How to Build Unconditionally Secure Quantum Bit Commitment Protocols

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I. INTRODUCTION

Bit commitment is a kind of a cryptographic protocol that can serve as a building block to achieve various cryptographic objectives, such as user authentication. There is a nearly universal acceptance of the general impossibility of secure quantum bit commitment (QBC), taken to be a consequence of the Einstein-Podolsky-Rosen (EPR) type entanglement cheating which supposedly rules out QBC and other quantum protocols that have been proposed for various cryptographic objectives [1]. In a bit commitment scheme, one party, Adam, provides another party, Babe, with a piece of evidence that he has chosen a bit b (0 or 1) which is committed to her. Later, Adam would open the commitment by revealing the bit b to Babe and convincing her that it is indeed the committed bit with the evidence in her possession and whatever further evidence Adam then provides, which she can verify. The usual concrete example is for Adam to write down the bit on a piece of paper, which is then locked in a safe to be given to Babe, while keeping for himself the safe key that can be presented later to open the commitment. The scheme should be binding, i.e., after Babe receives her evidence corresponding to a given bit value, Adam should not be able to open a different one and convince Babe to accept it. It should also be concealing, i.e., Babe should not be able to tell from her evidence what the bit b is. Otherwise, either Adam or Babe would be able to cheat successfully.

In standard cryptography, secure bit commitment is to be achieved either through a trusted third party, or by invoking an unproved assumption concerning the complexity of certain computational problems. By utilizing quantum effects, specifically the intrinsic uncertainty of a quantum state, various QBC schemes not involving a third party have been proposed to be unconditionally secure, in the sense that neither Adam nor Babe could cheat with any significant probability of success as a matter of physical laws. In 1995-1996, a supposedly general proof of the impossibility of unconditionally secure QBC, and the insecurity of previously proposed protocols, were presented [2-6]. Henceforth it has been generally accepted that secure QBC and related objectives are impossible as a matter of principle [7, 12].

There is basically just one impossibility proof (IP), which gives the EPR attacks for the cases of equal and unequal density operators that Babe has for the two different bit values. The proof purports to show that if Babe’s successful cheating probability $P_c^B$ is close to the value 1/2, which is obtainable from pure guessing of the bit value, then Adam’s successful cheating probability $P_A$ is close to the perfect value 1. This result is stronger than the mere impossibility of unconditional security, namely that it is impossible to have both $P_c^B \sim 1/2$ and $P_A^A \sim 0$. The impossibility proof describes the EPR attack on a specific type of protocols, and then argues that all possible QBC protocols are of this type.

Typically, one would expect that a proof of impossibility of carrying out some thing X would show that any possible way of doing X would entail a feature that is logically contradictory to given principles, as, for example, in the cases of quantum no-cloning [13, 14] and von Neumann’s no-hidden-variable theorem [15]. In the present case, one may expect a proof which shows, e.g., that any QBC protocol that is concealing is necessarily not binding. It is important for this purpose that the QBC protocol formulation be all-inclusive. In the absence of a proof that all possible QBC protocols have been included in its formulation, any impossibility proof is at best incomplete. Thus, a priori, there can be no general impossibility proof without a mathematical characterization or a definition of all QBC protocols. Within the framework of two-way quantum communications between Adam and Babe with no further constraints, which is the setting of the IP and its EPR attack, no such definition has ever been presented. Although one can judge whether or not a specific protocol is a QBC protocol, similar to whether a specific process of computation is “algorithmic” or not, it appears prohibitively difficult to characterize mathematically all QBC protocols. Just as there is no Church-
Turing theorem, just a Church-Turing thesis, there can be no impossibility theorem without a mathematical definition of a QBC protocol.

More concretely, there are many types of QBC protocols that are not captured by the IP formulation and differ from each other, similar to the existence of several types of algorithmic processes different from a Turing machine. Some of these types were described by this author previously [11-13], and secure protocols can actually be found among them. As those papers, and this one, should make clear, even if secure QBC were impossible in all of these types, the impossibility proof for each of them would be different and would bear no resemblance to the well-known IP. These different types of protocols arise because only certain techniques of protocol design, such as certain use of classical randomness in a quantum protocol, are included in the IP formulation, which does not show that all possible techniques have been included. Even just for classical randomness, the different ways it could affect a QBC protocol are not properly accounted for. In this paper, a systematic description of some gaps in the IP, and the corresponding opportunity for six new protocol types, would be identified and elaborated upon. The original IP formulation would be named Type 0, with the new ones named Type 1 to Type 6.

In Section III, the impossibility proof is reviewed and its scope delimited. The basis of previous incorrect claims on the security of several different QBC protocols would be discussed. See also Appendix A. In Section III, the incompleteness of the IP is analyzed from a variety of angles that lead to at least six different types of protocols not covered by the IP formulation. They are discussed in Section IV with security proofs given for QBC1 and QBC2. The security proofs for QBC4 and QBC5 are given in Refs. [19] and [20].

II. THE IMPOSSIBILITY PROOF: TYPE 0 PROTOCOLS

The impossibility proof, in its claimed generality, has never been systematically spelled out in one place, but the essential ideas that constitute this proof are generally agreed upon [8-12]. The formulation and the proof can be cast as follows. Adam and Babe have available to them two-way quantum communications that terminate in a finite number of exchanges, during which either party can perform any operation allowed by the laws of quantum physics, all processes ideally accomplished with no imperfection of any kind. During these exchanges, Adam would have committed a bit with associated evidence to Babe. It is argued that, at the end of the commitment phase, there is an entangled pure state \( |\Phi_b \rangle \), \( b \in \{0, 1\} \), shared between Adam who possesses state space \( \mathcal{H}^A \), and Babe who possesses \( \mathcal{H}^B \). For example, if Adam sends Babe one of \( M \) possible states \( \{ |\phi_{bi} \rangle \} \) for bit \( b \) with probability \( p_{bi} \), then

