AF LABELED GRAPH $C^*$-ALGEBRAS

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Abstract. It is known that a graph $C^*$-algebra $C^*(E)$ is approximately finite dimensional (AF) if and only if the graph $E$ has no loops. In this paper we consider the question of when a labeled graph $C^*$-algebra $C^*(E,L,B)$ is AF. A notion of loop in a labeled space $(E,L,B)$ is defined when $B$ is the smallest one among the accommodating sets that are closed under relative complements and it is proved that if a labeled graph $C^*$-algebra is AF, the labeled space has no loops. A sufficient condition for a labeled space to be associated to AF algebra is also given. For graph $C^*$-algebras $C^*(E)$, this sufficient condition is also a necessary one. Besides, we discuss other equivalent conditions for a graph $C^*$-algebra to be AF in the setting of labeled graphs and prove that these conditions are not always equivalent by invoking various examples.

1. Introduction

The class of graph $C^*$-algebras was introduced in [15, 16] as a generalization of the Cuntz-Krieger algebras of finite $\{0,1\}$ matrices [6]. The main benefit of working with graph algebras lies in the fact that many complex properties and structures of graph $C^*$-algebras can be explained in terms of conditions of graphs (see [2, 3, 13, 15, 16] among many others). For example, it is now well known [15] that a directed graph $E$ has no loops if and only if its graph $C^*$-algebra $C^*(E)$ is approximately finite dimensional (AF). Moreover the class contains all AF algebras up to Morita equivalence [7].

Besides the graph $C^*$-algebras, there have been various generalizations of Cuntz-Krieger algebras. The ultragraph algebras [17] and the Exel-Laca algebras [9] are those generalizations which also include the $C^*$-algebras of row-finite graphs with no sinks. In [14], conditions for an AF algebra to be realized as a graph $C^*$-algebra, an Exel-Laca algebra, and an ultragraph algebra are given, and then in [8] it is proved that if a higher-rank graph algebra $C^*(\Lambda)$ is AF, the higher-rank graph $\Lambda$ does not have an appropriate analogue of loop. The higher-rank graph algebras are of course another generalization of the Cuntz-Krieger algebras.

Recently a class of $C^*$-algebras $C^*(E,L,B)$ associated with labeled graphs $(E,L)$, more explicitly labeled spaces $(E,L,B)$, has been introduced in [4] to provide a common framework for working with some of the generalized Cuntz-Krieger algebras, and studied in [11, 15, 11, 12]. We investigate in this paper the question of when a labeled graph $C^*$-algebra $C^*(E,L,\overline{E^0})$ is AF, where $(E,L,\overline{E^0})$ is a labeled space such that the accommodating set $\overline{E^0}$, consisting of certain vertex subsets, is the

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smallest one that is closed under relative complements. To explore the question, we first find several conditions on a directed graph $E$ that are equivalent to the existence of a loop in $E$ (Proposition 3.1), and then extending one of these conditions we define a notion of loop for a labeled space (Definition 3.2). Each of the other equivalent conditions can also be restated in terms of labeled spaces, but it is not clear whether all these conditions, (a)-(d) stated below, are still equivalent to each other, especially to $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ being AF:

(a) For every finite set $\{A_1, \ldots, A_N\}$ of $\overline{\mathcal{E}^0}$ and every $K \geq 1$, there exists an $m_0 \geq 1$ such that $A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_n} = \emptyset$ for all $n > m_0$ and $A_{i_j} \in \{A_1, \ldots, A_N\}$.

(b) $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no repeatable paths.

(c) $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra.

(d) $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no loops (in the sense of Definition 3.2).

In (a), $A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_n}$ denotes the set of all paths $x = x_1 \cdots x_{n-1}$ consisting of subpaths $x_k$, from $A_{i_k}$ to $A_{i_{k+1}}$ in $E$, with length $|x_k| \leq K$ for $1 \leq k \leq n - 1$. A path $\alpha$ is repeatable if $\alpha^n$ appears in the (labeled) graph for all $n \geq 1$.

The conditions (a)-(d) above are all equivalent for graph $C^*$-algebras $C^*(E) \cong C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}^0})$ (Proposition 3.1) as already mentioned, where $\mathcal{L}_{id}$ is the trivial labeling. The purpose of this paper is to understand the relations of these conditions. Both of the implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (d) are immediate, but it will turn out in this paper that not all of them are equivalent. This shows an interesting contrast between the labeled graph $C^*$-algebras and the usual graph $C^*$-algebras.

The main results obtained in the paper are as follows.

**Theorem 1.1.** (Theorem 4.2) Let $(E, \mathcal{L})$ be a labeled graph. If $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra, the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no loops.

**Theorem 1.2.** (Theorem 4.3) Let $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ be a labeled space such that for every finite subset $\{A_1, \ldots, A_N\}$ of $\overline{\mathcal{E}^0}$ and every $K \geq 1$, there exists an $m_0 \geq 1$ for which

$$A_{i_1} E^{\leq K} A_{i_2} E^{\leq K} A_{i_3} \cdots E^{\leq K} A_{i_n} = \emptyset$$

for all $n > m_0$ and $1 \leq i_j \leq N$. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is an AF algebra.

**Theorem 1.3.** (Theorem 4.11) Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) = C^*(s_\alpha, p_\alpha)$ be the $C^*$-algebra of a labeled graph $(E, \mathcal{L})$ with no sinks or sources. Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ have a repeatable path $\alpha \in \mathcal{L}^*(E)$. If $p_{r(\alpha^m)}$ does not belong to the ideal generated by a projection $p_{r(\alpha^m)\setminus r(\alpha^{m+1})}$ for some $m \geq 1$, $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF.

Theorem 4.2 and Theorem 4.3 show that (c) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) hold true, respectively, and Theorem 4.11 can be regarded as a partial result for (c) $\Rightarrow$ (b). The converse of Theorem 4.2 may not be true (Example 4.3(iii)); (d) $\Rightarrow$ (c), in general. For other implications, we show in Example 4.9 that there is a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ with no repeatable paths whose $C^*$-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ is not AF. For this, we use the fact that the Morse sequence does not contain a block of the form...
ααα′ for a path α and its initial path α′; thus (b) ⇒ (c), in general. Furthermore
the labeled space of Example 4.9 does not satisfy the condition (a); thus (b) ⇒ (a).

To prove Theorem 4.2 we need a notion of exit of a loop in a labeled space. There
are, unlike in graph case, three possible types of exits of a loop (Definition 3.2). We
respectively).

\[ \mathbb{N} \hookrightarrow \mathbb{E} \]

maps \( v \) a vertex of a countable set of vertices \( C \). We write \( C \) to labeled spaces. From this, we define a notion of loop for labeled space. The main
results on AF labeled graph \( C^* \)-algebras are obtained in Section 4.

2. Preliminaries

2.1. Directed graphs and labeled spaces. We use notational conventions of \[15\] for graphs and graph \( C^* \)-algebras and of \[5\] for labeled spaces and their \( C^* \)-algebras. By a directed graph we mean a quadruple \( E = (E^0, E^1, r_E, s_E) \) consisting of a countable set of vertices \( E^0 \), a countable set of edges \( E^1 \), and the range, source maps \( r_E, s_E : E^1 \rightarrow E^0 \) (we often write \( r \) and \( s \) for \( r_E \) and \( s_E \), respectively). If a vertex \( v \in E^0 \) emits (receives, respectively) no edges, \( v \) is called a sink (source, respectively). \( E^0_{\text{sink}} \) denotes the set of all sinks of \( E \) and \( E^n \) denotes the set of all finite paths \( \lambda = \lambda_1 \cdots \lambda_n \) of length \( n \) \( |\lambda| = n \), \( (\lambda_i \in E^1, r(\lambda_i) = s(\lambda_{i+1}), 1 \leq i \leq n - 1) \). We write \( E^{\geq n} \) and \( E^{\geq n} \) for the sets \( \cup_{l=1}^n E^l \) and \( \cup_{l=n}^{\infty} E^l \), respectively. The maps \( r \) and \( s \) naturally extend to \( E^{\geq 0} \), where \( r(v) = s(v) = v \) for \( v \in E^0 \).

For a vertex subset \( A \subset E^0 \), \( A_{\text{sink}} \) denotes the sinks \( A \cap E^0_{\text{sink}} \) in \( A \), and for \( B \subset 2^{E^0} \) we simply denote the set \( \{ A_{\text{sink}} : A \in B \} \) by \( B_{\text{sink}} \). Also with abuse of notation, for \( B \subset 2^{E^0} \) and \( A \subset E^0 \), we write \( B \cap A := \{ B \in B : B \subset A \} \).

A labeled graph \(( E, \mathcal{L}) \) over a countable alphabet \( A \) consists of a directed graph \( E \) and a labeling map \( \mathcal{L} : E^1 \rightarrow A \). We assume that \( \mathcal{L}(E^1) = A \). Let \( A^* \) and \( A^\infty \) be the sets of all finite sequences (of length greater than or equal to 1) and infinite sequences with terms in \( A \), respectively. Then \( \mathcal{L}(\lambda) \) for \( \lambda = \lambda_1 \cdots \lambda_n \in E^n \), and \( \mathcal{L}(\delta) := L(\delta_1) \cdots \cdots \in L(E^\infty) \subset A^\infty \) for \( \delta = \delta_1 \delta_2 \cdots \in E^\infty \). We use notation \( L^*(E) := L(E^{\geq 1}) \). For \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_\|\| \in L^*_A(E) \), we denote the subsegment \( \alpha_1 \cdots \alpha_j \) of \( \alpha \) by \( [i,j] \) for \( 1 \leq i \leq j \leq |\alpha| \). A subsegment of the form \( \alpha_{[i,j]} \) is called an initial path of \( \alpha \). The range \( r(\alpha) \) and source \( s(\alpha) \) of a labeled path \( \alpha \in L^*(E) \) are subsets of \( E^0 \) defined by

\[ r(\alpha) = \{ r(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha \}, \]

\[ s(\alpha) = \{ s(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha \}. \]
The relative range of \( \alpha \in \mathcal{L}^*(E) \) with respect to \( A \subset 2^{E^0} \) is defined to be
\[
\{ r(\alpha) : \lambda \in E^\geq 1, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A \}.
\]

If \( B \subset 2^{E^0} \) is a collection of subsets of \( E^0 \) such that \( r(A, \alpha) \in B \) whenever \( A \in B \) and \( \alpha \in \mathcal{L}^*(E) \), \( B \) is said to be closed under relative ranges for \( (E, \mathcal{L}) \). We call \( B \) an accommodating set for \( (E, \mathcal{L}) \) if it is closed under relative ranges, finite intersections and unions and contains \( r(\alpha) \) for all \( \alpha \in \mathcal{L}^*(E) \). A set \( A \in B \) is called minimal (in \( B \)) if \( A \) does not have any proper subset in \( B \).

