Evaluation of Euler Number of Complex Grassmann Manifold $G(k, N)$ via Mathai-Quillen Formalism

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Abstract

In this paper, we provide a recipe for computing Euler number of Grassmann manifold $G(k, N)$ by using Mathai-Quillen formalism (MQ formalism) and Atiyah-Jeffrey construction. Especially, we construct path-integral representation of Euler number of $G(k, N)$. Our model corresponds to a finite dimensional toy-model of topological Yang-Mills theory which motivated Atiyah-Jeffrey construction. As a by-product, we construct free fermion realization of cohomology ring of $G(k, N)$.

1 Introduction

Our aim of this paper is to compute Euler number of finite dimensional manifold (Grassmann manifold $G(k, N)$) by using Mathai-Quillen formalism (MQ formalism) and Atiyah-Jeffrey construction. Mathai-Quillen formalism is a method for constructing Thom class of finite-dimensional vector bundle $E$ on a manifold $M$, that decreases like Gaussian along fiber direction. This Thom class plays the same role as the original Thom class which has compact support along fiber direction. The Euler class of $E$ is given by pull-back of the Thom class by the section $s : M \to E$. It does not depend on choice of the section of $s$ as a cohomology class because of homotopy invariance of de Rham
Therefore, we can compute the Euler number of $E$ by choosing a convenient section, which leads us to localization technique. In this paper, we choose zero-section as $s$. Suppose $M$ is given as an orbit space $X/G$ where a Lie group $G$ acts freely on a manifold $X$. Atiyah and Jefferey extended the MQ-formalism for an orbit space $X/G$ \[1\]. Strictly speaking, they extended it to the case when $X/G$ is given by infinite-dimensional space of gauge equivalence classes of connections of $SU(2)$ bundle on a 4-dimensional manifold, in order to study mechanism behind Witten’s construction of topological Yang-Mills theory. We call this method “Atiyah-Jeffrey construction”. In this paper, we apply Atiyah-Jeffrey construction to the case when $X/G$ is finite-dimensional complex Grassmann manifold $G(k, N)$ and $E$ is holomorphic tangent bundle of $G(k, N)$, and we construct path-integral representation of Euler number of $G(k, N)$. Our construction corresponds to a finite dimensional toy-model of topological Yang-Mills theory which motivated Atiyah-Jeffrey construction.

### 1.1 Our Model and Main Theorem

Complex Grassmann manifold $G(k, N)$ is a space which parametrizes $k$-dimensional linear subspaces of $N$-dimensional complex vector space. We denote by $U(k)$ unitary group that acts on complex $k$-dimensional vector space and by $V_k(\mathbb{C}^N)$ Stiefel manifold of orthonormal $k$-frames in $\mathbb{C}^N$. Then $G(k, N) \cong V_k(\mathbb{C}^N)/U(k)$. We introduce detailed informations of $G(k, N)$ in Section 3. By applying Atiyah-Jeffrey construction to the case when $X$ and $G$ are given by $V_k(\mathbb{C}^N)$ and $U(k)$, Euler number of $G(k, N)$ is represented by the following finite-dimensional path-integral (unless otherwise noted, we use the Einstein convention).

**Theorem 1. (Main Theorem)** Euler number of $G(k, N)$ is evaluated by finite dimensional path-integral:

\[
\chi(G(k, N)) = \chi(V_k(\mathbb{C}^N)/U(k)) = \beta \int_{V_k(\mathbb{C}^N)} Dz \int D\psi D\phi D\bar{\phi} DAD\chi DH \omega \exp(-\mathcal{L}_{MQ}),
\]

where the Lagrangian $\mathcal{L}_{MQ}$, the projection operator $\omega$ and the normalization factor $\beta$ are given by,

\[
\mathcal{L}_{MQ} = \delta <\chi, H> + \delta <\psi, A>, \quad \omega = k \prod_{i,j=1}^{k}(\psi_{s}^i z_s^i + \bar{z}_s^i \psi_s^j), \quad \beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k-k^2+N N!}(-\pi)^{k N} k N! (-1)^k 2^{k-1}}.
\]

In the above Lagrangian, $\delta$ represents supersymmetric transformation whose detailed construction will be explained in Section 2. In this section, we briefly introduce our notations and supersymmetric transformation. Let

\[
z^i := (z^1 \cdots z^k), \quad z^i := (z_1^i, z_2^i, \cdots, z_N^i) \quad (j = 1, 2, \cdots, k),
\]

be local complex coordinate system of $\mathbb{C}^{kN}$. Here, we introduce some notation for our model (this notation is used in this chapter, Chapters 3 and Chapter 4).
Definition 1. Let $X$ and $Y$ be matrix variables used in our Lagrangian. 

1. We represent complex conjugate of $X$ by $\bar{X}$ or $X^*$. We also use the notation $\bar{(X)}^i_j := X^j_i$ or $(\bar{X})^i_{ij} := X^j_{i}$. In our Lagrangian, $\phi$ and $\bar{\phi}$ are different independent fields, and we use $^*\phi$ and $^*\bar{\phi}$ to represent the complex conjugate of $\phi$ and $\bar{\phi}$.

2. We represent transpose of $X$ (resp. adjoint of $X$) by $^tX$ (resp. $X^\dagger$). $(X^\dagger := ^t\bar{X})$.

3. We define inner product of matrix variables $X$ and $Y$ of the same type by $<X,Y> := \text{tr}(X^\dagger Y)$.

Then $V_k(C^N)$ is given by a set of points in $C^{kn}$ that satisfy,

$$\sum_{s=1}^{N} \overline{z}_s^i z_s^j - \delta_{ij} = 0 \quad (i,j = 1, 2, \cdots, k),$$

where $\overline{z}_s^i$ represents $\overline{z}_s^i$, and $\delta_{ij}$ is the Kronecker’s delta. $\psi$s are complex grassmann variables that correspond to super-partner of $z$:

$$\psi := (\psi_1^1, \psi_2^2, \cdots, \psi^k), \quad \psi^j := (\psi_1^j, \psi_2^j, \cdots, \psi_N^j).$$

$\phi$ and $\bar{\phi}$ are $k \times k$ Hermite matrices ($\phi = \phi^\dagger$, $\bar{\phi} = \bar{\phi}^\dagger$). $i\phi$ and $i\bar{\phi}$ plays the role of generator of Lie algebra of the gauge group $U(k)$. Note that $\bar{\phi}$ is not complex conjugate of $\phi$. $\eta$ is super-partner of $\phi$. Hence $\eta$ is a grassmann and Hermite matrix ($\eta = \eta^\dagger$).

$$\phi := \left( \begin{array}{ccc} \phi_1^1 & \cdots & \phi_1^k \\ \vdots & \ddots & \vdots \\ \phi_k^1 & \cdots & \phi_k^k \end{array} \right), \quad \bar{\phi} := \left( \begin{array}{ccc} \bar{\phi}_1^1 & \cdots & \bar{\phi}_1^k \\ \vdots & \ddots & \vdots \\ \bar{\phi}_k^1 & \cdots & \bar{\phi}_k^k \end{array} \right).$$

$$\eta := \left( \begin{array}{ccc} \eta_1^1 & \cdots & \eta_1^k \\ \vdots & \ddots & \vdots \\ \eta_k^1 & \cdots & \eta_k^k \end{array} \right).$$

$H$ is $k \times N$ complex matrix which plays the role of auxiliary variable in MQ-formalism. $\chi$ is super-partner of $H$. $A$ is $k \times k$ complex matrix and $\psi_A$ is super-partner of $A$.

$$H := \left( \begin{array}{ccc} H_1^1 & \cdots & H_1^k \\ \vdots & \ddots & \vdots \\ H_k^1 & \cdots & H_k^k \end{array} \right), \quad H^\dagger := \left( \begin{array}{ccc} H_1^1 & \cdots & H_1^k \\ \vdots & \ddots & \vdots \\ H_k^1 & \cdots & H_k^k \end{array} \right).$$

$$\chi := \left( \begin{array}{ccc} \chi_1^1 & \cdots & \chi_1^k \\ \vdots & \ddots & \vdots \\ \chi_k^1 & \cdots & \chi_k^k \end{array} \right), \quad \chi^\dagger := \left( \begin{array}{ccc} \chi_1^1 & \cdots & \chi_1^k \\ \vdots & \ddots & \vdots \\ \chi_k^1 & \cdots & \chi_k^k \end{array} \right).$$

$$A := \left( \begin{array}{ccc} A_1^1 & \cdots & A_1^k \\ \vdots & \ddots & \vdots \\ A_k^1 & \cdots & A_k^k \end{array} \right), \quad \psi_A := \left( \begin{array}{ccc} \psi_A^1 & \cdots & \psi_A^k \\ \vdots & \ddots & \vdots \\ \psi_A^1 & \cdots & \psi_A^k \end{array} \right).$$

The supersymmetric transformation $\delta$ of our model is given as follows.

$$\delta z_i^j = \psi_i^j, \quad \delta \psi_i^j = i\phi_{im} z_m^j, \quad \delta \chi_i^j = H_i^j, \quad \delta H_i^j = (\gamma \delta_{ij} + i\phi_{ij}) \chi_i^j,$$

$$\delta A_i^j = \psi_{ij}^j, \quad \delta \phi_{ij} = 0, \quad \delta \bar{\phi}_{ij} = \eta_{ij}, \quad \delta \psi_A = \gamma A + [A, i\phi].$$

In the above transformation, $\gamma$ is central charge of central extension of standard supersymmetric transformation. We assume that $\gamma$ is a non-zero constant. Supersymmetric transformation for complex conjugate of $X$ is defined as complex conjugate of $\delta X$ ($\delta \bar{X} := \overline{(\delta X)}$). $\delta$ behaves like a grassmann variable. By applying (1.14) to (1.12), we can obtain explicit form of the Lagrangian (1.13). Note that the $z$ variables satisfy (1.14), the defining equations of the Stiefel manifold $V_k(C^N)$. On the other hand, $\psi$, the super-partner of $z$, plays the role of the 1-form $dz$ in the supersymmetric path-integral. Since $(z_i^j)^{1} z^j - \delta_{ij} = 0 \quad (i,j = 1, 2, \cdots, k)$, we also have the constraint for $dz$:

$$\sum_{s=1}^{N} (dz_s^i z_s^j + z_s^i dz_s^j) = 0 \quad (i,j = 1, 2, \cdots, k).$$
By identifying $dz^i_s$ and $dz^i_{s+1}$ with $\psi^i_s$ and $\bar{\psi}^i_s$, respectively, the above constraint is realized by insertion of the following projection operator $\omega$:

$$\omega = \prod_{i,j=1}^{k} \left[ \sum_{s=1}^{N} (\psi^i_s z^j_s + z^i_s \bar{\psi}^j_s) \right].$$  \hspace{1cm} (1.16)

$\beta$ is the normalization factor that normalizes volume of $G(k, N)$ into 1.

$$\beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k} (-\pi)^{k^2 + kN} \pi^{kN + \frac{k(k+1)}{2}} (-1)^{\frac{k}{2}(k-1)}}.$$  \hspace{1cm} (1.17)

Note that the volume $U(k)$ is given by,

$$\text{vol}(U(k)) = \prod_{j=1}^{k} \text{vol}(S^{2j-1}) = \frac{2^k \pi^{\frac{k(k+1)}{2}}}{\prod_{j=1}^{k-1} j!}.$$  \hspace{1cm} (1.18)

Then, we can rewrite $\beta$ into the following form:

$$\beta = \frac{1}{2^k (-\pi)^{k^2 + kN} \pi^{kN} (-1)^{\frac{k}{2}(k-1)} \text{vol}(U(k))}.$$  \hspace{1cm} (1.19)

Lastly, we remark that our Lagrangian is not strictly invariant under the supersymmetric transformation, i.e., $\delta L_{MQ} \neq 0$, because our supersymmetric transformation is not nilpotent. For example, we have the following relation.

$$\delta^2 < \psi_A, A > = \delta \{ \text{tr}(A^\dagger (\gamma A + [i\phi, A]) - \psi_A^\dagger \psi_A) \}$$

$$= \text{tr}(\psi_A^\dagger (\gamma A + [i\phi, A]) + A^\dagger (\gamma \psi_A + [i\phi, \psi_A]) - (\gamma A + [A, i\phi])^\dagger \psi_A + \psi_A^\dagger (\gamma A + [A, i\phi]))$$

$$= 2\gamma < \psi_A, A > .$$

This comes from central extension of the standard supersymmetry. The reason why we introduce central extension will be given in the next subsection. In some sense, we consider central extension of supersymmetric transformation in order to obtain top Chern class of tangent bundle of $G(k, N)$ from the toy model version of topological Yang-Mills theory.

