STRONG SOLUTIONS TO CAUCHY PROBLEM OF 2D COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS

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ABSTRACT. This paper studies the local existence of strong solutions to the Cauchy problem of the 2D simplified Ericksen-Leslie system modeling compressible nematic liquid crystal flows, coupled via $\rho$ (the density of the fluid), $u$ (the velocity of the field), and $d$ (the macroscopic/continuum molecular orientations). Notice that the technique used for the corresponding 3D local well-posedness of strong solutions fails treating the 2D case, because the $L^p$-norm ($p > 2$) of the velocity $u$ cannot be controlled in terms only of $\rho^2 u$ and $\nabla u$ here. In the present paper, under the framework of weighted approximation estimates introduced in [J. Li, Z. Liang, On classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum, J. Math. Pures Appl. (2014) 640–671] for Navier-Stokes equations, we obtain the local existence of strong solutions to the 2D compressible nematic liquid crystal flows.

1. Introduction. In this paper, we study the following simplified version of the 2D Ericksen-Leslie model

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= \mu \Delta u + (\lambda + \mu) \nabla \text{div} u - \nabla d \cdot \Delta d, \\
d_t + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d
\end{align*}
$$

in $\mathbb{R}^2 \times \mathbb{R}^+$, with the far-field behavior

$$
(\rho, u, d)(x, t) \to (0, 0, 1) \text{ as } |x| \to \infty, \ t > 0,
$$

and initial data

$$
\rho(x, 0) = \rho_0(x), \ \ (\rho u)(x, 0) = m_0(x), \ d(x, 0) = d_0(x), \ x \in \mathbb{R}^2.
$$

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Here $\rho : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}_+^+$ is the density of the fluid, $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ the velocity field, $P(\rho) = \rho^\gamma$ with $\gamma > 1$ the pressure, and $d : \mathbb{R}^2 \times [0, \infty) \to \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) the macroscopic/continuum molecular orientations. The constants $\mu$, $\lambda$ are viscosity coefficients satisfying the physical condition

$$\mu > 0, \quad \lambda + \mu \geq 0.$$  

(4)

Liquid crystals are substances with matter phases between conventional liquids and solid crystals [4]. The hydrodynamic flow of incompressible liquid crystals was first proposed by Ericksen and Leslie in 1960’s [3, 10]. Since then there have been remarkable developments in the field from both theoretical and applied aspects. Considering an incompressible, viscous fluid, Lin [15] first derived a simplified Ericksen-Leslie system (i.e. $\rho = \text{constant}$ and $\text{div}u = 0$ in (1)) to model liquid crystal flows in 1989. Consequently, some important results were obtained in a series of papers by Lin-Liu [14, 15, 16], such as the existence of weak and strong solutions, as well as the partial regularity of suitable weak solutions, where a Ginzburg-Landau approximation was considered. Refer to the survey [18] for more results on the incompressible liquid crystals. Recently, Liu-Zhang [23] established the global existence of strong solutions to the 2D Cauchy nonhomogeneous incompressible problem (i.e. $\text{div}u = 0$ in (1)) without vacuum as far field density. Liu-Liu-Tan-Zhong [21] extended the result of [23] to the vacuum case with large initial data, provided that the initial orientation $d_0 = (d_{01}, d_{02}, d_{03})$ satisfies a geometric condition

$$d_{03} \geq \varepsilon_0$$

in $\mathbb{R}^2$ for some positive $\varepsilon_0 > 0$.  

(5)

Li-Liu-Zhong [13] got the same result under small initial data without the additional geometric condition (5). Extended to the more complicated compressible case, the simplified Ericksen-Leslie system is strongly coupled via the compressible Navier-Stokes equation and the transported harmonic map heat flow to $\mathbb{S}^2$, with significant progresses made during past years. Among them, Ding-Lin-Wang-Wen [1, 2] obtained the global existence for weak and strong solutions of the 1D problem. For the 2D case, under the condition with the image of $d_0$ contained in the upper hemisphere $\mathbb{S}^2_+$, Jiang-Jiang-Wang [9] established the existence of global weak solutions. In the 3D case, Huang-Wang-Wen [6] studied the local existence of strong solutions of (1). Moreover, Huang-Wang-Wen [6, 7] and Huang-Wang [8] obtained some blow-up criteria. The local strong solution has been shown to be global for small initial energy in [11]. Very recently, Lin-Lai-Wang [19] established the existence of finite energy weak solutions with large initial data, provided the initial orientational director field $d_0$ lies in $\mathbb{S}^2_+$.  

Recently, Li-Liang [12] established the local existence of classical solutions to the Cauchy problem of Navier-Stokes equation

$$\left\{ \begin{array}{l}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u,
\end{array} \right.$$  

$(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+^+$  

(6)

via weighted estimates instead of the general ones. It can be found that the systems [1] and [6] share the same continuity equation. However, [1] is more complicated than [6] because of the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation [1], and the strong coupling nonlinear term $\Delta d \cdot \nabla d$ in the momentum equation [1] involved here.

The aim of this paper is to establish the local existence of strong solutions to the 2D Cauchy problem [1]. Notice that the local well-posedess of strong solutions
for the 3D case obtained by Huang-Wang-Wen \[6\] is not admitted for the 2D case. This is mainly due to that in the 2D case the \(L^p\)-norm \((p > 2)\) of the velocity \(u\) cannot be controlled in terms only of \(\rho^{1/2} u\) and \(\nabla u\). Moreover, the coupling of \(u\) and \(d\), and the presence of \(|\nabla d|^2 d\) bring additional difficulties. So, some new ideas and careful estimates are necessary to deal with the 2D case. In the present paper, we will use the framework of weighted approximation estimates introduced in \[12\] for Navier-Stokes equations to overcome these difficulties. It would be interesting to consider the global existence of the strong solutions to the problem \[1\], for which one should study the blow-up mechanism with the structure of possible singularities to the strong solutions of \([1]\). Along this direction, very recently, Liu-Wang \[22\] and Wang \[25\] obtained for the 2D isentropic compressible nematic liquid crystal flows \([1]\) that if \(T^* \in (0, \infty)\) is the maximal time of existence for strong solutions \((\rho, u, d)\), then

\[
\lim_{T \to T^*} \|\rho\|_{L^\infty(0, T; L^\infty)} = \infty.
\]

Throughout the paper, we use simplified notations

\[
\int f dx = \int f dx,
\]

and

\[
L^p = L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2},
\]

\[
\mathcal{D}^{1,2}(\Omega) := \{v \in H^1_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega)\},
\]

with \(p \in [1, \infty), k \geq 0\), and \(\Omega = \mathbb{R}^2\) or \(\Omega = B_R := \{x \in \mathbb{R}^2 | |x| < R\}\).