If \( B \) is accommodating for \( (E, \mathcal{L}) \), the triple \( (E, \mathcal{L}, B) \) is called a labeled space. For \( A, B, 2^{E^0} \) and \( n \geq 1 \), let
\[
AE^n = \{ \lambda \in E^n : s(\lambda) \in A \}, \quad E^nB = \{ \lambda \in E^n : r(\lambda) \in B \},
\]
and \( AE^nB = AE^n \cap E^nB \). We write \( E^nv \) for \( E^n\{v\} \) and \( vE^n \) for \( \{v\}E^n \), and will use notation like \( AE^{\geq k} \) and \( vE^\infty \) which should have their obvious meaning. For convenience we also take conventions like \( AE^0 = A \) and \( \mathcal{L}(A) = A \) for \( A \in B \), etc.

**Notation 2.1.** For \( A_i \in B, 1 \leq i \leq n, \) and \( K \geq 1 \), we will use the following notation
\[
A_1E^{\leq K}A_2 \cdots E^{\leq K}A_{n+1}
\]
for the set \( \{x_1 \cdots x_n \in E^{\geq 1} : x_i \in A_iE^{\leq K}A_{i+1}, 1 \leq i \leq n \} \) of paths in \( E^* \). To stress the fact that a path \( x = x_1 \cdots x_n \) belongs to \( A_1E^{\leq K}A_2 \cdots E^{\leq K}A_{n+1} \), we may write \( A_1x_1A_2 \cdots x_nA_{n+1} \) for \( x \).

A labeled space \( (E, \mathcal{L}, B) \) is said to be set-finite (receiver set-finite, respectively) if for every \( A \in B \) and \( l \geq 1 \) the set \( \mathcal{L}(AE^l) \) (\( \mathcal{L}(E^lA) \), respectively) is finite. A labeled space \( (E, \mathcal{L}, B) \) is weakly left-resolving if
\[
r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)
\]
holds for all \( A, B \in B \) and \( \alpha \in \mathcal{L}^*(E) \). A labeled space \( (E, \mathcal{L}) \) is left-resolving if the map \( \mathcal{L} : r^{-1}(v) \to A \) is injective for each \( v \in E^0 \), and label-finite if \( |\mathcal{L}^{-1}(a)| < \infty \) for each \( a \in A \). If \( (E, \mathcal{L}) \) is left-resolving, then it is label-finite if and only if \( r(\alpha) \) is finite for all \( a \in A \).

By \( \Omega_0(E) \) we denote the set of all vertices of \( E \) that are not sources. For \( v, w \in \Omega_0(E) \subset E^0 \), we write \( v \sim_l w \) if \( \mathcal{L}(E^{\leq l}v) = \mathcal{L}(E^{\leq l}w) \) as in [5]. Then \( \sim_l \) is an equivalence relation on the set \( \Omega_0(E) \). The equivalence class \( [v]_l \) of \( v \) is called a generalized vertex. Let \( \Omega_l(E) := \Omega_0(E)/\sim_l \) for \( l \geq 1 \). If \( k > l \), \( [v]_k \subset [v]_l \) is obvious and \( [v]_l = \bigcup_{i=1}^{m} [v_i]_{l+1} \) for some vertices \( v_1, \ldots, v_m \in [v]_l \) (see Proposition 2.4).

**Assumptions.** We assume that a labeled space \( (E, \mathcal{L}, B) \) considered in this paper always satisfies the following:
1. \( (E, \mathcal{L}, B) \) is weakly left-resolving.
2. \( (E, \mathcal{L}, B) \) is set-finite and receiver set-finite.

Also we assume that if \( v \in E^0 \) is a sink, it is not a source. A labeled space \( (E, \mathcal{L}, B) \) is said to be finite if there are only finitely many generalized vertices \( [v]_l \) for each \( l \geq 1 \), or equivalently there are only finitely many labels.
We denote by $\mathcal{E}^0$ the smallest accommodating set for $(E, \mathcal{L})$ (cf. [5] p.108);\
$$
\mathcal{E}^0 = \{ \cup_{k=1}^{n_0} \cap_{l=1}^{n_l} r(\beta_{l,k}) : \beta_{l,k} \in \mathcal{L}^*(E) \},
$$
and by $\mathcal{E}^0_{\alpha}$ the smallest accommodating set containing \{ $r(\alpha) : \alpha \in \mathcal{L}^*(E)$ \} and \{ $\{ v \} : v$ is a sink or a source \}.

If $E$ has no sinks or sources, $\mathcal{E}^0_{\alpha} = \mathcal{E}^0$ and every set in $\mathcal{E}^0$ can be expressed as a finite union of generalized vertices ([5] Remark 2.1 and Proposition 2.4.(ii));
$$
\mathcal{E}^0 \subseteq \{ \cup_{l=1}^{n} \{ v_{l} \} : v_{l} \in \Omega_0(E), \ n, l \geq 1 \}.
$$

Generalized vertices $[v]_l$ are not always members of the accommodating set $\mathcal{E}^0$ but always the relative complements of sets in $\mathcal{E}^0$, namely $[v]_l = X_l(v) \setminus r(Y_l(v))$, where $X_l(v), Y_l(v)$ are given by
$$
X_l(v) := \cap_{\alpha \in \mathcal{L}(E \leq v)} r(\alpha) \quad \text{and} \quad Y_l(v) := \cup_{w \in X_l(v)} \mathcal{L}(E \leq v) \setminus \mathcal{L}(E \leq v)
$$
so that $X_l(v), r(Y_l(v)) \in \mathcal{E}^0$ ([5] Proposition 2.4). One can easily check that the expression $[v]_l = X_l(v) \setminus r(Y_l(v))$ is valid even for a sink $v$ and $[v]_l \cap r(Y_l(v)) = \emptyset$.

The accommodating set $\mathcal{E}^0$ is not necessarily closed under relative complements. On the other hand, in the construction of the $C^*$-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ ([4] [5]) of a labeled space $(E, \mathcal{L}, \mathcal{B})$, to each nonempty set $A \in \mathcal{B}$ there is associated a nonzero projection $p_A$ in $C^*(E, \mathcal{L}, \mathcal{B})$ in such a manner that $p_A \leq p_B$ whenever $A \subset B$. Hence $p_B - p_A$ belongs to $C^*(E, \mathcal{L}, \mathcal{B})$ and it seems reasonable to write $p_B|A$ for $p_B - p_A$, which leads us to consider accommodating sets that are closed under relative complements.

**Notation 2.2.** Let $(E, \mathcal{L})$ be a labeled graph.

(i) For a labeled space $(E, \mathcal{L}, \mathcal{B})$, we denote by $\overline{\mathcal{B}}$ the smallest accommodating set that contains $\mathcal{B} \cup \mathcal{B}_{\text{sink}}$ and is closed under relative complements. The existence of $\overline{\mathcal{B}}$ clearly follows from considering the intersection of all those accommodating sets. $\overline{\mathcal{E}^0}$ will thus denote the smallest accommodating set that is closed under relative complements and contains the sets in $\overline{\mathcal{E}^0}_{\text{sink}} = \{ A_{\text{sink}} : A \in \overline{\mathcal{E}^0} \}$.

(ii) As in [1], $\mathcal{L}^*(E)$ will denote the union of all labeled paths $\mathcal{L}^*(E)$ and empty word $\epsilon$, where $\epsilon$ is a symbol such that $r(\epsilon) = E^0$, $r(A, \epsilon) = A$ for all $A \subset E^0$.

(iii) If $\mathcal{L}$ is the identity map $id : E^1 \to E^1$, it is called the trivial labeling and will be denote by $\mathcal{L}_{id}$. For a labeled graph $(E, \mathcal{L}_{id})$, the accommodating set $\overline{\mathcal{E}^0}$ is equal to the collection of all finite subsets of $E^0$.

**Proposition 2.3.** Let $(E, \mathcal{L})$ be a labeled graph and $A \in \overline{\mathcal{E}^0}$. Then $A$ is of the form
$$
A = ( \cup_{i=1}^{n_1} [v_i]_l ) \cup ( \cup_{j=1}^{n_2} ([u_j]_l)_{\text{sink}} ) \cup ( \cup_{k=1}^{n_3} [w_k]_l \setminus ([w_k]_l)_{\text{sink}} )
$$
for some $v_i, u_j, w_k \in \Omega_0(E)$ and $l \geq 1$, $n_1, n_2, n_3 \geq 0$.

**Proof.** Let $\mathcal{B}$ be the set of all such $A$’s. Then $\mathcal{B} \subset \overline{\mathcal{E}^0}$ is obvious since $\overline{\mathcal{E}^0}$ contains all generalized vertices. Now it suffices to show that $\mathcal{B}$ is an accommodating set that is closed under relative complements. By the proof of [5] Proposition 2.4], $r(\alpha) \in \mathcal{B}$
for all labeled paths \( \alpha \in \mathcal{L}^r(E) \). It is easy to see that \( \mathcal{B} \) is closed under finite unions, finite intersections and relative complements.

