INTEGRAL OPERATORS, EMBEDDING THEOREMS AND A LITTLEWOOD-PALEY FORMULA ON WEIGHTED FOCK SPACES

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Abstract. We obtain a complete characterization of the entire functions $g$ such that the integral operator $(T_g f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$ is bounded or compact, on a large class of Fock spaces $F_p^\phi$, induced by smooth radial weights that decay faster than the classical Gaussian one. In some respects, these spaces turn out to be significantly different than the classical Fock spaces. Descriptions of Schatten class integral operators are also provided.

En route, we prove a Littlewood-Paley formula for $\| \cdot \|_{F_p^\phi}$ and we characterize the positive Borel measures for which $F_p^\phi \subset L^q(\mu), 0 < p, q < \infty$.

In addition, we also address the question of describing the subspaces of $F_p^\phi$ that are invariant under the classical Volterra integral operator.

1. Introduction

Let $\mathbb{C}$ be the complex plane and denote by $H(\mathbb{C})$ the space of entire functions. Given $\phi : [0, \infty) \to \mathbb{R}^+$ a twice continuously differentiable function and $0 < p < \infty$, we extend $\phi$ to $\mathbb{C}$ setting $\phi(z) = \phi(|z|), z \in \mathbb{C}$. We consider the weighted Fock spaces,

$$F_p^\phi = \left\{ f \in H(\mathbb{C}) : \| f \|_{F_p^\phi}^p = \int_\mathbb{C} |f(z)|^p e^{-p\phi(z)} dm(z) \right\},$$

and

$$F_\infty^\phi = \left\{ f \in H(\mathbb{C}) : \| f \|_{F_\infty^\phi} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\phi(z)} \right\},$$

where $dm$ denotes the Lebesgue measure in $\mathbb{C}$.

As usual we write $F_p$ for the classical Fock spaces induced by the standard function $\phi(z) = \frac{|z|^2}{2}$. Moreover, for two real-valued functions $E_1, E_2$ we write $E_1 \asymp E_2$, or $E_1 \lesssim E_2$, if there exists a positive constant $k$ independent of the argument such that $\frac{1}{k} E_1 \leq E_2 \leq k E_1$, respectively $E_1 \leq k E_2$.

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We will deal with the integral operator

\[ T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) \, d\zeta \]

on weighted Fock spaces \( F_\phi^p \). This problem has been considered in [11] for the classical Fock spaces proving that \( T_g \) is bounded only for polynomials of degree \( \leq 2 \). We expect that for a stronger decay of the weight \( e^{-\phi} \) we shall find a wider class of symbols \( g \) such that \( T_g \) is bounded on \( F_\phi^p \).

With this aim, we first obtain a Littlewood-Paley type formula for \( F_\phi^p \) for a broad class of weights whose growth ranges from logarithmic (e.g. \( \phi(z) = a \log|z|, \ ap > 2 \)) to highly exponential (e.g. \( \phi(z) = e^{e|z|} \)). Under some mild assumptions on \( \phi \) (see Theorem 19 below), we prove that

\[
\int_{C} |f(z)|^p e^{-p\phi(z)} \, dm(z) \asymp |f(0)|^p + \int_{C} |f'(z)|^p (\psi_{p,\phi}(z))^p \, e^{-p\phi(z)} \, dm(z),
\]

for any entire function \( f \), where the distortion function \( \psi_{p,\phi} \) satisfies

\[
\psi_{p,\phi}(z) \asymp \frac{1}{\phi'(z)} \quad \text{for } |z| \geq r_0,
\]

for some \( r_0 > 0 \). The above formula in the particular case of the classical Fock space was proven in [11] using the explicit form of the reproducing kernel of \( F_2 \). The lack of precise information on the reproducing kernels for more general weights constrains us to employ a different method based on estimates for integral means of entire functions and their derivatives.

We then restrict our class of weights to rapidly increasing functions \( \phi \). More precisely, we consider twice continuously differentiable functions \( \phi : [0, \infty) \to \mathbb{R}^+ \) such that \( \Delta \phi > 0 \) and

\[
\tau(z) \asymp \begin{cases} 1, & 0 \leq |z| < 1 \\ (\Delta \phi(|z|))^{-1/2}, & |z| \geq 1 \end{cases},
\]

where \( \tau(z) \) is a radial positive differentiable function that decreases to zero as \( |z| \to \infty \) and \( \lim_{r \to \infty} \tau'(r) = 0 \). Furthermore, we suppose that either there exists a constant \( C > 0 \) such that \( \tau(r)r^C \) increases for large \( r \) or

\[
\lim_{r \to \infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.
\]

The class of rapidly increasing functions \( \phi \) will be denoted by \( \mathcal{I} \). It includes the power functions \( \phi(r) = r^\alpha \) with \( \alpha > 2 \) and exponential type functions such as \( \phi(r) = e^{\beta r}, \ \beta > 0 \) or \( \phi(r) = e^{e^r} \).

It turns out (see Theorem 19 below) that for \( \phi \in \mathcal{I} \), the Littlewood-Paley formula (1.1) can be written in the form

\[
||f||_{F_\phi^p}^p \asymp |f(0)|^p + \int_{C} |f'(z)|^p \frac{e^{-p\phi(z)}}{(1 + \phi'(z))^p} \, dm(z).
\]

Going further, we aim for a characterization of the Carleson measures for our Fock spaces which will subsequently be used to investigate the behavior of \( T_g \).
This problem was studied in [9, 10, 18] for the classical Fock space. The Hilbert space case $p = 2$ was considered by Seip and Youssfi [24] in several variables for a wide class of radial weights. However, our class of functions $I$ does not completely overlap with theirs, as shown by the examples $\phi(z) = |z|^m$ with $2 < m < 4$. Let $D(a, r)$ be the Euclidean disc centered at $a$ with radius $r > 0$, and for simplicity, we shall write $D(\delta \tau(a))$ for the disc $D(a, \delta \tau(a))$ with $\delta > 0$.

**Theorem 1.** Let $\phi \in I$ and let $\mu$ be a finite positive Borel measure on $\mathbb{C}$.

(I) Let $0 < p \leq q < \infty$.

(a) The embedding $I_d : F^\phi_p \to L^q(\mu)$ is bounded if and only if for some $\delta > 0$ we have

$$K_{\mu, \phi} := \sup_{a \in \mathbb{C}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} e^{q\phi(z)} \, d\mu(z) < \infty.$$  

Moreover, if any of the two equivalent conditions holds, then

$$K_{\mu, \phi} \propto \| I_d \|^q_{F^\phi_p \to L^q(\mu)}.$$  

(b) The embedding $I_d : F^\phi_p \to L^q(\mu)$ is compact if and only if for some $\delta > 0$ we have

$$\lim_{|a| \to \infty} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} e^{q\phi(z)} \, d\mu(z) = 0.$$  

(II) Let $0 < q < p < \infty$. The following conditions are equivalent:

(a) $I_d : F^\phi_p \to L^q(\mu)$ is compact;

(b) $I_d : F^\phi_p \to L^q(\mu)$ is bounded;

(c) For some $\delta > 0$, the function

$$z \mapsto \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} e^{q\phi(\zeta)} \, d\mu(\zeta)$$

belongs to $L^{q\over p-q}(\mathbb{C}, dm)$.

This theorem has an interesting consequence. Recall that, for the classical Fock spaces $F_p$, the following embeddings hold (see [29, 26])

$$F_p \subseteq F_q \quad \text{for } 0 < p \leq q < \infty.$$  

For the particular choice $d\mu(z) = e^{-q\phi(z)} \, dm(z)$, Theorem 1 shows under which conditions the Fock space $F^\phi_p$ is contained in $F^\phi_q$ for $p \neq q$, where $p, q > 0$. This never happens for rapidly increasing functions $\phi \in I$, as illustrated by the next corollary.

**Corollary 2.** If $\phi \in I$, the family of Fock spaces $\{F^\phi_p\}_{p>0}$ is not nested. In fact, $F^\phi_p \setminus F^\phi_q \neq \emptyset$ and $F^\phi_q \setminus F^\phi_p \neq \emptyset$ for all $p, q > 0$ with $p \neq q$.

This result shows a significant difference between the weighted Fock spaces $F^\phi_p$ with $\phi \in I$, and the classical Fock spaces $F_p$, which leads to additional technical difficulties in the study of $F^\phi_p$. 
We then apply our Carleson embedding theorem together with the Littlewood-Paley formula to the study of the integral operator

\[ T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) \, d\zeta. \]

The boundedness and compactness, as well as some spectral properties (such as Schatten class membership) of \( T_g \) acting on various spaces of analytic functions of the unit disc \( \mathbb{D} \) in \( \mathbb{C} \) have been extensively investigated (see e.g. \([3, 6]\) for Hardy spaces, \([5, 19, 23]\) for weighted Bergman spaces, or the survey \([2]\) and the references therein).

**Theorem 3.** Assume \( g \) is an entire function, and \( \phi \in \mathcal{I} \).

(I) For \( 0 < p \leq q < \infty \) we have

(a) \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) is bounded if and only if

\[ \sup_{z \in \mathbb{C}} \frac{|g'(z)|((\Delta \phi(z))^{\frac{q-p}{pq}}}{1 + \phi'(z)} < \infty. \]

(b) \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) is compact if and only if

\[ \lim_{|z| \to \infty} \frac{|g'(z)|((\Delta \phi(z))^{\frac{q-p}{pq}}}{1 + \phi'(z)} = 0. \]

(II) Let \( 0 < q < p < \infty \). The following conditions are equivalent:

(a) \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) is compact;

(b) \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) is bounded;

(c) The function \( \frac{g'(z)}{1 + \phi'(z)} \in L^r(\mathbb{C}, dm) \), where \( r = \frac{pq}{p-q} \).

Theorem 3 shows a much richer structure of \( T_g \) when acting on \( \mathcal{F}_p^\phi \) compared to the case of the classical Fock spaces, where this operator is bounded only for polynomial symbols of degree \( \leq 2 \) (see \([11]\)). On the other hand, Theorem 1, some other results in \([8, 19]\), and the similar techniques used to prove them, show some analogies between Bergman spaces with rapidly decreasing weights \( A_p^\omega \) and \( \mathcal{F}_p^\phi \), \( \phi \in \mathcal{I} \). Indeed, the family of test functions considered to study the boundedness of \( T_g \) on these spaces is basically the one introduced in \([8]\) to characterize sampling and interpolation sequences (such problems for Fock spaces with nonradial weights \( e^{-\phi} \) where \( \Delta \phi \) is a doubling measure were considered in \([15]\)).

However, there are some fundamental differences as well: the weighted Bergman spaces are nested and only constant symbols induce bounded operators \( T_g : A_p^\omega \to A_q^\omega \) when \( 0 < p < q < \infty \) (see \([19]\)), while Theorem 3 illustrates a rich structure of \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) for \( 0 < p < q < \infty \) (see Section 7 for a further analysis of Theorem 3).

Under a mild additional assumption on \( \phi \), we also provide a complete description of those entire symbols \( g \) for which the integral operator \( T_g \) belongs to the Schatten \( p \)-class \( S_p(\mathcal{F}_2^\phi) \).
Theorem 4. Let $g$ be an entire function and assume $\phi \in \mathcal{I}$ satisfies
\begin{equation}
\sup_{r>r_0} \frac{-\tau''(r)\tau(r)}{(\tau(r)\phi'(r))^2} < \infty,
\end{equation}
for some $r_0 > 0$, where $\tau$ is as in (1.3).

(a) If $1 < p < \infty$ then $T_g \in S_p(\mathcal{F}^\phi_p)$ if and only if $\frac{\phi'}{1+\phi'} \in L^p(\mathbb{C}, \Delta \phi dm)$.

(b) If $0 < p \leq 1$ then $T_g \in S_p(\mathcal{F}^\phi_p)$ if and only if $g$ is constant.

It is worth to comment that (1.7) is a technical and non-restrictive condition, because for natural examples from the class $\mathcal{I}$, the corresponding function $\tau$ is convex. In addition, it is only used to prove the sufficiency for $p \geq 1$.

In Section 8 we investigate the invariant subspaces of the Volterra operator $V : \mathcal{F}^\phi_p \to \mathcal{F}^\phi_p$, given by,
$$Vf(z) = \int_0^z f(\zeta) d\zeta.$$ Notice that $V = T_g$ for $g(z) = z$. For the weights $\phi(z) = |z|^m$, $m > 2$ we obtain a complete characterization of the invariant subspaces of $V$ showing that they are precisely the spaces
$$A^\phi_N = \text{Span}\{z^k : k \geq N\} \mathcal{F}^\phi_p, \quad N \geq 0.$$ This illustrates the applicability of an abstract result stated below in Theorem 29 whose proof presents an interesting feature: the characterization of the invariant subspaces of $V$ in the case $p \neq 2$ can be reduced to the case $p = 2$ via Theorem 3. The corresponding results for the classical Fock space can be found in [11]. The fact that the weighted Fock spaces $\mathcal{F}^\phi_p$, $\phi \in \mathcal{I}$, are not nested requires a more involved approach in comparison to the classical case.

We recall that a complete description of the invariant subspaces of $V$, when acting on various classical spaces of analytic functions on the unit disc (Hardy spaces, standard Bergman spaces, Dirichlet spaces) was obtained in [4]. Furthermore, the real variable analogue has a long tradition that goes back to Gelfand and Agmon [1, 13].

Finally, in Section 9 we point out that our approach on the Fock space also brings some improvements of the results in [19] on Bergman spaces with rapidly decreasing weights. We first deduce a natural asymptotic estimate analogous to (1.2) for the corresponding distortion function on the Bergman space. This observation leads us further to eliminate a hypothesis in Theorem 2 from [19], where a characterization of the boundedness and compactness of $T_g$ is provided. By doing this we extend this characterization to a wider class of weights. In particular, we allow for a considerably faster decay, including for example weights of the form
$$\omega(z) = \exp(-e^{\frac{1}{1-|z|}}).$$ The paper is organized as follows. In Section 2 we present some preliminary results including the existence of a covering of $\mathbb{C}$ in terms of discs $D(\delta \tau(z))$. We
deal with the Littlewood-Paley formula (1.1) and some useful results concerning
the behavior of the function \( \phi \) in Section 3. We prove Theorem 1 in Section 4,
Theorem 3 in Section 5 and Theorem 4 in Section 6. Moreover, we provide some
examples of functions in the class \( I \) in Section 7. Section 8 is devoted to the
study of invariant subspaces and in Section 9 we discuss Bergman spaces with
rapidly decreasing weights.

2. Preliminaries

2.1. Some technical tools. In this section we present some facts, which are
needed to prove the main results, but which may also be of independent interest.

Let \( \tau \) be a positive function on \( \mathbb{C} \). We say that \( \tau \in \mathcal{L} \) if there exist a constant \( c_1 > 0 \) such that
\[
|\tau(z) - \tau(\zeta)| \leq c_1 |z - \zeta|, \quad \text{for } \ z, \zeta \in \mathbb{C}.
\]

Throughout this paper, we will always use the notation
\[
m_\tau = \frac{\min\left(1, c_1^{-1}\right)}{4},
\]
where \( c_1 \) is the constant in (2.1).