\[
|\Phi_b \rangle = \sum_i \sqrt{p_{bi}} |e_i \rangle |\phi_{bi} \rangle
\]

with orthonormal \( |e_i \rangle \in \mathcal{H}^A \) and known \( |\phi_{bi} \rangle \in \mathcal{H}^B \). Adam would open by making a measurement on \( \mathcal{H}^A \), say \( \{ |e_i \rangle \} \), communicating to Babe his result \( i_0 \) and \( b \); then Babe would verify by measuring the corresponding projector \( |\phi_{bi_0} \rangle \langle \phi_{bi_0} | \) on \( \mathcal{H}^B \), accepting as correct only the result 1. More generally, one may consider the whole \( |\Phi_b \rangle \) of (1) as the state corresponding to the bit \( b \), with Adam sending \( \mathcal{H}^A \) to Babe upon opening, so she can verify by projection measurement on \( |\Phi_b \rangle \langle \Phi_b | \).

When classical random numbers known only to one party are used in the commitment, they are to be replaced by corresponding quantum entanglement purification. The commitment of \( |\phi_{bi} \rangle \) with probability \( p_{bi} \) in (1) is, in fact, an example of such purification. An example involving Babe may be a protocol [21-23] where \( |\phi_{bi} \rangle \) in (1) is to be obtained by Adam applying unitary operations \( U_{bi} \) on state \( |\psi_b \rangle \in \mathcal{H}^B_i \) sent to him by Babe with probability \( \lambda_k \), \( k \in K \), where \( |K| < \infty \). Generally, for any random \( k \) used by Babe, it is argued that from the doctrine of the “Church of the Larger Hilbert Space” [10], it is to be replaced by the purification \( |\Psi \rangle \) in \( \mathcal{H}^B_i \otimes \mathcal{H}^B_j \)

\[
|\Psi \rangle = \sum_k \sqrt{\lambda_k} |\psi_k \rangle |f_k \rangle,
\]

where \( |\psi_k \rangle \in \mathcal{H}^B_i \) and the \( |f_k \rangle \)’s are complete orthonormal in \( \mathcal{H}^B_j \) kept by Babe while \( \mathcal{H}^B_i \) would be sent to Adam. With such purification, it is claimed that any protocol involving classical secret parameters would become quantum-mechanically determinate, i.e., the shared state \( |\Phi_b \rangle \) at the end of commitment is completely known to both parties. Note that, from [24], this means that both \( \{ \lambda_k \} \) and \( \{ |f_k \rangle \} \) are taken to be known exactly to both Babe and Adam. The possibility that one can always purify a classically random situation as in (2) has never been proved, especially how it may be combined with the following [3]. It is elaborated later in this paper and in Ref. [20], in connection with QBC1 and QBC5, that this is generally not possible.

Why should Adam and Babe share a pure state at the end of commitment? Any measurement followed by a unitary operation \( U_l \) depending on the measurement result \( l \) can be equivalently described by an overall unitary operator. Thus, if the orthonormal \( \{ |g_l \rangle \} \) on \( \mathcal{H}^{C_2} \) is measured with result \( l \), and then \( U_l \) operates on \( \mathcal{H}^{C_1} \), it is equivalent to the unitary operation

\[
U = \sum_l U_l \otimes |g_l \rangle \langle g_l |
\]

on \( \mathcal{H}^{C_1} \otimes \mathcal{H}^{C_2} \). It is claimed that any actual measurement during commitment can be postponed until the opening and the verification phases of the protocol without affecting the protocol in any essential way. Actually, if
the measurement result is announced during commitment there is no need for \( (3) \) because the protocol state is just indexed by the measurement result \( k \) known to both parties. (With the delayed measurement description, the cheating and the opening would be quite involved and hard to describe when the possibility of aborting a protocol is allowed. They have never been explicitly spelled out.) In order to maintain quantum determinacy, the exact \( \{ |y_i \rangle \} \) in \( (3) \) are taken to be known to both parties. Let us use \( k \) to denote Babe’s secret parameter, and \( i \) to denote Adam’s secret parameter, such as the i with probabilities \( \{ p_i \} \) in \( (4) \). These crucial assumptions of openly known \( \{ p_i \} \), \( \{ \lambda_k \} \), \( \{ | f_k \rangle \} \), and \( \{ |y_i \rangle \} \) are made in the impossibility proof through the use of known fixed quantum computers or quantum machines for data storage and processing by either party \( (5) \). Even though the control of such machines belongs only to one of the parties. As it turns out, to cheat successfully, Adam does not need to know \( \{ | y_i \rangle \} \) in a careful formulation and he does not need to know \( \{ | f_k \rangle \} \) in one general class of protocols \( (6) \). However, he does need to know \( \{ | f_k \rangle \} \) in general, a fact which is exploited in our Type 4 protocols. The general possibility of such quantities being unknown or classically random to Adam is exploited in many of our protocols. A general limitation on the quantum purification of classical randomness is described in our discussion of Type 3 protocols in Section IV.C.

Protocols of the form \( (4) \), where \( \{ | \phi_{bi} \rangle \} \) are just sent from Adam to Babe, will be called single-stage protocols. In a multiple-stage protocol, \( | \phi_{bi} \rangle \) becomes, for \( i = \{ i_1, \ldots, i_n \} \) and \( k = \{ k_1, \ldots, k_{n-1} \} \) with \( 2n-1 \) stages in total

\[
| \phi_{bik} \rangle = U_{b_{ik}}^A \ldots U_{b_{i1}}^A U_{b_{k1}}^B U_{b_{k1}}^A | \phi_0 \rangle.
\]

(4)

The initial state \( | \phi_0 \rangle \) is openly known, and the alternate possible unitary operations by Adam and Babe, together with their respective probabilities, are also taken to be openly known. If the action is initiated by Babe instead of Adam, \( (5) \) can be replaced by, for a 2-stage protocol,

\[
| \phi_{bik} \rangle = U_{b_{ik}}^A U_{b_{i1}}^A | \phi_0 \rangle.
\]

