According to H. Poincaré, orbits of the famous three-body problem [2] are not integrable in general cases. Although chaotic orbits of three-body problems widely exist, three families of periodic orbits were found:

1. the Lagrange-Euler family, dating back to the analytical solutions in the 18th century (one recent orbit was given by Moore [3]);
2. the Broucke-Hadjidemetriou-Hénon family, dating back to the mid-1970s [4-9];
3. the Figure-8 family, discovered in 1993 by Moore [3] and extended to the rotating cases [10-13].

Note that nearly all of these reported periodic orbits are planar. In 2013, Šuvačok and Dmitrašinović [1] found by means of numerical approach (namely the Clean Numerical Simulation, CNS) that there exist four classes of planar periodic orbits of Newtonian three body with equal mass, with the above three families belonging to one class. Besides, they reported three new classes of planar periodic orbits and gave a few initial conditions for each class in 5-digit precision: they refined their initial conditions to the level of return proximity of less than 10⁻⁶ by using the gradient descent method. For the details of their 15 planar periodic orbits, please refer to the gallery [14]. Especially, Šuvačok and Dmitrašinović [1] expected their solutions “to be either stable or marginally unstable, as otherwise they probably would not have been found” by their numerical method.

Let the vector \( \mathbf{r}_i(t) \) denote the orbit of three body with equal mass, where \( i = 1, 2, 3 \), and \( t \) denotes the time, respectively. If the orbit is periodic with the period \( T \), it holds \( \mathbf{r}_i(t) = \mathbf{r}_i(t + nT) \) for arbitrary time \( t \geq 0 \) and arbitrary integer \( n \geq 1 \). If, given a tiny disturbance (for example at \( t = 0 \)), the three-bodies greatly depart their periodic orbits after a long enough time, then the corresponding periodic orbits are unstable.

Šuvačok and Dmitrašinović [1] used the gradient descent method to search for the initial conditions of the periodic orbits of three-body with equal mass. It is well-known that orbits of three-body problem are often chaotic, i.e. very sensitive to initial conditions. Thus, it is very important to gain reliable orbits of the three-body problem. However, Šuvačok and Dmitrašinović [1] employed it only in the normal precision (i.e. 5-digit) Thus, their reported periodic orbits should be checked carefully using a more reliable approach.

To gain mathematically reliable numerical simulations of orbits of Newtonian three body problem, we use the so-called “Clean Numerical Simulation” (CNS) [15, 16] that is based on arbitrary-order Taylor series method (TSM) [17, 18] and the arbitrary precision library [19] of the number of significant digits of multiple precision (MP) library [19]. Let \( M \) denote the order of TSM and \( N_s \) the number of significant digits of multiple-precision data, respectively. Unlike other numerical approaches, the CNS enforces that \( M \) increases together with \( N_s \), such as \( M = 2N_s \), as illustrated by Liao [15] for chaotic solutions of Lorenz equation. More importantly, the reliability of one CNS simulation in a given finite but long enough interval is guaranteed by means of other better CNS simulations using larger \( M \) and/or smaller time step \( \Delta t \). In this way, the numerical noises can be decreased to such a small level that both
truncation and round-off errors are negligible in a given finite but long enough interval. For example, Liao [15] employed the CNS to accurately and reliably simulate the propagation of physical uncertainty of initial positions (at the dimensionless level $10^{-60}$) of the chaotic Hamiltonian Hénon-Heiles system for motions of stars in a plane about the galactic center. Besides, using 1200 CPUs of the National Supercomputer TH-A1 and the modified parallel integral algorithm based on the CNS with the 3500th-order TSM and the 4180-digit multiple-precision data, Liao and Wang [20] currently gain, for the first time, a mathematically reliable simulation of chaotic solution of Lorenz equation in a rather long interval $[0,10000]$. All of these indicate that the CNS can indeed provide us a safe way to gain mathematically reliable simulations of chaotic dynamic systems in a finite but long enough interval.

All numerical simulations reported here are obtained by the CNS with high enough order of TSM and accurate enough multiple-precision data, whose validity in a given long enough interval is confirmed by other better CNS simulations with higher-order TSM, and/or more accurate MP data, and/or smaller time step $\Delta t$. For the detailed numerical algorithm, please refer to Liao [21].

Suvakov and Dmitrašinović [1] reported the initial conditions of the newly found periodic orbits in the 5-digit precision. Currently, they obtained the more accurate initial conditions in the 15-digit precision (the seven among them are listed in Table I) for the periodic orbits. In general, for chaotic dynamic systems, exact initial conditions of the periodic orbits should be irrational numbers, as illustrated by Viswanath [22], who reported the initial conditions of periodic solutions of Lorenz equation in accuracy of 500 significant digits.