Lemma 5. Suppose that \( \tau \in \mathcal{L}, \ 0 < \delta \leq m_\tau \) and \( a \in \mathbb{C} \). Then,
\[
\frac{1}{2} \tau(a) \leq \tau(z) \leq 2 \tau(a) \quad \text{if } \quad z \in D(a, \delta \tau(a)).
\]

Proof. Note that, by condition (2.1) we have
\[
\tau(a) \leq \tau(z) + c_1 |z - a| \leq \tau(z) + \frac{1}{4} \tau(a) \quad \text{if } \quad |z - a| \leq \delta \tau(a).
\]
Therefore \( \tau(a) \leq 2 \tau(z) \) if \( |z - a| \leq \delta \tau(a) \). Similarly it can be proved that \( \tau(z) \leq 2 \tau(a) \). \( \square \)

We shall now sketch the proof of a covering lemma that is obtained by adapting
an approach used by Oleinik [17] for bounded domains to our setting.

Lemma 6. Assume \( t : \mathbb{C} \rightarrow (0, \infty) \) is a continuous function such that
\[
|t(z) - t(\zeta)| \leq \frac{1}{4} |z - \zeta|, \quad z, \zeta \in \mathbb{C}.
\]
and \( \lim_{|z| \rightarrow \infty} t(z) = 0 \). Then there exists a sequence of points \( \{z_j\} \in \mathbb{C} \) such that the following conditions are satisfied:

(i) \( z_j \notin D(z_k, t(z_k)) \) for \( j \neq k \);
(ii) \( \bigcup_{j \geq 1} D(z_j, t(z_j)) = \mathbb{C} \);
(iii) \( \check{D}(z_j, t(z_j)) \subset D(z_j, 3t(z_j)) \), where \( \check{D}(z_j, t(z_j)) = \bigcup_{z \in D(z_j, t(z_j))} D(z, t(z)) \);
(iv) \( \{D(z_j, 3t(z_j))\}_{j \geq 1} \) is a covering of \( \mathbb{C} \) of finite multiplicity.
**Proof.** We construct a sequence \(\{z_j\}_{j \geq 1}\) inductively as follows: pick \(z_1 \in \mathbb{C}\) such that \(t(z_1) = \max_{z \in \mathbb{C}} t(z)\). Provided \(z_1, z_2, \ldots, z_{i-1}\) are chosen, we let \(z_i\) be one of those the points in \(C_i := \mathbb{C} \setminus \left( \bigcup_{k<i} D(z_k, t(z_k)) \right)\) such that \(t(z_i) = \max_{z \in C_i} t(z)\). This way, we obtain a sequence that satisfies condition \((i)\). Conditions \((iii),(iv)\) are of local nature and therefore the proofs of \((3) - (4)\) from "Lemma of coverings" in [17, p. 233] translate verbatim.

It remains to prove \((ii)\). We claim that \(|z_n| \to \infty\) as \(n \to \infty\). Indeed, if this did not hold, \(\{z_n\}_{n \geq 1}\) would possess a convergent subsequence \(\{z_{n_k}\}_{k \geq 1}\). The construction of \(\{z_n\}_{n \geq 1}\) would then imply
\[
|z_{n_k} - z_{n_l}| \geq t(z_{n_k}) \geq m > 0, \quad k \neq l,
\]
since, by continuity, the positive function \(t\) has a positive minimum on compacts. This contradicts the convergence of \(\{z_{n_k}\}_{k \geq 1}\) and the claim is proven. Consequently, we have \(t(z_n) \to 0\) as \(n \to \infty\). Now choose an arbitrary point \(y \in \mathbb{C}\). Since \(t(y) > 0\) there exists \(n_0 \geq 1\) with \(t(y) > t(z_{n_0})\), which implies
\[
y \in \bigcup_{i < n_0} D(z_i, t(z_i)),
\]
and hence \((ii)\) holds. \(\square\)

**Lemma 7.** Let \(\phi\) be a subharmonic function and let \(\tau \in \mathcal{L}\) such that \(\tau(z)^2 \Delta \phi(z) \leq c_2\) for some constant \(c_2 > 0\) and \(z \in \mathbb{C} \setminus \mathbb{D}\). If \(\beta \in \mathbb{R}\), there exists a constant \(M \geq 1\) such that
\[
|f(a)|^p e^{-\beta \phi(a)} \leq \frac{M}{\delta^2 (\tau(a))^2} \int_{D(\delta \tau(a))}|f(z)|^p e^{-\beta \phi(z)} \, dm(z), \quad a \in \mathbb{C},
\]
for all \(0 < \delta \leq m_{\tau}\) and \(f \in H(\mathbb{C})\).

We note that if a function \(\phi\) belongs to the class \(\mathcal{I}\), then its associated function \(\tau(z)^2 \Delta \phi(z)\) belongs to the class \(\mathcal{L}\). Thus Lemma 7 proves that for functions \(\phi\) in the class \(\mathcal{I}\), the point evaluations \(L_a\) are bounded linear functionals on \(\mathcal{F}^\phi_p\). Therefore, there are reproducing kernels \(K_a \in \mathcal{F}^\phi_2\) with \(\|L_a\| = \|K_a\|_{\mathcal{F}^\phi_2}\) and such that
\[
L_a f = \langle f, K_a \rangle = \int_{\mathbb{C}} f(z) \overline{K_a(z)} e^{-2\phi(z)} \, dm(z), \quad f \in \mathcal{F}^\phi_2.
\]

Another consequence is that norm convergence implies uniform convergence on compact subsets of \(\mathbb{C}\). It follows that also the space \(\mathcal{F}^\phi_p\) is complete.

### 2.2. Test Functions.

As a key tool for the study of the boundedness and compactness we use an appropriate family of test functions constructed in [8, Proposition 8.2] (see also [19]).
Proposition A. Let $\phi \in I$ and $R \geq 100$. There exists $\eta(R)$ such that for every $a \in \mathbb{C}$ with $|a| \geq \eta(R)$, there exists an entire function $F_{a,R}$ such that

\[
|F_{a,R}(\omega)| e^{-\phi(\omega)} \asymp e^{-\frac{|\omega-a|^2}{4\tau^2(a)}} \times 1, \quad \omega \in D(a, R\tau(a)),
\]

(2.3)

\[
|F_{a,R}(\omega)| e^{-\phi(\omega)} \leq C(\phi, R) \min \left\{ 1, \left[ \frac{\min\{\tau(a), \tau(\omega)\}}{|\omega - a|} \right]^{\frac{R^2}{4}} \right\}, \quad \omega \in \mathbb{C}.
\]

(2.4)

Corollary 8. Let $\phi \in I$, $0 < p < \infty$ and $R > \max\{100, \frac{2}{\sqrt{p}}\}$, and $\eta(R)$ that from Proposition A. Then

(i) for $0 < p < \infty$ the function $F_{a,R}$ in Proposition A belongs to $\mathcal{F}^\phi_p$ with

\[
\|F_{a,R}\|^p_{\mathcal{F}^\phi_p} \asymp \tau(a)^2, \quad \eta(R) \leq |a|.
\]

(ii) the reproducing kernel $K_a$ of $\mathcal{F}^\phi_2$ satisfies the estimate

\[
\|K_a\|^2_{\mathcal{F}^\phi_2} e^{-2\phi(a)} \asymp \tau(a)^{-2}, \quad \eta(R) \leq |a|.
\]

Proof. Let $a \in \mathbb{C}$ with $\eta(R) \leq |a|$, and consider the functions $F_{a,R}$. Write

\[
R_k(a) = \left\{ \omega \in \mathbb{C} : 2^{k-1}R \tau(a) < |\omega - a| \leq 2^kR \tau(a) \right\}, \quad k = 1, 2\ldots.
\]

Note that (2.3) gives

\[
\int_{|\omega-a|<R\tau(a)} |F_{a,R}(\omega)|^p e^{-p\phi(\omega)} \, dm(\omega) \asymp \tau(a)^2,
\]

and, by (2.4) and the fact that $R > \frac{2}{\sqrt{p}}$

\[
\int_{|\omega-a|>R\tau(a)} |F_{a,R}(\omega)|^p e^{-p\phi(\omega)} \, dm(\omega) \leq \sum_{k=1}^\infty \int_{R_k(a)} |F_{a,R}(\omega)|^p e^{-p\phi(\omega)} \, dm(\omega)
\]

\[
\lesssim \tau(a)^{pR^2} \sum_{k=1}^\infty \int_{R_k(a)} \frac{dm(\omega)}{|\omega-a|^{pR^2/2}}
\]

\[
\lesssim \sum_{k=1}^\infty 2^{-pR^2/2} m(R_k(a))
\]

\[
\lesssim \tau(a)^2.
\]

Therefore $F_{a,R} \in \mathcal{F}^\phi_p$ with $\|F_{a,R}\|^p_{\mathcal{F}^\phi_p} \asymp \tau(a)^2$, which gives (i).

The use of Lemma 7 (with $\beta = 2$) gives the upper estimate of (ii),

\[
\|K_a\|^2_{\mathcal{F}^\phi_2} e^{-2\phi(a)} \lesssim \tau(a)^{-2}.
\]
On the other hand, the functions \( F_{a,R} \) obtained from the previous proposition satisfy (by \( (i) \)) that \( F_{a,R} \in \mathcal{F}_2^\phi \) with \( \| F_{a,R} \|_{\mathcal{F}_2^\phi} \asymp \tau(a)^2 \), and by (2.3) this gives
\[
|F_{a,R}(a)|^2 \asymp e^{2\phi(a)} \asymp e^{2\phi(a)} \left( \tau(a)^2 \right)^{-1} \| F_{a,R} \|^2_{\mathcal{F}_2^\phi}.
\]
Since \( \| K_a \|_{\mathcal{F}_2^\phi} = \| L_a \| \), where \( L_a \) is the point evaluation functional at the point \( a \), this proves the lower estimate of \( (ii) \).

**Proposition 9.** Let \( \phi \in \mathcal{I} \), \( 0 < p < \infty \) and \( R > \max \left\{ 100, \frac{2}{\sqrt{p}}, 2\sqrt{p} \right\} \). If \( \eta(R) \) is the number given in Proposition A and \( \{ z_k \} \subset \mathbb{C} \) is the sequence from Lemma 6, the function
\[
F(z) = \sum_{z_k: |z_k| \geq \eta(R)} a_k \frac{F_{z_k,R}(z)}{\tau(z_k)^{2/p}}
\]
belongs to \( \mathcal{F}_p^\phi \) for every sequence \( \{ a_k \} \in \ell^p \). Moreover,
\[
\| F \|_{\mathcal{F}_p^\phi} \lesssim \left( \sum_k |a_k|^p \right)^{1/p}.
\]

**Proof.**
In what follows, we shall write
\[
F(z) = \sum_{z_k: |z_k| \geq \eta(R)} a_k \frac{F_{z_k,R}(z)}{\tau(z_k)^{2/p}} = \sum_k a_k \frac{F_{z_k,R}(z)}{\tau(z_k)^{2/p}},
\]
and for simplicity we shall denote \( \gamma = \gamma(R) = \frac{R^2}{4} \).

If \( 0 < p \leq 1 \), then bearing in mind Corollary 8, we have that
\[
\| F \|^p_{\mathcal{F}_p^\phi} = \int_{\mathbb{C}} \left[ \sum_k a_k \frac{F_{z_k,R}(z)}{\tau(z_k)^{2/p}} \right]^p e^{-p\phi(z)} \, dm(z)
\]
\[
\leq \sum_k \frac{|a_k|^p}{\tau(z_k)^{2}} \| F_{z_k,R} \|^p_{\mathcal{F}_p^\phi}
\]
\[
\leq C \sum_k |a_k|^p.
\]

If \( p > 1 \), an application of Hölder’s inequality yields
\[
|F(z)|^p \leq \sum_k \frac{|a_k|^p}{\tau(z_k)^{2p}} |F_{z_k,R}(z)|^{p(\mu - p + 1)} \left( \sum_k \tau(z_k)^2 |F_{z_k,R}(z)|^{p/\gamma} \right)^{-1}.
\]

Now, we claim that
\[
\sum_k \tau(z_k)^2 |F_{z_k,R}(z)|^{p/\gamma} \lesssim \tau(z)^{2} e^{\frac{p\phi(z)}{\gamma}}.
\]

In order to prove (2.6), fix \( \delta_0 \in (0, m_\tau) \) and observe that \( t(z) = \delta_0 \tau(z) \) satisfies the hypotheses of Lemma 6. Using the estimate (2.3), Lemma 5 and \( (iv) \) of
Lemma 6, we deduce that
\[
\sum_{\{z_k \in D(z, \delta \tau(z))\}} \tau(z_k)^2 |F_{z_k, R}(z)|^{p/\gamma} \leq e^{\frac{p\phi(z)}{\gamma}} \sum_{\{z_k \in D(z, \delta \tau(z))\}} \tau(z_k)^2 \leq \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}}.
\] (2.7)

On the other hand, an application of (2.4) gives
\[
\sum_{\{z_k \notin D(z, \delta \tau(z))\}} \tau(z_k)^2 |F_{z_k, R}(z)|^{p/\gamma} \leq \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}} \sum_{j=0}^{\infty} \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{2p}} = \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}} \sum_{j=0}^{\infty} \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{2p}},
\]
where
\[
R_j(z) = \{ \zeta \in \mathbb{C} : 2^j \delta_0 \tau(z) < |\zeta - z| \leq 2^{j+1} \delta_0 \tau(z) \}, \quad j = 0, 1, 2, \ldots
\]

By (2.1), we deduce that, for \( j = 0, 1, 2, \ldots \),
\[
D(z_k, \delta_0 \tau(z_k)) \subset D(z, 5\delta_0 2^j \tau(z)) \quad \text{if} \quad z_k \in D(z, 2^{j+1} \delta_0 \tau(z)).
\]
This fact together with the finite multiplicity of the covering (see Lemma 6) gives
\[
\sum_{z_k \in R_j(z)} \tau(z_k)^2 \leq m(D(z, 5\delta_0 2^j \tau(z))) \leq 2^{2j} \tau(z)^2.
\]
Therefore
\[
\sum_{\{z_k \notin D(z, \delta \tau(z))\}} \tau(z_k)^2 |F_{z_k, R}(z)|^{p/\gamma} \leq \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}} \sum_{j=0}^{\infty} \sum_{z_k \in R_j(z)} \frac{\tau(z_k)^2}{|z - z_k|^{2p}} \leq e^{\frac{p\phi(z)}{\gamma}} \sum_{j=0}^{\infty} 2^{-2pj} \sum_{z_k \in R_j(z)} \tau(z_k)^2 \leq \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}} \sum_{j=0}^{\infty} 2^{(2-2p)j} \leq \tau(z)^2 e^{\frac{p\phi(z)}{\gamma}},
\]
which together with (2.7), proves (2.6).

