Missing Value Imputation for Mixed Data Through Gaussian Copula

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Abstract

Missing data imputation forms the first critical step of many data analysis pipelines. The challenge is greatest for mixed data sets, including real, Boolean, and ordinal data, where standard techniques for imputation fail basic sanity checks: for example, the imputed values may not follow the same distributions as the data. This paper proposes a new semi-parametric algorithm to impute missing values, with no tuning parameters. The algorithm models mixed data as a Gaussian copula. This model can fit arbitrary marginals for continuous variables and can handle ordinal variables with many levels, including Boolean variables as a special case. We develop an efficient approximate EM algorithm to estimate copula parameters from incomplete mixed data. The resulting model reveals the statistical associations among variables. Experimental results on several synthetic and real datasets show superiority of our proposed algorithm to state-of-the-art imputation algorithms for mixed data.

1 INTRODUCTION

Mixed data sets, including real, Boolean, and ordinal data, are a fixture of modern data analysis. Ordinal data is particularly common in survey datasets. For example, Netflix users rate movies on a scale of 1-5. Social surveys may roughly bin respondents’ income or level of education as an ordinal variable, and ordinal Likert scales measure how strongly a respondent agrees with certain stated opinions. Binary variables may be considered a special case of an ordinal with two levels. Health data often contains ordinals that result from patient surveys or from coarse binning of continuous data into, e.g., cancer stages 0–IV or overweight vs obese patients.

In all of these settings, missing data is endemic due to nonresponse and usually represents a large proportion of the full dataset. Missing value imputation generally precedes other analysis, since most machine learning algorithms require complete observations. Imputation quality can strongly influence subsequent analysis. However, most available imputation methods treat ordinal data either as continuous or categorical. For example, much of the work on low rank matrix completion for movie rating datasets uses a quadratic loss [5, 20, 15, 13, 8], which implicitly treats ratings encoded as 1-5 as numerical values. However, for ordinal data, the differences between encoded values are misleading: is the difference between the ratings 4 and 5 really the same as the difference between ratings 3 and 4? On the other hand, methods that treat ordinal data as categorical [23, 2] throw away the ordering information. Further, imputation algorithms can usually afford only a limited number of categories. Hence users of these methods may be tempted to treat ordinal data with many levels as continuous, to their detriment. For example, the ordinal variable “Weeks Worked Last Year” from the General Social Survey takes 48 levels, but 74% of the population worked either 0 or 52 weeks. Imputation with the mean works terribly!

A more sensible, but still powerful, model, treats ordinal data as generated by thresholding continuous data, as in [21, 22]. In this paper, we associate each ordinal variable with a continuous latent variable. Each ordinal level corresponds to an interval of continuous values. When data follows this model, a naive interpretation of ordinal data can obscure correlation between variables, while correlations can be correctly computed if we can estimate the latent continuous values; see Figure 1.

To fully exploit the information in mixed data, imputation should take into account the interaction between continuous and ordinal variables. Thus imputation separately for each type is undesirable, while a direct model for the joint distribution can be complex. Existing parametric models [16] are too restrictive.
Missing Value Imputation for Mixed Data Through Gaussian Copula

Figure 1: Simulate 100 binormal points \((z_1, z_2)\) with correlation 0.8. Discretize \(z_1\) to \(x_1\), \(z_2\) to \(x_2\) on random cutoffs, respectively. Top two and bottom left panels plot one repetition. Dashed lines mark the cutoffs. Bottom right panel plots the sample correlation over 100 repetitions. Dashed line marks the truth.

2 RELATED WORK AND OUR CONTRIBUTION

Gaussian Copula for Mixed Data. Hoff et al. [14] proposes to model mixed data using a Gaussian copula and develops a Bayesian (MCMC) framework to fit the model with incomplete data. However, our numerical study shows that both imputed values and estimated parameters of this method can have large bias. Later, Fan et al. [7] and Feng and Ning [9] proposed more rigorous versions of this model and examined its theoretical properties, with particular focus on discovering a graphical model of the dependency structure. This model assumes that the observed ordinal variables are obtained by discretizing latent continuous variables and that the latent variables together with observed continuous variables follow the nonparamormal distribution[17]. When only ordinal variables are present, this model is equivalent to the probit graphical model [11]. However, this literature does not consider the presence of missing data, and their parameter estimation methods do not extend to incomplete data.

Mixed Data Imputation. Several other methods for mixed data imputation are available. Parametric methods [16, 27] make strong distributional assump-
tions that are generally unwarranted. While sophisticated parametric approaches are widely used [27], non-parametric methods such as Stekhoven and Bühlmann [23], an iterative imputation method based on random forests, tend to perform better.

In the low rank models literature, the generalized low rank models framework [26] handles missing values imputation for mixed data using a low rank model with appropriately chosen loss functions to ensure proper treatment of each data type. However, choosing the right loss functions for mixed data is challenging. A few papers in the low rank matrix completion literature share our motivation: for example, early papers by Rennie and Srebro [21, 22] proposed a thresholding model to generate ordinals from real low rank matrices. Ganti et al. [10] estimate monotonic transformations of a latent low rank matrix, but the method performs poorly in practice. Anderson-Bergman et al. [1] extends the low rank modeling framework to a Gaussian copula model, with careful attention to ordinal data; but their setup lacks a proper probabilistic foundation and cannot identify correlations between variables.

Contribution. In this paper, we full exploit the potential of Gaussian copula in missing value imputation for mixed data. First, we propose an imputation algorithm which utilizes different data types and their interaction. Numerical results show that our method highly improves the imputation accuracy compared to current Gaussian copula based imputation algorithm [14]. Second, we proposed an efficient EM-like algorithm to estimate copula correlation with incomplete mixed data. The problem setting we consider reduces to [7, 9] when there is no missing value; to [11] when there is neither missing value nor continuous dimension. Finally, our imputation method is invariant to coordinate monotonic transformation, free of tuning parameter, and interpretable. The fitted copula model can reveal the statistical association among variables, which is usually desirable in social survey studies.

3 METHODOLOGY

3.1 Notation

Define \([p] = \{1, \ldots, p\}\) for \(p \in \mathbb{Z}\). Let \(x = (x_1, \ldots, x_p) \in \mathbb{R}^p\) be a random vector. We use \(x_I\) to denote the subvector of \(x\) with entries in subset \(I \subset [p]\). Let \(\mathcal{M}, \mathcal{C}, \mathcal{D} \subset [p]\) denote missing, observed continuous, and observed discrete dimensions, respectively. The observed dimensions are \(\mathcal{O} = \mathcal{C} \cup \mathcal{D}\), so \(x = (x_\mathcal{C}, x_\mathcal{D}, x_\mathcal{M}) = (x_\mathcal{O}, x_\mathcal{M})\).

Let \(X \in \mathbb{R}^{n \times p}\) be a matrix whose rows correspond to observations and columns to variables. We refer to the
i-th row, j-th column, and (i, j)-th element as $x^i, X_j$ and $x^j_j$, respectively.

Two random variables $x$ and $y \in \mathbb{R}$ satisfy $x \overset{d}{=} y$ if their cumulative distribution functions (CDF) match. The set of correlation matrices is the elliptope $\mathcal{E} = \{ Z \succeq 0 : \text{diag}(Z) = 1 \}$.

3.2 Gaussian Copula

The Gaussian copula models complex multivariate distributions through transformations of a latent Gaussian vector. We call a random variable $x \in \mathbb{R}$ continuous when it is supported on an interval. We can match the marginals of any continuous random vector $x$ by applying a strictly monotone function to a random vector $z$ with standard normal marginals. Further, the required function is unique, as stated in Lemma 1.

**Lemma 1.** Suppose $x \in \mathbb{R}^p$ is a continuous random vector with CDF $F_j$ for each coordinate $j \in [p]$, and $z \in \mathbb{R}^p$ is a random vector with standard normal marginals. Then there exists a unique elementwise strictly monotone function $f(z) := (f_1(z_1), \ldots, f_p(z_p))$ such that

$$x_j \overset{d}{=} f_j(z_j) \quad \text{and} \quad f_j = F_j^{-1} \circ \Phi, \quad j \in [p] \quad (1)$$

where $\Phi$ is the standard normal CDF.

All proofs appear in the supplementary materials. Notice the functions $\{f_j\}_{j=1}^p$ in Eq. (1) are strictly monotone, so their inverses exist. Define $f^{-1} = (f_1^{-1}, \ldots, f_p^{-1})$. Then $z = f^{-1}(x)$ has standard normal marginals, but the joint distribution of $z$ is not uniquely determined. The Gaussian copula model (or equivalently nonparanormal distribution [17]) further assumes $z$ is jointly normal.

