Hamiltonian analysis of BHT massive gravity

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Abstract

We study the Hamiltonian structure of the Bergshoeff-Hohm-Townsend (BHT) massive gravity with a cosmological constant. In the space of coupling constants \((\Lambda_0, m^2)\), our canonical analysis reveals the special role of the condition \(\Lambda_0/m^2 \neq -1\). In this sector, the dimension of the physical phase space is found to be \(N^* = 4\), which corresponds to two Lagrangian degree of freedom. When applied to the AdS asymptotic region, the canonical approach yields the conserved charges of the BTZ black hole, and central charges of the asymptotic symmetry algebra.

1 Introduction

The new theory of massive gravity in three dimensions (3D), recently proposed by Bergshoeff, Hohm and Townsend (BHT) \([1, 2]\), is defined by adding the parity invariant, curvature-squared terms to the Einstein-Hilbert action. With the cosmological constant \(\Lambda_0\) and the sign of the Einstein-Hilbert term \(\sigma = \pm 1\), the action takes the form

\[
I = a \int d^3x \sqrt{\tilde{g}} \left( \sigma R - 2\Lambda_0 + \frac{1}{m^2} K \right), \quad K := \hat{R}_{ij} \hat{R}^{ij} - \frac{3}{8} \hat{R}^2, \tag{1.1}
\]

where \(\hat{R}_{ij}\) is the Ricci tensor and \(\hat{R}\) the scalar curvature. At the linearized level in asymptotically Minkowskian spacetime, the BHT gravity is equivalent to the Pauli-Fierz theory for a free massive spin-2 field. The action (1.1) ensures the absence of ghosts (negative energy modes), and the unitarity in flat space \([3]\); moreover, the theory is renormalizable \([4]\). In the AdS background and for generic values of the coupling constants, the unitarity of the massive gravitons is found to be in conflict with the positivity of central charges in the boundary CFT \([5, 2]\). One should also note that the BHT theory possesses a number of exact solutions \([6, 7, 8]\), its AdS sector is studied in \([5, 9]\), central charges are discussed in \([5, 2, 10]\), and supersymmetric extension in \([11]\).

It is interesting to observe that the particle content of the BHT gravity depends on the values of coupling constants. Thus, if we consider a maximally symmetric vacuum state defined by \(G_{ij} = \Lambda_{\text{eff}} \eta_{ij}\), where \(G_{ij}\) is the Einstein tensor and \(\Lambda_{\text{eff}}\) the effective cosmological constant, this configuration solves the BHT field equations if \(\Lambda_{\text{eff}}\) solves a simple quadratic...
equation. For $\Lambda_0/m^2 = -1$, two solutions for $\Lambda_{\text{eff}}$ coincide, and the two massive modes degenerate with each other [2, 5]. In that case, there is an extra gauge symmetry at the linearized level which allows massive modes to become partially massless [2, 12, 13]. The modes corresponding to $\Lambda_0/m^2 = 3$ are also found to be special, but they remain massive [2].

Motivated by the fact that the nature of physical modes in the BHT gravity has been studied only in the linear approximation, see also [14], we use here the constrained Hamiltonian approach to clarify the dynamical content of the BHT gravity nonperturbatively. In particular, we will find out a natural role of the condition $\Lambda_0/m^2 \neq -1$ in the canonical consistency procedure.

The paper is organized as follows. In section 2, we give a brief account of the basic dynamical features of the BHT gravity in the Lagrangian formalism, and describe the BTZ black hole solution. In section 3, we apply Dirac’s method for constrained dynamical systems [15] to make a consistent canonical analysis of the BHT gravity. Then, in section 4, we classify the constraints and find that the theory exhibits two local Lagrangian degrees of freedom. To obtain this result, we used a condition which, when applied to maximally symmetric solutions, takes the form $\Lambda_0/m^2 \neq -1$, corresponding to the case of massive gravitons. In section 5, we find the form of the gauge generator, showing thereby that the obtained classification of constraints is correct. In section 6, we briefly describe the AdS asymptotic structure by imposing the Brown-Henneaux asymptotic conditions, find the form of the improved generators and the corresponding conserved quantities, and calculate the central charges of the asymptotic symmetry. Finally, section 7 is devoted to concluding remarks, while appendices contain some technical details.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters $(i,j,k,...; \mu, \nu, \lambda,...)$ run over 0,1,2, the first letters of the Greek alphabet $(\alpha, \beta, \gamma,...)$ run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+,-,-)$; totally antisymmetric tensor $\varepsilon^{ijk}$ and the related tensor density $\varepsilon^{\mu
u\rho}$ are both normalized as $\varepsilon^{012} = 1$.

2 Lagrangian dynamics in the first order formalism

The BHT massive gravity with a cosmological constant is formulated as a gravitational theory in Riemannian spacetime. Instead of using the standard Riemannian formalism, with an action defined in terms of the metric as in (1.1), we find it more convenient to use the triad field and the spin connection as fundamental dynamical variables. Such an approach can be naturally described in the framework of Poincaré gauge theory [16], where basic gravitational variables are the triad field $b^i$ and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms), and the corresponding field strengths are the torsion $T^i$ and the curvature $R^{ij}$ (2-forms). After introducing the notation $A^{ij} = -\varepsilon^{ijk}\omega^k$ and $R^{ij} = -\varepsilon^{ijk} R^k$, we have:

$$T^i = db^i + \varepsilon^j_{jk} \omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \varepsilon^j_{jk} \omega^j \wedge \omega^k.$$  

The antisymmetry of $A^{ij}$ ensures that the underlying geometric structure corresponds to Riemann-Cartan geometry, in which $b^i$ is an orthonormal coframe, $g := \eta_{ij} b^i \otimes b^j$ is the metric of spacetime, $\omega^i$ is the Cartan connection, and $T^i, R^i$ are the torsion and the Cartan
curvature, respectively. For $T_i = 0$, this geometry reduces to Riemannian. In what follows, we will omit the wedge product sign $\wedge$ for simplicity.

