On Indecomposable Normal Matrices
in Spaces with Indefinite Scalar Product

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Abstract

Finite dimensional linear spaces (both complex and real) with indefinite scalar product $[\cdot, \cdot]$ are considered. Upper and lower bounds are given for the size of an indecomposable matrix that is normal with respect to this scalar product in terms of specific functions of $v = \min\{v_-, v_+\}$, where $v_-$ ($v_+$) is the number of negative (positive) squares of the form $[x, x]$. All the bounds except for one are proved to be strict.

1 Definitions and notation

Consider a complex (real) linear space $C^n (R^n)$ with an indefinite scalar product $[\cdot, \cdot]$. By definition, the latter is a nondegenerate sesquilinear (bilinear) Hermitian form. If the usual scalar product $(\cdot, \cdot)$ is fixed, then there exists a nonsingular Hermitian operator $H$ such that $[x, y] = (Hx, y) \forall x, y \in C^n (R^n)$. If $A$ is a linear operator, then the $H$-adjoint of $A$ (denoted by $A^\dagger$) is defined by the identity $[A^\dagger x, y] \equiv [x, Ay]$. An operator $N$ is called $H$-normal if $NN^\dagger = N^\dagger N$. An operator $U$ is called $H$-unitary if $UU^\dagger = I$, where $I$ is the identity transformation.

Let $V$ be a nontrivial subspace of $C^n (R^n)$. The subspace $V$ is called neutral if $[x, y] = 0 \forall x, y \in V$. If the conditions $x \in V$ and $[x, y] = 0 \forall y \in V$ imply $x = 0$, then $V$ is called nondegenerate. The subspace $V^{\perp}$ is defined as the set of all vectors $x$ from $C^n (R^n)$ such that $[x, y] = 0 \forall y \in V$. If $V$ is nondegenerate, then $V^{\perp}$ is also nondegenerate and $V + V^{\perp} = C^n (R^n)$, where $\perp$ stands for the direct sum.

A linear operator $A$ is called decomposable if there exists a nondegenerate proper subspace $V$ of $C^n (R^n)$ such that both $V$ and $V^{\perp}$ are invariant under $A$ or (it is the same) if $V$ is invariant both under $A$ and $A^\dagger$. Then $A$ is the $H$-orthogonal sum of $A_1 = A|_V$ and $A_2 = A|_{V^{\perp}}$. If an operator $A$ is not decomposable, it is called indecomposable.

By the rank of a space we mean $v = \min\{v_-, v_+\}$, where $v_-$ ($v_+$) is the number of negative (positive) squares of the form $[x, x]$, i.e., the number of negative (positive) eigenvalues of the operator $H$.

The problem is to find functions $f_1(\cdot), f_2(\cdot)$ such that $f_1(v) \leq n \leq f_2(v)$ for any indecomposable $H$-normal operator acting in a space of dimension $n$ and of rank $v$ and to find out whether these bounds are strict.

This problem arises in the classification of indecomposable $H$-normal matrices [2, 3]. The bounds for the size of an indecomposable $H$-normal matrix in a complex space are known [2]. In Section 2, we check their strictness. The bounds for matrices in real spaces are considered in Section 3.

As in [2] and [3], we denote by $I_r$ the $r \times r$ identity matrix, by $D_r$ the $r \times r$ matrix with 1’s on the trailing diagonal and zeros elsewhere, and by $A \oplus B \oplus \ldots \oplus Z$ the block diagonal matrix with blocks $A, B, \ldots, Z$. By $A^T$ we mean $A$ transposed.
2 Indecomposable normal matrices in complex spaces

The objective of this section is to prove the following theorem.

Theorem 1 Let an indecomposable $H$-normal operator $N$ act in a space $C^n$ of rank $k > 0$. Then either (A) or (B) holds:

(A) $N$ has only one eigenvalue and $2k \leq n \leq 4k$;

(B) $N$ has only two eigenvalues and $n = 2k$;

these bounds being strict.

Proof: Theorem 1 of [2] states that for an indecomposable $H$-normal operator $N$ there exist two alternatives: (A) and (B) so that it suffices to prove that these estimates are unimprovable.