(5)

Purification of the random state \( U \)'s is to be carried out as in \( (6) \). Thus, a multi-stage protocol is equivalent to one of the form

\[
| \Phi_b \rangle = \sum_{ik} \sqrt{p_{bi}} \lambda_k | e_i \rangle | f_k \rangle | \phi_{bik} \rangle,
\]

(6)

where \( \{ p_{bi} \} \), \( \{ \lambda_k \} \), \( \{ | \phi_{bik} \rangle \} \) are openly known, \( | e_i \rangle \in H^A \) controlled by Adam, \( | f_k \rangle \in H^{B_1} \) controlled by Babe, and \( | \phi_{bik} \rangle \in H^{B_2} \) is the evidence Babe possesses at the end of commitment. As in the case of quantum coin-tossing \( (2) \), formulation, in the IP the whole state space \( H^{B_2} \) is supposed to be passed on during each stage. As described later, this misses the nonuniqueness associated with passing back a portion of the space during commitment or opening, as in QBC1, and the problem of the very possibility of purification \( (2) \), as in QBC5. By writing \( | \phi_{bi} \rangle = \sum_{ik} \sqrt{\lambda_k} | e_i \rangle | f_k \rangle | \phi_{bik} \rangle \), \( (3) \) is also of the form \( (1) \) with \( H^B = H^{B_1} \otimes H^{B_2} \) and a multi-stage protocol is claimed to be equivalent to a single-stage one. Indeed, it is alternatively argued that \( | \Phi_b \rangle \) is always openly known at the end of commitment in any multi-stage protocol with the use of purification. Thus, it can always be represented by \( (11) \) with all the quantities involved openly known.

With such a formulation, Babe can try to identify the bit from \( \rho^B \), the marginal state of \( | \Phi_b \rangle \) on \( H^B \), by performing an optimal quantum measurement that yields the optimal cheating probability \( \bar{P}^B \) for her. Adam cheats by committing \( | \Phi_0 \rangle \) and making a measurement on \( H^A \) to open \( i_0 \) and \( b = 1 \). His probability of successful cheating is computed through \( | \Phi_b \rangle \), his particular measurement, and Babe’s verifying measurement; the one optimized over all of his possible actions will be denoted \( \bar{P}^A \). For a fixed measurement basis, Adam’s cheating can be described by a unitary operator \( U^A \) on \( H^A \). His general EPR attack goes as follows. If the protocol is perfectly concealing, i.e., \( \bar{P}^B = 1/2 \), then \( \rho^B = \rho^B \). By writing \( | \Phi_b \rangle \) as the Schmidt decomposition on \( H^A \otimes H^B \),

\[
| \Phi_b \rangle = \sum_j \sqrt{p_j} | e_{b,j} \rangle | \tilde{\phi}_j \rangle,
\]

(7)

where \( | \tilde{\phi}_j \rangle \) are the eigenvectors of \( \rho^B \) and \( \{ | e_{b,j} \rangle \} \) for each \( b \) are complete orthonormal in \( H^A \), it follows that Adam can obtain \( | \Phi_1 \rangle \) from \( | \Phi_b \rangle \) by a local cheating transformation \( U^A \) that brings \( \{ | e_{0,j} \rangle \} \) to \( \{ | e_{1,j} \rangle \} \). Thus his optimal cheating probability is \( P^A = 1 \) in this case. More generally, when Babe checks \( | \Phi_b \rangle \) on \( H^A \otimes H^B \), Adam still just cheats by applying a local transformation \( U^A \) to turn \( | \Phi_0 \rangle \) to \( | \Phi_1 \rangle \), although the terminology of EPR attack then becomes somewhat misleading.

For unconditional, rather than perfect, security, one demands that both cheating probabilities \( P^B - 1/2 \) and \( P^A \) can be made arbitrarily small when a security parameter \( n \) is increased \( (8) \). Thus, unconditional security is quantitatively expressed as

\[
\text{(US)} \quad \lim_{n} \bar{P}^B = \frac{1}{2}, \quad \lim_{n} \bar{P}^A = 0.
\]

(8)

The condition \( (8) \) says that, for any \( \epsilon > 0 \), there exists an \( n_0 \) such that for all \( n > n_0 \), \( P^B - 1/2 < \epsilon \) and \( P^A < \epsilon \), to which we may refer as \( \epsilon \)-concealing and \( \epsilon \)-binding. These cheating probabilities are to be computed purely on the basis of logical and physical laws, and thus would survive any change in technology, including an increase in computational power. In general, one can write down explicitly

\[
\bar{P}^B = \frac{1}{4} \left( 2 + || \rho^B_{0} - \rho^B_{1} ||_1 \right),
\]

(9)
where \( \| \cdot \|_1 \) is the trace norm, \( \| \tau \|_1 \equiv \text{tr}(\tau^\dagger \tau)^{1/2} \) for a trace-class operator \( \tau \), but the corresponding \( \tilde{P}_c^A \) is more involved. However, it may be shown that it satisfies

\[
4(1 - \tilde{P}_c^B)^2 \leq \tilde{P}_c^A \leq 2\sqrt{\tilde{P}_c^B(1 - \tilde{P}_c^B)}. \tag{10}
\]

The lower bound in (10) yields the following impossibility result given by the IP,

\[
\lim_{n} \tilde{P}_c^B = \frac{1}{2} \Rightarrow \lim_{n} \tilde{P}_c^A = 1 \tag{11}
\]

within its formulation. Condition or (11) is a continuity statement different from a point statement \( \tilde{P}_c^B = 1/2 \Rightarrow \tilde{P}_c^A = 1 \). Note that the impossibility proof makes a stronger statement than the mere impossibility of (US), i.e., (11) is stronger than (5) not being possible.

