The aim of this paper is to prove an inequality between relative entropy and the sum of average conditional relative entropies of the following form: For a fixed probability measure $q^n$ on $\mathcal{X}^n$, ($\mathcal{X}$ is a finite set), and any probability measure $p^n = \mathcal{L}(Y^n)$ on $\mathcal{X}^n$

$$D(p^n||q^n) \leq \text{Const.} \sum_{i=1}^{n} \mathbb{E}_{p^n} D(p_i(\cdot|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)||q_i(\cdot|Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)),$$

(*)

where $p_i(\cdot|y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ and $q_i(\cdot|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ denote the local specifications for $p^n$ resp. $q^n$, i.e., the conditional distributions of the $i$'th coordinate, given the other coordinates. The constant shall depend on the properties of the local specifications of $q^n$.

The inequality (*) is meaningful in product spaces, both in the discrete and the continuous case, and can be used to prove a logarithmic Sobolev inequality for $q^n$, provided uniform logarithmic Sobolev inequalities are available for $q_i(\cdot|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, for all fixed $i$ and all fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

(*) directly implies that the Gibbs sampler associated with $q^n$ is a contraction for relative entropy.

In this paper we derive inequality (*), and thereby a logarithmic Sobolev inequality, in discrete product spaces, by proving inequalities for an appropriate Wasserstein-like distance.

A logarithmic Sobolev inequality is, roughly speaking, a contractivity property of relative entropy with respect to some Markov semigroup. It is much easier to prove contractivity for a distance between measures, than for relative entropy, since distances satisfy the triangle inequality, and for them well known linear tools, like estimates through matrix norms can be applied.
1. Introduction and statement of some results.

Let $\mathcal{X}$ be a finite set, and $\mathcal{X}^n$ the set of $n$-length sequences from $\mathcal{X}$. Denote by $\mathcal{P}(\mathcal{X}^n)$ the space of probability measures on $\mathcal{X}^n$. For a sequence $x^n \in \mathcal{X}^n$ we denote by $x_i$ the $i$-th coordinate of $x^n$.

We consider a reference probability measure $q^n \in \mathcal{P}(\mathcal{X}^n)$ which will be fixed throughout Sections 1-3. In section 4 we still consider a fixed probability measure denoted by $q$, with some subscript.

The aim of this paper is to prove logarithmic Sobolev inequalities for measures on discrete product spaces, by proving inequalities for an appropriate Wasserstein-like distance. A logarithmic Sobolev inequality is, roughly speaking, a contractivity property of relative entropy with respect to some Markov semigroup. It is much easier to prove contractivity for a distance between measures, than for relative entropy, since a distance is symmetric and satisfies the triangle inequality. Our method shall be used to prove logarithmic Sobolev inequalities for measures satisfying a version of Dobrushin’s uniqueness condition, as well as Gibbs measures satisfying a strong mixing condition.

To explain the results, we need some definitions and some notation.

**Notation.** If $r$ and $s$ are two probability measures (on any measurable space) then we denote by $|r - s|$ their variational distance:

$$|r - s| = \sup_{A} |r(A) - s(A)|.$$

**Definition: $W_2$ distance.** (c.f. [B-L-M], Theorem 8.2)

For probability measures $r^n, s^n \in \mathcal{P}(\mathcal{X}^n)$ let $Z^n$ and $U^n$ represent $r^n$ resp. $s^n$, i.e., $Z^n$ and $U^n$ are random variables with distributions $\mathcal{L}(Z^n) = r^n$ and $\mathcal{L}(U^n) = s^n$, respectively. We define

$$W_2(r^n, s^n) = \min_{\pi} \sqrt{\sum_{i=1}^{n} Pr^2_{\pi}\{Z_i \neq U_i\}},$$

where the minimum is taken over all joint distributions $\pi = \mathcal{L}(Z^n, U^n)$ with marginals $r^n$ and $s^n$.

Note that $W_2$ is a distance on $\mathcal{P}(\mathcal{X}^n)$, but it cannot be defined by taking the minimum expectation of a distance (or some power of a distance) on $\mathcal{X}^n$. 
**Definition: Relative entropy, conditional relative entropy.** For probability measures \( r \) and \( s \) defined on a finite set \( Z \), we denote by \( D(r || s) \) the relative entropy of \( r \) with respect to \( s \):
\[
D(r || s) = \sum_{u \in Z} r(u) \log \frac{r(u)}{s(u)},
\]
with the convention \( 0 \log 0 = 0 \) and \( a \log 0 = \infty \) for \( a > 0 \). If \( Z \) and \( U \) are random variables with values in \( Z \) and distributed according to \( r = \mathcal{L}(Z) \) resp. \( s = \mathcal{L}(U) \), then we shall also use the notation \( D(Z || U) \) for the relative entropy \( D(r || s) \). If, moreover, we are given a probability measure \( \pi = \mathcal{L}(S) \) on another finite set \( S \), and conditional distributions \( \mu(\cdot | s) = \mathcal{L}(Z | S = s) \), \( \nu(\cdot | s) = \mathcal{L}(U | S = s) \) then we consider the average relative entropy
\[
\mathbb{E}_\pi D(\mu(\cdot | S) || \nu(\cdot | S)) = \sum_{s \in S} \pi(s) D(\mu(\cdot | s) || \nu(\cdot | s)).
\]
For \( \mathbb{E}_\pi D(\mu(\cdot | S) || \nu(\cdot | S)) \) we shall use either of the notations
\[
D(\mu(\cdot | S) || \nu(\cdot | S)), \quad D(\mu(\cdot | S) || U | S), \quad D(Z | S) || \nu(\cdot | S)), \quad D(Z | S) || U | S))
\]
(omitting the symbol of expectation as is usual in information theory).

**Notation.**
For \( y^n = (y_1, y_2, \ldots, y_n) \in \mathcal{X}^n \) and \( I \subset [1, n] \), we write
\[
y_I = (y_k : k \in I) \quad \text{and} \quad \bar{y}_I = (y_k : k \notin I).
\]
Moreover, if \( p^n = \mathcal{L}(Y^n) \) then
\[
p_I \triangleq \mathcal{L}(Y_I), \quad p_I(\cdot | \bar{y}_I) \triangleq \mathcal{L}(Y_I | \bar{Y}_I = \bar{y}_I), \quad \bar{p}_I \triangleq \mathcal{L}(\bar{Y}_I), \quad \bar{p}_I(\cdot | y_I) \triangleq \mathcal{L}(\bar{Y}_I | Y_I = y_I).
\]
If \( I = \{i\} \) then we write \( i \) instead of \( \{i\} \).

**Definition.** The conditional distributions \( q_i(\cdot | \bar{x}_i) \) are called the local specifications of the distribution \( q^n \).

**Theorem 1.**
Set
\[
\alpha \triangleq \min q_i(x_i | \bar{x}_i), \quad (1.1)
\]
where the minimum is taken over all \( x^n \in \mathcal{X}^n \) satisfying \( q(x^n) > 0 \) and all \( i \in [1, n] \). Fix a \( p^n = \mathcal{L}(Y^n) \in \mathcal{P}(\mathcal{X}^n) \) satisfying
\[
q^n(x^n) = 0 \quad \Rightarrow \quad p^n(x^n) = 0. \quad (1.2)
\]
Assume that $q^n \in \mathcal{P}(\mathcal{X}^n)$ satisfies all the inequalities

$$W_2^2(p_I(\cdot | \bar{y}_I), q_I(\cdot | \bar{y}_I)) \leq C \cdot \mathbb{E} \left\{ \sum_{i \in I} \left| p_i(\cdot | \bar{Y}_i) - q_i(\cdot | \bar{Y}_i) \right|^2 \left| \bar{Y}_I = \bar{y}_I \right\} \right.,$$

(1.3)

where $I \subset [1, n]$ and $\bar{y}_I \in \mathcal{X}^{[1,n] \setminus I}$ is a fixed sequence. Then

$$D(p^n||q^n) \leq \frac{4C}{\alpha} \cdot \sum_{i=1}^{n} \mathbb{E} \left| p_i(\cdot | \bar{Y}_i) - Q_i(\cdot | \bar{Y}_i) \right|^2$$

$$\leq \frac{2C}{\alpha} \cdot \sum_{i=1}^{n} D(Y_i|\bar{Y}_i||Q_i(\cdot | \bar{Y}_i)).$$

(1.4)

(Condition (1.2) is necessary, since otherwise $D(p^n||q^n)$ could be $\infty$, while the middle term is always finite. On the other hand, for the inequality between the first and last terms it is not necessary to assume (1.2), since if $D(p^n||q^n) = \infty$ then the last term is $\infty$ as well.)

