MODELLING AND ANALYSIS OF PREY-PREDATOR MODEL INVOLVING PREDATION OF MATURE PREY USING DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the modelling and analysis of prey-predator model involving predation of mature prey is done using DDE. Equilibrium points are calculated and stability analysis is performed about non-zero equilibrium point. Delay parameter destabilizes the system and triggers asymptotic stability when value of delay parameter is below the critical point. Hopf bifurcation is observed when the value of delay parameter crosses the critical point. Sensitivity analysis has also been performed to look into the effect of other parameters on the state variables. The numerical results are substantiated using MATLAB.

1. Introduction. Mathematical biology and population dynamics has been extensively studied using various mathematical models over the years. The most important contributions toward theoretical population biology were given by [10] and [18] which is also known as Lotka-Volterra model or a Prey-Predator model. It is a coupled non-linear system of ODE used to describe the dynamics of two populations out of which one is termed as prey and another is termed as predator. However, [18] soon realized that time delay needs to be included for realistic modelling and proposed an integro-differential equation model. But he could not analyse the stability of equilibrium point. The exact method to solve the modified model by [18] was given by [17] who used the simplest form of delay function. The limited prey population and limited predator appetite were considered in limit cycle prey-predator interaction models [11,12]. It has been proved by [8] that mass action predation has zero handling time where as Holling type-II has non-zero handling time. It has been shown that the predator can be affected by transmissible diseases under various physical conditions and circumstances [7]. The Hopf-bifurcation is always of supercritical nature if the functional responses of predation are taken as Holling type-II [3]. It has also been proved that there are many diseases such as rabies, hepatitis etc. which need incubation period before becoming infection vectors [9, 19]. It has been proved that susceptible predator consumes the prey

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as per mass action law, but the infected predator consumes as per Holling type-II response [13]. The ratio dependent model given by [1] evolved as an alternate to models given by [10] and [18]. Life history of many species compose of two stages: Immature and Mature. Each stage has distinct behavioural characteristics. Stage structure prey-predator models have also been proposed by many authors [2,5,20]. A mathematical model was proposed where the predator species consumed the immature prey extensively [6]. The stability properties for constant input of refuge preys in Lotka- Volterra prey-predator models have been studied in detail about positive equilibrium [4]. A detailed analysis was given for various kinds of stabilities like Absolute stability, conditional stability and bifurcation in predator-prey system with discrete delays [14]. The nature of zeros of exponential characteristic equation was analysed in detail [15]. The stability analysis of equilibrium points involving a non-linear system of delay differential equations is carried [16].

In the view of above, therefore, in this paper the analysis of prey-predator model with delay differential equations involving predation of mature prey is performed. In the beginning section, after formulation of the basic model, the interior non-zero equilibrium is calculated. In next section, the stability analysis about this non-zero equilibrium is performed, both in the absence of delay and in the presence of delay. Next, sensitivity analysis has also been performed to look into the role of other parameters on the state variables. At last, conclusion part has been written involving the major findings of the analysis and biological applications of the proposed model.

2. Mathematical Model. Let \( P_r \) and \( P_d \) be the prey and predator populations respectively. It is assumed that prey population has maturity time \( \tau \). It is further assumed that predators consume only mature prey. This prey-predator dynamic is governed by the following system:

\[
\frac{dP_r}{dt} = bP_r - \alpha P_r^2 - \beta P_r P_d
\]

\[
\frac{dP_d}{dt} = -dP_d - \delta P_d^2 + \gamma P_r(t-\tau)P_d
\]

where: \( P_r(0) > 0 \), \( P_d(0) > 0 \) for all \( t \) and \( P_r(t-\tau) = \text{constant} \ t \in [0, \tau] \).

