Crystal bases and quiver varieties

(Geometric construction of crystal base II)

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Abstract
We give a crystal structure on the set of all irreducible components of Lagrangian subvarieties of quiver varieties. One can show that, as a crystal, it is isomorphic to the crystal base of an irreducible highest weight representation of a quantized universal enveloping algebra.

1 Introduction
Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of a Kac-Moody Lie algebra $\mathfrak{g}$ with a symmetric Cartan datum and $U_q^-(\mathfrak{g})$ the minus part of it. In [8],[10], Lusztig gave a geometric realization of $U_q^-(\mathfrak{g})$ in terms of quivers and defined the canonical basis of it. On the other hand, motivated by the study of solvable lattice models, Kashiwara defined the crystal base and the global base of $U_q^-(\mathfrak{g})$ in an algebraic way [3]. Grojnowski and Lusztig proved the global base is coincide with the canonical basis due to Lusztig [1]. Afterwards Kashiwara and the author constructed the crystal base of $U_q^-(\mathfrak{g})$ in a geometric way [7]. More precisely they defined a crystal structure on the set of irreducible components of a Lagrangian variety and proved that it is isomorphic to the crystal base of $U_q^-(\mathfrak{g})$.

In the crystal base theory, there is another important example $B(\lambda)$ which is the crystal base of the irreducible highest weight module of $U_q(\mathfrak{g})$ with a highest weight $\lambda$. The purpose of this paper is to construct $B(\lambda)$ in a geometrical way. Nakajima defined a new family of hyper-Kähler manifolds,

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called quiver varieties [15], [16]. He also constructed a representation of $\mathfrak{g}$ on the middle homology groups of quiver varieties. Moreover he proved it is isomorphic to an irreducible highest weight representation by using Kac’s characterization of irreducible modules [2]. In this paper we consider a Lagrangian subvariety of a quiver variety, following Nakajima, and define a crystal structure on the set of irreducible components of it. Instead of Kac’s characterization, we give a crystal-theoretical characterization of $B(\lambda)$. By the aid of it, we show the crystal associated to the set of irreducible components is isomorphic to $B(\lambda)$.

In Kashiwara’s algebraic construction, the global base of an irreducible highest weight module of $U_q(\mathfrak{g})$ was also defined [3]. We remark that it has not constructed yet in a geometrical way, since a representation of $U_q(\mathfrak{g})$ has not.

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## 2 Preliminaries

### 2.1

Let $\mathfrak{g}$ be a symmetric Kac-Moody Lie algebra and $\mathfrak{t}$ its Cartan subalgebra. Let $\{\alpha_i | i \in I\} \subset \mathfrak{t}^*$ and $\{h_i | i \in I\} \subset \mathfrak{t}$ be the set of simple roots and simple coroots, respectively. We normalize the non-degenerate symmetric invariant bilinear form $(\ ,\ )$ on $\mathfrak{t}^*$ so that $(\alpha_i, \alpha_i) \in \mathbb{Z}_{>0}$. Let $P$ be the weight lattice and $P^*$ its dual lattice. Then $\alpha_i \in P$ and $h_i \in P^*$.

We define $U_q(\mathfrak{g})$ as the $\mathbb{Q}(q)$-algebra generated by $e_i, f_i (i \in I)$ and $q^h (h \in P^*)$ with the following defining relations:

1. $q^h = 1$ for $h = 0$ and $q^{h+h'} = q^h q^{h'}$,
2. $q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i$ and $q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i$,
3. $[e_i, f_j] = \delta_{i,j} (t_i - t_i^{-1})/(q_i - q_i^{-1})$ where $q_i = q^{(\alpha_i, \alpha_i)}$ and $t_i = q^{(\alpha_i, h_i)}$,
4. $\sum (-1)^n e_i^{(n)} e_j^{(b-n)} = \sum (-1)^n f_i^{(n)} f_j^{(b-n)} = 0$ where $i \neq j$ and $b = \langle h_i, \alpha_j \rangle$. 

2
Here we used the notations \([n]_i = (q^n_i - q^{-n}_i)/(q_i - q_i^{-1})\), [n]_i! = \prod_{k=1}^{n}[k]_i, e_i^{(n)} = e_i^n/[n]_i! and \(f_i^{(n)} = f_i^n/[n]_i!\). We understand \(e_i^{(n)} = f_i^{(n)} = 0\) for \(n < 0\).

We set \(Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, Q_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i\), and \(Q_- = -Q_+\). Let \(P_+\) be the set of dominant integral weights.

We denote by \(U_q^{-}(\mathfrak{g})\) the \(\mathbb{Q}(q)\)-subalgebra of \(U_q(\mathfrak{g})\) generated by \(f_i(i \in I)\).

For a fixed \(i \in I\), let \(U_q(\mathfrak{g}_i)\) be the \(\mathbb{Q}(q)\)-subalgebra of \(U_q(\mathfrak{g})\) generated by \(e_i, f_i\) and \(q^{\pm h_i}\). We say that \(U_q(\mathfrak{g})\)-module \(M\) is integrable if \(M\) has the weight space decomposition \(M = \bigoplus_{\nu \in P} M_\nu\) and \(M\) is \(U_q(\mathfrak{g}_i)\)-locally finite for any \(i \in I\).

### 2.2 crystals

The theory of crystal base developed in [3] provides very powerful method in the representation theory of \(U_q(\mathfrak{g})\). Motivated by the properties of the crystal base, Kashiwara defined crystals in the combinatorial way [4]. First we recall the definition of crystals. See [3],[4],[5],[6] and [7] for details.

