Effect of inter-sample spacing constraint on spectrum estimation with irregular sampling

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Abstract—A practical constraint that comes in the way of spectrum estimation of a continuous time stationary stochastic process is the minimum separation between successively observed samples of the process. When the underlying process is not band-limited, sampling at any uniform rate leads to aliasing, while certain stochastic sampling schemes, including Poisson process sampling, are rendered infeasible by the constraint of minimum separation. It is shown in this paper that, subject to this constraint, no point process sampling scheme is alias-free for the class of all spectra. It turns out that point process sampling under this constraint can be alias-free for band-limited spectra. However, the usual construction of a consistent spectrum estimator does not work in such a case. Simulations indicate that a commonly used estimator, which is consistent in the absence of this constraint, performs poorly when the constraint is present. These results should help practitioners in rationalizing their expectations from point process sampling as far as spectrum estimation is concerned, and motivate researchers to look for appropriate estimators of bandlimited spectra.

Index Terms—aliasing, non-uniform sampling, renewal processes, spectral density.

I. INTRODUCTION

Estimation of the power spectral density of a continuous time mean square continuous stationary stochastic process is a classical problem [1]. Estimates are usually based on finitely many observed points. If the spectral density of the process is bandlimited (i.e., compactly supported), then one can estimate it consistently by using uniformly spaced samples, provided the sampling rate is faster than the Nyquist rate [2]. On the other hand, if the spectral density is not bandlimited, then uniform sampling at any sampling rate leads to aliasing, i.e., there is a class of continuous processes whose spectral densities are indistinguishable from the sampled process. Thus, one can not estimate the spectral density of the original process consistently on the basis of uniformly spaced samples [3].

In such a situation, non-uniform or irregular sampling schemes have been explored. Silverman and Shapiro [4] introduced a notion of alias-free sampling. Beutler [4] further formalized this definition of alias-free sampling for different classes of power spectra. Masry [5] gave another definition of alias-free sampling. For each sampling scheme that is alias-free according to this definition, he provided a corresponding estimator of the spectral density, which would be consistent under certain conditions. Poisson sampling (i.e., sampling at the arrival times of a homogeneous Poisson process) turns out to be alias-free for the class of all spectra, according to both the definitions.

The existence of consistent estimators in the non-bandlimited case makes non-uniform sampling schemes, in particular Poisson process sampling, very attractive. However, consistency is only a large sample property of an estimator. Srivastava and Sengupta [6] showed that, if one has the ability to sample the process arbitrarily fast, then one can consistently estimate a non-bandlimited spectral density through uniformly spaced samples also, provided the sampling rate goes to infinity at a suitable rate as the sample size goes to infinity. By comparing the smoothed periodogram estimator with the corresponding estimator based on the Poisson process sampling, they found that, under certain regularity conditions, the rates of convergence for the two estimators are comparable and the constants associated with the rates of convergence have a trade-off in terms of bias and variance. Thus, the existence of a consistent estimator is not an exclusive advantage of point process sampling.

An attractive property of the spectrum estimator based on Poisson sampling [7] is that the estimator is consistent for any average sampling rate. This property implies that one can estimate bandlimited spectra consistently, even if the sampling is done at a sub-Nyquist average rate. Such estimators show that deficiencies in sampling rate can be made up by sample size, provided one is prepared to sample at irregular intervals. This fact gives rise to the hope that even when there is a constraint on the sampling rate, one can judiciously use non-uniform sampling to consistently estimate spectra with much larger bandwidth than what can be achieved through uniform sampling.

It is important to note that a small average sampling rate does not mean that any two successive samples are far apart. In the case of Poisson process sampling with any average sampling rate, it can be seen that as the sample size goes to infinity, there would be a large number of pairs of consecutive samples which are nearer to each other than any specified threshold. Thus, in order to use Poisson process sampling with any average sampling rate, sometimes one has to sample the process very fast. Many other non-uniform and alias-free sampling schemes also have this requirement. All these schemes become infeasible if there is a hard limit on
the minimum separation between successive samples. Such a constraint can arise because of technological limits as well as economic considerations.

Books on sampling \cite{8,9} give a clear picture of the limitations of uniform sampling in respect of a constraint on the minimum separation between successive samples. However, suitability of non-uniform sampling schemes in the presence of this constraint has not been studied so far \cite{10}.

In this paper, we consider the problem of consistent estimation of the power spectral density of a stationary stochastic process through non-uniform sampling, under a constraint on the minimum separation between successive samples. In Section \textbf{II} we describe the underlying set-up and discuss the notions of alias-free sampling provided by Shapiro and Silverman \cite{3} as well as by Masry \cite{5}. In Section \textbf{III} we consider the class of all power spectra, and show that under the above constraint, no stationary point process sampling scheme is alias-free for this class. Subsequently, we study the possibility of alias-free sampling for estimation of spectra that are known to be confined to a certain bandwidth. In Section \textbf{IV} we discuss the difficulties of obtaining a consistent estimator of the power spectrum even when it is known to be bandlimited. In Section \textbf{V} we report the results of a simulation study of the performance of a commonly used estimator based on Poisson process sampling, in the presence of the above constraint. We summarize the findings and provide some concluding remarks in Section \textbf{VI}.

\section{Notions of Alias-Free Sampling}

Let \(X = \{X(t), -\infty < t < \infty\}\) be a real, mean square continuous and wide sense stationary stochastic process with mean zero, covariance function \(C(\cdot)\) and spectral distribution function \(\Phi(\cdot)\). If \(\Phi(\cdot)\) has a density, we denote it by \(\phi(\cdot)\). Let \(\tau = \{t_n, n = \ldots, -2, -1, 0, 1, 2, \ldots\}\) be a sequence of real-valued, stochastic sampling times.

