TANGENT SPACES TO MOTIVIC COHOMOLOGY GROUPS
ON $CH^\bullet(X, 1)$ AND $CH^\bullet(X, 2)$

SEN YANG

Abstract. Using higher K-theory and tensor triangular geometry, we propose K-theoretic higher Chow groups at position 1 and 2, $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$, of derived categories of noetherian schemes and their Milnor variants for regular schemes and infinitesimal thickenings. We discuss functoriality and show that our higher Chow groups agree with the classical ones [6] for regular schemes.

As an application, we define tangent spaces to our higher Chow groups as usually and identify them with cohomology groups of absolute differentials. Moreover, Combining our results with Green-Griffiths’ results on tangent spaces to cycles [12], we put a concrete geometric meaning to the tangent space to $CH^2(X, 1)$, where $X$ is a smooth projective surface over a field $k$, $chark = 0$.

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1. Introduction

Motivic cohomology, relating algebraic geometry with algebraic K-theory, is in the center of modern study of algebraic cycles. Voevodsky’s construction of motivic cohomology via a triangulated category of motives, together with Bloch’s higher Chow groups, has put a solid foundation of the theory of motivic cohomology.

For a regular scheme $X$ over a field $k$, Voevodsky defines motivic cohomology $H^p_M(X, \mathbb{Z}(q))$ and proves the following identification (for $k$ a perfect field)

$$H^p_M(X, \mathbb{Z}(q)) = CH^q(X, 2q - p),$$

where $CH^q(X, 2q - p)$ is Bloch’s higher Chow group, recalled in section 2.1. After ignoring torsion, one has further identification

$$H^p_M(X, \mathbb{Z}(q)) \otimes \mathbb{Q} = CH^q(X, 2q - p) \otimes \mathbb{Q} = K_{2q-p}^q(X) \otimes \mathbb{Q},$$

where $K_{2q-p}^q(X)$ is the eigen-space of Adams’ operation $\psi^k = k^q$.

However, this theory of motivic cohomology has one obvious deficiency: it fails to detect the nilpotent which is very important for studying deformation problems. For example, for $X$ a regular scheme over a field $k$, the $j$-th infinitesimal thickening $X_j$, $X_j = (X, O_X[t]/(t^{j+1}))$, is a typical non-reduced scheme. Motivic cohomology can’t read the difference between $X$ and $X_j$.

As the first attempt to understand what motivic cohomology of infinitesimal thickening might mean, Bloch-Esnault [8] introduce additive version of higher Chow groups and show that $0$-cycles in this theory coincides with absolute differentials, see section 2.2. More results on this direction have been proved by Krishna-Levine, Park and Rülling.

In [7], Bloch suggests the following question.

**Question 1.1.** Assuming one has a good definition of motivic cohomology (of infinitesimal thickening), still denoted $H^p_M(X, \mathbb{Z}(q))$, what should the tangent space

$$TH^p_M(X, \mathbb{Z}(q)) := Ker\{H^p_M(X \times speck[\varepsilon], \mathbb{Z}(q)) \xrightarrow{\varepsilon=0} H^p_M(X, \mathbb{Z}(q))\}$$

mean? Here $speck[\varepsilon]$ denotes the dual number, $\varepsilon^2 = 0$. 
According to identification (1.1), we rewrite this question in term of higher Chow groups as follows.

**Question 1.2.** Can one have a good definition of higher Chow groups of infinitesimal thickening? If so, what should the tangent space

\[ TCH^q(X, p) := \text{Ker}\{CH^q(X \times \text{spec}[\varepsilon], p) \to CH^q(X, p)\} \]

mean?

In this paper, we focus on question 1.2 and propose an answer to it for \( p = 1, 2 \).

Our starting point is to look at the derived category \( D_{\text{perf}}(X) \) obtained from the exact category of perfect complexes of \( O_X \)-modules. It is obvious that the derived category \( D_{\text{perf}}(X) \) is different from \( D_{\text{perf}}(X_j) \).

In our approach, the derived category \( D_{\text{perf}}(X) \) is considered as a tensor triangulated category, see example 2.12. The following beautiful reconstruction theorem [1,Theorem 6.3], due to Balmer, says that a scheme can be reconstructed from its associated tensor triangulated category \( D_{\text{perf}}(X) \).

**Theorem 1.3.** [1]

Let \( X \) be a quasi-compact and quasi-separated scheme. We have an isomorphism \( \text{Spec}(D_{\text{perf}}(X)) \simeq X \) of ringed spaces.

Since one can reconstruct the scheme \( X \) from the tensor triangulated category \( D_{\text{perf}}(X) \), then it may be possible to define higher Chow groups in terms of \( D_{\text{perf}}(X) \), considered as a tensor triangulated category. The idea of constructing Chow groups in terms of \( D_{\text{perf}}(X) \) has been proposed in [3] by P.Balmer and followed by S.Klein [15] and the author [33]. Following the similar idea and guided by Bloch-type formulas, theorem 2.6, we propose new definitions of higher Chow groups at position 1 and 2. The definition of higher Chow groups at general position \( n \) remains to search.

Our main results are as follows.

- **Definitions.** We propose definitions of K-theoretic higher Chow groups, \( CH^q(D_{\text{perf}}(X), 1) \) and \( CH^q(D_{\text{perf}}(X), 2) \), of derived categories of noetherian schemes, see definition 3.3. And we also define their Milnor variants for regular schemes and infinitesimal thickenings, see definition 3.16 and 3.23.
- **Functoriality.** Flat pull-back and proper push-forward are discussed in section 3.3.
• Agreement. We show that our (Milnor)higher Chow groups of derived categories agree with the classical ones for regular schemes, see theorem 3.4, theorem 3.5 and theorem 3.24.

• Bloch-type formulas. We extend Bloch-type formulas from regular schemes to their thickenings, see theorem 3.19 and theorem 3.25.

• We define tangent spaces to our higher Chow groups as usually, while the classical higher Chow groups can’t do. We also identify tangent spaces to our Chow groups with cohomology groups of absolute differentials. See definition 3.20, theorem 3.21, definition 3.26 and theorem 3.27.

• Combining with Green-Griffiths’ work on studying tangent spaces to algebraic cycles, we put a concrete geometric meaning of the tangent space to $CH^2(X, 1)$ in section 4, where $X$ a smooth projective surface.

• We discuss the choice of tangent spaces to higher Chow groups in section 5 by looking at 3-folds.

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Notations and conventions. $X$ is a $d$-dimensional noetherian scheme of finite type over a field $k$, if not stated otherwise. $Speck[\varepsilon]$ denotes the dual number, $\varepsilon^2 = 0$.

2. Background

2.1. (Additive)Higher Chow group. We recall higher Chow groups and additive higher Chow groups briefly in this subsection.

Higher Chow group. Higher Chow groups of $X$, an algebraic analogue of simplicial homology, are defined in [6] via the cosimplicial scheme

$$\triangle^* : \triangle^0 \Rightarrow \triangle^1 \Rightarrow \ldots; \triangle^n := Speck[t_0, ..., t_n]/(\sum t_i - 1).$$
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To be precise, for the $n$-simplex $\Delta^n := Spec(k[t_0, \ldots, t_n]/(\sum t_i - 1)$, Bloch introduces
\[
Z^p(X, n) = \{ \xi \in Z^p(X \times \Delta^n) \mid 
\xi \text{ meets all faces } \{t_{i_1} = \cdots = t_{i_l} = 0, l \geq 1 \} \text{ properly} \}.
\]
Now set $\partial_j : Z^p(X, n) \to Z^p(X, n - 1)$, the restriction of $j$-th face given by $t_j = 0$. The boundary map
\[
\partial = \sum_{j=0}^n (-1)^j \partial_j : Z^p(X, n) \to Z^p(X, n - 1)
\]
satisfies $\partial^2 = 0$. So one obtains the following complex $(Z^p(X, \bullet), \partial)$:
\[
\cdots \to Z^p(X, n + 1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n - 1) \to \cdots
\]

Definition 2.1. [6]
Higher Chow group $CH^p(X, n)$ is defined to be homology of $(Z^p(X, \bullet), \partial)$ at position $n$
\[
CH^p(X, n) := H_n(Z^p(X, \bullet), \partial).
\]
Bloch proves several useful theorems for higher Chow groups, including homotopy, localization, Chern classes, Gersten conjecture etc. We collect the following ones for later use.

Theorem 2.2. [6]
Let $X$ be quasi-projective over a field $k$,
1). Local to global spectral sequence: there is a spectral sequence
\[
E_2^{p,q} = H^p(X, CH^r(O_X, -q)) \Rightarrow CH^r(X, -q - p),
\]
where $CH^r(O_X, -q)$ is the Zariski sheaf associated to the presheaf
\[
U \to CH^r(U, -q).
\]
2). Gersten conjecture. For $X$ regular over a field $k$, there exists flasque resolutions
\[
0 \to CH^p(O_X, r) \to CH^p(k(X), r) \to \bigoplus_{x \in X^{(1)}} i_*CH^{p-1}(k(x), r - 1) \to \cdots
\]
\[
\cdots \to \bigoplus_{x \in X^{(p-3)}} i_*CH^3(k(x), r - (p - 3)) \to \bigoplus_{x \in X^{(p-2)}} i_*CH^2(k(x), r - (p - 2)) \to \bigoplus_{x \in X^{(p-1)}} i_*CH^1(k(x), r - (p - 1)) \to \bigoplus_{x \in X^{(p)}} i_*CH^0(k(x), r - p) \to 0.
\]
3). Codimension 1. For $X$ regular over a field $k$, 
\[ CH^1(X, q) = \begin{cases} 
  Pic(X), & q = 0 \\
  H^0(X, O_X^*), & q = 1 \\
  0, & q \geq 2 
\end{cases} \]

The following two theorems, due to Nesterenko-Suslin and Kerz, establish the identification between \( CH^p(O_X, p) \) and Milnor K-theory sheaf \( K^M_p(O_X) \)

**Theorem 2.3.** [23]
For \( X \) regular over a field \( k \), we have the following identification
\[ CH^p(O_X, p) = K^M_p(O_X) \text{ mod torsion}. \]

We want to know whether this is also true without ignoring torsion.

**Theorem 2.4.** [18]
For \( A \) an essentially smooth local ring over a field and \( A \) has infinite residue field, one has the following isomorphism (don’t need to tensor with \( \mathbb{Q} \))
\[ K^M_n(A) = H^n_{mot}(A, \mathbb{Z}(n)), \]
where \( H^n_{mot}(A, \mathbb{Z}(n)) \) is Voevodsky’s motivic cohomology.