Now, joining (2.5) and (2.6), we obtain
\[
||F||^p_{F^p_{\phi}} \leq \sum_{k} \frac{|a_k|^p}{\tau(z_k)^{2p}} \int_{\mathbb{C}} |F_{z_k, R}(z)|^{\frac{p(1 - p + 1)}{2p - 2}} \tau(z)^{2p-2} e^{-p\phi(z)\left(\frac{n+1}{p\gamma}\right)} dm(z).
\]
we set

\[ \frac{p(\gamma - p + 1)}{\gamma} \tau(z)^{2p - 2} e^{-p\phi(z)\frac{\gamma - p + 1}{\gamma}} \] 

dm(z) \lesssim \tau(z_k)^{2p}

to obtain the desired estimate

\[ \|F\|_{L_p^\phi}^p \leq \sum_k |a_k|^p. \]

It follows from (2.3) and (2.1) that

\[ \int_{|z - z_k| < \tau(z_k)} |F_{z_k, R}(z)| \frac{p(\gamma - p + 1)}{\gamma} \tau(z)^{2p - 2} e^{-p\phi(z)\frac{\gamma - p + 1}{\gamma}} \] 

dm(z) \times \int_{|z - z_k| < \tau(z_k)} \tau(z)^{2p - 2} dm(z) \times \tau(z_k)^{2p}.

On the other hand, using (2.1), it follows that

\[ \tau(z) \leq C 2^j \tau(z_k) \quad \text{if} \quad |z - z_k| < 2^j \tau(z_k). \]

Thus, since \( \gamma > p \), bearing in mind (2.4), we deduce that

\[ \int_{|z - z_k| \geq \tau(z_k)} \frac{p(\gamma - p + 1)}{\gamma} \tau(z)^{2p - 2} e^{-p\phi(z)\frac{\gamma - p + 1}{\gamma}} dm(z) \]

\[ \lesssim \tau(z_k)^{2p(\gamma - p + 1)} \sum_{j=0}^{\infty} \int_{2^j \tau(z_k) \leq |z - z_k| < 2^{j+1} \tau(z_k)} \frac{\tau(z)^{2p - 2}}{|z - z_k|^{2p(\gamma - p + 1)}} dm(z) \]

\[ \lesssim \tau(z_k)^{2p} \sum_{j=0}^{\infty} 2^{-2jp(\gamma - p)} \lesssim \tau(z_k)^{2p}, \]

which together with (2.9) gives (2.8). This finishes the proof. \( \square \)

3. A Littlewood-Paley formula

Our aim will be to obtain a Littlewood-Paley formula for a large class of weighted Fock spaces \( \mathcal{F}_p^\phi \).

Let us introduce some useful notation. For an entire function \( f \) and \( 0 \leq r < \infty \), we set

\[ M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty, \]

\[ I_p(r, f) = M_p(r, f), \quad 0 < p < \infty, \]

\[ M_\infty(r, f) = \max_{|z|=r} |f(z)|. \]
Inspired by considerations in [25] and [21], for $\phi : [0, \infty) \to \mathbb{R}^+$ twice continuously differentiable function and $0 < p < \infty$, such that $\int_0^\infty s e^{-p\phi(s)} ds < \infty$, we define

$$
\psi_p(r) = \psi_{p,\phi}(r) = \frac{\int_r^\infty s e^{-p\phi(s)} ds}{(1 + r)e^{-p\phi(r)}} \quad 0 \leq r < \infty.
$$

The function $\psi_{p,\phi}$ will be called the $p$-distortion function of $\phi$.

We consider the equivalence of the following conditions,

$$
\int_0^\infty M^p_q(r,f) r e^{-p\phi(r)} dr < \infty,
$$

$$
\int_0^\infty M^p_q(r,f')(\psi_{p,\phi}(r))^p r e^{-p\phi(r)} dr < \infty,
$$

for functions $f \in H(\mathbb{C})$, where $0 < p < \infty$ and $0 < q \leq \infty$. The next theorem asserts that this equivalence holds if $\phi$ fulfills the $K_p$-condition below, which is a rather weak assumption that is satisfied, for instance, whenever $\lim_{r \to \infty} r \phi'(r) = +\infty$.

**Condition $K_p$:** The function $\phi$ is differentiable and there is a constant $K = K(p,\phi) \in \mathbb{R}$ such that

$$
\frac{d}{dr} \left( r e^{-p\phi(r)} \right) \int_r^\infty s e^{-p\phi(s)} ds \leq K, \quad 1 \leq r < \infty.
$$

**Theorem 10.** Let $0 < p < \infty$, $0 < q \leq \infty$. If $\phi$ is a function satisfying condition $K_p$, then

$$
\int_0^\infty M^p_q(r,f) r e^{-p\phi(r)} dr \asymp |f(0)|^p + \int_0^\infty M^p_q(r,f')(\psi_{p,\phi}(r))^p r e^{-p\phi(r)} dr
$$

for any entire function $f$.

In particular, we obtain the following.

**Corollary 11.** Assume that $0 < p < \infty$ and $\phi$ is a function satisfying the $K_p$-condition, then

$$
||f||_{F^p_\phi} \asymp |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p (\psi_{p,\phi}(z))^p e^{-p\phi(z)} dm(z)
$$

for any entire function $f$.

**3.1. Proof of Theorem 10.** We proceed in three steps: first we reformulate Theorem 10 as Theorem 12, then we present some preliminary material, and, finally, we prove Theorem 12.

**3.1.1. Reformulation.** First, we note that for any fixed $R_0 \in (0, \infty)$

$$
\int_0^\infty M^p_q(r,f) r e^{-p\phi(r)} dr \asymp \int_{R_0}^\infty M^p_q(r,f) r e^{-p\phi(r)} dr
$$

(3.2)

$$
\int_0^\infty M^p_q(r,f')(\psi_{p,\phi}(r))^p r e^{-p\phi(r)} dr \asymp \int_{R_0}^\infty M^p_q(r,f')(\tilde{\psi}_{p,\phi}(r))^p r e^{-p\phi(r)} dr
$$
where \( \tilde{w}_{p,\phi}(r) = \frac{\int_{r}^{\infty} se^{-p\phi(s)} \, ds}{re^{-p\phi(r)}} \) and the constants involved in (3.2) depend on \( \phi, p \) and \( R_0 \).

Given a function \( \phi \), and \( 0 < p < \infty \), we define the function \( \varphi \) by
\[
\varphi(r) = p \int_{r}^{\infty} se^{-p\phi(s)} \, ds.
\]

Since \( \phi \) is continuous
\[
\varphi'(r) - p^{-1} \varphi''(r) = re^{-p\phi(r)},
\]
and we define the measure \( dm_\varphi \) on \([0, \infty)\) by
\[
dm_\varphi(r) = \frac{\varphi'(r)}{\varphi(r)} \, dr.
\]

It is not difficult to see that condition \( K_p \) is equivalent to
\[
\sup_{1 \leq r < \infty} \frac{\varphi''(r) \varphi(r)}{\varphi'(r)^2} \leq M,
\]
where \( M \in \mathbb{R} \) is an appropriate constant.

Consequently, bearing in mind (3.2), we can reformulate Theorem 10 as follows.

**Theorem 12.** Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Let \( \varphi : [1, \infty) \to \mathbb{R} \) be a differentiable, positive and increasing function such that \( \lim_{r \to \infty} \varphi(r) = \infty \). For each entire function \( f \), we define \( F_1(r) := \frac{M_q(r,f)}{\varphi(r)} \), \( F_2(r) := \frac{M_q(r,f')}{\varphi(r)} \). If \( \varphi \) satisfies (3.4), then
\[
||F_1||_{L^p(dm_\varphi)} \asymp ||f(0)||^p + ||F_2||_{L^p(dm_\varphi)}
\]
for each \( f \in H(\mathbb{C}) \).

### 3.1.2. Preliminary results

We shall need some lemmas.

**Lemma 13.** If \( f \in H(\mathbb{C}) \), \( 0 < q \leq \infty \), then there is a constant \( C_q \) such that
\[
M_q(r,f') \leq C_q (\rho - r)^{-1} M_q(\rho, f), \quad 0 \leq r < \rho < \infty.
\]

**Proof.**

Let \( r \) and \( \rho \) be as in the statement. Fix \( R > \rho \) and define the analytic function in the unit disc \( \mathbb{D} \),
\[
f_R(z) = f(Rz), \quad z \in \mathbb{D}.
\]

By [21, Lemma 3.1], there is \( C_q > 0 \) such that
\[
M_q(s, (f_R)') \leq C_q \frac{M_q(t, f_R)}{t - s}, \quad \text{for any } 0 < s < t < 1.
\]

Choosing \( s = \frac{t}{R} \) and \( t = \frac{\rho}{R} \), the proof is finished. \( \square \)

The next result can be proved following the lines of [16, Lemma 2].

**Lemma 14.** If \( f \in H(\mathbb{C}) \), \( 0 < q \leq \infty \) and \( s = \min(q, 1) \), then there is a constant \( C_q \) such that
\[
M_q^s(\rho, f) - M_q^s(r, f) \leq C_q (\rho - r)^s M_q^s(\rho, f'), \quad 0 < r < \rho < \infty.
\]
The following lemma can be obtained by standard techniques (see [21, Lemma 3.4]), so its proof will be omitted.

**Lemma 15.** Let \( \{A_n\}_{n=-1}^\infty \) be a sequence of complex numbers, \( 0 < \gamma < \infty \), \( \alpha > 0 \). Set

\[
Q_1 = \sum_{n=-1}^\infty e^{-n\alpha}|A_n|^{\gamma},
\]

\[
Q_2 = |A-1|^{\gamma} + \sum_{n=-1}^\infty e^{-n\alpha}|A_{n+1} - A_n|^{\gamma}.
\]

Then the quantities \( Q_1 \) and \( Q_2 \) are equivalent in the sense that there is a positive constant \( C \) independent of \( \{A_n\}_{n=-1}^\infty \) such that \( (1/C)Q_1 \leq Q_2 \leq CQ_1 \).

The next lemma, which will be a key tool for the proofs of our main results, is essentially proved in [21, Lemma 3.5].

**Lemma 16.** Let \( \varphi : [1, \infty) \to \mathbb{R} \) be a differentiable, positive and increasing function such that \( \lim_{r \to \infty} \varphi(r) = \infty \) and \( \varphi(1) = 1 \). Define the sequence \( \{r_n\}_{n=0}^\infty \) by

\[
\varphi(r_n) = e^n, \quad n \geq 0.
\]

If \( \varphi \) satisfies \( (3.4) \), for every \( n \geq 0 \),

\[
\frac{\varphi'(y)}{\varphi'(x)} \leq e^{2M}, \quad r_n < x < y < r_{n+2}.
\]

**3.1.3. Proof of Theorem 12.** Let \( \{r_n\}_{n=0}^\infty \) be the sequence defined by (3.7). We may assume without loss of generality that \( \varphi(1) = 1 \).

First, we assume that \( F_1 \in L^p(dm_\varphi) \). Then,

\[
\infty > \|F_1\|_{L^p(dm_\varphi)}^p = \int_1^\infty M_q^p(r, f)\varphi(r)^{-p-1}\varphi'(r) \, dr
\]

\[
\geq \sum_{n=0}^\infty M_q^p(r_n, f) \int_{r_n}^{r_{n+1}} \varphi(r)^{-p-1}\varphi'(r) \, dr
\]

\[
= \sum_{n=0}^\infty M_q^p(r_n, f) \frac{\varphi(r_n)^{-p} - \varphi(r_{n+1})^{-p}}{p}
\]

\[
= C_p \sum_{n=0}^\infty M_q^p(r_n, f)e^{-np}.
\]

On the other hand,

\[
\|F_2\|_{L^p(dm_\varphi)}^p = \int_1^\infty M_q^p(r, f')\varphi'(r)^{-p}\varphi(r)^{-1} \, dr
\]

\[
\leq \sum_{n=0}^\infty M_q^p(r_{n+1}, f') \int_{r_n}^{r_{n+1}} \varphi(r)^{-p}\varphi(r)^{-1} \, dr
\]

\[
= \sum_{n=0}^\infty M_q^p(r_{n+1}, f')\varphi'(x_n)^{-p},
\]
where \( r_n < x_n < r_{n+1} \). Here we have used the formula \( \int_{r_n}^{r_{n+1}} \varphi'(r)\varphi(r)^{-1} \, dr = 1 \).

Now, taking into account (3.6) we obtain

\[
\|(r_{n+2}, f)(r_{n+2} - r_{n+1})^{-p}\varphi'(x_n)^{-p}.
\]

On the other hand, by Lagrange’s theorem,

\[
\varphi(r_{n+2}) - \varphi(r_{n+1}) = (r_{n+2} - r_{n+1})\varphi'(y_n), \quad \text{where } r_{n+1} < y_n < r_{n+2},
\]

whence

\[
r_{n+2} - r_{n+1} = (1 - e^{-1})e^{n+2} (\varphi'(y_n))^{-1}.
\]

Combining this with (3.9) and Lemma 16, we get

\[
|f(0)|^p + \|F_2||_{L^p(d\mu_r)}^p \lesssim |f(0)|^p + \sum_{n=0}^{\infty} \left|\frac{M_q^p(r_{n+2}, f)e^{-(n+2)p} \left(\varphi'(y_n)\right)^p}{\varphi'(x_n)}\right|
\]

\[
\lesssim |f(0)|^p + e^{2Mp} \sum_{n=0}^{\infty} M_q^p(r_{n+2}, f)e^{-(n+2)p} \lesssim \sum_{n=0}^{\infty} M_q^p(r_n, f)e^{-np},
\]

which together with (3.8) gives the inequality \(|f(0)|^p + \|F_2||_{L^p(d\mu_r)}^p \leq C\|F_1||_{L^p(d\mu_r)}^p\) in (3.5).

Now it will be proved the reverse inequality in (3.5). Assume that \( F_2 \in L^p(d\mu_r) \). We shall consider the case \( q < 1 \), the proof for \( q \geq 1 \) is similar.

Let \( q < 1 \) and \( \gamma = p/q \). Arguing as in (3.8) and choosing \( r_{-1} = 0 \), we get

\[
\|F_1||_{L^p(d\mu_r)}^p \lesssim C\sum_{n=0}^{\infty} M_q^p(r_n, f)e^{-np} = C\sum_{n=0}^{\infty} A_n e^{-np},
\]

where \( A_n = M_q^q(r_n, f) \). This together with Lemma 15 implies that

\[
\|F_1||_{L^p(d\mu_r)}^p \lesssim C|f(0)|^p + C\sum_{n=0}^{\infty} \left(M_q^p(r_{n+1}, f) - M_q^q(r_n, f)\right)^{p/q}e^{-np}.
\]

Hence, by Lemma 14, we have that

\[
(3.11) \|F_1||_{L^p(d\mu_r)}^p \leq C|f(0)|^p + C\sum_{n=0}^{\infty} M_q^p(r_{n+1}, f') (r_{n+1} - r_n)^p e^{-np},
\]

Now, we use Lagrange’s theorem, as in (3.10), to obtain

\[
\|F_1||_{L^p(d\mu_r)}^p \lesssim |f(0)|^p + \sum_{n=-1}^{\infty} M_q^p(r_{n+1}, f') \varphi'(x_n)^{-p},
\]

\[
\lesssim |f(0)|^p + |f'(1)|^p + \sum_{n=0}^{\infty} M_q^p(r_{n+1}, f') \varphi'(x_n)^{-p} \quad \text{where } r_n < x_n < r_{n+1}.
\]
On the other hand,
\[
\infty > \|F_2\|_{L^p(d\varphi)}^p = \int_1^\infty M^p_q(r, f') \varphi'(r)^{1-p} \varphi(r)^{-1} dr \\
\geq \sum_{n=0}^\infty M^p_q(r_{n+1}, f') \int_{r_{n+1}}^{r_{n+2}} \varphi'(r)^{1-p} \varphi(r)^{-1} dr \\
= \sum_{n=0}^\infty M^p_q(r_{n+1}, f') \varphi'(y_n)^{-p}, \quad \text{where } r_{n+1} < y_n < r_{n+2}.
\]

Finally, using subharmonicity, (3.11), Lemma 16 and the above inequality we deduce
\[
\|F_1\|_{L^p(d\varphi)}^p \leq C |f(0)|^p + |f'(1)|^p + C \sum_{n=0}^\infty M^p_q(r_{n+1}, f') \varphi'(x_n)^{-p} \\
\leq C |f(0)|^p + |f'(1)|^p + e^{2MpC} \sum_{n=0}^\infty M^p_q(r_{n+1}, f') \varphi'(y_n)^{-p} \\
\leq C \left( |f(0)|^p + \|F_2\|_{L^p(d\varphi)}^p \right).
\]

The proof is complete.