### 1.2 The New Feature of Our Model

Explicit evaluation of the above path-integral will be given in Section 3 and Section 4, but in this subsection, we briefly explain new feature of our model which has not appeared in the former literatures on MQ formalism. Let us introduce well-known facts on Chern classes of holomorphic tangent bundle $T'G(k, N)$ of $G(k, N)$. Let $S$ be tautological bundle of $G(k, N)$ whose fiber of $A \in G(k, N)$ is given by complex $k$-dimensional subspace $\Lambda \subset \mathbb{C}^N$ itself ($\text{rk}(S) = k$). Then universal quotient bundle $Q$ ( $\text{rk}Q = N - k$) is defined by the following exact sequence

$$0 \rightarrow S \rightarrow \mathbb{C}^N \rightarrow Q \rightarrow 0.$$  \hspace{1cm} (1.20)

where $\mathbb{C}^N$ means trivial bundle $G(k, N) \times \mathbb{C}^N$. Since $T'G(k, N)$ can be identified with $Q \otimes S^*$, we obtain the following exact sequence:

$$0 \rightarrow S \otimes S^* \rightarrow \mathbb{C}^N \otimes S^* \rightarrow T'G(k, N) \rightarrow 0.$$  \hspace{1cm} (1.21)

Hence total Chern class of $T'G(k, N)$ is given by $\left(\frac{c(S^*)}{c(S \otimes S^*)} \right)^N$ (Euler number of $G(k, N)$ is obtained from integration of top Chern class of $T'G(k, N)$). If we decompose $S^*$ formally by the line bundle $L_i$ ($i = 1, 2, \cdots, k$):

$$S^* = \bigoplus_{i=1}^{k} L_i,$$  \hspace{1cm} (1.22)
\(c(S^*)\) is written as \(\prod_{i=1}^{k}(1 + tx_i)\). (Here \(x_i := c_1(L_i)\)). From splitting principle of Chern classes, we also have \(c(S \otimes S^*) = \prod_{i>j}(1 - t^2(x_i - x_j)^2)\). Then we obtain,

\[
c(T'G(k, N)) = \prod_{i=1}^{k}(1 + tx_i)^N \prod_{i>j}(1 - t^2(x_i - x_j)^2).
\] (1.23)

Top chern class is given as coefficient of \(t^k(N-k)\). The new feature of our model is given as follows.

By introducing central extension of standard supersymmetry with central charge \(\gamma(\neq 0)\), we can produce top Chern class of \(T'G(k, N)\) via the total Chern class (1.23).

Precisely speaking, integration of \(H\) and \(\chi\) results in \(\prod_{i=1}^{k}(\gamma + x_i)^N\) and integration of \(A\) and \(\psi_A\) produces \(1/\gamma^k \prod_{i>j}(\gamma^2 - (x_i - x_j)^2)\). But after integration of the Grassmann variable \(\psi\), which corresponds to integration of differential form on \(G(k, N)\), only the contribution from the top Chern class survives and we obtain the Euler number of \(G(k, N)\). As we will see later, final result does not depend on \(\gamma\) as long as \(\gamma\) is non-zero. This construction was obtained after many try and errors, but we don’t know whether there exists more natural construction that evaluates Euler number of \(G(k, N)\) by using MQ formalism and Atiyah-Jeffrey construction.

1.3 Organization of the paper

This paper is organized as follows. In Section 2, we give an overview of MQ formalism and Atiyah-Jeffrey construction. In Section 3, we construct Lagrangian that counts Euler number of \(G(k, N)\) by applying these techniques. Then we integrate out fields except for \(\psi\) and show that the Euler number is represented by fermion integral of the Chern class represented by the matrix \(i\phi\) whose \((i, j)\)-element is given by \(\langle \psi^i, \psi^j \rangle\). In Section 4, we prove our main theorem by showing that the representation of the Chern class by the fermion variables give the desired Chern class as an elements of cohomology ring of \(G(k, N)\). Especially, validity of normalization factor \(\beta\) will be verified. We think that combinatorial aspects in the discussions in Section 4 is quite interesting for mathematicians.

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2 The Mathai Quillen formalism and the Atiyah Jeffrey construction

In this section, we explain outline of Mathai-Quillen formalism and Atiyah Jeffrey construction. For more details, see the literatures [2] [8] [12] [13].

2.1 Overview of Mathai Quillen formalism

Mathai-Quillen formalism (MQ formalism) provides us with a recipe to construct Thom from of a vector bundle. Here, we briefly explain outline of Mathai-Quillen formalism. Let \(\pi : E \to M\) be a vector bundle of rank \(n\) on \(n\)-dimensional compact manifold \(M\). We assume that each fiber \(\pi^{-1}(x)\) has metric (or inner product) that varies smoothly as \(x \in M\) varies. We denote by \(\{f_1, \cdots, f_n\}\) a local orthonormal frame of \(\pi^{-1}(U)\) (\(U\) is some open subset of \(M\)) with respect to this metric. Let \(\Omega^q(M, E)\) be vector space of \(E\)-valued differential \(q\)-form on \(M\) (\(\Omega^q(M, E) \simeq \Gamma(E)\) is vector space
of smooth section of $E$) and $\nabla^E : \Omega^q(M, E) \to \Omega^{q+1}(M, E)$ be a connection compatible with inner product on $E$. $\nabla^E$ satisfies Leibniz rule for $g \in \Omega^q(M), s \in \Gamma(E)$

$$\nabla^E(gs) = (d_M g)s + (-1)^q g \wedge (\nabla^E s), \quad (2.24)$$

where $d_M$ is exterior derivative on $M$ and $\Omega^q(M)$ is vector space of smooth $q$-form on $M$. Let us define a connection form $\omega_i^j$ by,

$$\nabla^E f_i = \omega_i^j f_j, \quad (2.25)$$

where $\omega_i^j = -\omega_j^i$. Curvature of $\nabla^E$ is given by $(\nabla^E)^2 := R^E$. $R^E$ for the local orthonormal frame is represented in the following form:

$$(R^E)^i_j f_j := (\nabla^E)^2 f_i = \nabla^E(\omega_i^j f_j) = d_M \omega_i^j f_j - \omega_i^j \wedge \omega_j^k f_k$$

$$= (d_M \omega_i^j - \omega_i^k \wedge \omega_j^k) f_j. \quad (2.26)$$

From now on, we assume $n = 2m$. Let $u = (u^1, \ldots, u^{2m})$ be coordinates of fiber $\pi^{-1}(x)$ of $E$ and $\chi$ be Grassmann variable: $\chi = t(\chi^1, \ldots, \chi^{2m})$, which correspond to super-partner of $u$. And let $R_{ij}$ be $(R^E)^i_j$ ($R_{ij}$ is skew-symmetric). Note that $R_{ij}$ is locally a 2-form on $M$. With this set-up, Thom form $\Phi_\nabla(E)$ constructed in MQ-formalism is given as follows.

$$\Phi_\nabla(E) = \frac{1}{(2\pi)^m} e^{-|u|^2/2} \int D\chi \exp \left( \frac{1}{2} t^i \chi R \chi + i t^i (\nabla u) \chi \right). \quad (2.27)$$

$$|u|^2 := \sum_{i=1}^{2m} (u^i)^2, \quad t^i (\nabla u) \chi := \sum_{i=1}^{2m} (\nabla u)^i \chi^i := \sum_{i=1}^{2m} (d_E u^i + \omega_j^i u^i) \chi^i, \quad (2.28)$$

where $d_E = d_M + \sum_{i=1}^{2m} du^i \frac{\partial}{\partial u^i}$ is exterior derivative on $E$. Let $\mathcal{L}_0$ be a Lagrangian defined by,

$$\mathcal{L}_0 := |u|^2/2 - \frac{1}{2} t^i \chi R \chi - i t^i (\nabla u) \chi. \quad (2.29)$$

Then we define supersymmetric transformation as follows:

$$\delta \chi^i := iu^i, \quad \delta u^i := \nabla u^i. \quad (2.30)$$

Here, we assume the following.

Assumptions for $\delta$

1. $\delta$ behaves like fermionic variable. Hence $\delta$ is anti-commutative with $d_E$.
2. $\delta$ acts only fiber variables and $\delta \omega = \delta R = 0$.

Then we can show that $\mathcal{L}_0$ invariant under $\delta$ transformation.

$$\delta(|u|^2/2) = \sum_{i=1}^{2m} u^i (\nabla u)^i. \quad (2.31)$$

$$\delta(-\frac{1}{2} t^i \chi R \chi) = \frac{1}{2} (-\delta(\chi^i) R_{ij} \chi^j + \chi^i R_{ij} \delta(\chi^j)) = -iu^i R_{ij} \chi^j. \quad (2.32)$$

$$\delta(-i t^i (\nabla u) \chi) = -i \sum_{i=1}^{2m} (\delta((\nabla u)^i) \chi^i - (\nabla u)^i \delta(\chi^i))$$

$$\delta(-i t^i (\nabla u) \chi) = -i \sum_{i=1}^{2m} ((-d_E \delta(u^i) - \omega_j^i \delta(u^j)) \chi^i - i(\nabla u)^i u^i) \quad (2.33)$$

$$= -i \sum_{i=1}^{2m} ((-d_E \delta(u^i) - \omega_j^i \delta(u^j)) \chi^i - i(\nabla u)^i u^i) \quad (2.34)$$
\[
\delta \mathcal{L}_0 = 0. \text{ Let us integrate out } \Phi(E) \text{ on a fiber } \pi^{-1}(x). \text{ Since } x \in M \text{ is fixed, } R = \omega = 0. \text{ Then we can derive, }
\int_{\pi^{-1}(x)} \Phi(E) = 1. \tag{2.39}
\]

Explicit derivation is given as follows.
\[
\int_{\pi^{-1}(x)} \Phi(E) = (2\pi)^{-m}(-1)^{m+\frac{(2m-1)(2m)}{2}} \int_{\pi^{-1}(x)} e^{-|u|^2/2} \int D\chi \prod_{a=1}^{2m} (1 + iu^a \chi^a)
\]
\[
= \frac{(-1)^{\frac{(2m-1)(2m)}{2}}}{(2\pi)^m} \int_{\pi^{-1}(x)} e^{-|u|^2/2} \int D\chi (du^1 \chi^1) \cdots (du^{2m} \chi^{2m})
\]
\[
= \frac{1}{(2\pi)^m} \int_{\pi^{-1}(x)} e^{-|u|^2/2} du^1 \cdots du^{2m} = 1. \tag{2.40}
\]

(2.39) is one of the two features that characterizes Thom form of \( E \). The other one is given by,
\[
s_0^*(\Phi(E)) = e_{0,\mathcal{V}}(E), \tag{2.41}
\]
where \( s_0 : M \to E \) is the zero section of \( E \) and \( e_{0,\mathcal{V}}(E) \) is Euler class of \( E \). This can be easily seen as follows:
\[
s_0^*(\Phi(E)) = \frac{1}{(2\pi)^m} \int D\chi \exp \left( \frac{1}{2} t \chi R \chi \right)
\]
\[
= \frac{1}{(2\pi)^m} \text{Pfaff}(R)
\]
\[
= e_{0,\mathcal{V}}(E), \tag{2.42}
\]
where we used that \( u^i(s_0(x)) = 0 \) and that \( R \) is skew symmetric. Integration of Euler class \( e_{0,\mathcal{V}}(E) \) on \( M \) gives Euler number of \( E \), which is denoted by \( \chi(E) \). Therefore, we have,
\[
\chi(E) = \int_M s_0^*(\Phi(E)). \tag{2.43}
\]

At this stage, we include auxiliary bosonic variable \( H^i \) and modify the supersymmetric transformation as follows.
\[
\delta \chi^i := H^i, \ \delta H^i := R_{ij} \chi^j. \tag{2.44}
\]

Now we introduce \( \Psi := \langle \chi, \frac{H}{2} - iu \rangle \) where \( \langle A, B \rangle := \langle AB \rangle \text{ is inner product of } A \text{ and } B \). We also use the notation \( |A|^2 := \langle A, A \rangle \). Then \( \delta \Psi \) is given as follows.
\[
\delta \Psi = \delta \left( \chi, \frac{H}{2} - iu \right) = \iota(\delta \chi) \left( \frac{H}{2} - iu \right) - \iota \chi \left( \frac{\delta H}{2} - i(\delta u) \right)
\]
\[
= \frac{1}{2} \sum_{i=1}^{2m} (H^i - iu^i)^2 + \frac{|u|^2}{2} - \frac{1}{2} \chi R \chi - i \iota(\nabla u) \chi = \frac{1}{2} |H - iu|^2 + \mathcal{L}_0. \tag{2.45}
\]
\(\delta \Psi\) can be identified with \(L_0\) modulo the relation \(H^a = iu^a\) (equation of motion of \(H\)). We can easily see that \(\Phi_{\nabla}(E)\) is obtained by integrating \(\exp(-\delta \Psi)\) by \(H\) and \(\chi\).

\[
\Phi_{\nabla}(E) := \frac{1}{(2\pi)^{2m}} \int D\chi \int D\mathcal{H} \exp \left( -\frac{1}{2} |H - iu|^2 - L_0 \right)
\]

\[
= \frac{1}{(2\pi)^{2m}} \int D\chi \int D\mathcal{H} \exp \left( -\delta \left( \mathcal{H}, \frac{H}{2} - iu \right) \right). \tag{2.46}
\]

Let \(s : M \to E\) is any smooth section of \(E\), then we can easily derive,

\[
e_{s,\nabla}(E) := s^* \Phi_{\nabla}(E) = \frac{1}{(2\pi)^{2m}} \int D\chi D\mathcal{H} \exp \left( -\delta \left( \mathcal{H}, \frac{H}{2} - is \right) \right). \tag{2.47}
\]

Since \(s\) is homotopic to the zero section \(s_0\), \(e_{s,\nabla}\) belongs to the same cohomology class as \(e_{0,\nabla}(E)\). Hence we obtain the following equality.

\[
\chi(E) = \int_M e_{0,\nabla}(E) = \int_M e_{s,\nabla}(E) = \frac{1}{(2\pi)^{2m}} \int_M D\mathcal{X} D\psi D\mathcal{H} \exp \left( -\delta \left( \mathcal{X}, \frac{H}{2} - is \right) \right). \tag{2.48}
\]

where \(x\) is local coordinate of \(M\) and \(\psi\) is fermion variable that plays the role of differential form \(dx\) on \(M\).

