Is the entropy $S_q$ extensive or nonextensive? *

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The cornerstones of Boltzmann-Gibbs and nonextensive statistical mechanics respectively are the entropies $S_{BG} \equiv -k \sum_{i=1}^W p_i \ln p_i$ and $S_q \equiv k_1 (1 - \sum_{i=1}^W p_i^q)/(q - 1)$ ($q \in \mathbb{R}$; $S_1 = S_{BG}$). Through them we revisit the concept of additivity, and illustrate the (not always clearly perceived) fact that (thermodynamical) extensivity has a well defined sense only if we specify the composition law that is being assumed for the subsystems (say $A$ and $B$). If the composition law is not explicitly indicated, it is tacitly assumed that $A$ and $B$ are statistically independent. In this case, it immediately follows that $S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B)$, hence extensive, whereas $S_q(A + B)/k = [S_q(A)/k] + [S_q(B)/k] + (1 - q)[S_q(A)/k][S_q(B)/k]$, hence nonextensive for $q \neq 1$. In the present paper we illustrate the remarkable changes that occur when $A$ and $B$ are specially correlated. Indeed, we show that, in such case, $S_q(A + B) = S_q(A) + S_q(B)$ for the appropriate value of $q$ (hence extensive), whereas $S_{BG}(A + B) \neq S_{BG}(A) + S_{BG}(B)$ (hence nonextensive). We believe that these facts substantially improve the understanding of the mathematical need and physical origin of nonextensive statistical mechanics, and its interpretation in terms of effective occupation of the $W$ a priori available microstates of the full phase space. In particular, we can appreciate the origin of the following important fact. In order to have entropic extensivity (i.e., $\lim_{N \to \infty} S(N)/N < \infty$, where $N \equiv$ number of elements of the system), we must use (i) $S_{BG}$, if the number $W^{eff}$ of effectively occupied microstates increases with $N$ like $W^{eff} \sim N^\mu$ ($\mu \geq 1$); (ii) $S_q$ with $q = 1 - 1/\rho$, if $W^{eff} \sim N^\rho < W$ ($\rho \geq 0$). We had previously conjectured the existence of these two markedly different classes. The contribution of the present paper is to illustrate, for the first time as far as we can tell, the derivation of these facts directly from the set of probabilities of the $W$ microstates.

I. INTRODUCTION

A quantity $X(A)$ associated with a system $A$ is said additive with regard to a (specific) composition of $A$ and $B$ if it satisfies

$$X(A + B) = X(A) + X(B),$$

where $+$ inside the argument of $X$ precisely indicates that composition. For example, suppose we partition the interior of a single closed bottle in two parts. If no chemical or other reactions occur between the gas molecules that might be inside the bottle, nor between these molecules and the bottle itself (and its internal physical partition), the number of gas molecules is an additive quantity with regard to the elimination of the partition surface. The same happens with the total energy of an ideal gas, where all interactions have been neglected, including the gravitational one. More trivially, the total height of various (rectangular) doors is, practically speaking, an additive quantity, if we pile them one above the other one. Not so if we put them side by side! On an abstract level, it is clear that this additivity just corresponds to the number of elements of the union of two sets $A$ and $B$ that have no common elements.

If, instead of two subsystems $A$ and $B$, we have $N$ of them ($A_1, A_2,..., A_N$), then we have that

$$X\left(\sum_{i=1}^N A_i\right) = \sum_{i=1}^N X(A_i).$$

If the subsystems happen to be all equal (a quite common case), then we have that

$$X(N) = NX(1),$$

with the notations $X(N) \equiv X(\sum_{i=1}^N A_i)$ and $X(1) \equiv X(A_1)$.

An intimately related concept is that of extensivity. It appears frequently in thermodynamics and elsewhere, and corresponds to a weaker demand, namely that of

$$\lim_{N \to \infty} \frac{X(N)}{N} < \infty.$$  

Clearly, all quantities that are additive with regard to a given composition, also are extensive with regard to

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that same composition (and \( \lim_{N \to \infty} X(N)/N = X(1) \)), whereas the opposite is not necessarily true. For example, the total energy, the total entropy and the total magnetization of the standard Ising ferromagnetic model with \( N \) spins on a square lattice are extensive but not additive quantities. In other words, they are asymptotically additive, but not strictly additive. Of course, there are quantities that are neither additive nor even extensive. They are called nonextensive. All types of behaviors can exist, such as \( X(N) \propto N^\gamma \) (\( \gamma \geq 0 \)). For instance, thermonuclear reactions that, with regard to some specific composition, exhibit \( \gamma = 0 \) are called intensive. Such is the case of the temperature, pressure, chemical potential and similar quantities in a great variety of (thermodynamically equilibrated) systems observed in nature. A less trivial example of nonextensive quantity emerges within a spatially homogeneous \( d \)-dimensional classical gas whose \( N \) particles (exclusively) interact through a two-body interaction potential that is strongly repulsive at short distances whereas it is attractive at long distances, decaying like \( 1/r^\alpha \) (\( r \equiv \text{distance between two particles} \)), \( 0 \leq \alpha/d \leq 1 \). The total potential energy of such a system corresponds \[ \text{to } \gamma = 2 - \alpha/d \] if \( 0 \leq \alpha/d \leq 1 \) (i.e., nonextensive), and to \( \gamma = 1 \) for \( \alpha/d > 1 \) (i.e., extensive). The total potential energy of this particular model has a logarithmic \( N \)-dependance (i.e., nonextensive) at the limiting value \( \alpha/d = 1 \). The Lennard-Jones model for gases corresponds to \((\alpha, d) = (6, 3)\), and has therefore an extensive total energy. In contrast, if we assume a cluster of stars gravitationally interacting (together with some physical mechanism effectively generating repulsion at short distances), we have \((\alpha, d) = (1, 3)\), hence nonextensivity for the total potential energy. The physical nonextensivity which naturally emerges in such anomalous systems is, in some theoretical approaches, disguised by artificially dividing the two-body coupling constant (which has in fact no means of “knowing” the total number of particles of the entire system) by \( N^{1-\alpha/d} \). For the particular case \( \alpha = 0 \) this yields the widely (and wildly!) used division by \( N \) of the coupling constant, typical for a variety of mean field approaches. See \ref{2} for more details.

