Spectral Asymptotics for Magnetic Schrödinger Operator with the Strong Magnetic Field

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Abstract

In this article we obtain eigenvalue asymptotics for 2D and 3D-Schrödinger, Schrödinger-Pauli and Dirac operators in the situations in which the role of the magnetic field is important. We have seen in Chapters 13 and 17 of [Ivr2] that these operators are essentially different and there is a significant difference between 2D and 3D-operators.

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1 2D-case. Introduction

In this article we obtain eigenvalue asymptotics for 2D-Schrödinger, Schrödinger-Pauli and Dirac operators in the situations in which the role of the magnetic field is important. We have seen in Chapters 13 and 17 of [Ivr2] that these operators are essentially different and they also differ significantly from the corresponding 3D-operators.

While we are trying to emulate results of Chapter ?? of [Ivr2], we find ourselves now in the very different situation. Indeed, for operators we study the remainder estimates in the local spectral asymptotics under non-degeneracy assumptions are better than for similar operators with the magnetic field. However, as we seen in Chapters 14 and 15 of [Ivr2] these remainder estimates deteriorate if the magnetic field degenerates or there is a boundary. This significantly limits our ability to consider the cases when magnetic field asymptotically degenerates at the point of singularity (finite or infinite) along some directions, or domains with the boundary.

We start from Section 2 in which we consider the case when the spectral parameter is fixed (τ = const) and study asymptotics with respect to µ, h exactly like in Section ?? of [Ivr2] we considered asymptotics with respect to h. However, since now we have two parameters, we need to consider an interplay between them: while always h → +0, we cover µ → +0, µ remains disjoint from 0 and ∞ and µ → ∞, which in turn splits into subcases µh → 0, µh remains disjoint from 0 and ∞ and µh → ∞.

In Section 3 we consider asymptotics with µ = h = 1 and with the spectral parameter τ tending to +∞ for the Schrödinger and Schrödinger-Pauli operators and to ±∞ for the Dirac operator. We consider bounded domains with the singularity at some point and external domains with the singularity at infinity. In the latter case the specifics of the 2D magnetic
Schrödinger and Schrödinger-Pauli operators manifest itself in the better remainder estimate (and in the larger principal part for the Schrödinger-Pauli operator) than in the non-magnetic case. Furthermore, in contrast to the non-magnetic case there are non-trivial results for the Dirac operator. This happens only in even dimensions.

In Section 4 we consider asymptotics with the singularity at infinity and $\mu = h = 1$ and with $\tau$ tending to $+0$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm (M - 0)$ for the Dirac operator. Again the specifics of the 2D magnetic Schrödinger and Schrödinger-Pauli operators manifest itself in the better remainder estimate (and in the larger principal part for the Schrödinger-Pauli operator) than in non-magnetic case.

It includes the most interesting case (see Subsection 4.1) when magnetic field is either constant or stabilizes fast at infinity and potential decays at infinity. As we know if magnetic field and potential were constant then the operator would have purely point spectrum of infinite multiplicity and each eigenvalue (Landau level) would be disjoint from the rest. Now we have a sequences of eigenvalues tending to the Landau level either from below, or from above, or from both sides and we are interested in their asymptotics. In contrast to the rest of the section we consider multidimensional case as well. In contrast to Section ?? of [Ivr2] there are non-trivial results for fast decaying potentials as well.

In Section 5 we consider asymptotics with respect to $\mu, h, \tau$, like in Section ?? of [Ivr2] again with significant differences mentioned above.

Finally, in Appendix 6.A the self-adjointness of the 2D-Dirac operator with a very singular magnetic field is proven.

The Schrödinger operator theory is more extensive than the Dirac operator theory: there are many more meaningful problems and questions for the Schrödinger operator than for the Dirac operator. Also, the 2D-theory is more extensive and provides more accurate remainder estimates than the 3D-theory. These circumstances are not due to the technical difficulties but are instead due to the fact that the spectrum of the Schrödinger operator is discrete more often than the spectrum of the Dirac operator in dimension.
$d = 2$ the spectrum is discrete more often than in dimension $d = 3$.

2 2D-case. Asymptotics with fixed spectral parameter

In this section we consider asymptotics with a fixed spectral parameter for 2-dimensional magnetic Schrödinger, Schrödinger-Pauli and Dirac operators and discuss some of the generalizations\(^1\).

As in Chapters 9 of [Ivr2] we will introduce a semiclassical zone and a singular zone, where $\rho \gamma \geq h$ and $\rho \gamma \leq h$ respectively. In the semiclassical zone we apply asymptotics of Chapters 13–22 (but mainly of 13 and 19 of [Ivr2]–in the multidimensional case). In the singular zone we need to apply estimates for a number of eigenvalues; usually it would be sufficient to use non-magnetic estimate\(^2\) for number of eigenvalues which trivially follows from standard one but if needed one can use more delicate estimates.

2.1 Schrödinger operator

2.1.1 Estimates of the spectrum

Consider first the Schrödinger operator (13.1.1) of [Ivr2])

\[
(2.1) \quad A = \sum_{j,k} P_j g^{jk} P_k + V, \quad \text{with } P_j = h D_j - \mu V_j
\]

where $g^{jk}, V_j, V$ satisfy (13.1.2) and (13.1.4) of [Ivr2]) i.e.

\[
(2.2) \quad \epsilon |\xi|^2 \leq \sum_{j,k} g^{jk} \xi_j \xi_k \leq c|\xi|^2 \quad \forall \xi \in \mathbb{R}^d.
\]

Without any loss of the generality we can fix $\tau = 0$ and then in the important function $V_{eff} F_{eff}^{-1}$ the parameters $\mu$ and $h$ enter as factors. Thus, we treat the operator (13.1.1) of [Ivr2]) assuming that it is self-adjoint.

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\(^1\) Mainly to higher dimensions with full-rank magnetic field.

\(^2\) With $V$ modified accordingly; for example, for the Schrödinger and Schrödinger-Pauli operators $V_\tau$ is replaced by $C \left( (1 - \epsilon) V - C \epsilon^2 |V|^2 \right)_\tau$. 

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We make assumptions similar to those of Chapter 9 of [Ivr2]

\[(2.3)_{1-3} \quad |D^\alpha g^{jk}| \leq c \gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \leq c \rho_1 \gamma^{-|\alpha|}, \quad |D^\alpha V| \leq c \rho^2 \gamma^{-|\alpha|}\]

where scaling function \(\gamma(x)\) and weight functions \(\rho(x), \rho_1(x)\) satisfy the standard assumptions (9.1.6)_{1,2} of [Ivr2]. Then

\[(2.4)_{1,2} \quad \mu_{\text{eff}} = \mu \rho_1 \gamma \rho^{-1}, \quad h_{\text{eff}} = h \rho^{-1} \gamma^{-1}.\]

Let us introduce a *semiclassical zone*

\[(2.5) \quad X' = \{x : \rho \gamma \geq h\} \]

and a *singular zone*

\[(2.6) \quad X'' = \{x : \rho \gamma \leq 2h\}.\]

Further, let us introduce two other overlapping zones

\[(2.7) \quad X'_1 = \{x \in X_{\text{sc}} : \mu \rho_1 \leq 2c \rho \gamma^{-1}\}\]

and

\[(2.8) \quad X'_2 = \{x \in X' : \mu \rho_1 \geq c \rho \gamma^{-1}\}\]

where the magnetic field \(\mu_{\text{eff}} = \mu \rho_1 \rho^{-1} \gamma\) is *normal* \((\mu_{\text{eff}} \leq 2c)\) and where it is *strong* \((\mu_{\text{eff}} \geq c)\) respectively. We assume that

\[(2.9) \quad |F| \geq \epsilon \rho_1 \quad \text{in} \quad X'_2\]

where \(F_{jk}\) and \(F\) are the tensor and scalar intensities of the magnetic field.

Moreover, let us assume that

\[(2.10)_1 \quad B(x, \gamma(x)) \subset X \quad \forall x \in X'_{2-},\]

and

\[(2.10)_2 \quad u|_{\partial X \cap B(x, \gamma(x))} = 0 \quad \forall x \in X' \quad \forall u \in \mathcal{D}(A)\]

where

\[(2.11) \quad X'_{2-} = \{x \in X'_2 : V + \mu hF \geq \epsilon \rho^2\}\]

and

\[(2.12) \quad X'_{2+} = \{x \in X' : V + \mu hF \leq 2\epsilon \rho^2\}.\]

Finally, let the standard boundary regularity condition be fulfilled:
For every $y \in X$, $\partial X \cap B(y, \gamma(y)) = \{x_k = \phi_k(x_k)\}$ with

$$|D^a \phi_k| \leq c \gamma^{-|a|}$$

and $k = k(y)$.

Due to (2.10), we need this condition only as $y \in X'_1 \cup X'_2$. Finally, in $X'_2$, let one of the following non-degeneracy conditions be fulfilled.

Recall that according to Chapter 13 of [Ivr2] the contribution of the partition element $\psi \in C^K_0(B(y, \frac{1}{2} \gamma(y)))$ to the principal part of asymptotics is

$$N^{-}(\mu, h) = N'^{-}(\mu, h) := h^{-2} \int N'MW(x, \mu h) \psi(x) \, dx$$

with $N'MW(x, \mu h)$ given by (13.1.9) of [Ivr2]) with $d = 2$.

On the other hand, its contribution to the remainder does not exceed $Ch^{-1} \rho \gamma$ if $\mu \rho_1 \gamma \leq c \rho$ and it does not exceed $C \mu^{-1} h^{-1} \rho^2 \rho_1^{-1}$ if $\mu \rho_1 \gamma \geq c \rho$ but $\mu \rho_1 \leq \rho^2$, $y \in X'_1$ and assumption (2.15) is fulfilled, and it does not exceed $C(\mu^{-s} \rho_1^{-s} \gamma^{-2s})$ if $C \mu \rho_1 \gamma \geq \rho$, $\mu h \rho_1 \leq \rho^2$, $y \in X'_2$.

Then we get estimate of $N^{-}$ from below by the magnetic Weyl approximation $N^{-}(\mu, h)$ minus corresponding remainder, and also from above

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3) It does not exceed the same expression with an extra logarithmic factor under assumption (2.16) but logarithmic factor could be skipped if we add corrections at the points with negative $\det \text{Hess}(VF^{-1})$. 

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by magnetic Weyl approximation plus corresponding remainder, provided
\( X = X' \) (so, there is no singular zone \( X'' = \emptyset \)):

\[
(2.18) \quad h^{-d} \int_{X'} N^{MW}(x, \mu h) \, dx - CR_1 \leq N^{-}(0) \leq h^{-d} \int_{X'} N^{MW}(x, \mu h) \, dx + CR_1 + C'R_2
\]

with

\[
(2.19) \quad R_1 = \mu^{-1} h^{1-d} \int_{X'_1} \rho^d \rho_1^{-1} \gamma^{-2} \, dx,
\]

\[
(2.20) \quad R_2 = \mu h^{s-d} \int_{X'_2} \rho_1 \rho^{d-s-1} \gamma^{1-s} \, dx
\]

provided

(2.21) \quad \mu \rho_1 \gamma \geq \rho

where the latter condition could be assumed without any loss of the generality, 
\( C' \) depens also on \( s \) and \( \epsilon \).

We leave to the reader the following not very challenging set of problems:

**Problem 2.1.** (i) Consider the case \( X'' \neq \emptyset \) and prove the estimate from
above with an extra term \( C_1R_0 \) like in estimate (9.1.29) (Theorem 9.1.7 of
[Ivr2]) with \( R_0 \) defined in the same way as in Section 9.1 of [Ivr2].

(ii) Consider multidimensional case; then we need to impose more sophisti-
cated non-degeneracy assumptions (see Chapter 19 of [Ivr2]).

(iii) Incorporate results of Chapters 14, 15, 18, 19 (in non-smooth settings),
21 and 22 of [Ivr2].

(iv) Using arguments and methods of Chapter 10 and results of Chapter 17
of [Ivr2] consider Dirac operator (in the full-rank case).

### 2.1.2 Basic results

In what follows \( h \to +0 \) and the semiclassical zone \( X' \) expands to \( X \) while \( \mu \)
is either bounded (then we can assume that the zone of the strong magnetic
field \( X_2' \) is fixed) or tends to \( \infty \) (then \( X_2' \) expands to \( X \)). We assume that all
conditions of the previous subsection are fulfilled with \( \mu = h = 1 \) but we will assume them fulfilled in the corresponding zones.

The other important question is whether \( \mu h \to 0 \), remains bounded and disjoint from 0 or tends to \( \infty \).

Finally, we should consider the singular zone \( X'' \). In order to avoid this task we assume initially that

\[
(2.22) \quad \rho_1 \gamma^2 + \rho \gamma \geq \epsilon.
\]

Then we obtain the following assertion from the arguments of the previous Subsection 2.1.1.1:

**Theorem 2.2.** Let \( d = 2 \) and let the Schrödinger operator \( A \) satisfy conditions (2.15), (2.2), (2.3)_{1-3}, (2.9) in the corresponding regions where \( \rho, \rho_1, \gamma \) satisfy (9.1.6)_{1-4} of [Ivri2], (2.10)_{1,2}, (2.22).

Let \( \rho_1 \geq \epsilon \rho \gamma^{-1} \) and

\[
(2.23) \quad \rho_2 \rho_1^{-1} \gamma^{-2} \in \mathcal{L}^1(X \cap \{ V + tF \leq \epsilon \rho^2 \}),
\]

\[
(2.24) \quad \rho_1^{-s} \gamma^{-2s-d} \in \mathcal{L}^1(X)
\]

with some \( s \geq 0 \) and \( t \geq 0 \). Then for \( h \to +0 \), \( 1 \leq \mu \) such that \( \mu h \leq t \) the “standard” asymptotics

\[
(2.25) \quad N^-(\mu, h) = N^-(\mu, h) + O(\mu^{-1} h^{-1})
\]

holds with

\[
(2.26) \quad N^-(\mu, h) := h^{-d} \int N^{\text{MW}}(x, \mu h) \, dx.
\]

### 2.1.3 Power singularities

**Example 2.3.** (i) Let 0 be an inner singular point\(^4\) and let conditions of Theorem 2.2 be fulfilled with \( \gamma = \epsilon |x|, \rho = |x|^m, \rho_1 = |x|^{m_1} \) and let \( m_{1} < \min(m - 2, 2m) \)^5.6.

\(^4\) I.e. 0 \( \in \bar{X} \) is an isolated point of \( \mathbb{R}^2 \setminus X \).

\(^5\) One can construct such potential \((V_1, V_2)\) easily; f.e. \( V_1 = -x_2 |x|^{m_1}, V_2 = x_1 |x|^{m_1} \) for \( m_1 \neq 2 \); for \( m_1 = -2 \) one needs to multiply \( V_1, V_2 \) by \( \log |x| \).

\(^6\) To have the non-degeneracy condition (2.15) fulfilled in the vicinity of 0 we assume that

\[
(2.27) \quad |\nabla V F^{-1}| \geq C \rho^2 \rho_1^{-1} \gamma^{-1} \quad \text{as} \quad |x| \leq \epsilon;
\]

in Statement (ii) one should replace \( |x| \leq \epsilon \) by \( |x| \geq \epsilon \).
Let $V + \mu h F \geq \epsilon \rho^2$ on $(\partial X \setminus 0) \cup \{x : |x| \geq c\}$.

Then conditions (2.23), (2.24) are fulfilled automatically and asymptotics (2.25)–(2.26) holds for $\mu$ disjoint from 0 and $h \to +0$.

Further,

$$\mathcal{N}^- (\mu, h) = \begin{cases} O(h^{-2}) & m > -1, \\ O(h^{-2}(|\log(\mu h)| + 1)) & m = -1, \\ O(h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}) & m < -1. \end{cases}$$

Furthermore, one can replace “$= O$” with “$\asymp$” if either $m > -1$, $\mu h \leq t$ and

$$\{X \setminus 0, V \leq -tF - \epsilon \} \neq \emptyset$$

or $m \leq -1$ and

$$V \leq -\epsilon \rho^2 \quad \text{in} \quad \Gamma \cap \{|x| \leq \epsilon\} \subset X$$

where $\Gamma$ is an open non-empty sector (cone) with vertex at 0, and $\mu h \leq t$ with a small enough constant $t > 0$.

(ii) Let infinity be an inner singular point and let conditions of Theorem 2.2 be fulfilled with $\gamma = \epsilon (x)$, $\rho (x)^m$, $\rho_1 = (x)^{m_1}$ and let $m_1 > \max(m-1, 2m)$.

Let $V + \mu h F \geq \epsilon \rho^2$ on $\partial X$.

Then conditions (2.23) and (2.24) are fulfilled automatically and asymptotics (2.25)–(2.26) holds for $\mu$ disjoint from 0 and $h \to +0$.

Further,

$$\mathcal{N}^- (\mu, h) = \begin{cases} O(h^{-2}) & m < -1, \\ O(h^{-2}(|\log(\mu h)| + 1)) & m = -1, \\ O(h^{-2}(\mu h)^{2(m+1)/(2m-m_1)}) & m > -1. \end{cases}$$

Furthermore, one can replace “$= O$” by “$\asymp$” if either $m < -1$, $\mu h \leq t$ and

$$\{X \setminus 0, V \leq -\epsilon \rho^2 \} \neq \emptyset$$

or $m \leq -1$,

$$V \leq -\epsilon \rho^2 \quad \text{in} \quad \Gamma \cap \{|x| \geq c\} \subset X$$

where $\Gamma$ is an open non-empty sector (cone) with vertex at 0, and $\mu h \leq t$ with a small enough constant $t > 0$.

7) I.e. $\mathbb{R}^2 \setminus X$ is compact.
(iii) One can easily see that for \( m > -1 \) in (i), \( m < -1 \) in (ii)

\[
(2.31) \quad \mathcal{N}^- (\mu, h) = \mathcal{N}^W^- (\mu, h) + \mathcal{O} \left( h^{-2} (\mu h)^{(2m+2)/(2m-m_1)} \right)
\]

under condition (2.15) where \( \mathcal{N}^W^- \) is the standard Weyl expression\(^8\). Therefore for \( \mu \leq h^p \) with \( p = (2 - m_1)/(4m - m_1 + 2) \) the asymptotics remain true with \( \mathcal{N}^- \) replaced by \( \mathcal{N}^W^- \).

On the other hand, under condition (2.30) or (2.30)# (respectively)

\[ \mathcal{N}^W^- - \mathcal{N}^W^- \geq \varepsilon h^{-2} (\mu h)^{(2m+2)/(2m-m_1)}. \]

Thus, for \( \mu > h^p \) one cannot replace \( \mathcal{N}^- \) by \( \mathcal{N}^W^- \) and preserve the remainder estimates. Note that \( p \) is not necessarily negative in our conditions! Therefore, due to the singularity, the magnetic field can be essential even for a fixed \( \mu \). In what follows we leave this type of analysis to the reader.

**Example 2.4.** (i) Assume now that \( m_1 \geq \min (m - 1, 2m), m_1 \neq 2m \)\(^9\), while all other assumptions of Example 2.3(i) are fulfilled. Since we want to have a finite \( \mathcal{N}^- \) and (2.30) to be fulfilled we need to assume that \( m > -1 \). Then \( \mathcal{N} \approx h^{-2} \). Let us calculate the remainder estimate.

(a) Assume first that \( m - 1 \leq m_1 < 2m \); then \( \int \rho_1^{-1} \rho_2 \, dx < \infty \) and then contribution of the semiclassical zone \( X' = \{ x: |x| \geq r_0 = h^{1/(m+1)} \} \) to the remainder is \( \mathcal{O} (\mu^{-1} h^{-1}) \) and we need to estimate the contribution of the singular zone \( X'' = \{ x: |x| \leq r_0 \} \). Without any loss of the generality we can assume that \( |V| \leq C \rho_1 \gamma \)\(^10\). Then LCR\(^11\) implies that the contribution of \( X'' \) to the asymptotics does not exceed \( Ch^{-2} \int_{X''} (\rho^2 + \mu^2 \rho_1^2 \gamma^2) \, dx \approx C \).

(b) Let now \( m_1 > 2m \); then we need to consider zones

\[
X'_2 = \{ x: |x| \geq r_1 = \mu^{-1/(m+1-m)} \}
\]

where \( \mu_{\text{eff}} \geq 1 \),

\[
X'_1 = \{ x: r_0 \leq |x| \leq r_1 \}
\]

where \( \mu_{\text{eff}} \leq 1 \) and \( h_{\text{eff}} \leq 1 \) and \( X'' \). Contributions of \( X'_2, X'_1 \) to the remainder do not exceed \( K_1 := C \mu^{-1} h^{-1} \int_{X'_2} \rho_1^{-1} \rho_2 \, dx \) and \( K_1 := Ch^{-1} \int_{X'_1} \rho_1 \gamma^{-1} \, dx \) respectively.

\(^8\) Rather than the magnetic Weyl expression \( \mathcal{N}^W^- \).

\(^9\) Otherwise we could not satisfy (2.27).

\(^10\) One can prove it easily taking \( V_1 = -\partial_2 \phi, V_2 = \partial_1 \phi \) with \( \phi \) solving \( \Delta \phi = F \).

\(^11\) Sure, LCR does not hold for \( d = 2 \) but we can use more complicated Rozenblioum’s estimate exactly like in Section ?? of [Ivr2].
Obviously, $K_1 \asymp K_2 \asymp Ch^{-1}t_1^{m+1} \asymp Ch^{-1}\mu^{-m/(m+1)}$ for $m_1 > 2m$. Finally, contribution of $X''$ to the asymptotics is $O(1)$.

Thus as $h \to +0$, $\mu$ is disjoint from 0 and $\mu h$ is disjoint from infinity, we have asymptotics

$$N^- (\mu, h) = \mathcal{N}^- (\mu, h) + \begin{cases} O(h^{-1}\mu^{-1}) & m_1 < 2m, \\ O(h^{-1}\mu^{-m+1/(m+1)}) & m_1 > 2m. \end{cases}$$

(ii) Assume now that $m_1 \leq \max(m - 1, 2m)$ while all other assumptions of Example 2.3(ii) are fulfilled and $m < -1$. Again, considering cases

(a) $2m < m_1 < m - 1$ and
(b) $m_1 < 2m$

we arrive to the asymptotics

$$N^- (\mu, h) = \mathcal{N}^- (\mu, h) + \begin{cases} O(h^{-1}\mu^{-1}) & m_1 > 2m, \\ O(h^{-1}\mu^{-m+1/(m+1)}) & m_1 < 2m. \end{cases}$$

Consider now fast increasing $\mu$ so that $\mu h \to \infty$. We will get non-trivial results only when domain defined by $\mu_{\text{eff}}h_{\text{eff}} \leq C_0$ shrinks but remains non-empty which happens only in the frameworks of subcases (b) of Example 2.4.

**Example 2.5.** (i) In the framework of Example 2.4(i) with $m_1 > 2m$ consider $\mu h \to \infty$. Then the allowed domain is

$$\{x: |x| \lesssim r_2 = (\mu h)^{-1/(m_1 - 2m)}\}$$

and we have $r_0 \leq r_1 \leq r_2$ if $\mu \lesssim h^{-(m_1 + 1 - m)/(m+1)}$ while for $\mu \gtrsim h^{-(m_1 + 1 - m)/(m+1)}$ inequalities go in the opposite direction.

Therefore as $h \to +0$, $ch^{-1} \leq \mu \leq h^{-(m_1 + 1 - m)/(m+1)}$ asymptotics (2.32) holds and one can see easily that $N^- (\mu, h) \asymp h^{-2}r_2^{2m+2}$:

$$N^- (\mu, h) \asymp \mu^{-2(m+1)/(m_1 - 2m)} h^{-2(m_1 + 1 - m)/(m_1 - 2m)}.$$
Let us consider $\mu \to \mu_0$ where $\mu_0 \geq 0$ is fixed.

**Example 2.6.** (i) Let us consider singularity at $0$. In this case we are in the framework of Section ?? of [Ivr2] provided $m > -1$, $m_1 > -2$. So we need to consider the case when either $m \leq -1$ or $m_1 \leq -2$ and $m_1 \neq 2m$.

(a) The contribution of the semiclassical zone to the remainder is $O(h^{-1})$ only if $m > -1$.

(b) Assume now that $m \leq -1$; then we need to assume that $m_1 < 2m$ and we have a normal magnetic field zone $X'_1 = \{x: |x| \geq r_1 = \mu^{-1/(m_1-m+1)}\}$, strong magnetic field zone $X'_2 = \{x: r_0 = (\mu h)^{-1/(m_1-2m)} \leq |x| \leq r_1\}$, and forbidden zone $X'_3 = \{x: |x| \leq r_1\}$. One can check that $r_0 < r_1$. Then the contribution to the remainder of $X'_1$ and $X'_2$ are both $O(h^{-1}r_1^{m+1}) = O(h^{-1}\mu^{-2(m+1)/(m_1-m+1)})$ as $m < -1$ while for $m = -1$ contribution of $X'_1$ is $O(h^{-1}(|\log \mu| + 1))$ and of $X'_2$ is $O(h^{-1})$. Contribution of $X'_3$ is smaller. So we get a remainder estimate

\[
N^-(\mu, h) = N^-(\mu, h) + \begin{cases} 
O(h^{-1}), & m > -1, \\
O(h^{-1}(|\log \mu| + 1)) & m = -1, \\
O(h^{-1}\mu^{-2(m+1)/(m_1-m+1)}) & m < -1.
\end{cases}
\]

Meanwhile, one can prove easily that

\[
N^-(\mu, h) \asymp \begin{cases} 
h^{-2} & m > -1, \\
h^{-2}(|\log \mu h| + 1) & m = -1, \\
h^{-2(m_1-m+1)/(m_1-2m)}\mu^{-2(m+1)/(m_1-2m)} & m < -1.
\end{cases}
\]

under assumption (2.30).

(ii) Similarly, consider the singularity at infinity. Then

(a) Let $m < -1$; then the remainder estimate is $O(h^{-1})$.

(b) Let $m \geq -1$, $m_1 > 2m$. Then

\[
N^-(\mu, h) = N^-(\mu, h) + \begin{cases} 
O(h^{-1}), & m < -1, \\
O(h^{-1}(|\log \mu| + 1)) & m = -1, \\
O(h^{-1}\mu^{-2(m+1)/(m_1-m+1)}) & m > -1.
\end{cases}
\]
and

\[
N^{-}(\mu, h) \simeq \begin{cases} 
  h^{-2} & M < -1, \\
  h^{-2(|\log \mu h| + 1)} & m = -1, \\
  h^{-2(m-m+1)/(m-2m)\mu^{-2(m+1)/(m-2)}} & m > -1.
\end{cases}
\]

under assumption (2.30)#.

### 2.1.4 Improved remainder estimates

Let us improve remainder estimates (2.32), (2.32)#, (2.35), (2.35)# under certain non-periodicity-type assumptions.

**Example 2.7.** (i) In the framework of Example 2.6(i) with \( m > -1 \) the contribution to the remainder of the zone \( \{x: |x| \leq \varepsilon\} \) does not exceed \( \sigma h^{-1} \) with \( \sigma = \sigma(\varepsilon) \to 0 \) as \( \varepsilon \to +0 \). Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian billiards\(^{12}\) with the Hamiltonian

\[
a(x, \xi, \mu_0) = \sum_{j,k} g_{jk}(\xi_j - \mu_0 V_j)(\xi_k - \mu_0 V_k) + V(x)
\]

the improved asymptotics

\[
N^{-}(\mu, h) = N^{-}(\mu, h) + \kappa_1 h^{-1} + o(h^{-1})
\]

holds as \( h \to +0, \mu \to \mu_0 \) where \( \kappa_1 h^{-1} \) is the contribution of \( \partial X \) calculated as \( \mu = \mu_0 \). However, in the general case we cannot replace \( N^{MW} \) by \( N^{W} \) even if \( \mu_0 = 0 \).

(ii) Similarly in the framework of Example 2.6(ii) with \( m < -1 \) under the standard non-periodicity assumption for Hamiltonian billiards\(^{12}\) with the Hamiltonian (2.37) asymptotics (2.38) holds as \( h \to +0, \mu \to \mu_0 \).

(iii) In the framework of Example 2.6(i) with \( m < -1 \) the contributions to the remainder of the zones \( \{x: |x| \leq \varepsilon r_1\} \) and \( \{x: |x| \geq \varepsilon^{-1} r_1\} \) do not exceed \( \sigma h^{-1} r_1^{m+1} \) with \( \sigma = \sigma(\varepsilon) \to 0 \) as \( \varepsilon \to +0 \). After scaling \( x \to x r_1^{-1} \) etc.

\(^{12}\) On the energy level 0.
the magnetic field in the zone \( \{ x : \varepsilon r_1 \leq |x| \leq \varepsilon^{-1} r_1 \} \) becomes disjoint from 0 and \( \infty \).

Assume that \( g^{jk}, V_j, V \) stabilize to positively homogeneous of degrees 0, \( m_1 + 1 \), 2 functions \( g^{jk}_0, V^0_j, V^0 \) as \( x \to 0 \):

\[
\begin{align*}
(2.39)_1 & \quad D^\alpha (g^{jk} - g^{jk}_0) = o(|x|^{-|\alpha|}), \\
(2.39)_2 & \quad D^\alpha (V_j - V^0_j) = o(|x|^{m_1 + 1 - |\alpha|}), \\
(2.39)_3 & \quad D^\alpha (V - V^0) = o(|x|^{2m - |\alpha|}) \quad \forall \alpha : |\alpha| \leq 1.
\end{align*}
\]

Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13}\) with the Hamiltonian

\[
(2.40) \quad a^0(x, \eta) = \sum_{j,k} g^{jk}_0 (\eta_j - V^0_j) (\eta_k - V^0_k) + V^0
\]

the improved asymptotics

\[
(2.41) \quad N^-(\mu, h) = N(\mu, h) + o(h^{-1} \mu^{-(m+1)/(m_1+1-m)})
\]

holds as \( h \to +0, \mu \to 0 \).

(iv) Similarly in the framework of Example 2.6(ii) with \( m > -1 \) let stabilization conditions \((2.39)_{1-3}\) be fulfilled. Then under the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13}\) with the Hamiltonian \((2.40)\) asymptotics \((2.41)\) holds as \( h \to +0, \mu \to 0 \).

(v) In the framework of Example 2.6(i) with \( m = -1 \) the contributions to the remainder of the zone \( \{ x : |x| \leq r_1 \} \) does not exceed \( Ch^{-1} \), while the contributions to the remainder of the zones \( \{ x : r_1 \leq |x| \leq r_1 \mu^{-\delta} \} \) and \( \{ x : |x| \geq \mu^\delta \} \) do not exceed \( C\delta h^{-1} |\log \mu| \) respectively. So, only zone \( \{ x : r_1 \mu^{-\delta} \leq |x| \leq \mu^\delta \} \) should be treated. After rescaling magnetic field in this zone is small.

Let stabilization conditions \((2.39)_{1,3}\) be fulfilled. Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13}\) with the Hamiltonian

\[
(2.42) \quad a^0(x, \eta) = \sum_{j,k} g^{jk}_0 \eta_j \eta_k + V^0
\]

\(^{13)\) In \( T^*(\mathbb{R}^2 \setminus 0) \).
the improved asymptotics

\[(2.43) \quad N^{-}(\mu, h) = N(\mu, h) + o(h^{-1}|\log \mu|)\]

holds as \(h \to +0, \mu \to 0\).

(vi) Similarly in the framework of Example 2.6(ii) with \(m = -1\) let stabilization conditions \((2.39)_{1,3}^{#}\) be fulfilled. Then under the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13)}\) with the Hamiltonian \((2.42)\) asymptotics \((2.43)\) holds as \(h \to +0, \mu \to 0\).

Consider now case of \(\mu \to \infty\); we would like to improve estimates \((2.32)\) for \(m_1 \geq 2m\) and \((2.32)^{#}\) for \(m_1 < 2m\). Recall that these estimates hold provided \(\mu \lesssim h^{-(m_1+1-m)/(m+1)}\).

**Example 2.8.** (i) In the framework of Example 2.4(i) with \(m_1 > 2m\) let stabilization conditions \((2.39)_{1-3}\) be fulfilled. Then using arguments of Example 2.7(iii) one can prove easily that under the the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13)}\) with the Hamiltonian \((2.40)\) asymptotics \((2.41)\) holds as \(h \to +0, \mu \to \infty\) provided \(\mu = o(h^{-(m_1+1-m)/(m+1)}).\)

(ii) Similarly in the framework of Example 2.4(ii) with \(m_1 < 2m\) let stabilization conditions \((2.39)_{1-3}^{#}\) be fulfilled. Then under the the standard non-periodicity assumption for Hamiltonian trajectories\(^{12),13)}\) with the Hamiltonian \((2.40)\) asymptotics \((2.41)\) holds as \(h \to +0, \mu \to \infty\) provided \(\mu = o(h^{-(m_1+1-m)/(m+1)}).\)

### 2.1.5 Degenerations

Consider magnetic field with degeneration.

**Example 2.9.** (i) Let \(0\) be an inner singular point of \(X\) and all assumptions of Example 2.3 (i) be fulfilled, except \((2.9)\) \(F \asymp \rho_1\) which is replaced now by

\[(2.44) \quad |F| + |\nabla F|_\gamma \asymp \rho_1.\]

let \(\Sigma = \{x : F_{12} = 0\}\) be the manifold of degeneration, \(Z = \{x : |F| \leq \epsilon \rho_1\}\) be the vicinity of the degeneration. Then we need to refer to Chapter 14 of [Ivr2]. To have \(\mu_{\text{eff}} \gg 1\), and also \(\mu_{\text{eff}} \geq Ch_{\text{eff}}^{-2}\) near singularity with as
before $\mu_{\text{eff}} = \mu|x|^{m_1+1-m}$, $h_{\text{eff}} = h|x|^{-m-1}$ we assume that $m_1 < 3m + 1$ if $m \leq -1$. Assume that $m_1 < \min(m - 1, 3m + 1)$.

We preserve the non-degeneracy assumption (2.27) in $X \setminus Z$ and replace it by\(^{14}\)

\[(2.45) \quad |\nabla \Sigma W| \approx \rho^2 \rho_1^{-3} \gamma^{-\frac{1}{3}}\]

where $W : |W| \lesssim \rho^2 \rho_1^{-\frac{2}{3}} \gamma^{\frac{2}{3}}$ is introduced according to Chapter 14 of [Ivr2]. Then in addition to $m_1 \neq 2m$ we assume also that $m_1 \neq 3m + 1$.

Recall that for the Schrödinger operator instead of $C_{m_1}^{-1} h_{\text{eff}}^{-1}$ the local remainder estimate now is $C_{m_1}^{-\frac{1}{2}} h_{\text{eff}}^{-1} \approx C_{m_1}^{-\frac{1}{2}} h^{-1}, h_{\text{eff}}^{-\frac{1}{2}} (\equiv m_1 + 3m + 1)$ and summation with respect to partition of unity returns $O(m_1^{-\frac{1}{2}} h^{-1})$ as $h \to +0$, $\mu$ is disjoint from 0 and $\mu^2 h$ is disjoint from infinity.

One can prove easily that $\mathcal{N}^{-}(\mu, h) \approx h^{-2}$ as $\mu \lesssim h^{-1}$ and $\mathcal{N}^{-}(\mu, h) \approx \mu^{-1} h^{-3}$ as $h^{-1} \lesssim \mu \lesssim h^{-2}$ exactly like in the regular case.

(ii) Similar results hold in the case when infinity is an inner singular point, $m_1 > \max(m - 1, 3m + 1)$.

The following problems are not very challenging but interesting:

**Problem 2.10.** (i) Consider the case of 0 being an inner singular point, $m > -1$, $m_1 \geq m - 1$. The threshold value is $m_1 = 3m + 1$. Assume that $h \to +0$.

(a) Consider cases $1 \lesssim \mu \lesssim h^{-1}$ and $h^{-1} \lesssim \mu \lesssim h^{-2}$ (cf. Example 2.4). Prove that in the case $m - 1 \leq m_1 < 3m + 1$ the remainder estimate is $O(-\mu^{\frac{1}{2}} h^{-1})$. Derive the remainder estimate in the case $m_1 > 3m + 1$. Calculate the magnitude of $\mathcal{N}^{-}$.

(b) Consider case $m_1 > 3m + 1$ and $\mu \gtrsim h^{-2}$ (cf. Example 2.5). Derive the remainder estimate. Calculate the magnitude of $\mathcal{N}^{-}$.

(c) Consider $\mu \to 0$ (cf. Example 2.6 and possibly Example 2.7). Derive the remainder estimate. Calculate the magnitude of $\mathcal{N}^{-}$.

(ii) Similarly consider the case of 0 being an inner singular point, $m > -1$, $m_1 \leq m - 1$.

\(^{14}\) We need it mainly to avoid correction terms.
Example 2.11. Stronger asymptotic degenerations are not easily accessible. F.e. even if Chapter 14 of [Ivr2] provides us with tools to consider $F_{12} = x^n|\xi|^m - n$ with $n = 2$, it does not provide us with a tool to deal with the “small perturbations” like $F_{12} = x^n|\xi|^m - n + bx^{n-2}|\xi|^m - n + 2$ with $m_2 > m_1$ ($m_2 < m_1$), if we consider vicinity of 0 (infinity respectively).

We want to recover whatever remainder estimates are possible.

(i) Case $n = 2$ and $b > 0$ should be the easiest as then the perturbation helps: the first rescaling is the standard $x \rightarrow 2(x - y)r^{-1}$, $r = |y|$ in $B(y, \frac{1}{2}r)$, transforming it to $B(0, 1)$, and as before $h \rightarrow h_{\text{eff}} = hr^{-m-1}$, $\mu \rightarrow \mu_{\text{eff}} = \mu r^{m+1}$ and perturbing field has the strength $\varepsilon \mu$ with $\varepsilon = r^{m_2 - m_1}$.

Then the second rescaling $x \rightarrow (x - y)\gamma^{-1}$ with $\gamma = |F|^{1/2}$ as long as $|F| \geq \bar{\gamma} = C_0 \max(\mu^{-1/3}, \varepsilon^{1/2})$ and $\bar{\gamma}$ otherwise. Therefore the contribution of $B(y, \gamma(y))$ to the remainder does not exceed $C_\mu h^{-1} \gamma^{-1}$ and the summation over partition results in $C_\mu h^{-1} \gamma^{-1}$, i.e. $Ch^{-1} \min(1, h^{-1}\varepsilon^{-\frac{1}{3}})$.

We leave to the reader to plug $h_{\text{eff}}, \mu_{\text{eff}}$ instead of $\mu, h$ and take a summation over the primary partition.

(ii) Cases $n = 3$ and $b > 0$ and $n = 2, 3$ and $b < 0$ are harder but for $m_2$ close enough to $m_1$ we refer to Chapter 14 of [Ivr2] after the second rescaling.

Then the second rescaling is the same $x \rightarrow (x - y)\gamma^{-1}$ with $\gamma = |F|^{1/n}$ as long as $|F| \geq \bar{\gamma} = C_0 \max(\mu^{-1/(n+1)}, \varepsilon^{1/2})$ and $\bar{\gamma}$ otherwise. Repeating the above arguments we conclude that the contribution of this zone is $Ch^{-1}$ for $\varepsilon \leq \mu^{2/(n+1)}$ and $Ch^{-1} \mu^{-1} \varepsilon^{-(n+1)/2}$ for $\varepsilon \leq \mu^{-2/(n+1)}$.

In the former case we are done, in the latter we need to explore zone $|F| \leq \varepsilon^{n/2}$. Rescaling we get $\mu' = \mu \varepsilon^{(n+1)/2}$ and $h' = h \varepsilon^{-1/2}$. Then we can refer to Chapter 14 of [Ivr2] rather than Chapter 13 of [Ivr2] and the contribution of $B(y, \varepsilon^{1/2})$ to the remainder does not exceed $C_\mu h^{-1/2} \varepsilon^{-3(n-1)/4}$ and summation over this zone results in $C_\mu h^{-1/2} \varepsilon^{-3(n+1)/4}$, which is greater than the contribution of the previous zone.

Again, we leave to the reader to plug $h_{\text{eff}}, \mu_{\text{eff}}$ instead of $\mu, h$ and take a summation over the primary partition.
2.1.6 Power singularities. II

Let us modify our arguments for the case $\rho_3 < 1$. Namely, in addition to (2.3)$_{1-3}$ we assume that

$$(2.46)_{1,2} \quad |D^\alpha g^{jk}| \leq c \rho_2 \gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \leq c \rho_2 \rho_1 \gamma^{-|\alpha|},$$

$$(2.46)_3 \quad |D^\alpha \frac{V}{F}| \leq c \rho_3 \rho^2 \rho_1^{-1} \gamma^{-|\alpha|} \quad \forall \alpha : 1 \leq |\alpha| \leq K$$

with $\rho_3 \leq \rho_2 \leq 1$ in the corresponding regions where $\rho, \rho_1, \gamma, \rho_3$ are scaling functions.

Recall that in Chapter 13 of [Ivr2] operator was reduced to the canonical form with the term, considered to be negligible, of magnitude $\rho_2 \mu_{\text{eff}}^{-2N}$. In this case impose non-degeneracy assumptions

$$(2.47)^* \quad |V + (2n + 1)\mu hF| \leq \epsilon_0 \rho_3 \rho^2, \quad n \in \mathbb{Z}^+ \implies |\nabla v^*| \geq \epsilon_0 \rho_3 \rho^2 \rho_1^{-1} \gamma^{-1}$$

and

$$(2.48) \quad \rho_3 \geq C_0 \rho_2 (\mu \rho_1 \gamma \rho^{-1})^{-N}$$

where $v^*$ is what this reduction transforms $VF^{-1}$ to and (2.48) means that “negligible” terms do not spoil (2.47)$^*$.

Then according to Chapter 13 of [Ivr2] the contribution of $B(x, \gamma)$ to the Tauberian remainder does not exceed

$$(2.49) \quad C (\rho_3 \mu_{\text{eff}}^{-1} h_{\text{eff}}^{-1} + 1)$$

while the approximation error does not exceed

$$(2.50) \quad C_2 \rho_2 \rho_3^{-1} \mu_{\text{eff}}^{-2N} h_{\text{eff}}^{-1} (\rho_3 \mu_{\text{eff}}^{-1} h_{\text{eff}}^{-1} + 1) \min((\mu_{\text{eff}} h_{\text{eff}} \rho_3^{-1})^s, 1)$$

with arbitrarily large $s$ and therefore selected factor could be skipped.

Example 2.12. (i) Let $0$ be an inner singular point and let assumptions (2.46)$_{1-3}$ be fulfilled with $\gamma = \epsilon |x|, \rho = |x|^m (|\ln |x|| + 1)^\alpha, \rho_1 = |x|^{2m (|\ln |x|| + 1)^\beta}, \rho_2 = 1$ and $\rho_3 = (||\ln |x|| + 1)^{-1}$.

Assume that
Then (2.48) is fulfilled with \( N = 1 \) and we can replace (2.47)* by
\[
|V + (2n + 1)\mu hF| \leq \epsilon_0\rho_3\rho^2, \quad n \in \mathbb{Z}^+ \implies
|\nabla VF^{-1}| \geq \epsilon_0\rho_3\rho^2\rho_1^{-1} \gamma^{-1}
\]
Then (2.49) becomes \( C(\mu^{-1}h^{-1}|\log r|^{2\alpha-\beta-1} + 1) \) and for \( s = 2 \) the summation over zone \( \{\mu_{\text{eff}} h_{\text{eff}} \lesssim 1\} = \{\mu h|\log r|^{\beta-2\alpha} \lesssim 1\} \) results in \( C((\mu h)^{-1} + (\mu h)^{-1/(\beta-2\alpha)}) \).

Meanwhile, (2.50) with \( N = 1 \) becomes \( C(\mu h)^s|\log r|^{s(\beta-2\alpha+1)+2} \) in the zone \( \{\mu h|\log r|^{\beta-2\alpha+1} \leq 1\} \) and the summation over this zone results in \( C(\mu h)^{-3/(\beta-2\alpha+1)} \).

On the other hand, (2.50) with \( N = 1 \) becomes \( C\mu^{-1}h^{-1}|\log r|^{1-\beta+2\alpha} \) in the zone \( \{\mu h|\log r|^{\beta-2\alpha+1} \geq 1, \mu h|\log r|^{\beta-2\alpha} \leq 1\} \) and the summation over this zone results in \( O(\mu^{-1}h^{-1}) \) as \( \beta > 2\alpha + 2 \), \( C(\mu h)^{-2/(\beta-2\alpha)} \) as \( \beta < 2\alpha + 2 \) and \( C\mu^{-1}h^{-1}|\log(\mu h)| \) as \( \beta = 2\alpha + 2 \). Therefore, we conclude that the remainder is \( O(R) \) with
\[
(2.52) \quad R := \begin{cases} 
(\mu h)^{-1} & \beta > 2\alpha + 2, \\
(\mu h)^{-1}|\log(\mu h)| & \beta = 2\alpha + 2, \\
(\mu h)^{-2/(\beta-2\alpha)} & 2\alpha < \beta < 2\alpha + 2.
\end{cases}
\]
In particular, for \( \beta \geq 2\alpha + 2 \) asymptotics (2.25)–(2.26) hold for \( h \to +0, \mu \) disjoint from 0 and \( \mu h \) disjoint from infinity\(^{15}\).

