On an inhomogeneous slip-inflow boundary value problem for a steady viscous compressible channel flow

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ABSTRACT
We prove the existence and uniqueness of a strong solution to the steady isentropic compressible Navier–Stokes equations with inflow boundary condition for density and mixed boundary conditions for the velocity around a shear flow. In particular, the Dirichlet boundary conditions on the inflow and outflow part of the boundary and the full Navier boundary conditions on the wall $\Gamma_0$ for the velocity field are considered. For our result, there are no restrictions on the amplitude of friction coefficients $\alpha$, and only the assumption that the viscosity coefficient $\mu$ is appropriately large is required. One of the substantial ingredients of our proof is an elegant transformation induced by the flow field. With the help of this transformation, we can overcome the difficulties caused by the hyperbolicity of the continuity equation, establish the a priori estimates for a linearized system and apply the fixed point argument.

ARTICLE HISTORY
Received 21 April 2022
Accepted 24 August 2022

COMMUNICATED BY
M. Mei

KEYWORDS
Inhomogeneous boundary conditions; compressible Navier–Stokes system; strong solution

MATHS
35-11

1. Introduction and main result

In this paper, we consider the steady isentropic compressible Navier–Stokes equations with inflow boundary condition in a two-dimensional tube $\Omega = (0, 1) \times (0, 1)$ near a shear flow. It is well known that the Navier–Stokes equations for a steady isentropic compressible viscous flow is a mixed system of hyperbolic–elliptic type. The momentum equations are an elliptic system in the velocity, while the continuity equation is hyperbolic in the density. Therefore, it is necessary to prescribe the density on the part of inflow boundary $(u \cdot n < 0)$ where $n$ is the outward unit normal to the part of inflow boundary. The inflow boundary value problem considered in this article reads as follows:

\begin{align*}
\text{div}(\rho u) &= 0 \quad \text{in } \Omega, \\
\text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \nu)\nabla \text{div} u + \nabla P &= 0 \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \Gamma_{in} \cup \Gamma_{out}, \\
\rho &= \rho_{in} \quad \text{on } \Gamma_{in}, \\
u \cdot n &= 0 \quad \text{on } \Gamma_0, \\
2\mu n \cdot D(u) \cdot \tau + \alpha u \cdot \tau &= b \quad \text{on } \Gamma_0,
\end{align*}

(1)

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is the unknown velocity filed of the fluid and $\rho : \mathbb{R}^2 \to \mathbb{R}$ is the unknown density. The $D(u)$ denotes the symmetric part of the velocity gradient, more precisely,

\[ D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T). \]
The constant viscosity coefficients $\mu, \nu$ satisfying $\mu > 0, \nu + \mu \geq 0$ and $\alpha \geq 0$ is a friction coefficient. The pressure $P$ is usually determined through the equation of states. In the case of isentropic flows, the pressure $P$ is given by $P(\rho) = \rho^\gamma$ with $\gamma > 1$ being the heat ratio. Next, $\mathbf{n}$ and $\tau$ are outer normal and tangent vectors to $\partial \Omega$. The boundary $\partial \Omega$ is naturally divided into the inflow part $\Gamma_{\text{in}}$, the outflow part $\Gamma_{\text{out}}$, and the impermeable wall $\Gamma_0$, more precisely:

$$
\begin{align*}
\Gamma_{\text{in}} &= \{ x_1 = 0, 0 \leq x_2 \leq 1 \}, \\
\Gamma_{\text{out}} &= \{ x_1 = 1, 0 \leq x_2 \leq 1 \}, \\
\Gamma_0 &= \{ 0 \leq x_1 \leq 0, x_2 = 0 \} \cup \{ 0 \leq x_1 \leq 0, x_2 = 1 \}.
\end{align*}
$$

The flow of a viscous, compressible fluid was governed by the compressible Navier–Stokes system, which has been extensively studied. The general existence results for weak solutions to stationary compressible Navier–Stokes system was given by the work of Lions [1] under the assumption that $\gamma > 1$ in two dimensions and $\gamma \geq \frac{5}{3}$ in three dimensions. Later, this result was improved with the weaker assumptions on the specific heat ratio $\gamma > \frac{3}{2}$ in three dimensions by Feireisl [2] for non-steady equations. By adapting Feireisl's nonsteady approach, the authors in [3] proved that $\gamma > \frac{3}{2}$ is also valid to the case of steady system. An overview of these results were given in the monograph [4, 5]. Several refined results for steady compressible flow were given in [6–8]. Let us remark that all those results were obtained under various homogeneous boundary conditions.

Strong solutions of the steady compressible Navier–Stokes equations with homogeneous boundary conditions has been studied in [9–13]. As to the existence of strong solutions to the stationary Navier–Stokes system with the inhomogeneous boundary conditions, the authors in [14] proved the existence of strong solutions to the stationary problems with an inflow boundary condition for the density and the Dirichlet boundary conditions for the velocity in a smooth two dimension domain $\Omega$ under the assumption that the Reynolds number is small. A mass of researches showed that the regularity of strong solutions is restricted by the geometry of the boundary [15]. In [16–18], the authors studied the existence and regularity of solutions to an inflow boundary value problem under the assumption that the viscosity coefficient $\mu$ is large enough on a polygon domain. In [19], T. Piasecki proved the existence of strong solutions around a constant equilibrium with an inflow boundary condition for the density and the full Navier boundary conditions for the velocity filed under the assumption that the friction coefficient $\alpha$ is large enough. Later, the authors obtained similar results in a cylinder domains [20]. It is worth mentioning that the validity for compressible perturbation of a Poiseuille-type flow under the same boundary conditions as before is also obtained in [21]. In contrast to the boundary conditions prescribed above, the authors established the existence of strong solutions near the constant state $u = 0, \rho = 1$ with the Dirichlet boundary conditions on $\Gamma_{\text{in}}$ and $\Gamma_{\text{out}}$ while slip without friction boundary conditions on the wall $\Gamma_0$ in [22]. Recently, the authors in [23] studied the existence of weak solutions to the stationary compressible Navier–Stokes system for arbitrarily large boundary data under additional physical hypotheses called molecular hypothesis and positive compressibility in 2D or 3D domains. One can refer to [24, 25] for more results concerning the existence of strong solutions with inhomogeneous boundary data.

The goal of this paper is to investigate the existence of strong solutions near the shear flow to the steady isentropic compressible Navier–Stokes system with inflow boundary condition in a square. We impose the Dirichlet boundary condition on inflow and outflow part of the boundary, and the full Navier boundary condition on the wall $\Gamma_0$ for the velocity. Due to the hyperbolicity of the density in the continuity equation, it is natural to prescribe the density on the inflow part of the boundary. Let us remark here that we do not need any restrictions on the amplitude of friction coefficients $\alpha$, and an assumption that $\mu > \frac{1}{\gamma^2}$ is enough. It is worth noting that most of the results discussed above investigated the structural stability around a constant equilibrium, while our background solution is a shear flow.
Let us introduce the perturbed flow. Consider the shear flow \((\rho_0 = 1, U_0)\), with \(U_0 = (1 + x_2, 0) = (\bar{U}, 0)\). It is obviously that the shear flow \((\rho, U_0)\) satisfies the following system:

\[
\begin{align*}
\text{div}(\rho_0 U_0) &= 0 \\
\text{div}(\rho_0 U_0 \otimes U_0) - \mu \Delta U_0 - (\mu + \nu) \nabla \text{div} U_0 + \nabla P_0 &= 0, \quad \text{in } \Omega, \\
U_0 &= U_0, \quad \text{on } \Gamma_{in} \cup \Gamma_{out}, \\
\rho_0 &= 1, \quad \text{on } \Gamma_{in}, \\
U_0 \cdot n &= 0, \quad \text{on } \Gamma_0, \\
2\mu n \cdot D(U_0) \cdot \tau + \alpha U_0 \cdot \tau &= \bar{b} \quad \text{on } \Gamma_0,
\end{align*}
\]

where \(\bar{b}\) is determined when \(U_0\) and \(\alpha\) are given. Our aim here is to study the existence and uniqueness of the strong solution in a square around the shear flow \((\rho_0, U_0)\). To formulate our main result, it is convenient to define the quantity \(D_0\) as

\[
D_0 := \|u_0 - U_0\|_{2, 1/2, p, \Gamma_{in} \cup \Gamma_{out}} + \|b - \bar{b}\|_{1, 1/2, p, \Gamma_0} + \|\rho_{in} - 1\|_{1, p, \Gamma_{in}}
\]

which measure the distance of the initial data away from the shear flow \(U_0\).

**Theorem 1.1:** Suppose that \(2 < p < \infty\), if \(D_0\) given by (2) is small enough and the viscosity coefficient \(\mu > \frac{1}{2\pi^2}\), then there exists a unique solution \((u, \rho) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)\) to the system (1) such that

\[
\|u - U_0\|_{2, p, \Omega} + \|\rho - 1\|_{1, p, \Omega} \leq E(D_0),
\]

where \(E(D_0)\) can be arbitrarily small provided that the quantity \(D_0\) is small enough.

This paper will be organized as follows. In Section 2, the linearized system corresponding to the original problem (1) around the shear flow \((\rho_0, U_0)\) is obtained. Next, we introduce an elegant transformation to overcome the difficulties caused by the hyperbolicity of the continuity equation. In Section 3, we establish the basic and higher-order energy estimate to the linearized equations. In order to overcome the difficulties caused by the nonlinearity for the Navier–Stokes system, we derive the uniform \(W^{2,p}(\Omega) \times W^{1,p}(\Omega)\), \((2 < p < \infty)\) estimates for the solution of the linearized system. An iteration scheme is also developed to study the existence and uniqueness of the strong solution. In Section 4, we construct an approximated solution by a Galerkin method and obtain the existence and regularity of the weak solution to the linearized system. In Section 5, we show that the iteration scheme designed in the Section 3 is a contraction and prove our main result Theorem 1.1 finally.