There have been quite a few incorrect claims on obtaining US QBC protocols, both before and after the appearance of the IP. In particular, two of the various approaches that were pursued by the present author do not work for reasons associated with the IP (see Appendix A for a summary). In the first case, also proposed in different forms by several others, simple use of classical randomness by Babe supposedly leads to different cheating transformations for Adam dependent on such randomness, hence a binding protocol is obtained after averaging over such randomness that has to be carried out in evaluating \( \tilde{P}_c^A \). The purification is not attended to, in view of the “equivalence” between classical and quantum randomness via the “Church of the Larger Hilbert Space” doctrine. However, this doctrine, often used in the IP as in (2), is incorrect. One simple way to see that is to observe that Adam does not entangle the classical randomness, e.g., if Adam sends \( |\phi_{bi}\rangle \) with probability \( p_{bi} \) instead of (11), he cannot launch entanglement cheating through \( \rho_B^0 = \rho_B^1 \) still applies. Even when the entanglement purification has been carried out, \( \rho_B^1 \) is equivalent to classically random \( \{|\psi_k\rangle\} \) only if the measurement of \( \{|f_k\rangle\} \) is first performed on \( \mathcal{H}^{B_2} \). Otherwise, the off-diagonal elements in \( \rho_B^1 \) involving \( |f_k\rangle\langle f_{k'}| \) may lead to better \( \tilde{P}_c^B \) in Babe’s cheating measurement on \( \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \), as compared to the case of purely classically random \( \{|\psi_k\rangle\} \) with zero off-diagonal elements.

As a specific example, consider the protocol preceding QBC5p discussed in the beginning of Ref. (21). It is perfectly concealing if Babe does not entangle, but not if she does. Other examples not involving teleportation can also be given. Thus, there is no equivalence between classical randomness and quantum purification. It is the possibility of entanglement cheating by Babe, not Church of the Larger Hilbert Space, which dictates that (11) is the correct representation in such a situation. Under such a stronger concealing condition, compared to just classically random \( \{|\psi_k\rangle\} \), Adam may indeed cheat in accordance with IP, depending on the protocol.

The second failed approach involves various attempts to turn a pure \( |\Phi_b\rangle \) into a mixed one through Adam’s action during the commitment phase before opening. The IP argues that a pure \( |\Phi_b\rangle \) can always be maintained in principle via perfect quantum computation. While there is no mathematical formalization on this issue that may lead to a rigorous proof, my different attempts indeed lead to different countermeasures by Adam, and I do not see what next attempt to try in this approach that may appear to have a possibility of success. However, this strategy works on Babe’s entanglement and leads to our QBC1. The reason is directly connected to the point of the last paragraph, namely that Babe’s measurement on \( \{|f_k\rangle\} \) first and then on \( \mathcal{H}^{B_2} \) can lead to a concealing protocol even though there may be a measurement on \( \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \) with which Babe can cheat. Thus, Babe’s entanglement may be effectively “destroyed” through, e.g., Adam’s questioning during commitment. See the following discussions related to QBC1 in Sections III and IV for details.

### III. PROBLEMS OF THE IMPOSSIBILITY PROOF

A plausible first reaction to the impossibility proof is: why are all possible QBC protocols covered by its formulation? More precisely, how may one capture mathematically the necessary feature of an unconditionally secure QBC protocol in a precise definition that is required for the formulation and proof of a mathematical theorem that says such a protocol is impossible? No such definition is available. An analogy to a QBC protocol is an “effectively computable” function, a function whose value for any specific argument can be “mechanically” obtained in a finite number of steps without the intervention of “intelligence.” The well-known Church-Turing thesis says that any effectively computable function can be computed recursively or by a Turing machine. It can be cast as an impossibility statement: there is no effective procedure that cannot be simulated by a Turing machine. It was found that a function that can be computed by a method that is clearly effective, such as Post machines or Markov algorithms, is indeed also Turing-computable. However, nobody calls the Church-Turing thesis the Church-Turing theorem. This is because there is no mathematical definition of an effective procedure. The logical possibility is open that someday a procedure is found that is intuitively or even physically effective, but which can compute a nonrecursive arithmetical function.

Thus, in the absence of a precise definition of a QBC protocol, one would have at best an “impossibility thesis,” not an impossibility theorem. (This view was emphasized to the author by Masanao Ozawa.) It is often difficult, if not impossible, to capture precisely by means of a mathematical characterization a given kind of physical operations that have unambiguous common sense meaning. For example, there is no definition that would characterize all classical cryptographic protocols, say for bit commitment. It is at least not clear why a definition in the more general quantum case can ever be
found. Just as there appear to be many different forms of effective procedures, there are many different QBC protocol types that appear not to be captured by the IP formulation. To uphold just the “impossibility thesis,” one would need to prove that US QBC is impossible in each of these types.

The problem of characterizing mathematically all QBC protocols, although quite difficult, does not seem to be as hopeless as that for an effectively computable function, if one believes that “bit commitment protocol” is less ambiguous than “effective procedure,” even though both concepts can presumably be recognized to be applicable or not when a particular instance is presented. In particular, the framework of two-way quantum communication, or the Yao model, without any constraints of relativity or superselection rules but with the possibility of the protocol being aborted as a result of cheating detection before opening, is an appropriate general setting. (The Yao model allows actual measurements during rounds but is often interpreted to imply used in coin-tossing formulation. That would exclude the possibility of sending back only part of a product space, which is utilized in many of our US protocols.) It is sometimes argued that every proof has to presuppose a “model,” but the question is whether the model used in the IP is general enough to capture all clear-cut QBC protocols within the above framework. It is also sometimes argued that the “community of experts in the field” have already agreed on a “definition” of what constitutes a QBC protocol, which would rule out some of our Type 3 protocols. But the question is why a clear-cut QBC protocol should be ruled out by legislation. Note that “definition” in this context does not mean an arbitrary choice of terminology, but a mathematical characterization of all instances where the concept is applicable. Observe also that this characterization problem does not arise in security proofs, because one should be able to exhaust all possible types of attack given a specific protocol.

