HIKITA-NAKAJIMA CONJECTURE FOR THE GIESEKER VARIETY

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Abstract. Let $Y, Y'$ be a pair of symplectically dual conical symplectic singularities. Assume that $Y$ has a symplectic resolution of singularities $X \to Y$. The equivariant Hikita conjecture (that we call Hikita-Nakajima conjecture) claims that there should be an isomorphism of (graded) algebras $H^*_S Y(X, \mathbb{C}) \cong \mathbb{C}[(Y', \text{univ}) \times \mathbb{C}^\times]$, here $S_Y \acts Y$ is the torus acting on $Y$ preserving the Poisson structure, $Y'$ is the universal deformation of $Y'$, $\mathbb{C}^\times$ is a generic one-dimensional torus acting on $Y'$, and $\mathbb{C}[(Y', \text{univ}) \times \mathbb{C}^\times]$ is the algebra of schematic $\mathbb{C}^\times$-fixed points of $Y'$. We prove the Hikita-Nakajima conjecture for $X = \mathcal{M}(n, r)$ Gieseker variety (ADHM space). We produce the isomorphism explicitly on generators. It turns out that if we realize $\mathcal{M}_0(n, r)^{\text{univ}}$ as a Coulomb branch (as spectrum of the algebra of equivariant homology of the variety of triples), then Chern classes of the tautological bundle on $\mathcal{M}(n, r)$ correspond to Chern classes of the tautological bundle on the variety of triples. We also describe the Hikita-Nakajima isomorphism above using realization of $\mathcal{M}_0(n, r)^{\text{univ}}$ as a Nakajima quiver variety and as the spectrum of the center of rational Cheredhik algebra corresponding to $S_n \times (\mathbb{Z}/r\mathbb{Z})^n$ and identify all the algebras that appear with the center of degenerate cyclotomic Hecke algebra (generalizing some results of Shan, Varagnolo and Vasserot). Finally, we formulate as a conjecture that when $X$ is a Nakajima quiver variety then the Hikita-Nakajima isomorphism should identify Chern classes of tautological bundles on $X$ with Chern classes of tautological bundles on the corresponding variety of triples and describe possible approaches towards the proof.

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(resp. Poisson) deformations of $X$ (resp. $Y$) over the base $t_Y := H^2(X, \mathbb{C})$ that we call universal deformation of $X$ (resp. $Y$) and denote by

$$X^{\text{univ}} \to t_Y, \ Y^{\text{univ}} \to t_Y.$$ 

**Remark 1.3.** One can show that the space $t_Y$ does not depend on the choice of the resolution $X$ of $Y$, this space can be defined even if $Y$ does not have a symplectic resolution $X$.

Let $\text{Aut}_{C^\times}(Y)$ be the group of Poisson automorphisms of $Y$ commuting with the contracting $C^\times$. This is a finite dimensional algebraic group. We denote by $S_Y \subset \text{Aut}_{C^\times}(Y)$ a maximal torus of $\text{Aut}_{C^\times}(Y)$ and set $s_Y := \text{Lie } S$. One can show that the action of $S_Y$ on $Y$ and the contracting action of $C^\times$ extend naturally to the action of $S_Y \times C^\times$ on $Y^{\text{univ}}$ (torus $S_Y$ acts fiberwise).

Often conical singularities come in “dual” pairs $Y, Y^!$.

We refer the reader to [BLPW14] for details on symplectic duality, let us just recall that for symplectically dual varieties $Y, Y^!$ one should have the natural identifications

$$t_Y \simeq s_Y^!, \ s_Y \simeq t_Y^!.$$ 

Also it is expected that the choice of $X$ (choice of the symplectic resolution of $Y$) should correspond on the dual side to a cocharacter $\nu_X : C^\times \to S_Y^!$ (actually to a choice of a certain chamber in $\text{Hom}(C^\times, T) \otimes \mathbb{Z} \mathbb{R}$) such that the set $(Y^!)^{\nu_X(C^\times)}$ consists of one point.

1.2. Schematic fixed points and Hikita-Nakajima conjectures.

1.2.1. Schematic fixed points. Given a variety with an action of some algebraic group $G$ we can define the functor

$$Y^G : \text{Schemes}_{\mathbb{C}} \to \text{Sets}$$

from the category $\text{Schemes}_{\mathbb{C}}$ of schemes over $\mathbb{C}$ to the category $\text{Sets}$ of sets as follows (see [Fog73, Dri13, Section 1.2]):

$$Y^G(S) := \text{Maps}^G(S, Y), \ S \in \text{Schemes}_{\mathbb{C}};$$

where the action of $G$ on $S$ is trivial and $\text{Maps}^G(S, Y)$ is the set of $G$-equivariant morphisms from $S$ to $Y$.

It turns out that in some cases functor $Y^G$ is represented by a scheme that we call schematic fixed points of $Y$ (for more details, see [Fog73, Theorem 2.3]).

We now consider the case $Y = \text{Spec } B$ for some $\mathbb{C}$-algebra $B$ and $G = C^\times$. Then the action $C^\times \curvearrowright Y$ corresponds to the $\mathbb{Z}$-grading $B = \bigoplus_{i \in \mathbb{Z}} B_i$. Compare the following Proposition with [Dri13, Example 1.2.3].

**Proposition 1.4.** If $Y = \text{Spec } B$ is an affine variety and $G = C^\times$, then $Y^{C^\times}$ is represented by an affine scheme whose ring of functions can be described in two equivalent ways:

$$(1.1) \quad \mathbb{C}[Y^{C^\times}] = B_0/ \sum_{i > 0} B_{-i} B_i = B/(b_i \in B_i, i \neq 0).$$

**Proof.** It is enough to show that the functor $Y^{C^\times}$ being restricted to the category of affine schemes over $\mathbb{C}$ is represented by the affine scheme with the algebra of functions as in (1.1).
Let $S = \text{Spec } C$ be an affine scheme with trivial $\mathbb{C}^\times$-action. The set $\text{Maps}_{\mathbb{C}^\times}(S,Y)$ identifies with the set of graded homomorphisms $B \to C$, where the grading on $C$ is the trivial one ($C = C_0$). Since $C = C_0$, we conclude that every such homomorphism $f: B \to C$ factors through $B/(b_i \in B_i, i \neq 0)$. Note now that every homomorphism $\overrightarrow{f}: B/(b_i \in B_i, i \neq 0) \to C$ induces the graded homomorphism $B \to B/(b_i \in B_i, i \neq 0) \to C$, so we must have $Y_{\mathbb{C}^\times} = \text{Spec}(B/(b_i \in B_i, i \neq 0))$.

It remains to note that the natural morphism

$$B_0/ \sum_{i>0} B_{-i}B_i \to B/(b_i \in B_i, i \neq 0),$$

given by

$$B_0/ \sum_{i>0} B_{-i}B_i \ni [b] \mapsto [b] \in B/(b_i \in B_i, i \neq 0)$$

is an isomorphism. $\square$

**Remark 1.5.** Proposition 1.4 can be easily generalized to the case when $G$ is a torus of arbitrary rank.

The following Proposition holds (see, for example, [KMc18]).

**Proposition 1.6.** If $Y$ is a smooth algebraic variety over $\mathbb{C}$ and $G$ is reductive, then $Y^G$ is smooth.

### 1.2.2. Hikita-Nakajima conjecture.

Let $Y, Y'$ be symplectically dual conical symplectic singularities such that $Y$ has a symplectic resolution $\overline{X} \to Y$. Let $H^{\bullet}_{S_{Y'}}(X, \mathbb{C})$ be the algebra of $S_{Y'}$-equivariant cohomology of $X$. This is an algebra over $H^{\bullet}_{S_{Y'}}(pt) = \mathbb{C}[s_{Y'}]$. Recall also that the choice of $X$ (resolution of $Y$) corresponds to a (generic) cocharacter $\nu_X: \mathbb{C}^\times \to S_{Y'}$. We can consider the algebra of functions of schematic fixed points $\mathbb{C}[[Y^{\text{univ}}]^{\nu_X(\mathbb{C}^\times)}]$ that is an algebra over $\mathbb{C}[s_{Y'}] = \mathbb{C}[s_{Y'}]$. Note also that the algebra $H^{\bullet}_{S_{Y'}}(X, \mathbb{C})$ has a natural cohomological grading and the algebra $\mathbb{C}[[Y^{\text{univ}}]^{\nu_X(\mathbb{C}^\times)}]$ is graded via the contracting $\mathbb{C}^\times$-action. Note now that the algebra $H^{\bullet}_{S_{Y'}}(X, \mathbb{C})$ is finitely generated over $\mathbb{C}[s_{Y'}]$. The algebra $\mathbb{C}[[Y^{\text{univ}}]^{S_{Y'}}]$ is considered as a set consists of one point. Since we want to identify the algebras above we must make the following assumption.

**Assumption 1.7.** The set $(Y')^{S_{Y'}}$ consists of one point.

**Remark 1.8.** Let us recall that symplectic duality predicts that $Y$ has a symplectic resolution iff $(Y')^{S_{Y'}}$ consists of one point.

The following conjecture belongs to Hikita in nonequivariant version and to Nakajima in general. We will call it Hikita-Nakajima conjecture (one can also call it equivariant Hikita conjecture).

**Conjecture 1.9.** There is an isomorphism of $\mathbb{Z}$-graded algebras over $\mathbb{C}[s_{Y'}]$: $H^{\bullet}_{S_{Y'}}(X, \mathbb{C}) \simeq \mathbb{C}[[Y^{\text{univ}}]^{\nu_X(\mathbb{C}^\times)}]$.

**Remark 1.10.** Actually Nakajima’s version of this conjecture is even more general: on the LHS we can consider the algebra $H^{\bullet}_{S_{Y'} \times \mathbb{C}^*_h}(X, \mathbb{C})$ and on the RHS we can consider the Rees algebra of “quantum schematic fixed points” (so-called $B$-algebra or Cartan subquotient) of the universal quantization of $\mathbb{C}[Y']$ (see, for example, [KMc22 Section 5.6]). In [KMcP21], another generalization of this conjecture is proposed.
Remark 1.11. Let us note that if $X$, $X'$ are two symplectic resolutions of $Y$, then the algebras $H^*_S(Y, \mathbb{C})$, $H^*_S(Y', \mathbb{C})$ are isomorphic. This follows from the fact that the universal deformations $X^{\text{univ}} \rightarrow S_Y$, $X'^{\text{univ}} \rightarrow S_Y$ are locally trivial in $C^\infty$-topology (see [Nam10] Section 1.2 and references therein), so $X$, $X'$ are both diffeomorphic to a generic fiber of $Y^{\text{univ}} \rightarrow S_Y$. We are grateful to Alexander Braverman for explaining this to us. Similarly, one can see that for any generic cocharacter $\nu: \mathbb{C}^\times \rightarrow S_Y$, the schematic fixed points $(Y^{\text{univ}})^{\nu}(\mathbb{C}^\times)$ are the same and are isomorphic to the schematic fixed points $(Y^{\text{univ}})^{S_{Y'}}$. Indeed, note that $Y^{\text{univ}}$ can be $S_{Y'}$-equivariantly embedded in some vector space $O$ with a linear action of $S_{Y'}$. Then $(Y^{\text{univ}})^{\nu}(\mathbb{C}^\times)$, $(Y^{\text{univ}})^{S_{Y'}}$ are (schematic) intersections

$$
(Y^{\text{univ}})^{\nu}(\mathbb{C}^\times) = Y^{\text{univ}} \cap O^{\nu}(\mathbb{C}^\times), \quad (Y^{\text{univ}})^{S_{Y'}} = Y^{\text{univ}} \cap O^{S_{Y'}}.
$$

This reduces the claim to showing that $O^{\nu}(\mathbb{C}^\times) = O^{S_{Y'}}$ for a generic $\nu: \mathbb{C}^\times \rightarrow S_{Y'}$, that easily follows from Proposition 1.10 since $O$ is smooth. We are grateful to Alexander Braverman for explaining this to us.

Let us briefly recall the current state of Hikita-Nakajima conjecture. The nonequivariant version of this conjecture was proven for the case of Hilbert scheme of points on $\mathbb{A}^2$, type A Slodowy slices and hypertoric varieties in [Hik17, Theorem 1.1, Theorem B.1]. In [Sh19] the second author has proven the case of $Y = \mathbb{C}^2/\Gamma$ where $\Gamma$ is a finite subgroup of SL$_2(\mathbb{C})$ with McKay “dual” of type $ADE$. In the paper [Hat21] Theorem 1.0.5, Hatano proved that $H^*(\mathcal{M}(n,r), \mathbb{C})$ and $\mathbb{C}[(\mathcal{M}_0(n,r))^{\mathbb{C}^\times}]$ are isomorphic as graded vector spaces. In [KTWWY1], Theorem 1.5], Kamnitzer, Tingley, Webster, Weekes and Yakobi has proven nonequivariant version of the conjecture for $ADE$ slices in affine Grassmanian, they have also equivariant and even quantized (see Remark 1.10) version of Conjecture 1.9 for type A quivers and some weak form of this conjecture for $DE$ quivers (see [KTWWY1], Section 8.3), [KTWWY2] Section 1.3 and Theorem 1.5, see also [Kam22] Section 6.6).

The main result of this paper is the following Theorem.

Theorem 1.12. Hikita-Nakajima conjecture holds for $X = \mathcal{M}(n,r)$, the Gieseker variety.

Actually we will describe the isomorphism explicitly using certain generators of the algebras, mentioned above (see Theorem 1.12 for more details).

Remark 1.13. Let us point out that our approach gives a new proof of the Hikita conjecture for the Hilbert scheme case (when $r = 1$) even in the non-equivariant setting (which was proved by Hikita himself). It also generalizes the results of [Hat21], where the author proves that $H^*(\mathcal{M}(n,r), \mathbb{C})$, $\mathbb{C}[(\mathcal{M}_0(n,r))^{\mathbb{C}^\times}]$ are isomorphic as graded vector spaces (Appendix 4 repeats some of the arguments of this paper). The main idea of our paper is that using deformations simplifies the picture.

1.3. Gieseker variety (moduli space of instantons). Gieseker variety $\mathcal{M}(n,r)$ depends on a pair $n,r \in \mathbb{Z}_{\geq 1}$ of positive integer numbers. It can be realized as a Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.4). Variety $\mathcal{M}(n,r)$ also has a realization as the moduli space of torsion free sheaves on $\mathbb{P}^2$ of rank $r$ with second Chern class being $n$ and with a fixed trivialization at the line at infinity (see [Nak99] for details). Variety $\mathcal{M}(n,r)$ is a very important object that originally came from physics (it has an interpretation as the moduli space of certain instantons on $\mathbb{R}^4$). Gieseker variety $\mathcal{M}(n,r)$ is a resolution of singularities of the variety $\mathcal{M}_0(n,r) = \text{Spec} \mathbb{C}[\mathcal{M}(n,r)]$. Variety $\mathcal{M}_0(n,r)$ has a realization as an (affine) Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.4).

So we are taking $X = \mathcal{M}(n,r)$, $Y = \mathcal{M}_0(n,r)$. Then the torus $S_Y$ can be described as follows. We have a natural symplectic action of SL$_r$ on $\mathcal{M}(n,r)$ (via changing the trivialization at infinity) and also the action of $T = \mathbb{C}^\times$ via the action on $\mathbb{P}^2$ that
multiplies one coordinate by \( t \) and another by \( t^{-1} \) (so-called “hyperbolic action”).

Let \( T_r \subset \text{SL}_r \) be a maximal torus. The torus \( S_Y \) is the image of \( A := T \times T_r \) in \( \text{Aut}_{\text{C}^n}^* (\mathfrak{M}(n, r)) \) and \( s_Y \) naturally identifies with \( a := \text{Lie} A. \) The space \( t_Y = H^2(\mathfrak{M}(n, r), \mathbb{C}) \) is known to be one-dimensional (follows, for example, from \( [MNTS] \)).

1.4. Symplectic dual to \( \mathfrak{M}(n, r) \). Let us now give a very sketchy description of the variety \( \mathfrak{M}_0(n, r) \text{^}{1} \) and its universal deformation \( \mathfrak{M}_0(n, r) \text{^}{1,\text{univ}} \). There are three different constructions of this variety.

1.4.1. Dual as the Coulomb branch and as the center of the rational Cherednik algebra. One way to construct dual to \( \mathfrak{M}_0(n, r) \) is via Coulomb branches introduced in \( [BFN] \). In this approach, \( \mathfrak{M}_0(n, r) \text{^}{1} = \mathcal{M}_{n, r} \) is equal to the spectrum of the algebra \( \mathcal{H}^*(\mathcal{R}_{n, r};\mathcal{C}) \) of \( \text{GL}_n \)-equivariant Borel-Moore homology of the variety of triples \( \mathcal{R}_{n, r} \) corresponding to the Jordan quiver (see Section \( 3.1.1 \) for details). The variety \( \mathfrak{M}_0(n, r) \text{^}{1,\text{univ}} = \mathcal{M}_{n, r} \text{^}{\text{univ}} \) is then the spectrum of \( A \times (\text{GL}_n)_{\text{C}} \) - equivariant homology of the same variety of triples \( \mathcal{R}_{n, r} \) (see Section \( 3.1 \) for details).

It is known (see \( [KNIS], [BEF20], [Weg19] \)) that the Coulomb branch \( \mathcal{M}_{n, r} \text{^}{\text{univ}} \) above can be realized as the spectrum of the center \( Z(H_{n, r}) \) of the rational Cherednik algebra \( H_{n, r} \) corresponding to the group \( \Gamma_n := S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n \) (see Section \( 3.1.2 \) for the details).

So summarizing we have

\[
\mathfrak{M}_0(n, r) \text{^}{1,\text{univ}} = \mathcal{M}_{n, r} \text{^}{\text{univ}} = \text{Spec } Z(H_{n, r}).
\]

The algebra \( H_{n, r} \) has a natural \( \mathbb{Z} \)-grading (see \( \{3.1\} \)) that induces the action of \( \mathbb{T} = \mathbb{C}^\times \) on \( \text{Spec } Z(H_{n, r}) = \mathcal{M}_{n, r} \text{^}{\text{univ}} \).

Remark 1.14. The action \( \mathbb{T} \ltimes \mathcal{M}_{n, r} \text{^}{\text{univ}} \) in Coulomb terms is described in \( [BFN] \) Section 3 (v).

1.4.2. Dual as a Nakajima quiver variety. One can also construct \( \mathfrak{M}_0(n, r) \text{^}{1} \) as the (affine) Nakajima quiver variety corresponding to the cyclic quiver with \( r \) vertices labeled by \( \mathbb{Z}/r\mathbb{Z} \) and having \( n \)-dimensional vector spaces placed at these vertices and one-dimensional framing at the vertex corresponding to zero (see Section \( 3.2 \) and Figure \( \#1 \)). We denote this quiver variety by \( \mathcal{X}_0(n, r) \). One can also consider its universal version that we denote by \( \mathcal{X}_0 \text{^}{\text{univ}}(n, r) \) (that is the quotient of the fiber of the moment map over the center of \( \mathfrak{g}_n^\text{aff} \)). We have

\[
\mathfrak{M}_0(n, r) \text{^}{1,\text{univ}} = \mathcal{X}_0 \text{^}{\text{univ}}(n, r).
\]

Torus \( \mathbb{T} = \mathbb{C}^\times \) acts naturally on \( \mathcal{X}_0(n, r), \mathcal{X}_0 \text{^}{\text{univ}}(n, r) \) (see \( \{3.6\} \)).

