A Fast Algorithm for Three-Dimensional Layers of Maxima Problem

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Abstract. We show that the three-dimensional layers-of-maxima problem can be solved in $o(n \log n)$ time in the word RAM model. Our algorithm runs in $O(n \log \log n)$ deterministic time or $O(n \log \log n^2)$ expected time and uses $O(n)$ space. We also describe a deterministic algorithm that uses optimal $O(n)$ space and solves the three-dimensional layers-of-maxima problem in $O(n \log n)$ time in the pointer machine model.

1 Introduction

A point $p$ dominates a point $q$ if each coordinate of $p$ is larger than or equals to the corresponding coordinate of $q$. A point $p$ is a maximum point in a set $S$ if no point of $S$ dominates $p$. The maxima set of $S$ is the set of all maximum points in $S$. In the layers-of-maxima problem we assign points of a set $S$ to layers $S_i$, $i \geq 1$, according to the dominance relation: The first layer of $S$ is defined as the maxima set of $S$, the layer 2 of $S$ is the maxima set of $S \setminus S_1$, and the $i$-th layer of $S$ is the maxima set of $S \setminus (\cup_{j=1}^{i-1} S_j)$. In this paper we show that the three-dimensional layers-of-maxima problem can be solved in $o(n \log n)$ time.

Previous and Related Work. The algorithm of Kung, Luccio, and Preparata [23] finds the maxima set of a set $S$ in $O(n \log n)$ time for $d = 2$ or $d = 3$ dimensions and $O(n \log^{d-2} n)$ time for $d \geq 4$ dimensions. The algorithm of Gabow, Bentley, and Tarjan [16] finds the maxima set in $O(n \log^{d-3} n \log\log n)$ time for $d \geq 4$ dimensions. Very recently, Chan, Larsen, and Pătraşcu [11] described a randomized algorithm that solves the $d$-dimensional maxima problem (i.e., finds the maxima set) for $d \geq 4$ in $O(n \log^{d-3} n)$ time. Numerous works are devoted to variants of the maxima problem in different computational models and settings: In [8], the authors describe a solution for the three-dimensional maxima problem in the cache-oblivious model. Output-sensitive algorithms and algorithms that find the maxima for a random set of points are described in [7,18,22]. The two-dimensional problem of maintaining the maxima set under insertions and deletions is considered in [21]: the problem of maintaining the maxima set for moving points is considered in [15].

The general layers-of-maxima problem appears to be more difficult than the problem of finding the maxima set. The three-dimensional layers-of-maxima problem can be solved in $O(n \log n \log \log n)$ time [11] using dynamic fractional
cascading \cite{24}. The algorithm of Buchsbaum and Goodrich \cite{9} runs in \(O(n \log n)\) time and uses \(O(n \log n \log \log n)\) space. Giyora and Kaplan \cite{17} described a data structure for point location in a dynamic set of horizontal segments and showed how it can be combined with the approach of \cite{9} to solve the three-dimensional layers-of-maxima problem in \(O(n \log n)\) time and \(O(n)\) space.

The \(O(n \log n)\) time is optimal even if we want to find the maxima set in two dimensions \cite{23} provided that we work in the infinite-precision computation model in which input values, i.e., point coordinates, can be manipulated with algebraic operations and compared. On the other hand, it is well known that it is possible to achieve \(o(n \log n)\) time (resp. \(o(\log n)\) time for searching in a data structure) for many one-dimensional as well as for some multi-dimensional problems and data structures in other computational models. For instance, the grid model, that assumes all coordinates to be integers in the range \([1, U]\) for a parameter \(U\), was extensively studied in computational geometry. Examples of problems that can be solved efficiently in the grid model are orthogonal range reporting queries \cite{25} and point location queries in a two- and three-dimensional rectangular subdivisions \cite{5}. In fact, we can use standard techniques to show that these queries can be answered in \(o(\log n)\) time when all coordinates are arbitrary integers. Recently, a number of other important geometric problems was shown to be solvable in \(o(n \log n)\) time (resp. in \(o(\log n)\) time) in the word RAM model. An incomplete list\footnote{1 We note that problems in this list are more difficult than the layers-of-maxima problem because in our case we process a set of axis-parallel segments.} includes Voronoi diagrams and three-dimensional convex hulls in \(O(n \cdot 2^{O(\sqrt{\log \log n})})\) time \cite{12}, two-dimensional point location in \(O(\log n / \log \log n)\) time \cite{20,10}, and dynamic convex hull in \(O(\log n / \log \log n)\) time \cite{14}. Results for the word RAM model are important because they help us better understand the structure and relative complexity of different problems and demonstrate how geometric information can be analyzed in algorithmically useful ways.

Our Results. In this paper we show that the three-dimensional layers-of-maxima problem can be solved in \(O(n(\log \log n)^3)\) deterministic time and \(O(n)\) space in the word RAM model. If randomization is allowed, our algorithm runs in \(O(n(\log \log n)^2)\) expected time. For comparison, the fastest known deterministic linear space sorting algorithm runs in \(O(n \log n)\) time \cite{19}. Our result is valid in the word RAM computation model, but the time-consuming operations, such as multiplications, are only used during the pre-processing step when we sort points by coordinates (see section \ref{sec:alg}). For instance, if all points are on the \(n \times n \times n\) grid, then our algorithm uses exactly the same model as \cite{25} or \cite{5}.

We also describe an algorithm that uses \(O(n)\) space and solves the three-dimensional layers-of-maxima problem in optimal \(O(n \log n)\) time in the pointer machine model \cite{27}. The result of Giyora and Kaplan \cite{17} that achieved the same space and time bounds is valid only in the RAM model. Thus we present the first algorithm that solves the three-dimensional layers-of-maxima problem in optimal time and space in the pointer machine model.
Overview. Our solution, as well as the previous results, is based on the sweep plane algorithm of [9] described in section 2. The sweep plane algorithm assigns points to layers by answering for each $p \in S$ a point location query in a dynamically maintained staircase subdivision. We observe that general data structures for point location in a set of horizontal segments cannot be used to obtain an $o(n \log n)$ time solution. Even in the word RAM model, no dynamic data structure that supports both queries and updates in $o(\log n)$ time is known.

Moreover, by the lower bound of [2] any data structure for a dynamic set of horizontal segments needs $\Omega(\log n / \log \log n)$ time to answer a point location query. We achieve a significantly better result using the methods described below.

In section 3 we describe the data structure for point location in a staircase subdivision that supports queries in $O((\log \log n)^3)$ time and updates in poly-logarithmic time per segment. This result may be of interest on its own.

The data structure of section 3 is not sufficient to obtain the desired runtime and space usage mainly due to high costs of update operations. To reduce the update time and space usage, we construct auxiliary staircases $B_i$, such that: 1. the total number of segments in $B_i$ and the total number of updates is $O(n/d)$ for a parameter $d = \log O(1)$; 2. locating a point $p$ among staircases $B_i$ gives us an approximate location of $p$ among the original staircases $M_i$ (up to $O(d)$ staircases). An efficient method for maintaining staircases $B_i$, described in section 4, is the most technically challenging part of our construction. In section 5 we show how the data structure of section 3 can be combined with the auxiliary staircases approach to obtain an $O(n \log n)$ time algorithm. We also sketch how the same approach enables us to obtain an $O(n \log n)$ time and $O(n)$ space algorithm in the pointer machine model.