### 2.2 The Case when \(M\) is an Orbit Space

In this subsection, we search for Lagrangian that produces Euler class of \(M\) when \(M\) is given as an orbit space, by using Atiyah-Jeffrey construction. Atiyah-Jeffrey construction is an extension of MQ formalism for vector bundle whose base space \(M\) is given as an orbit space \(X/G\) (\(G\): Lie group). Originally, Atiyah and Jeffrey constructed their formalism to study mathematical background to Witten’s construction of Lagrangian of Topological Yang-Mills theory \([1,10]\). In \([1]\), the orbit space is given by \(A/G\) where \(A\) is an infinite dimensional space of \(SU(2)\) connections \((A_\alpha)\) on a compact 4-dimensional manifold \(M_4\) and \(G\) is gauge transformation group that acts on \(A\). The vector bundle \(\pi : E \to A/G\) is not clearly stated in \([1]\), but the section \(s : A/G \to E\) is given by,

\[
s(A_\alpha) = F_{\alpha\beta} + \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta},
\]

\[
(F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]). \tag{2.49}
\]

Zero locus of the above section is nothing but the moduli space of anti self-dual instantons on \(M_4\), and it connects topological Yang-Mills theory with Donaldson invariants. Then how was the supersymmetric transformation \([2,44]\) modified to fit into the situation of an orbit space. Let us consider first behavior of \(\delta^2\) of the transformation \([2,44]\).

\[
\delta^2 \chi^i := R_{ij} \chi^j, \quad \delta^2 H^i := R_{ij} H^j. \tag{2.50}
\]

Therefore, \(\delta^2\) corresponds to “infinitessimal rotation of fiber coordinates generated by \(R_{ij}\)” . Then Atiyah and Jeffrey modified the above relation into the following:

\[
\delta^2 = \delta_\phi, \tag{2.51}
\]

where \(\delta_\phi\) is the infinitesimal gauge transformation generated by \(\phi\). It corresponds to infinitesimal rotation of infinite dimensional Lie group \(G\). Note that \(\delta\) is nilpotent when we consider orbit space.
$\mathcal{A}/G$. Then $\delta$ can be regarded as infinite dimensional version of equivariant derivative $d - \iota_\omega$, $\omega \in \text{Lie}(G)$ on $X/G$. 

$$\delta \leftrightarrow d - \iota_\omega, \quad (2.52)$$

$$\delta^2 = \delta_\phi \leftrightarrow (d - \iota_\omega)^2 = -d\iota_\omega - \iota_\omega d = -\mathcal{L}_\omega , \quad (2.53)$$

where $\mathcal{L}_\omega$ is the Lie derivative. With these considerations, $\delta$ transformation is modified as follows.

$$\delta x = \psi , \quad \delta \psi = \delta_\phi x \quad \delta \chi = H \quad \delta H = \delta_\phi \chi \quad \delta \phi = 0 , \quad (2.54)$$

where $x$ is coordinate of $\mathcal{A}$, $\psi$ is the fermion coordinate that plays the role of $dx$, $\chi$ is fermion coordinate of $\mathcal{E}$ and $H$ is auxiliary field and super-partner of $\chi$.

In the previous subsection, $\mathcal{L}_{MQ}$, the Lagrangian obtained from the MQ-formalism was represented as $\delta \Psi := \delta < \chi, \frac{H}{2} - is >$. In Atiyah-Jeffrey construction, Lagrangian $\mathcal{L}_{MQ}$, which is expected to produce Euler number $\chi(\mathcal{E})$ of the vector bundle $\mathcal{E}$ on $\mathcal{A}/G$,

$$\chi(\mathcal{E}) = \int \mathcal{D}x \mathcal{D}\psi \mathcal{D}H \mathcal{D}\chi \mathcal{D}\phi \exp(-\mathcal{L}_{MQ}) , \quad (2.55)$$

is given by,

$$\mathcal{L}_{MQ} = \delta(\Psi + \Psi_{\text{proj}}). \quad (2.56)$$

The term $\delta \Psi_{\text{proj}}$ plays the role of projecting out gauge horizontal direction (direction parallel to orbit of $G$) in integrating $\psi$ over $T^*\mathcal{A}$. In other words, $\exp(-\delta \Psi_{\text{proj}})$ can be regarded as projection operator from $T^*\mathcal{A}$ to $T^*(\mathcal{A}/G)$. In (2.55), we add $? \text{ above} = \text{because} \mathcal{E}$, the vector bundle of infinite rank, is not clearly stated in [1] and $\chi(\mathcal{E})$ is not well-defined. Indeed, the Lagrangian $\mathcal{L}_{MQ}$ is used to produce Donaldson invariants of $MQ$ in context of topological Yang-Mills theory [10].

Let us explain outline of construction of $\Psi_{\text{proj}}$. On $G$-principle bundle $\mathcal{A} \rightarrow \mathcal{A}/G$, group action for $x \in \mathcal{A}$ is given by $G$. Let $C : g$ (Lie algebra of $G$) $\rightarrow T_x \mathcal{A}$ be differentiation of group action on $x \in \mathcal{A}$. Let $\theta$ be an element of $g$. Then, $C\theta$ is given by,

$$C\theta = \delta_\theta x. \quad (2.57)$$

$C^\dagger$ is defined as adjoint operator of $C$,

$$\langle C^\dagger \psi, \theta \rangle = \langle \psi, C\theta \rangle , \quad (2.58)$$

where $\langle *, * \rangle$ in the l.h.s. is inner product of $g$ and $\langle *, * \rangle$ in the r.h.s is inner product of $T^+\mathcal{A}$. If $\psi \in \text{Ker}C^\dagger$, we obtain

$$C^\dagger \psi = 0 \leftrightarrow 0 = \langle C^\dagger \psi, \theta \rangle = \langle \psi, C\theta \rangle = \langle \psi, \delta_\theta x \rangle . \quad (2.59)$$

So, $\text{Ker}C^\dagger \subset T^*\mathcal{A}$ corresponds to vertical direction of gauge transformation. Then we have to restrict integration of $\psi$ into $\text{Ker}C^\dagger$. At this stage, we introduce additional boson field $\bar{\phi}$ and fermion field $\eta$. Supersymmetric transformation for these fields are defined by,

$$\delta \bar{\phi} = \eta \quad \delta \eta = \delta_\phi \bar{\phi} , \quad (2.60)$$

where $\phi$ is gauge transformation parameter (element of $g$). Then $\Psi_{\text{proj}}$ is defined in the following form.

$$\Psi_{\text{proj}} := \langle \psi, C\bar{\phi} \rangle , \quad (2.61)$$

We obtain

$$\delta \Psi_{\text{proj}} = \delta \langle C^\dagger \psi, \bar{\phi} \rangle = \langle \delta(C^\dagger \psi), \bar{\phi} \rangle - \langle C^\dagger \psi, \delta(\bar{\phi}) \rangle = \langle \delta(C^\dagger \psi), \bar{\phi} \rangle - \langle C^\dagger \psi, \eta \rangle . \quad (2.62)$$
From equation of motion of $\eta$, we obtain
\[
\delta \frac{\delta \Psi_{\text{proj}}}{\delta \eta} = 0 \leftrightarrow C^\dagger \psi = 0.
\] (2.63)
Hence multiplying $\exp(-\delta \Psi_{\text{proj}})$ can restrict $\psi$ integration to $\text{Ker}C^\dagger$. When we consider the case of zero section ($s \equiv 0$), the Lagrangian becomes,
\[
L_{MQ} = \delta \left< \chi, \frac{H}{2} \right> + i \delta \left< \psi, C \bar{\phi} \right>.
\] (2.64)
where supersymmetric transformation is given by,
\[
\delta x = \psi, \quad \delta \psi = \delta \phi x, \quad \delta \chi = H, \quad \delta H = \delta \phi \chi,
\] (2.65)
This is our starting point of construction of the Lagrangian (1.2) and of the supersymmetric transformation (1.14). But what we aim to compute is $\chi(G(k, N)) = \chi(T'G(k, N))$, we have to modify the above settings further. Some points of modification are already mentioned in Subsection 1.1 and Subsection 1.2. We will discuss again points of modification in Subsection 3.2.

3 Construction of Lagrangian and First Half of Evaluation of Path-Integral

3.1 The Grassmann Manifold

In order to apply MQ formalism on the Grassmann manifold, it is necessary to consider the Grassmann manifold $G(k, N)$ as an orbit space $X/G$ [11]. $G(k, N)$ is a space which parametrizes all $k$-dimensional linear subspaces of the $N$-dimensional complex vector space $\mathbb{C}^N$.
\[
G(k, N) := \{W \subset \mathbb{C}^N | \dim_{\mathbb{C}} W = k\}. \tag{3.66}
\]
Then we introduce the Stiefel manifold $V_k(\mathbb{C}^N)$. A point of $V_k(\mathbb{C}^N)$ is given by a set of $k$ unit vectors in $\mathbb{C}^N$ which are orthogonal to each other.
\[
V_k(\mathbb{C}^N) := \{(z^1, \cdots, z^k) \in \mathbb{C}^{kN} | z^i \in \mathbb{C}^N, \left< z^i, z^j \right> = \delta_{ij} (i, j = 1, \cdots, k)\} \tag{3.67}
\]
Let $< z^1, \cdots, z^k >_{\mathbb{C}}$ be a vector space spanned by $z^1, \cdots, z^k$. Two points $(z^1, \cdots, z^k)$ and $(z'^1, \cdots, z'^k)$ in $V_k(\mathbb{C}^N)$ satisfy the relation $< z^1, \cdots, z^k >_{\mathbb{C}} = < z'^1, \cdots, z'^k >_{\mathbb{C}}$ if and only if there exists $(U_{ij})_{1 \leq i, j \leq k} \in U(k)$ that satisfy $z^i = \sum_{j=1}^{k} U_{ij} z'^j$ $(i = 1, \cdots, k)$. Hence we can identify $G(k, N)$ with the orbit space $V_k(\mathbb{C}^N)/U(k)$.

Since $V_k(\mathbb{C}^N)$ is regraded as quotient space $U(N)/U(N - k)$ and volume of $U(N)$ is given by
\[
\prod_{j=1}^{N} \text{vol}(S^{2j-1}) \quad (S^{2j-1} \text{ is the (2j - 1)-dimensional unit sphere}) \tag{3.68}
\]
we obtain,
\[
\text{vol}(V_k(\mathbb{C}^N)) = \prod_{j=N-k+1}^{N} \text{vol}(S^{2j-1}) = \frac{2^k(\pi)^{kN - \frac{(k-1)}{2}}}{\prod_{j=N-k+1}^{N} (j-1)!}.
\]
3.2 Lagrange count Euler Number $\chi(G(k, N))$

In this section, we construct the Lagrangian for $\mathcal{L}_{MQ}$ by applying MQ-formalism and Atiyah-Jeffrey construction outlined in the previous section to the orbit space $G(k, N) = V_k(\mathbb{C}^N)/U(k)$. From now on, we set section $s$ used in Subsection 2.2 to zero section.

Let us mention again the fields used in our Lagrangian. The variable $z$ that describes a point of $V_k(\mathbb{C}^N)$ is given as follows.

$$z := (z^1 \cdots z^k) \in \mathbb{C}^N,$$

$$z^j := t(z^j_1, z^j_2, \cdots, z^j_k), \quad (j = 1, 2, \cdots, k), \quad (z^i)^t \cdot z^j - \delta^i_j = 0 \quad (i, j = 1, 2, \cdots, k).$$

This corresponds to the variable $x$ in Subsection 2.2. Other fields are represented in the following form.

Let us mention again the fields used in our Lagrangian. The variable $z$ that describes a point of $V_k(\mathbb{C}^N)$ is given as follows.

$$z := (z^1 \cdots z^k), \quad z^j := t(z^j_1, z^j_2, \cdots, z^j_k), \quad (j = 1, 2, \cdots, k), \quad (z^i)^t \cdot z^j - \delta^i_j = 0 \quad (i, j = 1, 2, \cdots, k).$$

This corresponds to the variable $x$ in Subsection 2.2. Other fields are represented in the following form.

$$\psi := (\psi^1, \psi^2, \cdots, \psi^k), \quad \psi^j := t(\psi^j_1, \psi^j_2, \cdots, \psi^j_N).$$

$$\phi := \begin{pmatrix} \phi_{11} & \cdots & \phi_{1k} \\ \vdots & \ddots & \vdots \\ \phi_{kk} & \cdots & \phi_{kk} \end{pmatrix}, \quad \bar{\phi} := \begin{pmatrix} \bar{\phi}_{11} & \cdots & \bar{\phi}_{1k} \\ \vdots & \ddots & \vdots \\ \bar{\phi}_{kk} & \cdots & \bar{\phi}_{kk} \end{pmatrix},$$

$$\eta := \begin{pmatrix} \eta_{11} & \cdots & \eta_{1k} \\ \vdots & \ddots & \vdots \\ \eta_{kk} & \cdots & \eta_{kk} \end{pmatrix},$$

$$\chi := (\chi^1, \cdots, \chi^k), \quad \chi^j := t(\chi^j_1, \cdots, \chi^j_N).$$

$$H := (H^1, \cdots, H^k).$$

$\phi$ and $\bar{\phi}$ are Hermite matrix ($\phi_{ij} = \phi_{ji}, \bar{\phi}_{ij} = \bar{\phi}_{ji}$). They are the generators of the elements of $U(k)$. $\delta \bar{\phi}$ is not the complex conjugate of $\phi$. These fields play the same roles as the corresponding fields in Subsection 2.2. Next, we introduce the field $A$ and its superpartner $\psi_A$ in order to produce the part $\frac{1}{c(S \otimes S^*)}$ in the total Chern class $\frac{c(S^*)^N}{c(S \otimes S^*)}$.

$$A := \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{kk} & \cdots & A_{kk} \end{pmatrix}, \quad \psi_A := \begin{pmatrix} \psi^1_{A1} & \cdots & \psi^k_{A1} \\ \vdots & \ddots & \vdots \\ \psi^1_{Ak} & \cdots & \psi^k_{Ak} \end{pmatrix}.$$  

At this stage, we define supersymmetry transformation for each variables. Supersymmetry are represent in following forms.

$$\delta z^i_s = \psi^i_s, \quad \delta \psi^i_s = i \phi_{im} z^m_s, \quad \delta \chi^i_s = H^i_s, \quad \delta H^i_s = (\gamma \delta_{ij} + i \bar{\psi}_{ij}) \chi^j_s,$$

$$\delta A^i_j = \psi^i_{A_j}, \quad \delta A^i_j = 0, \quad \delta \bar{\chi} = \eta_{ij}, \quad \delta \psi_A = \gamma A + [A, i \phi].$$

These are fundamentally obtained from applying the construction in Subsection 2.2 to the orbit space $G(k, N) = V_k(\mathbb{C}^N)/U(k)$ but we introduce central charge $\gamma(\neq 0)$ and modify the transformation of $H$ and $\psi_A$ from the standard version $\delta H^i_s = i \bar{\psi}_{ij} \chi^j_s$ and $\delta \psi_A = [A, i \phi]$. This modification corresponds to central extension of supersymmetry algebra.

