The balanced truncation bound is tight for SISO systems when the truncated system is state-space symmetric

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Abstract—Balanced truncation model reduction for linear dynamical systems yields a reduced-order model that satisfies a well-known error bound involving the system’s Hankel singular values. We identify a new class of single-input, single output systems for which this bound holds with equality; in this class the truncated systems exhibit a certain state-space symmetry. This result extends to singular perturbation balancing. We illustrate the result with an example from power-system modeling.

Index Terms—Balanced truncation, model reduction, error bound, power systems

I. INTRODUCTION

Balanced truncation [1], [2] is a powerful method for reducing linear time invariant (LTI) dynamical systems, yielding reduced-order models that are guaranteed to be asymptotically stable and satisfy a simple priori $H_{\infty}$ error bound involving the system’s Hankel singular values. This bound holds with equality under certain conditions, such as when the full-order system is state-space symmetric [3]. In such cases, the bound becomes an exact formula for the error in the reduction.

In this paper, we show that the balanced truncation error bound holds with equality for a larger family of single-input, single output (SISO) systems, those for which the truncated part of the model in the canonical balanced realization satisfies a particular sign symmetry. Section III reviews balanced truncation model reduction and describes a generalized version of state-space symmetric systems based on the canonical form of a balanced system. Section III contains the main result, showing that the balanced truncation error bound is tight for these systems when the truncated system has certain state-space symmetry. Lastly, Section IV provides examples of systems that exhibit such symmetry, including an example from power-system modeling that motivated our investigation. Numerical calculations confirm the tightness of the bound.

II. BALANCED TRUNCATION AND THE CANONICAL FORM

To establish notation, we briefly review balanced truncation, along with the canonical form of a balanced system.

A. The concept of balancing and balanced truncation

Consider the SISO LTI dynamical system

$$
\begin{align*}
G : \begin{cases}
x'(t) &= Ax(t) + bu(t) \\
y(t) &= cx(t) + du(t),
\end{cases}
\end{align*}
$$

with the transfer function

$$
G(s) = c(sI - A)^{-1}b + d,
$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$, and $d \in \mathbb{R}$. Throughout we assume that $G$ is asymptotically stable, i.e., the eigenvalues of $A$ have negative real parts, and that $G$ is minimal, i.e., it is reachable and observable. The $H_{\infty}$-norm of $G$ is defined as

$$
\|G\|_{H_{\infty}} := \sup_{\omega \in \mathbb{R}} |G(i\omega)|,
$$

where $i^2 = -1$. The Lyapunov equations

$$
AP + PA^\top + bb^\top = 0 \quad \text{and} \quad A^\top Q + QA + c^\top c = 0
$$

have unique positive definite solutions $P$ and $Q$ called the reachability and observability Gramians of $G$, respectively. The system $G$ is principal-axis balanced if $P = Q = \Sigma$, where

$$
\Sigma = \text{diag}(\sigma_1 I_{m_1}, \ldots, \sigma_q I_{m_q}),
$$

with $\sigma_1 > \sigma_2 > \cdots > \sigma_q > 0$ and $m_1 + \cdots + m_q = n$. Here $\sigma_1, \ldots, \sigma_q$ are the Hankel singular values of $G$. For brevity, throughout this paper we will use the term balanced as shorthand for principal-axis balanced. The state-space representation in balanced coordinates is called a balanced realization of $G$. In these balanced coordinates, states that are difficult to reach are simultaneously difficult to observe; such states are associated with the smallest Hankel singular values. Balanced truncation reduces the model order by removing these components of the state space. This reduction can be expressed compactly by writing the system in block form, as given in the following theorem. The asymptotic stability result is due to Pernebo and Silverman [4] and the error bound [9] is due to Enns [5].

