An Invariant-EKF VINS Algorithm for Improving Consistency

Kanzhi Wu*, Teng Zhang*, Daobilige Su, Shoudong Huang and Gamini Dissanayake

Abstract—The main contribution of this paper is an invariant extended Kalman filter (EKF) for visual inertial navigation systems (VINS). It is demonstrated that the conventional EKF based VINS is not invariant under the stochastic unobservable transformation, associated with translations and a rotation about the gravitational direction. This can lead to inconsistent state estimates as the estimator does not obey a fundamental property of the physical system. To address this issue, we use a novel uncertainty representation to derive a Right Invariant error extended Kalman filter (RIEKF-VINS) that preserves this invariance property. RIEKF-VINS is then adapted to the multi-state constraint Kalman filter framework to obtain a consistent state estimator. Both Monte Carlo simulations and real-world experiments are used to validate the proposed method.

I. INTRODUCTION

Visual-Inertial Navigation Systems (VINS) have been of significant interest to the robotics community in the past decade, as the fusion of information from a camera and an inertial measurement unit (IMU) provides an effective and affordable solution for navigation in GPS-denied environments. VINS algorithms can be classified into two categories, namely, filter based and optimization based. Although there has been recent progress in the development of optimization based algorithms [1][2], the extended Kalman filter (EKF) based solutions are still extensively used (e.g., [3][4][5][6]) mainly as a result of their efficiency and simplicity.

It is well known that conventional EKF based Simultaneous Localization and Mapping algorithms (EKF-SLAM) [7][8] suffer from inconsistency. Similarly, it has been shown that the conventional EKF VINS algorithm (ConEKF-VINS) using point features in the environment is also inconsistent resulting in the underestimation of the state uncertainty. This is closely related to the partial observability of these systems because conventional EKF algorithms do not necessarily guarantee this fundamental property [8][9] due to the linearized errors, which is the main reason for the overconfident estimates. This insight has been a catalyst for a number of observability-constraint algorithms (e.g., [10][11][12]), that explicitly enforces the unobservability of the system along specific directions via the modifications to the Jacobian matrices. Although the observability-constraint algorithms improve the consistency and accuracy of the estimator to some extent [13], extra computations in the update stage are required. Bloesch et al. in [5] propose a robot-centric formulation to alleviate the inconsistency. Under the robot-centric formulation, the filter estimates the locations of landmarks in the local frame instead of that in the global frame. As a result, the system becomes fully observable so that this issue is inherently avoided. However, this formulation can result in larger uncertainty and extra computations in the propagation stage, as discussed in [14][15].

Recently, the manifold and Lie group representations for three-dimensional orientation/pose have been utilized for solving SLAM and VINS. Both filter based algorithms (e.g., [15][16][17]) and optimization based algorithms (e.g., [1][18]) can benefit from the manifold representation and better accuracy can be achieved. The use of manifold does not only allow much easier algebraic computations (e.g., the computation of the Jacobian matrices) and avoid the representation singularity [19] but also have inspired a number of researchers to rethink the difference between the state representation and the state uncertainty representation, which is highlighted in [1][15]. In fact, this insight is also intrinsically understood in the well-known preintegration visual-inertial algorithm [20] although the algorithm does not use the manifold representation. From the viewpoint of control theory, Aghannan and Rouchon in [21] propose a framework for designing symmetry-preserving observers on manifolds by using a subtle geometrically adapted correction term. The fusion of the symmetry-preserving theory and EKF has resulted in the invariant-EKF (I-EKF), which possesses the theoretical local convergence property [22] and preserves the same invariance property of the original system. I-EKF based observers have been used in the inertial navigation [23] and the 2D EKF-SLAM [24][25]. Our recent work [15] also proves the significant improvement in the consistency through a 3D I-EKF-SLAM algorithm.

In this paper, we argue that the absence of the invariance affects the consistency of ConEKF-VINS estimates. There is a correspondence between this and the observability analysis reported in the previous literatures (e.g., [6][11]). The invariance in this refers to “the output of the filter is invariant under any stochastic unobservable transformation”. For the VINS system, the unobservable transformation is the rotation about the gravitational direction and the translations. Adopting the I-EKF framework, we propose the Right Invariant error EKF VINS algorithm (RIEKF-VINS) and prove that it is invariant. We then integrate RIEKF-VINS into the well-known visual-inertial odometry framework, i.e., the multi-state constraint Kalman filter (MSCKF) and remedy the inconsistency of the MSCKF algorithm. We show using extensive Monte Carlo simulations the proposed method outperforms the original MSCKF, especially in terms of the consistency. A preliminary real-world experiment also demonstrates the improved
accuracy of the proposed method.