One can see easily that under condition (2.30) for \( \mu h \leq t \) with small enough \( t > 0 \)
\[
(2.53)_1 \quad N^-(\mu, h) \asymp h^{-2}|\log(\mu h)| \quad \text{for} \quad m = -1, \alpha = -\frac{1}{2},
\]
\[
(2.53)_2 \quad N^-(\mu, h) \asymp h^{-2}(\mu h)^{(2\alpha+1)/(\beta-2\alpha)} \quad \text{for} \quad m = -1, \alpha > -\frac{1}{2}
\]
and
\[
(2.53)_3 \quad \epsilon h^{-2}\exp\left(\epsilon(\mu h)^{1/(2\alpha-\beta)}\right) \leq N^-(\mu, h) \leq C_1 h^{-2}\exp\left(C(\mu h)^{1/(2\alpha-\beta)}\right)
\]
\(^{15}\) Provided \( V + \mu h F \geq \epsilon \rho^2 \) on \( \partial X \).
for \( m < -1 \) where \( C_1 > \epsilon > 0 \) are constants.

On the other hand, for \( m = -1, \alpha < -\frac{1}{2} \) the equivalence \( \mathcal{N}^-(\mu, h) \asymp h^{-2} \)
holds.

(ii) Let infinity be an inner singular point⁷) and let \((2.46)_{1-3}\) be fulfilled
with \( \gamma = \epsilon \langle x \rangle, \rho = \langle x \rangle^m (\log \langle x \rangle + 1) \), \( \rho_1 = \langle x \rangle^{2m} (\log \langle x \rangle + 1)^\beta, \rho_2 = 1, \rho_3 = (\log \langle x \rangle + 1)^{-1} \).
Assume that

\[
(2.51) \quad \text{Either } m > -1 \text{ and } \beta > 2\alpha \text{ or } m = -1 \text{ and } \beta > \max(\alpha, 2\alpha).
\]

Then all the statements of (i) remain true with the obvious modification:
condition \((2.30)\) should be replaced by \((2.30)⁺\) and estimates \((2.53)_3\) hold for \( m > -1 \).

Let us investigate further the case of \( \beta \leq 2\alpha + 2 \). Assume the following
non-degeneracy assumption

\[
(2.54) \quad -\langle x, \nabla \rangle \mathcal{V}^{-1} \geq \epsilon |\log |x||^{2\alpha - \beta - 1} \quad \text{for } |x| \leq \epsilon.
\]

In fact, we need to check it for \( \mathcal{V}^* = \mathcal{V} + \mu^{-2} \omega_1 \), but this condition for \( \mathcal{V} \)
and \( \mathcal{V}^* \) are equivalent as long as \( \mu_{\text{eff}}^2 \geq C_0 \rho_3 \). This condition is fulfilled
automatically for all \( r \) provided either \( m < -1 \) or \( \beta > \alpha + \frac{1}{2} \).

Consider first the semiclassical error. Recall that the contribution of
the partition element does not exceed \((2.49)\) and only summation over zone
where \( \rho_3 \lesssim \mu_{\text{eff}} h_{\text{eff}} \lesssim 1 \) could bring an error, exceeding \( C \mu^{-1} h^{-1} \).

Assume first that \( m < -1 \). Then in this zone \( (\mu h)^{-1/(2m+2)} |\log \mu h|^a \lesssim \rho \lesssim (\mu h)^{-1/(2m+2)} |\log \mu h|^b \) and then its contribution is \( C |\log (\mu h)| = O(\mu^{-1} h^{-1}) \).

Consider the case \( m = -1 \). In this case the problematic zone is \( \mathcal{Z} := (\mu h)^{-1/(\beta - 2\alpha + 1)} \lesssim |\log r| \lesssim (\mu h)^{-1/(\beta - 2\alpha)} \). Observe that \( \int_{\mathcal{Z}} \gamma^{-2} \, dx \propto (\mu h)^{-1/(\beta - 2\alpha)} \) which is \( O(\mu^{-1} h^{-1}) \) provided \( \beta \geq 2\alpha + 1 \).

We can improve these arguments in the following way. Let us observe
that the zone \( \{|\log r| \leq C_0 (\mu h)^{-1/(\beta - 2\alpha)}\} \) consists of the spectral strips

\[
(2.55) \quad \Pi_n = \{ x : |\mathcal{V}^* + (2n+1)\mu h | \leq C_1 |\log r|^{-1} \}
\]

with \( \mathcal{V}^* := (\mathcal{V} + \omega_1 \mu^{-2}) F^{-1} \) with \( n = 0, \ldots, n_0 = C_1 (\mu h)^{-1/(\beta - 2\alpha + 1)} \), separated by the lacunary strips.
Condition (2.54) provides that for fixed \( n \) and \( x |x|^{-1} \) (thus, only \( |x| \) varies) on \( \Pi_n \) the inequality \( |\partial \log |x|| \leq C_1 \) holds where \( \partial f \) means the oscillation of \( f \) (on \( \Pi_n \)) and the constant \( C_1 \) does not depend on \( n, x |x|^{-1}, \mu, h \).

Therefore one can easily see that the contribution of each strip \( \Pi_n \) to the semiclassical remainder does not exceed \( C \) and their total contribution does not exceed \( C n_0 = O(\mu^{-1} h^{-1}) \).

Meanwhile, contribution of each lacunary strip does not exceed the contribution of the adjacent spectral strip, multiplied by \( C \rho^3 \) and one can see easily that their total contribution is \( O(\mu^{-1} h^{-1}) \). Therefore

(2.56) If in the framework of Example 2.12(i) either \( m < -1 \) or \( m = -1, \beta > \alpha + \frac{1}{2} \) then the semiclassical error is \( O(\mu^{-1} h^{-1}) \).

The same arguments work for the approximation error (2.50) with \( N = 1 \): the contribution of the spectral strip gains factor \( \rho^3 \) and the contribution of the lacunary strip is 0. Then the total approximation error does not exceed “improved” (2.52):

\[
R := \begin{cases} 
(\mu h)^{-1} & \beta > 2\alpha + 1, \\
(\mu h)^{-1} |\log(\mu h)| & \beta = 2\alpha + 1, \\
(\mu h)^{-1/(\beta-2\alpha)} & 2\alpha < \beta < 2\alpha + 1.
\end{cases}
\]

Then we arrive to

**Example 2.13.** (i) Let all the assumptions of Example 2.12(i) be fulfilled, in particular (2.51). Moreover, let us assume that (2.54) is fulfilled and either \( m < -1 \) or \( \beta > \alpha + \frac{1}{2} \). Then for \( h \to +0, \mu \) disjoint from 0 and \( \mu h \) disjoint from infinity the asymptotics

(2.58) \( N^- (\mu, h) = \mathcal{N}^-(\mu, h) + O(R) \)

holds with \( R \) defined by (2.57).

(ii) Let all the assumptions of Example 2.12(ii) be fulfilled. Moreover, let us assume that condition

\[
(2.54)'' \quad \langle x, \nabla \rangle VF^{-1} \geq \epsilon |\log x|^{2\alpha - \beta - 1} \quad \text{for} \ |x| \geq c.
\]

is fulfilled and either \( m > -1 \) or \( \beta \geq \alpha + \frac{1}{2} \). Then for \( h \to +0, \mu \) disjoint from 0 and \( \mu h \) disjoint from infinity the asymptotics (2.58) holds with \( R \) defined by (2.57).
If we want to improve the remainder estimate in the case \(\beta \leq 2\alpha + 1\) we should take \(N = 2\), which would make our formulae more complicated but still “computable”. Then expression (2.50) with \(N = 2\) acquires (in comparison with the same expression with \(N = 1\)) the factor \(\mu^{-2}r^{-2(m+1)}|\log r|^{-2(\beta-\alpha)}\) and becomes

\[
(2.59) \quad \mu^{-3}r^{-2(m+1)}|\log r|^{-3\beta+4\alpha+1}.
\]

Further, we can get gain a factor \(|\log r|^{-1}\) due to above arguments concerning spectral and lacunary zones. Then after the summation we get \(O(\mu^{-3}h^{-1})\) if either \(m < -1\) or \(m = -1, 3\beta > 4\alpha + 1\). However if \(m = -1\) this is the case due to \(\beta > 2\alpha\) and \(\beta \geq \alpha + \frac{1}{2}\). Then we arrive to

Example 2.14. (i) Let all the assumptions of Example 2.12(i) be fulfilled, in particular (2.51). Moreover, let us assume that (2.54) is fulfilled and either \(m < -1\) or \(\beta > \alpha + \frac{1}{2}\).

Then for \(h \to +0\), \(\mu\) disjoint from 0 and \(\mu h\) disjoint from infinity the asymptotics

\[
(2.60) \quad N^-(\mu, h) = N^{--}(\mu, h) + O(\mu^{-1}h^{-1})
\]

holds where

\[
(2.61) \quad N^{--}(\mu, h) = \int h^{-2}N^{MW}(x, \mu h) \, dx
\]

and \(N^{MW}\) is defined by (13.4.133) of [Iv2] with \(\psi = 1\).

(ii) Let all the assumptions of Example 2.12(ii) be fulfilled. Moreover, let us assume that condition (2.54)# is fulfilled and either \(m > -1\) or \(\beta > \alpha + \frac{1}{2}\). Then for \(h \to +0\), \(\mu\) disjoint from 0 and \(\mu h\) disjoint from infinity the asymptotics 2.60–2.61 holds.

Remark 2.15. Asymptotics (2.60)–(2.61) still holds if \(m = -1\) and \(2\alpha < \beta \leq \alpha + \frac{1}{2}\) (and therefore \(\alpha < \frac{1}{2}\)) under assumption (2.62) below.

Indeed, recall that we need \(\beta > \alpha + \frac{1}{2}\) only to ensure that inequality (2.54) for \(V\) implies the same inequality for \(V^*\). This conclusion should be checked as \(|\log r| \leq C_0(\mu h)^{-1/(\beta-2\alpha)}\) only. Observe that this conclusion still holds in the case under consideration if \(\mu^{-2}(\mu h)^{(2\beta-2\alpha-1)/(\beta-2\alpha)} \leq \epsilon\), i.e.

\[
(2.62) \quad \mu \geq Ch^{(2\beta-2\alpha-1)/(1-2\alpha)}.
\]
Then the semiclassical error does not exceed \( C\mu^{-1}h^{-1} \) and an approximation error does not exceed \( C\mu^{-3}h^{-1}(\mu h)^{-2(\beta-2\alpha)/(\beta-2\alpha)} \) (multiplied by \(|\log(\mu h)|\) if \( 4\alpha-3\beta+1 = 0 \)). One can prove easily that it is less than \( C\mu^{-1}h^{-1} \) under assumption (2.62).

We leave to the reader

**Problem 2.16.** (i) Consider cases of \( \mu \to \mu_0 > 0 \) and \( \mu \to \mu_0 = 0 \) like in Example 2.6.

(ii) Consider cases of \( m > -1 \) and singularity at 0, \( m < -1 \) and singularity at infinity, and \( m = -1, \alpha < 0 \) and singularity either at 0 or at infinity. In these three cases assumption \( \beta > 2\alpha \) is not necessary; therefore, one needs to consider also the case \( \beta < 2\alpha \).

(iii) Furthermore, in the framework of (ii) and \( \beta < 2\alpha \) consider the case of \( \mu h \to \infty \) like in Example 2.5.

### 2.1.7 Exponential singularities

Consider now singularities of the exponential type. The following example follows immediately from Theorem 2.2:

**Example 2.17.** (i) Let 0 be an inner singular point\(^4\) and let conditions of Theorem 2.2 be fulfilled with \( \gamma = \epsilon|x|^{1-\beta}, \rho = \exp(a|x|^\alpha), \rho_1 = \exp(b|x|^\beta) \) with either \( 0 > \alpha > \beta, a > 0, b > 0 \) or \( 0 > \alpha = \beta, b > 2a > 0 \).

Let \( V + \mu hF \geq \epsilon \rho^2 \) on \((\partial X \setminus 0) \cup \{x : |x| \geq c\}\). Then conditions (2.23) and (2.24) are fulfilled automatically and for \( \mu \geq 1, h \to +0 \) asymptotics (2.25)–(2.26) hold.

Moreover, under condition (2.30) for \( \alpha = \beta, \mu h < t \) with a small enough constant \( t > 0 \)

\[
(2.63) \quad \mathcal{N}^-(\mu, h) \asymp h^{-2}(\mu h)^{-2a/(b-2a)|\log \mu h|^2 - 1};
\]

otherwise the left-hand expression is “\( O \)” only.

(ii) Let infinity be an inner singular point\(^7\) and let conditions of Theorem 2.2 be fulfilled with \( \gamma = \epsilon<x>^{1-\beta}, \rho = \exp(a<x>^\alpha), \rho_1 = \exp(b<x>^\beta) \) where either \( 0 < \alpha < \beta, a > 0, b > 0 \) or \( 0 < \alpha = \beta, b > 2a > 0 \).
Let $V + \mu hF \geq \epsilon \rho^2$ on $\partial X$. Then conditions (2.23)–(2.24) are fulfilled automatically and asymptotics (2.25)–(2.26) hold.

Moreover, under condition (2.30) for $\alpha = \beta$, $\mu h < t$ with a small enough constant $t > 0$ (2.45) holds; otherwise the left-hand expression is “$O$” only.

We leave to the reader the following

**Problem 2.18.** Calculate magnitude of $N^- (\mu, h)$ as $\alpha < \beta$ in Example 2.17.

**Example 2.19.** (i) Let $0$ be an inner singular point$^4$. Assume that conditions of Theorem 2.2 are fulfilled with $\gamma = \epsilon |x|^{1-\beta}$, $\rho = |x|^m \exp(|x|^{\beta})$, $\rho_1 = |x|^{m_1} \exp(2|x|^{\beta})$, $\rho_2 = 1$, $\rho_3 = |x|^{-\beta}$ and let $\beta < 0$, $m_1 < 2m$.

Let $V + \mu hF \geq \epsilon \rho^2$ on $(\partial X \setminus 0) \cup \{x: |x| \geq c\}$. Then conditions (2.24), (2.48) are fulfilled automatically.

Moreover, condition

$$\int \rho_3^{-1} \rho_1^{-1} \rho^2 \, dx < \infty$$

is fulfilled provided $2m + 3\beta > m_1$; then asymptotics (2.25)–(2.26) holds. Moreover, under condition (2.30) while we cannot calculate magnitude of $N^- (\mu, h)$ itself, we conclude that

$$\log(N^- (\mu, h)) \asymp (\mu h)^{\beta/(2m-m_1)}.$$  

(ii) Let infinity be an inner singular point$^7$. Assume that conditions of Theorem 2.2 are fulfilled with $\gamma = \epsilon_0 \langle x \rangle^{1-\beta}$, $\rho = \langle x \rangle^m \exp(\langle x \rangle^{\beta})$, $\rho_1 = \langle x \rangle^{m_1} \exp(2\langle x \rangle^{\beta})$, $\rho_2 = 1$, $\rho_3 = \langle x \rangle^{-\beta}$ where $\beta > 0$, $m_1 > 2m$.

Let $V + \mu hF \geq \epsilon \rho^2$ on $\partial X$. Then condition (2.64) is fulfilled provided $2m + 3\beta < m_1$; then asymptotics (2.25)–(2.26) holds. Moreover, under condition (2.30)$^#$ estimates (2.65) hold.

Consider case of $2m + 3\beta \leq m_1$, $2m + 3\beta \geq m_1$ in the frameworks of Statements (i) and (ii) respectively. One can get some remainder estimates albeit less precise. We want to improve them using the same technique as in Subsubsection 2.1.6.6. **Power singularities. II.** However now things are a bit simpler. First of all, condition $\mu_{\mathrm{eff}}^2 \leq \rho_3$ is fulfilled automatically.
Contribution of the zone $Z_1 = \{ r \geq (\mu h)^{1/(-\beta-m_1+2m)} \}$ where $\rho_3 \geq \mu_{\text{eff}} h_{\text{eff}}$ to the semiclassical error does not exceed $(\mu h)^{-1} \int_{Z_1} \rho_3 \rho_1^{-1} \rho^2 \gamma^{-2} \, dx$. Contribution of the zone $Z_2 = \{ (\mu h)^{1/(-m_1+2m)} \leq r \leq (\mu h)^{1/(-\beta-m_1+2m)} \}$ where $\rho_3 \leq \mu_{\text{eff}} h_{\text{eff}} \lesssim 1$ to the semiclassical error does not exceed $\int_{Z_2} \rho_3 \gamma^{-2} \, dx$ due to the same “spectral and lacunary strips” arguments. Then the semiclassical error is does not exceed $C(\mu h)^{-1}$ provided $m_1 < 2m + \beta$.

Further, contributions of these zones $Z_j$ to the approximation error with $N = 1$ do not exceed $C \mu^{-1} h^{-1} \int_{Z_j} \rho^{-1} \rho^2 \gamma^{-2} \, dx$ and the approximation error is $O(\mu^{-1} h^{-1})$ provided $m_1 < 2m + 2\beta$.

On the other hand, one can see easily that the approximation error with $N = 2$ is $O(\mu^{-1} h^{-1})$ for sure. We arrive to the following

**Example 2.20.** (i) Let all the assumptions of Example 2.19(i) be fulfilled. Moreover, let us assume that condition

$$\langle x, \nabla \rangle VF^{-1} \geq \epsilon |x|^{2m-m_1} \quad \text{for } |x| \leq \epsilon.$$  

is fulfilled.

Then for $h \to +0$, $\mu$ disjoint from 0 and $\mu h$ disjoint from infinity the asymptotics (2.25)–(2.26) holds provided $m_1 < 2m + 2\beta$ and asymptotics (2.60)–(2.61) holds with $\mathcal{N}^{\text{MW'}}$ defined by (13.4.133) of [Ivr2]) with $\psi = 1$ provided $m_1 < 2m + \beta$.

(ii) Let all the assumptions of Example 2.12(ii) be fulfilled. Moreover, . Moreover, let us assume that condition

$$\langle x, \nabla \rangle VF^{-1} \geq \epsilon |x|^{2m-m_1} \quad \text{for } |x| \geq \epsilon.$$  

is fulfilled.

Then for $h \to +0$, $\mu$ disjoint from 0 and $\mu h$ disjoint from infinity the asymptotics (2.25)–(2.26) holds provided $m_1 > 2m + 2\beta$ and asymptotics (2.60)–(2.61) holds with $\mathcal{N}^{\text{MW'}}$ defined by (13.4.133) of [Ivr2]) with $\psi = 1$ provided $m_1 > 2m + \beta$.

**Example 2.21.** (i) In the framework of Example 2.17(i) assume that (2.66) is fulfilled. Moreover, let

$$|D^\sigma g^{jk}| \leq c|x|^{-|\sigma|}, \quad |D^\sigma \log F| \leq c|x|^{\beta-|\sigma|},$$

$$|D^\sigma V/F| \leq c|x|^{2m-m_1-|\sigma|} \quad \forall \sigma : 1 \leq |\sigma| \leq 2.$$
Then for $h \to +0$, $1 \leq \mu = O(h^{-1})$ asymptotics (2.25)–(2.26) holds provided $m_1 < 2m + \beta$.

Indeed, one can easily see that under condition (2.67)
\begin{equation}
|\omega_1| \leq C_1 F^{-1} |\rho|^{\beta + 4m - 2m_1 - 2} = \omega_1^*
\end{equation}
while the general theory yields only that
\[ |\omega_1| \leq C_1 F^{-1} |\rho|^{2\beta + 4m - 2m_1 - 2}. \]

Then the error $|N^- (\mu, h) - N^-' (\mu, h)|$ does not exceed
\[
C_1^{-1} h^{-1} \int_{\{\rho, \mu h \leq \rho, \rho^2\}} \rho_1^2 \rho_3^{-1} \omega_1^* \, dx + \int_{\{\rho, \mu h \leq \rho, \rho^2 \leq \rho_1 \rho^2\}} \rho_1 (\omega_1^* \mu^{-2} + \chi_n) \, dx
\]
where $\chi_n$ is the characteristic function of the set
\[ \Pi = \bigcup_{n \in \mathbb{Z}^+} \{x: |v + (2n + 1) \mu h| \leq C_1 \mu^{-2} \omega_1^*\}. \]

Applying estimate (2.68) and condition (2.66) one can prove that this error is $O(\mu^{-1} h^{-1})$.

(ii) Similarly, in the framework of Example 2.17(ii) assume that (2.66)* is fulfilled. Moreover, let
\begin{equation}
|D^\sigma g^{jk}| \leq c(x)^{-|\sigma|}, \quad |D^\sigma \log F| \leq c(x)^{\beta - |\sigma|} \quad |D^\sigma V/F| \leq c(x)^{2m - m_1 - |\sigma|} \quad \forall \sigma : 1 \leq |\sigma| \leq 2.
\end{equation}
Then for $h \to +0$, $1 \leq \mu = O(h^{-1})$ asymptotics (2.25)–(2.26) holds provided $m_1 > 2m + \beta$.

We leave to the reader:

**Problem 2.22.** Consider cases of $\mu \to \mu_0 > 0$ and $\mu \to \mu_0 = 0$ like in Example 2.6.
2.2 Schrödinger-Pauli operators

Consider now Schrödinger-Pauli operators, either genuine (0.34) or generalized (13.5.3) of [Ivr2]). The principal difference is that now \( F \) does not “tame” singularities of \( V \), on the contrary, it needs to be “tamed” by itself. As a result there are fewer examples than for the Schrödinger. Also we do not have a restriction \( \mu_{\text{eff}} h_{\text{eff}} = O(1) \) which we had in the most of the previous Subsection 2.1. Then we need to add \( 1 \) and \( \mu_{\text{eff}} h_{\text{eff}}^{-1} = \mu h^{-1} \rho_1 \gamma^2 \) to the contributions of this element to the remainder estimate and \( N^{-} (\mu, h) \) respectively.

Because of this, here we do not consider an abstract theorem like Theorem 2.2, but go directly to the examples.

Example 2.23 \( ^{16} \). (i) Let 0 be an inner singular point\(^4 \) and \( \gamma = \epsilon |x|, \rho = |x|^m, \rho_1 = |x|^{m_1} \) and let \( m > -1, 2m \neq m_1 > -2 \). Let our usual non-degeneracy assumptions be fulfilled. Then, in comparison with the theory of the previous Subsection 2.1, we need to consider also the the zone \( \mu_{\text{eff}} h_{\text{eff}} \gtrsim 1 \).

Its contribution to the remainder does not exceed \( C \int \gamma^{-2} \, dx \) while its contribution to \( N^{-} (\mu, h) \) does not exceed \( C h^{-1} \int \rho_1 \, dx \). Indeed, contributions of each \( \gamma \)-element do not exceed \( C \) and \( C \mu_{\text{eff}} h_{\text{eff}}^{-1} \simeq C h^{-1} \rho_1 \gamma^2 \).

(a) Let \( m_1 > 2m \); then this zone is disjoint from 0 and this expression is \( O(1) \). Furthermore, contribution of the zone \( \{ r \leq \mu_{\text{eff}}^{-1/(m_1+1-m)} \} \) where \( \mu_{\text{eff}} \lesssim 1 \) to the remainder does not exceed \( C h^{-1} \int \rho \gamma^{-1} \, dx \), taken over this zone, and it is \( \asymp h^{-1} \mu^{-(m+1)/(m_1+1-m)} \) and we arrive to (2.70)–(2.71) below; cf. (2.32).

(b) Consider now the case \( m_1 < 2m \). In this case zone where \( \mu_{\text{eff}} h_{\text{eff}} \gtrsim 1 \) is not disjoint from 0 and we need to consider also a singular zone \( \{ r \leq \bar{r} := (\mu h^{-1})^{-1/(m_1-2m)} \} \) where \( \mu_{\text{eff}}^{-1} h_{\text{eff}} \gtrsim 1 \).

However due to the variational estimates as before

\[(2.69) \quad \text{For the disk } \{ x : |x| \leq 2\bar{r} \} \text{ with the Dirichlet boundary conditions} \]

\[ N^{-} (\mu, h) \leq C h^{-2} \bar{r}^{2m+2} + C (\mu h^{-1})^2 \bar{r}^{2(m_1+2)} \]

which is \( O(1) \) due to the choice of \( \bar{r} \).

\(^{16} \) Cf. Example 2.3.
We leave to the reader to prove (2.69) and (2.69)# below and to justify the final result using methods of Section 10.1 of [Ivr2]:

\[(2.70)\]
\[N^-(\mu, h) = \mathcal{N}^-(\mu, h) + \begin{cases} O(h^{-1}\mu^{-1} + 1) & m_1 < 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)}) + 1 & m_1 > 2m. \end{cases} \]

with

\[(2.71)\]
\[\mathcal{N}^-(\mu, h) \asymp h^{-2} + \mu^{-1}. \]

(ii) Let infinity be an inner singular point and \(\gamma = \epsilon \langle x \rangle, \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}\) and let \(m < -1, 2m \neq m_1 < -2\). Let our usual non-degeneracy assumptions be fulfilled. Then again we need to consider cases (a) \(m_1 < 2m\) and (b) \(m_1 > 2m\) and in the latter case we need to consider contribution of zone \(\{r \leq \tilde{r} := (\mu^{-1}h)^{-1/(m_1-2m)}\}\).

However due to variational estimates

\[(2.69)^\#\] For the zone \(\{x: |x| \geq \frac{1}{2}\tilde{r}\}\) with the Dirichlet boundary conditions

\[N^-(\mu, h) \leq C h^{-2} \tilde{r}^{2m+2} + C (\mu h^{-1})^2 \tilde{r}^{2(m+2)}\]

which is again \(O(1)\) due to the choice of \(\tilde{r}\). Then

\[(2.70)^\#\]
\[N^-(\mu, h) = \mathcal{N}^-(\mu, h) + \begin{cases} O(h^{-1}\mu^{-1} + 1) & m_1 > 2m, \\ O(h^{-1}\mu^{-(m+1)/(m_1+1-m)}) + 1 & m_1 < 2m. \end{cases} \]

and (2.71) hold.

We leave to the reader the following problems:

Problem 2.24. (i) As 0 is an inner point consider both Schrödinger and Schrödinger-Pauli operators as

(a) \(\gamma = |x|, \rho = |x|^m, m > -1\) and \(\rho_1 = |x|^{-2} \log |x|^\beta\).

\[\tag{17} \text{One should take } \beta < -1 \text{ for the Schrödinger-Pauli operator.} \]
(b) $\gamma = |x|, \rho = |x|^{-1}|\log |x||^\alpha, \alpha < -\frac{1}{2}$, and $\rho_1 = |x|m_1, m_1 > -2$.

(ii) As infinity is an inner point consider both Schrödinger and Schrödinger Pauli operators as

(a) $\gamma = |x|, \rho = |x|^m, m < -1$ and $\rho_1 = |x|^{-2}|\log |x||_1^{\beta}\).

(b) $\gamma = |x|, \rho = |x|^{-1}|\log |x||^\alpha, \alpha < -\frac{1}{2}$, and $\rho_1 = |x|m_1, m_1 > -2$.

For the Schrödinger operator in cases (a) non-trivial results could be also obtained even as $\mu h \to +\infty$.

Problem 2.25. Let either $0$ or infinity be an inner singular point.

Using the same arguments and combining them with the arguments of Subsubsection 2.1.6.6, Power singularities. II consider both Schrödinger-Pauli and Schrödinger operators with $\gamma = |x|, \rho = |x|^{-1}|\log |x||\alpha, \rho_1 = |x|^{-2}|\log |x||_1^{\beta}$.

Take into account whether $\beta > 2\alpha$ or $\beta < 2\alpha$. For the Schrödinger operator in case $\beta < 2\alpha$ non-trivial results could be obtained even as $\mu h \to +\infty$.

2.3 Dirac operator

2.3.1 Preliminaries

Let us now consider the generalized magnetic Dirac operator (17.1.1) of [Ivr1,2]

$$A = \frac{1}{2} \sum_{i,j} \sigma_i(\omega^j P_j + P_j \omega^i) + \sigma_0 M + I \cdot V, \quad P_j = hD_j - \mu V_j$$

(2.72)

where $\sigma_0, \sigma_1, \sigma_2$ are $2 \times 2$-matrices.

We are interested in $N(\tau_1, \tau_2)$, the number of eigenvalues in $(\tau_1, \tau_2)$, with $\tau_1 < \tau_2$, fixed in this subsection.

The theory of the Dirac operator is more complicated than the theory of the Schrödinger operator because it is different in the cases when $V \pm M$.

---

18) One needs to consider cases $\alpha < -1, \alpha = -1, -1 < \alpha < -\frac{1}{2}$ separately.
19) Assuming that this interval does not contain essential spectrum; otherwise $N(\tau_1, \tau_2) := \infty$. It is more convenient for us to exclude both ends of the segment.
20) If $V \pm M \in [\tau_1 + \epsilon, \tau_2 - \epsilon]$ for both signs $\pm$, then infinity is not a singular point.
and $F$ tend to 0 and (or) $\infty$ and thus singularities at 0 and at infinity should be treated differently.

Furthermore, the theory of the magnetic Dirac operator is even more complicated. Indeed, the “pointwise Landau levels” are $V \pm \left(M^2 + 2j\mu hF\right)^{\frac{1}{2}}$ with $j = 0, 1, 2, \ldots$ but one of those is, in fact, excepted: with the same sign as $\varsigma_{F12}$ and $j = 0$. Recall that $\varsigma = \pm 1$ is defined by (17.1.14) of [Ivr2])

\begin{equation}
(2.73) \quad \sigma_0 \sigma_1 \sigma_2 = \varsigma i.
\end{equation}

Without any loss of generality one can assume that

\begin{equation}
(2.74) \quad \varsigma = 1, \quad F_{12} > 0.
\end{equation}

Then the Landau levels (at the point $x$) are $V + \left(M^2 + 2j\mu hF\right)^{\frac{1}{2}}$ with $j = 1, 2, 3, \ldots$ and $V - \left(M^2 + 2j\mu hF\right)^{\frac{1}{2}}$ with $j = 0, 1, 2, \ldots$. But then the negative $V$ could be “tamed” by a larger $F$. This allows us to get meaningful results in the situations impossible for non-magnetic Dirac operator: $V$ as singular at 0 as $|x|^m$ with $m \leq -1$ or $V + M$ as singular at infinity as $|x|^{2m}$ with $m \geq -1$ (as $M > 0$) provided it is negative there. Furthermore, if $F \to \infty$ as $|x| \to \infty$, we can get meaningful results even if $M = 0$.

On the other hand, we often should prove that the Dirac operator is essentially self-adjoint while the Schrödinger operator is obviously semibounded and therefore essentially self-adjoint. We do this in Appendix 6.A.

Therefore, we treat the operator given by (2.72) under the following assumptions

\begin{align}
(2.75)_{1-2} & \quad |D^\alpha \omega^{jk}| \leq c \gamma^{-|\alpha|}, \quad |D^\alpha F| \leq c \rho_1 \gamma^{-|\alpha|}, \\
(2.75)_{3} & \quad |D^\alpha V| \leq c \min(\rho, \frac{1}{M\rho^2}) \gamma^{-|\alpha|} \quad (\alpha \neq 0) \quad \forall \alpha: |\alpha| \leq K, \\
(2.75)_{4+} & \quad (V - \tau_2 - M)_+ \leq c \min(\rho, \frac{1}{M\rho^2}), \\
(2.75)_{4-} & \quad (V - \tau_1 + M)_- \leq c \min(\rho, \frac{1}{M\rho^2})
\end{align}

and also (2.3)_{1-3}, (2.2) for $g^{jk} = \sum_{l,r} \omega^l \omega^{kr} \delta_{lr}$ and (2.9) (with $F > 0$). In what follows (2.75)_{4} means the pair of conditions (2.75){4\pm}.

Moreover, let condition (2.13) be fulfilled and

\begin{align}
(2.76) & \quad \bar{X}'' \cap \partial X = \emptyset, \\
(2.77) & \quad |V_j| \leq c\rho, \quad |D_j \omega^{kl}| \leq c\rho \quad \text{in} \quad X''.
\end{align}

Finally, we assume that
Either \( \partial X = \emptyset \) or \( \mu = O(1) \) and \( \partial X \cap X' = \emptyset \) (in what follows).

### 2.3.2 Asymptotics. I

**Example 2.26**

Let condition (2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with \( \gamma = \epsilon_0 |x|, \rho = |x|^m, \rho_1 = |x|^{m_1}, m_1 < \min(m - 1, 2m) \).

Further, let

\[
(2.79) \quad - V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x : |x| \leq \epsilon
\]

and let non-degeneracy condition

\[
(2.80) \quad | V + \varsigma (M^2 + 2j \mu h F)^{\frac{1}{2}} - \tau_i | \leq \epsilon \rho \quad \implies \quad | \nabla v_i | \geq \epsilon \rho_1 \rho^2 \gamma^{-1} \quad \text{or} \quad \det \text{Hess} v_i \geq \epsilon \rho_1^{-1} \rho^4 \gamma^{-4}
\]

be fulfilled with a small enough constant \( \epsilon' = \epsilon(c, \epsilon, \epsilon_0) > 0 \); this assumption is a part of condition (2.80))

(a) Let \( m < 0, \tau_1 < \tau_2 \). Then for \( h \to +0, 1 \leq \mu = O(h^{-1}) \) asymptotics (2.25)–(2.26) holds with \( N \) defined by \((17.1.12)_2\) of [Ivr2] with \( d = 2 \). Moreover,

\[
(2.81) \quad N(\tau_1, \tau_2, \mu, h) \asymp \begin{cases} 
    h^{-2} & m \geq -1, \\
    h^{-2}(\mu h)^{2(m+1)/(2m-1)} & m < -1.
\end{cases}
\]

(b) Let \( m > 0, M > 0, \tau_1 \in (-M, M), \tau_2 = M \). Then for \( h \to +0, 1 \leq \mu = O(h^{-1}) \) asymptotics (2.25)–(2.26) holds with \( N \) defined by \((17.1.12)_2\) of [Ivr2] with \( d = 2 \). Moreover, equivalence (2.28) holds.

---

\(^{21}\) Cf. Example 2.3.

\(^{22}\) Recall that \( v_i = F^{-1}((V - \tau_i)^2 - M^2) \) and the conditions should be fulfilled for both \( i = 1, 2 \).
(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with \( \gamma = \epsilon_0(x), \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, m > \max(m - 1, 2m) \).

Further, let

\[
(2.79) \# \quad - V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x : |x| \geq c
\]

and let non-degeneracy condition (2.80) be fulfilled.

(a) Let \( m > 0, \tau_1 < \tau_2 \). Then for \( h \to +0, 1 \leq \mu = O(h^{-1}) \) asymptotics (2.25)–(2.26) holds with \( \mathcal{N} \) defined by (17.1.12) of [Ivr2] with \( d = 2 \). Moreover,

\[
(2.81) \# \quad \mathcal{N}^{-}(\tau_1, \tau_2, \mu, h) \asymp h^{-2} (\mu h)^{2(m+1)/(2m-m_1)}.
\]

(b) Let \( m < 0, M > 0, \tau_1 \in (-M, M), \tau_2 = M \). Then for \( h \to +0, 1 \leq \mu = O(h^{-1}) \) asymptotics (2.25)–(2.26) holds with \( \mathcal{N} \) defined by (17.1.12) of [Ivr2] with \( d = 2 \). Moreover, equivalence (2.28)\# holds.

Example 2.27\( ^{23} \). Let condition (2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with \( \gamma = \epsilon_0|x|, \rho = |x|^m, \rho_1 = |x|^{m_1}, m_1 > 2m, m > -1 \).

Further, let assumption (2.79) and non-degeneracy condition (2.80) be fulfilled. Let either

(a) \( m < 0, \tau_1 < \tau_2 \) or

(b) \( m > 0, M > 0, \tau_1 \in (-M, M), \tau_2 = M \).

Then for \( h \to +0, 1 \leq \mu \) asymptotics (2.32), (2.26) holds with \( \mathcal{N} \) defined by (17.1.12) of [Ivr2] with \( d = 2 \).

Moreover, \( \mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp h^{-2} \) as \( \mu h \lesssim 1 \) and (2.34) holds as \( \mu h \gtrsim 1 \).

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with \( \gamma = \epsilon_0(x), \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, m_1 < 2m, m < -1 \).

Further, let assumption (2.79)\# and non-degeneracy condition (2.80) be fulfilled. Let \( M > 0, \tau_1 \in (-M, M), \tau_2 = M \). Then for \( h \to +0, 1 \leq \mu \)

\( ^{23} \) Cf. Examples 2.4 and 2.5.
asymptotics (2.32)#, (2.26) holds with $N$ defined by (17.1.12) of [Ivr2] with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp h^{-2}$ as $\mu h \lesssim 1$ and (2.34) holds as $\mu h \gtrsim 1$.

We leave to the reader the following problems:

**Problem 2.28.** Modify for the Dirac operator under assumption (2.79) or (2.79)# (as 0 or infinity is an inner singular point, respectively) in the frameworks of

(a) Examples 2.6, 2.7, 2.8 , 2.9, 2.11 (power singularities) and Problem 2.10,

(b) Examples 2.12, 2.13, 2.14 and Problem 2.16 (power-logarithmic singularities),

(c) Examples 2.17, 2.19, 2.20, 2.21 and Problems 2.18, 2.22 (exponential singularities).

### 2.3.3 Asymptotics. II

While under assumptions (2.79), (2.79)# the Dirac operator behaves as the Schrödinger operator, under the same assumptions albeit with an opposite sign the Dirac operator behaves as the Schrödinger-Pauli operator.

**Example 2.29**\(^{24}\). Let condition (2.74) be fulfilled.

(i) Let 0 be an inner singular point and let all the above conditions, be fulfilled with $\gamma = \epsilon_0 |x|$, $\rho = |x|^m$, $\rho_1 = |x|^m_1$, $m_1 > 2m$, $m > -1$.

Further, let assumption

\[
(2.82) \quad V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x : |x| \leq \epsilon
\]

and non-degeneracy condition (2.80) be fulfilled.

(a) $m < 0$, $\tau_1 < \tau_2$ or

(b) $m > 0$, $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$.

\(^{24}\) Cf. Example 2.23.
Then for $h \to +0$, $1 \leq \mu$ asymptotics (2.32), (2.26) holds with $\mathcal{N}$ defined by (17.1.12) of [Ivr2] with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-1} + h^{-2}$.

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with $\gamma = \epsilon_0(x)$, $\rho = (x)^m$, $\rho_1 = (x)^{m_1}$, $m_1 < 2m$, $m < -1$.

Further, let assumption

$$(2.82)^\# \quad V \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x : |x| \geq c$$

and non-degeneracy condition (2.80) be fulfilled. Let $M > 0$, $\tau_1 = -M$, $\tau_2 \in (-M, M)$. Then for $h \to +0$, $1 \leq \mu$ asymptotics (2.32)$^\#$, (2.26) holds with $\mathcal{N}$ defined by (17.1.12) of [Ivr2] with $d = 2$.

Moreover, $\mathcal{N}(\tau_1, \tau_2, \mu, h) \asymp \mu h^{-1} + h^{-2}$.

**Problem 2.30.** (a) Modify Problem 2.27 under assumptions (2.82) or (2.82)$^\#$ (if 0 or infinity is an inner singular point, respectively).

(b) Consider degenerations like in Subsubsection 2.1.5.5. Degenerations.

Finally, we leave to the reader

**Problem 2.31.** Generalize results of this section to the even-dimensional full-rank case. In particular, consider power singularities$^{25}$.

## 3 2D-case. Asymptotics of large eigenvalues

In this section we consider the case when $\mu = h = 1$ are fixed and we consider the asymptotics of the eigenvalues, tending to $+\infty$ and for Dirac operator also to $-\infty$.

Here we consider the case of the spectral parameter tending to $+\infty$ (and for the Dirac operator we consider $\tau \to -\infty$ as well).

$^{25}$ Note that for $m_1 \neq -2$ one can construct $V_j$ positively homogeneous of degrees $m_1 + 1$ such that $F_{jk}$ is non-degenerate. F.e. one can take $V_{2j-1} = \frac{1}{2}x_{2j}|x|^{m_1}$, $V_{2j} = -\frac{1}{2}x_{2j-1}|x|^{m_1}$, $j = 1, 2, \ldots, d/2$, in which case $f_1 = ((m_1 + 2)/2)|x|^{m_1}$, $f_2 = \ldots = f_{d/2} = |x|^{m_1}$. See for details Appendix 13.C of [Ivr2].
3.1 Singularities at the point

We consider series of examples with singularities at the point.

3.1.1 Schrödinger operator

Example 3.1. (i) Let \( X \) be a compact domain and conditions (2.3)\(_{1-3} \) be fulfilled with \( \gamma = \epsilon |x|, \rho = |x|^m, \rho_1 = |x|^{m_1} \) with \( m_1 < 2m \). Let

\[
(3.1)_{1,2} \quad |F| \geq \epsilon_0 \rho_1, \quad |\nabla F| \geq \epsilon_0 \rho_1 \gamma^{-1} \quad \text{for } |x| \leq \epsilon.
\]

Then for the Schrödinger operator as \( \tau \to +\infty \)

\[
(3.2) \quad N^{-}(\tau) = N^{-}(\tau) + O(\tau^{(d-1)/2})
\]

while \( N^{-}(\tau) \approx \tau^{d/2} \).

Indeed, we need to consider only case \( m_1 \leq -2 \) (otherwise it is covered by Section ?? of [Ivr2]). Assume for simplicity, that \( V = 0 \) (modification in the general case is trivial). Then we need to consider zones \( X_1 = \{x: |x| \geq \tau^{1/2(m_1 + 1)} \} \) where \( \mu_{\text{eff}} = |x|^{m_1+1} \tau^{-1/2} \leq 1 \) and \( X_2 = \{x: |x| \leq \tau^{1/2(m_1 + 1)} \} \) where \( \mu_{\text{eff}} \geq 1 \). Meanwhile \( h_{\text{eff}} = \tau^{-1/2}|x|^{-1} \).

Contribution to the remainder of the \( \gamma \)-element from \( X_1 \) does not exceed \( C h_{\text{eff}}^{1-d} = C \tau^{(d-1)/2} \gamma^{d-1} \) while contribution to the remainder of the \( \gamma \)-element from \( X_2 \) does not exceed \( C \mu_{\text{eff}}^{-1} h_{\text{eff}}^{1-d} = C \tau^{d/2} \gamma^{d-2-m_1} \leq C \tau^{(d-1)/2} \gamma^{d-1} \) and the rest is easy.

(ii) Under proper assumptions the same proof is valid in the full-rank even-dimensional case.

(iii) The similar proof is valid for \( d = 3 \) and under proper assumptions it is valid in the maximal-rank odd-dimensional case (while contribution to the remainder of \( \gamma \)-element from \( X_2 \) is \( O(\tau^{(d-1)/2} \gamma^{d-1}) \)).

(iv) On the other hand, without assumption (3.1)\(_2 \) contribution to the remainder \( \gamma \)-element from \( X_2 \) is \( O(\tau^{(d-2)/2} \gamma^{d+1}) \) but we need to consider only \( \gamma \gtrsim \tau^{1/m_1} \) (otherwise \( \mu_{\text{eff}} h_{\text{eff}} \geq C_0 \)) and we arrive to the remainder estimates \( O(\tau^{(d-1)/2}) \) as \( m_1 \geq -2d \) and \( O(\tau^{d/2 + d/m_1}) \) as \( m_1 \leq -2d \).

Definitely these arguments are far from optimal for \( d = 3 \) or in the maximal-rank odd-dimensional case; we leave this case to the Sections 7–12.
Let us note that the case $m < -1$, $m_1 \geq m - 1$ is covered by Chapter ?? of [Ivr2]; we need to assume that
\begin{equation}
V \geq \epsilon_0 \rho^2 \quad \text{as } |x| \leq \epsilon.
\end{equation}

Example 3.2. (i) Let $X$ be a compact domain and conditions (2.3)_{1-3}, (3.1)_{1} be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and with $m < -1$, $2m \leq m_1 < m - 1$. Then we need to assume that
\begin{equation}
V + F \geq \epsilon_0 \rho^2 \quad \text{as } |x| \leq \epsilon
\end{equation}
which for $m_1 > 2m$ is equivalent to (3.3).

We need also to have some non-degeneracy assumption. Assume that\(^{26)}

\begin{equation}
\tau \geq V + F, \quad |\nabla(\tau - V)F^{-1}| \gamma \leq \epsilon_0 \tau F^{-1}
\implies |\nabla^2(\tau - V)F^{-1}| \gamma^2 \geq \epsilon_0 \tau F^{-1} \quad \text{as } |x| \leq \epsilon.
\end{equation}

Then asymptotics (3.2) holds while $N^{-}(\tau) \asymp \tau^{d/2}$.

Indeed, it follows from Chapter 13 of [Ivr2] (see condition (13.3.54)) that in this case the contribution to the remainder of the $\gamma$-element from $\mathcal{X}_2$ is also $O(h_{\text{eff}}^{-d+1})$.