The most important instances of incompleteness of the IP and quantum coin-tossing formulations, as presently understood by the author, are listed under four categories in the following. Some of these have been discussed previously. Some new protocol types made possible by such gaps are discussed in Section IV.

a. Freedom of Operation — In a two-party situation where either one can do anything, constrained only by physical laws, and has only his/her own interest to protect, neither can be trusted to be honest if an operation would lead to his/her advantage without penalty. On the other hand, a party is supposed to strive to achieve the aim of the protocol if his/her own security against cheating by the other party can be assured. These obvious considerations are codified as the Libertarian Principle and the Intent Principle of protocol formation, further elaborated in Ref. 14. The resulting freedom of action by either party is not accounted for in the QBC IP formulation, nor in the mathematical formulations of quantum coin-tossing protocols.

b. Honesty and Cheating — In a multi-stage protocol, where a state space is passed between Adam and Babe in rounds for operations, as 10, either party can substitute an entirely different space of his/her own at any stage. The possible advantage is clear in coin-tossing, and examples were given in Ref. 23 on bit commitment. There is no mechanism built into the protocol formulation to prevent such cheating. Fortunately, this problem can be alleviated in two different ways including cheating detection and the possibility of aborting the protocol, with perhaps a penalty imposed on the party that got caught cheating with the use of an ensemble, as described in Ref. 21 for QBC5. However, there is still a lot of freedom left which has not been accounted for, some to be discussed in the following points (c) and (d).

c. Random vs. Nonrandom Secret Parameters — Suppose a protocol has the property that it is concealing for every possible legal operation by Babe that can be checked as mentioned above. Then Babe is free to choose any such operation (or state) with whatever probability distribution unknown to Adam. This freedom, codified as the Secrecy Principle 15, 18, is a simple corollary of the Libertarian Principle and the Intent Principle. It directly contradicts the IP claim that a state is always openly known at the end of commitment in a QBC protocol. One consequence of this freedom is that Adam’s cheating transformations may depend on exactly what Babe’s choice is in order to succeed. Indeed, he may not even have a single density operator representation for each b due to the difference between a random parameter and an unknown parameter, a distinction well-known in statistics. Our Type 3 and Type 6 protocols arise from this freedom, but no concrete protocol has been found in these types that can be proved unconditionally secure. On the other hand, with additional features one can construct secure protocols utilizing this freedom, as in our QBC1, QBC2, and QBC4.

d. Generalizations to Imperfect Operations — Since the final criterion in a QBC protocol involves probabilities only, every step and requirement can also be relaxed to a probabilistic, rather than a perfect deterministic, one. For example, the verifying measurement by Babe need not succeed with probability 1. However, it seems that the relaxation should go only so far as to the case where the probability is arbitrarily close to 1, as in other quantum and classical algorithms. Even though there is no proof to the contrary, there is no known case where this particular generalization would ever lead to a US QBC protocol.
b. Generality of Quantum Purification — Adam needs to entangle his possible actions in 11 or 6 in order to launch an EPR attack as described in the IP. It has not been shown why all possible classically random elements in a protocol allow quantum entanglement purification. Indeed, when quantum teleportation involving one actual measurement among different possible spaces is utilized during commitment, as in the case of protocol QBC5 20A, there is no quantum purification. In a different way, this situation of no purification, or no unique purification as in QBC1, may also obtain when a random part of a tensor product space from one party is to be returned to the other party, which occurs in many of our US protocols.

c. Different Commitment Possibilities — Under this category one may consider almost all the restrictions of the IP formulation that can be removed. A particularly important example is the possible use of multiple evidence state spaces. We have two separate secure protocols, QBC2 and QBC4, that exploit this possibility in different ways. Another example is the use of quantum teleportation in our QBC5.

d. Nonuniqueness — There are various places in the IP where uniqueness of choice is implicitly assumed; otherwise the question would arise as to why a cheating transformation $U^A$ can be found which is successful for every possible choice. For example, the cheating probability $P^A_c$ depends on Babe’s verifying measurement. For an arbitrary protocol, the IP formulation does not, and in fact cannot, specify what the possible verifying measurements could be. There is no proof given that there cannot be more than one verifying measurement for which different cheating transformations are needed. When such a situation occurs, Adam may not know which one to use for a successful cheating. However, this gap can be closed when the verifying measurements are perfect, i.e., the bit is verified with probability 1 from the measurement.

A more serious situation occurs in the case of purification 6, when there is more than one way to purify a given classical random number. For example, the usual multi-stage formulation of QBC and coin tossing, exemplified in 4, 5, carries the implicit assumption (and restriction) that one fixed space is passed between the parties in the rounds. But there is great utility in splitting a classical or quantum correlated tensor product space for obtaining security. If a random sample of $m$ out of $n$ qubits are to be sent from the first party to the second after state modulation on the $m$ qubits, the first party can pick any $m$ of the $n$ qubits and entangle/purify the result with unitary permutation operators among all the $n$ qubits. Additional qubits apart from the $n$ given ones can also be employed with proper permutation. As a result, the $m$ qubits that are sent back can be any $m$-subset of the original $n$ qubits plus other auxiliary qubits. However, the resulting $m$-subset $|Φ_k⟩$ of 6 would be different for different choices because the qubits have been individuated by their positions, and there is no single overall purification. Thus, there arises again the question of existence of a uniformly successful cheating $U^A$ if concealing obtains in each of the purifications. If the $m$ qubits cannot be entangled, or if their entanglement cannot be maintained during the protocol, the possibility of uniform cheating $U^A$ becomes the issue of the existence of irreducible residual classical randomness in the protocol that is answered negatively in our following QBC1.

