Vector properties of radially self-accelerating beams

Marco Ornigotti and Alexander Szameit

Institut für Physik, Universität Rostock, Albert-Einstein-Straße 23, D-18059 Rostock, Germany

E-mail: marco.ornigotti@uni-rostock.de

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Abstract

We present a complete and consistent theory of vector radially self-accelerating beams (RSABs). We use this theory as a model for describing the properties of focused RSABs, and to show, in particular, that only circular polarised RSABs maintain their self-accelerating character upon focusing. Moreover, we also calculate the linear and angular momentum for paraxial vector RSABs, and discuss both their global and local properties.

Keywords: accelerating beams, vector fields, singular optics

(Some figures may appear in colour only in the online journal)

1. Introduction

Accelerating beams, i.e. electromagnetic fields that propagate along curved trajectories in free space without being subject to any external force, have been the subject of a thorough investigation over the last few years. The most famous representative of such a class of beams is, without doubt, the Airy beam. First introduced in the context of quantum mechanics by Berry and Balazs as an exotic solution to the Schrödinger equation [1], it was then introduced in optics in 2007 by Siviloglou et al [2, 3] as an exact solution of the paraxial equation propagating along a parabolic trajectory in free space. Due to their intriguing features, Airy beams were studied in different contexts, such as nonlinear optics [4] and particle manipulation [5], and were proposed as an efficient way to generate curved plasma channels [6].

Inspired by these results, the last few years have witnessed the emergence of many different types of accelerating beams in different coordinate systems, such as parabolic [7] and Weber [8] beams. Moreover, beams of light capable of propagating along curved [9–11] and arbitrary [12, 13] trajectories have also been proposed. Recently, two new classes of accelerating beams have been introduced, namely angular [14, 15] and radially self-accelerating beams [16, 17]. While the former acquire angular acceleration during rotation around their optical axis [14], the latter exhibit radial acceleration, a feature which makes them propagate along spiralling trajectories around their optical axis.

Radially self-accelerating beams (RSABs) can be understood in terms of superpositions of Bessel beams, where each single component is characterised by an angular velocity proportional to the amount of orbital angular momentum it carries. This ultimately results in an electromagnetic field, whose transverse field or intensity distribution rotates around the propagation direction with a given constant angular velocity $\Omega$ [16]. Among the vast zoology of RSABs, helicon beams in particular, i.e. a subclass of RSABs consisting of rotating diffraction-free beams based on the superposition of two Bessel beams with opposite orbital angular momentum, have attracted a lot of interest over the last decades [17–26]. Beyond helicon beams, RSABs have potentially significant applications in different areas of physics, such as sensing [6], material processing [27, 28], and particle manipulation [29, 30].

Despite this broad interest, RSABs have only been defined within the scalar electromagnetic theory, and their vector nature, as well as the effect of focusing on their self-accelerating character, has not yet been investigated. In this work, therefore, we introduce vector RSABs, and study their vector properties in terms of their linear and angular momentum content. Moreover, we carefully analyse what is the impact of focusing on the self-accelerating character of RSABs, and under which conditions the focusing process does not spoil this property.

This work is organised as follows: in section 2, we briefly recall the definition of RSABs, and recall some of their main properties. In section 3, we use the method of Hertz potentials to...
construct vector RSABs, and use these solutions as a model for focused RSABs. Then, we derive a condition on the polarisation that a scalar RSAB must possess, in order to maintain its self-accelerating character upon focusing. Section 4 is then devoted to calculating the linear and angular momentum for paraxial, intensity rotating RSABs. Conclusions are then drawn in section 5.

2. Radially self-accelerating beams

We start our analysis by considering scalar, monochromatic, free space solutions of the Helmholtz equation

\[(\nabla^2 + k_0^2)\psi(r) = 0,\]

where \(k_0\) is the vacuum wave vector. The most general solution of the above equation in cylindrical coordinates can be given in terms of the superposition of Bessel beams, i.e.,

\[\psi(r) = \sum_m \int d\xi C_m(\xi) J_m(\rho \sqrt{1 - \xi^2}) e^{i m \theta + i \xi l},\]

where \(\rho = k_0 R\), and \(\xi = k_0 z\) are normalised radial and longitudinal coordinates, \(J_m(\lambda)\) is the Bessel function of the first kind [31], and the integration variable \(\xi = \cos \theta_0\) plays the role of the Bessel cone angle \(\theta_0\) [32].

From the above solution, it is possible to extract RSABs by applying the requirements that equation (2) must fulfill to be an RSAB [16]. First, \(\psi(r)\) must propagate freely, and not under the action of a certain potential. Then, there should exist a suitable reference frame, in which \(\psi(r)\) is manifestly propagation invariant, i.e. no explicit \(\xi\)-dependence must appear. Finally, an observer at rest in such a reference frame should experience a fictitious force, which, ultimately, is at the core of the self-accelerating character of RSABs.

While the first requirement is automatically met by the fact that we are considering free space propagation, the second one is very useful to define RSABs properly. Once it is fulfilled, in fact, it is not hard to show that the third requirement follows accordingly. We therefore require that after a suitable coordinate transformation \(r' = S r\), the field \(\psi(r')\) in the new coordinate frame is manifestly propagation invariant, i.e. \(\partial \psi(r')/\partial \xi = 0\). To this aim, we introduce the co-rotating coordinate \(\Phi = \theta + \Lambda \zeta\), and choose the expansion coefficient as \(C_m(\xi) = D_m \delta(\xi - (m \Lambda + \beta))\), where \(\Lambda = \Omega/k_0 > 0\) is the normalised angular velocity of the RSAB, and \(\beta\) is a free (dimensionless) parameter with the physical meaning of a normalised propagation constant. Substituting this Ansatz in equation (2), we get the following result

\[\psi_{\text{RSAB}}(\rho, \Phi) = e^{i m \Phi} \sum_{m \in \mathcal{M}} D_m J_m(\alpha_m \rho) e^{i m \Phi},\]

where \(\alpha_m = \sqrt{1 - (m \Lambda + \beta)^2}\), and \(\mathcal{M} = \{m \in \mathbb{N} : \alpha_m > 0\}\). For \(\beta = 0\), the above field is manifestly propagation invariant, as no explicit \(\xi\)-dependence is present. Moreover, its amplitude and phase both rotate with normalised angular velocity \(\Lambda\) during propagation. For \(\beta \neq 0\), on the other hand, the field itself is not propagation invariant anymore, due to the presence of the global phase factor \(e^{i m \Phi}\). Nevertheless, the intensity \(|\psi_{\text{RSAB}}(\rho)|^2\) is also propagation invariant for \(\beta \neq 0\). In this case, however, while both the intensity and phase propagate describing spiralling trajectories, they are not synchronised anymore. These two classes of RSABs are called field rotating and intensity rotating, respectively [16].