Boltzmann-Gibbs (BG) statistical mechanics is based on the entropy

\[
S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i ,
\]

with

\[
\sum_{i=1}^{W} p_i = 1 ,
\]

where \( p_i \) is the probability associated with the \( i^{th} \) microscopic state of the system, and \( k \) is Boltzmann constant. In the particular case of equiprobability, i.e., \( p_i = 1/W \) (\( \forall i \)), Eq. (5) yields the celebrated Boltzmann principle (as named by Einstein \ref{3}):

\[
S_{BG} = k \ln W .
\]

From now on, and without loss of generality, we shall take \( k \) equal to unity.

Nonextensive statistical mechanics, first introduced in 1988 \cite{4,5,6} (see \cite{7,8,9,10,11,12,13,14,15} for reviews), is based on the so-called “nonextensive” entropy \( S_q \) defined as follows:

\[
S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathbb{R}; \quad S_1 = S_{BG}) .
\]  

(8)

For equiprobability (i.e., \( p_i = 1/W \), \( \forall i \)), Eq. (8) yields

\[
S_q = \ln_q W ,
\]

(9)

with the \( q \)-logarithm function defined \[\ref{10}\] as

\[
\ln_q z = \frac{z^{1-q} - 1}{1 - q} \quad (z \in \mathbb{R}; \quad z > 0; \quad \ln_1 z = \ln z) .
\]

(10)

The inverse function, the \( q \)-exponential, is given by

\[
e^q_z \equiv [1 + (1 - q)z]^{1/(1-q)} \quad (e^q_1 = e^z)
\]

(11)

if the argument \( 1 + (1 - q)z \) is positive, and equals zero otherwise.

The present paper is entirely dedicated to the analysis of the additivity or nonadditivity of \( S_{BG} \) and of its generalization \( S_q \). However, following a common (and sometimes dangerous) practice, we shall from now on cease distinguishing between additivity and extensive, and use exclusively the word extensive in the sense of strictly additive.

\section{II. THE CASE OF TWO SUBSYSTEMS}

Consider two systems \( A \) and \( B \) having respectively \( W_A \) and \( W_B \) possible microstates. The total number of possible microstates for the system \( A + B \) is then in principle \( W = W_{A+B} = W_A W_B \). We emphasized the expression “in principle” because, as we shall see, a more or less severe reduction of the full phase space might occur in the presence of strong correlations between \( A \) and \( B \).

We shall use the notation \( p_{ij}^{A+B} \) (\( i = 1, 2, ..., W_A; \ j = 1, 2, ..., W_B \)) for the joint probabilities, hence

\[
\sum_{i=1}^{W_A} \sum_{j=1}^{W_B} p_{ij}^{A+B} = 1 .
\]

(12)

The marginal probabilities are defined as follows:

\[
p_i^A \equiv \sum_{j=1}^{W_B} p_{ij}^{A+B} ,
\]

(13)

hence

\[
\sum_{i=1}^{W_A} p_i^A = 1 ,
\]

(14)
and
\[ p_i^B = \sum_{j=1}^{W_B} p_{ij}^{A+B} , \quad (15) \]
hence
\[ \sum_{j=1}^{W_B} p_j^B = 1 . \quad (16) \]

These quantities are indicated in the following Table.

| A \(\setminus\) B | 1   | 2   | \cdots | W_B |
|-------------------|-----|-----|--------|-----|
| 1                 | \(p_{11}^{A+B}\) | \(p_{12}^{A+B}\) | \cdots | \(p_{1W_B}^{A+B}\) |
| 2                 | \(p_{21}^{A+B}\) | \(p_{22}^{A+B}\) | \cdots | \(p_{2W_B}^{A+B}\) |
| \vdots            | \vdots | \vdots | \cdots | \vdots |
| \(W_A\)          | \(p_{W_A1}^{A+B}\) | \(p_{W_A2}^{A+B}\) | \cdots | \(p_{W_AW_B}^{A+B}\) |
| \(1\)            | \(p_1^A\) | \(p_2^A\) | \cdots | \(p_{W_B}^A\) |

We shall next illustrate the importance of the specification of the composition law. Let us consider two cases, namely independent and (specially) correlated subsystems.

**A. Two independent subsystems**

Consider a system composed by two independent subsystems A and B, i.e., such that the joint probabilities are given by
\[ p_{ij}^{A+B} = p_i^A p_j^B \quad (\forall (i,j)) . \quad (17) \]

With the definitions
\[ S_{BG}(A + B) \equiv - \sum_{i=1}^{W_A} \sum_{j=1}^{W_B} p_{ij}^{A+B} \ln p_{ij}^{A+B} , \quad (18) \]
\[ S_{BG}(A) \equiv - \sum_{i=1}^{W_A} p_i^A \ln p_i^A , \quad (19) \]
and
\[ S_{BG}(B) \equiv - \sum_{j=1}^{W_B} p_j^B \ln p_j^B , \quad (20) \]
we immediately verify that
\[ S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B) \quad (21) \]

and, analogously, that
\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) . \quad (22) \]

Therefore, \( S_{BG} \) is extensive. Consistently, \( S_q \) is, unless \( q = 1 \), nonextensive. It is in fact from property (22) that the \( q \neq 1 \) statistical mechanics we are referring to has been named nonextensive.

