Abstract

In this paper, we derive a Bayesian model order selection rule by using the exponentially embedded family (EEF) method, termed Bayesian EEF. Unlike many other Bayesian model selection methods, the Bayesian EEF can use vague proper priors and improper non-informative priors to be objective in the elicitation of parameter priors. Moreover, the penalty term of the rule is shown to be the sum of half of the parameter dimension and the estimated mutual information between parameter and observed data. This helps to reveal the EEF mechanism in selecting model orders and may provide new insights into the open problems of choosing an optimal penalty term for model order selection and choosing a good prior from information-theoretic viewpoints. The important example of linear model order selection is given to illustrate the algorithms and arguments. Lastly, the Bayesian EEF that uses Jeffreys’ prior coincides with the EEF rule derived by frequentist strategies. This shows another interesting relationship between the frequentist and Bayesian philosophies for model selection.

I. INTRODUCTION

Model order selection is an important problem of active research in signal processing. In model-based signal processing, one often needs to estimate both the number of unknown parameters and their values such as determining the order of autoregressive model [1], and the number of sinusoidal components in a noisy signal [7]. The determination of the number of sources in array signal processing, as an example,
is essentially a model order selection problem. Overestimating the order (the number of sources) fits the noise in the data; underestimating the order, on the other hand, fails to describe the data precisely [2]. Hence, a good model order selection rule is crucial for signal processing applications.

As a multiple hypotheses testing problem, model order selection lacks an optimal solution [12]. The generalized likelihood ratio test (GLRT) always favors the most complex model [10]. A typical model order selection algorithm introduces a penalty term to the GLRT, and it is the penalty term that makes one model order selection rule different from another. A model order selection rule derived from a Bayesian viewpoint typically tries to strike a balance between goodness of fit and model complexity [22].

Some leading model order selection algorithms, both frequentist and Bayesian, in literature [7] are Akaike’s information criterion (AIC) [4], the minimum description length (MDL) [5], Bayesian information criterion (BIC) [6] and maximum a posteriori (MAP) [12]. For example, AIC and BIC rules are respectively

\[
\ln p(x|\hat{\theta}) - k; \quad \text{AIC}
\]

\[
\ln p(x|\hat{\theta}) - \frac{k}{2} \ln N; \quad \text{BIC}
\]

where \( \theta \) is the model unknown parameter vector, \( \ln p(x|\hat{\theta}) \) is the maximum log-likelihood under a certain model, \( k \) is the dimension of the model parameter \( \theta \), \( N \) is the data record length. As seen the AIC penalty is \( k \), the dimension of the unknown parameter. And BIC has a penalty \( \frac{k}{2} \ln N \) which depends on the parameter dimension and data length.

As an alternative, an EEF model order selection rule derived from a frequentist viewpoint is introduced in [11]. It proves effective in model order selection and enjoys many great properties. It is consistent, i.e., the probability of correctly choosing the order goes to one as the signal-to-noise ratio (SNR) increases [3]. Its performance is superior to many others under several situations including low SNR regime [2] [11]. For example, it has been used to determine the degree of noncircularity of complex Gaussian random vectors and has been shown to outperform MDL substantially [15], to determine the source number in array processing [2], and to determine the order of AR processing [11]. It has also been successfully applied to many related areas such as classification and sensor fusion and shown great performances [16]-[18].
Fundamentally different from [11], we derive in this paper the EEF rule from a Bayesian viewpoint, termed the Bayesian EEF, as a novel Bayesian model order selection rule. The key difference lies in the philosophies of viewing the unknown parameters. The unknown model parameters are treated as deterministic in [11], but random variables in this paper. This is also a fundamental difference between the frequentist and Bayesian methods. Using Bayesian strategies allows us the possibilities to investigate the EEF mechanism in a new framework and from new viewpoints such as information theory and leads to the main contributions of this paper:

- A new Bayesian model order selection method, Bayesian EEF, is derived. It is proved that the Bayesian EEF can use both vague proper prior and improper non-informative prior for unknown parameters, both of which are usually forbidden for many Bayesian methods. The Bayesian EEF also does not have the Lindley’s paradox or the Information paradox.

- An intuitive justification is given in interpreting the Bayesian EEF penalty term. The penalty term is the sum of half the model parameter dimension and the estimated mutual information between model parameters and observed data. This not only helps to reveal the EEF mechanism in model order selection, but also sheds lights on the open problem of choosing a good penalty term in model order selection.

- In addition, it also shows that the Bayesian EEF using Jeffrey’s prior coincides with the EEF derived from a frequentist viewpoint. This is another case of the interesting interaction between the frequentist and Bayesian philosophies and may provide useful insights into the discussion on the difference between the two.

The paper is organized as follows. In Section II we derive the Bayesian EEF order selection rule that uses a vague proper prior for linear model and discuss some desirable properties of the Bayesian EEF. In Section III we justify the Bayesian EEF penalty term. In Section IV we derive the Bayesian EEF via improper non-informative prior, Jeffreys’ prior and discuss its interaction with frequentist EEF. Finally, some conclusions are given in Section V.
II. BAYESIAN EEF RULE FOR MODEL SELECTION VIA VAGUE PROPER PRIOR

