Using dark states to charge and stabilise open quantum batteries

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We introduce an open quantum battery protocol using dark states to achieve both superextensive capacity and power density, with only local interactions. Further, our power density actually scales with the of number of spins \( N \) in the battery. We show that the enhanced capacity and power is correlated with entanglement. Whilst connected to the charger, the charged state of the battery is a steady state, stabilised through quantum interference in the open system.

The recent interest in quantum technologies is driven by the potential power of quantum mechanics \cite{1,2}, and the push towards technological miniaturisation. Harnessing the unique properties of quantum mechanics, such as entanglement and superposition, promises to open new vistas in computing, sensing, cryptography, and other quantum technologies \cite{3,4}. The increasing rate of technology miniaturisation, in particular electronics, has meant that we need to account for quantum effects. This has driven the relatively new field of quantum thermodynamics, which tries to understand thermodynamic concepts such as work, heat, and entropy in a quantum context \cite{5,6,7,8,9,10,11,12,13,14,15}. Quantum batteries (QBs) aim to harness the unique properties of quantum thermodynamics to build batteries that are fundamentally different from conventional batteries \cite{26,27}.

Typically, QBs were modelled as a collection of \( N \) identical quantum subsystems to which an external field, which acted as the energy source, was applied \cite{28,29}. Alicki and Fannes \cite{28} sought to understand whether entanglement could enhance the amount of extractable work in this model. Under closed unitary evolution, they showed that one can extract more work with entanglement than without. Further work revealed that it may be possible to reduce the amount of entanglement without detrimentally affecting the maximal work extraction, with the caveat that with reduced entanglement one requires more operations \cite{25}. This then lead to the notion that entanglement boosted the charging rate of QBs, as it reduced that number of traversed states in the Hilbert space between the initial and final separable states \cite{25}. This conjecture was further supported by Binder \textit{et al.} \cite{26}, who showed that entangled spins can superextensively charge \( N \) time faster than non-interacting spins, where \( N \) is the number of spins. The main finding was that through global entangling operators, where all spins can interact with each other, can result in a speed-up of the charging power as compared to charging them individually.

All these studies assumed global operators, which in practice is difficult to implement. Ferraro \textit{et al.} \cite{28} overcame this problem by showing that, by locally coupling all of the spins coherently to the same quantum energy source in a photonic cavity, one can realize effective long-range interactions amongst all the spins. Known as the Dicke QB, after the Hamiltonian that describes it, they showed that the time taken to reach the maximum stored energy in the spin ensemble reduced as the ensemble got larger, such that the charging power scaled with \( \sqrt{N} \) for large \( N \). This increased the potential for QBs to be physically realised.

Recently, QBs have been considered in an open system context \cite{30,31,32}. This is important as QB must interact with its environment for the device to ever be practical. In particular, protocols are needed to stabilise the charged state of the QB in an open system. A recent attempt proposed the continual measurement of the system for stabilisation \cite{34}. However, this protocol requires continuous access to the battery, and the measurement process itself is costly, consuming energy.

Here we use dark states to achieve both superextensive capacity and power, that scales with \( N \), with only local interactions. We will show that the superextensive behaviour of the system is correlated with entanglement. Furthermore, the stored energy of the battery is stable without the need to continually access the battery.

\textit{Model.} In general, the QB charging protocol consists of a battery and an energy source or charger. Switching on (off) the coupling between the battery and charger initiates the charging (discharging) process. We consider a QB in an open system, modelled as an ensemble of \( N_B \) \( \frac{1}{2} \)-spins with transition energy \( \hbar \omega \), in a thermal reservoir Fig. 1. Initially, the QB is in thermal equilibrium with the reservoir. The charger is another ensemble of \( N_C \) \( \frac{1}{2} \)-spins, but in the excited (up) state. We will assume \( N_C \geq N_B \). The charging process is initiated by bringing the charger into the reservoir.