In order to show that \( \mathcal{B} \) is closed under relative ranges, it suffices to see that
\[
\text{for all labeled paths } \alpha \in \mathcal{L}^r(E) \text{ and } \alpha \in \mathcal{L}^r(E) \text{.}
\]
Since \( r([v]_l, \alpha) \cap r(Y_l(\alpha), \alpha) = r([v]_l \cap r(Y_l(\alpha)), \alpha) = r(\emptyset, \alpha) = \emptyset \), we have
\[
\text{which belongs to } \mathcal{B} \text{ since } r(X_l(\alpha), \alpha) \cap r(Y_l(\alpha), \alpha) \in \mathcal{E}^0 \subset \mathcal{B} \text{ and } \mathcal{B} \text{ is closed under relative complements.} \quad \square
\]

2.2. Labeled graph \( C^* \)-algebras.

**Definition 2.4.** (cf. [4 Definition 4.1] and [5 Remark 3.2]) Let \((E, \mathcal{L}, \mathcal{B})\) be a labeled space such that \( \mathcal{E}^0 \subset \mathcal{B} \). A representation of \((E, \mathcal{L}, \mathcal{B})\) consists of projections \( \{p_A : A \in \mathcal{B}\} \) and partial isometries \( \{s_a : a \in A\} \) such that for \( A, B \in \mathcal{B} \) and \( a, b \in A\),
\[
\begin{align*}
(i) & \quad p_\emptyset = 0, \quad p_Ap_B = p_\emptyset \cap B, \quad \text{and } p_A \cup B = p_A + p_B - p_A \cap B, \\
(ii) & \quad pAs_a = s_a p_{r(A,a)}, \\
(iii) & \quad s_a^*s_a = p_{r(a)} \text{ and } s_a^*s_b = 0 \text{ unless } a = b, \\
(iv) & \quad \text{for each } A \in \mathcal{B}, \\
\end{align*}
\]
\[
p_A = \sum_{a \in \mathcal{L}(AE^r)} s_a p_{r(A,a)} s_a^* + p_{\text{link}}.
\]

**Remark 2.5.** Let \((E, \mathcal{L}, \mathcal{B})\) be a labeled space such that \( \mathcal{E}^0 \subset \mathcal{B} \).

(i) The proof of [4 Theorem 4.5] shows that there exists a \( C^* \)-algebra \( C^*(E, \mathcal{L}, \mathcal{B}) \) generated by a universal representation \( \{s_a, p_A\} \) of \((E, \mathcal{L}, \mathcal{B})\); we need to modify the proof slightly, namely we should mod out the \( * \)-algebra \( k_{(E, \mathcal{L}, \mathcal{B})} \) by the ideal \( J \) generated by the elements \( q_{A \cup B} - q_A - q_B + q_A \cap B \) and \( q_A - \sum a \in \mathcal{L}(AE^r) s_a p_{r(A,a)} s_a^* - q_{\text{link}} \) for \( A, B \in \mathcal{B} \). If \( \{s_a, p_A\} \) is a universal representation of \((E, \mathcal{L}, \mathcal{B})\), we simply write \( C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A) \) and call \( C^*(E, \mathcal{L}, \mathcal{B}) \) the labeled graph \( C^* \)-algebra of a labeled space \((E, \mathcal{L}, \mathcal{B})\).

Note that \( s_a \neq 0 \) and \( p_A \neq 0 \) for \( a \in A \) and \( A \in \mathcal{B} \), \( A \neq \emptyset \), and that \( s_a p_A \neq \emptyset \) if and only if \( A \cap r(\alpha) \cap r(\beta) \neq \emptyset \). By Definition [2.4(iv)] and [4] Lemma 4.4] saying that with \( s_\alpha := p_\alpha \) for \( \alpha \in \mathcal{B} \),
\[
(s_\alpha p_A s_\beta^*)(s_\alpha p_B s_\delta^*) = \begin{cases} 
    s_\alpha \gamma' p_{r(A,\gamma') \cap B} s_\delta^*, & \text{if } \gamma = \beta' \\
    s_\alpha p_{A \cap (B,\beta')} s_\delta^*, & \text{if } \beta = \gamma' \\
    s_\alpha p_A \cap B s_\delta^*, & \text{if } \beta = \gamma \\
    0, & \text{otherwise,}
\end{cases}
\]
for \( \alpha, \beta, \gamma, \delta \in \mathcal{L}^r(E) \) and \( A, B \in \mathcal{B} \), it follows that
\[
C^*(E, \mathcal{L}, \mathcal{B}) = \text{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^r(E), A \in \mathcal{B}\},
\]
where \( s_\delta \) denotes the unit of the multiplier algebra of \( C^*(E, \mathcal{L}, \mathcal{B}) \) [4]. It is observed in [12] that if \( E \) has no sinks nor sources, then \( C^*(E, \mathcal{L}, \mathcal{E}^0) \cong C^*(E, \mathcal{L}, \overline{\mathcal{E}^0}) \).
(We wrote $\mathcal{E}$ for $\overline{\mathcal{E}}$ in the paper [12].)

(ii) Universal property of $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ defines a strongly continuous action $\gamma: \mathbb{T} \to \text{Aut}(C^*(E, \mathcal{L}, \mathcal{B}))$, called the gauge action, such that

$$\gamma_z(s_a) = zs_a \quad \text{and} \quad \gamma_z(p_A) = p_A$$

for $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$.

(iii) From Definition 2.4(iv), we have for each $n \geq 1$,

$$p_A = \sum_{\alpha \in \mathcal{L}(AE^n)} s_{\alpha}p_{r(A,\alpha)}s_{\alpha}^* + \sum_{\gamma \in \mathcal{L}(AE^{n-1})} s_{\gamma}p_{r(A,\gamma)}s_{\gamma}^*,$$

where $\sum_{\gamma \in \mathcal{L}(AE^n)} s_{\gamma}p_{r(A,\gamma)}s_{\gamma}^* := p_{A_{\text{sink}}}$. In fact,

$$p_A = \sum_{a \in \mathcal{L}(AE^1)} s_{a}p_{r(A,a)}s_{a}^* + p_{A_{\text{sink}}}$$

$$= \sum_{a \in \mathcal{L}(AE^1)} s_{a} \left( \sum_{b \in \mathcal{L}(r(A,a)E^1)} s_{b}p_{r(A,ab)}s_{b}^* + p_{r(A,a)_{\text{sink}}} \right) s_{a}^* + p_{A_{\text{sink}}}$$

$$= \sum_{\gamma \in \mathcal{L}(AE^2)} s_{\gamma}p_{r(A,\gamma)}s_{\gamma}^* + \sum_{a \in \mathcal{L}(AE^1)} s_{a}p_{r(A,a)_{\text{sink}}}s_{a}^* + p_{A_{\text{sink}}}$$

$$= \sum_{\gamma \in \mathcal{L}(AE^2)} s_{\gamma} \left( \sum_{c \in \mathcal{L}(r(A,ac)E^1)} s_{c}p_{r(A,ac)}s_{c}^* + p_{r(A,\gamma)_{\text{sink}}} \right) s_{\gamma}^* + \sum_{a \in \mathcal{L}(AE^1)} s_{a}p_{r(A,a)_{\text{sink}}}s_{a}^* + p_{A_{\text{sink}}}$$

$$= \sum_{\alpha \in \mathcal{L}(AE^2)} s_{\alpha}p_{r(A,a)}s_{\alpha}^* + \sum_{\gamma \in \mathcal{L}(AE^1)} s_{\gamma}p_{r(A,\gamma)_{\text{sink}}}s_{\gamma}^* + \sum_{a \in \mathcal{L}(AE^1)} s_{a}p_{r(A,a)_{\text{sink}}}s_{a}^* + p_{A_{\text{sink}}}$$

$$= \ldots$$

$$= \sum_{\alpha \in \mathcal{L}(AE^n)} s_{\alpha}p_{r(A,a)}s_{\alpha}^* + \sum_{\gamma \in \mathcal{L}(AE^{n-1})} s_{\gamma}p_{r(A,\gamma)_{\text{sink}}}s_{\gamma}^*.$$

(iv) For $B \in \overline{\mathcal{E}}^0$, one can easily show that the ideal $I_B$ of $C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0) = C^*(s_a, p_A)$ generated by the projection $p_B$ is equal to

$$I_B = \text{span}\{s_{\alpha}p_{A}^*s_{\beta}^* : \alpha, \beta \in \mathcal{L}^\#(E), A \in \overline{\mathcal{E}}^0 \cap r(\mathcal{L}(BE^0)) \}, \quad (2)$$

where $r(\mathcal{L}(BE^0)) := B$.

3. Loops in labeled spaces

3.1. Loops in directed graphs. Recall that a path $x \in E^{\geq 1}$ in a directed graph $E$ is called a loop (or a directed cycle) if $s(x) = r(x)$, that is, if it comes back to its source vertex. It is well known [13] Theorem 2.4] that for a graph $C^*$-algebra $C^*(E)$ to be AF it is a sufficient and necessary condition that $E$ has no loops.
Since the accommodating set $\overline{E^0}$ of a labeled graph $(E, \mathcal{L}_{id})$ with the trivial labeling $\mathcal{L}_{id}$ contains all the single vertex sets $\{v\}$, $v \in E^0$, the following are equivalent for a path $x = x_1 \cdots x_m \in E^{\geq 1}(= \mathcal{L}^*_w(E))$:

(i) $x$ is a loop in $E$,
(ii) $\{r(x)\} = r(\{r(x)\}, x)$,
(iii) $x$ is repeatable, that is, $x^n \in E^{\geq 1}$ for all $n \geq 1$,
(iv) $(A_1x_1A_2x_2 \cdots A_mx_m)^n(A_1x_1A_2x_2 \cdots A_i) \in \mathcal{L}^*_w(E)$ for all $n \geq 1$ and $1 \leq i \leq m$, where $A_i = \{s(x_i)\} \in \overline{E^0}$.

(See Notation 2.1 for the meaning of $A_1x_1A_2x_2 \cdots A_m$.)

From this we can obtain several equivalent conditions for a graph $C^*$-algebra $C^*(E)$ to be AF as follows.