Theorem 2.1: Let $G$ be an order-$n$ minimal and asymptotically stable dynamical system having the balanced realization

$$
\begin{bmatrix}
A & b \\
c & d
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & b_1 \\
A_{21} & A_{22} & b_2 \\
c_1 & c_2 & d
\end{bmatrix},
$$

where the state space matrices are partitioned according to the system Gramian $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, with

$$
\Sigma_1 = \text{diag}(\sigma_1 I_{m_1}, \ldots, \sigma_k I_{m_k}),
$$

$$
\Sigma_2 = \text{diag}(\sigma_{k+1} I_{m_{k+1}}, \ldots, \sigma_q I_{m_q}),
$$

where $r := m_1 + \cdots + m_k$. Then the $r$th order reduced model obtained via balanced truncation

$$
\begin{align*}
G_r : \begin{cases}
x'_r(t) &= A_{11} x_r(t) + b_1 u(t) \\
y_r(t) &= c_1 x_r(t) + d u(t)
\end{cases}
\end{align*}
$$

with the transfer function

$$
G_r(s) = c_1 (sI - A_{11})^{-1} b_1 + d
$$

is asymptotically stable and satisfies the error bound

$$
\|G - G_r\|_{H_{\infty}} \leq 2(\sigma_{k+1} + \cdots + \sigma_q).
$$

Equality is known to hold in (9) when only one Hankel singular value is truncated, that is, $\Sigma_2 = \sigma_q I_{m_q}$. We next explore other situations in which this bound (9) holds with equality, thus giving an exact formula for the error in the reduced order model.
B. The sign-symmetry of the balanced realizations

We next describe a crucial sign symmetry property of balanced systems characterized in [6], [7], which will be essential for our main results in Section III.

Theorem 2.2: Let \( G \) be an asymptotically stable and minimal SISO LTI system. Then \( G \) has a balanced realization satisfying
\[
  A = SA^T S \quad \text{and} \quad b = (cS)^T,
\]
where \( S = \text{diag}(s_1, s_2, \ldots, s_n) \) and \( s_i = \pm 1 \) for \( i = 1, \ldots, n \).

We refer to \( s_1, \ldots, s_n \) as the sign parameters corresponding to the Hankel singular values of \( G \). Ober [7], [8] shows that every asymptotically stable, minimal, SISO LTI system is equivalent to a balanced system with a realization satisfying (10), which we call the canonical form of a balanced system. Now suppose \( G \) is balanced with the canonical form (10), define \( r := m_1 + \cdots + m_k \) with \( 1 \leq k < q \), and partition the sign matrix as
\[
  S = \text{diag}(S_1, S_2),
\]
where
\[
  S_1 = \text{diag}(s_1, \ldots, s_r) \quad \text{and} \quad S_2 = \text{diag}(s_{r+1}, \ldots, s_n). \quad (11)
\]
Partition \( A, b, \) and \( c \) as in (5), giving the sign symmetries
\[
  A_{11} = S_1 A_{11}^T S_1, \quad A_{12} = S_1 A_{21}^T S_2, \quad A_{21} = S_2 A_{12}^T S_1, \quad (12)
\]
\[
  A_{22} = S_2 A_{22}^T S_2, \quad b_1 = (c_1 S_1)^T, \quad \text{and} \quad b_2 = (c_2 S_2)^T, \quad (13)
\]
which follow from direct multiplication in (10). Since the reduced model obtained via balanced truncation is independent of the initial system realization, for the rest of the paper we will assume, without loss of generality, that \( G \) is already balanced with the realization given in (10). Therefore, the Lyapunov equations in (3) become
\[
  AΣ + ΣA^T + bb^T = 0 \quad \text{and} \quad A^T Σ + ΣA + c^T c = 0. \quad (14)
\]

C. State-space symmetric systems

Definition 2.1: If \( s_i = 1 \) for all \( i = 1, \ldots, n \) in (10), then \( G \) is called state-space symmetric. In this case, \( A = A^T \) and \( b = c^T \).

As stated earlier, the \( H_\infty \) error bound (9) for balanced truncation holds with equality if only one singular value is truncated. However, state-space symmetric systems have the property that the error bound (9) holds with equality for any truncation order [3]. Before we extend this result to a more general setting in Section III, we prove an additional property regarding the Hankel singular values of state-space symmetric systems. An analogous result holds for zero interlacing pole (ZIP) systems, which are closely related to state-space symmetric systems in the SISO case [9].

Proposition 2.1: Let \( G \) be a stable, minimal, balanced SISO LTI system, and suppose \( G \) has a realization satisfying \( A = A^T \) and either \( b = c^T \) or \( b = -c^T \). Then the Hankel singular values of \( G \) must all have multiplicity one, \( m_1 = \cdots = m_n = 1 \).

