ON THE CAYLEY DIGRAPHS THAT ARE PATTERNS OF UNITARY MATRICES

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Abstract. Study the relationship between unitary matrices and their patterns is motivated by works in quantum chaology (see, e.g., [KS99]) and quantum computation (see, e.g., [M96] and [AKV01]). We prove that if a Cayley digraph is a line digraph then it is the pattern of a unitary matrix. We prove that for any finite group with two generators there exists a set of generators such that the Cayley digraph with respect to such a set is a line digraph and hence the pattern of a unitary matrix.

1. Definitions

The following standard definitions will be central:

Definition 1. Let $G = \langle S \rangle$ be a finite group. The (right) Cayley digraph $\text{Cay}(G, S)$ of $G$ with respect to $S$ is the digraph whose vertex set is $G$, and whose arc set is the set of all ordered pairs $\{(g, gs) : g \in G$ and $s \in S\}$.

Definition 2. Let $D = (V, A)$ be a digraph. The line digraph $L(D)$ of $D$ is the digraph whose vertex set is $A(D)$, and $((u, v), (w, z)) \in A(L(D))$ if and only if $v = w$, where $u, v, w, z \in V(D)$ and $(u, v), (w, z) \in A(D)$.

Definition 3. Let $M$ be a square matrix $M$ of size $n$. A digraph $D$ is said to be the pattern of $M$, if $D$ is on $n$ vertices and, for every $u, v \in V(D)$, $(u, v) \in A(D)$ if and only if the entry $M_{uv}$ is nonzero.

Definition 4. A square matrix $U$ with complex entries is said to be unitary if it is nonsingular and $U^\dagger U = I$, where $U^\dagger$ and $I$ denote the adjoint of $U$ and the identity matrix, respectively.

2. ON THE CAYLEY DIGRAPHS THAT ARE PATTERNS OF UNITARY MATRICES

Denote by $M(D)$ the adjacency matrix of a digraph $D$.

Lemma 1. If a regular digraph is a line digraph then it is the pattern of a unitary matrix.

Date: June, 2002.

1991 Mathematics Subject Classification. Primary 05C10.

Key words and phrases. Cayley digraph; line digraph; unitary matrices.

Thanks to Sonia Mansilla for bringing [MS01] to my attention. Thanks to Josef Lauri for our conversations on graph theory.
Proof. Recall the Richards characterization of line digraphs \[R67\]: a digraph is a line digraph if and only if: (i) the columns of its adjacency matrix are identical or orthogonal; (ii) the rows of its adjacency matrix are identical or orthogonal. Let $D$ be a regular line digraph. It is easy to see that, from the Richards characterization, $M(D)$ is composed of independent submatrices without zero entries. Since for any $n$ there exists a unitary matrix of size $n$ without zero entries, and since a matrix composed by independent unitary matrices is unitary, the lemma is proven. \qed

**Theorem 1.** If a Cayley digraph is a line digraph then it is the pattern of a unitary matrix.

Proof. By Lemma 1 and since a Cayley digraph is regular. \qed

**Remark 1.** The converse of Theorem 1 is not true. Denote by $\mathbb{Z}_n$ the group of integers modulo $n$. The adjacency matrix of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (0,0), (1,0), (0,1) \rangle$ is
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\] Then, since the matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 \\
0 & 1 & -1 & 1
\end{bmatrix}
\]
is real-orthogonal and hence unitary, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the pattern of a unitary matrix, but it is not a line digraph because it does not satisfy the Richard characterization.

**Lemma 2** (\[M01\], Proposition 4.3.1). If, for some $x \in S^{-1}$, $xS = H$, where $H$ is a subgroup of $G$ such that $|H| = r = |S|$, then $\text{Cay}(G, S) = L(D)$, where $D$ is an $r$-regular (multi)digraph. (See also \[MS01\], Corollary 7.)

Denote by $C_n$ the cyclic group of order $n$.

**Theorem 2.** For any finite group with two generators, there exists a set of generators such that the Cayley digraph with respect to such a set is a line digraph and hence the pattern of a unitary matrix.

