Generation of orbital-angular-momentum entangled biphotons in twisted nonlinear waveguides

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Abstract. We describe the generation of photon pairs entangled in optical angular momentum (OAM) through spontaneous four-wave mixing (SFWM) in circular arrays of nonlinear waveguides. We show that the incorporation of the array twist along the waveguides can enable robust generation even in presence of defects and inhomogeneities. We present analytical results and numerical simulations based on coupled-mode equations for the classical pump field and a discrete Schrödinger equation for the biphoton wave function.

1. Introduction
Quantum biphoton states with orbital-angular-momentum (OAM) entanglement is one of the most interesting phenomena for application in quantum communications, because they can be used to extend the alphabet of quantum cryptography, which can increase the transmission rate [1]. Also of interest are quantum walks [2], where interference of several walkers can be used to realize various simulations, including quantum teleportation [3,4] and quantum cryptography [5-7], enabling absolutely secure communications.

Quantum walks and generation of biphoton states with OAM in nonlinear straight arrays of waveguides through the effect of the SRS has been discussed in detail and analyzed in paper of Markin et al.[8].

Quantum walks and generation of photon pairs with OAM can be implemented in arrays of coupled waveguides with closed-loop boundary conditions [9-11]. However, in this paper the same OAM of pump photons can produce two different OAM states of biphotons, which may lead to an increase of errors number in quantum cryptography, based on such system. Recently, in a photonic-crystal fiber were demonstrated a new type of coupled waveguides with a twisted geometry [12], which are able to effectively provide the transfer of OAM states. The behavior and generation of biphotons in twisted waveguides can be described by the Schrödinger equation. Also, evolution of the pump profile in such waveguides can be described by the classical coupled-mode equations.

In this paper, we investigate the generation and distribution of photon pairs in nonlinear twisted waveguide arrays, as shown schematically in Fig. 1. In this case signal and idler photons will be strictly defined OAM according to the OAM of pump photons. Also, twisted waveguides allow to control the correlation functions of biphotons at different distances of propagation, depending on the twist angle of waveguides, that could have important applications in quantum cryptography and teleportation.
We consider a linear polarization of light and small twist of waveguides, because in this case the structure of the pump and biphoton modes is essentially the same as in the straight array.

Figure 1. Scheme of twisted circular array of coupled waveguides. Pairs of signal and idler photons are generated through spontaneous four-wave mixing from the pump.

2. Distribution and generation of photon pairs in twisted nonlinear waveguide arrays

We consider an array of closely spaced optical waveguides, which are twisted around a central axis along the propagation direction. Such structure composed of three waveguides is schematically shown in Fig. 1(a).

The normalized pump field profile evolution along the propagation distance \( z \) is defined through the classical coupled-mode equations [8]:

\[
i \frac{dE_n}{dz} + C(\omega)(\exp[-i\Delta \phi]E_{n+1} + \exp[i\Delta \phi]E_{n-1}) + \rho_{np} E_n = 0,
\]

where \( E_n \) is the complex field amplitude in the \( n \)th waveguide, and \( E_N \) maps to \( E_0 \), and \( E_{-1} \) maps to \( E_{N-1} \) due to closed-loop boundary conditions [8], \( n \) is the waveguide number, \( N \) is the total number of waveguides, \( \Delta \phi = \omega d\chi/dz \) is the additional phase, \( \omega \) - frequency, \( d\chi/dz \) - is the periodic waveguide bending profile, \( \chi(z) = \chi(z + L) \), \( L \) is the modulation period, coefficient \( C(\omega) \) defines a coupling strength between the neighboring waveguides (it characterizes diffraction strength in a straight waveguide array with \( C_\chi = 0 \) [9], and \( \rho_{np} \) is the mismatch between the waveguides. In our case \( d\chi/dz = \text{const} \) because waveguide bending profile is the same to all values of \( z \) and mismatch is equal to 0.

