On the computation of classical, boolean and free cumulants

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Abstract

This paper introduces a simple and computationally efficient algorithm for conversion formulae between moments and cumulants. The algorithm provides just one formula for classical, boolean and free cumulants. This is realized by using a suitable polynomial representation of Abel polynomials. The algorithm relies on the classical umbral calculus, a symbolic language introduced by Rota and Taylor in [11], that is particularly suited to be implemented by using software for symbolic computations. Here we give a MAPLE procedure. Comparisons with existing procedures, especially for conversions between moments and free cumulants, as well as examples of applications to some well-known distributions (classical and free) end the paper.

keywords: umbral calculus, classical cumulant, boolean cumulant, free cumulant, Abel polynomial

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1 Introduction

Among number sequences connected to random variables (r.v.’s), cumulants play a central role, characterizing many r.v.’s found in the practice. Moreover, due to their properties of additivity and invariance under translation, cumulants are not necessarily connected with moments of r.v.’s so that they

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can be analyzed by using an algebraic point of view. For these reasons, cumulants have been introduced not only in the context of classical probability theory, but also in the boolean [14] and the free context [15]. Since cumulants linearize convolutions of measures - no matter what framework they are referred to - this property allows us quick access to test whether a given probability measure is a convolution. The linearity of classical convolutions corresponds to independent r.v.'s [9] and the linearity of free convolutions allows us to recognize free r.v.'s [15]. The boolean convolution was constructed [14] starting exactly from the notion of partial cumulants, this last extensively used in the context of stochastic differential equations.

In this paper, we propose the classical umbral calculus of Rota and Taylor [11] as the most natural syntax for conversions between cumulants (classical, boolean and free) and moments, without any sophisticated programming.

The basic devices of the umbral calculus are essentially two. The first one is to represent a unital sequence of numbers by a symbol $\alpha$, called an umbra, that is, to represent the sequence $1, a_1, a_2, \ldots$ by means of the sequence $1, \alpha, \alpha^2, \ldots$ of powers of $\alpha$ via a linear operator $E$, resembling the expectation operator of r.v.'s. This setting is quite natural in free probability [15]. The second device consists in representing a sequence $1, a_1, a_2, \ldots$, by means of distinct umbrae, as it happens in probability theory too for independent and identically distributed (i.i.d.) r.v.'s. Thanks to these devices above all, the umbral calculus was already employed successfully as a syntax for symbolic computations in statistics [6, 7] by using Maple. Multivariate extensions have been given in [5] for cumulant estimators as an alternative to tensor methods [2].

The algorithm here proposed relies on umbral parametrizations of cumulants (classical, boolean and free) in terms of moments and viceversa, carried out in [8]. In the free context, such a parametrization involves umbral Abel polynomials. Here we show that it is sufficient to change a parameter in the formula in order to get same conversions in classical and boolean theory. So we trace back all the parametrizations to one umbral polynomial. We give a suitable expansion of such a polynomial in order to get a very simple algorithm for the whole matter.

The paper is organized as follows. Section 2 is provided for readers unaware of the classical umbral calculus. We resume terminology, notation and some basic definitions. Section 3 recalls the umbral theory of cumulants both classical, boolean and free, as well as their corresponding parametrizations. In Section 4, we show that all the parametrizations can be recovered through only one umbral polynomial for which a fruitful expansion is provided. The MAPLE algorithm for this expansion is also given. Comparisons,
with procedures available in the literature (see for example [1]), confirm the competitiveness of the umbral algorithm. The paper ends with some examples of how to use umbral calculus in order to compute classical, boolean and free cumulants for some classical probability laws.

2 The classical umbral calculus

In the following, we recall terminology, notation and some basic definitions of the classical umbral calculus, as introduced by Rota and Taylor in [11] and subsequently developed by Di Nardo and Senato in [3] and [4]. We also recall those results useful in developing the umbral algorithm, given in Section 4. We skip any proof: the reader interested in in-depth analysis is referred to [3] and [4].