Remark. In [M] a bound, analogous to the one relating the first and last terms of (1.4), was proved for measures on Euclidean spaces. (Under reasonable conditions.) That bound was used to derive a logarithmic Sobolev inequality, improving on an earlier result in [O-R]. In the present paper a logarithmic Sobolev inequality shall be deduced from the first inequality in (1.4) (Corollary 2 to Theorem 1).

Theorem 1 implies that the Gibbs sampler (or Glauber dynamics) defined by the local specifications of $q^n$ is a strict contraction for relative entropy.

Definition: Gibbs sampler.

For $i \in [1, n]$ let $\Gamma_i : \mathcal{P}(\mathcal{X}^n) \mapsto \mathcal{P}(\mathcal{X}^n)$ be the Markov kernel

$$\Gamma_i(z^n|y^n) = \delta(\bar{y}_i, \bar{z}_i) \cdot q_i(z_i|\bar{y}_i), \quad y^n, z^n \in \mathcal{X}^n.$$

(I.e., $\Gamma_i$ leaves all, but the $i$-th, coordinates unchanged, and updates the $i$-th coordinate according to $q_i(y_i|\bar{y}_i)$.) Finally, set

$$\Gamma = \frac{1}{n} \cdot \sum_{i=1}^{n} \Gamma_i.$$

I.e., $\Gamma$ selects an $i \in [1, n]$ at random, and applies $\Gamma_i$. It is easy to see that $\Gamma$ preserves, and is reversible with respect to, $q^n$. $\Gamma$ is called the Gibbs sampler governed by the local specifications of $q^n$. 
Corollary 1 to Theorem 1.
If $q^n$ on $X^n$ satisfies the conditions of Theorem 1 then

$$D(p^n \Gamma || q^n) \leq \left(1 - \frac{\alpha}{2nC}\right) \cdot D(p^n || q^n).$$

(1.5)

(1.5) follows from Theorem 1 by the inequality

$$D(p^n \Gamma || q^n) \leq \frac{1}{n} \sum_{i=1}^{n} D(p^n \Gamma_i || q^n)$$

(a consequence of the convexity of relative entropy), together with the identity

$$D(p^n || q^n) - D(p^n \Gamma_i || q^n) = D(p_i(\cdot | \bar{Y}_i) || q_i(\cdot | \bar{Y}_i)).$$

Theorem 1 also implies Gross’ logarithmic Sobolev inequality defined as follows:

**Definition: logarithmic Sobolev inequality for a Markov kernel.**
Let $(Z, \pi)$ be a finite probability space, and $G : Z \mapsto Z$ a Markov kernel with invariant measure $\pi$. The Dirichlet form associated with $G$ is

$$\mathcal{E}_G(f, f) = \langle (I - G)f, f \rangle_\pi.$$

We say that $G$ satisfies a logarithmic Sobolev inequality with logarithmic Sobolev constant $c$ if: for every probability measure $p$ on $Z$ we have

$$D(p||\pi) \leq c \cdot \mathcal{E}_G(\sqrt{f}, \sqrt{f}),$$

where $f(z) = p(z)/\pi(z)$.

The property expressed by the logarithmic Sobolev inequality was defined by L. Gross [Gr] in 1975. For an introduction to logarithmic Sobolev inequalities and their manifold interpretations and uses, c.f. [L] and [R].

Theorem 1 implies Gross’ logarithmic Sobolev inequality for the Gibbs sampler $\Gamma$. A simple calculation shows that

$$\mathcal{E}_\Gamma \left( \sqrt{\frac{p^n}{q^n}}, \sqrt{\frac{p^n}{q^n}} \right) = \frac{1}{n} \cdot \mathbb{E} \sum_{i=1}^{n} \left( 1 - \left( \sum_{y_i \in X} \sqrt{p_i(y_i | \bar{Y}_i) \cdot q_i(y_i | \bar{Y}_i)} \right)^2 \right).$$

(Using the fact that, for fixed $\bar{y}_i$, the measure $p^n\Gamma_i$ does not depend on $y_i$, we just calculate the Dirichlet form for a matrix with identical rows.)
Corollary 2 to Theorem 1.
If $q^n$ on $\mathcal{X}^n$ satisfies the conditions of Theorem 1 then
\[
\frac{1}{n} \cdot D(p^n \Gamma || q^n) \leq \frac{4C}{\alpha n} \cdot \sum_{i=1}^{n} \mathbb{E} \left( 1 - \left( \sum_{y_i \in \mathcal{X}} \sqrt{p_i(y_i|\bar{Y}_i) \cdot q_i(y_i|\bar{Y}_i)} \right)^2 \right)
\]
\[
= \frac{4C}{\alpha} \cdot \mathcal{E}_\Gamma \left( \sqrt{\frac{p^n}{q^n}} : \sqrt{\frac{p^n}{q^n}} \right).
\]
This can be considered a dimension free logarithmic Sobolev inequality, since $\Gamma$ only updates one coordinate.

Corollary 2 follows from Theorem 1 by the following

Lemma 1. (The proof is in Appendix A)
Let $r$ and $s$ be two probability measures on $\mathcal{X}$. Then
\[
|r - s|^2 \leq 1 - \left( \sum_{y \in \mathcal{X}} \sqrt{r(y)s(y)} \right)^2.
\]

Theorem 1 can be applied to distributions $q^n$ satisfying the following version of Dobrushin’s uniqueness condition:

Definition: Dobrushin’s uniqueness condition.
We say that $q^n$ satisfies (an $\mathbb{L}_2$-version of) Dobrushin’s uniqueness condition with coupling matrix
\[
A = (a_{k,i})_{k,i=1}^n,
\]
if: for any pair of integers $k, i \in [1,n], k \neq i$ and any two sequences $z^n, s^n \in \mathcal{X}^n$, differing only in the $k$’th coordinate,
\[
|q_i(\cdot|\bar{z}_i) - q_i(\cdot|\bar{s}_i)| \leq a_{k,i},
\]
and, setting $a_{i,i} = 0$ for all $i$,
\[
||A||_2 < 1.
\]
This differs from Dobrushin’s original uniqueness condition where the norm $||A||_1$ is assumed to be $< 1$. 

Theorem 2.
Assume that the measure \( q^n \) on \( \mathcal{X}^n \) satisfies Dobrushin’s uniqueness condition with coupling matrix \( A, \|A\|_2 < 1 \). Then the conditions of Theorem 1 are satisfied with \( C = 1/(1 - \|A\|)^2 \). Thus for any \( p^n \in \mathcal{P}(\mathcal{X}^n) \), satisfying \( (1.3) \):

\[
D(p^n||q^n) \leq \frac{4}{\alpha} \cdot \frac{1}{(1 - \|A\|)^2} \cdot \sum_{i=1}^{n} \mathbb{E}[p_i(\cdot|\bar{Y}_i) - Q_i(\cdot|\bar{Y}_i)]^2
\]

\[
\leq \frac{2}{\alpha} \cdot \frac{1}{(1 - \|A\|)^2} \cdot \sum_{i=1}^{n} D(Y_i|\bar{Y}_i||Q_i(\cdot|\bar{Y}_i)), \tag{1.7}
\]

and

\[
D(p^n\Gamma||q^n) \leq \left(1 - \frac{1}{n} \cdot \frac{\alpha}{2} \cdot (1 - \|A\|)^2\right) \cdot D(p^n||q^n). \tag{1.8}
\]

Remark. In [Z] a logarithmic Sobolev inequality is proved for discrete spin systems, where the title suggests that it uses Dobrushin’s uniqueness condition. However, the condition used there is reminiscent but not identical to Dobrushin’s uniqueness condition. Moreover, an inequality of the form relating the first and last terms of \( (1.4) \) has been recently proved in [C-M-T], assuming conditions slightly reminiscent of Dobrushin’s uniqueness condition.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

In Section 4 we are going to deduce a logarithmic Sobolev inequality from a strong mixing condition, for measures \( q \) on \( \mathcal{X}^{2^d} \). (Under the additional condition that the local specifications \( q_k(x_k|x_i, i \neq k) \), if not equal to 0, are bounded from below.) The strong mixing condition we use is the same as Dobrushin and Shlosman’s strong mixing conditions, but we do not assume that \( q \) is a Markov field. Our strong mixing condition can also be considered as a generalization of \( \Phi \)-mixing for (stationary) probability measures on \( \mathcal{X}^{Z} \). For non-Markov stationary probability measures on \( \mathcal{X}^{Z} \) it is more restrictive than usual strong mixing.