Moreover, it is worth noticing that while for \(\beta = 0\) the set \(\mathcal{M}\) contains only positive integers, for \(\beta \neq 0\) positive and negative values of \(m\) are allowed. Thus, helicon beams, for example, are a particular case of intensity rotating RSABs, where only two Bessel beams are participating in the sum in equation (3). An example of both classes of RSABs is given in figures 1 and 2, respectively.

Of particular interest are RSABs with \(\Lambda \ll 1\). Since \(\Lambda = \Omega/k_0\) this condition corresponds to RSABs whose actual angular velocity \(\Omega\) is much smaller than the beam’s wave vector \(k_0\). This ultimately corresponds to experimentally realisable RSABs. In the rest of this manuscript, if not specified otherwise, we will always implicitly assume that \(\Lambda \ll 1\) holds. Moreover, this assumption has different consequences for field and intensity rotating RSABs.

In the former case (i.e. for \(\beta = 0\)), \(\Lambda \ll 1\) implies that the (normalised) transverse momentum of each Bessel component is given by \(\alpha_m = \sqrt{1 - m^2 \Lambda^2} \approx 1 + O(m^2 \Lambda^2)\). If we recall that the transverse momentum of a Bessel beam is related to the Bessel cone angle by the relation \(k_2 = k_0 \sin \theta_0\), a value of the normalised transverse momentum \(\alpha_m \approx 1\) corresponds to \(\theta_0 \approx \pi/2\), i.e. to a highly nonparaxial Bessel beam.

Despite this fact, however, the nature of the resulting RSAB can be tuned at will between paraxial and nonparaxial, by simply changing the number of Bessel beams that participate in the sum in equation (3). To obtain nonparaxial RSABs, it is sufficient to limit the summation in equation (3) to \(m_{\text{max}} = \max |\mathcal{M}| < [\Lambda^{-1}]\). In this case, in fact, the transverse momentum of every Bessel component will be \(\alpha_m \approx 1\), and the resulting RSAB will be highly nonparaxial.

On the other hand, if one includes only values of \(m\) that are close to \([\Lambda^{-1}]\), i.e. if \(m \in [m_{\text{max}} - \tilde{m}, m_{\text{max}}]\) in equation (3), then \(\Lambda \approx 1\), and, correspondingly, \(\alpha_m \approx 0\). In this case, all Bessel components will be paraxial (i.e. the correspondent cone angle will be \(\theta_0 \ll 1\)), and the resulting RSAB can also be interpreted as a paraxial beam.

For intensity rotating RSABs (i.e. for \(\beta \neq 0\)), \(\alpha_m\) can instead be made arbitrarily small, independently from the value of \(\Lambda\), by suitably tuning the parameter \(\beta\). In this case, the paraxial limit is then simply obtained by choosing \(\beta\) such that \(\alpha_m \approx 0\), i.e. \(\beta \approx 1 - m_{\text{max}} \Lambda\), with \(m_{\text{max}} = \max |\mathcal{M}|\). Notice that with this choice of \(\beta\), \(\alpha_m = 0\), the sum in equation (3) extends to \(m_{\text{max}} - 1\) as \(J_m(\alpha_m \rho) = 0\).

This extra flexibility in tuning the propagation constant \(\beta\) and the angular velocity \(\Lambda\) independently makes intensity rotating RSABs easier to generate and manipulate experimentally than their field rotating counterparts [16, 17].

1 With \(\tilde{m}\) small compared to \(m_{\text{max}}\) such that \(\tilde{m}^2 \Lambda^2 \ll 1\) still holds.
3. Vector radially self-accelerating beams

The solution presented in equation (3) describes scalar RSABs. In many situations, however, a scalar representation of the electromagnetic field is not enough to fully describe its properties. A typical example is the focusing of a beam of light by means of a thick lens. On the focal plane of the lens, the transverse magnetic fields can be obtained straightforwardly from the TE ones by setting $E_{TM} \rightarrow B_{TE}$, and $B_{TM} \rightarrow -E_{TE}/c^2$.

To calculate the vector electric and magnetic fields corresponding to RSABs, we first rewrite equation (3) as

$$\psi_{RSAB}(r) = \sum_n D_n \phi_n(r),$$

with $\phi_n(r) = \exp[im\theta + i(m\Lambda + \beta)\xi]J_n(\alpha_n \rho)$ being the usual Bessel beam [32]. This allows us to define the Hertz potentials for RSABs in terms of the Hertz potentials for ordinary Bessel beams, i.e.

$$\Pi(r, t) = \sum_{m \in \mathcal{M}} D_m P^{(m)}(r, t)$$

where

$$P^{(m)}(r, t) = \phi_m(r) e^{-i\omega t} \hat{\mathbf{f}}$$

is the Hertz potential corresponding to a single Bessel beam $\phi_m(r)$, whose polarisation is defined by the unit vector $\hat{\mathbf{f}}$. Then, using equation (4), we can first calculate the electric and magnetic vector fields for a single Bessel beam, namely

$$E^{(m)}(r, t) = -\frac{\partial}{\partial t} [\nabla \times P^{(m)}(r, t)],$$

$$B^{(m)}(r, t) = \nabla \times \nabla \times P^{(m)}(r, t).$$

Then, the electric and magnetic fields of vector RSABs can be written as follows:

$$E(r, t) = \sum_{m \in \mathcal{M}} D_m E^{(m)}(r, t),$$

$$B(r, t) = \nabla \times \nabla \times E(r, t).$$
In all these distributions have been plotted in the region $0 \leq \rho \leq 1200$. The difference in the plotting range for the normalised radial coordinate $\rho$ with respect to figure 1 reflects the paraxial character of the intensity rotating RSABs, in contrast to the nonparaxial character of field rotating RSABs. In all these figures, $\Lambda = 10^{-5}$ (corresponding to an angular velocity of $\Omega \approx 75 \text{ rad m}^{-1}$ at $\lambda = 800 \text{ nm}$), $m_{\text{max}} = 4$, and $D_m = 1$ have been used. Moreover, $\beta = 1 - m_{\text{max}} \Lambda = 0.99996$ (corresponding to a value of a global propagation constant $\beta \approx 7.8 \mu\text{m}^{-1}$ for $\lambda = 800 \text{ nm}$) has been used. The white arrows in the intensity profiles show the direction of rotation.