Classical umbral calculus is a syntax consisting of the following data: a set \( A = \{\alpha, \beta, \ldots\} \), called the alphabet, whose elements are named umbrae; a commutative integral domain \( R \) whose quotient field is of characteristic zero\(^1\); a linear functional \( E : R[A] \rightarrow R \), called evaluation, such that \( E[1] = 1 \) and

\[
E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i] E[\beta^j] \cdots E[\gamma^k]
\]

for any set of distinct umbrae in \( A \) and for \( i, j, \ldots, k \) nonnegative integers (the so-called uncorrelation property); an element \( \epsilon \in A \), called augmentation\(^2\), such that \( E[\epsilon^n] = \delta_{0,n} \), for any nonnegative integer \( n \), where \( \delta_{i,j} = 1 \) if \( i = j \), otherwise being zero; an element \( u \in A \), called unity umbra, such that \( E[u^n] = 1 \), for any nonnegative integer \( n \).

A sequence \( a_0 = 1, a_1, a_2, \ldots \) in \( R \) is umbrally represented by an umbra \( \alpha \) when

\[
E[\alpha^i] = a_i, \quad \text{for } i = 0, 1, 2, \ldots.
\]

The elements \( a_i \) are called moments of the umbra \( \alpha \). This name recalls the device, familiar to statisticians, when \( a_i \) represents the \( i \)-th moment of a r.v. \( X \). Similarly, the factorial moments of an umbra \( \alpha \) are the elements

\[
a_{(0)} = 1, \quad a_{(n)} = E[(\alpha)_n] \quad \text{for nonnegative integers } n
\]

where \( (\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1) \) is the lower factorial. The following umbrae play a special role in the umbral calculus.

**Singleton umbra.** The singleton umbra \( \chi \) is the umbra such that \( E[\chi^1] = 1 \) and \( E[\chi^n] = 0 \) for \( n = 2, 3, \ldots \). Its factorial moments are \( x_{(n)} = (-1)^{n-1}(n-\)

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\(^1\)In many applications \( R \) is the field of real or complex numbers.
Bell umbra. The Bell umbra $\beta$ is the umbra whose factorial moments are all equal to 1, i.e. $E[(\beta)_n] = 1$ for $n = 0, 1, 2, \ldots$. Its moments are the Bell numbers, that is the $n$-th coefficient in the Taylor series expansion of the function $\exp(e^t - 1)$.

An umbral polynomial is a polynomial $p \in R[A]$. The support of $p$ is the set of all umbrae occurring in $p$. If $p$ and $q$ are two umbral polynomials, then $p$ and $q$ are uncorrelated if and only if their supports are disjoint; $p$ and $q$ are umbrally equivalent if and only if $E[p] = E[q]$, in symbols $p \simeq q$.

It is possible that two distinct umbrae represent the same sequence of moments, in such case they are called similar umbrae. More formally two umbrae $\alpha$ and $\gamma$ are similar when $\alpha^n$ is umbrally equivalent to $\gamma^n$, for all $n = 0, 1, 2, \ldots$ in symbols $\alpha \equiv \gamma \iff \alpha^n \simeq \gamma^n$.

Given a sequence $1, a_1, a_2, \ldots$ in $R$ there are infinitely many distinct, and thus similar umbrae, representing the sequence.

Thanks to the notion of similar umbrae, the alphabet $A$ can be extended by inserting the so-called auxiliary umbrae, resulting from operations among similar umbrae. This leads to construct a saturated umbral calculus, in which auxiliary umbrae are handled as elements of the alphabet.

In the following, we focus the attention on some special auxiliary umbrae. We assume $\{\alpha', \alpha'', \ldots, \alpha''''\}$ is a set of $n$ uncorrelated umbrae similar to an umbra $\alpha$.

Dot power. The symbol $\alpha^n$ is an auxiliary umbra denoting the product $\alpha' \alpha'' \cdots \alpha''''$. Moments of $\alpha^n$ can be easily recovered from its definition. Indeed, if the umbra $\alpha$ represents the sequence $1, a_1, a_2, \ldots$, then $E[(\alpha^n)_k] = a_k^n$ for all nonnegative integers $k$ and $n$.

