Hidden entanglement in the presence of random telegraph dephasing noise

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Abstract

Entanglement dynamics of two noninteracting qubits, locally affected by random telegraph noise at pure dephasing, exhibits revivals. These revivals are not due to the action of any nonlocal operation; thus their occurrence may appear paradoxical since entanglement is by definition a nonlocal resource. We show that a simple explanation of this phenomenon may be provided by using the (recently introduced) concept of hidden entanglement, which signals the presence of entanglement that may be recovered with the only help of local operations.

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1. Introduction

Entanglement is one of the most peculiar features of quantum mechanics and also plays the role of a fundamental resource in many applications of quantum information [1–4]. On the other hand, entangled systems unavoidably interact with their environments, causing decoherence and a loss of entanglement. Since entanglement is by definition a nonlocal resource, one expects that any attempt to restore it must involve the use of nonlocal operations.

We consider physical situations where two subsystems, for example two qubits, are prepared in an entangled state and subsequently decoupled [5, 6]. Due to the interaction with their local environment, entanglement dynamics may exhibit a nonmonotonic behaviour, with the occurrence of revivals alternating with dark periods [7–12]. In some cases, this phenomenon is due to the fact that entanglement is transferred to quantum environments and then back-transferred to the system [7, 8]. In the other cases, the environment can be modelled as a classical system [9, 11, 13] and no entanglement between the system and the environment is established at any time. In the latter cases, the occurrence of entanglement revivals may appear paradoxical, since the effect of the noise is analogous to a local operation on a subsystem. A first interpretation of this phenomenon has been given in terms of correlations present in a classical-quantum state of environments and qubits [11]. In a recent work [14], we have proposed to solve the apparent paradox by introducing the concept of hidden entanglement (HE), which measures the amount of entanglement that may be recovered without the help of any nonlocal operation. The definition of HE is based on the quantum trajectory description of the system dynamics, which allows to point out the presence of entanglement in the system even if the density operator formalism does not reveal it: this entanglement is thus not accessible (hidden) due to a lack of classical information [14].
Relevant examples of situations where the environment can be modelled as a classical system may be found in solid-state implementations of qubits. For instance, in superconducting nanocircuits, one of the most relevant sources of decoherence [15–17] is fluctuating background charges localized in the insulating materials surrounding superconducting islands [18]. Each impurity produces a bistable fluctuation of the island polarization. The collective effect of an ensemble of these random telegraph (RT) processes, with a proper distribution of switching rates, gives rise to 1/f-noise [20] routinely observed in nanodevices [17–19].

In this paper we exploit the concept of HE to explain the occurrence of entanglement revivals in a simple system. In particular, we consider two noninteracting qubits, one of them affected by an RT noise at pure dephasing [21–23]. The paper is organized as follows. In section 2, we introduce the Hamiltonian model. In section 3 we discuss the entanglement dynamics, showing that the revivals of entanglement are due to the presence of HE. In section 4 we summarize the obtained results and present some final comments.

2. The model

We consider two noninteracting qubits $A$ and $B$, initially prepared in a pure maximally entangled state $\rho(0) = |\psi|\langle\psi|$, evolving according to the Hamiltonian ($\hbar = 1$)

$$
\mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H},
$$

$$
\mathcal{H}_0 = -\frac{\Omega_A}{2}\sigma_{za} - \frac{\Omega_B}{2}\sigma_{zb}, \quad \delta \mathcal{H} = -\frac{\xi(t)}{2}\sigma_{za},
$$

(1)

where $\sigma_{za} = \sigma_z \otimes \mathbb{1}$, $\sigma_{zb} = \mathbb{1} \otimes \sigma_z$, and $\delta \mathcal{H}$ represents an RT process $\xi(t) \in [0, v]$ [24] acting on qubit $A$. The RT process induces a random switching of qubit $A$ frequency between $\Omega_A/(2\pi)$ and $\Omega_B/(2\pi)$, with an overall switching rate $\gamma$ (without loss of generality, we assume that $v > 0$). We consider a symmetric RT process where the transition rates between the two states are equal, that is, $\gamma_{0\rightarrow v} = \gamma_{v\rightarrow 0} = \gamma/2$.