Shapiro and Silverman’s \cite{3} notion of alias free sampling, which was further formalized by Beutler \cite{4}, is based on the following assumptions about the sampling process.

\begin{assumption}
The process \(\tau\) is independent of \(X\).
\end{assumption}

\begin{assumption}
The sequence of sampling times \(\tau\) constitutes a stationary point process, such that the probability distribution of \((t_{m+n} - t_m)\) does not depend on \(m\).
\end{assumption}

Under a sampling scheme that satisfies properties \textbf{A1} and \textbf{A2}, the sampled process \(X_s = \{X(t), t \in \tau\}\) is wide sense stationary. Denote the covariance sequence of the sampled process \(X_s\) by \(R = \{\ldots, r(-2), r(-1), r(0), r(1), r(2), \ldots\}\), where

\[
r(n) = E[X(t_{m+n})X(t_m)], \text{ for } m, n \text{ integers,}
\]

and the expectation is taken without conditioning on the sampling times. Beutler’s definition of alias-free sampling, based on Shapiro and Silverman’s earlier idea, is as follows.

\textbf{Definition 1.} The sampling process \(\tau\) satisfying assumptions \textbf{A1} and \textbf{A2} is alias-free relative to the class of spectra \(S\) if no two random processes with different spectra belonging to \(S\) yield the same covariance sequence \((R)\) of the sampled process.

Shapiro and Silverman \cite{3} had considered the special case where the sampling times constitute a renewal process, and \(S\) is the class of all spectra with integrable and square integrable densities. They referred to this scheme as additive random sampling, and showed that it is alias-free, provided the characteristic function of the inter-arrival distribution takes no value more than once on the real line. In particular, Poisson process (a renewal process having exponentially distributed inter-arrival times) sampling scheme is alias-free for the class of spectra \(S\).

The above definition has the drawback that it does not make use of the information contained in the sampling times. If one wishes to reconstruct \(\phi(\cdot)\) using a sampling scheme that is alias-free according to the above definition, then that would be done on the basis of the sequence \(R\) only. Beutler \cite{4} gave a procedure for this reconstruction, and indicated that this procedure may be used to estimate \(\phi(\cdot)\) from estimates of \(R\). However, Masry \cite{5} pointed out that the above definition does not lead to a spectrum estimator that is provably consistent.

From all these considerations, this approach appears to be rather restrictive. In practice, one would expect to use the information contained not only in the sampled values, but also in the sampling times, in order to estimate the power spectral density. In order to take into account the sampling times, Masry \cite{5} gave an alternative definition of alias-free sampling, while making Assumption \textbf{A1} and the following additional assumption about the sampling process.

\begin{assumption}
The process \(\tau\) constitutes a stationary orderly second-order point process on the real line.
\end{assumption}

Let \(\beta\) be the mean intensity and \(\mu_c\) be the reduced covariance measure of the process \(\tau\), and \(B\) be the Borel \(\sigma\)-field on the real line. Consider the compound process \(Z(B), B \in B\) defined by

\[
Z(B) = \sum_{t_i \in B} X(t_i).
\]

The process \(Z = \{Z(B), B \in B\}\) is second order stationary (i.e., the first and second moments of \(Z(B + t)\), for any real number \(t\), does not depend of \(t\)). Let \(\mu_z\) be the covariance measure of the process \(Z\). It can be shown that this measure is given by

\[
\mu_z(B) = \int_B C(u)[\beta^2 du + \mu_c(du)]. \tag{II.1}
\]

Masry’s notion of alias-free sampling is as follows.
**Definition 2.** The sampling process \( \tau \) satisfying assumptions \( A1 \) and \( A3 \) is alias-free relative to the class of spectra \( S \) if no two random processes with different spectra belonging to \( S \) yield the same covariance measure \( (\mu_z) \) of the compound process.

Note that this definition makes use of the information contained in the sampling times, as the covariance measure \( \mu_z \) involves the mean intensity \( \beta \) as well as the reduced covariance measure \( \mu_c \) of the sampling process. It has been shown that, according to Definition 2, Poisson process sampling is alias-free for the class of all spectra having integrable and square integrable densities [5].

**III. Sampling under Constraint**

As mentioned in Section II, the focus of the present work is on a sampling process \( \tau \) which satisfies the following constraint.

Assumption \( A4 \). The time separation between two successive sample points is at least \( d \) (i.e., \( t_{n+1} - t_n \geq d \) for any index \( n \)) for some fixed \( d > 0 \).

In this section, we investigate whether a sampling scheme satisfying this constraint can be alias-free.

**A. General spectra**

We present some negative results in the case when \( S \) is the class of all spectra – bandlimited or otherwise.

**Theorem 1.** No sampling point process satisfying Assumptions \( A1, A2 \) and \( A4 \) is alias-free according to Definition 1, for the class of all spectra.

**Theorem 2.** No sampling point process satisfying Assumptions \( A1, A3 \) and \( A4 \) is alias-free according to Definition 2, for the class of all spectra.

We prove these theorems in the appendix by constructing counter-examples, based on the following class of power spectral densities.

\[
\mathbb{H} = \left\{ \phi(\cdot) : \int_{-\infty}^{\infty} \phi(\lambda) e^{it\lambda} d\lambda = 0 \text{ for } |t| > d. \right\} \quad (III.1)
\]

The members of this class correspond to covariance functions supported over the interval \([-d, d]\). A member of this class is the power spectral density defined by

\[
\phi_a(\lambda) = \frac{1}{\pi a} \frac{1 - \cos(a\lambda)}{\lambda^2},
\]

for any arbitrary positive \( a \in (0, d] \). This density corresponds to the covariance function

\[
C_a(t) = \begin{cases} 
1 - \frac{|t|}{a} & \text{for } |t| \leq a, \\
0 & \text{for } |t| > a.
\end{cases}
\]

Some other members of \( \mathbb{H} \) can be constructed by convolving \( \phi_a(\cdot) \) with an arbitrary power spectral density. We show in the appendix that if \( X_1, X_2 \) and \( X_3 \) are independent mean square continuous stochastic processes such that \( X_2 \) and \( X_3 \) have spectra in \( \mathbb{A} \) and have the same variance, then the spectra of \( X_1 \) and \( X_2 \) and \( X_1 + X_3 \) cannot be distinguished from the sequence \( R \) or the measure \( \mu_z \), leading to aliasing according to Definitions 1 and 2.