Step 1. Show the strictness of the lower bound in (A), i.e., for any $k > 0$ point out a pair of $2k \times 2k$ matrices $\{N, H\}$, where $H$ has $k$ negative and $k$ positive eigenvalues, $N$ is $H$-normal and indecomposable and has only one eigenvalue $\lambda$. Let

$$
N = \begin{pmatrix} \lambda I_k & N_1 \\ 0 & \lambda I_k \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}.
$$

(1)

It can easily be checked that $N$ is $H$-normal. In addition, let the submatrix $N_1$ be nonsingular.

Proposition 1 from [2] and its Corollary may be restated as follows:

Let an $H$-normal operator $N$ acting in $C^n$ have $\lambda$ as its only eigenvalue. Let the subspace

$$
S_0 = \{x \in C^n : (N - \lambda I)x = (N^{[r]} - \lambda I)x = 0\}
$$

(2)

be neutral. Then there exists a decomposition of $C^n$ into a direct sum of subspaces $S_0, S, S_1$ such that

$$
N = \begin{pmatrix} N' = \lambda I & * \\ 0 & N_1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix},
$$

(3)

where $N' : S_0 \to S_0$, $N_1 : S \to S$, $N'' : S_1 \to S_1$, the internal operator $N_1$ is $H_1$-normal, and the pair $\{N_1, H_1\}$ is determined up to the unitary similarity. To go over from one decomposition $C^n = S_0 + S + S_1$ to another by means of a transformation $T$ it is necessary that $T$ be block triangular with respect to both decompositions.

(In Proposition 1 from [2] there are two conditions:

(a) $N$ is indecomposable

(b) $n > 1$

instead of the condition

(c) $S_0$ is neutral,

but the results are only derived from (c), which follows from (a) and (b)).

We see that (1) is a specific case of (3) corresponding to the decomposition $C^n = S_0 + S + S_1$ with $S = 0$. If there exists a nondegenerate subspace $V$ such that both $V_1 = V$ and $V_2 = V^{[r]}$ are invariant under $N$, then, according to our restatement of Proposition 1 from [2], for $i = 1, 2$ we have $V^{(i)} = S^{(i)}_0 + S^{(i)} + S^{(i)}_1$, where $S^{(i)}_0 = V^{(i)} \cap S_0$ and the pairs $\{N^{(i)}, H^{(i)}\}$ have the form (3). But (1) implies $S_0 = S^{(i)}_0$ so that for any $i = 1, 2$ the subspace $S^{(i)}$ is trivial. Thus, $N$ from (1) is decomposable if and only if there exists a transformation $T$ preserving $H$ and reducing $N$ to the form

$$
\tilde{N} = \begin{pmatrix} \lambda I_k & \tilde{N}_1 \\ 0 & \lambda I_k \end{pmatrix},
$$

(4)
where \( \tilde{N}_1 \) is block diagonal (that is, \( T \) is an \( H \)-unitary transformation of \( N \) to the form \( \tilde{N} \)). The matrix \( T \) is necessarily block triangular with respect to the decomposition \( C^n = S_0 + S_1 \), i.e.,

\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.
\]

For \( T \) to be \( H \)-unitary it is necessary to have \( T_3 = T_1^{-1} \). Then from the condition \( NT = T\tilde{N} \) it follows that \( N_1 = T_1\tilde{N}_1T_1^* \). Therefore, \( N \) will be indecomposable if \( N_1 \) is not congruent to any block diagonal matrix \( \tilde{N}_1 \).

If \( N_1 \) and a block diagonal matrix \( \tilde{N}_1 \) are congruent, then \( N_1N_1^{*-1} \) is similar to \( \tilde{N}_1\tilde{N}_1^{*-1} \). Since the latter is also block diagonal, for \( N_1 \) to be not congruent to \( \tilde{N}_1 \) it is sufficient that \( N_1N_1^{*-1} \) cannot be reduced to block diagonal form.

Let us prove that for any \( n = 1, 2, \ldots \) there exists a nonsingular real \( (2n - 1) \times (2n - 1) \) matrix \( N_1 \) such that

\[
N_1N_1^{*-1} = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}
\tag{4}
\]

and a nonsingular real \( 2n \times 2n \) matrix \( \tilde{N}_1 \) such that

\[
N_1N_1^{*-1} = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{pmatrix}
\tag{5}
\]

(now it is not necessary that \( N_1 \) be real, but this will be used in the next section). The matrices (4) and (5) are obviously not similar to any block diagonal ones because for each of them the subspace generated by their eigenvectors is one-dimensional.

