Hodge number polynomials for nearby and vanishing cohomology

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Abstract
We recall the construction of the Hodge character and we show, using a result due to F. Bittner, that these can be constructed using classical pure Hodge theory only, sidestepping Deligne’s construction of functorial mixed Hodge structures on the cohomology of algebraic varieties. We then give a proof, based on the weak factorization theorem, that the nearby fibre is well-defined in the Grothendieck group of varieties. Finally, various formulas for numerical invariants of Hodge-theoretic origin are presented which are associated to one-parameter degenerations of projective manifolds.

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1 Introduction
The behaviour of the cohomology of a degenerating family of complex projective manifolds has been intensively studied in the nineteen-seventies by Clemens, Griffiths, Schmid and others. See [Gr] for a nice overview. Recently, the theory of motivic integration, initiated by Kontsevich and developed by Denef and Loeser, has given a new impetus to this topic. In particular, in the case of a one-parameter degeneration it has produced an object $\psi_f$ in the Grothendieck group of complex algebraic varieties, called
the motivic nearby fibre \cite{B05}, which reflects the limit mixed Hodge structure of the family in a certain sense. The purpose of this paper is twofold. First, we prove that the motivic nearby fibre is well-defined without using the theory of motivic integration. Instead we use the Weak Factorization Theorem \cite{AKMW}. Second, we give a survey of formulas containing numerical invariants of the limit mixed Hodge structure, and in particular of the vanishing cohomology of an isolated hypersurface singularity, without using the theory of mixed Hodge structures or of variations of Hodge structure.

We hope that in this way this interesting topic becomes accessible to a wider audience.

2 Real Hodge structures

A real Hodge structure on a finite dimensional real vector space $V$ consists of a direct sum decomposition

$$V = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}, \text{ with } V^{p,q} = \overline{V^{q,p}}$$

on its complexification $V = V \otimes \mathbb{C}$. The corresponding Hodge filtration is given by

$$F^p(V) = \bigoplus_{r \geq p} V^{r,s}.$$ 

The numbers

$$h^{p,q}(V) := \dim V^{p,q}$$

are the Hodge numbers of the Hodge structure. If for some integer $k$ we have $h^{p,q} = 0$ for all $(p,q)$ with $p + q \neq k$ the Hodge structure is pure of weight $k$.

Any real Hodge structure is the direct sum of pure Hodge structures. The polynomial

$$P_{hn}(V) = \sum_{p,q \in \mathbb{Z}} h^{p,q}(V) u^p v^q$$

is its associated Hodge number polynomial. ¹ A classical example of a weight $k$ Hodge real structure is furnished by the rank $k$ (singular) cohomology group $H^k(X)$ (with $\mathbb{R}$-coefficients) of a compact Kähler manifold $X$.

Various multilinear algebra operations can be applied to Hodge structures as we now explain. Suppose that $V$ and $W$ are two real vector spaces with a Hodge structure of weight $k$ and $\ell$ respectively. Then:

¹There are other conventions in the literature, for instance, some authors put a sign $(-1)^{p+q}$ in front of the coefficient $h^{p,q}(V)$ of $u^p v^q$. 

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1. \( V \otimes W \) has a Hodge structure of weight \( k + \ell \) given by
\[
F^p(V \otimes W)_{\mathbb{C}} = \sum_m F^m(V_{\mathbb{C}}) \otimes F^{p-m}(W_{\mathbb{C}}) \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}
\]
and with Hodge number polynomial given by
\[
P_{hn}(V \otimes W) = P_{hn}(V)P_{hn}(W).
\]

2. On \( \text{Hom}(V,W) \) we have a Hodge structure of weight \( \ell - k \):
\[
F^p \text{Hom}(V,W)_{\mathbb{C}} = \{ f : V_{\mathbb{C}} \to W_{\mathbb{C}} \mid fF^n(V_{\mathbb{C}}) \subset F^{n+p}(W_{\mathbb{C}}) \quad \forall n \}
\]
with Hodge number polynomial
\[
P_{hn}(\text{Hom}(V,W))(u,v) = P_{hn}(V)(u^{-1},v^{-1})P_{hn}(W)(u,v).
\]

In particular, taking \( W = \mathbb{R} \) with \( W_{\mathbb{C}} = W^{0,0} \) we get a Hodge structure of weight \(-k\) on the dual \( V^\vee \) of \( V \) with Hodge number polynomial
\[
P_{hn}(V^\vee)(u,v) = P_{hn}(V)(u^{-1},v^{-1}).
\]

