Mellin-Barnes presentations for Whittaker wave functions

S. Kharchev⋆, S. Khoroshkin⋆◦,

*Institute for Theoretical and Experimental Physics, Moscow, Russia;
⃗⃗Institute for Information Transmission Problems RAS (Kharkevich Institute), Bolshoy Karetny per. 19, Moscow, 127994, Russia;
⃗⃗⃗National Research University Higher School of Economics, Moscow, Russia.

Abstract

We obtain certain Mellin-Barnes integrals which present Whittaker wave functions related to classical real split forms of simple complex Lie groups.

Keywords: Whittaker function, Mellin transform, Lusztig parametrization

1 Introduction

1. In this paper we obtain certain Mellin-Barnes integrals which present Whittaker functions related to classical real split forms of simple complex Lie groups.

Whittaker functions originally appeared as solutions of a special differential equation of hypergeometric type [WW]. They were then realized as eigenfunctions of Laplace operators of the group $GL(2, \mathbb{R})$, see e.g. [V]. This construction was generalized to real semisimple groups and led to a series of significant researches in representation theory, see e.g. [J, Sch]. B.Kostant related the group theory of Whittaker functions with a family of Toda integrable systems [T].

Whittaker functions admit several integral presentations. Particular examples were obtained more than forty years before, see e.g. [Bu]. S.Kharchev and D.Lebedev in [KL] found integral presentation of $GL(n, \mathbb{R})$ Whittaker wave functions using the machinery of inverse scattering method. Then in [GKL] the same presentation was obtained by calculating matrix elements in certain infinite dimensional ”Gelfand-Zetlin” representations. A.Givental found quite different integral presentation for the same Whittaker function using geometric arguments. A.Gerasimov et al. then realized in [GKLO, GLO] that Givental construction can be reformulated as the description of the matrix element in principal series representation using Gauss decomposition and Lusztig coordinates [L] on nilpotent subgroup. Moreover, Whittaker wave functions for all classical split real groups were described in [GLO] as certain integrals over positive cone in corresponding maximal nilpotent subgroup. Recently both presentations were generalized to a quantum group setting [SS] using the machinery of cluster mutations. In particular, the completeness and orthogonality of $q$-versions of $GL(n, \mathbb{R})$ Whittaker wave functions were proved there.

We start with the presentation of Whittaker wave function as of special matrix element [GLO]

$$
\Psi_\lambda(x) = e^{-\langle \rho, x \rangle} v^L_{\lambda - \rho} \exp (-x) v^R_{\lambda - \rho},
$$

(1.1)

see (2.17) and (2.18) for precise notations, and rewrite this matrix element as Barnes integral. Our technique is rather elementary. It contains four ingredients: use of Lusztig
coordinates, Berenstein-Zelevinsky transform, Plancherel formula for Mellin transform and linear algebra matrix calculations. Lusztig coordinates are convenient in the description of Whittaker functions by several reasons. They separate Lusztig positive cone $N_+$, which is original space of integration of the matrix element in consideration. Cluster type mutations between different Lusztig chats enable us to avoid the use of formulas for the action of the Lie algebra and observe elegant expressions for invariant forms and vectors.

The definition (1.1) of the matrix element exploit two special vectors in generalized principal series; they are called left and right Whittaker vector and the Whittaker functions is the matrix element of the Cartan flow related to these vectors. The right Whittaker vector has a simple direct expression in Lusztig coordinates, see Proposition 2.2. The definition of the left Whittaker vector requires the conjugation by the longest element $w_0$ of the Weyl group and the calculation of Gauss coordinates of the new matrix. This induces a birational map of the nilpotent subgroup $N$ which in slightly different setting was studied by A.Berenstein and A.Zelevinsky [BZ] and was crucial for their description of Lusztig coordinates via ‘generalised minors’ – matrix elements in fundamental representations. We find another expressions for this birational map (we call it BZ transform) just by elementary linear algebra calculations using the induction by the rank of the group. This is the main point of the construction and we suspect that the formulas for BZ maps which we found will serve elsewhere. The precise description of BZ maps is given in Theorems 3.1, 4.1, 5.1 and 6.1

The rest is the application of the convolution theorem for Mellin transform, which we use in a form of Plancherel formula [Tt]. It gives a presentation of Whittaker functions for classical split real groups $GL(n, \mathbb{R})$, $SO(n, n)$, $SO(n, n+1)$ and $Sp(2n, \mathbb{R})$ (which present root system of $A, D, B$ and $C$ series) by means of Barnes integrals. These presentations can be equivalently rewritten as Mellin transforms of Whittaker functions. See Section 7.

For the group $GL(n, \mathbb{R})$ we have now two different integral presentation of Whittaker function by Mellin–Barnes integral: Kharchev-Lebedev formulas [KL, GKL] and (1.2) of the present paper. They are quite different. We do not know how to derive one from another. On the other hand, Mellin transform for $GL(n, \mathbb{R})$ Whittaker function was calculated by E.Stade [St]. Our formula (7.1) also differs from that of [St].

The following are the main results of the paper.

2. Let $x = (x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$ be the coordinates on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n, \mathbb{R})$, $\lambda_i \in \mathbb{R}$ be dual coordinates on $\mathfrak{h}^*$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, so that the element $h(x) \in \mathfrak{h}$ is given by the diagonal matrix $h = \sum_{i=1}^n x_i e_i$; and $(x, \lambda) = \sum_{k=1}^n x_k \lambda_k$. Let $\Psi_\lambda(x)$ be the Whittaker wave function for $GL(n, \mathbb{R})$, defined by the relation (2.18).

**Theorem 3.2** The function $\Psi_\lambda(x)$ is given by the integral

$$\Psi_\lambda(x) = \frac{e^{-i(x, \lambda)}}{(2\pi i)^d} \int_C \exp \left( \sum_{1 \leq k \leq n} (\gamma_{1,n+2-k} - \gamma_{1,n+1-k})x_k \right) \prod_{1 \leq k < l \leq n} \Gamma(\gamma_{kl} - \gamma_{k+1,l} + i(\lambda_k - \lambda_{n+k-l+1})) \Gamma(\gamma_{kl} - \gamma_{k+1,l+1})d\gamma_{k,l}. \quad (1.2)$$

Here we set $\gamma_{k,l} = 0$ if the pair $(k, l)$ does not satisfies the condition $1 \leq k < l \leq n$; $d = n(n-1)/2$ is the dimension of the maximal unipotent subgroup of $GL(n, \mathbb{R})$. The
integration cycle $C$ is a deformation of the imaginary plain $\text{Re} \gamma_{k,l} = 0$ into the domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand. For instance one can use iterated integration, which starts with integration over $\gamma_{n-1,n}$ the over $\gamma_{n-2,n-1}$, then over $\gamma_{n-2,n}$ etc., which respects the conditions $\text{Re} \gamma_{k,l} > \text{Re} \gamma_{k+1,m}$ for all admissible triples $k, l, m$.

Let $\mathbf{x} = (x_1, \ldots, x_n), x_i \in \mathbb{R}$ be the coordinates on the Cartan subalgebra $\mathfrak{h}$ of $\text{so}(n, n)$, and $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i \in \mathbb{R}$ be dual coordinates on $\mathfrak{h}^*$, so that the element $h(\mathbf{x}) \in \mathfrak{h}$ is given by the diagonal matrix $h = \sum_{i=1}^n x_i (e_{ii} - e_{2n+1-i,2n+1-i})$ and $(\mathbf{x}, \lambda) = \sum_{k=1}^n x_k \lambda_k$.

Let $\Psi_\lambda(\mathbf{x})$ be the Whittaker wave function for $\text{SO}(n, n)$ defined by the relation (2.18).

**Theorem 4.2** The function $\Psi_\lambda(\mathbf{x})$ is given by the integral

$$
\Psi_\lambda(\mathbf{x}) = \frac{e^{-i(\mathbf{x}, \lambda)}}{(2\pi i)^d} \int_C \exp H(\mathbf{x}, \gamma) \prod_{k=1}^{n-2} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(\gamma_{k,n} + \delta_{k,n} - \gamma_{k+1,n} - \delta_{k+1,n} + 2i\lambda_k)} \prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}.
$$

Here we set $\gamma_{k,l} = \delta_{k,l} = 0$ if the pair $(k, l)$ does not satisfies the condition $1 \leq k < l \leq n, d = n(n-1)$ is the dimension of the maximal unipotent subgroup of $\text{SO}(n, n)$.

$$
H(\mathbf{x}, \gamma) = \sum_{j<n} (\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) + \gamma_{1,n}(x_n - x_{n-1}) - \delta_{1,n}(x_{n-1} + x_n),
$$

\begin{align*}
\xi_{i,j} &= -\gamma_{i+1,j+1} + \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad 1 \leq i < j < n, \\
\eta_{i,j} &= \gamma_{i+1,j+1} + \delta_{i+1,j+1} + \gamma_{i+1,j} - \delta_{i,j}, \quad 1 \leq i < j < n, \\
\xi_{i,n} &= \gamma_{i,n} - \delta_{i+1,n}, \quad \eta_{i,n} = \delta_{i,n} - \gamma_{i+1,n} \quad 1 \leq i < n.
\end{align*}

The integration cycle is a deformation of the imaginary plane $\text{Re} \gamma_{k,l} = 0$ into nonempty domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which is described by the relations

$$
\text{Re} \gamma_{i,j} > 0, \quad \text{Re} \delta_{i,j} > 0, \quad \text{Re} \xi_{i,j} > 0, \quad \text{Re} \eta_{i,j} > 0, \quad 1 \leq i < j \leq n.
$$

Let $\mathbf{x} = (x_1, \ldots, x_n), x_i \in \mathbb{R}$ be the coordinates on the Cartan subalgebra $\mathfrak{h}$ of $\text{so}(n + 1, n)$, and $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i \in \mathbb{R}$ be dual coordinates on $\mathfrak{h}^*$, so that the element $h(\mathbf{x}) \in \mathfrak{h}$ is given by the diagonal matrix $h = \sum_{i=1}^n x_i (e_{ii} - e_{2n+2-i,2n+2-i})$ and $(\mathbf{x}, \lambda) = \sum_{k=1}^n x_k \lambda_k$.

Let $\Psi_\lambda(\mathbf{x})$ be the Whittaker function for $\text{SO}(n + 1, n)$ defined by the relation (2.18).

**Theorem 5.2** The function $\Psi_\lambda(\mathbf{x})$ is given by the integral

$$
\Psi_\lambda(\mathbf{x}) = \frac{e^{-i(\mathbf{x}, \lambda)}}{(2\pi i)^d} \int_C \exp H(\mathbf{x}, \gamma) \Gamma(\gamma_n + 2i\lambda_n) \Gamma(\gamma_n) d\gamma_n.
$$

$$
\prod_{k=1}^{n-1} \Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k) \Gamma(\gamma_k - \gamma_{k+1}) d\gamma_k.
$$

$$
\prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}.
$$
Here we set $\gamma_{k,l} = \delta_{k,l} = 0$ if the pair $(k, l)$ does not satisfies the condition $1 \leq k < l \leq n$, and $\gamma_k = 0$ if $k = 1$ or $k = n + 1$, $d = n^2$ is the dimension of the maximal unipotent subgroup of $\text{SO}(n + 1, n)$

$$H(x, \gamma) = \sum_{j=2}^{n} (\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) - \gamma_1x_n.$$ (1.5)

$$\xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad 1 \leq i < j < n,$$
$$\eta_{i,j} = \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad 1 \leq i < j < n,$$
$$\xi_{i,n} = \gamma_{i+1,n} + \delta_{i,n} - \gamma_{i+1,n}, \quad \eta_{i,n} = -\gamma_{i+1,n} - \delta_{i,n} + \gamma_i, \quad 1 \leq i < n.$$ (1.6)

The integration cycle is a deformation of the imaginary plain $\text{Re} \gamma_{k,l} = \text{Re} \delta_{k,l} = \text{Re} \gamma_k = 0$ into nonempty domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which is described by the relations

$$\text{Re} \gamma_{i,j} > 0, \text{Re} \delta_{i,j} > 0, \text{Re} \xi_{i,j} > 0, \text{Re} \eta_{i,j} > 0, 1 \leq i < j \leq n, \quad \text{Re} \gamma_k > 0, k = 1, ..., n.$$

Let $x = (x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$ be the coordinates on the Cartan subalgebra $\mathfrak{h}$ of $\text{sp}(2n, \mathbb{R})$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R}$ be dual coordinates on $\mathfrak{h}^*$, so that the element $h(x) \in \mathfrak{h}$ is given by the diagonal matrix $h = \sum_{i=1}^{n} x_i e_{ii} - e_{2n+1-i,2n+1-i}$ and $(x, \lambda) = \sum_{k=1}^{n} x_k \lambda_k$. Let $\Psi_\lambda(x)$ be the Whittaker function for $\text{Sp}(2n, \mathbb{R})$ defined by the relation (2.18).

**Theorem 6.2.** The function $\Psi_\lambda(x)$ is given by the integral

$$\Psi_\lambda(x) = \frac{e^{-i(x,\lambda)}}{(2\pi i)^d} \int_\mathcal{C} \exp \left( H(x, \gamma) \prod_{k=1}^{n-1} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(2\gamma_k - 2\gamma_{k+1} + 2\lambda_k)} \right) \prod_{k<l} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}.$$ $$\prod_{k=1}^{n} \Gamma(\gamma_k - \gamma_{k+1} + i\lambda_k) \Gamma(\gamma_k - \gamma_{k+1}) d\gamma_k.$$

Here we set $\gamma_{k,l} = \delta_{k,l} = 0$ if the pair $(k, l)$ does not satisfies the condition $1 \leq k < l \leq n$, and $\gamma_k = 0$ if $k = 1$ or $k = n + 1$, $d = n^2$,

$$H(x, \gamma) = \sum_{j=2}^{n} (\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) - 2\gamma_1x_n.$$ (1.7)

$$\xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad 1 \leq i < j < n,$$
$$\eta_{i,j} = \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad 1 \leq i < j < n,$$
$$\xi_{i,n} = \gamma_{i+1,n} + \delta_{i,n} - 2\gamma_{i+1}, \quad 1 \leq i < n,$$
$$\eta_{i,n} = -\gamma_{i+1,n} - \delta_{i,n} + \gamma_i, \quad 1 \leq i < n.$$ (1.8)

The integration cycle is a deformation of the imaginary plain $\text{Re} \gamma_{k,l} = \text{Re} \delta_{k,l} = \text{Re} \gamma_k = 0$ into nonempty domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which is described by the relations

$$\text{Re} \gamma_{i,j} > 0, \text{Re} \delta_{i,j} > 0, \text{Re} \xi_{i,j} > 0, \text{Re} \eta_{i,j} > 0, 1 \leq i < j \leq n, \quad \text{Re} \gamma_k > 0, k = 1, ..., n.$$
2 Generalities

2.1 Whittaker vectors and Whittaker wave functions

Let $G$ be a split real form of a reductive group over $\mathbb{C}$, $B_{\pm}$ two opposite Borel subgroups of $G$, $N_{\pm}$ their maximal nilpotent subgroups and $H = B_{+}\cap B_{-}$ the Cartan subgroup. Let $D(G)$ be the ring of bi-invariant differential operators on $G$. Denote by $b_{\pm}$, $n_{\pm}$ and $h$ the corresponding Lie algebras. Let $\Delta_{\pm} \in h^{*}$ be the systems of positive and negative roots of $g$, $\Pi \subset \Delta_{+}$ be a subsystem of simple roots and $W$ the Weyl group of $G$. Denote by $G_{0}$ the big Bruhat cell $G_{0} = N_{-}HN_{+}$. It is dense open in $G$.

For each index $i$ of a simple root $\alpha_{i} \in \Pi$ denote by $e_{i} = e_{\alpha_{i}}$, $f_{i} = f_{\alpha_{i}} = e_{-\alpha_{i}}$ and $h_{i} = h_{\alpha_{i}}$ the corresponding Chevalley generators of $g$, so that

\[
\alpha_{i}(h_{i}) = 2
\]

and $e_{i}, f_{i}$ and $h_{i} = \alpha_{i}^{\vee}$ are standard generators of the embedded Lie algebra $\mathfrak{sl}_2$:

\[
[h_{i}, e_{i}] = 2e_{i}, \quad [h_{i}, f_{i}] = -2f_{i}, \quad [e_{i}, f_{i}] = h_{i}.
\]

Let $\zeta_{\pm} : N_{\pm} \rightarrow \mathbb{C}^{*}$ be nondegenerate characters, defined by the relations

\[
\zeta_{+}(\exp(te_{i})) := e^{-t}, \quad \zeta_{-}(\exp(tf_{i})) := e^{-t}
\]

for all simple roots $\alpha_{i}$. In this paper Whittaker function is a analytical function $\varphi$ on $G_{0}$ satisfying the conditions

\[
\varphi(g) \quad \text{is an eigenfunction for any} \quad D \in D(G);
\]

\[
\varphi(n_{-}gn_{+}) = \zeta_{-}(n_{-})\varphi(g)\zeta_{+}(n_{+}).
\]

for any $g \in G_{0}$, $n_{-} \in N_{-}$, $n_{+} \in N_{+}$. The condition (2.11) implies that Whittaker functions are completely determined by their restriction to Cartan subgroup $H$. B. Kostant noticed [K] that the restriction of the action of the center $Z(g)$ of universal enveloping algebra $U(g)$ to the space of Whittaker functions can be identified with Hamiltonians of the Toda chain related to the root system $\Delta = \Delta_{+} \cup \Delta_{-}$.

Whittaker functions can be constructed as matrix elements between a pair of dual Whittaker vectors. Let $V$ be a representation of Lie algebra $g$, such that its restriction to $b_{+}$ admits an extension to representation of the Borel group $B_{+}$ compatible with $g$-module structure on $V$, that is $bgb^{-1}u = Ad_{b}(g)u$ for any $u \in V$, $g \in g$ and $b \in B_{+}$. A vector $v \in V$ is called right Whittaker vector if

\[
e_{i}v = -v \quad \text{for any} \quad \alpha_{i} \in \Pi.
\]

Let also $V'$ be a representation of Lie algebra $g$, such that its restriction to $b_{-}$ admits an extension to representation of the Borel group $B_{-}$ compatible with $g$-module structure on $V'$. A vector $v' \in V'$ is called left Whittaker vector if

\[
f_{i}v' = -v' \quad \text{for any} \quad \alpha_{i} \in \Pi.
\]

\[1\] (g, B)-module in Harish-Chandra terminology
Assume now that $V$ and $V'$ are dual to each other, that is there is a nondegenerate pairing $(,): V' \otimes V \rightarrow \mathbb{C}$ such that $(xu', u) + (u', xu) = 0$ for any $u' \in V'$, $u \in V$ and $x \in \mathfrak{g}$. Then the matrix element

$$\varphi(g) = (v', gv)$$

is well defined for any $g \in G_0$ satisfies the condition (2.12).

If in addition the representations $V$ and $V'$ are quasii-simple, that is the center $Z(\mathfrak{g})$ acts on them by scalar operators, then the Whittaker functions (2.13) are eigenfunctions of generalized Toda Hamiltonians. It is common to use for this aim the representations of $G$ induced from Borel subalgebra $B_-$. Namely for any $\mu \in \mathfrak{h}^*$ let $\chi_\mu : B_- \rightarrow \mathbb{C}$ be a one-dimensional representation (character) of $B_-$ defined by the relation

$$\chi(e^h n) = e^{\mu(h)} \quad \text{for any } h \in \mathfrak{h} \text{ and } n \in N_-.$$

Denote by $V_\mu$ the space of analytical functions on $G_0$ satisfying the condition

$$f(bg) = \chi_\mu(b)f(g) \quad \text{for any } b \in B_-, g \in G_0 \quad (2.14)$$

The element of the group $B_+$ act on the functions from $V_\mu$ by the right shifts, $b \cdot f(g) = f(gb)$ for $g \in G_0$ and $b \in B_+$ and the elements of $\mathfrak{g}$ act by infinitesimal right shifts,

$$xf(g) = \frac{d}{dt}f(ge^{tx})|_{t=0}. \quad (2.15)$$

This induced module is quasisimple, see [Zh1], and can be regarded to wide extent as a representation of non-unitary principal series. If $\mu + \nu = -2\rho$ then the pairing

$$\langle f, g \rangle = \int_{N_+} f(n)g(n)dn, \quad f \in V_\mu, g \in V_\nu$$

is formally invariant. Here $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. The pairing is invariant under condition of the convergency of the integral. Here $dn$ is invariant measure on $N_+$. We use instead sesquilinear pairing

$$\langle f, g \rangle = \int_{N_*} \bar{f}(n)g(n)dn, \quad f \in V_\mu, g \in V_\nu \quad (2.16)$$

where $N_* \subset N_+$ is Lusztig positive cone, see (2.23). It is invariant when $\nu + \bar{\mu} = -2\rho$ and $f$ and $g$ rapidly vanish at the boundary of $N_*$. In the following we investigate right Whittaker vector $v_\mu^R$ in the space $V_\mu$, left Whittaker vector $v_\nu^L$ in $V_\nu$, where $\nu + \bar{\mu} = -2\rho$ and the Whittaker wave function

$$\tilde{\Psi}_{\mu,\nu}(x) = e^{-\langle \rho, x \rangle}(v_\nu^L, \exp(-x)v_\mu^R) \quad (2.17)$$

Here $x \in \mathfrak{h}$ is an element of Cartan subalgebra. When $\mu = \nu$ is a parameter of unitary principal series, $\mu = \nu = i\lambda - \rho$, where $\lambda$ is real, that is $\lambda(h_k) \in \mathbb{R}$ for all simple roots $\alpha_k$, see (2.9), the integral in the RHS of (2.17) definitely converges and has the form

$$\Psi_\lambda(x) := \tilde{\Psi}_{i\lambda-\rho, i\lambda-\rho}(x) = e^{-\langle \rho, x \rangle}(v_{i\lambda-\rho}^L, \exp(-x)v_{i\lambda-\rho}^R) \quad (2.18)$$
The functions $\tilde{\Psi}_{\mu,\nu}(x)$ and $\Psi_{\lambda}(x)$ are eigenfunctions for a family of Toda Hamiltonians $H_k$,

$$H_k \Psi_{\lambda}(x) = e^{-\langle \rho, x \rangle} C_k e^{\langle \rho, x \rangle} \Psi_{\lambda}(x) = e^{-\langle \rho, x \rangle} (v^L_{\lambda, -\rho}, \exp(-x) C_k v^R_{\lambda, -\rho})$$

where $C_k$ are generators of the ring $\mathbb{Z}(g)$. We will also call $\Psi_{\lambda}(x)$ Whittaker wave function despite it differs from the restriction to Cartan subgroup of the Whittaker function $(v^L_{\lambda, -\rho}, gv^R_{\lambda, -\rho})$ by the normalizing factor $e^{-\langle \rho, x \rangle}$ chosen for the agreement with Toda Hamiltonians.

