Viewing nonoscillatory second order linear differential equations from the angle of Riccati equations

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Abstract. We build an existence theory for second order linear differential equations of the form

\[(p(t)x')' = q(t)x,\] (A)

\(p(t)\) and \(q(t)\) being positive continuous functions on \([a, \infty)\), in which a crucial role is played by a pair of the Riccati differential equations

\[u' = q(t) - \frac{u^2}{p(t)} \quad (R1), \quad v' = \frac{1}{p(t)} - q(t)v^2 \quad (R2)\]

associated with (A). An essential part of the theory is the construction of a pair of linearly independent nonoscillatory solutions \(x_1(t)\) and \(x_2(t)\) of (A) enjoying explicit exponential-integral representations in terms of solutions \(u_1(t)\) and \(u_2(t)\) of (R1) or in terms of solutions \(v_1(t)\) and \(v_2(t)\) of (R2).

Keywords: Linear differential equation, non-oscillatory solutions, Riccati equation.

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1 Introduction

Consider the second order linear differential equation

\[(p(t)x')' = q(t)x,\] (A)

where \(p : [a, \infty) \to (0, \infty)\) and \(q : [a, \infty) \to \mathbb{R}, a \geq 0,\) are continuous functions.

We are concerned exclusively with nontrivial solutions of (A) defined in some neighborhood of infinity, that is, on an interval of the form \([T, \infty), T \geq a,\) Such a solution is called oscillatory if it has an infinite sequence of zeros clustering at infinity, and nonoscillatory otherwise.

It is known that (A) does not admit both oscillatory solutions and nonoscillatory solutions simultaneously, that is to say, all solutions of (A) are either oscillatory, in which case (A) is

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suggested to be oscillatory, or else nonoscillatory, in which case (A) is said to be nonoscillatory. For example, if \( q(t) > 0 \) on \([a, \infty)\), then equation (A) is always nonoscillatory, whereas if \( q(t) < 0 \) on \([a, \infty)\), then it is possible that (A) is either oscillatory or nonoscillatory depending on the delicate interrelations between \( p(t) \) and \( q(t) \). In order to detect criteria for oscillation of equation (A) with \( q(t) < 0 \) frequent use has been made of the first order nonlinear differential equation

\[
u' + \frac{u^2}{p(t)} - q(t) = 0, \quad (R1)
\]

which is referred to as the Riccati differential equation associated with (A). This (R1) is derived as a differential equation satisfied by the function \( u(t) = p(t)x'(t)/x(t) \) for any nonoscillatory solution \( x(t) \) of (A) defined in a neighborhood of infinity. A solution of (R1) existing for all large \( t \) is named a **global solution** of (R1). Conversely, if \( u(t) \) is a global solution of (R1) on \([T, \infty)\), then \( x(t) = \exp(\int_t^T u(s)/p(s)ds) \) gives a solution of (A) on that interval. Thus there exists a noteworthy relationship between (A) and (R1) as described in the following

**Proposition 1.1.** Equation (A) is nonoscillatory if and only if equation (R1) has a global solution.

On the basis of this fact a wealth of oscillation criteria for equation (A) have been obtained in the literature; see e.g., Swanson [9]. Those oscillation criteria for (A) represent one side of the usefulness of Proposition 1.1. However, there seems to be almost no past study shedding light on another side of its power, namely the possibility of a detailed qualitative study of nonoscillatory solutions of (A) with the aid of global solutions of (R1). A result (the only one we know of) in this regard can be found in the book of Hille [5, Theorem 9.4.3] concerning the nonoscillatory solutions of (A) with the aid of global solutions of (R1).

In this paper an attempt is made to develop a new nonoscillation theory for the purely nonoscillatory equation (A) with \( q(t) > 0 \) by means of the Riccati equation (R1) and its equivalent

\[
v' = \frac{1}{p(t)} - q(t)v^2. \quad (R2)
\]

Note that (R2) was recently found by Mirzov [7] as the equation satisfied by \( v(t) = x(t)/p(t)x'(t) \), where \( x(t) \) is any nonoscillatory solution of (A).

To specify what we are going to do with (A) and (R1) - (R2) we need to recollect some basic facts about the solution set \( S(A) \) of equation (A).

(1) \( S(A) \neq \emptyset \) and all of its members are nonoscillatory.

(2) \( S(A) \) is a two-dimensional linear space over the reals \( \mathbb{R} \). As a basis for \( S(A) \) one can take a pair of positive solutions \( \{x_1(t), x_2(t)\} \) of (A) satisfying

\[
\lim_{t \to \infty} \frac{x_1(t)}{x_2(t)} = 0, \quad (1.1)
\]

or

\[
\int_0^\infty \frac{dt}{p(t)x_1(t)^2} = \infty \quad \text{and} \quad \int_0^\infty \frac{dt}{p(t)x_2(t)^2} < \infty. \quad (1.2)
\]

See Hartman [4, Theorem 6.4; Corollary 6.4]. The functions \( x_1(t) \) and \( x_2(t) \) are called, respectively, a principal solution and a nonprincipal solution of (A). A principal solution is uniquely determined up to a constant factor. See [4, Theorem 6.4]. It may be assumed that \( x_1(t) \) is decreasing and \( x_2(t) \) is increasing; see [4, Corollary 6.4].

(3) If \( x(t) \) is any solution of (A) on \([T, \infty)\), then

\[
x_2(t) = x(t) \int_T^t \frac{ds}{p(s)x(s)^2} \quad (1.3)
\]
is a nonprincipal solution on $[T, \infty)$. If, in addition, $x(t)$ is a nonprincipal solution of (A), then

$$x_1(t) = x(t) \int_t^\infty \frac{ds}{p(s)x(s)^2} \tag{1.4}$$

is a principal solution of (A) on $[T, \infty)$. See [4, Corollary 6.3].

(4) Information on the possible asymptotic behavior of $x_1(t)$ and $x_2(t)$ forming a basis for $S(A)$ follows. We need to discuss the three cases $\{I_p = \infty\}$, $\{I_q = \infty\}$ and $\{I_p < \infty \land I_q < \infty\}$ separately, where

$$I_p = \int_a^\infty \frac{dt}{p(t)} \quad \text{and} \quad I_q = \int_a^\infty q(t)dt. \tag{1.5}$$

(i) Suppose that $I_p = \infty$. Then $x_1(t)$ and $x_2(t)$ satisfy either

$$\lim_{t \to \infty} x_1(t) = \text{const} > 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{x_2(t)}{P(t)} = \text{const} > 0, \tag{1.6}$$

or

$$\lim_{t \to \infty} x_1(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{x_2(t)}{P(t)} = \infty, \tag{1.7}$$

where $P(t)$ is defined by

$$P(t) = \int_t^1 \frac{ds}{p(s)}. \tag{1.8}$$

(ii) Suppose that $I_q = \infty$. Then $x_1(t)$ and $x_2(t)$ satisfy either

$$\lim_{t \to \infty} \frac{x_1(t)}{\pi(t)} = \text{const} > 0 \quad \text{and} \quad \lim_{t \to \infty} x_2(t) = \text{const} > 0, \tag{1.9}$$

or

$$\lim_{t \to \infty} \frac{x_1(t)}{\pi(t)} = 0 \quad \text{and} \quad \lim_{t \to \infty} x_2(t) = \infty, \tag{1.10}$$

where $\pi(t)$ is given by

$$\pi(t) = \int_t^\infty \frac{ds}{p(s)}. \tag{1.11}$$

(iii) Suppose that $I_p < \infty \land I_q < \infty$. Then $x_1(t)$ and $x_2(t)$ satisfy

$$\lim_{t \to \infty} \frac{x_1(t)}{\pi(t)} = \text{const} > 0 \quad \text{and} \quad \lim_{t \to \infty} x_2(t) = \text{const} > 0. \tag{1.12}$$

Solutions $x_1(t)$ and $x_2(t)$ satisfying (1.6), (1.9) and (1.12) are called moderate solutions of (A), while those satisfying (1.7) and (1.10) are called extreme solutions of (A). A pair of solutions $\{x_1(t), x_2(t)\}$ is termed a moderate basis or an extreme basis for $S(A)$ according to whether both $x_1(t)$ and $x_2(t)$ are moderate solutions or extreme solutions of (A). It will turn out later that no basis for $S(A)$ is allowed to be a combination of a moderate solution and an extreme solution.

Is it possible to establish criteria for judging the existence of a moderate basis or an extreme basis for equation (A)? Answering this question regarding moderate bases is relatively easy, and as is described in Section 3 one can find necessary and sufficient conditions for (A) to possess a moderate basis whose components $x_1(t)$ and $x_2(t)$ satisfy (1.6), (1.9) or (1.12). It is demonstrated that all moderate bases can be constructed by way of the Riccati equations (R1) and (R2), both of which are indispensable in the construction process. Although the Riccati differential equations have a long history (see e.g., Sansone [8]), their potential for reproduction of solutions of the associated second order linear differential equations has not been recognized until now. We note
that some results of Section 3 are known (see e.g.,[1], [2], [6]), but the derivation is essentially
different.

What about extreme bases for (A)? This question seems to be extremely difficult to answer
for now. As far as we are aware, historically no systematic study has been made of the existence
and asymptotic behavior of extreme solutions of (A). However, it is possible to detect several
nontrivial cases in which extreme solutions are produced by effective application of (R1) and
(R2). The details of how to develop a tiny theory of extreme solutions of (A) are provided in
Section 4. Examples illustrating all of our theorems are given in Section 5.

What is requisite in building our theory for (A) are an a priori knowledge of the overall
structure of solutions of (A), or equivalently, the classification of $S(A)$ into appropriate disjoint
subclasses according to the asymptotic behavior at infinity of its members, and understanding
of the Riccati equations (R1) and (R2) as an instrument for reproducing solutions of (A) from
their solutions. All these things together are explained in the preparatory Section 2.

2 Preparatory observations

2.1 Classification of nonoscillatory solutions of (A)

Let $x(t)$ be a (nonoscillatory) solution of equation (A). There exists $T \geq a$ such that $x(t)DX(t) \neq
0$ (i.e.,$x(t)x'(t) \neq 0$) for $t \geq T$. Since (A) implies that both $x(t)$ and $DX(t)$ are monotone on
$[T, \infty)$, they have the limits $x(\infty) = \lim_{t \to \infty} x(t)$ and $DX(\infty) = \lim_{t \to \infty} DX(t)$ in the extended
real number system. The pair $(x(\infty), DX(\infty))$ is a crucial indicator of the asymptotic behavior of
a solution $x(t)$ of (A) as $t \to \infty$. It is referred to as the terminal state of $x(t)$. Our purpose
here is to show that one can enumerate in advance all possible patterns of terminal states of
solutions of (A) by considering the three cases $I_p = \infty, I_q = \infty$ and $I_p < \infty \land I_q < \infty$ separately
(cf. (1,5)).

(I) We start with the case $I_p = \infty$. Let $x(t)$ be a solution of (A) such that $x(t)DX(t) > 0$ on
$[T, \infty)$. By (A) this means that $|x(t)|$ and $|DX(t)|$ are increasing for $t \geq T$. The limit $|DX(\infty)|$
may or may not be finite. In either case we have $|x(\infty)| = \infty$. In fact, if $x(t)$ is positive, then
from the inequality

$$DX(t) = p(t)x'(t) \geq DX(T) \quad \text{or} \quad x'(t) \geq \frac{DX(T)}{p(t)}, \quad t \geq T,$$

we see that $x(\infty) = \infty$. If $x(t)$ is negative, repeating the same argument with $x(t)$ replaced by
$-x(t)$, we are led to the conclusion that $|x(\infty)| = \infty$.

