Cowen-Douglas function and its application on chaos

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Abstract
In this paper, on \( \mathbb{D} \) we define Cowen–Douglas function introduced by the Cowen–Douglas operator \( M_\phi \) on Hardy space \( \mathcal{H}^2(\mathbb{D}) \), moreover, we give a necessary and sufficient condition to determine when \( \phi \) is a Cowen–Douglas function, where \( \phi \in \mathcal{H}^\infty(\mathbb{D}) \) and \( M_\phi \) is the associated multiplication operator on \( \mathcal{H}^2(\mathbb{D}) \). Then, we give some applications of Cowen–Douglas function on chaos, such as its application on the inverse problem of chaos for \( \phi(T) \), where \( \phi \) is a Cowen–Douglas function and \( T \) is the backward shift operator on the Hilbert space \( \mathcal{L}^2(\mathbb{N}) \).

Keywords Chaos · Hardy space · Rooted function · Cowen–Douglas function

Mathematics Subject Classification 47A16 · 47A65 · 30J99 · 37B99 · 37D45

1 Introduction

1.1 Inverse problem of chaos

If \( X \) is a metric space and \( T \) is a continuous self-map on \( X \), then the pair \( (X, T) \) is called a topological dynamic systems, which is induced by the iteration

\[
T^n = T \circ \cdots \circ T, \quad n \in \mathbb{N}, \text{ where } 0 \in \mathbb{N}.
\]
Let \((X, T)\) be a topological dynamic systems, a point \(x \in X\) is called a periodic point of \(T\), if there is some \(n \geq 1\) such that \(T^n x = x\). The least such number is called the period of \(x\) and the set of periodic points is denoted by \(\text{Per}(T)\). If for any pair \(U, V\) of nonempty open subsets of \(X\), there exists some \(n \geq 0\) such that \(T^n(U) \cap V \neq \emptyset\), then \((X, T)\) is called topologically transitive. If for any pair \(U, V\) of nonempty open subsets of \(X\), there is some \(N \geq 0\) such that
\[
T^n(U) \cap V \neq \emptyset, \text{ for all } n \geq N,
\]
then we say that \((X, T)\) is strongly mixing or just say that \(T\) is strongly mixing.

For a topological dynamic systems, there are many concepts and techniques to research its complexities, such as the following.

**Definition 1** [12] Let \((X, T)\) be a topological dynamic systems, if it satisfies:

(a) \(T\) is topologically transitive,

(b) \(T\) has a dense set of periodic points,

then we say that \(T\) is Devaney chaotic.

If the metric space \(X\) and the continuous self-map \(T\) are both linear, then we say that \((X, T)\) is a linear dynamic. Especially, if \(T\) is a continuous invertible self-map on \(X\), then we say that \((X, T)\) is an invertible dynamic. Similarly, for a Banach space \(\mathcal{B}\), if \(T\) is an invertible bounded linear operator on \(\mathcal{B}\), then we say that \((\mathcal{B}, T)\) is an invertible linear dynamic. In this paper, we say that is inverse problem of chaos to study the chaotic theory of invertible linear dynamic.

And Li–Yorke chaos is defined in [30], the following definition is its version in linear dynamics,

**Definition 2** (Li–Yorke chaos) Let \(T \in \mathcal{B}(\mathcal{B})\), if there exists \(x \in \mathcal{B}\) such that satisfies:

\[
\begin{align*}
\text{(1)} & \quad \lim_{n \to \infty} \|T^n(x)\| > 0, \\
\text{(2)} & \quad \lim_{n \to \infty} \|T^n(x)\| = 0,
\end{align*}
\]
then we say that \(T\) is Li–Yorke chaotic, and named \(x\) is a Li–Yorke chaotic point of \(T\), where \(x \in \mathcal{B}, n \in \mathbb{N}\).

The research of chaos in linear dynamics began with the work in [17]. After that, Li–Yorke chaotic operator have been extensively studied in [3–6, 16, 37, 42–45]. Also, there are many papers about the distributional chaos, Devaney chaos, mixing and hypercyclicity in linear dynamics, more details of these definitions can be find in [3, 16, 20, 45].

For invertible dynamics, the relationship between \((X, f)\) and \((X, f^{-1})\) is raised by Stockman as an open question [40]. And [22, 31] and [32] give counter examples for Li–Yorke chaos on noncompact space and compact space, respectively. However, there is no generally way for this research.
1.2 Cowen–Douglas operator and Cowen–Douglas function

**Definition 3** [10] For a connected open subset $\Omega \subset \mathbb{C}$, $n \in \mathbb{N}$, let $B_n(\Omega)$ be the set of all bounded linear operator $T$ on a Hilbert space $\mathbb{H}$ that satisfies:

1. $\Omega \in \sigma(T) = \{ \omega \in \mathbb{C} : T - \omega \text{ is not invertible} \}$;
2. $\text{ran}(T - \omega) = \mathbb{H}$ for $\omega \in \Omega$;
3. $\bigvee_{\omega \in \Omega} \ker(T - \omega) = \mathbb{H}$;
4. $\dim \ker(T - \omega) = n$ for $\omega \in \Omega$.

If $T \in B_n(\Omega)$, then we say that $T$ is a Cowen–Douglas operator.

