On Mass Spectrum in SQCD, and Problems with the Seiberg Duality. 
Another Scenario.

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Abstract

$\mathcal{N} = 1$ SQCD with $SU(N_c)$ colors and $N_F$ flavors of light quarks is considered within the dynamical scenario which assumes that quarks can be in two different phases only. These are: a) either the HQ (heavy quark) phase where they are confined, b) or they are higgsed, at the appropriate values of parameters of the Lagrangian.

The mass spectra of this (direct) theory and its Seiberg’s dual are obtained and compared, for quarks of equal or unequal masses. It is shown that in all cases when there is the additional small parameter at hand (it is $0 < (3N_c - N_F)/N_F \ll 1$ for the direct theory, or its analog $0 < (2N_F - 3N_c)/N_F \ll 1$ for the dual one), the mass spectra of the direct and dual theories are parametrically different.
1 Introduction.

The dynamics of 4-dimensional strongly coupled non-abelian gauge theories is complicated. It is well known that supersymmetry (SUSY) helps greatly to simplify some things, in comparison with the ordinary (i.e. non-SUSY) theories. Besides, it is widely believed that SUSY is relevant to the real world. So, in any case, it is of great interest to study the dynamics of the nearest SUSY-relative of the ordinary QCD, i.e. $\mathcal{N} = 1$ SQCD. But even with $\mathcal{N} = 1$ SQCD, there is no, at present, commonly accepted clear physical picture of even main non-trivial features of its dynamics.

It seems, the best what has been done before, was the Seiberg proposal [1] of the dual theory, which is weakly coupled when the direct one is strongly coupled, and vice versa (see e.g. the reviews [2][3] for $\mathcal{N} = 1$ SQCD, and [4][5] for Seiberg’s dual). The Seiberg duality passed some non-trivial checks (the ’t Hooft triangles and the behavior in the conformal regime), but up to now, unfortunately, no proof has been given whether the direct and dual theories are indeed equivalent (or not). The reason is that such a proof needs real understanding of the dynamics of both theories.

A definite dynamical scenario for $\mathcal{N} = 1$ SQCD has been proposed recently in [6]. The main idea of this scenario (#1) was that (when $m_Q = m(\mu = \Lambda_Q) \ll \Lambda_Q$ and $\mathcal{M}_{ch}^2 = \langle \overline{Q} Q(\mu = \Lambda_Q) \rangle \ll \Lambda_Q^2$) the quarks are not higgsed, but form in a vacuum a coherent condensate of colorless chiral pairs $\overline{Q} Q$ (the diquark condensate phase). As a result, they acquire the large dynamical constituent mass $\mu_C = \mathcal{M}_{ch}$, and there appear the light pseudo-Goldstone mesons $\pi_i$ (“pions”), with their masses $\mu_\pi \ll \mu_C$. The mass spectra of the direct and dual theories were obtained in this scenario #1, and they appeared to be quite different.

The purpose of this paper is to consider another dynamical scenario (#2), in which it is assumed that quarks can be in two different phases only. i) Either they are in the HQ (heavy quark) phase and so confined. Or (instead of forming the diquark condensate), ii) they are higgsed at $\mu = \mu_{gl} \ll \Lambda_Q$, at the appropriate values of parameters of the theory.

The direct and dual theories for quarks of equal masses are considered in sections 2-4. Other sections deal with quarks of unequal masses, when there are $N_l$ lighter flavors with masses $m_l$, and $N_h = N_F - N_l$ heavier flavors with masses $m_l < m_h \ll \Lambda_Q$. The mass spectra of both the direct and dual theories are described in sections 2-5. It is shown that in all cases when there is the additional small parameter at hand (it is $0 < b_o/N_F = (3N_c - N_F)/N_F \ll 1$ for the direct theory, and its analog $0 < b_o/N_F = (2N_F - 3N_c)/N_F \ll 1$ for the dual one), this allows to trace the parametrical differences in the mass spectra of the direct and dual theories. The sections 6-8 deal with the direct theory only, in some special regimes of interest. Finally, some conclusions are presented in the section 9.

2 Direct theory. Equal quark masses. $0 < b_o/N_F \ll 1$

The Lagrangian of the direct theory at scales $\mu > \Lambda_Q$ has the form:

$$L = \left[ \text{Tr} \left( Q^\dagger e^V Q + \overline{Q}^\dagger e^{-V} \overline{Q} \right) \right]_D + \left[ -\frac{2\pi}{\alpha(\mu)} S + \text{Tr} \left( m_Q(\mu) \overline{Q} Q \right) + \text{h.c.} \right]_F, \quad S = W_o^2/32\pi^2.$$
Here: $\alpha(\mu)$ is the running gauge coupling (with its scale parameter $\Lambda_Q$ independent of quark masses), $W_\alpha$ is the gluon field strength, $m_Q(\mu) \ll \Lambda_Q$ is the running current quark mass.

The theory is in the conformal regime at scales $\mu_H < \mu < \Lambda_Q$, where $\mu_H$ is the highest physical mass, and in this case it is the quark pole mass $m_Q^\text{pole}$:

$$\frac{m_Q^\text{pole}}{\Lambda_Q} = \frac{m_Q(\mu = m_Q^\text{pole})}{\Lambda_Q} = \frac{m_Q}{\Lambda_Q} \left( \frac{\Lambda_Q}{m_Q^\text{pole}} \right)^{\gamma_Q} = \left( \frac{m_Q}{\Lambda_Q} \right)^{\frac{1}{1+\gamma_Q}} ,$$

$$m_Q \equiv m_Q(\mu = \Lambda_Q) , \quad \frac{1}{1+\gamma_Q} = \frac{N_F}{3N_c} , \quad \frac{m_Q^\text{pole}}{\Lambda_Q} = \left( \frac{m_Q}{\Lambda_Q} \right)^{N_F/3N_c} , \quad (2.1)$$

where $\gamma_Q$ is the quark anomalous dimension [8]. After integrating out all quarks as heavy ones, there remains the pure YM-theory with $N_c$ colors at lower energies. Proceeding as in [6], its scale parameter $\Lambda_{YM}$ can be determined from matching couplings of the higher and lower energy theories. One obtains:

$$3N_c \ln \left( \frac{m_Q^\text{pole}}{\Lambda_{YM}} \right) \simeq \frac{2\pi}{\alpha^*} \simeq N_c \frac{N_F}{b_o} \rightarrow \Lambda_{YM} = m_Q^\text{pole} \exp \left( -\frac{N_F}{3b_o} \right) \ll m_Q^\text{pole} . \quad (2.2)$$

Therefore, there are two parametrically different scales in the mass spectrum of the direct theory in this case. There is a large number of colorless flavored hadrons made of (weakly confined, i.e. with the string tension $\sigma^{1/2} \sim \Lambda_{YM} \ll m_Q^\text{pole}$) non-relativistic heavy quarks $Q, \bar{Q}$, with masses $m_Q^\text{pole} \gg \Lambda_{YM}$, and a large number of gluonia with the mass scale $\Lambda_{YM} = m_Q^\text{pole} \exp(-N_F/3b_o) \ll m_Q^\text{pole}$.

To check the self-consistency, let us estimate a scale of the gluons masses due to a possible quark higgsing. The quark chiral condensate $\mathcal{M}_{ch}^2$ at $\mu = \Lambda_Q$ is given by [9]:

$$\frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2} \equiv \frac{1}{\Lambda_Q^2} \langle 0 | \bar{Q}Q(\mu = \Lambda_Q) | 0 \rangle = \langle S \rangle \equiv \Lambda_{YM}^2 = \left( \frac{m_Q}{\Lambda_Q} \right)^{\frac{N_F-N_c}{N_c}} \exp \left( -\frac{N_F}{b_o} \right) . \quad (2.3)$$

If the gluons were acquiring masses $\mu_{gl} > m_Q^\text{pole}$ due to higgsing of quarks, then the conformal RG-evolution will be stopped at $\mu = \mu_{gl}$. So, $\mu_{gl}$ can be estimated from:

$$\mu_{gl}^2 \sim a(\mu = \mu_{gl}) \langle 0 | \bar{Q}Q(\mu = \mu_{gl}) | 0 \rangle , \quad \mu_{gl}^2 \sim \mathcal{M}_{ch}^2 \left( \frac{\mu_{gl}}{\Lambda_Q} \right)^{\gamma_Q} , \quad a(\mu) \equiv \frac{N_c \alpha(\mu)}{2\pi} ,$$

$$\frac{\mu_{gl}}{\Lambda_Q} \sim \left( \frac{\mathcal{M}_{ch}^2}{\Lambda_Q^2} \right)^{\frac{1}{1-\gamma_Q}} \sim \frac{\Lambda_{YM}}{\Lambda_Q} \exp \left( -\frac{N_c}{2b_o} \right) \ll \frac{\Lambda_{YM}}{\Lambda_Q} \ll \frac{m_Q^\text{pole}}{\Lambda_Q} . \quad (2.4)$$

Therefore, the scale of possible gluon masses, $\mu = \mu_{gl}$, is parametrically smaller than the quark pole mass, $\mu_{gl} \ll m_Q^\text{pole}$, and the picture of $Q, \bar{Q}$ being in the HQ (heavy quark) -phase is self-consistent.

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1 At $b_o/N_F \ll 1$ the coupling $a^* = (N_c \alpha^*/2\pi)$ is weak, $\gamma_Q = (b_o/N_F) \simeq (1 - 1/N_c^2) a^* \simeq a^* \ll 1$.

Here and in what follows, we trace only the leading (i.e. exponential) dependence on the small parameter $b_o/N_F \ll 1$. Besides, it is always implied that even when $b_o/N_F$ or $b_o/N_F$ are $\ll 1$, these don’t compete in any way with the main small parameter $m_Q/\Lambda_Q$. 

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3 Dual theory. Equal quark masses. $0 < b_o/N_F \ll 1$

The Lagrangian of the dual theory (at the scale $\mu \sim \Lambda_Q$) is taken in the form [1]:

$$\mathcal{L} = \left[ \text{Tr} \left( q^\dagger e^{-V} q + \bar{q}^\dagger e^{-V} \bar{q} \right) + \frac{1}{(\mu_2)^2} \text{Tr} \left( M^\dagger M \right) \right]_D +$$

$$+ \left[ -\frac{2\pi}{\alpha(\mu)} \bar{s} + \frac{1}{\mu_1} \text{Tr} \left( \bar{q} M q \right) + \text{Tr} \left( \bar{m}_Q(\mu) M \right) + \text{h.c.} \right]_F, \quad \bar{s} = \bar{w}_\alpha^2/32\pi^2.$$

Here: $\alpha(\mu)$ is the dual running gauge coupling (with its scale parameter $\Lambda_q$), $\bar{w}_\alpha$ is the dual gluon field strength, and $M^2_{ch}$ are $N_F^2$ of mion fields.

The scale parameter $\Lambda_q$ of the dual theory can be taken as $|\Lambda_q| \sim \Lambda_Q$, so that both direct and dual theories entered the conformal regime simultaneously at $\mu \sim \Lambda_Q$, and $\mu_2 \lesssim \mu_1$ (the dual theory is considered as UV-free in the conformal window). Besides, the normalization of $\mathcal{O}(\mu = \Lambda_Q)$ and $M(\mu = \Lambda_Q)$ can always be matched, $M_0 \equiv \langle 0\left| \mathcal{O}(\mu = \Lambda_Q) \right| 0 \rangle = M^2_{ch}$. So, because the gluino condensates are also matched, then [9]: $\langle \mathcal{S} \rangle \equiv \Lambda^3_{YM} = m_Q M^2_{ch} = |\langle s \rangle| \equiv |\overline{\Lambda}_{YM}|^3 = \overline{m}_Q(\mu = \Lambda_Q) M_0$, and $\overline{m}_Q(\mu = \Lambda_Q) = m_Q$. Therefore, there remains only one free parameter $\mu_1 \equiv Z_q \Lambda_Q$ in the dual theory. It will be determined below from the explicit matching of gluino condensates in the direct and dual theories.