2 Sweep Plane Algorithm

Our algorithm is based on the three-dimensional sweep method that is also used in [9]. We move the plane parallel to the $xy$ plane from $z = +\infty$ to $z = 0$ and maintain the following invariant: when the $z$-coordinate of the plane equals $v$ all points $p$ with $p.z \geq v$ are assigned to their layers of maxima. Here and further $p.x$, $p.y$, and $p.z$ denote the $x$, $y$, and $z$-coordinates of a point $p$. Let $S_i(v)$ be the set of points $q$ that belong to the $i$-th layer of maxima such that $q.z > v$; let $P_i(v)$ denote the projection of $S_i(v)$ on the sweep plane, $P_i(v) = \{\pi(p) | p \in S_i(v)\}$ where $\pi(p)$ denotes the projection of a point $p$ on the $xy$-plane. For each value of $v$ maximal points of $P_i(v)$ form a staircase $M_i$; see Fig. 1. When the $z$-coordinate of the sweep plane is changed from $v + 1$ to $v$, we assign all points with $p.z = v$ to their layers of maxima. If $\pi(p)$, such that $p.z = v$, is dominated by a point from $P_i(v + 1)$, then $p$ belongs to the $j$-th layer of maxima and $j > i$. If $\pi(p)$, such that $p.z = v$, dominates a point on $P_k(v + 1)$, then $p$ belongs to the $j$-th layer of maxima and $j \leq k$. We observe that $\pi(p)$ dominates $P_i(v + 1)$ if and only

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2 We will describe update operations supported by our data structure in sections 2 and 3.

3 We assume that all points have positive coordinates.
if the staircase $M_i$ is dominated by $p$, i.e., the vertical ray shot from $p$ in $-y$ direction passes through $M_i$. Hence, the point $p$ belongs to the layer $i$, such that $\pi(p)$ is between the staircase $M_{i-1}$ and the staircase $M_i$. This means that we can assign a point to its layer by answering a point location query in a staircase subdivision. When all $p$ with $p.z = v$ are assigned to their layers, staircases are updated.

Thus to solve the layers of maxima problem, we examine points in the descending order of their $z$-coordinates. For each $v$, such that there is at least one $p$ with $p.z = v$, we proceed as follows: for every $p$ with $p.z = v$ operation $\text{locate}(p)$ identifies the staircase $M_i$ immediately below $\pi(p)$. If the first staircase below $\pi(p)$ has index $i$ ($\pi(p)$ may also lie on $M_i$), then $p$ is assigned to the $i$-th layer of maxima; if $\pi(p)$ is below the lowest staircase $M_j$, then $p$ is assigned to the new layer $j + 1$. When all points with $p.z = v$ are assigned to their layers, the staircases are updated. All points $p$ such that $p.z = v$ are examined in the ascending order of their $x$-coordinates. If a point $p$ with $p.z = v$ is assigned to layer $i$, we perform operation $\text{replace}(p, i)$ that removes all points of $M_i$ dominated by $p$ and inserts $p$ into $M_i$. If the staircase $i$ does not exist, then instead of $\text{replace}(p, i)$ we perform the operation $\text{new}(p, i)$; $\text{new}(p, i)$ creates a new staircase $M_i$ that consists of one horizontal segment $h$ with left endpoint $(0, p.y)$ and right endpoint $\pi(p)$ and one vertical segment $t$ with upper endpoint $\pi(p)$ and lower endpoint $(0, p.x)$. See Fig. 1 for an example.

![Fig. 1. Points a, b, c, d, and e have the same z-coordinate. (a) Points a, b and c are assigned to layer 2, d is assigned to layer 3, and e is assigned to a new layer 6. (b) Staircases after operations replace(a, 2), replace(b, 2), replace(c, 2), replace(d, 3), and new(e). Observe that b is not the endpoint of a segment in the staircase $M_2$ after updates.](image)

We can reduce the general layers of maxima problem to the problem in the universe of size $O(n)$ using the reduction to rank space technique [25,16]. The rank of an element $e \in S$ is defined as the number of elements in $S$ that are smaller than $e$: $\text{rank}(e, S) = |\{ a \in S | a < e \}|$; clearly, $\text{rank}(e, S) \leq |S|$. For a
point \( p = (p.x, p.y, p.z) \), \( p \in S \), let \( \tau(p) = (\text{rank}(p.x, S_x) + 1, \text{rank}(p.y, S_y) + 1, \text{rank}(p.z, S_z) + 1) \). Let \( S' = \{ \tau(p) \mid p \in S \} \). Coordinates of all points in \( S' \) belong to range \([1, n]\). A point \( p \) dominates a point \( q \) if and only if \( \text{rank}(p.x, S_x) \geq \text{rank}(q.x, S_x), \text{rank}(p.y, S_y) \geq \text{rank}(q.y, S_y), \) and \( \text{rank}(p.z, S_z) \geq \text{rank}(q.z, S_z) \) where \( S_x, S_y, S_z \) are sets of \( x \), \( y \), and \( z \)-coordinates of points in \( S \). Hence if a point \( p' \in S' \) is assigned to the \( i \)-th layer of maxima of \( S' \), then \( \tau^{-1}(p') \) belongs to the \( i \)-th layer of maxima of \( S \). We can find ranks of \( x \), \( y \), and \( z \)-coordinates of every point by sorting \( S_x, S_y, \) and \( S_z \). Using the sorting algorithm of \([19]\), \( S_x, S_y, \) and \( S_z \) can be sorted in \( O(n \log \log n) \) time and \( O(n) \) space. Thus the layers of maxima problem can be reduced to the special case when all point coordinates are bounded by \( O(n) \) in \( O(n \log \log n) \) time.

3 Fast Queries, Slow Updates

In this section we describe a data structure that supports \text{locate}(q)\) in \( O((\log \log n)^3) \) time and update operations \text{replace}(q, i)\) and \text{new}(q, i)\) in \( O(n (\log \log n)^2) \) time per segment. We will store horizontal segments of all staircases in a data structure that supports \text{ray shooting queries}: given a query point \( q \) identify the first segment \( s \) crossed by a vertical ray that is shot from \( q \) in \(-y\) direction; in this case we will say that the segment \( s \) precedes \( q \) (or \( s \) is the predecessor segment of \( q \)). In the rest of this paper, segments will denote horizontal segments. Identifying the segment that precedes \( q \) is (almost) equivalent to answering a query \text{locate}(q). Operation \text{replace}(q, i)\) corresponds to a deletion of all horizontal segments dominated by \( q \) and an insertion of at most two horizontal segments, see Fig.\text{I} Operation \text{new}(q, i)\) corresponds to an insertion of a new segment.

Our data structure is a binary tree on \( x \)-coordinates and segments are stored in one-dimensional secondary structures in tree nodes. The main idea of our approach is to achieve fast query time by binary search of the root-to-leaf path: using properties of staircases, we can determine in \( O((\log \log n)^2) \) time whether the predecessor segment of a point \( q \) is stored in the ancestor of a node \( v \) or in the descendant of a node \( v \) for any node \( v \) on the path from the root to \( q.x \). Our approach is similar to the data structure of \([5]\), but we need additional techniques to support updates.

For a horizontal segment \( s \), we denote by \text{start}(s)\) and \text{end}(s)\) the \( x \)-coordinates of its left and right endpoints respectively; we denote by \( y(s) \) the \( y \)-coordinate of all points of \( s \). An integer \( e \in S \) precedes (follows) an integer \( x \) in \( S \) if \( e \leq x \) (\( e \geq x \)). Let \( H \) be a set of segments and let \( H_y \) be the set of \( y \)-coordinates of segments in \( H \). We say that \( s \in H \) precedes (follows) an integer \( e \) if the \( y \)-coordinate of \( s \) precedes (follows) \( e \) in \( H_y \). Thus a segment that precedes a point \( q \) is a segment that precedes \( q.y \) in the set of all segments that intersect the vertical line \( x = q.x \).

We construct a balanced binary tree \( T \) of height \( \log n \) on the set of all possible \( x \)-coordinates, i.e., \( n \) leaves of \( T \) correspond to integers in \([1, n]\). The range of a
node \( v \) is the interval \( \text{rng}(v) = [\text{left}(v), \text{right}(v)] \) where \( \text{left}(v) \) and \( \text{right}(v) \) are leftmost and rightmost leaf descendants of \( v \).