The description of the BHT massive gravity can be technically simplified as follows.

(a) We use the triad field $b^i$ and the spin connection $\omega^i$ as independent dynamical variables. (b) The Riemannian nature of the connection is ensured by imposing the vanishing of torsion with the help of the Lagrange multiplier $\lambda^i = \lambda^i_\mu dx^\mu$.

(c) Finally, by introducing an auxiliary field $f^i = f^i_\mu dx^\mu$, we transform the term $K$ into an expression linear in curvature.

These modifications lead to a new formulation of the BHT massive gravity, classically equivalent to (1.1):

$$L = a \left( 2 \sigma R_i - \frac{1}{3} \lambda_0 \varepsilon_{ijk} b^j b^k + \frac{1}{m^2} L_K \right) + \lambda^i T_i .$$

(2.1a)

Here, the piece $L_K$ is linear in curvature and depends on the auxiliary field $f^i$:

$$L_K = R_i f^i - V_K , \quad V_K := \frac{1}{4} f^i \ast (f^i - f b^i) = \mathcal{V}_K \hat{\epsilon} ,$$

where $f = f^k_k$ and $\hat{\epsilon} = b^i b^j b^k$ is the volume 3-form. In the component notation, with $R_{imn} = G^k_i \varepsilon_{kmn}$, $L_K$ takes the well-known form [1]:

$$L_K = (f^i G^{ik} - \mathcal{V}_K) \hat{\epsilon} , \quad \mathcal{V}_K := \frac{1}{4} (f^i f^{ik} - f^2) .$$

The form of $V_K = V_K(b^i, f^i)$ ensures that after using the field equations to eliminate $f^i$, $L_K$ reduces to $K \hat{\epsilon}$ (Appendix A).

The field equations

Variation with respect to $b^i, \omega^i, f^i, \lambda^i$, yields the BHT field equations:

$$a \left( 2 \sigma R_i - \frac{1}{3} \lambda_0 \varepsilon_{ijk} b^j b^k - \frac{1}{m^2} \Theta_i \right) + \nabla \lambda_i = 0 ,$$

(2.2a)

$$a \left( 2 \sigma T_i + \frac{1}{m^2} \nabla f_i \right) + \varepsilon_{imn} \lambda^m b^n = 0 ,$$

(2.2b)

$$2 R_i - \ast (f^i - f b^i) = 0 ,$$

(2.2c)

$$T_i = 0 ,$$

(2.2d)

where $\Theta_i = - \partial L_K / \partial b^i$ is the energy-momentum current (2-form) associated to $L_K$, and $\nabla$ is the covariant derivative: for a 1-form $X_i$, $\nabla X_i = d X_i + \varepsilon_{ijk} \omega^j X^k$.

The last equation ensures that spacetime is Riemannian. The third equation implies:

$$f^i - f b^i = \ast R^i = 2 G^i_k b^k ,$$

$$2 f = R , \quad f^i = 2 L^i = 2 L^i_k b^k ,$$

(2.3)

where $G_{ij}$ is the Einstein tensor, and $L_{ij}$ the Schouten tensor:

$$G_{ij} := \hat{R}_{ij} - \frac{1}{2} \eta_{ij} R , \quad L_{ij} := \hat{R}_{ij} - \frac{1}{4} \eta_{ij} R .$$
Introducing the Cotton 2-form $C_i = \nabla L_i$, the second equation reads
\[
\frac{2a}{m^2} C_i + \varepsilon_{imn} \lambda^m b^n = 0.
\]
Next, we introduce the Cotton tensor $C_{ij}$ by $C_i = C^k_i \hat{\epsilon}_k$, where $\hat{\epsilon}_k = \frac{1}{2} \varepsilon_{kmn} b^m b^n$, and note that the previous equation, combined with $C_{ii} = 0$, implies:
\[
\lambda_{ij} = \frac{2a}{m^2} C_{ij}, \quad C_{ij} = \varepsilon_{imn} \nabla_m L_{nj},
\]
\[
\nabla \lambda_i = \frac{2a}{m^2} (\nabla_m C_{in}) b^m b^n.
\]
Now, the first field equation takes the form:
\[
2 \sigma R_i - \Lambda_0 \varepsilon_{imn} b^m b^n - \frac{1}{m^2} \Theta_i + \frac{2}{m^2} \left( \nabla_m C_{in} \right) b^m b^n = 0. \tag{2.4a}
\]
We can express the energy-momentum current $\Theta_i$ in terms of the corresponding energy-momentum tensor $T^n_i$ as (Appendix A)
\[
\Theta_i = T^n_i \hat{\epsilon}_n, \quad T^n_i := \delta^n_i \nabla_K - \frac{1}{2} f_{ik} (f^{kn} - f \eta^{kn}).
\]
Expanding (2.4a) in the dual basis $\hat{\epsilon}_j$, with $R_i = 2G_{ij} \hat{\epsilon}^j$, yields:
\[
\sigma G_{ij} - \Lambda_0 \eta_{ij} - \frac{1}{2m^2} K_{ij} = 0, \tag{2.4b}
\]
where
\[
K_{ij} := T_{ij} - 2(\nabla_m C_{in}) \varepsilon_{mnn} ,
\]
\[
= K_{\eta_{ij}} - 2L_{ik} G^{kj} - 2(\nabla_m C_{in}) \varepsilon_{mnn}.
\]
These equations coincide with those found in [5, 2] (Appendix A).
We display here a set of algebraic consequences of the field equations:
\[
f_{ij} = f_{ji}, \tag{2.5a}
\]
\[
\lambda_{ij} = \lambda_{ji}, \quad \lambda = 0, \tag{2.5b}
\]
\[
\sigma f + 3\Lambda_0 + \frac{1}{2m^2} \nabla_K = 0, \tag{2.5c}
\]
where we used $T^n_i = \nabla_K$. Consider now a maximally symmetric solution, for which
\[
R_{ijk} = \Lambda_{\text{eff}} \varepsilon_{ijk}, \quad \hat{R}_{ij} = -2\Lambda_{\text{eff}} \eta_{ij}, \quad R = -6\Lambda_{\text{eff}}. \tag{2.6}
\]
Equation (2.5c) with $f_{km} = 2L_{km}$ implies that the effective cosmological constant $\Lambda_{\text{eff}}$ satisfies the quadratic equation
\[
\Lambda_{\text{eff}}^2 + 4m^2 \sigma \Lambda_{\text{eff}} - 4m^2 \Lambda_0 = 0,
\]
which yields
\[
\Lambda_{\text{eff}} = -2m^2 \left( \sigma \pm \sqrt{1 + \Lambda_0/m^2} \right). \tag{2.7}
\]
BTZ black hole solution