**B. Two specially correlated subsystems**

Consider now that A and B are correlated, i.e.,
\[ p_{ij}^{A+B} \neq p_i^A p_j^B , \quad (23) \]
Assume moreover, for simplicity, that both A and B systems are equal, and that \( W_A = W_B = 2 \). Assume finally that the joint probabilities are given by the following Table (with \( 1/2 < p < 1 \)):

| A \(\setminus\) B | 1   | 2   | \cdots | \cdots |
|-------------------|-----|-----|--------|--------|
| 1                 | \(2p - 1\) | \(1 - p\) | \(p\) |
| 2                 | \(1 - p\) | \(0\) | \(1 - p\) |

It can be trivially verified that Eq. (21) is not satisfied. Therefore, for this special correlation, \( S_{BG} \) is nonextensive. It can also be verified that, for \( q = 0 \) and only for \( q = 0 \), the following additivity is satisfied:
\[ S_0(A + B) = S_0(A) + S_0(B) , \quad (24) \]
then we are using here and in the rest of the paper “extensive” to strictly mean “additive”) can be \( S_{BG} \) or a different one.

Before going on, let us introduce right away the distinction between \( a \text{ priori} \) possible states (in number \( W \)) and \( allowed \) or \( effective \) states (in number \( W_{\text{eff}} \)). Let us consider the above case of two equal binary subsystems \( A \) and \( B \) and consequently \( W = 4 \). If they are independent (i.e., the \( q = 1 \) case), their generic case corresponds to \( 0 < p < 1 \), hence \( W_{\text{eff}} = 4 \). But if they have the above special correlation (i.e., the \( q = 0 \) case), their generic case corresponds to \( 1/2 < p < 1 \), hence \( W_{\text{eff}} = 3 \) (indeed, the state (2,2), although possible a priori, has zero probability). This type of distinction is at the basis of this entire paper. Notice also that the \( q = 1 \) and \( q = 0 \) cases can be unified through \( W_{\text{eff}} = [2^{1-q} + 2^{1-q} - 1]^{1/(1-q)} = [2^{q} - 1]^{1/(1-q)} \). This specific unification will be commented later on.

Let us further construct on the above observations. Is it possible to unify, at the level of the joint probabilities, the case of independence (which corresponds to \( q = 1 \)), with the specially correlated case that we just analyzed (which corresponds to \( q = 0 \))? Yes, it is possible. Consider the following Table:
where \( f_q(p) \) is given by the following relation:

\[
2p^q + 2(1-p)^q - (f_q)^q - 2(p-f_q)^q - (1-2p+f_q)^q = 1, \tag{25}
\]

with \( f_q(1) = 1 \), and \( 0 \leq q \leq 1 \) (later on we shall comment on values outside this interval). Typical curves \( f_q(p) \) are indicated in Fig. 1. Since Eq. (25) is an implicit one, it can be checked, for instance, that \( f_q(1/2) \) smoothly increases from zero to unity, being very flat in the neighborhood of \( q = 0 \), and rather steep in the neighborhood of \( q = 1 \). The interesting point, however, is that it can be straightforwardly verified that, for the value of \( q \) chosen in \( f_q(p) \) defined through Eq. (25) (and only for that \( q \)),

\[
S_q(A + B) = 2S_q(A) = 2 \left( 1 - p^q - (1-p)^q \right), \tag{26}
\]

where we have used the fact that \( A = B \). In other words, we are facing a whole family of entropies that are extensive for the respective special correlations indicated in the Table just above.

Let us proceed and generalize the previous examples to two-state systems \( A \) and \( B \) that are not necessarily equal. The case of independence is trivial, and is indicated in the following Table:

\[
\begin{array}{c|c|c|c}
A \setminus B & 1 & 2 \\
1 & f_q(p) & p - f_q(p) & p \\
2 & p - f_q(p) & 1 - 2p + f_q(p) & 1 - p \\
\end{array}
\]

We verify that Eq. (24) is satisfied. Is it possible to unify the above anisotropic \( q = 1 \) and \( q = 0 \) cases? Yes, it is. The special correlations for these cases are indicated in the following Table:

\[
\begin{array}{c|c|c|c}
A \setminus B & 1 & 2 \\
1 & p_1^A & p_1^B & p_1^A \\
2 & p_2^A & p_2^B & p_2^A \\
\end{array}
\]

Of course, Eq. (21) is satisfied.