Let $x(t)$ be a solution of (A) satisfying $x(t)DX(t) < 0$ on $[T, \infty)$. Then, by (A) $|x(t)|$ is
decreasing and $|DX(t)|$ is increasing for $t \geq T$. The limit $|x(\infty)|$ may or may not be zero. In
either case we have $|DX(\infty)| = 0$. Suppose for contradiction that $|DX(\infty)| = k > 0$. If $x(t)$ is
positive, then since $DX(t)$ is negative, we obtain

$$DX(t) = p(t)x'(t) \leq -k \quad \text{or} \quad x'(t) \leq \frac{-k}{p(t)}, \quad t \geq T,$$

which implies $x(\infty) = -\infty$, an impossibility. If $x(t)$ is negative, it suffices to repeat the same
argument with $x(t)$ replaced by $-x(t)$.

From the above discussions it is concluded that for a solution $x(t)$ of (A) satisfying $x(t)DX(t) >
0$ for all large $t$ the following two types of its terminal state are possible

I(i) \quad \quad |x(\infty)| = \infty \quad \text{and} \quad 0 < |DX(\infty)| < \infty,
and that for a solution \( x(t) \) of (A) satisfying \( x(t)Dx(t) < 0 \) for all large \( t \) the following two types of terminal states are possible

I(iii) \( 0 < |x(\infty)| < \infty \quad \text{and} \quad |Dx(\infty)| = 0, \)

I(iv) \( |x(\infty)| = 0 \quad \text{and} \quad |Dx(\infty)| = 0. \)

(II) We turn to the case \( I_q = \infty \). Let \( x(t) \) be any solution of (A). If \( x(t)Dx(t) > 0 \) on \( [T, \infty) \), then \( |x(t)| \) and \( |Dx(t)| \) are increasing for \( t \geq T \). The limit \( |x(\infty)| \) may be finite or infinite, but in either case we have \( |Dx(\infty)| = \infty \). In fact, if \( x(t) \) is positive, then integrating (A) from \( T \) to \( t \)

\[
Dx(t) = Dx(T) + \int_T^t q(s)x(s)ds \geq x(T) \int_T^t q(s)ds \to \infty, \quad t \to \infty,
\]

which implies that \( Dx(\infty) = \infty \). The same argument as above applies to \( -x(t) \) leads us to \( Dx(\infty) = -\infty \).

If \( x(t)Dx(t) < 0 \) on \( [T, \infty) \), then \( |x(t)| \) is decreasing and \( |Dx(t)| \) is increasing. The limit \( |Dx(\infty)| \) may be finite or infinite, but in either case we have \( |x(\infty)| = 0 \). Suppose to the contrary that \( |x(\infty)| = c > 0 \). If \( x(t) \) is positive, then integrating (A) on \( [T, t] \), we obtain

\[
Dx(t) = Dx(T) + \int_T^t q(s)x(s)ds \geq Dx(T) + c \int_T^t q(s)ds \to \infty, \quad t \to \infty,
\]

which implies \( Dx(\infty) = \infty \). But this contradicts the negativity of \( Dx(t) \). Similarly, if \( x(t) \) is negative, we are led to the contradiction that \( Dx(\infty) \to -\infty \) as \( t \to \infty \).

Summarizing what is said above, we conclude that for a solution \( x(t) \) of (A) such that \( x(t)Dx(t) > 0 \) for all large \( t \) the following two types of its terminal state are possible

II(i) \( 0 < |x(\infty)| < \infty \quad \text{and} \quad |Dx(\infty)| = \infty, \)

II(ii) \( |x(\infty)| = \infty \quad \text{and} \quad |Dx(\infty)| = \infty, \)

and that a solution \( x(t) \) of (A) such that \( x(t)Dx(t) < 0 \) for all large \( t \) the following two types of its terminal state are possible

II(iii) \( |x(\infty)| = 0 \quad \text{and} \quad 0 < |Dx(\infty)| < \infty, \)

II(iv) \( |x(\infty)| = 0 \quad \text{and} \quad |Dx(\infty)| = 0. \)

(III) Finally we are concerned with the case \( I_p < \infty \land I_q < \infty \). It can be shown that if \( x(t) \) is a solution of (A) such that \( x(t)Dx(t) > 0 \) for all large \( t \), then its terminal state satisfies

III(i) \( 0 < |x(\infty)| < \infty \quad \text{and} \quad 0 < |Dx(\infty)| < \infty, \)

and that if \( x(t) \) is a solution of (A) such that \( x(t)Dx(t) < 0 \) for all large \( t \), then its terminal state satisfies either

III(ii) \( |x(\infty)| = 0 \quad \text{and} \quad 0 < |Dx(\infty)| < \infty, \)
III(iii) \[ 0 < |x(\infty)| < \infty \quad \text{and} \quad |Dx(\infty)| = 0. \]

It suffices to verify these facts for eventually positive solutions of (A). Let \( x(t) \) be a solution of (A) such that \( x(t) > 0 \) and \( Dx(t) > 0 \) on \([T, \infty)\), where \( T > a \) is chosen so that

\[
\int_T^\infty \frac{1}{p(s)} \int_T^s q(r) dr ds \leq \frac{1}{2}. \tag{2.1}
\]

Integrating (A) twice on \([T, t]\), we have

\[
Dx(t) = Dx(T) + \int_T^t q(s)x(s) ds, \tag{2.2}
\]

\[
x(t) = x(T) + \int_T^t Dx(T) p(s) ds + \int_T^t q(s)x(r) dr ds, \tag{2.3}
\]

for \( t \geq T \). From (2.1) and (2.3) we see that

\[
\frac{1}{2} x(t) \leq x(T) + Dx(T) \int_T^\infty \frac{ds}{p(s)}, \quad t \geq T,
\]

which implies that \( 0 < x(\infty) < \infty \). This combined with (2.2) gives

\[
Dx(t) \leq Dx(T) + x(t) \int_T^t q(s) ds \leq Dx(T) + x(\infty) \int_T^\infty q(s) ds, \quad t \geq T,
\]

which implies that \( 0 < Dx(\infty) < \infty \). Thus the terminal state of \( x(t) \) is of the type III(i).

Let \( x(t) \) be a solution of (A) such that \( x(t) > 0 \) and \( Dx(t) < 0 \) for all large \( t \). It is clear that \( 0 \leq x(\infty) < \infty \) and \( -\infty < Dx(\infty) \leq 0 \). But the possibility \( x(\infty) = Dx(\infty) = 0 \) should be excluded because if this would occur, integrating (A) twice from \( t \) to \( \infty \) would give

\[
x(t) = \int_t^\infty \frac{1}{p(s)} \int_s^\infty q(r)x(r) dr ds \leq x(t) \int_t^\infty \frac{1}{p(s)} \int_s^\infty q(r) dr ds,
\]

or

\[
1 \leq \int_t^\infty \frac{1}{p(s)} \int_s^\infty q(r) dr ds \to 0, \quad t \to \infty,
\]
a contradiction. Therefore the terminal state of \( x(t) \) must be either of the type III(ii) or of the type III(iii).

Solutions of (A) having the terminal states of the types I(i), I(iii), II(i), II(iii), III(i) - III(iii) are named moderate solutions of (A), whereas those having the terminal states of the types I(ii), I(iv), II(ii), II(iv) are named extreme solutions of (A).

Remark 2.1. (i) Suppose that \( I_p = \infty \wedge I_q = \infty \). Since both (I) and (II) apply to (A), we see that the only possible types of terminal states of its solutions are I(ii) (= II(ii)) and I(iv) (=II(iv)). In other words all solutions of (A) must be extreme solutions.

(ii) Suppose that \( I_p < \infty \wedge I_q < \infty \). In this case all solutions of (A) are moderate solutions which are bounded together with their quasi-derivatives for all large \( t \).
2.2 Riccati equations associated with (A)

Let \( x(t) \) be any nonoscillatory solution of equation (A). Then \( x(t)x'(t) \neq 0 \) on \([T, \infty)\) for some \( T \geq a \). Define

\[
  u(t) = \frac{p(t)x'(t)}{x(t)} \quad \text{and} \quad v(t) = \frac{x(t)}{p(t)x'(t)}.
\]

It is elementary to show that \( u(t) \) and \( v(t) \) satisfy, respectively, the first order nonlinear differential equations (R1) and (R2) on \([T, \infty)\) which are referred to as the first and second Riccati equations associated with (A).

Conversely, suppose that (R1) has a global solution \( u(t) \) on \([T, \infty)\). Bearing (2.1) in mind define \( x(t) \) by

\[
  x(t) = \exp\left(\int_{T}^{t} \frac{u(s)}{p(s)} ds\right) \quad \text{or} \quad x(t) = \exp\left(-\int_{t}^{\infty} \frac{u(s)}{p(s)} ds\right), \quad t \geq T,
\]

according to whether \( \int_{T}^{\infty} \frac{u(t)}{p(t)} dt \) is divergent or convergent. It is obvious that (2.5) gives a nonoscillatory solution of (A) defined for \( t \geq T \). Thus the truth of Proposition 1.1 is verified.

We may well call (2.5) a reproducing formula for solutions of (A) by way of Riccati equation (R1), or from solutions of (R1).

We note that an alternative reproducing formula for (A) by way of (R1)

\[
  x(t) = \frac{1}{u(t)} \exp\left(\int_{T}^{t} \frac{q(s)}{u(s)} ds\right) \quad \text{or} \quad x(t) = \frac{1}{u(t)} \exp\left(-\int_{t}^{\infty} \frac{q(s)}{u(s)} ds\right), \quad t \geq T,
\]

can be derived without difficulty. It suffices to obtain

\[
  \frac{(p(t)x'(t))'}{p(t)x'(t)} = \frac{q(t)}{u(t)}, \quad t \geq T,
\]

by combining (2.1) rewritten as \( x(t) = (p(t)x'(t))u(t) \) with (A) rewritten as \( x(t) = (p(t)x'(t))'/q(t) \), and then to integrate (2.7) over \([T, t]\) or \((t, \infty)\) according as \( q(t)/u(t) \) is non-integrable or integrable on \([T, \infty)\).

Similarly, reproducing solutions of (A) by way of the second Riccati equation (R2) can be carried out via the following formulas corresponding to (2.5) and (2.6):

\[
  x(t) = \exp\left(\int_{T}^{t} \frac{ds}{p(s)v(s)}\right) \quad \text{or} \quad x(t) = \exp\left(-\int_{t}^{\infty} \frac{ds}{p(s)v(s)}\right), \quad t \geq T,
\]

and

\[
  x(t) = v(t) \exp\left(\int_{T}^{t} q(s)v(s)ds\right) \quad \text{or} \quad x(t) = v(t) \exp\left(-\int_{t}^{\infty} q(s)v(s)ds\right), \quad t \geq T.
\]

One of the central problems studied in this paper is the reproduction of a pair of linearly independent solutions of equation (A) from suitable global solutions of the Riccati equations (R1) and/or (R2). Let \( \{x_1(t), x_2(t)\} \) be a pair of linearly independent solutions of (A) such that \( x_i(t)x_i'(t) \neq 0 \) for \( t \geq T, i = 1, 2 \). From the well-known identity

\[
  p(t)[x_2(t)x_1'(t) - x_1(t)x_2'(t)] = C \neq 0 \quad \text{(a constant)}, \quad t \geq T,
\]

it follows that

\[
  \frac{p(t)x_1'(t)}{x_1(t)} - \frac{p(t)x_2'(t)}{x_2(t)} = \frac{C}{x_1(t)x_2(t)}.\]
and
\[
\frac{x_1(t)}{p(t)x_1'(t)} - \frac{x_2(t)}{p(t)x_2'(t)} = \frac{C}{p(t)^2x_1'(t)x_2'(t)},
\]
for \(t \geq T\). This implies that if \(x_i(t), i = 1, 2\), are reproduced from a pair of solutions \(u_i(t), i = 1, 2\), of (R1), then \(u_1(t) \neq u_2(t)\) for all \(t \geq T\), and that the same is true of a pair of solutions \(v_i(t), i = 1, 2\), of (R2) which reproduce \(x_i(t), i = 1, 2\).