**Definition 2.2.1**  A crystal \(B\) is a set with

(2.2.1) maps \(\text{wt} : B \to P, \quad \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}\) and \(\varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}\),

(2.2.2) \(\tilde{e}_i : B \to B \sqcup \{0\}, \quad \tilde{f}_i : B \to B \sqcup \{0\}\).

They are subject to the following axioms:

(C 1) \(\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle\).

(C 2) If \(b \in B\) and \(\tilde{e}_i b \in B\) then,

\[\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i, \quad \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \quad \text{and} \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.\]

(C 2') If \(b \in B\) and \(\tilde{f}_i b \in B\), then

\[\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i, \quad \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1 \quad \text{and} \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1.\]

(C 3) For \(b, b' \in B\) and \(i \in I\), \(b' = \tilde{e}_i b\) if and only if \(b = \tilde{f}_i b'\).

(C 4) For \(b \in B\), if \(\varphi_i(b) = -\infty\), then \(\tilde{e}_i b = \tilde{f}_i b = 0\).

For two crystals \(B_1\) and \(B_2\), a morphism \(\psi\) from \(B_1\) to \(B_2\) is a map \(B_1 \to B_2 \sqcup \{0\}\) that satisfies the following conditions:
If $b \in B_1$ and $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$.

For $b \in B_1$, we have $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ provided $\psi(\tilde{e}_i b) \in B_2$.

For $b \in B_1$, we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ provided $\psi(\tilde{f}_i b) \in B_2$. A morphism $\psi : B_1 \to B_2$ is called strict, if it commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.

For two crystals $B_1$ and $B_2$, we define its tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{ b_1 \otimes b_2 ; b_1 \in B_1 \text{ and } b_2 \in B_2 \},$$

$$\varepsilon_i(b_1 \otimes b_2) = \max \{ \varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1) \},$$

$$\varphi_i(b_1 \otimes b_2) = \max \{ \varphi_i(b_1) + \mathrm{wt}_i(b_2), \varphi_i(b_2) \},$$

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Here $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}(b) \rangle$.

**Example 2.2.2** For $\lambda \in P_+, B(\lambda)$ is the crystal associated with the crystal base of the simple highest weight module with highest weight $\lambda$. For $b \in B(\lambda)$ we set $\varepsilon_i(b) = \max \{ k \geq 0 \mid \tilde{e}_i^k b \neq 0 \}$, $\varphi_i(b) = \max \{ k \geq 0 \mid \tilde{f}_i^k b \neq 0 \}$ and $\mathrm{wt}(b)$ is the weight of $b$. We denote by $b(\lambda)$ the unique element with weight $\lambda$.

**Example 2.2.3** $B(\infty)$ is the crystal associated with the crystal base of $U_q^{-}(\mathfrak{g})$. For $b \in B(\infty)$ we set $\varepsilon_i(b) = \max \{ k \geq 0 \mid \tilde{e}_i^k b \neq 0 \}$ and $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$.

**Example 2.2.4** Let $\lambda \in P_+$ be a dominant integral weight. Consider the set $T_\lambda = \{ t_\lambda \}$ with one element. Define $\mathrm{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ and $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$ for all $i \in I$. Then $T_\lambda$ is a crystal.
2.3

In this subsection we shall give a crystal-theoretical characterization of $B(\lambda)$. We will use this result in §4. Consider the tensor product of crystals $B(\infty) \otimes T_{\lambda}$. Let $\pi_{\lambda} : B(\infty) \otimes T_{\lambda} \to B(\lambda)$ be a strict morphism given by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_0 \otimes t_{\lambda} \mapsto \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b(\lambda)$. The morphism $\pi_{\lambda}$ has following properties:

\begin{equation}
\text{(2.3.1)}
\end{equation}

The set \{ $b \in B(\infty) \otimes T_{\lambda}$ | $\pi_{\lambda}(b) \neq 0$ \} is isomorphic to $B(\lambda)$ through $\pi_{\lambda}$.

**Proposition 2.3.1** Let $B$ be a crystal and $b_{\lambda}$ an element of $B$ with weight $\lambda \in P_+$. Assume the following conditions.

1. $b_{\lambda}$ is the unique element of $B$ with weight $\lambda$.
2. There is a strict morphism $\Phi : B(\infty) \otimes T_{\lambda} \to B$ such that $\Phi(b_0 \otimes t_{\lambda}) = b_{\lambda}$ and $\text{Im}\Phi = B \sqcup \{0\}$. Here $b_0$ is the unique element of $B(\infty)$ with weight zero.
3. Consider the set \{ $b \in B(\infty) \otimes T_{\lambda} | \Phi(b) \neq 0$ \}. Then it is isomorphic to $B$ through $\Phi$ as a set.
4. For any $b \in B$ and $i \in I$, $\varepsilon_i(b) = \max\{ k \geq 0 | \tilde{e}_i^k(b) \neq 0 \}$ and $\varphi_i(b) = \max\{ k \geq 0 | \tilde{f}_i^k(b) \neq 0 \}$.

Then $B$ is isomorphic to $B(\lambda)$.

**Proof.** By (1) and (2) any element of $B$ has the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_{\lambda}$ with $i_1, \cdots, i_l \in I$. The following lemma is a key of the proof of Proposition 2.3.1.

**Lemma 2.3.2** For a given $(i_1, \cdots, i_k) \in I^k$, the following two statements are equivalent.

1. $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b(\lambda) = 0$.
2. $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_{\lambda} = 0$.