Proof. Let $G = \langle S \rangle$, where $S = \{s_1, s_2\}$. Take $s_1^{-1} \in S^{-1}$ (or, equivalently, $s_2^{-1}$). Then
\[
s_1^{-1}S = \{s_1^{-1}s_1, s_1^{-1}s_2\} = \{e, s_1^{-1}s_2\}.
\]
Let $n$ be the order of $s_1^{-1}s_2$. Consider the group $C_n = \langle s_1^{-1}s_2 \rangle$. Write $T = s_1C_n$. Then $C_n = s_1^{-1}T$. Since $S \subset T$, $G = \langle T \rangle$, and since $|T| = n = |C_n|$ (in fact $T$ is a left coset of $C_n$), by Lemma 2, $\text{Cay}(G, T)$ is a line digraph, and hence, by Theorem 1, the pattern of a unitary matrix. \qed

**Corollary 1.** For any finite simple group there exists a set of generators such that the Cayley digraph of the group with respect to such a set is a line digraph and hence the pattern of a unitary matrix.

Proof. By Theorem 2, together with the fact that every finite simple group is generated by two elements \[AG84\]. \qed
3. Examples

**Example 1.** By Theorem 1, \( \text{Cay}(C_n, g) \), for any \( g \in C_n \), is the only 1-regular Cayley digraph that is the pattern of a unitary matrix.

**Remark 2.** By Theorem 1, a Cayley digraph \( \text{Cay}(G, \{s_1, s_2\}) \) is the pattern of a unitary matrix if and only if \( s_1 = s_2 s_1^{-1} s_2 \) and \( s_2 = s_1 s_2^{-1} s_1 \). Moreover, if \( G \) is abelian then \( s_1^2 = s_2^2 \).

Denote by \( D_n \) the dihedral group of order \( 2n \) and by \( e \) the identity element.

**Example 2.** \( \text{Cay}(D_n, \{s_1, s_2\}) \) is the pattern of a unitary matrix, because of Theorem 1 and since the standard presentation of \( D_n \) is \( \langle s_1, s_2 : s_1^2 = s_2^2 = e, s_2 s_1 s_2 = s_1^{-1} \rangle \) (see, e.g., [CM72], \S 1.5). Alternatively, by Theorem 1 and since

\[
\text{Cay}(D_n, \{s_1, s_2\}) \cong L(\text{Cay}(\mathbb{Z}_n, \{1, -1\}))
\]

(see, e.g., [BEFS95]).

**Example 3.** By Theorem 1 and by Remark 2, \( \text{Cay}(\mathbb{Z}_4, \{1, -1\}) \) is the pattern of a unitary matrix.

**Definition 5** ([F84]). Given the integers \( k \) and \( n, 1 \leq k \leq n-1 \), denote by \( P(n, k) \) the digraph whose vertices are the permutations on \( k \)-tuples from \( [n] = \{1, 2, ..., n\} \) and whose arcs are of the form \( (i_1 i_2 ... i_k), (i_2 i_3 ... i_k) \), where \( i \neq i_1, i_2, ..., i_k \) (see also [BFF97]).

Denote by \( S_n \) the symmetric group on \( [n] = \{1, 2, ..., n\} \).

**Example 4.** \( \text{Cay}(S_n, \{s_1, s_2\}) \), where \( s_1 = (1 \ 2 \ ... \ n) \) and \( s_2 = (1 \ 2 \ ... \ n-1) \), is the pattern of a unitary matrix, by Theorem 1 and since \( \text{Cay}(S_n, \{s_1, s_2\}) \cong L(P(n, n-2)) \) ([BFF97], Lemma 2.1).

**Example 5.** \( \text{Cay}(S_n, T) \), where \( T = (1 \ 2)C_{n-1} \), is the pattern of a unitary matrix. To show this, we use the construction in the proof of Theorem 1. Consider \( S_n = \langle S = \{(1 \ 2), (1 \ 2 \ ... \ n)\} \rangle \). Then \( S^{-1} = \{(1 \ 2), (1 \ n \ ... \ 2)\} \). Write \( x = (1 \ 2) \in S^{-1} \). Then \( (1 \ 2) S = \{e, (2 \ 3 \ ... \ n)\} \). Consider \( C_{n-1} = \{e, (2 \ 3 \ ... \ n)\} \). Write

\[
T = (1 \ 2)C_{n-1} = \{(1 \ 2), (1 \ 2) (2 \ 3 \ ... \ n) = (1 \ 2 \ ... \ n), \ldots\}.
\]

Since \( S \subset T \), \( S_n \subset \langle T \rangle \). Moreover \( x \in T^{-1} \), \( C_{n-1} = xT \) and \( |T| = n - 1 = C_{n-1} \). Then, by Theorem 1 and Lemma 1, \( \text{Cay}(S_n, T) \) is the pattern of a unitary matrix.

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