The first proof for the implication that Dobrushin and Shlosman’s strong mixing conditions imply a logarithmic Sobolev inequality for Markov fields was given by D. Stroock and B. Zegarlinski [S-Z1], [S-Z2] in 1992 (where the authors also proved the converse implication, i.e. that Dobrushin and Shlosman’s strong mixing conditions for Markov fields are equivalent to the logarithmic Sobolev inequality). The arguments in [S-Z2] are quite hard to follow. In 2001, F. Cesi proved that Dobrushin and Shlosman’s strong mixing conditions imply a logarithmic Sobolev inequality; his approach is quite different from the previous ones, and much simpler.

We feel that there is still room for alternative and perhaps simpler proofs in this important topic. Moreover, our proof is valid without the Markovity assumption.
(It may be, though, that the proofs in [S-Z2] and [C] can also be generalized for the non-Markovian case, just it has not been tried.)

We believe that the separate parts of our proof (Theorem 1 and the applicability of Theorem 1) are comprehensible in themselves, thus making the whole proof easier to follow.

2. Proof of Theorem 1.

We need the following

Lemma 2.
Let \( r \) and \( s \) be two probability measures on \( X \). Set
\[
\alpha_s = \min_{s(x) \neq 0} s(x).
\]
If \( D(r||s) < \infty \) then
\[
D(r||s) \leq \frac{4}{\alpha_s} \cdot |r - s|^2.
\] (2.1)

Remark.
Inequality (2.1) can be considered as a converse to the Pinsker-Csiszár-Kullback inequality which says that
\[
|r - s|^2 \leq \frac{1}{2} D(r||s).
\]
However, there is no uniform converse: the reverse inequality must depend on \( s \).

Proof.
Set \( X_+ = \{ x \in X : s(x) > 0 \} \). The following inequality is well known:
\[
D(r||s) \leq \sum_{X_+} \frac{|r(x) - s(x)|^2}{s(x)}.
\]
It follows that
\[
D(r||s) \leq \frac{1}{\alpha_s} \cdot \sum_{X_+} |r(x) - s(x)|^2 \leq \frac{1}{\alpha_s} \left( \sum_X |r(x) - s(x)| \right)^2 = \frac{4}{\alpha_s} \cdot |r - s|^2.
\]

We proceed to the proof of Theorem 1. Let \( \pi = \mathcal{L}(Y^n, X^n) \) be a coupling of \( p^n = \mathcal{L}(Y^n) \) and \( q^n = \mathcal{L}(X^n) \) that achieves \( W_2(p^n, q^n) \).

We apply induction on \( n \). Assume that the theorem holds for \( n - 1 \).
By the expansion formula for relative entropy we have

\[ D(p^n||q^n) = \frac{1}{n} \cdot \sum_{i=1}^{n} D(Y_i||X_i) + \frac{1}{n} \cdot \sum_{i=1}^{n} D(\bar{Y}_i|Y_i||\bar{q}_i|Y_i). \tag{2.2} \]

For each fixed \( y_i \), the measure \( \bar{q}_i(\cdot|y_i) \) satisfies the conditions of the theorem. By the induction hypothesis,

\[ \frac{1}{n} \cdot \sum_{i=1}^{n} D(\bar{Y}_i|Y_i||\bar{q}_i(\cdot|y_i)) \]

\[ \leq \frac{1}{n} \cdot \frac{4C}{\alpha} \cdot \sum_{i=1}^{n} \sum_{j \neq i} |p_j(\cdot|\bar{Y}_j) - Q_j(\cdot|\bar{Y}_j)|^2 \]

\[ = \left(1 - \frac{1}{n}\right) \cdot \frac{4C}{\alpha} \cdot \sum_{j=1}^{n} |p_j(\cdot|\bar{Y}_j) - Q_j(\cdot|\bar{Y}_j)|^2. \tag{2.3} \]

To estimate the first term in the right-hand-side of (2.2), observe that the definition of \( \alpha \) implies that for any \( i \in [1, n] \) and \( x \in \mathcal{X} \), \( Pr\{X_i = x\} \geq \alpha \). Thus by Lemma 2 we have

\[ D(Y_i||X_i) \leq \frac{4}{\alpha} \cdot |\mathcal{L}(Y_i) - \mathcal{L}(X_i)|^2. \tag{2.4} \]

Further, condition (1.3) implies

\[ \sum_{i=1}^{n} |\mathcal{L}(Y_i) - \mathcal{L}(X_i)|^2 \leq \sum_{i=1}^{n} Pr^n \{Y_i \neq X_i\} = W_2^2(p^n, q^n) \]

\[ \leq C \cdot \mathbb{E} \sum_{i=1}^{n} |p_i(\cdot|\bar{Y}_i) - q_i(\cdot|\bar{Y}_i)|^2. \tag{2.5} \]

Putting together (2.4) and (2.5), it follows that the first term on the right-hand-side of (2.2) can be bounded as follows:

\[ \frac{1}{n} \cdot \sum_{i=1}^{n} D(Y_i||X_i) \leq \frac{1}{n} \cdot \frac{2C}{\alpha} \cdot \sum_{i=1}^{n} \mathbb{E}|p_i(\cdot|\bar{Y}_i) - q_i(\cdot|\bar{Y}_i)|^2. \tag{2.6} \]

Substituting (2.3) and (2.6) into (2.2) we get the first inequality in (1.4). The second inequality follows from the Pinsker-Csiszár-Kullback inequality.
3. Proof of Proposition 3.

Let both $p^n$ and $q^n$ be fixed. We want to show that (1.3) holds with $C = 1/(1 - \|A\|)^2$, where $A$ is the coupling matrix for $q^n$. It is enough to prove this for $I = [1, n]$, since for any $I \subset [1, n]$ and $\tilde{y}_t$ the conditional distribution $q_I(\cdot|\tilde{y}_t)$ satisfies Dobrushin’s uniqueness condition with a minor of $A$ as a coupling matrix. (The idea of the proof for $I = [1, n]$ goes back to Dobrushin’s papers [D1], [D2], although he worked with another matrix norm.)

We are going to prove that Dobrushin’s uniqueness condition implies that the Gibbs sampler $\Gamma$ is a contraction with respect to the $W_2$-distance with rate $1 - 1/n \cdot (1 - \|A\|)$.

To achieve this, let $r^n$ and $s^n$ be two probability measures on $X^n$, and let $U^n$ and $Z^n$ be random sequences representing $r^n$ and $s^n$, respectively. (I.e., $r^n = L(U^n)$, $s^n = L(Z^n)$.)

Select an index $i \in [1, n]$ at random, and define

$$U_i' = U_i, \quad Z_i' = Z_i \quad \text{for} \quad k \neq i.$$ 

Then define $L(U'_i, Z'_i|\bar{U}_i = \bar{u}_i, \bar{Z}_i = \bar{z}_i)$ as that coupling of $q_i(\cdot|\bar{u}_i)$ and $q_i(\cdot|\bar{z}_i)$ that achieves $|q_i(\cdot|\bar{u}_i) - q_i(\cdot|\bar{z}_i)|$. It is clear that $L(U'^n) = r^n \Gamma$, and $L(Z'^n) = s^n \Gamma$.

By the definition of the coupling matrix we have

$$\Pr\{U_i' \neq Z_i'\} \leq (1 - 1/n) \cdot \Pr\{U_i \neq Z_i\} + 1/n \cdot \sum_{k \neq i} a_{k,i} \cdot \Pr\{U_k \neq Z_k\}.$$ 

It follows that

$$\sqrt{\sum_{i=1}^{n} \Pr^2\{U_i' \neq Z_i'\}} \leq \sqrt{B} \cdot \sqrt{\sum_{i=1}^{n} \Pr^2\{U_i \neq Z_i\}},$$

where

$$B = (1 - 1/n) \cdot I_n + 1/n \cdot A.$$ 

Thus

$$\sqrt{\sum_{i=1}^{n} \Pr^2\{U_i' \neq Z_i'\}} \leq \left(1 - \frac{1}{n} \cdot (1 - \|A\|)\right) \sqrt{\sum_{i=1}^{n} \Pr^2\{U_i \neq Z_i\}}.$$ 

This proves the contractivity of $\Gamma$ with rate $1 - 1/n \cdot (1 - \|A\|_2)$.

