ON TIDAL LOVE NUMBERS OF BRANEWORLD BLACK HOLES AND WORMHOLES

HAI SIONG TAN

Division of Physics and Applied Physics, School of Physical and Mathematical Sciences,
Nanyang Technological University,
21 Nanyang Link, Singapore 637371

ABSTRACT

We study the tidal deformations of various known black hole and wormhole solutions in a simple context of warped compactification — Randall-Sundrum theory in which the four-dimensional spacetime geometry is that of a brane embedded in five-dimensional Anti-de Sitter spacetime. The linearized gravitational perturbation theory generically reduces to either an inhomogeneous second-order ODE or a homogeneous third-order ODE of which indicial roots associated with an expansion about asymptotic infinity can be related to Tidal Love Numbers. We describe various tidal-deformed metrics, classify their indicial roots, and find that in particular the quadrupolar TLN is generically non-vanishing. Thus it could be a signature of a braneworld by virtue of its potential appearance in gravitational waveforms emitted in binary merger events.
1 Introduction

Recent spectacular advances in gravitational-wave (GW) detection have brought new optimism and excitement in using GW astronomy to explore a range of topics in black hole and neutron star physics. In particular, the GW signal emitted by a neutron-star binary carries information about the nuclear equation of state through Tidal Love Numbers which are fundamentally a set of quantities that measure the gravitational tidal deformation of the object. As was first explained in a seminal work by Flanagan and Hinderer in [1], the (electric, quadrupolar) TLN makes its appearance as a phase in the GW waveform at the fifth post-Newtonian order during the inspiral stage of a binary merger. Various detection events could then measure or set bounds on the TLN and thus the parameters (e.g., equation of state parameters, theory couplings, etc.) that it depends on (see for example [2, 3] for the most recent neutron binary (GW170817) data analysis performed by LIGO).
For black holes in general relativity, the TLN is conspicuously zero [4, 5]. In linearized perturbation theory, one can derive the differential equations which metric perturbations have to obey, and we can read off the TLNs from their large distance behavior. For the Schwarzschild black hole, the metric perturbations can be analytically solved and upon imposing regularity at the horizon, one could demonstrate the vanishing of its TLN. For neutron stars, we match the exterior spacetime to a suitable interior, and the TLN thus depends on the equation of state parameters associated with the stellar interior.

It is a natural and well-motivated question to explore how deviations from ordinary GR may lead to the non-vanishing of TLNs, especially in view of the large amount of work dedicated to various theoretical models for its UV completion. Let us briefly point out recent related results on this issue. In [6], black hole solutions in an effective field theory framework encompassing various higher-order curvature extensions of the Einstein-Hilbert action were studied and it was found that they have non-vanishing TLNs. A similar work in [7] found the same phenomenon for black holes in a theory with $R^3$ terms. In [8], black hole solutions in Brans-Dicke theory, Chern-Simons gravity and Einstein-Maxwell theory were studied for their TLNs, and non-vanishing quadrupolar and octopolar magnetic TLNs were found in the case of Chern-Simons gravity. The authors of [8] also computed the TLNs of compact exotic objects which are not black holes such as wormholes, boson stars, etc. which turn out to possess non-vanishing TLNs as well. Coupled with the increasing precision of GW detectors in the near future, this collection of recent results demonstrates that TLNs could be interesting and important indicators of new physics.

In this paper, we study the TLNs for various black hole and wormhole solutions in a simple context of warped compactification — Randall-Sundrum theory [9] in which our four-dimensional spacetime geometry is that of a brane embedded in a five-dimensional spacetime which has $AdS_5$ as its vacuum. The conception of this model was originally motivated by the gauge hierarchy problem, and it has been actively studied at various levels, including its embedding in string theory, relevance for AdS/CFT, etc. In our opinion, this braneworld scenario is a simple and well-motivated backdrop where one could start to probe how extra dimensions affect TLNs. In this case, the extra dimension is non-compact and gravity is localized on the brane by the warped geometry of the embedding spacetime.

Most of this paper will be devoted to the electric, quadrupolar TLN commonly termed as $k_2$, since as shown in [1], it is the phenomenologically relevant one that appears at the fifth post-Newtonian order in the waveform. We will also present some essential and useful results for the general TLN associated with polar perturbations, but we leave axial-type or magnetic TLNs [10] for future work. The background braneworld solutions include black holes, wormholes and naked singularities which were presented in the literature some time ago. They are exact solutions to a four-dimensional formulation of Randall-Sundrum model with an effective energy-momentum tensor (capturing the effects of the extra dimension) as first derived by Shiromizu, Maeda and Sasaki in [11]. The perturbation equations yield differential equations which appear difficult for analytic solutions but as we shall explain, the computation of TLNs in this case does not quite necessitate exact solutions but rather appropriate series expansions about infinity and the metric singularities, provided the singular points of the differential equations are regular. The families of solutions we consider are all parametrically connected to some well-defined limits of a particular black hole solution which share the same metric form with the Reissner-Nordström solution but
with a negative tidal charge. This ‘Tidal black hole’ will be used as an anchor solution for us to check the consistency of various series solutions. In the vanishing charge limit, it reduces to the Schwarzschild metric \([12]\). We should point out in \([13]\), there was an attempt to compute the TLN for the Tidal black hole, but upon review (see details in Section \([4.2]\)), we find that there is unfortunately a major error in that calculation. Nonetheless, we refer the reader to \([13]\) for a study of TLNs of braneworld stars \([14]\) which this paper does not cover.

Here is a summary of our main results: (i) TLNs of braneworld black holes and wormholes studied here are generically non-vanishing, (ii) they can be expressed for most cases as polynomials in the parameter that characterizes each family of solution, (iii) indicial roots associated with near-horizon and asymptotic expansions display some level of universality across various families of solutions.

The paper is organized as follows. In Sections \([2]\) and \([3]\), we furnish a review of the notions of TLNs and the effective field equations of the Randall-Sundrum braneworld model respectively, establishing some conventions along the way. In Section \([4]\), we derive the perturbation equations and show that we end up with either a homogeneous third-order ODE or an inhomogeneous second-order ODE to solve. In Sections \([5]\) and \([6]\), we study each braneworld solution in detail, staying mostly focussed on the case of the quadrupolar TLN. In each case, we will briefly state relevant aspects of the causal structure of each solution following the relevant references which the reader can refer to for the CP diagrams and other elaborations. Finally, we end with some concluding remarks in Section \([7]\).

2 On Tidal Love Numbers

We begin by reviewing the notion of tidal love numbers (TLN) following \([15]\). They are quantities which measure the effect of gravitational tidal deformation due to some external companion or field on the object and its appearance in gravitational waveforms was first studied seriously in \([1]\). In Newtonian gravity, the TLN is a constant of proportionality between the tidal field applied to the body and the resulting multipole moment of its mass distribution. One can characterize the tidal field by the tidal moment

\[
\xi_{ab}(t) = -\partial_a \partial_b U_{ext},
\]

(2.1)

where \(U_{ext}\) is the Newtonian potential of some external body. This is evaluated at the body’s CM and we are working in the object’s local asymptotic rest frame with the \(x^a\) in \((2.1)\) being asymptotically CM centered Cartesian coordinates. It is a symmetric and tracefree tensor which can be covariantly described by the Weyl tensor\(^1\). On the other hand, we define the quadrupole moment as

\[
Q_{ab} = \int \rho(x) (x^a x^b - \frac{1}{3} \delta^{ab} r^2) \, d^3x,
\]

(2.2)

where \(\rho(x)\) is the mass density within the body. In the absence of the tidal field, we assume the body to be spherical and \(Q_{ab}\) vanishes. In the presence of a weak tidal field, from dimensional analysis,

\[
Q_{ab} = -\frac{2}{3} k_2 R^5 \xi_{ab},
\]

(2.3)

\(^1\)In Fermi normal coordinates centered at \(r = 0\), \(\xi_{ab} = R_{0a0b}\). See for example \([16]\).
where $R$ is the body’s radius and the rest are conventions. $k_2$ is the dimensionless tidal love number for a quadrupolar deformation and is the object of focus in this paper.

More generally, we could have tidal moments of higher multipole orders and higher powers of $x^\alpha$. The set of tidal love numbers $k_l$ then measures the body’s response. A useful way of characterizing this definition can be obtained from the expression of the Newtonian potential

$$U = \frac{M}{r} - \frac{1}{(l-1)!} \left[ 1 + 2k_l \frac{R}{r} \frac{2l+1}{2l+1} \right] \xi_L(t)x^L,$$

(2.4)

where $L \sim a_1 a_2 ... a_l$ is a multi-index that contains $l$ individual indices. The tidal moment is now defined as $\xi_L(t) = -\frac{1}{(l-2)!} \partial_L U_{\text{ext}}$, and the $l-$pole moment of the mass distribution is the tensor

$$Q^L = \int \rho x^{(L)} d^3x, \quad Q_L = -\frac{2(l-1)!}{(2l-1)!!} k_l R^{2l+1} \xi_L,$$

(2.5)

where $\langle L \rangle$ denotes the removal of its trace. It is useful to formulate a working definition of the tidal love numbers from the following metric ansatz.

$$g_{tt} = -1 + \frac{2M}{r} + \sum_{l \geq 2} \frac{2}{r^{l+1}} (M_l Y^{(l)} + ...) - \frac{2}{l(l-1)} \frac{r^l}{l} (\xi_l Y^{(l)} + ...),$$

(2.6)

$$g_{t\varphi} = \frac{2J}{r} \sin^2 \theta + \sum_{l \geq 2} \frac{2}{r^l} \left( \frac{S_l}{l} Y^{(l)} + ... \right) + \frac{2r^{l+1}}{3l(l-1)} (B_l S_{\varphi}^{(l)} + ...),$$

(2.7)

where $S_{\varphi}^{(l)} = \sin(\theta) \partial_{\theta} Y^{(l)}$. The ‘electric’ and ‘magnetic’ tidal love numbers are defined as

$$k_l^{(E)} = -\frac{l(l-1)}{2M^{2l+1}} \frac{M_l}{\xi_l}, \quad k_l^{(B)} = -\frac{3l(l-1)}{2(l+1)M^{2l+1}} \frac{S_l}{B_l},$$

(2.8)

where we have absorbed a factor of $\sqrt{\frac{4\pi}{2l+1}}$ in $M_l, S_l$, with the two types of TLNs describing polar and axial perturbations. Typically, we use gravitational perturbation theory to construct the metric describing the tidal deformations. One can parametrize static perturbations of spherically symmetric and static geometries is as follows (see for example [17]). Let $g_{\mu\nu} = g_{\mu\nu}^{(bg)} + h_{\mu\nu}$, then in the chart of $\{t, r, \theta, \phi\}$, we can write

$$h_{tt} = g_{tt}^{(bg)} \sum_l H_0^{(l)} P_l, \quad h_{rr} = g_{rr}^{(bg)} \sum_l H_2^{(l)} P_l,$$

$$h_{\theta\theta} = r^2 \sum_l K^{(l)} P_l, \quad h_{\theta\phi} = r^2 \sin^2 \theta \sum_l K^{(l)} P_l, \quad h_{t\varphi} = \sum_l h_{0l}^{(l)} P_l \sin^2 \theta,$$