Without loss of generality, let us consider the case of BUTTERFLY-I, i.e. the Class I.A.1 defined in [1]. Using the 20th to 50th order TSM and the 300-digit multiple precision data with the time step $\Delta t = 10^{-5}$, we obtain the corresponding trajectories of the three bodies in the interval $[0,200]$. It is found that all of these trajectories agree well in the whole interval $[0,200]$ at least in 7 digits, for example as shown in Table I for the position of Body-1 at $t = 200$. Note that, for arbitrary $M \geq 40$, the CNS results given by the $M$th-order TSM with $\Delta t = 10^{-5}$ have at least the 14 significant digits in the whole interval $[0,200]$, whose reliability is confirmed by using the $M'$th-order ($M' \geq 12$) TSM with a smaller time step $\Delta t = 10^{-6}$. Therefore, all of these CNS numerical simulations are convergent to the same result in the interval $[0,200]$ and thus are reliable mathematically. However, the orbits of the three bodies are almost periodic only up to about $t = 130$, but thereafter depart the periodic ones far and far away, as shown in Figures 1 to 3. According to our reliable simulations in the interval $[0,200]$, we are quite sure that the orbits are completely non-periodic after $t > 130$, as shown in Figure 4. Body-1 and Body-3 escape together to become a binary-body system, while Body-2 escapes in the opposite direction. This counterexample clearly indicates that the given initial condition in 15-digit precision of the periodic orbit BUTTERFLY-I found by Suvakov and Dmitrašinović [1] is not accurate enough to guarantee periodic orbits.

The reliable simulations of orbits gained by means of the seven initial conditions listed in Table I and the CNS with the 300-digit multiple precision data (we will not repeat this point thereafter), high enough orders of TSM and small enough time step are gained in a similar way, respectively. All of these numerical simulations are guaranteed to be reliable in a finite but long enough interval. However, it is found that all of the seven initial conditions can not guarantee a periodic orbit.

In case of the moth-III, the orbits in the interval $[0,590]$ gained by means of the initial condition listed in Table I and the 20th-order TSM with the 300-digit multiple precision and the time step $\Delta t = 10^{-5}$ agree well in the 13 significant digits with those gained by the 25th-order TSM and the same time step. The orbits are almost periodic up to $t = 560$, i.e. about 22 periods ($T = 25.8406180475758$) of the moth-III, but thereafter depart the periodic ones far and far away. According to the CNS results given by the 25th-order TSM and the same time step, the collision occurs at about $t = 699.45068$, a little more than 27 periods of the periodic orbit reported by Suvakov and Dmitrašinović [1].

In the case of the goggles, the orbits in the interval $[0,90]$ given by the CNS with the 30th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 8 significant
TABLE I. The initial conditions of the 7 periodic orbits in 15-digit precision

| Class, number, name | $\dot{x}_1(0)$ | $\dot{y}_1(0)$ | $T$ |
|---------------------|----------------|----------------|-----|
| I.A.1 butterfly I   | 0.306892758965492 | 0.125506782829762 | 6.23564136316479 |
| I.B.4 moth III      | 0.383443534851074 | 0.377363693237305 | 25.8406180475758 |
| I.B.5 goggles       | 0.0833000564575194 | 0.127889282226563 | 10.4668176954385 |
| I.B.7 dragonfly     | 0.080584285736084 | 0.588836087036132 | 21.2709751966648 |
| II.B.1 yarn         | 0.559064247131347 | 0.349191558837891 | 55.5017624421301 |
| II.C.2a yin-yang I | 0.513938054919243 | 0.304736003875733 | 17.328369755004 |
| II.C.2b yin-yang I | 0.282698682308198 | 0.327208786129952 | 10.9625630756217 |

* These values are given by M. Šuvakov in an email communication

TABLE II. The position ($x_1$, $y_1$) of Body-1 at $t = 200$ in case of BUTTERFLY-I given by the different orders of TSM and 300-digit multiple-precision data with the different time steps. The initial condition is listed in Table I.

| Order | $\Delta t$ | $x_1(200)$ | $y_1(200)$ |
|-------|------------|------------|------------|
| 20    | $10^{-5}$  | -3.49138   | -12.017376 |
| 25    | $10^{-5}$  | -3.49137957| -12.0173712|
| 30    | $10^{-5}$  | -3.4913795772| -12.017371265|
| 40    | $10^{-5}$  | -3.491379571996| -12.017371265596|
| 45    | $10^{-5}$  | -3.491379571996| -12.017371265596|
| 50    | $10^{-5}$  | -3.491379571996| -12.017371265596|
| 12    | $10^{-6}$  | -3.491379571996| -12.017371265596|
| 15    | $10^{-6}$  | -3.491379571996| -12.017371265596|
| 20    | $10^{-6}$  | -3.491379571996| -12.017371265596|
| 12    | $10^{-7}$  | -3.491379571996| -12.017371265596|

digits with those by the 15th-order TSM and the smaller time step $\Delta t = 10^{-6}$. It is found that the orbits are almost periodic only up to $t = 55$ (i.e. a little more than 5 periods of the goggles reported in [1]), but thereafter depart from the periodic ones far and far away.

