Application of the variational method for studying stability of the 3D Lotka — Volterra system

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Abstract. The paper investigates the stability of the Lotka — Volterra dynamic system «predator — two preys» with the help of variational method. This method is applicable for studying equilibrium stability non-linear dynamical systems of any kind and in any dimension. It approves its effectiveness even in cases where the use of the Lyapunov function causes great difficulties (for example, when solving tasks of large dimension). The proposed method is based on a variation method for estimating the rate of phase-state metric change for the perturbed system when the variable $t \to \infty$. The disadvantage of the variation method is the fact that this method provides only the necessary conditions of stability. However, its it forms the first step in studying the stability of non-linear dynamical systems of any kind and any dimension.

1. Introduction

Let us consider the problem of studying the stability of the equilibrium positions of the Lotka — Volterra dynamic system «predator — two preys» (1)

$$
\begin{align*}
\dot{x}_1 &= -\gamma_1 \cdot x_1 \cdot (1 - x_2 - x_3); \\
\dot{x}_2 &= \gamma_2 \cdot x_2 \cdot (1 - x_1); \\
\dot{x}_3 &= \gamma_3 \cdot x_3 \cdot (\alpha - x_1).
\end{align*}
$$

Here $\alpha, \gamma_i$, with $i = 1, 2, 3$ are constants (parameters of the system), functions $x_i = x_i(t)$ denote the amount of the predator ($x_1$) and preys ($x_2, x_3$). System (1) has the following three equilibrium points $M_0(0, 0, 0)$, $M_1(1, 1, 0)$, $M_2(\alpha, 0, 1)$ respectively. For studying equilibrium points we apply the method proposed in the second point, which is based on checking necessary conditions of stability.

2. Main idea

The main idea of variational method is to determine the the maximum rate of change of Euclidean metric. $S = \frac{1}{2} \sum_{k=1}^{n} x_k^2$ in state-space, assuming that the solution sought does not leave $S(x) < \varepsilon$.

We analyse (1) in small neighbourhood $S(x_1, x_2, x_3) < \varepsilon$.

**Definition 1.** The solution of a system (1) is called $\varepsilon$ — stable, if there exist $\varepsilon > 0$ and $t_1 \in (0, +\infty)$ such that for all $t > t_1$ the trajectory of the system $x(t) = (x_1(t), x_2(t), x_3(t))$ does not leave $S(x_1, x_2, x_3) < \varepsilon$. 


If the zero solution of the system (1) is (asymptotically) stable, then the necessary conditions for stability (asymptotically stability) of zero solution perturbation in the vicinity of equilibrium point. Consider variational method for studying stability of system (1). Apply a variational method to equilibria at points in the vicinity of equilibrium point. Consider variational method for studying stability of system (1). Apply a variational method to equilibria at points in the vicinity of equilibrium point.

**Definition 2.** Solution of a system (1) is called asymptotically stable, if for all $\varepsilon > 0$ the trajectory $x(t)$ monotonically tends to zero in $S(x_1, x_2, x_3) < \varepsilon$ and reaches zero when $t \leq \infty$ (i.e. $t \to \infty$). At the same time trajectory satisfies the condition (derivative due to the system (1)) $\dot{S} = (\nabla S, \dot{x}) \leq 0$. The following statement holds true.

**Statement 1.** The fact that a given (perturbed) solution of dynamic system (1) is $\varepsilon$-stable, demands the following conditions are required:

- Time derivative $\dot{S}$ of function $S(x_1, x_2, x_3)$ due to the system (1) for the perturbed solution reaches maximum inside $S(x_1, x_2, x_3) < \varepsilon$.
- In case of monotonic asymptotic stability the maximum is reached on the zero solution.

**Corollary.** This statement, in particular yields, that the conditions

$$\frac{\partial S}{\partial x_i} = 0; \quad \frac{\partial^2 S}{\partial x_i^2} \leq 0; \quad \left( \frac{\partial S}{\partial x_i} = 0; \quad \frac{\partial^2 S}{\partial x_i^2} < 0 \right); \quad i = 1, 2, 3$$

are necessary conditions for stability (asymptotically stability) of zero solution perturbation in the vicinity of equilibrium point. Consider variational method for studying stability of system (1). Apply a variational method to equilibria at points $M_0(0, 0, 0)$, $M_1(1, 1, 0)$, $M_2(\alpha, 0, 1)$.