**Definition 1.** We say a continuous random vector $x \in \mathbb{R}^p$ follows the Gaussian copula $x \sim GC(\Sigma, f)$ with parameters $\Sigma$ and $f$ if there exists a correlation matrix $\Sigma$ and elementwise strictly monotone function $f : \mathbb{R}^p \rightarrow \mathbb{R}^p$ such that $f(z) = x$ for $z \sim N_p(0, \Sigma)$.

This model is semiparametric: it comprises nonparametric functions $f$ and parametric copula correlation matrix $\Sigma$. The monotone $f$ establishes the mapping between observed $x$ and latent $z$, while $\Sigma$ fully specifies the distribution of $z$. Further, the correlation $\Sigma$ is invariant to elementwise strictly monotone transformation of $x$. Concretely, if $x \sim GC(\Sigma, f)$ and $y = g(x)$ where $g$ is elementwise strictly monotone, then $y \sim GC(\Sigma, f \circ g^{-1})$. Thus the Gaussian copula separates the multivariate interaction $\Sigma$ from the marginal distribution $f$.

When $f_j$ is strictly monotone, $x_j$ must be continuous. On the other hand, when $f_j$ is monotone but not strictly monotone, $x_j$ takes discrete values in the range of $f_j$ and can model ordinals. Now $f_j$ is not invertible. For convenience, we define a set-valued inverse $f_j^{-1}(x_j) := \{ z_j : f_j(z_j) = x_j \}$. When the ordinal $x_j$ has range $[k]$, Lemma 2 states that the only monotone function $f_j$ mapping continuous $z_j$ to $x_j$ is a cutoff function, defined for some parameter $S \subset \mathbb{R}$ as

$cutoff(z; S) := 1 + \sum_{s \in S} I(z > s)$ for $z \in \mathbb{R}$.

**Lemma 2.** Suppose $x \in \mathbb{R}$ is an ordinal random variable with range $[k]$ and probability mass function $\{p_l\}_{l=1}^k$ and $z \in \mathbb{R}$ is a continuous random variable with CDF $F_z$. Then $f = cutoff(z; S)$ is the unique monotone function $f$ that satisfies $x \overset{d}{=} f(z)$, where $S = \{ s_l = F_z^{-1} \left( \sum_{t=1}^l p_t \right) : l \in [k-1] \}$.

For example, in recommendation systems we can think of the discrete ratings as obtained by rounding some ideal real valued score matrix. The rounding procedure amounts to apply a cutoff function. See Figure 2 for an example of a cutoff function.

![Cutoff function](image-url)

**Figure 2:** Cutoff function $f(\cdot)$ with cutoffs $\{-1,1\}$ maps continuous $z$ to ordinal $x \in \{1, 2, 3\}$.

To extend the Gaussian copula to mixed data, we simply specify that $f_j$ is strictly monotone for $j \in C$ and that $f_j$ is a cutoff function for $j \in D$. As before, the correlation $\Sigma$ remains invariant to elementwise strictly monotone transformations. The main difference is that while $f_j^{-1}(x_j)$ is a single number when $j \in C$, it is an interval when $j \in D$.

So far we have introduced a very flexible model for mixed data, which has been explored in graphical model with complete observation [7, 9]. Our interest is to investigate missing value imputation under this model. We assume the missing-data mechanism is...
missing at random (MAR) in this paper, which means the missingness only depends on the observed entries. This assumption allows us to obtain a consistent estimate for \( \Sigma \) using only observed values [16].

Suppose the data matrix \( \mathbf{X} \) has rows \( x_1, \ldots, x_n \) i.i.d. \( \text{GC}(\Sigma, f) \) and \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \) for \( i \in [n] \). Define \( f_j = (f_j)_{j \in I} \) for \( I \subset [p] \) and \( f_j^{-1}(x_j) = \mathbb{R} \) for \( j \in \mathcal{M}_i \). Given estimates for \( f \) and \( \Sigma \), we impute in three steps:

1. Compute the constraints \( z^i \in f^{-1}(x^i) \).
2. Impute \( \hat{z}_{M_i}^i \) using \( \Sigma \) and constraints on \( z^i_O \).
3. Impute \( \hat{x}_{M_i}^i = f_{M_i}(\hat{z}_{M_i}^i) \) using imputed \( \hat{z}_{M_i}^i \).

We show how to estimate \( f \) in Section 3.3, how to estimate \( \Sigma \) in Section 3.4, with details in Algorithm 2.

### 3.3 Monotonic Function Estimation

To map between \( \mathbf{x} \) and \( \mathbf{z} \), we require both \( f^{-1} \) and \( f \). It is easier to directly estimate \( f^{-1} \). For \( j \in \mathcal{C} \), we have \( f_j^{-1} = \Phi^{-1} \circ f_j \), as shown in Eq. (1). While the true CDF \( F_j \) is usually unavailable, it is natural to estimate it by the empirical CDF of \( \mathbf{X} \) on observed entries, denoted as \( \hat{F}_j \). Let \( n_j \) be the observed length of \( \mathbf{X}_j \). We use the following estimator:

\[
\hat{f}_j^{-1}(x_j^i) = \Phi^{-1} \left( \frac{n_j}{n_j + 1} \hat{F}_j(x_j^i) \right). \tag{2}
\]

(The scale constant \( n_j/(n_j + 1) \) ensures the output is finite.) Lemma 3 shows this estimator converges to \( f_j^{-1} \) in sup norm on observed domain.

**Lemma 3.** Suppose the continuous random variable \( x \in \mathbb{R} \) with CDF \( F_x \) and normal random variable \( z \in \mathbb{R} \) satisfy \( f(x) \stackrel{d}{=} x \) for a strictly monotone \( f \). Given \( x_1, \ldots, x_n \) i.i.d. \( F_x \), \( m = \min_i x_i \), and \( M = \max_i x_i \), the inverse \( \hat{f}^{-1} \) defined in Eq. (2) satisfies

\[
P \left( \sup_{m \leq x \leq M} |\hat{f}^{-1}(x) - f^{-1}(x)| > \epsilon \right) \leq 2e^{-c_1 n \epsilon^2}
\]

for any \( \epsilon \) in \( a_1 n^{-1} < \epsilon < b_1 \), where \( a_1, b_1, c_1 > 0 \) are constants depending on \( F_x(m) \) and \( F_x(M) \).

For an ordinal variable \( j \in \mathcal{D} \) with \( k \) levels, \( f_j(z_j) = \text{cutoff}(z_j; S^j) \). Since \( S^j \) is determined by the probability mass function \( \{p_l^j\} \) of \( x_j \), we may estimate cutoffs \( \hat{S}^j \) as a special case of Eq. (2) by replacing \( p_l^j \) with its sample mean:

\[
\hat{S}^j = \left\{ \Phi^{-1} \left( \frac{\sum_{i=1}^{n_j} \mathbb{I}(x_{ij}^j \leq l)}{n_j + 1} \right), l \in [k - 1] \right\}. \tag{3}
\]

Lemma 4 shows that \( \hat{S}^j \) consistently estimates \( S^j \).

**Lemma 4.** Suppose the ordinal random variable \( x \in [k] \) with probability mass function \( \{p_l\}_{l=1}^{k} \) and normal random variable \( z \in \mathbb{R} \) satisfy \( f(z) \stackrel{d}{=} x \). Given samples \( x_1, \ldots, x_n \) i.i.d. \( \{p_l\}_{l=1}^{k} \), the cutoff estimate \( \hat{S} \) from Eq. (3) satisfies

\[
P \left( \|\hat{S} - S\|_1 > \epsilon \right) \leq 2^{k} e^{-c_2 \epsilon^2/(k-1)^2}
\]

for any \( \epsilon \) in \( (k - 1)a_2 n^{-1} < \epsilon < (k - 1)b_2 \), where \( a_2, b_2, c_2 > 0 \) are constants depending on \( \{p_1, p_k\} \).

### 3.4 Copula Correlation Matrix Estimation

We consider maximum likelihood estimation for \( \Sigma \) in increasingly difficult settings by introducing missing and ordinal data sequentially.