**2. Reformulation of problem**

It is convenient to convert the inhomogeneous boundary conditions of the original problems into homogeneous one. To this end, we introduce a function \(\tilde{u} \in W^{2,p} (\Omega)\) such that

\[
\begin{align*}
\tilde{u} &= u_0 - U_0 \quad \text{on } \Gamma_{in} \cup \Gamma_{out}, \\
\tilde{u} \cdot n &= 0 \quad \text{on } \Gamma_0, \\
\mu n \cdot D(\tilde{u}) \cdot \tau + \alpha \tilde{u} \cdot \tau &= 0 \quad \text{on } \Gamma_0.
\end{align*}
\]

Indeed, one can construct \(\tilde{u}\) as a solution of the following Lamé system

\[-\mu \Delta \tilde{u} - (\mu + \nu) \nabla \text{div} \tilde{u} = 0 \quad \text{in } \Omega,
\]

with the boundary conditions (4). Consequently,

\[
\|\tilde{u}\|_{W^{2,p}(\Omega)} \leq C|u_0 - U_0|_{W^{2, 1/2, p}(\Gamma_{in} \cap \Gamma_{out})}.
\]
We denote the differences 
\[ v = u - \tilde{u} - U_0, \quad w = \rho - 1. \]

Then, one can yield \((\tilde{u}, \tilde{\rho})\) satisfying the following system by straightforward computation,

\[
\begin{align*}
\bar{U} \partial_1 w + (v + \tilde{u}) \cdot \nabla w + \text{div} v &= F(v, w) \quad \text{in } \Omega, \\
\bar{U} \partial_1 v + v \cdot \nabla U_0 + \gamma \nabla w - \mu \Delta v - (\mu + v) \nabla \text{div} v &= G(v, w) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\
w &= \rho_{\text{in}} - 1 \quad \text{on } \Gamma_{\text{in}}, \\
v \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_0, \\
2\mu \mathbf{n} \cdot D(v) \cdot \tau + \alpha v \cdot \tau &= B \quad \text{on } \Gamma_0.
\end{align*}
\]

where

\[
B = b - \tilde{b}, \\
F(v, w) = -\text{div} \tilde{u} - w \text{div} (v + \tilde{u}), \\
G(v, w) = -(w + 1)(v + \tilde{u} + U_0) \cdot \nabla \tilde{u} - (w + 1)(v + \tilde{u}) \cdot \nabla v \\
- w U_0 \cdot \nabla v - \tilde{u} \cdot \nabla U_0 - w (v + \tilde{u}) \cdot \nabla U_0 \\
+ \mu \Delta \tilde{u} + (\mu + v) \nabla \text{div} \tilde{u} + \gamma [(w + 1)^{\gamma-1} - 1] \nabla w.
\]

A straightforward computation gives the estimate of \(F\) and \(G\) as follows.

**Lemma 2.1:** Let \(F(v, w)\) and \(G(v, w)\) be defined by (7). Then we have

\[
\|F(v, w)\|_{W^{1,p}} + \|G(v, w)\|_{L^p} \\
\leq C [\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}}]^3 + (\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^2 \\
+ E(D_0)(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^2 + \|\tilde{u}\|_{W^{2,p}}.
\]

**Proof:** The estimate is almost obviously by the embedding \(W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)\), and one can easily check it term by term. In fact, the term \(w \text{div} v\) in \(F(v, w)\) is estimated as

\[
\|w \text{div} v\|_{1,p} \leq \|w\|_\infty \|\nabla v\|_p + \|\nabla w\|_p \|\nabla v\|_\infty + \|w\|_\infty \|\nabla^2 v\|_p \\
\leq C(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^2.
\]

The most subtle term in \(G(v, w)\) is controlled by

\[
\|w v \cdot \nabla v\|_p \leq \|w\|_\infty \|v\|_p \|\nabla v\|_\infty \\
\leq C \|v\|_{2,p}^2 \|w\|_{1,p} \\
\leq C(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^2.
\]

hence, the result holds. 

We want to prove the existence of solution by fixed-point theorem. However, a direct application of fixed-point argument may fail due to the term \(v \cdot \nabla w\) on the left-hand side of (6) which would cause the loss of regularity on the right-hand side of the system. Thanks to the strictly positivity of
\( \bar{U} + (v + \bar{u})^{(1)} \), we can get rid of the problematic term \( v \cdot \nabla w \) by introducing a change of variables 
\( \psi : (z_1, z_2) \mapsto (x_1, x_2) \) such that
\[
\partial z_1 = \partial x_1 + \bar{v} \partial x_2, \tag{11}
\]
where \( \bar{v} = \frac{(v + \bar{u})}{\bar{U} + (v + \bar{u})^{(1)}} \). The identity (11) implies that \( \psi = (\psi^1(z_1, z_2), \psi^2(z_1, z_2)) \) must satisfy
\[
\begin{cases}
\frac{\partial \psi^1}{\partial t}(t, z_2) = 1, \\
\frac{\partial \psi^2}{\partial t}(t, z_2) = \bar{v} \circ \psi(t, z_2). \tag{12}
\end{cases}
\]
For the simplicity, we seek \( \psi = \psi(z_1; z_2) \) satisfying the initial conditions
\[
(\psi^1(0; z_2), \psi^2(0; z_2)) = (0, z_2), \tag{13}
\]
which means that the entrance \( \Gamma_{\text{in}} \) is invariant under this transformation. It follows that the system (12)–(13) has a unique solution provided \( \| \nabla \bar{v} \|_{L^\infty} \) is bounded, and the explicit formula of \( \psi \) can be written as
\[
\begin{cases}
x_1 = \psi^1(z_1, z_2) = z_1, \\
x_2 = \psi^2(z_1, z_2) = z_2 + \int_0^{z_1} \bar{v}(t, \psi^2(t; z_2)) \, dt. \tag{14}
\end{cases}
\]
Furthermore, if we denote
\[
E = \begin{pmatrix}
0 & 0 \\
\bar{v}(\psi(z)) & \partial z_2 \int_0^{z_1} \bar{v}(\psi(t, z_2)) \, dt
\end{pmatrix}, \tag{15}
\]
then \( \nabla \psi = \text{Id} + E \), and we have the following result.

**Lemma 2.2:** Suppose that \( \psi \) is defined in (14), then \( \psi \) is a diffeomorphism such that \( \psi(\bar{\Omega}) = \Omega \), moreover
\[
\| E \|_{W^{1,p}(\bar{\Omega})} \leq E, \tag{16}
\]
here \( E = E(\| v \|_{2,p}, \| \bar{u} \|_{2,p}) \) can be arbitrary small provided \( \| v \|_{2,p}, \| \bar{u} \|_{2,p} \) small enough.

**Proof:** Since \( \bar{U} \) is strictly positive and \( \| v \|_{2,p}, \| \bar{u} \|_{2,p} \) is small, there exists a positive constant \( c \), such that
\[
\bar{U} + (v + \bar{u})^{(1)} \geq c > 0,
\]
it follows that \( \bar{v} \in W^{2,p} \) and
\[
\| \bar{v} \|_{L^\infty} \leq C \| \bar{v} \|_{2,p} \leq CE(\| v \|_{2,p}, \| \bar{u} \|_{2,p}), \tag{17}
\]
for some positive constant \( C \).
At this stage, the first part is standard, one can refer to [20], and the estimate (16) for $E_{21}$ is derived immediately. On the other hand,

$$
|E_{22}|^p \leq | \int_0^{z_1} \nabla \tilde{v}(\psi(t, z_2)) \cdot \partial_{z_2} \psi(t, z_2) \, dt |^p \\
\leq |z_1|^{p-1} \int_0^{z_1} |\nabla \tilde{v}(\psi(t, z_2)) \cdot \partial_{z_2} \psi(t, z_2)|^p \, dt \\
\leq C \| \nabla \tilde{v} \|_\infty^p \int_0^{z_1} |1 + E_{22}|^p \, dt \tag{18}
$$

Integrating it over $\Omega_1$ gives

$$
\|E_{22}\|_p \leq C \| \nabla \tilde{v} \|_\infty \|1 + E_{22}\|_p. \tag{19}
$$

Differentiate $E_{22}$ with respect to $z_1$ we have

$$
\partial_{z_1} E_{22} = \nabla \tilde{v}(\psi(z_1, z_2)) \cdot \partial_{z_2} \psi(z_1, z_2),
$$

which gives

$$
\|\partial_{z_1} E_{22}\|_p \leq C \| \nabla \tilde{v} \|_\infty \|1 + E_{22}\|_p \leq E, \tag{20}
$$

similarly, differentiate $E_{22}$ with respect to $z_2$ we have

$$
\partial_{z_2} E_{22} = \partial_{z_2} \int_0^{z_1} \partial_{x_i} \tilde{v}(\psi(t, z_2)) \partial_{z_2} \psi^i(t, z_2) \, dt \\
= \int_0^{z_1} \partial_{x_i, x_j} \tilde{v}(\psi(t, z_2)) \partial_{z_2} \psi^i(t, z_2) \partial_{z_2} \psi^j(t, z_2) \, dt \\
+ \int_0^{z_1} \partial_{x_i} \tilde{v}(\psi(t, z_2)) \partial_{z_2}^2 \psi^i(t, z_2) \, dt \\
:= I_1 + I_2,
$$

it follows that

$$
|I_1|^p = | \int_0^{z_1} \partial_{x_i, x_j} \tilde{v}(\psi(t, z_2)) \partial_{z_2} \psi^i(t, z_2) \partial_{z_2} \psi^j(t, z_2) \, dt |^p \\
\leq \| \partial_{z_2} \psi \|_\infty^{2p} |z_1|^{p-1} \int_0^{z_1} |\nabla_x^2 \tilde{v}|^p \, dt \\
\leq C \int_0^{z_1} |\nabla_x^2 \tilde{v}|^p \, dt
$$

and

$$
|I_2|^p = | \int_0^{z_1} \partial_{x_i} \tilde{v}(\psi(t, z_2)) \partial_{z_2}^2 \psi^i(t, z_2) \, dt |^p \\
\leq C \| \nabla \tilde{v} \|_\infty \int_0^{z_1} |\partial_{z_2} E_{22}|^p
$$

which implies

$$
\|\partial_{z_2} E_{22}\|_p \leq C(\|\tilde{v}\|_{L^p} + \| \nabla \tilde{v} \|_\infty \|\partial_{z_2} E_{22}\|_p). \tag{21}
$$