Proof: Suppose \( G \) has some Hankel singular value \( σ \) of multiplicity \( m \). The balanced realization of a system is unique up to orthogonal transformation. Then by [8, Corollary 2.1], there exists a unitary matrix \( U \in \mathbb{R}^{N \times n} \) such that the upper-left \( m \times m \) block of \( A := U^T AU \) has the form
\[
  \hat{A}(1 : m, 1 : m) = \begin{bmatrix}
    -\frac{1}{\sigma} & α_1 & \cdots & 0 \\
    0 & -α_1 & & \ddots \\
    & \ddots & \ddots & \ddots \\
    0 & & -α_{m-1} & \alpha_{m-1}
  \end{bmatrix}
\]
for \( α_1, \ldots, α_{m-1} > 0 \). However, note that \( A = A^T \) implies that \( \hat{A}^T = (U^T AU)^T = \hat{A} \), which is impossible if \( m > 1 \), given the form of \( \hat{A}(1 : m, 1 : m) \). Thus, all Hankel singular values of the system must have multiplicity \( m = 1 \).

III. MAIN RESULTS

In this section, we show that the \( H_\infty \) error bound for balanced truncation holds with equality for a more general class of systems than the state-space symmetric ones just described. In Section IV we will provide an example from this new class of systems that arises naturally in power system modeling.

In accordance with the partitioned balanced realization (5), we define the truncated system
\[
  \hat{G} : \begin{cases}
    \tilde{x}(t) = A_{22}\tilde{x}(t) + b_2 u(t), \\
    \tilde{y}(t) = c_2 \tilde{x}(t),
  \end{cases}
\]
with transfer function \( \hat{G}(s) = c_2(sI - A_{22})^{-1}b_2 \) and system Gramian (7). We call (15) the truncated system, since its state \( \tilde{x}(t) \in \mathbb{R}^{n-r} \) corresponds to the truncated states in the balanced realization. Note from (13) that \( \hat{G} \) satisfies the sign symmetries \( A_{22} = S_2 A_{22}^T S_2 \) and \( b_2 = (c_2 S_2)^T \). For the results below, we will allow the signs in \( S_1 \) to vary, but we will assume that either \( S_2 = -S_2 \) or \( S_2 = -S_2 \). In other words, we do not assume that \( \hat{G} \) is state-space symmetric; we only assume that the sign parameters corresponding to the truncated Hankel singular values are the same.

A. Result for Balanced Truncation

Theorem 3.1: Let \( G \) be an order-\( n \) asymptotically stable, minimal, balanced SISO system as in (1), with its matrix of Hankel singular values \( Σ = \text{diag}(Σ_1, Σ_2) \) partitioned as in (6) and (7), set \( r := m_1 + \cdots + m_k \). Let \( G_r \) denote the order-\( r \) approximation of \( G \) via balanced truncation, as in (3). Conformally partition the sign matrix, \( S = \text{diag}(S_1, S_2) \), with \( S_1 \in \mathbb{R}^{r \times r} \) and \( S_2 \in \mathbb{R}^{(n-r) \times (n-r)} \). If all the signs in \( S_2 \) are the same,
\[
  S_2 = \text{diag}(+1, \ldots, +1) \quad \text{or} \quad S_2 = \text{diag}(-1, \ldots, -1), \quad (16)
\]
then the truncated Hankel singular values are distinct,
\[
  Σ_2 = \text{diag}(σ_{k+1}, \ldots, σ_q), \quad (17)
\]
and \( G_r \) in (3) achieves the error bound (9):
\[
  \|G - G_r\|_{H_\infty} = 2(σ_{k+1} + \cdots + σ_q). \quad (18)
\]

