On the relations between osp(2,2) and the 
quasi exactly solvable systems

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Abstract

By taking a product of two sl(2) representations, we obtain the differential operators preserving some space of polynomials in two variables. This allows us to construct the representations of osp(2,2) in terms of matrix differential operators in two variables. The corresponding operators provide the building blocks for the construction of quasi exactly solvable systems of two and four equations in two variables. Some generalisations are also sketched. The peculiar labelling used for the generators allows

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us to elaborate a nice deformation of osp(2,2). This gives an appropriate basis for analyzing the quasi exactly solvable systems of finite difference equations.
1 Introduction

The number of quantum mechanical problems for which the spectral equation can be solved algebraically is rather limited. It is therefore not surprising that the quasi exactly solvable (QES) equations [1], [2] attract some attention. For these equations, indeed, a finite number of eigenvectors can be obtained by solving an algebraic equation. The study of QES equations has motivated the classification of finite dimensional real Lie algebras of first order differential operators. The algebras which, in a suitable representation, preserve a finite dimensional module of smooth real functions are particularly relevant for QES equations. The case of one variable was addressed and solved some years ago [3]. There is, up to an equivalence only one algebra, for instance sl(2), acting on the space of polynomials of degree at most $n$ in the variable. For two variables, the classification is more involved. It is described respectively in [4] and [5] for complex and real variables. The corresponding real QES operators finally emerge in seven classes summarized in table 7 of [5].

The natural next step is to classify the QES systems of two equations [6, 7]. It is known that, in general, the underlying algebraic structure is not a (super) Lie algebra; in those few cases where the algebra is a Lie algebra, it is generically sl(2)$\times$sl(2) or osp(2,2) [7]. The first case is rather trivial while the second one seems to be worth to be studied in more details. This was discussed in [6, 7]; the present paper is devoted to a further study of the relations of osp(2,2) with QES systems.

In the second section, we present some new aspects of the representations of the Lie algebra sl(2) in terms of differential operators. As a byproduct we obtain
2 Representations of sl(2)

We first discuss the realizations of sl(2) which will be useful in the next sections. In the case of one variable, the basic QES operators read \([1],[3]\)

\[
j_-(x, m) = \frac{d}{dx}, \quad j_0(x, m) = x \frac{d}{dx} - \frac{m}{2}, \quad j_+(x, m) = x^2 \frac{d}{dx} - mx \quad (1)
\]

When \(m\) is an integer these operators preserve the vector space, say \(P(m)\), of polynomials of degree at most \(m\) in the variable \(x\). They obey the commutation relations of the Lie algebra \(sl(2)\).

From (1), we can define the operators corresponding to the product of two representations. That is to say, if \(x\) and \(y\) denote two independent variables

\[
j_\epsilon(x, m; y, n) = j_\epsilon(x, m) + j_\epsilon(y, n) \quad , \quad \epsilon = \pm, 0 \quad (2)
\]

Clearly, the operators (2) obey the same commutation relations as (1). They act on the vector space, say \(P(m, n)\) of polynomials of degree at most \(m\) (resp. \(n\)) in the variable \(x\) (resp. \(y\)). However, their action on \(P(m, n)\) is not irreducible. One can show easily that the operators (2) preserve irreducibly the subspace of
$P(m, n)$ defined by

$$M(m, n) = \text{Span} \left\{ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^k x^m y^n , \quad 0 \leq k \leq m + n \right\} \quad (3)$$

This vector space is the eigenspace of the Casimir operator corresponding to the representation of highest spin in the decomposition of $P(m, n)$ in subspaces irreducible with respect to the action of $[2]$. The other irreducible representations result as similar structures with different values of $m$ and $n$; it is therefore sufficient to deal with (3).

Alternatively, $M(m, n)$ can be seen as the kernel of the operator

$$K(x, y, m, n) = (x - y) \frac{\partial}{\partial x} \frac{\partial}{\partial y} + n \frac{\partial}{\partial x} - m \frac{\partial}{\partial y} \quad (4)$$

acting on $P(m, n)$. Remark that $M(m, 0)$ (resp. $M(0, n)$) is isomorphic to $P(m)$ (resp. $P(n)$); it that case, the one dimensional operators (1) are recovered from (2) by ignoring the partial derivative $\frac{\partial}{\partial y}$ (resp. $\frac{\partial}{\partial x}$).