We say that a segment \( s \) spans a node \( v \) if \( \text{start}(s) < \text{left}(v) < \text{right}(v) < \text{end}(s) \); a segment \( v \) belongs to a node \( v \) if \( \text{left}(v) < \text{start}(s) < \text{end}(s) < \text{right}(v) \). A segment \( s \) l-cuts a node \( v \) if \( s \) intersects the vertical line \( x = \text{left}(v) \), but \( s \) does not span \( v \), i.e., \( \text{start}(s) \leq \text{left}(v) \) and \( \text{end}(s) < \text{right}(v) \); a segment \( s \) r-cuts a node \( v \) if \( s \) intersects the vertical line \( x = \text{right}(v) \) but \( s \) does not span \( v \), i.e., \( \text{start}(s) > \text{left}(v) \) and \( \text{end}(s) \geq \text{right}(v) \). A segment \( s \) such that \( [\text{start}(s), \text{end}(s)] \cap \text{rng}(v) \neq \emptyset \) either cuts \( v \), or spans \( v \), or belongs to \( v \). We store \( y \)-coordinates of all segments that \( l \)-cut (\( r \)-cut) a node \( v \) in a data structure \( \mathcal{L}_v (\mathcal{R}_v) \). Using exponential trees \([4]\), we can implement \( \mathcal{L}_v \) and \( \mathcal{R}_v \) in linear space, so that one-dimensional searching (i.e. predecessor and successor queries) is supported in \( O((\log \log n)^2) \) time. Since a segment cuts \( O(\log n) \) nodes (at most two nodes on each tree level), all \( \mathcal{L}_v \) and \( \mathcal{R}_v \) use \( O(n \log n) \) space. We denote by \( \text{index}(s) \) the index of the staircase \( \mathcal{M}_i \) that contains \( s \), i.e., \( s \in \mathcal{M}_{\text{index}(s)} \). The following simple properties are important for the search procedure:

**Fact 1** Suppose that an arbitrary vertical line cuts staircases \( \mathcal{M}_i \) and \( \mathcal{M}_j \), \( i < j \), in points \( p \) and \( q \) respectively. Then \( p.y > q.y \) because staircases do not cross.

**Fact 2** For any two points \( p \) and \( q \) on a staircase \( \mathcal{M}_i \), if \( p.x < q.x \), then \( p.y \geq q.y \).

**Fact 3** Given a staircase \( \mathcal{M}_i \) and a point \( p \), we can determine whether \( \mathcal{M}_i \) is below or above \( p \) and find the segment \( s \in \mathcal{M}_i \) such that \( p.x \in [\text{start}(s), \text{end}(s)] \) in \( O((\log \log n)^2) \) time. The data structure \( D_i \) that supports such queries uses linear space and supports finger updates in \( O(1) \) time.

**Proof:** The data structure \( D_i \) contains \( x \)-coordinates of all segment endpoints of \( \mathcal{M}_i \). \( D_i \) is implemented as an exponential tree so that it uses \( O(n) \) space. Using \( D_i \) we can identify \( s \in \mathcal{M}_i \) such that \( p.x \in [\text{start}(s), \text{end}(s)] \) in \( O((\log \log n)^2) \) time; \( \mathcal{M}_i \) is below \( p \) if and only if \( s \) is below \( p \). \( \square \)

Using Fact 3 we can determine whether a segment \( s \) precedes a point \( q \) in \( O((\log \log n)^2) \) time: Suppose that \( s \) belongs to a staircase \( \mathcal{M}_i \). Then \( s \) is the predecessor segment of \( q \) iff \( q.x \in [\text{start}(s), \text{end}(s)] \), \( q.y \geq y(s) \) and the staircase \( \mathcal{M}_{i-1} \) is above \( q \).

We can use these properties and data structures \( \mathcal{L}_v \) and \( \mathcal{R}_v \) to determine whether a segment \( b \) that precedes a point \( q \) spans a node \( v \), belongs to a node \( v \), or cuts a node \( v \). If the segment \( b \) we are looking for spans \( v \), then it cuts an ancestor of \( v \); if that segment belongs to \( v \), then it cuts a descendant of \( v \). Hence, we can apply binary search and find in \( O(\log \log n) \) iterations the node \( f \) such that the predecessor segment of \( q \) cuts \( f \). Observe that in some situations there may be no staircase \( \mathcal{M}_i \) below \( q \), see Fig 4 for an example. To deal with such situations, we insert a dummy segment \( s_d \) with left endpoint \((1, 0)\) and right endpoint \((n, 0)\); we set \( \text{index}(s_d) = +\infty \) and store \( s_d \) in the data structure \( \mathcal{L}_{v_0} \) where \( v_0 \) is the root of \( T \).
Let \( l_x \) be the leaf in which the predecessor of \( q.x \) is stored. We will use variables \( l, u \) and \( v \) to guide the search for the node \( f \). Initially we set \( l = l_x \) and \( u \) is the root of \( T \). We set \( v \) to be the middle node between \( u \) and \( l \); if the path between \( u \) and \( l \) consists of \( h \) edges, then the path from \( u \) to \( v \) consists of \( \lfloor h/2 \rfloor \) edges and \( v \) is an ancestor of \( l \).

Let \( r \) and \( s \) denote the segments in \( L_v \) that precede and follow \( q.y \). If there is no segment \( s \) in \( L_v \) with \( y(s) > q.y \), then we set \( s = \text{NULL} \). If there is no segment \( r \) in \( L_v \) with \( y(r) \leq q.y \), then we set \( r = \text{NULL} \). We can find both \( r \) and \( s \) in \( O((\log \log n)^2) \) time. If the segment \( r \neq \text{NULL} \), we check whether the staircase \( M_{\text{index}(r)} \) contains the predecessor segment of \( q \); by Fact 3 this can be done in \( O((\log \log n)^2) \) time. If \( M_{\text{index}(r)} \) contains the predecessor segment of \( q \), the search is completed. Otherwise, the staircase \( M_{\text{index}(r)-1} \) is below \( q \) or \( r = \text{NULL} \). In this case we find the segment \( r' \) that precedes \( q.y \) in \( R_v \). If \( r' \) is not the predecessor segment of \( q \) or \( r' = \text{NULL} \), then the predecessor segment of \( q \) either spans \( v \) or belongs to \( v \). We distinguish between the following two cases:

1. The segment \( s \neq \text{NULL} \) and the staircase that contains \( s \) is below \( q \). By Fact 1 a vertical line \( x = q.x \) will cross the staircase of \( s \) before it will cross a staircase \( M_i, i > \text{index}(s) \). Hence, a segment that spans \( v \) and belongs to the staircase \( M_i, i > \text{index}(s) \), cannot be the predecessor segment of \( q \). If a segment \( t \) spans \( v \) and \( \text{index}(t) < \text{index}(s) \), then the \( y \)-coordinate of \( t \) is larger than the \( y \)-coordinate of \( s \) by Fact 1. Since \( y(t) > y(s) \) and \( y(s) > q.y \), the segment \( t \) is above \( q \). Thus no segment that spans \( v \) can be the predecessor of \( q \).

2. The staircase that contains \( s \) is above \( q \) or \( s = \text{NULL} \). If \( r \) exists, the staircase \( M_{\text{index}(r)-1} \) is below \( q \). Hence, the predecessor segment of \( q \) belongs to a staircase \( M_i, \text{index}(s) < i \leq \text{index}(r) - 1 \). Since each staircase \( M_i \), \( \text{index}(s) < i \leq \text{index}(r) - 1 \), contains a segment that spans \( v \), the predecessor segment of \( s \) is a segment that spans \( v \). If \( r \) does not exist, then every segment below the point \( q \) spans the node \( v \). Hence, the predecessor segment of \( s \) spans \( v \). See Fig. 2 for an example.