Recall that Voevodsky proves that for \( k \) a perfect field and \( A \) a regular ring
\[ H^n_{mot}(A, \mathbb{Z}(n)) = CH^n(A, n). \]

**Corollary 2.5.** For \( X \) regular over a field \( k \) of characteristic 0, we have the following identification
\[ CH^p(O_X, p) = K^M_p(O_X). \]

According to [16] [20], one can relate higher Chow groups \( CH^n(X, m) \) with \( H^{p-m}_{Zar}(X, K^M_p(O_X)) \), for \( 0 \leq m \leq 2 \), in the following diagram

\[
\begin{array}{ccc}
Z^p(X, 2) & \xrightarrow{\text{Norm}} & \bigoplus_{x \in X^{(p-2)}} K^M_2(k(x)) \\
\downarrow & & \downarrow \text{Tame} \\
Z^p(X, 1) & \xrightarrow{\text{Norm}} & \bigoplus_{x \in X^{(p-1)}} K^M_1(k(x)) \\
\downarrow & & \downarrow \text{div} \\
Z^p(X) & \xrightarrow{=} & \bigoplus_{x \in X^{(p)}} K^M_0(k(x))
\end{array}
\]

This leads to the following identifications.
**Theorem 2.6.** [20, 21, 24, 27] Bloch-type formulas

Let \( X \) be a regular scheme of finite type over a field \( k \), \( \text{char} k = 0 \). Let \( \mathbf{K}_p(O_X)(\text{res. } \mathbf{K}_p^M(O_X)) \) denote the Quillen K-theory(res.Milnor K-theory) sheaf associated to the presheaf

\[
U \to K_p(O_X(U))(\text{res. } K_p^M(O_X(U))).
\]

After tensoring with \( \mathbb{Q} \), one has the following identifications

\[
CH^p(X, m) = H^{p-m}_{\text{Zar}}(X, K_p(O_X)), \text{ for } 0 \leq m \leq 2,
\]

and

\[
CH^p(X, m) = H^{p-m}_{\text{Zar}}(X, K_p^M(O_X)), \text{ for } 0 \leq m \leq 2.
\]

**Remark 2.7.** Most of the above identifications still hold true without tensoring with \( \mathbb{Q} \), since we already assume the ground field \( k \) to be of characteristic 0. For our purpose, we will use Adams' operations and need to ignore torsion later. So, to fix the idea, we tensor with \( \mathbb{Q} \) here.

**Proof.** When \( m = 0 \), the above formulas are the well-known Bloch's formulas, due to Bloch, Quillen and Soulé. Müller-Stach proves the cases of \( m = 1 \) and \( m = 2 \) (for Milnor K-theory) in [20, 21]. The main ingredients of the proofs are theorem 2.2 and theorem 2.3. Moreover, Müller-Stach points out that \( CH^p(X, 2) \to H^{p-2}_{\text{Zar}}(X, K_p(O_X)) \) is surjective.

We show that, after ignoring torsion, we have

\[
CH^p(X, 2) = H^{p-2}_{\text{Zar}}(X, K_p(O_X)).
\]

No originality is claimed.

There exists the following flasque resolution, due to Quillen [24],

\[
0 \to K_p(O_X) \to K_p(k(X)) \to \bigoplus_{x \in X^{(1)}} i_*K_{p-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^{(p-3)}} i_*K_3(k(x)) \to \bigoplus_{x \in X^{(p-2)}} i_*K_2(k(x)) \xrightarrow{d_{1}^{p-3,-p}} \bigoplus_{x \in X^{(p-1)}} i_*K_1(k(x)) \xrightarrow{d_{1}^{p-2,-p}} \bigoplus_{x \in X^{(p)}} i_*K_0(k(x)) \to 0.
\]

Adams' operations \( \psi^k \), due to Soulé [27], can decompose the above complex into direct sum of sub-complex. To be precise, there exists the following flasque resolution, each \( K_m^{(j)}(k(x)) \) is the eigen-space of \( \psi^k = k^j \).

\[
0 \to K_p^{(j)}(O_X) \to K_p^{(j)}(k(X)) \to \bigoplus_{x \in X^{(1)}} i_*K_{p-1}^{(j-1)}(k(x)) \to \cdots \to \bigoplus_{x \in X^{(p-2)}} i_*K_2^{(j-(p-2))}(k(x)) \xrightarrow{d_{1}^{p-3,-p}} \bigoplus_{x \in X^{(p-1)}} i_*K_1^{(j-(p-1))}(k(x)) \xrightarrow{d_{1}^{p-2,-p}} \bigoplus_{x \in X^{(p)}} i_*K_0^{(j-p)}(k(x)) \to 0.
\]
Letting $j = p$, after tensoring with $\mathbb{Q}$, one obtains the following one,

$$0 \to K^M_p(O_X) \to K^M_p(k(X)) \to \bigoplus_{x \in X^{(1)}} i_* K^M_{p-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^{(p-3)}} i_* K^M_{3}(k(x))$$

$$\xymatrix{0 \ar[r] & K^M_p(O_X) \ar[r]^-{d_{1,p}^{-1,-p}} & K^M_p(k(X)) \ar[r]^-{d_{1,p}^{-2,-p}} & \bigoplus_{x \in X^{(p-2)}} i_* K^M_2(k(x)) \ar[r]^-{d_{1,p}^{-3,-p}} & \bigoplus_{x \in X^{(p-1)}} i_* K^M_1(k(x))} \to \bigoplus_{x \in X^{(p)}} i_* K^M_0(k(x)) \to 0.$$

Letting $j = p - 1$, after tensoring with $\mathbb{Q}$, one obtains the following one,

$$0 \to K^{(p-1)}_p(O_X) \to K^{(p-1)}_p(k(X)) \to \bigoplus_{x \in X^{(1)}} i_* K^{(p-2)}_{p-1}(k(x)) \to \bigoplus_{x \in X^{(p-3)}} i_* K^{(2)}_3(k(x))$$

$$\xymatrix{0 \ar[r] & K^{(p-1)}_p(O_X) \ar[r]^-{d_{1,p}^{-1,-p}} & K^{(p-1)}_p(k(X)) \ar[r]^-{d_{1,p}^{-2,-p}} & \bigoplus_{x \in X^{(p-2)}} i_* K^{(1)}_2(k(x)) \ar[r]^-{d_{1,p}^{-3,-p}} & \bigoplus_{x \in X^{(p-1)}} i_* K^{(0)}_1(k(x))} \to \bigoplus_{x \in X^{(p)}} i_* K_0^{(p-1)}(k(x)) \to 0.$$

We know that $K^{(1)}_2(k(x)) = 0$, So $\text{Ker}(d_{1,p}^{-2,-p}) I_{\text{Im}(d_{1,p}^{-3,-p})} = 0$. This means

$$H^{p-2}(X, K^{(p-1)}_p(O_X)) = 0.$$

Same method shows that

$$H^{p-2}(X, K^{(j)}_p(O_X)) = 0, \text{ for } j < (p - 1).$$

Therefore, ignoring torsion, we have

$$H^{p-2}(X, K_p(O_X)) = H^{p-2}(X, K^M_p(O_X)) = CH^p(X, 2).$$

Based on this theorem, higher Chow groups $CH^p(X, m), 0 \leq m \leq 2,$ can be described as follows.

**Examples**

1.) $CH^p(X, 0) = CH^p(X)$, where $CH^p(X)$ is the classical Chow group of algebraic cycles.

2.) $CH^p(X, 1)$ is represented as a quotient:

$$CH^p(X, 1) = \frac{\text{Ker}(\text{div})}{\text{Im}(\text{Tame})}.$$ 

In other words, an element of $CH^p(X, 1)$ is of the form

$$\{ \sum_j \{ Y_j, f_j \mid \text{codim}(Y_j) = p - 1, f_j \in k(Y_j)^* \text{ and } \sum_j \text{div}(f_j) = 0 \}$$

and modulo the image of Tame symbol.

3.) $CH^p(X, 2)$ is represented by classes in the kernel of Tame symbol, modulo the image of higher Tame symbol.
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The Tame symbol map

$$Tame : \bigoplus_{z \in X^{(p-2)}} K_2^M(k(z)) \to \bigoplus_{y \in X^{(p-1)}} K_1^M(k(y))$$

can be described as follows, eg. see [16]. The Milnor K-group $K_2^M(k(z))$ is generated by symbols $\{f,g\}$, where $f,g \in k(z)^\times$. For the symbol $\{f,g\}$,

$$Tame(\{f,g\}) := \sum_y (-1)^{\nu_y(f)\nu_y(g)} \left( \frac{f^{\nu_y(g)}}{g^{\nu_y(f)}} \right)_y,$$

where $\nu_y(f)$ is the discrete valuation and $(*)_y$ means restriction to $y$.

**Additive Chow groups.** Now, let’s recall Bloch-Esnault’s additive Chow groups from [8]. One sets $Q^n = Spec(k[t_0, ..., t_n]/(\sum_{i=0}^n t_i)$, together with the faces

$$\partial_j : Q^{n-1} \to Q^n; \quad \partial_j^*(t_i) = \begin{cases} t_i & i < j \\ 0 & i = j \\ t_{i-1} & i > j \end{cases}$$

and degeneracies

$$\pi_j : Q^n \to Q^{n-1}; \quad \pi_j^*(t_i) = \begin{cases} t_i & i < j \\ t_i + t_{i+1} & i = j \\ t_{i+1} & i > j \end{cases}$$

Let $\{0\} \in Q^n$ denote the vertex defined by $t_i = 0$ and write $Q^n_X = X \times_{Spec(k)} Q^n$. The above faces and degeneracies make $Q^n_X$ a cosimplicial scheme

$$Q^n_X : \quad Q^n_X \Rightarrow Q^1_X \Rightarrow \cdots$$

**Definition 2.8.** Let $SZ_q(X,n)$ be the free abelian group on irreducible, dimension $q$ subvarieties in $Q^n_X$ with the property:

1. They do not meet $\{0\} \times X$.
2. They meet all the faces properly, that is in dimension $\leq q$.

Thus the face maps induce restriction maps

$$\partial_i : SZ_q(X,n) \to SZ_{q-1}(X,n-1); i = 0, 1, \ldots, n.$$ 

This yields the following complexes $SZ_{q-\bullet}(X, \bullet)$, where $\partial = \sum (-1)^i \partial_i$,

$$\cdots \partial \to SZ_{q+1}(X,n+1) \xrightarrow{\partial} SZ_q(X,n) \xrightarrow{\partial} SZ_{q-1}(X,n-1) \xrightarrow{\partial} \cdots$$
Definition 2.9. The additive higher Chow groups are given by for \( n \geq 1 \) by

\[
SH_q(X, n) := H_n(SZ_{q-\bullet}(X, \bullet))
\]

and (for \( X \) equidimensional)

\[
SH^p(X, n) := SH_{\dim X - p}(X, n).
\]

The groups are not defined for \( n = 0 \).

Bloch-Esnault showed the Chow group of 0-cycles on \( \mathbb{Q} \) in this theory is isomorphic to the group of absolute \((n-1)\)-Kähler forms \( \Omega^{n-1}_{k/\mathbb{Z}} \).

Theorem 2.10. \[8\]

\[
SH^n(k, n) \simeq \Omega^{n-1}_{k/\mathbb{Z}}.
\]

2.2. Tensor triangular geometry. Following Balmer [4], we briefly recall basic definitions and examples of tensor triangular geometry for later use.

Definition 2.11. [4] A tensor triangulated category \((\mathcal{L}, \otimes, \Pi)\) is a triangulated category \(\mathcal{L}\) equipped with a monoidal structure: \(\mathcal{L} \otimes \mathcal{L} \to \mathcal{L}\) with unit object \(\Pi\). We assume that \(- \otimes -\) exact in each variable, i.e. both functors \(a \otimes - : \mathcal{L} \to \mathcal{L}\) and \(- \otimes a : \mathcal{L} \to \mathcal{L}\) are exact for every \(a \in \mathcal{L}\). Let \(\sum\) denote the suspension of \(\mathcal{L}\), we assume that natural isomorphisms \((\sum a) \otimes b \cong \sum (a \otimes b)\) and \(a \otimes (\sum b) \cong \sum (a \otimes b)\) compatible in that the two ways from \((\sum a) \otimes (\sum b)\) to \(\sum^2 (a \otimes b)\) only differ by a sign.

