Non-Abelian Monopole Equations with Zero Curvature and Self-Dual Yang-Mills Theories

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Abstract
A version of non-Abelian monopole equations is explored through dimensional reductions, with often the addition of algebraic conditions. On zero curvature spaces, spinor related extensions of integrable systems have been generated, and certain reduced one-dimensional systems have been discussed with respect to integrability, as well as solutions found.

1 Introduction
The Abelian monopole, or Seiberg-Witten, equations have been found in relation with Donalson-Witten theories, which themselves can be interpreted as twisted \( N = 2 \) supersymmetric Yang-Mills systems (see for instance refs [1, 2, 3, 4]). These monopole equations can be formulated through a topological action [5, 6] and, as for the Donalson-Witten theories, can be useful in describing invariants as well as aspects of the topology of four manifolds [1, 2, 7].

Versions of extensions of the Seiberg-Witten equations to non-Abelian monopole equations have been obtained (see refs [8, 9, 10]). For the version of refs [9, 8], a topological action based on the Mathai-Quillen formalism has been shown, and found to be described as twisted \( N = 2 \) supersymmetric gauge theories with matter hypermultiplet [6, 8]. Topological Yang-Mills theories, or Donaldson-Witten theories, are known to explore the moduli space of (anti-) self-dual Yang-Mills equations, which are completely solvable. Moreover, the Abelian and non-Abelian monopole equations correspond to equations probed by the (above-mentioned) topological actions, and in the limit of vanishing spinors, coincide with the (anti-) self-dual Yang-Mills equations. One can also mention that Abelian monopole equations have been linked to integrable systems via prepotentials (see for example, refs [11] and [12]).
In what follows, the main objective is to present first steps in exploring non-
Abelian monopole equations, for which a Mathai-Quillen interpretation exists [6, 8], with respect to integrability, partial or complete, as well as finding solutions.

The method of reductions (e.g. [13]), here more precisely, dimensional re-
ductions accompanied by certain sets of algebraic constraints, of non-Abelian
monopole equations with gauge group $SU(n)$ is used. This approach has also
been carried out to exhibit solutions and show correspondences for the Abelian
monopole equations ([14, 15, 16, 17, 18]). For their integrability behaviour, in
addition to the Painlevé PDE test, it is for example of interest to recall that the
property of Painlevé has been conjectured to appear in the reductions of com-
pletely integrable systems (Painlevé ODE Test) [19], a result which has been
proved for certain cases. Thus, one could for instance imagine checking the
reductions of non-Abelian monopole equations for such property.

2 Non-Abelian Monopole Equations

On a four-dimensional oriented, closed manifold $M$, endowed with a Riemannian
metric $g$, a spin connection $\omega$ is defined. (When no spin structure exists, a spin
$c$ structure is introduced instead, but only spin manifold are here con-
sidered.) On the tensor product of the Dirac bundle $S = S^+ \oplus S^-$, with an associated bundle $E$ to a (non-Abelian) structure group $G$ and base manifold $M$, one provides
complete covariant derivatives, denoted $D_\mu$, on multiplets of (commuting) Weyl
spinors, $M^i_\alpha \in \Gamma(S^+ \otimes E), i = 1, \ldots, N$, transforming under the
$N$-dimensional representation of $G$ :

$$D_\mu M^i_\alpha = \partial_\mu M^i_\alpha - \frac{i}{2} \omega^m \eta^\alpha \beta M^i_\beta + i A^j_\mu M^j_\alpha,$$

where the connection $A$ has components $A^j_\mu$ on $E$. (Notations are detailed in
refs [6, 8].)

The non-Abelian equations on the spin manifold $M$ can then be expressed
with respect to a Hermitian basis $\{T^a\}, a = 1, \ldots, n^2 - 1$ of the Lie algebra of
$SU(n)$ as :

$$\left(F^{+ a}\right) + i \tilde{M}^i_\alpha (T^a)^{ij} M^j_\beta = 0,$$

$$D^\alpha M_\alpha = 0,$$

which involve the self-dual gauge field strength $F^{+ a}$, and the positive chirality
(Weyl) commuting spinor $M_\alpha$. Also : $M^{* a} = (a^*, b^*)$, $\tilde{M}_\alpha = M^\beta (s_2)_{\alpha \beta}$, and

$$M^i_\alpha M^j_\beta = M^i_\alpha M^j_\beta + M^j_\alpha M^i_\beta.$$

The Dirac operator can be expressed as follows in terms of the tetrads $e^{m\mu}$ on $M$ and $\Sigma_0 = \sigma_0, \Sigma_i = i \sigma_i, i = 1, 2, 3$ :

$$D^\alpha = (\Sigma_m)^\alpha e^{m\mu} D_\mu,$$

Let us add that $SU(n)$ gauge transformations will preserve the set of equations (2,3). There is moreover a corresponding topological action for the above
non-Abelian monopole equations \[6, 8\], preserved by \(Q\)– (or BRST-like) transformations.