During charging, the time evolution of the open spin ensemble is governed by the Lindblad master equation \cite{33},

\begin{equation}
\dot{\rho}(t) = -i\omega [J^z_C + J^z_B, \rho(t)] \\
+ \gamma(t) \left[ (\bar{n} + 1)\mathcal{L}(J^+_C + J^+_B) + \bar{n}\mathcal{L}(J^-_C + J^-_B) \right],
\end{equation}

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\end{equation}
where \( J^x_i \) are the usual collective spin operators on ensemble \( i \), with the usual collective raising and lowering operators defined as \( J^\pm_i = J^x_i \pm i J^y_i \). \( \mathcal{L}(O) = 2i[O, \rho] + \frac{\partial}{\partial t}\rho \) is the Lindblad superoperator, while \( \gamma(t) \) is the collective battery and charger coupling to the reservoir, with \( \bar{n} = 1/(\epsilon\hbar \omega/k_B T - 1) \) being the mean thermal population.

Energy transfer and stabilisation mechanism. Naively, one may expect any energy to be loss to the thermal reservoir at zero temperature. However, quantum interference can lead to steady states that are not the ground state. Consider for example, the two spin case at \( T = 0 \), which at initial time is \( |\psi_0\rangle = |\frac{1}{2}\rangle C |\frac{1}{2}\rangle B \). This can be expressed as

\[
\rho_0 = \frac{1}{2} (|\psi_+\rangle\langle \psi_+| + |\psi_+\rangle\langle \psi_-| + |\psi_-\rangle\langle \psi_+| + |\psi_-\rangle\langle \psi_-|),
\]

where \( |\psi_\pm\rangle = (|\frac{1}{2}\rangle C |\frac{1}{2}\rangle B \pm |\frac{1}{2}\rangle C |\frac{1}{2}\rangle B)/\sqrt{2} \). The anti-symmetric component does not couple to the reservoir, since \( \mathcal{L}(J^-_C + J^-_B) = 0 \) for \( |\psi_-\rangle\langle \psi_-| \), and therefore does not decay. Such states are known as dark or subradiant states \[33, 34\]. The other components decay to the ground state leading to a steady state of the form

\[
\rho_{ss} = \frac{1}{2} |\psi_+\rangle\langle \psi_+| + \frac{1}{2} |\psi_-\rangle\langle \psi_-|,
\]

where \( |\psi_\pm\rangle = (|\frac{1}{2}\rangle C |\frac{1}{2}\rangle B \pm |\frac{1}{2}\rangle C |\frac{1}{2}\rangle B)/\sqrt{2} \) (see appendix for a formal derivation). In this steady state the spin angular momenta of the charger and battery are

\[
\langle J^z_C \rangle = -\frac{\hbar}{4}, \quad \langle J^z_B \rangle = \frac{\hbar}{4},
\]

where \( \langle J^z_i \rangle = \text{Tr}(\rho_i J^z_i) \). We immediately observe that \( \hbar \omega/4 \) units of energy has been transferred from the charger to the battery, since initially \( \langle J^z_B \rangle = -\hbar/2 \). One notes that this transfer of energy cannot be viewed (semi-classically) as a transfer of energy due to the emission of a photon by the charger followed by the absorption of that photon by the battery. Instead, this is a purely quantum mechanical effect which arises out of the the collective behaviour of the battery, charger, and reservoir. As the steady state is decoupled from the environment, the stored energy of the battery is stable whilst the charger is present, even in the open system. This is the basis of how energy is transferred and stably stored in our open system protocol.

Superextensive capacity. The energy density of the charger and battery are

\[
E_i(t) = \omega \frac{\langle J^z_i \rangle}{N_i},
\]

with \( i = B, C \). Let use define the capacity of the battery as the energy in the steady state,

\[
E_{R,N_B} \equiv N_B \varepsilon_{ss} B ,
\]

where \( \varepsilon_{ss} B \) is the steady state energy density with \( R / N_C / N_B \) being the ratio of the number of spins in the charger to the battery. We have shown for the case where \( N_C / N_B = 1 \), that the steady state energy of the battery is \(-\hbar \omega/4\). If we had \( M \) of these systems isolated from each other, the energy density would not change, so that the total capacity would be \( E_{1,1} = M \varepsilon_{ss} B = -M \hbar \omega/4 \). However, we can improve on this by charging the batteries collectively.