Then the estimate (16) for $E_{22}$ follows from (19)–(21) and the proof thus is completed. □
Lemma 2.4: Let identities combined with (8) yield (27).

\[ \nabla \phi = \nabla \psi^{-1} = \text{Id} + \tilde{E}. \]  

Denote \( J = |\nabla \psi| \), then \( \nabla \psi^{-1} \) can be explicitly computed as

\[ \nabla \psi^{-1} = \frac{1}{J} \begin{pmatrix} 1 + \partial_{z_2} \int_0^{z_1} \nabla \psi(t, z_2) \, dt & 0 \\ -\nabla \psi(z) & 1 \end{pmatrix}, \]

hence, the estimate (16) also holds for \( \tilde{E} \). We also denote the transform \( \psi \) as \( \psi_{v+\tilde{u}} \) to emphasize that \( \psi_{v+\tilde{u}} \) is induced by \( \tilde{v} \) through (14).

Note that the most important property of the transform \( \psi \) is (11). Hence we can change variables and rewrite the system (6) in coordinates \( z \), which leads to

\[
\begin{align*}
((\tilde{U} + (v + \tilde{u})^{(1)} \circ \psi_{v+\tilde{u}}) \partial_{z_1} w + \text{div}_z v &= \tilde{F}(v, w) & \text{in } \tilde{\Omega}, \\
(\tilde{U} \circ \psi_{v+\tilde{u}}) \partial_{z_1} v + v \cdot \nabla_z U_0 + \gamma \nabla_z w - \mu \Delta_z v - (\mu + \nu) \text{div}_z v &= \tilde{G}(v, w) & \text{in } \tilde{\Omega}, \\
v &= 0 & \text{on } \tilde{\Gamma}_{\text{in}} \cup \tilde{\Gamma}_{\text{out}}, \\
w &= \rho_{\text{in}} - 1 & \text{on } \tilde{\Gamma}_{\text{in}}, \\
v \cdot n &= 0 & \text{on } \tilde{\Gamma}_{0}, \\
2\mu \nu \cdot D_z(v) \cdot \tau + \alpha v \cdot \tau &= \tilde{B}
\end{align*}
\]

Here \( \tilde{B} = B - 2\mu \nu \cdot R(v, D) \cdot \tau \), and the function \( R(\cdot, \cdot) \) with the first variable denotes a function and the second is a differential operator representing the differences of the differential operator acting on the function in \( x \)-coordinates and \( z \)-coordinates. For instance, \( R(v, D) = D_x v - D_z v \), and we also have

\[
\begin{align*}
\tilde{F}(v, w) &= F(v, w) - R(v, \text{div}) \\
\tilde{G}(v, w) &= G(v, w) - \tilde{U} R(v, \partial_1) - v \cdot R(U_0, \nabla) \\
&\quad - \gamma R(w, \nabla) + \mu R(v, \Delta) + (\mu + \nu)R(v, \text{div}).
\end{align*}
\]

Lemma 2.4: Let \( \tilde{F} \) and \( \tilde{G} \) be given by (25). Then we have

\[
\| \tilde{F}(v, w) \|_{W^{1,p}} + \| \tilde{G}(v, w) \|_{L^p} \\
\leq C[(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^3 + (\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}})^2] \\
+ E(D_0)(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}}) + \|\tilde{u}\|_{W^{2,p}}.
\]

Proof: A direct computation shows that under the change of variables \( \psi:(z_1, z_2) \mapsto (x_1, x_2) \), there holds

\[
\begin{align*}
\partial_{z_i} v^j &= \partial_{x_i} v^j + E_{ik} \partial_{x_k} v^j, \\
\partial_{z_1} v^j &= \partial_{x_1} v^j + E_{ik} \partial_{x_k} v^j + \partial_{x_j} E_{ik} \partial_{x_k} v^j \\
&\quad + E_{jk} \partial_{x_k} v^j + E_{jm} E_{ik} \partial_{x_k} v^j + E_{jm} \partial_{x_m} E_{ik} \partial_{x_k} v^j
\end{align*}
\]

where \( 1 \leq i, j, l \leq 2 \) are integers, and the repeated subscript always sum over its index set. The above identities combined with (8) yield (27).
3. A priori estimates

In this section, we construct a sequence that will converge to a solution of the nonlinear system (24). The solution sequence is defined as follows:

\[
\begin{align*}
((\bar{U} + (v^n + \bar{u})^{(1)} \circ \psi_\nu + \bar{u})) & w^{n+1} + \text{div}_2 v^{n+1} = \bar{F}(v^n, w^n) \quad \text{in } \bar{\Omega}, \\
(\bar{U} \circ \psi_\nu + \bar{u})_{\partial_1} v^{n+1} + v^{n+1} \cdot \nabla U_0 + \gamma \nabla w^{n+1} \\
- \mu \Delta v^{n+1} - (\mu + v) \nabla \text{div}_2 v^{n+1} = \bar{G}(v^n, w^n) \quad \text{in } \bar{\Omega}, \\
v^{n+1} &= 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\
w^{n+1} &= 0 \quad \text{on } \Gamma_{\text{in}}, \\
v^{n+1} \cdot n &= 0 \quad \text{on } \Gamma_0, \\
2\mu n \cdot D_z(v^{n+1}) \cdot \tau + \alpha v^{n+1} \cdot \tau &= \bar{B}(v^n) \quad \text{on } \Gamma_0, 
\end{align*}
\]

where \( \bar{B}(v^n) = B - 2\mu n \cdot R(v^n, D) \cdot \tau. \)

To show the existence of the solution to (28), we firstly deal with the following linear system:

\[
\begin{align*}
((\bar{U} + \bar{v}^{(1)} \circ \psi_\nu) & \partial_1 w + \text{div}_2 v = f \quad \text{in } \bar{\Omega}, \\
(\bar{U} \circ \psi_\nu)_{\partial_1} v + v \cdot \nabla U_0 + \gamma \nabla w - \mu \Delta v - (\mu + v) \nabla \text{div}_2 v = g \quad \text{in } \bar{\Omega}, \\
v &= 0 \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\
w &= 0 \quad \text{on } \Gamma_{\text{in}}, \\
v \cdot n &= 0 \quad \text{on } \Gamma_0, \\
2\mu n \cdot D_z(v) \cdot \tau + \alpha v \cdot \tau &= \bar{B} \quad \text{on } \Gamma_0, 
\end{align*}
\]

where \( f, g \in L^2(\Omega) \), \( w_{\text{in}} \in L^2(\Gamma_{\text{in}}) \), \( \bar{B} \in L^2(\Gamma_0) \) are given functions. Without the risk of confusing, we readily remove the superscript of the domains.

3.1. Solution of mass equation

One of the most important advantages of changing variables is that we can define an operator \( S : v \mapsto w \) as the solution of equation

\[
\begin{align*}
((\bar{U} + \bar{v}^{(1)} \circ \psi_\nu) & \partial_1 w = v \quad \text{in } \Omega, \\
w &= w_{\text{in}}(z_2) \quad \text{on } \Gamma_{\text{in}}.
\end{align*}
\]

Indeed, for a continuous function \( v \), setting

\[
S(v)(z) := w_{\text{in}}(z_2) + \int_0^{z_1} \frac{v}{(\bar{U} + \bar{v}^{(1)} \circ \psi_\nu)(t, z_2)} \ dt,
\]

then it is easy to verify that \( S(v) \) satisfies (30). Moreover, we have the following estimate.

Lemma 3.1: Let \( S \) be defined in (31), \( w_{\text{in}} \in L^2(\Gamma_{\text{in}}) \) is given then

\[
\|S(v)\|_{L^\infty(L^2)(\Omega)} \leq C(w_{\text{in}}|L^2(\Gamma_{\text{in}}) + \|v\|_{L^2(\Omega)})
\]


For the first term, we have
\[ |S(v)|_{L^2(\Omega_{z_1})}^2 = \int_0^1 [w_{in}(z_2) + \int_0^{z_1} \frac{\nu}{(\tilde{U} + \tilde{v}(t, z_2)) dt} dz_2
\leq C([w_{in}]_{L^2(\Gamma_1)}^2 + \|v\|_{L^2(\Omega)}^2), \tag{33} \]
where we have used
\[ \tilde{U} + (v + \tilde{u})^{(1)} \geq c > 0, \]
for some positive constant \( c > 0 \), and the estimate (32) follows. \( \square \)

**Remark 3.2:** By density arguments, one can easily extend \( S \) to \( L^2(\Omega) \) which also preserves the same estimate as (32).