Let $\Omega \subset \mathbb{C}$ be a bounded open subset. From the review of [10] by Atiyah, we know that if $T \in B_n(\Omega)$, then there is a holomorphic Hermitian vector bundle $E_T$ over $\Omega$ formed by the finite-dimensional null spaces of $T - \omega$, where $\omega \in \Omega$.

Cowen and Douglas [10] studied the connection between $B_n(\Omega)$ and $E_T$, the results have been refined 2 years later and they made the first systematic study for the Cowen–Douglas class $B_n(\Omega)$.

Curto and Salinas [11] linked various properties of operators in the class $B_n(\Omega)$ to the theory of generalized reproducing kernels. Jiang [26] made a characterization of the strongly irreducibility of Cowen–Douglas operators (more results see [27]).

Zhu [47] proved that for every $T \in B_n(\Omega)$, the Hermitian bundle $E_T$ possesses a spanning holomorphic cross-section $\gamma(z)$. And the generalizes classification results is given by [14].

In [28, 29], Korányi and Misra made a complete classification of homogeneous operators in the Cowen–Douglas class in terms of homogeneous holomorphic Hermitian vector bundles over the unit disc $\mathbb{D}$.

Chen et al. [8] studied the structure of the von Neumann algebra $V^*(T)$ consisting of operators commuting with both $T$ and $T^*$ from a geometric viewpoint as well as information on reducing subspaces of $T$ can be determined by the connection-preserving holomorphic Hermitian vector bundle maps on $E_T$.

Heiatian Naeini and Yousefi [18] made necessary and sufficient conditions under which the complement of the essential spectrum of $M_{\phi}$ in $\phi(\Omega)$ becomes nonempty and this gives conditions for the adjoint of the multiplication operator $M_{\phi}$ belongs to the Cowen–Douglas class of operators.

With this results of Cowen–Douglas class of operators, we give the following

**Definition 4** Let $\phi(z) \in \mathcal{H}^\infty(\mathbb{D})$, $M_{\phi}$ be the associated multiplication operator on $\mathcal{H}^2(\mathbb{D})$. If the adjoint multiplier $M_{\phi}^* \in B_n(\phi(\mathbb{D}))$, then we define that $\phi$ is a Cowen–Douglas function, where $n \in \mathbb{N}$.

In this paper, $\overline{\phi(\cdot)}$ means the conjugate of the holomorphic function $\phi(\cdot)$.

By Definition 4, we get that constant function is not Cowen–Douglas function.
1.3 Motivation and main results

Hardy space $\mathcal{H}^2(\mathbb{D})$ is a special Hilbert space that is relatively nice, not only for its theory but also for its computing by the holomorphic function. Hence, it is a good idea to construct some symbol operators on Hardy space to study the inverse problem of chaos.

Naturally, we have the following

Problem 1 For $\phi \in \mathcal{H}^\infty(\mathbb{D})$, what is the necessary and sufficient condition to determine when $\phi$ is a Cowen–Douglas function.

In Sect. 3, we give that $\phi$ is a Cowen–Douglas function if and only if $\phi$ is a $m$-folder holomorphic function and the rooter function of $\phi$ at $z_0$ is an outer function, where $\phi \in \mathcal{H}^\infty(\mathbb{D})$ and $z_0 \in \mathbb{D}$. And in Sect. 4, we give some applications of Cowen–Douglas function on chaos, such as application on the inverse problem of chaos for $\phi(T)$, where $\phi$ is a Cowen–Douglas function and $T$ is the backward shift operator on $L^2(\mathbb{N})$, i.e.,

$$T(x_1,x_2, \ldots) = (x_2, x_3, \ldots).$$

2 Some properties of Hardy space and chaos

For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ be the set of all bounded linear operator and compact operator, respectively. Firstly, we give some properties of Fredholm operator on Hilbert spaces.

Definition 5 If $\mathcal{H}$ is a Hilbert space, the natural homomorphism from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted by $\gamma$, then $T \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if $\gamma(T)$ is an invertible element of $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Moreover, the index of $T$ is defined by

$$\text{ind}(T) = \dim \ker T - \dim \ker T^*.$$

Proposition 1 Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be two Fredholm operators, then there is

$$\text{ind}(T_1T_2) = \text{ind}(T_1) + \text{ind}(T_2).$$

2.1 Properties of Hardy space

Then, we give some properties of Hardy space. For $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$, if $g$ is a holomorphic function on $\mathbb{D}$ and there is

$$\sup_{r < 1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 \, d\theta < +\infty,$$

then we denote $g \in \mathcal{H}^2(\mathbb{D})$. Obviously, $\mathcal{H}^2(\mathbb{D})$ is a Hilbert space with the norm
\[ \|g\|_{H^2}^2 = \sup_{r < 1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 \frac{d\theta}{2\pi}, \]

and \( H^2(D) \) is denoted as the Hardy space.

For any given holomorphic function \( g \), we get the Taylor expansion

\[ g(z) = \sum_{n=0}^{+\infty} a_n z^n, \]

so we get a natural equivalence from \( g \in H^2(D) \) to \( \sum_{n=0}^{+\infty} a_n^2 < +\infty \).