Suppose there are $M$ candidate models, $M_0, M_1, \cdots, M_{M-1}$, where $M_0$ is a null/reference model which has no unknown parameters and the model $M_i$ (for $i = 1, \cdots, M - 1$) has an unknown parameter vector $\theta_i$ of dimension $k_i \times 1$. The probability density functions (PDF) of the observed data $x$ of dimension $N \times 1$ for model $M_i$ is denoted as $p_i(x)$. From the frequentist viewpoint, the unknown parameters are deterministic. The EEF model order selection rule proposed in [11] adopts this assumption and hence is termed frequentist EEF. On the other hand, a Bayesian model order selection method views the parameter vectors as random. The Bayesian EEF adopts this philosophy. If we know the the model parameter priors, we can compare marginal PDFs of $x$ of different models or use a MAP rules to choose a model order. But in practice no prior information is available and the first question that arises for a Bayesian model order selection method is the specification of the prior distributions for the unknown parameter vector $\theta_i$. Which prior to choose is a controversial and difficult task [21]. Ideally we want to use a prior with minimal influence on the Bayesian inference. Improper non-informative priors such as uniform distribution and vague prior distributions (a proper prior with large spread) seem to be natural choices because they are objective in that they do not favor one parameter value over another. However, they can, unfortunately, lead to non-sensible answers when used in many Bayesian model selection methods. As shown next Bayesian EEF, on the other hand, can employ these two types of priors and still produce good results. This is a desirable property for a Bayesian model order selection algorithm. In this section, we derive the Bayesian EEF by assigning vague proper priors to unknown parameters. The resultant EEF is called the reduced Bayesian EEF. For illustration purposes, we focus on the normal linear model order selection problem. In Section IV, we give the Bayesian EEF that uses the improper non-informative Jeffreys’ prior.

Zellner’s g-prior is widely used in Bayesian inference because of its conjugacy and computational efficiency in computing the marginal likelihoods and its simple, understandable interpretation [20][25]. Losely speaking, the g-prior places less prior distribution mass in areas of the parameter space where the data is expected to be more informative about the unknown parameters. The vague proper prior adopted
herein is constructed by letting the hyperparameter $g$ of a g-prior goes to infinity and hence produce a “flat” and “non-informative” prior. Assume we want to choose a model from the following linear model candidates

$$\mathcal{M}_i : \mathbf{x} = \mathbf{H}_i \mathbf{\theta}_i + \mathbf{w}, \ i = 1, \cdots, M - 1.$$  

where $\mathbf{\theta}_i$ is a $k_i \times 1$ unknown parameter vector, $\mathbf{H}_i$ is a $N \times k_i$ design matrix, and $\mathbf{w} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$ is additive noise with $\mathbf{I}$ being a $N \times N$ identity matrix. There is also a null model $\mathcal{M}_0 : \mathbf{x} = \mathbf{w}$ which does not contain unknown parameters. Without loss of generality, we assume that $k_i \leq k_j$ for $i \leq j$.

We first assign $\mathbf{\theta}_i$ a vague proper prior, $\pi_i(\mathbf{\theta}_i)$, which is a g-prior with an infinite hypeparameter $g_i$ as

$$\pi_i(\mathbf{\theta}_i) = \mathcal{N}(0, g_i \sigma^2 (\mathbf{H}_i^T \mathbf{H}_i)^{-1}) \quad \text{and} \quad g_i \to \infty.$$

The marginal PDF $p_i(\mathbf{x})$ under the $\mathcal{M}_i$ model is then

$$p_i(\mathbf{x}) = \int p_i(\mathbf{x} | \mathbf{\theta}_i) \pi_i(\mathbf{\theta}_i) d\mathbf{\theta}_i = \mathcal{N}(0, \sigma^2 \mathbf{I} + g_i \sigma^2 \mathbf{P}_i) = \mathcal{N}(0, \mathbf{C}_i) \quad (1)$$

where $p_i(\mathbf{x} | \mathbf{\theta}_i) = \mathcal{N}(\mathbf{H}_i \mathbf{\theta}_i, \sigma^2 \mathbf{I})$ is the conditional PDF of $\mathbf{x}$ on $\mathbf{\theta}_i$ under model $\mathcal{M}_i$, $\mathbf{P}_i = \mathbf{H}_i (\mathbf{H}_i^T \mathbf{H}_i)^{-1} \mathbf{H}_i^T$ and the covariance matrix $\mathbf{C}_i = \sigma^2 \mathbf{I} + g_i \sigma^2 \mathbf{P}_i$. The PDF of $\mathbf{x}$ under the null model is

$$p_0(\mathbf{x}) = \mathcal{N}(0, \sigma^2 \mathbf{I}) = \mathcal{N}(0, \mathbf{C}_0), \quad (2)$$

where $\mathbf{C}_0 = \sigma^2 \mathbf{I}$. Then for each $p_i(\mathbf{x})$, $i = 1, \cdots, M - 1$, we can construct a new PDF, $p(\mathbf{x}; \eta_i)$ by exponentially embedding it with $p_0(\mathbf{x})$, which is parameterized by an embedding parameter $\eta_i$:

$$p(\mathbf{x}; \eta_i) = \frac{p_i^{\eta_i}(\mathbf{x}) p_0^{1-\eta_i}(\mathbf{x})}{\int p_i^{\eta_i}(\mathbf{x}) p_0^{1-\eta_i}(\mathbf{x}) d\mathbf{x}} = \exp(\eta_i T_i(\mathbf{x}) - K_0(\eta_i) + f_c(\mathbf{x})) \quad (3)$$
with

\[
T_i(x) = \ln \frac{p_i(x)}{p_0(x)}
\]

natural parameter: \(0 \leq \eta_i \leq 1\)

log-normalizer:
\[
K_0(\eta_i) = \ln \int p_i^{\eta_i}(x)p_0^{1-\eta_i}(x)dx
= \ln E_0 \left( e^{\eta_i T_i(x)} \right)
\]

carrier density:
\[
f_c(x) = \ln p_0(x)
\]

As shown the resulting PDF is an exponential family PDF and consequently inherits a multitude of mathematical and practical properties of the family. Note that the PDFs \(p_i(x)\) and \(p_0(x)\) do not necessarily to be members of the exponential family. The statistic \(T_i(x)\) is a minimal and complete sufficient statistic for \(\eta_i\); its moments can be easily found and \(K_0(\eta_i)\) is a convex function. The new PDF \(p(x; \eta_i)\) is called the Bayesian EEF for the model \(\mathcal{M}_i\) in that we employ both Bayesian philosophies and exponentially embedding to construct it. From the information-geometric viewpoints, the log-Bayesian EEF \(\ln p(x; \eta_i)\) can be viewed as a point on the geodesic that connects \(\ln p_i(x)\) and \(\ln p_0(x)\) \([11][13]\). As seen from \(3\), the Bayesian EEF \(p(x; \eta_i)\) reduces to \(p_0(x)\) when \(\eta_i = 0\) and \(p_i(x)\) when \(\eta_i = 1\).