**Remark 2.2.** Since a solution \(x(t)\) of (A) given by (2.5) or (2.6) satisfies \(x(t)Dx(t) = u(t)x(t)^2\), it is a principal solution or a nonprincipal solution according as \(u(t)\) is eventually positive or negative. Similarly, a solution \(x(t)\) of (A) given by (2.8) or (2.9) satisfies \(x(t)Dx(t) = x(t)^2/v(t)\), it is a principal solution or a nonprincipal solution according as \(v(t)\) is eventually positive or negative.

### 3 Moderate solutions of (A)

The main purpose of this section is to demonstrate that all moderate solutions of equation (A) can be reproduced from global solutions of the associated Riccati equations (R1) and (R2).

We first consider (A) with \(p(t)\) satisfying \(I_p = \infty\). Assume that (A) has a moderate solution \(x(t)\) such that \(x(t)Dx(t) \neq 0\) for \(t \geq T\). We may suppose that \(x(t)\) is positive on \([T, \infty)\). Let \(x_1(t)\) be a solution of the type I(i). Then it satisfies
\[
\lim_{t \to \infty} \frac{x_1(t)}{P(t)} = \lim_{t \to \infty} Dx_1(t) = d,
\]
for some \(d > 0\). Integrating the second equation of (A) from \(t\) to \(\infty\), we have
\[
Dx_1(t) = d - \int_t^\infty q(s)x_1(s)ds, \quad t \geq T,
\]
which means the integrability of \(q(t)x_1(t)\) on \([T, \infty)\). This fact combined with (3.1) gives
\[
\int_a^\infty P(t)q(t)dt < \infty. \tag{3.2}
\]

Let \(x_2(t)\) be a solution of class I(iii) of (A). It clearly satisfies
\[
\lim_{t \to \infty} x_2(t) = c \quad \text{and} \quad \lim_{t \to \infty} \frac{Dx_2(t)}{\rho(t)} = -c, \tag{3.3}
\]
for some \(c > 0\), where \(\rho(t)\) is defined by
\[
\rho(t) = \int_t^\infty q(s)ds, \quad t \geq a. \tag{3.4}
\]

Integrating (A) twice on \([t, \infty)\), we obtain
\[
x_2(t) = c + \int_t^\infty (P(s) - P(t))q(s)x_2(s)ds, \quad t \geq T,
\]
from which it follows that (3.2) must also be satisfied. Thus we see that (3.2) is a necessary condition for the existence of moderate solutions for (A).

A useful equivalent of (3.2) is given in the following remark.
Remark 3.1. If \( I_p = \infty \), then
\[
\int_a^\infty q(t)P(t)dt < \infty \iff \int_a^\infty \frac{\rho(t)}{p(t)}dt < \infty. \tag{3.5}
\]
In fact, by combining
\[
\int_a^t q(s)P(s)ds + \rho(t)P(t) = \int_a^t \frac{\rho(s)}{p(s)}ds, \quad t \geq a,
\]
with \( \rho(t)P(t) \leq \int_a^\infty q(s)P(s)ds \), we find that \( \int_a^\infty q(s)P(s)ds = \int_a^\infty \rho(s)/p(s)ds \).

It turns out that (3.2) is also a sufficient condition for the existence of moderate solutions of the types I(i) and I(iii) of (A). Suppose that (3.2) and hence (3.5) holds.

Choose \( T > a \) so that
\[
\int_T^\infty \frac{\rho(s)}{p(s)}ds \leq \frac{1}{4}, \tag{3.6}
\]
and denote by \( \mathcal{U} \) the set of functions
\[
\mathcal{U} = \{ u \in C_0[T, \infty) : -\rho(t) \leq u(t) \leq -\frac{1}{2}\rho(t), \quad t \geq T \}, \tag{3.7}
\]
where \( C_0[T, \infty) \) is the totality of continuous functions on \( [T, \infty) \) tending to zero as \( t \to \infty \). It is a Banach space with the norm \( \|u\|_0 = \sup \{|u(t)| : t \geq T\} \). Define the integral operator \( F \) by
\[
Fu(t) = -\rho(t) + \int_t^\infty \frac{u(s)^2}{p(s)}ds, \quad t \geq T, \tag{3.8}
\]
and let it act on \( \mathcal{U} \) which is a closed subset of \( C_0[T, \infty) \).

If \( u \in \mathcal{U} \), then, since
\[
\int_t^\infty \frac{u(s)^2}{p(s)}ds \leq \int_t^\infty \frac{\rho(s)^2}{p(s)}ds \leq \rho(t) \int_t^\infty \frac{\rho(s)}{p(s)}ds \leq \frac{1}{4}\rho(t), \quad t \geq T,
\]
we obtain \(-\rho(t) \leq Fu(t) \leq -\rho(t)/2\) for \( t \geq T \). This shows that \( F \) is a self-map of \( \mathcal{U} \). If \( u_1, u_2 \in \mathcal{U} \), then
\[
|Fu_1(t) - Fu_2(t)| \leq \int_t^\infty \frac{1}{p(s)}|u_1(s)^2 - u_2(s)^2|ds
\]
\[
\leq \int_t^\infty \frac{2\rho(s)}{p(s)}|u_1(s) - u_2(s)|ds \leq 2 \int_t^\infty \frac{\rho(s)}{p(s)}ds \|u_1 - u_2\|_0 \leq \frac{1}{2}\|u_1 - u_2\|_0,
\]
from which it follows that
\[
\|Fu_1 - Fu_2\|_0 \leq \frac{1}{2}\|u_1 - u_2\|_0.
\]
This means that \( F \) is a contraction on \( \mathcal{U} \). Therefore, there exists a unique fixed point \( u \in \mathcal{U} \) which satisfies
\[
u(t) = -\rho(t) + \int_t^\infty \frac{u(s)^2}{p(s)}ds, \quad t \geq T, \tag{3.9}
\]
and hence gives a global (negative) solution of (R1) on \( [T, \infty) \). Note that \(-u(t) \sim \rho(t)\) as \( t \to \infty \). Here the symbol \( \sim \) is used to mean the asymptotic equivalence of two positive functions \( f(t) \) and \( g(t) \);
\[
f(t) \sim g(t), \quad t \to \infty \iff \lim_{t \to \infty} \frac{f(t)}{g(t)} = 1.
\]
Since \(u(t)/p(t)\) is integrable on \([T, \infty)\), we can use the reproducing formula \(2.5\) to define a positive solution of (A) by

\[
x_1(t) = \exp\left(-\int_t^\infty \frac{u(s)}{p(s)} ds\right), \quad t \geq T.
\]  
(3.10)

It is clear that \(x_1(\infty) = 1\). The quasi-derivative of \(x_1(t)\) is given by

\[
 Dx_1(t) = u(t) \exp\left(-\int_t^\infty \frac{u(s)}{p(s)} ds\right), \quad t \geq T,
\]  
(3.11)

and satisfies \(Dx_1(t) \sim -p(t)\) as \(t \to \infty\). Thus \(x_1(t)\) is a positive decreasing solution of the type I(iii), and hence a principal solution of equation (A).

Moderate solutions of the type I(i) of (A) will be reproduced under condition (3.2) by way of the second Riccati equation (R2). Choose \(T > a\) so that

\[
\int_T^\infty q(s)P(s)ds \leq \frac{1}{4},
\]  
(3.12)

and define the set \(\mathcal{V}\) by

\[
\mathcal{V} = \{v \in C_p[T, \infty) : \frac{1}{2}P(t) \leq v(t) \leq P(t), \; t \geq T\},
\]  
(3.13)

where \(C_p[T, \infty)\) denotes the Banach space of continuous functions \(v(t)\) on \([T, \infty)\) such that \(\|v\|_p = \sup\{|v(t)|/P(t) : t \geq T\} < \infty\). Consider the integral operator \(G\) given by

\[
Gv(t) = P(t) - \int_T^t q(s)v(s)^2 ds, \quad t \geq T,
\]  
(3.14)

and let it act on \(\mathcal{V}\). If \(v \in \mathcal{V}\), then since

\[
\int_T^t q(s)v(s)^2 ds \leq \int_T^t q(s)P(s)^2 ds \leq P(t) \int_T^t q(s)P(s)ds \leq \frac{1}{4}P(t),
\]

for \(t \geq T\), we obtain \(P(t)/2 \leq Gv(t) \leq P(t)\) on \([T, \infty)\). This shows that \(G\) maps \(\mathcal{V}\) into itself. If \(v_1, v_2 \in \mathcal{V}\), then from the inequalities

\[
\frac{|Gv_1(t) - Gv_2(t)|}{P(t)} \leq \frac{1}{P(t)} \int_T^t q(s)|v_1(s)^2 - v_2(s)^2| ds
\]

\[
\leq 2 \int_T^t q(s)P(s) \cdot \frac{|v_1(s) - v_2(s)|}{P(s)} ds \leq \frac{1}{2} \|v_1 - v_2\|_p,
\]

we find that

\[
\|Gv_1 - Gv_2\|_p \leq \frac{1}{2} \|v_1 - v_2\|_p,
\]

that is, \(G\) is a contraction on \(\mathcal{V}\). Consequently, \(G\) has a unique fixed point \(v \in \mathcal{V}\), which satisfies

\[
v(t) = P(t) - \int_T^t q(s)v(s)^2 ds, \quad t \geq T,
\]  
(3.15)

and hence the Riccati equation (R2) on \([T, \infty)\). We claim that \(3.15\) implies \(v(t) \sim P(t)\) as \(t \to \infty\). This follows from \(3.15\) if it is confirmed that

\[
\lim_{t \to \infty} \frac{1}{P(t)} \int_T^t q(s)P(s)^2 ds = 0.
\]  
(3.16)
Let $\varepsilon > 0$ be given arbitrarily. Because of (3.2) there exists $t_\varepsilon > T$ such that
\[
\int_{t_\varepsilon}^{\infty} q(s)P(s)ds < \frac{\varepsilon}{2}.
\] (3.17)

Let this $t_\varepsilon$ be fixed and choose $T_\varepsilon > t_\varepsilon$ so that
\[
\frac{1}{P(t)} \int_{T}^{t_\varepsilon} q(s)P(s)^2ds < \frac{\varepsilon}{2}, \quad t \geq T_\varepsilon.
\] (3.18)

Then, using (3.17) and (3.18) we see that if $t > T_\varepsilon$, then
\[
\frac{1}{P(t)} \int_{T}^{t} q(s)P(s)^2ds = \frac{1}{P(t)} \left( \int_{t_\varepsilon}^{t} q(s)P(s)^2ds + \int_{T}^{t_\varepsilon} q(s)P(s)^2ds \right)
\leq \frac{\varepsilon}{2} + \int_{t}^{t_\varepsilon} q(s)P(s)ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\] This clearly implies (3.16), which guarantees the asymptotic equivalence of the solution $v(t)$ of (R2) obtained above and the function $P(t)$ as $t \to \infty$. Let us now construct a nonoscillatory solution $x_2(t)$ of (A)
\[
x_2(t) = v(t) \exp\left(- \int_{t}^{\infty} q(s)v(s)ds\right), \quad t \geq T,
\] (3.19)
according to the reproducing formula (2.9). Its quasi-derivative is given by
\[
Dx_2(t) = \exp\left(- \int_{t}^{\infty} q(s)v(s)ds\right), \quad t \geq T.
\] (3.20)

Since $x_2(t) \sim v(t) \sim P(t)$ and $Dx_2(t) \sim 1$ as $t \to \infty$, $x_2(t)$ is a positive moderate solution of the type I(i) and hence a nonprincipal solution of equation (A).

