FROM SPINOR GEOMETRY TO COMPLEX GENERAL RELATIVITY

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An attempt is made of giving a self-contained introduction to holomorphic ideas in general relativity, following work over the last thirty years by several authors. The main topics are complex manifolds, two-component spinor calculus, conformal gravity, $\alpha$-planes in Minkowski space-time, $\alpha$-surfaces and twistor geometry, anti-self-dual space-times and Penrose transform, spin-3/2 potentials, heaven spaces and heavenly equations.

Keywords: Two-component spinors; twistors; Penrose transform.

1. Introduction to Complex Space-Time

The physical and mathematical motivations for studying complex space-times or real Riemannian four-manifolds in gravitational physics are first described. They originate from algebraic geometry, Euclidean quantum field theory, the path-integral approach to quantum gravity, and the theory of conformal gravity. The theory of complex manifolds is then briefly outlined. Here, one deals with para-compact Hausdorff spaces where local coordinates transform by complex-analytic transformations. Examples are given such as complex projective space $\mathbb{P}_n$, non-singular sub-manifolds of $\mathbb{P}_n$, and orientable surfaces. The plan of the whole paper is eventually presented, with emphasis on two-component spinor calculus, Penrose transform and Penrose formalism for spin-$\frac{3}{2}$ potentials.

1.1. From Lorentzian to complex space-time

Although Lorentzian geometry is the mathematical framework of classical general relativity and can be seen as a good model of the world we live in ([29], [16], [17]), the theoretical-physics community has developed instead many models based on
a complex space-time picture. We postpone until Sec. 3.3 the discussion of real, complexified or complex manifolds, and we here limit ourselves to say that the main motivations for studying these ideas are as follows.

(1) When one tries to make sense of quantum field theory in flat space-time, one finds it very convenient to study the Wick-rotated version of Green functions, since this leads to well defined mathematical calculations and elliptic boundary-value problems. At the end, quantities of physical interest are evaluated by analytic continuation back to real time in Minkowski space-time.

(2) The singularity at $r = 0$ of the Lorentzian Schwarzschild solution disappears on the real Riemannian section of the corresponding complexified space-time, since $r = 0$ no longer belongs to this manifold ([17]). Hence there are real Riemannian four-manifolds which are singularity-free, and it remains to be seen whether they are the most fundamental in modern theoretical physics.

(3) Gravitational instantons shed some light on possible boundary conditions relevant for path-integral quantum gravity and quantum cosmology ([30], [27], [17]).

(4) Unprimed and primed spin-spaces are not (anti-)isomorphic if Lorentzian space-time is replaced by a complex or real Riemannian manifold. Thus, for example, the Maxwell field strength is represented by two independent symmetric spinor fields, and the Weyl curvature is also represented by two independent symmetric spinor fields (see (2.1.35) and (2.1.36)). Since such spinor fields are no longer related by complex conjugation (i.e. the (anti-)isomorphism between the two spin-spaces), one of them may vanish without the other one having to vanish as well. This property gives rise to the so-called self-dual or anti-self-dual gauge fields, as well as to self-dual or anti-self-dual space-times (Sec. 4.2).

(5) The geometric study of this special class of space-time models has made substantial progress by using twistor-theory techniques. The underlying idea ([55], [56], [57], [59], [62], [63], [64], [99], [100], [65], [101], [102], [32], [33], [107], [69], [70], [109], [45], [3], [46], [103], [48]) is that conformally invariant concepts such as null lines and null surfaces are the basic building blocks of the world we live in, whereas space-time points should only appear as a derived concept. By using complex-manifold theory, twistor theory provides an appropriate mathematical description of this key idea.

A possible mathematical motivation for twistors can be described as follows (papers 99 and 100 in [2]). In two real dimensions, many interesting problems are best tackled by using complex-variable methods. In four real dimensions, however, the introduction of two complex coordinates is not, by itself, sufficient, since no preferred choice exists. In other words, if we define the complex variables

$$z_1 \equiv x_1 + ix_2,$$

$$z_2 \equiv x_3 + ix_4,$$

we rely too much on this particular coordinate system, and a permutation of the four real coordinates $x_1, x_2, x_3, x_4$ would lead to new complex variables not well related
to the first choice. One is thus led to introduce three complex variables \((u, z^1, z^2)\): the first variable \(u\) tells us which complex structure to use, and the next two are the complex coordinates themselves. In geometric language, we start with the complex projective three-space \(P_3(C)\) (see Sec. 1.2) with complex homogeneous coordinates \((x, y, u, v)\), and we remove the complex projective line given by \(u = v = 0\). Any line in \((P_3(C) - P_1(C))\) is thus given by a pair of equations

\[
x = au + bv, \tag{1.1.3}
\]

\[
y = cu + dv. \tag{1.1.4}
\]

In particular, we are interested in those lines for which \(c = -b, d = a\). The determinant \(\Delta\) of (1.1.3) and (1.1.4) is thus given by

\[
\Delta = ab\bar{v} + b\bar{u} = |a|^2 + |b|^2, \tag{1.1.5}
\]

which implies that the line given above never intersects the line \(x = y = 0\), with the obvious exception of the case when they coincide. Moreover, no two lines intersect, and they fill out the whole of \((P_3(C) - P_1(C))\). This leads to the fibration \((P_3(C) - P_1(C)) \rightarrow R^4\) by assigning to each point of \((P_3(C) - P_1(C))\) the four coordinates \((\text{Re}(a), \text{Im}(a), \text{Re}(b), \text{Im}(b))\). Restriction of this fibration to a plane of the form

\[
\alpha u + \beta v = 0, \tag{1.1.6}
\]

yields an isomorphism \(C^2 \cong R^4\), which depends on the ratio \((\alpha, \beta) \in P_1(C)\). This is why the picture embodies the idea of introducing complex coordinates.

Such a fibration depends on the conformal structure of \(R^4\). Hence, it can be extended to the one-point compactification \(S^4\) of \(R^4\), so that we get a fibration \(P_3(C) \rightarrow S^4\) where the line \(u = v = 0\), previously excluded, sits over the point at \(\infty\) of \(S^4 = R^4 \cup \{\infty\}\). This fibration is naturally obtained if we use the quaternions \(H\) to identify \(C^4\) with \(H^2\) and the four-sphere \(S^4\) with \(P_1(H)\), the quaternion projective line. We should now recall that the quaternions \(H\) are obtained from the vector space \(R\) of real numbers by adjoining three symbols \(i, j, k\) such that

\[
i^2 = j^2 = k^2 = -1, \tag{1.1.7}
\]

\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{1.1.8}
\]

Thus, a general quaternion \(\bar{a} \in H\) is defined by

\[
x \equiv x_1 + x_2i + x_3j + x_4k, \tag{1.1.9}
\]

where \((x_1, x_2, x_3, x_4) \in R^4\), whereas the conjugate quaternion \(\bar{a}\) is given by

\[
\bar{a} \equiv x_1 - x_2i - x_3j - x_4k. \tag{1.1.10}
\]
Note that conjugation obeys the identities
\[ (xy) = yx, \]
\[ x = \sum_{\mu = 1}^{4} x_{\mu}^2 \equiv |x|^2. \]  
If a quaternion does not vanish, it has a unique inverse given by
\[ x^{-1} = \frac{x}{|x|^2}. \]  
Interestingly, if we identify \( i \) with \( \sqrt{-1} \), we may view the complex numbers \( \mathbb{C} \) as contained in \( H \) taking \( x_3 = x_4 = 0 \). Moreover, every quaternion \( x \) as in (1.1.9) has a unique decomposition
\[ x = z_1 + z_2 j, \]  
where \( z_1 \equiv x_1 + x_2 i, \; z_2 \equiv x_3 + x_4 i \), by virtue of (1.1.8). This property enables one to identify \( H \) with \( \mathbb{C}^2 \), and finally \( H^2 \) with \( \mathbb{C}^4 \), as we said following (1.1.6).

The map \( \sigma : P_3(\mathbb{C}) \longrightarrow P_3(\mathbb{C}) \) defined by
\[ \sigma(x, y, u, v) = (-y, x, -v, u), \]  
preserves the fibration because \( c = -b, d = \overline{u} \), and induces the antipodal map on each fibre. We can now lift problems from \( S^4 \) or \( R^4 \) to \( P_3(\mathbb{C}) \) and try to use complex methods.

### 1.2. Complex manifolds

Following [12], we now describe some basic ideas and properties of complex-manifold theory. The reader should thus find it easier (or, at least, less difficult) to understand the holomorphic ideas used in the rest of the paper.

We know that a manifold is a space which is locally similar to Euclidean space in that it can be covered by coordinate patches. More precisely ([29]), we say that a real \( C^r \)-n-dimensional manifold \( \mathcal{M} \) is a set \( \mathcal{M} \) together with a \( C^r \) atlas \( \{U_\alpha, \phi_\alpha\} \), i.e. a collection of charts \( \{U_\alpha, \phi_\alpha\} \), where the \( U_\alpha \) are subsets of \( \mathcal{M} \) and the \( \phi_\alpha \) are one-to-one maps of the corresponding \( U_\alpha \) into open sets in \( R^n \) such that

(i) \( \mathcal{M} \) is covered by the \( U_\alpha \), i.e. \( \mathcal{M} = \bigcup_\alpha U_\alpha \)

(ii) if \( U_\alpha \cap U_\beta \) is non-empty, the map
\[ \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \]

is a \( C^r \) map of an open subset of \( R^n \) into an open subset of \( R^n \). In general relativity, it is of considerable importance to require that the Hausdorff separation axiom should hold. This states that if \( p, q \) are any two distinct points in \( \mathcal{M} \), there exist disjoint open sets \( U, V \) in \( \mathcal{M} \) such that \( p \in U, \; q \in V \). The space-time manifold \( (M, g) \)
is therefore taken to be a connected, four-dimensional, Hausdorff \( C^\infty \) manifold \( M \) with a Lorentz metric \( g \) on \( M \), i.e. the assignment of a symmetric, non-degenerate bilinear form \( g_p : T_p M \times T_p M \to R \) with diagonal form \((-, +, +, +)\) to each tangent space. Moreover, a time orientation is given by a globally defined, timelike vector field \( X : M \to TM \). This enables one to say that a timelike or null tangent vector \( v \in T_p M \) is future-directed if \( g(X(p), v) < 0 \), or past-directed if \( g(X(p), v) > 0 \) ([16], [17]).

By a complex manifold we mean a paracompact Hausdorff space covered by neighbourhoods each homeomorphic to an open set in \( \mathbb{C}^m \), such that where two neighbourhoods overlap, the local coordinates transform by a complex-analytic transformation. Thus, if \( z^1, \ldots, z^m \) are local coordinates in one such neighbourhood, and if \( w^1, \ldots, w^m \) are local coordinates in another neighbourhood, where they are both defined one has \( w^i = w^i(z^1, \ldots, z^m) \), where each \( w^i \) is a holomorphic function of the \( z \)'s, and the determinant \( \partial(w^1, \ldots, w^m)/\partial(z^1, \ldots, z^m) \) does not vanish.

Various examples can be given as follows [12].

**E1.** The space \( \mathbb{C}^m \) whose points are the \( m \)-tuples of complex numbers \((z^1, \ldots, z^m)\). In particular, \( \mathbb{C}^1 \) is the so-called Gaussian plane.

**E2.** Complex projective space \( \mathbb{P}_m \), also denoted by \( \mathbb{P}_m(\mathbb{C}) \) or \( \mathbb{CP}^m \). Denoting by \( \{0\} \) the origin \((0, \ldots, 0)\), this is the quotient space obtained by identifying the points \((z^0, z^1, \ldots, z^m)\) in \( \mathbb{C}^{m+1} \setminus \{0\} \) which differ from each other by a factor. The covering of \( \mathbb{P}_m \) is given by \( m + 1 \) open sets \( U_i \) defined respectively by \( z^i \neq 0, 0 \leq i \leq m \). In \( U_i \), we have the local coordinates \( \zeta^k_i \equiv z^k/z^i, 0 \leq k \leq m, k \neq i \). In \( U_i \cap U_j \), transition of local coordinates is given by \( \zeta^h_j \equiv \zeta^h_i/\zeta^i_j, 0 \leq h \leq m, h \neq j \), which are holomorphic functions. A particular case is the Riemann sphere \( \mathbb{P}_1 \).

**E3.** Non-singular sub-manifolds of \( \mathbb{P}_m \), in particular, the non-singular hyperquadric

\[
(z^0)^2 + \ldots + (z^m)^2 = 0. \tag{1.2.1}
\]

A theorem of Chow states that every compact sub-manifold embedded in \( \mathbb{P}_m \) is the locus defined by a finite number of homogeneous polynomial equations. Compact sub-manifolds of \( \mathbb{C}^m \) are not very important, since a connected compact sub-manifold of \( \mathbb{C}^m \) is a point.

**E4.** Let \( \Gamma \) be the discontinuous group generated by \( 2m \) translations of \( \mathbb{C}^m \), which are linearly independent over the reals. The quotient space \( \mathbb{C}^m / \Gamma \) is then called the complex torus. Moreover, let \( \Delta \) be the discontinuous group generated by \( z^k \to 2z^k, 1 \leq k \leq m \). The quotient manifold \( (\mathbb{C}^m - \{0\})/\Delta \) is the so-called Hopf manifold, and is homeomorphic to \( S^1 \times S^{2m-1} \). Last but not least, we consider the group \( M_3 \).
of all matrices

\[
E_3 = \begin{pmatrix}
1 & z_1 & z_2 \\
0 & 1 & z_3 \\
0 & 0 & 1
\end{pmatrix},
\]

(1.2.2)

and let \( D \) be the discrete group consisting of those matrices for which \( z_1, z_2, z_3 \) are Gaussian integers. This means that \( z_k = m_k + in_k, 1 \leq k \leq 3 \), where \( m_k, n_k \) are rational integers. An Iwasawa manifold is then defined as the quotient space \( M_3 / D \).

**E5.** Orientable surfaces are particular complex manifolds. The surfaces are taken to be \( C^\infty \), and we define on them a positive-definite Riemannian metric. The Korn–Lichtenstein theorem ensures that local parameters \( x, y \) exist such that the metric locally takes the form

\[
g = \lambda^2 \left( dx \otimes dx + dy \otimes dy \right), \quad \lambda > 0,
\]

(1.2.3)

or

\[
g = \lambda^2 dz \otimes d\bar{z}, \quad z \equiv x + iy.
\]

(1.2.4)

If \( w \) is another local coordinate, we have

\[
g = \lambda^2 dz \otimes d\bar{z} = \mu^2 dw \otimes d\bar{w},
\]

(1.2.5)

since \( g \) is globally defined. Hence \( dw \) is a multiple of \( dz \) or \( d\bar{z} \). In particular, if the complex coordinates \( z \) and \( w \) define the same orientation, then \( dw \) is proportional to \( dz \). Thus, \( w \) is a holomorphic function of \( z \), and the surface becomes a complex manifold. Riemann surfaces are, by definition, one-dimensional complex manifolds.

Let us denote by \( V \) an \( m \)-dimensional real vector space. We say that \( V \) has a complex structure if there exists a linear endomorphism \( J : V \to V \) such that \( J^2 = -I \), where \( I \) is the identity endomorphism. An eigenvalue of \( J \) is a complex number \( \lambda \) such that the equation \( Jx = \lambda x \) has a non-vanishing solution \( x \in V \). Applying \( J \) to both sides of this equation, one finds \( -x = \lambda^2 x \). Hence \( \lambda = \pm i \). Since the complex eigenvalues occur in conjugate pairs, \( V \) is of even dimension \( n = 2m \). Let us now denote by \( V^* \) the dual space of \( V \), i.e. the space of all real-valued linear functions over \( V \). The pairing of \( V \) and \( V^* \) is \( \langle x, y^* \rangle \), \( x \in V, y^* \in V^* \), so that this function is \( R \)-linear in each of the arguments. Following Chern 1979, we also consider \( V^* \otimes C \), i.e. the space of all complex-valued \( R \)-linear functions over \( V \). By construction, \( V^* \otimes C \) is an \( n \)-complex-dimensional complex vector space. Elements \( f \in V^* \otimes C \) are of type \((1, 0)\) if \( f(Jx) = if(x) \), and of type \((0, 1)\) if \( f(Jx) = -if(x) \), \( x \in V \).

If \( V \) has a complex structure \( J \), an Hermitian structure in \( V \) is a complex-valued function \( H \) acting on \( x, y \in V \) such that

\[
H \left( \lambda_1 x_1 + \lambda_2 x_2, y \right) = \lambda_1 H(x_1, y) + \lambda_2 H(x_2, y), \quad x_1, x_2, y \in V, \quad \lambda_1, \lambda_2 \in R,
\]

(1.2.6)

\[
\overline{H(x, y)} = H(y, x),
\]

(1.2.7)
\[ H(Jx,y) = iH(x,y) \iff H(x,Jy) = -iH(x,y). \] (1.2.8)

By using the split of \( H(x,y) \) into its real and imaginary parts

\[ H(x,y) = F(x,y) + iG(x,y), \] (1.2.9)

conditions (1.2.7) and (1.2.8) may be re-expressed as

\[ F(x,y) = F(y,x), \quad G(x,y) = -G(y,x), \] (1.2.10)

\[ F(x,y) = G(Jx,y), \quad G(x,y) = -F(Jx,y). \] (1.2.11)

If \( M \) is a \( C^\infty \) manifold of dimension \( n \), and if \( T_x \) and \( T^*_x \) are tangent and cotangent spaces respectively at \( x \in M \), an \textit{almost complex structure} on \( M \) is a \( C^\infty \) field of endomorphisms \( J_x : T_x \to T_x \) such that \( J_x^2 = -1 I_x \), where \( I_x \) is the identity endomorphism in \( T_x \). A manifold with an almost complex structure is called \textit{almost complex}. If a manifold is almost complex, it is even-dimensional and orientable. However, this is only a necessary condition. Examples can be found (e.g. the four-sphere \( S^4 \)) of even-dimensional, orientable manifolds which cannot be given an almost complex structure.

\section*{1.3. An outline of this work}

Since this paper is devoted to the geometry of complex space-time in spinor form, Sec. 2 presents the basic ideas, methods and results of two-component spinor calculus. Such a calculus is described in terms of spin-space formalism, i.e. a complex vector space endowed with a symplectic form and some fundamental isomorphisms. These mathematical properties enable one to raise and lower indices, define the conjugation of spinor fields in Lorentzian or Riemannian four-geometries, translate tensor fields into spinor fields (or the other way around). The standard two-spinor form of the Riemann curvature tensor is then obtained by relying on the (more) familiar tensor properties of the curvature. The introductory analysis ends with the Petrov classification of space-times, expressed in terms of the Weyl spinor of conformal gravity.

Since the whole of twistor theory may be viewed as a holomorphic description of space-time geometry in a conformally invariant framework, Sec. 3 studies the key results of conformal gravity, i.e. \( C \)-spaces, Einstein spaces and complex Einstein spaces. Hence a necessary and sufficient condition for a space-time to be conformal to a complex Einstein space is obtained, following [37]. Such a condition involves the Bach and Eastwood–Dighton spinors, and their occurrence is derived in detail. The difference between Lorentzian space-times, Riemannian four-spaces, complexified space-times and complex space-times is also analyzed.

