Local Couplings and Sl(2,R) Invariance for Gauge Theories at One Loop

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The response of the one loop effective action for a gauge theory with local couplings $g(x), \theta(x)$ under a local Weyl rescaling of the background metric is calculated. Apart from terms which may be removed by local contributions to the effective action the result is compatible with $Sl(2,R)$ symmetry acting on $g, \theta$. Two loop effects are also discussed.

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Requiring that a renormalisable quantum field theory is extended to be finite for local couplings, so that the usual coupling constants are arbitrary functions of position, ensures that functional derivatives of the effective action with respect to the local couplings directly define finite correlation functions for all composite operators that appear in the basic lagrangian. In supersymmetric theories local couplings, which may be given by supergravity backgrounds, are directly relevant in constructing low energy effective theories and also for understanding the interplay of various anomalies [1,2,3]. The use of local couplings also avoids the requirement in more conventional treatments of introducing additional local counterterms which are necessary for finiteness of composite operator correlation functions since they are essentially included in the renormalised quantum action. Further the usual renormalisation group equations can be extended to equations corresponding to local Weyl rescalings of the metric. Imposing the necessary integrability conditions has proved to a convenient method for deriving non trivial relations amongst the $\beta$ and other functions that enter into the renormalisation of quantum field theories and the correlation functions of composite operators [4]. In two dimensions the necessary consistency relations involve identities which are equivalent, within the allowed ambiguities, to the Zamolodchikov $c$-theorem and in four dimensions similar results are obtained which are sufficient to show irreversibility of renormalisation group flow in the perturbative regime where the associated metric on the space of couplings can be calculated and shown to be positive (unlike in two dimensions there is no presently known connection to a two point function with manifest positivity properties). The resulting equations then reflect some of the fundamental aspects of quantum field theories.

Previously [5] we calculated the various renormalisation quantities that were necessary for a finite quantum field theory with local couplings for general renormalisable theories in four dimensions at one and two loops. In this note we extend these considerations to include the $\theta$ coupling which is of course present in a general four dimensional gauge theory. Assuming a Euclidean metric $\gamma_{\mu\nu}(x)$, and arbitrary local couplings $g(x), \theta(x)$ the gauge invariant Euclidean action has the usual form

$$S = \frac{1}{4} \int \! d^4 x \sqrt{\gamma} \left( \frac{1}{g^2} F^{\mu\nu} \cdot F_{\mu\nu} - i \theta F^{\mu\nu} \cdot (\ast F)_{\mu\nu} \right), \quad (\ast F)_{\mu\nu} = \frac{1}{2} \sqrt{\gamma} \gamma_{\mu\tau} \gamma_{\nu\omega} \epsilon^{\tau\omega\sigma\rho} F_{\sigma\rho},$$

(1)

with $F_{\mu\nu}$ the usual field strength and we define

$$\hat{\theta} = \frac{\theta}{8\pi^2}. \quad (2)$$

For calculational convenience we choose a gauge fixing term of the form

$$S_{g.f.} = \frac{1}{2} \int \! d^4 x \sqrt{\gamma} g^2 \nabla^\mu \left( \frac{1}{g^2} A_\mu \right) \cdot \nabla^\nu \left( \frac{1}{g^2} A_\nu \right),$$

(3)
and the associated ghost action is
\[ S_{\text{gh}} = \int d^4x \sqrt{-\gamma} \frac{1}{g^2} \nabla^\mu \hat{c} \cdot D_\mu c, \quad D_\mu c = \partial_\mu c + A_\mu \times c. \] (4)

Adding \( S \) and \( S_{\text{g.f.}} \) and expanding to quadratic order gives
\[ S_{\text{quadratic}} = \frac{1}{2} \int d^4x \sqrt{-\gamma} A^\mu \cdot (\Delta_1 A)_\mu, \] (5)
where, using form notation\(^1\),
\[ \Delta_1 = \delta \frac{1}{g^2} d + \frac{1}{g^2} d g^2 \delta \frac{1}{g^2} + \frac{1}{2} i i \delta i \hat{\vartheta} * d = \delta \frac{1}{g^2} d + \frac{1}{g^2} d g^2 \delta \frac{1}{g^2} + \frac{1}{2} i \delta i \hat{\vartheta} * . \] (6)