The identification \( \mathcal{X}_0 \text{^}{\text{univ}}(n, r) \simeq \text{Spec } Z(H_{n, r}) \) of two realizations of \( \mathfrak{M}_0(n, r) \text{^}{1,\text{univ}} \) is provided by the Etingof-Ginzburg isomorphism \( \text{EG} \) (see Section \( 3.3 \)).

1.5. Cyclotomic Hecke algebra and its center. Let \( Q_{n, r} \) be the algebra of functions on schematic \( \mathbb{T} \)-fixed points of \( \text{Spec } Z(H_{n, r}) \) (that can be also considered as an algebra of functions on \( (\mathcal{M}_{n, r})^\text{univ} \text{^}{\mathbb{T}} \)). Recall that by Proposition \( 1.3 \) explicitly we have

\[
Q_{n, r} = \frac{Z(H_{n, r})_0}{\sum_{i > 0} Z(H_{n, r})_{-i} Z(H_{n, r})_{i}} = \frac{Z(H_{n, r})/(b \in Z(H_{n, r}), i \neq 0)}{Z(H_{n, r})},
\]

the grading on \( H_{n, r} \) is as in \( \{3.3\} \):

\[
\deg x_j = 1, \deg y_j = -1, \deg \Gamma_n = \deg h = 0.
\]
The Etingof-Ginzburg isomorphism induces the identification $\mathbb{C}[(\mathcal{X}_0^{\text{univ}}(n,r))^T] \simeq \mathbb{Q}_{n,r}$ and our goal is to identify these algebras with the algebra $H^*_A(\mathcal{M}(n,r), \mathbb{C})$. It turns out that there is another algebra that is isomorphic to both of the algebras above. This algebra is the center $Z(R^r(n))^{JM}$ of the cyclotomic degenerate Hecke algebra $R^r(n)$ (see Section 5.1 for the definition). It was observed in [SVV17] that algebras $Z(R^r(n))^{JM}$, $H^*_A(\mathcal{M}(n,r), \mathbb{C})$ are isomorphic. We give an independent (but certainly similar) proof of this fact.

1.6. Main idea of the proof. It turns out that there exists one “universal” approach that allows us to identify algebras

\begin{equation}
H^*_A(\mathcal{M}(n,r), \mathbb{C}), \ Z(R^r(n))^{JM}, \ \mathbb{C}[(\mathcal{X}_0^{\text{univ}}(n,r))^T] \simeq \mathbb{Q}_{n,r} \simeq \mathbb{C}[(\mathcal{M}_n^{\text{univ}})^T]
\end{equation}

with each other simultaneously. The idea is simple: we embed all of algebras above inside the algebra

$$E := \bigoplus_{\lambda \in \mathcal{P}(r,n)} \mathbb{C}[\mathbb{A}^\lambda] = \mathbb{C}[\mathbb{A}^r|^{\mathcal{P}(r,n)}]$$

and show that their images coincide. Here $\mathcal{P}(r,n)$ is the set of $r$-multipartitions of $n$ (see Definition 1.3). In order to show that images are the same we consider natural generators of these algebras and show that their images in $E$ are the same. In particular, we obtain explicit descriptions for isomorphisms between algebras in (1.2).

Remark 1.15. Let $F$ be the function field of the parameter space $\mathbb{A}^r$. We will see that the embeddings above become isomorphisms after tensoring by $F$ or more precisely after localizing at certain finite set of elements of $\mathbb{C}[\mathbb{A}^r]$ (certain “walls”).

1.6.1. Embedding $Z(R^r(n))^{JM} \subset E$ and generators of $Z(R^r(n))^{JM}$. For every $\lambda \in \mathcal{P}(r,n)$ one can consider the corresponding “universal” Specht modules over $R^r(n)$ that we denote by $\tilde{S}_k(\lambda)$ (see Section 5 for details). Acting by the center $Z(R^r(n))^{JM}$ of $R^r(n)$ on these modules we obtain the desired embedding

$$\psi : Z(R^r(n))^{JM} \subset \bigoplus_{\lambda \in \mathcal{P}(r,n)} \text{End}_{R^r(n)}(\tilde{S}_k(\lambda)) = E.$$ 

It follows from [Bru08 Theorem 1] that natural generators of $Z(R^r(n))^{JM}$ are classes of elements

$$e_k(z_1, \ldots, z_n), \ k = 1, \ldots, n,$$

here $e_k \in \mathbb{C}[z_1, \ldots, z_n]^S_n$ are elementary symmetric polynomials.

1.6.2. Embedding $H^*_A(\mathcal{M}(n,r), \mathbb{C}) \subset E$ and generators of $H^*_A(\mathcal{M}(n,r), \mathbb{C})$. It is well-known that the set of $A$-fixed points $\mathcal{M}(n,r)$ can be parametrized by $\mathcal{P}(r,n)$ (see Section 4 for details).

We have the natural embedding

$$\iota : \mathcal{M}(n,r)^A \subset \mathcal{M}(n,r)$$

that induces the desired embedding

$$\iota^* : H^*_A(\mathcal{M}(n,r), \mathbb{C}) \subset \mathcal{M}(n,r)^A, \mathbb{C}) = E.$$ 

It remains to note that the algebra $H^*_A(\mathcal{M}(n,r), \mathbb{C})$ is generated by elements

$$c_k(V), \ k = 1, \ldots, n,$$

where $c_k(\bullet)$ is the $k$th $A$-equivariant Chern class and $V$ is the tautological $n$-dimensional vector bundle on $\mathcal{M}(n,r)$. This, for example, follows from [MN18 Corollary 1.5].
1.6.3. Embedding $Q_{n,r} \subset E$ and generators of $Q_{n,r}$. Recall that $Q_{n,r}$ is a quotient of $Z(H_{n,r}) \subset H_{n,r}$. To every $\lambda \in \mathcal{P}(r,n)$ one can associate the corresponding simple $H_{n,r}$-module $L(\lambda)$ on which $Z(H_{n,r})$ will act by some character and this action factors through the action of $Q_{n,r}$ (see Section 6.1). This gives us the desired embedding

$$\phi: Q_{n,r} \subset \bigoplus_{\lambda \in \mathcal{P}(r,n)} \text{End}_{H_{n,r}}(L(\lambda)) = E.$$ 

Generators of $Q_{n,r}$ are classes of

$$e_k(u_1, \ldots, u_n), \quad k = 1, \ldots, n,$$

where $u_i \in H_{n,r}$ are Dunkl-Opdam elements (see Lemma 6.5).

**Remark 1.16.** The identification $Z(H_{n,r}) \simeq H_n^{\Delta \times (\text{GL}_n)c}(\mathcal{R}_{n,r})$ sends $e_k(u_1, \ldots, u_n)$ to the Chern class $e_k(E)$ (see [Web19]), where $E$ is the tautological $(\text{GL}_n)c$-bundle on $\mathcal{R}_{n,r}$. This should be compared with the fact that generators on the dual side (i.e., generators of $H^\Delta_A(\text{M}(n,r), \mathbb{C})$) are Chern classes $c_i(V)$ of the tautological bundle $V$.

1.6.4. Embedding $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T] \subset E$ and generators of $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T]$. We do not use results of this Section in the main body of the paper. We are actually deducing all the claims about $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T]$ from the corresponding claims about $Q_{n,r}$ (see Section 8). The only case when we deal with $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T]$ without referring to the Cherednik side is $r = 1$. This particular case is considered in details in Section 2.

The main purpose of this Section is to point out that the embedding of the algebra of schematic fixed points $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T] \subset E$ and generators of $\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T]$ can be constructed without using of Etingof-Ginzburg isomorphism (i.e., without passing to the Cherednik side of the picture).

In [Prz16] Section 5, the parametrization of $T$-fixed points of smooth fibers of the family $X^\text{univ}_0(n,r) \to \mathbb{A}^r$ is given. It turns out that the set parametrizing these fixed points consists of $nr$-partitions with empty $r$-core and is denoted by $\mathcal{P}_\circ(nr)$ (see [Prz16] Section 4 for details). It also follows from [Prz16] Section 4.5 that we have the bijection $\mathcal{P}_\circ(nr) \xrightarrow{\sim} \mathcal{P}(r,n)$ given by $\mu \mapsto \text{Quot}(\mu)^T$, where $\circ$ corresponds to reversing the multipartition. Using the bijection above we identify $\mathcal{P}_\circ(nr) = \mathcal{P}(r,n)$ from now on.

**Remark 1.17.** The bijection above is nothing else but the bijection at the level of $T$-fixed points induced by the Etingof-Ginzburg isomorphism $\mathcal{E}\mathcal{G}$ at the fiber of a generic point (see [Prz16] Theorem 1.2).

It is easy to see that the construction of $T$-fixed points in [Prz16] can be done “universally”, i.e., it can be considered as a morphism of schemes

$$\bigcup_{\lambda \in \mathcal{P}(r,n)} \mathbb{A}^r \to (\lambda^\text{univ}_0(n,r))^T$$

which induces the desired embedding

$$\mathbb{C}[(\lambda^\text{univ}_0(n,r))^T] \subset \mathbb{C}

\bigcup_{\lambda \in \mathcal{P}(r,n)} \mathbb{A}^r = E.$$

**Remark 1.18.** Let us describe more general way to construct this embedding (which works for any symplectic resolution). We have a resolution $X^\text{univ}(n,r) \to X^\text{univ}_0(n,r)$ that induces morphism $(X^\text{univ}(n,r))^T \to (X^\text{univ}_0(n,r))^T$. Note now that $X^\text{univ}(n,r)$ is smooth, hence, $(X^\text{univ}(n,r))^T$ is also smooth and $\mathbb{C}[(X^\text{univ}(n,r))^T] = E$. Then the morphism above is nothing else but the induced homomorphism $\mathbb{C}[(X^\text{univ}_0(n,r))^T] \to \mathbb{C}[(X^\text{univ}(n,r))^T]$.

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Let us now describe natural generators of the algebra $\mathbb{C}[(\lambda_0^{\text{univ}}(n, r))^\mathbb{T}]$, we use the notations introduced in Section 3.2 (see Figure 1). Let us also point out that this Lemma is a particular case of a very general Proposition 3.7 See also Lemma 6.5 for an alternative argument.

**Lemma 1.19.** The algebra $\mathbb{C}[(\lambda_0^{\text{univ}}(n, r))^\mathbb{T}]$ is generated by $e_k(\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ is the multiset of eigenvalues of $Y_1X_0$ (in other words, ± coefficients of the characteristic polynomial of $Y_1X_0$).

**Proof.** We give a sketch of the proof. It is enough to show that restrictions of functions $(X, Y, \gamma, \delta) \mapsto \text{tr}((Y_1X_0)^i)$, $i \in \mathbb{Z}_{\geq 0}$ from $\mathcal{X}_0^{\text{univ}}(n, r)$ to $\mathcal{X}_0^{\text{univ}}(n, r))^\mathbb{T}$ generate the algebra $\mathbb{C}[(\mathcal{X}_0^{\text{univ}}(n, r))^\mathbb{T}]$. Moreover, by graded Nakayama lemma it is enough to show that restrictions of functions $(X, Y, \gamma, \delta) \mapsto \text{tr}((Y_1X_0)^i)$ from $\mathcal{X}_0(n, r)$ to $(\mathcal{X}_0(n, r))^\mathbb{T}$ generate the algebra $\mathbb{C}[(\mathcal{X}_0(n, r))^\mathbb{T}]$. In [Lus98 Theorem 1.3] the generators of $\mathbb{C}[\mathcal{X}_0(n, r)]$ are described.

Since we are passing to $\mathbb{T}$-fixed points then we are only interested in the generators of degree zero. It remains to show that the generators of degree zero are generated by functions $(X, Y, \gamma, \delta) \mapsto \text{tr}((Y_1X_0)^i)$. Since $\mathcal{X}_0(n, r)$ is reduced (see [GG05] and also [Los10 Section 4.4]) it is enough to check this on the stable locus of $\mathcal{X}_0(n, r)$. Note now that $\mathcal{X}_0(n, r)$ is resolved by the quiver variety $\mathcal{X}(n, r)$ and the resolution becomes isomorphism at the stable locus. Note also that $\mathcal{X}(n, r)$ is a closed subscheme of $\mathfrak{M}(nr, 1) = \text{Hilb}_n(\mathbb{A}^2)$. It then follows from [Nak99] that $\delta = 0$ for $(X, Y, \gamma, \delta) \in \mathcal{X}(n, r)$ so we conclude that

$$X_{i-1}Y_i = Y_{i+1}X_i$$

for every $i \in \mathbb{Z}/r\mathbb{Z}$. Then the claim follows from the description of generators in [Lus98 Theorem 1.3] using cyclic invariance of trace.

\[\Box\]

1.7. Main results and structure of the paper.

1.7.1. Identification of parameters and main Theorem. We have parameter spaces

$$\mathbb{C}[\kappa, c_1, \ldots, c_{l-1}], \mathbb{C}[\kappa, a_1, \ldots, a_r]/(a_1 + \ldots + a_r),$$

$$(1.3) \quad \mathbb{C}[h, H_0, H_1, \ldots, H_{r-1}]/(H_0 + \ldots + H_{r-1}), \mathbb{C}[\theta_0, \ldots, \theta_{r-1}]$$

$$(1.4) \quad \mathbb{C}[\kappa, c_1, \ldots, c_{l-1}], \mathbb{C}[\kappa, a_1, \ldots, a_r]/(a_1 + \ldots + a_r),$$

used in definitions of algebras (1.3), (1.4) as follows ($i = 1, \ldots, r$, $k = 1, \ldots, r - 1$):

$$a_i = p(\eta^{i-1}), \quad h = \kappa, \quad c_k = \sum_{p=0}^{r-1} \eta^{-kp}H_p, \quad \theta_0 = -h + H_0, \quad \theta_k = H_k,$$

where $p(q) = \frac{1}{r} \sum_{l=1}^{r-1} \frac{q^l}{1 - \eta^{-q^l}} \in \mathbb{C}[q], \quad \eta = e^{2\pi i/r}$. We will denote by $F$ the field of fractions of algebras (1.3), (1.4) identified via (1.5).

**Theorem 1.20.** After the identification of parameters (1.3) $\mathbb{Z}$-graded algebras

$$H_A^*(\mathfrak{M}(n, r)), Z(R^r(n)), Q_{n,r}, H_s^{\text{Ax}(\text{GL}_n)}(\mathcal{R}_{n,r}), \mathbb{C}[(\lambda_0^{\text{univ}}(n, r))^\mathbb{T}]$$

are isomorphic. Isomorphisms above identify generators as follows

$$c_k(\mathcal{V}) = [e_k(z_1, \ldots, z_n)] = [e_k(u_1, \ldots, u_n)] = [c_k(\mathcal{E})] = ((X, Y, \gamma, \delta) \mapsto e_k(\alpha_1, \ldots, \alpha_n)), \quad (1.6)$$
where \( \alpha_1, \ldots, \alpha_n \) is the multiset of eigenvalues of \( Y_i X_0 \), \( \mathcal{V} \) is the tautological rank \( n \) vector bundle on \( \mathcal{M}(n, r) \), \( \mathcal{E} \) is the tautological \( (\text{GL}_n)^\mathbb{C} \)-bundle on the variety of triples \( \mathcal{R}_{n,r} \), \( u_i \in H_{n,r} \) are Dunkl-Opdam elements (Definition (6.3)).

Remark 1.21. Note that isomorphisms above are automatically graded since they preserve the degree of generators (1.6). Note that the isomorphism \( H^*_A(\mathcal{M}(n, r)) \simeq H^*_A(\text{GL}_n)^\mathbb{C} (\mathcal{R}_{n,r}) \) is already an isomorphism of \( \mathbb{C}[k, a_1, \ldots, a_n]/(a_1 + \ldots + a_n) = \mathbb{C}[a]-\)algebras and there is no need in any identification of parameters.

1.7.2. Structure of the paper. The paper is organized as follows. In Section 2 we prove Hikita-Nakajima conjecture (equivariant version of Hikita conjecture) for \( r = 1 \), i.e., for Hilbert scheme \( \text{Hilb}_n(\mathbb{A}^2) \) using results of [Vas01]. In Section 3 we consider the case of arbitrary \( r \) and describe symplectically dual variety to \( \mathcal{M}_0(n, r) \) and its universal deformation using Coulomb branches, rational Cherednik algebras and Nakajima quiver varieties.

In Sections 4, 5, 6, we realize the idea of the proof of Hikita-Nakajima conjecture that we briefly explain in Section 1.6. More detailed, in Section 4 we describe the embedding \( H^*_A(\mathcal{M}(n, r), \mathbb{C}) \subset E \) and determine the image of generators \( c_i(\mathcal{V}) \in H^*_A(\mathcal{M}(n, r), \mathbb{C}) \) under this embedding. In Section 5 we define the cyclotomic degenerate Hecke algebra \( \Gamma^r(n) \) and recall its representation theory. We then describe the embedding \( Z(\Gamma^r(n))^{\text{univ}} \subset E \) (using representation theory of \( \Gamma^r(n) \)) and determine its image. In Section 6 we recall the representation theory of rational Cherednik algebra \( H_{n,r} \) and then describe the embedding \( Q_{n,r} \subset E \) (using the representation theory of \( H_{n,r} \)). In Lemma 6.5 we describe generators of \( Q_{n,r} \) and then determine their images under the embedding \( Q_{n,r} \subset E \). As a corollary of results of Sections 4, 5, 6, we almost obtain Theorem 1.20 except the explicit formula for the identification of \( H^*_A(\mathcal{M}(n, r), \mathbb{C}), Z(\Gamma^r(n)), Q_{n,r}, H^*_A(\text{GL}_n)^\mathbb{C} (\mathcal{R}_{n,r}) \) with \( \mathbb{C}[(\mathcal{M}^\text{univ}_{n,r})^\mathbb{C}] \). In Section 7 we explain the relation between \( H_{n,r} \) and the degenerate affine Hecke algebra corresponding to \( \Gamma_n \) (basically, we need this Section since we want to use the results of [Prz16]). We then use results of Section 7 in Section 8 to obtain the description of the isomorphism \( Q_{n,r} \simeq \mathbb{C}[(\mathcal{M}^\text{univ}_{n,r})^\mathbb{C}] \) (induced by Etingof-Ginzburg isomorphism). In Section 9 we prove Theorem 1.20 and then discuss possible approach to proving Hikita-Nakajima conjecture for more general quivers. Appendix A contains a short proof of the fact that \( Q_{n,r} \simeq \mathbb{C}[(\mathcal{M}^\text{univ}_{n,r})^\mathbb{C}] \) is flat over the space of parameters.

