The decision problem for normed spaces over any class of ordered fields

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Abstract

It is known that the theory of any class of normed spaces over \( \mathbb{R} \) that includes all spaces of a given dimension \( d \geq 2 \) is undecidable, and indeed, admits a relative interpretation of second-order arithmetic. The notion of a normed space makes sense over any ordered field of scalars, but such a strong undecidability result cannot hold in the more general case. Nonetheless, we find that the theory of any class of normed spaces in the more general sense that includes all spaces of a given dimension \( d \geq 2 \) over some ordered field admits a relative interpretation of Robinson’s theory \( Q \) and hence is undecidable.

Let \( \mathcal{L}_N \) be the natural two-sorted language for a normed space over an arbitrary ordered field of scalars: \( \mathcal{L}_N \) has sorts \( \mathcal{K} \) for the scalars and \( \mathcal{V} \) for the vectors together with the usual symbols of the appropriate sorts for a vector space over an ordered field equipped with a norm (see [2] for more details; we have adopted \( \mathcal{K} \) instead of \( \mathcal{R} \) for the scalar sort here). If \( K \) is an ordered field, a normed space over \( K \) is a structure for \( \mathcal{L}_N \) in which \( \mathcal{K} \) and the field symbols are interpreted in \( K \) and which satisfies the usual first-order axioms for a normed space. If \( \mathcal{C} \) is any non-empty class of ordered fields, let \( \text{NS}(\mathcal{C}) \) be the set of all sentences in \( \mathcal{L}_N \) that are valid in all normed spaces whose field of scalars belongs to \( \mathcal{C} \); let \( \text{NS}^n(\mathcal{C}) \) for \( n \in \mathbb{N} \), \( \text{NS}^\infty(\mathcal{C}) \), and \( \text{NS}^\infty(\mathcal{C}) \) denote the extensions of the theory \( \text{NS}(\mathcal{C}) \) comprising the

*Inspired by joint work with Robert M. Solovay and John Harrison.
sentences valid in all normed spaces whose field of scalars belongs to $C$ and that are, respectively, $n$-dimensional, finite-dimensional and infinite-dimensional; let $\text{NS}_n^+(C)$, $\text{NS}_F^+(C)$ and $\text{NS}_\infty^+(C)$ be the corresponding theories in the purely additive sublanguage $L_n^+$, i.e., the sublanguage in which scalar-scalar and scalar-vector multiplication are disallowed (although we may still use multiplication by rational constants as a shorthand).

[2] deals with the case $C = \{\mathbb{R}\}$ and shows that with the exception of the 1-dimensional case (which reduces trivially to the first-order theory of $\mathbb{R}$), all of the theories mentioned in the previous paragraph are undecidable with this choice of $C$ in the strong sense that they admit a relative interpretation of second-order arithmetic. In general, $C$ may be definable by a recursive set of axioms (the class of real closed fields is an example). In this case, $\text{NS}(C)$ is recursively axiomatizable, implying that we cannot interpret second-order arithmetic in it. Nonetheless, we shall see that for any non-empty $C$, even the purely additive theory $\text{NS}_+^+(C)$ is undecidable, as are all of $\text{NS}_n^+(C)$ for $1 < n \in \mathbb{N}$, $\text{NS}_F^+(C)$ and $\text{NS}_\infty^+(C)$.

We will use the classical method of proving that a theory $T$ is undecidable by giving a relative interpretation in $T$ of Raphael M. Robinson’s finitely axiomatizable and essentially undecidable theory $Q$. Recall, e.g., from [3], that $Q$ is the theory in the language $L_{PA}$ of Peano arithmetic comprising the deductive closure of the following axioms:

\begin{align*}
Q1: & \quad \forall x \ y \cdot S(x) = S(y) \Rightarrow x = y \\
Q2: & \quad \forall x \cdot 0 \neq S(x) \\
Q3: & \quad \forall x \cdot x \neq 0 \Rightarrow \exists y \cdot x = S(y) \\
Q4: & \quad \forall x \cdot x + 0 = x \\
Q5: & \quad \forall x \ y \cdot x + S(y) = S(x + y) \\
Q6: & \quad \forall x \cdot x \times 0 = 0 \\
Q7: & \quad \forall x \ y \cdot x \times S(y) = x \times y + x
\end{align*}

In $L_{PA}^+$, the intended interpretation of $K$ is as an ordered group $K$ with a distinguished positive element 1. If $K$ is such a group, let us write $\mathbb{N}_K$ for the semiring of natural numbers considered as a subset of $K$ by identifying $n$ with $\sum_{i=1}^n 1$. We then have the following theorem:

**Theorem 1** Let $L$ be a (many-sorted) first-order language including a sort $K$, together with a function symbol $+ : K \times K \to K$, a binary predicate symbol $< \in K$ and a constant symbol $1 : K$ whose intended interpretations are as some ordered abelian group with 1 as a distinguished positive element. Let $C$ be some class of structures for $L$, in which $K$ and these symbols have their intended
interpretations and let $T$ be the theory of $C$, i.e., the set of all sentences of $L$ valid in every member of $C$. Let $\mu(x, y, z)$ be a formula of $L$ with the indicated free variables all of sort $K$. Let $M$ be a structure in the class $C$, in which $K$ is interpreted as some ordered group $K$. and assume that in $M$, $\mu(x, y, z)$ defines the graph of multiplication in $\mathbb{N}_K$. Then $T$ is undecidable.

**Proof:** Define $\nu(x) := \mu(x, 0, 0)$, so that in $M$, $\nu(x)$ holds iff $x \in \mathbb{N}_K$. Define a translation $\phi \mapsto \phi^*$ from sentences of $L_{PA}$ to sentences of $L$, where $\phi^*$ is obtained from $\phi$ by the following sequence of transformations: (i) label all constants and variables with the sort $K$; (ii) unnest occurrences of $\times$ so that $\times$ only occurs in formulas of the form $z = x \times y$ where $x$, $y$ and $z$ do not involve $\times$; (iii) relativise with respect to $\nu$, i.e., replace all subformulas of the form $\forall x \cdot \psi$ (resp. $\exists x \cdot \psi$) by $\forall x \cdot \nu(x) \Rightarrow \psi$ (resp. $\exists x \cdot \nu(x) \land \psi$); (iv) replace all subterms of the form $S(x)$ by $x + 1$; and (v) replace subformulas of the form $z = x \times y$ by $\mu(x, y, z)$.

Define a sentence $OK$ of $L$ as follows:

\[
OK := \nu(0) \land \\
(\forall x \cdot \nu(x) \Rightarrow x \geq 0 \land \nu(x + 1)) \land \\
(\forall x \cdot \nu(x) \land x > 0 \Rightarrow \nu(x - 1)) \land \\
(\forall x \cdot y \cdot \nu(x) \land \nu(y) \Rightarrow \exists z \cdot \nu(x, y, z)) \land \\
(\forall x \cdot y \cdot z \cdot \mu(x, y, z) \Rightarrow \nu(x) \land \nu(y) \land \nu(z)) \land \\
(\forall x \cdot y \cdot z \cdot \mu(x, y, z + 1) \land \mu(x, y, z) \Rightarrow w = z + x)
\]

and write $OK_i$ for the $i$-th conjunct in $OK$.

It is easy to verify that the translations $Q1^* \ldots Q7^*$ of the axioms of $Q$ all hold in any normed space over any ordered field in which $OK$ holds. For example, $Q6^*$ is equivalent to the tautology $\forall x \cdot \nu(x) \Rightarrow \nu(x)$ and $Q3^*$ holds because if $x \neq 0$ and $\nu(x)$ holds, then, by $OK_2$, $x \geq 0$, whence $x > 0$, so that $\nu(x - 1)$ holds by $OK_3$, so that $(\exists y \cdot x = S(y))^*$ holds with $x - 1$ as witness. We have constructed a relative interpretation of the essentially undecidable and finitely axiomatizable theory $Q$ in the theory, $T_1$ say, obtained by adding the finite set of axioms $\{OK\}$ to $T$. Clearly $M$ is a model of $T_1$, so $T_1$ is consistent. It follows from Theorems I.8 and I.10 of [3] that $T$ is undecidable.

**Example:** The following is based on an idea of John Harrison. If $K$ is any ordered field, define a *metric space over $K$* to be a set $X$ equipped with a function $d : X \times X \to K$ satisfying the usual axioms for a metric space. Now assume that $K$ is a euclidean field, i.e., an ordered field such that every positive element has a square root, so that the vector space $K^n$ admits the euclidean norm $\|(x_1, \ldots, x_n)\|_e = \sqrt{x_1^2 + \ldots + x_n^2}$ and becomes a metric space over $K$ under $d_e(x, y) = \|y - x\|_e$. 