**Proof of Lemma 2.3.2.** We shall only show that (a) implies (b). For $\nu \in Q_-$ let $B(\lambda)_{\lambda+\nu}$ (resp. $B_{\lambda+\nu}$) be the set of all elements of $B(\lambda)$ (resp. $B$) with weight $\lambda + \nu$. We shall prove the statement by the induction on $|\text{ht}(\nu)|$. Here $\text{ht}(\nu) = \sum_{i \in I} n_i \in \mathbb{Z}$ for $\nu = \sum_{i \in I} n_i \alpha_i \in Q$. If $|\text{ht}(\nu)| = 0$, the
statement is clear. Assume the statement holds for $|\text{ht}(\nu)| \leq l - 1$. Take $f_{i_2} \cdots f_{i_l}(b(\lambda)) \in B(\lambda)_{\lambda + \nu}$ with $|\text{ht}(\nu)| = l - 1$ such that $f_{i_1}(f_{i_2} \cdots f_{i_l}(b(\lambda))) = 0$. By the definition of $B(\lambda)$ we have $\varphi_{i_1}(f_{i_2} \cdots f_{i_l}(b(\lambda))) = 0$. Since $\pi_\lambda$ is strict, we have $f_{i_2} \cdots f_{i_l}(b_0 \otimes t_\lambda) \neq 0$ and $\varphi_{i_1}(f_{i_2} \cdots f_{i_l}(b_0 \otimes t_\lambda)) = 0$. By the conditions (2), (3) in Proposition 2.3.1 and the induction hypothesis we have

$$\Phi(f_{i_2} \cdots f_{i_l}(b_0 \otimes t_\lambda)) = f_{i_2} \cdots f_{i_l}(b_\lambda) \neq 0$$

and

$$\varphi_{i_1}(f_{i_2} \cdots f_{i_l}(b_0 \otimes t_\lambda)) = \varphi_{i_1}(\Phi(f_{i_2} \cdots f_{i_l}(b_0 \otimes t_\lambda))) = \varphi_{i_1}(f_{i_2} \cdots f_{i_l}(b_\lambda)) = 0.$$

By the condition (4) in Proposition 2.3.1 this means $f_{i_1}f_{i_2} \cdots f_{i_l}(b_\lambda) = 0$. Therefore the statement holds for $|\text{ht}(\nu)| = l$. □

Let us return to the proof of Proposition 2.3.1. From (3), (2.3.1) and Lemma 2.3.2, we can define a bijection $\psi : B(\lambda) \to B$ by $f_{i_1} \cdots f_{i_l}(b(\lambda)) \mapsto f_{i_2} \cdots f_{i_l}(b_\lambda)$. By the construction it is easy to see that $\psi$ is an isomorphism of crystals. □

3 Quiver varieties

3.1

Let $A = (a_{ij})$ be the Cartan matrix of $\mathfrak{g}$. We shall define an oriented graph $(I, H)$ associated with $A$ as follows.

Let $I$ be the set of vertices and $H$ the set of arrows of our graph. For $i, j \in I$ ($i \neq j$), there are $|a_{ij}|$ arrows from $i$ to $j$. Let $\text{out}(\tau)$ (resp. $\text{in}(\tau)$) be the outgoing (resp. incoming) vertex of $\tau \in H$. For $\tau \in H$, we denote by $\bar{\tau}$ the same edge as $\tau$ with the reverse orientation. The map $\bar{\cdot}$ is a fixed free involution of $H$. An orientation of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$, $\Omega \cap \bar{\Omega} = \phi$. We call our oriented graph $(I, H)$ a quiver.
Let $\mathcal{V}$ be the family of $I$-graded vector spaces $V = \oplus_{i \in I} V_i$. For $V \in \mathcal{V}$, the dimension of $V$ is defined to be the vector $\dim V = (\dim_{C} V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$. For $\nu \in \mathbb{Z}_{\geq 0}^I$ let $\mathcal{V}_\nu$ be the family of $I$-graded vector spaces $V$ with $\dim V = \nu$.

Consider another $I$-graded complex vector space $W$ with $\dim W = \lambda$ and fix it throughout this paper. From now on we regard the dimension vectors $\nu$ and $\lambda$ as elements of $P$ by the following way:

\[ \nu \mapsto -\sum_{i=1}^n \dim_{C} V_i \alpha_i, \quad \lambda \mapsto \sum_{i=1}^n \dim_{C} W_i \Lambda_i. \]

Here $\Lambda_i$ is the fundamental weight of $g$.

We define a complex vector space $X(W; \nu)$ by

\[ X(W; \nu) = (\bigoplus_{\tau \in H} \text{Hom}_{C}(V_{\text{out}(\tau)}, V_{\text{in}(\tau)}) \oplus (\bigoplus_{i \in I} \text{Hom}_{C}(V_i, W_i)) \oplus (\bigoplus_{i \in I} \text{Hom}_{C}(W_i, V_i)). \]

For an element of $X(W; \nu)$ we denote its components by $(B, t, s)$. We write $B, t, s$ for the collection $(B_\tau, t_i, s_i)$. We fix a function $\varepsilon : H \to \mathbb{C}^*$ such that $\varepsilon(\tau) + \varepsilon(\overline{\tau}) = 0$ for any $\tau \in H$. We define the symplectic form $\omega$ on $X(W; \nu)$ by

\[ \omega((B, t, s), (B', t', s')) = \sum_{\tau \in H} \text{tr}(\varepsilon(\tau)B_\tau B'_\tau) + \sum_{i=1}^n \text{tr}(s_it'_i - s'_it_i) \]

(3.2.1)

