Sequential Rank Shiryaev-Roberts CUSUMs

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Abstract

We develop Shiryaev-Roberts type CUSUMs based on signed sequential ranks to detect changes in location and dispersion of a continuous distribution. The CUSUMs are distribution-free, hence do not require a parametric specification of an underlying density function. Tables of control limits are provided. The out-of-control average run length properties of the CUSUMs are gauged via theory-based calculations and Monte Carlo simulation. Implementation of the methodology is illustrated in an application to data from an industrial environment.

Keywords: Distribution free, Page CUSUM, Sequential ranks, Shiryaev-Roberts CUSUM
1 Introduction

CUSUM procedures are designed with the objective to signal a change in some characteristic of a process as soon as possible after its onset. The change manifests itself along a sequence of independent and identically distributed observations $X_1, X_2, \ldots$ as a sustained change in the distribution of the $X_i$. The change occurs at an index $\tau$, known as the changepoint. Thus, $X_1, X_2, \ldots, X_\tau$, the “in-control” observations, come from a distribution with density function $f_0$ while $X_{\tau+1}, X_{\tau+2}, \ldots$, the “out-of-control” observations, come from a distribution with density function $f_1 \neq f_0$. Perhaps the most widely known CUSUM is that designed by Page (1954). The Page CUSUM aims to detect a change in the mean away from a specified in-control value in a distribution from the exponential family. The special case in which $f_0$ is a standard normal density and $f_1(x) = f_0(x - \delta), \delta \neq 0$ is a simple shift alternative, has been the object of particularly intensive research. The CUSUM sequence $V_i, i \geq 0$ is defined by setting $V_0 = 0$ and applying the recursion

$$V_i = \max(1, V_{i-1}) \Lambda_i, \quad i \geq 1$$

where $\Lambda_i = f_1(X_i)/f_0(X_i)$ is the likelihood ratio at $X_i$ – see Moustakides et al. (2009). The CUSUM signals as soon as $V_i$ exceeds a control limit $h > 0$. The index $N$ at which the signal occurs is known as the run length. Even if no change occurs, i.e. when $f_0$ governs the data for all time, the CUSUM is sure to produce a (false) signal somewhere along the sequence. Because of this, the control limit is chosen in a manner that guarantees a predetermined average run length $E[N|f_0]$, known as the in-control ARL. Again, when the underlying densities are normal, Girschick and Rubin (1952), followed by Shiryaev (1963), proposed a different type of CUSUM which was further examined by Roberts (1966) with a view to practical application. This Shiryaev-Roberts CUSUM, henceforth referred to as the S-R CUSUM, uses the recursion

$$R_i = (1 + R_{i-1}) \Lambda_i, \quad i \geq 1. \quad (1)$$

The two CUSUMs can be shown to result from different optimality criteria involving the delay $N - \tau$, conditional upon $N > \tau$, from changepoint to signal point – see, for instance Lorden (1971) and Pollak (1985). A definitive comparison between the behaviours of the CUSUMs was made by Moustakides et al. (2009). Briefly, the main conclusions are that the out-of-control performances differ significantly only when the true shift in the mean is “small” and when the in-control ARL is short compared to the waiting time $\tau$ to the change. When the change is large, there is no practical difference between the performances of the two CUSUMs. The S-R CUSUM
has an advantage of very quick detection of a change occurring in the distant future but at the expense of more false signals prior to the change.

Lombard and Van Zyl (2018) developed Page-type CUSUMs that are distribution-free in the sense that their in-control behaviour does not depend upon a parametric specification of the underlying density. Unknown parameters need not be estimated. In fact, no moment conditions are required to assure the validity of the CUSUMs. This development frees one from the restrictions imposed by parametric CUSUMs and the difficulties inherent in designing CUSUMs that operate efficiently when unknown parameters must be estimated. See, for instance, Jones et al. (2004) and Bagshaw and Johnson (1974).

