Vertex Rings and their Pierce Bundles

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Abstract. In part I we introduce vertex rings, which bear the same relation to vertex algebras (or VOAs) as commutative, associative rings do to commutative, associative algebras over $\mathbb{C}$. We show that vertex rings are characterized by Goddard axioms. These include a generalization of the translation-covariance axiom of VOA theory that involves a canonical Hasse-Schmidt derivation naturally associated to any vertex ring. We give several illustratory applications of these axioms, including the construction of vertex rings associated with the Virasoro algebra. We consider some categories of vertex rings, and the role played by the center of a vertex ring. In part II we extend the theory of Pierce bundles associated to a commutative ring to the setting of vertex rings. This amounts to the construction of certain reduced étale bundles of vertex rings functorially associated to a vertex ring. We introduce von Neumann regular vertex rings as a generalization of von Neumann regular commutative rings; we obtain a characterization of this class of vertex rings as those whose Pierce bundles are bundles of simple vertex rings.

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Part I: Vertex Rings

1. Introduction

The raison d’être for the present paper stems from the simple observation that the axioms for a vertex operator algebra (VOA) are integral: there are no denominators. It is therefore meaningful to speak of a vertex ring which, roughly speaking, is a VOA with an additive structure but not necessarily a linear structure, and somewhat more precisely, it is an additive abelian group admitting a countable infinity of $\mathbb{Z}$-bilinear operations satisfying the same basic identity (sometimes called the Jacobi identity) as a VOA.

It is well-known that certain VOAs possess an integral structure, i.e., a basis with respect to which the structure constants are integers, and the $\mathbb{Z}$-span of such a basis is a vertex ring. (For example lattice theories have this property.) Dong and Griess have made a study of such integral forms invariant under a group action [2], [3].
If the VOA $V$ has an integral structure and if $\tilde{V}$ is the $\mathbb{Z}$-span of an integral basis, the base-change $k \otimes \tilde{V}$ is a vertex $k$-algebra for any commutative ring $k$, and the binary operations become $k$-linear. VOAs defined over base rings (or at least base fields) other than $\mathbb{C}$ occur frequently in the literature. One encounters base-changes such as $\mathbb{C}[t] \otimes V$ frequently, though they are often viewed as VOAs defined over $\mathbb{C}$. And in a slightly different direction, Dong and Ren [5] and Li and Mu [14] have made interesting studies of Virasoro VOAs and Heisenberg VOAs respectively over base fields other than $\mathbb{C}$.

All of these examples point to the desirability of having available a general theory of vertex rings, and more generally vertex $k$-algebras, and the purpose of the present paper is to make a start on such a theory. On the other hand, our original motivation for getting involved with such a project was quite different, and arose from the desire to extend some results in [4] to a more general setting. There, Chongying Dong and I described the decomposition of a VOA into blocks according to its (central) idempotents and I wanted to see what this theory would look if the VOA had a lot of idempotents. In order to even formulate precisely what this means one needs the general notion of a vertex ring.

In the rest of this Introduction we will describe some of the content and main ideas of the present paper, which has two quite different parts. The first part deals with the axiomatics of vertex rings, the second with their so-called Pierce bundles. As is well-known, one may obtain an important characterization of vertex algebras (over $\mathbb{C}$) using the Goddard axioms [9], [12], [18]. The general idea is to show that the Jacobi identity for a VOA is equivalent to a collection of mutually local, translation-covariant, creative fields. Part I is mainly devoted to a generalization of this result to vertex rings and giving some applications. Most of the needed proofs dealing with locality already exist in the literature and carry over to the setting of vertex rings. However the same is not true of translation-covariance. Translation-covariance for VOAs deals with a certain natural derivation, often denoted by $T$. For a vertex ring $V$, $T$ must be replaced by what we call the canonical Hasse-Schmidt derivation of $V$, which is a certain sequence $D=(D_0:=Id_V, D_1, D_2, \ldots)$ of endomorphisms $D_i$ of $V$ satisfying Leibniz, or divided power, identities. We formulate a general translation-covariance axiom for vertex rings using the canonical HS derivation. This is carried out in Section 3. The introduction of the canonical HS derivation is very natural, and not without precedent. There is an extensive literature dealing with commutative rings with either a derivation or HS-derivation [7], [17]. Indeed, pairs $(k, D)$ consisting of a (unital) commutative ring $k$ equipped with HS-derivation $D$ provide perhaps the easiest examples of vertex rings that are not VOAs.

Section 4 is taken up with the characterization of vertex rings a la Goddard, using locality and our more general notion of translation-covariance. Here the exposition has been influenced by the presentation of Matsuo and Nagatomo [18]. We make several subsequent applications of this characterization. The first, in Section 4, deals with generating fields for a vertex ring. This is the most transparent way to construct VOAs and vertex rings alike. The remainder of Part I is concerned with categories of vertex rings and related topics. Of paramount importance for everything that comes later is the idea of the center $C(V)$ of a vertex ring $V$. This is concept is known in VOA theory [4], [12],
but its importance diminishes in the presence of denominators. One way to define $C'(V)$, which is naturally a commutative ring and a vertex subring of $V$, is as the group of $D$-constants of $V$. It is also the set of states with constant vertex operator (cf. Theorem \ref{thm:constant}). There is a categorical explanation for the importance of $C'(V)$ that runs as follows: a unital, commutative ring $k$ is a vertex ring. Indeed, it corresponds to a pair $(k,D)$ where $D = (\text{Id}_k, 0, 0, \ldots)$ is the trivial HS-derivation. Thus there is a functorial insertion

$$K : \text{Comm} \to \text{Ver}$$

of the category $\text{Comm}$ of unital, commutative rings into the category $\text{Ver}$ of vertex rings. It is a basic fact that in this way, $\text{Comm}$ becomes a coreflective subcategory of $\text{Ver}$, i.e., the insertion $K$ has a right adjoint (see \cite{Mason1993} for background). Indeed, the right adjoint is the center functor $C : \text{Ver} \to \text{Comm}$ that associates $C(V)$ to $V$. Section 6 is taken up with these issues and some other aspects of vertex rings that depend on the center functor. These include idempotents and units of $V$, all of which turn out to lie in $C(V)$. We also formally introduce vertex $k$-algebras. This could have been done from the outset in Section 1, but since we want to think of a commutative ring $k$ as a vertex ring it is more natural to define a vertex $k$-algebra as an object in the comma category $(k, \text{Ver})$ of objects under $k$. We treat tensor products of vertex rings using our Goddard axioms; this is the coproduct in $\text{Ver}$. This construction, which can be awkward even in the setting of VOA over $\mathbb{C}$ (cf. \cite{Goddard1986}), includes base changes such as $R \otimes_k V$ ($R$ is a commutative $k$-algebra) that we mentioned before.

Section 7 gives a more substantial application of the Goddard axioms to the construction of Virasoro vertex $k$-algebras over an arbitrary base ring $k$. The Virasoro $k$-algebra ($k$ a commutative ring) is the Lie $k$-algebra $\text{Vir}$ with $k$-basis $L(n)$ ($n \in \mathbb{Z}$) together with $K$, and where the nontrivial brackets are

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{6} \delta_{m+n,0} c' K$$

$(c' \in k$ is called the quasizentral charge of $\text{Vir}$). Compared to the usual definition of the Virasoro algebra \cite{Goddard1986}, this makes sense for any $k$. It amounts to a rescaling of the central element $K$ by a factor of 2. To show that $\sum_n L(n)z^{-n-2}$ is a generating field for a vertex $k$-algebra, thanks to the Goddard axioms one only needs to demonstrate the existence of a suitable HS derivation $D = (\text{Id}, D_1, D_2, \ldots)$. We show that $D$ exists and that

$$L(-1)^m = m!D_m (m \in \mathbb{Z}_{\geq 0}).$$

This construction allows us to define VOA over an arbitrary commutative ring $k$ as vertex $k$-algebras with a compatible $k$-grading and Virasoro vector whose modes satisfy \cite{Goddard1986}. Vertex algebras over $\mathbb{C}$ and $\text{Vir}$ itself are the basic examples. We include \cite{Lepowsky1985} as one of our VOA axioms. This seems natural, though experts may demur. In any case, we terminate our presentation of the axiomatics of vertex rings at this point.

The coreflective property of \cite{Goddard1986} suggests that $\text{Ver}$ may be regarded as a natural extension of $\text{Comm}$, and that certain theorems and/or theories that hold in $\text{Comm}$ might extend to $\text{Ver}$. This point of view motivates Part II, where the main idea is to demonstrate that some of the constructions and results in the remarkable paper \cite{Pierce1981} of Pierce do indeed extend to $\text{Ver}$. Pierce’s paper concerns
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Certain sheaves of rings functorially associated to a commutative ring $k$. Actually, in keeping with standard practice at the time, Pierce did not deal with sheaves *per se* but rather with *bundles*, and more precisely an equivalent category $\text{redCommbun}$ whose objects are reduced étale bundles of rings. ‘Reduced’ means that the bundles have a *Boolean base space* (Hausdorff and totally disconnected) and indecomposable fibers. One of the main results of [19] is an equivalence of categories $\text{Comm} \xrightarrow{\sim} \text{redCommbun}$. Similarly, one of our main results in Part II is the extension of this result to vertex rings: thus every vertex ring $V$ is canonically associated with a reduced étale bundle $\mathcal{R} \to X$ of vertex rings. An important point is that the base $X$ is none other than the base of the Pierce bundle $E \to X$ associated to $C(V)$, namely $X := \text{Spec}(B(C(V)))$ where $B(k)$ for a commutative ring $k$ is a certain Boolean ring whose elements comprise the idempotents of $k$. We call $X$ the *Stone space* of $k$ (or $V$), being closely related to the duality theory of Marshall Stone. It is sometimes called the *Boolean spectrum* of $k$. The upshot is that there is a diagram of categories and functors that is discussed in Section 11:

![Diagram](image)

where $\text{redVerbun}$ is the category of reduced étale bundles of vertex rings and the horizontal functors are equivalences.

For the purposes of the present paper it is crucial to deal with bundles rather than the corresponding sheaf of sections. This is because the infinitely many operations in a vertex ring may lead to problems with sections over open sets, and one does not necessarily obtain a sheaf of vertex rings as the sheaf of sections but rather something weaker - what we call a sheaf of *nonassociative vertex rings*. On the other hand the local sections over a *closed* set do carry the structure of a vertex ring. Since the base spaces we deal with are Boolean there is a basis of clopen sets, and this makes a sheaf perspective viable. Most of this is explained (with plenty of background) in the first two Sections of Part II.

Pierce’s theory works particularly well for commutative *von Neumann regular rings*, and something similar holds true for vertex rings. Thus in Section 10 we introduce *von Neumann regular* (vNr) vertex rings. These are vertex rings $V$ such that every principal 2-sided ideal has the form $e(-1)V$ for an idempotent $e$. We establish various properties of vNr vertex rings. In particular we show that in the upper equivalence of (1.4), the full subcategory of $\text{Ver}$ whose objects are vNr vertex rings corresponds to the category of reduced étale bundles of *simple vertex rings*. This result is the main motivation for considering vNr vertex rings. Indeed, if $V$ is a simple vertex ring then $C(V)$ is a field, so that simple vertex rings are the more familiar vertex algebras over a field, and vNr vertex rings are exactly those vertex rings whose Pierce bundle has such vertex algebras as stalks. As a special case, applying this when $V$ is a commutative vNr ring recovers Pierce’s Theorem ([19], Theorem 10.3). Pierce used this result to study modules over a vNr ring, however we do not pursue the representation theory of vertex rings here.

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1. In fact, the phrase ‘étale bundle’ never occurs in [19], where such things are called ‘sheaves’.
The paper is expository in nature, though proofs are almost always complete. (Section 11 is an exception.) It should be possible for nonexperts to follow the material, while experts will find much that is familiar. This approach is more-or-less forced upon us by the nature of the material: some of the existing proofs in the literature concerning VOAs work perfectly well in the setting of vertex rings, some work only with modification, some do not work at all. Under the circumstances, it seemed better to give a presentation starting from scratch. Part II is written assuming that the reader has no prior knowledge of bundleology. We have explained the basic constructions in the context of vertex rings, though this is hardly different from bundles of commutative rings as discussed, for example, in [16]. Pierce’s original paper [19] is also an excellent place to read about his construction (indeed, about bundles too), and we have borrowed shamelessly from this source, going so far as to use the same notation in some places.

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2. Basic properties of vertex rings

In this and the following few Sections we introduce vertex rings and show that they consist of mutually local, creative, translation-covariant fields.

2.1. Definition of vertex ring.

Definition 2.1. A vertex ring is an additive abelian group $V$ equipped with biadditive products $(u, v) \mapsto u(n)v$ ($u, v \in V$) defined for all $n \in \mathbb{Z}$, together with a distinguished element $1 \in V$ (the vacuum element). The following identities are required to hold for all $u, v, w \in V$:

\begin{enumerate}
\item[(a)] there is an integer $n_0(u, v) \geq 0$ such that $u(n)v = 0$ for all $n \geq n_0$.
\item[(b)] $u(-1)1 = u; \ u(n)1 = 0$ for $n \geq 0$.
\item[(c)] $\forall r, s, t \in \mathbb{Z}$,
\begin{equation}
\sum_{i \geq 0} \binom{r}{i} (u(t + i)v)(r + s - i)w = \sum_{i \geq 0} (-1)^i \binom{t}{i} \{ u(r + t - i)v(s + i)w - (-1)^i v(s + t - i)u(r + i)w \}.
\end{equation}
\end{enumerate}

We refer to (2.1) as the Jacobi identity. Thanks to (a), the two sums in (c) make sense inasmuch as there are only finitely many nonzero terms. Similar comments will apply in numerous contexts in what follows, and we will generally not make this explicit. We call $u(n)v$ the $n^{th}$ product of $V$.

For an additive abelian group $V$, $\text{End}(V)$ denotes the set of endomorphisms of $V$. It is an associative ring with respect to composition, and a $\mathbb{Z}$-Lie algebra with respect to the usual bracket $[a, b] := ab - ba$.

If $V$ is a vertex ring, we often refer to elements of $V$ states, and call $u(n)$ the $n^{th}$ mode of the state $u$. Because $u(n)v$ is additive in $v$, we may, and shall, regard $u(n)$ as an endomorphism in $\text{End}(V)$ for all $u \in V$ and $n \in \mathbb{Z}$. Then (2.1) (c) can be regarded as an identity in $\text{End}(V)$.
The vertex operator corresponding to $u \in V$ is the formal generating function defined by

$$Y(u, z) := \sum_{n \in \mathbb{Z}} u(n) z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].$$

for an indeterminate $z$. Identities between endomorphisms of $V$ are conveniently written as identities involving vertex operators. To facilitate this we use some 'obvious' notations when dealing with vertex operators. For example, if $u, v \in V$ then

$$Y(u, z)v := \sum_{n \in \mathbb{Z}} u(n)v z^{-n-1} \in V[[z]][z],$$

the last containment being a consequence of (2.1)(a). Similarly, (2.1)(b) says that

$$Y(u, z)1 = u + zV[[z]].$$

(2.2)

**Definition 2.1.** We paraphrase (2.2) by saying that $Y(u, z)$ is creative with respect to 1 and creates the state $u$. (A refinement is discussed in Theorem 3.5.)

The additivity of $u(n)v$ in $u$ as well as $v$ permits us to promote $Y$ to a morphism of abelian groups

$$Y: V \rightarrow \text{End}(V)[[z, z^{-1}]], \quad u \mapsto Y(u, z).$$

$Y$ is then called the state-field correspondence. (Discussion of the word field as it used here is deferred until Section 4.1.)

**Remark 2.2.** The state-field correspondence is injective.

**Proof.** Use the creativity of $Y(u, z)$. □

### 2.2. Commutator, associator, and locality formulas

We emphasize some particularly useful special cases of (2.1). The first two, the commutator formula and associator formula, are obtained simply by setting $t=0$ and $r=0$ respectively. As identities in $\text{End}(V)$, they read as follows:

(2.3) $$[u(r), v(s)] = \sum_{i \geq 0} \binom{r}{i} (u(i)v)(r+s-i),$$

(2.4) $$(u(t)v)(s) = \sum_{i \geq 0} (-1)^i \binom{t}{i} \left\{ u(t - i)v(s + i) - (-1)^i v(s + t - i)u(i) \right\}.$$ 

The third special case arises by choosing $t \geq 0$ large enough so that, for a given pair of states $u, v$, we have $u(t+i)v = 0$ for all $i \geq 0$. The existence of $t$ is guaranteed by (2.1)(a), and with such a choice the left-hand-side of (2.1)(c) vanishes. What pertains is the following formula, which holds for $t \gg 0$:

(2.5) $$\sum_{i \geq 0} (-1)^i \binom{t}{i} \left\{ u(r + t - i)v(s + i) - (-1)^i v(s + t - i)u(r + i) \right\} = 0.$$ 

This is more compelling when formulated in terms of vertex operators:
Lemma 2.3. (Locality formula) If $u, v$ are states in a vertex ring then there is an integer $t \gg 0$ (depending on $u$ and $v$) such that
\[(z - w)^t[Y(u, z), Y(v, w)] = 0.\]

In other words,
\[(z - w)^tY(u, z)Y(v, w) = (z - w)^tY(v, w)Y(u, z).\]

Proof. We have
\[(z - w)^t[Y(u, z), Y(v, w)] = \left\{ \sum_{i=0}^{t} (-1)^i \binom{t}{i} z^i w^{t-i} \right\} \left\{ \sum_{m,n} [u(m), v(n)] z^{-m-1} w^{-n-1} \right\},\]
and the coefficient of $z^{-a-1}w^{-b-1}$ in this expression is
\[
\sum_{i=0}^{t} (-1)^i \binom{t}{i} |u(i + a), v(t - i + b)|
\]
\[
= \sum_{i=0}^{t} (-1)^{t-i} \binom{t}{i} u(t - i + a)v(i + b) - \sum_{i=0}^{t} (-1)^i \binom{t}{i} v(t - i + b)u(i + a).
\]
But this vanishes on account of (2.5) if $t$ is large enough. \qed

Definition 2.4. We paraphrase property (2.6) by saying that $Y(u, z)$ and $Y(v, z)$ are mutually local or order $t$, or simply mutually local if we do not wish to emphasize $t$.

Remark 2.5. (a) The identity (2.7) is also sometimes called weak commutativity. There is an analog, called weak associativity or duality, which can be proved by a similar argument that requires only slightly more effort. We will just state the result, which will not be used below: For fixed $u, v \in V$ and $t \gg 0$ we have
\[(z + w)^t[Y(u, z)v, w] = (z + w)^tY(u, z+w)Y(v, w),\]
where we are observing the convention (12.5) for the binomial expansion of $(z+w)^n$.

(b) As we will see, the Jacobi identity is more-or-less equivalent to the conjunction of weak commutativity and weak associativity. As in the classical theory of rings, it can be fruitful to consider axiomatic set-ups where weak associativity (but not weak commutativity) is assumed, leading to a theory of associative (but not commutative) vertex rings. One might even consider nonassociative vertex rings where neither weak commutativity nor weak associativity pertain (but other relevant axioms hold). Such objects arise naturally as sheaves of sections of bundles of vertex rings. We discuss this further in Section 8. The present work focuses almost exclusively on vertex rings (aka weak commutative, and associative vertex rings) as we have defined them.

We record a result which will be useful in several places.

Lemma 2.6. The following are equivalent for a vertex ring $V$ and states $u, v \in V$.

(a) $[u(r), v(s)] = 0$ for all $r, s \in \mathbb{Z}$,
(b) $u(n)v = 0$ for all $n \geq 0$.

Proof. This follows easily from (2.3). \qed
2.3. Vacuum vector. We will prove

**Theorem 2.7.** For all $n \in \mathbb{Z}$ we have $1(n) = \delta_{n,-1} \text{Id}_V$. That is,

$$Y(1, z) = \text{Id}_V.$$  

**Proof.** First take $v = w = 1, r = -1, t = 0$ in (2.1) and use (b) to obtain

$$u(-1)1(s)1 = 1(s)u(-1)1 = 1(s)u$$

for all $u \in V$. Thus in order to prove the Theorem, it suffices to show that $1(s)1 = \delta_{s,-1}1$. By (2.1)(b) we have $1(s)u = 0$ for $s \geq 0$ and any $u$, so certainly $1(s) = 0$ for $s \geq 0$. Similarly, $1(-1)1 = 1$.

We prove that $1(n)1 = 0$ for $n \leq -2$ by induction on $-n$. First take $u = v = w = 1$ and $t = -1$ in (2.1) together with (b) to see that

$$(2.9) \quad 1(r+s)1 = \sum_{i \geq 0} \{1(r-1-i)1(s+i)1 + 1(s-1-i)1(r+i)1\}$$

for all $r, s \in \mathbb{Z}$. If we first take $r = s = -1$ in (2.10) we obtain

$$1(-2)1 = \sum_{i \geq 0} \{1(-2-i)1(-1+i)1 + 1(-2-i)1(-1+i)1\} = 21(-2)1,$$

whence $1(-2)1 = 0$. This begins the induction. Let $r + s = n \leq -2$ where we choose $0 \leq r < -s - 1$. (2.10) then reads

$$1(n)1 = \sum_{i \geq 0} 1(r-1-i)1(s+i)1.$$  

Note that all of the modes $1(r), 1(s)$ commute thanks to (2.3). By induction, it follows that in the previous display, the only possible nonzero terms on the right-hand-side come from $i = r$ and $i = -s - 1$, and in both cases these are equal to $1(n)1$. Thus we obtain $1(n)1 = 21(n)1$ and therefore $1(n)1 = 0$. This completes the proof of the Theorem. \qed

3. Derivations

Derivations play a ubiquitous rôle in the theory of vertex rings.

3.1. Hasse-Schmidt derivations. By *nonassociative ring* we will always mean a not-necessarily associative ring $V$ that may not have an identity, i.e., an additive abelian group equipped with a biadditive product $uv$ ($u, v \in V$). The main examples we use are commutative rings, which will always mean commutative, associative rings with an identity; and vertex rings $V$ equipped with their $n^{th}$ product.

**Definition 3.1.** (a) Let $V$ be a nonassociative ring. A *derivation* of $V$ is an endomorphism $f \in \text{End}(V)$ such that $f(uv) = uf(v) + f(u)v$ ($u, v \in V$). (b) Let $V$ be a vertex ring. A *derivation* of $V$ is an endomorphisms $f \in \text{End}(V)$ such that $f$ is a derivation of each of the nonassociative rings defined by $V$ together with any of its $n^{th}$ products. In other words, we have for all $n \in \mathbb{Z}$ and $u, v \in V$,

$$f(u(n)v) = u(n)f(v) + f(u(n)v)$$

In each case we let $\text{Der}(V)$ denote the set of all derivations of $V$. By a standard argument, $\text{Der}(V) \subseteq \text{End}(V)$ is a Lie subalgebra.
DEFINITION 3.2. Let $V$ be an additive abelian group, and suppose that $\mathcal{D} = (D_0, D_1, \ldots)$ is a sequence of endomorphisms $D_i \in \text{End}(V)$ with $D_0 = \text{Id}_V$.