The algebraic group $G(\nu) = \prod_{i=1}^n GL(V_i)$ acts on $X(W; \nu)$ by

(3.2.2) \[ (B, t, s) \mapsto (g_{\text{out}(\tau)}B_\tau g_{\text{in}(\tau)}^{-1}, t_ig_i^{-1}, s_is_i) \]

where $g = (g_i) \in G(\nu)$. The action of $G(\nu)$ preserve the symplectic form $\omega$. Let $\mu : X(W; \nu) \to g(\nu)$ be the associated moment map. Its $i$-th component $\mu_i : X(W; \nu) \to \text{End}(V_i)$ is given by

\[ \mu_i((B, t, s)) = \sum_{\tau \in H, i = \text{out}(\tau)} \varepsilon(\tau)B_\tau B_i + s_it_i. \]

### 3.3

We recall basic properties of Nakajima’s quiver variety. The results of this subsection originally proved by Nakajima [15],[16].
Definition 3.3.1 We denoted by $X(W; \nu)^{st}$ the set of all elements $(B, t, s) \in X(W; \nu)$ satisfying the following property; if there is a family of subspaces $V'_i$ in $V_i$ such that they are $B$-invariant (i.e., for each $\tau \in H$, $B\tau(V'_{\text{out}(\tau)} \subset V'_{\text{in}(\tau)})$ and contained to $\text{Ker}(t_i)$ for each $i \in I$, then $V'_i = 0$ for each $i \in I$. We call an element of $X(W; \nu)^{st}$ a stable point of $X(W; \nu)$.

It is easy to see that $X(W; \nu)^{st}$ is an open subset of $X(W; \nu)$. Clearly $G(\nu)$ acts on $X(W; \nu)^{st}$.

Lemma 3.3.2 The action of $G(\nu)$ on $X(W; \nu)^{st}$ is fixed point free.

By this lemma we can consider the quotient variety of $X(W; \nu)^{st}$ by $G(\nu)$.

Proposition 3.3.3 Assume $X(W; \nu)^{st} \neq \phi$.

1. $\mathfrak{X}(W; \nu)$ is a quasi-projective smooth variety of dimension $\| \lambda \|^2 - \| \lambda + \nu \|^2$.

2. $\mathfrak{X}(W; \nu)$ has the symplectic structure induced by $\omega$.

4 Construction of crystal base

4.1

Let $\nu, \bar{\nu} \in Q_-$ such that $\nu - \bar{\nu} \in \mathbb{Z}_{\geq 0} \alpha_i$ for some $i \in I$. Assume that $V \in \mathcal{V}_\nu, \bar{V} \in \mathcal{V}_{\bar{\nu}}$. Now we consider the diagram

\[(4.1.1) \quad X(W; \bar{\nu}) \overset{q_1}{\leftarrow} X(W; \bar{\nu}, \nu) \overset{q_2}{\rightarrow} X(W; \nu).\]

Here $X(W; \bar{\nu}, \nu)$ is the variety of $(B, t, s, \phi)$ where $(B, t, s) \in X(W; \nu)$ and $\phi = (\phi_i) : V \rightarrow V$ is an injective morphism of $I$-graded vector spaces such that $\text{Im}\phi = (\text{Im}\phi_i)$ is $B$-stable and contains $\text{Im}s = (\text{Im}s_i)$. Hence $B, t, s$ induce $\bar{B} : \bar{V} \rightarrow \bar{V}, \bar{t}_i : \bar{V}_i \rightarrow W_i$ and $\bar{s}_i : \bar{W}_i \rightarrow \bar{V}_i$ respectively. We define $q_1(B, t, s, \phi) = (\bar{B}, \bar{t}, \bar{s})$ and $q_2(B, t, s, \phi) = (B, t, s)$. The following lemma is proved easily.
Lemma 4.1.1 We use the above notations.

(1) The following two conditions are equivalent;
   (a) \((B, t, s)\) is a stable point.
   (b) \((\bar{B}, \bar{t}, \bar{s})\) is a stable point and the map
        \[ V_i \xrightarrow{(B_r, t_i)} \bigoplus_{\tau; \out(\tau) = i} V_{\in(\tau)} \oplus W_i \]
        is injective.

(2) The following two conditions are equivalent;
   (a) \(\mu((B, t, s)) = 0\). (b) \(\mu((\bar{B}, \bar{t}, \bar{s})) = 0\).

By this lemma we can restrict the diagram to \(\mu^{-1}(0)\) and stable points and take its quotient by \(G(\nu), G(\bar{\nu})\). Then we get the diagram

\[ (4.1.2) \quad \mathfrak{X}(W; \bar{\nu}) \xleftarrow{\mathfrak{X}(W; \nu)} \mathfrak{X}(W; \nu) \xrightarrow{\mathfrak{X}(W; \nu)} \mathfrak{X}(W; \nu). \]

Here the variety of \([B, t, s], \text{Im}(\phi)\) where \([B, s, t] \in \mathfrak{X}(W; \nu)\) and \(\phi\) is given above.

4.2

For \(i \in I\) and \(c \in \mathbb{Z}_{\geq 0}\) we consider

\[ \mathfrak{X}(W; \nu)_{i,c} = \{ [B, t, s] \in \mathfrak{X}(W; \nu) | \varepsilon_i((B, t, s)) = c \} \]

where

\[ \varepsilon_i((B, t, s)) = \dim_{\mathbb{C}} \text{Coker}(\bigoplus_{\tau; \in(\tau) = i} V_{\out(\tau)} \oplus W_i \xrightarrow{(B_r, s_i)} V_i). \]

Since \([B, t, s]\) is a \(G(\nu)\)-orbit the above definition is well defined. It is clear that \(\mathfrak{X}(W; \nu)_{i,c}\) is a locally closed subvariety of \(\mathfrak{X}(W; \nu)\).