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2. HIKITA-NAKAJIMA CONJECTURE FOR HILBERT SCHEME

In this Section, we give an “elementary” proof of the equivariant version of Hikita conjecture for Hilbert scheme of points on \( \mathbb{A}^2 \). As a corollary (setting the equivariant
We also consider the cotangent space $M$ to obtain Hikita-Nakajima conjecture for $\text{Hilb}^n(A^2)$ (see [Hik17]) using different methods.

**Remark 2.1.** If reader is not interested in “elementary” proof for Hikita-Nakajima conjecture in the Hilbert scheme case, then feel free to skip this Section since it is not used in the rest of the paper. We have decided to include this Section since it gives some feeling of what is going on.

The idea is the following: we identify the algebra of schematic fixed points with the Rees algebra of the center $Z(CS_n)$ of $CS_n$ and then use the identification

\begin{equation}
\text{Rees}(Z(CS_n)) \simeq H^*_T(\text{Hilb}_n(A^2)) \quad (\text{see [Vas01]})
\end{equation}

to obtain Hikita-Nakajima conjecture for $\text{Hilb}_n(A^2)$. An alternative approach to the identification (2.1) appears in Sections 4, 5.

2.1. **Nakajima quiver varieties.** Let $I = (I_0, I_1)$ be a finite quiver, where $I_0$ is the set of vertices, and $I_1$ is the set of oriented edges. Let $v = (v_i)_{i \in I_0}$, $w = (w_i)_{i \in I_0}$ be $I_0$-tuples of nonnegative integer numbers. Let $V = \bigoplus_{i \in I_0} V_i$, $W = \bigoplus_{i \in I_0} W_i$ be $I_0$-graded vector spaces with $\dim V_i = v_i$, $\dim W_i = w_i$.

Consider the representation space

$$N = N(v, w) = N_I(v, w) := \bigoplus_{(i \rightarrow j) \in I_1} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I_0} \text{Hom}(V_i, V_i).$$

We also consider the cotangent space $M_I(v, w) = M(v, w) = M := T^*N$ that can be identified with

$$\bigoplus_{(i \rightarrow j) \in I_1} \text{Hom}(V_i, V_j) \oplus \bigoplus_{(i \rightarrow j) \in I_1} \text{Hom}(V_j, V_i) \oplus \bigoplus_{i \in I_0} \text{Hom}(V_i, V_i) \oplus \bigoplus_{i \in I_0} \text{Hom}(V_i, W_i).$$

We can represent elements of $M(v, w)$ as quadruples $(X, Y, \gamma, \delta)$, here

$$X \in \bigoplus_{(i \rightarrow j) \in I_1} \text{Hom}(V_i, V_j), \quad Y \in \bigoplus_{(i \rightarrow j) \in I_1} \text{Hom}(V_j, V_i),$$

$$\gamma \in \bigoplus_{i \in I_0} \text{Hom}(W_i, V_i), \quad \delta \in \bigoplus_{i \in I_0} \text{Hom}(V_i, W_i).$$

The space $M(v, w) = T^*N$ carries a natural symplectic form. We set

$$G_v := \prod_{i \in I_0} \text{GL}(V_i), \quad g_v := \bigoplus_{i \in I_0} \text{gl}(V_i).$$

The group $G_v$ acts naturally on the vector space $M(v, w)$. This action is symplectic with the moment map

$$\mu: M(v, w) \rightarrow g_v, \quad (X, Y, \gamma, \delta) \mapsto [X, Y] + \gamma \delta.$$ 

**Definition 2.2.** A quadruple $(X, Y, \gamma, \delta) \in M(v, w)$ is called stable if for every $X, Y$-invariant graded subspace $S \subset V$ such that $S$ contains $\text{im} \gamma$ we have $S = V$. We denote by $M(v, w)^{st} \subset M(v, w)$ the (open) subset of stable quadruples.

**Definition 2.3.** The Nakajima quiver varieties $M(v, w)$, $M_0(v, w)$ are defined as the following quotients

$$M(v, w) := \mu^{-1}(0)^{st}/G_v, \quad M_0(v, w) := \mu^{-1}(0)/G_v.$$

We have the natural (projective) morphism $M(v, w) \rightarrow M_0(v, w)$. Let $z_v \subset g_v$ be the center of $g_v$. Varieties $M(v, w)$, $M_0(v, w)$ admit certain natural deformations over the space $z_v$. 

Definition 2.4. The universal quiver varieties $\mathcal{M}^{\text{univ}}(v, w)$, $\mathcal{M}_0^{\text{univ}}(v, w)$ are defined as follows:

$$\mathcal{M}^{\text{univ}}(v, w) := \mu^{-1}(3_v)^{\text{st}} / G_v, \quad \mathcal{M}_0^{\text{univ}}(v, w) := \mu^{-1}(3_v) / G_v.$$ 

For $a \in 3_v$, we denote by $\mathcal{M}^a(v, w)$ the fibers of these families over $a$.

2.2. Hilbert scheme $\text{Hilb}_n(\mathbb{A}^2)$ and $S^n(\mathbb{A}^2)$ as Nakajima quiver varieties. Our main object of study in this section is $\text{Hilb}_n(\mathbb{A}^2)$ the Hilbert scheme of $n$ points on $\mathbb{A}^2$.

Definition 2.5. The variety $\text{Hilb}_n(\mathbb{A}^2)$ is the variety whose $\mathbb{C}$-points are ideals $J \subset \mathbb{C}[x, y]$ of codimension $n$. The (affine) variety $S^n(\mathbb{A}^2)$ is the categorical quotient $(\mathbb{A}^2^n / S_n$).

Recall that we have the Hilbert-Chow morphism

$$\text{Hilb}_n(\mathbb{C}^2) \to S^n(\mathbb{A}^2), \quad J \mapsto \text{Supp}(\mathbb{C}[x, y] / J).$$

This morphism is a symplectic resolution of singularities.

Let us now recall the description of $\text{Hilb}_n(\mathbb{C}^2)$ as a Nakajima quiver variety (corresponding to the Jordan quiver).

Definition 2.6. We denote by $\mathcal{M}(n, r), \mathcal{M}_0(n, r)$ the Nakajima quiver varieties corresponding to the quiver $I$ consisting of one vertex and one loop with $\dim V = n, \dim W = r$.

The following Proposition holds by [Nak99].

Proposition 2.7. There exist isomorphisms

$$(2.2) \quad \mathcal{M}(n, 1) \cong \text{Hilb}_n(\mathbb{A}^2), \quad \mathcal{M}_0(n, 1) \cong S^n(\mathbb{A}^2)$$

compatible with natural morphisms $\mathcal{M}(n, 1) \to \mathcal{M}_0(n, 1), \text{Hilb}_n(\mathbb{A}^2) \to S^n(\mathbb{A}^2)$.

We conclude that points of $\text{Hilb}_n(\mathbb{A}^2), S^n(\mathbb{A}^2)$ can be represented as certain quadruples $(X, Y, \gamma, \delta)$ that can be considered as representations of the following quiver (here we identify $V = \mathbb{C}^n, W = \mathbb{C}$):

$$\begin{array}{c}
\mathbb{C} \\
\delta \downarrow \gamma \\
X \hookrightarrow \mathbb{C}^n \hookleftarrow Y
\end{array}$$

2.3. Calogero-Moser space, deformations of $\text{Hilb}_n(\mathbb{A}^2)$ and torus actions.

We see that varieties $\mathcal{M}^{\text{univ}}(n, 1), \mathcal{M}_0^{\text{univ}}(n, 1)$ (see Definition 2.4) are one-parameter deformations of $\text{Hilb}_n(\mathbb{A}^2)$ and $S^n(\mathbb{A}^2)$, where the base of the deformation is the center $3_n \subset \mathfrak{gl}(V)$ that can be identified with $\mathbb{A}^1$ via the map $\mathbb{A}^1 \ni c \mapsto c \text{Id}_V \in \mathfrak{gl}(V)$.

Let us now discuss torus actions. Let $T, C^\times$ be copies of $\mathbb{C}^\times$. We have an action of $T \times C^\times$ on $\text{Hilb}_n(\mathbb{A}^2), S^n(\mathbb{A}^2)$ that is induced by the action $T \times C^\times \hookrightarrow A^2$ given by

$$(t, h) \cdot (x, y) = (t^{-1} h^{-1} x, t h^{-1} y), \quad t \in T, \quad h \in C^\times.$$

After identifications $(2.2)$ the action of $T \times C^\times$ can be described as follows: it is induced from the following action on $\mathcal{M}(n, 1)$:

$$(2.3) \quad (t, h) \cdot (X, Y, \gamma, \delta) = (t^{-1} h^{-1} X, t h^{-1} Y, h^{-1} \gamma, h^{-1} \delta).$$

Remark 2.8. Note that $T$ acts symplectically, while $C^\times$ scales the symplectic form with the weight 2.
Formula (2.3) induces actions
\[ T \times \mathbb{C}_n^\times \curvearrowright \mathcal{M}^\text{univ}(n, 1), \]
\[ T \times \mathbb{C}_n^\times \curvearrowright \mathcal{M}^\text{univ}_0(n, 1). \]

Consider \( a \in \mathbb{C}^\times \). Let us describe the fibers \( \mathcal{M}^a(n, 1), \mathcal{M}^a_0(n, 1) \) of the families \( \mathcal{M}^\text{univ}(n, 1), \mathcal{M}^\text{univ}_0(n, 1) \) over \( a \). First of all, note that the action of \( \mathbb{C}_n \) induces identifications
\[ \mathcal{M}^a(n, 1) \simeq \mathcal{M}^1(n, 1), \]
\[ \mathcal{M}^a_0(n, 1) \simeq \mathcal{M}^1_0(n, 1). \]

**Definition 2.9.** Recall that \( V \) is a vector space of dimension \( n \). We define Calogero-Moser variety \( \mathcal{C}(n) \) as the following quotient:
\[ \mathcal{C}(n) := \{(X, Y) \in \text{End}(V)^{\otimes 2} \mid \text{rk} \left( [X, Y] - \text{Id}_V \right) = 1\}/\text{GL}(V). \]

The following Proposition is well-known (see, for example, [Wil98, Section 1]).

**Proposition 2.10.** Natural morphisms
\[ \mathcal{M}^1(n, 1) \to \mathcal{M}^1_0(n, 1) \to \mathcal{C}(n) \]
given by
\[ [(X, Y, \gamma, \delta)] \mapsto [X, Y, \gamma, \delta] \mapsto [(X, Y)] \]
are isomorphisms.

So families \( \mathcal{M}^\text{univ}(n, 1), \mathcal{M}^\text{univ}_0(n, 1) \) are \( \mathbb{C}_n^\times \)-equivariant deformations of \( \text{Hilb}_n(\mathbb{A}^2) \), \( S^n(\mathbb{A}^2) \) over \( \mathbb{A}^1 \). Over a non-zero parameter their fibers are isomorphic to the Calogero-Moser variety \( \mathcal{C}(n) \).

2.4. **Hikita-Nakajima conjecture for** \( \text{Hilb}_n(\mathbb{A}^2) \). We denote by \( H_T^*(\text{Hilb}_n(\mathbb{A}^2)) \) the \( T \)-equivariant cohomology of \( \text{Hilb}_n(\mathbb{A}^2) \). This is a \( \mathbb{Z} \)-graded algebra over \( H_T^*(\text{pt}) = \mathbb{C}[\text{Lie } T] \).

We are now ready to state the equivariant version of Hikita conjecture for \( \text{Hilb}_n(\mathbb{A}^2) \) that we will refer to as Hikita-Nakajima conjecture.

**Theorem 2.11.** We have an isomorphism of \( \mathbb{Z} \)-graded algebras over \( \mathbb{C}[3_n] \simeq \mathbb{C}[\text{Lie } T] \) (the identification induced by the isomorphism \( 3_n \simeq \mathbb{A}^1 \simeq \text{Lie } T \)):
\[ \mathbb{C}\left[ \left( \mathcal{M}^\text{univ}_0(n, 1) \right)^T \right] \simeq H_T^*(\text{Hilb}_n(\mathbb{A}^2)). \]

Our goal is to prove this Theorem. To do so we will show that both of algebras in the Theorem are isomorphic to the Rees algebra of the center \( Z(\mathbb{C}S_n) \) of the group algebra of \( S_n \).

2.5. **Equivariant cohomology of** \( \text{Hilb}_n(\mathbb{A}^2) \) **and the center of** \( \mathbb{C}S_n \). Let \( Z(\mathbb{C}S_n) \) in \( \mathbb{C}S_n \) be the center of the group algebra \( \mathbb{C}S_n \). Consider the grading on the vector space \( \mathbb{C}S_n \) defined in the following way: pick a permutation \( \sigma \in S_n \) and let \( \ell(\sigma) \) be the number of cycles in the decomposition of \( \sigma \) as a product of disjoint cycles. We then define
\[ \deg \sigma := 2(n - \ell(\sigma)). \]

The grading above induces the increasing \( \mathbb{Z}_{\geq 0} \)-filtration on \( Z_n \):
\[ C = F_0Z_n = F_1Z_n \subset F_2Z_n = F_3Z_n \subset \ldots \subset Z_n = F_{2n-2}Z_n = F_{2n-1}Z_n = \ldots \]
We denote by \( \text{Rees}(Z(CS_n)) \) the Rees algebra corresponding to the filtration (2.4). Recall that the algebra \( \text{Rees}(Z(CS_n)) \) is defined as follows:

\[
\text{Rees}(Z(CS_n)) := \bigoplus_{k \geq 0} \kappa^k F_{2k} Z(CS_n) \subset Z(CS_n)[\kappa],
\]

where \( \kappa \) is a formal parameter of degree 2. We consider \( \text{Rees}(Z(CS_n)) \) as an algebra over \( \mathbb{C}[A^1] = \mathbb{C}[\kappa] \).

The following result holds by [Vas01] or Sections 4, 5 (see also [SVV17, Theorem 4.7 and Corollary 4.8]).

**Proposition 2.12.** There is an isomorphism of \( \mathbb{Z} \)-graded algebras over \( \mathbb{C}[\text{Lie } T] \simeq \mathbb{C}[A^1] \) (the identification is induced by the isomorphism \( \text{Lie } T \simeq A^1 \)):

\[
H^*_T(\text{Hilb}_n(A^2)) \simeq \text{Rees}(Z(CS_n)).
\]

Let us now recall the description of the center \( Z(CS_n) \). To every \( k \in 1, \ldots, n \) we can associate the corresponding Jucys–Murphy element \( JM_k \) defined as follows:

\[ JM_k := (1 \ k) + (2 \ k) + \ldots + (k-1 \ k) \in CS_n, \]

where \( (i \ k) \in S_n \) is the transposition switching \( i, k \).

**Remark 2.13.** Note that \( JM_1 = 0 \).

The following Proposition is classical (see, for example, [Mur83, Theorem 1.9]).

**Proposition 2.14.** The center \( Z(CS_n) \) is generated (as a vector space) by elements \( f(JM_1, \ldots, JM_n) \), where \( f \) runs through symmetric functions on \( n \) variables.

### 2.6. Construction of the isomorphism between schematic fixed points of \( \mathcal{M}_0^{\text{univ}}(n, 1) \) and \( \text{Rees}(Z(CS_n)) \)

We will prove the following Theorem and using Proposition 2.12 obtain Theorem 2.11 as a corollary.

**Theorem 2.15.** There is an isomorphism of \( \mathbb{Z} \)-graded algebras over \( \mathbb{C}[A^1] = \mathbb{C}[\text{Lie } T] \)

\[
(2.5) \quad \text{Rees}(Z(CS_n)) \simeq \mathbb{C}\left[\left(\mathcal{M}_0^{\text{univ}}(n, 1)\right)^T\right]
\]

that sends \( f(JM_1, \ldots, JM_n) \in Z(CZ_n) \) to the restriction of the function

\[
\mathcal{M}_0^{\text{univ}}(n, 1) \ni [(X, Y, \gamma, \delta)] \mapsto f(\alpha_1, \ldots, \alpha_n)
\]

to \( \left(\mathcal{M}_0^{\text{univ}}(n, 1)\right)^T \). Here \( f \) is a symmetric function on \( n \) variables and \( \alpha_1, \ldots, \alpha_n \)

are roots of the characteristic polynomial of \( YX \in \text{End}(V) \), \( \mathbb{Z} \)-grading on the LHS of (2.5) is the natural grading on \( \text{Rees}(Z(CS_n)) \), \( \mathbb{Z} \)-grading on the RHS is the one induced by the action of \( \mathbb{C}_k^* \).

The rest of the Section is devoted describing the idea of the proof of Theorem 2.15. We start from the following Proposition which proof is given in Appendix A.

**Proposition 2.16.** The algebra \( \mathbb{C}\left[\left(\mathcal{M}_0^{\text{univ}}(n, 1)\right)^T\right] \) is flat (hence, free) over \( \mathfrak{g}_n = A^1 \). In particular, we have an isomorphism of \( \mathbb{Z} \)-graded algebras

\[
\mathbb{C}\left[\left(\mathcal{M}_0^{\text{univ}}(n, 1)\right)^T\right] \simeq \text{Rees}(\mathbb{C}[\mathcal{M}_0^1(n, 1)^T]) = \text{Rees}(\mathbb{C}[C(n)^T]).
\]
We conclude that to prove Theorem 2.15 it is enough to construct the isomorphism of filtered algebras $\mathbb{C}[C(n)^T] \simeq Z(\mathbb{C}S_n)$. To do so we first need to describe the algebra $\mathbb{C}[C(n)^T]$. Let us note that the variety $C(n)$ is smooth, hence, the scheme $C(n)^T$ is also smooth (see Proposition 1.6) and in particular reduced.

The description of the set of fixed points $C(n)^T$ was given by Wilson in [Wil98 Proposition 6.11], we recall it in Section 2.7. The set $C(n)^T$ is finite and can be parametrized by the set $P(n)$ of partitions of $n$, we denote by $[(X^\lambda, Y^\lambda)]$ the fixed point corresponding to $\lambda \in P(n)$ (see Definition 2.18). Since every finite reduced scheme over $\mathbb{C}$ is just the spectrum of the direct sum of copies of $\mathbb{C}$ then we must have

$$C[C(n)^T] = \bigoplus_{\lambda \in P(n)} \mathbb{C}\chi_\lambda,$$

where $\chi_\lambda \in \mathbb{C}[C(n)^T]$ is the characteristic function of the $T$-fixed point $[(X^\lambda, Y^\lambda)]$ corresponding to $\lambda \in P(n)$.

Recall now that we have the natural identification

$$Z(\mathbb{C}S_n) = \bigoplus_{\mathcal{P}(n)} \mathbb{C}e_\lambda,$$

where $e_\lambda \in Z(\mathbb{C}S_n)$ is the idempotent corresponding to the Specht module $S(\lambda)$ (in other words, for $\nu \in \mathcal{P}(n)$ the element $e_\lambda \in Z(\mathbb{C}S_n)$ acts on $S(\nu)$ via $\delta_{\lambda\nu} \cdot Id_{S(\nu)}$).