One can easily construct two integrable and square integrable power spectral densities that are indistinguishable from \( R \) or \( \mu_z \). Therefore, the statements of Theorems 1 and 2 also hold in respect of all spectra having integrable and square integrable densities (rather than all spectra). Thus, the alias-free property of Poisson process sampling mentioned in Section II become inapplicable, once the inter-sample spacings are adjusted in accordance with the constraint \( A4 \).

These two theorems show that, under the constraint of a minimum inter-sample spacing, any point process sampling scheme would be inadequate for the identification of a completely unrestricted power spectral density – according to the existing notions of alias-free sampling. If the power spectral density of the original continuous time process is not identifiable from the sequence \( R \) or the covariance measure \( \mu_z \), then one cannot expect to consistently estimate the power spectral density on the basis of estimates of either of these.

Note that the Assumption \( A4 \) comes from a practical consideration, and it is difficult to think of an implementable sampling scheme that would not require it (i.e., a scheme that can have arbitrarily closely spaced samples).

It is well known that estimators based on uniformly spaced samples, irrespective of the sampling rate, also suffer from the limitation of non-identifiability. In fact, it is this limitation of uniform sampling that has been historically used as one of the major arguments in favor of non-uniform sampling schemes. The above theorems show that the same difficulty applies to practical non-uniform sampling schemes as well.

**B. Bandlimited Spectra**

In the case of uniform sampling, it is well known that a bandlimited process would not lead to aliasing provided that the sampling is done at the Nyquist rate or faster. On the other hand, uniform sampling at any fixed rate would be free from the problem of aliasing if and only if the spectrum of the continuous time process is known to be confined to an appropriate band. This fact, together with the limitation of point process sampling in the case of non-bandlimited spectra, gives rise to the question: Can point process sampling under Assumption \( A4 \) be alias-free for the class of bandlimited spectra? If so, it would be interesting to compare the maximum allowable spectral bandwidths for alias-free sampling, arising from uniform and point process sampling schemes under Assumption \( A4 \).
It turns out that alias-free sampling under Assumption A4 is possible for an important class of stochastic sampling schemes, namely, renewal process sampling. This is a special case of point process sampling, which has received much attention from researchers [3]–[5], [11]. Poisson process sampling is a further special case of renewal process sampling. However, it is an ideal sampling scheme, in contrast with implementable schemes that would require Assumption A4.

The benchmark for the present study would be the fastest possible rate of uniform sampling under Assumption A4, which is 1/d. Note that uniform sampling at this rate is alias-free for the class of spectra supported on [−π/d, π/d].

First, we present a general result that would be useful in answering the foregoing question, as far as Definition 1 of alias-free sampling is concerned.

**Theorem 3.** A renewal process sampling scheme satisfying Assumptions A1 and A2, and having characteristic function of the inter-sample spacing denoted by $f''$, is alias-free relative to a class of spectra supported on the closed and finite interval $I$ according to Definition 1 if and only if the graph of $f''(\lambda)$ on the complex plane, for $\lambda \in I$, does not divide the complex plane.

Theorem 3 relates the alias-free property of a renewal process sampling scheme to the geometry of the characteristic function of the inter-sample spacing. It may be noted that the distribution of $d + X$, where $d$ is fixed and $X$ has the gamma distribution with any combination of parameters, does not satisfy the necessary and sufficient condition given in Theorem 3 for $I = [−\pi/d, \pi/d]$. It follows that the corresponding renewal process sampling schemes, including the case of inter-sample spacing having a left-truncated exponential distribution, are not alias-free according to Definition 1, relative to a class of spectra limited to the band $[−\pi/d, \pi/d]$. The graph of $f''(\lambda)$ for the left-truncated exponential distribution with mean $2d$ and truncation point $d$, for $−\pi/d < \lambda \leq \pi/d$, is shown in Figure 1. For such sampling schemes, aliasing can be avoided only if the continuous time process is confined to a bandwidth that is even smaller than the maximum allowable bandwidth in the case of uniform sampling.

However, there are some other renewal process sampling schemes that satisfy Assumption A4 and are alias-free for the class of spectra limited to a band larger than $[−\pi/d, \pi/d]$, as the next theorem shows.

**Theorem 4.** There exists a closed and finite interval $I$, which contains the interval $[−\pi/d, \pi/d]$, and a renewal process sampling scheme satisfying Assumptions A1, A3 and A4 which is alias-free relative to the class of spectra supported on $I$, according to Definition 1.

The proof of Theorem 4, given in the appendix, invokes an example, for which $I$ is more than 10% larger than the interval $[−\pi/d, \pi/d]$, while the average inter-sample spacing is about 35% more than the minimum allowable spacing (d). The graph of $f''(\lambda)$ for this inter-sample spacing distribution, for $−1.1\pi/d < \lambda < 1.1\pi/d$, is shown in Figure 2.

We now turn to Definition 2. Since this notion of alias-free sampling is weaker than that of Definition 1, one can expect a stronger result.

**Theorem 5.** Any renewal process sampling scheme, satisfying Assumptions A1, A3 and A4 and the further assumption that the inter-sample spacing distribution has a density that is positive over a semi-infinite interval, is alias-free according to Definition 2, for the class of spectra limited to the band $[−\lambda_0, \lambda_0]$ for every finite $\lambda_0 > 0$.