Denote

\[
\bar{x} := (e + |x|^2)^{1/2} \log^{1+\eta_0}(e + |x|^2),
\]

with \(\eta_0 > 0\), \(B_N := \{x \in \mathbb{R}^2 | |x| < N\}\). The main result of this paper is stated as the following theorem:

**Theorem 1.1.** Suppose that the initial data \((\rho_0, u_0, d_0)\) satisfy

\[
\rho_0 \geq 0, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \rho_0^{1/2} u_0 \in L^2,
\]

\[
\nabla^2 d_0 \in L^2, \quad \bar{x}^a \nabla d_0 \in L^2, \quad m_0 = \rho_0 u_0, \quad |d_0| = 1,
\]

with \(q > 2\) and \(1 < a < 2\). Then there exist \(T_0, N > 0\) such that the problem \([1]-[3]\) has a unique strong solution \((\rho, u, d)\) on \(\mathbb{R}^2 \times (0, T_0)\) satisfying

\[
\left\{
\begin{array}{l}
\rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \quad \bar{x}^a \rho \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{\bar{x}} \nabla u \bar{u} \in L^\infty(0, T_0; L^2), \\
\nabla u \in L^2(0, T_0; H^1) \cap L^{2+} (0, T_0; W^{1,q}), \quad \sqrt{\nabla} u \in L^2(0, T_0; W^{1,q}), \\
\bar{x} d, |\nabla d|^2, \bar{x}^2 \nabla d, \nabla^2 d, \sqrt{\bar{x}} \nabla d \sqrt{\bar{x}} \nabla^2 d \in L^\infty(0, T_0; L^2), \\
\nabla^2 d \in L^2(0, T_0; H^1), \quad \sqrt{\nabla} u \in L^2(0, T_0; W^{1,q}), \\
\sqrt{\rho} u_t, \bar{x}^a \nabla^2 d, \sqrt{\nabla} u_t, \sqrt{\bar{x}} \nabla^2 d_t, \sqrt{\bar{x}^{-1} u_t} \in L^2(\mathbb{R}^2 \times (0, T_0)), \\
\sqrt{\bar{x}} \nabla^2 d \in L^2(\mathbb{R}^2 \times (0, T_0)),
\end{array}
\right.
\]

and

\[
\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx.
\]


Notice that in the framework of the paper, some weighted assumptions on \( \rho_0 \) and \( \nabla \rho_0 \), differently from the case of \( \mathbb{R}^3 \), are required in Theorem 1.1.

The rest of the paper is organized as follows. In Section 2, we recall some elementary facts and inequalities used in the sequel. Sections 3 deals with an approximation problem on \( B_R \) to derive uniform estimates for the unique strong solution with respect to \( R \). Finally, we give the proof of Theorem 1.1 in Section 4.

2. Preliminaries. In this section, we recall some known results as preliminaries.

Consider an approximation problem for (1)

\[
\begin{aligned}
\rho^R_t + \text{div}(\rho^R u^R) &= 0, \\
(\rho^R u^R)_t + \text{div}(\rho^R u^R \otimes u^R) + \nabla P(\rho^R) &= \mu \Delta u^R + (\lambda + \mu) \nabla \text{div} u^R - \nabla d^R \cdot \Delta d^R, \\
d^R_t + u^R \cdot \nabla d^R &= \Delta d^R + |\nabla d^R|^2 d^R
\end{aligned}
\]

in \( B_R \times \mathbb{R}^+ \), with

\[
u^R = \frac{\partial d^R}{\partial \nu} = 0 \text{ on } \partial B_R \times \mathbb{R}^+, \quad (\rho^R, u^R, d^R)(x, 0) = (\rho_0^R, u_0^R, d_0^R)(x) \text{ in } B_R.
\]

By an argument similar to that in [6], it is easy to establish the local existence and uniqueness of classical solutions to (9). We give this result as a lemma below without proof. Then we will prove in the next two sections that the classical solutions \((\rho^R, u^R, d^R)\) of (9) converge to the strong solution of the original Cauchy problem (1) by letting \( R \to \infty \).

Lemma 2.1. For any given \( R > 0 \), assume that \((\rho_0^R, u_0^R, d_0^R)\) satisfies

\[
(\rho_0^R, u_0^R, \nabla d_0^R) \in H^3(B_R) \text{ with } |d_0^R| = 1, \quad \inf_{x \in B_R} \rho_0^R > 0,
\]

\[
u^R = \frac{\partial d_0^R}{\partial \nu} = 0, \quad x \in \partial B_R.
\]

Then there exist \( T_R > 0 \) and a unique classical solution \((\rho^R, u^R, d^R)\) to (9) on \( B_R \times (0, T_R) \) such that

\[
\begin{aligned}
\rho^R \in C([0, T_R]; H^3), \quad \rho^R_t \in L^\infty(0, T_R; H^2), \quad \sqrt{\rho^R}u^R_t \in L^\infty(0, T_R; L^2), \\
u^R \in C([0, T_R]; H^3) \cap L^2(0, T_R; H^4), \quad u^R_t \in L^\infty(0, T_R; H^1) \cap L^2(0, T_R; H^2), \\
\nabla d^R \in C([0, T_R]; H^3) \cap L^2(0, T_R; H^1), \\
d^R \in C([0, T_R]; H^3) \cap L^2(0, T_R; H^2), \quad u_t^R \in L^\infty(0, T_R; H^2), \quad u_t^{R, 1} \in L^\infty(0, T_R; H^1), \\
\n\sqrt{\rho^R}u^R_{tt} \in (0, T_R; L^2), \quad tu^R_t \in L^\infty(0, T_R; H^3), \\
tu^R_{tt} \in L^\infty(0, T_R; H^1) \cap L^2(0, T_R; H^2), \quad t\sqrt{\rho^R}u^R_{tt} \in L^\infty(0, T_R; L^2), \\
t^2u^R_{tt} \in L^\infty(0, T_R; H^2), \quad t^2u^R_{tt} \in L^\infty(0, T_R; H^1), \\
t^2d^R \in L^\infty(0, T_R; H^2), \quad t^2d^R_t \in L^\infty(0, T_R; H^3), \\
t^2d^R_{tt} \in L^\infty(0, T_R; H^2), \quad td^R_t \in L^\infty(0, T_R; H^2).
\end{aligned}
\]

In addition, we cite a lemma involving estimates on weighted bounds for functions in \( D^{1,2}(\Omega) \).

Lemma 2.2. [12 Lemma 2.4] Let \( \bar{x} \) and \( \eta_0 \) be as in Theorem 1.1 with \( \Omega = \mathbb{R}^2 \) or \( \Omega = B_R \) and \( \rho \in L^1(\Omega) \cap L^\gamma(\Omega) \) with \( \gamma > 1 \) be a non-negative function satisfying

\[
\int_{B_{N_1}} \rho \, dx \geq M_1, \quad \int \rho^2 \, dx \leq M_2,
\]
with $M_1, M_2 > 0$, and $B_{N_1} \subset \Omega$ ($N_1 \geq 1$). Then for every $v \in \tilde{D}^{1,2}(\Omega)$, there is $C = C(M_1, M_2, N_1, \gamma, \eta_0) > 0$ such that

$$\|v\tilde{x}^{-1}\|_{L^2} \leq C\|\rho^{\frac{1}{2}}v\|_{L^2} + C\|\nabla v\|_{L^2}. \quad (13)$$

Moreover, there is $C = C(\varepsilon, \eta, M_1, M_2, N_1, \gamma, \eta_0) > 0$ such that

$$\|v\tilde{x}^{-\eta}\|_{L^{(2+\varepsilon)\eta}} \leq C\|\rho^{\frac{1}{2}}v\|_{L^2} + C\|\nabla v\|_{L^2} \quad (14)$$

with $\tilde{\eta} = \min\{1, \eta\}$.

3. **Uniform estimates for approximation problem.** In this section, we will derive some uniform estimates for the solution $(\rho^R, u^R, d^R)$ to the approximation problem (9)-(10) ensured by Lemma 2.1, independent of the lower bound of the initial density and the size of the domain $B_R$, which are crucial to prove the local existence of strong solutions to the Cauchy problem (1) with the initial vacuum permitted. For simplicity, denote $(\rho^R, u^R, d^R)$ by $(\rho, u, d)$.

Without loss of generality, assume that there exists $N_0 > 0$ such that

$$\frac{1}{2} \leq \int_{B_{N_0}} \rho_0(x)dx \leq \int_{B_R} \rho_0(x)dx \leq \frac{3}{2} \quad (15)$$

with $R > 4N_0 \geq 4$, due to the assumption (11).