Section 4 is a pedagogical introduction to twistor spaces, from the point of view of mathematical physics and relativity theory. This is obtained by defining twistors as \( \alpha \)-planes in complexified compactified Minkowski space-time, and as \( \alpha \)-surfaces in curved space-time. In the former case, one deals with totally null two-surfaces,
in that the complexified Minkowski metric vanishes on any pair of null tangent vectors to the surface. Hence such null tangent vectors have the form \( \lambda^A \pi^{A'} \), where \( \lambda^A \) is varying and \( \pi^{A'} \) is covariantly constant. This definition can be generalized to complex or real Riemannian four-manifolds, provided that the Weyl curvature is anti-self-dual. An alternative definition of twistors in Minkowski space-time is instead based on the vector space of solutions of a differential equation, which involves the symmetrized covariant derivative of an unprimed spinor field. Interestingly, a deep correspondence exists between flat space-time and twistor space. Hence complex space-time points correspond to spheres in the so-called projective twistor space, and this concept is carefully formulated. Sheaf cohomology is then presented as the mathematical tool necessary to describe a conformally invariant isomorphism between the complex vector space of holomorphic solutions of the wave equation on the forward tube of flat space-time, and the complex vector space of complex-analytic functions of three variables. These are arbitrary, in that they are not subject to any differential equation. Eventually, Ward’s one-to-one correspondence between complex space-times with non-vanishing cosmological constant, and sufficiently small deformations of flat projective twistor space, is presented.

An example of explicit construction of anti-self-dual space-time is given in Sec. 5, following [98]. This generalization of Penrose’s non-linear graviton ([60], [61]) combines two-spinor techniques and twistor theory in a way very instructive for beginning graduate students. However, it appears necessary to go beyond anti-self-dual space-times, since they are only a particular class of (complex) space-times, and they do not enable one to recover the full physical content of (complex) general relativity. This implies going beyond the original twistor theory, since the three-complex-dimensional space of \( \alpha \)-surfaces only exists in anti-self-dual space-times. After a brief review of alternative ideas, attention is focused on the recent attempt by Roger Penrose to define twistors as charges for massless spin-\( \frac{3}{2} \) fields. Such an approach has been considered since a vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless helicity-\( \frac{3}{2} \) fields in curved space-time. Moreover, in Minkowski space-time the space of charges for such fields is naturally identified with the corresponding twistor space. The resulting geometric scheme in the presence of curvature is as follows. First, define a twistor for Ricci-flat space-time. Second, characterize the resulting twistor space. Third, reconstruct the original Ricci-flat space-time from such a twistor space. One of the main technical difficulties of the program proposed by Penrose is to obtain a global description of the space of potentials for massless spin-\( \frac{3}{2} \) fields. The corresponding local theory is instead used, for other purposes, in [18].

Last, Sec. 6 reviews the Plebanski contributions to complex general relativity, i.e. heaven spaces and heavenly equations, while concluding remarks are presented in Sec. 7.
2. Two-Component Spinor Calculus

Spinor calculus is presented by relying on spin-space formalism. Given the existence of unprimed and primed spin-space, one has the isomorphism between such vector spaces and their duals, realized by a symplectic form. Moreover, for Lorentzian metrics, complex conjugation is the (anti-)isomorphism between unprimed and primed spin-space. Finally, for any space-time point, its tangent space is isomorphic to the tensor product of unprimed and primed spin-spaces via the Infeld–van der Waerden symbols. Hence the correspondence between tensor fields and spinor fields.

Euclidean conjugation in Riemannian geometries is also discussed in detail. The Maxwell field strength is written in this language, and many useful identities are given. The curvature spinors of general relativity are then constructed explicitly, and the Petrov classification of space-times is obtained in terms of the Weyl spinor for conformal gravity.

2.1. Spin-spaces

Two-component spinor calculus is a powerful tool for studying classical field theories in four-dimensional space-time models. Within this framework, the basic object is spin-space, a two-dimensional complex vector space $S$ with a symplectic form $\varepsilon$, i.e. an antisymmetric complex bilinear form. Unprimed spinor indices $A, B, ...$ take the values $0, 1$ whereas primed spinor indices $A', B', ...$ take the values $0', 1'$ since there are actually two such spaces: unprimed spin-space $(S, \varepsilon)$ and primed spin-space $(S', \varepsilon')$. The whole two-spinor calculus in Lorentzian four-manifolds relies on three fundamental properties ([96], [86], [54], [67], [16], [17]):

(i) The isomorphism between $(S, \varepsilon_{AB})$ and its dual $(S^*, \varepsilon^{A'B'})$. This is provided by the symplectic form $\varepsilon$, which raises and lowers indices according to the rules

$$\varepsilon^{AB} \varphi_B = \varphi^A \in S,$$

$$\varphi^B \varepsilon_{BA} = \varphi_A \in S^*.$$  \hspace{1cm} (2.1.1) (2.1.2)

Thus, since

$$\varepsilon_{AB} = \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$  \hspace{1cm} (2.1.3)

one finds in components $\varphi^0 = \varphi_1, \varphi^1 = -\varphi_0$.

Similarly, one has the isomorphism $(S', \varepsilon_{A'B'}) \cong (S'^*, \varepsilon^{A'B'})$, which implies

$$\varepsilon^{A'B'} \varphi_{B'} = \varphi^{A'} \in S',$$  \hspace{1cm} (2.1.4)

$$\varphi^{B'} \varepsilon_{B'A'} = \varphi_{A'} \in (S')^*,$$  \hspace{1cm} (2.1.5)

where

$$\varepsilon_{A'B'} = \varepsilon^{A'B'} = \begin{pmatrix} 0' & 1' \\ -1' & 0' \end{pmatrix}.$$  \hspace{1cm} (2.1.6)
(ii) The (anti-)isomorphism between \((S, \varepsilon_{AB})\) and \((S', \varepsilon'_{A'B'})\), called complex conjugation, and denoted by an overbar. According to a standard convention, one has
\[
\overline{\psi^A} \equiv \overline{\psi^A'} \in S',
\]
\[
\overline{\psi^{A'}} \equiv \overline{\psi^A} \in S.
\]
(2.1.7)
(2.1.8)

Thus, complex conjugation maps elements of a spin-space to elements of the complementary spin-space. Hence some authors say it is an anti-isomorphism. In components, if \(w^A\) is thought as \(w^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\), the action of (2.1.7) leads to
\[
\overline{w^A} \equiv \overline{w^A'} \equiv \begin{pmatrix} \overline{\alpha} \\ \overline{\beta} \end{pmatrix},
\]
whereas, if \(z^{A'} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}\), then (2.1.8) leads to
\[
\overline{z^{A'}} \equiv \overline{z^A} \equiv \begin{pmatrix} \overline{\gamma} \\ \overline{\delta} \end{pmatrix}.
\]
(2.1.9)
(2.1.10)

With our notation, \(\overline{\alpha}\) denotes complex conjugation of the function \(\alpha\), and so on. Note that the symplectic structure is preserved by complex conjugation, since \(\varepsilon_{A'B'} = \varepsilon_{A'B'}\).

(iii) The isomorphism between the tangent space \(T\) at a point of space-time and the tensor product of the unprimed spin-space \((S, \varepsilon_{AB})\) and the primed spin-space \((S', \varepsilon'_{A'B'})\):
\[
T \cong (S, \varepsilon_{AB}) \otimes (S', \varepsilon'_{A'B'}).
\]
(2.1.11)

The Infeld–van der Waerden symbols \(\sigma^a_{AA'}\) and \(\sigma_a^{AA'}\) express this isomorphism, and the correspondence between a vector \(v^a\) and a spinor \(v^{AA'}\) is given by
\[
v^{AA'} \equiv v^a \sigma_a^{AA'},
\]
\[
v^a \equiv v^{AA'} \sigma_a^{AA'}.
\]
(2.1.12)
(2.1.13)

These mixed spinor-tensor symbols obey the identities
\[
\overline{\sigma_a^{AA'}} = \sigma_a^{AA'},
\]
\[
\sigma_a^{AA'} \sigma_b^{AA'} = \delta^b_a,
\]
\[
\sigma_a^{AA'} \sigma_a^{BB'} = \varepsilon_{A'B'} \varepsilon_{A'B'},
\]
\[
\sigma_a^{AA'} \sigma^b_{[A} B' = -\frac{i}{2} \varepsilon_{abcd} \sigma^c^{AA'} \sigma^d_{A'} B'.
\]
(2.1.14)
(2.1.15)
(2.1.16)
(2.1.17)
Similarly, a one-form $\omega$ has a spinor equivalent

$$\omega_{AA'} \equiv \omega_a \sigma^a_{AA'}, \quad (2.1.18)$$

whereas the spinor equivalent of the metric is

$$\eta_{ab} \sigma^a_{AA'} \sigma^b_{BB'} \equiv \varepsilon_{AB} \varepsilon_{A'B'}. \quad (2.1.19)$$

In particular, in Minkowski space-time, the above equations enable one to write down a coordinate system in $2 \times 2$ matrix form

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \quad (2.1.20)$$

More precisely, in a (curved) space-time, one should write the following equation to obtain the spinor equivalent of a vector:

$$u^{AA'} = u^a e^c_a \sigma^{AA'},$$

where $e^c_a$ is a standard notation for the tetrad, and $e^c_a \sigma^{AA'} = e^a_{AA'}$ is called the **soldering form**. This is, by construction, a spinor-valued one-form, which encodes the relevant information about the metric $g$, because $g_{ab} = e^d_\xi e^c_d \eta_{cd}$, $\eta$ being the Minkowskian metric of the so-called “internal space”.

In the Lorentzian-signature case, the Maxwell two-form $F = F_{ab} dx^a \wedge dx^b$ can be written spinorially [103] as

$$F_{AA'BB'} = \frac{1}{2} \left( F_{AA'BB'} - F_{BB'AA'} \right) = \varphi_{AB} \varepsilon_{A'B'} + \varphi_{A'B'} \varepsilon_{AB}, \quad (2.1.21)$$

where

$$\varphi_{AB} \equiv \frac{1}{2} F_{AC'B'} C' = \varphi_{(AB)}, \quad (2.1.22)$$

$$\varphi_{A'B'} \equiv \frac{1}{2} F_{C'B'} A' = \varphi_{(A'B')}. \quad (2.1.23)$$

These formulae are obtained by applying the identity

$$T_{AB} - T_{BA} = \varepsilon_{AB} T^C_C \quad (2.1.24)$$

to express $\frac{1}{2} \left( F_{AA'BB'} - F_{AB'BA} \right)$ and $\frac{1}{2} \left( F_{AB'BA} - F_{BB'AA'} \right)$. Note also that round brackets $(AB)$ denote (as usual) symmetrization over the spinor indices $A$ and $B$, and that the antisymmetric part of $\varphi_{AB}$ vanishes by virtue of the antisymmetry of $F_{ab}$, since $\varepsilon_{AB} \frac{1}{2} F_{CC'C'} C' = \frac{1}{2} \varepsilon_{AB} \eta^{cd} F_{cd} = 0$. Last but not least, in the Lorentzian case

$$\varphi_{AB} = \varphi_{A'B'} = \varphi_{A'B'}'. \quad (2.1.25)$$

The symmetric spinor fields $\varphi_{AB}$ and $\varphi_{A'B'}$ are the anti-self-dual and self-dual parts of the curvature two-form, respectively.
Similarly, the Weyl curvature $C_{a b c d}$, i.e. the part of the Riemann curvature tensor invariant under conformal rescalings of the metric, may be expressed spinorially, omitting soldering forms for simplicity of notation, as

$$C_{a b c d} = \psi_{A B C D} \varepsilon_{A' B'} \varepsilon_{C' D'} + \overline{\psi}_{A'B'C'D'} \varepsilon_{A B} \varepsilon_{C D}. \quad (2.1.26)$$

In Hamiltonian gravity, two-component spinors lead to a considerable simplification of calculations. On denoting by $n^\mu$ the future-pointing unit timelike normal to a spacelike three-surface, its spinor version obeys the relations

$$n_{A A'} e^{A A'}_i = 0, \quad \text{(2.1.27)}$$

$$n_{A A'} n^{A A'} = 1, \quad \text{(2.1.28)}$$

where $e^{A A'}_\mu \equiv e^a_\mu \sigma_a^{A A'}$ is the two-spinor version of the tetrad, i.e. the soldering form introduced before. Denoting by $h$ the induced metric on the three-surface, other useful relations are [17]

$$h_{i j} = - e_{A A'} i e^{A A'}_j, \quad \text{(2.1.29)}$$

$$e^{A A'}_0 = N n^{A A'} + N^i e^{A A'}_i, \quad \text{(2.1.30)}$$

$$n_{A A'} n^{B A'} = \frac{1}{2} \varepsilon^{B A'}, \quad \text{(2.1.31)}$$

$$n_{A A'} n^{A B'} = \frac{1}{2} \varepsilon^{A B'}, \quad \text{(2.1.32)}$$

$$n_{[E B']} n_{A] A'} = \frac{1}{4} \varepsilon_{E A} \varepsilon_{B' A'}, \quad \text{(2.1.33)}$$

$$e_{A A'}^j e^{A B'}_k = - \frac{1}{2} h_{j k} \varepsilon_{A A'} B' - i \varepsilon_{j k l} \sqrt{\det h} n_{A A'} e^{A B'}_l. \quad \text{(2.1.34)}$$

In Eq. (2.1.30), $N$ and $N^i$ are the lapse and shift functions respectively [17].

To obtain the space-time curvature, we first need to define the spinor covariant derivative $\nabla_{A A'}$. If $\theta, \phi, \psi$ are spinor fields, $\nabla_{A A'}$ is a map such that ([67], [88])

1. $\nabla_{A A'}(\theta + \phi) = \nabla_{A A'}\theta + \nabla_{A A'}\phi$ (i.e. linearity).

2. $\nabla_{A A'}(\theta \psi) = \left(\nabla_{A A'}\theta\right)\psi + \theta \left(\nabla_{A A'}\psi\right)$ (i.e. Leibniz rule).

3. $\psi = \nabla_{A A'}\theta$ implies $\overline{\psi} = \nabla_{A A'}\overline{\theta}$ (i.e. reality condition).

4. $\nabla_{A A'} \varepsilon_{B C} = \nabla_{A A'} \varepsilon^{B C} = 0$, i.e. the symplectic form may be used to raise or lower indices within spinor expressions acted upon by $\nabla_{A A'}$, in addition to the usual metricity condition $\nabla g = 0$, which involves instead the product of two $\varepsilon$-symbols.

5. $\nabla_{A A'}$ commutes with any index substitution not involving $A, A'$. 

(6) For any function $f$, one finds $(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 2 S_{ab}^c \nabla_c f$, where $S_{ab}^c$ is the torsion tensor.

(7) For any derivation $D$ acting on spinor fields, a spinor field $\xi^{AA'}$ exists such that $D \psi = \xi^{AA'} \nabla^{AA'} \psi, \forall \psi$.

As proved in [67], such a spinor covariant derivative exists and is unique.

If Lorentzian space-time is replaced by a complex or real Riemannian four-manifold, an important modification should be made, since the (anti-)isomorphism between unprimed and primed spin-space no longer exists. This means that primed spinors can no longer be regarded as complex conjugates of unprimed spinors, or vice versa, as in (2.1.7) and (2.1.8). In particular, Eqs. (2.1.21) and (2.1.26) should be re-written as

$$F_{AA'BB'} = \varphi_{AB} \varepsilon_{A'B'} + \tilde{\varphi}_{A'B'} \varepsilon_{AB}, \quad (2.1.35)$$

$$C_{abcd} = \psi_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \tilde{\psi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD}. \quad (2.1.36)$$

With our notation, $\varphi_{AB}, \tilde{\varphi}_{A'B'}$, as well as $\psi_{ABCD}, \tilde{\psi}_{A'B'C'D'}$ are completely independent symmetric spinor fields, not related by any conjugation.

Indeed, a conjugation can still be defined in the real Riemannian case, but it no longer relates $\left(S, \varepsilon_{AB}\right)$ to $\left(S', \varepsilon_{A'B'}\right)$. It is instead an anti-involutory operation which maps elements of a spin-space (either unprimed or primed) to elements of the same spin-space. By anti-involutory we mean that, when applied twice to a spinor with an odd number of indices, it yields the same spinor with the opposite sign, i.e. its square is minus the identity, whereas the square of complex conjugation as defined in (2.1.9) and (2.1.10) equals the identity. Following [107] and [17], Euclidean conjugation, denoted by a dagger, is defined by

$$\left(w^A\right)^\dagger \equiv \left(\begin{array}{c} \beta \\ -\alpha \end{array}\right), \quad (2.1.37)$$

$$\left(z^{A'}\right)^\dagger \equiv \left(\begin{array}{c} -\delta \\ \gamma \end{array}\right). \quad (2.1.38)$$

This means that, in flat Euclidean four-space, a unit $2 \times 2$ matrix $\delta_{BA'}$ exists such that

$$\left(w^A\right)^\dagger \equiv \varepsilon^{AB} \delta_{BA'} w^{A'}. \quad (2.1.39)$$

We are here using the freedom to regard $w^A$ either as an $SL(2, C)$ spinor for which complex conjugation can be defined, or as an $SU(2)$ spinor for which Euclidean conjugation is instead available. The soldering forms for $SU(2)$ spinors only involve spinor indices of the same spin-space, i.e. $\tilde{e}_i^{AB}$ and $\tilde{e}_i^{A'B'}$. More precisely, denoting
by $E_i^a$ a real triad, where $i = 1, 2, 3$, and by $\tau^a_A B$ the three Pauli matrices, the $SU(2)$ soldering forms are defined by

$$\tilde{e}_A^j B \equiv -\frac{i}{\sqrt{2}} E_j^a \tau^a_A B. \quad (2.1.40)$$

The soldering form in (2.1.40) provides an isomorphism between the three-real-dimensional tangent space at each point of $\Sigma$, and the three-real-dimensional vector space of $2 \times 2$ trace-free Hermitian matrices. The Riemannian three-metric on $\Sigma$ is then given by

$$h_{ij} = -\tilde{e}_A^j B \tilde{e}_B^j A. \quad (2.1.41)$$

2.2. Curvature in general relativity

At this stage, following [67], we want to derive the spinorial form of the Riemann curvature tensor in a Lorentzian space-time with vanishing torsion, starting from the well-known symmetries of Riemann. In agreement with the abstract-index translation of tensors into spinors, soldering forms will be omitted in the resulting equations.

Since $R_{abcd} = -R_{bacd}$ we may write

$$R_{abcd} = R_{AA'B'B'}CC'D'D' = \frac{1}{2} R_{AF'F'}^B CD \varepsilon_{A'B'} + \frac{1}{2} R_{F'A'}^F B'cd \varepsilon_{AB}. \quad (2.2.1)$$

Moreover, on defining

$$X_{ABCD} = \frac{1}{4} R_{AF'F'} C L'D', \quad (2.2.2)$$

$$\Phi_{ABC'D'} = \frac{1}{4} R_{AF'F'} L C'D', \quad (2.2.3)$$

the anti-symmetry in $cd$ leads to

$$R_{abcd} = X_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + \Phi_{ABC'D'} \varepsilon_{A'B'} \varepsilon_{CD}$$

$$+ \Phi_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{C'D'} + X_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD}. \quad (2.2.4)$$

According to a standard terminology, the spinors (2.2.2) and (2.2.3) are called the curvature spinors. In the light of the (anti-)symmetries of $R_{abcd}$, they have the following properties:

$$X_{ABCD} = X_{(AB)(CD)}, \quad (2.2.5)$$

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}, \quad (2.2.6)$$

$$X_{ABCD} = X_{C' D'AB}, \quad (2.2.7)$$

$$\overline{X}_{ABC'D'} = \overline{X}_{ABC'D'}. \quad (2.2.8)$$
Remarkably, Eqs. (2.2.6) and (2.2.8) imply that $\Phi_{A'A'B'B'}$ corresponds to a trace-free and real tensor:

$$\Phi^a_a = 0, \quad \Phi_{A'A'B'B'} = \Phi_{ab} = \overline{\Phi}_{ab}. \quad (2.2.9)$$

Moreover, from Eqs. (2.2.5) and (2.2.7) one obtains

$$X^{A(BC)} = 0. \quad (2.2.10)$$

Three duals of $R_{abcd}$ exist which are very useful and are defined as follows:

$$R^*_{abcd} \equiv \frac{1}{2} \varepsilon_{cd}^{pq} R_{abpq} = i R_{A'A'B'B'CD'D'D'}, \quad (2.2.11)$$

$$*R_{abcd} \equiv \frac{1}{2} \varepsilon_{ab}^{pq} R_{pqcd} = i R_{AB'BA'CC'D'C'D'}, \quad (2.2.12)$$

$$*R^*_{abcd} \equiv \frac{1}{4} \varepsilon_{ab}^{pq} \varepsilon_{cd}^{rs} R_{pqrs} = -R_{A'B'B'B'C'D'C'D'}. \quad (2.2.13)$$

For example, in terms of the dual (2.2.11), the familiar equation $R_{a[bc]} = 0$ reads

$$R^*_{ab} = 0. \quad (2.2.14)$$

Thus, to derive the spinor form of the cyclic identity, one can apply (2.2.14) to the equation

$$R^*_{abcd} = -i X_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + i \Phi_{A'B'C'D'} \varepsilon_{A'B'} \varepsilon_{CD}$$

$$-i \overline{\Phi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{C'D'} + i X_{AB'C'D'} \varepsilon_{AB} \varepsilon_{CD}. \quad (2.2.15)$$

By virtue of (2.2.6) and (2.2.8) one thus finds

$$X_{AB'C'} \varepsilon_{A'C'} = X_{A'B'C'} \varepsilon_{A'B'C'}, \quad (2.2.16)$$

which implies, on defining

$$\Lambda \equiv \frac{1}{6} X_{AB}^{AB}, \quad (2.2.17)$$

the reality condition

$$\Lambda = \overline{\Lambda}. \quad (2.2.18)$$

Equation (2.2.1) enables one to express the Ricci tensor $R_{ab} \equiv R_{abc} \epsilon^c$ in spinor form as

$$R_{ab} = 6\Lambda \varepsilon_{AB} \varepsilon_{A'B'} - 2\Phi_{ABA'B'}. \quad (2.2.19)$$

Thus, the resulting scalar curvature, trace-free part of Ricci and Einstein tensor are

$$R = 24\Lambda, \quad (2.2.20)$$

$$R_{ab} - \frac{1}{4} R g_{ab} = -2\Phi_{ab} = -2\Phi_{ABA'B'}. \quad (2.2.21)$$
respectively.