Note that by putting $u(t) = 1/v(t)$, (3.19) and (3.20) are rewritten as
\[
x_2(t) = \frac{1}{u(t)} \exp\left(- \int_{t}^{\infty} \frac{q(s)}{u(s)}ds\right) \quad \text{and} \quad Dx_2(t) = \exp\left(- \int_{t}^{\infty} \frac{q(s)}{u(s)}ds\right).
\]

Since $u(t)$ is a solution of (R1), this allows us to assert that a solution $x_2(t)$ of the type (I)(i) of (A) can also be reproduced from a global solution of the first Riccati equation (R1).

Summarizing what are discussed above, we obtain the following theorem which characterizes the structure of the totality of moderate solutions of (A) for the case $I_p = \infty$.

**Theorem 3.2.** Assume that $I_p = \infty$. All solutions of equation (A) are moderate if and only if the condition
\[
\int_{a}^{\infty} q(t)P(t)dt < \infty \quad \text{or equivalently,} \quad \int_{a}^{\infty} \frac{\rho(t)}{P(t)}dt < \infty
\]
is satisfied. In this case (A) has a moderate basis consisting of solutions $x_1(t)$ and $x_2(t)$ which are reproduced from global solutions of (R1) (or (R2)) and satisfy, as $t \to \infty$,
\[
x_1(t) \sim 1, \quad Dx_1(t) \sim -\rho(t), \quad x_2(t) \sim P(t), \quad Dx_2(t) \sim 1.
\] (3.21) (3.22)

Let us now turn our attention to equation (A) with $q(t)$ satisfying $I_q = \infty$. Use is made of the notation
\[
Q(t) = \int_{a}^{t} q(s)ds, \quad t \geq a.
\] (3.23)

The situation in which (A) has moderate solutions of the types II(i) and II(iii) can be characterized as the following theorem shows.
Theorem 3.3. Assume that $I_q = \infty$. All solutions of equation (A) are moderate if and only if the condition
\[
\int_a^\infty q(t) \pi(t) dt < \infty \quad \text{(or equivalently, } \int_a^\infty \frac{Q(t)}{p(t)} dt < \infty) ,
\]  
(3.24)
is satisfied. In this case (A) has a moderate basis consisting of solutions $x_1(t)$ and $x_2(t)$ which are reproduced from global solutions of (R1) or (R2) and satisfy, as $t \to \infty$,
\[
x_1(t) \sim \pi(t), \quad Dx_1(t) \sim -1
\]
(3.25)
\[
x_2(t) \sim 1, \quad Dx_2(t) \sim Q(t).
\]
(3.26)

Proof. (The "only if" part) Suppose that (A) has a positive solution $x_1(t)$ of the type II(i) on $[T, \infty)$. It is bounded since $x_1(\infty) = c$ for some $c > 0$. Integrating (A) twice from $T$ to $t$ one obtains
\[
x_1(t) = x_1(T) + \int_T^t \frac{Dx_1(T)}{p(s)} ds + \int_T^t \frac{1}{p(s)} \int_T^s q(r)x_1(r) dr ds, \quad t \geq T,
\]
from which, in view of the boundedness of $x_1(t)$, it follows that $I_p < \infty$ and
\[
\int_T^\infty \frac{1}{p(s)} \int_T^s q(r) dr ds < \infty.
\]
This clearly implies $\int_a^\infty Q(s)/p(s) ds < \infty$.

Let (A) possess a positive solution $x_2(t)$ of the type II(iii) on $[T, \infty)$. Note that $Dx_2(\infty) = -d$ for some $d > 0$ and this implies
\[
\lim_{t \to \infty} \frac{x_2(t)}{\pi(t)} = d.
\]
(3.27)
Integrating (A) on $[T, \infty)$ one gets $\int_T^\infty q(s)x_2(s) ds = -d - D(T) < \infty$, which combined with (3.27) shows that $\int_T^\infty q(s) \pi(s) ds < \infty$.

(The "if" part) Assume that (3.24) holds. We first construct a principal solution of (A) by using a solution of the second Riccati equation (R2). Choose $T > a$ so that
\[
\int_T^\infty q(s) \pi(s) ds \leq \frac{1}{4},
\]
(3.28)
and look for a solution of the integral equation
\[
v(t) = -\pi(t) + \int_t^\infty q(s)v(s)^2 ds, \quad t \geq T,
\]
(3.29)
lying in the set
\[
V = \{ v \in C_0[T, \infty) : -\pi(t) \leq v(t) \leq -\frac{1}{2} \pi(t), \quad t \geq T \}.
\]
(3.30)
Consider the integral operator $G$ given by
\[
Gv(t) = -\pi(t) + \int_t^\infty q(s)v(s)^2 ds, \quad t \geq T.
\]
(3.31)
Let $v \in V$. Since
\[
\int_t^\infty q(s)v(s)^2 ds \leq \int_t^\infty q(s)\pi(s)^2 ds \leq \pi(t) \int_t^\infty q(s)\pi(s) ds \leq \frac{1}{4} \pi(t), \quad t \geq T,
\]
(3.32)
we have $-\pi(t) \leq Gv(t) \leq -\pi(t)/2, \ t \geq T$, which shows that $G$ is a self-map of $\mathcal{V}$. If $v_1, \ v_2 \in \mathcal{V}$, then,

$$|Gv_1(t) - Gv_2(t)| \leq \int_t^{\infty} q(s)|v_1(s)^2 - v_2(s)^2|ds \leq 2 \int_t^{\infty} q(s)|v_1(s) - v_2(s)|ds \leq 2 \int_t^{\infty} q(s)\pi(s)|v_1(s) - v_2(s)|ds \leq \frac{1}{2}\|v_1 - v_2\|_0, \ t \geq T,$$

which implies that

$$\|Gv_1 - Gv_2\|_0 \leq \frac{1}{2}\|v_1 - v_2\|_0.$$

This means that $G$ is a contraction of $\mathcal{V}$. Consequently, $G$ has a unique fixed point $v \in \mathcal{V}$, which satisfies (3.29) and hence gives a solution of (R2) on $[T, \infty)$. Note that (3.29) and (3.30) guarantee that $-v(t) \sim \pi(t)$ as $t \to \infty$. We now use the second reproducing formula of (2.9) to form the function

$$x_1(t) = -v(t) \exp\left(-\int_t^{\infty} q(s)v(s)ds\right), \ t \geq T. \tag{3.32}$$

Then, $x_1(t)$ is a positive solution of (A) on $[T, \infty)$ and satisfies $x_1(t) \sim \pi(t)$ as $t \to \infty$. It is easy to see that $Dx_1(t) \sim -1$ as $t \to \infty$.

Put $u(t) = 1/v(t)$. Then, $u(t)$ is a solution of (R1) and (3.32) is rewritten as

$$x_1(t) = -\frac{1}{u(t)} \exp\left(-\int_t^{\infty} \frac{q(s)}{u(s)}ds\right).$$

So we may assert that $x_1(t)$ can be reproduced from a global solution of (R1).

To construct a nonprincipal solution of (A) under condition (3.24) use is made of the first Riccati equation (R1). Choose $T > a$ so that

$$\int_T^{\infty} \frac{Q(s)}{p(s)}ds \leq \frac{1}{4}, \tag{3.33}$$

and define the integral operator $F$ and the set $\mathcal{U}$ by

$$Fu(t) = Q(t) - \int_T^t \frac{u(s)^2}{p(s)}ds, \ t \geq T, \tag{3.34}$$

and

$$\mathcal{U} = \{u \in C_Q[T, \infty) : \frac{1}{2}Q(t) \leq u(t) \leq Q(t), \ t \geq T\}. \tag{3.35}$$

respectively. Here $C_Q[T, \infty)$ denotes the Banach space of continuous functions $u(t)$ on $[T, \infty)$ satisfying $\|u\|_Q = \sup\{|u(t)|/Q(t) : t \geq T\} < \infty$. Let $u \in \mathcal{U}$. Since by (3.33)

$$\int_T^t \frac{u(s)^2}{p(s)}ds \leq \int_T^{\infty} \frac{Q(s)^2}{p(s)}ds \leq Q(t) \int_T^{\infty} \frac{Q(s)}{p(s)}ds \leq \frac{1}{4}Q(t), \ t \geq T,$$

one sees that $Q(t) \geq Fu(t) \geq Q(t)/2, \ t \geq T$. This shows that $Fu \in \mathcal{U}$, implying that $F$ maps $\mathcal{U}$ into itself. Let $u_1, \ u_2 \in \mathcal{U}$. Then, using (3.32) again, we obtain

$$\frac{|Fu_1(t) - Fu_2(t)|}{Q(t)} \leq \frac{1}{Q(t)} \int_T^t \frac{u_1(s) + u_2(s)}{p(s)}|u_1(s) - u_2(s)|ds \leq 2 \int_T^t \frac{Q(s)}{p(s)}\frac{|u_1(s) - u_2(s)|}{Q(s)}ds \leq \frac{1}{2}\|u_1 - u_2\|_Q, \ t \geq T,$$
which implies that
\[ \| Fu_1 - Fu_2 \|_Q \leq \frac{1}{2} \| u_1 - u_2 \|_Q. \]

Therefore, \( F \) has a unique fixed point \( u \in \mathcal{U} \) which satisfies
\[ u(t) = Q(t) - \int_T^t \frac{u(s)^2}{p(s)} ds, \quad t \geq T, \]
and hence gives a global solution of the Riccati equation (R1) on \([T, \infty)\). To obtain a positive solution of the type II(i) of (A) it suffices to construct the function
\[ x_2(t) = \exp \left( - \int_t^\infty \frac{u(s)}{p(s)} ds \right), \quad t \geq T, \quad (3.36) \]
according to the second reproducing formula in (2.5). It is clear that \( x_2(t) \sim 1 \) as \( t \to \infty \). Its quasi-derivative is given by
\[ Dx_2(t) = u(t) \exp \left( - \int_t^\infty \frac{u(s)}{p(s)} ds \right). \]
Since
\[ \lim_{t \to \infty} \frac{1}{Q(t)} \int_T^\infty \frac{u(s)^2}{p(s)} ds = 0, \]
(see the second half of the proof of Theorem 3.1), it holds that \( u(t) \sim Q(t) \) as \( t \to \infty \). This fact can be used to confirm that \( Dx_2(t) \sim Q(t) \) as \( t \to \infty \). This completes the proof.