As a example, let us consider the case with \( R = 5 \) during charging. Solving the master equation \[\text{Eq. [1]}\] at zero temperature, we plot \( E_i(t) \) for \( N_B = 1, 2, 3 \) in Fig. 2(a). Firstly, the plots show that \( \varepsilon_{ss} C(t) \) monotonically decreases as \( \varepsilon_{ss} B \) correspondingly increases, indicating a transfer of energy from charger to the battery. Secondly, \( \varepsilon_{ss} B \) increases with \( N_B \). This is shown in Fig. 2(b) where we plot \( \varepsilon_{ss} B(N_B) \) for \( R = 2, 5, 10 \). As \( \varepsilon_{ss} B(N_B) \) increases monotonically, the capacity of the battery scales superextensively. Fig. 2(c) plots \( \varepsilon_{ss} B(R) \) for \( N_B = 1, 2, 3 \). In the thermodynamic limits, \( \lim_{N_B \to \infty} \varepsilon_{ss} B = \lim_{N_B \to \infty} \varepsilon_{ss} B = \hbar \omega/2, \forall R > 1 \).

The superextensive scaling of \( \varepsilon_{ss} B \) means that the capacity of one battery with \( M \) spins is greater than \( M \) batteries with one spin, i.e. \( E_{R,M} > M E_{R,1} \), \( \forall M > 1 \). This improves upon the Dicke QB, where the capacity does not superextensively scale with the number of spins.

One notes that not all stored energy may be extractable as work. In the open system, the thermal state energy \( \langle E_B^{th} \rangle \) represents a natural limit on extractable work as

\[
\mathcal{W}_{\text{open}} = E_B - \langle E_B^{th} \rangle.
\]

For zero temperature \( \langle E_B^{th} \rangle = 0 \), and so \( \mathcal{W}_{\text{open}} = E_B \). Another class of extractable work occurs under unitary evolution of the battery, and is known as ergotropy. Here extractable work is upper bounded by

\[
\mathcal{W}_{\text{closed}} = E_B - \min_{U_B} E_B ,
\]

where the second term is read as the minimum battery energy under all possible unitary evolution of the battery \( U_B \). It is conjectured that \( \mathcal{W}_{\text{closed}} \to E_B \) in the large \( N_B \) limit \[35\]. This is a particular useful conjecture as this would mean that in principle nearly all the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{model.png}
\caption{Model. The spin-charged QB is modelled as an ensemble of spins in a reservoir. Initially, the QB and charger is in thermal equilibrium with the reservoir. The charger is another ensemble of spins, but in the excited state. The charging process is initiated by bringing the charger into the reservoir.}
\end{figure}
stored energy could be extracted as work, in most practical applications. Our system is indeed consistent with this conjecture, as shown in the appendix. In addition, we find that $W_{\text{closed}} \rightarrow \mathcal{E}_B$ in the large $R$ limit also.

Superextensive charging. The power density of the battery is

$$P_B(t) = \frac{d\mathcal{E}_B(t)}{dt},$$

which we plot in Fig. 2(d). The plot shows that maximum power density $P_B^{\text{max}}$ increases with $N_B$. This is clearly shown in Fig. 2(e) where we observe that $P_B^{\text{max}}(N_B) \propto N_B$. Up until now, charging protocols have required global interactions to achieve $N$ scaling [25, 37]. Protocols with local interactions have not exceeded $\sqrt{N}$ scaling [28, 30, 31]. Here we have shown for the first time that one can achieve $N$ power scaling with local interactions. Fig. 2(f) shows that $P_B^{\text{max}}(R)$ also scales with $R$.

As the battery superextensively charges, if one were to simply disconnect the charger, it would also superextensively discharge. The reason for this is that the coherent spins would superradiantly decay [35]. However, if a slow discharge is desired, we propose an intermediated process of dephasing to destroy spin coherence, before disconnecting the charger. This could be achieved with a dephasing pulse, for example. With no coherence, the battery would discharge at the single-spin relaxation rate.