Plugging \(p_i(x)\) of \(1\) and \(p_0(x)\) of \(2\) into \(3\) produces the Bayesian EEF \(p(x; \eta_i)\) for the linear model as follows.

\[
p(x; \eta_i) = \frac{p_i^{\eta_i}(x)p_0^{1-\eta_i}(x)}{\exp(K_0(\eta_i))}
= \frac{1}{\exp(K_0(\eta_i))} \left[ \frac{1}{\sqrt{2\pi\eta_i}} \exp\left( -\frac{1}{2} x^T C_\eta^{-1} x \right) \right]^{\eta_i}
\cdot \left[ \frac{1}{\sqrt{2\pi C_0}} \exp\left( -\frac{1}{2} x^T C_0^{-1} x \right) \right]^{1-\eta_i}
= c_1 \exp \left[ -\frac{1}{2} x^T \left( \eta_i C_\eta^{-1} + (1 - \eta_i) C_0^{-1} \right) x \right]
\]

where \(c_1\) is a constant normalization term and \(C_\eta = (\eta_i C_i^{-1} + (1 - \eta_i) C_0^{-1})^{-1}\). It shows that the constructed Bayesian EEF is also a zero mean normal distribution with a covariance matrix \(C_\eta\) depending
on \( \eta_i \). Explicitly,

\[
C_{\eta_i} = \eta_i (\sigma^2 I + g_i \sigma^2 P_i)^{-1} + (1 - \eta_i)(\sigma^2 I)^{-1}
\]

\[
= \sigma^2 \left[ \frac{I - \frac{g_i}{g_i + 1} P_i}{(1 - \eta_i)I} \right]^{-1}
\]

\[
= \sigma^2 \left( I + \frac{\eta_i}{1 - \eta_i + \frac{1}{g_i} P_i} \right)
\]

\[
\rightarrow \sigma^2 I + \frac{\eta_i}{1 - \eta_i} \sigma^2 P_i \text{ as } g_i \to \infty
\]

So the reduced Bayesian EEF (the Bayesian EEF using vague proper priors) for \( M_i \) is

\[
p(x; \eta_i) = \mathcal{N}(0, \sigma^2 I + \frac{\eta_i}{1 - \eta_i} \sigma^2 P_i).
\]

(4)

Then a model order selection algorithm based on the Bayesian EEF in (4) consists of two steps.

- **Step1**: Find the MLE of \( \eta_i \), \( 0 \leq \hat{\eta}_i \leq 1 \), which maximizes \( p(x; \eta_i) \);

For the linear model from (4) we have

\[
\hat{\eta}_i = \begin{cases} 
0 & \text{if } x^T P_i x < k_i \sigma^2 \\
\frac{x^T P_i x - k_i \sigma^2}{x^T P_i x} & \text{otherwise}
\end{cases}
\]

(5)

where \( k_i \) is the dimension of \( \theta_i \).

- **Step2**: Compare the values of the \( M - 1 \) maximized EEF \( p(x; \hat{\eta}_i) \) or equivalently the log-likelihood ratio (LLR) \( \ln \frac{p(x; \hat{\eta}_i)}{p_0(x)} \) and choose the model which is associated with the maximum value.

For the linear model, plugging \( \hat{\eta}_i \) into (4) produces the maximized LLR

\[
\ln \frac{p(x; \hat{\eta}_i)}{p_0(x)} = \left( \frac{x^T P_i x}{2 \sigma^2} - \frac{k_i}{2} \right) \ln \left( \frac{x^T P_i x}{\sigma^2 k_i} \right) + \ln \left( \frac{x^T P_i x}{\sigma^2 k_i} \right)
\]

\[
\cdot \ln \left( \frac{x^T P_i x}{2 \sigma^2} - \frac{k_i}{2} \right).
\]
where \( u(\cdot) \) is a unit step function. In fact, the term \( \frac{x^T P_i x}{2\sigma^2} \) is the maximized LRT of the conditional PDF \( p_i(x|\theta_i) \) and \( p_0(x) \), termed as \( l_{G_i} \):

\[
l_{G_i} = \ln \frac{\max_{\theta_i} p_i(x|\theta_i)}{p_0(x)} = \ln \frac{p_i(x|\theta_i)}{p_0(x)} \quad \text{with} \quad \hat{\theta}_i = (H_i^T H_i)^{-1}H_i^T x = \frac{x^T P_i x}{2\sigma^2}
\]

where the MLE \( \hat{\theta}_i \) is the value of \( \theta_i \) that maximizes \( p_i(x|\theta_i) \) or explicitly,

\[
p_i(x|\theta_i) = \exp \left( -\frac{1}{2\sigma^2} (x - H_i \theta_i)^T (x - H_i \theta_i) \right) \sqrt{\frac{1}{2\pi\sigma^2 I}} \]

In summary, we can write the linear model Bayesian EEF as

\[
\ln \frac{p(x; \hat{\eta}_i)}{p_0(x)} = \left( l_{G_i} - \frac{k_i}{2} - \frac{k_i}{2} \ln \frac{l_{G_i}}{k_i/2} \right) u \left( l_{G_i} - \frac{k_i}{2} \right).
\]

(6)