By the triangle inequality

$$W_2(p^n, q^n) \leq W_2(p^n, p^n \Gamma) + W_2(p^n \Gamma, q^n).$$
PROOF BY A TRANSPORTATION COST DISTANCE

By contractivity of $\Gamma$, and since $q^n$ is invariant with respect to $\Gamma$, it follows that

$$W_2(p^n, q^n) \leq W_2(p^n, p^n\Gamma) + (1 - 1/n \cdot (1 - ||A||)) \cdot W_2(p^n, q^n),$$

i.e.,

$$W_2(p^n, q^n) \leq \frac{n}{1 - ||A||} \cdot W_2(p^n, p^n\Gamma).$$

But it is easy to see that

$$W_2(p^n, p^n\Gamma) = \frac{1}{n} \cdot \sqrt{\mathbb{E} \sum_{i=1}^{n} |p_i(\cdot | Y_i) - q_i(\cdot | Y_i)|^2}.$$

By the last two inequalities, (1.3) (for $I = [1, n]$), and hence Theorem 2, is proved. □

4. Gibbs measures with the strong mixing property.

4.1. Definitions, notation and statement of Theorem 3.

In this section we work with measures on $\mathcal{X}^\Lambda$, where $\Lambda$ is a subset of the $d$-dimensional cubic lattice $\mathbb{Z}^d$. Most of the time $\Lambda$ shall be finite.

The lattice points in $\mathbb{Z}^d$ shall be called sites. The distance $\rho$ on $\mathbb{Z}^d$ is

$$\rho(k, i) = \max_{\nu} |k_\nu - k_\nu|, \text{ where } k = (k_1, k_2, \ldots, k_d), \text{ } i = (i_1, i_2, \ldots, i_d).$$

The notation $\Lambda \subset \subset \mathbb{Z}^d$ expresses that $\Lambda$ is a finite subset of $\mathbb{Z}^d$.

The elements of $\mathcal{X}$ are called spins, and the elements of the set $\mathcal{X}^\Lambda$ ($\Lambda \subset \mathbb{Z}^d$, possibly infinite) are called spin configurations, or just configurations, over $\Lambda$.

We consider an ensemble of conditional distributions $q_\Lambda(\cdot | x_{\Lambda})$, where $\Lambda \subset \subset \mathbb{Z}^d$, and $\Lambda$ is the complement of $\Lambda$. We prefer to write $\tilde{x}_{\Lambda}$ in place of $x_{\Lambda}$, and, accordingly, $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$ in place of $q_\Lambda(\cdot | x_{\Lambda})$. The measure $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$ is considered as the conditional distribution of a random spin configurations over $\Lambda$, given the spin configuration outside of $\Lambda$. For a site $i \in \mathbb{Z}^d$ we use the notation $q_i(\cdot | \tilde{x}_i)$.

The conditional distribution $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$ ($\Lambda \subset \subset \mathbb{Z}^d$, $\tilde{x}_{\Lambda} \in \mathcal{X}^\Lambda$) naturally defines the conditional distributions $q_M(\cdot | \tilde{x}_M)$ for any $M \subset \Lambda$. We assume that the conditional distributions $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$ satisfy the natural compatibility conditions. The conditional distribution $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$ also defines, for $M \subset \Lambda$, the conditional distribution $q_M(\cdot | \tilde{x}_M)$.

If the compatibility conditions hold then there exists at least one probability measure $q = \mathcal{L}(X)$ on the space of configurations $\mathcal{X}^{\mathbb{Z}^d}$, compatible with the conditional distributions $q_\Lambda(\cdot | \tilde{x}_{\Lambda})$:

$$\mathcal{L}(X_{\Lambda} | \tilde{X}_{\Lambda} = \tilde{x}_{\Lambda}) = q_\Lambda(\cdot | \tilde{x}_{\Lambda}).$$
Here $X_\Lambda$ denotes the marginal of the random configuration $X$ for the sites in $\Lambda$, and $\bar{x}_\Lambda$ is called an outside configuration for $\Lambda$. The conditional distributions $q_\Lambda(\cdot|\bar{x}_\Lambda)$ are called the local specifications of $q$, and $q$ is called a Gibbs measure compatible with the local specifications $q_\Lambda(\cdot|\bar{x}_\Lambda)$.

We say that the ensemble of local specifications $q_\Lambda(\cdot|\bar{x}_\Lambda)$ has finite range of interaction $R$ (or is Markov of order $R$) if $q_\Lambda(\cdot|\bar{x}_\Lambda)$ only depends on those coordinates $x_k$ ($k \in \bar{\Lambda}$) that are in the $R$-neighborhood of $\Lambda$.

In general, the local specifications do not uniquely determine the Gibbs measure. The question of uniqueness has been extensively studied in the case of local specifications with finite range of interaction, and a sufficient condition for uniqueness was given by R. Dobrushin and S. Shlosman [D-Sh1]. But the general question of uniqueness is open, even for measures with finite range of interaction.

A property stronger than uniqueness is strong mixing.

In their celebrated paper [D-Sh2] in 1987, R. Dobrushin and S. Shlosman gave a characterization of complete analyticity of Markov Gibbs measures over $\mathbb{Z}^d$. Their characterization was formulated in twelve conditions which were proved to be equivalent, and are referred to as Dobrushin and Shlosman’s strong mixing conditions. The following definition is the same as one of these twelve (III C), except that we do not assume Markovity, and replace the function $K \cdot \exp(-\gamma r)$ by a more general function $\varphi(r)$. In the Markov case $\varphi(r)$ necessarily shall have the form $K \cdot \exp(-\gamma r)$.

In order to define strong mixing, let $\varphi : \mathbb{Z}^+ \mapsto \mathbb{R}^+$ be a function satisfying

$$\sum_{i \in \mathbb{Z}^d} \varphi(\rho(0, i)) < \infty. \quad (4.1.1)$$

**Definition: Strong mixing.** The Gibbs measure $q$ is called strongly mixing with coupling function $\varphi$ if for any sets $M \subset \Lambda \subset \subset \mathbb{Z}^d$ and any two outside configurations $\bar{y}_\Lambda$ and $\bar{z}_\Lambda$ differing only at one single site $k \notin \Lambda$:

$$|q_M(\cdot|\bar{y}_\Lambda) - q_M(\cdot|\bar{z}_\Lambda)| \leq \varphi(\rho(k, M)). \quad (4.1.2)$$

For stationary probability measures on $\mathcal{X}^\mathbb{Z}$, this definition is more restrictive than usual strong mixing, and is equivalent to $\Phi$-mixing. On $\mathbb{Z}^d$ the term strong mixing has been only used for Markov fields, and for simplicity we extend its use without adding any qualification.

Our aim in this section is to prove the following
**Theorem 3.**
Assume that the ensemble $q_\Lambda(\cdot | \bar{x}_\Lambda)$ satisfies the strong mixing condition with coupling function $\varphi$. Moreover, assume that
\[
\alpha \triangleq \inf q_i(x_i | \bar{x}_i) > 0,
\]
where the infimum is taken for all $x \in \mathcal{X}^{\mathbb{Z}_d}$ and $i \in \mathbb{Z}_d$ such that $q_i(x_i | \bar{x}_i) > 0$. Then, for fixed $\Lambda \subset \subset \mathbb{Z}_d$ and outside configuration $\bar{y}_\Lambda$, the conditional distribution $q_\Lambda(\cdot | \bar{y}_\Lambda)$, as a measure on $\mathcal{X}^\Lambda$, satisfies condition (1.3) of Theorem 1, with a constant $C$, independent of $\Lambda$ and $\bar{y}_\Lambda$. Moreover, it is enough to assume (4.1.2) for sets $\Lambda$ of diameter at most $m_0$, where $m_0$ depends on the dimension $d$ and the function $\varphi$. The constant $C$ depends on the dimension $d$, the function $\varphi$ and on $\alpha$.

**Remark.** If $q$ has finite range of interaction then Theorem 3 implies that condition (4.1.2) is constructive, in the sense of Dobrushin and Shlosman.

There is another approach to strong mixing, for measures $q$ on $\mathcal{X}^{\mathbb{Z}_d}$ with finite range of interaction. This approach was developed by E. Olivieri, P. Picco and F. Martinelli; c.f. [M-O1]. Their aim was to replace the above condition of strong mixing ((4.1.2)) by a milder one, requiring (4.1.2) only for "non-pathological" sets $\Lambda$, i.e. for sets whose boundary is much smaller than their volume. Martinelli and Olivieri [M-O2] proved a logarithmic Sobolev inequality under this modified condition, for measures $q$ with finite range of interaction. In Appendix B we briefly sketch the Olivieri-Picco-Martinelli approach, and how to modify Theorem 1 and the Auxiliary Theorem (below), to get logarithmic Sobolev inequalities under this weaker assumption.