The distribution-free CUSUMs are based on the series of signed sequential ranks \( s_i r_i^+ \), \( i \geq 1 \), of the observations. Here, \( s_i = \text{sign}(X_i) \) and \( r_i^+ \) is the rank of \( |X_i| \) in the sequence \( |X_1|, \ldots, |X_i| \), that is, the number among \( |X_1|, \ldots, |X_i| \) less than or equal to \( |X_i| \). When the process is in control, the \( r_i^+ \) form a series of independent random variables with \( r_i^+ \) uniformly distributed on the integers 1, \( \ldots, i \) – see Barndorff-Nielsen (1963, Theorem 1.1). Furthermore, since the \( s_i \) are mutually independent and independent of the \( |X_i| \) series, \( s_i r_i^+ \), \( i \geq 1 \) is a sequence of independent random variables with \( s_i r_i^+ \) uniformly distributed on \( \pm 1, \ldots, \pm i \), no matter what the common distribution of the \( X_i \) is. The independence, distribution freeness and naturally sequential nature of signed sequential ranks makes them ideally suited to the construction of CUSUMs for independently distributed time ordered data. Pollak (2009) commented on the possibility of designing distribution-free S-R CUSUMs. Earlier, Gordon and Pollak (1995) constructed a nonparametric generalized CUSUM based on ranks and sequential ranks. The present paper is a further step towards designing an easily implementable distribution-free S-R CUSUM.

A question that arises quite naturally is how distribution-free versions of the S-R CUSUM can be constructed and, if so, whether their out-of-control behaviour viz. á viz. the distribution-free Page CUSUM mimics the behaviour of the corresponding parametric CUSUMs in the normal distribution case. It is this question that we investigate in the present paper, which is structured as follows.

In Section 2 the parametric Shiryaev-Roberts CUSUM for detecting a change in the mean of a standard normal distribution is briefly discussed. We show that the CUSUM is not robust against deviations from an underlying normal distribution. In Section 3, we define a class of sequential rank S-R CUSUMs to detect a change in the median from zero to a non-zero value in a symmetric distribution. Control limits guaranteeing a pre-specified in-control ARL are provided. In Section 4 we discuss the out-of-control properties of
the CUSUMs and formulate an informal result that is useful in gauging a priori their out-of-control behaviour. Section 5 compares by Monte Carlo simulation the performances of the S-R and Page-type CUSUMs. Finally, in Section 6 a data set analyzed previously by Lombard and Van Zyl (2018) is considered from the standpoint of the Wilcoxon S-R CUSUM. The implementation of a two-sided S-R CUSUM together with estimation of a changepoint is discussed and the results are compared to those found from the Wilcoxon Page-type CUSUM.

2 The Shiryaev-Roberts CUSUM

Suppose we wish to detect a shift from a mean of zero to a mean $\delta \neq 0$ in a standard normal distribution. Then the likelihood ratio is

$$f_1(X_i)/f_0(X_i) = \exp\left(2\zeta (X_i - \zeta)\right)$$

where $\zeta = \delta/2$. Setting $C_i = \log(R_i)$ in (1), we get

$$C_i = \log(1 + \exp C_{i-1}) + 2\zeta (X_i - \zeta).$$

where $\zeta = \delta/2$. This form of the recursion is especially convenient for the purpose of graphical representation. The CUSUM signals a change as soon as $C_i$ exceeds the control limit $h$, that is, at the index

$$N = \min\{i \geq 1 : C_i \geq h\}.$$

Given a desired in-control average run length $ARL_0$, an excellent approximation to the appropriate value of the appropriate control limit is (Pollak and Siegmund, 1991)

$$h \approx \log(ARL_0) - 0.583|\delta|$$

provided, of course, that the data come from a standard normal distribution. Table 1 shows a range of control limits for various values of $ARL_0$ and $\delta$. Extensive Monte Carlo simulations indicated that the approximation in (3) is sufficient to guarantee an in-control ARL to within 3% of the specified $ARL_0$.