This paper is organized as follows. Section II recalls the VINS system and gives an introduction of the ConEKF-VINS under the general continuous-discrete EKF. Section III performs the consistency analysis of the general EKF algorithm based on the invariance theory and proves the absence of the invariance of ConEKF-VINS. Section IV proposes RIEKF-VINS with the extension to the MSCKF framework. Section V reports both the simulation and experiment results. Finally, Section VI includes the main conclusions of this work and future work. Appendix provides some necessary formulas used in the proposed algorithms and the proofs of the theorems.

Notations: Throughout this paper bold lower-case and upper-case letters are reserved for column vectors and matrices/tuples, respectively. To simplify the presentation, the vector transpose operators are omitted for the case $A = [a^T,b^T,\ldots,c]^T$. The notation $S(\cdot)$ denotes the skew-symmetric operator that transforms a 3-dimensional vector into a skew symmetric matrix: $S(x)y = x \times y$ for $x,y \in \mathbb{R}^3$, where the notation $\times$ refers to the cross product.

II. BACKGROUND KNOWLEDGE

In this section, we first provide an overview of the VINS system and then describe the ConEKF-VINS algorithm based on the framework of the general continuous-discrete EKF.

A. The VINS system

The VINS system is used to estimate the state denoted as the tuple below

$$X = (R,v,p,b_g,b_a,f)$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ are the orientation and position of the IMU sensor, respectively, $v \in \mathbb{R}^3$ is the IMU velocity expressed in the global frame, $b_g \in \mathbb{R}^3$ is the gyroscope bias, $b_a \in \mathbb{R}^3$ is the accelerometer bias and $f \in \mathbb{R}^3$ is the coordinates of the landmark in the global frame. Note that only one landmark is included in the state system $[1]$ for a more concise notation.

1) The continuous-time motion model: The IMU measurements are usually used for state evolution due to its high frequency. The continuous-time motion model of the VINS system is given by the following ordinary differential equations (ODEs):

$$\dot{X} = f(X,u,n) = (RS(w-b_g-n_c),R(a-b_a-n_a)+g,v,n_{bg},n_{ba},0)$$

where $w \in \mathbb{R}^3$ is the gyroscope reading, $a \in \mathbb{R}^3$ is the accelerometer reading, $g \in \mathbb{R}^3$ is the global gravity vector (constant), and $n = [n_g,n_{bg},n_a] \in \mathbb{R}^3$ is the system noise modeled as a white Gaussian noise with the covariance matrix $Q$: $E(n(t)n(t)^\top) = Q\delta(t-\tau)$. Note that $u = (w,a,g)$ is the time-varying system input and the IMU noise covariance $Q$ is a constant matrix as prior knowledge.

2) The discrete-time measurement model: The visual measurement as the system output is discrete due to the low frequency of camera. After data association and rectification, the visual measurement of the landmark at time-step $k \in \mathbb{N}$ is available and given by

$$z_k = h(X_k,n_c) = h(R_k^T(f-p_k))+n_c$$

where $n_c \sim \mathcal{N}(0,V_k)$ is the measurement noise. Note that $h(\cdot) := \pi \circ T_CI$, where $\pi$ denotes the projection function and $T_CI$ is the transformation from the IMU frame to the camera frame.

B. The general continuous-discrete EKF

Being a natural extension of the standard EKF, the general EKF allows more flexible uncertainty representation by the following:

$$X = \hat{X} + e \text{ and } e \sim \mathcal{N}(0,P)$$

where $(\hat{X},P)$ can be regarded as the mean estimate and the covariance matrix, $e$ is a white Gaussian noise vector and the notation $\oplus$ is called retraction in differentiable geometry [26], coupled with the inverse mapping $\ominus$. Note that the user-defined operators $\oplus$ and $\ominus$ need to be designed such that $X = X \oplus 0$ and $e = X \ominus \hat{X}$. Here we also highlight that the choice of the retraction $\oplus$ has a significant contribution to the performance of the filter, as discussed in our previous work [15].

Once determining the retraction $\oplus$, the process of the general continuous-discrete EKF is similar to conventional continuous-discrete EKF, as summarized in Alg. I For propagation, we first calculate the time-varying Jacobians matrices $F$ and $G$ from the linearized error-state propagation model:

$$\dot{e} = Fe + Gn + o(||e|| ||n||).$$

We then compute the state transition matrix $\Phi := \Phi(t_{n+1},t_n)$ that is the solution at time $t_{n+1}$ of the following ODE:

$$\frac{d}{dt}\Phi(t,t_n) = F(t)\Phi(t,t_n)$$

with the condition $\Phi(t_n,t_n) = I$ at time $t_n$. The matrix $Q_{d,n}$ can be computed as

$$Q_{d,n} = \int_{t_n}^{t_{n+1}} \Phi(t_{n+1},\tau)G(\tau)GG^\top(\tau)\Phi^\top(t_{n+1},\tau)d\tau.$$
Algorithm 1: The general continuous-discrete EKF