Entanglement. The role of entanglement has been studied in closed unitary QB systems [24, 25, 27, 36, 37]. Here we systematically investigate the role of entanglement in our open QB protocol. For mixed systems, the logarithmic negativity [39, 40] provides a convenient measure of entanglement. It is defined using the trace norm as

$$S_B(t) = \log_2 \| \rho_B(t) \|.$$

We plot $S_B(t)$ in Fig. 3(a), with the same parameters as Fig. 2(a). A comparison of these two plots shows higher entanglement to correspond to higher energy, supporting the idea that entanglement drives the superextensive capacity of the battery. Their relationship is shown in Fig. 3(b), where we plot $\mathcal{E}_B(t)$ and $S_B(t)$ parametrised over $t$. In Fig. 3(c), we plot $S_B^{ss}(N_B)$, showing that steady state entanglement scales positively with $N_B$. In Fig. 3(d) we plot $\mathcal{E}_B^{ss}(N_B)$ and $S_B^{ss}(N_B)$ parametrised over $N_B$, showing the positive correlation between the battery capacity and entanglement.

One notes in Fig. 2(b) that with increasing $R$, the scaling $\mathcal{E}_B^{ss}$ with $N_B$ decreases, i.e., with increasing $R$, the plot tends to flatten out. This indicates a decrease in the superextensive capacity of the battery with $R$. Revealingly, Fig. 3(d) shows that entanglement decreases with increasing $R$ (for a given $\mathcal{E}_B^{ss}$), inline with the decreased superextensive scaling of $S_B^{ss}$ in Fig. 2(b). In other words, as $R$ increases we have less entanglement to drive the system, and hence the ability of the battery capacity to superextensively increase, diminishes. If energy correlates with entanglement, then it follows that power should correlate with entanglement rate. In Fig. 4(a) and (b) we plot $P_B(t)$ and $\dot{S}_B(t)$ for $N_B = 1, 2, 3$ at $R = 50$. Periods of non-zero $P_B(t)$ corresponds to periods of non-zero $\dot{S}_B(t)$. In Fig. 4(c) we plot the local maximum entanglement rate $S_B^{\text{max}}$ for various $R$ (when there are more than one local maxima, such is the case for $N_B = 1$, we choose the largest value). The plot shows that $S_B^{\text{max}}$ linearly scales with $N_B$. As $P_B^{\text{max}}$ also linearly scales with $N_B$, the scaling of $P_B^{\text{max}}$ and $S_B^{\text{max}}$ is reduced.
FIG. 3. Entanglement and capacity. (a) Logarithmic negativity \( S_B(t) \). Comparing this plot with \( E_B(t) \) in Fig. 2(a), shows that higher entanglement corresponds to higher energy density. (b) The relationship between entanglement and energy density is shown in this parametrised plot of \( E_B(t) \) and \( S_B(t) \). (c) \( S_B^\max \) scales positively with \( N_B \). Comparing this plot with \( E_B^\max (N_B) \) in Fig. 2(b), shows that higher entanglement corresponds to higher energy density, in the steady state. (d) \( E_B^\max \) and \( S_B^\max \) parametrised over \( N_B \), shows the positive correlation between the capacity and entanglement. Parameters: (a),(b) \( R = 5, N_B = 1 \) (blue), 2 (orange), 3 (green). (c),(d) \( R = 2 \) (blue), 3 (orange), 5 (green).

FIG. 4. Entanglement rate and power density. (a) Power density \( P_B(t) \). (b) Entanglement rate \( \dot{S}_B(t) \). Comparing (a) and (b) shows that periods of non-zero \( P_B(t) \) approximately corresponds to periods of non-zero \( \dot{S}_B(t) \). (c) Local maximum entanglement rate \( \dot{S}_B^{\max} \) linearly scales with \( N_B \). (d) The lag time between \( P_B^{\max} \) and \( \dot{S}_B^{\max} \) decreases with \( N_B \). Parameters: (a),(b) \( R = 50, N_B = 1 \) (blue), 2 (orange), 3 (green). (c),(d) \( R = 3 \) (blue), 5 (orange), 10 (green).