In the AdS sector of the BHT gravity, with $\Lambda_{\text{eff}} = -1/\ell^2$, there exists a maximally symmetric solution, locally isomorphic to the BTZ black hole [11, 17, 18].

In the Schwarzschild-like coordinates $x^\mu = (t, r, \phi)$, the BTZ black hole solution is defined in terms of the lapse and shift functions, respectively:

$$ N^2 = \left( -8Gm_0 + \frac{r^2}{\ell^2} + \frac{16G^2J_0^2}{r^2} \right), \quad N_\phi = \frac{4GJ_0}{r^2}, $$

where $m_0, J_0$ are the integration parameters and $\Lambda_{\text{eff}} = -1/\ell^2$. The triad field has the simple diagonal form

$$ b^0 = N dt, \quad b^1 = N^{-1} dr, \quad b^2 = r \left( d\phi + N_\phi dt \right), \quad (2.8a) $$

while the Riemannian connection reads:

$$ \tilde{\omega}^0 = -Nd\phi, \quad \tilde{\omega}^1 = N^{-1}N_\phi dr, \quad \tilde{\omega}^2 = -\frac{r}{\ell^2} dt - rN_\phi d\phi. \quad (2.8b) $$

Then, using (2.6) and $C_{ij} = 0$, the field equations imply that the Lagrange multiplier $\lambda^i$ vanishes, while the auxiliary field $f^i$ is proportional to the triad field:

$$ \lambda^i = 0, \quad f^i = \frac{1}{\ell^2} b^i. \quad (2.8c) $$

3 Hamiltonian and constraints

In local coordinates $x^\mu$, the component form of the Lagrangian density reads:

$$ \mathcal{L} = a\varepsilon^{\mu\nu\rho} \left( \sigma b^\mu b^\rho R_{\nu\rho} - \frac{1}{3} A_0 \varepsilon^{ijk} b_{i\mu} b_{j\nu} b_{k\rho} \right) + \frac{a}{m^2} \mathcal{L}_K + \frac{1}{2} \varepsilon^{\mu\nu\rho} \lambda^i \mathcal{V}_{i\mu\nu\rho}, \quad (3.1a) $$

where the term $\mathcal{L}_K$ is conveniently represented in the first order formalism as

$$ \mathcal{L}_K = \frac{1}{2} \varepsilon^{\mu\nu\rho} f^i_{\mu} R_{i\nu\rho} - b \mathcal{V}_K, \quad (3.1b) $$

where $b = \text{det}(b^\mu_i)$.

**Primary constraints.** From the definition of the canonical momenta $(\pi^\mu_i, \Pi^\mu_i, p^\mu_i, P^\mu_i)$, conjugate to the basic dynamical variables $(b^\mu_i, \omega^i, \lambda^i, f^i)$, respectively, we obtain the primary constraints:

$$ \phi^0_i := \pi^0_i \approx 0, \quad \phi^\alpha_i := \pi^\alpha_i - \varepsilon^{\alpha\beta} \lambda_{i\beta} \approx 0, $$

$$ \Phi^0_i := \Pi^0_i \approx 0, \quad \Phi^\alpha_i := \Pi^\alpha_i - 2a \varepsilon^{\alpha\beta} \left( \sigma b_{i\beta} + \frac{1}{2m^2} f_{i\beta} \right) \approx 0. $$

$$ p^\mu_i \approx 0, \quad P^\mu_i \approx 0. \quad (3.2) $$

The PB algebra of the primary constraints is displayed in Appendix B.
After noting that the term $bV_K$ is bilinear in the variables $b^i_0$ and $f^i_0$, one can conveniently represent the canonical Hamiltonian as

$$
\mathcal{H}_c = b^i_0 \mathcal{H}_i + \omega^i_0 \mathcal{K}_i + f^i_0 \mathcal{R}_i + \lambda^i_0 \mathcal{T}_i + \frac{a}{m^2} bV_K + \partial_\alpha D^\alpha,
$$

where

$$
\mathcal{H}_i = -\varepsilon^{0\alpha\beta}(a\sigma R_{i\alpha\beta} - a\Lambda_{0\varepsilon_{ijk}b^j_\alpha b^k_\beta} + \nabla_\alpha \lambda_{i\beta}) ,
\mathcal{K}_i = -\varepsilon^{0\alpha\beta}(a\sigma T_{i\alpha\beta} + \frac{a}{m^2}\nabla_\alpha f_{i\beta} + \varepsilon_{ijk}b^j_\alpha \lambda^k_\beta) ,
\mathcal{R}_i = -\frac{a}{2m^2}\varepsilon^{0\alpha\beta}R_{i\alpha\beta} ,
\mathcal{T}_i = -\frac{1}{2}\varepsilon^{0\alpha\beta}T_{i\alpha\beta} ,
D^\alpha = \varepsilon^{0\alpha\beta}\left[\omega^i_0 \left(2a\sigma b_{i\beta} + \frac{a}{m^2} f_{i\beta}\right) + b^i_0 \lambda_{i\beta}\right].
$$