Theorem 5 shows that, under the constraint of a minimum allowable separation between successive samples, renewal
process sampling is alias-free (according to Definition 2) for a wider range of power spectra than uniform sampling. It is interesting to note that sampling schemes following the assumptions of Theorem 5 are alias-free according to Definition 2 when the spectral density of the underlying continuous-time process is known to be confined to any finite bandwidth (no matter how large), but according to Theorem 2, these are not alias-free when the process is non-bandlimited.

It transpires from the foregoing discussion that there are contrasting scopes of alias-free renewal process sampling under the constraint of a minimum allowable inter-sample spacing, according to Definitions 1 and 2. The limited scope of alias-free sampling in the case of Definition 1 stems from spacing, according to Definitions 1 and 2. The limited scope of alias-free renewal process sampling is not alias-free when the process is non-bandlimited.

Theorem 5 gives us a reason to look for estimators of \( \phi(\cdot) \) based on constrained sampling schemes that are alias-free for bandlimited processes according to Definition 2. Such an estimator would be based on the measure \( \mu_z \). It can be shown that the power spectral density \( \phi(\cdot) \) is related to the characteristic functions \( \phi_z \) and \( \phi_c \) of the measures \( \mu_z \) and \( \mu_c \) as defined

\[
\phi_z(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda u} \mu_z(du), \\
\phi_c(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda u} \mu_c(du),
\]

respectively, through the integral equation

\[
\phi_z(\lambda) = \beta^2 \phi(\lambda) + \int_{-\infty}^{\infty} (\lambda - \omega) \phi_c(\omega) d\omega, \tag{IV.3}
\]

provided that the measure \( \mu_c \) is totally finite. Masry [5] and Brillinger [12] gave a unique and explicit solution to (IV.3) under the following additional assumption about the sampling process \( \tau \).

Assumption B1. The reduced covariance measure \( \mu_c \) has the density \( f_c(\cdot) \) satisfying the conditions

(i) \( f_c(u) + \beta^2 > 0 \) for all \( u \in (-\infty, \infty) \);

(ii) \( \int_{-\infty}^{\infty} \frac{f_c(u)}{f_c(u) + \beta^2} \) is integrable.

Masry [5] also showed that an empirical version of this solution is a consistent estimator of \( \phi(\cdot) \).

Note that, in the case of renewal processes, \( (f_c(\cdot) + \beta^2) / \beta \) is the renewal density. Under Assumption A4, this density would be necessarily zero over the interval \([-d, d] \). Therefore, Assumption B1 is violated, and the solution to (IV.3), given in [5], [12], is not applicable to the present situation. No explicit solution is available in general. Consequently, there is no scope of using Masry’s plug-in estimator.

It emerges from this discussion that all the estimators of a bandlimited power spectral density, which have been proposed so far on the basis of renewal process sampling, are either inapplicable in the present context, or cannot be shown to be consistent.

In summary, even though renewal process sampling, subject to the constraint of a minimum inter-sample spacing, is alias-free for bandlimited power spectral densities, there is no estimator in the literature that is known to be consistent. One has to look for new estimators that may be appropriate in this situation.

V. SIMULATION

The foregoing discussion leads us to an interesting question: How would the available estimators perform under the constraint of a minimum inter-sample spacing? The performance
of various estimators based on uniform sampling have been studied both theoretically and empirically, and their limitations arising from aliasing have been exposed. In this section, we consider the performance of a well-known estimator based on non-uniform sampling, under the constraint of a minimum inter-sample spacing.

We consider a continuous time stationary stochastic process \( X \) with mean 0 and power spectral density \( \phi(\cdot) \) given by

\[
\phi(\lambda) = \begin{cases} 
\frac{2\sqrt{2}}{\pi} \left( 4e^{-8(4\lambda-3\pi)^2/\pi} + 3e^{-9(4\lambda-7\pi)^2/2\pi} \right) & \text{if } -2\pi \leq \lambda \leq 2\pi, \\
0 & \text{otherwise.}
\end{cases}
\]

We consider sampling with a stationary renewal process \( \tau \) whose inter-sample spacing is distributed as \( d + T \), where the random variable \( T \) has the exponential distribution with mean \( \theta \). Note that for \( d = 0 \), this sampling scheme reduces to Poisson sampling. We assume that \( n \) consecutive samples, denoted by \( X(t_1), X(t_2), \ldots, X(t_n) \) are available for estimation.

An estimator of the power spectral density based on the above data, which is consistent in the special case \( d = 0 \), is given as follows.

\[
\hat{\phi}_n(\lambda) = \frac{1}{\pi b_n} \sum_{m=1}^{n-1} \sum_{k=1}^{n-m} X(t_k)X(t_{k+m}) w(b_n(t_{k+m} - t_k)) \cos(\lambda(t_{k+m} - t_k)),
\]

(V.1)

where \( w(\cdot) \) is a covariance averaging kernel, and \( b_n \) is the bandwidth of the kernel. Note that this is the estimator proposed by Masry [5], which is an empirical version of the solution to (IV.3) if one assumes \( d = 0 \) (that is, disregards the constraint A4).

We study the performance of the estimator, \( \hat{\phi}_n(\cdot) \) for the choices

\[
n = 1000, \\
b_n = 1/50
\]

and \( w(x) = \begin{cases} 
\frac{1}{2} \{1 + \cos(\pi x)\} & \text{if } -1 \leq x \leq 1, \\
0 & \text{otherwise,}
\end{cases} \]

under the constraint A4.