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Let $V = X \cup Y \subseteq K^2$ where $X = \{(x, y)|x \in \mathbb{N}_K \land y = x^2\}$ and $Y = \{(x, y)|x \in \mathbb{N}_K \land y = x^2\}$. In $K^n$ just as in $\mathbb{R}^n$, a point $q$ lies on the line segment $[p, r]$ iff $d_\mathbb{C}(p, r) = d_\mathbb{C}(p, q) + d_\mathbb{C}(q, r)$. Using this fact, it is not difficult to give a first-order formula $\phi(x, y)$ in the natural two-sorted language $L_M$ for metric spaces with the indicated free variables of sort $K$ that holds in $V$ iff $x \in \mathbb{N}_K$ and $y = x^2$ (Design $\phi(x, y)$ to assert that $x = d(0, x)$ and $y = d(x, y)$ where $y \in Y$ and $x$ is the point of $X$ nearest to $y$. Cf. the proof of Theorem 5 in [2].) Now if we put $\mu(x, y, z) \equiv \exists a \ b \ c \cdot \phi(x, a) \land \phi(y, b) \land \phi(x + y, c) \land a + b + 2z = c$, then $\mu(x, y, z)$ defines the graph of multiplication in $\mathbb{N}_K$. Applying the theorem, we obtain the undecidability of the theory $MS(C)$ of metric spaces over any class $C$ of ordered fields that includes at least one euclidean field. (In fact, $\phi$ can be defined without using multiplication, so the additive theories $MS_{+}(C)$ are also undecidable.)

We will give a construction inspired by the proof of theorem 41 in [2], where we found normed spaces over $\mathbb{R}$ in which there are definable consecutive pairs of line segments inscribed in the unit circle whose lengths are in the ratio $1 : m$ for $m$ in the set $\mathbb{N}_{>1}$ of natural numbers greater than 1. Now, working over an arbitrary ordered field $K$, we will construct normed spaces $\mathbb{J}^d$ in which there are definable consecutive quadruples of line segments inscribed in the unit circle $S$ whose lengths are in the ratio $1 : m : mn : n$ for $m, n \in \mathbb{N}_{>1}$. Thus for positive $r \in K$ if one of the corresponding quadruples $(x_1, x_2, x_3, x_4)$ in the circle $rS$ has $x_1 = 1$ then $x_2$ and $x_4$ are in $\mathbb{N}_K$ and $x_3 = x_2x_4$; moreover, for any $x_2, x_4 \in \mathbb{N}_K$, such a quadruple exists for some $r > 0$. This will allow us to apply theorem [1] to conclude the undecidability of any class of normed spaces that includes $\mathbb{J}^d$.

The proof of theorem 41 in [2] was based on the convergence of the power series for the exponential function. We need a replacement for this series that has consecutive quadruples of terms in the ratios $1 : m : mn : n$ for all $m, n \in \mathbb{N}_{>1}$. The following lemma gives us this.

**Lemma 2** There are $p, q, m_i, n_i \in \mathbb{N}_{>1}$ and $a, a_k \in \mathbb{Q}_{>0}$, $i = 0, 1, \ldots, k = 1, 2, \ldots$ satisfying the following conditions:

(i) the function $(m, n) \mapsto p^mq^n$ is an injection of $\mathbb{N}_{>1} \times \mathbb{N}_{>1}$ into $\mathbb{N}_{>1}$;

(ii) the pairs $(m_i, n_i)$, $i = 0, 1, \ldots$ enumerate $\mathbb{N}_{>1} \times \mathbb{N}_{>1}$;

(iii) $p^{m_0}q^{n_0} < p^{m_1}q^{n_1} < p^{m_2}q^{n_2} < \ldots$;
(iv) for \(k = 1, 2, \ldots\),

\[
a_k := \begin{cases} 
p^{-m_i}q^{-n_i} & \text{if } k = 4i + 1 \\
m_i p^{-m_i}q^{-n_i} & \text{if } k = 4i + 2 \\
m_i n_i p^{-m_i}q^{-n_i} & \text{if } k = 4i + 3 \\
n_i p^{-m_i}q^{-n_i} & \text{if } k = 4i + 4; 
\end{cases}
\]

(v) the sum \(\sum_{k=1}^\infty a_k\) converges to \(a < 1\).

**Proof:** If \(p, q \in \mathbb{N}_{>1}\) are coprime then (i) certainly holds, in which case (ii), (iii) and (iv) uniquely determine the \(m_i, n_i\) and \(a_k\). I claim that for all sufficiently large rational \(p\) and \(q\), the sum in (v) converges to a rational limit \(a < 1\). We may then take \(p, q \in \mathbb{N}_{>1}\) large and coprime to complete the proof.