**Secondary constraints.** Going over to the total Hamiltonian,

$$
\mathcal{H}_T = \mathcal{H}_c + u^i_\mu \phi_{i\mu} + v^i_\mu \Phi_{i\mu} + w^i_\mu p_{i\mu} + z^i_\mu P_{i\mu} ,
$$

where $(u^i_\mu, v^i_\mu, w^i_\mu, z^i_\mu)$ are canonical multipliers, we find that the consistency conditions of the primary constraints $\pi_{i0}, \Pi_{i0}, p_{i0}^i$ and $P_{i0}^i$ yield the secondary constraints:

$$
\hat{\mathcal{H}}_i := \mathcal{H}_i + \frac{a}{m^2} bT_i^0 \approx 0 ,
\mathcal{K}_i \approx 0 ,
\hat{\mathcal{R}}_i := \mathcal{R}_i + \frac{a}{2m^2} b(f_i^0 - fh_i^0) \approx 0 ,
\mathcal{T}_i \approx 0 .
$$

They correspond to the $\mu = 0$ components of the field equations (2.2). Using the relation

$$
V_K = b^0_0 T_i^0 + f^0_0 \frac{1}{2}(f_i^0 - fh_i^0) ,
$$

the canonical Hamiltonian can be rewritten in the form

$$
\mathcal{H}_c = b^i_0 \hat{\mathcal{H}}_i + \omega^i_0 \mathcal{K}_i + f^i_0 \hat{\mathcal{R}}_i + \lambda^i_0 \mathcal{T}_i + \partial_\alpha D^\alpha .
$$

The consistency of the remaining primary constraints $X_A := (\phi_{i,0}^\alpha, \Phi_{i,0}^\alpha, p_{i,0}^\alpha, P_{i,0}^\alpha)$ leads to the determination of the multipliers $(u_{i,0}^\alpha, v_{i,0}^\alpha, w_{i,0}^\alpha, z_{i,0}^\alpha)$ (Appendix B). However, we find it more convenient to continue our analysis in the reduced phase space formalism. Using the second class constraints $X_A$, we can eliminate the momenta $(\pi_{i,0}^\alpha, \Pi_{i,0}^\alpha, p_{i,0}^i, P_{i,0}^i)$ and construct the reduced phase space $R_1$, in which the basic nontrivial Dirac brackets (DB) take the following form (Appendix B):

$$
\{b^i_\alpha, \lambda^j_\beta\}^*_1 = \eta^{ij}\varepsilon_{0\alpha\beta}\delta ,
\{\omega^i_\alpha, f^j_\beta\}^*_1 = \frac{m^2}{a}\eta^{ij}\varepsilon_{0\alpha\beta}\delta ,
\{\lambda^i_\alpha, f^j_\beta\}^*_1 = -2m^2\sigma\eta^{ij}\varepsilon_{0\alpha\beta}\delta .
$$
The remaining DBs are the same as the corresponding Poisson brackets.

In $R_1$, the total Hamiltonian takes the simpler form:

$$H_T = H_c + u^i_0 \phi^0_i + v^i_0 \Phi^0_i + w^i_0 p^0_i + z^i_0 P^0_i,$$

(3.6)

$H_c$ is given by (3.4), and the consistency conditions (3.3) remain unchanged.

**Tertiary constraints.** The consistency conditions of the secondary constraints can be written in the form:

$$\{\hat{H}_i, H_T\}_1^* \approx \frac{a}{m^2} \nabla_\mu (b T^\mu_i) - \frac{1}{2} \varepsilon_{imn} b (f^{mj} - f h^{mj}) \lambda^n_i,$$

$$\{K_i, H_T\}_1^* \approx 0,$$

$$\{T_i, H_T\}_1^* \approx -\frac{1}{4} b \varepsilon_{ijk} f^{jk},$$

$$\{\hat{R}_i, H_T\}_1^* \approx \frac{a}{2m^2} \nabla_\mu \left[ b (f^i_\mu - f h^i_\mu) \right],$$

(3.7)

where, on the right-hand side, we use the symbolic notation $\dot{\phi} := \{\phi, H_T\}_1^*$. The result is obtained with the help of the canonical algebra of constraints, displayed in Appendix C. By using $\nabla_\mu (b T^\mu_i) \approx 0$, the divergence of $b T^\mu_i$ can be represented in the form

$$\nabla_\mu (b T^\mu_i) \approx \frac{1}{4} b h^\mu_i \nabla_\mu (f_{mn} f^{mn} - f^2) - \frac{1}{2} \nabla_\mu \left[ b (f ji f^{j\mu} - f f^{\mu j}) \right].$$

The third relation in (3.7) yields the following tertiary constraints:

$$\theta_{0\beta} := f_{0\beta} - f_{\beta 0} \approx 0,$$

(3.8a)

$$\theta_{\alpha\beta} := f_{\alpha\beta} - f_{\beta\alpha} \approx 0.$$

(3.8b)

They represent Hamiltonian counterparts of the Lagrangian relations (2.5a).