\[
\mathbf{B}(\mathbf{r}, t) = \sum_{m \in \mathcal{M}} D_m \mathbf{B}^{(m)}(\mathbf{r}, t). \tag{8b}
\]

### 3.1. The role of polarisation of the Hertz potential in determining the properties of vector RSABs

Vector beams are frequently used as a model to describe focused light. From this perspective, the method of the Hertz potential offers an intuitive and insightful perspective on the process of focusing of a beam of light by a lens, or an objective, for example. In fact, one can interpret the Hertz potential $\mathbf{H}$ as the electromagnetic field before the focusing system, consisting of a scalar field distribution, and a given polarisation $\hat{f}$. The vectorialisation procedure described in equations (4) then represent the full vector field after the focusing process (for example, in the focal plane of a lens). Because of the structures of equations (4), it is not difficult to see that the initial polarisation $\hat{f}$ possessed by the field will contribute in determining all the components of the focused field.

In RSABs, it is interesting to see whether the vectorialisation procedure described above (i.e. the focusing process) preserves their self-accelerating character, or, in case it does not, under which conditions the self-accelerating character of RSABs is preserved. To do so, we first introduce the polarisation vector $\hat{f} = f_p \hat{x} + f_q \hat{y}$ (where $f_{p,q} \in \mathbb{C}$, and $|f_p|^2 + |f_q|^2 = 1$). Then, we use equations (6) and (7) to calculate the vector electric and magnetic fields corresponding to arbitrary polarised Bessel beams. Because of the intrinsic cylindrical symmetry of RSABs, we also introduce a (normalised) cylindrical reference frame $(\hat{\rho}, \hat{\theta}, \hat{\zeta})$. In this reference frame, the electric and magnetic fields of a single vector Bessel component can be written as

\[
\mathbf{E}^{(m)}(\mathbf{r}, t) = \mathbf{E}^{(m)}_p(\rho, \theta) \hat{\rho} + \mathbf{E}^{(m)}_q(\rho, \theta) \hat{\theta}, \tag{9a}
\]

\[
\mathbf{B}^{(m)}(\mathbf{r}, t) = \mathbf{B}^{(m)}_p(\rho, \theta) \hat{\rho} + \mathbf{B}^{(m)}_q(\rho, \theta) \hat{\theta}. \tag{9b}
\]

where $\Phi = \theta + \Lambda \zeta$ is the co-rotating coordinate defined in the previous section, and the field components are given by

\[
E_p^{(m)}(\rho) = \omega (\beta + m \Lambda) (f_p \cos \theta - f_q \sin \theta) J_m(\alpha_m \rho), \tag{10a}
\]

\[
E_q^{(m)}(\rho) = -\omega (\beta + m \Lambda) (f_p \cos \theta + f_q \sin \theta) J_m(\alpha_m \rho), \tag{10b}
\]

\[
E_{c}^{(m)}(\rho) = \frac{m \omega}{\rho} J_m(\alpha_m \rho) (f_p \cos \theta + f_q \sin \theta)
+ i \omega (\alpha_m J_{m-1}(\alpha_m \rho) - \frac{m}{\rho} J_m(\alpha_m \rho)) (f_p \cos \theta - f_q \sin \theta), \tag{10c}
\]

for the electric field, and
\[ B_{\rho}^{(m)}(\rho) = -\frac{\alpha_m}{\rho}[(f_0 - i\mu'f_p)\cos \theta + (f_p + i\mu'f_0)\sin \theta]J^\prime_m(\alpha_m \rho) - 2i\mu_m(\alpha_m \rho)[(f_0 + i\mu'f_p)\cos \theta - (f_p - i\mu'f_0)\sin \theta], \]

(11a)

\[ B_{\theta}^{(m)}(\rho) = \frac{im}{\rho}(f_0 \cos \theta + f_p \sin \theta)[\alpha_m J^\prime_m(\alpha_m \rho) - J_m(\alpha_m \rho)] + (f_p \cos \theta - f_0 \sin \theta)(\beta + \mu \Lambda)J_m(\alpha_m \rho) - \alpha^2 J^\prime_m(\alpha_m \rho), \]

(11b)

\[ B_{\phi}^{(m)}(\rho) = -\frac{m(\beta + \mu \Lambda)}{2}(f_0 \cos \theta - f_p \sin \theta)J_m(\alpha_m \rho) + i\mu_m(\beta + \mu \Lambda)(f_p \cos \theta + f_0 \sin \theta)J^\prime_m(\alpha_m \rho), \]

(11c)

for the magnetic field. In the equations above, \( J^\prime_m(\alpha_m \rho) \) and \( J_m(\alpha_m \rho) \) are the first and second derivatives of the Bessel function with respect to their arguments, respectively [31]. The electric and magnetic fields of arbitrary polarised RSABs can then be constructed by substituting the expressions above into equation (10). Their explicit expression is reported in appendix A, for completeness.

3.2. Polarisation constraint for vector RSABs

In section 2, we have described the requirements that a scalar field must fulfil to be an RSAB. In particular, the most important requirement is the existence of a suitable co-rotating reference frame, in which the field appears propagation invariant. If such a reference frame exists, an observer at rest in such a reference frame would then experience a fictitious centrifugal force.

For scalar fields, however, this condition is independent of polarisation as it only applies to the field distribution and not to the constant polarisation pattern possessed by the field. For vector fields, on the other hand, this assumption may not be valid anymore, as different polarisation states are focused in different ways, thus resulting in a mixing of the various polarisation states [36]. In this case, it is then necessary to investigate under which condition the polarisation coefficients \( f_0 \) and \( f_p \) preserve the self-accelerating character of vector RSABs. A natural way to prove this is to impose that vector RSABs fulfill the same requirements described in section 2.