Let \( \mathbb{T} = \partial \mathbb{D} \), if \( H^2(\mathbb{T}) \) is the closed span of functions which have Taylor expansions in \( L^2(\mathbb{T}) \), then \( H^2(\mathbb{T}) \) is a closed subspace of \( L^2(\mathbb{T}) \). From the natural isomorphism between \( H^2(D) \) and \( H^2(T) \) by the properties of holomorphic function, sometimes we say that \( H^2(T) \) is the Hardy space.

Let \( H^\infty(D) \) denote the set of all bounded holomorphic function on \( D \), and \( H^\infty(T) \) denote the closed span of functions which have Taylor expansions in \( L^\infty(T) \). Then we get that \( H^\infty(D) \) and \( H^\infty(T) \) are naturally isomorphic by the properties of holomorphic function associated with the Dirichlet Problem (see [7, p. 55, p. 97, p. 103]).

For any given \( \phi \in H^\infty(D) \), it is easy to get that

\[ \|\phi\|_{\infty} = \sup\{ |\phi(z)|; |z| < 1 \} \]

is a norm on \( H^\infty(D) \).

And for any given \( \phi \in H^\infty(D) \), the multiplication operator

\[ M_\phi(g) = \phi g, \quad g \in H^2(D), \]

is a bounded linear operator, and on \( H^2(D) \) we get

\[ \|M_\phi(g)\| \leq \|\phi\|_{\infty} \|g\|_{H^2}. \]

By [9, p. 6], we get that any Cauchy sequence with the norm \( \| \cdot \|_{H^2} \) on \( H^2(D) \) is an uniformly Cauchy sequence on any closed disk in \( D \), in particular, we get that the point evaluation \( f \to f(z) \) is continuous linear functional on \( H^2(D) \). By the Riesz Representation [9, p. 13], for any \( g(s) \in H^2(D) \), there is an unique \( f_z(s) \in H^2(D) \) such that

\[ g(z) = \langle g(s), f_z(s) \rangle. \]

And then we define this \( f_z \) is the reproducing kernel at \( z \in D \). There are many properties about Hardy space in [1, p.106], [3, p. 7], [13, p. 133], [15, p. 48] and [19, p.39].

### 2.2 Properties of chaos

Let \( \mathcal{B}(\mathbb{B}) \) be the set of all bounded linear operator on \( \mathbb{B} \). For prepare, we give some definitions and properties of chaos on Banach space \( \mathbb{B} \).
Definition 6  Let $T \in \mathcal{B}(\mathbb{H})$, if there is some $x \in \mathbb{H}$ such that $\{T^n x; n \in \mathbb{N}\}$ is dense in $\mathbb{H}$, then we say that $T$ is hypercyclic, and named that $x$ is a hypercyclic point for $T$.

It is easy to get that $T$ is hypercyclic if and only if $(\mathbb{B}, T)$ is topologically transitive. Following [39], there is a distributional function

$$F^n_x(\tau) = \frac{1}{n} \#\{0 \leq i \leq n : \|T^n(x)\| < \tau\},$$

where $T \in \mathcal{B}(\mathbb{B}), x \in \mathbb{B}$ and $n \in \mathbb{N}$. Also, there are

$$F_x(\tau) = \liminf_{n \to \infty} F^n_x(\tau), \quad F^*_x(\tau) = \limsup_{n \to \infty} F^n_x(\tau),$$

and

Definition 7  Let $T \in \mathcal{B}(\mathbb{B})$, if there exists $x \in \mathbb{B}$ and

1. If $F_x(\tau) = 0$ for some $\tau > 0$ and $F^*_x(\varepsilon) = 1, \forall \varepsilon > 0$, then we say that $T$ is distributional chaos or I-distributionally chaotic.
2. If $F^*_x(\varepsilon) > F_x(\tau), \forall \tau > 0$, and $F^*_x(\varepsilon) = 1, \forall \varepsilon > 0$, then we say that $T$ is II-distributionally chaotic.
3. If $F^*_x(\varepsilon) > F_x(\tau), \forall \tau > 0$, then we say that $T$ is III-distributionally chaotic.

Easily, we get the following

Lemma 1  I-distributionally chaotic $\implies$ II-distributionally chaotic $\implies$ III-distributionally chaotic $\implies$ Li–Yorke chaos.

Theorem 1 [20] For a connected open subset $\Omega \subset \mathbb{C}, T \in \mathcal{B}_n(\Omega)$, we get

1. If $\Omega \cap \mathbb{T} \neq \emptyset$, then $T$ is Devaney chaos.
2. If $\Omega \cap \mathbb{T} \neq \emptyset$, then $T$ is I-distributionally chaotic.
3. If $\Omega \cap \mathbb{T} \neq \emptyset$, then $T$ is strongly mixing.

For an invertible dynamic, $(X, f)$ is transitive if and only if $(X, f^{-1})$ is. And by the counter examples for Li–Yorke chaos [22, 31, 32], we get that, in generally, there is no relation between transitivity and Li–Yorke chaos. Devaney chaos does not imply distributional chaos [36, 38]. However, strong mixing, Devaney chaos and distributional chaos do imply Li–Yorke chaos, respectively [25, 33, 41].