(a) If $V$ is a nonassociative ring, we call $\mathcal{D}$ a Hasse-Schmidt (HS) derivation of $V$ if, for all $u, v \in V$ and all $m \geq 0$, we have

$$D_m(uv) = \sum_{i+j=m} D_i(u)D_j(v).$$

(b) If $V$ is a vertex ring, we call $\mathcal{D}$ a HS derivation of $V$ if, for every $n \in \mathbb{Z}$, $\mathcal{D}$ is a HS derivation of the nonassociative ring defined by $V$ together with its $n^{th}$ product.

(c) $\mathcal{D}$ is called iterative if, for all $i, j \geq 0$, we have

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

\text{(3.1)}

EXAMPLE 3.3. (a) The trivial HS derivation of $V$ is $\mathcal{D} = (\text{Id}_V, 0, 0, \ldots)$, in which all higher $D_m$ ($m \geq 1$) are zero.

(b) If $\mathcal{D}$ is iterative then $D_m^n = m! D_m$ ($m \geq 0$).

Using the binomial theorem, we obtain

\text{LEMMA 3.4. Suppose that $\mathcal{D} = (\text{Id}_V, D_1, \ldots)$ is an iterative derivation. Then}

$$\left\{ \sum_{m=0}^{\infty} D_m z^m \right\} \left\{ \sum_{m=0}^{\infty} D_m (-z)^m \right\} = \text{Id}_V.$$\[\square\]

For HS derivations in the theory of commutative rings, see [17]. Iterative HS derivations arise naturally in vertex rings, as we now show.

\text{THEOREM 3.5. Let $V$ be a vertex ring, and for $m \geq 0$ define $D_m \in \text{End}(V)$ by the formula $D_m(u) := u(-m-1)1$, i.e.,}

$$Y(u, z)1 = \sum_{m \geq 0} D_m(u)z^m.$$

Then $\mathcal{D} = (D_0, D_1, \ldots)$ is an iterative, HS derivation of $V$.

\text{PROOF. We use Theorem 2.7 repeatedly in what follows. The identification $D_0 = \text{Id}_V$ follows from (2.1)(b). For the iterative property, we have}

$$D_\ell \circ D_m(u) = \binom{(-m-1)1}{\ell}(-\ell-1)1$$

$$= \sum_{i \geq 0} (-1)^i \binom{-m-1}{i} u(-m-1-i)1(-\ell-1+i)1$$

$$= (-1)^\ell \binom{-m-1}{\ell} u(-m-1-\ell)1$$

$$= \binom{-m-1}{\ell} \text{Id}_V.$$

As for the Hasse-Schmidt property, we first record

\footnote{To refer to $\mathcal{D}$ as a derivation is a convenient misnomer as only $D_1$ is a true derivation. $\mathcal{D}$ is called a \textit{differentiation} in [17].}
Lemma 3.6. We have
\[ D_i(u)(n) = (-1)^i \binom{n}{i} u(n-i). \]

Proof. We have
\[
(D_i u)(n) = (u(-i - 1)1)(n)
= \sum_{j \geq 0} (-1)^j \binom{n-j-1}{j} \{ u(-i - 1 - j)1(n + j) + (-1)^i 1(n - i - 1 - j)u(j) \}
= \sum_{j \geq 0} \binom{i+j}{j} \{ u(-i - 1 - j)1(n + j) + (-1)^i 1(n - i - 1 - j)u(j) \}
= (-1)^i \binom{n-i}{i} u(n-i)
\]
(check the cases \( n \geq 0 \) and \( n < 0 \) separately). The Lemma is proved. □

To complete the proof of Theorem 3.5, use Lemma 3.6 and (2.4) to obtain
\[
D_m u(n) v = (u(n)v)(-m-1)1
= \sum_{i \geq 0} (-1)^i \binom{n}{i} u(n-i)v(-m-1+i)1
= \sum_{i \geq 0} (D_i u)(n)D_{m-i}v.
\]
This is the required Hasse-Schmidt property. □

Definition 3.7. If \( V \) is a vertex ring, we call \( D \) defined as in Theorem 3.5 the canonical HS derivation of \( V \).

A first example of the utility of the canonical HS derivation is the skew-symmetry formula.

Lemma 3.8. Let \( V \) be a vertex ring with canonical HS derivation \( D \). Then for all \( u, v \in V \) and \( n \in \mathbb{Z} \) we have
\[
v(n)u = (-1)^{n+1} \sum_{i \geq 0} (-1)^i D_i(u(n+i)v).
\]

In terms of vertex operators, this reads
\[
Y(v, z)u = \sum_{m \geq 0} z^m D_m Y(u, -z)v.
\]

Proof. Take \( w=1 \) and \( r = -1, s=0 \) in (2.4) (c) to obtain
\[
\sum_{i \geq 0} (-1)^i (u(t+i)v)(-1-i)1 = \sum_{i \geq 0} (-1)^i \binom{t+i}{i} (-1)^{i+1} v(t-i)u(-1+i)1
= (-1)^{i+1} v(t)u.
\]
Since the left-hand-side is equal to \( \sum_{i \geq 0} (-1)^i D_i(u(t+i)v) \), the Lemma follows. □
A standard way to consider the iterative property of the canonical HS derivation involves the ring of divided powers \( \mathbb{Z}(x) \). This is the commutative ring generated by symbols \( x^n \) \((n \geq 0)\) subject to the identity
\[
x^m x^n = \binom{m+n}{n} x^{m+n}.
\]
\( \mathbb{Z}(x) \) can be realized as the subring of \( \mathbb{Q}[x] \) generated by \( \frac{x^n}{n!} \) \((n \geq 0)\). The iterative property of \( D \) immediately implies

**Lemma 3.9.** Suppose that \( V \) is a vertex ring with canonical HS derivation \( D \). Then the association \( D_n \mapsto x^n \) \((n \geq 0)\) makes \( V \) into a left \( \mathbb{Z}(x) \)-module. \( \Box \)

### 3.2. Translation-covariance.

Let \( V \) be a vertex ring with canonical HS derivation \( D \). For a state \( u \in V \) we set
\[
[\delta^i z] Y(u, z) := \frac{1}{i!} \partial_z^i Y(u, z),
\]
where \( \partial_z \) is formal differentiation with respect to \( z \). Despite the appearance of \( i! \) in the denominator, we have \( [\delta^i z] Y(u, z) \in \text{End}(V)[[z, z^{-1}]] \), because
\[
[\delta^i z] \left( \sum_n u(n) z^{-n-1} \right) = \sum_n \binom{-n-1}{i} u(n) z^{-n-1-i} = (-1)^i \sum_n \binom{n}{i} u(n - i) z^{-n-1}.
\]
In fact, \( [\delta^i z] Y(u, z) \) is the vertex operator for a state in \( V \). This is part of the next result.

**Theorem 3.10.** The following hold for all \( u \in V \) and \( m \geq 1 \).

(a) \( D_m 1 = 0 \),
(b) \( Y(D_m(u), z) = [\delta^m z] Y(u, z) \),
(c) \( [D_m, Y(u, z)] = \sum_{i=1}^{m} [\delta^i z] Y(u, z) D_{m-i} \).

**Proof.** (a) amounts to \( 1 (-i - 1) 1 = 0 \) for \( i \geq 1 \), which follows from Theorem 2.7. Part (b) follows from Lemma 3.6 and (3.4).

As for (c), use Theorem 3.3 to see that
\[
[D_m, Y(u, z)] w = \sum_n (D_m(u(n)w) - u(n)D_m(w)) z^{-n-1}
\]
\[= \sum_n \sum_{i=0}^{m} (D_i(u)(n)D_{m-i}(w) - u(n)D_m(w)) z^{-n-1}
\]
\[= \sum_n \sum_{i=1}^{m} D_i(u)(n)D_{m-i}(w) z^{-n-1}.
\]
This shows that
\[
[D_m, Y(u, z)] = \sum_{i=1}^{m} Y(D_i(u), z) D_{m-i},
\]
and then (c) follows from (b). This completes the proof of the Theorem. \( \Box \)
DEFINITION 3.11. We say that \( Y(u, z) \) is translation covariant with respect to \( D \) if property (c) of Theorem 3.10 holds for all \( m \geq 0 \).

4. Characterizations of vertex rings

In Sections 2 and 3 we have shown that the vertex operators in a vertex ring are mutually local (Definition 2.4), creative (Definition 2.1), and translation-covariant (Definition 3.11). In this Section we show that vertex rings can be characterized by these properties. This amounts to an extension of the Goddard axioms [9] for vertex algebras to the general setting of vertex rings. To carry this through we need to develop machinery to facilitate calculations with quantum fields on an arbitrary abelian group.

4.1. Fields on an abelian group.

DEFINITION 4.1. Let \( V \) be an additive abelian group. We set

\[
\mathcal{F}(V) := \left\{ a(z) := \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \in \text{End}(V)[[z, z^{-1}]] \mid a(n)b = 0 \text{ for } n \gg 0, \text{ all } b \in V \right\}.
\]

In this definition, it is understood that the integer \( t \) such that \( a(n)b = 0 \) for \( n \geq t \) depends on the states \( a \) and \( b \). Clearly, \( \mathcal{F}(V) \) is an additive abelian group. We say that \( a(z) \in \mathcal{F}(V) \) is a field on \( V \), and call \( \mathcal{F}(V) \) the space of fields on \( V \).

Definition 4.1 is, of course, motivated by the corresponding axiom (2.1)(a) for vertex operators in a vertex ring. Indeed, if \( V \) is a vertex ring, the state-field correspondence defines a morphism of abelian groups \( Y: V \to \mathcal{F}(V) \).

We now carry over to fields in \( \mathcal{F}(V) \) the main properties that we previously considered for vertex operators in a vertex ring. To be clear, we repeat the relevant definitions in this more general setting.

DEFINITION 4.2. Let \( V \) be an additive abelian group. Let \( a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \) and \( b(z) \in \mathcal{F}(V) \) be fields on \( V \), \( v_0 \in V \) be a fixed state, and \( D := (Id_V, D_1, ...) \) a sequence of endomorphisms of \( V \).

(a) \( a(z) \) is creative with respect to \( v_0 \) and creates the state \( a \in V \), if \( a(n)v_0 = 0 \) for \( n \geq 0 \), \( a(-1)v_0 = a \), i.e., \( a(z)v_0 \in a + zV[[z]] \).

We say that \( a(z) \) is merely creative (with respect to \( v_0 \)) if \( a(n)v_0 = 0 \) for \( n \geq 0 \), in which case the state \( a(-1)v_0 \) that is created is unspecified.

(b) \( a(z) \) is translation covariant with respect to \( D \) if, for all \( m \geq 0 \), we have

\[
[D_m, a(z)] = \sum_{i=1}^{m} \delta_z^{(i)} a(z) D_{m-i}.
\]

(c) \( a(z) \) and \( b(z) \) are mutually local (of order \( t \)) if there is \( t \geq 0 \) such that

\[
(z - w)^t[a(z), b(w)] = 0.
\]

We write this as \( a(z) \sim_t b(z) \), or simply \( a(z) \sim b(z) \) if we do not wish to emphasize \( t \).

In (b), the operator \( \delta_z^{(i)} \) on fields is defined as in [3,4]. As in Section 2.2 the mutual locality of \( a(z) \) and \( b(z) \) is equivalent to the analog of the locality formula (2.6) for all integers \( r, s \).
The thrust of our earlier arguments is that if \( V \) is a vertex ring then the set of vertex operators \( \{ Y(u, z) | u \in V \} \) is a set of mutually local fields on \( V \) that are creative with respect to the vacuum vector \( \mathbf{1} \) and translation covariant with respect to the canonical Hasse-Schmidt derivation of \( V \).

4.2. Statement of the existence Theorem. In this Subsection we state the main existence Theorem and make a start on its proof.

Theorem 4.3. Let \( (V, Y, v_0, \mathcal{D}) \) consist of an additive abelian group \( V \), a state \( v_0 \in V \), a sequence of endomorphisms \( \mathcal{D} := (\text{Id}_V, D_1, \ldots) \) in \( \text{End}(V) \) satisfying \( D_m(v_0) = 0 \) for \( m \geq 1 \), and a morphism of abelian groups \( Y : V \to \mathcal{F}(V) \), \( u \mapsto Y(u, z) := \sum_n u(n)z^{-n-1} \).

Suppose that the following assumptions hold for all states \( u, v \in V \):

\[
Y(u, z) \sim Y(v, z),
Y(u, z)v_0 \in u + zV[[z]],
[D_m, Y(u, z)] = \sum_{i=1}^{m} \delta_z^{(i)}Y(u, z)D_{m-i} \ (m \geq 0).
\]

(In short, \( \{ Y(u, z) | u \in V \} \) is a set of mutually local, creative and translation-covariant fields on \( V \).) Then \( V \) is a vertex ring with state-field correspondence \( Y \), vacuum vector \( v_0 \), and canonical HS derivation \( \mathcal{D} \).

Given the translation-covariance assumption as in the statement of the Theorem, the creativity assumption \( Y(u, z)v_0 \in u + zV[[z]] \) is equivalent to the stronger assertion

\[
Y(u, z)v_0 = \sum_{m \geq 0} D_m(u)z^m.
\]

Indeed, translation-covariance implies that

\[
D_mY(u, z)v_0 = \delta_z^{(m)}Y(u, z)v_0,
\]

so that \( D_m(u) = D_mu(-1)v_0 = u(-m - 1)v_0 \). Then we deduce that \( Y(u, z)v_0 = \sum_{m \geq 0} u(-m - 1)v_0z^m = \sum_{m \geq 0} D_m(u)z^m \), as asserted. In particular, in the context of Theorem 4.4 once it is known that \( V \) is a vertex ring the statement that \( \mathcal{D} \) is the canonical HS derivation of \( V \) follows automatically.

Let \( u, v \in V \). Since \( Y(u, z)v := \sum_n u(n)vz^{-n-1} \in V[[z, z^{-1}]] \), we have bilinear products \( u(n)v \) for all integers \( n \), and because \( Y(u, z) \in \mathcal{F}(V) \) then \( u(n)v = 0 \) for \( n > 0 \). Furthermore, the creativity assumption means that \( u(n)v_0 = 0 \) for \( n \geq 0 \) and \( u(-1)v_0 = u \). Thus \( Y(u, z)v \) is a vertex ring and thereby complete the proof of Theorem 4.3 it only remains to establish the Jacobi identity.

As a first step we have the following result.

Lemma 4.4. Suppose that \( V \) is an additive abelian group, with \( a(z) := \sum_n a(n)z^{-n-1} \), \( b(z) := \sum_n b(n)z^{-n-1} \) a pair of fields on \( V \). Then the modes of \( a(z) \) and \( b(z) \) satisfy the associativity formula \( \langle 2.3 \rangle \) and the locality formula \( \langle 2.7 \rangle \) for all integers \( r, s, t \) if, and only if, they satisfy the Jacobi identity \( \langle 2.4 \rangle \) for all integers \( r, s, t \).
Proof. In Section 2.2 we derived associativity and locality (cf. Lemma 2.3) of vertex operators in a vertex ring as a purely formal consequence of (2.1)(c), and the proof in the more general set-up we are now in is exactly the same.

It remains to show that, conversely, (2.1)(c) is a consequence of the conjunction of associativity and locality of fields in \( F(V) \). In fact, standard proofs of this assertion for vertex algebras defined over \( \mathbb{C} \) (e.g. [18], Proposition 4.4.3) remain valid in the present setting. We sketch the details following the proof of Matsuo-Nagatomo (loc. cit).

For any \( r,s,t \in \mathbb{Z} \) we introduce the notation

\[
A(r,s,t) = \sum_{i \geq 0} \binom{r}{i} (a(t+i)b(r+s-i)),
\]

\[
B(r,s,t) = \sum_{i \geq 0} (-1)^i \binom{t}{i} (r+i)b(s+i),
\]

\[
C(r,s,t) = \sum_{i \geq 0} (-1)^{t+i} \binom{t}{i} b(s+t-i)a(r+i).
\]

In these terms, the Jacobi identity (2.1)(c) for the fields \( a(z), b(z) \) just says that for all \( r,s,t \) we have

\[
(4.2) \quad A(r,s,t) = B(r,s,t) - C(r,s,t).
\]

On the other hand, as we discussed in Subsection 2.2, the associativity formula (2.4) is nothing but the case \( r = 0 \) of (4.2), while locality in the form of (2.5) is just (4.2) for \( t \gg 0 \). So we have to deduce the general case of (4.2) on the basis of these two special cases.

We can do this by first using (12.2) in the Appendix to observe that

\[
(4.3) \quad A(r+1,s,t) = A(r,s+1,t) + A(r,s,t+1).
\]

Furthermore, exactly the same formula holds if we replace \( A \) by \( B \) or \( C \).

Because (4.2) holds for \( r = 0 \) (and any \( s,t \)), an induction using (4.3) shows that it holds for all \( r \geq 0 \). Since it also holds for all big enough \( t \) independently of \( r,s \), if it is false in general then there is a pair \( (r,t) \) for which it is false and for which \( r+t \) is maximal. But we have

\[
A(r,s,t) = A(r+1,s-1,t) - A(r,s-1,t+1)
\]

\[
= B(r+1,s-1,t) - C(r+1,s-1,t) - B(r,s-1,t+1) + C(r,s-1,t+1)
\]

\[
= B(r,s,t) - C(r,s,t).
\]

So in fact (4.2) holds for all \( r,s,t \), and the proof of the Lemma is complete. \( \square \)

4.3. Residue products. Because locality of fields is one of the hypotheses of Theorem 4.3, in order to complete the proof of the Theorem we are reduced (thanks to Lemma 4.1) to establishing the associativity formula (2.4). A good way to approach this is through the use of residue products in \( F(V) \).

Definition 4.5. Let \( V \) be an additive abelian group with \( a(z) = \sum_n a(n) z^{-n-1} \) and \( b(z) = \sum_n b(n) z^{-n-1} \) a pair of fields in \( F(V) \). Let \( m \) be any integer. The \( m \)th
residue product of \(a(z)\) and \(b(z)\) is the field in \(F(V)\), denoted by \(a(z)_m b(z)\), whose \(n^{th}\) mode is given by the following formula:

\[
(a(z)_m b(z))_{n} := \sum_{i \geq 0} (-1)^i \binom{m}{i} \{a(m-i)b(n+i) - \delta_{m,n} b(m+n-i) a(i)\}.
\]

It is easy to see that because \(a(z)\) and \(b(z)\) are fields on \(V\), then for any state \(u \in V\) we have \((a(z)_m b(z))_n u = 0\) for all large enough \(n\). Hence, \(a(z)_m b(z)\) is a field on \(V\). Thus for any integer \(m\), \(F(V)\) equipped with its \(m^{th}\) residue product is a nonassociative ring.

Motivation for introducing this field stems from the nature of the associativity formula \((2.3)\). Indeed, for a vertex ring we can restate the associativity formula in the following compact and highly suggestive form:

\[
Y(u(t)v, z) = Y(u, z) Y(v, z).
\]

In proving Theorem \(4.3\) we of course do not know that \(V\) is a vertex ring. Nevertheless our goal is to establish \((4.5)\) for the fields \((u, z)_m\), \(F(V)\) equipped with its \(m^{th}\) residue product is a nonassociative ring.

**Lemma 4.6.** Suppose that \(a(z), b(z)\) are creative with respect to \(v_0\) and that \(b(z)\) creates \(v\). Then \(a(z)_m b(z)\) is creative with respect to \(v_0\) and creates \(a(m)v\).

**Proof.** Let \(n \geq -1\). Since \(a(n)v_0 = b(n)v_0 = 0\) (\(n \geq 0\)) and \(b(-1)v_0 = v\), we have

\[
(a(z)_m b(z))_{n} v_0 = \sum_{i \geq 0} (-1)^i \binom{m}{i} \{a(m-i)b(n+i) - \delta_{m,n} b(m+n-i) a(i)\} v_0 = \delta_{n,-1} a(m)v.
\]

This completes the proof of the Lemma. \(\square\)

**Lemma 4.7.** Suppose that \(a(z), b(z), c(z) \in F(V)\) are pairwise mutually local fields. Then \(a(z)_m b(z)\) and \(c(z)\) are also mutually local fields for all integers \(m\).

**Proof.** Standard proofs of this result for vertex algebras over \(\mathbb{C}\) (e.g., [18], Proposition 2.1.5) also hold for vertex rings with the proof unchanged. \(\square\)

### 4.4. The relation between residue products and translation-covariance.

The main result of this Subsection is

**Theorem 4.8.** Let \(V\) be an additive abelian group with a sequence of endomorphisms \(\mathcal{D} = (\text{Id}_V, D_1, \ldots)\) in \(\text{End}(V)\). Suppose that \(a(z)\) and \(b(z)\) are fields on \(V\) that are translation-covariant with respect to \(\mathcal{D}\). Then \((a(z)_m b(z))\) is also translation-covariant with respect to \(\mathcal{D}\) for all integers \(m\).

In order to establish this result we first prove a result of independent interest.