It remains to analyze the asymptotic behavior of solutions of (A) with \( p(t) \) and \( q(t) \) satisfying \( I_p < \infty \wedge I_q < \infty \). We know that in this case all solutions of (A) are moderate and their terminal states are classified into the three types III(i), III(ii) and III(iii). It will be shown that all of these types of solutions of (A) can be reproduced from suitable global solutions of the Riccati equations (R1) and (R2).

Let \( x(t) \) be a solution of the type III(i) on \([T, \infty)\). We may assume that \( x(t) > 0 \) and \( Dx(t) > 0 \) for \( t \geq T \), so that \( x(\infty) = c \) and \( Dx(\infty) = d \) for some constants \( c > 0 \) and \( d > 0 \). To reproduce this solution from a solution of the Riccati equation (R1) we proceed as follows. Let \( \omega = d/c \).

Choose \( T > a \) so that
\[ \rho(T) \leq \frac{\omega}{4} \quad \text{and} \quad \pi(t) \leq \frac{1}{9}\omega, \]
define the integral operator \( F_1 \) by
\[ F_1 u(t) = \omega - \rho(t) + \int_T^\infty \frac{u(s)^2}{p(s)} ds, \quad t \geq T, \quad (3.37) \]
and let \( F_1 \) act on the set
\[ \mathcal{U}_1 = \{ u \in C_b[T, \infty) : |u(t) - \omega| \leq \frac{\omega}{2}, \quad t \geq T \}. \quad (3.38) \]
As is readily shown that \( F_1(\mathcal{U}_1) \subset \mathcal{U}_1 \) and that if \( u_1, u_2 \in \mathcal{U}_1 \), then
\[ \| F_1 u_1 - F_1 u_2 \|_b \leq \frac{1}{3} \| u_1 - u_2 \|_b. \]
By the contraction principle there exists a unique \( u_1 \in \mathcal{U}_1 \) satisfying the integral equation
\[ u_1(t) = \omega - \rho(t) + \int_T^\infty \frac{u_1(s)^2}{p(s)} ds, \quad t \geq T, \quad (3.39) \]
and hence the Riccati equation (R1) on $[T, \infty)$. With this $u_1(t)$ construct a function $x_1(t)$ by

$$x_1(t) = c \exp\left(- \int^t_u \frac{u_1(s)}{p(s)} ds\right), \quad t \geq T.$$  

(3.40)

It is clear that $x_1(t)$ is a solution of (A) such that $x_1(\infty) = c$. This $x_1(t)$ is the desired solution of the type III(i) of (A) since its quasi-derivative

$$Dx_1(t) = cu_1(t) \exp\left(- \int^t_u \frac{u_1(s)}{p(s)} ds\right), \quad t \geq T,$$  

(3.41)

satisfies $Dx_1(\infty) = cw = d$.

Let $x(t)$ be any type-III(ii) positive solution of (A) on $[T, \infty)$. It satisfies $x(\infty) = 0$ and $Dx(\infty) = -d$ for some $d > 0$. Our aim is to reproduce $x(t)$ from a solution of the Riccati equation (R2). Let $T > a$ be large enough that $\pi(T)\rho(T) \leq 1/4$, define the integral operator $G$ and the set $V$ by

$$Gv(t) = -\pi(t) + \int^\infty q(s)v(s)^2 ds, \quad t \geq T,$$  

(3.42)

and

$$V = \{v \in C_0[T, \infty) : -\pi(t) \leq v(t) \leq -\frac{1}{2}\pi(t), \quad t \geq T\}.$$  

(3.43)

Show that $G$ is a self-map of $V$ and that

$$\|Gv_1 - Gv_2\|_0 \leq \frac{1}{2}\|v_1 - v_2\|_0 \quad \text{for any} \quad v_1, v_2 \in V.$$  

Let $v$ denote the unique fixed point of $G$ in $V$. Then, it is a solution of (R2) on $[T, \infty)$. With this $v(t)$ form a solution of (A)

$$x_2(t) = -dv(t) \exp\left(- \int^t_u q(s)v(s) ds\right), \quad t \geq T,$$  

(3.44)

according to the reproducing formula (3.40). Clearly, $x_2(t)$ is a positive solution of (A) on $[T, \infty)$ satisfying $x_2(\infty) = 0$. Since

$$Dx_2(t) = -d\exp\left(- \int^t_u q(s)v(s) ds\right), \quad t \geq T,$$  

(3.45)

we see that $Dx_2(\infty) = -d$. Therefore $x_2(t)$ is a desired solution of the type III(ii).

Finally let $x(t)$ be any positive III(iii)-type solution on $[T, \infty)$. There is a constant $c > 0$ such that $x(\infty) = c$ and $Dx(\infty) = 0$. Choose $T > a$ so that $\rho(T)\pi(T) \leq 1/4$. Then, one proves that the integral operator $F_2$ defined by

$$Fu_2(t) = -\rho(t) + \int^\infty u(s)^2 \frac{q(s)}{p(s)} ds, \quad t \geq T.$$  

(3.46)

is a contraction on the set

$$U_2 = \{u \in C_0[T, \infty) : -\rho(t) \leq u(t) \leq -\frac{1}{2}\rho(t), \quad t \geq T\}.$$  

(3.47)

Therefore $F_2$ has a unique fixed point $u_2 \in U_2$ which reproduces a solution of (A)

$$x_3(t) = c \exp\left(- \int^t_u u_2(s) \frac{q(s)}{p(s)} ds\right), \quad t \geq T,$$  

(3.48)
whose quasi-derivative is given by
\[
Dx_3(t) = cu_2(t) \exp\left(-\int_t^\infty \frac{u_2(s)}{p(s)} ds\right), \quad t \geq T,
\]
(3.49)
This solution \(x_3(t)\) is of the type III(iii) since \(x_3(\infty) = c\) and \(Dx_3(\infty) = 0\).

The above discussions are summarized in the following theorem.

**Theorem 3.4.** Assume that \(I_p < \infty \land I_q < \infty\). All solutions of (A) are divided into the three classes consisting of moderate solutions of the types III(i), III(ii) and III(iii). These classes are represented, respectively, by the solutions \(x_1(t), x_2(t)\) and \(x_3(t)\) of (A) which are reproduced from global solutions of (R1) (or (R2)) having the specified asymptotic behavior as \(t \to \infty\),
\[
x_1(t) \sim 1, \quad Dx_1(t) \sim 1, \quad (3.50)
\]
\[
x_2(t) \sim \pi(t), \quad Dx_2(t) \sim -1, \quad (3.51)
\]
\[
x_3(t) \sim 1, \quad Dx_3(t) \sim \rho(t). \quad (3.52)
\]

**Remark 3.5.** Of the above three solutions, \(x_2(t)\) is a principal solution, while \(x_1(t)\) and \(x_3(t)\) are nonprincipal solutions. It seems natural to adopt \(\{x_2(t), x_3(t)\}\) as a basis for the solution space \(S(A)\) of (A) in the case \(I_p < \infty \land I_q < \infty\).

4 Extreme solutions of equation (A)

By definition a solution \(x(t)\) of (A) is extreme if it has the terminal state
\[
(a) \quad (|x(\infty)| = 0, |Dx(\infty)| = 0) \quad \text{or} \quad (b) \quad (|x(\infty)| = \infty, |Dx(\infty)| = \infty).
\]
For simplicity a solution \(x(t)\) satisfying (a) or (b) is termed, respectively, a decaying extreme solution or a growing extreme solution of (A).

All that are known at this stage about the existence of extreme solutions of (A) are listed in the following

**Theorem 4.1.** (i) If \(I_p < \infty \land I_q < \infty\), then (A) has no extreme solutions.

(ii) If \(I_p = \infty \land I_q = \infty\), then all solutions of (A) are extreme and there exist both decaying and growing extreme solutions.

(iii) Let \(I_p = \infty \land I_q < \infty\). All solutions of (A) are extreme, and there exist both decaying and growing extreme solutions if and only if
\[
\int_a^\infty q(t)p(t) dt = \infty \quad \left(\text{or equivalently,} \int_a^\infty \frac{p(t)}{p(t)} dt = \infty\right). \quad (4.1)
\]

(iv) Let \(I_p < \infty \land I_q = \infty\). All solutions of (A) are extreme and there exist both decaying and growing extreme solutions if and only if
\[
\int_a^\infty q(t)\pi(t) dt = \infty \quad \left(\text{or equivalently,} \int_a^\infty \frac{Q(t)}{p(t)} dt = \infty\right). \quad (4.2)
\]
This theorem has little substance. For example, the propositions (iii) and (iv) automatically follow from Theorems 3.2 and 3.3 based on the fact that extreme solutions and moderate solutions cannot coexist for (A), and no information is available about how to construct and determine their asymptotic behavior of such solutions. As far as we know, no serious asymptotic analysis seems to have ever been made of extreme solutions of (A) in the existing literature.

Unlike moderate solutions, it is very difficult to have a good grasp of extreme solutions of (A) presumably because of lack of a priori information about their precise asymptotic behaviors as \( t \to \infty \). However, we are able to indicate some special cases of (A) for which the existence of extreme solutions can actually be established with the help of the Riccati equations (R1) and (R2). The first result concerns growing extreme solutions of (A) satisfying (4.1).

**Theorem 4.2.** Assume that \( I_p = \infty \land I_q < \infty \). In addition to (4.1) suppose that there is a constant \( \gamma \in (0, 1) \) such that

\[
\int_t^q q(s)P(s)^2 ds \leq \gamma P(t) \quad \text{for all large } t, \tag{4.3}
\]

Then, equation (A) possesses a positive growing extreme solution \( x(t) \) satisfying \( x(\infty) = D x(\infty) = \infty \).

**Proof.** Choose \( T > a \) so that (4.3) holds for \( t \geq T \). Let \( \mathcal{V} \) denote the set

\[
\mathcal{V} = \{ v \in C[T, \infty) : (1 - \gamma)P(t) \leq v(t) \leq P(t), \ t \geq T \}, \tag{4.4}
\]

which is a closed convex subset of the locally convex space \( C[T, \infty) \) with the topology of uniform convergence on compact subintervals of \([T, \infty)\). Define the integral operator \( G : \mathcal{V} \to C[T, \infty) \) by

\[
Gv(t) = P(t) - \int_t^q q(s)v(s)^2 ds, \quad t \geq T. \tag{4.5}
\]

It can be shown that \( G \) is a continuous self-map of \( \mathcal{V} \) sending \( \mathcal{V} \) into a relatively compact subset of \( C[T, \infty) \).

(i) Since by (4.4) \( v \in \mathcal{V} \) implies

\[
\int_T^q q(s)v(s)^2 ds \leq \int_T^q q(s)P(s)^2 ds \leq \gamma P(t), \quad t \geq T,
\]

we see that \( P(t) \geq Gv(t) \geq (1 - \gamma)P(t) \) for \( t \geq T \). This implies \( G(\mathcal{V}) \subset \mathcal{V} \).

(ii) Let \( \{v_n(t)\} \) be a sequence in \( \mathcal{V} \) such that \( v_n(t) \to v(t) \) as \( n \to \infty \) uniformly on compact subintervals of \([T, \infty)\). Noting that \( |v_n(t)^2 - v(t)^2| \leq 2P(t)^2, \ t \geq T, \) and \( v_n(t) - v(t) \to 0 \) at every point \( t \in [T, \infty) \) as \( n \to \infty \), and using the Lebesgue dominated convergence theorem we see that

\[
|Gv_n(t) - Gv(t)| \leq \int_T^q q(s)|v_n(s)^2 - v(s)^2| ds \to 0, \quad t \to \infty,
\]

uniformly on any compact subinterval of \([T, \infty)\). This establishes the continuity of \( G \).