In linear dynamics, there is no hypercyclic operator on finite-dimensional Fréchet space, in fact, our study is really for infinite-dimensional Fréchet space [3, p. 1] [5, 24]. There are hypercyclic operators on Banach spaces that are not strongly mixing, but strong mixing does imply hypercyclicity [16, p. 48,p. 55]. There is strongly mixing operator that is not Devaney chaotic and the first example of Devaney chaotic operator which is not strongly mixing was constructed by Bedea and Grivaux [2]
[16, p. 46, p. 66]. Distributional chaos was originally introduced in [39] and from [34] we get that distributional chaos does not imply hypercyclicity, that is, distributional chaos does not imply Devaney chaos. Also, from [35] we get that hypercyclicity and strong mixing do not imply distributional chaos, respectively.

Briefly, in view of the different types of definitions, we summarize the links between them as the following relations.

\[
\text{strong mixing} \quad \Downarrow \quad \text{hypercyclicity} \quad \Uparrow \\
\text{Li-Yorke chaos} \quad \Leftarrow \quad \text{Devaney chaos} \quad \Uparrow \\
\text{distributional chaos}
\]

### 3 Cowen–Douglas function on Hardy space

In this section, we give the definition of rooter function and we prove that the rooter function of a \( m \)-folder holomorphic function \( \phi \) is an outer function if and only if \( \phi \) is a Cowen–Douglas function.

**Definition 8** [13, p. 141] Let \( \mathcal{P}(z) \) be the set of all polynomial about \( z \), where \( z \in \mathbb{T} \). Define a function \( h(z) \in \mathcal{H}^2(\mathbb{T}) \) is an outer function if

\[
\text{cl}[h(z)\mathcal{P}(z)] = \mathcal{H}^2(\mathbb{T}).
\]

**Lemma 2** [13, p. 141] A function \( h(z) \in \mathcal{H}^\infty(\mathbb{T}) \) is invertible on the Banach algebra \( \mathcal{H}^\infty(\mathbb{T}) \), if and only if \( h(z) \in \mathcal{L}^\infty(\mathbb{T}) \) and \( h(z) \) is an outer function.

**Theorem 2** [15, p. 81] Let \( \mathcal{P}(z) \) be the set of all polynomial about \( z \), where \( z \in \mathbb{D} \). then \( h(z) \in \mathcal{H}^2(\mathbb{D}) \) is an outer function if and only if

\[
\mathcal{P}(z)h(z) = \{p(z)h(z); p \in \mathcal{P}(z)\} \text{ is dense in } \mathcal{H}^2(\mathbb{D})
\]

Let \( \phi \) be a non-constant holomorphic function on \( \mathbb{D} \), from [7, p. 29] we get that for any given \( z_0 \in \mathbb{D} \), there exists \( \delta_{z_0} > 0 \) such that

\[
\phi(z) - \phi(z_0)|_{|z-z_0|<\delta_{z_0}} = p_{n_{z_0}}(z)h_{z_0}(z)|_{|z-z_0|<\delta_{z_0}},
\]

where \( h_{z_0}(z) \) is complex analytic on a neighbourhood of \( z_0 \) and

\[
h_{z_0}(z_0) \neq 0,
\]

\[
p_{n_{z_0}}(z) = \sum_{k=0}^{n_{z_0}} a_k(z-z_0)^k, \quad a_{n_{z_0}} = 1.
\]

By the Analytic Continuation Theorem in [7, p. 28], we get that
Definition 9 There is an unique holomorphic function $h_{z_0}(z)$ on $\mathbb{D}$ such that

$$\phi(z) - \phi(z_0) = p_{n_{z_0}}(z)h_{z_0}(z),$$

and we define that $h_{z_0}(z)$ is the rooter function of $\phi$ at $z_0$. If for any given $z_0 \in \mathbb{D}$, the rooter function $h_{z_0}(z)$ has non-zero point on $\mathbb{D}$, but the roots of $p_{n_{z_0}}(z)$ are all in $\mathbb{D}$ and $n_{z_0} \in \mathbb{N}$ is a constant $m$ on $\mathbb{D}$, then we define that $\phi$ is a $m$-folder holomorphic function on $\mathbb{D}$.

Now we give the main results of this paper, that is the following

Theorem 3 Let $\phi(z) \in \mathcal{H}^\infty(\mathbb{D})$ be a $m$-folder holomorphic function, $M_\phi$ be the associated multiplication operator on $\mathcal{H}^2(\mathbb{D})$. If for any given $z_0 \in \mathbb{D}$, the rooter function of $\phi$ at $z_0$ is an outer function, then $\phi$ is a Cowen-Douglas function, that is, the adjoint multiplier $M_\phi^* \in \mathcal{B}_m(\overline{\phi}(\mathbb{D}))$.

Proof By Definition 9, we get that $\phi$ is not a constant function. For any given $z \in \mathbb{D}$, let $f_z \in \mathcal{H}^2(\mathbb{D})$ be the reproducing kernel at $z$.

We claim that $M_\phi^*$ is valid the conditions (1) (3) (4) of Definition 3 and in the end we confirm that $M_\phi^*$ is valid the condition (2) of Definition 3.