To do so, we first define a suitable co-rotating reference frame in which the electric and magnetic fields of a vector RSAB appear propagation invariant. If such a reference frame exists, this automatically implies that the self-accelerating character has been preserved by the vectorisation procedure. This means that the electric (magnetic) field described by the first (second) of equation (8) must be propagation invariant in a co-rotating reference frame \( \rho' = \hat{S}\rho \) defined by the following coordinate transformation

\[ \begin{cases} 
\rho' = \rho, \\
\Phi = \theta + \Lambda \zeta, \\
\zeta' = \zeta,
\end{cases} \]

(12)

In principle, one should check that both the electric and the magnetic field are independently propagation invariant in this reference frame. However, Maxwell’s equation imposes that if one field fulfils the requirement, the other must fulfill it too. For this reason, we limit our analysis to the electric field only.

The same condition that we will derive for the polarisation coefficients \( f_0 \) and \( f_p \) will also apply to the magnetic field, and can also be derived using the same approach with the magnetic, rather than electric, field.

We then start by separating the electric field into its transverse and longitudinal parts, namely \( E(\rho') = E_t(\rho') + E_\rho(\rho') \), and require that they are both propagation invariant, i.e. \( \partial E(\rho') \partial \zeta = 0 \). Instead of dealing directly with this condition, however, we can require that the transverse and longitudinal intensities, rather than the amplitudes, are propagation invariant. By doing this, we are formally requiring that only intensity rotating RSABs remain propagation invariant upon focusing. However, if the intensity of a field is independent from \( \zeta \), its amplitude will be \( \zeta \)-independent as well, and the \( \zeta \) dependence can be at most contained into a phase factor. Once the condition of the intensity has been met, one could then look at the phase of the corresponding field and check whether it remains synchronised with its corresponding intensity profile.

The transverse \( |E_t(\rho)|^2 = |E_t|^2 + |E_\rho|^2 \) and longitudinal \( |E_\rho(\rho)|^2 = |E_\rho|^2 \) intensities can be calculated using the expressions given in appendix A, thus obtaining

\[ |E_t(\rho)|^2 = \sum_{n=m} E_{t}^{(n)}(\rho)^2 + \sum_{n=m} E_{t}^{(1)}(\rho)^2 \cos(m(m-n)\Phi), \]

(13a)

\[ |E_\rho(\rho)|^2 = G_1(\rho, \theta) + \sum_{n=m} E_{\rho}^{(1)}(\rho)^2 \cos(m(m-n)\Phi) \times [a_n^2 \cos^2 \theta \sin(m(m-n)\Phi - \Delta) - (a_n^2 - a_n^2) \sin^2 \theta \cos \sin(m(m-n)\Phi) - a_n^2 \sin^2 \theta \sin(m(m-n)\Phi + \Delta)] \]

(13b)

where we have rewritten the polarisation coefficients as \( f_0 = a_p, f_p = a_e \cos(\Delta) \) (with \( a_p, a_e, \Delta \in \mathbb{R} \)), with the \( \Delta \) in the relative phase between the two polarisation components, and

\[ G_1(\rho, \theta) = \sum_{n=m} a_n^2 \cos^2 \theta |E_{\rho}^{(n)}(\rho)|^2 + \sin^2 \theta |E_{\rho}^{(1)}(\rho)|^2 + a_n^2 \cos^2 \theta |E_{\rho}^{(n)}(\rho)|^2 + a_n^2 \cos^2 \theta |E_{\rho}^{(1)}(\rho)|^2 \]

\[ + 2a_n a_c (|E_{\rho}^{(n)}(\rho)|^2 - |E_{\rho}^{(1)}(\rho)|^2) \sin \theta \cos \Delta.] \]

(14)

Equation (13) already contains important information. No matter the polarisation, the transverse intensity always remains propagation invariant, as no explicit \( \zeta \)-dependence appears in the expression of \( |E_t(\rho)|^2 \).

The longitudinal part of the intensity on the other hand contains terms that depend on \( \sin \theta \) and \( \cos \theta \). Once transformed in the co-rotating frame, these terms become \( \zeta \)-dependent, as \( \theta' = \Phi - \Lambda \zeta \). To avoid this problem, the polarisation coefficients must be chosen in such a way to guarantee the propagation invariance of the longitudinal intensity as well. The condition on \( a_p, a_e, \Delta \) can be then
be found by requiring that
\[ \frac{\partial |\mathbf{E}(\rho)|^2}{\partial \zeta} = (a^2_\rho - a^2_\theta) \{ [F_2(\rho) - F_3(\rho)] \sin 2\theta 
+ F_4(\rho) \cos 2\theta \} + a_\rho a_\theta \cos \Delta \{ F_2(\rho) \sin 2\theta 
- [F_2(\rho) - F_3(\rho)] \cos 2\theta \} = 0, \]
where the functions \( F_k(\rho) \) (with \( k = \{1, 2, 3, 4\} \)) can be determined from equations (13b) and (14). It is not difficult to see that the above equation is satisfied if and only if \( a_\rho = a_\theta \), and \( \Delta = \pm \pi/2 \). Moreover, since \( |f_\rho|^2 + |f_\theta|^2 = 1 \), this condition implies that \( f_\rho = 1/\sqrt{2} \) and \( f_\theta = \pm i/\sqrt{2} \), which correspond to left-handed (+) and right-handed (−) circular polarisation, respectively.

This is the main result of our work. Vector RSABs only maintain their self-accelerating character if the polarisation of the Hertz vector is chosen to be circular. In other words, when focusing polarised RSABs, only circular polarisation is allowed, to preserve the self-accelerating character of the focused RSABs.

A simple explanation of this result can be given by looking at the symmetry of the scalar and vector beams, respectively. In the scalar case, in fact, RSABs naturally possess cylindrical symmetry due to their transverse profile. By virtue of this symmetry, the co-rotating coordinate can be chosen as a \( \zeta \)-dependent azimuthal coordinate, namely \( \Phi = \theta + \Lambda \zeta \). Upon focusing, the overall cylindrical symmetry must be preserved, in order for the vector RSAB to maintain its self-accelerating character. This ultimately constrains the polarisation to be chosen as circular.