\( S_B \), \( \dot{S}_B^{\max} \) and \( P_B^{\max} \) are positively correlated. Interestingly, \( P_B^{\max} \) and \( \dot{S}_B^{\max} \) do not occur at the same time: \( \dot{S}_B^{\max} \) lags \( P_B^{\max} \) by \( \Delta t \). Fig. 4(d) plots this lag time; it shows that the lag time decreases with increasing \( N_B \) or \( R \). In the large \( N_B \) or \( R \) limit, the lag time vanishes.

Another important feature revealed by the plots is that \( \dot{S}_B^{\max} \) increases with \( R \), whilst \( S_B^{\max} \) decreases. This correlates with the observation that \( P_B^{\max} \) superextensively increase with \( R \) [Fig. 2(d)], whilst the superextensivity of \( S_B^\min \) diminishes with \( R \) [Fig. 2(b)]. These correlations provide further evidence that entanglement underpins the superextensive properties of the battery.

In unitary systems with global interaction, it has been shown that entangled states reduce the number of operations required to reach a passive state, thereby increasing power [25, 27]; the rate at which entangled states are generated does not seem to play a part. Here we show something different. In our non-unitary system with local interactions, we show that for a given \( R \), energy is correlated with the level of entanglement, and power is related to the rate at which this entanglement is generated. This suggests that open local systems may have a different mechanism for driving superextensive behaviour with entanglement.

Temperature. As previously mentioned, a non-zero temperature lowers the upper bound on extractable work. Beyond this, the effects of thermal fluctuation on the battery provides a rich area of investigation; here we point out some interesting properties. At low temperature the battery obtains its energy primarily from the charger, but as the temperature increases the energy source shifts from the charger to the reservoir. However there is a trade-off between the infusion of energy from the reservoir, and the destruction of dark states caused by thermal fluctuations. As shown in the appendix, for \( R = 1 \), \( E_B^SS \) increases with temperature, as the infusion of energy from the reservoir more than compensates for the loss of energy from the destruction of dark states. Conversely, for \( R > 3 \), \( E_B^SS \) decreases with temperature, with the greatest decline occurring at low temperature, as the infusion of energy from the reservoir cannot compensate for the destruction of dark states. \( R = 2 \) is an interesting intermediary case, as \( E_B^SS \) can both increase or decrease, depending on the temperature.

Implementation. Our protocol can be implemented with atomic or artificial two-level systems, including superconducting qubits, semiconductor quantum dots, ultracold atoms, trapped ions, and nitrogen-vacancy (NV) centres. We propose that experimental verification should be conducted in two regimes. Our protocol should be investigated deep in the quantum regime with few spins and at low temperature, but with a high level of control and measurement. As such, superconducting qubits coupled to a broad band resonator, which acts as the reservoir, would be suitable [41]. However, this platform typically is limited to few qubits.

Although QB capacity on small energy scales may find application in quantum technologies, verifying the ability to scale up capacity is important for wider adoption. Therefore, we propose that the protocol should also be investigated in the semi-classical regime with many spins and high temperature. NV centres coupled to a broad band resonator, would be a suitable platform to achieve this. Large coherent ensembles of NV-centre spins (> 10^16) coupled to superconducting circuits have been used to demonstrate the collective behaviour of superperradiance [42], and the coherent coupling between two
macroscopically separated spin ensembles has also been realised [43].

**Conclusion.** Our protocol is major step towards the experimental realisation of a QB that achieves superextensive capacity and charging: it uses only local interactions, and is intrinsically stable in an open system - two critical features for practical applications. This rich protocol opens the way for further theoretical investigation, including a deeper understanding of the correlation between entanglement rate and power.

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**Appendix A: Ergotropy**

The ergotropy of a system is the maximal amount of work that can be extracted acting cyclically under thermal isolation. This is an important measure, as not all the energy stored in a system can be unitarily extracted as work. The ergotropy is given by [44]

$$\mathcal{W}^\text{closed} = \mathcal{E}_B - \min_{U_B} \mathcal{E}_B ,$$  \hspace{1cm} (A1)

where the second term is the minimum battery energy under all possible unitary evolution of the battery $U_B$. The $\min_{U_B} \mathcal{E}_B = \omega \min_{U_B} \text{Tr}(J_B^2 U_B \rho_B U_B^\dagger)$ term can be found by ordering the eigenvalues of $J_B^2$ in increasing order ($\epsilon_1 < \epsilon_2 < \cdots < \epsilon_n$), and the eigenvalues of $\rho_B$ in decreasing order ($r_1 < r_2 < \cdots < r_n$). From this we get that [44]

$$\min_{U_B} \mathcal{E}_B = \omega \sum_i r_i \epsilon_i .$$  \hspace{1cm} (A2)