**Theorem 4.9.** Let \(V\) be an additive abelian group. Then \((\text{Id}_V, \delta_z, \delta_z^2, \ldots)\) is an iterative HS derivation of the nonassociative ring consisting of \(F(V)\) equipped with its \(m^{th}\) residue product.
PROOF. The iterative property (which we do not use) is straightforward to prove, and we skip the details. As for the HS property, let \( a(z) = \sum_n a(n)z^{-n-1} \) and \( b(z) = \sum_n b(n)z^{-n-1} \) lie in \( \mathcal{F}(V) \). By (4.4) we have

\[
\delta_z^{(i)} a(z) = (-1)^i \sum_n A_i(n) z^{-n-1}, \text{ with } A_i(n) = \binom{n}{i} a(n-i).
\]

With analogous notation for \( \delta_z^{(j)} b(z) \), we see that if \( i + j = \ell \geq 0 \) then

\[
((\delta_z^{(i)} a(z))_m(\delta_z^{(j)} b(z)))_n
= (-1)^\ell \sum_{t \geq 0} (-1)^t \binom{m}{t} \{ A_i(m-t) B_j(n+t) - (-1)^m B_j(m+n-t) A_i(t) \},
\]

and similarly

\[
\delta_z^{(i)}(a(z)_m b(z)) = (-1)^\ell \sum_{n} \binom{n}{\ell} (a(z)_m b(z))_{n-\ell} z^{-n-1}
\]

\[
= (-1)^\ell \sum_{n} \binom{n}{\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} \{ a(m-t) b(n-\ell+t) - (-1)^m b(m+n-\ell-t) a(t) \} z^{-n-1}.
\]

So it suffices to show for all integers \( m, n \) that

\[
\sum_{i+j=\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} \{ A_i(m-t) B_j(n+t) - (-1)^m B_j(m+n-t) A_i(t) \} = \]

\[
(4.6) \quad \sum_{i+j=\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} \{ A_i(m-t) B_j(n+t) - (-1)^m B_j(m+n-t) A_i(t) \}.
\]

Note that

\[
A_i(m-t) B_j(n+t) = \binom{m-t}{i} \binom{n+t}{j} a(m-t-i) b(n+t-j),
\]

\[
B_j(m+n-t) A_i(t) = \binom{m+n-t}{j} \binom{t}{i} b(m+n-t-j) a(t-i).
\]

Thus (4.6) will follow from

\[
(4.7) \quad \sum_{i+j=\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} a(m-t) b(n-\ell+t)
\]

\[
= \sum_{i=0}^{\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} \binom{n+t}{\ell-i} a(m-t-i) b(n+t-\ell+i)
\]

and

\[
(4.8) \quad \sum_{i+j=\ell} \sum_{t \geq 0} (-1)^t \binom{m}{t} b(m+n-\ell-t) a(t)
\]

\[
= \sum_{i=0}^{\ell} \sum_{t \geq 0} (-1)^t \binom{m+n-t}{\ell-i} \binom{t}{i} b(m+n-t-\ell+i) a(t-i).
\]
To prove (4.8), notice that for $p \geq 0$ the coefficient of $b(m + n - \ell - p)a(p)$ on the right-hand-side is equal to

$$
\sum_{i=0}^{\ell} (-1)^{p+i}\binom{m}{p+i}(m + n - p - i)\binom{p+i}{\ell - i}
$$

$$
= (-1)^{p}\binom{m}{p}\sum_{i=0}^{\ell} (-1)^i\binom{m - p}{i}(m + n - p - i)
$$

$$
= (-1)^{p}\binom{m}{p}\binom{n}{\ell}
$$

where the last equality follows from (12.4). This proves (4.8), and we can establish (4.7) in exactly the same way. The proof of Theorem 4.9 is complete. □

We turn to the proof of Theorem 4.8. Choose $\ell \geq 0$ and use the operator identity $[D_\ell, AB] = [D_\ell, A]B + A[D_\ell, B]$ to obtain

$$
[D_\ell, a(z)m b(z)] = \sum_{n} [D_\ell, (a(z)m b(z))_n] z^{-n-1}
$$

$$
= \sum_{n} \sum_{i \geq 0} (-1)^i \binom{m}{i} [D_\ell, \{a(m-i)b(n+i) - ((-1)^i m b(m + n - i)a(i))\}] z^{-n-1}
$$

$$
= [D_\ell, a(z)]_m b(z) + a(z)_m [D_\ell, b(z)]
$$

$$
= \sum_{i=1}^{\ell} \left\{ (\delta^{(i)}_z a(z) D_{\ell-i})_m b(z) + a(z)_m (\delta^{(i)}_z b(z) D_{\ell-i}) \right\}.
$$

On the other hand, by Theorem 4.9 we have

$$
\sum_{i=1}^{\ell} \delta^{(i)}_z (a(z)m b(z)) D_{\ell-i} = \sum_{i=1}^{\ell} \sum_{j=0}^{i} (\delta^{(j)}_z a(z))_m (\delta^{(i-j)}_z b(z)) D_{\ell-i}
$$

$$
= \sum_{r=0}^{\ell-1} \sum_{p+q+r=\ell} (\delta^{(p)}_z a(z))_m (\delta^{(q)}_z b(z)) D_r.
$$

Thus we must establish the following identity:

(4.9) $$
\sum_{i=1}^{\ell} \left\{ (\delta^{(i)}_z a(z) D_{\ell-i})_m b(z) + a(z)_m (\delta^{(i)}_z b(z) D_{\ell-i}) \right\}
$$

$$
= \sum_{r=0}^{\ell-1} \sum_{p+q+r=\ell} (\delta^{(p)}_z a(z))_m (\delta^{(q)}_z b(z)) D_r.
$$
For various fields \( c(z) = \sum_n c(n)z^{-n-1} \) we have to consider expressions of the form

\[
((c(z)D_t)_m b(z))_n \quad = \quad \sum_{r \geq 0} (-1)^r \binom{m}{r} \{ c(m-r)D_t b(n+r) - \((-1)^m b(m+n-r)c(r)D_t \}
\]

and similarly

\[
a(z)_m (d(z)D_t)_n \quad = \quad \sum_{r \geq 0} (-1)^r \binom{m}{r} \{ a(m-r)d(n+r)D_t - \((-1)^m d(m+n-r)d_t a(r) \}
\]

As a result, we obtain

\[
\sum_{i=1}^\ell \left\{ (\delta_z^{(i)} a(z) D_{\ell-i})_m b(z) + a(z)_m (\delta_z^{(i)} b(z) D_{\ell-i}) \right\}
\]

\[
= \sum_{i=1}^\ell \sum_{r \geq 0} \sum_n (-1)^r \binom{m}{r} \{ c_i(m-r)[D_{\ell-i}, b(n+r)] - (-1)^m d_i(m+n-r)[D_{\ell-i}, a(r)] \} z^{-n-1}
\]

\[
= \sum_{i=1}^\ell \sum_{r \geq 0} \sum_n (-1)^r \binom{m}{r} \sum_{j=1}^{\ell-i} (-1)^j \left\{ c_i(m-r) \binom{n+r}{j} b(n+r-j) - (-1)^m d_i(m+n-r) \binom{r}{j} a(r-j) \right\} D_{\ell-i-j} z^{-n-1},
\]

where we have set \( c_i(z) = \delta_z^{(i)} a(z) \) and \( d_i(z) = \delta_z^{(i)} b(z) \).
Comparing this with \([4.3]\), we are reduced to proving the following equality:

\[
\sum_{i=1}^{\ell} \sum_{r=0}^{\ell-i} (-1)^r \binom{m}{r} \sum_{j=1}^{\ell-j} (-1)^j \left\{ c_i(m-r) \binom{n+r}{j} b(n+r-j) - (-1)^m d_i(m+n-r) \binom{r}{j} a(r-j) \right\} D_{\ell-i-j} z^{-n-1} = \\
\sum_{r=0}^{\ell-1} \sum_{p+q+r=\ell} (\delta_z^{[p]} a(z))_m (\delta_z^{[q]} b(z)) D_r \\
= \sum_{n} \sum_{r=0}^{\ell-1} \sum_{s \geq 0} (-1)^s \binom{m}{s} \left\{ c_i(m-s)d_j(n+s) - (-1)^m d_j(m+n-s)c_i(s) \right\} z^{-n-1} D_r,
\]

that is

\[
\sum_{i=1}^{\ell} \sum_{s \geq 0} (-1)^s \binom{m}{s} \sum_{j=1}^{\ell-i} (-1)^j \left\{ c_i(m-s) \binom{n+s}{j} b(n+s-j) - (-1)^m d_i(m+n-s) \binom{s}{j} a(s-j) \right\} D_{\ell-i-j} = \\
\sum_{r=0}^{\ell-1} \sum_{i+j+r=\ell} \sum_{s \geq 0} (-1)^s \binom{m}{s} \left\{ c_i(m-s)d_j(n+s) - (-1)^m d_j(m+n-s)c_i(s) \right\} D_r.
\]

Now

\[
\sum_{i=1}^{\ell} \sum_{s \geq 0} (-1)^s \binom{m}{s} \sum_{j=1}^{\ell-i} (-1)^j \left\{ c_i(m-s) \binom{n+s}{j} b(n+s-j) - (-1)^m d_i(m+n-s) \binom{s}{j} a(s-j) \right\} D_{\ell-i-j} = \\
\sum_{u=0}^{\ell-1} \sum_{i+j+u=\ell} \sum_{s \geq 0} (-1)^s \binom{m}{s} (-1)^j \left\{ c_i(m-s) \binom{n+s}{j} b(n+s-j) - (-1)^m d_i(m+n-s) \binom{s}{j} a(s-j) \right\} D_u,
\]

so we need for fixed \(1 \leq v \leq \ell\) that

\[
\sum_{i+j=v} \sum_{s \geq 0} (-1)^s \binom{m}{s} (-1)^j \left\{ c_i(m-s) \binom{n+s}{j} b(n+s-j) - (-1)^m d_i(m+n-s) \binom{s}{j} a(s-j) \right\} = \\
\sum_{i+j=v} \sum_{s \geq 0} (-1)^s \binom{m}{s} \left\{ c_i(m-s)d_j(n+s) - (-1)^m d_j(m+n-s)c_i(s) \right\}.
\]

But this follows directly from the definition of the fields \(c_i(z), d_j(z)\), and the proof of Theorem \([4.8]\) is complete.
4.5. Completion of the proof of Theorem 4.3 We have already seen in Subsection 4.3 that only the associativity formula (2.4) for the fields $Y(u, z) (u \in V)$ remains to be proved, and that furthermore this is equivalent to proving the identity (4.5). We have now assembled all of the pieces that allow us to carry this out. We first record a Lemma that we will need again later.

**Lemma 4.10.** Suppose that $d(z)=\sum_{n} d(n)z^{-n-1} \in \mathcal{F}(V)$ is translation-covariant, mutually local with all fields $Y(u, z) (u \in V)$ and creative with respect to $v_0$. Then $d(z)=0$ if, and only if, $d(z)$ creates $0$, i.e., $d(-1)v_0=0$.

**Proof.** We have to prove that $d(z)=0$ on the basis of the assumption that $d(-1)v_0=0$. To see this, we first show that $d(z)v_0=0$. Let $m \geq 1$. Then $D_m(v_0)=0$, so that

$$[D_m, d(z)]v_0=D_m d(z)v_0 = \sum_{n<0} D_m d(n) v_0 z^{-n-1}.$$

On the other, by translation-covariance we also have

$$[D_m, d(z)]v_0 = \sum_{i=1}^{m} \delta^{(i)}_z d(z) D_{m-i} v_0 = \delta^{(m)}_z d(z) v_0 = \sum_{n<0} \left(-\frac{n-1}{m}\right) d(n) v_0 z^{-n-m-1}.$$

This shows that for $m \geq 1, n<0$ we have

$$D_m d(n)v_0 = \left(\frac{m-n-1}{m}\right) d(n-m)v_0.$$

Because $d(-1)v_0=0$ by hypothesis, we obtain for all $m \geq 1$ that

$$0=D_m d(-1)v_0 = d(-1-m)v_0,$$

and therefore $d(z)v_0=0$, as claimed.

Now by assumption, $d(z)$ is mutually local with every field $Y(u, z) (u \in V)$. Suppose that $(z-w)^N [d(z), Y(u, z)]=0$ for some $N \geq 0$. Then

$$z^N d(z)u = Res_w w^{-1}(z-w)^N d(z) Y(u, w)v_0 = Res_w w^{-1}(z-w)^N Y(u, w)d(z)v_0 = 0.$$

This shows that $d(z)u=0$ for all $u \in V$, so $d(z)=0$, and the proof of the Lemma is complete.

To complete the proof of Theorem 4.3, choose $u, v \in V, t \in \mathbb{Z}$ and set

$$d(z)=Y(u, z)Y(v, z) - Y(u(t)v, z).$$

We have to show that $d(z)=0$. Indeed, $u(t)v \in V$, so that $Y(u(t)v, z)$ is creative, translation-covariant and mutually local with all fields $Y(a, z) (a \in V)$ by hypothesis, and it creates $u(t)v$. On the other hand, $Y(u, z)Y(v, z)$ is also creative, mutually local with all fields $Y(a, z)$, and translation-covariant by Lemmas 4.6, 1.7, and Theorem 4.8 respectively, and it also creates $u(t)v$ (loc. cit). Therefore, $d(z)$ satisfies the assumptions of the previous Lemma, and the conclusion $d(z)=0$ follows. This completes the proof of Theorem 4.3.
4.6. Generators of a vertex ring and a refinement of Theorem 4.3

In this Subsection we deduce a refinement of Theorem 4.3 which only involves generators of $V$.

**Definition 4.11.** Let $U \subseteq V$ be a subset of a vertex ring $V$. We say that $U$ generates $V$ if $V$ is generated (as an additive abelian group) by the states

$$\{u_1(i_1)\ldots u_k(i_k)1 | u_j \in U, i_j \in \mathbb{Z}, j = 1,\ldots,k\}.$$  

If this holds, we write $V = \langle U \rangle$.

We shall prove

**Theorem 4.12.** Let $(V,v_0,D)$ consist of an additive abelian group $V$, a state $v_0 \in V$, and a sequence of endomorphisms $D = (Id_V, D_1, \ldots)$ in $\text{End}(V)$ satisfying $D_m(v_0) = 0 \ (m \geq 1)$. Suppose that $U \subseteq V$, that to each $u \in U$ is attached a field $u(z) = \sum_{n} u(n) z^{-n-1} \in \mathcal{F}(V)$, and that the following assumptions hold for all $u,v \in U$:

$$u(z)v_0 \in u + zV \{z\},$$

$$V \text{ is generated (as an additive abelian group) by } \{u_1(m_1)\ldots u_k(m_k)v_0 | u_j \in U, m_j \in \mathbb{Z}, j = 1,\ldots,k\}.$$

Then $V = (V,Y,v_0,D)$ is a vertex ring generated by $U$ and with state-field correspondence $Y$ satisfying $Y(u,z) = u(z) \ (u \in U)$, vacuum vector $v_0$, and canonical HS derivation $D$.

**Proof.** We associate to each state $a := u_1(m_1)\ldots u_k(m_k)v_0 \ (u_j \in U)$ the field

$$a(z) := u_1(z)^{m_1}(u_2(z)^{m_2}\ldots(u_k(z)^{m_k}Id_V)\ldots),$$

and to each such sum of states the corresponding sum of fields. Thanks to the hypotheses of the Theorem together with Lemmas 4.5, 4.7, and Theorem 4.8, we create in this way a set of mutually local, translation-covariant, creative fields associated to the states of $V$.

We aver that this process is well-defined, i.e., each state is associated to a unique field on $V$. If this is so, the linear extension of the association $a \mapsto a(z)$ is the state-field correspondence we are after, and the present Theorem follows directly from Theorem 4.3.

As for well-definedness, it suffices to show that if we have a relation of the form

$$\sum_{j=1}^{t} u_{1,j}(i_{1,j})\ldots u_{k,j}(i_{k,j})v_0 = 0 \ (u_{m,j} \in U),$$

then the field associated to the state on the left-hand-side by the process described above also vanishes. Let the field in question be denoted by $d(z)$. Now $d(z)$ is mutually local with all other fields associated to the states of $V$ and translation-covariant by Lemma 4.7 and Theorem 4.8 respectively, and it is creative and creates 0 by Lemma 4.6. Then $d(z) = 0$ holds by Lemma 4.10. This completes the proof of Theorem 4.12. \hfill \Box
Remark 4.13. The proof shows that if \( u = u_1(m_1) \ldots u_k(m_k)v_0 \) then
\[
Y(u, z) = Y(u_1, z)_{m_1} \ldots (Y(u_k, z)_{m_k} Id_V) \ldots .
\]

4.7. Formal Taylor expansion and an alternate existence Theorem. In this Subsection we illustrate some additional techniques dealing with formal series, in particular the formal Taylor expansion, by proving an alternate characterization of vertex rings. The result we are after, the proof of which follows a line of argument due to Haisheng Li [13], is the following.

**Theorem 4.14.** Let \((V, Y', v_0, D)\) consist of an additive abelian group \(V\), a state \(v_0 \in V\), an iterative sequence of endomorphisms \(D := (Id_V, D_1, \ldots)\) in \(\text{End}(V)\) satisfying \(D_m(v_0)=0\) for \(m \geq 1\), and a morphism of abelian groups
\[
Y' : V \to \text{End}(V)[[z, z^{-1}]], \quad u \mapsto Y'(u, z) := \sum_n u(n)z^{-n-1}.
\]
Suppose that the following assumptions hold for all states \(u, v \in V\):
\[
Y'(u, z) \sim Y'(v, z),
Y'(u, z)v_0 \in u + zV[[z]],
[D_m, Y'(u, z)] = \sum_{i=1}^m \delta^{(i)}_z Y'(u, z)D_{m-i} \quad (m \geq 0).
\]
Then \(V\) is a vertex ring with state-field correspondence \(Y'\), vacuum vector \(v_0\), and canonical HS derivation \(D\). (In particular, each \(Y'(u, z) \in \mathcal{F}(V)\).)

Remark 4.15. The statement of Theorem 4.14 is very similar to that of Theorem 4.3, but it differs in two crucial ways: first, the formal series \(Y'(u, z)\) are not assumed to be fields on \(V\), merely series of operators on \(V\). For this reason, we have used \(Y'\) to denote such series; the prime indicates that \(Y'(u, z)\) is not necessarily a field. (At the end of the day, of course, the conclusion of the Theorem is that \(Y'(u, z)\) is a field.) The second difference is that \(D\) is assumed to be iterative. Note that the other assumptions of locality, creativity and translation-covariance do not require \(Y'(u, z)\) to be a field on \(V\) - they make sense for all series.

We turn to the proof of Theorem 4.14, which we give in a sequence of Lemmas. We adopt the notation and assumptions of the Theorem until further notice. In the following, \(y\) and \(z\) are a pair of formal variables.

**Lemma 4.16.** We have
\[
Y'(u, z)v_0 = \sum_{m \geq 0} D_m(u)z^m.
\]

**Proof.** The proof follows from translation-covariance, just as (4.1) is deduced from the assumptions of Theorem 4.3. \(\square\)

The expression \(\delta^{(i)}_z = \frac{1}{i!}\partial_z^i (i \geq 0)\) introduced in Subsection 3.2 may be applied to series in \(V[[z, z^{-1}]]\), as may the exponential \(e^{y\delta_z}\). However they cannot generally be applied to states in any meaningful way. As a substitute we may use \(D_i\) in place of \(\delta^{(i)}_z\), when the exponential becomes \(\sum_{m \geq 0} D_my^m\).
Lemma 4.17. We have
\[ \left\{ \sum_{m \geq 0} D_m y^m \right\} Y'(u, z) \left\{ \sum_{m \geq 0} (-y)^m D_m \right\} = e^{y\partial_z} Y'(u, z). \]

Proof. We have to prove that if \( n \geq 0 \) then
\[ \sum_{m=0}^{n} (-1)^{n-m} D_m Y'(u, z) D_{n-m} = \delta_z^{(n)} Y'(u, z). \]

But the lhs of this equation is equal to
\[
\sum_{m=0}^{n} (-1)^{n-m} \left( [D_m, Y'(u, z)] + Y'(u, z) D_m \right) D_{n-m} \\
= \sum_{m=0}^{n} (-1)^{n-m} \left( \sum_{i=0}^{m} \delta_z^{(i)} Y'(u, z) D_{m-i} \right) D_{n-m} \\
= \sum_{i=0}^{n} \delta_z^{(i)} Y'(u, z) \sum_{m=0}^{n} (-1)^{n-m} D_{m-i} D_{n-m} \\
= \sum_{i=0}^{n} \delta_z^{(i)} Y'(u, z) \sum_{m=0}^{n} (-1)^{n-m} \binom{n-i}{n-m} D_{n-i} \\
= \delta_z^{(n)} Y'(u, z).
\]

Lemma 4.18. (Formal Taylor expansion.) We have
\[ Y'(u, z + y) = \left\{ \sum_{m \geq 0} D_m y^m \right\} Y'(u, z) \left\{ \sum_{m \geq 0} (-y)^m D_m \right\}. \]

(Convention (12.5) for binomial expansions is in effect here.)

Proof. We have
\[
e^{y\partial_z} Y'(u, z) = \sum_{i \geq 0} y^i \delta_z^{(i)} Y'(u, z) = \sum_{i \geq 0} \sum_{n \in \mathbb{Z}} \frac{y^i}{i!} a(n-1) \ldots (n-i+1) u(-n-1) z^{n-i} \\
= \sum_{i \geq 0} \sum_{n \in \mathbb{Z}} \binom{n}{i} u(-n-1) y^i z^{n-i} = Y'(u, z + y).
\]

Now apply Lemma 4.17.

Lemma 4.19. If \( Y'(u, z) \sim_t Y'(v, z) \) (\( t \geq 0 \)) then \( u(n) v = 0 \) for \( n \geq t \). In particular, \( Y'(u, z) \) is a field on \( V \).
Proof. Using Lemma 4.16, Corollary 3.4 (applicable thanks to iterativity of D), Lemma 4.18 and locality, we have

\[
(z - y)^t Y'(u, y) Y'(v, z) v_0
= (z - y)^t Y'(v, z) Y'(u, y) v_0
= (z - y)^t Y'(v, z) \left( \sum_{m \geq 0} D_m y^m \right) u
= (z - y)^t \left( \sum_{m \geq 0} D_m y^m \right) Y'(v, z - y) u.
\]

In this sequence of equalities the first expression contains no negative powers of z, whereas in the last expression there are no negative powers of y thanks to our binomial convention. So both sides lie in \( V[[y, z]] \), and setting \( z = 0 \) in the first expression returns the pure powers of y, equal to

\[
(-y)^t Y'(u, y) v.
\]

We deduce that

\[
u(n)v = 0 \text{ for } n \geq t,
\]

which completes the proof of the Lemma. \( \square \)

Now that we know that \( Y'(u, z) \in \mathcal{F}(V) \) for \( u \in V \), Theorem 4.14 follows from Theorem 4.3.

Remark 4.20. It is possible to avoid Theorem 4.3 altogether and give an independent proof of Theorem 4.14 (13). We shall not use this fact, so we skip the details, leaving them to the reader as a (nontrivial) exercise.

5. Categories of vertex rings

We consider the category of vertex rings and various subcategories. A particularly useful result (Theorem 6.5) says that the category of commutative rings is a coreflective subcategory of the category of vertex rings. For relevant background in category theory, see [15].

5.1. The category of vertex rings. Let \( U \) and \( V \) be two vertex rings with state-field correspondences \( Y_U, Y_V \) respectively. We frequently abuse notation by using \( 1 \) for the vacuum element in either \( U \) or \( V \), denoting products in both \( U \) and \( V \) by \( a(n)b \), and writing \( Y \) for either \( Y_U \) or \( Y_V \), making the distinction between \( U \) and \( V \) explicit only when necessary to avoid confusion.

Definition 5.1. A morphism \( f: U \rightarrow V \) is a morphism of additive abelian groups that preserves all products in the sense that \( f(u(n)v) = f(u(n))f(v) \) \((u, v \in U, n \in \mathbb{Z})\) and also preserves vacuum elements, i.e., \( f(1) = 1 \). In terms of vertex operators,

\[
f Y_U (u, z) v = Y_V (f(u), z) f(v).
\]

With this definition, vertex rings and their morphisms define a large category that we denote by \( \text{Ver} \).
Definition 5.2. A left ideal in \( V \) is an additive subgroup \( I \subseteq V \) such that \( a(n)u \in I \) for all \( a \in V, u \in I, n \in \mathbb{Z} \). Similarly, a right ideal satisfies \( u(n)a \in I \) for all \( a \in V, u \in I, n \in \mathbb{Z} \). A 2-sided ideal is one that is both a left and right ideal.

\( V \) is called simple if it has no nontrivial 2-sided ideals.

In a vertex algebra over \( \mathbb{C} \) the notions of left, right and 2-sided ideals coincide, but that is not quite right for a general vertex ring. Recall (cf. Lemma 5.9) the left action of the ring \( \mathbb{Z}\langle x \rangle \) of divided powers.

Lemma 5.3. Let \( V \) be a vertex ring. The following are equivalent for a subgroup \( I \subseteq V \).

(a) \( I \) is a 2-sided ideal,

(b) \( I \) is a left ideal and a left \( \mathbb{Z}\langle x \rangle \)-submodule.

Proof. In order for \( I \) to be a left \( \mathbb{Z}\langle x \rangle \)-submodule of \( V \), it is necessary and sufficient that \( D_su=a(-n+1)1 \in I \) for all \( a \in I, n \geq 0 \). So clearly (a) \( \Rightarrow \) (b). On the other hand, suppose that (b) holds. Use skew-symmetry (Lemma 5.5) to see that if \( a \in I, u \in V, n \in \mathbb{Z} \) then

\[ a(n)u = (-1)^{n+1} \sum_{i \geq 0} (-1)^i Du(n+i) a \in I, \]

whence \( I \) is a right as well as a left ideal, and (a) holds. The Lemma is proved.