**Proposition 3.1.** Let $(E, \mathcal{L}_{id}, \overline{E^0})$ be a labeled space with the trivial labeling $\mathcal{L}_{id}$ so that $C^*(E, \mathcal{L}_{id}, \overline{E^0}) \cong C^*(E)$. Then the following are equivalent:

(i) $C^*(E, \mathcal{L}_{id}, \overline{E^0})$ is AF,
(ii) $E$ has no loops,
(iii) $A \not\subset r(A, x)$ for all $A \in \overline{E^0}$ and $x \in \mathcal{L}^*_w(E)$,
(iv) there are no repeatable paths in $\mathcal{L}^*_w(E)$,
(v) if $\{A_1, \ldots, A_m\}$ is a finite collection of sets from $\overline{E^0}$ and $K \geq 1$, there is an $m_0 \geq 1$ such that $A_{i_1}E^{\leq K}A_{i_2} \cdots E^{\leq K}A_{i_{n+1}} = \emptyset$ for all $n > m_0$.

**Proof.** We only prove that (ii) and (v) are equivalent since the equivalence of (i) and (ii) is well known and the other implications are rather obvious. Suppose $x$ is a loop in $E$, then with $A = \{s(x)\} \in \overline{E^0}$ and $K := |x| \geq 1$, it is immediate that (v) does not hold. For the converse, suppose that (v) does not hold and so there are finitely many sets $A_1, \ldots, A_m$ in $\overline{E^0}$ and $K \geq 1$ such that $A_{i_1}E^{\leq K}A_{i_2} \cdots E^{\leq K}A_{i_{n+1}} \neq \emptyset$ for all $n \geq 1$. Since every set in $\overline{E^0}$ is finite, the number of vertices in $\bigcup_{i=1}^m A_i$ is also finite. Choose an integer $N$ with $|\bigcup_{i=1}^m A_i| < N$. Then for any path in $A_{i_1}E^{K}A_{i_2} \cdots E^{K}A_{i_{N+1}}(\neq \emptyset)$, there is a vertex in $\bigcup_{i=1}^m A_i$ the path passes through at least two times, which proves the existence of a loop (at that vertex) in $E$. \qed

### 3.2. Loops in labeled spaces.

If $(E, \mathcal{L}_{id}, \overline{E^0})$ is a labeled space with the trivial labeling, by Proposition 3.1 the condition that there is a set $A \in \overline{E^0}$ satisfying $A \subset r(A, x)$ for a path $x$ (in fact, $A = \{s(x)\} = \{r(x)\}$) is equivalent to the existence of a loop in $E$. This equivalent condition can be extended to a labeled space as follows.

**Definition 3.2.** Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $\alpha \in \mathcal{L}^*(E)$ be a labeled path.

(a) $\alpha$ is called a generalized loop at $A \in \mathcal{B}$ if $\alpha \in \mathcal{L}(AE^{\geq 1}A)$.
(b) $\alpha$ is called a loop at $A \in \mathcal{B}$ if it is a generalized loop such that $A \subset r(A, \alpha)$.
(c) A loop $\alpha$ at $A \in \mathcal{B}$ has an exit if one of the following holds:

(i) $\{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\} \subseteq \mathcal{L}(AE^{\leq |\alpha|})$,
(ii) $r(A, \alpha_{[1,i]})_{\text{sink}} \neq \emptyset$ for some $i = 1, \ldots, |\alpha|$,
(iii) $A \subset r(A, \alpha)$. 


Note that every loop $\alpha$ is \textit{repeatable}, that is, $\alpha^n \in \mathcal{L}^*(E)$ for all $n \geq 1$ (Definition 6.6), and every repeatable path is a generalized loop at its range. Not every repeatable path is a loop as we can see in Example 4.3(iii).

**Remark 3.3.** Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $A \in \mathcal{B}$.

(i) A generalized loop $\alpha$ at a minimal set $A \in \mathcal{B}$ is always a loop because $A \subset r(A, \alpha)$ follows from the minimality of $A$ since $\emptyset \neq A \cap r(A, \alpha) \subset A$. A labeled graph $(E, \mathcal{L})$ might have a loop $\alpha$ even when the graph $E$ itself has no loops at all as we will see in Example 4.3(i) and (ii).

(ii) If $\alpha$ is a loop at $A$, then evidently $p_A \leq p_{r(A, \alpha)}$.

**Example 3.4.** We give examples of labeled graphs with a loop that has an exit.

(i) The loop $\alpha := b_1b_2$ at $A := \{v\} \in \overline{E^0}$ has an exit of type (i) of Definition 3.2(c) because $\{\alpha_{[1,k]} : 1 \leq k \leq 2\} = \{b_1, b_1b_2\}$ while $\mathcal{L}(AE \leq [\alpha]) = \{b_1, b_1b_2, b_1c\}$.

(ii) Let $A := \{v, w\}$. Since $A = r(A, b)$, $b$ is a loop at $A$ and has an exit of type (ii) of Definition 3.2(c); $r(A, b)_{\text{sink}} = \{w\} \neq \emptyset$.

(iii) The loop $\alpha := bc$ at $A := \{v\} \in \overline{E^0}$ has an exit of type (iii) of Definition 3.2(c) because $A \subset r(A, \alpha)$.

The following proposition is an extended version of the fact that if a directed graph $E$ has a loop with an exit, its graph $C^*$-algebra has an infinite projection.

**Proposition 3.5.** Let $(E, \mathcal{L})$ be a labeled graph and $\alpha$ be a loop at $A \in \overline{E^0}$ with an exit. Then $p_A$ is an infinite projection in $C^*(E, \mathcal{L}, \overline{E^0})$.

**Proof.** If $A \subset r(A, \alpha)$, the projection $p_{r(A, \alpha)}$ is infinite because

$$p_{r(A, \alpha)} > p_A \geq s_\alpha p_{r(A, \alpha)} s^*_\alpha \sim p_{r(A, \alpha)}.$$
If either $\mathcal{L}(AE^{\leq |\alpha|}) \supseteq \{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\}$ or $r(A, \alpha_{[1,i]})_{\text{sink}} \neq \emptyset$ for some $i$, $1 \leq i \leq |\alpha|$, by Remark 2.5(iii) we have

$$p_A = \sum_{\beta \in \mathcal{L}(AE^{\leq |\alpha|})} s_\beta p_{r(A,\beta)} s_\beta^* + \sum_{|\gamma| \leq |\alpha|-1} s_{\gamma} p_{r(A,\gamma)_{\text{sink}}} s_\gamma^* \geq s_{\alpha} p_{r(A,\alpha)} s_{\alpha}^*.$$ 

Thus $p_A > s_{\alpha} p_{r(A,\alpha)} s_{\alpha}^* \sim p_{r(A,\alpha)} \geq p_A$ and we see that the projection $p_{r(A,\alpha)}$ (hence $p_A$) is infinite. Now it remains to prove the assertion in case $r(A,\alpha)_{\text{sink}} \neq \emptyset$ and $A = r(A,\alpha)$. The set $A_0 := A \setminus A_{\text{sink}} (\neq \emptyset)$ then satisfies $A_0 \subseteq A = r(A,\alpha) = r(A_0,\alpha)$, and by the first argument of the proof $p_{A_0}$ is infinite. Hence $p_A (\geq p_{A_0})$ is infinite. 

**Remark 3.6.** A generalized version of Proposition 3.5 is also true: Let $(E, \mathcal{L})$ be a labeled graph and $\alpha_1, \ldots, \alpha_n$ be distinct labeled paths with the same length, say $l \geq 1$, such that $A \subseteq \bigcup_{i=1}^n r(A,\alpha_i)$. Then $p_A$ is an infinite projection in $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ if one of the following holds:

(i) $\bigcup_{i=1}^n \{\alpha_i' : \alpha_i' \text{ is an initial path of } \alpha_i\} \subseteq \mathcal{L}(AE^{\leq l})$

(ii) $r(A,\alpha_i')_{\text{sink}} \neq \emptyset$ for some $i$ and an initial path $\alpha_i'$ of $\alpha_i$

(iii) $A \subseteq \bigcup_{i=1}^n r(A,\alpha_i)$.

To prove this, first assume the case (iii) and set $A_1 := r(A,\alpha_1)$ and $A_i := r(A,\alpha_i) \setminus \bigcup_{j=1}^{i-1} r(A,\alpha_j)$, $i = 2, \ldots, n$, so that $\bigcup_{i=1}^n r(A,\alpha_i) = \bigcup_{i=1}^n A_i$ is the union of disjoint sets $A_i$’s. Then we have

$$p_A \geq \sum_{i=1}^n s_{\alpha_i} p_{r(A,\alpha_i)} s_{\alpha_i}^* \geq \sum_{i=1}^n s_{\alpha_i} p_{\alpha_i} s_{\alpha_i}^* \sim \sum_{i=1}^n p_{\alpha_i} = p_{\bigcup_{i=1}^n r(A,\alpha_i)} \geq p_A$$

and so the projection $p_A$ is infinite, where the equivalence is given by the partial isometry $u := \sum_{i=1}^n s_{\alpha_i} p_{\alpha_i}$. It is not hard to see that the same argument in the proof of Proposition 3.5 shows the assertion for the rest cases.

**Proposition 3.7.** Let $(E, \mathcal{L}, \overline{\mathcal{E}^0})$ be a labeled space such that $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no infinite projections. Let $A \in \overline{\mathcal{E}^0}$ admit a loop. Then there exists a loop $\alpha$ at $A$ such that $A = r(A,\alpha)$ and $\mathcal{L}(AE^{\geq 1}) = \{\alpha^k \alpha' : k \geq 0, \alpha' \text{ is an initial path of } \alpha\}$.