The parameters considered in this model are interpreted in Table 1:

| Parameter | Description |
|-----------|-------------|
| \( b \)  | Intrinsic birth rate of prey population |
| \( \alpha \) | Intra specific competition rate among prey |
| \( \beta \) | Inter specific competition rate |
| \( d \)  | Death rate of predator population |
| \( \gamma \) | Inter specific competition rate |
| \( \delta \) | Intra specific competition rate among predator |
| \( \tau \) | Delay parameter |

It is justified to assume all the parameters as positive constants.
3. **Non-Zero Equilibrium.** At the steady state $P_r (t-\tau) \cong P_r(t)$. The non-zero equilibrium $E^* (P_r^*, P_d^*)$ is calculated as:

$$\frac{dP_r^*}{dt} = 0 \Rightarrow P_d^* = \frac{b - \alpha - P_r^*}{\beta}$$

$$\frac{dP_d^*}{dt} = 0 \Rightarrow P_d^* = \frac{\gamma P_r^* - d}{\delta}$$

On comparing these values of $P_d^*$, we get:

$$P_r^* = \frac{b\delta - \alpha\delta - b\beta}{(\gamma\beta + \delta)}$$

Thus, we have non-zero equilibrium:

$$E^* (P_r^*, P_d^*) = E^* \left(\frac{b\delta - \alpha\delta - d\beta}{(\gamma\beta + \delta)}, \frac{\gamma\beta b - \alpha\beta\gamma + d\beta}{(\gamma\beta + \delta)\beta}\right)$$

4. **Stability of Equilibrium $E^* (P_r^*, P_d^*)$ and Hopf-bifurcation.** The system of equations governing the plant population competition mechanism at the equilibrium $E^* (P_r^*, P_d^*)$ is given by:

$$\frac{dP_r^*}{dt} = bP_r^* - \alpha P_r^{*2} - \beta P_r^* P_d^*$$

$$\frac{dP_d^*}{dt} = -dP_d^* - \delta P_d^{*2} + \gamma P_r^* (t - \tau) P_d^*$$

The characteristic equation associated with the system of equations (3) - (4) is given by:

$$\lambda^2 + a\lambda + b + ce^{-\lambda\tau} = 0$$

where

$$a = (2\alpha P_r^* + \beta P_d^* + d + 2\delta P_d^* - b), b = (d + 2\delta P_d^*)(2\alpha P_r^* + \beta P_d^* - b), c = \beta\gamma P_r^* P_d^*$$

When $\tau = 0$, the equation (5) becomes:

$$\lambda^2 + a\lambda + b + c = 0$$

By Routh-Hurwitz criteria, roots of equation (6) will have negative real part i.e. the system is stable if:

$$(3H_1) : a > 0;$$

$$(3H_2) : (b + c) > 0;$$

which obviously is true.

Now, we would like to check the shifting of negative real part of the roots to positive real parts with variations in the values of $\tau$.

Let $\lambda = i\omega$ be a root of equation (5), then equation (5) becomes:

$$(i\omega)^2 + a(i\omega) + b + ce^{-(i\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + a(i\omega) + b + c(\cos\omega\tau - i\sin\omega\tau) = 0$$

Separating real and imaginary parts:

$$\omega^2 - b = c\cos\omega\tau$$

$$a\omega = c\sin\omega\tau$$

(7)

(8)
Squaring and adding:

\[ \omega^4 + (a^2 - 2b)\omega^2 + (a^2 - c^2) = 0 \quad (9) \]

The two roots of equation (9) are:

\[ \omega_{1,2}^2 = \frac{(2b - a^2) \pm \sqrt{(a^2 - 2b)^2 - 4(a^2 - c^2)}}{2} \quad (10) \]

None of the two roots \( \omega_{1,2}^2 \) is positive if:

\[ (H_3) : (2b - a^2) < 0 \text{ and } (a^2 - c^2) > 0 \text{ or } (a^2 - 2b)^2 < 4(a^2 - c^2) \]

That means equation (10) does not have positive roots if condition \((H_3)\) holds.

We have the following lemma [14].

Lemma 4.1. If \((H_1) - (H_2)\) hold, then all the roots of equation (5) have negative real parts for all \( \tau \geq 0 \).