If the predecessor segment spans \( v \), we search for \( f \) among ancestors of \( v \); if the predecessor segment belongs to \( v \), we search for \( f \) among descendants of \( v \). Hence, we set \( l = v \) in case 2, and we set \( u = v \) in case 1. Then, we set \( v \) to be the middle node between \( u \) and \( l \) and examine the new node \( v \). Since we examine \( O((\log \log n)^2) \) time in each node, the total query time is \( O((\log \log n)^3) \).

If the predecessor segment is the dummy segment \( s_d \), then there is no horizontal segment of any \( M_i \) below \( q \). In this case we must identify the staircase to the left of \( q.x \). Let \( m_i \) denote the rightmost point on the staircase \( M_i \), i.e., \( m_i \) is a point on \( M_i \) such that \( m_i.y = 0 \). Then \( q \) is between staircases \( M_{i-1} \) and \( M_i \), such that \( m_i.x < q.x < m_{i-1}.x \). We can find \( m_i \) in \( O((\log \log n)^2) \) time.

When a segment \( s \) is deleted, we delete it from the corresponding data structure \( D_i \). We also delete \( s \) from all data structures \( L_v \) and \( R_w \), for all nodes \( v \) and \( w \), such that \( s \) \( l \)-cuts \( v \) (respectively \( r \)-cuts \( w \)). Since a segment cuts \( O(\log n) \)

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4 To simplify the description, we assume that \( \text{index}(s) = 0 \) if \( s = \text{NULL} \).
nodes and exponential trees support updates in $O((\log \log n)^2)$ time, a deletion takes $O(\log n(\log \log n)^2)$ time. Insertions are supported in the same way. Operation \texttt{new}(q, l) is implemented by inserting a segment with endpoints $(0, q.y)$ and $(q.x, q.y)$ into $T$, incrementing by one the number of staircases $l$, and creating the new data structure $D_l$. To implement \texttt{replace}(q, i) we delete the segments “covered” by $q$ from $T$ and $D_i$ and insert the new segment (or two new segments) into $T$ and $D_i$.

Lemma 1. We can store $n$ horizontal staircase segments with endpoints on $n \times n$ grid in a $O(n \log n)$ space data structure that answers ray shooting queries in $O((\log \log n)^3)$ time and supports operation \texttt{replace}(q, i) in $O(m \log n(\log \log n)^2)$ time where $m$ is the number of segments inserted into and deleted from the staircase $M_i$, and operation \texttt{new}(q) in $O(\log n(\log \log n)^2)$ time.

The data structure of Lemma 1 is deterministic. We can further improve the query time if randomization is allowed.

Fact 4 Given a staircase $M_i$ and a point $p$, we can determine whether $M_i$ is below or above $p$ and find the segment $s \in M_i$ such that $p.x \in [\text{start}(s), \text{end}(s)]$ in $O((\log \log n))$ time. The data structure $D_i$ that supports such queries uses linear space and supports finger updates in $O(1)$ expected time.

Proof: The data structure is the same as in the proof of Fact 3 but we use the y-fast tree data structure \cite{28} instead of the exponential tree.

Lemma 2. We can store $n$ horizontal staircase segments with endpoints on $n \times n$ grid in a $O(n \log n)$ space data structure that answers ray shooting queries in $O((\log \log n)^2)$ time and supports operation \texttt{replace}(q, i) in $O(m \log n \log \log n)$ expected time where $m$ is the number of segments inserted into and deleted from the staircase $M_i$, and operation \texttt{new}(q) in $O(\log n \log \log n)$ expected time.

The update time can be slightly improved using fractional cascading and similar techniques, but this is not necessary for our presentation.
Proof: Our data structure is the same as in the proof of Lemma 1. But we implement $D_i$ using Fact 4. Data structures $L_v$ and $R_v$ are implemented using the y-fast tree [28]. Hence, the search procedure spends $O(\log \log n)$ time in each node of $T$ and a query is answered in $O((\log \log n)^2)$ time. □

Although this is not necessary for further presentation, we can prove a similar result for the case when all segment endpoints are on a $U \times U$ grid; the query time is $O(\log \log U + (\log \log n)^3)$ and the update time is $O(\log^3 n(\log \log n)^2)$ per segment. See Appendix D for a proof of this result.

4 Additional Staircases

The algorithm in the previous section needs $O(n \log n (\log \log n)^2)$ time to construct the layers of maxima: $n$ ray shooting queries can be performed in $O(n(\log \log n)^3)$ time, but $O(n)$ update operations take $O(n \log n (\log \log n)^2)$ time. To speed-up the algorithm and improve the space usage, we reduce the number of updates and the number of segments in the data structure of Lemma 1 to $O(n/\log^2 n)$.

Let $D$ denote the data structure of Lemma 1. We construct and maintain a new sequence of staircases $B_1, B_2, \ldots, B_m$, where $m \leq n/d$ and the parameter $d$ will be specified later. All horizontal segments of $B_1, \ldots, B_m$ are stored in $D$. The new staircases satisfy the following conditions:

1. There are $O(n/d)$ horizontal segments in all staircases $B_i$.
2. $D$ is updated $O(n/d)$ times during the execution of the sweep plane algorithm.
3. For any point $q$ and for any $i$, if $q$ is between $B_{i-1}$ and $B_i$, then $q$ is situated between $M_k$ and $M_{k+1}$ for $(i - 3/2)d \leq k \leq (i + 1/2)d$.

Conditions 1 and 2 imply that the data structure $D$ uses $O(n)$ space and all updates of $D$ take $O(n)$ time if $d \geq \log n (\log \log n)^2$. Condition 3 means that we can use staircases $B_i$ to guide the search among $M_k$: we first identify the index $i$, such that the query point $q$ is between $B_{i-1}$ and $B_i$, and then locate $q$ in $M_{i-3/2}d, \ldots, M_{i+1/2}d$. It is not difficult to construct $B_i$ that satisfy conditions 1 and 3. The challenging part is maintaining the staircases $B_i$ with a small number of updates.

**Lemma 3.** The total number of inserted and deleted segments in all $B_i$ is $O(n/d)$. The number of segments stored in $B_i$ is $O(n/d)$.

We describe how staircases can be maintained and prove Lemma 3 in Appendix B.

5 Efficient Algorithms for the Layers-of-Maxima Problem

**Word RAM Model.** To conclude the description of our main algorithm, we need the following simple
Lemma 4. Using a $O(m)$ space data structure, we can locate a point in a group of $d$ staircases $M_1, M_{j+1}, \ldots, M_{j+d}$ in $O(\log d \cdot (\log \log m)^2)$ time, where $m$ is the number of segments in $M_j, M_{j+1}, \ldots, M_{j+d}$. An operation replace($q, i$) is supported in $O((\log \log m)^2 + m_q)$ time, where $m_q$ is the number of inserted and deleted segments in the staircase $M_i$, $j \leq i \leq j + d$.

Proof: We can use Fact 3 to determine whether a staircase is above or below a staircase $M_k$ for any $j \leq k \leq j + d$. Hence, we can locate a point in $O(\log d \cdot (\log \log m)^2)$ time by a binary search among $d$ staircases.

We set $d = \log^2 n$. The data structure $F_i$ contains all segments of staircases $M_{(i-1)d+1}, M_{(i-1)d+2}, \ldots, M_{id}$ for $i = 1, 2, \ldots, j$, where $j = \lfloor l/d \rfloor$ and $l$ is the highest index of a staircase; the data structure $F_{j+1}$ contains all segments of staircases $M_{jd+1}, \ldots, M_l$. We can locate a point $q$ in each $F_i$ in $O((\log \log n)^3)$ time by Lemma 4. Since each staircase belongs to one data structure, all $F_i$ use $O(n)$ space. We also maintain additional staircases $B_i$ as described in section 3. All segments of all staircases $B_i$ are stored in the data structure $D$ of Lemma 4 since $D$ contains $O(n/d)$ segments, the space usage of $D$ is $O(n)$.

