Research on vibration characteristics of resonator for the HRG

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Abstract. Resonator is the core structure of hemispherical resonator gyro (HRG), and its vibration characteristics play an important role in the output accuracy of gyro. A method combining dynamics model and finite element method (FEM) is proposed to study vibration characteristics of resonator. Firstly, this paper established dynamics equations of ideal resonator and dynamics equations of resonator with density distributed nonuniformity, and gave the mathematical expression of frequency split. Then, accomplished modal analysis and obtained natural frequency, mode shapes of resonator based on ANSYS Workbench. Finally, established the resonator model in consideration of density distributed nonuniformity by FEM, and investigated the influence of the 4th harmonic of density nonuniformity on resonator’s frequency split. The finite element analysis was in good agreement with the results calculated by dynamics equations, which verified the accuracy of dynamics model.

1. Introduction

HRG is a new type of solid vibration gyro based on the theory of Coriolis vibration, which appeared in 1960s. Because of the advantages of simple structure, high measuring precision, long life, high working stability and reliability, it is widely used in aeronautics, astronautics, military defense, earth exploration and other fields, and has become one of the research hotspots in the field of inertial technology at home and abroad. HRG is mainly composed of resonator, excitation device and sensitive readout device. Its basic principle is to measure angle of rotation and angular velocity of the base by using the precession effect of the radial vibration standing wave at the edge of resonator [1].

The resonator is a hemispherical shell with a central support rod, is also a typical thin-walled surface with high gradient. Fused quartz is mostly selected as the processing material, which has characteristics of high hardness and brittleness. Limited by the current level of ultra-precision machining and testing, it is difficult to obtain high precision and good surface quality. When mass, radius and other parameters of resonator have the 4th harmonic which is not uniformly distributed along the circumferential direction, the working mode of resonator will have two intrinsic rigid axes of 45°, along which natural frequency reach the maximum and minimum respectively, and the difference between these two natural frequencies is called frequency split [2]. Frequency split will cause slow drift in the antinode’s azimuth angle of the working mode, which will cause gyro error.

Scholars at home and abroad have done a lot of research on dynamics model and finite element analysis of resonator. Fox briefly analyzed frequency split of rings with mass defects by using energy...
model [3]. Achong studied vibration characteristics of plates and shells with added mass [4]. Rouke proposed the theory of eliminating frequency split of rings by removing part of mass [5]. William studied the influence of arbitrary mass distribution for ring resonator on frequency split and the rule of frequency split [6]. Choi gave the expression of natural frequency and frequency split for the resonator with additional mass point by Rayleigh-Ritz method and energy equations, also studied the influence of additional mass point on azimuth angle and the influence of parameters such as thickness, radius of hemispherical shell on natural frequency, gave the numerical simulation curves [7]. Fan established mathematical model of resonator based on the principle of energy conservation, and derived mathematical expressions of natural frequency and precession factor for the hemispherical shell with variable thickness [8-9]. Gao used the energy model established by the former to analyze the influence of the non-ideal of parameters such as wall thickness, bottom angle of resonator and input angular velocity on the performance of gyro [10]. Li studied the influence of structural parameters such as diameter and length of support rod and thickness of shell on the performance of HRG [11]. Yu established the model of mass tuning and mass perturbation for metal cylinder-shaped resonator, verified the frequency tuning algorithm by simulation and modified frequency split to 0.04 Hz [12].

Therefore, a method combining dynamics model and finite element analysis is proposed to study vibration characteristics of resonator and the influence of density distributed nonuniformity on the frequency split. It is important to modify the mass of resonator to further reduce frequency split and improve the output accuracy of gyro.

2. Dynamics model

The resonator is composed of a hemispherical shell and a central supporting rod. In theory their central axes are coaxially aligned and they are all rotational symmetric. In the process of establishing dynamics equations, the vertex of hemispherical shell is considered to be fixed on the cylindrical supporting rod, which is simplified to the analysis of an ideal hemispherical shell. The establishment is based on the Kirchhoff-Liaf hypothesis, which includes [1]:

1) Any line perpendicular to the middle surface of the shell before deformation is still perpendicular to the surface after deformation;
2) The length of the normal section along the thickness of the shell remains constant during the deformation process;
3) The normal stress generated between two adjacent surfaces parallel to the middle surface of the shell is small compared with other components of the stress tensor and can be ignored.