If we introduce a modified connection
\[ \tilde{\nabla}_\sigma A_\mu = \nabla_\sigma A_\mu + \frac{1}{2} i g^2 \frac{1}{\sqrt{-\gamma}} \gamma_\sigma \gamma_\mu \omega^{\rho \nu} \partial_\rho \hat{\vartheta} A_\nu, \] (7)
then \( \Delta_1 \) may be written in the form,
\[ \Delta_{1\mu}^\nu = - \tilde{\nabla}_\sigma \frac{1}{g^2} \tilde{\nabla}_\sigma \delta_\mu^\nu + \frac{1}{g^2} X_{\mu}^\nu, \]
\[ X_{\mu\nu} = g^4 \nabla_\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} - g^2 \nabla_\mu \nabla_\nu \frac{1}{g^2} + \frac{1}{2} g^4 (\partial^\sigma \hat{\vartheta} \partial_\sigma \hat{\vartheta} \gamma_{\mu\nu} - \partial_\mu \hat{\vartheta} \partial_\nu \hat{\vartheta}) + R_{\mu\nu}. \] (8)

The corresponding ghost operator obtained from (4) is
\[ \Delta_0 = \delta \frac{1}{g^2} d = - \nabla_\sigma \frac{1}{g^2} \nabla_\sigma. \] (9)

Using
\[ g \nabla_\sigma \frac{1}{g^2} \nabla_\sigma g = \nabla^2 - X, \quad X = \frac{1}{2} g^2 \nabla^2 \frac{1}{g^2} - \frac{1}{4} g^4 \nabla^\sigma \frac{1}{g^2} \nabla_\sigma \frac{1}{g^2}, \] (10)
we may easily see from (8) and (9),
\[ \tilde{\Delta}_1 = g \Delta_1 g = - \tilde{\nabla}^2 1 + Y_1, \quad \tilde{\Delta}_0 = g \Delta_0 g = - \tilde{\nabla}^2 + Y_0, \] (11)
with \( Y_{1\mu\nu} = X_{\mu\nu} + X_{\gamma_{\mu\nu} 1}, \ Y_0 = X_1. \)

The one loop effective action is then defined by
\[ W^{(1)} = - \frac{1}{2} \ln \det \tilde{\Delta}_1 + \ln \det \tilde{\Delta}_0. \] (12)

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\(^1\) Thus \( d \) is the exterior derivative with \( (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and for a scalar \( \phi \) \( (d\phi)_\mu = \partial_\mu \phi \) while \( \delta \) is its adjoint with \( (\delta F)_\mu = - \frac{1}{\sqrt{-\gamma}} \gamma_{\mu\nu} \partial_\nu (\sqrt{-\gamma} \gamma^{\sigma\rho} F_{\sigma\rho}), \) \( \delta A = - \frac{1}{\sqrt{-\gamma}} \partial_\sigma (\sqrt{-\gamma} \gamma^{\sigma\rho} A_\rho). \) Also for a vector \( u^\mu \) and a \( n \)-form \( F_{\mu_1 \ldots \mu_n} \) we define \( (i_u F)_{\mu_1 \ldots \mu_n-1} = u^\mu F_{\mu_1 \ldots \mu_n-1}. \)
For an operator $\Delta$ the functional determinant may be defined in terms of the heat kernel,

$$-\ln \det \Delta = \zeta_\Delta'(0), \quad \zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \text{Tr}(e^{-\tau \Delta}).$$

(13)

We here consider the response to a local Weyl rescaling of the metric $\delta_\sigma \gamma_{\mu\nu} = 2\sigma \gamma_{\mu\nu}$ for this classically conformally invariant theory. Using (6) and (9) we have

$$\delta_\sigma \tilde{\Delta}_1 = -2\sigma \tilde{\Delta}_1 + 2\sigma \frac{d}{g} g^2 \delta \frac{1}{g} + \frac{1}{g} d g^2 \delta \frac{1}{g} 2\sigma - \frac{1}{g} d 4\sigma g^2 \delta \frac{1}{g},$$

$$\delta_\sigma \tilde{\Delta}_0 = -4\sigma g \delta \frac{1}{g} d g + g \delta 2\sigma \frac{1}{g^2} d g,$$

so that, since $\delta \Delta_1 = \Delta_0 g^2 \delta \frac{1}{g}$, $\Delta_1 d = \frac{1}{g^2} g^2 \Delta_0$ and using cyclicity of the functional trace,

$$\delta_\sigma \left( \text{Tr}(e^{-\tau \tilde{\Delta}_1}) - 2 \text{Tr}(e^{-\tau \tilde{\Delta}_0}) \right) = -2\tau \frac{d}{d\tau} \left( \text{Tr}(\sigma e^{-\tau \tilde{\Delta}_1}) - 2 \text{Tr}(\sigma e^{-\tau \tilde{\Delta}_0}) \right).$$