With these set-up’s Lagrangian is defined as follows.

$$\mathcal{L}_{MQ} = \delta < \chi, H > + \delta < \psi_A, A > + \frac{i}{2} \delta \{ < \psi, C \bar{\phi} > + * < \bar{\psi}, C \phi > \}.$$  

where $C \bar{\phi} = \delta_{\phi} z = iz^t \bar{\phi}$. Except for the term $\delta < \psi_A, A >$, this Lagrangian is obtained from applying discussion of Subsection 2.2 with $s = 0$ to the orbit space $G(k, N) = V_k(\mathbb{C}^N)/U(k)$. Then we can
Lemma 1. Proof. First, we note that projection operator \( V \) along tangent space of \( \theta \).

\[ Z_{MQ} := \beta \int_{V_k(\mathbb{C}^N)} Dz \int D\psi D\phi D\bar{\phi} DAD\eta D\chi DH \omega \exp(-\mathcal{L}_{MQ}) \]

\[ \mathcal{L}_{MQ} = \delta <\chi, H> + \delta <\psi, A> + \frac{i}{2} \delta \left[ <\psi, iz\bar{\phi}> + <\bar{\psi}, iz\phi> \right] \]

\[ = \sum_{s=1}^{N} \sum_{i=1}^{k} H^i_s H^s_i - \left( \chi^i_s (\gamma \delta_{ij} + i\phi_{ij}) \chi^j_s \right) + \psi^i_s \bar{\phi}_i \psi^j_s + \text{itr}(\phi) \]

\[ + \frac{1}{2} \sum_{s=1}^{N} \left( \psi^i_s z^j_s - \psi^j_s z^i_s \right) \eta_{ij} + \text{tr}(A^4 (\gamma A + [i\phi, A]) - \psi^i_A \psi^j_A). \]

As we have already mentioned, \( \beta \) is normalization factor for Chern class. and \( \omega \) is projective operator along tangent space of \( V_k(\mathbb{C}^N) \). They are given as follows.

\[ \omega = \prod_{i,j=1}^{k} \left( \sum_{s=1}^{N} \left( \psi^i_s z^j_s + z^i_s \psi^j_s \right) \right), \quad \beta = \frac{\prod_{j=0}^{k-1} j!}{2^{2k}(\pi)^{k^2 + kN} \pi^{kN} + \frac{2(k+1)}{2} (-1)^{\frac{k}{2}(k-1)}}. \] (3.79)

3.3 First Half of Evaluation of the Path-Integral

3.3.1 \( U(N) \times U(k) \) Symmetry of the Lagrangian

Lemma 1.

\[ Z_{MQ} = \beta \int_{V_k(\mathbb{C}^N)} Dz \int D\psi D\phi D\bar{\phi} DAD\eta D\chi DH \omega \exp(-\mathcal{L}_{MQ}) \]

\[ = \beta \text{vol}(V_k(\mathbb{C}^N)) \int D\psi D\phi D\bar{\phi} DAD\eta D\chi DH \omega' \exp(-\mathcal{L}'_{MQ}). \]

\[ \mathcal{L}'_{MQ} = \sum_{s=1}^{N} \sum_{i=1}^{k} H^i_s H^s_i - \left( \chi^i_s (\gamma \delta_{ij} + i\phi_{ij}) \chi^j_s \right) + \psi^i_s \bar{\phi}_i \psi^j_s + \text{itr}(\phi) \]

\[ + \frac{1}{2} \sum_{i,j=1}^{k} \left( \psi^i_j - \psi^j_i \right) \eta_{ij} + \text{tr}(A^4 (\gamma A + [i\phi, A]) - \psi^i_A \psi^j_A). \]

\[ \omega' = \prod_{i=1}^{k} \prod_{j=1}^{k} \left( \psi^i_j + \psi^j_i \right). \] (3.80)

Proof. First, we note that projection operator \( \omega \) is rewritten as follows:

\[ \omega = \prod_{i=1}^{k} \prod_{j=1}^{k} \left( \sum_{s=1}^{N} \left( \psi^i_s z^j_s + \psi^j_s z^i_s \right) \right) \]

\[ = (-1)^{\frac{k^2}{2}} \int D\theta \exp \left( <\psi, z^\dagger \theta> + <\bar{\psi}, z \theta> \right), \] (3.81)

where \( \theta \) is fermionic Hermite matrix.

Hence we can rewrite \( Z_{MQ} \) by,

\[ Z_{MQ} = (-1)^{\frac{k^2}{2}} (k^2 + 1) \beta \int_{V_k(\mathbb{C}^N)} Dz \int D\psi D\phi D\bar{\phi} DAD\eta D\chi DH \omega' \exp(-\mathcal{L}'_{MQ}(z, \psi, \phi, \bar{\psi}, A, \eta, \chi, H, \theta)) \]

\[ \mathcal{L}'_{MQ}(z, \psi, \phi, \bar{\psi}, A, \eta, \chi, H, \theta) = \mathcal{L}_{MQ}(z, \psi, \phi, \bar{\psi}, A, \eta, \chi, H) + <\psi, z^\dagger \theta> + <\bar{\psi}, z \theta>. \] (3.82)
If we transform $\chi, \psi, \phi, \bar{\phi}, \eta, A, z, H, \theta$ in the following way,

\begin{align*}
\chi &= U^N \chi' U^k, & \psi &= U^N \psi' U^k, & \phi &= \imath U^k \phi' \ast U^k, & \bar{\phi} &= \imath U^k \bar{\phi}' \ast U^k, \\
\eta &= \imath U^k \eta' \ast U^k, & A &= \imath U^k A' \ast U^k, & z &= U^N z' U^k, & H &= U^N H' U^k, \\
\theta &= \imath U^k \theta' \ast U^k,
\end{align*}

we can easily see that $\widetilde{\mathcal{L}}_{MQ}$ has $U(N) \times U(k)$ symmetry, i.e.,

\begin{align*}
\widetilde{\mathcal{L}}_{MQ}(z', \psi', \phi', \bar{\phi}', A', \eta', \chi', H', \theta') \\
= \widetilde{\mathcal{L}}_{MQ}(z, \psi, \phi, \bar{\phi}, A, \eta, \chi, H, \theta).
\end{align*}

(3.83)

On the other hand, we can confirm that the integral measures of each variable is also $U(N) \times U(k)$ invariant: $DX = DX'$ ($X$ is each variable). Let us define $F_{MQ}(z)$ by,

\begin{align*}
F_{MQ}(z) := (-1)^{\frac{k^2}{2}} (k^2 + 1) \beta \int D\psi D\phi D\bar{\phi} D\eta D\chi DHD \theta \exp\left( -\widetilde{\mathcal{L}}_{MQ}(z, \psi, \phi, \bar{\phi}, A, \eta, \chi, H, \theta) \right)
\end{align*}

(3.84)

Then we obviously have,

\begin{align*}
Z_{MQ} &= \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z).
\end{align*}

(3.85)

Then by using (3.84) and invariance of integral measure, we obtain,

\begin{align*}
F_{MQ}(z) &= (-1)^{\frac{k^2}{2}} (k^2 + 1) \beta \int D\psi D\phi D\bar{\phi} D\eta D\chi DHD \theta \exp\left( -\widetilde{\mathcal{L}}_{MQ}(z, \psi, \phi, \bar{\phi}, A, \eta, \chi, H, \theta) \right) \\
= (-1)^{\frac{k^2}{2}} (k^2 + 1) \beta \int D\psi D\phi D\bar{\phi} D\eta D\chi DHD \theta \exp\left( -\widetilde{\mathcal{L}}_{MQ}(z', \psi', \phi', \bar{\phi}', A', \eta', \chi', H', \theta') \right) \\
= (-1)^{\frac{k^2}{2}} (k^2 + 1) \beta \int D\psi' D\phi' D\bar{\phi}' D\eta' D\chi' DHD \theta' \exp\left( -\widetilde{\mathcal{L}}_{MQ}(z', \psi', \phi', \bar{\phi}', A', \eta', \chi', H', \theta') \right) \\
= F_{MQ}(z')
\end{align*}

(3.86)

For each $z \in V_k(\mathbb{C}^N)$, we can choose $U^N \in U(N)$ and $U^k \in U(k)$ that satisfy,

\begin{align*}
U^N z U^k &= \begin{pmatrix} I_k \\ 0_{N-k,k} \end{pmatrix} =: z_0,
\end{align*}

(3.87)

where $I_k$ is $k \times k$-type unit matrix and $0_{N-k,k}$ is $(N-k) \times k$-type zero matrix. Hence we obtain,

\begin{align*}
Z_{MQ} = \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z) = \int_{V_k(\mathbb{C}^N)} Dz F_{MQ}(z_0) = \text{vol}(V_k(\mathbb{C}^N)) F_{MQ}(z_0).
\end{align*}

(3.88)

This is nothing but the assertion of the Lemma. \qed
3.3.2 Integration of Fields except for $\psi$

Note that we have integrated out $z$ in the previous subsubsection. Next, we integrate out $\psi_A$, $H$ and $\chi$.

\[
\int D\psi_A \exp\left(\text{tr}(\psi_A^\dagger \psi_A)\right)
\]
\[
= \int d\psi_A^1 d\psi_A^1 \cdots d\psi_A^k d\psi_A^k \cdots d\psi_A^1 d\psi_A^1 \cdots d\psi_A^k d\psi_A^k \prod_{i=1}^k \left(1 + \psi_A^i \psi_A^i_1 \cdots (1 + \psi_A^i \psi_A^i_1)\right)
\]
\[
= (-1)^{k^2}.
\]

\[
\int D\psi \exp\left(-\sum_{s=1}^N \sum_{i=1}^k H_s^i H_s^i\right)
\]
\[
:= \int \prod_{s=1}^N \left(\frac{i}{2}\right)^k dH_s^1 dH_s^1 \cdots dH_s^k dH_s^k \exp\left(-\sum_{s=1}^N \sum_{i=1}^k H_s^i H_s^i\right) = \pi^{kN}.
\]

\[
\int D\chi \exp\left(\sum_{s=1}^N \chi_s^i \left(\gamma \delta_{ij} + i\phi_{ij}\right) \chi_s^j\right)
\]
\[
:= \int \prod_{s=1}^N d\chi_s^1 d\chi_s^1 \cdots d\chi_s^k d\chi_s^k \exp\left(\sum_{s=1}^N \chi_s^i \left(\gamma \delta_{ij} + i\phi_{ij}\right) \chi_s^j\right)
\]
\[
= (-1)^{kN} \int \prod_{s=1}^N d\chi_s^1 d\chi_s^1 \cdots d\chi_s^k d\chi_s^k \exp\left(\sum_{s=1}^N \chi_s^i \left(\gamma \delta_{ij} + i\phi_{ij}\right) \chi_s^j\right)
\]
\[
= (-1)^{kN} \left(\text{det}(\gamma I_k + i\phi)\right)^N.
\]

Then let us integrate out $\eta$. We set,

\[
I_\eta := \int D\eta \exp\left(\frac{-1}{2} \sum_{i,j=1}^k \left\{ \psi_i^\dagger - \psi_j^\dagger \right\} \eta_{ij}\right)
\]

We abbreviate $\left\{ \psi_i^\dagger - \psi_j^\dagger \right\}$ as $\alpha^{ij}$ and obtain,

\[
I_\eta = \int D\eta \left(\frac{-1}{2}\right)^{k^2} \prod_{i=1}^k \prod_{j=1}^k \alpha^{ij} \eta_{ij} = \left(\frac{-1}{2}\right)^{k^2} \left(1 + \frac{1}{2}\right)^{k^2} \int D\eta \left[ \prod_{i=1}^k \prod_{j=1}^k \alpha^{ij}\right]
\]
\[
= \left(\frac{-1}{2}\right)^{k^2} \left(1 + \frac{1}{2}\right)^{k^2} \alpha^{ij} \left[ \prod_{i=1}^k \prod_{j=1}^k \alpha^{ij}\right].
\]