### 3.2. \( H^1 \) estimates for linear system

**Lemma 3.3:** Let \( \tilde{v} \in W^{2,p}(\Omega) \) and \( \|\tilde{v}\|_{L^p} \) be small enough, the viscous coefficients \( \mu > \frac{1}{2\pi^2} \). Suppose that \( (v, w) \) be a solution to the system (29) with given \( (f, g, \tilde{B}, w_{in}) \in L^2(\Omega) \times V^* \times L^2(\Gamma_0) \times L^2(\Gamma_1) \), then
\[ \|v\|_{H^1} + \|w\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^2} + \|g\|_{V^*} + \|\tilde{B}\|_{L^2(\Gamma_0)} + \|w_{in}\|_{L^2(\Gamma_1)}) \tag{34} \]
where
\[ V := \{ v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_{in} \cup \Gamma_{out}, v \cdot n|\Gamma_0 = 0 \} \]
and \( V^* \) is the dual space of \( V \).

**Proof:** For any \( \phi \in V \) we have
\[ \int_\Omega (-\mu \Delta v - (\mu + \nu) \nabla \text{div} v) \cdot \phi \, dz = \int_\Omega (2\mu D(v) : \nabla \phi + v \text{div} \nabla \phi \text{div} \phi) \, dz
- \int_{\partial \Omega} 2\mu n \cdot D(v) \cdot \phi \, ds. \]

Then multiplying (29) by \( v \) and integrating over \( \Omega \) we have
\[ -\frac{1}{2} \int_\Omega \partial_{z_1} (\tilde{U} \circ \psi_{\tilde{v}})|v|^2 \, dz + \int_\Omega (\nabla U_0) \cdot v \, dz - \gamma \int_\Omega w \text{div} v \, dz + 2\mu \int_\Omega |D(v)|^2 \, dz + \nu \int_\Omega |\text{div} v|^2 \, dz + \int_{\Gamma_0} \alpha |v|^2 \, ds = \int_{\Gamma_0} \tilde{B} v \cdot \tau + \int_\Omega g \cdot v. \tag{35} \]

For the first term, we have
\[ |\partial_{z_1} (\tilde{U} \circ \psi_{\tilde{v}})| = |\partial_{x_1} \tilde{U} + E_{ik} \partial_{x_k} \tilde{U}| = |E_{12}| \leq E(\|\tilde{v}\|_{W^{2,p}}). \]

As to the second term in the left-hand side of (35), by applying the Poincaré’s inequality we get
\[ \left| \int_\Omega (v \cdot \nabla U_0) \cdot v \, dz \right| = \left| \int_\Omega (v^{(1)} v^{(2)}) \right| \leq \frac{1}{2} \|v\|_{L^2}^2 \leq \frac{1}{2} \|\nabla v\|_{L^2}^2. \]

The well-known Korn inequality in [21, 26] shows that
\[ \mu \pi^2 \|\nabla v\|_{L^2}^2 \leq 2\mu \int_\Omega |D(v)|^2 \, dz + \nu \int_\Omega |\text{div} v|^2 \, dz, \tag{37} \]
where the appearance of constant $\pi$ concerning the best constant in Korn inequality. Involving in the continuity equation in (29) we have

$$
- \int_{\Omega} w \text{div} v \, dz = \int_{\Omega} w((\tilde{U} + \tilde{v}^{(1)}) \circ \psi_{\tilde{Q}}) \partial_{z_{1}} w \, dz - \int_{\Omega} w f \, dz
$$

$$
= -\frac{1}{2} \int_{\Omega} \partial_{z_{1}}((\tilde{U} + \tilde{v}^{(1)}) \circ \psi_{\tilde{Q}}) w^{2} \, dz - \frac{1}{2} \int_{\Gamma_{0}} ((\tilde{U} + \tilde{v}^{(1)}) \circ \psi_{\tilde{Q}}) w_{\Gamma_{0}}^{2} \, ds
$$

$$
+ \frac{1}{2} \int_{\Gamma_{0}} ((\tilde{U} + \tilde{v}^{(1)}) \circ \psi_{\tilde{Q}}) w^{2} \, ds - \int_{\Omega} w f \, dz
$$

(38)

Due to the smallness assumption of $\tilde{v}$, the integral over $\Gamma_{out}$ will be nonnegative, and we also have

$$
|\partial_{z_{1}}(\tilde{v}_{\Omega}) \circ \psi_{\tilde{Q}}| = |\partial_{z_{1}} \tilde{v}_{\Omega}^{(1)} + E_{\Omega} \partial_{z_{1}} \tilde{v}_{\Omega}^{(1)}| \leq E(\|\tilde{v}\|_{W^{2,2}}).
$$

(39)

Combing all those estimates together, we arrive at

$$
\left[ \frac{\mu \pi^{2}}{2} - \frac{1}{2} \cdot \frac{E(\|\tilde{v}\|_{W^{2,2}})}{2} \right] \|\nabla v\|_{L^{2}}^{2}
$$

$$
\leq \|g\|_{V^{2}} \|v\|_{L^{2}} + |\tilde{B}|_{L^{2}(\Gamma_{0})} \|v\|_{L^{2}(\Gamma_{0})} + E\|w\|_{L^{2}}^{2}
$$

$$
+ C\|w_{\Omega}\|_{L^{2}(\Gamma_{0})}^{2} + \|w\|_{L^{2}} \|f\|_{L^{2}}^{2}
$$

(40)

Observe that the left-hand side of (40) will be positive provided $\|v\|_{W^{2,2}}$ is small enough and the viscosity efficiencies $\mu > \frac{1}{2\pi^{2}}$. Finally by substituting $v = f - \text{div} v$ in (32) yields

$$
\|w\|_{L^{\infty}(L^{2})} \leq C(\|w_{\Omega}\|_{L^{2}(\Gamma_{0})}^{2} + \|\nabla v\|_{L^{2}(\Omega)}^{2} + \|\tilde{B}\|_{H^{\frac{1}{2},2}(\Gamma_{0})} + \|w_{\Omega}\|_{H^{1}(\Gamma_{0})}^{2}).
$$

(41)

Combing (40) and (41) by using trace theorem and Poincaré’s inequality we get the estimate (34). 

Next, we want to show higher regularity up to boundary by the method of difference quotient.

**Lemma 3.4:** Suppose that $(v, w) \in H^{1} \times L^{\infty}(L^{2})$ be a solution to the system (29) with $(f, g, \tilde{B}, w_{\Omega}) \in H^{1}(\Omega) \times L^{2}(\Omega) \times H^{\frac{1}{2},2}(\Gamma_{0}) \times H^{1}(\Gamma_{in})$, then we have

$$
\|w\|_{H^{1}(\Omega)} + \|v\|_{H^{2}(\Omega)} \leq C(\|f\|_{H^{1}} + \|g\|_{L^{2}} + |\tilde{B}|_{H^{\frac{1}{2},2}(\Gamma_{0})} + w_{\Omega})_{H^{1}(\Gamma_{in})}.
$$

(42)

**Proof:** Let us firstly focus on the interior estimates. By using the standard $L^{2}$ theory of elliptic system, we only need to find a bound of $\|\partial_{z_{2}} w\|_{L^{2}}$. To this end, we define the difference quotient operator along $z_{2}$-direction $\Delta_{z_{2}}^{-h}$ as

$$
\Delta_{z_{2}}^{-h} \phi := -\frac{\phi(z_{1}, z_{2} - h) - \phi(z_{1}, z_{2})}{h}.
$$

Effecting $\Delta_{z_{2}}^{h}$ on the mass equation in (29), we have

$$
(\tilde{U} + \tilde{v}^{(1)}) \partial_{z_{1}}(\Delta_{z_{2}}^{h} w) + \partial_{z_{1}} w_{z_{2}}^{h} \Delta_{z_{2}}^{h} (\tilde{U} + \tilde{v}^{(1)}) + \text{div}(\Delta_{z_{2}}^{h} v) = \Delta_{z_{2}}^{h} f,
$$

(43)

where $w_{z_{2}}^{h}(z_{1}, z_{2}) = w(z_{1}, z_{2} + h)$. Hence, in order to find a bound of $\|\partial_{z_{2}} w\|_{L^{2}}$, it is sufficient to find a bound of $\|\partial_{z_{2}} w\|_{L^{2}}$. Since $(v, w) \in H^{1} \times L^{\infty}(L^{2})$ is a solution of the system (29), which means

$$
\int_{\Omega} \tilde{U} \partial_{z_{1}} v \cdot \phi \, dz + \int_{\Omega} (v \cdot \nabla U_{0}) \cdot \phi \, dz - \gamma \int_{\Omega} w \text{div} \phi \, dz + \mu \int_{\Omega} \nabla v \cdot \nabla \phi \, dz
$$

$$
+ (\mu + v) \int_{\Omega} \text{div} v \text{div} \phi \, dz = \int_{\Omega} g \cdot \phi \, dz
$$

(44)
holds for any \( \phi \in C_c^\infty(\Omega) \). Without the risk of confusing, we may remove the superposition of \( \psi \). Replace \( \phi \) in (44) by \( \Delta_2^{-h} \phi \), then we have

\[
- \gamma \int_\Omega \text{wdiv}(\Delta_2^{-h} \phi) \, dz + \mu \int_\Omega \nabla \phi : \nabla (\Delta_2^{-h} \phi) \, dz + (\mu + \nu) \int_\Omega \text{div}\text{div}(\Delta_2^{-h} \phi) \, dz \\
= \gamma \int_\Omega (\Delta_2^h w)\text{div}\phi \, dz - \mu \int_\Omega (\Delta_2^h \nabla \phi) : \nabla \phi \, dz - (\mu + \nu) \int_\Omega \text{div}(\Delta_2^h \phi)\text{div}\phi \, dz \\
= - \int_\Omega \partial_1 \phi \cdot (\Delta_2^{-h} \phi) \, dz - \int_\Omega (\nabla \cdot U_0) \cdot (\Delta_2^{-h} \phi) \, dz + \int_\Omega g \cdot \Delta_2^{-h} \phi. \tag{45}
\]