To find an explicit form of the consistency conditions for $\hat{H}_i$ and $\hat{R}_i$, we have to replace the time derivatives $\dot{\phi}$ by their canonical expressions $\{\phi, H_T\}$. To do that, we introduce the following change of variables in $H_T$:

$$\pi^0_i := \pi^0_i + f^i_k P^0_k,$$

$$z^0_i := z^0_i - f^i_k u^k_0,$$

(3.9)

whereupon the $(\pi^0_i, P^0_i)$ piece of $H_T$ takes the form

$$u^i_0 \pi^0_i + z^0_i P^0_i = u^i_0 \pi^0_i + z^0_i P^0_i.$$

Besides, we introduce the generalized multipliers

$$U^i_\mu = u^i_\mu + \varepsilon^{imn} \omega_{m0} b_{n\mu},$$

$$Z^i_\mu = z^i_\mu + \varepsilon^{imn} \omega_{m0} f_{n\mu},$$

which correspond, on-shell, to $\nabla_0 b^i_\mu$ and $\nabla_0 f^i_\mu$, respectively; moreover, we define

$$Z^i_\mu = Z^i_\mu - f^i_m U^m_\mu.$$
The consistency condition of $\mathcal{R}_i$, multiplied first by $b^i_0$ and then by $b^i_\beta$, yields:

\[ U^\nu \nu (f^0_0 - f) - f^0_0 U^0_\mu - (Z'_\alpha - f U^0_\alpha) + b^{-1} b^i_0 \nabla \alpha [b(f^i_\alpha - f h^i_\alpha)] = 0, \quad (3.10a) \]

\[ g^{0\mu} Z'_\beta \mu + f^0_0 U^0_\alpha - (f^0_\alpha - f b^\alpha_\beta) U^0_\alpha + b^{-1} b^i_\beta \nabla \alpha [b(f^i_\alpha - f h^i_\alpha)] = 0. \quad (3.10b) \]

The first relations, in which the arbitrary multipliers $U_{k0}$ and $Z'_{k0}$ are cancelled, contains only the determined multipliers $U_{k\alpha}$ and $Z'_{k\alpha}$. Using the expressions for $U_{k\alpha}$ and $Z'_{k\alpha}$ calculated with the help of Appendix B, one finds that this relation reduces to an identity (Appendix D). The second relation defines the two components $Z'_{\beta0} = b^k_\beta Z'_{k0}$ of $Z'_{\beta0}$.

The consistency condition of $\mathcal{H}_i$ in conjunction with (3.10) yields:

\[(f^0_0 h^i_\alpha - f^j_\alpha h^0_0) Z'_{j\alpha} + f^j_\alpha \nabla \alpha f^j_\alpha - f^j_\alpha h^i_\alpha \nabla \alpha f^j_\alpha + \frac{m^2}{a} \varepsilon_{ijk} (f^j_\mu f^j_\mu) \lambda_n \approx 0.\]

Substituting here the expression for the determined multiplier $Z'_{j\alpha}$, we find:

\[ f \varepsilon_{ijk} \lambda^j_k = 0. \]

Thus, for $f \neq 0$, we obtain three tertiary constraints:

\[ \psi_{0\beta} := \lambda_{0\beta} - \lambda_{\beta0} \approx 0, \quad (3.11a) \]

\[ \psi_{\alpha\beta} := \lambda_{\alpha\beta} - \lambda_{\beta\alpha} \approx 0, \quad (3.11b) \]

The consistency conditions of the secondary constraints determine $Z'_{\beta0}$ and produce the tertiary constraints $\theta_{\mu\nu}$ and $\psi_{\mu\nu}$.

**Quartic constraints.** The consistency of $\varepsilon^{0\alpha\beta} \theta_{\alpha\beta}$ reads

\[ \{\varepsilon^{0\alpha\beta} \theta_{\alpha\beta}, H_T\}_1^* \approx \frac{4m^2}{a} b \lambda \approx 0, \]

and we have a new, quartic constraint, the canonical counterpart of (2.5b):

\[ \chi := \lambda \approx 0. \quad (3.12a) \]

The consistency condition of $\theta_{0\beta}$ is identically satisfied (Appendix D):

\[ \{\theta_{0\beta}, H_T\}_1^* = \varepsilon'_{0\beta} - \varepsilon'_{\beta0} \approx 0. \quad (3.12b) \]

The consistency of $\varepsilon^{0\alpha\beta} \psi_{\alpha\beta}$ reads:

\[ \{\varepsilon^{0\alpha\beta} \psi_{\alpha\beta}, H_T\}_1^* \approx -4ab \left(\sigma f + 3A_0 + \frac{1}{2m^2 V_K}\right) \approx 0. \]

Thus, we have a new quartic constraint:

\[ \varphi := \sigma f + 3A_0 + \frac{1}{2m^2 V_K} \approx 0, \quad (3.13a) \]

as expected from (2.5c).
To interpret the consistency condition for $\psi_{0\beta}$, we introduce the notation

$$\pi_i^{0\prime\prime} := \pi_i^0 + \lambda_k^i p_k^0, \quad w_i' := w_i^0 - u_k^0 \lambda_i^k.$$  

The, the $(\pi_i^0, P_i^0, p_i^0)$ piece of the Hamiltonian takes the form

$$u_i^0 \pi_i^0 + w_i^0 p_i^0 + z_i^0 P_i^0 = u_i^0 \pi_i^{0\prime\prime} + w_i^0 p_i^0 + z_i^0 P_i^0,$$

and we have:

$$\{\psi_{0\beta}, H_T\}^*_1 = w'_{0\beta} - \bar{w}'_{0\beta} \approx 0. \quad (3.13b)$$

Hence, the multipliers $w'_{0\beta}$ are determined.

- The consistency conditions of the tertiary constraints determine $w'_{0\beta}$ and produce the quartic constraints $\chi$ and $\varphi$.

**End of the consistency procedure.** The consistency condition of the quartic constraint $\chi$ determines the multiplier $w'_{00}$:

$$\{\chi, H_T\}^*_1 = w''_{\mu} \approx 0,$$

$$g^{00} w'_{00} + g^{03} \bar{w}'_{00} + h^{i\alpha} \bar{w}'_{i\alpha} = 0, \quad (3.14)$$

where $\bar{w}'_{i\alpha} = \bar{w}_{i\alpha} - \lambda_{ik} \bar{u}^k_{\alpha}$.