The dual theory is also in the conformal regime at $b_o/N_F \ll 1$ and $\mu_H < \mu < \Lambda_Q$, where $\mu_H$ is the corresponding highest physical mass.

Assuming that the dual quarks $\bar{q}, q$ are in the HQ-phase and so $\mu_H = \mu^\text{pole}_q$, let us find their pole mass ($N_c = N_F - N_c$, $b_o = 3N_c - N_F$, $\gamma_q = b_o/N_F$):

$$\frac{\mu^\text{pole}_q}{\Lambda_Q} = \mu_q \left( \frac{\Lambda_Q}{\mu^\text{pole}_q} \right)^{\gamma_q} = \left( \frac{M^2_{ch}}{Z_q \Lambda_Q^2} \right)^{N_F/3N_c}, \quad \mu_q \equiv \frac{M_0}{\mu_1} = \frac{M^2_{ch}}{Z_q \Lambda_Q^2}. \quad (3.2)$$

Now, let us integrate out all dual quarks as heavy ones and determine the scale factor $\overline{\Lambda}_{YM}$ of the remained dual YM-theory (the dual coupling is $\alpha^* = \overline{N}_c/\alpha^* \sim 1$):

$$3N_c \ln \left( \frac{\mu^\text{pole}_q}{-\overline{\Lambda}_{YM}} \right) \sim \frac{\overline{N}_c}{\alpha^*} \sim N_c \quad \rightarrow \quad -\overline{\Lambda}_{YM} \sim \mu^\text{pole}_q. \quad (3.3)$$

Equating the gluino condensates (i.e. $(\overline{\Lambda}_{YM})^3 = \Lambda^3_{YM}$) of the direct and dual theories, one obtains:

$$-\overline{\Lambda}_{YM} = \Lambda_{YM} \quad \rightarrow \quad Z_q = \exp \left( \frac{-N_c}{b_o} \right) \ll 1. \quad (3.4)$$

Let us find now the mass $\mu_M$ of the dual mesons $M$ (mions). It can be found from their effective Lagrangian, obtained by integrating out all dual quarks and gluons. Proceeding as

\textsuperscript{2} The phase of $\Lambda_q$ has to be chosen appropriately [4]. This is always implied and will not be shown explicitly in what follows.
in [6][7] and integrating out all quarks as heavy ones and all gluons through the Veneziano-Yankielowicz (VY) procedure [10], one obtains the Lagrangian of mions (the fields $M$ are normalized at $\mu = \Lambda_Q$ in (3.5) and (3.6)):

$$L_M = \left[ \frac{z_M(\Lambda_Q, \mu^\text{pole}_q)}{\mu^2} \text{Tr} M^1 M \right]_D + \left[ -\n_c \left( \frac{\text{det} M}{Z^N_F \Lambda_Q^{1/N_c}} \right)^{1/N_c} + \text{Tr} m_Q M \right]_F , \quad (3.5)$$

$$z_M(\Lambda_Q, \mu^\text{pole}_q) = \left( \frac{\Lambda_Q}{\mu^\text{pole}_q} \right)^{2\gamma_q} = \left( \frac{\Lambda_Q}{\Lambda_{YM}} \right)^{2\beta_0/N_F} \gg 1 , \quad m_Q \langle M \rangle = m_Q M^2 = \Lambda_{YM}^3 .$$

Besides, in a number of cases it is also convenient to write the Lagrangian of mions in the form:

$$L_M = \left[ \frac{z_M(\Lambda_Q, \mu^\text{pole}_q)}{\mu^2} \text{Tr} M^1 M \right]_D + \left[ -\n_c \Lambda_{YM}^3 \left( \frac{M}{\langle M \rangle} \right)^{1/N_c} + \text{Tr} m_Q M \right]_F . \quad (3.6)$$

It follows from (3.5):

$$\mu_M = m_Q \left( \frac{\mu^2}{Z M^2 M_0} \right) \sim m_Q \frac{Z^2 \Lambda_{YM}^2}{z_M M^2 M_0} \sim \Lambda_{YM} . \quad (3.7)$$

To check that the above assumption that dual quarks $q, \bar{q}$ are in the HQ phase (i.e. not higgsed) is not self-contradictory, let us estimate the possible mass $\mu_{gl}$ of dual gluons. The condensate of dual quarks at $\mu = \Lambda_Q$ is [9]:

$$\mu^2 \equiv \mu^2_C(\mu = \Lambda_Q) = |\langle \bar{q}q(\mu = \Lambda_Q)\rangle| = \mu_1 m_Q = Z_q \Lambda_Q m_Q , \quad \mu^2_C(\mu < \Lambda_Q) = \mu^2_0 \left( \frac{\mu}{\Lambda_Q} \right)^{\gamma_q} . \quad (3.8)$$

Therefore, the possible mass of dual gluons can be estimated as:

$$\mu_{gl}^2 \sim \left( \alpha^* \sim 1 \right) Z_q \Lambda_Q m_Q \left( \frac{\mu_{gl}}{\Lambda_Q} \right)^{\beta_0/N_F} \rightarrow \mu_{gl} \sim \Lambda_{YM} . \quad (3.9)$$

It is seen that it is, at least, not parametrically higher than the pole mass of the dual quark, $\mu_{gl} \sim \mu^\text{pole}_q$. So, it may be that it is really even somewhat smaller, $\mu_{gl} = \mu^\text{pole}_q/(\text{several})$ (because we have no real control over possible non-parametrical factors $O(1)$), and the dual quarks are indeed in the HQ-phase. Here and below in similar situations we will assume that this is the case. 3

3 The opposite case, i.e. that quarks are really higgsed, will result in a genuine spontaneous breaking of the flavor symmetry (and complete breaking of the gauge group), and appearance of a large number of exactly massless Nambu-Goldstone particles. In what follows, it will be assumed that this variant is not realized (at least, in cases considered in this paper). Besides, if one believes that there are only $N_c$ isolated supersymmetric vacua in this theory, then either this opposite case is excluded, or this will exclude the whole scenario #2.
On the whole, there is only one scale $\sim \Lambda_{YM}$ in the mass spectrum of the dual theory in this case. The masses of the dual quarks $q, \overline{q}$, mions $M$ and dual gluonia are all $\sim \Lambda_{YM}$.

Comparing the mass spectra of the direct and dual theories it is seen that they are parametrically different. The direct quarks have large pole masses $m_Q^\text{pole}/\Lambda_{YM} \sim \exp\{N_c/b_o\} \gg 1$ and are parametrically weakly interacting and non-relativistic inside hadrons (and weakly confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll m_Q^\text{pole}$), so that the mass spectrum of low-lying mesons is Coulomb-like, with small mass differences between nearest hadrons, $\Delta \mu_H/\mu_H = O(b_o^2/N_F^2) \ll 1$. All gluonia have masses $\sim \Lambda_{YM}$, i.e. parametrically smaller than the masses of flavored hadrons. At the same time, all hadron masses in the dual theory are of the same scale $\sim \Lambda_{YM}$, and all interaction couplings are $O(1)$, so that there is no reason for parametrically small mass differences between hadrons. We conclude that the direct and dual theories are not equivalent.

4 Direct and dual theories. Equal quark masses.  

$0 < b_o/N_F \ll 1$.

This case is considered analogously to those in sections 2 and 3, so that the emphasis will be only on differences with sections 2 and 3. The main difference is that at $\mu/\Lambda_Q$ the direct coupling is not small now, $a^* \simeq 1$, while both the gauge and Yukawa couplings of the dual theory are parametrically small, $a^* \sim a^*_f \sim b_o/N_F \ll 1$.

The pole mass of direct quarks, $m_Q^\text{pole}$, is the same as in (2.1). But $\Lambda_{YM}$ is now parametrically the same: $\Lambda_{YM} \sim m_Q^\text{pole}$ (as well as the scale of the gluon mass due to possible quark higgsing, $\mu_{gl} \sim m_Q^\text{pole} \sim \Lambda_{YM}$). Assuming that quarks are not higgsed, but are in the HQ-phase (see the footnote 3), one obtains that there is only one mass scale $\mu_{YM} \sim \Lambda_{YM} \sim \Lambda_Q(m_Q/\Lambda_Q)^{N_F/3N_c}$ in the mass spectrum of hadrons.

Let us consider now the (weakly coupled) dual theory. It is entered already the conformal regime at $\mu < \Lambda_Q$, so that the pole mass of dual quarks is now:

$$\frac{\mu_q^\text{pole}}{\Lambda_Q} = \left(\frac{\mu_q}{\Lambda_Q}\right)^{N_F/3N_c}, \quad \mu_q \equiv \mu_q(\mu = \Lambda_Q) = \frac{M^2_{\text{ch}}}{Z_q\Lambda_Q^2}, \quad \frac{M^2_{\text{ch}}}{\Lambda_Q^2} = \frac{3\Lambda_{YM}^2}{m_Q^\text{pole} \Lambda_Q} = \left(\frac{m_Q}{\Lambda_Q}\right)^{N_c/N_c}$$  (4.1)

After integrating out all dual quarks as heavy ones at $\mu = \mu_q^\text{pole}$, there remains the dual YM-theory (together with the mions $M$). Its scale parameter $\bar{\Lambda} = \langle \Lambda_L(M) \rangle$ can be found from [11]:

$$3N_c \ln \left(\frac{\mu_q^\text{pole}}{-\bar{\Lambda}}\right) \simeq 2\pi \frac{2\pi}{\alpha} \simeq N_F \frac{N_c - 1}{2N_F + N_c} \simeq \frac{3N_c^2}{7b_o}. \quad (4.2)$$

Now, from matching the gluino condensates in the direct and dual theories, one obtains:

$$|\bar{\Lambda}| = \Lambda_{YM} \quad \rightarrow \quad Z_q \sim \exp\left(-\frac{N_c}{7b_o}\right) \ll 1, \quad \mu_q^\text{pole} \sim \Lambda_{YM} \exp\left(-\frac{N_c}{7b_o}\right) \gg \Lambda_{YM}. \quad (4.3)$$

6
On the whole, the expressions for $\Lambda_{YM}$ and $Z_q$ can be written in a general case as:

$$
\frac{\Lambda_{YM}}{\Lambda_Q} \sim \exp \left\{ - \frac{N_c}{|b_o|} \left( \frac{\det m_Q}{\Lambda_Q^{2NF}} \right)^{1/3N_c} \right\}, \quad Z_q \sim \exp \left\{ - \left( \frac{N_c}{|b_o|} + \frac{N_c}{7|b_o|} \right) \right\}.
$$

(4.4)

Here, the sign $\sim$ implies the exponential accuracy in dependence on large parameters $N_c/|b_o| \gg 1$ or $\overline{N}_c/|\overline{b}_o| \gg 1$, i.e. the pre-exponential factors like powers of $\sim (|b_o|/N_F)$ or $\sim (|\overline{b}_o|/N_F)$ are ignored in comparison with the exponential ones. So, if $N_c/|b_o|$ or $\overline{N}_c/|\overline{b}_o|$ are $O(1)$, the dependence on these has to be omitted from (4.4). For our purposes this exponential accuracy in (4.4) will be sufficient. Finally, (4.4) is written in such a form that it is applicable at both large positive or negative $N_c/|b_o|$ or $\overline{N}_c/|\overline{b}_o|$.