Now we deal with the required estimates to the approximation solution $(\rho, u, d)$. For simplicity, denote

$$\psi(t) := 1 + \|\rho^{\frac{1}{2}}u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla^2 \tilde{x}\|_{L^2} + \|\tilde{x}^2 \nabla \tilde{x}\|_{L^2} + \|\tilde{x}^2 \nabla^2 \tilde{x}\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2},$$

$$E_0 := \|\tilde{x}^2 \nabla \rho_0\|_{L^2} + \|\tilde{x}^2 \nabla u_0\|_{L^2} + \|\tilde{x}^2 \nabla^2 d_0\|_{L^2} + \|\tilde{x}^2 \nabla^2 d_0\|_{L^2} + \|\tilde{x}^2 \nabla^2 d_0\|_{L^2} + \|\tilde{x}^2 \nabla^2 d_0\|_{L^2}. \quad (17)$$

**Proposition 1.** Assume that $(\rho_0, u_0, d_0)$ satisfies (11) and (15). Then there exist $T_0, M > 0$, both depending only on $\rho, \gamma, q, a, \eta_0, N_0$, and $E_0$, such that

$$\sup_{0 \leq t \leq T_0} \psi(t) + \int_0^{T_0} (\|\nabla^2 u\|_{L^4}^{\frac{4}{3}} + t\|\nabla^2 u\|_{L^4} + \|\nabla^2 u\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2} + \|\tilde{x}^2 \nabla^2 d\|_{L^2})dt \leq M. \quad (16)$$

Proposition 1 will be proved via the next three lemmas.

**Lemma 3.1.** Under the conditions of Proposition 1, let $(\rho, u, d)$ be a smooth solution to the initial-boundary value problem (9)-(10). Then there exist $T_1 = T_1(N_0, E_0) > 0$ and $\alpha = \alpha(\gamma, q) > 1$ such that for all $t \in (0, T_1)$

$$\sup_{0 \leq s \leq t} \|\tilde{x}^2 \nabla d\|_{L^2}^2 + \int_0^t \|\tilde{x}^2 \nabla^2 d\|_{L^2}^2 ds \leq C \exp \left\{ C \int_0^t \psi^{\alpha} ds \right\}, \quad (17)$$

$$\sup_{0 \leq s \leq t} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) \quad (18)$$

$$+ \int_0^t (\|\rho^{\frac{1}{2}} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\},$$

where and throughout the paper, denote by $C$ generic positive constants independent of $R$. 


Proof. First, [9] Theorem 1.1] says that the solution satisfies the energy inequality
\[
\sup_{0 \leq s \leq t} (\|\rho^s u\|_{L^2}^2 + \|\rho\|_{L^\infty}^2 + \|\nabla d\|_{L^2}^2) + \int_0^t (\|\nabla u\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2) ds \leq C. \quad (19)
\]
The conservation of \(\rho\) with (19) yields that there exists \(T_1 > 0\) such that
\[
\inf_{0 \leq t \leq T_1} \int_{B_{2\mathcal{N}_0}} \rho dx \geq \frac{1}{4}, \quad (20)
\]
that is [12] (3.8). Furthermore, corresponding to (3.10) obtained in [12], we have by (20) that
\[
\|\rho^\eta u\|_{L^{2+\eta}} + \|u\bar{x}^{-\eta}\|_{L^{2+\eta}} \leq C\varepsilon \gamma^1 \psi^1, \quad t \in (0, T_1]
\]
with \(\bar{\eta} = \min\{1, \eta\}\). Next, we always assume that \(t \leq T_1\).

To obtain [17], apply the gradient operator to [9]_3,
\[
\nabla d - \nabla \Delta d = -\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d).
\]
Multiplying (22) by \(\bar{x}^a \nabla d\) and integrating by parts yield
\[
\frac{1}{2} \left( \int |\nabla d|^2 \bar{x}^a dx \right)_1 + \int |\nabla d|^2 \bar{x}^a dx 
\]
\[
\leq \frac{1}{2} \int |\nabla d|^2 \Delta \bar{x}^a dx + \int |\nabla d|^2 \bar{x}^a dx + \int |\nabla u| |\nabla d|^2 \bar{x}^a dx + \frac{1}{2} \int |u| |\nabla d|^2 \nabla \bar{x}^a dx 
\]
\[
=:\sum_{i=1}^4 J_i. \quad (23)
\]
By virtue of [19] and (21), we have
\[
J_1 \leq C \int |\nabla d|^2 \bar{x}^a \bar{x}^{-2} \log^{2(1+\gamma_0)} \gamma (e + |x|^2) dx \leq C \int |\nabla d|^2 \bar{x}^a dx,
\]
\[
J_2 = \int |\nabla d|^2 |\nabla d|^2 \bar{x}^a dx \leq \|\nabla d\|_{L^2}^2 \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2 
\]
\[
\leq C(1 + \|\nabla d\|_{L^2}^2)(\|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2} + \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^\infty}) \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2} 
\]
\[
\leq C\varepsilon \gamma^3 \gamma \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2 + \varepsilon \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2,
\]
\[
J_3 = \int |\nabla u| |\nabla d|^2 \bar{x}^a dx \leq \|\nabla u\|_{L^2} \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2 
\]
\[
\leq C\varepsilon \gamma^3 \gamma \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2 + \varepsilon \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2,
\]
\[
J_4 = \int |\nabla d| \bar{x}^{\frac{3}{2}} |\nabla d| \bar{x}^{\frac{3}{2}} |u| \bar{x}^{-\frac{3}{2}} \bar{x}^{-\frac{1}{2}} \log^{(1+\gamma_0)} \gamma (e + |x|^2) dx 
\]
\[
\leq C \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^4} \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2} \|\bar{x}^{-\frac{3}{2}} u\|_{L^4} 
\]
\[
\leq C \left( \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2} + \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^\infty} \right)^{\frac{1}{2}} \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2} \psi^\alpha(t) 
\]
\[
\leq C\varepsilon \gamma^3 \gamma \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2 + \varepsilon \|\bar{x}^{\frac{3}{2}} \nabla d\|_{L^2}^2,
\]
where Young’s inequality and the fact from the Gagliardo-Nirenberg inequality are used that
\[
\|\nabla d\|_{L^4}^2 \leq C \|\nabla d\|_{L^2} \|\nabla d\|_{H^1} \leq C + C\psi^\alpha, \quad p \in (2, +\infty).
\]
Multiplying (22) by $4$ and integrating over $(0, t)$, we get (17) by Gronwall’s inequality. To prove (18), multiply (22) by $u_t$ and integrate by parts over $B_R$,

$$\frac{d}{dt} \int (\mu|\nabla u|^2 + (\mu + \lambda)\text{div} u^2) dx + \int \rho |u_t|^2 dx \leq C \int \rho |u|^2 |\nabla u|^2 dx + 2 \int P \text{div} u_t dx + \int M(d) : \nabla u_t dx,$$

(24)

where

$$M(d) := \nabla d \circ \nabla d - \frac{1}{2} |\nabla d|^2_{L^2}, \quad (\nabla d \circ \nabla d)_{ij} := \frac{\partial d}{\partial x_i} \cdot \frac{\partial d}{\partial x_j}, \quad i, j = 1, 2,$$

and thus $\text{div}(M(d)) = \nabla d \cdot \Delta d$, while the notation “$:\cdot$” represents the operation between two matrices that

$$A : B := \sum_{i,j=1}^{2} a_{i,j}b_{i,j} \quad \text{for} \quad A = (a_{i,j})_{2 \times 2} \quad \text{and} \quad B = (b_{i,j})_{2 \times 2}.$$