The 2-sided ideals of \( V \) are the kernels of morphisms in \( \text{Ver} \). If \( I \subseteq V \) is a 2-sided ideal then the quotient group \( V/I \) inherits the structure of vertex ring in the natural way, and the standard isomorphism theorems apply.

Example 5.4. Many elementary constructions in commutative ring theory have vertex ring analogs. We give some examples that will arise later on.

(i) Maximal 2-sided ideals. These are proper 2-sided ideals \( I \subseteq V \) (proper means not equal to \( V \)) that are maximal in the lattice of all 2-sided ideals. They exist by dint of a standard application of Zorn’s Lemma in which one considers the 2-sided ideals that do not contain \( 1 \). \( I \) is maximal if, and only if, \( V/I \) is a simple vertex ring.

(ii) The torsion ideal. This is the torsion subgroup \( T(V) \subseteq V \) of the underlying abelian group. Because all products in \( V \) are bilinear, \( T(V) \) is a 2-sided ideal of \( V \) whose quotient \( V/T(V) \) is a torsion-free vertex ring.

(iii) The principal 2-sided ideal generated by \( v \in V \), i.e. the smallest 2-sided ideal of \( V \) containing \( v \), consists of all finite sums \( u_1(n_1) \ldots u_k(n_k)v(n)1 \) with \( u_1, \ldots, u_k \in V \) and integers \( n_1, \ldots, n_k, n \). This is straightforward to check using Lemma 5.3.

(iv) The ring of rational integers \( \mathbb{Z} \) is a vertex ring (see the following Subsection for more on this), indeed \( \mathbb{Z} \) is an initial object in the category \( \text{Ver} \). The characteristic map \( \mathbb{Z} \to V, n \mapsto n1 \) is the unique morphism in \( \text{Hom}_V(\mathbb{Z}, V) \). The kernel is \( p\mathbb{Z} \) for an integer \( p \geq 0 \) that we naturally call the characteristic of \( V \), denoted by \( \text{char}V \). Note that \( T(V)=V \) if, and only if, \( \text{char}V>0 \).

The image \( f(U) \) of a morphism of vertex rings \( f:U \to V \) is a vertex subring of \( V \) in the usual sense. That is, it contains \( 1 \) and is closed with respect to all \( n^{th} \) products of its elements.

Suppose that \( U \subseteq V \), and let \( \langle U \rangle \) be the additive subgroup of \( V \) generated by all states \( u_1(m_1) \ldots u_k(m_k)1 \) for all \( u_j \in U, m_j \in \mathbb{Z}, k \geq 0 \). We assert that \( \langle U \rangle \) is a
vertex subring of $V$. Indeed, the case $k = 0$ shows that $1 \in \langle U \rangle$. Moreover, by the associativity formula in the form \[4.13\] we have

\[ Y(u_1(m_1) \ldots u_k(m_k)1, z) = Y(u_1, z)_{m_1} \ldots Y(u_k, z)_{m_k}Id_Y \ldots \]

and from the very nature of residue product (cf. Definition \[4.5\]) the modes of this field are sums of compositions of the modes of the states $u_i$. Thus each $n^{th}$ product of any pair of generating states $u_1(m_1) \ldots u_k(m_k)1$ and $u'_1(m'_1) \ldots u'_k(m'_k)1$ lies in $\langle U \rangle$, and the fact that $\langle U \rangle$ is a vertex subring of $V$ follows immediately. We remark that use of the symbol $\langle U \rangle$ here is consistent with Definition \[4.11\] inasmuch as $U$ generates the vertex subring $\langle U \rangle$ of $V$.

5.2. Commutative rings with HS derivation.

**Theorem 5.5.** Suppose that $A$ is a commutative ring with identity 1 that admits an iterative Hasse-Schmidt derivation $D\equiv(Id_A, D_1, \ldots)$. For $u, v \in A$ define

\[ u(n)v= \begin{cases} D_{-n-1}(u)v, & n < 0 \\ 0, & n \geq 0 \end{cases} \]

Equipped with these products, $A$ is a vertex ring with vacuum element 1, canonical HS derivation $D$, and vertex operators

\[ Y(u, z) = \sum_{n \geq 0} (D_n u)z^n, \]

where $D_n(u)$ is the endomorphism of $A$ corresponding to multiplication by $D_n(u)$.

**Proof.** We will verify the axioms of associativity, locality, and translation-covariance. Then Theorem \[4.3\] applies, and the present Theorem follows.

First we check the associativity formula \[2.4\]. Because all $n^{th}$ products are 0 for $n \geq 0$, this reduces to showing that for $s, t \leq -1$ we have

\[ (u(t)v)(s) = \sum_{i \geq 0} (-1)^i \binom{i}{i} \{u(t-i)v(s+i)\}, \]

i.e.,

\[ D_{-s-1}(D_{-t-1}(u)v) = \sum_{i \geq 0} (-1)^i \binom{i}{i} D_{-t+i-1}(u)D_{-s-i-1}(v). \]

To prove this, set $m=-s-1, n=-t-1 \geq 0$. Then the left-hand-side of the previous display is equal to

\[ D_m(D_n(u)v) = \sum_{i+j=m} (D_i \circ D_n)(u)D_j(v) = \sum_{i+j=m} \binom{i+n}{i} (D_{i+n})(u)D_j(v) = \sum_{i \geq 0} (-1)^i \binom{i}{i} D_{-t+i-1}(u)D_{-s-i-1}(v). \]

This establishes associativity.

Next, because $A$ is commutative then $D_m(u)$ and $D_n(v)$ commute as multiplication operators on $A$. Therefore, $[Y(u, z), Y(v, w)] = 0$, so that locality \[2.0\] is obvious.
As for translation-covariance, use the previous displayed calculation to see that
\[
[D_m, Y(u, z)]v = \sum_{n \geq 0} [D_m, D_n(u)]vz^n = \sum_{n \geq 0} (D_m D_n(u)v - D_n(u)D_m(v))z^n
\]
\[
= \sum_{n \geq 0} \sum_{i \geq 1} \binom{n + i}{i} D_{n+i}(u)D_{m-i}(v)z^n
\]
\[
= \sum_{i \geq 1} \delta_i^{(1)} Y(u, z) D_{m-i}v.
\]
This completes the proof of the Theorem.

All vertex rings with the property that all modes \(u(n)\) vanish for \(n \geq 0\) arise from the construction in Theorem 5.5.

**Theorem 5.6.** Suppose that \(V\) is a vertex ring such that \(u(n)=0\) for all \(u \in V\) and all \(n \geq 0\). Then \(V\) is a commutative ring with identity \(1\) with respect to the \(-1\)th product, and \(V\) is a vertex ring of the type described in Theorem 5.5.

**Proof.** We have \(u(-1)1 = 1(-1)u = u\), so \(1\) is an identity with respect to the \(-1\)th operation. Moreover, by (2.3) we see that
\[
\sum_{i \geq 0} \left( \frac{(-1)^i}{i} \right) (u(i)v)(-2 - i)1 = 0.
\]
This shows that \(V\) is commutative with respect to the \(-1\)th product, and similarly using (2.4) we find that it is associative. So \(V\) is indeed a commutative ring with respect to the \(-1\)th product.

That \(D\) is an iterative Hasse-Schmidt derivation of this commutative ring is a special case of Theorem 5.5. Finally, we have for \(m \geq 0\),
\[
D_m(u)(-1)v = (u(-m-1)1)(-1)v = \sum_{i \geq 0} \left( \frac{(-1)^i}{i} \right) \binom{m-1 + i}{i} u(-m-1-i)1(-1+i)v
\]
\[
= u(-m-1)v.
\]
which says that
\[
Y(u, z)v = \sum_{m \geq 0} u(-m-1)vz^m = \sum_{m \geq 0} D_m(u)(-1)vz^m.
\]
This shows that \(V\) is the type of vertex ring described in Theorem 5.5, thus completing the proof of the present Theorem.

As a special case of Theorem 5.5 consider any commutative ring \(A\), equipped with the trivial HS derivation (cf. Example 3.3). Then by Theorems 5.5 and 5.6 we deduce

**Theorem 5.7.** Let \(A\) be a commutative ring with identity \(1\). Then \(A\) is a vertex ring with vacuum element \(1\) and with canonical HS derivation that is trivial. All \(n\)th products \(u(n)v\) are zero for \(n \neq -1\) and \(u(-1)v = uv\) is the product in \(A\).
Let $\text{Comm}$ be the category of unital commutative rings and unital ring morphisms. Let $\text{CommHS}$ be the category of unital commutative rings equipped with an iterative HS derivation $D$. If $(A, D), (A', D')$ are two objects in $\text{CommHS}$, a morphism of $f : (A, D) \to (A', D')$ is a unital ring morphism $f : A \to A'$ such that $fD_m = D'_m f$ for all $m \geq 0$. $\text{Comm}$ is isomorphic to the full subcategory of $\text{CommHS}$ whose objects consist of those $(A, D)$ for which $D = (\text{Id}_A, 0, 0, ...)$ is trivial. We have the following commuting diagram of functors, each of which are insertions.

$$\text{Ver} \quad \downarrow \quad \Downarrow \quad \text{Comm} \quad \text{CommHS}$$

6. The center of a vertex ring

6.1. Basic properties.

**Definition 6.1.** The center $C(V)$ of a vertex ring $V$ consists of all $u \in V$ such that $Y(u, z) = u$.

**Example 6.2.** (i) $1 \in C(V)$. Therefore, the image of the characteristic map (Example 5.4(iv)) is contained in $C(V)$. Since $C(V)$ is a commutative ring with respect to the $-1$ product (see Lemma 6.7 for the easy proof) then $\text{char} V = \text{char} C(V)$.

(ii) If $a \in C(V)$ then $a(-1)V$ is the 2-sided ideal of $V$ generated by $a$.

**Proof.** That $1 \in C(V)$ is a restatement of Theorem 2.7, and the remaining assertions (i) follow. As for (ii), it suffices to show that $a(-1)V$ is a 2-sided ideal. But because $Y(a, z) = a(-1)$, it is readily checked using (2.3) and (2.4) that $a(-1)$ commutes with modes $b(n)$ and associates in the sense that $a(-1)b(n) = a(-1)b(n)$ for all $b \in V$ and all $n \in \mathbb{Z}$, and our assertion about $a(-1)V$ follows.

The next result is very useful. In alternate parlance it says that $C(V)$ consists of the $D$-constants.

**Theorem 6.3.** Let $V$ be a vertex ring with canonical Hasse-Schmidt derivation $D = (\text{Id}_V, D_1, ...)$, and let $u \in V$. Then the following are equivalent:

(a) $u \in C(V)$,

(b) $D_i u = 0$ for $i \geq 1$.

**Proof.** If (a) holds then $D_i(u) = u(-i - 1)1 = 0$ for $i \geq 1$, so (b) holds. The implication (b)$\Rightarrow$(a) requires a bit more effort. By Lemma 3.6 if (b) holds then

$$\left(\frac{i + k}{i}\right) u(k) = 0 \quad (k \in \mathbb{Z}, i \geq 1). \quad (6.1)$$

Suppose first that $k \leq -2$. Then we may take $i = -k - 1$ in (6.1) to see that $u(k) = 0$. Now suppose that $k \geq 0$. In order to prove that $u(k) = 0$, we use the following result: the integers $\left(\frac{k + i}{i}\right)$, where $i$ ranges over all positive integers, have greatest common divisor 1. To see this, we have

$$(1 - x)^{-(k + 1)} = 1 + \sum_{i \geq 1} \left(\frac{k + i}{i}\right) x^i.$$
So if all \((k+1)^i\) for \(i > 0\) are divisible by a prime \(p\) then \((1-x)^{-(k+1)}\), and therefore also its inverse \((1-x)^{k+1}\), are congruent to \(1\) (mod \(p\)). But this is false because the coefficient of \(x^{k+1}\) in the latter polynomial is \(\pm 1\).

It follows that for any \(k \geq 0\), we can find an integral linear combination of the binomial coefficients \(\binom{i+k}{i}\) (with \(i \geq 1\) varying) equal to \(1\). Then from (6.1) we deduce that in fact \(u(k) = 0\). This completes the proof of the Theorem. \(\Box\)

We give some first applications of Theorem 5.3.

**Corollary 6.4.** Suppose that \(V\) is a vertex ring whose canonical HS derivation is trivial. Then \(V\) is of the type described in Theorem 5.7.

**Proof.** We have \(D_m = 0\) for \(m \geq 1\), so by Theorem 5.7 we find that \(Y(u, z) = u(-1)\) for all \(u \in V\), and therefore \(V = C(V)\). Now the Corollary follows from Theorem 5.6. \(\square\)

We may state the Corollary is as follows: if \(\mathbf{Comm}'\) is the full subcategory of \(\mathbf{Ver}\) consisting of objects which are vertex rings whose canonical HS derivation is trivial, then \(\mathbf{Comm}'\) is isomorphic to the category \(\mathbf{Comm}\) of commutative rings. As a result, we may, and shall, identify the two categories \(\mathbf{Comm}\) and \(\mathbf{Comm}'\).

The next result says that thus identified, \(\mathbf{Comm}\) is coreflective in \(\mathbf{Ver}\) (cf. [15, Chapter IV, Section 3]).

**Theorem 6.5.** The functorial insertion \(K: \mathbf{Comm} \rightarrow \mathbf{Ver}\) has a right adjoint \(C: \mathbf{Ver} \rightarrow \mathbf{Comm}\). \(C\) is the center functor \(V \rightarrow C(V)\) that assigns to each vertex ring \(V\) its center \(C(V)\).

We discuss the proof of the Theorem in the next few Lemmas.

Theorem 6.5 may be reformulated in several ways (loc. cit.) One particularly useful variant, which is more or less equivalent to the Theorem, is the following:

**Lemma 6.6.** Suppose that \(A\) is a commutative ring, \(V\) a vertex ring, and \(f: A \rightarrow V\) a morphism of vertex rings. Then \(f(A) \subseteq C(V)\).

**Proof.** Suppose to begin with, and in somewhat more generality than we need, that \((A, D)\) is an object in \(\mathbf{CommHS}\) and that \(D'\) is the canonical HS derivation of \(V\). Then for \(u \in A\) we have

\[
f(D_i u) = f(u(-i - 1)1) = f(u)(-i - 1)1 = D'_i f(u).
\]

In the case at hand, all higher endomorphisms \(D_i\) \((i \geq 1)\) of \(A\) vanish. Thus the last calculation shows that \(f(A)\) is annihilated by all \(D'_i\) \((i \geq 1)\). By Theorem 6.3 we deduce that \(f(A) \subseteq C(V)\), and the Lemma is proved. \(\Box\)

**Lemma 6.7.** Let \(V\) be a vertex ring. Then \(C(V)\) is a commutative ring with identity \(1\) with respect to the \(-1^{th}\) product. The assignment \(V \mapsto C(V)\) is the object map of a functor \(C: \mathbf{Ver} \rightarrow \mathbf{Comm}\).

**Proof.** We have \(1 \in C(V)\) by Example 5.2 and because \(u(-1)1 = 1(-1)u = u\), \(1\) is the identity element for \(C(V)\). Moreover, if \(u, v \in C(V)\) and \(n \geq 1\) then \(D_i u = D_i v = 0\) for \(i \geq 1\) and hence

\[
D_n u(-1)v = \sum_{i+j=n} (D_i u)(-1)D_j v = 0.
\]
This shows that \( C(V) \) is closed with respect to the \(-1^{th}\) product.

Taking \( r=s=-1 \) in (2.3), applying both sides to 1, and using \( u(i)=0 \) for \( i\geq 0 \) shows that \( u(-1)v=v(-1)u \), and a similar application of (2.4) shows that multiplication in \( C(V) \) is associative. This completes the proof of the first assertion of the Lemma.

To complete the proof that \( C:\text{Ver}\rightarrow\text{Comm} \) is a functor, suppose that \( f:U\rightarrow V \) is a morphism of vertex rings. Restriction of \( f \) to \( C(U) \) is a morphism of vertex rings \( \text{res}f:C(U)\rightarrow V \), and by Lemma 6.6 the image of this morphism lands in \( C(V) \). This says that by defining \( C(f) := \text{res}f \) we obtain a commuting diagram

\[
\begin{array}{ccc}
U & \rightarrow & C(U) \\
\downarrow f & & \downarrow \text{C}(f) \\
V & \rightarrow & C(V)
\end{array}
\]

That \( C \) is a functor then follows immediately.

To complete the proof of Theorem 6.5, let \( A \) be a commutative ring and \( V \) a vertex ring. What we must show is that there is a natural bijection of Hom-sets \( \varphi: \text{Hom}_{\text{Ver}}(K(A),V)\rightarrow \text{Hom}_{\text{Comm}}(A,C(V)) \). Indeed, \( K(A) \) is just \( A \) regarded as a vertex ring, and application of Lemma 6.6 shows that the two displayed Hom-sets consist of the same functions (with different codomains). This gives us the required bijection, and naturality follows.

\[\square\]

\[\square\]

6.2. Vertex \( k \)-algebras. Fix a vertex ring \( U \). As usual (cf. Chapter II, Section 6 of [15]), the comma category \((U\downarrow\text{Ver})\) consists of the \( \text{Ver} \)-objects under \( U \). Precisely, it has objects consisting of morphisms \( U\rightarrow V \) in \( \text{Ver} \). A morphism \( f \) in \((U\downarrow\text{Ver})\) from \( U\rightarrow V_1 \) to \( U\rightarrow V_2 \) is a morphism \( f:V_1\rightarrow V_2 \) in \( \text{Ver} \) such that the following diagram in \( V \) commutes:

\[
\begin{array}{ccc}
U & \stackrel{f}{\rightarrow} & V_1 \\
\downarrow & & \downarrow f \\
& V_2
\end{array}
\]

Definition 6.8. Let \( k \) be a commutative ring, considered as an object in \( \text{Ver} \). We call the objects in \((k\downarrow\text{Ver})\) vertex \( k \)-algebras.

Suppose that \( \varphi:k\rightarrow V \) is a vertex \( k \)-algebra. By Lemma 6.6 we have \( \varphi(k)\subseteq C(V) \). We claim that the left action of \( k \) on \( V \) by the \(-1\) operation induces a left action

\[
k\times V \rightarrow V, (a,v) \mapsto \varphi(a)(-1)v \quad (a\in k)
\]

which turns \( V \) into a unital left \( k \)-module such that all \( n^{th} \) products in \( V \) are \( k \)-linear. Indeed, this amounts to the identities

\[
t(-1)(u(n)v) = (t(-1)u)(n)v = u(n)(t(-1)v) \quad (t\in C(V), u,v\in V, n\in \mathbb{Z}),
\]

and these are easily proved using (2.4) and (2.5) and the fact that \( Y(t,z) = t(-1) \) for \( t \in C(V) \).

We have a category \( k\text{Ver} \) whose objects are vertex rings which are also unital left \( k \)-modules such that all \( n^{th} \) products are \( k \)-linear. Morphisms in \( k\text{Ver} \) are
morphisms in $\text{Ver}$ that are also $k$-linear. Our remarks in the previous paragraph then amount to showing that there is a functor $(k \downarrow \text{Ver}) \to k\text{Ver}$. On the other hand, given an object $V$ in $k\text{Ver}$, the map $a \mapsto a.1$ ($a \in k$) defines a morphism $k \to V$ in $\text{Ver}$ and thereby an object in the comma category $(k \downarrow \text{Ver})$ and thereby a functor $k\text{Ver} \to (k \downarrow \text{Ver})$. These two functors are inverse to each other up to natural equivalence, and we obtain the following result.

**Theorem 6.9.** Let $k$ be a commutative ring. There is an equivalence of categories

$$(k \downarrow \text{Ver}) \cong k\text{Ver}.$$ 

We complete this Subsection with some examples of vertex $k$-algebras related to base-change. Suppose we have a unital morphism of commutative rings $\varphi : k \to R$ and a vertex $k$-algebra $V$. The base-change

$$V_R := R \otimes_k V$$

produces a left $R$-module where $b(a \otimes u) = (ba) \otimes u$ ($a,b \in R, u \in V$). We easily check that $V_R$ is a vertex $R$-algebra if we define $n^{th}$ modes in the obvious way, i.e.,

$$(a \otimes u)(b \otimes v) := ab \otimes (u(n)v).$$

For example, if we take $R := k[[t]]$ with an indeterminate $t$ and the natural embedding $k \to R$, we obtain the power series vertex $k[[t]]$-algebra with coefficients in $V$:

$$V[[t]] := k[[t]] \otimes_k V.$$ 

This vertex algebra and similar ones with $R = k[[t]], k[[t, t^{-1}]],$ or $k[t, t^{-1}]$, for example, and $k = \mathbb{C}$, are commonly used in VOA theory, where they are usually regarded as vertex $\mathbb{C}$-algebras.

$V[[t]]$ plays a rôle in an alternate treatment of HS derivations which is standard in the theory of rings with derivation (cf. Section 1 of [17]). When pursued, this line of argument leads to connections between vertex rings and formal group laws. We will not carry this out here.

**Lemma 6.10.** Suppose that $V$ is a vertex $k$-algebra with canonical HS derivation $\mathcal{D} = (1d_V, D_1, \ldots)$. The map

$$\alpha : V \to V[[t]], \quad u \mapsto \sum_{m \geq 0} D_m(u)t^m$$

is a morphism of vertex $k$-algebras. Moreover, it has a unique extension to an automorphism $\alpha : V[[t]] \to V[[t]]$ satisfying $\alpha(1 \otimes t) = 1 \otimes t$.

**Proof.** The argument in [17] is essentially unchanged. To prove the first part of the Lemma, notice that $\alpha$ fixes the vacuum $1$ because $D_m(1) = 0$ ($m \geq 1$). Moreover

$$\alpha(u(n)v) = \sum_{m \geq 0} D_m(u(n)v)t^m = \sum_{m \geq 0} \sum_{i+j = m} D_i(u)(n)D_j(v)t^m = \left(\sum_{i \geq 0} D_i(u)t^i\right)(n)\sum_{j \geq 0} D_j(v)t^j = \alpha(u(n))\alpha(v).$$
This shows that $\alpha$ is a morphism of vertex $k$-algebras.

We assert that there is a unique extension of $\alpha$ to a morphism $\alpha: V[[t]] \to V[[t]]$ of vertex $k[[t]]$-algebras satisfying $\alpha(t) = t$. Indeed, because powers of $t$ associate (because $t \in C(V[[t]])$) it is easy to see that if the extension exists then we must have $\alpha(t^n) = \alpha(t)^n = t^n$ ($n \geq 1$), so that

$$\alpha \left( \sum_{i \geq 0} a_i t^i \right) = \sum_{i \geq 0} \alpha(a_i) t^i.$$  

Thus there is only one possible extension, and the defined action of $\alpha$ on $V[[t]]$ does work, because

$$\alpha \left( \left( \sum_i a_i t^i \right) \left( \sum_j b_j t^j \right) \right) = \alpha \left( \sum_k \left( \sum_{i+j=k} a_i(n)b_j \right) t^k \right)$$

$$= \sum_k \left( \sum_{i+j=k} \alpha(a_i(n))\alpha(b_j) \right) t^k = \alpha \left( \sum_i a_i t^i \right) \left( \sum_j b_j t^j \right).$$

In fact, we assert that so defined, $\alpha$ is surjective, hence is an automorphism of $V[[t]]$. Given any $\sum_j b_j t^j$, we have to solve recursively for $a_i \in V$ that satisfy

(6.2)  
\[ \sum_i \alpha(a_i) t^i = \sum_j b_j t^j. \]

Since $\alpha(a_0) = a_0 + O(t)$ we must have $a_0 = b_0$. Suppose we have found $a_0, \ldots, a_{n-1}$ such that (6.2) holds modulo $t^n$. Let $a(a_i) = \sum_{m} a_{im} t^m$ ($0 \leq i \leq n - 1$). By (6.2) we must have

$$b_n = \sum_{i=0}^{n-1} a_{im} + a_n,$$

so $a_n$ is uniquely determined. The Lemma is proved. \hfill \Box

### 6.3. Idempotents. We consider idempotents in a vertex ring. They have a rôle to play in Part II.

**Definition 6.11.** Let $V$ be a vertex ring. An idempotent is an element $e \in V$ such that $Y(e, z)e=e$, i.e., $e(n)e=\delta_{n,-1}e$.