The Whittaker function in a form of matrix coefficient (2.18) and (2.16) was studied in [GKLO, GLO]. The paper [GKLO] contains integral presentations of Whittaker functions generalizing Givental formula [G] for $\mathfrak{gl}_n$. The integration over positive cone $N_+$ implies the important property of this construction: the function $\Psi_{\lambda}(x)$ rapidly decreases in the region

$$\mathfrak{h}_+ = \{h \in \mathfrak{h}^*, (\alpha, h) > 0 \quad \text{for all} \quad \alpha \in \Delta_+ \}.$$

For $G = \text{GL}(n, \mathbb{R})$ this Whittaker function is known to be symmetric on parameters $\lambda$, see [Si, SS].

### 2.2 Lusztig coordinates

In this subsection we describe Lusztig parametrization of the group $N_+$ in slightly different notation.

Let $w_0$ be the longest element of the Weyl group $W$ and

$$w_0 = s_{i_1}s_{i_2} \cdots s_{i_{N-1}} s_{i_N}$$

be its reduced decomposition. Here $s_j$ is the simple reflection in $\mathfrak{h}^*$ corresponding to the root $\alpha_j$. We associate to (2.19) the following normal (or convex in other terminology) ordering of the set $\Delta_+$:

$$\gamma_1 = \alpha_{i_1}, \gamma_2 = s_{i_1}(\alpha_{i_2}), \gamma_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \ldots \gamma_N = s_{i_1}s_{i_2} \cdots s_{i_{N-1}}(\alpha_N).$$

A normal orderings $<$ of the system $\Delta_+$ of positive roots of a reductive Lie algebra over $\mathbb{C}$ is characterized by the condition

$$\alpha < \alpha + \beta < \beta \quad \text{or} \quad \beta < \alpha + \beta < \alpha$$

if $\alpha, \beta, \alpha + \beta$ are all in $\Delta_+$. The rule (2.20) establishes a bijective correspondence between reduced decompositions of $w_0$ and normal orderings of positive roots [Zh2].

Elementary transformations of reduced decompositions are performed by means of braid group relations

$$s_i s_j \cdots s_j s_i \cdots, \quad i \neq j,$$

where $n_{i,j} = a_{i,j}a_{j,i} + 1$ and $a_{i,j}$ is the entry of Cartan matrix, are reformulated into changes of normal orderings inside subsystems of the rank two:

$$\alpha, \alpha + \beta, \beta \rightarrow \beta, \alpha + \beta, \alpha \quad \text{if} \quad \alpha, \beta \in A_2$$

$$\alpha, \alpha + \beta, \alpha + 2\beta, \beta \rightarrow \beta, \alpha + 2\beta, \alpha + \beta, \alpha \quad \text{if} \quad \alpha, \beta \in B_2$$

$$\alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta \rightarrow \beta, \alpha + 3\beta, \alpha + 2\beta, 2\alpha + 3\beta, \alpha + \beta, \alpha \quad \text{if} \quad \alpha, \beta \in G_2$$
and

\[ \alpha, \beta \rightarrow \beta, \alpha \quad \text{if} \quad \alpha, \beta \in A_1 \times A_1. \]

The reverse of the normal ordering of positive roots does not destroy its defining property (2.21) and thus is the normal ordering as well. Moreover the reverse respects the above transformations of normal orderings and thus defines an involutive automorphism of the root system preserving the subsystem of positive roots. Thus it is induced by an automorphism \( \theta \) of Dynkin diagram \(^\text{2}\). In particular this means that the last root \( \gamma_N \) is simple, as well as the first root \( \gamma_1 = \alpha_{i_1} \), and

\[ \gamma_N = \theta(\alpha_{i_N}) \]

Following Lusztig \([L]\) we associate to each reduced decomposition (2.19) (or, equivalently, to the related normal ordering (2.20)) the group element \( X(t) \in N_+ \)

\[ X(t) = \exp(t_{\gamma_1}e_{i_1}) \cdots \exp(t_{\gamma_N}e_{i_N}) \quad (2.22) \]

The correspondence (2.22) establishes a birational isomorphism of the varieties \( N_+ \) and \( \mathbb{R}^{\mid\Delta_+\mid} \). Denote by \( N_* \subset N_+, \ N_* \sim \mathbb{R}^{\mid\Delta_+\mid} \) the open subset of \( N \) defined by the conditions \( t_\gamma > 0 \) for all \( \gamma \in \Delta_+ \)

\[ N_* = \{ X(t) \mid t_\gamma > 0, \ \gamma \in \Delta_+ \} \quad (2.23) \]

The definition of \( N_* \) does not depend on the choice of reduced decomposition of \( w_0 \), see (2.24), (2.25), (2.26). The passage to another reduced decomposition defines the involutive transition map

\[ X(t') = X(t) \]

where

\[ t'_\alpha = \frac{t_\alpha t_{\alpha+\beta}}{t_\alpha + t_\beta}, \quad t'_{\alpha+\beta} = t_\alpha + t_\beta, \quad t'_{\alpha+2\beta} = \frac{t_\beta t_{\alpha+\beta}}{t_\alpha + t_\beta} \quad (2.24) \]

for the changes of the normal order

\[ \ldots, \alpha, \alpha + \beta, \alpha + 2\beta, \ldots \rightarrow \ldots, \beta, \alpha + \beta, \alpha, \ldots \quad \text{or} \quad \ldots, \beta, \alpha + \beta, \alpha, \ldots \rightarrow \ldots, \alpha, \alpha + \beta, \beta, \ldots \]

of \( A_2 \) subsystem, see \([L]\);

\[ t'_\alpha = \frac{t_{\alpha+2\beta} t_{\alpha+\beta} t_\alpha}{\pi_2}, \quad t'_{\alpha+\beta} = \frac{\pi_2}{\pi_1}, \quad t'_{\alpha+2\beta} = \frac{\pi_1^2}{\pi_2}, \quad t'_\beta = \frac{t_\beta t_{\alpha+\beta} t_{\alpha+2\beta}}{\pi_1} \quad (2.25) \]

where

\[ \pi_1 = t_\beta t_{\alpha+2\beta} + (t_\beta + t_{\alpha+\beta}) t_\alpha, \quad \pi_2 = t_\beta t_{\alpha+2\beta} + (t_\beta + t_{\alpha+\beta})^2 t_\alpha \]

for the changes of the normal order

\[ \ldots, \alpha, \alpha + \beta, \alpha + 2\beta, \beta, \ldots \rightarrow \ldots, \beta, \alpha + 2\beta, \alpha + \beta, \alpha, \ldots \quad \text{or} \quad \ldots, \beta, \alpha + 2\beta, \alpha + \beta, \alpha, \ldots \rightarrow \ldots, \alpha, \alpha + \beta, \alpha + 2\beta, \beta, \ldots \]

\(^2\)We further preserve the notation \( \theta \) for the corresponding automorphism of Lie algebra \( g \) and group \( G \).
of B₂ subsystem, see [BZ, Theorem 3.1]; and

\[ t'_{\alpha} = \frac{t_\alpha t^3 t_{a+3} t_{a+3} t_{a+3} t_{a+3}}{\pi_3}, \quad t'_{a+\beta} = \frac{\pi_3}{\pi_2}, \quad t'_{2a+3\beta} = \frac{\pi^3_2}{\pi_3 \pi_4}, \quad (2.26) \]

where

\[
\begin{align*}
\pi_1 &= t_\beta t_{a+3\beta} t^2_{a+2\beta} t_{2a+3\beta} + t_\beta t_{a+3\beta} (t_\alpha + t_\beta + t_{a+3\beta}) t_{2a+3\beta} t^2_{a+3\beta} t_{a+3}, \\
\pi_2 &= t^2_\beta t^2_{a+3\beta} t^2_{a+2\beta} t_{2a+3\beta} + t^2_\beta t^2_{a+3\beta} (t_\alpha + t_\beta + t_{a+3\beta}) t_{2a+3\beta} t^3_{a+3\beta} t_{a+3} + \left( t_\beta t_{a+3\beta} t_{2a+3\beta} t^2_{a+3\beta} t_{a+3} + 2t^2_{a+3\beta} + 2t_{a+3\beta} t_{a+3} + 2t_\beta t_{a+3} \right), \\
\pi_3 &= t^3_\beta t^2_{a+3\beta} t^2_{a+2\beta} t_{2a+3\beta} + t^3_\beta t^2_{a+3\beta} (t_\alpha + t_\beta + t_{a+3\beta}) t_{2a+3\beta} t^3_{a+3\beta} t_{a+3} + \left( t^2_\beta t_{a+3\beta} t_{2a+3\beta} t^3_{a+3\beta} t_{a+3} + 3t^2_{a+2\beta} + 3t_{a+2\beta} t_{a+3} + 2t_\beta t_{a+3} \right), \\
\pi_4 &= t^2_\beta t^3_{a+3\beta} t^2_{a+2\beta} t_{2a+3\beta} + t^2_\beta t^3_{a+3\beta} (t_\alpha + t_\beta + t_{a+3\beta}) t_{2a+3\beta} t^3_{a+3\beta} t_{a+3} + \left( t^3_\beta t_{a+3\beta} t^2_{a+3\beta} t_{2a+3\beta} t^2_{a+3\beta} t_{a+3} + 3t^2_{a+2\beta} + 3t_{a+2\beta} t_{a+3} + 2t_\beta t_{a+3} \right) t_{2a+3\beta} t^2_{a+3} t_{a+3} \\
&\quad + \left( t^2_\beta t_{a+3\beta} (t_\alpha + t_\beta + t_{a+3\beta}) t^2_{a+3} t_{a+3} + (t_\beta t_{a+3\beta} t_{a+3} t^2_{a+3} t_{a+3} + 2t_\beta t_{a+3} \right) t_{2a+3}\beta t^2_{a+3} t_{a+3} t^3 \]
\]

for the changes of the normal order

\[ \ldots, \alpha, \alpha + \beta, 2a + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta, \ldots \to \beta, \alpha + 3\beta, \alpha + 2\beta, 2a + 3\beta, \alpha + \beta, \ldots; \]

\[ \ldots, \beta, \alpha + 3\beta, \alpha + 2\beta, 2a + 3\beta, \alpha + \beta, \alpha, \ldots \to \alpha, \alpha + 2\beta, 2a + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta, \ldots \]

of G₂ subsystem, see [BZ, Theorem 3.1].

Let (, ) be W-invariant bilinear form on h*. We use the common notation

\[ \gamma^\vee = \frac{2\gamma}{(\gamma, \gamma)}, \quad \gamma^\vee \in h^* \]

for coroots. When identifying h and h* by means of the form (, ) we denote them by h_γ, so that \( \mu(h_\gamma) = (\mu, \gamma^\vee) \). This notation is in agreement with (2.9). Let \( \mu \in h^* \) be a weight (an integer weight), that is \( \mu(h_i) \in \mathbb{Z} \) for any simple root \( \alpha_i \). Set

\[ t^\mu = \prod_{\gamma \in \Delta^+} t^\mu_{\gamma} \gamma^\vee \]

(2.27)

**Lemma 2.1** The product (2.27) does not depend on the choice of the reduced decomposition of w₀.

Thus \( t^\mu \) is a well defined rational function on \( N_+ \).

**Proof.** This is a direct consequence of [BZ, Theorem 4.3], which states that the matrix element \( \Delta_k(g) = (v_k^+, g v_k^-) \) (generalized minor), restricted to \( N_+ \), admits a presentation

\[ \Delta_k(X(t)) = t^\omega_k \]
where \((\omega_k, \alpha_i) = \delta_{ik}\) for each simple root \(\alpha_i\). Here \(v_k^+\) and \(v_k^-\) are highest and lowest weight vectors of the fundamental representation \(V_{\omega_k}\). Another way to see that is to notice that the transition maps (2.24), (2.25) and (2.26) leave invariant the following monomials:

\[
\begin{align*}
t^\alpha &= t_\alpha t_{\alpha+\beta} \\
t^\alpha &= t_\alpha t_{\alpha+\beta}^2 t_{\alpha+2\beta} \\
t^\alpha &= t_\alpha t_{\alpha+\beta}^3 t_{\alpha+2\beta}^2 t_{\alpha+3\beta}
\end{align*}
\]

\[
\begin{align*}
t^\beta &= t_\beta t_{\alpha+\beta} \\
t^\beta &= t_\beta t_{\alpha+2\beta} t_{\alpha+\beta} \\
t^\beta &= t_\beta t_{\alpha+3\beta}^2 t_{\alpha+2\beta} t_{\alpha+3\beta} t_{\alpha+\beta}
\end{align*}
\]

which ensure the invariance of the product (2.27) under all transition maps.

**Lemma 2.2** (see [GLO, Proposition 2.1]) The invariant measure \(dn\) on \(N_+\) is

\[
dn = t^\rho \prod_{\gamma \in \Delta_+} \frac{dt_\gamma}{t_\gamma}
\]

**Proof [GLO].** First one checks by direct calculation that the measure

\[
\frac{dt}{t} = \prod_{\gamma \in \Delta_+} \frac{dt_\gamma}{t_\gamma}
\]

is invariant with respect to transition maps (2.24), (2.25) and (2.26). Thus the measure \(t^\rho \frac{dt}{t}\) is invariant as well. Second we check that this measure is invariant with respect to multiplication of the nilpotent matrix \(X(t)\) by group element \(\exp(se_\alpha)\) from the right, where \(\alpha\) is arbitrary simple root.

\[
X(t) \to X(t) \cdot \exp(se_\alpha)
\]

To this end we choose a normal ordering which ends by the simple root \(\theta(\alpha)\) (such ordering surely exists, see [Zh2]). In these Lusztig coordinates the map (2.29) becomes a translation \(t_{\theta(\alpha)} \to t_{\theta(\alpha)} + s\), the measure in the right hand side of (2.28) factorises to the product \(\omega \land dt_{\theta(\alpha)}\), where \(\omega\) does not depend on \(t_{\theta(\alpha)}\) and thus is invariant under that translation.

### 2.3 Structure of Whittaker vectors

To describe Whittaker vectors, we need some more invariants of transition maps (2.24)–(2.26).

**Proposition 2.1**

(i) The sum \(t_\alpha + t_{\alpha+\beta} + t_\beta\) is invariant with respect to transition map (2.24);

(ii) The sums \(t_\alpha + t_{\alpha+2\beta}\) and \(t_{\alpha+\beta} + t_\beta\) are invariant with respect to transition map (2.25);

(iii) The sums \(t_\alpha + t_{2\alpha+3\beta} + t_{\alpha+3\beta}\) and \(t_{\alpha+\beta} + t_{\alpha+2\beta} + t_\beta\) are invariant with respect to transition map (2.26).

**Proof.** Direct Maple check.

\[\square\]

**Corollary 2.1** The sum \(\sum_{\gamma \in \Delta_+} t_\gamma\) does not depend on the choice of coordinates \(t_\gamma\).
Moreover, the sums of coordinates over the roots of the same length are invariant as well due to statements (ii) and (iii) of Proposition 2.1.

Since Whittaker vectors are functions on $G_0$ from the spaces $V_\mu$ and $V_\nu$, they are completely determined by their restrictions to the subgroup $N_+$. Denote respectively the restriction $v^R_\mu(X(t))$ of the functions $v^R_\mu(g)$ to $N_+$ by $\omega^R_\mu(t)$ and the restriction $v^L_\nu(X(t))$ of the functions $v^L_\nu(g)$ to $N_+$ by $\omega^L_\nu(t)$.

**Proposition 2.2** The right Whittaker vector $v^R_\mu$ is given by the function $\omega^R_\mu(t)$ on $N_+$

$$\omega^R_\mu(t) = \exp \left( - \sum_{\gamma \in \Delta_+} t_\gamma \right)$$

**Proof.** Choose a simple root $\alpha$. The group element $g \exp(se_\alpha)$ has the same Gauss components from $N_-$ and $H$. Thus it is enough to prove that for any simple root $\alpha$

$$v^R_\mu(X(t) \exp(se_\alpha)) = e^{-s}v^R_\mu(X(t))$$

Again we choose a normal ordering which ends by $\theta(\alpha)$. Then $X(t) \exp(se_\alpha)$ has the same coordinates as $X(t)$ except $t_\theta(\alpha)$ which shifts to $t_\theta(\alpha) + s$. Then the sum $\sum_{\gamma \in \Delta_+} t_\gamma$ changes to $s + \sum_{\gamma \in \Delta_+} t_\gamma$ and the function $\omega^R_\mu(t)$ transmits to $e^{-s}\omega^R_\mu(t)$.

Note also that for any real number $\epsilon$ the function $\omega^R_\mu(\epsilon t)$ gives rise to a Whittaker vector with a renormalized character $\zeta^*_+$, see (2.10). In particular, the function $\bar{v}^R_\mu(g)$ on $g \in G_0$, where $g = YTX(t)$, $Y \in N_-$, $T \in H$, $X(t) \in N_+$

$$\bar{v}^R_\mu(g) = \chi_\mu(H)\omega^R_\mu(X(-t))$$

is the Whittaker vector from the space $V_\mu$ with respect to the character $\zeta^*_+$, that is

$$\bar{v}^R_\mu(g \exp(se_\alpha)) = e^{s} \cdot \bar{v}^R_\mu(g)$$

for each simple root $\alpha$.

We use in the following the standard lifts $\bar{s}_{\alpha_i}$ of simple reflections $s_{\alpha_i} \in W$ to the group elements which we denote by the same letter,

$$\bar{s}_{\alpha_i} = \exp(e_i) \exp(-f_i) \exp(e_i)$$  \hspace{1cm} (2.30)$$

so that the products $\bar{w} = \bar{s}_{\alpha_i} \cdots \bar{s}_{\alpha_i}$ do not depend on the choice of reduced decomposition of any element $w \in W$. Then this lift to $G$ of the longest element group element $w_0$ has the property

$$\bar{w}_0e_\alpha\bar{w}_0^{-1} = \text{Ad}_{\bar{w}_0}(e_\alpha) = -f_{-w_0(\alpha)}, \quad \bar{w}_0f_\alpha\bar{w}_0^{-1} = \text{Ad}_{\bar{w}_0}(f_\alpha) = -e_{-w_0(\alpha)}$$  \hspace{1cm} (2.31)$$

for each simple root $\alpha$. This fact is a direct consequence of Tits’s result [T, Proposition 2.1 (3)] which states that if for a simple root $\alpha_i$ and $w \in W$ there exists a simple root $\alpha_j$ such that $ws_{\alpha_i}w^{-1} = s_{\alpha_j}$, then the same is true for their lifts. See e.g. [BB, Lemme 4.9],
Proposition 2.3 The function

\[ v^L_\nu(g) = \bar{v}^R_\nu(g \bar{w}_0) \]

is the left Whittaker vector in the space \( V_\nu \) and character \( \zeta_- \).

**Proof.** By the construction, the function \( u(g) = \bar{v}^R_\nu(g \bar{w}_0) \) belongs to the space \( V_\nu \). Choose a simple root \( \alpha_i \) and consider the function \( u(g \exp(\varepsilon f)) \). We have

\[ u(g \exp(s f_i)) = \bar{v}^R_\nu(g \exp(s f_i) \bar{w}_0) = \bar{v}^R_\nu(g \bar{w}_0 \exp(-se_i)) = \exp(-s) \cdot \bar{v}^R_\nu(g \bar{w}_0) = \exp(-s) \cdot u(g). \]

\[ \square \]

The left Whittaker vector, as a function on \( G_0 \), is completely determined by its restriction \( \omega^L_\nu \) to \( N_+ \). One of the goals of this work is a proper description of this restriction, applicable for the study of Mellin transform. The proposition 2.3 describes this restriction as follows. We start with the group element \( X(-t) \bar{w}_0 \). Take its Gauss decomposition

\[ X(-t) \bar{w}_0 = Y \cdot T \cdot \bar{X}, \quad Y \in N_-, \quad T \in H, \quad \bar{X} \in N_+. \]

Let \( p_\gamma(t) \) be Lusztig coordinates of the element \( \bar{X} \in N_+ \). Then the restriction of the left Whittaker vector to \( N_+ \) is given by the function

\[ \omega^L_\nu(t) = \chi_\nu(T) \cdot \exp \left( - \sum_{\gamma \in \Delta^+} p_\gamma(t) \right) \]

Denote by \( g_\pm \) and \( g_0 \) the parts of Gauss decomposition of the group element \( g \),

\[ g = g_- g_0 g_+, \quad g_\pm \in N_\pm, \quad g_0 \in H, \]

and by \( g_0^+ \) the product \( g_0 g_+ \). We see that the problem of writing precise expressions for the left Whittaker vector leads to the study of birational isomorphism \( \tau \) of the manifold \( N_+ \) and the map \( \sigma : N_+ \to H \) given by

\[ \tau(X(t)) = (X(-t) \bar{w}_0)_+, \quad \sigma(X(t)) = (X(-t) \bar{w}_0)_0 \]

Another possibility, is to use for the construction of left Whittaker vector slightly different maps \( \tilde{\tau} : N_+ \to N_+ \) and \( \tilde{\sigma} : N_+ \to H \) given by

\[ \tilde{\tau}(X(t)) = (X^{-1}(t) \bar{w}_0)_+, \quad \tilde{\sigma}(X(t)) = (X^{-1}(t) \bar{w}_0)_0. \]

Below we present explicit description of the maps \( \tau \) and \( \sigma \) in Lusztig coordinates and use this description for the derivation of Mellin transforms of Whittaker functions.

The maps \( \tilde{\tau} \) and \( \tau \) are closely related to the birational transform \( \eta_{\bar{w}_0} \) by Berenstein and Zelevinsky. The latter plays the crucial role in their derivation of the factorized expression of Lusztig coordinates via generalized minors [BZ, Theorem 1.4]. The map \( \eta_{\bar{w}_0} \) is

\[ \eta_{\bar{w}_0}(X) = (\bar{w}_0 X^T)_+ \quad (2.32) \]
where $T$ is an anti-automorphism of $G$, trivial on $H$, and satisfying the relations
\[
(\exp(te_\alpha))^T = \exp(tf_\alpha), \quad \text{and} \quad (\exp(tf_\alpha))^T = \exp(te_\alpha)
\]
for each simple root $\alpha$. Compute first $\bar{w}_0 X(t) \bar{w}_0^{-1}$. We have
\[
\bar{w}_0 X(t) \bar{w}_0^{-1} = \bar{w}_0 \prod_k \exp\left(t_{\gamma_k} f_{\alpha_k}\right) \bar{w}_0^{-1} = \prod_k \exp\left(-t_{\gamma_k} e^{-w_0(\alpha_k)}\right) \tag{2.33}
\]
This expression coincides with $(\theta(X))^{-1}(t)$. We then have
\[
\eta_{\bar{w}_0}(X) = (\theta(X)\bar{w}_0)_+ = \tilde{\tau}(\theta(X)) \tag{2.34}
\]
so that the maps $\eta_{\bar{w}_0}$ and $\tilde{\tau}$ differ by Dynkin automorphism $\theta$. The relation (2.33) implies also slightly more complicated relation between $\eta_{\bar{w}_0}$ and $\tau$,
\[
\eta_{\bar{w}_0}(X) = \tau(\iota \theta(X)) \tag{2.35}
\]
where $\iota$ is the birational automorphism of the manifold $N_+$, reversing the order of the product in Lusztig presentation (2.22) of the group element of $N_+$. Note that the map $\iota$ defined first for a given normal ordering, respects the transition maps (2.24)–(2.26) and thus does not depend on a choice of Lusztig coordinates.