**Proof.** Choose a loop $\alpha$ at $A$ with the smallest length; $|\alpha| \leq |\gamma|$ for all loops $\gamma$ at $A$. Since $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no infinite projections, $\alpha$ does not have an exit by Proposition 3.5 hence $A = r(A,\alpha)$ and

$$\mathcal{L}(AE^{\leq |\alpha|}) = \{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\}. \quad (3)$$

Now let $\beta \in \mathcal{L}(AE^{\geq 1})$ be a path with $|\beta| > |\alpha|$. Then by (3), $\mathcal{L}(AE^{\leq |\alpha|}) = \{\alpha\}$ and so we can write $\beta = \alpha \beta'$ for a path $\beta'$. But then from $A = r(A,\alpha)$, $\beta'$ must be either an initial path of $\alpha$ or of the form $\alpha \beta''$ for some path $\beta''$. Applying the argument repeatedly, we finally end up with $\beta = \alpha^k \alpha'$ for some $k \geq 1$ and an initial path $\alpha'$ of $\alpha$. 

\[\square\]
4. AF Labeled Graph $C^*$-Algebras

Remark 4.1. We will consider the following properties (a)-(d) of a labeled space $(E, \mathcal{L}, \overline{E_0})$ and its $C^*$-algebra $C^*(E, \mathcal{L}, \overline{E_0})$. These properties are equivalent if $\mathcal{L}$ is the trivial labeling $\mathcal{L}_{id}$ as we have seen in Proposition 3.1.1

(a) For every finite set \( \{A_1, \ldots, A_N\} \) of $\overline{E_0}$ and every $K \geq 1$, there exists an $m_0 \geq 1$ such that $A_{i_1}E^{\leq K}A_{i_2} \cdots E^{\leq K}A_{i_n} = \emptyset$ for all $n > m_0$ and $A_{i_j} \in \{A_1, \ldots, A_N\}$.

(b) $(E, \mathcal{L}, \overline{E_0})$ has no repeatable paths.

(c) $C^*(E, \mathcal{L}, \overline{E_0})$ is an AF algebra.

(d) $(E, \mathcal{L}, \overline{E_0})$ has no loops (in the sense of Definition 3.2).

Note that (a) $\Rightarrow$ (b) follows from a simple observation that if $\alpha$ is a repeatable path, then with $A := r(\alpha)$ one has $A_{i_1}E^{[\alpha]}A_{i_2} \cdots E^{[\alpha]}A_{i_n} \neq \emptyset$ for all $n \geq 1$, where $A_{i_j} = A$. $j = 1, \ldots, n$. The implication (b) $\Rightarrow$ (d) is obvious.

For the other implications, we shall see (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c), in general throughout Example 4.9. Consequently (d) $\Rightarrow$ (c) follows although it can also be seen from Example 4.3(iii). We will show that (c) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) hold true in Theorem 4.11 and Theorem 4.8, respectively.

It would be interesting to know whether the remaining implication (c) $\Rightarrow$ (b) is true, that is, whether $C^*(E, \mathcal{L}, \overline{E_0})$ will never be AF whenever $(E, \mathcal{L}, \overline{E_0})$ contains a repeatable path. In Theorem 4.11, we obtain a partial affirmative answer.

Theorem 4.2. Let $(E, \mathcal{L})$ be a labeled graph. If $C^*(E, \mathcal{L}, \overline{E_0})$ is an AF algebra, the labeled space $(E, \mathcal{L}, \overline{E_0})$ has no loops.

Proof. Suppose, for contradiction, that $(E, \mathcal{L}, \overline{E_0})$ has a loop $\alpha$ at $A \in \overline{E_0}$. By Proposition 3.5 $A = r(A, \alpha)$ and so $p_AS_\alpha = s_\alpha D_r(A, \alpha) = s_\alpha p_A$. Then $U := s_\alpha p_A$ satisfies

$$p_A = U^*U \sim UU^* = s_\alpha p_A s_\alpha = s_\alpha D_r(A, \alpha) s_\alpha \leq p_A.$$

Since $p_A$ is a finite projection, it follows that $U$ is a unitary of the unital hereditary subalgebra $p_A C^*(E, \mathcal{L}, \overline{E_0}) p_A$. Since $\gamma_z(p_A) = p_A$ for any $z \in \mathbb{T}$, the algebra $p_A C^*(E, \mathcal{L}, \overline{E_0}) p_A$ admits an action of $\mathbb{T}$ which is the restriction of the gauge action $\gamma$ on $C^*(E, \mathcal{L}, \overline{E_0})$. Then the fact that $\gamma_z(U) = \gamma_z(s_\alpha)p_A = z^{[\alpha]}U$ shows that $U$ is not in the unitary path connected component of the unit $p_A$ (Proposition 3.9), which is a contradiction to the assumption that $C^*(E, \mathcal{L}, \overline{E_0})$ (hence any nonzero hereditary subalgebra) is an AF algebra.

In Example 4.3(iii) below, we see that the converse of Theorem 4.2 may not be true, in general.

Example 4.3. (i) For the following labeled graph $(E, \mathcal{L})$

$$
\begin{array}{cccccccccc}
\cdots & \bullet & \overset{a}{\rightarrow} & \bullet & \overset{a}{\rightarrow} & \bullet & \overset{a}{\rightarrow} & \bullet & \overset{a}{\rightarrow} & \bullet & \cdots \\
& v_{-2} & & v_{-1} & & v_0 & & v_1 & & v_2 & \\
\end{array}
$$
we have $\overline{E^0} = \{r(a)\} = \{E^0\}$ and the path $a$ is a loop at $r(a)$. By Theorem 4.2, $C^*(E, L, \overline{E^0}) := C^*(s_a, p_A)$ is not AF. Actually $C^*(E, L, \overline{E^0}) \cong C(\mathbb{T})$ is the universal $C^*$-algebra generated by the unitary $s_a$.

(ii) $\overline{E^0}$ of the following labeled graph consists of three sets $r(a) = E^0$, $r(a)_{\text{sink}} = \{v_0\}$, and $A := r(a) \setminus r(a)_{\text{sink}} = \{v_{-1}, v_{-2}, \ldots\}$.

\[
\begin{array}{cccccc}
& a & a & a & a & a \\
v_{-4} & \cdot & \cdot & v_{-3} & v_{-2} & v_{-1} & v_0 \\
\end{array}
\]

Since $A \not\subset r(A, a)$, the loop $a$ at $A$ has an exit and $C^*(E, L, \overline{E^0})$ contains an infinite projection by Proposition 3.5.

(iii) If $(E, L)$ is as follows

\[
\begin{array}{cccccc}
& a & a & a & a & a \\
v_0 & v_1 & v_2 & v_3 & v_4 & \cdots \\
\end{array}
\]

it is not hard to see that $\overline{E^0}$ consists of all finite sets $F$ with $v_0 \not\in F$ and all sets of the form $F \cup \{v_k, v_{k+1}, \ldots\}$ for some $k \geq 1$. It is also easy to see that every $A \in \overline{E^0}$ containing at least two vertices always admits a generalized loop. But there does not exist a loop at any $A \in \overline{E^0}$. Nevertheless we shall see that $C^*(E, L, \overline{E^0})$ contains an infinite projection and so the $C^*$-algebra is not AF. Let $C^*(E, L, \overline{E^0}) = C^*(p_A, s_a)$. Then for $B := r(a)$, we have $r(B, a) \not\subset B$, and a similar argument as in (ii) shows that the projection $p_B$ is infinite; $p_{r(a)} = s_a p_{r(r(a), a)} s_a = s_a p_{r(a)^2} s_a \sim p_{r(a^2)} < p_{r(a)}$.

The $C^*$-algebra $C^*(p_A, s_a)$ is unital with the unit $s_a^* s_a$: $(s_a^* s_a)p_A = s_a p_{r(A, a)} s_a^* = p_A$ for all $A \in \overline{E^0}$ and $(s_a^* s_a^*) s_a = s_a = s_a p_{r(a)} = s_a p_{r(a)} (s_a s_a^*) = s_a (s_a s_a^*)$. Also we have $s_a s_a^* \geq p_{r(a)} = s_a^* s_a$ since $s_a^* s_a \geq s_a p_{r(v_1)} s_a^* (\neq 0)$ and $(s_a p_{r(v_1)} s_a^*) p_A = s_a p_{r(v_1)} p_{r(A, a)} s_a^* = s_a p_{r(v_1)} (r(A, a) s_a^* = 0$ because $\{v_1\} \cap r(A, a) = \emptyset$ for all $A \in \overline{E^0}$. Moreover every projection $p_A$ belongs to the $\ast$-algebra generated by $s_a$. Therefore $C^*(E, L, \overline{E^0})$ is the universal $C^*$-algebra generated by a proper coisometry $s_a$, and thus $C^*(E, L, \overline{E^0})$ is the Toeplitz algebra. The ideal $I_{\{v_1\}}$ generated by the projection $p_{\{v_1\}}$ is in fact isomorphic to the the $C^*$-algebra of compact operators on an infinite dimensional separable Hilbert space as $I_{\{v_1\}} = \text{span}\{s_a^n p_{\{v_1\}} (s_a^*)^m : m, n \geq 0$ and $i \geq 1\}$ (see (2)). The quotient algebra $C^*(E, L, \overline{E^0})/I_{\{v_1\}}$ is isomorphic to $C(\mathbb{T})$.

For a labeled graph $(E, L_E)$, $v \sim w$ if and only if $v \sim_l w$ for all $l \geq 1$ defines an equivalence relation on $E^0$. We denote the equivalence class of $v \in E^0$ by $[v]_\infty$. If $(E, L_E)$ has no sinks or sources, there exists a labeled graph $(F, L_F)$ called the merged labeled graph of $(E, L_E)$ with vertices $F^0 := \{[v]_\infty : v \in E^0\}$ and edges $F^1 := \{e_\lambda : \lambda \in E^1\}$, where $e_\lambda$ is a path with $s_F(e_\lambda) = [s(\lambda)]_\infty$, $r_F(e_\lambda) = [r(\lambda)]_\infty$, and $L_F(e_\lambda) = L_E(\lambda)$. The range of $a \in L_F(F^1)$ is defined by $r_F(a) = \{r_F(e_\lambda) : L_F(e_\lambda) = a\}$. Here we use notation $r_F$ to denote both the range map of paths
in \( F^* \) and of labeled paths in \( L_F^*(F) \). It is known in [12, Theorem 6.10] that if \( [v]_\infty \in \mathcal{E}_0 \) for all \( v \in E^0 \), then \( \{[v]_\infty\} \in \mathcal{F}_0 \) for all \( [v]_\infty \in F^0 \) and moreover \( C^*(E, \mathcal{L}_E, \mathcal{E}_0) \cong C^*(F, \mathcal{L}_F, \mathcal{E}_0) \). Even when \((E, \mathcal{L}_E)\) has sinks or sources, we can obtain \( C^*(E, \mathcal{L}_E, \mathcal{E}_0) \cong C^*(F, \mathcal{L}_F, \mathcal{E}_0) \) whenever \( [v]_\infty \in \mathcal{E}_0 \) for all \( v \in E^0 \) without significant modification of the proof of [12, Theorem 6.10].