We also abbreviate $\left\{ \psi_i^\dagger + \psi_j^\dagger \right\}$ as $\beta^{ij}$ and evaluate $I_\eta \omega$. Since

\[
\left[ \prod_{i=1}^k \prod_{j=1}^k \alpha^{ij}\right] \left[ \prod_{i=1}^k \prod_{j=1}^k \beta^{ij}\right] = (-1)^{k^2} \left(1 + \frac{1}{2}\right)^{k^2} \alpha^{ij} \beta^{ij} \left[ \prod_{i=1}^k \prod_{j=1}^k \alpha^{ij}\right].
\]
and \( \alpha^j \beta^j = (\psi_j^i - \psi_j^i)(\psi_j^i + \psi_j^i) = -2\psi_j^i \psi_j^i \), we obtain,

\[
I_{\eta}\omega = \left( -\frac{1}{2} \right)^k \prod_{i=1}^k \prod_{j=1}^k \alpha^i \beta^j = \prod_{i=1}^k \prod_{j=1}^k \psi_i^j \psi_j^i = \left( \prod_{i=1}^k \psi_i^j \psi_j^i \right) \left( \prod_{i<j}^k \psi_i^j \psi_j^i \right)
\]

\[
= \left( \prod_{i=1}^k \psi_i^j \psi_j^i \right) \left( \prod_{i<j}^k \psi_i^j \psi_j^i \right) = \left( \prod_{i=1}^k \psi_i^j \psi_j^i \right) \left( \prod_{i<j}^k (-\psi_i^j \psi_j^i) \right)
\]

\[
= (-1)^{\frac{k(k-1)}{2}} \prod_{i=1}^k \prod_{j=1}^k \psi_i^j \psi_j^i.
\]

(3.97)

Then the result of these integrations is given as follows.

\[
Z_{MQ} = \beta \int_{V_\delta(CN)} Dz D\phi D\bar{\phi} DA \left( -1 \right)^{k+\frac{k(k-1)}{2}} (-\pi)^k \left( \det(\gamma I_k + i\phi) \right) N \exp \left( -\sum_{s=1}^N \psi_s^i \bar{\phi}_s + i\text{tr}(\phi\bar{\phi}) + \text{tr}(A^l(\gamma A + [i\phi, A])) \right) \prod_{i=1}^k \prod_{j=1}^k \psi_i^j \psi_j^i.
\]

(3.98)

In order to integrate out \( \phi \) and \( \bar{\phi} \), we we decompose complex fields \( \phi, \bar{\phi} \) and \( \psi \) into real parts and imaginary parts:

\[
\phi_{ij} = \phi_{ij}^R + i\phi_{ij}^I,
\]

\[
\bar{\phi}_{ij} = \bar{\phi}_{ij}^R + i\bar{\phi}_{ij}^I,
\]

\[
\psi^i_s = \psi^i_{Rs} + i\psi^i_{Is},
\]

(3.99)

(3.100)

(3.101)

and define integration measures for \( \bar{\phi} \) and \( \phi \) as,

\[
D\bar{\phi} := \prod_{i=1}^k d\bar{\phi}_i^R \prod_{j=i+1}^k d\bar{\phi}_{ij}^I,
\]

(3.102)

\[
D\phi := \prod_{i=1}^k d\phi_i^R \prod_{j=i+1}^k d\phi_{ij}^I.
\]

(3.103)

Note that \( \phi_{ii}^I = 0 \) and \( \phi_{ij}^R = \phi_{ij}^R, \bar{\phi}_{ij}^R = \bar{\phi}_{ij}^R, \phi_{ij}^I = -\phi_{ij}^I, \phi_{ij}^I = -\phi_{ij}^I(i \neq j) \) since both \( \bar{\phi} \) and \( \phi \) are both Hermitian matrices. Then integration of \( \phi \) and \( \bar{\phi} \) results in the following lemma:

**Lemma 2.** Let \( \lambda_i (i = 1, 2, \cdots, k) \) be eigenvalues of the Hermite matrix \( \phi \). Then we have,

\[
\int D\bar{\phi} \exp \left( -i\text{tr}(\phi\bar{\phi}) - \sum_{s=1}^N \psi_s^i \bar{\phi}_s \psi_s^i \right)
\]

\[
= \left( \frac{2\pi}{2} \right)^{\frac{k^2}{2}} \left\{ \prod_{i=1}^k \delta \left( \phi_{ii} + 2 \sum_{s=1}^N \psi_{Rs}^i \psi_{Is}^i \right) \right\}
\]

\[
\times \left\{ \prod_{i=1}^{k-1} \prod_{j=i+1}^k \delta \left( \phi_{ij}^R + \sum_{s=1}^N (\psi_{Rs}^i \psi_{Is}^j - \psi_{Is}^j \psi_{Rs}^i) \right) \right\}
\]

\[
\times \delta \left( \phi_{ji}^I - \sum_{s=1}^N (\psi_{Rs}^j \psi_{Is}^j - \psi_{Is}^j \psi_{Rs}^j) \right). \]

(3.104)

\[
\int DA \exp \left( -\text{tr}(\gamma A^l A + A^l[i\phi, A]) \right) = \frac{\pi^{k^2}}{\gamma^k \prod_{l<j} \left( \gamma^2 - (i\lambda_l - i\lambda_j)^2 \right)}
\]

(3.105)
This follows from straightforward computation, and we leave the proof to readers as exercises. By using the above lemma, we obtain,

\[ Z_{MQ} = \beta \int D\psi^* Dz D\phi (-1)^{\frac{1}{2}(k-1)} (2\pi)^{k^2} (-\pi)^{k^2+kN} \left( \text{det}(\gamma I_k + \Phi) \right)^N \frac{1}{2^{k(k-1)} \gamma^k \prod_{l>j} (\gamma^2 - (\lambda_l - \lambda_j)^2)} \]

\[
\left\{ \prod_{s=1}^k \delta \left( \phi_{is} + 2 \sum_{i=1}^N \psi_{Rs}^i \psi_{Is} \right) \right\} \left\{ \prod_{i=1}^{k-1} \prod_{j=1}^k \delta \left( \phi_{ji}^* + \sum_{s=1}^N (\psi_{Rs}^i \psi_{Is} - \psi_{Is}^i \psi_{Rs}) \right) \right\} \times \delta \left( \phi_{ji}^* - \sum_{s=1}^N (\psi_{Rs}^i \psi_{Is} + \psi_{Is}^i \psi_{Rs}) \right) \left\{ \prod_{i=1}^k \prod_{j=1}^k \psi_{i}^j \psi_{i}^j \right\} (3.106)
\]

Then, integration of \( \phi \) results in replacement of \( \psi \) fields by the following composite of \( \psi \) fields.

\[
\phi_{ii} \rightarrow -2 \sum_{s=1}^N \psi_{Rs}^i \psi_{Is} = - \sum_{s=1}^N (\psi_{Rs}^i \psi_{Is} - \psi_{Is}^i \psi_{Rs})
\]

\[
= -i \sum_{s=1}^N (\psi_{Rs}^i + i \psi_{Is}^i) (\psi_{Rs}^i - i \psi_{Is}^i) = -i \sum_{s=1}^N \psi_{Rs}^i \psi_{Is}^i, \quad (3.107)
\]

\[
\phi_{ji} = \phi_{ji}^* + i \phi_{ji}^* \rightarrow \sum_{s=1}^N \left\{ - (\psi_{Rs}^i \psi_{Is} - \psi_{Is}^i \psi_{Rs}) + i (\psi_{Rs}^i \psi_{Rs}^j + \psi_{Rs}^j \psi_{Rs}^i) \right\}
\]

\[
= i \sum_{s=1}^N (\psi_{Rs}^i \psi_{Rs}^j + \psi_{Rs}^j \psi_{Rs}^i) + i (\psi_{Rs}^i \psi_{Rs}^j - \psi_{Rs}^j \psi_{Rs}^i)
\]

\[
= i \sum_{s=1}^N \psi_{Rs}^i \psi_{Rs}^j - i \sum_{s=1}^N \psi_{Rs}^j \psi_{Rs}^i, \quad (3.108)
\]

\[
\Phi_s := \left( \begin{array}{ccc}
\psi_{1s}^1 & \cdots & \psi_{1s}^k \\
\vdots & \ddots & \vdots \\
\psi_{ks}^1 & \cdots & \psi_{ks}^k
\end{array} \right), \quad \Phi := \sum_{s=1}^N \Phi_s, \quad (3.110)
\]

Then the result of integration of \( \phi \) is summarized by replacement of \( \phi \) by \( -i \Phi \). Let \( \lambda_s' \) (\( i = 1, \cdots, k \)) be eigenvalues of the \( \Phi \). Then integration of \( \phi \) and \( \Phi \) results in,

\[
Z_{MQ} = \beta \text{vol}(V_k(\mathbb{C}^N)) \int D\psi \left( -1 \right)^{\frac{1}{2}(k-1)} (2\pi)^{k^2} (-\pi)^{k^2+kN} \left( \text{det}(\gamma I_k + \Phi) \right)^N \prod_{i=1}^k \prod_{j=1}^k \psi_{i}^j \psi_{i}^j. \quad (3.111)
\]

On the other hand, we obtain from (3.68),

\[
\beta \text{vol}(V_k(\mathbb{C}^N)) \left( -1 \right)^{\frac{1}{2}(k-1)} (2\pi)^{k^2} (-\pi)^{k^2+kN} = \prod_{j=0}^{k-1} \frac{j!}{\prod_{j=N-k}^N j!},
\]

and reach the final expression of \( Z_{MQ} \) in this section:

\[
Z_{MQ} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^N j!} \int D\psi \left( -1 \right)^{\frac{1}{2}(k-1)} (2\pi)^{k^2} (-\pi)^{k^2+kN} \left( \text{det}(\gamma I_k + \Phi) \right)^N \prod_{i=1}^k \prod_{j=1}^k \psi_{i}^j \psi_{i}^j. \quad (3.113)
\]
4 Second Half: Proof of the Main Theorem

4.1 Free Fermion Realization of Cohomology Ring of $G(k, N)$

In the previous section, we reached the expression (3.113). Then what remains to prove is the following equality:

$$Z_{MQ} = \prod_{j=N-k}^{N-k-1} \frac{(\det(\gamma I_k + \Phi))^N}{\gamma_k \prod_{i>j} (\gamma^2 - (\lambda_i - \lambda_j)^2)} \prod_{i=1}^{k} \psi_i^{\bar{j}} \psi_i^{j}$$

(4.114)

$$= \int_{G(k,N)} c(T'G(k, N))$$

(4.115)

$$= \int_{G(k,N)} \prod_{i=1}^{k} (1 + x_i)$$

(4.116)

$$= \binom{N}{k},$$

(4.117)

where $c(S^*) = \prod_{i=1}^{k} (1 + x_i)$. First, we note that the factor $\prod_{i=1}^{k} \psi_i^{\bar{j}} \psi_i^{j}$ allows us to neglect $\psi_i^{\bar{j}} \psi_i^{j}$ $(i, j = 1, 2, \cdots, k)$ in the integral measure and the remaining part of the integrand. Hence we only have to consider the fields $\psi_i^{\bar{j}}, \psi_i^{j}$ $(i = k + 1, \cdots, N, j = 1, \cdots, k)$. At this stage, we redefine $\psi_{k+i}^{\bar{j}}, \psi_{k+i}^{j}$ by $\psi_i^{\bar{j}}, \psi_i^{j}$ $(i = 1, \cdots, N - k, j = 1, \cdots, k)$ and introduce

$$\Phi' := \sum_{s=1}^{N-k} \begin{pmatrix} \omega_s^{1\bar{1}} & \cdots & \omega_s^{1k} \\ \vdots & \ddots & \vdots \\ \omega_s^{k\bar{1}} & \cdots & \omega_s^{kk} \end{pmatrix} \quad (\omega_s^{ij} := \psi_s^{i} \psi_s^{j}),$$

(4.118)

$$D\psi' = \prod_{s=1}^{N-k} d\psi_s^{1} d\psi_s^{1} \cdots d\psi_s^{k} d\psi_s^{k}.$$  

(4.119)

Let $\lambda_i$ $(i = 1, \cdots, k)$ be eigenvalues of the Hermite matrix $\Phi'$. Then (3.113) is rewritten as follows.

$$Z_{MQ} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int_{G(k,N)} \frac{(\det(\gamma I_k + \Phi'))^N}{\gamma_k \prod_{i>j} (\gamma^2 - (\lambda_i - \lambda_j)^2)}.$$  

(4.120)

**Theorem 2.** Let us define $b_i$ $(i = 0, 1, 2, \cdots)$ by,

$$\frac{1}{\det(I_k + t\Phi')} = \sum_{m=0}^{\infty} b_m t^m.$$  

(4.121)

Then, $b_m = 0$ if $m > N - k$.