Therefore, similarly to the case of the weakly coupled direct theory in section 2, the dual quark pole mass $\mu^\text{pole}_q$ is now parametrically larger than $\Lambda_{YM}$.

Then, proceeding as in [6][7] and integrating out the dual gluons, one obtains the Lagrangian of mions $M$, which can be written in the form (3.6). So, the mion masses are now:

$$
\mu_M = m_Q \left( \frac{\mu^2_g}{z_M^2 M_{ch}^2} \right) \sim m_Q \left( \frac{Z_q^2 \Lambda^2_Q}{z_M^2 M_{ch}^2} \right), \quad z_M = z_M(\Lambda_Q, \mu^\text{pole}_q) = \left( \frac{\Lambda_Q}{\mu^\text{pole}_q} \right)^{2\overline{b}_o/N_F} \gg 1.
$$

(4.5)

One obtains from (4.5):

$$
\mu_M \sim Z_q^2 \Lambda_{YM} = \Lambda_{YM} \exp \left( \frac{-2N_c}{7\overline{b}_o} \right) \ll \Lambda_{YM}.
$$

(4.6)

So, the mion masses are parametrically smaller than $\Lambda_{YM}$.

To check that there are no self-contradictions, it remains to estimate the gluon masses due to a possible higgsing of dual quarks. One has:

$$
\frac{\mu^2_g}{\Lambda^2_Q} \sim Z_q m_Q \left( \frac{\mu^2_g}{\Lambda_Q} \right) \overline{b}_o/N_F \implies \overline{\mu}_g \sim \Lambda_{YM} \exp \left( \frac{-2N_c}{14\overline{b}_o} \right) \ll \Lambda_{YM} \ll \mu^\text{pole}_q.
$$

(4.7)

Therefore, there are three parametrically different mass scales in the dual theory in this case. a) There is a large number of flavored hadrons made of weakly coupled non-relativistic (and weakly confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mu^\text{pole}_q$) dual quarks with the pole masses $\mu^\text{pole}_q/\Lambda_{YM} = \exp(\overline{N}_c/7\overline{b}_o) \gg 1$. The mass spectrum of low lying mesons is Coulomb-like, with small mass differences $\Delta \mu_{H}/\mu_{H} \ll 1$. b) There is a large number of gluons with the mass scale $\sim \Lambda_{YM}$. c) There are $N_F^2$ lightest mions $M$ with parametrically smaller masses, $\mu_M/\Lambda_{YM} \sim \exp(-2\overline{N}_c/7\overline{b}_o) \ll 1$. At the same time, there is only one mass scale $\sim \Lambda_{YM}$ of all hadron masses in the direct theory. Clearly, the mass spectra of the direct and dual theories are parametrically different.
On the whole, it is seen that when there is the appropriate small parameter at hand (0 < b_o/N_F ≪ 1 when the direct theory is weakly coupled, or 0 < \bar{b}_o/N_F ≪ 1 when weakly coupled is the dual theory), the mass spectra of the direct and dual theories are parametrically different. So, there are no reasons for these mass spectra to become exactly the same when b_o/N_F or \bar{b}_o/N_F become O(1).

5 Direct and dual theories. Unequal quark masses.

0 < \bar{b}_o/N_F ≪ 1, \ N_c < N_l < 3N_c/2.

The standard consideration which "checks" that the duality works properly for quarks of unequal masses looks as follows [1][4]. Let us take, for instance, N_h quarks of the direct theory to be heavier than N_c < N_l = N_F - N_h of others. Then, after integrating out these h-flavored quarks as heavy ones, there remains at lower energies the direct theory with <l flavors. By duality, it is equivalent to the dual theory with N_c colors and N_l flavors. From the other side, the original theory is equivalent to the dual theory with \bar{N}_c = N_F - N_c colors and N_F flavors. In this dual theory the h-flavored dual quarks are assumed to be higgsed, so that at lower energies one remains with the dual theory with N_F - N_c - N_h = N_l - N_c colors and N_F - N_h = N_l of l-flavors. Therefore, all looks self-consistent.

However, let us consider this variant in more detail. Let us start with the left end of the conformal window, 0 < \bar{b}_o/N_F ≪ 1. At \mu < \Lambda_Q the dual theory entered already the conformal regime, and so both its gauge and Yukawa couplings are close to their parametrically small frozen values, \pi \sim a^*_f \sim \bar{b}_o/N_F ≪ 1. Let us take N_c < N_l < 3N_c/2 of direct quarks Q_l, \bar{Q}^l to have at \mu = \Lambda_Q smaller masses m_i, while other N_h = N_F - N_l quarks Q_h, \bar{Q}^h have larger masses m_h, r ≡ m_l/m_h < 1.

A. Direct theory. We start with the direct theory, because, in a sense, its mass spectrum is simpler to calculate. It is in the conformal regime at \mu < \Lambda_Q, and the highest physical mass scale \mu_H is equal here to the pole mass of heavier Q_h, \bar{Q}^h quarks:

\[
\frac{m_h^\text{pole}}{\Lambda_Q} = \frac{m_h}{\Lambda_Q} \left( \frac{\Lambda_Q}{m_h^\text{pole}} \right)^{\gamma_Q=(b_o/N_F)} = \left( \frac{m_h}{\Lambda_Q} \right)^{N_F/3N_c}.
\] (5.1)

After integrating these out, there remains the direct theory with N_c colors and N_c < N_l < 3N_c/2 of lighter Q_l, \bar{Q}^l quarks. 4 Its coupling at \mu = m_h^\text{pole} is ≃ 1, and so its scale parameter is: \Lambda'_Q \sim m_h^\text{pole}. The current masses of Q_l, \bar{Q}^l quarks at \mu = m_h^\text{pole} are:

\[
\hat{m}_l \equiv m_l(\mu = m_h^\text{pole}) = r m_h^\text{pole} ≪ \Lambda'_Q = m_h^\text{pole}.
\]

Now, how to deal further with this theory at lower energies? As was described in [6] (see section 7), there are two variants, "a" and "b". All hadron masses \mu_H are much smaller than \Lambda'_Q in the variant "a", \mu_H ≪ \Lambda'_Q, and the direct theory and its dual are not equivalent (see

4 The value of (3N_c - 2N_l)/N_l is considered as O(1) in this section, to simplify all formulae.
eqs. (7.2–7.4) in [6]). So, we don’t consider this variant ”a” in this section, as our purpose here is to check the duality in the most favorable for its validity variant. This is the variant ”b” : ”confinement without chiral symmetry breaking”. This implies (because all original direct degrees of freedom can’t ”dissolve in a pure air”) that due to strong non-perturbative confining effects, the direct $Q_l, \overline{Q}^l$ -quarks and gluons form a large number of heavy hadrons with masses $\mu_H \sim \Lambda'_Q$, made of these $Q_l, \overline{Q}^l$ quarks and gluons.

Instead, there appear new light composite particles (special solitons), with their masses parametrically smaller than $\Lambda'_Q$. These are the dual quarks $q^l, \overline{q}$, the dual gluons with $\overline{N}_c = N_l - N_c$ of dual colors, and the mions $M_l \equiv M_l^\dagger$. Their interactions at scales $\mu < \Lambda'_Q$ are described by the Seiberg dual Lagrangian. Their dual gauge coupling at $\mu \sim m_{h}^{\text{pole}}$ is $O(1)$, so that the scale parameter of this dual gauge coupling is $\sim \Lambda'_Q \sim m_{h}^{\text{pole}}$.

The dual Lagrangian (at the scale $\mu \sim \Lambda'_Q$, $(\mathcal{M}_c^l)^2 \equiv \langle \overline{Q}^l Q_l(\mu = \Lambda_Q) \rangle = \langle \lambda \lambda \rangle / m_l = \Lambda_{YM}^3 / m_l$) is:

$$\mathcal{L} = \left[ \text{Tr}_I \left( \hat{q}^l \partial^\dagger \hat{q} + \hat{q}^\dagger \partial \hat{q} \right) + \frac{1}{(\Lambda'_Q)^2} \text{Tr} \left( \hat{M}_l^\dagger \hat{M}_l \right) \right] +$$

$$\frac{2\pi}{\alpha(\mu)} \overline{s} + \frac{1}{N_c} \text{Tr}_I \left( \hat{q} \hat{M}_l \hat{q} \right) + m_l \text{Tr} \hat{M}_l \right] _F ; \quad \overline{s} = \overline{w}_a^2 / 32 \pi^2 , \quad \langle \overline{q} q \rangle = - \delta_l \delta_M \Lambda'_Q ,$$

$$m_l = m_l z^{-1}_Q , \quad \langle \hat{M}_l \rangle = (\mathcal{M}_c^l)^2 z Q , \quad z Q = \left( \frac{m_{h}^{\text{pole}}}{\Lambda_Q} \right) ^{\gamma_Q = (b_0 / N_c)} \quad = \left( \frac{m_h}{\Lambda_Q} \right) ^{b_0 / 3 N_c} \ll 1 . \quad (5.3)$$

This theory is in the HQ-phase [6] (see section 7). At lower scales $\mu_{q, l}^{\text{pole}} < \mu < \Lambda'_Q$ both its gauge and Yukawa couplings decrease logarithmically ( $\overline{b}_0 = 3 N_c' - N_l < 0$ ) and become $\ll 1$ at $\mu = \mu_{q, l}^{\text{pole}} \ll \Lambda_Q$, at $r \ll 1$. The pole mass $\mu_{q, l}^{\text{pole}}$ of $q^l, \overline{q}$ quarks is:

$$\mu_{q, l}^{\text{pole}} = \mu_{q, l} / z'_Q = \langle \hat{M}_l \rangle / z'_Q \Lambda'_Q = \left( r = \frac{m_l}{m_h} \right) \frac{N_l - N_c}{N_c} m_{h}^{\text{pole}} / z'_Q \ll m_{h}^{\text{pole}} , \quad (5.4)$$

where $z'_Q = z'_Q (\Lambda'_Q, \mu_{q, l}^{\text{pole}}) \ll 1$ is the logarithmic renormalization factor of dual $l$-quarks.

At scales $\mu < \mu_{q, l}^{\text{pole}}$ all quarks $q^l, \overline{q}$ can be integrated out as heavy ones, and there remains the dual YM theory with $\overline{N}_c = N_l - N_c$ colors and mions $\hat{M}_l$. The scale factor $\overline{\Lambda}_{YM} = \langle \overline{\Lambda}_L (\hat{M}_l) \rangle$ (with mions $\hat{M}_l$ sitting down on $\overline{\Lambda}_L (\hat{M}_l)$ ) of its gauge coupling is determined from

$$3 \overline{N}_c \ln \left( \frac{\mu_{q, l}^{\text{pole}}}{- \overline{\Lambda}_{YM}} \right) = \overline{b}_0 \ln \left( \frac{\mu_{q, l}^{\text{pole}}}{\Lambda_Q} \right) + N_l \ln \left( \frac{\mu_{q, l}^{\text{pole}}}{\mu_{q, l}} \right) \rightarrow$$

$$- \overline{\Lambda}_{YM} = - \langle \overline{\Lambda}_L (\hat{M}_l) \rangle = \Lambda_{YM} = \left( \Lambda_Q^b \Lambda_{}\Lambda_{\Lambda} \right) ^{1 / 3 N_c} . \quad (5.5)$$
Proceeding as in [6] and going through the VY-procedure, one obtains the lowest energy Lagrangian of mions $\hat{M}_i$:

$$L_M = \left[ \frac{z'_M}{(N_c^i)^2} \frac{\text{Tr}(\hat{M}_i^\dagger \hat{M}_i)}{D} \right] + \left[ - (N_l - N_e) \left( \frac{\det \hat{M}_i}{(N_q^i)^{3(N_{c} - N_l)}} \right)^{1/(N_l - N_e)} + \hat{m}_i \text{Tr} \hat{M}_i \right]$$

(5.6)

where $z'_M = z'_M(N^i_q, \mu_{q, i}^\text{pole}) \gg 1$ is the logarithmic renormalization factor of mions. From (5.6), the masses of mions $M_i$ are:

$$\mu(M_i) \sim \hat{m}_i \left( \frac{N_c^i}{z'_M(M_i)} \right)^{2(N_l - N_e) / N_c} m_h^\text{pole} / z_M^i, \quad r \ll 1, \quad (5.7)$$

$$\frac{\Lambda_{YM}}{\mu_q^\text{pole}} \sim z'_q r^\Delta \ll 1, \quad \frac{\mu(M_i)}{\Lambda_{YM}} \sim r^{2\Delta} / z'_M \ll 1, \quad 0 < \Delta = \frac{3N_e - 2N_l}{3N_c} < \frac{1}{3}. \quad$$

So, on the whole for 'A', when we started with the direct theory, integrated out the heaviest $Q_h, \overline{Q}^h$ quarks at $\mu = m_h^\text{pole}$, and dualized the remained theory of the direct $l$-flavors and direct gluons in the variant "b", the mass spectrum looks as follows.

