A NONUNIFORM ANISOTROPIC FEM FOR ELLIPTIC BOUNDARY LAYER OPTIMAL CONTROL PROBLEMS

HONGBO GUAN
College of Mathematics and Information Science
Zhengzhou University of Light Industry
Zhengzhou 450002, China

YONG YANG
College of Science
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

HUIQING ZHU∗
School of Mathematics and Natural Sciences
The University of Southern Mississippi
Hattiesburg, MS 39406, USA

(Communicated by Zhimin Zhang)

Abstract. In this paper, an anisotropic bilinear finite element method is constructed for the elliptic boundary layer optimal control problems. Supercloseness properties of the numerical state and numerical adjoint state in a ϵ-norm are established on anisotropic meshes. Moreover, an interpolation type post-processed solution is shown to be superconvergent of order O(N−2), where the total number of nodes is of O(N2). Finally, numerical results are provided to verify the theoretical analysis.

1. Introduction. Optimal control problems (OCPs) governed by partial differential equations (PDEs) play an important role in many sciences and engineering applications (cf. [3, 4]). Since analytical solutions of such problems are generally unavailable, much work has been devoted to the design, implementation, and analysis of numerical methods. While many numerical methods are known to PDE-governed control problems, finite element method (FEM) became one of the most commonly used numerical methods in the past two decades. Falk [8] proposed for the first time the conforming FEM for the OCPs governed by elliptic type PDEs and obtained error estimates in L2-norm. In [15], Hinze investigated a discretization concept which utilizes for the discretization of the control variable the relation between adjoint state and control, in which they used the linear elements to discretize the state equations and obtained optimal error estimates in L2-norm. On the other hand, some a posteriori error estimates of FEM solutions for OCPs governed by elliptic equations, Stokes equations, and parabolic equations were established in [9], [23]

2020 Mathematics Subject Classification. Primary: 65M60, 49K20; Secondary: 65J10.

Key words and phrases. Anisotropic meshes, elliptic boundary layer equations, optimal control problems, superconvergence.

The first author is supported by National Natural Science Foundation of China (No. 11501527).

* Corresponding author: Huiqing Zhu.
and [10], respectively. For a detailed discussion of nonconforming FEMs applied to OCPs, we refer the readers to [11, 12, 13].

In this paper, we are interested in solving the following elliptic control problem governed by an elliptic boundary layer equation: find \((y, u) \in H^1_0(\Omega) \times L^2(\Omega)\), such that

\[
\min_{(y, u) \in H^1_0(\Omega) \times L^2(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} (y - y_d)^2 \, dx + \frac{\alpha}{2} \int_{\Omega} u^2 \, dx \right\} \tag{1}
\]

subject to

\[
\begin{align*}
-\text{div}(A \nabla y) + y &= f + u, \quad \text{in } \Omega = (0, 1)^2, \\
y &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\tag{2}
\]

where \(\alpha\) is a positive constant parameter, \(y_d \in C^0(\Omega)\) and \(f \in L^2(\Omega)\) are given functions. We consider the coefficient matrix \(A = (\epsilon^2, 0; 0, 1)\) that contains a small parameter \(0 < \epsilon \ll 1\). Throughout this paper, the standard Sobolev spaces and norms defined in [5] are adopted.

The control problem (1)-(2) exhibits rapid changes near the domain boundary \(x_1 = 0\) and \(x_1 = 1\) as \(\epsilon \to 0^+\). It is well known that classical FEMs defined on isotropic meshes will produce numerical solutions with wild oscillations throughout the whole domain so that numerical solutions are no longer convergent to exact solution. To overcome this difficulty, many advanced numerical schemes and anisotropic meshes have been introduced for solving singularly perturbation PDEs [14, 16, 17, 22, 6, 28, 30, 29]. More details can be found in the above mentioned papers and references therein. The standard FEM was proven to be uniformly convergent on the so-called Shishkin type meshes (see [26, 25, 6]), but analysis on Shishkin meshes is quite complicated since it is based closely on a regularity decomposition, as indicated in [18].