On the other hand, if:

\[ (H_4) : (a^2 - c^2) < 0 \text{ or } (2b - a^2) > 0 \text{ and } (a^2 - 2b)^2 = 4(a^2 - c^2) \]

Then, positive root of equation (7) is \( \omega_1^2 \).

On the same basis, if:

\[ (H_5) : (a^2 - c^2) > 0 \text{ or } (2b - a^2) > 0 \text{ and } (a^2 - 2b)^2 > 4(a^2 - c^2) \]

Then, two positive roots of equation (7) are \( \omega_{1,2}^2 \).

In both- \((H_4)\)and \((H_5)\), the equation (5) has purely imaginary roots when \( \tau \) takes certain values. The critical values \( \tau_j^\pm \) of \( \tau \) can be calculated from the system of equations (7)-(8), given by:

\[ \tau_j^\pm = \frac{1}{\omega_{1,2}} \cos^{-1} \left( \frac{\omega_{1,2}^2 - b}{c} \right) + \frac{2j\pi}{\omega_{1,2}}, j = 0, 1, 2, ... \quad (11) \]

The above discussion can be condensed in succeeding lemma [14].

Lemma 4.2. (I) If \((H_1) - (H_2)\) and \((H_4)\) hold and \( \tau = \tau_j^+ \), then equation (5) has a pair of purely imaginary roots \( \pm i\omega_1 \).

(II) If \((H_1) - (H_2)\) and \((H_5)\) hold and \( \tau = \tau_j^- \) (\( \tau = \tau_j^- \) respectively), then equation (5) has a pair of purely imaginary roots \( \pm i\omega_1 \pm i\omega_2 \) respectively.

Our expectation is the shifting of negative real part of some roots of equation (5) to positive real part when \( \tau > \tau_j^+ \) and \( \tau < \tau_j^- \). To look into this possibility, let us denote:

\[ \tau_j^+ = \mu_j^+ (\tau) + i\omega_j^+ (\tau); j = 0, 1, 2, ... \]

The roots of equation (5) satisfy. \( \mu_j^+ (\tau_j^+) = 0 \), \( \omega_j^+ (\tau_j^+) = \omega_{1,2} \)

We can verify that the following transversality condition holds:

\[ \frac{d}{d\tau} (\text{Re}\lambda_j^+ (\tau_j^+)) > 0 \quad \text{and} \quad \frac{d}{d\tau} (\text{Re}\lambda_j^- (\tau_j^-)) < 0 \]

It concludes that \( \tau_j^\pm \) are bifurcating values. The succeeding postulate gives the scattering of the zeros of the equation (5) [14].
Theorem 4.3. Let $\tau_j^+ (j = 0, 1, 2, 3,)$ be defined by equation (11).

(I) If ($\mathcal{H}_1, \mathcal{H}_2$) hold, then all the roots of equation (5) have negative real parts for all $\tau \geq 0$.

(II) If ($\mathcal{H}_1, \mathcal{H}_2$) and ($\mathcal{H}_4$) hold and when $\tau \in [0, \tau_0^+]$, then all the roots of equation (5) have negative real parts. When $\tau = \tau_0^+$, then equation (5) has a pair of purely imaginary roots $\pm i\omega_1$. When $\tau > \tau_0^+$, equation (5) has at least one root with positive real part.

(III) If ($\mathcal{H}_1, \mathcal{H}_2$) and ($\mathcal{H}_5$) hold, then there is a positive integer $m$ such that $0 < \tau_0^- < \tau_1^- < \tau_1^+ < \tau^-_{m-1} < \tau^+_m$ and there are $m$ switches from stability to instability. This means, when $\tau \in [0, \tau_0^+), (\tau_0^+, \tau_1^+), ..., (\tau^-_{m-1}, \tau^+_m)$ all the roots of equation (5) have negative real parts. When $\tau \in [\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), ..., (\tau^-_{m-1}, \tau^+_m)$ and $\tau > \tau^-_m$, equation (5) has at least one root with positive real part.