### 2.1 QES operators in two variables

The QES operators in two real variables are classified in Ref. [5]; the authors summarize the seven possible hidden algebras in their table 7. The operators labelled (1.4), (1.10) and (2.3) in this table are studied independently in Refs. [8], [12]. The operators labelled (1.1) in the table appear to be new; in particular they lead to the only case for which the invariant module is not manifestly a space of polynomials in the two variables. In the following we show that the formulation (1.1) can be simplified and related to the algebra (2) by means of a suitable change of function.

The operators (1.1) in table 7 of ref.[4] read

$$\tilde{j}_- = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$\tilde{j}_0 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

$$\tilde{j}_+ = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + \frac{n}{2} (x - y) \quad (5)$$
They preserve the space $\tilde{M}(m, n)$ defined as

$$\tilde{M}(m, n) = \text{Span} \left\{ (x - y)^{m + \frac{2}{\epsilon} - k} R_k^{m,n} \left( \frac{x + y}{x - y} \right), \quad 0 \leq k \leq 2m + n \right\} \quad (6)$$

$$R_k^{m,n}(t) = \frac{d^k}{dt^k} (t - 1)^{m+n}(t + 1)^m \quad (7)$$

Our observation is summarized by the following two formulas

$$(x - y)^{m + \frac{2}{\epsilon}} j_{\epsilon} (x - y)^{-m - \frac{2}{\epsilon}} = j_{\epsilon}(x, m) + j_{\epsilon}(y, m + n) \quad (8)$$

$$(x - y)^{m + \frac{2}{\epsilon}} \tilde{M}(m, n) = M(m, m + n) \quad (9)$$

In other words the algebra (3) is equivalent to the algebra (2) (up to a suitable redefinition of $n$ into $m + n$).

The advantage of the new formulation is twofold. First the relevant operators form an $\mathfrak{sl}(2)$ diagonal subalgebra of the $\mathfrak{sl}(2) \otimes \mathfrak{sl}(2)$ algebra generated by

$$j_\epsilon(x, m) \quad , \quad j_\epsilon(y, n) \quad , \quad \epsilon = \pm, 0 \quad (10)$$

In this respect the form (4) of the operators (3) is clearly related to the fundamental operators (1) and is easy to generalize to the case of $V$ variables. Second, the vector space of the representation, i.e. $M(m, n)$, is a space of polynomials like in all the other cases of the classification of Ref.[5]. In the next sections, we discuss some possible extensions of the algebra (2) into graded algebras. The corresponding operators are related to systems of QES equations.

### 2.2 Imbeddings into gl(2)

In the next sections, we will put the emphasis on the classification of the operators which preserve the direct sum

$$M(m, n) \oplus M(m + \Delta, n + \Delta') \quad (11)$$

where $m, n, m + \Delta, n + \Delta'$ are positive integers. The operators (2) are crucial for this task; however we find it convenient to label them as follows. For $\mu, \nu \in \{0, 1\}$
we define

$$J_\mu^\nu(x, m, m + \Delta; y, n, n + \Delta') = \text{diag}(j_\mu^\nu(x, m; y, n), j_\mu^\nu(x, m + \Delta; y, n + \Delta')) - \frac{T}{2}\delta_\mu^\nu$$

where $T$ is a constant $2 \times 2$ diagonal matrix while

$$j_0^0(x, m; y, n) = j_0(x, m; y, n)$$

$$j_1^1(x, m; y, n) = -j_0(x, m; y, n)$$

$$j_0^1(x, m; y, n) = j_-(x, m; y, n)$$

$$j_1^0(x, m; y, n) = -j_+(x, m; y, n)$$

The operators (12) obey the commutation relations of the Lie algebra $\mathfrak{gl}(2)$:

$$[J_\mu^\nu, J_\alpha^\beta] = \delta_\beta^\alpha J_\mu^\nu - \delta_\mu^\alpha J_\beta^\nu,$$

(for shortness we drop the dependence on $x, m, \ldots$) defining a (reducible) representation of this Lie algebra on the vector space (11).