Composing (2.6) and (2.7) we obtain the isomorphism of algebras

$$\Theta: Z(\mathbb{C}S_n) \rightarrow \mathbb{C}[C(n)^T], \; e_\lambda \mapsto \chi_\lambda.$$

To prove Theorem 2.15 it remains to show that the isomorphism $\Theta$ is the one that we need, i.e., it sends element $f(JM_1, \ldots, JM_n)$ to the restriction of the function $C(n) \ni [(X, Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ to $C(n)^T$ and then we can conclude from this that the isomorphism $\Theta$ is filtration-preserving.

To prove that $\Theta$ sends element $f(JM_1, \ldots, JM_n)$ to the function $C(n) \ni [(X, Y)] \mapsto f(\alpha_1, \ldots, \alpha_n)$ we just need to show that for every symmetric function $f$ on $n$ variables we have

$$f(JM_1, \ldots, JM_n)|_{S(\lambda)} = f(\alpha_1, \ldots, \alpha_n) \cdot Id_{S(\lambda)},$$

where $\alpha_1, \ldots, \alpha_n$ is the multiset of eigenvalues of $Y^\lambda X^\lambda$. Recall that $f(JM_1, \ldots, JM_n)$ acts on $S(\lambda)$ via the multiplication by $f(c_1, \ldots, c_n)$, where $c_1, \ldots, c_n$ is the multiset of contents of boxes of the Young diagram $Y(\lambda)$ corresponding to $\lambda$ (see (2.3)). So it remains to check that the multiset of eigenvalues of $Y^\lambda X^\lambda$ is precisely the multiset of contents of boxes of $Y(\lambda)$.

2.7. Description of $C(n)^T$ and eigenvalues of $X^\lambda Y^\lambda$. The parametrization of $C(n)^T$ by the elements of $P(n)$ goes as follows (the description was obtained in [Wil98, we follow Prz10]). Pick $m \in \mathbb{Z}_{\geq 1}$ and $1 \leq k \leq m$.

**Definition 2.17.** By $D_m$ we will denote the $m \times m$ matrix with 1’s on the first diagonal and 0’s elsewhere, i.e., $D_m = \sum_{i=1}^{m-1} E_{i+1}$. Now, let $Y(m, k)$ be the $m \times m$ matrix such that its only non-zero entries are on the $-1$st diagonal, and it satisfies the relation $[Y(m, k), D_m] = mE_{kk}$. In other words, the numbers below the diagonal
Theorem 2.15 it remains to show that the isomorphism \( \Theta \) is well-defined. Following [Prz16, Section 4.1] we denote by

\[
Y(m, k) = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 2 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\cdots & \cdots & k-1 & 0 & \cdots \\
\cdots & \cdots & \cdots & -m+k & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0 \\
\end{pmatrix}
\]

Pick \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n) \).

\[
\mathbb{Y}(\lambda) := \{(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}
\]

the corresponding Young diagram. If \( \Box = (i, j) \in \mathbb{Y}(\lambda) \) is a cell let \( c(\Box) := j - i \) be the content of \( \Box \). By a hook associated to the cell \((i, j)\) we mean the set

\[
\mathbb{H}_{(i,j)} := \{(i,j)\} \cup \{(i', j) \in \mathbb{Y}(\lambda) \mid i' > i \} \cup \{(i, j') \in \mathbb{Y}(\lambda) \mid j' > j \}.
\]

Box \((i, j)\) is called the root of the hook \( \mathbb{H}_{(i,j)} \). By a Frobenius hook of \( \mathbb{Y}(\lambda) \) we mean a hook of the form \( \mathbb{H}_{(i,j)} \). Diagram \( \mathbb{Y}(\lambda) \) is the disjoint union of its Frobenius hooks. Suppose that \((1, 1), (2, 2), \ldots, (s, s)\) are cells of \( \mathbb{Y}(\lambda) \) with zero content. Let \( \mathbb{H}_i \) be the Frobenius hook with root \((i, i)\). Let \( k_i \) be the height of \( \mathbb{H}_i \) and \( n_i \) be the size of \( \mathbb{H}_i \).

We are now ready to describe the tuple \([ (X^\lambda, Y^\lambda) ] \in \mathcal{C}(n)^T \) corresponding to \( \lambda \).

**Definition 2.18.** Tuple \((X^\lambda, Y^\lambda)\) is defined as follows. We have \( X^\lambda = D_n \). The \( n_i \times n_i \) diagonal blocks of \( Y^\lambda \) are given by the matrices \( Y(n_i, k_i) \), and the off-diagonal blocks satisfy the following property: For \( i \neq j \) \( (Y^\lambda)_{ij} \) is the unique \( n_i \times n_j \) matrix with non-zero entries on the diagonal \( k_i - k_j \), satisfying the following property:

\[
(Y^\lambda)_{ij}D_{n_j} - D_{n_i}(Y^\lambda)_{ij} = n_iE_{k_i, k_j}.
\]

**Remark 2.19.** If \( \lambda \in \mathcal{P}(n) \) is a hook of height \( k \), then we have \( X^\lambda = X(n, r) \). Note that diagonal matrix elements of \( Y^\lambda X^\lambda = Y(n, r)D_n \) are precisely contents of cells of the hook \( \lambda \).

**Proposition 2.20.** The eigenvalues of \( Y^\lambda X^\lambda = Y^\lambda D_n \) are the same as the eigenvalues of blocks \( (Y^\lambda)_{ii}D_{n_i} = Y(n_i, k_i)D_{n_i} \) as if off-diagonal blocks in \( Y^\lambda X^\lambda \) were not present. So the eigenvalues of \( Y^\lambda X^\lambda \) are diagonal elements of \( Y(n_i, k_i)D_{n_i} \) that are exactly the multiset of contents of boxes of \( \lambda \).

**Proof.** Follows from the proof of [Wil98, Proposition 6.13]. \( \square \)

**Corollary 2.21.** The isomorphism \( \Theta: Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)^T] \) sends \( f(JM_1, \ldots, JM_n) \) to the restriction of the function \([X, Y]) \mapsto f(\alpha_1, \ldots, \alpha_n) \) to \( \mathcal{C}(n)^T \), here \( f \) is a symmetric function on \( n \) variables.

2.8. **Proof of Theorem 2.15.** We have already constructed (see Section 2.6) the isomorphism

\[
\Theta: Z(\mathbb{C}S_n) \xrightarrow{\sim} \mathbb{C}[\mathcal{C}(n)^T]
\]

and have shown that this isomorphism sends generators \( f(JM_1, \ldots, JM_n) \in Z(\mathbb{C}S_n) \) to functions \( \mathcal{C}(n)^T \ni (X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n) \) (see Corollary 2.21). To finish the proof of Theorem 2.15 it remains to show that the isomorphism \( \Theta \) is filtered. Let us first of all note that if \( f \) has degree \( k \) then \( \deg f(JM_1, \ldots, JM_n) = 2k \) and the degree of the function \( (X, Y) \mapsto f(\alpha_1, \ldots, \alpha_n) \) is also equal to \( 2k \). It follows that in order to show
that \( \Theta \) is filtration-preserving it is enough to check that \( F_{2k}Z(\mathbb{C}S_n) \) is generated (as a vector space) by
\[
\{ f(JM_1, \ldots, JM_n) \mid f \text{ is a homogeneous symmetric polynomial of degree } \leq k \}.
\]
This is a direct corollary of the following (classical) Proposition.

**Proposition 2.22.** The algebra \( \text{gr} Z(\mathbb{C}S_n) \) is generated by elements
\[
\{ \text{gr} (f(JM_1, \ldots, JM_n)) \mid f \text{ is a homogeneous symmetric polynomial} \}.
\]

**Proof.** The claim follows from the proof of [Mur83, Theorem 1.9]. \( \square \)

We finish this Section with the following Conjecture.

**Conjecture 2.23.** The isomorphism \( \text{gr} \Theta : \text{gr} Z(\mathbb{C}S_n) \cong C[\left( S^n(A^2) \right)^\mathfrak{g}] \) coincides with the isomorphism constructed in [Hik17].

### 3. TWO DESCRIPTIONS OF SYMPLECTICALLY DUAL TO GIESEKER VARIETY

Recall that the Gieseker variety is the Nakajima quiver variety corresponding to the Jordan quiver (see Definition 2.6). It depends on the pair \( n, r \in \mathbb{Z}_{\geq 1} \) and is denoted by \( \mathcal{M}_0(n, r) \). The corresponding affine Poisson variety is denoted by \( \mathfrak{M}_0(n, r) \). In this Section, we give to different ways to describe the symplectically dual variety \( \mathfrak{M}_0(n, r) \) and its (universal) deformation \( \mathfrak{M}_0(n, r)^\text{univ} \).

#### 3.1. Description of \( \mathfrak{M}_0(n, r) \) as Coulomb branch.

In the paper [BFN], the candidate for symplectically dual variety to every Nakajima quiver variety was constructed.

##### 3.1.1. Construction of Coulomb branch.

Let us recall the construction in our case (when we are starting from the Jordan quiver with the dimension vector \( n \in \mathbb{Z}_{\geq 1} \) and framing \( r \in \mathbb{Z}_{\geq 1} \)).

Recall the vector space \( \mathbf{N} = \text{Hom}(V, V) \oplus \text{Hom}(V, W) \) and the group \( G_n = \text{GL}(V) \) acting on \( \mathbf{N} \) (see Section 2.1).

**Definition 3.1.** We define \( \text{Gr}_{\text{GL}_V} \) as the moduli space of the data \((\mathcal{P}, \varphi)\), where
(a) \( \mathcal{P} \) is a \( \text{GL}_V \)-bundle on \( \mathbb{P}^1 \);
(b) \( \varphi : \mathcal{P}^{\text{str}}|_{\mathbb{P}^1\setminus\{0\}} \rightarrow \mathcal{P}|_{\mathbb{P}^1\setminus\{0\}} \) is a trivialization of \( \mathcal{P} \) restricted to \( \mathbb{P}^1 \setminus \{0\} \).

We then consider the moduli space of triples \( \mathcal{R}_{n, r} \) (corresponding to the Jordan quiver, dimension vector \( n \) and framing \( r \)) defined as follows.

**Definition 3.2.** Let \( \mathcal{R}_{n, r} \) be the moduli space of triples \( \{(\mathcal{P}, \varphi, s)\} \), where \( (\mathcal{P}, \varphi) \) is a point of \( \text{Gr}_{\text{GL}_V} \) and \( s \) is a section of the associated vector bundle \( \mathcal{P}_N = \mathcal{P} \times_{\text{GL}_V} \mathbf{N} \) such that it is sent to a regular section of a trivial bundle under \( \varphi \).

We can consider the equivariant Borel-Moore homology \( H_*(\text{GL}_V)^{\mathcal{O}}(\mathcal{R}_{n, r}) \) (see [BFN Section 2(ii)] for the definition and detailed discussion), this vector space is equipped with an algebra structure via convolution \( * \) (see [BFN Section 3]). It follows from [BFN Proposition 5.15] that the algebra \( (H_*^{\text{GL}_V})^{\mathcal{O}}(\mathcal{R}_{n, r}), *) \) is commutative.

**Definition 3.3.** Coulomb branch \( \mathcal{M}_{n, r} \) is defined as the spectrum of the algebra \( H_*^{\text{GL}_V}(\mathcal{R}_{n, r}) \):
\[
\mathcal{M}_{n, r} := \text{Spec}(H_*^{\text{GL}_V}(\mathcal{R}_{n, r})).
\]
Let us now give more explicit description of the algebra $\mathcal{M}_{n,r}$ being zero, i.e., $\mathcal{M}_{n,r} = \mathbb{C}$.

We can identify $\text{Lie} \mathbb{T} \simeq \mathbb{A}^1$, so $\mathbb{C}[\text{Lie} \mathbb{T}] = \mathbb{C}[\kappa]$ for some variable $\kappa$. We can also identify $\mathfrak{t}_r := \text{Lie} T_r$ with the subspace of $\mathbb{C}^r$ consisting of points with sum of coordinates being zero, i.e., $\mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$.

**Definition 3.4.** We define

$$\mathcal{M}_{n,r}^{\text{univ}} := \text{Spec}(H_{s}^A \times \text{GL}(V)) \circ (\mathcal{R}_{n,r}).$$

Let us now give more explicit description of the algebra $\mathbb{C}[\mathcal{M}_{n,r}^{\text{univ}}]$, we will realize it as a spherical subalgebra of the double affine rational Cherednik algebra $H_{n,r}$ corresponding to the group $\Gamma_n = \mathbb{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

3.1.2. **Double affine Rational Cherednik algebra corresponding to $\mathbb{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.** We start from recalling some definitions and notations. Consider the subgroup $\Gamma_n \subset \text{GL}_n$ of monomial matrices with entries being $r$th roots of unity. Let $\eta \in \mathbb{C}^\times$ be a $r$th primitive root of unity. We set $\epsilon_j = \text{diag}(1, \ldots, 1, \eta, 1, \ldots, 1)$. Note that we have the natural embedding $\mathbb{S}_n \subset \Gamma_n$. We obtain the natural identification $\mathbb{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n \rightarrow \Gamma_n$.

Consider the standard representation $\Gamma_n \rtimes \mathfrak{h} := \mathbb{C}^n$ induced by the embedding $\Gamma_n \subset \text{GL}_n$. Let $y_1, \ldots, y_n$ be the standard basis in $\mathfrak{h} = \mathbb{C}^n$ and denote by $x_1, \ldots, x_n \in \mathfrak{h}^*$ the dual basis.

Let $t, \kappa, c_1, \ldots, c_{r-1}$ be formal parameters. Let $(h, H_0, H_1, \ldots, H_{t-1})$ be formal parameters such that (compare with [1,5])

$$h = \kappa, \ c_k = \sum_{p=0}^{r-1} \eta^{-kp} H_p, \ H_0 + H_1 + \ldots + H_{r-1} = 0.$$ 

We will switch between parameters $(\kappa, c_1, \ldots, c_{r-1}), (h, H_0, \ldots, H_{r-1})$ freely. We set $h := \mathbb{C}[h, H_0, \ldots, H_{r-1}]/(H_0 + H_1 + \ldots + H_{r-1}) = \mathbb{C}[\kappa, c_1, \ldots, c_{r-1}]$.

We set $c(q) := \sum_{i=1}^{r-1} c_i q^i$, here $q$ is a formal variable. Following [Web19] and [Prz16] we define rational Cherednik algebra $\mathcal{H}_{\Gamma_n}$ in the following way.

**Definition 3.5.** Algebra $\mathcal{H}_{\Gamma_n} = \mathcal{H}_{n,r}$ is a quotient of the semi-direct product

$$\left( \mathbb{C} \Gamma \ltimes \mathfrak{t}^*(\mathfrak{h} \oplus \mathfrak{h}^*) \right) \otimes h[h]$$

subject to the relations

$$[x_i, x_j] = [y_i, y_j] = 0,$$

$$[x_i, y_i] = -h + \kappa \sum_{j \neq i}^{r-1} (ij) \epsilon_i^p \epsilon_j^{-p} + c(\epsilon_i),$$

$$[x_i, y_j] = -\kappa \sum_{p=0}^{r-1} \eta^p (ij) \epsilon_i^p \epsilon_j^{-p} \quad (i \neq j)$$

We also set $H_{n,r} := \mathcal{H}_{n,r}/h$. 

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Remark 3.6. Note that our parameters \((h, \kappa, c_1, \ldots, c_r)\) and the parameters \((\xi, \zeta, \ldots, \zeta_{r-1})\) of [Mar14] are related in the following way: \(\xi = h, \zeta = \kappa, \zeta = \frac{c_i}{1-\eta}\) (note that to get the relations used to define rational Cherednik algebra in [Mar14] we need to replace \(\eta\) by \(\eta^{-1}\)).

Definition 3.7. Set \(e := \frac{1}{|V|} \sum_{g \in V} g\). We denote by \(\mathcal{H}_{n,r}^{\text{sph}}\) the spherical subalgebra \(e \mathcal{H}_{n,r} \subset \mathcal{H}_{n,r}\). We denote by \(\mathcal{H}_{n,r}^{\text{univ}}\) the subalgebra \(e \mathcal{H}_{n,r} e \subset \mathcal{H}_{n,r}\).

Recall that \(Z(H_{n,r}) \subset \mathcal{H}_{n,r}\) is the center. The following Proposition holds by [EC02].

Proposition 3.8. The composition \(Z(H_{n,r}) \subset \mathcal{H}_{n,r} \rightarrow e \mathcal{H}_{n,r} e\) induces the identification \(Z(H_{n,r}) \cong e \mathcal{H}_{n,r} e\).

Proposition 3.10. There exists the isomorphism of algebras

\[
\mathcal{H}_{n,r}^{\text{sph}} \cong H^{A \times GL(V)}_{\mathcal{O}}(\mathcal{R}_{n,r})
\]

that identifies parameters in the following way \(\kappa = \kappa, a_i = p(\eta^{-i-1})\). The isomorphism above sends \(c_i(u_1, \ldots, u_n)\) to \(c_i(E)\), where \(E\) is the tautological \(GL(V)_{\mathcal{O}}\)-bundle on \(\mathcal{R}_{n,r}\).

So (using Proposition 3.8) we conclude that the universal deformation of the variety \(\mathcal{M}_0(n, r)^!\) can be described as

\[
\mathcal{M}_{n,r}^{\text{univ}} = \text{Spec}(Z(H_{n,r})).
\]

3.2. Description of \(\mathcal{M}_0(n, r)^!\) as Nakajima quiver variety. In this Section we describe \(\mathcal{M}_0(n, r)^!\) and its universal deformation as certain (affine) Nakajima quiver varieties.

Definition 3.11. Let \(X_0(n, r)\) be the (affine) Nakajima quiver variety corresponding to the cyclic quiver \(I\) with \(r\) vertexes labeled by the numbers \(0, 1, \ldots, r - 1\) and such that \(v_0 = v_1 = \ldots = v_{r-1} = n, w_0 = 1, w_1 = \ldots = w_{r-1} = 0\) (see Figure 1). Let \(\mathcal{X}_0^{\text{univ}}(n, r)\) be the corresponding universal quiver variety (see Definition 2.4).

We will denote by \((X, Y, \gamma, \delta) = (X_0, X_1, \ldots, X_{r-1}, Y_0, Y_1, \ldots, Y_{r-1}, \gamma, \delta)\) the elements of

\[
\mathbb{M} = \bigoplus_{i=0}^{r-1} \text{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i=0}^{r-1} \text{Hom}(V_i, V_{i-1}) \oplus \text{Hom}(C, V_0) \oplus \text{Hom}(V_0, C).
\]
Remark 3.12. This picture was originally made by Tomek Przedziecek and we adapted for our needs.