(2.9)

where $P_l' = \frac{dP_l(\cos \theta)}{d(\cos \theta)}$, and $P_l = P_l(\cos \theta)$ are Legendre polynomials used as basis functions for the angular dependence and due to the spherical symmetry of the ansatz, we suppress the degenerate azimuthal numbers. The above ansatz for static linearized perturbations was first studied by Thorne and Campolattaro [18] and has since been frequently invoked in studies of TLN as well as quasi-Schwarzschild solutions [17]. For solutions relevant for TLN, the metric is not asymptotically flat and is valid within a restricted domain that lies in the compact object’s exterior and which captures the tidal effects of an external matter source.
In the remaining of our paper, we will focus mainly on $k_2^{(E)}$, the dimensionless quadrupolar TLN which carries crucial phenomenological value due to its appearance as a tidal phase correction in the gravitational waveform associated with the inspiral stage. Taking $l = 2$ in (2.8), we have

$$k_2^{(E)} = -\frac{1}{M^5} \frac{M_2}{\xi_2}, \quad k_2^{(B)} = -\frac{1}{M^5} \frac{S_2}{B_2}. \quad (2.10)$$

We have suppressed other angular harmonics in our ansatz for the perturbation. Let us quickly note that upon restoring them, we can write the metric component $g_{tt}$ in the form

$$g_{tt} = -1 + \frac{2M}{r} + \frac{3Q_{ij}}{r^3} (n^i n^j) - \xi_{ij} x^i x^j + \ldots \quad (2.11)$$

where $n^i = x^i/r$ is the unit vector and the quarupole moment $Q_{ij}$ is traceless. The $r^2$ term represents the tidal force and the $1/r^3$ term characterizes the compact object’s response to the tidal force. To linear order in $\xi_{ij}$, the induced moment takes the form (see for example [19])

$$Q_{ij} = -\lambda \xi_{ij} \quad (2.12)$$

for some constant $\lambda$. The dimensionless TLN $k_2^{(E)}$ is related by $k_2^{(E)} = \frac{3\lambda}{2M}$. Using the ansatz for $l = 2$ and expanding in spherical harmonics, we can write

$$Q_{ij} n^i n^j = \sum_{m=-2}^{2} Q_{m} Y_{2m}, \quad \xi_{ij} n^i n^j = \sum_{m=-2}^{2} \tilde{\xi}_m Y_{2m}. \quad (2.13)$$

Comparing between (2.11) and (2.6) yields

$$\lambda = -\frac{2M_2}{3\xi_2}, \quad k_2^{(E)} = \frac{3\lambda}{2M^5} = -\frac{1}{M^5} \frac{M_2}{\xi_2}. \quad (2.14)$$

To extract $k_2^{(E)}$, we need the asymptotic series expansion of $h_{tt}$ from which we read off any non-vanishing pair of $r^2$ and $1/r^3$ terms. Since we are seeking the induced response, the relevant term of the form $1/r^3$ should vanish together with the tidal force term. In the following, for each braneworld solution, we will focus on computing $\lambda = \frac{1}{3} \frac{C_2}{C_3}$, where $C_3, C_2$ are the coefficients of the $1/r^3, r^2$ terms in an asymptotic series expansion of $h_{tt}$. We will refer to $\lambda$ as the Tidal Love Number (TLN) from now on.

3 On Randall-Sundrum theory with warped compactification

In this section, we furnish some essential points concerning the background gravitational theory — Randall-Sundrum theory with a brane of positive tension on which gravity is localized via curvature rather than compactification [9]. We work with a five-dimensional bulk with $AdS_5$ as the vacuum and an energy-momentum tensor that is confined on the 3 + 1D brane, in which the effective cosmological constant can be set to vanish with a choice of brane tension. Following [11], we find it useful to formulate the gravitational dynamics purely in terms of an effective four-dimensional theory which we review below. We will also derive how various metric components (including the perturbations) which are functions of the 3+1D brane-worldvolume coordinates appear in the line element obtained by expanding the metric about the brane.
We begin with a parent 5D metric ansatz that reads

$$ds^2 = dy^2 + g_{\mu\nu}(x,y)dx^\mu dx^\nu,$$

where $y$ is a Gaussian normal coordinate that is orthogonal to the brane. In the neighborhood of $y = 0$ where the confining brane lies, we can express the metric perturbation as

$$g_{\mu\nu}(x,y) = g_{\mu\nu}^B(x,y) + h_{\mu\nu}(x,y) = g_{\mu\nu}^B(x,y) + h_{\mu\nu}(x,0) + \frac{1}{2} h''_{\mu\nu}(x,0)y^2 + \ldots,$$  

(3.2)

where we denote $\partial_y$ with a prime for notational simplicity in this section. From (3.1), the extrinsic curvature describing the embedding of constant $y$ surfaces reads

$$K_{\mu\nu} = \left(\delta^{\alpha}_\mu - n^{\alpha}\eta_{\mu} \right) \left(\delta^{\beta}_\nu - n^{\beta}\eta_{\nu} \right) \nabla_\alpha n_\beta = \frac{1}{2} g'_{\mu\nu}.$$  

(3.3)

We can fine-tune the brane cosmological constant to vanish, with

$$\Lambda_5 = -\frac{\kappa_5^2 \Lambda^2}{6}, \kappa_4^2 = \frac{1}{6} \kappa_5^4 \lambda,$$

(3.4)

where $\lambda$ is the brane tension, and $\kappa_{4,5}^2 = 8\pi G_{4,5}$. In this paper, we will adopt this condition for simplicity. The ordinary 4D limit involves taking $\kappa_5 \to 0, \lambda \sim \kappa_5^{-4} \to \infty$ such that $\kappa_4$ is finite. Henceforth, we denote $\kappa_4$ simply as $\kappa$. The 5D field equations read

$$G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 \left( T_{AB} + T_{AB}^B \delta(y) \right).$$  

(3.5)

For the energy content, we set

$$T_{AB} = 0, \ T_{AB}^B = -\lambda g_{AB} \delta^{A}_\mu \delta^{B}_\nu.$$  

(3.6)

Together with the fine-tuned vanishing of $\Lambda_4$, it can be shown using Gauss-Codazzi equations (see for example [21]) that the field equations on the brane read

$$G_{\mu\nu} = -E_{\mu\nu}, \ E_{\mu\nu} = C^{\alpha}_{\beta\rho\sigma} n_\alpha n_\beta q^\rho q^\sigma = K_{\mu\alpha} K^{\alpha}_{\nu} - \eta_y K_{\mu\nu} - \frac{\Lambda_5}{6} g_{\mu\nu}.$$  

(3.7)

Apart from the continuity of metric across $y = 0$, the Israel junction conditions impose the discontinuity of the extrinsic curvature to be

$$K^+_{\mu\nu} - K^-_{\mu\nu} = -\kappa_5^2 \left( T_{\mu\nu}^B - \frac{1}{3} T_{\mu\nu}^B g_{\mu\nu} \right) = -\frac{1}{3} \kappa_5^2 \lambda g_{\mu\nu},$$  

(3.8)

which implies that on the brane,

$$K_{\mu\nu} = -\frac{1}{6} \kappa_5^2 \lambda g_{\mu\nu} = \frac{1}{4} K g_{\mu\nu}.$$  

(3.9)

By virtue of the metric ansatz, we also have (3.3) which leads to the following useful relation on the brane

$$g'_{\mu\nu}(x,0) = \frac{1}{2} K g_{\mu\nu}(x,0).$$  

(3.10)

With the metric perturbation switched on, the scalar $K = -\frac{2}{3} \kappa_5^2 \lambda$ remains invariant and specifies the constraint on the brane

$$g^{\mu\nu}(x,0) g''_{\mu\nu}(x,0) = K^2.$$  

(3.11)
The perturbations have to satisfy (from (3.10))

$$h_{\mu\nu}'(x,0) = \frac{1}{2} Kh_{\mu\nu}(x,0). \quad (3.12)$$

Under the perturbation, we find

$$\delta E_{\mu\nu} = -\frac{1}{2} h''_{\mu\nu} + \frac{1}{6} \Lambda h_{\mu\nu} + \frac{1}{2} h'_{\mu\alpha} K^\alpha_{\nu} + \frac{1}{2} K_{\mu}^\beta h'_{\beta\nu}. \quad (3.13)$$

When restricted on the brane, we find

$$\delta E_{\mu\nu}(x,0) = -\frac{1}{2} h''_{\mu\nu}(x,0) + \frac{3}{16} K^2 h_{\mu\nu}(x,0), \quad (3.14)$$

with a slightly different background relation

$$E_{\mu\nu}(x,0) = -\frac{1}{2} g''_{\mu\nu}(x,0) + \frac{1}{8} K^2 g_{\mu\nu}(x,0). \quad (3.15)$$

We note that equations (3.14) and (3.15) imply that up to second-order in $y$, we have the following expansion of the metric components.

$$g_{\mu\nu}^{bg} = g_{\mu\nu}^{bg}(x,0) + \frac{1}{2} K g_{\mu\nu}^{bg}(x,0) y + \left[ \frac{3}{16} K^2 g_{\mu\nu}^{bg}(x,0) - E_{\mu\nu}(x,0) \right] y^2 + \ldots \quad (3.16)$$

$$h_{\mu\nu} = h_{\mu\nu}(x,0) + \frac{1}{2} K h_{\mu\nu}(x,0) y + \left[ \frac{3}{16} K^2 h_{\mu\nu}(x,0) - \delta E_{\mu\nu}(x,0) \right] y^2 + \ldots \quad (3.17)$$

In subsequent sections, we will work with various classical solutions in an effective 3+1D description as backgrounds. We will not seek their 4+1D completion beyond what they imply for the bulk extension via (3.16) and (3.17). Although it has proven difficult to find the full solution in the bulk, the effective 4D field equations were shown in [11] to be fully consistent.