For any interval \( I \subset (0, 1) \), select a cut-off function \( \eta(t) \in C_c^\infty(0, 1) \), such that \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) in \( I \). Substituting \( \phi = \eta^2(z_2) \Delta_2^h \phi \) in (45) and combing with (43), a direct computation shows that

\[
\gamma \int_\Omega (\Delta_2^h w)\text{div}\phi \, dz = \gamma \int_\Omega (\Delta_2^h w)[2\eta(\Delta_2^h \nabla \phi) \cdot \nabla \eta + \eta^2 \text{div}\Delta_2^h \phi] \, dz \\
= \gamma \int_\Omega (\Delta_2^h w)[2\eta(\Delta_2^h \nabla \phi) \cdot \nabla \eta - \eta^2 (\bar{U} + \bar{v}) \partial_1 (\Delta_2^h w)] \, dz \\
+ \gamma \int_\Omega (\Delta_2^h w)[\eta^2 \Delta_2^h f - \eta^2 \partial_1 w \Delta_2^h (\bar{U} + \bar{v})] \, dz \\
= \gamma \int_\Omega (\Delta_2^h w)[2\eta(\Delta_2^h \nabla \phi) \cdot \nabla \eta + \eta^2 \Delta_2^h f - \eta^2 \partial_1 w \Delta_2^h (\bar{U} + \bar{v})] \, dz \\
+ \frac{\gamma}{2} \int_\Omega \partial_1 (\bar{U} + \bar{v}) |\eta \Delta_2^h w|^2 \, ds - \frac{\gamma}{2} \int_{\Gamma_{\text{out}}} (\bar{U} + \bar{v}) |\eta \Delta_2^h w|^2 \, ds, \tag{46}
\]

and

\[
- \mu \int_\Omega \nabla (\Delta_2^h \nabla \phi) : \nabla \phi \, dz - (\mu + \nu) \int_\Omega \text{div}(\Delta_2^h \nabla \phi)\text{div}\phi \, dz \\
= - \mu \int_\Omega \eta^2 |\nabla (\Delta_2^h \nabla \phi)|^2 \, dz - (\mu + \nu) \int_\Omega \eta^2 |\text{div}(\Delta_2^h \nabla \phi)|^2 \, dz \\
+ 2\mu \int_\Omega \eta |\nabla (\Delta_2^h \nabla \phi)| \, dz + (2\mu + \nu) \int_\Omega \eta \text{div}(\Delta_2^h \nabla \phi) \Delta_2^h \nabla \phi \cdot \nabla \eta \, dz, \tag{47}
\]

Combing (45)–(47), we obtain

\[
\int_\Omega \eta^2 |\nabla (\Delta_2^h \nabla \phi)|^2 \leq C(\delta |\nabla (\Delta_2^h \nabla \phi)|^2 + \|v\|^2_{L^2} + \|\partial_2 f\|^2_{L^2} + \|g\|^2_{L^2} + \|w_{\text{in}}\|^2_{L^2(\Gamma_{\text{in}})}) \\
\leq C(|f|^2_{H^1} + |g|^2_{L^2} + \bar{B}_0^2 + \|w_{\text{in}}\|^2_{H^1(\Gamma_{\text{in}})}), \tag{48}
\]

here \( \delta \) is an arbitrary constant. Letting \( h \to 0 \) in (48), we get

\[
\|\eta \partial_2 v\|^2_{H^1(\Omega)} \leq C(|f|^2_{H^1(\Omega)} + \|g\|^2_{L^2(\Omega)} + \|\bar{B}_0\|^2_{L^2(\Gamma_0)} + \|w_{\text{in}}\|^2_{H^1(\Gamma_{\text{in}})}),
\]

denote \( \Omega' = (0, 1) \times I \), which implies that

\[
\|\partial_2 v\|^2_{H^1(\Omega')} \leq C(|f|^2_{H^1(\Omega')} + \|g\|^2_{L^2(\Omega')} + \|\bar{B}_0\|^2_{L^2(\Gamma_0)} + \|w_{\text{in}}\|^2_{H^1(\Gamma_{\text{in}})}). \tag{49}
\]
On the other hand, (43) implies that

$$\|\partial_z w\|_{L^2(\Omega')} \leq C(\|\partial_z v\|_{H^1(\Omega')} + \|f\|_{H^1(\Omega')} + |\partial_z w(0, z_2)|_{L^2(\Gamma^-)}),$$

(50)

which combining with the standard $H^2$ estimates of elliptic system gives the interior estimate of (42) in $\Omega'$.

As to the boundary estimate near $\Gamma^-_0 := \{(z_1, z_2)|0 < z_1 < 1, z_2 = 0\}$, we extend the domain $\Omega$ to $\Omega^\circ = (0, 1) \times (-1, 1)$, and denote the even extension of $w, v_1, v_2$ with respect to $z_1 = 0$ as $\hat{w}, \hat{v}_1, \hat{v}_2$, respectively. More precisely,

$$v_1^*(z_1, z_2) = \begin{cases} v_1(z_1, z_2) & 0 < z_2 < 1, \\ \hat{v}_1(z_1, z_2) = v_1(z_1, -z_2) & -1 < z_2 < 0, \end{cases}$$

$$v_2^*(z_1, z_2) = \begin{cases} v_2(z_1, z_2) & 0 < z_2 < 1, \\ \hat{v}_2(z_1, z_2) = -v_2(z_1, -z_2) & -1 < z_2 < 0. \end{cases}$$

Due to the boundary condition in (29), we have $v_2 = 0$ on $\Gamma^-_0$ and the Navier boundary condition preserve this symmetry, that is to say

$$2\mu \hat{n} \cdot D(v) \cdot \tau + \alpha v \cdot \tau = 2\mu \hat{n} \cdot D(\hat{v}) \cdot \tau + \alpha \hat{v} \cdot \tau \quad \text{on } \Gamma^-_0,$$

where $\hat{n}$ is the outer normal vector of the domain $(0, 1) \times (-1, 0)$ on $\Gamma^-_0$, which implies that

$$\partial_{z_2} v_1^* = -\frac{\partial_z v_1}{\mu} + \alpha v_1 + \partial_1 v_2 \quad \text{on } \Gamma^-_0.$$

Hence the extended function $(v_1^*, v_2^*)$ still belongs to $H^1(\Omega^\circ)$, and it is easy to check that $(v^*, w^*)$ satisfies the system (29) in the senses of $O'(\Omega^\circ)$ with corresponding extension of source terms. It follows that the estimate near the boundary $\Gamma^-_0$ is converted to an interior one, a slight modification of the above proof derives the desired estimates. An identical argument can be applied on $\Gamma^+_0 := \{(z_1, z_2)|0 < z_1 < 1, z_2 = 1\}$ and we get the global regularity as claimed.

### 3.3. $W^{2,p}$ estimates of linear system

The crucial point to establish the $W^{2,p}$ estimates is to find a bound of $\|\partial_z w\|_{L^p}$. A key observation shows that it can be bounded by $\|\text{curl } \partial_z v\|_{L^p}$. To this end, we introduce the following lemma concerning the so-called Bogovskii operator which is proved by Bogovskii in [27].

**Lemma 3.5**: Suppose that $\Omega \subset \mathbb{R}^n$ is starlike with respect to some ball contained in it, and that $1 < p < \infty$ and $n \geq 2$. Then, the exists a constant $C > 0$, depending only on $n, p$ and $\Omega$, such that for any $h \in L^p(\Omega)$ with $\int_\Omega h(x)dx = 0$, there is a vector field $\omega \in W^{1,p}_0$ satisfying

$$\begin{cases} \text{div } \omega = h & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial \Omega. \end{cases}$$

and

$$\|\omega\|_{W^{1,p}(\Omega)} \leq C\|h\|_{L^p(\Omega)},$$

where the constant $C$ depends only on $p$ and $\Omega$.

By constructing a proper test function defined by Bogovskii operator, the following result can be established.
Lemma 3.6: Assume that \((v, w)\) be a solution of (29) with \((f, g, \ddot{B}, w_{in}) \in W^{1,p}(\Omega) \times L^p(\Omega) \times H^{1,2}(\Gamma_0) \times W^{1,p}(\Gamma_{in})\), then for any \(p \in (2, \infty)\) we have

\[
\|\partial_{z_2} w\|_{L^p} + |\partial_{z_2} w|_{L^p(\Gamma_{out})} \\
\leq C(\|\text{curl}\partial_{z_2} v\|_{L^p} + \|f\|_{W^{1,p}} + \|g\|_{L^p} + |\dot{B}|^2_{H^{1,2}(\Gamma_0)} + |w_{in}|_{W^{1,p}(\Gamma_{in})})
\]

(51)

Proof: Let \(\phi = \mathfrak{B}(\partial_{z_2} w|\partial_{z_2} w|^{p-2} - M)\), then by the properties of Bogovskii operator we have

\[
\|\phi\|_{W^{1,p-1}_\Omega(\omega)} \leq C\|\partial_{z_2} w\|_{L^p(\Omega)}^{p-1},
\]