**3. Point $M_0(0, 0, 0)$**

For the singular point $M_0(0, 0, 0)$ construct the make metric function:

$$\dot{S} = x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 + x_3 \cdot \dot{x}_3 = -\gamma_1 \cdot x_1^2 \cdot (1 - x_2 - x_3) + \gamma_2 \cdot x_2^2 \cdot (1 - x_1) + \gamma_3 \cdot x_3^2 \cdot (\alpha - x_1).$$

For the first partial derivatives of metric function $\dot{S}$ the necessary conditions of stability of perturbed solution in the vicinity of zero equilibrium positions are as follows:

$$\begin{align*}
\frac{\partial \dot{S}}{\partial x_1} &= -2 \cdot x_1 \cdot \gamma_1 \cdot (1 - x_2 - x_3) - \gamma_2 \cdot x_2^2 \cdot \gamma_3 \cdot x_3^2 = 0; \\
\frac{\partial \dot{S}}{\partial x_2} &= \gamma_1 \cdot x_1^2 + 2 \cdot \gamma_2 \cdot x_2 \cdot (1 - x_1) = 0; \\
\frac{\partial \dot{S}}{\partial x_3} &= \gamma_1 \cdot x_1^2 + 2 \cdot \gamma_3 \cdot x_3 \cdot (\alpha - x_1) = 0.
\end{align*}$$

For second partial derivatives of metric function $\dot{S}(x_1, x_2, x_3)$ the necessary conditions for stability of perturbed solution in the vicinity of zero equilibrium positions are as follows:

$$\begin{align*}
\frac{\partial^2 \dot{S}}{\partial x_1^2} &= -2 \cdot \gamma_1 \cdot (1 - x_2 - x_3) = -2 \cdot \gamma_1 + o(\rho) < 0 \quad \Leftrightarrow \quad \gamma_1 > 0; \\
\frac{\partial^2 \dot{S}}{\partial x_2^2} &= 2 \cdot \gamma_2 \cdot (1 - x_1) = 2 \cdot \gamma_2 + o(\rho) < 0 \quad \Leftrightarrow \quad \gamma_2 < 0; \\
\frac{\partial^2 \dot{S}}{\partial x_3^2} &= 2 \cdot \gamma_3 \cdot (\alpha - x_1) = 2 \cdot \alpha \cdot \gamma_3 + o(\rho) < 0 \quad \Leftrightarrow \quad \gamma_3 < 0.
\end{align*}$$

**Statement 2.** If the zero solution of the system (1) is (asymptotically) stable in the vicinity of equilibrium positions $(0, 0, 0)$, then the following inequalities hold true:

$$\gamma_1 > 0, \quad \gamma_2 < 0, \quad \alpha \cdot \gamma_3 < 0 \quad \text{(or } \Leftrightarrow \gamma_3 < 0).$$

In the other words, this solution is not stable, if at least one of the indicated inequalities is violated (that contradicts a condition $\gamma_1 > 0$), therefore the perturbed solution of zero equilibrium positions is not stable.
4. Point $M_1(1,1,0)$

In order to study the stability of equilibrium position of the point $M_1(1,1,0)$ of system (1), we let: $x_1 = z_1 + 1$; $x_2 = z_2 + 1$; $x_3 = z_3$. Thus we obtain the following system:

$$\begin{cases}
\dot{z}_1 = -\gamma_1 \cdot (z_1 + 1) \cdot (-z_2 - z_3); \\
\dot{z}_2 = -\gamma_2 \cdot z_1 \cdot (z_2 + 1); \\
\dot{z}_3 = \gamma_3 \cdot z_3 \cdot (\alpha - 1 - z_1).
\end{cases} \tag{2}$$