(ii) Under proper assumptions the same proof is valid in the maximal-rank multidimensional case.

(iii) Without assumption (3.4) we can apply arguments of Example 3.1(iv), however cutting off as $|x| \leq \tau^{1/(2m)}$ and we arrive to the remainder estimate $O(\tau^{(d-1)/2})$ as $d + m_1 \geq m$ and $O(\tau^{(d-1/2+1/(2m))+(m_1-2m)/(2m)})$ as $d + m_1 \leq m$.

Again, these arguments are far from optimal for $d = 3$ or in the maximal-rank odd-dimensional case.; we leave this case to the Sections 7–12.

### 3.1.2 Schrödinger-Pauli operator

Next, consider Schrödinger-Pauli operators. We will need to impose (3.3) and the related non-degeneracy assumption
\begin{equation}
|\nabla V| \geq \epsilon_0 \rho^2 \gamma^{-1} \quad \text{for } |x| \leq \epsilon.
\end{equation}

\(^{26)}\text{One can check easily that this condition or a similar condition in the multidimensional case is fulfilled provided } F \text{ and } V \text{ stabilize as } |x| \to 0 \text{ to } V^0 \text{ and } F^0, \text{ positively homogeneous of degrees } 2m \text{ and } m_1 \text{ respectively.}\)}}
Example 3.3. (i) Let $X$ be a compact domain, $d = 2$. Let conditions (2.3)$_{1-3}$, (3.1)$_{1,2}$, (3.3) and (3.5) be fulfilled with $\gamma = \epsilon, \rho = |x|^m, \rho_1 = |x|^{m_1}$.

Let either $m_1 < \min(2m, -2)$ or $2m \leq m_1 < m - 1$ and condition (3.4) be fulfilled.

Then for the Schrödinger-Pauli operator as $\tau \to +\infty$ asymptotics (3.2) holds while

\[ N^- (\tau) \asymp \tau^{(m_1+2)/(2m)} + \tau. \]

Indeed, if $m_1 \geq 2m$ then no modification to the arguments of Examples 3.1 and 3.2 is needed; if $m_1 < 2m$ we also need to consider the zone \{ $\tau^{1/(2m)} \lesssim |x| \lesssim \tau^{1/m_1}$ \}. The contribution of the corresponding partition element to the principal part of the asymptotics is $O(r^{m_1}\gamma^2)$ ($r = |x| \asymp \gamma$) while its contribution to the remainder is $O(\tau^{1/2}\gamma + 1)$.

(ii) One can generalize this example to the even-dimensional full-rank case; then

\[ N^- (\tau) \asymp \tau^{(m_1+2)d/(4m)} + \tau^{d/2} \]

and the remainder is $O(R)$ with

\[ R = \tau^{(m_1+2)(d-2)/(4m)} + \tau^{(d-1)/2}. \]

3.1.3 Dirac operator

Finally, consider Dirac operator. We want to consider either $N(0, \tau)$ with $\tau \to +\infty$ and $N(\tau, 0)$ as $\tau \to -\infty$.

Example 3.4. Let $X$ be a compact domain, $d = 2$ and conditions (2.75)$_{1-2}$, (3.1)$_{1,2}$ and

\[ |D^\alpha V| \leq c \rho \gamma^{-|\alpha|} \]

be fulfilled with $\gamma = \epsilon |x|, \rho = |x|^m, \rho_1 = |x|^{m_1}$ and $m_1 < \min(2m, -2)$. Let assumption (2.74) be fulfilled as $|x| \leq \epsilon$.

(i) Let $V < M$ in the vicinity of 0. Then for the Dirac operator asymptotics

\[ N(0, \tau) = N(0, \tau) + O(\tau) \]
holds as \( \tau \to +\infty \) and \( \mathcal{N}(0, \tau) \asymp \tau^2 \).

Indeed, assumption \( V < M \) guarantees that \( V - (M^2 + 2jF)^{1/2} \) with 
\( j = 0, 1, \ldots \) do not contribute.

(ii) Let \( V^2 < M^2 + F \) in the vicinity of 0 and

\( V \leq -\epsilon_0 \rho, \quad |\nabla V| \geq \epsilon_0 \rho \gamma^{-1} \quad \text{as} \quad |x| \leq \epsilon. \)

Then for the Dirac operator asymptotics

\[
N(\tau, 0) = \mathcal{N}(\tau, 0) + O(|\tau|^{1/2})
\]

and

\[
\mathcal{N}(\tau, 0) \asymp \tau^2 + |\tau|^{(m_1 + 2)/m}
\]

holds as \( \tau \to -\infty \).

Indeed, assumption \( V^2 < M^2 + F \) guarantees that \( V + (M^2 + 2jF)^{1/2} \) with 
\( j = 1, \ldots \) do not contribute.

We leave to the reader

**Problem 3.5.** (i) Extend results of Example 3.4(i) to the case \( 2m \leq m_1 < m - 1 \).

(ii) Expand results of Example 3.4(ii) to the case \( m_1 = 2m < -2 \).

In both Statements (i) and (ii) one needs to formulate the analogue of the non-degeneracy assumption (3.4).

(iii) Consider the full-rank even-dimensional case.

### 3.1.4 Miscellaneous singularities

Consider now miscellaneous singularities in the point, restricting ourselves to \( d = 2 \):

**Example 3.6.** Let \( \mathcal{X} \) be a compact domain, \( d = 2 \) and conditions (2.3)_{1-3},
(3.1)_{1,2} and (3.3) be fulfilled with \( \gamma = \epsilon |x|, \rho = |\log |x||^\sigma, \rho_1 = |x|^{m_1}, \sigma > 0, \)
\( m_1 < -2. \)

Then for the Schrödinger-Pauli operator asymptotics (3.2) holds and

\[
\log(\mathcal{N}^-(\tau)) \asymp \tau^{-(m_1 + 2)/2\sigma}.
\]
Example 3.7. Let $X$ be a compact domain, $d = 2$ and conditions $(2.3)_{1-3}$, $(3.1)_{1,2}$ be fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho_1 = \exp(b|x|^\beta)$, $\beta < 0$, $b > 0$ and with $\rho = \exp(a|x|^\alpha)$ where either $\beta < \alpha < 0$ or $\beta = \alpha$ and $b > 2a$.

(i) Then for Schrödinger operator asymptotics (3.2) holds and $\mathcal{N}^-(\tau) \asymp \tau$ as $\tau \to +\infty$.

Indeed, as $\beta > -1$ it is easy, and as $\beta \leq -1$ one can apply the same arguments as in Example 3.23 below (in which case Remark 3.11(iii) does not apply.

(ii) Let also conditions $(2.3)_{1,3}$, $(3.3)$ and $(3.5)$ be fulfilled with $\gamma = \epsilon|x|^{1-\alpha}$. Then for the Schrödinger-Pauli operator asymptotics (3.2) holds with

\begin{align}
\mathcal{N}^-(\tau) &\asymp \tau^{b/2a}|\log \tau|^{(2-\beta)/\beta} \quad \text{as } \alpha = \beta, \\
\log(\mathcal{N}^-(\tau)) &\asymp |\log \tau|^{\beta/\alpha} \quad \text{as } \beta < \alpha < 0.
\end{align}

Indeed, in this case the forbidden zone is $\{x: |x| \leq r^*\}$ with $r^* = \epsilon|\log \tau|^{1/\alpha}$ and contribution of the zone $\{x: |x| \geq r^*\}$ to the “extra” remainder does not exceed $\int r^{-1-2\beta} r \, dr$, taken over this zone, which is $r^{-2\beta} \asymp |\log \tau|^{2\beta/\alpha}$.

Example 3.8. and conditions $(2.3)_{1-3}$, $(3.1)_{1,2}$ be fulfilled with $\gamma = \epsilon|x|^{1-\beta}$, $\rho_1 = \exp(b|x|^\beta)$, $\beta < 0$, $b > 0$ and with $\rho = |x|^{2m}$.

(i) Then Schrödinger operator is covered by the previous Example 3.7..

(ii) Let conditions $(2.3)_{1,3}$, $(3.3)$, $(2.23)$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $m < 0$.

Then for the Schrödinger-Pauli operator the following asymptotics holds:

\begin{align}
\mathcal{N}^-(\tau) &= \mathcal{N}^-(\tau) + O(\tau^{1/2} + \tau^{\beta/(2m)}) \\
\log(\mathcal{N}^-(\tau)) &\asymp \tau^{\beta/(2m)}.
\end{align}

Indeed, in this case $r^* = \tau^{1/(2m)}$ (cf. Example 3.7(i)).

Indeed, the contributions of the zone where $\mu_{\text{eff}}h_{\text{eff}} \leq C$ to the main term of the asymptotics and to the remainder estimates are $\int_{\mathcal{X}} \rho_1 r \, dr$ and $\int_{\mathcal{Z}} \gamma^{-2} r \, dr$ where $\mathcal{X} = \{x: V(x) \leq \tau\}$ and $\mathcal{Z}$ is a $\gamma$-vicinity of $\partial \mathcal{X}$; so we get (3.18) (we cannot get magnitude of $\mathcal{N}^-(\tau)$ itself precisely) and $\bar{r}$ with $\bar{r} = \tau^{1/(2m)}$. 39
The following problem seems to be very challenging:

**Problem 3.9.** Using the fact that singularities propagate along the drift lines, and the length of the drift line is \( \approx \bar{r} \) rather than \( \approx \bar{\gamma} = \bar{r}^{1-\beta} \) prove that the contribution of \( Z \) to the remainder is in fact \( O(1) \) and thus improve the remainder estimate (3.17) to \( O(\tau^{1/2}) \).

**Problem 3.10.** Extend results of Examples 3.6, 3.7 and 3.8 to the Dirac operator.

**Remark 3.11.** (i) Observe that the contribution to the remainder of the zone \( \{ x : |x| \leq \varepsilon \} \) does not exceed \( \varepsilon^\sigma \tau^{(d-1)/2} \) with \( \sigma > 0 \) in the frameworks of Example 3.1(i), Example 3.1(iv) with \( m_1 > -2d \), Example 3.2(i), Example 3.2(iii) with \( d + m_1 > m \) and Example 3.3.

The same is correct in Example 3.6 and Examples 3.7 and 3.8 with \( 0 > \beta > -1 \).

Therefore, in these cases under the standard non-periodicity condition to the geodesic flow with reflections from \( \partial X \) the asymptotics

\[
N(\tau) = \mathcal{N}(\tau) + \kappa_1 \tau^{\frac{1}{2}} + o(\tau^{\frac{1}{2}})
\]

holds with the standard coefficient \( \kappa_1 \).

(ii) The similar statement (with \( \tau \) replaced by \( \tau^2 \)) is true in the framework of Example 3.4.

(iii) Since we used local estimates \( O(h_{\text{eff}}^{-1}) \) rather than \( O(\mu_{\text{eff}}^{-1} h_{\text{eff}}^{-1}) \) as \( \mu_{\text{eff}} \geq 1 \) (the latter gave us no advantage) we do not need 0 to be an inner singular point; the same results hold for \( 0 \in \partial X \) under Dirichlet or Neumann boundary condition.

**Problem 3.12.** In the frameworks of Examples 3.1(i), 3.1(iv), 3.2(i), 3.2(iii) and 3.3 estimate \( |\mathcal{N}^- (\tau) - \kappa_0 \tau^{d/2}| \).

Furthermore, in the frameworks of Examples 3.4 (i), (ii) estimate \( |\mathcal{N}(0, \tau) - \kappa_0 \tau^2| \) and \( |\mathcal{N}(\tau, 0) - \kappa_0 \tau^2| \) respectively.

Finally, consider the case when the singularity is located on the curve.
Example 3.13. Let $X$ be a compact domain, $d = 2$ and conditions $(2.3)_{1-3}$, $(3.1)_{1,2}$ be fulfilled with $\gamma = \epsilon \delta(x), \rho_1 = \delta(x)^m, \rho_1 = \delta(x)^{m_1}$ with $m_1 < \min(2m, -2)$ where $\delta(x) = \text{dist}(x, L)$, $m < 0$, $L$ is either a closed curve ($q = 1$) or a closed set of Minkowski dimension $q < 1$.

(i) Then for the Schrödinger operator asymptotics (3.2) holds for $\tau \to +\infty$ and $N^- (\tau) \asymp \tau$.

Indeed, using the same arguments as before we can get a remainder estimate $O(\tau^{1/2})$ if $q < 1$ and $O(\tau^{q/2} |\log \tau|)$ if $q = 1$ but in the latter case we can get rid off logarithm using standard propagation arguments.

(ii) Let also conditions (3.3) and (3.5) be fulfilled with $m > -1$. Then for the Schrödinger-Pauli operator asymptotics holds

$$N^-(\tau) = N^-(\tau) + O\left(\tau^{1/2 + \frac{q-2}{(2m)}}\right)$$

holds while

$$N^-(\tau) \asymp \tau + \tau^{\frac{m+q}{(2m)}}.$$ 

Indeed, in this case a forbidden zone is $\{x: \delta(x) \leq \delta_* = \epsilon \tau^{1/(2m)}\}$ and contributions of the zone $\{x: \delta(x) \geq \delta_*\}$ to the main term of the asymptotics and the remainder are $\asymp \delta_*^{m+q}$ and $\delta_*^{q-2}$ respectively.

Problem 3.14. (i) Explore, if we can using propagation arguments improve remainder estimate (3.20).

(ii) Extend results of Example 3.13 to different types of the singularities along $L$ and/or Dirac operator.

3.2 Singularities at infinity

Let us consider unbounded domains:

3.2.1 Power singularities: Schrödinger operator

Let us start from the power singularities.
Example 3.15. (i) Let $X$ be a connected exterior domain$^{28}$ with $C^K$ boundary, $d = 2$. Let conditions (2.2) and (2.3)$_{1-3}$ be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $m_1 > 2m$. Further, let

$$|F| \geq \epsilon_0 \rho_1, \quad |\nabla F| \geq \epsilon_0 \rho_1 \gamma^{-1} \quad \text{for } |x| \geq c.$$  

Then for the Schrödinger operator the following asymptotics holds:

$$N^- (\tau) = N^- (\tau) + O(\tau^{(m_1+2)/2(m_1+1)})$$

with

$$N^- (\tau) \approx \tau^{(m_1+2)/m_1}.$$  

Indeed, there will be zone $X'_1 = \{|x| \leq \tau^{1/(2(m_1+1))}\}$ with $\mu_{\text{eff}} = |x|^{m_1+1} \tau^{-1/2} \lesssim 1$ and a zone $X'_2 = \{|x| \geq \tau^{1/(2(m_1+1))}\}$ with $\mu_{\text{eff}} \geq 1$. Contribution of the partition element in $X'_1$ to the remainder is $O(h_{\text{eff}}^{-1}) = O(\tau^{-1/2})$ and the total contribution of $X'_1$ is $O(\tau^{(m_1+2)/(2(m_1+1))})$. On the other hand, contribution of the partition element in $X'_2$ to the remainder is $O(\mu_{\text{eff}}^{-1} h_{\text{eff}}^{-1}) = O(\tau^{-m})$ and the total contribution of this zone is $O(\tau^{(m_1+2)/(2(m_1+1))})$ again.

(ii) Under proper assumptions the similar asymptotics holds in the full-rank even-dimensional case:

$$N^- (\tau) = N^- (\tau) + O(\tau^{(d-1)(m_1+2)/2(m_1+1)})$$

with

$$N^- (\tau) \approx \tau^{d(m_1+2)/(2m_1)}.$$  

Example 3.16. (i) Let $X$ be a connected exterior domain$^{28}$ with $C^K$ boundary. Let conditions (2.2), (2.3)$_{1-3}$ and (3.1)$_1$ be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $m > 0$, $m - 1 < m_1 \leq 2m$.

Then we need to assume that

$$V + F \geq \epsilon_0 \rho^2 \quad \text{as } |x| \geq c$$

which for $m_1 < 2m$ is equivalent to

$$V \geq \epsilon_0 \rho^2 \quad \text{as } |x| \geq c$$

---

$^{27}$ Cf. Example 3.1.

$^{28}$ i.e. a domain with compact complement $\bar{C}X$ in $\mathbb{R}^d$.

$^{29}$ Cf. Example 3.2.
We need also to have some non-degeneracy assumption. Assume that (cf. (2.16))\(^30\)

\[(3.27) \quad \tau \geq V + F, \quad |\nabla (\tau - V)F^{-1}| \gamma \leq \epsilon_0 \tau F^{-1}\]

\[\implies |\text{det Hess}(\tau - V)F^{-1}| \gamma^2 \geq \epsilon_0 \tau F^{-1} \quad \text{as } |x| \geq c.\]

Then for the Schrödinger operator asymptotics

\[(3.28) \quad N^{-}(\tau) = N^{-}(\tau) + O(R),\]

holds with

\[(3.29) \quad R = \begin{cases} \tau^{(m+2)/2(m+1)} & m_1 > 0, \\ \tau |\log \tau|^2 & m_1 = 0, \\ \tau^{1-m_1/(2m)} |\log \tau| & m - 1 < m_1 < 0, \end{cases}\]

and

\[(3.30) \quad N^{-}(\tau) \asymp \tau^{(m+1)/m}.\]

Indeed, the contribution of $\gamma$-element in $X'_1$ (see Example 3.15) to the remainder does not exceed $C\mu \log \mu \tau^{-m_1} \log r_\ast^{-1}$, $r_\ast = \tau^{1/(2(m+1))}$. Then summation with respect to partition returns $R$; we need to take into account that $\{|x| \geq C\tau^{1/(2m)}\}$ is a forbidden zone. Meanwhile, contribution of $X'_1$ does not exceed $C\tau^{(d-1)/2} r_\ast^{d-1}$.

(ii) As $m_1 < 0$ we can get rid off the logarithmic factor in the remainder estimate, if we define $N^{-}(\tau)$ by the corrected magnetic Weyl formula; similarly, for $m_1 = 0$ we can then replace $|\log \tau|^2$ by $|\log \tau|$.

Also for $m_1 = 0$ we can $|\log \tau|^2$ by $|\log \tau|$ under assumption

\[(3.31) \quad \tau \geq V + F, \quad |\nabla (\tau - V)F^{-1}| \gamma \leq \epsilon_0 \tau F^{-1}\]

\[\implies \text{det Hess}(\tau - V)F^{-1} \gamma^2 \geq \epsilon_0 \tau F^{-1} \quad \text{as } |x| \geq c.\]

\(^30\) One can see easily, that if $F, V$ stabilize at infinity to functions $F^0, V^0$, positively homogeneous of degrees $m_1 \geq 0, 2m$ respectively, then even stronger non-degeneracy assumption (cf. (2.14)) holds:

\[(3.26) \quad \tau \geq V + F \implies |\nabla (\tau - V)F^{-1}| \gamma \geq \epsilon_0 \tau \quad \text{as } |x| \geq c.\]

On the other hand, for $m_1 < 0$ condition (3.26) is fulfilled if $w(\theta) = F^{0-2m} \nu^{0m}$ has only nongenerate critical points on $\mathbb{S}^1$.  

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(iii) Under proper assumptions the similar asymptotics holds in the full-rank even-dimensional case:

\[
R = \begin{cases} 
\tau^{(d-1)(m_1+2)/2(m_1+1)} & m_1 > d - 2, \\
\tau^{d/2} |\log \tau|^2 & m_1 = d - 2, \\
\tau^{d/2+(d-2-m_1)/(2m)} |\log \tau| & m - 1 < m_1 < d - 2,
\end{cases}
\]

with

\[
N^- (\tau) \asymp \tau^d(m_1+2)/(2m_1).
\]

Moreover, we can get rid off one logarithmic factor under assumption similar to (3.31).

Let is improve (3.29) using arguments associated with long-range dynamics:

**Example 3.17.** Let \( m_1 > 0 \) in the framework of Examples 3.15(i) or 3.16(i). Moreover, let the stabilization conditions

\[
(3.34)_1 \quad D^\sigma(g^{jk} - g^{jk0}) = o(|x|^{-|\sigma|}), \\
(3.34)_2 \quad D^\sigma(V_j - V^0_j) = o(|x|^{m_1+1-|\sigma|}) \quad \forall \sigma : |\sigma| \leq 1
\]

be fulfilled for \(|x| \to \infty\) with the positively homogeneous functions \( g^{jk0}, V^0_j \in \mathcal{C}^K(\mathbb{R}^2 \setminus 0) \) of degrees 0 and \( m_1 + 1 \) respectively. Let the standard non-periodicity condition be fulfilled for the Hamiltonian

\[
(3.35) \quad H^0 = |x| \left( \sum_{j,k} g^{jk0}(\xi_j - V^0_j)(\xi_k - V^0_k) - 1 \right)
\]

on the energy level \( 0 \). Then for the Schrödinger operator as \( \tau \to +\infty \) the following asymptotics holds:

\[
N^- (\tau) = N^-(\tau) + o(\tau^{1/2(m_1+2)/(m_1+1)}).
\]

Indeed, let us observe that for \( m_1 > 0 \) the main contribution to the remainder estimate (3.29) is given by the zone \( \{ \varepsilon \leq |x| \tau^{-1/2(m_1+1)} \leq \varepsilon^{-1} \} \) with an arbitrarily small constant \( \varepsilon > 0 \). In this zone the magnetic field is normal, \( \mu_{\text{eff}} \lesssim 1 \) (for every fixed \( \varepsilon > 0 \)) and \( V \ll \tau, |\nabla V| \ll \tau|x|^{-1} \). Applying the improved Weyl asymptotics here we obtain (3.36).

**Remark 3.18.** Similar improvements are possible in the full-rank even-dimensional case.
3.2.2 Power singularities: Schrödinger-Pauli operator

Next, consider Schrödinger-Pauli operators. We will need to impose (3.3) and the related non-degeneracy assumption

\[(3.5)^\# \quad \left| \nabla V \right| \geq \epsilon_0 \rho^2 \gamma^{-1} \quad \text{for } |x| \geq c.\]

**Example 3.19** \(^{31}\). Let (3.3) and (3.5) be fulfilled. Then for the Schrödinger-Pauli operator

(i) In the framework of Examples 3.15 and 3.16 asymptotics (3.22) and (3.29)–(3.29) hold, respectively.

(ii) In the framework of Example 3.19(i), asymptotics (3.36) holds.

(iii) Further,

\[(3.37) \quad N^{-}(\tau) \asymp \tau^{(m+1)/m + (m_1+2)/(2m)}.\]

(iv) Finally, under proper assumptions one can consider the full-rank even-dimensional case and prove asymptotics with the remainder estimate \(O(R)\), with \(R := R_1 + R_2\) where \(R_1\) is the remainder estimate for the Schrödinger operator,

\[(3.38) \quad R_2 = \tau^{(m_1+2)(d-2)/(4m)}\]

and

\[(3.39) \quad N^{-}(\tau) \asymp \tau^{d(m+1)/(2m_1)} + \tau^{d(m_1+2)/(4m)}.\]

3.2.3 Power singularities: Dirac operator

Finally, consider the Dirac operators. We want to explore either \(N(0, \tau)\) with \(\tau \to +\infty\) and \(N(\tau, 0)\) as \(\tau \to -\infty\).

**Example 3.20.** Let \(X\) be a connected exterior domain with \(C^K\) boundary, \(d = 2\). Let conditions (2.2), (2.75)\(^{1-2}\), (3.1)\(^{\#}\) and (3.9) be fulfilled with \(\gamma = \epsilon_0 \langle x \rangle\), \(\rho = \langle x \rangle^m\), \(\rho_1 = \langle x \rangle^{m_1}\), \(m_1 > \max(2m, -2)\). Further, let assumption (2.74) be fulfilled as \(|x| \leq \epsilon.\)

\(^{31}\) Cf. Example 3.3.
(a) Schrödinger operator

\[ \mu_{\text{eff}} \lesssim 1, \quad \mu_{\text{eff}} h_{\text{eff}} \lesssim 1 \]

(b) Schrödinger-Pauli operator

\[ \mu_{\text{eff}} \gtrsim 1, \quad \mu_{\text{eff}} h_{\text{eff}} \gtrsim 1 \]

Figure 1: \( m_1 > 2m \); dots show the forbidden zone

(i) Let \( V < M \) in the vicinity of infinity. Then as \( \tau \to +\infty \) for the Dirac operator asymptotics

\[ N^{-}(\tau) = N^{-}(\tau) + O(\tau^{(m_1+2)/(m_1+1)}) \]

holds with

\[ N^{-}(\tau) \sim \tau^{2(m_1+2)/m}. \]

(ii) Let \( V^2 < M^2 + F \) in the vicinity of infinity and

\[ \mathcal{V} \leq -\epsilon_0 \rho, \quad |\nabla \mathcal{V}| \geq \epsilon_0 \rho \gamma^{-1} \]

as \( |x| \geq c \).

Then as \( \tau \to -\infty \) for the Dirac operator asymptotics (3.40) holds with

\[ N^{-}(\tau) \sim \tau^{2(m+1)/m} + \tau^{2(m+2)/(2m)}. \]

We leave to the reader

**Problem 3.21.** (i) Using arguments of Example 3.16 extend results of Example 3.20(i) to the case \( 2m \geq m_1 > m - 1 \).

(ii) Using arguments of Examples 3.16 and 3.19 expand results of Example 3.20(ii) to the case \( m_1 = 2m < -2 \).

In both Statements formulate the analogue of the non-degeneracy assumption (3.4)#.

(iii) Consider the full-rank even-dimensional case.

**Problem 3.22.** Extend to the Dirac operator results of Example 3.17; one still defines Hamiltonian \( H^0 \) by (3.35).
3.2.4 Exponential singularities

Consider now an exponential growth at infinity.

**Example 3.23.** Let $X$ be a connected exterior domain with $C^K$ boundary. Let conditions (2.2), (2.3)$_{1-3}$, (3.1)$_{12}$ be fulfilled with $\gamma = \epsilon_0 \langle x \rangle^{1-\beta}$, $\rho = \exp(a \langle x \rangle^\alpha)$, $\rho_1 = \exp(b \langle x \rangle^\beta)$, $\beta > 0$ and either $\beta > \alpha$ or $\beta = \alpha$ and $b > 2a > 0$.

(i) Then for the Schrödinger operator the following asymptotics holds:

\begin{equation}
N^- (\tau) = N^-(\tau) + O(\tau^2 |\log \tau|^{1/\beta})
\end{equation}

with

\begin{equation}
N^- (\tau) \asymp \tau |\log \tau|^{2/\beta}.
\end{equation}

Indeed, using described $\gamma$, consider zone $\{|x| \leq C|\log \tau|^{1/\beta}\}$, where $\mu_{\text{eff}} = F \gamma \tau^{-1/2} \geq 1$ and $F \leq c\tau$; here $\tau$ is defined by $\tau^1 \exp(b \bar{r} \beta) = \tau^{1/2}$. Then contribution of $\gamma$-element to the remainder does not exceed $C \mu_{\text{eff}}^{-1} \bar{r} \beta$\exp(-br^\beta)$ with $r = |x|$ and the total contribution of this zone does not exceed $CR$ with

\begin{equation*}
R = \int_\bar{r} \tau \exp(-br^\beta) r^{-2+2\beta} r \, dr,
\end{equation*}

which is equal to the integrand, multiplied by $r^{1-\beta}$ and calculated as $r = \bar{r}$: $R \asymp \tau \exp(-b \bar{r} \beta) \bar{r}^\beta = \tau^{1/2} \bar{r}$ with $\bar{r} \asymp |\tau|^{1/\beta}$.

On the other hand, consider zone $\{|x| \leq \bar{r}\}$, where we can redefine $\gamma = (\tau^{1/2} \tau^{1-\beta} \delta \exp(-br^\beta))^{1/(1+\delta)}$ with $\delta > 0$; then its contribution to the remainder does not exceed $CR$ with

\begin{equation}
R = \int_\bar{r} \tau^{1/2} (\tau^{1/2} \tau^{1-\beta} \delta \exp(-br^\beta))^{-1/(1+\delta)} r \, dr,
\end{equation}

which is also equal to the integrand, multiplied by $r^{1-\beta}$ and calculated as $r = \bar{r}$: $R \asymp \tau^{1/2} (\tau^{1/2} \bar{r}^{1-\beta} \delta \exp(-b \bar{r} \beta))^{-1/(1+\delta)} \bar{r}^{2-\beta} = \tau^{1/2} \bar{r}$ again.

(ii) Let also conditions (2.3)$_{13}$, (3.3)$_{\#}$ and (3.5)$_{\#}$ be fulfilled with $\gamma = \epsilon |x|^{1-\alpha}$. Then for the Schrödinger-Pauli operator asymptotics (3.43) holds with

\begin{equation}
N^- (\tau) \asymp \tau^{b/(2a)} |\log \tau|^{(2-\beta)/\beta}
\end{equation}

$\beta = \alpha$,

\begin{equation}
\log(N^- (\tau)) \asymp |\log \tau|^{\beta/\alpha}
\end{equation}

$\beta > \alpha$. 

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Indeed, in this case the forbidden zone is \( \{ x : |x| \geq r_* = c|\log \tau|^{1/\alpha} \} \) (cf. Example 3.7 (ii)).

**Example 3.24.** Let \( X \) be a connected exterior domain with \( C^K \) boundary. Let conditions (2.2), (2.3)\(_{1,3} \), (3.1)\(_{1,2} \) be fulfilled with \( \gamma = \epsilon_0 \langle x \rangle^{1-\beta} \), \( \rho = \exp(a \langle x \rangle^\alpha) \), \( \rho_1 = \exp(b \langle x \rangle^\beta) \), \( \beta > 0 \) and \( \beta = \alpha \) and \( 2a > b > a > 0 \). Let conditions (3.3)\(^\# \) and (3.5)\(^\# \) be also fulfilled.

Then for both Schrödinger and Schrödinger-Pauli operators (3.43) and (3.44) hold. Indeed, the forbidden zone is the same as in the previous example.

**Example 3.25.** Let \( X \) be a connected exterior domain with \( C^K \) boundary. Let conditions (2.2), (2.3)\(_{1,3} \), (3.1)\(_{1,2} \) be fulfilled with \( \gamma = \epsilon_0 \langle x \rangle^{1-\beta} \), \( \rho_1 = \exp(b \langle x \rangle^\beta) \), \( \beta > 0 \) and \( \rho = \langle x \rangle^m \), \( m > 0 \).

(i) Then the Schrödinger operator is covered by Example (i).

(ii) Let also conditions (2.3)\(_{1,3} \), (3.3)\(^\# \) and (3.5)\(^\# \) be fulfilled with \( \gamma = \epsilon |x| \). Then for the Schrödinger-Pauli operator the following asymptotics holds:

\[
N^{-}(\tau) = N^{-}(\tau) + O(\tau^{1/2} |\log \tau|^{1/\beta + \beta/(2m)})
\]

and

\[
\log(N^{-}(\tau)) \simeq |\tau|^{\beta/(2m)}.
\]

Indeed, in this case the forbidden zone is \( \{ x : |x| \geq r_* = c|\tau|^{1/(2m)} \} \) (cf. Example 3.8 (ii)).

The following problem seems to be very challenging:

**Problem 3.26\(^{32} \).** Using the fact that singularities propagate along the drift lines, and the length of the drift line is \( \simeq \bar{r} \) rather than \( \simeq \bar{\gamma} = \bar{r}^{1-\beta} \) prove that the contribution of \( Z \) to the remainder is in fact \( O(1) \) and thus improve the remainder estimate (3.17) to \( O(\tau^{1/2}) \).

**Problem 3.27.** (i) Consider Dirac operator in the same settings as in Example 3.23.

\(^{32} \) Cf. Problem 3.9
(ii) Since for Schrödinger, Schrödinger-Pauli and Dirac operators the main contribution to the remainder is delivered by the zone where \( \varepsilon \leq \mu_{\text{eff}} \leq \varepsilon^{-1} \) (and \( V \ll \tau \)), while the contribution of the zones of the zones where \( \mu_{\text{eff}} \leq \varepsilon \) and \( \mu_{\text{eff}} \geq \varepsilon^{-1} \) do not exceed \( \sigma(\varepsilon) \tau |\log \tau|^{1/\beta} \) with \( \sigma = o(1) \) as \( \varepsilon \to 0 \), derive remainder estimate \( o(\tau |\log \tau|^{1/\beta}) \) under non-periodicity condition for the Hamiltonian similar to (3.35).

(iii) Consider Schrödinger-Pauli and Dirac operators in the same settings as in Example 3.11 albeit with \( V \) of the logarithmic growth at infinity (i.e. with \( \rho = |\log |x||^\alpha, \gamma = \epsilon|x| \)).

More challenging is the following

Problem 3.28. (i) In the frameworks of Examples 3.1, 3.2, 3.3 and 3.4 allow degenerations of \( F \).

(ii) In the frameworks of Examples 3.2, 3.3 and 3.4 allow degenerations of \( V \).

4 2D-case. Asymptotics of small eigenvalues

Now we consider external domains and asymptotics of eigenvalues tending to some finite limit.

4.1 Operators stabilizing at infinity

We begin with the analysis of the Schrödinger operator \( A \) defined by (2.1) under assumption (2.2) assuming that

\[(4.1)_{1-3} \quad g \to g_\infty, \quad F \to F_\infty, \quad V \to 0 \quad \text{as} \quad |x| \to \infty.\]

Recall that \( F := (F_{jk}) \) with \( F_{jk} = \partial_k V_j - \partial_j V_k, \quad g := (g^{jk}). \)

We start from the theorem, describing the essential spectrum of \( A \):

Theorem 4.1. Let \( X \) be an exterior domain\(^{33}\) with \( C^K \) boundary. Let the Schrödinger operator \( A \) satisfy conditions (2.1), (2.2), and (4.1)\(_{1-3} \). Then

\(^{33}\) I.e. with a compact complement. If \( X \neq \mathbb{R}^d \), then the appropriate boundary condition are given on \( \partial X \) such that operator is self-adjoint. In other words, infinity is an isolated singular point; see \(^7\).
(i) If \( \text{rank } F_\infty = 2r = d \) then

\[
\text{Spec}_{\text{ess}}(A) = \left\{ \sum_j \delta_j f_\infty,j : \delta = (\delta_1, \ldots, \delta_r) \in (2\mathbb{Z}^+ + 1)' \right\}
\]

where \( \pm \delta_\infty,j \) are eigenvalues of \( g_\infty F_\infty \), \( f_\infty,j > 0, j = 1, \ldots, r \).

(ii) If \( \text{rank } F_\infty = 2r < d \) then \( \text{Spec}_{\text{ess}}(A) = [f^*, \infty) \) with \( f^* = f_{\infty,1} + \ldots + f_{\infty,r} \).

Proof. Indeed, one can see easily that \( \text{Spec}_{\text{ess}}(A) \) coincides with \( \text{Spec}(A_\infty) \) where \( A_\infty \) is a toy-model operator in \( \mathbb{R}^d \) with \( g = g_\infty, F = F_\infty \) and \( V = 0 \). For such operator we calculated spectrum in Theorem 13.1.1 of [Ivr2].

Remark 4.2. (i) Similarly, for Schrödinger-Pauli operator \( \text{Spec}_{\text{ess}}(A) \) is defined by (4.2) albeit with \( \delta \) running \( (2\mathbb{Z}^+)' \) if \( \text{rank } F_\infty = 2r = d \) and \( \text{Spec}_{\text{ess}}(A) = [0, \infty) \) if \( \text{rank } F_\infty = 2r < d \).

(ii) Further, for the Dirac operator \( \text{Spec}_{\text{ess}}(A) \) also coincides with \( \text{Spec}(A_\infty) \), calculated in Theorem 17.1.2 of [Ivr2].

In this section we assume that

\[
\text{rank } F_\infty = d;
\]

very different and a more complicated case of \( \text{rank } F_\infty < d \) is left for the next Sections 7–12.

According to Theorem 4.1(i) under assumption (4.3) the essential spectrum consists of separate points, which are points of the pure point spectrum (of infinite multiplicity) of the limiting operator \( A_\infty \). We are interested in the asymptotics of eigenvalues of \( A \) tending to some fixed \( \tau^* \in \text{Spec}_{\text{ess}}(A) \). Namely, let us introduce

\[
\begin{align*}
\mathcal{N}^- (\eta) &= \mathcal{N}(\tau^* - \epsilon, \tau^* - \eta) \\
\mathcal{N}^+ (\eta) &= \mathcal{N}(\tau^* + \eta, \tau^* + \epsilon)
\end{align*}
\]

with a small constant \( \epsilon > 0 \) and a small parameter \( \eta \rightarrow +0 \). We also introduce

\[
\mathcal{W} := \{ \delta \in (2\mathbb{Z}^+ + 1)' : \sum_j \delta_j f_\infty,j = \tau^* \}.
\]
To characterize the rate of the decay at infinity we assume that

$$1 - 3 |\nabla \alpha (g - g_\infty)| = o(\rho^2 \gamma^{-|\alpha|}), \quad |\nabla \alpha (F - F_\infty)| = o(\rho^2 \gamma^{-|\alpha|})$$

$$|\nabla \alpha V| = O(\rho^2 \gamma^{-|\alpha|}) \quad \text{as} \quad |x| \to \infty \quad \forall \alpha.$$

### Theorem 4.3

Let $X$ be a connected exterior domain with $C^K$ boundary. Let the Schrödinger operator $A$ satisfy conditions (2.1), (2.2) and (4.6)$_{1-3}$ with scaling functions$^{34}$ such that $\gamma \to \infty$ and $\rho \to 0$ as $|x| \to \infty$.

Let $\text{rank} F_\infty = 2r = d$. Moreover let

(4.7) $\mp V \geq -\epsilon \rho^2 \implies |\nabla V| \geq \epsilon_0 \rho^2 \gamma^{-1}$ \quad as \quad $|x| \geq c$.

(i) Then

(4.8) $|N^\mp(\eta) - N^\mp(\eta)| \leq C \int _{Z(\eta)} \gamma^{-2} \, dx + C \int \gamma^{-s} \, dx$

where

(4.9) $N^\mp(\eta) := (2\pi)^{-r} \sum _{j \in \mathbb{N}} \int _{\{x: \mp V_j(x) \geq \eta\}} f_1 f_2 \cdots f_r \sqrt{g} \, dx$

$g = \det g^{-1}$, $\pm i f_j$ are eigenvalues of $gF$, $f_j > 0$, $j = 1, \ldots, r$, and

(4.10) $V_j(x) := V(x) + \sum _j \delta_j (f_j(x) - f_\infty, j)$.

$Z(\eta)$ is $\epsilon \gamma$-vicinity$^{35}$ of $\Sigma(\eta) = \{ x: \mp V_j(x) = \eta \}$.

(ii) Further, under assumption

(4.11) $\mp V \geq \epsilon_0 \rho^2$

$\tau^* \pm 0$ is not a limit point of the discrete spectrum.

**Proof.** Indeed, in the zones $Z(\eta)$ and

(4.12) $\Omega(\eta) := \{ x: \mp V(x) - \eta \geq \epsilon (\rho^2 + \eta) \}$,

\footnote{Recall that this means that $|\nabla \gamma| \leq c$ and $|\nabla \rho \leq c \rho \gamma^{-1}$.}

\footnote{I.e. $Z(\eta) = \bigcup _{x \in \Sigma(\eta)} B(x, \epsilon \gamma(x))$.}
it suffices to make $\gamma$-admissible partition of unity and observe that after rescaling $B(x, \gamma(x)) \mapsto B(0, 1)$ we have $\mu \mapsto \mu_{\text{new}} = \mu \gamma \rho^{-1}$, $h \mapsto h_{\text{new}} = h \gamma^{-1} \rho^{-1}$ and therefore $\mu h \mapsto \mu h / \rho^2$, $\mu^{-1} h \mapsto \mu^{-1} h \gamma^{-2}$ and before rescaling $\mu = h = 1$.

Applying Theorem 13.4.32 for $d = 2$ and Theorem 19.6.25 of [Ivr2] for $d \geq 4$ we estimate contribution of $Z(\eta)$ to the remainder by the first term in the right-hand expression of (4.8).

Further, applying Theorem 13.5.6 for $d = 2$ and similar results of Section 19.6 of [Ivr2] for $d \geq 4$ case we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \geq \eta\}$ to the remainder by the second term in the right-hand expression of (4.8).

In the same way we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \leq \eta\}$ to the remainder by the second term in the right-hand expression of (4.8) albeit now we use scale $\mu \mapsto \mu_{\text{new}} = \mu \gamma \eta^{-\frac{1}{2}}$, $h \mapsto h_{\text{new}} = h \gamma^{-1} \eta^{-\frac{1}{2}}$.

We discuss possible generalizations later; right now we want just get two simple corollaries which follow immediately from Theorem 4.3:

**Example 4.4.** (i) In the framework of Theorem 4.3 with $\gamma = \langle x \rangle$, $\rho = \langle x \rangle^m$, $m < 0$

\begin{equation}
|N(\eta) - N(\eta)| \leq C \begin{cases} 
| \log \eta | & \text{for } d = 2, \\
\eta^{(d-2)/(2m)} & \text{for } d \geq 4 
\end{cases}
\end{equation}

with $N(\eta) = O(\eta^{d/(2m)})$. Further, $N(\eta) \asymp \eta^{d/(2m)}$ if condition (4.11) is fulfilled in some non-empty cone.

(ii) Furthermore, if condition (4.11) is fulfilled, then for $d = 2$

\begin{equation}
N(\eta) = N(\eta) + O(1).
\end{equation}

**Example 4.5.** (i) In the framework of Theorem 4.3 with $\gamma = \langle x \rangle^{1-\sigma}$, $\rho \leq \exp(-\epsilon \langle x \rangle^\sigma)$, $0 < \sigma < 1$

\begin{equation}
N(\eta) = N(\eta) + O(| \log \eta |^{2+/(d-2)/\sigma})
\end{equation}

with $N(\eta) = O(| \log \eta |^{d/\sigma})$. Further, $N(\eta) \asymp | \log \eta |^{d/\sigma}$ if condition (4.11) is fulfilled in some non-empty cone and $\rho \geq \exp(-c \langle x \rangle^\sigma)$.  

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(ii) Furthermore, if condition \((4.11)_\pm\) is fulfilled then the remainder estimate \((4.15)\) could be improved to
\[
N^\pm(\eta) = \mathcal{N}^\pm(\eta) + O(|\log \eta|^{2+(d-2)/\sigma}).
\]

The following problem seems to be very challenging:

**Problem 4.6**\(^{36}\). Let \(d = 2, \log(\pm V(x)) = |x|^\beta \phi(x)\) where \(|\nabla \phi| \leq C\). Then the drift line is of the length asymptotic \(r\). Improve the remainder estimate \((4.15)\) to \(O(1)\).

**Remark 4.7.** (i) We need conditions \((4.6)_1-3\) only for \(|\alpha| \leq 3\) due to Section 19.6 of [Ivr2] and we need ‘‘\(o\)’’ in this condition only for \(|\alpha| \leq 1\). Further, if \(\epsilon_0\) in conditions \((4.7)_\pm\) and \((4.11)_\pm\) is fixed we can replace ‘‘\(o(\rho^2 \gamma^{-|\alpha|})\)’’ by ‘‘\(\leq \epsilon_1 \rho^2 \gamma^{-|\alpha|}\)’’ with \(\epsilon_1 = \epsilon_1(\epsilon_0)\).

(ii) If \(#W = 1\) we can have ‘‘\(O\)’’ but replace \(V\) in \((4.7)_\pm\) and \((4.11)_\pm\) by \(V_\delta\).

We leave to the reader the series of the following not challenging but interesting problems:

**Problem 4.8.** (i) Consider even faster decaying \(\rho \leq \exp(-|x|\gamma^{-1}(|x|))\) with monotone increasing \(\gamma(t)\) such that \(\gamma'(t) = o(\gamma(t)t^{-1})\) and \(\gamma(t) \to \infty\) as \(t \to \infty\) and prove the remainder estimate
\[
\begin{align*}
(\text{a}) & \quad O(t^{d-\gamma(t)^{-2}}) \text{ in the general case and} \\
(\text{b}) & \quad O(t^{d-1-\gamma(t)^{-1}}) \text{ under assumption } (4.11)_\pm
\end{align*}
\]
while \(\mathcal{N}^+(\eta) = O(t^d)\) in the general case and \(\mathcal{N}^+(\eta) \asymp t^d\) under assumption \((4.11)_\pm\) fulfilled some non-empty cone \(\Gamma\) as \(|x| \geq c\). Here \(t = t(\eta)\) recovered from \(t \gamma(t)^{-1} \asymp |\log \eta|\).

While proof of Theorem 4.3 provides proper estimates of the contributions to the remainder of the zones \(\mathcal{Z}(\eta)\) and \(\Omega^+(\eta) \setminus \mathcal{Z}(\eta)\) it fails in the zone \(\Omega^- (\eta) \setminus \mathcal{Z}(\eta) \) where \(\Omega^\pm (\eta) := \{x: |V_\delta(x)| \geq \eta\}\). However one can use here \(\gamma_\eta = \frac{1}{2}(r - r(\eta))\) instead of \(\gamma\).

(ii) For example, consider \(\gamma(t) = (\log(n) t)^\sigma\), where \(\log(n) t\) is \(n\)-tuple logarithm\(^{37}\) with \(\sigma > 0\). Then \(t(\eta) = |\log \eta||\log(n+1) \eta|^\sigma\).