The category \( \mathfrak{hs} \) of real Hodge structures leads to a ring, the Grothendieck ring \( K_0(\mathfrak{hs}) \) which is is the free group on the isomorphism classes \([V]\) of real Hodge structures \( V \) modulo the subgroup generated by \([V] - [V'] - [V'']\) where
\[
0 \to V' \to V \to V'' \to 0
\]
is an exact sequence of pure Hodge structures and where the complexified maps preserve the Hodge decompositions. Because the Hodge number polynomial\(^1\) is clearly additive and by \(^2\) behaves well on products the Hodge number polynomial defines a ring homomorphism
\[
P_{hn} : K_0(\mathfrak{hs}) \to \mathbb{Z}[u,v,u^{-1},v^{-1}].
\]

As remarked before, pure Hodge structures of weight \( k \) in algebraic geometry arise as the (real) cohomology groups \( H^k(X) \) of smooth complex projective varieties. We combine these as follows:
\[
\chi_{\text{Hdg}}(X) := \sum (-1)^k[H^k(X)] \in K_0(\mathfrak{hs}); \quad (5)
\]
\[
e_{\text{Hdg}}(X) := \sum (-1)^k P_{hn}(H^k(X)) \in \mathbb{Z}[u,v,u^{-1},v^{-1}] \quad (6)
\]
which we call the Hodge-Grothendieck character and the Hodge-Euler polynomial of \( X \) respectively.

Let us now recall the definition of the naive Grothendieck group \( K_0(\text{Var}) \) of (complex) algebraic varieties. It is the quotient of the free abelian group
on isomorphism classes \([X]\) of algebraic varieties over \(\mathbb{C}\) with the so-called \textit{scissor relations} \([X] = [X - Y] + [Y]\) for \(Y \subset X\) a closed subvariety. The cartesian product is compatible with the scissor relations and induces a product structure on \(K_0(\text{Var})\), making it into a ring. There is a nice set of generators and relations for \(K_0(\text{Var})\). To explain this we first recall:

\textbf{Lemma 1} Suppose that \(X\) is a smooth projective variety and \(Y \subset X\) is a smooth closed subvariety. Let \(\pi : Z \to X\) be the blowing-up with centre \(Y\) and let \(E = \pi^{-1}(Y)\) be the exceptional divisor. Then

\[
\chi_{\text{Hdg}}(X) - \chi_{\text{Hdg}}(Y) = \chi_{\text{Hdg}}(Z) - \chi_{\text{Hdg}}(E);
\]

\[
e_{\text{Hdg}}(X) - e_{\text{Hdg}}(Y) = e_{\text{Hdg}}(Z) - e_{\text{Hdg}}(E).
\]

\textit{Proof:} By \cite[605]{GH} \[\] is exact. \(\square\)

\textbf{Theorem 2} (\cite[Theorem 3.1]{B04}) The group \(K_0(\text{Var})\) is isomorphic to the free abelian group generated by the isomorphism classes of smooth complex projective varieties subject to the relations \([\emptyset] = 0\) and \([Z] - [E] = [X] - [Y]\) where \(X, Y, Z, E\) are as in Lemma 1.

It follows that for every complex algebraic variety \(X\) there exist projective smooth varieties \(X_1, \ldots, X_r, Y_1, \ldots, Y_s\) such that

\[
[X] = \sum_i [X_i] - \sum_j [Y_j]\text{ in } K_0(\text{Var})
\]

and so, using Lemma \[\] we have:

\textbf{Corollary 3} The Hodge Euler character extends to a ring homomorphism

\[
\chi_{\text{Hdg}} : K_0(\text{Var}) \to K_0(\mathfrak{h})
\]

and the Hodge number polynomial extends to a ring homomorphism

\[
e_{\text{Hdg}} : K_0(\text{Var}) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]
\]

\textbf{Remark 4} By Deligne’s theory \cite{Del71}, \cite{Del74} there is a mixed Hodge structure on the real vector spaces \(H^k(X)\). For our purposes, since we are working with real coefficients, a mixed Hodge structure is just a real Hodge structure, i.e. a direct sum of real Hodge structures of various weights, and so the Hodge character and Hodge number polynomial are defined for any real mixed Hodge structure. However, ordinary cohomology does not behave well with respect to the scissor relation; we need compactly supported
cohomology $H^k_c (X; \mathbb{R})$. But these also carry a Hodge structure and we have the following explicit expression for the above characters.