On the other hand, Apel et al. [2] developed another kind of anisotropic mesh, named A-mesh, which could fully separate the inner subdomain and boundary layer subdomains, and obtained a suboptimal order of convergence in \(L^2\)-norm. Later on, Li et al. [18, 19] derived some global superconvergence properties for singularly perturbed problems on this kind of meshes of FEMs and mixed FEMs. As for OCPs governed by singularly perturbed PDEs, Roos et al. [27] obtained optimal convergence order in the energy norm for the linear FEM approximation on one-dimensional Shishkin type meshes, but the \(L^2\)-norm error estimate was not optimal. Allendes [1] proved a quasi-optimal order of convergence for bilinear FEMs on a graded meshes developed in [7] for solving OCPs governed by convection-reaction-diffusion equations. Lube et al. [24] analyzed the local projection stabilization approach of Galerkin discretization and obtained some a priori estimates for reaction-diffusion-advection control problems with mixed boundary value conditions. To our knowledge, the application of FEM to optimal control problems governed by elliptic boundary layer equations has not been undertaken.

This paper will study the convergence and superconvergence of the bilinear FEM on anisotropic meshes for boundary layer control problems. The rest of this paper is organized as follows. In section 2, the discrete formulation of (1) and some important lemmas are presented. In Section 3, some supercloseness and superconvergence results are derived. Finally, some numerical experiments are provided in Section 4 to verify the efficiency of the proposed method.
2. The optimality conditions and anisotropic meshes. It is known from [20] that (1) has a unique solution \((y, u)\) if and only if there is an adjoint state \(p \in H^1_0(\Omega)\) such that \((y, p, u)\) satisfies the following optimality conditions:

\[
\begin{aligned}
& a(y, v) + (y, v) = (f + u, v), \quad \forall v \in H^1_0(\Omega), \\
& a(p, v) + (p, v) = (y - y_d, v), \quad \forall v \in H^1_0(\Omega), \\
& (\alpha u + p, v) = 0, \quad \forall v \in L^2(\Omega),
\end{aligned}
\]

(3)

where \(a(y, v) := \int_\Omega \nabla y \nabla v dx\), \((y, v) := \int_\Omega yv dx\).

Before introducing the discretization of the above optimality conditions, let us define the partition of the domain \(\Omega\). Note that the state \(y\) and adjoint state \(p\) have boundary layers along edges \(x_1 = 0\) and \(x_1 = 1\), the domain \(\Omega\) is therefore divided into three subdomains \(\Omega_i\) \((i = 1, 2, 3)\) as

\[\Omega_1 := (0, \sigma_x) \times (0, 1), \quad \Omega_2 := (\sigma_x, 1 - \sigma_x) \times (0, 1), \quad \Omega_3 := (1 - \sigma_x, 1) \times (0, 1),\]

(4)

where the transition parameter \(\sigma_x\) is selected to be \(\sigma_x := \min\left\{\frac{1}{3}, 2\epsilon|\ln \epsilon|\right\}\). Then, we divide each subdomain \(\Omega_i\) equally into \(N \times N\) rectangles. As a consequence, we obtain a partition \(T_h\) with highly stretched rectangular mesh elements in \(\Omega_1\) and \(\Omega_3\), which is anisotropically refined along the direction of the boundary layers.

We denote by \(V_h\) the bilinear finite element space with homogeneous boundary condition on the anisotropic mesh \(T_h\). Fig. 2 provides a comparison between an anisotropic rectangular mesh and a uniform rectangular mesh.

Then the finite element discretization of the optimality conditions (3) reads: find \((y_h, p_h, u_h) \in V_h \times V_h \times L^2(\Omega)\) such that

\[
\begin{aligned}
& a(y_h, v_h) + (y_h, v_h) = (f + u_h, v_h), \quad \forall v_h \in V_h, \\
& a(p_h, v_h) + (p_h, v_h) = (y_h - y_d, v_h), \quad \forall v_h \in V_h, \\
& (\alpha u_h + p_h, v_h) = 0, \quad \forall v_h \in L^2(\Omega),
\end{aligned}
\]

(5)

To estimate the approximation errors between the solutions of (3) and (5), we introduce the following reasonable assumption:

**Assumption 2.1.** If a function \(\phi \in C^4(\Omega)\) satisfy the following two conditions for any \(0 \leq i \leq 4\) that:

1. \(|\phi_{x_1}^i(x_1, x_2)| \leq c(1 + \epsilon^{-i}e^{-x_1/\epsilon} + \epsilon^{-i}e^{-(1-x_1)/\epsilon}), \text{ on } \Omega_1,\)
2. \(|\phi_{x_2}^i(x_1, x_2)| \leq c, \text{ on } \Omega_1,\)

Figure 1. Anisotropic mesh (left) and uniform mesh (right).
then we call this function \( \phi \) satisfies the Assumption 2.1.