Lemma 4.2.1 Suppose \(\mathfrak{X}(W; \nu)_{i,c} \neq \phi\). Then,

(1) \(c + \langle h_i, \lambda + \nu \rangle \geq 0\).
(2) \(\mathfrak{X}(W; \nu - l\alpha_i)_{i,c+l} \neq \phi\) if and only if \(-c \leq l \leq c + \langle h_i, \lambda + \nu \rangle\).
Proof. First we shall show (1). Let \([B, t, s] \in {\mathcal{X}}(W; \nu)_{i,c}\). We consider the following diagram:

\[
V_i \xrightarrow{(e(\tau)B_r, t_i)} \bigoplus_{\tau; \text{out}(\tau) = i} V_{\text{in}(\tau)} \oplus W_i \xrightarrow{(B_r, s_i)} V_i.
\]

Since \((B, t, s)\) is a stable point the first map is injective. On the other hand \((B, t, s) \in \mu^{-1}(0)\) implies that the composition of these two maps equal to zero. Therefore we have

\[
dim_c V_i \leq \dim_c \text{Coker}((B_r, s_i)) - \dim_c V_i + \sum_{\tau; \text{out}(\tau) = i} V_{\text{in}(\tau)} + \dim_c W_i.
\]

Rewriting the above inequality, we get the statement.

Let us prove the sufficient part of (2). Assume \(\mathcal{X}(W; \nu - l\alpha_i)_{i,c+l} \neq \phi\). By (1) we have \(c + l + \langle h_i, \lambda + \nu - l\alpha_i \rangle \geq 0\). Therefore we have \(c + \langle h_i, \lambda + \nu \rangle \geq l\).

By the definition of \(\varepsilon_i\) we have \(c + l \geq 0\). We get the statement.

We shall show the necessary part of (2). Let \([B, t, s] \in {\mathcal{X}}(W; \nu)_{i,c}\). First we assume \(-c \leq l \leq 0\) and let \(\bar{\nu} = \nu - l\alpha_i\). We introduce \(\bar{V} \in \mathcal{V}_{\bar{\nu}}\) in the following way. If \(j \neq i\), define \(\bar{V}_j = V_j\). \(\bar{V}_i\) is \(\langle h_i, \lambda + \bar{\nu} \rangle\)-dimensional subspace of \(V_i\) such that

\[
\text{Im}(\bigoplus_{\tau; \text{in}(\tau) = i} V_{\text{out}(\tau)} \oplus W_i \xrightarrow{(B_r, s_i)} V_i) \subset \bar{V}_i.
\]

Let \((\bar{B}, \bar{t}, \bar{s}) \in p_1p_2^{-1}(B, t, s)\). Then by Lemma 4.1.1 we have \((\bar{B}, \bar{t}, \bar{s})\) is a stable point and \((\bar{B}, \bar{t}, \bar{s}) \in \mu^{-1}(0)\). Moreover \(\varepsilon_i(\bar{B}, \bar{t}, \bar{s}) = c + l\). Therefore \([\bar{B}, \bar{t}, \bar{s}] \in {\mathcal{X}}(W; \bar{\nu})_{i,c+l}\). That is \({\mathcal{X}}(W; \bar{\nu})_{i,c+l} \neq \phi\).

Assume \(0 \leq l \leq c + \langle h_i, \lambda + \nu \rangle\). Let \(\nu' = \nu - l\alpha_i\). We introduce \(V' \in \mathcal{V}_{\nu'}\) in the following way. If \(j \neq i\) define \(V'_j = V_j\). \(V'_i\) is \((l + \dim_c \bar{V}_i)\)-dimensional subspace of \(\text{Ker}(\bigoplus_{\tau; \text{in}(\tau) = i} V_{\text{out}(\tau)} \oplus W_i \xrightarrow{(B_r, s_i)} V_i)\). Note that the dimension of the kernel is \(\langle h_i, \lambda + \nu \rangle + c + \dim_c \bar{V}_i\). Therefore such \(V'\) exists. Let \((B', t', s') \in p_2p_1^{-1}(B, t, s)\) such that the map

\[
V_i' \xrightarrow{(B', t', s')} \bigoplus_{\tau; \text{out}(\tau) = i} V_{\text{in}(\tau)} \oplus W_i
\]

is injective. By the definition of \(V'\) such \((B', t', s')\) exists. By Lemma 4.1.1 and (4.2.1) we have \((B', t', s')\) is a stable point and \(\mu^{-1}(B', t', s') = 0\).
From the construction we have $\varepsilon_i(B', t', s') = c + l$. Therefore we conclude $[B', t', s'] \in \mathcal{X}(W; \nu')_{i,c+l}$. That is $\mathcal{X}(W; \nu')_{i,c+l} \neq \emptyset$.

Assume $\mathcal{X}(W; \nu)_{i,c} \neq \emptyset$. By the above lemma we have $\mathcal{X}(W; \bar{\nu})_{i,0} \neq \emptyset$ for $\bar{\nu} = \nu + c_\alpha_i$. Moreover, from the definition of the diagram (4.1.2), we have

$$\omega_1^{-1}(\mathcal{X}(W; \bar{\nu})_{i,0}) = \omega_2^{-1}(\mathcal{X}(W; \nu)_{i,c}).$$

We set

$$\mathcal{X}(W; \bar{\nu}, \nu)_{i,0} = \omega_1^{-1}(\mathcal{X}(W; \bar{\nu})_{i,0}) = \omega_2^{-1}(\mathcal{X}(W; \nu)_{i,c}).$$

Then we have the following diagram

$$(4.2.2). \quad \mathcal{X}(W; \bar{\nu})_{i,0} \leftarrow \mathcal{X}(W; \bar{\nu}, \nu)_{i,0} \rightarrow \mathcal{X}(W; \nu)_{i,c}$$

It is easy to see that the restriction of $\omega_2$ to $\mathcal{X}(W; \bar{\nu}, \nu)_{i,0}$ is an isomorphism and $\mathcal{X}(W; \bar{\nu})_{i,0}$ is a open subvariety of $\mathcal{X}(W; \bar{\nu})$.