(52)

where \(M = \int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2} \, dz\). Next, differentiate the momentum equation in (29) with respect to \(z_2\) and multiply the resulting equations by \(\phi\) we have

\[
-\gamma \int_\Omega |\partial_{z_2} w|^p \, dz + \gamma M \int_\Omega |\partial_{z_2} w| \, dz + (2\mu + \nu) \int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2} \text{div}\partial_{z_2} v \, dz \\
+ \mu \int_\Omega \partial_{z_2}(\text{curl} v) : \nabla \phi \, dz + (\mu + \nu) M \int_\Omega \text{div}\partial_{z_2} v \, dz + \int_\Omega (\partial_{z_2} \ddot{U})\partial_{z_2} v \cdot \phi \, dz \\
+ \int_\Omega \ddot{U}(\partial_{z_1,z_2} v) \cdot \phi \, dz + \int_\Omega \partial_{z_2} v \nabla U_0 \phi \, dz + \int_\Omega v \cdot \nabla (\partial_{z_2} U_0) \phi \, dz = -\int_\Omega g\partial_{z_2} \phi \, dz
\]

(53)

Consequently,

\[
\gamma \int_\Omega |\partial_{z_2} w|^p \, dz - (2\mu + \nu) \int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2} \text{div}\partial_{z_2} v \, dz \\
\leq (\|\text{curl}\partial_{z_2} v\|_{L^p} + \|g\|_{L^p} + \|v\|_{W^{1,p}})\|\partial_{z_2} w\|_{L^p}^{p-1} \\
+ (\|\partial_{z_2} w\|_{L^1} + \|\nabla^2 v\|_{L^1})\|\partial_{z_2} w\|_{L^p}^{p-1}.
\]

(54)

To control the second term of the l.h.s of (54), we differentiate the mass equation in (29) to obtain

\[
\partial_{z_2}(\ddot{U} + \ddot{v}(1))\partial_{z_1} w + (\ddot{U} + \ddot{v}(1))\partial_{z_1,z_2} w + \text{div}\partial_{z_2} v = \partial_{z_2} f
\]

(55)

and hence

\[
-\int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2} \text{div}\partial_{z_2} v \, dz
\]

\[
= \int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2}[\partial_{z_2} (\ddot{U} + \ddot{v}(1))\partial_{z_1} w + (\ddot{U} + \ddot{v}(1))\partial_{z_1,z_2} w - \partial_{z_2} f] \, dz
\]

\[
= \int_\Omega \partial_{z_2} (\ddot{U} + \ddot{v}(1))\partial_{z_2} w|\partial_{z_2} w|^{p-2} \partial_{z_1} w \, dz - \int_\Omega \partial_{z_2} w|\partial_{z_2} w|^{p-2} \partial_{z_2} f
\]

\[
- \frac{1}{p} \int_\Omega \partial_{z_1} (\ddot{U} + \ddot{v}(1))|\partial_{z_2} w|^p \, dz + \int_{\Gamma_{out}} (\ddot{U} + \ddot{v}(1))|\partial_{z_2} w|^p \, ds - \int_{\Gamma_{in}} (\ddot{U} + \ddot{v}(1))|\partial_{z_2} w|^p \, ds.
\]

(56)

Recall that \(\ddot{U}(z_1, z_2) = \ddot{U} \circ \psi\) and the definition of \(\psi\), due to the smallness of \(\|\partial_{z_1} \ddot{v}(1)\|_\infty\), we have

\[
|\partial_{z_1}(\ddot{U} + \ddot{v}(1))| \leq E.
\]

(57)
The above bound (57) combined with (54) and (56), after using Young’s inequality and (42), implies that
\[
\int_{\Omega} |\partial_2 w|^p \, dz + \int_{\Gamma_{\text{out}}} |\partial_2 w|^p \, ds \\
\leq (\|\text{curl} \partial_2 v\|_{L^p}^p + \delta \|\partial_1 w\|_{L^p}^p + \|g\|_{L^p}^p + \|f\|_{W^{1,p}}^p + \|w_{\text{in}}\|_{W^{1,p}(\Gamma_{\text{in}})})^p, \tag{58}
\]
here \(\delta\) is a small constant depending on \(\|\bar{v}\|_{L^2}\), which yields (51).

The \(W^{2,p}\) estimates are done if \(\|\text{curl} \partial_2 v\|_{L^p}\) is well controlled. Roughly speaking, one can find that \(\text{curl} v\) satisfies a Laplace systems by taking \(\text{curl}\) on the both side of momentum equations in (29). Unfortunately, it is not clear about the behavior of \(\text{curl} v\) near \(\Gamma_{\text{in}}\) and \(\Gamma_{\text{out}}\). We can only expect to have a good interior estimate, but this is not enough. Nevertheless, the authors have proved the following beautiful result by a delicate construction in [22],
\[
\|\text{curl} v\|_{W^{1,p}(\Omega)} \leq C(\epsilon \|v\|_{W^{2,p}(\Omega)} + \|f\|_{W^{1,p}} + \|g\|_{L^p} + \|w_{\text{in}}\|_{W^{1,p}(\Gamma_{\text{in}})}), \tag{59}
\]
where \(\epsilon\) is a sufficiently small constant. Since (29) has exactly the same main terms as the linear system in [22], one checks easily that the estimate (59) still holds true for the system (29) by adapting the constructions developed in [22]. It is immediately to get the \(W^{2,p}\) estimates by using (59).

**Lemma 3.7:** For any \(p \in (2, \infty)\), assume that \((v, w)\) be a solution of (29) with \((f, g, \bar{B}, w_{\text{in}}) \in W^{1,p}(\Omega) \times L^p(\Omega) \times W^{1-\frac{1}{p}, p}(\Gamma_0) \times W^{1,p}(\Gamma_{\text{in}})\), then we have
\[
\|w\|_{W^{1,p}(\Omega)} + \|\partial_1 w\|_{W^{1,p}(\Omega)} + \|v\|_{W^{2,p}(\Omega)} \\
\leq C(\|f\|_{W^{1,p}} + \|g\|_{L^p} + \|\bar{B}\|_{W^{1-\frac{1}{p}, p}(\Gamma_0)} + \|w_{\text{in}}\|_{W^{1,p}(\Gamma_{\text{in}})}), \tag{60}
\]

**Proof:** The same extension introduced in the proof of Lemma 3.4 shows that the boundary estimates can be converted into the interior one, hence we only demonstrate the interior estimates for the convenience. By standard \(W^{2,p}\) estimates of elliptic system, we have
\[
\|v\|_{W^{2,p}(\Omega)} \leq C(\|v\|_{W^{1,p}} + \|g\|_{L^p} + \|\bar{B}\|_{W^{1-\frac{1}{p}, p}(\Gamma_0)} + \|\nabla w\|_{L^p}) \tag{61}
\]
By virtue of mass equation in (29) we have
\[
\|v\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{W^{1,p}} + \|g\|_{L^p} + \|\bar{B}\|_{W^{1-\frac{1}{p}, p}(\Gamma_0)} + \|v\|_{W^{1,p}} + \|\partial_2 w\|_{L^p}) \tag{62}
\]
which combing with the estimates (51) and (59) after using interpolation inequalities to give the desired inequality.

4. Solution of linear system

In this section, the existence of weak solution to the system (29) is given by Galerkin method. Having the weak solution at hand, we can show easily that this solution is also strong if the data have the appropriate regularity.
4.1. Weak solution

Definition 4.1: We call \((w, v)\) is a weak solution of (29) if for any \(\phi \in V\) there holds

\[
\int_{\Omega} [\bar{U}_z w + v \cdot \nabla U_0] \phi \, dz = \gamma \int_{\Omega} w \text{div} \phi \, dz - \int_{\Gamma_0} \alpha(v \cdot \tau)(\phi \cdot \tau) \, ds
+ \int_{\Omega} (2\mu D(v) : \nabla \phi + \nu v \text{div} \phi) \, dz = \int_{\Omega} g \phi \, dz + \int_{\Gamma_0} \bar{B}(\phi \cdot \tau) \, ds.
\]

and for any \(\eta \in \tilde{C}^\infty(\Omega)\), there holds

\[
- \int_{\Omega} \eta \partial_z (\bar{U} + \tilde{v}^{(1)}) w \, dz - \int_{\Omega} \partial_z \eta (\bar{U} + \tilde{v}^{(1)}) w \, dz
= \int_{\Omega} [f - \text{div} v] \eta \, dz + \int_{\Gamma_\text{in}} (\bar{U} + \tilde{v}^{(1)}) w_{\text{in}} \eta \, ds
\]

where \(\eta \in \tilde{C}^\infty(\Omega)\) means that \(\eta \in C^\infty(\Omega)\) and \(\eta|_{\Gamma_{\text{out}}} = 0\).

We want to apply the Galerkin method to prove the existence of weak solution. To this end, we introduce an orthonormal basis of \(\{\omega_k\} \subset V\) and finite-dimensional subspace \(V^N = \{\sum_{i=1}^N c_i \omega_i : c_i \in \mathbb{R}\}\). We look for the approximate velocity field of the form \(v^N = \sum_{i=1}^N c_i \omega_i\). Note that the solution of the continuity equation has been given by the operator \(S\) defined in (31). Now we proceed with the Galerkin scheme. Taking \(g = g^N, v = v^N = \sum_{i=1}^N c_i \omega_i, \phi = \omega_k, k = 1, 2 \cdots N\) and \(w = w^N = S(f^N - \text{div} v^N)\), where \(f^N, g^N\) are orthogonal projections of \(f, g\) on \(V^N\). We arrive at a system of \(N\) equations

\[
B^N(v^N, \omega_k) = 0, \quad k = 1, 2 \cdots N,
\]

where \(B^N : V^N \times V^N \rightarrow \mathbb{R}\) is defined as

\[
B^N(v^N, \phi) = \int_{\Omega} \{v^N \partial_z \phi + (v^N \cdot \nabla \bar{U}_0) \cdot \phi + 2\mu D(v^N) : \nabla \phi + v \text{div} v^N \text{div} \phi\} \, dz
- \int_{\Omega} g^N \cdot \phi \, dz - \gamma \int_{\Omega} S(f^N - \text{div} v^N) \text{div} \phi \, dz
+ \int_{\Gamma_0} [\alpha(v^N \cdot \tau) - \bar{B}](\phi \cdot \tau) \, d\sigma.
\]

Now, if \(v^N\) satisfies (65) for \(k = 1, 2 \cdots N\), then the pair \((\phi, \eta)\) satisfies (29) for \((\phi, \eta) \in V \times \tilde{C}^\infty\) with \(w = w^N\). we call such a pair an approximation solution. In order to show the existence of approximation solution, we apply the following result to finite-dimensional Hilbert space [5].