Generation of photon pairs in cubic nonlinear WGAs through SFWM in the absence of multiple photon pairs can be characterized by the evolution of a bi-photon wave function \( \psi_{s,ni}(z) \) in a Schrödinger-type equation. The equation is obtained from the Hamiltonian, and it has a form similar to that of quadratic media [8]:

\[
i \frac{d\psi_{n_s,n_i}}{dz} = -C(\exp[i\Delta \phi]\psi_{n_s-1,n_i} + \exp[i\Delta \phi]\psi_{n_s,n_i-1} + \exp[-i\Delta \phi]\psi_{n_s+1,n_i} + \\
+ \exp[-i\Delta \phi]\psi_{n_s,n_i+1}) + i\gamma g_{s,n_i} E_{n_s}^{(p)}(z)E_{n_i}^{(p)}(z)\delta_{n_s,n_i} \exp[i\Delta \beta^{(0)} z] + (\rho_{s,n_i}^{(x)} + \rho_{n_i}^{(i)})\psi_{n_s,n_i},
\]

where \( n_s \) and \( n_i \) are the waveguide numbers describing the positions of the signal, and the idler photons, and \( E_{n_i}^{(p)}(z) \) is the pump amplitude in waveguide number \( n_s \). \( \Delta \beta \) - the linear four-wave
mixing phase-mismatch in a single waveguide, \( \gamma \) - is a nonlinear coefficient, \( \delta \) - Kronecker delta, and \( \rho_n^{(s)}, \rho_n^{(i)} \) is the mismatch between the waveguides [7]. In our case mismatch is equal to 0.

3. **Analysis of photon propagation modes depending on the input state of the pump**

A pump beam with a particular OAM will generate only pairs of signal and idler photons which satisfy the angular momentum conservation expressed through the Kronecker delta function. We illustrate this relation by summarizing in Table I the possible states of generated biphotons for different OAM in a three-waveguide structure. In Table I, the first column represents the OAM of a pump beam coupled to the array, and the second column shows the corresponding pump propagation constants. The third column lists the possible OAM states of the generated biphotons, and the fourth column presents the corresponding biphoton propagation constants. For clarity, we explicitly write down the equivalence of pump states due to the discrete periodic symmetry. It then becomes easy to see how the selection rules work based on the OAM conservation for the generated biphotons: \( m_s + m_i = 2m_p \) up to multiples of N (number of waveguides). We see that two different momentum states of biphotons can be generated for each pump OAM. Further selectivity in excitation of particular photon states can be achieved by adjusting the phase mismatch.

| Pump state \( |m_p\rangle \) | Propagation constant \( \beta_{m_p} \) | Biphoton state \( |m_s, m_i\rangle \) | Propagation constant \( \beta_{m_s, m_i} \) |
|----------------------|----------------------|----------------------|----------------------|
| \( |0\rangle \) | \( 2C \cos \varphi_0 \) | \( |0, 0\rangle \) | \( 4C \cos \varphi_0 \) |
| \( |{-2}\rangle \equiv |1\rangle \) | \( -C \left[ \cos(\varphi_0) - \sqrt{3} \sin(\varphi_0) \right] \) | \( |0, -1\rangle + |1, 0\rangle \) | \( C \left[ \cos(\varphi_0) - \sqrt{3} \sin(\varphi_0) \right] \) |
| \( |2\rangle \equiv |-1\rangle \) | \( -C \left[ \cos(\varphi_0) + \sqrt{3} \sin(\varphi_0) \right] \) | \( |0, 1\rangle + |1, 0\rangle \) | \( C \left[ \cos(\varphi_0) + \sqrt{3} \sin(\varphi_0) \right] \) |

At the Fig. 2 we present the biphoton correlations \( \left| \Psi_{m_s, m_i} \right|^2 \), in case of photon generation by SFWM from a pump inside the straight array (1st and 2nd columns, \( d\chi/dz = 0 \)) and twisted array (3rd and 4th columns, \( d\chi/dz = 1 \)) for different distances: at 1st string \( z = 1.05 \); at 2nd string \( z = 2.1 \); at 3rd string \( z = 3.15 \). 1st and 3rd columns corresponds the case without mismatch (\( \rho = 0 \)). 2nd and 4th columns corresponds the case with the mismatch (\( \rho = 0.5 \)). At \( z = 0 \) there are no photons. \( |m_p\rangle = 2 \).
There are strong anti-bunching behavior of photons in the Fig. 2 (intersection of 1st column and 2nd string), which corresponds the violating of Bell-inequality. But in the Fig. 2 (intersection of 2nd column and 2nd string) anti-bunching behavior of photons is vanished because of mismatch. Similarly, at the Fig. 2 (3rd and 4th columns) the behavior of photons is conserved in all the distances, in despite of the mismatch. But strength of correlations may be smaller in the case of twisted waveguides.