Our first aim is to find the system density matrix at any time $\rho(t)$. The two qubits independently evolve under the Hamiltonian (1): qubit $B$ freely evolves whereas qubit $A$ displays a pure dephasing dynamics due to the effect of the stochastic process $\xi(t)$. The dynamics of a single qubit subject to RT noise at pure dephasing has been solved in [21–23]. A possible way to obtain $\rho(t)$ is to solve a stochastic Schrödinger equation which gives the following formal expression for $\rho(t)$:

$$
\rho(t) = \int \mathcal{D}[\xi(t)] P[\xi(t)] \rho_0(t),
$$

(2)

where $\rho_0(t) = |\psi_0(t)|\langle\psi_0(t)|$ with $|\psi_0(t)| = e^{\xi(t)v/2\gamma} |\psi| \times e^{-i\mathcal{H}_0 t}|\psi|$, and the probability of the realization $\xi(t)$ can be written as

$$
P[\xi(t)] = \lim_{m \to \infty} \eta_{m+1}(\xi_0, t_0; \ldots; \xi_1, t_1; \xi_0, t_0),
$$

(3)

where $\eta_{m+1}$ is an $(m+1)$ joint probability for the sampled $\xi(t)$ at regular intervals $\Delta t = (t - t_0)/m$, $t_k = t_0 + k\Delta t$, $\xi_k \equiv \xi(t_k)$ ($k = 0, \ldots, m$) [25]. Since the qubits evolve independently, the above procedure leads to a simple form depending on the single qubit coherences. In the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, where $|ij\rangle \equiv |i\rangle \otimes |j\rangle$, with $\sigma_z |i\rangle = (-1)^i |i\rangle$ and $i \in \{0, 1\}$, and assuming an initial Bell state $|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, we obtain

$$
\rho(t) = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(4)

where the coherence decay factor $q(t)$ reads [21, 23]

$$
q(t) = e^{-\frac{\alpha}{2} \left[ A e^{-\gamma(1+\gamma/2)} + (1-A) e^{-\gamma(1+\gamma/2)} \right]},
$$

(5)

with $\alpha = \sqrt{1 - g^2}$, $A = \frac{1}{2} (1 + \frac{v}{\gamma})$ and $g = v/\gamma$. In the following, we shall exploit $\rho(t)$ given by equation (4) to analyse the two-qubit entanglement dynamics.

3. Entanglement dynamics

To quantify the degree of entanglement of the system state $\rho(t)$, we use the entanglement of formation $E_f$ [26] that can be readily calculated by the formula [27]

$$
E_f(\rho(t)) = f(C(\rho(t))) = \frac{1 + \sqrt{1 - C(\rho(t))^2}}{2},
$$

(6)

where $C(\rho(t))$ is the concurrence and $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$. For the state $\rho(t)$ of equation (4) we obtain $C(\rho(t)) = q(t)$, where $q(t)$ is given in equation (5). It is worth noting that the evolved state $\rho(t)$ belongs to the Hilbert space spanned by the Bell states $|\phi_{\pm}\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)$; therefore the entanglement of formation equals the entanglement cost [28]. In the strong coupling regime, $g = v/\gamma > 1$ [21, 23], entanglement revivals occur during the system dynamics [9, 13].

3.1. Dephasing under a ‘static’ RT process

To understand the nature of this phenomenon we initially consider the limiting case of an extraordinarily slow RT process, $\gamma \to 0$ ($g \to \infty$). This regime physically describes situations where the stochastic process is slow enough to be considered static during the system time evolution lasting $t$, i.e. we assume that $1/\gamma \gg t$ [15]. The evolution expressed by equation (4) describes an average resulting from the collection of several time evolutions each lasting $t$. The average includes the possibility that the RT process takes any of the two values $\xi = 0$ or $v$ at time $t = 0$ with equal probability. By a straightforward calculation, we find that the concurrence in this case is given by $C(\rho(t)) = |\cos(\nu t/2)|$. Under these conditions we do not find any entanglement decay, rather the concurrence is equal to one at times $t_n = 2n\pi/\nu$ and it vanishes at times $t_n = t_0 + \pi/\nu$, where $n$ is a non-negative integer. The entanglement revivals (see the top solid curve in figure 1) are not due to periodic entanglement death and rebirth by nonlocal operations. Indeed, the Hamiltonian evolution described by equation (1) only includes local operations. Since local operations cannot
increase entanglement [3, 4], its increase during the intervals $[\tilde{t}_n, t_n]$ must be attributed to the manifestation of quantum correlations that were already present at times $\tilde{t}_n$, but were hidden, meaning that the density operator formalism does not capture them.