### 3 Representations of $\mathfrak{osp}(2,2)$

We first consider the $2 \times 2$ matrix operators preserving the direct sum

$$M(m, n) \oplus M(m + 1, n).$$

In addition to the diagonal generators $J_\mu^\nu(x, m, m + 1; y, n, n)$, defined above, we have to construct the off diagonal ones. They can be formulated in terms of the matrices $\sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/2$ ($\sigma_a$, $a=1,2,3$ are the Pauli matrices) and of the following differential operators

$$q_0(x, m; y, n) = \frac{1}{m+1}(m+n+1+(x-y)\frac{\partial}{\partial y})$$

$$q_1(x, m; y, n) = -\frac{1}{m+1}\left((m+1)x + ny + y(x-y)\frac{\partial}{\partial y}\right)$$

$$\bar{q}_1(x, m) = \frac{\partial}{\partial x}$$

$$\bar{q}_0(x, m) = (x\frac{\partial}{\partial x} - m - 1)$$

(16)
The coming proposition allows to classify the operators under consideration.

**Proposition**

The linear operators preserving the vector space \((13)\) are the elements of the enveloping algebra generated by the eight operators

\[
J_{\mu}^{\nu}(m,n,1,0) \quad , \quad \mu, \nu = 0, 1
\]

\[
Q_{\mu} = q_{\mu}(x,m;y,n)\sigma_- \quad , \quad \mu = 0, 1
\]

\[
\overline{Q}^{\mu} = \overline{q}^{\mu}(x,m)\sigma_+ \quad , \quad \mu = 0, 1
\]

The eight operators \((18),(19),(20)\) acting on the space \((15)\) lead to an irreducible representation (with dimension = \(2m + 2n + 3\)) of the graded Lie algebra \(osp(2,2)\). The labelling of the generators, together with the matrix \(T\) in \((12)\) of the form

\[
T = \text{diag}(m + n + \Delta + \Delta', 1, m + n + 1)
\]

results in a particularly concise form of the structure constants. In addition to \((14)\), we find

\[
[J_{\mu}^{\nu}, Q_{\alpha}] = \delta_{\mu}^{\nu}Q_{\alpha} - \delta_{\alpha}^{\nu}Q_{\mu}
\]

\[
[J_{\mu}^{\nu}, \overline{Q}^{\alpha}] = \delta_{\mu}^{\nu}\overline{Q}^{\alpha} - \delta_{\alpha}^{\nu}\overline{Q}^{\mu}
\]

\[
\{Q_{\mu}, \overline{Q}^{\nu}\} = J_{\mu}^{\nu}
\]

\[
\{Q_{\mu}, Q_{\nu}\} = \{\overline{Q}^{\mu}, \overline{Q}^{\nu}\} = 0
\]

In particular \(Q_{\mu}\) (and \(\overline{Q}^{\nu}\)) transforms as a doublet under the adjoint action of the \(\text{gl}(2)\) subalgebra generated by the four \(J\)'s.

The relationship between the super Lie algebra \(osp(2,2)\) and the differential operators (of one variable) preserving \(P(m) \oplus P(m+1)\) was first noticed in \([6]\). The differential operators used in \([6]\) can be recovered from ours by setting \(n = 0\) and dropping all derivatives \(\frac{\partial}{\partial y}\). Our labelling of the generators is, however, different.
The finite dimensional representations of \( \text{osp}(2,2) \) \cite{13} can be expressed in terms of differential operators of one variable \cite{11}. In this kind of approach, the generic, finite dimensional irreducible representation appears as acting on the vector space

\[ P(m) \oplus P(m + 1) \oplus P(m - 1) \oplus P(m). \] (26)

The representations of \( \text{osp}(2,2) \) can be formulated also in terms of the two variables differential operators constructed above. In particular the generic irreducible representation can be constructed equally well on the vector spaces

\[ V_I = M_{m,n} \oplus M_{m+1,n} \oplus M_{m-1,n} \oplus M_{m,n} \] (27)

\[ V_{II} = M_{m,n} \oplus M_{m+1,n} \oplus M_{m,n+1} \oplus M_{m,n} \] (28)

The associated representations are equivalent (in agreement with \cite{13}) but the expressions of the generators in terms of the partial derivatives is quite different. The proof of the equivalence between the representations acting on \( V_I \) and on \( V_{II} \) relies on the fact that the operators \( \text{(12)} \) are invariant under the double substitution \( m \leftrightarrow n \) and \( x \leftrightarrow y \). Therefore the operators \( q_\mu(y, n; x, m) \) (resp. \( \bar{q}_\mu(y, n) \)) behave exactly as \( q_\mu(x, m; y, n) \) (resp. \( \bar{q}_\mu(x, m) \)) under the adjoint action of \( \text{(12)} \).