(15)

Hence with the definition (13) we have

$$\delta_\sigma W^{(1)} = \text{Tr}(\sigma e^{-\tau \tilde{\Delta}_1}) \big|_{\tau_0} - 2 \text{Tr}(\sigma e^{-\tau \tilde{\Delta}_0}) \big|_{\tau_0},$$

(16)

where $|_{\tau_0}$ denotes the term $O(\tau^0)$ in the heat kernel expansion. For operators of the form $\Delta = -\nabla^2 + Y$ we have

$$\text{Tr}(\sigma e^{-\tau \Delta}) \big|_{\tau_0} = \frac{1}{16\pi^2} \int d^4 x \sqrt{\sigma} \text{tr}(a_2^\Delta),$$

(17)

where the diagonal DeWitt coefficient is given by [6]

$$a_2^\Delta = \frac{1}{36\pi^2} (3F - G) + \frac{1}{12} F^{\sigma \rho} F_{\sigma \rho} + \frac{1}{2} (Y - \frac{1}{6} R)^2 - \frac{1}{6} \nabla^2 Y + \frac{1}{36} \nabla^2 R 1.$$  

(18)

with $F$ the square of the Weyl tensor, $G$ the Euler density, both quadratic in the Riemann tensor$^2$, and $[\nabla_\sigma, \nabla_\rho] = F_{\sigma \rho}$.

Since according (11) both $\tilde{\Delta}_1$ and $\tilde{\Delta}_0$ are of the required form we may readily obtain $a_{2_1}^\Delta$ and $a_{2_0}^\Delta$ using (18). For the former we may note that

$$\text{tr}_v \left( (Y - \frac{1}{6} R)^2 \right) = g^4 \nabla^\mu \nabla^\nu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2} - 2g^6 \nabla^\mu \frac{1}{g^2} \nabla^\nu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2}$$

$$+ \frac{1}{2} g^6 \nabla^\mu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2} + \frac{3}{4} g^8 \nabla^\mu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2}$$

$$+ \frac{1}{2} g^6 \nabla^\nu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2} \nabla_\mu \nabla_\nu \frac{1}{g^2}$$

$$+ \frac{1}{2} g^6 \nabla^\mu \nabla_\nu \frac{1}{g^2} \nabla_\nu \nabla_\mu \nabla_\nu \nabla_\mu \nabla_\nu \frac{1}{g^2}$$

$$- g^8 \nabla^\mu \frac{1}{g^2} \nabla_\nu \nabla_\nu \nabla_\mu \nabla_\nu \nabla_\nu \nabla_\mu \nabla_\nu \nabla_\nu \frac{1}{g^2}$$

$$+ G^{\mu \nu} \left( 2g^4 \nabla^\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} - 2g^2 \nabla^\mu \nabla_\nu \frac{1}{g^2} - g^4 \nabla_\mu \nabla_\nu \nabla_\mu \nabla_\nu \frac{1}{g^2} \right)$$

$$+ R \left( \frac{1}{2} g^4 \nabla^\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} - \frac{1}{2} g^2 \nabla^2 \frac{1}{g^2} \right) + R^{\mu \nu} R_{\mu \nu} - \frac{2}{9} R^2,$$

(19)

$^2 F = R^{\mu \nu \sigma \rho} R_{\mu \nu \sigma \rho} - 2R^{\mu \nu} R_{\mu \nu} + \frac{1}{3} R^2, \quad G = R^{\mu \nu \sigma \rho} R_{\mu \nu \sigma \rho} - 4R^{\mu \nu} R_{\mu \nu} + R^2.$
where $\text{tr}_v$ denotes the trace over vector indices and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} R$ is the Einstein tensor. In addition $F^{\sigma\rho} F_{\sigma\rho} \rightarrow -\mathcal{R}^{\mu\nu\sigma\rho} \mathcal{R}_{\mu\nu\sigma\rho}$ where $[\hat{\nabla}_\sigma, \hat{\nabla}_\rho] A_\mu = \mathcal{R}_{\mu\nu\sigma\rho} A^\nu$ and