**Example 6.12.** An idempotent in the commutative ring $C(V)$ is an idempotent of $V$.

**Proof.** If $e \in C(V)$ is an idempotent then $Y(e, z)e=e(-1)$ and $e=e(-1)e$. \hfill \Box

**Lemma 6.13.** Let $V$ be a vertex ring with $e \in V$. Then the following are equivalent.

(a) $e(n)e=0$ for $n \geq 0$ and $e(-1)e=e$,
(b) $e$ is an idempotent in $V$,
(c) $e$ is an idempotent in $C(V)$.  

Proof. After Example 6.12 the implications $(c)\Rightarrow(b)\Rightarrow(a)$ are obvious, so we only need to prove that $(a)\Rightarrow(c)$. We adapt a standard argument.

First note that if $e(n)e=0$ for $n\geq 0$ then all modes of $e$ commute with each other thanks to Lemma 2.6. Let $D=(Id,D_1,\ldots)$ be the canonical HS derivation of $V$. By Theorem 6.3 it suffices to show that $D_n(e)=0$ for $n\geq 1$. We prove this by induction on $n$. First we have

$$D_1(e)=D_1(e(-1)e)=D_1(e)(-1)e+e(-1)D_1(e)=2e(-1)D_1(e),$$

where we have used that modes of $e$ commute. Therefore

$$e(-1)D_1(e)=2e(-1)^2D_1(e)=2e(-1)D_1(e),$$

leading to $0=e(-1)D_1(e)=2e(-1)D_1(e)=D_1(e)$.

Similarly, if $n\geq 2$ then $D_n(e)=D_n(e)\sum_{i=0}^n D_i(e)(-1)D_{n-i}(e)$, and by induction it follows that $D_n(e)=2e(-1)D_n(e)$. We deduce that $D_n(e)=0$ just as in the case $n=1$, and the proof of the Lemma is complete. 

Let $k$ be a commutative ring and $V$ a vertex $k$-algebra with canonical HS derivation $D=(Id,D_1,\ldots)$. The endomorphism algebra of $V$ is defined as follows:

$$E(V)=\{f\in End_k(V)|fY(v,z)=Y(v,z)f \ (v \in V), \ f D_m=D_mf \ (m\geq 1)\}.$$ 

Here, $End_k(V)$ is the $k$-algebra of $k$-linear endomorphisms of the $k$-module $V$. The next result follows an argument in [4].

Lemma 6.14. There is an isomorphism of $k$-algebras

$$\varphi:C(V) \rightarrow E(V), \ a \mapsto \varphi_a:v \mapsto a(-1)v \ (a\in C(V), v\in V).$$

Proof. Define $\varphi_a \ (a\in C(V))$ as in the statement of the Lemma. If $a\in C(V)$ then $a(-1)$ commutes with all modes $v(n)$, and it follows immediately that $\varphi_a$ also commutes with all $v(n)$. Moreover,

$$D_m\varphi_a v=(a(-1)v)(-m-1)1=a(-1)v(-m-1)1=\varphi_a D_m v,$$

showing that $\varphi$ commutes with each $D_m$. Therefore, $\varphi_a \in E(V)$. Thus $\varphi:a\mapsto \varphi_a$ defines a map $C(V)\rightarrow E(V)$, and because we have $(a(-1)b(-1)v=a(-1)b(-1)v$ for $a,b\in C(V)$ then $\varphi$ is a morphism of rings.

To see that $\varphi$ is surjective, let $f\in E(V)$ and $v\in V$. By creativity we have $f(v)=f v(-1)1=v(-1)f(1)$. Furthermore $D_n f(1)=f D_n (1)=0$ for $n\geq 1$, so $f(1)\in C(V)$ by Theorem 6.3. Finally, we have $\varphi_f(v)=f(1)(-1)v=v(-1)f(1)=f(v)$, showing that $f=\varphi_f(1)$. The Lemma now follows.

Idempotents in a vertex $k$-algebra $V$ define decompositions of $V$ into direct sums of ideals, just as for commutative rings. If $e$ is an idempotent in $V$ then $e \in C(V)$ by Lemma 6.13 and it follows easily that

$$(6.3) \quad V=e(-1)V\oplus (1-e)(-1)V$$

is a decomposition of $V$ into the direct sum of ideals, each of which is itself a vertex $k$-algebra. (The corresponding vacuum elements are $e$ and $1-e$.) The projection $V\rightarrow e(-1)V$ is the idempotent in $E(V)$ that corresponds to $\varphi_e$ in the isomorphism described in Lemma 6.14. In particular, $V$ is indecomposable as a vertex $k$-algebra if, and only if, $C(V)$ is an indecomposable commutative ring.
Conversely, given a pair of vertex \( k \)-algebras \( U, V \), their \textit{direct sum} \( U \oplus V \) is a vertex \( k \)-algebra with \( Y(u \oplus v, z) = Y(u, z) \oplus Y(v, z) \) \((u \in U, v \in V)\) and \( 1_U \oplus 1_V = 1_U \oplus 1_V \). The vacuum elements \( 1_U, 1_V \) become idempotents in \( U \oplus V \). This construction defines a \textit{product} in \( \text{Ver} \). Moreover, the abelian group \( 0 \) is a terminal object, so that \( \text{Ver} \) has all finite products.

6.4. \textit{Units.} 

\textbf{Definition 6.15.} Let \( V \) be a vertex ring. A \textit{unit} in \( V \) is an element \( a \in V \) such that for some \( b \in V \) we have \( Y(a, z)b=1 \), i.e., \( a(n)b=\delta_{n,-1}1 \).

\textbf{Example 6.16.} A unit in the commutative ring \( C(V) \) is a unit of \( V \).

\textbf{Proof.} If \( a \in C(V) \) is a unit, \( Y(a, z)=a(-1) \) and \( a(-1)b=1 \) for some \( b \in C(V) \).

\textbf{Lemma 6.17.} Let \( V \) be a vertex ring, and suppose that \( a, b \in V \) satisfy \( Y(a, z)b=1 \). Then \( Y(b, z)a=1 \) and \( a, b \in C(V) \).

\textbf{Proof.} Let \( (Id, D_1, \ldots) \) be the canonical HS derivation of \( V \). We have \( a(n)b=\delta_{n,-1}1 \), and by skew-symmetry (Lemma 6.8) it follows that 

\[
\begin{align*}
 b(n)a & = (-1)^{n+1} \sum_{i \geq 0} (-1)^i D_i(a(n + i)b) \\
 & = (-1)^{n+1} \sum_{i \geq 0} (-1)^i \delta_{n+i,-1} D_i(1) = \delta_{n,-1}1.
\end{align*}
\]

This proves the first assertion of the Lemma.

Next we observe that \([a(r), b(s)] = 0\) for all \( r, s \in \mathbb{Z} \). This follows from Lemma 2.6. Now let \( m \geq 1 \). Then using (2.4) we have

\[
0 = (b(-m - 1)a)(-1)b = \sum_{i \geq 0} (-1)^i \binom{-m-1}{i} b(-m - 1 - i)a(-1 + i) + (-1)^m a(-m - 2 - i)b(i) b = b(-m - 1)a(-1)b = b(-m - 1)1 = D_m(b).
\]

This shows that \( D_m(b)=0 \) \((m \geq 1)\), and similarly \( D_m(a)=0 \) \((m \geq 1)\). Therefore \( a \) and \( b \) lie in \( C(V) \) by Theorem 6.3. This completes the proof of the Lemma.

\textbf{Example 6.18.} (i) Let \( V \) be a vertex ring with \( a \in C(V) \). The 2-sided ideal \( a(-1)V \) of \( V \) generated by \( a \) (cf. Example 6.2(ii)) satisfies \( a(-1)V = V \) if, and only if, \( a \) is a unit.

(ii) If \( V \) is \textit{simple} then \( C(V) \) is a \textit{field}.

\textbf{Proof.} (i) \( a(-1)V = V \iff a(-1)b=1 \) \((\exists b \in V)\iff Y(a, z)b=1 \). (ii) follows immediately from (i).

6.5. \textit{Tensor product of vertex rings.} The category \( \text{Ver} \) of vertex rings has a \textit{coproduct} given by the tensor product \( U \otimes V \) of a pair of vertex rings \( U, V \). (It also has a product, given by the direct sum construction explained in the previous Subsection.) We give a proof using Theorem 4.3.
The underlying abelian group is the tensor product of the abelian groups $U$ and $V$, and the vacuum element for $U \otimes V$ is taken to be $1 \otimes 1$. The vertex operators for $U \otimes V$ are defined in the obvious manner, i.e., if $u \in U, v \in V$ we define

$$ Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z). $$

In terms of modes, this means that

$$ (u \otimes v)(n) = \sum_{i+j=n-1} u(i) \otimes v(j), $$

which shows in particular that $Y(u \otimes v, z) \in \mathcal{F}(U \otimes V)$. Moreover,

$$ Y(u \otimes v, z)(1 \otimes 1) = \sum_{m \geq 0} \sum_{i+j=m} u(-i-1) \otimes v(-j-1) 1z^m. $$

This proves the creative property, moreover it shows that the canonical HS-derivation $D'_m = (Id, D'_{1, \ldots})$ for $U \otimes V$ must necessarily be defined by

$$ (6.4) \quad D'_m = \sum_{i+j=m} D_i \otimes D_j, $$

where we have abused notation by setting $(Id, D_{1, \ldots})$ for the canonical HS derivations of both $U$ and $V$. The mutual locality of the fields $Y(u \otimes v, z)$ for $u, v \in V$ is an easy consequence of the locality of the $Y(u, z)$, so it remains to establish translation-covariance.

Although it is not necessary to know that $D'$ is an HS-derivation in order to apply Theorem 4.3, in the present instance it will be convenient to know in advance that it is. To see this, we calculate

$$ D'_m(u \otimes v)(n)(a \otimes b) = \sum_{i+j=n-1} D'_m(u(i) a \otimes v(j) b) $$

$$ = \sum_{p=0}^m \sum_{i+j=n-1} (D_p u(i) a) \otimes (D_{m-p} v(j) b) $$

$$ = \sum_{p=0}^m \sum_{i+j=n-1} \sum_{r=0}^m \sum_{s=0}^{m-p} (D_r u(i) (D_{p-r} a) \otimes (D_s v(j) (D_{m-p-s} b)) $$

$$ = \sum_{p=0}^m \sum_{r=0}^m \sum_{s=0}^{m-p} (D_r u \otimes D_s v)(n)(D_{p-r} a \otimes D_{m-p-s} b) $$

$$ = \sum_{i=0}^m D'_i(u \otimes v)(n) D'_{m-i}(a \otimes b), $$

and this is what we required.
Turning to the proof of translation-covariance, we calculate

\[ Y(D_m'(u \otimes v), z) = Y \left( \sum_{p+q=m} D_p(u) \otimes D_q(v), z \right) \]

\[ = \sum_{p+q=m} Y(D_p(u), z) \otimes Y(D_q(v), z) = \sum_{p+q=m} \delta_z^{(p)} Y(u, z) \otimes \delta_z^{(q)} Y(v, z) \]

\[ = (-1)^m \sum_{p+q=m} \sum_{\ell} \binom{\ell}{p} \binom{n}{q} u(\ell - p) \otimes v(n - q) z^{-\ell - n - 2} \]

The coefficient of \( z^{-t-1} \) in this expression is equal to

\[ (-1)^m \sum_{p+q=m} \sum_{\ell} \binom{\ell}{p} \left( \frac{t - \ell - 1}{q} \right) u(\ell - p) \otimes v(t - \ell - q - 1) \]

\[ = (-1)^m \sum_{p+q=m} \sum_{i} \left( \frac{p+i}{p} \right) \left( \frac{t - p - i - 1}{q} \right) u(i) \otimes v(t - i - m - 1) \]

\[ = (-1)^m \left( \frac{t}{m} \right) \sum_{i} u(i) \otimes v(t - i - m - 1) \quad \text{ (use [12.1] and [12.4])} \]

\[ = (-1)^m \left( \frac{t}{m} \right) (u \otimes v)(t - m), \]

and this establishes that

\[ Y(D_m'(u \otimes v), z) = (-1)^m \sum_{t} \left( \frac{t}{m} \right) (u \otimes v)(t - m) z^{-t-1} = \delta_z^{(m)} Y(u \otimes v, z). \]

Having already shown that \( D' \) is an HS derivation, the proof that \( Y(u \otimes v, z) \) is translation-covariant with respect to \( (Id, D'_1, D'_2, \ldots) \) now follows, using the previous display, in exactly the same way that part (c) of Theorem 6.19 is deduced from part (b). This completes the proof that \( U \otimes V \), equipped with vacuum vector \( 1 \otimes 1 \), vertex operators \( Y(u, z) \otimes Y(v, z) \), and endomorphisms \( D'_m \), is indeed a vertex ring.

We also observe that the same proof works \textit{mutatis mutandis} if \( U \) and \( V \) are both vertex \( k \)-algebras for some commutative ring \( k \) and \( U \otimes V \) is taken to mean the tensor product \( U \otimes_k V \) of \( k \)-modules.

We have the usual diagram

\[ (6.5) \quad U \xleftarrow{i} U \otimes V \xrightarrow{j} V \]

where \( i: u \mapsto u \otimes 1 \) and \( j: v \mapsto 1 \otimes v \). Both \( i \) and \( j \) are morphisms of vertex \( k \)-algebras, and the images of \( i \) and \( j \) jointly generate \( U \otimes V \) on account of the formula \( u \otimes v = (u \otimes 1)(-1)(1 \otimes v) \). The universal property of (6.5) that shows it is a coproduct in \( k \text{Ver} \) is then easy to see, and we have proved

\textbf{Theorem 6.19.} Suppose that \( U \) and \( V \) are a pair of vertex \( k \)-algebras. Then \( U \otimes_k V \) carries a natural structure of vertex \( k \)-algebra with vertex operators defined by \( Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z) \), vacuum vector \( 1 \otimes 1 \), and canonical HS-derivation \( D'_m \) defined by (6.4). \( U \otimes_k V \) defines a coproduct in \( k \text{Ver} \). \( \square \)
Example 6.20. If \( k \to R \) is a morphism of commutative rings and \( V \) a vertex \( k \)-algebra, the tensor product \( R \otimes_k V \) is a vertex \( k \)-algebra where \( R \) has the trivial HS-derivation. Since \( (a \otimes 1)(n) = a(n) \otimes 1 = \delta_{n,-1}a(-1) \otimes 1 \) (\( a \in R \)) then \( i(R) \) is contained in \( C(R \otimes_k V) \) and \( R \otimes_k V \) is then a vertex \( R \)-algebra. This is the base-change that we discussed in Subsection 6.2.

7. Virasoro vertex \( k \)-algebras

The Virasoro algebra \( \text{Vir} \) (over \( C \)) is a well-known Lie algebra giving rise to vertex algebras over \( C \) in a standard manner (\([9, 12, 13]\)). In this Section we show how to construct Virasoro vertex \( k \)-algebras associated to any commutative ring \( k \) and any quasicentral charge in \( k \) (cf. Definition \([7, 4]\)). This leads to the Definition and first examples of vertex operator algebras over a commutative ring \( k \).

7.1. The Lie algebra \( \text{Vir}_k \). Fix a commutative ring \( k \). We consider a free \( k \)-module

\[
\text{Vir}_k := \oplus_{n \in \mathbb{Z}} kL(n) + kK
\]

with \( k \)-basis consisting of \( L(n) \) (\( n \in \mathbb{Z} \)) and \( K \). We make \( \text{Vir}_k \) into a Lie algebra over \( k \) by defining the bracket relations:

\[
\begin{align*}
[L(m), L(n)] &= (m - n)L(m + n) + \delta_{m+n,0} \left( \frac{m+1}{3} \right) K, \\
[L(m), K] &= 0.
\end{align*}
\]

Remark 7.1. If \( k = \mathbb{C} \), one usually replaces \( \left( \frac{m+1}{3} \right) \) with \( \frac{m^2 - m}{12} \) in this definition. This amounts to a rescaling of the central element \( K \). We prefer to use \([7, 1] \) because it makes sense for all rings \( k \), whereas the other choice is problematic if 2 is not a unit in \( k \).

\( \text{Vir}_k \) has a triangular decomposition into Lie \( k \)-subalgebras

\[
\text{Vir}_k = \text{Vir}_k^+ + \text{Vir}_k^- + \text{Vir}_k^0,
\]

where

\[
\begin{align*}
\text{Vir}_k^+ &= \oplus_{m > 0} kL(m), \quad \text{Vir}_k^- := \oplus_{m < 0} kL(m), \\
\text{Vir}_k^0 &= kL(0) \oplus kK.
\end{align*}
\]

\( \text{Vir}_k^\geq := \text{Vir}_k^+ \oplus \text{Vir}_k^0 \) is also Lie \( k \)-subalgebra of \( \text{Vir}_k \).

For each \( c' \in k \), there is a 1-dimensional free \( \text{Vir}_k^\geq \)-module \( U_{c'} := kv_0 \) such that \( L(m).v_0 = 0 \) (\( m \geq 0 \)) and \( K.v_0 = c'.v_0 \). We consider the induced (Verma) module

\[
\text{Ver}_{c'} := \text{Ind}_{\text{Vir}_k^\geq}^{\text{Vir}_k} kv_0 = \text{Vir}_k \otimes_{\text{Vir}_k^\geq} U_{c'}.
\]

\( \text{Ver}_{c'} \) is a free \( k \)-module with a basis consisting of states

\[
\{ u(n_1, \ldots, n_k) := L(-n_1) \cdots L(-n_k).v_0 \mid n_1 \geq n_2 \geq \ldots \geq n_k \geq 1 \}.
\]

Adopting standard notation, we set

\[
\omega(z) := \sum_n \omega(n) z^{-n-1} = \sum_n L(n) z^{-n-2},
\]

so that \( \omega(n) = L(n - 1) \).
Lemma 7.2. The following hold:

(a) \( \omega(z) \in \mathcal{F}(\text{Ver}_{c'}) \),

(b) \( \omega(z) \sim_4 \omega(z) \).

Proof. If \( n \) and \( n_1 \geq \ldots \geq n_k \geq 1 \) are integers with \( n \geq n_1 + \ldots + n_k \), an induction based on the relations in \( \text{Vir}_k \) shows that \( L(n).u(n_1, \ldots, n_k) = 0 \). Part (a) of the Lemma follows easily from this statement.

Part (b) asserts that \((z - y)^4[\omega(z), \omega(y)] = 0\) (cf. Definition \([12] \))\). This is a famous relation whose proof in the case \( k = C \) carries over unchanged to the present context. We skip the details and refer the reader to [13], Section 9.4 and [12], Section 6.1. \( \square \)

7.2. The Virasoro vertex ring \( M_k(c', 0) \). We set

\[ M_k(c', 0) := M(c', 0) := \text{Ver}_{c'}/I_c, \]

where \( I_c := \text{Vir}_k.L(-1) \) is the \( \text{Vir}_k \)-submodule of \( \text{Ver}_{c'} \) generated by \( L(-1) \). \( M(c', 0) \) is a free \( k \)-module with basis consisting of states

\[ \{ u'(n_1, \ldots, n_k) := L(-n_1) \ldots L(-n_k)v_0 + I_c \mid n_1 \geq n_2 \geq \ldots \geq n_k \geq 2 \}. \]

There is an induced action of operators and fields such as \( L(n) \) and \( \omega(z) \) on \( M(c', 0) \). We will often use the same symbol for such operators on both the Verma module and its quotient. This should cause no confusion.

The bulk of this Section is taken up with the proof of the next result.

Theorem 7.3. Set \( \omega := L(-2)v_0 + I_c \). Then \( M_k(c', 0) \) is a vertex \( k \)-algebra generated by \( \omega \), with \( Y(\omega, z) = \omega(z) \).

Definition 7.4. Continuing the discussion in Remark \([7, 14]\) we call \( c' \) the quasi-central charge, and we call \( c = 2c' \) the central charge of \( M_k(c', 0) \).

We begin the proof of Theorem 7.3 by first noting that as an immediate consequence of Lemma 7.2 \( \omega(z) \) is a self-local field on \( M(c', 0) \). Moreover,

\[ \omega(z)v_0 = \sum_n (L(n)v_0 + I_c)z^{-n-2} = \omega + \sum_{n \geq 3} (L(-n)v_0 + I_c)z^{n-2}, \]

so that \( \omega(z) \) is creative and creates \( \omega \).

We are going to apply Theorem \([11, 12]\). We need a sequence of endomorphisms \( D = (\text{Id}, D_1, \ldots) \) of \( M(c', 0) \) satisfying \( D_m(v_0) = 0 \) \( (m \geq 1) \), and with respect to which \( \omega(z) \) is translation covariant. If \( k \) is a \( \mathbb{Q} \)-algebra we could take \( D_m = \frac{L(-1)^m}{m!} \), but this is not defined for general \( k \). The strategy for proving the Theorem is to first prove it when \( k \) is torsion-free, then deduce the general case by a base-change.

The way in which \( D \) must be defined is dictated by the requirement that it should be an HS derivation. Thus for \( k, m \geq 0 \) we inductively define

\[ D_0(v_0) := v_0, D_m(v_0) := 0 \ (m \geq 1) \]
and

(7.3) \[ D_m u'(n_1, \ldots, n_k) := \sum_{i=0}^{m} (D_i \omega)(-n_1 + 1) D_{m-i} u'(n_2, \ldots, n_k) \quad (m \geq 0), \]

\[ D_i(\omega)(n) := (-1)^i \binom{n}{i} L(n - i - 1) \quad (i \geq 0, n \in \mathbb{Z}). \]

This defines each \( D_m \) on the \( k \)-base of states \( \{u'(n_1, \ldots, n_k)\} \), and we extend the definition by \( k \)-linearity to \( M(c', 0) \).