1) There is a large number of heavy direct $h$-mesons $M_{\text{dir}}^i$, made of $Q_h, \overline{Q}^h$ quarks (and/or antiquarks and direct gluons) with masses $\sim m_h^\text{pole} \ll \Lambda_Q$.

1') In the variant "b" considered here, the dualization of $Q_l, \overline{Q}^l$ quarks and direct gluons leaves behind a large number of heavy direct $l$-mesons $M_{\text{dir}}^i$ with different spins, made of $Q_l, \overline{Q}^l$ quarks (and direct gluons), also with masses $\sim m_h^\text{pole}$. Besides, there will be also a large number of heavy direct hybrid mesons $M_{\text{dir}}^{h_j}$, baryons $B_h$ and gluonia, all also with masses $\sim m_h^\text{pole}$. All these heavy particles are strongly interacting, with couplings $O(1)$.

All other particles are parametrically lighter and originate from the dual quarks $q^l, \overline{q}^l$, the dual gluons and mions $M_i$.

2) There is a large number of $l$-flavored dual mesons, made of non-relativistic (and weakly confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mu_{q, i}^\text{pole}$) $q^l, \overline{q}^l$ - quarks, with their pole masses $\mu_{q, i}^\text{pole} \sim (r)^{(N_l - N_e) / N_c} m_h^\text{pole} / z_q^i \ll m_h^\text{pole}.

3) There is a large number of gluonia with masses $\sim \Lambda_{YM} \ll \mu_{q, i}^\text{pole}$.

4) The lightest are $N_f^2$ of scalar mions $M_l$ with masses $\mu(M_l) \sim (r)^{2\Delta} \Lambda_{YM} / z_M^i \ll \Lambda_{YM}$. \quad

B. Dual theory. Let us return now to the beginning of this section and start directly with the dual theory with $N_c$ dual colors, $N_F$ dual quarks $q$ and $\overline{q}$, and $N_F^2$ mions $M_i^\overline{q}$. At the scale $\mu < \Lambda_Q$ it is already entered the weak coupling conformal regime, i.e. its coupling $\alpha(\mu)$ is close to $\alpha^* = N_c \alpha / 2\pi \sim 3 \bar{N}_0 / 3N_c \ll 1$, see (4.2). Let us consider first the most favorable for the dual theory case when the parameter $r = m_l / m_h$ is taken to be already sufficiently small (see below). Then, in the scenario #2 considered in this paper, the highest physical mass scale $\mu_H$ in the dual theory is determined by masses of
dual gluons due to higgsing of $q_h$, $q^h$ quarks: $\mu_{gl,h}^2 = \overline{\mu}_{gl,h}^2 \sim \overline{\mu}_h |(q_h q^h(\mu = \overline{\mu}_{gl,h})|$. The mass spectrum of the dual theory in this phase can be obtained in a more or less standard way, proceeding similarly to [6][7]. For this reason, from now on we will skip everywhere below some intermediate relations in similar situations. The emphasis will be made on new elements which did not appear before.

1) The masses of $(2\overline{\mathcal{N}}_c N_h - N_h^2)$ massive dual gluons and their scalar superpartners are:

$$
\left(\overline{\mu}_{gl,h}\right)^2 \sim |(\overline{\mu}_h q^h)| \left(\overline{\mu}_{gl,h} \Lambda_Q\right)^{\gamma_q} \sim \mu_1 m_h \left(\overline{\mu}_{gl,h} \Lambda_Q\right)^{\gamma_q} \sim Z_q \Lambda_Q m_h \left(\overline{\mu}_{gl,h} \Lambda_Q\right)^{\overline{\nu}_0/N_F}, \quad (5.8)
$$

$$
\overline{\mu}_{gl,h} \Lambda_Q \sim \exp\{-\overline{\mathcal{N}}_c/14\overline{\nu}_0\} \left(\frac{m_h}{\Lambda_Q}\right)^{\overline{\nu}_0/3N_c} \ll \left(\frac{m_h}{\Lambda_Q}\right)^{\overline{\nu}_0/3N_c}, \quad Z_q \sim \exp\{-\overline{\mathcal{N}}_c/7\overline{\nu}_0\} \ll 1. \quad (5.9)
$$

2) $N_l N_h$ hybrid mions $M_{hl}$ and $N_l N_h$ nions $N_h$ (these are those dual $l$-quarks which have higgsed colors, their partners $M_{ll}$ and $N_{ll}$ are implied and are not shown explicitly) can be considered independently of other degrees of freedom, and their masses are determined mainly by their mutual interactions:

$$
L_{hybr} \simeq \left[ \hat{z}_M \mathrm{Tr} \left( \frac{M_{hl}^2 M_{hl}}{Z_q \Lambda_Q^2} \right) + \hat{z}_q \mathrm{Tr} \left( N_{hl}^t N_{hl} \right) \right] D + \left[ \left( Z_q \Lambda_Q m_h \right)^{1/2} \mathrm{Tr} \left( \frac{M_{hl} N_{hl}}{Z_q \Lambda_Q} \right) \right] F, \quad (5.10)
$$

where $\hat{z}_q$ and $\hat{z}_M$ are the perturbative renormalization factors of dual quarks and mions,

$$
\hat{z}_q = \hat{z}_q(\Lambda_Q, \overline{\mu}_{gl,h}) = \left(\frac{\overline{\mu}_{gl,h}}{\Lambda_Q}\right)^{\overline{\nu}_0/N_F} \sim \left(\frac{m_h}{\Lambda_Q}\right)^{\overline{\nu}_0/3N_c} \ll 1, \quad \hat{z}_M = \hat{z}_M(\Lambda_Q, \overline{\mu}_{gl,h}) = 1/\hat{z}_q^2 \gg 1.
$$

Therefore,

$$
\frac{\mu(M_{hl})}{\Lambda_Q} \sim \frac{\mu(N_{hl})}{\Lambda_Q} \sim \exp\{-\overline{\mathcal{N}}_c/14\overline{\nu}_0\} \left(\frac{m_h}{\Lambda_Q}\right)^{\overline{\nu}_0/3N_c} \sim \frac{\overline{\mu}_{gl,h}}{\Lambda_Q}. \quad (5.11)
$$

3) Because $q_h$ and $q^h$ - quarks are higgsed, there appear $N_h^2$ pseudo-Goldstone mesons $N_{hh}$ (nions). After integrating out the heavy gluons and their superpartners, the Lagrangian of remained degrees of freedom takes the form:

$$
L \simeq \left[ \hat{z}_M \mathrm{Tr} \left( \frac{M_{hh}^2 M_{hh} + M_{ll}^2 M_{ll}}{Z_q \Lambda_Q^2} \right) + \hat{z}_q 2 \mathrm{Tr} \sqrt{N_{hh}^t N_{hh}} + \hat{z}_q \mathrm{Tr} l \left( q^t e^{-\nu} q + \overline{q}^t e^{-\overline{\nu}} \overline{q} \right) \right] D
$$

$$
+ \left[ -\frac{2\pi}{\alpha'(\mu)} \overline{\mathcal{F}} + \mathrm{Tr} \left( \frac{M_{hh} N_{hh}}{Z_q \Lambda_Q} \right) + \mathrm{Tr} l \left( \overline{q} M_{ll} q \right) + \mathrm{Tr} \left( m_l M_{ll} + m_h M_{hh} \right) \right] F, \quad (5.12)
$$

where $\overline{\mathcal{F}}$ includes the field strengths of remained $SU(\overline{\mathcal{N}}_c)$ gluons, with $\overline{\mathcal{N}}_c = \overline{\mathcal{N}}_c - N_h = N_l - N_c$, and the nions $N_{hh}$ are ”sitting down” inside $\overline{\mathcal{F}}'(\mu)$. 

11
At lower scales $\mu < \mu_{gl,h}$, the mions $M_{hh}$ and mions $N_{hh}$ are frozen and don’t evolve, while the gauge coupling decreases logarithmically in the interval $\mu < \mu_{gl,h}$. The numerical value of the pole mass of $\piT$, $q^l$ - quarks is:

$$\frac{\mu_{pole}}{\Lambda_Q} = \langle M_\mu \rangle \frac{1}{Z_q^2 \Lambda_Q^2} \sim \exp \left[ \frac{N_c}{T_{D_0}} \left( \frac{m_h}{\Lambda_Q} \right)^{N_F/3N_c} \right] / z''_q,$$  \hspace{1cm} (5.13)

where $z''_q \ll 1$ is the logarithmic renormalization factor of $q^l$, $\piT$ quarks.

Integrating the $\piT$, $q^l$-quarks as heavy ones at $\mu = \mu_{pole}$, there remains the $SU(N_l - N_c)$ Yang-Mills theory with the scale factor of its gauge coupling $\langle -\Lambda_L \rangle = \Lambda_{YM}$ (and with mions $M_{ll}$ “sitting down” on $\bar{\Lambda}_L$). Finally, integrating the gluons through the VY-procedure, one obtains (let us recall that all fields in (5.12) and (5.14) are normalized at $\mu = \Lambda_Q$, and $\Lambda_{YM}/\Lambda_Q = (r)^{N_l/3N_c} (m_h/\Lambda_Q)^{N_f/3N_c}$, $\langle N_{hh} \rangle = -Z_q m_h \Lambda_Q$):

$$L \sim \left[ \hat{z}_M \text{Tr} \left( \frac{M_{hh}^\dagger M_{hh} + z''_m M_{ll}^\dagger M_{ll}}{Z_q^2 \Lambda_Q^2} \right) + \hat{z}_q 2 \text{Tr} \left( \frac{N_{hh}}{N_{hh}} \Lambda_{YM} \right) \right] + \left[ \text{Tr} \left( \frac{M_{hh} N_{hh}}{Z_q \Lambda_Q} \right) \right] - (N_l - N_c) \Lambda_{YM}^2 \left( \det \frac{\langle N_{hh} \rangle}{N_{hh}} \right)^{1/(N_l - N_c)} \left( \det \frac{\langle M_\mu \rangle}{\langle M_\mu \rangle} \right)^{1/2} + \text{Tr} \left( m_l M_{ll} + m_h M_{hh} \right), \hspace{1cm} (5.14)$$

where $z''_M = z''_M(\mu_{gl,h}, \pi_{pole}) \approx z''_M(\mu_{gl,h}, \mu_{pole}) = z''_M \gg 1$ is the logarithmic renormalization factor of $M_{ll}$ mions.

