The generalized Ermakov conservative system: a discussion

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Abstract Using older and recent results on the integrability of two-dimensional (2d) dynamical systems, we prove that the results obtained in a recent publication concerning the 2d-generalized Ermakov system can be obtained as special case of a more general approach. This approach is geometric and can be used to study efficiently similar dynamical systems.

1 Introduction

The two-dimensional (2d) generalized Ermakov system has attained attention in 90’s [1–6] where most of its properties have been revealed. A review of these studies can be found in [7]. However, the interest in the topic is still alive and a recent article has appeared in this journal [8] presenting new results. The purpose of the present discussion is to show that these latter results can be obtained as special cases of older and more recent results on the integrability of 2d dynamical systems.

2 The 2d-generalized Ermakov system

The 2d-generalized Ermakov system is defined by the equations

\begin{align*}
\ddot{x} &= -\omega^2(t)x + \frac{1}{x^2y} f\left(\frac{y}{x}\right) \\
\ddot{y} &= -\omega^2(t)y + \frac{1}{xy^2} g\left(\frac{x}{y}\right)
\end{align*}

where \( f, g \) are arbitrary functions. This system admits the Ermakov first integral (FI)

\[ I_0 = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int_{y/x}^{y} f(u)du + \int_{x/y}^{x/y} g(v)dv \]

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where \( u = v^{-1} = \frac{v}{x} \). It is well-known that the 2d-generalized Ermakov system generalizes the one-dimensional (1d) time-dependent oscillator.

Introducing the functions \( F, G \) by the relations

\[
 f\left(\frac{y}{x}\right) = \frac{y}{x} F\left(\frac{y}{x}\right), \quad g\left(\frac{x}{y}\right) = \frac{x}{y} G\left(\frac{y}{x}\right)
\]

Equations (1), (2) take the equivalent form

\[
 \ddot{x} = -\omega^2(t)x + \frac{1}{x^3} F\left(\frac{y}{x}\right)
\]

\[
 \ddot{y} = -\omega^2(t)y + \frac{1}{y^3} G\left(\frac{y}{x}\right)
\]

while the Ermakov FI (3) becomes

\[
 I_0 = \frac{1}{2} (xy - y\dot{x})^2 + \int_{y/x} \left[ uF(u) - u^{-3} G(u) \right] du.
\]

If one introduces the variables [4]

\[
 T = \int \rho^{-2} dt, \quad X = \rho^{-1} x, \quad Y = \rho^{-1} y
\]

where \( \rho(t) \) is a solution of the 1d time-dependent oscillator

\[
 \dot{\rho} + \omega^2(t)\rho = 0
\]

then Eqs. (4), (5) become the autonomous system

\[
 X'' = \frac{1}{x^3} F\left(\frac{Y}{X}\right)
\]

\[
 Y'' = \frac{1}{y^3} G\left(\frac{Y}{X}\right)
\]

and the Ermakov FI

\[
 I_0 = \frac{1}{2} (XY' - YX')^2 + \int_{y/x}^{Y/X} \left[ uF(u) - u^{-3} G(u) \right] du
\]

where we use the notation \( f' \equiv \frac{df(T)}{dT} \) and \( \dot{f} \equiv \frac{df(t)}{dt} \). For general functions \( F, G \) the autonomous dynamical system (9)–(10) is not conservative. In the following, we determine the family of conservative Ermakov systems together with their FIs using collineations of the metric defined by these equations.

### 3 Integrability of the 2d-generalized Ermakov system

Since the system (9)–(10) is autonomous, the second FI will be the Hamiltonian \( H \). To find \( H \), we do not have to do any new calculations, because in [9,10] all the integrable and superintegrable 2d autonomous conservative systems have been determined. From these results, we find (see e.g., Sect. 8 case (1) in [10]) that the 2d integrable potential

\[
 V_{21} = \frac{F_1(u)}{X^2 + Y^2} + F_2(X^2 + Y^2)
\]
where \( u = \frac{Y}{X} \) and \( F_1, F_2 \) are arbitrary functions of their arguments, admits the FI

\[
I_{11} = \frac{1}{2} (XY' - YX')^2 + F_1(u).
\]  

(13)

If we consider \( F_1(u) = (u^2 + 1)N(u) \) and \( F_2 = 0 \), we find that \( V_{21} = \frac{N(u)}{X^2} \) while the resulting equations are

\[
\begin{align*}
X'' &= \frac{2N + u \frac{dN}{du}}{X^3} \\
Y'' &= -\frac{1}{X^3} \frac{dN}{du}.
\end{align*}
\]

(14)

(15)

Therefore, if we define the functions

\[
F(u) = 2N + u \frac{dN}{du}, \quad G(u) = -u^3 \frac{dN}{du}
\]

then Eqs. (14), (15) become the Ermakov equations (9), (10) while \( I_0 = I_{11} \).