Having in mind relations (2.34) and (2.35) we further use the name Berenstein–Zelevinsky (BZ) transform for the birational map $\tau : N_+ \to N_+$ and for the corresponding change of Lusztig coordinates. The map $\sigma$ will be referred as (Cartan) twist.

### 2.4 Using Plancherel formula

The convolution property for the Mellin transform admits the following reading. Let $F_1(s)$ and $F_2(s)$ be Mellin transforms of integrable functions $f_1(t)$ and $f_2(t)$,
\[
F_i(s) = \int_0^\infty f_i(t) t^s \frac{dt}{t}.
\]
Let $S_1$ and $S_2$ be the strips of analyticity of $F_1(s)$ and $F_2(s)$,
\[
S_i : \alpha_i < \text{Re } s < b_i.
\]
Assume that the intersection of strips $-S_1$ and $S_2$ is nonzero and both functions $F_1(s)$ and $F_2(s)$ rapidly decrease when $s$ goes to $\pm i\infty$ along the contour $C = \{\text{Re } s = c\} \subset -S_1 \cap S_2$. Then
\[
\int_0^\infty f_1(t) f_2(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_C F_1(-s) F_2(s) ds \tag{2.36}
\]
Here is a check:
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(-s) F_2(s) ds = \frac{1}{2\pi i} \int_C F_1(-s) ds \int_0^\infty f_2(t) t^s \frac{dt}{t} = \\
\int_0^\infty f_2(t) \frac{dt}{t} \int_{c-i\infty}^{c+i\infty} t^s F_1(-s) ds = \frac{1}{2\pi i} \int_0^\infty f_2(t) \frac{dt}{t} \int_{-c-i\infty}^{-c+i\infty} t^{-s} F_1(s) ds = \int_0^\infty f_1(t) f_2(t) \frac{dt}{t}.
\]
The first equality is due to the inclusion $C \subset S_2$, the second uses the decreasing at
infinities, the third is the change of variables $s \to -s$, the fourth is the Mellin inversion
theorem due to the inclusion $-C \subset S_1$. Since Mellin transform of the function $\tilde{F}_1(t)$ equals
to $\tilde{F}_1(\tilde{s})$, we can rewrite the convolution property (2.36) in a form of Plancherel formula
\[
\int_0^\infty \tilde{f}_1(t)f_2(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_C F_1(-\tilde{s})F_2(s) ds
\] (2.37)
under the same assumptions on the functions $F_1(s)$ and $F_2(s)$.

Denote by $\hat{\omega}_\mu^R(\gamma)$ and $\hat{\omega}_\mu^L(\gamma)$ the Mellin transforms of $t^\rho \omega_\mu^L(t)$ and $\omega_\mu^R(t)$ as of functions
on $N_+$,
\[
\hat{\omega}_\mu^R(\gamma) = \int_{t > 0} \omega_\mu^R(t) t^\gamma \frac{dt}{t}, \quad \hat{\omega}_\mu^L(\gamma) = \int_{t > 0} t^\rho \omega_\mu^L(t) t^\gamma \frac{dt}{t}
\]
Here $t^\gamma$ means the product
\[
t^\gamma = \prod_{\gamma \in \Delta^+} (t_\gamma)^{\gamma},
\]
where $(t_\gamma)^{\gamma}$ is the power of the variable $t_\gamma$, and $t^\rho$ is defined by (2.27). We now use the
Plancherel formula (2.37) to rewrite Whittaker wave functions (2.17) and (2.18) in terms
of the Plancherel formula (2.37) to rewrite Whittaker wave functions (2.17) and (2.18) in terms
of $\hat{\omega}_\mu^L(\gamma)$ and $\hat{\omega}_\mu^R(\gamma)$. First we note that by definition (2.14), (2.15) of the action of the
Lie algebra $\mathfrak{g}$ in the space $V_\mu$, the restriction of the function \(\exp(-x)v_\mu^R\) for any \(x \in \mathfrak{h}\) is
given by the relation
\[
\exp(-x)v_\mu^R|_{N_+} = \omega_\mu^R(t) \exp(-x)|_{N_+} = \exp(-x) \left( \exp(x)\omega_\mu^R(t) \exp(-x) \right) = e^{-(x,\mu)} e^{ad_x} (\omega_\mu^R(t))
\]
Mellin transform turns first order differential operators $ad_x$ into operators of multiplications
on functions $\tilde{H}(x, \gamma)$, see (3.17), (4.10), (5.10), (6.4). Then by (2.37) and Lemma
2.2 we have
\[
\tilde{\Psi}_{\mu,\nu}(x) = e^{-(\rho+\mu, x)} \int_{N_+} \omega_\nu^L(t) e^{ad_x} (\omega_\mu^R(t)) \frac{dt}{t} = e^{-(\rho+\mu, x)} \left( \int_{C} \frac{1}{(2\pi i)^d} \frac{\tilde{\omega}_\nu^L(-\gamma)}{\gamma} \exp \tilde{H}(x, \gamma) \hat{\omega}_\mu^R(\gamma) d\gamma \right) (2.38)
\]
and
\[
\Psi_\lambda(x) = e^{-i(\lambda, x)} \left( \int_{C} \frac{1}{(2\pi i)^d} \frac{\tilde{\omega}_\mu^L(-\gamma)}{\gamma} \exp \tilde{H}(x, \gamma) \hat{\omega}_\mu^R(\gamma) d\gamma \right) (2.39)
\]
Here $d$ is the dimension of $N_+$, the contour $C$ is a deformation of imaginary plane $\text{Re} \gamma = 0$
to the intersection of strips of analyticity of $\hat{\omega}_\mu^R(\gamma)$ and $\tilde{\omega}_\mu^L(-\gamma)$ under the assumption
of nonemptiness of their intersections and their vanishing on imaginary infinities.

3 \textbf{GL}(n, \mathbb{R})

3.1 BZ transform

Let $v_i$ be a fixed basis of $\mathbb{R}^n$ and $e_{i,j} \in \text{Mat}_{n \times n}$ be matrix units, $e_{i,j}(v_k) = \delta_{j,k} v_i$. Denote by
$\varepsilon_i \in \mathfrak{h}^*$ the basic elements of $\mathfrak{h}$ defined by the condition $\varepsilon_i(e_{j,j}) = \delta_{i,j}$ and by $s_i$ the group
elements, related to simple reflection \(s_{\varepsilon_i, \varepsilon_{i+1}}\), that is \(s_i = (e_i, i+1 - e_{i+1}, i) + \sum_{j \neq i, i+1} e_{j,j}\) as element of \(\text{Mat}_{n \times n}\). We use the following decomposition of the Weyl group element \(w_0 = w_0^{(n)}\),
\[
w_0^{(n)} = s_{n-1}(s_{n-2}s_{n-1}) \cdots (s_1s_2 \cdots s_{n-1}).
\]
Under the so that the corresponding ordering (2.20) of positive roots is
\[
\varepsilon_{n-1,n}, \varepsilon_{n-2,n}, \varepsilon_{n-2,n-1}, \varepsilon_{n-3,n}, \varepsilon_{n-3,n-1}, \varepsilon_{n-3,n-2}, \ldots, \varepsilon_{1,n}, \varepsilon_{2,n} \cdots \varepsilon_{1,2},
\]
where \(\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j\). Let \(X^{(n)}(t)\) be a group element of the subgroup \(N_+\), given by (2.22),
\[
X^{(n)}(t) = \exp(t_{n-1,n}e_{n-1,n}) \exp(t_{n-2,n}e_{n-2,n-1}) \exp(t_{n-2,n-1}e_{n-1,n}) \cdots \\
\exp(t_{1,n}e_{1,2}) \exp(t_{1,n-1}e_{2,3}) \cdots \exp(t_{1,2}e_{n-1,n}).
\]
Here we regard \(e_{i,i+1}\) as Chevalley generators of Lie algebra \(\mathfrak{g}_l_n\). In fundamental representation they are given by the matrices
\[
X^{(n)}(t) = (\text{Id} + t_{n-1,n}e_{n-1,n}) (\text{Id} + t_{n-2,n}e_{n-2,n}) (\text{Id} + t_{n-2,n-1}e_{n-1,n}) \cdots \\
(\text{Id} + t_{1,n}e_{1,2}) (\text{Id} + t_{1,n-1}e_{2,3}) \cdots (\text{Id} + t_{1,2}e_{n-1,n}).
\]
where \(\text{Id} \in \text{Mat}_{n \times n}\) is the identity matrix. We are going to solve the following equation
\[
TX^{(n)}(p) = \left(X^{(n)}(-t)w_0^{(n)}\right)_{0+}
\]
where
\[
X^{(n)}(p) = (\text{Id} + p_{n-1,n}e_{n-1,n}) (\text{Id} + p_{n-2,n}e_{n-2,n}) (\text{Id} + p_{n-2,n-1}e_{n-1,n}) \cdots \\
(\text{Id} + p_{1,n}e_{1,2}) (\text{Id} + p_{1,n-1}e_{2,3}) \cdots (\text{Id} + p_{1,2}e_{n-1,n}).
\]
and \(T\) is a diagonal matrix. In other word, we have to find out the unknown variables \(p_{i,j}\) as function of \(t_{i,j}\) and the diagonal matrix \(T\) depending on \(p_{i,j}\) (and thus on \(t_{i,j}\)) such that relation (3.2) holds. Denote further in this section
\[
\hat{k} = k + 1 - i.
\]

**Theorem 3.1** a) BZ map \(\tau\), see (2.32) and (3.1), has the form
\[
p_{i,j} = \frac{1}{t_{i,i+j} t_{i-1,i+j} \cdots t_{i-1,i+j-1}} t_{i,i+j-1} t_{i-2,i+j-1} \cdots t_{1,i+j-1}
\]

b) The twisting matrix \(T \in H\) reads
\[
T = t_{1,n} \cdots t_{1,2} \cdot e_{1,1} + \frac{t_{2,n} \cdots t_{2,3}}{t_{1,2}} \cdot e_{2,2} + \frac{t_{3,n} \cdots t_{3,4}}{t_{1,3}t_{2,3}} \cdot e_{3,3} + \ldots \\
+ \frac{t_{n,n} \cdots t_{n,n-1}}{t_{1,n} \cdots t_{n-1,n}} \cdot e_{n,n}.
\]

c) BZ transform preserves the measure \(\text{d}t\),
\[
\prod_{i < j} \frac{\text{d}t_{i,j}}{t_{i,j}} = \prod_{i < j} \frac{\text{d}p_{i,j}}{p_{i,j}}.
\]
Remark 1. BZ transform $\tau$ is involutive, that is the inverse relation

$$t_{i,j} = \frac{1}{p_{i,i+j}} \frac{p_{i-1,i+j-1}}{p_{i-1,i+j}} \frac{p_{i-2,i+j-1}}{p_{i-2,i+j}} \cdots \frac{p_{1,i+j-1}}{p_{1,i+j}}$$

has the same form as (3.3). Indeed, the map $\tau: X^{(n)}(t) \to X^{(n)}(p)$ is defined by the relation $X^{(n)}(p) = \left( X^{(n)}(-t)\bar{w}_0^{(n)} \right)_0$. Since $\bar{w}_0^{(n)} = (-1)^{n+1} \text{Id}$, this is equivalent to $X^{(n)}(-t) = \left( X^{(n)}(p)w_0 \right)_0$. This means that we can interchange in (3.3) $t_{i,j}$ to $-p_{i,j}$ and $p_{i,j}$ to $-t_{i,j}$. After cancelation of signs we get (3.5). \(\square\)

Remark 2. The Cartan twist $T$ can be as well written in shorthand notation as $T(t) = T(p^{-1})$, that is

$$T = \frac{1}{p_{1,n} \cdots p_{1,2}} \cdot e_{1,1} + \frac{p_{1,2}}{p_{2,n} \cdots p_{2,3}} \cdot e_{2,2} + \frac{p_{1,3}p_{2,3}}{p_{3,n} \cdots p_{3,4}} \cdot e_{3,3} + \cdots$$

Indeed, $T$ is specified by the condition

$$TX^{(n)}(p) = \left( X^{(n)}(-t)\bar{w}_0^{(n)} \right)_0.$$ Again, since $\left( \bar{w}_0^{(n)} \right)^2 = (-1)^{n+1} \text{Id}$, this is equivalent to

$$T^{-1}X^{(n)}(-t) = (-1)^{n+1} \left( X^{(n)}(p)\bar{w}_0^{(n)} \right)_0,$$

which implies the equality $T^{-1}(-p) = (-1)^{n+1}T(t)$. The cancelation of signs gives (3.6). \(\square\)

Corollary 3.1 The restriction of left Whittaker vector to $N_+$ is given by the function

$$\omega^L_\nu(t) = t^\nu \cdot \exp \left( -\sum_{i<j} p_{i,j} \right) = p^{-\nu} \cdot \exp \left( -\sum_{i<j} p_{i,j} \right)$$

where $p_{i,j}$ are given by (3.3)

The formula (3.7) is well known, see [GLO] an references therein. However, we got it in a form convenient for the study of Mellin transform. Further on we find analogous form for other classical groups.

The proof of Theorem 3.1 is based on inductive calculation of BZ transform $\tau$ and related map $\sigma$. Namely, the group element $X^{(n)}(-t)$ admits a factorization

$$X^{(n)}(-t) = \tilde{X}^{(n-1)}(-t) \cdot A^{(n)}(-t)$$

where

$$A^{(n)}(-t) = (\text{Id} - t_{1,n}e_{1,2})(\text{Id} - t_{1,n-1}e_{2,3}) \cdots (\text{Id} - t_{1,2}e_{n-1,n}).$$
and
\[
\bar{X}^{(n-1)}(-t) = (\text{Id} - t_{n-1,n}e_{n-1,n})(\text{Id} - t_{n-2,n}e_{n-2,n})(\text{Id} - t_{n-3,n}e_{n-3,n}) \cdots (\text{Id} - t_{2,n}e_{2,n})(\text{Id} - t_{1,n}e_{1,n})(\text{Id} - t_{3,n}e_{3,n} \cdots (\text{Id} - t_{2,3}e_{2,3} \cdots (\text{Id} - t_{2,3}e_{2,2} \cdots (\text{Id} - t_{2,2}e_{2,1})\cdots (\text{Id} - t_{2,1}e_{2,1}) \cdots (\text{Id} - t_{1,1}e_{1,1}) \cdot (\text{Id} - t_{1,1}e_{1,1}) \cdots (\text{Id} - t_{1,1}e_{1,1}).
\]

represents a group element of the unipotent subgroup of embedded $\text{GL}(n - 1)$,

\[
\bar{X}^{(n-1)}(-t) = \begin{pmatrix}
1 & 0 & \ast & \ast & \ast \\
0 & 1 & \ast & \ast & \ast \\
& 0 & \ddots & \ast & \ast \\
& & 0 & 1 & 1
\end{pmatrix}
\]

The matrix $X^{(n)}(p)$ has the same structure,

\[
X^{(n)}(p) = \bar{X}^{(n-1)}(p) \cdot A^{(n)}(p).
\]

In the induction step we first express Lusztig coordinates of $A^{(n)}(p)$ via that of $A^{(n)}(-t)$; compute the input of $A^{(n)}(-t)$ into Cartan twist $T$ and reduce the rest of calculations to the computation of BZ transform and twist of $\text{GL}(n - 1)$ matrix $X^{(n-1)}(-t)$, which is given by a certain gauge transform of $\bar{X}^{(n-1)}(-t)$.

Denote by $\Lambda^{(n)}$ the diagonal matrix with diagonal entries

\[
\Lambda_{1,1}^{(n)} = t_{1,n}t_{1,n-1} \cdots t_{1,2}, \quad \Lambda_{j,j}^{(n)} = \frac{1}{t_{1,j+1}} = \frac{1}{t_{1,n-j+2}}, \quad j > 1
\]

and define the variables $\tilde{t}_{i,j}$, $1 < i < j \leq n$ by the relation

\[
\tilde{t}_{i,j} = t_{i,j} \cdot \frac{t_{1,j-i+2}}{t_{1,j-i+1}}
\]

**Proposition 3.1**

a) The parameters $p_{1,j}$, $j = 2, \ldots, n$ of $A^{(n)}(p)$ are equal to

\[
p_{1,j} = \frac{1}{t_{1,n-j+2}}
\]

b) Cartan twist $T$ and the matrix $\bar{X}^{(n-1)}(p)$ satisfy the relation

\[
T \bar{X}^{(n-1)}(p) = \Lambda^{(n)} \cdot \left(\bar{X}^{(n-1)}(-\tilde{t}) \bar{w}_0^{(n-1)}\right)_{0+}
\]

**Proof** of part a) of Proposition 3.1. Matrix element $A_{i,j}$ of upper triangular unipotent matrix $A^{(n)}(-t)$ is

\[
A^{(n)}_{i,j} = A^{(n)}_{i,j}(-t) = (-1)^{j-i}t_{1,j+1}t_{1,j+2} \cdots t_{1,i-1}, t_{j,j} \quad i < j
\]

Due to the structure (3.10) of the matrix $\bar{X}^{(n-1)}(-t)$ the first row of $X^{(n)}(-t)$ coincides with that of $A^{(n)}(-t)$ and is equal to

\[
((-1)^{n-1}A^{(n)}_{1,n}, (-1)^{n-2}A^{(n)}_{2,n}, \ldots, A^{(n)}_{1,1}) = (t_{1,n}t_{1,n-1} \cdots t_{1,2}, \ldots, t_{1,3}t_{1,2}, t_{1,2}, 1).
\]

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Since the right multiplication by \( u_0^{(n)} = e_{1,n} - e_{2,n} + \ldots + (-1)^{n-1} e_{n,1} \) leaves the first row of the matrix \( X^{(n)}(t) \) invariant, permuting its entries, the parameters of \( A^{(n)}(p) \) and matrix element \( T_{1,1} \) can be completely determined from the parameters of \( A^{(n)}(-t) \). Namely, \( T_{1,1} = t_{1,n} t_{2,n} \ldots t_{n-1,n} \) and we have two presentations for the first row of the matrix \( A^{(n)}(p) \):

\[
(1, p_{1,n}, p_{1,n} p_{1,n-1}, \ldots, p_{1,n} p_{2,n} \ldots p_{1,2}) = \left(1, \frac{1}{t_{1,2}}, \frac{1}{t_{1,2} t_{1,3}}, \ldots, \frac{1}{t_{1,2} t_{1,3} \ldots t_{1,n}}\right)
\]

so that the parameters \( p_{i,n} \) of the matrix \( A^{(n)}(p) \) are equal to \( p_{1,j} = \frac{1}{t_{1,n+2-j}} \).

Let \( \bar{w}_0^{(n-1)} \) be the element of embedded group \( GL(n-1) \) representing the corresponding longest element of the Weyl group, \( \bar{w}_0^{(n-1)} = e_{1,1} + e_{2,n} - e_{3,n-1} + \ldots + (-1)^n e_{n,2} \). Let \( U \) be the matrix

\[
U = \left(\bar{w}_0^{(n-1)}\right)^{-1} A^{(n)}(-t) \bar{w}_0^{(n)} (A^{(n)}(p))^{-1}
\]

The following lemma is the crucial technical step in the calculation of the transformation (3.1). It states that the matrix \( U \) has the block structure

\[
U = \begin{pmatrix} \Lambda' & 0 \\ 0 & \Lambda' \end{pmatrix}
\]

where \( \Lambda' \) is \( (n-1) \times (n-1) \) diagonal matrix. More precisely

**Lemma 3.1** Matrix elements \( U_{i,j} \) equal zero for \( i \neq j \), \( j > 1 \). Matrix element \( U_{1,1} \) equals \( (p_{1,2} p_{1,3} \ldots p_{1,n})^{-1} = t_{1,2} t_{1,3} \ldots t_{1,n} \). The element \( U_{i,i} \) equals \( p_{1,i+1} = t_{1,i}^{-1} \) for \( i > 1 \).

The proof of Lemma 3.1 is given in Appendix A.

**Proof** of part b) of Proposition 3.1. Denote by \( \Lambda'^{(n)} = \bar{w}_0^{(n-1)} \Lambda^{(n)} \left(\bar{w}_0^{(n-1)}\right)^{-1} \) the diagonal matrices with nonzero entries

\[
\Lambda'^{(n)}_{1,1} = t_{1,2} t_{1,3} \ldots t_{1,n}, \quad \Lambda'^{(n)}_{i,i} = \frac{1}{t_{1,i}}, \quad i > 1
\]

Since \( A^{(n)}(p) \in N \), the relation (3.1) can be equivalently rewritten as

\[
T \tilde{X}^{(n-1)}(p) = \left(\tilde{X}^{(n-1)}(-t) A^{(n)}(-t) \bar{w}_0^{(n)} (A^{(n)}(p))^{-1}\right)_{0+}
\]

or

\[
T \tilde{X}^{(n-1)}(p) = \left(\tilde{X}^{(n-1)}(-t) \bar{w}_0^{(n)} U\right)_{0+} = \left(\tilde{X}^{(n-1)}(-t) \bar{w}_0^{(n)} \Lambda'^{(n)} V\right)_{0+} =
\]

\[
\Lambda^{(n)} \left(\tilde{X}^{(n-1)}(-t) \bar{w}_0^{(n)} V\right)_{0+},
\]

where

\[
V = \left(\Lambda'^{(n)}\right)^{-1} U \quad \text{and} \quad \tilde{X}^{(n-1)}(-t) = (\Lambda^{(n)})^{-1} \tilde{X}^{(n-1)}(-t) \Lambda^{(n)}.
\]

(3.14)
The matrices $Y = \bar{X}^{(n-1)}(-t)w_0^{(n-1)}$ and $V$ have the following block structure due to Lemma 3.1:

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ Z & \text{Id}_{n-1} \end{pmatrix}$$

where $Y$ and $\text{Id}_{n-1}$ are $(n - 1) \times (n - 1)$ matrices and $Z$ is $1 \times (n - 1)$ matrix. Then the product $YV$ also has a block structure and can be rewritten as

$$YV = \begin{pmatrix} 1 & 0 \\ YZ & \text{Id}_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}$$

The first factor of the latter product is unipotent lower triangular matrix so that the upper triangular part of $YV$ coincides with upper triangular part of $Y$. Thus we proved the equality

$$T\bar{X}^{(n-1)}(p) = \Lambda^{(n)}(\bar{X}^{(n-1)}(-t)w_0^{(n-1)})_{0+}$$

where the matrix $\bar{X}^{(n-1)}(-t)$ is given by the relation (3.14).