The masses obtained from (5.14) look then as:

$$\frac{\mu(M_{hh})}{\Lambda_Q} \sim \frac{\mu(M_{hh})}{\Lambda_Q} \sim \left( \frac{\langle N_{hh} \rangle}{\Lambda_Q^2} \right)^{1/2} \sim \exp \left( -\frac{N_c}{14b_0} \right) \left( \frac{m_h}{\Lambda_Q} \right)^{N_F/3N_c} \sim \frac{\pi_{gl,h}}{\Lambda_Q}, \hspace{1cm} (5.15)$$

$$\frac{\mu(M_{ll})}{\Lambda_Q} \sim \frac{Z_q^2 m_l \Lambda_Q}{\hat{z}_m^\dagger M_{ll}} \sim \exp \left[ \frac{2N_c}{T_{D_0}} \right] \left( \frac{2N_c - N_l}{N_c \Lambda_Q} \right)^{N_F/3N_c} / z''_M.$$  \hspace{1cm} (5.16)

On the whole for 'B', when we started directly with the dual theory with $\bar{N}_c = (N_F - N_c)$ colors, $N_F = (N_l + N_h)$ quarks $\piT$, $q$ and $N_F^2$ mions $M$, its mass spectrum looks as follows.

\footnote{The second term of the superpotential in (5.14) can equivalently be written as:

$$\frac{-\left( N_l - N_c \right)}{\Lambda_Q^2 \det (-N_{hh}/Z_q \Lambda_Q)}^{1/(N_l - N_c)}$$}
1) The sector of heavy masses includes: a) \((2N_c N_h - N_b^2)\) of massive gluons and their scalar superpartners, b) \(2N_h N_t\) of hybrid scalar monions \(M_{ht} + M_{th}\), c) \(2N_h N_t\) of hybrid scalar monions \(N_{ht} + N_{th}\) (these are \(q^f, \bar{q}^f\) - quarks with higgsed colors), d) \(N_b^2\) of scalar monions \(M_{bh}\) and \(N_b^2\) of scalar monions \(N_{bh}\). All these particles, with specific numbers of each type, definite spins and other quantum numbers, have definite masses of the same scale \(\sim \bar{m}_{gl,h} \sim \exp\{-\frac{N_c}{14\bar{b}_0}\}(m_h/L_{Q})^{N_F/3N_c}L_{Q} = \exp\{-\frac{N_c}{14\bar{b}_0}\} m^\text{pole}_h \ll m^\text{pole}_h\). All these particles are interacting only weakly, with both their gauge and Yukawa couplings \(\bar{q}^* \sim a^*_q, \bar{b}_o/N_F \ll 1\).

1') Because the dual quarks \(q^h, \bar{q}^h\) are higgsed, one can imagine that there also appear solitonic excitations in the form of monopoles of the dual group (its broken part) (see e.g. [12], section 3 and the footnote 6 therein). These dual monopoles will be confined and can form, in principle, a number of additional hadrons \(H'_h\). Because the dual theory is weakly coupled at the scale of higgsing \(\mu \sim \bar{m}_{gl,h} \sim \exp\{-\frac{N_c}{14\bar{b}_0}\} m^\text{pole}_h\), the masses of these monopoles, as well as the tension \(\sqrt{\sigma}\) of strings confining them (with our exponential accuracy in parametrical dependence on \(\frac{N_c}{\bar{b}_0} \gg 1\)), will be also \(\sim \bar{m}_{gl,h}\). So, the mass scale of these hadrons \(H'_h\) will be also \(\sim \bar{m}_{gl,h} \ll m^\text{pole}_h\). Let us even assume (in favor of the duality) that, with respect to their quantum numbers, these hadrons \(H'_h\) can be associated in some way with the direct hadrons made of \(Q^h, \bar{Q}_h\) - quarks. But even so, the masses of \(H'_h\) are parametrically smaller than those of various direct hadrons made of \(Q^h, \bar{Q}_h\) - quarks, \(\mu(H'_h) \sim \exp\{-\frac{N_c}{14\bar{b}_0}\} m^\text{pole}_h \ll m^\text{pole}_h\).

Besides, because the chiral \(\bar{t}, l\)-flavors and the \(R\) - charges of the lower energy theory at \(\mu \lesssim \bar{m}_{gl,h}\) remain unbroken, no possibility is seen for the appearance in this dual theory \('B'\) of such \(\bar{t}, l\)-flavored chiral hadrons \(H'_l\) with the heavy masses \(\sim \bar{m}_{gl,h}\), which can be associated with a large number of various direct \(\bar{t}, l\) - flavored chiral hadrons made of \(\bar{Q}^q, \bar{Q}_l\) quarks (and \(W_o\) in \('A'\) dualized in the variant \"b\") (not even speaking about their parametrically different mass scales, see the point 1' in \('A'\)'). This very fact shows the self-contradictory character of duality in the variant \"b\".

All other particles in the mass spectrum of \('B'\) constitute the sector of lighter particles, with their masses being parametrically smaller than \(\bar{m}_{gl,h}\).

2) The next mass scale is constituted by the large number of \(l\)-flavored dual mesons and \(b_l, \bar{b}_l\) baryons made of non-relativistic (and weakly confined, the string tension is \(\sqrt{\sigma} \sim \Lambda_{YM} \ll \bar{m}_{q,l}\)) dual \(q^f, \bar{q}^f\) quarks with \(\bar{N}_c = (N_l - N_c)\) of unhiggsed colors. The pole masses of these \(q^f, \bar{q}^f\) quarks are: \(\bar{m}_{q,l}^\text{pole} \sim \exp\{3\frac{N_c}{14\bar{b}_0}\}(r)^{N_l-N_c}/N_c \bar{m}_{gl,h}/z^l_q \ll \bar{m}_{gl,h}\).

3) Next, there is a large number of gluonia with the mass scale \(\sim \Lambda_{YM} \ll \bar{m}_{q,l}^\text{pole}\).

4) Finally, the lightest are \(N_b^2\) scalar monions \(M_{ll}\) with masses \(\mu(M_{ll}) \sim \exp\{-2\frac{N_c}{\bar{b}_0}\}(r)^{2\Delta} \Lambda_{YM}/z^l_M \ll \Lambda_{YM}\).

Comparing the mass spectra of two supposedly equivalent descriptions \('A'\) and \('B'\) above, it is seen that masses are clearly different parametrically, in powers of the parameter \(Z_q = \exp\{-\frac{N_c}{\bar{b}_0}\} \ll 1\). Besides, in \('B'\) there are very specific definite numbers of fields with fixed quantum numbers and spins and with the definite masses \(\sim \bar{m}_{gl,h}\), all parametrically weakly interacting (see point 1 in \('B'\')). No analog of these specific particles is seen in
'A'. Instead, there is a large number of various $h$-flavored hadrons with different spins, all strongly interacting, with the coupling $a(\mu \sim m_h^{\text{pole}}) \sim 1$. Finally, and we consider this as of special importance, there are no heavy $\bar{Q}, Q_l$-flavored hadrons $H_l^\prime$ in 'B', which have the appropriate conserved $\bar{t}, l$-chiral flavors and $R$-charges, such that they can be associated with a large number of various heavy chiral hadrons made of $\bar{Q}, Q_l$-quarks (and $W_\alpha$), which are present in 'A' dualized in the variant "b". This shows that the duality in the variant "b" is not self-consistent. 6 This agrees with some general arguments presented in [6] (see section 7, it is also worth noting that these arguments are not connected with the use of the scenario #1 with the diquark condensate), that the duality in the variant "b" can’t be realized, because masses of chiral hadron superfields with different spins and with unbroken chiral flavors and $R$-charges can’t be made "of nothing".

This is not the whole story, though. For the mass $\bar{\mu}_{gl,h}$ of gluons in the dual theory to be the largest physical mass $\mu_H$ (as was used in 'B' above), the parameter $r = m_l/m_h$ has to be taken sufficiently small (see (5.9),(5.13), from now on the non-leading effects due to factors like $z_q^\prime \simeq z_q^\mu$ will be ignored):

$$\frac{\bar{\mu}_{gl,h}^{\text{pole}}}{\bar{\mu}_{gl,h}} \ll 1 \rightarrow r \ll r_l = \left( z_q^\prime \exp\left(-\frac{3N_c}{14b_0}\right) \right)^{\frac{N_c}{N_l-N_c}} \sim \exp\left(-\frac{3N_c}{14b_0} \frac{N_c}{N_l-N_c}\right) \ll 1. \quad (5.16)$$

Let us trace however the behavior of the direct and dual theories in the whole interval $r_l < r < 1$. 7

As for the direct theory (see 'A' above), its regime and all hierarchies in the mass spectrum remain the same for any value of $r < 1$, i.e. the pole mass $m_h^{\text{pole}}$ of $Q_h, \bar{Q}$ quarks becomes the largest physical mass $\mu_H$ already at $r < 1/(\text{several})$, etc.

This is not the case however for the dual theory (the case 'B' above). At $r_l < r < 1/(\text{several})$, the pole mass $\hat{\mu}_{q,l}^{\text{pole}}$ of $q', \bar{Q}$ quarks remains the largest physical mass:

$$\frac{\hat{\mu}_{q,l}^{\text{pole}}}{\Lambda_Q} = \frac{\langle M_{ll} \rangle}{Z_q \Lambda_Q^2} \left( \frac{\Lambda_Q}{\hat{\mu}_{q,l}^{\text{pole}}} \right)^{\frac{b_0}{N_F}} \sim \exp\left(\frac{N_c}{7b_0}\right) \left( r \frac{N_l-N_c}{N_c} \right)^{\frac{N_F}{3N_c}} \left( \frac{m_h}{\Lambda_Q} \right) \sim \frac{\bar{\mu}_{gl,h}^{\text{pole}}}{\Lambda_Q}. \quad (5.17)$$

Already this is sufficient to see a qualitative difference between 'A' and 'B'. The $hh$-flavored hadrons in 'A' have largest masses, while the $ll$-flavored hadrons are the heaviest ones in 'B'.

Let us make some rough estimates in 'B' at $r > r_l$. After integrating out the heaviest quarks $q', \bar{Q}$ at the scale $\mu \sim \hat{\mu}_{q,l}^{\text{pole}}$, there remains the dual theory with $\bar{N}_c$ colors and $N_h / \bar{N}_c$ of dual quarks $\bar{Q}, q^h$ (and mions $M$), and with $b_0 = (3N_c - N_h) > 0$. It will be now in the weak coupling logarithmic regime at $\mu_H' = \mu < \hat{\mu}_{q,l}^{\text{pole}}$, where $\mu_H'$ is the highest mass scale in the remained theory. The new scale factor $\Lambda_q'$ of its gauge coupling can be

---

6 It is not difficult to see that a similar situation with the $\bar{t}, l$-flavored chiral hadrons which can be made of $\bar{Q}, Q_l$-quarks (and $W_\alpha$) occurs in the variant "b" also in the scenario #1 considered in [6][7] (see section 3b in [7] for a similar regime). 'A' therein is exactly the same as in this paper. 'B' is different, as the $h$-flavors are not higgsed, but form instead the diquark condensate, but qualitatively the situation is the same, and the difference between 'A' and 'B' is only more prominent in the scenario #1.