4 More graded algebras

We now put the emphasis on the operators preserving the vector space

\[ M_{m,n} \oplus M_{m+\Delta,n+\Delta'} \] (29)

The relevant diagonal operators can be chosen according to \( \text{(12)} \). The construction of the off diagonal ones depends on the relative signs of \( \Delta \) and \( \Delta' \).
4.1 case $\Delta, \Delta' \geq 0$

The operators connecting the vector space $M(m, n)$ with $M(m + \Delta, n + \Delta')$ (and vice versa) can be formulated in terms of products of operators (16) (and (17)) where the indices $m$ and $n$ in the different factors are appropriately shifted. The following identities allows one to deal with the ambiguities of ordering of the different factors:

$$q_b(x, m + 1; y, n) q_a(x, m; y, n) = q_a(x, m + 1; y, n) q_b(x, m; y, n)$$
$$q_b(y, n; x, m + 1) q_a(x, m; y, n) = q_a(y, n; x, m + 1) q_b(x, m; y, n)$$
$$q_b(y, n; x, m + 1) q_a(x, m; y, n) = q_a(x, m; y, n + 1) q_b(y, n; x, n)$$  \hspace{1cm} (30)

One can use these identities in order to define the operators

$$q(x, [a_k]; y, [b_l]) \equiv \prod_{l=1}^{\Delta'} \prod_{k=1}^{\Delta} q_{b_l}(y, n + l - 1; x, m + \Delta) q_{a_k}(x, m + k - 1, n)$$  \hspace{1cm} (31)

symmetrically in the multi indices

$$[a_k] \equiv (a_1, \ldots, a_{\Delta'}) \hspace{1cm} [b_l] \equiv (b_1, \ldots, b_{\Delta})$$  \hspace{1cm} (32)

The operators (31) connect $M(m, n)$ with $M(m + \Delta, n + \Delta')$ and

$$Q([a_k], [b_l]) = q(x, [a_k]; y, [b_l]) \sigma_-$$  \hspace{1cm} (33)

are the counterparts of the operators (16). Using the remarks made at the end of the previous section, one can convince that the operators (31) transform according to the representation of spin $\Delta + \Delta'$ under the adjoint action of the $\text{gl}(2)$ represented via (12).

The operators $\overline{Q}([a_k], [b_l])$, proportional to $\sigma_+$, can be constructed in exactly the similar way as for the $Q([a_k], [b_l])$. Identities like (30) exist among the $\overline{q}_a$. The complete algebra (which is non linear) can be obtained following the same lines as in [14].
4.2 The case $M(m,n) \oplus M(m+1,n-1)$

If we consider $\Delta$ and $\Delta'$ of opposite signs, the operators preserving (11) do not represent a Lie super algebra even for $|\Delta| = |\Delta'| = 1$. We studied the operators preserving the vector space $M(m, n) \oplus M(m + 1, n - 1)$ and observed that the underlying algebraic structure is different from those obtained in [14]. Again, $J^\nu_\mu(x, m, m+1; y, n, n-1)$ can be used as a starting point. We find it convenient to set $T = 0$ in (12) and add separately the grading operator $T \equiv \sigma_3$. As far as the off diagonal operators are concerned, we choose

$$R^\nu_\mu = \bar{q}_\mu(y, n - 1, x, m + 1)q^\nu(x, m, y, n)\sigma_-$$
$$\overline{R}^\nu_\mu = \bar{q}_\mu(x, m, y, n)q^\nu(y, n - 1, x, m + 1)\sigma_+$$

These tensors are not irreducible with respect to the adjoint action of the $J$ generators. The traces

$$R^\mu_\mu = \frac{m + n + 2}{m + 1}((y - x)\frac{\partial}{\partial y} - n), \quad \overline{R}^\mu_\mu = \frac{m + n + 2}{n + 1}((x - y)\frac{\partial}{\partial x} - m)$$

are operators which intertwines the equivalent representations carried by the spaces $M(m, n)$ and $M(m + 1, n - 1)$; they commute with the four operators (12).