We conclude that the family of the conservative 2d Ermakov systems is defined by the potential \( V = \frac{N(u)}{X^2} \) where \( N(u) \) is an arbitrary function while the Hamiltonian is given by the expression

\[
H = \frac{1}{2} \left( X'^2 + Y'^2 \right) + \frac{N(u)}{X^2}.
\]

(17)

In the original coordinates, the system (14)–(15) becomes

\[
\begin{align*}
\ddot{x} &= -\omega^2(t)x + \frac{2N(u) + u \frac{dN}{du}}{x^3} \\
\ddot{y} &= -\omega^2(t)y - \frac{1}{y^3} \frac{dN}{du}
\end{align*}
\]

(18)

(19)

where \( u = \frac{y}{x} = \frac{Y}{X} \).

For \( N(u) = \frac{u^2}{2} \), we find, respectively,

\[
\begin{align*}
\ddot{x} &= -\omega^2(t)x \\
\ddot{y} &= -\omega^2(t)y + \frac{1}{y^3}
\end{align*}
\]

(20)

(21)

while the Ermakov FI becomes the well-known Lewis invariant \[11\]

\[
I_0 = \frac{1}{2} (x\dot{y} - y\dot{x})^2 + \frac{1}{2} \left( \frac{x}{y} \right)^2.
\]

(22)

Equation (20) is the 1d time-dependent harmonic oscillator and (21) is the auxiliary equation with which one determines the frequency \( \omega(t) \) for a given function \( y(t) \). These justify the characterization of the Ermakov system as a generalization of the harmonic oscillator.

### 4 The FIs of the conservative Ermakov system

There are two ways to find the FIs of the Ermakov system. One way is to use the results of [9, 10] and read the FIs for the potential \( V_{21} \) given in (12) for \( F_1(u) = (u^2 + 1)N(u) \), \( F_2 = 0 \). Here, we shall follow another way which can be useful in many similar problems. We shall use Theorem 2 of [12] where it is stated that the generators of the Noether point
symmetries of autonomous conservative systems are the elements of the homothetic algebra of the metric defined by the kinetic energy (kinetic metric). In the Ermakov case, this metric is the Euclidean 2d metric \( \gamma_{ab} = \text{diag}(1, 1) \).

For the convenience of the reader, we state Theorem 2 of [12].

**Theorem 1**

Autonomous conservative dynamical systems of the form

\[
\ddot{q}^a = -\Gamma^a_{bc} \dot{q}^b \dot{q}^c - V^a(q)
\]

where \( \Gamma^a_{bc} \) are the Riemannian connection coefficients determined from the kinetic metric \( \gamma_{ab} \) (kinetic energy) and \( V(q) \) the potential of the system, admit the following point Noether symmetries.

**Case 1.** The point Noether symmetry

\[
A_1 = \partial_t, \quad f_1 = \text{const} \equiv 0
\]

which produces the Noether FI (Hamiltonian)

\[
H = \frac{1}{2} \gamma_{ab} \dot{q}^a \dot{q}^b + V(q).
\]

**Case 2.** The point Noether symmetry

\[
A_2 = 2\psi_B t \partial_t + B^a \partial_{q^a}, \quad f_2 = c_1 t
\]

where \( c_1 \) is an arbitrary constant and \( B^a \) is a KV (\( \psi_B = 0 \)) or the HV (\( \psi_B = 1 \)) such that

\[
B_a V^a + 2\psi_B V + c_1 = 0.
\]

The associated Noether FI is

\[
I_2 = 2\psi_B t H - B_a \dot{q}^a + c_1 t.
\]