Proof: First note that the truncated system \( \hat{G} \) satisfies the conditions of Proposition 2.1 and so we can conclude that the associated Hankel singular values in (17) are distinct. To prove that the bound is tight, we begin by repeating the necessary details of the balanced truncation error bound proof; see [5] and [10, Thm. 7.3]. Recall the definition of the \( H_\infty \)-norm in (3). Because \( G_r \) is obtained via balanced truncation, the error system satisfies the upper bound (9), so it suffices to show that this bound is attained for the particular frequency \( ω = 0 \), i.e., \( s := iω = 0 \). Assume without loss of generality that \( \hat{G} \) is balanced and in the canonical form (10). Then the state-space matrices satisfy the sign symmetries given in (12) and (13). Following [10, Thm. 7.3], the transfer function for the error system can be written as
\[
  G(s) - G_r(s) = c(sI - A)^{-1}b - c_1(sI - A_{11})^{-1}b_1 = c(s)τ(s)^{-1}b_1(s),
\]
where
\[ \psi(s) = sI - A_{22} - A_{21}(sI - A_{11})^{-1}A_{12} \]
\[ \hat{b}(s) = A_{21}(sI - A_{11})^{-1}b_1 + b_2 \]
\[ \hat{c}(s) = c_1(sI - A_{11})^{-1}A_{12} + c_2. \]

We claim that the error bound is achieved at \( s = 0 \). Observe that
\[ \psi(0) = (-A_{22} + A_{21}A_{11}^{-1}A_{12}) \]
\[ = -A_{22} + (A_{21}A_{11}^{-1}A_{12})^\top \]
\[ = -A_{22} + A_{11}^{-1}A_{12}, \]

since \( A_{22} = S_2A_2^\top S_2 \) by \([13]\), and \( S_2 = \pm I_{n-r} \). Now by \([12]\),
\[ \psi(0) = -A_{22} + S_2A_2S_1S_1^{-1}S_1A_1A_2S_2 \]
\[ = -A_{22} + A_{11}^{-1}A_{12} = \psi(0), \]

and hence \( \psi(0) \) is a symmetric matrix. (The sign assumption \( S_2 = \pm I_{n-r} \) is only needed to establish this symmetry.) Now evaluate the error at \( s = 0 \):
\[ |G(0) - G_r(0)|^2 = (\hat{c}(0)\psi(0)^{-1}\hat{b}(0) - \hat{c}(0)(0)^{-1}\hat{b}(0)) \]
\[ = \hat{b}(0)\psi(0)^{-1}\hat{c}(0)(0)^{-1}\hat{b}(0) \]
\[ = \text{tr}(\hat{b}(0)^\top \psi(0)^{-1}\hat{c}(0)(0)^{-1}\hat{b}(0)) \]
\[ = \text{tr}(\hat{b}(0)\hat{b}(0)^\top \psi(0)^{-1}\hat{c}(0)(0)^{-1}\hat{c}(0)(0)^{-1} \psi(0)^{-1}) \]
\[ = \lambda_{\text{max}}(\hat{b}(0)\hat{b}(0)^\top \psi(0)^{-1}\hat{c}(0)(0)^{-1} \psi(0)^{-1}). \]

One can show that \( \hat{b}(0)\hat{b}(0)^\top \psi(0)^{-1}\hat{c}(0)(0)^{-1} \psi(0)^{-1} \) are the right-hand sides of Lyapunov equations for which \( \Sigma_2 \) is the solution:
\[ \Sigma_2\psi(0)^{-1} + \psi(0)^{-1}\Sigma_2 = \hat{b}(0)\hat{b}(0)^\top \]
\[ \Sigma_2\psi(0)^{-1} + \psi(0)^{-1}\Sigma_2 = \hat{c}(0)^\top \hat{c}(0). \]
(See [10, Sect. 7.1] for further details.) Substituting the left-hand sides into the last expression for \( |G(0) - G_r(0)|^2 \) gives
\[ |G(0) - G_r(0)|^2 = \lambda_{\text{max}}((\Sigma_2 + \psi(0)\Sigma_2\psi(0)^{-1})(\Sigma_2 + \psi(0)\Sigma_2\psi(0)^{-1})). \]

Since \( \psi(0) \) is symmetric,
\[ |G(0) - G_r(0)|^2 = \lambda_{\text{max}}((\Sigma_2 + \psi(0)\Sigma_2\psi(0)^{-1})^2) \]
\[ = (\lambda_{\text{max}}(\Sigma_2 + \psi(0)\Sigma_2\psi(0)^{-1}))^2. \]