(1) For any given $z \in \mathbb{D}$, $f_z$ is an eigenvector of $M_\phi^*$ associated with eigenvalue $\lambda = \overline{\phi}(z)$. Because for any $g \in \mathcal{H}^2(\mathbb{D})$ we get

$$<g, M_\phi^*(f_z)>\mathcal{H}^2 = <\phi g, f_z>\mathcal{H}^2 = \phi(z) f(z) = <g, \overline{\phi}(z)f_z>\mathcal{H}^2.$$

By the Riesz Representation in [9, p.13] of bounded linear functional in the form of inner product on Hilbert space, we get

$$M_\phi^*(f_z) = \phi(z)f_z = \lambda f_z,$$

that is, $f_z$ is an eigenvector of $M_\phi^*$ associated with eigenvalue $\lambda = \overline{\phi}(z)$.

(3) Suppose that

$$\text{span}\{f_z; z \in \mathbb{D}\} = \text{span}\left\{\frac{1}{1 - \overline{z}s}; z \in \mathbb{D}\right\}$$

is not dense in $\mathcal{H}^2(\mathbb{D})$.

Then there exists nonzero $g \in \overline{\text{span}\{f_z; z \in \mathbb{D}\}^\perp}$, for any given $z \in \mathbb{D}$, we have

$$0 = \langle g, f_z \rangle_{\mathcal{H}^2} = g(z).$$

So we get $g = 0$ by the Analytic Continuation Theorem in [7, p. 28], that is a contradiction for $g \neq 0$.

Therefore, we get that $\text{span}\{f_z; z \in \mathbb{D}\}$ is dense in $\mathcal{H}^2(\mathbb{D})$, that is,

$$\bigvee_{\lambda \in \overline{\phi}(\mathbb{D})} \text{ker}_{\lambda \in \overline{\phi}(\mathbb{D})}(M_\phi^* - \lambda) = \mathcal{H}^2(\mathbb{D}).$$
(4) With Definition 9 and the conditions of this theorem, for any given \( \lambda \in \phi(\mathbb{D}) \), there are \( z_0 \in \mathbb{D} \), \( m \)-th polynomial \( p_m(z) \) and outer function \( h(z) \) such that
\[
\phi(z) - \lambda = \phi(z) - \phi(z_0) = p_m(z)h(z),
\]
We will prove
\[
\dim \ker(M^*_{\phi(z)} - \lambda) = m
\]
by the following (i)–(iii) assertions.

(i) Let the roots of \( p_m(z) \) be \( z_1, z_2, \ldots, z_m \), then there exists decomposition
\[
p_m(z) = (z - z_1)(z - z_2) \cdots (z - z_m).
\]
Let \( p_{m,z_1,z_2,\ldots,z_m}(z) \) be the decomposition of \( p_m(z) \) by any given permutation of \( z_1, z_2, \ldots, z_m \), we will prove
\[
\dim \ker(M^*_{p_m; z_1, z_2, \ldots, z_m}) = m.
\]
By the Taylor expansion of function in \( \mathcal{H}^2(\mathbb{D}) \), we get a natural isomorphism
\[
F_s : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D} - s), F_s(g(z)) \to g(z + s), \quad s \in \mathbb{C}.
\]
It is easy to get that \( G = \{ F_s; s \in \mathbb{C} \} \) is an abelian group by the composite operation \( \circ \).

Hence, for \( 1 \leq n \leq m \), there is
\[
\begin{array}{ccc}
\mathcal{H}^2(\mathbb{D}) & \xrightarrow{M_{z-z_n}} & \mathcal{H}^2(\mathbb{D}) \\
\mathcal{H}^2(\mathbb{D} - z_n) & \xrightarrow{M^*_z} & \mathcal{H}^2(\mathbb{D} - z_n) \\
F_{z_n} & \downarrow & F_{z_n} \\
\end{array}
\]
Let \( T \) be the backward shift operator on the Hilbert space \( L^2(\mathbb{N}) \), that is,
\[
T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).
\]
By the natural isomorphism from \( \mathcal{H}^2(\mathbb{D} - z_n) \) to \( \mathcal{H}^2(\partial(\mathbb{D} - z_n)) \), \( M^*_z \) is equivalent to the backward shift operator \( T \) on \( \mathcal{H}^2(\partial(\mathbb{D} - z_n)) \), that is, \( M^*_z \) is surjective and
\[
\dim \ker(M^*_z) = 1.
\]
Hence, \( M^*_z \) is surjective and
\[
\dim \ker(M^*_{z-z_n}) = 1,
\]
where \( 1 \leq n \leq m \).

By the composition of
\[
F_{z_m} \circ F_{z_{m-1}} \circ \cdots \circ F_{z_1},
\]
\[\vspace{12pt}\]
\( \mathcal{M}^*_{p_m(z_1 \cdots z_m)} \) is equivalent to \( T^m \).

That is, \( \mathcal{M}^*_{p_m(z_1 \cdots z_m)} \) is surjective and

\[
\dim \ker \mathcal{M}^*_{p_m(z_1 \cdots z_m)} = m.
\]

Hence, \( \mathcal{M}^*_{p_m(z_1 \cdots z_m)} \) is surjective and

\[
\dim \ker \mathcal{M}^*_{p_m(z_1 \cdots z_m)} = m.
\]

(ii) Because \( \mathcal{H}^\infty \) is an abelian Banach algebra, \( \mathcal{M}^*_{p_m} \) is independent to the permutation of 1-th factors of \( p_m(z) \), that is, \( \mathcal{M}^*_{p_m} \) is independent to the position of 1-th factors of \( p_m(z) = (z - z_1)(z - z_2) \cdots (z - z_m) \).