Under reductions, sets of \(Q\)– transformations, left untouched by the symmetry or algebraic conditions, will be allowed if they commute with translations along the coordinates \(x^\mu\) \[20\]. Rotations can be such symmetries leaving (residual) \(Q\)– transformations. However, the presence of rotations in the isotropy subgroup could impose vanishing invariant Weyl spinors \(M_i^\alpha\), and therefore the reduced systems probed by the reduced topological action would correspond to the reduced (anti-) self-dual Yang-Mills equations.

3 Example of Reduced Non-Abelian Monopole Equations

In this example, reductions by translations along the coordinates \(x^1, x^2, x^3\) are performed. The reduced equations have the following general form with respect to a Hermitian basis of the Lie algebra of the gauge group \(SU(n)\), and with gauge choice \(A_0 = 0\):

\[
\begin{align*}
(D_0A_1 + i[A_2, A_3])^a &= (a^{ij}(T^a)^{ij}b^j + b^{ij}(T^a)^{ij}a^j), \\
(D_0A_2 + i[A_3, A_1])^a &= -i(a^{ij}(T^a)^{ij}b^j - b^{ij}(T^a)^{ij}a^j), \\
(D_0A_3 + i[A_1, A_2])^a &= (a^{ij}(T^a)^{ij}a^j - b^{ij}(T^a)^{ij}b^j),
\end{align*}
\]

(5)

for the reduced equations from the (anti-) self-dual Yang-Mills system with spinor extensions. For the Dirac equation in the background of the gauge fields, one finds:

\[
\begin{bmatrix}
D_0 - A_3 & -A_1 + iA_2 \\
-A_1 - iA_2 & D_0 + A_3
\end{bmatrix}^{ij}
\begin{bmatrix}
[a]^j \\
[b]^j
\end{bmatrix} = 0,
\]

(6)

where the covariant derivative \(D_\mu\) involves only the spin connection. The whole system of equations (5,6) could be seen (since in 1d) as a “spinor” extension of the Nahm’s equations.

If one chooses in addition to translation invariance, the conditions \(A_2 = A_3 = 0\), and reverts the gauge choice on \(A_0\), to put: \(A_0 = -iN\), and \(A_1 = L\), with the constraint on spinor contribution components: \(a = \pm b\), then a spinor extended form of integrable systems related to Lax pairs \((L, N)\), is derived:

\[
\begin{align*}
\left(\frac{d}{dx} + M, L\right)^a &= \pm 2a^i T^a a, \\
\left(\frac{d}{dx} + M \mp L\right)^{ij} a^j &= 0,
\end{align*}
\]

(7,8)

where (7) is a Lax pair equation with spinor contributions. Different examples of reductions can be found in ref. \[21\].
4 Solution on Flat Space

In this section, one example of solution to a set of reduced non-Abelian monopole equations on flat space is shown. The o.d.e’s are obtained by imposing $SU(2)$ as gauge group, along with the following gauge fields:

$$A_1 = (-X)\sigma_1, \quad A_2 = (-X)\sigma_2, \quad A_3 = (-Y)\sigma_3,$$

(9)

where $X, Y$ are functions of $x^0$ only. The spinor contributions are given as: $a^1 = b^2 = A$, and $a^2 = b^1 = B$, where $A$ and $B \in \mathbb{R}$.

This system of o.d.e’s can be interpreted as an extension of the $SU(2)$ Toda integrable equations, which can be written as:

$$d_0 Y + 2X^2 = 2(B^2 - A^2),$$

$$d_0 X + 2XY = -2(A^2 + B^2),$$

$$d_0 X + 2XY = 2(A^2 - B^2),$$

(10)

deduced from the self-dual gauge fields equations; and for its reduced Dirac part:

$$(d_0 + Y)A = 0; \quad (d_0 + 2X - Y)B = 0.$$

(11)

Then, from the solution $A = 0$, and the supplementary condition: $X = -Y$, one derives that:

$$d_0 Y + 2Y^2 = 2B^2,$$

$$d_0 B - 3YB = 0.$$

(12)

(13)

The latter equations (12,13) lead to the following o.d.e. for $Y$:

$$d_0^2 Y - 2Yd_0 Y - 12Y^3 = 0,$$

(14)

for which solutions can be found in terms of elliptic integrals, but which has not been found to be of the Painlevé type. When the spinor contribution $B$ vanishes, the corresponding equation for $Y$ is of the Painlevé type, as expected from the integrability of the self-dual Yang-Mills equations.

5 Conclusion

More examples of reductions and solutions are presented in ref. [21]. Further developments could be for instance in the direction of non-Abelian monopole equations on curved spaces, whether or not a spin structure exists (e.g. as generalizations of compactifications [22]). Their relations to integrable systems, as extended systems based on spinor components, and the study of their integrability property could also be the focus of different activities.

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