We have still to obtain a more suitable form of the Riemann curvature. For this purpose, following again [67], we point out that the curvature spinor $X_{ABCD}$ can be written as

$$X_{ABCD} = \frac{1}{3} (X_{ABCD} + X_{ACDB} + X_{ADBC}) + \frac{1}{3} (X_{ABCD} - X_{ACBD}) = X_{(ABCD)} + \frac{1}{3} \epsilon_{BC} X_{AFD} + \frac{1}{3} \epsilon_{BD} X_{AFC}. \quad (2.2.23)$$

Since $X_{AFC} = 3\Lambda \epsilon_{AF}$, Eq. (2.2.23) leads to

$$X_{ABCD} = \psi_{ABCD} + \Lambda \left( \epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC} \right), \quad (2.2.24)$$

where $\psi_{ABCD}$ is the Weyl spinor.

Since $\Lambda = \Lambda$ from (2.2.18), the insertion of (2.2.24) into (2.2.4), jointly with the identity

$$\epsilon_{A'B'} \epsilon_{C'D'} + \epsilon_{A'D'} \epsilon_{B'C'} - \epsilon_{A'C'} \epsilon_{B'D'} = 0, \quad (2.2.25)$$

yields the desired decomposition of the Riemann curvature as

$$R_{abcd} = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \overline{\psi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}$$

$$+ \Phi_{AB'C'D'} \epsilon_{A'B'} \epsilon_{C'D'} + \overline{\Phi}_{A'B'CD} \epsilon_{AB} \epsilon_{C'D'}$$

$$+ 2\Lambda \left( \epsilon_{AC} \epsilon_{BD} \epsilon_{A'C'} \epsilon_{B'D'} - \epsilon_{AD} \epsilon_{BC} \epsilon_{A'D'} \epsilon_{B'C'} \right). \quad (2.2.26)$$

With this standard notation, the conformally invariant part of the curvature takes the form $C_{abcd} = (-)C_{abcd} + (+)C_{abcd}$, where

$$(-)C_{abcd} \equiv \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'}, \quad (2.2.27)$$

$$(+)^{C_{abcd}} \equiv \overline{\psi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}, \quad (2.2.28)$$

are the anti-self-dual and self-dual Weyl tensors, respectively.

### 2.3. Petrov classification

Since the Weyl spinor is totally symmetric, we may use a well known result of two-spinor calculus, according to which, if $\Omega_{AB...L}$ is totally symmetric, then there exist univalent spinors $\alpha_A, \beta_B, ..., \gamma_L$ such that [88]

$$\Omega_{AB...L} = \alpha_A \beta_B ... \gamma_L, \quad (2.3.1)$$
where $\alpha, ..., \gamma$ are called the principal spinors of $\Omega$, and the corresponding real null vectors are called the principal null directions of $\Omega$. In the case of the Weyl spinor, such a theorem implies that

$$\psi_{ABCD} = \alpha(A \beta_B \gamma_C \delta_D). \quad (2.3.2)$$

The corresponding space-times can be classified as follows [88].

1. **Type I.** Four distinct principal null directions. Hence the name algebraically general.

2. **Type II.** Two directions coincide. Hence the name algebraically special.

3. **Type D.** Two different pairs of repeated principal null directions exist.

4. **Type III.** Three principal null directions coincide.

5. **Type N.** All four principal null directions coincide.

Such a classification is the Petrov classification, and it provides a relevant example of the superiority of the two-spinor formalism in four space-time dimensions, since the alternative ways to obtain it are far more complicated.

Within this framework (as well as in Sec. 3) we need to know that $\psi_{ABCD}$ has two scalar invariants:

$$I \equiv \psi_{ABCD} \psi^{ABCD}, \quad (2.3.3)$$

$$J \equiv \psi_{AB}^{CD} \psi_{CD}^{EF} \psi_{EF}^{AB}. \quad (2.3.4)$$

Type-II space-times are such that $I^3 = 6J^2$, while in type-III space-times $I = J = 0$. Moreover, type-D space-times are characterized by the condition

$$\psi_{PQR}^{(A} \psi_{BC}^{PQ} \psi_{DEF}^{R)} = 0, \quad (2.3.5)$$

while in type-N space-times

$$\psi_{(AB}^{EF} \psi_{CDE)}^{EF} = 0. \quad (2.3.6)$$

These results, despite their simplicity, are not well known to many physicists and mathematicians. Hence they have been included also in this paper, to prepare the ground for the more advanced topics of the following sections.

### 3. Conformal Gravity

Since twistor theory enables one to reconstruct the space-time geometry from conformally invariant geometric objects, it is important to know the basic tools for studying conformal gravity within the framework of general relativity. This is achieved by defining and using the Bach [6] and Eastwood–Dighton tensors, here presented in two-spinor form (relying on previous work by Kozameh, Newman and Tod [37]). After defining $C$-spaces and Einstein spaces, it is shown that a space-time
is conformal to an Einstein space if and only if some equations involving the Weyl spinor, its covariant derivatives, and the trace-free part of Ricci are satisfied. Such a result is then extended to complex Einstein spaces. The conformal structure of infinity of Minkowski space-time is eventually introduced.

3.1. $C$-spaces

Twistor theory may be viewed as the attempt to describe fundamental physics in terms of conformally invariant geometric objects within a holomorphic framework. Space-time points are no longer of primary importance, since they only appear as derived concepts in such a scheme. To understand the following sections, almost entirely devoted to twistor theory and its applications, it is therefore necessary to study the main results of the theory of conformal gravity. They can be understood by focusing on $C$-spaces, Einstein spaces, complex space-times and complex Einstein spaces, as we do from now on in this section.

To study $C$-spaces in a self-consistent way, we begin by recalling some basic properties of conformal rescalings. By definition, a conformal rescaling of the space-time metric $g$ yields the metric $\hat{g}$ as

$$\hat{g}_{ab} \equiv e^{2\omega} g_{ab},$$  \hspace{1cm} (3.1.1)

where $\omega$ is a smooth scalar. Correspondingly, any tensor field $T$ of type $(r,s)$ is conformally weighted if

$$\hat{T} \equiv e^{k\omega} T$$  \hspace{1cm} (3.1.2)

for some integer $k$. In particular, conformal invariance of $T$ is achieved if $k = 0$.

It is useful to know the transformation rules for covariant derivatives and Riemann curvature under the rescaling (3.1.1). For this purpose, defining

$$F_{ab}^m \equiv 2\delta_a^m \nabla_b \omega - g_{ab} g_{mn} \nabla_n \omega,$$  \hspace{1cm} (3.1.3)

one finds

$$\hat{\nabla}_a V_b = \nabla_a V_b - F_{ab}^m V_m,$$  \hspace{1cm} (3.1.4)

where $\hat{\nabla}_a$ denotes covariant differentiation with respect to the metric $\hat{g}$. Hence the Weyl tensor $C_{abc}^d$, the Ricci tensor $R_{ab} \equiv R_{cab}^c$ and the Ricci scalar transform as

$$\hat{C}_{abc}^d = C_{abc}^d,$$  \hspace{1cm} (3.1.5)

$$\hat{R}_{ab} = R_{ab} + 2\nabla_a \omega_b - 2\omega_a \omega_b + g_{ab} \left(2\omega^c \omega_c + \nabla^c \omega_c\right),$$  \hspace{1cm} (3.1.6)

$$\hat{R} = e^{-2\omega}\left[R + 6\left(\nabla^c \omega_c + \omega^c \omega_c\right)\right].$$  \hspace{1cm} (3.1.7)

With our notation, $\omega_c \equiv \nabla_c \omega = \omega_{,c}$. 

We are here interested in space-times which are conformal to $C$-spaces. The latter are a class of space-times such that

$$\hat{\nabla}^f \hat{C}_{abcf} = 0.$$  \hspace{1cm} (3.1.8)

By virtue of (3.1.3) and (3.1.4) one can see that the conformal transform of Eq. (3.1.8) is

$$\nabla^f C_{abcf} + \omega^f C_{abcf} = 0.$$  \hspace{1cm} (3.1.9)

This is the necessary and sufficient condition for a space-time to be conformal to a $C$-space. Its two-spinor form is

$$\nabla^F A \psi^{ABCD} + \omega^F A \psi^{ABCD} = 0.$$  \hspace{1cm} (3.1.10)

However, note that only a real solution $\omega^F A$ of Eq. (3.1.10) satisfies Eq. (3.1.9). Hence, whenever we use Eq. (3.1.10), we are also imposing a reality condition [37].

On using the invariants defined in (2.3.3) and (2.3.4), one finds the useful identities

$$\psi_{ABCD} \psi^{ABCE} = \frac{1}{2} I \delta^E_D,$$  \hspace{1cm} (3.1.11)

$$\psi_{ABCD} \psi^{AB} \psi^{PQCE} = \frac{1}{2} J \delta^E_D.$$  \hspace{1cm} (3.1.12)

The idea is now to act with $\psi_{ABCD}$ on the left-hand side of (3.1.10) and then use (3.1.11) when $I \neq 0$. This leads to

$$\omega^{AA'} = -\frac{2}{I} \psi^{ABCD} \nabla^F A' \psi_{FBCD}. $$  \hspace{1cm} (3.1.13)

By contrast, when $I = 0$ but $J \neq 0$, we multiply twice Eq. (3.1.10) by the Weyl spinor and use (3.1.12). Hence one finds

$$\omega^{AA'} = -\frac{2}{J} \psi^{CD} \psi^{EFGA} \nabla^{BA'} \psi_{BCDG}. $$  \hspace{1cm} (3.1.14)

Thus, by virtue of (3.1.13), the reality condition $\omega^{AA'} = \overline{\omega^{AA'}} = \overline{\omega}^{AA'}$ implies

$$T \psi^{ABCD} \nabla^F A' \psi_{FBCD} - I \psi^{AB'C'D'} \nabla^{AF'} \psi_{F'B'C'D'} = 0.$$  \hspace{1cm} (3.1.15)

We have thus shown that a space-time is conformally related to a $C$-space if and only if Eq. (3.1.10) holds for some vector $\omega^{DD'} = K^{DD'}$, and Eq. (3.1.15) holds as well.

### 3.2 Einstein spaces

By definition, Einstein spaces are such that their Ricci tensor is proportional to the metric: $R_{ab} = \lambda g_{ab}$. A space-time is conformal to an Einstein space if and only if a function $\omega$ exists (see (3.1.1)) such that (cf. (3.1.6))

$$R_{ab} + 2\nabla_a \omega_b - 2\omega_a \omega_b - \frac{1}{4} T g_{ab} = 0.$$  \hspace{1cm} (3.2.1)
where
\[ T \equiv R + 2\nabla^c \omega_c - 2\omega^c \omega_c. \tag{3.2.2} \]

Of course, Eq. (3.2.1) leads to restrictions on the metric. These are obtained by deriving the corresponding integrability conditions. For this purpose, on taking the curl of Eq. (3.2.1) and using the Bianchi identities, one finds
\[ \nabla^f C_{abcf} + \omega^f C_{abcf} = 0, \]
which coincides with Eq. (3.1.9). Moreover, acting with \( \nabla^a \) on Eq. (3.1.9), applying the Leibniz rule, and using again (3.1.9) to re-express \( \nabla^f C_{abcf} \) as \(-\omega^f C_{abcf} \), one obtains
\[ \left[ \nabla^a \nabla^d + \nabla^a \omega^d - \omega^a \omega^d \right] C_{abcd} = 0. \tag{3.2.3} \]

We now re-express \( \nabla^a \omega^d \) from (3.2.1) as
\[ \nabla^a \omega^d = \omega^a \omega^d + \frac{1}{8} T g^{ad} - \frac{1}{2} R^{ad}. \tag{3.2.4} \]

Hence Eqs. (3.2.3) and (3.2.4) lead to
\[ \left[ \nabla^a \nabla^d - \frac{1}{2} R^{ad} \right] C_{abcd} = 0. \tag{3.2.5} \]

This calculation only proves that the vanishing of the *Bach tensor*, defined as
\[ B_{bc} \equiv \nabla^a \nabla^d C_{abcd} - \frac{1}{2} R^{ad} C_{abcd}, \tag{3.2.6} \]
is a necessary condition for a space-time to be conformal to an Einstein space (jointly with Eq. (3.1.9)). To prove sufficiency of the condition, we first need the following Lemma [37]:

**Lemma 3.2.1** Let \( H^{ab} \) be a trace-free symmetric tensor. Then, providing the scalar invariant \( J \) defined in (2.3.4) does not vanish, the only solution of the equations
\[ C_{abcd} H^{ad} = 0, \tag{3.2.7} \]
\[ C^*_{abcd} H^{ad} = 0, \tag{3.2.8} \]
is \( H^{ad} = 0 \). As shown in Kozameh et al. (1985), such a Lemma is best proved by using two-spinor methods. Hence \( H_{ab} \) corresponds to the spinor field
\[ H_{AA'B'B'} = \phi_{ABA'B'} = \overline{\phi}_{(A'B')(AB)}, \tag{3.2.9} \]
and Eqs. (3.2.7) and (3.2.8) imply that
\[ \psi_{ABCD} \phi^{CD}_{A'B'} = 0. \tag{3.2.10} \]

Note that the extra primed spinor indices \( A'B' \) are irrelevant. Hence we can focus on the simpler eigenvalue equation
\[ \psi_{ABCD} \phi^{CD} = \lambda \phi_{AB}. \tag{3.2.11} \]
The corresponding characteristic equation for $\lambda$ is
\begin{equation}
-\lambda^3 + \frac{1}{2} I \lambda + \det(\psi) = 0,
\end{equation}
by virtue of (2.3.3). Moreover, the Cayley–Hamilton theorem enables one to re-write Eq. (3.2.12) as
\begin{equation}
\psi_{\alpha\beta}^P \psi_{\rho\sigma}^{QR} \psi_{\tau\upsilon}^{RS} C^D = \frac{1}{2} I \psi_{\alpha\beta} C^D + \det(\psi) \delta^C_A \delta^D_B,
\end{equation}
and contraction of $\alpha\beta$ with $CD$ yields
\begin{equation}
\det(\psi) = \frac{1}{3} J.
\end{equation}
Thus, the only solution of Eq. (3.2.10) is the trivial one unless $J = 0$ [37].

We are now in a position to prove sufficiency of the conditions (cf. Eqs. (3.1.9) and (3.2.5))
\begin{equation}
\nabla^f C_{abcdef} + K_f C_{abcdef} = 0,
\end{equation}
\begin{equation}
B_{bc} = 0.
\end{equation}
Indeed, Eq. (3.2.15) ensures that (3.1.9) is satisfied with $\omega_f = \nabla f \omega$ for some $\omega$. Hence Eq. (3.2.3) holds. If one now subtracts Eq. (3.2.3) from Eq. (3.2.16) one finds
\begin{equation}
C_{abcd} \left( R^{ad} + 2 \nabla^a \omega^d - 2 \omega^a \omega^d \right) = 0.
\end{equation}
This is indeed Eq. (3.2.7) of Lemma 3.2.1. To obtain Eq. (3.2.8), we act with $\nabla^a$ on the dual of Eq. (3.1.9). This leads to
\begin{equation}
\nabla^a \nabla^d C^*_{abcd} + \left( \nabla^a \omega^d - \omega^a \omega^d \right) C^*_{abcd} = 0.
\end{equation}
Following [37], the gradient of the contracted Bianchi identity and Ricci identity is then used to derive the additional equation
\begin{equation}
\nabla^a \nabla^d C^*_{abcd} - \frac{1}{2} R^{ad} C^*_{abcd} = 0.
\end{equation}
Subtraction of Eq. (3.2.19) from Eq. (3.2.18) now yields
\begin{equation}
C^*_{abcd} \left( R^{ad} + 2 \nabla^a \omega^d - 2 \omega^a \omega^d \right) = 0,
\end{equation}
which is the desired form of Eq. (3.2.8).

We have thus completed the proof that (3.2.15) and (3.2.16) are necessary and sufficient conditions for a space-time to be conformal to an Einstein space. In two-spinor language, when Einstein’s equations are imposed, after a conformal rescaling the equation for the trace-free part of Ricci becomes (see Sec. 2.2)
\begin{equation}
\Phi_{ABA'B'} - \nabla_{BB'} \omega_{AA'} - \nabla_{BA'} \omega_{AB'} + \omega_{AA'} \omega_{BB'} + \omega_{AB'} \omega_{BA'} = 0.
\end{equation}
Similarly to the tensorial analysis performed so far, the spinorial analysis shows that the integrability condition for Eq. (3.2.21) is
\[ \nabla^{AA'} \psi_{ABCD} + \omega^{AA'} \psi_{ABCD} = 0. \] (3.2.22)

The fundamental theorem of conformal gravity states therefore that a space-time is conformal to an Einstein space if and only if \[ \nabla^{DD'} \psi_{ABCD} + k^{DD'} \psi_{ABCD} = 0, \] (3.2.23)
\[ T \psi^{ABCD} \nabla^{FA'} \psi_{FBCD} - I \psi^{A'B'C'D'} \nabla^{AF'} \psi_{F'B'C'D'} = 0, \] (3.2.24)
\[ B_{AFA'F'} = 2 \left( \nabla^{C} \nabla^{D} \psi_{AFCD} + \Phi^{CD}_{A'F'} \psi_{AFCD} \right) = 0. \] (3.2.25)

Note that reality of Eq. (3.2.25) for the Bach spinor is ensured by the Bianchi identities.

### 3.3. Complex space-times

Since this paper is devoted to complex general relativity and its applications, it is necessary to extend the theorem expressed by (3.2.23)–(3.2.25) to complex space-times. For this purpose, we find it appropriate to define and discuss such spaces in more detail in this section. In this respect, we should say that four distinct geometric objects are necessary to study real general relativity and complex general relativity, here defined in four-dimensions ([68], [17]).