(iii) To prove the relative compactness of \( G(\mathcal{V}) \) it suffices to show that \( G(\mathcal{V}) \) is locally uniformly bounded and locally equicontinuous on \([T, \infty)\). The local uniform boundedness is a consequence of the inclusion \( G(\mathcal{V}) \subset \mathcal{V} \), while the local equicontinuity follows from the inequality

\[
|(Gv)'(t)| \leq \frac{1}{P(t)} + q(t)P(t)^2, \quad t \geq T,
\]

holding for all \( v \in \mathcal{V} \).
Therefore, by the Schauder-Tychonoff fixed point theorem (cf. Coppel [3]), $G$ has a fixed element $v \in \mathcal{V}$ which satisfies
\[ v(t) = P(t) - \int_T^t q(s)v(s)^2ds, \quad t \geq T, \]
and hence the Riccati equation (R2) on $[T, \infty)$. Using this $v(t)$ and the formula (2.9), we reproduce a solution of equation (A)
\[ x(t) = v(t) \exp \left( \int_T^t q(s)v(s)ds \right), \quad t \geq T, \quad (4.6) \]
which has the quasi-derivative
\[ Dx(t) = \exp \left( \int_T^t q(s)v(s)ds \right), \quad t \geq T. \quad (4.7) \]
It is clear that $x(\infty) = Dx(\infty) = \infty$ since $v(\infty) = \infty$ and the integral $\int_T^t q(s)v(s)ds$ diverges as $t \to \infty$. This proves Theorem 4.2.

A decaying extreme solution of (A) satisfying (4.1) can be reproduced from a solution of the Riccati equation (R1) as is described in the following theorem.

**Theorem 4.3.** Assume that $I_p = \infty \land I_q < \infty$. Suppose in addition to (4.1) that there is a constant $\delta \in (0, 1)$ such that
\[ \int_t^{\infty} \frac{\rho(s)^2}{p(s)}ds \leq \delta \rho(t) \quad \text{for all large} \quad t. \quad (4.8) \]
Then, equation (A) possesses a positive decaying extreme solution $x(t)$ satisfying $x(\infty) = Dx(\infty) = 0$.

**Proof.** Choose $T > a$ so that (4.8) holds for $t \geq T$, define the integral operator $F$ by
\[ Fu(t) = -\rho(t) + \int_t^{\infty} \frac{u(s)^2}{p(s)}ds, \quad t \geq T, \quad (4.9) \]
and let it act on the set
\[ \mathcal{U} = \{ u \in C[T, \infty) : -\rho(t) \leq u(t) \leq -(1 - \delta)\rho(t), \quad t \geq T \}. \quad (4.10) \]
As in the proof of the preceding theorem it can be shown that $F$ is a continuous self-map of $\mathcal{U}$ with the property that $F(\mathcal{U})$ is a relatively compact subset of $C[T, \infty)$. Therefore, the Schauder-Tychonoff theorem ensures the existence of a function $u \in \mathcal{U}$ such that $u = Fu$, i.e.,
\[ u(t) = -\rho(t) + \int_t^{\infty} \frac{u(s)^2}{p(s)}ds, \quad t \geq T, \quad (4.11) \]
so that $u(t)$ is a negative solution of the Riccati equation (R1) on $[T, \infty)$. Using the reproducing formula (2.5) with this $u(t)$ construct a positive solution of equation (A)
\[ x(t) = \exp \left( \int_T^t \frac{u(s)}{p(s)}ds \right), \quad t \geq T, \quad (4.12) \]
whose quasi-derivative is given by
\[ Dx(t) = u(t) \exp \left( \int_T^t \frac{u(s)}{p(s)} ds \right), \quad t \geq T. \] (4.13)

Since by (4.1)
\[ \int_T^t \frac{u(s)}{p(s)} ds \leq -(1 - \delta) \int_T^t \frac{\rho(s)}{p(s)} ds \to -\infty, \quad t \to \infty, \]
it follows that \( x(\infty) = Dx(\infty) = 0 \), which means that \( x(t) \) is a decaying extreme solution of (A). This proves Theorem 4.3.

There exists a situation in which equation (A) certainly possesses a pair of decaying and growing extreme solutions.

**Corollary 4.4.** Assume that \( I_p = \infty \land I_q < \infty \). If conditions (4.1), (4.3) and (4.8) are satisfied, then equation (A) has an extreme solution basis \( \{ x_1(t), x_2(t) \} \) such that \( x_1(\infty) = Dx_1(\infty) = 0 \) and \( x_2(\infty) = Dx_2(\infty) = \infty \).

Our next task is to the existence of decaying extreme solutions \( x(t) \) of for equation (A) satisfying (4.2).

**Theorem 4.5.** Assume that \( I_p < \infty \land I_q = \infty \). Suppose in addition to (4.2) that there is a constant \( \gamma \in (0, 1) \) such that
\[ \int_t^\infty q(s)\pi(s)^2 ds \leq \gamma \pi(t) \quad \text{for all large } t. \] (4.14)

Then, equation (A) possesses a positive decaying extreme solution \( x(t) \) satisfying \( x(\infty) = Dx(\infty) = 0 \).

**Theorem 4.6.** Assume that \( I_p < \infty \land I_q = \infty \). Suppose in addition to (4.2) that there is a constant \( \delta \in (0, 1) \) such that
\[ \int_a^t \frac{Q(s)^2}{p(s)} ds \leq \delta Q(t) \quad \text{for all large } t. \] (4.15)

Then, equation (A) possesses a positive growing extreme solution \( x(t) \) satisfying \( x(\infty) = Dx(\infty) = \infty \).

It will suffice to outline the proof of the above theorems. To prove Theorem 4.5 choose \( T > a \) so that (4.14) holds for \( t \geq T \), define the operator
\[ Gv(t) = -\pi(t) + \int_t^\infty q(s)v(s)^2 ds, \quad t \geq T, \]
and let \( G \) act on the set
\[ \mathcal{V} = \{ v \in C[T, \infty) : -\pi(t) \leq v(t) \leq -(1 - \gamma)\pi(t), \quad t \geq T \}. \]

It can be shown routinely that \( G \) is a self-map of \( \mathcal{V} \), that \( G \) is a continuous map and that \( G(\mathcal{V}) \) is relatively compact in \( C[T, \infty) \). Therefore, by the Schauder-Tychonoff theorem \( G \) has a fixed point \( v \in \mathcal{V} \) which gives a solution of the Riccati equation (R2) on \( [T, \infty) \). With this \( v(t) \) form a function
\[ x(t) = -v(t) \exp \left( \int_T^t q(s)v(s) ds \right), \quad t \geq T. \]
Then, it is easily checked that \( x(t) \) is a positive extreme solution of (A) satisfying \( x(\infty) = 0 \) and \( Dx(\infty) = 0 \).

To prove Theorem 4.6 choose \( T > a \) so that (4.15) holds for \( t \geq T \) and consider the set

\[
\mathcal{U} = \{ u \in C[T,\infty) : (1 - \delta)Q(t) \leq u(t) \leq Q(t), \ t \geq T \}.
\]

Then, letting the integral operator

\[
Fu(t) = Q(t) - \int_T^t \frac{u(s)^2}{p(s)} ds, \quad t \geq T,
\]

act on \( \mathcal{U} \), one can verify that \( F \) is a continuous self-map of \( \mathcal{U} \) such that \( F(\mathcal{U}) \) is a relatively compact subset of \( C[T,\infty) \). Therefore, \( F \) has a fixed point \( u \in \mathcal{U} \) by the Schauder-Tychonoff theorem. Clearly, \( u(t) \) is a solution of the Riccati equation (R1) on \( [T,\infty) \). With this \( u(t) \) form a positive function by

\[
x(t) = \exp\left( \int_T^t \frac{u(s)}{p(s)} ds \right), \quad t \geq T.
\]

Then, one easily sees that it is an extreme solution of (A) satisfying \( x(\infty) = Dx(\infty) = \infty \).

**Corollary 4.7.** Assume that \( I_p < \infty \land I_q = \infty \). If conditions (4.2), (4.14) and (4.15) are satisfied, then equation (A) has a positive extreme solution basis \( \{ x_1(t), x_2(t) \} \) such that \( x_1(\infty) = Dx_1(\infty) = 0 \) and \( x_2(\infty) = Dx_2(\infty) = \infty \).

The case of equation (A) with \( I_p = \infty \land I_q = \infty \) remains to be examined. In this case all members \( x(t) \) of \( \mathcal{S}(A) \) are extreme solutions. A simple example of such equations is

\[
(p(t)x')' = \frac{k^2}{p(t)} x, \quad (4.16)
\]

where \( k > 0 \) is a constant. This is a special case of (A) with \( q(t) = k^2 / p(t) \). Note that \( I_q = k^2 I_p \). The first Riccati equation associated with (4.16) is

\[
u' = \frac{k^2 - u^2}{p(t)}. \quad (4.17)
\]

Since (4.17) has the exact global solutions

\[
u_1(t) \equiv k, \quad \nu_2(t) \equiv -k, \quad (4.18)
\]

using the formula (2.5) we see that if \( I_p = \infty \), equation (4.16) has linearly independent extreme solutions

\[
x_1(t) = \exp(kP(t)), \quad x_2(t) = \exp(-kP(t)), \quad t \geq a. \quad (4.19)
\]

One can use the second Riccati equation (R2) \( v' = (1 - k^2v^2) / p(t) \) to reach the same conclusion.

It is desirable to find a nontrivial class of equations of the form (A) with \( (p, q) \) satisfying \( I_p = \infty \land I_q = \infty \) whose extreme solutions, either growing or decaying or both, can be reproduced by way of the Riccati equations. However, we are still far from solving this problem, and so we choose to close this section by presenting an artificial method of making special equations of the form (A) possessing exact extreme solutions that are reproduced from exact global solutions of the associated Riccati equation (R1) or (R2).
Theorem 4.8. (i) Let $I_p = \infty$. If $\varphi(t)$ is a positive $C^1$-function on $[a, \infty)$ satisfying $\varphi'(t) > 0$ and $\varphi(\infty) = \infty$, then the differential equation

$$(p(t)x')' = \left(\frac{\varphi(t)^2}{p(t)} + \varphi'(t)\right)x$$

(4.20)

has an extreme solution $x(t)$ such that $x(\infty) = Dx(\infty) = \infty$.

(ii) Let $I_p = \infty$. If $\Phi(t)$ is a positive $C^1$-function on $[a, \infty)$ satisfying $\Phi'(t) < 0$, $\Phi(\infty) = 0$ and

$$\int_a^\infty \frac{\Phi(t)^2}{p(t)} dt = \infty,$$

(4.21)

then the differential equation

$$(p(t)x')' = \left(\frac{\Phi(t)^2}{p(t)} - \Phi'(t)\right)x$$

(4.22)

has an extreme solution $x(t)$ such that $x(\infty) = Dx(\infty) = 0$.

Theorem 4.9. (i) Let $I_p = \infty$. If $\varphi(t)$ is a positive $C^1$-function on $[a, \infty)$ satisfying $\varphi'(t) < 0$ and $\varphi(\infty) = 0$, then the differential equation

$$(p(t)x')' = \frac{1}{\varphi(t)^2}\left(\frac{1}{p(t)} - \varphi'(t)\right)x$$

(4.23)

has an extreme solution $x(t)$ such that $x(\infty) = Dx(\infty) = \infty$.