4 On the perturbation equations

From (3.7), we see that $E_{\mu\nu}$ should be interpreted as an energy-momentum tensor satisfying the Bianchi identity. We proceed by adopting the following ansatz for it

$$-\frac{1}{\kappa^2} E_{\mu\nu} = \rho \left( U_{\mu} U_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + q_{(\mu} U_{\nu)} + \Pi \left( \frac{1}{3} h_{\mu\nu} - r_{\mu} r_{\nu} \right) \quad (4.1)$$

where $r_{\mu}, U_{\mu}$ are unit radial and time-like vectors respectively, $h_{\mu\nu} = g_{\mu\nu} + U_{\mu} U_{\nu}$ and the quantities $\rho, \Pi$ are the effective density and stress induced on the brane. We also set $\kappa^2 = 1$ from now on. This ansatz has turned out to be very useful in seeking classical solutions (see for example [21] for an extensive review). For the solutions we consider, we take $q_{\mu}$ to vanish and $\rho = \rho(r), \Pi = \Pi(r)$.

\[\text{In [11], it was shown that one can integrate into the bulk by complementing the effective 3+1D description by two more differential equations involving the Lie derivative of } E_{\mu\nu} \text{ and the Weyl tensor as explained in the Appendix of [11]. We also refer the interested reader to [20] which contains discussions of how Campbell-Magaard type embedding theorems in differential geometry imply the existence of bulk solutions from solutions in Shiromizu-Maeda-Sasaki formulation.}\]
which generally correspond to spherically symmetric and static geometries. We adopt the metric ansatz
\[ ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \] (4.2)
The Bianchi identity can be further simplified to be a single ODE in the radial coordinate
\[ \rho' + \frac{f'(r)}{f(r)} (-\Pi + 2\rho) - 2\Pi' - \frac{6}{r} \Pi = 0. \] (4.3)
The background field equations read
\[ G^t_t = -\rho, \quad G^r_r = -\frac{1}{3}(2\Pi - \rho), \quad G^\phi_\phi = \frac{1}{3}(\Pi + \rho). \] (4.4)
In terms of the functions \( f, g, \)
\[ G^t_t = \frac{g(r) - g(r)^2 - rg'(r)}{r^2g(r)^2}, \quad G^r_r = \frac{f(r)(1 - g(r)) + rf'(r)}{r^2f(r)g(r)}, \] \[ G^\phi_\phi = G^\phi_\phi = \frac{-rg(r)(f'(r))^2 - 2f^2(r)g'(r) + f(r)(-rf'(r)g'(r) + 2g(r)(f'(r) + rf''(r))))}{4rf^2(r)g^2(r)}. \] (4.5)
We express the perturbations in the basis of Legendre polynomials \( P_l(\cos \theta). \)
\[ f(r) \to f(r)(1 + \sum_l H^l_0(r)P_l), \quad g(r) \to g(r)(1 + \sum_l H^l_2(r)P_l), \] \[ g_{\theta\theta} = g_{\phi\phi}/\sin^2 \theta = r^2 \to r^2 \sum_l K^l(r)P_l, \] \[ \rho(r) \to \rho(r) + \sum_l \delta \rho^l(r)P_l, \quad \Pi(r) \to \Pi(r) + \sum_l \delta \Pi^l(r)P_l. \] (4.7)
Also the Bianchi identity translates into an ODE for the perturbations of \( \rho, \Pi \) as follows. From (4.3)
\[ \delta \rho' + 2\frac{f'(r)}{f(r)} \delta \rho + (2\rho - \Pi)H^l_0' = 2\delta \Pi' + \delta \Pi \left( \frac{f'(r)}{f(r)} + \frac{6}{r} \right). \] (4.8)
Since \( \rho, \Pi \) parametrize the 5D graviton perturbations, we take \( \delta \rho, \delta \Pi \) to be of the same fluctuation order as the 4D metric perturbations \( \{H_0(r), H_2(r), K(r)\}. \)

### 4.1 Decoupling of angular modes and a third-order ODE

In the following, we show that the angular polar modes in the basis of Legendre polynomials \( P_l(\cos \theta) \) decouple in the linearized equations, and derive the ODE that determines \( H_0(r), H_2(r), K(r) \). Form now on, we omit the superscript \( (l) \) on these functions and their dependence on \( r \) for notational simplicity.

From \( \delta G^\phi_\phi = 0 \), we find the relation
\[ K' = -H^l_0' + \frac{1}{r}(H_0 + H_2) + \frac{f'}{2f}(H_2 - H_0), \] (4.9)
whereas from $\delta G^0_0 - \delta G^0_0$, we obtain

$$\delta G^0_0 - \delta G^0_0 = \frac{1 + l}{r^2 \sin^2 \theta} (H_0 + H_2) \left[ (4 + l + (2 + l) \cos(2\theta)) P_{1+t} - 2(5 + 2l) \cos(\theta) P_{1+l} + 2(2 + l) P_{2+t} \right],$$

(4.10)

which implies for $l > 2$ that

$$H_2 = -H_0, \quad K' = -H_0' - \frac{f'}{f} H_0.$$

(4.11)

We find that $K$ in eliminated in $\delta G^r_r - \delta G^t_t$ which reads

$$\delta G^r_r - \delta G^t_t = -\frac{1 + l}{r^2 \sin^2 \theta} H_0 \left[ (3 + 2l) \cos(\theta) P_{1+t} - (2 + l) P_{2+t} \right]$$

$$+ \frac{P_t}{2r^2 f^2 g^2} \left[ -3r^2 g H_0 f'^2 + rf \left( -r H_0 f' g' + g(6 H_0 f')^2 \right) + f^2 \left( (1 + l)(2 + l + l \cos(2\theta)) \frac{g^2 H_0}{\sin^2 \theta} \right. \right.$$

$$\left. + r g'(2 H_0 - r H_0') + 2r g(2 H_0' + r H_0'' \right)].$$

(4.12)

The appearance of $P_{1+t}, P_{2+t}$ may suggest possible couplings among different polar modes, but at this point we invoke Bonnet’s recursion formula

$$(l + 2) P_{2+t} - (2l + 3) \cos(\theta) P_{t+1} + (l + 1) P_{t} = 0$$

to simplify eqn. (4.12) to be

$$\delta G^r_r - \delta G^t_t = -\frac{l(l + 1)}{r^2} P_t H_0 + \frac{P_t}{2r^2 f^2 g^2} \left[ -3r^2 g H_0 f'^2 + rf \left( -r H_0 f' g' \right. \right.$$

$$\left. + g(6 H_0 f' + r f'H_0' + 2r H_0 f'') \right) + f^2 \left[ r g'(2 H_0 - r H_0') + 2 r g(2 H_0' + r H_0'') \right].$$

(4.13)

Thus we see there is decoupling of the polar modes since the angular dependence lies purely in $P_l(\cos(\theta))$. This should be equated to $\frac{2}{3}(2\delta \rho - \delta \Pi) P_t$. We are left with the tracelessness condition which however involves $K(r)$. To proceed, we can invoke the Bianchi identity again to reduce the equations to a single third-order ODE. First, we find the relation

$$\delta G^0_0 = \delta G^0_0 = \frac{1}{2g f^2} H_0 (g f)' P_t = \frac{1}{3} (\delta \Pi + \delta \rho) P_t,$$

(4.14)

which allows one to express $\delta \Pi$ in terms of $\delta \rho$. This equation is identical for all $l > 2$. Substituting into

$$\delta \rho + 2 \frac{f''}{f} \delta \rho + (2 \rho - \Pi) H_0' = 2 \delta \Pi' + \delta \Pi \left( \frac{f'}{f} + \frac{6}{r} \right),$$

(4.15)

$^3$For $l = 0, 1$, the RHS vanishes identically, but $K$ decouples from all equations and can be solved by the same quadrature once we solve two coupled ODE in $H_0, H_2$. 

10
we find the following third-order ODE

\[ C_3H'''_0 + C_2H''_0 + C_1H'_0 + C_0H_0 = 0, \]  

(4.16)

where

\[ C_3 = \frac{r^2 f}{2g}, \quad C_2 = \frac{r}{4g^2} \left[ 3rgf' + f(8g - 3rg') \right], \]
\[ C_1 = \frac{1}{4fg^3} \left( - 3r^2g^2 f'^2 + rfg(-3rf'g' + g(16f' + 3rf'')) \right) \]
\[ - f^2 \left[ - 4g^2 + 2l(l+1)g^3 - 2r^2g^2 + r^2g'' \right], \]
\[ C_0 = \frac{1}{4fg^3} \left( 3r^2g^2 f'^3 + 3rgf f'(3rf'g' - 2g(4f' + 3rf'')) \right) \]
\[ + 4f^3(-2rg^2 + g(2g' + rg'')) - f^2 \left[ 2l(l+1)g^3 f' - 2r^2f'g'^2 \right] \]
\[ + rg(3rg'f'' + f'(8g' + rg'')) - 2g^2(6f' + r(6f'' + rf''')) \]. \hspace{1cm} (4.17)

### 4.2 The decoupled case \( \Pi = 2\rho, \ f g = 1 \)

There is a distinguished case of \( \Pi = 2\rho = -\frac{q}{r}, \ f = 1/g = 1 - \frac{2m}{r} + \frac{q}{r} \) which corresponds to the important example of the Tidal black hole. We will study this solution in detail in Section 5. In this case we find that \( \delta \rho \) decouples from the differential equations and (4.15) implies that

\[ \delta \rho \sim \frac{1}{r^2 f} = -\delta \Pi. \]  

(4.18)

We should point out that in [13], while studying this black hole solution, the authors unfortunately took the (correct) relation \( \Pi = 2\rho \) to imply the (incorrect) relation \( \delta \Pi = 2\delta \rho \). Although it is true that there is no a priori constraints between \( \rho, \Pi \), the linearized perturbations of these functions have to satisfy the Bianchi identity for consistency of the field equations, and (4.15) simply leads to (4.18) for this solution (without imposing any further constraints by hand).

In this case, equation (4.13) simplifies to read

\[ H''_0 + \frac{P(r)}{r} H'_0 + \frac{Q(r)}{r^2} H_0 = -l(l+1)\alpha \frac{r^2}{(r(r-2m) + q)^2}, \]  

(4.19)

where \( \alpha \) parametrizes the arbitrary constant for \( \delta \rho \)\(^4\) and

\[ P(r) = -\frac{2r(m-r)}{r(r-2m) + q}, \]
\[ Q(r) = -\frac{qr(l^2 + l - 2) r - 4m) + r^2(-2l(l+1)mr + l(l+1)r^2 + 4m^2) + 2q^2}{(r(r-2m) + q)^2}. \]  

(4.20)

In Section 5, we will discuss the near-horizon and asymptotic series solutions to (4.19), focusing mostly on the more phenomenologically relevant \( l = 2 \) case.

\(^4\)The factor of \( l(l+1) \) is introduced for a slightly simpler notations for related computations in Section 5.
5 On the Tidal Black Hole

In this section, we study the tidal deformation of the following black hole solution first presented in [22] with metric
\[ f(r) = g(r)^{-1} = 1 - \frac{2m}{r} + \frac{q}{r^2}, \quad q < 0. \] (5.1)
This is of the usual Reissner-Nordström form but we note that the corresponding energy-momentum tensor is of course not of Maxwell’s theory. The parameter \( q \) is known as the tidal charge parameter and is of the negative sign which can be interpreted as a strengthening of the gravitational field.

We will henceforth refer to this solution as the ‘Tidal Black Hole’. There is a regular horizon at
\[ r = r_h = m \left[ 1 + \sqrt{1 - \frac{q}{m^2}} \right]. \] (5.2)
The other zero of \( f \) lies at a negative \( r_− = m \left[ 1 - \sqrt{1 - \frac{q}{m^2}} \right] \). The background fields read
\[ G_t^t = G_r^r = -G_φ^φ = -G_θ^θ = -\frac{q}{r^4}, \quad ρ = -\frac{q}{r^4} = Π. \] (5.3)

Setting \( l = 2 \) in (4.19), we obtain the equation
\[ H''_0 + \frac{P(r)}{r} H'_0 + \frac{Q(r)}{r^2} H_0 = \frac{-6αr^2}{(q + r^2 - 2mr)^2} \equiv S(r), \] (5.4)
where
\[ P(r) = -\frac{-2r(m - r)}{q + r(r - 2m)}, \quad Q(r) = \frac{(-2(q^2 + 2qr(-m + r) + r^2(2m^2 - 6mr + 3r^2))}{(q + r(-2m + r))^2}. \] (5.5)

In the following, we first study the series expansion about the horizon which has a useful and manifest GR limit. We then study the asymptotic series solution to (5.4) which turns out to reveal a non-vanishing TLN that is a simple function of mass and tidal charge.