For the terms involved in (24), by Hölder’s and Young’s inequalities,

$$\int \rho |u|^2 |\nabla u|^2 dx \leq C_\delta \psi^\alpha + \delta \psi^{-2} ||\nabla^2 u||^2_{L^2}, \quad (25)$$

$$\int P \text{div} u_t dx \leq \frac{d}{dt} \int P \text{div} u dx + C_\delta \psi^\alpha + \delta \psi^{-1} ||\nabla^2 u||^2_{L^2}, \quad (26)$$

$$\int M(d) : \nabla u_t dx = \frac{d}{dt} \int M(d) : \nabla u dx - \int M(d)_t : \nabla u dx$$

$$\leq \frac{d}{dt} \int M(d) : \nabla u dx + \int |\nabla d||\nabla d_t||\nabla u| dx$$

$$\leq \frac{d}{dt} \int M(d) : \nabla u dx + \varepsilon ||\nabla d_t||^2_{L^2} + C_\varepsilon ||\nabla d||^2_{L^2} ||\nabla u||^2_{L^2}, \quad (27)$$

where it is used that

$$||\nabla u||_{L^p} \leq C_p ||\nabla u||^\frac{2}{p} \left(\frac{2}{p} - \frac{1}{2}\right) \leq C_p \psi + C_p \psi ||\nabla^2 u||^\frac{1}{2} \left(\frac{2}{p} - \frac{1}{2}\right), \quad p \in (2, +\infty).$$

(28)

Now consider the orientation field $d$. From (22) we have

$$\frac{d}{dt} \int |\Delta d|^2 dx + \int (|\nabla d_t|^2 + |\nabla \Delta d|^2) dx = \int |\nabla d_t - \nabla \Delta d|^2 dx = \int |-\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d)|^2 dx \quad (29)$$

$$\leq C \int (|\nabla d|_0^4 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2) dx.$$

Multiplying (22) by $4|\nabla d|^2 \nabla d$ and integrating by parts lead to

$$\frac{d}{dt} \int |\nabla d|^4 dx + 4 \int (|\nabla d|^2 |\nabla^2 d|^2 + 2|\nabla d|^2 |\nabla (\nabla d)|^2) dx$$

$$= 4 \int |\nabla d|^2 \nabla d (-\nabla (u \cdot \nabla d) + \nabla (|\nabla d|^2 d)) dx.$$
\[
\begin{align*}
&\leq C \int (|\nabla d|^4|\nabla u| + |\nabla d|^3|\nabla^2 d||u| + |\nabla d|^4|\nabla^2 d| + |\nabla d|^6) dx \\
&\leq \int |\nabla d|^2|\nabla^2 d|^2 dx + C \int (|\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6) dx,
\end{align*}
\]
and hence
\[
\frac{d}{dt} \int |\nabla d|^4 dx + 3 \int |\nabla d|^2|\nabla^2 d|^2 dx \leq C \int |\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6 dx.
\]
Adding (29) to (30) leads to
\[
\frac{d}{dt} \int (|\Delta d|^2 + |\nabla d|^4) dx + \int (|\nabla d|^2 + |\nabla^3 d|^2 + |\nabla d|^2|\nabla^2 d|^2) dx \\
\leq C \int (|\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2) dx
\]
\[
= \sum_{i=1}^{4} \tilde{J}_i.
\]
By Hölder’s and Young’s inequalities,
\[
\begin{align*}
|\tilde{J}_1| &\leq 2C\|\nabla d\|^\frac{1}{2} \|\nabla d\|^\frac{3}{2} \leq C + C\|\nabla^2 d\|^2 \leq C\psi^\alpha, \\
|\tilde{J}_2| &\leq \|\nabla d\|^\frac{1}{2} \|\nabla^2 d\|^2, \\
&\leq C\|\nabla d\|^\frac{1}{2} \left(\|\nabla d\|_L^2 + \|\nabla^2 d\|_L^2\right) \frac{1}{2} \|\nabla^2 d\|^2 \left(\|\nabla^2 d\|_L^2 + \|\nabla^3 d\|_L^2\right)^\frac{1}{2} \\
&\leq C(1 + \|\nabla^2 d\|_L^2)^\frac{1}{2} \|\nabla^2 d\|^2 \left(\|\nabla^2 d\|_L^2 + \|\nabla^3 d\|_L^2\right)^\frac{1}{2} \\
&\leq \varepsilon \|\nabla^3 d\|^2 + C\varepsilon \psi^\alpha, \\
|\tilde{J}_3| &\leq \|\nabla d\|^\frac{1}{2} \|\nabla u\|^2 \leq C \|\nabla d\|_L^2 \|\nabla u\|_H^2 \left(\psi^2 + \psi^2 \|\nabla^2 u\|_L^2\right) \\
&\leq \delta \psi^{-1} \|\nabla^2 u\|^2 + C\delta \psi^\alpha, \\
|\tilde{J}_4| &\leq \int |u|^2 |\tilde{x}|^{-\frac{1}{2}} |\nabla^2 d| |\tilde{x}|^{\frac{1}{2}} |\nabla^2 d| dx \\
&\leq \varepsilon \|\tilde{x}^{\frac{1}{2}} \nabla^2 d\|_L^2 \|\nabla^2 d\|_{L^2} \|\nabla^{-\frac{1}{2}} u\|_L^2, \\
&\leq \varepsilon \|\tilde{x}^{\frac{1}{2}} u\|_{L^2} \|\nabla^2 d\|_{L^2} + C\varepsilon \|\tilde{x}^{\frac{1}{2}} \nabla^2 d\|_{L^2}^2 \\
&\leq \varepsilon \|\nabla^3 d\|_{L^2}^2 + C\varepsilon \psi^\alpha + \|\tilde{x}^{\frac{1}{2}} \nabla^2 d\|_{L^2}^2.
\end{align*}
\]
Substitute the estimates for \(\tilde{J}_1, \ldots, \tilde{J}_4\) into (31), and (25)–(27) into (24), and then sum them together,
\[
\frac{d}{dt} \int (\mu|\nabla u|^2 + (\mu + \lambda)\text{div}u^2 + |\nabla^2 d|^2 + |\nabla d|^4 - M(d)\nabla u - 2P\text{div}u) dx \\
+ \int (\rho|u|^2 + |\nabla d|^2 + |\nabla^3 d|^2 + |\nabla^2 d|^2|\nabla^2 d|^2) dx \\
\leq 3\delta \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + \varepsilon \|\nabla^2 d\|_{L^2}^2 + \varepsilon \|\nabla^3 d\|_{L^2}^2 + C\delta \psi^\alpha + C\varepsilon \|\tilde{x}^{\frac{1}{2}} \nabla^2 d\|_{L^2}^2.
\]
Next, we deal with estimates for the term \(\psi^{-1} \|\nabla^2 u\|_{L^2}^2\). By (1), \(u\) satisfies the elliptic problem
\[
\begin{align*}
\left\{\begin{array}{ll}
\mu \Delta u + (\mu + \lambda)\nabla \text{div}u = \rho u_t + \rho u \cdot \nabla u + \nabla P + \text{div}(M(d)) & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R.
\end{array}\right.
\end{align*}
\]
The standard $L^p$-estimate to (33) yields that
\[ \|\nabla^2 u\|_{L^p} \leq C(\|\rho u_t\|_{L^p} + \|p u \cdot \nabla u\|_{L^p} + \|\nabla P\|_{L^p} + \|\nabla d\|\|\nabla^2 d\|_{L^p}), \quad p \in (1, \infty). \] (34)

Combining with (25), by Hölder’s, Young’s and Gagliardo-Nirenberg inequalities, we have
\[ \|\nabla^2 u\|_{L^2} \leq C\psi^{\frac{1}{2}} \|\nabla^3 u\|_{L^2} + C\psi^{\alpha} + \|\nabla d\|\|\nabla^2 d\|_{L^2} \]
\[ \leq C\psi^{\frac{1}{2}} \|\nabla^3 u\|_{L^2} + C\psi^{\alpha} + \|\nabla d\|_{L^p}\|\nabla^2 d\|_{L^3} \]
\[ \leq C\psi^{\frac{1}{2}} \|\nabla^3 u\|_{L^2} + C\psi^{\alpha} + \varepsilon \psi^{\frac{1}{2}} \|\nabla^3 d\|_{L^2}. \] (35)