The last step is the computation of the conjugation in (3.14). To perform it we note that the exponent in the products (3.4) containing the variable $t_{i,j}$ is

$$\exp(-t_{i,j}e_{n-j+i,n-j+i+1}) = (1 - t_{i,j}e_{n-j+i,n-j+i+1})$$

which means that the conjugation (3.14) results to multiplication of the coefficient at $e_{n-j+i,n-j+i+1}$ by $\Lambda_{n-j+i+1,n-j+i+1}^{(n)}/\Lambda_{n-j+i,n-j+i}^{(n)}$ which is due to (3.8), the rescaling

$$t_{i,j} \rightarrow t_{i,j} \frac{t_{1,j-i+2}}{t_{1,j-i+1}}.$$

**Proof** of Theorem 3.1 follows from the inductive application of Proposition 3.1. At the first step we find the variables $p_{i,j}$, see (3.10) and the input $\Lambda^{(n)}$ of the first step to the diagonal matrix $T$. In particular, we find that $T_{1,1} = t_{1,2} \cdots t_{1,n}$. Then we pass to the second step, where we deal with $(n - 1)$ square matrix but with rescaled by (3.9) matrix elements $t_{i,j}$. Here after the corresponding shift of indices we find out the next portion of variables,

$$p_{2,j} = \frac{1}{t_{2,n-j+3}} = \frac{1}{t_{2,n-j+3}t_{1,n-j+3}} \frac{t_{1,n-j+2}}{t_{1,n-j+3}}$$

due to (3.9) and (3.10). The Cartan matrix $T$ gains the new income, equal to $\Lambda^{(n-1)}$ with matrix entries

$$\Lambda_{1,1}^{(n-1)} = 1, \quad \Lambda_{2,2}^{(n-1)} = \bar{t}_{2,3} \cdots \bar{t}_{2,n} = t_{2,3} \cdots t_{2,n} \cdot \frac{t_{1,n}}{t_{1,2}},$$

$$\Lambda_{j,j}^{(n-1)} = \frac{1}{\bar{t}_{2,n-j+3}} = \frac{1}{t_{2,n-j+3}t_{1,n-j+3}} \frac{t_{1,n-j+2}}{t_{1,n-j+3}}, \quad j > 2$$

and the renormalization of the variables $\bar{t}_{i,j}$ for new $(n - 2) \times (n - 2)$ task,

$$\bar{t}_{i,j} \rightarrow \bar{t}_{i,j} = \frac{\bar{t}_{2,j-i+3}}{\bar{t}_{2,j-i+2}}.$$
Following this procedure we get both a) and b) statements of the Theorem. The part c) may be observed as follows. Denote by \( \Omega_t \) and \( \Omega_p \) the skew forms \( \Omega_t = \wedge_{i<j} dt_{i,j} \) and \( \Omega_p = \wedge_{i<j} dp_{i,j} \). They admit the factorizations \( \Omega_t' = \Omega_t \wedge \Omega_{t''} \) and \( \Omega_p' = \Omega_p \wedge \Omega_{p''} \), where

\[
\Omega_t' = \wedge_{1<j} \frac{dt_{1,j}}{t_{1,j}}, \quad \Omega_t'' = \wedge_{1<i<j} \frac{dt_{i,j}}{t_{i,j}},
\]

and analogously for \( \Omega_p' \) and \( \Omega_p'' \). Looking at the induction step and relation (3.10) we see that the skew forms \( \Omega_t' \) and \( \Omega_p' \) coincide up to sign. But then the relations (3.9) say that in the wedge product

\[
\Omega' \wedge_{1<i<j} \frac{dt_{i,j}}{t_{i,j}}
\]

the renormalization fractions \( \frac{t_{1,i-i+2}}{t_{1,i-i+1}} \) should be regarded as constants which so do not contribute to the wedge product, so that

\[
\Omega'' = \wedge_{1<i<j} \frac{dt_{i,j}}{t_{i,j}}
\]

and we may further use the same equalities for the next induction steps. \( \square \)

### 3.2 Whittaker function

Using the relation (3.15) we immediately describe the action of Cartan generators on the right Whittaker vector. By definition, \( \exp(\sum_k x_k e_{k,k}) \cdot f(g) = f(g \exp(\sum_k x_k e_{k,k})) \) for any function \( f : G \to \mathbb{C} \), so the vector \( \exp(-\sum_k \mu_k x_k) v^R_\mu \) is presented by the function \( \exp(-\sum_k \mu_k x_k) \omega^R_\mu(t') \), where

\[
t_{i,j}' = t_{i,j} \cdot \exp(x_{n-j+i} - x_{n-j+i+1}) \tag{3.16}
\]

Denote by \( \hat{\omega}^L_\mu(\gamma) \) and \( \hat{\omega}^R_\mu(\gamma) \) the Mellin transforms of the functions on \( N_+ t^n \omega^L_\mu(t) \) and \( \omega^R_\mu(t) \),

\[
\hat{\omega}^R_\mu(\gamma) = \int_{t>0} \omega^R_\mu(t) t^n \frac{dt}{t} := \int_{t_{i,j}>0} \omega^R_\mu(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}} \frac{dt_{i,j}}{t_{i,j}},
\]

\[
\hat{\omega}^L_\mu(\gamma) = \int_{t>0} t^n \omega^L_\mu(t) t^n \frac{dt}{t} := \int_{t_{i,j}>0} \omega^L_\mu(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}+j-i} \frac{dt_{i,j}}{t_{i,j}}.
\]

Here \( \rho = -\sum_{k=1}^n k \varepsilon_k \). Due to (3.16) the action of Cartan subgroup on the right Whittaker vector \( v^R_\mu \) transforms the function \( \hat{\omega}^R_\mu(\gamma) \) to the product

\[
\exp \left( \sum_k -x_k e_{k,k} \right) : \hat{\omega}^R_\mu(\gamma) \mapsto \exp(-\sum_k \mu_k x_k) \cdot \exp(\tilde{H}(x, \gamma)) \cdot \hat{\omega}^R_\mu(\gamma)
\]

where

\[
\tilde{H}(x, \gamma) = \sum_{i<j} \gamma_{i,j} (x_{n-j+i+1} - x_{n-j+i}) \tag{3.17}
\]
Due to Proposition 2.2 we have:

\[ \hat{\omega}^R_{\mu}(\gamma) = \prod_{i<j} \Gamma(\gamma_{i,j}). \]

For the calculation of \( \hat{\omega}^L_{\nu}(\gamma) \) we pass in the integral

\[ \int_{t_{i,j}>0} \exp \left( - \sum_{i<j} p_{i,j} \right) \cdot t^{\nu'} \prod_{i<j} t^{\gamma_{i,j}-\nu'_i} \frac{dt}{t}, \]

where

\[ \nu'_i = \nu_i - i \]

from the integration variables \( t_{i,j} \) to \( p_{i,j} \). Substitution of (3.5) gives the relation

\[ \prod_{i<j} t^{\gamma_{i,j}} := \prod_{i<j} t^{\gamma_{i,j}+\nu'_i-\nu'_j} \]

where

\[ \varphi_{i,j} = -\sum_{k>i} \gamma_{k,k+1} + \hat{j} - 1 + \sum_{k\geq i} \gamma_{k,k+1} \]

This implies the relation

\[ \hat{\omega}^L_{\nu}(\gamma) = \prod_{i<j} \Gamma(-\varphi_{i,j} + \nu'_j - \nu'_i), \quad \hat{\omega}^L_{\nu}(\gamma) = \prod_{i<j} \Gamma(\varphi_{i,j} + \nu'_i - \nu'_j) \]

Then by (2.38) and (2.39) we have

\[ \tilde{\Psi}_{\mu,\nu}(x) = \frac{e^{-i(x,\rho+\mu)}}{(2\pi i)^d} \int_C \exp \hat{H}(x, \gamma) \prod_{i<j} \Gamma(\varphi_{i,j} + \nu'_i - \nu'_j) \Gamma(\gamma_{i,j}) \, d\gamma_{i,j}. \]

where now \( \nu' + \bar{\mu} = -\rho \), \( d = n(n - 1)/2 \) and the contour \( C \) is a deformation of imaginary plane \( \text{Re} \gamma_{ij} = 0 \) to the strip of analyticity of the integrand. In particular, for \( \mu = i\lambda - \rho \) we have

\[ \Psi_{\lambda}(x) = \frac{e^{-i(x,\lambda)}}{(2\pi i)^d} \int_C \exp \hat{H}(x, \gamma) \prod_{k<l} \Gamma(\varphi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\gamma_{k,l}) \, d\gamma_{k,l}. \]}

(3.18)

Performing the following change of variables in the integral in (3.18):

\[ \gamma_{i,j} = \gamma_{i,j} + \gamma_{i+1,j+1} + \cdots + \gamma_{n-j,i,n}, \]

we finally arrive to

**Theorem 3.2**

\[ \Psi_{\lambda}(x) = \frac{e^{-i(\lambda,x)}}{(2\pi i)^d} \int_C \exp \sum_{1\leq k\leq n} (\gamma_{1,n+2-k} - \gamma_{1,n+1-k}) x_k \prod_{k<l} \Gamma(\gamma_{kl} - \gamma_{k+1,l+1} + i(\lambda_k - \lambda_{n+k-l+1})) \Gamma(\gamma_{kl} - \gamma_{k+1,l+1}) \, d\gamma_{k,l} \]
Here we assume that $\gamma_{k,l} = 0$ unless $1 \leq k < l \leq n$. The integration cycle $C_n$ is a deformation of the imaginary plain $\text{Re} \gamma_{k,l} = 0$ into nonzero strip $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which can be described by inequalities

$$\text{Re} \gamma_{k,l} > 0, \text{Re} \gamma_{k,l} > \gamma_{k+1,l}$$

for all admissible pairs $(k, l)$ of indices.

4 $\text{SO}(n, n)$

4.1 BZ transform

The split real form of the group $\text{SO}(2n, \mathbb{C})$ is the group $\text{SO}(n, n) \subset \text{SL}(2n, \mathbb{R})$ preserving symmetric form $\sum_{i=1}^{2n} x_i^2$, where as before $i = 2n+1-i$. Gauss decomposition is induced from that of $\text{GL}(2n, \mathbb{R})$. Positive roots are $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$, where $\varepsilon_k \in \mathfrak{h}^*$ are now defined by the condition $\varepsilon_k (e_{j,j} - e_{j,j}) = \delta_{j,k}$. We denote Chevalley generator of the Lie algebra $\mathfrak{g} = \mathfrak{so}(2n)$ by $e_i$ and $f_i$, $i = 1, \ldots, n-2$ and $e_{n-1}^\pm, f_{n-1}^\pm$. Here

$$e_i = e_{i,i+1} - e_{2n-i,2n+1-i}, \quad f_i = e_{i+1,i} - e_{2n+1-i,2n-i}, \quad i = 1, \ldots, n-2,$$

$$e_{n-1}^- = e_{n-1,n} - e_{n+1,n+2}, \quad f_{n-1}^- = e_{n,n-1} - e_{n+2,n+1},$$

$$e_{n-1}^+ = e_{n-1,n+1} - e_{n,n+2}, \quad f_{n-1}^+ = e_{n+1,n} - e_{n+2,n}.$$

Denote by $s_i$, $i = 1, \ldots, n-1$ and $s_{n-1}^\pm$ the corresponding generators of the Weyl group and their lifts to the group $\text{SO}(n, n)$ according to (2.30). We choose the following normal ordering of the system $\Delta_+$ of positive roots:

$$(\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n), (\varepsilon_{n-2} + \varepsilon_{n-1}, \varepsilon_{n-2} - \varepsilon_{n-1}), \ldots,$$

$$(\varepsilon_1 + \varepsilon_2, \varepsilon_1, \varepsilon_3, \ldots, \varepsilon_1 + \varepsilon_n, \varepsilon_1 - \varepsilon_n, \ldots, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_2).$$

It corresponds by (2.20) to the following reduced decomposition of the longest element $w_0^{(n)}$ of the Weyl group $W$:

$$w_0^{(n)} = (s_{n-1}^- s_{n-1}^+) (s_{n-2} s_{n-1}^+) \cdots (s_1 s_{n-2} s_{n-1} \varepsilon^{(n)}_{n-1}) \varepsilon^{(n)}_{n-1} s_{n-2} s_{n-1} \cdots s_2 s_1),$$

where $\varepsilon(k) = (-1)^k$. Note that as element of $\text{SO}(n, n)$,

$$w_0^{(n)} = - \sum_{k=1}^{2n} e_{k,k}, \quad \text{for even } n,$$

$$\bar{w}_0^{(n)} = \sum_{k \neq n,n+1} e_{k,k} + (e_{n,n} + e_{n+1,n+1}), \quad \text{for odd } n.$$  

(4.1)

Denote by $t_{i,j}$ the Lusztig parameter corresponding to the root $\varepsilon_i - \varepsilon_j$, and by $s_{i,j}$ the Lusztig parameter corresponding to the root $\varepsilon_i + \varepsilon_j$. Here $1 \leq i < j \leq n$. Then the group
element $X^{(n)}(t)$ looks as

$$X^{(n)}(t) = \left( \exp(s_{n-1,n}e_{n-1}^\pm) \exp(t_{n-1,n}e_{n-1}^\pm) \right) \cdot \ldots$$

$$\cdot \left( \exp(s_{2,3}e_2) \cdots \exp(s_{2,n}e_{n-2}^{e(n-2)}) \exp(t_{2,n}e_{n-2}^{e(n-1)}) \cdots \exp(t_{2,4}e_3) \exp(t_{2,3}e_2) \right) \cdot$$

$$\cdot \left( \exp(s_{1,2}e_1) \exp(s_{1,3}e_2) \cdots \exp(s_{1,n}e_{n-1}^{e(n-1)}) \exp(t_{1,n}e_{n-1}^{e(n)}) \cdots \exp(t_{1,3}e_2) \exp(t_{1,2}e_1) \right).$$

(4.2)

BZ transform for $SO(n,n)$ is the solution of the equation (3.1), where in $X^{(n)}(p)$ we use the notation $p_{i,j}$ for Lusztig parameter, corresponding to the root $\varepsilon_i - \varepsilon_j$, and $q_{i,j}$ for the Lusztig parameter, corresponding to the root $\varepsilon_i + \varepsilon_j$,

$$X^{(n)}(p) = \left( \exp(q_{n-1,n}e_{n-1}^\pm) \exp(p_{n-1,n}e_{n-1}^\pm) \right) \cdots$$

$$\cdot \left( \exp(q_{1,2}e_1) \exp(q_{1,3}e_2) \cdots \exp(q_{1,n}e_{n-1}^{e(n-1)}) \exp(p_{1,n}e_{n-1}^{e(n)}) \cdots \exp(p_{1,3}e_2) \exp(p_{1,2}e_1) \right).$$

Set in addition $s_{j,j} = t_{j,j} = p_{j,j} = q_{j,j} = u_{j,j} = 1$, and for all $i, j$, $1 \leq i < j \leq n$ put

$$u_{i,j} = \begin{cases} t_{i,j} + s_{i,j}, & j < n, \\
               t_{i,j}, & j = n, \text{ } n - i \text{ odd}, (\text{i.e. } \epsilon(n-i) = -1), \\
               s_{i,j}, & j = n, \text{ } n - i \text{ even}, (\text{i.e. } \epsilon(n-i) = 1), \\
               r_{i,j} = \frac{u_{i,j-1}}{u_{i,j}s_{i,j-1}}, & j < n \end{cases}$$

Theorem 4.1 a) BZ map for $SO(n,n)$ looks as

$$p_{i,j} = r_{i,j} \cdot \prod_{k < l} t_{k,l-1} \prod_{k < l} \frac{s_{k,l}}{t_{k,l}}, \quad q_{i,j} = r_{i,j} \cdot \prod_{k < l} t_{k,l-1} \prod_{k < l} \frac{s_{k,l}}{t_{k,l}}, \quad j < n$$

$$p_{i,n} = r_{i,n} \cdot \prod_{k < l} t_{k,n-1} \prod_{k < l} \left( \frac{t_{k,n}}{s_{k,n}} \right)^{\epsilon(n-i)}, \quad q_{i,n} = r_{i,n} \cdot \prod_{k < l} t_{k,n-1} \prod_{k < l} \left( \frac{s_{k,n}}{t_{k,n}} \right)^{\epsilon(n-i)} \quad (4.3)$$

b) The twisting matrix $T \in H$ is $T = \sum_{k=1}^{n} \left( T_k e_{k,k} + T_k^{-1} e_{k,k} \right)$, where

$$T_k = \prod_{j < k} s_{j,k} t_{j,k} \cdot \prod_{j > k} s_{k,j} t_{k,j}$$

c) BZ transform preserves the measure $\frac{dt}{t}$:

$$\prod_{i < j} \frac{dt_{i,j}ds_{i,j}}{t_{i,j}s_{i,j}} = \prod_{i < j} \frac{dp_{i,j}dq_{i,j}}{p_{i,j}q_{i,j}}.$$

We supply theorem 4.1 with remarks identical to those related to Theorem 3.1. Their proofs are similar without any troubles with signs since here $\left( \bar{w}_0^{(n)} \right)^2 = 1$. Namely

Remark 1. BZ transform $\tau$ is involutive, that is the inverse relations

$$t_{i,j} = \tilde{r}_{i,j} \cdot \prod_{k < l} \frac{p_{k,j-1}}{q_{k,j-1}} \prod_{k < l} \frac{q_{k,j}}{p_{k,j}}, \quad s_{i,j} = \tilde{r}_{i,j} \cdot \prod_{k < l} \frac{p_{k,j-1}}{q_{k,j-1}} \prod_{k < l} \frac{q_{k,j}}{p_{k,j}}, \quad j < n$$

$$t_{i,n} = \tilde{r}_{i,n} \cdot \prod_{k < l} \frac{p_{k,n-1}}{q_{k,n-1}} \prod_{k < l} \left( \frac{p_{k,n}}{q_{k,n}} \right)^{\epsilon(n-i)}, \quad s_{i,n} = \tilde{r}_{i,n} \cdot \prod_{k < l} \frac{p_{k,n-1}}{q_{k,n-1}} \prod_{k < l} \left( \frac{q_{k,n}}{p_{k,n}} \right)^{\epsilon(n-i)} \quad (4.4)$$
have the same form as (4.3). Here

\[ v_{i,j} = \begin{cases} p_{i,j} + q_{i,j}, & j < n, \\ p_{i,j}, & j = n, \quad n - i \text{ odd,} \quad (\text{i.e. } \epsilon(n-i) = -1), \\ q_{i,j}, & j = n, \quad n - i \text{ even,} \quad (\text{i.e. } \epsilon(n-i) = 1) \end{cases} \]

\[ \tilde{r}_{i,j} = \frac{v_{i,j-1}}{v_{i,j}q_{i,j-1}}. \]

**Remark 2.** The Cartan twist \( T \) can be as well written as \( T(t) = T(p^{-1}) \), that is

\[ T = \sum_{k=1}^{n} \left( T_k e_{k,k} + T_k^{-1} e_{k,k}^{-1} \right) \]

where

\[ T_k = \prod_{j<k} q_{j,k} \cdot \prod_{j>k} \frac{1}{p_{j,k}p_{k,j}} \]

**Corollary 4.1** The restriction of the left Whittaker vector to \( N_+ \) is given by the function

\[ \omega^L_\nu(t) = t^{\nu} \cdot \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) \right) = p^{-\nu} \cdot \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) \right) \]

where \( p_{i,j} \) are given by (3.1) and \( t^{\nu} \) for a weight \( \nu = \sum_{k=1}^{n} \nu_k \epsilon_k \) means the product

\[ t^{\nu} = \prod_{i<j} t^{\nu_i - \nu_j} s_{i,j}^{\nu_i + \nu_j} \]

The proof of Theorem 4.1 follows the same scheme as for \( \text{GL}(n) \). The matrix \( X^{(n)}(-t) \) admits a factorization \( X^{(n)}(-t) = \tilde{X}^{(n-1)}(-t) \cdot A^{(n)}(-t) \), where

\[ A^{(n)}(-t) = \exp(-s_{1,2}e_1) \exp(-s_{1,3}e_2) \cdots \exp(-s_{1,n}e_{n-1}), \]

\[ \exp(-t_{1,n}e_{n-1}) \cdots \exp(-t_{1,3}e_2) \exp(-t_{1,2}e_1) \]

and

\[ \tilde{X}^{(n-1)}(-t) = \left( \exp(-s_{n-1,n}e_{n-1}^{-1}) \exp(-t_{n-1,n}e_{n-1}^{+1}) \right) \cdot \]

\[ \cdots \]

\[ (-\exp(s_{2,3}e_2) \cdots \exp(-s_{2,n}e_{n-1}^{(n-2)}) \exp(-t_{2,n}e_{n-1}^{(n-1)}) \cdots \exp(-t_{2,4}e_3) \exp(-t_{2,3}e_2)) \]

represents a group element of the unipotent subgroup of embedded \( \text{SO}(n-1, n-1) \),

\[ \tilde{X}^{(n-1)}(-t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
The matrix $X^{(n)}(\mathbf{p})$ has the same structure, $X^{(n)}(\mathbf{p}) = \bar{X}^{(n-1)}(\mathbf{p}) \cdot A^{(n)}(\mathbf{p})$. Denote by $\Lambda^{(n)}$ the diagonal matrix $\Lambda^{(n)} = \sum_{k=1}^{n} (\Lambda^{(n)}_{k,k} e_{k,k} + \Lambda^{(n)}_{n,k} e_{k,k})$, where

$$
\Lambda^{(n)}_{1,1} = \prod_{k=1}^{n-1} t_{1,k} s_{1,k}, \quad \Lambda^{(n)}_{k,k} = \frac{t_{1,k}}{s_{1,k}}, \quad 1 < k < n, \quad \Lambda^{(n)}_{n,n} = \left( \frac{t_{1,n}}{s_{1,n}} \right)^{e_{(n-1)}} (4.5)
$$

and define the variables $\tilde{t}_{i,j}$ and $\tilde{s}_{i,j}$, $1 < i < j \leq n$ by the relations

$$
\tilde{t}_{i,j} = t_{i,j} \cdot \frac{s_{1,j-1} t_{1,j}}{s_{1,j}}, \quad j < n, \quad \tilde{t}_{i,n} = t_{i,n} \cdot \frac{s_{1,n-1}}{s_{1,n}} \left( \frac{t_{1,n}}{s_{1,n}} \right)^{e_{(n-1)}}
$$

$$
\tilde{s}_{i,j} = s_{i,j} \cdot \frac{s_{1,j-1} t_{1,j}}{s_{1,j}}, \quad j < n, \quad \tilde{s}_{i,n} = s_{i,n} \cdot \frac{s_{1,n-1}}{s_{1,n}} \left( \frac{t_{1,n}}{s_{1,n}} \right)^{e_{(n)}} (4.6)
$$

**Proposition 4.1**

a) The parameters $p_{1,j}$ and $q_{1,j}$, $j = 2, \ldots, n$ of $A^{(n)}(\mathbf{p})$ are equal to

$$
p_{1,j} = \frac{u_{1,j-1}}{u_{1,j} s_{1,j-1}}, \quad q_{1,j} = \frac{u_{1,j-1} s_{1,j}}{u_{1,j} s_{1,j-1} t_{1,j}}, (4.7)
$$

b) Cartan twist $T$ and the matrix $\bar{X}^{(n-1)}(\mathbf{p})$ satisfy the relation

$$
T \bar{X}^{(n-1)}(\mathbf{p}) = \Lambda^{(n)} \cdot \left( \bar{X}^{(n-1)}(-\mathbf{t}) \bar{w}^{(n-1)} \right)_{0+}
$$

**Proof** of part a) of Proposition 4.1 consists as before in comparison of the first rows of matrices $B^{(n)} = A^{(n)}(-\mathbf{t}) \bar{w}^{(n)}$ and $A^{(n)}(\mathbf{p})$. We have

$$
B^{(n)}_{1,j} = (-1)^{j+1} s_{1,j+1} \cdots s_{1,j-1} u_{1,j}, \quad B^{(n)}_{1,j} = -(s_{1,2} \cdots s_{1,i+1} t_{1,j+1} \cdots t_{1,n}), \quad j < n, \\
B^{(n)}_{1,n} = s_{1,2} \cdots s_{1,n}, \quad B^{(n)}_{1,n} = s_{1,2} \cdots s_{1,n-1} t_{1,n}
$$

By this we see first that the first diagonal entry $\Lambda^{(n)}_{1,1}$ in the Gauss decomposition of the right hand side of (3.1) equals to $B^{(n)}_{1,1} = \prod_{j>1} t_{1,j} s_{1,j}$ and since the first row of the left hand side of (3.1) coincides with the first row of $A^{(n)}(\mathbf{p})$,

$$
A^{(n)}_{1,j}(\mathbf{p}) = q_{1,j+1} \cdots q_{1,j-1} v_{1,j}, \quad A^{(n)}_{1,j}(\mathbf{p}) = (-1)^{j} (q_{1,2} \cdots q_{1,i+1}) (p_{1,j+1} \cdots p_{1,n}),
$$

the variables $p_{1,j}$ and $q_{1,j}$ then can be found via ratios of coefficients $B^{(n)}_{1,k}$. Thus we get (4.7).