$$\chi_{\text{Hdg}}(X) = \sum (-1)^k [H^k_c (X)];$$

$$e_{\text{Hdg}}(X) = \sum (-1)^k P_{\text{hn}}(H^k_c (X)).$$

**Example 5**

1) Let $U$ be a smooth, but not necessarily compact complex algebraic manifold. Such a manifold has a good compactification $X$, i.e. $X$ is a compact complex algebraic manifold and $D = X - U$ is a normal crossing divisor, say $D = D_1 \cup \cdots D_N$ with $D_j$ smooth and irreducible. We introduce

$$D_I = D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_m}, \quad I = \{i_1, \ldots, i_m\};$$

$$a_I : D_I \hookrightarrow X$$

and we set

$$D(0) = X;$$

$$D(m) = \coprod_{|I|=m} D_I, \quad m = 1, \ldots, N;$$

$$a_m = \coprod_{|I|=m} a_I : D(m) \to X.$$

Then each connected component of $D(m)$ is a complex submanifold of $X$ of codimension $m$. Note that

$$[U] = \sum_m (-1)^m [D(m)] \in K_0(\text{Var}).$$

Hence

$$\chi_{\text{Hdg}}(U) = \sum_m (-1)^m \chi_{\text{Hdg}}(D(m));$$

$$e_{\text{Hdg}}(U) = \sum_m (-1)^m e_{\text{Hdg}}(D(m)).$$

2) If $X$ is compact the construction of cubical hyperresolutions $(X_I)_{\varnothing \neq I \subset A}$ of $X$ from [GNPP] leads to the expression

$$[X] = \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} [X_I].$$

and we find:

$$\chi_{\text{Hdg}}(X) = \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} \chi_{\text{Hdg}}(X_I);$$

$$e_{\text{Hdg}}(X) = \sum_{\varnothing \neq I \subset A} (-1)^{|I|-1} e_{\text{Hdg}}(X_I).$$

5
The scissor-relations imply that the inclusion-exclusion principle can be applied to a disjoint union $X$ of locally closed subvarieties $X_1, \ldots, X_m$:

$$\chi_{\text{Hdg}}(X) = \sum_{i=1}^{m} \chi_{\text{Hdg}}(X_i)$$

and a similar expression holds for the Hodge Euler polynomials.

**Example 6** Let $T^n = (\mathbb{C}^*)^n$ be an $n$-dimensional algebraic torus. Then $e_{\text{Hdg}}(T^1) = uv - 1$ so $e_{\text{Hdg}}(T^n) = (uv - 1)^n$. Consider an $n$-dimensional toric variety $X$. It is a disjoint union of $T^n$-orbits. Suppose that $X$ has $s_k$ orbits of dimension $k$. Then

$$e_{\text{Hdg}}(X) = \sum_{k=0}^{n} s_k e_{\text{Hdg}}(T^k) = \sum_{k=0}^{n} s_k (uv - 1)^k.$$

If $X$ has a pure Hodge structure (e.g. if $X$ is compact and has only quotient singularities) then this formula determines the Hodge numbers of $X$.

### 3 Nearby and vanishing cohomology

In this section we consider a relative situation. We let $X$ be a complex manifold, $\Delta \subset \mathbb{C}$ the unit disk and $f : X \to \Delta$ a holomorphic map which is smooth over the punctured disk $\Delta^*$. We say that $f$ is a one-parameter degeneration. Let us assume that $E = \bigcup_{i \in I} E_i = f^{-1}(0)$ is a divisor with strict normal crossings on $X$. We have the specialization diagram

$$
\begin{array}{ccc}
X_\infty & \xrightarrow{k} & X \\
\downarrow f & & \downarrow f \\
\s & \xleftarrow{e} & \Delta \\
\end{array}
$$

where $\s$ is the complex upper half plane, $e(\z) := \exp(2\pi i \z)$ and where

$$
X_\infty := X \times_{\Delta^*} \s.
$$

We let $e_i$ denote the multiplicity of $f$ along $E_i$ and choose a positive integer multiple $e$ of all $e_i$. We let $\tilde{f} : \tilde{X} \to \Delta$ denote the normalization of the pull-back of $X$ under the map $\mu_e : \Delta \to \Delta$ given by $\tau \mapsto \tau^e = t$. It fits into a commutative diagram describing the $e$-th root of $f$:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\mu} & X \\
\downarrow \tilde{f} & & \downarrow f \\
\Delta & \xrightarrow{\mu_e} & \Delta.
\end{array}
$$
We put
\[ D_i = \rho^{-1}(E_i), \quad D_J = \bigcap_{i \in J} D_j, \quad D(m) = \bigsqcup_{|J|=m} D_J. \]