Here and later, \( c \) denotes a generic positive constant independent of \( N \) or \( \epsilon \).

Note that it has been proven in Theorem 4.2 of [1] that if \( f \) and \( y_d \) both satisfy the Assumption 2.1, then the state \( y \) and adjoint state \( p \) also satisfy this Assumption 2.1.

**Lemma 2.2.** Let \( f, y_d \in H^1(\Omega) \), then we have the following regularity properties of the solution:

\[
y, p \in Y
\]

and

\[
u \in U,
\]

where \( Y = H_0^1(\Omega) \cap H^3(\Omega) \) and \( U = H^1(\Omega) \).

**Proof.** If \( f, y_d \in L^2(\Omega) \), by the regularity theory of PDEs, we have

\[
y, p \in H_0^1(\Omega) \cap H^2(\Omega),
\]

then by the third equation of (3), we can get that

\[
u \in H^1(\Omega).
\]

Furthermore, noticing the above result (9) and the assumption of \( f, y_d \in H^1(\Omega) \), we can obtain \( y, p \in H_0^1(\Omega) \cap H^3(\Omega) \) by using again the regularity theory of PDEs. The proof is thus completed. \( \square \)

We now recall some existing anisotropic properties of the bilinear finite element space \( V_h \). Let \( \Pi_h \) be the associated interpolation operator, for which the following interpolation error estimates have been proved in [18]: for any \( y \in Y \), there hold

\[
\| y - \Pi_h y \|_{0,\Omega} \leq c N^{-2},
\]

\[
| (\epsilon(y - \Pi_h y)_{x_1}, v_{x_1}) | \leq c N^{-2} \| v_{x_1} \|_{0,\Omega}, \quad \forall v \in V_h,
\]

and

\[
| ((y - \Pi_h y)_{x_2}, v_{x_2}) | \leq c N^{-2} \| v_{x_2} \|_{0,\Omega}, \quad \forall v \in V_h.
\]

Based on the above results, one can easily obtain the following finite element error estimate for elliptic boundary layer problems:

**Lemma 2.3.** ([18]) Let \( \varphi \in Y \) and \( \varphi_h \in V_h \) be the solutions of

\[
a(\varphi, v) + (\varphi, v) = (f, v), \quad \forall v \in H_0^1(\Omega),
\]

and

\[
a(\varphi_h, v_h) + (\varphi_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,
\]

respectively. Suppose that Assumption 2.1 holds true for \( \varphi \), then there exists a constant \( c \) such that

\[
\epsilon N^{-1} \| (\varphi - \varphi_h)_{x_1} \|_{0,\Omega} + N^{-1} \| (\varphi - \varphi_h)_{x_2} \|_{0,\Omega} + \| \varphi - \varphi_h \|_{0,\Omega} \leq c N^{-2}.
\]
3. Convergence and superconvergence analysis. In this section, we will first derive the error estimate for the control $u$. Then, some superclose properties for state $y$ and adjoint state $p$ will be shown. Lastly, the global superconvergence will be proved by the so-called interpolated postprocessing technique.

**Theorem 3.1.** Let $f$ and $y_d$ satisfy the Assumption 2.1, and $u$, $u_h \in U$ be the solutions of (3) and (5), respectively. Then we have

$$\|u - u_h\|_{0, \Omega} \leq cN^{-2}. \quad (14)$$

*Proof.* By the last equations of (3) and (5), there holds

$$\alpha\|u - u_h\|_{0, \Omega}^2 \quad (15)$$

so that

$$\|u - u_h\|_{0, \Omega} \leq \frac{\alpha M_1 + M_2}{\alpha}. \quad (16)$$

where $p(u_h) \in Y$ is the solution of the following system:

$$\begin{cases}
  a(y(u_h), v) + (y(u_h), v) = (f + u_h, v), \quad \forall v \in H_0^1(\Omega), \\
  a(p(u_h), v) + (p(u_h), v) = (y(u_h) - y_d, v), \quad \forall v \in H_0^1(\Omega).
\end{cases} \quad (17)$$