The following proposition is a slightly generalized version of the result well known for graph \( C^* \)-algebras. Actually in case \( \mathcal{L} \) is the trivial labeling, \( C^*(E, \mathcal{L}, \mathcal{E}_0) = C^*(E) \) and the minimal sets in \( \mathcal{E}_0 \) are the single vertex sets \( \{v\} \), \( v \in E^0 \).

**Proposition 4.4.** Let \((E, \mathcal{L})\) be a row-finite labeled graph with no sinks or sources such that every generalized vertex is a finite union of minimal sets in \( \mathcal{E}_0 \). Then \( C^*(E, \mathcal{L}, \mathcal{E}_0) \) is AF if and only if no minimal set of \( \mathcal{E}_0 \) admits a loop.

**Proof.** Let \((F, \mathcal{L}_F)\) be the merged labeled graph of \((E, \mathcal{L})\). We first show that \( C^*(E, \mathcal{L}, \mathcal{E}_0) \) is isomorphic to the graph \( C^* \)-algebra \( C^*(F) \).

Our assumption implies \( [v]_\infty \in \mathcal{E}_0 \) for all \( v \in E^0 \), so \( \{[v]_\infty\} \in \mathcal{F}_0 \) for all \( v \in E^0 \) and \( C^*(E, \mathcal{L}, \mathcal{E}_0) \) is isomorphic to \( C^*(F, \mathcal{L}_F, \mathcal{E}_0) \) ([12, Theorem 6.10]). For each \( a \in \mathcal{L}(E^1) \), its range \( r(a) \) can be written as the union \( r(a) = \bigcup_{i=1}^{n} [w_i]_{\infty} \) of finitely many minimal sets \( [w_i]_{\infty} \) by the assumption, but minimality of each \( [w_i]_{\infty} \) implies that \( [w_i]_{\infty} = [w_i]_\infty \) for \( 1 \leq i \leq n \). Hence \( r_F(a) = [r(a)]_\infty := \{[w]_\infty : w \in r(a)\} = \{[w_1]_\infty, \ldots, [w_n]_\infty\} \) is finite for each \( a \in \mathcal{L} \). But from the construction ([12, Definition 6.1]), the merged labeled graph \((F, \mathcal{L}_F)\) is left-resolving. Thus the finiteness of each range set \( r_F(a) \) implies that \((F, \mathcal{L}_F)\) is label-finite. Then by [4, Theorem 6.6], we have \( C^*(F, \mathcal{L}_F, \mathcal{E}_0) \cong C^*(F) \).

Suppose that there is no loop at any minimal set \( [v]_\infty \) in \( \mathcal{E}_0 \). Since \( \mathcal{L}_E([v]_\infty E^k v') = \mathcal{L}_F([v]_\infty F^k [v]_\infty) \) for all \( v' \in [v]_\infty \) and \( k \geq 1 \) ([12, Lemma 6.7]), if \( F \) has a loop \( \alpha \) at a vertex \( [v]_\infty \in E^0 \), \( \alpha \in \mathcal{L}_E([v]_\infty E^k v') \) for all \( v' \in [v]_\infty \). This means that \( [v]_\infty \in \mathcal{E}_0 \) satisfies \( [v]_\infty \subset r([v]_\infty, \alpha) \), a contradiction. Hence \( F \) has no loops and the \( C^* \)-algebra \( C^*(F) \) is AF. The converse was proved in Theorem 4.2. \( \square \)

**Example 4.5.** In the following labeled graph \((E, \mathcal{L})\)

\[
\begin{array}{cccccccc}
  \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \cdot & b & c & \cdot & \cdot \cdot & \cdot & \cdot \cdot & \cdot \\
  \cdot & u_3 & u_2 & u_1 & u_0 & \cdot & \cdot & \cdot \\
  v_3 & v_2 & v_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
  v & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
  \cdot & b & c & b & c & \cdot & \cdot & \cdot \\
  \cdot & u_{-3} & u_{-2} & u_{-1} & u_0 & \cdot & \cdot & \cdot \\
  \cdot & v_{-3} & v_{-2} & v_{-1} & v_0 & \cdot & \cdot & \cdot \\
\end{array}
\]

the path \( \alpha := a^2 \) is a loop at \( \{v_{2k} : k \in \mathbb{Z}\} \) and also at \( \{v_{2k+1} : k \in \mathbb{Z}\} \). By Theorem 4.2, the \( C^* \)-algebra \( C^*(E, \mathcal{L}, \mathcal{E}_0) \) is not AF. In fact, \( C^*(E, \mathcal{L}, \mathcal{E}_0) \) is isomorphic to the graph algebra \( C^*(F) \), where \( F \) is the underlying graph of the merged labeled graph \((F, \mathcal{L}_F)\) of \((E, \mathcal{L})\) by Proposition 4.4.
Example 4.6. The following labeled graph \((E, \mathcal{L})\) does not have any infinite paths, but it has a repeatable path a.

\[
\begin{align*}
\bullet & \quad a \quad \bullet \\
\bullet & \quad a \quad \bullet \\
\bullet & \quad a \quad \bullet \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]

Note that each finite path \(a^n\) is not a loop at any \(A \in \overline{\mathcal{E}}^0\) but it is a generalized loop at \(r(a^k)\) for all \(k \geq 1\). \((E, \mathcal{L}, \overline{\mathcal{E}}^0)\) has the generalized vertices as follows:

\[
[v_{ij}]_k = \begin{cases} 
\{ r(a^k), & \text{if } 1 \leq k \leq j \\
\{ r(a^j) \setminus r(a^{j+1}), & \text{if } 1 \leq j < k 
\end{cases}
\]

\[
([v_{ij}]_k)_{\text{sink}} = \begin{cases} 
\{ v_{mn} : m \geq k \}, & \text{if } 1 \leq k \leq j \\
\{ v_{jj} \}, & \text{if } 1 \leq j < k 
\end{cases}
\]

\[
[v_{ij}]_k \setminus ([v_{ij}]_k)_{\text{sink}} = \begin{cases} 
\{ v_{mn} : m > n \geq k \}, & \text{if } 1 \leq k \leq j \\
\{ v_{mj} : m \geq j \}, & \text{if } 1 \leq j < k 
\end{cases}
\]

and every \(A \in \overline{\mathcal{E}}^0\) is a finite union of these sets.

Let \(J\) be the ideal of \(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0) = C^*(p_A, s_a)\) generated by the projection \(p_{[v_{11}]}\). Then (2) shows that

\[
J = \text{span}\{ s_a^mp_Bs_a^n : B \in [v_{kk}]_{k+1} \cap \overline{\mathcal{E}}^0, \ m, n \geq 0, \ k \geq 1 \}.
\]

From \(p_{r(a)} - p_{r(a^2)} = p_{r(a)\setminus r(a^2)} = p_{[v_{11}]} \in J\), we have

\[
s_a + J = s_ap_{r(a)} + J = p_{r(a)}s_a + J.
\]

Thus \(s_ap_{r(a)} + J\) is a unitary of the unital hereditary subalgebra (with unit \(p_{r(a)} + J\)) of the quotient algebra \(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0)/J\). The ideal \(J\) is obviously invariant under the gauge action \(\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0))\). Hence there exists an induced action \(\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0)/J)\) such that \(\gamma_z(s_ap_{r(a)} + J) = z(s_ap_{r(a)} + J)\) for \(z \in \mathbb{T}\). Thus the unitary \(s_ap_{r(a)} + K\) does not belong to the unitary path connected component of the unit of the hereditary subalgebra of \(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0)/J\), which implies as in the proof of Theorem 4.2 that \(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0)/J\) and hence \(C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0)\) is not AF.
Notation 4.7. If \( x_i \in A_i E \leq K A_{i+1} \) is a path with \( \alpha_i = \mathcal{L}(x_i) \) for \( i = 1, \ldots, n \) such that \( x_1 \cdots x_n \in A_1 E \leq K \cdots E \leq K A_{n+1} \), then we set
\[
\bar{r}(A_1 \alpha_1 A_2) := r(A_1, \alpha_1) \cap A_2 \\
\bar{r}(A_1 \alpha_1 A_2 A_3) := r(\bar{r}(A_1 \alpha_1 A_2), \alpha_2) \cap A_3 = r(r(A_1, \alpha_1) \cap A_2, \alpha_2) \cap A_3,
\]
and so on, thus for \( 3 \leq i \leq n + 1, \)
\[
\bar{r}(A_1 \alpha_1 A_2 \cdots \alpha_{i-1} A_i) := r(\bar{r}(A_1 \alpha_1 A_2 \cdots A_{i-1}), \alpha_{i-1}) \cap A_i.
\]
Note that \( \bar{r}(A_1 \alpha_1 A_2 \cdots \alpha_{i-1} A_i) \) belongs to \( \overline{\mathcal{E}}^0 \) whenever \( A_j \notin \overline{\mathcal{E}}^0 \) for \( 1 \leq j \leq i \). The notation \( \bar{r}(A_1 E \leq K A_2 \cdots E \leq K A_{n+1}) \) will then be used for the collection of all sets \( \bar{r}(A_1 \alpha_1 A_2 \cdots \alpha_{n-1} A_{n+1}) \) for \( \alpha_1 \cdots \alpha_n \in \mathcal{L}(A_1 E \leq K A_2 \cdots E \leq K A_{n+1}) \).

Theorem 4.8. Let \( (E, \mathcal{L}, \overline{\mathcal{E}}^0) \) be a labeled space such that for every finite subset \( \{A_1, \ldots, A_N\} \) of \( \overline{\mathcal{E}}^0 \) and every \( K \geq 1 \), there exists an \( m_0 \geq 1 \) for which
\[
A_{i_1} E \leq K A_{i_2} E \leq K A_{i_3} \cdots E \leq K A_{i_n} = \emptyset
\]
for all \( n > m_0 \) and \( 1 \leq i_j \leq N \). Then \( C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0) \) is an AF algebra.