The consistency condition for the quartic constraint $\varphi$ has the form:

$$\{\varphi, H_T\}^*_1 = \Omega^{\mu\nu} z'_{\mu\nu} \approx 0,$$

$$\Omega^{\mu\nu} := \sigma g^{\mu\nu} + \frac{1}{4m^2} (f^{\mu\nu} - f^{\mu\nu}). \quad (3.15)$$

This relation determines the multiplier $z'_{00}$, provided the coefficient $\Omega^{00}$ does not vanish.

- The consistency conditions for the quartic constraints determine $w'_{00}$ and $z'_{00}$.

This finally completes the consistency procedure. At the end, we wish to stress that the completion of this process is achieved by employing the following *extra conditions*:

$$f \neq 0, \quad (3.16a)$$

$$\Omega^{00} \neq 0. \quad (3.16b)$$

Dynamical interpretation of these conditions is discussed in the next section.

### 4 Classification of constraints

Among the primary constraints, those that appear in $H_T$ with arbitrary multipliers are first class (FC):

$$\pi_i^{0\prime\prime}, \Pi_i^0 = FC, \quad (4.1a)$$

while the remaining ones, $p_i^0$ and $P_i^0$, are second class.

Going to the secondary constraints, we use the following simple theorem:
If $\phi$ is a FC constraint, then $\{\phi, H_T\}^*_i$ is also a FC constraint.

The proof relies on using the Jacobi identity. The theorem implies that the secondary constraints $\mathcal{H}_i := -\{\pi_i^{0n}, H_T\}^*_i$ and $\mathcal{K}_i := -\{\Pi_i^0, H_T\}^*_i$ are FC. After a straightforward but lengthy calculation, we obtain:

$$
\mathcal{H}_i = \mathcal{H}_i^{\prime\prime} + h_i^\mu (\nabla_\mu \lambda_{jk}) b_j^k 0 j^0 + h_i^\mu (\nabla_\mu f_{jk}) b_j^k 0 P_j^0,
$$

$$
\mathcal{K}_i = \mathcal{K}_i - \varepsilon_{ijk} (\lambda^j_0 p^k 0 - b_j^0 \lambda^k n b^0 n) - \varepsilon_{ijk} (f_j^0 P^k 0 - b_j^0 f^k n P^0 n),
$$

where $\mathcal{H}_i^{\prime\prime} := \dot{\mathcal{H}}_i + f_k^i \mathcal{R}_k + \lambda^k T_k$. As before, the time derivative $\dot{\phi}$ is a short for $\{\phi, H_T\}^*_i$.

The total Hamiltonian can be expressed in terms of the FC constraints (up to an ignorable square of constraints) as follows:

$$
\dot{\mathcal{H}}_T = b_i^0 \mathcal{H}_i + \omega_i^0 \mathcal{K}_i + u_i^0 \pi_i^{0n} + v_i^0 \Pi_i^0.
$$

In what follows, we will show that the complete classification of constraints in the reduced space $R_1$ is given as in Table 1, provided the conditions (3.16) are satisfied.

| Table 1. Classification of constraints in $R_1$ |
|-----------------------------------------------|
| First class | Second class |
| Primary | $\pi_i^{0n}, \Pi_i^0$ | $p_i^0, P_i^0$ |
| Secondary | $\mathcal{H}_i, \mathcal{K}_i$ | $\mathcal{T}_i, \dot{\mathcal{T}}_i$ |
| Tertiary | $\theta_{0\beta}, \theta_{\alpha\beta}, \psi_0, \psi_{\alpha\beta}$ |
| Quartic | $\chi, \varphi$ |

Here, $\dot{\mathcal{T}}_i$ is a suitable modification of $\mathcal{T}_i$, defined so that it does not contain $f_{ab}$:

$$
\dot{\mathcal{T}}_i = \mathcal{T}_i + \frac{ab}{2n^2} [(g^{00} h_i^0 - g^{0\alpha} h_i^0) f_{0\alpha} + g^{0\alpha} f_{i\alpha} - h_i^0 f_{i\alpha}] .
$$

To prove the content of Table 1, we need to verify the second-class nature of the constraints in the last column. This can be done by calculating the determinant of their DBs. In order to simplify the calculation, we divide the procedure into three simpler steps, as described in Appendix E: (i) we start with the subset of 6 constraints $Y_A := (\theta_{0\beta}, \varphi, P_{\alpha0}, P_{00})^*$ and show that they are second class since the determinant of $\{Y_A, Y_B\}^*_i$ is nonsingular; then, (ii) we extend our considerations to $Z_A := (\psi_{0\beta}, \chi, P_{\alpha0}, P_{00})^*$, and show that these 6 constraints are also second class; finally, (iii) we show in the same manner that the remaining 8 constraints $W_A := (\mathcal{T}_i, \dot{\mathcal{T}}_i, \frac{1}{2} \varepsilon^{0\alpha\beta} \psi_{\alpha\beta}, \frac{1}{2} \varepsilon^{0\alpha\beta} \theta_{\alpha\beta})$ are second class.

Thus, all 20 constraints $(Y_A, Z_B, W_C)$ are second class.

Note, however, that this result is valid only if the condition (3.16b) is satisfied, as shown in Appendix E.

When the classification of constraints is complete, the number of independent dynamical degrees of freedom in the phase space $R_1$ is given by the formula:

$$
N^* = N - 2N_1 - N_2,
$$
where $N$ is the number of phase space variables in $R_1$, $N_1$ is the number of FC, and $N_2$ the number of second class constraints. According to the results in Table 1, we have $N = 4 \times 9 + 4 \times 3 = 48$ (4 $\times$ 6 momentum variables are already eliminated from $R_1$), $N_1 = 12$ and $N_2 = 20$. Consequently:

- the number of physical modes in the phase space $R_1$ is $N^* = 4$, and the theory exhibits 2 local Lagrangian degree of freedom.