Put (35) into (32) integrate over $(0, t)$, and choose $\varepsilon, \delta$ suitably small,
\[ \|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \int_0^t \int \rho |u_t|^2 + |\nabla d_t|^2 + |\nabla^3 d|^2 dxds \]
\[ \leq C + C\|P\|_{L^2}^2 + C\|\nabla d\|_{L^4}^2 + C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}, \] (36)
where it is used that
\[ \|P\|_{L^2}^2 \leq \|P(\rho_0)\|_{L^2}^2 + \int_0^t \|P\|_{L^2}\|P\|_{L^\infty}\|\nabla u\|_{L^2} \leq C + C \int \psi^\alpha ds. \]

The proof is complete. \qed

**Lemma 3.2.** Under the conditions of Proposition 4, let $(\rho, u, d)$ be a smooth solution to (9)-(10) with $T_1 = T_1(N_0, E_0) > 0$ determined by Lemma 3.1. Then it holds for all $t \in (0, T_1)$ that
\[ \sup_{0 \leq s \leq t} (s\|\nabla^2 u\|_{L^2}^2) + \int_0^t s\|\nabla^3 u\|_{L^2}^2 dt \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}, \] (37)
\[ \sup_{0 \leq s \leq t} (s\|\nabla^3 u\|_{L^2}^2) \]
\[ + \int_0^t s\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}^2 ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \] (38)

**Proof.** Multiply (22) by $\bar{x}^a \nabla d$ and integrate by parts to get
\[ \frac{1}{2} \frac{d}{dt} \int \bar{x}^a|\nabla^2 d|^2 dx + \int \bar{x}^a|\nabla^3 d|^2 dx \]
\[ \leq \frac{1}{2}\|\nabla\bar{x}^a \nabla^3 d\|_{L^2}^2 + C \int |\nabla u||\nabla d||\nabla\bar{x}^a| dx + C \int |u||\nabla^2 d|^2|\nabla\bar{x}^a| dx \]
\[ + C \int |\nabla d|^3|\nabla^2 d||\nabla\bar{x}^a| dx + C \int |\nabla|^2|\nabla^2 d|^2|\nabla\bar{x}^a| dx \]
\[ + C \int |\nabla u||\nabla^2 d|^2|\nabla\bar{x}^a| dx + C \int |\nabla^2 d|^2|\nabla\bar{x}^a| dx \]
\[ =: \frac{1}{2}\|\nabla\bar{x}^a \nabla^3 d\|_{L^2}^2 + \sum_{i=1}^8 I_i. \] (39)

By Hölder’s and Young’s inequalities,
\[ I_1 \leq C \int |\nabla u||\nabla d||\nabla^2 d||\bar{x}^a (\bar{x}^{-1} |\nabla\bar{x}|) dx \]
\[ I_2 \leq C \int |\nabla^2 d|^2 d^2 x |\nabla u| |\nabla^2 d| d^2 x \]
\[ \leq C\|\nabla^2 d\|_{L^2}^2 \|\nabla u\|_{L^\infty} \|\nabla^2 d\|_{L^2}^\infty \]
\[ \leq C_\varepsilon \psi \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2, \]
\[ I_3 \leq C \int |\nabla d|^2 d^2 x |\nabla^2 d| \nabla^2 d |\nabla x| d^2 x \]
\[ \leq C \|\nabla^2 d\|_{L^2}^2 \|\nabla^2 d\|_{L^2} \|\nabla^2 d\|_{L^2} \]
\[ \leq C \psi \|\nabla^2 d\|_{L^2}^2, \]
\[ I_4 \leq C \int |\nabla d|^2 d^2 x |\nabla^2 d| |\nabla x| d^2 x \]
\[ \leq C \|\nabla^2 d\|_{L^2} \|\nabla^2 d\|_{L^2}^2 \]
\[ \leq C \|\nabla^2 d\|_{L^2} \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2, \]
\[ I_5 + I_6 \leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2}^2 + C \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2}^2 \]
\[ \leq C \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^2}^2 \]
\[ \leq C \psi \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2, \]
\[ I_7 + I_8 \leq C \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \|\nabla^2 d\|_{L^2} \]
\[ \leq C \psi \|\nabla^2 d\|_{L^2}^2. \]

Putting the estimates for \( I_1, \ldots, I_8 \) into (39) and using (35), we have after taking \( \varepsilon \) small that
\[
\frac{d}{dt} \int \nabla^3 d^2 dx + \int \nabla^3 d^2 dx \leq C \psi + C \psi \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2. \tag{40}
\]

Multiplying (40) by \( t \), and then integrating over \((0, t)\), we arrive at (37) by using Gronwall’s inequality immediately.

Now, turn attention to (38). Differentiate (9) with respect to \( t \),
\[
\rho u_t + \rho u \cdot \nabla u_t - \mu \Delta u_t - (\mu + \lambda) \nabla \div u_t = -\rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t - \div (M(d))_t. \tag{41}
\]

Multiplying (41) by \( u_t \), and then integrating over \( B_R \), we obtain by (9) that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int (\mu |u_t|^2 + (\mu + \lambda) \div u_t^2) dx \]
\[
= -2 \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \]
\[
- \int \rho u_t \cdot \nabla u \cdot u_t dx + \int P_t \div u_t dx + \int (M(d))_t : \nabla u_t dx
\]
\[
\begin{align*}
&\leq C \int \rho |u| |u_t| (|\nabla u_t| + |\nabla u| + |u| |\nabla^2 u|) dx + C \int \rho |u|^2 |\nabla u| |\nabla u_t| dx \\
&+ C \int \rho |u_t|^2 |\nabla u| dx + C \int |P_t| |\nabla u_t| dx + C \int (|M(d)|_t) |\nabla u_t| dx \\
&=: \Psi(t) + C \int (|M(d)|_t) |\nabla u_t| dx.
\end{align*}
\]

By the arguments \cite{12, 3.27–3.31} for Lemma 3.3, we know from (19), (21), and \cite{28} for \(\varepsilon, \kappa \in (0, 1)\) that
\[
\Psi(t) \leq \varepsilon \|\nabla u_t\|^2_{L^2} + C_\varepsilon \psi^\alpha (\|\nabla^2 u\|^2_{L^2} + \|\rho^\frac{1}{2} u_t\|^2_{L^2} + 1).
\]

On the other hand,
\[
\int (|M(d)|_t) |\nabla u_t| dx \leq \int |\nabla d||\nabla d_t| |\nabla u_t| dx
\]
\[
\leq \varepsilon \|\nabla u_t\|^2_{L^2} + C_\varepsilon \|\nabla d\|^2_{L^4} \|\nabla d_t\|^2_{L^4}
\]
\[
\leq \varepsilon \|\nabla u_t\|^2_{L^2} + C_\varepsilon \psi^\alpha \|\nabla d_t\|^2_{L^2} + \kappa \|\nabla^2 d_t\|^2_{L^2}.
\]

Combining (42)–(44) with \(\varepsilon\) small enough, we have by using \cite{35} that
\[
\frac{d}{dt} \int \rho |u_t|^2 dx + \int \frac{\mu}{2} |\nabla u|^2 dx
\]
\[
\leq C \psi^\alpha (\|\nabla^2 u\|^2_{L^2} + \|\rho^\frac{1}{2} u_t\|^2_{L^2} + 1) + C_\varepsilon \psi^\alpha \|\nabla d_t\|^2_{L^2} + \kappa \|\nabla^2 d_t\|^2_{L^2}.
\]

Next, we should estimate \(\|\nabla d_t\|^2_{L^2}\) and \(\|\nabla^2 d_t\|^2_{L^2}\). Differentiate \cite{9}\(3\) with respect to \(t\),
\[
\nabla d_t - \nabla \Delta d_t = -\nabla (u \cdot \nabla d)_t + \nabla (|\nabla d|^2 d)_t.
\]