Dot product. The symbol $n.\alpha$ denotes an auxiliary umbra similar to the sum $\alpha' + \alpha'' + \cdots + \alpha''''$. So $n.\alpha$ is the umbral counterpart of a sum of i.i.d. r.v.'s. Moments of $n.\alpha$ can be expressed using the notions of integer partition\(^2\) and dot-power. By using an umbral version of the well-known

\(^2\)Recall that a partition of an integer $i$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $\lambda_j$ are weakly decreasing positive integers such that $\sum_{j=1}^{t} \lambda_j = i$. The integers $\lambda_j$ are named parts of $\lambda$. The length of $\lambda$ is the number of its parts and will be indicated by $\nu_\lambda$. A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, where $r_j$ is the number of parts of $\lambda$ equal to $j$ and $r_1 + r_2 + \cdots = \nu_\lambda$. Note that $r_j$ is said to be the multiplicity of $j$. We use the classical notation $\lambda \vdash i$ to denote "$\lambda$ is a partition of $i$".
multinomial expansion theorem, we have
\[(n,α)^i \simeq \sum_{λ\vdash i}(n)_νd_λα,\]
where the sum is over all partitions \(λ = (1^r_1, 2^r_2, \ldots)\) of the integer \(i\), \((n)_ν = 0\) for \(ν > n\),
\[d_λ = \frac{i!}{r_1!r_2!\ldots} \frac{1}{(1!)^{r_1}(2!)^{r_2}\ldots} \text{ and } α_λ = [α^{r_1}]^{r_1}[α^{r_2}]^{r_2}\ldots\] (2)

A feature of the classical umbral calculus is the construction of new auxiliary umbrae by suitable symbolic substitutions. For example, in \(n.α\) replace the integer \(n\) by an umbra \(γ\). From (1), the new auxiliary umbra \(γ.α\) is such that
\[(γ.α)^i \simeq \sum_{λ\vdash i}(γ)_νd_λα,\]
and it is the umbral counterpart of a so-called random sum. By using (3), we have \(β.χ ≡ χ.β ≡ u\) as well as \((α.χ)^i ≃ (α)^i\). In the next section, we will see that \(χ.α\) has a different meaning. Moreover we have
\[(γ.β.α)^i \simeq \sum_{λ\vdash i} γ^{α_λ}d_λα,\]
(4)

The umbra \(γ.β.α\) is called composition umbra of \(α\) and \(γ\) and it is such that \((γ.β).α ≡ γ.(β.α)\). The compositional inverse \(α^{<−1>}\) of an umbra \(α\) is such that \(α^{<−1>}.β.α ≡ χ ≡ α.β.α^{<−1>}\).

3 Classical, boolean and free cumulants

In this section, we recall some results given in [8] about connections between moments and cumulants (classical, boolean and free).

\(α\)-cumulant umbra. The umbra \(χ.α\), where \(χ\) is the singleton umbra, is called \(α\)-cumulant umbra. Replacing \(γ\) by \(χ\) and by virtue of (3), we have
\[(χ.α)^i \simeq \sum_{λ\vdash i} x_νd_λα \simeq \sum_{λ\vdash i} (-1)^{ν_α−1}(ν_α - 1)!d_λα,\]
(5)

Since the second equivalence in (4) recalls the well-known expression of cumulants in terms of moments of a r.v., it is natural to refer to moments of \(χ.α\) as cumulants of \(α\). The \(α\)-cumulant umbra, usually denoted by \(κ_α\), is deeply studied in [4]. In particular, if \(κ_α\) is the \(α\)-cumulant umbra, then
\( \alpha \equiv \beta. \kappa_\alpha \). Moreover, by recalling equivalence (3) and \( E[(\beta)_i] = 1 \) for all nonnegative integers \( i \), we have

\[
\alpha^i \simeq (\beta. \kappa_\alpha)^i \simeq \sum_{\lambda \vdash i} (\beta)_i \nu_{\lambda} (\kappa_\alpha)_{\lambda} \simeq \sum_{\lambda \vdash i} d_{\lambda} (\kappa_\alpha)_{\lambda}.
\] (6)

**Theorem 3.1** (Parametrizations). Let \( \kappa_\alpha \) be the \( \alpha \)-cumulant umbra. For \( i = 1, 2, \ldots \) we have

\[
\alpha^i \simeq \kappa_\alpha -(\kappa_\alpha + \beta. \kappa_\alpha) + 1 \quad \kappa_\alpha^i \simeq \alpha -(\alpha - 1. \alpha)^{i-1}.
\] (7)

The symbol \(-1. \alpha\) denotes the inverse of an umbra \( \alpha \), that is the umbra such that \( \alpha + -1. \alpha \equiv \varepsilon \).