Now we can describe how operations locate, replace, new can be implemented in $O((\log \log n)^3)$ time per segment.

- locate($q$): We find the index $k$, such that $q$ is between $B_{k-1}$ and $B_k$ in $O((\log \log n)^3)$ time. As described in section 3, $q$ is between $M_{kd+g}$ and $M_{(k-1)d-g}$. Hence, we can use data structures $F_{k+1}, F_k,$ and $F_{k-1}$ to identify $j$ such that $q$ is between $M_j$ and $M_{j+1}$. Searching $F_{k+1}, F_k,$ and $F_{k-1}$ takes $O((\log \log n)^3)$ time, and the total time for locate($q$) is $O((\log \log n)^3)$.

- replace($q, i$): let $m_q$ be the number of inserted and deleted segments. The data structure $F_{(i/d)}$ can be updated in $O(m_q + (\log \log n)^2)$ time. We may also have to update $B_{(i/d)}$, $B_{(i/d)+1}$, and the data structure $D$.

- new($q, l$): If $l = kd + 1$ for some $k$, a new data structure $F_{k+1}$ is created. We add the horizontal segment of the new staircase into the data structure $F_{k+1}$. If $l = kd$, we create a new staircase $B_k$ and add the segments of $B_k$ into the data structure $D$.

There are $O(n/d)$ update operations on the data structure $D$ that can be performed in $O((n/d) \log n(\log \log n)^2) = O(n)$ time. If we ignore the time to update $D$, then replace($q, i$) takes $O(m_q(\log \log n)^2)$ time and new($q, l$) takes $O((\log \log n)^2)$ time. Since $\sum_{q \in S} m_q = O(n)$ and new($q, l$) is performed at most $n$ times, the algorithm runs in $O(n(\log \log n)^3)$ time. We thus obtain the main result of this paper.

Theorem 1. The three-dimensional layers-of-maxima problem can be solved in $O(n(\log \log n)^3)$ deterministic time in the word RAM model. The space usage of the algorithm is $O(n)$.

If we use Fact 4 instead of Fact 3 in the proof of Lemma 4 and Lemma 2 instead of Lemma 1 in the proof of Theorem 4, we obtain a slightly better randomized algorithm.
Theorem 2. The three-dimensional layers-of-maxima problem can be solved in \(O(n(\log \log n)^2)\) expected time. The space usage of the algorithm is \(O(n)\).

**Pointer Machine Model.** We can apply the idea of additional staircases to obtain an \(O(n \log n)\) algorithm in the pointer machine model. This time, we set \(d = \log n\) and maintain additional staircases \(B_i\) as described in section 4. Horizontal segments of all \(B_i\) are stored in the data structure \(D\) of Giyora and Kaplan [17] that uses \(O(m \log^{\varepsilon} m)\) space and supports queries and updates in \(O(\log m)\) and \(O(\log^{1+\varepsilon} m)\) time respectively, where \(m\) is the number of segments in all \(B_i\) and \(\varepsilon\) is an arbitrarily small positive constant. Using dynamic fractional cascading [24], we can implement \(F_i\) so that \(F_i\) uses linear space and answers queries in \(O(\log n + \log \log n \log d) = O(\log n)\) time. Updates are supported in \(O(\log n)\) time; details will be given in the full version of this paper. Using \(D\) and \(F_1\), we can implement the sweep plane algorithm in the same way as described in the first part of this section. The space usage of all data structures \(F_i\) is \(O(n)\), and all updates of \(F_i\) take \(O(n \log n)\) time. By Lemma 3, the data structure \(D\) is updated \(O(n/\log n)\) times; hence all updates of \(D\) take \(O(n \log n)\) time. The space usage of \(D\) is \(O(m \log^{\varepsilon} m) = O(n)\). Each new point is located by answering one query to \(D\) and at most three queries to \(F_i\); hence, a new point is assigned to its layer of maxima in \(O(\log n)\) time.

Theorem 3. A three-dimensional layers-of-maxima problem can be solved in \(O(n \log n)\) time in the pointer machine model. The space usage of the algorithm is \(O(n)\).

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**References**

1. P.K. Agarwal. Personal communication.
2. S. Alstrup, T. Husfeldt, T Rauhe, *Marked Ancestor Problems*, Proc. FOCS 1998, 534-544.
3. M. J. Atallah, M. T. Goodrich, K. Ramaiyer, *Biased Finger Trees and Three-dimensional Layers of Maxima*, Proc. SoCG 1994, 150-159.
4. A. Andersson, M. Thorup, *Dynamic Ordered Sets with Exponential Search Trees*, J. ACM 54,Article No. 13 (2007).
5. M. de Berg, M. J. van Kreveld, J. Snoeyink, *Two- and Three-Dimensional Point Location in Rectangular Subdivisions*, J. Algorithms 18, 256-277 (1995).
6. M. A. Bender, R. Cole, E. D. Demaine, M. Farach-Colton, J. Zito, *Two Simplified Algorithms for Maintaining Order in a List*, Proc. ESA 2002, 152-164.
7. J. L. Bentley, K. L. Clarkson, and D. B. Levine, *Fast Linear Expected-Time Algorithms for Computing Maxima and Convex Hulls*, Proc. SODA 1990, 179-187.
8. G. S. Brodal, R. Fagerberg, *Cache Oblivious Distribution Sweeping*, Proc. ICALP 2002, 426-438.
9. A. L. Buchsbaum, M. T. Goodrich, *Three-Dimensional Layers of Maxima*, Algorithmica 39, 275-286 (2004).
10. T. M. Chan, *Point Location in $o(\log n)$ Time, Voronoi Diagrams in $o(n \log n)$ Time, and Other Transdichotomous Results in Computational Geometry*, Proc. FOCS 2006, 333-344.
11. T. M. Chan, K. Larsen, M. Pătraşcu, *Orthogonal Range Searching on the RAM, Revisited*, to be published in SoCG 2011.
12. T. M. Chan, M. Pătraşcu, *Voronoi Diagrams in $n^{2^{\Omega(\sqrt{\log n})}}$ time*, Proc. STOC 2007, 31-39.
13. K. L. Clarkson, *More Output-Sensitive Geometric Algorithms*, Proc. FOCS 1994, 695-702.
14. E. D. Demaine, M. Pătraşcu, *Tight Bounds for Dynamic Convex Hull Queries (Again)*, Proc. SoCG 2007, 354-363.
15. P. G. Franciosa, C. Gaibisso, and M. Talamo, *An Optimal Algorithm for the Maxima Set Problem for Data in Motion*, Proc. CG 1992, 17-21.
16. H. N. Gabow, J. L. Bentley, R. E. Tarjan, *Scaling and Related Techniques for Geometry Problems*, Proc. STOC 1984, 135-143.
17. Y. Giyora, H. Kaplan, *Optimal Dynamic Vertical Ray Shooting in Rectilinear Planar Subdivisions*, ACM Transactions on Algorithms 5, (2009).
18. M. J. Golin, *A Provably Fast Linear-Expected-Time Maxima-Finding Algorithm*, Algorithmica 11, 501-524 (1994).
19. Y. Han, *Deterministic Sorting in $O(n \log \log n)$ time and linear space*, J. Algorithms 50, 96-105 (2004).
20. A. Itai, A. G. Konheim, M. Rodeh, *A Sparse Table Implementation of Priority Queues*, Proc. ICALP 1981, 417-431.
21. S. Kapoor, *Dynamic Maintenance of Maximas of 2-d Point Sets*, Proc. SoCG 1994, 140-149.
22. D. G. Kirkpatrick, R. Seidel, *Output-Size Sensitive Algorithms for Finding Maximal Vectors*, Proc. SoCG 1985, 89-96.
23. H. T. Kung, F. Luccio, F. P. Preparata, *On Finding the Maxima of a Set of Vectors*, J. ACM 22, 469-476 (1975).
24. K. Mehlhorn, S. Näher, *Dynamic Fractional Cascading*, Algorithmica 5, 215-241 (1990).
25. M. H. Overmars, *Efficient Data Structures for Range Searching on a Grid*, J. Algorithms 9(2), 254-275 (1988).
26. M. Pătraşcu, *Planar Point Location in Sublogarithmic Time*, Proc. FOCS 2006, 325-332.
27. R. E. Tarjan, *A Class of Algorithms which Require Nonlinear Time to Maintain Disjoint Sets*, J. Comput. Syst. Sci. 18(2), 110-127 (1979).
28. Dan E. Willard, *Log-Logarithmic Worst-Case Range Queries are Possible in Space $\Theta(N)$*, Information Processing Letters 17(2), 81-84 (1983).
29. D. E. Willard, *A Density Control Algorithm for Doing Insertions and Deletions in a Sequentially Ordered File in Good Worst-Case Time*, Information and Computation 97, 150-204 (1992).
Appendix A. Figures