Although some of the theory holds without further assumption, we are going to assume moreover that is symmetric monoidal: \(a \otimes b = b \otimes a\).

Examples of tensor triangulated categories can be found from algebraic geometry, motivic theory, modular representation theory and etc. For our main interest, we recall the following standard example from algebraic geometry. More examples have been discussed in Balmer’s ICM talk [4].

Example 2.12. [4] Let \(X\) be a scheme, here always assumed quasi-compact and quasi-separated (i.e. \(X\) admits a basis of quasi-compact open subsets). A complex of \(O_X\)-modules is called perfect if it is locally quasi-isomorphic to a bounded complex of finite generated projective modules. Then \(\mathcal{L} = D^{perf}(X)\), the derived category of perfect complexes over \(X\), is a tensor triangulated category. See SGA6 [25] or Thomason [29]. The tensor \(\otimes = \otimes_{O_X}^L\) is the left derived tensor product and the unit \(\Pi\) is \(O_X\), considered as a complex concentrated in degree 0.
When $X = \text{Spec}(A)$ is affine, $\mathcal{L} = D^{\text{perf}}(X) \cong K^b(A - proj)$, is the homotopy category of bounded complexes of finite generated projective $A$-modules.

The basic idea for studying tensor triangulated categories is to construct a topological space for every tensor triangulated category $\mathcal{L}$, called the tensor spectrum of $\mathcal{L}$, in which every object $b$ of $\mathcal{L}$ would have a support.

**Definition 2.13.** A non-empty full subcategory $\mathcal{J} \subset \mathcal{L}$ is a triangulated subcategory if for every distinguished triangle $a \to b \to c \to \sum a$ in $\mathcal{L}$, when two out of $a, b, c$ belong to $\mathcal{J}$, so does the third.

$\mathcal{J}$ is called thick if it is stable by direct summands: $a \oplus b \in \mathcal{J} \Rightarrow a, b \in \mathcal{J}$ and triangulated.

$\mathcal{J}$ is $\otimes$-ideal if $\mathcal{L} \otimes \mathcal{J} \subset \mathcal{J}$; it is called radical if $a^\otimes \in \mathcal{J} \Rightarrow a \in \mathcal{J}$.

**Definition 2.14.** A thick $\otimes$-ideal $\mathcal{P} \subset \mathcal{L}$ is called prime if it is proper($\prod \notin \mathcal{P}$) and if $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$, $b \in \mathcal{P}$.

The spectrum of $\mathcal{L}$ is the set of primes:

$$\text{Spc}(\mathcal{L}) = \{\mathcal{P} \subset \mathcal{L} \mid \mathcal{P} \text{ is a prime}\}.$$  

The support of an object $a \in \mathcal{L}$ is defined as:

$$\text{supp}(a) := \{\mathcal{P} \in \text{Spc}(\mathcal{L}) \mid a \notin \mathcal{P}\}.$$  

The complement $U(a) := \{\mathcal{P} \in \text{Spc}(\mathcal{L}) \mid a \in \mathcal{P}\}$, for all $a \in \mathcal{L}$, defines an open basis of the topology of $\text{Spc}(\mathcal{L})$.

We recall the following useful condition on $\mathcal{L}$ for later use.

**Definition 2.15.** A tensor triangulated category $\mathcal{L}$ is rigid if there exists an exact functor $D : \mathcal{L}^{\text{op}} \to \mathcal{L}$ and a natural isomorphism $\text{Hom}_\mathcal{L}(a \otimes b, c) \cong \text{Hom}_\mathcal{L}(b, Da \otimes c)$ for every $a, b, c \in \mathcal{L}$.

**Hypothesis 2.16.** From now on, we assume our tensor triangulated category $\mathcal{L}$ to be essentially small, rigid and idempotent complete.

**Definition 2.17.** A rigid tensor triangulated category $\mathcal{L}$ is called local if $a \otimes b = 0$ implies $a = 0$ or $b = 0$.

**Example 2.18.** For every prime $\mathcal{P} \in \text{Spc}(\mathcal{L})$, the following tensor triangulated category is local in the above sense:

$$\mathcal{L}_P := (\mathcal{L}/\mathcal{P})^\#,$$

where $\mathcal{L}/\mathcal{P}$ denote the Verdier quotient and $(-)^\#$ the idempotent completion.
Definition 2.19. Assuming that $\mathcal{L}$ is local and that $\text{Spc}(\mathcal{L})$ is noetherian, the open complement of the unique closed point \{\ast\} in $\text{Spc}(\mathcal{L})$ is quasi-compact. This one-point subset corresponds to the minimal non-zero thick $\otimes$-ideal

$$\text{Min}(\mathcal{L}) := \{a \in \mathcal{L} \mid \text{supp}(a) \subset \{\ast\}\}.$$ 

These are the objects with minimal possible support (empty or a point).

Definition 2.20. A dimension function on the space $\text{Spc}(\mathcal{L})$ is a map $\dim: \text{Spc}(\mathcal{L}) \to \mathbb{Z} \cup \{\pm \infty\}$ satisfying the following two conditions:

- $\mathcal{P} \subseteq \mathcal{Q}$ implies $\dim(\mathcal{P}) \leq \dim(\mathcal{Q})$.
- $\mathcal{P} \subseteq \mathcal{Q}$ and $\dim(\mathcal{P}) = \dim(\mathcal{Q}) \in \mathbb{Z}$ imply $\mathcal{P} = \mathcal{Q}$.

Examples are Krull dimension of $\{\mathcal{P}\}$ in $\text{Spc}(\mathcal{L})$, or the opposite of its Krull codimension.

Assuming $\dim(\cdot)$ is clear from the context, we shall use the notation

$$\text{Spc}(\mathcal{L})(p) := \{\mathcal{P} \in \text{Spc}(\mathcal{L}) \mid \dim(\mathcal{P}) = p\}.$$ 

Theorem 2.21. For a closed subset $Y \subset \text{Spc}(\mathcal{L})$, we set $\dim(Y) = \text{Sup}\{\dim(\mathcal{P}) \mid \mathcal{P} \in Y\}$ and consider the filtration $\cdots \subset \mathcal{L}(p) \subset \mathcal{L}(p+1) \subset \cdots \subset \mathcal{L}$ by dimension of support

$$\mathcal{L}(p) := \{a \in \mathcal{L} \mid \dim(\text{supp}(a)) \leq p\}.$$ 

For every integer $p \in \mathbb{Z}$, we have the following equivalence induced by localization

$$(\mathcal{L}(p)/\mathcal{L}(p-1))^\# \simeq \bigsqcup_{\mathcal{P} \in \text{Spc}(\mathcal{L})(p)} \text{Min}(\mathcal{L}_{\mathcal{P}})$$

where $\mathcal{L}(p)/\mathcal{L}(p-1)$ is the Verdier quotient and $(\cdot)^\#$ the idempotent completion.

3. K-theoretic higher Chow groups of derived categories of schemes

In section 3.1, we propose K-theoretic higher Chow groups, at position 1 and 2, of derived categories of noetherian schemes. We show that our K-theoretic higher Chow groups recover the classical ones for regular schemes in section 3.2. Functoriality is discussed in section 3.3. In section 3.4, for regular schemes and infinitesimal thickenings, we refine our definitions and introduce Milnor K-theoretic higher Chow groups at position 1 and 2. In section 3.5, we extend Bloch-type formulas.
from regular schemes to their infinitesimal thickenings. We also define tangent spaces to (Milnor)K-theoretic higher Chow groups as usually and identify them with cohomology groups of absolute differentials.

3.1. Definition. Let $X$ be a noetherian scheme of finite Krull dimension $d$. As explained in Example 2.12, the derived category $\mathcal{L} = D^{\text{perf}}(X)$ is a tensor triangulated category. With chosen dimension function on $D^{\text{perf}}(X)$, one can filter this category $\cdots \subset \mathcal{L}(p) \subset \mathcal{L}(p+1) \subset \cdots \subset \mathcal{L}$ by dimension of support

$$\mathcal{L}(p) := \{ a \in \mathcal{L} \mid \text{dim}(\text{supp}(a)) \leq p \}.$$ 

The Verdier quotient $\mathcal{L}(p)/\mathcal{L}(p-1)$ doesn’t have good description when $X$ is singular. However, theorem 2.21 guides us to look at the idempotent completion $(\mathcal{L}(p)/\mathcal{L}(p-1))^\#$.

To fix some notations, for every $i \in \mathbb{Z}$, we define $X(i) = \{ x \in X \mid \text{dim}\{x\} = i \}$. We further assume the dimension function satisfy: $-d \leq \text{dim}(-) \leq d$. The following theorem is a corollary of theorem 2.21.

**Theorem 3.1.** For each $p \in \mathbb{Z}$, localization induces an equivalence

$$(\mathcal{L}(p)/\mathcal{L}(p-1))^\# \simeq \bigsqcup_{x \in X(p)} D_x^{\text{perf}}(X)$$

between the idempotent completion of the quotient $\mathcal{L}(p)/\mathcal{L}(p-1)$ and the coproduct over $x \in X(p)$ of the derived category of the $O_{X,x}$-modules with homology supported on the closed point $x \in \text{spec}(O_{X,x})$.

The short sequence

$$\mathcal{L}(p-1) \rightarrow \mathcal{L}(p) \rightarrow (\mathcal{L}(p)/\mathcal{L}(p-1))^\#,$$

which is exact up to summand, induces a long exact sequence:

$$\cdots \rightarrow K_n(\mathcal{L}(p-1)) \xrightarrow{i} K_n(\mathcal{L}(p)) \xrightarrow{j} K_n((\mathcal{L}(p)/\mathcal{L}(p-1))^\#) \xrightarrow{k} K_{n-1}(\mathcal{L}(p-1)) \rightarrow \cdots$$

As pointed in [2], the above long exact sequence produces an exact couple as usually and then give rise to the associated coniveau spectral sequence with $E_1$-term:

$$E_1^{p,q} = K_{-p-q}((\mathcal{L}(-p)/\mathcal{L}(-p-1))^\#).$$

The differential $d$ is the composition $d = j \circ k$ as usual

$$d_1^{p,q} : K_{-p-q}((\mathcal{L}(-p)/\mathcal{L}(-p-1))^\#) \xrightarrow{L} K_{-p-q-1}(\mathcal{L}(-p-1)) \xrightarrow{J} K_{-p-q-1}((\mathcal{L}(-p-1)/\mathcal{L}(-p-2))^\#).$$
Definition 3.2. \cite{2}

For each integer $q$ satisfying $-d + 1 \leq q \leq d + 1$, the $q^{th}$ Gersten complex $G_q$ is defined to be the $-q^{th}$ line of $E_1$ page of the above coniveau spectral sequence,

$$G_q : 0 \to \bigoplus_{x \in X(d)} K_{d+q}(O_{X,x} \text{ on } x) \to \bigoplus_{x \in X(d-1)} K_{d+q-1}(O_{X,x} \text{ on } x) \to \cdots$$

$$\to \bigoplus_{x \in X(-(q-1))} K_3(O_{X,x} \text{ on } x) \xrightarrow{d_{1}^{-q}-q} \bigoplus_{x \in X(-(q-2))} K_2(O_{X,x} \text{ on } x) \xrightarrow{d_{1}^{-q+2}-q}$$

$$\cdots \to \bigoplus_{x \in X(-d)} K_{q-d}(O_{X,x} \text{ on } x) \to 0.$$ 

With the above preparation, we are ready to propose our K-theoretic definitions of higher Chow group $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$.