Let us recall that $X^\text{univ}_0(n,r)$ is a family over the center of $\mathfrak{g}_{n}^{\text{op}}$ that we denote by $\mathfrak{g}_{n}^{\text{op}}$. We identify $\mathfrak{g}_{n}^{\text{op}} \simeq \mathbb{A}^r$ via $\mathbb{A}^r \ni (\theta_0, \theta_1, \ldots, \theta_{r-1}) \mapsto (\theta_0 \text{ Id}, \theta_1 \text{ Id}, \ldots, \theta_{r-1} \text{ Id})$. Torus $\mathbb{T} = \mathbb{C}^\times$ acts on $X^\text{univ}_0(n,r)$ in the following way:

\begin{equation}
(3.6) \quad t \cdot (X, Y, \gamma, \delta) = (t^{-1} X, t Y, \gamma, \delta).
\end{equation}

3.3. **Relation between two approaches to $\mathcal{M}_0(n,r)$**. Recall that we gave three descriptions of symplectically dual to the variety $\mathcal{M}_0(n,r)$. First description is via Coulomb branches, second description is via the center of $\text{RCA} H_{n,r}$ and the third one realizes symplectically dual variety as a certain Nakajima quiver variety. The relation between first and second descriptions is formulated in Proposition 3.10. The relation between second and third descriptions is given by the Etingof-Ginzburg isomorphism (that goes back to the paper [EG02]). We identify the parameters $(\theta_0, \ldots, \theta_{r-1})$, $(h, H_1, \ldots, H_{r-1})$ in the following way:

\[(\theta_0, \theta_1, \ldots, \theta_{r-1}) = (-h - H_1 - \ldots - H_{r-1}, H_1, \ldots, H_{r-1}).\]

**Proposition 3.13.** There exist a $\mathbb{T}$-equivariant isomorphism of algebras over $\mathbb{C}[\mathbb{A}^r]$ extending isomorphisms described in [Prz16, Section 3.3]

\[\mathbb{E}G: \mathbb{C}[X^\text{univ}_0(n,r)] \simeq Z(H_{n,r}).\]

**Proof.** It follows from [Los10 Theorem 6.2.1] (see also [Obl07] together with [Gor05]) that we indeed have an isomorphism $\mathbb{C}[X^\text{univ}_0(n,r)] \simeq Z(H_{n,r})$. In [Prz16 Section 3.3] it is explained (following the proof of [EG02 Theorem 11.16]) how to construct the isomorphism between the smooth fibers of the families $X^\text{univ}_0(n,r)$, Spec $Z(H_{n,r})$ over $\mathbb{A}^r$. The only thing that we need to show is that these isomorphisms extend to the isomorphism $\mathbb{C}[X^\text{univ}_0(n,r)] \simeq Z(H_{n,r})$. We are grateful to Pavel Etingof for explaining the argument to us. Note that the morphism defined in [Prz16 Section 3.3] actually extends (using the same formula) to smooth locus over fiber over *every* $c \in \mathbb{A}^r$. In other words we obtain a morphism

\[\mathbb{E}G^\text{smooth}: \text{Spec}(Z(H_{n,r}))^\text{smooth} \to (X^\text{univ}_0(n,r))^\text{smooth}\]
extending $\mathbb{E}\mathbb{G}$ (here $\text{Spec}(\mathbb{Z}(H_{n,r}))^{\text{smooth}}$ is the smooth locus of $\text{Spec}(\mathbb{Z}(H_{n,r}))$ considered as scheme over $\mathbb{A}^r$ and similarly $(\lambda_0^{\text{univ}}(n,r))^{\text{smooth}} \subset \lambda_0^{\text{univ}}(n,r)$). Note now that the complement of $\text{Spec}(\mathbb{Z}(H_{n,r}))^{\text{smooth}} \subset \text{Spec}(\mathbb{Z}(H_{n,r}))$ has codimension at least two (since every fiber is a Poisson variety with finite number of symplectic leaves which are even dimensional) and the same holds for $(\lambda_0^{\text{univ}}(n,r))^{\text{smooth}} \subset \lambda_0^{\text{univ}}(n,r)$. Using that $Z(H_{n,r}) \simeq \mathbb{C}[\lambda_0^{\text{univ}}(n,r)]$ are normal (see $\mathbb{E}\mathbb{G}02$ Lemma 3.5) we conclude that

$$\mathbb{C}[\text{Spec}(\mathbb{Z}(H_{n,r}))^{\text{smooth}}] = Z(H_{n,r}), \mathbb{C}[(\lambda_0^{\text{univ}}(n,r))^{\text{smooth}}] = \mathbb{C}[\lambda_0^{\text{univ}}(n,r)].$$

Same argument as in the proof of $\mathbb{E}\mathbb{G}02$ Theorem 11.16 then shows that $\mathbb{E}\mathbb{G}^{\text{smooth}}$ induces isomorphism at the level of functions i.e. induces the desired isomorphism $\mathbb{C}[\lambda_0^{\text{univ}}(n,r)] \simeq Z(H_{n,r})$.

\[ \square \]

4. EQUIVARIANT COHOMOLOGY $H^*_A(\mathfrak{M}(n, r), \mathbb{C})$

We start from recalling the combinatorial parametrization of the $A = T \times T_r$-fixed points of the Gieseker variety $\mathfrak{M}(n, r)$.

We start with the case $r = 1$. In this case the variety $\mathfrak{M}(n, 1)$ coincides with the Hilbert scheme $\text{Hilb}_n(\mathbb{A}^2)$ (see $\text{Nak99}$ Section 2.2). Recall that this Hilbert scheme parameterizes the codimension $l$ ideals in $\mathbb{C}[x, y]$.

The torus $T_1$ is zero-dimensional thus, $(\text{Hilb}_n(\mathbb{A}^2))^{T \times T_1} = (\text{Hilb}_n(\mathbb{A}^2))^T$. The fixed point set $(\text{Hilb}_n(\mathbb{A}^2))^T$ is the set of monomial ideals $J \in \text{Hilb}_n(\mathbb{A}^2)$. Such ideals are parametrized by the set $\mathcal{P}(n)$ of partitions of $n$ as follows. Recall that (following notations of $\text{Prz16}$ Section 4.1) we associate to $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(n)$ the Young diagram

$$\mathcal{Y}(\lambda) = \{(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}.$$

We fill $\mathcal{Y}(\lambda)$ with monomials putting $x^{i-1}y^{j-1}$ into the box $(i, j) \in \mathcal{Y}(\lambda)$. The ideal $J_\lambda$ that corresponds to $\lambda$ is spanned by the monomials outside the diagram. We have the following result.

**Proposition 4.1.** The fixed point set $(\text{Hilb}_n(\mathbb{A}^2))^T$ is identified with the set $\mathcal{P}(n)$ via the map $\lambda \mapsto J_\lambda$ described above.

Now we proceed to the case of arbitrary $r$. Let us introduce some notation. Let $w_1, \ldots, w_r \in W$ be a basis of $W$ consisting of eigenvectors of $T_r$.

The following lemma is classical.

**Lemma 4.2.** The variety $\mathfrak{M}(n, r)^T$ is isomorphic to the disjoint union

$$\bigcup_{\sum_{l=1}^r n_l = n} \prod_{l=1}^r \mathfrak{M}(n_l, 1) \simeq \bigcup_{\sum_{l=1}^r n_l = n} \prod_{l=1}^r \text{Hilb}_{n_l}(\mathbb{A}^2).$$

**Definition 4.3.** We say that an ordered $r$-tuple $\lambda = (\lambda^0, \ldots, \lambda^{r-1})$ of partitions defines an $r$-multipartition of $n \in \mathbb{Z}_{\geq 0}$ if $\sum_{l=0}^{r-1} |\lambda^l| = n$. Let $\mathcal{P}(r, n)$ denote the set of $r$-multipartitions of $n$.

**Proposition 4.4.** The set of fixed points $\mathfrak{M}(n, r)^A$ is identified with the set $\mathcal{P}(r, n)$. A multipartition $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$, corresponds to the following quiver data $V, W, X^\lambda, Y^\lambda \in \text{End}(V), \gamma^\lambda \in \text{Hom}(W, V), \delta^\lambda \in \text{Hom}(V, W)$:

$$V := \bigoplus_{l=0}^{r-1} \mathbb{C}[x, y]/J_\lambda, \quad W = \bigoplus_{l=0}^{r-1} \mathbb{C}w_l;$$
\[ A^\lambda := \bigoplus_{i=0}^{r-1} L_x^i, \quad B^\lambda := \bigoplus_{l=0}^{r-1} L_y^l; \]

\[ \gamma^\lambda \text{ sends } w_i \in W \text{ to } [1] \text{ in } \mathbb{C}[x,y]/J_N, \quad \delta^\lambda = 0, \]

by \( L_x^l, L_y^l \) we denote operators of multiplications by \( x, y \) on \( \mathbb{C}[x,y]/J_N \).

**Proof.** This follows from Lemma 4.2 and the description of \( \mathbb{T} \)-fixed points of Hilbert schemes above. \( \square \)

We have the \( A \)-equivariant embedding

\[ \iota : \mathcal{M}(n,r)^A \subset \mathcal{M}(n,r). \]

This embedding induces the homomorphism

\[ \iota^* : H_A^*(\mathcal{M}(n,r)) \rightarrow H_A^*(\mathcal{M}(n,r)^A) = \mathbb{C}[a]^{\oplus |P(r,n)|} = E. \]

**Lemma 4.5.** The homomorphism \( \iota^* \) is an embedding that becomes isomorphism after tensoring by \( F = \mathbb{C}(a) \).

**Proof.** Follows from the Atiyah-Bott localization theorem (see [ABS4]) together with the fact that \( H_A^*(\mathcal{M}(n,r), \mathbb{C}) \) is a free \( \mathbb{C}[a] \)-module (see [Nak01, Theorem 7.3.5]). \( \square \)

**Remark 4.6.** Actually we do not need to tensor by the whole \( \mathbb{C}(a) \) but only need to localize at elements \( h \in \mathbb{C}[a] \) corresponding to the cocharacters \( \nu : \mathbb{C}^* \rightarrow A \) such that \( \mathcal{M}(n,r)^{\nu(\mathbb{C}^*)} \) is infinite. One can describe these elements by explicitly.

Recall now that by [MN18] the algebra \( H_A^*(\mathcal{M}(n,r), \mathbb{C}) \) is generated by the \( A \)-equivariant Chern classes \( c_k(V) \), \( k = 1, \ldots, n \), where \( V \) is the tautological rank \( n \) vector bundle on \( \mathcal{M}(n,r) \).

**Lemma 4.7.** The image \( \iota^*(c_k(V)) \) is equal to the collection

\[ (c_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|X|}^0 + a_1, \ldots, \kappa c_{|X|}^{r-1} + a_r), \ldots, \kappa c_{|X|}^0 + a_r))_{\lambda \in \mathcal{P}(r,n)}, \]

where for \( l = 0, 1, \ldots, r-1 \) \( c_1, c_2, \ldots, c_{|X|}^l \) is the multiset of contents of boxes of the diagram \( \nu(\lambda^l) \).

**Proof.** The homomorphism \( \iota^* \) sends \( c_k(V) \) to

\[ c_k(V)|_{\mathcal{M}(n,r)^A} = (c_k(V)|_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)})_{\lambda \in \mathcal{P}(r,n)}. \]

Note now that \( c_k(V)|_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)} \) is nothing else but \( c_k(\alpha_1, \ldots, \alpha_n) \), where \( \alpha_1, \ldots, \alpha_n \) is the multiset of \( a \)-weights of \( \nu |_{(X^\lambda, Y^\lambda, \gamma^\lambda, 0)}. \)

Recall that \( \mathbb{C}[\text{Lie } \mathbb{T}] = \mathbb{C}[\kappa], \mathbb{C}[t_r] = \mathbb{C}[a_1, \ldots, a_r]/(a_1 + \ldots + a_r) \) and \( a = \text{Lie } \mathbb{T} \oplus t_r. \)

We claim that the \( a \)-weight of \( [x^y, y^z] \in \mathbb{C}[x,y]/J_N \) is equal to \( \kappa (j-i) + a_i \) and then the claim follows. Indeed taking \( g = (t^e, t^{a_1}, \ldots, t^{a_r}) \in \mathbb{T} \times T_r \) we see that this element acts on \((X^\lambda, Y^\lambda, \gamma^\lambda, 0)\) as follows

\[ g \cdot A^\lambda = \bigoplus_{l=0}^{r-1} t^{e_l} L_x^l, \quad g \cdot B^\lambda = \bigoplus_{l=0}^{r-1} t^{-\epsilon_l} L_y^l, \quad (g \cdot \gamma^\lambda)(w_1) = t^{-a_1} w_1. \]

Consider the element \( g' \in \prod_{l=0}^{r-1} \text{GL}(\mathbb{C}[x,y]/J_N) \) that acts on \( x^y, y^z \in \mathbb{C}[x,y]/J_N \) via the multiplication by \( e^{a_i + \kappa_j - a_i} \). Directly from the definitions we see that

\[ g \cdot (A^\lambda, B^\lambda, \gamma^\lambda, 0) = g' \cdot (A^\lambda, B^\lambda, \gamma^\lambda, 0). \]

\( \square \)
Proposition 4.8. The homomorphism \( \iota^*: \mathcal{H}^*_A(\mathfrak{H}(n, r)) \rightarrow \mathcal{H}^*_A(\mathfrak{H}(n, r)^A) = E \) is injective and becomes isomorphism after tensoring by \( F \). On generators \( c_k(\mathcal{V}), k = 1, \ldots, n \) it is given by
\[
c_k(\mathcal{V}) \mapsto (c_k(\kappa e_1^0 + a_1, \ldots, \kappa e_{|\lambda|}^0 + a_1, \ldots, \kappa e_r^{r-1} + a_r, \ldots, \kappa e_{|\lambda|}^{r-1} + a_r))_{\lambda \in \mathcal{P}(r, n)}.
\]

5. The center \( Z(R^r(n))^{JM} \) of the cyclotomic degenerate Hecke algebra

Recall the space \( k = \mathbb{C}[\kappa, a_1, \ldots, a_r] / (a_1 + \cdots + a_r) \).

Definition 5.1. The degenerate affine Hecke algebra \( R(n) = R(S_n) \) is generated by \( \mathbb{C}S_n \) and \( k[z_1, \ldots, z_n] \) subject to relations
\[
s_i z_j = z_{s_i(j)} s_i + \kappa (\delta_{i+1,j} - \delta_{i,j}).
\]
The cyclotomic degenerate Hecke algebra \( R^r(n) \) is the quotient of \( R(n) \) by the ideal generated by \( \prod_{i=1}^r (z_i - a_i) \).

Definition 5.2. Let \( Z(R^r(n))^{JM} \subset R^r(n) \) be the image of \( \mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(n) \) in \( R^r(n) \).

Remark 5.3. By [Lus88] Theorem 6.5 subalgebra \( \mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(n) \) is nothing else but the center of \( R(n) \). It follows from [Bru08] Theorem 1 that the algebra \( Z(R^r(n))^{JM} \) is the center of \( R^r(n) \).

To every \( \lambda \in \mathcal{P}(r, n) \) one can associate the “universal” Specht module \( \widetilde{S}_k(\lambda) \) over \( R^r(n) \) (compare with [Bru08] Section 4). Let us recall the construction of this module.

Recall that \( \lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1}) \). For \( i = 0, 1, \ldots, r - 1 \) we set \( n_i := |\lambda^i| \). We slightly modify the algebras \( R(n), R^r(n) \) first.

Definition 5.4. Let \( R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n) \) be the algebra generated by \( \mathbb{C}S_n \) and \( \mathbb{C}[z_1, \ldots, z_n] \)

over \( \mathbb{C}[\kappa, a_1, \ldots, a_r] \) subject to the relations
\[
s_i z_j = z_{s_i(j)} s_i + \kappa (\delta_{i+1,j} - \delta_{i,j}).
\]
We denote by \( R^r_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n) \) the quotient of \( R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n) \) by the ideal generated by
\[
(z_1 - a_1) \cdots (z_r - a_r).
\]

For every \( i = 0, 1, \ldots, r - 1 \) consider the usual Specht module \( S_{n_i} \subset S(\lambda^i) \). We can extend \( S(\lambda^i) \otimes \mathbb{C}[\kappa, a_i] \) to the \( R_{\mathbb{C}[\kappa, a_i]}(n_i) \)-module by letting \( z_1 \) act via the multiplication by \( a_i \). Note that we have the natural embedding
\[
R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n_1, \ldots, n_{r-1}) := R_{\mathbb{C}[\kappa, a_1]}(n_1) \otimes_{\mathbb{C}[\kappa]} \cdots \otimes_{\mathbb{C}[\kappa]} R_{\mathbb{C}[\kappa, a_r]}(n_{r-1}) \subset R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n)
\]
induced by the embedding \( S_{n_0} \times \cdots \times S_{n_{r-1}} \subset S_n \). Then we define
\[
\widetilde{S}_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(\lambda) :=
R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n) \otimes_{R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n_1, \ldots, n_{r-1})} (S(\lambda^0) \otimes_{\mathbb{C}[\kappa, a_1]} \cdots \otimes_{\mathbb{C}[\kappa]} (S(\lambda^{r-1}) \otimes_{\mathbb{C}[\kappa, a_r]})
\]
that is an \( R_{\mathbb{C}[\kappa, a_1, \ldots, a_r]}(n) \)-module. Modding out by \( a_1 + \cdots + a_r = 0 \) we obtain the desired \( R^r(n) \)-module \( \widetilde{S}_k(\lambda) \).

Let us now compute the action of \( Z(R^r(n))^{JM} \) on \( \widetilde{S}_k(\lambda) \). We start from the following Lemma. Let \( \mu \) be a partition of some \( m \in \mathbb{Z}_{\geq 1} \) and consider the action \( R_{\mathbb{C}[\kappa, a]}(m) \subset S(\mu) \otimes \mathbb{C}[\kappa, a] \).
Lemma 5.5. Let $B$ be a Young tableaux on $\mu$ and recall that $p_B \in S(\mu)$ is the corresponding vector. The element $z_j \in R_{C[k,a]}(m)$ acts on $p_B$ via the multiplication by $\kappa \cdot (B(i)) + a$.

Proof. Recall the Jucys-Murphy element

$$JM_i = \sum_{j<i}(ij) \in \mathbb{C} S_m.$$  

Recall now that $s_i z_{i+1} = z_i s_i + \kappa$ so we have

$$z_{i+1} = s_i z_i s_i + s_i \kappa.$$  

It follows that

$$z_j = s_{j-1}s_{j-2} \ldots s_i z_i s_1 \ldots s_{j-1} + \kappa JM_i$$  

so the action of $z_j$ on $S(\mu) \otimes \mathbb{C}[k,a]$ coincides with the action of $\kappa JM_i + a$ and then the claim follows. \qed

The following Proposition follows from Lemma 5.5 (see also \cite[Section 4]{Bru08}).

**Proposition 5.6.** The class $[f(z_1, \ldots, z_n)] \in Z(R^r(n))^{JM}$ of the element $f(z_1, \ldots, z_n)$ in $\mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(n)$ acts on the representation $\tilde{S}_k(\lambda)$ via the multiplication by

$$f(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda|}^0 + a_1, \ldots, \kappa c_{|\lambda|-1}^r + a_r, \ldots, \kappa c_{|\lambda|-1}^r + a_r),$$  

where $c_1^1, c_2^1, \ldots, c_{|\lambda|}^1$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda')$.