3.2. The distortion function. We now consider a class of weights for which the statement of Corollary 11 becomes more transparent.

**Lemma 17.** Assume \( \phi : [0, \infty) \to \mathbb{R}^+ \) is twice continuously differentiable and there exists \( r_0 > 0 \) such that \( \phi'(r) \neq 0 \) for \( r > r_0 \). Let \( p > 0 \) and suppose
\[
\lim_{r \to \infty} r e^{-p\phi(r)} = 0 \\
\limsup_{r \to \infty} \frac{1}{r} \left( \frac{r}{\phi'(r)} \right)' < p \\
\liminf_{r \to \infty} \frac{1}{r} \left( \frac{r}{\phi'(r)} \right)' > -\infty.
\]
(3.12)

Then \( \phi \) satisfies the condition \( K_p \) and there exists \( r_1 > 0 \) such that
\[
\psi_p(r) \asymp \frac{1}{\phi'(r)} \text{ for } r \geq r_1,
\]
where the involved constants might depend on \( p > 0 \).

**Proof.** By hypothesis there is \( \alpha < 1 \) and \( r_2 \geq r_0 \) such that
\[
\left( \frac{r}{p\phi'(r)} \right)' \leq \alpha r \text{ on } [r_2, +\infty).
\]
(3.13)
So an integration by parts on $(r_2, r]$ gives
\[
\int_{r_2}^{r} se^{-p\phi(s)} \, ds = \int_{r_2}^{r} -p\phi'(s)e^{-p\phi(s)} \left( \frac{-s}{p\phi'(s)} \right) \, ds
\]
\[
= \frac{-re^{-p\phi(r)}}{p\phi'(r)} + \frac{r_2 e^{-p\phi(r_2)}}{p\phi'(r_2)} + \int_{r_2}^{r} \left( \frac{s}{p\phi'(s)} \right)' e^{-p\phi(s)} \, ds
\]
\[
\leq \frac{-re^{-p\phi(r)}}{p\phi'(r)} + \frac{r_2 e^{-p\phi(r_2)}}{p\phi'(r_2)} + \alpha \int_{r_2}^{r} se^{-p\phi(s)} \, ds,
\]
that is
\[
\int_{r_2}^{r} se^{-p\phi(s)} \, ds \leq \frac{1}{1-\alpha} \left( \frac{-re^{-p\phi(r)}}{p\phi'(r)} + \frac{r_2 e^{-p\phi(r_2)}}{p\phi'(r_2)} \right),
\]
so taking the limit as $r \to \infty$ and bearing in mind (3.12), we can assert that there is $C_1 = C_1(p, \phi) > 0$ such that for any $r \geq r_2$
\[
(3.14) \quad \int_{r}^{\infty} se^{-p\phi(s)} \, ds \leq C_1 \frac{re^{-p\phi(r)}}{\phi'(r)}.
\]
In particular, $\int_{0}^{\infty} se^{-p\phi(s)} \, ds < \infty$. We note that $\phi$ satisfies the $K_p$ condition if
\[
(3.15) \quad \limsup_{r \to \infty} \left( \frac{\int_{r}^{\infty} se^{-p\phi(s)} \, ds}{r^2e^{-p\phi(r)}} - \frac{p\phi'(r) \int_{r}^{\infty} se^{-p\phi(s)} \, ds}{re^{-p\phi(r)}} \right) \leq K < \infty.
\]
Now, by (3.13)
\[
\frac{r}{\phi'(r)} = \int_{r_2}^{r} \left( \frac{s}{p\phi'(s)} \right)' \, ds + \frac{r_2}{\phi'(r_2)} \leq \frac{pr_2^2}{2} + \frac{r_2}{r^2\phi'(r_2)},
\]
which together with (3.14) implies that
\[
\frac{\int_{r}^{\infty} se^{-p\phi(s)} \, ds}{r^2e^{-p\phi(r)}} \leq \frac{1}{\phi'(r)} \leq \frac{p}{2} + \frac{r_2}{r^2\phi'(r_2)} < \infty,
\]
so the first addend in (3.15) is bounded. On the other hand, a straight-forward application of L'Hospital’s rule gives
\[
\liminf_{r \to \infty} \frac{p \int_{r}^{\infty} se^{-p\phi(s)} \, ds}{re^{-p\phi(r)} \phi'(r)} \geq \liminf_{r \to \infty} \frac{1}{1 - \frac{1}{p\phi'(r)'}} > -\infty,
\]
consequently (3.15) holds. Finally, another application of L'Hospital’s rule gives
\[
\liminf_{r \to \infty} \frac{1}{p - \frac{r}{\phi'(r)}} \leq \liminf_{r \to \infty} \frac{\psi_p(r)}{(\phi'(r))^{-1}} \leq \limsup_{r \to \infty} \frac{\psi_p(r)}{(\phi'(r))^{-1}} \leq \limsup_{r \to \infty} \frac{1}{p - \frac{r}{\phi'(r)}},
\]
which proves the lemma.

**Lemma 18.** Assume $\phi : [0, \infty) \to \mathbb{R}^+$ is a twice continuously differentiable function such that $\Delta \phi > 0, (\Delta \phi(z))^{-1/2} \asymp \tau(z)$, where $\tau(z)$ is a radial positive.
function that decreases to zero as $|z| \to \infty$ and $\lim_{r \to \infty} \tau'(r) = 0$. Then

(a) $\lim_{r \to \infty} \frac{\phi'(r)}{r} = \infty$.

(b) $\lim_{r \to \infty} \tau(r)\phi'(r) = \infty$, or, equivalently, $\lim_{r \to \infty} \frac{\phi''(r)}{\phi'(r)^2} = 0$.

(c) For any $p > 0$ we have $\psi_p(r) \asymp \frac{1}{\phi(r) + 1}$, for $r \geq 0$.

(d) $\liminf_{r \to \infty} \frac{r\Delta \phi(r)}{\phi'(r)} \geq C > 0$.

Proof. A simple calculation shows that

$$\Delta \phi(r) = \phi''(r) + \frac{1}{r} \phi'(r) = \frac{(r\phi'(r))'}{r}.$$  \hfill (3.16)

By L’Hospital’s rule we obtain

$$\lim_{r \to \infty} \frac{r\phi'(r)}{r^2} = \lim_{r \to \infty} \frac{(r\phi'(r))'}{2r} = \lim_{r \to \infty} \frac{\Delta \phi(r)}{2} = \infty,$$

which proves (a).

Let us now prove (b). Taking into account (3.16) we obtain

$$\tau(r)\phi'(r) = \frac{\tau(r)}{r} \int_0^r s \Delta \phi(s) \, ds \geq \frac{\tau(r)}{r} \int_0^r \frac{s}{\tau^2(s)} \, ds.$$

Again by L’Hospital’s rule we get

$$\lim_{r \to \infty} \frac{r\tau^{-1}(r)}{\int_0^r \frac{s}{\tau^2(s)} \, ds} = \lim_{r \to \infty} \frac{\tau(r) - r\tau'(r)}{r} = 0,$$

which implies $\lim_{r \to \infty} \tau(r)\phi'(r) = \infty$. By (a) and relation (3.16) this last fact is equivalent to $\lim_{r \to \infty} \frac{\phi''(r)}{(\phi'(r))^2} = 0$.

Taking into account (a) – (b) it is straight-forward to check that the hypotheses in Lemma 17 are satisfied, indeed

$$\lim_{r \to \infty} \frac{1}{r} \left( \frac{r}{\phi'(r)} \right) = 0,$$

and hence $\psi_p(r) \asymp \frac{1}{\phi(r)}$ for $r \geq r_0$. Since $\phi' \geq 0$ and $\lim_{r \to \infty} \phi'(r) = \infty$, we obtain (c).

We now turn to (d). By (3.16)

$$\frac{r\Delta \phi(r)}{\phi'(r)} \asymp \frac{r\tau^{-2}(r)}{\phi'(r)} \asymp \frac{\tau^2 \tau^{-2}(r)}{\int_0^r \frac{s}{\tau^2(s)} \, ds},$$

so L’Hospital’s rule implies

$$\liminf_{r \to \infty} \frac{r^2 \tau^{-2}(r)}{\int_0^r \frac{s}{\tau^2(s)} \, ds} \geq \liminf_{r \to \infty} 2 \left( 1 - \frac{r \tau'(r)}{\tau(r)} \right) \geq 2,$$

and we are done. \hfill \Box

The previous considerations lead us to the following Littlewood-Paley formula.
Theorem 19. Assume that $0 < p < \infty$ and $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ is twice continuously differentiable satisfying the hypotheses of Lemma 17. Then we have

$$\|f\|_{F_p^\phi}^p \approx \|f(0)^p + \int_{\mathbb{C}} |f'(z)|^p (\psi_{p,\phi}(z))^p e^{-p\phi(z)} dm(z),$$

for any entire function $f$, where the distortion function $\psi_{p,\phi}$ satisfies

$$\psi_{p,\phi}(z) \approx \frac{1}{\phi'(z)} \quad \text{for } |z| \geq r_0,$$

for some $r_0 > 0$. In particular, if $\phi \in \mathcal{I}$, the following holds

$$\|f\|_{F_p^\phi}^p \approx \|f(0)^p + \int_{\mathbb{C}} |f'(z)|^p \frac{e^{-p\phi(z)}}{(1 + \phi'(z))^p} dm(z),$$

for any entire function $f$.

It is worth to point out that the class of functions satisfying the hypotheses of Lemma 17 is quite large, containing functions whose growth ranges from logarithmic (e.g. $\phi(r) = a \log(1 + r)$, $ap > 2$) to highly exponential (e.g. $\phi(r) = e^{e^r}$).

The following result which is deduced from Lemma 5 shows that the distortion function is ”almost constant” on sufficiently small discs whose radii depend on $\Delta \phi$.

Lemma 20. Assume that $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ is a twice continuously differentiable function such that $\Delta \phi > 0$, $(\Delta \phi(z))^{-1/2} \approx \tau(z)$, where $\tau(z)$ is a radial positive function that decreases to zero as $|z| \rightarrow \infty$ and $\lim_{r \rightarrow \infty} \tau'(r) = 0$. Then, there exists $r_0 > 0$ such that

$$\phi'(a) \times \phi'(z), \quad z \in D(a, \delta \tau(a)),$$

(3.17)

for all $a \in \mathbb{C}$ with $|a| > r_0$. Moreover,

$$1 + \phi'(a) \times 1 + \phi'(z), \quad z \in D(a, \delta \tau(a)),$$

for all $a \in \mathbb{C}$.

Proof. Since $\phi$ is radial, it is enough to show that there exists $r_0 > 0$ such that for $a \geq r_0$

$$\phi'(a) \times \phi'(r), \quad r \in (a - \delta \tau(a), a + \delta \tau(a)).$$

Recall that

$$\phi'(r) = \frac{1}{r} \int_0^r s \Delta \phi(s) ds, \quad r > 0.$$

Our assumptions on $\tau$ imply

$$a \times r, \quad r \in (a - \delta \tau(a), a + \delta \tau(a)),$$

for $a > 2\delta \max_{s \in [0, \infty)} \tau(s)$. Hence, proving (3.18) reduces to showing that

$$\int_0^r s \Delta \phi(s) ds \approx \int_0^a s \Delta \phi(s) ds, \quad r \in (a - \delta \tau(a), a + \delta \tau(a)).$$
We have

\[ \frac{\int_0^{a-\delta\tau(a)} \frac{s}{\tau^2(s)} \, ds}{\int_0^a \frac{s}{\tau^2(s)} \, ds} \lesssim \frac{\int_0^r s \Delta \phi(s) \, ds}{\int_0^a s \Delta \phi(s) \, ds} \lesssim \frac{\int_0^{a+\delta\tau(a)} \frac{s}{\tau^2(s)} \, ds}{\int_0^a \frac{s}{\tau^2(s)} \, ds} \]

(3.19)

By L’Hospital’s rule and Lemma 5 we deduce

\[
\limsup_{a \to \infty} \frac{\int_0^{a+\delta\tau(a)} \frac{s}{\tau^2(s)} \, ds}{\int_0^a \frac{s}{\tau^2(s)} \, ds} \lesssim \limsup_{a \to \infty} \frac{(1 + \delta\tau’(a))(a + \delta\tau(a))\tau^{-2}(a + \delta\tau(a))}{a\tau^{-2}(a)}
\sim \limsup_{a \to \infty} \frac{(1 + \delta\tau’(a))(a + \delta\tau(a))}{a} = 1.
\]

Analogously one can show that

\[
\liminf_{a \to \infty} \frac{\int_0^{a-\delta\tau(a)} \frac{s}{\tau^2(s)} \, ds}{\int_0^a \frac{s}{\tau^2(s)} \, ds} \geq c > 0.
\]

and (3.17) follows in view of (3.19). Since \( \lim_{r \to \infty} \phi'(r) = \infty \) last assertion is now straight-forward.

\[ \square \]

4. Proof of Theorem 1

At the end of this section we provide a proof of Corollary 2.

Fix \( R > \max \left\{ 100, \frac{2}{\sqrt{p}}, 2\sqrt{p} \right\} \), \( \delta \in (0, m_r) \) and consider the covering \( \{D(\delta \tau(z_j))\} \) given by Lemma 6 for \( t(z) = \delta \tau(z) \).

4.1. Proof of (I): boundedness. Suppose first that \( I_d : \mathcal{F}_p^\phi \to L^q(\mu) \) is bounded. For \( a \in \mathbb{C} \) with \( |a| \geq \eta(R) \), consider the function \( F_{a,R} \) obtained in Proposition A. By Corollary 8, we have \( \|F_{a,R}\|_{\mathcal{F}_p^\phi} \approx \tau(a)^2 \). Then, using (2.3) we get

\[
\int_{D(\delta \tau(a))} e^{\phi(z)} \, d\mu(z) \lesssim \int_{D(\delta \tau(a))} |F_{a, R}(z)|^q \, d\mu(z) 
\leq \int_C |F_{a, R}(z)|^q \, d\mu(z) \lesssim \|I_d\|_{\mathcal{F}_p^\phi \to L^q(\mu)} \tau(a)^{\frac{2q}{p}},
\]

which implies that \( K_{\mu, \phi} \leq C\|I_d\|_{A^p(w) \to L^q(\mu)} \).
Conversely, suppose that (1.4) holds. The idea of the proof goes back to [17].