Let us consider now the following Table (with \( p_1^A + p_1^B > 1 \)):

\[
\begin{array}{c|c|c|c}
A \setminus B & 1 & 2 \\
1 & p_1^A + p_1^B - 1 & 1 - p_1^B & p_1^A \\
2 & 1 - p_1^A & 0 & 1 - p_1^A \\
\end{array}
\]

where \( f_q(p_1^A, p_1^B) = f_q(p_1^B, p_1^A) \), \( f_q(p, 1) = p \), \( f_q(p, p) = f_q(p) \), \( f_q(p_1^A, 1, p_1^B) = p_1^A p_1^B \), and \( f_0(p_1^A, p_1^B) = p_1^A + p_1^B - 1 \). For any value of \( q \) in the interval \([0, 1]\), and for any probabilistic pair \((p_1^A, p_1^B)\), the function \( f_q(p_1^A, p_1^B) \) is (implicitly) defined through

\[
(p_1^A)^q + (1 - p_1^A)^q + (p_1^B)^q + (1 - p_1^B)^q - [f_q(p_1^A, p_1^B)]^q - [p_1^A - f_q(p_1^A, p_1^B)]^q - [1 - p_1^A - p_1^B + f_q(p_1^A, p_1^B)]^q = 1 \tag{27}
\]

(We remind that, for the \( q = 0 \) particular case, it must be \( p_1^A + p_1^B > 1 \)). We notice that the special correlations we are addressing here make that all joint probabilities can be expressed as functions of only one of them.

\[
S_q(A + B) = 2S_q(A) = 2 \left( 1 - p^q - (1-p)^q \right), \tag{26}
\]
say $p_{11}^{A+B}$, which is determined once for ever. More explicitly, we have that $p_{12}^{A+B} = p_1^A - p_{11}^{A+B}$, $p_{21}^{A+B} = p_1^B - p_{11}^{A+B}$, $p_{22}^{A+B} = 1 - p_1^A - p_1^B - p_{11}^{A+B}$.

Eq. (27) recovers Eq. (25) as the particular instance $p_1^A = p_1^B$. And we can easily verify that, for $0 \leq q \leq 1$,

$$S_q(A + B) = S_q(A) + S_q(B).$$

So, we still have extensivity for the appropriate value of $q$, i.e., the value of $q$ which has been chosen in Eq. (27) to define the function $f_q(x, y)$ reflecting the special type of correlations assumed to exist between $A$ and $B$. In other words, when the marginal probabilities have all the information, then the appropriate entropy is $S_{BG}$. But this happens only when $A$ and $B$ are independent. In all the other cases addressed within the above Table, the important information is by no means contained in the marginal probabilities, and we have to rely on the full set of joint probabilities. In such cases, $S_{BG}$ is nonextensive, whereas $S_q$ is extensive.

Before closing this section dedicated to the case of two systems, let us indicate the Table associated to the $q = 0$ entropy for arbitrary systems $A$ and $B$:

| $A^\backslash B$ | $W_B$ |
|------------------|-------|
| 1                |       |
| 2                |       |
| $\vdots$         |       |
| $W_A$            |       |

We easily verify that Eq. (24) is satisfied. For example, the generic case corresponds to all probabilities in the Table being nonzero, excepting those explicitly indicated in the Table. For this case we have $S_0(A) = W_A - 1$, $S_0(B) = W_B - 1$, and $S_0(A + B) = W_A + W_B - 2$. This is a neat illustration of the fact that, although the full space admits in principle $W = W_A W_B$ microstates, the strong correlations reflected in the Table make that the system uses appreciably less, namely, in this example, $W^{eff} = W_A + W_B - 1$. It is tempting to conjecture the generalization of this expression into $W^{eff} = [W_A^{1-q} + W_B^{1-q} - 1]^{1/(1-q)}$ for $0 \leq q \leq 1$. It is clear that $W^{eff} \leq W_A W_B$, the equality holding only for $q = 1$. Since, strictly speaking, $W_A$, $W_B$ and $W^{eff}$ are integer numbers, this expression for $W^{eff}$ can only be generically valid for real $q \neq 0, 1$ in some appropriate asymptotic sense. This sense has to be for $W_A, W_B >> 1$, which however are not fully addressed in the present paper for $q \neq 0, 1$. For the particular instance $A = B$, we have $W^{eff} = [2W_A^{1-q} - 1]^{1/(1-q)}$.

We also verify another interesting aspect. If $A$ and $B$ are independent, equal values in the marginal probabilities are perfectly compatible with equal values in the joint probabilities. In the most general independent two-system case, we can simultaneously have $p_i^A = 1/W_A$ ($\forall i$), $p_i^B = 1/W_B$ ($\forall j$), and $p_{ij}^{A+B} = 1/(W_A W_B)$ ($\forall (i,j)$). This is not possible in the above Table. Indeed, equal probability values for all allowed microstates in the Table imply $p_{ij}^{A+B} = 1/(W_A + W_B - 1)$ ($\forall (i,j)$), which is incompatible with equal values for the marginal probabilities. This fact starts pointing into what kind of (irreducibly correlated) situation, the usual BG microcanonical hypothesis “equal probability occupation of the entire phase space” for thermal equilibrium might become inadequate. It is very plausible that a variety of microscopic dynamical situations must exist (e.g., long-range-interacting Hamiltonian systems) for which the standard equilibrium hypothesis is an oversimplification for physically relevant stationary states that do not correspond to thermal equilibrium.

### III. THE CASE OF THREE SUBSYSTEMS

Consider now three systems $A$, $B$, and $C$, having respectively $W_A$, $W_B$, and $W_C$ possible microstates. The total number of possible microstates for the system $A + B + C$ is then in principle $W = W_A W_B W_C$. As for the case of two systems, we shall see that strong collective correlations between $A$, $B$, and $C$ may cause a severe reduction of the allowed phase space.