Prove the statement by induction for odd numbers. If \( n = 1 \), let \( N_1^{(1)} = (1) \). Suppose we have found a nonsingular real \( (2n - 1) \times (2n - 1) \) matrix \( N_1^{(n)} \) with the property required. Let

\[
N_1^{(n+1)} = \begin{pmatrix} 0 & A_{n+1} & B_{n+1} \\ C_{n+1} & N_1^{(n)} & 0 \\ D_{n+1} & 0 & 0 \end{pmatrix},
\]

where the submatrices \( A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} \) will be shortly specified. If we denote by \( \Lambda_n \) the \( (2n - 1) \times (2n - 1) \) matrix (4), by \( \Lambda_n' \) the \( 1 \times (2n - 1) \) matrix

\[
\Lambda_n' = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix},
\]

and by \( \Lambda_n'' \) the \((2n - 1) \times 1 \) matrix

\[
\Lambda_n'' = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \end{pmatrix}^T,
\]

then the condition \( N_1^{(n+1)} = A_{n+1}N_1^{(n+1)*} \) may be rewritten as follows:

\[
0 = \Lambda_n'A_{n+1}^*,
\tag{6}
\]

\[
A_{n+1} = C_{n+1} + \Lambda_n'A_{n+1}^{(n)*},
\tag{7}
\]

\[
B_{n+1} = D_{n+1}^*,
\tag{8}
\]

\[
C_{n+1} = \Lambda_n'A_{n+1} + \Lambda_n''B_{n+1}^*
\tag{9}
\]
(by the inductive hypothesis, \( N_1^{(n)} = \Lambda_n N_1^{(n+1)} \)). Taking
\[
A_{n+1} = \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_{2n-1} \\ a_2, & a_3, & \ldots, & a_{2n-1}, & b \end{pmatrix}, \quad B_{n+1} = (b),
\]
\( C_{n+1} = \Lambda_n A_{n+1}^* + \Lambda_n^* B_{n+1}^* \), \( D_{n+1} = B_{n+1}^* \), one can satisfy the conditions \( \text{[3, 8, 9]} \). Substituting the expresion for \( C_{n+1} \) in \( \text{[4]} \), we get the only condition to be satisfied:
\[
- N_1^{(n)} A_n^* = (\Lambda_n - I) A_{n+1}^* + \Lambda_n^* B_{n+1}^*.
\]
(10)
Since its right hand side is equal to
\[
( a_2 \quad a_3 \quad \cdots \quad a_{2n-1} \quad b )^T,
\]
it always can be satisfied by choosing the appropriate values of \( a_2, a_3, \ldots, a_{2n-1}, b \). By construction, the elements of each matrix \( N_1^{(n)} \) below the trailing diagonal are zeros and those on the trailing diagonal are equal to \( \pm 1 \) so that \( N_1^{(n)} \) is nonsingular. Thus, for odd numbers our statement is proved.

To prove it for even numbers one can take
\[
N_1^{(1)} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}
\]
and construct \( N_1^{(n+1)} \) from \( N_1^{(n)} \) in the same way as before (the details are left to the reader). Step 1 is completed.

**Step 2.** Show the strictness of the upper bound in (A). The example of the pair \( \{N, H\} \) is
\[
N = \begin{pmatrix} \lambda I_k & I_k & 0 & 0 \\ 0 & \lambda I_k & 0 & N_1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & I_k \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_k & 0 \\ I_k & 0 & 0 & 0 \end{pmatrix}, \tag{11}
\]
where
\[
N_1 = \begin{pmatrix} 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{k-1} \\ r_k & 0 & 0 & \cdots & 0 \end{pmatrix},
\]
\[
N_2 = \begin{pmatrix} \sqrt{1-r_1^2} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{1-r_2^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{1-r_k^2} \end{pmatrix},
\]
r_i \in (0, 1) \forall i = 1, \ldots, k, \text{ and } r_i \neq r_j \text{ if } i \neq j. \text{ The matrix } H \text{ has } k \text{ negative and } 3k \text{ positive eigenvalues.}