What is the dynamical meaning of the extra conditions (3.16)? To clarify this issue, let us consider their content for maximally symmetric solutions. When the first condition is violated, that is when $f = 0$, we have $R = 0$, $\Lambda_{\text{eff}} = 0$, and $\hat{R}_{ij} = 0$. This is possible only when $\Lambda_0 = 0$, as follows from the field equation (2.4b), and we have a completely trivial dynamics. This motivates us to accept $f \neq 0$ as a natural dynamical assumption.

Turning to the second condition, we use $f_{\mu\nu} = 2L_{\mu\nu}$ to rewrite $\Omega_{\mu\nu}$ in the form

$$\Omega_{\mu\nu} = \sigma g^{\mu\nu} + \frac{1}{2m^2} G_{\mu\nu} = g^{\mu\nu} \left( \sigma + \frac{1}{2m^2} \Lambda_{\text{eff}} \right).$$

Thus, $\Omega^{00}$ vanishes when $\Lambda_{\text{eff}} = -2m^2\sigma$, or equivalently, when $\Lambda_0/m^2 = -1$, as follows from (2.7). At this point, the mass spectrum of the BHT gravity, in the linearized approximation, undergoes a serious transition, whereby the massive sector of gravitons becomes partially massless [2, 12, 13]. At the canonical level, this phenomenon is reflected in the fact that, for $\Omega^{00} = 0$, the multiplier $z_0'$ remains undetermined, and consequently, some of the second class constraints become first class. Thus, using $\Omega^{00} \neq 0$, we stay in the massive sector of the BHT gravity. In particular, the special case $\Lambda_0/m^2 = 3$ also belongs to this sector. The canonical structure of the complementary sector $\Omega^{00} = 0$ is left for future studies.

### 5 Gauge generator

After completing the Hamiltonian analysis, we now employ the Castellani procedure [19] to construct the canonical gauge generator. Starting with the primary FC constraints $\pi_i^{0\mu}$ and $\Pi_i^0$, we find:

$$G[\tau] = \dot{\tau}^i \pi_i^{0\mu} + \tau^i \left[ \hat{H}^i - \epsilon_{ijk} \omega_j^0 \pi_k^{0\mu} - \epsilon_{ijk} b^j_0 (f^{kn} - f^n \eta^{kn}) \Pi_n^0 \right],$$
$$G[\sigma] = \dot{\sigma}^i \Pi_i^0 + \sigma^i \left( \hat{K}^i - \epsilon_{ijk} \omega_j^0 \Pi_k^0 - \epsilon_{ijk} b^j_0 \pi_k^{0\mu} \right).$$

The complete gauge generator has the form $G = G[\tau] + G[\sigma]$, its action on the fields is defined by the DB operation $\delta_0 \phi = \{ \phi, G \}_1$, but the resulting gauge transformations do not have the Poincaré form. The standard Poincaré content of the gauge transformations is obtained by introducing the new parameters [20]

$$\tau^i = -\xi^\rho b^i_\rho, \quad \sigma^i = -\theta^i - \xi^\rho \omega^i_\rho.$$
Expressed in terms of these parameters (and after neglecting some trivial terms, quadratic in the constraints), the gauge generator takes the form:

\[ G = -G_1 - G_2 , \]
\[ G_1 = \dot{\xi}^\mu (b^i_\mu \pi^0_i + \omega^i_\mu \Pi^0_i + \lambda^i_\mu \pi^0_i + f^i_\mu P^0_i) \]
\[ + \xi^\mu \left[ b^i_\mu \hat{H}_i + \omega^i_\mu \hat{K}_i + \lambda^i_\mu \hat{T}_i + f^i_\mu \hat{R}_i \right] + (\partial_\mu b^i_0) \pi^0_i + (\partial_\mu \omega^i_0) \Pi^0_i + (\partial_\mu \lambda^i_0) \pi^0_i + (\partial_\mu f^i_0) P^0_i , \]
\[ G_2 = \dot{\theta}^i \Pi^0_i + \theta^i \left[ K_i - \varepsilon_{ijk} (b^j_0 \pi^k_0 + \omega^j_0 \Pi^k_0 + \lambda^j_0 \pi^k_0 + f^j_0 P^k_0) \right]. \] (5.2)

Looking at the related gauge transformations, we find a complete agreement with the Poincaré gauge transformations on shell.

6 Asymptotic structure in the AdS sector

Asymptotic conditions imposed on dynamical variables determine the form of asymptotic symmetries, and consequently, they are closely related to the gravitational conservation laws. In this section, we focus our attention to the AdS sector of the theory, with \( \Lambda_{\text{eff}} = -1/\ell^2 \).

Asymptotic conditions. The AdS asymptotic conditions are defined by demanding that (a) the asymptotic configurations include the BTZ black hole solution, (b) they are invariant under the action of the AdS group \( SO(2, 2) \), and (c) the corresponding conserved charges are well defined. These requirements are realized by the Brown-Henneaux type of asymptotic conditions on the triad field \( b^i_\mu \) and the Riemannian connection \( \omega^i_\mu \), which have the same form as in the topologically massive gravity \[20\]. In the BHT massive gravity, there are two more Lagrangian variables, the Lagrange multiplier \( \lambda^i_\mu \) and the auxiliary field \( f^i_\mu \), whose asymptotic behavior is defined by generalizing (2.8c):

\[ \lambda^i_\mu = \hat{O} , \quad f^i_\mu = \frac{1}{\ell^2} b^i_\mu + \hat{O} , \] (6.1)

where \( \hat{O} \) denotes terms with arbitrarily fast asymptotic decrease.