Since every element of $Z(R^r(n))^{JM}$ is central we can consider the homomorphism

$$\psi: Z(R^r(n))^{JM} \rightarrow \bigoplus_{\lambda} \text{End}_{R^r(n)}(\tilde{S}_k(\lambda))$$  

induced by the action of $Z(R^r(n))^{JM}$ on representations $\tilde{S}_k(\lambda)$.

**Proposition 5.7.** The homomorphism $\psi: Z(R^r(n))^{JM} \rightarrow \bigoplus_{\lambda} \text{End}_{R^r(n)}(\tilde{S}_k(\lambda))$ becomes isomorphism after tensoring by $F = \text{Frac}(k)$. This homomorphism is injective.

It sends generators $[e_k(z_1, \ldots, z_n)]$ to the collection

$$(e_k(\kappa c_1^0 + a_1, \ldots, \kappa c_{|\lambda|}^0 + a_1, \ldots, \kappa c_{|\lambda|-1}^r + a_r, \ldots, \kappa c_{|\lambda|-1}^r + a_r))_{\lambda \in \mathcal{P}(r,n)},$$  

where $c_1^1, c_2^1, \ldots, c_{|\lambda|}^1$ is the multiset of contents of boxes of the diagram $\mathbb{Y}(\lambda')$.

Proof. The last part of the claim is Proposition 6.1. Recall now that by \cite[Theorem 1]{Bru08} $Z(R^r(n))^{JM}$ is a free $k$-module of rank $|\mathcal{P}(n,r)|$. So injectivity of $\psi$ would follow if we show that $\psi$ becomes isomorphism after tensoring by $F$. In order to show that it is enough to show that $\psi$ becomes injective after tensoring by $F$ (use the equality of dimensions of $Z(R^r(n))^{JM}, \bigoplus_{\lambda} \text{End}_{R^r(n)}(\tilde{S}_k(\lambda))$ over $k$). Surjectivity is a corollary of Proposition 5.6 and Proposition 4.8. \qed

6. Schematic fixed points of Spec $Z(H_{n,r})$

6.1. Standard and simple representations of $H_{n,r}$, grading on them. We set

$$\zeta_{i,j} := \frac{1}{r} \sum_{p=0}^{r-1} \epsilon_i^p \epsilon_j^{-p}$$  

(projector to the invariants under $\epsilon_i \epsilon_j^{-1}$). The Jucys-Murphy elements are

$$JM_{\Gamma_{n,i}} := \sum_{j<i} \zeta_{i,j} (ij) \in \mathbb{C} \Gamma_n.$$  

(6.1)
Recall that \( \mathcal{P}(r, n) \) is the set of \( r \)-multipartitions of \( n \) (Definition 4.3). Pick \( \lambda \in \mathcal{P}(r, n) \) and consider the corresponding \( r \)-tuple of Young diagrams
\[
(6.2) \quad \Psi(\lambda) = (\Psi(\lambda^0), \ldots, \Psi(\lambda^{r-1})).
\]
Given a cell \( b \in \Psi(\lambda) \), define \( \beta(b) = k \) if \( b \in \Psi(\lambda^k) \) and \( \mathrm{ct}(b) = j - i \) if \( b \) is in the \( i \)th row and \( j \)th column of \( \Psi(\lambda^k) \). There is a bijection \( \lambda \mapsto S(\lambda) \) from the set of \( r \)-partitions of \( n \) to the set of irreducible \( \Gamma_n \)-modules such that \( S(\lambda) \) has a basis \( p_B \) indexed by standard Young tableaux \( B \) on \( \lambda \), and \( p_B \) is determined up to scalars by the equations (see for example [SG18, Equation (2.16)])
\[
(6.3) \quad JM_{\Gamma_n,i} \cdot p_B = \mathrm{ct}(B(i))p_B, \quad \epsilon_i \cdot p_B = \eta^{\beta(B(i))}p_B.
\]
We can consider \( S(\lambda) \otimes \mathfrak{h} \) as a module over \((\mathfrak{h} \otimes \mathbb{C}\Gamma_n) \ltimes S^*\mathfrak{h}^* \) via the trivial action of \( S^*\mathfrak{h}^* \). Let \( \Delta(\lambda) := \text{Ind}_{(\mathfrak{h} \otimes \mathbb{C}\Gamma_n) \ltimes S^*\mathfrak{h}^*}(S(\lambda) \otimes \mathfrak{h}) \) be the induced module (sometimes called the standard module corresponding to \( \lambda \)). We then define \( L(\lambda) \) to be the (unique) simple quotient of \( \Delta(\lambda) \). Recall now that the algebra \( H_{n,r} \) is graded via
\[
\deg x_j = 1, \quad \deg y_j = -1, \quad \deg \Gamma_n = \mathfrak{h} = 0.
\]
This grading induces (nonpositive) grading on \( \Delta(\lambda), L(\lambda) \).

Note that we have the natural embedding \( S(\lambda) \otimes \mathfrak{h} \subset L(\lambda) \) that identifies \( S(\lambda) \otimes \mathfrak{h} \) with the degree zero elements \( L(\lambda)_0 \subset L(\lambda) \).

The following Lemma is standard.

**Lemma 6.1.** We have \( \text{End}_{H_{n,r}}(L(\lambda)) = \mathfrak{h} \).

Recall the algebra
\[
Q_{n,r} := Z(H_{n,r})_0/ \bigoplus_{i>0} Z(H_{n,r})^{-i}Z(H_{n,r})_i = Z(H_{n,r})/(b \in Z(H_{n,r}), i \neq 0)
\]
of functions on schematic fixed points \((\text{Spec } Z(H_{n,r}))^\mathbb{Z}\). The grading on \( H_{n,r} \) used in the definition of \( Q_{n,r} \) is given by
\[
\deg x_i = 1, \quad \deg y_i = -1, \quad \deg \Gamma_n = \deg \mathfrak{h} = 0.
\]

**Remark 6.2.** This is exactly the grading corresponding to the \( T \)-action.

Recall now that the action of \( Z(H_{n,r})_0 \) on \( L(\lambda)_0 \) factors through \( Q_{n,r} \) so we obtain a homomorphism
\[
\phi: Q_{n,r} \to \bigoplus_{\lambda} \text{End}_{H_{n,r}}(L(\lambda)).
\]
We have constructed the homomorphism \( \phi \), let us now describe generators of the algebra \( Q_{n,r} \).

**Definition 6.3.** Dunkl-Opdam operators \( u_i, i = 1, \ldots, n \) are the following elements of \( H_{n,r} \):
\[
(6.4) \quad u_i := \frac{1}{r}y_ix_i + p(\epsilon_i) + \kappa JM_{\Gamma_n,i} = \frac{1}{r}(x_iy_i + \mathfrak{h}) + p(\eta^{-1}\epsilon_i) - \kappa \sum_{j>i} (ij)\zeta_{i,j},
\]
where the last equality follows from the fact that \([y_i, x_i] = \mathfrak{h} - \kappa \sum_{j \neq i} \sum_{p=0}^{r-1} (ij)\epsilon_i^p \epsilon_j^{r-p} - c(\epsilon_i)\) together with the equality \( c(\epsilon_i) = rp(\epsilon_i) - rp(\eta^{-1}\epsilon_i) \). We will denote by the same symbol \( u_i \) the class of \( u_i \) in \( H_{n,r} \).

**Lemma 6.4.** The subalgebra \( \mathbb{C}[u_1, \ldots, u_n]^{S_n} \subset H_{n,r} \) is central.

**Proof.** This follows from [Mar14, Theorem 3.4], see also Lemma 7.6 below. \( \square \)
The following Lemma describes generators of the algebra \( Q_{n,r} \), see Proposition 9.17 and Lemma [11.19] for alternative arguments.

**Lemma 6.5.** Classes of elements \( e_k(u_1, \ldots, u_n) \), \( k = 1, \ldots, n \) generate the algebra \( Q_{n,r} \).

**Proof.** We are grateful to Gwyn Bellamy for explaining this Lemma to us. The claim follows from the proof of [Mar14, Theorem 5.5]. We recall the argument for reader convenience. Recall that \( Q_{n,r} \) is the quotient of the algebra \( Z(H_{n,r}) = eH_{n,r}e \). Consider the following filtration on \( H_{n,r} \):

\[
\text{deg } x_i = \text{deg } y_i = 1, \text{deg } \kappa = \text{deg } c_j = \text{deg } \Gamma_n = 0.
\]

We have

\[
\text{gr } H_{n,r} = \left( \mathbb{C} \Gamma_n \ltimes S^* (\mathfrak{h} \oplus \mathfrak{h}^*) \right) \otimes \mathfrak{h}.
\]

It is enough to show that classes of the elements \( \sum_{i=1}^n u_i^{a+c} \), \( a, c \in \mathbb{Z}_{\geq 0} \) do generate the algebra \( Q_{n,r} \). To see that it is enough to show that the elements \( \text{gr } \left( \sum_{i=1}^n u_i^{a+c} \right) = \text{gr } \left( \sum_i (x_i y_i)^{a+c} \right) \) do generate \( \text{gr } Q_{n,r} \). Note that \( \text{gr } Q_{n,r} \) is the quotient of

\[
\text{gr } eH_{n,r}e = \mathfrak{h}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n \ltimes (Z/2Z)^n} =
\]

\[
= \mathfrak{h}[x_1 y_1, \ldots, x_n y_n, x_1^r, \ldots, x_n^r, y_1^r, \ldots, y_n^r]
\]

that is generated by \( \sum_{i=1}^n (x_i^r)^a(y_i^r)^b(x_i y_i)^c \). The claim follows. \( \square \)

**Lemma 6.6.** Let \( B \) be a Young tableaux on \( \lambda \) and recall that \( p_B \in S(\lambda) \) is the corresponding vector. The element \( u_i \) acts on \( p_B \) via the multiplication by

\[
\kappa \text{ct}(B(i)) + \eta^{y(B(i))}.
\]

**Proof.** Follows from the definition of \( u_i \) (see [6.4]) together with [6.3]. \( \square \)

**Proposition 6.7.** The homomorphism \( \phi \colon Q_{n,r} \to \bigoplus_{\lambda} \text{End}(L(\lambda)) \) becomes isomorphism after tensoring by \( F = \text{Frac}(\mathfrak{h}) \). This homomorphism is injective. It sends generators \([e_k(u_1, \ldots, u_n)]\) to the collection

\[
(\kappa c_i^0 + p(1), \ldots, \kappa c_i^0_{|\lambda|} + p(1), \ldots, \kappa c_i^{-1} + p(\eta^{-1}), \ldots, \kappa c_i^{-1}_{|\lambda|-1} + p(\eta^{-1}))_{\lambda \in \mathcal{P}(r,n)},
\]

where \( c_1, c_2, \ldots, c_{|\lambda|} \) is the multiset of contents of boxes of the diagram \( \mathbb{Y}(\lambda) \).

**Proof.** The proof is the same as the proof of Proposition [5.7] The only difference is that we use Appendix A (flatness of \( Q_{n,r} \) over \( \mathfrak{h} \)) together with the fact that the fiber of \( Q_{n,r} \) over a generic point is just \( \mathbb{C}^{\oplus |P(r,n)|} \) (follows from [Gor02] instead [Bru08, Theorem 1] and Lemma 6.6 instead of Proposition 5.6). \( \square \)

7. DAHA \( R(\Gamma_n) \) vs RCA \( H_{n,r} \)

In this Section we discuss the relation between degenerate affine Hecke algebra corresponding to \( \Gamma_n \) and the rational Cherednik algebra \( H_{n,r} \) and also relate their central subalgebras.
7.1. **DAHA for** $\Gamma_n$, **and the embedding** $R(\Gamma_n) \subset H_{n,r}$. Recall the Dunkl-Opdam elements $u_i \in H_{n,r}$ (see Definition 6.3). Recall also the idempotents $\zeta_{i,j} := \frac{1}{r} \sum_{p=0}^{r-1} \epsilon_i^p \epsilon_j^{r-p}$. The following Lemma can be easily checked (see [Mar14, Lemma 3.2], [Web19]).

**Lemma 7.1.** Elements $u_i, \epsilon_j$ commute with each other. The following equality holds in $H_{n,r}$:

\[(7.1) \quad s_i f(u) - f(s_i \cdot u) s_i = \kappa \zeta_{i,i+1} \frac{f^{s_i} - f}{u_{i+1} - u_i},\]

where $f(u)$ is an element of $\mathbb{C}[u_1, \ldots, u_n] \subset H_{n,r}$.

Let us now define the degenerate affine Hecke algebra corresponding to the group $\Gamma_n$. Recall that $k = \mathbb{C}[\kappa, a_1, \ldots, a_r]/(a_1 + \ldots + a_r)$.

**Definition 7.2.** The algebra $R(\Gamma_n)$ (called degenerate affine Hecke algebra corresponding to $\Gamma_n$) is generated by a copy of $\mathbb{C}\Gamma_n$ and $k[z_1, \ldots, z_n]$ subject to relations

\[(7.2) \quad s_i z_j = z_{s(i,j)} s_i + \kappa \zeta_{i,i+1} (\delta_{i+1,j} - \delta_{i,j}), \quad \epsilon_j z_k = z_k \epsilon_j.\]

The cyclotomic quotient $R^e(\Gamma_n)$ is the quotient of $R(\Gamma_n)$ by the ideal generated by $\prod_{i=1}^r (z_1 - a_i)$.

We obtain the following Proposition.

**Proposition 7.3.** There is an embedding $R(\Gamma_n) \subset H_{n,r}$ given by

\[z_i \mapsto u_i, \quad \kappa \mapsto \kappa, \quad a_j \mapsto p(\eta^{-1}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, r\]

and which is identity on $\Gamma_n$.

**Remark 7.4.** Note that since $p$ is a polynomial of degree $r - 1$ without constant term then we have $p(1) + p(\eta) + \ldots + p(\eta^{r-1}) = 0$ so we indeed obtain the identification $k \xrightarrow{\sim} h$ given by $\kappa \mapsto \kappa, \quad a_i \mapsto p(\eta^{i-1})$ (compare with (1.3)).

7.2. **Central subalgebra** $Z(R(\Gamma_n))^J M \subset R(\Gamma_n)$ and homomorphism from $Z(R(\Gamma_n))^J M$ to $Z(H_{n,r})$. It follows from [Dez06, Proposition 2.3] that the center $Z(R(\Gamma_n))$ is equal to $\mathbb{C}[z_1, \ldots, z_n]^{S_n} \otimes \mathbb{C}[\epsilon_1, \ldots, \epsilon_n]^{S_n}$.

**Definition 7.5.** We denote by $Z(R(\Gamma_n))^J M \subset R(\Gamma_n)$ the subalgebra $\mathbb{C}[z_1, \ldots, z_n]^{S_n} \subset R(\Gamma_n)$.

The following Lemma already appears in Section 6 (see Lemma 6.4).

**Lemma 7.6.** The subalgebra $\mathbb{C}[u_1, \ldots, u_n]^{S_n} \subset H_{n,r}$ is central.

**Proof.** Note that by [Web19] the algebra $H_{n,r}$ is generated by $R(\Gamma_n)$ together with two additional elements $\sigma, \tau$. It is clear that every element of $\mathbb{C}[u_1, \ldots, u_n]^{S_n}$ commutes with $R(\Gamma_n)$ so it remains to show that $\sigma, \tau$ commute with $R(\Gamma_n)$. Again by [Web19] we have

\[u_i \sigma = \sigma u_{i-1}, \quad u_i \tau = \tau u_{i+1},\]

for every $i \in \mathbb{Z}$ (we use the convention that $u_k = u_{k-n}$). It then follows that every element of $\mathbb{C}[u_1, \ldots, u_n]^{S_n}$ commutes with $\sigma, \tau$ as desired.

**Corollary 7.7.** Image of $Z(R(\Gamma_n))^J M \subset R(\Gamma_n)$ in $H_{n,r}$ lies in $Z(H_{n,r})$.

**Proof.** The image of $Z(R(\Gamma_n))^J M = \mathbb{C}[z_1, \ldots, z_n]^{S_n}$ in $H_{n,l}$ is $\mathbb{C}[u_1, \ldots, u_n]^{S_n}$. It remains to note that $\mathbb{C}[u_1, \ldots, u_n]^{S_n} \subset Z(H_{n,r})$ by Lemma 6.4. □
So we obtain the homomorphism \( Z(R(\Gamma_n))^{JM} \to Z(H_{n,r}) \). Let 
\[
\rho_1 : \text{Spec} \ Z(H_{n,r}) \to \text{Spec} \ Z(R(\Gamma_n))^{JM}
\]
be the corresponding morphism of schemes. Passing to \( T \)-fixed points we obtain a morphism 
\[
\rho_1^\mathbb{T} : (\text{Spec} \ Z(H_{n,r}))^\mathbb{T} \to \text{Spec} \ Z(R(\Gamma_n))^{JM}.
\]

**Lemma 7.8.** The morphism \( \rho_1^\mathbb{T} \) is the closed embedding.

**Proof.** The corresponding homomorphism \( (\rho_1^\mathbb{T})^* : Z(R(\Gamma_n))^{JM} \to Q_{n,r} \) is just the composition 
\[
Z(R(\Gamma_n))^{JM} \xrightarrow{\rho_1} Z(H_{n,r}) \to Q_{n,r}
\]
that is surjective by Lemma 6.5. It follows that the morphism \( \rho_1^\mathbb{T} \) is a closed embedding. \( \square \)

8. **Explicit formula for the identification** \( \text{EG} : Q_{n,r} \simeq C[(\mathcal{X}_0^{univ})^\mathbb{T}] \)

Let us recall the Nakajima quiver variety \( \mathcal{X}_0(n, r) \) and its universal version 
\[
\varpi : \mathcal{X}_0^{univ}(n, r) \to \mathbb{A}^r.
\]

Recall that \( \mathbb{T} \) is a copy of \( \mathbb{C}^\times \) that acts on \( \mathcal{X}_0^{univ}(n, r) \) via its action on \( \mathcal{M} \) given by 
\[
t \cdot (X, Y, \gamma, \delta) = (t^{-1}X, tY, \gamma, \delta).
\]

We have the \( \mathbb{T} \)-equivariant isomorphism \( \text{EG} : \text{Spec} \ Z(H_{n,r}) \simeq \mathcal{X}_0^{univ}(n, r) \) that induces the identification \( \text{EG}^\mathbb{T} : (\text{Spec} \ Z(H_{n,r}))^\mathbb{T} \simeq (\mathcal{X}_0^{univ}(n, r))^\mathbb{T} \). The goal of this Section is to describe the isomorphism \( \text{EG}^\mathbb{T} \) explicitly. All the results that we need appear in the paper \([\text{Prz}16]\).