(ii) Let $I_p = \infty$. If $\Phi(t)$ is a positive $C^1$-function on $[a, \infty)$ satisfying $\Phi'(t) > 0$, $\Phi(\infty) = \infty$ and

$$\int_a^\infty \frac{dt}{p(t)\Phi(t)^2} = \infty,$$

(4.24)

then the differential equation

$$(p(t)x')' = \frac{1}{\Phi(t)^2}\left(\frac{1}{p(t)} + \Phi'(t)\right)x$$

(4.25)

has an extreme solution $x(t)$ such that $x(\infty) = Dx(\infty) = 0$.

To prove Theorem 4.8 it suffices to notice that $\varphi(t)$ (or $-\Phi(t)$) is a solution of (R1) for (4.20) (or (R1) for (4.22)) and to apply the formula (2.5) to obtain a growing extreme solution $\exp\left(\int_a^t (\varphi(s)/p(s))ds\right)$ of (4.20) and a decaying extreme solution $\exp\left(-\int_a^t (\Phi(s)/p(s))ds\right)$ of (4.22). In the statement (ii) condition (4.21) is needed to assure that $q(t)$ satisfies $I_q = \infty$. Theorem 4.9 can be proved analogously.

5 Examples

We present some examples illustrating all the results obtained in Sections 3 and 4. In the first example, $p$ is assumed to satisfy $I_p = \infty$ and $P$ is given by $P(t) = \int_a^t ds / p(s)$. 

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Example 5.1. Let \( p(t) \) be a positive continuous function on \([a, \infty)\) such that \( I_p = \infty \) and consider the linear differential equation

\[
(p(t)x')' = q(t)x, \quad q(t) = \frac{k}{p(t)p(t)\lambda(\log P(t))^\mu},
\]

(5.1)

where \( k > 0, \lambda > 0 \) and \( \mu \in \mathbb{R} \) are constants. Solutions of this equations are sought on intervals of the form \([T, \infty)\), where \( T \geq a \) should be such that \( P(T) \geq e \).

Our attention is focused on the case \( I_q < \infty \). This occurs if and only if \((\lambda, \mu)\) is a member of the set

\[
\{(\lambda, \mu) : \lambda > 1, \mu \in \mathbb{R}\} \cup \{(\lambda, \mu) : \lambda = 1, \mu > 1\},
\]

(5.2)

which is abbreviated as \(\{\lambda > 1, \mu \in \mathbb{R}\}\) or \(\{\lambda = 1, \mu > 1\}\). In this case \(\rho(t)\) has the asymptotic property

\[
\rho(t) \sim \frac{k}{(\lambda - 1)p(t)\lambda - 1(\log P(t))^\mu}, \quad t \to \infty, \quad \text{if } \{\lambda > 1, \mu \in \mathbb{R}\},
\]

(5.3)

\[
\rho(t) \sim \frac{k}{(\mu - 1)(\log P(t))^\mu - 1}, \quad t \to \infty, \quad \text{if } \{\lambda = 1, \mu > 1\}.
\]

Condition (5.5) holds for (5.1) if \((\lambda, \mu)\) satisfies

\[
\{\lambda > 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, \mu > 1\},
\]

(5.4)

in which case Theorem 3.2 ensures the existence of a moderate basis \(\{x_1(t), x_2(t)\}\) for (5.1) on some interval \([T, \infty)\) which exhibit the asymptotic behavior

\[
x_1(t) \sim 1, \quad Dx_1(t) \sim \rho(t), \quad x_2(t) \sim P(t), \quad Dx_2(t) \sim 1, \quad t \to \infty.
\]

(5.5)

These solutions are reproduced by way of the Riccati equations (R1) and (R2) associated with (5.1) as follows:

\[
x_1(t) = \exp\left(-\int_t^\infty \frac{u(s)}{p(s)}ds\right), \quad x_2(t) = v(t)\exp\left(\int_T^t q(s)v(s)ds\right),
\]

(5.6)

where \(u(t)\) and \(v(t)\) are, respectively, solutions of (R1) and (R2) on some interval \([T, \infty)\) which satisfy \(-\rho(t) \leq u(t) \leq -\rho(t)/2\) and \(P(t)/2 \leq v(t) \leq P(t)\) there.

Turning to extreme solutions of (5.1), we first note that condition (4.1) holds for (5.1) if \((\lambda, \mu)\) satisfies

\[
\{\lambda = 1, \mu > 1\} \quad \text{or} \quad \{1 < \lambda < 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, \mu \leq 1\}.
\]

(5.7)

We then compute to see that

\[
\int_T^t q(s)p(s)^2ds \sim \frac{kP(t)^{3-\lambda}}{(3-\lambda)(\log P(t))^\mu}, \quad t \to \infty,
\]

from which it follows that

\[
\lim_{t \to \infty} \frac{1}{P(t)} \int_T^t q(s)p(s)^2ds = \begin{cases} 0 & \text{for all } k > 0 \text{ if } 0 < \mu \leq 1, \\ k & \text{for all } k < 1 \text{ if } \mu = 0. \end{cases}
\]

(5.8)

Consequently, by Theorem 4.2 equation (5.1) possesses a positive extreme solution \(x(t)\) such that \(x(\infty) = Dx(\infty) = \infty\) for all \(k\) if \(\{\lambda = 2, 0 < \mu \leq 1\}\) and for all \(k < 1\) if \(\{\lambda = 2, \mu = 0\}\). In either case the solution \(x(t)\) can be represented in the form

\[
x(t) = v(t)\exp\left(\int_T^t q(s)v(s)ds\right), \quad t \geq T,
\]

(5.9)
in terms of a global solution \( v(t) \) of (R2) for (5.1) satisfying \((1 - l)P(t) \leq v(t) \leq P(t) \) for \( t \geq T \), where \( l > 0 \) is any constant such that \( l < 1 \) if \( 0 < \mu \leq 1 \) and such that \( l < k \) if \( \mu = 0 \).

To examine the applicability of Theorem 4.3 it should be noted that the set of \((\lambda, \mu)\) in (5.7) for which \( \rho(t)^2/p(t) \) is integrable on \([a, \infty)\) is

\[
\{ \lambda = 2, \mu \leq 1 \} \text{ or } \{ 2 > \lambda > \frac{3}{2}, \mu > \frac{1}{2} \} \text{ or } \{ \lambda = \frac{3}{2}, \mu \in \mathbb{R} \}. \tag{5.10}
\]

Then we compute:

\[
\{ \lambda = \frac{3}{2}, \mu \in \mathbb{R} \} \quad \Rightarrow \quad \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)^2}{p(s)} ds \sim \frac{2k}{2\mu - 1} \frac{P(t)^{\lambda - 1}}{\log P(t)^\mu} \to \infty, \quad t \to \infty,
\]

\[
\{ 2 > \lambda > \frac{3}{2}, \mu > \frac{1}{2} \} \quad \Rightarrow \quad \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)^2}{p(s)} ds \sim \frac{k}{(\lambda - 1)(2\lambda - 3)} \frac{P(t)^{2 - \lambda}}{\log P(t)^\mu}, \quad t \to \infty,
\]

and

\[
\{ \lambda = 2, \mu \leq 1 \} \quad \Rightarrow \quad \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)^2}{p(s)} ds \sim \frac{k}{(\log P(t)^\mu}, \quad t \to \infty,
\]

which implies

\[
\lim_{t \to \infty} \frac{1}{\rho(t)} \int_t^\infty \frac{\rho(s)^2}{p(s)} ds = \begin{cases} 0 & \text{for all } k \text{ if } 0 < \mu \leq 1, \\ k & \text{for all } k < 1 \text{ if } \mu = 0 \end{cases} \tag{5.11}
\]

This shows that Theorem 4.3 is applicable to (5.1) only in the case where \( \lambda = 2 \) and \( 0 \leq \mu \leq 1 \) and guarantees the existence of a positive extreme solution \( x(t) \) of equation (5.1) such that \( x(\infty) = Dx(\infty) = 0 \) for all \( k \) if \( 0 < \mu \leq 1 \) and for all \( k < 1 \) if \( \mu = 0 \). In either case the solution \( x(t) \) is represented in the form

\[
x(t) = \exp \left( \int_t^\infty \frac{u(s)}{p(s)} ds \right), \quad t \geq T, \tag{5.12}
\]

by using a global solution \( u(t) \) of the Riccati equation (R1) for (5.1) satisfying \(-P(t) \leq u(t) \leq -(1 - l)P(t) \) for \( t \geq T \), where \( l > 0 \) is any constant such that \( l < 1 \) if \( \mu \leq 1 \) and such that \( l < k \) if \( \mu = 0 \).

Thus it is concluded that equation (5.1) possesses both growing and decaying extreme solutions for all \( k \) if \( \lambda = 2 \) and \( 0 < \mu \leq 1 \) and for all \( k < 1 \) if \( \lambda = 2 \) and \( \mu = 0 \).

**Remark 5.2.** The particular case \( \{ \lambda = 2, \mu = 0 \} \) of (5.1), i.e.,

\[
(p(t)x')' = \frac{kx}{p(t)P(t)^2}, \quad k > 0, \tag{5.13}
\]

has a pair of exact extreme solutions \( \{ P(t)^{\alpha_1}, P(t)^{\alpha_2} \} \), where

\[
\alpha_1 = \frac{1}{2}(1 + \sqrt{1 + 4k}), \quad \alpha_2 = \frac{1}{2}(1 - \sqrt{1 + 4k}). \tag{5.14}
\]

In the second example, \( p \) is assumed to satisfy \( I_p < \infty \) and \( \pi \) is given by \( \pi = \int_1^\infty ds/p(s) \).

**Example 5.3.** Let \( p(t) \) be a positive continuous function on \([a, \infty)\) such that \( I_p < \infty \). Let \( k > 0 \), \( \lambda \) and \( \mu \) are constants and consider the linear differential equation

\[
(p(t)x')' = q(t), \quad q(t) = \frac{k}{p(t)} \left( \frac{1}{\pi(t)^2} \right)^\lambda \left( \log \left( \frac{1}{\pi(t)} \right) \right)^\mu, \tag{5.15}
\]

}\]
on \([T, \infty)\), where \(T \geq a\) is chosen so that \(\pi(T) \leq e\).