Definition 3.3. With chosen dimension function on $D^{perf}(X)$, the $q^{th}$ K-theoretic higher Chow group of $(X, O_X)$ at position 1 and 2, denoted by $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$ respectively, are defined to be

$$CH_q(D^{perf}(X), 1) := \frac{\text{Ker}(d_{1}^{-q+1}-q)}{\text{Im}(d_{1}^{-q+2}-q)}$$

$$CH_q(D^{perf}(X), 2) := \frac{\text{Ker}(d_{1}^{-q+2}-q)}{\text{Im}(d_{1}^{-q+3}-q)}.$$ 

It is clear that our K-theoretic higher Chow groups $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$ are cohomology groups of Gersten complexes.

3.2. Agreement. We show that $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$ recover the classical ones for regular schemes.

Theorem 3.4. Let $X$ be a regular scheme of finite type over a field $k$ and let the tensor triangulated category $D^{perf}(X)$ be equipped with $-\text{codim}_{K_{\text{rull}}}$ as a dimension function, the K-theoretic higher Chow group $CH_q(D^{perf}(X), 1)$ and $CH_q(D^{perf}(X), 2)$ agrees with the classical ones \cite{6} respectively

$$CH_q(D^{perf}(X), 1) = CH^q(X, 1)$$

$$CH_q(D^{perf}(X), 2) = CH^q(X, 2).$$ 

Proof. When the tensor triangulated category $D^{perf}(X)$ is equipped with $-\text{codim}_{K_{\text{rull}}}$ as a dimension function, $X(-i) = \{ x \in X \mid -\text{codim}_{K_{\text{rull}}} \{ x \} = i \}.$
Theorem 3.5. Let $X$ be a $d$-dimensional regular scheme of finite type over a field $k$ and let the tensor triangulated category $D^{perf}(X)$ be equipped with $\dim_{Krull}$ as a dimension function, then we have the following identifications

$$CH_q(D^{perf}(X), 1) = CH^{d+q}(X, 1)$$

$$CH_q(D^{perf}(X), 2) = CH^{d+q}(X, 2).$$
Proof. When the tensor triangulated category $D^{perf}(X)$ is equipped with $dim_{Krull}$ as a dimension function, the augmented Gersten complex $G_q$ is

$$G_q : 0 \to K_{d+q}(X) \to \cdots \to \bigoplus_{x \in X_{(-q-3)}} K_3(O_{X,x} \text{ on } x) \xrightarrow{d_q^{3,-q}}$$

$$\bigoplus_{x \in X_{(-q-2)}} K_2(O_{X,x} \text{ on } x) \xrightarrow{d_q^{2,-q}} \bigoplus_{x \in X_{(-q-1)}} K_1(O_{X,x} \text{ on } x) \xrightarrow{d_q^{1,-q}}$$

$$\bigoplus_{x \in X_{(-q)}} K_0(O_{X,x} \text{ on } x) \to \ldots.$$

Noting $X(i) = \{ x \in X \mid dim_{Krull}(x) = i \} = \{ x \in X \mid codim_{Krull}(x) = d-i \}$, $G_q$ is

$$G_q : 0 \to K_{d+q}(X) \to \cdots \to \bigoplus_{x \in X^{(d+q-3)}} K_3(O_{X,x} \text{ on } x) \xrightarrow{d_q^{3,-q}}$$

$$\bigoplus_{x \in X^{(d+q-2)}} K_2(O_{X,x} \text{ on } x) \xrightarrow{d_q^{2,-q}} \bigoplus_{x \in X^{(d+q-1)}} K_1(O_{X,x} \text{ on } x) \xrightarrow{d_q^{1,-q}}$$

$$\bigoplus_{x \in X^{(d+q)}} K_0(O_{X,x} \text{ on } x) \to \ldots.$$

The remaining proof is the same as the above one.

\[ \square \]

3.3. Functoriality. In this subsection, we discuss functoriality of $K$-theoretic higher Chow groups, flat pull-back and proper push-forward. The two main results, theorem 3.8 and 3.12, are very similar to [33, theorem 3.12 and 3.16] and the proofs are almost the same. We repeat the proofs for the readers convenience.

Flat pull-back. Let $X$ and $Y$ be noetherian schemes with an ample family of line bundles([SGA 6] II 2.2.3, or [29] 2.1.1]) and $f : X \to Y$ a flat morphism. For a noetherian scheme with an ample family of line bundles $S(S = X,Y)$, $D^{perf}(S)$ is equivalent to the derived category obtained from strict perfect complexes of $O_S$-modules(see [29, lemma 3.8]). We use the later in the following and also assume $D^{perf}(S)$ equipped with the dimension function $-\text{codim}_{Krull}$.

Lemma 3.6. ([SGA 6] 1.2, [29] 2.5.1]) Let $f : X \to Y$ be a map of schemes, $L_f^*$ sends (strict)perfect complexes to (strict)perfect complexes

$$L_f^* : D^{perf}(Y) \to D^{perf}(X).$$
TANGENT SPACES TO MOTIVIC COHOMOLOGY GROUPS ON $CH^\bullet(X,1)$ AND $CH^\bullet(X,2)$

Proof. For $E^\bullet$ a strict perfect complex on $Y$, $f^*E^\bullet$ is clearly a strict perfect complex on $X$. This complex represents $Lf^*E^\bullet$ as the vector bundles $E^i$ are flat over $O_Y$ and hence deployed for $Lf^*$.

Moreover, for $f : X \to Y$ a flat morphism, $Lf^*$ respects the filtration of dimension of support. This has been proved by Klein in [15, lemma 4.2.1] for regular schemes $X$ and $Y$. In fact, Klein’s proof also works in our setting.

Lemma 3.7. [15, lemma 4.2.1]

The functor $Lf^* : D^{perf}(Y) \to D^{perf}(X)$ respect the filtration of dimension of support

$$Lf^* : D^{perf}(Y)_{(p)} \to D^{perf}(X)_{(p)}.$$  

Theorem 3.8. Let $f : X \to Y$ be a flat morphism, then $Lf^*$ induces group homomorphisms

$$CH(Lf^*) : CH_p(D^{perf}(Y), m) \to CH_p(D^{perf}(X), m), m = 1, 2.$$  

Proof. $Lf^*$ respects the filtration of dimension of support

$$Lf^* : D^{perf}(Y)_{(p)} \to D^{perf}(X)_{(p)}$$  

$$Lf^* : D^{perf}(Y)_{(p-1)} \to D^{perf}(X)_{(p-1)}.$$  

According to universal property of Verdier quotient, we have

$$Lf^* : D^{perf}(Y)_{(p)}/D^{perf}(Y)_{(p-1)} \to D^{perf}(X)_{(p)}/D^{perf}(X)_{(p-1)}.$$  

Furthermore, according to [4], we have

$$Lf^* : (D^{perf}(Y)_{(p)}/D^{perf}(Y)_{(p-1)})^\# \to (D^{perf}(X)_{(p)}/D^{perf}(X)_{(p-1)})^\#.$$  

This induces maps between coniveau spectral sequences $E_1^{p,-q}(Y)$ and $E_1^{p,-q}(X)$

$$Lf^* : K_{p+q}((D^{perf}(Y)_{(p)}/D^{perf}(Y)_{(p-1)})^\#) \to K_{p+q}((D^{perf}(X)_{(p)}/D^{perf}(X)_{(p-1)})^\#).$$  

Therefore, we have the following commutative diagram

$$\cdots \xrightarrow{d_1^{p-3,-p}} \bigoplus_{y \in Y^{(p-2)}} K_2(O_{Y,y} \text{ on } y) \xrightarrow{d_1^{p-2,-p}} \bigoplus_{y \in Y^{(p-1)}} K_1(O_{Y,y} \text{ on } y) \xrightarrow{d_1^{p-1,-p}} \cdots$$

$$\xrightarrow{Lf^*} \cdots \xrightarrow{d_1^{p-3,-p}} \bigoplus_{x \in X^{(p-2)}} K_2(O_{X,x} \text{ on } x) \xrightarrow{d_1^{p-2,-p}} \bigoplus_{x \in X^{(p-1)}} K_1(O_{X,x} \text{ on } x) \xrightarrow{d_1^{p-1,-p}} \cdots$$

Hence, $Lf^*$ induces group homomorphisms

$$CH(Lf^*) : CH_p(D^{perf}(Y), m) \to CH_p(D^{perf}(X), m), m = 1, 2.$$  

\[\square\]
II: proper push-forward. Let $X$ and $Y$ still be noetherian schemes with an ample family of line bundles. We assume $D^{\text{perf}}(X)$ and $D^{\text{perf}}(Y)$ equipped with the dimension function $\dim_{\text{Krull}}$.

Let’s recall a theorem from SGA6 firstly. For the definition of pseudo-coherent and perfect, we refer to [SGA6] III or [29, section 2].

Lemma 3.9. ([SGA6] III 2.5, 4.8.1 or [29] theorem 2.5.4). Let $f : X \to Y$ be a proper map of schemes. Suppose either that $f$ is projective, or that $Y$ is locally noetherian. Suppose that $f$ is a pseudo-coherent (respectively, a perfect) map. Then if $E'$ is a pseudo-coherent (resp. perfect) complex on $X$, $Rf_*(E')$ is pseudo-coherent (resp. perfect) on $Y$.

Examples of perfect maps are smooth maps, regular closed immersion, locally complete intersection and etc. We refer the readers to [SGA6] VII for more discussions.

Corollary 3.10. Let $f : X \to Y$ be a proper morphism between noetherian schemes. Suppose that $f$ is a perfect map. Then $Rf_*$ sends perfect complexes to perfect complexes

$$Rf_* : D^{\text{perf}}(Y) \to D^{\text{perf}}(X).$$

We expect that $Rf_*$ act like $Lf^*$, respecting the filtration by dimension of support. In fact, it does if we allow some assumptions. The following lemma has been proved by Klein in [15] for both $X$ and $Y$ integral, non-singular, separated schemes of finite type over an algebraically closed field. His proof also works in our setting.

Lemma 3.11. [15, lemma 4.3.1] Let $f : X \to Y$ be a proper morphism between algebraic varieties defined over an algebraically closed field. Suppose that $f$ is a perfect map (so the above corollary applies). The functor $Rf_* : D^{\text{perf}}(X) \to D^{\text{perf}}(Y)$ respects the filtration by dimension of support

$$Rf_* : D^{\text{perf}}(X)(p) \to D^{\text{perf}}(Y)(p).$$

Consequently, we have the following theorem

Theorem 3.12. With the above assumption, $Rf_*$ induces group homomorphisms

$$CH(Rf_*) : CH_p(D^{\text{perf}}(X), m) \to CH_p(D^{\text{perf}}(Y), m), m = 1, 2.$$ 

Proof. Similiar to the above theorem 3.8. □
3.4. Milnor variant. Keeping Bloch-type formulas (theorem 2.6) in mind, we would like to define Milnor K-theoretic higher Chow groups of derived categories of schemes. However, we don’t have Milnor K-theory with support $K^M_m(O_{X,x} \text{ on } x)$ directly. In [33], we use suitable eigen-space (of Adams' operations $\psi^k$) of $K^M_m(O_{X,x} \text{ on } x)$ to propose Milnor K-theory with support $K^M_m(O_{X,x} \text{ on } x)$ as follows.