We plot in Fig. 5(a) the ergotropy for $R = 5$ for various $N_B$. For $N_B = 1$ the ergotropy is zero until the stored energy is positive. As $N_B$ increases, the ergotropy approaches the stored energy. Fig. 5(b) plots the ergotropy for $N_B = 1$ various $R$. As $R$ increases, the ergotropy approaches the stored energy. The figures shows that work can only be extracted in a cyclic manner when there is a net positive spin angular moment.

**Appendix B: Derivation of the steady state of two spins in a thermal reservoir**

Here we derive the steady state of two spins in a thermal reservoir, which gives Eq. (3) in the main text. We begin by defining the following spin basis:

$$|\psi_1\rangle \equiv |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A \equiv |1\rangle$$  \hspace{1cm} (B1)

$$|\psi_+\rangle \equiv |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A + |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A \equiv |2\rangle$$  \hspace{1cm} (B2)

$$|\psi_-\rangle \equiv |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A - |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A \equiv |3\rangle$$  \hspace{1cm} (B3)

$$|\psi_4\rangle \equiv |\frac{1}{2}\rangle_C |\frac{1}{2}\rangle_A \equiv |4\rangle$$  \hspace{1cm} (B4)

From the Lindblad master equation, we write down the equations of motion for the elements of the Hermitian density matrix in the spin basis defined above $[\rho_{ij} \equiv \langle i|\rho|j\rangle]$:

$$\dot{\rho}_{11} = -2\gamma (\bar{n} + 1) \rho_{11} + 2\gamma \bar{n} \rho_{22}$$  \hspace{1cm} (B5)

$$\dot{\rho}_{22} = 2\gamma (\bar{n} + 1) \rho_{11} - 2\gamma (2\bar{n} + 1) \rho_{22} + 2\gamma \bar{n} \rho_{44}$$  \hspace{1cm} (B6)

$$\dot{\rho}_{33} = 0$$  \hspace{1cm} (B7)

$$\dot{\rho}_{44} = 2\gamma (\bar{n} + 1) \rho_{22} - 2\gamma \bar{n} \rho_{44}$$  \hspace{1cm} (B8)

$$\dot{\rho}_{12} = -[\gamma (3\bar{n} + 2) - i\omega] \rho_{12}$$  \hspace{1cm} (B9)

$$\dot{\rho}_{13} = -[\gamma (\bar{n} - i\omega)] \rho_{13}$$  \hspace{1cm} (B10)

$$\dot{\rho}_{14} = -[\gamma (2\bar{n} + 1) - i2\omega] \rho_{14}$$  \hspace{1cm} (B11)

$$\dot{\rho}_{23} = -[\gamma (2\bar{n} + 1)] \rho_{23}$$  \hspace{1cm} (B12)

$$\dot{\rho}_{24} = -[\gamma (3\bar{n} + 1) - i\omega] \rho_{24}$$  \hspace{1cm} (B13)

$$\dot{\rho}_{34} = -[\gamma (\bar{n} - i\omega)] \rho_{34}$$  \hspace{1cm} (B14)

Solving these equations one finds that in the steady state, the off-diagonal terms vanish, leaving only the diagonal terms given by:

$$\rho_{11}^{ss} = \frac{\bar{n}^2 [1 - \rho_{33}(0)]}{1 + 3\bar{n}(\bar{n} + 1)}$$  \hspace{1cm} (B15)

$$\rho_{22}^{ss} = \frac{\bar{n}^2(\bar{n} + 1) [1 - \rho_{33}(0)]}{1 + 3\bar{n}(\bar{n} + 1)}$$  \hspace{1cm} (B16)

$$\rho_{33}^{ss} = \rho_{33}(0)$$  \hspace{1cm} (B17)