Also, because \( G = \{ F_s : s \in \mathbb{C} \} \) is an abelian group by the composite operation \( \circ \), for \( 1 \leq n \leq m \), \( F_{z_n} \circ F_{z_{n+1}} \cdots \circ F_{z_1} \) is independent to the permutation of composition. Hence \( \mathcal{M}^*_{p_m} \) is surjective and

\[
\dim \ker \mathcal{M}^*_{p_m} = \dim \ker \mathcal{M}^*_{p_m(z_1 \cdots z_m)} = m.
\]

(iii) By Definitions 8, 9, Lemma 2 and Theorem 2, also by the conditions of this theorem and with \([9, \text{p. 93, p. 305}]\) and \([46, \text{p. 97}]\), we get that the multiplication operator \( \mathcal{M}^*_{\phi} \) associated with the outer function \( \phi \) is surjective.

Hence, we get

\[
\ker \mathcal{M}^*_{\phi(z)} = (\text{ran}\mathcal{M}^*_{\phi(z)})^\perp = (\mathcal{H}^2(\mathbb{D}))^\perp = 0.
\]

Because there exists decomposition

\[
\mathcal{M}^*_{p_m(z)\phi(z)} = \mathcal{M}^*_{\phi(z)} \mathcal{M}^*_{p_m(z)}
\]

on \( \mathcal{H}^2(\mathbb{D}) \), we get

\[
\dim \ker (\mathcal{M}^*_{\phi} - \bar{\lambda}) = \dim \ker \mathcal{M}^*_{p_m(z)\phi(z)} = \dim \ker \mathcal{M}^*_{p_m(z)} = m.
\]

(2) For any given \( \bar{\lambda} \in (\mathbb{D}) \), because of \( 0 \neq \phi \in \mathcal{H}^\infty (\mathbb{D}) \), we get that the multiplication operator \( \mathcal{M}^*_{\phi} - \bar{\lambda} \) is injective by the properties of holomorphic function, hence

\[
\ker (\mathcal{M}^*_{\phi} - \bar{\lambda}) = 0.
\]

Because of

\[
\mathcal{H}^2(\mathbb{D}) = \ker (\mathcal{M}^*_{\phi} - \bar{\lambda})^\perp = \text{cl}[\text{ran}(\mathcal{M}^*_{\phi} - \bar{\lambda})],
\]

with the proof (iii) of (4) in this theorem we get

\[
\text{ran}(\mathcal{M}^*_{\phi} - \bar{\lambda}) = \mathcal{H}^2(\mathbb{D}).
\]

By (1)(2)(3)(4), we get \( \mathcal{M}^*_{\phi} \in B_m(\phi(\mathbb{D})). \)
With Lemma 2 and the proof of Theorem 3, we get the following

**Corollary 1** Let $\phi \in \mathcal{H}^\infty(\mathbb{D})$ be a m-folder holomorphic function, for any given $z_0 \in \mathbb{D}$, if the rooter function of $\phi$ at $z_0$ is invertible in the Banach algebra $\mathcal{H}^\infty(\mathbb{D})$, then $\phi$ is a Cowen–Douglas function. Especially, for any given $n \geq 1$, the non-constant polynomial $p_n(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{H}^\infty(\mathbb{D})$ is a Cowen–Douglas function.

**Theorem 4** Let $\phi(z) \in \mathcal{H}^\infty(\mathbb{D})$, $M_\phi$ be the associated multiplication operator on $\mathcal{H}^2(\mathbb{D})$. The adjoint multiplier $M_\phi^* \in \mathcal{B}_m(\phi(\mathbb{D}))$ if and only if $\phi$ is a m-folder holomorphic function and the rooter function of $\phi$ at $z_0 \in \mathbb{D}$ is an outer function.

**Proof** Following Theorem 3, we get the sufficient condition $\iff$. Hence, it is enough to prove the necessary condition $\implies$.

From Definitions 3, 9, and the proof (1) and (2) of Theorem 3, we get that for any given $z_0 \in \mathbb{D}$, there is a constant $m \in \mathbb{N}$ such that

$$\phi(z) - \phi(z_0) = p_m(z) h_{z_0}(z), \quad \text{ran} M_{p_m(z) h_{z_0}(z)}^* = \mathcal{H}^2(\mathbb{D}),$$

and

$$\text{ind}(M_{p_m(z) h_{z_0}(z)}) = \dim \ker M_{p_m(z) h_{z_0}(z)} - \dim \ker M_{p_m(z) h_{z_0}(z)}^* = -m.$$  

Where the rooter function $h_{z_0}(z)$ has non-zero point on $\mathbb{D}$, the roots of $p_m(z)$ are all in $\mathbb{D}$,

$$\ker M_{h_{z_0}(z)} = 0 \quad \text{and} \quad \text{ind}(M_{p_m(z)}) = -m.$$  

By Proposition 1 and the proof (iii) of (4) in Theorem 3, with a simple computing it is easy to get

$$\text{ind}(M_{h_{z_0}(z)}) = 0 \quad \text{and} \quad \text{ran} M_{h_{z_0}(z)}^* = \mathcal{H}^2(\mathbb{D}).$$