A. Rationale of Bayesian EEF model order selection algorithm

We now present the rationale for Bayesian EEF model order selection algorithm given above. First, when \( \eta_i \) is chosen as its MLE \( \hat{\eta}_i \),

\[
\frac{\partial \ln p(x; \eta_i)}{\partial \eta_i} = T_i(x) - K_0'(\eta_i) = 0
\]

follows from (B). That is \( T_i(x) = K_0'(\eta_i) \) evaluated at \( \eta_i = \hat{\eta}_i \). Moreover, it holds in general \( \int p(x; \eta_i) T_i(x) dx = K_0'(\eta_i) \) for the exponential family \([11]\). Therefore

\[
\left[ \int p(x; \eta_i) T_i(x) dx \right]_{\eta_i \to \hat{\eta}_i} = T_i(x)
\]

(7)

And consequently we have
\[
\text{KL}(p(\mathbf{x}; \hat{\eta}_i) || p_0(\mathbf{x})) \\
= \int p(\mathbf{x}; \hat{\eta}_i) \ln \frac{p(\mathbf{x}; \hat{\eta}_i)}{p_0(\mathbf{x})} d\mathbf{x} \\
= \int p(\mathbf{x}; \hat{\eta}_i) [\hat{\eta}_i T_i(\mathbf{x}) - K_0(\hat{\eta}_i)] d\mathbf{x} \\
= \hat{\eta}_i T_i(\mathbf{x}) - K_0(\hat{\eta}_i) \\
= \ln \frac{p(\mathbf{x}; \hat{\eta}_i)}{p_0(\mathbf{x})}
\]
where \(\text{KL}(\cdot || \cdot)\) denotes Kullback Libler divergence (KLD).

Moreover, a Pythagorean-like relationship holds asymptotically for large data record among KLD quantities for EEF \cite{11}

\[
\text{KL}(p_t(\mathbf{x}) || p(\mathbf{x}; \hat{\eta}_i)) = \text{KL}(p_t(\mathbf{x}) || p_0(\mathbf{x})) - \text{KL}(p(\mathbf{x}; \hat{\eta}_i) || p_0(\mathbf{x})),
\]
where \(p_t(\mathbf{x})\) denotes the true PDF of the data, which is unknown but fixed. The distance \(\text{KL}(p_t(\mathbf{x}) || p_0(\mathbf{x}))\) is fixed, hence the model that maximizes the distance \(\text{KL}(p(\mathbf{x}; \hat{\eta}_i) || p_0(\mathbf{x}))\) or equivalently \(\ln \frac{p(\mathbf{x}; \hat{\eta}_i)}{p_0(\mathbf{x})}\), among all models has the minimum \(\text{KL}(p_t(\mathbf{x}) || p(\mathbf{x}; \hat{\eta}_i))\)-the “distance” from the true PDF \(p_t(\mathbf{x})\). This is the reason why the Bayesian EEF model selection rule chooses the model with the maximum of the maximized EEF’s of all models.

**B. Discussion on paradoxes**

The EEF model order selection algorithm has many desirable properties such as consistency \cite{2} and better performances than many other algorithms in the low signal-to-noise ratio regime \cite{11}. In addition to these properties, we now show that the newly derived Bayesian EEF has additional desirable properties-it does not have *Lindley’s paradox* nor the *Information paradox*. On the contrary, many other Bayesian model selection methods based on marginal bayes factor (BF) may suffer from these paradoxes \cite{20}.

*Lindley’s paradox* can be understood as: “large spread of the prior induced by the non-informative choice of hyper-parameter has the unintended consequence of forcing the BF to favor the null model, the smallest
model, regardless of the information in the data \[20\]. As shown in \(6\), the reduced Bayesian EEF does not necessarily favor the null model even if we let the hyper-parameter \(g_i \to \infty\). This indicates that the reduced Bayesian EEF rule has no “Lindley’s paradox”. The Information paradox is “a paradox related to the limiting behavior of the BF. The BF yields a constant even when there is infinite amount of information supporting to choose a model \[20\].” For instance, the linear model BF resulted from assigning the parameter \(\theta_i\) a g-prior with a certain \(g_i\) is\[20\]

\[
BF(M_i : M_0) = \frac{(1 + g_i)^{(N-1-k_i)/2}}{(1 + g_i(1-R^2_r))^{(N-1)/2}}
\]

where \(R^2_r\) is the ordinary coefficient of determination of the regression model \(M_i\). When there is overwhelming information supporting to choose \(M_i\) instead of \(M_0\), \(R^2_r \to 1\); however, the BF yields a constant \((1 + g_i)^{(N-1-k_i)/2}\) instead of infinity. This information limiting behavior is called the information paradox. When \(R^2_r \to 1\) or equivalently \(x^T P_i x \gg k_i \sigma^2\) we have \(\hat{\eta}_i \to 1\) from \(5\). In this case, the reduced Bayesian EEF \(\ln p(x; \hat{\eta}_i) / p_0(x)\) in \(6\) also goes to infinity. This shows that the Bayesian EEF has no information limiting behavior and hence no Information paradox. In fact, these two nice properties of the Bayesian EEF model selection rule are due to its mechanism of choosing the value of \(\eta_i\). It uses the MLE \(\hat{\eta}_i\) which is dependent on data.