1. **Lorentzian** space-time \((M, g_L)\). This is a Hausdorff four-manifold \(M\) jointly with a symmetric, non-degenerate bilinear form \(g_L\) to each tangent space with signature \((+, -, -, -)\) (or \((-,-,+,+)\)). The latter is then called a Lorentzian four-metric \(g_L\).

2. **Riemannian** four-space \((M, g_R)\), where \(g_R\) is a smooth and \textit{positive-definite} section of the bundle of symmetric bilinear two-forms on \(M\). Hence \(g_R\) has signature \((+, +, +, +)\).

3. **Complexified** space-time. This manifold originates from a real-analytic space-time with real-analytic coordinates \(x^a\) and real-analytic Lorentzian metric \(g_L\) by allowing the coordinates to become complex, and by an holomorphic extension of the metric coefficients into the complex domain. In such manifolds the operation of complex conjugation, taking any point with complexified coordinates \(z^a\) into the point with coordinates \(\bar{z}^a\), still exists. Note that, however, it is not possible to define reality of tensors at \textit{complex points}, since the conjugate tensor lies at the complex conjugate point, rather than at the original point.

4. **Complex** space-time. This is a \textit{four-complex-dimensional} complex-Riemannian manifold, and no four-real-dimensional subspace has been singled out to give it a
real structure \[68\]. In complex space-times no complex conjugation exists, since such a map is not invariant under holomorphic coordinate transformations.

Thus, the complex-conjugate spinors \( \lambda^{A\ldots M} \) and \( \bar{\lambda}^{A'\ldots M'} \) of a Lorentzian space-time are replaced by independent spinors \( \lambda^{A\ldots M} \) and \( \tilde{\lambda}^{A'\ldots M'} \). This means that unprimed and primed spin-spaces become unrelated to one another. Moreover, the complex scalars \( \phi \) and \( \tilde{\phi} \) are replaced by the pair of independent complex scalars \( \phi \) and \( \tilde{\phi} \). On the other hand, quantities \( X \) that are originally real yield no new quantities, since the reality condition \( X = \bar{X} \) becomes \( X = \tilde{X} \). For example, the covariant derivative operator \( \nabla_a \) of Lorentzian space-time yields no new operator \( \tilde{\nabla}_a \), since it is originally real. One should instead regard \( \nabla_a \) as a complex-holomorphic operator.

The spinors \( \psi_{ABCD}, \Phi_{ABC'D'} \) and the scalar \( \Lambda \) appearing in the Riemann curvature (see (2.2.26)) have as counterparts the spinors \( \tilde{\psi}_{A'B'C'D'}, \tilde{\Phi}_{ABC'D'} \) and the scalar \( \tilde{\Lambda} \). However, by virtue of the original reality conditions in Lorentzian space-time, one has \[68\]

\[
\tilde{\Phi}_{ABC'D'} = \Phi_{ABC'D'}, \tag{3.3.1}
\]

\[
\tilde{\Lambda} = \Lambda, \tag{3.3.2}
\]

while the Weyl spinors \( \psi_{ABCD} \) and \( \tilde{\psi}_{A'B'C'D'} \) remain independent of each other. Hence one Weyl spinor may vanish without the other Weyl spinor having to vanish as well. Correspondingly, a complex space-time such that \( \tilde{\psi}_{A'B'C'D'} = 0 \) is called right conformally flat or conformally anti-self-dual, whereas if \( \psi_{ABCD} = 0 \), one deals with a left conformally flat or conformally self-dual complex space-time. Moreover, if the remaining part of the Riemann curvature vanishes as well, i.e. \( \Phi_{ABC'D'} = 0 \) and \( \Lambda = 0 \), the word conformally should be omitted in the terminology described above (cf. Sec. 4). Interestingly, in a complex space-time the principal null directions (cf. Sec. 2.3) of the Weyl spinors \( \psi_{ABCD} \) and \( \tilde{\psi}_{A'B'C'D'} \) are independent of each other, and one has two independent classification schemes at each point.

### 3.4. Complex Einstein spaces

In the light of the previous discussion, the fundamental theorem of conformal gravity in complex space-times can be stated as follows [4].

**Theorem 3.4.1** A complex space-time is conformal to a complex Einstein space if and only if

\[
\nabla^{DD'} \psi_{ABCD} + k^{DD'} \psi_{ABCD} = 0, \tag{3.4.1}
\]

\[
\bar{I} \psi_{ABCD} \nabla^{FA'} \psi_{FBCD} - I \psi_{A'B'C'D'} \nabla^{AF'} \tilde{\psi}_{F'B'C'D'} = 0, \tag{3.4.2}
\]

\[
B_{AFA'F'} = 2 \left( \nabla^C_{A'} \nabla^D_{F'} \psi_{AFCD} + \Phi^{CD}_{A'F'} \psi_{AFCD} \right) = 0, \tag{3.4.3}
\]
where $I$ is the complex scalar invariant defined in (2.3.3), whereas $\tilde{I}$ is the independent invariant defined as

$$\tilde{I} \equiv \psi_{A'B'C'D'} \overline{\psi}^{A'B'C'D'}.$$  \hfill (3.4.4)

The left-hand side of Eq. (3.4.2) is called the Eastwood–Dighton spinor, and the left-hand side of Eq. (3.4.3) is the Bach spinor.

### 3.5. Conformal infinity

To complete our introduction to conformal gravity, we find it helpful for the reader to outline the construction of conformal infinity for Minkowski space-time. Starting from polar local coordinates in Minkowski, we first introduce (in $c = 1$ units) the retarded coordinate $w \equiv t - r$ and the advanced coordinate $v \equiv t + r$. To eliminate the resulting cross term in the local form of the metric, new coordinates $p$ and $q$ are defined implicitly as \[ \tan p \equiv v, \tan q \equiv w, \, p - q \geq 0. \]  \hfill (3.5.1)

Hence one finds that a conformal-rescaling factor $\omega \equiv \cos p \cos q$ exists such that, locally, the metric of Minkowski space-time can be written as $\omega^{-2} \tilde{g}$, where

$$\tilde{g} \equiv -dt' \otimes dt' + \left[ dr' \otimes dr' + \frac{1}{4} (\sin(2r'))^2 \Omega_2 \right].$$  \hfill (3.5.2)

where $t' = \frac{(p+q)}{2}, r' = \frac{(p-q)}{2}$, and $\Omega_2$ is the metric on a unit two-sphere. Although (3.5.2) is locally identical to the metric of the Einstein static universe, it is necessary to go beyond a local analysis. This may be achieved by analytic extension to the whole of the Einstein static universe. The original Minkowski space-time is then found to be conformal to the following region of the Einstein static universe:

$$(t' + r') \in [-\pi, \pi], \quad (t' - r') \in [-\pi, \pi], \quad r' \geq 0.$$  \hfill (3.5.3)

By definition, the boundary of the region in (3.5.3) represents the conformal structure of infinity of Minkowski space-time. It consists of two null surfaces and three points, i.e. \[ \{t' - r' = q = -\frac{\pi}{2}\}, \quad \{t' + r' = p = \frac{\pi}{2}\}. \]

(i) The null surface $\text{SCRI}^- \equiv \{t' - r' = q = -\frac{\pi}{2}\}$, i.e. the future light cone of the point $r' = 0, t' = -\frac{\pi}{2}$.

(ii) The null surface $\text{SCRI}^+ \equiv \{t' + r' = p = \frac{\pi}{2}\}$, i.e. the past light cone of the point $r' = 0, t' = \frac{\pi}{2}$.

(iii) Past timelike infinity, i.e. the point

$$\iota^- \equiv \left\{ r' = 0, t' = -\frac{\pi}{2} \right\} \Rightarrow p = q = -\frac{\pi}{2}. \quad \text{(iii)}$$

(iv) Future timelike infinity, defined as

$$\iota^+ \equiv \left\{ r' = 0, t' = \frac{\pi}{2} \right\} \Rightarrow p = q = \frac{\pi}{2}. \quad \text{(iv)}$$
(v) Spacelike infinity, i.e. the point
\[ t^0 \equiv \left\{ r' = \frac{\pi}{2}, t' = 0 \right\} \Rightarrow p = -q = \frac{\pi}{2}. \]

The extension of the SCRI formalism to curved space-times is an open research problem, but we limit ourselves to the previous definitions in this section.

4. Twistor Spaces

In twistor theory, \( \alpha \)-planes are the building blocks of classical field theory in complexified compactified Minkowski space-time. The \( \alpha \)-planes are totally null two-surfaces \( S \) in that, if \( p \) is any point on \( S \), and if \( v \) and \( w \) are any two null tangent vectors at \( p \in S \), the complexified Minkowski metric \( \eta \) satisfies the identity \( \eta(v, w) = v_a w^a = 0 \). By definition, their null tangent vectors have the two-component spinor form \( \lambda^A \pi_A' \), where \( \lambda^A \) is varying and \( \pi_A' \) is fixed. Therefore, the induced metric vanishes identically since \( \eta(v, w) = (\lambda^A \pi_A') (\mu_A \pi_A') = 0 = \eta(v, v) = (\lambda^A \pi_A') (\lambda A \pi A') \). One thus obtains a conformally invariant characterization of flat space-times. This definition can be generalized to complex or real Riemannian space-times with non-vanishing curvature, provided the Weyl curvature is anti-self-dual. One then finds that the curved metric \( g \) is such that \( g(v, w) = 0 \) on \( S \), and the spinor field \( \pi_A' \) is covariantly constant on \( S \). The corresponding holomorphic two-surfaces are called \( \alpha \)-surfaces, and they form a three-complex-dimensional family. Twistor space is the space of all \( \alpha \)-surfaces, and depends only on the conformal structure of complex space-time.

Projective twistor space \( PT \) is isomorphic to complex projective space \( CP^3 \). The correspondence between flat space-time and twistor space shows that complex \( \alpha \)-planes correspond to points in \( PT \), and real null geodesics to points in \( PN \), i.e. the space of null twistors. Moreover, a complex space-time point corresponds to a sphere in \( PT \), and a real space-time point to a sphere in \( PN \). Remarkably, the points \( x \) and \( y \) are null-separated if and only if the corresponding spheres in \( PT \) intersect. This is the twistor description of the light-cone structure of Minkowski space-time.

A conformally invariant isomorphism exists between the complex vector space of holomorphic solutions of \( \Box \phi = 0 \) on the forward tube of flat space-time, and the complex vector space of arbitrary complex-analytic functions of three variables, not subject to any differential equation. Moreover, when curvature is non-vanishing, there is a one-to-one correspondence between complex space-times with anti-self-dual Weyl curvature and scalar curvature \( R = 24\Lambda \), and sufficiently small deformations of flat projective twistor space \( PT \) which preserve a one-form \( \tau \) homogeneous of degree 2 and a three-form \( \rho \) homogeneous of degree 4, with \( \tau \wedge d\tau = 24\Lambda \). Thus, to solve the anti-self-dual Einstein equations, one has to study a geometric problem, i.e. finding the holomorphic curves in deformed projective twistor space.
4.1. $\alpha$-planes in Minkowski space-time

The $\alpha$-planes provide a geometric definition of twistors in Minkowski space-time. For this purpose, we first complexify flat space-time, so that real coordinates $(x^0, x^1, x^2, x^3)$ are replaced by complex coordinates $(z^0, z^1, z^2, z^3)$, and we obtain a four-dimensional complex vector space equipped with a non-degenerate complex-bilinear form \[ (z,w) \equiv z^0w^0 - z^1w^1 - z^2w^2 - z^3w^3. \] (4.1.1)

The resulting matrix $z^{A\bar{A}'}$, which, by construction, corresponds to the position vector $z^a = (z^0, z^1, z^2, z^3)$, is no longer Hermitian as in the real case. Moreover, we compactify such a space by identifying future null infinity with past null infinity ([58], [68], [17]). The resulting manifold is here denoted by $CM^\#$, following [68].

In $CM^\#$ with metric $\eta$, we consider two-surfaces $S$ whose tangent vectors have the two-component spinor form

\[ v^a = \lambda^A \pi^A', \] (4.1.2)

where $\lambda^A$ is varying and $\pi^A'$ is fixed. This implies that these tangent vectors are null, since $\eta(v,v) = v_av^a = \left(\lambda^A \lambda_{\bar{A}}\right)\left(\pi^{A'} \pi_{A'}\right) = 0$. Moreover, the induced metric on $S$ vanishes identically since any two null tangent vectors $v^a = \lambda^A \pi^A'$ and $u^a = \mu^A \pi^A'$ at $p \in S$ are orthogonal:

\[ \eta(v,w) = \left(\lambda_{\bar{A}} \mu_{A}\right)\left(\pi^{A'} \pi_{A'}\right) = 0, \] (4.1.3)

where we have used the property $\pi^{A'} \pi_{A'} = \epsilon^{A'B'} \pi_{A'} \pi_{B'} = 0$. By virtue of (4.1.3), the resulting $\alpha$-plane is said to be totally null. A twistor is then an $\alpha$-plane with constant $\pi_{A'}$ associated to it. Note that two disjoint families of totally null two-surfaces exist in $CM^\#$, since one might choose null tangent vectors of the form

\[ u^a = \nu^A \pi^A', \] (4.1.4)

where $\nu^A$ is fixed and $\pi^A'$ is varying. The resulting two-surfaces are called $\beta$-planes [69].

Theoretical physicists are sometimes more familiar with a definition involving the vector space of solutions of the differential equation

\[ D_{A'}(A\omega^{B'}) = 0, \] (4.1.5)

where $D$ is the flat connection, and $D_{A'A}$ the corresponding spinor covariant derivative. The general solution of Eq. (4.1.5) in $CM^\#$ takes the form ([68], [17])

\[ \omega^A = (\omega^a)^A - i x^{A'A'} \pi_{A'}, \] (4.1.6)

\[ \pi_{A'} = \pi^a_{A'}, \] (4.1.7)

where $\omega^a$ and $\pi^a_{A'}$ are arbitrary constant spinors, and $x^{A'A'}$ is the spinor version of the position vector with respect to some origin. A twistor is then represented...
by the pair of spinor fields \((\omega^A, \pi_{A'})\) \(\Leftrightarrow\) \(Z^\alpha\) \([59]\). The twistor equation (4.1.5) is conformally invariant. This is proved bearing in mind the spinor form of the flat four-metric

\[
\eta_{ab} = \varepsilon_{AB} \varepsilon^{A'B'}, \quad (4.1.8)
\]

and making the conformal rescaling

\[
\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}, \quad (4.1.9)
\]

which implies

\[
\widehat{\varepsilon}_{AB} = \Omega \varepsilon_{AB}, \quad \widehat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}, \quad \widehat{\varepsilon}^{AB} = \Omega^{-1} \varepsilon^{AB}, \quad \widehat{\varepsilon}^{A'B'} = \Omega^{-1} \varepsilon^{A'B'}. \quad (4.1.10)
\]

Thus, defining \(T_a \equiv D_a \left( \log \Omega \right)\) and choosing \(\widehat{\omega}^B = \omega^B\), one finds \([68], [17]\)

\[
\widehat{D}_{AA'} \widehat{\omega}^B = D_{AA'} \omega^B + \varepsilon^B_A T_{CA} \omega^C, \quad (4.1.11)
\]

which implies

\[
\widehat{D}_{A'}^{\omega^B} = \Omega^{-1} D_A^{\omega^B}. \quad (4.1.12)
\]

Note that the solutions of Eq. (4.1.5) are completely determined by the four complex components at \(O\) of \(\omega^A\) and \(\pi_{A'}\) in a spin-frame at \(O\). They are a four-dimensional vector space over the complex numbers, called twistor space \([68], [17]\).

Requiring that \(\nu_A\) be constant over the \(\beta\)-planes implies that \(\nu^A \pi^{A'} D_{AA'} \nu_B = 0\), for each \(\pi_{A'}\), i.e. \(\nu^A D_{AA'} \nu_B = 0\). Moreover, a scalar product can be defined between the \(\omega^A\) field and the \(\nu_A\)-scaled \(\beta\)-plane: \(\omega^A \nu_A\). Its constancy over the \(\beta\)-plane implies that \([69]\)

\[
\nu^A \pi^{A'} D_{AA'} \left( \omega^B \nu_B \right) = 0, \quad (4.1.13)
\]

for each \(\pi_{A'}\), which leads to

\[
\nu_A \nu_B \left( D_A^{\omega^B} \right) = 0, \quad (4.1.14)
\]

for each \(\beta\)-plane and hence for each \(\nu_A\). Thus, Eq. (4.1.14) becomes the twistor equation (4.1.5). In other words, it is the twistor concept associated with a \(\beta\)-plane which is dual to that associated with a solution of the twistor equation \([69]\).

Flat projective twistor space \(PT\) can be thought of as three-dimensional complex projective space \(CP^3\) (cf. example E2 in Sec. 1.2). This means that we take the space \(C^4\) of complex numbers \((z^0, z^1, z^2, z^3)\) and factor out by the proportionality relation \((\lambda z^0, ..., \lambda z^3) \sim (z^0, ..., z^3)\), with \(\lambda \in C - \{0\}\). The homogeneous coordinates \((z^0, ..., z^3)\) are, in the case of \(PT \cong CP^3\), as follows: \((\omega^0, \omega^1, \pi_0', \pi_1') \equiv (\omega^A, \pi_{A'})\). The \(\alpha\)-planes defined in this section can be obtained from the equation (cf. (4.1.6))

\[
\omega^A = i x^{AA'} \pi_{A'}, \quad (4.1.15)
\]
where \((\omega^A, \pi_A')\) is regarded as fixed, with \(\pi_{A'} \neq 0\). This means that Eq. (4.1.15), considered as an equation for \(x^{AA'}\), has as its solution a complex two-plane in \(CM^\#\), whose tangent vectors take the form in Eq. (4.1.2), i.e. we have found an \(\alpha\)-plane. The \(\alpha\)-planes are self-dual in that, if \(v\) and \(u\) are any two null tangent vectors to an \(\alpha\)-plane, then \(F \equiv v \otimes u - u \otimes v\) is a self-dual bivector since

\[
F^{AA'B'B'} = \varepsilon^{AB} \phi(A'B'),
\]

where \(\phi(A'B') = \sigma \pi^{A'} \pi^{B'}\), with \(\sigma \in C - \{0\}\) (Ward 1981b). Note also that \(\alpha\)-planes remain unchanged if we replace \((\omega^A, \pi_A')\) by \((\lambda \omega^A, \lambda \pi_A')\) with \(\lambda \in C - \{0\}\), and that all \(\alpha\)-planes arise as solutions of Eq. (4.1.15). If real solutions of such equation exist, this implies that \(x^{AA'} = \bar{\pi}^{AA'}\). This leads to

\[
\omega^A \pi_A + \bar{\omega}'^{A'} \pi_{A'} = i x^{AA'} (\bar{\pi}^{A} \pi_A - \pi_{A'} \bar{\pi}_{A'}) = 0,
\]

where overbars denote complex conjugation in two-spinor language, defined according to the rules described in Sec. 2.1. If (4.1.17) holds and \(\pi_{A'} \neq 0\), the solution space of Eq. (4.1.15) in real Minkowski space-time is a null geodesic, and all null geodesics arise in this way (Ward 1981b). Moreover, if \(\pi_{A'}\) vanishes, the point \((\omega^A, \pi_{A'}) = (\omega^A, 0)\) can be regarded as an \(\alpha\)-plane at infinity in compactified Minkowski space-time. Interestingly, Eq. (4.1.15) is the two-spinor form of the equation expressing the incidence property of a point \((t, x, y, z)\) in Minkowski space-time with the twistor \(Z^\alpha\), i.e. [65]

\[
\left( \begin{array}{c} Z^0 \\ Z^1 \end{array} \right) = \frac{i}{\sqrt{2}} \left( \begin{array}{cc} t + z & x + iy \\ x - iy & t - z \end{array} \right) \left( \begin{array}{c} Z^2 \\ Z^3 \end{array} \right), \tag{4.1.18}
\]

The left-hand side of Eq. (4.1.17) may be then re-interpreted as the twistor pseudo-norm [65]

\[
Z^\alpha \bar{Z}_\alpha = Z^0 \bar{Z}^0 + Z^1 \bar{Z}^1 + Z^2 \bar{Z}^2 + Z^3 \bar{Z}^3 = \omega^A \pi_A + \pi_{A'} \bar{\omega}'^{A'},
\]

by virtue of the property \((Z_0, Z_1, Z_2, Z_3) = (\bar{Z}^0, \bar{Z}^1, \bar{Z}^2, \bar{Z}^3)\). Such a pseudo-norm makes it possible to define the top half \(PT^+\) of \(PT\) by the condition \(Z^\alpha \bar{Z}_\alpha > 0\), and the bottom half \(PT^-\) of \(PT\) by the condition \(Z^\alpha \bar{Z}_\alpha < 0\).