### 3.3. Vector fields from circularly polarised RSABs

We now apply the polarisation constraints derived above and investigate the form of the electric and magnetic fields generated by focusing circularly polarised RSABs. By substituting \( f_\rho = 1/\sqrt{2} \) and \( f_\theta = \pm i/\sqrt{2} \) into equations (10) and (11), and using equations (8), the electric and magnetic fields of a circularly polarised focused RSAB can be written as

\[
\mathbf{E}(\rho, t) = \sum_{m \in M} \mathbf{e}_m(\rho) e^{i[(m+\sigma)\theta + (\beta + m\Lambda)\zeta - \omega t]},
\]

\[
\mathbf{B}(\rho, t) = \sum_{m \in M} \mathbf{b}_m(\rho) e^{i[(m+\sigma)\theta + (\beta + m\Lambda)\zeta - \omega t]},
\]

where \( \sigma = \pm 1 \) is the helicity index, which distinguishes between left-handed (+) and right-handed (−) circular polarisation [37]. \( \mathbf{e}_m(\rho) \) and \( \mathbf{b}_m(\rho) \) are radially dependent vector fields, whose explicit expression is given by

\[
\mathbf{e}_m(\rho) = \sigma D_m \omega \{ i(\beta + m\Lambda)J_m(\alpha_m \rho) \mathbf{h}_\sigma \n- \left[ \alpha_m J_{m-1}(\alpha_m \rho) - \frac{m}{\rho} J_m(\alpha_m \rho) \right] \mathbf{\hat{\zeta}} \},
\]

\[
\mathbf{b}_m(\rho) = \frac{D_m}{\sqrt{2}} \left\{ (1 + m\sigma) \left[ \frac{\alpha_m}{\rho} J'_m(\alpha_m \rho) + 2m\sigma J_m(\alpha_m \rho) \right] \mathbf{\hat{\rho}} 
+ i \left[ \frac{m}{\rho} J_m(\alpha_m \rho) - J'_m(\alpha_m \rho) \right] \mathbf{\hat{\theta}} 
+ \sigma [i(\beta + m\Lambda)^2 J_m(\alpha_m \rho) - \alpha_m^2 J''_m(\alpha_m \rho)] \mathbf{\hat{\zeta}} \right\},
\]

where \( \mathbf{h}_\sigma = (\mathbf{\hat{\rho}} + i\sigma \mathbf{\hat{\theta}}) / \sqrt{2} = (\mathbf{\hat{x}} + i\sigma \mathbf{\hat{y}}) / \sqrt{2} \) is the helicity basis [37]. For experimentally realisable RSABs, \( \Lambda \ll 1 \). Within this approximation, one should distinguish between field rotating and intensity rotating vector RSABs. For the former, \( \beta = 0 \), and the radial and azimuthal components of the electric field, as well as the longitudinal component of the magnetic field, are \( O(\Lambda) \) and can therefore be neglected, leaving a purely longitudinal electric field and a purely transverse magnetic field, namely

\[
\mathbf{E}(\rho, t) \simeq \frac{-\sigma \omega}{\sqrt{2}} \sum_{m \in M} D_m \{ (1 + m\sigma) \left[ \frac{\alpha_m}{\rho} J'_m(\alpha_m \rho) + \frac{m}{\rho} J_m(\alpha_m \rho) \right] \mathbf{\hat{\rho}} 
- \frac{m}{\rho} J_m(\alpha_m \rho) \mathbf{\hat{\zeta}} \},
\]

\[
\mathbf{B}(\rho, t) \simeq \frac{1}{\sqrt{2}} \sum_{m \in M} D_m \{ (1 + m\sigma) \left[ \frac{\alpha_m}{\rho} J'_m(\alpha_m \rho) + \frac{m}{\rho} J_m(\alpha_m \rho) \right] \mathbf{\hat{\rho}} 
+ i \left[ \frac{m}{\rho} J_m(\alpha_m \rho) - \frac{m}{\rho} J'_m(\alpha_m \rho) \right] \mathbf{\hat{\theta}} 
- \sigma \frac{m^2}{2} J''_m(\alpha_m \rho) \mathbf{\hat{\zeta}} \} e^{i[(m+\sigma)\theta + (\beta + m\Lambda)\zeta - \omega t]}.
\]
for TM fields, in fact, the result would be the same, with the magnetic field retaining the original polarisation and the electric field being mixed up. In the most general case, where both TE and TM waves are present, each field has these two components of polarisation, thus resulting in a more complex polarisation pattern.

The intensities and phases of the electric field components for field rotating and intensity rotating RSABs are reported in figures 3 and 4, respectively. As can be seen from figure 3(d), upon focusing, field rotating vector RSABs lose their property, so that the intensity and phase profiles are synchronised in rotation during propagation. This is ultimately due to the fact that while the field intensity contains terms of the form \(\cos[(m - n)(\theta + \Lambda \zeta)]\), the phase contains terms that oscillate such as \(\cos[m(\theta + \Lambda \zeta) + \sigma \theta]\). The presence of the extra term \(\sigma \theta\) (which disappears in the intensity) is then responsible for the different evolution of the amplitude and phase of the field, as it corresponds to a \(\zeta\)-dependent term, once transformed in the co-rotating frame.

4. Linear and angular momentum densities of vector RSABs

In this section, we calculate the linear and angular momentum for intensity rotating vector RSABs. We limit ourselves to the paraxial case, as within this approximation, we can separate the angular momentum in its spin and orbital parts. This gives us the possibility for distinguishing between intrinsic and extrinsic orbital angular momentum of vector RSABs, and to then isolate the extrinsic contribution given by the fact that the intensity rotates with angular velocity \(\Lambda\).

Following Jackson, the linear and angular momentum per unit length of the electromagnetic field are defined as follows [35]:

\[
\mathbf{P} = \int d^2 \rho \, \mathbf{p}(\rho), \tag{20a}
\]

\[
\mathbf{J} = \int d^2 \rho \, \mathbf{j}(\rho), \tag{20b}
\]

where

\[
\mathbf{p}(\rho) = \frac{\varepsilon_0}{2} \text{Re}\{\mathbf{E}(\rho) \times \mathbf{B}^*(\rho)\}, \tag{21a}
\]

\[
\mathbf{j}(\rho) = \frac{\varepsilon_0}{2} \text{Re}\{\rho \times \mathbf{p}(\rho)\}, \tag{21b}
\]

are the corresponding densities, \(d^2 \rho = \rho d\rho d\theta\), and the integrals are extended over the whole space.