**\( \alpha \)-boolean cumulant umbra.** Let \( M(t) \) be the ordinary generating function (g.f.) of a r.v. \( X \), that is \( M(t) = 1 + \sum_{i \geq 1} a_i t^i \) where \( a_i = E[X^i] \). We have

\[
M(t) = \frac{1}{1 - H(t)}, \quad \text{where} \quad H(t) = \sum_{i \geq 1} h_i t^i,
\]

and \( h_i \) are called boolean cumulants of \( X \). The umbral theory of boolean cumulants has been introduced in [8]. In particular, the \( \alpha \)-boolean cumulant umbra \( \eta_\alpha \) is such that \( E[\eta_\alpha^i] = h_i \). This umbra corresponds to the composition of the umbra \( \bar{\alpha} \), having moments \( \bar{\alpha}^i \simeq \bar{i} \alpha_i \), and the compositional inverse of \( \bar{\alpha} \), having moments \( n! \), in symbols \( \bar{\eta}_\alpha \equiv \bar{u}^{< -1 >}. \beta. \bar{\alpha} \). By this last equivalence, we get

\[
\bar{\eta}_\alpha^i \simeq \sum_{\lambda \vdash i} (-1)^\gamma \lambda^{-1} \nu_{\lambda}! d_{\lambda} \bar{\alpha}_{\lambda}.
\]

The previous equivalence allows us to express boolean cumulants in terms of moments. In order to get the inverted expressions, we need of the following equivalence \( \bar{\alpha} \equiv \bar{u}^{< -1 >}. \beta. \bar{\alpha} \), that has been proved in the Boolean Inversion Theorem (cfr. [8]). Again we have

\[
\bar{\alpha}^i \simeq \sum_{\lambda \vdash i} \nu_{\lambda}! d_{\lambda} (\bar{\eta}_\alpha)_{\lambda}.
\]

Observe the analogy between the similarity \( \bar{\eta}_\alpha \equiv \bar{u}^{< -1 >}. \beta. \bar{\alpha} \), and the one characterizing the \( \alpha \)-cumulant umbra \( \kappa_\alpha \equiv \chi. \alpha \equiv u^{< -1 >}. \beta. \alpha \).

**Theorem 3.2** (Parametrizations). Let \( \eta_\alpha \) be the \( \alpha \)-boolean cumulant umbra. For \( i = 1, 2, \ldots \), we have

\[
\bar{\alpha}^i \simeq \bar{\eta}_\alpha (\bar{\eta}_\alpha + 2. \bar{\alpha} \beta. \bar{\eta}_\alpha)^{i-1} \quad \bar{\eta}_\alpha^i \simeq \bar{\alpha} (\bar{\alpha} - 2. \bar{\alpha})^{i-1}.
\] (8)
**$\alpha$-free cumulant umbra.** Let us consider a noncommutative r.v. $X$, i.e. an element of an unital noncommutative algebra $A$. Suppose $\phi : A \to \mathbb{C}$ is an unital linear functional. The $i$-th moment of $X$ is the complex number $m_i = \phi(X^i)$ while its g.f. is the formal power series $M(t) = 1 + \sum_{i \geq 1} m_i t^i$. The noncrossing (or free) cumulants of $X$ are the coefficients $r_i$ of the ordinary power series $R(t) = 1 + \sum_{i \geq 1} r_i t^i$ such that $M(t) = R[tM(t)]$. The umbral theory of free cumulants has been introduced in [8]. As befor e, the $\bar{\alpha}$-free cumulant $K_{\bar{\alpha}}$ has been characterized so that $E[K_{\bar{\alpha}}^i] = i! r_i$. In particular, by using the Lagrange inversion formula [3], the $\bar{\alpha}$-free cumulant umbra is such that $(-1)^{i-1} K_{\bar{\alpha}}^i \equiv \frac{1}{i!} \frac{\partial^i}{\partial \alpha^i} \bar{\alpha}$, where $\alpha^i$ is the derivative umbra of $\alpha$, i.e. such that $\alpha^i \simeq n \alpha^{n-1}$. It is quite obvious to observe the difference between the previous equivalence and the ones characterizing both the $\alpha$-classical and the $\alpha$-boolean cumulant umbrae. This is why the computation of free cumulants is quite difficult compared with the classical and boolean ones. Speicher has found a way to expressing free cumulants $\{r_n\}_{n \geq 1}$ in terms of moments $\{m_n\}_{n \geq 1}$ (and vice versa) by using non-crossing partitions of a set [12] [13]. However, the resulting algorithm is quite difficult to implement. Bryc in [1] uses a different approach. In the next section, we propose an unifying algorithm, which relies on the following parametrizations.