That is

$$\ker M_{h_{z_0}(z)}^* = 0.$$  

With Definition 8, it is easily to get that $\text{cl}[h_{z_0}(z) \mathcal{P}(z)] = \mathcal{H}^2(\mathbb{D})$, that is, $h_{z_0}(z)$ is an out function, where $\mathcal{P}(z)$ is the set of all polynomial about $z$. \qed

### 4 Applications of Cowen–Douglas function and chaos

In this section, we give some properties about the adjoint multiplier of Cowen–Douglas function and give some applications of which on the inverse problem of chaos, where $\phi$ is a Cowen–Douglas function and $T$ is the backward shift operator on $L^2(\mathbb{N})$, i.e.,
Theorem 5  If $\phi \in H^\infty(\mathbb{D})$ is a Cowen–Douglas function, $M_\phi$ is the associated multiplication operator on $H^2(\mathbb{D})$. Then the following assertions are equivalent

(1) $M_\phi^*$ is Devaney chaotic;
(2) $M_\phi^*$ is I-distributionally chaotic;
(3) $M_\phi^*$ is II-distributionally chaotic;
(4) $M_\phi^*$ is III-distributionally chaotic;
(5) $M_\phi^*$ is strongly mixing;
(6) $M_\phi^*$ is Li–Yorke chaotic;
(7) $M_\phi^*$ is hypercyclic;
(8) $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Proof  By [3, p. 138] we get that $M_\phi^*$ is Devaney chaotic if and only if it is hypercyclic, i.e., $\phi$ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, hence

$$ (1) \iff (7) \iff (8). $$

With the fact that strong mixing implies Li–Yorke chaos and the fact that $\hat{\phi}(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ are mutually equivalent. By Definition 4, Lemma 1 and Theorem 1, it is enough to prove that (6) implies (8).

If $M_\phi^*$ is Li–Yorke chaotic, then $\phi$ is non-constant and by [21, Theorem 3.5] we get

$$ \sup_{n \to +\infty} \|M_\phi^n\| \to \infty, $$

hence,

$$ \|M_\phi\| = \|M_\phi^*\| > 1, \quad \text{that is,} \quad \sup_{z \in \mathbb{D}} |\phi(z)| > 1. $$

Moreover, we get

$$ \inf_{z \in \mathbb{D}} |\phi(z)| < 1, $$

Indeed, if

$$ \inf_{z \in \mathbb{D}} |\phi(z)| \geq 1, $$

then

$$ \frac{1}{\phi} \in H^\infty \quad \text{and} \quad \|M_\phi^*\| = \|M_{\frac{1}{\phi}}\| \leq 1. $$

Hence, for any $0 \neq f \in H^2(\mathbb{D})$, we get

$\mathbb{B}$ Birkhäuser
\[ \|M_{\phi}^n f\| \geq \frac{1}{\|M_{\phi}^{-n}\|} \|f\| \geq \frac{1}{\|M_{\phi}^*\|} \|f\| \geq \|f\|. \]

It is a contradiction to that \( M_{\phi}^* \) is Li–Yorke chaotic.

Therefore, the Li–Yorke chaos of \( M_{\phi}^* \) implies

\[ \inf_{z \in \mathbb{D}} |\phi(z)| < 1 < \sup_{z \in \mathbb{D}} |\phi(z)|. \]

Because of the properties of simple connected of \( \phi(\mathbb{D}) \), we get \( \phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \), that is, (6) implies (8).

\[ \square \]

**Corollary 2** If \( \phi \in \mathcal{H}^\infty(\mathbb{D}) \) is an invertible Cowen–Douglas function in the Banach algebra \( \mathcal{H}^\infty(\mathbb{D}) \), and let \( M_{\phi} \) be the associated multiplication operator on \( \mathcal{H}^2(\mathbb{D}) \). Then \( M_{\phi}^* \) has property \( \alpha \) if and only if \( M_{\phi}^{-1} \) has, where \( \alpha \in \{ \text{Devaney chaotic, I-distributionally chaotic, II-distributionally chaotic, III-distributionally chaotic, strongly mixing, Li–Yorke chaotic, hypercyclic} \} \).

**Proof** Because of \( T = (T^{-1})^{-1} \), it is enough to prove that \( M_{\phi}^* \) has property \( \alpha \) implies that \( M_{\phi}^{-1} \) has property \( \alpha \).