Table 1: Control limits for the normal mean S-R CUSUM of the form (2).

| $|\delta|$ | 100   | 200   | 300   | 400   | 500   | 1000  | 2000  |
|---------|-------|-------|-------|-------|-------|-------|-------|
| 0.05    | 4.58  | 5.27  | 5.67  | 5.96  | 6.19  | 6.88  | 7.37  |
| 0.10    | 4.55  | 5.24  | 5.65  | 5.93  | 6.16  | 6.85  | 7.54  |
| 0.20    | 4.49  | 5.18  | 5.59  | 5.87  | 6.10  | 6.79  | 7.48  |
| 0.30    | 4.43  | 5.12  | 5.53  | 5.82  | 6.04  | 6.73  | 7.43  |
| 0.40    | 4.37  | 5.07  | 5.47  | 5.76  | 5.98  | 6.67  | 7.37  |
| 0.50    | 4.31  | 5.01  | 5.41  | 5.70  | 5.92  | 6.62  | 7.31  |
The out-of-control mean \( \delta \) which figures in the CUSUM recursion is known as the target out-of-control value and \(|\delta|\) is often thought of as the smallest change size that would be tolerable. However, \( \delta \) (and therefore \( \zeta \)) is in fact a free parameter which the analyst can choose with a view of designing a "fit for purpose" CUSUM.

Next, we look at the degree to which the in-control behaviour of the S-R CUSUM is affected by deviations from an assumed normal distribution. The last three columns in Table 2 show Monte Carlo estimates (10,000 runs per cell) of the true in-control ARL values of the CUSUM when the data come from a logistic distribution, which has moderately heavy tails, and a heavy-tailed Student \( t_3 \) distribution, both standardised to unit variance. The change sizes are \( \delta = 0.2 \) and \( \delta = 0.5 \) (first column) and the pre-specified \( ARL_0 \) values are 200, 500 and 1000 (second column). The substantial differences between the estimated true in-control ARLs and the desired \( ARL_0 \) values clearly indicate the need for a distribution-free CUSUM.

Table 2: Estimated in-control ARLs of the standard normal S-R CUSUM for data arising from standardised logistic and \( t_3 \) distributions.

| \( \delta \) | \( ARL_0 \) | \( h \) | Estimated in-control ARL |
|---|---|---|---|
|   |   |   | \( t_3 \) | Logistic |
| 0.2 | 200 | 5.18 | 228 | 320 |
|    | 500 | 6.10 | 581 | 1120 |
|    | 1000 | 6.79 | 1156 | 3116 |
| 0.5 | 200 | 5.01 | 250 | 604 |
|    | 500 | 5.92 | 560 | 2234 |
|    | 1000 | 6.62 | 970 | 6270 |

3 Signed sequential rank S-R CUSUMs

A signed sequential rank analogue of (2) is obtained upon replacing \( X_i \) there by

\[ \xi_i = J \left( \frac{s_i r_i^+}{i + 1} \right) / \nu_i \]  \hspace{1cm} (4)

where \( s_i = \text{sign}(X_i) \), where \( J(u), -1 < u < 1 \) is an odd, square-integrable, function on the interval \((-1, 1)\) and where

\[ \nu_i = \sqrt{\frac{1}{i} \sum_{j=1}^{i} J^2 \left( \frac{j}{i + 1} \right)} . \]

Thus,

\[ C_i = \log(1 + \exp C_{i-1}) + 2\zeta (\xi_i - \zeta) \]  \hspace{1cm} (5)
with $\xi_i$ from (4). Here $\delta$, hence $\zeta$, is interpreted in units of an underlying scale parameter $\sigma$, typically the standard deviation if it exists, otherwise an interquartile range, for instance. Since the signs and sequential ranks are scale invariant, the numerical value of $\sigma$ is immaterial. This contrasts with the parametric setup in which the data are not scale invariant and $\sigma$ must be known in order to implement the S-R CUSUM.

It is customary in this context to refer to $J(u)$ as a score function. Under an in-control regime, the successive $\xi_i$ are independently and uniformly distributed on the sets

$$\left\{ J\left( \frac{k}{i+1} \right) / \nu_i, \ k = 0, \pm 1, \ldots, \pm i \right\},$$

regardless of the form of the underlying symmetric distribution. The resulting CUSUM is therefore distribution-free.

The Wilcoxon score $J(u) = u$ for which $\nu_i = \sqrt{(2i+1)(i+1)/6}$ leads to a particularly useful CUSUM. This score has a simple form which makes for particularly simple computations. It is bounded, hence particularly robust against outliers. Furthermore, it has a high correlation with the efficient scores of distributions with heavy tails such as certain Student $t$-distributions, see Lombard and Van Zyl (2018), hence can be expected to be very efficient in a variety of circumstances. In this case

$$\xi_i = \sqrt{\frac{6}{(2i+1)(i+1)}} s_{i^+} i^+$$

replaces $X_i$ in (2).