**Proof.** By using Gaussian integral of complex variables $X_1, \cdots, X_k$, we obtain the equality:

$$\frac{1}{\det(I_k + t\Phi')} = \int_{\mathbb{C}^k} \mathcal{D}X \exp\{(- t \bar{X}(I_k + t\Phi')X)\}.$$  

(4.122)
where \(X = \{X_1, \ldots, X_k\}\) and integral measure is given by \(DX := \frac{1}{(2\pi)^{\frac{N-k}{2}}} dX_1 dX_2 \cdots dX_k dX_k\).

\[
(R.H.S.) = \int_{\mathbb{C}^n} DX \exp\left\{ (-X_i(\delta_{ij} + t \sum_{\mu=1}^{N-k} \omega^{(j)}_{\mu} X_j) \right\}
\]

\[
= \int_{\mathbb{C}^n} DX \left\{ e^{-|X|^2} e^{-t \sum_{\mu=1}^{N-k} \omega^{(j)}_{\mu} X_j} \right\}
\]

\[
= \int_{\mathbb{C}^n} DX e^{-|X|^2} \sum_{m=0}^{\infty} \frac{1}{m!} (-t \sum_{\mu=1}^{N-k} \omega^{(j)}_{\mu} X_j)^m
\]

\[
= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{(\mu_1, \ldots, \mu_m)} \sum_{(i_1, \ldots, i_m)} Sym(i_1, \ldots, i_m) \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} \int_{\mathbb{C}^n} DX e^{-|X|^2} X_{i_1} X_{i_2} \cdots X_{i_m}.
\]

(4.123)

Here, \(|X|^2\) represents \(\sum_{j=1}^{k} X_j \cdot X_j\). We remark that \(1 \leq \mu_{l} \leq N - k\), \(1 \leq i_l \leq k\), \((l = 1, \ldots, m)\). By Wick’s theorem of Gaussian integral, we can easily see that the integral in the last line of (4.124) does not vanish if and only if \(\{i_1, \ldots, i_m\} = \{j_1, \ldots, j_m\}\). Hence we obtain,

\[
\int_{\mathbb{C}^n} DX \exp\left\{ (-X(I_{k} + t \Phi') \right\} X)
\]

\[
= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{(\mu_1, \ldots, \mu_m)} \sum_{(i_1, \ldots, i_m)} Sym(i_1, \ldots, i_m) \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} \int_{\mathbb{C}^n} DX e^{-|X|^2} \prod_{j=1}^{m} |X_{i_j}|^2.
\]

(4.124)

where \(Sym(i_1, \ldots, i_m)\) is symmetric factor of the \(m\)-tuple \((i_1, \ldots, i_m)\) given by,

\[
Sym(i_1, \ldots, i_m) = \frac{1}{\text{mul}((i_1, \ldots, i_m); j)!},
\]

\[
\text{mul}((i_1, \ldots, i_m); j) := \text{number of } l\text{'s that satisfy } i_l = j.
\]

(4.125)

Then let us fix \((\mu_1, \ldots, \mu_m)\) and \((i_1, \ldots, i_m)\), and assume that there exists a pair \((i,j)\) \((1 \leq i < j \leq m)\) that satisfy \(\mu_i = \mu_j\). Without loss of generality, we can further assume that \(\mu_1 = \mu_2 = \mu\). Obviously, for any \(\sigma \in S_m\) we can uniquely take \(\sigma' \in S_m\) that satisfy \(\sigma'(1) = \sigma(2), \sigma'(2) = \sigma(1)\), \(\sigma'(i) = \sigma(i)\)(i = 3, 4, \ldots m). Then, we can easily see,

\[
\omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} = \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} + \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m}
\]

\[
= \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} + \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m}
\]

\[
= 0
\]

(4.126)

because \(\omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m} = \psi^{(i_1)}_{\mu_1} \psi^{(i_2)}_{\mu_2} \cdots \psi^{(i_m)}_{\mu_m} = -\psi^{(i_1)}_{\mu_1} \psi^{(i_2)}_{\mu_2} \cdots \psi^{(i_m)}_{\mu_m} = -\omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m}\). Hence

\[
\sum_{\sigma \in S_m} Sym(i_1, \ldots, i_m) \omega^{(i_1)}_{\mu_1} \cdots \omega^{(i_m)}_{\mu_m}
\]

in the last line of (4.124) vanishes if some \(\mu_j\)'s in the \(m\)-tuple \((\mu_1, \ldots, \mu_m)\) coincide. Since \(1 \leq \mu_j \leq N - k\), it follows that the summand in the last line of (4.124) vanishes if \(m > N - k\).

Then we introduce another theorem which will be proved in the next subsection.

**Theorem 3.**

\[
\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi' \left( (\det(\Phi'))^{N-k} \right) = 1.
\]

(4.127)
Now we recall the well-known facts on the cohomology ring of $G(k, N)$ [3]. As we have mentioned in Subsection 1.2, we have the following exact sequence of vector bundles on $G(k, N)$.

$$0 \to S \to \mathbb{C}^N \to Q \to 0,$$

where $S$ is the tautological bundle of rank $k$, $\mathbb{C}^N$ is trivial bundle $G(k, N) \times \mathbb{C}^N$ and $Q$ is the quotient bundle of rank $N - k$. Then the cohomology ring of $G(k, N)$ is given by,

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S), \ldots, c_k(S), c_1(Q), \ldots, c_{N-k}(Q)]}{(c(S)c(Q) = 1)}.$$  

(4.129)

Since relation between $i$-th Chern class of a vector bundle $E$ and the one of its dual bundle $E^*$ is given by, $c_i(E^*) = (-1)^i c_i(E)$, we can take $c_i(S^*)'$s and $c_i(Q^*)'$s as generators of $H^*(G(k, N))$.

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \ldots, c_k(S^*), c_1(Q^*), \ldots, c_{N-k}(Q^*)]}{(c(S^*)c(Q^*) = 1)}.$$  

(4.130)

On the other hand, the relation $c_i(S^*)c_i(Q^*) = 1$ is rewritten by,

$$c_i(Q^*) = \frac{1}{c_i(S^*)}.$$  

(4.131)

If we expand $\frac{1}{c_i(S^*)} = 1/(1 + c_1(S^*)t + c_2(S^*)t^2 + \cdots + c_k(S^*)t^k)$ in powers of $t$,

$$\frac{1}{1 + c_1(S^*)t + c_2(S^*)t^2 + \cdots + c_k(S^*)t^k} = \sum_{i=0}^{\infty} a_i t^i,$$  

(4.132)

we can rewrite [4.131] as follows:

$$c_i(Q^*) = a_i \quad (i = 1, 2, \ldots, N - k), \quad a_i = 0 \quad (i > N - k).$$  

(4.133)

Note that $a_i$ is degree $i$ homogeneous polynomial of $c_j(S^*)$'s ($j = 1, 2, \ldots, k$). Hence we can eliminate generators $c_j(Q^*)$'s from [4.130] and obtain another representation of $H^*(G(k, N))$.

$$H^*(G(k, N)) = \frac{\mathbb{R}[c_1(S^*), \ldots, c_k(S^*)]}{(a_i = 0 \quad (i > N - k))}.$$  

(4.134)

At this stage, we look back at Theorem [2] Let us define $\sigma_j$ ($j = 1, 2 \cdots, k$) by,

$$1 + \sigma_1 t + \cdots + \sigma_k t^k := \text{det}(I_k + t\Phi') = \prod_{j=1}^{k} (1 + \lambda_j t).$$  

(4.135)

In other words, $\sigma_j$ is the $j$-th fundamental symmetric polynomial of $\lambda_1, \cdots, \lambda_k$. Note that $\sigma_k$ is identified with $\text{det}(\Phi')$. Then let us consider the ring $\mathbb{R}[\sigma_1, \cdots, \sigma_k]$. Since we have set,

$$\sum_{i=0}^{N-k} b_i t^i = \frac{1}{\text{det}(I_k + t\Phi')} = \frac{1}{1 + \sigma_1 t + \cdots + \sigma_k t^k},$$  

(4.136)

assertion of Theorem [2] $b_i = 0 \quad (i > N - k)$ tells us that we have a ring homomorphism $f : H^*(Gr(k, N)) \to \mathbb{R}[\sigma_1, \cdots, \sigma_k]$ defined by,

$$f(c_j(S^*)) = \sigma_j \quad (j = 1, 2, \cdots, k).$$  

(4.137)

**Theorem 4.** The ring homomorphism $f : H^*(Gr(k, N)) \to \mathbb{R}[\sigma_1, \cdots, \sigma_k]$ is an isomorphism.
Proof. Since \cite{[7]} and Theorem \cite{[7]} holds true, \( f \) is surjective. Then it is enough for us to show that \( f \) is injective. Let us assume that \( \ker(f) \neq \{0\} \). Then we have \( a \neq 0 \) that satisfies \( f(a) = 0 \). Since \( f \) preserves degree (of cohomology ring), we can assume that \( a \in H^{2j}(Gr(k, N)) \). By Poincaré duality theorem, we have \( b \in H^{2(N-k)-2j}(Gr(k, N)) \) that satisfies \( ab = (c_k(S^*))^{N-k} \in H^{2k(N-k)}(Gr(k, N)) \). On the other hand, \( f(c_k(S^*))^{N-k} = f(c_k(S^*))^{N-k} = (\det(\Phi))^{N-k} \) and Theorem \cite{[7]} tells us that \( f(c_k(S^*))^{N-k} \neq 0 \). But from the assumption \( f(a) = 0 \), \( f(ab) = f(a)f(b) = f(c_k(S^*))^{N-k} = 0 \) follows. This contradiction leads us to conclude that \( f \) is injective. \( \square \)

In Subsection 1.2, we have introduced formal line bundle decomposition \( S^* = \otimes_{i=1}^k L_i \) and the relation:

\[
c(S^*) = \prod_{j=1}^k (1 + x_j l_j) \quad (x_j = c_1(L_j)).
\]  

(4.138)

Then \( f(x_j) = \lambda_j \), i.e., \( f \) identifies \( x_j \) with \( \lambda_j \). Moreover, according to \cite{[7]}, normalization condition of integration on \( Gr(k, N) \) is given by,

\[
\int_{Gr(k, N)} c_k(S^*)^{N-k} = 1.
\]  

(4.139)

Therefore, Theorem \cite{[7]} and Theorem \cite{[7]} lead us to the following equality:

\[
\int_{Gr(k, N)} g(x_1, \ldots, x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int_{Gr(k, N)} \frac{\det(\gamma^i)}{\gamma^k \prod_{i>j} (\gamma^i - (\lambda_i - \lambda_j)^2)}
\]

(4.140)

where \( g(x_1, \ldots, x_k) \) is a symmetric polynomial of \( x_1, \ldots, x_k \) that represents an element of \( H^{2k(N-k)}(Gr(k, N)) \). By combining (4.120) with (4.140), we obtain,

\[
Z_{MQ} = \prod_{j=0}^{N-k} j! \int_{Gr(k, N)} \frac{(\det(\gamma^i + \Phi')}{(\gamma^k \prod_{i>j} (\gamma^i - (\lambda_i - \lambda_j)^2))} \prod_{i>j} \frac{(1 + \frac{1}{2}\lambda_i)^N}{(1 - \frac{1}{2}\lambda_i - \lambda_j)^2)}
\]

(4.141)

This completes proof of the main theorem. \( \square \)
4.2 Proof of Theorem

Definition 2. Let $\mathcal{M}_l$ be set of $k \times k$ matrix $M_l$:

$$M_l := \begin{pmatrix}
m_{1,1}^l & \cdots & m_{1,k}^l \\
\vdots & \ddots & \vdots \\
m_{k,1}^l & \cdots & m_{k,k}^l
\end{pmatrix},$$

(4.142)

whose $(i,j)$-element $m_{i,j}^l$ is given by non-negative integer that satisfies the following conditions:

$$\sum_{i=1}^{k} m_{i,j}^l = l, \quad (j = 1, \cdots, k), \quad \sum_{j=1}^{k} m_{i,j}^l = l \quad (i = 1, \cdots, k).$$

(4.143)

Definition 3. Let $S_k$ be symmetric group of size $k$. For $\sigma \in S_k$, we define $k \times k$ matrix $R(\sigma)$:

$$R(\sigma) := \begin{pmatrix}
\delta_{\sigma(1),1} & \cdots & \delta_{\sigma(1),k} \\
\vdots & \ddots & \vdots \\
\delta_{\sigma(k),1} & \cdots & \delta_{\sigma(k),k}
\end{pmatrix},$$

(4.144)

where $\delta_{i,j}$ is Kronecker’s delta symbol.

Proposition 1. We denote by $(n_{\sigma})_{\sigma \in S_k}$ a sequence of $k!$ non-negative integers labeled by $\sigma \in S_k$. Let $\mathcal{N}_1$ be set of $(n_{\sigma})_{\sigma \in S_k}$ that satisfy $\sum_{\sigma \in S_k} n_{\sigma} = l$. Then $\phi : \mathcal{N}_1 \rightarrow \mathcal{M}_l$ defined by,

$$\phi((n_{\sigma})_{\sigma \in S_k}) = \sum_{\sigma \in S_k} n_{\sigma} R(\sigma) \in \mathcal{M}_l,$$

(4.145)

is a surjection.

Proof. In the $l = 1$ case, assertion of the proposition is obvious because $\phi : \mathcal{N}_1 \rightarrow \mathcal{M}_1$ is a bijection. Then we can prove the proposition by induction of $l$.

Remark 1. For general $k$ and $l$, $\phi$ is not injective. For example, in the $k = l = 3$ case, we have the following equalities:

$$\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

(4.146)

Definition 4. Let $X$ be $k \times k$ matrix whose $(i,j)$-element is given by $x_{i,j}$. For $M_l \in \mathcal{M}_l$, we define integer $\widetilde{\text{mul}}(M_l)$ by the following expansion:

$$|X|^l = \sum_{M_l \in \mathcal{M}_l} \widetilde{\text{mul}}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^l}.$$  

(4.147)

Proposition 2. $\widetilde{\text{mul}}(M_l)$ is explicitly evaluated as follows.

$$\widetilde{\text{mul}}(M_l) = \sum_{\phi((n_{\sigma})_{\sigma \in S_k}) = M_l} \frac{l! \prod_{\sigma \in S_k} (\text{sgn}(\sigma))^{n_{\sigma}}}{\prod_{\sigma \in S_k} n_{\sigma}!}.$$  

(4.148)
\textbf{Proof.} First, we explicitly expand $|X|^l$ by using definition of $|X|:

\begin{align*}
|X|^l &= \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{a=1}^{k} x_{a,\sigma(a)} \right)^l \\
&= \sum_{(n_\sigma) \in S_k} \prod_{\sigma \in S_k} n_\sigma! \prod_{\sigma \in S_k} \left( \text{sgn}(\sigma)^{n_\sigma} \prod_{a=1}^{k} x_{a,\sigma(a)} \right) \\
&= \sum_{(n_\sigma) \in S_k} \prod_{\sigma \in S_k} n_\sigma! \prod_{\sigma \in S_k} \left( \text{sgn}(\sigma)^{n_\sigma} \prod_{a,b=1}^{k} x_{a,b}^{n_\sigma \delta_{\sigma(a),b}} \right) \\
&= \sum_{(n_\sigma) \in S_k} \prod_{\sigma \in S_k} n_\sigma! \prod_{\sigma \in S_k} \left( \text{sgn}(\sigma)^{n_\sigma} \right) \prod_{a,b=1}^{k} \sum_{\sigma \in S_k} n_\sigma \delta_{\sigma(a),b}. \tag{4.149}
\end{align*}

$\sum_{\sigma \in S_k} n_\sigma \delta_{\sigma(a),b}$ is nothing but the $(a,b)$-element of $\sum_{\sigma \in S_k} n_\sigma R(\sigma)$ and $\sum_{\sigma \in S_k} n_\sigma R(\sigma) = \varphi((n_\sigma)_{\sigma \in S_k}) \in M_l$. Then assertion of proposition immediately follows from (4.149).