**4.2. Proof of Theorem 3**
Consider the infinite symmetric matrix
\[
\Phi = \left( \varphi(\rho(k,i)) \right)_{k,i \in \mathbb{Z}_d}.
\]
Since the entries are non-negative, and the row-sums equal, $||\Phi||$ equals the row-sum:
\[
||\Phi|| = \sum_{i \in \mathbb{Z}_d} \varphi(\rho(0,i)).
\]

Fix a $\Lambda \subset \subset \mathbb{Z}_d$, an outside configuration $\bar{y}_\Lambda$ and a $p_\Lambda \in \mathcal{P}(\mathcal{X}^\Lambda)$. It is enough to prove that
\[
W_2^2(p_\Lambda, q_\Lambda(\cdot | \bar{y}_\Lambda)) \leq C \cdot \mathbb{E} \sum_{i \in \Lambda} W_2^2(p_i(\cdot | \bar{Y}_i), q_i(\cdot | \bar{Y}_i)), \tag{4.2.1}
\]
(with $C$ independent of $\Lambda$ and $\bar{y}_\Lambda$), since for any $M \subset \Lambda$ and any fixed $\bar{y}_{\Lambda \setminus M}$, the conditional distribution $q_M(\cdot | \bar{y}_M)$ (where $\bar{y}_M = (y_{\Lambda \setminus M}, \bar{y}_\Lambda)$) satisfies the strong mixing condition with the same function $\varphi$.

We start with a weaker version of (4.2.1).
Notation.
Let $I_m = I_m(\Lambda)$ denote the set of $m$-sided cubes in $\mathbb{Z}^d$ that intersect $\Lambda$. Set

$$\Theta_m \triangleq \min_R \left[ ||\Phi|| \cdot \frac{d \cdot R}{m} + 2d \cdot \sum_{r=R}^{\infty} (2r + 1)^{d-1} \varphi(r) \right].$$  \hfill (4.2.2)

Note that we can achieve

$$\Theta_m < 1,$$  \hfill (4.2.3)

by selecting $R$ large enough to make the second term in (4.2.2) small, and then selecting $m$.

**Auxiliary Theorem.** If $m$ is so large that $\Theta_m < 1$ then

$$\begin{align*}
W_2^2(p_\Lambda, q_\Lambda(\cdot | \bar{y}_\Lambda)) &\leq \frac{1}{m^d} \cdot \frac{1}{(1 - \Theta_m)^2} \cdot \sum_{I \in I_m} \mathbb{E}W_2^2(p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}), q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda})) \\
&\leq \frac{1}{(1 - \Theta_m)^2} \cdot \sum_{I \in I_m} \mathbb{E}\left| p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}) - q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}) \right|^2. \tag{4.2.4}
\end{align*}$$

If the ensemble $q_\Lambda(\cdot | \bar{x}_\Lambda)$ has finite range of interaction $R$ then the Auxiliary Theorem holds with $||\Phi|| \cdot \frac{d \cdot R}{m}$ in place of $\Theta_m$.

The second inequality in (4.2.4) follows from the first one by the trivial inequality

$$W_2^2(r^n, s^n) \leq n \cdot |r^n - s^n|^2 \quad \text{for} \quad r^n, s^n \in \mathcal{P}(\mathcal{X}^n).$$

The proof of the Auxiliary Theorem follows that of Theorem 2, but we use a more general Gibbs sampler, updating (the intersection of $\Lambda$ with) an $m$-sided cube at a time, not just one site. Let us extend the definition of $p_\Lambda$ so that on $\Lambda$ it be concentrated on the fixed $\bar{y}_\Lambda$.

**Definition.**
For $I \in I_m$ let $\Gamma_I : \mathcal{P}(\mathcal{X}^\Lambda) \mapsto \mathcal{P}(\mathcal{X}^\Lambda)$ be the Markov kernel:

$$\Gamma_I(z_\Lambda | y_\Lambda) = \delta_{y_\Lambda \setminus I, z_\Lambda \setminus I} \cdot q_{I \cap \Lambda}(z_{I \cap \Lambda} | \bar{Y}_{I \cap \Lambda}).$$

(For $k \in \bar{\Lambda}$, $y_k$ is defined by the fixed $\bar{y}_\Lambda$). Then set

$$\Gamma_{I_m} = \frac{1}{|I_m|} \cdot \sum_{I \in I_m} \Gamma_I.$$
Then $\Gamma_{I_m}$ preserves, and is reversible with respect to, $q_{\Lambda}(\cdot|\bar{y}_{\Lambda})$. We call $\Gamma_{I_m}$ the Gibbs sampler for measure $q_{\Lambda}(\cdot|\bar{y}_{\Lambda})$, defined by the local specifications $q_{I\cap \Lambda}(\cdot|\bar{y}_{I\cap \Lambda})$, $I \in I_m$.

**Proof of the Auxiliary Theorem.**

To estimate $W_2^2(\rho_{\Lambda}, q_{\Lambda}(\cdot|\bar{y}_{\Lambda}))$, we are going to prove that if (4.2.3) holds then the Gibbs sampler $\Gamma_{I_m}$ is a contraction with respect to the $W_2$-distance, with rate $1 - m^d/|I_m| \cdot (1 - \Theta_m)$.

To achieve this, let $r$ and $s$ be two probability measures on $\mathcal{X}^\Lambda$, and let $Y$ and $Z$ be random variables representing $r$ and $s$, respectively. (I.e., $r = \mathcal{L}(Y)$, $s = \mathcal{L}(Z)$.) Let the coupling $\mathcal{L}(Y, Z)$ of $r$ and $s$ achieve $W_2(r, s)$. We extend the definition of $\mathcal{L}(Y, Z)$, letting $\bar{Y}_{\Lambda} = \bar{Z}_{\Lambda} = \bar{y}_{\Lambda}$, where $\bar{y}_{\Lambda}$ is the fixed outside configuration. Let $Y'$ and $Z'$ be random variables representing $r \Gamma_{I_m}$ and $s \Gamma_{I_m}$.

Suppose that, when carrying out one step in the Gibbs sampler $\Gamma_{I_m}$, we have selected a certain $I \in I_m$. Then we can assume that

$$Y_i' = Y_i \text{ and } Z_i' = Z_i \text{ for all } i \in \Lambda \setminus I.$$  

Moreover,

$$\mathcal{L}(Y_{I\cap \Lambda}' \mid Y_{\Lambda \setminus I} = y_{\Lambda \setminus I}) = q_{I\cap \Lambda}(\cdot|\bar{y}_{I\cap \Lambda}),$$

and

$$\mathcal{L}(Z_{I\cap \Lambda}' \mid Z_{\Lambda \setminus I} = z_{\Lambda \setminus I}) = q_{I\cap \Lambda}(\cdot|\bar{z}_{I\cap \Lambda}).$$

At this point we need the following

**Lemma 3.** (The proof is in Appendix A.)

Let us fix the set $M \subseteq \mathbb{Z}^d$, together with two outside configurations $\bar{y}_M$ and $\bar{z}_M$, differing only at site $k \notin M$. Let $Y$ and $Z$ be random variables realizing $q_M(\cdot|\bar{y}_M)$ and $q_M(\cdot|\bar{z}_M)$. Define

$$J_i = J_{k, M, i} = \{j \in M : \rho(k, j) \geq \rho(k, i)\}. \quad (4.2.5)$$

Then there exists a coupling $\pi = \mathcal{L}(Y, Z|\bar{y}_M, \bar{z}_M)$ of $\mathcal{L}(Y)$ and $\mathcal{L}(Z)$, satisfying

$$\Pr_\pi\{Y_i \neq Z_i\} = |q_{J_i}(\cdot|\bar{y}_M) - q_{J_i}(\cdot|\bar{z}_M)|, \quad i \in M.$$  

If $q$ satisfies the strong mixing condition with function $\varphi$ then, for this coupling,

$$\Pr_\pi\{Y_i \neq Z_i\} \leq \varphi(\rho(k, i)) \quad \text{for all} \quad i \in M.$$  

By Lemma 3, for fixed $I$, $\bar{y}_{I\cap \Lambda}$ and $\bar{z}_{I\cap \Lambda}$, we can define a coupling

$$\pi_{I\cap \Lambda}(\cdot|\bar{y}_{I\cap \Lambda}, \bar{z}_{I\cap \Lambda}) = \mathcal{L}(Y_{I\cap \Lambda}', Z_{I\cap \Lambda}' \mid \bar{Y}_{I\cap \Lambda} = \bar{y}_{I\cap \Lambda}, \bar{Z}_{I\cap \Lambda} = \bar{z}_{I\cap \Lambda}),$$