**Theorem 3.3 (Parametrizations).** Let $K_{\bar{\alpha}}$ be the $\bar{\alpha}$-free cumulant umbra. For $i = 1, 2, \ldots$ we have

$$\bar{\alpha}^i \simeq K_{\bar{\alpha}}(K_{\bar{\alpha}} + i \cdot K_{\bar{\alpha}})^{i-1} \quad K_{\bar{\alpha}}^i \simeq \bar{\alpha}(\bar{\alpha} - i \cdot \bar{\alpha})^{i-1}.$$  

(9)

The umbral polynomials $x(x - n \cdot \alpha)^{n-1}$ are known as umbral Abel polynomials.

**4 The umbral algorithm**

The algorithm we propose relies on an efficient expansion of the following umbral polynomial $\gamma(\gamma + \delta \cdot \gamma)^{i-1}$ for $i = 1, 2, \ldots$ and $\delta, \gamma$ umbrae.

**Proposition 4.1.** If $\delta, \gamma \in A$ then

$$\gamma(\gamma + \delta \cdot \gamma)^{i-1} \simeq \sum_{\mu \vdash \lambda} (\delta)_{\nu\mu} d_{\mu} \gamma^\lambda.$$  

(10)

**Proof.** By using the binomial expansion and equivalence (11), we have

$$\gamma(\gamma + \delta \cdot \gamma)^{i-1} \simeq \sum_{s=1}^i \binom{i}{s-1} \gamma^s \sum_{\lambda \vdash \gamma^s} d_{\lambda} \gamma^\lambda.$$  

(11)
Suppose we consider the partition \( \mu \) of the integer \( i \), obtained by adding the integer \( s \) to the partition \( \lambda \). Then we have \( \gamma^s \gamma_\lambda \equiv \gamma_\mu \) and \( \nu_\lambda = \nu_\mu - 1 \). If \( c_s \) denotes the multiplicity of \( s \) in \( \lambda \) and \( m_s \) denotes the multiplicity of \( s \) in \( \mu \), then \( m_s = c_s + 1 \). Therefore, we have

\[
\begin{align*}
\left( \frac{i - 1}{s - 1} \right) d_\lambda &= \frac{s}{i} \frac{i!}{(1!)^{c_1} \cdots (s!)^{c_s+1} \cdots c_1! \cdots c_s!} = s m_s \frac{d_\mu}{i},
\end{align*}
\]

where the last equality comes by multiplying numerator and denominator for \( m_s > 0 \). Recalling that \( \sum s m_s = i \), equivalence (10) follows.

In order to evaluate \( \gamma (\gamma + \delta \gamma)^{i-1} \) via (10), we need the factorial moments of \( \delta \) and the moments of \( \gamma \). Recall that, if we just have information on moments \( \delta \), the factorial moments can be recovered by using the well-known change of bases:

\[
(\delta)_i \simeq \sum_{k=1}^i s(i, k) \delta^k,
\]

where \( \{s(i, k)\} \) are the Stirling numbers of I kind. In particular equivalence (10) allows us to give any expression of cumulants (classical, boolean, free) in terms of moments and vice versa.

i) For classical cumulants in terms of moments, due to the latter of (7), we set \( \delta = -1.u \) and \( \gamma = \alpha \). Here we find \( E[(-1.u)_i] = (-1)_i = (-1)^i! \).

ii) For moments in terms of classical cumulants, due to the first of (7), we set \( \delta = \beta \) and \( \gamma = \kappa_\alpha \). Here we know \( E[(\beta)_i] = 1 \).

iii) For boolean cumulants in terms of moments, due to the latter of (8), we set \( \delta = -2.u \) and and \( \gamma = \bar{\alpha} \). Here we find \( E[(-2.u)_i] = (-1)^i (i + 1)! \).

iv) For moments in terms of boolean cumulants, due to the first of (8), we set \( \delta = 2.\bar{u}.\beta \) and \( \gamma = \bar{\eta}_\alpha \). Here we have \( E[(2.\bar{u}.\beta)_i] = E[(2.\bar{u}.\beta.\chi)^i] = E[(2.\bar{u})^i] = E[(\bar{u} + \bar{u}')^i] = (i + 1)! \).

v) For free cumulants in terms of moments, due to the latter of (9), we set \( \delta = -n.u \) and \( \gamma = \bar{\alpha} \). Here we have \( E[(-n.u)_i] = (-n)_i \).

vi) Finally, for moments in terms of free cumulants, due to the first of (9), we set \( \delta = n.u \) and \( \gamma = \mathfrak{R}_\alpha \). Here we have \( E[(n.u)_i] = (n)_i \).