7 For this, it is convenient to keep $m_h$ intact, while $m_l$ will be decreasing, starting with $m_l = m_h$. 

found from
\[
\overline{B}'_0 \ln \left( \frac{\hat{\mu}_{q,t}^{\text{pole}}}{\Lambda'_q} \right) \simeq \frac{2\pi}{\alpha'} = \frac{3N_c}{7b_0} \quad \rightarrow \quad \frac{\Lambda'_q}{\hat{\mu}_{q,t}^{\text{pole}}} \sim \exp \left\{ -\frac{3N_c^2}{7b_0} \right\} \ll 1. \quad (5.18)
\]

If (see below) \( r_l \ll r_h < r < 1/(\text{several}) \), then \( \mu'_H = \hat{\mu}_{q,h}^{\text{pole}} > \Lambda'_q \), where \( \hat{\mu}_{q,h}^{\text{pole}} \) is the pole mass of \( q^h, \bar{q}^h \) quarks, which will be in the HQ-phase (i.e. not yet higgsed). Roughly, \( \mu_{q,h}^{\text{pole}} \sim r \hat{\mu}_{q,t}^{\text{pole}} \), so that
\[
\frac{\hat{\mu}_{q,h}^{\text{pole}}}{\Lambda'_q} > 1 \quad \rightarrow \quad r > r_h \sim \exp \left\{ -\frac{3N_c^2}{7b_0} \right\} \gg r_l. \quad (5.19)
\]

In this interval \( r_h < r < 1/(\text{several}) \), the mass spectrum of 'B' is qualitatively not much different from the case \( r = 1 \) (see section 4). The heaviest are \( ll \)- hadrons, then \( hh \)-hadrons, then gluonia, and besides, there are the mions \( M_{hh}, M_{hl}, M_{ll} \) with their masses: \( \mu(M_{hh}) \sim m_h^2/\mu_o \), \( \mu(M_{hl}) \sim m_h m_l/\mu_o \), \( \mu(M_{ll}) \sim m_l^2/\mu_o \), \( \mu_o = \Lambda_M^3/Z_5^2 \Lambda_Q^2 \).

As \( r \) decreases further, there is a phase transition from the HQ\(_h\) phase to the Higgs\(_h\) phase at \( r \sim r_h \), i.e. after \( \hat{\mu}_{q,h}^{\text{pole}} > \Lambda'_q \) becomes \( \hat{\mu}_{q,h}^{\text{pole}} < \Lambda'_q \), and the quarks \( q^h, \bar{q}^h \) are higgsed. \(^8\) But even now, the \( q^l, \bar{q}^l \) quarks remain the heaviest ones. And only when \( r \) becomes \( r < r_l \ll r_h \ll 1 \), the gluon mass \( \overline{\psi} g_{l,h} \) becomes the largest one, and the mass spectrum of 'B' becomes that described above in this section.

On the whole, it seems that a sufficient number of examples were presented above in sections 2-5 to show that the direct and dual theories are not equivalent (and besides, the duality in the variant "b" is not self-consistent). So, because we are interested mainly in the direct theory, as the original microscopic UV - free theory, we don’t consider below the dual theory any more. The next sections will deal with the direct theory in some different regimes.

### 6 Direct theory. Unequal quark masses.

\[
3N_c/2 < N_F < 3N_c, \quad b_o/N_F = O(1), \quad N_l > N_c.
\]

This section continues the preceding one, but now we forget about any dualizations and deal with the direct theory as it is, i.e. in the variant "a" (see [6], section 7). This means, that after the heavy quarks \( Q^h, \overline{Q}^h \) have been integrated out at \( \mu = m_h^{\text{pole}} \), all particle masses in the lower energy theory with \( N_c \) colors and \( N_c < N_l < 3N_c/2 \) of lighter \( Q_l, \overline{Q}_l \) quarks are parametrically smaller than the scale \( \Lambda_Q \sim m_h^{\text{pole}} \) (see (5.1) above), and at \( \mu \ll \Lambda_Q \) the lower energy theory enters the strong coupling regime \( a(\mu \ll \Lambda_Q) \gg 1. \(^9\)

\(^8\) As was argued in [12] (see section 3 therein), the transition proceeds through formation of the mixed phase in the threshold region \( \Lambda'_q/\text{(several)} < \hat{\mu}_{q,h}^{\text{pole}} < \text{(several)}\Lambda'_q \) (i.e. \( r_h/\text{(several)} < r < \text{(several)}r_h \) here).

\(^9\) To have definite answers, from now on and everywhere below, we use the anomalous quark dimension \( 1 + \gamma_Q(N_F, N_c, a(\mu) \gg 1) = N_c/(N_F - N_c) \), and strong coupling \( a(\mu) \gg 1 \) given in eq.(7.4) in [6].
This lower energy theory is assumed to be in the HQ-phase, and the pole mass of $Q_l, \overline{Q}^\gamma$ quarks is:

$$
\frac{m_i^{\text{pole}}}{\Lambda'_Q} = \left( \frac{m_i(\mu = \Lambda'_Q)}{\Lambda'_Q} = r \right) \left( \frac{\Lambda_Q}{m_i^{\text{pole}}} \right)^\gamma_Q \rightarrow \frac{m_i^{\text{pole}}}{\Lambda'_Q} = \left( r \right)^{(N_i - N_c)/N_c} \ll 1, \ r \ll 1,
$$

$$
1 + \gamma_Q = \frac{N_c}{N_i - N_c}, \ \nu = \frac{3N_c - 2N_i}{N_i - N_c}. \quad (6.1)
$$

The coupling $a_+(\mu = m_i^{\text{pole}})$ is large:

$$
a_+(\mu = m_i^{\text{pole}}) = \left( \frac{\Lambda_Q}{\mu = m_i^{\text{pole}}} \right)^\nu = \left( \frac{1}{r} \right)^{(3N_c - 2N_i)/N_c} \gg 1. \quad (6.2)
$$

So, after integrating out all $Q_l, \overline{Q}^\gamma$ quarks as heavy ones at $\mu = m_i^{\text{pole}}$, one remains with the pure $SU(N_c)$ Yang-Mills theory, but it is now in the strong coupling regime. This is somewhat unusual, but there is no contradiction, as this perturbative strong coupling regime with $a_{YM}(\mu) \gg 1$ is realized in the restricted interval of scales only, $\Lambda_{YM} \ll \mu < m_i^{\text{pole}} \ll \Lambda'_Q$. It follows from the NSVZ $\beta$-function [8] that its coupling $a_{YM}(\mu \gg \Lambda_{YM}) \gg 1$ looks then as:

$$
a_{YM}(\mu = m_i^{\text{pole}}) = \left( \frac{\mu = m_i^{\text{pole}}}{\Lambda_{YM}} \right)^3 = a_+(\mu = m_i^{\text{pole}}) \rightarrow \frac{\Lambda_{YM}}{\Lambda_Q} = \left[ \left( \frac{m_i}{\Lambda_Q} \right)^{N_i} \left( \frac{m_h}{\Lambda_Q} \right)^N \right]^{1/3N_c},
$$

$$
a_{YM}(\Lambda_{YM} \ll \mu < m_i^{\text{pole}}) = a_{YM}(\mu = m_i^{\text{pole}}) \frac{\mu}{m_i^{\text{pole}}} = \left( \frac{\mu}{\Lambda_{YM}} \right)^3, \quad (6.3)
$$

and it decreases now with lowering $\mu$ from $a_{YM}(\mu = m_i^{\text{pole}}) \gg 1$ down to $a_{YM}(\mu \sim \Lambda_{YM}) \sim 1$, after which the non-perturbative effects come into a game.

So, lowering the scale $\mu$ from $\mu = m_i^{\text{pole}}$ down to $\mu < \Lambda_{YM}$, integrating out all gauge degrees of freedom except for the one whole field $S \sim W_2^2$, and using then the VY-form for the superpotential of the field $S$ [10], one obtains the standard gluino condensate, $\langle S \rangle = \Lambda_{YM}^3$.

To check self-consistency, one also has to estimate the scale $\mu_{gl}$ of possible higgsing of $Q_l, \overline{Q}^\gamma$ quarks. This estimate looks as:

$$
\mu_{gl}^2 \sim a_+(\mu = \mu_{gl}) \langle \overline{Q} Q_l \rangle_{\mu = \mu_{gl}} \sim (m_i^{\text{pole}})^2, \ \ a_+(\mu = \mu_{gl}) = \left( \frac{\Lambda'_Q = m_h^{\text{pole}}}{\mu_{gl}} \right)^\nu, \quad (6.4)
$$

$$
\langle \overline{Q} Q_l \rangle_{\mu = \mu_{gl}} = \langle \overline{Q} Q_l \rangle_{\mu = \Lambda_Q} \left( \frac{\Lambda_Q}{\Lambda'_Q} \right)^{\gamma_Q = 10\nu/N \gamma_p} \left( \frac{\mu_{gl}}{\Lambda'_Q} \right)^{\gamma'_Q}, \quad \langle \overline{Q} Q_l \rangle_{\mu = \Lambda_Q} = \frac{\Lambda_{YM}^3}{m_i}. 
$$

Therefore, as usually happens in this scenario #2 in the strong coupling region, $\mu_{gl}$ and $m_i^{\text{pole}}$ are parametrically the same, and this is "a point of tension". As before, we assume
that really $\mu_{gl} = m_{l_{\text{pole}}}/(\text{several})$, so that the $Q_l, \overline{Q}$ quarks are not higgsed and the $HQ_l$-phase is self-consistent.

On the whole, all quarks of the direct theory are in the HQ-phase (and so confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM}$). The mass spectrum includes in this case: 1) a large number of heavy $hh$-flavored mesons with the mass scale $\sim m_{h_{\text{pole}}} \ll \Lambda_Q$; 2) a large number of hybrid $hl$-mesons and baryons $B_{hl}, \overline{B}_{hl}$ with the same mass scale $\sim m_{h_{\text{pole}}}$; 3) a large number of $ll$-flavored mesons and baryons $B_{l}, \overline{B}_{l}$ with the mass scale $\sim m_{l_{\text{pole}}} \ll m_{l_{\text{pole}}}$, and finally, 4) a large number of lightest gluonia with masses $\sim \Lambda_{YM} \ll m_{l_{\text{pole}}}$. 

7 Direct theory. Unequal quark masses.

$N_c < N_F < 3N_c/2$, $N_l > N_c$.

As this case with quarks of equal masses has already been described in the preceding section, we consider here the case $\mu = m_{l_{\text{pole}}}$. Really, the mass spectrum in this case is not much different.