Note that \(I_q = \infty\) if

\[
\{\lambda > 1, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 1, \mu \geq -1\},
\]
and that the function \(Q(t)\) has the asymptotic properties

\[
Q(t) \sim \frac{k}{\lambda - 1} \left(\frac{1}{\pi(t)}\right)^{\lambda-1} \left(\log \left(\frac{1}{\pi(t)}\right)\right)^\mu, \quad t \to \infty, \quad \text{if} \quad \{\lambda > 1, \mu \in \mathbb{R}\},
\]

\[
Q(t) \sim \frac{k}{\mu + 1} \left(\log \left(\frac{1}{\pi(t)}\right)\right)^{\mu+1}, \quad t \to \infty, \quad \text{if} \quad \{\lambda = 1, \mu > -1\},
\]

\[
Q(t) \sim k \log \log \left(\frac{1}{\pi(t)}\right), \quad t \to \infty, \quad \text{if} \quad \{\lambda = 1, \mu = -1\}.
\]

Since condition (3.24) is satisfied if

\[
\{\lambda < 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, \mu \geq -1\},
\]

by Theorem 3.3 there exists a moderate basis \(\{x_1(t), x_2(t)\}\) for equation (5.15) having the asymptotic behavior

\[
x_1(t) \sim \pi(t), \quad Dx_1(t) \sim -1, \quad x_2(t) \sim \pi(t), \quad Dx_2(t) \sim Q(t), \quad t \to \infty.
\]

These solutions are reproduced by way of the Riccati equations (R2) and (R1) for (5.13) as follows:

\[
x_1(t) = -v(t) \exp \left(-\int_t^\infty q(s)v(s)ds\right), \quad x_2(t) = \exp \left(-\int_t^\infty u(s)p(s)ds\right),
\]

where \(v(t)\) and \(u(t)\) are, respectively, solutions of (R2) and (R1) on some interval \([T, \infty)\) which satisfy \(-\pi(t) \leq v(t) \leq -\pi(t)/2\) and \(Q(t)/2 \leq u(t) \leq Q(t)\). There.

Turning to extreme solutions of (5.15), we must notice that condition (4.2) holds for (5.15) if \((\lambda, \mu)\) satisfies

\[
\{\lambda > 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, \mu \geq -1\}.
\]  

Then we find that

\[
\int_t^\infty q(s)\pi(s)^2ds \sim \begin{cases} 
2/(3-\lambda) \left(\frac{1}{\pi(t)}\right)^{\lambda-3} \left(\log \left(\frac{1}{\pi(t)}\right)\right)^\mu & \text{if} \quad \lambda < 3, \mu \in \mathbb{R}, \\
\frac{k}{\lambda} \left(\frac{1}{\pi(t)}\right)^{\mu+1} \left(\log \left(\frac{1}{\pi(t)}\right)\right)^\mu & \text{if} \quad \lambda = 3, \mu < -1,
\end{cases}
\]

as \(t \to \infty\), from which it follows that

\[
\lim_{t \to \infty} \frac{1}{\pi(t)} \int_t^\infty q(s)\pi(s)^2ds = \begin{cases} 
0 & \text{if} \quad \{\lambda > 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, -1 \leq \mu < 0\}, \\
k & \text{if} \quad \{\lambda = 2, \mu = 0\}.
\end{cases}
\]

Therefore, it is concluded from Theorem 4.5 that equation (5.15) possesses a decaying extreme solution for all \(k > 0\) if \(\{\lambda > 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda, -1 \leq \mu < 0\}\) and for all \(k < 1\) if \(\{\lambda = 2, \mu = 0\}\). In either case the solution \(x(t)\) is expressed in the form

\[
x(t) = -v(t) \exp \left(\int_T^t q(s)v(s)ds\right), \quad t \geq T,
\]

where \(v(t)\) is a solution of (R2) for (5.15) satisfying \(-\pi(t) \leq v(t) \leq -\pi(t)/2\) for \(t \geq T\), where \(l > 0\) is a constant such that \(l < 1\) if \(\{\lambda < 2, \mu \in \mathbb{R}\} \quad \text{or} \quad \{\lambda = 2, \mu < 0\}\) and such that \(l < k\) if \(\{\lambda = 2, \mu = 0\}\).

It should be noted that an extreme basis exists for equation (5.15) for all \(k\) if \(\{\lambda, \mu < 0\}\) and for all \(k < 1\) if \(\{\lambda = 2, \mu = 0\}\).
Remark 5.4. The simplest case \( \{ \lambda = 2, \mu = 0 \} \) of (5.15), i.e.,

\[
(p(t)x')' = \frac{kx}{p(t)\pi^2},
\]  

(5.24)

has two exact extreme solutions \( P(t)^{a_1} \) and \( P(t)^{a_2} \), where \( a_1 \) and \( a_2 \) are given by (5.14).

Our next step is to reproduce growing extreme solutions of (5.15) on the basis of Theorem 4.6. Necessary is the precise information about the asymptotic behavior of \((1/Q(t)) \int_T^t Q(s)^2 / p(s) ds\) as \( t \to \infty \). It can be verified that if \( \{ \lambda > 3/2, \mu \in \mathbb{R} \} \), then

\[
\frac{1}{Q(t)} \int_T^t \frac{Q(s)^2}{p(s)} ds \sim \frac{k}{(\lambda - 1)(2\lambda - 3)} \left( \frac{1}{\pi(t)} \right)^{\lambda - 2} \left( \log \left( \frac{1}{\pi(s)} \right) \right)^2, \tag{5.25}
\]

as \( t \to \infty \), and that if \( \{ \lambda = 3/2, \mu \in \mathbb{R} \} \), then

\[
\frac{1}{Q(t)} \int_T^t \frac{Q(s)^2}{p(s)} ds \sim \begin{cases} 
2k(\frac{1}{\mu + 1})^{\frac{1}{2}} \left( \log \left( \frac{1}{\pi(t)} \right) \right)^{\mu + 1} & \text{if } \mu \neq -\frac{1}{2}, \\
2k(\frac{1}{\pi(t)})^{\frac{1}{2}} \left( \log \left( \frac{1}{\pi(t)} \right) \right)^{\frac{3}{2}} & \text{if } \mu = -\frac{1}{2},
\end{cases} \tag{5.26}
\]

as \( t \to \infty \). Since (5.25) and (5.26) implies that

\[
\lim_{t \to \infty} \frac{1}{Q(t)} \int_T^t \frac{Q(s)^2}{p(s)} ds = \begin{cases} 
k & \text{if } \{ \lambda = 2, \mu = 0 \}, \\
0 & \text{if } \{ \lambda = 2, \mu < 0 \} \text{ or } \{ \frac{3}{2} \leq \lambda < 2, \mu \in \mathbb{R} \},
\end{cases} \tag{5.27}
\]

Theorem 4.6 shows that equation (5.15) possesses a growing extreme solution for all \( k < 1 \) if \( \{ \lambda = 2, \mu = 0 \} \) or \( \{ 3/2 \leq \lambda < 2, \mu \in \mathbb{R} \} \) and for all \( k \) if \( \{ \lambda = 2, \mu = 0 \} \). In either case the solution \( x(t) \) is expressed in the form

\[
x(t) = \exp \left( \int_T^t \frac{u(s)}{p(s)} ds \right) ds, \quad t \geq T, \tag{5.28}
\]

in terms of a solution \( u(t) \) of the Riccati equation (R1) for (5.15) satisfying \((1 - \delta)Q(t) \leq u(t) \leq Q(t)\) on \([T, \infty)\) for some \( \delta < k \) if \( \{ \lambda = 2, \mu = 0 \} \) or for some \( \delta < 1 \) if \( \{ \lambda = 2, \mu < 0 \} \) or \( \{ 3/2 \leq \lambda < 2, \mu \in \mathbb{R} \} \).

From the aforementioned we conclude that there exists an extreme basis for equation (5.15) for all \( k \) if \( \{ \lambda = 2, \mu < 0 \} \) and for all \( k < 1 \) if \( \{ \lambda = 2, \mu = 0 \} \).

Remark 5.5. The simplest case \( \{ \lambda = 2, \mu = 0 \} \) of (5.15), i.e.,

\[
(p(t)x')' = \frac{kx}{p(t)\pi^2},
\]  

(5.29)

has two exact extreme solutions \( \pi(t)^{-a_1} \) and \( \pi(t)^{-a_2} \), where \( a_1 \) and \( a_2 \) are defined by (5.14).

Example 5.6. As an example illustrating Theorem 3.4 concerning the type-III moderate solutions of (A) we consider equation (5.5) in which \( q \) satisfies \( l_q < \infty \). It is clear that this holds if and only if \( \{ \lambda < 1, \mu \in \mathbb{R} \} \) or \( \{ \lambda = 1, \mu < -1 \} \), in which case \( \rho \) is given asymptotically by

\[
\rho(t) \sim \begin{cases} 
k \left( \frac{1}{\pi(t)} \right)^{\lambda - 1} \left( \log \left( \frac{1}{\pi(t)} \right) \right)^\mu & \text{if } \lambda < 1 \mu \in \mathbb{R}, \\
k \left( \frac{1}{\pi(t)} \right)^{\mu + 1} & \text{if } \lambda = 1 \mu < -1 \end{cases}, \tag{5.30}
\]
According to Theorem 3.4, there exist under these circumstances three types of bounded moderate solutions $x_i(t), i = 1, 2, 3$, on some interval $[T, \infty)$ such that

$$x_1(t) \sim 1, \quad Dx_1(t) \sim 1, \quad x_2(t) \sim \pi(t), \quad Dx_2(t) \sim 1, \quad x_3(t) \sim 1, \quad Dx_3(t) \sim -\rho(t),$$
as $t \to \infty$. All of them are reproduced from suitable global solutions of (R1) or (R2) associated with (5.15).

The final example is given to illustrate Theorem 4.9.

**Example 5.7.** (i) The equation

$$(e^{-t}x')' = (e^t + e^{3t})x$$

is a special case of (4.20) with $p(t) = e^{-t}$ and $\varphi(t) = e^{-t}$, and so from (i) of Theorem 4.9 we conclude that (5.31) has a growing extreme solution

$$x_1(t) = \exp\left(\frac{1}{2}e^{2t}\right), \quad t \geq 0.$$  

(5.32)

Using the formula (1.4) we then have a decaying extreme solution of (5.31) given explicitly by

$$x_2(t) = \exp\left(\frac{1}{2}e^{2t}\right) \int_0^\infty \exp(s - e^{2s})ds, \quad t \geq 0.$$  

(5.33)

(ii) Since the equation

$$(e^{-3t}x')' = (e^{-t} + e^t)x$$

is a special case of (4.22) with $p(t) = e^{-3t}$ and $\Phi(t) = e^t$, by (ii) of Theorem 4.8 this equation has a decaying extreme solution

$$x_1(t) = \exp\left(-\frac{1}{2}e^{2t}\right), \quad t \geq 0.$$  

(5.35)

Using the formula (1.3) we obtain a growing extreme solution of (5.34) expressed as

$$x_2(t) = \exp\left(-\frac{1}{2}e^{2t}\right) \int_0^t \exp(3s + e^{2s})ds, \quad t \geq 0.$$  

(5.36)

**Remark 5.8.** (i) The solutions $x_1(t)$ and $x_2(t)$ of (5.31) defined by (5.32) and (5.33) satisfy

$$p(t)(x_1(t)x_2'(t) - x_1'(t)x_2(t)) \equiv 1,$

which can be rewritten as

$$\frac{x_2(t)}{Dx_2(t)} - \frac{x_1(t)}{Dx_1(t)} = \frac{1}{Dx_1(t)Dx_2(t)}, \quad t \geq 0.$$  

(5.37)

Note that $v_i(t) = x_i(t)/Dx_i(t), i = 1, 2$, are solutions of the Riccati equation (R2) that reproduce the solutions $x_i(t), i = 1, 2$, of (5.31). Since $v_1(t) = e^{-t}$ is already known, (5.37) can also be used to determine the second solution $v_2(t)$ of (R2).

(ii) Likewise, the solutions $x_i(t), i = 1, 2$, of (5.34) given by (5.35) and (5.36) must satisfy

$$\frac{x_2(t)}{Dx_2(t)} - \frac{x_1(t)}{Dx_1(t)} = -\frac{1}{Dx_1(t)Dx_2(t)}, \quad t \geq 0.$$  

(5.38)
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