Bearing in mind Lemma 6, Lemma 7 and Lemma 5, it follows that

\[ \int_C |f(z)|^q d\mu(z) \leq \sum_j \int_{D(\delta r(z_j))} |f(z)|^q d\mu(z) \]

\[ \leq \sum_j \int_{D(\delta r(z_j))} \left( \frac{1}{\tau(z)}^2 \int_{D(\delta r(z))} |f(\zeta)|^p e^{-p \phi(\zeta)} d\mu(\zeta) \right)^{\frac{q}{p}} e^{q \phi(z)} d\mu(z) \]

(4.1)

\[ \leq \sum_j \left( \int_{D(3\delta r(z_j))} |f(\zeta)|^p e^{-p \phi(\zeta)} d\mu(\zeta) \right)^{\frac{q}{p}} \int_{D(\delta r(z_j))} \frac{e^{q \phi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu(z) \]

\[ \leq K_{\mu,\phi} \sum_j \left( \int_{D(3\delta r(z_j))} |f(\zeta)|^p e^{-p \phi(\zeta)} d\mu(\zeta) \right)^{\frac{q}{p}} \]

Now, using Minkowski inequality and the finite multiplicity \( N \) of the covering \( \{D(3\delta r(z_j))\} \) (see Lemma 6), we have

\[ \int_C |f(z)|^q d\mu(z) \leq K_{\mu,\phi} \left( \sum_j \int_{D(3\delta r(z_j))} |f(\zeta)|^p e^{-p \phi(\zeta)} d\mu(\zeta) \right)^{\frac{q}{p}} \]

\[ \leq K_{\mu,\phi} N^{q/p} \|f\|_{F_p^q}^q, \]

proving that \( I_d : F_p^q \rightarrow L^q(\mu) \) is continuous with \( \|I_d\|_{F_p^q \rightarrow L^q(\mu)} \lesssim K_{\mu,\phi}. \)

4.2. Proof of (I): compactness. Suppose that (1.5) holds and let \( \{f_n\} \) be a bounded sequence in \( F_p^q \). By Lemma 7, Montel’s theorem and Fatou’s lemma, we may extract a subsequence \( \{f_{n_k}\} \) converging uniformly on compact sets of \( C \) to some function \( f \in F_p^q \). Given \( \epsilon > 0 \), fix \( r_0 \in (0, \infty) \) with

\[ \sup_{|a|>r_0} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta r(a))} e^{q \phi(z)} d\mu(z) < \epsilon. \]

(4.2)

Observe that there is \( r_0' \in (0, \infty) \) with \( r_0 \leq r_0' \) such that if a point \( z_k \) of the sequence \( \{z_j\} \) belongs to \( \{|z| \leq r_0\} \), then \( D(\delta r(z_k)) \subset \{|z| \leq r_0'\} \). So, take \( n_k \) big enough such that \( \sup_{|z| \leq r_0'} |f_{n_k}(z) - f(z)| < \epsilon \). Then, setting \( g_{n_k} = f_{n_k} - f \), and arguing as in (4.1), it follows that

\[ \|g_{n_k}\|_{L^q(\mu)}^q \leq \int_{|z| \leq r_0'} |g_{n_k}(z)|^q d\mu(z) + \sum_{|z| > r_0} \int_{D(\delta r(z_j))} |g_{n_k}(z)|^q d\mu(z) \]

\[ \leq C\epsilon + C\|g_{n_k}\|_{F_p^q}^q \sup_{|z| > r_0} \frac{1}{\tau(z)^{2q/p}} \int_{D(\delta r(z_j))} e^{q \phi(z)} d\mu(z) < C\epsilon. \]

This proves that \( I_d : F_p^q \rightarrow L^q(\mu) \) is compact.
Conversely, suppose that $I_d : \mathcal{F}_p^\phi \to L^q(\mu)$ is compact. Take $f_{a,R}(z) = \frac{F_{a,R}(z)}{\tau(a)^{2/p}}$, $\eta(R) \leq |a|$, where $\eta(R)$ and $F_{a,R}$ are obtained from Proposition A. By Corollary 8,

$$\sup_{|a| \geq \eta(R)} \|f_{a,R}\|_{F_\phi} \leq C < \infty$$

which together with the compactness of the identity operator implies that \{f_{a,R} : \eta(R) \leq |a|\} is a compact set in $L^q(\mu)$. Thus

$$\lim_{r \to \infty} \int_{|z| < r} |f_{a,R}(z)|^q \, d\mu(z) = 0 \text{ uniformly in } a. \tag{4.3}$$

On the other hand, if $\gamma = \frac{R^2}{3}$ the estimate (2.4) gives

$$|f_{a,R}(z)|^p e^{-p\phi(z)} \lesssim \frac{\tau(a)^{2p/2}}{r^{2p}}, \quad |z| \leq r, \quad |a| \geq 2r.$$  

Thus $f_{a,R} \to 0$ as $|a| \to \infty$ uniformly on compact subsets of $\mathbb{C}$, which together with (4.3) implies that $\lim_{|a| \to \infty} \|f_{a,R}\|_{L^q(\mu)} = 0$. Therefore, using the estimate (2.3) of Proposition A we obtain

$$\tau(a)^{-2q/p} \int_{D(\delta \tau(a))} e^{q\phi(z)} \, d\mu(z) \lesssim \int_{D(\delta \tau(a))} |f_{a,R}(z)|^q \, d\mu(z) \leq \|f_{a,R}\|_{L^q(\mu)}^q.$$  

Now let $|a| \to \infty$ above to complete the proof.

4.3. Proof of (II). The implication $(a) \Rightarrow (b)$ is obvious. To prove that $(b) \Rightarrow (c)$, we use an adaptation of an argument due to Luecking (see [14]), where, instead of reproducing kernels, we employ the test functions $F_{a,R}$. For an arbitrary sequence $\{a_k\} \in \ell^p$, consider the function

$$G_t(z) = \sum_{z_k : |z_k| \geq \eta(R)} a_k r_k(t) \frac{F_{z_k,R}(z)}{\tau(z_k)^{2/p}}, \quad 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions (see page 336 of [14], or Appendix A of [12]). By Proposition 9 and condition (b)

$$\int_{\mathbb{C}} |G_t(z)|^q \, d\mu(z) \leq C \|G_t\|_{F_\phi}^q \leq C \left( \sum_k |a_k|^p \right)^{q/p} \quad \text{for } 0 < t < 1.$$  

Integrating with respect to $t$ from 0 to 1, applying Fubini’s theorem, and invoking Khinchine’s inequality (see [14]), we obtain

$$\int_{\mathbb{C}} \left( \sum_{z_k : |z_k| \geq \eta(R)} |a_k|^2 \frac{|F_{z_k,R}(z)|^2}{\tau(z_k)^{4/p}} \right)^{q/2} \, d\mu(z) \leq C \left( \sum_k |a_k|^p \right)^{q/p} \tag{4.4}$$
If $\chi_E(z)$ denotes the characteristic function of a set $E$, bearing in mind the estimate (2.3), and the finite multiplicity $N$ of the covering \{D(3\delta \tau(z_k))\} (see (iv) of Lemma 6), we have
\[
\sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q/p}} \int_{D(3\delta \tau(z_k))} e^{q \phi(z)} \, d\mu(z)
\lesssim \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q/p}} \int_{D(3\delta \tau(z_k))} |F_{z_k,R}(z)|^q \, d\mu(z)
= \int_{\mathbb{C}} \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q/p}} |F_{z_k,R}(z)|^q \chi_{D(3\delta \tau(z_k))}(z) \, d\mu(z)
\lesssim \max\{1, N^{1-q/2}\} \int_{\mathbb{C}} \left( \sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|F_{z_k,R}(z)|^2}{\tau(z_k)^{q/p}} \right)^{q/2} \, d\mu(z)
\]
This, together with (4.4) yields
\[
\sum_{|z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q/p}} \int_{D(3\delta \tau(z_k))} e^{q \phi(z)} \, d\mu(z) \leq C \left( \sum_{k} |a_k|^p \right)^{q/p}.
\]
Then using the duality between $\ell^p_q$ and $\ell^{p/q}$ we conclude that
\[
(4.5) \sum_{|z_k| \geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(3\delta \tau(z_k))} e^{q \phi(z)} \, d\mu(z) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty.
\]
Note that there is $\rho_1 \in (0, \infty)$, with $\eta(R) \leq \rho_1$ such that if a point $z_k$ of the sequence $\{z_j\}$ belongs to $\{ |z| < \eta(R) \}$, then $D(\delta \tau(z_k)) \subset \{ |z| < \rho_1 \}$. Therefore, using Lemma 5, (ii) and (iii) of Lemma 6, and (4.5) we deduce that
\[
\int_{|z| \geq \rho_1} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} e^{q \phi(\zeta)} \, d\mu(\zeta) \right)^{\frac{p}{p-q}} \, dm(z)
\leq \sum_{|z_k| \geq \eta(R)} \int_{D(\delta \tau(z_k))} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} e^{q \phi(\zeta)} \, d\mu(\zeta) \right)^{\frac{p}{p-q}} \, dm(z)
\lesssim \sum_{|z_k| \geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(3\delta \tau(z_k))} e^{q \phi(\zeta)} \, d\mu(\zeta) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty.
\]
This, together with the fact that the integral
\[
\int_{|z| < \rho_1} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} e^{q \phi(\zeta)} \, d\mu(\zeta) \right)^{\frac{p}{p-q}} \, dm(z)
\]
is clearly finite, proves that (c) holds.
Finally, we are going to prove that (c) implies (a). It is enough to prove that if \( \{ f_n \} \) is a bounded sequence in \( F_p^\phi \) that converges to 0 uniformly on compact subsets of \( C \) then \( \lim_{n \to \infty} \| f_n \|_{L^q(dm)} = 0 \).

Bearing in mind (2.2) for \( t(z) = \delta \tau(z) \), and the fact that \( \tau(r) \) is a decreasing function with \( \lim_{r \to \infty} \tau(r) = 0 \), we assert that there is \( r_0' \)

\[
D \left( \frac{\delta}{2} \tau(z) \right) \subset \left\{ \zeta \in C : |\zeta| > \frac{\tau}{2} \right\}, \quad \text{if } |z| > r \geq r_0'.
\]

On the other hand, it follows from Lemma 7 that

\[
|f_n(z)|^q \leq C \frac{e^{q\phi(z)}}{\tau(z)^{2q}} \int_{D(\delta \tau(z))} |f_n(\zeta)|^q e^{-q\phi(\zeta)} \, dm(\zeta).
\]

Integrate with respect to \( dm \), apply Fubini’s theorem, use (4.6) and Lemma 5 to obtain

\[
\int_{\{ z \in C : |z| > r \}} |f_n(z)|^q \, d\mu(z)
\leq C \int_{\{ \zeta \in C : |\zeta| > \frac{\tau}{2} \}} |f_n(\zeta)|^q e^{-q\phi(\zeta)} \left( \frac{1}{\tau(\zeta)^2} \int_{D(\delta \tau(\zeta))} e^{q\phi(\zeta)} \, d\mu(\zeta) \right) \, dm(\zeta), \quad r \geq r_0'.
\]

By condition (c) for any fixed \( \varepsilon > 0 \), there is \( r_0 \geq r_0' \in (0, \infty) \), such that

\[
\int_{\{ \zeta \in C : |\zeta| > \frac{\tau}{2} \}} \left( \frac{1}{\tau(\zeta)^2} \int_{D(\delta \tau(\zeta))} e^{q\phi(\zeta)} \, d\mu(\zeta) \right) \frac{\rho}{p-q} \, dm(\zeta) < \varepsilon \frac{\rho}{p-q}.
\]

Then (4.7) and an application of Hölder’s inequality yields

\[
\int_{\{ z \in C : |z| > r_0 \}} |f_n(z)|^q \, d\mu(z)
\leq C \| f_n \|^q_{F_p^\phi} \left( \int_{\{ \zeta \in C : |\zeta| > \frac{\tau}{2} \}} \left( \frac{1}{\tau(\zeta)^2} \int_{D(\delta \tau(\zeta))} e^{q\phi(\zeta)} \, d\mu(\zeta) \right) \frac{\rho}{p-q} \, dm(\zeta) \right) \frac{p}{p-q} \leq C \varepsilon.
\]

Moreover, we have \( \lim_{n \to \infty} \int_{|z| \leq r_0} |f_n(z)|^q \, d\mu(z) = 0 \), which together with (4.8), gives \( \lim_{n \to \infty} \| f_n \|_{L^q(dm)} = 0 \). This completes the proof of Theorem 1.

**Proof of Corollary 2.** Fix \( p \in (0, \infty) \) and put \( \mu(z) = e^{-q\phi(z)} \, dm(z) \) in Theorem 1. For \( q > p \), taking into account that \( \tau \) decreases to 0 as \( a \to \infty \), we deduce that for any \( \delta > 0 \)

\[
\frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} e^{q\phi(z)} \, d\mu(z) \propto \tau(a)^{2(1-q/p)} \to \infty, \quad \text{if } a \to \infty,
\]

so by Theorem 1 (I), \( F_p^\phi \not\subset F_q^\phi \). On the other hand, if \( 0 < q < p \), using Theorem 1 (II) and the fact that \( f \equiv 1 \not\in L^p_{\rho-q}(C, dm) \), we deduce that \( F_p^\phi \not\subset F_q^\phi \). \( \square \)
5. Proof of Theorem 3

Let us first notice that, by Theorem 19, we have

\[
\|T_g f\|_{L^q_p}^q \sim \int_{\mathbb{C}} |f(z)|^q |g'(z)|^q (1 + \phi'(z))^{-q} e^{-q\phi(z)} \, dm(z),
\]

for \( q > 0 \) and for any entire function \( f \). This relation shows that the boundedness (compactness) of the integration operator \( T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi \) is equivalent to

\[
\text{Proof of 5.1.}
\]

Concerning the compactness part \( (b) \) of \( (I) \), note that, by part \( (I) \) of Theorem 1, the embedding \( I_d : \mathcal{F}_p^\phi \to L^q(\mu_g, \phi) \) is compact if and only if

\[
\sup_{a \in \mathbb{C}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} |g'(z)|^q (1 + \phi'(z))^{-q} \, dm(z) < \infty.
\]

5.1. Proof of (I). Assume \( 0 < p \leq q < \infty \) and let \( \delta \in (0, m_T) \). By Theorem 1, the embedding \( I_d : \mathcal{F}_p^\phi \to L^q(\mu_g, \phi) \) is continuous if and only if

\[
g'(a)^q (1 + \phi'(a))^{-q} \tau(a)^{2-2q/p} \lesssim (1 + \phi'(a))^{-q} \tau(a)^{-2q/p} \int_{D(\delta \tau(a))} |g'(z)|^q \, dm(z)
\]

\[
\lesssim \tau(a)^{-2q/p} \int_{D(\delta \tau(a))} |g'(z)|^q (1 + \phi'(z))^{-q} \, dm(z),
\]

and the proof of part \( (a) \) of \( (I) \) is complete.