These correlations are evident in the quantum trajectory description of the system dynamics [29]. The system evolution, in fact, results from averaging on only two possible quantum trajectories. The first trajectory corresponds to $\xi(t) = 0$ and the Bloch vector of qubit $A$ rotates around its $z$-axis with frequency $\Omega_A/(2\pi)$. The second trajectory corresponds to $\xi(t) = v$ and the Bloch vector of qubit $A$ rotates around the $z$-axis with a different frequency $(\Omega_A + v)/(2\pi)$. The Bloch vector of qubit $B$ instead rotates in both cases around its $z$-axis with frequency $\Omega_B/(2\pi)$. Thus, during the first trajectory the system state evolves, up to an irrelevant global phase factor, as

$$|\psi_0(t)\rangle = \frac{1}{\sqrt{2}} \left( \langle 00| + e^{-i\Omega_A t}\langle 0\rangle|11\rangle \right).$$

while during the second trajectory the system evolves, apart from an irrelevant global phase factor, as

$$|\psi_v(t)\rangle = \frac{1}{\sqrt{2}} \left( \langle 00| + e^{-i\Omega_A t} e^{-i\Omega_B t}\langle 0\rangle|11\rangle \right).$$

The two quantum trajectories only differ by the fact that the basis states $|00\rangle$ and $|11\rangle$ acquire the additional relative phase $i\gamma t$ in the quantum superpositions of equations (7) and (8). Since the two quantum trajectories occur with equal probability, the system’s state is described by the quantum ensemble

$$\mathcal{A} = \left\{ \left( p_0, |\psi_0(t)\rangle \right), \left( p_v, |\psi_v(t)\rangle \right) \right\},$$

where $p_0 = p_v = \frac{1}{2}$. In contrast with the entanglement associated with the density matrix $\rho(t) = \sum_{i\in[0, 1]} p_i |\psi_i(t)\rangle\langle\psi_i(t)|$, $E_t(\rho(t))$, we consider the average entanglement of the quantum ensemble $\mathcal{A}$ as the average entanglement associated with the states in the ensemble $\mathcal{A}$ [26, 30–32]

$$E_{av}(A, t) = \sum_{i\in[0, 1]} p_i E(|\psi_i(t)\rangle) = 1,$$

since both states $|\psi_0(t)\rangle$ and $|\psi_v(t)\rangle$ are maximally entangled at any time ($E$ is the entropy of entanglement [3, 4]).

The HE [14] of the ensemble $\mathcal{A}$ is defined as the difference between the average entanglement of the ensemble $\mathcal{A}$, equation (10), and the entanglement of the corresponding density operator $\rho(t)$:

$$E_h(A, t) = E_{av}(A, t) - E_t(\rho(t)).$$

The meaning of HE is simple: it measures the entanglement that cannot be exploited as a resource owing to the lack of classical knowledge about which state in the ensemble $\mathcal{A}$ we are dealing with. Once this classical information is provided, the entanglement can be recovered.

Coming back to the interpretation of entanglement revivals, the ensemble description (the average entanglement of $\mathcal{A}$ is equal to 1 at any time, $E_{av}(A, t) = 1$) tells us that at times $t_n$ the system is always in a maximally entangled state $\langle |\psi_0(t_n)\rangle |\psi_0(t_n)\rangle$ or $\langle |\psi_v(t_n)\rangle |\psi_v(t_n)\rangle$ but the lack of classical knowledge about which of the two states in the ensemble $\mathcal{A}$ we are dealing with prevents us from distilling any entanglement: in fact, entanglement is hidden being $E_t(\rho(t_n)) = 0$ and $E_h(A(t_n)) = 0$. At times $t_n$ this lack of knowledge is irrelevant since the random relative phase becomes meaningless at $\nu t_n = 2\pi n$: all the initial entanglement is recovered, $E_t(\rho(t_n)) = 1$ and $E_h(A(t_n)) = 0$.