III. THE PENALTY TERM OF REDUCED BAYESIAN EEF

The penalty term is the key term for a model order selection rule. Its function is to penalize the maximum log-likelihood with a measure of model complexity so that the model order selection rule can strike a tradeoff between goodness-of-fit and model complexity. In light of the general relationship KLD=SNR-MI \[8\], the reduced Bayesian EEF penalty term is found to possess a very intuitive and enlightening interpretation. This not only helps further understanding EEF’s mechanism in model selection but also provides new insights into the problem of choosing a good penalty term for model selection. As shown next, the EEF penalty term can be viewed as the sum of a term proportional to the parameter dimension, \(k_i\), and estimated mutual information between the parameter and received data, \(\frac{k_i}{2} \ln \frac{2G_i}{k_i} \).
First note that if assigning the unknown parameter \( \theta_i \) a prior that depends upon the embedding parameter \( \eta_i \):

\[
\pi'(\theta_i; \eta_i) = \mathcal{N}(0, \frac{\eta_i}{1 - \eta_i} \sigma^2 (H_i^T H_i)^{-1}),
\]

the marginal PDF for model \( M_i \) becomes the reduced Bayesian EEF in (4)

\[
p_i(x) = \int p_i(x|\theta_i)\pi'(\theta_i; \eta_i)\,d\theta_i = \mathcal{N}\left(0, \sigma^2 I + \frac{\eta_i}{1 - \eta_i} \sigma^2 P_i\right) = p(x; \eta_i).
\]

Note that this new \( p_i(x) \) is in fact parameterized by \( \eta_i \) because \( \pi'(\theta_i; \eta_i) \) depends upon \( \eta_i \). To strengthen this point, we denote \( p_i(x) \) as \( p_{\eta_i}(x) \). Then we can write \( p_{\eta_i}(x) = p(x; \eta_i) \). Together with the relationship of (8) and the decomposition \( \text{KLD} = \text{SNR} - \text{MI} \) established in [8] (see also [9] for some illustrative examples of this decomposition), we have the following for \( \eta_i = \hat{\eta}_i \)

\[
\ln \frac{p(x; \eta_i)}{p_0(x)} \approx \text{KL}(p(x; \eta_i)||p_0(x))
\]

\[
= \text{KL}(p_{\eta_i}(x)||p_0(x))
\]

\[
= \int p_{\eta_i}(x) \ln \frac{p_{\eta_i}(x)}{p_0(x)}\,dx
\]

\[
= \int \left[ \int p_{\eta_i}(x, \theta_i) \ln \frac{p_{\eta_i}(x|\theta_i)}{p_0(x)}\,d\theta_i \,dx \right]
\]

\[
= \int \left[ \int p_{\eta_i}(x, \theta_i) \ln \frac{p_{\eta_i}(x|\theta_i)}{p_0(x)}\,d\theta_i \,dx \right]
\]

\[
= \int \left[ \int p_{\eta_i}(x, \theta_i) \ln \frac{p_{\eta_i}(x|\theta_i)}{p_0(x)}\,d\theta_i \,dx \right]
\]

\[
= \int \left[ \int p_{\eta_i}(x, \theta_i) \ln \frac{p_{\eta_i}(x|\theta_i)}{p_0(x)}\,d\theta_i \,dx \right]
\]

\[
= \text{SNR} \approx \hat{\text{SNR}}
\]

\[
- \int \left[ \int p_{\eta_i}(x, \theta_i) \ln \frac{p_{\eta_i}(x|\theta_i)}{p_0(x)}\,d\theta_i \,dx \right]
\]

\[
= \text{MI} \approx \hat{\text{MI}}
\]

(9)

where \( p_{\eta_i}(x, \theta_i) \) denotes the joint PDF of \( x \) and \( \theta_i \) and \( p_{\eta_i}(x|\theta_i) = \mathcal{N}(H_i \theta_i, \sigma^2 I) \) is the conditional PDF. This says that the reduced EEF can be decomposed into two terms. As shown next, the first term is an estimated SNR and hence is denoted as \( \hat{\text{SNR}} \) and the second term is an estimated MI between parameter \( \theta_i \) and data \( x \), denoted as \( \hat{\text{MI}} \). Note they are estimated terms in the sense that \( \eta_i \) is replaced by its MLE \( \hat{\eta}_i \).
A. The estimated SNR term

First, we have for $\eta_i = \hat{\eta}_i$

$$\tilde{\text{SNR}} = \int \int p_{\hat{\eta}_i}(x, \theta_i) \ln \frac{p_{\hat{\eta}_i}(x|\theta_i)}{p_0(x)} d\theta_i dx \quad (10)$$

$$= \int \pi'_{\theta_i} \left[ \text{KL}(p_{\hat{\eta}_i}(x|\theta_i)||p_0(x)) \right] d\theta_i$$

$$= \int \pi'_{\theta_i} \left[ \text{KL} \left( \mathcal{N}(H_i \theta_i, \sigma^2 I)|| \mathcal{N}(0, \sigma^2 I) \right) \right] d\theta_i$$

$$= \int \pi'_{\theta_i} \left( \frac{1}{2} \theta_i^T H_i H_i^T \theta_i \right) d\theta_i$$

$$= \int \left[ e^{\frac{1}{2} \theta_i^T \left( \frac{\eta_i}{\hat{\eta}_i} \sigma^2 (H_i^T H)^{-1} \right) \theta_i} \right] \frac{\left( \frac{1}{2} \theta_i^T H_i H_i^T \theta_i \right)^{\frac{1}{2}}}{2\pi^{\frac{1}{2}} \eta_i^{\frac{1}{2}} \sigma^2 (H_i^T H)^{-1}} d\theta_i$$

$$= \frac{1}{2} x^T P_i x - \frac{k_i}{2}$$

$$= l_{G_i} - \frac{k_i}{2} \quad (12)$$

where we have used the $\hat{\eta}_i$ in (5), treated as a constant, to replace $\eta_i$. The eqn (11) indicates that the first term is an average ratio of signal energy $||H_i \theta_i||^2$ and the noise power $\sigma^2$, and indeed is a measure of SNR; furthermore by (13) we see that $\tilde{\text{SNR}}$ has introduced a penalty term $k_i/2$, which is proportional to the parameter dimension. In fact, (13) not only holds for linear model but also is approximately valid in general. First, we can rewrite the $\tilde{\text{SNR}}$ term as