So far, we have seen that an \(\alpha\)-plane corresponds to a point in \(PT\), and null geodesics to points in \(PN\), the space of null twistors. However, we may also interpret (4.1.15) as an equation where \(x^{AA'}\) is fixed, and solve for \((\omega^A, \pi_A')\). Within this framework, \(\pi_{A'}\) remains arbitrary, and \(\omega^A\) is thus given by \(ix^{AA'} \pi_{A'}\). This yields a complex two-plane, and factorization by the proportionality relation \((\lambda \omega^A, \lambda \pi_{A'}) \sim (\omega^A, \pi_A')\) leads to a complex projective one-space \(CP^1\), with two-sphere topology. Thus, the fixed space-time point \(x\) determines a Riemann sphere \(L_x \cong CP^1\) in \(PT\). In particular, if \(x\) is real, then \(L_x\) lies entirely within \(PN\), given by those twistors whose homogeneous coordinates satisfy Eq. (4.1.17). To sum up, a complex
space-time point corresponds to a sphere in $PT$, whereas a real space-time point corresponds to a sphere in $PN$ ([65], [102]).

In Minkowski space-time, two points $p$ and $q$ are null-separated if and only if there is a null geodesic connecting them. In projective twistor space $PT$, this implies that the corresponding lines $L_p$ and $L_q$ intersect, since the intersection point represents the connecting null geodesic. To conclude this section it may be now instructive, following [33], to study the relation between null twistors and null geodesics. Indeed, given the null twistors $X^\alpha, Y^\alpha$ defined by

$$X^\alpha \equiv \left( i \ x_0^{AC'} \ X_{C'}, X_{A'} \right) ,$$  
$$Y^\alpha \equiv \left( i \ x_1^{AC'} \ Y_{C'}, Y_{A'} \right) ,$$

the corresponding null geodesics are

$$\gamma_X : \ x^{AA'} = x_0^{AA'} + \lambda \overline{X}^A X^{A'},$$
$$\gamma_Y : \ x^{AA'} = x_1^{AA'} + \mu \overline{Y}^A Y^{A'} .$$

If these intersect at some point $x_2$, one finds

$$x_2^{AA'} = x_0^{AA'} + \lambda \overline{X}^A X^{A'} = x_1^{AA'} + \mu \overline{Y}^A Y^{A'},$$

where $\lambda, \mu \in R$. Hence

$$x_2^{AA'} \overline{Y}_A X_{A'} = x_0^{AA'} \overline{Y}_A X_{A'} = x_1^{AA'} \overline{Y}_A X_{A'},$$

by virtue of the identities $X^{A'} X_A = \overline{Y}^A \overline{Y}_A = 0$. Equation (4.1.25) leads to

$$X^\alpha \overline{Y}_\alpha = i \left( x_0^{AA'} \overline{Y}_A X_{A'} - x_1^{AA'} \overline{Y}_A X_{A'} \right) = 0 .$$

Suppose instead we are given Eq. (4.1.26). This implies that some real $\lambda$ and $\mu$ exist such that

$$x_0^{AA'} - x_1^{AA'} = -\lambda \overline{X}^A X^{A'} + \mu \overline{Y}^A Y^{A'},$$

where signs on the right-hand side of (4.1.27) have been suggested by (4.1.24). Note that (4.1.27) only holds if $X_{A'} Y^{A'} \neq 0$, i.e. if $\gamma_X$ and $\gamma_Y$ are not parallel. However, the whole argument can be generalized to this case as well [33], and one finds that in all cases the null geodesics $\gamma_X$ and $\gamma_Y$ intersect if and only if $X^\alpha \overline{Y}_\alpha$ vanishes.

### 4.2. $\alpha$-surfaces and twistor geometry

The $\alpha$-planes defined in Sec. 4.1 can be generalized to a suitable class of curved complex space-times. By a complex space-time $(M, g)$ we mean a four-dimensional Hausdorff manifold $M$ with holomorphic metric $g$. Thus, with respect to a holomorphic coordinate basis $x^\alpha$, $g$ is a $4 \times 4$ matrix of holomorphic functions of $x^\alpha$, and its determinant is nowhere-vanishing ([100], [103]). Remarkably, $g$ determines a unique
holomorphic connection $\nabla$, and a holomorphic curvature tensor $R^a_{\ bcd}$. Moreover, the Ricci tensor $R_{ab}$ becomes complex-valued, and the Weyl tensor $C^a_{\ bcd}$ may be split into independent holomorphic tensors, i.e. its self-dual and anti-self-dual parts, respectively. With our two-spinor notation, one has (see (2.1.36))

$$C_{abcd} = \psi_{ABCD} \varepsilon^{A'B'C'D'} + \tilde{\psi}_{A'B'C'D'} \varepsilon_{AB} \varepsilon_{CD},$$

(4.2.1)

where $\psi_{ABCD} = \psi_{(ABCD)}$, $\tilde{\psi}_{A'B'C'D'} = \tilde{\psi}_{(A'B'C'D')}$. The spinors $\psi$ and $\tilde{\psi}$ are the anti-self-dual and self-dual Weyl spinors, respectively. Following [60], [61], [103], complex vacuum space-times such that

$$\tilde{\psi}_{A'B'C'D'} = 0, \ R_{ab} = 0,$$

(4.2.2)

are called right-flat or anti-self-dual, whereas complex vacuum space-times such that

$$\psi_{ABCD} = 0, \ R_{ab} = 0,$$

(4.2.3)

are called left-flat or self-dual. Note that this definition only makes sense if space-time is complex or real Riemannian, since in this case no complex conjugation relates primed to unprimed spinors (i.e. the corresponding spin-spaces are no longer anti-isomorphic). Hence, for example, the self-dual Weyl spinor $\psi_{A'B'C'D'}$ may vanish without its anti-self-dual counterpart $\psi_{ABCD}$ having to vanish as well, as in Eq. (4.2.2), or the converse may hold, as in Eq. (4.2.3) (see Sec. 1.1 and problem 2.3).

By definition, $\alpha$-surfaces are complex two-surfaces $S$ in a complex space-time $(M, g)$ whose tangent vectors $v$ have the two-spinor form (4.1.2), where $\lambda^A$ is varying, and $\pi^A$ is a fixed primed spinor field on $S$. From this definition, the following properties can be derived (cf. Sec. 4.1).

(i) tangent vectors to $\alpha$-surfaces are null;

(ii) any two null tangent vectors $v$ and $u$ to an $\alpha$-surface are orthogonal to one another;

(iii) the holomorphic metric $g$ vanishes on $S$ in that $g(v, u) = g(v, v) = 0, \forall v, u$ (cf. (4.1.3)), so that $\alpha$-surfaces are totally null;

(iv) $\alpha$-surfaces are self-dual, in that $F \equiv v \otimes u - u \otimes v$ takes the two-spinor form (4.1.16);

(v) $\alpha$-surfaces exist in $(M, g)$ if and only if the self-dual Weyl spinor vanishes, so that $(M, g)$ is anti-self-dual.

Note that properties (i)–(iv), here written in a redundant form for pedagogical reasons, are the same as in the flat-space-time case, provided we replace the flat metric $\eta$ with the curved metric $g$. Condition (v), however, is a peculiarity of curved space-times. We here focus on the sufficiency of the condition, following [103].

We want to prove that, if $(M, g)$ is anti-self-dual, it admits a three-complex-parameter family of self-dual $\alpha$-surfaces. Indeed, given any point $p \in M$ and a spinor $\mu_A$ at $p$, one can find a spinor field $\pi_A$ on $M$, satisfying the equation

$$\pi_{A'} \left( \nabla_{AA'} \pi_{B'} \right) = \xi_A \pi_{B'},$$

(4.2.4)
and such that
\[ \pi_A'(p) = \mu_A'(p). \]  
(4.2.5)

Hence \( \pi_A' \) defines a holomorphic two-dimensional distribution, spanned by the vector fields of the form \( \lambda^A \pi_A' \), which is integrable by virtue of (4.2.4). Thus, in particular, there exists a self-dual \( \alpha \)-surface through \( p \), with tangent vectors of the form \( \lambda^A \mu_A' \) at \( p \). Since \( p \) is arbitrary, this argument may be repeated \( \forall p \in M \). The space \( \mathcal{P} \) of all self-dual \( \alpha \)-surfaces in \((M,g)\) is three-complex-dimensional, and is called twistor space of \((M,g)\).

### 4.3. Geometric theory of partial differential equations

One of the main results of twistor theory has been a deeper understanding of the solutions of partial differential equations of classical field theory. Remarkably, a problem in analysis becomes a purely geometric problem ([102], [103]). For example, in [5] it was shown that the general real-analytic solution of the wave equation \( \Box \phi = 0 \) in Minkowski space-time is
\[ \phi(x,y,z,t) = \int_{-\pi}^{\pi} F(x \cos \theta + y \sin \theta + iz, y + iz \sin \theta + t \cos \theta, \theta) \, d\theta, \]  
(4.3.1)

where \( F \) is an arbitrary function of three variables, complex-analytic in the first two. Indeed, twistor theory tells us that \( F \) is a function on \( \mathcal{PT} \). More precisely, let \( f(\omega^A, \pi_{A'}) \) be a complex-analytic function, homogeneous of degree \(-2\), i.e. such that
\[ f(\lambda \omega^A, \lambda \pi_{A'}) = \lambda^{-2} f(\omega^A, \pi_{A'}), \]  
(4.3.2)

and possibly having singularities [102]. We now define a field \( \phi(x^a) \) by
\[ \phi(x^a) \equiv \frac{1}{2\pi i} \oint f(i x^{A'} \pi_{A'}, \pi_B') \pi_{C'} \, d\pi_{C'}, \]  
(4.3.3)

where the integral is taken over any closed one-dimensional contour that avoids the singularities of \( f \). Such a field satisfies the wave equation, and every solution of \( \Box \phi = 0 \) can be obtained in this way. The function \( f \) has been taken to have homogeneity \(-2\) since the corresponding one-form \( f \pi_{C'} \, d\pi_{C'} \) has homogeneity zero and hence is a one-form on projective twistor space \( \mathcal{PT} \), or on some subregion of \( \mathcal{PT} \), since it may have singularities. The homogeneity is related to the property of \( f \) of being a free function of three variables. Since \( f \) is not defined on the whole of \( \mathcal{PT} \), and \( \phi \) does not determine \( f \) uniquely, because we can replace \( f \) by \( f + \tilde{f} \), where \( \tilde{f} \) is any function such that
\[ \oint \tilde{f} \pi_{C'} \, d\pi_{C'} = 0, \]  
(4.3.4)

we conclude that \( f \) is an element of the sheaf-cohomology group \( H^1\left(\mathcal{PT}^+, O(-2)\right) \), i.e. the complex vector space of arbitrary complex-analytic functions of three variables, not subject to any differential equations ([63], [102], [103]). Remarkably,
a conformally invariant isomorphism exists between the complex vector space of holomorphic solutions of \( \square \phi = 0 \) on the forward tube \( CM^+ \) (i.e. the domain of definition of positive-frequency fields), and the sheaf-cohomology group \( H^1 \left( PT^+, O(-2) \right) \).

It is now instructive to summarize some basic ideas of sheaf-cohomology theory and its use in twistor theory, following [63]. For this purpose, let us begin by recalling how Čech cohomology is obtained. We consider a Hausdorff paracompact topological space \( X \), covered with a locally finite system of open sets \( U_i \). With respect to this covering, we define a cochain with coefficients in an additive Abelian group \( G \) (e.g. \( Z, R \) or \( C \)) in terms of elements \( f_i, f_{ij}, f_{ijk}, \ldots \in G \). These elements are assigned to the open sets \( U_i \) of the covering, and to their non-empty intersections, as follows: \( f_i \) to \( U_i \), \( f_{ij} \) to \( U_i \cap U_j \), \( f_{ijk} \) to \( U_i \cap U_j \cap U_k \) and so on. The elements assigned to non-empty intersections are completely antisymmetric, so that \( f_{i...p} = f_{[i...p]} \). One is thus led to define

zero cochain \( \alpha \equiv (f_1, f_2, f_3, ...) \),

one cochain \( \beta \equiv (f_{12}, f_{23}, f_{13}, ...) \),

two cochain \( \gamma \equiv (f_{123}, f_{124}, ...) \),

and the coboundary operator \( \delta \):

\[
\delta \alpha \equiv (f_2 - f_1, f_3 - f_2, f_3 - f_1, ...) \equiv (f_{12}, f_{23}, f_{13}, ...),
\]

\[
\delta \beta \equiv (f_{12} - f_{13} + f_{23}, f_{12} - f_{14} + f_{24}, ...) \equiv (f_{123}, f_{124}, ...).
\]

By virtue of (4.3.8) and (4.3.9) one finds \( \delta^2 \alpha = \delta^2 \beta = \ldots = 0 \). Cocycles \( \gamma \) are cochains such that \( \delta \gamma = 0 \). Coboundaries are a particular set of cocycles, i.e. such that \( \gamma = \delta \beta \) for some cochain \( \beta \). Of course, all coboundaries are cocycles, whereas the converse does not hold. This enables one to define the \( p \)th cohomology group as the quotient space

\[
H^p \left\{ \left\{ U_i \right\} \right\} (X, G) \equiv G^p_{CC}/G^p_{CB},
\]

where \( G^p_{CC} \) is the additive group of \( p \)-cocycles, and \( G^p_{CB} \) is the additive group of \( p \)-coboundaries. To avoid having a definition which depends on the covering \( \left\{ U_i \right\} \), one should then take finer and finer coverings of \( X \) and settle on a sufficiently fine covering \( \left\{ U_i \right\}^* \). Following [63], by this we mean that all the \( H^p \left( U_i \cap \ldots \cap U_k, G \right) \) vanish \( \forall p > 0 \). One then defines

\[
H^p \left\{ \left\{ U_i \right\} \right\} (X, G) \equiv H^p(X, G).
\]
We always assume such a covering exists, is countable and locally finite. Note that, rather than thinking of $f_i$ as an element of $G$ assigned to $U_i$, of $f_{ij}$ as assigned to $U_{ij}$ and so on, we can think of $f_i$ as a function defined on $U_i$ and taking a constant value $\in G$. Similarly, we can think of $f_{ij}$ as a $G$-valued constant function defined on $U_i \cap U_j$, and this implies it is not strictly necessary to assume that $U_i \cap U_j$ is non-empty.

The generalization to sheaf cohomology is obtained if we do not require the functions $f_i, f_{ij}, f_{ijk}...$ to be constant (there are also cases when the additive group $G$ is allowed to vary from point to point in $X$). The assumption of main interest is the holomorphic nature of the $f$’s. A sheaf is so defined that the Čech cohomology previously defined works as well as before [63]. In other words, a sheaf $S$ defines an additive group $G_u$ for each open set $U \subset X$. Relevant examples are as follows.

(i) The sheaf $O$ of germs of holomorphic functions on a complex manifold $X$ is obtained if $G_u$ is taken to be the additive group of all holomorphic functions on $U$.

(ii) Twisted holomorphic functions, i.e. functions whose values are not complex numbers, but are taken in some complex line bundle over $X$.

(iii) A particular class of twisted functions is obtained if $X$ is projective twistor space $PT$ (or $PT^+$, or $PT^-$), and the functions studied are holomorphic and homogeneous of some degree $n$ in the twistor variable, i.e.

$$f(\lambda \omega^A, \lambda \pi_A') = \lambda^n f(\omega^A, \pi_A').$$

(4.3.12)

If $G_u$ consists of all such twisted functions on $U \subset X$, the resulting sheaf, denoted by $O(n)$, is the sheaf of germs of holomorphic functions twisted by $n$ on $X$.

(iv) We can also consider vector-bundle-valued functions, where the vector bundle $B$ is over $X$, and $G_u$ consists of the cross-sections of the portion of $B$ lying above $U$.

Defining cochains and coboundary operator as before, with $f_i \in G_{U_i}$ and so on, we obtain the $p$th cohomology group of $X$, with coefficients in the sheaf $S$, as the quotient space

$$H^p(X, S) \equiv G^p(S)/G^p_{\text{CB}}(S),$$

(4.3.13)

where $G^p(S)$ is the group of $p$-cochains with coefficients in $S$, and $G^p_{\text{CB}}(S)$ is the group of $p$-coboundaries with coefficients in $S$. Again, we take finer and finer coverings $\{U_i\}$ of $X$, and we settle on a sufficiently fine covering. To understand this concept, we recall the following definitions [63].

**Definition 4.3.1** A coherent analytic sheaf is locally defined by $n$ holomorphic functions factored out by a set of $s$ holomorphic relations.

**Definition 4.3.2** A Stein manifold is a holomorphically convex open subset of $C^m$.

Thus, we can say that, provided $S$ is a coherent analytic sheaf, sufficiently fine means that each of $U_i, U_i \cap U_j, U_i \cap U_j \cap U_k...$ is a Stein manifold. If $X$ is Stein and
$S$ is coherent analytic, then $H^p(X, S) = 0, \forall p > 0$.

We can now consider again the remarks following Eq. (4.3.4), i.e. the interpretation of twistor functions as elements of $H^1\left(PT^+, O(-2)\right)$. Let $X$ be a part of $PT$, e.g. the neighbourhood of a line in $PT$, or the top half $PT^+$, or the closure $\overline{PT^+}$ of the top half. We assume $X$ can be covered with two open sets $U_1, U_2$ such that every projective line $L$ in $X$ meets $U_1 \cap U_2$ in an annular region. For us, $U_1 \cap U_2$ corresponds to the domain of definition of a twistor function $f(Z^\alpha)$, homogeneous of degree $n$ in the twistor $Z^\alpha$ (see (4.3.12)). Then $f = f_{12} \equiv f_2 - f_1$ is a twisted function on $U_1 \cap U_2$, and defines a one-cochain $\epsilon$, with coefficients in $O(n)$, for $X$. By construction $\delta \epsilon = 0$, hence $\epsilon$ is a cocycle. For this covering, the one-coboundaries are functions of the form $l_2 - l_1$, where $l_2$ is holomorphic on $U_2$ and $l_1$ on $U_1$. The equivalence between twistor functions is just the cohomological equivalence between one-cochains $\epsilon, \epsilon'$ that their difference should be a coboundary: $\epsilon' - \epsilon = \delta \alpha$, with $\alpha = (l_1, l_2)$. This is why we view twistor functions as defining elements of $H^1\left(X, O(n)\right)$. Indeed, if we try to get finer coverings, we realize it is often impossible to make $U_1$ and $U_2$ into Stein manifolds. However, if $X = \overline{PT^+}$, the covering $\{U_1, U_2\}$ by two sets is sufficient for any analytic, positive-frequency field [63].