Lemma 4.2: Let \(X\) be a finite-dimensional Hilbert space and let \(P : X \rightarrow X\) be a continuous operator satisfying

\[\exists M > 0 : (p(\xi), \xi) > 0 \quad \text{for} \quad \|\xi\| = M,\]

then there is at least one \(\xi^*\) such that \(\|\xi^*\| \leq M\) and \(P(\xi^*) = 0\).

The existence of approximation solution is guaranteed by the following lemma.

Lemma 4.3: Let \(f, g \in L^2(\Omega), w_{\text{in}} \in L^2(\Gamma_{\text{in}}), \bar{B} \in L^2(\Gamma_0)\). If \(\|\tilde{v}\|_{W^{2,p}}\) is small enough and the viscous coefficient \(\mu > \frac{1}{2\pi^2}\), then there exists \(v^N \in V^N\) fulfilling (65). Furthermore, there exists a positive constant \(\tilde{M}\) indecent of \(N\) such that

\[
\|v^N\|_{H^1} \leq \tilde{M}.
\]
**Proof:** Define $P^N : V^N \to V^N$ as
\[ P^N(\xi^N) = \sum_{k=1}^N B^N(\xi^N, \omega_k) \omega_k \quad \text{for} \ \xi^N \in V^N. \] (67)

According to Lemma 4.3, we need to show that $(p(\bar{x}^N), \bar{x}^N) > 0$ on some sphere in $V^N$. Since $B^N(\cdot, \cdot)$ is linear with respect to the second variable, we have
\[
(p^N(\bar{x}^N), \bar{x}^N) = B^N(\bar{x}^N, \bar{x}^N) = 2\mu \int_{\Omega} |D(\bar{x}^N)|^2 \, dz + \nu \int_{\Omega} \left| \frac{\partial}{\partial z_1}(\bar{\psi} \cdot \bar{x}^N) \right|^2 \, dz - \frac{1}{2} \int_{\Omega} \partial_{z_1}(\bar{U} \cdot \psi \bar{x}^N) \, dz - \frac{1}{2} \int_{\Omega} B\bar{x}^N \cdot \tau + \int_{\Omega} \alpha|\bar{x}^N|^2 \, ds - \gamma \int_{\Omega} S(\bar{x}^N - \bar{\xi}^N) \, div \bar{x}^N. \] (68)

Due to Korn inequality similar as in the proof of Lemma 3.3, we only need to find a bound on the last term in (68). Denote $\eta^N = S(\bar{x}^N - \bar{\xi}^N)$, then
\[
\int_{\Omega} \eta^N \, div \bar{x}^N = \int_{\Omega} f^N \eta^N \, dz - \int_{\Omega} (\bar{U} + \bar{\psi}^{(1)}) \partial_{z_1} \eta^N \eta^N \, dz,
\]
the first term is controlled by
\[
\int_{\Omega} f^N \eta^N \, dz \leq \|f\|_{L^2(\Omega)} \|\eta^N\|_{L^2} \leq \|f\|_{L^2} (\|\bar{x}^N\|_{H^1} + \|w_{in}\|_{L^2}) \] (69)

where we have used (32). With the second integral, we have
\[
- \int_{\Omega} (\bar{U} + \bar{\psi}^{(1)}) \partial_{z_1} \eta^N \eta^N \, dz = \frac{1}{2} \int_{\Gamma_{in}} (\bar{U} + \bar{\psi}^{(1)})(\eta^N)^2 \, d\sigma - \frac{1}{2} \int_{\Gamma_{out}} (\bar{U} + \bar{\psi}^{(1)})(\eta^N)^2 \, d\sigma \] 
\[+ \frac{1}{2} \int_{\Gamma_{in}} \partial_{z_1} (\bar{\psi} + \bar{\psi}^{(1)})(\eta^N)^2 \, d\sigma \leq C|w_{in}|_{L^2}^2 + E\|\eta^N\|_{L^2}^2.
\]

Combine all those estimates together, we get
\[
(p^N(\bar{x}^N) \geq C(\|\bar{x}^N\|_{H^1}^2 - D \|\bar{x}^N\|_{H^1}^2 - D^2),
\]
where $D = \|f\|_{L^2} + \|g\|_{V^*} + \|\tilde{B}\|_{L^2(\Gamma_{in})} + \|w_{in}\|_{L^2(\Gamma_{in})}$. Hence there exists a constant $M$ such that $p^N(\bar{x}^N) > 0$ for some $\|\bar{x}^N\| = M$. Applying Lemma 4.3, we conclude that there is a $\bar{x}^* \in V^*$ such that $p^N(\bar{x}^*) = 0$ and $\|\bar{x}^*\| \leq M$. Finally, since $\{\omega_k\}$ is the basis of $V^N$, $p^N(\bar{x}^*) = 0$ implies $B^N(\bar{x}^*, \omega_k) = 0$, that is to say $\bar{x}^*$ is an approximation solution. 

Since the system is linear, the uniform estimate (66) immediately gives the existence of weak solution. The result is as follows.

**Lemma 4.4:** Let $f, g \in L^2(\Omega)$, $w_{in} \in L^2(\Gamma_{in})$, $\tilde{B} \in L^2(\Gamma_0)$. If $\|\bar{\psi}\|_{W^{2,\rho}}$ is small enough and the viscous coefficients $\mu > \frac{1}{2\pi}$, then the system (29) has a unique weak solution $(\mathbf{v}, w) \in H^1 \times L^\infty(L^2)$, which satisfies the estimate (34).
Proof: The above lemma shows that the approximation solution \( v^n \) satisfying
\[
\|v^n\|_{H^1} \leq M
\]
which, combined with (32), gives
\[
\|v^n\|_{H^1} + \|w\|_{L^\infty(L^2)} \leq M.
\]
Hence there exists a pair \((v, w) \in H^1 \times L^\infty(L^2)\) such that
\[
v^n \rightharpoonup v \quad \text{in} \quad H^1,
\]
and
\[
w^n \rightharpoonup w \quad \text{in} \quad L^\infty(L^2).
\]
Since the system (29) is linear, passing to the limit in (63) for \(v^n, w^n\), it follows that \(v\) satisfies (63) with \(w\). Similarly, taking the limit in (64) it is easy to verify \(w = S(f - \text{div} v)\), the boundary condition on \(w\) is guaranteed to hold by the definition of the operator \(S\). It is obvious that \((v, w)\) satisfies the estimate (34). The proof is thus completed.

4.2. Strong solution

In this section, we will show that the weak solution given above is also strong by using symmetric extension methods.

Lemma 4.5: Let \( f \in W^{1,p}(\Omega), g \in L^p(\Omega), w_{in} \in W^{1,p}(\Gamma_{in}) \bar{B} \in W^{1-p,p}(\Gamma_0) \). If \( \|\bar{v}\|_{W^{2,p}} \) is small enough and the viscous coefficients \( \mu > \frac{1}{2\pi^2} \), then the system (29) has a unique strong solution \((v, w) \in W^{2,p} \times W^{1,p}\), which satisfies the estimate (60).

Proof: Since (29) is a linear system, the a priori estimate (60) will deduce the regularity of the weak solution in the interior. In order to deal with the singularity of the boundary at the junctions of \(\Gamma_0\) with \(\Gamma_{in}\) and \(\Gamma_{out}\), we still apply the symmetric extension methods introduced in the proof of lemma (3.4). Hence we can extend the weak solution on the negative values of \(z_2\), and use the estimate (60) show that the extended solution has the same regularity.

5. Proof of main result

According to the analysis in Section 4, we can define the solution operator of linear systems as
\[
T: W^{2,p}(\Omega) \times W^{1,p}(\Omega) \rightarrow W^{2,p}(\Omega) \times W^{1,p}(\Omega)
\]
(70)
such that \((v^{n+1}, w^{n+1}) = T(v^n, w^n)\) is the solution of the linear system (28). In the following, we want to show that \(T\) is a contraction mapping in some proper subspace, hence \(T\) has a unique solution of (24).

Lemma 5.1: There is a positive constant \(R > 0\) such that the operator \(T\) mapping \(B_R\) into itself provided \(D_0\) is small enough, where
\[
B_R = \{(v, w) \in W^{2,p}(\Omega) \times W^{1,p}: \|v\|_{W^{2,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)} \leq R\}.
\]
Lemma 5.3: Therefore, the proof is completed.

Proof: By (26) and (60) we have
\[
\|T(\mathbf{v}^n, \mathbf{w}^n)\|_{W^{2,p} \times W^{1,p}} = \|(v^{n+1}, w^{n+1})\|_{W^{2,p} \times W^{1,p}} \\
\leq C\left(\|v^n\|_{2,p} + \|w^n\|_{1,p}\right)^3 + \left(\|v^n\|_{2,p} + \|w^n\|_{1,p}\right)^2 \\
+ E(\|v\|_{W^{2,p}} + \|w\|_{W^{1,p}}) + \|\tilde{u}\|_{W^{2,p}} \\
\leq C\left(R^2 + R^2\right) + E(D_0) \tag{71}
\]
for \(R < 1\), and hence for \(R \leq 2E(D_0)\) with \(E(D_0) \leq \frac{1}{16C}\), we have
\[
\|(v^{n+1}, w^{n+1})\|_{W^{2,p} \times W^{1,p}} \leq 2E(D_0),
\]
the proof thus is completed. 