Input: \( \hat{X}_n, P_n, u_{n+1,i}, z_{n+1} \);
Output: \( \hat{X}_{n+1}, P_{n+1} \);

Propagation:
Turn off the system noise and compute \( \hat{X}_{n+1|n} \) with \( \hat{X}_n \) and the ODEs (2);
\( P_{n+1|n} \leftarrow \Phi_n P_n \Phi_n^T + Q_{n+1} \);
Update:
\( H_{n+1} = \frac{\partial h(\hat{X}_{n+1|n} + \epsilon; 0)}{\epsilon} |_{\epsilon = 0} \);
\( S \leftarrow H_{n+1} P_{n+1|n} H_{n+1}^T + R_{n+1} \), \( K \leftarrow P_{n+1|n} H_{n+1}^T S^{-1} \);
\( \hat{z} \leftarrow h(\hat{X}_{n+1|n}, 0) - z_{n+1} \);
\( \hat{X}_{n+1} \leftarrow \hat{X}_{n+1|n} + K \hat{z} \); \( P_{n+1} \leftarrow (I - KH_{n+1}) P_{n+1|n} \);

III. CONSISTENCY ANALYSIS

In this section, we first introduce the concepts of unobservable transformation, invariance and observability. We then perform the consistency analysis for the general EKF filter and prove that ConEKF-VINS does not have the expected invariance property. Moreover, we also discuss the relationship between invariance and consistency.

A. Unobservability, unobservable transformation and invariance of the VINS system

The concept observability of nonlinear systems can be traced to the early literature [27]. As discussed in [6][28][29], the state \((\hat{X}_0, \hat{P}_0)\) of the VINS system is not locally observable. To make it more intuitive, we introduce the unobservability of the VINS system based on the unobservable transformation rather than the observability rank criterion reported in [27].

Definition 1: The transformation \( T \) is called to be an unobservable transformation for the VINS system and the output of the VINS system \((\hat{X}_0, \hat{P}_0)\) is invariant under \( T \) when the following condition is satisfied: For arbitrary \( t_i \) such that \( Y(t_i) = T(X(t_i)) \), we have \( h(X(t_i), 0) = h(Y(t_i), 0) \) \( \forall n \geq i \) where the notations \( X(\cdot) \) and \( Y(\cdot) \) denote the two evoluted trajectories that follow the same ODEs (2) with the conditions \( X(t_i) \) and \( Y(t_i) \) at time \( t_i \), respectively. On the other hand, the system is called to be unobservable if there exists an unobservable transformation.

Remark 1: One can see that an unobservable system is always accompanied by an unobservable transformation. And the invariance to the unobservable transformation is a more detailed description of the unobservability.

Definition 2: For the system state \((\hat{X}_0, \hat{P}_0)\), a stochastic transformation of translation and rotation (about the gravitational direction) \( T_\Sigma \) is a mapping:

\[
T_\Sigma(X) = (\exp(g(\epsilon_1 + \theta_1)) R, \exp(g(\epsilon_1 + \theta_1)) v, \\
\exp(g(\epsilon_2 + \theta_2)) p + \theta_2 + e_2, \\
b_\Sigma, b_\Sigma, \exp(g(\epsilon_2 + \theta_2)) f + \theta_2 + e_2)
\]

where \( S = (\theta, \epsilon), \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}^3, \theta = [\theta_1, \theta_2] \in \mathbb{R}^4, \epsilon_1 \in \mathbb{R}, \epsilon_2 \in \mathbb{R}^3 \) and \( e = [\epsilon_1, \epsilon_2] \in \mathbb{R}^4 \) is a white Gaussian noise with the covariance \( \Sigma \). \( T_\Sigma \) degenerates into the deterministic transformation \( T_D \) \( (D = (\theta, 0)) \) under the condition \( \Sigma = 0 \). \( T_\Sigma \) degenerates into a stochastic identity transformation under the condition \( \theta = 0 \).

Theorem 1: The stochastic transformation \( T_\Sigma \) is an unobservable transformation to the VINS system \((\hat{X}_0, \hat{P}_0)\).

Proof: It can be straightforwardly verified.

Remark 2: Theorem 1 corresponds to the conclusion in [6][30] that the IMU yaw angle and the IMU position are (locally) unobservable.