**Definition 3.13.** Let $X$ be a $d$-dimensional noetherian scheme and $x \in X$ satisfy $\dim O_{X,x} = j$. After tensoring with $\mathbb{Q}$, Milnor K-theory with support $K^M_m(O_{X,x} \text{ on } x)$ is defined to be

$$K^M_m(O_{X,x} \text{ on } x) := K^{(m+j)}_m(O_{X,x} \text{ on } x),$$

where $K^{(m+j)}_m$ is the eigen-space of $\psi^k = k^{m+j}$.

Our definition was inspired by the following two theorem of Soulé [27].

**Theorem 3.14.** [27]

For $X$ a regular scheme of finite type over a field $k$ with characteristic 0, let $K^M_m(O_X)$ (resp. $K^M_m(O_X)$) denote the sheaf associated to the presheaf

$$U \to K^M_m(O_X(U))$$

(resp. $K^M_m(O_X(U))$).

After ignore torsion, we have the following identification

$$K^{(m)}_m(O_X) = K^M_m(O_X).$$

**Theorem 3.15.** Riemann-Roch without denominator [27]

For $X$ regular scheme of finite type and $\eta \in X^{(j)}$, we have (for any integer $m$ and $i$)

$$K^{(i)}_m(O_{X,\eta} \text{ on } \eta) = K^{(i-j)}_m(k(\eta)).$$

These two theorem says that our definition of Milnor K-theory with support is a honest generalization of the classical one, at least for regular case.

Next, we would like to define Milnor K-theoretic higher Chow groups of derived categories of schemes by mimicking definition 3.3. In order to do that, we need to detect whether the differentials of the Gersten complex respect Adams’ operations.

To fix the idea, we let the tensor triangulated category $D^{perf}(X)$ be equipped with $-\text{codim}_{K_{	ext{rull}}}$ as a dimension function. If the differentials $d_1^{p-q}$ of the Gersten complex (definition 3.2) respect Adams’ operations
For every $i \in \mathbb{Z}$, then there exists the following refiner augmented complex

$$G_q^{(i)}: 0 \to K_q^{(i)}(X) \to \cdots \to \bigoplus_{x \in X^{(q-3)}} K_3^{(i)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,i}^{q-3,-q}} \bigoplus_{x \in X^{(q-2)}} K_2^{(i)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-3,-q}} \bigoplus_{x \in X^{(q-1)}} K_1^{(i)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-2,-q}} \cdots \xrightarrow{d_{1,q}^{q-i,-q}} \bigoplus_{x \in X^{(q-d)}} K_{q-d}^{(i)}(O_{X,x} \text{ on } x) \to 0.$$

We are particularly interested in the “Milnor” part. One obtains the following refiner complex by taking $i = q$

$$G_q^{(q)}: 0 \to K_q^{(q)}(X) \to \cdots \to \bigoplus_{x \in X^{(q-3)}} K_3^{(q)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-3,-q}} \bigoplus_{x \in X^{(q-2)}} K_2^{(q)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-2,-q}} \bigoplus_{x \in X^{(q-1)}} K_1^{(q)}(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-i,-q}} \cdots \xrightarrow{d_{1,q}^{q-i,-q}} \bigoplus_{x \in X^{(q-d)}} K_{q-d}^{(q)}(O_{X,x} \text{ on } x) \to 0.$$

After tensoring with $\mathbb{Q}$, this complex can be written as

$$G_q^{(q)}: 0 \to K_q^M(X) \to \cdots \to \bigoplus_{x \in X^{(q-3)}} K_3^M(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-3,-q}} \bigoplus_{x \in X^{(q-2)}} K_2^M(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-2,-q}} \bigoplus_{x \in X^{(q-1)}} K_1^M(O_{X,x} \text{ on } x) \xrightarrow{d_{1,q}^{q-i,-q}} \cdots \xrightarrow{d_{1,q}^{q-i,-q}} \bigoplus_{x \in X^{(q-d)}} K_{q-d}^M(O_{X,x} \text{ on } x) \to 0.$$

**Definition 3.16.** If the differentials $d_{1,q}^{q-i,-q}$ of the Gersten complex respect Adams’ operations, then the $q$th Milnor $K$-theoretic higher Chow groups of $(X, O_X)$ at position 1 and 2, denoted $CH_q^M(D_{perf}^T(X), 1)$ and $CH_q^M(D_{perf}^T(X), 2)$, are defined to be

$$CH_q^M(D_{perf}^T(X), 1) := \frac{\text{Ker}(d_{1,q}^{q-1,-q})}{\text{Im}(d_{1,q}^{q-2,-q})},$$

$$CH_q^M(D_{perf}^T(X), 2) := \frac{\text{Ker}(d_{1,q}^{q-2,-q})}{\text{Im}(d_{1,q}^{q-3,-q})}.$$  

We shall show that the above definitions work for regular schemes and their infinitesimal thickenings in the following.

3.5. **Bloch-type formulas and tangent spaces to higher Chow groups.** In this subsection, $X$ is a $d$-dimensional regular scheme of finite type over a field $k$, where $\text{Char}k = 0$. Let $T_j$ denote the spectrum of the truncated polynomial $\text{Spec}(k[t]/(t^{j+1}))$ and $X_j$ denote the $j$-th infinitesimal thickening, i.e. $X_j = X \times T_j$. The tensor triangulated category $D_{perf}^T(X)$ and $D_{perf}^T(X_j)$ are equipped with $-\text{codim}_{K_{runt}}$ as a dimension function.
The aim of this subsection is to show the above definitions works for \( X \) and its infinitesimal thickenings \( X_j \) and to extend Bloch-type formula, theorem 2.6, from \( X \) to its infinitesimal thickening \( X_j \). As an application, we define tangent spaces to higher Chow groups and identify them with cohomology groups of absolute differentials.

**Definition 3.17.** [2, definition 4] or similar definition

For any integer \( q \), there exists the following augmented Gersten complex \( G_{q,j} \) on the \( j \)-th infinitesimal neighborhood \( X_j \)

\[
G_{q,j} : 0 \rightarrow K_q(X_j) \rightarrow K_q(k(X)_j) \rightarrow \bigoplus_{x_j \in X_j^{(1)}} K_{q-1}(O_{X_j,x_j} \text{ on } x_j) \rightarrow \ldots
\]

\[
\rightarrow \ldots \rightarrow \bigoplus_{x_j \in X_j^{(d)}} K_{q-d}(O_{X_j,x_j} \text{ on } x_j) \rightarrow 0
\]

where \( X_j = X \times T_j, \ k(X)_j = k(X) \times T_j, \ x_j = x \times T_j. \)

We have proved that the sheafification of this Gersten complex is indeed a flasque resolution [33, section 4.4]. Moreover, we have proved

**Theorem 3.18.** [33, section 4.4]

For any integer \( q \) satisfying \( 1 \leq q \leq d + 1 \), there exists the following splitting commutative diagram in which the sheafification of each
column is a flasque resolution.

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
(\Omega_{X/Q}^\bullet)^{\oplus j} & K_q(X_j) & K_q(X) \\
\downarrow & \downarrow & \downarrow \\
(\Omega_{k(X)/Q}^\bullet)^{\oplus j} & K_q(k(X)_j) & K_q(k(X)) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(1)}} H_2^1((\Omega_{X/Q}^\bullet)^{\oplus j}) & \bigoplus_{x \in X^{(1)}} K_{q-1}(O_{X_j,x} \text{ on } x_j) & \bigoplus_{x \in X^{(1)}} K_{q-1}(O_{X,x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots \\
\bigoplus_{x \in X^{(q-3)}} H_2^{-3}((\Omega_{X/Q}^\bullet)^{\oplus j}) & \bigoplus_{x \in X^{(q-3)}} K_3(O_{X_j,x} \text{ on } x_j) & \bigoplus_{x \in X^{(q-3)}} K_3(O_{X,x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(q-2)}} H_2^{-2}((\Omega_{X/Q}^\bullet)^{\oplus j}) & \bigoplus_{x \in X^{(q-2)}} K_2(O_{X_j,x} \text{ on } x_j) & \bigoplus_{x \in X^{(q-2)}} K_2(O_{X,x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(q-1)}} H_2^{-1}((\Omega_{X/Q}^\bullet)^{\oplus j}) & \bigoplus_{x \in X^{(q-1)}} K_1(O_{X_j,x} \text{ on } x_j) & \bigoplus_{x \in X^{(q-1)}} K_1(O_{X,x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\bigoplus_{x \in X^{(d)}} H_2^d((\Omega_{X/Q}^\bullet)^{\oplus j}) & \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X_j,x} \text{ on } x_j) & \bigoplus_{x \in X^{(d)}} K_{q-d}(O_{X,x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

where

\[
\begin{align*}
\Omega_{X/Q}^\bullet &= \Omega_{X/Q}^{q-1} \oplus \Omega_{X/Q}^{q-3} \oplus \ldots \\
\Omega_{k(X)/Q}^\bullet &= \Omega_{k(X)/Q}^{q-1} \oplus \Omega_{k(X)/Q}^{q-3} \oplus \ldots
\end{align*}
\]

Now we can extend Bloch-type formulas (Theorem 2.6) from \( X \) to its infinitesimal thickening \( X_j \).

**Theorem 3.19. Bloch-type formulas**

We have the following identifications

\[
CH_q(D_{\text{perf}}(X_j), 1) = H^{q-1}(X, K_q(O_{X_j}))
\]

\[
CH_q(D_{\text{perf}}(X_j), 2) = H^{q-2}(X, K_q(O_{X_j}))
\]
In particular, for $j = 1$,

$$
CH_q(D^{perf}(X[\varepsilon]), 1) = H^{q-1}(X, K_q(O_X[\varepsilon]))
$$

$$
CH_q(D^{perf}(X[\varepsilon]), 2) = H^{q-2}(X, K_q(O_X[\varepsilon])).
$$

**Proof.** For $m = 1, 2$, the definition of $CH_q(D^{perf}(X_j), m)$ says that it equals to the $(q - m)$-th cohomology of the Gersten complex $G_{q,j}$

$$
CH_q(D^{perf}(X_j, m) = H^{q-m}(G_{q,j}).
$$

It follows because the sheafification of $G_{q,j}$ is a flasque resolution of $K_q(O_{X_j})$.

Now, we consider the K-theoretic higher Chow group as a functor on $X$ and define the tangent space to it as usually.

**Definition 3.20.** For $m = 1, 2$, the tangent spaces to $CH_q(D^{perf}(X), m)$, denoted $TCH_q(D^{perf}(X), m)$, are defined to be

$$
TCH_q(D^{perf}(X), m) := \text{Ker}\{CH_q(D^{perf}(X[\varepsilon], m)) \stackrel{\varepsilon=0}{\longrightarrow} CH_q(D^{perf}(X), m)\}
$$

We can identify this tangent space with cohomology group of absolute differentials.

**Theorem 3.21.**

$$
TCH_q(D^{perf}(X), m) = H^{q-m}(\Omega^\bullet_{O_X/Q}), m = 1, 2,
$$

where $\Omega^\bullet_{O_X/Q} = \Omega^{q-1}_{O_X/Q} \oplus \Omega^{q-3}_{O_X/Q} \oplus \ldots$

**Proof.** Diagram chasing. Immediately follows from theorem 3.18.

Furthermore, we have the following refiner diagram involving the “Milnor K-theory”.