Then $\Gamma_{I_m}$ preserves, and is reversible with respect to, $q_{\Lambda}(\cdot|\bar{y}_{\Lambda})$. We call $\Gamma_{I_m}$ the Gibbs sampler for measure $q_{\Lambda}(\cdot|\bar{y}_{\Lambda})$, defined by the local specifications $q_{I\cap \Lambda}(\cdot|\bar{y}_{I\cap \Lambda})$, $I \in I_m$.
satisfying

\[ \Pr_{\pi_I \cap \Lambda} \{ Y'_i \neq Z'_i \mid \bar{Y}_{I \cap \Lambda} = \bar{y}_{I \cap \Lambda}, \bar{Z}_{I \cap \Lambda} = \bar{z}_{I \cap \Lambda} \} \]
\[ \leq \sum_{k \in \Lambda \setminus I} \delta(y_k, z_k) \cdot \varphi(\rho(k, i)), \quad \text{for all} \quad i \in I \cap \Lambda. \]

Thus

\[ \Pr \{ Y'_i \neq Z'_i \mid I \text{ selected} \} \]
\[ \leq \sum_{k \in \Lambda \setminus I} \Pr \{ Y_k \neq Z_k \} \cdot \varphi(\rho(k, i)) \quad \text{for all} \quad i \in I \cap \Lambda. \quad (4.2.6) \]

We calculate \( \Pr \{ Y'_i \neq Z'_i \} \) by averaging for \( I \in \mathcal{I}_m \). Set \( N = |\mathcal{I}_m| \). Since each \( i \in \Lambda \) is covered by exactly \( m^d \) cubes from \( \mathcal{I}_m \), (4.2.6) implies

\[ \Pr \{ Y'_i \neq Z'_i \} \]
\[ \leq (1 - \frac{m^d}{N}) \cdot \Pr \{ Y_i \neq Z_i \} + \frac{1}{N} \cdot \sum_{I \ni i} \sum_{k \in \Lambda \setminus I} \Pr \{ Y_k \neq Z_k \} \cdot \varphi(\rho(k, i)). \quad (4.2.7) \]

Consider the vectors

\[ u = \left( \Pr \{ Y_k \neq Z_k \} \right)_{k \in \Lambda} \quad \text{and} \quad v = \left( \Pr \{ Y'_i \neq Z'_i \} \right)_{i \in \Lambda}, \]

and let \( D \) denote the matrix with entries

\[ d_{k,i} = \varphi(\rho(k, i)) \cdot \sum_{I \ni i, \Lambda \setminus I \ni k} 1, \quad k, i \in \Lambda. \]

With this notation, (4.2.7) means that

\[ v \leq \left( (1 - \frac{m^d}{N}) \cdot \text{Id} + \frac{1}{N} \cdot D \right) \cdot u \]
coordinatewise, thus

\[ ||v|| \leq \left( (1 - \frac{m^d}{N}) + ||D|| \right) \cdot ||u||. \quad (4.2.8) \]

We claim that

\[ \sum_{I : k \notin I, I \ni i} 1 \leq \min \{ d \cdot m^{d-1} \cdot \rho(k,i), \quad m^d \}. \]
Indeed, there are $d$ lattice-hyperplanes separating $k$ and $i$, and there is exactly one among these that intersects the line segment (in $\mathbb{R}^d$) connecting $k$ and $i$. These facts imply that an $m$-sided cube can be placed in at most $d \cdot m^{d-1} \cdot \rho(k,i)$ ways so as to satisfy both conditions $k \notin I$ and $I \ni i$. It follows that

$$d_{k,i} \leq m^d \cdot \varphi(\rho(k,i)) \cdot \min\left\{ \frac{d \cdot \rho(k,i)}{m}, 1 \right\}. \quad (4.2.9)$$

Since the right-hand-side of (4.2.9) is symmetric in $k$ and $i$, we have

$$||D|| \leq m^d \cdot \sum_i \varphi(\rho(k,i)) \cdot \min\left\{ \frac{d \cdot \rho(k,i)}{m}, 1 \right\}. \quad (4.2.10)$$

Now fix an $R$, and divide the sum in (4.2.10) into two parts, for $i$ satisfying $\rho(k,i) \leq R$ and $(\rho(k,i) > R$, respectively. We see that

$$||D|| \leq m^d \cdot \left( ||\Phi|| \cdot \frac{d \cdot R}{m} + \sum_{\rho(k,i) > R} \varphi(\rho(k,i)) \right).$$

Taking minimum in $R$, we get

$$||D|| \leq m^d \cdot \Theta_m. \quad (4.2.11)$$

By (4.2.8) and the definition of the vectors $u$ and $v$, (4.2.11) implies that

$$\sqrt{\sum_{i \in \Lambda} P_{r^2} \{Y_i' \neq Z_i'\}} \leq \left( 1 - \frac{m^d}{N} \cdot (1 - \Theta_m) \right) \cdot \sqrt{\sum_{k \in \Lambda} P_{r^2} \{Y_k \neq Z_k\}},$$

i.e.,

$$W_2(r\Gamma_m, s\Gamma_m) \leq \left( 1 - \frac{m^d}{N} \cdot (1 - \Theta_m) \right) \cdot W_2(r, s). \quad (4.2.12)$$

The stated contractivity is proved.

By the triangle inequality it follows that

$$W_2(p\Lambda, q\Lambda(\cdot | \bar{y}_\Lambda)) \leq W_2(p\Lambda, p\Lambda \Gamma_m) + W_2(p\Lambda \Gamma_m, q\Lambda(\cdot | \bar{y}_\Lambda))$$

$$\leq W_2(p\Lambda, p\Lambda \Gamma_m) + \left( 1 - \frac{m^d}{N} \cdot (1 - \Theta_m) \right) \cdot W_2(p\Lambda, q\Lambda(\cdot | \bar{y}_\Lambda)),$$

whence

$$W_2(p\Lambda, q\Lambda(\cdot | \bar{y}_\Lambda)) \leq \frac{N}{m^d} \cdot \frac{1}{(1 - \Theta_m)} \cdot W_2(p\Lambda, p\Lambda \Gamma_m). \quad (4.2.13)$$
To complete the proof if the Auxiliary Theorem, we have to estimate \( W_2(p_{\Lambda}, p_{\Lambda \Gamma_{I_m}}) \) in terms of the quantities

\[
\mathbb{E} W_2^2(p_{I \cap \Lambda}(\cdot | Y_{I \cap \Lambda}), q_{I \cap \Lambda}(\cdot | Y_{I \cap \Lambda})).
\]

To do this, fix an \( I \in \mathcal{I}_m \), together with a sequence \( y_{\Lambda \backslash I} \in \mathcal{X}^\Lambda \backslash I \), and define a coupling \( \pi_{I \cap \Lambda}(\cdot | y_{\Lambda \backslash I}) \) of \( p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}) \) and \( q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}) \) that achieves \( W_2 \)-distance. We extend \( \pi_{I \cap \Lambda}(\cdot | y_{\Lambda \backslash I}) \) to a measure on \( \mathcal{X}^\Lambda \times \mathcal{X}^\Lambda \) concentrated on the diagonal \((y_{\Lambda \backslash I}, y_{\Lambda \backslash I})\), for coordinates outside of \( I \). Finally, we define the coupling \( \pi \) of \( p_{\Lambda} \) and \( p_{\Lambda \Gamma_{I_m}} \) by averaging the distributions \( \pi_{I \cap \Lambda}(\cdot | y_{\Lambda \backslash I}) \) with respect to \( I \) and \( y_{\Lambda \backslash I} \).

Using this construction, an easy computation (using the Cauchy-Schwarz inequality) shows that

\[
W_2^2(p_{\Lambda}, p_{\Lambda \Gamma_{I_m}}) \leq \frac{m^d}{N^2} \sum_{I \in \mathcal{I}_m} \mathbb{E} W_2^2(p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}), q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda})).
\]  

Substituting (4.2.14) into (4.2.13), we get the first inequality in (4.2.4). Understanding the proof one easily sees that the statement for Gibbs measures with finite range of interaction holds true. The Auxiliary Theorem is proved.