B. The invariance of the general EKF based filter

The general EKF based filter is not a linear system for the estimated state \( \hat{X} \). However, the invariance of the filter can be described as the following:

Definition 3: The output of a general EKF framework based filter (Alg. 1) for the VINS system is invariant under any stochastic unobservable transformation \( T_\Sigma \) if the following condition is satisfied: for any two estimates \((\hat{X}_i, P_i)\) and \((\hat{Y}_i, P_Y)\) at time-step \( i \), where \( \hat{Y}_i = T_\Sigma(\hat{X}_i) \) and \( P_Y = M_i P_i M_i^T + N_i \Sigma N_i^T \) in which

\[
M_i := \frac{\partial T_\Sigma(\hat{X}_i) \odot T_\Sigma(\hat{X}_i)}{\partial \epsilon} \bigg|_{\epsilon = 0}
\]

and

\[
N_i := \frac{\partial T_\Sigma(\hat{X}_i)}{\partial \epsilon} \odot T_\Sigma(\hat{X}_i) \bigg|_{\epsilon = 0}
\]

we have \( h(\hat{X}_i, 0) = h(\hat{Y}_i, 0) \) for all \( n \geq i \). The notations \( \hat{X}_n \) and \( \hat{Y}_n \) above represent the mean estimate of this filter at time-step \( n \) by using the same input \( u \) from time \( t_i \) to \( t_n \), from the conditions \((\hat{X}_i, P_i)\) and \((\hat{Y}_i, P_Y)\) at time-step \( i \), respectively.

As shown in Def. 1 and Def. 2, the invariance to any stochastic transformation \( T_\Sigma \) can be divided into two properties: 1) the invariance to any deterministic transformation \( T_D \) and 2) the invariance to any stochastic identity transformation. The following two theorems analytically provide the methods to judge whether a general EKF based filter has the two invariances properties above.

Theorem 2: The output of the general EKF based filter for the VINS system is invariant under any deterministic unobservable transformation only if for each deterministic unobservable transformation \( T_D \), there exists an invertible matrix \( W_D \) (unrelated to \( X \)) such that

\[
T_D(X \odot e) = T_D(X) \odot W_D e.
\]

Proof: See Appendix C.

Theorem 3: The output of the general EKF based filter for the VINS system is invariant under any stochastic identity transformation only if

\[
H_{n+i+1} \Phi_n + \Phi_{n+i+1} \cdots \Phi_i N_i = 0 \ \forall n \text{ and } i \geq 0.
\]

Proof: See Appendix C.

By using the theorems above, we can easily determine the invariance properties of ConEKF-VINS.

Theorem 4: ConEKF-VINS satisfies (12) but does not satisfy (13). Hence, ConEKF-VINS has the invariance to any deterministic unobservable transformation \( T_D \) but not the
invariance to stochastic identity transformations. In all, the output of ConEKF-VINS is not invariant under stochastic unobservable transformation $T_S$.

Proof: For the ConEKF-VINS algorithm, the invariance to the deterministic unobservable transformation $T_D$ can be verified by using Theorem 2. The absence of invariance of ConEKF-VINS to stochastic identity transformations can be verified by using Theorem 3. More details are omitted here.

Remark 3: The previous literatures [9][10][11][12] directly perform the observability analysis of the filter on the linearized error-state model. However, Theorem 2 and Theorem 3 clarifies the relationship between the filter and the linearized error-state model.

C. Consistency and invariance

The unobservability in terms of stochastic unobservable transformation $T_S$ is a fundamental property of the VINS system. Therefore a consistent filter (as a system for the estimated state $\hat{X}$) is expected to mimic this property, i.e., the output of a consistent estimator is invariant under any stochastic unobservable transformation. The invariance to the deterministic transformation $T_D$ implies that the estimates from the filter do not depend on the selection of the (initial) mean estimate of the unobservable variables, i.e., the IMU yaw angle and the IMU position, essentially. Similarly, the invariance to stochastic identity transformation implies that the uncertainty w.r.t. these unobservable variables does not affect the subsequent mean estimates. We can conclude that the consistency of a filter is tightly coupled with the invariance to stochastic unobservable transformation. A filter that does not have the invariance property will gain the unexpected information and produce inconsistent (overconfident) estimates. Note that ConEKF-VINS is a typical example due to the absence of the invariance property.

IV. THE PROPOSED METHOD: RIEKF-VINS

In this section, we propose RIEKF-VINS by using a new uncertainty representation and prove it has the expected invariance properties. We then apply RIEKF-VINS to the MSCKF framework.

A. The Uncertainty representation and Jacobians

RIEKF-VINS also follows the framework (Alg. 1). The uncertainty representation of RIEKF-VINS is defined as below

\[
X = \hat{X} + e = (\exp(e_\theta) \hat{X}, \exp(e_\theta) \hat{v} + J_f(-e_\theta) e_v, \\
\exp(e_\theta) \hat{p} + J_r(-e_\theta) e_p, \hat{b}_s + e_{b_s}, \hat{b}_a + e_{b_a}, \\
\exp(e_\theta) \hat{f} + J_f(-e_\theta) e_f)
\]

where $e = [e_\theta, e_v, e_p, e_{b_s}, e_{b_a}, e_f] \sim \mathcal{N}(0, P)$ and the right Jacobian operator $J_f(\cdot)$ is given in (13). Note that this uncertainty representation intrinsically employs the Lie group so that the recent result (Theorem 2 of [22]) can be used to easily compute the Jacobians $F$ and $G$ of the propagation