Recall that by Section 7 we have the embedding \( R(\Gamma_n) \subset H_{n,r} \) which induces the embedding \( Z(R(\Gamma_n))^{JM} \subset Z(H_{n,r}) \). We recall that 
\[
\rho_1 : \text{Spec} \ Z(H_{n,r}) \to \text{Spec} \ Z(R(\Gamma_n))^{JM} = S^n(\mathbb{A}^1) \times \mathbb{A}^r
\]
is the corresponding morphism. Here we identify \( C[z_1, \ldots, z_n]^{S_n} = C[S^n(\mathbb{A}^1)] \) and \( \text{Spec} \ k = \mathbb{A}^r \).

**Remark 8.1.** Recall that \( \text{Spec} \ Z(H_{n,r}) = \mathcal{M}^{univ}_{n,r} \) so we can consider \( \rho_1 \) as a morphism \( \mathcal{M}^{univ}_{n,r} \to S^n(\mathbb{A}^1) \times \mathbb{A}^r \).

We also have the morphism \( \rho_2 : \mathcal{X}_0^{univ}(n, r) \to S^n(\mathbb{A}^1) \times \mathbb{A}^r \) given by 
\[
(X, Y, \gamma, \delta) \mapsto ((\alpha_1, \ldots, \alpha_n), \varpi(X, Y, \gamma, \delta)),
\]
where \( \alpha_1, \ldots, \alpha_n \) is the multiset of eigenvalues of \( Y_1 X_0 \).

The following Proposition holds by \([\text{Prz}16] , \text{Proposition 6.10}]\).

**Proposition 8.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Spec} \ Z(H_{n,r}) & \xrightarrow{\text{EG}} & \mathcal{X}_0^{univ}(n, r) \\
\rho_1 \downarrow & & \rho_2 \downarrow \\
S^n(\mathbb{A}^1) \times \mathbb{A}^r
\end{array}
\]

Passing to \( T \)-fixed points we conclude that the following diagram is commutative:

\[
\begin{array}{ccc}
(\text{Spec} \ Z(H_{n,r}))^\mathbb{T} & \xrightarrow{\text{EG}^\mathbb{T}} & (\mathcal{X}_0^{univ}(n, r))^\mathbb{T} \\
\rho_1^\mathbb{T} \downarrow & & \rho_2^\mathbb{T} \downarrow \\
S^n(\mathbb{A}^1) \times \mathbb{A}^r
(8.1)
\end{array}
\]
where we denote by $\rho_1^T$, $\rho_2^T$, $\mathrm{EG}^T$ the restrictions of $\rho_1$, $\rho_2$, $\mathrm{EG}$ to the $T$-fixed points.

Recall now that by Lemma 7.8 $\rho_1^T$ is a closed embedding. Since $\rho_2^T$ identifies with $\rho_1^T$ via $\mathrm{EG}^T$ we conclude that $\rho_2^T$ is a closed embedding (this also follows from Lemma 1.19). We see that the closed subschemes

$$\rho^T_2: (X_0^\text{univ}(n, r))^T \subset \text{Spec } Z(R(\Gamma_n))^{JM} \supset \text{Spec } Q_{n,r}: \rho^T_1$$

are the same. In other words, the following diagram is commutative with $(\rho^T_1)^*$, $(\rho^T_2)^*$ being surjective:

$$\begin{array}{ccc}
Q_{n,r} & \xrightarrow{(\rho^T_1)^*} & \mathrm{EG}^T \\ & \map{Z(R(\Gamma_n))^{JM}} & \map{C[(X_0^\text{univ}(n, r))^T]} \\
\end{array}$$

Recall now that the surjection $(\rho^T_1)^*: Z(R(\Gamma_n))^{JM} \rightarrow Q_{n,r}$ sends $f \in \mathbb{C}[z_1, \ldots, z_n]^{S_n} = Z(R(\Gamma_n))^{JM}$ to the class of $f(u_1, \ldots, u_n)$ and directly from the definitions $(\rho^T_2)^*(f)$ is nothing else but the function on $(X_0^\text{univ}(n, r))^T$ given by

$$(X, Y, \gamma, \delta) \mapsto f(\alpha_1, \ldots, \alpha_n),$$

where $\alpha_1, \ldots, \alpha_n$ is the multiset of eigenvalues of $Y_1X_0$.

We obtain the following Proposition as a corollary.

**Proposition 8.3.** The identification $\mathrm{EG}^T: C[(X_0^\text{univ})^T] \simeq Q_{n,r}$ sends $[f(u_1, \ldots, u_n)]$ to the function

$$(X, Y, \gamma, \delta) \mapsto f(\alpha_1, \ldots, \alpha_n),$$

where $f \in \mathbb{C}[z_1, \ldots, z_n]^{S_n}$ and $\alpha_1, \ldots, \alpha_n$ is the multiset of eigenvalues of $Y_1X_0$.

**Remark 8.4.** Note that another way to think about the identification $C[(X_0^\text{univ})^T] \simeq Q_{n,r}$ is the following. We have embeddings $C[(X_0^\text{univ})^T] \subset E$, $Q_{n,r} \subset E$ (see Sections 1.6.4, 4) and in order to identify $C[(X_0^\text{univ})^T]$ with $Q_{n,r}$ we just need to show that the images of $f(u_1, \ldots, u_n)$ and $(X, Y, \gamma, \delta) \mapsto f(\alpha_1, \ldots, \alpha_n)$ are the same. It is enough to check this at the smooth locus so we are in the situation considered in [Prz16] and do not actually need to worry about extending the Etingof-Ginzburg isomorphism from the smooth locus (Proposition 3.13).

**9. Proof of Theorem 1.20 and Possible Generalizations**

**9.1. Proof of Theorem 1.20.** Let us now recall the statement of Theorem 1.20.

**Theorem 9.1.** After the identification of parameters (1.3) graded algebras

$$H^*_A(\mathfrak{M}(n, r)), Z(R^r(n)), Q_{n,r}, H^*_\times(\mathfrak{GL}_n)_{\mathcal{O}}(\mathcal{R}_{n,r}), C[(X_0^\text{univ}(n, r))^T]$$

are isomorphic. Isomorphisms above identify generators as follows

$$c_k(\mathcal{Y}) = [e_k(z_1, \ldots, z_n)] = [e_k(u_1, \ldots, u_n)] = [c_k(\mathcal{E})] = ((X, Y, \gamma, \delta) \mapsto e_k(\alpha_1, \ldots, \alpha_n)).$$

**Proof.** The claim follows from Propositions 1.8, 5.7, 6.7 together with Propositions 3.10, 8.3.

**9.2. Possible generalizations.**
9.2.1. Changing the quiver. Let $I = (I_0, I_1)$ be a finite quiver and let $v = (v_i)_{i \in I_0}$ be a dimension vector for $I$ and $w = (w_i)_{i \in I_0}$ be a framing. We can consider the corresponding (smooth) Nakajima quiver variety that we denote $\mathcal{M}_I(v, w)$ (see Section 2.1). There is a natural torus $A$ acting on $\mathbb{N}_I$. We can consider the symplectically dual variety $\mathcal{M}^\text{inv}_I(v, w)$ that can be described as the spectrum of the algebra $H^{A \times (G_v)\circ}_*(\mathcal{R}_I(v, w))$ of equivariant homology of the variety of triples $\mathcal{R}_I = \mathcal{R}_I(v, w)$ corresponding to $I$, $v$, $w$ (see [BFN]). Let $\mathcal{T}$ be $(\mathbb{C}^*)^{I_0}$. We have a natural action of $\mathcal{T}$ on $H^{A \times (G_v)\circ}_*(\mathcal{R}_I(v, w))$ (see [BFN] Section 3(v)). For $i \in I_0$ let $V_i, E_i$ be the corresponding tautological bundles on $\mathcal{M}_I(v, w)$, $\mathcal{R}_I(v, w)/(G_v)\circ$ respectively (one should think about $E_i$ as about the trivial vector bundle $V_i \times \mathcal{R}_I(v, w)$ on $\mathcal{R}_I(v, w)$ that has interesting $(G_v)\circ$-equivariant structure).

**Conjecture 9.2.** There exists the isomorphism of $\mathbb{Z}$-graded $\mathbb{C}[a]$-algebras

$$H^*_A(\mathcal{M}_I(v, w), \mathcal{C}) \simeq \mathbb{C}[\text{Spec}(H^{A \times (G_v)\circ}_*(\mathcal{R}_I(v, w)))^T]$$

that sends $c_k(V_i) \in H^*_A(\mathcal{M}_I(v, w), \mathcal{C})$ to $[c_k(E_i)] \in \mathbb{C}[\text{Spec}(H^{A \times (G_v)\circ}_*(\mathcal{R}_I(v, w)))^T]$, here $i \in I_0$, $k = 1, \ldots, v_i$.

**Remark 9.3.** If $I$ is a Jordan quiver then Conjecture 9.2 follows from Theorem 8.10. If $I$ is a finite quiver of type $ADE$ then the conjecture above is compatible with [KTWWY1] Conjecture 8.10, indeed recall that we use the notations of loc. cit.) in [KTWWY1] it is conjectured that the homomorphism $\mathcal{H} \to H^*(\mathfrak{M}(w, W, \mathcal{R}))$ given by $A_{i}^{(s)} \mapsto c_s(V_i)$ induces the desired isomorphism $B(Y_{i}^{(s)}(\mathcal{R})) \to H^*(\mathfrak{M}(w, W, \mathcal{R}))$. Note now that by [BFN] Theorem B.18 the identification of $Y_{i}^{(s)}(\mathcal{R})$ with the Coulomb realization sends $A_{i}^{(s)}$ to $(-1)^s c_s(w_{i,v})$ that is the $s$th Chern class of the tautological bundle $E_i$.

**Remark 9.4.** Conjecture 9.2 can be reformulated as follows. Recall that $\mathcal{M}_I(v, w) = \mu^{-1}(0)^{\text{st}}/G_v$ so $H^*_A(\mathcal{M}_I(v, w), \mathcal{C}) = H^*_A(\mu^{-1}(0)^{\text{st}}, \mathcal{C})$ and we have the natural surjective (see [MN18]) homomorphism

$$H^*_A(\mathcal{M}_I(v, w), \mathcal{C}) \to H^*_A(\mu^{-1}(0)^{\text{st}}, \mathcal{C}) = H^*_A(\mathfrak{M}(v, w), \mathcal{C}).$$

Similarly we can consider the natural composition

$$H^*_A(\mathcal{M}_I(v, w), \mathcal{C}) \subset H^*_A(\mathfrak{M}(v, w), \mathcal{C}) \to \mathbb{C}[\text{Spec}(H^{A \times (G_v)\circ}_*(\mathcal{R}_I(v, w)))^T]$$

that is also surjective (see Proposition 9.4 below). Conjecture 9.2 is then equivalent to the fact that the kernel of (9.2) is equal to the kernel of (9.3) i.e. that they define the same closed subschemes of $\text{Spec} H^*_A(\mathcal{M}_I(v, w), \mathcal{C})$.

The approach used in this paper has a chance to be generalized to (some) other quivers. The following Conjecture is the first step towards the proof of Conjecture 9.2 using methods of this paper.

**Conjecture 9.5.** The algebra $\mathbb{C}[\text{Spec}(H^{A \times (G_v)\circ}_*(\mathcal{R}_I)))^T]$ is flat over $\mathbb{C}[a]$.

We plan to return to the proof of this Conjecture in the future following the suggestion of Ben Webster that this conjecture might follow by combining results of papers [Web13], [Web∞].

**Remark 9.6.** Note that when $I$ is the Jordan quiver then Conjecture 9.2 follows from Appendix A, note also that the “dual” statement to the Conjecture 9.3 is indeed true and holds by [Nak01] Theorem 7.3.5.

Recall now that by [MN18] the algebra $H^*_A(\mathcal{M}_I(v, w))$ that appears at the LHS of Conjecture 9.2 is generated over $H^*_A(\text{pt})$ by the Chern classes of tautological bundles $c_k(V_i)$. It turns out that the “dual” statement can be also proved (without any
restrictions on the quiver $I$) i.e. that the algebra $\mathbb{C}[[\text{Spec}(H_{\mathbb{A}^{\times}(G_{\mathbb{C}})}(\mathcal{R}_{I}(v, w)))^T]$ of schematic fixed points is generated over $H^*_A(\text{pt})$ by the classes of $c_k(E_i)$. This is equivalent to the following Proposition which proof was explained to us by Ben Webster and Alex Weekes and which detailed proof will appear in their joint work with Joel Kamnitzer and Oded Yacobi.

**Proposition 9.7.** The natural embedding $H^*_A(G_{\mathbb{C}})(\text{pt}) \subset H^*_A(G_{\mathbb{C}})(\mathcal{R}_{I})$ induces surjection $H^*_A(G_{\mathbb{C}})(\text{pt}) \twoheadrightarrow \mathbb{C}[[\text{Spec}(H_{\mathbb{A}^{\times}(G_{\mathbb{C}})})](\mathcal{R}_{I}))^T].$

**Proof.** The claim follows from [Wee19, Proposition 3.1] using that the dressed minus-cule monopole operators have a nonzero degree with respect to $\mathbb{T}$ so their images in $\mathbb{C}[[\text{Spec}(H_{\mathbb{A}^{\times}(G_{\mathbb{C}})})](\mathcal{R}_{I}))^T]$ are zero. $\square$

**Remark 9.8.** Recall that $H^*_A(G_{\mathbb{C}})(\text{pt}) \subset H^*_A(G_{\mathbb{C}})(\mathcal{R}_{I})$ is called Cartan subalgebra (see Section 3(vi)). The spectrum of this algebra is nothing else but $(t_v/W_v) \times a$, where $t_v \subset G_{\mathbb{C}}$ is a maximal torus and $W_v$ is the Weyl group of $G_v$. The embedding above corresponds to the integrable system $\varpi : \mathcal{M}^\text{inv}(v, w) \to (t_v/W_v) \times a$ (see [BFN (3.17)]) and Proposition 9.7 is equivalent to the fact that after passing to (schematic) $\mathbb{T}$-fixed points, morphism $\varpi$ induces the closed embedding

$$(\mathcal{M}^\text{inv}_I(v, w))^T \subset (t_v/W_v) \times a.$$  

Assume now that Conjecture 9.5 holds. Assume also that there exists a resolution of singularities $\mathcal{M}^\text{inv}_I(v, w) \to \mathcal{M}^\text{inv}_I(v, w)$ (see [Wee22] for the discussion). It induces the morphism $(\mathcal{M}^\text{inv}_I(v, w))^T \to (\mathcal{M}^\text{inv}_I(v, w))^T$ that gives us the embedding $\mathbb{C}[(\mathcal{M}^\text{inv}_I(v, w))^T] \subset (\mathcal{M}^\text{inv}_I(v, w))^T$, the fact that this is indeed an embedding follows from Conjecture 9.5. We can also consider the embedding $H^*_A(\mathcal{M}_I(v, w)) \subset H^*_A(\mathcal{M}_I(v, w))$ that corresponds to the restriction $\mathcal{M}_I(v, w) \subset \mathcal{M}_I(v, w)$. Using that $c_k(V_i), c_k(E_i)$ are generators of our algebras it then remains to show that the images of $c_k(V_i), c_k(E_i)$ under these embeddings coincide.

Let us finally note that in order to show that the images of $c_k(V_i), c_k(E_i)$ coincide it is enough to do the following. For a generic $a \in a$ we need to construct a bijection

$$\mathcal{M}_I(v, w)^A \sim \mathcal{M}_I(v, w)^T, \ p \mapsto p'$$

such that

$$c_k(\alpha_1, \ldots, \alpha_{n_v}) = c_k(E_i)(p'),$$

where $\alpha_1, \ldots, \alpha_{n_v}$ are eigenvalues of $a \in a$ acting on $V_i|_{p'}$ and $c_k(E_i)$ is considered as a function on $\mathcal{M}_I(v, w)$. This approach might be related to the “enumerative” approach to symplectic duality (3d mirror symmetry) discussed in [AO21], see also [RSVZ19, RSVZ20].

9.2.2. $K$-theoretic and quantum Hikita-Nakajima conjecture for Gieseker variety. Instead of replacing the quiver we can still consider the Jordan quiver but replace algebras that we consider by their “quantum” or “deformed” analogs.

One way to do this is to replace cohomology by $K$-theory on both sides of (9.1). In the case of Gieseker variety the algebra on the LHS was studied in [BEF20, Section 3]. Following the suggestion of Pavel Etingof, we plan to generalize our approach to the $K$-theoretic case using [BEF20].

Another possible quantization is to replace the algebra $H^*_A(\mathcal{M}(n, r))$ by the algebra $H^*_{C_k^* \times A}(\mathcal{M}(n, r))$, where the action of $C_k^* = C^*$ on $\mathcal{M}(n, r)$ is induced by the scaling $C^*$-action on $T^*N$. The algebra $H^*_A(GL_n)(\mathcal{R}_{n,r})$ is replaced by $H^*_{C_k^* \times A}(GL_n)(\mathcal{R}_{n,r})$, 

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where $\mathbb{C}_h^\times = \mathbb{C}^\times$ acts on $R_{n,r}$ via the loop rotation. The corresponding $B$-algebra (analog of the algebra $\mathbb{C}[\{(\mathcal{M}_{n,r}^\text{univ})^T\}]$) is

\begin{equation}
(H^*_{c_h} \mathbb{C}^\times \times A^\times (GL_n)^\circ (R_{n,r}))_0 / \sum_{i>0} (H^*_{c_h} \mathbb{C}^\times \times A^\times (GL_n)^\circ (R_{n,r}))_{-i} (H^*_{c_h} \mathbb{C}^\times \times A^\times (GL_n)^\circ (R_{n,r}))_i.
\end{equation}

It is still true that the algebra $H^*_{c_h} \mathbb{C}^\times \times A^\times (GL_n)^\circ (R_{n,r})$ is isomorphic to the spherical subalgebra of the RCA (see [BEF20, Web19]).

**Proposition 9.9.** There exists the isomorphism of algebras

\[ \mathcal{H}_{n,r}^{\text{sph}} \simeq H^*_{c_h} \mathbb{C}^\times \times A^\times (GL(V)^\circ (R_{n,r}) \]  

that identifies parameters in the following way $h = h$, $\kappa = \kappa$, $a_i = p(n_i^{-1})$. The isomorphism above sends $c_i(u_1, \ldots, u_n)$ to $c_i(\mathcal{E})$, where $\mathcal{E}$ is the tautological $(GL_n)^\circ$-bundle on $R_{n,r}$.