We first estimate $M_1$. Setting $v = y$ in the second equation of (3), we have

$$a(p, y) + (p, y) = (y - y_d, y), \quad (18)$$

which together with the first equation of (3) yields

$$(y - y_d, y) = (f + u, p). \quad (19)$$

Similarly, in the second equation of (16), setting $v = y$ yields

$$a(p(u_h), y) + (p(u_h), y) = (y(u_h) - y_d, y). \quad (20)$$

So, using the first equation of (3) again, we get

$$(y(u_h) - y_d, y) = (f + u, p(u_h)). \quad (21)$$

Subtracting (20) from (18), we have

$$(y(u_h) - y, y) = (f + u, p(u_h) - p). \quad (22)$$

On the other hand, setting $v = y(u_h)$ in the second equation of (3) yields

$$a(p, y(u_h)) + (p, y(u_h)) = (y - y_d, y(u_h)), \quad (23)$$

which together with the first equation of (16) gives

$$(y - y_d, y(u_h)) = (f + u_h, p). \quad (24)$$

Similarly, setting $v = y(u_h)$ in the second equation of (16), we have

$$a(p(u_h), y(u_h)) + (p(u_h), y(u_h)) = (y(u_h) - y_d, y(u_h)). \quad (25)$$

Once again, we get from the first equation of (16) that

$$(y(u_h) - y_d, y(u_h)) = (f + u_h, p(u_h)). \quad (26)$$

Subtracting (25) from (23), we have

$$(y(u_h) - y, y(u_h)) = (f + u_h, p(u_h) - p). \quad (27)$$

Therefore, subtracting (26) from (21) yields

$$M_1 = (p(u_h) - p, u - u_h) = (y(u_h) - y, y - y(u_h)) \leq 0. \quad (28)$$
Next, we consider $M_2$. Introducing the following symbols:

$$M_2 = (p_h - p(y_h), u - u_h) + (p(y_h) - p(u_h), u - u_h) \triangleq M_{21} + M_{22}.$$  \hfill (28)

where $p(y_h)$ is the solution of

$$a(p(y_h), v) + (p(y_h), v) = (y_h - y_d, v), \quad \forall v \in H_0^1(\Omega).$$  \hfill (29)

Noticing that $p_h$ is the standard finite element approximation of $p(y_h)$, we immediately get from Lemma 2.3 that

$$|M_{21}| \leq \|p_h - p(y_h)\|_{0,\Omega} \|u - u_h\|_{0,\Omega} \leq cN^{-2} \|u - u_h\|_{0,\Omega}.$$  \hfill (30)

Subtracting (16) from (29), we have the following error equation:

$$a(p(y_h) - p(u_h), v) = (y_h - y(u_h), v), \quad \forall v \in \hat{V}.$$  \hfill (31)

By the regularity property of the solutions of elliptic equations [5], we derive that

$$\|p(y_h) - p(u_h)\|_{0,\Omega} \leq c \|y_h - y(u_h)\|_{0,\Omega}.$$  \hfill (32)

Noticing that $y_h$ is the standard finite element approximation of $y(u_h)$, using again Lemma 2.3, we have

$$|M_{22}| \leq c \|y_h - y(u_h)\|_{0,\Omega} \|u - u_h\|_{0,\Omega} \leq cN^{-2} \|u - u_h\|_{0,\Omega}.$$  \hfill (33)

Combining the results of (15), (27), (28) (30) and (33), this yields the desired result.

**Theorem 3.2.** Let $f$ and $y_d$ satisfy the Assumption 2.1, $y \in Y$ and $y_h \in V_h$ be the solutions of (3) and (5), respectively. Then, the numerical state $y_h$ has the following supercloseness property:

$$|||y_h - \Pi_h y||| \leq cN^{-2},$$  \hfill (34)

where $|||v||| = (c^2||v_x||^2_{0,\Omega} + ||v_{x_2}||^2_{0,\Omega})^{1/2}$.

**Proof.** Subtracting (3) from (5) and setting the test function $v_h = y_h - \Pi_h y$, we get the following error equation

$$a(y_h - y, y_h - \Pi_h y) + (y_h - y, y_h - \Pi_h y) = (u_h - u, y_h - \Pi_h y).$$  \hfill (35)