The order of the factors $q$ and $\bar{q}$ entering in $R$ and $\overline{R}$ can be reversed by using the identity

$$q_\mu(y, n - 1, x, m + 1)q^\nu(x, m, y, n) - q^\nu(x, m, y, n - 1)q_\mu(y, n - 1, x, m) = \frac{1}{m + n + 2}R^\mu_\mu$$

The generators $J^\nu_\mu, R^\nu_\mu, \overline{R}^\nu_\mu$ obey the following commutation and anticommutation relations

$$[J^\nu_\mu, R^\beta_\alpha] = \delta^\nu_\beta R^\mu_\alpha - \delta^\mu_\alpha R^\beta_\nu,$$
$$[J^\nu_\mu, \overline{R}^\beta_\alpha] = \delta^\nu_\beta \overline{R}^\mu_\alpha - \delta^\mu_\alpha \overline{R}^\beta_\nu$$
\[
\{ R^\nu_{\mu}, \mathcal{R}^\beta_\alpha \} = \frac{1}{2} \{ J^\beta_\mu, J^\nu_\alpha \} + \frac{T}{2} (\delta^\beta_\mu J^\nu_\alpha - \delta^\nu_\alpha J^\beta_\mu) \\
- \frac{1}{2} (\delta^\nu_\mu J^\beta_\alpha + \delta^\beta_\alpha J^\nu_\mu) - \frac{1}{2} \delta^\nu_\mu \delta^\beta_\alpha
\]

(38)

These relations do not define an abstract algebra. In order to fulfil the associativity conditions (i.e. the Jacobi identities), the anticommutators between two \( R \) (two \( \mathcal{R} \)), which vanish for (34) have to be implemented in a non trivial way \[7,14\]; equation (38) indicates that the underlying algebra is non linear.

5 Deformation of \( \text{osp}(2,2) \)

If we want to describe in abstract terms the algebraic structure underlying the QES finite difference equations, some deformations of the algebras discussed above seem to emerge in a natural way. For scalar equations the relevant deformation was pointed out some time ago \[3\]. In a previous paper \[11\] we presented some (finite dimensional) representations of a deformation of \( \text{osp}(2,2) \) in terms of finite difference operators. However, we have taken advantage of the “\( \text{gl}(2) \)-labelling” used here for the generators of \( \text{osp}(2,2) \) to construct a new deformation of this super Lie algebra.

A deformation of \( \text{osp}(2,2) \) is known \[15\] (for more general graded algebra see \[16\]); it is such that the (anti) commutators of some generators close, within the enveloping algebra, in terms of transcendental functions of the generators. The deformation of \( \text{osp}(2,2) \) that we present here is constructed in the same spirit as the so called “Witten type II” deformation of \( \text{sl}(2) \) \[8,10\]. That is to say that the structure of the algebra relates q-commutators of the generators to linear combinations of them. The q-commutators and q-anti-commutators are defined respectively as

\[
[A, B]_q = AB - qBA \quad , \quad \{ A, B \}_q = AB + qBA
\]

(39)

where \( q \) is the deformation parameter.
Our deformation of osp(2,2) is expressed as follows,

\[ \{ Q_\mu, Q_\nu \} q^{\nu-\mu} = 0 \quad , \quad \{ \overline{Q}^\nu, \overline{Q}^\nu \} q^{\nu-\mu} = 0 \quad \text{(40)} \]

\[ \{ Q_\mu, \overline{Q}^\nu \} = J_\mu^\nu \quad \text{(41)} \]

\[ [J_\mu^\nu, Q_\alpha] q^{\nu-\alpha} = q^{\frac{\alpha-\nu-1}{2}} (\delta_\mu^\nu Q_\alpha - \delta_\alpha^\nu Q_\mu) \quad \text{(42)} \]

\[ [J_\mu^\nu, \overline{Q}^\alpha] q^{\nu-\alpha} = q^{\frac{\mu-\alpha-1}{2}} (\delta_\nu^\alpha \overline{Q}^\mu - \delta_\mu^\nu \overline{Q}^\alpha) \quad \text{(43)} \]

\[ [J_\mu^\nu, J_\alpha^\beta] q^s = q^{\frac{s-1}{2}} (\delta_\mu^\beta J_\nu^\alpha - q^r \delta_\alpha^\nu J_\mu^\beta) + \frac{q-1}{q^2} (Q_0 \overline{Q}^0 + Q_1 \overline{Q}^1) \delta_\alpha^\beta \delta_\mu^\nu (1 - \delta_\mu^\nu \delta_\alpha^\beta) \quad \text{(44)} \]

with \( s \equiv \nu + \alpha - \mu - \beta \) and \( r \equiv (\nu - \beta)(\mu - \alpha) \).

All Jacobi identities are obeyed by these relations. The last commutator indicates that the bosonic generators do not close into a gl(2) subalgebra (similarly to the deformation of ref.[15]). Neglecting all fermionic operators in the above formulas leads to one deformation of gl(2) found in [17].