For the proof of part b) we again need the crucial technical lemma which says that the matrix $U = \left( \bar{w}^{(n-1)} \right)^{-1} A^{(n)}(-\mathbf{t}) \bar{w}^{(n)} (A^{(n)}(\mathbf{p}))^{-1}$, see (3.13) has the block structure

$$
U = \begin{pmatrix}
* & 0 & 0 \\
* & 0 & 0 \\
* & 0 & * \\
* & * & *
\end{pmatrix}
$$

and specializes its diagonal entries.
Lemma 4.1  Nondiagonal matrix elements $U_{i,j}$ $(i \neq j)$ equal zero if $j > 1$ and $i < 2n$. Matrix element $U_{1,1}$ equals $\prod_{j>1} t_{1,j} s_{1,j}$. Matrix element $U_{k,k}$ equals $\frac{s_{k}}{t_{1,k}}$ for $1 < k \leq n$.

The proof of Lemma 4.1 is sketched in Appendix B.

As well as in the proof of Proposition (3.1), Lemma 4.1 implies the equality

$$T \tilde{X}^{(n-1)}(p) = \Lambda^{(n)}( \tilde{X}^{(n-1)}(-t) \bar{u}^{(n-1)}_0 )_{0+}$$

where the diagonal matrix $\Lambda^{(n)}$ is given in (4.5) and

$$\tilde{X}^{(n-1)}(-t) = (\Lambda^{(n)})^{-1} \tilde{X}^{(n-1)}(-t) \Lambda^{(n)}.$$  \hspace{1cm} (4.8)

Then the structure of the group element $X^{(n)}(t)$, see (4.2) says that the parameters $t_{i,j}$ and $s_{i,j}$ are the coefficients at $e_{j-1}$ for $j < n$; $t_{i,n}$ is the coefficient at $e^{(n-1-i+1)}_n$, and $s_{i,n}$ is the coefficient at $e^{(n-1)}_n$ in Lusztig presentation of $X(-t)$. This enables us to rewrite the conjugation (4.8) as the change of variables (4.6) and finish the proof of Proposition 4.1. Then the proof of part b) of Theorem 4.1 follows by induction on $n$. The inductive proof of part c) is analogous to that of Theorem 3.1. \hspace{1cm} \Box

4.2 Whittaker function

Using the arguments of the end of the previous subsection, we describe the action of Cartan generators on the right Whittaker vector. Namely, the vector $\exp(\sum_{k=1}^{n} x_k (e_{k,k} - e_{k,k})) \cdot \omega^R_\mu$ is presented by the function $\exp(-\sum_k \mu_k x_k) \omega^R_\mu(t')$, where

$$t'_{i,j} = t_{i,j} \cdot \exp(x_j - x_{j-1}), \quad s'_{i,j} = s_{i,j} \cdot \exp(x_j - x_{j-1}), \quad j < n,$$

$$t'_{i,n} = t_{i,n} \cdot \exp(\epsilon(n-i+1)x_n - x_{n-1}), \quad s'_{i,n} = s_{i,n} \cdot \exp(\epsilon(n-i)x_n - x_{n-1}).$$  \hspace{1cm} (4.9)

Denote by $\hat{\omega}^L_\rho(\gamma)$ and $\hat{\omega}^R_\rho(\gamma)$ the Mellin transforms of the functions on $N_+ \omega^L_\rho(t)$ and $\omega^R_\rho(t)$,

$$\hat{\omega}^R_\rho(\gamma) = \int_{t>0} \omega^R_\rho(t) t^\gamma dt = \int \omega^R_\rho(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}} s_{i,j}^{\delta_{i,j}} dt_{i,j} ds_{i,j},$$

$$\hat{\omega}^L_\rho(\gamma) = \int_{t>0} t^\rho \omega^L_\rho(t) t^\gamma dt = \int \omega^L_\rho(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}+\epsilon_{i,j}} s_{i,j}^{\delta_{i,j}+2n-i-j} dt_{i,j} ds_{i,j}.$$

Here $\rho = \sum_{k=1}^{n} (n-k)(\varepsilon_{k,k} - k_{k,k})$. Due to (4.9) the action of Cartan subgroup on the right Whittaker vector $v^R_\rho$ transforms the function $\hat{\omega}^R_\rho(\gamma)$ to the product $\exp(-\sum_k \mu_k x_k) \cdot \hat{H}(x, \gamma) \hat{\omega}^R_\rho(\gamma)$, where

$$\hat{H}(x, \gamma) = \sum_{i<j<n} (\gamma_{i,j} + \delta_{i,j})(x_j - x_{j-1}) + (\gamma_{n-1,n} + \delta_{n-2,n} + \ldots)(x_n - x_{n-1}) +$$

$$\hspace{2cm} (\delta_{n-1,n} + \delta_{n-2,n} + \delta_{n-3,n} + \ldots)(-x_{n-1} - x_n).$$  \hspace{1cm} (4.10)
Due to Proposition 2.2 we have:

\[ \hat{\omega}_\mu^R(\gamma) = \prod_{i<j} \Gamma(\gamma_{i,j})\Gamma(\delta_{i,j}). \]

For the calculation of \( \hat{\omega}_\nu^L(\gamma) \) we pass in the integral

\[
\int_{t_{i,j}, s_{i,j} > 0} \exp \sum_{i<j} -(p_{i,j} + q_{i,j}) \cdot \prod_{i<j} t_{i,j}^{\gamma_{i,j} + \nu'_{i,j} - \nu'_{j,i}} s_{i,j}^{\delta_{i,j} + \nu'_{j,i} + \nu'_{i,j}} dt_{i,j} ds_{i,j}
\]

from the integration variables \( t_{i,j} \) and \( s_{i,j} \) to \( p_{i,j} \) and \( q_{i,j} \). Here

\[ \nu' = \nu + \rho, \quad \nu'_i = \nu_i + n - i. \]

Substitution of (4.4) gives the relation

\[
\prod_{i<j} t_{i,j}^{\gamma_{i,j}} s_{i,j}^{\delta_{i,j}} = \prod_{1<i<j<n} p_{i,j}^{\varphi_{i,j}} q_{i,j}^{\psi_{i,j}} \cdot \prod_{1<i<j<n} (p_{i,j} + q_{i,j})^{\theta_{i,j}}
\]

where

\[
\varphi_{i,j} = -\delta_{i,j} + \sum_{k>i} (\gamma_{k,j+1} + \delta_{k,j+1} - \gamma_{k,j} - \delta_{k,j})
\]

\[
\psi_{i,j} = \delta_{i,j} - \sum_{k>i} (\gamma_{k,j+1} + \delta_{k,j+1} - \gamma_{k,j} - \delta_{k,j}) - \gamma_{i,j+1} - \delta_{i,j+1}, \tag{4.11}
\]

\[
\theta_{i,j} = -\delta_{i,j} - \gamma_{i,j} + \delta_{i,j+1} + \gamma_{i,j+1},
\]

if \( j < n \) and \( \theta_{i,n} = 0 \),

\[
\varphi_{i,n} = \begin{cases} 
-\delta_{i,n} + \sum_{k=1}^{n-i-1} (-1)^{k} (\gamma_{i+k,n} - \delta_{i+k,n}), & \varepsilon(n - i) = 1 \\
-\gamma_{i,n} + \sum_{k=1}^{n-i-1} (-1)^{k} (\delta_{i+k,n} - \gamma_{i+k,n}), & \varepsilon(n - i) = -1
\end{cases} \tag{4.12}
\]

\[
\psi_{i,n} = \begin{cases} 
-\gamma_{i,n} + \sum_{k=1}^{n-i-1} (-1)^{k} (\delta_{i+k,n} - \gamma_{i+k,n}), & \varepsilon(n - i) = 1 \\
-\delta_{i,n} + \sum_{k=1}^{n-i-1} (-1)^{k} (\gamma_{i+k,n} - \delta_{i+k,n}), & \varepsilon(n - i) = -1
\end{cases}
\]

This means that the Mellin transform of the left Whittaker vector is described by the integral

\[
\int_{p_{i,j}, q_{i,j} > 0} \prod_{i<j} p_{i,j}^{\varphi_{i,j} - \nu'_{i,j}} q_{i,j}^{\psi_{i,j} - \nu'_{i,j}} (p_{i,j} + q_{i,j})^{\theta_{i,j}} dp_{i,j} dq_{i,j}
\]

Integration over the variables \( p_{i,n} \) and \( q_{i,n} \) produced the product

\[
\prod_{i=1}^{n-1} \Gamma(\varphi_{i,n} - \nu_i + \nu_n) \Gamma(\psi_{i,n} - \nu'_i - \nu'_n) \tag{4.13}
\]

of corresponding \( \Gamma \) functions. For the calculation of integrals over the variables \( p_{i,j} \) and \( q_{i,j} \) with \( j < n \) we use the integral

\[
\int_{x,y > 0} x^{a-1} y^{b-1} e^{-x-y} dxdy = \frac{\Gamma(a)\Gamma(b)\Gamma(a+b+c)}{\Gamma(a+b)}. \tag{4.14}
\]
Its evaluation is based on the change of variables \( u = x + y, \ t = \frac{x}{x+y} \). This gives the product over \( 1 < i < j < n \) of the factors

\[
\prod_{1 \leq i < j \leq n} \frac{\Gamma(\varphi_{i,j} - \nu_i' + \nu_j') \Gamma(\psi_{i,j} - \nu_i' - \nu_j') \Gamma(\varphi_{i,j} + \theta_{i,j} + \psi_{i,j} - 2\nu_i)}{\Gamma(\varphi_{i,j} + \psi_{i,j} - 2\nu_i')}. \tag{4.15}
\]

Due to (4.11)

\[
\frac{\Gamma(\varphi_{i,j} + \theta_{i,j} + \psi_{i,j} - 2\nu_i)}{\Gamma(\varphi_{i,j} + \psi_{i,j} - 2\nu_i')} = \frac{\Gamma(-\gamma_{i,j} - \delta_{i,j} - 2\nu_i)}{\Gamma(-\gamma_{i,j+1} - \delta_{i,j+1} - 2\nu_i')}
\]

which implies cancelations of ratios in the product (4.15) so that it becomes equal to

\[
\prod_{1 \leq i < j < n} \frac{\Gamma(\varphi_{i,j} - \nu_i' + \nu_j') \Gamma(\psi_{i,j} - \nu_i' - \nu_j')}{\Gamma(-\gamma_{i,j+1} - \delta_{i,j+1} - 2\nu_i')} \prod_{i=1}^{n-2} \frac{\Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2\nu_i')}{\Gamma(-\gamma_{i,i} - \delta_{i,i} - 2\nu_i')}. \tag{4.16}
\]

Using (4.13) we arrive to the following answers

\[
\varphi^L_\nu(\gamma) = \prod_{i=1}^{n-2} \frac{\Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2\nu_i')}{\Gamma(-\gamma_{i,i} - \delta_{i,i} - 2\nu_i')} \prod_{1 \leq i < j \leq n} \Gamma(\varphi_{i,j} - \nu_i' + \nu_j') \Gamma(\psi_{i,j} - \nu_i' - \nu_j'), \tag{4.17}
\]

and

\[
\Psi_\lambda(x) = \frac{e^{-i(\lambda,x)}}{(2\pi i)^d} \int_C \exp \tilde{H}(x, \gamma) \prod_{k=1}^{n-2} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(\gamma_{k,k} + \delta_{k,k} + 2i\lambda_k)} \prod_{k<l} \Gamma(-\varphi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(-\psi_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l}) \Gamma(\delta_{k,l}) d\gamma_{k,l} d\delta_{k,l}. \tag{4.18}
\]

Here \( \tilde{H}(x, \gamma) \) is given in (4.10), \( \psi_{i,j} \) and \( \varphi_{i,j} \) are given in (4.11) and (4.12), \( d = n(n-1) \) and the contour \( C \) is a deformation of the imaginary plane \( \text{Re} \gamma_{ij} = \text{Re} \delta_{ij} = 0 \) into the strip of analyticity of the integrand.

Perform now the following change of variables:

\[
\gamma_{i,j} = \tilde{\gamma}_{i,j} - \tilde{i}_{i+1,j}, \quad \delta_{i,j} = \tilde{\delta}_{i,j} - \tilde{i}_{i+1,j}, \quad 1 \leq i < j < n,
\]

\[
\gamma_{i,n} = \begin{cases} \tilde{\gamma}_{i,n} - \tilde{i}_{i+1,n}, & \varepsilon(n-i) = 1, \\ \tilde{\gamma}_{i,n} - \tilde{i}_{i+1,n}, & \varepsilon(n-i) = -1, \end{cases}
\]

\[
\delta_{i,n} = \begin{cases} \tilde{\delta}_{i,n} - \tilde{i}_{i+1,n}, & \varepsilon(n-i) = 1, \\ \tilde{\delta}_{i,n} - \tilde{i}_{i+1,n}, & \varepsilon(n-i) = -1, \end{cases}
\]

that is

\[
\tilde{\gamma}_{i,j} = \gamma_{i,j} + \ldots + \gamma_{j-1,j}, \quad \tilde{\delta}_{i,j} = \delta_{i,j} + \ldots + \delta_{j-1,j}, \quad 1 \leq i < j < n,
\]

\[
\tilde{\gamma}_{i,n} = \delta_{n} + \gamma_{i+1,n} + \ldots + \gamma_{n-1,n} \quad \text{if } n - i \text{ is even},
\]

\[
\tilde{\gamma}_{i,n} = \gamma_{n} + \delta_{i+1,n} + \ldots + \gamma_{n-1,n} \quad \text{if } n - i \text{ is odd},
\]

\[
\tilde{\delta}_{i,n} = \gamma_{n} + \delta_{i+1,n} + \ldots + \delta_{n-1,n} \quad \text{if } n - i \text{ is even},
\]

\[
\tilde{\delta}_{i,n} = \delta_{n} + \gamma_{i+1,n} + \ldots + \delta_{n-1,n} \quad \text{if } n - i \text{ is odd},
\]

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In this variables we have
\[ \varphi_{i,j} = \tilde{\gamma}_{i+1,j+1} + \delta_{i+1,j+1} - \tilde{\delta}_{i+1,j}, \quad 1 \leq i < j < n, \]
\[ \psi_{i,j} = -\tilde{\gamma}_{i+1,j} + \delta_{i,j+1} + \tilde{\delta}_{i,j}, \quad 1 \leq i < j < n, \]
\[ \varphi_{i,n} = -\tilde{\gamma}_{i,n} + \delta_{i+1,n}, \quad \psi_{i,n} = -\delta_{i,n} + \tilde{\gamma}_{i+1,n} \quad 1 \leq i < n, \]
and \[ H(x, \gamma) = \sum_{j<n} (\tilde{\gamma}_{1,j} + \delta_{1,j})(x_j - x_{j-1}) + \tilde{\gamma}_{1,n}(x_n - x_{n-1}) + \delta_{1,n}(-x_{n-1} - x_n). \]

Then we have

**Theorem 4.2** The function \( \Psi_{\lambda}(x) \) is given by the integral
\[
\Psi_{\lambda}(x) = \frac{e^{-i\langle x, \lambda \rangle}}{(2\pi i)^d} \int_{C_n} \exp H(x, \gamma) \prod_{k=1}^{n-2} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(\gamma_{k,n} + \delta_{k,n} - \gamma_{k+1,n} - \delta_{k+1,n} + 2i\lambda_k)} \\
\times \prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}.
\]

Here we set \( \gamma_{k,l} = \delta_{k,l} = 0 \) if the pair \((k, l)\) does not satisfy the condition \( 1 \leq k < l \leq n, \)
\( d = n(n-1), \) \( H(x, \gamma) = \sum_{j<n} (\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) + \gamma_{1,n}(x_n - x_{n-1}) - \delta_{1,n}(x_{n-1} + x_n), \)
\[ \xi_{i,j} = -\gamma_{i+1,j+1} + \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad 1 \leq i < j < n, \]
\[ \eta_{i,j} = \gamma_{i+1,j} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad 1 \leq i < j < n, \]
\[ \xi_{i,n} = \gamma_{i,n} - \delta_{i+1,n}, \quad 1 \leq i < n, \]
\[ \eta_{i,n} = \delta_{i,n} - \gamma_{i+1,n} \quad 1 \leq i < n, \]
The integration cycle is a deformation of the imaginary plain \( \text{Re} \gamma_{k,l} = \text{Re} \delta_{k,l} = 0 \) into nonempty domain \( D \subset \mathbb{C}^d \) of the analyticity of the integrand, which is described by the relations \( \text{Re} \gamma_{i,j} > 0, \) \( \text{Re} \delta_{i,j} > 0, \) \( \text{Re} \xi_{i,j} > 0, \) \( \text{Re} \eta_{i,j} > 0, \) \( 1 \leq i < j \leq n. \)

## 5 \( \text{SO}(n + 1, n) \)

### 5.1 BZ transform

The split real form of the group \( \text{SO}(2n + 1, \mathbb{C}) \) is the group \( \text{SO}(n + 1, n) \subset \text{SL}(2n, \mathbb{R}) \) preserving symmetric form \( \sum_{i=1}^{2n}(x_iy_i + x_iy_i + x_ny_n), \) where \( i = 2n + 2 - i. \) Gauss decomposition is induced from that of \( \text{GL}(2n + 1, \mathbb{R}). \) Positive roots are elements \( \varepsilon_i \pm \varepsilon_j \) for \( 1 \leq i < j \leq n \) and \( \varepsilon_i, \) where \( \varepsilon_k \in \mathfrak{h}^*, \) \( k = 1, \ldots, n \) are defined by the condition \( \varepsilon_k(e_{j,j} - e_{j,j}) = \delta_{j,k}. \) We denote Chevalley generator of the Lie algebra \( \mathfrak{g} = \mathfrak{so}(2n + 1) \) by \( e_i \) and \( f_i, i = 1, \ldots, n \) Here
\[
e_i = e_{i,i+1} - e_{2n-i,2n+1-i}, \quad f_i = e_{i+1,i} - e_{2n+1-i,2n-i}, \quad i = 1, \ldots, n - 1, 
\]
\[ e_n = \sqrt{2}(e_{n,n+1} - e_{n+1,n+2}), \quad f_n = \sqrt{2}(e_{n+1,n} - e_{n+2,n+1}). \]

Denote by \( s_i, i = 1, \ldots, n \) the corresponding generators of the Weyl group and by \( \tilde{s}_i \) their lifts to the group \( \text{SO}(n + 1, n) \) according to (2.30). We choose the following normal
ordering of the system $\Delta_+$ of positive roots:
\[
\varepsilon_n, (\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n), \ldots, \\
(\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \ldots, \varepsilon_1 + \varepsilon_n, \varepsilon_1 - \varepsilon_n, \ldots, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_2).
\]

It corresponds by (2.20) to the following reduced decomposition of the longest element $w_0^{(n)}$ of the Weyl group $W$:
\[
w_0^{(n)} = s_n(s_{n-1}s_ns_{n-1})(s_{n-2}s_{n-1}s_ns_{n-1}s_{n-2}) \cdots (s_1s_2\ldots s_{n-1}s_n s_{n-1} \cdots s_2 s_1).
\] (5.1)

Note that as element of $SO(n+1, n)$,
\[
\tilde{w}_0^{(n)} = (-1)^n \left( \sum_{k=1}^{2n} e_{k,k} + e_{n+1,n+1} \right)
\]

Denote by $t_{i,j}$ Lusztig parameter corresponding to the root $\varepsilon_i - \varepsilon_j$, and by $s_{i,j}$ Lusztig parameter corresponding to the root $\varepsilon_i + \varepsilon_j$, and by $t_k$ the parameter, corresponding to $\varepsilon_k$. Here $1 \leq i < j \leq n$, $k = 1, \ldots, n$. Then the group element $X^{(n)}(t)$ looks as
\[
X^{(n)}(t) = \exp(t_ne_n) \left( \exp(s_{n-1,n}e_{n-1}) \exp(t_{n-1,n}e_{n-1}) \right) \cdots \\
\left( \exp(s_{1,2}e_1) \exp(s_{1,3}e_2) \cdots \exp(s_{1,n}e_{n-1}) \exp(t_{1,n}e_{n-1}) \cdots \exp(t_{1,3}e_2) \exp(t_{1,2}e_1) \right).
\] (5.2)

BZ transform for $SO(n+1, n)$ is the solution of the equation (3.1), where in $X^{(n)}(p)$ we use the notation $p_{i,j}$ for Lusztig parameter, corresponding to the root $\varepsilon_i - \varepsilon_j$, $q_{i,j}$ for the Lusztig parameter, corresponding to the root $\varepsilon_i + \varepsilon_j$, and $p_i$ for the parameters, corresponding to $\varepsilon_i$,
\[
X^{(n)}(p) = \exp(p_ne_n) \left( \exp(q_{n-1,n}e_{n-1}) \exp(p_{n-1,n}e_{n-1}) \right) \cdots \\
\left( \exp(q_{1,2}e_1) \exp(q_{1,3}e_2) \cdots \exp(q_{1,n}e_{n-1}) \exp(p_{1,n}e_{n-1}) \cdots \exp(p_{1,3}e_2) \exp(p_{1,2}e_1) \right).
\] (5.3)