**Lemma 4.2.2**

1. For any $i \in I$,

$$\mathcal{X}(W; 0)_{i,c} = \begin{cases} \{p.t.\}, & c = 0, \\ \phi, & c > 0. \end{cases}$$

2. Suppose $\mathcal{X}(W; \nu)_{i,c} \neq \emptyset$ and let $\bar{\nu} = \nu + c_\alpha_i$. Consider the restriction of $\omega_1$ to $\mathcal{X}(W; \bar{\nu})_{i,0}$. Then the fiber of this map is isomorphic to $\text{Grass}_c(\mathbb{C}^{h_i \lambda+\phi})$. Here we denote by $\text{Grass}_c(\mathbb{C}^p)$ the Grassmanian variety of $c$-dimensional subspaces of $\mathbb{C}^p$.

**Proof.** We have (1) immediately from the definition. Let $[B, \bar{t}, \bar{s}] \in \mathcal{X}(W; \bar{\nu})_{i,0}$ and $[B, t, s, \phi] \in \mathcal{X}(W; \bar{\nu}, \nu)_{i,0}$ such that $q_1((B, t, s, \phi)) = (\bar{B}, \bar{t}, \bar{s})$. By the definition we have the following commutative diagram;

$$\begin{array}{cccc}
V_i & \xrightarrow{(\varepsilon(\tau)B_{(\tau),t_i})} & \bigoplus_{(\tau:\text{out}(\tau)=i)} V_{\text{in}(\tau)} & \xrightarrow{(B_{(\tau),s_i})} & \bar{V_i} \\
\downarrow \phi & & \downarrow \phi \oplus \text{id}_{W_i} & & \downarrow \phi \\
V_i & \xrightarrow{(\varepsilon(\tau)B_{(\tau),t_i})} & \bigoplus_{(\tau:\text{out}(\tau)=i)} V_{\text{in}(\tau)} & \xrightarrow{(B_{(\tau),s_i})} & V_i.
\end{array}$$

We remark that the second vertical map is an isomorphism. By the similar argument in the proof of lemma 4.2.1 we have the map $(\varepsilon(\tau)B_{(\tau),t_i})$ is injective.
and \( \sum \varepsilon(\tau)B_\tau s_i = 0 \). On the other hand the map \( \bar{B}, \bar{s} \) is subjective because \( [\bar{B}, \bar{t}, \bar{s}] \) is the element of \( \mathfrak{X}(W; \bar{\nu})_{i,0} \). Therefore, for given \([B, t, s, \phi] \in \mathfrak{X}(W; \bar{\nu}, \nu)_{i,0}\) such that
\[
\bar{V}_i \xrightarrow{\phi} V_i \xrightarrow{\forall} \text{Ker} \left( \bigoplus_{\tau : \text{out}(\tau) = i} V_{\text{in}(\tau)} \twoheadrightarrow \bar{V}_i \right).
\]
Hence we have \( \bar{\omega}_1^{-1}([\bar{B}, \bar{t}, \bar{s}]) \) is isomorphic to
\[
\text{Grass}_c(\mathbb{C}^{\dim C-W_i - \sum_{i \neq j}^{\dim C} Y_j - 2 \dim C V_i}) = \text{Grass}_c(\mathbb{C}(h_i, \lambda + \bar{\nu})).
\]

By Lemma 4.2.2 (2) we have the following corollary immediately.

**Corollary 4.2.3** Suppose \( \mathfrak{X}(W; \nu)_{i,c} \neq \phi \). There is one to one correspondence between the set of all irreducible components of \( \mathfrak{X}(W; \bar{\nu})_{i,0} \) and the set of all irreducible components of \( \mathfrak{X}(W; \nu)_{i,c} \).

### 4.3

An element of \( \bar{B} = (B_\tau) \) is said to be nilpotent if there exists an integer \( n \geq 2 \) such that the following holds; for any sequence \( \tau_1, \tau_2, \ldots, \tau_n \) in \( H \) such that \( \text{in}(\tau_1) = \text{out}(\tau_2), \text{in}(\tau_2) = \text{out}(\tau_3), \ldots, \text{in}(\tau_{n-1}) = \text{out}(\tau_n) \), the composition \( B_{\tau_n} \cdots B_{\tau_2}B_{\tau_1} : V_{\text{out}(\tau_1)} \rightarrow V_{\text{in}(\tau_n)} \) is zero. Let us define a subvariety \( \Lambda(W; \nu) \) of \( \mathfrak{X}(W; \nu) \) by
\[
\Lambda(W; \nu) = \{ [B, t, s] \in \mathfrak{X}(W; \nu) | s = 0 \text{ and } B \text{ is nilpotent} \}.
\]

The following is due to Nakajima [15].

**Proposition 4.3.1** \( \Lambda(W; \nu) \) is a Lagrangian subvariety of \( \mathfrak{X}(W; \nu) \).

Now we denote by \( B(W; \nu) \) the set of all irreducible components of \( \Lambda(W; \nu) \). Take \( \Lambda \in B(W; \nu) \). For a generic point \( [B, t, s] \) of \( \Lambda \) we define \( \varepsilon_i(\Lambda) = \varepsilon_i((B, t, s)) \). For \( c \in \mathbb{Z}_{\geq 0} \) we set \( B(W; \nu)_{i,c} \) the set of all elements of \( B(W; \nu) \) such that \( \varepsilon_i(\Lambda) = c \).