The most striking application of twistor theory to partial differential equations is perhaps the geometric characterization of anti-self-dual space-times with a cosmological constant. For these space-times, the Weyl tensor takes the form

$$C^{(A.S.D.)}_{abcd} = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'}, \quad (4.3.14)$$

and the Ricci tensor reads

$$R_{ab} = -2\Phi_{ab} + 6\Lambda g_{ab}. \quad (4.3.15)$$

With our notation, $\epsilon_{AB}$ and $\epsilon_{A'B'}$ are the curved-space version of the $\varepsilon$-symbols (denoted again by $\varepsilon_{AB}$ and $\varepsilon_{A'B'}$ in Eqs. (2.1.36) and (4.2.1)), $\Phi_{ab}$ is the trace-free part of Ricci, $24\Lambda$ is the trace $R = R^a_a$ of Ricci [100]. The local structure in projective twistor space which gives information about the metric is a pair of differential forms: a one-form $\tau$ homogeneous of degree 2 and a three-form $\rho$ homogeneous of degree 4. Basically, $\tau$ contains relevant information about $\epsilon_{A'B'}$ and $\rho$ tells us about $\epsilon_{AB}$, hence their knowledge determines $g_{ab} = \epsilon_{AB} \epsilon_{A'B'}$. The result proved in [100] states that a one-to-one correspondence exists between sufficiently local anti-self-dual solutions with scalar curvature $R = 24\Lambda$ and sufficiently small deformations of flat projective twistor space which preserve the one-form $\tau$ and the three-form $\rho$, where $\tau \wedge d\tau = 2\Lambda \rho$. We now describe how to define the forms $\tau$ and $\rho$, whereas the explicit construction of a class of anti-self-dual space-times is given in Sec. 5.

The geometric framework is twistor space $\mathcal{P}$ defined at the end of Sec. 4.2, i.e. the space of all $\alpha$-surfaces in $(M, g)$. We take $M$ to be sufficiently small and convex to ensure that $\mathcal{P}$ is a complex manifold with topology $R^4 \times S^2$, since every point in
an anti-self-dual space-time has such a neighbourhood [100]. If \( Q \), represented by the pair \((\alpha^A, \beta^{A'})\), is any vector in \( \mathcal{P} \), then \( \tau \) is defined by

\[
\tau(Q) \equiv e^{A'B'} \pi_{A'}^{\beta} \beta_{B'}.
\]

(4.3.16)

To make sure \( \tau \) is well defined, one has to check that the right-hand side of (4.3.16) remains covariantly constant over \( \alpha \)-surfaces, i.e. is annihilated by the first-order operator \( \lambda^A \pi^{A'} \nabla_{AA'} \), since otherwise \( \tau \) does not correspond to a differential form on \( \mathcal{P} \). It turns out that \( \tau \) is well defined provided the trace-free part of Ricci vanishes. This is proved using spinor Ricci identities and the equations of local twistor transport as follows [100].

Let \( v \) be a vector field on the \( \alpha \)-surface \( Z \) such that \( \epsilon v^a \) joins \( Z \) to the neighbouring \( \alpha \)-surface \( Y \). Since \( \epsilon v^a \) acts as a connecting vector, the Lie bracket of \( v^a \) and \( \lambda^B \pi^{B'} \) vanishes for all \( \lambda^B \), i.e.

\[
\lambda^B \pi^{B'} \nabla_{BB'} v^{AA'} - v^{BB'} \nabla_{BB'} \lambda^A \pi^{A'} = 0.
\]

(4.3.17)

Thus, after defining

\[
\beta_{A'} \equiv v^{BB'} \nabla_{BB'} \pi_{A'},
\]

(4.3.18)

one finds

\[
\pi_{A'} \lambda^B \pi^{B'} \nabla_{BB'} v^{AA'} = \lambda^A \beta_{A'} \pi^{A'}. \tag{4.3.19}
\]

If one now applies the torsion-free spinor Ricci identities ([66], [67]), one finds that the spinor field \( \beta_{A'}(x) \) on \( Z \) satisfies the equation

\[
\lambda^B \pi^{B'} \nabla_{BB'} \beta_{A'} = -i \lambda^A \pi^{A'} \beta_{A'}, \tag{4.3.20}
\]

where \( P_{ab} = \Phi_{ab} - \Lambda g_{ab} \) and \( \alpha^A = iv^{AC'} \pi_{C'} \). Moreover, Eq. (4.3.19) and the Leibniz rule imply that

\[
\lambda^B \pi^{B'} \nabla_{BB'} \alpha^A = -i \lambda^A \pi^{A'} \beta_{A'}, \tag{4.3.21}
\]

since \( \pi^{B'} \nabla_{BB'} \pi_{C'} = 0 \). Equations (4.3.20) and (4.3.21) are indeed the equations of local twistor transport, and Eq. (4.3.20) leads to

\[
\lambda^A \pi^{A'} \nabla_{CC'} \left( e^{A'B'} \pi_{A'}^{\beta} \beta_{B'} \right) = e^{A'B'} \pi_{A'} \left( \lambda^A \pi^{A'} \nabla_{CC'} \beta_{B'} \right),
\]

\[
= -i \lambda^B \pi^{B'} \pi_{C'} e^{C'A'} \alpha^A \left( \Phi_{AB'B'} - \Lambda \epsilon_{AB} \epsilon_{A'B'} \right) = i \lambda^B \pi^{A'} \pi^{B'} \alpha^A \Phi_{ABA'B'},
\]

(4.3.22)

since \( \pi^{A'} \pi^{B'} \epsilon_{A'B'} = 0 \). Hence, as we said before, \( \tau \) is well defined provided the trace-free part of Ricci vanishes. Note that, strictly, \( \tau \) is a twisted form rather than a form on \( \mathcal{P} \), since it is homogeneous of degree 2, one from \( \pi_{A'} \) and one from \( \beta_{B'} \). By contrast, a one-form would be independent of the scaling of \( \pi_{A'} \) and \( \beta_{B'} \) [100].
We are now in a position to define the three-form $\rho$, homogeneous of degree 4. For this purpose, let us denote by $Q^h$, $h = 1, 2, 3$ three vectors in $P$, represented by the pairs $(\alpha^A_h, \beta^A_{h'})$. The corresponding $\rho(Q_1, Q_2, Q_3)$ is obtained by taking

$$\rho_{123} \equiv \frac{1}{2} (\epsilon^{A'B'} \pi^{A'}_B \beta_1 B') (\epsilon_{AB} \alpha^A_2 \alpha^B_3),$$

and then anti-symmetrizing $\rho_{123}$ over 1, 2, 3. This yields

$$\rho(Q_1, Q_2, Q_3) \equiv \frac{1}{6} \left( \rho_{123} - \rho_{132} + \rho_{231} - \rho_{213} + \rho_{312} - \rho_{321} \right).$$

The reader can check that, by virtue of Eqs. (4.3.20) and (4.3.21), $\rho$ is well defined, since it is covariantly constant over $\alpha$-surfaces:

$$\lambda^A \pi^{A'} \nabla_{AA'} \rho(Q_1, Q_2, Q_3) = 0.$$

5. Penrose Transform for Gravitation

Deformation theory of complex manifolds is applied to construct a class of anti-self-dual solutions of Einstein’s vacuum equations, following the work of Penrose and Ward. The hard part of the analysis is to find the holomorphic cross-sections of a deformed complex manifold, and the corresponding conformal structure of an anti-self-dual space-time. This calculation is repeated in detail, using complex analysis and two-component spinor techniques.

If no assumption about anti-self-duality is made, twistor theory is by itself insufficient to characterize geometrically a solution of the full Einstein equations. After a brief review of alternative ideas based on the space of complex null geodesics of complex space-time, and Einstein-bundle constructions, attention is focused on the attempt by Penrose to define twistors as charges for massless spin-$\frac{3}{2}$ fields. This alternative definition is considered since a vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless spin-$\frac{3}{2}$ fields in curved space-time, whereas in Minkowski space-time the space of charges for such fields is naturally identified with the corresponding twistor space.

The two-spinor analysis of the Dirac form of such fields in Minkowski space-time is carried out in detail by studying their two potentials with corresponding gauge freedoms. The Rarita–Schwinger form is also introduced, and self-dual vacuum Maxwell fields are obtained from massless spin-$\frac{3}{2}$ fields by spin-lowering. In curved space-time, however, the local expression of spin-$\frac{3}{2}$ field strengths in terms of the second of these potentials is no longer possible, unless one studies the self-dual Ricci-flat case. Thus, much more work is needed to characterize geometrically a Ricci-flat (complex) space-time by using this alternative concept of twistors.

5.1. Anti-self-dual space-times

Following [98], we now use twistor-space techniques to construct a family of anti-self-dual solutions of Einstein’s vacuum equations. Bearing in mind the space-time
twistor-space correspondence in Minkowskian geometry described in Sec. 4.1, we take a region $R$ of $CM^4$, whose corresponding region in $PT$ is $\tilde{R}$. Moreover, $N$ is the non-projective version of $R$, which implies $N \subset T \subset C^4$. In other words, as coordinates on $N$ we may use $(\omega^A, \pi_{A'})$. The geometrically-oriented reader may like it to know that three important structures are associated with $N$:

(i) the fibration $(\omega^A, \pi_{A'}) \to \pi_{A'}$, which implies that $N$ becomes a bundle over $C^2 - \{0\}$;

(ii) the two-form $\frac{1}{2} d\omega^A \wedge d\omega^A$ on each fibre;

(iii) the projective structure $N \to \tilde{R}$.

Deformations of $N$ which preserve this projective structure correspond to right-flat metrics (see Sec. 4.2) in $R$. To obtain such deformations, cover $N$ with two patches $Q$ and $\hat{Q}$. Coordinates on $Q$ and on $\hat{Q}$ are $(\omega^A, \pi_{A'})$ and $(\hat{\omega}^A, \hat{\pi}_{A'})$ respectively. We may now glue $Q$ and $\hat{Q}$ together according to

$$\hat{\omega}^A = \omega^A + f^A(\omega^B, \pi_{B'}), \quad (5.1.1)$$

$$\hat{\pi}_{A'} = \pi_{A'}, \quad (5.1.2)$$

where $f^A$ is homogeneous of degree 1, holomorphic on $Q \cap \hat{Q}$, and satisfies

$$\det \left( \varepsilon^{AB} + \frac{\partial f^B}{\partial \omega^A} \right) = 1. \quad (5.1.3)$$

Such a patching process yields a complex manifold $N^D$ which is a deformation of $N$. The corresponding right-flat space-time $G$ is such that its points correspond to the holomorphic cross-sections of $N^D$. The hard part of the analysis is indeed to find these cross-sections, but this can be done explicitly for a particular class of patching functions. For this purpose, we first choose a constant spinor field $p^{A'A'B'} = p^{A(A'B')}$, and a homogeneous holomorphic function $g(\gamma, \pi_{A'})$ of three complex variables:

$$g(\lambda^3 \gamma, \lambda \pi_{A'}) = \lambda^{-1} g(\gamma, \pi_{A'}) \quad \forall \lambda \in C - \{0\}. \quad (5.1.4)$$

This enables one to define the spinor field

$$p^A \equiv p^{A'A'B'} \pi_{A'} \pi_{B'}, \quad (5.1.5)$$

and the patching function

$$f^A \equiv p^A g(p_B \omega^B, \pi_{B'}), \quad (5.1.6)$$

and the function

$$F(x^a, \pi_{A'}) \equiv g(i p_A x^{AC'} \pi_{C'}, \pi_{A'}). \quad (5.1.7)$$

Under suitable assumptions on the singularities of $g$, $F$ may turn out to be holomorphic if $x^a \in R$ and if the ratio $\tilde{\pi} \equiv \frac{\pi_{A'}}{\pi_{A}} \in \{1, \infty\}$. It is also possible to express $F$
as the difference of two contour integrals after defining the differential form

\[
\Omega \equiv \left( 2\pi i\rho^A\pi_{A'} \right)^{-1} F(x^b, \rho_{B'}) \rho_C d\rho^{C'}. \tag{5.1.8}
\]

In other words, if \( \Gamma \) and \( \hat{\Gamma} \) are closed contours on the projective \( \rho_A' \)-sphere defined by \( |\tilde{\rho}| = 1 \) and \( |\tilde{\rho}| = 2 \) respectively, we may define the function

\[
h \equiv \oint_{\Gamma} \Omega, \tag{5.1.9}
\]

holomorphic for \( \tilde{\pi} < 2 \), and the function

\[
\hat{h} \equiv \oint_{\hat{\Gamma}} \Omega, \tag{5.1.10}
\]

holomorphic for \( \tilde{\pi} > 1 \). Thus, by virtue of Cauchy’s integral formula, one finds (cf. [98])

\[
F(x^a, \pi_{A'}) = \hat{h}(x^a, \pi_{A'}) - h(x^a, \pi_{A'}). \tag{5.1.11}
\]

The basic concepts of sheaf-cohomology presented in Sec. 4.3 are now useful to understand the deep meaning of these formulae. For any fixed \( x^a \), \( F(x^a, \pi_{A'}) \) determines an element of the sheaf-cohomology group \( H^1(P_1(C), O(-1)) \), where \( P_1(C) \) is the Riemann sphere of projective \( \pi_{A'} \) spinors and \( O(-1) \) is the sheaf of germs of holomorphic functions of \( \pi_{A'} \), homogeneous of degree \(-1\). Since \( H^1 \) vanishes, \( F \) is actually a coboundary. Hence it can be split according to (5.1.11).

In the subsequent calculations, it will be useful to write a solution of the Weyl equation \( \nabla^{AA'}\psi_A = 0 \) in the form

\[
\psi_A \equiv i\pi^A \nabla_{AA'} h(x^a, \pi_{C'}). \tag{5.1.12}
\]

Moreover, following again [98], we note that a spinor field \( \xi_A^{B'}(x) \) can be defined by

\[
\xi_A^{B'} \pi_{B'} \equiv i \ p^{A'B'C'} \pi_{B'} \pi_{C'} \nabla_{AA'} h(x, \pi_{D'}), \tag{5.1.13}
\]

and that the following identities hold:

\[
i \ p^{AA'B'} \pi_{B'} \nabla_{AA'} h(x, \pi_{C'}) = \xi \equiv \frac{1}{2} \xi_A^{A'}, \tag{5.1.14}
\]

\[
\psi_A \ p^{AA'B'} = -\xi^{(A'B')}. \tag{5.1.15}
\]

We may now continue the analysis of our deformed twistor space \( N^D \), written in the form (cf. (5.1.1) and (5.1.2))

\[
\hat{\omega}^A = \omega^A + p^A g(p_B \omega^B, \pi_{B'}), \tag{5.1.16a}
\]

\[
\hat{\pi}_{A'} = \pi_{A'}. \tag{5.1.16b}
\]

In the light of the split (5.1.11), holomorphic sections of \( N^D \) are given by

\[
\omega^A(x^b, \pi_{B'}) = i \ x^{AA'} \pi_{A'} + p^A h(x^b, \pi_{B'}) \text{ in } Q, \tag{5.1.17}
\]
\[ \hat{\omega}(x^b, \pi_{B'}) = i x^{AA'} \pi_A + p^A \hat{h}(x^b, \pi_{B'}) \text{ in } \hat{Q}, \]  
(5.1.18)

where \( x^b \) are complex coordinates on \( G \). The conformal structure of \( G \) can be computed as follows. A vector \( U = U^{BB'} \nabla_{BB'} \) at \( x^a \in G \) may be represented in \( \mathcal{N}^D \) by the displacement

\[ \delta \omega^A = U^h \nabla_b \omega^A(x^c, \pi_{C'}). \]  
(5.1.19a)

By virtue of (5.1.17), Eq. (5.1.19a) becomes

\[ \delta \omega^A = U^{BB'} \left( i \varepsilon^A B \pi_{B'} + p^A \nabla_{BB'} h(x^c, \pi_{C'}) \right). \]  
(5.1.19b)

The vector \( U \) is null, by definition, if and only if

\[ \delta \omega^A(x^b, \pi_{B'}) = 0, \]  
(5.1.20)

for some spinor field \( \pi_{B'} \). To prove that the solution of Eq. (5.1.20) exists, one defines (see (5.1.14))

\[ \theta \equiv 1 - \xi, \]  
(5.1.21)

\[ \Omega_{BB'}^{AA'} \equiv \theta \varepsilon_A^B \varepsilon_{A'}^{B'} - \psi_A p_A^{BB'}. \]  
(5.1.22)

We are now aiming to show that the desired solution of Eq. (5.1.20) is given by

\[ U^{BB'} = \Omega_{BB'}^{AA'} \lambda^A \pi_{A'}. \]  
(5.1.23)

Indeed, by virtue of (5.1.21)–(5.1.23) one finds

\[ U^{BB'} = (1 - \xi) \lambda^B \pi_{B'} - \psi_A p_A^{BB'} \lambda^A \pi_{A'}. \]  
(5.1.24)

Thus, since \( \pi_{B'} \pi_{B'} = 0 \), the calculation of (5.1.19b) yields

\[ \delta \omega^A = -\psi_C \lambda^C \pi^A \left[ i p_A^{BB'} \pi_{B'} + p_A^{BB'} p^A \nabla_{BB'} h(x, \pi) \right] \]
\[ + (1 - \xi) \lambda^B \pi_{B'} p^A \nabla_{BB'} h(x, \pi). \]  
(5.1.25)

Note that (5.1.12) may be used to re-express the second line of (5.1.25). This leads to

\[ \delta \omega^A = -\psi_C \lambda^C \Gamma^A, \]  
(5.1.26)

where

\[ \Gamma^A \equiv \pi^A \left[ i p_A^{BB'} \pi_{B'} + p_A^{BB'} p^A \nabla_{BB'} h(x, \pi) \right] + i(1 - \xi)p^A \]
\[ = -i p_A^{AA'} \pi_A \pi_{B'} + i p^A + p^A \left[ -p^{BB'} A \pi_{A'} \nabla_{BB'} h(x, \pi) - i \xi \right] \]
\[ = \left[ -i + i \xi - i \xi \right] p^A = 0, \]  
(5.1.27)

in the light of (5.1.5) and (5.1.14). Hence the solution of Eq. (5.1.20) is given by (5.1.23).
Such null vectors determine the conformal metric of $\mathcal{G}$. For this purpose, one defines\cite{98}
\begin{equation}
\nu_{A'B'} \equiv \varepsilon_{A'B'} - \xi_{A'B'},
\end{equation}
\begin{equation}
\Lambda \equiv \frac{1}{2} \nu_{A'B'} \nu_{A'B'},
\end{equation}
\begin{equation}
\Sigma_{BB'}^{CC'} \equiv \theta^{-1} \varepsilon_{B'C'} + \Lambda^{-1} \psi_B \nu_{A'C'} \nu_{B'A'}.
\end{equation}
Interestingly, $\Sigma^c_b$ is the inverse of $\Omega^a_b$, since
\begin{equation}
\Omega^b_a \Sigma^c_b = \delta^c_a.
\end{equation}
Indeed, after defining
\begin{equation}
H_{CC'}^{AA'} \equiv \varepsilon_{C'B'} - \psi_B \psi_{B'} \xi_{A'A'},
\end{equation}
\begin{equation}
\Phi_{CC'}^{AA'} = \left[ \theta \Lambda^{-1} H_{CC'}^{AA'} - \Lambda^{-1} \psi_B H_{B'C'}^{CC'} - \theta^{-1} p_{A'C'} \right],
\end{equation}
a detailed calculation shows that
\begin{equation}
\Omega_{BB'}^{AA'} \Sigma_{BB'}^{CC'} = \psi_A \xi_{A'A'} + \psi_{B'B'} \xi_{B'B'}.
\end{equation}
One can now check that the right-hand side of (5.1.34) vanishes (see problem 5.1). Hence (5.1.31) holds. For our anti-self-dual space-time $\mathcal{G}$, the metric $g = g_{ab} dx^a \otimes dx^b$ is such that
\begin{equation}
g_{ab} = \Xi(x) \Sigma^c_a \Sigma_{bc}.
\end{equation}
Two null vectors $U$ and $V$ at $x \in \mathcal{G}$ have, by definition, the form
\begin{equation}
U^{AA'} \equiv \Omega^{AA'}_{BB'} \lambda^B \alpha^B',
\end{equation}
\begin{equation}
V^{AA'} \equiv \Omega^{AA'}_{BB'} \lambda^B \beta^B',
\end{equation}
for some spinors $\lambda^B, \chi^B, \alpha^B, \beta^B$. In the deformed space $\mathcal{N}^D$, $U$ and $V$ correspond to two displacements $\delta_1 \omega^A$ and $\delta_2 \omega^A$ respectively, as in Eq. (5.1.19). If one defines the corresponding skew-symmetric form
\begin{equation}
S_{\pi}(U, V) \equiv \delta_1 \omega_A \delta_2 \omega^A,
\end{equation}
the metric is given by
\begin{equation}
g(U, V) \equiv \left( \alpha^{A'} \beta_A \right) \left( \alpha^{B'} \pi_{B'} \right)^{-1} \left( \beta^{C'} \pi_{C'} \right)^{-1} \Sigma_{\pi}(U, V).
\end{equation}
However, in the light of (5.1.31), (5.1.35)--(5.1.37) one finds
\begin{equation}
g(U, V) = g_{ab} U^a V^b = \Xi(x) \left( \lambda^A \chi_A \right) \left( \alpha^{A'} \beta_A \right).
\end{equation}
By comparison with (5.1.39) this leads to
\begin{equation}
S_{\pi}(U, V) = \Xi(x) \left( \lambda^A \chi_A \right) \left( \alpha^{B'} \pi_{B'} \right) \left( \beta^{C'} \pi_{C'} \right).
\end{equation}
If we now evaluate (5.1.41) with $\beta^A = \alpha^A$, comparison with the definition (5.1.38) and use of (5.1.12), (5.1.13), (5.1.19b) and (5.1.36) yield

$$\Xi = \Lambda.$$  \hspace{1cm} (5.1.42)

The anti-self-dual solution of Einstein’s equations is thus given by (5.1.30), (5.1.35) and (5.1.42).