5. Numerical Simulation. The following set of parametric values is taken to represent graphically the dynamics depicted by the system of equations (1)-(2).

\[ b = 1.7, \alpha = 1, \beta = 1, d = 0.9, \delta = 1, \gamma = 1.2 \]

The change of behaviour of the system of equations (1)-(2) from being stable to complex dynamics about the equilibrium $E^*(P_r^*, P_d^*)$ for different values of delay parameter $\tau$ is shown below:

![Figure 1](image)

**Figure 1.** The equilibrium $E^*(P_r^*, P_d^*)$ is stable in the absence of delay i.e. when $\tau = 0$

6. Sensitivity Analysis of State Variables with respect to Model Parameters. Sensitivity analysis enables to understand the role and effect of other parameters involved in the system on the stability of the dynamic system. Here the effect of Intra specific competition rate among prey $\alpha$ on predator population $P_d$ and the effect of Intra specific competition rate among predator $\delta$ on prey population $P_r$ has been depicted with the help of Figure 6 and Figure 7.

7. Conclusion. The analysis of prey-predator model involving predation of mature prey is performed using DDE with the help of proposed model. Delay disrupts the system and triggers the complex behaviour with limit cycles and stable periodic solutions through Hopf-bifurcation. The stability analysis of the equilibrium $E^*$ is carried out. In the absence of delay, the equilibrium $E^*$ is stable as shown in Figure
The equilibrium $E^*(P^*_r, P^*_d)$ is asymptotically stable when delay is less than critical value i.e. when $\tau < 8.23$

Figure 3. Phase plane of equilibrium $E^*(P^*_r, P^*_d)$ when the delay is less than critical value i.e. $\tau < 8.23$

The equilibrium $E^*(P^*_r, P^*_d)$ losses stability and shows Hopf-bifurcation when delay is crosses the critical value i.e. when $\tau \geq 8.23$
The same fact is also supplemented by ($\mathcal{H}_1$) - ($\mathcal{H}_2$) as in lemma 1. When the value of delay parameter $\tau$ is below the critical point i.e. $\tau < 8.23$, the equilibrium $E^*$ starts losing stability and leads to asymptotical stability as shown in Figure 2 and 3. The moment, the delay parameter $\tau$ crosses the critical value i.e. $\tau \geq 8.23$, the equilibrium $E^*$ exhibits the complex dynamics in the form of Hopf bifurcation. The stable periodic solutions having large amplitude and limit cycle trajectory are demonstrated in Figure 3 and 4. This observation of complex behaviour shown by the system (1)-(2) is in agreement with ($\mathcal{H}_4$) - ($\mathcal{H}_5$) as in lemma 2.

Sensitivity analysis of the proposed model has been done. The Intra specific competition rate among prey and predator has been chosen as the parameters whose effect is seen on prey and predator populations. Figure 6 shows that as the Intra specific competition rate among prey increases from 1.0 to 1.2, the predator population tends to move toward stability. Figure 7 shows that as the Intra specific competition rate among predators increases from 1.0 to 1.2, the prey population tends to move toward stability.

Figure 5. Phase plane of equilibrium $E^*(P^*_r, P^*_d)$ when the delay crosses the critical value i.e. $\tau \geq 8.23$

Figure 6. Time series graph between partial changes in $P_d$ (predator populations) for different values of the parameter $\alpha$. 
8. **Biological Application of proposed model:** The effect of time delays due to gestation period, incubation, mating or maturity etc. on the stability of predator-prey ecosystem is an important problem in population biology. The presence of predator may significantly change the behaviour and physiology of prey to such an extent that it could affect the prey population more effective than direct predation. In the presence of predators, many prey species change their behaviour due to predation risk and show a variety of antipredator responses, which incorporates foraging activity, habitat alterations, vigilance and some physiological changes, etc. Very few researchers have investigated the impact of prey maturation in a predator-prey model with the help of mathematical modelling. Many interesting results have been obtained using this proposed model since time delay have very complicated impact on the dynamical behaviour of the system, such as stability switches, attractivity, periodic oscillation, bifurcation and so on.

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