It is possible to construct two independent expressions, quadratic in the generators, which q-commute with the generators. These “q-Casimir” operators read

\[ C_1 = Q_0 \overline{Q}^0 + Q_1 \overline{Q}^1 + q J_0^0 J_1^1 - J_0^0 J_1^1 - J_0^0 \quad \text{(45)} \]

\[ C_2 = (q - 1)^2 (Q_0 \overline{Q}^0 + Q_1 \overline{Q}^1) + q(q - 1)^2 J_1^0 J_0^1 + (q - 1) J_1^1 - q(q - 1) J_0^0 \]

and obey the following commutation properties (for i=1,2)

\[ [C_i, J_\mu^\nu] q^{2(\mu - \nu)} = 0 \quad , \]

\[ [C_i, Q_\mu] q^{2\mu - 1} = 0 \quad , \]

\[ [C_i, Q_\mu] q^{1-2\mu} = 0 \quad \text{(46)} \]

It follows that any function of the ratio \( C_1/C_2 \) commute with the generators and constitute a conventional Casimir.
We further constructed the representations of the algebra above which are relevant for systems of finite difference QES equations. To describe them we define a finite difference operator $D_q$.

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad D_q x^n = [n]_q x^{n-1}, \quad [n]_q = \frac{1 - q^n}{1 - q} \quad (47)$$

The simplest of these realizations are characterized by a positive integer $n$ and act on the vector space $P(n - 1) \oplus P(n)$ .

Adopting $x$ as the variable, the fermionic generators are represented by

$$Q_0 = q^{-\frac{n}{2}} \sigma_-, \quad Q_1 = -x \sigma_-$$

$$\overline{Q}^0 = q^{-\frac{n}{2}} (xD_q - [n]_q) \sigma_+, \quad \overline{Q}^1 = D_q \sigma_+ \quad (49)$$

The bosonic operators can be constructed easily from (41) but we write them for completeness

$$J_0^0 = q^{-n} ((xD_q - [n]_q) \Pi_2, \quad J_1^1 = (-1) \text{diag}(qxD_q + 1, xD_q) \quad (50)$$

$$J_0^1 = q^{-\frac{n}{2}} D_q \Pi_2, \quad J_1^0 = (-1) q^{-\frac{n}{2}} \text{diag}(qx(xD_q - [n - 1]_q), x(xD_q - [n]_q)) \quad (51)$$

The enveloping algebra constructed with the eight generators above contains all finite difference operators preserving $P(n - 1) \oplus P(n)$.

As in the underformed case, there also exist representations which act on the vector space

$$P(n) \oplus P(n + 1) \oplus P(n - 1) \oplus P(n) \quad (52)$$

The fermionic generators are realized as follows

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ D_q & 0 & 0 & 0 \\ 0 & -D_q & 1 & 0 \end{pmatrix} \quad (53)$$
where $\lambda$ is an arbitrary complex parameter and, for shortness, we used $\delta(n) \equiv xD_q - [n]_q$. The invariance of the operators under the similarity transformation has been exploited to set $Q_0$ in a form as simple as possible. In the limit $\lambda \to q^{-n-1}$ the representation (56) becomes reducible and decomposes into atypical ones of the form (53).

The deformation of osp(2,2) presented above leads to simple normal ordering rules for the generators. In this respect, it is very appropriate for the classification of the finite difference operators preserving the space (58) or (52) and of the corresponding QES systems. The normal ordering rules associated with the deformation used in ref.[11] are not as transparent since they depend non polynomially of some operators.

6 Concluding remarks

The most interesting examples of QES systems are related to the algebra osp(2,2), e.g. the relativistic Coulomb problem and the stability of the sphaleron in the
abelian Higgs model [7]. Therefore, further realizations of this algebra in terms of differential operators deserve some attention. Here, we present realizations formulated in terms of differential operators in two variables. The labelling used for the generators clearly exhibits their tensorial structure under the gl(2) subalgebra and provides very naturally the building blocks for the construction of series of (non linear) graded algebras preserving some vector spaces of polynomials.

Witten type deformations attracted recently some attention (see e.g. [18] for osp(1,2)). The deformation of osp(2,2) presented here is of this type and it admits representations which are directly relevant for the study of QES finite difference systems. The existence of a coproduct for this type of deformation would allow to adapt the construction of section 2 to finite difference equations. This is a nice application of the coproduct that we plan to address in a subsequent paper.
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