Fig. 3. Example of a just constructed additional staircase $B_i$ for $d = 6$. The staircase $B_i$ is shown with dashed red lines. Staircases are denoted by their indexes.

Fig. 4. There are no staircases below $p$ and $q$. 
Appendix B. Proof of Lemma 3

In the first part of this section we describe the construction procedure of a boundary $B_i$. Then, we will prove some facts about $B_i$ and describe the update procedure. In the last part of this section we will prove that all $B_i$ are updated $O(1)$ times for $d$ updates of $M_i$.

Construction of Additional Staircases. We construct one staircase $B_i$ for $d$ staircases $M_{id-1}, \ldots, M_{id}$. Let $p$ be the starting point of the staircase $M_{id}$, i.e., $p \in M_i$ and $p.x = 1$. The staircase $B_i$ is the path traced by $p$ as we alternatively move $p$ in the $+x$ and $-y$ direction until it hits the $x$-axis.

A segment $s$ covers a point $p$ if $\text{start}(x) \leq p.x \leq \text{end}(s)$. A segment $r$ is related to a segment $s$ if $s$ covers the left endpoint of $r$; a segment $s$ covers a segment $r$ if $\text{start}(s) \leq \text{start}(r)$ and $\text{end}(r) \leq \text{end}(s)$. A point $p$ dominates a segment $s$ if $p$ dominates the left endpoint of $s$. A segment $s$ follows the segment $r$ in a staircase $B_i$ or $M_i$ (resp. $r$ precedes $s$) if both $r$ and $s$ belong to the same staircase and $\text{end}(r) = \text{start}(s)$.
Let $g = d/2$. For convenience we assume that each point $q \in S$ has even $x$-coordinate. This is achieved by replacing each point $q = (q.x, q.y)$ with a point $q' = (2q.x, q.y)$. Endpoints of all segments of $B_i$ will have odd $x$-coordinates. The set $G_i$ contains all segments of $M_{id-g}, \ldots, M_{id}$. The staircase is constructed by repeating the following steps until $p$ hits the $x$-axis or the $x$-coordinate of $p$ is maximal possible, i.e. until $p.y = 0$ or $p.x = 2n$:
(1) We move $p$ in the $+x$ direction until $p$ cuts $M_{id-g}$, i.e. until $p.x = \text{start}(s) + 1$ for a segment $s \in M_{id-g}$ such that $y(s) < p.y$
(2) If $p.x < 2n$, we move $p$ in $-y$ direction until it hits a segment of $M_{id}$ or $p.y = 0$.
Observe that at the beginning of step (1) the point $p$ always belongs to a horizontal segment of $M_{id}$. Hence, a point on $M_{id+1}$ does not dominate a segment of $B_i$. Since each horizontal segment of $B_i$ cuts $M_{id-g}$ it also cuts $M_{id-j}$, $0 < j < g$. Hence, there are at least $g$ segments of $G_i$ related to each horizontal segment of $B_i$ and the total number of segments in all $B_i$ is $O(\frac{g}{2})$. An example of a (just constructed) additional staircase is shown on Fig. 3.

**Updates.** When we update a staircase $M_{id+j}$ for $g/2 \geq j \geq -g/2$ by operation $\text{replace}$, the staircase is moved in the north-east direction. As a result, a point on a staircase $M_{id+j}$, $j > 0$, may dominate a segment of $B_i$. Therefore we maintain a weaker property: no segment of $M_{id+g}$ dominates $B_i$ and each point of $B_i$ is dominated by a point on $M_{id-g}$. Our goal is to update $B_i$ $O(1)$ times for $\Omega(g)$ updates of $M_j$ (in average). We achieve this by maintaining the following invariants

**Invariant 1** Each segment $s \in B_i$ is dominated by the right endpoint of a segment $r \in M_{id}$.

**Invariant 2** No point of $B_i$ is dominated by a point of $M_{id+g/2}$.

**Invariant 3** No segment $s \in B_i$ cuts $M_{id-g+1}$.

We say that a segment $s$ is empty if it does not cut $M_{id-g/2}$.

**Invariant 4** If a segment $s_2$ follows $s_1$ in $B_i$, then either $s_2$ or $s_1$ is not empty.

If Invariants 1 and 3 are true when $B_i$ is constructed, they will not be violated after updates of $M_{id}, \ldots, M_{id-g}$. We update $B_i$ if Invariants 2 or 4 are violated: If a segment $s \in B_i$, such that $s$ was not empty when $s$ was inserted into $B_i$, does not cut $M_{id-g/2}$ after an operation $\text{replace}(q, id - g/2)$, we call the procedure $\text{Rectify}(B_i, s)$ that will be described later in this section. If a segment $s$ of $B_i$ is dominated by a point of $M_{id+g/2}$ after $\text{replace}(q, id + g/2)$, we also call the procedure $\text{Rectify}(B_i, s)$.

**Fact 5** If a point $q$ dominates a segment of $M_{id-j}$, then $q$ dominates at least one segment of $M_{id-k}$ for each $k < j$.

**Fact 6** If a point $q$ dominates more than two segments of $B_i$, then $q$ dominates at least one segment of $M_{id-j}$ for each $j = 1, 2, \ldots, g/2$.  

Proof: If \( q \) dominates three segments of \( \mathcal{B}_i \) then \( q \) dominates the right endpoint of at least one non-empty segment \( s \in \mathcal{B}_i \). Since \( s \) cuts \( \mathcal{M}_{id-g/2} \), the right endpoint of \( s \) dominates a segment of \( \mathcal{M}_{id-g/2} \). Hence, the right endpoint of \( s \) dominates at least one segment of \( \mathcal{M}_{id-j} \) for \( j = 1, 2, \ldots, g/2 - 1 \) by Fact \( 5 \).

Since \( q \) dominates the right endpoint of \( s \), \( q \) also dominates at least one segment of \( \mathcal{M}_{id-j} \) for each \( j = 1, 2, \ldots, g/2 \). \( \square \)

**Fact 7** Any point \( q \) on \( \mathcal{M}_{id+j} \), \( j \geq 0 \), dominates at most two segments of \( \mathcal{B}_i \).