The umbral algorithm in MAPLE is the following:

```maple
y := combinat['partition'](i):
umbralg := proc(i, fm, y)
    i! * add(fm[j] * mul(((g[x[1]]/x[1]!)-x[2])/x[2]!),
```

```maple
```
x=convert(y[j],multiset)),
  j=1..nops(y));
end:

In the MAPLE procedure, the factorial moments \( E[(\delta)_{j-1}] \) are referred by the vector \( fm[j] \) and the moments \( E[\gamma^k] \) are referred by the vector \( g[k] \).

Table 1 refers to computational times (in seconds) reached by using the umbral algorithm and the Bryc’s procedure \(^1\), both implemented in MAPLE, release 7, when we need free cumulants in terms of moments \(^3\).

| \( i \) | MAPLE (umbral) | MAPLE (Bryc) |
|-------|----------------|-------------|
| 15    | 0.015          | 0.016       |
| 18    | 0.031          | 0.062       |
| 21    | 0.078          | 0.141       |
| 24    | 0.172          | 0.266       |
| 27    | 0.375          | 0.703       |

Table 1: Comparisons of computational times needed to compute free cumulants in terms of moments. Tasks performed on Intel (R) Pentium (R), CPU 3.00 GHz, 512 MB RAM.

**Moments of Wigner semicircle distribution.** In free probability, the Wigner semicircle distribution is analogous to the Gaussian r.v. in the classical probability. Indeed, free cumulants of degree higher than 2 of the Wigner semicircle r.v. are zero. The first column in Table 2 shows moments of the Wigner semicircle r.v. \( X \), computed by the umbral algorithm. They are compared with Catalan numbers \( C_i \) (second column), since it is well-known that \( E[X^{2i}] = C_i \) and \( E[X^{2i+1}] = 0 \). By using equivalence \(^4\) it is straightforward to prove that the umbra corresponding to the Wigner semicircle distribution is \( \zeta.\beta.\delta \), where \( \zeta \) is the umbra whose moments are the Catalan numbers and \( \delta \) is an umbra having only second moment equal to 1, the others being zero. In the next section, we will use again this umbra in describing the umbral syntax of a Gaussian r.v.

**Moments of Marchenko-Pastur distribution.** In free probability, the Marchenko-Pastur distribution is analogous to the Poisson r.v. in the classical probability. Indeed, the free cumulants are all equal to a parameter \( \lambda \). The last column in Table 2 shows moments of the Marchenko-Pastur distribution computed by the umbral algorithm.

\(^3\)The output is in the same form of the one given by Bryc’s procedure.
Table 2: Moments of some special free distributions.

| i | Wigner r.v. | Catalan numbers | Marchenko-Pastur r.v. |
|---|---|---|---|
| 2 | 1 | 2 | $\lambda^2 + \lambda$ |
| 3 | 0 | 5 | $\lambda^3 + 3\lambda^2 + \lambda$ |
| 4 | 2 | 14 | $\lambda^4 + 6\lambda^3 + 6\lambda^2 + \lambda$ |
| 6 | 5 | 132 | $\lambda^6 + 15\lambda^5 + 50\lambda^4 + 50\lambda^3 + 15\lambda^2 + \lambda$ |
| 8 | 14 | 1430 | $\lambda^8 + 28\lambda^7 + 196\lambda^6 + 490\lambda^5 + 490\lambda^4 + 196\lambda^3 + 28\lambda^2 + \lambda$ |

5 Computing cumulants of some known laws through the umbral algorithm

An umbra looks like the framework of a r.v. with no reference to any probability space, just looking at moments. The way to recognize the umbra corresponding to a r.v. is to characterize the sequence of moments \( \{a_n\} \). When the sequence exists, this can be done by comparing the moment generating function (m.f.g.) of a r.v. with the so-called generating function (g.f.) of an umbra. The g.f. of an umbra has been defined \[4\] as the following formal power series

\[
f(\alpha, t) = 1 + \sum_{n \geq 1} a_n t^n n!
\]