To complete the proof of Theorem 3 we have to deduce (4.2.1) from the Auxiliary Theorem. To do this we need the following

**Lemma 4.** (The proof is in Appendix A.)

Let \( p^n = \mathcal{L}(Y^n) \) and \( q^n \) be two measures on \( \mathcal{X}^n \). Let \( \alpha \) be defined by (1.1). Then

\[
|p^n - q^n|^2 \leq \left( \frac{2}{(|\mathcal{X}| \cdot \alpha)^2} \right)^{n + \log_2 n} \cdot \sum_{i=1}^{n} \mathbb{E} |p_i(\cdot | Y_i) - q_i(\cdot | Y_i)|^2.
\]

Using Lemma 4, we estimate the terms in the last sum in (4.2.5). We get

\[
W_2^2(p_{\Lambda}, q_{\Lambda}(\cdot | \bar{Y}_{\Lambda})) \leq \frac{m^d}{(1 - \Theta_m)^2} \cdot \left( \frac{2}{(|\mathcal{X}| \cdot \alpha)^2} \right)^{m + \log_2 m} \cdot \mathbb{E} \sum_{i \in \Lambda} |p_i(\cdot | Y_{\Lambda \backslash i}) - q_i(\cdot | Y_{\Lambda \backslash i}, \bar{Y}_{\Lambda})|^2.
\]

Thus (4.2.1) is fulfilled with

\[
C = \frac{m^d}{(1 - \Theta_m)^2} \cdot \left( \frac{2}{(|\mathcal{X}| \cdot \alpha)^2} \right)^{m + \log_2 m},
\]
as soon as $m$ is large enough for $\Theta_m < 1$.

We used the strong mixing condition (4.1.2) in proving Lemma 3, and Lemma 3 was used for subsets of $m$-sided cubes. It was enough to consider $m$-sided cubes with $m$ so large that $\Theta_m < 1$ holds, a condition depending on $d$ and $\varphi$. This proves the last two statements of Theorem 3.

\[ \square \]

Remark. An argument similar to the use of Lemma 4 was also there in [S-Z2].

Acknowledgement
The author thanks M. Raginsky for providing a simple proof of Lemma 1.

Appendix A

Proof of Lemma 1. (This proof was suggested to the author by M. Raginsky [R].)

We use the notions of Hellinger distance and Hellinger affinity:

\[ H(r, s) = \left( \sum_{x \in \mathcal{A}} \left| \sqrt{r(x)} - \sqrt{s(x)} \right|^2 \right)^{1/2} \quad \text{and} \quad A(r, s) = \sum_{x \in \mathcal{A}} \sqrt{r(x) \cdot s(x)}. \]

The statement of the lemma can be formulated as

\[ |r - s|^2 \leq 1 - A^2(r, s). \quad (A.1) \]

To prove (A.1), we use the identity

\[ H^2(r, s) = 2(1 - A(r, s)). \]

(A.1) is now proved by the following chain of equalities and inequalities:

\[ |r - s|^2 = \left( \frac{1}{2} \cdot \sum_{x \in \mathcal{A}} |r(x) - s(x)| \right)^2 \]
\[ = \frac{1}{4} \left( \sum_{x \in \mathcal{A}} |\sqrt{r(x)} - \sqrt{s(x)}| \cdot |\sqrt{r(x)} + \sqrt{s(x)}| \right)^2 \]
\[ \leq \frac{1}{4} \cdot \sum_{x \in \mathcal{A}} \left| \sqrt{r(x)} - \sqrt{s(x)} \right|^2 \cdot \sum_{x \in \mathcal{A}} \left| \sqrt{r(x)} + \sqrt{s(x)} \right|^2 \]
\[ = H^2(r, s) \cdot 2(1 + A(r, s)) \]
\[ = (1 - A(r, s)) \cdot (1 + A(r, s)) = 1 - A^2(r, s). \]
(The inequality follows from the Cauchy-Schwarz inequality.)

Proof of Lemma 3.
Order the elements of $\Lambda$ so that

$$\rho(k, i_1) \leq \rho(k, i_2) \leq \cdots \leq \rho(k, i_{|\Lambda|}),$$

i.e., the sequence of sets $J_i = J_{k,M,i}$ (c.f. (4.1.2)) is decreasing in $i$. Let $Y_{J_i}$ and $Z_{J_i}$ denote the marginals of $Y$ and $Z$, respectively, for the sites in $J_i$. Then $(Y_{J_1}, \ldots, Y_{J_{|\Lambda|}})$ and $(Z_{J_1}, Z_{J_2}, \ldots, Z_{J_{|\Lambda|}})$ are Markov chains. (In fact, $Y_{J_{i+1}}$ is a function of $Y_{J_i}$.) Therefore, by a theorem of Goldstein [Go], there exists a coupling $\pi = \mathcal{L}(Y, Z|\bar{y}_M, \bar{z}_M)$ of $\mathcal{L}(Y)$ and $\mathcal{L}(Z)$, satisfying

$$\Pr_{\pi}\{Y_{J_i} \neq Z_{J_i}\} = |\mathcal{L}(Y_{J_i}) - \mathcal{L}(Z_{J_i})| = |q_{J_i}(\cdot|\bar{y}_M) - q_{J_i}(\cdot|\bar{z}_M)|.$$

Since $i \in J_i$, and $\rho(k, i) = \rho(k, J_i)$, the statement of Lemma 4 follows.

Proof of Lemma 4.
Note first that if $r$ and $s$ are probability measures on $\mathcal{X}$, and $r(x), s(x) \geq \alpha$ then

$$|r - s| \leq 1 - |\mathcal{X}| \cdot \alpha.$$

Now consider measures $p^2 = \mathcal{L}(Y_1, Z_2)$ and $q^2$ on a product space $\mathcal{Y} \times \mathcal{Z}$, where $q_2(z_2|y_1) \geq \alpha_2$, and $q_1(y_1|z_2) \geq \alpha_1$ for all $y_1, z_2 \in \mathcal{Y} \times \mathcal{Z}$. Then

$$|q_2(\cdot|y_1) - q_2(\cdot|y_1')| \leq 1 - |\mathcal{Z}| \cdot \alpha_2, \quad \text{and} \quad |q_1(\cdot|z_2) - q_2(\cdot|z_2')| \leq 1 - |\mathcal{Y}| \cdot \alpha_1$$

for all $y_1, y_1' \in \mathcal{Y}$ and $z_2, z_2' \in \mathcal{Z}$.

Thus in this case Dobrushin’s uniqueness condition is satisfied with a $2 \times 2$ coupling matrix, with entries $1 - |\mathcal{Y}| \cdot \alpha_1$ and $1 - |\mathcal{Z}| \cdot \alpha_2$ outside the diagonal. (It does not matter that $\mathcal{Y}$ and $\mathcal{Z}$ may be different.) The coupling matrix has norm

$$\leq \max\{1 - |\mathcal{Y}| \cdot \alpha_1, 1 - |\mathcal{Z}| \cdot \alpha_2\}.$$

By the argument proving Theorem 2, it follows that

$$W_2(p^2, q^2) \leq \max\left\{\frac{1}{(|\mathcal{Y}| \cdot \alpha_1)^2}, \frac{1}{(|\mathcal{Z}| \cdot \alpha_2)^2}\right\} \cdot \mathbb{E}\left(|p_1(\cdot|Y_2) - q_1(\cdot|Y_2)|^2 + |p_2(\cdot|Y_1) - q_2(\cdot|Y_1)|^2\right),$$

and, consequently,

$$|p^2 - q^2|^2 \leq \max\left\{\frac{2}{(|\mathcal{Y}| \cdot \alpha_1)^2}, \frac{2}{(|\mathcal{Z}| \cdot \alpha_2)^2}\right\} \cdot \mathbb{E}\left(|p_1(\cdot|Y_2) - q_1(\cdot|Y_2)|^2 + |p_2(\cdot|Y_1) - q_2(\cdot|Y_1)|^2\right).$$
Lemma 4 follows from (A1) by a recursive argument, dividing the index set into two possibly equal parts of size $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$, and applying (A1) for the two parts. Then

$$\max \left\{ \frac{2}{(|Y| \cdot \alpha_1)^2}, \frac{2}{(|Z| \cdot \alpha_2)^2} \right\}$$

shall be replaced by

$$\left( \frac{2}{(|X| \cdot \alpha)^2} \right)^{\lceil \frac{n}{2} \rceil}.$$

Repeating this step about $\log_2 n$ times we get the statement of the lemma. □

Appendix B

Let $\mathbb{Z}^d/l$ ($l \geq 1$ integer) denote the sub-lattice in $\mathbb{Z}^d$, consisting of points whose coordinates are all multiples of $l$, and let $C_l$ denote the set of finite unions of $l$-sided cubes with vertices in $\mathbb{Z}^d/l$.