The matrix \( N \) is \( H \)-normal, since the condition \( N_1^* N_1 + N_2^* N_2 = I \) is satisfied. As before, we see that \( \text{[11]} \) is a specific case of \( \text{[3]} \). Suppose a nondegenerate subspace \( V \) is invariant both under \( N \) and under \( N^{[s]} \). Denote the basis vectors of \( S_0 \) by \( \{v_i\}_{i=1}^k \), the basis vectors of \( S_1 \) by \( \{w_i\}_{i=1}^k \) (here the basis corresponds to \( \text{[11]} \)). Let \( \tilde{V} \) be the range of the projection of \( V \) onto \( S_1 \) along \( S_0^{[\perp]} \). It is a subspace of dimension \( m > 0 \), since \( V \) necessarily contains at least one nontrivial vector from \( S_0 \) and, therefore, at least one vector with nontrivial projection onto \( S_1 \) (otherwise \( V \) would be degenerate). Let \( \{\sum_{j=1}^k a_{ij} w_j\}_{i=1}^m \) be a basis of \( \tilde{V} \). From the condition \( (N - \lambda I)(N^{[s]} - \Sigma J)V \subseteq V \) it follows that \( \{\sum_{j=1}^k a_{ij} v_j\}_{i=1}^m \subset V \). If \( V \) is nondegenerate, \( \tilde{V} \) and \( \tilde{S}_0 = S_0 \cap V \) are necessarily of the same dimension. Therefore, \( \{\sum_{j=1}^k a_{ij} v_j\}_{i=1}^m \) is a basis of \( \tilde{S}_0 \).
As \((N - \lambda I)^2 \subseteq V\), \((N^* - \overline{\lambda} I)^2 \subseteq V\), we obtain \(\{\sum_{j=1}^{k} \alpha_{ij} N_1 v_j\}_{i=1}^{m} \subseteq V\), \(\{\sum_{j=1}^{k} \alpha_{ij} N_1^* v_j\}_{i=1}^{m} \subseteq V\).

As \(V^{[\perp]} \cap S_0 \neq \{0\}\), we have \(m(= \text{dim}S_0) < k\). Thus, for \(N\) to be decomposable it is necessary that the subspace \(S_0\), which is of dimension more than zero and less than \(k\), be invariant under \(N_1\) and under \(N_1^*\). This means the existence of an orthogonal projection \(P(\neq 0, I)\) commuting with \(N_1\). But it can easily be checked by direct calculation that from the conditions \(N_1 P = P N_1\) and \(P = P^*\) it follows that \(P = \mu I\). Since \(P^2 = P\), we have \(\mu = 0\) or \(\mu = 1\) so that \(P = 0\) or \(P = I\). The contradiction obtained shows that \(N\) is indecomposable. Step 2 is completed.

**Step 3.** Now for any \(k > 0\) let us point out a pair of \(2k \times 2k\) matrices \(\{N, H\}\), where \(H\) has \(k\) negative and \(k\) positive eigenvalues, \(N\) is \(H\)-normal and indecomposable and has the two eigenvalues \(\lambda_1, \lambda_2\) \((\lambda_1 \neq \lambda_2)\). Let

\[
N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},
\]

where the \(k \times k\) matrices \(N_i\) \((i = 1, 2)\) are as follows:

\[
N_1 = \begin{pmatrix} \lambda_1 & 1 & 0 & \ldots & 0 \\ 0 & \lambda_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_1 \end{pmatrix}, \quad N_2 = \lambda_2 I_k.
\]

The matrix \(N\) is \(H\)-normal, for it satisfies the condition \(N_1 N_2^* = N_2^* N_1\). Suppose that \(N\) is similar to

\[
\tilde{N} = \begin{pmatrix} N|_V = \tilde{N}_1 & 0 \\ 0 & N|_{V^{[\perp]} = \tilde{N}_2} \end{pmatrix},
\]

where \(V\) is a nondegenerate subspace. Since the subspace generated by the eigenvectors of \(N\) corresponding to the eigenvalue \(\lambda_1\) is one-dimensional, one of the submatrices \(\tilde{N}_1, \tilde{N}_2\) (for example, \(\tilde{N}_2\)) has \(\lambda_2\) as its only eigenvalue, hence \(\tilde{N}_2 = \lambda_2 I\). But any subspace generated by eigenvectors corresponding to the eigenvalue \(\lambda_2\) is neutral so that \(V\) cannot be nondegenerate. This contradiction shows the indecomposability of \(N\). Step 3 is completed. The theorem is proved. \(\square\)

### 3 Indecomposable normal matrices in real spaces

The objective of this section is to prove the following theorem.