The construction of an anti-self-dual space-time described in this section is a particular example of the so-called non-linear graviton [60], [61]. In mathematical language, if $\mathcal{M}$ is a complex three-manifold, $\mathcal{B}$ is the bundle of holomorphic three-forms on $\mathcal{M}$ and $\mathcal{H}$ is the standard positive line bundle on $\mathbb{P}^1$, a non-linear graviton is the following set of data [31]:

(i) $\mathcal{M}$, the total space of a holomorphic fibration $\pi : \mathcal{M} \to \mathbb{P}^1$;

(ii) a four-parameter family of sections, each having $\mathcal{H} \oplus \mathcal{H}$ as normal bundle (see e.g. [33] for the definition of normal bundle);

(iii) a non-vanishing holomorphic section $s$ of $\mathcal{B} \otimes \pi^* \mathcal{H}^4$, where $\mathcal{H}^4 \equiv \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, and $\pi^* \mathcal{H}^4$ denotes the pull-back of $\mathcal{H}^4$ by $\pi$;

(iv) a real structure on $\mathcal{M}$ such that $\pi$ and $s$ are real. $\mathcal{M}$ is then fibred from the real sections of the family.

5.2. Beyond anti-self-duality

The limit of the analysis performed in Sec. 5.1 is that it deals with a class of solutions of (complex) Einstein equations which is not sufficiently general. In [108] and [109] the authors have examined in detail the limits of the anti-self-dual analysis. The two main criticisms are as follows:

(a) a right-flat space-time (cf. the analysis in [38]) does not represent a real Lorentzian space-time manifold. Hence it cannot be applied directly to classical gravity [100];

(b) there are reasons for expecting that the equations of a quantum theory of gravity are much more complicated, and thus are not solved by right-flat space-times.

However, an alternative approach due to LeBrun has become available in the eighties [40]. LeBrun’s approach focuses on the space $G$ of complex null geodesics of complex space-time $(\mathcal{M}, g)$, called ambitwistor space. Thus, one deals with a standard rank-2 holomorphic vector bundle $E \to G$, and in the conformal class determined by the complex structure of $G$, a one-to-one correspondence exists between non-vanishing holomorphic sections of $E$ and Einstein metrics on $(\mathcal{M}, g)$ [40]. The bundle $E$ is called Einstein bundle, and has also been studied in [14]. The work by Eastwood adds evidence in favour of the Einstein bundle being the correct generalization of the non-linear-graviton construction to the non-right-flat case (cf. [38], [52], [43], [53]). Indeed, the theorems discussed so far provide a characterization of the vacuum Einstein equations. However, there is not yet an independent way
of recognizing the Einstein bundle. Thus, this is not yet a substantial progress in solving the vacuum equations. Other relevant work on holomorphic ideas appears in [41], where the author proves that, in the case of four-manifolds with self-dual Weyl curvature, solutions of the Yang–Mills equations correspond to holomorphic bundles on an associated analytic space (cf. [97], [104], [101]).

5.3. Twistors as spin-$\frac{3}{2}$ charges

In this section, we describe a proposal by Penrose to regard twistors for Ricci-flat space-times as (conserved) charges for massless helicity-$\frac{3}{2}$ fields ([71], [72], [73], [74]). The new approach proposed by Penrose is based on the following mathematical results [73]:

(i) A vanishing Ricci tensor provides the consistency condition for the existence and propagation of massless helicity-$\frac{3}{2}$ fields in curved space-time ([11], [13]);

(ii) In Minkowski space-time, the space of charges for such fields is naturally identified with the corresponding twistor space.

Thus, Penrose points out that if one could find the appropriate definition of charge for massless helicity-$\frac{3}{2}$ fields in a Ricci-flat space-time, this should provide the concept of twistor appropriate for vacuum Einstein equations. The corresponding geometric program may be summarized as follows:

1. Define a twistor for Ricci-flat space-time $(M, g)_{RF}$;
2. Characterize the resulting twistor space $\mathcal{F}$;
3. Reconstruct $(M, g)_{RF}$ from $\mathcal{F}$.

We now describe, following [71], [72], [73], [74], properties and problems of this approach to twistor theory in flat and in curved space-times.

5.3.1 Massless spin-$\frac{3}{2}$ equations in Minkowski space-time

Let $(M, \eta)$ be Minkowski space-time with flat connection $\mathcal{D}$. In $(M, \eta)$ the gauge-invariant field strength for spin $\frac{3}{2}$ is represented by a totally symmetric spinor field

$$\psi_{A'B'C'} = \psi_{(A'B'C')}$$

obeying a massless free-field equation

$$\mathcal{D}^{AA'} \psi_{A'B'C'} = 0.$$  \hspace{1cm} (5.3.2)

With the conventions of Penrose, $\psi_{A'B'C'}$ describes spin-$\frac{3}{2}$ particles of helicity equal to $\frac{3}{2}$ (rather than $-\frac{3}{2}$). The Dirac form of this field strength is obtained by expressing locally $\psi_{A'B'C'}$ in terms of two potentials subject to gauge freedoms involving a primed and an unprimed spinor field. The first potential is a spinor field symmetric
in its primed indices
\[ \gamma_{B'C'}^A = \gamma_{(B'C')}^A, \]  
subject to the differential equation
\[ D^{BB'} \gamma_{B'C'}^A = 0, \]  
and such that
\[ \psi_{A'B'C'} = D_{AA'} \gamma_{B'C'}^A. \]  
The second potential is a spinor field symmetric in its unprimed indices
\[ \rho_{(AB)}^{C'} = \rho_{AB}^{C'}, \]  
subject to the equation
\[ D^{CC'} \rho_{AB}^{C'} = 0, \]  
and it yields the \[ \gamma_{B'C'}^A \] potential by means of
\[ \gamma_{B'C'}^A = D_{BB'} \rho_{C'}^{AB}. \]  
If we introduce the spinor fields \[ \nu_{C'} \] and \[ \chi^B \] obeying the equations
\[ D^{AC'} \nu_{C'} = 0, \]  
\[ D_{AC'} \chi^A = 2i \nu_{C'}, \]  
the gauge freedoms for the two potentials enable one to replace them by the potentials
\[ \tilde{\gamma}_{B'C'}^A = \gamma_{B'C'}^A + D_{B'B}^A \nu_{C'}, \]  
\[ \tilde{\rho}_{C'}^{AB} = \rho_{C'}^{AB} + \varepsilon^{AB} \nu_{C'} + i D_{C'}^A \chi^B, \]  
without affecting the theory. Note that the right-hand side of (5.3.12) does not contain antisymmetric parts since, despite the explicit occurrence of the antisymmetric \[ \varepsilon^{AB}, \] one finds
\[ D_{C'[A}^A \chi_{B]} = \frac{\varepsilon^{AB}}{2} D_{LC'} \chi^L = i \varepsilon^{AB} \nu_{C'}, \]  
by virtue of (5.3.10). Hence (5.3.13) leads to
\[ \tilde{\rho}_{C'}^{AB} = \rho_{C'}^{AB} + i D_{C'}(A \chi^B). \]  
The gauge freedoms are indeed given by Eqs. (5.3.11) and (5.3.12) since in our flat space-time one finds
\[ D^{AA'} \tilde{\gamma}_{A'B'}^C = D^{AA'} D^{C'B'}_{(B'} \nu_{A')} = D^{C'B'}_{(B'} D^{AA'}_{(A')} \nu_{A')} = 0, \]  
by virtue of (5.3.4) and (5.3.9), and
\[ D^{AA'} \tilde{\rho}_{A'B'}^{BC} = D^{AA'} D^{C'B} A' \chi_B = D^{C'A'} D_{A'}^A \chi_B \]
which implies
\[ \mathcal{D}^{AA'} \hat{\rho}^{BC} A = 0. \] (5.3.16b)

The result (5.3.16b) is a particular case of the application of spinor Ricci identities to flat space-time.

We are now in a position to show that twistors can be regarded as charges for helicity-\( \frac{3}{2} \) massless fields in Minkowski space-time. For this purpose, following [72], [74] let us suppose that the field \( \psi \) satisfying (5.3.1) and (5.3.2) exists in a region \( \mathcal{R} \) of \((M, \eta)\), surrounding a world-tube which contains the sources for \( \psi \). Moreover, we consider a two-sphere \( S \) within \( \mathcal{R} \) surrounding the world-tube. To achieve this we begin by taking a dual twistor, i.e. the pair of spinor fields
\[ W_\alpha \equiv (\lambda_A, \mu_A'), \] (5.3.17)
obeying the differential equations
\[ \mathcal{D}_{AA'} \mu^{B'} = i \varepsilon_{A'B'} \lambda_A, \] (5.3.18)
\[ \mathcal{D}_{AA'} \lambda_B = 0. \] (5.3.19)

Hence \( \mu^{B'} \) is a solution of the complex-conjugate twistor equation
\[ \mathcal{D}^{(A'} \mu^{B')} = 0. \] (5.3.20)

Thus, if one defines
\[ \varphi_{A'B'} \equiv \psi_{A'B'C'} \mu^{C'}, \] (5.3.21)
one finds, by virtue of (5.3.1), (5.3.2) and (5.3.20), that \( \varphi_{A'B'} \) is a solution of the self-dual vacuum Maxwell equations
\[ \mathcal{D}^{AA'} \varphi_{A'B'} = 0. \] (5.3.22)

Note that (5.3.21) is a particular case of the spin-lowering procedure [33], [68]. Moreover, \( \varphi_{A'B'} \) enables one to define the self-dual two-form
\[ F \equiv \varphi_{A'B'} \, dx_A' \wedge dx^{AB'}, \] (5.3.23)
which leads to the following charge assigned to the world-tube:
\[ Q \equiv \frac{i}{4\pi} \oint F. \] (5.3.24)

For some twistor
\[ Z_\alpha \equiv (\omega^A, \pi_{A'}), \] (5.3.25)
the charge \( Q \) depends on the dual twistor \( W_\alpha \) as
\[ Q = Z_\alpha W_\alpha = \omega^A \lambda_A + \pi_{A'} \mu^{A'}. \] (5.3.26)
These equations describe the strength of the charge, for the field $\psi$, that should be assigned to the world-tube. Thus, a twistor $Z^\alpha$ arises naturally in Minkowski space-time as the charge for a helicity $+\frac{3}{2}$ massless field, whereas a dual twistor $W_\alpha$ is the charge for a helicity $-\frac{3}{2}$ massless field [74].

Interestingly, the potentials $\gamma^{C}_{A'B'}$ and $\rho^{BC}_{A'}$ can be used to obtain a potential for the self-dual Maxwell field strength, since, after defining

$$\theta^{C}_{A'} \equiv \gamma^{C}_{A'B'} \mu^{B'} - i \rho^{BC}_{A'} \lambda_{B}, \quad (5.3.27)$$

one finds

$$D_{CB'} \theta^{C}_{A'} = \left(D_{CB'} \gamma^{C}_{A'D'}\right) \mu^{D'} + \gamma^{C}_{A'D'} \left(D_{CB'} \mu^{D'}\right) - i \left(D_{CB'} \rho^{BC}_{A'}\right) \lambda_{B}$$

$$= \psi_{A'B'D'} \mu^{D'} + i \varepsilon_{B'C'} \gamma^{C}_{A'D'} \lambda_{C} - i \gamma^{C}_{1A'B'} \lambda_{C}$$

$$= \psi_{A'B'D'} \mu^{D'} = \varphi_{A'B'}, \quad (5.3.28)$$

$$D_{B'} A' \theta^{C}_{A'} = \left(D_{B'} \gamma^{C}_{A'B'}\right) \mu^{B'} + \gamma^{C}_{A'B'} \left(D_{B'} \mu^{B'}\right) - i \left(D_{B'} \rho^{BC}_{A'}\right) \lambda_{D}$$

$$- i \rho^{DC}_{A'} \left(D_{B'} \lambda_{D}\right) = 0. \quad (5.3.29)$$

Eq. (5.3.28) has been obtained by using (5.3.5), (5.3.8), (5.3.18) and (5.3.19), whereas (5.3.29) holds by virtue of (5.3.3), (5.3.4), (5.3.7), (5.3.18) and (5.3.19). The one-form corresponding to $\theta^{C}_{A'}$ is defined by

$$A \equiv \theta_{B'B'} \; dx^{B'B'}, \quad (5.3.30)$$

which leads to

$$F = 2 \; dA, \quad (5.3.31)$$

by using (5.3.23) and (5.3.28).

The Rarita–Schwinger form of the field strength does not require the symmetry (5.3.3) in $B'C'$ as we have done so far, and the $\gamma^{A}_{B'C'}$ potential is instead subject to the equations [72], [73], [73]

$$\varepsilon^{B'C'} \; D_{A'(A' B'C')} = 0, \quad (5.3.32)$$

$$D^{B'} (B \gamma^{A}_{B'C'}) = 0. \quad (5.3.33)$$

Moreover, the spinor field $\nu^{C'}_{C}$ in (5.3.11) is no longer taken to be a solution of the Weyl equation (5.3.9).

The potentials $\gamma$ and $\rho$ may or may not be global over $S$. If $\gamma$ is global but $\rho$ is not, one obtains a two-dimensional complex vector space parametrized by the spinor field $\pi_{A'}$. The corresponding subspace where $\pi_{A'} = 0$, parametrized by $\omega^{A}$, is called $\omega$-space. Thus, following [74], we regard $\pi$-space and $\omega$-space as quotient spaces defined as follows:

$$\pi - \text{space} \equiv \text{space of global } \psi/\text{s}\text{/space of global } \gamma/\text{s}, \quad (5.3.34)$$
\[
\omega - \text{space} \equiv \text{space of global } \gamma' s/\text{space of global } \rho' s.
\] (5.3.35)

### 5.3.2 Massless spin-\(3\over2\) field strengths in curved space-time

The conditions for the local existence of the \(\rho_{A'}^{BC}\) potential in curved space-time are derived by requiring that, after the gauge transformation (5.3.12) (or, equivalently, (5.3.14)), also the \(\hat{\rho}_{A'}^{BC}\) potential should obey the equation

\[
\nabla^{AA'} \hat{\rho}_{A'}^{BC} = 0, \tag{5.3.36}
\]

where \(\nabla\) is the curved connection. By virtue of the spinor Ricci identity \([103]\)

\[
\nabla_{M'}(A \nabla_{M'_B}) \chi_C = \psi_{ABDC} \chi^D - 2\Lambda \chi_{(A} \varepsilon_{B)C}, \tag{5.3.37}
\]

the insertion of (5.3.14) into (5.3.36) yields, assuming for simplicity that \(\nu_{C'} = 0\) in (5.3.10), the following conditions:

\[
\psi_{ABCD} = 0, \quad \Lambda = 0, \tag{5.3.38}
\]

which imply we deal with a vacuum self-dual (or left-flat) space-time, since the anti-self-dual Weyl spinor has to vanish \([74]\).

Moreover, in a complex anti-self-dual vacuum space-time one finds \([74]\) that spin-\(3\over2\) field strengths \(\psi_{A'B'C'}\) can be defined according to (cf. (5.3.5))

\[
\psi_{A'B'C'} = \nabla_{C'C'} \psi_{A'B'C'}, \tag{5.3.39}
\]

are gauge-invariant, totally symmetric, and satisfy the massless free-field equations (cf. (5.3.2))

\[
\nabla^{AA'} \psi_{A'B'C'} = 0. \tag{5.3.40}
\]

In this case there is no obstruction to defining global \(\psi\)-fields with non-vanishing \(\pi\)-charge, and a global \(\pi\)-space can be defined as in (5.3.34). It remains to be seen whether the twistor space defined by \(\alpha\)-surfaces may then be reconstructed (Sec. 4.2, \([60]\), \([61]\), \([103]\), \([74]\)).

Interestingly, in \([73]\) it has been proposed to interpret the potential \(\gamma\) as providing a bundle connection. In other words, one takes the fibre coordinates to be given by a spinor \(\eta_{A'}\) and a scalar \(\mu\). For a given small \(\epsilon\), one extends the ordinary Levi-Civita connection \(\nabla\) on \(M\) to bundle-valued quantities according to \([73]\)

\[
\nabla_{PP'} \left( \eta_{A'} \mu \right) = \left( \nabla_{PP'} \eta_{A'} \mu \right) - \epsilon \left( 0 \quad \gamma_{PP'A'} \gamma_{PP'} \mu \quad 0 \right) \left( \eta_{B'} \mu \right), \tag{5.3.41}
\]

with gauge transformations given by

\[
\left( \hat{\eta}_{A'} \hat{\mu} \right) = \left( \hat{\eta}_{A'} \mu \right) + \epsilon \left( 0 \quad \nu_{A'} \mu \quad 0 \right) \left( \hat{\eta}_{B'} \mu \right). \tag{5.3.42}
\]
Note that terms of order $\epsilon^2$ have been neglected in writing (5.3.42). However, such gauge transformations do not close under commutation, and to obtain a theory valid to all orders in $\epsilon$ one has to generalize to $SL(3, C)$ matrices before the commutators close. Writing $(A)$ for the three-dimensional indices, so that $\eta_{(A)}$ denotes $\left( \eta_{\alpha}^A \right)$, one has a connection defined by
\[
\nabla_{P, P'} \eta_{(A)} \equiv \left( \nabla_{P, P'} \eta_{\alpha}^A \right) - \gamma_{P, P'}^{(B)} \eta_{(B)}, \tag{5.3.43}
\]
with gauge transformation
\[
\hat{\eta}_{(A)} \equiv \eta_{(A)} + \nu_{(A)}^{(B)} \eta_{(B)}. \tag{5.3.44}
\]
With this notation, the $\nu_{(A)}^{(B)}$ are $SL(3, C)$-valued fields on $M$, and hence
\[
\mathcal{E}^{(P)} \mathcal{E}^{(Q)} = \nu_{(P)}^{(A)} \nu_{(Q)}^{(B)} \nu_{(R)}^{(C)} = \mathcal{E}^{(A)} \mathcal{E}^{(B)} \mathcal{E}^{(C)}, \tag{5.3.45}
\]
where $\mathcal{E}^{(P)} \mathcal{E}^{(Q)}$ are generalized Levi–Civita symbols. The $SL(3, C)$ definition of $\gamma$-potentials takes the form [73]
\[
\gamma_{P, P'}^{(B)} \equiv \left( \frac{\alpha_{P, P'}{A}^{B'}}{\gamma_{P, P'}{B'}}, \frac{\beta_{P, P'}{A'}^{B'}}{\delta_{P, P'}}, \frac{\epsilon_{P, P'}}{\delta_{P, P'}} \right), \tag{5.3.46}
\]
while the curvature is
\[
K_{pq}^{(B)} \equiv 2\nabla_{[p} \gamma_{q]}^{(B)} + 2 \gamma_{[p}^{(C)} \gamma^{q(C)} = 0, \tag{5.3.47}
\]
Penrose has proposed this as a generalization of the Rarita–Schwinger structure in Ricci-flat space-times, and he has even speculated that a non-linear generalization of the Rarita–Schwinger equations (5.3.32) and (5.3.33) might be
\[
(-)K_{pq}^{(B)} = 0, \tag{5.3.48}
\]
\[
(+)K_{pq}^{(B)} \mathcal{E}^{p(A)} \mathcal{E}^{q(A)} = 0, \tag{5.3.49}
\]
where $(-)K$ and $(+)K$ are the anti-self-dual and self-dual parts of the curvature respectively, i.e.
\[
K_{pq}^{(B)} = \varepsilon_{pq}^{(B)} (-)K_{pq}^{(B)} + \varepsilon_{pq}^{(B)} (+)K_{pq}^{(B)}. \tag{5.3.50}
\]
Following [73], one has
\[
\mathcal{E}^{p(A)} \mathcal{E}^{q(A)} \equiv \mathcal{E}^{(P)} \mathcal{E}^{(Q)} \epsilon_{(P)}^{(Q)}, \tag{5.3.51}
\]
\[
\mathcal{E}^{q(X)} \mathcal{E}^{r(X)} \equiv \mathcal{E}^{(P)} \mathcal{E}^{(Q)} \epsilon_{(P)}^{(Q)}, \tag{5.3.52}
\]
the $\epsilon_{(P)}^{(Q)}$ relating the bundle directions with tangent directions in $M$. 