All quarks are in the HQ - phase, and the highest physical mass is the pole mass of $h$-quarks $(1 + \gamma_+ = N_c/(N_F - N_c))$:

$$m_{h_{\text{pole}}} = m_h \left( \frac{\Lambda_Q}{m_{h_{\text{pole}}}} \right)^{\gamma_+} \rightarrow \frac{m_{h_{\text{pole}}}}{\Lambda_Q} = \left( \frac{m_h}{\Lambda_Q} \right)^{(N_F - N_c)/N_c}.$$ (7.1)

After integrating $Q_h, \overline{Q}^h$ - quarks as heavy ones at $\mu = m_{h_{\text{pole}}}$, there remains the lower energy theory with $N_c$ colors and $N_c < N_l < 3N_c/2$ of $l$-quarks. The next physical scale is the pole mass of $l$-quarks $(1 + \gamma_- = N_c/(N_l - N_c))$:

$$m_{l_{\text{pole}}} = r m_{h_{\text{pole}}} \left( \frac{m_{l_{\text{pole}}}}{m_{h_{\text{pole}}}} \right)^{\gamma_-} \rightarrow \left( \frac{m_{l_{\text{pole}}}}{m_{h_{\text{pole}}}} \right) = \left( \frac{r}{N_l - N_c} \right)^{N_l - N_c}/N_c \ll 1.$$ (7.2)

After integrating $l$-quarks as heavy ones at $\mu = m_{l_{\text{pole}}}$, there remains $SU(N_c)$ YM - theory in the strong coupling regime. Its scale factor $\Lambda'_{YM}$ is determined from:

$$\left( \frac{m_{l_{\text{pole}}}}{\Lambda'_{YM}} \right)^3 = \left( \frac{\Lambda_Q}{m_{l_{\text{pole}}}} \right)^{\nu_+} \left( \frac{m_{h_{\text{pole}}}}{m_{l_{\text{pole}}}} \right)^{\nu_-} \rightarrow \Lambda'_{YM} = \Lambda_{YM} = \left( \frac{\Lambda_{b_{0}} N_l m_h N_l}{m_{l_{\text{pole}}}^2} \right)^{1/3N_c}.$$ (7.3)

$$\nu_+ = \frac{3N_c - 2N_F}{N_F - N_c}, \quad \nu_- = \frac{3N_c - 2N_l}{N_l - N_c} > \nu_+.$$

To check self-consistency, let us also estimate the gluon masses due to possible higgsing of $Q_h$ and/or $Q_l$ - quarks. As for $Q_h$ - quarks,

$$\frac{\mu_{gl,h}^2}{\Lambda_Q^2} \sim \left[ a_+(\mu = \mu_{gl,h}) = \left( \frac{\Lambda_Q}{\mu_{gl,h}} \right)^{\nu_+} \right] \left( \frac{\overline{Q}^h Q_h}{\Lambda_Q^2} \right) \left( \frac{\mu_{gl,h}}{\Lambda_Q} \right)^{\gamma_+} \rightarrow \mu_{gl,h} \sim \left( r \right)^{N_l/N_c} \ll 1.$$ (7.4)
so that there is no problem, but as for $Q_l$ - quarks, this looks now as

$$
\frac{\mu_{gl,l}^2}{\Lambda_Q^2} \sim \left[ a_-(\mu = \mu_{gl,l}) = \left( \frac{\Lambda_Q}{m_h^\text{pole}} \right)^\nu_+ \left( \frac{m_h^\text{pole}}{\mu_{gl,l}} \right)^\nu_- \right] \left( \frac{\langle Q_l^+ Q_l \rangle}{\Lambda_Q^2} \right) \left( \frac{m_h^\text{pole}}{\Lambda_Q} \right)^\gamma_+ \left( \frac{\mu_{gl,l}}{m_h^\text{pole}} \right)^\gamma_- \right)
$$

(7.5)

$$
\rightarrow \frac{\mu_{gl,l}}{\Lambda_Q} \sim \left( \frac{\langle Q_l^+ Q_l \rangle}{\Lambda_Q^2} \right) \sim \frac{m_h^\text{pole}}{\Lambda_Q},
$$

(7.6)

and this is also ”a point of tension”. As before, we assume that it is in favor of $m_l^\text{pole}$, i.e. $m_l^\text{pole} = (\text{several}) \mu_{gl,l}$.

On the whole, all quarks are in the HQ - phase and the mass spectrum consists of: a) a large number of $hh$ and $hl$ - mesons with the mass scale $\sim m_h^\text{pole} \sim \Lambda_Q (m_h/\Lambda_Q)^{(N_F-N_c)/N_c} \ll \Lambda_Q$, b) a large number of $ll$ - mesons with the mass scale $\sim m_h^\text{pole} \sim m_h^\text{pole} (r)(N_f-N_c)/N_c \ll m_h^\text{pole}$, c) a large number of gluonia with masses $\sim \Lambda_{YM} \ll m_l^\text{pole}$.

8 Direct theory. Unequal quark masses.

$N_c < N_F < 3N_c/2, \ N_l < N_c - 1$.

In this regime, at $r = m_l/m_h \ll 1$, the highest physical scale $\mu_H$ is determined by the gluon masses $\mu_{gl,l}$ which arise due to higgsing of $Q_l$, $\overline{Q}$ - quarks: \footnote{Here and below, the value of $r$ is taken to be not too small, so that $\mu_{gl,l} \ll \Lambda_Q$. At $r$ so small that $\mu_{gl,l} \gg \Lambda_Q$, the quarks $Q_l$, $\overline{Q}$ are higgsed in the logarithmic weak coupling region (as in the scenario #1 in [7]), and the form of the RG - flow will be different, but qualitatively the regime is the same for $\mu_{gl,l} \ll \Lambda_Q$ or $\mu_{gl,l} \gg \Lambda_Q$, and nothing happens as $\mu_{gl,l}$ overshoots $\Lambda_Q$ in the scenario #2 considered here.

Besides, we neglect everywhere below the additional dependence of Kahler terms on the quantum pion fields $\pi_l^\dagger/\mathcal{M}_{ch}^l$ (originating from the dependence on $\pi/\mathcal{M}_{ch}^l$ of the quark renormalization factor $z_Q(\Pi_l^\dagger, \Pi_l)$), because this will influence the particle mass values with non-parametric factors $O(1)$ only.}

$$
\frac{\mu_{gl,l}^2}{\Lambda_Q^2} \sim \left[ a_+(\mu = \mu_{gl,l}) = \left( \frac{\Lambda_Q}{\mu_{gl,l}} \right)^\nu_+ \right] \left( \frac{\langle Q_l^+ Q_l \rangle}{\Lambda_Q^2} \right) \left( \frac{m_h}{\Lambda_Q} \right)^\gamma_+ \left( \frac{\mu_{gl,l}}{m_h} \right)^\gamma_- \right)
$$

(8.1)

The lower energy theory includes: $\hat{N}_c = (N_c - N_l)$ of unbroken colors, $2N_lN_h$ scalar hybrids $\Pi_{hl} + \Pi_{lh}$ (these are $Q_h$ and $\overline{Q}^h$ - quarks with broken colors), $N_h$ flavors of active $Q_h$, $\overline{Q}^h$ - quarks with unbroken colors, and finally, $N_l^2$ pions $\Pi_{ll}$, $\langle \Pi_{ll} \rangle = \langle Q_l^+ Q_l \rangle = (\mathcal{M}_{ch}^l)^2$. Their Lagrangian at $\mu < \mu_{gl,l}$ takes the form (all fields are normalized at $\mu = \Lambda_Q$):

$$
L = \left[ z_Q \text{Tr} \sqrt{\Pi_{ll}^\dagger \Pi_{ll}} + z_Q \text{Tr}_h \left( Q_l^+ \overline{Q} + \overline{Q}^\dagger e^{-\overline{Q}} Q_l \right) + z_Q \text{Tr} \left( \Pi_{hl}^\dagger \Pi_{hl} + \Pi_{lh}^\dagger \Pi_{lh} \right) + \cdots \right]_D
$$
\begin{equation}
\left[ -\frac{2\pi}{\hat{\alpha}(\mu)}\hat{S} + m_t \text{Tr} \Pi \Pi_{\mu} + m_h \text{Tr}_{h} \overline{Q}Q + m_h \text{Tr} \Pi_{h} \Pi_{\mu}\right]_F, \tag{8.2}
\end{equation}

where \( \hat{V} \) are the \( SU(\hat{N}_c) \) gluons and \( \hat{\alpha}(\mu) \) is their gauge coupling (with the pions \( \Pi_{\mu} \) sitting down inside). The dots in D-terms denote residual interactions. It is assumed that these play no essential role for what follows. All fields in (8.2) are normalized at \( \mu = \Lambda_Q \). Besides, \( z_Q = z_Q(\Lambda_Q, \mu_{gl,t}) \) is the quark renormalization factor due to the perturbative evolution in the interval of scales \( \mu_{gl,t} < \mu < \Lambda_Q \),

\[
z_Q = z_Q(\Lambda_Q, \mu_{gl,t}) = \left( \frac{\mu_{gl,t}}{\Lambda_Q} \right) ^{(2N_c - N_F)/(N_F - N_c)} \ll 1.
\]

All pion fields \( \Pi_{h} \), \( \Pi_{\mu} \) and \( \Pi_{\mu} \) are frozen and don't evolve any more at \( \mu < \mu_{gl,t} \). The numbers of colors and flavors have already changed in the threshold region \( \mu_{gl,t}/(\text{several}) < \mu < \mu_{gl,t}^{(\text{several})} \), \( N_F \rightarrow \hat{N}_F = N_F - N_t = N_h \), \( N_c \rightarrow \hat{N}_c = N_c - N_t \), while the coupling \( \hat{\alpha}(\mu) \) did not change essentially and remains \( \sim \alpha(\mu = \mu_{gl,t}) \gg 1 \). So, the new quark anomalous dimension \( \gamma_-(\hat{N}_c, \hat{N}_F = N_h, \hat{\alpha} \gg 1) \) and the new \( \beta \)-function have the form:

\[
\frac{d\hat{a}(\mu)}{d\ln \mu} = - \nu_- \hat{a}(\mu), \quad \nu_-= \frac{\hat{N}_F \gamma_- - \hat{b}_o}{\hat{N}_c} = \frac{3\hat{N}_c - 2\hat{N}_F}{\hat{N}_F - \hat{N}_c} = \nu_+ - \frac{N_t}{\hat{N}_F - \hat{N}_c}; \tag{8.3}
\]

\[
\hat{b}_o = (3\hat{N}_c - \hat{N}_F) = (b_o - 2N_t), \quad \gamma_- = \frac{2\hat{N}_c - \hat{N}_F}{\hat{N}_F - \hat{N}_c} = \gamma_+ - \frac{N_t}{\hat{N}_F - \hat{N}_c}.
\]

Depending on the value of \( \hat{N}_F/\hat{N}_c \), the lower energy theory will be in different regimes. We consider below two cases only.

i). \( 1 < \hat{N}_F/\hat{N}_c < 3/2 \): In this case \( \nu_- > 0 \), so that the coupling \( \hat{a}(\mu) \) continues to grow with decreasing \( \mu \), but more slowly than before.

The next physical scale is given by the pole mass of \( Q_h, \overline{Q}^\top \) - quarks:

\[
\frac{m_{h}^{\text{pole}}}{\Lambda_Q} = \left[ \frac{m_h(\mu = \mu_{gl,t})}{\Lambda_Q} = \frac{m_h(\Lambda_Q)}{\mu_{gl,t}} \right] ^{\gamma_-} \to \frac{m_{h}^{\text{pole}}}{\mu_{gl,t}} = r \ll 1, \quad \frac{m_{h}^{\text{pole}}}{\Lambda_Q} = \left( r \right)^{N_t/N_c} \left( \frac{m_h}{\Lambda_Q} \right)^{(N_F - N_c)/N_c} \gg \frac{\Lambda_{YM}}{\Lambda_Q}. \tag{8.4}
\]

After integrating out \( Q_h, \overline{Q}^\top \) - quarks as heavy ones at \( \mu = m_{h}^{\text{pole}} \), there remains the \( SU(\hat{N}_c) \) YM - theory in the strong coupling regime \( a_{YM}(\mu = m_{h}^{\text{pole}}) \gg 1 \) (and pions). The scale of its gauge coupling \( \hat{\Lambda}_{YM} = \left( \Lambda_L(\Pi_{\mu}) \right) \) is determined from:

\[
a_{YM}(\mu = m_{h}^{\text{pole}}) = \left( \frac{m_{h}^{\text{pole}}}{\Lambda_{YM}} \right)^3 = \hat{a}(\mu = m_{h}^{\text{pole}}) = \left( \frac{\Lambda_Q}{\mu_{gl,t}} \right)^{\nu_-} \left( \frac{\mu_{gl,t}}{m_{h}^{\text{pole}}} \right)^{\nu_+} \to \hat{\Lambda}_{YM} = \Lambda_{YM}. \tag{8.5}
\]
Finally, tracing the fate of pions $\Pi_{ll}$ and going through the VY - procedure at $\mu < \Lambda_{YM}$, one obtains their Lagrangian:

$$L = \left[ z_Q \text{Tr} \sqrt{\Pi_{ll}^\dagger \Pi_{ll} + z_Q \text{Tr} \left( \Pi_{hl}^\dagger \Pi_{hl} + \Pi_{lh}^\dagger \Pi_{lh} \right)} \right]_D + \left[ \left( N_c - N_l \right) \left( \frac{\Lambda^b_Q}{\text{det} \Pi_{ll}} \right)^{1/(N_c - N_l)} \right]_F \left[ \text{Tr} \Pi_{ll} + m_l \text{Tr} \Pi_{hl} \Pi_{lh} \right].$$