The function $J(u) = \Phi^{-1}(u)$, the inverse normal cdf, more commonly known as the Van der Waerden score, is also deserving of attention. Besides being distribution free, the resulting S-R CUSUM is asymptotically just as efficient as the original normal distribution S-R CUSUM if the underlying distribution happens to be normal. Table 3 gives control limits for a matrix of $(\zeta, ARL_0)$ pairs for the Wilcoxon S-R CUSUM. The table was generated by Monte Carlo simulation using the method detailed in Lombard and Van Zyl (2018). For completeness, we again describe the method in the Appendix. Table 1 can be used as is when the Van der Waerden S-R CUSUM is employed.

Table 3: Control limits for the Wilcoxon SSR S-R CUSUM.
4 Out-of-control properties

Consider a situation in which the out-of-control observations $X_{\tau+1}, X_{\tau+2}, \ldots$ have a median $\mu \neq 0$, again specified in units of the underlying scale parameter $\sigma$. Then, with $\xi_i$ defined in (4), for a fixed $i \geq 1$ and a “large” $\tau$ (see Lombard and Van Zyl (2018)),

$$E[\xi_{\tau+i}] \approx \theta_0 \mu,$$

with

$$\theta_0 = \frac{1}{\sqrt{\eta}} \int_{-\infty}^{\infty} J'(2F_0(x) - 1)f_0^2(x)dx,$$

(7)

where $f_0$ and $F_0$ denote the pdf and cdf of the in-control $X/\sigma$, and where $\eta = \int_0^1 J^2(u)du$. The last two relations, together with the fact that the CUSUM was designed to detect a change of size $\delta$, suggest that the choice $\zeta = \delta \theta_0/2$ would be an appropriate reference value. The values of $\theta_0$ that are likely to be encountered in practice are given in Table 1 of Lombard and Van Zyl (2018) for Student $t$-distributions with a range of tail thicknesses. For completeness, we include these here in Table 4. Even though the density function underlying the data is unknown, we can use these values of $\theta_0$ to make an informed choice of $\theta_0$ based on the expected heaviness of the tails.

On the other hand, if some preliminary data are available, $\theta_0$ can be estimated as indicated in Lombard and Van Zyl (2018, Section 3.1).

Table 4: Values of $\theta_0$ for a range of $t_\nu$ distributions.

|       | normal | $t_4$ | $t_3$ | $t_2$ | $t_1$ |
|-------|--------|-------|-------|-------|-------|
| Wilcoxon | 0.98   | 1.18  | 1.38  | 1.18  | 1.10  |
| Van der Waerden | 1.00   | 1.12  | 1.29  | 1.06  | 0.93  |

Because the distribution of the partial sums of the $\xi_i$ tends to a normal distribution, we can expect the S-R CUSUM to behave in an out-of-control situation more or less as would a standard normal S-R CUSUM. We
formulate this result as an heuristic, which is an analogue of the heuristic derived in Lombard and Van Zyl (2018).

**Heuristic result** Let $\zeta$ be “small” and let a persistent shift of “small” size $\mu$ (in units of $\sigma$) occur at a “large” changepoint $\tau$. Then the signed sequential rank S-R CUSUM behaves approximately as does a normal distribution S-R CUSUM with the same $\zeta$ and $h$ when a persistent shift of size $\mu \theta_0$ commences at $\tau$.

The usefulness of this result can be assessed to some extent via Monte Carlo simulation, as follows. A useful measure of the performance of a CUSUM is the average number of observations required to signal an out-of-control state after the onset of a change,

$$E[N - \tau | N \geq \tau].$$

This is called the out-of-control (OOC) ARL. We show some Monte Carlo simulated OOC ARL results in Table 5. In the table the numbers to the right of $W(\mu)$ are the estimated OOC ARLs of a Wilcoxon S-R CUSUM at a shift $\mu$ while those to the right of $N(\mu \theta_0)$ denote the estimated OOC ARLs of a standard normal distribution S-R CUSUM at a shift $\mu \theta_0$. The heuristic says that these two OOC ARLs should not differ by too much. We run the Wilcoxon CUSUM under standardised normal and $t_3$ distributions, shifted by $\mu$ from observation $\tau = 100$ onward. The normal S-R CUSUM is run under a standard normal distribution, shifted by $\mu \theta_0$. The design parameters $(\delta, \zeta, h)$ used in each simulation run are shown in the third column of the table with $\delta$ denoting the target shift size. Inspection of the results in the last six columns of Table 5 indicates that the approximation

$$W(\mu) \approx N(\mu \theta_0)$$

is indeed acceptably accurate in the cases considered.