Concerning the compactness part \( (b) \) of \( (I) \), note that, by part \( (I) \) of Theorem 1, the embedding \( I_d : \mathcal{F}_p^\phi \to L^p(\mu_g, \phi) \) is compact if and only if

\[
\lim_{|a| \to \infty} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} |g'(z)|^q (1 + \phi'(z))^{-q} \, dm(z) = 0.
\]

Proceeding as in the boundedness part, we see that this is equivalent to

\[
\lim_{|a| \to \infty} |g'(a)|^q (1 + \phi'(a))^{-q} \tau(a)^{2-2q/p} = 0.
\]

5.2. Proof of (II). The equivalence \( (a) \Leftrightarrow (b) \) follows from part \( (II) \) of Theorem 1.

Let us prove that \( (b) \Rightarrow (c) \). From part \( (II) \) of Theorem 1 we deduce that \( (b) \) is equivalent to

\[
\int_{\mathbb{C}} \left( \frac{1}{\tau(z)^2} \int_{D(\tau(z))} |g'(\zeta)|^q (1 + \phi'(\zeta))^{-q} \, dm(\zeta) \right)^{\frac{p}{p-q}} \, dm(z) < \infty.
\]
Thus Parseval’s identity gives
\begin{equation}
\int_{\mathbb{C}} |g'(z)|^q (1 + \phi'(z))^{-r} \, dm(z)
\end{equation}

Now, by Lemma 20 and the subharmonicity of $|g'|^q$,
\begin{align*}
\lesssim & \int_{\mathbb{C}} \left( \frac{1}{\tau(z)^2} \int_{D(\tau(z))} |g'(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{p}} \, dm(z) \\
\lesssim & \int_{\mathbb{C}} \left( \frac{1}{\tau(z)^2} \int_{D(\tau(z))} |g'(\zeta)|^q (1 + \phi'(\zeta))^{-q} \, dm(\zeta) \right)^{\frac{1}{p}} \, dm(z) < \infty.
\end{align*}

(c) $\Rightarrow$ (b). If $\frac{q}{1+\varphi} \in L^r(\mathbb{C}, dm)$, then (5.1) and Hölder’s inequality a gives
\begin{align*}
\|T_g f\|^q_{\mathcal{F}^q_p} \lesssim & \left( \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} \, dm(z) \right)^{q/p} \left( \int_{\mathbb{C}} |g'(z)|^q (1 + \phi'(z))^{-q} \, dm(z) \right)^{q/r} \\
\lesssim & \| \frac{g'}{1+\varphi} \|^q_{L^r(\mathbb{C}, dm)} \|f\|^q_{\mathcal{F}^q_p}.
\end{align*}

Thus $T_g : \mathcal{F}^\varphi_p \to \mathcal{F}^\varphi_q$ is bounded with $\|T_g\| \lesssim \|\frac{g'}{1+\varphi}\|_{L^r(\mathbb{C}, dm)}$. This finishes the proof. \qed

6. Schatten classes on $\mathcal{F}^\varphi_2$

Given a separable Hilbert space $H$, the Schatten $p$-class of operators on $H$, $\mathcal{S}_p(H)$, consists of those compact operators $T$ on $H$ with its sequence of singular numbers $\lambda_n$ belonging to $\ell^p$, the $p$-summable sequence space.

If $\{e_n\}$ is an orthonormal basis of a Hilbert space $H$ of analytic functions in $\mathbb{C}$ with reproducing kernel $K_z$, then
\begin{equation}
K_z(\zeta) = \sum_n \langle K_z, e_n \rangle e_n(\zeta) = \sum_n e_n(\zeta) \overline{e_n(z)}
\end{equation}

Also, by (6.1) we have
\begin{equation}
\frac{\partial}{\partial \bar{z}} K_z(\zeta) = \sum_n \overline{e'_n(z)} e_n(\zeta), \quad z, \zeta \in \mathbb{C}.
\end{equation}

Thus Parseval’s identity gives
\begin{equation}
\|K_z\|^2_H = \sum_n |e_n(z)|^2 \quad \text{and} \quad \left\| \frac{\partial}{\partial \bar{z}} K_z \right\|^2_H = \sum_n |e'_n(z)|^2.
\end{equation}

Now, we are going to give the proof of Theorem 4 on the description of the Schatten classes $\mathcal{S}_p := \mathcal{S}_p(\mathcal{F}^\varphi_2)$. First we consider the sufficiency part of the case $1 < p < \infty$. For this we need the following two lemmas. The first one is a $L^\infty$ version of Theorem 12 (see also [20, Theorem 2.1]).

Lemma 21. Let $0 < p \leq \infty$ and $\varphi : [1, \infty) \to \mathbb{R}$ be a twice differentiable, positive and increasing function such that $\lim_{r \to \infty} \varphi(r) = \infty$. If $\varphi$ satisfies (3.4), then for each entire function $f$, the following conditions are equivalent,
(i) \( M_p(r, f) = O(\varphi(r)), \quad r \to \infty. \)
(ii) \( M_p(r, f') = O(\varphi'(r)), \quad r \to \infty. \)

**Proof.** Let \( \{r_n\} \) the sequence defined by \( \varphi(r_n) = e^n. \)
(i) \( \Rightarrow \) (ii). By Lagrange’s theorem for each \( n \in \mathbb{N} \)

\[
\varphi(r_{n+1}) - \varphi(r_n) = \varphi'(t_n)(r_{n+1} - r_n), \quad t_n \in (r_n, r_{n+1})
\]

so by Lemma 13 and Lemma 16

\[
M_p(r_n, f') \leq \frac{C_p}{r_{n+1} - r_n} M_p(r_{n+1}, f) = \frac{C_p \varphi'(t_n)}{\varphi(r_{n+1}) - \varphi(r_n)} M_p(r_{n+1}, f)
\]

\[
\leq \frac{C_p \varphi'(t_n) \varphi(r_{n+1})}{\varphi(r_{n+1}) - \varphi(r_n)} \leq C \varphi'(t_n) \leq C \varphi'(r_n).
\]

Now, for \( r \in [r_0, \infty) \) choose \( n \) such that \( r \in [r_n, r_{n+1}) \). Then

\[
M_p(r, f') \leq M_p(r_{n+1}, f') \leq C \varphi'(r_{n+1}) \leq C \varphi'(r),
\]

where the last inequality follows from Lemma 16.

(ii) \( \Rightarrow \) (i). We assume that \( 0 < p \leq 1 \) (the proof for \( p > 1 \) is analogous).

Bearing in mind Lemma 14, (6.3) and Lemma 16, we get

\[
M^p_p(r_{j+1}, f) - M^p_p(r_j, f) \leq C_p \varphi'(r_j) (r_{j+1} - r_j)^p M^p_p(r_{j+1}, f') \leq C (r_{j+1} - r_j)^p \varphi'(r_{j+1})^p \leq C \varphi'(r_{j+1})^p \leq C e^{jp},
\]

so a summation gives

\[
M^p_p(r_{n+1}, f) \leq C \sum_{j=0}^{n} e^{jp} + M^p_p(r_0, f) \leq C e^{np} = C \varphi(r_n)^p
\]

and the proof follows. \( \square \)

**Lemma 22.** If \( \phi \in \mathcal{I} \) satisfies (1.7), then

\[
\left| \frac{\partial}{\partial z} K_z \right|_{\mathcal{F}^\phi_2} = O \left( \|K_z\|_{\mathcal{F}^\phi_2} (|z|) \right), \quad |z| \to \infty.
\]

**Proof.**

Let \( \{e_n\} \) be the orthonormal basis of \( \mathcal{F}^\phi_2 \) given by

\[
e_n(z) = z^n \delta_n^{-1}, \quad n \in \mathbb{N},
\]

where \( \delta_n^2 = 2\pi \int_0^\infty r e^{-2\phi(r)} dr \). By Corollary 8 we have that

\[
\sum_{n=0}^{\infty} r^{2n} \delta_n^{-2} = \sum_{n=0}^{\infty} |e_n(z)|^2 = \left| K_z \right|_{\mathcal{F}^\phi_2}^2 \times \frac{e^{2\phi(r)}}{r^2(r)}, \quad |z| = r.
\]

So, if we consider the entire function defined by

\[
f(z) = \sum_{n=0}^{\infty} z^n \delta_n^{-1},
\]
then \(M_2(r, f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta\right)^{1/2} \propto \Phi(r), \text{ as } r \to \infty, \) where
\[
\Phi(r) = \frac{e^{\phi(r)}}{r(r)}. \]

By Lemma 18
\[
\Phi'(r) \asymp \Phi(r)\phi'(r), \quad \text{as } r \to \infty.
\]

Moreover, in view of (1.7) and Lemma 18, a calculation shows that
\[
\limsup_{r \to \infty} \frac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} < \infty.
\]

Thus by Lemma 21
\[
M_2(r, f') = O \left(\Phi'(r)\right),
\]

Finally, since for \(r = |z|,\)
\[
\left\|\frac{\partial}{\partial \bar{z}} K_z\right\|_{\mathcal{F}_2^\phi}^2 = \sum_{n=0}^{\infty} |e_n'(z)|^2 = \sum_{n=1}^{\infty} n^2 r^{2n-2} \delta_n^{-2} = M_2^2(r, f')
\]
we obtain
\[
\left\|\frac{\partial}{\partial \bar{z}} K_z\right\|_{\mathcal{F}_2^\phi} = M_2(r, f') = O \left(\Phi'(r)\right) \asymp \Phi(r)\phi'(r) \asymp \|K_z\|_{\mathcal{F}_2^\phi}\phi'(r), \quad r \to \infty.
\]

\[\square\]

**Proposition 23.** Let \(g \in H(\mathbb{C}), 1 < p < \infty\) and \(\phi \in \mathcal{I}\) satisfying (1.7). If \(\frac{g'}{1+\phi'} \in L^p(\mathbb{C}, \Delta \phi \, dm),\) then \(T_g \in S_p(\mathcal{F}_2^\phi).\)

**Proof.**

By Theorem 19, the inner product
\[
\langle f, g \rangle_* = f(0)\overline{g(0)} + \int_{\mathbb{C}} f'(z)\overline{g'(z)} (1 + \phi'(z))^{-2} e^{-2\phi(z)} \, dm(z)
\]
gives a norm on \(\mathcal{F}_2^\phi\) equivalent to the usual one. If \(1 < p < \infty,\) the operator \(T_g\) belongs to the Schatten \(p\)-class \(S_p\) if and only if
\[
\sum_n |\langle T_g e_n, e_n \rangle_*|^p < \infty
\]
for any orthonormal basis \(\{e_n\}\) (see [28, Theorem 1.27]). Let \(\{e_n\}\) be an orthonormal set of \((\mathcal{F}_2^\phi, \langle \cdot, \cdot \rangle_*)\). Next, applying Theorem 19 for \(p = 1\) and \(\phi = 2\phi\) we deduce
\[
1 = \|e_n\|_{\mathcal{F}_2^\phi}^2 = \|e_n\|_{\mathcal{F}_2^\phi}^2 \geq \int_{\mathbb{C}} |e_n(z)e_n'(z)|(1 + \phi'(z))^{-1} e^{-\phi(|z|)} \, dm(z)
\]
\[
\times \int_{\mathbb{C}} |e_n(z)e_n'(z)|(1 + \phi'(z))^{-1} e^{-2\phi(|z|)} \, dm(z)
\]
This together with Hölder’s inequality yields

\[
\sum_n |\langle Tg, e_n \rangle|^p \leq \sum_n \left( \int_C |g'(z)e_n(z)e_n'(z)|(1 + \phi'(z))^{-2}e^{-2\phi(|z|)} \, dm(z) \right)^p
\]

\[
\lesssim \sum_n \int_C |g'(z)|^p |e_n(z)e_n'(z)|(1 + \phi'(z))^{-(p+1)}e^{-2\phi(|z|)} \, dm(z)
\]

\[
= \int_C |g'(z)|^p \left( \sum_n |e_n(z)e_n'(z)| \right) (1 + \phi'(z))^{-(p+1)}e^{-2\phi(|z|)} \, dm(z),
\]

and since \( \|Kz\|_{L^p_{\mathcal{F}_2}}e^{-\phi(|z|)} \asymp \Delta \phi(z) \) as \(|z| \to \infty\) (see Corollary 8), the result will be proved if we are able to show that

\[
(6.4) \quad \sum_n |e_n(z)e_n'(z)| \lesssim \|Kz\|^2_{\mathcal{F}_2}(1 + \phi'(z)).
\]

To prove (6.4), we use the Cauchy-Schwarz inequality to obtain

\[
\sum_n |e_n(z)e_n'(z)| \leq \left( \sum_n |e_n(z)|^2 \right)^{1/2} \left( \sum_n |e_n'(z)|^2 \right)^{1/2}
\]

\[
= \|Kz\|_{\mathcal{F}_2} \left\| \frac{\partial}{\partial \bar{z}} Kz \right\|_{\mathcal{F}_2^\phi}.
\]

Now, the inequality (6.4) follows from Lemma 22. This completes the proof of the Proposition.

Now we turn to prove the necessity for \(0 < p < \infty\).

**Proposition 24.** Let \(g \in H(\mathbb{C})\), \(0 < p < \infty\) and \(\phi \in \mathcal{I}\). If \(Tg \in \mathcal{S}_p(\mathcal{F}_2^\phi)\), then \(\frac{g'}{1+\phi} \in L^p(\mathbb{C}, \Delta \phi \, dm)\).

**Proof.** We split the proof in two cases.

**Case 2 \(\leq p < \infty\).** Suppose that \(Tg\) is in \(\mathcal{S}_p\), and let \(\{e_k\}\) be an orthonormal basis in \(\mathcal{F}_2^\phi\) and \(R > 100\). Let \(\{z_k\}\) be the sequence from Lemma 6 for \(t(z) = \delta \tau(z)\), \(\delta \in (0, m_\tau)\), and consider the operator \(A\) taking \(e_k(z)\) to \(f_{z_k}(z) = F_{z_k,R}(z)/\tau(z_k)\). It follows from Proposition 9 that the operator \(A\) is bounded on \(\mathcal{F}_2^\phi\). Then \(TgA\) belongs to \(\mathcal{S}_p\) (see [28, p.27]), and by [28, Theorem 1.33]

\[
\sum_k \|Tg(f_{z_k})\|_{\mathcal{F}_2^\phi}^p = \sum_k \|TgAe_k\|_{\mathcal{F}_2^\phi}^p < \infty.
\]
This together with Proposition A and Theorem 19 gives
\[ \sum_k \frac{1}{\tau(z_k)^p} \left( \int_{D(\tau(z_k))} |g'(z)|^2 (1 + \phi'(z))^{-2} \, dm(z) \right)^{p/2} \]
\[ \times \sum_k \left( \int_{D(\tau(z_k))} |f_{z_k}(z)|^2 |g'(z)|^2 (1 + \phi'(z))^{-2} e^{-2\phi(|z|)} \, dm(z) \right)^{p/2} \]
\[ \lesssim \sum_k \|T_g(f_{z_k})\|_p^p < \infty. \]

On the other hand, if \( \delta \) is sufficiently small, applying Lemma 7, Lemma 5, Lemma 20 and Lemma 6, it follows that
\[ \int_{\mathbb{C}} |g'(z)|^p (1 + \phi'(z))^{-p} \Delta \phi(z) \, dm(z) \]
\[ \lesssim \sum_k \int_{D(\delta \tau(z_k))} \left( \frac{1}{\tau^2(z)} \right) \left( \int_{D(\delta \tau(z))} |g'(\zeta)|^2 \, dm(\zeta) \right) \left( 1 + \phi'(z) \right)^{p/2} \, dm(z) \]
\[ \lesssim \sum_k \frac{1}{\tau(z_k)^p} \int_{D(\delta \tau(z_k))} \left( \int_{D(\delta \tau(z))} |g'(\zeta)|^2 (1 + \phi'(z))^{-2} \, dm(\zeta) \right)^{p/2} \, dm(z) \]
\[ \lesssim \sum_k \frac{1}{\tau(z_k)^p} \left( \int_{D(3\delta \tau(z_k))} |g'(\zeta)|^2 (1 + \phi'(z))^{-2} \, dm(\zeta) \right)^{p/2}. \]

This together with the previous inequality concludes the proof.