5.1 Expansion about \( r = r_h \)

We now carry out an expansion about \( r = r_h \), and impose regularity at the horizon. Let \( x = (r - r_h)/m \) and recast the ODE in variable \( x \), writing (5.4) as
\[ H''_0 + \frac{\tilde{P}(x)}{x} H'_0 + \frac{\tilde{Q}(x)}{x^2} H_0 = m^2 S(x) \] (5.6)
where \( \tilde{P}(x) = xP(x)/(x + r_h), \tilde{Q}(x) = x^2Q(x)/(x + r_h)^2 \). The indicial equation now reads
\[ R(R - 1) + R\tilde{P}(0) + \tilde{Q}(0) = R(R - 1) + R - 1 = 0, \quad \tilde{P}(0) = 1, \quad \tilde{Q}(0) = -1, \] (5.7)
which has roots being ±1 independent of \( q \). The two homogeneous solutions are thus of the form
\[ y_1(x) = x \left( 1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \right), \quad y_2(x) = \mathcal{N} y_1(x) \log(x) + \frac{1}{x} \left( 1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots \right). \] (5.8)
Since $y_2(x)$ diverges as $x \to 0$, this implies that $g_{rr}$ is divergent at the horizon - a feature which nullifies the series as part of the general solution which is thus the sum of $y_1(x)$ and the particular solution. We also find the particular solution of the following form

$$y_p(x) = C_0 + (C_1 + K_1 \log(x)) x + (C_2 + K_2 \log(x)) x^2 + \ldots$$  (5.9)

In the $q = 0$ limit, these series solutions read

$$y_1(x) = x \left( 1 + \frac{1}{2} x \right) = \left( \frac{r}{m} - 2 \right) \frac{r}{2m},$$

$$y_2(x) = -3y_1(x) \log(x) + \frac{1}{x} \left( 1 - \frac{5}{2} x + \frac{13}{4} x^3 + \ldots \right),$$

$$y_p(x) = \alpha \left( 1 + (\log(x))x + \left( -\frac{13}{12} + \frac{1}{2} \log(x) \right) x^2 - \frac{5}{48} x^3 + \ldots \right)$$  (5.10)

We can compare them with the Schwarzschild case in ordinary GR for which there is an exact solution for $H_0$ first presented by Hinderer to be

$$H_0 = c_1 \left( \frac{r}{m} \right)^2 \left( 1 - \frac{2m}{r} \right) \left[ - \frac{m(m-r)(2m^2 + 6mr - 3r^2)}{r^2(m-r)^2} - \frac{3}{2} \log(1 - \frac{2m}{r}) \right] + 3c_2 \left( \frac{r}{m} \right)^2 \left( 1 - \frac{2m}{r} \right).$$  (5.11)

In terms of the expansion variable $x$, we found that (5.11) can be equivalently expressed as

$$H_0(x) = c_1 \left( y_2(x) - \frac{13}{4} y_1(x) \right) + 6c_2 y_1(x).$$

Since the solution parametrized by $c_1$ diverges at the horizon, we are left with $y_1(x)$ which implies that the Schwarzschild black hole in ordinary GR has vanishing TLN. Also the particular solution vanishes in the ordinary GR limit, since $\alpha$ parametrizes fluctuations of the matter density that effectively descend purely from the 5D brane embedding (and not from 4D matter).

Although the expansion about the horizon allows us to match the general form of regular solution to the Schwarzschild case in the limit of ordinary GR, we need the expansion about infinity to study the TLN since the latter is most conveniently read off from such a series. Nonetheless, we have picked up the crucial fact that the most general regular solution is parametrized by two arbitrary constants and we can use the $q = 0, \alpha = 0$ limit to identify them when we carry out the series expansion about infinity. Finally, for the higher values of $l$, we find that we have the same indicial equation as in the $l = 2$ case, with the roots being $\pm 1$. There is also a particular solution of the same form as in the $l = 2$ case. This once again implies that there are two regular branches of solutions, with one of them relevant for capturing the tidal response.

### 5.2 Expansion about $r = \infty$

We now perform a series expansion about $r = \infty$ in order to pick up any non-vanishing TLN. After switching to coordinate $u = m/r$, we find the ODE

$$H_0'' + \frac{2 - P(u)}{u} H_0' + \frac{Q(u)}{u^2} H_0 = \frac{m^2 S(u)}{u^4}. \quad (5.12)$$
We find the expansion behavior
\[
2 - P(u) = -2u + (2q - 4)u^2 + \ldots, \\
Q(u) = -6 - 12u + \ldots, \\
m^2 S(u) = \frac{1}{u^4} + \frac{4}{u} + (12 - 2q) + \ldots,
\]
which reveal \( r = \infty \) or \( u = 0 \) to be a regular point. The indicial equation reads
\[
R(R - 1) + R(2 - P(0)) + Q(0) = R(R - 1) - 6 = 0 \quad (5.14)
\]
which has roots 3 or \(-2\). The two independent homogeneous solutions are thus of the form
\[
Y_d(u) = u^3 \left( 1 + 3u + \frac{1}{5m^2}(-50m^2 - 7q)u^2 + \frac{1}{42} \left( 660 - 217 \frac{q}{m^2} \right) u^3 + \ldots \right), \\
Y_r(u) = \frac{1}{u^2} - \frac{2}{u} + \frac{2q}{3m^2} + \frac{2qu}{3m^4} - \frac{16}{15m^4} (q^2 \log(u))u^3 + \frac{2q^2}{45m^4} \left( 31 + 15 \frac{q}{m^2} - 72 \log(u) \right) u^4 + \ldots
\]
\( (5.15) \) \( (5.16) \)

We also find the particular solution to be of the following form
\[
\frac{Y_p(u)}{\alpha} = 1 + 2u + (4 - q)u^2 + \left( \frac{q^2}{m^4} - 8 \right) u^4 + O(u^5).
\]
\( (5.17) \)
For these series solutions in \( u \), it is not apparent which one is regular at the horizon while we gathered from Section 5.1 that it is parametrized by two arbitrary constants. This is where taking the \( q = 0 \) limit comes in useful. In the \( q = 0 \) limit, we have
\[
\lim_{q=0} Y_d = u^3(1 + 3u + \frac{50}{7} u^2 + \frac{110}{7} u^3 + \ldots), \\
\lim_{q=0} Y_r = \frac{1}{u^2}(1 - 2u), \\
\lim_{q=0} Y_p = \alpha (1 + 2u + 4u^2 - 8u^4 + \ldots).
\]
\( (5.18) \)

We note that the homogeneous solution \( Y_r \) is the one that correctly reduces to the regular solution branch for the Schwarzschild black hole in \( (5.11) \). On the other hand, for \( Y_d(u) \), we find that in the same limit, it reduces precisely to the corresponding solution parametrized by \( c_1 \) in \( (5.11) \) and is the one that is divergent at the horizon. Thus, we identify \( Y_r(u), Y_p(u) \) to be the regular solutions. The most general regular tidal-deformed solution is of the form
\[
H_0 = CY_r(u) + Y_p(u),
\]
\( (5.19) \)
where \( C \) and the free parameter \( \alpha \) (in \( Y_p(u) \)) are arbitrary constants. We note that the particular solution does not contain the tidal moment \( (r^2 \text{-term}) \) although there is a non-vanishing quadrupolar moment due to the 5D embedding. Since we are considering the induced response of the body to some tidal force, when reading off the TLN, we consider purely \( Y_r(u) \) which contains the tidal moment term and possibly some induced quadrupolar moment. Thus, taking \( \alpha = 0 \), we have
\[
H_0 = C \left( \frac{r}{m} \right)^2 + \frac{16}{15} q^2 \log \left( \frac{r}{m} \right) \frac{m^3}{r^3} - 2 \frac{r}{m} + \frac{2q}{3m^2} - \frac{2q^2}{3m^2 r^2} + \ldots + \sum_{k=4} \left( \tilde{C}_k + \tilde{D}_k \log(u) \right) u^k,
\]
\( (5.20) \)
where $\tilde{C}_k, \tilde{D}_k$ are constant coefficients which can be computed straightforwardly whenever required. This expression does not contain $1/r^3$ term on its own, but taking into account the undeformed metric component, we find

$$\lambda = \frac{2mq^2}{3}. \quad (5.21)$$

This is our first example of a braneworld black hole solution that has a non-vanishing TLN which clearly increases with mass and the tidal charge $q$.

For higher values of $l$, from the expansion about infinity, we find the same solution for $2 - P(u)$ but $Q(u)$ is $l$-dependent and reads

$$Q(u) = -l(1 + l) - 2l(1 + l)u + (l^2(q - 4) + l(q - 4) + 2(q - 2)) u^2 + \ldots \quad (5.22)$$

The indicial equation reads

$$\mathcal{R}(\mathcal{R} - 1) - l(l + 1) = 0, \quad (5.23)$$

which has roots $\{-l, 1 + l\}$ with the particular solution being identical for all $l \geq 2$. We note that the most general tidal solution regular at the horizon is of the form

$$H_0 = CY_r(u) + Y_p(u), \quad (5.24)$$

where $Y_r(u)$ is the solution associated with the indicial root of $-l$ which contains the correct leading order tidal moment $r^l$. In this series, one can read off the higher $l$-order TLN. The solution associated with the other root of $1 + l$ diverges at the horizon.

### 6 Other examples of braneworld black holes and wormholes

We now proceed to consider other black hole and wormhole solutions with a similar computational approach focusing on the $l = 2$ case. As we noted earlier, in the generic case, there is no decoupling of $\delta \rho$ from the perturbation equations and one has to solve a homogeneous third-order ODE for $H_0$. Expanding about a metric singularity in each case (this is typically a black hole horizon or wormhole throat) reveals that the general regular solution should contain two arbitrary constants. We also require an expansion about $r = \infty$ to read off the TLN in the absence of analytic solutions. Although such series solutions do not manifestly allow us to impose regularity at finite horizon radii, for each family of solutions considered in this work, there is nevertheless a limiting procedure to take it to a well-defined limit of the Tidal black hole. For the latter, we have invoked the GR limit to determine the appropriate tidal-deformed solution $Y_r$, and similarly for each case, we will adopt suitable limits to seek the regular tidal-deformed solution that correctly reduces to $Y_r$ of the Tidal black hole. This yields tidal-deformed solutions that, within each of their parameter spaces, reduce smoothly to the appropriate one in the Schwarzschild case.