$$
-\mathcal{R}^{\mu\nu\sigma\rho} \mathcal{R}_{\mu\nu\sigma\rho} = 2g^4 \nabla^\mu \nabla^\nu \hat{\theta} \nabla_\mu \nabla_\nu \hat{\theta} + g^4 \nabla^2 \hat{\theta} \nabla^2 \hat{\theta} - 4g^6 \nabla^\mu \frac{1}{g^2} \nabla^\nu \hat{\theta} \nabla_\mu \nabla_\nu \hat{\theta} - 2g^6 \nabla^\mu \frac{1}{g^2} \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta} \\
+ 2g^8 \nabla^\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla^\nu \hat{\theta} \nabla_\nu \hat{\theta} + g^8 \nabla^\mu \frac{1}{g^2} \nabla_\mu \nabla_\nu \hat{\theta} - \frac{4}{3} g^8 \nabla^\mu \hat{\theta} \nabla_\mu \nabla^\mu \hat{\theta} \nabla_\nu \hat{\theta} \\
+ G^{\mu\nu} 2g^4 \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta} - R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho}.
$$

(20)

With these results it is straightforward to obtain from (16),

$$
16\pi^2 \delta_\sigma W^{(1)} = \int d^4 x \sqrt{\gamma} \sigma \left( \text{tr}(a_2^{\hat{\Delta}_1}) - 2 \text{tr}(a_2^{\hat{\Delta}_0}) \right),
$$

(21)

where, for $n_\mathcal{V}$ vector fields, we may write

$$
\text{tr}(a_2^{\hat{\Delta}_1}) - 2 \text{tr}(a_2^{\hat{\Delta}_0}) = c F - a G - h \nabla^2 R + n_\mathcal{V} (\mathcal{L} - \nabla_\mu \mathcal{Z}_\mu + \nabla^2 \mathcal{Y}),
$$

(22)

which may be identified with $16\pi^2 \gamma^{\mu\nu} \langle T_{\mu\nu} \rangle$ at one loop. $\mathcal{L}$ can be reduced to the form

$$
\mathcal{L} = \frac{1}{3} g^4 \left( \nabla^2 \frac{1}{g^2} \nabla^2 \frac{1}{g^2} + \nabla^2 \hat{\theta} \nabla^2 \hat{\theta} - 2G^{\mu\nu} \left( \nabla_\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} + \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta} \right) \right) \\
- \frac{1}{3} R \left( \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} + \nabla_\mu \hat{\theta} \nabla_\mu \hat{\theta} \right) \\
- \frac{1}{3} g^6 \left( \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla^2 \frac{1}{g^2} + 2 \nabla_\mu \frac{1}{g^2} \nabla_\mu \hat{\theta} \nabla^2 \hat{\theta} - \nabla_\mu \frac{1}{g^2} \nabla_\mu \hat{\theta} \nabla_\mu \hat{\theta} \right) \\
+ g^8 \left( \frac{5}{16} \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} + \frac{13}{12} \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\nu \hat{\theta} \nabla_\nu \hat{\theta} \right) \\
+ \frac{11}{24} \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\nu \hat{\theta} \nabla_\nu \hat{\theta} + \frac{5}{16} \nabla_\mu \hat{\theta} \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta} \nabla_\nu \hat{\theta} \right).
$$

(23)

In (22) we should take $c = \frac{1}{10} n_\mathcal{V}$, $a = \frac{31}{180} n_\mathcal{V}$, $h = \frac{1}{10} n_\mathcal{V}$ but they may be left general since they are altered by one loop contributions for scalar and spinor fields. $\mathcal{L}$, $\mathcal{Z}_\mu$ and $\mathcal{Y}$, which depend on $\partial_\mu \frac{1}{g^2}$, $\partial_\mu \hat{\theta}$, are not so affected. For $\mathcal{Z}_\mu, \mathcal{Y}$ we also obtain

$$
\mathcal{Z}_\mu = G^{\mu\nu} g^2 \nabla_\nu \frac{1}{g^2} + \frac{1}{2} g^4 \nabla^\mu \frac{1}{g^2} \nabla^2 \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\nu \hat{\theta} \nabla^2 \hat{\theta} \\
- \frac{1}{2} g^6 \nabla^\mu \frac{1}{g^2} \nabla^\nu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla_\nu \hat{\theta} + \frac{1}{12} g^6 \nabla^\mu \frac{1}{g^2} \nabla^\nu \hat{\theta} \nabla_\nu \hat{\theta} - \frac{1}{2} g^6 \nabla^\mu \frac{1}{g^2} \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta},
$$

(24)

$$
\mathcal{Y} = \frac{1}{6} g^4 \nabla^\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} - \frac{1}{6} g^4 \nabla^\mu \hat{\theta} \nabla_\mu \hat{\theta}.
$$
We now demonstrate that $L$ may be expressed in a form which exhibits manifest $Sl(2, \mathbb{R})$ invariance. We define as usual