We set $s_{j,j} = t_{j,j} = p_{j,j} = q_{j,j} = p_j = u_{j,j} = 1$, and for all $i, j$, $1 \leq i < j \leq n$ put
\[
u_{i,j} = t_{i,j} + s_{i,j}, \quad r_{i,j} = \frac{u_{i,j-1}}{u_{i,j}s_{i,j-1}}.
\] (5.4)

**Theorem 5.1**

a) BZ map for $SO(n, n)$ looks as
\[
p_{i,j} = r_{i,j} \prod_{k<i} \frac{t_{k,j}^{-1}}{s_{k,j-1}} \prod_{k<i} \frac{s_{k,j}}{t_{k,j}}, \quad q_{i,j} = r_{i,j} \prod_{k<i} \frac{t_{k,j}^{-1}}{s_{k,j-1}} \prod_{k<i} \frac{s_{k,j}}{t_{k,j}}, \quad p_i = \frac{u_{i,n}}{t_{i,s_{i,n}}^2} \prod_{k<i} \frac{t_{k,n}}{s_{k,n}}
\] (5.5)

b) The twisting matrix $T \in H$ is $T = e_{n+1,n+1} + \sum_{k=1}^n \left( T_{k}e_{k,k} + T_{k}^{-1}e_{k,k} \right)$, where
\[
T_k = t_k^2 \prod_{j<k} \frac{s_{j,k}}{t_{j,k}} \prod_{j>k} s_{j,k}t_{j,k}
\]
c) BZ transform preserves the measure $\frac{dt}{t}$:
\[
\prod_{i<j} dt_{i,j} ds_{i,j} \prod_{k} dt_{k} = \prod_{i<j} \frac{dp_{i,j} dq_{i,j}}{p_{i,j} q_{i,j}} \prod_{k} dp_{k}.
\]
Again, we have the same remarks with the same proofs:

**Remark 1.** BZ transform $\tau$ is involutive, that is the inverse relation

$$t_{i,j} = \bar{r}_{i,j} \prod_{k<i} \frac{p_{k,j} - 1}{q_{k,j} - 1} \frac{g_{k,j}}{p_{k,j}}, \quad s_{i,j} = \bar{r}_{i,j} \prod_{k<i} \frac{p_{k,j-1}}{q_{k,j-1}} \frac{q_{k,j}}{p_{k,j}}, \quad t_i = \frac{v_{i,n}}{p_i q_i} \prod_{k<i} \frac{p_{kn}}{q_{kn}} \quad (5.6)$$

has the same form as (4.3). Here

$$v_{i,j} = p_{i,j} + q_{i,j}, \quad \bar{r}_{i,j} = \frac{v_{i,j-1}}{v_{i,j} q_{i,j-1}}.$$

**Remark 2.** The Cartan twist $T$ can be as well written as $T(t) = T(p^{-1})$, that is $T = e_{n+1,n+1} + \sum_{k=1}^{n} \left( T_k e_{k,k} + T_k^{-1} e_{k,k} \right)$, where

$$T_k = \frac{1}{p_k} \prod_{j<k} q_{j,k} \prod_{j>k} \frac{1}{q_{j,k} p_{j,k}}$$

**Corollary 5.1** The restriction of left Whittaker vector to $\mathbb{N}_+$ is given by the function

$$\omega_{\nu}(t) = t^{\nu} \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) - \sum_k p_k \right) = p^{-\nu} \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) - \sum_k p_k \right)$$

where $p_{i,j}$ and $p_k$ are given by (5.5) and $t^{\nu}$ for a weight $\nu = \sum_{k=1}^{n} \nu_k e_k$ means the product

$$t^{\nu} = \prod_{i<j} \left( \begin{array}{c} \nu_i - \nu_j \\ s_{i,j} \end{array} \right) \prod_k \left( t_k^{\nu_k} \right).$$

The proof of Theorem 5.1 follows the same scheme as before. The matrix $X^{(n)}(-t)$ admits a factorization $X^{(n)}(-t) = \tilde{X}^{(n-1)}(-t) \cdot A^{(n)}(-t)$, where

$$A^{(n)}(-t) = \exp(-s_{1,2} e_1) \exp(-s_{1,3} e_2) \cdots \exp(-s_{1,n} e_{n-1}) \exp(-t_1 e_n),$$

$$\exp(-t_{1,n} e_{n-1}) \cdots \exp(-t_{1,3} e_2) \exp(-t_{1,2} e_1) \quad (5.7)$$

and

$$\tilde{X}^{(n-1)}(t) = \exp(t_n e_n) \left( \exp(s_{n-1,n} e_{n-1}) \exp(t_{n-1} e_n) \exp(t_{n-1,n} e_{n-1}) \right) \cdot$$

$$\cdots$$

$$\exp(s_{2,3} e_2) \cdots \exp(s_{2,n} e_{n-1}) \exp(t_2 e_n) \exp(t_{2,n} e_{n-1}) \cdots \exp(t_{2,3} e_2).$$

represents a group element of the unipotent subgroup of embedded $\text{SO}(n,n-1)$. The matrix $X^{(n)}(p)$ has the same structure, $X^{(n)}(p) = \tilde{X}^{(n-1)}(p) \cdot A^{(n)}(p)$. Denote by $\Lambda^{(n)}$ the diagonal matrix $\Lambda^{(n)} = e_{n+1,n+1} + \sum_{k=1}^{n} \left( \Lambda^{(n)}_{k,k} e_{k,k} + \Lambda^{(n-1)}_{k,k} e_{k,k} \right)$, where

$$\Lambda^{(n)}_{1,1} = t_1^2 \prod_{k=1}^{n-1} t_{1,k} s_{1,k}, \quad \Lambda^{(n)}_{k,k} = \frac{t_1 e_k}{s_{1,k}}, \quad 1 < k \leq n,$$

and define the variables $\tilde{t}_{i,j}$ and $\tilde{s}_{i,j}$, $1 < i < j \leq n$ by the relations

$$\tilde{t}_{i,j} = t_{i,j} \cdot s_{i,j} t_{1,j-1} s_{1,j}, \quad \tilde{s}_{i,j} = s_{i,j} \cdot s_{i,j-1} t_{1,j} s_{1,j}, \quad \tilde{t}_i = t_i \frac{s_{1,n}}{t_{1,n}}$$

The induction step is given by the following lemma and proposition.
Lemma 5.1  Nondiagonal matrix elements $U_{i,j}$ ($i \neq j$) of the matrix (3.13) equal zero if $j > 1$ and $i < 2n + 1$. Matrix element $U_{1,1}$ equals $t_1^2 \prod_{j=1}^{n} t_{1,j} s_{1,j}$. Matrix element $U_{k,k}$ equals $\frac{s_{1,k}}{t_{1,k}}$ for $1 < k \leq n$. Matrix element $U_{k,1}$ equals $\frac{t_{1,k}}{s_{1,k}}$ for $1 < k \leq n$. Element $U_{n+1,n+1} = 1$.

Proposition 5.1 a) The parameters $p_{1,j}$, $q_{1,j}$, $j = 2, \ldots, n$ and $p_1$ of $A^{(n)}(p)$ are equal to

$$p_{1,j} = \frac{u_{1,j-1}}{u_{1,j}s_{1,j-1}}, \quad q_{1,j} = \frac{u_{1,j-1}s_{1,j}}{u_{1,j}s_{1,j-1}t_{1,j}}, \quad p_1 = \frac{u_{1,n}}{t_{1}s_{1,n}}\$$

b) Cartan twist $T$ and the matrix $\tilde{X}^{(n-1)}(p)$ satisfy the relation

$$T \tilde{X}^{(n-1)}(p) = \Lambda^{(n)} \cdot \left( \tilde{X}^{(n-1)}(-t) \tilde{w}_0^{(n-1)} \right)_{0+}$$

All the proofs repeat that of the previous sections \qed

5.2 Whittaker function

Describe first the action of Cartan generators on the right Whittaker vector. The vector $\exp(\sum_{k=1}^{n} x_k(e_{k,k} - e_{j,k})) \cdot v^R_\mu$ is presented by the function $\exp(-\sum_{k} \mu_k x_k) \cdot \omega^R(t')$, where

$$t'_{i,j} = t_{i,j} \cdot \exp(x_j - x_{j-1}), \quad s'_{i,j} = s_{i,j} \cdot \exp(x_j - x_{j-1}), \quad t'_i = t_i \cdot \exp(-x_n). \quad (5.9)$$

Denote by $\tilde{\omega}^L(\gamma)$ and $\tilde{\omega}^R(\gamma)$ the Mellin transforms of the functions $\omega^L_\mu(t)$ and $\omega^R_\mu(t)$,

$$\tilde{\omega}^R(\gamma) = \int_{t>0} \omega^R_\mu(t) t^{\gamma} \frac{dt}{t} := \int_{t_{i,j}>0} \omega^R_\mu(t) \prod_{i<j} t_{i,j}^{\delta_{i,j}} s_{i,j}^{\delta_{j,i}} \frac{dt_{i,j}}{t_{i,j}} \frac{ds_{i,j}}{s_{i,j}} \frac{dt_i}{t_i},$$

$$\tilde{\omega}^L(\gamma) = \int_{t>0} t^\rho \omega^L_\nu(t) t^{\gamma} \frac{dt}{t} := \int_{t_{i,j}>0} \omega^L_\nu(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j} + 1} s_{i,j}^{\delta_{i,j} + 1} \frac{dt_{i,j}}{t_{i,j}} \frac{ds_{i,j}}{s_{i,j}} \frac{dt_i}{t_i}.$$  

Here $\rho = \sum_{k=1}^{n} (n + \frac{1}{2} - k)(e_{k,k} - e_{j,k})$. Due to (5.9) the action of Cartan subgroup on the right Whittaker vector $v^R_\mu$ transforms the function $\tilde{\omega}^R_\mu(\gamma)$ to the product $\exp(-\sum_{k} \mu_k x_k) \cdot \exp(\tilde{H}(x, \gamma) \cdot \tilde{\omega}^R_\mu(\gamma))$, where

$$\tilde{H}(x, \gamma) = \sum_{i<j} (\gamma_{i,j} + \delta_{i,j})(x_j - x_{j-1}) - x_n \sum_i \gamma_i. \quad (5.10)$$

As before, we have

$$\tilde{\omega}^R_\mu(\gamma) = \prod_{1<i<j\leq n} \Gamma(\gamma_{i,j}) \Gamma(\delta_{i,j}) \prod_{k=1}^{n} \Gamma(\gamma_k). \quad (5.11)$$
For the calculation of \( \hat{\omega}_\nu^L(\gamma) \) we pass in the integral
\[
\int_{t_{i,j}, s_{i,j} > 0, t_i > 0} \prod_{i < j} \prod_{i = 1} \int \gamma_{i,j} t_{i,j}^\gamma s_{i,j}^{\gamma_{i,j}} e^{-p_{i,j} - q_{i,j}} \prod_{i} t_i^{\gamma_i + 2\nu'_{i}} e^{-p_i} dt_i/t_i
\]
from the integration variables \( t_{i,j}, s_{i,j} \) and \( t_i \) to \( p_{i,j}, q_{i,j} \) and \( p_i \). Here
\[
\nu' = \nu + \rho, \quad \nu'_i = \nu_i + n - i + \frac{1}{2}.
\]
Substitution of (5.6) gives the relation
\[
\prod_i t_i^{\gamma_i} \prod_{i < j} t_{i,j}^{\gamma_{i,j}} = \prod_i p_i^{\psi_i} \prod_{i < j} p_{i,j}^{\psi_{i,j}} (p_{i,j} + q_{i,j})^{\theta_{i,j}}
\]
where
\[
\begin{align*}
\phi_{i,j} &= -\delta_{i,j} + \sum_{k > i} (\gamma_{k,j + 1} + \delta_{k,j + 1}) - \sum_{k > i} (\gamma_{k,j} + \delta_{k,j}), \\
\psi_{i,j} &= \delta_{i,j} - \sum_{k > i} (\gamma_{k,j + 1} + \delta_{k,j + 1}) + \sum_{k > i} (\gamma_{k,j} + \delta_{k,j}), \\
\theta_{i,j} &= -\delta_{i,j} - \gamma_{i,j} + \delta_{i,j + 1} + \gamma_{i,j + 1},
\end{align*}
\]
for \( 1 \leq i < j < n \), and
\[
\begin{align*}
\phi_{i,n} &= -\delta_{i,n} - \sum_{k > i} (\gamma_{k,n} + \delta_{k,n} - \gamma_k), \\
\psi_{i,n} &= \delta_{i,n} - \gamma_i + \sum_{k > i} (\gamma_{k,n} + \delta_{k,n} - \gamma_k), \\
\theta_{i,n} &= \gamma_i - \delta_{i,n} - \gamma_{i,n}, \quad \varphi_i = -\gamma_i
\end{align*}
\]
for \( 1 \leq i < n \). Using (4.14) we present the multiple integral
\[
\int_{p_{i,j}, q_{i,j}, p_i > 0} \prod_{i < j} p_{i,j}^{\phi_{i,j} - \nu'_{i} - \nu'_{j}} q_{i,j}^{\psi_{i,j} - \nu'_{i} - \nu'_{j}} (p_{i,j} + q_{i,j})^\theta_{i,j} \frac{dp_{i,j}}{p_{i,j}} \frac{dq_{i,j}}{q_{i,j}} \prod_i p_i^{\phi_i - 2\nu'_i} \frac{dp_i}{p_i}
\]
as the product
\[
\prod_{k=1}^n \Gamma(-\gamma_k - 2\nu'_k) \prod_{i < j} \Gamma(\phi_{i,j} - \nu'_i + \nu'_j) \Gamma(\psi_{i,j} - \nu'_i - \nu'_j) \frac{\Gamma(\phi_{i,j} + \theta_{i,j} + \psi_{i,j} - 2\nu'_i)}{\Gamma(\phi_{i,j} + \psi_{i,j} - 2\nu'_i)}.
\]
Substitution of (5.12) and (5.13) into the product over \( 1 \leq i < j < n \) of ratios of \( \Gamma \) functions in the latter expression results, just as for \( \text{SO}(n,n) \) to the product
\[
\prod_{i=1}^{n-1} \Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2\nu'_i) \Gamma(-\gamma_{i,n} - \delta_{i,n} - 2\nu'_i).
\]
while the same product over \( j = n \), \( i = 1, \ldots, n - 1 \) is

\[
\prod_{i=1}^{n-1} \frac{\Gamma(-\delta_{i,n} - \gamma_{i,n} - 2\nu'_i)}{\Gamma(-\gamma_i - 2\nu'_i)}.
\]

Thus we have the following expressions for the left Whittaker vector

\[
\hat{\omega}_L^\nu(\gamma) = \Gamma(-\gamma_n - 2\nu'_n) \prod_{i=1}^{n-1} \Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2\nu'_i) \prod_{1 \leq i < j \leq n} \Gamma(\varphi_{i,j} - \nu'_i + n'u_j) \Gamma(\psi_{i,j} - \nu'_i - \nu'_j).
\]

which results after substitution \( \nu' = i\lambda \) in the following expression for Whittaker wave function:

\[
\Psi_\lambda(x) = \frac{e^{-i(\lambda, x)}}{(2\pi i)^d} \int_{C_n} \exp \tilde{H}(x, \gamma) \prod_{k=1}^{n-1} \Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k) \Gamma(\gamma_k) d\gamma_k \prod_{k<l} \Gamma(-\varphi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(-\psi_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\varphi_{k,l}) \Gamma(\psi_{k,l}) d\gamma_{k,l} d\delta_{k,l} \Gamma(\gamma_n + 2i\lambda_n) d\gamma_n
\]

Here \( \tilde{H}(x, \gamma) \) is given in (5.10), \( \psi_{k,l} \) and \( \varphi_{k,l} \) are given in (5.12) and (5.13), \( d = n^2 \). The contour \( C \) is a deformation of imaginary plane \( \text{Re} \gamma_{i,j} > 0, \text{Re} \delta_{i,j} > 0, \text{Re} \gamma_i > 0 \) into the strip of analyticity of the integrand.

The formula for \( \Psi_\lambda(x) \) can be simplified by using the change of variables

\[
\gamma_{i,j} = \tilde{\gamma}_{i,j} - \gamma_{i+1,j}, \quad \delta_{i,j} = \tilde{\delta}_{i,j} - \delta_{i+1,j}, \quad 1 \leq i < j \leq n, \\
\gamma_i = \tilde{\gamma}_i - \gamma_{i+1}, \quad 1 \leq i \leq n,
\]

that is

\[
\tilde{\gamma}_{i,j} = \gamma_{i,j} + \ldots + \gamma_{j-1,j}, \quad \tilde{\delta}_{i,j} = \delta_{i,j} + \ldots + \delta_{j-1,j}, \quad \tilde{\gamma}_i = \gamma_i + \ldots + \gamma_n
\]

In this variables we have

\[
\varphi_{i,j} = \tilde{\gamma}_{i+1,j+1} + \tilde{\delta}_{i+1,j+1} - \tilde{\gamma}_{i+1,j} - \tilde{\delta}_{i,j}, \quad 1 \leq i < j < n, \\
\psi_{i,j} = -\tilde{\gamma}_{i+1,j+1} - \tilde{\delta}_{i+1,j+1} + \tilde{\gamma}_{i+1,j} + \tilde{\delta}_{i,j}, \quad 1 \leq i < j < n, \\
\varphi_{i,n} = -\tilde{\gamma}_{i+1,n} - \tilde{\delta}_{i,n} + \tilde{\gamma}_{i+1}, \quad \psi_{i,n} = \tilde{\gamma}_{i+1,n} + \tilde{\delta}_{i,n} - \tilde{\gamma}_i. \quad i = 1, \ldots, n.
\]

so that each Gamma function in the integrand depends now not more than of four summands. The exponential factor is now \( \tilde{H}(x, \tilde{\gamma}) = \sum_{j=2}^{n}(\tilde{\gamma}_{1,j} + \tilde{\delta}_{1,j})(x_j - x_{j-1}) - \tilde{\gamma}_1 x_n \).

Finally we have
Theorem 5.2 The function $\Psi_\lambda(x)$ is given by the integral
\[
\Psi_\lambda(x) = \frac{e^{-i(x, \lambda)}}{(2\pi i)^d} \int_C \exp(H(x, \gamma)) \Gamma(\gamma_n + 2i\lambda_n) \Gamma(\gamma_n) d\gamma_n.
\]
\[
\prod_{k=1}^{n-1} \Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k) \Gamma(\gamma_k - \gamma_{k+1}) d\gamma_k.
\]
\[
\prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}.
\]

Here we set $\gamma_{k,l} = \delta_{k,l} = 0$ if the pair $(k, l)$ does not satisfy the condition $1 \leq k < l \leq n$, and $\gamma_k = 0$ if $k = 1$ or $k = n + 1$, $d = n^2$, $H(x, \gamma) = \sum_{j=2}^{n} (\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) - \gamma_1 x_n$, and
\[
\xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad 1 \leq i < j < n,
\]
\[
\eta_{i,j} = \gamma_{i+1,j+1} + \delta_{i+1,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad 1 \leq i < j < n,
\]
\[
\xi_{i,n} = \gamma_{i+1,n} + \delta_{i,n} - \gamma_{i+1}, \quad \eta_{i,n} = -\gamma_{i+1,n} - \delta_{i,n} + \gamma_i, \quad 1 \leq i < n.
\]

The integration cycle is a deformation of the imaginary plain $\text{Re} \gamma_{k,l} = \text{Re} \delta_{k,l} = \text{Re} \gamma_k = 0$ into nonempty domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which is described by the relations $\text{Re} \gamma_{i,j} > 0$, $\text{Re} \delta_{i,j} > 0$, $\text{Re} \xi_{i,j} > 0$, $\text{Re} \eta_{i,j} > 0$, $1 \leq i < j \leq n$, $\text{Re} \gamma_k > 0$, $k = 1, \ldots, n$.

6 $\text{Sp}(2n, \mathbb{R})$

6.1 BZ transform

The split real form of the group $\text{Sp}(2n, \mathbb{C})$ is the group $\text{Sp}(2n, \mathbb{R}) \subset \text{SL}(2n, \mathbb{R})$ preserving skew-symmetric form $\sum_{i=1}^{n} (x_i y_i - x_i y_i)$, where $i = 2n + 1 - i$. Positive roots are elements $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$ and $2\varepsilon_i$, where $\varepsilon_k \in \mathfrak{h}^*$, $k = 1, \ldots, n$ are defined by the condition $\varepsilon_k(e_{i,j} - e_{j,i}) = \delta_{j,k}$. We denote Chevalley generator of the Lie algebra $g = \mathfrak{so}_{2n}$ by $e_i$ and $f_i$, $i = 1, \ldots, n$ Here
\[
e_i = e_{i,i+1} - e_{2n-i,2n+1-i}, \quad f_i = e_{i+1,i} - e_{2n+1-i,2n-i}, \quad i = 1, \ldots, n - 1,
\]
\[
e_n = e_{n,n+1}, \quad f_n = e_{n+1,n}.
\]
Denote by $s_i$ $i = 1, \ldots, n$ the corresponding generators of the Weyl group and their lifts to the group $\text{Sp}(2n, \mathbb{R})$ according to (2.30). The normal ordering of the system $\Delta_+$ copies that of $SO(n + 1, n)$:
\[
2\varepsilon_n, (\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_n), \ldots,
\]
\[
(\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \ldots, \varepsilon_1 + \varepsilon_n, 2\varepsilon_1, 2\varepsilon_1, \varepsilon_1 - \varepsilon_n, \ldots, \varepsilon_1 - \varepsilon_3, \varepsilon_1 - \varepsilon_2).
\]

It corresponds to reduced decomposition of the longest element $w_0^{(n)}$ of the Weyl group $W$ literally the same as (5.1):
\[
w_0^{(n)} = s_n(s_{n-1}s_n s_{n-1}) (s_{n-2}s_n s_{n-1}s_{n-2}) \cdots (s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1).
Note that as element of $\text{Sp}(2n, \mathbb{R})$,

$$w_0^{(n)} = (-1)^{n+1} \left( \sum_{k=1}^{n} e_{kk} - e_{k,k} \right)$$

Denote by $t_{i,j}$ Lusztig parameter corresponding to the root $\varepsilon_i - \varepsilon_j$, and by $s_{i,j}$ Lusztig parameter corresponding to the root $\varepsilon_i + \varepsilon_j$, and by $t_k$ the parameter, corresponding to $2\varepsilon_k$. Here $1 \leq i < j \leq n$, $k = 1, \ldots, n$. Then the group element $X^{(n)}(t)$ has a form (5.2) so that BZ transform for $\text{Sp}(2n, \mathbb{R})$ is the solution of the equation (3.1), where in $X^{(n)}(p)$ we use the notation $p_{i,j}$ for Lusztig parameter, corresponding to the root $\varepsilon_i - \varepsilon_j$, $q_{i,j}$ for the Lusztig parameter, corresponding to the root $\varepsilon_i + \varepsilon_j$, and $p_i$ for the parameters, corresponding to $2\varepsilon_i$, see (5.3).