**Theorem 3.22.** [33, theorem 4.29]

For any integer $q$ satisfying $1 \leq q \leq d + 1$, there exists the following splitting commutative diagram in which the sheafification of each
column is a flasque resolution.

\[
\begin{array}{c c c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
(\Omega_{X/Q}^{q-1})^{\oplus j} & K_q^M(X_j) & K_q^M(X) \\
\downarrow & \downarrow & \downarrow \\
(\Omega_{k(X)/Q}^{q-1})^{\oplus j} & K_q^M(k(X)_j) & K_q^M(k(X)) \\
\oplus_{x \in X^{(1)}} H_1^j((\Omega_{X/Q}^{q-1})^{\oplus j}) & \oplus_{x \in X^{(q-3)}} K^M_3(O_{X, x} \text{ on } x_j) & \oplus_{x \in X^{(q-3)}} K^M_3(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots \\
\oplus_{x \in X^{(q-2)}} H_2^{-2}((\Omega_{X/Q}^{q-1})^{\oplus j}) & \oplus_{x \in X^{(q-2)}} K^M_2(O_{X, x} \text{ on } x_j) & \oplus_{x \in X^{(q-2)}} K^M_2(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\oplus_{x \in X^{(q-1)}} H_2^{-1}((\Omega_{X/Q}^{q-1})^{\oplus j}) & \oplus_{x \in X^{(q-1)}} K^M_1(O_{X, x} \text{ on } x_j) & \oplus_{x \in X^{(q-1)}} K^M_1(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
\oplus_{x \in X^{(d)}} H_d^2((\Omega_{X/Q}^{q-1})^{\oplus j}) & \oplus_{x \in X^{(d)}} K^M_{q-d}(O_{X, x} \text{ on } x_j) & \oplus_{x \in X^{(d)}} K^M_{q-d}(O_{X, x} \text{ on } x) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 
\end{array}
\]

The middle and right columns are complexes, so the definition 3.16 applies.

**Definition 3.23.** The \(q\)-th Milnor K-theoretic higher Chow group of \((X, O_X)\) at position 1 and 2, are defined to be

\[
CH_q^M(D^{perf}(X), 1) := \frac{\text{Ker}(d_{1,q}^{q-1,-q})}{\text{Im}(d_{1,q}^{q-2,-q})},
\]

\[
CH_q^M(D^{perf}(X), 2) := \frac{\text{Ker}(d_{1,q}^{q-2,-q})}{\text{Im}(d_{1,q}^{q-3,-q})}.
\]
The $q^{th}$ Milnor K-theoretic higher Chow group of $(X, O_{X_j})$ at position 1 and 2, are defined to be

$$CH^M_q(D^\text{perf}(X_j), 1) := \frac{\text{Ker}(d_{1,q,j}^{q-1,-q})}{\text{Im}(d_{1,q,j}^{q-2,-q})}$$

$$CH^M_q(D^\text{perf}(X_j), 2) := \frac{\text{Ker}(d_{1,q,j}^{q-2,-q})}{\text{Im}(d_{1,q,j}^{q-3,-q})}.$$

**Agreement.** We now prove that our Milnor K-theoretic higher Chow groups agree with the classical ones for regular schemes, after tensoring with $\mathbb{Q}$.

**Theorem 3.24.** For $X$ is a regular scheme of finite type over a field $k$, $\text{chark} = 0$, let $CH^q(X, 1)$ and $CH^q(X, 2)$ denote the higher Chow groups defined in [6], after tensoring with $\mathbb{Q}$, then we have the following identifications

$$CH^M_q(D^\text{perf}(X), 1) = CH^q(X, 1)$$

$$CH^M_q(D^\text{perf}(X), 2) = CH^q(X, 2).$$

**Proof.** Since $X$ is a regular, Soulé’s Riemann-Roch without denominator, theorem 3.15, shows that the right column of theorem 3.22 agrees with the following classical sequence, ignoring torsion,

$$G^{(q)} : 0 \to K^M_1(X) \to K^M_1(k(X)) \to \cdots \to \bigoplus_{x \in X^{(q-3)}} K^M_3(k(x)) \xrightarrow{d_{1,q,j}^{q-3,-q}} \bigoplus_{x \in X^{(q-2)}} K^M_2(k(x)) \xrightarrow{d_{1,q,j}^{q-2,-q}} \bigoplus_{x \in X^{(q-1)}} K^M_1(k(x)) \xrightarrow{d_{1,q,j}^{q-1,-q}} \bigoplus_{x \in X^{(q)}} K^M_0(k(x)) \to 0.$$

Hence,

$$CH^M_q(D^\text{perf}(X), 1) = H^{q-1}(G^{(q)}_q) = H^{q-1}(X, K^M_q(O_X)) = CH^q(X, 1).$$

The first identity is the definition, the second one follows from that the sheafification of $G^{(q)}_q$ is a flasque resolution, the third one is theorem 2.6.

The same idea works for $CH^M_q(D^\text{perf}(X), 2) = CH^q(X, 2)$. □

Also we extend Bloch-type formulas for Milnor K-theory from $X$ to its infinitesimal neighborhood $X_j$.

**Theorem 3.25.** Bloch-type formula
After tensoring with $\mathbb{Q}$, for $m = 1, 2$, we have the following identifications:

$$\text{CH}_M^q(D_{\text{perf}}(X), m) = H^{q-m}(X, K^M_q(O_X)).$$

$$\text{CH}_M^q(D_{\text{perf}}(X_j, m)) = H^{q-m}(X, K^q_j(O_{X_j})).$$

Proof. Immediately from the flasqueness of sheafifications of columns in theorem 3.22.

Now, we define the tangent space to Milnor K-theoretic higher Chow groups.

**Definition 3.26.** For $m = 1, 2$, the tangent spaces to $\text{CH}_M^q(D_{\text{perf}}(X), m)$, denoted by $T\text{CH}_M^q(D_{\text{perf}}(X), m)$, are defined to be

$$T\text{CH}_M^q(D_{\text{perf}}(X), m) := \ker \{ \text{CH}^q(D_{\text{perf}}(X[z]), m) \rightarrow \text{CH}^q(D_{\text{perf}}(X), m) \}.$$

Recall that one can formally define tangent spaces to higher Chow groups via Bloch-type formulas

$$T_fCH^q(X, m) = H^{q-m}(X, TK^M_q(O_X)) = H^{q-m}(X, \Omega^{q-1}_{O_X/\mathbb{Q}}).$$

Our definition agrees with it:

**Theorem 3.27.**

$$T\text{CH}_M^q(D_{\text{perf}}(X), m) = H^{q-m}(X, \Omega^{q-1}_{O_X/\mathbb{Q}}) = T_fCH^q(X, m).$$

Proof. Diagram chasing. Immediately from theorem 3.22.

4. Geometric meaning of tangent spaces to $CH^2(X, 1)$

In this section, we shall put a concrete geometric meaning to the tangent space $TCH^M_2(D_{\text{perf}}(X), 1) = TCH^2(X, 1)$ by using Green-Griffiths’ results in [12].

To fix the idea, $X$ is a smooth projective surface over a field $k$, $\text{char} k = 0$. The key to our approach is the following splitting commutative diagram.

**Theorem 4.1.** Letting $q = 2$ and $j = 1$ in theorem 3.22, we have the following splitting commutative diagram at the sheaf level (each column...
is a flasque resolution): 

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\Omega^1_{X/Q} & \xleftarrow{\text{Chern-1}} & K_2^M(O_X[\epsilon]) \\
\downarrow & & \downarrow \\
\Omega^1_{k(X)/Q} & \xleftarrow{\text{Chern-2}} & K_2^M(k(X)[\epsilon]) \\
\downarrow & & \downarrow \\
\oplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}) & \xleftarrow{\text{Chern-3}} & \bigoplus_{y[\epsilon] \in X^{\epsilon}[1]} K_1^M(O_{X,y}[\epsilon] \text{ on } y[\epsilon]) \\
\downarrow & & \downarrow \\
\oplus_{x \in X^{(2)}} H^2_x(\Omega^1_{X/Q}) & \xleftarrow{\text{Chern-4}} & \bigoplus_{x[\epsilon] \in X^{\epsilon}[2]} K_0^M(O_{X,x}[\epsilon] \text{ on } x[\epsilon]) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

In the following, combining the above diagram with Green-Griffiths’ results [12, section 8.3], we shall describe $K_2^M(k(X))$ and $\bigoplus K_1^M(O_{X,y}[\epsilon] \text{ on } y[\epsilon])$ explicitly and also describe Chern-2 and Chern-3 explicitly.

### 4.1. Nenashev’s result.

In this subsection, we recall a result of Nenashev on describing $K_1$ of any exact category, which will be used later.

Inspired by Gillet and Grayson’s work on attaching a simplicial set to any exact category, A.Nenashev in [22] provides a way to describe $K_1$ of any exact category in terms of generators and relations. We recall Nenashev’s method briefly and refer the readers to his paper [22] for more details. Let $\mathcal{E}$ denote an exact category in the following.

**Definition 4.2.** [22] A double short exact sequence in $\mathcal{E}$ is a pair of short exact sequences on the same objects:

\[
\begin{align*}
\left\{ 
0 & \to A \xrightarrow{f_1} B \xrightarrow{\eta} C \to 0 \\
0 & \to A \xrightarrow{f_2} B \xrightarrow{\eta_2} c \to 0
\end{align*}
\]

In particular, if $A \in \mathcal{E}$ and $\alpha \in \text{Aut}A$, we can associate a double short exact sequence to $\alpha$:

\[
\begin{align*}
\left\{ 
0 & \to A \xrightarrow{1} A \\
0 & \to A \xrightarrow{\alpha} A
\end{align*}
\]
Now one defines an abelian group generated by these *double short exact sequences*.

**Definition 4.3.** [22] We define $A(\mathcal{E})$ to be the abelian group generated by all *double short exact sequences* subject to the following two relations:

- (1). If $f_1 = f_2$ and $g_1 = g_2$, then the *double short exact sequence* is 0.
- (2). $3 \times 3$ relations defined in proposition 2.1 in [22].

The main theorem in [22] by Nenashev says

**Theorem 4.4.** [22] For any exact category $\mathcal{E}$, there is an isomorphism between the following two abelian groups:

$$A(\mathcal{E}) \cong K_1(\mathcal{E}).$$

4.2. *Green-Griffiths’ arcs.* Recall that $X$ is a smooth projective surface over a field $k$, $\text{chark} = 0$. Suppose $Y$ is an irreducible curve with generic point $y$. We are interested in describing the stalk of $K^M_1(O_{X,y}[\varepsilon])$ on $y[\varepsilon]$ at any point $x$. We may assume $x \in Y$ and let $f$ be the local defining equations for $Y$. Then the local ring of $Y$, $O_{X,y}$, can be identified with $(O_{X,x})_f$, the localization of the local ring of $x$ with respect to $(f)$

$$O_{X,y} = (O_{X,x})_f.$$

Let’s introduce the following definition, used by Green-Griffiths [12, chap 8, page 127].