**Figure 2.** Biphoto correlations $|\Psi_{n,m}|^2$, in case of photon generation by SFWM from a pump inside the straight array (1st and 2nd columns, $d\chi/dz = 0$) and twisted array (3rd and 4th columns, $d\chi/dz = 1$) for different distances: at 1st string $z = 1.05$; at 2nd string $z = 2.1$; at 3rd string $z = 3.15$. 1st and 3rd columns corresponds the case without mismatch ($\rho = 0$). 2nd and 4th columns corresponds the case with the mismatch ($\rho = 0.5$). At $z = 0$ there are no photons. $\langle m_p \rangle = 2$.

Thus, the twist of waveguides can effectively provide the generation of photons with certain OAM and its propagation even through the array with the mismatch.

4. **Conclusion**

Building on a recent experimental demonstration of classical OAM state transmission through twisted multi-core photonic-crystal fibers, we predict that the correlation properties of entangled photon pairs propagating in the regime of quantum walks in closed-loop waveguide arrays can be controlled by the introduction of twist. We also analyze integrated photon generation through spontaneous four-wave mixing and predict that the input pump profile also can control the features of correlation properties of entangled photons. We also show that in both cases photons can demonstrate bunching behavior at some distances and anti-bunching behavior at other distances. These features can be controlled by the amount of twist and the input pump profile.

Thus, our results suggest a potential for developing quantum communications for cryptography and other applications by combining robust photon-pair generation in twisted structures and their subsequent transmission through twisted optical fibers.
5. Acknowledgements
This work was financially supported by the Government of Russian Federation, Grant 074-U01.

References
[1] Hamilton C S, Kruse R, Sansoni L, Silberhorn C, and Jex I 2014 Driven Quantum Walks. *Phys. Rev. Lett.* **113**, 083602-5
[2] Owens J O et al. 2011 Two-photon quantum walks in an elliptical direct-write waveguide array. *New J. Phys.* **13**, 075003-13
[3] Garanovich I L, Longhi S, Sukhorukov A A, and Kivshar Yu S 2012 Light propagation and localization in modulated photonic lattices and waveguides. *Phys. Rep.* **518**, 1-79
[4] Ekert A K, Rarity J G, Tapster P R, and Palma G M 1992 Practical quantum cryptography based on two-photon interferometry. *Phys. Rev. Lett.* **69**, 1293
[5] D N Vavulin, A A Sukhorukov 2015 Quantum walks of photon pairs in twisted waveguide arrays *Journal of Physics: Conference Series* **643**, 012050
[6] Egorov V I, Vavulin D N, Latypov I Z, Gleim A V, Rupasov A V 2013 Analysis of a sidebands based quantum cryptography system with different detector types *NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS*, **4**(2), pp. 190–195
[7] D N Vavulin, V I Egorov, A V Gleim, S A Chivilikhin 2014 Determining influence of four-wave mixing effect on quantum key distribution *Journal of Physics: Conference Series* **541**, 012066
[8] Markin D M, Solntsev A S, and Sukhorukov A A 2013 Generation of orbital-angular-momentum entangled biphotons in triangular quadratic waveguide arrays. *Phys. Rev. A* **87**, 063814-5
[9] Christodoulides D N, Lederer F, and Silberberg Y 2003 Discretizing light behaviour in linear and nonlinear waveguide lattices *Nature* **424**, 817–823
[10] Solntsev A S, Sukhorukov A A, Neshev D N, and Kivshar Yu S 2012 Photon-pair generation in arrays of cubic nonlinear waveguides. *OPTICS EXPRESS* **20**, No. 24, 27441
[11] Jonathan C F, M and M G Thompson 2012 Quantum optics: An entangled walk of photons. *Nature* **484**, 47–48
[12] Xi X M, Wong G K L, Frosz M H, Babic F, Ahmed G, Jiang X, Euser T G, and Russell P St.J 2014 Orbital-angular-momentum-preserving helical Bloch modes in twisted photonic crystal fiber. *Optica* **1**, 165-169