For this system the point $(0,0,0)$ is an equilibrium point. Following variational method we construct the metric function:

$$S(z_1, z_2, z_3) = z_1 \cdot \dot{z}_1 + z_2 \cdot \dot{z}_2 + z_3 \cdot \dot{z}_3 =$$

$$= \gamma_1 \cdot z_1 \cdot (z_1 + 1) \cdot (z_2 + z_3) - \gamma_2 \cdot z_1 \cdot z_2 + \gamma_3 \cdot z_3^2 \cdot (\alpha - 1 - z_1).$$

Necessary conditions for stability of perturbed solution in the vicinity of zero equilibrium position $(0,0,0)$ of system (2) for the first partial derivatives of function $S(z_1, z_2, z_3)$ are as follows:

$$\begin{aligned}
\frac{\partial S}{\partial z_1} &= \gamma_1 \cdot (2 \cdot z_1 + 1) \cdot (z_2 + z_3) - \gamma_2 \cdot (z_2^2 + z_2) - \gamma_3 \cdot z_3^2 = 0; \\
\frac{\partial S}{\partial z_2} &= \gamma_1 \cdot (z_1^2 + z_1) - \gamma_2 \cdot z_1 \cdot (2 \cdot z_2 + 1) = 0; \\
\frac{\partial S}{\partial z_3} &= \gamma_1 \cdot (z_1^2 + z_1) + 2 \cdot \gamma_3 \cdot z_3 \cdot (\alpha - 1 - z_1) = 0.
\end{aligned}$$

Next we need to define necessary conditions for stability of perturbed solution for the second partial derivatives of function $S(z_1, z_2, z_3)$ in the vicinity of zero equilibrium positions of system (2). These conditions are as follows:

$$\begin{aligned}
\frac{\partial^2 S}{\partial z_1^2} &= 2 \cdot \gamma_1 \cdot (z_2 + z_3) \leq 0; \quad \Leftrightarrow \gamma_1 < 0; \\
\frac{\partial^2 S}{\partial z_2^2} &= -2 \cdot \gamma_2 \cdot z_1 \leq 0; \quad \Leftrightarrow \gamma_2 > 0; \\
\frac{\partial^2 S}{\partial z_3^2} &= 2 \cdot \gamma_3 \cdot (\alpha - 1 - z_1) \leq 0; \quad \Leftrightarrow \gamma_3 \cdot (\alpha - 1) < 0 \quad (\gamma_3 > 0, \alpha < 1).
\end{aligned}$$

**Statement 3.** Perturbed solution in the vicinity of equilibrium positions $(1,1,0)$ of system (1) (or, which is also the same, zero equilibrium positions of system (2)) *is necessarily stable*, if the following conditions hold true:

$$\gamma_1 < 0; \quad \gamma_2 > 0; \quad \gamma_3 \cdot (\alpha - 1) < 0 \quad (\text{or} \gamma_3 > 0, \alpha < 1).$$

In the other words, if at least one of these conditions is violated, the specified solution *is not stable*.

5. Point $M_2(\alpha,0,1)$

In order to study the stability of equilibrium position of the point $M_2(\alpha,0,1)$ of system (1), we let: $x_1 = z_1 + \alpha$; $x_2 = z_2$; $x_3 = z_3 + 1$.

Now the equilibrium position $M_2(\alpha,0,1)$ becomes the zero equilibrium position of the following:

$$\begin{cases}
\dot{z}_1 = \gamma_1 \cdot (z_1 + \alpha) \cdot (z_2 + z_3); \\
\dot{z}_2 = \gamma_2 \cdot z_2 \cdot (1 - \alpha - z_1); \\
\dot{z}_3 = -\gamma_3 \cdot z_1 \cdot (1 + z_3).
\end{cases} \tag{3}$$
Following the variational method we get the following metric function:

\[ \dot{S}(z_1, z_2, z_3) = \gamma_1 \cdot z_1 \cdot (z_1 + \alpha) \cdot (z_2 + z_3) + \gamma_2 \cdot z_2^2 \cdot (1 - \alpha - z_1) - \gamma_3 \cdot z_1 \cdot z_3 \cdot (1 + z_3). \]