We shall use the notation $p_{ijk}^{A+B+C}$ ($i = 1, 2, \ldots, W_A$; $j = 1, 2, \ldots, W_B$; $k = 1, 2, \ldots, W_C$) for the joint probabilities, hence

$$\sum_{i=1}^{W_A} \sum_{j=1}^{W_B} \sum_{k=1}^{W_C} p_{ijk}^{A+B+C} = 1.$$  \hspace{1cm} (29)

The $AB$–marginal probabilities are defined as follows:

$$p_{ij}^{A+B} \equiv \sum_{k=1}^{W_C} p_{ijk}^{A+B+C},$$  \hspace{1cm} (30)

hence

$$\sum_{i=1}^{W_A} \sum_{j=1}^{W_B} p_{ij}^{A+B} = 1.$$  \hspace{1cm} (31)

Similar expressions exist for the $AC$– and $BC$–marginal probabilities. The joint probabilities for the $W_A = W_B = W_C = 2$ case are indicated in the following Table, where the numbers without parentheses correspond to system $C$ in state 1, and the numbers within parentheses correspond to system $C$ in state 2.
The corresponding \( AB \)-marginal probabilities are indicated in the Table below:

\[
\begin{array}{c|cc}
A \backslash B & 1 & 2 \\
\hline
1 & p_{11}^{A+B+C} & p_{112}^{A+B+C} \\
2 & p_{211}^{A+B+C} & p_{212}^{A+B+C} \\
\end{array}
\]

which of course reproduces the situation we had for the two-system \((A + B)\) problem. This is to say \( p_{11}^{A+B} = p_{111}^{A+B} + p_{112}^{A+B+C} \), and so on.

### A. Three independent subsystems

Consider first the case where all three subsystems \( A \) and \( B \) are binary and statistically independent, i.e., such that the joint probabilities are given by

\[ p_{ijk}^{A+B+C} = p_i^A p_j^B p_k^C \quad (\forall (i, j, k)) \tag{32} \]

The corresponding Table is of course as follows

\[
\begin{array}{c|cc}
A \backslash B & 1 & 2 \\
\hline
1 & p_{11}^{A+B+C} p_i^A p_j^B p_k^C & p_{12}^{A+B+C} p_i^A p_j^B p_k^C \\
2 & p_{21}^{A+B+C} p_i^A p_j^B p_k^C & p_{22}^{A+B+C} p_i^A p_j^B p_k^C \\
\end{array}
\]

We immediately verify that

\[ S_{BG}(A + B + C) = S_{BG}(A) + S_{BG}(B) + S_{BG}(C) \tag{33} \]

Therefore, \( S_{BG} \) is extensive. Consistently, \( S_q \) is, unless \( q = 1 \), nonextensive.

### B. Three specially correlated subsystems

Consider now that the three binary subsystems are correlated as indicated in the next Table (with \( p_i^A + p_i^B + p_i^C > 2 \)):

\[
\begin{array}{c|cc}
A \backslash B & 1 & 2 \\
\hline
1 & p_i^A + p_i^B + p_i^C - 2 & 1 - p_i^B \tag{34} \\
2 & 1 - p_i^A & 0 \\
\end{array}
\]

We easily verify that

\[ S_0(A + B + C) = S_0(A) + S_0(B) + S_0(C) \]

For example, if \( A = B = C \) and \( 2/3 < p < 1 \), we have that \( S_0(A + B + C) = 3S_0(A) = 3 \).

Let us next unify the \( q = 1 \) and the \( q = 0 \) cases. We heuristically found the solution. It is indicated in the following Table:

\[
\begin{array}{c|cc}
A \backslash B & 1 & 2 \\
\hline
1 & f_q(p_i^A, p_i^B) + f_q(p_i^A, p_i^C) - f_q(p_i^A, p_i^B) - f_q(p_i^A, p_i^C) & \tag{34} \\
2 & \tag{34} \\
\end{array}
\]

where the function \( f_q(x, y) \) is defined in Eq. (27). Interestingly enough, it has been possible to find a three-subsystem solution in terms of the two-subsystem and one-system ones. More explicitly, we have, for example, that

\[ p_{111}^{A+B+C} = f_q(p_i^A, p_i^C) + f_q(p_i^A, p_i^C) - f_q(p_i^A, p_i^B) - f_q(p_i^A, p_i^C) \]

and similarly for the other seven three-subsystem joint probabilities. Of course, all eight joint probabilities associated with the above Table are nonnegative; whenever the values of \( p_i^A, p_i^B, p_i^C \) replaced within one or the other of these analytic expressions yield negative numbers, the corresponding probabilities are to be taken equal to zero.

The \( AB \)-marginal probabilities precisely recover the joint probabilities of the previously discussed two-system \((A + B)\) Table. For example, \( f_q(p_i^A, p_i^C) + f_q(p_i^A, p_i^C) - f_q(p_i^A, p_i^B) - f_q(p_i^A, p_i^C) \) and similarly for the other seven

\[
\begin{array}{c|cc}
A \backslash B & 1 & 2 \\
\hline
1 & p_i^A + p_i^B + p_i^C - 2 & 1 - p_i^B \tag{34} \\
2 & 1 - p_i^A & 0 \\
\end{array}
\]

Finally, we verify that

\[ S_q(A + B + C) = \frac{1}{2} [S_q(A + B) + S_q(A + C) + S_q(B + C)] \]
\[ = S_q(A) + S_q(B) + S_q(C) \]  
(35) or, equivalently,

\[ 1 + (1 - q)S_q(\sum_{r=1}^{N} A_r) = \prod_{r=1}^{N} [1 + (1 - q)S_q(A_r)] \]  
(39)

For the particular case \( A = B = C \), the above Table becomes

| A | B | 1 | 2 |
|---|---|---|---|
| 1 | \(2(f_q(p) - p^2) + p f_q(p)\) | \(2p^2 - f_q(p) - p f_q(p)\) | \(p(1 - 2p + f_q(p))\) |
| 2 | \(2p^2 - f_q(p) - p f_q(p)\) | \(p(1 - 2p + f_q(p))\) | \((1 - p)(1 - 2p + f_q(p))\) |

where we have used \( f_q(p, p) = f_q(p) \).