\(^{36}\) Cf. Problem 3.9.
\(^{37}\) I.e. \(\log(1) t = \log t\) and \(\log(n) t = \log(\log(n-1) t)\).
(iii) Consider even $\exp(-c\varepsilon |x|) \leq \rho \leq \exp(-\varepsilon |x|)$, $\gamma = \varepsilon^{-1}$ with sufficiently small $\varepsilon \leq \varepsilon(c, \varepsilon_0, \Gamma)$ and condition $(4.11)_\mp$ fulfilled in some non-empty cone $\Gamma$. Then

$$N_\mp(\eta) \asymp \varepsilon^{-d} |\log \eta|^d.$$  

Remark 4.9. Asymptotics in the case of $\rho \leq \exp(-\varepsilon_0 |x|)$ or even compactly supported $V$ is out of reach of our methods.

Amazingly such asymptotics (without remainder estimate) were obtained in papers M. Melgaard and G. Rozenblum [MR], G. Rozenblum and G. Tashchiyan [RT1, RT2] G. Raikov and S. Warzel [RaV] by completely different methods.

Problem 4.10. Consider slowly decreasing potentials with $\gamma \asymp |x|$ and $\rho = |\log(\eta) |x||^{-\sigma}$ with $\sigma > 0$.

In this case we need to replace assumptions $(4.6)_{1-3}$ with $|\alpha| \geq 1$ and $(4.7)_{\mp}$ by

$$(4.6)^{1-3}_{\pm} |\nabla^{\alpha}(g-g_\infty)| = o(\rho^{2\gamma^{-|\alpha|}}), \quad |\nabla^{\alpha}(F-F_\infty)| = o(\rho^{2\gamma^{-|\alpha|}}),$$

$$|\nabla^{\alpha} V| = O(\rho^{2\gamma^{-|\alpha|}}) \quad \text{as} \quad |x| \to \infty \quad \forall \alpha : |\alpha| \geq 1.$$  

and

$$(4.7)^{\mp}_{\mp} V \geq -\varepsilon \rho^2 \implies |\nabla V| \geq \varepsilon_0 \rho^2 \gamma^{-1} \quad \text{as} \quad |x| \geq c.$$  

respectively where $\rho$ is another $\gamma$-admissible scaling function; $\rho \leq 1$.

Here again we apply Theorems 13.4.32 and 13.6.6 for $d = 2$ and results of Section 19.6 of [Ivr2] for $d \geq 4$.

The first of the following problems seems to be challenging enough while the second one is rather easy:

Problem 4.11. Using results of Chapter 15 of [Ivr2] consider 2-dimensional domains $X$ with are $\gamma$-admissible boundaries, i.e. domains which are conical outside of the ball $B(0, c)$. Neumann boundary conditions would be especially interesting and challenging.

Problem 4.12. Generalize results of this subsection to genuine Schrödinger-Pauli and Dirac operators. While in the former case no modifications is needed (except the Landau levels), in the latter case we need to consider two cases.
(a) \( M^2 + 2jF_{\infty} > 0 \) and potential \( V \sim \rho^2 \) at infinity.

(b) \( M^2 + 2jF_{\infty} = 0 \) and potential \( V \sim \rho \) at infinity.

It is so because \( M^2 + 2jF_{\infty} \) plays a role of the mass.

### 4.2 Operators stabilizing at infinity. II

Assume now that \( g \) and \( F \) stabilize at infinity to \( g_\infty = g_\infty(\theta), F_\infty = F_\infty(\theta) \), positively homogeneous of degree 0, and \( V \to 0 \). Then one can see easily that the for the Schrödinger and Schrödinger-Pauli operators essential spectrum of \( A \) consists of (possibly overlapping) spectral bands \( \Pi_j \)

\[
\text{Spec}_{\text{ess}}(A) = \bigcup_{j \in \mathbb{Z}} \Pi_j, \quad \Pi_j := \{ \sum_j j f_{\infty,j}(\theta), : \theta \in [0, 2\pi] \}.
\]

with the spectral gaps between them.

In particular, for the Schrödinger operator all spectral bands in the generic case have non-zero width while for the Schrödinger-Pauli operator \( \Pi_0 = \{0\} \). Then under proper assumptions for the eigenvalues tending to +0 or −0 the results of the previous Subsection 4.1 hold. In this subsection we are interested in the asymptotics of the eigenvalues tending to the border between a spectral gap and a spectral band of non-zero width.

Further, for \( d = 2 \) in the generic case there could be an infinite number of spectral gaps, but for \( d \geq 4 \) there is only finite number of them.

Similarly, for the Dirac operator essential spectrum consists of the spectral bands, one of them consisting of a single point \( M \) or \(-M\).

**Theorem 4.13.** Let \( S \) be a Schrödinger or Schrödinger-Pauli operator. Let conditions (4.6)\(_{1-3}\) are fulfilled with \( g_\infty, F_\infty \) positively homogeneous of degree 0, \( \gamma = \epsilon |x|, \rho = |x|^m, m < 0 \). Let \( x = r \theta \) with \( r = |x|, \) and \( \theta \in \mathbb{S}^{d-1} \).

Assume for simplicity\(^{38}\) that

\[
\tau^* = \sum_j j f_{\infty,j}(\theta) \text{ if and only if } \dot{z} = \ddot{z} \text{ and } \theta = \ddot{\theta},
\]

\[
\text{In the vicinity of } \ddot{\theta} \text{, } f_{\infty,1}(\theta), \ldots, f_{\infty,s}(\theta) \text{ are disjoint},
\]

\(^{38}\) Otherwise we will get the sums of asymptotics.
\( (4.21)_\pm \pm \sum_j \bar{z}_j f_{\infty,j}(\theta) \geq \epsilon |\theta - \tilde{\theta}|^{2n} \)

and
\( (4.22) |\partial^\alpha \sum_j \bar{z}_j f_{\infty,j}(\theta)| \leq c_{\alpha} |\theta - \tilde{\theta}|^{2n-|\alpha|} \quad \forall \alpha : |\alpha| \leq 2n. \)

Under assumption \( (4.21)_+ \) let \( N^-(\eta) := N(\tau^* - \epsilon, \tau^* - \eta) \) and under assumption \( (4.21)_- \) let \( N^+(\eta) := N(\tau^* + \eta, \tau^* + \epsilon) \).

(i) Let
\( (4.23) \mp, 1, 2 \mp V(r \bar{\theta}) \geq \epsilon r^{2m}, \quad \mp \partial_r V(r \bar{\theta}) \geq \epsilon r^{2m-1} \quad \text{as} \ r \geq c \)

and \( m + n > 0 \). Then as \( \eta \to +0 \)
\( (4.24) N^\pm(\eta) = N^\pm(\eta) + O(\eta^{((m+n)(d-3)+n)/2mn}) \)

with
\( (4.25) N^\pm(\eta) = (2\pi)^{-r} \int_{\{x: \mp \bar{V}_j \geq \eta\}} f_1 \cdots f_r \, dx \asymp \eta^{((m+n)(d-1)+n)/2mn}) \),
\( (4.26) \bar{V}_j(x) := V(x) + \sum_j \bar{z}_j (f_j(x) - f_{\infty,j}(\bar{\theta})). \)

(ii) On the other hand, under assumption \( (4.23) \pm N^\pm(\eta) = O(1) \).

Proof. Assume that \( \bar{\theta} = (1, 0, \ldots, 0), x = (x_1; x') = (x_1; x_2, \ldots, x_d) \). Observe that outside of \( \mathcal{X}(\eta) = \{x: |x'| \leq cr^{1+m/n}, 0 < x_1 = r \leq c\eta^{1/(2m)}\} \) is a forbidden zone and one can prove easily that its contribution to the remainder is \( O(1) \). On the other hand, contribution to the remainder of \( \gamma'(r) \)-partition element in \( \mathcal{X}'(\tau) \) is \( O(\gamma^{d-2}) \) and the total remainder does not exceed \( \int_{\{r \leq c\eta^{1/(2m)}\}} \gamma'^{-3}(r) \, dr \) which results in \( (4.24) \). \( \square \)

The following problem is rather easy:

**Problem 4.14.** Derive similar results for the Dirac operator.

The following problem looks challenging:

**Problem 4.15.** Investigate what happens if \( m + n \leq 0 \). Our methods provide only \( N^\pm(\eta) = O(\eta^{-1/(2n)}) \). Probably methods of Section 12.2 of [Ivr2] could provide an answer.
4.3 Case $F \to \infty$ as $|x| \to \infty$

In this subsection we consider cases of $F \to \infty$ and $V \to 0$ as $|x| \to \infty$. In this case the Schrödinger operator does not have any essential spectrum at all and thus is not the subject of our analysis, while for the Schrödinger-Pauli and Dirac operators essential spectrum consists of just one point: $0$ and $\pm M$ respectively (see Theorem 17.1.2 of [Ivr2] to find out which; if $d = 2$ it is determined by signs of $F_{12}$ and $\varsigma$). Again due to the specifics of the problem we can consider the multidimensional case without any modifications.

It turns out that for $d = 2$ the remainder estimate is as in the previous Subsection 4.1, while the magnitude of the principal part is larger but it is still given by the same formula).

Let for the Schrödinger-Pauli operator $N^-\left(\eta\right)$ be a number of eigenvalues in $(-\epsilon, -\eta)$ and $N^+\left(\eta\right)$ be a number of eigenvalues in $(\eta, \epsilon)$.

**Theorem 4.16** 39). (i) Let $X$ be a connected exterior domain with $\mathcal{C}^K$ boundary. Let conditions (2.2), (2.3)$_{1-3}$ and (4.7)$_{\pm}$ be fulfilled with scaling functions $\gamma$, $\rho$ and $\rho_1$, $\rho \to 0$, $\rho_1 \to \infty$ and $\rho_1 \gamma^2 \to \infty$ as $|x| \to \infty$. Assume that

\begin{equation}
|F^{-1}| \leq c \rho_1^{-1} \quad \text{for } |x| \geq c
\end{equation}

and

\begin{equation}
\text{For each } j \neq k \text{ either } f_j = f_k \text{ or } |f_j - f_k| \geq \epsilon \rho_1 \text{ for all } |x| \geq c.
\end{equation}

Then for the Schrödinger-Pauli operator

\begin{equation}
N^\mp(\eta) = \mathcal{N}^\mp(\eta) + O(R)
\end{equation}

where

\begin{equation}
\mathcal{N}^\mp(\eta) := (2\pi)^{-r} \int_{\{x: \mp V(x) \geq \eta\}} f_1 f_2 \cdots f_r \sqrt{g} \, dx
\end{equation}

\begin{equation}
R = C \int_{\mathcal{Z}(\eta)} \rho_1^{r-1} \gamma^{-2} \, dx + C \int \rho_1^{r-s} \gamma^{-2s} \, dx
\end{equation}

holds, $r = d/2$, $\mathcal{Z}(\eta)$ is $\epsilon \gamma$-vicinity$^{35}$ of $\Sigma(\eta) = \{x: \mp V(x) = \eta\}$.

\textit{39) Cf. Theorem 4.3.}
(ii) Further, under assumption \((4.11)_{\mp}\) \(0 \pm 0\) is not a limit point of the discrete spectrum.

Example 4.17\(^{40}\). (i) In the framework of Theorem 4.16 with \(\gamma = \langle x \rangle\), \(\rho = \langle x \rangle^m\), \(\rho_1 = \langle x \rangle^{m_1}\), \(m < 0 < m_1\) estimate \((4.29)\) holds with

\[
R = \begin{cases}
|\log \eta| & \text{for } d = 2, \\
\eta^{(d-2)k/(2m)} & \text{for } d \geq 4
\end{cases}
\]

and with \(N^\pm(\eta) = O(\eta^{dk/(2m)})\). Further, \(N^\pm(\eta) \approx \eta^{dk/(2m)}\) if condition \((4.11)_{\mp}\) is fulfilled in some non-empty cone.

(ii) Furthermore, if condition \((4.11)_{\mp}\) is fulfilled, then even for \(d = 2\) \(R = 1\).

Example 4.18. (i) Let conditions \((2.2), (2.3)_2, (4.24), (4.28)\) be fulfilled with \(\gamma = \langle x \rangle^{1-\beta}\), \(\rho = \langle x \rangle^m\), \(\rho_1 = \exp(b \langle x \rangle^\beta)\), \(m < 0\), \(\beta > 0\).

Further, let conditions \((2.3)_{1,3}\) and \((4.7)_{\mp}\) be fulfilled with \(\rho = \langle x \rangle^m\), \(\gamma = \epsilon \langle x \rangle\). Then the following asymptotics holds:

\[
N^\pm(\eta) = N^\mp(\eta) + O(\eta^{(d-2+2\beta)/(2m)})
\]

and

\[
\log(N^\pm(\eta)) = O(\eta^{\beta/(2m)}).
\]

(ii) Let conditions \((2.2), (2.3)_{1,-3}\) and \((4.7)_{\mp}\) be fulfilled with \(\gamma = \langle x \rangle^{1-\alpha}\), \(\rho = \exp(a \langle x \rangle^\alpha)\), \(\rho_1 = \langle x \rangle^{m_1}\), \(a < 0\), \(\alpha > 0\), \(m_1 > 0\), \(m_1 + 2(1 - \alpha) > 0\). Then following asymptotics holds:

\[
N^\pm(\eta) = N^\mp(\eta) + O(|\log \eta|^{(d-2+2\alpha)/\alpha})
\]

and

\[
N^\pm(\eta) = O(|\log \eta|^{(d+m_1)/\alpha}).
\]

(iii) Moreover, if condition \((4.11)_{\mp}\) is fulfilled, then the remainder estimate \((4.35)\) could be improved to

\[
N^\pm(\eta) = N^\mp(\eta) + O(|\log \eta|^{(d-2+\alpha)/\alpha})
\]

\(^{40}\) Cf. Example 4.4.
(iv) Let conditions (2.2) and (2.3) be fulfilled with \( \gamma = \langle x \rangle^{1-\sigma}, \rho = \exp(a\langle x \rangle^{\alpha}), \rho_1 = \exp(b\langle x \rangle^{\beta}), a < 0 < b, \alpha > 0, \beta > 0, \sigma = \max(\alpha, \beta) \).

Further, let conditions (2.3) and (4.7) be fulfilled with \( \gamma = \langle x \rangle^{1-\alpha}, \rho = \exp(a\langle x \rangle^{\alpha}). \) Then the following asymptotics holds:

\begin{align}
N^\pm(\eta) &= N^{\mp}(\eta) + \mathcal{O}(|\log \eta|^{(d-2+2\sigma)/\alpha}) \\
\log(N^{\pm}(\eta)) &= \mathcal{O}(|\log \eta|^{\beta/\alpha}).
\end{align}

(v) Moreover, if condition (4.11) is fulfilled, then the remainder estimate (4.38) could be improved to

\begin{align}
N^{\mp}(\eta) &= N^{\pm}(\eta) + \mathcal{O}(|\log \eta|^{(d-2+2\sigma-\alpha)/\alpha})
\end{align}

(vi) Furthermore, if condition (4.11) is fulfilled in some non-empty cone then there is \( \asymp \) rather than \( \mathcal{O}(\cdot) \) in (4.34), (4.36) and (4.39).

**Problem 4.19.** Again, one can hope to improve estimates (4.37) and (4.40) in the same way as specified in Problem 4.6.

We leave to the reader **Problem 4.20**. Consider the Dirac operator. In this case \( N^-(\eta) \) is a number of eigenvalues in \( (\pm M - \epsilon, \pm M - \eta) \) and \( N^+(\eta) \) is a number of eigenvalues in \( (\pm M + \eta, \pm M + \epsilon), 0 < \eta < \epsilon \) and \( \pm M \) is a point of the essential spectrum.

We need to distinguish two cases

(a) \( M > 0 \) and potential \( V \sim \rho^2 \) at infinity.

(b) \( M = 0 \) and potential \( V \sim \rho \) at infinity.

\[\text{Cf. Problem 4.12.}\]
4.4 Case $F \to 0$ as $|x| \to \infty$

In this subsection we consider cases of $F \to 0$ and $V \to 0$ as $|x| \to \infty$. In this case the essential spectra of the Schrödinger and Schrödinger-Pauli operators are $[0, \infty)$; however, as $V = o(F)$ as $|x| \to \infty$ the Schrödinger operator has only a finite number of the negative eigenvalues and thus is not a subject of our analysis while the Schrödinger-Pauli operator is.

Further, the Dirac operator has its essential spectrum $(-\infty, -M] \cup [M, \infty)$ and we need to assume that $M > 0$.

It turns out that the remainder estimate is as in Subsection 4.1, while the magnitude of the principal part is smaller but it is still given by the same formula).

**Theorem 4.21** \(^\text{42}\). (i) Let $X$ be a connected exterior domain with $C^K$ boundary. Let conditions (2.2), (2.3)\(_{1-3}\), (2.19), (4.24) and (4.28) and (4.7)\(_*\) (with sign “−”) be fulfilled with scaling functions $\gamma$, $\rho$ and $\rho_1$, $\rho \to 0$, $\rho_1 \to 0$ and $\rho_1 \gamma^2 \to \infty$, $\rho_1 \rho^{-2} \to \infty$ as $|x| \to \infty$. Then for the Schrödinger-Pauli operator (4.29)–(4.32) holds\(^\text{43}\).

(ii) Further, under assumption (4.11)\(_*\) (with sign “−”) $0 - 0$ is not a limit point of the discrete spectrum.

**Example 4.22** \(^\text{44}\). (i) Let conditions of Theorem 4.21 be fulfilled with $\gamma = \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $m < 0$, $\max(2m, -2) < m_1 < 0$.

Then estimate (4.29)\(^\text{43}\) holds with $R$ defined by (4.32) and with $N^- (\eta) = O(\eta^{d/(2m)})$. Further, $N^- (\eta) \asymp \eta^{d/(2m)}$ if condition (4.11)\(_*\) is fulfilled in some non-empty cone.

(ii) Furthermore, if condition (4.11)\(_*\) is fulfilled, then even for $d = 2$ $R = 1$.

**Example 4.23.** (i) Let conditions of Theorem 4.21 be fulfilled with $\gamma = \epsilon \langle x \rangle$, $\rho_1 = \langle x \rangle^{-2} |\log \langle x \rangle|^{\beta}$, $\beta > 0$. Let either $\rho = \langle x \rangle^m$ with $m < -1$ or $\rho = \langle x \rangle^{-1} |\log \langle x \rangle|^{\alpha}$ with $2\alpha < \beta$. Then the remainder estimate is $O(R)$ with $R$ defined by (4.32) and $N^- (\eta) = O(S)$ with

\[
S = \begin{cases}
|\log \eta|^{\beta+1} & d = 2, \\
|\eta^{(d-2)/(2m)}| |\log \eta|^{\beta} & d \geq 4.
\end{cases}
\]

\(^{42}\) Cf. Theorems 4.3 and 4.16.

\(^{43}\) With the sign “−”.

\(^{44}\) Cf. Examples 4.4 and 4.17.
Further, $N^- (\eta) \asymp S$ if condition (4.11) is fulfilled in some non-empty cone.

(ii) Furthermore, if condition (4.11) is fulfilled, then even for $d = 2$, $R = 1$.

Example 4.24. (i) Let conditions of Theorem 4.21 be fulfilled with $\gamma = \langle x \rangle^{1-\alpha}$, $\rho = \exp (a \langle x \rangle^\alpha)$, $\rho_1 = \langle x \rangle^{m_1}$, $a < 0$, $\alpha > 0$, $m_1 < 0$, $m_1 + 2 - 2\alpha > 0$. Then the remainder estimate is $O(R)$ with $R$ defined by (4.35) and (4.36) holds.

(ii) Further, if condition (4.11) is fulfilled, then the remainder estimate (4.35) could be improved to (4.37).

(iii) Furthermore, if condition (4.11) is fulfilled in some non-empty cone then there is $\asymp$ in (4.36).

Problem 4.25. Again, one can hope to improve estimates (4.37) and (4.40) in the same way as specified in Problem 4.6.

We also leave to the reader

Problem 4.26. Consider in this framework the Dirac operator. In this case $N^- (\eta) = N^- (\eta) + O(\eta^{1-m_1/(2m)})$.

Example 4.27. (i) Let $X$ be a connected exterior domain with $\mathcal{C}^K$ boundary and $d = 2$. Let conditions (2.2), (2.3) and (3.1) be fulfilled with $\gamma = \rho_0 (x)$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $-1 < m < 0$, $m - 1 < m_1 \leq 2m$. Then the Schrödinger operator asymptotics

$$N^- (\eta) = N^- (\eta) + O(\eta^{1-m_1/(2m)})$$

---

45) Cf. Example 4.18.
46) Cf. Problem 4.12 and 4.20.
47) Cf. Example 3.16.
holds as $\eta \to +0$ with

$$N^- (\eta) = O(\eta^{(m+1)/m}). \tag{4.43}$$

Indeed, it follows from the arguments of Example 3.15; we need to take into account that $\{ |x| \geq C \tau^{1/(2m)} \}$ is a forbidden zone.

(ii) Similar results hold in the full-rank even-dimensional case:

$$N^- (\eta) = N^- (\eta) + O(\eta^{1-m/(2m)} + (d-2)(m+1)/(2m)) \tag{4.44}$$

holds as $\eta \to +0$ with

$$N^- (\eta) = O(\eta^{d(m+1)/(2m)}). \tag{4.45}$$

We leave to the reader:

Problem 4.28. Consider the case of $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^{-2} |\log x|^\alpha$, $\rho_1 = \langle x \rangle^{-2} |\log x|^{\beta}$, $2\alpha \geq \beta > \alpha$.

We also leave to the reader

Problem 4.29. Consider in this framework the Dirac operator with $M > 0$. In this case both points $M - 0$ and $-M + 0$ could be limits of the discrete spectrum simultaneously.

5 2D-case. Multiparameter asymptotics

In this section we consider asymptotics with respect to three parameters $\mu$, $h$ and $\tau$; here spectral parameter $\tau$ tends either to $\pm \infty$ or to the border of the essential spectrum or to $-\infty$ (for Schrödinger and Schrödinger-Pauli operators) or to the border of the spectrum. In two last cases presence of $h \to +0$ is crucial. We consider here only $d = 2$ and $h \ll 1$.

\(^{48}\) Cf. Problems 4.12, 4.20 and 4.26.
5.1 Asymptotics of large eigenvalues

In this subsection \( \tau \to +\infty \) for the Schrödinger and Schrödinger-Pauli operators and \( \tau \to \pm \infty \) for the Dirac operator. We consider the Schrödinger and Schrödinger-Pauli operators, leaving the Dirac operator to the reader.

**Example 5.1.** Assume first that \( \psi \in \mathcal{C}_0^\infty \) and there are no singularities on \( \text{supp}(\psi) \). We consider

\[
N^-_\psi(\tau) = \int e(x, x, \tau) \psi(x) \, dx.
\]

Then for scaling \( A \mapsto \tau^{-1} A \) leads to \( h \mapsto h \tau^{-1/2} \) and \( \mu \to \mu \tau^{-1/2} \).

(i) If \( \mu \lesssim \tau^{1/2} \) then we can apply the standard theory with the “normal” magnetic field; we need to assume that \( h \ll \tau^{1/2} \) and we need neither condition \( d = 2 \), nor \( F \geq \epsilon_0 \), nor \( \partial X = \emptyset \); the principal part of the asymptotics has magnitude \( h^{-d} \tau^{d/2} \) and the remainder estimate is \( O(h^{1-d} \tau^{(d-1)/2}) \) which one can even improve to \( o(h^{1-d} \tau^{(d-1)/2}) \) under proper non-periodicity assumption.

(ii) Let \( \mu \gtrsim \tau^{1/2}, \mu h \lesssim \tau \). Then we can apply the standard theory with the “strong” magnetic field; we assume that \( d = 2 \), \( \partial X = \emptyset \) and \( F \geq \epsilon_0 \). Then the principal part of the asymptotics has magnitude \( h^{-2} \tau \) and under non-degeneracy assumptions

\[
\nabla F = 0 \implies \det \text{Hess} F \geq \epsilon
\]

and

\[
\nabla F = 0 \implies |\det \text{Hess} F| \geq \epsilon
\]

fulfilled on \( \text{supp}(\psi) \) the remainder estimate is \( O(\mu^{-1} h^{-1} \tau) \) and \( O(\mu^{-1} h^{-1} \tau(|\log(\mu \tau^{-1/2})| + 1)) \) respectively\(^{49}\). Without non-degeneracy assumption the remainder estimate is \( O(\mu h^{-1}) \).

(iii) If \( \mu \gtrsim \tau^{1/2}, \mu h \geq c \tau \) than \( N^-(\tau) = 0 \) for the Schrödinger operator; for the Schrödinger-Pauli operator the principal part of the asymptotics has magnitude \( \mu h^{-1} \) and under non-degeneracy assumptions (5.2) and (5.3) the remainder estimate is \( O(1) \) and \( O(\log \mu) \) respectively (or better for \( \tau \) belonging to the spectral gap).

\(^{49}\) In the latter case logarithmic factor could be removed by adding a correction term.
Example 5.2. Let $X$ be a connected exterior domain with $\mathcal{C}^K$ boundary. Let conditions (2.2), (2.3)$_{1-3}$, (3.1)$_{1,2}$, be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^m_1$, $m_1 > 2m \geq 0$. Consider the Schrödinger operator and assume that

$$\tau \geq \mu h, \quad \tau^{2m - m_1} \leq \epsilon(h)^{2m}. \quad (5.4)_{1,2}$$

Then

$$\mathcal{N}^- (\tau, \mu, h) \asymp \tau^{(2+m_1)/m_1} h^{-2(1+m_1)/m_1} \mu^{-2/m_1}. \quad (5.5)$$

(i) Further, if $\tau \gtrsim \mu^2$, then the zone of the strong magnetic field $\mu_{\text{eff}} = \mu \langle x \rangle^{m_1+1} \tau^{-1/2} \geq C$ is contained in $\{x: |x| \geq c\}$ and here we have non-degeneracy condition fulfilled. Then the remainder estimate is $O(R)$ with

$$R = \tau^{(m+2)/2(m+1)} \mu^{-1/(m+1)} h^{-1}, \quad (5.6)$$

which could be improved under non-periodicity assumption; see Example 3.18.

(ii) On the other hand, if $\mu^2 \gg \tau$, then the contribution of the zone $\{x: |x| \leq c\}$ to the remainder is $O(\mu^{-1} h^{-1})$. The contribution of the zone $\{x: |x| \geq c\}$ to the remainder is $O(\mu^{-1} h^{-1})$ provided $X = \mathbb{R}^2$ and non-degeneracy assumption (5.2) is fulfilled (etc) and $O(\mu h^{-1})$ in the general case.

(iii) Let us replace (5.4)$_2$ by the opposite inequality, and assume (3.3)$_\#$.

Then (5.5) is replaced by $\mathcal{N}^- (\tau, \mu, h) \asymp h^{-2} \tau^{(m+1)/m}$. Let us discuss $R$.

(a) If $\mu \tau^{(m_1+1-m)/(2m)} \lesssim 1$, then $\mu_{\text{eff}} \lesssim 1$ as $|x| \lesssim \tau^{1/(2m)}$ and $R = h^{-1/(m+1)}$. 

(b) If $\mu \tau^{(m_1+1-m)/(2m)} \gtrsim 1$, but $\mu^2 \tau \lesssim 1$, then $R$ is given by (5.6).

(c) If $\mu^2 \gg \tau$, then we are in the framework of (ii).

Example 5.3. In the framework of Example 5.2 for the Schrödinger-Pauli operator under assumption (3.3)$_\#$ the remainder estimate is the same as in Statement (iii) while

$$\mathcal{N}^- (\tau, \mu, h) \asymp h^{-2} \tau^{(m+1)/m} + \mu h^{-1} \tau^{(m_1+2)/(2m)}. \quad (5.7)$$

50) Cf. Example 3.15.

51) Cf. Example 3.19.
We leave to the reader

**Problem 5.4.** Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition \( m_1 > 2m \) replaced by \( 2m \geq m_1 \geq 0 \). Assume that \( (3.3)^\# \) is fulfilled.

Then magnitude of \( N^- (\tau, \mu, h) \) is described in Examples 5.2 and 5.3. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

(ii) In the same framework as in (i) albeit with \( m_1 < 0 \) (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

(iii) With other types of the behaviour at infinity.

**Problem 5.5.** For the Dirac operators derive similar results as \( \tau \to \pm \infty \).

### 5.2 Asymptotics of the small eigenvalues

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to \(-0\).

**Example 5.6.** Let \( X \) be a connected exterior domain with \( \mathcal{C}^K \) boundary. Let conditions (2.2), (2.3)\(_{1-3}\), (3.1)\(^\#\) be fulfilled with \( \gamma = \epsilon_0 \langle x \rangle \), \( \rho = \langle x \rangle^m \), \( \rho_1 = \langle x \rangle^{m_1} \), \( -1 < m < 0 \), \( m_1 > m - 1 \).

Consider the Schrödinger operator and assume that

\[
(5.8)_{1,2} \quad 1 \geq \mu h, \quad |\tau|^{2m-m_1} \leq \epsilon (\mu h)^{2m}.
\]

Then \( N^- (\tau, \mu, h) = O(h^{-2}|\tau|^{(m+1)/m}) \) as \( \tau \to -0 \) with “\( \asymp \)” instead of “\( = O \)” if condition \( (4.11)^\mp \) (with the sign “\( - \)” ) fulfilled in some non-empty cone.

Further, under non-degeneracy assumption \( (3.27) \) the contribution to the remainder of zone \( \{ x : |x| \geq c \} \) is \( O(R) \) with

(i) If \( \mu|\tau|^{(m+1-m)/(2m)} \lesssim 1 \) then \( R = h^{-1}|\tau|^{(m+1)/(2m)} \).

(ii) Let \( \mu|\tau|^{(m_1+1-m)/(2m)} \gtrsim 1 \). Then

\[
52) \text{ Cf. Example 4.22.}
\]
(a) If $m_1 < 2m$ then $R = \mu^{-1}h^{-1}|\tau|(2m-m_1)/(2m)$.
(b) If $m_1 = 2m$ then $R = \mu^{-1}h^{-1}|\log \mu|$.
(c) If $m_1 > 2m$ then $R = h^{-1}\mu^{(m+1)/(m_1+1-m)}$ for $\mu \lesssim 1$ and $R = \mu^{-1}h^{-1}$ for $\mu \gtrsim 1$.

Example 5.7\textsuperscript{53}). In the framework of Example 5.6 for the Schrödinger-Pauli operator under assumption (3.3)\textsuperscript{*} the contribution to the remainder of the zone $\{x: |x| \geq c\}$ the same as in Example 5.6(ii) while

$$(5.9) \quad N^-(\tau, \mu, h) = O(h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)})$$

$\asymp$ if condition (4.11)\textsuperscript{\mp} (with the sign “−”) fulfilled in some non-empty cone.

Problem 5.8. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) If condition (5.8)\textsubscript{1} is violated (then there could be a forbidden zone in the center).

(ii) $m_1 \leq m - 1$.

(iii) With other types of the behaviour at infinity.

Problem 5.9. Consider the Schrödinger and Schrödinger-Pauli operators in

the framework of Subsections 4.1 and 4.2 if

(i) $\mu h = 1$; then the essential spectrum does not change.

(ii) $\mu h \to \infty$; then only point 0 of the essential spectrum is preserved for
the Schrödinger-Pauli operators, while others go to $+\infty$. Consider $N^\mp(\eta)$ with $\eta \to 0$.

(iii) $\mu h \to 0$; then only point 0 of the essential spectrum is preserved for
the Schrödinger-Pauli operators, while others move towards it. Consider $N^- (\eta)$ with $\eta \to 0$ for both Schrödinger and Schrödinger-Pauli operators.

Problem 5.10. (i) Consider the Schrödinger-Pauli operators in the framework of Subsection 4.3.

(ii) Consider the Schrödinger and Schrödinger-Pauli operators in the framework of Subsection 4.4.

Problem 5.11. For the Dirac operators derive similar results as $M \neq 0$ and $\tau \to M - 0$ and $-M + 0$; or as $M = 0$ albeit $m_1 \geq 0$.

\textsuperscript{53)} Cf. Example 4.22.
5.3 Case of $\tau \to +0$

In this subsection $\tau \to +0$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm M \pm 0$ for the Dirac operator. Consider the Schrödinger and Schrödinger-Pauli operators first.

Example 5.12$^{54}$). Let $V > 0$ everywhere except $V(0) = 0$. Let conditions (2.2), (2.3)$_{1-3}$, (3.1)$^\#$, be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m \geq 0$. Consider the Schrödinger operator and assume that $\tau \to +0$.

(i) Let

$$(5.10)_{1,2} \quad \mu \ll \tau^{(m_1+2)/2}h^{-(m_1+1)}, \quad \tau^{2m-m_1} \leq \epsilon(\mu h)^{2m}.$$ 

Then (5.5) holds.

Then the remainder estimate is $O(R)$ with defined by (5.6) which could be improved under non-periodicity assumption; see Example 3.18.

(ii) Let us replace $(5.10)_2$ by the opposite inequality, and assume $(3.3)^\#$. Then (5.5) is replaced by $N^- (\tau, \mu, h) \prec h^{-2\tau^{(m_1+1)/m}}$. Let us discuss $R$.

(a) If $\mu\tau^{(m_1+1-m)/(2m)} \ll 1$, then $\mu_{\text{eff}} \ll 1$ as $|x| \lesssim \tau^{1/(2m)}$ and $R = \frac{1}{h^{-1}\tau^{(m_1+1)/(2m)}}$.

(b) If $\mu\tau^{(m_1+1-m)/(2m)} \gg 1$, but $\mu^2\tau \ll 1$, then $R$ is given by (5.6).

Example 5.13$^{55}$). In the framework of Example 5.12 for the Schrödinger-Pauli operator under assumption $(3.3)^\#$ the remainder estimate is the same as in Statement (ii) while (5.7) holds.

We leave to the reader

Problem 5.14$^{56}$). Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition $m_1 > 2m$ replaced by $2m \geq m_1 \geq 0$. Assume that $(3.3)^\#$ is fulfilled.

Then magnitude of $N^- (\tau, \mu, h)$ is described in Examples 5.12 and 5.13. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

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$^{54}$ Cf. Example 5.2.

$^{55}$ Cf. Example 5.3.

$^{56}$ Cf. Problem 5.4.
(ii) In the same framework as in (i) albeit in with \( m_1 < 0 \) (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

(iii) With other types of the behaviour at infinity.

**Problem 5.15.** For the Dirac operators derive similar results as \( \tau \to \pm (M + 0) \).

### 5.4 Case of \( \tau \to -\infty \)

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics with \( \tau \to -\infty \).

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to \(-0\).

**Example 5.16** \(^{57}\). Let \( X \ni 0 \) and let conditions \((2.2), (2.3)_{1-3}, (3.1)^\# \) be fulfilled with \( \gamma = \epsilon_0(x), \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, -1 < m < 0, m_1 > m - 1 \).

Consider the Schrödinger operator and assume that

\[
(5.11)_{1,2} \quad h \ll |\tau|^{(m+1)/(2m)}, \quad |\tau|^{m_1-2m} \leq c(\mu h)^{2m}.
\]

Then \( \mathcal{N}^- (\tau, \mu, h) = O(h^{-1}|\tau|^{(m+1)/m}) \) as \( \tau \to -0 \) with \( \asymp \) if condition \((4.11)_{\mp} \) (with the sign “\(-\)” ) fulfilled in some non-empty cone.

(i) If \( \mu |\tau|^{(m+1-m)/(2m)} \lesssim 1 \) then \( R = h^{-1}|\tau|^{(m+1)/(2m)} \).

(ii) Let \( \mu |\tau|^{(m+1-m)/(2m)} \gtrsim 1 \). Then

(a) If \( m_1 < 2m \) then \( R = \mu^{-1}h^{-1}|\tau|^{(2m-m_1)/(2m)} \).

(b) If \( m_1 = 2m \) then \( R = \mu^{-1}h^{-1}|\log \mu| \).

(c) If \( m_1 > 2m \) then \( R = h^{-1}\mu^{(m+1)/(m_1+1-m)} \) for \( \mu \lesssim 1 \) and \( R = \mu^{-1}h^{-1} \) for \( \mu \gtrsim 1 \).

---

\(^{57}\) Cf. Example 4.22.

\(^{58}\) Cf. Example 4.22.
Example 5.17. In the framework of Example 5.16 for the Schrödinger-Pauli operator under assumption \( (3.3)^\# \) the contribution to the remainder of the zone \( \{ x: |x| \geq c \} \) the same as in Example 5.16(ii) while

\[
N^-(\tau, \mu, h) = O(h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)})
\]

\( \asymp \) if condition \( (4.11) \) (with the sign “−”) fulfilled in some non-empty cone.

Problem 5.18. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) \( m_1 \leq m - 1 \).

(ii) With other types of the behaviour at \( 0 \).

6 Appendices

6.A On the self-adjointness of the Dirac operator

The Dirac operators treated in this Chapter are surely self-adjoint in the case of an exterior domain with the singularity at infinity. However, the same fact should be proven for an interior domain with singular points. We consider a single singular point at \( 0 \).

Theorem 6.1. Let \( X \subset \mathbb{R}^2/(\bar{\gamma}_1 \mathbb{Z} \times \bar{\gamma}_2 \mathbb{Z}) \) \( (0 < \bar{\gamma}_j \leq \infty) \) be an open domain. Let conditions \( (2.2), (2.3)_{1-3}, (2.13) \) be fulfilled. Further, let

\[
F \geq \epsilon \rho_1 \quad \text{and} \quad |V| \geq \epsilon \rho \quad \text{for} \quad |x| \leq \epsilon.
\]

Let us assume that there exists a neighborhood of \( \partial X \), denoted by \( Y \), such that for every \( r > 0 \) the inequalities

\[
\rho_1 \geq \epsilon \rho^2, \quad (\rho \gamma)^s \geq \epsilon \rho_1, \quad \rho \geq 1,
\]

\[
|V + \sqrt{2\mu \rho F}| \geq \epsilon \sqrt{\rho_1} - \epsilon^{-1} \quad \forall j \in \mathbb{Z}^+ \setminus 0,
\]

\[
|V| \geq \epsilon \rho - \epsilon^{-1}
\]

are fulfilled on \( Y \cap X \cap \{|x| \leq r\} \) with appropriate \( \epsilon = \epsilon(r) > 0 \).

Then for \( \mu > 0, h > 0 \) the operator \( A \) with domain \( \mathcal{D}(A) = \mathcal{C}_0^1(X, \mathbb{H}) \) is essentially self-adjoint in \( \mathcal{L}^2(X, \mathbb{H}) \).
Proof. Let us consider the adjoint operator $A^*$. This operator is defined by the same formula with $\mathcal{D}(A^*) = \{u \in L^2, Au \in L^2\}$ with $Au$ calculated as a distribution. We should prove that $\ker(A^* \pm il) = 0$ for both signs. So, here and below let $u \in L^2$ and $(A^* \pm il)u = 0$ for some sign.

The microlocal canonical form of Section 17.2 of [Ivr2] yields the inequality

\begin{equation}
\|\rho v\| \leq M(\|Av\| + \|v\| + \|\gamma^{-1}v\|)
\end{equation}

with a constant $M = M(r)$; all the constants now depend on $\mu$ and $h$.

Let us prove that $(\rho \gamma)^n u \in L^2(X_r)$ for every $r > 0$ by induction on $n$. This is true for $n = 0$ by the assumption $u \in L^2$.

Let $(\rho \gamma)^n u \in L^2(X_r)$ for some $n$; then $v = \gamma (\rho \gamma)^n u$ also belongs to $L^2(X_r)$ and

$$Av = [A, \gamma (\rho \gamma)^n]u \mp iv.$$ 

One can easily see that $[A, \gamma (\rho \gamma)^n]$ is a matrix-valued function the matrix norm of which does not exceed $M(\rho \gamma)^n$. Therefore, taking a $\gamma$-admissible partition of unity in a neighborhood of $X_r$ we see that $\sum_\nu \|A\psi_\nu v\|^2 < \infty$ and therefore (6.5) yields that $\sum_\nu \|\rho \psi_\nu v\|^2 < \infty$. Therefore $\rho v = (\rho \gamma)^{n+1} u \in L^2(X_r)$ and the induction step is complete.

Thus we have proven that $(\rho \gamma)^n u \in L^2(X_r)$ for every $r > 0$. Then $\rho_1 u \in L^2(X_r)$ by (6.2). The ellipticity of $A$ yields that $D_j u \in L^2(X_r)$.

Therefore, if $\psi \in \mathcal{C}_0^1(\mathbb{R}^2/(\bar{\gamma}_1 \mathbb{Z} \times \bar{\gamma}_2 \mathbb{Z})))$ then $\psi u \in \mathcal{D}(\widetilde{A})$ where $\widetilde{A}$ is the closure of $A$. Moreover,

$$\langle \widetilde{A} \pm il \rangle \psi u = [\widetilde{A}, \psi]u.$$ 

Calculating the real part of the inner product with $i\psi u$ we obtain the inequality

$$\|\psi u\| \leq \max |\nabla \psi^2| \cdot \|u\|^2.$$ 

Let us take $\psi = \psi^0(x/r)$ where $\psi^0 \in \mathcal{C}_0^1$ is a fixed function equal to 1 in a neighborhood of 0. Then for $r \rightarrow +\infty$ we see that the left-hand expression of this inequality tends to $\|u\|^2$ and the right-hand expression tends to 0. Therefore $u = 0$. \qed
Theorem 6.2. Let all of the conditions of Theorem 6.1 excluding condition (6.2)_{1-3} be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^m_1$, $m_1 \leq 2m$, $m_1 < -2$. Then the Dirac operator has a self-adjoint extension for $\mu > 0$, $h > 0$.

Proof. For $m < -1$ condition (6.2)_{1-3} is also fulfilled and therefore the operator $A$ is essentially self-adjoint. So, let us treat the case $m \geq -1$. Let us consider the Dirac operator $A_D$ with the potential $V_t = V + tW$ where $t > 0$, $W$ is a potential which is regular away from $\{x = 0\}$, and $W = \pm |x|^{m'}$ in a neighborhood of $x = 0$ with $-1 > m' > \frac{1}{2}m_1$. This operator is essentially self-adjoint by Theorem 6.1. For $m < 0$ let us choose the sign of $W$ coinciding with the sign of $V$ on a neighborhood of $0$ (condition (6.1)_{2} yields that this is possible).

For $m \geq 0$ let us choose an appropriate interval $[\tau_1, \tau_2]$ with $\tau_1 < \tau_2$. Then applying the results of Section 3 we see that the number of eigenvalues of the operator $\tilde{A}_t$ lying in the interval $[\tau_1, \tau_2]$ is bounded uniformly with respect to $t > 0$. Then there exists a sequence $t_k \to +0$ such that there exist $\tau'_1 < \tau'_2$ which do not depend on $k$ and such that $[\tau'_1, \tau'_2] \cap \text{Spec}(\tilde{A}_t) = \emptyset$ for $t = t_k$.

Let $\tau = \frac{1}{2}(\tau' + \tau'_2)$; then all the operators $(\tilde{A}_t - \tau)^{-1}$ are uniformly bounded and $\|A_t u\| \geq \epsilon \|u\| \forall u \in D(A_t)$ for $t = t_k$. Then the same is true for $A$. Therefore $A$ has a self-adjoint extension $\tilde{A}$ satisfying the same estimate.

7 3D-case. Introduction

In this chapter we obtain eigenvalue asymptotics for 3D-Schrödinger, Schrödinger-Pauli and Dirac operators in the situations in which the role of the magnetic field is important. We have seen in Chapters 13 and 17 of [Ivr2] that these operators are essentially different and they also differ significantly from the corresponding 2D-operators which we considered in the Sections 1–5.

Now we usually find ourselves in the situation much closer to Chapter ?? than Sections 1–5 was. Indeed, our local asymptotics now are the same as without magnetic field, under very week non-degeneracy assumptions. We also allow boundaries and a singular points, finite or infinite, belonging to the boundary.

We start from Section 8 in which we consider the case when the spectral parameter is fixed ($\tau = \text{const}$) and study asymptotics with respect to $\mu, h$.
exactly like in Section ?? of [Ivr2] we considered asymptotics with respect to $h$. However, since now we have two parameters, we need to consider an interplay between them: while always $h \to +0$, we cover $\mu \to +0$, $\mu$ remains disjoint from 0 and $\infty$ and $\mu \to \infty$, which in turn splits into subcases $\mu h \to 0$, $\mu h$ remains disjoint from 0 and $\infty$ and $\mu h \to \infty$.

In Section 9 we consider asymptotics with $\mu = h = 1$ and with $\tau$ tending to $+\infty$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm \infty$ for the Dirac operator. We consider bounded domains with the singularity at some point and unbounded domains with the singularity at infinity.

In Section 10 we consider asymptotics with the singularity at infinity and $\mu = h = 1$ and with $\tau$ tending to $+0$ for the Schrödinger and Schrödinger-Pauli operators and to $\pm (M - 0)$ for the Dirac operator.