#### 3.4.1 Continuous Observations

**Complete Observations.** We begin by considering continuous, fully observed data: \( \mathcal{D} = \mathcal{M} = \emptyset \). The density of the observed variable \( \mathbf{x} \) is

\[
p(\mathbf{x}; \Sigma, f) \, d\mathbf{x} = \phi_p(\mathbf{z}; \Sigma) \, d\mathbf{z}
\]

where \( \mathbf{z} = f^{-1}(\mathbf{x}) \), \( d\mathbf{z} = \left| \frac{\partial f}{\partial \mathbf{z}} \right| \, d\mathbf{x} \), \( \phi_p(\cdot; \Sigma) \) is the PDF of the \( p \)-dimensional normal with mean \( \mathbf{0} \) and covariance \( \Sigma \). The maximum likelihood estimator (MLE) of \( \Sigma \) maximizes the observed likelihood function

\[
\ell_1(\Sigma; \mathbf{x}^i) = \frac{1}{n} \sum_{i=1}^{n} \log \phi_p(f^{-1}(\mathbf{x}_O^i); \Sigma_{O,O}) \tag{4}
\]

over \( \Sigma \in \mathcal{E} \). (We omit here and later the constant arising from \( \frac{\partial f}{\partial \mathbf{z}} \) after the log transformation.) Thus the MLE of \( \Sigma \) is the sample covariance of \( \mathbf{Z} := f(\mathbf{X}) = (f_1(\mathbf{X}_1), \ldots, f_p(\mathbf{X}_p)) \). When substituting \( f \) for its empirical estimation in Eq. (2), the resulting covariance matrix of \( \hat{\mathbf{Z}} := \hat{f}(\mathbf{X}) \) is still consistent and asymptotically normal under some regularity conditions [24], which justifies the use of our estimator \( \hat{f} \). To simplify notation, we assume \( f \) is known below.

**Incomplete Observations.** With missing values \( \mathcal{M} \neq \emptyset \), we observe only \( \mathbf{x}_O \) with density

\[
p(\mathbf{x}_O; \Sigma, f) \, d\mathbf{x}_O = \int_{\mathbf{x}' \in \{\mathbf{x}_O\} \times \mathbb{R}^{|M|}} p(\mathbf{x}'; \Sigma, f) \, d\mathbf{x}'
\]

\[
= \int_{\mathbf{z} \in f^{-1}(\mathbf{x}_O)} \phi_p(\mathbf{z}; \Sigma) \, d\mathbf{z} = \phi_p(\mathbf{z}_O; \Sigma_{O,O}) \, d\mathbf{z}_O.
\]

where \( \mathbf{z}_O = f^{-1}(\mathbf{x}_O) \). Define \( p_l = |O_l| \), the observed length of \( \mathbf{x}_O \), to see the MLE of \( \Sigma \) maximizes the observed likelihood function

\[
\ell_2(\Sigma; \mathbf{x}_O^i) = \frac{1}{n} \sum_{i=1}^{n} \log \phi_p(f^{-1}(\mathbf{x}_O^i); \Sigma_{O,O}) \tag{4}
\]

where \( \mathbf{x}_O^i \in O_i \).
Suppose we know the values of the unobserved $z'$. Then we may rewrite the likelihood as
\[
\ell_2(\Sigma; x'_{iO}, z^i) = \frac{1}{n} \sum_{i=1}^{n} \log \phi_p(z'; \Sigma). \tag{5}
\]
Since the values of $z'$ are unknown, we proceed in an iterative fashion, substituting $\ell_2(\Sigma; x'_{iO}, z')$ by its expected value given observations $x'_{iO}$ and estimate $\hat{\Sigma}$:

\[
Q(\hat{\Sigma}; x'_{iO}) := \mathbb{E}[\ell_2(\Sigma; x'_{iO}, z^i)|x'_{iO}, \hat{\Sigma}]
= c - \frac{1}{2} \left( \log \det(\Sigma) + \text{Tr} \left( \Sigma^{-1} G(\hat{\Sigma}, x'_{iO}) \right) \right)
= \hat{Q}(\hat{\Sigma}; G(\hat{\Sigma}, x'_{iO}))
\]

where the sufficient statistic $G$ is defined as
\[
G(\hat{\Sigma}, x'_{iO}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(z'z'^{\top}|x'_{iO}, \hat{\Sigma})
\]
and $c$ is a universal constant. EM theory [6, 19] guarantees the updated $\Sigma = \text{argmax}_{\Sigma \in \mathcal{E}} Q(\Sigma; \hat{\Sigma}, x'_{iO})$ improves the likelihood compared to $\hat{\Sigma}$,
\[
\ell_2(\Sigma; x'_{iO}) \geq \ell_2(\hat{\Sigma}; x'_{iO}),
\]
and that by iterating this update, we produce a sequence $\{\Sigma^{(t)}\}$ that converges monotonically to a local maximizer of $\ell_2(\Sigma; x'_{iO})$. At the $t$-th iteration, for the E step we compute $\mathbb{E}(z'z'^{\top}|x'_{iO}, \Sigma^{(t)})$ to express $\hat{Q}(\Sigma; G(\hat{\Sigma}, x'_{iO}))$ in terms of $\Sigma$; we find $\Sigma^{(t+1)} = \arg\max_{\Sigma \in \mathcal{E}} \hat{Q}(\Sigma; G(\hat{\Sigma}, x'_{iO}))$ for the M step. We summarize the resulting EM algorithm here in Algorithm 1.

### Algorithm 1 EM algorithm for Gaussian Copula

**Input:** observed entries $x_{iO}$

**Initialize:** $t = 0$, $\Sigma^{(0)}$

For $t = 0, 1, 2, \ldots$

1. **E-step:** Compute $G^{(t)} = G(\Sigma^{(t)}, x_{iO})$
2. **M-step:** $\Sigma^{(t+1)} = \arg\max_{\Sigma \in \mathcal{E}} \hat{Q}(\Sigma; G^{(t)})$

until convergence.

**Output:** $\Sigma = \Sigma^{(t)}$

In the E-step, suppressing the index $i$ for simplicity, the conditional distribution of $z$ given $\{x_{iO}, \Sigma\}$ is normal $\mathcal{N}_p(\mathbf{0}, \Sigma)$ but constrained to the region $z \in \mathcal{F}^{-1}(x)$. That is, $z_{iO}$ is constrained to the point $f_{z_{iO}}^{-1}(x_{iO})$ and $z_{D-i}$ is unconstrained. Since the conditional distribution of $z_{D-i}$ given $z_{iO}$ is also normal, with mean $\Sigma_{M,O} \Sigma_{O,O}^{-1} z_{iO} + \Sigma_{M,O} \Sigma_{O,M}^{-1} \Sigma_{M,M}^{-1} z_{D-M}$, computing $\mathbb{E}[zz'x|x, \Sigma]$ is easy.

For the M-step, we resort to an approximation, as in [11]. Notice that the unconstrained maximizer is $\Sigma = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(z'z')|x'_{iO}, \Sigma]$. We update $\Sigma^{(t+1)} = P_{(t)} \Sigma$, where $P_{(t)}$ scales its argument to output a correlation matrix: for $D = \text{diag}(\Sigma)$, $P_{(t)}(\Sigma) = D^{-1/2} \Sigma D^{-1/2}$.

In more challenging observation settings with missing and discrete data, we use the same EM algorithm. Only the E-step must be modified, as shown below.

### 3.4.2 Complete Mixed Observations

Now let’s consider complete mixed observations: $M = \emptyset, D \neq \emptyset$. Now the observed likelihood function
\[
\ell_4(\Sigma; x') = \frac{1}{n} \sum_{i=1}^{n} \int_{z' \in \mathcal{F}^{-1}(x')} \phi_p(z'; \Sigma) \, dz'
\tag{6}
\]
features a truncated Gaussian integral, since $f_{z_{iO}}^{-1}(x_{iO})$ is an interval for $j \in D_i$. However, we may still use the EM approach, Algorithm 1, to estimate $\Sigma$. The joint likelihood in terms of $z'$ and $x'$ is still given by Eq. (5). Due to the interval constraints, $\mathbb{E}[z'z'|x', \Sigma]$ in Section 3.4.1 requires evaluating a challenging truncated Gaussian integral. We now suppress index $i$ and show how to compute $\mathbb{E}[zz'|x, \Sigma]$.

Observed continuous entries are easy to handle: $\mathbb{E}[z_{iO}|x, \Sigma] = f_{z_{iO}}^{-1}(x_{iO})$. For observed discrete entries, we must compute $\mathbb{E}[z_{D}|x, \Sigma]$ and Cov[$z_{D}|x, \Sigma$], the mean and covariance of a $|D|$-dimensional normal truncated to $f_{z_{D}}^{-1}(x_{D})$, a Cartesian product of intervals. The computation involves multiple integrals of a nonlinear function and admits a closed form expression only when $|D| = 1$. Direct computational methods [4] are very expensive and can be inaccurate even for moderate $|D|$. Instead, we use an approximate method that scales well to large datasets, following [11].