Then \( D_J \to E_J \) is a cyclic cover of degree \( \gcd(e_j \mid j \in J) \). The maps \( D_J \to E_J \) do not depend on the choice of the integer \( e \) and so in particular this is true for the varieties \( D_J \). See e.g. [Ste77] or [B05] for a detailed study of the geometry of this situation. The special fibre \( \tilde{X}_0 = \tilde{f}^{-1}(0) \) is now a complex variety equipped with the action of the cyclic group of order \( e \). Let us introduce the associated Grothendieck-group:

**Definition 7** We let \( \hat{K}_0^\mu(\text{Var}) \) denote the Grothendieck group of complex algebraic varieties with an action of a finite order automorphism modulo the subgroup generated by expressions \([\mathbb{P}(V)] - [\mathbb{P}^n \times X]\) where \( V \) is a vector bundle of rank \( n+1 \) over \( X \) with action which is linear over the action on \( X \). See [B05, Sect. 2.2] for details.

As a motivation, we should remark that in the ordinary Grothendieck group the relation \([\mathbb{P}(V)] = [\mathbb{P}^n \times X]\) follows from the fact that any algebraic vector bundle is trivial over a Zariski-open subset and the fact that the scissor-relations hold. The above relation extend to the case where one has a group action.

**Definition 8** Suppose that the fibres of \( f \) are projective varieties. Following [B05] Ch. 2] we define the **motivic nearby fibre** of \( f \) by
\[
\psi_f := \sum_{m \geq 1} (-1)^{m-1} [D(m) \times \mathbb{P}^{m-1}] \in \hat{K}_0^\mu(\text{Var})
\]
and the **motivic vanishing fibre** by
\[
\phi_f := \psi_f - [E] \in \hat{K}_0^\mu(\text{Var}) \quad (7)
\]

**Remark 9** If we let \( E_I^0 \) be the open subset of \( E_I \) consisting of points which are exactly on the \( E_j \) with \( j \in I \), \( D_I^0 \) the corresponding subset of \( D_I \) and \( D^0(i) = \bigsqcup_{|I|=1} D_I^0 \). We have \( D_I = \bigsqcup_{J \supset I} D_J^0 \) and the scissor relations imply that
\[
\sum_{i \geq 1} (-1)^{i-1} [D(i) \times \mathbb{P}^{i-1}] = \sum_{i \geq 1} (-1)^i \sum_{j \geq i} \binom{j}{i} [D^0(j) \times \mathbb{P}^{j-1}]
\]
\[
= \sum_{j \geq 1} [D^0(j)] \times \sum_{i=1}^j (-1)^{i-1} \binom{j}{i} [\mathbb{P}^{i-1}]
\]
\[
= \sum_{j \geq 1} (-1)^{j-1} [D^0(j) \times (\mathbb{C}^*)^{j-1}].
\]

This expression for the motivic nearby fibre has been used in [L].
The motivic nearby fibre turns out to be a relative bimeromorphic invariant:

**Lemma 10** Suppose that \( g : Y \to X \) is a bimeromorphic proper map which is an isomorphism over \( X - E \). Assume that \( (f \circ g)^{-1}(0) \) is a divisor with strict normal crossings. Then

\[
\psi_f = \psi_{fg}.
\]

**Proof:** In [B05], the proof relies on the theory of motivic integration [DL]. We give a different proof, based on the weak factorization theorem [AKMW]. This theorem reduces the problem to the following situation: \( g \) is the blowing-up of \( X \) in a connected submanifold \( Z \subset E \), with the following property. Let \( A \subset I \) be those indices \( i \) for which \( Z \subset E_i \). Then \( Z \) intersects the divisor \( \bigcup_{i \notin A} E_i \) transversely, hence \( Z \cap \bigcup_{i \notin A} E_i \) is a divisor with normal crossings in \( Z \).

We fix the following notation. We let \( L := [A^1] \) and \( P_m := [\mathbb{P}^m] = \sum_{i=0}^{m} L_i \). For \( J \subset I \) we let \( j = |J| \). So, using the product structure in \( K_0(\text{Var}) \), we have

\[
\psi_f = \sum_{\emptyset \neq J \subset I} (-1)^{j-1} [D_J] P_{j-1}.
\]