**Case 0 < p < 2.** If \( T_g \in \mathcal{S}_p \) then the positive operator \( T_g^* T_g \) belongs to \( \mathcal{S}_{p/2} \). Without loss of generality we may assume that \( g' \neq 0 \). Suppose
\[ T_g^* T_g f = \sum_n \lambda_n <f, e_n> e_n \]
is the canonical decomposition of \( T_g^* T_g \). Then a standard argument gives that \( \{e_n\} \) is an orthonormal basis. So relation (6.2) together with Corollary 8 and Hölder’s inequality yields
\[ \int_{\mathbb{C}} |g'(z)|^p (1 + \phi'(z))^{-p} \Delta \phi(z) \, dm(z) \]
\[ \times \int_{\mathbb{C}} |g'(z)|^p (1 + \phi'(z))^{-p} \|K_z\|^2 e^{-2\phi(|z|)} \, dm(z) \]
\[ = \sum_n \int_{\mathbb{C}} |g'(z)|^p (1 + \phi'(z))^{-p} |e_n(z)|^2 e^{-2\phi(|z|)} \, dm(z) \]
\[ \lesssim \sum_n \left( \int_{\mathbb{C}} |g'(z)|^2 (1 + \phi'(z))^{-2} |e_n(z)|^2 e^{-2\phi(|z|)} \, dm(z) \right)^{p/2} \]
\[ \lesssim \sum_n \langle T_g^* T_g e_n, e_n \rangle^{p/2} = \sum_n \lambda_n^{p/2} = \|T_g^* T_g\|_{\mathcal{S}_{p/2}}^{p/2}. \]
This completes the proof.

Finally, we shall prove the main result in this section.

**Proof of Theorem 4.**

Part (a) follows directly from Proposition 23 and Proposition 24. Moreover, if $0 < p \leq 1$ and $T_g \in \mathcal{S}_p(\mathcal{F}_2^\phi) \subset \mathcal{S}_1(\mathcal{F}_2^\phi)$, then Proposition 24 and Lemma 18 (d) imply that for some $r_0 \in (0, \infty)$

$$\int_{r_0 < |z| < \infty} \frac{|g'(z)|}{|z|} \, dm(z) \leq \int_{r_0 < |z| < \infty} \frac{|g'(z)|}{(1 + \phi'(z))} \Delta \phi(z) \, dm(z)$$

$$\leq \int_{C} \frac{|g'(z)|}{|1 + \phi'(z)|} \Delta \phi(z) \, dm(z) < \infty.$$

Therefore, it follows that $g' \equiv 0$, which gives (b). The proof is complete. \qed

7. **Examples**

In this section, several examples of rapidly increasing functions are given. We offer the corresponding description for the boundedness and compactness of the integration operator $T_g$ in each case.

**Example 1:** The functions $\phi(|z|) = |z|^\alpha$, $\alpha > 2$ belong to $\mathcal{I}$ with $\Delta \phi \asymp |z|^\alpha - 2$, so we obtain the following as a byproduct of Theorem 3.

**Corollary 25.** Let $0 < p, q < \infty$, $g$ an entire function, $\phi(|z|) = |z|^\alpha$, $\alpha > 2$.

(I) (a) Let $0 < p \leq q < \infty$. Then,

- If $1 + (\alpha - 2) \left( 1 - \frac{1}{p} + \frac{1}{q} \right) < 0$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is constant.
- If $1 + (\alpha - 2) \left( 1 - \frac{1}{p} + \frac{1}{q} \right) \geq 0$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is a polynomial with

$$\deg(g) \leq 2 + (\alpha - 2) \left( 1 - \frac{1}{p} + \frac{1}{q} \right).$$

(b) Let $0 < p \leq q < \infty$. Then,

- If $1 + (\alpha - 2) \left( 1 - \frac{1}{p} + \frac{1}{q} \right) \leq 0$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is compact if and only if $g$ is constant.
- If $1 + (\alpha - 2) \left( 1 - \frac{1}{p} + \frac{1}{q} \right) > 0$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is compact if and only if $g$ is a polynomial with

$$\deg(g) < (\alpha - 1) - \left( \frac{1}{p} - \frac{1}{q} \right) (\alpha - 2) + 1.$$

(II) Let $0 < q < p < \infty$.

- If $q \leq \frac{2p}{p(\alpha - 1) + 2}$, $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is constant.
- If $q > \frac{2p}{p + 2\alpha - 1}$. The following conditions are equivalent:

  (a) $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is compact;
  (b) $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded;
Corollary 26. Let $0 < p, q < \infty$, $g$ an entire function, $\phi(|z|) = e^{\beta|z|}$, $\beta > 0$.

(I) (a) Let $0 < p < q < \infty$. Then,
- If $\frac{1}{p} - \frac{1}{q} > 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is constant.
- If $\frac{1}{p} - \frac{1}{q} = 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is a polynomial with $\deg(g) \leq 1$.
- If $\frac{1}{p} - \frac{1}{q} < 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if
  $$\sup_{z \in \mathbb{C}} |g'(z)| e^{\left(\frac{1}{p} - \frac{1}{q} - 1\right)|z|} < \infty.$$ 

(b) Let $0 < p \leq q < \infty$. Then,
- If $\frac{1}{p} - \frac{1}{q} \geq 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is compact if and only if $g$ is constant.
- If $\frac{1}{p} - \frac{1}{q} < 1$, then $T_g$ is compact if the "little oh" versions of the boundedness conditions in (a) hold.

(II) Let $0 < q < p < \infty$. The following conditions are equivalent:
  (a) $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is compact;
  (b) $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded;
  (c) $\int_{\mathbb{C}} (|g'(z)||e^{-\beta|z|}|)^r \, dm(z) < \infty$, where $r = \frac{pq}{p-q}$.

Moreover, Theorem 4 shows that $T_g \in S_p$ if and only if
$$g'(z)e^{-\beta \left(1 - \frac{1}{q}\right)|z|} \in L^p(\mathbb{C}, dm), \quad p > 1.$$ 

The above example shows that for any $r > 0$ in Theorem 3 (II) (c) $g$ can grow exponentially.

Example 3: The function $\phi(|z|) = e^{e|z|}$ belongs to $\mathcal{I}$ with $\Delta \phi \asymp e^{e|z|}$, so Theorem 3 provides the following

Corollary 27. Suppose $g$ is an entire function, $0 < p, q < \infty$ and $\phi(|z|) = e^{e|z|}$.

(I) (a) Let $0 < p \leq q < \infty$. Then,
- If $\frac{1}{p} - \frac{1}{q} \geq 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if $g$ is constant.
- If $\frac{1}{p} - \frac{1}{q} < 1$, then $T_g : \mathcal{F}_p^\phi \to \mathcal{F}_q^\phi$ is bounded if and only if
  $$\sup_{z \in \mathbb{C}} |g'(z)| e^{|z| + \left(\frac{1}{p} - \frac{1}{q} - 1\right)(2|z| + e^{|z|})} < \infty.$$
Let $0 < p \leq q < \infty$. Then $T_g$ is compact if the "little oh" versions of the boundedness conditions in (a) hold.

(II) Let $0 < q < p < \infty$. The following conditions are equivalent:

(a) $T_g : \mathcal{F}_p^0 \rightarrow \mathcal{F}_q^0$ is compact;
(b) $T_g : \mathcal{F}_p^0 \rightarrow \mathcal{F}_q^0$ is bounded;
(c) $\int_C \left( |g'(z)| e^{-|z| - e^{z}} \right)^r \, dm(z) < \infty$, where $r = \frac{pq}{p-q}$.

Moreover, Theorem 4 shows that $T_g \in S_p$ if and only if $g'(z) e^{\left( \frac{2}{p} - 1 \right) |z| - \left( 1 - \frac{1}{p} \right) e^{|z|}} \in L^p(C, dm)$, $p > 1$.

8. Invariant subspaces of the Volterra operator

In the particular case $g(z) = z$, the operator $T_g$ becomes the Volterra operator

$$Vf(z) = \int_0^z f(\zeta) \, d\zeta, \quad z \in \mathbb{C}, \quad f \in \mathcal{F}_p^0, \quad p > 0.$$ 

Recall that a closed subspace $\mathcal{M}$ in $\mathcal{F}_p^0$ is called invariant for $V$ if $VM \subseteq \mathcal{M}$. In this section we aim to characterize the invariant subspaces of $V$. For this we begin by showing that polynomials are dense in $\mathcal{F}_p^0$, by two different methods. The first one is based on some smooth polynomials and the second one follows by a dilation argument.

8.1. Density of polynomials. For the first proof, we need some background on certain smooth polynomials defined in terms of Hadamard products. Let $T$ be the boundary of the unit disc $D = \{ z : |z| < 1 \}$. If $W(e^{i\theta}) = \sum_{k \in J \subset \mathbb{Z}} b_k e^{ik\theta}$ is a trigonometric polynomial and $f(e^{i\theta}) = \sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \in L^p(T)$, then the Hadamard product

$$(W * f)(e^{i\theta}) = \sum_{k \in J} a_k b_k e^{ik\theta}$$

is well defined.

If $\Phi : \mathbb{R} \rightarrow \mathbb{C}$ is a $C^\infty$-function with compact support $\text{supp}(\Phi)$, we set

$$A_{\Phi,m} = \max_{s \in \mathbb{R}} |\Phi(s)| + \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)|,$$

and we consider the polynomials

$$W_N^\Phi(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \Phi \left( \frac{k}{N} \right) e^{ik\theta}, \quad N \in \mathbb{N}.$$ 

With this notation we can state the next result on smooth partial sums.

**Theorem B.** Let $\Phi : \mathbb{R} \rightarrow \mathbb{C}$ be a $C^\infty$-function with compact support $\text{supp}(\Phi)$. Then the following assertions hold:

(i) There exists a constant $C > 0$ such that

$$\left| W_N^\Phi(e^{i\theta}) \right| \leq C \min \left\{ N \max_{s \in \mathbb{R}} |\Phi(s)|, N^{1-m} \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)| \right\},$$
for all $m \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $0 < \theta < \pi$.

(ii) If $0 < p \leq 1$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C(p) > 0$ such that

$$\left( \sup_{N} \left| (W_{N}^{\Phi} * f)(e^{i\theta}) \right| \right)^{p} \leq CA_{\Phi,m}M(|f|^{p})(e^{i\theta})$$

for all $f \in H^{p}$. Here $M$ denotes the Hardy-Littlewood maximal-operator

$$M(|f|)(e^{i\theta}) = \sup_{0 < h < \pi} \frac{1}{2h} \int_{\theta-h}^{\theta+h} |f(e^{it})| \, dt.$$  

(iii) For each $p \in (0, \infty)$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that

$$\|W_{N}^{\Phi} * f\|_{H^{p}} \leq CA_{\Phi,m}\|f\|_{H^{p}}$$

for all $f \in H^{p}$.

Moreover, if $\Phi(0) = 1$, then

(iv) $\lim_{N \to \infty} \|f - W_{N}^{\Phi} * f\|_{H^{p}} = 0$, for any $f \in H^{p}$, $0 < p < \infty$.

Proof 2. For $f \in F_{p}^{\phi}$ and $r \in (0, 1)$ put $f_{r}(z) = f(rz)$, $z \in \mathbb{C}$. Then

$$\lim_{r \to 1} \|f_{r} - f\|_{F_{p}^{\phi}} = 0.$$
This follows by a standard argument that we sketch for the sake of completeness. Let $R > 0$ be such that \( \int_{|z|>R} |f|^p e^{-p\phi} \, dA < \varepsilon \). Hence
\[
\int_C |f_r - f|^p e^{-p\phi} \, dm \leq \int_{|z|\leq R} |f_r - f|^p e^{-p\phi} \, dm + 2^p \int_{|z|>R} |f|^p e^{-p\phi} \, dm \\
+ 2^p \int_{|z|>R} |f_r|^p e^{-p\phi} \, dm.
\]
It is clear that the first term in the above sum goes to zero as \( r \to 1^- \). Now using polar coordinates together the fact that the integral means \( M_p(\rho, f) \) are increasing in \( \rho \) we deduce that
\[
\int_{|z|>R} |f_r|^p e^{-p\phi} \, dm \leq \int_{|z|>R} |f|^p e^{-p\phi} \, dm < \varepsilon,
\]
and now (8.2) follows.

We now show that \( f_r \) can be approximated by its Taylor polynomials in the \( \mathcal{F}_p^\phi \)-norm.

Suppose first that \( p \geq 1 \). If \( a_n \) is the \( n \)-th coefficient in the Taylor expansion of \( f \), then by the Cauchy formula and Hölder’s inequality we get
\[
|a_n|^p r_n^p \leq c_p \int_0^{2\pi} |f(re^{it})|^p \, dt, \quad r \geq 0.
\]
Multiplying both sides of the above inequality by \( re^{-p\phi(r)} \) and integrating on \([0, \infty)\) we obtain
\[
|a_n|\|z^n\|_{\mathcal{F}_p^\phi} \leq c_p\|f\|_{\mathcal{F}_p^\phi}.
\]
Using this we obtain
\[
\left\| \sum_{n=N}^{\infty} a_n r_n^p z^n \right\|_{\mathcal{F}_p^\phi} \leq \sum_{n=N}^{\infty} |a_n|\|z^n\|_{\mathcal{F}_p^\phi} r_n^p \leq c_p\|f\|_{\mathcal{F}_p^\phi} \sum_{n=N}^{\infty} r_n^p,
\]
which shows that \( f_r \) can be approximated by its Taylor polynomials in the \( \mathcal{F}_p^\phi \)-norm.