\textbf{Lemma 3.} The following equality holds,

$$
\widetilde{\text{mul}}(M_{l+1}) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \widetilde{\text{mul}}(M_{l+1} - R(\sigma)), \tag{4.150}
$$

where we set $\widetilde{\text{mul}}(M_{l+1} - R(\sigma)) = 0$ if $M_{l+1} - R(\sigma) \notin M_l$.

\textbf{Proof.}

\begin{align*}
|X|^{l+1} &= \sum_{M_{l+1} \in M_{l+1}} \widetilde{\text{mul}}(M_{l+1}) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}} \\
&= |X|^l \cdot |X| \\
&= \left[ \sum_{M_l \in M_l} \widetilde{\text{mul}}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l}} \right] \times \left[ \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{a=1}^{k} x_{a,\sigma(a)} \right] \\
&= \sum_{M_l \in M_l} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \widetilde{\text{mul}}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l} + \delta_{\sigma(a),b}} \\
&= \sum_{M_{l+1} \in M_{l+1}} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \widetilde{\text{mul}}(M_{l+1} - R(\sigma)) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}. \tag{4.151}
\end{align*}

\textbf{Definition 5.} Let $\psi_s^i$ $(s = 1, \cdots, l, \ i = 1, \cdots, k)$ be complex Grassmann variable and $\bar{\psi}_s^i$ be its complex conjugate. We denote by $\Phi$ a $k \times k$ matrix whose $(i,j)$-element is given by $\sum_{s=1}^{l} \psi_s^i \psi_s^j$. Then we define $C(l, k)$ as follows.

$$
C(l, k) := \int \left( \prod_{s=1}^{l} \prod_{i=1}^{k} d\psi_s^i d\bar{\psi}_s^i \right) \det(\Phi)^l. \tag{4.151}
$$

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Proof. By substituting $\Phi'$ for $X$ in (4.147), we rewrite (4.151) as follows.

$$C(l, k) = \sum_{M_l \in M_l} \left[ \prod_{a, b = 1}^{k} \left( m_{a, b}^l \right)! \right] \mul(M_l)^2. \quad (4.154)$$

In going from the first line to the second line, we used an equality:

$$\left( \sum_{s=1}^{m_{a, b}} \omega_{s}^{a, b} \right)^{m_{a, b}} = \left( m_{a, b}^l \right)! \left( \sum_{1 \leq s_{a, b}^{1, \cdots, m_{a, b}} \leq l} \omega_{s_{a, b}^{1, \cdots, m_{a, b}}}^{a, b} \cdots \omega_{s_{a, b}^{m_{a, b}}^{1, \cdots, m_{a, b}}}^{a, b} \right), \quad (4.156)$$

which follows from $\left( \omega_{s}^{a, b} \right)^2 = 0$. Let us define $S_{a, b} := \{s_{a, b}^{1}, \cdots, s_{a, b}^{m_{a, b}}\}$ ($|S_{a, b}| = m_{a, b}^l$) associated with the sequence $1 \leq s_{a, b}^{1, \cdots, m_{a, b}} \leq l$ in the last line of (4.155). Since $\omega_{s}^{a, b} \omega_{\bar{s}}^{a, b} = \omega_{s}^{a, b} \omega_{\bar{s}}^{b, a} = 0$, $S_{a, b}$'s associated with non-vanishing $\prod_{a, b = 1}^{k} \omega_{s_{a, b}^{1, \cdots, m_{a, b}}}^{a, b} \cdots \omega_{s_{a, b}^{m_{a, b}}^{1, \cdots, m_{a, b}}}^{a, b}$ satisfy the following condition:

$$\prod_{a=1}^{k} S_{a, b} = \{1, \cdots, l\} \quad (b = 1, \cdots, k),$$

$$\prod_{b=1}^{k} S_{a, b} = \{1, \cdots, l\} \quad (a = 1, \cdots, k). \quad (4.157)$$

We denote by $\{S_{a, b}\}$ set of $S_{a, b}$'s $(a, b = 1, \cdots, k)$ that satisfy $|S_{a, b}| = m_{a, b}^l$ and the condition (4.157). We also denote by $S(M_l)$ set of $\{S_{a, b}\}$'s associated with the matrix $\{m_{a, b}^l\} = M_l \in M_l$. Then we can further rewrite $C(l, k)$ into the following form:

$$C(l, k) = \sum_{M_l \in M_l} \mul(M_l) \left[ \prod_{a, b = 1}^{k} \left( m_{a, b}^l \right)! \right] \int D\psi \left( \sum_{\{S_{a, b}\} \in S(M_l)} \prod_{a, b = 1}^{k} \omega_{s_{a, b}^{1, \cdots, m_{a, b}}}^{a, b} \cdots \omega_{s_{a, b}^{m_{a, b}}^{1, \cdots, m_{a, b}}}^{a, b} \right). \quad (4.158)$$
We take a closer look at the sum \( \sum_{\{S_{a,b}\}\in S(M_l)} \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \). For a fixed element \( \{a,b\} \in S(M_l) \), we can construct a sequence \( (\sigma_1, \sigma_2, \ldots, \sigma_l) \) of permutations \( \sigma_s \in S_k \) (\( s = 1, \ldots, l \)). This is because for each \( s \in \{1, 2, \ldots, l\} \), we can fix unique permutation \( \sigma_s \in S_k \) that satisfies \( \sigma_s \in S_{a, \sigma_s(a)} \) (\( a = 1, \ldots, k \)) by using the condition (4.157). Obviously, the sequence \( (\sigma_1, \sigma_2, \ldots, \sigma_l) \) satisfy the following condition:

\[
\sum_{s=1}^l R(\sigma_s) = M_l. \tag{4.159}
\]

Conversely, for a sequence \( (\sigma_1, \ldots, \sigma_l) \) that satisfy (4.159), we can construct unique \( \{a,b\} \in S(M_l) \) that satisfies \( s \in S_{a, \sigma_s(a)} \) (\( a = 1, \ldots, k \)). Hence we have one to one correspondence between \( \{a,b\} \in S(M_l) \) and a sequence \( (\sigma_1, \ldots, \sigma_l) \) that satisfy (4.159). Since we can construct \( \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \) different elements of \( S(M_l) \) from a fixed \( (n_\sigma)_{\sigma \in S_k} \) that satisfies \( \varphi((n_\sigma)_{\sigma \in S_k}) = M_l \), the following equality holds.

\[
\int D\psi \left( \sum_{\{S_{a,b}\}\in S(M_l)} \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \right) = \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

Then we obtain,

\[
C(l, k) = \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

\[
= \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

\[
= \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

\[
= \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

\[
= \sum_{M_l \in M_l} \sum_{a,b=1}^k \omega^{a,b}_{S_{a,b}} \int D\psi \prod_{a,b=1}^k \omega^{a,b}_{S_{a,b}}
\]

In going from the third line to the last line, we used (4.148).

**Lemma 5.** (cf. [6])

\[
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \prod_{a=1}^k \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} = \frac{(k+l)!}{l!} |X|^l. \tag{4.161}
\]

**Proof.** We start from the following identity:

\[
|X|^{l+1} = \int \prod_{a=1}^{l+1} d\psi_a^0 d\psi_a^1 \exp \left( \sum_{a=1}^{l+1} \sum_{i,j=1}^k \psi_a^i x_{i,j} \psi_a^j \right)
\]

\[
= \int D\psi \exp \left( \sum_{i,j=1}^k x_{i,j} \omega^{i,j} \right). \tag{4.162}
\]
In going from the first line to the second line, we set \( D\psi = \prod_{i=1}^{k} d\psi^i_a d\psi^j_a \), \( \omega^{i,j} = \sum_{a=1}^{l+1} \psi^i_a \psi^j_a \). Then we obtain,

\[
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} \\
= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) \int D\psi \exp \left( \sum_{i,j=1}^{k} x_{i,j} \omega^{i,j} \right) \\
= \int D\psi \exp \left( \sum_{i,j=1}^{k} x_{i,j} \omega^{i,j} \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{a=1}^{k} \omega^{a,\sigma(a)} \right). \tag{4.163}
\]

Since the integrand in the last line is invariant under coordinate change \( \psi^i_a \to P^i_j \psi^j_a \), \( \psi^j_a \to (P^{-1})^j_i \psi^i_a \) (\( P \) is arbitrary \( k \times k \) invertible matrix), we can assume that \( X \) is a diagonal matrix (\( x_{i,j} = \lambda_i \delta_{i,j} \)) from the start. Hence we obtain,

\[
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} \\
= \int D\psi \exp \left( \sum_{i=1}^{k} \lambda_i \omega^{i,i} \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right) \\
= \int D\psi \left( \sum_{i=1}^{k} \lambda_i \omega^{i,i} \right) \left( \prod_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right) \\
= \int D\psi \left( \prod_{i=1}^{k} \left( \lambda_i \omega^{i,i} \right) \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right). \tag{4.164}
\]

In going from the fourth line to the last line, we used the condition that \( \psi^j_a \) and \( \psi^i_a \) must appear exactly at \( l + 1 \) times for each \( j \) for non vanishing result. We can further proceed as follows.

\[
\int D\psi \left( \prod_{i=1}^{k} \left( \lambda_i \omega^{i,i} \right) \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right) \\
= \prod_{i=1}^{k} (\lambda_i)^l \int D\psi \left( \prod_{i=1}^{k} \left( \omega^{i,i} \right)^l \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right) \\
= |X|^l \int D\psi \left( \prod_{i=1}^{k} \left( \omega^{i,i} \right)^l \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{j=1}^{k} \omega^{j,\sigma(j)} \right). \tag{4.165}
\]

At this stage, we recall the following equalities:

\[
(\omega^{i,j})^l = \left( \sum_{a=1}^{l+1} \psi^i_a \psi^j_a \right)^l = \sum_{a=1}^{l+1} \left( \prod_{b \neq a} \psi^i_b \psi^j_b \right).
\]

\(^1\)Precisely speaking, \( X \) is only reduced to Jordan normal form, but this does not affect the remaining part of the proof.
By combining (4.161), (4.165) and (4.166), we obtain,

\[
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1}
\]

\[
= |X|^{l} \left( \sum_{\sigma \in S_k} \frac{1}{a_1 = 1} \cdots \frac{1}{a_k = 1} \int D\psi \left( \prod_{i=1}^{l+1} \left( \sum_{a_1 = 1} a_1 b \neq a \right) \prod_{i=1}^{l+1} \psi_{b_i}^{(j)} \right) \right) \text{sgn}(\sigma) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

\[
= |X|^{l} \left( \sum_{\sigma \in S_k} \frac{1}{a_1 = 1} \cdots \frac{1}{a_k = 1} \int D\psi \left( \prod_{i=1}^{l+1} \left( \sum_{a_1 = 1} a_1 b \neq a \right) \prod_{i=1}^{l+1} \psi_{b_i}^{(j)} \right) \right) \text{sgn}(\sigma) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

\[
= |X|^{l} \left( \sum_{\sigma \in S_k} \frac{1}{a_1 = 1} \cdots \frac{1}{a_k = 1} \int D\psi \left( \prod_{i=1}^{l+1} \left( \sum_{a_1 = 1} a_1 b \neq a \right) \prod_{i=1}^{l+1} \psi_{b_i}^{(j)} \right) \right) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

In going from the second line to the third line, we used the relation,

\[
\prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)} = \text{sgn}(\sigma) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

At this stage, we fix \( \sigma \in S_k \) and consider the integral:

\[
\sum_{a_1 = 1}^{l+1} \cdots \sum_{a_k = 1}^{l+1} \int D\psi \left( \prod_{i=1}^{l+1} \left( \sum_{a_1 = 1} a_1 b \neq a \right) \prod_{i=1}^{l+1} \psi_{b_i}^{(j)} \right) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

We can easily see that it counts number of sequences \((a_1, a_2, \cdots, a_k)\) \((1 \leq a_j \leq l + 1)\) that satisfy the following condition:

\[
a_j = a_{\sigma(j)} \quad (j = 1, 2, \cdots, k).
\]

Let \( P_k \) be set of partition of \( k \),

\[
P_k := \{ \tau = (n_1, \cdots, n_{l(\tau)}) \mid n_1 + n_2 + \cdots + n_{l(\tau)} = k, \ n_1 \geq n_2 \geq \cdots \geq n_{l(\tau)} > 0 \}.
\]