By Definition 4, we get

\[ M_{\phi}^* \in \mathcal{B}_n(\bar{\phi}(\mathbb{D})), \]

with a simple computing we get

\[ M_{\phi}^{-1} \in \mathcal{B}_n\left(\frac{1}{\bar{\phi}}(\mathbb{D})\right). \]

If \( M_{\phi}^* \) has property \( \alpha \), by Theorem 5, we get

\[ \phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset. \]

And by the properties of holomorphic function, we get

\[ \frac{1}{\bar{\phi}}(\mathbb{D}) \cap \mathbb{T} \neq \emptyset. \]

Because of

\[ M_{\phi}^{-1} \in \mathcal{B}_n\left(\frac{1}{\bar{\phi}}(\mathbb{D})\right) \]

and by Theorem 5, we get that \( M_{\phi}^{-1} \) has property \( \alpha \). \[ \square \]

We now study some properties about scalars perturbation of an operator inspired by [4, 24] which research some properties about the compact perturbation of scalars.
In [23], the authors defined Li–Yorke chaos translation set $S_{LY}(T)$ for linear operators $T \in B(\mathbb{H})$, similarly we can give the following

**Definition 10** Let $T \in B(\mathbb{H})$ and we define

(i) $S_{H}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is hypercyclic} \}$;
(ii) $S_{DV}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is Devaney chaotic} \}$;
(iii) $S_{LY}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is Li-Yorke chaotic} \}$;
(iv) $S_{DC1}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is I-distributionally chaotic} \}$;
(v) $S_{DC2}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is II-distributionally chaotic} \}$;
(vi) $S_{DC3}(T) = \{ \lambda \in \mathbb{C}; \lambda + T \text{ is III-distributionally chaotic} \}$.

It is easy to get the following properties

\[
S_{H}(\lambda + T) = S_{H}(T) - \lambda, \\
S_{DV}(\lambda + T) = S_{DV}(T) - \lambda, \\
S_{LY}(\lambda + T) = S_{LY}(T) - \lambda, \\
S_{DC1}(\lambda + T) = S_{DC1}(T) - \lambda, \\
S_{DC2}(\lambda + T) = S_{DC2}(T) - \lambda, \\
S_{DC3}(\lambda + T) = S_{DC3}(T) - \lambda.
\]

Now, we give the following

**Example 1** Let $T$ be the backward shift operator on $L^2(\mathbb{N})$,

\[
T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).
\]

Then

\[
S_{LY}(T) = S_{DC1}(T) = S_{DC2}(T) = S_{DC3}(T) = S_{DV}(T) = S_{H}(T) = 2\mathbb{D}\setminus\{0\}, \\
S_{LY}(2T) = S_{DC1}(2T) = S_{DC2}(2T) = S_{DC3}(2T) = S_{DV}(2T) = S_{H}(2T) = 3\mathbb{D}.
\]

**Proof** By [9, p. 209], we get $\sigma(T) = cl\mathbb{D}$ and $\sigma(2T) = cl2\mathbb{D}$, by the properties of $\sigma(T)$, we get

\[
\sigma(\lambda + T) = \lambda + cl\mathbb{D}.
\]

Because the method is similarly for $T$ and $2T$, we only prove the conclusion for $T$.

Let

\[
L^2(\mathbb{N}) = \mathcal{H}^2(\mathbb{T}),
\]

with the definition of $T$ and the natural isomorphism between $\mathcal{H}^2(\mathbb{T})$ and $\mathcal{H}^2(\mathbb{D})$, we get that $(\lambda + T)^*$ is the multiplication operator $M_f$ associated with $f(z) = \lambda + z$ on the Hardy space $\mathcal{H}^2(\mathbb{T})$.

By the Dirichlet Problem [7, p.103], we get that $f(z)$ is associated with the holomorphic function
\[ \phi(z) = \bar{\lambda} + z \in \mathcal{H}^\infty(D) \]
determined by the boundary condition \( \phi(z)|_T = f(z) \).

From Corollary 1, we get that \( \phi \) is a Cowen–Douglas function. Therefore, \( \lambda + T \) is naturally equivalent to the operator \( M_\phi^* \) on \( \mathcal{H}^2(D) \).

Because of
\[
\sigma(\lambda + T) = \sigma(\bar{\lambda} + T^*),
\]
we get
\[
\sigma(\lambda + T) = \sigma(M_\phi^*) = \sigma(M_\phi) \supseteq \phi(\mathbb{D}).
\]

With Theorem 5, we get
\[
S_{LY}(T) = S_{DC1}(T) = S_{DC2}(T) = S_{DC3}(T) = S_{DV}(T) = S_{H}(T) = 2\mathbb{D} \setminus \{0\}.
\]

Corollary 3 If \( \phi \in \mathcal{H}^\infty(D) \) is an invertible Cowen–Douglas function in the Banach algebra \( \mathcal{H}^\infty(D) \), and let \( T \) be the backward shift operator on \( L^2(\mathbb{N}) \), Then \( \phi(T) \) has property \( \alpha \) if and only if \( \phi^{-1}(T) \) has, where \( \alpha \in \{ \text{Devaney chaotic}, \text{I-distributionally chaotic}, \text{II-distributionally chaotic}, \text{III-distributionally chaotic}, \text{strongly mixing}, \text{Li–Yorke chaotic}, \text{hypercyclic} \} \).

Corollary 4 Let \( T \) be the backward shift operator on \( L^2(\mathbb{N}) \). For any given \( n \in \mathbb{N} \), if non-constant \( p_n(T) = \sum_{k=0}^{n} a_k T^k \) is invertible, then \( p_n(T) \) has property \( \alpha \) if and only if \( (p_n(T))^{-1} \) has, where \( \alpha \in \{ \text{Devaney chaotic}, \text{I-distributionally chaotic}, \text{II-distributionally chaotic}, \text{III-distributionally chaotic}, \text{strongly mixing}, \text{Li–Yorke chaotic}, \text{hypercyclic} \} \).

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