It includes the most interesting case (see Subsection 4.2 of [Ivr2]) when magnetic field is either constant or stabilizes fast at infinity and potential fast decays at infinity in the direction of magnetic field. In this case we consider a reduced one-dimensional operator which has just one negative eigenvalue $\Lambda(x')$ and it turns out that the asymptotics of the eigenvalues tending to the bottom of the continuous spectrum for 3D-operator coincides with the asymptotics obtained in Subsection 4.1 for 2D-operator with the potential $\Lambda(x')$. In contrast to the rest of the section we consider multidimensional case as well.

In Section 11 we consider asymptotics with respect to $\mu, h, \tau$, like in Section ?? of [Ivr2] and 5 again with significant differences mentioned above.

Further, in Section 12 we consider Riesz means for the 3-dimensional Schrödinger operator with the strong magnetic field, which will be useful in the Part 8 of [Ivr2].

Finally, in Appendices 13.A and 13.B we investigate 1D-Schrödinger operators and in Appendix 13.C we construct examples of vector potentials with different rates of growth of the magnetic field at infinity.

8 3D-case. Asymptotics with fixed spectral parameter

In this section we consider asymptotics with a fixed spectral parameter for 3-dimensional magnetic Schrödinger, Schrödinger-Pauli and Dirac operators
and discuss some of the generalizations\(^{59}\).

As in Chapters 9–12 of [Ivr2] and Sections 1–5 we will introduce a semiclassical zone and a singular zone, where \(\rho \gamma \geq h\) and \(\rho \gamma \leq h\) respectively. In the semiclassical zone we apply asymptotics of Chapters 13, 18 and 20 of [Ivr2]–in the multidimensional case. In the singular zone we need to apply estimates for a number of eigenvalues; usually it would be sufficient to use non-magnetic estimate\(^{60}\) for number of eigenvalues which trivially follows from standard one but if needed one can use more delicate estimates.

8.1 Schrödinger operator

8.1.1 Estimates of the spectrum

Consider first the Schrödinger operator (13.1.1 of [Ivr2]) where \(g^{jk}, V_j, V\) satisfy (13.1.2) and (13.1.4) of [Ivr2] i.e.

\[
\epsilon |\xi|^2 \leq \sum_{j,k} g^{jk} \xi_j \xi_k \leq c |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.
\]

Without any loss of the generality we can fix \(\tau = 0\) and then in the important function \(V_{\text{eff}} F_{\text{eff}}^{-1}\) the parameters \(\mu\) and \(\hbar\) enter as factors. Thus, we treat the operator (13.1.1) of [Ivr2] assuming that it is self-adjoint.

We make assumptions the same assumptions 2.3 of [Ivr2] i.e.

\[
(8.2)_{1-3} |D^\alpha g^{jk}| \leq c \gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \leq c \rho_1 \gamma^{-|\alpha|}, \quad |D^\alpha V| \leq c \rho^2 \gamma^{-|\alpha|}
\]

where scaling function \(\gamma(x)\) and weight functions \(\rho(x), \rho_1(x)\) satisfy the standard assumptions (9.1.6)\(_{1,2}\) of [Ivr2]. Then

\[
(8.3)_{1,2} \quad \mu_{\text{eff}} = \mu \rho_1 \gamma \rho^{-1}, \quad \hbar_{\text{eff}} = \hbar \rho^{-1} \gamma^{-1}.
\]

Let us introduce a semiclassical zone \(X' = \{x: \rho \gamma \geq h\}\) and a singular zone \(X'' = \{x: \rho \gamma \leq 2h\}\) by (2.5) and (2.6) of [Ivr2] respectively.

Further, let us introduce two other overlapping zones \(X'_1 = \{x \in X_{\text{scl}}: \mu \rho_1 \leq 2c \rho \gamma^{-1}\}\) and \(X'_2 = \{x \in X': \mu \rho_1 \geq c \rho \gamma^{-1}\}\) where the magnetic field \(\mu_{\text{eff}} = \mu \rho_1 \rho^{-1} \gamma\) is normal (\(\mu_{\text{eff}} \leq 2c\)) and where it is strong

\(^{59}\) Mainly to higher dimensions with maximal-rank magnetic field.

\(^{60}\) With \(V\) modified accordingly; for example, for the Schrödinger and Schrödinger-Pauli operators \(V\) is replaced by \(C((1-\epsilon)V - C_\epsilon \mu^2 |V|^2)_-\).
\((\mu_{\text{eff}} \geq c)\) respectively (see (2.7) and (2.8) of [Ivr2]). We also assume that
\[(8.4)\quad |F| \geq \epsilon \rho_1 \quad \text{in} \quad X_2'\]
where \(F_{jk}, F^j\) and \(F\) are the tensor, vector (as \(d = 3\)) and scalar intensities of the magnetic field respectively. Moreover, let us assume that
\[(8.5)\quad u|_{\partial X \cap B(x, \gamma(x))} = 0 \quad \forall x \in X_2' \quad \forall u \in \mathcal{D}(A);\]
we do not need (2.10) since in 3D the boundary does not lead to the deterioration of the remainder estimate. We define \(X_2' - = \{x \in X_2': V + \mu h F \geq \epsilon \rho_2\}\) and \(X_2' + = \{x \in X': V + \mu h F \leq 2\epsilon \rho_2\}\) by (2.10) and (2.11) of [Ivr2] respectively.

Finally, let the standard boundary regularity condition be fulfilled:
\[(8.6)\quad \text{For every } y \in X, \partial X \cap B(y, \gamma(y)) = \{x_k = \phi_k(x_k)\} \quad \text{with} \quad |D^\alpha \phi_k| \leq c\gamma^{-|\alpha|} \quad \text{and} \quad k = k(y).\]

Recall that according to Chapter 13 of [Ivr2] the contribution of the partition element \(\psi \in C^0_0(B(y, \frac{1}{2}\gamma(y)))\) to the principal part of asymptotics is
\[(8.7)\quad \mathcal{N}^- (\mu, h) = \mathcal{N}^{MW}^- (\mu, h) := h^{-3} \int \mathcal{N}^{MW}(x, \mu h)\psi(x) \, dx\]
with \(\mathcal{N}^{MW}(x, \mu h)\) given by (13.1.9) of [Ivr2] with \(d = 3\).

On the other hand, its contribution to the remainder does not exceed \(Ch^{-2}\rho^2\gamma^{-2}\) if \(\mu \rho_1 \gamma \leq c\rho\) or \(\mu \rho_1 \gamma \geq c\rho\) but \(\mu h \rho_1 \leq \rho^2\), \(y \in X_2'\) and non-degeneracy assumption
\[(8.8)\quad \sum_{|\alpha| \leq k} |\nabla^\alpha (VF^{-1} + (2n + 1)\mu h)|\gamma^{|\alpha|} \geq \epsilon \rho^2 \rho_1^{-1} \quad \forall n \in \mathbb{Z}^+\]
is fulfilled\(^{61}\), and it does not exceed \(C(\mu^{-s}\rho_1^{-s}\gamma^{-2s})\) if \(C\mu \rho_1 \gamma \geq \rho, \mu h \rho_1 \leq \rho^2, \ y \in X_2'\).

\(^{61}\) It does not exceed the same expression plus \(\mu h^{-1-\delta} \rho_1^{1+\delta} \gamma^{2+\delta}\) in the general case; here \(\delta > 0\) is arbitrarily small but \(K\) in (8.2)\(_{1-3}\) depends on it.
Then we get an estimate of \( N^- \) from below by the magnetic Weyl approximation \( N^- (\mu, h) \) minus corresponding remainder, and also from above by magnetic Weyl approximation plus corresponding remainder, provided \( X = X' \) (so, there is no singular zone \( X'' = \emptyset \)):

\[
(8.9) \quad h^{-d} \int_{X'} N^{MW} (x, \mu h) \, dx - CR_1 \leq N^-(0) \leq h^{-d} \int_{X'} N^{MW} (x, \mu h) \, dx + CR_1 + CR_2
\]

with

\[
(8.10) \quad R_1 = \mu^{-1} h^{1-d} \int_{X'_1} \rho^{d-1} \gamma^{-1} \, dx,
\]

\[
(8.11) \quad R_2 = \mu h^{s-d} \int_{X'_2} \rho_1 \rho^{d-s-1} \gamma^{1-s} \, dx
\]

provided

\[
(8.12) \quad \mu \rho \gamma \geq \rho
\]

where the latter condition could be assumed without any loss of the generality, \( C' \) depends also on \( s \) and \( \epsilon \). 62)

We leave to the reader the following not very challenging set of problems:

**Problem 8.1.** (i) Consider in the current framework Problems 2.1 (i), (ii), (iv) of [Ivr2].

(ii) Using results of Subsubsection 13.7.2.1 of [Ivr2] Case \( d = 3 \) of [Ivr2] replace condition (8.8)\(_k\) by (13.7.19)\(_m\).

In what follows \( h \to +0 \) and the semiclassical zone \( X' \) expands to \( X \) while \( \mu \) is either bounded (then we can assume that the zone of the strong magnetic field \( X'_2 \) is fixed) or tends to \( \infty \) (then \( X'_2 \) expands to \( X \)). We assume that all conditions of the previous subsection are fulfilled with \( \mu = h = 1 \) but we will assume them fulfilled in the corresponding zones.

The other important question is whether \( \mu h \to 0 \), remains bounded and disjoint from 0 or tends to \( \infty \).

Finally, we should consider the singular zone \( X'' \). In order to avoid this task we assume initially that

\[
(8.13) \quad \rho_1 \gamma^2 + \rho \gamma \geq \epsilon.
\]

62) Cf. (2.18)–(2.21) of [Ivr2].
8.1.2 Power singularities

Example 8.2 \(^{63}\). (i) Let \(X\) be a compact domain, \(0 \in \bar{X}\) and let conditions (8.8), (8.1), (8.2)\(_{1-3}\), (8.4) and (8.6) be fulfilled with \(\gamma = \epsilon_0 |x|, \rho = |x|^m, \rho_1 = |x|^{m_1}, m_1 < 2m \leq -2\(^{64}\),\(^{65}\)).

Then, for the Schrödinger operator the following asymptotic holds as \(h \to +0\), \(\mu h\) bounded:

\[
N^-(\mu, h) = N^-(\mu, h) + \begin{cases} 
O(h^{-2} (\mu h)^{2(m+1)/(2m-m_1)}) & m < -1, \\
O(h^{-2} (|\log \mu h| + 1)) & m = -1
\end{cases}
\]
with

\[
N^-(\mu, h) = \begin{cases} 
O(h^{-3} (\mu h)^{2(m+1)/(2m-m_1)}) & m < -1, \\
O(h^{-3} (|\log \mu h| + 1)) & m = -1
\end{cases}
\]

Furthermore, one can replace in (8.15) "\(= O\)" with "\(\asymp\)" if

\[\phi\leq -\epsilon\rho^2 \quad \text{in} \quad \Gamma \cap \{|x| \leq \epsilon\} \subset X\]

where \(\Gamma\) is an open non-empty sector (cone) with vertex at 0, and \(\mu h \leq t\) with small enough \(t > 0\).

(ii) Let \(X\) be unbounded domain and let conditions (8.8), (8.1), (8.2)\(_{1-3}\), (8.4) and (8.6) be fulfilled with \(\gamma = \epsilon_0 \langle x \rangle, \rho \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, m_1 > 2m \geq -2\(^{64}\),\(^{65}\)).

Then for the Schrödinger operator asymptotics (8.14) holds as \(h \to +0\), \(\mu h\) bounded.

Further, (8.15) holds and one can replace "\(= O\)" by "\(\asymp\)" if

\[\phi\leq -\epsilon\rho^2 \quad \text{in} \quad \Gamma \cap \{|x| \geq c\} \subset X\]

where \(\Gamma\) is an open non-empty sector (cone) with vertex at 0, and \(\mu h \leq t\) with small enough \(t > 0\).

\(^{63}\) Cf. Example 2.3 of [Ivr2].

\(^{64}\) Such potential \((V_1, V_2, V_3)\) exists, see Appendix 13.C.

\(^{65}\) The non-degeneracy condition (8.8)\(_2\) is fulfilled in the vicinity of 0 if \(V, F\) stabilize as \(x \to 0\) to \(V^0, F^0\) positively homogeneous of degrees \(2m, m_1\) respectively.
Example 8.3. (i) Assume now that \( m > -1 \) while all other assumptions of Example 8.2(i) are fulfilled. Then \( \mathcal{N}^-(\mu, h) = O(h^{-3}) \). Let us calculate the remainder estimate and prove that

\[
\tag{8.17} \quad \mathcal{N}^-(\mu, h) = \mathcal{N}^-(\mu, h) + O(h^{-2}).
\]

Obviously, the contribution of the regular zone \( \mathcal{X}' = \{ x: |x| \geq r^* = h^{1/(m+1)} \} \) is \( O(h^{-2}) \) and we need to consider the contribution of the singular zone \( \mathcal{X}'' = \{ x: |x| \leq r_2 \} \).

(a) Assume first that \( m_1 \geq m - 1 \). If \( \mu r_1^{m_1+1-m} \leq c \), then by virtue of LCR we estimate contribution of \( \mathcal{X}'' \)

\[
\tag{8.18} C h^{-3} \int_{\mathcal{X}''} \left( r^{3m} + \mu^3 r^{3(m_1+1)} \right) dx \approx h^{-3} \left( r_1^{3m+3} + \mu^3 r_1^{3m+6} \right) = O(1).
\]

Indeed, we can take \( \vec{V} = O(r_1^{m_1+1}) \).

On the other hand, if \( \mu r_1^{m_1+1-m} \geq c \), the same estimate would work for \( \mathcal{X}''' = \{ x: |x| \leq r_* = \mu^{-1/(m_1+1-m)} \} \) while contribution of \( \mathcal{X}'' \setminus \mathcal{X}''' \) does not exceed due to Chapter 13 of [Ivr2]

\[
\tag{8.19} C \int_{\mathcal{X}'' \setminus \mathcal{X}'''} (\mu r_1^{m_1+1-m})^{-s} r^{-3} dx = O(|\log h|)
\]

(actually it is \( O(1) \) if \( m_1 > m - 1 \)).

(b) Let now \( m_1 < m - 1 \). If \( \mu r_1^{m_1+1-m} \geq c \), then we can apply estimate (8.19) in the whole zone \( \mathcal{X}'' \).

On the other hand, if \( \mu r_1^{m_1+1-m} \leq c \), then we can apply estimate (8.19) in the zone \( \mathcal{X}''' \) and estimate (8.18) in \( \mathcal{X}'' \setminus \mathcal{X}'''. \)

(ii) Assume now that \( m < -1 \) while all other assumptions of Example 8.2(ii) are fulfilled. Again, considering cases

(a) \( m_1 \leq m - 1 \) and

(b) \( m_1 > m - 1 \)

we arrive to the asymptotics (8.17) and \( \mathcal{N}^-(\mu, h) = O(h^{-3}). \)

\[\text{66) Cf. Example 2.4 of [Ivr2].}\]
Consider now fast increasing $\mu$ so that $\mu h \to \infty$. We will get non-trivial results only when domain defined by $\mu_{\text{eff}} h_{\text{eff}} \leq C_0$ shrinks but remains non-empty which happens only if $m_1 > 2m$, $m_1 < 2m$ in the frameworks of Example 8.3(i) and (ii) respectively.

**Example 8.4** (i) In the framework of Example 8.3(i) with $m_1 > 2m$ consider $\mu h \to \infty$. Then the allowed domain is

\[(8.20) \{ x : |x| \lesssim r_2 = (\mu h)^{1/(m-2m)} \}\]

and we have $r_1 \leq r_2$ if $\mu \approx h^{-(m_1+1-m)/(m+1)}$ while for $\mu \gtrsim h^{-(m_1+1-m)/(m+1)}$ inequalities go in the opposite direction.

Therefore as $h \to +0$, $ch^{-1} \leq \mu \leq h^{-(m_1+1-m)/(m+1)}$ asymptotics

\[(8.21) \quad N^-(\mu, h) = N^-(\mu, h) + O(\mu^{-2(m+1)/(m-2m)}h^{-2(m_1+1-m)/(m-2m)})\]

holds and one can see easily that $N^-(\mu, h) \approx h^{-2}r_2^{2m+2}$.

\[(8.22) \quad N^-(\mu, h) \approx \mu^{-3(m+1)/(m-2m)}h^{-3(m_1+1-m)/(m-2m)}.\]

(ii) Similarly, in the framework of Example 8.3(ii) with $m_1 < 2m$ asymptotics (8.21) and (8.22) hold as $h \to +0$, $ch^{-1} \leq \mu \leq h^{-(m_1+1-m)/(m+1)}$.

### 8.1.3 Improved remainder estimates

Let us improve remainder estimates under certain non-periodicity-type assumptions.

**Example 8.5** (i) In the case of the singularity at $0$ with $m > -1$ the contribution to the remainder of the zone $\{ x : |x| \leq \varepsilon \}$ does not exceed $\sigma h^{-1}$ with $\sigma = \sigma(\varepsilon) \to 0$ as $\varepsilon \to +0$. Then the standard arguments imply that under the standard non-periodicity assumption for Hamiltonian billiards with the Hamiltonian

\[(8.23) \quad a(x, \xi, \mu_0) = \sum_{j,k} g_{jk}^0(\xi_j - \mu_0 V_j)(\xi_k - \mu_0 V_k) + V(x)\]

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67) Cf. Example 2.5 of [Ivr2].
68) Cf. Example 2.7 of [Ivr2].
69) On the energy level $0$. 

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the improved asymptotics

\[ N^{-}(\mu, h) = \mathcal{N}^{-}(\mu, h) + \kappa_1 h^{-2} + o(h^{-2}) \]

holds as \( h \to +0, \mu \to \mu_0 \) where \( \kappa_1 h^{-2} \) is the contribution of \( \partial X \) calculated as \( \mu = \mu_0 \).

(ii) Similarly in the case of the singularity at infinity with \( m < -1 \) under the standard non-periodicity assumption for Hamiltonian billiards\(^{69}\) with the Hamiltonian (8.23) asymptotics (8.24) holds as \( h \to +0, \mu \to \mu_0 \).

(iii) In the case of the singularity at \( 0 \) with \( m < -1 \) (and thus \( m_1 < 2m \)) and \( h \to 0, \mu h \to 0 \) the contributions to the remainder of the zone \( \{ x : |x| \geq \varepsilon^{-1} r_2 \} \) with \( r_2 = (\mu h)^{-1/(2m-m_1)} \) do not exceed \( \sigma h^{-2} r_2^{m+1} \) with \( \sigma = \sigma(\varepsilon) \to 0 \) as \( \varepsilon \to +0 \). After scaling \( x \mapsto x r_2^{-1} \) etc the magnetic field in the zone \( \{ x : |x| \leq \varepsilon^{-1} r_2 \} \) becomes strong.

Assume that \( g^{jk}, V_j, V \) stabilize to positively homogeneous of degrees \( 0, m_1 + 1, 2m \) functions \( g^{jk}, j_0, V^0 \) as \( x \to 0 \): namely, assume that (2.39)\(_{1-3} \) of [Ivr2] are fulfilled. Then the standard arguments imply that under the standard non-periodicity assumption for 1-dimensional Hamiltonian movement along magnetic lines (see Subsection \( 13.6.2 \) of [Ivr2]) the improved asymptotics

\[ N^{-}(\mu, h) = \mathcal{N}(\mu, h) + o(h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}) \]

holds as \( h \to +0, \mu h \to 0 \). Furthermore, if \( 0 \) is not an inner point and domain stabilizes to the conical \( X^0 \) near \( 0 \), we need to consider movement along magnetic lines with reflection at \( \partial X^0 \) and include into asymptotics the term \( \kappa_1 h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)} \) which comes out from the contribution of \( \partial X^0 \):

\[ N^{-}(\mu, h) = \mathcal{N}(\mu, h) + \kappa_1 h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)} + o(h^{-2}(\mu h)^{-2(m+1)/(m_1-2m)}) \]

(iv) In the case of the singularity at \( 0 \in X \) with \( m_1 < 2m = -2 \) the main contribution to the remainder comes from the zone \( \{ x : \varepsilon^{-1} r_2 \leq |x| \leq \varepsilon \} \). After rescaling magnetic field in this zone is strong.
Let stabilization conditions \((2.39)_{1,3}\) of \([Ivr2]\) be fulfilled. Then the standard arguments imply that under the same non-periodicity assumption as in (iii) the improved asymptotics

\[
N^{-}(\mu, h) = \mathcal{N}(\mu, h) + o(h^{-2}|\log(\mu h)|)
\]

holds as \(h \to +0, \mu h \to 0\). Furthermore, if \(0\) is not an inner point and domain near it stabilizes to the conical \(X^0\) near it,

\[
N^{-}(\mu, h) = \mathcal{N}(\mu, h) + \kappa_1 h^{-2}\log(\mu h) + o(h^{-2}|\log(\mu h)|);
\]

again, an extra term comes out from the contribution of \(\partial X^0\).

**Remark 8.6.** Statements, similar to (iii), (iv) but with the singularity at infinity seem to have impossible conditions.

### 8.1.4 Power singularities. II

Let us modify our arguments for the case \(\rho_3 < 1\). Namely, in addition to (8.2) we assume that

\[
|D^\alpha g^{jk}| \leq c\rho_2 \gamma^{-|\alpha|}, \quad |D^\alpha F_{jk}| \leq c\rho_2 \rho_1 \gamma^{-|\alpha|},
\]

\[
|D^\alpha \frac{V}{F}| \leq c\rho_3 \rho_2 \rho_1^{-1} \gamma^{-|\alpha|} \quad \forall \alpha : 1 \leq |\alpha| \leq K
\]

with \(\rho_3 \leq \rho_2 \leq 1\) in the corresponding regions where \(\rho, \rho_1, \gamma, \rho_3\) are scaling functions.

Recall that in Chapter 13 of \([Ivr2]\) operator was reduced to the canonical form with the term, considered to be negligible, of magnitude \(\rho_2 \mu^{-2N}\). In this case impose non-degeneracy assumptions

\[
(8.30)^* \sum_{\alpha : |\alpha| \leq k} |\nabla^\alpha (v^* + (2n + 1)\mu h)\gamma^{|\alpha|} | \geq \epsilon \rho_3 \rho^2 \quad \forall n \in \mathbb{Z}^+
\]

and

\[
(8.31) \quad \rho_3 \geq C_0 \rho_2 (\mu \rho_1 \gamma \rho^{-1})^{-N}
\]

where \(v^*\) is what this reduction transforms \(VF^{-1}\) to and (8.31) means that “negligible” terms do not spoil \((8.30)^*_k\).

Then according to Chapter 13 of \([Ivr2]\) the contributions of \(B(x, \gamma)\) to the both to the Tauberian remainder and an approximation error do not exceed \(C\rho^2 \gamma^{-1} h^{-2}\).
Example 8.7\textsuperscript{70).} (i) Let \(0 \in \bar{X}\) be a singular point and let assumptions (8.29)\textsubscript{1-3} be fulfilled with \(\gamma = \epsilon_0|x|, \rho = |x|^{-m}(|\ln |x||+1)^{\alpha}, \rho_1 = |x|^{-2m}(|\ln |x||+1)^{\beta}, \rho_2 = 1\) and \(\rho_3 = (|\ln |x||+1)^{-1}\).

Assume that

\[(8.32) \quad m = -1, \beta > \max(\alpha, 2\alpha).\]

Then (8.31) is fulfilled with \(N = 1\) and we can replace (8.30)\textsuperscript{*} by

\[(8.30)_m \quad \sum_{\alpha:|\alpha| \leq k} |\nabla^\alpha(VF^{-1} + (2n + 1)\mu h)|^{\alpha} \geq \epsilon \rho_3 \beta^2 \quad \forall n \in \mathbb{Z}^+.\]

Therefore, we conclude that the remainder is \(O(R)\) with

\[(8.33) \quad R := \begin{cases} 
  h^{-2} & \alpha < -\frac{1}{2}, \\
  h^{-2} |\log(\mu h)| & \alpha = -\frac{1}{2}, \\
  h^{-2}(\mu h)^{-(2\alpha+1)/(\beta-2\alpha)} & 2\alpha < \beta < 2\alpha + 2.
\end{cases}\]

One can see easily that under condition (8.16) for \(\mu h \leq t\) with small enough \(t > 0\)

\[(8.34) \quad \mathcal{N}^-(\mu, h) \asymp \begin{cases} 
  h^{-3} & \alpha < -\frac{1}{3}, \\
  h^{-3}(|\log(\mu h)| + 1) & \alpha = -\frac{1}{3}, \\
  h^{-3}(\mu h)^{-(3\alpha+1)/(\beta-2\alpha)} & \alpha > -\frac{1}{3}.
\end{cases}\]

(ii) Let infinity be a singular point and let conditions (8.29)\textsubscript{1-3} be fulfilled with \(\gamma = \epsilon_0\langle x \rangle, \rho = \langle x \rangle^m(\langle \log \langle x \rangle \rangle + 1)^{\alpha}, \rho_1 = \langle x \rangle^{2m}(\langle \log \langle x \rangle \rangle + 1)^{\beta}, \rho_2 = 1, \rho_3 = (\langle \log \langle x \rangle \rangle + 1)^{-1}\). Assume that (8.32) is fulfilled.

Then all the statements of (i) remain true with the obvious modification: condition (8.16) should be replaced by (8.16)\textsuperscript{#}.

**Problem 8.8.** \textsuperscript{70) Consider}

\[70) \text{Cf. Example 2.12 of [Ivr2].}\]

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(i) Case of a singular point at 0, \( m < -1, \beta > 2\alpha. \)

(ii) Case of a singular point at infinity, \( m < -1, \beta > 2\alpha. \)

In both cases \( \log(\mathcal{N}^{-}(\mu, h)h^{3}) \approx (\mu h)^{-1/(\beta - 2\alpha)} \) and \( \log(\mathcal{N}^{-}(\mu, h)/R^{3/2}) \sim 1. \)

8.1.5 Exponential singularities

Consider now singularities of the exponential type.

Since we assume only that either \( 0 \in \bar{X} \) or \( X \) is a unbounded domain and \( \gamma(x) \ll |x| \), it can happen that

\[
\text{mes}\{x \in X: |x| \leq r\} \approx r^{n}
\]

with \( n \neq d \) (more precisely, \( n \geq d \) if \( 0 \) is a singular point and \( n \leq d \) is infinity is a singular point). See Figure ?? of [Ivr2] for unbounded domains; for singularity at 0 it is a spike:

Figure 2: Domains of the type (8.35)

Example 8.9\textsuperscript{71). (i) Let \( 0 \in \bar{X} \) be a singular point and let our standard assumptions be fulfilled with \( \gamma = \epsilon_{0}|x|^{1-\beta}, \rho = \exp(a|x|^{\beta}), \rho_{1} = \exp(b|x|^{\beta}) \) with \( \beta < 0, b > 2a > 0. \)

Then for \( \mu h < t \) with a small enough constant \( t > 0 \)

\[
\mathcal{N}^{-}(\mu, h) = \mathcal{N}^{-}(\mu, h) + O(h^{-2}(\mu h)^{-2a/(b-2a)}|\log(\mu h)|^{(n-1)/\beta}),
\]

\[
\mathcal{N}^{-}(\mu, h) = O(h^{-3}(\mu h)^{-3a/(b-2a)}|\log(\mu h)|^{n/\beta-1}).
\]

(ii) Let infinity be a singular point and let our standard assumptions be fulfilled with \( \gamma = \epsilon_{0}<x>^{1-\beta}, \rho = \exp(a<x>^{\beta}), \rho_{1} = \exp(b<x>^{\beta}) \) where \( \beta > 0, b > 2a > 0. \)

Then asymptotics (8.36)–(8.37) holds.

We leave to the reader the following

\textsuperscript{71)} Cf. Example 2.17 of [Ivr2].

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Problem 8.10. (i) Consider the case of a singular point at 0 and $\gamma = \epsilon_0 |x|^{1-\beta}$, $\rho = |x|^m \exp(|x|^\beta)$, $\rho_1 = |x|^{m_1} \exp(2 |x|^\beta)$, $\rho_2 = 1$, $\rho_3 = |x|^{-\beta}$, $\beta < 0$, $m_1 < 2m$.

(ii) Consider the case of a singular point at infinity and $\gamma = \epsilon_0 \langle x \rangle^{1-\beta}$, $\rho = \langle x \rangle^m \exp(\langle x \rangle^\beta)$, $\rho_1 = \langle x \rangle^{m_1} \exp(2 \langle x \rangle^\beta)$, $\rho_2 = 1$, $\rho_3 = \langle x \rangle^{-\beta}$, $\beta > 0$, $m_1 > 2m$.

We leave to the reader:

Problem 8.11. Consider cases of $\mu \to \mu_0 > 0$ and $\mu \to \mu_0 = 0$.

8.2 Schrödinger-Pauli operators

Consider now Schrödinger-Pauli operators, either genuine (0.34) or generalized (13.5.3) of [Irv2]. The principal difference is that now $F$ does not “tame” singularities of $V$, on the contrary, it needs to be “tamed” by itself. As a result there are fewer examples than for the Schrödinger. Also we do not have a restriction $\mu_{\text{eff}}^h \text{eff} = O(1)$ which we had in the most of the previous Subsection 8.1.

Example 8.12. (i) Let 0 be a singular point and let our standard assumptions be fulfilled with $\gamma = \epsilon_0 |x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and let $m > -1$, $2m \neq m_1 > -2$. Then, in comparison with the theory of the previous Subsection 8.1, we need to consider also the the zone $\mu_{\text{eff}}^h \text{eff} \gtrsim 1$, and also a singular zone where $h_{\text{eff}} \gtrsim 1$.

The contribution to the remainder of the regular part (defined by $\mu_{\text{eff}}^h \text{eff} \gtrsim 1$, $h_{\text{eff}} \lesssim 1$) does not exceed $C \mu h^{-1} \int \rho_1 \gamma^{-1} dx$ while its contribution to $N^- (\mu, h)$ does not exceed $C \mu h^{-2} \int \rho_1 \rho dx$. Indeed, contributions of each $\gamma$-element do not exceed $C \mu_{\text{eff}}^h \text{eff}^{-1} \times C \mu h^{-1} \rho_1 \gamma^2$ and $C \mu_{\text{eff}}^h \text{eff}^{-2} \times C \mu h^{-2} \rho_1 \rho \gamma^3$.

Consider now the singular zone $\{x : |x| \leq h^{1/(m+1)}\}$. We claim that

(8.38) Contribution of the singular zone to the asymptotics does not exceed $C (\mu h + 1)$.

Indeed, if $\mu h \leq 1$ it suffices to apply LCR as in 2D-case. If $\mu h \geq 1$, then LCR returns $\mu^{3/2} h^{1/2}$ and we need to be more tricky. Without any loss of

\footnote{Cf. Example 8.2.}
the generality we can assume that $V_1 = 0$ and then apply (variant of) LCR for 1D-operator $\hbar^2 D_1^2 - |x|^{2m}$ and for 2D-operator $\hbar^2 |D|^2 - \mu^2|x|^{m_1} - |x|^{2m}$.
We leave details to the reader.

\begin{equation}
N^-(\mu, h) = N^-(\mu, h) + O(\mu h^{-1} + h^{-2})
\end{equation}

with

\begin{equation}
N^-(\mu, h) = O(\mu h^{-2} + h^{-3})
\end{equation}

where under condition (8.16) fulfilled in non-empty cone “$= O(\cdot)$” could be replaced by “$\asymp \cdot$”.

(ii) Let infinity be a singular point and let our standard assumptions be fulfilled with $\gamma = \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$ and let $m < -1$, $2m \neq m_1 < -2$. Then the same arguments yield (8.39)–(8.40) again.

We leave to the reader the following problems:

**Problem 8.13.** (i) As 0 is a singular point consider both Schrödinger and Schrödinger-Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^m$, $m > -1$ and $\rho_1 = |x|^{-2}||x|^\beta$.
(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log |x||^{\alpha}$, $\alpha < -\frac{1}{2}$ and $\rho_1 = |x|^m$, $m_1 > -2$.

(ii) As infinity is a point consider both Schrödinger and Schrödinger Pauli operators as

(a) $\gamma = |x|$, $\rho = |x|^m$, $m < -1$ and $\rho_1 = |x|^{-2}||x|^\beta$.
(b) $\gamma = |x|$, $\rho = |x|^{-1}|\log |x||^{\alpha}$, $\alpha < -\frac{1}{2}$ and $\rho_1 = |x|^m$, $m_1 > -2$.

For the Schrödinger operator in cases (a) non-trivial results could be obtained even as $\mu h \to +\infty$.

**Problem 8.14.** Let either 0 or infinity be a singular point.

Using the same arguments and combining them with the arguments of Subsubsection 8.1.4.6. Power singularities. II consider both Schrödinger-Pauli and Schrödinger operators with $\gamma = |x|$, $\rho = |x|^{-1}|\log |x||^{\alpha}$, $\rho_1 = |x|^{-2}|\log |x||^{\beta}$, $\alpha < -\frac{1}{2}$, $\beta < -1$.

For the Schrödinger operator in case $\beta < 2\alpha$ non-trivial results could be obtained even as $\mu h \to +\infty$.

The following problem seems to be rather challenging:

**Problem 8.15.** Investigate $\rho_1 = |x|^{-2}$.

\footnote{One should take $\beta < -1$ for the Schrödinger-Pauli operator.}
8.3 Dirac operator

8.3.1 Preliminaries

Let us now consider the generalized magnetic Dirac operator (17.1.1) of [Ivr2] either

\begin{equation}
A = \frac{1}{2} \sum_{i,j} \sigma_i (\omega^{ij} P_j + P_j \omega^{ij}) + \sigma_0 M + I \cdot V, \quad P_j = hD_j - \mu V_j
\end{equation}

where \( \sigma_0, \sigma_1, \sigma_2, \sigma_3 \) are 4 \( \times \) 4-matrices and \( M > 0 \) or

\begin{equation}
A = \frac{1}{2} \sum_{i,j} \sigma_i (\omega^{ij} P_j + P_j \omega^{ij}) + I \cdot V, \quad P_j = hD_j - \mu V_j
\end{equation}

where \( \sigma_1, \sigma_2, \sigma_3 \) are 2 \( \times \) 2-matrices and \( M = 0 \).

We are interested in \( N(\tau_1, \tau_2) \), the number of eigenvalues in \( (\tau_1, \tau_2)^{74} \) with \( \tau_1 < \tau_2 \), fixed in this subsection.

In contrast to 2D-case relations similar to (2.73)–(2.74) of [Ivr2] do not matter. The Landau levels (at the point \( x \)) are \( V \pm (M^2 + 2j\mu hF)^{\frac{1}{2}} \) with \( j = 0, 1, 2, 3, ... \) and magnetic Dirac operator always behaves like Schrödinger-Pauli operator.

Therefore, we treat the operator given by (8.41) under the following assumptions

\begin{align}
8.43)_{1-2} & \quad |D^\alpha \omega^{jk}| \leq c \gamma^{-|\alpha|}, \quad |D^\alpha F| \leq c \rho_1 \gamma^{-|\alpha|}, \\
8.43)_{3} & \quad |D^\alpha V| \leq c \min(\rho, \frac{1}{M^2}) \gamma^{-|\alpha|} \quad (\alpha \neq 0) \quad \forall \alpha : |\alpha| \leq K, \\
8.43)_{4+} & \quad (V - \tau_2 - M)_+ \leq c \min(\rho, \frac{1}{M^2}), \\
8.43)_{4-} & \quad (V - \tau_1 + M)_- \leq c \min(\rho, \frac{1}{M^2})
\end{align}

and also (8.2)_{1-3}, (8.1) for \( g^{jk} = \sum_{l,r} \omega^{jl} \omega^{kr} \delta_{lr} \) and (8.4) (with \( F > 0 \)). In what follows (8.43)_{4} means the pair of conditions (8.43)_{4\pm}.

\footnote{Assuming that this interval does not contain essential spectrum; otherwise \( N(\tau_1, \tau_2) := \infty \). It is more convenient for us to exclude both ends of the segment.}
Moreover, let condition (8.6) be fulfilled and

\( \bar{X}'' \cap \partial X = \emptyset, \)  
\( |V_j| \leq c\rho, \ |D_j\omega^{kl}| \leq c\rho \quad \text{in} \ X'' \).

Finally, we assume that

(8.46) Either \( \partial X = \emptyset \) or \( \mu = O(1) \) and \( \partial X \cap X' = \emptyset \) (in what follows).

### 8.3.2 Asymptotics

**Example 8.16** \(^{75} \). (i) Let 0 be an inner singular point and let all the above conditions be fulfilled with \( \gamma = \epsilon_0|x|, \ \rho = |x|^m, \ \rho_1 = |x|^{m_1}, \ m_1 > 2m, \ m > -1. \)

Further, let assumption

(8.47) \( |V| \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x: |x| \leq \epsilon \)

and non-degeneracy condition (8.8) \( k \) be fulfilled.

(a) \( m < 0, \ \tau_1 < \tau_2 \) or

(b) \( m > 0, \ M > 0, \ \tau_1 = -M, \ \tau_2 \in (-M, M). \)

Then for \( h \to +0, \ 1 \leq \mu \) asymptotics

(8.48) \( N(\tau_1, \tau_2; \mu, h) = N(\tau_1, \tau_2; \mu, h) + O(\mu h^{-1} + h^{-2}) \)

holds with \( N \) defined by (17.1.12) \( 1 \) with \( d = 3, \ r = 1. \) Moreover, \( N(\tau_1, \tau_2; \mu, h) \asymp \mu h^{-2} + h^{-3}. \)

(ii) Let infinity be an inner singular point and let all the above conditions be fulfilled with \( \gamma = \epsilon_0 \langle x \rangle, \ \rho = \langle x \rangle^m, \ \rho_1 = \langle x \rangle^{m_1}, \ m_1 < 2m, \ m < -1. \)

Further, let assumption

(8.47)' \( |V| \geq \epsilon \min(\rho, \frac{\rho^2}{M}) \quad \forall x: |x| \geq c \)

and non-degeneracy condition (8.8) \( k \) be fulfilled. Let \( M > 0, \ \tau_1 = -M, \ \tau_2 \in (-M, M). \) Then for \( h \to +0, \ 1 \leq \mu \) asymptotics (8.48) holds with \( N \) defined by (17.1.12) \( 1 \) with \( d = 3, \ r = 1. \) Moreover, \( N(\tau_1, \tau_2; \mu, h) \asymp \mu h^{-2} + h^{-3}. \)

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\(^{75} \) Cf. Example 8.12.
Problem 8.17. Generalize results of this section to the odd-dimensional maximal-rank case. In particular, consider power singularities.

9 3D-case. Asymptotics of large eigenvalues

In this section we consider the case when $\mu$ and $h$ are fixed and we consider the asymptotics of the eigenvalues, tending to $+\infty$ and for Dirac operator also to $-\infty$.

Here we consider the case of the spectral parameter tending to $+\infty$ (and for the Dirac operator we consider $\tau \to -\infty$ as well).

9.1 Singularities at the point

We consider series of example with singularities at the point.

9.1.1 Schrödinger operator

Example 9.1. (i) Let $X$ be a compact domain, $0 \in \bar{X}$ and let conditions $(8.2)_{1-3}$ and $(8.6)$ be fulfilled with $\gamma = \epsilon |x|$, $\rho = |x|^m$, $\rho_1 = |x|^m$, $m_1 < 2m$. Let

\begin{equation}
|F| \geq \epsilon_0 \rho_1 \quad \text{for} \quad |x| \leq \epsilon.
\end{equation}

Then for the Schrödinger operator as $\tau \to +\infty$

\begin{equation}
N^-(\tau) = N^-(\tau) + O(\tau^{(d-1)/2})
\end{equation}

while $N^-(\tau) \asymp \tau^{d/2}$.

Indeed, we need to consider only case $m_1 \leq -2$ (otherwise it is covered by Section ?? of [Ivr2]). Assume for simplicity, that $V = 0$ (modification in the general case is trivial). Recall that $\mu_{\text{eff}} = |x|^{m_1+1} \tau^{-1/2}$ and $h_{\text{eff}} = \tau^{-1/2} |x|^{-1}$. Then under week non-degeneracy assumption\(^{76}\) contribution to the remainder of any element with $\mu_{\text{eff}} h_{\text{eff}} \leq c$ does not exceed $C h_{\text{eff}}^{1-d} = C \tau^{(d-1)/2} \gamma^{d-1}$ while contribution to the remainder of the $\gamma$-element $\mu_{\text{eff}} h_{\text{eff}} \geq c$ does not exceed $C \mu_{\text{eff}} h_{\text{eff}}^{d-s}$ and the rest is easy. Without non-degeneracy assumption.

\(^{76}\) Which we do not, however, assume fulfilled at this example.
assumption we need to add to the contribution of the $\gamma$-element $C_{\mu_{\text{eff}}} h_{\text{eff}}^{1-d-\delta}$, which after summation results in $O((\tau^{d-1-\delta^2})/2)$.

(ii) Under proper assumptions the same proof is valid in the odd-dimensional maximal-rank case.

Let us note that the case $m < -1$, $m_1 \geq m - 1$ is covered by Chapter ?? of [Ivr2]; we need to assume that
\begin{equation}
0 \geq \epsilon_0 \rho^2 \quad \text{as} \quad |x| \leq \epsilon.
\end{equation}

Example 9.2. (i) Let $X$ be a compact domain, $0 \in \bar{X}$ and let conditions (8.2)$_{1-3}$, (8.6) and (9.1)$_1$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$ and with $m < -1$, $2m \leq m_1 < m - 1$. Then we need to assume that
\begin{equation}
0 \geq \epsilon_0 \rho^2 \quad \text{as} \quad |x| \leq \epsilon
\end{equation}
which for $m_1 > 2m$ is equivalent to (9.3).

By the same reason as in the previous Example 9.1 we do not need any non-degeneracy assumption. Then asymptotics (9.2) holds while $\mathcal{N}^{-}(\tau) \asymp \tau^{d/2}$.

(ii) Under proper assumptions the same proof is valid in the odd-dimensional maximal-rank case.

9.1.2 Schrödinger-Pauli operator

Next, consider Schrödinger-Pauli operators. We will need to impose (9.3) and the related non-degeneracy assumption
\begin{equation}
\sum_{\alpha: 1 \leq |\alpha| \leq k} |\nabla^\alpha V|_{\gamma^{|\alpha|}} \geq \epsilon \rho^2 \quad \text{as} \quad |x| \leq \epsilon
\end{equation}

Example 9.3. (i) Let $X$ be a compact domain, $0 \in \bar{X}$ and let conditions (8.2)$_{1-3}$, (8.6), (9.1), (9.3) and (9.4)$_k$ be fulfilled with $\gamma = \epsilon|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m < -1$.

Then for the Schrödinger-Pauli operator as $\tau \to +\infty$ asymptotics
\begin{equation}
\mathcal{N}^{-}(\tau) = \mathcal{N}^{-}(\tau) + O((\tau^{m_1+2})/(2m) + \tau)
\end{equation}
holds while
\[ N^-(\tau) \approx \tau^{(m_1 + m + 3)/(2m)} + \tau^{3/2}. \]

Indeed, if \( m_1 \geq 2m \) then no modification to the arguments of Examples 9.1 and 9.2 is needed; if \( m_1 < 2m \) we also need to consider the zone where \( \mu_{\text{eff}} h_{\text{eff}} \gtrsim 1 \).

The contribution of the corresponding partition element to the principal part of the asymptotics is \( \mu_{\text{eff}} h_{\text{eff}}^{-2} \) while its contribution to the remainder is \( O(\mu_{\text{eff}} h_{\text{eff}}^{-1}) \); also, zone \( \{ x : |x| \leq \epsilon \tau^{-1/(2m)} \} \) is forbidden.

(ii) Without condition (9.4), one needs to add \( \mu_{\text{eff}} h_{\text{eff}}^{-1-\delta} \) to the contribution of the partition element to the remainder; it does not affect the remainder estimate \( O(\tau) \) if \( m_1 > 2m + 2 \); if \( m_1 \leq 2m + 2 \) we arrive to the remainder estimate \( O(\tau^{(m_1 + 2)/(2m + \delta)}) \).

(iii) One can generalize this example to the odd-dimensional maximal-rank case; then
\[ N^-(\tau) \approx \tau^{(m_1 + m + 3)(d-1)/(4m)} + \tau^{d/2} \]
and the remainder is \( O(R) \) with
\[ R = \tau^{(m_1 + 2)(d-1)/(4m)} + \tau^{(d-1)/2}. \]

Problem 9.4. (i) Investigate the case of \( \gamma = \epsilon |x|, \rho = |x|^{-1} |\log |x||^{\alpha}, \alpha > 0, \rho_1 = |x|^{m_1}, m_1 < -2. \)

(ii) Investigate the case of \( \gamma = \epsilon |x|, \rho = |x|^{-1} |\log |x||^{\alpha}, \rho_1 = |x|^{-2} |\log |x||^{\beta}, \beta > \alpha > 0. \)

The following problem seems to be challenging; we don’t know even if \( N^-(\tau) < \infty \) for \( \tau > 0 \).