It follows from interpolation error estimates (10)-(12) and Theorem 3.1 that

$$|||y_h - \Pi_h y|||^2 + ||y_h - \Pi_h y||^2_{0,\Omega}$$

$$= a(y - \Pi_h y, y_h - \Pi_h y) + (y - \Pi_h y, y_h - \Pi_h y) + (u_h - u, y_h - \Pi_h y)$$

$$= (c^2(y - \Pi_h y), (y_h - \Pi_h y)_{x_1}) + ((y - \Pi_h y)_{x_2}, (y_h - \Pi_h y)_{x_2})$$

$$+ (y - \Pi_h y, y_h - \Pi_h y) + (u_h - u, y_h - \Pi_h y)$$

$$\leq cN^{-2} \|(y_h - \Pi_h y)_{x_1}\|_{0,\Omega} + cN^{-2} \|(y_h - \Pi_h y)_{x_2}\|_{0,\Omega} + \|y_h - \Pi_h y\|_{0,\Omega}$$

$$\leq cN^{-4} + \frac{1}{2} |||y_h - \Pi_h y|||^2 + \frac{1}{2} |||y_h - \Pi_h y|||_{0,\Omega}^2,$$  \hfill (36)

which implies (34). The proof is completed.

**Remark 1.** By using the interpolation error estimate (10) and the above estimate (36), we get a by-product of error estimate for $y - y_h$ in $L^2$-norm:

$$\|y - y_h\|_{0,\Omega} \leq \|y - \Pi_h y\|_{0,\Omega} + \|y_h - \Pi_h y\|_{0,\Omega} \leq cN^{-2}.$$  \hfill (37)
Theorem 3.3. Let $f$ and $y_d$ satisfy the Assumption 2.1, $p \in Y$ and $p_h \in V_h$ be the solutions of (3) and (5), respectively. Then the following error estimate holds for the adjoint state $p$:

$$|||p_h - \Pi_h p||| \leq cN^{-2}. \quad (38)$$

Proof. By a similar approach as in the proof of Theorem 3.2, we get the error equation

$$a(p_h - p, p_h - \Pi_h p) + (p_h - p, p_h - \Pi_h p) = (y_h - y, p_h - \Pi_h p), \quad (39)$$

and

$$|||p_h - \Pi_h p|||^2 + ||p_h - \Pi_h p||^2_{0, \Omega} = a(p - \Pi_h p, p_h - \Pi_h p) + (p - \Pi_h p, p_h - \Pi_h p) + (y_h - y, p_h - \Pi_h y)
$$

$$= (c^2(p - \Pi_h p) x_1, (p_h - \Pi_h p) x_1) + ((p - \Pi_h p) x_2, (p_h - \Pi_h p) x_2)
+ (p - \Pi_h p, p_h - \Pi_h p) + (y_h - y, p_h - \Pi_h p)
\leq cN^{-2}||p_h - \Pi_h p||_{0, \Omega} + cN^{-2}||p_h - \Pi_h p||_{0, \Omega} + ||p_h - \Pi_h p||_{0, \Omega}
\leq cN^{-4} + \frac{1}{2}||p_h - \Pi_h p||_{0, \Omega}^2 + \frac{1}{2}||p_h - \Pi_h p||_{0, \Omega}^2, \quad (40)$$

where the result (37) was used in the second last step. The proof is thus completed.

Remark 2. Using the same technique as in Remark 1, we get an error estimate for $p - p_h$ in $L^2$-norm from the interpolation error estimate (10) and the above estimate (40):

$$||p - p_h||_{0, \Omega} \leq ||p - \Pi_h p||_{0, \Omega} + ||p_h - \Pi_h p||_{0, \Omega} \leq cN^{-2}. \quad (41)$$

In order to obtain the global superconvergence of $y_h$ and $p_h$, we combine four neighbour elements $K_1, K_2, K_3, K_4 \in T_h$ into a larger rectangular element $K_0$ (see Fig. 2). $T_{2h}$ presents the corresponding new partition [21]. We construct the interpolated postprocessing operator $\Pi_{2h}$ as:

$$\begin{cases}
\Pi_{2h}w|_{K_0} \in Q_{22}(K_0), \forall K_0 \in T_{2h}, \\
\Pi_{2h}w(Z_i) = w(Z_i), \quad i = 1, 2, ..., 9.
\end{cases}$$
It is worth mentioning that the following properties for \( \Pi_{2h} \) have been established in [21]:

\[
\begin{aligned}
\Pi_{2h}\Pi_h w &= \Pi_{2h} w, \quad \forall w \in H^2(\Omega), \\
\| (\Pi_{2h} v)_{x1} \|_{0, \Omega} &\leq c \| v_{x1} \|_{0, \Omega}, \quad \forall v \in V_h, \\
\| (\Pi_{2h} v)_{x2} \|_{0, \Omega} &\leq c \| v_{x2} \|_{0, \Omega}, \quad \forall v \in V_h, \\
c \| (w - \Pi_{2h} w)_{x1} \|_{0, \Omega} + \| (w - \Pi_{2h} w)_{x2} \|_{0, \Omega} &\leq c N^{-2}, \quad \forall w \in H^3(\Omega).
\end{aligned}
\]  

**Theorem 3.4.** Under the assumptions of Theorems 3.2-3.3, we have the following global superconvergence results:

\[
\| | | y - \Pi_{2h} y_h | | \| \leq c N^{-2}
\]

and

\[
\| | | p - \Pi_{2h} p_h | | \| \leq c N^{-2},
\]

for \( y \) and \( p \), respectively.