Proof: Suppose that a point \( q \) on \( \mathcal{M}_{id+j} \) dominates more than two segments of \( \mathcal{B}_i \). Then, there is a point \( q' \) on \( \mathcal{M}_{id} \) that also dominates more than two segments of \( \mathcal{B}_i \). By Fact \( 6 \) \( q' \) dominates a segment of \( \mathcal{M}_{id-j} \) for each \( j = 1, 2, \ldots, g/2 \). Since a point on \( \mathcal{M}_{id} \) cannot dominate a point on \( \mathcal{M}_{id-g/2} \), we obtain a contradiction. \( \square \)

Fact \( 7 \), which is a corollary of Invariant \( 3 \), guarantees us that each operation \( \text{replace}(q, id+j) \) such that \( q \) dominates \( \mathcal{B}_i \) affects at most two segments of \( \mathcal{B}_i \). This will be important in our analysis of the number of updates of \( \mathcal{B}_i \). Now we are ready to describe the update procedure.

The procedure \( \text{Rectify}(\mathcal{B}_i, s) \) deletes a segment \( s \) and a number of preceding and following segments and replaces them with new segments. We say that a segment \( s' \) is the child of \( s \) if \( s \) was removed by an operation \( \text{replace}(q) \), such that \( q \) is the right endpoint of \( s' \); \( s' \) is a descendant of \( s \) if \( s' \) is a child of \( s \) or a descendant of a child of \( s \). Let \( s_0 \) be the segment that precedes \( s \) in \( \mathcal{B}_i \). Let \( s_1, s_2, \ldots \) be segments of \( \mathcal{B}_i \) such that \( s_1 \) follows \( s \) and \( s_i \) follows \( s_{i-1} \) for \( i > 1 \). Suppose that \( s \) contained the right endpoint of a segment \( r_B \in \mathcal{M}_{id} \) that belonged to \( \mathcal{M}_{id} \) when \( s \) was inserted into \( \mathcal{B}_i \), and let \( r_u \) be the descendant of \( r_B \) that belongs to \( \mathcal{M}_{id} \) when the procedure \( \text{Rectify}(\mathcal{B}_i, s) \) is performed. By Fact \( 7 \), \( r_u \) may dominate the segment \( s_0 \) that precedes \( s \) in \( \mathcal{B}_i \), but \( r_u \) does not dominate the segment that precedes \( s_0 \) in \( \mathcal{B}_i \). Let \( r_{max} \) be a descendant of a segment \( r \in \mathcal{M}_{id} \) related to \( s \) with the largest \( x \)-coordinate of its right endpoint.

By Fact \( 7 \) \( r_{max} \) may dominate \( s_1 \) and \( s_2 \) but it cannot dominate \( s_3 \).

Now we must decide which segments are to be deleted from \( \mathcal{B}_i \) and how to construct new segments. We delete segments that are dominated by \( r_u \) or \( r_{max} \). As shown above, there are at most three such segments (except of \( s \) itself). If \( s_f \), \( f \leq 2 \), is the last segment dominated by \( r_{max} \), we may also remove some segments that follow \( s_f \). But our guarantee is that all removed segments \( s_{f+1}, \ldots, s_m \) do not cut \( \mathcal{M}_{id-3g/4} \). We insert new segments into \( \mathcal{B}_i \) by moving a point \( p \) in \( +x \) and \( -y \) directions. A more detailed description follows.

To simplify the description, we will use set \( \mathcal{V}_i \) that contains some horizontal segments that currently belong to \( \mathcal{M}_{id} \) and some segments that belonged to \( \mathcal{M}_{id} \) but are already deleted. When a staircase \( \mathcal{B}_i \) is constructed, \( \mathcal{V}_i \) contains all horizontal segments of \( \mathcal{M}_{id} \). When the procedure \( \text{Rectify}(\mathcal{B}_i, s) \) is called, we delete all segments of \( \mathcal{V}_i \) dominated by \( r_u \) or \( r_{max} \) and insert all segments \( r \in \mathcal{M}_{id} \) such that \( \text{end}(r_u) \leq \text{start}(r) \leq \text{start}(r_{max}) \). Segments of \( \mathcal{V}_i \) are used to “bound the staircase \( \mathcal{B}_i \) from below”, i.e. the left endpoint of each horizontal
segment in $B_i$ belongs to a segment from $V_i$.

(1) Let $p$ be the point on $B_i$ such that $p.y = y(r_u)$. This is the left endpoint of the first inserted segment of $\text{Rectify}(B_i, s)$.

(2) We move $p$ in the $+x$ direction until $p.x = \text{end}(r_{max})$ or $p$ cuts $M_{id-g}$. While $p.x < \text{end}(r_{max})$, we repeat the following steps: we move $p$ in the $-y$ direction until it hits “new” $M_{id}$; then, we move $p$ in $+x$ direction until it cuts $M_{id-g}$. Observe that all horizontal segments inserted in step 2 cut.

(3) When $p.x = \text{end}(r_{max})$, we move $p$ in $+x$ direction until it cuts $M_{id-3g/4}$ and $p.x \geq \text{end}(r_{max}) + 1$. Suppose that now $\text{start}(s_m) < p.x \leq \text{end}(s_m)$ for some $s_m$ in $B_i$. We continue moving $p$ in $+x$ direction until $p.x = \text{end}(s_m)$ or $p$ cuts $M_{id-g}$. Then, we move $p$ in $-y$ direction until $p$ hits $V_i$.

(4) We insert a new segment $t$ instead of the segment $s_m$. It is possible that now $p.y < y(s_m)$. We move $p$ in $+x$ direction until $p.x = \text{end}(s_m)$ and move $p$ in $-y$ direction until $p$ hits $V_i$.

(5) Now we must pay attention that Invariant 4 is maintained: all inserted segments except of may be the last one are not empty. Let $t$ denote the last inserted segment, and suppose that both $t$ and $s_{m+1}$ are empty. We can replace two empty segments either with one empty segment, or one non-empty and one empty segment as follows. We replace $t$ with a new segment $t'$: the left endpoint of $t'$ coincides with the left endpoint of $t$ and either $\text{end}(t') = \text{end}(s_{m+1})$ or $t'$ cuts $M_{id-g}$. If $t'$ cuts $M_{id-g}$, we replace $s_{m+1}$ with a new segment $t''$ such that $\text{start}(t'') = \text{end}(t')$, $\text{end}(t'') = \text{end}(s_{m+1})$, and $y(t'') = y(s_{m+1})$. See Fig. 5 for an example of step (5).

Observe that all but one non-empty segments constructed by $\text{Rectify}(B_i, s)$ cut $M_{id-g}$. The only exception is the segment constructed in step (3) that cuts $M_{id-3g/4}$. We prove in Appendix C that the total number of updates of $B_i$ during the execution of the sweep plane algorithm is $O(\frac{n}{g})$.

This completes the Proof of Lemma 3.

Appendix C. Analysis of Update Operations for Additional Staircases

We will show below that the data structure $D$ is updated $O(\frac{n}{g})$ times during the execution of the sweep-plane algorithm. First, we will estimate the number of deleted segments. We will estimate the number of insertions in the end of this section. We assign $c$ credit points to each segment of $M_{id-j}$ and $3c$ credit points to every segment of $M_{id+j}$ for $j = 1, 2, \ldots, g$ and $c = 24$. Insertion of a new segment into $B_i$ is free and deletion costs $q$ credit points.

Every time when we perform operation $\text{replace}(q, id + j)$ for $j > 0$ we distribute the credit points of the newly inserted segment $r$ with right endpoint $q$ among several segments of $B_i$. We evenly distribute credits of $r$ among segments $s_j \in B$ such that either $q$ dominates $s_j$ or $q.x > \text{start}(s_j)$ and $q.y \geq y(s_{j+1})$ where $s_{j+1}$ denotes the segment that follows $s_j$ in $B_i$. By Fact 6 there are at most three such segments $s_j$; hence each $s_j$ obtains at least $c$ credits. When a segment $r$ of $M_{id-j}$, $j \geq 0$, is deleted, we assign credits of $r$ to $s \in B_i$, such
that \( r \) is related to \( s \). We say that a segment \( s \) is initially non-empty if \( s \) cuts \( \mathcal{M}_{id-g/2} \) when \( s \) is inserted into \( \mathcal{B}_i \). We will show below that we can pay \( g \) credit points for each deleted segment of \( \mathcal{B}_i \) and maintain the following property.