Assume now \( 0 < p < 1 \). We have (see [12, Theorem 6.4])
\[
r^{np} |a_n|^p \leq C_p n^{1-p} \int_0^{2\pi} |f(re^{it})|^p \, dt,
\]
and proceeding as in the case \( p \geq 1 \) we deduce
\[
|a_n|\|z^n\|_{\mathcal{F}_p^\phi} \leq C_p n^{1/p-1} \|f\|_{\mathcal{F}_p^\phi}.
\]
Thus
\[
\left\| \sum_{n=N}^{\infty} a_n r_n^p z^n \right\|_{\mathcal{F}_p^\phi} \leq \sum_{n=N}^{\infty} |a_n|^p\|z^n\|_{\mathcal{F}_p^\phi} r_n^{np} \leq C_p\|f\|_{\mathcal{F}_p^\phi} \sum_{n=N}^{\infty} r_n^{np} n^{1-p},
\]
and the conclusion follows, since the last expression above is the tail of a convergent series. \( \square \)
8.2. Invariant subspaces. Notice that, with respect to the standard orthonormal basis $e_n(z) = z^n/\|z^n\|_2^\phi$, $n \geq 0$, the operator $V : \mathcal{F}_2^\phi \rightarrow \mathcal{F}_2^\phi$ is a weighted shift, i.e. $Ve_n = \omega_n e_{n+1}$ with weight sequence
\begin{equation}
\omega_n = \frac{\|z^{n+1}\|_2^\phi}{(n+1)\|z^n\|_2^\phi}, \quad n \geq 0.
\end{equation}

A simple argument using integration by parts and Lemma 18 shows that, for $\phi \in \mathcal{I}$, we have $\omega_n \rightarrow 0$ as $n \rightarrow \infty$.

**Theorem 29.** Assume $p > 0$ and let $\phi \in \mathcal{I}$ If the sequence $(\omega_n)_{n \geq 1}$ given by (8.3) is eventually decreasing to zero, then a closed subspace $\mathcal{M} \subset \mathcal{F}_p^\phi$ is a proper invariant subspace of $V : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_p^\phi$ if and only if there exists a positive integer $N$ such that
\[ \mathcal{M} = \{ f \in \mathcal{F}_p^\phi : f^{(k)}(0) = 0 \text{ for } 0 \leq k \leq N - 1 \} = \text{Span}\{ z^k : k \geq N \}. \]

**Proof.** Clearly, the sets
\[ A_N^p := \{ f \in \mathcal{F}_p^\phi : f^{(k)}(0) = 0 \text{ for } 0 \leq k \leq N - 1 \} \]
are invariant subspaces for $V$.

Let us now prove that these are all the invariant subspaces of $V$. The case $p = 2$ follows directly from a result of Yakubovich on weighted shifts (see [27] Theorems 3-4). We are now going to show that the result for $p \neq 2$ can be obtained via the case $p = 2$ using the boundedness of $T_p$ on Fock spaces $\mathcal{F}_p^\phi$ (see Theorem 3 above). In order to do this, let us first prove the following

**Claim:** For each $p > 0$ there exist positive integers $M_1 = M_1(p), M_2 = M_2(p)$ such that $V^{M_1} : \mathcal{F}_2^\phi \rightarrow \mathcal{F}_2^\phi$ and $V^{M_2} : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_p^\phi$ are bounded.

Let us first notice that by Theorem 3 and by (a), (b) of Lemma 18 we have
\begin{equation}
V : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_q^\phi \text{ is bounded if } F(q) \leq p' \leq q,
\end{equation}
\begin{equation}
V : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_q^\phi \text{ is bounded if } F(p') < q < p',
\end{equation}
where $F(s) = \frac{2s}{s+2}$ for $s > 0$. From this it is easy to see that $V : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_2^\phi$ and $V : \mathcal{F}_2^\phi \rightarrow \mathcal{F}_p^\phi$ are bounded if $p \geq 2$, and hence we can take $M_1 = M_2 = 1$ in this case.

Let us now study the case $0 < p < 2$. Consider sequence $(x_n)$ given by $x_0 = 2, x_n = F(x_{n-1})$, $n \geq 1$ and notice that $(x_n)$ decreases to zero as $n \rightarrow \infty$. We aim to show that
\begin{equation}
V^M : \mathcal{F}_p^\phi \rightarrow \mathcal{F}_2^\phi \text{ is bounded if } x_{M-1} \geq p \geq x_M,
\end{equation}
\begin{equation}
V^M : \mathcal{F}_2^\phi \rightarrow \mathcal{F}_p^\phi \text{ is bounded if } x_{M-1} \geq p > x_M, \quad M \geq 1.
\end{equation}
Both assertions follow by induction. We only prove (8.7), since (8.6) is shown in a similar way. For $M = 1$ relation (8.7) follows directly from (8.4-8.5). Now assume that $V^M : \mathcal{F}_2^\phi \rightarrow \mathcal{F}_p^\phi$ is bounded if $x_{M-1} \geq p > x_M$. By (8.5) we deduce that
\[ V : F_{x_M + \varepsilon} \to F^\phi_q \] is bounded if \( x_M + \varepsilon > q > F(x_M + \varepsilon) \). For \( 0 < \varepsilon < x_{M-1} - x_M \) we now obtain
\[ V^{M+1} : F^\phi_2 \to F^\phi_q \] is bounded if \( x_M + \varepsilon > q > F(x_M + \varepsilon) \).

Let \( \varepsilon \to 0 \) above to get
\[ V^{M+1} : F^\phi_2 \to F^\phi_q \] is bounded if \( x_M \geq q > F(x_M) = x_{M+1} \), and with this the claim is proven.

Now let \( M \) be an invariant subspace for \( V : F^\phi_p \to F^\phi_p \). Then let \( M_1, M_2 \in \mathbb{N} \) be such that \( V^{M_1} : F^\phi_2 \to F^\phi_p \) and \( V^{M_2} : F^\phi_p \to F^\phi_2 \) are bounded. Then \( V^{M_2} M \subset F^\phi_2 \) and \( V^{M_2} M \) is an invariant subspace for \( V \) on \( F^\phi_2 \), so it is of the form
\[ V^{M_2} M = A^2_N. \]

Now let \( f \in A^2_N \). Then there exists a sequence \(( f_n ) \subset M\) with
\[ V^{M_2} f_n \to f \text{ in } F^\phi_2. \]

But \( V^{M_1} : F^\phi_2 \to F^\phi_p \) is bounded, hence
\[ V^{M_1+M_2} f_n \to V^{M_1} f \text{ in } F^\phi_p, \]
which gives \( V^{M_1} f \in M \), since \( V^{M_1+M_2} f_n \in M \). Put \( f = z^N \) to deduce \( z^{N+M_1} \in M \) and by applying \( V \) indefinitely to \( z^{N+M_1} \) we get
\[ A^p_{N+M_1} \subseteq M. \]

If \( M \neq A^p_{N+M_1} \), let \( N_1 \) be the smallest nonnegative integer such that there exists \( f \in M \) with \( f^{(N_1)}(0) \neq 0 \) (clearly \( 0 \leq N_1 < N + M_1 \)). But, for this particular \( f \), we have \( V^{N+M_1-M_1-1} f \in M \), which in view of the above inclusion implies \( z^{N+M_1-M_1-1} \in M \), and therefore \( A^p_{N+M_1-M_1-1} \subseteq M \). We repeat this procedure until we obtain \( A^p_{N_1} \subseteq M \), and then the choice of \( N_1 \) forces \( M = A^p_{N_1} \), so that the proof is done.

**Corollary 30.** For \( \phi(x) = |x|^\alpha \) with \( \alpha > 2 \), the proper invariant subspaces of \( V \) on \( F^\phi_p \) are precisely the spaces
\[ \{ f \in F^\phi_p : f^{(k)}(0) = 0 \text{ for } 0 \leq k \leq N - 1 \} = \text{Span}\{ z^k : k \geq N \} F^\phi_p. \]

**Proof.** By Theorem 29 it suffices to show that the sequence
\[ \omega_n = \frac{\|z^{n+1}\|_{F^\phi_2}}{(n+1)\|z^n\|_{F^\phi_2}}, \quad n \geq 0 \]
is eventually decreasing to zero. By Stirling’s formula we deduce that
\[ \omega_n^2 = 2^{-2/\alpha} \frac{\Gamma(\frac{\alpha}{2}(n + 2))}{(n + 1)^2 \Gamma(\frac{\alpha}{2}(n + 1))} \asymp n^{2/\alpha - 2}, \]
and hence $\omega_n \to 0$ as $n \to \infty$. From the proof of Proposition 7 in [7] it follows that the function

$$f(x) = \frac{\Gamma(x + 2/\alpha)}{x^2 \Gamma(x)}$$

is eventually decreasing for $\alpha > 1$. Thus $\{\omega_n\}_{n=0}^{\infty}$ is eventually decreasing to zero. \qed

9. Remarks on the Bergman space case

In this section, we would like to illustrate that some of the methods we employed in the context of Fock spaces provide additional insight into the above mentioned results from [19, 21].

Given a positive radial weight $w$, the distortion function is defined as follows (see [19, 21])

$$\psi_w(r) = \frac{1}{w(r)} \int_r^1 w(u) \, du, \quad 0 \leq r < 1.$$  

Analogues of Theorem 3 and Corollary 11 for the setting of Bergman spaces with rapidly decreasing weights were obtained in [19, 21]. In particular, the following holds.

**Theorem C.** Suppose that $w$ is a radial differentiable weight, and there is $L > 0$ such that

$$\sup_{0 < r < 1} \frac{w'(r)}{w(r)^2} \int_r^1 w(x) \, dx \leq L,$$  

then for each $p \in (0, \infty)$ and $g$ analytic on $\mathbb{D}$

$$\int_{\mathbb{D}} |g(z)|^p w(z) \, dm(z) \asymp |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p \psi_w(z)^p w(z) \, dm(z).$$

Let

$$\phi(r) = -\log w(r), \quad 0 \leq r < 1.$$  

The following disc analogue of Lemmas 17, 18 and 20 can be easily proven.

**Lemma 31.** Assume $\phi : [0, 1) \to \mathbb{R}$ is twice continuously differentiable and there exists $r_0 \in [0, 1)$ such that $\phi'(r) \neq 0$ for $1 > r > r_0$. Suppose

$$\lim_{r \to 1} \frac{e^{-\phi(r)}}{\phi'(r)} = 0$$

$$\liminf_{r \to 1} \frac{\phi''(r)}{\phi'(r)^2} > -1$$

$$\limsup_{r \to 1} \frac{\phi''(r)}{\phi'(r)^2} < \infty.$$  

Then (9.1) holds and there exists $r_1 \in [0, 1)$ such that

$$\psi_w(r) \asymp \frac{1}{\phi'(r)} \quad \text{for} \quad r \in [r_1, 1)$$
Proof. By hypotheses there is $\alpha > -1$ and $r_2 \geq r_0$ such that $\frac{\phi''(r)}{\phi'(r)^2} \geq \alpha$ on $[r_2, 1)$. So, an integration by parts on $(r_2, r) \subset (r_2, 1)$, gives

$$
\int_{r_2}^{r} e^{-\phi(s)} \, ds = \int_{r_2}^{r} \frac{-\phi'(s)e^{-\phi(s)}}{-\phi'(s)} \, ds - \frac{e^{-\phi(r_2)}}{\phi'(r_2)} \int_{r_2}^{r} \frac{\phi''(s)}{\phi'(s)^2} e^{-\phi(s)} \, ds
$$

that is

$$
\int_{r_2}^{r} e^{-\phi(s)} \, ds \leq \frac{1}{\alpha + 1} \left( \frac{e^{-\phi(r)}}{\phi'(r)} + \frac{e^{-\phi(r_2)}}{\phi'(r_2)} \right)
$$

so taking limits as $r \to 1^-$, we deduce that $\int_{1}^{1} e^{-\phi(s)} \, ds < \infty$. Next, arguing as in the proof Lemma 17, (9.2) follows. These calculations also give (9.1). This finishes the proof. \hfill \Box

It is worth noticing that there are weights $\omega$ satisfying (9.1) but such that (9.2) does not hold. The weight $\omega(r) = e^{-(1-r)^\alpha}$, $1 > \alpha > 0$, gives a concrete example.

Lemma 32. Assume $\phi : [0, \infty) \to \mathbb{R}^+$ is a twice continuously differentiable function such that $\Delta \phi > 0$, $(\Delta \phi(z))^{-1/2} \propto \tau(z)$, where $\tau(z)$ is a radial positive function that decreases to zero as $|z| \to 1^-$ and $\lim_{r \to 1^-} \tau'(r) = 0$. Then

\begin{enumerate}
\item $\lim_{r \to 1^-} (1-r)\phi'(r) = \infty$.
\item $\lim_{r \to 1^-} \tau(r)\phi'(r) = \infty$, or, equivalently, $\lim_{r \to 1^-} \frac{\phi''(r)}{\phi'(r)^2} = 0$.
\item $\psi_\omega(r) \propto \frac{1}{\phi'(r)^2} + 1$, for $r \in [0, 1)$.
\item There exists $r_0 \in [0, 1)$ such that for all $a \in \mathbb{D}$ with $1 > |a| > r_0$, and any $\delta > 0$ small enough we have $\phi'(a) \propto \phi'(z)$, $z \in D(a, \delta \tau(a))$.
\end{enumerate}

Proof. Since by L'Hospital's rule

$$
\lim_{r \to 1^-} \frac{1-r}{\tau(r)} = \lim_{r \to 1^-} \frac{-1}{\tau'(r)} = +\infty
$$

bearing in mind (3.16) and using again L'Hospital's rule,

$$
\lim_{r \to 1^-} r(1-r)\phi'(r) = \lim_{r \to 1^-} \frac{\int_{0}^{r} s\phi'(s) \, ds}{(1-r)^{1-}} \geq C \lim_{r \to 1^-} \frac{r(1-r)^2}{\tau'(r)} = +\infty,
$$

which gives (a).

Arguing as in the proof of Lemma 18 we obtain $\lim_{r \to 1^-} \tau(r)\phi'(r) = +\infty$. By (a) and relation (3.16) this last fact is equivalent to $\lim_{r \to 1^-} \frac{\phi''(r)}{\phi'(r)^2} = 0$.

Taking into account $(a) - (b)$ it is clear that the hypotheses in Lemma 31 are satisfied, and hence $\psi_\omega(r) \propto \frac{1}{\phi'(r)^2}$ for $r \geq r_0$. Since $\phi' \geq 0$ and $\lim_{r \to 1^-} \phi'(r) = \infty$, we obtain (c).
Part (d) can be proved following the steps in the proof of Lemma 20. □

As a byproduct of the previous lemmas, we obtain the following improvements:

- In view of (9.2) and [19, Theorem B] the more transparent Littlewood-Paley formula

\[(9.3) \quad ||f||_{A^p}^p \asymp |f(0)|^p + \int_D \frac{|f'(z)|^p}{(1 + \phi'(z))^p} \omega(z)dA(z)\]

can be obtained for Bergman spaces with weights \(\omega = e^{-\phi}\) belonging to the class \(I\) considered in [19].

- Part (d) of Lemma 32 shows that the hypothesis (6) on the distortion function in [19, Theorem 2] can be omitted. Going further, this observation allows us to extend the description of the boundedness and compactness of \(T_g\) to a wider class of weights. Especially, we admit for a considerably faster decay, including triple exponential weights of the form

\[\omega(z) = \exp(-e^{e^{-|z|}}).\]

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