We emphasize that \( \Sigma_2 + \psi(0)\Sigma_2\psi(0)^{-1} \) must be rank-one by equations \([18]\) and \([19]\), and thus has one nonzero eigenvalue. Additionally note that \( \Sigma_2 \) and \( \psi(0)\Sigma_2\psi(0)^{-1} \) have identical eigenvalues, since the latter is a similarity transformation of the former. This fact and the aforementioned rank-one structure allow us to conclude
\[ |G(0) - G_r(0)| = \lambda_{\text{max}}(\Sigma_2 + \psi(0)^{-1}\Sigma_2\psi(0)) \]
\[ = \text{tr}(\Sigma_2 + \psi(0)^{-1}\Sigma_2\psi(0)) \]
\[ = \text{tr}(\Sigma_2) + \text{tr}(\psi(0)^{-1}\Sigma_2\psi(0)) \]
\[ = 2\text{tr}(\Sigma_2) \]
\[ = 2(\sigma_{k+1} + \cdots + \sigma_q), \]
completing the proof.

### B. Extension to Singular Perturbation Balancing

As opposed to truncating the state \( \tilde{x}(t) \) corresponding to \( \Sigma_2 \) in the balanced form \([5]\), one can perform model reduction via singular perturbation balancing \([11]\) by setting \( \tilde{x}(t) = 0 \). Starting with the balanced realization of \( \mathcal{G} \) in \([5]\), the order-\( r \) singular perturbation balancing approximation of \( \mathcal{G} \) is
\[ \mathcal{G}_r^{sp} : \begin{cases} x_r(t) = A_r^{sp}x_r(t) + b_r^{sp}u(t) \\ y_r(t) = c_r^{sp}x_r(t) + d_r^{sp}u(t), \end{cases} \]
(20)
where
\[ A_r^{sp} = A_{11} - A_{21}A_{11}^{-1}A_{12} \]
\[ b_r^{sp} = b_1 - A_{12}A_{11}^{-1}b_2 \]
\[ c_r^{sp} = c_1 - c_2A_{11}^{-1}A_{12} \]
\[ d_r^{sp} = d - c_2A_{11}^{-1}b_2. \]
(21)
(22)
The reduced model \( \mathcal{G}_r^{sp} \) satisfies the same \( H_\infty \) error bound \([9]\); see \([11, \text{Thm. 3.2}] \).

As a consequence of Theorem 3.1, we will show that the \( H_\infty \) error bound \([9]\) also holds with equality when performing singular perturbation balancing, provided the sign parameters of the system \( \mathcal{G} \) satisfy \([16]\).

**Theorem 3.2:** Let \( \mathcal{G} \) be an order-\( n \) asymptotically stable, minimal, balanced SISO system as in \([1]\) with its matrices of Hankel singular values \( \Sigma = \text{diag}(\Sigma_1, \Sigma_2) \) and sign parameters \( S = \text{diag}(S_1, S_2) \) partitioned as in Theorem 3.1. Let \( \mathcal{G}_r^{sp} \) be the singular perturbation approximation of \( \mathcal{G} \) given by \([20]\), truncated after the \( k \)th distinct Hankel singular value and having order \( r := m_1 + \cdots + m_k \). If all the signs in \( S_2 \) are the same, as in \([16]\), then \( \mathcal{G}_r^{sp} \) in \([20]\) achieves the error bound \([9]\):
\[ \|\mathcal{G} - \mathcal{G}_r^{sp}\|_{H_\infty} = 2(\sigma_{k+1} + \cdots + \sigma_q). \]