**Theorem 2** Let an indecomposable \(H\)-normal operator \(N\) act in a space \(R^n\) of rank \(k > 0\). Then one of the conditions (A) - (E) holds:

(A) \(N\) has only one real eigenvalue and \(2k \leq n \leq 4k\);

(B) \(N\) has only two real eigenvalues and \(n = 2k\);

(C) \(N\) has only two complex conjugate eigenvalues and \(n = 2k\) if \(k = 1\) and \(2k \leq n \leq 10|k/2| - 2\) if \(k > 1\);

(D) \(N\) has only one real and one pair of complex conjugate eigenvalues and \(n = 2k\);

(E) \(N\) has only two pairs of complex conjugate eigenvalues and \(n = 2k\).

The alternatives (D) and (E) are possible only if \(k\) is even. The estimates (A), (B), (D), (E), and the low bound in (C) are strict.

**Proof:** That an indecomposable \(H\)-normal matrix has one of the five sets of eigenvalues is proved in [2, Lemma 1]. Bounds (A) and (B) are proved in [2, Theorem 1], their strictness in Theorem 1 from the previous section (since the matrices constructed in Theorem 1 are real and any matrix that is indecomposable
in a complex space is also indecomposable in a real one). The condition $n \geq 2k$ is obvious. Indeed, since $k = \min \{v_-, v_+\}$ and $n = v_- + v_+$, we have $n \geq 2k$. Thus, we must consider the cases (C) - (E) only, keeping in mind that $n \geq 2k$.

**Step 1.** Consider the case (C). Let $N$ have the two distinct eigenvalues $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. The equality $n = 2$ for $k = 1$ is proved in [3] Theorem 1). In case when $k = 2$ Theorem 2 of [3] states that $n < 8$. So, it remains to prove the inequality $n \leq 10[k/2] - 2$ for $k \geq 3$. To this end recall Proposition 2 from [3]:

Let an indecomposable $H$-normal operator $N$ acting in $\mathbb{R}^n$ ($n > 2$) have the two distinct eigenvalues $\lambda = \alpha + i\beta$, $\bar{\lambda} = \alpha - i\beta$. Let

$$S_0' = \{z = x + iy \ (x, y \in \mathbb{R}^n) : Nz = \lambda z, N^{|n|}z = \bar{\lambda}z\},$$

$$S_0'' = \{z = x + iy \ (x, y \in \mathbb{R}^n) : Nz = \lambda z, N^{|n|}z = \lambda z\},$$

$\{z_j\}^p_{j=1} (\{z_j\}^{p+q})$ be a basis of $S_0'$ ($S_0''$), and

$$S_0 = \sum_{j=1}^{p+q} \text{span}\{x_j, y_j\}.$$  \hfill (13)

Then there exists a decomposition of $\mathbb{R}^n$ into a direct sum of subspaces $S_0$, $S$, $S_1$ such that

$$N = \begin{pmatrix} N' & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix},$$  \hfill (14)

where

$$N': S_0 \to S_0, \quad N'' = \underbrace{N_1' \oplus \ldots \oplus N''_{p+q}},$$

$$N_j' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad j = 1, \ldots, p+q,$$  \hfill (15)

$$N'' = S_1 \to S_1, \quad N'' = \underbrace{N''_1 \oplus \ldots \oplus N''_{p+q}},$$

$$N''_j = N''_j \text{ if } 1 \leq j \leq p, \quad N''_j = N''_j \text{ if } p < j \leq p+q,$$  \hfill (16)

the internal operator $N_1$ is $H_1$-normal and the pair $\{N_1, H_1\}$ is determined up to unitary similarity. To go over from one decomposition $\mathbb{R}^n = S_0 + S + S_1$ to another by means of a transformation $T$ it is necessary that the matrix $T$ be block triangular with respect to both decompositions.

According to this proposition, for an indecomposable operator $N$ the subspace $S_0$ defined in (13) is neutral so that its dimension does not exceed $k$. Therefore, if we prove that for $n > 10[k/2] - 2$ the condition $\dim S_0 \leq k$ fails, this will mean the decomposability of $N$.