Let \( E' = \bigcup_{i \in I} E'_i \) be the zero fibre of \( fg \). We have \( I = I' \cup \{\ast\} \) where \( E'_i \) is the exceptional divisor of \( g \) and \( E'_i \) is the proper transform in \( Y \) of \( E_i \). Form \( \rho' : D' \to E' \), the associated ramified cyclic covering. For \( J \subset I \) we let \( J' = J \cup \{\ast\} \). Note that \( E'_i \) has multiplicity equal to \( \sum_{i \in A} e_i \). Without loss of generality we may assume that \( e \) is also a multiple of this integer. We have two kinds of \( j'\)-uple intersections \( D_{J'} \): those which only contain \( D_{j'} \), \( j \neq \ast \) and those which contain \( D_{j'} \), \( j = \ast \).

\[
\psi_{fg} = [D'_j] + \sum_{\emptyset \neq J' \subset I} (-1)^{j-1} ( [D'_j] P_{j-1} - [D'_{j'}] P_{j'} ) .
\]

We are going to calculate the difference between \( \psi_{fg} \) and \( \psi_f \). Let \( B \subset I - A \). Let \( c = \operatorname{codim}(Z, X) \). Then for all \( K \subset A \) we have that \( \operatorname{codim}(Z \cap E_{K \cup B}, E_{K \cup B}) = c - k \), and \( D'_{K \cup B} \) is the blowing up of \( D_{K \cup B} \) with centre \( Z \cap E_{K \cup B} = Z \cap E_B =: Z_B \) and exceptional divisor \( D'_{K \cup B} \). Hence if we let \( W_B = \rho'^{-1}(Z_B) \) we have

\[
[D'_{K \cup B}] = [D_{K \cup B}] + [D'_{K \cup B}] - [W_B] = [D_{K \cup B}] + [W_B](P_{c-k-1} - 1).
\]

Hence

\[
\psi_{fg} - \psi_f = \sum_B c_B [W_B]
\]

for suitable coefficients \( c_B \).
Note that \( Z_\emptyset = Z \) and that \( D_*' = W_\emptyset \times P_{c-1} \) and so if \( B = \emptyset \) we get

\[
c_\emptyset = P_{c-1} + \sum_{\emptyset \neq K \subset A} (-1)^{k-1} ((P_{c-k-1} - 1)P_{k-1} - P_{c-k-1}P_k) = P_{c-1} + \sum_{\emptyset \neq K \subset A} (-1)^k (L^kP_{c-k-1} + P_{k-1}) = P_{c-1} \sum_{K \subset A} (-1)^k = 0.
\]

In case \( B \neq \emptyset \) we get

\[
c_B = \sum_{K \subset A} (-1)^{k+b-1} ((P_{c-k-1} - 1)P_{k+b-1} - P_{c-k-1}P_{k+b}) = \sum_{K \subset A} (-1)^{k+b} (L^{k+b}P_{c-k-1} + P_{k+b-1}) = P_{c+b-1} \sum_{K \subset A} (-1)^{b+k} = 0. \quad \Box
\]

Let us now pass to the nearby fibre in the Hodge theoretic sense. In [Schm] and [Ste76] a mixed Hodge structure on \( H^k(X_\infty) \) was constructed; its weight filtration is the monodromy weight filtration which we now explain. The loop winding once counterclockwise around the origin gives a generator of \( \pi(\Delta^*, *) \), \( * \in \Delta^* \). Its action on the fibre \( X_* \) over \( * \) is well defined up to homotopy and on \( H^k(X_*) \simeq H^k(X_\infty) \) it defines the monodromy automorphism \( T \). Let \( N = \log T_u \) be the logarithm of the unipotent part in the Jordan decomposition of \( T \). Then \( W \) is the unique increasing filtration on \( H^k(X_\infty) \) such that \( N(W_j) \subset W_j - 2 \) and \( N_j : \text{Gr} \, W^k \to \text{Gr} \, W^{k+j} \) is an isomorphism for all \( j \geq 0 \). In fact, from [Schm, Lemma 6.4] we deduce:

**Lemma 11** There is a Lefschetz-type decomposition

\[
\text{Gr} \, W^k H^k(X_\infty) = \bigoplus_{\ell=0}^k \bigoplus_{r=0}^\ell \text{Gr} \, P_{k+\ell},
\]

where \( P_{k+\ell} \) is pure of weight \( k + \ell \). The endomorphism \( N \) has \( \dim P_{\ell+m-1} \) Jordan blocks of size \( m \).