\[
F = \begin{bmatrix}
0_{3,3} & 0_{3,3} & 0_{3,3} & -\hat{R} & 0_{3,3} & 0_{3,3} \\
S(\hat{g}) & 0_{3,3} & 0_{3,3} & -S(\hat{v}) \hat{R} & -\hat{R} & 0_{3,3} \\
0_{3,3} & I_3 & 0_{3,3} & -S(\hat{p}) \hat{R} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3}
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
\hat{R} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
S(\hat{v}) \hat{R} & 0_{3,3} & \hat{R} & 0_{3,3} \\
S(\hat{p}) \hat{R} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & I_3 & 0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
S(\hat{f}) \hat{R} & 0_{3,3} & 0_{3,3} & 0_{3,3}
\end{bmatrix}
\]

The measurement Jacobian is

\[
H_{n+1} = \partial h(\hat{X}_{n+1|n}) \begin{bmatrix} 0_{3,6} & -\hat{R}^T_{n+1|n} & 0_{3,6} & \hat{R}^T_{n+1|n} \end{bmatrix} \in \mathbb{R}^{3}
\]

where $\hat{X}_{n+1|n} = \hat{R}^T_{n+1|n} \hat{X}_{n+1|n} - \hat{p}_{n+1} \in \mathbb{R}^{3}$.

B. Invariance proof

Theorem 5: The output of RIEKF-VINS is invariant under any stochastic unobservable transformation $T_S$.

Proof: For the retraction defined in (14), we have

\[
W_D = \begin{bmatrix}
\delta R & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
0_{3,3} & \delta R & 0_{3,3} & 0_{3,3} & 0_{3,3} \\
S(\theta_2) \delta R & 0_{3,3} & \delta R & 0_{3,3} & 0_{3,3} \\
0_{3,3} & S(\theta_2) \delta R & 0_{3,3} & I_3 & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & I_3 & 0_{3,3} \\
0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,3} & \delta R
\end{bmatrix}
\]

and $\delta R := \exp(g\theta_1)$. According to Theorem 2 the output of RIEKF-VINS is invariant under any deterministic transformation $T_D$. On the other hand, for all $i$, we have

\[
\Phi_i = \begin{bmatrix}
I_3 & 0_{3,3} & * & 0_{3,3} \\
\Delta t_i & S(g) & 0_{3,3} & * & 0_{3,3} \\
\Delta t_i & S(g) & * & 0_{3,3} \\
0_{3,3} & 0_{3,3} & * & 0_{3,3} \\
0_{3,3} & 0_{3,3} & * & 0_{3,3} \\
0_{3,3} & 0_{3,3} & * & I_3
\end{bmatrix}
\]

and

\[
N_i = \frac{\partial T_S(\hat{X}_i) \oplus T_D(\hat{X}_i)}{\partial \epsilon} \bigg|_{\epsilon = 0} = \begin{bmatrix}
g & 0_{3,3} \\
0_{3,3} & I_3 \\
0_{3,3} & I_3 \\
0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3} \\
0_{3,3} & I_3
\end{bmatrix}
\]

where $\Delta t_i := t_{i+1} - t_i$ and the elements denoted by the notation $*$ are omitted here because these do not have any contribution to the computation of $\Phi_i N_i$. Note that $\Phi_i N_i = N_{i+1}$ and $H_{i+1} N_{i+1} = 0$ for all $i$ and then we can easily verify that RIEKF-VINS satisfies (13). According to Theorem 3
the output of RIEKF-VINS is invariant under any stochastic identity transformation.

Remark 4: The observability-constraint filters proposed in [9][10][11][12] artificially modify the transition matrix $\Phi_n$ and the measurement Jacobian $H_{n+1}$ to meet the condition (13) such that they have the invariance to stochastic identity transformation. As a comparison, our proposed RIEKF-VINS employs the uncertainty representation (14) such that the “natural” matrices $\Phi_n$ and $H_{n+1}$ can elegantly meet the condition (13).

C. Application to MSCKF

A drawback of ConEKF-VINS and RIEKF-VINS is the expensive cost of maintaining the covariance matrix for a number of landmarks. Especially, RIEKF-VINS suffers from the complexity quadratic to the number of landmarks in the propagation stage. On the other hand, the well known MSCKF [3] that has the complexity linear to the number of landmarks inherits the inconsistency of ConEKF-VINS. One can see that the uncertainty w.r.t the global yaw has effects on the mean estimates in the MSCKF algorithm, unexpectedly. Due to the reasons above, we integrate RIEKF-VINS into the MSCKF framework such that the modified algorithm has the linear complexity and better consistency. For convenience, we call the modified filter as RI-MSCKF. In this subsection, we do not state all details of RI-MSCKF but point out the modifications.