To prove the claim, apply standard facts about series of non-negative terms to show that the sum \(a = \sum_{k=1}^\infty a_k\) converges for \(p, q > 1\) and may be rearranged as follows:

\[
a = \sum_{i=0}^\infty (p^{-m_i}q^{-n_i} + m_i p^{-m_i}q^{-n_i} + m_i n_i p^{-m_i}q^{-n_i} + n_i p^{-m_i}q^{-n_i})
\]

\[
= \left(\sum_{i=0}^\infty p^{-m_i}q^{-n_i} + \sum_{i=0}^\infty m_i p^{-m_i}q^{-n_i} + \sum_{i=0}^\infty n_i p^{-m_i}q^{-n_i}\right)
\]

\[
= \left(\sum_{m=2}^\infty p^{-m}\sum_{n=2}^\infty q^{-n} + \sum_{m=2}^\infty mp^{-m}\sum_{n=2}^\infty q^{-n}\right)
\]

\[
= f(p)f(q) + g(p)f(q) + g(p)g(q) + f(p)g(q)
\]

where, using the formulas \(\sum_{i=0}^\infty x^i = (1-x)^{-1}\) and \(\sum_{i=1}^\infty ix^{i-1} = (1-x)^{-2}\) for the sums of a geometric series and its derivative, we have \(f(r) := (1-r^{-1})^{-1} - 1 - r^{-1}\) and \(g(r) := r^{-1}(1-r^{-1})^{-2} - 1\). Thus \(\sum_{k=1}^\infty a_k\) exists and is a rational function, \(a(p, q)\) say, with rational coefficients, of the numbers \(p\) and \(q\) that tends to \(0\) as \(p+q\) tends to \(\infty\). Hence for all sufficiently large rational \(p\) and \(q\), the sum \(\sum_{k=1}^\infty a_k\) converges to a rational \(a = a(p, q) < 1\). (In fact, for \(p, q \in \mathbb{N}_{>1}\), \(a(p, q) < 1\) unless \(p = q = 2\) or \(\{p, q\} = \{2, 3\}\).)

We plan to encode the sequence \(a_1, a_2, \ldots\) as the lengths of line segments \([v_1, v_2], [v_2, v_3], \ldots\) inscribed in the unit circle of a 2-dimensional normed space \(\mathbb{J}\). So that the construction works over an arbitrary field of scalars, we will arrange
for the $v_k$ to have rational coordinates with respect to a basis $e_1, e_2$. We will also arrange for the norm of the vectors $v_{k+1} - v_k$ on $\mathbb{J}$ to agree with the 1-norm with respect to this basis. The following lemma will give us gradients for the vectors $v_{k+1} - v_k$ that let the $v_k$ fit conveniently in the unit circle of $\mathbb{J}$.

**Lemma 3** Let $a_1, a_2, \ldots \in \mathbb{Q}_{>0}$ be as in lemma 2. Then there are $b_1, b_2, \ldots \in \mathbb{Q}_{>0}$ such that $1 > b_1 > b_2 > \ldots$ and $b = \sum_{k=1}^{\infty} b_k a_k \in \mathbb{Q}_{>0}$.

**Proof:** Let $p$, $q$, etc. be as in lemma 2. As a first approximation to the $b_k$, let $b'_{i+j} = p^{-m_i} q^{-n_i}$ for $i = 0, 1, \ldots, j = 1, 2, 3, 4$. Then $b = \sum_{k=1}^{\infty} b'_{i+k} a_k = a(p^2, q^2) \in \mathbb{Q}_{>0}$ (where the rational function $a(r, s)$ is as in the proof of lemma 2).

Now the sequence $b_1, b_2, \ldots$ is not strictly decreasing, but we can derive a suitable strictly decreasing sequence $b_1, b_2, \ldots$ from it. We do this in stages: at the $i$-th stage we construct a block of 4 new values $b_{4i+1} > b_{4i+2} > b_{4i+3} > b_{4i+4}$ such that $\sum_{j=1}^{4} b_{4i+j} a_{4i+j} = \sum_{j=1}^{4} b'_{4i+j} a_{4i+j}$ given a strict upper bound $U_i$ for $b_{4i+1}$ and a strict lower bound $L_i$ for $b_{4i+4}$. In stage 0, $U_0 = 1$. Thereafter $U_i$ is the value $b_{4(i-1)+4}$ that ends the block constructed in the previous stage. We will arrange for $U_i > p^{-m_i} q^{-n_i}$ (this is true for $i = 0$ since $p^{-m_0} q^{-n_0} = (pq)^{-1} < 1$).