**Proof:** Without loss of generality assume that \( \mathcal{G} \) is balanced and given as in the canonical form \([10]\). It is shown in \([11, \text{Thm. 3.2}] \) that the model reduction error expression from the \( r \)th order singular perturbation approximation to \( \mathcal{G} \) can be written as
\[ \mathcal{G} - \mathcal{G}_r^{sp} \mid_{H_\infty} = \|\mathcal{G} - \mathcal{G}_r\|_{H_\infty}, \]
where \( \mathcal{G} \) is the reciprocal system of \( \mathcal{G} \) given by the realization
\[ \hat{A} = A^{-1}, \quad \hat{b} = A^{-1}b, \quad \hat{c} = cA^{-1}, \]
and \( \mathcal{G}_r \) is the \( r \)th order balanced truncation reduced model for \( \mathcal{G} \). Then \([11, \text{Lemma 3.1}] \) states that the given realization of \( \mathcal{G} \) is balanced and has Gramian \( \Sigma \), and so the Hankel singular values of \( \mathcal{G} \) are the same as those of the original system \( \mathcal{G} \). Notice that the reciprocal system obeys the same sign symmetry as the original system, that is, \( \hat{A} = SA^{-1}S \) and \( \hat{b} = \hat{c}(ES)^\top \), where \( S \) is the sign matrix of \( \mathcal{G} \); inverting both sides of \( \hat{A} = SA^{-1}S \) shows that \( \hat{A} = S\hat{A}S \); additionally we see
\[ \hat{b} = A^{-1}b = A^{-1}(cS)^\top = A^{-1}Sc^\top = SA^{-1}c^\top = S\hat{c}. \]
It follows that the submatrices of the reciprocal system partitioned according to \([6]\) satisfy the same sign symmetries as in \([12]\) and \([13]\). Thus applying the result of Theorem 3.1 to \( \mathcal{G} \) and \( \mathcal{G}_r \), we conclude
\[ \|\mathcal{G} - \mathcal{G}_r^{sp}\|_{H_\infty} = \|\mathcal{G} - \mathcal{G}_r\|_{H_\infty} = 2(\sigma_{k+1} + \cdots + \sigma_q). \]

### IV. NUMERICAL RESULTS AND EXAMPLES

We illustrate Theorem 3.1 through two numerical examples: the first a simple academic example of order \( n = 4 \), the second arising from a model of power system dynamics.
A. An example with flipped signs and a strict bound
We provide a synthetic example to show that the balanced truncation error bound holds with equality when the truncated system obeys the sign consistency in \((16)\). We construct the system \(G\) of order \(n = 4\) in its canonical form. Start by specifying the Hankel singular values \(\Sigma = \text{diag}(10^3, 10^0, 10^{-1}, 10^{-2})\), and the corresponding sign parameters \(S = \text{diag}(1, 1, -1, -1)\).

Since the Hankel singular values of \(G\) are distinct, we can apply the formula \([12, \text{eq. (7.24)}]\) to construct the realization:

\[
a_{ij} = \frac{-\gamma_i \gamma_j}{s_i s_j \sigma_i + \sigma_j}, \quad c_i = s_i \gamma_i, \quad i, j = 1, \ldots, n, \tag{23}
\]

where \(\gamma_i\) denotes the \(i\)th entry of \(b\). The system is completely determined by the distinct Hankel singular values \(\sigma_1, \ldots, \sigma_4 > 0\), the signs \(s_1, \ldots, s_4 \in \{\pm 1\}\), and the vector \(b \in \mathbb{R}^4\). (The entries of \(b\) are given below.) Then the associated system \(\hat{G}\) is asymptotically stable, minimal, and balanced, having the canonical form

\[
A = \begin{bmatrix}
-0.05 & -0.18 & 0.30 & 0.40 \\
-0.18 & -2.00 & 6.67 & 8.08 \\
-0.30 & -6.67 & -45.00 & -109.09 \\
-0.40 & -8.08 & -109.09 & -800.00
\end{bmatrix},
\]

\[
b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad c^T = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}.
\]

We compute reduced order models via balanced truncation of orders \(r = 1, 2, 3\). We highlight the partitioning of \(G\) with respect to the truncation order \(r = 1\) to expose the sign symmetry of the truncated system. Table I compares the \(H_\infty\) norm of the error system to the balanced truncation upper bound \((9)\). When performing reduction for orders \(r = 2\) and \(r = 3\), the condition \((16)\) is met, and the balanced truncation bound holds with equality, as guaranteed by Theorem 3.1. However when performing reduction for \(r = 1\), the truncated system does not obey the required sign consistency \((16)\), and the upper bound \((9)\) holds with a strict inequality.

### TABLE I

| \(\|G - \hat{G}\|_{H_\infty}\) for \(r = 2\) and \(3\), but not for \(r = 1\) | \(2(\sigma_r+1 + \cdots + \sigma_n)\) |
|---|---|
| \(r = 1\) | \(1.780 \times 10^0\) | \(2.220 \times 10^0\) |
| \(r = 2\) | \(2.200 \times 10^{-1}\) | \(2.200 \times 10^{-1}\) |
| \(r = 3\) | \(2.000 \times 10^{-2}\) | \(2.000 \times 10^{-2}\) |