1) System state and retraction: The system state $\chi_n$ at time-step $n$ in RI-MSCKF is

$$\chi_n = (\hat{X}_n, C_{t_1}, \ldots, C_{t_j}, \ldots, C_{t_k}, \ldots, C_{t_m})$$

where $X_n = (R_n, v_n, p_n, b_{k,n}, b_{k,n})$ denotes the IMU state at time-step $n$, $C_{t_j} = (R_{t_j}^c, p_{t_j}^c) \in SE(3)$ denotes the camera pose at the time $t_j (t_j < t_n)$. According to the IMU state uncertainty in RIEKF-VINS, the uncertainty representation of $X_n$ are defined as below

$$\chi_n = \tilde{X}_n \oplus e$$

$$= (\hat{X}_n \oplus_{imu} e_t, \tilde{X}_n \oplus_{pose} e^c_1, \ldots, \tilde{X}_n \oplus_{pose} e^c_m)$$

where $e = [e_t, e_c] \in \mathbb{R}^{15+6m}$, $e_t \in \mathbb{R}^{15}$ and $e_c = [e^c_1, \ldots, e^c_m] \in \mathbb{R}^{6m}$. Note that $\oplus_{imu}$ and $\oplus_{pose}$ are given in Appendix A.

2) Propagation: The mean propagation $\chi_{n+1|n}$ of RI-MSCKF also follows that of MSCKF while the covariance $P_{n+1|n}$ is calculated by

$$P_{n+1|n} = \Phi_n P_n \Phi_n^T + Q_{d,n}$$

where $\Phi_n = \text{Diag}(\Phi_n', I_{15m})$, $Q_{d,n} = \text{Diag}(Q_{d,n}', 0_{6m,6m})$. Note that $\Phi_n'$ and $Q_{d,n}'$ are the matrices from the first 15 rows and 15 columns of $\Phi_n$ and $Q_{d,n}$, respectively, where $\Phi_n$ and $Q_{d,n}$ are the matrices of RI-MSCKF.

3) State augment: Once a new image is captured at time-step $n+1$, we augment the system state and the covariance matrix as the following:

$$\hat{X}_{n+1|n} \leftarrow (\hat{X}_{n+1|n}, \tilde{X}_{n+1})$$

where $\tilde{X}_{n+1} = (\hat{R}_{n+1|n} \Delta R, \hat{p}_{n+1|n} \Delta p, \hat{p}_{n+1|n}) \in SE(3)$ is the mean estimate of camera pose at the time $t_{n+1}$, $(\Delta R, \Delta p) \in SE(3)$ denotes the transformation from the camera to the IMU. Due to the new uncertainty representation (22), the Jacobian $J$ needs to be changed as below

$$J = \begin{bmatrix} I_3 & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,6} & 0_{3,6m} \end{bmatrix}.$$ (26)

4) Update: Note that the landmark uncertainty is coupled with the IMU pose in RIEKF-VINS. In RI-MSCKF, we describe the landmark uncertainty coupled with the camera pixel $C_{t_j}$ that earliest captures the landmark within the current system state $\chi_n$ as below

$$(\hat{C}_{t_j}, \hat{f}) \oplus \hat{e}_c = (\hat{C}_{t_j} \oplus_{pose} e^c, \hat{f} + \hat{J}_r(-e^c \hat{e}_f))$$

where $\hat{e}_c = [e^c_1, e^c_2, e^c_3] = e^c \in \mathbb{R}^9$. From the uncertainty representations (22) and (27), we can compute the linearized measurement model for the visual measurement at time-step $k (t_k \leq t_n)$. With a slight abuse of notations, the linearized measurement model can be represented as below

$$\pi(\hat{R}_{n}^c \hat{f} - p_{t_k}^c) - z_k \approx \partial_\pi H_{f,k}^c e_{n+1|n} + \partial_\pi H_{f,k}^c e_{f} + V_k$$

$$\hat{z}_k \approx \partial_\pi H_{f,k}^c e_{n+1|n} + \partial_\pi H_{f,k}^c e_{f} + V_k$$

$$\hat{z}_k \approx H_{f,k}^c e_{n+1|n} + H_{f,k}^c e_{f} + V_k$$

where $\partial_\pi := \partial \pi(\hat{R}_{n}^c \hat{f} - p_{t_k}^c)$, $z_k$ is the measurement captured at the time $t_k$. Here the matrices $H_{f,k}^c$ and $H_{f,k}$ are given by

$$H_{f,k}^c = \hat{R}_{f,k}^c$$

$$H_{f,k} = \hat{R}_{f,k}$$

$$H_{f,k} = [\cdots \cdots A \cdots B \cdots \cdots]$$

where $A = [-\hat{R}_{n}^c S(f), 0_{3,3}]$ and $B = [\hat{R}_{n}^c S(f), -\hat{R}_{n}^c ]$. Due to the absence of the covariance of landmark, RI-MSCKF also uses the null-space trick on (28) and the resulting residual equation

$$H_{f,k}^c \hat{z}_k \approx H_{f,k}^c H_{f,k}^c e_{n+1|n} + H_{f,k}^c V_k$$

$$\hat{z}_k \approx H_{f,k}^c e_{n+1|n} + V_k$$

is employed for update.