**Proposition 4.3.2**
\[
B(W; \bar{\nu})_{i,0} \cong B(W; \nu)_{i,c}.
\]

**Proof.** We use the notations of Lemma 4.1.1. It is easy to see that \( s = 0 \) if and only if \( \bar{s} = 0 \) and \( B \) is nilpotent if and only if \( \bar{B} \) is nilpotent. By these equivalences we can restrict diagram (4.1.2) to nilpotent elements and \( s = 0 \). Therefore we obtain the desired result from Corollary 4.2.3.
4.4

**Definition 4.4.1** Suppose that $\bar{\Lambda} \in B(W; \bar{\nu})_{i,0}$ corresponds to $\Lambda \in B(W; \nu)_{i,c}$ by the isomorphism in Proposition 4.3.2. Then we define maps $\tilde{f}_i^c : B(W; \nu)_{i,c} \rightarrow B(W; \nu)_{i,0}$ and $\tilde{e}_i^c : B(W; \nu)_{i,c} \rightarrow B(W; \nu)_{i,0}$ by

$$
\tilde{f}_i^c (\bar{\Lambda}) = \Lambda,
$$

$$
\tilde{e}_i^c (\Lambda) = \bar{\Lambda}.
$$

Furthermore we define the maps

$$
\tilde{e}_i, \tilde{f}_i : \bigsqcup_{\nu} B(W; \nu) \rightarrow \bigsqcup_{\nu} B(W; \nu) \sqcup \{0\}
$$

by

$$
\tilde{e}_i : B(W; \nu)_{i,c} \xrightarrow{\tilde{e}_i^c} B(W; \nu + c\alpha_i)_{i,0} \xrightarrow{\tilde{f}_i^c} B(W; \nu + \alpha_i)_{i,c-1},
$$

$$
\tilde{f}_i : B(W; \nu)_{i,c} \xrightarrow{\tilde{e}_i^c} B(W; \nu + c\alpha_i)_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(W; \nu - \alpha_i)_{i,c+1}
$$

We put $\tilde{e}_i (\Lambda) = 0$ for $\Lambda \in B(W; \nu)_{i,0}$ and $\tilde{f}_i (\Lambda) = 0$ for $\Lambda \in B(W; \nu)_{i,c}$ for $c < -\langle h_i, \lambda + \nu \rangle$.

Then the maps $\tilde{e}_i^c$ (resp. $\tilde{f}_i^c$) which is constructed in the definition may be considered as a $c$-th power of $\tilde{e}_i$ (resp. $\tilde{f}_i$). Let us define a map $wt : \bigsqcup_{\nu} B(W; \nu) \rightarrow P$ by $wt(\Lambda) = \lambda + \nu \in P$ for $\Lambda \in B(W; \nu)$. We set $\varphi_i(\Lambda) = \varepsilon(\Lambda) + \langle h_i, wt(\Lambda) \rangle$.

**Theorem 4.4.2** $\bigsqcup_{\nu} B(W; \nu)$ is a crystal in the sense of Definition 2.1.1.

**Proof.** It is enough to see that the axioms of the crystal are satisfied. But it is obvious from the definition. $\square$

**Lemma 4.4.3** For any $\Lambda \in \bigsqcup_{\nu} B(W; \nu)$, $\varphi_i(\Lambda) = \max\{k \geq 0 | \tilde{f}_i^k (\Lambda) \neq 0\}$.

**Proof.** Suppose $B(W; \nu) \neq \phi$ and $\Lambda \in B(W; \nu)_{i,c}$. By the definition $\varphi_i(\Lambda) = c + \langle h_i, \lambda + \nu \rangle$. By Lemma 4.2.1 $B(W; \nu - l\alpha_i)_{i,c+l} \neq \phi$ if and only if $-c < l \leq c + \langle h_i, \lambda + \nu \rangle$. Therefore we have the lemma. $\square$
4.5

We recall the result of [7]. See [7] for details.

Let $V \in \mathcal{V}_\nu$. Define

$$X(\nu) = ( \bigoplus_{\tau \in H} \text{Hom}_C(V_{\text{out}}(\tau), V_{\text{in}}(\tau)))$$

and the symplectic form $\omega$ on $X(\nu)$ by

$$\omega(B, B') = \sum_{\tau \in H} \text{tr}(\varepsilon(\tau)B\tau B).$$

The algebraic group $G(\nu)$ acts on $X(\nu)$ in the same way to (3.2.2). Let $\mu$ be the corresponding moment map. For $B \in X(\nu)$ we set

$$\varepsilon_i(B) = \dim_C \text{Coker}(\bigoplus_{\tau; \text{in}(\tau) = i} V_{\text{out}}(\tau) \xrightarrow{(B\tau)} V_i).$$

Let

$$\Lambda(\nu) = \{B \in X(\nu)| \mu(B) = 0 \text{ and } B \text{ is nilpotent}\}$$

and $B(\infty; \nu)$ the set of irreducible components of $\Lambda(\nu)$. For a generic point $B$ of $\Lambda \in B(\infty; \nu)$ we set $\varepsilon_i(\Lambda) = \varepsilon_i(B)$. We can define the operators

$$\tilde{e}_i : \bigsqcup_{\nu} B(\infty; \nu) \to \bigsqcup_{\nu} B(\infty; \nu) \sqcup \{0\}, \quad \tilde{f}_i : \bigsqcup_{\nu} B(\infty; \nu) \to \bigsqcup_{\nu} B(\infty; \nu)$$

by the similar way to Definition 4.4.1. Set $\text{wt}(\Lambda) = \nu$ and $\varphi_i(\Lambda) = \varepsilon_i(\Lambda) + \langle h_i, \text{wt}(\Lambda) \rangle$ for $\Lambda \in B(\infty; \nu)$.