Keep the notation (5.4). We have

**Theorem 6.1**

a) BZ map for $\text{Sp}(2n, \mathbb{R})$ looks as

$$p_{i,j} = r_{i,j} \prod_{k<i} t_{k,j-1} s_{k,j-1} t_{k,j} \frac{s_{k,j}}{t_{k,j}}, \quad q_{i,j} = r_{i,j} \prod_{k<i} t_{k,j-1} s_{k,j-1} \frac{s_{k,j}}{t_{k,j}}, \quad p_i = \frac{v_{i,n}^2}{t_i} \prod_{k<i} t_{k,j} \frac{s_{k,j}}{t_{k,j}} (6.1)$$

b) The twisting matrix $T \in H$ is

$$T = \sum_{k=1}^{n} \left( T_k e_{k,k} + T_k^{-1} e_{k,k} \right),$$

where

$$T_k = t_k \prod_{j<k} \frac{s_{j,k}}{t_{j,k}} \cdot \prod_{j>k} s_{k,j} t_{k,j}$$

c) BZ transform preserves the measure $\frac{dt}{t}$:

$$\prod_{i<j} dt_{i,j} ds_{i,j} \prod_k \frac{dt_k}{t_k} = \prod_{i<j} \frac{dp_{i,j} dq_{i,j}}{p_i q_{i,j}} \prod_k \frac{dp_k}{p_k}.$$

Again, we have the same remarks with the same proofs:

**Remark 1.** BZ transform $\tau$ is involutive, that is the inverse relation

$$t_{i,j} = \tilde{r}_{i,j} \prod_{k<i} \frac{p_{k,j-1}}{q_{k,j-1}} \prod_{k<i} \frac{q_{k,j}}{p_{k,j}}, \quad s_{i,j} = \tilde{r}_{i,j} \prod_{k<i} \frac{p_{k,j-1}}{q_{k,j-1}} \prod_{k<i} \frac{q_{k,j}}{p_{k,j}}, \quad t_i = \frac{v_{i,n}^2}{p_i q_{i,n}^2} \prod_{k<i} \frac{p_{k,n}^2}{q_{k,n}} (6.2)$$

has the same form as (4.3). Here

$$v_{i,j} = p_{i,j} + q_{i,j}, \quad \tilde{r}_{i,j} = \frac{v_{i,j-1}}{v_{i,j} q_{i,j-1}}.$$

**Remark 2.** The Cartan twist $T$ can be as well written as $T(t) = T(p^{-1})$, that is

$$T = \sum_{k=1}^{n} \left( T_k e_{k,k} + T_k^{-1} e_{k,k} \right),$$

where

$$T_k = \frac{1}{p_k} \prod_{j<k} \frac{q_{j,k}}{p_j p_k} \prod_{j>k} \frac{1}{q_{j,k} p_{j,k}}$$
Corollary 6.1 The restriction of left Whittaker vector to $N_+$ is given by the function
\[ \omega^L_\nu(t) = t^\nu \cdot \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) - \sum_k p_k \right) = p^{-\nu} \cdot \exp \left( - \sum_{i<j} (p_{i,j} + q_{i,j}) - \sum_k p_k \right) \]
where $p_{i,j}$ and $p_k$ are given by (6.1) and $t^\nu$ for a weight $\nu = \sum_{k=1}^n \nu_k \varepsilon_k$ means the product
\[ t^\nu = \prod_{i<j} t_{i,j}^{\nu_i - \nu_j} s_{i,j}^{\nu_i + \nu_j} \prod_k t_k^{\nu_k} \]
For inductive proof Theorem 6.1 we use the factorization $X^{(n)}(-t) = \tilde{X}^{(n-1)}(-t) \cdot \tilde{A}^{(n)}(-t)$, where $\tilde{X}^{(n-1)}(-t)$ and $\tilde{A}^{(n)}(-t)$ are given by the expressions (5.7) and (5.8). Denote by $\Lambda^{(n)}$ the diagonal matrix $\Lambda^{(n)} = \sum_{k=1}^n (\Lambda_{k,k}^{(n)} e_k e_k + \Lambda_{k,k}^{(n-1)} e_k^2)$, where
\[ \Lambda_{1,1}^{(n)} = t_1 \prod_{k=1}^{n-1} t_{1,k} s_{1,k}, \quad \Lambda_{k,k}^{(n)} = \frac{t_{1,k}}{s_{1,k}}, \quad 1 < k \leq n, \]
and define the variables $\tilde{t}_{i,j}$ and $\tilde{s}_{i,j}$, $1 < i < j \leq n$ by the relations
\[ \tilde{t}_{i,j} = t_{i,j} \cdot \frac{s_{1,j-1}}{s_{1,j}}, \quad \tilde{s}_{i,j} = s_{i,j} \cdot \frac{s_{1,j-1}}{s_{1,j}} \cdot \frac{t_{1,j}}{t_{1,j-1}}, \quad \tilde{t}_i = t_i \cdot \frac{s_{1,n}^2}{t_{1,n}^2} \]
Then we have again

**Lemma 6.1** Non-diagonal matrix elements $U_{i,j}$ ($i \neq j$) of the matrix (3.13) equal zero if $j > 1$ and $i < 2n + 1$. Matrix element $U_{1,1}$ equals $t_1 \prod_{j>1} t_{1,j} s_{1,j}$. Matrix element $U_{k,k}$ equals $\frac{s_{1,k}}{t_{1,k}}$ for $1 < k \leq n$. Matrix element $U_{k,k}$ equals $\frac{t_{1,k}}{s_{1,k}}$ for $1 < k \leq n$.

**Proposition 6.1** a) The parameters $p_{1,j}$, $q_{1,j}$, $j = 2, \ldots, n$ and $p_1$ of $A^{(n)}(p)$ are equal to
\[ p_{1,j} = \frac{u_{1,j-1}}{u_{1,j} s_{1,j-1}}, \quad q_{1,j} = \frac{u_{1,j-1} s_{1,j}}{u_{1,j} s_{1,j-1} t_{1,j}}, \quad p_1 = \frac{u_{1,n}^2}{t_1 s_{1,n}} \]
b) Cartan twist $T$ and the matrix $\tilde{X}^{(n-1)}(p)$ satisfy the relation
\[ T \tilde{X}^{(n-1)}(p) = \Lambda^{(n)} \cdot \left( \tilde{X}^{(n-1)}(-\tilde{t}) \tilde{w}_0^{(n-1)} \right)_{0+} \]
These statement are proved in the same manner as in Section 3. They are sufficient for inductive proof of Theorem 6.1. \[ \square \]
6.2 Whittaker function

We start again with the action of Cartan generators on the right Whittaker vector. The vector \( \exp(\sum_{k=1}^{n} x_k(e_{k,k} - e_{k,k})) \cdot v_R^\mu \) is presented by the function \( \exp(-\sum_k \mu_k x_k) \omega^R_\mu(t') \), where

\[
t'_i = t_i \cdot \exp(x_i - x_{i-1}), \quad s'_{i,j} = s_{i,j} \cdot \exp(x_j - x_{j-1}), \quad t_i' = t_i \cdot \exp(-2x_n). \tag{6.3}
\]

Denote by \( \hat{\omega}^L_\nu(\gamma) \) and \( \hat{\omega}^R_\mu(\gamma) \) the Mellin transforms of the functions \( \omega^L_\nu(t) \) and \( \omega^R_\mu(t) \),

\[
\hat{\omega}^R_\mu(\gamma) = \int_{t>0} \frac{\omega^R_\mu(t) t^\gamma dt}{t} := \int_{t_{i,j}>0, s_{i,j}>0, t_i>0} \omega^R_\mu(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}} s_{i,j}^{\delta_{i,j}} t_i^{\gamma_i} ds_{i,j} dt_i,
\]

\[
\hat{\omega}^L_\nu(\gamma) = \int_{t>0} \frac{t^\nu \omega^L_\nu(t) t^{\gamma} dt}{t} := \int_{t_{i,j}>0, s_{i,j}>0, t_i>0} \omega^L_\nu(t) \prod_{i<j} t_{i,j}^{\gamma_{i,j}+\nu_{i,j}-\nu_{j}} s_{i,j}^{\delta_{i,j}+\nu_{j}} t_i^{\gamma_{i,j}+\nu_{i,j}+\nu_{j}+\nu_j} ds_{i,j} dt_i.
\]

Here \( \rho = \sum_{k=1}^{n} (n + 1 - k) (\varepsilon_{k,k} - \varepsilon_{k,k}) \). Due to (6.3) the action of Cartan subgroup on the right Whittaker vector \( v_R^\mu \) transforms the function \( \hat{\omega}^R_\mu(\gamma) \) to the product \( \exp(-\sum_k \mu_k x_k) \cdot \exp(\hat{H}(x, \gamma) \hat{\omega}^R_\mu(\gamma)) \), where

\[
\hat{H}(x, \gamma) = \sum_{i<j} (\gamma_{i,j} + \delta_{i,j})(x_j - x_{j-1}) - 2x_n \sum_i \gamma_i. \tag{6.4}
\]

The right Whittaker vector is presented by the function \( \hat{\omega}^R_\mu(\gamma) \) of the form (5.11). For the calculation of \( \hat{\omega}^L_\nu(\gamma) \) we pass in the integral

\[
\int_{t_{i,j}>0, s_{i,j}>0, t_i>0} \prod_{i<j} t_{i,j}^{\gamma_{i,j}+\nu_{i,j}+\nu_j} s_{i,j}^{\delta_{i,j}+\nu_{j}} e^{-p_{i,j} q_{i,j}} ds_{i,j} dt_i
\]

from the integration variables \( t_{i,j}, s_{i,j} \) and \( t_i \) to \( p_{i,j}, q_{i,j} \) and \( p_i \). Here

\[
\nu' = \nu + \rho, \quad \nu'_i = \nu_i + n - i + 1.
\]

Substitution of (6.2) gives the relation

\[
\prod_i t_i^{\gamma_i} \prod_{i<j} t_{i,j}^{\delta_{i,j}} s_{i,j}^{\gamma_{i,j}} = \prod_i p_i^{\varphi_i} \prod_{i<j} p_{i,j}^{\psi_{i,j}} q_{i,j}^{\theta_{i,j}} (p_{i,j} + q_{i,j})^{\delta_{i,j}}
\]

where as before

\[
\varphi_{i,j} = -\delta_{i,j} + \sum_{k>i} (\gamma_{k,j+1} + \delta_{k,j+1}) - \sum_{k>i} (\gamma_{k,j} + \delta_{k,j}),
\]

\[
\psi_{i,j} = \delta_{i,j} - \sum_{k>i} (\gamma_{k,j+1} + \delta_{k,j+1}) + \sum_{k>i} (\gamma_{k,j} + \delta_{k,j}), \tag{6.5}
\]

\[
\theta_{i,j} = -\delta_{i,j} - \gamma_{i,j} + \delta_{i,j+1} + \gamma_{i,j+1},
\]

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for \(1 \leq i < j < n\), and

\[
\varphi_{i,n} = -\delta_{i,n} - \sum_{k>i} (\gamma_{kn} + \delta_{kn} - 2\gamma_k),
\]

\[
\psi_{i,n} = \delta_{i,n} - 2\gamma_i + \sum_{k>i} (\gamma_{kn} + \delta_{kn} - 2\gamma_k),
\]

\[
\theta_{i,n} = 2\gamma_i - \delta_{i,n} - \gamma_{i,n}, \quad \varphi_i = -\gamma_i
\]

for \(1 \leq i < n\). Using the (4.14) we present the multiple integral

\[
\int_{\pi_{i,j},q_{i,j},p_{i,j}>0} \prod_{i<j} \frac{\varphi_{i,j}-\nu'+\nu'_j}{q_{i,j}} \varphi_{i,j}-\nu'-\nu'_j (p_{i,j} + q_{i,j}) \frac{dp_{i,j}}{p_{i,j}} \frac{dq_{i,j}}{q_{i,j}} \prod_i \frac{\varphi_i-\nu'}{p_i}
\]

as the product

\[
\prod_{k=1}^{n} \frac{\Gamma(-\gamma_k - \nu_k)}{\Gamma(-\gamma_k - \delta_{i,n} - 2\gamma_k)}.
\]

Substitution of (6.5) into the product over \(1 \leq i < j < n\) of ratios of \(\Gamma\) functions in the latter expression results, just as for \(\text{SO}(n,n)\) to the product

\[
\prod_{i=1}^{n-1} \frac{\Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2n'\nu_i)}{\Gamma(-\gamma_i - \delta_{i,n} - 2\gamma_i)}.
\]

while due to (6.6) the same product over \(j = n, i = 1, ..., n - 1\) is

\[
\prod_{i=1}^{n-1} \frac{\Gamma(-\gamma_i - \delta_{i,n} - 2\gamma_i)}{\Gamma(-2\gamma_i - 2\gamma_i)}.
\]

Thus we have the following expressions for the left Whittaker vector

\[
\hat{\phi}^L_{\nu'}(\gamma) = \Gamma(-\gamma_n - \nu'_n) \prod_{i=1}^{n-1} \frac{\Gamma(-\gamma_{i,i+1} - \delta_{i,i+1} - 2\nu'_i)}{\Gamma(-2\gamma_i - 2\gamma_i)} \prod_{1\leq i < j \leq n} \frac{\Gamma(\varphi_{i,j} - \nu'_i + \nu'_j) \Gamma(\psi_{i,j} - \nu'_i - \nu'_j)}{\Gamma(\varphi_{i,j} - \nu'_i - \nu'_j) \Gamma(\psi_{i,j} - \nu'_i + \nu'_j)}
\]

and for the Whittaker wave function:

\[
\Psi_\lambda(x) = \frac{e^{-i(\lambda x)}}{(2\pi i)^d} \int_C \exp H(x, \gamma) \prod_{k=1}^{n} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(2\gamma_k + 2i\lambda_k)} \cdot \prod_{k=1}^{n} \Gamma(\gamma_k + i\lambda_k) \Gamma(\gamma_k) d\gamma_k.
\]

\[
\prod_{1\leq k < l \leq n} \Gamma(-\varphi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(-\psi_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l}) \Gamma(\delta_{k,l}) d\gamma_{k,l} d\delta_{k,l}.
\]

Here \(H(x, \gamma)\) is given in (5.10), \(\varphi_{i,j}\) and \(\phi_{i,j}\) are given in (5.12) and (5.13), \(d = n^2\). The contour \(C\) is the same as in the previous section.
The change of variables (5.14) simplifies the formula for Whittaker wave functions. Now we have

$$\varphi_{i,j} = \bar{\gamma}_{i+1,j+1} + \bar{\delta}_{i+1,j+1} - \bar{\gamma}_{i,j} - \bar{\delta}_{i,j}, \quad 1 \leq i < j \leq n,$$

$$\psi_{i,j} = -\bar{\gamma}_{i,j+1} - \bar{\delta}_{i,j+1} + \bar{\gamma}_{i+1,j} + \bar{\delta}_{i+1,j}, \quad 1 \leq i < j \leq n,$$

$$\varphi_{i,n} = -\bar{\gamma}_{i+1,n} - \bar{\delta}_{i,n} + 2\bar{\gamma}_{i+1} + \bar{\delta}_{i,n} - 2\bar{\gamma}_i, \quad 1 \leq i < n,$$

Finally we have

**Theorem 6.2** The function $\Psi_\lambda(x)$ is given by the integral

$$\Psi_\lambda(x) = \frac{e^{-i(x,\lambda)}}{(2\pi i)^d} \int_C \exp H(x, \gamma) \prod_{k=1}^{n-1} \frac{\Gamma(\gamma_{k,k+1} + \bar{\delta}_{k,k+1} + 2i\lambda_k)}{\Gamma(2\gamma_k - 2\gamma_{k+1} + 2\lambda_k)} \prod_{k<l} (\Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}) d\gamma_{k,l} d\delta_{k,l}).$$

Here we set $\gamma_{k,l} = \delta_{k,l} = 0$ if the pair $(k, l)$ does not satisfies the condition $1 \leq k < l \leq n$, and $\gamma_k = 0$ if $k = 1$ or $k = n+1$, $d = n^2$ is the dimension of the maximal unipotent subgroup of $\text{Sp}(2n, \mathbb{R})$, $H(x, \gamma) = \sum_{j=2}^{n}(\gamma_{1,j} + \delta_{1,j})(x_j - x_{j-1}) - 2\gamma_1 x_n$,

$$\xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i,j} + \delta_{i,j}, \quad 1 \leq i < j < n,$$

$$\eta_{i,j} = \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i+1,j}, \quad 1 \leq i < j < n,$$

$$\xi_{i,n} = \gamma_{i+1,n} + \delta_{i,n} - 2\gamma_{i+1}, \quad 1 \leq i < n,$$

$$\eta_{i,n} = -\gamma_{i+1,n} - \delta_{i,n} - 2\gamma_i.$$

The integration cycle is a deformation of the imaginary plain $\text{Re} \, \gamma_{k,l} = \text{Re} \, \delta_{k,l} = \text{Re} \, \gamma_k = 0$ into nonempty domain $D \subset \mathbb{C}^d$ of the analyticity of the integrand, which is described by the relations

$$\text{Re} \, \gamma_{i,j} > 0, \text{ Re} \, \delta_{i,j} > 0, \text{ Re} \, \xi_{i,j} > 0, \text{ Re} \, \eta_{i,j} > 0, \quad 1 \leq i < j \leq n, \quad \text{Re} \, \gamma_k > 0, \quad k = 1, \ldots, n.$$

### 7 Mellin transforms of Whittaker functions

The presentations of Whittaker functions given in Theorems 3.2, 4.2, 5.2, 6.2, have a form which is easy to interpret as an inverse Mellin transform. This enables one to write down precise expressions for direct Mellin transforms of Whittaker functions.

1. **GL(n)** Using the notations of Section 3.2 we denote $z_k = \exp x_k$, $z_{k,k+1} = \exp(x_k - x_{k+1})$, $z = \{z_1, \ldots, z_n\}$. Set

$$\Lambda_i = \frac{(n-1)\lambda_i - \sum_{j \neq i} \lambda_j}{n}, \quad \Lambda_{i,j} = \Lambda_i + \Lambda_{i+1} + \ldots + \Lambda_j.$$
and put \( s_{k,k+1} = \tilde{\gamma}_{1,n+1-k} + \Lambda_{1,k} \). Then the formula (1.2) can be written as follows

\[
\Psi_{\lambda}(z) = \frac{(z_1 \cdots z_n)^{-i(\lambda_1 + \cdots + \lambda_n)}}{(2\pi i)^{n-1}} \int_{\Re s_{k,k+1}=0^+} z_1^{-s_1,2} \cdots z_n^{-s_{n-1,n}} M^n_{\lambda}(s_1, \ldots, s_{n-1,n}) ds_1 \cdots ds_{n-1,n}
\]

where

\[
M^n_{\lambda}(s_1, \ldots, s_{n-1,n}) = \frac{1}{(2\pi i)^{(n-1)(n-2)/2}} \int_C \prod_{k=1}^{n-1} \frac{\Gamma(s_{k,k+1} - \gamma_{2,n+1-k} + i(\lambda_1 - \Lambda_{1,k})) \Gamma(s_{k,k+1} - \gamma_{2,n+1-k} - i\Lambda_{1,k})}{\prod_{1<k<l\leq n} \Gamma(\gamma_{kl} - \gamma_{k+1,l} + i(\lambda_k - \Lambda_{n+k-l+1}))} ds_{k,k+1}^{\lambda_{n+1-k} - 1} \cdots ds_{n-1,n}^{n-1}
\]

The integration contour \( C \) is a deformation of the imaginary plane \( \Re \gamma_{ij} = 0 \) to the region of analyticity of the integrand. The function \( M^n_{\lambda}(s_1, \ldots, s_{n-1,n}) \) is then equal to the Mellin transform of \( SL(n) \) Whittaker function \( \tilde{\Psi}_{\lambda}(z) = \Psi_{\lambda}(z) \cdot (z_1 \cdots z_n)^{i(\lambda_1 + \cdots + \lambda_n)} \):

\[
M^n_{\lambda}(s_1, \ldots, s_{n-1,n}) = \int_{z_{i,j}>0} \tilde{\Psi}_{\lambda}(z) z_1^{s_1,2-1} \cdots z_n^{s_{n-1,n}-1} dz_1 \cdots dz_{n-1,n}
\]

The relation (7.1) can be taken as a starting point for iterative construction of the Mellin transform of \( SL(n) \) part of the Whittaker function \( \tilde{\Psi}_{\lambda}(z) \).