According to [11] the proof of Lemma 1], if an $H$-normal operator $N$ acting in $\mathbb{C}^n$ has the two distinct eigenvalues $\lambda$, $\bar{\lambda}$, then there exists a decomposition of $\mathbb{C}^n$ into a direct sum of subspaces $V_1$, $V_2$, $V_3$, $V_4$ such that

$$N = \begin{pmatrix} N|_{V_1} = N_1 & 0 & 0 & 0 \\ 0 & N|_{V_2} = N_2 & 0 & 0 \\ 0 & 0 & N|_{V_3} = N_3 & 0 \\ 0 & 0 & 0 & N|_{V_4} = N_4 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix}.$$  \hfill (14)

Here $N_1$ and $N_3$ have only one eigenvalue $\lambda$, $N_2$ and $N_4$ only one eigenvalue $\bar{\lambda}$, and $\dim V_1 = \dim V_2$. In our case $\mathbb{C}^n$ is $\mathbb{R}^n$ complexified, therefore, $\dim V_3 = \dim V_4$ too. Either $V_1$ or $V_3$ may be equal to zero.
Let \( n > 10[k/2] - 2 \), i.e., \( n \geq 10[k/2] \). Consider the following three cases: (a) \( V_1 = V_2 = 0 \), (b) \( V_3 = V_4 = 0 \), (c) \( \dim V_3 > 0 \) and \( \dim V_3 = 0 \).

(a) If \( V_3 = 0 \), then \( \dim V_3 (= \dim V_4) \geq 5 [k/2] \). Let \( H_3 \) \((H_4)\) have \( v_{-(3)} \) \((v_{-(4)}\) negative eigenvalues. Without loss of generality it can be assumed that \( k = v_+ = v_{-(3)} + v_{-(4)} \) so that \( \min \{v_{-(3)}, v_{-(4)}\} \leq [k/2] \).

Let \( v_{-(3)} \leq [k/2] \). Decompose \( N_3 \) into an \( H \)-orthogonal sum of indecomposable operators \( N_3^{(1)}, N_3^{(2)}, \ldots, N_3^{(m)} \): \( N_3 = N_3^{(1)} \oplus N_3^{(2)} \oplus \cdots \oplus N_3^{(m)} \), \( H_3 = H_3^{(1)} \oplus H_3^{(2)} \oplus \cdots \oplus H_3^{(m)} \), \( V_3 = V_3^{(1)} + V_3^{(2)} + \cdots + V_3^{(m)} \). Denote by \( v^{(j)}_{-(3)} \) the number of negative eigenvalues of \( H_3^{(j)} \) \((j = 1, \ldots, m)\).

\[
V'_3 = \sum_{v^{(j)}_{-(3)} > 0} V_3^{(j)}, \quad V''_3 = \sum_{v^{(j)}_{-(3)} = 0} V_3^{(j)},
\]

\( H_3' \) \(H_3'' \) and \( V'_3 \) \(V''_3 \) be the corresponding sums of \( H_3^{(j)} \) and \( V_3^{(j)} \). Since for \( v^{(j)}_{-(3)} > 0 \) the condition \( \dim V_3^{(j)} \leq 4 v^{(j)}_{-(3)} \) holds \([2] \) Theorem 1\), we have \( \dim V'_3 \leq 4 v^{(j)}_{-(3)} \leq 4 [k/2] \). If \( v^{(j)}_{-(3)} = 0 \), then \( \dim V_3^{(j)} = 1 \), \( H_3^{(j)} = (1) \), and \( N_3^{(j)} = (\lambda) \). Thus, \( \dim V''_3 \geq [k/2] \) and \( N_3 = \lambda z, N^{(i)}z = \overline{\lambda} z \) for all \( z \in V''_3 \). The operators \( N_3^* \) and \( N_3^{(3)} \) commute so that if \( \dim V'_3 \geq 1 \), there exists at least one vector \( z_0 \in V''_3 \) such that \( N_3 z_0 = \lambda z_0 \) and \( N^{(i)} z_0 = \overline{\lambda} z_0 \). If \( \dim V'_3 = 0 \), all nontrivial vectors from \( V_3 \) are eigenvectors of \( N_3 \) corresponding to the eigenvalue \( \lambda \) and those of \( N^{(i)} \) corresponding to the eigenvalue \( \overline{\lambda} \). Therefore, in either case there exist at least \( p = [k/2] + 1 \) linearly independent vectors \( \{z_i\}_{i=1}^p \) such that \( N z_i = \lambda z_i, N^{(i)} z_i = \overline{\lambda} z_i \). Therefore, \( \dim S_0 \geq 2 [(k/2) + 1] > k \).