$L_i$ is $p^{-m_{i+1}} q^{-n_{i+1}}$ at the $i$-th stage for every $i$. In the $i$-th stage, for $j = 1, 2, 3$, define $c_j(\delta) = b'_{4i+j} + (4-j)\delta = p^{-m_i} q^{-n_i} + (4-j)\delta$ and define $c_4(\delta)$ to be the rational function of $\delta$ that makes the following hold:

$$\sum_{j=1}^{4} c_j(\delta) a_{4i+j} = \sum_{j=1}^{4} b'_{4i+j} a_{4i+j}.$$

Then for $\delta \in \mathbb{Q}_{>0}$, each $c_j(\delta) \in \mathbb{Q}$. Also $c_1(\delta) > c_2(\delta) > c_3(\delta) > p^{-m_i} q^{-n_i} > c_4(\delta)$ and each $c_j(\delta)$ tends to $p^{-m_i} q^{-n_i}$ as $\delta$ tends to 0. So we may choose $\delta \in \mathbb{Q}_{>0}$ such that with $b_{4i+j} = c_j(\delta)$ for $j = 1, 2, 3, 4$, we have $U_i > b_{4i+1} > b_{4i+2} > b_{4i+3} > p^{-m_i} q^{-n_i} > b_{4i+4} > L_i$.

We then have $U_i+1 = b_{4i+4} > L_i = p^{-m_{i+1}} q^{-n_{i+1}}$ so the precondition for the next stage is satisfied. Clearly, we have $\sum_{k=1}^{4i+4} b_k a_k = \sum_{k=1}^{4i+4} b'_k a_k$ for all $i$, so that $\sum_{k=1}^{\infty} b_k a_k = \sum_{k=1}^{\infty} b'_k a_k = b \in \mathbb{Q}_{>0}$ and the sequence $b_k$ is as required.

We will now construct a 2-dimensional normed space $\mathbb{J}$ over an arbitrary ordered field $K$ in which the graph of natural number multiplication is definable.

Before embarking on the construction, note that the the basic ideas of affine geometry and convexity theory all carry over to a vector space over an arbitrary ordered field $K$. If $V$ is a vector space over $K$, we may define a norm on $V$ by specifying its unit disc, which can be any convex subset $D$ of $V$ that meets every
line through the origin in a closed line segment $[-x, x]$ where $x \neq 0$. Given such a $D$ and $v \in V$, there is a unique non-negative $r \in K$ such that $v \in rD$ and $v \not\in sD$ for $0 \leq s < r$. We set $||v|| = r$ and verify that this satisfies the norm axioms in exactly the same way as when $K = \mathbb{R}$. In examples over $\mathbb{R}$, the requirements on $D$ are often simple consequences of the Heine-Borel theorem. But that theorem holds in $K$ iff $K$ is isomorphic to $\mathbb{R}$, so this method of proof does not apply in general. With $K = \mathbb{Q}$, for example, if we take $D = \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 \leq 1\}$, then $D$ is closed, convex and bounded, but with $v = (1, 1)$, the set of $s$ such that $v \not\in sD$ is non-empty and bounded but has no least upper bound.

Let $e_1$ and $e_2$ be the standard basis for $K^2$. Define vectors $v_k$, $k = 0, 1 \ldots$ in the north-west quadrant as follows:

\[
\begin{align*}
 v_0 & := -e_1 \\
 v_1 & := -e_1 + (1 - b)e_2 \\
 v_{k+1} & := v_k + (1 - b_k)a_k e_1 + b_k a_k e_2
\end{align*}
\]
where the $a_k$, $b_k$ and $b$ are the rational numbers of lemmas 2 and 3. See Figure 1.

If $l$ is any line in $K^2$, we define its gradient to be the symbol $\infty$ if $l$ is parallel to $e_2$ or to be the unique $g \in K$ such that $l$ is parallel to $e_1 + ge_2$ otherwise.

Let the line $l_k$ through $v_k$ and $v_{k+1}$ have gradient $g_k$. So $g_0 = \infty$, and $g_k = b_k/(1 - b_k) = (b_k^{-1} - 1)^{-1}$, for $k = 1, 2, \ldots$. By lemma 3, $1 > b_1 > b_2 > \ldots > 0$, so that $g_1 > g_2 > \ldots > 0$. Also, if we write $v_k = x_k e_1 + y_k e_2$, the sequence $(x_k, y_k)$ tends to a limit $(a - b - 1, 1)$ in $\mathbb{Q}^2$ as $k$ tends to $\infty$, where $a$ is as in lemma 2 so that $-1 < a - b - 1 < 0$.