In the case of dragonfly, the orbits in the interval [0,950] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 4 significant digits with those given by the 30th-order TSM and the same time step. The orbits are almost periodic only up to $t = 720$, and thereafter depart the periodic ones far and far away.

In the case of yarn, the orbits in the interval [0,560] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 7 significant digits with those given by the 30th-order TSM and the same time step. The orbits are almost periodic only up to $t = 440$, and thereafter depart the periodic ones far and far away.

In the case of yin-yang I (II.C.2a), the orbits in the interval [0,320] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 16 significant digits with those given by the 30th-order TSM and the same time step. The orbits are almost periodic only up to $t = 250$, thereafter depart from the periodic ones far and far away.

FIG. 2. Orbits of Body-2 in case of BUTTERFLY-I in the interval [0,200] gained by means of the CNS using 40th-order TSM condition in 15-digit precision in Table I is used. Greed line: periodic orbit reported in [1].

Besides, it is found that the orbits are sensitive to the initial conditions: adding a small disturbance at the level $10^{-17}$ to the initial conditions in Table I, we gain a non-periodic orbit that departs considerably from the original non-periodic ones for a long enough time. Thus, the orbits given by the seven initial conditions in Table I are unstable.

Using the original initial conditions (in 5-digit precision) of the considered seven orbits reported by Šuvakov and Dmitrašinović [1], we gain the same conclusion: the seven corresponding orbits are non-periodic and unstable.
Therefore, according to our reliable numerical simulations based on the CNS, either the initial conditions in Table I of the reported seven “periodic” orbits are not accurate enough to predict a periodic ones, or the corresponding orbits are unstable. Thus, at least seven “periodic” orbits (listed in Table I) of the Newtonian three-body problem found currently by Šuvakov and Dmitrašinović [1] are doubtful and should be checked very carefully.

The periodicity and stability of the other orbits reported by Šuvakov and Dmitrašinović [1] also should be doubt checked in details.

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[1] M. Šuvakov and V. Dmitrašinović, Phys. Rev. Lett. 110, 114301 (2013)
[2] M. Valtonen and H. Karttunen, The Three Body Problem (Cambridge University Press, 2005)
[3] C. Moore, Phys. Rev. Lett. 70, 3675 (1993)
[4] R. Broucke and D. Boggs, Celest. Mech. 11, 13 (1975)
[5] J. D. Hadjidemetriou and T. Christides, Celest. Mech. 12, 175 (1975)
[6] J. D. Hadjidemetriou, Celest. Mech. 12, 255 (1975)
[7] R. Broucke, Celest. Mech. 12, 439 (1975)
[8] M. Hénon, Celest. Mech. 13, 267 (1976)
[9] M. Hénon, Celest. Mech. 15, 243 (1977)
[10] M. Nauenberg, Phys. Lett. A 292, 93 (2001)
[11] J. F. A. Chenciner and R. Montgomery, Nonlinearity 18, 1407 (2005)
[12] A. E. R. Broucke and A. Riaguas, Chaos, Solitons and Fractals 30, 513 (2006)
[13] M. Nauenberg, Celest. Mech. 97, 1 (2007)
[14] http://suki.ipb.ac.rs/3body/
[15] S. Liao, Chaos, Solitons and Fractals 47, 1 (2013)
[16] S. Liao, Tellus-A 61, 550 (2009)
[17] G. Corliss and Y. Chang, ACM Trans. Math. Software 8, 114 (1982)
[18] F. B. R. Barrio and M. Lara, Comput. and Math. with Appli. 50, 93 (2005)
[19] P. Oyanarte, Comput. Phys. Commun. 59, 345 (1990)
[20] S. Liao and P. Wang, Science China - Physics, Mechanics & Astronomy (accepted)(2013), (see also arXiv:1305.4222)
[21] S. Liao, Communications in Nonlinear Science and Numerical Simulations (accepted) (see also arXiv:1305.6094)
[22] D. Viswanath, Physica D 190, 115 (2004)
FIG. 4. Orbits of three bodies in case of BUTTERFLY-I in the interval [0,200] gained by means of the CNS using 40th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table I is used. Blue line: orbit of Body-1; Red line: orbit of Body-2; Black line: orbit of Body-3.

FIG. 5. Orbits of three bodies in case of II.C.2b (yin-yang I) in the interval [0,190] gained by means of the CNS using 25th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table I is used. Blue line: orbit of Body-1; Red line: orbit of Body-2; Black line: orbit of Body-3.