IV. SIX NEW TYPES OF PROTOCOLS

In this section we describe six different types of protocols, together with specific examples for five of them, that are not covered by the IP for reasons expounded in the preceding section. For four protocols, namely QBC1, QBC2, QBC4, and QBC5, full unconditional security proofs are available. The situation for Type 3 and Type 6 protocols is not certain. Together they should make clear the many possibilities that are open for developing US QBC protocols. All our protocols assume for simplicity that Adam opens perfectly, i.e., with probability one, for $b = 0$, as in the IP.

A. Type 1 protocols — residual classical randomness

Type 1 protocols are defined to be those in which there is inherent classical randomness that cannot be quantum-mechanically purified and maintained. This classical randomness is distinguished from that of a Type 3 protocol that arises from $\{λ_k⟩$ and $\{⟨j_k\}$ randomness in 2. Our three-stage protocol QBC1 provides such an example and can be motivated as follows. It is possible to create protocols that are clearly binding; the question becomes how to make them concealing. The main difficulty in this connection is Babe’s entanglement (cheating) over the random choices. This, it turns out, can be prevented during cheating detection by Adam. Thus, the overall protocol becomes both concealing and binding.

Consider a protocol in which Adam sends $n_0$ qubits to Babe, each randomly drawn from a set of BB84 states $S = \{|1⟩, |2⟩, |3⟩, |4⟩\}, \{1⟩3⟩ = 2⟩4⟩ = 0, \{1⟩2⟩ = 3⟩4⟩ = 1/√2\}$. Babe randomly picks one and sends it back to Adam, who modulates it by $U_0 = I$ or $U_1 = R_π$, the rotation by $π$ on the great circle containing $S$, and commits it as evidence. He opens by telling Babe the state of each qubit, which she verifies, telling $b = 1$ if the one she sent was moved by $R_π$. In a more complete protocol, Babe would check that Adam indeed sends her states from $S$, and Adam would check that Babe is sending back one of the qubits from him. This can be carried out either in a classical game-theoretic formulation or through an ensemble approach described in Appendices A and B of Ref. 20. In any event, according to IP and all coin-tossing formulations, both parties are assumed honest.
except that they can (and should) entangle all possibilities.

The protocol is \(\epsilon\)-binding regardless of whether Babe entangles for the following reason: Adam has to know exactly which qubit Babe sent back in order to cheat successfully, regardless of whether or not he uses his initial entanglement, which may involve permutations among the \(n_0\) qubits. As just discussed in [111], IP does not apply because he does not know which qubit Babe sends back. He needs to turn by \(R_x\) just the one qubit via \(U^A\) that depends on exactly which qubit it is, i.e., on the exact way Babe chose to purify and/or the exact \(|f_k\rangle\) she will measure. But if Adam knows which qubit it is, he could just cheat by declaring a state appropriately different from the original one, e.g., if he sent \(|3\rangle\) he could declare he sent \(|1\rangle\) instead. Without knowing which qubit it is, let \(m(\leq n)\) qubits be turned by him, each by an amount that would be accepted as 1 with probability \(p < 1\) by Babe upon her verification, while the other \(n_0 - m\) become a permutation of the original. (It can be shown that with his full entanglement, the best he can do is to turn a small fraction and re-permute the others, but this result is not needed for the present argument.) Thus, his probability of successful cheating is \((m/n_0)(1-p)^{m-1}\), the maximum of which over \(m\) can be made arbitrarily small for large \(n_0\).

If Babe does not entangle, this protocol is perfectly concealing. Since she may be able to cheat with permutation entanglement, Adam can defeat that as follows. He sends her originally \(n \gg n_0\) qubits from which Babe returns \(n - n_0 + 1\) qubits. Adam randomly asks Babe to reveal \(n - n_0\) of the returned qubits and check that they are indeed in states sent by him. The only entanglement Babe can employ is permutation among the qubits as she could not respond to Adam perfectly with additional entanglement. She may entangle a minimum of two qubits at a time between one in the set of \(n - n_0 + 1\) elements she sends back to Adam and one in the set of \(n_0 - 1\) elements she keeps, in order to maximize the probability that the last one retained by Adam remains entangled to at least one qubit in her possession for her entanglement cheating. The probability of having the entanglement surviving on the one remaining in Adam’s possession is \((n_0 - 1)(n_0 - n + 1)^{-1}\), which can be made arbitrarily small. Hence, \(P^B \leq \frac{1}{2} + \epsilon\) for any \(\epsilon\) and fixed \(n_0\) with large \(n\). This use of \(n \gg n_0\) qubits gives Adam new possibility of entanglement, which he could not use under the protocol condition that he has to open the bit on the one remaining qubit whose name Babe knows and would verify upon. Thus, we have proved that the following protocol is \(\epsilon\)-concealing and \(\epsilon\)-binding.

### PROTOCOL QBC1

1. Adam sends Babe \(n\) qubits, each drawn at random from the set \(S\) of four BB84 states and named by its temporal position.
2. Babe randomly selects \(n - n_0 + 1\) of them and sends them in a random order back to Adam, who asks Babe to reveal the names for \(n - n_0\) of them. After verifying them, he modulates the remaining qubit for \(U_0 = I, U_1 = R_e\).
3. Adam opens by declaring \(b\) and the states of all the \(n_0\) remaining qubits; Babe checks by corresponding projective measurements.

To recapitulate the logic of its success, this protocol allows many different purifications by Babe with different results \(|\Phi_b\rangle\) and which may not be concealing, so Adam cannot cheat anyway. However, by checking an ensemble Adam can force Babe to measure and destroy her entanglement cheating possibility. The resulting protocol becomes a classically randomized one, in which Adam of course still cannot cheat.