B. A model from power system dynamics
We first encountered the phenomenon characterized in Theorem 3.1 while studying a model from power systems, which we introduce here for the purpose of numerical study. A common technique for modeling the frequency dynamics of a network of coherent generators is to aggregate the system response into a single effective machine. It is shown in \([13, 14]\) that for a network of \(N\) coherent generators, the aggregate frequency dynamics are well-approximated by a system \(G\) having the transfer function

\[
\hat{g}(s) = \frac{1}{\hat{m} s + \hat{d} + \sum_{i=1}^N \frac{r_i^{-1}}{\tau_i s + 1}}, \tag{24}
\]

for generators given by the swing model with first-order turbine control. Here \(\hat{m}\) and \(\hat{d}\) denote the aggregate inertia and damping coefficients of the generators in the network, while \(\tau_i\) and \(r_i^{-1}\) denote the time constant and droop coefficient of the \(i\)th generator. For details on the derivation and justification of the model, see \([14, \text{Sect. 2}]\). For these systems we observed numerically that the balanced truncation error bound was typically tight, and then noticed that the truncated system observed the sign consistency specified in \((16)\).

We illustrate this property with a specific example. Consider the case of \(N = 4\) coherent generators, which leads to a SISO LTI system of order \(n = N + 1 = 5\). Take \(\hat{m} = 0.044, \hat{d} = 0.038, \quad (r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) = (0.013, 0.014, 0.022, 0.025)\) and

\[
(\tau_1, \tau_2, \tau_3, \tau_4) = (5.01, 6.82, 7.38, 7.79).
\]

We first calculate the Hankel singular values of the system \(G\) with transfer function \(\hat{g}(s)\) in \((23)\)

\[
\Sigma = \text{diag}(11.63, 7.13, 3.53 \times 10^{-2}, 8.48 \times 10^{-5}, 4.12 \times 10^{-8})
\]

We compute a balanced realization of \(G\) satisfying the symmetry condition \((10)\), giving the sign matrix

\[
S = \text{diag} (1, -1, -1, -1, -1)
\]

Next we compute order-\(r\) balanced truncation approximations to \(G\) for \(r = 2, 3, 4\). Under these conditions the truncated systems obey the sign consistency in \((16)\). As in the previous example we highlight this symmetry by partitioning the system for \(r = 3:\)

\[
A = \begin{bmatrix}
-0.9913 & 0.5924 & -0.0467 & 0.0020 & 0.0000 \\
-0.5924 & -0.0216 & 0.0087 & -0.0004 & -0.0000 \\
0.0467 & 0.0087 & -0.1800 & 0.0157 & 0.0003 \\
-0.0020 & -0.0004 & 0.0157 & -0.1437 & -0.0062 \\
-0.0000 & -0.0000 & 0.0003 & -0.0062 & -0.1372
\end{bmatrix},
\]

\[
b = \begin{bmatrix} -4.8009 \\ -0.5552 \\ 0.1126 \\ -0.0049 \\ -0.0001 \end{bmatrix}, \quad c^T = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}.
\]

Table II compares the \(H_\infty\) norm of the error system to the balanced truncation upper bound \((9)\). Because the last four diagonal entries of \(S\) are all \(-1\), we can perform truncation at any order \(r \geq 1\), and the truncated system will satisfy the sign requirements \((16)\) of Theorem 3.1. Thus the balanced truncation error bound holds with equality for approximations of all orders.

### TABLE II

| \(\|G - \hat{G}\|_{H_\infty}\) for \(r = 2\), \(3\), \(4\), \(N = 4\) | \(2(\sigma_r+1 + \cdots + \sigma_n)\) |
|---|---|
| \(r = 2\) | \(7.067 \times 10^{-2}\) | \(7.067 \times 10^{-2}\) |
| \(r = 3\) | \(1.697 \times 10^{-4}\) | \(1.697 \times 10^{-4}\) |
| \(r = 4\) | \(8.248 \times 10^{-8}\) | \(8.248 \times 10^{-8}\) |

V. Conclusion
Through analysis and numerical examples, we have demonstrated that balanced truncation reduced-order models for SISO LTI systems that obey the sign consistency condition \((16)\) satisfy an explicit error formula involving the Hankel singular values. This result generalizes the class of systems for which the balanced truncation error bound is known to hold with equality. Systems with this property arise in a model for coherent generators in power system dynamics.
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