Define a subset $D$ of $K^2$ as follows:

$$D := \bigcap_{k=0}^{\infty} H_k \cap A \cap B \cap \bigcap_{k=0}^{\infty} -H_k \cap -A \cap -B$$

where $H_k$ is the closed half-plane that contains the origin and has the line $l_k$ as boundary and where $A$ and $B$ are the closed half-planes defined by the formulas $y \leq 1$ and $x + y \leq 1$ respectively. Clearly $D$ is convex and symmetric about the origin.

I claim that $D$ meets every line through the origin in a line segment $[-x, x]$ where $x \neq 0$. To prove this, first note that by routine algebra, if $h_k = y_k/x_k$ is the gradient of the line through $0$ and $v_k$, then $0 = h_0 > h_1 > h_2 > \ldots$. Also from the remarks above about the $x_k$ and $y_k$, the $h_k$ tend in $\mathbb{Q}$ to the limit $h = (a - b - 1)^{-1}$. Now let $l$ be a line through the origin with gradient $g$. We need to exhibit an $x \neq 0$ such that $l \cap D = [-x, x]$. By symmetry, it is enough to find $x \neq 0$ such that $l \cap D_u = [0, x]$ where $D_u = D \cap H_u$, $H_u$ being the half-plane defined by the formula $y \geq 0$. We identify three cases as follows (see Figure 1):

(i) $h_k > g \geq h_{k+1}$ for some $k$: clearly the $v_i$ lie in the interior of the half-planes $A$ and $B$; also, the conditions on the gradients $g_j$ imply that $v_i \in H_j$ for all $i, j$, and so as $D$ is convex, $[v_k, v_{k+1}] \subseteq D$. As $h_k > g \geq h_{k+1}$, $l$ meets $[v_k, v_{k+1}]$ at some point $x \neq 0$ and then as every neighbourhood of $x$ meets both $D$ and its complement, we must have $l \cap D_u = [0, x]$.

(ii) $g \neq \infty$ and $h_k > g$ for all $k$: in this case, $l$ meets the line $y = 1$ at a point $x = pe_1 + e_2 \neq 0$ where either $a - b - 1 \leq p < 0$ or $a - b - 1 - p$ is a positive infinitesimal. $x$ then lies on the boundary of half-plane $A$ and in the interior of half-plane $B$. Since the gradients $g_k = (y_{k+1} - y_k)/(x_{k+1} - x_k)$ are rational and satisfy $g_1 > g_2 > \ldots > 0$ and since the points $(x_k, y_k)$ converge in $\mathbb{Q}^2$ to the limit $(a - b - 1, 1)$, the line $l_k$ must meet the line $y = 1$ at a point $re_1 + e_2$ where $r < a - b - 1$ is rational. But this implies that $r < p$ so that $l_k$ and $l$ meet at a point $y = se_1 + te_2$ where $r < s < p$ and $t > 1$ so that $x$ is to the south and east of the
point \( y \in \ell_k \). Thus \( x \in H_k \) for every \( k \) and every neighbourhood of \( x \) meets both \( D \) and its complement, whence \( l \cap D_u = [0, x] \).

(iii) \( g \geq 0 \) or \( g = \infty \): it is easy to see that the intersection of \( D \) with the north-east quadrant is the triangle \( \triangle 0e_1e_2 \), so that, if \( g \geq 0 \) or \( g = \infty \), \( l \) meets \( D_u \) in the interval \([0, x]\), where \( x = e_2 \) if \( g = \infty \) and \( x = (e_1 + ge_2)/(1 + g) \) otherwise. In both cases \( x \neq 0 \).

We have proved that \( D \) is convex and meets each line through the origin in a line segment \([-x, x]\) where \( x \neq 0 \). Hence there is a norm on \( K^2 \) having \( \overline{D} \) as its unit disc. Define \( \mathbb{J} \) to be \( K^2 \) equipped with that norm.