The approach by Olivieri and Picco is based on the following definition of strong mixing:

**Definition by Olivieri and Picco.**

The Gibbs measure $q$ on $\mathcal{X}^{\mathbb{Z}^d}$ with finite range of interaction is called strongly mixing over $C_l$, if there exist numbers $\gamma > 0$, $K > 0$ such that: for any sets $\Lambda \in C_l$, $M \subset \Lambda$ and any two outside configurations $\bar{y}_\Lambda$ and $\bar{z}_\Lambda$ differing only at a single site $k \notin \Lambda$, we have

$$\left| q_M(\cdot | \bar{y}_\Lambda) - q_M(\cdot | \bar{z}_\Lambda) \right| \leq K \cdot \exp \left( -\gamma \cdot \rho(k, M) \right).$$

(B.1)

In force of the following theorem, if $l$ is sufficiently large then it is enough to require (B.1) just for cubes in $C_l$, to get (B.1) for all $\Lambda \in C_l$, however, with a different $\gamma$ and $K$.

**Olivieri and Picco’s Effectivity Theorem, [O-P], [M-O1].**

Assume that the Gibbs measure $q$ on $\mathcal{X}^{\mathbb{Z}^d}$ has finite range of interaction. For any $\gamma, K > 0$ there exists an $l_0$ such that: if for some $l \geq l_0$ (B.1) holds for all $l$-sided cubes $\Lambda \in C_l$, all $M \subset \Lambda$ and all $k \notin \Lambda$, then (B.1) also holds for all $\Lambda \in C_l$, and $M$ and $k$ as above, with different $\gamma$ and $K$.

We use a slightly more general definition, although we cannot justify it with some analog of the above Effectivity Theorem:
Definition: Strong mixing over $\mathcal{C}_l$.
Let $\varphi : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ be a function satisfying (4.1.1). Fix an integer $l \geq 1$. The ensemble of conditional distributions $q_\Lambda(\cdot | \bar{x}_\Lambda)$ on $\mathcal{X}^{Z^n}$ is called strongly mixing over $\mathcal{C}_l$, with coupling function $\varphi$, if for any sets $\Lambda \in \mathcal{C}_l$, $M \subset \Lambda$, and any two outside configurations $\bar{y}_\Lambda$ and $\bar{z}_\Lambda$ differing only at the single site $k$, (4.1.2) holds. (We do not assume finite range of interaction.)

For measures strongly mixing over $\mathcal{C}_l$ one can prove a logarithmic Sobolev inequality by means of the following modifications of Theorem 1 and the Auxiliary Theorem:

**Theorem 1’.**
Consider a measure $q^\Lambda$ on $\mathcal{X}^\Lambda = \prod_{j=1}^n \mathcal{X}_{\Lambda_j}$, where

$$\Lambda = \bigcup_{j=1}^n \Lambda_j, \quad \Lambda_j \cap \Lambda_k = \emptyset \quad \text{for} \quad j \neq k, \quad |\Lambda_j| = m.$$ 

Set

$$\alpha = \min \{ q_i(x_i|\bar{x}_i) : \ q_\Lambda(x_\Lambda) > 0, \ i \in \Lambda \}.$$ 

Fix a $p_\Lambda = L(\bar{Y}_\Lambda)$ on $\mathcal{X}^\Lambda$ satisfying

$$q_\Lambda(x_\Lambda) = 0 \implies p_\Lambda(x_\Lambda) = 0.$$ 

Assume that $q_\Lambda$ satisfies all the inequalities

$$W_2^2(p_I(\cdot | \bar{y}_I), q_I(\cdot | \bar{y}_I)) \leq C \cdot \mathbb{E} \left\{ \sum_{\Lambda_j \subset I} W_2^2(p_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j}), q_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j})) \bigg| \bar{Y}_I = \bar{y}_I \right\},$$

where $I \subset \Lambda$ is the union of some of the sets $\Lambda_j$, and $\bar{y}_I \in \mathcal{X}^{\Lambda \setminus I}$ is a fixed sequence. Then

$$D(p_\Lambda||q_\Lambda) \leq \frac{4Cm}{\alpha^m} \cdot \sum_{j=1}^n \mathbb{E} W_2^2(p_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j}), q_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j})).$$

This can be proved by the same argument as Theorem 1, using Lemma 1, the inequalities

$$\left| p_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j}) - q_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j}) \right|^2 \leq m \cdot W_2^2(p_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j}), q_{\Lambda_j}(\cdot | \bar{Y}_{\Lambda_j})), $$

and, in each induction step, fixing a whole new block $Y_{\Lambda_j}$. 

Auxiliary Theorem for measures strongly mixing over $C_l$.

Fix an integer $l$, and assume that the ensemble of conditional distributions $q_{\Lambda}(\cdot | \bar{x}_\Lambda)$ on $X^{Z^d}$ satisfies the strong mixing condition over $C_l$, with coupling function $\varphi$. Let $\Lambda \in C_l$, and fix an outside configuration $\bar{y}_\Lambda$. For fixed $m$ let $I_{ml}$ denote the set of $m \cdot l$-sided cubes from $C_l$ intersecting $\Lambda$. Then for large enough $m$ and any measure $p_{\Lambda}$ on $X^\Lambda$

$$W_2^2(p_{\Lambda}, q_{\Lambda}(\cdot | \bar{y}_\Lambda))$$

$$\leq C \cdot \sum_{I \in I_{ml}} \mathbb{E}W_2^2(p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}), q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}))$$

$$\leq C \cdot m^d \cdot \sum_{I \in I_{ml}} \mathbb{E}|p_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda}) - q_{I \cap \Lambda}(\cdot | \bar{Y}_{I \cap \Lambda})|^2,$$

where $C$ and $m$ depend on the dimension $d$ and on the function $\varphi$.

The proof uses a Gibbs sampler, updating (intersections with $\Lambda$ of) randomly chosen cubes of side $m \cdot l$ from $C_l$. (For an appropriate $m$.)

References

[B-L-M] S. Boucheron, G. Lugosi, P. Massart, Concentration inequalities, Oxford University Press, 2013.

[Gr] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975), 1061-1083.

[L] M. Ledoux, Séminaire de Probabilités XXXIII. Lecture Notes in Math. 1709 Springer, Berlin, 1999, pp. 120-216.

[R] G. Royer, Une Initiation aux Inégalités de Sobolev Logarithmiques, Soc. Math. de France, 1999.

[M] K. Marton, An inequality for relative entropy and logarithmic Sobolev inequalities in Euclidean spaces, Journal of Functional Analysis 264 (2013), 3461.

[O-R] F. Otto, M. Reznikoff, A new criterion for the logarithmic Sobolev inequality and two applications., J. Funct. Anal. 243 (2011), 121157.

[Z] B. Zegarlinski, Dobrushin uniqueness theorem and logarithmic Sobolev inequalities., J. Funct. Anal. 105(1) (1992), 77111.

[C-M-T] P Caputo, G Menz, P Tetali, Approximate tensorization of entropy at high temperature, preprint, arXiv:1405.0608 (2014).

[D1] R. L. Dobrushin, The description of a random field by means of conditional probabilities and condition of its regularity (in Russian), Theory Probab. Appl. 13 (1968), 197224.

[D2] R. L. Dobrushin, Prescribing a system of random variables by conditional distributions, Theory Probab. Appl. 15 (1970), 458486.

[D-Sh1] R. L. Dobrushin, S.B. Shlosman, Statistical Physics and Dynamical Systems, Jaffe, Fritz, Szász editors, 1985, pp. 371-403.

[D-Sh2] R. L. Dobrushin, S.B. Shlosman, Completely analytical interactions: Constructive description, J. Statist. Phys 46 (1987), 9831014.

[S-Z] D. W. Stroock B. Zegarlinski, The Equivalence of the Logarithmic Sobolev Inequality and the Dobrushin Shlosman Mixing Condition, Commun. Math. Phys. 144 (1992), 303-323.

[S-Z] D. W. Stroock B. Zegarlinski, The logarithmic Sobolev inequality for discrete spin systems on the lattice, Comm. Math. Phys. 149 (1992), 175193.

[C] F. Cesi, Quasi-factorization of the entropy and logarithmic Sobolev inequalities for Gibbs random fields, Probability Theory and Related Fields 120 (2001), 569-584.

[R] M. Raginsky, private communication.
[O] Olivieri, E., *On a cluster expansion for lattice spin systems a finite size condition for the convergence*, J. Stat. Phys. **50**, (1988), 1179-1200.

[O-P] Olivieri, E., Picco, P., *Clustering for D-dimensional lattice systems and finite volume factorization properties*, J. Stat. Phys. **59** (1990), 221-256.

[M-O1] Martinelli, F., Olivieri, E., *Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case*, Commun. Math. Phys. **161** (1994), 447-486.

[M-O2] Martinelli, F., Olivieri, E., *Approach to equilibrium of Glauber dynamics in the one phase region. II. The general case*, Commun. Math. Phys. **161** (1994), 487-514.

H-1364 POB 127, Budapest, tel. 36 (1) 4838300, Hungary

E-mail address: marton@renyi.hu