(A, B)$ is similar to $(C, D)$.

(b) If the involution on $\mathbb{F}$ is not the identity and $\varepsilon = 1$, then the second matrix in (3) is Hermitian. If the involution on $\mathbb{F}$ is the identity and $\varepsilon = \pm 1$, then the second matrix in (3) is symmetric or skew-symmetric. This proves the statement (b) for the pairs (i)–(iii) from Section 1. \hfill \square

### 3 Algebras

An algebra (without the identity) is a vector space $R$ over $\mathbb{F}$ with multiplication $(u, w) \mapsto uw \in R$ being bilinear and associative; this means that

$$(au + bv)w = a(uw) + b(vw), \quad u(aw + bw) = a(uw) + b(uw),$$

$$(uv)w = u(vw)$$

for all $a, b \in \mathbb{F}$ and all $u, v, w \in R$. Denote by $R^2$ and $R^3$ the vector spaces spanned by all $uv$ and, respectively, by all $uvw$.

An algebra $\Lambda$ that contains the identity 1 is called local if the set of its noninvertible elements is closed under addition. Then this set is the radical, is denoted by $\text{Rad} \Lambda$, and $\Lambda/\text{Rad} \Lambda$ is isomorphic to $\mathbb{F}$ (see [2, Section 5.2]).
Theorem 3. Let $\mathbb{F}$ be an algebraically closed field of characteristic not two.

(a) The problem of classifying algebras $R$ (without the identity) over $\mathbb{F}$ with $R^3 = 0$ and $\dim R^2 = 2$ is wild.

(b) The problem of classifying local algebras $\Lambda$ over $\mathbb{F}$ with $(\text{Rad } \Lambda)^3 = 0$ and $\dim(\text{Rad } \Lambda)^2 = 2$ is wild.

Due to the next lemma, the statement (a) ensures (b).

Lemma 4. If $R$ is an algebra from Theorem 3(a), then $R$ is the radical of some local algebra $\Lambda$ from Theorem 3(b), and $\Lambda$ is fully determined by $R$.

Proof. Let $R$ be an algebra for which $R^3 = 0$ and $\dim R^2 = 2$. We “adjoin” the identity 1 by considering the algebra $\Lambda$ consisting of the formal sums

$$a1 + u \quad (a \in \mathbb{F}, \ u \in R)$$

with the componentwise addition and scalar multiplication and the multiplication

$$(a1 + u)(b1 + v) = ab1 + (av + bu + uv).$$

The algebra $\Lambda$ is local since $R$ is the set of its noninvertible elements.

The next lemma reduces the problem of classifying algebras from Theorem 3(a) to a matrix problem.

Lemma 5. Every algebra $R$ for which $R^3 = 0$ and $\dim R^2 = 2$ is isomorphic to exactly one algebra on $\mathbb{F}^{2+n}$ for some $n \geq 2$ with multiplication

$$uv = \left( u^T \begin{bmatrix} 0 & 2 & 0 \\ 0 & A \end{bmatrix} v, u^T \begin{bmatrix} 0 & 2 \\ 0 & B \end{bmatrix} v, 0, \ldots, 0 \right)^T$$

given by $n$-by-$n$ matrices $A$ and $B$ that are linearly independent:

$$aA + bB = 0 \quad \implies \quad a = b = 0.$$ 

The pair $(A, B)$ is determined by $R$ uniquely up to congruence and linear substitutions

$$(A, B) \mapsto (r_{11}A + r_{12}B, r_{21}A + r_{22}B), \quad (7)$$

in which the matrix $[r_{ij}]$ must be nonsingular.
Proof. Let $R$ be an algebra of dimension $n+2$ such that $R^3 = 0$ and $\dim R^2 = 2$. Choose a basis $e_1, e_2$ of $R^2$ and complete it to a basis

$$e_1, e_2, f_1, \ldots, f_n$$

of $R$. Since $e_1, e_2 \in R^2$ and $R^3 = 0$,

$$e_i e_j = 0, \quad e_i f_j = 0, \quad f_i f_j = a_{ij} e_1 + b_{ij} e_2,$$

in which $A = [a_{ij}]$ and $B = [b_{ij}]$ are some $n$-by-$n$ matrices. Representing the elements of $R$ by their coordinate vectors with respect to the basis (8) and using (9), we obtain (6). A change of the basis $e_1, e_2$ of $R^2$ reduces $(A, B)$ by congruence transformations. The matrices $A$ and $B$ are linearly independent due to (9) and the condition $\dim R^2 = 2$. □

Proof of Theorem 3. Due to Lemmas 4 and 5, it suffices to prove that the problem of classifying pairs of matrices up to congruence and substitutions (7) is wild. Its wildness is proved in much the same way as [1, Theorem 4.5].