So we can identify the $B$-algebra [9.4] of quantum schematic fixed points with

\[ Q_{n,r} := (\mathcal{H}_{n,r}^{\text{sph}})_0 / \sum_{i>0} (\mathcal{H}_{n,r}^{\text{sph}})_{-i} (\mathcal{H}_{n,r}^{\text{sph}})_i, \]

where the grading on $\mathcal{H}_{n,r}^{\text{sph}}$ is induced by the following grading on $\mathcal{H}_{n,r}$:

\[ \deg x_i = 1, \quad \deg y_i = -1, \quad \deg \Gamma_n = \deg t = \deg h = 0. \]

Note that $\mathfrak{M}(n, r) \mathbb{C}^\times \times A = \mathfrak{M}(n, r)^A$ and we still have the embedding

\begin{equation}
H^*_{c_h} \mathbb{C}^\times \times A(\mathfrak{M}(n, r), \mathbb{C}) \subset H^*_{c_h} \mathbb{C}^\times \times A(\mathfrak{M}(n, r)^A, \mathbb{C}) = E[h].
\end{equation}

It still follows from [MN18, Corollary 1.5] that the elements $c_i(\mathcal{V})$ do generate the algebra $H^*_{c_h} \mathbb{C}^\times \times A(\mathfrak{M}(n, r))$. To finish the proof we need to construct an embedding $Q_{n,r} \subset E[h]$ compatible with the embedding (9.5). The natural way to do this is to use the representation theory of $\mathcal{H}_{n,r}^{\text{sph}}$. Recall that by [BEF20] (see also [Web16, Theorem 3.3]) the algebra $\mathcal{H}_{n,r}^{\text{sph}}$ is Morita equivalent to the algebra $e'\mathcal{H}_{n,r} e'$, where $e'$ is the symmetrizer with respect to $\mathbb{Z}/r\mathbb{Z}^n \subset \Gamma_n$. Algebra $e'\mathcal{H}_{n,r} e'$ has a presentation via generators and relations (see [BEF20]) and should have a collection of irreducible graded representations $\mathcal{L}'(\lambda)$ parametrized by $\mathcal{P}(r, n)$. Then the corresponding irreducible representations of $\mathcal{H}_{n,r}^{\text{sph}}$ are $\mathcal{L}'(\lambda)^{\mathbb{C}^n}$ and we can construct the embedding $Q_{n,r} \subset E[h]$ by acting on the highest weight components of $\mathcal{L}'(\lambda)$. In order to make this idea into an actual proof, one needs to describe representations $\mathcal{L}'(\lambda)$ and to compute actions of $e_k(u_1, \ldots, u_n)$ on the highest weight components of $\mathcal{L}'(\lambda)^{\mathbb{C}^n}$.

**Appendix A. Flatness of schematic fixed points: approach of Hikita and Hatano**

The goal of this Appendix is to give self-contained proof of the fact that the algebra $Q_{n,r}$ of functions on the schematic fixed points

\[ (\text{Spec } \mathcal{Z}(H_{n,r}))^T = (\mathcal{M}_{n,r}^{\text{univ}})^T = (\mathcal{X}_{0}^{\text{univ}}(n, r))^T \]

is a flat (hence, free) $h$-module of rank $|\mathcal{P}(r, n)|$. Let us first of all note that by graded Nakayama lemma together with the fact that $\dim_F(Q_{n,r} \otimes h F) = |\mathcal{P}(r, n)|$ (this follows from [Gor02], see also [Prz16, Section 5]) in order to prove this fact it is enough to show that

\[ \dim_{\mathbb{C}} Q_{n,r}/(\kappa, c_1, \ldots, c_{r-1}) \leq |\mathcal{P}(r, n)| \]
i.e. that
\[(A.1) \quad \dim \mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]^T \leq |\mathcal{P}(r, n)|.\]

The goal of this Section is to prove the inequality \((A.1)\). Our argument simply follows papers [Hik17] (for \(r = 1\) case) and [Hat21] (in general) but is much shorter since we do not need any explicit formulas for the multiplication rule of the elements of \(\mathbb{C}[\mathbb{A}^{2n}/\Gamma_n]^T\) and only need to estimate the dimension of this algebra from above since the estimate from below follows from the deformation argument (so we only need [Hik17] Lemma 2.5] for \(r = 1\) case and [Hat21] Lemma 2.1.4] for general \(r\). We start from the case \(r = 1\) i.e. from the case when \(\Gamma_n = S_n\).

A.1. Hilbert scheme case \((r = 1)\). Let us recall some notation (we follow [Hik17]).

**Definition A.1.** An unordered sequence \(\Lambda = (a_1, b_1) \ldots (a_l, b_l)\) with \((a_i, b_i) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}\) is called bipartite partition of \((a, b) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}\) if \(\sum_{i=1}^l a_i = a, \sum_{i=1}^l b_i = b\). We set \(\ell(\Lambda) = l, |\Lambda| = (a, b)\).

We have a natural surjection
\[\mathbb{C}[S^{n+1}(\mathbb{A}^2)] = \mathbb{C}[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]^{S_{n+1}} \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n} = \mathbb{C}[S^n(\mathbb{A}^2)]\]
and denote by \(S\) the inverse limit
\[S := \lim_{\leftarrow} \mathbb{C}[S^n(\mathbb{A}^2)].\]

We denote by \(m_\Lambda \in S\) the symmetrization of the monomial \(x_i^{a_i} y_i^{b_i} \ldots x_i^{a_l} y_i^{b_l}\).

We have
\[S = \mathbb{C}[m_{(a, b)} | (a, b) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}]\]

For \((a, b) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}\) we set \((a, b)\Lambda := (a, b)(a_1, b_1) \ldots (a_l, b_l)\). If \((a, b) = (a_i, b_i)\) for some \(i \in \{1, 2, \ldots, l\}\) we set \(\Lambda \setminus (a, b) := (a_1, b_1) \ldots (a_{i-1}, b_{i-1})(a_{i+1}, b_{i+1}) \ldots (a_l, b_l)\). We set \(\overline{S} := S/(m_{(a, b)}, a \neq b)\) and denote by \(\overline{m}_\Lambda\) the image of \(m_\Lambda\) in \(\overline{S}\). Note that directly from the definitions for every \(n \in \mathbb{Z}_{\geq 1}\) we have a surjective homomorphism \(\overline{S} \twoheadrightarrow \mathbb{C}[S^n(\mathbb{A}^2)]\) which sends every \(\overline{m}_\Lambda\) with \(\ell(\Lambda) > n\) to zero.

The following Lemma is clear.

**Lemma A.2.** Let \(\Lambda\) be a bipartite partition. We have
\[m_{(a, b)} m_\Lambda = km_{(a, b)\Lambda} + \sum_{(i, j) \in \Lambda} k_{(i, j)} m_{(a+i, b+j)\Lambda \setminus (i, j)}\]
for some \(k, k_{(i, j)} \in \mathbb{Z}_{>0}\).

For a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)\) we denote by \((\lambda, 0)\) the bipartite partition \((\lambda_1, 0), \ldots, (\lambda_l, 0)\).

This Lemma is [Hik17] Lemma 2.5].

**Lemma A.3.** \(\{\overline{m}_{(\lambda, 0)(0, 1)^{|\lambda|}} | \lambda \text{ is partition}\}\) spans \(\overline{S}\).

**Proof.** Let us first of all note that the functions \(\overline{m}_{(a, b)(b, c)} \ldots \text{span} \overline{S}\). Indeed to prove this it is enough to show that every \(\overline{m}_\Lambda\) can be obtained as a linear combination of \(\overline{m}_{(a, b)b}(b, c)\ldots\). This can be proved by the induction on \(\ell(\Lambda)\) using Lemma A.2 together with the fact that \(\overline{m}_{(a, b)} = 0\) for \(a \neq b\).

It remains to show that every \(\overline{m}_{(a, b)(b, c)} \ldots \text{can be expanded in terms of} \overline{m}_{(\lambda, 0)(0, 1)^{|\lambda|}}\).

To see that it is enough to show that every \(\overline{m}_{(a_1, b_1)(a_2, b_2)(0, 1)^m} \text{ with } a_i \geq b_i\) can be
obtained as a linear combination of $\overline{m}_{(\lambda,0)(0,1)}$. We prove this by the induction on $d = \sum_{i=0}^{l} b_i$. For $d = 0$ the claim is clear.

For the induction step without losing the generality we can assume that $b_1 > 0$. Using Lemma A.2 we obtain:

$$m_{(a_1,b_1-1)m_{(a_2,b_2)...(a_l,b_l)(0,1)}^m} = k_1 m_{(a_1,b_1)...(a_l,b_l)(0,1)}^m + k_0 m_{(a_2,b_2)...(a_l,b_l)(0,1)}^m + \sum_{i=2}^{l} k_i m_{(a_2,b_2)...(a_i+b_1+b_2-1)...(a_l,b_l)(0,1)}^m$$

for some $k_0, k_1, \ldots, k_l \in \mathbb{Z}$ with $k_1 \neq 0$. Induction hypothesis together with the fact that $\overline{m}_{(a_1,b_1-1)}$ finish the proof. □

**Corollary A.4.** The image of the set $\{\overline{m}_{(\lambda,0)(0,1)} | \ell(\lambda) + |\lambda| \leq n\}$ spans $\mathbb{C}[S^n(\mathbb{A}^2)] = \mathbb{C}[\mathbb{A}^2n/S_n]$. In particular we have

$$\dim \mathbb{C}[\mathbb{A}^2n/S_n] \leq |\mathcal{P}(n)|.$$  

**Proof.** Clearly the elements $\overline{m}_{(\lambda,0)(0,1)}$ with $\ell(\lambda) + |\lambda| > n$ lie in the kernel of $\mathcal{S} \rightarrow \mathbb{C}[S^n(\mathbb{A}^2)]$. Now the first claim follows from Lemma A.3. It remains to note that we have a bijection

$$\{\lambda | \ell(\lambda) + |\lambda| \leq n\} \xrightarrow{\sim} \mathcal{P}(n)$$

that sends a partition $\lambda = (1^{a_1}2^{a_2} \ldots)$ to the partition $\hat{\lambda} \in \mathcal{P}(n)$ given by

$$\hat{\lambda} = 1^{n-\ell(\lambda)-|\lambda|}2^{a_1}3^{a_2} \ldots i^{a_i-1} \ldots.$$ □

A.2. **General case ($r$ is arbitrary).** Let us now generalize the arguments of Section A.1 to the of arbitrary $r \in \mathbb{Z}_{\geq 1}$. We follow [Hat21].

We start from some notation. Recall that $\mathbb{C}[\mathbb{A}^2n/\Gamma_n]$ is nothing else but

$$\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_{\mathbb{Z}/r\mathbb{Z}}^{S_n} \simeq \left(\mathbb{C}[x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z'_1, \ldots, z'_n]/(x'_1y'_1 - (z'_1)^r, \ldots, x'_ny'_n - (z'_n)^r)\right)^{S_n}$$

where the isomorphism is given by

$$x'_i \mapsto x_i, \ y'_i \mapsto y_i, \ z'_i \mapsto x_iy_i.$$  

**Remark A.5.** Geometrically, isomorphism above corresponds to the identification $\mathbb{C}[\mathbb{A}^2n/\Gamma_n] \simeq S^n(\mathbb{A}^2/(\mathbb{Z}/r\mathbb{Z}))$.

Set $I_n := (x'_1y'_1 - (z'_1)^r, \ldots, x'_ny'_n - (z'_n)^r) \subset \mathbb{C}[x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z'_1, \ldots, z'_n]$. Set $S' := \lim_{\leftarrow} \mathbb{C}[x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z'_1, \ldots, z'_n]$, $I := \lim_{\leftarrow} I_n$, $S := S/I$.

For every tripartition $\Lambda = (a_1, b_1, c_1)(a_2, b_2, c_2), \ldots, (a_l, b_l, c_l)$ let $m'_{\Lambda} \in S'$ be the symmetrization of the monomial $(x'_1)^{a_1}(y'_1)^{b_1}(z'_1)^{c_1} \ldots (x'_n)^{a_n}(y'_n)^{b_n}(z'_n)^{c_n}$, we set $\ell(\Lambda) := I$. We denote by $m_{\Lambda} \in S$ the image of $m'_{\Lambda}$. The following Lemma is clear.

**Lemma A.6.** The set $\{m_{\Lambda} | \Lambda$-tripartition}$ spans $S'$. So

$$\{m_{\Lambda} | \Lambda = (a_1, b_1, c_1), \ldots, c_i \leq r - 1\}$$

spans $S$.  

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Lemma A.7. Let $\Lambda$ be a tripartition and $(a, b, c) \in \mathbb{Z}_{\geq 0}^3 \setminus \{(0, 0, 0)\}$. Then we have

$$m'_{(a,b,c)}m'_{\Lambda} = \cdot m'_{(a,b,c)\Lambda} + \sum_{(i,j,k) \in \Lambda} \cdot m'_{(a+i,b+j,c+k)\Lambda \setminus (i,j,k)}$$

with $\cdot$ being some positive numbers. As a corollary we have

$$m_{(a,b,c)}m_{\Lambda} = \cdot m_{(a,b,c)\Lambda} + \sum_{(i,j,k) \in \Lambda, c+k < r-1} \cdot m_{(a+i,b+j,c+k)\Lambda \setminus (i,j,k)} + \sum_{(i,j,k) \in \Lambda, c+k \geq r} \cdot m_{(a+i+1,b+j+1,c+k-r)\Lambda \setminus (i,j,k)}$$

with $\cdot$ being some positive numbers.

For $\Lambda = (a_1, b_1, c_1) \ldots (a_l, b_l, c_l)$ we set $\deg \Lambda := \sum_{i=1}^l a_i - \sum_{i=1}^l b_i$. Let $J \subseteq S$ be the ideal generated by $\{m_{\Lambda} \mid \deg \Lambda \neq 0\}$. We set $\overline{S} := S/J$. We denote by $\overline{m}_{\Lambda} \in \overline{S}$ the image of $m_{\Lambda}$. For an $r$-tuple of partitions $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ we define tripartition to be denoted by the same symbol

$$\lambda := (\lambda^0_l, 0, 0) \ldots (\lambda^0_l(\lambda^0_{l-1}), 0, 0), \ldots, (\lambda^0_1, 0, r-1), \ldots, (\lambda^0_1, 0, r-1).$$

Recall that $\ell(\lambda) = \sum_{i=0}^{r-1} \ell(\lambda^i)$, $|\lambda| = \sum_{i=0}^{r-1} |\lambda^i|$. This Lemma is [Hat21, Lemma 2.1.4].

Lemma A.8. The set $\{\overline{m}_{\lambda(0,1,0)\lambda} \}$ spans $\overline{S}$.

Proof. We start from proving that elements $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)} \ldots \overline{m}_{(a_l,a_l,c_l)}$ span $\overline{S}$. It is enough to show that every $m_{\Lambda}$ can be presented as a linear combination of $\overline{m}_{(a_1,a_1,c_1)(a_2,a_2,c_2)} \ldots \overline{m}_{(a_l,a_l,c_l)}$. We can assume that there exists $(a, b, c) \in \Lambda$ such that $a \neq b$ (otherwise there is nothing to prove). From Lemma A.7 it follows that

$$0 = \overline{m}_{(a,b,c)}\overline{m}_{\Lambda \setminus (a,b,c)} = k\overline{m}_{\Lambda} + \sum_{\lambda', \ell(\lambda') = \ell(\Lambda)-1} \cdot \overline{m}_{\lambda'}$$

and the claim follows by the induction on the length of $\Lambda$.

It remains to show that every element $\overline{m}_{(a_1,a_1,c_1) \ldots (a_l,a_l,c_l)}$ can be written as a linear combination of $\overline{m}_{(a_1,a_1,c_1) \ldots (a_l,a_l,c_l)}$. We prove more general statement: that every element $\overline{m}_{(a,b,c_1) \ldots (a,b,c_l)(0,1,0)^k}$ with $a_i \geq b_i$ can be written as a linear combination of $\overline{m}_{(a_1,a_1,c_1) \ldots (a_l,a_l,c_l)}$. We prove this claim by the induction on $b + l$, where $b := \sum_{i=1}^l b_i$. Let us first of all note that we can assume that $\sum_{i=1}^l a_i = b + k$. For $b = 0$ we must have $b_i = 0$ for every $i$ and then there is nothing to prove. Suppose now that $b > 0$. Without loosing the generality we can assume that $b_1 > 0$. By Lemma A.7 we have

$$0 = \overline{m}_{(a_1,b_1-1,c_1)}\overline{m}_{(a_2,b_2,c_2) \ldots (a_l,b_l,c_l)(0,1,0)^{k+1}} =$$

$$= \cdot \overline{m}_{(a_1,b_1-1,c_1)}\overline{m}_{(a_2,b_2,c_2) \ldots (a_l,b_l,c_l)(0,1,0)^{k+1}} + u \overline{m}_{(a_1,b_1,c_1) \ldots (a_l,b_l,c_l)(0,1,0)^{k+1}} +$$

$$+ \sum_{c_1+c_i < r} \cdot \overline{m}_{(a_2,b_2,c_2) \ldots (a_i+a_1,b_i+b_1,c_i+c_1) \ldots (a_l,b_l,c_l)(0,1,0)^{k+1}} +$$

$$\sum_{c_1+c_i \geq r} \cdot \overline{m}_{(a_2,b_2,c_2) \ldots (a_i+a_1,b_i+b_1,c_i+c_1-r) \ldots (a_l,b_l,c_l)(0,1,0)^{k+1}}$$

with $u \in \mathbb{Z}_{\geq 0}$.

Now the claim follows from the induction hypothesis. \qed
Corollary A.9. The image of the set \( \{ \overline{m}_{\lambda(0,1,0)} | \ell(\lambda) + |\lambda| \leq n \} \) spans \( \mathbb{C}[A^{2n}/\Gamma_n] \). In particular
\[
\dim \mathbb{C}[A^{2n}/\Gamma_n] \leq |\mathcal{P}(r,n)|.
\]

Proof. Note that \( \ell(\lambda(0,1,0)) = \ell(\lambda) + |\lambda| \) so the elements \( \overline{m}_{\lambda(0,1,0)} \) with \( \ell(\lambda) + |\lambda| > n \) lie in the kernel of \( \overline{\mathcal{F}} \to \mathbb{C}[A^{2n}/\Gamma_n] \). Now the first claim follows from Lemma A.8. It remains to note that we have a bijection
\[
\{ \lambda | \ell(\lambda) + |\lambda| \leq n \} \overset{\sim}{\rightarrow} \mathcal{P}(r,n)
\]
that sends an \( r \)-partition \( \lambda \) with \( \lambda^0 = 1^{\alpha_1}2^{\alpha_2} \ldots \) to the \( r \)-partition \( \hat{\lambda} \in \mathcal{P}(r,n) \) given by
\[
\hat{\lambda}^0 = 1^{n-\ell(\lambda)-|\lambda|}2^{\alpha_1}3^{\alpha_2} \ldots k^{\alpha_{k-1}} \ldots, \hat{\lambda}^i = \lambda^i \text{ for } i = 1, \ldots, r - 1.
\]
The inverse map sends \( \mu \in \mathcal{P}(r,n) \) with \( \mu^0 = 1^{\beta_1}2^{\beta_2} \ldots \) to the partition \( \lambda \) given by
\[
\lambda^0 = 1^{\beta_2}2^{\beta_3} \ldots k^{\beta_{k+1}} \ldots, \lambda^i = \mu^i \text{ for } i = 1, \ldots, r - 1.
\]
\[\square\]

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