6. The Plebanski Contributions

The analysis of (conformally) right-flat space-times of the previous sections has its counterpart in the theory of heaven spaces developed by Plebanski. This section reviews weak heaven spaces, strong heaven spaces, heavenly te trads and heavenly equations.

6.1. Outline

One of the most recurring themes of this paper is the analysis of complex or real Riemannian manifolds where half of the conformal curvature vanishes and the vacuum Einstein equations hold. Section 5 has provided an explicit construction of such anti-self-dual space-times, and the underlying Penrose-transform theory has been presented in Sec. 4. However, alternative ways exist to construct these solutions of the Einstein equations, and hence this section supplements the previous sections by describing the work in [78]. By using the tetrad formalism and some basic results in the theory of partial differential equations, the so-called heaven spaces and heavenly tetrads are defined and constructed in detail.

6.2. Heaven spaces

In his theory of heaven spaces, Plebanski studies a four-dimensional analytic manifold $M_4$ with metric given in terms of tetrad vectors as [78]

$$ g = 2e^1 e^2 + 2e^3 e^4 = g_{ab} e^a e^b \in \Lambda^1 \otimes \Lambda^1. \quad (6.2.1) $$

The definition of the $2 \times 2$ matrices

$$ \tau_{AB}' = \sqrt{2} \begin{pmatrix} e^4 \\ e^1 \\ -e^3 \end{pmatrix} \quad (6.2.2) $$

enables one to re-express the metric as

$$ g = -\det \tau_{AB}' = \frac{1}{2} \varepsilon_{AB} \varepsilon_{C'D'} \tau^{AC'} \tau^{BD'}. \quad (6.2.3) $$

Moreover, since the manifold is analytic, there exist two independent sets of $2 \times 2$ complex matrices with unit determinant: $L_A' \in SL(2, C)$ and $\tilde{L}_B' \in \tilde{SL}(2, C)$. On defining a new set of tetrad vectors such that

$$ \sqrt{2} \begin{pmatrix} e^{3'} \\ e^{1'} \\ -e^{4'} \end{pmatrix} = L_A' L_B' \tau_{AB}', \quad (6.2.4) $$

the metric is still obtained as $2e^{1'} e^{2'} + 2e^{3'} e^{4'}$. Hence the tetrad gauge group may be viewed as

$$ G \equiv SL(2, C) \times \tilde{SL}(2, C). \quad (6.2.5) $$

A key role in the following analysis is played by a pair of differential forms whose spinorial version is obtained from the wedge product of the matrices in (6.2.2), i.e.

$$ \tau_{AB}' \wedge \tau^{CD'} = S^{AC} \varepsilon_B D' + \varepsilon^{AC} \tilde{S}^{BD'}. \quad (6.2.6) $$
where

\[ S^{AB} \equiv \frac{1}{2} \varepsilon_{RS}^{S'} \tau^{AR'} \wedge \tau^{BS'} = \frac{1}{2} e^a \wedge e^b S_{ab}^{AB}, \quad (6.2.7) \]

\[ \tilde{S}^{A'B'} \equiv \frac{1}{2} \varepsilon_{RS} \tau^{RA'} \wedge \tau^{SB'} = \frac{1}{2} e^a \wedge e^b \tilde{S}_{ab}^{A'B'}. \quad (6.2.8) \]

The forms \( S^{AB} \) and \( \tilde{S}^{A'B'} \) are self-dual and anti-self-dual respectively, in that the action of the Hodge-star operator on them leads to [78]

\[ *S^{AB} = S^{AB}, \quad (6.2.9) \]

\[ *\tilde{S}^{A'B'} = -\tilde{S}^{A'B'}. \quad (6.2.10) \]

To obtain the desired spinor description of the curvature, we introduce the anti-symmetric connection forms \( \Gamma_{ab} = \Gamma_{[ab]} \) through the first structure equations

\[ de^a = e^b \wedge \Gamma^a_{\ b}. \quad (6.2.11) \]

The spinorial counterpart of \( \Gamma_{ab} \) is given by

\[ \Gamma_{AB} \equiv -\frac{1}{4} \Gamma_{ab} S_{ab}^{AB}, \quad (6.2.12) \]

\[ \Gamma_{A'B'} \equiv -\frac{1}{4} \Gamma_{ab} \tilde{S}_{ab}^{A'B'}, \quad (6.2.13) \]

which implies

\[ \Gamma_{ab} = -\frac{1}{2} S_{ab}^{AB} \Gamma_{AB} - \frac{1}{2} \tilde{S}_{ab}^{A'B'} \Gamma_{A'B'}. \quad (6.2.14) \]

To appreciate that \( \Gamma_{AB} \) and \( \Gamma_{A'B'} \) are actually independent, the reader may find it useful to check that [78]

\[ \Gamma_{AB} = -\frac{1}{2} \begin{pmatrix} 2\Gamma_{42} & \Gamma_{12} + \Gamma_{34} \\ \Gamma_{12} + \Gamma_{34} & 2\Gamma_{31} \end{pmatrix}, \quad (6.2.15) \]

\[ \Gamma_{A'B'} = -\frac{1}{2} \begin{pmatrix} 2\Gamma_{41} & -\Gamma_{12} + \Gamma_{34} \\ -\Gamma_{12} + \Gamma_{34} & 2\Gamma_{32} \end{pmatrix}. \quad (6.2.16) \]

The action of exterior differentiation on \( \tau^{AB'}, S^{AB}, \tilde{S}^{A'B'} \) shows that

\[ d\tau^{AB'} = \tau^{AL'} \wedge \Gamma_{BL'}^{L'} + \tau^{LB'} \wedge \Gamma_{AL}^{L}, \quad (6.2.17) \]

\[ dS^{AB} = -3S^{(AB} \Gamma_{C)}^{C}, \quad (6.2.18) \]

\[ d\tilde{S}^{A'B'} = -3\tilde{S}^{(A'B'} \Gamma^{C')}_{C')}, \quad (6.2.19) \]

and two independent curvature forms are obtained as

\[ R^A_B \equiv d\Gamma^A_B + \Gamma^A_L \wedge \Gamma^L_B = \frac{1}{2} \psi^A_{BCD} S^{CD}. \]
$$+ \frac{R}{24} S_{AB} + \frac{1}{2} \Phi_{BCD'} \tilde{S}^{C'D'}, \quad (6.2.20)$$

$$\tilde{R}_{A'B'} \equiv d\tilde{\Gamma}_{A'B'} + \tilde{\Gamma}_{A'L'} \wedge \tilde{\Gamma}_{L'B'} = -\frac{1}{2} \tilde{\psi}_{B'C'D'} \tilde{S}^{C'D'}$$

$$+ \frac{R}{24} \tilde{S}_{A'B'} + \frac{1}{2} \Phi_{CD} A'_{B'} S^{CD}. \quad (6.2.21)$$

The spinors and scalars in (6.2.20) and (6.2.21) have the same meaning as in the previous sections. With the conventions in [78], the Weyl spinors are obtained as

$$\psi_{ABCD} = \frac{1}{16} S_{AB} C_{abcd} S_{CD} = \psi_{(ABCD)}, \quad (6.2.22)$$

$$\tilde{\psi}_{A'B'C'D'} = \frac{1}{16} \tilde{S}_{A'B'} C_{abcd} \tilde{S}_{C'D'} = \tilde{\psi}_{(A'B'C'D')}, \quad (6.2.23)$$

and conversely the Weyl tensor is

$$C_{abcd} = \frac{1}{4} S_{AB} \psi_{ABCD} S_{CD} + \frac{1}{4} \tilde{S}_{A'B'} \tilde{\psi}_{A'B'C'D'} \tilde{S}_{C'D'}. \quad (6.2.24)$$

The spinor version of the Petrov classification (Sec. 2.3) is hence obtained by stating that $k^A$ and $\omega^A$ are the two types of P-spinors if and only if the independent conditions hold:

$$\psi_{ABCD} k^A k^B k^C k^D = 0, \quad (6.2.25)$$

$$\tilde{\psi}_{A'B'C'D'} \omega^A \omega^B \omega^C \omega^D = 0. \quad (6.2.26)$$

For our purposes, we can omit the details about the principal null directions, and focus instead on the classification of spinor fields and analytic manifolds under consideration. Indeed, Plebanski proposed to call all objects which are $\tilde{S}L(2,C)$ scalars and are geometric objects with respect to $S\tilde{L}(2,C)$, the heavenly objects (e.g. $S^{AB}, \Gamma_{AB}, \psi_{ABCD}$). Similarly, objects which are $SL(2,C)$ scalars and behave like geometric objects with respect to $\tilde{S}\tilde{L}(2,C)$ belong to the complementary world, i.e. the set of hellish objects (e.g. $\tilde{S}^{A'B'}, \tilde{\Gamma}_{A'B'}, \tilde{\psi}_{A'B'C'D'}$). Last, spinor fields with (abstract) indices belonging to both primed and unprimed spin-spaces are the earthly objects.

With the terminology of Plebanski, a weak heaven space is defined by the condition

$$\tilde{\psi}_{A'B'C'D'} = 0, \quad (6.2.27)$$

and corresponds to the conformally right-flat space of Sec. 3. Moreover, a strong heaven space is a four-dimensional analytic manifold where a choice of null tetrad exists such that

$$\tilde{\Gamma}_{A'B'} = 0. \quad (6.2.28)$$
One then has *a fortiori*, by virtue of (6.2.21), the conditions [78]
\[
\tilde{\psi}_{A'B'C'D'} = 0, \quad \Phi_{ABC'D'} = 0, \quad R = 0.
\]  
(6.2.29)

The vacuum Einstein equations are then automatically fulfilled in a strong heaven space, which turns out to be a right-flat space-time in modern language. Of course, strong heaven spaces are non-trivial if and only if the anti-self-dual Weyl spinor \(\psi_{ABCD}\) does not vanish, otherwise they reduce to flat four-dimensional space-time.

### 6.3. First heavenly equation

A space which is a strong heaven according to (6.2.28) is characterized by a key function \(\Omega\) which obeys the so-called first heavenly equation. The basic ideas are as follows. In the light of (6.2.19) and (6.2.28), \(d\tilde{S}'\) vanishes, and hence, in a simply connected region, an element \(U_{A'B'}\) of the bundle \(\Lambda^1\) exists such that locally
\[
\tilde{S}' = dU_{A'B'}.
\]  
(6.3.1)

Thus, since
\[
\tilde{S}'_{11'} = 2e^4 \wedge e^1, \\
\tilde{S}'_{22'} = 2e^3 \wedge e^2, \\
\tilde{S}'_{33'} = -e^1 \wedge e^2 + e^3 \wedge e^4,
\]  
(6.3.2-6.3.4)

Equation (6.3.1) leads to
\[
2e^4 \wedge e^1 = dU_{11'}, \\
2e^3 \wedge e^2 = dU_{22'}.
\]  
(6.3.5-6.3.6)

Now the Darboux theorem holds in our complex manifold, and hence scalar functions \(p, q, r, s\) exist such that
\[
2e^4 \wedge e^1 = 2dp \wedge dq = 2d(p \, dq + d\tau), \\
2e^3 \wedge e^2 = 2dr \wedge ds = 2d(r \, ds + d\sigma), \\
e^1 \wedge e^2 \wedge e^3 \wedge e^4 = dp \wedge dq \wedge dr \wedge ds.
\]  
(6.3.7-6.3.9)

The form of the *heavenly tetrad* in these coordinates is
\[
e^1 = Adp + B \, dq, \\
e^2 = G \, dr + H \, ds, \\
e^3 = E \, dr + F \, ds, \\
e^4 = -C \, dp - D \, dq.
\]  
(6.3.10-6.3.13)
If one now inserts (6.3.10)–(6.3.13) into (6.3.7)–(6.3.9), one finds that
\[ AD - BC = EH - FG = 1, \]  
(6.3.14)
which is supplemented by a set of equations resulting from the condition \( dS_1^2 = 0 \). These equations imply the existence of a function, the first key function, such that
\[ AG - CE = \Omega_{pr}, \]  
(6.3.15)
\[ BG - DE = \Omega_{qr}, \]  
(6.3.16)
\[ AH - CF = \Omega_{ps}, \]  
(6.3.17)
\[ BH - DF = \Omega_{qs}. \]  
(6.3.18)
Thus, \( E, F, G, H \) are given by
\[ E = B \Omega_{pr} - A \Omega_{qr}, \]  
(6.3.19)
\[ F = B \Omega_{ps} - A \Omega_{qs}, \]  
(6.3.20)
\[ G = D \Omega_{pr} - C \Omega_{qr}, \]  
(6.3.21)
\[ H = D \Omega_{ps} - C \Omega_{qs}. \]  
(6.3.22)
The request of compatibility of (6.3.19)–(6.3.22) with (6.3.14) leads to the first heavenly equation
\[ \det \begin{pmatrix} \Omega_{pr} & \Omega_{ps} \\ \Omega_{qr} & \Omega_{qs} \end{pmatrix} = 1. \]  
(6.3.23)

6.4. Second heavenly equation

A more convenient description of the heavenly tetrad is obtained by introducing the coordinates
\[ x \equiv \Omega_p, \ y \equiv \Omega_q, \]  
(6.4.1)
and then defining
\[ A \equiv -\Omega_{pp}, \ B \equiv -\Omega_{pq}, \ C \equiv -\Omega_{qq}. \]  
(6.4.2)
The corresponding heavenly tetrad reads [78]
\[ e^1 = dp, \]  
(6.4.3)
\[ e^2 = dx + A dp + B dq, \]  
(6.4.4)
\[ e^3 = -dy - B dp - C dq. \]  
(6.4.5)
\[ e^4 = -dq. \quad (6.4.6) \]

Now the closure condition for \( \tilde{S}^{}_{y'z'} : d\tilde{S}^{}_{y'z'} = 0 \), leads to the equations

\[ A_x + B_y = 0, \quad (6.4.7) \]
\[ B_x + C_y = 0, \quad (6.4.8) \]
\[ \left( AC - B^2 \right)_x + B_q - C_p = 0, \quad (6.4.9) \]
\[ \left( AC - B^2 \right)_y - A_q + B_p = 0. \quad (6.4.10) \]

By virtue of (6.4.7) and (6.4.8), a function \( \theta \) exists such that

\[ A = -\theta_{yy}, \quad B = \theta_{xy}, \quad C = -\theta_{xx}. \quad (6.4.11) \]

On inserting (6.4.11) into (6.4.9) and (6.4.10) one finds

\[ \partial_w \left( \theta_{xx} \theta_{yy} - \theta_{x}^{2} + \theta_{xp} + \theta_{yq} \right) = 0, \quad (6.4.12) \]

where \( w = x, y \). Thus, one can write that

\[ \theta_{xx} \theta_{yy} - \theta_{x}^{2} + \theta_{xp} + \theta_{yq} = f_p(p,q), \quad (6.4.13) \]

where \( f \) is an arbitrary function of \( p \) and \( q \). This suggests defining the function

\[ \Theta \equiv \theta - xf, \quad (6.4.14) \]

which implies

\[ f_p = \Theta_{xx} \Theta_{yy} - \Theta_{x}^{2} + \Theta_{xp} + \Theta_{yq} + f_p, \]

and hence

\[ \Theta_{xx} \Theta_{yy} - \Theta_{x}^{2} + \Theta_{xp} + \Theta_{yq} = 0. \quad (6.4.15) \]

Equation (6.4.15) ensures that all forms \( \tilde{S}^{}_{A'B'} \) are closed, and is called the second heavenly equation. Plebanski was able to find heavenly metrics of all possible algebraically degenerate types. An example is given by the function

\[ \Theta \equiv \frac{\beta}{2\alpha(\alpha - 1)} x^\alpha y^{1-\alpha}. \quad (6.4.16) \]

The reader may check that such a solution is of the type \([2 - 2] \otimes [-] \) if \( \alpha = -1, 2 \), and is of the type \([2 - 1 - 1] \otimes [-] \) whenever \( \alpha \neq -1, 2 \) [78]. More work on related topics and on yet other ideas in complex general relativity can be found in [79], [19], [50], [80], [35], [10], [28], [91], [92], [20], [36], [87], [15], [42], [7], [81], [82], [83], [95].
7. Concluding Remarks

It has been our aim to give a pedagogical introduction to twistor theory, with emphasis on the topics and methods that a general relativist is more familiar with. Much more material can be found, for example, in [18], especially spin-3/2 potentials ([1], [23], [47], [24], [25]) and various definitions of twistors in curved space-time [59]. Moreover we should acknowledge that complex (Riemannian) manifolds have been investigated from the point of view of the corresponding real structure in [8], [9]. These are the so-called Norden–Kähler or anti-Kähler manifolds. With spinor notation, they are treated in [84], [85]. Older references on this subject which deserve mention are [26], [34], [39], [21], [22], while very recent results can be found in [49], [89].

For a long time the physics community has thought that twistor theory is more likely to contribute to mathematics, e.g. powerful geometric methods for solving non-linear partial differential equations ([102], [93], [44], [75], [94], [51], [76]). However, the work of Penrose on spin-3/2 potentials ([74], [75]) and the work in [105] on perturbative gauge theory has changed a lot the overall perspectives. In particular, Witten points out that perturbative scattering amplitudes in Yang–Mills theories have remarkable properties such as holomorphy of the maximally helicity violating amplitudes. To interpret this result, he considers the Fourier transform of scattering amplitudes from momentum space to twistor space, and finds that the transformed amplitudes are supported on certain holomorphic curves. Hence he suggests that this might result from an equivalence between the perturbative expansion of N=4 super Yang–Mills theory and the D-instanton expansion of the topological B-model the target space of which is the Calabi–Yau supermanifold $\mathbb{CP}^{3/4}$. The subject of twistor-string theory (see Twistor String Theory URL [106] in the References) has evolved out of such a seminal paper, showing once more the profound importance of holomorphic ideas in quantum gravity.

Moreover, from the point of view of classical general relativity, it appears very encouraging that general asymptotically flat (neither necessarily self-dual nor anti-self-dual) vacuum four-spaces can be described within a new twistor-geometric formalism [77].

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