Proof. Let \( F := \{s_{\alpha_i} P A_i s_{\beta_i}^* : A_i \subset r(\alpha_i) \cap r(\beta_i), i = 1, \ldots, N\} \) be a finite set in the \( C^* \)-algebra \( C^*(E, \mathcal{L}, \overline{\mathcal{E}}^0) = C^*(s_{\alpha_i} P A_i) \) with \( F = F^* \). We shall show that \( F \) generates a finite dimensional \( C^* \)-algebra. Set \( K := \max\{|\alpha_i|, |\beta_i| : i = 1, \ldots, N\} \).

By Remark 2.5(i), we have
\[
(s_{\alpha_i} P A_i s_{\beta_i}^*)(s_{\alpha_j} P A_j s_{\beta_j}^*) =
\begin{cases}
  s_{\alpha_i} \gamma' P r(A_i, \gamma') \cap A_j s_{\beta_j}^*, & \text{if } \alpha_j = \beta_j \gamma' \\
  s_{\alpha_i} P A_i \cap r(A_j, \beta_j) s_{\beta_j}^*, & \text{if } \beta_j = \alpha_j \beta' \\
  s_{\alpha_i} P A_i \cap A_j s_{\beta_j}^*, & \text{if } \alpha_i = \alpha_j \\
  0, & \text{otherwise,}
\end{cases}
\]
and so if, for example, \( \alpha_j = \beta_j \gamma' \) and \( \alpha_k = \beta_j \gamma'' \), we get
\[
(s_{\alpha_i} P A_i s_{\beta_i}^*)(s_{\alpha_j} P A_j s_{\beta_j}^*)(s_{\alpha_k} P A_k s_{\beta_k}^*) = (s_{\alpha_i} \gamma' P r(A_i, \gamma') \cap A_j s_{\beta_j}^*)(s_{\alpha_k} P A_k s_{\beta_k}^*) = s_{\alpha_i} \gamma' P r(A_i, \gamma') \cap A_j s_{\beta_j}^* \cap A_k s_{\beta_k}^*.
\]
Here note that \( \gamma' \gamma'' \) belongs to \( \mathcal{L}(A_i E^{[\gamma']} A_j E^{[\gamma'']} A_k) \) and the set \( r(r(A_i, \gamma') \cap A_j, \gamma'') \cap A_k \) is equal to \( \bar{r}(A_i \gamma'A_j, \gamma'' A_k) \). Continuing a similar computation once more, for example with \( \beta_k = \alpha_l \beta' \), we have
\[
(s_{\alpha_i} P A_i s_{\beta_i}^*)(s_{\alpha_j} P A_j s_{\beta_j}^*)(s_{\alpha_k} P A_k s_{\beta_k}^*)(s_{\alpha_l} P A_l s_{\beta_l}^*)
= (s_{\alpha_i} \gamma' P r(A_i, \gamma') \cap A_j s_{\beta_j}^*)(s_{\alpha_l} P A_l s_{\beta_l}^*)
= s_{\alpha_i} \gamma' P r(A_i, \gamma') \cap A_j \cap r(A_l, \beta') s_{\beta_j}^* \beta_l \beta'.
\]
which is nonzero only when \( \gamma' \gamma'' \in \mathcal{L}(A_i E^{[\gamma']} A_j E^{[\gamma'']} A_k) \) and \( \beta' \in \mathcal{L}(A_i E^{[\beta']} A_k) \). If this is the case, we have
\[
s_{\alpha_i} \gamma' \gamma'' P r(A_i, \gamma') \cap A_j \cap r(A_l, \beta') s_{\beta_j}^* \beta_l \beta' = s_{\alpha_i} \gamma' \gamma'' P r(A_i, \gamma' A_j, \gamma'' A_k) \cap r(A_l, \beta' A_k) s_{\beta_j}^* \beta_l \beta'.
\]
as before. Repeating the process of multiplying any finite elements from the set $F$ actually produces an element of the form $s_{\alpha, \mu} p A^* s_{\beta, \nu}$, where $A$ is a finite intersection of sets in 

$$A(F) := \bigcup_{n \geq 1} \bigcap_{1 \leq i, j \leq N} \tilde{r}(A_{i_1} E \leq K A_{i_2} \cdots E \leq K A_{i_n})$$

and $\mu$ and $\nu$ are paths in 

$$\mathcal{L}(F) := \bigcup_{n \geq 1} \mathcal{L}(A_{i_1} E \leq K A_{i_2} \cdots E \leq K A_{i_n}).$$

By our assumption, we find an $m_0 \geq 1$ such that $\mathcal{L}(A_{i_1} E \leq K A_{i_2} \cdots E \leq K A_{i_n}) = \emptyset$ for all $n > m_0$, so that $\mathcal{L}(F)$ turns out to be a finite set since our labeled space is always assumed receiver set-finite. Then the finiteness of the set $A(F)$ is immediate, and so we conclude that $F$ generates the finite dimensional $\ast$-algebra;

$$\text{span}\{s_{\alpha, \mu} p A^* s_{\beta, \nu} : A = \cap B_k, \; B_k \in A(F), \; \mu, \nu \in \mathcal{L}(F), \; 1 \leq i, j \leq N \}.$$ 

\[\square\]

In the following example, we see that the condition that $(E, \mathcal{L}, \overline{E^0})$ has no repeatable paths is not a sufficient condition for $C^*(E, \mathcal{L}, \overline{E^0})$ to be AF.

**Example 4.9.** Consider the following labeled graph $(E, \mathcal{L})$:

\[
\begin{array}{cccccccc}
\cdots & 0 & \bullet & 1 & \bullet & 0 & x_0 = 0 & x_1 = 1 \\
v_4 & v_3 & v_2 & v_1 & v_0 & v_2 & v_3 & v_4
\end{array}
\]

where the \{0, 1\} sequence is the Morse sequence

$$x = \cdots x_{-2} x_{-1} x_0 x_1 x_2 \cdots$$

given by $x_0 = 0$, $x_1 = 0 := 1$ ($\bar{1} := 0$), $x_{[0,3]} := x_0 x_1 x_0 x_1 = 0110$, $x_{[0,7]} := x_0 x_1 x_2 x_3 x_0 x_1 x_2 x_3 = 01101001$ and so on, and then $x_{-i} := x_{i-1}$ for $i \geq 1$. It is known that $x$ contains no block (no finite subsequence) of the form $\beta \beta \beta_1$ for $\beta = \beta_1 \ldots \beta_{[3]} \in \mathcal{L}^*(E)$. (See [10] for the Morse sequence.) Thus $(E, \mathcal{L}, \overline{E^0})$ has no repeatable paths satisfying (b) in Remark 4.1. But, the set $A := r(0)$, with $K := 2$, satisfies 

$$A_{i_1} E \leq A_{i_2} \cdots E \leq A_{i_n} \neq \emptyset$$

for all $n \geq 1$, where $A_{i_j} = A$, $j \geq 1$. This is because the block 111 does not appear in the sequence $x$. Thus $(E, \mathcal{L}, \overline{E^0})$ does not meet the condition (a) in Remark 4.1. To see that $C^*(E, \mathcal{L}, \overline{E^0})$ (equivalently, $M_2 \otimes C^*(E, \mathcal{L}, \overline{E^0})$) is not AF, it is enough to show that $M_2 \otimes C^*(E, \mathcal{L}, \overline{E^0})$ contains a unitary $U$ such that $(id_{M_2} \otimes \gamma)_{t}(U) = zU$ for all $z \in \mathbb{T}$, where $\gamma$ is the gauge action of $\mathbb{T}$ on $C^*(E, \mathcal{L}, \overline{E^0}) = C^*(s_i, p_A)$ ([8 Proposition 3.9]). Actually one can easily check that the unitary $U = (u_{ij})$, with entries $u_{ij} = \delta_{ij} s_0 + (1 - \delta_{ij}) s_1$, is a desired one.
Now we prove a partial result about the implication (c) ⇒ (b) of Remark 4.1.
For a C^*-algebra C^*(E, L, E̅) = C^*(s_α, p_A) and a set A ∈ E̅, we denote by I_A the ideal of C^*(E, L, E̅) generated by the projection p_A as before.

**Lemma 4.10.** Let C^*(E, L, E̅) = C^*(s_α, p_A) be the C^*-algebra of a labeled graph (E, L) with no sinks or sources. For A, B ∈ E̅, we have p_A ∈ I_B if and only if there exist n ≥ 1 and finitely many paths \{μ_i\}_{i=1}^n in L(BE̅^0) such that
\[
\bigcup_{|β|=n} r(A, β) ⊂ \bigcup_{i=1}^n r(B, μ_i).
\]

**Proof.** If p_A ∈ I_B, we can approximate p_A, within a small enough ε > 0, by an element \(\sum_{i=1}^n c_is_βp_{B_i}r(B, μ_i)s_β^*\) of I_B, where c_i ∈ C, β_i, γ_i ∈ L(AE̅^0), B_i ∈ E̅, and μ_i ∈ L(BE̅^0) for 1 ≤ i ≤ n (see (3)). We assume (β_i, μ_i, γ_i) ≠ (β_j, μ_j, γ_j) if i ≠ j. Considering the image of \(X := p_A - \sum_{i=1}^n c_is_βp_{B_i}r(B, μ_i)s_β^*\), under the conditional expectation onto the AF core (the fixed point algebra of the gauge action), we may assume that |β_i| = |γ_i|, 1 ≤ i ≤ n, since p_A is in the core. Moreover, since (E, L) has no sinks, we can also assume that |β_i| = |β_1| for all i. Put N := |β_1|, 1 ≤ i ≤ n. From \(p_A = \sum_{|β|=N} s_βp_{r(A, β)}s_β^*\), we have
\[
\|X\| = \left\| \sum_{|β|=N} s_βp_{r(A, β)}s_β^* - \sum_{i=1}^n c_is_βp_{B_i}r(B, μ_i)s_β^* \right\| < ε.
\]
If r(A, β) ∉ \(\bigcup_{i=1}^n r(B, μ_i)\) for some β ∈ L(AE̅^N), then, A' := r(A, β) \(\bigcup_{i=1}^n r(B, μ_i)\) ≠ ∅, one obtains a contradiction, ε > \(\|p_{A'}(s_β^*Xs_β)p_{A'}\| = \|p_{A'}\| = 1\).