$$\tilde{\text{SNR}} = \int \int p_{\hat{\eta}_i}(x, \theta_i) \ln \frac{p_{\hat{\eta}_i}(x|\theta_i)}{p_0(x)} d\theta_i dx$$

$$= \int p_{\hat{\eta}_i}(x) \int \pi(\theta_i|x) \left[ \ln \frac{p_{\hat{\eta}_i}(x|\theta_i)}{p_0(x)} \right] d\theta_i dx$$

$$\pi(\theta_i|x) = \mathcal{N}(\hat{\theta}_i, I^{-1}(\hat{\theta}_i)), \quad (14)$$

where $\pi(\theta_i|x)$ is the posterior distribution of $\theta_i$ after observing x. For large data records we have approximately [19]
where $I(\hat{\theta}_i)$ is the Fisher information matrix (FIM) of $\theta_i$ evaluated at its MLE $\hat{\theta}_i$. And using the Laplace approximation we have

$$
\int_{\theta_i} \pi(\theta_i | x) \ln \frac{p_{\eta_i}(x | \theta_i)}{p_0(x)} d\theta_i \\
\approx \int_{\theta_i} \pi(\theta_i | x) \left[ \ln \frac{p_{\eta_i}(x | \hat{\theta}_i)}{p_0(x)} \right] \\
\qquad - \frac{1}{2} (\theta_i - \hat{\theta}_i)^T I(\hat{\theta}_i) (\theta_i - \hat{\theta}_i) \right] d\theta_i \\
= l_{G_i} - \frac{k_i}{2}.
$$

Therefore from (14)

$$\widehat{SNR} \approx \int p_{\eta_i}(x) \left[ l_{G_i} - \frac{k_i}{2} \right] dx \\
= \int p(x; \hat{\eta}_i) (T_i(x) - \frac{k_i}{2}) dx \\
= T_i(x) - \frac{k_i}{2} \\
= l_{G_i} - \frac{k_i}{2} \quad (15)
$$

where we have used $\left[ \int p(x; \eta_i) T_i(x) dx \right]_{\eta_i = \hat{\eta}_i} = T_i(x)$ in (7). This shows that the difference between $l_{G_i}$ and the estimated SNR is asymptotically half of the parameter dimension.

**B. The estimated mutual information term**

We now consider the second term $\widehat{MI}$ in the decomposition (9). For linear model it is $\frac{k_i}{2} \ln \frac{2l_{G_i}}{k_i}$ as given in (6). It is shown next that in general it is the estimated MI between $\theta_i$ and $x$. First we have from
definition of mutual information,

\[\hat{MI} = \int \int p_{\hat{\eta}}(x, \theta_i) \ln \frac{p_{\hat{\eta}}(x|\theta_i)}{p_{\hat{\eta}}(x)} dx d\theta_i\]  \hspace{1cm} (16)

\[= \int_{\theta_i} \pi'(\theta_i) \int_x p_{\hat{\eta}}(x|\theta_i) \ln \frac{p_{\hat{\eta}}(x|\theta_i)}{p_{\hat{\eta}}(x)} dx d\theta_i\]

\[= \int_{\theta_i} \pi'(\theta_i)\text{KL} \left( p_{\hat{\eta}}(x|\theta_i) || p_{\hat{\eta}}(x) \right) d\theta_i\]

\[= \int_{\theta_i} \pi'(\theta_i)\]

\[\cdot \text{KL} \left( \mathcal{N}(H_i \theta_i, \sigma^2 I) \| \mathcal{N}(0, \sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_i) \right) d\theta_i\]

\[= \int_{\theta_i} \pi'(\theta_i) \left[ \frac{1}{2} \ln |\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H| 
\quad + \frac{1}{2} \text{tr} \left( \sigma^2 (\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H)^{-1} - I \right) 
\quad + \frac{1}{2} (H \theta)^T (\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H)^{-1} H \theta \right] d\theta_i\]

\[= \frac{1}{2} \ln |\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H| 
\quad + \frac{1}{2} \text{tr} \left( \sigma^2 (\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H)^{-1} - I \right) 
\quad + \int_{\theta_i} \left[ \pi'(\theta_i) \frac{1}{2} (H \theta)^T (\sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_H)^{-1} H \theta \right] d\theta_i\]

\[= \frac{1}{2} \ln \left| \sigma^2 I + \frac{\hat{\eta}_i}{1 - \hat{\eta}_i} \sigma^2 P_i \right|\]

\[= k_i \ln \left( \frac{1}{1 - \hat{\eta}_i} \right) \quad \hspace{1cm} (17)\]

\[= k_i \ln \left( \frac{x^T P_i x}{k_i \sigma^2} \right) \quad \hspace{1cm} (18)\]

\[= \frac{k_i}{2} \ln \frac{2l_{G_i}}{k_i} \quad \hspace{1cm} (19)\]

This verifies that the term \(\frac{k_i}{2} \ln \frac{2l_{G_i}}{k_i}\) of (6) is indeed an estimated mutual information term. As a measure of the statistical dependence of the parameter and observed data, the estimated MI is a reasonable measure of model complexity. First, the estimated MI can be viewed as averaged KLD distance between the \(p_{\hat{\eta}}(x|\theta_i)\) and \(p_{\hat{\eta}}(x)\), see (16), which assesses the “modeling potential” of the conditional distribution. Second, the
estimated MI also measures the difference between the prior and posterior distributions of the unknown parameter and thus relates to the “difficulty of estimation” [24]. From (17) we see that for linear model \( \hat{MI} \) is monotonic with both the parameter dimension \( k_i \) and the embedding parameter \( \hat{\eta}_i \). As \( \hat{\eta}_i \) goes to zero, \( \hat{MI} \to 0 \). This is in agreement with the expectation from (3) in that when \( \eta_i \to 0 \), the Bayesian EEF \( p(x; \eta_i) \) reduces to the null model PDF \( p_0(x) \). When \( \hat{\eta}_i \) increases, the resulting Bayesian EEF \( p(x; \eta_i) \) moves closer towards \( p_i(x) \) as shown in (3). The estimated MI simultaneously increases to reflect the increasing model complexity.