We first investigate how this estimator performs when \( d > 0 \). We conduct multiple simulation runs for each of the choices \( d = 0, 0.5, 1, 2, 4 \) and 8, together with \( \theta = 1 \). Figure 3 shows spectrum estimates from five typical simulation runs, along with the true power spectral density. The plots show how the estimator begins to perform poorly as one moves away from \( d = 0 \). For larger values of \( d \), the inter-sample spacing is dominated by the constant part. Therefore, the sampled data resemble that from uniform sampling, which have the problem of aliasing. As a result, for larger values of \( d \), spurious peaks in greater numbers begin to show in the estimates. The estimator also assumes larger negative values when \( d \) is larger.

Figure 4 shows the mean squared error (in log-scale) of the estimate computed in each of the above cases from 500 simulated runs, along with the squared power spectral density. It is clear that the mean squared error around the peaks of the power spectral density are of the same order as the squared power spectral density, and the mean squared error around the valleys are much larger for \( d > 0 \) than for \( d = 0 \).

This simulation study indicates that the estimator, which is consistent in the absence of the constraint on the minimum inter-sample spacing, can perform poorly in the presence of the constraint.

The next question we try to answer is: Given the constraint \( d = 1 \) (so that uniform sampling at any sampling rate would necessarily lead to aliasing), is there an appropriate choice of \( \theta \) that would produce a reasonable estimate of the power spectral density? In order to answer this question, we again run multiple simulations for \( \theta = 0, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10 \) and 20. In Figure 5, we present spectrum estimates from five typical simulation runs in each of these cases, together with the true power spectral density. For \( \theta = 0 \), i.e., the case of uniform sampling at sub-Nyquist rate, there is clear evidence of spurious peaks in the spectrum estimates. A similar occurrence is observed for small positive values of \( \theta \). On the other hand, large values of \( \theta \) give rise to large variability in the estimates.

Figure 6 shows mean squared errors (in log scale) of the estimates computed in each of the above cases from 500 simulated runs, along with the squared power spectral density. It transpires that irrespective of the trade-off between bias and variance observed in Figure 5, the mean squared errors in all the cases are comparable. The mean squared error is of the order of the squared value of the true power spectral density around the peaks, and several orders of magnitude larger around the valleys.

These findings indicate that the estimator (V.1) does not perform well for any choice of \( \theta \).

VI. CONCLUDING REMARKS

The constraint of a specified minimum inter-sample spacing is a natural one, in view of technological and economic constraints. We have come across some interesting findings after formally incorporating this constraint in the study of aliasing in the context of spectrum estimation. The most important finding is that under this constraint, no point process sampling scheme is alias-free for the class of all spectra – according to any definition. It should be noted that the possibility of alias-free sampling, leading to consistent estimation of the power spectral density, has traditionally been a major argument in favour of point process sampling (in contrast with uniform sampling). This argument does not hold at all in the presence of the above constraint.

We have shown in Section III that when the inter-sample spacing is constrained to be larger than a threshold, renewal
Fig. 3. Estimates of the power spectral density for $\theta = 1$ and different values of $d$. The bold line represents the true power spectral density, while the thinner lines represent five typical estimates.

Fig. 4. Plot of the true squared power spectral density and the mean squared errors of spectrum estimates (in log scale) based on 500 simulation runs for $\theta = 1$ and different values of $d$. 

Fig. 5. Estimates of the power spectral density for $d = 1$ and different values of $\theta$. The bold line represents the true power spectral density, while the thinner lines represent five typical estimates.
process sampling schemes are alias-free for suitably bandlimited spectra according to Definition 1. The range of bandwidths for alias-free renewal process sampling for some inter-sample spacing distributions is smaller than the corresponding range for regular sampling, while it is larger for some other distributions. On the other hand, according to Definition 2, all renewal process samplings schemes satisfying the conditions of Theorem 5 are alias-free for the class of spectra limited to any finite band. 

Masry [5] pointed out that the plug-in estimator suggested by Beutler [4] is not provably consistent, and provided another estimator which is consistent under certain conditions. The discussion of Section IV shows that these conditions do not hold under the constraint considered in this paper – even when the power spectrum is known to be bandlimited. This brings us back to square one as far as estimation of power spectral density is concerned. Since the standard methods would not work well in the presence of the constraint, as demonstrated through the simulations of Section V, there is ample scope for further research in the area of estimation.

APPENDIX

Proof of Theorem 1. Consider independent, zero mean, mean square continuous stationary stochastic processes $X_1$, $X_2$ and $X_3$, having covariance functions $C_i(\cdot), i = 1, 2, 3,$ respectively, such that $C_2(0) = C_3(0)$ and $X_2$ and $X_3$ have different spectral densities belonging to the class $\mathcal{H}$ defined in (III). Consider a sampling point process $\tau = \{t_n, n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ satisfying the Assumptions A1, A2 and A4. Let the processes $X_1 + X_2$ and $X_1 + X_3$ have spectral distributions $\Phi_{12}(\cdot)$ and $\Phi_{13}(\cdot)$, respectively, and covariance sequences of sampled processes $R_{12} = \{r_{12}(n), n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ and $R_{13} = \{r_{13}(n), n = \ldots, -2, -1, 0, 1, 2, \ldots\}$, respectively. We have

$$r_{12}(0) = C_1(0) + C_2(0) = C_1(0) + C_3(0) = r_{13}(0).$$

For arbitrary integers $m$ and $n$, let $F_n(x)$ be the distribution function of $(t_{m+n} - t_m)$. Assumption A4 implies that $F_n(x)$ is supported on the interval $[|n|d, \infty)$. It follows that, for $n \neq 0$,

$$r_{12}(n) = E[X_1(t_{m+n})X_1(t_m)] + E[X_2(t_{m+n})X_2(t_m)]$$

$$= \int_0^\infty C_1(u)dF_n(u) + \int_0^\infty C_2(u)dF_n(u)$$

$$= \int_{|n|d}^\infty C_1(u)dF_n(u) + \int_{|n|d}^\infty C_2(u)dF_n(u)$$

$$= \int_{|n|d}^\infty C_1(u)dF_n(u),$$

since $C_2(\cdot)$ is supported on $[-d, d]$. Likewise, $r_{13}(n)$ is also equal to the last expression. This completes the proof. □