**Property 1.** Every initially non-empty segment \( s \in \mathcal{B}_i \) that does not cut \( \mathcal{M}_{id-g+k} \), \( k \geq g/4 \), accumulated at least \( c \cdot k \) credit points. Every segment \( s \in \mathcal{B}_i \) that is dominated by a point on \( \mathcal{M}_{id+j} \) accumulated \( j \cdot c \) credit points.

*Proof:* Property [1] is obviously true for a just constructed staircase \( \mathcal{B}_i \). Suppose that Property [1] is true after the procedure \texttt{Rectify}(\( \mathcal{B}_i, s \)) was called for segments of \( \mathcal{B}_i \), \( f \geq 0 \) times. We will show that this property is maintained after the \((f+1)\)-th call of the procedure \texttt{Rectify}(\( \mathcal{B}_i, s \)). If an initially non-empty segment \( s \) does not cut \( \mathcal{M}_{id-g+k} \) after the \( f \)-th call of \texttt{Rectify} is completed, then \( s \) accumulated \( k'c \) credits. If the segment \( s \) does not cut \( \mathcal{M}_{id-g+k} \) at some point after the \( f \)-th call of \texttt{Rectify}, then at least \( k - k' \) segments of \( \mathcal{M}_{id-g+j} \), \( k' < j \leq k \), that are related to \( s \) are already deleted. Hence, \( s \) accumulated at least \( ck' + c(k - k') = ck \) credits. Therefore if an initially non-empty segment \( s \) does not cut \( \mathcal{M}_{id-g/2} \), then \( s \) accumulated \( cg/2 \) credits.

If a point of \( \mathcal{M}_{id+k'} \) dominates a segment \( s \in \mathcal{B}_i \) when the \( f \)-th call of the procedure \texttt{Rectify} is completed, then \( s \) has \( k'c \) credit points. If a point of \( \mathcal{M}_{id+k} \) dominates a segment \( s \in \mathcal{B}_i \), then we performed at least one operation \texttt{replace}(\( q, id + j \)) such that \( q \) dominates \( s \) for each \( k' < j \leq k \). Hence, \( s \) accumulated \( kc \) credits. Therefore if a segment of \( \mathcal{B}_i \) is dominated by a point on \( \mathcal{M}_{id+g/2} \), then \( s \) has \( cg/2 \) credits.

Hence, when we start the procedure \texttt{Rectify}(\( \mathcal{B}_i, s \)), the segment \( s \) has \( cg/2 \) credit points. In addition to \( s \), we may have to remove segments \( s_0, s_1, s_2 \) because a descendant of some segment \( r \in \mathcal{M}_{id} \) such that \( r \) was related to \( s \) when \( s \) was constructed, dominates \( s_0, s_1, \) or \( s_2 \). If segments \( s_3, s_4, \ldots, s_m \) are removed by \texttt{Rectify}(\( \mathcal{B}_i, s \)), then each non-empty segment among \( s_3, \ldots, s_m \) does not cut \( \mathcal{M}_{id-3g/4} \). By Property [1] every such segment has \( cg/4 \) credits. Since there are at least \((m-4)/2\) non-empty segments among \( s_3, \ldots, s_m \), we can use \( cg(m-4)/4 \) credits accumulated by non-empty segments to remove \( s_4, \ldots, s_m \). We use \( cg/4 = 6g \) credits accumulated by \( s \) to remove \( s \) and to remove \( s_0, s_1, s_2, s_3, \) and \( s_{m+1} \) if necessary. If a segment \( s' \) inserted after the procedure \texttt{Rectify} cuts only \( 3g/4 \) staircases \( \mathcal{M}_{id+j} \), then we transfer to \( s' \) the remaining credit points accumulated by \( s \). Recall that there is at most one such segment \( s' \) that may be inserted during step (3) of the update procedure. Since \( s \) accumulated at least \( cg/2 \) credit points and at most \( cg/4 \) are spent for removing the segments \( s, s_0, s_1, \) and \( s_2 \), the segment \( s' \) obtains at least \( cg/4 \) credit points after the update procedure.

We must also take care of the segment \( t \) resp. segments \( t' \) and \( t'' \). Since it is possible that \( y(t) < y(s_m) \), \( t \) can be dominated by a point of \( \mathcal{M}_{id+j} \) for some \( j > 0 \) when it is constructed. Remaining credit points of segment \( s_m \) are transferred to \( t \) (resp. to \( t' \)): if \( t'' \) is constructed, then credit points of \( s_{m+1} \) are transferred to \( t'' \). If \( t' \) is constructed but \( t'' \) is not constructed (i.e. if \( t' \) replaces both \( s_m \) and \( s_{m+1} \)), then credits of \( s_{m+1} \) are also transferred to \( t' \). If \( t \) is dominated
by a point of $\mathcal{M}_{id+k}$ for some $0 < k < g/2$, then we performed an operation replace($q, id + j$) for each $j \leq k$, such that $q.y \geq y(t)$ and $q.x \geq \text{start}(t)$. Since $\text{start}(t) > \text{start}(s_m)$ and $y(t) > y(s_m + 1)$, $s_m$ was assigned $c$ credits for each $1 \leq j \leq k$. Hence, if $t$ is dominated by $\mathcal{M}_{id+j}$, then $t$ has at least $cq$ credit points. The same is also true for $t'$ and $t''$. □

We can conclude from Property 1 that we can always pay $g$ credit points for a deleted segment of $B_i$; hence, the total number of deleted segments is $O(n)$. Let $N_f$ be the number of segments in all $B_i$ when the algorithm is finished. Since for every second segment $s$ in $B_i$ there are at least $g/2$ segments of $\mathcal{M}_{id}, \ldots, \mathcal{M}_{id-g}$ related to $s$, the total number of segments in $B_i$ is $O(n_i/g)$ where $n_i$ denotes the total number of segments in $\mathcal{M}_{id}, \ldots, \mathcal{M}_{id-g}$. Hence, $N_f = O(\sum n_i/g) = O(n/g)$. Clearly $N_i = N_f + N_d$ where $N_i$ is the number of inserted segments and $N_d$ is the number of deleted segments. Hence, $N_i = O(n/g)$ and the total number of inserted and deleted segments in all $B_i$ is $N_i + N_d = O(n/g) = O(n/d)$.

Appendix D. Staircases on $U \times U$ Grid

Lemma 5. We can store $n$ horizontal staircase segments with endpoints on $U \times U$ grid in an $O(n \log n)$ space data structure that answers ray shooting queries in $O(\log \log U + (\log \log n)^3)$ time and supports operation replace($q, i$) in $O(m \log^2 n(\log \log n)^2)$ time where $m$ is the number of segments inserted into and deleted from the staircase $\mathcal{M}_i$, and operation new($q$) in $O(\log^3 n(\log \log n)^2)$ time.

Proof: Instead of storing point coordinates of segment endpoints in the data structure, we store labels of point coordinates: each $x$- and $y$-coordinate is assigned an $x$-label ($y$-label), so that the $x$-label ($y$-label) of $q$ is smaller than the $x$-label ($y$-label) of $p$ if and only if $q.x < p.x$ ($q.y < p.y$). All labels belong to range $[1, O(n)]$ and are maintained using the technique of [20,29]. When a new segment is inserted or deleted, $O(\log^2 n)$ labels may change, and we have to delete and re-insert into data structures those segments whose labels are changed. Since each segment is stored in $O(\log n)$ secondary data structures, a deleted/inserted segment leads to $O(\log^3 n)$ updates in $L_v$ and $R_v$. Hence the update time is $O(\log^3 n(\log \log n)^2)$. The query procedure is exactly the same as in the proof of Lemma 1. □