### B. Type 2 protocols — bit-value dependent evidence state space

As developed in Ref. [15], it is possible to have secure protocols for which the evidence state space \(H^B\) in \(H^A \otimes H^B\), which is in Babe’s possession at the end of commitment, depends on the bit \(b\) and becomes \(H^B_b\) as it appears to Adam, but is indistinguishable for the two bit values to Babe. Type 2 protocols are defined to be those in which Babe sends Adam \(H^B_0\) and \(H^B_1\) which she does not need to entangle to her kept spaces although she may choose to. Adam returns \(H^B_b\) to commit \(b\) while keeping the other space. It is distinguished from Type 4 protocols that employ split entangled pairs to individuate \(H^B_b\), and is easier to implement practically.

Protocol QBC2 goes as follows. Let Babe send Adam two sets of named states \(S_0 = \{\phi_{01}, \ldots, \phi_{0n}\}\), \(S_1 = \{\phi_{11}, \ldots, \phi_{1n}\}\), each \(\phi_{bi}\) drawn randomly from the set \(S\) of four BB84 states on a qubit. Adam does not know and cannot determine perfectly what each state \(\phi_{bi}\) is. To commit \(b\), he sends back randomly one of \(\phi_{bi}\), revealing \(b\) and \(i\) at opening. In order to cheat, Babe has to distinguish the two sets \(S_0\) and \(S_1\) and then measure on the committed state. It is readily checked that this protocol is \(\epsilon\)-concealing for sufficiently large \(n\), such that the four states in \(S\) appear in nearly equal fractions among \(S_0\) and \(S_1\), even if Babe entangles the states. One may impose the condition that each state in \(S\) appears equally often in \(S_0\) and \(S_1\), which yields perfect concealing if Babe does not entangle, but again only \(\epsilon\)-concealing if she does. Such a condition may be obtained, e.g., by sending in \(n\) sets of four randomly permuted states instead, with Adam picking one from a set. On the other
hand, Adam cannot cheat perfectly given that Babe does not entangle since that does not really help her (or that she does but with $\{f_k\}$ of even unknown to Adam; in the latter case it shares the feature of QBC4 [19]). For QBC2 we just ake the case Babe does not entangle. If Adam entangles the $\phi_0$ by way of permutations and commits a qubit $\mathcal{H}_0$ or his own qubit $\mathcal{H}_0$, he could not change by local transformation the state in $\mathcal{H}_0$ (or $\mathcal{H}_0$) to any of those in $\mathcal{H}_1$. That is, the state in $\mathcal{H}_0$ (or $\mathcal{H}_0$) is entangled with the states in $\mathcal{H}_0$ where these states may go from local transformations. This is because of the invariance of the position of a qubit within $S_0$ and $S_1$ under entanglement that assures perfect $b = 0$ opening. The IP does not apply because $U^A$ cannot be determined without knowing the state of the committed $\phi_b$. See the following Section V for further discussion. Let $p_A < 1$ be Adam’s optimal cheating probability. As usual, this protocol QBC2.p can be extended to an $\epsilon$-binding one, QBC2, in an $N$-sequence, making $P_c = p_A^N$ arbitrarily small.

C. Type 3 protocols — anonymous states

Type 3 protocols are defined to be those where concealing is obtained for each of Babe’s possible choices of $\{\lambda_k\}$ and/or $\{f_k\}$ in [2] at any stage of the protocol. Each choice thereby results in an anonymous state on $\mathcal{H}_2 \otimes \mathcal{H}_2^B$ as it is unknown to Adam. To explain how such a situation may arise in view of the committed [2], observe that the unknown $|\psi_k\rangle$ without purification is merely replaced by the unknown $|f_k\rangle$ in [2] even for known $\{\lambda_k\}$. How would the other party, say Adam, know $\{f_k\}$? Babe can use any orthonormal $\{f_k\}$ without affecting the protocol security, assuming that the protocol is perfectly concealing for any orthonormal $\{f_k\}$, as it usually is. The Secrecy Principle [18] mentioned above ensures that Adam cannot demand to know exactly what $\{f_k\}$ Babe uses in any instance of the protocol execution. As a matter of fact, in reality [2] may just be an abstract representation such that even Babe does not and cannot know $\{f_k\}$, as for example when Babe generates the $|\psi_k\rangle$ in a classically random fashion. It is clear that IP would not go through unless Adam’s cheating transformation $U^A$ is independent of $\{f_k\}$. This issue has not been examined in the literature, but a proof that such independence is obtained was given in Ref. [23] for protocols that do not involve what we call the switching of evidence state space that produces the $\{f_k\}$ dependence on $U^A$, or Babe’s checking over the entire entangled $U^B|\Psi\rangle$ upon verification that makes Theorem 3 of Ref. [23] inapplicable. Moving the boundary to further entanglement on $\mathcal{H}_2^B$ does not work because Adam cannot operate on $\mathcal{H}_2^B$. Thus, the proof breaks down in general, and the above scenario of $\{f_k\}$ dependence of $U^A$ with corresponding $U^B|\Psi\rangle$ verification is carried out for the development of a secure protocol QBC4.

In the early anonymous-state protocols [17, 22, 23], the use of $b$-dependent evidence state spaces has not been discovered and it was thought that $U^A$ is always independent of $\{f_k\}$ in [2], even though a proof is only available for a special class of protocols describe by $U^B|\Psi\rangle$ or Eq. (26) of Ref. [23]. Furthermore, the use of split entangled pair verification on $U^B|\Psi\rangle$ has also not been discovered, and a theorem was proved in Ref. [23] that $U^A$ is independent of $\{\lambda_k\}$ in the perfect concealing case. The question then becomes whether the $\{\lambda_k\}$ freedom alone would yield a secure protocol in the $\epsilon$-concealing case.

It is sometimes argued that such freedom cannot be automatized and thus cannot lead to a definite QBC protocol. However, since each party can clearly keep its own secret mechanism of choosing the $\{\lambda_k\}$, similar to other cases of a kept secret in cryptography, this kind of protocols are perfectly well-defined QBC protocols. In this connection, one should avoid the confusion between a probability distribution $\{\lambda_k\}$ and a definite $|\Psi\rangle$ of [2], and between a random and an unknown parameter. See [10, 18] for further elaborations on these points. See also Ref. [20] for a general classification of many anonymous states protocols that, however, does not include our four protocols of this paper.