For the generic case of three subsystems with \( W_A, W_B \) and \( W_C \), states respectively, we have that \( W = W_A W_B W_C \), whereas in the appropriate asymptotic sense we expect \( W^{eff} = [W_A^{1 - q} W_B^{1 - q} W_C^{1 - q} - 2]^{1/(1-q)} \leq W \) for \( 0 \leq q \leq 1 \) (the equality generically holds only for \( q = 1 \)). In the particular instance \( A = B = C \), this expression becomes \( W^{eff} = [3W_A^{1 - q} - 2]^{1/(1-q)} \).

### IV. ENLARGING THE SCENARIO

#### A. The case of \( N \) subsystems

The three-system case discussed above is a generic one under the assumption that \( W_A = W_B = W_C = 2 \). We have not attempted to generalize its corresponding special correlation Table to the generic \((W_A, W_B, W_C)\) case, and even less to the even more general case of \( N \) such systems \((A_1, A_2, ..., A_N)\). It is clear however that, assuming that this (not necessarily trivial) task was satisfactorily accomplished, the result would lead to

\[ S_q(\sum_{r=1}^{N} A_r) = \sum_{r=1}^{N} S_q(A_r), \]  
(36)

where \( q = 1 \) if all \( N \) systems are mutually independent, i.e.,

\[ p_{i_1 i_2 ... i_N}^{A_1 + A_2 + ... + A_N} = \prod_{r=1}^{N} p_{i_r}^{A_r} \quad (\forall (i_1, i_2, ..., i_N)), \]  
(37)

and \( q \neq 1 \) otherwise. This is to say, if we have independence, the only entropy which is extensive is \( S_{BG} \). If we do not have independence but the special type of (collective) correlations focused on in this paper instead, then only \( S_q \) for a special value of \( q \) is extensive.

For the case of independence, the generic composition law for \( S_q \) is given by

\[ \ln[1 + (1 - q)S_q(\sum_{r=1}^{N} A_r)] = \sum_{r=1}^{N} \ln[1 + (1 - q)S_q(A_r)], \]  
(38)

Eq. (38) exhibits in fact the well known (monotonic) connection between \( S_q \) and the Renyi entropy \( S^R_q \).

We have generically \( W = \prod_{i=1}^{N} W_{A_i} \), which corresponds of course to the total number of a priori possibly occupied states (i.e., whose joint probabilities are generically nonzero) for the generic \( q = 1 \) case. In contrast, the generic \( q = 0 \) case has only \( W^{eff} = (\sum_{i=1}^{N} W_{A_i}) - (N - 1) \) nonzero joint probabilities. These are \( p_{i_1 i_2 ... i_N}^{A_1 + A_2 + ... + A_N} = \prod_{i=1}^{N - 1} p_{i_i}^{A_i} \) \((i_1 = 2, 3, ..., W_{A_1}), p_{i_1 i_2 ... i_{N-1}} = p_{i_2}^{A_2} \) \((i_2 = 2, 3, ..., W_{A_2}), p_{i_11 ... i_N} = p_{i_N}^{A_N} \) \((i_N = 2, 3, ..., W_{A_N})\). The generic \( q = 1 \) and \( q = 0 \) cases can, analogously to what has been done before, be unified through \( W^{eff} = [(\sum_{i=1}^{N} W_{A_i}^{1-q}) - (N - 1)]^{1/(1-q)} \). In a given interaction \( A_1 = A_2 = ... = A_N = A \), this expression becomes \( W^{eff} = [W_A^{1-q} - (N - 1)]^{1/(1-q)} \).

Furthermore, for \( N \) equal subsystems (a quite frequent case, as already mentioned), Eq. (36) becomes

\[ S_q(N) = N S_q(1), \]  
(40)

where the change of notation is transparent. This is an extremely interesting relation since it already has the shape that accommodates well within standard thermodynamics, even if the entropic index \( q \) is not necessarily the usual one, i.e., \( q = 1 \). It is allowed to think that Clausius would perhaps have been as satisfied with this relation as he surely was with the same relation but with \( S_{BG} \)! One might also quite safely speculate that if the system is such that its Table of joint probabilities is not exactly of the type we have discussed here, but close to it, then we might have, not exactly relation (40) but rather only asymptotically \( S_q(N) \sim N \). In other words, as long as the system belongs to what we may refer to as the \( q \)-universality class, we should expect \( \lim_{N \to \infty} S_q(N)/N < \infty \), in total analogy with the usual \( BG \) case.

To geometrically interpret Eq. (40), we may consider the case of equal probabilities in the allowed phase space, i.e., in that part of phase space which is expected to have, not necessarily \( W \) microstates, but generically \( W^{eff} \) macrostates (with \( W^{eff} < W \)). The effective number \( W^{eff} \) is expected (at least in the \( N >> 1 \) limit) to be precisely the number of all those states that the special collective correlations allow to visit. So, if we assume equal probabilities in Eq. (40) (i.e., \( p_{i_1 i_2 ... i_N}^{A_1 + A_2 + ... + A_N} = 1/W^{eff} \), we
obtain
\[
\ln_q W^{\text{eff}} \equiv \frac{(W^{\text{eff}})^{1-q} - 1}{1 - q} = NS_q(1),
\]
(41)
or, equivalently
\[
W^{\text{eff}} = e^{NS_q(1)} = [1 + (1 - q)NS_q(1)]^{1/(1 - q)}. \tag{42}
\]
Two cases are possible for this relation, namely \( q = 1 \) and \( q < 1 \). In the first case, we have the usual result
\[
W^{\text{eff}} = W = \mu^N,
\]
(43)
with
\[
\mu \equiv e^{S_{BG}(1)} \geq 1. \tag{44}
\]
In the second case, we have an unusual result, namely
\[
W^{\text{eff}} = [1 + NS_q(1)/\rho]^\rho, \tag{45}
\]
with
\[
\rho \equiv 1/(1 - q) \geq 0. \tag{46}
\]
In the \( N \to \infty \) limit, this relation becomes the following one:
\[
W^{\text{eff}} \propto N^\rho. \tag{47}
\]
This (physically quite appealing) possibility was informally advanced by us long ago, and formally in [12]. It has now been obtained along an appropriate probabilistic path.