### 3.2. Dephasing due to an RT process: the dynamic case

We now investigate the case when the RT process evolves during the system evolution time, i.e. we consider the regime $1/\gamma \gtrsim \tau$. This situation is illustrated in figure 1 where we observe that the amplitude of revivals decreases as $\gamma$ increases ($g$ decreases). Also in this case there is HE. The possible quantum trajectories the system undergoes are now infinite.

The system state is described by the quantum ensemble $\mathcal{A}(t) = \{P[\xi(t)], |\psi(t)\rangle\}$ and the average entanglement of $\mathcal{A}$ is calculated by solving the path integral

$$E_{av}(A, t) = \int \mathcal{D}[\xi(t)] P[\xi(t)] E(|\psi(t)\rangle).$$

Once again, we obtain $E_{av}(A, t) = 1$ since during each trajectory the state remains in a maximally entangled state at any time. On the other hand, the entanglement of formation assumes lower values with respect to the static noise case, $\gamma \to 0$. In particular, the amplitude of revivals does not reach anymore the initial maximum value. This is due to the fact that, in general, the action of the RT process during the time evolution makes the coherences $\langle |\psi_0(t)\rangle |\psi_0(t)\rangle$ (in the basis $\{00\}, \{11\}$) no longer in phase at times $t_n$. In the time interval $[0, t_n]$ one or more transitions can occur between the two RT states, such that we can have a random extra phase at the times $t_n$ given by

$$\vartheta(t_n) = \int_0^{t_n} dt' \tilde{\xi}(t') - 2\pi n,$$

7 Note that in [30] the expression ‘HE’ is used with a different meaning.
where $n$ is a non-negative integer. This unknown phase difference is responsible for the decay of the absolute values of coherences $|q(t_n)|$ in the evolved two-qubit state $\rho(t_n)$ of equation (4); if we knew the phase difference $\vartheta(t_n)$ for each state $|\psi_n(t_n)|$, we would be able to restore the coherence absolute value to 1, and therefore recover all the initial entanglement, by simply applying the unitary local operation $e^{-i\vartheta(t_n)\sigma_z}$. For completeness we point out that the relative maxima of the entanglement of formation occur at $t^*_n = t_n/\sqrt{1 - \frac{1}{n^2}}$, as one can derive from equation (5).

Note that the amount of decay of the amplitude of entanglement revivals is monotonically related to the normalized autocorrelation function of the symmetric RT process $R(\tau) = \langle \xi(\tau)\xi(0)/|\xi(0)|^2 \rangle = e^{-\gamma t}$ [24]. Indeed, as we have already mentioned, the reduction of the amplitude revivals is related to the transitions of the RT occurring in $[0,t_n]$, whose mean number is $\gamma t_n/2$. From a quantitative point of view, for $g > 1$ the coherence decay factor equation (5) can be approximated as $|q(t_n)| \approx e^{-|\gamma t_n/2|\cos(\nu t_n/2) + 1/g|\sin(\nu t_n/2)|}$, so that $C(\rho(t_n)) \approx e^{-\gamma t_n/2}$ and $E_\tau(\rho(t_n/2)) = f(-\gamma t_n/2)$, with $f$ defined in equation (6). This clearly shows that the decay of the entanglement amplitude revivals is due to the decrease of the RT correlations, or in other words, to the memory loss of the stochastic process $\xi(t)$ itself.

4. Conclusions

In this paper we have exploited the concept of HE to interpret the occurrence of entanglement revivals in a particular system where back-action from the environment is absent. Namely, we have considered a system composed of two noninteracting qubits where one qubit is subject to RT noise at pure dephasing. During the dynamics, entanglement vanishes and revives always ‘remaining’ inside the system, as it is signalled by the average entanglement of the quantum ensemble describing the system dynamics, $E_\omega(t_n) = 1$ at any time. At certain times $t_n$ this entanglement is completely hidden, meaning that the entanglement of formation $E_\xi(t_n) = 0$, while the hidden entanglement $E_\omega(t_n) = E_\omega(t_n) - E_\xi(t_n) = 1$. For this reason, the two-qubit entanglement can be simply recovered at subsequent times without the help of any nonlocal operation: in the considered case, in fact, the Hamiltonian only involves local operations.

Finally, we remark that the concept of HE can be of practical relevance in solid-state devices, where dominant noise sources typically have large-amplitude components at low frequencies. In these systems, entanglement revivals may also be induced by applying local pulses to the qubits [13, 14, 33].

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A D’Arrigo et al