Necessary conditions of stability of disturbed solution of zero equilibrium position of system (3) (and therefore equilibrium position \(M_2(\alpha, 0, 1)\) of system (1)) for first partial derivatives of metric function \(S(z_1, z_2, z_3)\) are the following:

\[
\begin{align*}
\frac{\partial S}{\partial z_1} &= \gamma_1 \cdot (2 \cdot z_1 + \alpha) \cdot (z_2 + z_3) - \gamma_2 \cdot z_2^2 - \gamma_3 \cdot (z_3 + z_3^2) = 0; \\
\frac{\partial S}{\partial z_2} &= \gamma_1 \cdot (z_1^2 + \alpha \cdot z_1) + 2 \cdot \gamma_2 \cdot z_2 \cdot (1 - \alpha - z_1) = 0; \\
\frac{\partial S}{\partial z_3} &= \gamma_1 \cdot (z_1^2 + \alpha \cdot z_1) - \gamma_3 \cdot z_1 \cdot (2 \cdot z_3 + 1) = 0.
\end{align*}
\]

Necessary conditions of stability of disturbed solution for second partial derivatives of metric function \(\ddot{S}(z_1, z_2, z_3)\) are the following:

\[
\begin{align*}
\frac{\partial^2 S}{\partial z_1^2} &= 2 \cdot \gamma_1 \cdot (z_2 + z_3) \leq 0; \quad \Leftrightarrow \quad \gamma_1 < 0; \\
\frac{\partial^2 S}{\partial z_2^2} &= 2 \cdot \gamma_2 \cdot (1 - \alpha - z_1) \leq 0; \quad \Leftrightarrow \quad \gamma_2 \cdot (1 - \alpha) < 0 \quad \Leftrightarrow \quad (\gamma_2 > 0, \alpha < 1); \\
\frac{\partial^2 S}{\partial z_3^2} &= -2 \cdot \gamma_3 \cdot z_1 \leq 0; \quad \Leftrightarrow \quad \gamma_3 > 0.
\end{align*}
\]

Hence we come to the following statement:

**Statement 4.** Perturbed solution in the vicinity of equilibrium position \(M_2(\alpha, 0, 1)\) of system (1) (or, which is also the same, zero equilibrium positions of system (3)) is necessarily stable, if the following conditions hold true:

\[ \gamma_1 < 0; \gamma_2 \cdot (1 - \alpha) < 0; \quad (\text{or} \quad \gamma_2 > 0, \alpha < 1); \quad \gamma_3 > 0. \]

In the other words, if at least one of these conditions is violated then the specified solution is not stable.

6. Theorem

Thus, the paper proposes a variation method for studying the stability of the equilibrium positions of Lotka – Volterra system((1)). This study shows that the variational tool is an effective method of optimization for essentially non-linear dynamical systems also. In particular, this method could be effectively applied to dynamic systems of large dimension.

It is shown that this method allows us to formulate the necessary stability conditions of the perturbed solution for each of the three equilibrium positions and for all of incoming parameters. At the same time, we found necessary conditions of stability of disturbed solution for each equilibrium position and for each incoming parameters, that is studying in researching optimization of solutions in systems. The following theorem is obtained:

**Theorem.** The necessary conditions of stability for bifurcation points of system (1) are:

- For stability of equilibrium position of \(M_0(0, 0, 0)\):

\[ \gamma_1 > 0; \quad \gamma_2 < 0; \quad \alpha \cdot \gamma_3 < 0 \quad (\text{or} \quad \gamma_3 < 0). \]
• For **stability** of equilibrium position of \( M_1(1, 1, 0) \):

\[
\gamma_1 < 0; \quad \gamma_2 > 0; \quad \gamma_3 \cdot (\alpha - 1) < 0 \quad (\text{or} \quad \gamma_3 > 0, \alpha < 1).
\]

• For **stability** of equilibrium position of \( M_2(\alpha, 0, 1) \):

\[
\gamma_1 < 0; \quad \gamma_2 \cdot (1 - \alpha) > 0; \quad (\text{or} \quad \gamma_2 < 0, \alpha < 1); \quad \gamma_3 > 0.
\]

• On the other hand, if at least one of the above conditions is not true, then perturbed solution in neighbourhood of the respective equilibrium position equilibrium positions is **not stable**.