Remark 5: RI-MSCKF does not need any extra computation to maintain the expected invariance while the observability-constraint algorithms need to explicitly project the measurement Jacobians onto the observable space.

V. SIMULATION AND EXPERIMENT

A. Simulation Result

In order to validate the theoretical contributions in this paper, we perform 50 Monte Carlo simulations and compare RI-MSCKF to MSCKF for a Visual-Inertial Odometry (VIO) scenario without loop closure.

Consider that a robot equipped with an IMU and a camera moves in a specific trajectory (average speed is 3m/s) with the sufficient 6-DOFs motion, shown as the blue circles in Fig. [1]. In this environment, 675 landmarks...
are distributed on the surface of a cylinder with radius 6.5m and height 4m shown as the green stars in Fig. 1. Under the simulated environment, the camera is able to observe sufficiently overlapped landmarks between consecutive frames. The standard deviation of camera measurement is set as 1.5 pixels. The IMU noise covariance \( \mathbf{Q} \) is set as \( \text{Diag}(0.008^2 \mathbf{I}_3, 0.0004^2 \mathbf{I}_3, 0.019^2 \mathbf{I}_3, 0.05^2 \mathbf{I}_3) \) (the International System of Units). In each round of Monte Carlo simulation, the initial estimate is set as the ground truth. And the measurements from IMU and camera are generated from the same trajectory with random noises. The maximal number of camera poses in the system state of RI-MSCKF and MSCKF is set as 10. For robust estimation, we use the landmarks for the update step only when the landmarks are captured more than 5 times by the cameras within the current system state.

The results of 50 Monte Carlo simulations are plotted in Fig. 2. We use the root mean square error (RMS) and the average normalized estimation error squared (NEES) to evaluate both accuracy and consistency, respectively. Note that the ideal NEES of orientation is 3 and that of pose is 6. As shown in Fig. 2, RI-MSCKF clearly outperforms MSCKF especially for the consistency. This phenomenon can be explained as RI-MSCKF has the invariance property to stochastic rotation about the gravitational direction and thus it can reduce the unexpected information gain when compared to MSCKF. In addition, the RMS of orientation and position of both filters increase with the time because the loop closure in this simulation is turned off.

B. Preliminary Experiment

In order to validate the performance of the proposed RI-MSCKF algorithm under practical environments, we evaluate the algorithm on Euroc dataset [31] which is collected onboard a macro aerial vehicle in the indoor environments. Without a dedicated designed front-end which handles the feature extraction and tracking perfectly, we selected sequence V2_01_easy in this section to demonstrate the performance of the RI-MSCKF algorithm where the features can be tracked correctly and thus making it perfect to compare our algorithm against the MSCKF algorithm.

In this preliminary experiment, we designed a front-end based on ORB-SLAM [32] while only keeping the feature tracking sub-module. Without knowing the map points, new keyframe is inserted once there is \( n_{frames} \) frames have passed since the insertion of the last keyframe. One sample image with the tracked landmarks is shown in Fig. 4. The uncertainty of the IMU sensor is set as instructed in the dataset. The maximal number of the camera poses in the system state is set as 10 and the minimal observed times for a landmark is set as 5.

Fig. 5 shows the estimated trajectories using MSCKF and RI-MSCKF. As shown in Fig. 5 and indicated in Fig. 6, RI-MSCKF shows the similar accuracy of position compared with MSCKF but also avoids the drift in the last few frames of the sequence, however, RI-MSCKF shows significant better results in terms of orientation estimation accuracy compared with the original MSCKF algorithm. Even without a robust front-end to handle feature tracking perfectly, this preliminary experiment is able to demonstrate the superiority of RI-MSCKF compared with MSCKF algorithm in terms of the estimation accuracy.

VI. CONCLUSION AND FUTURE WORK

In this work, we proposed the RIEKF-VINS algorithm and stressed that the consistency of a filter is tightly coupled with the invariance property. We proved that RIEKF-VINS has the expected invariance property while ConEKF-VINS does not satisfy this property. We also provided the methods to check whether a general EKF based filter has the invariance properties. After theoretical analysis, we integrated RIEKF-VINS into the MSCKF framework such that the resulting RI-MSCKF algorithm can achieve better consistency relative to the original MSCKF. Monte Carlo simulations illustrated the significantly improved performance of RI-MSCKF, especially for the consistency. The real-world experiments also validated its improved accuracy. Future work includes improving the front end to achieve more robust estimation. We will also compare RIEKF-VINS to the observability-constraint algorithms in both simulations and real-world experiments.