Dynamics analysis of hemispherical shell is carried out in the polar coordinate system, as shown in Figure 1. Set the centre of sphere as the origin of the coordinate system, the corresponding parameters are $e$, $\theta$, and $\varphi$, where $\theta$ ranges $[-\pi/2, \pi/2]$ and $\varphi$ ranges $[0, 2\pi]$. Considering the structure of resonator, the boundary condition of $\theta=0$ is set with a fixed support, and there is no external input when resonator is in free harmonic state. According to the Bubnov-Galerkin method, the dynamics equations of four antinode mode for resonator are [13]:

![Figure 1. Coordinate system of the hemisphere](image-url)
\begin{align}
\begin{cases}
    m_0 \ddot{x} - 2\Omega b \dot{y} + k_0 x = 0 \\
    m_0 \ddot{y} - 2\Omega b \dot{x} + k_0 y = 0
\end{cases}
\end{align}
(1)

where

\begin{align*}
    m_0 &= R^2 \rho h \int_0^{\pi/2} \left( U^2 + V^2 + W^2 \right) \sin \theta d\theta \\
    b &= 2R^2 \rho h \int_0^{\pi/2} \left( U \cos \theta + W \sin \theta \right) V \sin \theta d\theta \\
    k_0 &= \frac{12Eh^3}{(1+\mu)R^2} \int_0^{\pi/2} \frac{\tan^4 \theta}{\sin^3 \theta} d\theta
\end{align*}

U, V and W are Rayleigh functions, namely

\begin{align*}
    U = V = \sin \theta \tan^2 \frac{\theta}{2}, W = -\left( 2 + \cos \theta \right) \tan^2 \frac{\theta}{2}
\end{align*}

Where \( \rho \) is the material density, \( h \) is the thickness of hemispherical shell, \( R \) is the radius of the middle surface, \( E \) is the elastic modulus and \( \mu \) is the Poisson's ratio.

The resonator's natural frequency of the working mode is \([1]\):

\begin{align}
    \omega_0 = \frac{1.52h}{R^2} \left( \frac{E}{(1+\mu)\rho} \right)^{\frac{1}{2}}
\end{align}
(2)

Assuming that the structure of resonator is an ideal hemisphere shape, when the density of resonator is not uniformly distributed along the circumference of hemispherical shell, the mass distribution will be uneven, leading to frequency split and affecting the output accuracy of gyro. The density nonuniformity of resonator can be expressed as:

\begin{align}
    \rho(\varphi) = \rho_0 \left[ 1 + \sum_{n=1}^{\infty} \varepsilon_n \cos n(\varphi - \varphi_n) \right] = \rho_0 + \sum_{n=1}^{\infty} \rho_n \cos n(\varphi - \varphi_n)
\end{align}
(3)

Where \( \rho_0 \) is the mean density of resonator, \( \varepsilon_n \) is the relative amplitude of the \( n \)th harmonic, \( \varphi_n \) is the phase of the \( n \)th harmonic, \( \rho_n \) is the amplitude of the \( n \)th harmonic.