In the classical umbral calculus, the convergence of the formal power series \( f(\alpha, t) \) is not relevant. This means that we can define the umbra whose moments are the same as the moments of a lognormal r.v., even if this r.v. does not admit a m.g.f. So the umbral algorithm allows us to compute classical\[4\], boolean or free cumulants of any r.v. having sequence of moments \( \{m_n\} \). Once the umbra corresponding to the r.v. has been characterized, we choose \( \delta = -1.u, -2.u, -n.u \) in equivalence \[10\], depending on whether we need classical, boolean or free cumulants.

In the following, we take up again some of the examples given in \[1\], showing how they can be recovered through the umbral algorithm by a suitable characterization of the involved umbrae.

**Poisson r.v.** A Poisson r.v. of parameter \( \lambda \) is umbrally represented by

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\[4\] Recall that, due to their properties of additivity and invariance under translation, the cumulants are not necessarily connected with the moments of probability distributions. We can define cumulants of any r.v. disregarding the question whether its m.g.f. converges \[9\].
the umbra \(\lambda, \beta\), because \(f(\lambda, \beta, t) = \exp[\lambda(\exp(t) - 1)]\), which is the m.g.f. of a Poisson r.v. The moments are \(a_i = E[(\lambda, \beta)^i] = \sum_{k=1}^i S(i, k)\lambda^k\), where \(S(i, k)\) are the Stirling numbers of second type. So cumulants of a Poisson r.v. can be computed via the umbral algorithm, taking as input the sequence of moments \(E[(\lambda, \beta)^i]\). If the input is the sequence of factorial moments \((-1)^i!\), we get classical cumulants; if the input is the sequence of factorial moments \((-1)^i(i+1)!\), we get boolean cumulants; if the input is the sequence of factorial moments \((-n)_i\) we get free cumulants (Tables 3 and 4 in [1]).

**Compound Poisson r.v.** A compound Poisson r.v. \(S_N = X_1 + \cdots + X_N\), where \(X_i\) are i.i.d. r.v.’s and \(N\) is a Poisson r.v. of parameter \(\lambda\), is umbrally represented by the umbra \(\lambda, \beta\), because \(f(\lambda, \beta, t) = \exp[\lambda f(\alpha, t) - 1]\), which is the m.g.f. of a compound Poisson r.v. The formal power series \(f(\alpha, t)\) corresponds to the m.g.f. of \(X_i\). Due to equivalence (1), the moments are \(a_i = E[(\lambda, \beta)^i] = \sum_k \lambda^k d_{\lambda, \alpha}\).

**Exponential r.v.** An exponential r.v. is umbrally represented by the umbra \(\lambda, \beta\), because \(f(\lambda, \beta, t) = (1 - \frac{t}{\lambda})^{-1}\), which is the m.g.f. of an exponential r.v. with parameter \(\lambda > 0\). So its moments are \(a_i = E[(\lambda, \beta)^i] = \sum_{k=1}^i \lambda^k d_{\lambda, \alpha}\). In order to obtain the second column of Table 3 in [1], choose in (10) \(\lambda = 1\) and \{(-n)_i\} as factorial moments.

**Uniform r.v.** An uniform r.v. on the interval \([a, b]\) has m.g.f.

\[
M(t) = \frac{e^{tb} - e^{ta}}{t(b - a)} = \sum_{s=0}^{t(b-a)} \left[ e^s - 1 \right]_{s=t(b-a)}.
\]

The umbra, with g.f. \(\frac{e^s - 1}{s}\), is the inverse of the Bernoulli umbra [11], i.e. \(-1.\). So an uniform r.v. on the interval \([a, b]\) is umbrally represented by the umbra \(a + (b - a)(-1.\)). Recalling that \(E[(-1.\)^i] = \frac{1}{i+1}\), we get

\[
a_i = E\{[(a + (b - a)(-1.\))]^i\} = \sum_{j=0}^i \binom{i}{j} a^{i-j} (b-a)^j.
\]

In order to obtain the last column of Table 3 in [1], choose in [10] \(a = -1, b = 1\) and \{((-n)_i)\} as factorial moments.