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2},$$

(25)

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{R})$ acts according to $\tau \to (a\tau + b)/(c\tau + d)$. The quadratic terms in $L$ correspond to the invariant line interval on the upper half complex plane defined by

$$ds^2 = \frac{1}{(\text{Im}\, \tau)^2} \, d\tau d\bar{\tau},$$

(26)

which describes the constant negative curvature hyperboloid. With this metric the Christoffel connections are just $\Gamma^\tau_{\tau\tau} = i/\text{Im}\, \tau$, $\Gamma^\tau_{\bar{\tau}\bar{\tau}} = -i/\text{Im}\, \tau$, and we may then define

$$D^2 \tau = \nabla^2 \tau + \frac{i}{\text{Im}\, \tau} \nabla^\mu \nabla_\mu \tau, \quad D^2 \bar{\tau} = \nabla^2 \bar{\tau} - \frac{i}{\text{Im}\, \tau} \nabla^\mu \nabla_\mu \bar{\tau}.$$  

(27)

and consequently rewrite (23) in the form

$$L = \frac{1}{4(\text{Im}\, \tau)^2} (D^2 \tau D^2 \bar{\tau} - 2G^{\mu\nu} \nabla_\mu \tau \nabla_\nu \bar{\tau} - \frac{1}{3} R \nabla^\mu \tau \nabla_\mu \bar{\tau})$$

$$+ \frac{1}{16(\text{Im}\, \tau)^4} (\nabla^\mu \tau \nabla_\mu \bar{\tau})^2 + \frac{1}{48(\text{Im}\, \tau)^4} \left( \nabla^\mu \tau \nabla^{\nu} \tau \nabla_\mu \bar{\tau} \nabla_\nu \bar{\tau} - (\nabla^\mu \tau \nabla_\mu \bar{\tau})^2 \right).$$

(28)

The remaining terms given by (24) cannot be expressed in an $Sl(2, \mathbb{R})$ invariant form but they can be removed by taking $W^{(1)} \to W^{(1)} + W_{\text{loc}}$ with,

$$16\pi^2 W_{\text{loc}} = -\frac{nV}{8} \int d^4x \sqrt{\gamma} \ln g^2 G$$

$$- nV \int d^4x \sqrt{\gamma} \left( \frac{1}{3} g^4 \nabla^2 \frac{1}{g^2} \nabla^2 \frac{1}{g^2} + \frac{1}{24} g^4 \nabla^2 \hat{\theta} \nabla^2 \hat{\theta} - \frac{1}{6} \hat{R} \hat{\gamma} \right)$$

$$- \frac{1}{4} g^6 \nabla^\mu \frac{1}{g^2} \nabla_\mu \frac{1}{g^2} \nabla^2 \frac{1}{g^2} - \frac{1}{4} g^6 \nabla^\mu \frac{1}{g^2} \nabla_\mu \hat{\theta} \nabla^2 \hat{\theta} + \frac{1}{24} g^6 \nabla^2 \frac{1}{g^2} \nabla^2 \hat{\theta} + \frac{1}{24} g^6 \nabla^2 \frac{1}{g^2} \nabla^2 \hat{\theta},$$

(29)

where we may easily calculate $\delta_\sigma W_{\text{loc}}$ using $\delta \nabla^2 = -2\sigma \nabla^2 + 2\partial_\mu \sigma \nabla^\mu$, $\delta R = -2\sigma R - 6\nabla^2 \sigma$ and $\delta G = -4\sigma G + 8G^{\mu\nu} \nabla_\mu \partial_\nu \sigma$. Save for the $\ln g^2 G$ term, which cancels the term involving $G^{\mu\nu} \nabla_\nu \frac{1}{g^2}$ in (24), the terms present in $W_{\text{loc}}$ reflect the usual arbitrariness up to local expressions in the effective action arising from renormalisation scheme dependence.

\textsuperscript{3} In [5] a calculation for $\theta = 0$ based on dimensional regularisation gave different results for coefficients of the terms in $\mathcal{Z}^\mu$, $\mathcal{Y}$ except for the one involving $G^{\mu\nu}$. 