As shown, the Bayesian EEF penalty term takes into account three levels of model complexity, namely, parameter dimension, the prior of the unknown parameter \( \pi'_i(\theta_i) \) and the functional form on how the model is parameterized, the latter two of which contribute to the estimated MI. On the other hand, AIC only accounts for the dimension of unknown parameters \( k_i \); BIC takes into consideration the parameter dimension \( k_i \) and the number of independently identical distributed (IID) data samples [4], [6] and [23].

C. An alternative interpretation of the estimated mutual information term

A closer look at the estimated mutual information term in (19) leads to an alternative intuition. Using the approximate relationship of \( \hat{SNR} \) and \( l_{Gi} \) [15] in (19) we have

\[
\hat{MI} = \frac{k_i}{2} \ln \frac{2l_{Gi}}{k_i} \\
= k_i \left[ \frac{1}{2} \ln \left( 1 + \frac{\hat{SNR}}{k_i/2} \right) \right] \quad \text{per dim}
\]

The estimated mutual information term is the multiplicative result of parameter dimension \( k_i \) and the estimated MI per parameter dimension \( \frac{1}{2} \ln \left( 1 + \frac{\hat{SNR}}{k_i/2} \right) \). As an example, for the normal linear model we
have from (18) that \( \hat{MI} = \frac{k_i}{2} \ln \left( \frac{x^T P_i x}{k_i \sigma^2} \right) \) and

\[
x^T P_i x = x^T H_i (H_i^T H_i)^{-\frac{1}{2}} x
\]

\[
= \frac{1}{2} \left( \frac{1}{k_i \sigma^2} \right) \ln \left( \frac{1}{k_i \sigma^2} \right)
\]

\[
\hat{MI} = \frac{k_i}{2} \ln \left( \frac{x^T P_i x}{k_i \sigma^2} \right)
\]

\[
= \frac{k_i}{2} \ln \left( \frac{1}{k_i \sigma^2} \sum_{j=1}^{k_i} (\theta_i[j] + w'[j])^2 \right)
\]

where we have denoted \( \theta_i' = (H_i^T H_i)^{-\frac{1}{2}} H_i \theta_i = (H_i^T H_i)^{\frac{1}{2}} \theta_i \). It is of dimension \( k_i \times 1 \) and can be viewed as a signal coordinate vector. Also \( w' = (H_i^T H_i)^{-\frac{1}{2}} H_i \theta_i \). It is of dimension \( k_i \times 1 \) and is a noise coordinate vector. Finally we denote \( y = \theta_i' + w' \), which is of dimension \( k_i \times 1 \).

With these notations, the estimated MI can be rewritten as

\[
\hat{MI} = \frac{k_i}{2} \ln \left( \frac{1}{k_i \sigma^2} \sum_{j=1}^{k_i} (\theta_i[j] + w'[j])^2 \right)
\]

Furthermore, we have the distributions of \( \theta_i' \) and \( w' \) based on the PDFs of \( \theta_i \) and \( w \), as

\[
\theta_i' \sim N(0, C_{\theta_i'})
\]

with

\[
C_{\theta_i'} = (H_i^T H_i)^{\frac{1}{2}} \frac{\eta_i}{1 - \eta_i} \sigma^2 (H_i^T H_i)^{-1} (H_i^T H_i)^{\frac{1}{2}}
\]

\[
= \frac{\eta_i \sigma^2}{1 - \eta_i} I_{k_i},
\]

where \( I_{k_i} \) denotes the identity matrix of dimension \( k_i \) and we have introduced \( \sigma^2 = \frac{m}{1 - m} \sigma^2 \) to simply the notation. This shows that by using the g-prior on \( \theta_i \), the coordinate vector \( \theta_i' \) has a scaled identity.
matrix as its covariance matrix; that is each element of the resulting vector $\mathbf{\theta}_i$ is identically independently distributed (IID). The g-prior equalizes the distribution of each parameter of $\mathbf{\theta}_i$.

Similarly, we have the distribution of $\mathbf{w}'$ as

$$\mathbf{w}' \sim \mathcal{N}(0, \mathbf{C}_w')$$

with

$$\mathbf{C}_w' = (\mathbf{H}_i^T \mathbf{H}_i)^{-\frac{1}{2}} \mathbf{H}_i^T \sigma^2 \mathbf{I}_N \mathbf{H}_i (\mathbf{H}_i^T \mathbf{H}_i)^{-\frac{1}{2}}$$

$$= \sigma^2 \mathbf{I}_{k_i}$$

This shows that $\mathbf{w}'$ still has a zero mean normal distribution with a covariance matrix being $\sigma^2 \mathbf{I}_{k_i}$. Then we have the PDF of $\mathbf{y} = \mathbf{\theta}_i' + \mathbf{w}'$, $p(\mathbf{y})$ as

$$p(\mathbf{y}) = \mathcal{N}(0, \mathbf{C}_{\theta_i}' + \mathbf{C}_w')$$

$$= \mathcal{N}(0, (\sigma^2 + \sigma_{\theta_i}^2) \mathbf{I}_{k_i})$$

In fact the term $\frac{1}{k_i} \sum_{j=1}^{k_i} (\mathbf{\theta}_i'[j] + \mathbf{w}'[j])^2$ in (21) is the estimate of $\sigma^2 + \sigma_{\theta_i}^2$ and the hence (21) can be expressed alternatively as

$$\hat{\text{MI}} = k_i \frac{1}{2} \ln \frac{\sigma^2 + \sigma_{\theta_i}^2}{\sigma^2}$$

The term $\hat{\text{MI}}$ per dim is the standard estimated mutual information for the case of Gaussian signal in additive Gaussian noise [26] for each signal component/parameter dimension. Since by employing the g-prior each element of the signal $\mathbf{\theta}_i'$ is IID, the total estimated MI is simply a multiplication of the $\hat{\text{MI}}$ per dim and the parameter dimension $k_i$. This provides another intuition on how the estimated MI depends on the parameter dimensions and the mechanism of the g-prior.