For example, we have \( D_0(\omega) = D_0(\omega)(-1).v_0 = L(-2).v_0 = \omega \), and it follows from (7.3) that \( D_0 = Id \). More generally,

**Lemma 7.5.** For all \( m \geq 0 \) we have

\[ L(-1)^m = D_1^m = m! D_m. \]

**Proof.** By construction,

\[ D_m u'(n_1, \ldots, n_k) = \sum_{i=0}^{m} (-1)^i \binom{-n_1 + 1}{i} L(-n_1 - i) D_{m-i} u'(n_2, \ldots, n_k) \]

\[ = \sum_{i=0}^{m} \binom{n_1 + i - 2}{i} L(-n_1 - i) D_{m-i} u'(n_2, \ldots, n_k), \]

so if we set \( D_{[m]} := m! D_m \), then

\[ D_{[m]} u'(n_1, \ldots, n_k) = \sum_{i=0}^{m} \binom{m}{i} (n_1 + i - 2) \ldots (n_1 - 1) L(-n_1 - i) D_{[m-i]} u'(n_2, \ldots, n_k). \]

Now in order to show that \( L(-1)^m = m! D_m \), it suffices to show that \( L(-1)^m \) satisfies the same recursive identity as \( D_{[m]} \), that is

\[ L(-1)^m u'(n_1, \ldots, n_k) = \sum_{i=0}^{m} \binom{m}{i} (n_1 + i - 2) \ldots (n_1 - 1) L(-n_1 - i) L(-1)^{m-i} u'(n_2, \ldots, n_k). \]
To do this, use induction on \( m \) to see that the left-hand-side is equal to
\[
\sum_{i=0}^{m-1} \binom{m-1}{i} (n_1 + i - 2)(n_1 - 1)L(-1)L(-n_1 - i)L(-1)^{m-1-i}u'(n_2, \ldots, n_k)
\]
\[
= \sum_{i=0}^{m-1} \binom{m-1}{i} (n_1 + i - 2)(n_1 - 1)
\]
\[
\{ (n_1 + i - 1)L(-n_1 - i - 1) + L(-n_1 - i)L(-1) \} L(-1)^{m-1-i}u'(n_2, \ldots, n_k)
\]
\[
= \sum_{i=0}^{m-1} \binom{m-1}{i} (n_1 + i - 1)(n_1 - 1)L(-n_1 - i)\}
\]
\[
L(-1)^{m-1-i}u'(n_2, \ldots, n_k) +
\]
\[
\sum_{j=1}^{m} \binom{m-1}{j-1} (n_1 + j - 2)(n_1 - 1)L(-n_1 - j)L(-1)^{m-j}u'(n_2, \ldots, n_k)
\]
\[
= \sum_{i=0}^{m-1} \binom{m-1}{i} (n_1 + i - 2)(n_1 - 1)L(-n_1 - i)L(-1)^{m-1-i}u'(n_2, \ldots, n_k)
\]
\[
+ \sum_{j=1}^{m} \binom{m-1}{j-1} (n_1 + j - 2)(n_1 - 1)L(-n_1 - j)L(-1)^{m-j}u'(n_2, \ldots, n_k)
\]
\[
= \sum_{i=0}^{m} \binom{m}{i} (n_1 + i - 2)(n_1 - 1)L(-n_1 - i)L(-1)^{m-1-i}u'(n_2, \ldots, n_k).
\]

This establishes the identity \( L(-1)^m = m!D_m \). In particular, \( L(-1)=D_1 \), whence also \( L(-1)^m = D_m^m \). This completes the proof of the Lemma.

We can now prove Theorem 7.3 in the case when \( k \) is torsion-free. We have to show that
\[
[D_m, \omega(z)] = \sum_{i=1}^{m} \delta^{(i)}\omega(z)D_{m-i},
\]
i.e., for all \( n \in \mathbb{Z} \),
\[
[D_m, L(n)] = \sum_{i=1}^{m} (-1)^i \binom{n+1}{i} L(n-i)D_{m-i}.
\]

Because \( k \) is torsion-free, it suffices to prove the identity that results upon multiplying each side by \( m! \). Then by Lemma 7.3, it suffices to show that
\[
[L(-1)^m, L(n)] = \sum_{i=1}^{m} (-1)^i(n+1)\ldots(n-i+2) \binom{m}{i} L(n-i)L(-1)^{m-i}.
\]
This can be proved by induction on $m$, as follows:

$$
[L(-1)^m, L(n)] = L(-1)[L(-1)^{m-1}, L(n)] + [L(-1), L(n)]L(-1)^{m-1}
$$

$$
= \sum_{i=1}^{m-1} (-1)^i(n + 1)(n - i + 2) \binom{m - 1}{i} L(-1)L(n - i)L(-1)^{m-i-1}
+ (-1 - n)L(n - 1)L(-1)^{m-1}
$$

$$
= \sum_{i=1}^{m-1} (-1)^i(n + 1)(n - i + 2) \binom{m - 1}{i} \{ [L(-1), L(n - i)]L(-1)^{m-i-1} + L(n - i)L(-1)^{m-i} \}
+ (-1 - n)L(n - 1)L(-1)^{m-1}
$$

$$
= \sum_{i=1}^{m-1} (-1)^i(n + 1)(n - i + 2) \binom{m - 1}{i} L(n - i)L(-1)^{m-i}
+ \sum_{i=1}^{m} (-1)^i(n + 1)(n - i + 2) \binom{m - 1}{i - 1} L(n - i)L(-1)^{m-i}
$$

$$
= \sum_{i=1}^{m} (-1)^i(n + 1)(n - i + 2) \binom{m}{i} L(n - i)L(-1)^{m-i}.
$$

This completes the proof that $M_k(c',0)$ is a vertex ring when $k$ is torsion-free. By construction, all products in $\text{Vir}_k$ are $k$-linear, so that it is a vertex $k$-algebra.

Now suppose that $k$ is an arbitrary commutative ring. We can find a torsion-free commutative ring $R$ and a surjective ring morphism $\psi: R \to k$. (E.g., take $R = \mathbb{Z}[x_a | a \in k]$ with $\psi : x_a \mapsto a$.) Because $R$ is torsion-free, the case of Theorem 7.3 already established shows that $M := M_R(c', 0)$ is a vertex $R$-algebra for any $c' \in R$.

Let $I = \text{ker} \psi$. We claim that $IM = \{ \sum_i Iu'(n_1, \ldots, n_k) \mid n_1 \geq \ldots \geq n_k \geq 2 \}$ is a 2-sided ideal of $M$. Indeed, this follows immediately (cf. Lemma 5.3) because the operators $L(n)$ are $R$-linear. Thus, $M/IM$ carries the structure of a vertex $R$-algebra. On the other hand, the very construction of $M_k(c', 0)$ (as $\text{Vir}_k$-module) shows that it arises as the base change $k \otimes_R M$, where $c'$ is an element of $R$ that projects onto $c'$.

There is an isomorphism of $R$-modules

$$
M/IM \xrightarrow{\sim} (R/I) \otimes_R M, \quad m + IM \mapsto 1 \otimes m \ (m \in M),
$$

and by transporting the vertex structure of $M/IM$ using this isomorphism, we obtain the desired vertex $k$-algebra structure on $(R/I) \otimes_R M = k \otimes_R M = M_k(c', 0)$. This completes the proof of Theorem 7.3. 

7.3. Virasoro vectors. Virasoro vectors in vertex rings are ubiquitous and useful. 

DEFINITION 7.6. Suppose that $k$ is a commutative ring, $V$ a vertex $k$-algebra, and $c\in k$. A Virasoro element (vector) of quasicentral charge $c'$ in $V$ is a state $\omega \in V$ such that if $Y(\omega, z) := \sum_n L(n) z^{-n-2}$ is the vertex operator for $\omega$, then the modes $L(n)$ satisfy

$$[L(m), L(n)] = (m - n)L(m + n) + \delta_{m+n,0} \left( \frac{m+1}{3} \right) c' \text{Id}_V.$$ 

In other words, the $L(n)$ furnish a representation of the Virasoro Lie $k$-algebra $Vir_k$ on $V$ in which the central element $K$ acts as multiplication by $c'$. We call $V$ a Virasoro vertex $k$-algebra of quasicentral charge $c'$ if $V$ is generated by a Virasoro element of quasicentral charge $c'$.

The category $k\text{Ver}_{c'}$ of vertex $k$-algebras of quasicentral charge $c'$ is defined as follows: objects are pairs $(V, \omega)$ where $V$ is a vertex $k$-algebra and $\omega$ is a Virasoro element of quasicentral charge $c'$ in $V$; a morphism $\alpha: (U, \nu) \rightarrow (V, \omega)$ is a morphism of vertex $k$-algebras $\alpha: U \rightarrow V$ such that $\alpha(\nu) = \omega$.

**THEOREM 7.7.** Fix a commutative ring $k$ and an element $c' \in k$. Then $M_k(c', 0)$ is an initial object in $k\text{Ver}_{c'}$.

**PROOF.** By Theorem 7.3, $M_k(c', 0)$ is an object in $k\text{Ver}_{c'}$.

Let $(U, \nu)$ be an object in $k\text{Ver}_{c'}$. We have to show that there is a unique morphism of vertex $k$-algebras $\alpha: (M(c', 0), \omega) \rightarrow (U, \nu)$. Because $\langle \omega \rangle = M_k(c', 0)$ we may, and shall, assume without loss that $U = \langle \nu \rangle$.

First we prove that there is at most one such $\alpha$. Write $Y(\omega, z) = \sum_n L(n) z^{-n-2}$ and $Y(\nu, z) = \sum_n L'(n) z^{-n-2}$. The construction of $M(c', 0)$ in Subsection 7.2 shows that it has a free $k$-basis given by states $L(-n_1) \ldots L(-n_k) v_0$ for $n_1 \geq \ldots \geq n_k \geq 2$. Furthermore, the relations satisfied by the modes $L'(n)$ together with the creativity statement $L'(n) 1 = 0$ ($n \geq -1$) show that $U$ is spanned by states $L'(-n_1) \ldots L'(-n_k) 1$ for $n_1 \geq \ldots \geq n_k \geq 2$.

Now because $\alpha$ is a morphism of vertex $k$-algebras and $\alpha(\omega) = \nu$ then we have $\alpha L(n)v = L'(n) \alpha(v)$ for $v \in M(c', 0)$, $n \in \mathbb{Z}$. It follows that $\alpha L(-n_1) \ldots L(-n_k) v_0 = L'(-n_1) \ldots L'(-n_k) 1$ for $n_1 \geq \ldots \geq n_k \geq 2$, and therefore $\alpha$ is uniquely determined.

It remains to show that, so defined, $\alpha$ really is a morphism of vertex $k$-algebras. But this is clear (in principle) because all of the relations satisfied by the operators in $M(c', 0)$ are consequences of the Virasoro relations \[4], and they will therefore also hold in $U$. Since $\alpha$ merely exchanges $L'(n)$ for $L(n)$ then it is a morphism of vertex $k$-algebras. We leave further details to the reader. \[\square\]

7.4. Graded vertex rings. The appropriate notion of $\mathbb{Z}$-grading for vertex algebras over $\mathbb{C}$ is well-known, and carries over unchanged to vertex rings.

**DEFINITION 7.8.** Let $k$ be a commutative ring and $V$ a vertex $k$-algebra. We say that $V$ is $\mathbb{Z}$-graded (or simply graded, since we will not consider other kinds of gradings) if there is a decomposition of $V$ into $k$-submodules

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$

with the following property: if $u \in V_k$, $v \in V_l$ and $n \in \mathbb{Z}$, then

$$u(n)v \in V_{k+l-n-1}.$$ 

We say that homogeneous states $u \in V_k$ have weight $k$, written $wt(u) = k$.
Lemma 7.9. Suppose that $V$ is a graded vertex $k$-algebra. Then $1 \in V_0$.

Proof. Write $1 = \sum_k 1_k$ where $1_k \in V_k$. Then for $u \in V_k$ we have
\[ u = 1(-1)u = \left( \sum_k 1_k \right)(-1)u = \sum_k 1_k(-1)u, \]
and $wt(1_k(-1)u) = k + \ell$. Therefore, we must have $1_k(-1)u = 0$ whenever $k \neq 0$, and since this holds for all homogeneous $u$ then $1_k(-1)1 = 0$ for $k \neq 0$. Therefore $1_k = 1_k(-1)1 = 0$ for $k \neq 0$, and consequently $1 = 1_0$, as required. \hfill \Box

Example 7.10. (i) Suppose that $V$ is a graded vertex $k$-algebra such that $V = V_0$. Then $V$ has a trivial HS derivation and is a commutative ring as in Theorem 5.7.
(ii) Suppose that $V \ell = 0$ for $\ell < 0$. Then $V_0$ is a commutative $k$-algebra with respect to the $-1$ operation, and $(V_0)$ is a vertex $k$-subalgebra of $V$ of the type prescribed in Theorem 5.6.

Proof. Let $u, v \in V_0$. Then $u(n)v \in V_{-n-1}$, and this is 0 if $n \geq 0$ in the situation of either (i) or (ii). Indeed, if $V = V_0$ then $u(n)v = 0$ for $n \neq 1$, and the assertion of (i) follows.
(ii) In this case we can still conclude that $u(r)v(s) = v(s)u(r)$ for all integers $r, s$ by (2.3), so that $(V_0)$ consists of sums of states of the form $w = u(n_1) \ldots u(n_k)1$ with $u_i \in V_0$ and $n_i < 0$. We assert that if $w = u(n_1) \ldots u(n_r)1$ is another such state ($v_i \in V_0, m_i < 0$) then $u(n)v = 0$ for $n \geq 0$. Indeed by (2.3) we have
\[ u(n)v = (u(n_1) \ldots u(n_k)1)(n)v = \sum_{i \geq 0} (-1)^{n_1} \binom{n_1}{i} \{ u_1(n - i)w(n + i) \} v \]
where we have set $w = u_2(n_2) \ldots u_k(n_k)1$, and our assertion follows by induction on $k$. Now we see that the hypotheses, and therefore also the conclusions, of Theorem 5.6 hold, and the assertions of (ii) follow easily. \hfill \Box

Lemma 7.11. Suppose that $V$ is a graded vertex $k$-algebra with canonical HS derivation $D = (D_0, D_1, \ldots)$. Then $D_m$ has weight $m$ as an operator, i.e.,
\[ D_m : V_k \rightarrow V_{k+m}. \]

Proof. Let $u \in V_k$. By definition of $D_m$ (cf. Theorem 5.10), $D_m(u) = u(-1-m)1$. By Lemma 7.9 it then follows that $wt(D_m(u)) = wt(u) + wt(1) + m = k + m$. \hfill \Box

Example 7.12. $M = M_k(c', 0)$ is a graded vertex $k$-algebra
\[ M = \oplus_{\ell \geq 0} M_{\ell} \]
where (by definition, and using the notation of Section 7) $M_{\ell}$ is spanned by those states $w'(n_1, \ldots, n_k)$ satisfying $n_1 + \ldots + n_k = \ell$. Furthermore, $L(0)$ leaves each $M_{\ell}$ invariant and acts as multiplication by $\ell$ on $M_{\ell}$.

Proof. Recall that $M$ is generated as a vertex ring by the state $\omega = L(-2)1$, with $Y(\omega, z) = \sum_n L(n)z^{-n-2}$, and has a $k$-basis consisting of the states
\[ u'(n_1, \ldots, n_k) = L(-n_1) \ldots L(-n_k)1 = \omega(1 - n_1) \ldots \omega(1 - n_k)1 \]
with $n_1 \geq \ldots \geq n_k \geq 2$. For example, $\omega \in M_2$. 


First we show that \( L(-n) \) has weight \( n \) as an operator, i.e., \( L(-n)u'(n_1,\ldots,n_k) \) has weight \( n+1\ldots+n_k \). (Because \( M \) has no nonzero states of negative weight, this includes the statement that if \( n_1+\ldots+n_k < 0 \) then \( L(-n)u'(n_1,\ldots,n_k) = 0 \).) Suppose to begin with that \( k=0 \), so that \( u' = 1 \). If also \( n \leq 1 \) then \( L(-n)1 = 0 \) by the creation axiom, and there is nothing to prove. On the other hand, if \( n \geq 2 \) then \( L(-n)1 = \omega(1-n)1 = D_{n-2}(\omega) \), and by Lemma 7.11 this has weight \( wt(\omega) + n - 2 = n \), as required. This completes the case \( k=0 \). We proceed by induction on \( n_1 + \ldots + n_k \).

If \( n \geq n_1 \) then \( L(-n)u'(n_1,\ldots,n_k) = u'(n,n_1,\ldots,n_k) \) has weight \( n+n_1 + \ldots + n_k \) by definition. If \( n < n_1 \) then

\[
L(-n)u'(n_1,\ldots,n_k) = L(-n)L(-n_1)u'(n_2,\ldots,n_k)
\]

\[
= \left((n_1-n)L(-n-n_1) + L(-n_1)L(-n) + \delta_{n+n_1,0} \frac{1-n}{3} \right) u'(n_2,\ldots,n_k),
\]

and the desired result follows by induction. This completes the proof of our assertion that \( L(-n) \) has weight \( n \) as an operator.

Now according to Remark 4.13,

\[
Y(u'(m_1,\ldots,m_r),z) = Y(\omega,z)_{1-m_1} \ldots (Y(\omega,z)_{1-m_r}Id_M) \ldots
\]

is a composition of residue products. Using (4.4), it follows easily from the special case previously established that the \( n \)th product \( u'(m_1,\ldots,m_r)(n)u'(n_1,\ldots,n_k) \) has weight \( n_1 + \ldots + n_k + n_1 + \ldots + n_k + n - 1 \), as required.

Finally, we prove the assertions about \( L(0) \). Indeed, taking \( n = 0 \) in (7.4) yields

\[
L(0)u'(n_1,\ldots,n_k) = L(0)L(-n_1)u'(n_2,\ldots,n_k)
\]

\[
= \{n_1 L(-n_1) + L(-n_1)L(0)\} u'(n_2,\ldots,n_k),
\]

and the proof that \( L(0)u'(n_1,\ldots,n_k) = (n_1 + \ldots + n_k)u'(n_1,\ldots,n_k) \) follows by induction. \( \square \)

### 7.5. Vertex operator \( k \)-algebras

Our definition is modeled on the definition of a VOA over \( C \) and the example of \( M_k(c',0) \).

**Definition 7.13.** Let \( (V,\omega) \) be an object in \( k\text{-Ver}c' \), i.e., a vertex \( k \)-algebra with Virasoro element \( \omega \) of quasicialeral charge \( c' \) and \( Y(\omega,z) := \sum_n L(n)z^{-n-2} \), and let the canonical HS derivation be \( D = (Id,D_1,\ldots) \). We call \( (V,\omega) \) a **vertex operator \( k \)-algebra of quasicialeral charge \( c' \)** if \( V \) carries a \( \mathbb{Z} \)-grading in the sense of Definition 7.8

\[
V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell
\]

that satisfies the following additional assumptions:

(a) \( V_\ell \) is a finitely generated \( k \)-module,

(b) \( V_\ell = 0 \) for \( \ell \ll 0 \),

(c) \( L(0) \) acts on \( V_\ell \) as multiplication by \( \ell \),

(d) \( L(-1)^m = m! D_m \) (\( m \geq 0 \)).

**Example 7.14.** (i) \( M_k(c',0) \) (and any of its graded quotients) are vertex operator \( k \)-algebras of quasi-central charge \( c' \). This follows from the construction of \( M_k(c',0) \), Lemma 7.5, Theorem 7.7 and Example 7.12.

(ii) A vertex operator algebra over \( C \) of central charge \( c \) is a vertex operator \( C \)-algebra of quasicialeral charge \( 2c \).
(iii) If \( V \) is a vertex operator \( \mathbb{Z} \)-algebra then the base change \( k \otimes V \) is a vertex operator \( k \)-algebra.

**Lemma 7.15.** Suppose that \((V, \omega)\) is a vertex operator \( k \)-algebra. Then \( \omega \in V_2 \).

**Proof.** By creativity we have \( \omega = L(-2) \mathbf{1} \). Then \( L(0) \mathbf{1} = 0 \) by Lemma 7.9 and hypothesis (c), and \( L(0) \omega = L(0) L(-2) \mathbf{1} = ([L(0), L(-2)] + L(-2) L(0)) \mathbf{1} = 2 L(-2) \mathbf{1} = 2 \omega \), as required.

**Part II: Pierce bundles of Vertex Rings**

The second part of this paper revolves around the idea of a Pierce bundle of vertex rings, which is a special kind of \( \acute{e}tale \) bundle. It transpires that Pierce’s theory for commutative rings \[19\] may be carried over \textit{en bloc} to the setting of vertex rings.

8. \( \acute{e}tale \) bundles of vertex rings

8.1. Basic definitions. In this Subsection we give the basic definitions concerning \( \acute{e}tale \) bundles of vertex rings \( p : E \to X \), regardless of whether they are Pierce bundles or not. We refer the reader to the relevant sections of \[15\] and \[16\] for additional background on bundles and categories.

An \( \acute{e}tale \) bundle in the category \( \text{Top} \) of topological spaces and continuous maps is a surjective morphism \( p : E \to X \) that is a \textit{local homeomorphism} in the sense that \( E \) is covered by open sets \( U_i \) such that the restriction \( p|U_i \) of \( p \) to each \( U_i \) is a homeomorphisms whose image \( p(U_i) \) is open in \( X \).

The \textit{stalks} of the bundle are defined by \( E_x := p^{-1}(x) \) \( (x \in X) \).

An \( \acute{e}tale \) bundle of abelian groups is an \( \acute{e}tale \) bundle \( p : E \to X \) such that each stalk \( E_x \) has the structure of an additive abelian group. Moreover, the operations in the stalks are \textit{continuous} in the following sense: if \( \Delta := \{(t, u) \in E \times E | p(t) = p(u)\} \) then the map \( \mu : \Delta \to E, (t, u) \mapsto t - u \) is \textit{continuous}. The relevant diagram is

\[
\begin{array}{ccc}
E & \overset{\mu}{\longrightarrow} & \Delta \\
\downarrow{\pi_1} & & \downarrow{p} \\
E_x & \longrightarrow & E \\
\end{array}
\]

where the square is a pullback in \( \text{Top} \). In particular, \( \pi_i \) is projection on the \( i^{th} \) coordinate, \( E \times E \) has the product topology and \( \Delta \subseteq E \times E \) the subspace topology. Note that \( (t, u) \in \Delta \) if, and only if, \( t, u \) lie in the \textit{same stalk} \( E_{p(t)} \), in which case \( t - u \) is the group operation in \( E_{p(t)} \).

We set \( 0_x \) \( (x \in X) \) as the \textit{zero element} in \( E_x \). It follows from the definitions that the map \( 0_X : X \to E, x \mapsto 0_x \) is \textit{continuous}.

**Definition 8.1.** An \( \acute{e}tale \) bundle of vertex rings is an \( \acute{e}tale \) bundle of abelian groups \( p : E \to X \) such that each stalk \( E_x \) has the structure of a vertex ring, and the operations \( u(n)v \) \( (u, v \in E, n \in \mathbb{Z}) \) are continuous in the same sense as before. Furthermore, if \( 1_x \) is the vacuum element of \( E_x \) then the map \( 1_X : X \to E, x \mapsto 1_x \) must be continuous.
Example 8.2. (a) If each \( E_x \) is a commutative ring regarded as a vertex ring with trivial canonical HS derivation, the last definition reduces to the usual definition of an étale bundle of commutative rings.

(b) Each \( E_x \) is a discrete subspace of \( E \). On the other hand, if \( V \) is any vertex ring equipped with the discrete topology and \( X \times V \) has the product topology, then the canonical projection \( p: X \times V \to X \) is an étale bundle of vertex rings with constant stalk \( V \).