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The possible presence of such additional local contributions may be seen by considering an alternative one loop effective action instead of (12)

\[ \hat{W}^{(1)} = -\frac{1}{2} \ln \det \Delta_1 + \ln \det \Delta_0, \]

where the operators \( \Delta_1, \Delta_0 \) are given by (6) and (9), without the rescaling in (11) which is equivalent in the original action to taking \( A_\mu = g a_\mu \) and correspondingly for the ghost fields. With the zeta function definition (13) we have for arbitrary \( \lambda(x) \)

\[ \ln \det \lambda \Delta - \ln \det \Delta = \left. \Tr(\ln \lambda e^{-\tau \Delta}) \right|_{\tau_0}. \]

Hence we have

\[ 16\pi^2 (\hat{W}^{(1)} - W^{(1)}) = \frac{1}{2} \int d^4 x \sqrt{\gamma} \ln g^2 \left( c F - a G - h \nabla^2 R + n V (\mathcal{L} - \nabla_\mu Z^\mu + \nabla^2 Y) \right). \]  

The \( \ln g^2 G \) term in (32) matches that in (29) if \( a = \frac{1}{4} n_V. \) This is exactly the result for \( \mathcal{N} = 4 \) supersymmetric gauge theories (when \( c = a \)). Under a Weyl rescaling \( \delta \sigma = 2 \sigma \gamma_{\mu \nu} \) the remaining terms just lead to an expression for \( \delta \sigma \hat{W}^{(1)} \) which is of the form (21) with the same result (23) for \( \mathcal{L} \) although a modified \( Z^\mu. \) Exactly the same set of independent terms appear in the new \( Z^\mu \) as are present in the formula in (24). As before these may then be cancelled by a simple local counterterm of the same form as (29), with differing coefficients, but without the need now for a \( \ln g^2 G \) term. To show this we need to note that

\[ \delta \sigma \mathcal{L} = -4 \sigma \mathcal{L} + \nabla_\mu (\partial_\nu \sigma U^{\mu \nu}), \]

\[ U_{\mu \nu} = g^4 \left( \nabla_\mu \frac{1}{g^2} \nabla_\nu \frac{1}{g^2} + \nabla_\mu \hat{\theta} \nabla_\nu \hat{\theta} \right) - \frac{1}{2} \gamma_{\mu \nu} g^4 \left( \nabla_\rho \frac{1}{g^2} \nabla_\rho \frac{1}{g^2} + \nabla_\rho \hat{\theta} \nabla_\rho \hat{\theta} \right). \]

The form for \( \delta \sigma \mathcal{L} \) with \( U_{\mu \nu} = U_{\nu \mu} \) follows from the consistency relations in [4], this condition constrains the quadratic and cubic terms in \( \mathcal{L} \) although it is not fully determined.

It would of course be interesting to extend these considerations beyond one loop and to see whether \( SL(2, \mathbb{R}) \) invariance is maintained. In general we may write

\[ 16\pi^2 D_\sigma W = \int d^4 x \sqrt{\gamma} \sigma \left( c F - a G - h \nabla^2 R - \frac{1}{9} b R^2 + L - \nabla_\mu Z^\mu + \nabla^2 Y \right), \]

\[ D_\sigma = \int d^4 x \sigma \left( -2 \gamma^{\mu \nu} \frac{\delta}{\delta \gamma^{\mu \nu}} + \beta^i \frac{\delta}{\delta g^i} \right), \]

for local couplings \( g^i, \) with corresponding \( \beta \)-functions \( \beta^i, \) and where \( L, Z^\mu, Y \) depend on their derivatives. Various consistency conditions were derived in [4] from \([D_\sigma, D_\sigma^\prime] = 0.\) For a simple gauge coupling \( g, \) with \( \theta = 0, \) we may write

\[ L = n_V \left\{ \frac{1}{g^2} (\alpha (\nabla^2 g)^2 - 2 \delta G^{\mu \nu} \partial_\mu g \partial_\nu g - \frac{1}{2} \epsilon R \partial^\mu g \partial_\mu g) \right. \]

\[ \left. - 2 \kappa \frac{1}{g^3} \partial^\mu g \partial_\mu g \nabla^2 g + 2 \lambda \frac{1}{g^4} \partial^\mu g \partial_\mu g \partial^\nu g \partial_\nu g \right\}, \]
and to two loop order using dimensional regularisation, for $\hat{g}^2 = g^2/16\pi^2$, extending the results in [5]4,

\[
\alpha = \delta = 1 + \frac{1}{3}(51C - 20R_\psi - \frac{2}{3}R_\phi)\hat{g}^2, \\
\epsilon = 1 + \frac{1}{3}(29C - 12R_\psi - \frac{2}{3}R_\phi)\hat{g}^2, \\
\kappa = 1 + \frac{4}{3}(11C - 4R_\psi - \frac{1}{2}R_\phi)\hat{g}^2, \\
\lambda = 1 + \frac{1}{16}(323C - 76R_\psi - \frac{20}{9}R_\phi)\hat{g}^2, \\
\]