For the reverse inclusion, it is enough to note that \(p_{\bigcup_{i=1}^n r(B, μ_i)} \in I_B\) (see [12, Lemma 3.5]).

If α is a repeatable path in a directed graph E, then α is a loop with the range r(α) consisting of a single vertex and every repetition α^n also has the same range as α, r(α^m) = r(α), m ≥ 1. The projection p_r(α)r(α^m) is then equal to 0 in the C^*-algebra C^*(E, L_id, E̅), and so the (zero) ideal generated by the projection p_r(α)r(α^m) can not have the nonzero projection p_r(α). In this case, we already know that C^*(E) = C^*(E, L_id, E̅) is not AF.

**Theorem 4.11.** Let C^*(E, L, E̅) = C^*(s_α, p_A) be the C^*-algebra of a labeled graph (E, L) with no sinks or sources. Let C^*(E, L, E̅) have a repeatable path α ∈ L^*(E). If p_r(α^m) does not belong to the ideal generated by a projection p_r(α^m), then for some m ≥ 1, C^*(E, L, E̅) is not AF.

**Proof.** Let \(A_m := r(α^m) \setminus r(α^{m+1})\). Then \(\{I_{A_m}\}_{m=1}^∞\) is a decreasing sequence of ideals because the generator \(p_{r(α^{m+2})}\) of \(I_{A_{m+1}}\) belongs to \(I_{A_m}\);
\[
p_{r(α^{m+1})\setminus r(α^{m+2})} = s_α^*s_αp_{r(α^{m+1})\setminus r(α^{m+2}), α} = s_α^*p_{r(α^{m+1})\setminus r(α^{m+2})}s_α ∈ I_{A_m}.
\]
We first show the following claim.

**Claim:** If p_r(α) does not belong to the ideal generated by p_r(α)r(α^2), then the C^*-algebra C^*(E, L, E̅) is not AF.
To prove the claim, it is enough to show that the quotient algebra \( C^*(E, \mathcal{L}, \overline{E^0})/I_{A_1} \) is not AF. Note that \( p_{r(a)} + I_{A_1} = p_{r(a^2)} + I_{A_1} \) is a nonzero projection in the quotient algebra \( C^*(E, \mathcal{L}, \overline{E^0})/I_{A_1} \) and that
\[
I_{A_1} = \text{span}\{ s_\beta p_B s_\gamma^*: \beta, \gamma \in \mathcal{L}(E^{\leq 0}) \text{ and } B \in r(\mathcal{L}(A_1E^{\leq 0})) \cap \overline{E^0} \}
\]
by (2). If \( s_\alpha^* s_\alpha + I_{A_1} = s_\alpha^* s_\alpha + I_{A_1} \), the hereditary subalgebra of \( C^*(E, \mathcal{L}, \overline{E^0})/I_{A_1} \) with the unit projection \( p_{r(a)} + I_{A_1} \) is not AF since it contains a unitary \( s_\alpha + I_{A_1} \), satisfying \( \gamma_z(s_\alpha + I_{A_1}) = z^{[a]}(s_\alpha + I_{A_1}) \) for each \( z \in \mathbb{C} \). Thus the hereditary subalgebra (hence \( C^*(E, \mathcal{L}, \overline{E^0}) \)) is not an AF algebra. (The fact that \( s_\alpha + I_{A_1} \) belongs to the hereditary subalgebra follows from \( p_{r(a)} s_\alpha + I_{A_1} = s_\alpha p_{r(a)} + I_{A_1} = s_\alpha + I_{A_1} \). If \( s_\alpha^* s_\alpha + I_{A_1} \neq s_\alpha^* s_\alpha + I_{A_1} \), then \( s_\alpha^* s_\alpha + I_{A_1} = p_{r(a)} + I_{A_1} \geq s_\alpha p_{r(a)} s_\alpha^* + I_{A_1} = s_\alpha s_\alpha^* + I_{A_1} \), and this shows that \( s_\alpha^* s_\alpha + I_{A_1} \) is finite, and the quotient algebra is not AF as claimed.

Now suppose that \( p_{r(a^m)} \notin I_{A_m} \) for some \( m \geq 2 \). Since \( \delta := \alpha^m \) is a repeatable path, by the above claim, we only need to show that \( p_{r(\delta)} \) does not belong to the ideal, say \( J \), generated by the projection \( p_{r(\delta)} \cap \mathcal{L}(r(\delta^2)) = p_{r(a^m)} \cap \mathcal{L}(r(a^{2m})) \). For this, assuming \( p_{r(\delta)} \in J \) we have from Lemma 4.10 that there exist an \( N \geq 1 \) and paths \( \{\mu_j\}^N_{j=1} \) such that
\[
 r(r(\delta), \beta) \subset \bigcup_{i=1}^N r(r(\alpha^m), \mu_i) \cap r(\alpha^{2m})
\]
for all \( \beta \in \mathcal{L}(r(\delta)E^N) \). Since each set \( r(r(\alpha^m) \cap r(\alpha^{2m}), \mu_i) \) coincides with
\[
 \bigcup_{j=0}^{m-1} r(r(\alpha^m+j), \mu_i) = \bigcup_{j=0}^{m-1} r(r(\alpha^m) \cap r(\alpha^{m+j}), \mu_i),
\]
we can write the set \( \bigcup_{i=1}^N r(r(\alpha^m) \cap r(\alpha^{2m}), \mu_i) \) as \( \bigcup_{j=1}^N r(r(\alpha^m) \cap r(\alpha^{m+j}), \mu_i) \) for some finitely many paths \( \mu_j \) which is of the form \( \alpha^j \mu_i \). This means that \( p_{r(a^m)} = p_{r(\delta)} \in I_{A_m} \) again by Lemma 4.10 which is a contradiction.

Assume that \( E \) is a graph with no sinks or sources. Recall from [4] Definition 5.1 and [12] Proposition 3.9 that a labeled space \( (E, \mathcal{L}, \overline{E^0}) \) is disagreeable if \( [v]_1 \) is disagreeable for all \( v \in E^0 \) and \( l \geq 1 \); a generalized vertex \([v]_l \) is not disagreeable if and only if there is an \( N > 0 \) such that every path \( \alpha \in \mathcal{L}([v]_l E^{\geq N}) \) is agreeable, namely is of the form \( \alpha = \beta^k \gamma^l \) for some \( k \geq 0 \) and some paths \( \beta, \gamma \in \mathcal{L}(E^{\leq l}) \), where \( \beta \) is an initial path of \( \beta \). In case of trivial labeling, \( (E, \mathcal{L}_{id}, \overline{E^0}) \) is disagreeable exactly when the graph \( E \) satisfies condition (L) [4] Lemma 5.3. Condition (L) meaning that every loop has an exit was introduced as an essential hypothesis for the Cuntz-Krieger uniqueness theorem for graph \( C^*-\)algebras in [15]. More generally, in [4] Theorem 5.5 it is known that if \( (E, \mathcal{L}, \overline{E^0}) \) is disagreeable, Cuntz-Krieger uniqueness Theorem holds, that is, if \( \{S_a, P_A\} \) is a representation of a labeled space \( (E, \mathcal{L}, \overline{E^0}) \) such that \( S_a \neq 0 \) and \( P_A \neq 0 \) for all \( a \in A \) and \( A \in \overline{E^0} \), the \( C^*-\)algebra \( C^*(S_a, P_A) \) generated by \( \{S_a, P_A\} \) is isomorphic to \( C^*(E, \mathcal{L}, \overline{E^0}) \).

As pointed out in [5], a disagreeable labeled space \( (E, \mathcal{L}, \overline{E^0}) \) contains lots of aperiodic paths and in fact, as can be seen in the following proposition, \( (E, \mathcal{L}, \overline{E^0}) \) is disagreeable whenever it has no repeatable paths.
Proposition 4.12. Let $E$ be a directed graph with no sinks or sources. If the labeled space $(E, L, E^0)$ has no repeatable paths, it is always disagreeable.

Proof. Assuming that $(E, L, E^0)$ is not disagreeable, one can pick a generalized vertex $[v]_l$ that is not disagreeable. Then there is an $N > 0$ such that every $\alpha$ in $L([v]_l E_{\geq N})$ is agreeable and of the form $\alpha = \beta k \beta'$ for some $\beta \in L([v]_l E_{\leq l})$ and its initial path $\beta'$. On the other hand, there are only finitely many labeled paths in $L([v]_l E_{\leq l})$ while $L([v]_l E_{\geq N})$ has infinitely many labeled paths. This shows that there should exist a path $\beta$ in $L([v]_l E_{\leq l})$ such that its repetitions $\beta^n$ appear in $L([v]_l E_{\geq N})$ for all sufficiently large $n$. □

One might expect that a labeled space to be disagreeable if it has no loops, but this is not necessarily true; see the following example.

Example 4.13. Consider the following labeled graph $(E, L)$:

```
\[
\cdots \quad v_{-3} \quad -3 \quad v_{-2} \quad -2 \quad v_{-1} \quad -1 \quad v_0 \quad 0 \quad v_1 \quad 0 \quad v_2 \quad 0 \quad v_3 \quad \cdots
\]
```

Then $E^0$ is the collection of all finite sets $F$ of $E^0$ and sets of the form $F \cup \{v_n, v_{n+1}, \ldots\}$, $n \geq 1$. For the generalized vertex $\{v_0\} = [v_0]_1$, every path $\alpha \in L(v_0 E_{\geq N})$ is agreeable since it must be equal to $a^m$ for some $m \geq N$, so the labeled space $(E, L, E^0)$ is not disagreeable, whereas it is obvious that $(E, L, E^0)$ has no loops.

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