$$\rho_{44}^{ss} = \frac{\bar{n}^2 (\bar{n} + 1) [1 - \rho_{33}(0)]}{1 + 3\bar{n}(\bar{n} + 1)}$$  \hspace{1cm} (B18)
The initial state in the spin basis has non-zero elements: \( \rho_{22}(0) = \rho_{33}(0) = \rho_{23}(0) = \rho_{32}(0) = 1/2 \). It is then straightforward to show the steady state density matrix has the form

\[
\rho_{\text{ss}} = \frac{1}{2} \left[ n^2 |\psi_\uparrow\rangle \langle \psi_\uparrow| + n(n+1)|\psi_\downarrow\rangle \langle \psi_\downarrow| + |\psi_\downarrow\rangle \langle 
\right. \\
\left. \left( 1 + 3n(n+1) \right), \quad \forall n > 0 ,
\right]
\]

Eq. (3) of the main text is obtained by setting \( \langle J^z_i \rangle = \text{Tr}(\rho_i J^z_i) \).

**Appendix C: Temperature**

The effects of thermal fluctuations on the battery provides a rich area of investigation; here we show some interesting properties. Let us begin by considering two spins at non-zero temperature. From Eq. (B19) we can determine the spin expectation values of the charger and battery for non-zero temperature,

\[
\langle J^z_i \rangle = \langle J^z_B \rangle = -\frac{2n + 1}{12n(n+1) + 4} \hbar . \tag{C1}
\]

At high temperature \( \lim_{T \to \infty} \langle J^z_i \rangle = 0 \), meaning thermal fluctuations dominate so that spins are equally as likely to found in the spin-up as spin-down state. At low temperature the battery obtains its energy primarily from the charger, but as the temperature increases the energy source shifts from the charger to the reservoir.

This behaviour is generalised to various \( R \) as shown in Fig. 6 where we we plot \( \mathcal{E}(t) \) for increasing \( T \). Fig. 6(a) shows that as the temperature increases, less energy is transferred from the charger to the battery. In Fig. 6(b) we plot \( \mathcal{E}_{SS}B(T) \). It shows that all states converge to \( \lim_{T \to \infty} \mathcal{E}_{SS}B = 0 \). For states where \( \mathcal{E}_{SS} < 0 \) at \( T = 0 \), thermal fluctuations increases the battery capacity. Conversely for states where \( \mathcal{E}_{SS} > 0 \) at \( T = 0 \), thermal fluctuations decreases battery capacity. However there is a trade-off between the infusion of energy from the reservoir, and the destruction of dark states caused by thermal fluctuations. This is manifest at low temperatures before thermal fluctuations dominant, as shown in the plot of \( d\mathcal{E}_{SS}/dT \) in Fig. 6(c). As we increase the temperature, we see a rapid decline in \( \mathcal{E}_{SS} \) at low temperatures, as thermal fluctuation destroy the dark states. This is followed by a deceleration in the loss of energy as the system thermalises. The exception is for \( R = 1 \), where the infusion of energy from the reservoir more than compensates for the loss of energy from the destruction of the dark state, even at low temperatures.
FIG. 6. Charger and battery performance at non-zero temperature. (a) Energy density of the charger $E_C(t)$ and battery $E_B(t)$ during the charging process for various temperatures. As the temperature increases, less energy is transferred from the charger to that battery. (b) Steady state energy density $E_{SS}^B(T)$ for various $R$. All states converge in the thermodynamic limit to $E_{SS}^B = 0$. (c) A plot of the rate of change of the steady state energy density against temperature $dE_{SS}^B / dT$. There is a decline in $E_{SS}^B$ at low temperatures, as thermal fluctuation destroy the dark states. This is followed by a deceleration in the loss of energy as the system thermalises. The exception is for $R = 1$, where the infusion of energy from thermal reservoir more than compensates for the loss of energy from the destruction of the dark state. Parameters: (a) $T = 0$ (blue), 2 (orange), 4 (green), $R = 10$, $N_B = 1$. (b), (c) $R = 1$ (blue), 2 (orange), 3 (green), 4 (red), $N_B = 1$. The vertical axes are in units of k$\hbar$.

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