Table 5: S-R CUSUM OOC ARL approximations where $S := W$ and $S := V$ are the S-R CUSUMs based on the Wilcoxon SSR and Van der Waerden SSR scores, respectively. The target shift and reference value are abbreviated $\delta$ and $\zeta$, while $\mu$ is the true out-of-control mean in units of $\sigma$. The control limit is $h$. 
5 Comparison with a sequential rank Page-type CUSUM

The conclusion arrived at Moustakides et al. (2009) when comparing the standard normal S-R and Page CUSUMs is that the only marked difference in out-of-control performance occurs at small shifts $\mu$. In view of this, and taking into account the heuristic result formulated in Section 4, one would expect a similar conclusion to hold in respect of the signed sequential rank equivalents. We compare the Wilcoxon S-R CUSUM from Section 3 with the Wilcoxon SSR CUSUM developed in Lombard and Van Zyl (2018). Table 6 shows estimated OOC ARLs from 20 000 independent Monte Carlo trials per case, of both CUSUMs for data from a standardised normal ($\theta_0 = 0.98$) and $t_3$ ($\theta_0 = 1.38$) distribution shifted by an amount $\mu$ after the changepoints $\tau = 50$ and $\tau = 250$. The target shift sizes are $\delta = 0.25$, 0.5 and 1.0 and we use $\zeta = \delta \theta_0 / 2$ as the reference value. In the table we abbreviate the Page-type CUSUM by “P” and the Shiryaev-Roberts CUSUM by “S-R”. The reference value $\zeta$ and the corresponding control limit $h$ that guarantees an $ARL_0$ of 500 are shown in the bottom rows of each table. Clearly, the S-R CUSUM performs much better than the counterpart Page CUSUM when the true shift $\mu$ is small, while there is little or no difference between the OOC ARLs at large shifts $\mu$.

| Distribution | $\theta_0$ | $(\delta; \zeta; h)$ | Approx. | $\mu$ | $0.125$ | $0.25$ | $0.375$ | $0.50$ | $0.75$ | $1.00$
|--------------|----------|-----------------|---------|------|-------|-------|-------|-------|-------|-------|
| normal       | 0.98     | (0.25; 0.12; 6.08) | $W(\mu)$ | 116  | 56    | 36    | 26    | 18    | 14    |       |
|              |          |                 | $N(\mu \theta_0)$ | 114  | 55    | 35    | 26    | 17    | 13    |       |
|              |          | (0.50; 0.245; 5.92) | $W(\mu)$ | 144  | 63    | 35    | 24    | 15    | 11    |       |
|              |          |                 | $N(\mu \theta_0)$ | 142  | 62    | 35    | 24    | 15    | 11    |       |
| $t_3$        | 1.38     | (0.25; 0.173; 6.03) | $W(\mu)$ | 87   | 38    | 24    | 18    | 12    | 10    |       |
|              |          |                 | $N(\mu \theta_0)$ | 88   | 37    | 23    | 17    | 11    | 8     |       |
|              |          | (0.50; 0.345; 5.77) | $W(\mu)$ | 115  | 43    | 24    | 16    | 10    | 8     |       |
|              |          |                 | $N(\mu \theta_0)$ | 110  | 42    | 23    | 15    | 9     | 7     |       |

Table 6: OOC ARL comparison of the SSR Page CUSUM with the SSR S-R CUSUM, both using the Wilcoxon score.
The data and the particular application from which they arose are the same as those considered by Lombard and Van Zyl (2018, Section 7). In the latter paper, the Page-type Wilcoxon CUSUM signaled at $i = 235$ and the changepoint was estimated as $\hat{\tau} = 214$. We now apply the Wilcoxon S-R CUSUM to these data.