As can be seen from equations (3) and (16), RSABs are defined in terms of superpositions of Bessel beams. Therefore, as Bessel beams carry infinite energy, the above integrals diverge, and linear and angular momentum per unit length (as well as energy) are not well defined quantities for RSABs. This problem, however, can be overcome in different ways, by introducing different forms of regularisation. For example, one could limit the radial integration, up to a maximum radius. Alternatively, one could insert a regularisation function, such as a Gaussian function, in the radial integrals to make them finite. Physically speaking, both regularisations can be implemented. The former, in fact, corresponds to using a pupil of a fixed diameter to filter the field. The latter, on the other hand, corresponds to describing RSABs in terms of Bessel-Gauss beams, which, de facto, are the closest approximation to Bessel beams that can be realised experimentally.

In the remainder of this section, we calculate the explicit expressions for both the momentum densities and their integrated counterpart. For the sake of simplicity, however, we will not compute the radial integrals. These, in fact, only contribute to a multiplicative constant, and do not carry any valuable information for the purpose of investigating the properties of linear and angular momentum of RSABs.

4.1. Linear momentum

If we substitute the expressions of the electric and magnetic fields of a paraxial RSAB as given by equations (16) into equation (21a), the linear momentum density can be written

\[
\mathbf{p}(\rho) = \frac{\varepsilon_0}{2} \text{Re}\{\mathbf{E}(\rho) \times \mathbf{B}^*(\rho)\}, \tag{21a}
\]

\[
\mathbf{j}(\rho) = \frac{\varepsilon_0}{2} \text{Re}\{\rho \times \mathbf{p}(\rho)\}, \tag{21b}
\]
as follows:

\[
p(\rho) = \sum_{m,n\in\mathbb{M}} \frac{|D_m D_n|}{4} |P_{\rho}^{(m,n)}(\rho)|^2 \\
\times \sin[(m-n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \hat{\rho} \\
+ P_{\theta}^{(m,n)}(\rho) \cos[(m-n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \hat{\theta} \\
+ P_{\zeta}^{(m,n)}(\rho) \cos[(m-n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \hat{\zeta},
\]

(22)

where \(\phi_{m,n} = \arg[D_{m,n}]\), and

\[
P_{\rho}^{(m,n)}(\rho) = \sigma J_m(\alpha_m \rho) \left\{ \frac{mn}{\rho^2} - \frac{m\sigma(\beta + n\Lambda)^2}{\rho} \\
- \frac{n(\beta + m\Lambda)(\beta + n\Lambda)}{2} J_n(\alpha_n \rho) - \frac{mn}{\rho^2} J_n(\alpha_n \rho) \right\},
\]

(23a)

\[
P_{\theta}^{(m,n)}(\rho) = \sigma \left[ \frac{2mn(n + \sigma)}{\rho} \\
+ \frac{n\sigma(\beta + m\Lambda)(\beta + n\Lambda)}{2} J_n(\alpha_n \rho) J_n(\alpha_n \rho) \right],
\]

(23b)

\[
P_{\zeta}^{(m,n)}(\rho) = (\beta + m\Lambda) J_m(\alpha_m \rho) \left\{ \frac{[(\beta + n\Lambda)^2}{2} \\
+ 2n(n + \sigma) - \frac{n\sigma}{\rho} J_n(\alpha_n \rho) + \frac{n}{\rho} J_n(\alpha_n \rho) \right\},
\]

(23c)

where the terms of order \(O(\alpha_m)\) have been neglected, since in the paraxial regime \(\alpha_m \ll 1\). The components of the linear momentum density are shown in figure 5. Notice, that the transverse part of the linear momentum presents an unusual characteristic. While it rotates clockwise along the propagation direction, as the intensity distribution of the corresponding RSAB does, the local orientation of the transverse momentum is purely azimuthal (despite \(p(\rho)\) having a nonzero radial component), and is always directed in the opposite direction with respect to the rotation direction of the RSAB, as can be seen from the white arrows in figure 5(a). This has an interesting consequence for applications such as particle manipulation and material processing, where the local, rather than the global, behaviour of the momentum plays an important role. While the RSAB (and, with it, the transverse momentum density) rotates clockwise during propagation, a particle placed in the vicinity of an RSAB will experience a local momentum, that will tend to push it in the opposite direction. This effect, however, is purely local, and it disappears when considering the whole momentum. To understand this, let us integrate equation (22) over the transverse space. The linear momentum can then be written as follows

\[
P = \sum_{m,n\in\mathbb{M}} |D_m|^2 [P_{\theta}^{(m,n)} \hat{\theta} + P_{\zeta}^{(m,n)} \hat{\zeta}],
\]

(24)
rotating vector RSAB are given as follows:

\[
\mathbf{s}(\rho) = \frac{c_0}{2 \omega^2} \sum_{m,n \in M} \left\{ -S_r^{(m,n)}(m - n) \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \right\} \hat{\boldsymbol{\rho}} + \frac{c_0}{2 \omega^2} \sum_{k=1}^{3} \text{Re} \left\{ \epsilon_k^{\rho} \cdot (\mathbf{E}^* \times \mathbf{E}) \right\},
\]

where \( k = \{1, 2, 3\} \) indicates the \{x, y, z\} field components, respectively and \(-i \mathbf{\rho} \times \nabla\) is the angular momentum operator [35] in the normalised cylindrical reference frame \((\hat{\rho}, \hat{\theta}, \hat{\zeta})\).

Using the expression for the vector electric field given in appendix A, the SAM and OAM of a paraxial, intensity rotating vector RSAB are given as follows:

\[
\mathbf{l}(\rho) = \frac{c_0}{2 \omega^2} \sum_{m,n \in M} \left\{ L_r^{(m,n)}(m - n) \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \right\} \hat{\boldsymbol{\rho}} - L_z^{(m,n)}(m - n) \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \hat{\boldsymbol{\theta}}.
\]

4.2. Spin and orbital angular momentum

To calculate the spin and orbital angular momentum for intensity rotating, paraxial RSABs, we make use of the usual decomposition of the total angular momentum in its spin (SAM) and orbital (OAM) components, namely \( \mathbf{J} = \mathbf{S} + \mathbf{L} \). Following [38], the angular momentum density then assumes the following gauge invariant form

\[
\mathbf{j}(\rho) = \mathbf{s}(\rho) + \mathbf{l}(\rho) = \frac{c_0}{2 \omega^2} \sum_{m,n \in M} \left\{ \mathbf{E}^* \cdot (\rho \nabla) \mathbf{E} \right\} \hat{\rho} + \frac{c_0}{2 \omega^2} \sum_{k=1}^{3} \text{Im} \left\{ \epsilon_k^{\rho} \cdot (\mathbf{E}^* \times \mathbf{E}) \right\},
\]

where \( \lambda \in \{ \theta, \zeta \} \). Notice that the radial integrals (once regularised) amount to a positive constant. Moreover, there is no radial component of the momentum, since the radial part of \( p(\rho) \) depends on \( \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \), which zero once integrated with respect to the azimuthal coordinate \( \theta \).