Remark 5.2: The same proof also shows that \(\|\partial_z w_{n+1}\|_{1,p} \leq E(D_0)\), thanks to the second term in the r.h.s of (60).

In the next lemma, we will show that \(T\) has a contraction property.

Lemma 5.3: There is a constant \(0 < E_0 < 1\) independent of \(n\), such that
\[
\|v^{n+1} - v^{m+1}\|_{2,p;\Omega} + \|w^{n+1} - w^{m+1}\|_{1,p;\Omega} \leq E_0[\|v^n - v^m\|_{2,p;\Omega} + \|w^n - w^m\|_{1,p;\Omega}] \tag{72}
\]
holds for all \((v^n, w^n) \in B_R\).

Proof: Making difference to derive the system of \((v^{n+1} - v^{m+1}, w^{n+1} - w^{m+1})\), we have
\[
\begin{align*}
((\tilde{U} + (\mathbf{v}^n + \tilde{u}^{(1)})) \circ \psi_{v^n + \tilde{u}})\partial_z (w^{n+1} - w^{m+1}) + \text{div}_z(v^{n+1} - v^{m+1}) \\
= \tilde{F}(v^n, w^n) - \tilde{F}(v^m, w^m) \\
- \partial_z(w^{m+1}[((\tilde{U} + (\mathbf{v}^n + \tilde{u}^{(1)})) \circ \psi_{v^n + \tilde{u}}) - ((\tilde{U} + (\mathbf{v}^m + \tilde{u}^{(1)})) \circ \psi_{v^m + \tilde{u}})], \\
(\tilde{U} \circ \psi_{v^n + \tilde{u}})\partial_z(v^{n+1} - v^{m+1}) + (v^{n+1} - v^{m+1}) \cdot \nabla_z U_0 + \gamma \nabla_z(v^{n+1} - w^{m+1}) \\
- \mu \Delta_z(v^{n+1} - v^{m+1}) - (\mu + \nu)\nabla \text{div}_z(v^{n+1} - v^{m+1}) = \tilde{G}(v^n, w^n) - \tilde{G}(v^m, w^m) \\
- \tilde{v}^{m+1}(\tilde{U} \circ \psi_{v^n + \tilde{u}} - \tilde{U} \circ \psi_{v^m + \tilde{u}}),
\end{align*}
\]
\[
v^{n+1} - v^{m+1} = 0, \\
w^{n+1} - w^{m+1} = 0, \\
(v^{n+1} - v^{m+1}) \cdot \mathbf{n} = 0, \\
2\mu \mathbf{n} \cdot D_z(v^{n+1} - v^{m+1}) \cdot \tau + \alpha(v^{n+1} - v^{m+1}) \cdot \tau = 2\mu \mathbf{n} \cdot [R(v^m, D) - R(v^n, D)] \cdot \tau.
\]
Applying \(W^{2,p}\) estimates to the (73), we obtain
\[
\begin{align*}
\|v^{n+1} - v^{m+1}\|_{2,p;\Omega} + \|w^{n+1} - w^{m+1}\|_{1,p;\Omega} \\
\leq C\|\partial_z(w^{m+1}[((\tilde{U} + (\mathbf{v}^n + \tilde{u}^{(1)})) \circ \psi_{v^n + \tilde{u}}) - ((\tilde{U} + (\mathbf{v}^m + \tilde{u}^{(1)})) \circ \psi_{v^m + \tilde{u}})]\|_{1,p} \\
+ \|\tilde{F}(v^n, w^n) - \tilde{F}(v^m, w^m)\|_{1,p} + \|v^{m+1}(\tilde{U} \circ \psi_{v^n + \tilde{u}} - \tilde{U} \circ \psi_{v^m + \tilde{u}})\|_p \\
+ \|\tilde{G}(v^n, w^n) - \tilde{G}(v^m, w^m)\|_p + 2\mu \mathbf{n} \cdot [R(v^m, D) - R(v^n, D)] \cdot \tau\|_{1-\frac{p}{p;\Gamma_0}} \\
:= I_1 + I_2 + I_3 + I_4 + I_5 \tag{74}
\end{align*}
\]
Estimate of $I_1$. By the definition of $\psi$, it continuously dependent on the parameters $v$, which implies that
\[
|\psi_{v^m+\bar{u}} - \psi_{v^m+\bar{u}}| \leq C|v^n - v^m| \tag{75}
\]
for some positive constant $C$. Hence,
\[
|(v^n + \bar{u})^{(1)} \circ \psi_{v^n+\bar{u}} - (v^m + \bar{u})^{(1)} \circ \psi_{v^m+\bar{u}}| \leq C||\nabla (v^n + \bar{u})^{(1)}||v^n - v^m| + |v^n - v^m|| \tag{76}
\]
Consequently,
\[
I_1 \leq \|\partial_2 w^{m+1}\|_{1,p}\|v^n - v^m\|_{2,p} \leq E(D_0)\|v^n - v^m\|_{2,p} \tag{77}
\]
It is easy to show that the same estimate holds for $I_3$.

Estimates of $I_2$. Note that (27) implies that
\[
|R(\cdot, \partial)| \leq E\|1_{1,p}, \tag{78}
\]
\[
|R(\cdot, \partial^2)| \leq E\|1_{2,p}.
\]
Hence, to show the bounds of $I_2$, it is sufficient to estimate $\|F(v^n, w^n) - F(v^m, w^m)\|_{1,p}$.

\[
F(v^n, w^n) - F(v^m, w^m) = w^n \text{div}(v^n + \bar{u}) - w^m \text{div}(v^m + \bar{u}) = w^n \text{div}v^n - w^m \text{div}v^m + (w^n - w^m)\text{div}\bar{u}, \tag{79}
\]
and
\[
w^n \text{div}v^n - w^m \text{div}v^m
= w^n[\text{div}_z v^n + R(v^n, \text{div})] - w^m[\text{div}_z v^m + R(v^m, \text{div})]
= (w^n - w^m)\text{div}v^n + w^m \text{div}(v^n - v^m)
= (w^n - w^m)R(v^n, \text{div}) + w^mR(v^n - v^m, \text{div}), \tag{80}
\]
then it is obtained that
\[
\|F(v^n, w^n) - F(v^m, w^m)\|_{1,p} \leq C(D_0)(\|w^n - w^m\|_{1,p} + \|v^n - v^m\|_{2,p}) \tag{81}
\]
Estimates of $I_4$. Due to the same reason, we just need to estimate $\|G(v^n, w^n) - G(v^m, w^m)\|_p$. For the convenience, denote $\delta P'(w) = P'(w+1) - P'(w)$, then
\[
\delta P'(w^n)\nabla_x w^n - \delta P'(w^m)\nabla_x w^m
= \delta P'(w^n)\nabla_x (w^n - w^m) - \nabla_x w^m(\delta P'(w^m) - \delta P'(w^n))
= \delta P'(w^n)[\nabla_z (w^n - w^m) + R(w^n - w^m, \nabla)]
+ [\nabla_z w^m + R(w^m, \nabla)](\delta P'(w^m) - \delta P'(w^n)) \tag{82}
\]
which gives
\[
\|\delta P'(w^n)\nabla_x w^n - \delta P'(w^m)\nabla_x w^m\|_p \leq E(\|w^n\|_{1,p} + \|w^m\|_{1,p})\|w^n - w^m\|_{1,p} \tag{83}
\]
The estimates of reminder terms are similar, so we show one of them as an example.

\[ w^n \nabla x^n - w^m \nabla x^m = w^n \nabla x^n - [\nabla z v^n + R(v^n, \nabla)] - w^m \nabla x^m - [\nabla z v^m + R(v^m, \nabla)] \]

\[ = (w^n - w^m) \nabla x^n + w^m (v^n - v^m) \cdot \nabla z (v^n - v^m), \]

\[ + (w^n - w^m) \nabla x^n \cdot R(v^n, \nabla) + w^m (v^n - v^m) \cdot R(v^m, \nabla) + w^m v^m R(v^n - v^m, \nabla), \]

(84)

then, we have

\[ \| G(v^n, w^n) - G(v^m, w^m) \|_p \leq C(D_0)(\| w^n - w^m \|_{1,p} + \| v^n - v^m \|_{2,p}). \]  

(85)

Estimates of \( I_5 \).

\[ \| 2 \mu n \cdot [R(v^m, D) - R(v^n, D)] \cdot r \|_{1, \frac{1}{p}, p; \Gamma_0} \]

\[ \leq \| R(v^m - v^n, D) \|_{1,p} \]

\[ \leq C(E) \| v^n - v^m \|_{2,p}. \]

(86)

Due to the smallness assumption of \( C(\cdot) \), one can choose a small constant \( 0 < E_0 < 1 \) independent of \( n \) fulfilling (72).

**Proof:** we have proved that the operator \( T \) is a contraction mapping on \( B_R \in W^{2,p} \times W^{1,p} \) for some small \( R = R(D_0) \). Hence the Banach fixed-point theorem gives existence of a unique fixed point in the ball \( B_R \). By the definition of \( T \), the fixed point is exactly the solution of the system (24). Finally, we can change variables to the original coordinate, and in \( x \)-coordinates our solution satisfies the system (6).

The proof is thus completed.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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