In particular, for one class of black hole solutions, there exists a limit in which they correspond to the massless Tidal black hole - an exotic solution that can be recognized as the Reissner-Nordström metric with imaginary charge and vanishing mass. In the massless limit, we have

$$\lim_{m \to 0} Y_d(u) = u^3 - qu^5 + q^2u^7 + \ldots$$
\[
\lim_{m=0} Y_r(u) = \frac{1}{u^2} + \frac{2}{3} q - \frac{2}{3} q^2 u^2 + \frac{2}{3} q^3 u^4 + \ldots
\]
\[
\lim_{m=0} \frac{Y_p(u)}{\alpha} = 1 - q u^2 - q^2 u^4 + \ldots
\]  
(6.1)

This corresponds to a particular limit of a class of black hole solutions that we shall consider shortly. The TLN for that class of black holes is then computed from the unique solution branch that reduces smoothly to \(Y_r(u)\) in (5.16). Note that in (6.1), the logarithmic terms in \(Y_r(u)\) all vanish. For the cases we consider in this work, we find that logarithmic terms are generically present in the regular solutions although they play no role formally in the conventionally defined TLNs. To our knowledge, the inclusion of such terms and their effects on the tidal phase correction of gravitational waveforms in the inspiral regime has not been studied. It would be interesting to examine how such terms may affect the tidal phase correction and if they turn up at the same or lower post-Newtonian order. For a general braneworld solution other than the distinguished case in the preceding section, we have to solve a third-order ODE. In the following, we focus on the expansion about \(r = \infty\) which is the relevant series for picking up the TLN.

For expanding about \(r = \infty\), we define \(u = L/r\) where \(L\) is some length parameter in the undeformed solution, and recast the ODE in (4.16) into the following form

\[
H''''_0 + \frac{1}{u} P_1 H''_0 + \frac{1}{u^2} P_2 H'_0 + \frac{1}{u^3} P_3 H_0 = 0,
\]  
(6.2)

where

\[
P_1 = 6 - \frac{LC_2}{C_3u}, \quad P_2 = 6 - \frac{2LC_2}{C_3u} + \frac{L^2C_1}{C_3u^2}, \quad P_3 = -\frac{L^3C_0}{C_3u^3}.
\]  
(6.3)

The indicial equation reads

\[
\mathcal{R} (\mathcal{R} - 1)(\mathcal{R} - 2) + \mathcal{R} (\mathcal{R} - 1)P_{10} + \mathcal{R} P_{20} + P_{30} = 0
\]  
(6.4)

of which roots \(\mathcal{R}_1 > \mathcal{R}_2 > \mathcal{R}_3\) determine the form of the general solution.

In the following, we consider various black hole and wormhole solutions which are static and spherically symmetric. For all the solutions considered in this work, we find the universal values

\[
P_{10} = 2, \quad P_{20} = -6, \quad P_{30} = 0, \quad R_1 = 3, \quad R_2 = 0, \quad R_3 = -2.
\]  
(6.5)

This leads to the general solution being the linear combination of

\[
y_1(u) = u^3(1 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots),
\]
\[
y_2(u) = V_1 y_1(u) \log(u) + (1 + b_1 u + b_2 u^2 + \ldots),
\]
\[
y_3(u) = \frac{1}{u^4} \left(1 + d_1 u + d_2 u^2 + \ldots + d_5 u^5 + \ldots\right) + W_2 \log(u) \left(c_0 + c_1 u + c_2 u^2 + \ldots\right) + W_3 (\log(u))^2 y_1(u).
\]  
(6.6)

Each family of solutions has a certain limit within its moduli space which reduces uniquely to the Schwarzschild solution or certain limits of the regular tidal-deformed metric of the Tidal black hole. This yields consistency checks for regularity of each solution at the horizon or wormhole throat. As we shall demonstrate shortly, we find that \(y_3(u)\) is the series relevant for picking the TLN. We also need the full metric component \(h_{tt}\) in each case which is collected separately in the Appendix.
For higher values of $l$, in all other braneworld solutions we consider in this work, we find the following universal values for the roots of the indicial equation:

$$P_{10} = 2, \ P_{20} = -l(1 + l), \ P_{30} = 0,$$
$$R_1 = l + 1, \ R_2 = 0, \ R_3 = -l.$$ \hspace{1cm} (6.7)

Similar to the specific case of $l = 2$, this leads to the general solution being the linear combination of

$$y_1(u) = u^{l+1}(1 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots),$$
$$y_2(u) = V_1 y_1(u) \log(u) + \left(1 + b_1 u + b_2 u^2 + \ldots\right),$$
$$y_3(u) = \frac{1}{u^l} \left(1 + d_1 u + d_2 u^2 + \ldots + d_5 u^5 + \ldots\right) + W_2 \log(u) \left(1 + c_1 u + c_2 u^2 + \ldots\right) + W_3 (\log(u))^2 y_1(u).$$ \hspace{1cm} (6.8)

6.1 CFM black holes

This family of black hole solutions was found by Casadio, Fabbri and Mazzacurati in [23]. The metric components read

$$f(r) = 1 - \frac{2m}{r}, \ g(r) = \frac{1 - \frac{3m}{2r}}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{m}{2r}(4\beta - 1)\right)}.$$

The horizon is located at $r = r_h = 2m$ and the limit $\beta \to 1$ corresponds to the Schwarzschild ansatz. The TLN is read off from series expansion about $r = \infty$ in which we take the perturbation variable to be $u = L/r = m/r$. As discussed earlier, for the expansion about the horizon $r = r_h$, we find that for the CFM black holes, the indicial equation has roots $\{-1, 0, 1\}$ which indicate that there are two regular solutions and one which blow up at the horizon. In the $\beta \to 1$ limit, the solution reduces to the $q = 0$ (Schwarzschild) limit of the Tidal black hole. Thus in this limit, each branch of solution to (6.2) should reduce to some linear combinations of those in the Schwarzschild case.

To read off the TLN, we solve for series solutions in $u = m/r$. For $y_1(u)$, we find

$$y_1(u) = u^3 + \frac{1}{4}(13\beta - 1)u^4 + \frac{1}{70}(533\beta^2 - 112\beta + 79)u^5 + \ldots$$ \hspace{1cm} (6.9)

which, in the limit $\beta \to 1$, becomes

$$y_1(u) \to u^3 \left(1 + 3u + \frac{50}{7}u^2 + \ldots\right) = \lim_{q=0} Y_d(u).$$ \hspace{1cm} (6.10)

This is precisely the series that we identify to be divergent at the horizon. We should only consider the remaining two general solutions which are regular at the horizon. The $y_2(u), y_3(u)$ series read

$$y_2(u) = -(5 - 8\beta + 3\beta^2)y_1(u) \log(u) + 1 + 2u - \frac{1}{2}(3\beta - 11)u^2 + \ldots$$
$$y_3(u) = \frac{1}{4}(\beta - 1)^2 \left(33\beta^2 - 64\beta + 15\right) y_1(u) \log^2(u) + \log(u) \left(\frac{1}{2}(1 - \beta)(11\beta - 3)(1 + 2u - \frac{1}{2}(3\beta - 11)u^2 + \ldots)\right)$$
$$+ \frac{1}{u^2}(1 - \frac{1}{2}(11\beta - 7)u + \frac{1}{12} (55\beta^3 - 180\beta^2 + 158\beta - 33) u^3 + \ldots \quad (6.11)$$

In the $\beta \to 1$ limit, the log$(u)$ series in $y_2(u)$ vanishes and

$$y_2(u) \to \lim_{q=0} Y_p(u)/\alpha \quad (6.12)$$

which is the particular solution in the Schwarzschild case. We should note that in solving for the series $y_2(u)$, there is an arbitrary constant $b_3$ which parametrizes $y_1(u)$ and setting it to vanish implies that in the $q \to 0$ limit, we have $y_2(u) \to 0$.

In the latter, the $\delta \rho$ equation decouples and we have an arbitrary constant characterizing the arbitrary strength of the quadrupole moment. But this is not induced by some tidal moment (there is no accompanying $1/u^2$). The coefficient for $u^3$ in $y_2(u)$ is not what we seek in the traditional TLN definition. We should focus on the solution branch with $1/u^2$ representing the presence of the tidal moment due to an external body. Now in the same limit, both log$(u)$ and (log$(u)$)$^2$ terms in $y_3(u)$ vanish and we have

$$y_3(u) \to \frac{1}{u^2} - \frac{2}{u} = \lim_{q=0} Y_\rho(u) \quad (6.13)$$

which is the $q = 0$ limit of the regular tidal-deformed Schwarzschild solution which has vanishing TLN in the absence of any $\rho^3$ term. Again, we should note that (6.13) is obtained by setting $d_2 = d_5 = 0$ and these parameters are otherwise arbitrary. From $h_{tt}$ (see Appendix), we read off the TLN to be (see Figures 1 and 2)

$$\lambda = m^5 \frac{1}{72} (\beta - 1) \left(220\beta^3 - 1963\beta^2 + 3358\beta - 1155\right). \quad (6.14)$$

Phenomenological interest lies in the small neighborhood of the critical value $\beta = 1$ where for solutions with $\beta > 1$, the TLN is positive whereas it is negative for $\beta < 1$. The $\beta < 1$ solutions are black holes that are Schwarzschild-like in nature with an event horizon at $r_h = 2m$, whereas the $\beta > 1$ solutions are non-singular wormhole geometries. For $1 < \beta < \frac{5}{4}$, their Carter-Penrose
Figure 2: Plot of the TLN $\lambda$ (in units of $m^5$) vs the parameter $\beta$ near $\beta \approx 1$. Near this transition value, solutions with $\beta > 1$ are non-singular in nature while those with $\beta < 1$ contain curvature singularities.

diagrams resemble the form of the Kerr black hole, and for $\beta \geq \frac{5}{4}$, we have symmetric traversable wormholes. In this family of solutions, near the critical point, the sign of the TLN for solutions indicates the presence (-) or absence (+) of the black hole curvature singularity.

6.2 $\gamma$-wormholes

This family of solutions was studied by Casadio, Fabbri and Mazzacurati in [23], and contains both pathological naked singularities as well as regular wormhole geometries. The metric components read

$$f(r) = \frac{1}{\gamma^2} \left( \gamma - 1 + \sqrt{1 - 2\gamma m r} \right)^2, \quad g(r) = \left( 1 - \frac{2\gamma m r}{r} \right)^{-1},$$

with $\gamma = 1$ being the Schwarzschild limit. For $\gamma < 1$, the metric is singular at

$$r_s = \begin{cases} \frac{2m}{2-\gamma} \equiv r_h, \\ 2m\gamma \equiv r_0, \end{cases}$$

where $r_h = \frac{2m}{2-\gamma}$ is a null and singular surface along which the Ricci scalar $R$ diverges as $R \sim 1/ (\sqrt{r - r_0} - \sqrt{r_h - r_0})$. For $\gamma > 1$, the only metric singularity lies at $r_0 = 2m\gamma$. There is a turning/minimum point (for all timelike geodesics) at $r = 2m\gamma$ where all curvature invariants are regular. The causal interpretation is that of a wormhole solution as explained in [23].