\[
\mathcal{P}_{\lambda}^{(m,n)} = \frac{\pi \xi_0}{2} \int_0^{\infty} d\rho \rho P_{\lambda}^{(m,n)}(\rho),
\]

where \( \lambda \in \{ \theta, \zeta \} \). Notice that the radial integrals (once regularised) amount to a positive constant. Moreover, there is no radial component of the momentum, since the radial part of \( p(\rho) \) depends on \( \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \), which zero once integrated with respect to the azimuthal coordinate \( \theta \).

\[
\mathcal{P}_{\lambda}^{(m,n)} = \frac{\pi \xi_0}{2} \int_0^{\infty} d\rho \rho \left\{ -S_r^{(m,n)}(m - n) \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \right\} \hat{\boldsymbol{\rho}} + \frac{\pi \xi_0}{2} \sum_{k=1}^{3} \text{Im} \left\{ \epsilon_k^{\rho} \cdot (\mathbf{E}^* \times \mathbf{E}) \right\}.
\]

Figure 5. Transverse (a) and longitudinal (b) components of the linear momentum density, as given by equation (22), in the plane \( \zeta = 0 \), for \( \sigma = \pm 1 \). The white arrows in (a) represent the flow of the transverse component of the linear momentum density. As can be seen, the transverse momentum density always points in the opposite direction with respect to the field rotation (red arrow). These plots are made assuming \( 0 \leq \rho \leq 1200 \). The plot parameters are the same as the one chosen for figure 2. The red arrow in both panels shows the direction of rotation of the RSAB intensity. Moreover, notice that the orientation of the momentum density is spin-independent, as choosing \( \sigma = 1 \) or \( \sigma = -1 \) only lightly tilts the local orientation of the momentum, but globally it gives the same result as depicted in (a).

\[
\mathbf{s}(\rho) = \frac{c_0}{2 \omega^2} \sum_{m,n \in M} \left\{ -S_r^{(m,n)}(m - n) \sin[(m - n)(\theta + \Lambda \zeta)] + \phi_m - \phi_n \right\} \hat{\boldsymbol{\rho}} + \frac{c_0}{2 \omega^2} \sum_{k=1}^{3} \text{Im} \left\{ \epsilon_k^{\rho} \cdot (\mathbf{E}^* \times \mathbf{E}) \right\}.
\]
are the components of the OAM density and $D_{m,n}(\rho) = |D_mD_n|\int\int J_m(\alpha_m\rho)J_n(\alpha_n\rho)/2$. In the above expressions, the terms of order $O(\alpha_m)$ have been neglected, since, for paraxial fields, $\alpha_m \ll 1$. The longitudinal and transverse SAM densities are plotted in figure 6. As can be seen, the SAM density can become negative. This means that locally, the helicity of the vector RSAB can change sign. However, a close comparison between the SAM density distribution in figure 6 and the transverse and longitudinal intensity distributions depicted in figure 4 reveals that regions of negative SAM density occur where the RSAB intensity is very low, or even zero. These correspond to the central region of the intensity distribution for the radial component (see figures 4(a) and 6(a)), and the outer regions of the beam for the longitudinal one (see figures 4(c), and 6(b)). In the regions where the intensity is different from zero, on the other hand, both the radial and the longitudinal SAM densities depicted in figure 6 have positive values. This ultimately means that in these regions, the SAM density retains the same sign as $\sigma$ both globally and locally.

The spin and orbital angular momenta of paraxial vector RSABs are then obtained by integrating equations (27) over the transverse space. By doing so, we obtain

$$ S = \sigma \sum_{m \in M} |D_m|^2 S^{(m)}_{\zeta}, $$

$$ L = \sum_{m \in M} |D_m|^2 [L^{(m)}_{\theta} + L^{(m)}_{\zeta}], $$

where

$$ S^{(m)}_{\zeta} = \frac{\xi_0}{4} \int_0^\infty d\rho \rho S^{(m)}_{\zeta}(\rho), $$

and

$$ L^{(m)}_{\zeta} = \frac{\xi_0}{4} \int_0^\infty d\rho \rho L^{(m)}_{\zeta}(\rho), $$

being $\lambda \in \{\theta, \zeta\}$. The total SAM is purely longitudinal and is given as the sum of the longitudinal components of the individual Bessel components. This is not surprising since we are dealing with paraxial fields, for which the SAM is only directed along the propagation direction [38].

The OAM, on the other hand, can be seen as the sum of two contributions: an intrinsic component relative to the intrinsic OAM carried by Bessel beams, and an extrinsic one, connected to the fact that the beam rotates around the $\zeta$-axis during propagation. Their explicit expressions read then as follows:

$$ L^{(\text{int})}_{\zeta} = \sum_{m \in M} |D_m|^2 L^{(m)}_{\text{int},\zeta}(\hat{\theta} + \hat{\zeta}), $$

$$ L^{(\text{ext})}_{\zeta} = \sum_{m \in M} |D_m|^2 [\sigma L^{(m)}_{\text{ext},\zeta} \hat{\theta} + L^{(m)}_{\text{ext},\zeta} \hat{\zeta}], $$

where $D_m = D_m \sqrt{\beta + m\Lambda}$, and

$$ L^{(m)}_{\text{int},\zeta} = \frac{\xi_0}{4} m(m + \sigma) \int_0^\infty d\rho \rho^2 \mathcal{I}_m(\alpha_m\rho), $$

$$ L^{(m)}_{\text{ext},\zeta} = \frac{\xi_0}{4} (\beta + m\Lambda)^2 \int_0^\infty d\rho \rho^2 \mathcal{I}_m(\alpha_m\rho), $$

The intrinsic part of the OAM has the standard spin-orbit interaction form, through the mixed term $(m + \sigma)$ [38]. The extrinsic part, on the other hand, depends on $(\beta + m\Lambda)$, which
is, essentially, the angular velocity of the beam along the $\zeta$-axis. Moreover, the beam rotation also induces a longitudinal OAM, which is also proportional to the angular velocity ($\beta + m\Lambda$).