(b) If \( \dim V_3 = 0 \), then \( n = 2 \dim V_1 \). Since no neutral subspace of a space of rank \( k \) can be of dimension more than \( k \), \( \dim V_1 \leq k \) so that \( n \leq 2k \). But it was proved before that \( n \geq 2k \). Thus, in this case \( n = 2k \) \(< 10 [k/2] - 2 \).

(c) If \( \dim V_1, \dim V_3 > 0 \), we can assume, as in the case (a) above, that \( v_{-(3)} \leq [k/2] - \dim V_1 \) (the notation here is also as in (a)). Since \( \dim V_3 \geq 5 [k/2] - \dim V_1 \), there are \( p \) linearly independent vectors \( \{z_i\}_{i=1}^p \) such that \( N z_i = \lambda z_i, N^{(i)} z_i = \overline{\lambda} z_i \) \((p = \text{equal to} \ [k/2] + 3 \dim V_1 + 1 \) if \( \dim V'_3 > 0 \) or to \( 5 [k/2] - \dim V_1 \) if \( \dim V'_3 = 0 \)). Since \( \dim V_1 \leq k \) and \( k \geq 3 \), we have \( 5 [k/2] - \dim V_1 \geq [k/2] + 1 \). Thus, again \( \dim S_0 > k \) so that \( N \) is decomposable. The upper bound in (C) is proved. Step 1 is completed.

Step 2. Let us show the strictness of the low bound in (C) for even numbers \( k \). Consider the pair of \( 2k \times 2k \) matrices

\[
N = \begin{pmatrix}
A & I_2 & 0 & \ldots & 0 & 0 \\
0 & A & I_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A & I_2 \\
0 & 0 & 0 & \ldots & 0 & A \\
\end{pmatrix}, \quad H = \begin{pmatrix}
0 & 0 & \ldots & 0 & I_2 \\
0 & 0 & \ldots & I_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & I_2 & \ldots & 0 & 0 \\
I_2 & 0 & \ldots & 0 & 0 \\
\end{pmatrix},
\]

where

\[
A = \begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha \\
\end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}, \beta > 0)
\]

(17)

(throughout what follows, by \( A \) we will denote the matrix \([17] \)). It is seen that \( H \) has \( k \) negative and \( k \) positive eigenvalues. It can easily be checked by direct calculation that \( N \) is \( H \)-normal. The number of linearly independent vectors \( z_i \) satisfying the condition \( N z_i = \lambda z_i \) \((\lambda = \alpha + i \beta)\) is equal to 1, hence \( \dim S_0 = 2 \). By \([3] \) Proposition 3, if the subspace \( S_0 \) is two-dimensional, the operator \( N \) is indecomposable. So, the statement is proved and Step 2 is completed.
Step 3. For the case when $k$ is odd consider the following pair of $2k \times 2k$ matrices

$$N = \begin{pmatrix} A & X & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A & X & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & A & X & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & A^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & A^* & X \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & A^* \\ \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_2 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & D_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & I_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & I_2 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ I_2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \end{pmatrix},$$

where

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

As $N^{[s]} = N$, the matrix $N$ is $H$-normal. Since the condition $Nz = \lambda z$ ($\lambda = \alpha + i\beta$) implies

$$z = \begin{pmatrix} z_1 & iz_1 & 0 & \cdots & 0 \end{pmatrix}^T,$$

the subspace $S_0$ is two-dimensional and, according to [3, Proposition 3], the matrix $N$ is indecomposable. Step 3 is completed.