The Hodge filtration is constructed in [Schm] as a limit of the Hodge filtrations on nearby smooth fibres in a certain sense, and in [Ste76] using the relative logarithmic de Rham complex. With \( X_t \) a smooth fibre of \( f \) it follows that

\[
\dim F^m H^k(X_t) = \dim F^m H^k(X_\infty)
\]

and hence

\[
e_{\text{Hdg}}(X_\infty)|_{v=1} = e_{\text{Hdg}}(X_t)|_{v=1}.
\] (8)

We next remark that the nearby cycle sheaf \( i^* R_* k^* \mathbb{R}_X \) can be used to put a Hodge structure on the cohomology groups \( H^k(X_\infty) \), while the hypercohomology of the vanishing cycle sheaf \( \phi_f = \text{Cone}[i^* \mathbb{R}_X \to \psi_f \mathbb{R}_X] \) (it is
a complex of sheaves of real vector spaces on $E$) likewise admits a Hodge structure. In fact, the spectral sequence of [Ste76, Cor. 4.20] shows that

$$\chi_{\text{Hdg}}(X_\infty) = \chi_{\text{Hdg}}(\psi_f)$$

and then the definition shows that

$$\chi_{\text{Hdg}}(X_\infty) - \chi_{\text{Hdg}}(E) = \chi_{\text{Hdg}}(\phi_f).$$

In fact this formula motivates the nomenclature “motivic nearby fibre” and “motivic vanishing cycle”.

As a concluding remark, the semisimple part $T_s$ of the monodromy is an automorphism of the mixed Hodge structure on $H^k(X_\infty)$.

We can use these remarks to deduce information about the Hodge numbers on $H^k(X_\infty)$ from information about the geometry of the central fibre as we shall illustrate now.

**Example 12**

1) Let $F, L_1, \ldots, L_d \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous forms with $\deg F = d$ and $\deg L_i = 1$ for $i = 1, \ldots, d$, such that $F \cdot L_1 \cdots L_d = 0$ defines a reduced divisor with normal crossings on $\mathbb{P}^2(\mathbb{C})$. We consider the space

$$X = \{(x_0, x_1, x_2, t) \in \mathbb{P}^2 \times \Delta \mid \prod_{i=1}^d L_i(x_0, x_1, x_2) + tF(x_0, x_1, x_2) = 0\}$$

where $\Delta$ is a small disk around $0 \in \mathbb{C}$. Then $X$ is smooth and the map $f : X \rightarrow \Delta$ given by the projection to the second factor has as its zero fibre the union $E_1 \cup \cdots \cup E_d$ of the lines $E_i$ with equation $L_i = 0$. These lines are in general position and have multiplicity one. We obtain

$$\psi_f = (d - \binom{d}{2})[\mathbb{P}^1]$$

so

$$e_{\text{Hdg}}(\psi_f) = (1 - \binom{d-1}{2})(1 + uv)$$

and substituting $v = 1$ in this formula we get the formula $g = \binom{d-1}{2}$ for the genus of a smooth plane curve of degree $d$. The monodromy on $H^1(X_\infty)$ has $g$ Jordan blocks of size 2, so is “maximally unipotent”.

2) If we consider a similar example, but replace $\mathbb{P}^2$ by $\mathbb{P}^3$ and curves by surfaces, lines by planes, then the space $X$ will not be smooth but has ordinary double points at the points of the zero fibre where two of the planes meet the surface $F = 0$. There are $d\binom{d}{2}$ of such points, $d$ on each line of intersection. If we blow these up, we obtain a family $f : \tilde{X} \rightarrow \Delta$ whose zero fibre $D = E \cup F$ is the union of components $E_i$, $i = 1, \ldots, d$ which are
copies of \( \mathbb{P}^2 \) blown up in \( d(d-1) \) points, and components \( F_j, j = 1, \ldots, d(d) \) which are copies of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Thus
\[
e_{\text{Hdg}}(D(1)) = d(1 + (d^2 - d + 1)uv + u^2v^2) + d \begin{pmatrix} d \end{pmatrix} \frac{1}{2}(1 + uv)^2.
\]
The double point locus \( D(2) \) consists of the \( \begin{pmatrix} d \end{pmatrix} \) lines of intersections of the \( E_i \) together with the \( d^2(d-1) \) exceptional lines in the \( E_i \). So
\[
e_{\text{Hdg}}(D(2)) = d(d-1)(d + \frac{1}{2})(1 + uv).
\]
Finally \( D(3) \) consists of the \( \begin{pmatrix} d \end{pmatrix} \) intersection points of the \( E_i \) together with one point on each component \( F_j \), so
\[
e_{\text{Hdg}}(D(3)) = \begin{pmatrix} d \end{pmatrix} + \begin{pmatrix} d \end{pmatrix} = \frac{1}{3}d(d-1)(2d-1)
\]
We get
\[
e_{\text{Hdg}}(\psi f) = \left( \begin{pmatrix} d-1 \end{pmatrix} + 1 \right)(1 + u^2v^2) + \frac{1}{3}d(2d^2 - 6d + 7)uv
\]
in accordance with the Hodge numbers for a smooth degree \( d \) surface:
\[
h^{2,0} = h^{0,2} = \begin{pmatrix} d-1 \end{pmatrix}, \quad h^{1,1} = \frac{1}{3}d(2d^2 - 6d + 7)
\]
The monodromy on \( H^2(X_\infty) \) has \( \begin{pmatrix} d-1 \end{pmatrix} \) Jordan blocs of size 3 and \( \frac{1}{2}d^3 - d^2 + \frac{1}{2}d + 1 \) blocks of size 1.