Thus, the dynamics equations of resonator are:

\begin{align}
\begin{cases}
    m_0 \left( 1 + \frac{I}{I_0} \varepsilon_4 \cos 4\varphi_4 \right) \ddot{x} + m_0 \frac{I}{I_0} \varepsilon_4 \sin 4\varphi_4 \dot{y} - 2\Omega b \dot{y} + k_0 x = 0 \\
    m_0 \left( 1 - \frac{I}{I_0} \varepsilon_4 \cos 4\varphi_4 \right) \ddot{y} + m_0 \frac{I}{I_0} \varepsilon_4 \sin 4\varphi_4 \dot{x} + 2\Omega b \dot{x} + k_0 y = 0
\end{cases}
\end{align}
(4)

where

\begin{align}
\begin{cases}
    I_0 = \int_0^{\pi/2} \left( U^2 + V^2 + W^2 \right) \sin \theta d\theta \\
    I_1 = \int_0^{\pi/2} \left( U^2 - V^2 + W^2 \right) \sin \theta d\theta
\end{cases}
\end{align}
(5)
From Equation (4), the 4th harmonic of density nonuniformity will cause a small difference of equivalent mass and vibration coupling between the two modes of resonator, causing frequency split and orthogonal errors. The effect of the 1st-3rd harmonics on frequency split is small and can be ignored. Thus, frequency split $\Delta \omega$ has the following relationship with relative amplitude of the 4th harmonic of density nonuniformity:

$$\Delta \omega = 0.35 \omega_0 \varepsilon_d$$  \hspace{1cm} (6)

3. Finite element analysis

Finite element method (FEM) is a practical, efficient and widely used numerical analysis method to simulate real physical systems by means of mathematical approximation, which can provide reliable analysis results. So vibration mode, natural frequency and frequency split value of resonator with mass defect can be obtained based on FEM. ANSYS Workbench provides a variety of analysis types, covering structural, temperature, fluid, electromagnetic, and other areas, meanwhile the simulation process is well customizable.

3.1 Modal analysis

Mode is damped or undamped vibration of mechanical system in free state, and is a natural vibration characteristic of mechanical system. Each mode corresponds to a certain natural frequency, damping ratio and mode shape. Modal analysis is the process of analyzing the above modal parameters. In the process of analysis, the motion equation of the system can be expressed as [14]:

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t)$$  \hspace{1cm} (7)

Where $\ddot{x}(t)$ and $\dot{x}(t)$ are the node acceleration vector and velocity vector of the system, $M$ is the mass matrix of the system, $C$ is the damping matrix of the system, $K$ is the stiffness matrix of the system, and $F(t)$ is the load vector of the node.

Ignoring the damping and removing the applied load, the free vibration equation of the system can be simplified as:

$$M\ddot{x}(t) + Kx(t) = 0$$  \hspace{1cm} (8)

Natural frequency and mode shapes of the system can be obtained by solving the equations based on the boundary conditions.
Figure 3. Resonator’s mode shapes of first 10 modes.

The finite element model of resonator is established, and its size parameters are shown in Figure 2. The material parameters of fused quartz are as follows: $E = 7.67 \times 10^9 \text{ Pa}$, $\rho = 2200 \text{ kg/m}^3$, $\mu = 0.17$. The mesh of the model should be symmetrically generated, and the mesh nodes should be evenly
distributed along the hemispherical shell to ensure high solution accuracy. Because of the complex surface in the model, the outer circle surface of supporting rod is used to segment the model, and then the joint surface of supporting rod and hemispherical shell is treated as common nodes.

The first 10 natural frequency values and mode shapes of resonator are obtained based on modal analysis. Mode shapes are shown in Figure 3 and natural frequency values are shown in Table 1. In the 1\textsuperscript{st} and 2\textsuperscript{nd} modes, the hemispherical shell oscillates from side to side, and the axis angle between the two oscillations is 90°; The 3\textsuperscript{rd} and 4\textsuperscript{th} modes are four antinodes vibration of the hemispherical shell, and the angle between the two intrinsic rigid axes is 45°; The 5\textsuperscript{th} mode shows the stretching motion of the hemispherical shell along the circumferential direction. The 6\textsuperscript{th} and 7\textsuperscript{th} modes show that the support rod oscillates from side to side, and the angle of the two oscillations is 45°. The 8\textsuperscript{th} and 9\textsuperscript{th} modes are characterized by six antinodes vibration of the hemispherical shell, and the angle of the inherent rigid axis is 30°. The 10\textsuperscript{th} mode shows the axial stretching motion of the hemispherical shell along the support rod.