**Theorem 4.5.1**

(1) $\bigsqcup_{\nu} B(\infty; \nu)$ is a crystal.

(2) $\bigsqcup_{\nu} B(\infty; \nu)$ is isomorphic to $B(\infty)$.

4.6

From Proposition 4.3.1 we have the following statement.

**Lemma 4.6.1** $B(W; \nu)$ is isomorphic to the set of irreducible $G(\nu)$-invariant Lagrangian subvarieties of $X(W; \nu)^{st}$.
Define
\[ \tilde{\Lambda}(W; \nu) = \{(B, t, s) \in X(W; \nu) | s = 0 \text{ and } B \text{ is nilpotent}\} \]
and let \( \tilde{B}(W; \nu) \) be the set of all irreducible components of \( \tilde{\Lambda}(W; \nu) \). From Lemma 4.6.1 there exist an injective map
\[ \iota : B(W; \nu) \rightarrow \tilde{B}(W; \nu). \]
We introduce the map \( \kappa : \tilde{\Lambda}(W; \nu) \rightarrow \Lambda(\nu) \) by \( (B, t, s) \mapsto B \). It is easy to see that \( \kappa \) induce the following isomorphism;
\[ \kappa : \tilde{B}(W; \nu) \cong B(\infty; \nu). \]
Therefore we have the following.

Lemma 4.6.2 There is an injective map \( \kappa \circ \iota : B(W; \nu) \rightarrow B(\infty; \nu) \).

Let us introduce the map \( \Psi : T_\lambda \otimes \bigsqcup_\nu B(\infty; \nu) \rightarrow \bigsqcup_\nu B(W; \nu) \sqcup \{0\} \) by
\[
\Psi(t_\lambda \otimes \Lambda) = \begin{cases} 
(\kappa \circ \iota)^{-1}(\Lambda), & \text{if } \Lambda \in \text{Im}(\kappa \circ \iota), \\
0, & \text{otherwise.}
\end{cases}
\]

Lemma 4.6.3 \( \Psi \) is a strict morphism of crystal with the following properties.

1. \( \text{Im}\Phi = \bigsqcup_\nu B(W; \nu) \sqcup \{0\} \).
2. The set \( \{t_\lambda \otimes \Lambda \in (T_\lambda \otimes \bigsqcup_\nu B(\infty; \nu)) \mid \Psi(t_\lambda \otimes \Lambda) \neq 0\} \) is isomorphic to \( B(\lambda) \) through \( \Phi \).

Proof. By the definition of \( \Phi \) (1) and (2) are clear. Therefore it is enough to show that \( \Phi \) is a strict morphism. From the construction \( \Phi \) preserves weights. Let \( \Lambda \in B(W; \nu) \) and \( [B, t, s] \in \Lambda \). Since \( s = 0 \) and \( \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \) the map \( \kappa \circ \iota \) preserves the values of \( \varepsilon_i \) and \( \varphi_i \). Therefore \( \Phi \) also preserves them. By the definition \( \kappa \circ \iota \) commutes with \( \tilde{e}_i \) and \( \tilde{f}_i \). This means \( \Phi \) commutes with them.

Theorem 4.6.4 \( \bigsqcup_\nu B(W; \nu) \) is isomorphic to \( B(\lambda) \) as a crystal.

Proof. By Theorem 4.5.1 we can consider \( \Phi \) is a strict morphism from \( T_\lambda \otimes B(\infty) \) to \( \bigsqcup_\nu B(W; \nu) \). It is enough to see that the conditions in Proposition 2.3.1 are satisfied. (1) follows from the definition. (2) and (4) are just the above lemma. (3) is already proved in Lemma 4.4.3.
Recall that $\Lambda(W; \nu)$ is a Lagrangian subvariety of $\mathfrak{X}(W; \nu)$. It is homotopic to $\mathfrak{X}(W; \nu)$ (See [16].), hence the top homology group of $\Lambda(W; \nu)$ is isomorphic to the middle homology group of $\mathfrak{X}(W; \nu)$. Since the set of all irreducible components of $\Lambda(W; \nu)$ is isomorphic to $B(\lambda)_{\lambda+\nu}$, we have

$$\dim H_{\text{middle}}(\mathfrak{X}(W; \nu)) = \sharp B(\lambda)_{\lambda+\nu}.$$ 

Here $\nu = -\sum \dim V_i \alpha_i \in Q_-$ and $B(\lambda)_{\lambda+\nu}$ is the set of all elements of $B(\lambda)$ with weight $\lambda + \nu$. As an application we can compute the dimension of the middle homology groups of quiver varieties by using Kac’s character formula [2]. That is,

$$\sum_{\nu \in Q_-} \dim H_{\text{middle}}(\mathfrak{X}(W; \nu)) e^{\lambda+\nu} = \sum_{w \in W} \frac{(-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})_{\text{mult}\alpha}}.$$ 

Here $W$ is the Weyl group of $\mathfrak{g}$, $l(w)$ the length of $w \in W$, $\rho$ a Weyl vector, $\Delta_+$ the set of all positive roots and $\text{mult}\alpha$ the multiplicity of a root $\alpha$. This result was already given by Lusztig and Nakajima. (See [12] and [16].) We can reprove it in a crystal-theoretical way.

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