Multiply \cite{46} by \(\nabla d_t\), and integrate the resulting equality over \(B_R\),
\[
\frac{1}{2} \frac{d}{dt} \|\nabla d_t\|^2_{L^2} + \|\nabla^2 d_t\|^2_{L^2}
\]
\[
\leq C \int |\nabla u_t||\nabla d_t| |\nabla d_t| dx + C \int |\nabla u||\nabla d_t|^2 dx + C \int |u_t||\nabla^2 d| |\nabla d_t| dx
\]
\[
+ C \int |\nabla d|^2 |d_t||\nabla^2 d_t| dx + C \int |\nabla d||\nabla d_t||\nabla^2 d_t| dx
\]
\[
=: \sum_{i=1}^{5} \tilde{I}_i.
\]

By Hölder’s, Young’s and Gagliardo-Nirenberg inequalities, we have
\[
\tilde{I}_1 \leq C \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla d\|_{L^4}
\]
\[
\leq \varepsilon \|\nabla u_t\|^2_{L^2} + C_\varepsilon \|\nabla d\|_{L^2} \|\nabla d\|_{H^1} \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{H^1}
\]
\[
\leq \varepsilon \|\nabla u_t\|^2_{L^2} + \delta \|\nabla^2 d_t\|^2_{L^2} + C_\varepsilon \delta \psi^\alpha \|\nabla d_t\|^2_{L^2},
\]

and
\[
\tilde{I}_2 \leq C \|\nabla u\|_{L^2} \|\nabla d_t\|^2_{L^4} \leq C \|\nabla u\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{H^1}
\]
\[
\leq \delta \|\nabla^2 d_t\|^2_{L^2} + C_\delta \psi^\alpha \|\nabla d_t\|^2_{L^2},
\]

\[
\tilde{I}_3 \leq C \int |u_t x^{-\frac{3}{4}}| |\nabla^2 d|^{\frac{1}{2}} x^{\frac{1}{4}} |\nabla^2 d|^{\frac{1}{2}} |\nabla d_t| dx
\]
\[
\leq C \|u_t x^{-\frac{3}{4}}\|_{L^4} \|\nabla^2 d x^{\frac{1}{2}} x^{\frac{1}{4}}\|_{L^2} \|\nabla^2 d|^{\frac{1}{2}} \|\nabla d_t\|_{L^4}
\]
Again by Hölder’s, Young’s and Gagliardo-Nirenberg inequalities, we get with the
\begin{align*}
\leq C \|u_t \bar{x}^p \|_{L^2} \| \nabla^2 d \bar{x}^p \|_{L^2} \| \nabla^3 d \|_{L^2} \| \nabla d_t \|_{L^2} \| \bar{x}^p \|_{L^2} \| \nabla d_t \|_{L^2}^2 \\
\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \|u_t \bar{x}^p \|_{L^2}^2 + C \| \nabla^2 d \bar{x}^p \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla d_t \|_{L^2}^2 \\
\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \|u_t \bar{x}^p \|_{L^2}^2 + \varepsilon \| \nabla u_t \|_{L^2}^2 + C_{\varepsilon, \delta} \| \nabla^2 d \bar{x}^p \|_{L^2} \| \nabla^2 d \|_{L^2}^2 \| \nabla d_t \|_{L^2}^2.
\end{align*}

(50)

To estimate \( \tilde{I}_4 \), we have
\begin{align*}
\|d_t\|_{L^2} &= \| - u \cdot \nabla d + |\nabla d|^2 d + \Delta d\|_{L^2} \\
&\leq \| \bar{x}^p u_t \bar{x}^p \|_{L^2} + \| \nabla d\|_{L^2}^2 + \| \nabla^2 d\|_{L^2} \\
&\leq C(\| \bar{x}^p u\|_{L^\infty} \| \nabla d\|_{L^2} + \| \nabla d\|_{L^2} \| \nabla d\|_{H^1} + \| \nabla^2 d\|_{L^2}) \\
&\leq C \psi^\alpha \| \bar{x}^p u\|_{L^\infty} + C \psi^\alpha \leq C(\psi^\alpha + \| \nabla^2 u\|_{L^2}), \quad (51)
\end{align*}

where it is used with \( 0 < \sigma < 1 \) that
\begin{align*}
\|u \bar{x}^{-\sigma}\|_{L^\infty} &\leq C(\|u \bar{x}^{-\sigma}\|_{L^\frac{1}{2}} + \| \nabla (u \bar{x}^{-\sigma})\|_{L^2}) \\
&\leq C(\|u \bar{x}^{-\sigma}\|_{L^\frac{1}{2}} + \| \nabla u\|_{L^2} + \|u \bar{x}^{-\sigma}\|_{L^\frac{1}{2}} \| \bar{x}^{-1} \nabla \bar{x}\|_{L^\frac{1}{2}}) \\
&\leq C(\psi^\alpha + \| \nabla^2 u\|_{L^2}).
\end{align*}

Again by Hölder’s, Young’s and Gagliardo-Nirenber inequalities, we get with the help of \((51)\) that
\begin{align*}
\tilde{I}_4 &\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \| \nabla d\|_{L^2} \| \nabla d_t \|_{L^2}^2 \\
&\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \| \nabla d\|_{L^2} \| \nabla d_t \|_{L^2} \| \nabla d_t \|_{H^1} \\
&\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \| \nabla d\|_{L^2} \| \nabla d_t \|_{H^1} \\
&\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \psi^\alpha \| \nabla d\|_{L^2} \| \nabla d_t \|_{H^1} \\
&\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \psi^\alpha \| \nabla d\|_{L^2} \| \nabla d_t \|_{H^1} \\
&\leq \delta \| \nabla^2 d_t \|_{L^2} + C_\delta \psi^\alpha \| \nabla d\|_{L^2} + C_\delta \psi^\alpha + C_\delta \| \nabla^2 u\|_{L^2}. \\
&\quad (52)
\end{align*}

and similarly,
\begin{align*}
\tilde{I}_5 &\leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \| \nabla d\|_{L^2} \| \nabla d_t \|_{L^2}^2 \leq \delta \| \nabla^2 d_t \|_{L^2}^2 + C_\delta \psi^\alpha \| \nabla d_t \|_{L^2}^2. \quad (53)
\end{align*}

By the estimates of \( \tilde{I}_1, \ldots, \tilde{I}_5 \) with \( \varepsilon, \delta \) small enough, we have from \((47)\) that
\begin{align*}
\frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d|^2 dx \\
\leq C \psi^\alpha (\| \rho \bar{x}^2 u_t \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2 + \| \bar{x}^p \| \nabla^2 d \|_{L^2}^2 + 1) + C \| \nabla^2 u\|_{L^2}^2 + \frac{1}{2} \| \nabla u_t\|_{L^2}^2. \\
&\quad (54)
\end{align*}

Combine \((54)\) and \((45)\) with \( \kappa \) small to get
\begin{align*}
\frac{d}{dt} \int (\rho |u_t|^2 + |\nabla d_t|^2) dx + \int (\frac{\mu}{2} |\nabla u_t|^2 + |\nabla^2 d_t|^2) dx \\
\leq C \psi^\alpha (\| \rho \bar{x}^2 u_t \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2 + \| \bar{x}^p \| \nabla^2 d \|_{L^2}^2 + 1) + C \| \nabla^2 u\|_{L^2}^2. \\
&\quad (55)
\end{align*}