Proof of Theorem 2. Consider independent, zero mean, mean square continuous stationary stochastic processes $X_1$, $X_2$ and $X_3$, having covariance functions $C_i(\cdot), i = 1, 2, 3,$ respectively, such that $C_2(0) = C_3(0)$ and $X_2$ and $X_3$ have different spectral densities belonging to the class $\mathcal{H}$ defined in (III). Consider a sampling point process $\tau = \{t_n, n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ satisfying the Assumptions A1, A3 and A4, and having mean intensity $\beta$ and reduced covariance measure $\mu_c$. Let the processes $X_1 + X_2$ and $X_1 + X_3$ have spectral distributions $\Phi_{12}(\cdot)$ and $\Phi_{13}(\cdot)$, respectively. As in Section II consider the compound processes

$$Z_{1j} = \left\{Z_{1j}(B) = \sum_{t_i \in B} X_1(t_i) + X_j(t_i), \ B \in \mathbb{B} \right\}, \ j = 2, 3,$$
The interchange of the integration is possible by Fubini’s theorem, which have covariance measures \( \mu_{z_{12}} \) and \( \mu_{z_{13}} \) given by
\[
\mu_{z_{1j}}(B) = \int_B \{ C_1(u) + C_j(u) \} [\beta^2 du + \mu_c(du)], \quad j = 2, 3,
\]
respectively.

The reduced covariance measure \( \mu_c \) of the point process \( \tau \) can be expressed as
\[
\mu_c(B) = \beta \delta_0(B) + \beta \int_B [dK(u) - \beta du], \quad B \in \mathbb{B},
\]
where
\[
K(u) = \sum_{n=1}^{\infty} F_n(\{u\}),
\]
\( F_n(\cdot) \) is the conditional probability
\[
F_n(\cdot) = \lim_{\epsilon \downarrow 0} P \left[ N(t, t + u) \geq n \mid N(t - \epsilon, t] \geq 1 \right],
\]
and \( \{ N(B), B \in \mathbb{B} \} \) is the counting process induced by the process \( \tau \) [13], [14]. Assumption A4 implies that \( K(u) = 0 \) for \( u \in [-d, d] \).

It follows from the above representation of \( \mu_c \) that, for each Borel set \( B \), the covariance measures \( \mu_{z_{12}} \) is given by
\[
\mu_{z_{12}}(B) = \int_B C_1(u)[\beta^2 du + \mu_c(du)] + \int_B C_2(u)[\beta^2 du + \mu_c(du)]
\]
\[
= \int_B C_1(u)[\beta^2 du + \mu_c(du)] + \beta C_2(0) \delta_0(B \cap [-d, d]).
\]

Since \( C_2(0) = C_3(0) \), it is clear that the measures \( \mu_{z_{12}} \) and \( \mu_{z_{13}} \) agree on all Borel sets. This completes the proof. \( \square \)

**Proof of Theorem 3.** Here, the sampling process \( \tau = \{ t_n, n = \ldots, -2, -1, 0, 1, 2, \ldots \} \) is such that the inter-sample spacing \( t_{n+1} - t_n \) for different values of \( n \) are independent and identically distributed, say with distribution function \( F(\cdot) \).

Let \( S \) be the class of spectra supported on the closed and finite interval \( I \). Let \( X \) be a process as defined in the theorem, and have the power spectral distribution \( \Phi(\cdot) \) belonging to \( S \).

The covariance sequence \( R \) of the sampled process is given by
\[
r(n) = E[X(t_{m+n})X(t_m)] = E\left[ E\left[ X(t_{m+n})X(t_m) \mid \tau \right] \right] = E\left[ C(t_{m+n} - t_m) \right]
\]
\[
= E\left[ \frac{1}{2\pi} \int_I e^{i\lambda(t_{m+n} - t_m)} d\Phi(\lambda) \right] = \frac{1}{2\pi} \int_I E\left( e^{i\lambda(t_{m+n} - t_m)} \right) d\Phi(\lambda).
\]

The interchange of the integration is possible by Fubini’s theorem, since the power spectral distribution \( \Phi(\cdot) \) and the probability distribution of \( t_{m+n} - t_m \) are both finite. Since the latter distribution is the \( n \)-fold convolution of the inter-sample spacing distribution, we have
\[
r(n) = \frac{1}{2\pi} \int_I \left| f'(\lambda) \right|^n d\Phi(\lambda), \quad (A.1)
\]
where \( f'(\cdot) \) is the characteristic function of inter-sample spacing distribution, i.e.,
\[
f'(\lambda) = \int_0^\infty e^{i\lambda y} dF(y), \quad -\infty < \lambda < \infty.
\]

The sampling scheme \( \tau \) is alias-free relative to the class of spectra \( S \) according to Definition 1, if no two different spectra \( \Phi_1 \) and \( \Phi_2 \) belonging to \( S \) produce the same covariance sequence \( R \). Since the sequence \( R \) satisfies \( r(-n) = r(n) \), the foregoing condition is equivalent to the statement:
\[
\int_I \left| f'(\lambda) \right|^n (d\Phi_1(\lambda) - d\Phi_2(\lambda)) = 0 \quad \text{for } n = 0, 1, 2, \ldots
\]
implies that \( \Phi_1(\cdot) = \Phi_2(\cdot) \). \( (A.2) \)