We find the following series solutions. Corresponding to the highest root of the indicial equation,

$$y_1(u) = u^3 + \frac{1}{4}(13\gamma - 1)u^4 + \frac{1}{70}(533\gamma^2 - 42\gamma + 9)u^5 + O(u^6),$$

which is the solution that in the limit $q \rightarrow 1$ reduces precisely to the divergent branch of the tidal-deformed Schwarzschild solution, i.e.

$$\lim_{\gamma \rightarrow 1} y_1(u) = \lim_{q \rightarrow 0} Y_d(u).$$
Thus, we discard this solution branch for ensuring regularity at horizon. We also find the following solution

\[ y_2(u) = \frac{1}{5} (-11\gamma^2 + 34\gamma - 23) y_1(u) \log(u) + 1 + 2u + \frac{9 - \gamma}{2} u^2 + \ldots , \]  

(6.18)

We note that in solving for the series \( y_2(u) \), there is an arbitrary constant \( b_3 \) which parametrizes \( y_1(u) \) and setting it to vanish implies that in the \( q \to 1 \) limit, this solution reduces exactly to the particular solution of the tidal-deformed Schwarzschild solution, i.e.

\[ \lim_{\gamma \to 1} y_2(u) = \lim_{q \to 0} Y_p(u). \]

Finally, corresponding to the only negative root of the indicial equation, we have

\[
y_3(u) = \frac{1}{20} (\gamma - 1)^2 (121\gamma^2 - 286\gamma + 69) y_1(u) \log^2(u) \\
\quad \frac{1}{2} (-11\gamma^2 + 14\gamma - 3) \log(u) \left( 1 + 2u + \frac{9 - \gamma}{2} u^2 + \ldots \right) \\
\quad \frac{1}{u^2} \left( 1 + \frac{1}{2} (7 - 11\gamma) u + \frac{1}{12} (55\gamma^3 - 196\gamma^2 + 189\gamma - 48) u^3 + \ldots \right) \]

(6.19)

where we found that setting the arbitrary constants \( d_2 = d_5 = 0 \) allow us to identify \( y_3(u) \) with the regular solution of the tidal-deformed Schwarzschild solution, i.e.

\[ \lim_{\gamma \to 1} y_3(u) = \lim_{q \to 0} Y_r(u). \]

Thus, just as in the case of the CFM black holes, we found that the three series solutions can be parametrically connected to their corresponding series solutions of the \( q = 0 \) limit of the tidal-deformed Tidal black hole in a unique way, and that the one corresponding to the negative root of the indicial equation \( y_3(u) \) is the one relevant for computing TLN.

Taking into account the background metric, from the series expansion of \( h_{tt} = f(r)y_3(u) \) we can read off the TLN to be (see Figures 3 and 4)

\[ \lambda = m^5 \frac{1}{144} (\gamma - 1) \left( 233\gamma^3 - 1284\gamma^2 + 2232\gamma - 711 \right) . \]

(6.20)

In the neighborhood of the critical value \( \gamma = 1 \), we find that \( \lambda > 0 \) for \( \gamma > 1 \) which corresponds to non-singular wormhole geometries whereas \( \lambda < 0 \) for \( \gamma < 1 \) which corresponds to naked singularities. This is similar to the scenario in our study of tidal-deformed CFM black hole solutions where a positive \( \lambda \) labels the non-singular wormhole branch of the solution class whereas negative \( \lambda \) pertains to Schwarzschild-type black holes near the critical Schwarzschild point. In both cases, \( \lambda \) is a degree-four polynomial in the parameter with one of the polynomial roots associated with the Schwarzschild limit.

### 6.3 Bronnikov-Kim wormholes

In [24], Bronnikov, Melnikov and Dehnen presented a powerful solution-generating technique to construct exact braneworld black hole and wormhole solutions for Randall-Sundrum theory considered in our work, covering the previous two families of solutions. In this and the subsequent sections, we study the two concrete examples mentioned in their seminal work.
Figure 3: Plot of $\lambda$ (in units of $m^5$) vs $\gamma$ for the family of $\gamma-$ wormhole solutions. The solutions contain naked singularities for $\gamma < 1$ whereas we have regular wormhole geometries for $\gamma > 1$.

Figure 4: Plot of $\lambda$ (in units of $m^5$) vs $\gamma$ near Schwarzschild limit $\gamma = 1$, showing clearly that in the vicinity of the transition point, the sign of $\gamma$ corresponds to the presence (-) / absence (+) of curvature singularities.

We first study example 3 of [24] where the line element reads

$$f(r) = \left(1 - \frac{2\tilde{m}}{r}\right)^2, \quad g(r) = \left(1 - \frac{r_0}{r}\right)^{-1} \left(1 - \frac{r_1}{r}\right)^{-1}, \quad r_1 = \frac{\tilde{m}r_0}{r_0 - \tilde{m}},$$

with the parameter $r_0$ determining the causal structure as follows:

- $r_0 < \tilde{m}$: naked singularity at $r = 2\tilde{m}$,
- $\tilde{m} < r_0 < 2\tilde{m}$: wormhole with throat at $r_1 > 2\tilde{m}$,
- $r_0 = 2\tilde{m}$: extremal Reissner-Nordström,
- $r_0 > 2\tilde{m}$: wormhole with throat at $r_0$.

Unlike the previous two cases, for this solution, there is no natural limiting procedure to send it to Schwarzschild, but to the $q = m^2, m = 2\tilde{m}$ limit of the Tidal black hole (which originally has negative tidal charge) or equivalently the extremal Reissner-Nordström (in ordinary GR) at some
finite value of the radial coordinate \( r_0 = 2\tilde{m} \). We find the following series solutions (taking the expansion variable to be \( u = \frac{\tilde{m}}{r} \), and defining \( R = r_0/\tilde{m} \))

\[
y_1(u) = u^3 + \frac{13R^2 - 4R + 4}{8(R-1)}u^4 + \frac{533R^4 - 448R^3 + 592R^2 - 288R + 144}{280(R-1)^2}u^5 + \ldots
\]

\[
y_2(u) = -\frac{(R-2)^2(15R^2 - 92R + 92)}{10(R-1)^2}y_1(u)\log(u) + 1 + 4u + \frac{3R^2 - 36R + 36}{2-2R}u^2 + \ldots
\]

\[
y_3(u) = \frac{(R-2)^4(165R^4 - 1192R^3 + 2296R^2 - 2208R + 1104)}{160(R-1)^4}y_1(u)\log^2(u) - \frac{((-2 + R)^2(12 - 12R + 11R^2))/(8(-1 + R)^2)}{2-2R}y_1(u)\log(u)(1 + 4u + \frac{3R^2 - 36R + 36}{2-2R}u^2 + \ldots)
\]

\[
+ \frac{1}{u^2}(1 + \frac{11R^2 - 28R + 28}{4 - 4R})u + \frac{8}{3}u^2 + \ldots
\]

(6.21)

where for \( y_2(u) \) and \( y_3(u) \), we have picked uniquely appropriate values of \( b_3 = 0, d_2 = \frac{8}{3}, d_5 = 0 \) such that in the \( R = 2 \) limit, each reduces to a corresponding series solution in the \( q = m^2 = 4\tilde{m}^2 \) limit of the Tidal black hole. Specifically, we find in this extremal limit

\[
\lim_{R \to 2} y_1(u) = \lim_{q \to m^2} Y_d,
\]

\[
\lim_{R \to 2} y_2(u) = \lim_{q \to m^2} Y_p/\alpha,
\]

\[
\lim_{R \to 2} y_3(u) = \lim_{q \to m^2} Y_r,
\]

(6.22)

with the last series solution being the regular solution from which one determines the TLN. Taking into account the background metric, from the series expansion of \( h_{tt} = f(r)y_3(u) \), we read off the TLN to be (see Figures 5 and 6)

\[
\lambda = \tilde{m}^{\frac{55}{144}}R^8 - 2073R^7 + 19389R^6 - 83064R^5 + 200948R^4 - 298640R^3 + 278448R^2 - 153344R + 38336
\]

(6.23)

We note that the vertical asymptote at \( R = 1 \) corresponds to the limit in which the wormhole geometry degenerates into a naked singularity. Another notable point is at \( R = 2 \) which corresponds to the extremal Reissner-Nordström solution. This turns out to be a local minimum point, with

\[
\lim_{R \to 2} \lambda = \frac{64}{5}m^5 = \frac{2}{3}m^5
\]

in agreement with the \( q = m^2 \) limit of (5.16). Unlike the extremal black hole solution which has a time-like curvature singularity behind a horizon, the wormhole geometries are globally regular.

### 6.4 Massless Geometries

Finally, we study a family of solutions which is example 2 of Bronnikov-Melnikov-Dehnen solution-generating algorithm in [24]. There is no natural limits to Schwarzschild, but to the massless limit of the Tidal black hole. This family of spacetimes is interesting as it has a subset of solutions which admit interpretations of wormholes, just like the CFM family of solutions. It is parametrized by a
Figure 5: Plot of $\lambda$ (in units of $\tilde{m}^5$) vs $R$. We note that $R = 1$ marks the transition to a naked singularity with a vertical asymptote, the left of which pertains to naked singularities and the right of which is associated with regular wormhole geometries. At $R = 2$, we have the extremal Reissner-Nordström metric.

Figure 6: Plot of $\lambda$ (in units of $\tilde{m}^5$) vs $R$ near the local minimum point $R = 2$ limit which pertains to the extremal Reissner- Nordström metric. This is a solution which is also part of the Tidal black hole family, albeit an extremal solution with positive tidal charge $q = m^2$. The value of $\lambda$ at $R = 2$ was checked to agree with (5.21) in such a limit.

dimensionless constant $C$ and a length parameter $h$ which in the $C = 1$ limit can be interpreted as an imaginary charge of the massless Reissner-Nordström.

$$f(r) = 1 - \frac{h^2}{r^2}, \quad g^{-1}(r) = f(r) \left(1 + h \frac{C - 1}{\sqrt{2r^2 - h^2}}\right), \quad h > 0. \quad (6.24)$$

Metric singularity arises at the following radii

$$r_h = \begin{cases} h, & C \geq 0{\text{ (horizon)}} \\ \sqrt{\frac{1}{2}h^2 + h^2(1-C)^2}, & C < 0{\text{ (wormhole throat)}} \end{cases} \quad (6.25)$$
In particular, we note that $C = 1, f = g^{-1}$ is the Reissner-Nordström metric with zero mass and imaginary charge or the massless limit of the Tidal black hole. We find the series solutions

$$y_1(u) = u^3 \left(1 - \frac{13(C-1)}{8\sqrt{2}} u + \frac{1}{560} (533C^2 - 1066C + 1093) u^2 + \ldots\right)$$

$$y_2(u) = \frac{\sqrt{2}}{10} (1 - C) u^3 y_1(u) \log(u) + 1 + u^2 + \frac{1}{960} (-91C^2 + 182C + 869) u^4 + \ldots$$

$$y_3(u) = \frac{11(C-1)^3}{160\sqrt{2}} y_1(u) \log^2(u) - \frac{11(C-1)^2}{16} \log(u) \left(1 + u^2 - \frac{1375C^4 - 5500C^3 + 46912C^2 - 82824C + 68557}{13200\sqrt{2}(C-1)} u^3 + \ldots\right)$$

$$\frac{1}{u^2} \left(1 + \frac{11(C-1)}{4\sqrt{2}} u - \frac{2}{3} u + \ldots\right)$$

We find that the various series solutions reduce to those in (6.1) upon setting $C = 1$. We find that with $q = -h^2$,

$$\lim_{C \to 1} y_1 = \lim_{m \to 0} Y_d,$$

$$\lim_{C \to 1} y_2 = \lim_{m \to 0} Y_p/\alpha,$$

$$\lim_{C \to 1} y_3 = \lim_{m \to 0} Y_r,$$

(6.27)

where for $y_3(u)$ we have set $d_2 = -2h^2/3 = 2q/3$ and $d_5 = 0$. Taking into account the background metric, we find (see Figure 7 below.)

$$\lambda = h^5 \frac{(C - 1) (55C^2 - 110C - 1)}{576\sqrt{2}}.$$ 

(6.28)

The vanishing value $\lambda$ corresponds to the massless Reissner-Nordström black hole with imaginary charge/massless Tidal black hole. For negative $C$, we have a wormhole solution. For $C \in [0, 1]$, we obtain Kerr-like regular black hole. For $C > 1$, we have Schwarzschild-like causal structure with the singularity at some finite value of $r = h/\sqrt{2}$.