Step 4. Consider the case (D). Let $N$ have one real eigenvalue $\lambda$ and two complex conjugate eigenvalues $\alpha \pm i\beta$. According to [3, Proposition 1], $R^n$ is a direct sum of neutral subspaces $Q_1, Q_2$ such that $\dim Q_1 = \dim Q_2$, $NQ_1 \subseteq Q_1$, $NQ_2 \subseteq Q_2$, $N|_{Q_1}$ has $\lambda$ as its only eigenvalue, $N|_{Q_2}$ has the two eigenvalues $\alpha \pm i\beta$. From the last condition it follows that $\dim Q_2$ is even. As in Step 1, case (b), we have $n = 2 \dim Q_1 \leq 2k$, hence $n = 2k$ and $\dim Q_2 = k$ is necessarily an even number.

Now suppose that $k$ is even and consider the following pair $\{N, H\}$:

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},$$

where the $k \times k$ submatrices $N_1$ and $N_2$ are as follows:

$$N_1 = \begin{pmatrix} A & I_2 & 0 & \cdots & 0 & 0 \\ 0 & A & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A & I_2 \\ 0 & 0 & \cdots & 0 & A \end{pmatrix}, \quad N_2 = \lambda I_k.$$

It is clear that the condition

$$N_1N_2^* = N_2^*N_1$$

(18)
is satisfied so that $N$ is $H$-normal. Suppose that there exists a nondegenerate subspace $V$ such that $N$ is similar to the matrix

$$
\tilde{N} = \begin{pmatrix}
N|_V = \tilde{N}_1 & 0 \\
0 & N|_{V^\perp} = \tilde{N}_2
\end{pmatrix}.
$$

(19)

Since the subspace generated by the eigenvectors corresponding to the eigenvalue $\alpha + i\beta$ is one-dimensional (in the complexified space), one of the submatrices $\tilde{N}_1$ and $\tilde{N}_2$ has $\lambda$ as its only eigenvalue, therefore, either $\tilde{N}_1$ or $\tilde{N}_2$ is equal to $\lambda I$. As in Theorem 1, Step 3, we conclude that under this condition $V$ cannot be nondegenerate so that $N$ is indecomposable. Step 4 is completed.

Step 5. Consider the case (E). That $k$ is necessarily even and $n = 2k$ can be proved just as in Step 4 before. So, it remains to construct a pair $\{N, H\}$ satisfying the $H$-normality condition, where $H$ has $k$ negative and $k$ positive eigenvalues, $N$ is indecomposable, and $N$ has only two pairs of complex conjugate eigenvalues $\alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2$ ($\beta_1, \beta_2 > 0, (\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$). Let

$$
N = \begin{pmatrix}
N_1 & 0 \\
0 & N_2
\end{pmatrix}, \quad H = \begin{pmatrix}
0 & I_k \\
I_k & 0
\end{pmatrix},
$$

where the $k \times k$ submatrices $N_1$ and $N_2$ are as follows:

$$
N_1 = \begin{pmatrix}
A_1 & I_2 & 0 & \ldots & 0 & 0 \\
0 & A_1 & I_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & A_1 & I_2 \\
0 & 0 & 0 & \ldots & 0 & A_1
\end{pmatrix}, \quad N_2 = \begin{pmatrix}
A_2 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_2
\end{pmatrix}.
$$

Here

$$
A_1 = \begin{pmatrix}
\alpha_1 & \beta_1 \\
-\beta_1 & \alpha_1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\alpha_2 & \beta_2 \\
-\beta_2 & \alpha_2
\end{pmatrix}.
$$

It can easily be checked that the $H$-normality condition (19) is satisfied. As in Step 4 before, the assumption that $N$ is similar to (19) implies that either $\tilde{N}_1$ or $\tilde{N}_2$ (suppose $\tilde{N}_2$) has two eigenvalues $\alpha_2 \pm i\beta_2$ only. Therefore, there are $m = \frac{\dim N_2}{2}$ complex linearly independent eigenvectors $\{z_j = x_j + iy_j\}_{j=1}^m$ of $\tilde{N}_2$ corresponding to the eigenvalue $\alpha_2 + i\beta_2$. Consequently, the set $\{z_j\}_{j=1}^m \cup \{y_j\}_{j=1}^m$ is a basis of $\tilde{N}_2$. But $[x_j, x_l] = [y_j, y_l] = [x_j, y_l] = 0$ for all $j, l = 1, \ldots, m$. Therefore, the subspace $V$ cannot be nondegenerate and hence $N$ is indecomposable. Step 5 is completed. The theorem is proved.

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