### Conditional Expectation Approximation.

Suppose all but one element of $z_{D}$ is known. Then we can easily compute the resulting one-dimensional truncated normal mean: for $j \in D$, if $z_{j}$ is unknown and $z_{D-j}$ is known, let $\mathbb{E}[z_{j}|z_{D-j}, x, \Sigma] = g_1(z_{D-j}, x_j, \Sigma)$ define the nonlinear function $g_1: \mathbb{R}^{|D|-1} \rightarrow \mathbb{R}$, parameterized by $x_j$ and $\Sigma$. We may also use $g_1$ to estimate $\mathbb{E}[z_{j}|x, \Sigma]$ if $\mathbb{E}[z_{D-j}|x, \Sigma]$ is known:

\[
\mathbb{E}[z_{j}|x, \Sigma] = \mathbb{E}[\mathbb{E}[z_{j}|z_{D-j}, x, \Sigma]|x, \Sigma]
=E[g_1(z_{D-j}|x_j, \Sigma)|x, \Sigma] = g_1(\mathbb{E}[z_{D-j}|x, \Sigma]|x_j, \Sigma)
\]
if \( g_1 \) is approximately linear. Suppose we have an estimate \( \hat{z}_t^{(t)} \approx \mathbb{E}[z_T|x, \Sigma^{(t)}] \) at iteration \( t \). Then at iteration \( t + 1 \) we compute

\[
\mathbb{E}[z_j|x, \Sigma^{(t+1)}] \approx \hat{z}_j^{(t+1)} := g_1(\hat{z}_{D\setminus j}^{(t)}; x_j, \Sigma^{(t+1)}). \tag{7}
\]

We use a diagonal approximation for \( \text{Cov}[z_D|x, \Sigma] \); we approximate \( \text{Cov}[z_j, z_k|x, \Sigma] \) as 0 for \( j \neq k \in D \). This approximation performs well when \( z_j \) and \( z_k \) are nearly independent given all observed information. We approximate the diagonal entries \( \text{Var}[z_j|x, \Sigma^{(t+1)}] \) for \( j \in D \) using a recursion similar to Eq. (7). Computational details appear in the supplement.

### 3.4.3 Incomplete Mixed Observations

Finally we consider incomplete mixed observation: \( M \) and \( D \) are both nonempty. The observed likelihood function is still Eq. (6) and we can still use Algorithm 1 to estimate \( \Sigma \). However, computing \( \mathbb{E}[z, z^T|x_O, \Sigma] \) is more complicated due to both the interval constraints for discrete observations \( D \) and the missing dimensions \( M \). Again suppressing index \( i \), in addition to the two terms \( \mathbb{E}[z_D|x_O, \Sigma], \text{Cov}[z_D|x_O, \Sigma] \) discussed in Section 3.4.2, we must also compute

- the conditional mean and covariance of missing dimensions \( \mathbb{E}[z_M|x_O, \Sigma], \text{Cov}[z_M|x_O, \Sigma] \).
- the conditional covariance between missing and observed ordinal dimensions \( \text{Cov}[z_M, z_D|x_O, \Sigma] \).

Suppose we can estimate the ordinal values \( z_D \) and thus \( z_O \). The conditional mean \( \mathbb{E}[z_M|x_O, \Sigma] = \Sigma_M, \Sigma^{-1} \Sigma_{O, O} \Sigma_O \) is a linear function of \( z_O \). We can use this observation to compute the expected function of \( z \):

\[
\mathbb{E}[z_M|x_O, \Sigma] = \mathbb{E} [\mathbb{E}[z_M|x_O, \Sigma]|x_O, \Sigma] = \Sigma_M, \Sigma^{-1} \Sigma_{O, O} \mathbb{E}[z_O|x_O, \Sigma] \tag{8}
\]

Eq. (8) also shows how to impute \( z_{M_i}^i \) using its conditional mean given observed \( x_{M_i}^i \) and estimated \( \hat{\Sigma} \). One can compute \( \text{Cov}[z_M|x_O, \Sigma] \) and \( \text{Cov}[z_M, z_D|x_O, \Sigma] \) similarly: deferring details to the supplement, we find

\[
\text{Cov}[z_M, z_O|x_O, \Sigma] = \Sigma_M, \Sigma^{-1} \Sigma_{O, O} \Sigma_O \text{Cov}[z_O|x_O, \Sigma],
\]

\[
\text{Cov}[z_M|x_O, \Sigma] = \Sigma_M - \Sigma_M, \Sigma^{-1} \Sigma_{O, O} \Sigma_M + \Sigma_M - \Sigma^{-1} \Sigma_{O, O} \Sigma_M \cdot \Sigma^{-1} \Sigma_{O, O} \Sigma_M,
\]

where \( \text{Cov}[z_O|x_O, \Sigma] \) has value \( \text{Var}(z_j|x_O, \Sigma) \) at \( j \)-th diagonal entry for \( j \in D \) and 0 elsewhere. Notice \( \text{Cov}[z_M, z_D|x_O, \Sigma] \) is a submatrix of \( \text{Cov}[z_M, z_O|x_O, \Sigma] \).

**Computational Cost.** The complexity of each EM iteration is \( O(\alpha n p^2) \) with observed entry ratio \( \alpha \). Thus the overall complexity is \( O(T \alpha n p^2) \), where \( T \) is the number of EM steps required for convergence. We found \( T \leq 50 \) in most of our experiments when using relative tolerance. On a laptop with Inter-i5-3.1GHz Core and 8 GB RAM, it takes 1.1min for our algorithm to converge on a dataset with size 2538 × 18 and 25% missing entries.

**Complete Imputation.** We have shown how to estimate the model parameters \( f \) and \( \Sigma \). We summarize the complete imputation approach in Algorithm 2.

**Algorithm 2 Imputation via Gaussian Copula**

**Input:** \( x_O \), observed entries of \( x \in \mathbb{R}^{n \times p} \).

1. Estimate \( \hat{f} \) using Eqs. (2) and (3).
2. Compute constraints \( z^i_O \in f^{-1}_O(x^i_O), i \in [n] \).
3. Compute \( \hat{\Sigma} \) using Algorithm 1.
4. For \( i = 1, \ldots, n \),
   - Impute \( \hat{z}^i_M = \mathbb{E}[z^i_M|x^i_D, \hat{\Sigma}] \).
   - Impute \( \hat{x}^i_M = \hat{f}(\hat{z}^i_M) \).

**Output:** \( \hat{x}^i_M \) for \( i \in [n] \) and \( \hat{\Sigma} \).

### 4 EXPERIMENTS

We compare our method with sbgcop[14], a state-of-the-art imputation algorithm using the Gaussian copula; missForest[23], xPCA[1] and imputeFAMD[2], state-of-the-art nonparametric imputation algorithms for mixed data; and the low rank matrix completion algorithms softImpute[18] and GLRM[26], which scale to large datasets. Of these, only sbgcop can estimate correlations. All tuning parameters such as rank and regularization are selected through 5-fold cross validation (5CV) unless further mentioned, as detailed in the supplement.

#### 4.1 Synthetic Data

We evaluate the performance of imputation and correlation estimation. We designed one simple and one complex experimental setting. For both settings, generate rows of \( Z \in \mathbb{R}^{n \times p} \) as \( z^1, \ldots, z^n \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma) \), then generate \( X = f(Z) \) through monotone functions \( f \). \( X \) consists of \( n \) observations and \( p \) variables. We generate 100 datasets from each model.

- **Simple:** \( p = 3 \) and \( n = 1000 \). \( \Sigma \) has \( \sigma_{12} = \sigma_{13} = 0.8 \) and \( \sigma_{23} = 0.64 \). Use \( f \) such that \( X_1 \) has stan-
standard normal distribution, \( X_2 \) is binary and \( X_3 \) is ordinal with 5 levels.

- **Complicated:** \( p = 15 \) and \( n = 2000 \). Randomly generate \( \Sigma \) at each repetition. Use \( f \) such that \( X_1, \ldots, X_5 \) have exponential distributions, \( X_6, \ldots, X_{10} \) are binary and \( X_{11}, \ldots, X_{15} \) are ordinal with 5 levels.

We define a scaled mean absolute error: 
\[
\text{SMAE} := \frac{1}{|I|} \sum_{j \in I} \frac{||\hat{X}_j - X_j||_1}{||X_j^{\text{med}} - X_j||_1}
\]

to measure the imputation error on columns in \( I \), where \( \hat{X}_j, X_j^{\text{med}} \) are the imputed values and observed median for \( j \)-th column, respectively. For each data type, we compute the SMAE on corresponding columns. Evaluation using root mean squared error (RMSE) is in supplement. The estimator’s SMAE is smaller than 1 if it outperforms column median imputation. To evaluate the estimated correlation, we use relative error 
\[
||\hat{\Sigma} - \Sigma||_F/||\Sigma||_F,
\]
where \( \hat{\Sigma} \) is the estimated correlation matrix.