Problem 9.5. (i) Investigate the case of \( \gamma = \epsilon |x|, \rho = |x|^{-1} |\log |x||^{\alpha}, \alpha \leq 0, \rho_1 = |x|^{m_1}, m_1 < -2. \)

(ii) Investigate the case of \( \gamma = \epsilon |x|, \rho = |x|^{m_1}, \rho_1 = |x|^{m_1}, m_1 < -2, -1 < m < 0. \)
9.1.3 Miscellaneous singularities

Consider now miscellaneous singularities in the point.

Example 9.6. Let $0 \in \bar{X}$ be a compact domain, and let conditions (8.2)$_{1-3}$, (8.6) and (9.1) be fulfilled with $\gamma = \epsilon |x|^{1-\beta}$, $\rho_1 = \exp(b|x|^{\beta})$, $\beta < 0$, $b > 0$ and with $\rho = \exp(a|x|^{\beta})$ where $b > 2a$. Assume also that

$$
(9.9)_{k} \sum_{\alpha; 1 \leq |\alpha| \leq k} |\nabla^\alpha F| \gamma^{|\alpha|} \geq \epsilon \rho_1 \quad \text{as} \quad |x| \leq \epsilon.
$$

(i) Then for the Schrödinger operator $N^-(\tau) = N^- + O(R)$ with $N^- \asymp \tau^{3/2}$ and

$$
(9.10) \quad R = \begin{cases} 
\tau & \beta \geq -2, \\
|\log \tau|^{(2+\beta)/\beta} & \beta < -2.
\end{cases}
$$

Indeed, this is trivial unless $\beta > -2$. If $\beta \leq -2$ we need to take into account that the forbidden zone where $\mu_{\text{eff}} \mu_{\text{eff}} \geq C_0$ is $\{x: |x| \leq \epsilon_0 |\log \tau|^{1/\beta}\}$. Then we get the remainder $O(\tau r^{2+\beta})$ for $\beta < -2$ and $O(\tau |\log r|)$ for $\beta = -2$.

Consider $\beta = -2$. Since $\mu_{\text{eff}} \lesssim 1$ in the zone $\{x: |x| \geq r^*: C|\log \tau|^{1/\beta}\}$, we can take $\gamma \asymp |x|^{1-\beta'}$ with $\beta' > -2$ there and the contribution of this zone to the remainder is $O(\tau)$. Meanwhile, the contribution of the zone $\{x: r_* \leq |x| \leq r^*\}$ to the remainder is $O(\tau |\log (r_* / r^*)|) = O(\tau)$.

(ii) Let conditions (9.3) and (9.4)$_{k}$ be fulfilled. Then for the Schrödinger-Pauli operator asymptotics $N^-(\tau) = N^- + O(R)$ holds with

$$
(9.11) \quad R = \tau^{b/2a} |\log \tau|^{2/\beta}
$$

and

$$
(9.12) \quad N^- \asymp \tau^{b/2a+1/2} |\log \tau|^{(3-\beta)/\beta}.
$$

(iii) On the other hand, let $a < b \leq 2a$ and conditions (9.3) and (9.4)$_{k}$ be fulfilled. Then for both Schrödinger and Schrödinger-Pauli operators $N^- \asymp \tau^{3/2}$ and $R$ is defined by (9.10).

Moreover, for $b < 2a$ we do not need the non-degeneracy assumption (9.9)$_{k}$ because in the allowed zone $\{x: V(x) \leq C\tau\}$ we have $\mu_{\text{eff}} \leq h_{\text{eff}}^{b-1}$.

Problem 9.7. Extend results of Example 9.6 to the Dirac operator.
Remark 9.8. (i) Observe that the contribution to the remainder of the zone \( \{ x : |x| \leq \varepsilon \} \) does not exceed \( \varepsilon^\sigma \tau^{(d-1)/2} \) with \( \sigma > 0 \) in the frameworks of some above examples (sometimes under certain additional assumptions).

Therefore, in these cases under the standard non-periodicity condition to the geodesic flow with reflections from \( \partial X \) the asymptotics

\[
N(\tau) = N(\tau) + \kappa_1 \tau + o(\tau) \tag{9.13}
\]

holds with the standard coefficient \( \kappa_1 \).

(ii) The similar statement (with \( \tau \) replaced by \( \tau^2 \)) is true for the Dirac operator.

Problem 9.9. In the frameworks of the examples above estimate \( |N^- (\tau) - \kappa_0 \tau^{d/2}| \).

Finally, consider the case when the singularity is located on the curve or a surface (or a more general set).

Example 9.10. Let \( 0 \in \bar{X} \) be a compact domain, and let conditions (8.2)\(_{1-3}\), (8.6) and (9.1) be fulfilled with \( \gamma = \epsilon_0 \delta(x) \), \( \rho_1 = \delta(x)^m \), \( \rho_1 = \delta(x)^m \) with \( m_1 < \min(2m, -2) \) where \( \delta(x) = \text{dist}(x, L) \), \( m < 0 \), \( L \) is a set of Minkowski codimension \( p > 1 \) or a smooth surface; in the latter case we assume also that (9.9)\(_k\) is fulfilled.

(i) Then for the Schrödinger operator asymptotics (9.2) holds for \( \tau \to +\infty \) and \( N^- (\tau) \asymp \tau^{3/2} \).

Indeed, using the same arguments as before we can get a remainder estimate \( O(\tau) \) if \( p > 1 \) or \( O(\tau |\log \tau|) \) if \( p = 1 \) but in the latter case we can get rid off logarithm using standard propagation arguments.

(ii) Let also conditions (9.3) and (9.4)\(_k\) be fulfilled. Then for the Schrödinger-Pauli operator asymptotics

\[
N^- (\tau) = N^- (\tau) + O(\tau + \tau^{(m+2-p)/(2m)}) \tag{9.14}
\]

holds while

\[
N^- (\tau) \asymp \tau + \tau^{(m_1+m+3-p)/(2m)}. \tag{9.15}
\]

Problem 9.11. Extend results of Example 9.10 to different types of the singularities along \( L \).
9.2 Singularities at infinity

Let us consider *unbounded domains*:

9.2.1 Power singularities: Schrödinger operator

Let us start from the power singularities.

*Example 9.12* \(^{77}\). (i) Let \(X\) be an unbounded domain. Let conditions (8.1), (8.6), (8.2)\(_{1-3}\), (9.1) and (9.9)\(_k\) be fulfilled with \(\gamma = \epsilon_0 \langle x \rangle\), \(\rho = \langle x \rangle^m\), \(\rho_1 = \langle x \rangle^{m_1}\), \(m_1 > 2m\).

Then for the Schrödinger operator the following asymptotics holds:

\(N^- (\tau) = N^- (\tau) + O(\tau^{(m_1+2)/m_1})\) \(\text{with}\)

\(N^- (\tau) \asymp \tau^{3(m_1+2)/(2m_1)}\).

The proof is standard.

(ii) Under proper assumptions the similar asymptotics holds in the odd-dimensional maximal-rank case:

\(N^- (\tau) = N^- (\tau) + O(\tau^{(d-1)(m_1+2)/(2m_1)})\) \(\text{with}\)

\(N^- (\tau) \asymp \tau^{d(m_1+2)/(2m_1)}\).

*Example 9.13* \(^{78}\). (i) Let \(X\) be an unbounded domain. Let conditions (8.1), (8.6), (8.2)\(_{1-3}\) and (9.1)\(^*\) be fulfilled with \(\gamma = \epsilon_0 \langle x \rangle\), \(\rho = \langle x \rangle^m\), \(\rho_1 = \langle x \rangle^{m_1}\), \(m > 0\), \(m - 1 < m_1 \leq 2m\).

Let us assume that either \(m_1 = 2m\) and 

\((9.3)^*\)  \(V + F \geq \epsilon_0 \beta^2\) as \(|x| \geq c\)

or \(m_1 < 2m\) and 

\((9.3)^\#\)  \(V \geq \epsilon_0 \beta^2\) as \(|x| \geq c\)

\(^{77}\) Cf. Example 9.1.

\(^{78}\) Cf. Example 9.2.
Assume that if $m_1 = 2m$ then

\[(9.20)_k \quad \tau \geq V + F \implies \sum_{\alpha:1 \leq |\alpha| \leq k} |\nabla (\tau - V) F^{-1} |\gamma|^{\alpha}| \geq \epsilon_0 \tau \rho_1^{-1}\]

as $|x| \geq c$.

Then for the Schrödinger operator asymptotics

\[(9.21) \quad N^-(\tau) = N^-(\tau) + O(\tau^{(d-1)(m+1)/(2m)}),\]

holds with

\[(9.22) \quad N^-(\tau) \asymp \tau^{d(m+1)/(2m)}.\]

(ii) Under proper assumptions the similar asymptotics holds in the odd-dimensional maximal-rank case.

**9.2.2 Power singularities: Schrödinger-Pauli operator**

Next, consider Schrödinger-Pauli operators. We will need to impose \((9.3)^\#\) and the related non-degeneracy assumption

\[(9.4)^\# \quad \sum_{\alpha:1 \leq |\alpha| \leq k} |\nabla^\alpha V|\gamma|^{\alpha}| \geq \epsilon \rho_1 \quad \text{as } |x| \geq \epsilon.\]

**Example 9.14**\(^{79}\). Let \((9.3)^\#\) and \((9.4)^\#\) be fulfilled. Then for the Schrödinger-Pauli operator

(i) In the framework of Example 9.12

\[(9.23) \quad N^-(\tau) = N^-(\tau) + O(\tau^{(m_1+2)/(2m)})\]

with

\[(9.24) \quad N^-(\tau) \asymp \tau^{(m_1+m+3)/(2m)}.\]

(ii) In the framework of Example 9.13

\[(9.25) \quad N^-(\tau) = N^-(\tau) + O(\tau^{(m+1)/m})\]

with

\[(9.26) \quad N^-(\tau) \asymp \tau^{3(m+1)/(2m)}.\]

\(^{79}\) Cf. Example 9.3.
(iii) Finally, under proper assumptions one can consider the odd-dimensional maximal-rank case and prove asymptotics with the remainder estimate $O(R)$, with

$$
R(\tau) = \begin{cases} 
\tau^{(d-1)(m_1+2)/(4m)} & m_1 > 2m, \\
\tau^{(d-1)(m+1)/(2m)} & m_1 \leq 2m,
\end{cases}
$$

and

$$
N^-(\tau) \simeq \begin{cases} 
\tau^{(d-1)(m_1+2)/(4m)+(m+1)/(2m)} & m_1 > 2m, \\
\tau^{d(m+1)/(2m)} & m_1 \leq 2m,
\end{cases}
$$

9.2.3 Exponential singularities

Consider now an exponential growth at infinity.

**Example 9.15.** Let $X$ be an unbounded domain. Let conditions (8.1), (8.2)$_{1-3}$, (8.4), (8.6) and (9.9)$_k$ be fulfilled with $\gamma = \epsilon_0(x)^{1-\beta}$, $\rho = \exp(a(x)^{\alpha})$, $\rho_1 = \exp(b(x)^{\beta})$, $\beta > 0$ and either $\beta > \alpha$ or $\beta = \alpha$ and $b > 2a > 0$.

(i) Then for the Schrödinger operator the following asymptotics holds:

$$
N^-(\tau) = N^-(\tau) + O(\tau^{(2+\beta)/\beta})
$$

with

$$
N^-(\tau) \simeq \tau^{3/2} |\log \tau|^{3/\beta}.
$$

(ii) Let $\alpha = \beta$ and conditions (9.3)$_\#$ and (9.4)$_k^\#$ be fulfilled. Then for the Schrödinger-Pauli operator asymptotics holds

$$
R(\tau) = \tau^{b/2a} |\log \tau|^{2/\beta}
$$

and

$$
N^-(\tau) \simeq \tau^{(b+a)/2a} |\log \tau|^{3/\beta}.
$$

(iii) On the other hand, let $\beta = \alpha$, $a < b \leq 2a$ and conditions (9.3)$_\#$ and (9.4)$_k^\#$ be fulfilled. Then for both Schrödinger and Schrödinger-Pauli operators $N^- (\tau) \asymp \tau^{3/2}$ and $R$ is defined by (9.10).

Moreover, for $b < 2a$ we do not need the non-degeneracy assumption (9.9)$_k$ because in the allowed zone $\{x : V(x) \leq C\tau\}$ we have $\mu_{eff} \leq \hbar^{\delta-1}$.
We leave to the reader

*Problem 9.16.* (i) Consider the Schrödinger-Pauli operators in the same settings as in Example 9.15, albeit with $\rho$ of the power growth at infinity.

(ii) Consider the Schrödinger-Pauli operators in the same settings as in Example 9.14 albeit with $V$ of the logarithmic growth at infinity (i.e. with $\rho = |\log |x||^{\alpha}$, $\gamma = \epsilon|x|$).

### 10 3D-case. Asymptotics of small eigenvalues

In this section we need to consider first miscellaneous asymptotics (cf. Subsections 4.3 and 4.4 of [Ivr2]) and only after case of $F$ stabilizing at infinity (cf. Subsections 4.1 and 4.2 of [Ivr2]).

#### 10.1 Miscellaneous asymptotics

**10.1.1 Case $F \gtrsim 1$ as $|x| \to \infty$**

In this subsection we consider cases of either $F \to \infty$ or $F \asymp 1$ and $V \to 0$ as $|x| \to \infty$. If $F \to \infty$ the Schrödinger operator either does not have any essential spectrum “at infinity”\(^{80}\); if $F \asymp 1$ we do not assume any stabilization conditions so far. Anyway, the Schrödinger operator is not the subject of our analysis, while for the Schrödinger-Pauli and Dirac operators essential spectrum “at infinity” equals $[0, \infty)$ and $(-\infty, M] \cup [M, \infty)$ respectively\(^{81}\).

Again due to the specifics of the problem we can consider the multidimensional case with minimal modifications. In this case we assume that

\[(10.1) \text{rank } F = 2p \text{ as } |x| \geq c \text{ and } f_1 \asymp f_2 \asymp \ldots \asymp f_p \asymp \rho_1 \text{ as } |x| \geq c\]

and

\(^{80}\) More precisely, the lowest Landau level tends to $+\infty$ as $|x| \to \infty$.

\(^{81}\) Again, understood in the sense of the Landau levels.
(10.2) For each \( j \neq k \) either \( f_j = f_k \) or \( |f_j - f_k| \geq \epsilon \rho_1 \) for all \( |x| \geq c \).

Let for the Schrödinger-Pauli operator \( N^- (\eta) \) be a number of eigenvalues in \((-\epsilon, -\eta)\) and \( N^+ (\eta) \) be a number of eigenvalues in \((\eta, \epsilon)\).

**Theorem 10.1**\(^{82}\). Let \( X \) be an unbounded domain. Let conditions (8.1), (8.2)\(^{1-3}\), (10.1), (10.2) and (9.4)\(^{\#}\) be fulfilled with scaling functions \( \gamma, \rho \) and \( \rho_1, \rho \to 0, \rho_1 \gamma/\rho \to \infty \) and \( \rho \gamma \to \infty \) as \( |x| \to \infty \). Then for the Schrödinger-Pauli operator the following asymptotics holds \( N^- (\eta) = N^- (\eta) + \mathcal{O}(R) \) where

\[
(10.3) \quad N^- (\eta) := (2\pi)^{-d+\rho} \omega_{d-2p} \int_{\{x: -V(x) \geq \eta\}} f_1 f_2 \cdots f_p (-V - \eta)^{(d-2p)/2} \sqrt{g} \, dx
\]

and

\[
(10.4) \quad R = C \int_{\{x: -V(x) \geq \eta\}} \rho_1^p \rho^{d-2p-1} \gamma^{-1} \, dx + C \int \rho_1^{p-s} \gamma^{-2s} \, dx.
\]

**Example 10.2.** Let \( X \) be an unbounded domain. Let conditions (8.1), (8.2)\(^{1-3}\), (8.6), (10.1), (10.2) and (9.4)\(^{\#}\) be fulfilled with scaling functions \( \gamma = \epsilon \langle x \rangle, \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, -1 < m < 0 \leq m_1 \).

Then for the Schrödinger-Pauli operator

\[
(10.5) \quad R = \eta^{(m+2)(d-1)/(4m)}
\]

and

\[
(10.6) \quad N^- (\eta) = \mathcal{O}(\eta^{(m+2)(d-1)/(4m)+(m+1)/(2m)}).
\]

Further, we can replace “\( = \mathcal{O}(\cdot) \)” by “\( \asymp \cdot \)” if condition \( V \leq -\epsilon \rho^2 \) is fulfilled in some non-empty cone.

We leave to the reader

**Problem 10.3.** (i) Consider the case of \( \gamma = \epsilon \langle x \rangle, \rho = \langle x \rangle^{-1} |\log \langle x \rangle|^{\alpha}, \rho_1 = \langle x \rangle^{m_1}, m_1 \geq 0, \alpha > 0 \).

(ii) Consider the case of \( \gamma = \epsilon \langle x \rangle, \rho = |\log \langle x \rangle|^{\alpha}, \rho_1 = \langle x \rangle^{m_1}, m_1 \geq 0, \alpha < 0 \).\(^{82}\) Cf. Theorem 4.16 of [Ivr2].
(iii) Consider the case of \( \gamma = \langle x \rangle^{1-\beta}, \rho_1 = \exp(b\langle x \rangle^\beta), \beta > 0 \) while conditions to \( V, g^j \) are fulfilled with \( \gamma = \epsilon \langle x \rangle \), and either \( \rho = \langle x \rangle^{-1}|\log \langle x \rangle|^\alpha, \alpha > 0 \) or \( \rho = |\log \langle x \rangle|^\alpha, \alpha < 0 \).

Problem 10.4. Consider the Dirac operator. In this case \( N^- (\eta) \) is a number of eigenvalues in \( (M - \epsilon, M - \eta) \) and \( N^+ (\eta) \) is a number of eigenvalues in \( (-M + \eta, -M + \epsilon), 0 < \eta < \epsilon, M > 0 \).

10.1.2 Case \( F \to 0 \) as \( |x| \to \infty \)

In this subsection we consider cases of \( F \to 0 \) and \( V \to 0 \) as \( |x| \to \infty \). In this case the essential spectra of the Schrödinger and Schrödinger-Pauli operators are \([0, \infty)\); however, as \( V = o(F) \) as \( |x| \to \infty \) the Schrödinger operator has only a finite number of the negative eigenvalues and thus is not a subject of our analysis while the Schrödinger-Pauli operator is.

Further, the Dirac operator has its essential spectrum \((-\infty, -M] \cup [M, \infty)\) and we need to assume that \( M > 0 \).

Theorem 10.5 \(^{83} \). Let \( X \) be an unbounded domain. Let conditions (8.1), (8.2)\(_{1-3}\), (8.6), (10.1), (10.2) and (9.4)\\(^{\#}\) be fulfilled with scaling functions \( \gamma, \rho \) and \( \rho_1 \), \( \rho \to 0, \rho_1 \to 0, \rho_1 \rho/\rho^2 \geq C_0 \) and \( \rho \gamma \to \infty \) as \( |x| \to \infty \).

Then for the Schrödinger-Pauli operator \( N^- (\eta) = N^-(\eta) + O(R) \) with \( N^- (\eta) \) and \( R \) defined by (10.3) - (10.4).

Example 10.6. Let conditions of Theorem 10.5 be fulfilled with \( \gamma = \langle x \rangle, \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, m < 0, \max(2m, -2) < m_1 < 0 \).

Then all statements of the Example 10.2 remain true.

Example 10.7. Let conditions of Theorem 10.5 be fulfilled with \( \gamma = \epsilon \langle x \rangle, \rho_1 = \langle x \rangle^{-2} |\log \langle x \rangle|^\beta, \rho = \langle x \rangle^{-1} |\log \langle x \rangle|^\alpha \) with \( 0 < 2\alpha < \beta \). Then

\[
R = |\log \eta|^{(\beta \rho + \alpha(d - 2\rho - 1) + 1)/2}
\]
and \( N^- (\eta) = O(S) \) with

\[
S = |\log \eta|^{(\beta \rho + \alpha(d - 2\rho) + 1)/2}
\]

Further, \( N^- (\eta) \asymp S \) if condition \( V \leq -\epsilon \rho^2 \) is fulfilled in some non-empty cone.

\(^{83} \) Cf. Theorem 10.1.
We also leave to the reader

**Problem 10.8.** Consider in this framework the Dirac operator. In this case $N^-(\eta)$ is a number of eigenvalues in $(M - \epsilon, M - \eta)$ and $N^+(\eta)$ is a number of eigenvalues in $(-M + \eta, -M + \epsilon)$, $0 < \eta < \epsilon$. We need to assume that $M > 0$ and potential $V \sim \rho^2$ at infinity.

Consider now the case when condition $\rho^2 = o(\rho_1)$ as $|x| \to \infty$ is not fulfilled. Then the results will be similar to those of Section 9.

**Example 10.9.** (i) Let $X$ be an unbounded domain with $C^K$ boundary. Let conditions (8.1), (8.2), (8.6), (10.1), (210.2) and (9.4) be fulfilled with scaling functions $\gamma = \epsilon_0(x)$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $-1 < m < 0$, $m - 1 < m_1 < 2m$.

Then for the Schrödinger and Schrödinger-Pauli operators asymptotics

\begin{equation}
R = \eta^{(m+1)(d-1)/2m}
\end{equation}

and

\begin{equation}
S = \eta^{(m+1)(d-1)/2m}
\end{equation}

(ii) Similar results hold if $m_1 = 2m$ but as $\rho = 1$ one needs to assume a non-degeneracy assumption (we leave it to the reader) and for the Schrödinger operator $\mathcal{N}^-(\eta) \asymp S$ provided $V + F \leq -\epsilon\rho^2$ in some non-empty cone.

We leave to the reader:

**Problem 10.10.** Consider the case of $\gamma = \epsilon_0(x)$, $\rho = \langle x \rangle^{-2}|\log x|^\alpha$, $\rho_1 = \langle x \rangle^{-2}|\log x|^\beta$, $2\alpha \geq \beta > \alpha$.

**Problem 10.11.** Consider in this framework the Dirac operator with $M > 0$.

### 10.2 Case rank $F_\infty = d - 1$. Fast decaying potential

Consider case of stabilization at infinity (cf. Subsection 4.1 of [Ivr2])

\begin{equation}
(10.11)_{1-3} \quad g \to g_\infty, \quad F \to F_\infty, \quad V \to 0 \quad \text{as} \quad |x| \to \infty.
\end{equation}

Recall that $F := (F_{jk})$ with $F_{jk} = \partial_k V_j - \partial_j V_k$, $g := (g^{jk})$.

Assume now that
(10.12) \( \text{rank } F_\infty = 2p \) while \( d = 2p + 1 \)

and potential \( V \) decays faster than in the previous subsection—at least in the direction of the magnetic field.

### 10.2.1 Preliminary analysis

Observe first that

(10.13) If \( F \) and \( g \) are constant than without any loss of the generality we can assume that \( g^{jk} = \delta_{jk} \) and \( \text{Ker } F = \mathbb{R}^{d-2p} \times \{0\} \).

Indeed we can achieve it by a linear change of the coordinates. In the general case under assumption (10.12) we can assume that

\[
(10.14) \quad \text{Ker } F = \mathbb{R} \times \{0\} \quad \text{as } |x| \geq c
\]

and

\[
(10.15) \quad V_1 = 0 \quad \text{as } |x'| \geq c
\]

where \( x = (x_1; x') \). Indeed, we can achieve (10.14) by the change of the coordinate system which straightens magnetic lines\(^{84}\) and we can achieve (10.15) by the gauge transformation.

These two assumptions imply \( V_j = v_j(x') \) for \( j = 2, \ldots, d \) and together with stabilization as \( x_1 \to \infty \) we conclude that \( F \) is constant. Without any loss of the generality we can assume that

\[
(10.16) \quad g_\infty = \delta_{jk}, \ F_{jk} = 0 \text{ unless either } j = 2l, \ k = 2l + 1 \text{ when } F_{jk} = f_{\infty,l}, \text{ or } j = 2l + 1, \ k = 2l \text{ when } F_{jk} = -f_{\infty,l}.
\]

(10.17) By means of the allowed change of the coordinates\(^{84}\) on each magnetic line\(^{85}\) \( \{x: x' = y'\} \) with \( |y'| \geq c \) we can achieve

\[
(10.18) \quad g^{jd} = 0 \quad j = 0, \ldots, d.
\]

**Remark 10.12.** One can prove easily that in this reduction \( V \) is perturbed by \( \mathcal{O}(\rho^4 \gamma^{-2}) \) which would not affect the principal part and an error estimate.

---

\(^{84}\) After this we are allowed only changes \( x \mapsto y \) with \( y' = y'(x') \) and \( y_d = y_d(x) \).

\(^{85}\) But not necessarily on all of them in simultaneously.
As (10.14), (10.15) and (10.18)\(^{86}\) are fulfilled consider 1D-operator on \(\mathbb{R} \ni z\)

(10.19) \[ \mathcal{L}(y') := D_z g_{11}(z; y') D_z + V^*(z; y'), \]

(10.20) \[ V^*(x) := V(x) + \sum_j (f_j(x) - f_\infty). \]

Let us consider operator for which (10.14) and (10.15) are fulfilled allowing instead some anisotropy:

(10.21) \[ |\nabla \alpha (g - g_\infty)| = o(\rho^2 \gamma^{-|\alpha'| \gamma_1^{-\alpha_1}}), \]

(10.21) \[ |\nabla \alpha V| = O(\rho^2 \gamma^{-|\alpha'| \gamma_1^{-\alpha_1}}), \]

(10.21) \[ |\nabla \alpha F_{jk}| = O(\rho^2 \gamma^{-|\alpha'| \gamma_1^{-\alpha_1}}) \text{ as } |x| \to \infty \forall \alpha \]

with scaling functions

(10.22) \[ \gamma_1(x') \geq 1, \quad \gamma = \gamma(x') \to \infty \text{ as } |x'| \to \infty \]

(10.23) \[ \rho(x) = \rho(x') \gamma_1(x'/\gamma_1) \]

such that

(10.24) \[ |\nabla \gamma| \leq \frac{1}{2}, \quad |\nabla \gamma_1| \leq \frac{1}{2} \gamma_1^{-1}, \]

(10.25) \[ |\rho_1| \leq 1, \quad \int_{\mathbb{R}} |t| \rho_1^2(t) \, dt < \infty, \]

(10.26) \[ |\nabla \rho| \leq \rho \gamma_1^{-1}, \quad |\nabla \rho_1| \leq \rho_1 \gamma_1^{-1} \]

(10.27) \[ \zeta := \rho^2 \gamma_1 \to 0 \text{ as } |x'| \to \infty. \]

In virtue of Proposition 13.1 operator \(\mathcal{L}(x')\) has a finite number of negative eigenvalues for all \(x'\) and no more than one negative eigenvalue as \(|x'| \geq c\); further, under assumption

(10.28) \[ W(x') := -\frac{1}{2} \int_{\mathbb{R}} V^*(x_1; x') \, dx_1 > 0 \text{ and } W(x') \simeq \zeta \]

as \(|x'| \geq c\),

\(^{86}\) Only as \(x' = y'\).
there is exactly one negative eigenvalue \( \lambda(x') \) and

\[
\nabla^{\alpha'}(\lambda(x') + W(x')^2) = o(\zeta^2 \gamma^{-|\alpha'|})
\]

while

\[
\nabla^{\alpha'} W = O(\zeta \gamma^{-|\alpha'|})
\]

and

\[
|\nabla^{\alpha} v| = O(\gamma^{-|\alpha'|} \eta_1^{-|\alpha|}).
\]

Here \( v = v(x_1; x') \) is a corresponding eigenfunction.

### 10.2.2 The main theorem

The principal result of this subsection is the following theorem:

**Theorem 10.13.** Let conditions (10.11)\(_{1-3}\), (10.14), (10.15), (10.21)\(_{1-3}\), (10.22)\(_{1-2}\), (10.24)–(10.28) be fulfilled. Moreover, let

\[
|\nabla W| \geq \epsilon_0 \zeta^2 \gamma^{-1} \quad \text{as } |x| \geq c.
\]

Then

\[
|N^-(\eta) - N^-_{\eta}(\eta)| \leq C \int_{\mathcal{Z}(\eta)} \gamma^{-2} dx' + C \int \gamma^{-s} dx'
\]

where

\[
N^-_{\eta}(\eta) := (2\pi)^{-\frac{r}{2}} \int \{x: \lambda(x') = \eta\} f_{\infty,1} f_{\infty,2} \cdots f_{\infty,r} dx'
\]

\(\mathcal{Z}(\eta)\) is \(\epsilon \gamma\)-vicinity of \(\Sigma(\eta) = \{x': \lambda(x') = \eta\}\); cf. \(34\).

**Proof.** (a) We know that

\[
N^-_{\eta}(\eta) = N^-_{\eta}(A - f_{\infty}^* + \eta) = N^-_{\eta}(A_{\eta}) = \text{Tr}(E_{\eta}(0)),
\]

where

\[
A_{\eta} := J^{-\frac{1}{2}}(A - f_{\infty}^* + \eta)J^{-\frac{1}{2}}
\]

is a self-adjoint operator in \(L^2(\mathbb{R}^d)\) and \(E_{\eta}(\tau)\) is the spectral projector of this operator and \(J \simeq \rho^2\) is such that

\[
|\nabla^{\alpha} J| = O(J^{-|\alpha'|} \eta_1^{-|\alpha|}) \quad \forall \alpha.
\]
We consider only $\eta > 0$ and for any fixed $\eta > 0$ and $\tau$ this projector is finite-dimensional and its Schwartz kernel belongs to $\mathcal{S}(\mathbb{R}^{2d})$ uniformly on $\tau \leq \tau_0$. Let us note that

\[(10.38) \quad ((A - f_\infty^*)v, v) \geq (1 - \epsilon)((A_\infty - f_\infty^*)v, v) - C\|\rho v\|^2 \quad \forall v \in C^2_0(\mathbb{R}^d)\]

where $(A_\infty - f_\infty^*)$ is non-negative.

Indeed, without any loss of the generality one can assume that

\[(10.39) \quad A_\infty = D_1^2 + \sum_{1 \leq j \leq p} (D_{2j}^2 + (D_{2j+1} + f_j x_{2j})^2).\]

Then

\[(10.40) \quad A_\infty - f_\infty = D_1^2 + \sum_{1 \leq j \leq p} Z_j^* Z_j\]

with

\[(10.41) \quad Z_j = iD_{2j} + (D_{2j+1} + f_j x_{2j}).\]

On the other hand, $A - A_\infty$ is a linear combination of $D_1^2$, $D_1 Z_j$, $D_1 Z_j^*$, $Z_j Z_k$, $Z_j Z_k^*$, $Z_j$, $Z_j^*$ and 1 with the coefficients $\beta_s$ satisfying

\[(10.42) \quad |\nabla^\alpha \beta_s| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}) \quad \forall \alpha.\]

Then extra terms in $(Av, v)$ do not exceed

\[C\left(\|\rho D_1 v\|^2 + \sum_j \|\rho Z_j v\|^2 + \sum_j \|\rho Z_j^* v\|^2 + \|\rho v\|^2\right),\]

where obviously

\[\|\rho Z_j^* v\|^2 \leq C\left(\|\rho Z_j v\|^2 + \|\rho v\|^2\right).\]

This inequality (10.38) immediately yields estimates

\[(10.43)_{1,2} \quad \|D_1 J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C, \quad \|Z_j J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \quad j = 1, \ldots, p,\]

\[(10.43)_{3,4} \quad \|J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \eta^{-\frac{1}{2}}, \quad \|E_\eta(\tau)\| \leq 1\]

and therefore

\[(10.43)_{5,6} \quad \|Z_j^* J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \eta^{-\frac{1}{2}}, \quad \|Z_j^* E_\eta(\tau)\| \leq C\]

for operator norms where here and below $\tau \leq \tau_0$.

Then one can prove easily that
(10.44) Let $Q$ be a product of several factors $D_1, Z_\bullet$ and $Z_\bullet^*$. Then
\[ \|QJ^{-\frac{1}{2}}E_\eta(\tau)\| \leq C \] provided there are more of factors $D_1, Z_\bullet$ than of $Z_\bullet^*$, and
\[ \|QJ^{-\frac{1}{2}}E_\eta(\tau)\| \leq C\eta^{-\frac{1}{2}}, \|QE_\eta(\tau)\| \leq C \] provided there as many of factors $D_1, Z_\bullet$ as of $Z_\bullet^*$.

Then this claim remains true for $Q$ replaced by $Q' := D_\alpha^s Q$ for any $\alpha$ and then in virtue of the embedding theorem this is also true for the operator norm from $\mathcal{L}^2(\mathbb{R}^d) \to \mathcal{L}^2(\mathbb{R}^d)$ replaced by the operator norm from $\mathcal{L}^2(\mathbb{R}^d) \to \mathcal{L}^2(\mathbb{R})$ taken over any magnetic line \{\(x: x' = y'\}\} uniformly with respect to $y'$. In particular,
\[ |D_1J^{-\frac{1}{2}}E_\eta(\tau)v|_{\mathcal{L}^2(\mathbb{R})} \leq C\|v\| \quad \text{and} \quad |E_\eta(\tau)v|_{\mathcal{L}^2(\mathbb{R})} \leq C\|v\| \]
and therefore
\[ |(E_\eta(\tau)v)(x)| \leq C\lambda \left( J_0^{-\frac{1}{2}}(x') \gamma_1^{-1} + \langle x_1 \rangle^{-\frac{1}{2}} \right) \|v\| \]
with $J_0(x') = \max_{x_1} J(x_1, x')$. So, we estimated the operator norm of $w \to (E_\eta(\tau)w)(x)$ from $\mathcal{L}^2(\mathbb{R}^d)$ to $\mathbb{C}$; therefore
\[ |e_\eta(x, x, \tau)| \leq C\lambda \left( J_0^{-1}(x') \gamma_1^{-1} + \langle x_1 \rangle^{-\frac{1}{2}} \right) \]
and therefore $|\int e_\eta(x, x, \tau) \, dx_1| \leq C(1 + \rho^2(\langle x' \rangle) \gamma_1^2) \leq C$ due to properties of $J_0$, $\rho$ and $\gamma$. Therefore we have proven:

**Proposition 10.14.** In the framework of Theorem 10.13 and the definitions of $J$ and $e_\eta(x, y, \tau)$,
\[ \int_{|x'| \leq r} e_\eta(x, x, \tau) \, dx = O(1) \]
for all $\eta > 0$, $\tau \leq \tau_0$ and for any fixed $r$ and $\tau_0$.

Recall that $e_\eta(x, y, \tau)$ is the Schwartz kernel of $E_\eta(\tau)$. So, we only need to treat the contribution of the zone \{\(x: |x'| \geq r\}\).

(b) Let $y' \in \mathbb{R}^d$ and consider $\psi(x')$, $\psi \in \mathcal{C}_0^\infty(B(y', \frac{1}{2}\gamma))$ with $\gamma = \gamma(y')$ such that $|D^\alpha \psi| \leq c\gamma^{-|\alpha|} \forall \alpha : |\alpha| \leq K$. We want to derive asymptotics of
\[ \int \psi(x')e_\eta(x, x, 0) \, dx = \int \psi \text{Tr}_{\mathbb{H}}(e_\eta(x', x', 0)) \, dx' = \text{Tr}(\psi E_\eta(0)), \]

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where \( e_\eta(x', y', \tau) \) is the family of operators in \( \mathbb{H} \) with Schwartz kernel \( e_\eta(x, y, \tau) \). Let us rescale \( x'_{\text{new}} = (x' - y')\gamma^{-1}, x_{1\text{new}} = x_1\gamma_1^{-1} \). Then we obtain the standard LSSA problem for an operator with an operator-valued symbol, with the semiclassical parameters \( h = \gamma^{-1} \) and \( h_1 = \gamma_1^{-1} \) and with magnetic field intensity parameter \( \mu = \gamma \). Recall that the rescaled operator is

\[
(10.46) \quad h_1^2 D_1^2 + \sum_{1 \leq j \leq \rho} \left( h^2 D_{2j}^2 + \left( hD_{2j+1} - h^{-1}f_j x_{2j} \right)^2 \right) + \rho^2 A',
\]

where \( A' = a'(x, h_1 D_1, hD', h) \) is an operators with uniformly smooth symbol \( a' \) (we consider it more carefully later). Let \( U(x, y, t) \) be the Schwartz kernel of the operator \( \exp(ihD^2tA_0) \). Later we rescale \( t \). Then

\[
(10.47) \quad (h^2D_t - A_0)U = 0, \quad U|_{t=0} = \gamma^{2\rho} \delta(x - y)I
\]

So let \( \psi \) be a \( \gamma \)-admissible partition element. It follows from (10.44) that the operator norm (from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d) \)) of \( Q\psi E_\eta(\tau)Q^* \) does not exceed \( C \) for the operators \( Q \) which are products of several factors \( h_1 D_1, Z_\star \) and \( Z_\star \) and there are more factors of \( h_1 D_1, Z_\star \) and than of \( Z_\star \). Then the operator norm of \( F_{x \rightarrow h^{-2}x} \chi_T(t)Q\psi UQ^* \) does not exceed \( CT \) for the operators \( Z \) listed above where \( \chi \in \mathcal{C}_0^\infty(\mathbb{R}) \) is fixed and \( T \geq T_0 \) with constant \( T_0 > 0 \).

Let us apply the transformation \( T = T_0^{-1}T_1 T_0 \) where \( T_0 = F_{x^{''\prime} \rightarrow h^{-2}x^{''\prime}} \), \( x' = (x'', x^{''\prime}) \), \( x'' = (x_3, x_4, ..., x_{d-1}) \), \( x^{''\prime} = (x_3, x_5, ..., x_{d}) \) and the same partition for \( \xi' = (\xi'', \xi^{''\prime}) \), and

\[
(10.48) \quad T_1\nu(x_2, \xi_3, x_4, ..., \xi_d) = \nu(x_2 - f_1^{-1}\xi_3, \xi_3, ..., x_{d-1} - f_p^{-1}\xi_d, \xi_d).
\]

Then instead of \( hD_{2j} \) and \( (hD_{2j+1} - h^{-1}f_j x_{2j}) \) we obtain \( hD_{2j} \) and \( -h^{-1}x_{2j+1} \) respectively. Let \( \psi \) be the corresponding linear symplectic transformation. Let \( \tilde{U} = \tilde{T}_x\psi U^T \tilde{T}_y \) where \( \psi \) is supported in \( B(0, 1 - \epsilon) \) and equals 1 in \( B(0, 1 - 2\epsilon) \).

Let us decompose \( U(x, y, t) \) in terms of the functions

\[
(10.49) \quad \tilde{T}_\zeta(x'') = h^{-\frac{3}{2}}v_{\zeta_1}(x_2h^{-1})h^{-\frac{3}{2}}v_{\zeta_2}(x_4h^{-1}) \cdots h^{-\frac{3}{2}}v_{\zeta_p}(x_{2p}h^{-1})
\]

and \( \tilde{T}_\nu(y'') \):

\[
(10.50) \quad \tilde{U}(x, y, t) = \sum_{\zeta, \nu \in \mathbb{Z}^+} \tilde{T}_\zeta(x'') \tilde{T}_\nu(y'') U_{\zeta\nu}(x_1, x'', y_1, y''; t).
\]

---

(87) Due to (10.41) now \( Z_\star = iD_{2j} + (hD_{2j+1} + h^{-1}f_j x_{2j}) \).

(88) We shifted the coordinate system so that our partition element is supported there.
We make the same decomposition for \( E(x, y, \tau) \). Then the above estimates yield that

(10.51) The operator norm of \( F_{t\to h^{-2}rT}(t)U_{\varsigma\nu} \) does not exceed \( CT \).

Next, the standard ellipticity arguments show that

(10.52) The operator norm of \( F_{t\to h^{-2}rT}(t)J^{-\frac{1}{2}}(x_1)U_{\varsigma\nu} \) does not exceed \( C_0|\varsigma|^{-1}T \) for \( \varsigma \neq 0 \), and also the operator norm of \( F_{t\to h^{-2}rT}(t)J^{-\frac{1}{2}}(y_1)U_{\varsigma\nu} \) does not exceed \( C_0|\nu|^{-1}T \) for \( \nu \neq 0 \), and, finally, the operator norm of \( F_{t\to h^{-2}rT}(t)J^{-\frac{1}{2}}(x_1)J^{-\frac{1}{2}}(y_1)U_{\varsigma\nu} \) does not exceed \( C_0|\varsigma|+|\nu|^{-2}T \) for \( \varsigma \neq 0 \) and \( \nu \neq 0 \) and the same is true if we apply \( D_{x_1}^k \) and \( D_{y_1}^l \).

Moreover, for \( \varsigma = \nu = 0 \) we have

(10.53) The operator norm of \( F_{t\to h^{-2}rT}(t)D_{x_1}^kJ^{-\frac{1}{2}}(x_1)U_{00} \) does not exceed \( CT \) for \( k \geq 1 \), and also the operator norm of \( F_{t\to h^{-2}rT}(t)D_{y_1}^lJ^{-\frac{1}{2}}(y_1)U_{00} \) does not exceed \( CT \) for \( l \geq 1 \), and, finally, the operator norm of \( F_{t\to h^{-2}rT}(t)D_{x_1}^kD_{y_1}^lJ^{-\frac{1}{2}}(x_1)J^{-\frac{1}{2}}(y_1)U_{00} \) does not exceed \( CT \) for \( k \geq 1 \) and \( l \geq 1 \).

Then

(10.54) \[ \text{Tr}(\psi E) = \sum_{\varsigma, \nu} h^{\varsigma-\nu} \text{Tr}(\psi_{\varsigma\nu} E_{\varsigma, \nu}) + O(h^s) \]

where we have the original expression on the left-hand side, \( \psi_{\varsigma\nu} = \psi_{\varsigma\nu}(x'', h^2D'', h^2) \), \( \psi_{\varsigma\nu}(x'', \xi'', 0) = \psi(x'', \xi'') \). Moreover, \( \text{supp}(\psi_{\varsigma\nu}) \subset \text{supp}(\psi) \) and one can replace \( \psi_{\varsigma\nu} - \delta_{\varsigma\nu} \psi \) by a linear combination of the derivatives of \( \psi \) of non-zero order.

(c) It follows from Proposition 13.1 that operator \( (A - f_{\varsigma\nu}^*)J^{-1} \) is elliptic outside of \( \mathcal{Z}(\eta) \) and then one can prove easily that the total contribution of \( \mathbb{R}^d \setminus \mathcal{Z}(\eta) \) to the remainder does not exceed

(10.55) \[ C \int \gamma^{-s} d\alpha' \]

while its contribution to the principal part of asymptotics is given by the Tauberian expression.

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89) In the obvious situations we do not distinguish operators and their Schwartz kernels.
(d) From now on \( \psi' \) is a partition element in \( Z(\eta) \). Recall that

\[
(h^2 D_t - J^{-\frac{1}{2}} A J^{-\frac{1}{2}}) = 0, \quad U|_{t=0} = \delta(x-y).
\]

and the dual equation with respect to \( y \). Then using ellipticity arguments we can express \( U_\varsigma \) with \(|\varsigma| + |\nu| \geq 1\) via \( U_{00} \) via some \((h_1, h^2)\)-pseudodifferential operators (with respect to \((x_1, x'')\)) and \( h_t^2 \) and then plugging back into equation we get

\[
(h^2 D_t - J^{-\frac{1}{2}} A_0 J^{-\frac{1}{2}}) = 0, \quad U_{00}|_{t=0} = M \delta(x-y).
\]

where \( A_0 \) differs from \( h_t^2 D_1 + V^* + \eta \), with \( V^* = V + f^* - f_\infty^* \) by \( o(\rho^2) \); from the beginning we could assume that \( g^{11} = 1 \). Here \( J_0 \) and \( A_0 \) are \((h_1, h^2)\)-pseudodifferential operators (with respect to \((x_1, x'')\))-pseudodifferential operators. Let \( h := h^2 \). Let us observe that in virtue of Proposition 13.1 the operator \( J^{-\frac{1}{2}} A_0 J^{-\frac{1}{2}} \) has discrete spectrum in \( \mathbb{H} \) and all the eigenvalues of this operator excluding at most one are positive and uniformly disjoint from \( 0 \) and there is one (the lowest) eigenvalue \( \Lambda = \Lambda(x'', \xi'', \eta) \) which is \( O(1) \); moreover, due to (10.32) it satisfies the microhyperbolicity condition

\[
|\Lambda| + |\nabla \Lambda| \asymp 1.
\]

Then there exists a symbol \( q(x'', \xi'', h) : \mathbb{H} \rightarrow \mathbb{C} \oplus \mathbb{H} \) such that for the operator \( Q = q(x'', hD'', h) \) and for \( U' = QU_0Q^* \) we obtain separate equations for all four blocks of \( U' = \begin{pmatrix} U'_{00} & U'_{01} \\ U'_{10} & U'_{11} \end{pmatrix} \). Moreover, for the blocks \( U'_{10} \) and \( U'_{11} \) the equations are elliptic for \( \tau \leq \epsilon_1 h \) and for \( U'_{00} \) and \( U'_{11} \) this is true for the dual equations. Therefore \( U' \equiv \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \) with \( u = U_{00} \) and

\[
F_{t \rightarrow h^{-1} \tau} \chi(t) \text{Tr}(\psi U) \equiv F_{t \rightarrow h^{-1} \tau} \chi(t) \text{Tr}(\psi'' u)
\]

for \( \tau \leq \epsilon_0 h^2 \), \( T \in (h^{-\delta}, \epsilon_0) \) where \( \psi'' = \psi''(x'', h''D'', h) \). We have an equation for \( u \):

\[
(hD_t - \Lambda)u = 0
\]

where \( \Lambda \) is an \( h \)-pseudodifferential operator. More precisely: due to the microhyperbolicity we conclude that
The contribution of the partition element to the final answer is given by a Tauberian expression with \( T = \hbar^{1-\delta} \) with an error \( O(1) \).