**Definition 4.5.** *Green-Griffiths’ arcs*

*Green-Griffiths’ arcs* are defined to be pairs of the form

$$\{\text{div}(f + \varepsilon f_1), g + \varepsilon g_1 |_{\text{div}(f+\varepsilon f_1)}\},$$

where $f_1 \in O_{X,x}$, $g + \varepsilon g_1 \in (K(Y)[\varepsilon])^*$ is a rational function on a irreducible component of $\text{div}(f + \varepsilon f_1)$ and furthermore we assume $\text{div}(f + \varepsilon f_1)$ and $\text{div}(g + \varepsilon g_1)$ have no common curve components.

**Remark 4.6.** Considering $\text{speck}[\varepsilon] = \text{speck}[t]/(t^2)$, one can write *Green-Griffiths’ arcs* as the form

$$\{\text{div}(f + tf_1), g + tg_1 |_{\text{div}(f+tf_1)}\}.$$

This is the notation used in [12]. Intuitively, one can think of $\text{div}(f + tf_1)$ as the $1^{st}$ order deformation of $\text{div}(f)$ and $g + tg_1$ is a deformation of $g$. $\text{div}(f + tf_1)$ is understood to be Puiseaux series in $t$, see [12, page 27-28].
One sees that \((O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1)\) is a \((O_{X,x})_{(f)}[\varepsilon]\)-module, i.e. \(O_{X,y}[\varepsilon]\)-module, supported at the closed point \(y[\varepsilon]\) in \(\text{Spec}(O_{X,x})_{(f)}[\varepsilon]\). Note that the closed point \(y[\varepsilon]\) corresponds to the maximal ideal of \(\text{Spec}(O_{X,x})_{(f)}[\varepsilon]\).

We shall identify the above Green-Griffiths’ \(\text{arcs} \) as elements of the stalk of \(K_1^M(O_{X,y}[\varepsilon]\) on \(y[\varepsilon]\) at point \(x\). The bridge is the following theorem of Thomason, Exercise 5.7 of Thomason-Trobaugh [29].

**Theorem 4.7.** Thomason-Trobaugh [29]

Let \(X\) be a scheme with an ample family of line bundles. Let \(i : Y \to X\) be a regular closed immersion ([SGA 6] VII Section 1) defined by ideal \(J\). Suppose \(Y\) has codimension \(k\) in \(X\). Then show that \(K(X on Y)\) is homotopy equivalent to the Quillen K-theory of the exact category of pseudo-coherent \(O_X\)-modules supported on the subspace \(Y\) and of \(\text{Tor-dimension} \leq k\) on \(X\).

According to this theorem, \(K_1(O_{X,y}[\varepsilon]\) on \(y[\varepsilon]\)) can be considered as a \(K\)-group of the exact category of pseudo-coherent \(O_{X,y}[\varepsilon]\)-modules supported on the subspace \(y[\varepsilon]\) and of \(\text{Tor-dimension} \leq 1\) on \(O_{X,y}[\varepsilon]\).

\((O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1))\) is such a module.

Then we can use Nenashev’s theorem 4.4 to identify Green-Griffiths’ \(\text{arcs} \) \(\{\text{div}(f+\varepsilon f_1), g+\varepsilon g_1 \mid \text{div}(f+\varepsilon f_1)\}\) as an element of the stalk of \(K_1^M(O_{X,y}[\varepsilon]\) on \(y[\varepsilon]\)) at the point \(x\) as follows. Considering \(g+\varepsilon g_1\) as an automorphism of \((O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1)\), we can associate a double short exact sequence to \(\{\text{div}(f+\varepsilon f_1), g+\varepsilon g_1 \mid \text{div}(f+\varepsilon f_1)\}\)

\[
\begin{align*}
0 & \to (O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1) \xrightarrow[]{1} (O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1) \\
0 & \to (O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1) \xrightarrow{g+\varepsilon g_1} (O_{X,x})_{(f)}[\varepsilon]/(f+\varepsilon f_1)).
\end{align*}
\]

(4.4)

According to Nenashev’s theorem 4.4, this double short exact sequence is an element of the stalk of \(K_1^M(O_{X,y}[\varepsilon]\) on \(y[\varepsilon]\)) at \(x\).

### 4.3. Green-Griffiths’ maps and Chern character.

In this subsection, we recall Green-Griffiths’ geometric descriptions of tangent maps from Green-Griffiths’ \(\text{arcs} \) to local cohomology groups. We shall show that Green-Griffiths’ tangent maps agree with Chern character maps in theorem 4.1.

**Describing Chern-2.** An element of \(K_2^M(k(X)[\varepsilon]\)) is given by Steinberg symbol \(\{f+\varepsilon f_1, g+\varepsilon g_1\}\), where \(f, g \in k(X)^*\), \(f_1, g_1 \in k(X)\). Now we describe the Chern-2 map from \(K_2^M(k(X)[\varepsilon]\)) to \(\Omega^1_{k(X)/Q}\) in theorem 4.1.

\(\text{Chern} – 2 : K_2^M(k(X)[\varepsilon]\)) \to \Omega^1_{k(X)/Q}\).

According to [17, section 8.4, page 275], the following composition, call it \(Ch_i\),
$Ch : K^M_2(k(X)[\varepsilon]) \to K_2(k(X)[\varepsilon]) \xrightarrow{Chern} HN_2(k(X)[\varepsilon]) \to \Omega^2_{k(X)[\varepsilon]/Q}$
sends Steinberg symbol $\{s, t\}$ to $s^{-1}t^{-1}dsdt$. On the other hand, there is a truncation map from $\Omega^2_{k(X)[\varepsilon]/Q}$ to $\Omega^1_{k(X)/Q}$

$$\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} : \Omega^2_{k(X)[\varepsilon]/Q} \to \Omega^1_{k(X)/Q}$$

$$(a + b\varepsilon)(x + y\varepsilon)(z + w\varepsilon) \to a(wdx - ydz).$$

**Theorem 4.8.** The Chern-2 map in theorem 4.1 which can be described as the composition of $Ch$ and the truncation $\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0}$, is of the following form:

$$Chern - 2 : K^M_2(k(X)[\varepsilon]) \xrightarrow{Ch} \Omega^2_{k(X)[\varepsilon]/Q} \xrightarrow{\frac{\partial}{\partial \varepsilon} |_{\varepsilon=0}} \Omega^1_{k(X)/Q}$$

$$\{f + \varepsilon f_1, g + \varepsilon g_1\} \to \frac{g_1df - f_1dg}{fg}.$$

Now, we recall Green-Griffiths’ tangent map [12, page 130], call it $tangent 2$, from $K^M_2(k(X)[\varepsilon])$ to $\Omega^1_{k(X)/Q}$.

**Definition 4.9.** [12, page 130]
Green-Griffiths’ tangent map, $tangent 2$, from $K^M_2(k(X)[\varepsilon])$ to $\Omega^1_{k(X)/Q}$ is defined to be

$$tangent 2 : K^M_2(k(X)[\varepsilon]) \to \Omega^1_{k(X)/Q}$$

$$\{f + \varepsilon f_1, g + \varepsilon g_1\} \to \frac{g_1df - f_1dg}{fg}.$$

**Corollary 4.10.** The Chern-2 map in theorem 4.1 agrees with Green-Griffiths’ tangent 2 map.

**Green-Griffiths’ map.** Next, we want to describe Chern-3 map in theorem 4.1 explicitly. We us the following splitting commutative diagram, taken from theorem 4.1 for our purpose.
We begin with recalling Green-Griffiths’ description [12, page 105] of \( \partial_{1,2}^{0,-2} \),

\[
\partial_{1,2}^{0,-2} : \Omega^1_{k(X)/\mathbb{Q}} \to \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}).
\]

It suffices to look at \( \partial_{1,2}^{0,-2} \) from stalk to stalk. We work in a Zariski neighborhood of a point \( x \), and write an element \( \beta \in \Omega^1_{k(X)/\mathbb{Q}} \) as

\[
\beta = \frac{h}{f_1^{l_1} \cdots f_k^{l_k}} dg
\]

where \( f_1, \ldots, f_k, h, g \in O_{X,x} \) are relatively prime and \( f_i^{l_i}'s \) are irreducible. Set \( Y_i = \{ f_i = 0 \} \) and let \( \beta_i \) denote

\[
\beta_i = \frac{h}{f_1^{l_1} \cdots \hat{f_i}^{l_i} \cdots f_k^{l_k}}
\]

where \( \hat{f_i}^{l_i} \) means to omit the \( i \)th term. By abuse of notations, we still use \( \beta_i \) denote the following diagram

(4.5)

\[
\begin{align*}
(O_{X,x})_{(f_i^{l_i})} & \xrightarrow{f_i^{l_i}} (O_{X,x})_{(f_i^{l_i})} \\
(O_{X,x})_{(f_i^{l_i})} & \xrightarrow{\beta_i} \Omega^1_{(O_{X,x},(f_i^{l_i})/\mathbb{Q})}
\end{align*}
\]

**Theorem 4.11.** [12, page 105]

With above notations, \( \partial_{1,2}^{0,-2} \) can be described as follows:

\[
\partial_{1,2}^{0,-2} : \Omega^1_{k(X)/\mathbb{Q}} \to \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}})
\]

\[
\beta \mapsto \sum_i \beta_i.
\]

Now we recall Green-Griffiths’ description [12, page 130] of \( d_{1,2}^{0,-2} \). An element of \( K_2^M(k(X)[\varepsilon]) \) is given by Steinberg symbol \( \{ f + \varepsilon f_1, g + \varepsilon g_1 \} \), where \( f, g \in k(X)^* \), \( f_1, g_1 \in k(X) \). We work in a Zariski neighborhood of a point \( x \), factor \( f, g \) into irreducible factors in \( O_{X,x} \) and write \( \{ f + \varepsilon f_1, g + \varepsilon g_1 \} \) as a product of symbols of the form \( \{ a + \varepsilon a_1, b + \varepsilon b_1 \} \) or its inverse, where \( a, b, a_1, b_1 \in O_{X,x}, a \neq 0, b \neq 0 \).

For simplicity, we assume an element of \( K_2^M(k(X)[\varepsilon]) \) is given by Steinberg symbol \( \{ f + \varepsilon f_1, g + \varepsilon g_1 \} \), where \( f, g, f_1, g_1 \in O_{X,x}, f \neq 0 \) and \( g \neq 0 \).

**Theorem 4.12.** [12, page 130]
The differential
\[ d_{1,2,\varepsilon}^{0,-2} : K^M_2(k(X)_\varepsilon) \to \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \]
can be described as follows,
\[
\{f + \varepsilon f_1, g + \varepsilon g_1\} \to \{(\text{div } (f + \varepsilon f_1), g + \varepsilon g_1 |_{\text{div } (f + \varepsilon f_1)})\} \]
\[-\{(\text{div } (g + \varepsilon g_1), f + \varepsilon f_1 |_{\text{div } (g + \varepsilon g_1)})\}, \]
and where we assume \(\text{div } (f + \varepsilon f_1)\) and \(\text{div } (g + \varepsilon g_1)\) have no common curve components, otherwise, the image defined to be 0.

Green-Griffiths [12, page 127] define a tangent map, call it \textit{tangent 3}, from the set of Green-Griffiths’ arcs, \{GG'\text{’s arcs}\} for short, to \(\bigoplus H^1_y(\Omega^1_{X/Q})\)
\[
\text{tangent 3} : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \{GG'\text{’s arcs}\} \longrightarrow \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}).
\]

**Definition 4.13.** [12] Working in a Zariski neighborhood of a point \(x\), one defines \textit{tangent 3} from stalk to stalk as follows.