For the simple setting, we randomly remove 10% of entries in each column but ensure no row has more than one missing entry. Figure 3 plots the estimated correlation and imputation error. The imputation error of our method is clearly lower than that of \texttt{sbgcop} for all data types. Our method estimates correlation more accurately than \texttt{sbgcop} and produces fewer outliers. Notably, our method accurately recovers continuous-ordinal correlations from only 5 ordinal levels.

For the complicated setting, we randomly remove some portions of the entries of \( X \). Figure 4 shows the imputation performance and estimated correlation error.

Our method performs the best for all data types. For binary and ordinal data, the semiparametric (correctly specified) model \texttt{sbgcop} performs worse than nonparametric algorithms \texttt{missForest} and \texttt{imputeFAMD}. As for estimated correlation, \texttt{sbgcop} gives inaccurate and unstable estimates, while our method gives accurate and stable estimates.

![Figure 3: Simple Setting: Top panels plot estimated correlation. Dashed lines indicate truth. Bottom panels plot imputation error for different data types.](image1)

![Figure 4: Complex Setting: Bottom right panel plots error of estimated correlation matrix, and other three panels plot imputation error for each data type.](image2)

### 4.2 General Social Survey (GSS) Data

We chose 18 variables with 2538 observations from GSS dataset in year 2014. 24.9% of the entries are missing. The dataset consists of 1 continuous and 17 ordinal variables with 2 to 48 levels. We investigate the imputation accuracy on five selected variables: \texttt{INCOME}, \texttt{LIFE}, \texttt{HEALTH}, \texttt{CLASS} \(^1\) and \texttt{HAPPY}. For each variable, we sample 1500 observation and divide them into 20 folds. We mask one fold of only one variable as test data in each experiment. We report the SMAE for each variable in Table 1. Our method performs the best for all variables. Further our method always performs better than median imputation while the other two methods perform worse than for some variable. Our method also provides estimated variable correlation, which is usually interested in social survey study. We plot high correlations from the copula correlation matrix in Figure 5. One can also use the correlation of the imputed complete dataset.

\(^1\)Subjective class identification from lower to upper class
4.3 MovieLens 1M Data

Recall our method scales cubicly in the number of variables. Hence for this experiment, we sample the subset of the MovieLens 1M data [12] consisting of the 207 movies with at least 1000 ratings and all users who rate at least one of those 207 movies. On this subset, 75.6% of entries are missing. The selected rank using 5CV is 99 for softImpute and 6 for xPCA. For imputeFAMD, we select the best rank 3 under time limit 5 hours. For GLRM, we use the bigger-vs-smaller loss and select rank 8 as detailed in the supplement. Then we manually mask 10% of the data for the test set and use the remaining data to train the model, and repeat 20 times. We round the imputed values to a 1-5 integer for softImpute. We calculate both mean absolute error (MAE) and RMSE, reported in Table 2. Our method outperforms all other methods in terms of both MAE and RMSE. Interestingly, GLRM with rank 8 performs comparably to softImpute with rank 99.

4.4 More Ordinal Data and Mixed Data

We compare the top methods on two more ordinal classification datasets, Lecturers Evaluation (LEV) and Employee Selection (ESL), and two more mixed datasets, German Breast Cancer Study Group (GBSG) and Restaurant Tips (TIPS). Dataset descriptions appear in Table 3. More details appear in the supplement. All datasets are completely observed.

For each dataset, we randomly remove 30% entries as a test set and repeat 100 times. For ordinal classification datasets, we evaluate the SMAE for the label and for the features, respectively. For mixed datasets, we evaluate the SMAE for ordinal dimensions and for continuous dimensions, respectively. We report results in Table 4. Our method outperforms the others in all but one setting, often by a substantial margin.

5 SUMMARY AND DISCUSSION

In this paper, we proposed an imputation algorithm that models mixed data with a Gaussian copula model, together with an effective approximate EM algorithm to estimate the copula correlation with incomplete mixed data. Our algorithms have no tuning parameter and are easy to implement. Our experiments demonstrate the success of proposed method. Scal-
Table 4: Imputation Error on More Ordinal and Mixed Datasets.

| Dataset:Type | EM(our) | sbgcop | missForest | xPCA | imputeFAMD |
|-------------|---------|--------|------------|------|------------|
| ESL: Label  | 0.374(0.03) | 0.695(0.07) | 0.552(0.08) | 0.404(0.04) | 0.503(0.06) |
| ESL: Feature| 0.587(0.02) | 0.773(0.033) | 0.877(0.07) | 0.668(0.03) | 0.687(0.03) |
| LEV: Label  | 0.759(0.03) | 0.965(0.04) | 0.994(0.08) | 0.860(0.06) | 0.882(0.05) |
| LEV: Feature| 0.910(0.01) | 0.975(0.01) | 0.800(0.04) | 1.037(0.02) | 1.085(0.04) |
| GBSG: Ordinal| 0.797(0.03) | 1.039(0.04) | 0.881(0.05) | 0.890(0.04) | 0.840(0.03) |
| GBSG: Continuous| 0.878(0.01) | 0.972(0.02) | 1.030(0.03) | 1.100(0.04) | 1.038(0.03) |
| TIPS: Ordinal| 0.812(0.05) | 1.020(0.06) | 0.927(0.09) | 0.928(0.08) | 0.891(0.09) |
| TIPS: Continuous| 0.754(0.04) | 0.788(0.05) | 0.838(0.05) | 1.011(0.11) | 0.892(0.13) |

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Supplementary Material for
"Missing Value Imputation for Mixed Data
Through Gaussian Copula"

1 Proof of Lemmas

1.1 Proof of Lemma 1

Proof: For any $j \in [p]$, $x_j \overset{d}{=} f_j(z_j)$ if and only if $x_j$ and $f_j(z_j)$ have the same CDF. Since $f_j^{-1}$ exists for any strictly monotone function $f_j$, we can calculate the CDF of $f_j(z_j)$:

$$F_{f_j(z_j)}(t) = P(f_j(z_j) \leq t) = P(z_j \leq f_j^{-1}(t)) = \Phi(f_j^{-1}(t)).$$

The last equality is because $z_j$ has standard normal marginal. Then $x_j \overset{d}{=} f_j(z_j)$ if and only if $\Phi \circ f_j^{-1} = F_j$, which is equivalent to $f_j = F_j^{-1} \circ \Phi$. Thus there exists a unique strictly monotone function $f$ such that $f(z) = (f_1(z_1), \ldots, f_p(z_p))$ have the same marginals as $x$ and the unique $f$ satisfies $f_j = F_j^{-1} \circ \Phi$ for $j \in [p]$.

1.2 Proof of Lemma 2

Proof: The aim is to show for monotone function $f$, $x \overset{d}{=} f(z)$ if and only if $f(z) = \text{cutoff}(z; S)$ with $S = \{s_l = F_z^{-1}\left(\sum_{t=1}^{l} p_t\right) : l \in [k-1]\}$. First notice $x \overset{d}{=} f(z)$ if and only if the range of $f(z)$ is $[k]$ and $p_l = P(f(z) = l)$ for any $l \in [k]$.

When $f(z) = \text{cutoff}(z; S)$ with $S = \{s_l = F_z^{-1}\left(\sum_{t=1}^{l} p_t\right) : l \in [k-1]\}$, further define $s_k = \infty$ and $s_0 = -\infty$. Since $z$ is continuous with CDF $F_z$, $P(f(z) = l) = P(s_{l-1} < z \leq s_l) = F_z(s_l) - F_z(s_{l-1}) = p_l$, for $l \in [k]$

thus $x \overset{d}{=} f(z)$.

When $x \overset{d}{=} f(z)$, random variable $f(z)$ has range $[k]$. For $l \in [k]$, define $A_l = \{z : f(z) = l\}, s_l = \sup_{z \in A_l} z$ and $s_0 = \inf_{z \in A_l} z$. Since $P(f(z) = l) = p_l > 0$, we have $\inf_{z \in A_l} z < s_l$. Since $f$ is monotone, we have $s_{l-1} \leq \inf_{z \in A_l} z$. We claim $s_{l-1} = \inf_{z \in A_l} z$. If not, there exists $s_{l-1} < z^* < \inf_{z \in A_l} z$ satisfying $(l-1) \leq f(z^*) \leq l$. Since $f(z)$ has range $[k]$, $f(z^*)$ can only be $l$ or $l-1$, or equivalently $z^* \in A_l$ or $z^* \in A_{l-1}$. Either
will contradict \( s_{l-1} < z^* < \inf z \). Thus \( s_{l-1} = \inf z \) holds. We can then express \( f(z) \) as \( f(z) = 1 + \sum_{l=1}^{k-1} 1(z > s_l) \). Further,
\[
p_l = P(f(z) = l) = P(z \in A_l) = P(s_{l-1} \leq z \leq s_l) = F_z(s_l) - F_z(s_{l-1}),
\]
Thus we have
\[
F_z(s_l) = \sum_{l=1}^{l} p_l \Rightarrow s_l = F_z^{-1}(\sum_{l=1}^{l} p_l).
\]
which completes our proof.