(8.6)

From this, the masses of $\Pi_{hl}$, $\Pi_{lh}$ and $\Pi_{ll}$ are:

$$\mu(\Pi_{hl}) = \mu(\Pi_{lh}) \sim (r)^{\gamma_-} m^\text{pole}_{h}, \quad \mu(\Pi_{ll}) \sim \frac{m_l}{z_Q} \sim r \mu(\Pi_{hl}).$$

(8.7)

To check self-consistency, i.e. that $Q_h \rarr \overline{Q}^+$ quarks are indeed in the HQ - phase and are not higgsed, let us estimate the possible value of the gluon mass $\mu_{gl,h}$:

$$\mu^2_{gl,h} \sim \left[ \hat{a}(\mu = \mu_{gl,h}) = \left( \frac{\Lambda_Q}{\mu_{gl,l}} \right)^{\nu_+} \left( \frac{\mu_{gl,l}}{\mu_{gl,h}} \right)^{\nu_-} \right] \left( \overline{Q}^h Q_h \right)_{\mu = \mu_{gl,h}},$$

(8.8)

$$\frac{\left( \overline{Q}^h Q_h \right)_{\mu = \mu_{gl,h}}}{\Lambda^2_Q} = r \frac{\mu_{gl,l}}{\Lambda_Q} \left( \frac{\mu_{gl,l}}{\mu_{gl,h}} \right)^{\gamma_+} \left( \frac{\mu_{gl,h}}{\mu_{gl,l}} \right)^{\gamma_-} \rightarrow \mu_{gl,h} \sim m^\text{pole}_{h},$$

(8.9)

as one could expect beforehand. This is ”the standard point of tension” in the scenario #2.

ii). $3/2 < \bar{N}_F / N_c < 3$. In this case $\nu_- < 0$, so that the RG - flow is reversed and the coupling $\hat{a}(\mu)$ begins to decrease with decreasing $\mu$ at $\mu < \mu_{gl,l}$, approaching its fixed point value $\hat{a} < 1$ from above (if not stop before at $\mu = m^\text{pole}_{h}$). Until $\hat{a}(\mu) \gg 1$, it behaves as:

$$\hat{a}(\mu < \mu_{gl,l}) = a_+(\mu = \mu_{gl,l}) \left( \frac{\mu}{\mu_{gl,l}} \right)^{(-\nu_-) > 0} = \left( \frac{\Lambda_Q}{\mu_{gl,l}} \right)^{\nu_+} \left( \frac{\mu_{gl,l}}{\mu} \right)^{\nu_-}, \quad \nu_- < 0.$$

(8.10)

So, it will decrease down to $\sim 1$ at $\mu \sim \Lambda_o$:

$$\hat{a}(\Lambda_o) \sim 1 \rightarrow \frac{\Lambda_o}{\Lambda_Q} \sim \left( \frac{\mu_{gl,l}}{\Lambda_Q} \right)^\Delta, \quad \Delta = \frac{N_l}{2N_F - 3N_c} > 1.$$

(8.11)

Let us consider first the case $m^\text{pole}_{h} \gg \Lambda_o$. The value of $m^\text{pole}_{h}$ is given then by (8.4), and this requires

$$m^\text{pole}_{h} = r \mu_{gl,l} \gg \Lambda_o \rightarrow \left( \frac{\mu_{gl,l}}{\Lambda_Q} \right)^{(\Delta - 1) > 0} \ll r \ll 1.$$

(8.12)

Then, the running of $\hat{a}(\mu)$ will stop at $\hat{a}(m^\text{pole}_{h}) \gg 1$, so that it will not enter the conformal regime. The situation then is similar to those described above at the point 'i'. The quarks
$Q_h, \overline{Q}$ decouple at $\mu < m_h^{\text{pole}}$, and there remains the $SU(\hat{N}_c)$ YM - theory (and pions) in the strong coupling regime, $\alpha_{YM}(\mu) = (\mu/\Lambda_M)^3 \gg 1$ at $\Lambda_M \ll \mu < m_h^{\text{pole}}$, etc. One can deal with it as before at the point 'i'.

The new regime is realized for such values of parameters that $m_h^{\text{pole}} \ll \Lambda_o$, but still $\mu_{gl.1} \ll \Lambda_Q$. In this case, with lowering the scale $\mu$ below $\mu_{gl.1} \ll \Lambda_Q$, the large but decreasing coupling $a(\mu) \gg 1$ crosses unity at $\mu = \Lambda_o$ and becomes $a(\mu < \Lambda_o) < 1$, and theory enters the conformal regime, but with $a(\mu)$ approaching its fixed point value $\hat{a}^* = 1$ from above. The self-consistency of this regime requires then very specific behavior of the quark anomalous dimension $\hat{\gamma}(\mu) = \gamma_-(\hat{N}_F, \hat{N}_c, \hat{a}(\mu))$ in the region $\mu \sim \Lambda_o$, when decreasing $\hat{a}(\mu)$ undershoots unity. Qualitatively, the behavior has to be as follows: a) $\hat{\gamma}(\mu)$ stays nearly intact at its value $(2\hat{N}_c - \hat{N}_F)/(\hat{N}_F - \hat{N}_c) < \hat{b}_o/\hat{N}_F$, as far as the coupling remains large, $\hat{a}(\mu \gg \Lambda_o) \gg 1$; b) $\hat{\gamma}(\mu)$ changes rapidly in the threshold region $\Lambda_o/(\text{several}) < \mu < (\text{several}) \Lambda_o$. It begins to increase at $\mu = (\text{several}) \Lambda_o$ and crosses the value $\hat{b}_o/\hat{N}_F$ right at the point $\mu = \Lambda_o$ where $\hat{a}(\mu)$ crosses unity, so that the $\beta$ - function remains smooth and does not change its sign; c) $\hat{\gamma}(\mu)$ continues to increase at $\mu < \Lambda_o$ and reaches its maximal positive value at $\mu = \Lambda_o/(\text{several})$, and then begins to decrease with further decreasing $\mu$, approaching its limiting value ($\hat{b}_o/\hat{N}_F$ or $\hat{b}_o > 0$, or zero at $\hat{b}_o < 0$) from above at $\mu \ll \Lambda_o$.

It will be useful to confirm this very specific behavior of $\hat{\gamma}(\mu)$ independently from elsewhere. But once this is accepted, one can trace then the lower energy behavior proceeding similarly as above for the conformal regime (but additionally taking into account the presence of pions, which are remnants of $l$ - quarks higgsed before at the higher scale $\mu_{gl.1}$).

9 Conclusions.

The mass spectra of $N = 1$ SQCD with $SU(N_c)$ colors and $N_F$ flavors of light quarks $Q, \overline{Q}$ (with masses $m_i \ll \Lambda_Q$) have been described above, within the dynamical scenario #2. This scenario implies that quarks can be in two different phases only: either this is the HQ (heavy quark) phase where they are confined, or this is the Higgs phase. Besides, a comparison has been made of this (direct) theory with its Seiberg’s dual variant [1|4], which contains $SU(N_F - N_c)$ dual colors, $N_F$ dual quarks $q, \overline{q}$, and $N_F^2$ additional mesons $M$ (mions).

As was shown above in the text, in all cases when there is the additional small parameter at hand (this is 0 < $b_o/N_F = (3N_c - N_F)/N_F \ll 1$ in the direct theory, or its analog 0 < $\overline{b}_o/N_F = (2N_F - 3N_c)/N_F \ll 1$ in the dual one), there are parametrical differences in the mass spectra of direct and dual theories, so that they are clearly not equivalent. Really, this implies that even when $b_o/N_F$ or $\overline{b}_o/N_F$ are $O(1)$, there are no reasons for these two theories to become exactly the same. \footnote{But to see more clearly possible differences, it is insufficient in this case to make rough estimates of particle masses, as has been done in this paper. One has either to resolve the mass spectra in more detail, or to calculate also some Green functions in both theories, and to compare. At present, it is unclear how to do this.}

Besides, as was shown in section 5, one can trace unavoidable internal inconsistencies of the Seiberg duality in the variant "b", i.e. "confinement without chiral symmetry breaking" (this implies that at $N_c < N_F < 3N_c/2$ the direct quarks and gluons form a large number of massive hadrons with masses $\sim \Lambda_Q$, while there appear new light particles with masses
$\mu_i \ll \Lambda_Q$, described by the dual theory). \footnote{Similar situation with this variant "b" can be traced also in the scenario \#1 (but in this scenario the differences between the direct and dual theories are much more prominent), see also the footnote 6.}

As for the mass spectra of the direct theory in this scenario \#2, their main features look as follows (for $b_0/N_F = O(1)$, for $b_0/N_F \ll 1$ see section 2).

1) In all cases considered, there is a large number of gluonia with masses $\sim \Lambda_{YM} = (\Lambda_Q^b \det m_Q)^{1/3N_c}$.

2) When all quark masses are equal, they are (most probably) in the HQ - phase (i.e. not higgsed, but confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM}$), for the whole interval $N_c < N_F < 3N_c$, and so form a large number of hadrons with masses: a) $\sim m_Q^{\text{pole}} \sim \Lambda_{YM}$ at $3N_c/2 < N_F < 3N_c$, b) $\sim m_Q^{\text{pole}} \sim \Lambda_Q (m_Q/\Lambda_Q)^{(N_F-N_c)/N_c} \gg \Lambda_{YM}$ at $N_c < N_F < 3N_c/2$.

There are no additional lighter pions $\pi_i$ with masses $\mu_\pi \ll m_Q^{\text{pole}}$, for all $N_c < N_F < 3N_c$.

3) Also considered was the case with $N_l$ flavors of smaller masses $m_l$ and $N_h = N_F - N_l$ flavors with larger masses $m_h$, $m_l < m_h \ll \Lambda_Q$. When $N_l > N_c$, all quarks are also in the HQ - phase for all $N_c < N_F < 3N_c$, and form a large number of hadrons, whose masses depend on their flavor content (see the text), but also there are no any additional lighter pions.

4) Only when $N_l < N_c$, the $l$ - flavored quarks $Q_l, \overrightarrow{Q}_l$ are higgsed, $SU(N_c) \to SU(N_c - N_l)$, and there appear $N_l^2$ lighter pions $\pi_l^\tau$, while heavier $h$ - flavored quarks $Q_h, \overrightarrow{Q}_h$ remain always in the HQ - phase. For this case, the mass spectra and some new regimes with unusual properties of the RG - flow were presented in sections 6-8.

We have considered in this paper not all possible regimes, only those which reveal some qualitatively new features. We hope that, if needed, a reader can deal with other regimes by himself, using the methods from \cite{6}\cite{7} and from this paper.

On the whole, the mass spectra have been obtained for both dynamical scenarios for $\mathcal{N} = 1$ SQCD (\#1 considered in \cite{6}\cite{7}, and \#2 considered in this paper). Both scenarios look possible, i.e. no unavoidable internal inconsistences are seen. So, with our present very restricted abilities to trace the dynamics in more detail, it remains unclear - which one of them (if any) is right. The time will show. But, in any case, the direct and dual theories are not equivalent in both scenarios.

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