**Case 3.** The point Noether symmetry

\[
A_3 = 2\psi \int C(t) dt \partial_t + C(t) \Phi^a \partial_{q^a}, \quad f_3 = C, t \Phi(q) + D(t)
\]

where \( \Phi^a(q) \) is a gradient KV (\( \psi = 0 \)) or a gradient HV (\( \psi = 1 \)) such that (\( c_2, c_3 \) are arbitrary constants)

\[
\Phi_a V^a + 2\psi V = c_2 \Phi + c_3
\]

and the functions \( C(t), D(t) \) are determined by the relations (\( C, t \neq 0 \))

\[
C, tt = -c_2 C, \quad D, t = -c_3 C.
\]

The associated Noether FI is

\[
I_3 = 2\psi H \int C(t) dt - C(t) \Phi^a \dot{q}^a + C, t \Phi - c_3 \int C(t) dt.
\]

We apply Theorem 1 in the case of the autonomous integrable Ermakov system (14)–(15) which has potential \( V = \frac{N(u)}{X^2} \), where \( u = Y/X \), and kinetic metric \( \gamma_{ab} = \text{diag}(1, 1) \).

The homothetic algebra of \( \gamma_{ab} \) consists of two gradient Killing vectors (KVs) \( \partial_X, \partial_Y \), one non-gradient KV (rotation) \( Y \partial_X - X \partial_Y \) and the gradient homothetic vector (HV) \( X \partial_X + Y \partial_Y \).

For each case of Theorem 1, we have the following.
4.1 The vector $\partial_T$

Case 1 In this case, the point Noether symmetry $A_1 = \partial_T$, $f_1 = 0$ produces the Hamiltonian (as expected)

$$H = \frac{1}{2} (X'^2 + Y'^2) + V = \frac{1}{2} (X'^2 + Y'^2) + \frac{N(u)}{X^2}. \quad (33)$$

4.2 The gradient $HV X \partial_X + Y \partial_Y$

Case 2 Consider the gradient $HV B^a = (X, Y)$ with homothety factor $\psi_B = 1$.

Substituting in condition (27), we find that

$$XV_X + YV_Y + 2V + c_1 = 0 \Rightarrow -2N + u \frac{dN}{du} + \frac{u}{X^2} \frac{dN}{du} + 2\frac{N}{X^2} + c_1 = 0 \Rightarrow c_1 = 0.$$ 

Therefore, the point Noether symmetry is 

$$A_2 = 2T \partial_T + X \partial_X + Y \partial_Y, \quad f_2 = 0 \quad (34)$$

and the associated Noether FI

$$I_2 = 2TH - (XX' + YY'). \quad (35)$$

It can be shown that the three FIs $I_0$, $H$, $I_2$ are independent; therefore, the conservative generalized Ermakov system is superintegrable.

Although the remaining FIs will be expressible in terms of the $I_0$, $H$, $I_2$, we continue in order to show that we recover the results of [5] which were obtained using Lie symmetries.

Case 3 The point Noether symmetry is

$$A_3 = T^2 \partial_T + TX \partial_X + TY \partial_Y, \quad f_3 = \frac{X^2 + Y^2}{2} \quad (36)$$

with associated Noether integral

$$I_3 = T^2H - T (XX' + YY') + \frac{X^2 + Y^2}{2} = \frac{I_2^2 + 2I_0}{4H}. \quad (37)$$

We observe that the Lie symmetries (2.9a), (2.9b), (2.9c) found in [5] are the point Noether symmetries $A_1$, (34), (36). Concerning the remaining FIs of [5], we have: (4.11) $I' = 2I_0$, (4.12) $J'_1 = 2H$, (4.13) $J'_2 = I_2$ and (4.14) $J'_3 = 2I_3$. Using these relations, equation (4.18) is equivalent to the expression (37).

4.3 The gradient $KV b_1 \partial_X + b_2 \partial_Y$

The potential becomes$^1$ $V_I = \frac{k}{(b_1 Y - b_2 X)^2}$ where $k$, $b_1$, $b_2$ are arbitrary constants.

Case 2 The Noether generator, the Noether function and the FI are

$$A_{21} = b_1 \partial_X + b_2 \partial_Y, \quad f_{21} = 0, \quad I_{21} = b_1 X' + b_2 Y'.$$

Case 3 The Noether generator, the Noether function and the FI are

$$A_{31} = Tb_1 \partial_X + Tb_2 \partial_Y, \quad f_{31} = b_1 X + b_2 Y, \quad I_{31} = b_1 (-TX' + X) + b_2 (-TY' + Y).$$