APPENDIX

A. Some Formulas

\[
\exp(y) = \mathbf{I}_3 + \frac{\sin(\|y\|)}{\|y\|} \mathbf{S}(y) + \frac{1 - \cos(\|y\|)}{\|y\|^2} \mathbf{S}^2(y) \tag{32}
\]

\[
J_r(y) = \mathbf{I}_3 - \frac{1 - \cos(\|y\|)}{\|y\|^2} \mathbf{S}(y) + \frac{\|y\| - \sin(\|y\|)}{\|y\|^3} \mathbf{S}^2(y) \tag{33}
\]

for \( y \in \mathbb{R}^3 \).

The notation \( \oplus_{\text{imu}} \) is defined as

\[
\mathbf{X} \oplus_{\text{imu}} \mathbf{e}_{\text{f}} = (\exp(\mathbf{e}_p) \mathbf{R}, \exp(\mathbf{e}_v) \mathbf{v} + J_r(\mathbf{e}_g) \mathbf{b}_g, \mathbf{e}_g, \mathbf{e}_n, \mathbf{e}_{bg}, \mathbf{e}_{bu}) \tag{34}
\]

\[
\exp(\mathbf{e}_p) \mathbf{p} + J_r(\mathbf{e}_g) \mathbf{b}_g + \mathbf{e}_g, \mathbf{b}_b + \mathbf{e}_{bg}, \mathbf{e}_{bu}
\]
Fig. 2: 50 Monte Carlo simulation results. The proposed RI-MSCKF outperforms the original MSCKF, both in terms of accuracy (RMS) and consistency (NEES).

Fig. 3: The estimated trajectories from MSCKF and RI-MSCKF in V2_01_easy.

Fig. 4: Sample image with landmarks in the experiment. The green dots represent the tracked key points and the red dots represent the new key points.

Fig. 5: The RMS of orientation and position estimate from MSCKF and RI-MSCKF in V2_01_easy.

\[ \bar{X} = (R, v, p, b_g, b_a) \] and \( e_I = [e_\theta, e_v, e_p, e_{b_g}, e_{b_a}] \in \mathbb{R}^{15} \)

The notation \( \oplus_{\text{pose}} \) is defined as

\[ C \oplus_{\text{pose}} e'_c = (\exp(e'_\theta)R, \exp(e'_\theta)p + J_r(-e'_\theta)e'_p) \] (35)

where \( C = (R, p) \in SE(3) \) and \( e'_c = [e'_\theta, e'_v, e'_p] \in \mathbb{R}^6 \).

B. Proof of Theorem 2

Here we only prove the sufficient condition. It is assumed that this filter satisfies: for each deterministic unobservable transformation \( T_D \) there exists \( W_D \) such that \( T_D(X \oplus e) = T_D(X) \oplus W_De \).

For any estimate \( (\hat{X}_i, P_i) \) at time-step \( i \), we have another estimate \( (\hat{Y}_i, P_Y) = (T_D(\hat{X}_i), W_D P_i W_D^T) \) after applying the deterministic transformation \( T_D \). After one step propagation, we have \( (\hat{X}_{i+1|i}, P_{i+1|i}) \) and \( (\hat{Y}_{i+1|i}, P_{Y_{i+1|i}}) \) where \( \hat{Y}_{i+1|i} = \)
The covariance matrix after update becomes $P_{i+1} = (I - K_i H_i) P_{i|j-1} + W_i P_{i|j-1} W_i^T$. In all, $\hat{Y}_{i+1} = T_D(\hat{X}_{i+1})$ and $P_{i+1} = T_D(\hat{X}_{i+1})$. By mathematical induction, we can see $\hat{Y}_{n} = T_D(\hat{X}_{n})$ for $n \geq i$ and hence the output of this filter is invariant under any deterministic transformation $T_D$.

C. Proof of Theorem 3

Here we only prove the sufficient condition. It is assumed that this filter satisfies: $H_{n+1} = \Phi N_i = 0 \forall i \geq 0$ and $n \geq i$.

For any estimate $\hat{X}_i, P_i$ at time-step $i$, we have another estimate $\hat{Y}_n, P_n = (\hat{X}_i, P_i + N_i N_i^T)$ after applying the stochastic identity transformation $T_{\phi}$ where $\delta = (0, \varepsilon)$ and $\varepsilon \sim \mathcal{N}(0, \Sigma)$. After one step propagation, we have $\hat{X}_{i+1} = \hat{X}_{i} P_{i+1} + \Phi N_i N_i^T$. Note that $H_{n+1} = \Phi N_i = 0$, we can easily get $\hat{Y}_{i+1} = (\hat{X}_{i+1} P_{i+1} + \Phi N_i N_i^T)$. By mathematical induction, we have $\hat{Y}_n = (\hat{X}_n P_n + \Phi N_i N_i^T)^{i-1}$ Therefore, the output of this filter is invariant under any stochastic identity transformation.

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