Multiplying \((55)\) by \( t \) and integrating over \((0, t)\), we have by Gronwall’s inequality that
\begin{align*}
\sup_{0 \leq s \leq t} s (\| \sqrt{\rho} u_t \|_{L^2}^2 + \| \nabla d_t \|_{L^2}^2) + \int_0^t s (\| \nabla u_t \|_{L^2}^2 + \| \nabla^2 d_t \|_{L^2}^2) ds \\
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \\
&\quad (56)
\end{align*}
It remains to estimate the upper bound for \( \|\nabla^3 d\|_{L^2}^2 \). By virtue of (22) with the Gagliardo-Nirenberg and Young’s inequalities, we have
\[
\|\nabla^3 d\|_{L^2} \leq C (\|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{H^1})
\]
\[
\leq C (\|\nabla d\|_{L^2} + \|\nabla (u \cdot \nabla d)\|_{L^2} + \|\nabla (|\nabla d|^2 d)\|_{L^2} + \|\nabla d\|_{H^1})
\]
\[
\leq C \|\nabla d\|_{L^2} + C \|\nabla u\|_{L^4} \|\nabla d\|_{L^4} + C \|\alpha\|_{L^\infty} \|\nabla^2 d\|_{L^2} + C \|\nabla d\|_{L^1} \|\nabla^2 d\|_{L^4} + \|\nabla d\|_{H^1}
\]
\[
\leq C \|\nabla d\|_{L^2} + C \varepsilon \|\psi\|_{\alpha} + \varepsilon \|\nabla^3 d\|_{L^2}. 
\quad \text{(57)}
\]
Combining (56) and (57) with \( \varepsilon \) small yields (38). The proof is complete. \( \square \)

**Lemma 3.3.** Let \((\rho, u, d)\) with \( T_1 \) in Lemma 3.1. Then for all \( t \in (0, T_1) \),
\[
\sup_{0 \leq s \leq t} \|\rho \bar{\xi}\|_{L^1 \cap H^1 \cap W_{1,4}} \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \right\}. \quad \text{(58)}
\]

**Proof.** Notice that there is the common continuity equation between the nematic liquid crystal flows \([\text{1]}\) and the Navier-Stokes equations \([\text{2]}\), with different velocity field \(u\) involved. So, observing the framework of \([\text{12}]\) Lemma 3.4] for proving an estimate similar to (58), it suffices to verify the following estimate:
\[
\int_0^t (\|\nabla^2 u\|_{L^2}^{\frac{q+1}{2}} + s \|\nabla^2 u\|_{L^2}^2) \, ds \leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\}. \quad \text{(59)}
\]
It follows from (35), (18), and (38) that
\[
\int_0^t (\|\nabla^2 u\|_{L^2}^{\frac{q}{2}} + s \|\nabla^2 u\|_{L^2}^2) \, ds
\]
\[
\leq C \int_0^t (\|\rho \bar{u}\|_{L^q}^{\frac{2}{q-1}} + \psi^\alpha + \|\nabla^3 d\|_{L^2}^2) \, ds + C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \int_0^t \psi^\alpha \, ds
\]
\[
\leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\}. \quad \text{(60)}
\]
Choosing \( p = q \) in (34), with notation \( \dot{u} := u_t + u \cdot \nabla u \), we have
\[
\|\nabla^2 u\|_{L^q} \leq C (\|\rho \bar{u}\|_{L^q} + \|\nabla P\|_{L^q} + \|\nabla d\| \|\nabla^2 d\|_{L^q})
\]
\[
\leq C (\|\rho \bar{u}\|_{L^q} + \psi^\alpha + \|\nabla d\| \|\nabla^2 d\|_{L^q}), \quad \text{(61)}
\]
where
\[
\|\rho \bar{u}\|_{L^q} \leq \|\rho u_t\|_{L^q} + \|\rho u \cdot \nabla u\|_{L^q}
\]
\[
\leq \|\rho u_t\|_{L^q}^{\frac{2(q-1)}{q-2}} + \|\rho u_t\|_{L^q}^{\frac{2}{q-2}} + \|\rho u\|_{L^q} \|\nabla u\|_{L^q}
\]
\[
\leq C \psi^\alpha (\|\nabla^2 u\|_{L^2}^{\frac{2(q-1)}{q-2}} + \|\nabla^2 u\|_{L^q}^{\frac{2}{q-2}} + \|\nabla^2 u\|_{L^q})
\]
\[
+ C \psi^\alpha (1 + \|\nabla^2 u\|_{L^2}^{1-\frac{2}{q}}) \quad \text{(62)}
\]
by (21) and (28). Combine (60)–(62) with (38) and (18) to get that
\[
\int_0^t \|\rho \bar{u}\|_{L^q}^{\frac{q+1}{2}} \, dt \leq C \int_0^t \psi^\alpha t^{-\frac{q+1}{2}} \left( t \|\rho \bar{u}\|_{L^q}^{\frac{2(q-1)}{q-2}} \right) \left( t \|\nabla u\|_{L^q}^2 \right) \left( t \|\nabla u\|_{L^q}^{\frac{2}{q-2}} \right) \, dt
\]
\[
+ C \int_0^t \|\rho \bar{u}\|_{L^2}^2 \, dt + C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\}
\]
\[
\leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \int_0^t (\psi^\alpha + t \frac{\kappa_1^2 + \kappa_2 + \kappa_3}{\kappa_1^2 - \kappa_1} + t \| \nabla u_t \|_{L^2}^2) \, dt \\
+ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\},
\]

and that
\[
\int_0^t \| \rho \dot{u} \|_{L^2}^2 \, dt \leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\}.
\]

On the other hand, we have by \(18\), \(19\) and the Gagliardo-Nirenberg inequality that
\[
\int_0^t \left( \| \nabla d \|_{L^2}^{2r \frac{q+1}{2q}} + s \| D^2 d \|_{L^2}^{2r \frac{q}{2q+1}} \right) \, ds \\
\leq C \int_0^t \psi^\alpha \left( \| \nabla^3 d \|_{L^2}^{2r \frac{q}{q+1}} + s \| D^3 d \|_{L^2}^{2r \frac{q+1}{2q}} \right) \, ds \\
\leq C \int_0^t \psi^\alpha (1 + s^q + \| \nabla^3 d \|_{L^2}^2) \, dt \leq C \int_0^t \psi^\alpha \, ds.
\]

The desired (59) follows from (60)–(65).

Now, we can deal with the proof of Proposition 1.

**Proof of Proposition 1.** We have from \(17\)–\(19\), \(58\), and \(59\) that
\[
\psi(t) \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \right\}, 
\]

\(t \in (0, T_1]\).

Let \(\hat{M} := e^{Ce} \) and \(T_0 := \min\{T_1, (CM^{\alpha})^{-1}\} \).

Then,
\[
\sup_{0 \leq s \leq T_0} \psi(t) \leq \hat{M}.
\]

Together with \(18\), \(35\), \(37\) and \(38\), this concludes \(16\).

**4. Proof of Theorems 1.1**

Now we make the approximation procedure to prove Theorem 1.1.