3) Consider a similar smoothing of the union of two transverse quadrics in \( \mathbb{P}^3 \). The generic fibre is a smooth K3-surface and after blowing up the 16 double points of the total space we obtain the following special fibre:

1. \( E(1) \) has two components which are blowings up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in 16 points, and 16 components isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \); hence \( e_{\text{Hdg}}(E(1)) = 18(1 + uv)^2 + 32uv \).

2. \( E(2) \) consists of the 32 exceptional lines together with the strict transform of the intersection of the two quadrics, which is an elliptic curve; hence \( e_{\text{Hdg}}(E(2)) = 33(1 + uv) - u - v \);

3. \( E(3) \) consists of 16 points: one point on each exceptional \( \mathbb{P}^1 \times \mathbb{P}^1 \), so \( e_{\text{Hdg}}(E(3)) = 16 \).

We get
\[
e_{\text{Hdg}}(X_\infty) = 1 + u + v + 18uv + u^2v + uv^2 + u^2v^2
\]
Putting \( v = 1 \) we get \( 2 + 20u + 2u^2 \), in agreement with the Hodge numbers \( (1,20,1) \) on the \( H^2 \) of a K3-surface. The monodromy has two Jordan blocs of size 2 and 18 blocs of size 1.
4 Equivariant Hodge number polynomials

We have seen that the mixed Hodge structure on the cohomology of the nearby fibre of a one-parameter degeneration comes with an automorphism of finite order. This leads us to consider the category $\hat{\mathfrak{hs}}^\mu_\mathbb{R}$ of pairs $(H, \gamma)$ consisting of a real Hodge structure (i.e. direct sum of pure real Hodge structures of possibly different weights) $H$ and an automorphism $\gamma$ of finite order of this Hodge structure.

We are going to consider a kind of tensor product of two such objects, which we call convolution (see [SchS], where this operation was defined for mixed Hodge structures and called join). We will explain this by settling an equivalence of categories between $\hat{\mathfrak{hs}}^\mu_\mathbb{R}$ and a category $\mathfrak{fhs}$ of so-called fractional Hodge structures. (called Hodge structures with fractional weights in [L]; it is however not the weights which are fractional, but the indices of the Hodge filtration!)

**Definition 13** (See [L]). A fractional Hodge structure of weight $k$ is a real vector space $H$ of finite dimension, equipped with a decomposition

$$H_C = \bigoplus_{a+b=k} H^{a,b}$$

where $a, b \in \mathbb{Q}$, such that $H^{b,a} = \overline{H^{a,b}}$. A fractional Hodge structure is defined as a direct sum of pure fractional Hodge structures of possibly different weights.

**Lemma 14** We have an equivalence of categories $G: \hat{\mathfrak{hs}}^\mu \to \mathfrak{fhs}$.

**Proof:** Let $(H, \gamma)$ be an object of $\hat{\mathfrak{hs}}^\mu$ pure of weight $k$. We define $H_a = \ker(\gamma - \exp(2\pi i a); H_C)$ for $0 \leq a < 1$ and for $0 < a < 1$ put

$$\tilde{H}_{p+a,k+1-a} = H^{p,k-p}_a, \quad \tilde{H}_{p,k-p} = H^{p,k-p}_0$$

This transforms $(H, \gamma)$ into a direct sum $\tilde{H} =: G(H, \gamma)$ of fractional Hodge structures of weights $k$ and $k + 1$ respectively. Conversely, for a fractional Hodge structure $\tilde{H}$ of weight $k$ one has a unique automorphism $\gamma$ of finite order which is multiplication by $\exp(2\pi ib)$ on $\tilde{H}^{b,k-b}$.