As before, the target change size is $\delta = 0.25$ and we use $\theta_0 = 1.18$. We run two S-R CUSUMs $C^+$ and $C^-$ – see (5) – with reference values $\zeta = 0.15$ and $\zeta = -0.15$, respectively. Each CUSUM is run at an $ARL_0$ value of 2000, which gives an overall $ARL_0$ of approximately 1000. The appropriate control limit is $h = 7.42$. The CUSUMs are shown in Figure 1 which shows the plots $(i, C^+_i)$, $1 \leq i \leq 275$, the upper CUSUM, and $(i, -C^-_i)$, $1 \leq i \leq 275$, the lower CUSUM. An change from zero to a positive median is detected at observation $N = 239$, quite close to that detected by the Wilcoxon Page-type CUSUM.

Finally, we come to estimation of the changepoint. A distinguishing feature of a two-sided Page-type CUSUM is that the upper (lower) CUSUM is at zero whenever the lower (upper) CUSUM is non-zero. The usual changepoint estimator is then the last index at which the hitting CUSUM sequence, upper or lower, was at zero. The same feature is not present in an S-R CUSUM, so that an alternative estimator must be sought.

A straightforward approach is to look upon the stopped sequence
$X_1, X_2, \ldots, X_N$ as a fixed set of data from two normal distributions and to estimate the changepoint by maximum likelihood (forgetting that $N$ is in fact a random variable). The maximum likelihood estimator is

$$\hat{\tau} = \text{arg} \max_{1 \leq i \leq N-1} |T_i|$$

where

$$T_i = \sum_{k=i}^{N} X_k / \sqrt{N + 1 - i},$$

see Pignatiello and Samuel (2001). This suggests that we use the same estimator after replacing $X_i$ by $\xi_i$ from (4). The plot of this $|T_i|$ against $i$ is shown in Figure 2. The maximum occurs at $\hat{\tau} = 218$, again quite close to the value 214 from the Wilcoxon Page-type CUSUM.

Figure 1: The Wilcoxon S-R CUSUMs. The control limit and changepoint estimate is indicated by the dashed horizontal and vertical barriers, respectively.

Figure 2: The changepoint estimator statistic $|T_i|$. 

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7 Summary

We develop Shiryaev-Roberts CUSUMs based on signed sequential ranks to detect a change in the median of an unspecified symmetric distribution. The CUSUMs are distribution-free, meaning that the control limits apply regardless of the functional form of the underlying distribution. We also show that the out-of-control behaviour of the CUSUMs are adequately approximated by those of a standard normal Shiryaev-Roberts CUSUM with appropriately adjusted out-of-control means. Monte Carlo simulation results indicate that a Wilcoxon Shiryaev-Roberts CUSUM performs quite well under a broad range of circumstances. In particular, it seems to be more adept at detecting small changes than the corresponding Wilcoxon Page-type CUSUM.

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8 Appendix

The computations used to obtain the control limits in Table 3 were as follows. Since the partial sums of the $\xi_i$ are approximately normally distributed, it is not difficult to imagine that the control limits $h$ of the CUSUM will correspond closely to those of a standard normal CUSUM. Given a set of reference values and nominal in-control ARL values $ARL_0$, denote by $h_1$ the corresponding set of control limits from a standard normal CUSUM. The first step of an iterative process was to estimate the in-control ARL of the S-R CUSUM on a $(\zeta, h_1)$ grid using, for instance, 10 000 independent Monte Carlo generated realizations, with a uniform distribution on $[-1, 1]$ serving as in-control distribution. Denote the resulting estimates by $\hat{A}(\zeta, h_1)$. Cubic spline interpolation between $(\zeta, \hat{A}(\zeta, h_1))$ and $(\zeta, h)$ then yielded new estimates, $h_2$, of the correct control limits. A further 10 000 independent Monte Carlo generated realizations using $h_2$ produced new estimated in-control ARLs $\hat{A}(\zeta, h_2)$. This process was repeated until all the differences $|\hat{A}(\zeta, h) - ARL_0|$ were less than 3. For $\zeta \leq 0.25$, no more that three iterations were required, while for $\zeta \geq 0.25$, six iterations sufficed. Finally, the control limits were all checked independently in 100 000 Monte Carlo runs. The largest difference between nominal and estimated in-control ARLs was 3.