We may assume \(x \in Y = \{y\}\) and let \(f\) be the local defining equations for \(Y\). Recall that Green-Griffiths’ arcs are defined to be a pair
\[
\{\text{div } (f + \varepsilon f_1), g + \varepsilon g_1 |_{\text{div } (f + \varepsilon f_1)}\},
\]
where \(f_1 \in O_{X,x}\), \(g + \varepsilon g_1 \in (K(Y)_\varepsilon)^*\) is a rational function on a irreducible component of \(\text{div } (f + \varepsilon f_1)\) and furthermore we assume \(\text{div } (f + \varepsilon f_1)\) and \(\text{div } (g + \varepsilon g_1)\) have no common curve components.

For the Green-Griffiths’ arcs \(\{\text{div } (f + \varepsilon f_1), g + \varepsilon g_1 |_{\text{div } (f + \varepsilon f_1)}\}\), the following diagram
\[
\begin{align*}
(O_{X,x}(f)) \quad &\xrightarrow{f} \quad (O_{X,x}(f)) \quad \longrightarrow \quad (O_{X,x}(f))/(f) \quad \longrightarrow \quad 0 \\
(O_{X,x}(f)) \quad &\xrightarrow{g_1 df - f_1 dg} \quad \Omega^1_{\text{Ext}_1^{r}(O_{X,x}(f), (O_{X,x}(f))/(f), \Omega^1_{(O_{X,x}(f))/(f)}))}
\end{align*}
\]
gives an element \(\alpha\) in \(\text{Ext}_1^{r}(O_{X,x}(f), (O_{X,x}(f))/(f), \Omega^1_{(O_{X,x}(f))/(f)}))\). Noting that
\[
H^1_y(\Omega^1_{X/Q}) = \lim_{n \to \infty} \text{Ext}_1^{n}(O_{X,x}(f), (O_{X,x}(f))/(f)^n, \Omega^1_{(O_{X,x}(f))/(f)})
\]
the image \([\alpha]\) of \(\alpha\) under the limit is in \(H^1_y(\Omega^1_{X/Q})\) and it is the tangent to \(\{\text{div } (f + \varepsilon f_1), g + \varepsilon g_1 |_{\text{div } (f + \varepsilon f_1)}\}\).
Remark 4.14. One can check that the above Green-Griffiths’ tangent map factors out the two relations in definition 4.3. So it is well-defined on Green-Griffiths’ arcs, considered as elements of K-groups. Many thanks to B.Dribus-J.W.Hoffman, M.Schlichting and C.Soulé for helpful discussions on this.

Let $\bigoplus <GG's arcs>$ denote the subgroup of $\bigoplus K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$, generated by Green-Griffiths’ arcs. Green-Griffiths’ tangent map, is well-defined on the subgroup $\bigoplus <GG's arcs>$

$$\text{tangent } 3 : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs> \longrightarrow \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}).$$

Moreover, Green-Griffiths proved that the tangent 3 map is surjective.

Theorem 4.15. Green-Griffiths [12, page 127, theorem 8.22]

The tangent map from Green-Griffiths’ arcs to local cohomology groups

$$\text{tangent } 3 : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} \{GG's arcs\} \longrightarrow \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}).$$

is surjective.

Consequently, we know the tangent 3 map from the group $\bigoplus <GG's arcs>$ to local cohomology groups

$$\text{tangent } 3 : \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs> \longrightarrow \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}).$$

is surjective.

Corollary 4.16. $\bigoplus K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ is generated by Green-Griffiths’ arcs.

$$\bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs>.$$

Proof. Let $K$ denote the kernel of the tangent 3 map, it is obvious that $\bigoplus_{y \in X^{(1)}} K^M_1(k(y)) \subseteq K$. This results in

$$\frac{\bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs>}{\bigoplus_{y \in X^{(1)}} K^M_1(k(y))} \supset \frac{\bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs>}{K}.$$

Since tangent 3 map is surjective, the right side of the above inequality

$$\frac{\bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <GG's arcs>}{K} = \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/Q}).$$
According to Theorem 4.1, we know
\[ \bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) = \bigoplus_{y \in X^{(1)}} K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) / \bigoplus_{y \in X^{(1)}} K^M_1(k(y)). \]

Hence, we have
\[ \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <\text{GG}'s arcs > \bigoplus_{y \in X^{(1)}} K^M_1(k(y)) = \bigoplus_{y \in X^{(1)}} K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \bigoplus_{y \in X^{(1)}} K^M_1(k(y)). \]

Therefore,
\[ \bigoplus_{y \in X^{(1)}} K^M_1(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(1)}} <\text{GG}'s arcs >. \]

One can easily check that we have the following commutative diagram

**Theorem 4.17.** There exists the following commutative diagram

\[ \begin{array}{ccc}
\bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{\text{tangent 2}} & K^M_2(k(X)[\varepsilon]) \\
\bigoplus_{y \in X^{(1)}} K^M_1(k(y)) & \xrightarrow{\varepsilon=0} & K^M_2(k(X)) \\
\end{array} \]

Recall that we also have the following commutative diagram

\[ \begin{array}{ccc}
\bigoplus_{y \in X^{(1)}} H^1_y(\Omega^1_{X/\mathbb{Q}}) & \xleftarrow{\text{Chern-2}} & K^M_2(k(X)[\varepsilon]) \\
\bigoplus_{y \in X^{(1)}} K^M_1(k(y)) & \xrightarrow{\varepsilon=0} & K^M_2(k(X)) \\
\end{array} \]

We have shown that the Chern-2 map agrees with Green-Griffiths' tangent 2 map on page 31. Then we have the following theorem.

**Theorem 4.18.** The Chern-3 map agrees with Green-Griffiths' tangent 3 map, at least on the image of \(d_{1,2,\varepsilon}^{-2}\).

We conclude this section with the following summary.

**Theorem 4.19.** With all the above interpretation, the K-theoretic higher Chow group \(CH^M_2(D_{\text{perf}}(X[\varepsilon]), 1)\) is of concrete geometric meaning. It can be considered as a deformation of \(CH^M_2(D_{\text{perf}}(X), 1)\) which is isomorphic to \(CH^2(X, 1)\).
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Therefore, the tangent space to $CH^M_2(D^\text{perf}(X),1)$, defined formally in definition 3.26

$$TCH^M_2(D^\text{perf}(X),1) := \text{Ker}\{CH^M_2(D^\text{perf}(X[\varepsilon]),1) \xrightarrow{\varepsilon=0} CH^M_2(D^\text{perf}(X),1)\},$$

carries concrete geometric meaning.

5. Choice of tangent spaces

In this section, we discuss the choice of defining tangent spaces to higher Chow groups $CH^q(X,m)$, where $m = 1,2$. After tensoring with $\mathbb{Q}$, we have shown the following identification

$$CH^q(X,m) = CH_q(D^\text{perf}(X),m) = CH^M_q(D^\text{perf}(X),m).$$

Recall that, in section 3, we have defined tangent spaces to $CH_q(D^\text{perf}(X),m)$ and $CH^M_q(D^\text{perf}(X),m)$ as follows respectively.

**Definition 5.1.** The tangent spaces to $CH_q(D^\text{perf}(X),m)$ and $CH^M_q(D^\text{perf}(X),m)$, are defined to be

$$TCH_q(D^\text{perf}(X),m) := \text{Ker}\{CH_q(D^\text{perf}(X[\varepsilon],m)) \xrightarrow{\varepsilon=0} CH_q(D^\text{perf}(X),m)\}$$

$$TCH^M_q(D^\text{perf}(X),m) := \text{Ker}\{CH^M_q(D^\text{perf}(X[\varepsilon],m)) \xrightarrow{\varepsilon=0} CH^M_q(D^\text{perf}(X),m)\}.$$  

We also have identified these tangent spaces with cohomology group of absolute differentials.

**Theorem 5.2.**

$$TCH^M_q(D^\text{perf}(X),m) = H^{q-m}(\Omega_{X/\mathbb{Q}}^{q-1})$$

$$TCH_q(D^\text{perf}(X),m) = H^{q-m}(\Omega_{X/\mathbb{Q}}^\bullet).$$

where $\Omega_{X/\mathbb{Q}}^\bullet = \Omega_{X/\mathbb{Q}}^{q-1} \oplus \Omega_{X/\mathbb{Q}}^{q-3} \oplus \ldots$

When $q \leq 2$, $TCH_q(D^\text{perf}(X),m)$ agree with $TCH^M_q(D^\text{perf}(X),m)$. When $q \geq 3$, $TCH_q(D^\text{perf}(X),m)$ is different from $TCH^M_q(D^\text{perf}(X),m)$. So we now have the following question.

**Question 5.3.** When $q \geq 3$, which is the right choice of tangent spaces to $CH^q(X,m)$?

Of course, the answer to this question depends on the definition of “right choice”. Apparently, $TCH_q(D^\text{perf}(X),m)$ contains more information than $TCH^M_q(D^\text{perf}(X),m)$ does. However, we want the tangent space to $CH^q(X,m)$ to carry concrete geometric meaning, like we have shown in chapter 4. So we choose $TCH^M_q(D^\text{perf}(X),m)$ to be the definition of $TCH^q(X,m)$. 

Definition 5.4.

\[ TCH^q(X, m) := TCH^M_q(D^{perf}(X), m). \]

Now, we explain why the tangent spaces \( TCH^q(D^{perf}(X), m) \) may not carry geometric meaning. In the remaining of this section, \( X \) is a smooth projective 3-fold over a field \( k, \text{char} k = 0. \)

Theorem 5.5. Taking \( q = 3 \) and \( j = 2, 3 \) in [33, theorem 4.28], we obtain the following two splitting commutative diagrams. The highest eigen-component

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\Omega^2_{X/Q} & \xleftarrow{\oplus z \in X^{(1)} H^1_z(\Omega^2_{X/Q})} & K_3^{(3)}(X[\varepsilon]) \\
\downarrow & & \downarrow \\
\Omega^2_{k(X)/Q} & \xleftarrow{\oplus y \in X^{(2)} H^2_y(\Omega^2_{X/Q})} & K_3^{(3)}(k(X)[\varepsilon]) \\
\downarrow & & \downarrow \\
\mathbb{R}^3 & \xleftarrow{\oplus x \in X^{(3)} H^3_x(\Omega^2_{X/Q})} & K_3^{(3)}(O_{X,x} \text{ on } x[\varepsilon]) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

and the second highest eigen-component
Noting $K_1^{(2)}(O_X, y \text{ on } y) = K_2^{(2)}(O_X, z \text{ on } z) = 0$, we have

$K_1^{(2)}(O_X, y[\varepsilon] \text{ on } y[\varepsilon]) = H_y^2(O_X)$

$K_2^{(2)}(O_X, z[\varepsilon] \text{ on } z[\varepsilon]) = H_z^1(O_X)$.

Because of depth condition, $H_y^2(O_X) \neq 0$ and $H_z^1(O_X) \neq 0$, so both $K_1^{(2)}(O_X, y[\varepsilon])$ and $K_2^{(2)}(O_X, z[\varepsilon])$ are non-zero. Therefore, in the second highest eigen-component diagram, the middle column can’t be a deformation of the right column in the usual sense.

On the other hand, we hope the highest eigen-component diagram carry geometric information. We continue exploring geometry behind this diagram in future work.

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Mathematics Department, University of Louisiana, Baton Rouge, Louisiana

E-mail address: senyangmath@gmail.com