The above integral with respect to the real variable \( \lambda \) can be written as a complex integral over the contour
\[
\Omega = \{ z : z = f'(\lambda), \quad \lambda \in I \}. \quad (A.3)
\]

Thus, we can conclude that the sampling scheme \( \tau \) is alias free relative to the class of spectra \( S \) according to Definition 1 if and only if

For any signed measure \( \nu \) defined on the Borel \( \sigma \)-field on \( \Omega \),
\[
\int_\Omega z^n \nu(dz) = 0 \quad \text{for } n = 0, 1, 2, \ldots \quad \Rightarrow \quad \nu = 0. \quad (A.4)
\]

Note that, since \( \Omega \) is the image of the continuous function \( f'(\lambda) \) on the closed and finite interval \( I \), the contour \( \Omega \) is compact. Let \( C(\Omega) \) be the Banach space of all complex-valued continuous functions on \( \Omega \) equipped with the supremum norm. Let \( M \) be the set of all signed measures defined on the Borel \( \sigma \)-field on \( \Omega \). For any \( \nu \in M \), define the complex valued bounded linear functional \( L_\nu \) defined on \( C(\Omega) \) as
\[
L_\nu(g) = \int_\Omega g(z) \nu(dz) \quad \text{for all } g \in C(\Omega). \quad (A.5)
\]

In terms of these notations, we rewrite (A.1) as for any \( \nu \in M \), “\( L_\nu(z^n) = 0 \) for \( n = 0, 1, 2, \ldots \)” \( \Rightarrow \nu = 0 \). \( (A.6) \)

By the Riesz representation theorem, *every* bounded linear functional \( L \) on \( C(\Omega) \) can be represented as
\[
L(g) = \int_\Omega g(z) \nu_1(dz) + i \int_\Omega g(z) \nu_2(dz) \quad \text{for all } g \in C(\Omega), \quad (A.7)
\]
for a unique pair of measures \( \nu_1 \) and \( \nu_2 \) in \( M \) [15]. It follows that the necessary and sufficient condition (A.6) is equivalent to the condition:

For any bounded linear functional \( L \) on \( C(\Omega) \),
“\( L(z^n) = 0 \) for \( n = 0, 1, 2, \ldots \)” \( \Rightarrow \) \( L = 0 \). \( (A.8) \)
The above condition is a statement about the sequence \{1, z, z^2, \ldots \} in relation to the Banach space \( C(\Omega) \) \cite[pp. 257 of \cite{16}]{}. By Theorem 11.1.7 of \cite{16}, \( (A.8) \) is equivalent to the condition:

“This linear span of the sequence \{1, z, z^2, \ldots \} is dense in \( C(\Omega) \).”

The above condition can be rephrased as: “Any \( g \in C(\Omega) \) can be expanded in a uniformly convergent sequence of polynomials.” By a result of \cite{17} (see also \cite{18}), we get the further equivalent condition:

“The set \( \Omega \) is nowhere dense and does not divide the plane.”

\[ \text{(A.10)} \]

Since the set \( \Omega \) is a curve in the complex plane, it is always a nowhere dense set. This completes the proof. \( \square \)

**Proof of Theorem 4.** Let \( I = [-1.1\pi/d, 1.1\pi/d] \). Consider the two-point discrete distribution \( F \), given by

\[
F(t) = \begin{cases} 
0 & \text{for } t < d, \\
0.68 & \text{for } 1 \leq t < 2.1d, \\
1 & \text{for } t \geq 2.1.
\end{cases} \tag{A.11}
\]

It follows that the average inter-sample spacing is 1.352\( d \). Also,

\[ f'(\lambda) = 0.68e^{i\lambda d} + 0.32e^{2.1i\lambda d}. \]

The plot of the imaginary part of \( f'(\lambda) \) against the real part, for \( \lambda \in I \), is given in Figure 2. It can be verified that the graph does not divide the complex plane. The result follows from Theorem 3. \( \square \)

**Proof of Theorem 5.** Here, the sampling process \( \tau = \{t_n, n = \ldots, -2, -1, 0, 1, 2, \ldots \} \) is such that the inter-sample spacing \( t_{n+1} - t_n \) for different values of \( n \) are independent and identically distributed having probability density function \( f(\cdot) \). Let \( \beta \) and \( \mu_c \) be the mean intensity and the reduced covariance measure, respectively, of the process \( \tau \). The measure \( \mu_c \) can be expressed as

\[
\mu_c(B) = \beta \delta_0(B) + \int_B \beta |h(u) - \beta| du \text{ for each } B \in \mathbb{B}, \tag{A.12}
\]

where \( h(u) \) is the renewal density function, i.e.,

\[ h(u) = \sum_{n=1}^{\infty} f^{(n)}(|u|). \]

Note that Assumption \( A4 \) implies that \( f(\cdot) \) is supported on \([d, \infty)\), and so \( h(\cdot) \) is supported on \((-\infty, d] \cup [d, \infty)\). Let us assume, without loss of generality, that \( f(u) > 0 \) for \( u \geq ld \) for some \( l \geq 1 \).