The working mode of resonator is the four-antinodes mode. By analyzing the natural frequency values of each mode, the larger the frequency’s difference between four-antinodes mode and neighbouring mode is, the better performance of resonator will be. According to Equation (2), natural frequency can be changed by changing its radius, wall thickness and other parameters. Modal analysis can provide reference for the structure size design of resonator, and verify the rationality of each parameter selection.

| Mode | 1    | 2    | 3    | 4    | 5    |
|------|------|------|------|------|------|
| Frequency /Hz | 7188.4712 | 7188.5639 | 7943.8329 | 7943.8396 | 14491.7914 |

Table 1. Resonator’s natural frequency values of first 10 modes.

3.2 Error model analysis

According to the above, the most important harmonic affecting frequency split of resonator is the 4\textsuperscript{th} harmonic. Therefore, by establishing the finite element error model with density distributed nonuniformity, FEM is used to analyze the influence of the 4\textsuperscript{th} harmonic with density nonuniformity on frequency split.

As shown in Figure 4, the blue curve is the standard density distribution curve and the red curve is the density distribution curve of each harmonic. The red density curve in Figure 4. (d) is uniformly dispersed into \( n \) points based on Matlab, and the function values of these points are the density values at each point from 0° to 360° of the hemispherical shell. Accordingly, the hemispherical shell of resonator is evenly divided into \( n \) parts to establish the finite element model based on ANSYS Workbench, and density parameter is assigned to the corresponding part in the model. Take \( n=64 \) for modal analysis and obtain the values of frequency split.

| \( \rho \) (kg / m\(^3\)) | 1.00 | 10.00 | 20.00 | 50.00 | 100.00 | 150.00 | 200.00 |
|--------------------------|------|-------|-------|-------|--------|--------|--------|
| \( \Delta \omega_1 \) /Hz | 1.24 | 12.80 | 25.48 | 63.42 | 126.67 | 190.01 | 253.28 |
| \( \Delta \omega_2 \) /Hz | 1.26 | 12.64 | 25.28 | 63.20 | 126.38 | 189.57 | 252.76 |
| \( e \) (%) | 1.59 | 1.27 | 0.79 | 0.32 | 0.23 | 0.23 | 0.21 |

Table 2. Results of finite element analysis and equations.
According to Equation (3), take the 4th harmonic amplitude $\rho_4$ from 1 to 200 $kg/m^3$. Then values of frequency split $\Delta\omega_1$ are obtained when the 4th harmonic amplitude $\rho_4$ is taken at different values. Meanwhile, the values of frequency split $\Delta\omega_2$ are calculated by Equation (6) with different values of $\rho_4$, where the value of natural frequency $\omega_0$ is 7943.83 Hz of the above modal analysis.

$$e = \frac{|\Delta\omega_1 - \Delta\omega_2|}{\Delta\omega_2} \times 100\% \quad (9)$$

Finally, the bias $e$ is calculated according to Equation (9) by analyzing the results obtained by the two ways. And the results are shown in Table 2 and Figure 5. In the case of the same amplitude of 4th harmonic, the finite element analysis is highly consistent with the frequency split values calculated by the equations and the bias $e$ are all less than 2%. The mesh division in finite element analysis affects the solution accuracy to some extent, so the smaller $\rho_4$ is, the more significant the influence is and the bigger the bias $e$ is. At the same time, the accuracy of the dynamics model established above this paper is verified by finite element analysis.

Figure 5. Results of finite element analysis and equations.
4. Conclusions
By establishing the dynamics model of hemispherical resonator with density distributed nonuniformity, obtained the mathematical expression of frequency split. The modal analysis of resonator is accomplished and the influence of 4th harmonic of density nonuniformity on frequency split is analyzed by FEM. The results of FEM are in good agreement with theoretical calculation, which verifies the accuracy of the dynamics model. And summarized the linear relationship between frequency split and density nonuniformity, which can guide frequency tuning of resonator to reduce frequency split and improve performance of HRG.

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