5. Conclusions

In this work, we have analysed the properties of vector RSABs, generated by focusing a scalar, polarised RSAB. Using the method of Hertz potentials as a model for the focusing process, we have demonstrated that only circularly polarised scalar RSABs, when focused, maintain their self-accelerating character. For this case, we have given explicit expressions of the TE vector electric and magnetic fields for both field and intensity rotating RSABs. In particular, we have shown that the vectorisation (focusing) process does not allow the amplitude and phase of field rotating RSABs to rotate synchronously during propagation anymore. Moreover, within the paraxial approximation, we have presented explicit expressions for the linear and angular momentum densities of intensity rotating RSABs. For the SAM, in particular, we have shown that locally the SAM density can be negative, thus meaning a local inversion of the helicity axis. Moreover, for the OAM, we have distinguished between the intrinsic and extrinsic contributions, and shown how the rotation of the RSAB around the propagation axis is connected with the extrinsic OAM.

Our work represents a useful guideline for investigating experimentally focused RSABs and their properties. Moreover, the properties highlighted in this work represent a useful toolbox for studying the interaction of RSABs with matter and dielectric particles. In particular, the fact that locally the linear momentum density flows in the opposite direction with respect to the overall beam rotation during propagation, could open new possibilities for particle manipulation.

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Appendix A. Explicit form of RSAB electric and magnetic fields

The vector electric and magnetic fields for single Bessel beams defined in equations (10) and (11) can be used to write the expressions for the RSAB vector electric and magnetic fields explicitly. Substituting these expressions into equations (8), we then get

\[
\mathbf{E}(\rho) = e^{i(\beta - \omega t)} \sum_{m \in M} e^{im\phi} \left\{ E_m^{(1)}(\rho) \{ f_p \cos \theta - f_p \sin \theta \} \hat{\rho} \\
- (f_p \cos \theta + f_p \sin \theta) \hat{\theta} \right\} + [E_m^{(2)}(\rho) (f_p \cos \theta + f_p \sin \theta) \hat{\zeta},
\]

where

\[
E_m^{(1)}(\rho) = D_m \omega (\beta + m\Lambda) J_m(\alpha_m \rho),
\]

\[
E_m^{(2)}(\rho) = D_m (m\omega / \rho) J_m(\alpha_m \rho),
\]

\[
E_m^{(3)}(\rho) = iD_m \omega \left[ \alpha_m J_{m-1}(\alpha_m \rho) - \frac{m}{\rho} J_m(\alpha_m \rho) \right].
\]

are the radially dependent expansion coefficients for the electric field

\[
\mathbf{B}(\rho) = e^{i(\beta - \omega t)} \sum_{m \in M} e^{im\phi} \left\{ [B_m^{(1)}(\rho)(f_p \cos \theta + f_p \sin \theta) + [B_m^{(2)}(\rho)(f_p \cos \theta + f_p \sin \theta)] \hat{\rho} \\
+ [B_m^{(3)}(\rho)(f_p \cos \theta + f_p \sin \theta) \hat{\theta} + [B_m^{(4)}(\rho)(f_p \cos \theta + f_p \sin \theta)] \hat{\zeta},
\]

where

\[
B_m^{(1)}(\rho) = D_m \left[ -\frac{\alpha_m}{\rho} J_m(\alpha_m \rho) + 2m^2 J_m(\alpha_m \rho) \right],
\]

\[
B_m^{(2)}(\rho) = D_m \left[ \frac{m\alpha_m}{\rho} J_m(\alpha_m \rho) + 2mJ_m(\alpha_m \rho) \right],
\]

\[
B_m^{(3)}(\rho) = \frac{mD_m}{\rho} [\alpha_m J_m(\alpha_m \rho) - J_m(\alpha_m \rho)],
\]

\[
B_m^{(4)}(\rho) = D_m [(\beta + m\Lambda)^2 J_m(\alpha_m \rho) - J_m(\alpha_m \rho)],
\]

\[
B_m^{(5)}(\rho) = \frac{-mD_m(\beta + m\Lambda) J_m(\alpha_m \rho)}{2},
\]

\[
B_m^{(6)}(\rho) = iD_m \alpha_m(\beta + m\Lambda) J_m(\alpha_m \rho),
\]

are the radially dependent expansion coefficients for the magnetic field.

Appendix B. Explicit expression for the vector potential for RSABs

The vector potential can be defined from the Hertz potential as $\mathbf{A}(\rho, t) = \nabla \times \mathbf{E}(\rho, t)$ [33]. Using equations (5) and (6), the vector potential for an arbitrary polarised vector RSAB is given, in cylindrical coordinates, as follows:

\[
\mathbf{A}(\rho, t) = \sum_{m \in M} D_m e^{i(m\rho + \Lambda \zeta + \beta - \omega t)} [ -i(\beta + m\Lambda) J_m(\alpha_m \rho) \\
\times \{ (f_p \cos \theta - f_p \sin \theta) \hat{\rho} + i(f_p \cos \theta + f_p \sin \theta) \hat{\theta} + \left[ \alpha_m (f_p \cos \theta + f_p \sin \theta) J_m(\alpha_m \rho) - \frac{im}{\rho} (f_p \cos \theta + f_p \sin \theta) \hat{\zeta} \right].
\]
For the case of circular polarisation, the above expression simplifies to

\[ A(\rho, t) = \sum_{m \in M} \frac{D_m}{\sqrt{2}} e^{i(m\theta + \lambda_m + \sigma\phi + 3\lambda - \omega t)} \times [A_p^{(m)}(\rho) \hat{P} + A_\theta^{(m)}(\rho) \hat{\Theta} + A_\zeta^{(m)}(\zeta) \hat{\zeta}], \tag{51} \]

where

\[ A_p^{(m)}(\rho) = \sigma(\beta + \mu\lambda)J_m(\alpha_m\rho), \tag{52a} \]

\[ A_\theta^{(m)}(\rho) = i(\beta + \mu\lambda)J_m(\alpha_m\rho), \tag{52b} \]

\[ A_\zeta^{(m)}(\rho) = i\sigma\alpha_m J_m'(\alpha_m\rho) - \frac{m}{\rho} J_m(\alpha_m\rho). \tag{52c} \]

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