2. \( SO(n, n) \). Using the notations of Section 4.2 denote \( z_n = \exp(x_{n-1} + x_n) \) and \( z_i = \exp(x_i - x_{i+1}) \) for \( 1 \leq i \leq n-1 \). Set

\[
\Lambda_i = \lambda_1 + \cdots + \lambda_i, \quad 1 \leq i = 1 \leq n - 2,
\]

\[
\Lambda_{n-1} = \frac{\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n}{2}, \quad \Lambda_n = \frac{\lambda_1 + \cdots + \lambda_{n-1} - \lambda_n}{2},
\]

\[
s_{n-1} = \tilde{\gamma}_{1,n} + i\Lambda_{n-1}, \quad s_n = \tilde{\delta}_{1,n} + i\Lambda_{n},
\]

\[
s_i = \tilde{\gamma}_{1,i+1} + \tilde{\delta}_{1,i+1} + i\Lambda_i, \quad t_i = \tilde{\gamma}_{1,i+1} - \tilde{\delta}_{1,i+1}, \quad 1 \leq i \leq n - 2,
\]

\[
\tilde{s}_j = s_i - i\Lambda_j = \tilde{\gamma}_{1,j} + \tilde{\delta}_{1,j}, \quad 1 \leq j \leq n.
\]

In this notations the formula of Theorem 4.2 for the Whittaker function is written as the inverse Mellin transform:

\[
\Psi_{\lambda}(z) = \frac{1}{(2\pi i)^n} \int_{\Re s_i=0^+} z_1^{-s_1,2} \cdots z_n^{-s_n} M^n_{\lambda}(s_1, \ldots, s_n) ds_1 \cdots ds_n,
\]

where

\[
M^n_{\lambda}(s_1, \ldots, s_n) = \frac{1}{(2\pi i)^{(n-1)^2}} \int_{C_n} \frac{\Gamma(s_1 + 2i\lambda_1) \Gamma(s_{n-1} - \gamma_{2,n}) \Gamma(s_n - \delta_{2,n})}{\Gamma(s_{n-1} + s_n - \gamma_{2,n} - \delta_{2,n} + 2i\lambda_1)} ds_{k,k+1}^{n-2} \prod_{k=1}^{n-2} \Gamma\left(\frac{s_k + t_k}{2} - \gamma_{2,k+1}\right) \Gamma\left(\frac{s_k - t_k}{2} - \delta_{2,k+1}\right) L_{\lambda}^{n-2} dt_k \prod_{2 \leq k < l \leq n} d\gamma_{k,l} d\delta_{k,l}
\]
The contour $C$ is a deformation of the imaginary plane to the region of analyticity of the integrand. Here

$$
L^1_{\lambda} = \prod_{k=2}^{n} \frac{\Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k)}{\Gamma(\gamma_{k,n} + \delta_{k,n} - \gamma_{k+1,n} - \delta_{k+1,n} + 2i\lambda_k)} \prod_{1<k<l\leq n} \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l})
$$

$$
\prod_{k=2}^{n} \Gamma(\gamma_k - \gamma_{k+1}) \prod_{1\leq k<l\leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l))
$$

with

$$
\xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad i < j < n,
$$

$$
\eta_{i,j} = \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad i < j < n,
$$

$$
\xi_{i,n} = \gamma_{i,n} - \delta_{i+1,n}, \quad \eta_{i,n} = \delta_{i,n} - \gamma_{i+1,n}, \quad i < n,
$$

for $i > 1$ and

$$
\xi_{1,j} = -\gamma_{2,j+1} - \delta_{2,j+1} + \gamma_{2,j} + \frac{s_{j-1} - t_{j-1}}{2}, \quad 1 < j < n,
$$

$$
\eta_{1,j} = s_j - \gamma_{2,j} - \frac{s_{j-1} + t_{j-1}}{2}, \quad 1 < j < n,
$$

$$
\xi_{1,n} = s_{n-1} - \delta_{2,n}, \quad \eta_{1,n} = s_n - \gamma_{2,n}.
$$

3. $\text{SO}(n+1, n)$. Using the notations of Section 5.2 denote $z_n = \exp(x_n)$ and $z_i = \exp(x_i - x_{i+1})$ for $1 \leq i \leq n - 1$. Set

$$
\Lambda_i = \lambda_1 + \ldots + \lambda_i, \quad 1 \leq i = 1 \leq n,
$$

$$
s_i = \gamma_{1,i+1} + \delta_{1,i+1} + i\Lambda_i, \quad t_i = \gamma_{1,i+1} - \delta_{1,i+1}, \quad 1 \leq i \leq n - 1,
$$

$$
s_n = \gamma_1 + i\Lambda_n, \quad s_i = s_i - i\Lambda_i \quad 1 \leq i \leq n.
$$

In this notations the formula of Theorem 5.2 for the Whittaker function looks as follows:

$$
\Psi_{\lambda}(z) = \frac{1}{(2\pi i)^n} \int_{\text{Re } s_i = 0^+} z_1^{-s_1} \cdots z_n^{-s_n} M_{\lambda, n}^{\lambda}(s_1, \ldots, s_n) ds_1 \cdots ds_n,
$$

where

$$
\Psi_{\lambda}(x) = \frac{1}{(2\pi i)^{n(n-1)}} \int_{C} \prod_{k=1}^{n-1} \Gamma \left( \frac{s_k + t_k}{2} - \gamma_{2,k+1} \right) \Gamma \left( \frac{s_k - t_k}{2} - \delta_{2,k+1} \right).
$$

$$
\Gamma(s_1 + 2i\lambda_1) \Gamma(s_n - \gamma_2) \cdot L^n_{\lambda} \prod_{k=1}^{n-1} dt_k \prod_{2 \leq k \leq l \leq n} d\gamma_{k,l} d\delta_{k,l} \prod_{k=2}^{n} d\gamma_k
$$

The contour $C$ is a deformation of imaginary plane to the region of analyticity of the integrand. Here

$$
L^n_{\lambda} = \Gamma(\gamma_n + 2i\lambda_n) \prod_{k=2}^{n} \Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k) \prod_{k=2}^{n} \Gamma(\gamma_k - \gamma_{k+1}).
$$

$$
\prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \prod_{2 \leq k < l \leq n} \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l})
$$
with

\[ \xi_{i,j} = -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad i < j < n, \]
\[ \eta_{i,j} = \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad i < j < n, \]
\[ \xi_{i,n} = \gamma_{i,n} - \delta_{i+1,n}, \quad \eta_{i,n} = \delta_{i,n} - \gamma_{i+1,n} \quad i < n, \]

for \( i > 1 \) and

\[ \xi_{1,j} = -\gamma_{2,j+1} - \delta_{2,j+1} + \gamma_{2,j} + \frac{s_{j-1} - t_{j-1}}{2}, \quad 1 < j < n, \]
\[ \eta_{1,j} = s_{j} + \gamma_{2,j} - \frac{s_{j-1} + t_{j-1}}{2}, \quad 1 < j < n, \]
\[ \xi_{1,n} = \frac{s_{1} - t_{1}}{2} - \gamma_{2,n}, \quad \eta_{1,n} = -\frac{s_{1} - t_{1}}{2} + s_{1} - \gamma_{2,n}. \]

**4. \( \mathbf{Sp}(2n, \mathbb{R}) \).** Using the notations of Section 6.2 denote again \( z_n = \exp(x_n) \) and \( z_i = \exp(x_i - x_{i+1}) \) for \( 1 \leq i \leq n - 1 \). Set

\[ \Lambda_i = \lambda_1 + \ldots + \lambda_i, \quad 1 \leq i = 1 \leq n, \]
\[ s_i = \gamma_{1,i+1} + \delta_{1,i+1} + i\Lambda_i, \quad t_i = \gamma_{1,i+1} - \delta_{1,i+1}, \quad 1 \leq i \leq n - 1, \]
\[ s_n = 2\gamma_1 + i\Lambda_n, \quad s_i = s_i - i\Lambda_i, \quad 1 \leq i \leq n. \]

In this notations the formula of Theorem 6.2 looks as follows:

\[ \Psi_{\Lambda}(z) = \frac{1}{(2\pi i)^n} \int_{\Re s_i = 0^+} z_1^{-s_1} \cdots z_n^{-s_n} M_{\Lambda}^n(s_1, \ldots, s_n) ds_1 \cdots ds_n, \]

where

\[ \Psi_{\Lambda}(x) = \frac{1}{(2\pi i)^{n(n-1)}} \int_C \prod_{k=1}^{n-1} \Gamma \left( \frac{s_k + t_k}{2} - \gamma_{2,k+1} \right) \Gamma \left( \frac{s_k - t_k}{2} - \delta_{2,k+1} \right). \]

\[ \frac{\Gamma(s_n - \gamma_2 + i\lambda_1)}{\Gamma(s_n - 2\gamma_2 + 2i\lambda_1)} \Gamma(s_1 + 2i\lambda_1) \Gamma(s_n - \gamma_2) \cdot L_{\Lambda}^n \prod_{k=1}^{n-1} dt_k \prod_{2 \leq k < l \leq n} d\gamma_{k,l} \prod_{k=2}^{n} d\gamma_{k} \]

The contour \( C \) is a deformation of the imaginary to the region of analyticity of the integrand. Here

\[ L_{\Lambda}^n = \Gamma(\gamma_n + 2i\lambda_n) \prod_{k=2}^{n-1} \Gamma(\gamma_{k,k+1} + \delta_{k,k+1} + 2i\lambda_k) \prod_{k=2}^{n} \Gamma(\gamma_k - \gamma_{k+1}). \]

\[ \prod_{1 \leq k < l \leq n} \Gamma(\xi_{k,l} + i(\lambda_k - \lambda_l)) \Gamma(\eta_{k,l} + i(\lambda_k + \lambda_l)) \prod_{2 \leq k < l \leq n} \Gamma(\gamma_{k,l} - \gamma_{k+1,l}) \Gamma(\delta_{k,l} - \delta_{k+1,l}). \]

\[ \Gamma(\gamma_n + i\lambda_n) \prod_{k=2}^{n-1} \frac{\Gamma(\gamma_k - \gamma_{k+1} + i\lambda_k)}{\Gamma(\gamma_k - \gamma_{k+1} + 2i\lambda_k)} \]

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with
\[
\begin{align*}
\xi_{i,j} &= -\gamma_{i+1,j+1} - \delta_{i+1,j+1} + \gamma_{i+1,j} + \delta_{i,j}, \quad i < j < n, \\
\eta_{i,j} &= \gamma_{i,j+1} + \delta_{i,j+1} - \gamma_{i+1,j} - \delta_{i,j}, \quad i < j < n, \\
\xi_{i,n} &= \gamma_{i,n} - \delta_{i+1,n}, \quad \eta_{i,n} = \delta_{i,n} - \gamma_{i+1,n} \quad i < n,
\end{align*}
\]
for \(i > 1\) and
\[
\begin{align*}
\xi_{1,j} &= -\gamma_{2,j+1} - \delta_{2,j+1} + \gamma_{2,j} + \frac{s_{j-1} - t_{j-1}}{2}, \quad 1 < j < n, \\
\eta_{1,j} &= s_j + \gamma_{2,j} - \frac{s_{j-1} + t_{j-1}}{2}, \quad 1 < j < n, \\
\xi_{1,n} &= \frac{s_1 - t_1}{2} - 2\gamma_2 + \gamma_{2,n}, \quad \eta_{1,n} = -\frac{s_1 - t_1}{2} + s_1 - \gamma_{2,n}.
\end{align*}
\]

**Appendix**

**A  Proof of Lemma 3.1**

The proof essentially consists of calculation of the product of two matrices, \(U = \tilde{A} \cdot (A^{(n)}(p))^{-1}\), where \(\tilde{A} = (\bar{w}_0^{(n-1)})^{-1}A^{(n)}(-t)\bar{w}_0^{(n)}\) with matrix elements
\[
\tilde{A}_{i,j} = \begin{cases} 
(-1)^{n-j}A^{(n)}_{1,n+1-j}(-t), & i = 1, \\
(-1)^{i+j}A^{(n)}_{n+2-i,n+1-j}(-t) & i > 1,
\end{cases}
\]
and
\[
(A^{(n)})^{-1}(p) = \begin{pmatrix}
1 & -p_{1,n} & 0 & \ldots & 0 \\
0 & 1 & -p_{1,n-1} & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 & -p_{1,2} \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]
so that \((A^{(n)})^{-1}i,j(p) = -p_{1,i}\), if \(j = i + 1\), and \((A^{(n)})^{-1}i,j(p) = 0\), if \(j > i + 1\). We then have
\[
U_{1,j} = (-1)^{n-j}A^{(n)}_{1,n+1-j}(-t) + (-1)^{n-j+1}A^{(n)}_{1,n+2-j}(-t) \cdot (-p_{1,j+1}), \quad i = 1,
\]
\[
U_{i,j} = (-1)^{i+j}A^{(n)}_{n+2-i,n+1-j}(-t) + (-1)^{i+j+1}A^{(n)}_{n+2-i,n+2-j}(-t) \cdot (-p_{1,j+1}), \quad i > 1
\]
so that the relations (3.12) and (3.10) imply the equalities
\[
U_{i,1} = t_{1,2}t_{1,3} \cdots t_{1,n} = \frac{1}{p_{1,2}p_{1,3} \cdots p_{1,n}}, \quad U_{ii} = p_{1,i+1} = \frac{1}{t_{1,i}}, \quad i > 1,
\]
\[
U_{ij} = 0, \quad i \neq j, \quad j > 1.
\]
\[\square\]
B Proof of Lemma 4.1

This proof is analogous to that of Lemma 3.1 with more technical details which differ for \( n \) even and odd. Assume first that \( n \) is even. Denote for simplicity of notations entries of the matrix \( A^{(n)}(-t) \) by \( A_{i,j} \), entries of the matrix \( A^{(n)}(p) \) by \( C_{i,j} \), variables \( t_{1,j}, s_{1,j}, u_{1,j}, p_1, q_1, v_{1,j} \) by \( t_j, s_j, u_j, p_j, q_j \), and \( v_j \) correspondingly. The upper triangular matrix \( A^{(n)}(-t) \) has a natural block structure with matrix coefficients equal to

\[
A_{i,j} = (-1)^{i+j}s_is_{i+1} \cdots s_{j-1}u_j, \quad 1 \leq i \leq j \leq n,
A_{i,j} = t_{j+1}t_{j+2} \cdots t_{i-1}u_i, \quad 1 \leq j \leq i \leq n,
A_{i,j} = (-1)^j(s_ns_{n-1} \cdots s_{i+1})(t_{n}t_{n-1} \cdots t_{j+1}) \quad 1 \leq i < n, \quad 1 \leq j \leq n, \tag{B.1}
A_{n,n} = 0, \quad A_{n,j} = (-1)^j(s_nt_{n-1} \cdots t_{j+1}), \quad 1 \leq j < n,
A_{i,j} = 0, \quad 1 \leq i, j \leq n
\]

Using (4.1) we can express the matrix \( \tilde{A} = \left( \tilde{w}_0^{(n-1)} \right)^{-1} A^{(n)}(-t) \tilde{w}_0^{(n)} \) as the sum

\[
-\tilde{A} = \sum_{i,j; i \neq 1} (A_{ik}e_{i\bar{k}} + A_{i\bar{k}}e_{ik} + A_{ik}e_{\bar{k}i}) + \sum_k (A_{1,k}e_{1\bar{k}} + A_{1\bar{k}}e_{1,k} + A_{nk}e_{nk} + A_{nk}e_{\bar{n}k}) + e_{11} + e_{n\bar{n}}.
\]

Next, for any matrix \( X \in SO(n,n) \) we have the relation

\[
(X^{-1})_{i,j} = X_{ji}
\]

so the matrix coefficients \( U_{i,j} \) of the matrix \( U = \left( \tilde{w}_0^{(n-1)} \right)^{-1} A^{(n)}(-t) \tilde{w}_0^{(n)} (A^{(n)}(p))^{-1} \) are

\[
U_{i,j} = -\sum_k A_{ik}C_{j\bar{k}}, \quad U_{i,j} = -\sum_k A_{ik}C_{\bar{j}k},
U_{i,j} = -\sum_k A_{ik}C_{j\bar{k}}, \quad U_{i,j} = -\sum_k (A_{ik}C_{j\bar{k}} + A_{ik}C_{j\bar{k}}).
\]

and

\[
U_{1,j} = \sum_k (A_{1,k}C_{j,k} + A_{1\bar{k}}C_{j\bar{k}}), \quad U_{n,j} = C_{jn} + \sum_k A_{nk}C_{j\bar{k}}, \quad U_{1,j} = C_{\bar{j}1},
U_{1,j} = \sum_k A_{1,k}C_{\bar{j}k}, \quad U_{n,j} = \sum_k A_{nk}C_{\bar{j}k}, \quad U_{1,j} = C_{j1},
U_{n,j} = \sum_k A_{n,k}C_{\bar{j}k}, \quad U_{n,j} = \sum_k A_{n,k}C_{\bar{j}k},
\]

Then the proof reduces to the check of identities

\[
\sum_k A_{ik}C_{j\bar{k}} = 0, \quad \sum_k A_{ik}C_{\bar{j}k} = 0, \quad \sum_k A_{ik}C_{j\bar{k}} = 0, \quad \sum_k A_{ik}C_{\bar{j}k} = -\delta_{i,j} \frac{s_i}{t_i}, \quad \sum_k (A_{ik}C_{j\bar{k}} + A_{ik}C_{\bar{j}k}) = -\delta_{i,j} \frac{t_i}{s_i} \tag{B.2}
\]

(4.1)
for \( i, j \neq 1, n \), and of special cases

\[
U_{1,j} = U_{i,j} = U_{j,1} = U_{j,1} = 0, \quad j \neq 1, \quad U_{n,n} = -\frac{s_n}{t_n}, \quad U_{n,n} = -\frac{t_n}{s_n}
\]

\( U_{n,j} = 0, \quad j \neq 1, n, \quad U_{n,j} = 0, U_{n,j} = 0, \quad j \neq n, \quad U_{n,j} = 0, \quad \hat{j} \neq n. \)

We check here several of them. First (B.2). According to (4.7),

\[
p_i = -\frac{u_{i-1}}{u_is_{i-1}}, \quad q_i = -\frac{u_{i-1}s_i}{u_is_{i-1}}, \quad v_i = -\frac{u_{i-1}}{t_is_{i-1}}
\]  

(B.4)

Here we assume \( s_1 = t_1 = u_1 = 1 \). The relations (B.2) reduce to

\[
1 + u_ip_i (1 + p_{i-1}t_{i-1} + p_{i-2}p_{i-1}t_{i-1}t_i + \cdots + (p_2 \cdots p_{i-1})(t_2 \cdots t_{i-1})) = 0 \quad (B.5)
\]

Substitute (B.4):

\[
1 + \left( \frac{-u_{i-1}}{s_{i-1}} + \frac{u_{i-2}t_{i-1}}{s_{i-1}s_{i-2}} + \cdots + (-1)^{i-1} \frac{t_{i-1}t_2}{s_{i-1} \cdots s_1} \right) = \]

\[
1 - \left( 1 + \frac{t_{i-1}}{s_{i-1}} \right) + \frac{t_{i-1}}{s_{i-1}} \left( 1 + \frac{t_{i-2}}{s_{i-2}} \right) + \cdots + (-1)^{i-1} \frac{t_{i-1}t_2}{s_{i-1} \cdots s_1} = 0
\]

Compute the diagonal entries of \( U \) for \( i \neq 1 \):

\[
U_{ii} = 1 + u_iv_i (1 + t_{i-1}p_{i-1} + \cdots + (t_2 \cdots t_{i-1})(p_2 \cdots p_{i-1}))
\]

By the previous calculation, \( 1 + t_{i-1}p_{i-1} + \cdots + (t_2 \cdots t_{i-1})(p_2 \cdots p_{i-1}) = -1/(u_ip_i) \), so that

\[
U_{ii} = 1 - \frac{u_iv_i}{u_ip_i} = -\frac{s_i}{t_i}
\]

Vanishing of nondiagonal entries \( U_{i,j} \) for \( i, j \neq 1, n \), as well as of \( U_{1,j} \) for \( j \neq 1 \) also uses the equality (B.5). Next, the element \( U_{n,n} \) equals to the sum

\[
\sum_k A_{nk}C_{nk} = 0 + s_nv_n(1 + t_{n-1}p_{n-1} + \cdots + (t_2 \cdots t_{n-1})(p_2 \cdots p_{n-1})).
\]

Again use (B.5) and get

\[
-\frac{s_nv_n}{p_nu_n} = -\frac{s_n}{t_n}.
\]

Calculations for odd \( n \) are analogous with slightly different initial description of matrix elements \( A_{i,j} \) of the matrix \( A^{(n)}(-t) \):

\[
A_{i,j} = (-1)^{i+j}s_{i+1}s_{i+2} \cdots s_{j-1}u_j, \quad 1 \leq i \leq j \leq n,
\]

\[
A_{i,j} = t_{j+1}t_{j+2} \cdots t_{i-1}u_i, \quad 1 \leq j \leq i \leq n,
\]

\[
A_{i,j} = (-1)^i(s_n s_{n-1} \cdots s_{i+1}) (t_n t_{n-1} \cdots t_{j+1}), \quad 1 \leq i < n, 1 \leq j \leq n,
\]

\[
A_{n,n} = 0, \quad A_{n,j} = (-1)^j(t_n t_{n-1} \cdots t_{j+1}), \quad 1 \leq j < n,
\]

\[
A_{i,j} = 0, \quad 1 \leq i, j \leq n
\]
C  GL(3) example

Theorem 3.2 for GL(3) reads as follows:

\[ \Psi_\lambda(x) = \frac{e^{-i(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)}}{(2\pi i)^3} \int_{\text{Re } \gamma = 0^+} \exp \left( \gamma_{1,3}(x_2 - x_1) + (\gamma_{1,2} + \gamma_{2,3})(x_3 - x_2) \right) \cdot \frac{\Gamma(\gamma_{1,2} + \gamma_{2,3} + i(\lambda_1 - \lambda_3)) \Gamma(\gamma_{2,3} + i(\lambda_2 - \lambda_3)) \Gamma(\gamma_{1,3} - \gamma_{2,3} + i(\lambda_1 - \lambda_2))}{\Gamma(\gamma_{1,2}) \Gamma(\gamma_{1,3}) \Gamma(\gamma_{2,3}) d\gamma_{1,2} d\gamma_{1,3} d\gamma_{2,3}} \]

Using the notation

\[ \Lambda_1 = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{3}, \quad \Lambda_2 = \frac{2\lambda_2 - \lambda_1 - \lambda_3}{3}, \quad \Lambda_3 = \frac{2\lambda_3 - \lambda_1 - \lambda_2}{3}. \]

we rewrite (C.1) as

\[ \Psi_\lambda(x) = \frac{e^{-i\sum_{k,l=1}^3 \lambda_k x_l / 3}}{(2\pi i)^3} \int_{\text{Re } \gamma = 0^+} \exp \left( \left( (\gamma_{1,3} + i\Lambda_1)(x_2 - x_1) + (\gamma_{1,2} + \gamma_{2,3} - i\Lambda_3)(x_3 - x_2) \right) \cdot \frac{\Gamma(\gamma_{1,2} + \gamma_{2,3} + i(\lambda_1 - \lambda_3)) \Gamma(\gamma_{2,3} + i(\lambda_2 - \lambda_3)) \Gamma(\gamma_{1,3} - \gamma_{2,3} + i(\lambda_1 - \lambda_2))}{\Gamma(\gamma_{1,2}) \Gamma(\gamma_{1,3}) \Gamma(\gamma_{2,3}) d\gamma_{1,2} d\gamma_{1,3} d\gamma_{2,3}} \right) \]

Denote \( q_1 = \exp(x_1 - x_2), q_2 = \exp(x_2 - x_3), s_1 = \gamma_{1,3} + i\Lambda_{1}, s_2 = \gamma_{1,2} + \gamma_{2,3} - i\Lambda_{3}. \) Then (6.1) can be presented as the inverse Mellin transform

\[ \Psi_\lambda(x) = \frac{e^{-i\sum_{k,l=1}^3 \lambda_k x_l / 3}}{(2\pi i)^3} \int_{\text{Re } s_i = 0^+} q_1^{-s_1} q_2^{-s_2} M(s_1, s_2) ds_1 ds_2, \]

where

\[ M(s_1, s_2) = \Gamma(s_1 - i\Lambda_1) \Gamma(s_2 + i\Lambda_1) \cdot \frac{1}{2\pi i} \int_{\text{Re } \gamma_{2,3} = 0} \frac{\Gamma(\gamma_{2,3} + i(\lambda_2 - \lambda_3)) \Gamma(\gamma_{1,3} - \gamma_{2,3} - i\Lambda_2) \Gamma(s_2 - \gamma_{2,3} + i\Lambda_3)}{\Gamma(s_1 - i\Lambda_1) \Gamma(s_2 + i\Lambda_2) \Gamma(s_1 + s_2)} d\gamma_{2,3}. \]

Applying first Barnes lemma, we arrive to Bump [Bu] formula

\[ M(s_1, s_2) = \frac{\prod_{j=1}^3 \Gamma(s_1 - i\Lambda_j) \Gamma(s_2 + i\Lambda_j)}{\Gamma(s_1 + s_2)} \] (C.2)

Note that Bump formula (C.2) can be also derived from ‘Gelfand-Tsetlin’ presentation of GL(n, \( \mathbb{R} \)) Whittaker function studied in [GKL]. The derivation uses an integral calculated by de Branges and Wilson [Br, W]

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