Note that this equivalence of categories does not preserve tensor products! Hence it makes sense to make

**Definition 15** The convolution $(H', \gamma') \ast (H'', \gamma'')$ of two objects in $\hat{\mathfrak{hs}}^\mu$ is the object corresponding to the tensor product of their images in $\mathfrak{fhs}$:

$$G((H', \gamma') \ast (H'', \gamma'')) = G(H', \gamma') \otimes G(H'', \gamma'').$$
Note that the Hodge number polynomial map $P_{hn}$ extends to a ring homomorphism
$$P_{hn}^\hat{\mu} : K_0(f\mathfrak{hs}) \to R := \lim_{\to} \mathbb{Z}[u^{\frac{1}{\mu}}, v^{\frac{1}{\mu}}, u^{-1}, v^{-1}]$$
We denote its composition with the functor $G$ by the same symbol. Hence
$$P_{hn}^\hat{\mu} : K_0^\hat{\mu}(\mathfrak{hs}) \to R$$
transforms convolutions into products. We equally have an equivariant Hodge-Grothendieck character
$$\hat{\chi}^\mu_{Hdg} : K_0^\hat{\mu}(\text{Var}) \to K_0^\hat{\mu}(\mathfrak{hs})$$
and an equivariant Hodge-Euler characteristic
$$\hat{e}^\mu_{Hdg} : K_0^\hat{\mu}(\text{Var}) \to R.$$

Let $f : X \to \mathbb{C}$ be a projective morphism where $X$ is smooth, of relative dimension $n$, with a single isolated critical point $x$ such that $f(x) = 0$. Construct $\psi_f$ by replacing the zero fibre $X_0$ by a divisor with normal crossings as above. Then the Milnor fibre of $F$ of $f$ at $x$ has the homotopy type of a wedge of spheres of dimension $n$. Its cohomology is also equipped with a mixed Hodge structure. Recalling definition 7 of the motivic vanishing fibre, it can be shown that
$$P_{hn}^\hat{\mu}(\tilde{H}^n(F)) = (-1)^n \hat{e}^\mu_{Hdg}(\phi_f).$$
Write $P_{hn}^\hat{\mu}(\tilde{H}^n(F)) = \sum_{\alpha \in \mathbb{Q}, w \in \mathbb{Z}} m(\alpha, w) u^{\alpha} v^{w}\cdot (n - \alpha, w)$. In the literature several numerical invariants have been attached to the singularity $f : (X, x) \to (\mathbb{C}, 0)$. These are all related to the numbers $m(\alpha, w)$ as follows:

1. The characteristic pairs [Ste77 Sect. 5].
$$\text{Chp}(f, x) = \sum_{\alpha, w} m(\alpha, w) \cdot (n - \alpha, w).$$

2. The spectral pairs [N-S]:
$$\text{Spp}(f, x) = \sum_{\alpha \not\in \mathbb{Z}} m(\alpha, w) \cdot (\alpha, w) + \sum_{\alpha \in \mathbb{Z}, w} m(\alpha, w) \cdot (\alpha, w + 1).$$

3. The singularity spectrum in Saito’s sense [Sa]:
$$\text{Sp}_{Sa}(f, x) = P_{hn}^\hat{\mu}(\tilde{H}^n(F))(t, 1).$$

4. The singularity spectrum in Varchenko’s sense [Var]:
$$\text{Sp}_{V}(f, x) = t^{-1} P_{hn}^\hat{\mu}(\tilde{H}^n(F))(t, 1).$$
Note that the object $\phi_f$ depends only on the germ of $f$ at the critical point $x$. Let us as a final remark rephrase the original Thom-Sebastiani theorem (i.e. for the case of isolated singularities):

**Theorem 16** Consider holomorphic germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with isolated singularity. Then the germ $f \oplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, (0,0)) \to \mathbb{C}$ with $(f \oplus g)(x,y) := f(x) + g(y)$ has also an isolated singularity, and

$$\chi^\mu_{\text{Hdg}}(\phi_{f \oplus g}) = -\chi^\mu_{\text{Hdg}}(\phi_f) \ast \chi^\mu_{\text{Hdg}}(\phi_g)$$

so

$$e^\mu_{\text{Hdg}}(\phi_{f \oplus g}) = -e^\mu_{\text{Hdg}}(\phi_f) \cdot e^\mu_{\text{Hdg}}(\phi_g) \in \mathbb{R}.$$ 

**Remark 17** This theorem has been largely generalized, for functions with arbitrary singularities, and even on the level of motives. Denef and Loeser [DL] defined a convolution product for Chow motives, and Looijenga [L] defined one on $\mathcal{M}^\mu = K^\mu_0(\text{Var})[L^{-1}]$, both with the property that the Hodge Euler polynomial commutes with convolution and that the Thom-Sebastiani property holds already on the level of varieties/motives.

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