where it $t^a_\alpha, t^\psi_\alpha$ are the gauge group generators acting on scalar, fermion fields, $\text{tr}(t^a_\alpha t^\psi_\beta) = -\delta_{ab}R_\phi$, $\text{tr}(t^\psi_\alpha t^\psi_\beta) = -\delta_{ab}R_\psi$. The results are scheme dependent. For supersymmetric theories it is more natural to transform to a dimensional reduction scheme by letting $1/\hat{g}^2 \to 1/\hat{g}^2 + \frac{1}{3}C$. For $\mathcal{N} = 1$ supersymmetry we let $2R_\psi = C + R$, $R_\phi = 2R$ and also add $\frac{2}{3}R\hat{g}^2$ to $\alpha, \delta, \epsilon$ since there are Yukawa couplings proportional to $g$ (for more details see [7]). This gives

\[
\alpha = \delta = 1 + (13C - 5R)\hat{g}^2, \quad \epsilon = 1 + (7C - 3R)\hat{g}^2, \\
\kappa = 1 + 4(3C - R)\hat{g}^2, \quad \lambda = 1 + \frac{1}{2}(31C - 7R)\hat{g}^2. \\
\]

For $\mathcal{N} = 2$ theories $R \to C + 2R$ and Yukawa couplings add a further $2C\hat{g}^2$ to $\alpha, \delta, \epsilon, \kappa$, giving now

\[
\alpha = \delta = 1 + 10(C - R)\hat{g}^2, \quad \epsilon = 1 + 6(C - R)\hat{g}^2, \\
\kappa = 1 + 8(C - R)\hat{g}^2, \quad \lambda = 1 + (12C - 7R)\hat{g}^2. \\
\]

For $\mathcal{N} = 4$, when there is a single adjoint hypermultiplet, $C = R$, as is necessary for a zero $\beta$-function. In this case there are no corrections to $\alpha, \delta, \epsilon, \kappa$, in accord with consistency relations, although $\lambda$ remains non zero. Even for $\mathcal{N} = 4$ there are thus additional perturbative contributions beyond one loop. It is natural to suppose that for $\mathcal{N} = 4$ $L$ in (35) extends with the inclusion of $\theta$ to a form which is invariant under $Sl(2,\mathbb{Z})$, where $\alpha = \delta = \epsilon = \kappa = 1$ and $\lambda$ becomes an appropriate modular form.

**Appendix A. Two loop Calculations with Local Couplings**

We here revisit old calculations for the divergences at two loops [5] for pure gauge theories using just the local coupling $g$. Using integration by parts and Ward identities van der Ven [8] showed that the two loop vacuum amplitude on flat space can be reduced to just

\[
W^{(2)} = \int \int gg' \left( - (D_\alpha G_{\beta\gamma}, G_{\beta\gamma} \overrightarrow{D}_\delta, G_{\alpha\delta}) + 2(D_\alpha G_{\beta\gamma}, G_{\alpha\gamma} \overrightarrow{D}_\delta, G_{\beta\delta}) \right) \\
- \frac{1}{4} \int g^2 (2 \text{tr}(T_\alpha G_{(\alpha\beta)} T_\alpha G_{\alpha\beta}) - \text{tr}(T_\alpha G_{\alpha\alpha} T_\alpha G_{\beta\beta}) + \text{tr}(T_\alpha G_{\alpha\beta}) \text{tr}(T_\alpha G_{\alpha\beta})) \right), \\
\]

(A.1)

4 The coefficient of $C$ in $\lambda$ is corrected from [5].
where the ghost contribution is cancelled. In the first line of (A.1), involving integrations over \(x, x'\), \(G_{\alpha\beta}(x, x')\) is the vector propagator, \((X, Y, Z) = f_{abc} f_{a'b'c'} X_{aa'} Y_{bb'} Z_{cc'}\), \(g = g(x), g' = g(x')\) are the local gauge couplings and

\[
D_\alpha = \partial_\alpha + v_\alpha(x), \quad \overline{D}_\delta = \overline{\partial}_\delta + v_\delta(x'), \quad v_\alpha = \frac{1}{g} \partial_\alpha g . \quad (A.2)
\]