$^1$ This is a superintegrable potential of the form $F(b_1 Y - b_2 X)$ (see Sect. 7 in [10]).
In order to compare these results with the ones of [8], we use polar coordinates \( X = r \cos \theta \) and \( Y = r \sin \theta \). We find:

\[
V_1 = \frac{k}{r^2 (b_1 \sin \theta - b_2 \cos \theta)^2}
\]

\[
A_{21} = (b_1 \cos \theta + b_2 \sin \theta) \partial_\theta + \frac{1}{r} (b_2 \cos \theta - b_1 \sin \theta) \partial_r, \quad f_{21} = 0
\]

\[
A_{31} = (b_1 T \cos \theta + b_2 T \sin \theta) \partial_r + \frac{1}{r} (b_2 T \cos \theta - b_1 T \sin \theta) \partial_\theta, \quad f_{31} = r (b_1 \cos \theta + b_2 \sin \theta)
\]

and

\[
I_{21} = b_1 (r' \cos \theta - r \theta' \sin \theta) + b_2 (r' \sin \theta + r \theta' \cos \theta)
\]

\[
= b_1 \left( \tilde{p}_1 \cos \theta - \frac{\tilde{p}_2 \sin \theta}{r} \right) + b_2 \left( \tilde{p}_1 \sin \theta + \frac{\tilde{p}_2 \cos \theta}{r} \right)
\]

\[
I_{31} = b_1 (-T r' \cos \theta + T r \theta' \sin \theta + r \cos \theta) + b_2 (-T r' \sin \theta - T r \theta' \cos \theta + r \sin \theta)
\]

\[
= b_1 \left( -T \tilde{p}_1 \cos \theta + \frac{T \tilde{p}_2 \sin \theta}{r} + r \cos \theta \right) + b_2 \left( -T \tilde{p}_1 \sin \theta - \frac{T \tilde{p}_2 \cos \theta}{r} + r \sin \theta \right)
\]

where \( \tilde{p}_a = \tilde{\gamma}_{ab} \tilde{q}^{b'} \) are the generalized momenta. Replacing with \( \tilde{q}_{ab} = diag(1, r^2) \), we find that \( \tilde{p}_1 = r' \) and \( \tilde{p}_2 = r^2 \theta' \).

It is straightforward to show that the point Noether symmetries \( A_{21}, A_{31} \), the Noether functions \( f_{21}, f_{31} \) and the FIs \( I_{21}, I_{31} \) are the symmetries \( X_9, X_{10} \), the functions \( B_9, B_{10} \) and the FIs \( I_9, I_{10} \), respectively, of [8], while

- For \( b_1 = 0, b_2 = 1 \), they reduce to the symmetries \( X_5, X_6 \), the functions \( B_5, B_6 \) and the FIs \( I_5, I_6 \), respectively, of [8] and

- For \( b_1 = 1, b_2 = 0 \), they reduce to the symmetries \( X_7, X_8 \), the functions \( B_7, B_8 \) and the FIs \( I_7, I_8 \), respectively, of [8].

As expected, the non-gradient KV (rotation) \( Y \partial_X - X \partial_Y \) leads to the linear first integral of angular momentum.

Finally, using the three FIs \( H, I_0, I_2 \), we integrate the system (14)–(15) and find that in polar coordinates the solution is

\[
r^2(T) = \frac{1}{2H} (2HT - I_2)^2 + \frac{I_0}{H}
\]

\[
\int \frac{d\theta}{\sqrt{I_0 - F(\theta)}} = \pm \int \frac{\sqrt{2}}{r^2(T)} dT = \pm \frac{1}{\sqrt{I_0}} \tan^{-1} \left[ \frac{1}{\sqrt{2I_0}} (2HT - I_2) \right]
\]

where \( \tilde{F}(\theta) = (\tan^2 \theta + 1)N(\tan \theta) \). In Table 3 of [8], the corresponding formula of (39) gives \( \tan^{-1} \) instead of \( \tan^{-1} = \arctan \) which is the correct result.

5 Conclusions

Using recent results on the integrability of 2d conservative dynamical systems, we proved that the generalized Ernackov system is superintegrable and determined all the quadratic FIs. We showed that the recent results of [8] can be obtained from the more general method outlined in [10,12]. Obviously, the methods discussed in the present work can be used by
other authors in the study of similar dynamical systems. As a final remark, we note that an extension of this method, where one computes the time-dependent and autonomous quadratic FIs without the use of Noether symmetries, has appeared recently in [13].

Data availability All data that support the findings of this study are available within the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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