Therefore, the total contribution of \( Z(\eta) \) to the remainder does not exceed \( C \int_{Z(\eta)} \gamma^{-2} \, dx \) (in the original coordinates).

(e) Employing the method of the successful approximations and picking \( \psi = 1 \), and we conclude that the final answer is given by (10.34) since since \( \Lambda < 0 \iff \lambda < -\eta \). We leave easy details to the reader.

\[ \Box \]

Remark 10.15. If \( V^+(x) \approx |x|^{-2} \) then (formally) \( W(x') \approx |x'|^{-1} \) and \( \lambda(x') \approx |x'|^{-2} \) and \( N(\eta) \approx \eta^{-\epsilon} \) as follows from the results of Subsection 10.1.

10.2.3 Generalizations

Remark 10.16. (i) For the Schrödinger-Pauli operator Theorem 10.13 obviously holds albeit with \( f^* = f^*_\infty = 0 \).

(ii) The same is true for the Dirac operator. The proof is essentially the same. We need to assume that the mass \( M \neq 0 \), otherwise the spectral gap \((-M, M)\) is empty. Then we consider \( N^-(\eta) = N(0, M - \eta) \) and \( N^+(\eta) = N(-M + \eta, 0) \). Instead of \( 0 \) we can take any \( \tilde{\tau} \in (-M, M) \) which preserves the result modulo \( O(1) \).

Let us consider \( N^-(\eta) \). Modulo \( O(1) \) it equals to \( \tilde{N}(\eta; -\epsilon_2, 0) \), where \( \tilde{N}(\eta; \tau_1, \tau_2) \) is the number of eigenvalues of the problem

\[
(A - M + \eta)\nu + \tau J\nu = 0
\]

belonging to the interval \((\tau_1, \tau_2)\) and \( J = \frac{1}{2}(I + \sigma_0)J \) where \( J \) was introduced in the proof of Theorem 10.13. This problem is equivalent to the problem

\[
(A_\eta - \tau J)w = 0, \quad A_\eta = L^*(2M - \eta - V)^{-1}L + V + \eta,
\]

where we assume that

\[
\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & \sigma_j' \\ \sigma_j' & 0 \end{pmatrix}, \quad \sigma_j^* = \sigma_j' (j = 1, \ldots, d)
\]
and
\[
\mathcal{L} = \sum_{1 \leq j \leq d} \frac{1}{2} (P_k \omega^{jk} + \omega^{jk} P_k) \sigma'_j.
\]

One can easily transform $\mathcal{A}_\eta$ to the form of the Schrödinger-Pauli operator with the metric $\tilde{g}^{jk} = (2M - \eta - V)^{-1} g^{jk}$.

**Example 10.17.** (i) In the standard isotropic case $\gamma_1 = \gamma = \langle x' \rangle$ and as $\rho(x) = \langle x \rangle^l$ with $l < -2$; then for $W$ defined by (10.28) satisfies similar conditions with $m = 2l + 1$ and thus we are in the framework of Example 4.4 of [Ivr2].

(ii) However, we can also consider $\rho(x) = \langle x \rangle^l \langle x' \rangle^p$ with $l < -2$; then $m = 2l + 2p + 1$.

(iii) We can also consider faster and slower decaying potentials, as soon as $W^2$ satisfies conditions imposed on $V$ in Subsection 10.1. See Example 4.5, Problems 4.8 and 4.10 of [Ivr2].

**Example 10.18.** Let us consider the case when at infinity $f_j$ stabilize not to $f_\infty = \text{const}$ but to $f_\infty(\theta), \theta = x'/|x'| \in S^{d-2}$.

Then as long as other assumptions are fulfilled, we arrive to asymptotics described in Theorem 4.13 of [Ivr2]; again, instead of $V(x')$ we have $\Lambda(x')$ in the conditions and in the expression for $N^- (\eta)$.

**Remark 10.19.** Let us consider an auxiliary operator with potential $V$ which is $\asymp |x_1|^{-2}$ as $x_1 \to \infty$. One can easily prove (see Proposition 13.3 below) that if
\[
V^* \geq -\frac{1}{4} |x|^{-2} \quad \forall x: |x| \geq C,
\]
then the number of negative eigenvalues is finite and there is no more than one negative eigenvalue if this inequality holds for all $x$. Moreover, under the conditions
\[
V^* \geq (\epsilon - \frac{1}{4}) |x|^{-2} \quad \forall x: |x| \geq C
\]
and (13.12) with arbitrarily small $\epsilon > 0$ all the statements of Proposition 13.1 remain true. Furthermore, under condition (10.66)*
\[
\langle av, v \rangle \geq \frac{\epsilon}{2} |\langle x \rangle^{-1} v|^2 - C |\langle x \rangle^{-s} v|^2 \quad \forall v
\]
with arbitrarily large $s$. Therefore we can cover the case

\begin{equation}
\rho(x) = \langle x \rangle^{-2} \langle x' \rangle^{p+2}, \quad p < -1
\end{equation}

provided

\begin{equation}
V^* \geq (\epsilon - \frac{1}{4})|x|^{-2} \quad \forall x: |x_1| \geq c|x'|
\end{equation}

with arbitrarily small $\epsilon > 0$. The remainder estimate is the same $O(1)$ as above. The details are left to the reader.

**Remark 10.20.** Let $\text{rank} F(x) = 2r \leq d - 2$ (as $|x| \geq c$). Then the auxiliary operator is $(d - 2r)$-dimensional and does not have negative eigenvalues at all in the assumptions of this subsection.

Then one can prove easily that $N^-(\eta) = O(1)$. In particular, if $\gamma_1 = \gamma = \langle x' \rangle$ and $\rho = \langle x' \rangle^m$, $N^-(\eta) = O(1)$ for $m < -1$ (and even for $m = -1$ under assumption (10.69) but there is a non-trivial asymptotics for $m > -1$; see Subsection 10.3 below.

### 10.2.4 Possible generalizations

Consider the case when condition (10.28) is not fulfilled. We believe that while the Part (i) is not extremely challenging, the Part (ii) is:

**Problem 10.21.** (i) Prove that the main part of the asymptotics is still given by (10.34).

(ii) Prove that (10.33) still holds.

### 10.3 Case $\text{rank } F_\infty = d - 1$. Slow decaying potential

Now we consider the case as in the previous Subsection 10.2 but we assume that the potential $V$ which either decays slower than $x_1^{-2}$ or as $x_1^{-2}$ but fails condition (10.66)*. We only sketch the main arguments.
10.3.1 Main theorem (statement)

We assume that

\begin{align*}
|\nabla \alpha (g - g_{\infty})| &= o(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-a_1}), \\
|\nabla \alpha V| &= O(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-a_1}), \\
|\nabla \alpha (F_{jk} - F_{\infty, jk})| &= O(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-a_1}) \quad \text{as } |x| \to \infty \quad \forall \alpha
\end{align*}

where

\begin{align*}
\rho &= \langle x \rangle^{-q} \langle x' \rangle^{m+q}
\end{align*}

\(q > 0, m < 0\). Further, as \(m + q = 0\) we assume in addition that

\begin{align*}
|\nabla \alpha (g - g_{\infty})| &= O(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-a_1}), \\
|\nabla \alpha V| &= O(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-a_1}), \\
|\nabla \alpha (F_{jk} - F_{\infty, jk})| &= O(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-a_1}) \quad \text{as } |x| \to \infty \quad \forall \alpha
\end{align*}

Theorem 10.22. Let conditions (10.70)\(_1\)–\(_3\) be fulfilled. Let one of two assumptions be fulfilled:

(i) \(m + q < 0\) and

\begin{align*}
-\langle x', \nabla' V^* \rangle &\geq \epsilon \rho^2 \quad \forall x: |x'| \geq C_0.
\end{align*}

(ii) \(m + q = 0\), conditions (10.72)\(_1\)–\(_3\) be also fulfilled

\begin{align*}
-\langle x', \nabla' V^* \rangle &\geq \epsilon \rho^2 \langle x' \rangle^2 \langle x \rangle^{-2} \quad \forall x: |x'| \geq C_0.
\end{align*}

Then

\begin{align*}
N^-(\eta) &= N^-(\eta) + O(R(\eta)) \\
N^-(\eta) &= (2\pi)^{-r} \int n(x', \eta) f_{\infty, 1} f_{\infty, 2} \cdots f_{\infty, r} \, dx' \\
R(\eta) &= \int_{\Lambda(\eta)} (n(x', \eta) + 1) \langle x' \rangle^{-2} \, dx' + \int \gamma^{-s} \, dx'
\end{align*}

where \(n(x', \eta)\) is the number of eigenvalues of the operator \(L(x')\) which are less than \(-\eta\) and \(\Lambda(\eta)\) is \(\epsilon\gamma\)-vicinity \(\Sigma(\eta) = \{x': n(x', \eta) > 0\}\), \(\gamma = \langle x' \rangle\) and \(L(x')\) is defined by (10.19)–(10.20).
10.3.2 Proof of Theorem 10.22: Propagation of singularities

Again let us consider the number of negative eigenvalues of operator $A_\eta$, defined by (10.36) with

\begin{equation}
J(x) = j(x') \langle x \rangle^{-2p} \langle x' \rangle^{2p} \quad p = \begin{cases} 
q & \text{if } m + q < 0, \\
q + 1 & \text{if } m + q = 0
\end{cases}
\end{equation}

and $\gamma$-admissible function $j(x')$. Let $\rho = \langle x' \rangle^{2m}$. As in the previous Subsection 10.2 we consider $A_\eta$ as operator in $L^2(\mathbb{R}^2, \mathbb{H})$ with $\mathbb{H} = L^2(\mathbb{R}, \mathbb{C})$. As usual after proper scaling $h = \rho^{-1} \gamma^{-1}$ and $\mu = \rho^{-1} \gamma$.

Again let consider the corresponding propagator. Our first goal is to estimate the propagation speed with respect to $x'$ from above and then to estimate it under the microhyperbolicity condition also from below.

**Proposition 10.23.** Let assumptions (10.70)_1−3 be fulfilled with $m < 0$ and $0 < q \leq 1$. Further, let for $m + q = 0$ assumptions (10.72)_1−3 be fulfilled as well.

Then the propagation speed with respect to $x'$ does not exceed $C \rho^2 \gamma^{-1}$ (before scaling) and therefore singularity initially supported in $B(y', \frac{1}{2} \gamma)$ is confined to $B(y', \gamma)$ for $T \leq T^* = c j \rho^2 \gamma^{-1}$ calculated at $y'$:

\begin{equation}
| F_{t \to h^{-1} \tau} \chi_T(t) \psi(x')(1 - \psi_0(y')) u(x, y, t) | \leq CT \gamma^{-s}
\end{equation}

provided $\psi \in C^\infty_0 B(y', \frac{1}{2} \gamma)$, $\psi_0 \in C^\infty_0 B(y', \gamma)$, $\psi_0 = 1$ in $B(y', \gamma)$, $\gamma = \gamma(y')$, $|\tau| \leq \epsilon$, $\| \cdot \|$ is a standard operator norm from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

**Proposition 10.24.** In the framework of Proposition 10.23 assume that the microhyperbolicity assumption (10.73) is fulfilled for $m + q < 0$ and the microhyperbolicity assumption (10.74) is fulfilled for $m + q = 0$. Then the propagation speed with respect to $x'$ in an appropriate direction is greater than $c j^{-1} \rho^2 \gamma^{-1}$ (before scaling) and therefore

\begin{equation}
| F_{t \to h^{-1} \tau} \chi_T(t) \Gamma'(u \psi) | \leq C \gamma^{-1} (T/T_*)^{-s} \quad \text{for } |\tau| \leq \epsilon
\end{equation}

where, as usual, $\chi \in C^\infty_0([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$, $\psi \in C^\infty_0 (B(x', \frac{1}{2} \gamma(x'))$ and $T \in [T_*, T^*]$, $T_* = c j^{-1} \rho^2$ and $\gamma = \gamma(y')$.  

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Proofs of Propositions 10.23 and 10.24. Standard proofs are left to the reader. We just observe the following:

If \( m + q < 0 \) then due to \((10.70)_1-3\) and \((10.78)\) the drift speed with respect to \( x' \) does not exceed \( c_j^{-1} \rho^2 \gamma^{-1} \) (Proposition 10.23) and is exactly of this magnitude due to \((10.73)\) (Proposition 10.23). This is also true for \( m + q = 0 \) due to \((10.72)_1-3\) and \((10.78)\), and \((10.74)\).

\[ \square \]

10.3.3 Proof of Theorem 10.22: Estimates

Now we can use the standard successive approximations method leading us to the estimate

\[ |F_{t \rightarrow h^{-1} \tau} \tilde{\chi}_T(t) \Gamma'(u_\psi)| \leq C \gamma^{d-1} \]  

and then to estimate

\[ |F_{t \rightarrow h^{-1} \tau} \tilde{\chi}_T(t) \Gamma(u_\psi)| \leq C \gamma^{d-1} N \]  

as \( T_\epsilon \leq T \leq T^* \) and \( |\tau| \leq \epsilon \), where \( N = \sup_{B(y', (1+\epsilon)\gamma)} n(x', \eta) \). The latter estimate leads us to the estimate \((10.77)\) to the Tauberian error and in the conjugation with successive approximation method to the estimate \((10.77)\) itself.

10.3.4 Discussion

Remark 10.25. If we replace \( n(x', \eta) \) by the corresponding 1D-Weyl expression, we will get

\[ N^{-}(\eta) = N^{MW^{-}}(\eta) + O(R(\eta) + R_1(\eta)) \]  

with

\[ N^{MW^{-}}(\eta) = (2\pi)^{-r} \int (-V^* - \eta)^{1/r} \sqrt{g_{\infty,1} f_{\infty,2} \cdots f_{\infty,r}} \, dx \]

and

\[ R_1(\eta) = \int_{\mathcal{N}(\eta)} \, dx'. \]

Remark 10.26. (i) Obviously \( V(x) = -\langle x \rangle^{-2q} U(x') \) with \( U \) positively homogeneous of degree \( 2(m + q) \) satisfies \((10.70)_1-3\), and for \( q + m = 0 \) it also satisfies \((10.72)_1-3\).

(ii) Furthermore, if \( U \asymp \langle x' \rangle^{2(m+q)} \) this \( V \) satisfies \((10.73)\) if \( q + m < 0 \) and \((10.74)\) if \( q + m = 0 \).
Theorem 10.13 as would imply (c) If \( n \in \mathbb{R} \) and \( \gamma \) and \( n \) except the case \( \| \gamma \| < \) and \( n \in \mathbb{R} \gamma \). Let us evaluate magnitudes of (10.87) \( m, n \) except when \( n \in \mathbb{R} \gamma \). Then contribution of \( \gamma = \langle x' \rangle \) as \( \gamma \leq \epsilon \min(\tilde{\gamma}_1, \tilde{\gamma}_2) \) with

\[
(10.86) \quad \tilde{\gamma}_1 = \eta(1-q)/(2(m+q)), \quad \tilde{\gamma}_2 = \eta^{1/(2m)}
\]

and relying upon Proposition 13.4 we conclude that

\[
(10.87) \quad \mathbf{n}(x', \eta) \asymp \eta^{(q-1)/(2m+q)} \gamma^{(m+q)/q} \quad \text{as} \quad \gamma \leq \epsilon \min(\tilde{\gamma}_1, \tilde{\gamma}_2)
\]

provided \( m + q < 0 \). On the other hand, obviously \( \mathbf{n}(x', \eta) = 0 \) as \( \gamma \geq C\tilde{\gamma}_2 \). Recall that \( m < 0 \). Observe that \( \tilde{\gamma}_1 \geq \tilde{\gamma}_2 \) for any \( \eta \leq 1 \) if and only if \( m \in [-1, 0) \).

(a) Let \( m \in [-1, 0) \). Then (10.87) holds as \( \gamma \leq \epsilon \tilde{\gamma}_2 \) and we conclude that

\[
(10.88)_{1-3} \quad \mathcal{N}^- (\eta) \asymp \eta^{(d+m)/(2m)}, \quad \mathcal{R}(\eta) \asymp \eta^{(d+m-2)/(2m)}, \quad R_1(\eta) = \eta^{(d-1)/(2m)}.
\]

(b) Let \( m < -1 \). Then (10.87) holds as \( \gamma \leq \epsilon \tilde{\gamma}_1 \) and contributions of the zone \( \mathcal{X}_0 = \{ x : \| x' \| \leq \tilde{\gamma}_1 \} \) to \( \mathcal{N}^- (\eta) \) and \( \mathcal{R}(\eta) \) are respectively of the magnitudes \( \eta^{(1-q)(d-1)/(2m+q)} \) and \( \eta^{(1-q)(d-3)/2(m+q)} \).

We need to consider the zone \( \mathcal{X}_1 := \{ x : \tilde{\gamma}_1 \leq \| x' \| \leq \tilde{\gamma}_2 \} \) separately.

(c) If \( 1 > q \) \( \geq \frac{1}{2} \) we in virtue of Proposition 13.6 \( \mathbf{n}(x', \eta) \approx 1 \) if \( \gamma \leq \epsilon \tilde{\gamma}_3 \) and \( \mathbf{n}(x', \eta) = 0 \) if \( \gamma \geq C\tilde{\gamma}_3 \) with \( \tilde{\gamma}_3 = \eta^{1/(2m+1)} \); one can see easily that \( \tilde{\gamma}_1 \leq \tilde{\gamma}_3 \leq \tilde{\gamma}_2 \). Then contribution of \( \mathcal{X}_1 \) to \( \mathcal{N}^- (\eta) \) and \( \mathcal{R}(\eta) \) are of magnitudes \( \tilde{\gamma}_3^{d-1} \) and \( \tilde{\gamma}_3^{d-3} \) respectively except the case \( d = 3 \) when the contribution to \( \mathcal{R}(\eta) \) is of magnitude \( | \log(\eta) | \).

Combining with Statement (b) we conclude that \( \mathcal{X}_1 \) contributes more to \( \mathcal{N}^- (\eta) \) and \( \mathcal{R}(\eta) \) than \( \mathcal{X}_0 \) and therefore

\[
(10.89)_{1,2} \quad \mathcal{N}^- (\eta) \asymp \eta^{(d-1)/(2m+1)}, \quad \mathcal{R}(\eta) \asymp \eta^{(d-3)/2(2m+1)}
\]

except the case \( d = 3 \) when \( \mathcal{R}(\eta) \) is of magnitude \( | \log(\eta) | \). Meanwhile, \( R_1(\eta) \) is of the same magnitude as \( \mathcal{N}^- (\eta) \).
(d) If $0 < q \leq \frac{1}{2}$ then in virtue of Proposition 13.8 with $\varepsilon = \gamma^{2(m+q)}$ we conclude that $n(x', \eta) \approx 1$ if $\gamma \leq \epsilon_1$ and $n(x', \eta) = 0$ if $\gamma \geq C_1$. Combining with Statement (b) we conclude that in this case

$$(10.90)_{1,2} \quad N^-(\eta) \approx \eta^{(1-q)(d-1)/2(m+q)}, \quad R(\eta) \approx \eta^{-(q-1)(d-3)/2(m+q)}.$$

Meanwhile, $R_1(\eta)$ is of the same magnitude as $N^-(\eta)$.

**Problem 10.28.** As $d = 3$ derive remainder estimate $O(1)$ in the framework of Example 10.27(c). This analysis should be done in the zone $X_1$ and we need to consider the spectral stripes $\Lambda_k = \{x': \lambda_k(x') \approx \eta\}$ and zone $X_1 \setminus (\Lambda_1 \cup \Lambda_2 \cup ... \cup \Lambda_K)$ separately.

**Example 10.29.** (a) If $q > 1$ then $n(x', \eta) \leq C \gamma^{m+1}$ with $m + 1 < 0$ since we need to assume that $m + q \leq 0$ in virtue Remark 10.26(iii) which puts us in the framework of Subsection 10.2.

(b) If $q = 1, m < -1$ then $n(x', \eta) = 0$ for $|x'| \geq c$ which leads only to the estimate $N^-(\eta) \leq C|\log \eta|$ rather than asymptotics.

(c) Thus, consider $q = 1, m = -1$. Then $n(x', \eta) \approx \gamma^{m+1}|\log(\gamma/\bar{\gamma})|$ and $n(x', \eta) = 0$ if $\gamma \geq C_1, \bar{\gamma} = \eta^{-1/2}$.

This leads to

$$(10.91)_{1,2} \quad N^-(\eta) \approx \eta^{-(d-1)/2}, \quad R(\eta) \approx \eta^{-(d-3)/2}$$

again except $d = 3$, in which case $R(\eta) \approx |\log \eta|^2$. However, under assumption $V \leq -C_0(\gamma)^{-2}$ with sufficiently large $C_0$, we conclude that $n(x', \eta) \approx \gamma^{m+1}|\log(\gamma/\bar{\gamma})|$ if $\gamma \leq \epsilon_1$ and then $N^-(\eta) \approx \eta^{-(d-1)/2}$.

**Problem 10.30.** Let us consider the case $m + q > 0$. In this case we do not have a microhyperbolicity condition and we can apply only more simple and less precise approach of Subsection 10.1. So, we leave to the reader to derive the remainder estimate in the following cases:

(a) Let $0 < q < 1$.

\footnote{And then $m \in (-1, 0)$.} Then, exactly as in Example 10.27(a) (10.87) holds if $\gamma \leq \epsilon_2 = \eta^{1/(2m)}$ and $n(x', \eta) = 0$ if $\gamma \geq \bar{\gamma}_2$ and therefore (10.88) \footnote{And then $m \in (-1, 0)$.} holds.
Let $q = 1^{(90)}$. Then we have the same magnitude of $n(x', \eta)$ as in Example 10.29(c) and $\mathcal{N}^-(\eta) \propto \eta^{(d+m)/(2m)}$.

(c) Let $q > 1$. Then for $m < -1$ we are in the framework of Subsection 10.2, and for $m = -1$ we are either in the framework of that Subsection or close to it.

So, let us assume that $m \in (-1, 0)$. In this case $n(x', \eta) \lesssim \gamma^{m+1}$, $n(x', \eta) \asymp \gamma^{m+1}$ for $\gamma \leq c\gamma_2$ and $n(x', \eta) = 0$ for $\gamma \geq C\gamma_2$. Then (10.88) holds.

Finally, we leave to the reader

Problem 10.31. Consider the case of $\text{rank } F = d - 2r \leq d - 2$. To do so we need to modify Theorem 10.22 in rather obvious way, in this case $x' = (x''; x') = (x_1, \ldots, x_n; x_{n+1}, \ldots, x_d)$ with $n = d - 2r$ and $L(x')$ is $n$-dimensional Schrödinger operator.

11 3D-case. Multiparameter asymptotics

In this section we consider asymptotics with respect to three parameters $\mu$, $h$ and $\tau$; here spectral parameter $\tau$ tends either to $\pm \infty$ or to the border of the essential spectrum or to $-\infty$ (for Schrödinger and Schrödinger-Pauli operators) or to the border of the spectrum. In two last cases presence of $h \to +0$ is crucial. We consider here only $d = 2$ and $h \ll 1$.

11.1 Asymptotics of large eigenvalues

In this subsection $\tau \to +\infty$ for the Schrödinger and Schrödinger-Pauli operators and $\tau \to \pm \infty$ for the Dirac operator. We consider the Schrödinger and Schrödinger-Pauli operators, leaving the Dirac operator to the reader.

Example 11.1. Assume first that $\psi \in C_0^\infty$ and there are no singularities on $\text{supp}(\psi)$. We consider

\begin{equation}
N^-_\psi(\tau) = \int e(x, x, \tau)\psi(x) \, dx.
\end{equation}

Then for scaling $A \mapsto \tau^{-1}A$ leads to $h \mapsto h_{\text{eff}} = h\tau^{-1/2}$ and $\mu \mapsto \mu_{\text{eff}} = \mu\tau^{-1/2}$.
(i) If \( \mu \lesssim \tau^{1/2} \) then we can apply the standard theory with the “normal” magnetic field; we need to assume that \( h \ll \tau^{1/2} \) and we need neither condition \( d = 3 \), nor \( F \geq \epsilon_0 \), nor \( \partial X = \emptyset \); the principal part of the asymptotics has magnitude \( h^{-d_1 \tau^{d/2}} \) and the remainder estimate is \( O(h^{1-d_1 \tau^{(d-1)/2}}) \) which one can even improve to \( o(h^{1-d_1 \tau^{(d-1)/2}}) \) under proper non-periodicity assumption.

(ii) Let \( \mu \gtrsim \tau^{1/2}, \mu h \lesssim \tau \). Then we can apply the standard theory with the “strong” magnetic field; we assume that \( d = 3 \) and \( F \geq \epsilon_0 \). Then the principal part of the asymptotics has magnitude \( h^{-3 \tau^{3/2}} \) and under weak non-degeneracy assumption (which is needed, only in the case \( \mu_{\text{eff}} \leq h_{\text{eff}}^{1/2} \)) fulfilled on \( \text{supp}(\psi) \) the remainder estimate is \( O(h^{-2 \tau}) \) and marginally worse without non-degeneracy assumption

(iii) If \( \mu \gtrsim \tau^{1/2}, \mu h \gtrsim c \tau \) than \( N - (\tau) = 0 \) for the Schrödinger operator; for the Schrödinger-Pauli operator the principal part of the asymptotics has magnitude \( \mu h^{-1} \) and under weak non-degeneracy assumptions the remainder estimate is \( O(\mu h^{-1}) \).

Example 11.2\(^{91}\). Let \( X \) be an unbounded domain. Let conditions (8.1), (8.6), (8.2)\(_{1-3} \), (9.1) and (9.9)\(_k \) be fulfilled with \( \gamma = \epsilon_0(x), \rho = \langle x \rangle^m, \rho_1 = \langle x \rangle^{m_1}, m_1 > 2m \). Consider the Schrödinger operator and assume that

\[
(11.2)_{1,2} \quad \tau \geq \mu h, \quad \tau^{2m-m_1} \leq c(\mu h)^{2m}.
\]

Then

\[
(11.3) \quad N^-(\tau, \mu, h) \approx \tau^{3(2+m_1)/(2m_1)} h^{-3(1+m_1)/m_1} \mu^{-3/m_1}.
\]

(i) Further, if \( \tau \gtrsim \mu^2 \), then the zone of the strong magnetic field \( \mu_{\text{eff}} = \mu \langle x \rangle^{m_1+1} \tau^{-1/2} \geq C \) is contained in \( \{x: |x| \geq c\} \) and here we have non-degeneracy condition fulfilled. Then the remainder estimate is \( O(R) \) with

\[
(11.4) \quad R = \tau^{(2+m_1)/m_1} h^{-2(1+m_1)/m_1} \mu^{2/m_1}.
\]

(ii) On the other hand, if \( \mu^2 \gg \tau \), then the contribution of the zone \( \{x: |x| \geq c\} \) to the remainder is \( O(R) \) with \( R \) defined by (11.4). The contribution of the zone \( \{x: |x| \leq c\} \) to the remainder is \( O(h^{-2 \tau}) \) under weak non-degeneracy assumption fulfilled there.

\(^{91}\) Cf. Example 9.12.
(iii) Let us replace (11.2) by the opposite inequality, and assume (9.3)#. Then (11.3) is replaced by $N - (\tau, \mu, h) \asymp h^{-3}\tau^{3(m+1)/(2m)}$ while under non-degeneracy assumption

$$R = h^{-2}\tau^{(m+1)/m}.$$  

**Example 11.3**\(^{92}\). In the framework of Example 11.2 for the Schrödinger-Pauli operator under assumption (3.3)# of [Ivr2]

$$N - (\tau, \mu, h) \asymp h^{-3}\tau^{3(m+1)/(2m)} + \mu h^{-2}\tau^{(m_1+m+3)/(2m)}.$$  

$$R = h^{-2}\tau^{(m+1)/m} + \mu h^{-1}\tau^{(m_1+2)/(2m)}$$

We leave to the reader

**Problem 11.4.** Consider the Schrödinger and Schrödinger-Pauli operators with other types of the behaviour at infinity.

**Problem 11.5.** For the Dirac operators derive similar results as $\tau \to \pm \infty$.

### 11.2 Asymptotics of the small eigenvalues

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to $-0$.

**Example 11.6**\(^{93}\). Let $X$ be an unbounded domain. Let conditions (8.1), (8.3)\(_{1,2}\), (9.1)# be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^{m_1}$, $-1 < m < 0$, $m_1 > m - 1$.

Consider the Schrödinger operator and assume that

$$(11.8)_{1,2} \quad 1 \geq \mu h, \quad |\tau|^{2m-m_1} \leq \epsilon (\mu h)^{2m}.$$  

Then $N - (\tau) = O(h^{-3}|\tau|^{3(m+1)/(2m)})$ as $\tau \to -0$ with "$\asymp"$ instead of "$= O"$ if condition $V \leq -\epsilon \rho^2$ fulfilled in some non-empty cone.

Further, under the non-degeneracy assumption (9.4)# the contribution to the remainder of the zone $\{x: |x| \geq c\}$ is $O(h^{-2}|\tau|^{(m+1)/m})$.

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\(^{92}\) Cf. Example 9.14.  
^{93}\) Cf. Example 10.2.
Example 11.7\(^{94})\). In the framework of Example 11.6 for the Schrödinger-Pauli operator under assumption (9.3)\(^{\#}\) the contribution to the remainder of the zone \(\{x: |x| \geq c\}\) is \(O(R)\) with

\[
R = h^{-2} |\tau|^{(m+1)/m + \mu h^{-1}|\tau|^{(m+2)/(2m)}},
\]

while

\[
N^{-}(\tau) = O(h^{-3} |\tau|^{3(m+1)/(2m)} + \mu h^{-2} |\tau|^{(m+m+3)/(2m)}),
\]

with “\(\asymp\)” instead of “\(\sim\)” if condition \(V \leq -\epsilon \rho^2\) fulfilled in some non-empty cone.

Problem 11.8. Consider the Schrödinger and Schrödinger-Pauli operators if

(i) If condition (11.8)\(_1\) is violated (then there could be a forbidden zone in the center).

(ii) With other types of the behaviour at infinity.

Problem 11.9. Consider the Schrödinger and Schrödinger-Pauli operators in the framework of Subsections 10.2 and 10.3 if

(i) \(\mu h = 1\); then the essential spectrum does not change.

(ii) \(\mu h \to \infty\); then the limit of the essential spectrum is \(\emptyset\) (for the Schrödinger operator) and \([0, \infty)\) (for the Schrödinger-Pauli operator). Consider \(N^{-}(\eta)\) with \(\eta \to 0\).

(iii) \(\mu h \to 0\); then the limit of the essential spectrum is \([0, \infty)\) (for both the Schrödinger and Schrödinger-Pauli operators). Consider \(N^{-}(\eta)\) with \(\eta \to 0\).

Problem 11.10. For the Dirac operators derive similar results as \(M \neq 0\) and \(\tau \to M - 0\) and \(-M + 0\).

11.3 Case of \(\tau \to +0\)

In this subsection \(\tau \to +0\) for the Schrödinger and Schrödinger-Pauli operators and \(\tau \to \pm M \pm 0\) for the Dirac operator. Consider the Schrödinger and Schrödinger-Pauli operators first.

\(^{94)}\) Cf. Example 10.2.
Example 11.11\footnote{95).}. Let $V > 0$ everywhere except $V(0) = 0$. Let conditions (8.1), (8.2)$_{1-3}$ and (9.1) be fulfilled with $\gamma = \epsilon_0|x|$, $\rho = |x|^m$, $\rho_1 = |x|^{m_1}$, $m_1 > 2m \geq 0$. Consider the Schrödinger operator and assume that $\tau \to +0$.

(i) Let

\begin{equation}
(11.11)_{1,2} \quad \mu \ll \tau^{(m+2)/2}h^{-(m+1)}, \quad \tau^{2m-m_1} \leq \epsilon(\mu h)^{2m}.
\end{equation}

Then (11.3) holds while the remainder estimate is $O(R)$ with defined by (11.4).

(ii) Let us replace (11.11)$_2$ by the opposite inequality, and assume (9.3)#. Then (11.3) is replaced by $N^{-}(\tau) \propto h^{-3}r^{-3(m+1)/(2m)}$ while $R = h^{-2}r^{(m+1)/2m}$.

Example 11.12\footnote{96).}. In the framework of Example 11.11 for the Schrödinger-Pauli operator under assumption (9.3)# both (11.6) and (11.7) hold.

We leave to the reader

Problem 11.13\footnote{97).}. Consider the Schrödinger and Schrödinger-Pauli operators

(i) In the same framework albeit with condition $m_1 > 2m$ replaced by $2m \geq m_1 \geq 0$. Assume that (9.3)# is fulfilled.

Then magnitude of $N^{-}(\tau)$ is described in Examples 11.11 and 11.12. Under proper non-degeneracy assumption (which we leave to the reader to formulate) derive the remainder estimate.

(ii) In the same framework as in (i) albeit in with $m_1 < 0$ (magnetic field is stronger in the center but there is no singularity), in which case the center can become a classically forbidden zone.

(iii) With other types of the behaviour at infinity.

Problem 11.14. For the Dirac operators derive similar results as $\tau \to \pm(M + 0)$.

\footnote{95) Cf. Example 11.2.}
\footnote{96) Cf. Example 11.3.}
\footnote{97) Cf. Problem 11.4.}
11.4 Case of $\tau \to -\infty$

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics with $\tau \to -\infty$.

In this subsection for the Schrödinger and Schrödinger-Pauli operators we consider asymptotics of eigenvalues tending to $-0$.

**Example 11.15**\(^{98}\). Let $X \ni 0$ and let conditions (8.1), (8.2)\(_{1-3}\), (9.1) and a week non-degeneracy assumption be fulfilled with $\gamma = \epsilon_0 \langle x \rangle$, $\rho = \langle x \rangle^m$, $\rho_1 = \langle x \rangle^m$, $-1 < m < 0$, $m_1 > m - 1$.

Consider the Schrödinger operator and assume that

\[
(11.12)_{1,2} \quad h \ll |\tau|^{(m+1)/(2m)}, \quad |\tau|^{m_1-2m} \leq \epsilon(\mu h)^{2m}.
\]

Then $\mathcal{N}^- (\tau) = O(h^{-3}|\tau|^{3(m+1)/(2m)})$ as $\tau \to -0$ with “$\asymp$” instead of “$= O$” if condition $V \leq -\epsilon \rho^2$ as $|x| \leq \epsilon$ fulfilled in some non-empty cone and $R = h^{-2}|\tau|^{(m+1)/m}$.

**Example 11.16**\(^{99}\). In the framework of Example 11.15 for the Schrödinger-Pauli operator under assumption (9.3)\(^*\)

\[
(11.13) \quad \mathcal{N}^- (\tau) = O(h^{-3}|\tau|^{3(m+1)/(2m)} + \mu h^{-2}|\tau|^{(m_1+m+3)/(2m)})
\]

and

\[
(11.14) \quad R = h^{-2}|\tau|^{(m+1)/m} + \mu h^{-1}|\tau|^{(m_1+2)/(2m)}
\]

with “$\asymp$” instead of “$= O$” if condition $V \leq -\epsilon \rho^2$ as $|x| \leq \epsilon$ fulfilled in some non-empty cone.

**Problem 11.17.** Consider the Schrödinger and Schrödinger-Pauli operators if with other types of the behaviour at 0.

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\(^{98}\) Cf. Example 10.2.

\(^{99}\) Cf. Example 10.2.
12 3D-case. Asymptotics of Riesz means for Schrödinger operators singular at a point with strong magnetic field

In this section we consider Riesz means for the 3-dimensional Schrödinger-Pauli operator. We leave it to the reader to treat the easier case of the Schrödinger operator.

12.1 The regular case

We would like to recall the results of Chapter 13 of [Ivr2] for the regular case, under some more restrictive assumptions. Namely, let us assume that

\[(12.1) \quad B(0, 1) \subset X,\]

in \(B(0, 1)\) conditions \((8.3)_{1,2}\) are fulfilled and

\[(12.2) \quad F \geq \epsilon_0.\]

Then in Chapter 13 of [Ivr2] the following statement was proven:

**Theorem 12.1.** Let \(d = 3\), let the Schrödinger-Pauli operator be self-adjoint and let conditions \((8.3)_{1,2}\), \((12.1)\) and \((12.2)\) be fulfilled. Moreover, let us assume that in \(B(0, 1)\)

\[(12.3) \quad V \leq \epsilon_0 \implies |\nabla \frac{V}{F}| \geq \epsilon_0.\]

Finally, let \(\psi \in C^K_0 (B(0, \frac{1}{2}))\). Then

\[(12.4) \quad \left| \int \psi(x) \left( e_\theta(x, x, \tau) - h^{-3} S_\phi(x, \mu h, \tau) \right) dx \right| \leq Ch^{-2+\vartheta}(1 + \mu h) \quad \forall \tau \leq \frac{1}{2} \epsilon_0\]

where

\[(12.5) \quad S(x, \mu h, \tau) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^+} \left( \tau - V - 2n\mu h F \right) \frac{1}{2} F \mu h \sqrt{g} \]
and for \( \vartheta \in [0, 1] \)

\[
(12.6) \quad S_\vartheta(x, \mu h, \tau) = \vartheta \tau_+^{\vartheta - 1} * S = \text{const} \sum_{n \in \mathbb{Z}^+} (\tau - V - 2n\mu hF)^{\frac{1}{2} + \vartheta} F \mu h \sqrt{g}.
\]

Remark 12.2. (i) This theorem also remains true without certain conditions for \( \mu \leq 1 \). This follows from routine semiclassical asymptotics.

(ii) Condition (12.3) is too restrictive; a weaker condition was used in Chapter 13 of [Ivr2]. In fact, no condition was necessary for \( \vartheta > 0 \). However, that is not very important in this Section and it could lead to more complicated expression for \( S_\vartheta(x, \mu, h, \tau) \).

(iii) For \( \vartheta > 1 \) it is difficult to provide explicit formula for \( S_\vartheta(x, \mu, h, \tau) \).

### 12.2 The singular case. I

Now let us consider the Schrödinger-Pauli in the case when there is a singularity at 0. Namely, we replace assumptions (8.3)\(_{1,2} \), (12.2) and (12.3) by

\[
(12.7)_{1,2} \quad |D^\sigma g^{jk}| \leq c|x|^{-|\sigma|}, \quad |D^\sigma V_j| \leq c|x|^{\beta-|\sigma|+1},
\]

\[
(12.7)_3 \quad |D^\sigma V| \leq c|x|^{2\alpha-|\sigma|} \quad \forall \sigma : |\sigma| \leq K
\]

and

\[
(12.8) \quad F \geq \epsilon_0|x|^\beta,
\]

\[
(12.9) \quad V \leq \epsilon_0|x|^{2\alpha} \implies |\nabla \frac{V}{F}| \geq \epsilon_0|x|^{2\alpha-\beta-1}
\]

where we assume that \( \alpha \in (-1, 0] \), \( \beta > 2\alpha \).

Our standard goal is to derive the asymptotics of

\[
(12.10) \quad \int \psi(x) e_\vartheta(x, x, 0) \, dx
\]

but instead we derive the asymptotics of

\[
(12.11) \quad \int \psi(x) \left( e_\vartheta(x, x, 0) - e_\vartheta^0(x, x, 0) \right) \, dx
\]
where $A^0$ is the same Schrödinger-Pauli operator albeit with $\mu = 0$ in $B(0, 1)$. The corresponding asymptotics for this operator are due to Section 12.5 of [Ivr2] under appropriate conditions. First of all we treat the general situation without propagation of singularities arguments.

Let us consider the operator $A$ in $B(\bar{x}, \frac{1}{2} r)$ with $r = |\bar{x}|$. Then the routine rescaling procedure results in the operator with $\hbar = hr^{-\alpha - 1}$ and $\mu' = \mu r^{\beta + 1 - \alpha}$ and one should compare these quantities with 1. For $h \leq 1$ one can apply the semiclassical approach to $\int \bar{\psi} e_c(x, x, 0)\ dx$ which provides the following remainder estimate

(12.12) \[ R_1 = r^{2\alpha \vartheta} (h^{-1} + \mu') h^{-1+\vartheta} = h^{-2+\vartheta} r^{\alpha \vartheta - \vartheta + 2\alpha + 2} + \mu h^{-1+\vartheta} r^{\alpha \vartheta - \vartheta + \beta + 2} \]

where $\bar{\psi}$ is an element of the partition of unity.

On the other hand, let us apply semiclassical asymptotics to

(12.13) \[ \int \bar{\psi} Be^{\eta}_{\beta-1}(x, x, 0)\ dx \]

where $B = A - A^0$ and $A^\eta = \eta A^0 + (1 - \eta)A$. Then the remainder estimate is $\mu' h^{-1} R_1$ (provided $\mu' \leq 1$). This is justified for $\vartheta \geq 1$; for $0 < \vartheta < 1$ one should apply the arguments of Subsection 12.5.4 of [Ivr2] to get the remainder estimate $(\mu' h^{-1})^\vartheta R_1$ (see the rigorous analysis below).

Both of these remainder estimates are better than $R_1$ if $\mu' < h'$. Therefore one should introduce $r_*$ with $\mu' = h$ and apply the first and second approaches for $r > r_*$ and for $r < r_*$ respectively. Thus, $r_* = (h/\mu)^{1/(\beta + 2)}$. However, one should remember the condition $h \leq 1$ which yields two different situations:

(A) The magnetic field is not very strong:

(12.14) \[ \mu^{\alpha + 1} h^{\beta + 1 - \alpha} \leq 1. \]

(B) The magnetic field is rather strong:

(12.15) \[ \mu^{\alpha + 1} h^{\beta + 1 - \alpha} \geq 1. \]

In the former case (A) surely $h \leq 1$ for $r \geq r_*$. Now we want a better implementation of the above idea. Let us introduce $r_2 = \mu^{-1/(\beta + 1 - \alpha)}$ from the condition $\mu' = 1$. Due to (12.14) $r_* \geq r_2$. 123
Let us apply straightforward semiclassical approximation only in the zone \( \{ r^* \leq |x| \leq \varepsilon \} \). Then the contribution of this zone to the remainder estimate is \( O(R_1^*) \) with

\[
R_1^* = \begin{cases} 
  h^{-2+\vartheta} + \mu h^{-1+\vartheta} & \text{for } \vartheta < \tilde{\vartheta}_0, \\
  h^{-2+\vartheta} \log \mu + \mu h^{-1+\vartheta} & \text{for } \vartheta = \tilde{\vartheta}_0, \\
  h^{-2+\vartheta} \mu (1-\alpha)(\vartheta-\tilde{\vartheta}_0)/(\beta+1-\alpha) + \mu h^{-1+\vartheta} & \text{for } \vartheta > \tilde{\vartheta}_0
\end{cases}
\]

with the same \( \tilde{\vartheta}_0 = (2\alpha + 2)/(1 - \alpha) \) as in Section 12.5 of [Ivr2].

Let us now treat the contribution of \( B(0, r^*) \). Making the rescaling \( x_{\text{new}} = r^*-1 x \) we are in exactly the framework of Section 12.5 of [Ivr2] with \( h = h\mu^{(\alpha+1)/(\beta+1-\alpha)} \leq 1 \). Let us apply the results of this section. We should consider the general situation and the situation of escape condition (see Definition 12.5.14 of [Ivr2]).

Let us first consider the general situation:

(i) First of all, one can easily see that the contribution of \( B(0, r^*) \) to the remainder is \( O(h^{-2+\vartheta}) \) for \( \vartheta \leq \tilde{\vartheta}_0 \).

(ii) On the other hand, for \( \tilde{\vartheta}_0 \leq \vartheta \leq \bar{\vartheta} \) (with \( \bar{\vartheta} = \infty \) for \( \alpha + \beta + 1 \geq 0 \)) the contribution of \( B(0, r^*) \) to the remainder estimate is given by the first or second line of (12.5.74) of [Ivr2] and is \( O(r^* h^{(\alpha+\beta+1)}/(\beta+2) \in \mathbb{Z}^+ \).

(iii) Finally, for \( \vartheta > \bar{\vartheta} \) the contribution of \( B(0, r^*) \) to the remainder is given by the third line in (12.5.74) of [Ivr2] and is \( O(h^{(\alpha+\beta+1)}/(\alpha+1) \mu^{\vartheta}) \).

On the other hand, under escape condition (see Definition 12.5.14 of [Ivr2]) there is no adjustment in the cases (i) and (iii) but in the case (ii) the contribution of \( B(0, r^*) \) is \( O(r^* h^{(\alpha+\beta+1)}/(\alpha+1) \mu^{\vartheta}) \).