In the second line of (A.1) \((T_a)_{bc} = -f_{abc}\) and \(G_{\alpha\beta}\) denotes the coincident limit \(x' = x\) and the trace is over group indices \(a, b = 1, \ldots, n_V\). The counterterms necessary to subtract sub-divergences are given by \((\text{tr}\{T_a T_b\} = -C\delta_{ab})\),

\[
W^{(2)}_{\text{c.t.}} = -\frac{C}{16\pi^2 \varepsilon} \int g^2 \text{tr}\left\{ \frac{5}{3} D_\alpha G_{\alpha\beta} \overline{D}_\beta - \frac{14}{3} D_\alpha G_{\alpha\beta} \overline{D}_\beta + 2(v_\beta D_\alpha G_{\alpha\beta} + v_\alpha G_{\alpha\beta} \overline{D}_\beta) \right\} + 2\partial \cdot v G_{\beta\beta} - 4(\partial_\alpha v_\beta + v_\alpha v_\beta) G_{\alpha\beta} \right\} . \quad (A.3)
\]

This form is in accord with that expected according to [5], combining the ghost contribution here with the vector piece,\(^5\) if we note the identity

\[
D_\alpha G_{\alpha\beta} \overline{D}_\beta - 2(v_\beta D_\alpha G_{\alpha\beta} + v_\alpha G_{\alpha\beta} \overline{D}_\beta) + 4v_\alpha v_\beta G_{\alpha\beta} = 0 . \quad (A.4)
\]

Using an expansion of the propagators in terms of Seeley-DeWitt coefficients \(a_n(x, x')\) then in [9] a simple algorithm was given for calculating the poles in \(\varepsilon = 4 - d\),

\[
W^{(2)}_{\text{div}} = \frac{C}{(16\pi^2 \varepsilon)^2} \int g^2 \text{tr}\left\{ \left(\frac{5}{3} + \frac{\varepsilon}{36}\right) D_\alpha a_{1\alpha\beta} \overline{D}_\beta - \frac{14}{3} D_\alpha a_{1\alpha\beta} \overline{D}_\beta + (1 - \frac{\varepsilon}{4}) a_{2\beta\beta} \right\} + (2 - \frac{\varepsilon}{2})(v_\beta D_\alpha a_{1\alpha\beta} + v_\alpha a_{1\alpha\beta} \overline{D}_\beta) + (2 + \frac{\varepsilon}{2}) \partial \cdot v a_{1\beta\beta} \right\} - (4 + \varepsilon) \partial_\alpha v_\beta a_{1\alpha\beta} - 4v_\alpha v_\beta a_{1\alpha\beta} + \frac{\varepsilon}{4} a_{1\alpha\alpha} a_{1\beta\beta} \right\} \right\} . \quad (A.5)
\]

Simplifying (A.5), using standard results for \(a_1, a_2\), we get

\[
W^{(2)}_{\text{div}} = -\frac{C n_V}{(16\pi^2 \varepsilon)^2} \int g^2 \left\{ \frac{22}{3} ((\partial \cdot v)^2 + 2v^2 \partial \cdot v + 2v^2 v^2) - \varepsilon \left(\frac{17}{2} (\partial \cdot v)^2 + \frac{7}{9} v^2 \partial \cdot v + \frac{106}{9} v^2 v^2 \right) \right\} . \quad (A.6)
\]

Correspondingly at one loop

\[
W^{(1)}_{\text{div}} = \frac{n_V}{16\pi^2 \varepsilon} \int ((\partial \cdot v)^2 + v^2 v^2) . \quad (A.7)
\]

---

\(^5\) We note here the following misprints, in (5.7) the result for \(Z^{(1)}_\beta\) should have a – sign, in (5.1) \(Y_{sh} = -\nabla^\sigma v_\sigma + \nabla^\sigma v_\sigma\). Note also that the first two terms on the r.h.s. of (2.20) and the first three on the r.h.s. of (2.21) should have the opposite sign.
To two loop order this gives

\[
(\varepsilon - \hat{\beta}(g) \frac{\partial}{\partial g}) W_{\text{div}} = \frac{nV}{16\pi^2} \int \left\{ (1 + 17C\hat{g}^2) \frac{1}{g^2} (\partial^2 g)^2 - 2(1 + \frac{44}{3} C\hat{g}^2) \frac{1}{g^3} (\partial g)^2 \partial^2 g \\
+ 2(1 + \frac{323}{18} C\hat{g}^2) \frac{1}{g^4} (\partial g)^2 (\partial g)^2 \right\},
\]

(A.8)

for \( \hat{g}^2 = g^2 / 16\pi^2 \).

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