Let \( \mathbb{S} \) be the class of bandlimited spectra supported on \([-\lambda_0, \lambda_0]\). If the sampling scheme \( \tau \) is not alias-free relative to the class of spectra \( \mathbb{S} \), then there exist two zero mean, mean square continuous stationary stochastic processes \( X_1 \) and \( X_2 \) with different power spectral distributions \( \Phi_1(\cdot) \) and \( \Phi_2(\cdot) \) such that compound processes

\[
Z_j = \left\{ Z_j(B) = \sum_{t_i \in B} X_j(t_i), \ B \in \mathbb{B} \right\}, \quad j = 1, 2,
\]

have the covariance measures \( \mu_{z_1} \) and \( \mu_{z_2} \), respectively, satisfying \( \mu_{z_1} = \mu_{z_2} \). Here, for \( B \in \mathbb{B} \), the covariance measures are given by (see \( (L.1) \) and \( (A.12) \))

\[
\mu_{z_j}(B) = \beta C_j(0)\delta_0(B) + \beta \int_B C_j(u)h(u)du, \quad j = 1, 2,
\]

where \( C_1(\cdot) \) and \( C_2(\cdot) \) are the covariance functions of the processes \( X_1 \) and \( X_2 \) respectively. In order that the covariance measures \( \mu_{z_1} \) and \( \mu_{z_2} \) are the same, the point masses at zero, as well as the absolutely continuous parts, must agree. The equality of the point masses requires

\[
C_1(0) = C_2(0). \tag{A.13}
\]

On the other hand, equality of the absolutely continuous parts means

\[
C_1(u)h(u) = C_2(u)h(u) \text{ for } -\infty < u < \infty.
\]

Since \( h(u) > 0 \) for the \( |u| \geq ld \), we have

\[
C_1(u) = C_2(u) \text{ for } |u| \geq ld. \tag{A.14}
\]

If the processes \( X_1 \) and \( X_2 \) have spectra limited to the band \([-\lambda_0, \lambda_0]\), then the covariance function \( C_j(\cdot) \) for \( j = 1, 2 \) can be expressed as \cite{19}

\[
C_j(u) = \frac{1}{T} \sum_{n=-\infty}^{\infty} C_j(nT) \text{sinc} \left( \frac{\pi}{T}(u - nT) \right), \tag{A.15}
\]

where \( T = \frac{2\pi}{2\pi_0} \) and

\[
\text{sinc}(x) = \begin{cases} \sin(x) \quad \text{if } x \neq 0, \\
1 \quad \text{if } x = 0.
\end{cases}
\]

Let \( k = \lfloor ld/T \rfloor \), where \( \lfloor u \rfloor \) represents the integer part of the real number \( u \). It follows from \( (A.13) \) - \( (A.15) \) that

\[
C_1(u) - C_2(u)
\]

\[
= \sum_{n=1}^{k} \{ C_1(nT) - C_2(nT) \} \text{sinc} \left( \frac{\pi}{T}(u - nT) \right)
\]

\[
+ \sum_{|n| > k} \{ C_1(nT) - C_2(nT) \} \text{sinc} \left( \frac{\pi}{T}(u - nT) \right)
\]

\[
= \sum_{n=1}^{k} \{ C_1(nT) - C_2(nT) \} \text{sinc} \left( \frac{\pi}{T}(u - nT) \right) + \text{sinc} \left( \frac{\pi}{T}(u + nT) \right).
\]
By using the fact that \( \sin(k\pi + \theta) = (-1)^k \sin \theta \) for all integer \( k \), we have for \( \alpha = u/T - [u/T] > 0 \),
\[
C_1(u) - C_2(u) = \sum_{n=1}^{k} (C_1(nT) - C_2(nT)) \times \left( \frac{\sin \left\{ (-n + \frac{\pi}{T}) \pi + \alpha \pi \right\}}{\phi(u - nT)} + \frac{\sin \left\{ (n + \frac{\pi}{T}) \pi + \alpha \pi \right\}}{\phi(u + nT)} \right) = (-1)^{|u/T|} \sin(\alpha \pi)
\]
\[
\times \sum_{n=1}^{k} (-1)^n (C_1(nT) - C_2(nT)) \left[ (\frac{(-1)^{-n}}{\phi(u - nT)} + \frac{(-1)^n}{\phi(u + nT)}) \right].
\]
Since \((-1)^n = (-1)^{-n}\) for each integer \( n \), we have
\[
C_1(u) - C_2(u) = (-1)^{|u/T|} \frac{2uT}{\pi} \sin(\alpha \pi)
\]
\[
\times \sum_{n=1}^{k} (-1)^n (C_1(nT) - C_2(nT)) \frac{1}{u^2 - n^2T^2}.
\]
Let \( v_n = (-1)^n (C_1(nT) - C_2(nT)) \). In view of (A.14), the above equation implies that
\[
\sum_{n=1}^{k} \frac{v_n}{u^2 - n^2T^2} = 0, \quad (A.16)
\]
for \( u \in \{ (ld, (k+1)T) \} \cup \{ \cup_{m=k+1}^\infty (mT, (m+1)T) \} \).

Note that the function on the left hand side of (A.16) is a ratio of polynomials. The polynomial in the numerator has degree \( 2k - 2 \), while the denominator is bounded over the domain of the function. Thus, the ratio of the polynomials can be zero at most at \( 2k - 2 \) points. Therefore, the fact that this function assumes the value 0 everywhere on the interval \( (k+1)T, (k+2)T \) implies that the polynomial in the numerator is identically equal to zero. Thus, the ratio of the polynomials is identically zero. Hence,
\[
\sum_{n=1}^{k} \frac{v_n}{u^2 - n^2T^2} = 0, \text{ for } u \in \bigcup_{m=0}^\infty (mT, (m+1)T).
\]

By considering the limit of the left hand side as \( u \downarrow nT \), it is found that \( v_n = 0 \) for \( n = 1, \ldots, k \), that is,
\[
C_1(nT) = C_2(nT) \text{ for } |n| = 1, \ldots, k.
\]

According to (A.13), the above equality holds for \( n = 0 \), while (A.15) and (A.14) imply that it holds for \( |n| = k+1, k+2, \ldots \). Thus, \( C_1(nT) = C_2(nT) \) for all \( n \). It follows from (A.15) that \( C_1(u) = C_2(u) \) for each \( u \), which contradicts the assumption that \( C_1 \) and \( C_2 \) are different. So the sampling scheme \( \tau \) is alias-free for the class of the spectra \( S \). \[\square\]

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