The case analysis on the gradient of the line \( l \) in the argument above shows that the upper half of the unit circle in \( \mathbb{J} \) comprises: (i) the line segments \([v_0, v_1]\), \([v_1, v_2] \ldots \), (ii) the set \( E \) of all points on the line segment \([-e_1 + e_2, e_2] \) that lie to the east of every \( v_k \) and (iii) the line segment \([e_1, e_2] \). If \( K \) is archimedean, \( E \) is the line segment \([v, e_2] \) where \( v = (a - b - 1)e_1 + e_2 \), but in the non-archimedean case, \( E \) also contains every point \( v - \epsilon e_1 \) where \( \epsilon \) is a positive infinitesimal.

Having defined \( \mathbb{J} \) and described its unit circle, let us develop the formula \( M(x, y, z) \). First we define:

\[
\begin{align*}
\text{EP}(p) & := \forall u \ w. \ ||u|| = ||p|| = ||w|| \wedge p = \frac{1}{2}(u + w) \Rightarrow u = p = w \\
\text{SEP}(p) & := \text{EP}(p) \wedge \exists u \ w. \ \text{EP}(u) \wedge \text{EP}(w) \wedge ||u|| = ||w|| = ||p|| \wedge 0 \neq ||p - u|| \neq ||p - w|| \neq 0 \\
\text{ADS}(p, q) & := \text{SEP}(p) \wedge \text{SEP}(q) \wedge p \neq q \wedge ||p|| = ||q|| = ||(p + q)/2||.
\end{align*}
\]

Thus in any normed space \( \text{EP}(p) \) holds iff \( p \) is an extreme point of the sphere \( S_{||p||} \) of radius \( ||p|| \) centred on the origin and \( \text{SEP}(p) \) holds iff \( p \) is an extreme point of \( S_{||p||} \) that is not equidistant from every other extreme point of \( S_{||p||} \). Let us call points \( p \) satisfying \( \text{SEP}(p) \) \textit{special} extreme points of \( S_{||p||} \). In \( \mathbb{J} \), the special extreme points are just the non-zero ones. In any normed space, \( \text{ADS}(p, q) \) holds iff \( p, q \) are adjacent special extreme points of \( S_{||p||} \). Next we define:

\[
\begin{align*}
\text{HPV}(p_1, \ldots, p_n) & := \bigwedge_{i=1}^{n-1} \text{ADS}(p_i, p_{i+1}) \wedge \bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} p_i \neq p_j \\
\text{HPL}(x_1, \ldots, x_n) & := \exists p_1, \ldots, p_{n+1}. \ \text{HPV}(p_1, \ldots, p_{n+1}) \wedge \bigwedge_{i=1}^{n} x_i = ||p_{i+1} - p_i||.
\end{align*}
\]

\[9\]
So in any normed space, HPV($p_1, \ldots, p_n$) holds iff $p_1, \ldots, p_n$ is a sequence of special extreme points of $S_{||p_1||}$ forming the vertices of a Hamiltonian path made up of straight line segments inscribed in $S_{||p_1||}$. HPV($x_1, \ldots, x_n$) holds iff $x_1, \ldots, x_n$ are the lengths of the successive edges of such a path. Finally we define:

\[
M_{>1}(x, y, z) := \exists u. \text{HPL}(1, x, z, y, u) \land 1 < x < z > y > u < 1
\]

\[
M(x, y, z) := M_{>1}(x + 2, y + 2, 4 + 2x + 2y + z).
\]

Identifying $\mathbb{N}$ and $\mathbb{N}_K$, I claim that $M_{>1}(x, y, z)$ holds in $\mathbb{J}$ iff $x, y, z \in \mathbb{N}_{>1}$ and $z = xy$. This implies that $M(x, y, z)$ defines the graph of multiplication in $\mathbb{N}$.

To prove the claim, first note that for vectors in the north-east quadrant, the $\mathbb{J}$-norm is the same as the 1-norm, so $||v_{k+1} - v_k||_\mathbb{J} = (1 - b_k)a_k + b_k a_k = a_k$, for $k > 0$. Also $||e_2 - e_1||_\mathbb{J} > 1$ because $e_2 - e_1 \not\in D$. So $||e_2 - e_1||_\mathbb{J}$ is greater than the $\mathbb{J}$-length of any line segment that can be inscribed in the north-west quadrant of the unit circle of $\mathbb{J}$.