### IV. BAYESIAN EEF VIA JEFFREYS’ PRIOR

Jeffreys’ prior is another compelling non-informative prior [21] due to its property of invariance to reparameterization. In this section, we use the Jeffreys’ prior in Bayesian EEF and derive the asymptotic
Bayesian EEF. For each model $\mathcal{M}_i$ we assign a Jeffreys’ prior $\pi_i(\theta_i)$ to the unknown $\theta_i$. The Jeffreys’ prior PDF of $\theta$ is proportional to the square root of the determinant of FIM of $\theta_i$; that is, $\pi_i(\theta_i) \propto \sqrt{|I(\theta_i)|}$.

A motivation for the Jeffreys’ prior is that Fisher information $I(\theta_i)$ is an indicator of the amount of information brought by the model/observations about unknown parameter $\theta_i$. Favoring the values of $\theta_i$ for which $I(\theta_i)$ is large, is equivalent to minimizing the influence of the prior [21]. By the Laplace approximation we have

$$p_i(x|\theta_i) \approx p_i(x|\hat{\theta}_i)e^{-\frac{1}{2}(\theta_i-\hat{\theta}_i)^T I(\hat{\theta}_i)(\theta_i-\hat{\theta}_i)}.$$ 

Moreover when assuming that $\pi_i(\theta_i)$ is flat around $\hat{\theta}_i$, which is valid for large data records, we have approximately

$$p_i(x) = \int_{\theta_i} p_i(x|\theta_i)\pi_i(\theta_i)d\theta_i$$

$$\approx p_i(x|\hat{\theta}_i)\pi_i(\hat{\theta}_i)\int e^{-\frac{1}{2}(\theta_i-\hat{\theta}_i)^T I(\hat{\theta}_i)(\theta_i-\hat{\theta}_i)}d\theta_i$$

$$= \frac{p_i(x|\hat{\theta}_i)\pi_i(\hat{\theta}_i)}{(2\pi)^{-\frac{n_i}{2}}\sqrt{|I(\hat{\theta}_i)|}}$$

Substituting this approximation into the EEF definition, we have

$$\ln \frac{p(x;\eta_i)}{p_0(x)}$$

$$= \eta_i \ln \frac{p_i(x)}{p_0(x)} - K_0(\eta_i)$$

$$\approx \eta_i \ln \frac{(2\pi)^{-\frac{n_i}{2}}\sqrt{|I(\hat{\theta}_i)|}}{p_0(x)}$$

$$- \ln E_0 \exp \left( \eta_i \ln \frac{p_i(x|\hat{\theta}_i)\pi_i(\hat{\theta}_i)}{p_0(x)} \right)$$

$$= \eta_i \ln \frac{p_i(x|\hat{\theta}_i)}{p_0(x)} - \ln E_0 \exp \left( \eta_i \ln \frac{p_i(x|\hat{\theta}_i)}{p_0(x)} \right)$$

Assigning $\theta_i$ a Jeffreys’ prior, the term $\frac{\pi_i(\hat{\theta}_i)}{(2\pi)^{-\frac{n_i}{2}}\sqrt{|I(\hat{\theta}_i)|}}$ becomes a constant and thus the marginal PDF $p_i(x)$ becomes the multiplication of the maximized conditional PDF $p_i(x|\hat{\theta}_i)$ with the constant. From the derivation, it shows that by employing EEF mechanism, the resulting Bayesian model selection rule does
not suffer from problems when \( \int \sqrt{I(\theta_i)} d\theta_i \rightarrow \infty \) as the FIM term is eliminated by the log-normalization term \( K_0(\eta_i) \) using the Jeffreys’ prior. This is one of many examples showing that the embedded family derives many of its useful properties from the use of the normalization term \( K_0(\eta_i) \). And it is this property that makes the approximate Bayesian EEF yield the same result as the frequentist EEF in [11].

For the normal linear model problem, the reduced Bayesian EEF, approximate Bayesian EEF methods and the reduced frequentist EEF all coincide with each other. This coincidence stems from the fact that the FIM for all \( \theta_i \) are the same under a certain model \( \mathcal{M}_i \) in that \( I(\theta_i) = \frac{H_i^T H_i}{\sigma^2} \). In this case the Jeffreys’ prior, \( \pi(\theta_i) \propto \sqrt{|I(\theta_i)|} \), becomes an improper uniform distribution, \( \pi(\theta_i) = c > 0 \), where \( c \) is a positive constant. This example also shows that Bayesian EEF can employ improper uniform prior without suffering from integration problems.

V. CONCLUSION

We have derived the Bayesian EEF, a new Bayesian model order selection rule, by using the EEF strategy in a Bayesian framework. The Bayesian EEF is shown to possess some desirable properties. To avoid introducing subjectivity in choosing parameter priors, the Bayesian EEF can utilize a vague proper prior as well as an improper non-informative prior, both of which are natural choices of non-informative priors but are usually forbidden by Bayesian model selection methods. It is also demonstrated that the EEF model order selection rule has a very intuitive penalty term as the sum of the parameter dimension and the estimated MI between parameter and received data. This interpretation not only helps in understanding the mechanisms at work in the EEF method but also provides new insights into the open question of designing an optimal penalty term for model selection. Some interesting interactions and coincidences between the EEF model order selection rules derived from Bayesian and frequentist viewpoints are also explained.

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