**Proof.** By (34) and (42), there hold

\[
\| | | \Pi_{2h} \Pi_h y - \Pi_{2h} y_h | | \| = \| | | \Pi_{2h} (\Pi_h y - y_h) | | \| \leq c \| | | \Pi_h y - y_h | | \| \leq c N^{-2}
\]

and

\[
\| | | y - \Pi_{2h} \Pi_h y | | \| = \| | | y - \Pi_{2h} y | | \| \leq c N^{-2}.
\]

So we have

\[
\| | | y - \Pi_{2h} y_h | | \| = \| | | y - \Pi_{2h} \Pi_h y + \Pi_{2h} \Pi_h y - \Pi_{2h} y_h | | \| \leq \| | | y - \Pi_{2h} \Pi_h y | | \| + \| | | \Pi_{2h} \Pi_h y - \Pi_{2h} y_h | | \| \leq c N^{-2}.
\]

Similarly, (44) directly follows from (38) and (42). The proof is completed.

4. **Numerical results.** We consider the optimal control problem (1)-(2) and choose \( \alpha = 1 \). Based on the resultant optimality system (3), the source term \( f \) and the target state \( y_d \) are determined correspondingly to the exact solution

\[
y = x_2(x_2 - 1) \left( 1 - e^{-\frac{r_1}{x_1}} + e^{-\frac{x_1 - x_2}{x_1}} - 2e^{-\frac{1}{x_2}} \right),
\]

\[
p = x_2(1 - x_2)(1 - e^{-\frac{r_1}{x_1}})(1 - e^{-\frac{x_1 - x_2}{x_1}}),
\]

\[
u = -p.
\]

In the computation, we set \( \epsilon = 0.001 \).

For comparison purposes, we first test the proposed method on uniform meshes. Table 1 presents the errors in different norms and orders of convergence. All these orders are less than optimal order of convergence achieved by using nonuniform anisotropic meshes. In addition, Fig. 3 plots the numerical state \( y_h \) and the numerical adjoint state \( p_h \), and Fig. 4 plots the error between the numerical approximations \( (p_h, u_h) \) and exact solutions \( (p, u) \) in the domain \( \Omega \). These two figures illustrate that the error near the boundary \( x = 0 \) and \( x = 1 \) deteriorates the accuracy of the numerical method.

Next, we present the numerical results obtained on nonuniform anisotropic meshes for the same boundary layer control problem. It can be seen from Table 2 that the finite element solution and the post-processed solution on anisotropic meshes exhibit optimal convergence in \( L^2 \) norm and superconvergence of the second order.
Table 1. Errors and convergence rates on uniform meshes.

| $N$ | $\|u - u_h\|_0$ | $\|\Pi_h y - y_h\|$ | $\|\Pi_h p - p_h\|$ | $\|y - \Pi_{2h}y_h\|$ | $\|p - \Pi_{2h}p_h\|$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
|     | 4.1470E-02      | 5.0533E-02      | 4.8403E-02      | 1.7466E-01      | 1.7207E-01      |
|     | order           | order           | order           | order           | order           |
|     | 0.5869          | 0.5225          | 0.4745          | 0.8424          | 0.8276          |
|     |                 |                 |                 |                 |                 |
|     | 2.7610E-02      | 3.5215E-02      | 3.4837E-02      | 9.7410E-02      | 9.6955E-02      |
|     | order           | order           | order           | order           | order           |
|     | 0.6478          | 0.5242          | 0.5126          | 0.7515          | 0.7468          |
|     |                 |                 |                 |                 |                 |
|     | 1.7622E-02      | 2.4487E-02      | 2.4420E-02      | 5.7862E-02      | 5.7777E-02      |
|     | order           | order           | order           | order           | order           |
|     | 0.7250          | 0.5616          | 0.5596          | 0.6899          | 0.6888          |
|     |                 |                 |                 |                 |                 |
|     | 1.0661E-02      | 1.6591E-02      | 1.6569E-02      | 3.5868E-02      | 3.5844E-02      |
|     | order           | order           | order           | order           | order           |
|     | 0.9258          | 0.7398          | 0.7419          | 0.7323          | 0.7332          |
|     |                 |                 |                 |                 |                 |
|     | 5.6118E-03      | 9.9351E-03      | 9.9076E-03      | 2.1591E-02      | 2.1562E-02      |
|     | order           | order           | order           | order           | order           |
|     | 0.9258          | 0.7398          | 0.7419          | 0.7323          | 0.7332          |