However, to be able to enjoy these remainder estimates in cases (ii) and (iii) one should assume that either \( \vartheta \leq \tilde{\vartheta}_1 \) or the homogeneity condition is fulfilled. For the sake of simplicity in the second case we assume that

\[
(12.17) \quad g^{jk} = g^{0jk} + g^{1jk}, \quad V = V^0 + V^1 \text{ and } g^{0jk}, g^{1jk}, V^0, V^1, V_j \text{ are } C^K \text{ functions (outside of } 0), \text{ positively homogeneous of degrees } 0, \beta + 1 - \alpha, 2\alpha, \alpha + \beta + 1, \beta + 1 \text{ respectively.}
\]

100) With \( hr^* - \alpha - 1 \) instead of \( h \) and with the additional factor \( r^* 2^{\alpha \vartheta} \).
In the first case we only assume that

\[(12.18) \quad g_{jk} = g_{0jk} + g_{1jk}, \quad V = V^0 + V^1 \quad \text{and} \quad g_{0jk}, \quad V^0, \quad \text{are } \mathcal{C}^K \quad \text{functions (outside of 0), positively homogeneous of degrees } 0, 2α \quad \text{respectively and}
\]

\[
|D^\sigma g_{1jk}| \leq c|x|^{β+1-α-|σ|}, \quad |D^\sigma V^1| \leq c|x|^{β+1+α-|σ|} \quad \forall σ : |σ| \leq K
\]

in addition to (12.14).

Then the terms generated by the singularity are

\[(12.19) \quad \sum_{j,k} \omega_{jk} h^{(2α+(j+k)(β-α+1))(α+1)^{-1}} \mu^j
\]

while the other terms are semiclassical (but not necessary Weyl). To avoid computational difficulties we will only consider the case \(ϑ ≤ 1\) in the final statement. The calculations for \(ϑ > 1\) are left to the reader.

**Theorem 12.3.** (i) Let the Schrödinger-Pauli operator be self-adjoint and let conditions (8.2), (12.7)_{1-3}, (12.8), (12.9) and (12.18) be fulfilled. Let \(1 ≥ ϑ ≥ ϑ_0\). Then for \(μ^{α+1}h^{β+1-α} ≤ 1\) the following estimate holds:

\[(12.20) \quad R_ϑ := \left| \int ψ(x)\left(e_ϑ(x,x,0) - e_0^0(x,x,0) - h^{-3}S_ϑ(x,μh,0) + h^{-3}ξ_{1,0}(x)\right) dx\right| ≤
\]

\[
Ch^{-2+ϑ}(\frac{h}{μ})^{(αϑ-ϑ+2α+2)/(β+2)} + Cμh^{-1+ϑ}.
\]

(ii) Moreover, under escape condition (see Definition 12.5.14 of [Ivr2]) the following estimate holds

\[(12.21) \quad R_ϑ ≤ Ch^{-2+ϑ} μ^{-(αϑ-ϑ+2α+2)/(β+1-α)} + Cμh^{-1+ϑ}
\]

**Remark 12.4.** Let us compare the remainder estimates with the Scott correction \(ωh^l\) with \(l = 2αϑ/(α + 1)\). One can see easily that for \(μ^{α+1}h^{β+1-α} ≤ 1\) the first terms in these remainder estimates are surely less than the Scott term. Therefore the remainder estimate is less than the Scott term if and only if \(μ ≤ h^l\) which means exactly that \(μ ≤ h^{(αϑ+α+1-ϑ)/(α+1)}\) and the idea to treat \((e_ϑ - e_0^0)\) instead of \(e_ϑ\) is reasonable only in this case.
12.3 The singular case. II

Let us consider the case (12.15). In this case we should treat

\[ R_\vartheta^* = | \int \psi(x) \left( e_\vartheta(x, x, 0) - h^{-3} S_\vartheta(x, \mu h, 0) \right) dx |. \]

Let us pick \( r_0 = h^{1/(\alpha+1)} \). In the zone \( \{|x| \geq r_0\} \) the routine semiclassical technique is applicable and the contribution of this zone to the remainder estimate does not exceed \( CR \) with

\[
R = \begin{cases} 
\mu h^{-1+\vartheta} & \text{for } (3\alpha + 1)\vartheta + \beta - 2\alpha > 0 \\
\mu h^{-1+\vartheta} \log \mu & \text{for } (3\alpha + 1)\vartheta + \beta - 2\alpha = 0 \\
\mu h^{2\alpha\vartheta/(\alpha+1)} & \text{for } (3\alpha + 1)\vartheta + \beta - 2\alpha < 0
\end{cases}
\]

So we need to estimate the contribution of \( B(0, r_0) \). The routine estimate due to Chapter 8 and standard variational estimates gives a very poor result (which the reader can derive if desired). So we want to improve the result under certain conditions. Namely, let us assume that conditions \((12.7)_{1,2}\) are fulfilled.

The rest of this subsection is devoted to the proof of

**Proposition 12.5.** Let the Schrödinger-Pauli operator \( A \) be self-adjoint and let conditions (8.2), (12.1) (12.7)\(_{1-3}\) and (12.8) be fulfilled with \( \beta = 0 \). Let \( h = 1 \), \( \mu \geq 1 \). Then the operator \( A \) in \( B(0, 1) \) (with the Dirichlet boundary conditions) is semibounded from below uniformly with respect to \( \mu \) and the estimate

\[ N(\tau) \leq C \mu \tau^{1/2} + C \tau^{3/2} \]

holds for \( \tau \geq 1 \).

Then the standard arguments of Chapter 9 of [Ivr2] yield the following theorem (details are left to the reader):

**Theorem 12.6.** Let the Schrödinger-Pauli operator \( A \) be self-adjoint and let conditions (8.2), (12.1) (12.7)\(_{1-3}\) and (12.8) be fulfilled with \( \beta = 0 \). Then for \( \mu^{\alpha+1} h^{1-\alpha} \geq 1 \) the following estimate holds:

\[ R_\vartheta \leq C \mu h^{-1+\vartheta}. \]
Proof of Proposition 12.5. Without any loss of the generality one can assume that $F_1 = F_2 = 0$ and $V_2 = V_3 = 0$, $g^{13} = g^{23} = 0$. Then $V_1 = V_1(x')$, $x' = (x_1, x_2)$. Surely in this case the standard variational estimates are much better than in the general case but we also need to improve them.

Let $H = L^2(B(0, 1))$ and on $K \subset C^2_0(B(0, 1))$ let the operator $A - \tau I$ be negative definite. We should prove that

$$\dim K = 0$$

for $\tau \leq -C_0$ and $\dim K \leq C\mu \tau^{\frac{1}{2}} + C\tau^\frac{3}{2}$ for $\tau \geq 1$.

Let us consider

$$A - \tau I = g^{33}D_3^2 + \sum_{1 \leq j,k \leq 2} P_j g^{jk} P_k - \mu F + V - \tau I.$$  

Observe that the operator

$$A' = \sum_{1 \leq j,k \leq 2} P_j g^{jk} P_k - \mu F$$

is semibounded from below on $H$ and, moreover, that $A' + C_0I \geq \epsilon_0 A''$ where

$$A'' = P_1^2 + P_2^2 - \mu F''$$

and the scalar intensity $F''$ is calculated for the Euclidean metrics; then the operator $A''$ does not contain $x_3$. On the other hand, let us note that $V \geq -c_0|x_3|^{2m}$ and the quadratic form $\epsilon\|D_3^2 u\|^2 - c\|W u\|^2$ is semibounded from below on $L^2([-1, 1])$ for every $\epsilon > 0$, $W = |x_3|^{\alpha}$ since $\alpha > -1$.

Therefore for appropriate $\epsilon_0 > 0$ and $C_0$ the operator $\epsilon_0 D_3^2 + A'' - \tau - C_0$ should be negative definite on $K$. One can replace the ball by the cylinder $\Omega \times [-1, 1]$ where $\Omega \subset \mathbb{R}^2$ is a unit circle and separate variables $x'$ and $x_3$. So, our original problem is reduced to a problem for the operator $A''$ in the circle (and for the operator $\epsilon_0 D_3^2$ on $[-1, 1]$) and estimate (12.25) follows from the well known estimate

$$N(\tau) \leq C\mu + C\tau$$

for this operator for $\tau > 0$ (and the routine estimate for the one-dimensional operator).
12.4 The case $d = 2$

Let us discuss briefly the case $d = 2$. Not going into details of the regular case, recall that in the framework of Subsection 12.1 the remainder estimate is $O(\mu^{-1-\vartheta}h^{-1+\vartheta})$. Let us attack the singular case assuming that conditions of Subsection 12.2 are fulfilled (including condition (12.14)). Then $r^* \geq r_s$, where $r_0, r_s, r^*$ are introduced in Subsection 12.2. There are again two possibilities:

(i) For $0 \leq \vartheta \leq \vartheta^* = (\beta + 2)(\beta + 2 - 2\alpha)^{-1}$ the contribution of the zone $\{r^* \leq |x| \leq r_1\}$ to the remainder is $O(\mu^{-1-\vartheta}h^{-1+\vartheta})$ with the factor $\log \mu$ for $\vartheta = \vartheta^*$. However, since $\vartheta^* < \bar{\vartheta}_0$ we can apply the standard semiclassical asymptotics and no Scott correction term appears. The final remainder estimate is $O(\mu^{-1-\vartheta}h^{-1+\vartheta})$ with the factor $\log \mu$ for $\vartheta = \vartheta^*$.

(ii) Further, for $\vartheta^* < \vartheta \leq \bar{\vartheta}_0$ the best possible remainder estimate is $O(h^{-1-\vartheta}\mu^{(\alpha+\beta+1-\vartheta)(\beta+1-\alpha)^{-1}})$ which is also appears as a contribution of $B(0, r_3)$ to the remainder and there is still no Scott correction term.

(iii) For $\vartheta > \bar{\vartheta}_0$ it is not important which regular remainder estimate (namely, $O(\mu^{-1-\vartheta}h^{-1+\vartheta})$ or $O(h^{-1+\vartheta})$) was applied in the zone $\{r^* \leq |x| \leq r_1\}$. So, the arguments of Subsection 12.2 remain valid and for $\vartheta \geq \bar{\vartheta}_0$ we can obtain the same type of remainder estimate as before with the Scott correction term(s).

We have not succeeded in getting a variational estimate in the case $h = 1$, $\mu \geq 1$ and thus we have no extension of the results of Subsection 12.3.

13 Appendices

13.A 1D Schrödinger operator

Operators of the type we consider here studied by many authors. Related statements could be found in many books, including Chapter XIII, Part 2 of N. Danford and J. T. Schwarz [DS], M. S. Birman and M. Z. Solomyak [Bir] and V. Maz’ya and I. Verbitsky [MV].

**Proposition 13.1.** Let us consider the operator

\[ a_\varepsilon = D_t g_\varepsilon(t) D_t + \varepsilon^{-1} V_\varepsilon(t) \]
in $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ with $D = D_t$, $V_\epsilon(x) = V(\frac{x}{\epsilon})$, etc., $t \in \mathbb{R}$,

$$(13.2)\quad \epsilon_0 \leq g \leq c, \quad |V| \leq \rho^2, \quad 0 \leq \rho \leq c, \quad \|\rho\|_{L^1} + \|\rho^2 t\|_{L^1} \leq c.$$ 

Then

(i) The number of negative eigenvalues of the operator $a$ does not exceed $C_0$ for $|\epsilon| \leq 1$.

(ii) The number of negative eigenvalues of the operator $a$ does not exceed 1 for $|\epsilon| \leq \epsilon$ with a small enough constant $\epsilon > 0$.

(iii) Further, let us assume that

$$(13.3)\quad W = -\frac{1}{2} \int_{-\infty}^{+\infty} V(t) \, dt \geq \epsilon_0.$$ 

Then for $\epsilon \in (0, \epsilon]$ there is exactly one negative eigenvalue $\lambda(\epsilon)$ and

$$(13.4)\quad -\epsilon_2 \geq \lambda \geq -c_1.$$ 

(iv) Furthermore, let us assume that (13.3) holds and

$$(13.5)\quad |g - 1| \leq c\rho.$$ 

Then

$$(13.6)\quad |\lambda + W^2| \leq C_0\epsilon.$$ 

(v) Moreover, let us assume that (13.3) holds and that $g$ and $V$ depend on the parameter $z \in \Omega$ and

$$(13.7)\quad |D^\alpha_z g| \leq c, \quad |D^\alpha_z V| \leq c\rho^2 \quad \forall \alpha : |\alpha| \leq K.$$ 

Then

$$(13.8)\quad |D^\alpha_z \lambda| \leq C_0;$$ 

moreover, under the condition

$$(13.9)\quad |D^\alpha_z g| \leq c\rho \quad \forall \alpha : |\alpha| \leq K$$

we obtain that

$$(13.10)\quad |D^\alpha_z (\lambda + W^2)| \leq C_0\epsilon.$$
Finally, let \( v \in H, |v| = 1 \) be an appropriate eigenfunction of \( a \) with eigenvalue \( \lambda \). Then

\[
(13.11)_{1-3} \quad |D_0 v| \leq C_0, \quad |D_0 D_t v| \leq C_0, \quad |D_0 v|_\infty \leq C_0,
\]

where \( |.|_p \) means the \( L^p \)-norm and we skip \( p = 2 \) in this notation.

**Proof.** Statement (i) follows from the fact that the operator \( \rho^s(D_0^2 + 1)^{-s} \) is compact in \( H \) for any \( s > 0 \).

In order to prove Statement (ii) let us consider the quadratic form \( Q(u) = \langle au, u \rangle \) on the subspace \( H_1 = \{ u \in H, \int_{-\varepsilon}^{\varepsilon} u \ dt = 0 \} \) of codimension 1.\(^{101}\) Obviously \( |u(t)| \leq 3(|t| + \varepsilon)^{\frac{1}{2}} |D_t u| \) for \( u \in H_1 \) and therefore

\[
|\langle V_e u, u \rangle| \leq C_0 \varepsilon^2 |D_t^2 u|^2
\]

in virtue of (13.2). Then (13.2) yields that the quadratic form \( Q(u) \) is positive definite on \( H_1 \) for \( |\varepsilon| \leq \sigma_2 \) and therefore \( a \) has no more than one negative eigenvalue \( \lambda \). Moreover, for arbitrary \( u \in H \) the inequality

\[
|u(t)| \leq \sigma |D_t u| + C_\sigma |u|
\]

with arbitrarily small \( \varepsilon > 0 \) yields that

\[
|\langle V_e u, u \rangle| \leq c_0 \sigma \varepsilon |D_t u|^2 + C_\sigma \varepsilon |u|^2
\]

and hence \( Q(u) \) is uniformly semibounded from below and therefore

\[
(13.12) \quad \lambda \geq -C_0.
\]

Let \( v \) be the corresponding eigenfunction with \( |v| = 1 \) (if there exists a negative eigenvalue). Then obviously \( |v(t)| \leq C_0 \) and then \( |D_t v(t)| \leq C_0 \) and hence \( |v(t) - v(0)| \leq C_0 |t| \). Then (13.2), (13.5) yield that

\[
|Q(v) - \tilde{Q}(v)| \leq C_1 \varepsilon
\]

for the quadratic form \( \tilde{Q}(u) = |D_t u|^2 - W|u(0)|^2 \). Therefore \( \lambda \geq \lambda - C_1 \varepsilon \) where \( \lambda \) is the lower bound of \( \tilde{Q}(u)|u|^{-2} \) at \( H \).

One can apply the same arguments to the eigenvalues and eigenfunctions of \( \tilde{Q} \); as a result we obtain that \( \lambda \geq \lambda - C_1 \varepsilon \). On the other hand, one can see easily that \( \lambda = -\frac{1}{3} W^2 \) if \( W > 0 \) (otherwise \( \tilde{Q} \) is non-negative definite) and \( \tilde{v}(t) = \frac{W}{2} \exp(-\frac{1}{2} W|t|) \) and hence we obtain that for \( W > 0 \) there is a negative eigenvalue and (13.6) holds.

\(^{101}\) One can consider the subspace \( \{ u \in D(a), u(0) = 0 \} \) as well.
Moreover, if (13.5) is violated then one can treat the quadratic form
\[ C_0|D^2_t| + \varepsilon^{-1}(V_\varepsilon u, u) \]
instead of the original one and since (13.4) holds for this form it remains true for the original one.

So all the statements excluding those associated with derivatives on \( z \) are proven\(^{102}\).

The proof of (13.8), (13.11) is standard, due to K. O. Friedrichs [Fr]. Let these estimates be proven for \(|\alpha| \leq n\); then applying the operator \( \partial^\alpha_z \) with \(|\alpha| = n\) to the equation
\[ (a - \lambda)v = 0 \]
we obtain
\[ \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!}(a - \lambda)^{(\alpha - \beta)}v^{(\beta)} = 0 \]
with \( u^{(\alpha)} = \partial^\alpha_z u \).

Let us multiply this equation by \( v \). Then terms with \( v^{(\alpha)} \) disappear and we obtain terms with \(|\beta| < n\)
\[ \langle g_\varepsilon^{(\alpha - \beta)}D_t v^{(\beta)}, D_t v \rangle, \quad \varepsilon^{-1} \langle V_\varepsilon^{(\alpha - \beta)}v^{(\beta)}, v \rangle, \quad \lambda^{(\alpha - \beta)} \langle v^{(\beta)}, v \rangle. \]

Terms of the first and second types are bounded in virtue of (13.11)\(^2\), (13.11)\(^3\) respectively for \(|\beta| < n\). Terms of the third type are bounded for \( \beta \neq 0 \) by (13.8), (13.11)\(_1\). Therefore the remaining term
\[ -\lambda^{(\alpha)}|v|^2 \]
should also be bounded and (13.8) holds for \(|\alpha| = n\). Let us consider equation (13.14); we now multiply it by \( w = v^{(\alpha)} \). We obtain terms with \(|\beta| \leq n\)
\[ \langle g_\varepsilon^{(\alpha - \beta)}v^{(\beta)}, w \rangle, \quad \varepsilon^{-1} \langle V_\varepsilon^{(\alpha - \beta)}v^{(\beta)}, w \rangle, \quad \lambda^{(\alpha - \beta)} \langle v^{(\beta)}, w \rangle. \]

For \(|\beta| < n\) terms of the first and second type do not exceed \( C|D_t w| \) and \( C|w|_\infty \) due to (13.11)\(_2\), (13.11)\(_3\). Finally, terms of the third type for \(|\beta| < n\) do not exceed \( C\|w\| \) due to (13.8) and (13.11)\(_1\). Thus
\[ |\langle (a - \lambda)w, w \rangle| \leq C|D_t w| + C|w| \]

\(^{102}\) One can easily prove that for \( W < 0 \) and small enough \( \varepsilon \) the operator \( a \) is non-negative definite. We think that it would be nice to treat the case \( W = 0 \). However we are not an expert here.
because \(|w|_\infty \leq C|D_t w| + C|w|\). Taking into account that \(\lambda \leq -\epsilon_0\) we obtain from this inequality that

\[
|D_t w|^2 + |w|^2 \leq C\epsilon^{-1}|\langle V_\epsilon w, w \rangle| + C.
\]

Let us assume that \(v(0) = 1\). Surely, we should reject the condition \(|v| = 1\) but our above arguments yield that \(|v| \asymp |v(0)|\). Then \(w(0) = 0\) and \(|w(t)| \leq |t|^{3/2}|D_t w|\) and therefore \(|\langle V_\epsilon w, w \rangle| \leq C\epsilon^2|D_t w|\) and therefore (13.16) yields (13.11) for \(n = 1\) and (13.11) for \(n \neq 0\). Moreover, if we replace \(v(t)\) by \(v(0)\) in this term with \(\beta = 0\); we then obtain \(W^{(n)}|v(0)|^2\) and under additional the restriction \(W = \text{const}\) this term vanishes. Then induction on \(n\) yields that \(|\lambda| \leq C_0\epsilon\) under this restriction. So under this restriction (13.10) holds. However one can reduce the general case to the case \(W = 1\) by introducing \(t' = tW^{-1}\) and multiplying \(a\) by \(W^2\).

\section*{Remark 13.2}

Applying the above results one can find \(v\) in the form

\[
v = \exp\left(\int_0^t \phi_\epsilon(t') dt'\right) \cdot (1 + \epsilon^2 \psi_\epsilon + ...)
\]

where the number of terms depends on \(m\) and \(\lambda = -W^2 + \mu\epsilon + \cdots\) with \(\partial_t \phi = V\) and one can obtain \(\mu \neq 0\) in the generic case; so estimate (13.6) is the best possible estimates without this correction term. Therefore (10.33) remains true with \(\lambda(x')\) replaced by \(-W(x')^2\) provided \(m \leq -2\) and \(\rho(x) = \langle x \rangle^m, \gamma(x) = \gamma_1(x) = \langle x \rangle\).

For \(m > -2\) this is correct with the remainder estimate \(O(\eta^{(m+2)(2m+1)-1})\) coinciding with the principal part for \(m = -1\) (in the framework of Remark 10.19).
Proposition 13.3. (i) Under condition (10.66) the operator $L$ has a finite number of negative eigenvalues.

(ii) Moreover, if this condition is fulfilled for all $x$ then there is at most one negative eigenvalue.

(iii) On the other hand, if

\[(13.18) \quad W \leq -\left(\frac{1}{4} + \epsilon\right)|x|^{-2} \quad \forall x: x \geq C\]

then there is an infinite number of negative eigenvalues.

Proof. To prove Statements (i) and (ii) one needs to prove the estimate

\[(13.19) \quad |u'|^2 \geq \frac{1}{4}||x|^{-1}u|^2 \quad \forall u : u(0) = 0\]

where $|u|$ is the $L^2(\mathbb{R}^+)$-norm. However, the left side is equal to

\[|x^{\frac{1}{2}}(ux^{-\frac{1}{2}})' + \frac{1}{2}x^{-1}u|^2 = |x^{\frac{1}{2}}(ux^{-\frac{1}{2}})'|^2 + \frac{1}{4}||x|^{-1}u|^2\]

provided $u = o(x^{\frac{1}{2}})$ as $x \to 0$.

To prove Statement (iii) it is sufficient to prove that the inequality $|u'|^2 \leq (\frac{1}{4} + \epsilon)||x|^{-1}u|^2$ is fulfilled on some subspace of $L^2([1, \infty))$ of infinite dimension. It is sufficient to prove that for any $n$ this inequality is fulfilled on some function supported in $[L^n, L^{n+1}]$ with sufficiently large $L$.

Further, due to homogeneity it is sufficient to consider only $n = 0$. Substituting $u = x^{\frac{1}{2}}v, x = e^t$ we obtain that it is sufficient to fulfill the inequality $|v'|^2 \leq \epsilon|v|^2$ with some $v$ such that $v(0) = v(\log L) = 0$. But this is obvious provided $L$ is large enough. $\square$

13.B 1D Schrödinger operator. II

We consider operator

\[(13.20) \quad b_\epsilon = D^2 + \epsilon V(x)\]

with

\[(13.21) \quad |V| \leq \rho^2 = \langle x \rangle^{-2q}, \quad V \leq -\epsilon_0 \rho^2 \quad \text{for } |x| \geq c,\]

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0 < q ≤ 1, and ε > 0.

We are interested in \( n_\varepsilon(\eta) \), the number of eigenvalues of \( b_\varepsilon \) which are less than \( -\eta \). Consider first the corresponding Weyl’s expression

\[
(13.22) \quad n_\varepsilon^W(\eta) := (2\pi)^{-1} \int (\varepsilon V(x) - \eta)^{\frac{1}{2}} \, dx.
\]

**Proposition 13.4.** (i) If \( n_\varepsilon^W(\eta) \geq C_0 \) then \( n_\varepsilon(\eta) \approx n_\varepsilon^W(\varepsilon, \eta) \).

(ii) If \( n_\varepsilon^W(\varepsilon, \eta) \leq C_0 \) then \( n_\varepsilon(\eta) \leq C_1 \).

**Remark 13.5.** Obviously \( n_\varepsilon^W(\eta) \approx \varepsilon^{(2q-1)/2q} \eta^{-(1-q)/2q} \) and \( n_\varepsilon^W(\eta) \leq C_0 \) if and only if \( \eta \geq c_0 \varepsilon^{(2q-1)/(1-q)} \).

**Proof of Proposition 13.3.** One can easily prove Statement (i) using our semiclassical theory.

On the other hand, one can easily prove Statement (ii) using variational methods, and covering \( \mathbb{R} \) by a finite number of intervals \([L_k, L_{k+1}]\) and \([-L_{k+1}, -L_k]\) with \( k = 1, \ldots, n - 1, [L_n, \infty] \) and \([-\infty, -L_0]\) and \([-L_0, L_0]\) such that \( L_{k+1} = \varepsilon_0 L_k^{1/2}, L_0 = 1, L_n \geq c_0 \varepsilon^{-1/2q} \).

We leave the easy details to the reader.

Now we need to figure out when \( n_\varepsilon(\eta) \geq 1 \). To do so we need to evaluate the lowest eigenvalue \( \lambda(\varepsilon) < 0 \) of operator (13.20).

**Proposition 13.6.** Let \( V \in \mathcal{L}^1(\mathbb{R}) \) and \( W > 0 \). Then

\[
(13.23) \quad \lambda(\varepsilon) = -\varepsilon^2(W^2 + o(1)) \quad \text{as } \varepsilon \to 0
\]

with \( W \) defined by (13.3).

**Remark 13.7.** (i) Since after scaling \( x \mapsto x/\varepsilon \) and multiplication by \( \varepsilon^{-1} \) with \( \varepsilon \) operator (13.20) becomes (13.1), this is consistent with (13.6).

(ii) For \( V = -\langle x \rangle^{-2q} \) we have \( W < \infty \) if and only if \( q > \frac{1}{2} \).

**Proof of Proposition 13.6.** (a) We can apply Proposition 13.1 for \( V \geq 0, V = 0 \) as \( |t| \geq \varepsilon^{-1} \). Therefore \( b_\varepsilon' = D^2 + V'(x) \geq -C\varepsilon^2 \) where \( V' = V(x) \) as \( |x| \leq t, V(x) = 0 \) as \( |x| \geq t \). Indeed it is true for \( V \) replaced by \( -C \rho(x)^2 \).
Consider $t_n = 2^n$ and consider $V_0(x) = V(x)$ as $|x| \leq t_0$, $V_n(x) = V(x)$ as $t_{n-1} \leq |x| \leq t_n$ and vanishing on all other segments. Consider

(13.24) \[ \sigma_n > 0, \quad \sum_n \sigma_n \leq 1 \]

Then

$$b_\varepsilon \geq \sum_n b_{n,\varepsilon}, \quad b_{n,\varepsilon} = \sigma_n D^2 + \varepsilon V_n(x).$$

Scaling $x \mapsto x/t_n$ we have $b_{n,\varepsilon} \mapsto \sigma_n t_n^{-2}[D^2 + \sigma_n^{-1} \varepsilon t_n^2 \rho(t_n)^2 U_n(x)]$, with $U_n(x) = \rho(t_n)^{-2} V_n(x/t_n)$ and if $\sigma_n^{-1} \varepsilon t_n^2 \rho(t_n)^2 \leq 1$ we can apply the above estimate to the operator in the brackets. On the other hand, it is greater than $-C \varepsilon \sigma_n^{-1} \rho(t_n)^2$ and we can apply this estimate even without this condition; so we arrive to $b_{n,\varepsilon} \geq \sigma_n^{-1} \varepsilon t_n^2 \rho(t_n)^4$ and therefore $\|b_\varepsilon\| \geq -C \varepsilon^2$ provided

(13.25) \[ \sum_n \sigma_n^{-1} \varepsilon^2 t_n^2 \rho(t_n)^4 \leq C_0. \]

Picking up $\sigma_n = \varepsilon_0 t_n \rho(t_n)^2$ we satisfy both (13.24) and (13.25).

(b) Consider now $t$ such that $t \rho(t)^2 \leq \delta^2$. Then $|W_1 - W| \leq C \delta^2$ with $W_1 = -\frac{1}{2} \int_t^\infty V(x) \, dx$. Therefore $b_\varepsilon = b_{1,\varepsilon} + b_{2,\varepsilon}$ with $b_{1,\varepsilon} = (1 - \delta) D^2 + \varepsilon V_1(x)$, $b_{2,\varepsilon} = \delta D^2 + \varepsilon V_2(x)$. Applying Proposition 13.1 to $b_{1,\varepsilon}$ and the results of Part (a) to $b_{2,\varepsilon}$ we conclude that $b_\varepsilon \geq -\varepsilon^2 (W^2 + C \delta) \implies \lambda(\varepsilon) \geq -\varepsilon^2 (W^2 + C \delta)$.

Similarly, $b_{3,\varepsilon} = (1 + \delta) D^2 - V_1(x) \geq b_\varepsilon + b_{4,\varepsilon}$ and applying Proposition 13.1 to $b_{3,\varepsilon}$ and the results of Part (a) to $b_{4,\varepsilon}$ we conclude that $\lambda(\varepsilon) \leq -\varepsilon^2 (W^2 - C \delta)$.

Since we can take $\delta > 0$ arbitrarily small we arrive to (13.23).

Let $0 < q \leq \frac{1}{2}$. Then the integral defining $W$ in (13.23), diverges (logarithmically, as $q = \frac{1}{2}$).

**Proposition 13.8.** Let $0 < q < \frac{1}{2}$. Then

(i) $\lambda \geq -\varepsilon^{1/(1-q)}$.

(ii) Assume that $V(x) \sim V^0(x)$ as $|x| \to \infty$ where $V^0(x) = V \pm |x|^{-2q}$ as $\pm x > 0$. Let either $V_+ < 0$ or $V_- < 0$ and let $\mu < 0$ be the lowest eigenvalue of the operator $a^0 = D^2 + V^0(x)$. Then

(13.26) \[ \lambda = \varepsilon^{1/(1-q)} (\mu + o(1)) \quad \text{as} \quad \varepsilon \to 0. \]
Proof. Observe first that for $0 < q < \frac{1}{2}$ operator $a^0$ is properly defined and semibounded from below and in the framework of Statement (ii) it has an infinite number of negative eigenvalues.

(i) Replacing $V$ by $-C|x|^{-2q}$ and using scaling $x \mapsto c_\epsilon^{-1/2(1-q)}x$ we arrive to operator $\epsilon^{1/1-q}a^0 \geq -C\epsilon^{1/1-q}$. Thus we arrive to Statement (i).

(ii) Observe that in the framework of Statement (ii)

$$b_\epsilon \geq b_{1,\epsilon} + b_{2,\epsilon}$$

with

$$b_{1,\epsilon} = (1 - \delta)D^2 - \epsilon(V^0(x) + \delta|x|^{-2q}), \quad b_{1,\epsilon} = \sigma D^2 - \epsilon U(x)$$

with arbitrarily small $\sigma > 0$ and $U$ supported in $[-t, t]$ with $t = t(\delta)$. Then $b_{1,\epsilon} \geq (\mu_1 - C\delta)\epsilon^{1/(1-q)}$, $b_{2,\epsilon} \geq C(t, \delta)\epsilon^2$ and therefore $\lambda(\epsilon) \geq (\mu_1 - 2C\delta)\epsilon^2$.

Similarly, one can prove that $\lambda(\epsilon) \leq (\mu_1 + 2C\delta)\epsilon^2$. Since we can take $\delta > 0$ arbitrarily small we arrive to (13.26).

Problem 13.9. (i) Using arguments of the proof of Part (a) of Proposition 13.6 prove that if $\int_\mathbb{R} \rho^2(x) \, dx = \infty$ then $\lambda(\epsilon) \geq -C\eta$ where $\eta = \eta(\epsilon)$ is defined from

$$\eta^\frac{1}{2} = \epsilon \int_{x: \varepsilon \rho(x)^2 \geq \eta} \rho^2(x) \, dx$$

which is consistent with $\epsilon^{1/(1-q)}$ in the framework of Proposition 13.8 but also works for $\rho(x) = |x|^{-q} \log |x|^p$ with either $0 < q < \frac{1}{2}$ or $q = \frac{1}{2}$, $p \geq -\frac{1}{2}$.

(ii) Derive asymptotics of $\lambda(\epsilon)$ in the framework of Statement (i); the most interesting and difficult case seems to be $q = \frac{1}{2}$.

(iii) Provide a better error estimate in (13.23) and (13.26).

13.C Examples of vector potential

In this Appendix we prove that the results of this Chapter are meaningful. The only questionable part of their conditions is the existence of vector potentials with the given properties of the scalar intensities $F$ and the doubts
are only in the three-dimensional case. The construction of a conformal asymptotic Euclidean metric tensor\textsuperscript{103)} \( g^{jk} \) and the construction of a scalar potential \( V \) are obvious in all cases and are left to the reader. Analysis of the two-dimensional case is also obvious. In what follows \( g^{jk} = \delta^{jk} \).

In what follows \( \Lambda_{(2n)} \) be a block-diagonal \( 2n \times 2n \)-matrix with \( n \) diagonal \( 2 \times 2 \)-blocks \( \lambda = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \) (and non-diagonal blocks \( 0 \)). Further, let \( \Lambda_{(2n+1)} \) be a block-diagonal \( (2n+1) \times (2n+1) \)-matrix with \( n \) diagonal \( 2 \times 2 \)-blocks \( \Lambda \) and \( 1 \times 1 \)-block and all non-diagonal blocks equal \( 0 \).

**Lemma 13.10.** (i) Let \( d = 2n \) and \( V_j = (\Lambda x)_j \sigma(|x|) \). Then eigenvalues of \( (F_{jk}) \) are \( \pm f_1(x), \ldots, \pm f_n(x) \) with

\[
(13.28)_{1,2} \quad f_1(x) = 2\sigma(|x|) + |x|\sigma'(|x|), \quad f_2(x) = \ldots = f_n(x) = 2\sigma(|x|).
\]

(ii) Let \( d = 2n + 1 \) and \( V_j = (\Lambda x)_j \sigma(|x|) \). Then eigenvalues of \( (F_{jk}) \) are \( \pm f_1(x), \ldots, \pm f_n(x), 0 \) with \( f_2, \ldots, f_n \) defined by (13.28)\textsubscript{1,2} and

\[
(13.29) \quad f_1(x)^2 = (2\sigma(|x|) + |x'|^2|x|^{-1}\sigma'(|x|))^2 + x_0^2|x'|^2|x|^{-4}\sigma(|x|)^2,
\]

\( x' = (x_1, \ldots, x_{d-1}) \).

**Proof.** (i) Without any loss of the generality one can assume that \( x_j = 0 \) for \( j = 3, \ldots, d \) since we can always reach it by a rotation, commuting with \( \Lambda \). Then \( F_{jk} = 2\sigma\Lambda_{jk} \) if either \( j \geq 3 \) or \( k \geq 3 \) and \( F_{jk} = (2\sigma + |x|\sigma')\Lambda_{jk} m \) for \( j, k \leq 2 \) which implies Statement (i).

(ii) Without any loss of the generality one can assume that \( x_j = 0 \) for \( j = 3, \ldots, d - 1 \). Then again \( F_{jk}(x) = 2\sigma\Lambda_{jk} \) if either \( j = 3, \ldots, d - 1 \) or \( k = 3, \ldots, d - 1 \) and therefore again \( f_2 = \ldots = f_n \) are defined by (13.28)\textsubscript{2}.

Meanwhile \( F_{12} = (2\sigma + |x'|^2|x|^{-1}\sigma') \), \( F_{13} = x_0x_0|x|^{-1}\sigma' \), \( F_{12} = -x_0x_0|x|^{-1}\sigma' \), and \( f_1^2 = F_{12}^2 + F_{13}^2 + F_{23}^2 \) which implies (13.29).

We start from power singularities:

**Example 13.11.** (i) In the framework of Lemma 13.10(i) with \( \sigma = |x|^m \)

\( f_1 = (2 + m)|x|^m \). In particular, \( |f_1| \asymp |x|^m \) if \( m \neq -2 \).

\textsuperscript{103)} I.e., a tensor \( g^{jk} = \delta^{jk}(1 + \varphi) \) with \( |1 + \varphi| \geq \epsilon > 0 \) such that \( D^\alpha \varphi = o(\gamma^{-|\alpha|}) \) as \( |x| \to \infty \) or \( |x| \to 0 \) for all \( \alpha \).
(ii) In the framework of Lemma 13.10(ii) with $\sigma = |x|^m$

$$f_1^2 = \left((2 + m|x'|^2|x|^{-2})^2 + m^2|x'|^2x_0|x|^{-4}\right)|x|^{2m}.$$  

In particular, $|f_1| \asymp |x|^m$ if $m \neq -2$.

**Example 13.12.** In the framework of Lemma 13.10(ii) with $\sigma = |x|^m$ let us define $V_d = a|x|^{m+1}$. Again, without any loss of the generality we can assume that $x_j = 0, j = 3, \ldots, d - 1$. In this case the only $F_{jk}$ to change are $F_{1d}, F_{2d}$ (and $F_{a1}, F_{a2}$) and therefore $f_2, \ldots, f_n$ are still defined by (13.28)$_2$. One can prove easily that

$$f_1^2 = \left((2 + m|x'|^2|x|^{-2})^2 + m^2|x'|^2x_0|x|^{-4} + (m + 1)^2a^2|x'|^2|x|^{-2}\right)|x|^{2m}.$$  

In particular, $f_1 \asymp |x|^m$ if $a \neq 0$.

**Example 13.13.** (i) In the framework of Lemma 13.10(i) with $\sigma = \langle x \rangle^m$

$$f_1 = (2 + m|x|\langle x \rangle^{-2})\langle x \rangle^m.$$  

In particular, $|f_1| \asymp \langle x \rangle^m$ if $m > -2$.

(ii) In the framework of Lemma 13.10(ii) with $\sigma = \langle x \rangle^m$

$$f_1^2 = \left((2 + m|x'|^2\langle x \rangle^{-2})^2 + m^2|x'|^2x_0\langle x \rangle^{-4}\right)\langle x \rangle^{2m}.$$  

In particular, $f_1 \asymp \langle x \rangle^m$ if $m > -2$.

(iii) In the framework of Lemma 13.10(ii) with $\sigma = \langle x \rangle^m$ and $V_d = a\langle x \rangle^{m+1}$

$$f_1^2 = \left((2 + m|x'|^2\langle x \rangle^{-2})^2 + m^2|x'|^2x_0\langle x \rangle^{-4} + (m + 1)^2a^2|x'|^2\langle x \rangle^{-2}\right)|x|^{2m}.$$  

In particular, $f_1 \asymp \langle x \rangle^m$ if $a \neq 0$.

Consider now power-log singularities.

**Example 13.14.** (i) In the framework of Lemma 13.10(i) with $\sigma = |x|^m\ell(x)^\beta$, $\ell(x) = |\log |x|| + C_0$ (with sufficiently large $C_0$) $f_1 \asymp \sigma$ if $m \neq -2$ and $f_1 = \beta|x|^{-2}\ell^{-1}$ if $m = -2$.

(ii) In the framework of Lemma 13.10(ii) with $\sigma = |x|^m\ell(x)^\beta$, $\ell(x) = |\log |x|| + C_0$ (with sufficiently large $C_0$) $f_1 \asymp \sigma$ if $m \neq -2$. 

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(iii) In the framework of Lemma 13.10(ii) with $\sigma = \langle x \rangle^m$ and $V_d = a|x|^{m+1}\ell(x)^\beta f_1 \asymp \sigma$ if $a \neq 0$.

**Example 13.15.** (i) In the framework of Lemma 13.10(i) with $\sigma = \langle x \rangle^m \ell(x)^\beta$, $\ell(x) = \log \langle x \rangle + C_0$ (with sufficiently large $C_0$) $f_1 \asymp \sigma$ if $m > -2$.

(ii) In the framework of Lemma 13.10(ii) with $\sigma = \langle x \rangle^m \ell(x)^\beta$, $\ell(x) = \log \langle x \rangle + C_0$ (with sufficiently large $C_0$) $f_1 \asymp \sigma$ if $m > -2$.

(iii) In the framework of Lemma 13.10(ii) with $\sigma = \langle x \rangle^m \ell(x)^\beta$ and $V_d = a\langle x \rangle^{m+1}\ell(x)^\beta f_1 \asymp \sigma$ if $a \neq 0$.

Consider now exponential potentials.

**Example 13.16.** Let $d = 2$. Then

(i) $V_1 = -x_2|x|^m \exp(|x|^\beta)$, $V_2 = x_1|x|^m \exp(|x|^\beta)$ with $\beta > 0$ provide $f \asymp |x|^{m+\beta-1} \exp(|x|^\beta)$ as $|x| \geq c$.

(ii) The same example with $\beta < 0$ provide $f \asymp |x|^{m+\beta-1} \exp(|x|^\beta)$ as $|x| \leq \epsilon$.

For $d = 3$ we need to be more crafty.

**Example 13.17.** Let $d = 3$.

(i) Consider

$$\begin{align*}
(13.34)_1 & \quad V_1 = \exp(\nu(x)) \cos(\psi(x)) |x|^m, \\
(13.34)_{2,3} & \quad V_2 = \exp(\nu(x)) \sin(\psi(x)) |x|^m, \quad V_3 = 0
\end{align*}$$

with $\nu(x) = |x|^\beta$, $\beta > 0$. Then

$$\nabla^\alpha V_j \leq c_\alpha \exp(a|x|^\beta)|x|^{|\beta-1|\alpha|+m} \quad \forall \alpha$$

as $|x| \geq 1$ provided $|\nabla \psi| \lesssim |x|^\beta-1$. Moreover, one can see easily that

$$\nabla \psi \geq b|x|^{\beta-1}$$

provided $|\partial \psi| \geq b|x|^{\beta-1}$ as $|x| \geq c$. One can take

$$\psi(x', x_3) = \int_0^{x_3} (|x'|^2 + y^2)^{(\beta-1)/2} dy, \quad x' = (x_1, x_2)$$

satisfying these restrictions.
(ii) Similarly, for $\beta < 0$ this constructions work for $|x| \leq \epsilon$.

Consider now quasihomogeneous case. In what follows $L = (l_1, l_2)$ for $d = 2$, $L = (l_1, l_2, l_3)$ for $d = 3$

\[ [x]_L = (\sum_j x_j^{2n/l_j})^{1/2n} \]

is $L$-quasihomogeneous length, and $n$ is large so functions are smooth in $\mathbb{R}^d \setminus 0$.

**Example 13.18.** (i) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 < l_2$. Let

\[ V_1 = -a x_2 [x]_L^m, \quad V_2 = x_1 [x]_L^m. \]

Then $F \asymp [x]_L^m$ for $|x|_L \geq c$ provided $m \neq -(1 + l_2)$ and $a$ is properly chosen.

(ii) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 > l_2 > 0$. Let $V_1, V_2$ are defined by (13.38). Then $F \asymp [x]_L^m$ for $|x|_L \leq \epsilon$ provided $m \neq -(1 + l_2)$ and $a$ is properly chosen.

**Example 13.19.** (i) Let $d = 3$ and $L = (l_1, l_2, l_3)$ with $1 = l_1 \leq l_2 \leq l_3$. Let

\[ V_1 = 0, \quad V_2 = x_1 [x]_L^m, \quad V_3 = x_1 [x]_L^{m+1}, \]

\[ V_1 = -x_2 [x]_L^m, \quad V_2 = x_1 [x]_L^m, \quad V_3 = 0 \]

if $m \neq -1$, $m = -1$ respectively. Then $F \asymp [x]_L^m$ for $|x|_L \geq c$.

(ii) Let $d = 2$ and $L = (l_1, l_2)$ with $1 = l_1 \geq l_2 \geq l_3 > 0$. Let $V_1, V_2, V_3$ are defined by (13.39) or (13.39) if $m \neq -1$, $m = -1$ respectively. Then $F \asymp [x]_L^m$ for $|x|_L \leq \epsilon$.

**Example 13.20.** Let $d = 3$, $X = (\mathbb{R}^2 \setminus 0) \times \mathbb{R}/\mathbb{Z} \ni (x', x_3) = (x_1, x_2, x_3)$, $m \neq -2$ and

\[ V_1 = x_2 |x'|^m, \quad V_2 = -x_3 |x'|^m, \quad V_3 = a |x'|^{m+2} \]

positively homogeneous on $x'$ of degrees $m + 1$, $m + 1$, $m + 2$ respectively\(^{104}\). Then $F^3 \neq 0$, $|x'| = \text{const}$ along integral curves of the vector field $\frac{1}{F^3} (F^1, F^2, F^3)$ and for irrational $a/\pi$ these curves are not closed.

\(^{104}\) Then $F^1, F^2, F^3$ are positively homogeneous of degrees $m + 1$, $m + 1$, $m$ respectively.
Comments

In addition to papers, mentioned in Remark 4.9 I would like also mention S. Solnyshkin [Sol], A. Sobolev [Sob1, Sob2, Sob3, Sob4], Y. Colin de Verdiere [CdV1, CdV2], A. Morame [Mar1, Mar2], A. Morame & J. Nourrigat [MN] and H. Tamura [Tam1, Tam2, Tam3, Tam4], M. Birman & G. Raikov [BR], G. Raikov [Rai2, Rai3, Rai4, Rai5, Rai6, Rai7, Rai8, Rai9].

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