**Bernoulli r.v.** A Bernoulli r.v. of parameter \(p \in (0, 1)\) has m.g.f. \(M(t) = q + pe^t = 1 + p(e^t - 1)\). The umbra with g.f. \(M(t)\) is \(\chi, p, \beta\) (see [4] for more details), whose moments are \(a_i = p\).

**Binomial r.v.** A binomial r.v. of parameters \(n\), a positive integer, and \(p \in (0, 1)\) is a sum of \(n\) i.i.d. Bernoulli r.v.’s. Similarly a binomial r.v. is umbrally represented by \(n.\chi, p, \beta\). Due to equivalence [1], we have \(a_i = \)
\[ E\{n.(\chi.p.\beta)^i\} = \sum_{\lambda \vdash i} (n)_\lambda d\lambda p^\lambda. \] In order to obtain the second column of Table 4 in [1], choose \(\{(-n)\}_i\) as factorial moments in (10).

**Gaussian r.v.** A gaussian r.v. with real parameter \(\mu\) and \(\sigma > 0\) has m.g.f. \(M(t) = \exp\left(\mu t + \sigma^2 t^2/2\right)\). This power series is the g.f. of the umbra \(\mu + \beta.(\sigma\delta)\), where \(\delta\) is an umbra such that \(E[\delta^2] = 1\) whereas \(E[\delta^i] = 0\) for positive integers \(i \neq 2\). The following proposition gives the expression of the \(n\)-th moment of the umbra representing the Gaussian r.v.

**Proposition 5.1.** For \(n=1,2,\ldots\) we have

\[ a_n = E[(\mu + \beta.(\sigma\delta))^n] = \sum_{k=0}^{[n/2]} \left(\frac{\sigma^2}{2}\right)^{k} \frac{(n)_{2k}}{k!} \mu^{n-2k}. \quad (12) \]

**Proof.** By using the binomial expansion, we have

\[ (\mu + \beta.(\sigma\delta))^n \simeq \sum_{j=0}^{n} \binom{n}{j} \mu^{n-j} [\beta.(\sigma\delta)]^j \simeq \sum_{j=0}^{n} \binom{n}{j} \mu^{n-j} \sum_{\lambda \vdash j} d\lambda(\sigma\delta)\lambda. \quad (13) \]

Suppose \(\lambda = (1^{r_1}, 2^{r_2}, \ldots)\), then \(\beta.(\sigma\delta)\lambda \equiv (\sigma\delta')^{r_1}[\sigma^2(\delta')^2]^{r_2} \cdots \equiv \sigma^j \delta_{\lambda}.\) On the other hand, due to the definition of the umbra \(\delta\), we have \(E[\delta_{\lambda}] \neq 0\) iff the partition \(\lambda\) is of type \((2^{r_2})\). Thus, if \(j\) is odd, there does not exist any partition of this type and so \([\beta.(\sigma\delta)]^j \simeq 0\). Instead, if \(j\) is even, say \(j = 2k\), there exists a unique partition of type \((2^{r_2})\) which corresponds to \(r_2 = k\). For this partition, we have \(d_{\lambda} = \frac{(2k)!}{(2!)^k}\), so that \([\beta.(\sigma\delta)]^{2k} \simeq \frac{(2k)!}{(2!)^k} \sigma^{2k}\). Replacing this last equivalence in (13), we have

\[ (\mu + \beta.(\sigma\delta))^n \simeq \sum_{k=0}^{[n/2]} \frac{n!}{(2k!)(n-2k)!} \frac{(n)_{2k}}{k!} \mu^{n-2k} \frac{(2k)!}{(2!)^k} \sigma^{2k} \]

by which the result follows.

We have proved by umbral tools that moments of a Gaussian r.v. can be expressed by using a particular sequence of orthogonal polynomials, the **Hermite polynomials**:

\[ H_n^{(\nu)}(x) \simeq \sum_{k=0}^{[n/2]} \left(\frac{-\nu}{2}\right)^{k} \frac{(n)_{2k}}{k!} x^{n-2k}, \]

with \(x = \mu\) and \(\nu = -\sigma^2\). All the properties of Hermite polynomials can be easily recovered by using the Gaussian umbra. We skip the details. As before, the first column of Table 3 in [1] can be recovered from the umbral algorithm by using (12) with \(\mu = 0\) and \(\sigma = 1\) and choosing \(\{(-n)\}_i\) as factorial moments in (10).
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