Figure 3. The profile of $y_h$ (left plot) and $p_h$ (right plot) on a uniform mesh with $N = 8$.

Figure 4. Pointwise errors of $|y - y_h|$ (left plot) and $|p - p_h|$ (right plot) on a uniform mesh with $N = 8$. 
Table 2. Errors and convergence rates on anisotropic meshes.

| $N$ | 4     | 8     | 16    | 32    | 64    |
|-----|-------|-------|-------|-------|-------|
| $\|u - u_h\|_0$ | 1.1762E-02 | 3.0109E-03 | 7.5308E-04 | 1.8747E-04 | 4.6824E-05 |
| order | /     | 1.9659 | 1.9993 | 2.0062 | 2.0013 |
| $||\Pi_h y - y_h||$ | 8.5488E-03 | 2.9582E-03 | 8.8988E-04 | 2.3807E-04 | 6.2920E-05 |
| order | /     | 1.5310 | 1.7330 | 1.9022 | 1.9198 |
| $||\Pi_h p - p_h||$ | 6.3618E-03 | 2.5354E-03 | 7.8790E-04 | 2.1187E-04 | 5.8477E-05 |
| order | /     | 1.3272 | 1.6861 | 1.8949 | 1.8572 |
| $||y - \Pi^{2h} y_h||$ | 1.2562E-02 | 4.3649E-03 | 1.2406E-03 | 3.0455E-04 | 7.6278E-05 |
| order | /     | 1.5250 | 1.8149 | 2.0263 | 1.9974 |
| $||p - \Pi^{2h} p_h||$ | 1.0955E-02 | 4.0558E-03 | 1.1598E-03 | 2.8267E-04 | 7.3036E-05 |
| order | /     | 1.4335 | 1.8061 | 2.0367 | 1.9524 |

Figure 5. The profile of $y_h$ (left plot) and $p_h$ (right plot) on an anisotropic mesh with $N = 8$.

in the $\epsilon$-norm respectively, which confirm the theoretical analysis. Fig.5 shows the numerical state $y_h$ and the numerical adjoint state $p_h$ on anisotropic meshes when $N = 8$, while errors between the numerical approximations $(p_h, u_h)$ and exact solutions $(p, u)$ are plotted in Fig.6. We observe that the finite element solution on this anisotropic mesh successfully resolve the boundary scales near the domain boundary $x = 0$ and $x = 1$ so that the supercloseness and superconvergence properties of the finite element approximation can be restored.

Acknowledgments. The authors thank the anonymous referees for their critical and constructive comments which improved the presentation and the quality of this paper.

REFERENCES

[1] A. Allendes, E. Hernández and E. Otárola, A robust numerical method for a control problem involving singularly perturbed equations, Computers and Mathematics with Applications, 72 (2016), 974–991.
[2] T. Apel and G. Lube, Anisotropic mesh refinement for a singularly perturbed reaction diffusion model problem, Applied Numerical Mathematics, 26 (1998), 415–433.
Figure 6. Pointwise errors of $|y - y_h|$ (left plot) and $|p - p_h|$ (right plot) on an anisotropic mesh with $N = 8$.

[3] R. Becker, H. Kapp and R. Rannacher, Adaptive finite element methods for optimal control of partial differential equations: Basic concept, *SIAM Journal on Control and Optimization*, 39 (2000), 113–132.

[4] J. Bonnans and H. Zidani, Optimal control problems with partially polyhedric constraints, *SIAM Journal on Control and Optimization*, 37 (1999), 1726–1741.

[5] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Berlin, 1994.

[6] P. Das, An a posteriori based convergence analysis for a nonlinear singularly perturbed system of delay differential equations on an adaptive mesh, *Numerical Algorithms*, 81 (2019), 465–487.