It is well-known that conjugacy class of symmetric group \( S_k \) is labeled by \( \tau \in P_k \). If \( \sigma \in S_k \) belongs to the conjugacy class labeled by \( \tau = (n_1, \cdots, n_{l(\tau)}) \), number of sequences \((a_1, a_2, \cdots, a_k)\) that satisfy (4.170) is given by \((l + 1)^{l(\tau)}\). It is well-known that number of permutations that belong to conjugacy class labeled by \( \tau = (n_1, \cdots, n_{l(\tau)}) \) is given by,

\[
\frac{k!}{\prod_{j=1}^{l(\tau)} n_j \prod_{j=1}^{k} \text{mul}(j; \tau)!}
\]

where \( \text{mul}(j; \tau) \) is number of \( i \)'s that satisfy \( n_i = j \). Hence we have obtain,

\[
\sum_{\sigma \in S_k} \frac{1}{a_1 = 1} \cdots \frac{1}{a_k = 1} \int D\psi \left( \prod_{i=1}^{l+1} \left( \sum_{a_1 = 1} a_1 b \neq a \right) \prod_{i=1}^{l+1} \psi_{b_i}^{(j)} \right) \prod_{j=1}^{k} \psi_{a_j}^{(j)} \psi_{a_{\sigma(j)}}^{(j)}
\]

\[
= \sum_{\tau \in P_k} \frac{k!}{\prod_{j=1}^{l(\tau)} n_j \prod_{j=1}^{k} \text{mul}(j; \tau)!} (l + 1)^{l(\tau)}.
\]
On the other hand, we have the following combinatorial identity:

$$
\sum_{\tau \in P_k} \frac{k!}{\prod_{l=1}^{k} n_l} \prod_{l=1}^{k} \text{mul}(j_l; \tau)! x^{l(\tau)} = x(x + 1) \cdots (x + k - 1),
$$

(4.174)

that follows from the following computation,

$$
\sum_{\tau \in P_k} \frac{k!}{\prod_{l=1}^{k} n_l} \prod_{l=1}^{k} \text{mul}(j_l; \tau)! x^{l(\tau)} \\
\quad = \frac{d^k}{dq^k} \left( 1 + \sum_{m=1}^{\infty} \sum_{\tau \in P_m} \prod_{l=1}^{m} \frac{1}{n_l} \prod_{l=1}^{m} \text{mul}(j_l; \tau)! q^{m_l} x^{l(\tau)} \right) \bigg|_{q=0} \\
\quad = \frac{d^k}{dq^k} \left( \prod_{n=1}^{\infty} \exp \left( \frac{q^n}{n} x \right) \right) \bigg|_{q=0} \\
\quad = \frac{d^k}{dq^k} \left( \exp \left( \sum_{n=1}^{\infty} \frac{q^n}{n} x \right) \right) \bigg|_{q=0} \\
\quad = \frac{d^k}{dq^k} \left( \exp(-x \log(1-q)) \right) \bigg|_{q=0} \\
\quad = x(x + 1) \cdots (x + k - 1).
$$

(4.175)

By combining (4.167), (4.173) and (4.174), we obtain,

$$
\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_a, \sigma(a)} \right) |X|^{l+1} \\
\quad = (l+1)(l+2) \cdots (l+k) |X|^l \\
\quad = \frac{(l+k)!}{l!} |X|^l.
$$

(4.176)

This completes the proof of the lemma. \hfill \Box

**Lemma 6.**

$$
C(l+1, k) = \frac{(l+k)!}{l!} C(l, k).
$$

(4.177)

**Proof.** We start from Lemma 4 applied to \(C(l+1, k)\).

$$
C(l+1, k) = \sum_{M_{l+1}, \text{mul}(M_{l+1})} \left[ \prod_{a,b=1}^{k} (m_{a,b}^{l+1})! \right] \text{mul}(M_{l+1})^2
$$

(4.178)

By combining the above equality with (4.150), we obtain,

$$
C(l+1, k) = \sum_{M_{l+1}, \text{mul}(M_{l+1})} \sum_{\sigma, \tau \in S_k} \left[ \prod_{a,b=1}^{k} (m_{a,b}^{l+1})! \right] \text{sgn}(\sigma) \text{sgn}(\tau) \\
\quad \times \text{mul}(M_{l+1} - R(\sigma)) \text{mul}(M_{l+1} - R(\tau)).
$$

(4.179)
We set $M_l = M_{l+1} - R(\sigma)$ and rewrite further the above equality.

$$C(l+1,k) = \sum_{M_l \in M_1} \sum_{\sigma,\tau \in S_k} \left( \prod_{a,b=1}^{k} (m_{a,b}^l + \delta_{\sigma(a),b})! \right) \text{sgn}(\sigma) \text{sgn}(\tau) \mul(M_l) \times \mul(M_l + R(\sigma) - R(\tau))$$

$$= \sum_{M_l \in M_1} \left( \prod_{a,b=1}^{k} (m_{a,b}^l)! \right) \mul(M_l) \sum_{\sigma,\tau \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \text{sgn}(\sigma) \text{sgn}(\tau) \times \mul(M_l + R(\sigma) - R(\tau)).$$

Hence in order to prove the lemma, we only have to confirm the following equality:

$$\sum_{\sigma,\tau \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \text{sgn}(\sigma) \text{sgn}(\tau) \mul(M_l + R(\sigma) - R(\tau)) = \frac{(k+l)!}{l!} \mul(M_l).$$

(4.180)

If $\sigma = \tau$, $\text{sgn}(\sigma) \text{sgn}(\tau) \mul(M_l + R(\sigma) - R(\tau))$ obviously equals $\mul(M_l)$. Let us use the representation $M_l = \sum_{\sigma' \in S_k} n_{\sigma'} R(\sigma')$. If $\sigma \neq \tau$, we have the following representation:

$$M_l + R(\sigma) - R(\tau) = \sum_{\sigma' \neq \sigma, \tau} n_{\sigma'} R(\sigma') + (n(\sigma) + 1)R(\sigma) + (n(\tau) - 1)R(\tau).$$

(4.182)

Applying (4.180) carefully to these two cases, we obtain,

$$\sum_{\sigma,\tau \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \text{sgn}(\sigma) \text{sgn}(\tau) \mul(M_l + R(\sigma) - R(\tau))$$

$$= \left\{ \begin{array}{cl}
\sum_{\varphi((n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}} & (\sigma = \tau), \\
\sum_{\varphi((n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \times \frac{n_{\sigma}}{n_{\sigma} + 1} \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}} & (\sigma \neq \tau). 
\end{array} \right. $$

(4.183)

Hence we can rewrite the l.h.s. of (4.181) as follows.

$$\sum_{\sigma,\tau \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \text{sgn}(\sigma) \text{sgn}(\tau) \mul(M_l + R(\sigma) - R(\tau))$$

$$= \sum_{\sigma \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \left\{ \sum_{(n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}} \right\}$$

$$+ \sum_{\tau \neq \sigma} \sum_{\varphi((n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \times \frac{n_{\sigma}}{n_{\sigma} + 1} \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}} \right\}$$

$$= \sum_{\sigma \in S_k} \left( \prod_{a=1}^{k} (m_{a,\sigma(a)}^l + 1) \right) \left\{ \sum_{(n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}} \right\}$$

$$+ \sum_{\varphi((n_{\sigma'}),\sigma' \in S_k) = M_l} l! \prod_{\sigma' \in S_k} n_{\sigma'}! \times \frac{l - n_{\sigma}}{n_{\sigma} + 1} \prod_{\sigma' \in S_k} \left( \text{sgn}(\sigma') \right)^{n_{\sigma'}}. $$

(4.184)
In going from the second expression to the third expression, we used the equality:

\[
\sum_{\tau \neq \sigma} n_\tau = \sum_{\tau \in S_k} n_\tau - n_\sigma = l - n_\sigma. \tag{4.185}
\]

At this stage, we explicitly compute the l.h.s. of Lemma 3: \(\sum_{\sigma \in S_k} \sgn(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} \) by using the expansion \(|X|^{l+1} = \sum_{M_l+1 \in M_{l+1}} \text{mul}(M_{l+1}) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}\).

\[
\sum_{\sigma \in S_k} \sgn(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} = \sum_{\sigma \in S_k} \sgn(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) \sum_{M_l+1 \in M_{l+1}} \varphi((n_\tau)_{\tau \in S_k}) = \sum_{M_l+1} \prod_{\tau \in S_k} n_\tau! \left( \prod_{\tau \in S_k} \sgn(\tau) \right)^{n_\tau} \prod_{\tau \in S_k} x_{a,b}^{m_{a,b}^{l+1}} \times \prod_{a=1}^{k} \prod_{b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}. \tag{4.186}
\]

In the last expression, \(\varphi((n_\tau)_{\tau \in S_k}) = M_{l+1}\) that corresponds to non-vanishing summand satisfies the condition \(n_\sigma \geq 1\). Hence we can set \(M_{l+1} = M_l + R(\sigma)\) with \(\varphi((m_\tau)_{\tau \in S_k}) = M_l\). Then we can further rewrite the above expression,

\[
= \sum_{\sigma \in S_k} \sum_{M_l \in M_l} \sum_{\varphi((m_\tau)_{\tau \in S_k}) = M_l} \frac{(l+1)!}{(m_\sigma + 1) \prod_{\tau} m_\tau!} \prod_{\tau \in S_k} \left( \prod_{\tau \in S_k} \sgn(\tau) \right)^{m_\tau} \left( \prod_{a=1}^{k} \prod_{b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}} \right) \times \prod_{a=1}^{k} \prod_{b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}. \tag{4.187}
\]

By combining, the above derivation with assertion of Lemma 3:

\[
\sum_{\sigma \in S_k} \sgn(\sigma) \left( \prod_{a=1}^{k} \frac{\partial}{\partial x_{a,\sigma(a)}} \right) |X|^{l+1} = \sum_{M_l \in M_l} \sum_{\varphi((m_\tau)_{\tau \in S_k}) = M_l} \sum_{\sigma \in S_k} \frac{(l+1)!}{(m_\sigma + 1) \prod_{\tau} m_\tau!} \prod_{\tau \in S_k} \left( \prod_{\tau \in S_k} \sgn(\tau) \right)^{m_\tau} \left( \prod_{a=1}^{k} \prod_{b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}} \right) = \frac{(l+1)!}{l!} \sum_{M_l \in M_l} \text{mul}(M_l) \prod_{a,b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}}, \tag{4.188}
\]

we obtain,

\[
\sum_{M_l \in M_l} \sum_{\sigma \in S_k} \frac{(l+1)!}{(m_\sigma + 1) \prod_{\tau} m_\tau!} \prod_{\tau \in S_k} \left( \prod_{\tau \in S_k} \sgn(\tau) \right)^{m_\tau} \left( \prod_{a=1}^{k} \prod_{b=1}^{k} x_{a,b}^{m_{a,b}^{l+1}} \right) = \frac{(l+1)!}{l!} \sum_{M_l \in M_l} \text{mul}(M_l). \tag{4.189}
\]
By comparing (4.184) with (4.189), we reach the equality:

\[
\sum_{\sigma, \tau \in S_k} \prod_{i=1}^{k} \left( m_{\alpha_i, \sigma(\alpha_i)} + 1 \right) \text{sgn}(\sigma) \text{sgn}(\tau) \tilde{\text{mul}}(M_i + R(\sigma) - R(\tau)) = \frac{(k + l)!}{l!} \tilde{\text{mul}}(M_i),
\]

which completes the proof of the lemma. \(\square\)

**Proof of Theorem 3**

Proof. \(C(1, k)\) is calculated as follows.

\[
C(1, k) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \int D\psi \psi_1^{\sigma(1)} \ldots \psi_k^{\sigma(k)}
\]

\[
= \sum_{\sigma \in S_k} (\text{sgn}(\sigma))^2 \int D\psi \psi_1^{\sigma(1)} \ldots \psi_k^{\sigma(k)} = \sum_{\sigma \in S_k} 1 = k!.
\]

(4.191)

Then successive use of Lemma 6 leads us to,

\[
C(l, k) = \frac{(l + k - 1)!}{(l - 1)!} \ldots \frac{(k + 1)!}{1!} C(1, k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=0}^{l-1} j! \prod_{j=0}^{l-1} j!}.
\]

(4.192)

Hence in the case of Theorem 3, the l.h.s. \(\frac{\prod_{j=0}^{k-1} j!}{\prod_{i=0}^{N-k} j!} \int D\psi' (\text{det}(\Phi'))^{N-k}\) equals \(\frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{N-k-1} j!}{\prod_{j=0}^{l} j! \prod_{j=0}^{l} j!} C(N - k, k) = 1.\)

\(\square\)

**References**

[1] M. F. Atiyah and L. Jeffrey, Topological Lagrangians and cohomology, *J. Geom. Phys.*, 7, 119-136 (1990).

[2] M. Blau, The Mathai–Quillen formalism and topological field theory, *J. Geom. Phys.*, 11, 95-127 (1993).

[3] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, (1st ed., Springer-Verlag New York, 1982).

[4] G. Camano–Garcia, Statistics on Stiefel manifolds, *Retrospective Theses and Dissertations*, 1493, (2006).

[5] K. Fujii, Introduction to Grassmann manifolds and quantum computation, *J. Appl. Math.*, 2, 371-405 (2002).

[6] K. Fujii, T. Kashiwa, S. Sakoda, Coherent states over Grassmann manifolds and the WKB exactness in path integral, *Journal of Mathematical Physics*, 37, 567 (1996).

[7] P. Griffiths; J. Harris, Principles of algebraic geometry, Pure and Applied Mathematics. Wiley–Interscience, New York (1978).  

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[8] J. M. F. Labastida and C. Lozano, Lectures on topological quantum field theory, *AIP Conf. Proc.*, **419**, 54-93 (1998), arXiv:hep-th/9709192.

[9] V. Mathai and D. Quillen, Superconnections, Thom classes, and equivariant differential forms, *Topology*, **25**, 85-110 (1986).

[10] E. Witten, Topological quantum field theory, *Commun.Math. Phys.*, **117**, 353-386 (1988).

[11] E. Witten, The Verlinde algebra and the cohomology of the Grassmannian, *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 357–422 (1993), arXiv:hep-th/9312104.

[12] A. Sako, Supersymmetric Gauge Theory and Geometry: Non-perturbative approach, (in Japanese, printed in Japan), Nihon Hyouronsha (2007).

[13] W. Zhang, Lectures on Chern–Weil theory and Witten deformations, Nankai Tracts in Mathematics-Vol.4, World Scientific (2002).