1.3 Proof of Lemma 3

Before we prove Lemma 3, we introduce the Dvoretzky-Kiefer-Wolfowitz inequality proposed in [3], also introduced in [4].

**The Dvoretzky-Kiefer-Wolfowitz Inequality.** For any i.i.d. sample \( x^1, \ldots, x^n \) with distribution \( F \),
\[
P \left( \sup_{t \in \mathbb{R}} |\bar{F}_n(t) - F(t)| \geq \epsilon \right) \leq 2e^{-2n\epsilon^2}, \quad \epsilon > 0.
\]
where \( \bar{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1\{x^i \leq t\} \).

**Proof:** Applying the Dvoretzky-Kiefer-Wolfowitz inequality, for any \( \epsilon > 0 \) with probability at least \( 1 - 2e^{-2n\epsilon^2} \), \( \sup_{t \in \mathbb{R}} |\bar{F}_n(t) - F(t)| < \epsilon \). Then take \( \epsilon > n^{-1} \),
\[
\sup_{t \in \mathbb{R}} \left| \frac{n}{n+1} \bar{F}_n(t) - F(t) \right| \leq \sup_{t \in \mathbb{R}} \left| \frac{n}{n+1} \bar{F}_n(t) - \bar{F}_n(t) \right| + \sup_{t \in \mathbb{R}} |\bar{F}_n(t) - F(t)|
\]
\[
\leq \frac{1}{n+1} + \epsilon < 2\epsilon.
\]
Thus for any \( t \in \mathbb{R} \), we have \( F(t) - 2\epsilon < \frac{n}{n+1} \bar{F}_n(t) < F(t) + 2\epsilon \). When \( t \in [m, M] \), we have \( F(t) \in [F(m), F(M)] \). Further let \( \epsilon < \min \{ \frac{F(m)}{4}, \frac{1-F(M)}{4} \} \), we have \( \frac{n}{n+1} \bar{F}_n(t) \in \left[ \frac{F(m)}{2}, \frac{1+F(M)}{2} \right] \) for \( t \in [m, M] \). Then we can bound the error:
\[
\sup_{t \in [m, M]} \left| \hat{f}^{-1}(t) - f^{-1}(t) \right| = \sup_{t \in [m, M]} \left| \Phi^{-1} \left( \frac{n}{n+1} \bar{F}_n(t) \right) - \Phi^{-1}(F(t)) \right|
\]
\[
\leq \sup_{r \in \left[ \frac{F(m)}{2}, \frac{1+F(M)}{2} \right]} \left| \Phi^{-1}(r) \right|^\prime \cdot \sup_{t \in [m, M]} \left| \frac{n}{n+1} \bar{F}_n(t) - F(t) \right|
\]
\[
< 2\epsilon \cdot \sup_{r \in \left[ \frac{F(m)}{2}, \frac{1+F(M)}{2} \right]} \left| \Phi^{-1}(r) \right|^\prime.
\]
Since \( \left( \Phi^{-1}(r) \right)^\prime = \frac{1}{\phi \Phi^{-1}(r)} \) and using the property of functions \( \phi \) and \( \Phi \), we can get
\[
\sup_{r \in \left[ \frac{F(m)}{2}, \frac{1+F(M)}{2} \right]} \left| \left( \Phi^{-1}(r) \right)^\prime \right| = 1/ \min \left\{ \phi \left( \Phi^{-1} \left( \frac{F(m)}{2} \right) \right), \phi \left( \Phi^{-1} \left( \frac{F(M)+1}{2} \right) \right) \right\}.
\]
Let $K_1 = \min \left\{ \frac{F(m) - 1}{4}, 1 - \frac{F(M)}{4} \right\}$ and $K_2 = \min \left\{ \phi \left( \Phi^{-1} \left( \frac{F(m)}{2} \right) \right), \phi \left( \Phi^{-1} \left( \frac{F(M) + 1}{2} \right) \right) \right\}$. Adjusting the constants we have

$$P \left( \sup_{t \in [m, M]} \left| \hat{f}^{-1}(t) - f^{-1}(t) \right| > \epsilon \right) \leq 2 \exp \left\{ - \frac{K_2^2}{2} n \epsilon^2 \right\}$$

when $\frac{2}{K_2} n^{-1} < \epsilon < \frac{2K_1}{K_2}$.

### 1.4 Proof of Lemma 4

Before we prove Lemma 4, we introduce the Bretagnolle-Huber-Carol inequality introduced in [6].

**The Bretagnolle-Huber-Carol Inequality.** If the random vector $(N_1, \ldots, N_k)$ is multinomially distributed with parameters $n$ and $(p_1, \ldots, p_k)$, then

$$P \left( \sum_{i=1}^k \left| N_i/n - p_i \right| \geq \epsilon \right) \leq 2^k e^{-\frac{1}{2} n \epsilon^2}, \quad \epsilon > 0.$$

**Proof:** According to Lemma 2, the cutoff function $f(z) = \text{cutoff}(z; S)$ is unique and $S = \{ s_l : s_l = \Phi^{-1}(\sum_{t=1}^l p_t), l \in [k-1] \}$.

Define $s_0^* = -\infty, s_k^* = \infty$ and further define $\Delta_l^* = \Phi(s_l^*) - \Phi(s_{l-1}^*) = \sum_{i=1}^n 1(x^i = l)/n$. Notice $(n\Delta_1^*, \ldots, n\Delta_k^*)$ is multinomially distributed with parameters $n$ and $(p_1, \ldots, p_k)$, applying the Bretagnolle-Huber-Carol inequality, for any $\epsilon > 0$, with probability at least $1 - 2^k e^{-\frac{1}{2} n \epsilon^2}$, $\sum_{i=1}^k |\Delta_i^* - p_i| < \epsilon$. First for each $l \in [k]$,

$$|\Phi(s_l^*) - \Phi(s_l)| = \left| \sum_{t=1}^l \Delta_t^* - \sum_{t=1}^l p_t \right| \leq \sum_{t=1}^k |\Delta_t^* - p_t| < \epsilon$$

Take $\epsilon > n^{-1}$, we have

$$\left| \Phi(s_l^*) \cdot \frac{n}{n+1} - \Phi(s_l) \right| \leq |\Phi(s_l^*) - \Phi(s_l)| + \frac{\Phi(s_l^*)}{n+1} < 2 \epsilon$$

Thus for any $l \in [k-1]$, we have

$$\Phi(s_l) - 2\epsilon < \Phi(s_l^*) \cdot \frac{n}{n+1} = \sum_{i=1}^n 1(x^i \leq l) \frac{n}{n+1} \leq \Phi(s_l) + 2\epsilon$$

When $l \in [k-1]$, we have $p_1 \leq \Phi(s_l) \leq \sum_{t=1}^{k-1} p_t$. Further let $\epsilon < \min \{ \frac{\rho_1}{4}, \frac{\rho_k}{4} \}$, we
have \( \Phi(s^*) \cdot \frac{n}{n+1} \leq 1 - \frac{p}{2} \). Thus we can bound the error:

\[
||\hat{S} - S||_1 = \sum_{l=1}^{n} |\hat{s}_l - s_l| = \sum_{l=1}^{n} \left| \Phi^{-1} \left( \frac{\sum_{i=1}^{n} \mathbb{1}(x^i \leq l)}{n + 1} \right) - \Phi^{-1}(\Phi(s_l)) \right|
\]

\[
\leq \sup_{r \in [\frac{p}{2}, 1 - \frac{p}{2}]} \left( (\Phi^{-1}(r))' \right) \cdot \sum_{l=1}^{k-1} \left| \sum_{i=1}^{n} \mathbb{1}(x^i \leq l) - \Phi(s_l) \right|
\]

\[
\leq \frac{1}{\min \{ \phi(\Phi^{-1}(\frac{p}{2})), \phi(\Phi^{-1}(1 - \frac{p}{2})) \}} \cdot 2(k - 1)\epsilon
\]

Let \( K_1 = \min \{ \frac{p}{2}, \frac{p}{4} \} \) and \( K_2 = \min \{ \phi(\Phi^{-1}(\frac{p}{2})), \phi(\Phi^{-1}(1 - \frac{p}{2})) \} \). Adjusting the constants we have

\[
P\left(||\hat{S} - S||_1 > \epsilon\right) \leq 2 \exp \left\{-\frac{K_2^2}{8} \cdot \frac{n \epsilon^2}{(k-1)^2}\right\}
\]

when \((k - 1)^2 \frac{2}{K_2^2} n^{-1} < \epsilon < (k - 1)^2 \frac{2K_1}{K_2} \).