6.5 The Near-Horizon Regime

We now consider series solutions relevant for the ‘near-horizon’ regime. The solutions that we study in this paper include wormholes and naked singularities apart from black hole geometries, and in those cases, we consider the expansion about the throat and singular loci. By the ‘near-horizon regime’, we generally refer to the small neighborhood of spherical surfaces along which the metric appears to be singular. Although the series solutions constructed in the near-horizon regime will not be useful for directly reading off the TLN, obtaining their general form by computing the indicial roots serves as a consistency check of the necessary regularity at the horizon or throat. Apart from requiring them to be finite, we note that if all roots are negative, then no asymptotic series solution is regular at the horizon/throat of the black hole/wormhole tidal-deformed geometries. In the following, we compute and collect the indicial roots of all the braneworld geometries that we examine in this work by further studying (4.16). The near-horizon expansion in the decoupled case (Tidal black hole) have been studied separately in Section 5.
Figure 7: Plot of $\lambda$ (in units of $h^5$) vs $C$. This is a cubic curve with the zero at $C = 1$ corresponding to the transition case of the massless Tidal black hole with $q = -h^2$. For $C > 1$, the causal structure is identical to that of Schwarzschild. For $C < 0$, we have globally regular wormholes, but $C = 0$ is not a polynomial root.

Let the expansion parameter be $x = \frac{r - r_h}{L}$ where $r_h$ could be either the black hole throat or the wormhole throat and $L$ is some length parameter of each family of solutions. We will also be treating a couple of cases of naked singularities, in which case, $r_h$ is simply the singular surface. In all cases, $r = r_h$ describes a codimension-two metric singularity which may or may not cloak physical curvature singularities. Writing (4.16) in the form

$$H_0''' + \frac{P_1(x)}{x}H_0'' + \frac{P_2(x)}{x^2}H_0' + \frac{P_3(x)}{x^3}H_0 = 0,$$

we compute the following limits from the metric

$$\lim_{x \to 0} P_1 \equiv P_{10}, \quad \lim_{x \to 0} P_2 \equiv P_{20}, \quad \lim_{x \to 0} P_3 \equiv P_{30},$$

from which we compute the indicial roots or the roots of the cubic equation

$$R(R - 1)(R - 2) + R(R - 1)P_{30} + RP_{20} + P_{10} = 0.$$  

In Table 1 below, we present their values for all the spacetime geometries considered in our paper. Foremost, the finiteness of the set of $\{P_{10}, P_{20}, P_{30}\}$ implies that $x = 0$ is a regular singular point for all cases. The roots of the indicial equation are, apart from a couple of cases, independent of $l$ and for all solutions, there is at least one positive non-negative root which is a necessary condition for the existence of series solutions regular at the metric singularity associated with either a horizon or throat.

For each family of solution characterized by some real parameter $\alpha$ (as summarized in Table 1 $\alpha \in \{r_0, h, \gamma, \beta\}$) the coefficient functions $\{P_1(x, \alpha), P_2(x, \alpha), P_3(x, \alpha)\}$ are discontinuous as bivariate functions of $x$ and $\alpha$ at the regular singular point $x = 0$ at values of $\alpha$ which mark transitions
Table 1: In the above, we use the abbreviations WH (wormholes), Sch-BH (Schwarzschild black holes), NS (Naked Singularities), RN (extremal Reissner-Nordström). Kerr-WH refers to completely regular wormhole geometries of which CP diagram resembles the form of that of Kerr. RN-WH refers to spacetimes with the casual structure of extremal Reissner-Nordström with a horizon cloaking a time-like singularity. The radius parameters \( r_s, r_{th} \) refer to the singular surface of a naked singularity and the radius of a wormhole throat respectively. This table excludes the roots \((\pm 1)\) for the Tidal black hole which are associated with a second-order ODE as was presented in Section 5.

between different spacetime interpretations. This leads to different sets of roots characterizing different types of spacetime geometries. Interestingly, we find that for the various solutions we study, the indicial roots tend to display some level of universality — similar sets of values are associated with the solutions belonging to different families but sharing identical causal structures. From Table 1 one can easily recognize the following pairings between the indicial roots and spacetime interpretations.

- \( \{-1,0,1\} \sim \) Schwarzschild black holes and globally regular Kerr-like wormholes
- \( \{-2,0,2\} \sim \) naked singularities
- \( \{0,\frac{1}{2},1\} \sim \) globally regular and traversable wormholes
The only solutions with $l$-dependent roots are the extremal Reissner-Nordström solution which is a member of the Bronnikov-Kim family of solutions and a member of the CFM family of black hole solutions of which CP diagram is identical to that of extremal Reissner-Nordström but which are geometrically different. A subset of solutions defined by $C \leq 0$ in the ‘Massless geometries’ family appears to fall out of this classification. The case of $C = 0$ has extremal Reissner-Nordström-like causal structure and metrics with $C < 0$ are globally regular and traversable wormholes. It would be interesting to study the relation between indicial roots and spacetime causal structures with more examples and in a deeper systematic classification. These roots reflect the tidal response of the object in the near-horizon regime and appear to capture aspects of causal structures of the various objects beyond their local horizon geometries.

7 Concluding remarks

For each of the five classes of static and spherically symmetric braneworld solutions examined in this work, we computed the (quadrupolar) Tidal Love Number, each being some rational function of the parameter that characterizes the family of solutions. They are generically non-vanishing and we have derived them essentially by performing an asymptotic series expansion about the radial infinity. The gravitational perturbation equations were shown to reduce to a single third-order homogeneous ODE and we found that the indicial equation associated with the regular singular point at infinity has a universal set of roots $\{3, 0, -2\}$ across all the braneworld solutions, with the tidal-deformed geometry described by the series solution with the negative root.

Among the braneworld solutions, there is a distinguished case in which the effective density fluctuation can be solved analytically and the perturbation equation reduces to an inhomogeneous second-order ODE of which indicial roots associated with the asymptotic expansion are $\{3, -2\}$. This corresponds to the Tidal black hole which is one of the more popularly studied black hole solutions in Randall-Sundrum theory and its TLN reads simply as $\lambda = \frac{2mq^2}{T}$ where $q$ is the tidal charge. A crucial aspect of our TLN computation is that each of the classes of braneworld solutions can be parametrically connected to either the Schwarzschild or other limits of the Tidal black hole which thus serves as an anchor point. For the Tidal black hole, taking the $q = 0$ limit allows us to identify the branches of solution corresponding to the regular and divergent ones in the Schwarzschild case. (For the latter, the existence of exact solutions allows us to impose regularity easily.) The series solution associated with the negative root is the one that reduces to the regular series in the Schwarzschild limit, and in it we saw that the tidal charge implies that the tidal deformation comes with a non-vanishing TLN. Once we established the case for Tidal black holes, we performed various limits (specifically, (i)$q = 0$, (ii)$m = 0$ and (iii)$q = m^2$) to relate the series solutions in each family of solutions to those for Tidal black holes and identify the one relevant for picking up the TLN.

As a necessary consistency check, we also constructed series solutions by expanding about the metric singularity in each family of solutions, which can be interpreted as either horizon or throat surfaces. In Table 1 we tabulated the indicial roots (for general $l$ and not just the quadrupolar case). Apart from the demonstrating that there exists regular tidal-deformed solutions by virtue of at least one non-negative root in each case, we found that certain sets of roots are shared by
different families of solutions when these solutions share the same global causal structure. This fact warrants a deeper study as it appears to indicate that the indicial roots may contain physical information about the spacetime geometry beyond the near-horizon regime. For the Tidal black hole, we checked that the series solutions associated with the indicial roots \( \{1, -1\} \) reduce in the vanishing tidal charge limit to the regular and divergent solution branches respectively in the Schwarzschild case which can be obtained just by performing a near-horizon expansion of the exact solution.

Although we have focussed on the computation of the quadrupolar \( l = 2 \) TLN for various braneworld solutions, this work also contains several results with regards to the general \( l \) case as described by (4.16) and (4.19). For the near-horizon expansion, we found that the indicial roots are almost exclusively independent of \( l \), the exceptions being those with the causal structure of extremal Reissner-Nordström. For the asymptotic expansion, the set of indicial roots \( \{3, 0, -2\} \) generalizes to \( \{l + 1, 0, -l\} \) whereas for the Tidal black hole, \( \{3, -2\} \) generalizes to \( \{l + 1, -l\} \). The methodology to compute the higher TLNs is similar - the relevant series solution is the one associated with the negative indicial root and we can adopt the identical limiting procedure to check the consistency of this approach via the parametric connection of the various braneworld solutions to the Tidal black hole. From the phenomenological point of view, the higher TLNs may not be as important since they should remain beyond the detectors’ precision reach. But our study of the indicial roots in the near-horizon expansion suggests that certain aspects of the indicial equation may contain physical data related to global causal structures beyond horizon geometry. Thus, it is nevertheless an interesting conjecture to explore as it may yield further insights into the nature of gravitational degrees of freedom in braneworld black hole solutions.

A natural future work is to study the implication of the logarithmic terms in the tidal-deformed metric (as collected in the Appendix). We have computed only the traditionally defined TLNs but a more careful study may reveal the signature of these logarithmic terms in gravitational waves emitted in binary systems of these objects. Notably, these terms were absent in a separate study of black hole solutions in an effective theoretic framework collecting various extensions of GR by higher curvature terms nor mentioned in other studies of TLNs in our knowledge.