Even if a theorem is proved with the purification [2] for any $\{\lambda_k\}$ and a fixed $\{f_k\}$ that says $\epsilon$-concealing for all $\{\lambda_k\}$ yields a good cheating $U^A$ independently of $\{\lambda_k\}$, it does not apply to our QBC2 where no $\{f_k\}$ is used (or equivalently $U^A$ needs to succeed for all $\{f_k\}$) in Adam’s cheating. This is because with known $\{f_k\}$, Adam does not need to really switch the state from $S_0$ to $S_1$. Thus, our QBC2 protocol is a Type 3 protocol also if one observes that Babe can entangle the $\mathcal{H}_2^B$ states $|\psi_k\rangle$ with any $\mathcal{H}_2^B$ state $|f_k\rangle$ without affecting $\epsilon$-concealing, but the cheating $U^A$ depends on $\{f_k\}$.

D. Type 4 protocols — split entangled pair to individuate evidence space

The use of bit-value dependent evidence state space leads to $\epsilon$-concealing and $\epsilon$-binding protocol QBC2 described above. It is possible to obtain perfectly concealing protocols when the evidence space is entangled by Babe, which is indistinguishable to Babe when presented
to her by Adam as committed evidence. This is described in Ref. [10] for protocol QBC4. These two protocols, although relying on the same basic idea, illustrate in different ways the diverse manifestation of $b$-dependent $H_b^\epsilon$ and classical randomness in protocol design.

E. Type 5 protocols — utilizing quantum teleportation

Type 5 protocols are defined to be those where quantum teleportation is utilized during commitment. An example is a two-stage protocol in which Babe randomly sends an entangled pair to Adam, who uses it to teleport a single possible state $|b\rangle$ for each $b$ to one qubit and sends it to Babe at opening. The Bell measurement result is committed as evidence. If Babe does not or cannot entangle, it is readily seen that such a protocol is perfectly concealing, while Adam cannot cheat perfectly. If Babe entangles her choice, the protocol can be made $\epsilon$-concealing if Babe first sends in many qubit pairs. The resulting protocol QBC5p and its US extension QBC5 are fully described in Ref. [20]. The main reason for the failure of IP in this case is that Adam’s measurement cannot be postponed to after commitment while the possible measurement results cannot be entangled as the actual reading is the committed evidence.

F. Type 6 protocols — necessary condition on concealing or binding

Once the freedom of operation is opened up, one may no longer assume that anything has to be known to both parties. For example, even in the original IP formulation, one may allow Adam to use different possible $\{p_b_i\}$ in [11]. Thus, Babe has to decide on $b$ by some strategy different from the one that assumes $\{p_b_i\}$ are known. The issue of unknown versus random parameters enters again, which greatly complicates the situation. It is not clear how one may formulate necessary conditions for concealing, or binding, that are needed to yield an impossibility proof that says if the necessary concealing (or binding) condition is satisfied, then the protocol cannot be binding (or concealing). However, it is also not clear how to formulate a protocol this way that can be proved secure. We just reserve the name “Type 6 protocols” for this approach without an example.

V. CONCLUDING REMARKS

We have tried to explain and to dissect the various ways a QBC protocol can be designed, and to show the many possibilities that the “impossibility proof” formulation misses. Specifically, we list four categories of gaps and six types of new protocol formulations that exploit such gaps. In four of these types, there are protocols QBC1, QBC2, QBC4, and QBC5 that can be proved unconditionally secure. This also solves the quantum coin-tossing problem, in which much work has been done assuming secure quantum bit commitment is impossible. Also, one of our protocols, QBC2, can be readily implemented with coherent states and is close to being practical even with the limited quantum memory and quantum communication capabilities we have at present. The challenge remains to find fully practical, secure QBC schemes including system imperfections, as well as efficient ways to utilize them for various cryptographic objectives.

APPENDIX A: PREVIOUS PROTOCOLS

I discuss briefly here the security status of the protocols I previously proposed and claimed unconditionally secure.

Protocol “QBC2” in Ref. [21], the only one claimed to be US there, is not proved perfectly concealing because entanglement cheating by Babe is not accounted for. It was assumed to be the same as a classically random protocol. While a proof is not available, it appears that the protocol cannot be made perfectly concealing or $\epsilon$-concealing without allowing Adam to cheat, even with the modification described in Section 5 of Ref. [10].

Ref. [22] is a preliminary version of Ref. [23], in which Type 3 protocols are introduced. However, the binding argument from no-cloning there is not valid. Thus, if Adam knows the (purified) anonymous state, the protocol is insecure due to the fact that all perfectly verifying measurements under the IP formulation lead to insecure protocols against a single cheating $U^A$, although that is a fact not proved in the IP itself. If Adam does not know the anonymous state, the situation is not yet completely resolved. However, it appears that Adam can probably cheat for operator-theoretic reasons on tensor product spaces, in this case as well as in QBC2.1p of v2 of this paper.

Protocols “QBC1” in Ref. [16] and “QBC4” in Ref. [17], as well as the preliminary version of Ref. [21], involve attempts to force Adam to measure on $H^A$, thus effectively “destroying” his entanglement. They fail as indicated in Section II. However, such an attempt can succeed when applied to destroy Babe’s entanglement, in the sense discussed in this paper, which leads to our present QBC1 that was first briefly discussed in Ref. [13]. The protocol “QBC2” in Ref. [17] and v1 of this paper, which is a “Type 2” protocol of Ref. [18], protocol “QBC4” in v1 of Ref. [24], and protocol “QBC5” in v1 of Ref. [19] are all insecure because they can be brought into the form of a single $|\Phi_b\rangle$ so that IP and the results of Section III in Ref. [23] apply.
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