B. The \( q \to -\infty \) case

From Eq. (46) we expect the \( q \to -\infty \) case to correspond to the limiting situation where \( W^{\text{eff}} \) is constant. To realize this situation, let us first consider the \( A = B \) two-system case with the following Table (\( W_A = W_B \)):

| \( A \backslash B \) | 1     | 2     | ... | \( W_A \) |
|-----------------|-------|-------|-----|---------|
| 1               | \( p_1 \) | 0     | 0   | \( p_1 \) |
| 2               | 0     | \( p_2 \) | 0   | \( p_2 \) |
| ...             | ...   | ...   | ... | ...     |
| \( W_A \)       | 0     | 0     | \( pW_A \) | \( pW_A \) |
| \( p_1 \)       | \( p_2 \) | ...   | 1   |         |

This Table corresponds to \( p_{ij}^{A+B} = p_i \delta_{ij} \). Its generalization to \( N \) equal systems is trivial: \( p_{i_1,i_2,...,i_N} = p_{i_1} \) if all \( N \) indices coincide, and zero otherwise. The corresponding entropy therefore asymptotically approaches the relation
\[
S_{-\infty}(N) = S_{-\infty}(1) \quad (\forall N), \tag{48}
\]
thus corresponding to \( \rho = 0 \) as anticipated. It appears then that all cases equivalent (through permutations) to the above Table, should yield the same limit \( q \to -\infty \).

C. Connection with the Borges-Nivanen-Le Mehaute-Wang \( q \)-product

Let us mention at this point an interesting connection that can be established between the present problem and the \( q \)-product introduced by L. Nivanen, A. Le Mehaute and Q.A. Wang and by E.P. Borges [17]. It is defined as follows:
\[
x \times_q y \equiv (x^{1-q} + y^{1-q} - 1)^{1/(1-q)} \quad (x \times_1 y = xy). \tag{49}
\]
It has the elegant, extensive-like, property
\[
\ln_q(x \times_q y) = \ln_q x + \ln_q y, \tag{50}
\]
to be compared with the by now quite usual, nonextensive-like, property
\[
\ln_q(xy) = \ln_q x + \ln_q y + (1 - q)(\ln_q x)(\ln_q y). \tag{51}
\]
This type of structure was since long (at least since 1999) being informally discussed by A.K. Rajagopal, E.K. Lenzi, S. Abe, myself, and probably others. But only very recently it was beautifully formalized [17]. It has immediately been followed and considerably extended by Suyari in a relevant set of papers [13].

Let us now go back to the main topic of the present paper. Consider the following joint probabilities associated with \( N \) generic subsystems:
\[
p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} = \left[1 - N + \phi^{(q)}_{i_1 i_2 ... i_N} + \sum_{r=1}^{N} (p_{i_r})^{q-1} \right]^{1/(q-1)}, \tag{52}
\]
where \( \phi^{(q)}_{i_1 i_2 ... i_N} \) is a nontrivial function which ensures that
\[
\sum_{i_1 i_2 ... i_N} p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} = 1 \tag{53}
\]
In the limit \( q \to 1 \), Eq. (52) must recover the independent-systems one, namely
\[
p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} = \prod_{r=1}^{N} p_{i_r}^{A_r}, \tag{54}
\]
which implies \( \phi^{(1)}_{i_1 i_2 ... i_N} = 0 \).

Notice that, excepting for the function \( \phi^{(q)}_{i_1 i_2 ... i_N} \), Eq. (52) associates \( 1/p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} \) with
\[
\prod_{r=1}^{N} x_r^{(q)} = \left[ x_1^{1-q} + x_2^{1-q} + ... + x_N^{1-q} - N + 1 \right]^{1/(1-q)}.
\]
It follows from Eq. (52) that
\[
(p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N})^{q} = (1 - N + \phi^{(q)}_{i_1 i_2 ... i_N}) p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} + p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} \sum_{r=1}^{N} (p_{i_r})^{q-1} \tag{55}
\]

hence
\[
\sum_{i_1 i_2 ... i_N} (p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N})^{q} = (1 - N) + \sum_{i_1 i_2 ... i_N} p^{A_1 + A_2 + ... + A_N}_{i_1 i_2 ... i_N} \sum_{r=1}^{N} (p_{i_r})^{q-1}, \tag{56}
\]
where we have imposed one more nontrivial condition on $\phi_{i_1 i_2 \ldots i_N}^{(q)}$, namely that
\[
\sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} \phi_{i_1 i_2 \ldots i_N}^{(q)} = 0 .
\] (57)