Now, if $M_{>1}(x, y, z)$ holds, there are special extreme points $p_1, \ldots, p_6$ forming the vertices of a Hamiltonian path inscribed in $S_r$ for some $r > 0$ such that the edges $[p_2, p_1], \ldots, [p_6, p_5]$ have $\mathbb{J}$-lengths $1, x, y, z, u$ respectively where $1 < x < z > y > u < 1$ for some $u$. But then the local maximum $z$ can only be the $\mathbb{J}$-length of $\pm [rv_{4i+3}, rv_{4i+4}]$ for some $i$ (it cannot be $||re_2 - re_1||$ since, even when $\mathcal{K}$ is archimedean, there is no Hamiltonian path in $S_r$ with 6 extreme points as vertices such that $\pm [re_1, re_2]$ is the 3rd edge). As $u < 1$, $p_1, \ldots, p_6$, are $\pm v_{4i+1}, \ldots, \pm v_{4i+6}$ in that order, so that $1 = ||p_2 - p_1|| = ra_{4i+1} = rp^{-m_i}q^{-n_i}$ and $r = p^{m_i}q^{n_i}$. Hence $x = rm_i p^{-m_i}q^{-n_i} = m_i$ and, similarly, $z = m_i n_i$ and $y = n_i$, so $x, y, z \in \mathbb{N}_{>1}$ and $z = xy$.

Conversely, if $x, y, z \in \mathbb{N}_{>1}$ and $z = xy$, then for some $i, x = m_i$ and $y = n_i$. Let $r = p^{m_i}q^{n_i}$ so that $rv_{4i+1}, \ldots, rv_{4i+6}$ are the vertices of a Hamiltonian path inscribed in $S_r$, whose edges have $\mathbb{J}$-lengths $1, x, y, rp^{-m_i+1}q^{-n_i+1}$ respectively. By lemma 2, we have $1 < x < z > y > rp^{-m_i+1}q^{-n_i+1} < 1$, and so $M_{>1}(x, y, z)$ holds in $\mathbb{J}$ with $rp^{-m_i+1}q^{-n_i+1}$ as the witness for $u$.

**Theorem 4** There is a formula $M(x, y, z)$ in the purely additive language of normed spaces $\mathcal{L}_N^+$ such that for any ordered field $\mathcal{K}$ and any $d \in \{2, 3, 4, \ldots\} \cup \{\infty\}$, there is a normed space $\mathbb{J}^d$ of dimension $d$ in which $M(x, y, z)$ defines the graph of natural number multiplication on $\mathbb{N}_K$.

**Proof:** Let $M(x, y, z)$ and $\mathbb{J}$ be as above. Let $W$ be a $(d - 2)$-dimensional vector space over $\mathcal{K}$ equipped with the 1-norm with respect to some basis $b_1, b_2, \ldots$
(\| \sum_i c_i b_i \|_W = \sum_i |c_i|) and let \( J^d \) be the 1-sum \( J \times W \) \((\| (v, w) \|_{J^d} = \| v \|_J + \| w \|_W)\). Identify \( J \) and \( W \) with \( J \times 0 \) and \( 0 \times W \) respectively. Then the extreme points of \( S_r \) in \( J^d \) comprise the extreme points of \( S_r \cap J \) together with the extreme points \( \pm r b_1, \pm r b_2, \ldots \) of \( S_r \cap W \). Moreover, \( \| \pm r b_n - p \| = 2r \) for every extreme point \( p \) of \( S_r \) with \( p \neq \pm r b_n \). Thus, in the sense defined above, the only special extreme points of \( S_r \) in \( J^d \) are those of \( J \). It follows from the discussion above that \( M(x, y, z) \) defines the graph of natural number multiplication in \( J^d \).

**Corollary 5** If \( C \) is any non-empty class of ordered fields, then the theories \( \text{NS}_+^n(C) \), \( \text{NS}_+^\infty(C) \) for \( 1 < n \in \mathbb{N}, \) \( \text{NS}_F(C) \) and \( \text{NS}_\infty(C) \) are all undecidable.

**Proof:** This is immediate from theorems [1] and [4].

**Corollary 6** If \( C \) is any non-empty class of ordered fields, then the theories \( \text{NS}(C), \text{NS}_n(C) \) for \( 1 < n \in \mathbb{N}, \) \( \text{NS}_F(C) \) and \( \text{NS}_\infty(C) \) are all undecidable.

**Proof:** By the corollary, we cannot decide sentences in the additive fragments of these theories.

When \( C \) is any class of ordered fields including \( \mathbb{Q} \), the theory of \( C \) is then undecidable by a classic result of Julia Robinson [1]. Corollary 6 tells us nothing new for such a \( C \), but as the additive fragment of the theory of ordered fields is decidable, Corollary 5 is significant. Both corollaries have force when the theory of the fields in \( C \) is decidable, e.g., when the fields in \( C \) are real closed.

**References**

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