2 \ Conditional Expectation Computation

Write \( z = (z_C, z_M) \) with \( z_C = (z_c, z_D) \). Given information \( x_C \) is equivalent to given information \( \{z_c = f_c^{-1}(x_C), z_D \in f_D^{-1}(x_D)\} \), denoted as \( \{*\} \). We also know \( z \sim \mathcal{N}(\mu, \Sigma) \) with \( \Sigma \) given. We want to compute \( E[zz^\top|\{*\}] \). We have shown how to compute it for continuous observation in our paper and now present more details on how to compute it for mixed observation.

2.1 \ Complete Mixed Observation

For complete mixed observation, \( M = \emptyset \) and the computation reduces to compute \( E[z_D|\{*\}] \) and \( \text{Cov}[z_D|\{*\}] \). Closed form expression is no longer available unless \(|D| = 1\).

The main idea is to use the law of total expectation and first assume we know \( z_{D-j} \). The conditional distribution of \( z_j \) given information \( \{*, z_{D-j}\} \) is one dimensional normal truncated to the region \( f_j^{-1}(x_j) \):

\[
z_j|z_{D-j}, z_C \sim \mathcal{N}(\Sigma_{j-j}^{-1} \Sigma_{j-D}^{-1} z_{D-j}, 1 - \Sigma_{j,j} \Sigma_{j-D}^{-1} \Sigma_{j,D}^{-1}) \quad \text{and} \quad z_j \in f_j^{-1}(x_j)
\]

where indexing \(-j\) means all dimensions but \( j \) i.e. \([p] - j\). The constraint \( f_j^{-1}(x_j) \) is always an interval. Write \( f_j^{-1}(x_j) = (a_j, b_j) \). Here are three cases: (1) \( a_j, b_j \in \mathbb{R} \); (2) \( a_j \in \mathbb{R}, b_j = \infty \); (3) \( a_j = -\infty, b_j \in \mathbb{R} \). The computation for all cases are very similar.

We take the first case as an example. First we introduce a lemma to compute the mean and variance of 1-dimensional truncated normal distribution.

**Lemma 2.1.** Suppose a random variable \( z \in \mathbb{R} \) follows the normal distribution with mean \( \mu \) and variance \( \sigma^2 \). For constants \( a < b \), let \( \alpha = (a - \mu)/\sigma \) and \( \beta = (b - \mu)/\sigma \). Then the mean and variance of \( z \) truncated to the interval \((a, b)\) are given by:

\[
E(z|a < z \leq b) = \mu + \frac{\phi(\alpha) - \phi(\beta)}{\Phi(\beta) - \Phi(\alpha)} \cdot \sigma
\]
\begin{equation}
\text{Var}(z|a < z \leq b) = \left(1 + \frac{\alpha \phi(\alpha) - \beta \phi(\beta)}{\Phi(\beta) - \Phi(\alpha)} - \left(\frac{\phi(\alpha) - \phi(\beta)}{\Phi(\beta) - \Phi(\alpha)}\right)^2\right) \cdot \sigma^2
\end{equation}

According to Eq 1 and Lemma 2.1, we have

\begin{equation}
E[z_j|\mathbf{z}_{D-j}, \ast] = \Sigma_{j, -j}^{-1} \Sigma_{j, -j}^{-1} \mathbf{z}_{j, -j} + (1 - \Sigma_{j, -j}^{-1} \Sigma_{j, -j}^{-1}) \cdot \frac{\phi(\alpha_j) - \phi(\beta_j)}{\Phi(\beta_j) - \Phi(\alpha_j)}
\end{equation}

where

\begin{align*}
\alpha_j &= a_j - \Sigma_{j, -j}^{-1} \Sigma_{j, -j}^{-1} \mathbf{z}_{j, -j} \\
\beta_j &= b_j - \Sigma_{j, -j}^{-1} \Sigma_{j, -j}^{-1} \mathbf{z}_{j, -j}.
\end{align*}

Define \(g_j(\mathbf{z}_{D-j}; x_j, \Sigma) = E[z_j|\mathbf{z}_{D-j}, \ast]: \mathbb{R}^{p-1} \rightarrow \mathbb{R}\), which is a nonlinear function parameterized by \(\{x_j, \Sigma\}\). Approximate \(E[g_j(\mathbf{z}_{D-j}; x_j, \Sigma)]\) using delta method i.e. approximating \(g_j(\mathbf{z}_{D-j}; x_j, \Sigma)\) using its first order expansion at \(\mathbf{z}_{D-j} = E[\mathbf{z}_{D-j}|\ast]\), we obtain

\begin{equation}
E[z_j|\ast] = E[E[z_j|\mathbf{z}_{D-j}, \ast]|\ast] = E[g_j(\mathbf{z}_{D-j}; x_j, \Sigma)|\ast] \approx g_j(E[\mathbf{z}_{D-j}|\ast]; x_j, \Sigma)
\end{equation}

Specifically, the approximation we make is the nonlinear part of \(g_j(\mathbf{z}_{D-j}; x_j, \Sigma)\):

\begin{equation}
E\left[\frac{\phi(\alpha_j) - \phi(\beta_j)}{\Phi(\beta_j) - \Phi(\alpha_j)}\right] \approx \frac{\phi(E[\alpha_j|\ast]) - \phi(E[\beta_j|\ast])}{\Phi(E[\beta_j|\ast]) - \Phi(E[\alpha_j|\ast])}.
\end{equation}

As discussed in our paper, Eq 5 provides a way to approximate \(E[\mathbf{z}_D|\ast]\) during EM iterations. Next, we approximate \(\text{Cov}[\mathbf{z}_D|\ast] = \text{diag}\{\text{Var}[z_j|\ast]\}_{j \in D}\). Using the law of total variance,

\begin{equation}
\text{Var}[z_j|\ast] = E[\text{Var}[z_j|\mathbf{z}_{D-j}, \ast]|\ast] + E[\text{Var}[E[z_j|\mathbf{z}_{D-j}, \ast]|\ast]
\end{equation}

For term 1, define \(h_j(\mathbf{z}_{D-j}; x_j, \Sigma) = \text{Var}[z_j|\mathbf{z}_{D-j}, \ast]\), which is a nonlinear function parameterized by \(\{x_j, \Sigma\}\). Approximate a nonlinear term of \(h_j(\mathbf{z}_{D-j}; x_j, \Sigma)\):

\begin{equation}
E\left[\frac{\alpha_j \phi(\alpha_j) - \beta_j \phi(\beta_j)}{\Phi(\beta_j) - \Phi(\alpha_j)}\right] \approx \frac{E[\alpha_j|\ast] \cdot \phi(E[\alpha_j|\ast]) - E[\beta_j|\ast] \cdot \phi(E[\beta_j|\ast])}{\Phi(E[\beta_j|\ast]) - \Phi(E[\alpha_j|\ast])}
\end{equation}

Similarly to Eq 5, With approximation in Eq 6 and Eq 8, we obtain

\begin{equation}
E[h_j(\mathbf{z}_{D-j}; x_j, \Sigma)|\ast] \approx h_j(E[\mathbf{z}_{D-j}|\ast]; x_j, \Sigma)
\end{equation}