[7] R. Durán and A. Lombardi, Error estimates on anisotropic $Q_1$ elements for functions in weighted Sobolev spaces, *Mathematics of Computation*, 74 (2005), 1679–1706.

[8] R. S. Falk, Approximation of a class of optimal control problems with order of convergence estimates, *Journal of Mathematical Analysis and Applications*, 44 (1973), 28–47.

[9] W. Gong and N. N. Yan, Adaptive finite element method for elliptic optimal control problems: Convergence and optimality, *Numerische Mathematik*, 135 (2017), 1121–1170.

[10] W. Gong, H. P. Liu and N. N. Yan, Adaptive finite element method for parabolic equations with Dirac measure, *Computer Methods in Applied Mechanics and Engineering*, 328 (2018), 217–241.

[11] H. B. Guan and D. Y. Shi, A high accuracy NFEM for constrained optimal control problems governed by elliptic equations, *Applied Mathematics and Computation*, 245 (2014), 382–390.

[12] H. B. Guan, D. Y. Shi and X. F. Guan, High accuracy analysis of nonconforming MFEM for constrained optimal control problems governed by Stokes equations, *Applied Mathematics Letters*, 53 (2016), 17–24.

[13] H. B. Guan and D. Y. Shi, An efficient NFEM for optimal control problems governed by a bilinear state equation, *Computers and Mathematics with Applications*, 77 (2019), 1821–1827.

[14] W. Hackbusch, *Multigrid Methods and Applications*, Springer-Verlag, Berlin, 1985.

[15] M. Hinze, A variational discretization concept in control constrained optimization: The linear-quadratic case, *Computational Optimization and Applications*, 30 (2005), 45–63.

[16] S. Kumar and M. Kumar, An analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems, *Journal of Computational and Applied Mathematics*, 281 (2015), 250–262.

[17] S. Kumar and S. C. S. Rao, A robust domain decomposition algorithm for singularly perturbed semilinear systems, *International Journal of Computer Mathematics*, 94 (2017), 1108–1122.

[18] J. C. Li, Convergence and superconvergence analysis of finite element methods on highly nonuniform anisotropic meshes for singularly perturbed reaction-diffusion problems, *Applied Numerical Mathematics*, 36 (2001), 129–154.
[19] J. C. Li and M. F. Wheeler, Uniform convergence and superconvergence of mixed finite element methods on anisotropically refined grids, *SIAM Journal on Numerical Analysis*, 38 (2000), 770–798.

[20] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.

[21] Q. Lin, L. Tobiska and A.H. Zhou, Superconvergence and extrapolation of non-conforming low order finite elements applied to the Poisson equation, *IMA Journal of Numerical Analysis*, 25 (2005), 160–181.

[22] L. B. Liu and Y. P. Chen, An adaptive moving grid method for a system of singularly perturbed initial value problems, *Journal of Computational and Applied Mathematics*, 274 (2015), 11–22.

[23] W. B. Liu and N. N. Yan, A posteriori error estimates for control problems governed by Stokes equations, *SIAM Journal on Numerical Analysis*, 40 (2002), 1850–1869.

[24] G. Lube and B. Tews, Optimal control of singularly perturbed advection-diffusion-reaction problems, *Mathematical Models and Methods in Applied Sciences*, 20 (2010), 375–395.

[25] J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1995.

[26] H.-G. Roos, Layer-adapted grids for singular perturbation problems, *Journal of Applied Mathematics and Mechanics*, 78 (1998), 291–309.

[27] H.-G. Roos and C. Reibiger, Numerical analysis of a system of singularly perturbed convection-diffusion equations related to optimal control, *Numerical Mathematics: Theory, Methods and Applications*, 4 (2011), 562–575.

[28] Z. M. Zhang, Finite element superconvergence on Shishkin mesh for 2-D convection-diffusion problems, *Mathematics of Computation*, 72 (2003), 1147–1177.

[29] Z. M. Zhang and H. Q. Zhu, Uniform convergence of the LDG method for a singularly perturbed problem with the exponential boundary layer, *Mathematics of Computation*, 83 (2014), 635–663.

[30] H. Q. Zhu and Z. M. Zhang, Convergence analysis of the LDG method applied to singularly perturbed problems, *Numerical Methods for Partial Differential Equations*, 29 (2013), 396–421.

Received May 2019; revised February 2020.

E-mail address: guanhongbo@zzuli.edu.cn
E-mail address: yongyang@nuaa.edu.cn
E-mail address: Huiqing.Zhu@usm.edu