With regards to phenomenology, it was suggested in [8] that LIGO’s precision could potentially bound the quadrupolar TLN \( k_2 \) to be of the order of ten or less, with the Einstein Telescope possibly improving it by a factor of a hundred, and more with LISA’s capabilities. The discovery of a non-vanishing \( k_2 \) in the GW waveform of merger event presumed to be that of a black hole binary would provide exciting grounds for further study of whether the deviation is a signature of a braneworld among other possibilities. It would be interesting to generalize our work to include realistic braneworld solutions arising from gravitational collapse and in the broader context of warped compactifications in string theory beyond the specific Randall-Sundrum model we considered.

Acknowledgments

I am very grateful to Ori Ganor, A.T.Phan and Neal Snyderman for their moral support over the years, and acknowledge a research fellowship at the School of Physical and Mathematical Sciences, NTU during the course of completion of this work.
A On the metric component $h_{tt}$ for various tidal-deformed braneworld solutions

In this Appendix, we collect various expressions for the metric perturbation $h_{tt} = f(r)H_0$ for $l = 2$ perturbations. The quadrupolar TLN is read off from the $1/u^2$ and $u^3$ coefficients in each case.

(I) CFM black holes

$$h_{tt} = \frac{1}{u^2} + \frac{(3-11\beta)}{2u} + \left(-\frac{1}{2}\left(11\beta^2 - 14\beta + 3\right)\log(u) + (11\beta - 7)\right) + \frac{1}{12} u \left(55\beta^3 - 180\beta^2 + 158\beta - 33\right)$$
$$+ \frac{1}{48} u^2 (-\beta - 1) \left(220\beta^3 - (36(11\beta^2 - 14\beta + 3))\log(u) - 1523\beta^2 + 2358\beta - 891\right)$$
$$+ \frac{1}{120} u^3 (-\beta - 1) \left(-1100\beta^3 + 9815\beta^2 - (30(33\beta^3 - 97\beta^2 + 79\beta - 15))\log^2(u)\right)$$
$$+ (1100\beta^4 - 11047\beta^3 + 25706\beta^2 - 23235\beta + 7191)\log(u) - 16790\beta + 5775\right) + O\left(u^4\right).$$

(A.1)

(II) $\gamma$-wormholes

$$h_{tt} = \frac{2}{u^2} + \frac{5 - 11\gamma}{u} + \left(10\gamma + \left(-11\gamma^2 + 14\gamma - 3\right)\log(u) - 7\right)$$
$$+ \frac{1}{6} u \left(55\gamma^3 - 169\gamma^2 + 141\gamma - 6(11\gamma^2 - 14\gamma + 3)\log(u) - 15\right)$$
$$- \frac{1}{24} u^2 \left((\gamma - 1)(220\gamma^3 - 1453\gamma^2 + 2106\gamma - 24(11\gamma^2 - 36\gamma + 9)\log(u) - 639)\right)$$
$$- \frac{1}{600} (\gamma - 1) \left(-25(233\gamma^3 - 1284\gamma^2 + 2232\gamma - 711) + 2\left(12327 - 56665\gamma + 93611\gamma^2\right)\right)$$
$$+ 5500\gamma^4 - 45683\gamma^3 \log(u) - 60(121\gamma^3 - 407\gamma^2 + 355\gamma - 69)\log^2(u))\right) u^3 + O\left(u^4\right).$$

(A.2)

(III) Bronnikov-Kim wormholes

$$h_{tt} = \frac{1}{u^2} + \frac{11R^2 - 12R + 12}{(4 - 4R)u}$$
$$+ \frac{1}{24(R - 1)^2} \frac{8(33R^3 - 97R^2 + 128R - 64) - 3(R - 2)^2(11R^2 - 12R + 12)\log(u)}{(55R^6 - 720R^5 + 2304R^4 - 3552R^3 + 2736R^2 - 1152R + 384)u}$$
$$+ \frac{1}{192(R - 1)^4} \left(-24000 + 96000R - 176592R^2 + 193776R^3 - 134772R^4 + 58584R^5 - 14739R^6\right)$$
$$+ 1743R^7 - 55R^8 + 36(-2 + R)^4(-12 + 24R - 23R^2 + 11R^3)Log[u] u^2$$
$$+ \frac{1}{4800(R - 1)^5} \left[100(55R^9 - 2128R^8 + 21462R^7 - 102453R^6 + 284012R^5 - 499588R^4\right]$$
$$+ 577088R^3 - 431792R^2 + 191680R - 38336\right) + \left(-1375R^{10} + 60735R^9 - 718471R^8\right).$$
\[
+4174672R^7 - 14597128R^6 + 33515840R^5 - 52485312R^4 + 56357888R^3 - 40272512R^2
+17455360R - 3491072 \log(u)
\]
\[
+30(R-2)^4 \left( 165R^5 - 1357R^4 + 3488R^3 - 4504R^2 + 3312R - 1104 \log^2(u) \right) u^3 + O(u^4) \] (A.3)

(IV) Massless geometries

\[
h_{tt} = \frac{1}{u^2} + \frac{11(C-1)}{(4\sqrt{2})u} + \frac{1}{48} \left( -(33(C-1)^2) \log(u) - 80 \right) + \frac{(-55C^3 + 165C^2 - 637C + 527)}{192\sqrt{2}} u
\]
\[-\frac{1}{768} (\left( (C-1)^2(55C^2 - 110C + 1319) \right) u^2
+ \frac{1}{19200\sqrt{2}} (C-1)^3 \left( 100 \left( 55C^2 - 110C - 1 \right) + \left( 1375C^4 - 5500C^3 + 46912C^2
-82824C + 68557 \right) \log(u) + (1320(C-1)^2) \log^2(u) \right) + O(u^4) \] (A.4)

References

[1] E. E. Flanagan and T. Hinderer, “Constraining neutron star tidal Love numbers with gravitational wave detectors,” Phys. Rev. D 77, 021502 (2008) doi:10.1103/PhysRevD.77.021502 [arXiv:0709.1915 [astro-ph]].

[2] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,” Phys. Rev. Lett. 119, no. 16, 161101 (2017) doi:10.1103/PhysRevLett.119.161101 [arXiv:1710.05832 [gr-qc]].

[3] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “GW170817: Measurements of neutron star radii and equation of state,” Phys. Rev. Lett. 121, no. 16, 161101 (2018) doi:10.1103/PhysRevLett.121.161101 [arXiv:1805.11581 [gr-qc]].

[4] R. A. Porto, “The Tune of Love and the Nature(ness) of Spacetime,” Fortsch. Phys. 64, no. 10, 723 (2016) doi:10.1002/prop.201600064 [arXiv:1606.08895 [gr-qc]].

[5] T. Damour and A. Nagar, “Relativistic tidal properties of neutron stars,” Phys. Rev. D 80, 084035 (2009) doi:10.1103/PhysRevD.80.084035 [arXiv:0906.0096 [gr-qc]].

[6] V. Cardoso, M. Kimura, A. Maselli and L. Senatore, “Black Holes in an Effective Field Theory Extension of General Relativity,” Phys. Rev. Lett. 121, no. 25, 251105 (2018) doi:10.1103/PhysRevLett.121.251105 [arXiv:1808.08962 [gr-qc]].

[7] S. Cai and K. D. Wang, “Non-vanishing of tidal Love numbers,” [arXiv:1906.06850 [hep-th]].
[8] V. Cardoso, E. Franzin, A. Maselli, P. Pani and G. Raposo, “Testing strong-field gravity with tidal Love numbers,” Phys. Rev. D 95, no. 8, 084014 (2017) Addendum: [Phys. Rev. D 95, no. 8, 089901 (2017)] doi:10.1103/PhysRevD.95.089901, 10.1103/PhysRevD.95.084014 [arXiv:1701.01116 [gr-qc]].

[9] L. Randall and R. Sundrum, “An Alternative to compactification,” Phys. Rev. Lett. 83, 4690 (1999) doi:10.1103/PhysRevLett.83.4690 [hep-th/9906064].

[10] P. Pani, L. Gualtieri, T. Abdelsalhin and X. Jimenez-Forteza, “Magnetic tidal Love numbers clarified,” Phys. Rev. D 98, no. 12, 124023 (2018) doi:10.1103/PhysRevD.98.124023 [arXiv:1810.01094 [gr-qc]].

[11] T. Shiromizu, K. i. Maeda and M. Sasaki, “The Einstein equation on the 3-brane world,” Phys. Rev. D 62, 024012 (2000) doi:10.1103/PhysRevD.62.024012 [gr-qc/9910076].

[12] A. Chamblin, S. W. Hawking and H. S. Reall, “Brane world black holes,” Phys. Rev. D 61, 065007 (2000) doi:10.1103/PhysRevD.61.065007 [hep-th/9909205].

[13] K. Chakravarti, S. Chakraborty, S. Bose and S. SenGupta, “Tidal Love numbers of black holes and neutron stars in the presence of higher dimensions: Implications of GW170817,” Phys. Rev. D 99, no. 2, 024036 (2019) doi:10.1103/PhysRevD.99.024036 [arXiv:1811.11364 [gr-qc]].

[14] C. Germani and R. Maartens, “Stars in the brane world,” Phys. Rev. D 64, 124010 (2001) doi:10.1103/PhysRevD.64.124010 [hep-th/0107011].

[15] T. Binnington and E. Poisson, “Relativistic theory of tidal Love numbers,” Phys. Rev. D 80, 084018 (2009) doi:10.1103/PhysRevD.80.084018 [arXiv:0906.1366 [gr-qc]].

[16] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation,” San Francisco 1973, 1279p.

[17] G. Raposo, P. Pani and R. Emparan, “Exotic compact objects with soft hair,” Phys. Rev. D 99, no. 10, 104050 (2019) doi:10.1103/PhysRevD.99.104050 [arXiv:1812.07615 [gr-qc]].

[18] K. S. Thorne and A. Campolattaro, “Non-radial pulsation of general-relativistic stellar models. I. Analytic analysis for L 2,” Astrophys.J., 149, 591, 1967.

[19] T. Hinderer, “Tidal Love numbers of neutron stars,” Astrophys. J. 677, 1216 (2008) doi:10.1086/533487 [arXiv:0711.2420 [astro-ph]].

[20] S. S. Seahra and P. S. Wesson, “Application of the Campbell-Magaard theorem to higher dimensional physics,” Class. Quant. Grav. 20, 1321 (2003) doi:10.1088/0264-9381/20/7/306 [gr-qc/0302015].

[21] R. Maartens, “Brane world gravity,” Living Rev. Rel. 7, 7 (2004) doi:10.12942/lrr-2004-7 [gr-qc/0312059].

[22] N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezania, “Black holes on the brane,” Phys. Lett. B 487, 1 (2000) doi:10.1016/S0370-2693(00)00798-X [hep-th/0003061].
[23] R. Casadio, A. Fabbri and L. Mazzacurati, “New black holes in the brane world?,” Phys. Rev. D 65, 084040 (2002) doi:10.1103/PhysRevD.65.084040 [gr-qc/0111072].

[24] K. A. Bronnikov, V. N. Melnikov and H. Dehnen, “On a general class of brane world black holes,” Phys. Rev. D 68, 024025 (2003) doi:10.1103/PhysRevD.68.024025 [gr-qc/0304068].