**Proof of Theorem 1.1** For simplicity, assume
\[
\int_{\mathbb{R}^2} \rho_0 \, dx = 1.
\]

So, there exists \(N_0 > 0\) such that
\[
\int_{B_{N_0}} \rho_0 \, dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 \, dx = \frac{3}{4}.
\]

Construct \(\rho_0^R = \rho_0^R + R^{-1} e^{-|x|^2}\) with \(0 \leq \rho_0^R \in C_0^\infty (\mathbb{R}^2)\) to satisfy
\[
\int_{B_{N_0}} \rho_0^R \, dx \geq \frac{1}{2},
\]

and
\[
\bar{x}^a \rho_0^R \to \bar{x}^a \rho_0 \text{ in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \ R \to \infty.
\]

With \(\bar{x}^a \nabla d_0 \in L^2(\mathbb{R}^2)\) and \(\nabla^2 d_0 \in L^2(\mathbb{R}^2)\), define
\[
d_0^R = \begin{cases} 
  d_0(x) & \text{in } B_R, \\
  d_0 \left( \frac{R x}{2|x|} \right) & \text{in } \mathbb{R}^2 \setminus B_R.
\end{cases}
\]

(69)
It is standard (cf. [17]) that there exists \( d_0^R \in H^4(\mathbb{R}^2, \mathbb{S}^2) \) such that
\[
\nabla^2 d_0^R \to \nabla^2 d_0, \quad \bar{x} \cdot \nabla d_0^R \to \bar{x} \cdot \nabla d_0 \quad \text{in} \quad L^2(\mathbb{R}^2), \quad R \to \infty. \tag{70}
\]
Since \( \nabla u_0 \in L^2(\mathbb{R}^2) \), choose \( v^R_i \in C_0^\infty(B_R) \) (\( i = 1, 2 \)) such that
\[
\lim_{R \to \infty} \| v^R_i - \partial_t u_0 \|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2, \tag{71}
\]
and let smooth \( u^R_0 \) uniquely solve
\[
\begin{cases}
-\Delta u^R_0 + R^{-1} u^R_0 = -\rho^0 u^R_0 + \sqrt{\rho^0} h^R - \partial_t v^R_i & \text{in } B_R, \\
\rho^R_0 \frac{\partial u^R_0}{\partial B} = 0 & \text{on } \partial B_R,
\end{cases}
\tag{72}
\]
where \( h^R := (\sqrt{\rho^0} v^R_i)^* + (\bar{j}_1 / R) \) with the standard mollifying kernel \( j_\delta \), \( \delta > 0 \). Extend \( u^R_0 \) to \( \mathbb{R}^2 \) by defining 0 outside \( B_R \), denoted by \( u^R_0 \). By the same arguments as those for the proof of [12] Theorem 1.1, we obtained that
\[
\lim_{R \to \infty} \left( \| \nabla (u^R_0 - u_0) \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\rho^0} u^R_0 - \sqrt{\rho^0} u_0 \|_{L^2(\mathbb{R}^2)} \right) = 0. \tag{73}
\]

It is easy to check that \((\rho^R, u^R, d^R)\) satisfies the conditions of Lemma 2.1 and thus there exists a unique classical solution \((\rho^R, u^R, d^R)\) to the initial-boundary problem (9)–(10) on \( B_R \times (0, T_R] \). Moreover, Proposition 1 says that there exists \( T_0 > 0 \) independent of \( R \) such that (10) holds for \((\rho^R, u^R, d^R)\). By \( [16, 17, 70, 73 \) and \( 37 \), after taking a subsequence, \((\rho^R, u^R, d^R)\) locally and weakly (in the corresponding spaces) converges to a strong solution \((\rho, u, d)\) of (1)–(3) on \( \mathbb{R}^2 \times (0, T_0] \), satisfying (7) and (3).

Next prove the uniqueness of the strong solutions. Take two strong solutions \((\rho_i, u_i, d_i)\) (\( i = 1, 2 \)) sharing the same initial data with (7) and (3), and denote \( \bar{\rho} = \rho_2 - \rho_1, \bar{u} = u_2 - u_1, \bar{d} = d_2 - d_1 \). Then
\[
\begin{cases}
\bar{\rho} + (u_2 \cdot \nabla) \bar{\rho} + \bar{u} \cdot \nabla \rho_1 + \ddot{\rho} \ \text{div} \ u_2 + \rho_1 \ \text{div} \ \bar{u} = 0, \\
\rho_1 \bar{u}_i + \rho_1 \bar{u}_1 \cdot \nabla \bar{u} + \nabla (P(\rho_2) - P(\rho_1)) = \mu \Delta \bar{u}, \\
+(\mu + \lambda) \nabla \Delta \bar{u} - \bar{\rho} (\nabla u_2 + u_2 \cdot \nabla u_2) - \rho_1 \bar{u} \cdot \nabla u_2 - \Delta \bar{d} \cdot \nabla d_2 - \Delta d_1 \cdot \nabla d, \\
\bar{\dot{d}} - \Delta \bar{d} = \nabla \cdot (\nabla d_2 + \nabla d_1) d_1 + |\nabla d_2|^2 \bar{d} - \bar{u} \cdot \nabla d_2 - u_1 \cdot \nabla d
\end{cases}
\tag{74}
\]
for \((x, t) \in \mathbb{R}^2 \times (0, T_0] \) with
\[
\bar{\rho}(x, 0) = \bar{u}(x, 0) = \bar{d}(x, 0) = 0, \quad x \in \mathbb{R}^2. \tag{75}
\]
Firstly, multiply (74) by \( 2 \bar{\rho} \bar{x} r \), and integrate by parts. Based on the common continuity equation contained in (1) and (3), [12] Inequality (5.32) says that
\[
\| \bar{\rho} \bar{x} r \|_{L^2} \leq C \int_0^t (\| \nabla \bar{u} \|_{L^2} + \| \sqrt{\rho^0} \bar{u} \|_{L^2}) ds, \quad t \in (0, T_0]. \tag{76}
\]
Secondly, multiply (74) by \( \bar{u} \), and integrate by parts,
\[
\frac{1}{2} \frac{d}{dt} \int \rho_1 |\bar{u}|^2 dx + \int (2\mu + \lambda) \ |\nabla \bar{u}|^2 + \mu |\omega|^2 dx
\]
\[
= - \int \bar{\rho} \nabla (u_2 \cdot \nabla u_2) \cdot \bar{u} dx - \int \rho_1 \bar{u} \cdot \nabla u_2 \cdot \bar{u} dx + \int \left( P(\rho_2) - P(\rho_1) \right) \text{ div } \bar{u} dx
\]
\[
+ \int (\nabla \bar{d} \cdot \nabla \bar{d} + |\nabla \bar{d}|^2 - \bar{u} \cdot \nabla \bar{d} + \bar{d} \cdot \nabla d_2 + \nabla u_2) dx - \int \Delta d_1 \cdot \nabla \bar{d} \cdot \bar{u} dx
\]
\[
\leq C \| \nabla u_2 \| \int_0^t \rho_1 |\bar{u}|^2 dx + C \int |\bar{\rho}| |\bar{u}| (|u_2| + |u_2| |\nabla u_2|) dx
\]
This concludes \( \bar{y} \) implies that \( \bar{y} \), By (77) and (80) with \( \varepsilon \)

\[
\bar{G}(x,t) = \| \sqrt{\bar{\rho}} \bar{u} \|_{L^2}^2 + \int_0^t (\| \sqrt{\bar{\rho}} \bar{u} \|_{L^2}^2 + \| \Delta \bar{d} \|_{L^2}^2) ds.
\]

By (77) and (80) with \( \varepsilon \) suitably small, we know

\[
G(t) \leq C \left( 1 + \| \bar{u} \|_{L^\infty} + t \| \nabla^2 \bar{u} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 + \| \Delta \bar{d} \|_{L^2}^2 \right)
\]

Together with Gronwall’s inequality and (7), we obtain \( G(t) = 0 \), and hence \( \bar{u}(x,t) = \nabla \bar{d}(x,t) = 0 \) for almost all \( (x,t) \in \mathbb{R}^2 \times (0,T_0) \). Consequently, (76) implies that \( \bar{\rho}(x,t) = 0 \) almost everywhere in \( \mathbb{R}^2 \times (0,T_0) \). Next, by (74), we have

\[
\bar{d}(x,t) = 0 \] almost everywhere in \( \mathbb{R}^2 \times (0,T_0) \), \( \bar{d}(0,t) = 0 \) on \( \mathbb{R}^2 \).

This concludes \( \bar{d}(x,t) \) is a stable solution of (75) with \( \bar{d}(0,t) = 0 \), almost everywhere in \( \mathbb{R}^2 \times (0,T_0) \). The proof is complete.
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