8.2. Nonassociative vertex rings and sections. Let \( f: Y \to X \) be continuous. The inverse image of an étale bundle \( p: E \to X \) of (say) abelian groups or commutative rings, with respect to \( f \), denoted \( f^*(E) \), is the pullback

\[
\begin{array}{ccc}
f^* (E) & \xrightarrow{\pi_2} & E \\
\pi_1 \downarrow & & \downarrow p \\
Y & \xrightarrow{f} & X 
\end{array}
\]

Thus \( f^*(E) := \{ (y, e) \in Y \times E | f(y) = p(e) \} \) with coordinate projections \( \pi_i \) as before.

It is a standard fact that \( \pi_1: f^*(E) \to Y \) is again an étale bundle of abelian groups or commutative rings. However, if \( p \) is a bundle of vertex rings then \( f^*(E) \to Y \) may not be. We discuss this in further detail below, in the context of local sections. Note that the \( n^{th} \) product on stalks \( f^*(E)_y \) are (like the other operations) defined in the natural way, i.e.,

\[
(y, t)(n)(y, u) := (y, t(n)u) \quad (t, u \in E, n \in \mathbb{Z}).
\]

We record

Lemma 8.3. If \( p: E \to X \) is an étale bundle of vertex rings and \( f: Y \to X \) a continuous map, then the pullback \( f^*(E) \) is an étale bundle of abelian groups.

Let \( p: E \to X \) be an étale bundle with \( U \subseteq X \) any subset. The set of local sections over \( U \) is defined by

\[
\Gamma(U, E) := \{ \sigma: U \to E | \sigma \text{ is continuous and } p \circ \sigma = Id_U \}.
\]

The set of global sections is \( \Gamma(X, E) \). If \( i: U \to X \) is insertion, then \( \Gamma(U, X) \) is isomorphic (as a set, or abelian group) to \( \Gamma(U, i^*(E)) \).

Example 8.4. The zero section \( 0_X \) and vacuum section \( 1_X \) both lie in \( \Gamma(X, E) \).

If \( p: E \to X \) is a bundle of vertex rings, the algebraic operations enjoyed by the stalks may be transported to sections in a pointwise fashion. Explicitly, the \( n^{th} \) product of sections \( \sigma, \tau \in \Gamma(U, E) \) is defined by

\[
(\sigma(n) \tau)(u) := \sigma(u) \tau(u) \quad (u \in U).
\]

This is meaningful inasmuch as \( \sigma(u), \tau(u) \in E_u \), so that their \( n^{th} \) product is defined. The group operations are defined similarly.

It is immediate that \( \Gamma_U := \Gamma(U, E) \) is an additive abelian group. It is equipped with \( n^{th} \) products, so we may define formal series \( Y(\sigma, z) \in \text{End}(\Gamma_U)[[z, z^{-1}]] \) in the expected way:

\[
Y'(\sigma, z) \tau := \sum_{n \in \mathbb{Z}} \sigma(n) \tau z^{-n-1}.
\]
The $n^{th}$ products on $\Gamma_U$ being defined in a pointwise fashion, some of their basic properties carry over unchanged from the corresponding property in the vertex rings that comprise the stalks of the bundle. Thus we have a vacuum element $1_U: u \mapsto 1_u (u \in U)$, $Y'(1_U, z) = Id_{1_U}$, and the creativity formula $Y'(\sigma, z)1_U = \sigma + \ldots$ also holds.

The sequence of endomorphisms $D := (Id, D_1, \ldots)$ defined, as in Theorem 3.5 by $D_m(\sigma) := \sigma(-m-1)1_U$ is iterative and HS with respect to all products, moreover $D_m(1_U) = 0_U$ $(m \geq 1)$. Additionally, each $Y'(\sigma, z)$ is translation-covariant with respect to $D$.

Motivated by Theorem 4.14 and the preceding remarks, we make the

**Definition 8.5.** A nonassociative vertex ring $(V, Y', v_0, D)$ consists of an additive abelian group $V$, a state $v_0 \in V$, an iterative sequence of endomorphisms $D := (Id_U, D_1, \ldots)$ in $End(V)$ satisfying $D_m(v_0) = 0$ for $m \geq 1$, and a morphism of abelian groups

$$Y': V \to \text{End}(V)[[z, z^{-1}]], \quad u \mapsto Y'(u, z) := \sum_n u(n)z^{-n-1},$$

satisfying for all $u, v \in V$:

$$Y'(u, z)v_0 \in u + zV[[z]],$$

$$[D_m, Y'(u, z)] = \sum_{i=1}^m \delta_z^{(i)}Y'(u, z)D_{m-i} (m \geq 0).$$

We might have incorporated more of the properties enjoyed by $\Gamma_U$ in this Definition, for example the HS property of $D$. It is arguably more natural to also assume that each $Y'(u, z) \in F(V)$. As it is, we can paraphrase Theorem 4.14 as follows:

**Lemma 8.6.** A nonassociative vertex ring $V$ is a vertex ring if, and only if, the locality condition $Y'(u, z) \sim Y'(v, z)$ holds for all $u, v \in V$. \(\square\)

We encapsulate the earlier discussion in the following

**Lemma 8.7.** Suppose that $p: E \to X$ is an étale bundle of vertex rings. Then for any subset $U \subseteq X$, $\Gamma_U$ is a nonassociative vertex ring. \(\square\)

The question of whether $\Gamma_U$ is in fact a vertex ring is murkier. At issue is the locality property. Certainly for $u \in U$ we have $Y'(\sigma, z)(u) \sim Y'(\tau, z)(u)$ for $t \gg 0$, however the choice $t$ depends on $u$, whereas we need a $t$ that works uniformly for $u \in U$. We state a standard fact used repeatedly in what follows.

**Lemma 8.8.** Let $p: E \to X$ be an étale bundle (of sets) with $U \subseteq X$ open and $\sigma, \tau \in \Gamma(U, E)$. Then the set $\{x \in U | \sigma(x) = \tau(x)\}$ is open in $X$. In particular, taking $\tau = 0_U$, the zero set of $\sigma$, i.e., $\{x \in U | \sigma(x) = 0_x\}$ is open. \(\square\)

**Theorem 8.9.** Let $p: E \to X$ be an étale bundle of vertex rings over a compact space $X$. Then $\Gamma_Y$ is a vertex ring if $Y \subseteq X$ is closed.

**Proof.** First consider the case $Y = X$, and let $\sigma, \tau \in \Gamma_X$. By Lemmas 8.6 and 8.7, it suffices to show that $Y'(\sigma, z) \sim Y'(\tau, z)$. 



Let $x \in X$. By locality in the stalk $E_x$, there is a nonnegative integer $t_x$ such that
\begin{equation}
(z - w)^{t_x}[Y'(\sigma, z), Y'(\tau, w)](x) = 0_x.
\end{equation}
By Lemma 8.3 (8.2) continues to hold throughout an open neighborhood $U_x \subseteq U$ of $x$ for all integers $\geq t_x$. As $x$ ranges over $X$ we obtain an open cover $\{U_x\}$ of $X$, and by compactness there is a finite subcover $X = \bigcup_{i=1}^n U_{x_i}$. Then if we choose $t$ to be the maximum of $t_{x_1}, \ldots, t_{x_n}$ then $Y'(\sigma, z) \sim Y'(\tau, z)$ holds throughout $X$, as required.

Finally, if $i:Y \to X$ is a closed subset then $Y$ is compact, and we have the pullback bundle $i^*(E) \to Y$ (cf. Lemma 8.3) which is a bundle of vertex rings thanks to compactness. Now (b) follows from the special case $Y = X$. 

\section{Pierce bundles of vertex rings}

So far, we have not discussed any interesting examples of étale bundles of vertex rings. We will rectify this situation in the present Section by constructing Pierce bundles. We follow [19] closely.

\subsection{The Stone space of a vertex ring}

Fix a vertex ring $V$ with center $C = C(V)$. Let $B = B(V) = B(C)$ be the set of idempotents in $V$. By Lemma 6.13 these are precisely the idempotents of $C$. By a standard procedure we can equip $B$ with the structure of a Boolean ring (cf. [19] or [11], Chapter 8). This means that $B$ is a commutative ring in which every element is idempotent. Indeed, multiplication in $B$ is that of $C$ and addition $\oplus$ is defined by
\[e \oplus f := e + f - 2ef \quad (e, f \in B).\]

$B$ is also a Boolean algebra (complemented, distributive lattice with 0, 1) if we define $e \vee f := e + f - ef, e \wedge f := ef$ and $e' := 1 - e$. This fact will not play much of a rôle in what follows.

\textbf{Definition 9.1.} The Stone space of $V$ is the topological space $X = \text{Spec}(B)$.

Here, and below, $\text{Spec}(R)$ for a commutative ring $R$ denotes the set of prime ideals of $R$ equipped with its Zariski topology. As is well-known, the Boolean property of $B$ implies that $X$ has additional topological properties beyond those that hold in a general prime spectrum. We state some of the main properties of $X = \text{Spec}(B)$ that we use.

It is easy to see that all prime ideals of $B$ are maximal, so that the points of $X$ are the maximal ideals of $B$. Moreover $X$ is a Boolean space, i.e., it is a totally disconnected, compact Hausdorff space. (Totally disconnected means that the connected components of $X$ are single points.) For further details, see [10.1].

A basis for the Zariski topology on $X$ consists of the sets
\[N(e) := \{M \in X | e \notin M\} \quad (e \in B).\]
Note that $N(e') = N(e)'$ is the complement of $N(e)$ in $X$, so that $N(e)$ is a clopen (closed and open) subset in $X$. Indeed, the sets $\{N(e) | e \in B\}$ comprise all of the clopen subsets of $X$. Because $X$ is Hausdorff, clopen is the same as compact and open. We have in addition that
\begin{equation}
N(e) \cup N(f) = N(e \vee f), \quad N(e) \cap N(f) = N(e \wedge f).
\end{equation}
The map \(e \mapsto N(e)\), which induces a lattice isomorphism between \(B\) and clopen subsets of \(B\) thanks to (9.1), is the Stone duality. This explains our nomenclature in Definition 9.1.

The following property of \(X\) is very powerful.

**Lemma 9.2.** (Partition property) Suppose that \(\{X_i\}\) is an open cover of \(X\). Then there are clopen sets \(N_1, \ldots, N_k\) such that each \(N_i\) is contained in some \(X_j\) and the \(N_i\) partition \(X\) in the sense that \(\bigcup_i N_i = X\) and \(N_i \cap N_j = \emptyset\) \((i \neq j)\).

In terms of idempotents, the partition property means that if \(N_i = N(e_i)\) \((e_i \in B)\), then
\[
e_i e_j = \delta_{ij} e_i, \quad \sum_i e_i = 1.
\]

The following Example is telling.

**Example 9.3.** Recall that an associative ring \(R\) is von Neumann regular if, for every \(x \in R\), there is \(y \in R\) such that \(xyx = x\). If \(R\) is a commutative von Neumann regular ring then there is a homeomorphism \(\text{Spec}(R) \to \text{Spec}(B(R)), P \mapsto P \cap B(R)\). (For further discussion about von Neumann regular rings and vertex rings, see Subsection 10.)

**Proof.** See [10], Theorem 8.25.

**9.2. The Pierce bundle of a vertex ring.** Let \(V\) be a vertex ring with Stone space \(X = \text{Spec}(B(V))\). Following Pierce [19], we construct an étale bundle \(\mathcal{R} \to X\) of vertex rings associated to this data.

For each \(M \in X\) set
\[
\overline{M} := \cup_{e \in B \cap M} e(-1)V.
\]

**Lemma 9.4.** \(\overline{M}\) is a 2-sided ideal in \(V\).

**Proof.** For an idempotent \(e \in M \in B\) we know that \(e(-1)V\) is a 2-sided ideal of \(V\) (Example 6.2(ii)), hence so is \(\sum_{e \in M} e(-1)V\). The main point of the Lemma is that this sum coincides with the displayed union. This follows from the identity \((e \oplus f)e = e\) in \(B\), which implies that if \(u, v \in V\) then
\[
e(-1)u + f(-1)v = (e \oplus f)(-1)(e(-1)u + f(-1)v).
\]

The Lemma follows.

By Lemma 9.4, each \(V/\overline{M}\) is a vertex ring, and we let \(0_{\overline{M}}, 1_{\overline{M}}\) denote the zero element and vacuum element respectively. Introduce the disjoint union
\[
\mathcal{R} := \bigcup_{M \in B} V/\overline{M},
\]
and define
\[
\pi : \mathcal{R} \to X, \quad v + \overline{M} \mapsto M.
\]

The result we are after is

**Theorem 9.5.** Let \(V\) be a vertex ring with Stone space \(X\). Then \(\pi : \mathcal{R} \to X\) is an étale bundle of vertex rings.
First we introduce a section $\sigma_v$ of the set map $\pi$ for each $v \in V$. These will be instrumental in everything that follows, and are defined in the natural way, namely

\begin{equation}
\sigma_v: X \to \mathcal{R}, \quad M \mapsto v + \overline{M}.
\end{equation}

We may, and shall, topologize $\mathcal{R}$ by taking the family of sets $\{\sigma_v(N(e))\}$ for $v \in V$ and $e \in B$ as a basis. The justification for this is the same as the case of commutative rings ([19], pp 16-17). We give details to highlight some common techniques and results that we use.

**Lemma 9.6.** Suppose that $u, v \in V$ satisfy $\sigma_u(M) = \sigma_v(M)$ for some $M \in X$. Then there is $e \in B \setminus M$ such that $\sigma_u(L) = \sigma_v(L)$ for all $L \in N(e)$.

**Proof.** We have $0_{\overline{M}} = \sigma_u(M) - \sigma_v(M) = (u - v) + \overline{M}$. So $u - v \in \overline{M} \Rightarrow u - v = f(-1)w$ ($f \in B \cap M, w \in V$), and we take $c = 1 - f \in B \setminus M$. Then if $L \in N(e)$ we have $f \in L$ and we obtain $\sigma_u(L) = u + \overline{L} = v + f(-1)w + \overline{L} = v + L = \sigma_v(L)$.

Now consider idempotents $e, f \in B$ such that $M \in N(e) \cap N(f)$, and suppose further that $u, v \in V$ satisfy $\sigma_u(M) = \sigma_v(M)$. By Lemma 9.6, $\sigma_u, \sigma_v$ agree on a clopen neighborhood $N(g)$ of $M$ for some $g \in B$. But we have $M \in N(e) \cap N(f) \cap N(g)$, so that $\sigma_u(M) \cap \sigma_u(N(e)) \cap \sigma_v(N(f))$. This suffices to establish our assertion about the topology on $\mathcal{R}$.

Now it is clear that $\pi$ is a continuous surjection onto $X$ and a local homeomorphism. Indeed, $\sigma_v(N(e))$ is mapped homeomorphically onto the open set $N(e)$. Thus the bundle $\pi : \mathcal{R} \to X$ is étale.

**Example 9.7.** $\sigma_v \in \Gamma(X, \mathcal{R})$ for all $v \in V$. $\sigma_0 : M \mapsto 0_{\overline{M}}, \sigma_1 : M \mapsto 1_{\overline{M}}$ $(M \in X)$ are the zero and vacuum sections respectively.

**Proof.** We need the continuity of $\sigma_v$. But $N(1) = X$, so $\sigma_v$ is the inverse of the restriction of $\pi$ to $\sigma_v(X)$, which is a homeomorphism.

The stalks of the bundle $\pi : \mathcal{R} \to X$ are the vertex rings $\mathcal{R}_M = V/\overline{M}$. So we can define $n^{th}$ products on sections in the usual way. In particular, if $u, v \in V$ and $M \in X$ we have

\begin{equation}
(\sigma_u(n) \sigma_v(M))(n) \sigma_v(M) = (u + \overline{M})(n)(v + \overline{M}) = u(n) v + \overline{M} = \sigma_u(n) v(M).
\end{equation}

This shows that

\begin{equation}
\sigma_u(n) \sigma_v = \sigma_u(n) v.
\end{equation}

Now we can show that $\pi$ is a bundle of vertex rings for which we require the continuity of the various algebraic operations (cf. Subsection 8.1). Precisely, set

\begin{equation}
\Delta := \{(a, b) \in \mathcal{R} \times \mathcal{R} | \pi(a) = \pi(b)\}.
\end{equation}

We have to show that the maps $\Delta \to \mathcal{R}$, $(a, b) \mapsto a \pm b, a(n)b$ are continuous.

For example, fix $(a, b) \in \Delta, n \in \mathbb{Z}$. There is $t \in V$ such that $a(n)b \in \sigma_t(X)$. So there are $u, v \in V, M \in X$ such that $a = u + \overline{M}, b = v + \overline{M}$, and $a(n)b = t + \overline{M}$. Now we have $\sigma_t(M) = \sigma_u(n)v(M)$, so (Lemma 9.6) $\sigma_t$ and $\sigma_u(n)v$ agree on a basic open neighborhood $N(e)$ of $M$. Write $u(n) \Delta \to \mathcal{R}$ for the $n^{th}$ product. So $\sigma_t(N(e))$ is a basic open neighborhood of $a(n)b$ and

\begin{equation}
\mu_n(\sigma_a(N(e))) \times \sigma_v(N(e)) \cap \Delta = \bigcup_{M \in N(e)} u(n) v + \overline{M} = \sigma_u(n) v(N(e)) = \sigma_t(N(e)).
\end{equation}
The continuity of $\mu_n$ follows. The other operations are treated similarly, and the proof of Theorem 9.5 is complete. \(\square\)

**Definition 9.8.** The étale bundle $\pi:R \to X$ associated to the vertex ring $V$ is the Pierce bundle associated to $V$.

### 9.3. Some local sections.

We continue with previous notation, in particular $V$ is a vertex ring with Stone space $X$ and associated Pierce bundle $\pi:R \to X$ (Theorem 9.5). According to Theorem 9.9, the local sections $\Gamma_Y$ for closed $Y \subseteq X$ carry the structure of a vertex ring. We shall explicitly realize these vertex rings as quotients of $V$. The main result is the case when $Y = X$, and may be stated as follows.

**Theorem 9.9.** There is an isomorphism of vertex rings

$$\xi: V \xrightarrow{\sim} \Gamma(X, R), \ v \mapsto \sigma_v.$$ 

**Proof.** The main issue is to show that $\xi$ is surjective, and we do this first. Pick any $\alpha \in \Gamma(X, R)$ and $M \in X$. Then $\alpha(M) \in \mathcal{R}_M$, so $\alpha(M) = a + \mathcal{M}$ for some $a \in V$. The sections $\alpha, \sigma_a$ agree at $M$, so by Lemma 9.8 they agree on a basic open neighborhood of $M$. As $M$ ranges over all points of $X$ we obtain an open cover of $X$, and on each open set in the cover $\alpha$ agrees with some $\sigma_a$ ($a \in V$).

By Lemma 9.2 there is a partition $X = N(e_1) \cup \ldots \cup N(e_r)$ into clopen sets $(e_j \in B)$ such that on each $N(e_j)$ $\alpha$ agrees with some $\sigma_{a_j} (a_j \in V)$. Set

$$b_v = \sum_j e_j(-1)a_j.$$ 

We will show that $\alpha = \sigma_b$. Let $L \in X$. Then there is a unique index $i$ such that $L \in N(e_i)$, and we have to show that $\alpha(L) = \sigma_b(L)$.

The partition property implies that $\sum_j e_j = 1$, so that $a_i = \sum_j e_j(-1)a_i$. Now observe that $i \neq j \Rightarrow L \notin N(e_j) \Rightarrow e_j \in L \Rightarrow e_j(-1)a_i \not\in L$. It follows that $a_i \equiv e_i(-1)a_i \pmod \mathcal{L}$, and therefore $\sigma_{a_i}(L) = \sigma_{e_i(-1)a_i}(L)$. Furthermore we have $\sigma_{b}(L) = \sigma_{\sum_j e_j(-1)a_j}(L)$.

On the other hand because $\alpha$ agrees with $\sigma_a$, on $N(e_i)$ then $\alpha(L) = \sigma_a(L)$. Therefore, $\alpha(L) = \sigma_{b}(L)$ follows from the previous paragraph, and the surjectivity of $\xi$ is established.

The field property $\sigma(n)\tau = 0$ for $n > 0$ ($\sigma, \tau \in \Gamma(X, R)$) now follows because $\sigma = \sigma_u$ and $\tau = \sigma_v$ for some $u, v \in V$, and $\sigma_u(n)\sigma_v = \sigma_{u(n)v} = 0$ for $n > 0$. The Jacobi identity and vacuum identities follow because they hold in each stalk, and the first assertion of the Theorem, that $\Gamma(X, R)$ is a vertex ring, is proved.

$\xi$ is a morphism of vertex rings because

$$\xi(u(n)v) = \sigma_{u(n)v} = \sigma_u(n)\sigma_v = \xi(u)(n)\xi(v)$$

and $\xi(1) = \sigma_1$. Finally, if $\sigma_v \in \text{ker} \xi$ then $v \in \cap_{M \in X} \mathcal{M}$, and this intersection is 0 by Proposition 1.7 of [19]. This completes the proof of the Theorem. \(\square\)

To describe $\Gamma_Y$ for an arbitrary closed subset $Y \subseteq X$ we introduce some additional notation.

The *support* of $\sigma \in \Gamma_X$ is $\text{supp}(\sigma) := \{ M \in X | \sigma(M) \neq 0 \}$.

For $U \subseteq X$, $J[U] := \{ \sigma \in \Gamma_X | \text{supp}(\sigma) \subseteq U \}$.

For $J \subseteq \Gamma_X$, $U[J] := \cup_{\sigma \in J} \text{supp}(\sigma)$. 

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**Lemma 9.10.** Let $Y \subseteq X$ be closed with complement $Y'$. Then there is a short exact sequence

$$0 \rightarrow J[Y'] \rightarrow \Gamma(X,\mathcal{R}) \overset{res}{\rightarrow} \Gamma(Y,\mathcal{R}) \rightarrow 0$$

(res is restriction from $X$ to $Y$.)

**Proof.** To show that $res$ is surjective, let $\alpha \in \Gamma_Y$. If $M \in Y$ then $\alpha(M) = a + M$ for some $a \in V$, so $\sigma_a$ and $\alpha$ agree on a basic open neighborhood of $M$. If $M \notin Y$, choose a basic open neighborhood of $M$ whose intersection with $Y$ is empty. As $M$ ranges over $X$, we obtain in this way an open cover $\{N_i\}$ of $X$ such that either $\alpha$ agrees with some $\sigma_{a_i}$ on $N_i$ or else $Y \cap N_i = \emptyset$. There is a partition $X = N(e_1) \cup \ldots \cup N(e_k)$ such that each $N(e_j)$ is contained in some $N_i$. Define $\sigma : X \rightarrow \mathcal{R}$ as follows: if $N(e_j) \cap Y = \emptyset$ then $\sigma|N(e_j) = 0$, and otherwise $\sigma|N(e_j) = \alpha|N(e_j)$. Because the $N(e_i)$'s partition $X$ and the restriction of $\sigma$ to $N(e_i)$ belongs to $\Gamma(N(e_i),\mathcal{R})$, then $\sigma \in \Gamma(X,\mathcal{R})$. Moreover by construction we have $\sigma Y = \alpha$. This completes the proof of the surjectivity of $res$.