One might naturally have the impression that no function $\phi_{i_1 i_2 \ldots i_N}^{(q)}$ might exist satisfying simultaneously Eqs. (53) and (57). This is not so however, at least for particular cases, since we have explicitly shown in the present paper that

\[
(1 - q) S_q \left( \sum_{r=1}^{N} A_r \right) = \sum_{i_1 i_2 \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} \sum_{r=1}^{N} \left( p_{i_1 i_2 \ldots i_N}^{A_r} \right)^{-1} - N .
\] (58)

Let us now introduce in Eq. (58) the definition of marginal probabilities, namely
\[
p_{i_1 i_2 \ldots i_N}^{A_r} = \sum_{i_1 i_2 \ldots i_r - 1 i_{r+1} \ldots i_N} p_{i_1 i_2 \ldots i_N}^{A_1 + A_2 + \ldots + A_N} .
\] (59)

We obtain
\[
(1 - q) S_q \left( \sum_{r=1}^{N} A_r \right) = \sum_{r=1}^{N} \left( p_{i_1 i_2 \ldots i_N}^{A_r} \right)^{-1} - N .
\] (60)

Using once again the definition of $S_q$ on the right-hand member, we finally obtain
\[
S_q \left( \sum_{r=1}^{N} A_r \right) = \sum_{r=1}^{N} S_q(A_r)
\] (61)
as desired.

It should, however, be clear that this remarkable mathematical fact by no means exhausts the problem of the search of explicit Tables of joint probabilities that would lead to extensivity of $S_q$ for nontrivial values of $q$. The constraints imposed by the definition itself of the concept of marginal probabilities are of such complexity that the search of solutions is by no means trivial, at least at our present degree of knowledge. Indeed, one easily appreciates this fact by looking at the explicit solutions indicated in Sections II.B and III.B.

V. CONCLUSIONS

Let us summarize the obvious conclusion of the present paper: Unless the composition law is specified, the question whether an entropy (or some similar quantity) is or is not extensive has no sense. Allow us a quick digression. The situation is in fact quite analogous to the quick or slow motion of a body. Ancient Greeks considered the motion to be an absolute property. It was not until Galileo that it was clearly perceived that motion has no sense unless the referential is specified. In Galileo's time, and even now, when no referential is indicated, one tacitly assumes that the referential is the Earth. In total analogy, when no composition law is indicated for analyzing the extensivity of an entropy, one tacitly assumes that the subsystems that we are composing are independent. It is only — a big only! — in this sense that we can say that $S_{BG}$ is extensive, and that $S_q$ (for $q \neq 1$) is nonextensive.

Once we have established the point above, the next natural question is: Are there classes of collective correlations for which we know which is the specific entropy to be extensive? (knowing, of course, that absence of all correlations leads to $S_{BG}$). For this operationally important question, nontrivial illustrations on how the entropic form is dictated by the type of special collective correlations that might (or might not) exist in the system have explicitly presented in Section II.B and III.B. From this discussion, two vast categories of systems are identified (at the most microscopic possible level, i.e., that of the joint probabilities), namely those whose allowed phase space increases (in size) with $N$ like an exponential or like a power-law, corresponding respectively to $q = 1$ and to $q < 1$.

However, it should be clear that the present paper is only exploratory in what concerns this hard task. Indeed, we have not found the generic answer for $N$ (not necessarily equal) systems, and we have basically concentrated only on the interval $0 \leq q \leq 1$. We do not even know without doubt if the answer is unique (excepting of course for trivial permutations), or if it admits a variety of forms all belonging to the same universality class of nonextensivity (i.e., sharing the same value of the entropic index $q$). Even worse, we still do not know what specifically happens in the structure of the allowed phase space in the (thermodynamically) most important limit $N \to \infty$, or in the frequent limit $W_A \to \infty$ (which would provide a precise geometrical interpretation to a formula such as $W_{\text{eff}} = (W_A^{1-q} - (N-1))^{1/(1-q)}$ for say $0 \leq q \leq 1$). It is precisely this structure which is crucial for fully understanding nonextensive statistical mechanics and its related applications in terms on nonlinear dynamical systems. For example, an interesting situation might occur if we compare the distribution which optimizes $S_q(N)$ and then consider $N \gg 1$, with the distribution corresponding to having first considered $N \gg 1$ in $S_q(N)$ and then only optimizing. We certainly expect the thermodynamic limit and the optimization operation to commute for a system composed by $N$ independent (or nearly independent) subsystems. But the situation seems to be more subtle if our system was composed by $N$ subsystems correlated in that special, collective manner which demands $q \neq 1$ in order to have entropy extensivity. Such a situation would be consistent with a property which emerges again and again [7, 8, 9, 10, 11, 12, 13, 14, 15].
for nonextensive systems, namely that the $N \to \infty$ and the $t \to \infty$ limits do not necessarily commute. One more relevant issue concerns what specific dynamical nature is required for a physical system to “live”, in phase space, within a structure close to one of those that we have presently analyzed. It is our conjecture that this would occur for nonlinear dynamical systems whose Lyapunov spectrum is either zero or close to it, i.e., under circumstances similar to the edge of chaos, where many of the so called complex systems are expected to occur. We leave all these questions as open points needing further progress.

Let us finally mention the following point. It is by no means trivial to find sets of joint probabilities (associated to relevant statistical correlations) that produce no means trivial to find sets of joint probabilities (associated to relevant statistical correlations) that produce very simple marginal probabilities (such as $c_i$) which simultaneously (i) admits such solutions, (ii) is concave ($\forall q > 0$), (iii) is Lesche-stable, and (iv) leads to finite entropy production per unit time, constitutes — we believe — a strong mathematical basis for being physically meaningful in the thermostatistical sense.

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