**Remark.** While proving the theorem, it was not assumed that the system parameters are strictly positive, as it is in the Lotka — Volterra model «predator — two preys». Turning back to this model one can formulate the following result.

**Corollary.** If all the coefficients \( \gamma_i \) are positive, then all equilibrium positions \( M_i (i = 0, 1, 2) \) are not stable.

### 7. Stability by Lyapunov

Let’s compare our results with classical methods of finding stability by Lyapunov approach. Henceforth (for certainty) we will assume that all parameters \( \gamma_i > 0 \). We will consider the stability of the equilibrium positions with respect to the first approximation.

Then let \( M_i(x_1, x_2, x_3), \quad i = 0, 1, 2 \) - be equilibrium positions (point of bifurcation) of system (1). To determine the nature of these points’ stability, we introduce the Jacobian matrix.

\[
J(M) = \frac{\partial f}{\partial x}(M) = \begin{bmatrix}
-\gamma_1 \cdot (1 - x_2 - x_3) & x_1 \cdot \gamma_1 & x_1 \cdot \gamma_1 \\
-\gamma_2 \cdot x_2 & \gamma_2 \cdot (1 - x_1) & 0 \\
-\gamma_3 \cdot x_3 & 0 & \gamma_3 \cdot (\alpha - x_1)
\end{bmatrix}
\]

(4)

Characteristic determinant of this matrix at the point in \( M_0(0, 0, 0) \)

\[
|J(M_0) - \lambda \cdot E| = \begin{vmatrix}
-(\gamma_1 + \lambda) & 0 & 0 \\
0 & \gamma_2 - \lambda & 0 \\
0 & 0 & \alpha \cdot \gamma_3 - \lambda
\end{vmatrix}
\]

It has eigenvalues with different signs. \( \lambda_1 = \alpha \cdot \gamma_3 > 0; \quad \lambda_2 = \gamma_2 > 0; \gamma_3 = -\gamma_1 < 0 \). Then we get a conclusion that zero equilibrium position \( M_0(0, 0, 0) \) of system (1) is not stable by Lyapunov.

Let’s consider the bifurcation point \( M_1(1, 1, 0) \). At this point the characteristic matrix (4) equals:

\[
J(M_1) = \frac{\partial f}{\partial x}(M_1) = \begin{bmatrix}
0 & \gamma_1 & \gamma_1 \\
-\gamma_2 & 0 & 0 \\
0 & 0 & \gamma_3 \cdot (\alpha - 1)
\end{bmatrix}
\]

(5)

Eigenvalues of matrix are: \( \lambda_1 = (\alpha - 1) \cdot \gamma_3 < 0 \) (if \( \alpha < 1 \)), and two other are imaginary \( \lambda_{2,3} = \pm i \cdot \sqrt{\gamma_1 \cdot \gamma_2} \) , and \( (\gamma_1 \cdot \gamma_2) > 0 \). In this case, the study on the first approximation cannot determine anything. Determining the nature of stability requires further research. The method that we propose solves this problem definitely: the equilibrium position is not stable.

Let’s consider equilibrium position \( M_2(\alpha, 0, 1) \) and determine the nature of its stability. Jacobian matrix in this point is:

\[
J(M_2) = \frac{\partial f}{\partial x}(M_2) = \begin{bmatrix}
0 & \alpha \cdot \gamma_1 & \alpha \cdot \gamma_1 \\
0 & \gamma_2 \cdot (1 - \alpha) & 0 \\
-\gamma_3 & 0 & 0
\end{bmatrix}
\]

Eigenvalues of this matrix are: real one \( \lambda_1 = \gamma_2 \cdot (1 - \alpha) > 0 \) and two imaginary \( \lambda_{2,3} = \pm i \cdot \sqrt{\alpha \cdot (\gamma_1 \cdot \gamma_3)} \) bearing in mind that \( (\gamma_1 \cdot \gamma_3) > 0 \). In this case equilibrium position \( M_2(\alpha, 0, 1) \), obviously, is not stable.
Key words
Variational method, research of stability, non-linear dynamic system, necessary conditions of stability, partial derivative, Jacobi matrix, eigenvalues.

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