Clearly $\sigma \in \ker res \iff \supp(\sigma) \subseteq Y' \iff \sigma \in J[Y']$, and the Lemma is proved. \qed

**Example 9.11.** If $M \in X$ then $\{M\}$ is closed because $X$ is Hausdorff, and there is an isomorphism of short exact sequences, where $\eta : \sigma \mapsto \sigma(M)$,

$$0 \longrightarrow J[X \setminus \{M\}] \longrightarrow \Gamma(X,\mathcal{R}) \overset{res}{\rightarrow} \Gamma(\{M\},\mathcal{R}) \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow V \longrightarrow \mathcal{R}_M \longrightarrow 0$$

\qed

We establish an additional property of the Pierce sheaf $\mathcal{R} \rightarrow X$.

**Lemma 9.12.** Each stalk $\mathcal{R}_M$ is indecomposable, i.e. (cf. Subsection $\S.3$) the only idempotents in $V/M$ are $0_M$ and $1_M$.

**Proof.** Suppose that $e \in V$ is such that $e + M \in B(\mathcal{R}_M(V))$. Then $e(-1)e - e \in M$ and $e(m)e \in M$ for $m \neq 0$. Then $\sigma_e$ and $\sigma_{e(-1)}e$ agree at $M \in X$, and therefore they agree on a basic open neighborhood $N_{-1}$ of $M$ by Lemma 9.7. Similarly, $\sigma_{e(m)e}$ vanishes on a basic open neighborhood $N_m$ of $M$ for $m \neq 0$.

Now $e(m)e = 0$ for $m \gg 0$. So intersecting $N_{-1}$ and a sufficient finite number of $N_m$'s for $m \geq 0$, we obtain a basic open neighborhood $N(f)$ of $M$ ($f \in B$) with the property that $\sigma_{e(m)e}$ vanishes on $N(f)$ for all $m \geq 0$ and $\sigma_{e(-1)}e = \sigma_e$ on $N(f)$. Because $N(f)$ is a clopen set, we can find a global section $\tau \in \Gamma(X,\mathcal{R})$ that agrees with $\sigma_e$ on $N(f)$ and vanishes on $N(f)'$.

Consider $\tau(m)\tau$ for $m \geq 0$: it vanishes on $N(f)'$ because $\tau$ vanishes there, whereas it agrees with $\sigma_e(m)\sigma_e = \sigma_{e(m)e}$ on $N(f)$ and therefore again vanishes because $\sigma_{e(m)e}$ does. So $\tau(m)\tau = 0$ for $m \geq 0$. Similarly, $\tau(-1)\tau$ vanishes on $N(f)'$ and coincides with $\sigma_{e(-1)}e = \sigma_e$ on $N(f)$. As a consequence, we have $\tau(-1)\tau = \tau$. Now by Lemma 6.11 we conclude that $\tau$ is an idempotent in $\Gamma(X,\mathcal{R})$, and by Theorem 9.9 it follows that $\tau = \sigma_g$ for some idempotent $g \in B$. Finally, because $\tau$ agrees with $\sigma_e$ on $N(f)$ we have

$$e + M = \sigma_e(M) = \tau(M) = \sigma_g(M) = g + M = 0_M \text{ or } 1_M.$$
according to whether \( g \in M \) or \( g \notin M \). This completes the proof of the Lemma. \( \square \)

**Definition 9.13.** Following Pierce [19], we call an étale bundle of vertex rings \( \mathcal{R} \to X \) **reduced** if (i) \( X \) is a Boolean space and (ii) each stalk is an indecomposable vertex ring.

By Theorem [9.10] and Lemma [9.12] we conclude

**Corollary 9.14.** Suppose that \( V \) is a vertex ring with Stone space \( X \). Then the Pierce bundle \( \mathcal{R} \to X \) of \( V \) is reduced. \( \square \)

### 10. Von Neumann regular vertex rings

#### 10.1. Regular ideals.

**Definition 10.1.** Let \( V \) be a vertex ring with Boolean ring \( B = B(V) \). A 2-sided ideal \( I \subseteq V \) is called regular if it satisfies \( I = \cup_{e \in I \cap B} e(-1)V \).

For example, the ideal \( \mathcal{M} \) in \( \mathcal{O}(X) \) is regular. Our aim in this Subsection is to describe the lattice of all regular 2-sided ideals in a vertex ring \( V \).

Let \( X \) be the Stone space of \( V \), with \( \mathcal{R} \to X \) the associated Pierce bundle of \( V \).

As usual \( \mathcal{O}(X) \) denotes the set of open sets in \( X \).

Thanks to Theorem [9.10] we have an isomorphism of vertex rings \( \xi: V \to \Gamma_X \), so for our purposes it suffices to describe the regular 2-sided ideals of \( \Gamma_X \). We prove

**Theorem 10.2.** If \( U \in \mathcal{O}(X) \) then \( J[U] \subseteq \Gamma_X \) is a regular 2-sided ideal, and the map

\[
\mathcal{O}(X) \to \{ \text{regular 2-sided ideals in } \Gamma_X \}, \quad U \mapsto J[U]
\]

is an isomorphism of lattices with inverse \( J \to U[J] \). (So the regular 2-sided ideals are the kernels of the projections described in Lemma [9.10].)

Theorem 10.2 is a consequence of the following Lemmas 10.3-10.5.

**Lemma 10.3.** If \( \sigma \in \Gamma(X, \mathcal{R}) \) then \( \text{supp}(\sigma) \subseteq X \) is compact, and it is a clopen subset if \( \sigma \) is an idempotent.

**Proof.** If \( \sigma \) vanishes at \( M \in X \) then it vanishes on an open neighborhood of \( M \). This shows that the zero set of \( \sigma \) is open, so that \( \text{supp}(\sigma) \) is closed in \( X \). Furthermore, because \( \mathcal{R} \to X \) is reduced by Corollary [9.14] it follows that if \( \sigma \in B(\Gamma(X, \mathcal{R})) \) then \( \sigma(M) = 0_{\mathcal{M}} \) or \( 1_{\mathcal{M}} \) for all \( M \). Since \( \{ M \in X | \sigma(M) = 1_{\mathcal{M}} \} \) is open then \( \text{supp}(\sigma) \) is also closed. \( \square \)

**Lemma 10.4.** If \( U \in \mathcal{O}(X) \) then \( J[U] \) is a regular 2-sided ideal of \( \Gamma_X \), and \( U[J[U]] = U \).

**Proof.** That \( J[U] \) is a 2-sided ideal follows from Lemma [9.10]. Let \( \sigma \in J[U] \), so that \( \sigma \in \Gamma_X \) with \( \text{supp}(\sigma) \subseteq U \). Because \( \text{supp}(\sigma) \) is closed by Lemma [10.3] there is a clopen set \( N \) such that \( \text{supp}(\sigma) \subseteq N \subseteq U \). (Consider the open cover \( \{ U, \text{supp}(\sigma)' \} \) of \( X \). Then there is a partition of \( X \) into clopen sets, each contained in either \( U \) or \( \text{supp}(\sigma)' \), and \( N \) is the union of those contained in \( U \).

We get an idempotent \( \tau \in J[U] \) by setting \( \tau(M) = 1_{\mathcal{M}} (M \in N) \) together with \( \tau(M) = 0_{\mathcal{M}} (M \notin N) \). We have \( \tau(-1)\sigma = \sigma \) because \( \text{supp}(\sigma) \subseteq \text{supp}(\tau) \), proving that \( J[U] \) is regular.
Finally, $U[J[U]]$ is the union of $\text{supp}(\sigma)$ for $\sigma \in J[U]$, which coincides with the union of $\text{supp}(\tau)$ for the idempotents $\tau \in J[U]$. By the argument of the first paragraph, there is a clopen set $N' \subseteq U$ containing a given point of $U$ (points are closed because $X$ is Hausdorff), and by the argument of the second paragraph $N'$ is the support of an idempotent in $J[U]$. This shows that $U[J[U]]=U$, and the Lemma is proved. □

**Lemma 10.5.** If $J \subseteq \Gamma(X, \mathcal{R})$ is a regular 2-sided ideal then $U[J] \subseteq X$ is open and $J[U[J]]=J$.

**Proof.** If $\sigma=\tau\sigma$ then $\text{supp}(\sigma) \subseteq \text{supp}(\tau)$. Thus if $J$ is regular then the union of the supports $\text{supp}(\sigma)$ for $\sigma \in J$, i.e., $U[J]$, coincides with the union of the corresponding supports for the idempotents in $J$. But this is open thanks to Lemma 10.3.

Now if $\sigma \in J[U[J]]$ then $\text{supp}(\sigma) \subseteq U[J] = \bigcup_{\tau \in B(J)} \text{supp}(\tau)$. By compactness we get $\text{supp}(\sigma) = \bigcup_{i=1}^{n} \text{supp}(\tau_i)$ where the $\tau_i \in B(J)$. Then $\sigma=\sigma(\tau_1 \oplus ... \oplus \tau_n) \in J$. This proves that $J[U[J]] \subseteq J$, and since the opposite inclusion is obvious then $J=J[U[J]]$ and the proof of the Lemma is complete. □

**10.2. Von Neumann regular vertex rings.** Suppose that $R$ is a commutative, unital ring, and let $X=\text{Spec}(R)$. To provide some background and context, we recall (1.10) that the following properties of $R$ are equivalent:

- $R/\text{Nil}(R)$ is a von Neumann regular ring (cf. Example 9.3),
- Every principal ideal in $R$ is generated by an idempotent,
- Every f. g. ideal in $R$ is generated by an idempotent,
- $R$ has Krull dimension 0
- $X$ is a Hausdorff space
- $X$ is totally disconnected

(Krull dimension 0 means that every prime ideal is maximal; $\text{Nil}(R)$ is the nilpotent radical of $R$.) Topological spaces enjoying the last two properties are called Boolean.

**Example 10.6.** Let $V$ be a vertex ring with associated Boolean ring $B=B(V)$. Then we showed in Section 9.1 that $B$ has Krull dimension 0. Since every element is idempotent then $\text{Nil}(R)=0$, so $B$ is a commutative von Neumann ring. Therefore its prime spectrum $X=\text{Spec}(B)$ is Boolean. □

**Definition 10.7.** A vertex ring $V$ is called von Neumann regular if every principal 2-sided ideal of $V$ is generated by an idempotent. In other words, every principal 2-sided ideal of $V$ is equal to $e(-1)V$ for some $e \in B(V)$.

**Example 10.8.** If $R$ is a commutative von Neumann regular ring then by (10.1) $R$ is a von Neumann regular vertex ring with trivial HS derivation.

**Lemma 10.9.** Let $V$ be a von Neumann regular vertex ring. Then every 2-sided ideal $I \subseteq V$ is regular in the sense of Definition 10.7.

**Proof.** $I$ is certainly a sum of principal 2-sided ideals, hence a sum of ideals generated by idempotents. That it is a union of such ideals follows as in the proof of Lemma 9.4. □
**Remark 10.10.** This argument shows that in a von Neumann regular vertex ring, every finitely generated 2-sided ideal is generated by an idempotent.

**Corollary 10.11.** Let $V$ be a von Neumann regular vertex ring with Stone space $X$. Then there is an isomorphism of lattices

$$\mathcal{O}(X) \cong \{\text{2-sided ideals of } V\}.$$

**Proof.** By Corollary 9.14 the Pierce bundle $\mathcal{R} \to X$ associated to $V$ is reduced. Then the present Corollary follows from Theorem 10.9 and Lemma 10.9.

**Lemma 10.12.** Let $V$ be a von Neumann regular vertex ring. Then the Pierce bundle $\mathcal{R} \to X$ of $V$ is a bundle of simple vertex rings. That is, each stalk $\mathcal{R}_M$ is a simple vertex ring.

**Proof.** According to Corollary 10.11 the maximal ideals of $V$ correspond to those open subsets of $X$ that are maximal (with respect to containment) among all proper open subsets of $X$. Since $X$ is Hausdorff, these are precisely the subsets $X\setminus\{M\}$ ($M\in X$). By Lemma 10.4 the maximal ideal of $\Gamma_X$ that corresponds to $X\setminus\{M\}$ is $J[X\setminus\{M\}]$. Therefore $\Gamma_X/J[X\setminus\{M\}] \cong \mathcal{R}_M$ is simple, where the isomorphism follows from Example 9.11.

**Theorem 10.13.** Let $V$ be a von Neumann regular vertex ring. Then the center $C(V)$ is a commutative von Neumann regular ring.

**Proof.** Let $B:=B(V), C:=C(V)$ with $X$ the Stone space of $V$. First we show that if $M\in X$ then $C+\overline{M}/\overline{M}$ is a field. Equivalently, setting $I:=C \cap \overline{M}$, we show that $I$ is a maximal ideal of $C$. It is clear that $I$ is a proper ideal of $C$. Now $\overline{M}$ is generated by the idempotents in $M$, and these all lie in $C$. Thus $\overline{M}$ is generated by the idempotents in $I$. Because $\overline{M}$ is a maximal 2-sided ideal of $V$, it follows that the 2-sided ideal $N:=\sum a(e)(1)V$ of $V$ generated by $I$ is equal to either $V$ or $\overline{M}$.

We show that $N=V$ leads to a contradiction. To achieve this we need the following result: if $a\in I$ then $a(1)V=e(1)V$ for some $e\in B \cap I$. (We know that we can choose $e\in B$ because $V$ is von Neumann regular. The main point is that in fact $e\in I$.)

To prove this assertion, and starting from $a(1)V=e(1)V$ with $e\in B$, there are $u, v\in V$ such that $a(1)u=e, e(1)v=a$. Consider the decomposition $V=e(1)V \oplus (1-e)(1)V$. Since $a\in e(1)V$ then we may, and shall, also assume that $u\in e(1)V$. Then from $a(1)u=e$ it follows from Lemma 6.17 (with $e(1)V$ in place of $V$) that $u\in C(e(1)V) \subseteq C$. Since $I$ is an ideal in $C$ and $a\in I$ we deduce that $e=a(1)u\in I$, as required.

From what we have just proved, we have $N=\sum e\in B \cap N e(1)V=\bigcup_{e\in B \cap N} e(1)V$. If $V=N$ we have $1\in e(1)V$ for some $e\in B \cap I$. Thus $e(1)u=1$ for some $u\in V$, so that $e$ is a unit in $C$ by Lemma 6.17. Since $e\in I$ this is a contradiction.

Having shown that $N\neq V$, we must conclude that $N=\overline{M}$. Let $I \subseteq J \subseteq C$ where $J$ is a maximal ideal of $C$. We show that $I=J$. We can proceed as before, setting $N':=\sum_{b\in J} b(1)V$. This is a 2-sided ideal of $V$ that contains $\overline{J}$, so either $N'=V$ or $N'=\overline{M}$. If $N'=V$ we get a contradiction just as before, so $N'=\overline{M}$. But then $N'$ is generated by the idempotents in $M$, and therefore it must be equal to $\overline{M}$, as asserted.
We have established that \( C+\mathbb{M}/\mathbb{M} \) is a field for each \( \mathbb{M} \in \mathcal{X} \). Therefore, setting \( S:= \bigcup_{\mathbb{M} \in \mathcal{X}} C+\mathbb{M}/\mathbb{M} \subseteq \mathcal{R} \), it follows that \( S \to X \) is an étale bundle of commutative fields over \( X \). Then we can invoke one of the main results of [19] (loc. cit. Theorem 10.3) to conclude that \( C \) is a von Neumann regular ring. This completes the proof of the Theorem. \( \square \)

**Corollary 10.14.** Let \( V \) be a von Neumann regular vertex ring with Stone space \( X \) and center \( C \). Then the lattices of 2-sided ideals in \( V \) and \( C \) are each isomorphic to \( \mathcal{O}(X) \).

**Proof.** \( X \) is the Stone space of \( C \) as well as \( V \), and because \( C \) is von Neumann regular by Theorem 10.13 we can apply the commutative ring case of Corollary 9.10 with \( C \) in place of \( V \). Thus \( \mathcal{O}(X) \) is also isomorphic to the lattice of 2-sided ideals of \( C \), and the Corollary is proved. \( \square \)

**11. Equivalence of some categories of vertex rings**

In this Section we will discuss the following diagram of categories and functors. The lower half gives expression to some theorems of Pierce [19] while the upper half reflects the extensions of these theorems to the corresponding categories of vertex rings.

The notation is supposed to be self-explanatory. \( \text{Ver} \to \text{Comm} \) is the center functor from vertex rings to commutative rings that is left adjoint to the insertion \( K: \text{Comm} \to \text{Ver} \). See Section 6, especially Theorem 6.5.

\( \text{regVer} \) and \( \text{regComm} \) are the full subcategories of \( \text{Ver} \) and \( \text{Comm} \) whose objects are the von Neumann regular vertex rings and von Neumann regular commutative rings respectively. That the inner square commutes then means that the center of a von Neumann regular vertex ring is a von Neumann regular commutative ring, and this is the content of Theorem 10.13.

The four diagonals are categorical equivalences and require more discussion. The objects of \( \text{redVerbun} \) are reduced étale bundles of vertex rings. Recall (Definition 9.13) that these are étale bundles \( \mathcal{R} \to X \) of indecomposable vertex rings over a Boolean base space \( X \). We will define morphisms shortly. Then \( \text{simpVerbun} \), \( \text{redCommbun} \) and \( \text{simpCommbun} \) are the full subcategories whose objects are bundles of simple vertex rings, indecomposable commutative rings, and fields (i.e., simple commutative rings) respectively. Thus the sides of the outer square are functorial insertions.
Morphisms in categories such as \textbf{redVerbun} are standard, and are described as follows. Given a pair of objects \( R \xrightarrow{\nu} X, S \xrightarrow{\pi} Y \), a morphism from the first object to the second is a pair of continuous maps \((f,g)\) as in the diagram

\[
\begin{array}{ccc}
X & \times & Y \\
\downarrow f & & \downarrow g \\
S & \rightarrow & S \\
\nu & & \pi \\
R & \rightarrow & Y
\end{array}
\]

where the square is a pull-back in \textbf{Top} and where the following property holds: for each \( x \in X \),

\[ g(x,*) : S_{f(x)} \rightarrow R_x \]

is a morphism of vertex rings.

The lower left diagonal equivalence \textbf{Comm} \(\xrightarrow{\sim} \textbf{redCommbun}\) is a main result of Pierce ([19], Theorem 10.1). Moreover upon restricting to the subcategory of \textsc{vNr} commutative rings, Pierce shows ([19], Theorem 10.3) that the corresponding bundles have stalks which are fields (i.e., simple commutative rings), thereby giving rise to the lower right categorical equivalence \textbf{regComm} \(\xrightarrow{\sim} \textbf{simpCommbun}\)

These results extend to equivalences of the corresponding categories of vertex rings. The first equivalence, namely \textbf{Ver} \(\xrightarrow{\sim} \textbf{redVerbun}\) may be established by a proof parallel to that of Pierce (loc. cit.) The object map assigns to a vertex ring \( V \) its Pierce bundle \( R \rightarrow X \) described in Subsection 9.2, while the inverse is the global sections functor that assigns to a reduced bundle \( E \rightarrow X \) of vertex rings the vertex ring \( \Gamma(X,E) \). Theorem 9.9 shows that the composition of these two functors is equivalent to the identity functor, at least on objects. There are a multitude of additional details to check, most of them very similar to [19], and we skip the details here. The fourth and final equivalence is \textbf{regVer} \(\xrightarrow{\sim} \textbf{simpVerbun}\). The main point, and the result that we will actually prove, is the following.

**Theorem 11.1.** Suppose that \( R \rightarrow X \) is an étale bundle of vertex rings over a Boolean space \( X \). Then the vertex ring of global sections \( \Gamma(X,R) \) is von Neumann regular if, and only if, \( R \) is a bundle of simple vertex rings.

**Proof.** Set \( V := \Gamma(X,R) \). Then \( R \rightarrow X \) is isomorphic to the Pierce bundle of \( V \), so the implication \( \Rightarrow \) follows from Lemma [10.12]

Conversely, suppose that \( R \rightarrow X \) is a bundle of simple rings. Then certainly each stalk is an indecomposable vertex ring, so the sheaf is reduced (cf. Definition 9.13) and we have to show that \( V \) is von Neumann regular. Let \( \sigma \in \Gamma(X,R) \). It suffices to show that \( \sigma = e(-1)\sigma \) for some idempotent \( e \), because then the 2-sided ideal generated by \( \sigma \) is equal to \((e(-1)V)\).

Suppose that \( M \in \text{supp}(\sigma) \). Then \( 0 \neq \sigma(M) \in R_M \), and since \( R_M \) is simple then it coincides with the 2-sided ideal generated by \( \sigma(M) \). Therefore there is an equation of the form

\[
\sum v_i^M(n_1) \ldots v_k^M(n_k)\sigma(M)(i)1_{\overline{M}} = 1_{\overline{M}} \quad (v_j^M \in R_M)
\]
We can find global sections $\tau^M_1, \ldots, \tau^M_k \in \Gamma(X, \mathcal{R})$ such that $\tau^M_j(M) = v^M_j$ ($1 \leq j \leq k$). Then $\sum \tau^M_1(n_1) \ldots \tau^M_k(n_k)\sigma(i)1$ takes the value $1_M$ at $M$, and it follows that $\sum \tau^M_1(n_1) \ldots \tau^M_k(n_k)\sigma(i)1$ and $1$ agree on an open neighborhood $N_M$ of $M$.

As $M$ ranges over $\text{supp}(\sigma)$ we get an open cover $\{N_M \cup \text{supp}(\sigma)\}'$ of $X$, and by the partition property we can find a partition of $X$ into clopen sets, each of which is contained in some $N_M$ or in $\text{supp}(\sigma)'$. It follows that there is a continuous function $\nu$ of the form $\nu = \sum t_1(n_1) \ldots t_k(n_k)\sigma(i)1$ with the property that $\nu(L) = 1_T$ for all $L \in \text{supp}(\sigma)$; and if $L \notin \text{supp}(\sigma)$ then $\sigma(L) = 0$, so that also $\nu(L) = 0$. This shows that $\nu$ is idempotent. Finally, to check that $\nu(-1)\sigma = \sigma$, we only have to check it locally. But this is clear, because $(\nu(-1)\sigma)(L) = \nu(L)(-1)\sigma(L)$ and this is $\sigma(L)$ or 0 according to whether $L \in \text{supp}(\sigma)$ or not. The proof of Theorem 11.1 is complete. \(\square\)

12. Appendix

If $m, n \in \mathbb{Z}$ we define

$$\binom{m}{n} = \begin{cases} m(m-1) \ldots (m-n+1)/n! & \text{if } n \geq 1 \\ 1 & \text{if } n = 0, \\ 0 & \text{if } n < 0 \end{cases}$$

We have $\binom{m}{n} \in \mathbb{Z}$, and the following identities hold for all $m, n, r$.

$$\binom{m}{n} = (-1)^n \binom{n-m-1}{n}, \quad (12.1)$$
$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}, \quad (12.2)$$
$$\binom{m}{n} = \sum_{i=0}^{n} \binom{r}{i} \binom{m-r}{n-i}, \quad (12.3)$$
$$\binom{m}{n} = \sum_{i=0}^{n} (-1)^i \binom{r}{i} \binom{r+m-i}{n-i}. \quad (12.4)$$

Care is warranted when dealing with these binomial coefficients. For example, the ‘familiar’ identity $\binom{m}{n} = \binom{m-n}{n}$ is not universally true.

We use the following binomial expansion for all $m \in \mathbb{Z}$:

$$\binom{m}{n} = \sum_{n \geq 0} \binom{m}{n} z^{m-n} w^n. \quad (12.5)$$

That is, we always expand binomial expressions in nonnegative powers of the second variable.

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