This term is the one dimensional truncated normal variance with \(\mathbf{z}_{D-j} = E[\mathbf{z}_{D-j}|\ast]\). For term 2, further approximate a nonlinear term:

\begin{equation}
E\left[\frac{\phi(\alpha_j) - \phi(\beta_j)}{\Phi(\beta_j) - \Phi(\alpha_j)}\right] \approx E[\mathbf{z}_{D-j}|\ast] \cdot \frac{\phi(E[\alpha_j|\ast]) - \phi(E[\beta_j|\ast])}{\Phi(E[\beta_j|\ast]) - \Phi(E[\alpha_j|\ast])}
\end{equation}
then we obtain
\[
\begin{align*}
\text{Var}\left[ E[z_j|z_{D^{-j}},*]\right] & = E \left[ (E[z_j|z_{D^{-j}},*])^2 \right] - (E \left[ E[z_j|z_{D^{-j}},*]\right])^2 \quad (11) \\
\approx E \left[ (E[z_j|z_{D^{-j}},*])^2 \right] - (g_j(E[z_{D^{-j}},*]; x_j, \Sigma))^2 \quad (12) \\
\approx \Sigma_{j,-j}\Sigma_{j,-j}^{-1}\text{Cov}[z_{-j},*]\Sigma_{j,-j}^{-1} \Sigma_{j,j} \quad (13)
\end{align*}
\]

where Cov[z_{-j},*] is diagonal. It has positive diagonal entries for dimensions $D - j$ and 0 for dimensions $C$. Specically, from Eq 11 to 12, we plugged in Eq 5. From Eq 12 to 13, we plugged in Eq 8 and 10. To put everything together, at $(t + 1)$-th EM iteration, with $E[z_{D^{-j}},*]$ and $\text{Cov}[z_{D^{-j}},*]$ from $t$-th iteration, we can proceed as:

\[
\begin{align*}
\text{Var}[z_j|x, \Sigma^{(t+1)}] & \approx h_j(E[z_{D^{-j}},x, \Sigma^{(t)}]; x_j, \Sigma^{(t+1)}) \\
& + \Sigma_{j,-j}\left(\Sigma_{j,-j}^{(t+1)}\right)^{-1}\cdot \text{Cov}[z_{-j},x, \Sigma^{(t)}]\cdot \left(\Sigma_{j,j}^{(t+1)}\right)^{-1}\Sigma_{j,j}^{(t+1)}
\end{align*}
\]

### 2.2 Incomplete Mixed Observation

In our paper, we have discussed how to compute $E[z_D,*], E[z_M,*]$ and Cov[z_D,*]. Now we provide more details on how to compute Cov[z_M,z_D,*] and Cov[z_M,*]. Notice Cov[z_M, z_D,*] is a submatrix of Cov[z_M,z,*], we compute Cov[z_M,z,*] directly.

\[
\text{Cov}[z_M,z_*] = E[z_Mz_*^T] = E[z_M]* - E[z_M]*E[z_*^T]. \quad (14)
\]

Using the law of total expectation and first assume $z_*$ is known,
\[
E[z_Mz_*^T] = E\left[ E[z_Mz_*^T|z_*,*]\right] = E\left[ E[z_M|z_*,*] \cdot z_*^T \right] = E\left[ \Sigma_{M,O} \Sigma_{*,*}^{-1} z_* \cdot z_*^T \right] = \Sigma_{M,O} \Sigma_{*,*}^{-1} E[z_*z_*^T]. \quad (15)
\]

Showed in our paper $E[z_M,*] = \Sigma_{M,O} \Sigma_{*,*}^{-1} E[z_*,*]$. Plug it and Eq 14 into Eq 15, we obtain
\[
\text{Cov}[z_M,z_*] = \Sigma_{M,O} \Sigma_{*,*}^{-1} \text{Cov}[z_*|x_*,\Sigma].
\]

Similarly,
\[
\text{Cov}[z_M,*] = E[z_Mz_*^T] = E[z_M]* - E[z_M]*E[z_*^T] \quad (16)
\]

Again using the law of total expectation and first assume $z_*$ is known,
\[
E[z_Mz_*^T] = E\left[ E[z_Mz_*^T|z_*,*]\right] = E\left[ \Sigma_{M,O} \Sigma_{*,*}^{-1} z_* \cdot z_*^T \right] = \Sigma_{M,M} - \Sigma_{M,O} \Sigma_{*,*}^{-1} \Sigma_{O,M} + \Sigma_{M,O} \Sigma_{*,*}^{-1} \Sigma_{O,O} \cdot \text{Cov}[z_*|*] \cdot \Sigma_{*,*}^{-1} \Sigma_{O,M}. \quad (17)
\]

Plug Eq 17 into Eq 16, we obtain
\[
\text{Cov}[z_M,*] = \Sigma_{M,M} - \Sigma_{M,O} \Sigma_{*,*}^{-1} \Sigma_{O,M} + \Sigma_{M,O} \Sigma_{*,*}^{-1} \Sigma_{O,O} \cdot \text{Cov}[z_*|*] \cdot \Sigma_{*,*}^{-1} \Sigma_{O,M}.
\]
3 More Experiments Details

3.1 Synthetic Data

We further provide evaluation results for continuous data imputation using relative RMSE for both simple and complex experiment settings in Fig 1. The relative RMSE is defined as $\frac{||\hat{X} - X||_F}{||X||_F}$ where $X$ deontes the continuous columns and $\hat{X}$ denotes its imputed values.

For the simple setting, our algorithm clearly outperforms sbgcop. For the complex setting, our algorithm performs similar to sbgcop when missing ratio is no more than 20% and slightly worse than sbgcop for high missing ratio. Both algorithms clearly outperform all other methods. The optimal rank selected using 5CV is 3 for xPCA and 6 for imputeFAMD.

![Boxplots showing relative RMSE for continuous data imputation](image)

(a) Simple Setting  (b) Complex Setting

Figure 1: Imputation error for continuous dimensions in terms of relative RMSE.

3.2 MovieLens 1M Data

The MovieLens 1M dataset originally has size 6040 * 207. After we select the 207 movies with more 1000 users’ rating, there is only one user who didn’t rate any of 207 movies. We deleted that user and used the subset with size 6039 * 207.

To select tuning parameters for softImpute, we apply 5 cross validation using all data. For GLRM, we need to select the loss function, the regularization function and the regularization parameters. Exhaustive search using 5CV is very expensive. We split all available data into 80/20 training/test sets. After a small grid search, we choose bigger-vs-smaller loss, rank 8, ordinal regularization on $Y$ factors, and quadratic regularization with parameter $n_{obs} * 1.2 * 10^{-4}$ on $X$ factors where $n_{obs}$ is
Table 1: Imputation error on 207 movies.

| Algorithm     | MAE       | RMSE      |
|---------------|-----------|-----------|
| softImpute    | 0.646(0.003) | 0.833(0.005) |

the number of observed entries. We fit the model with SVD initialization and offset term. We implement GLRM using LowRankModels.jl [5].

We further provide evaluation results of softImpute without rounding its’ results to 1 − 5 integer scales in Table 1. As we can see, without the rounding, softImpute achieves the best performance in terms of RMSE. It is reasonable since softImpute minimizes RMSE to train its model. However, its MAE increases without the rounding.

3.3 More Ordinal Data and Mixed Data

The ESL dataset is used in [2]. The GBSG dataset and the TIPS dataset are used in [1]. The ESL and LEV datasets are available at https://www.cs.waikato.ac.nz/ml/weka/datasets.html. The GBSG dataset is available in R package mfp https://cran.r-project.org/web/packages/mfp. The TIPS dataset is available at http://www.ggobi.org/book/. We provide more description about the four datasets we used below.

- **ESL** This data set contains 488 profiles of applicants for certain industrial jobs. Expert psychologists of a recruiting company, based upon psychometric test results and interviews with the candidates, determined the values of the input attributes. The output is the an overall score corresponding to the degree of fitness of the candidate to this type of job.

- **LEV** This data set contains 1000 examples of anonymous lecturer evaluations, taken at the end of MBA courses. Before receiving the final grades, students were asked to score their lecturers according to four attributes such as oral skills and contribution to their professional/general knowledge. The single output was a total evaluation of the lecturer’s performance.

- **GBSG** This data set contains the information of 686 women with breast cancer concerning the status of the tumours and the hormonal system of the patient. It has 6 continuous variables and 4 ordinal variables.

- **TIPS** This data set concerns the tips given to a waiter in a restaurant collected from 244 customers. There are 2 continuous and 5 ordinal variables containing the price of the meal, the tip amount and the conditions of the restaurant meal(number of guests, time of data, etc.).

We further provide evaluation results of continuous dimensions using relative RMSE for mixed datasets in Table 2. Our algorithm performs similar to sbgcop and both algorithms outperform all other imputation algorithms. We also provide the optimal ranks selected using 5CV for xPCA and imputeFAMD in Table 3.
Table 2: Imputation error in terms of relative RMSE (standard deviation).

| Dataset:Type | EM(our)     | sbgcop     | missForest | xPCA | imputeFAMD |
|--------------|-------------|------------|------------|------|------------|
| GBSG: Continuous | 0.263(0.01) | 0.262(0.01) | 0.278(0.01) | 0.426(0.01) | 0.272(0.01) |
| TIPS: Continuous | **0.177(0.02)** | **0.177(0.02)** | 0.192(0.02) | 0.256(0.02) | 0.211(0.08) |

Table 3: Optimal ranks selected using 5CV.

|             | ESL | LEV | GBSG | TIPS |
|-------------|-----|-----|------|------|
| xPCA        | 1   | 1   | 2    | 2    |
| imputeFAMD  | 5   | 5   | 2    | 6    |

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