Consider the pair

$$\mathcal{P}(x, y) := (I_{20}, 0_{20}) \oplus (0_{10}, I_{10}) \oplus (I_1, I_1) \oplus T_0(x, y)$$

of 35-by-35 matrices, in which $T_0(x, y)$ is defined in (2). Let us prove that matrix pairs $(A, B)$ and $(C, D)$ are similar if and only if $\mathcal{P}(A, B)$ reduces to $\mathcal{P}(C, D)$ by transformations of congruence and substitutions (7).

If $(A, B)$ is similar to $(C, D)$, that is, $S^{-1}(A, B)S = (C, D)$ for some nonsingular $S$, then $T_0(A, B)$ is congruent to $T_0(C, D)$ by (4), and so $\mathcal{P}(A, B)$ is congruent to $\mathcal{P}(C, D)$.

Conversely, assume that $\mathcal{P}(A, B)$ reduces to $\mathcal{P}(C, D)$ by congruence transformations and substitutions (7); we need to prove that $(A, B)$ is similar to $(C, D)$. These transformations are independent: we can first produce substitutions and obtain

$$(r_{11} M_1 + r_{12} M_2(A, B), \quad r_{21} M_1 + r_{22} M_2(A, B))$$

where $M_1$ and $M_2(A, B)$ are the first and the second matrices of the pair $\mathcal{P}(A, B)$ and $[r_{ij}]$ is nonsingular, and then congruence transformations and obtain

$$\mathcal{P}(C, D) = (M_1, M_2(C, D)).$$
Clearly,
\[
\begin{align*}
\text{rank } (r_{11}M_1 + r_{12}M_2(A, B)) &= \text{rank } M_1, \\
\text{rank } (r_{21}M_1 + r_{22}M_2(A, B)) &= \text{rank } M_2(C, D).
\end{align*}
\]

Since \(\mathcal{P}(x, y)\) is defined by (10), these equalities imply \(r_{ij} = 0\) if \(i \neq j\); that is, \(\mathcal{P}(C, D)\) is congruent to \(M_1, r_{22}M_2(A, B)\), which is congruent to \(r_{11}^{-1}(r_{11}M_1, r_{22}M_2(A, B))\) because \(\mathbb{F}\) is algebraically closed. We have that
\[
\mathcal{P}(C, D) \text { is congruent to } (M_1, aM_2(A, B)), \tag{12}
\]
where \(a = r_{22}/r_{11}\).

We say that a pair \(P\) has a direct summand \(D\) if \(P\) is congruent to a direct sum with the summand \(D\). By (10), \((M_1, M_2(A, B))\) has the direct summand \((1, 1) := (I_1, I_1)\), and so \((M_1, aM_2(A, B))\) has the direct summand \((1, a)\). By (12), \(\mathcal{P}(C, D)\) has the direct summand \((1, a)\) too.

Assume that \(a \neq 1\). The pair \(\mathcal{P}(C, D)\) is congruent to a direct sum of \(\mathcal{T}_0(C, D)\) and pairs of the form \((1, 0)\), \((0, 1)\), and \((1, 1)\). Since \(a \neq 1\) and by Lemma 2(b), \((1, a)\) is a direct summand of \(\mathcal{T}_0(C, D)\). Then the same holds for their first matrices; that is, \(I_1\) is a direct summand of
\[
F := \begin{bmatrix}
0 & 1 & 0 \\
2 & 1 & 0 \\
0 & 2 & 0
\end{bmatrix}.
\]

This means that \(S^TFS = I_1 \oplus G\) for some \(G\) and a nonsingular \(S\). Hence,
\[
S^T(F - F^T)S = (I_1 - I_1^T) \oplus (G - G^T) = 0_1 \oplus (G - G^T);
\]
this is impossible since \(F - F^T\) is nonsingular.

Hence \(a = 1\) and by (12) \(\mathcal{P}(A, B)\) is congruent to \(\mathcal{P}(C, D)\). Due to (10), all the direct summands of \(\mathcal{P}(A, B)\) and \(\mathcal{P}(C, D)\) coincide except for \(\mathcal{T}_0(A, B)\) and \(\mathcal{T}_0(C, D)\). By Lemma 2(b), the pairs \(\mathcal{T}_0(A, B)\) and \(\mathcal{T}_0(C, D)\) are congruent. By Theorem 1, \((A, B)\) is similar to \((C, D)\).

\[\square\]

References

[1] G. R. Belitskii, V. V. Sergeichuk, Complexity of matrix problems, Linear Algebra Appl. 361 (2003) 203–222.
[2] R. S. Pierce, *Associative Algebras*. Springer-Verlag, 1982.

[3] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, in: Lecture Notes in Mathematics, vol. 1099, Springer, Berlin, 1984.

[4] V. V. Sergeichuk, *Classification problems for systems of linear mappings and sesquilinear forms*. Manuscript No. 196 Uk-D84, deposited at the Ukrainian NIINTI, 1984 (in Russian); R. Zh. Mat. 1984, 7A331.

[5] V. V. Sergeichuk, Classification problems for systems of forms and linear mappings, *Math. USSR, Izvestiya* 31 (no. 3) (1988) 481–501.