Having chosen the asymptotic conditions, one should find the subset of gauge transformations that respect these conditions. It turns out that the parameters of the restricted gauge transformations are defined in terms of two functions, \( T^+ (x^+) \) and \( T^- (x^-) \), in the same way as in \[20\]. The resulting asymptotic symmetry of spacetime coincides with the conformal symmetry.

The improved generator. The canonical generator acts on dynamical variables via the Dirac bracket operation, hence, it should have well-defined functional derivatives. In order to ensure this property, we have to improve the form of \( G \) by adding a suitable surface term \( \Gamma \), such that \( \tilde{G} = G + \Gamma \) is a well-defined canonical generator. The surface term of the improved canonical generator \( \tilde{G} \) takes the form

\[ \Gamma = - \int_0^{2\pi} d\varphi \left( \xi^0 \mathcal{E}^1 + \xi^2 \mathcal{M}^1 \right) , \] (6.2a)
where
\[ E^\alpha := 2a\varepsilon^{0\alpha\beta} \left[ \left( \sigma + \frac{1}{2m^2\ell^2} \right) \omega^{0}_\beta + \frac{1}{\ell} b^{0}_\beta + \frac{1}{2m^2\ell} f^{0}_\beta \right] b^{0}_0, \]
\[ M^\alpha := -2a\varepsilon^{0\alpha\beta} \left[ \left( \sigma + \frac{1}{2m^2\ell^2} \right) \omega^{2}_\beta + \frac{1}{\ell} b^{0}_\beta + \frac{1}{2m^2\ell} f^{0}_\beta \right] b^{2}_2. \] (6.2b)

**Conserved charges.** The values of the surface terms, calculated for \( \xi^0 = 1 \) and \( \xi^2 = 1 \), define the energy and angular momentum of the system, respectively. In particular, the energy and angular momentum for the BTZ black hole are:
\[ E = \left( \sigma + \frac{1}{2m^2\ell^2} \right) m_0, \quad M = \left( \sigma + \frac{1}{2m^2\ell^2} \right) J_0. \] (6.3)
This results is verified by using Nester’s general covariant formalism [21], see also [6].

**Central charges.** Using the notation \( \tilde{G}_{(i)} := \tilde{G}[T^+_i, T^-_i] \), the main theorem of [22] states that the canonical algebra of the improved generators has the general form:
\[ \left\{ \tilde{G}_{(2)}, \tilde{G}_{(1)} \right\} = \tilde{G}_{(3)} + C_{(3)}, \] (6.4)
where \( C_{(3)} \) is the central term. Introducing the Fourier modes for the improved generator, \( L^\pm_n = -\tilde{G}[T^\pm = e^{inx\pm}] \), the above canonical algebra is found to have the form of two independent Virasoro algebras with identical central charges,
\[ c^- = c^+ = \frac{3\ell}{2G} \left( \sigma + \frac{1}{2m^2\ell^2} \right), \] (6.5)
see [5, 2, 6]. Once we have the central charges, we can use Cardy’s formula to calculate the black hole entropy:
\[ S = \left( \sigma + \frac{1}{2m^2\ell^2} \right) \frac{2\pi r_+}{4G}, \] (6.6)
where \( r_+ \) is the radius of the outer black hole horizon.

In order to have a unitary boundary CFT, the central charge (6.5) has to be positive. On the other hand, one also expects that massive gravitons, defined as small excitations around the AdS background, should carry positive energy. Now, relying on the analysis performed in [2], one can conclude that for generic values of the coupling constants, these two requirements are in conflict with each other. For possible resolutions of this conflict, see [2]. Note, however, that the positivity of the central charge and the BTZ black hole energy are in agreement with each other.

### 7 Concluding remarks

In this paper, we studied the BHT massive gravity as a constrained dynamical system.

Our basic goal was to obtain and classify the constraints and deduce the dimension of the physical phase space \( N^* \). In the process of completing Dirac’s consistency procedure, we
discovered the essential role of the extra condition $\Omega^{00} \neq 0$, Eq. (3.16b). When applied to maximally symmetric solutions, this condition describes the sector of massive gravitons with $\Lambda_0/m^2 \neq -1$. In this sector, the dimension of the phase space is found to be $N^* = 4$, which means that the theory exhibits 2 Lagrangian degrees of freedom. The canonical structure of the complementary sector $\Omega^{00} = 0$ with partially massless gravitons is left for future studies.

As a particular application of our results, we examined the AdS asymptotic structure of the theory. Using the Brown-Henneaux type of asymptotic conditions, we calculated energy and angular momentum of the BTZ black hole, and central charges of the asymptotic symmetry algebra. Our results are in agreement with those existing in the literature.

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**A On the first order form of $L_K$**

In this appendix, we display several interesting relations related to the first order formulation of $L_K$, defined in (2.1b).

The variation of $L_K$ with respect to $f_i$ yields $2R_i - * (f_i - fb_i) = 0$. This equation can be solved for $f^i$ as in (2.3), which implies

$$L_K = \frac{1}{2} R_i f^i = R_i L^i = G_{ki} L^{ik} \hat{\epsilon} = K \hat{\epsilon}.$$  
(A.1)

Thus, the expression for $L_K$ is classically equivalent to $K$.

Following the analogy with electrodynamics, we rewrite the term $V_K$ in $L_K$ as:

$$V_K = \frac{1}{2} f_i \mathcal{H}^i = V_K \hat{\epsilon}, \quad \mathcal{H}^i := \frac{1}{2} * (f^i - fb^i).$$

The energy-momentum current (density) associated to $L_K$ is given by:

$$\Theta_i := -\frac{\partial L_K}{\partial b^i} = \frac{\partial V_K}{\partial b^i} = b_i \mid V_K - \mathcal{H}^k (b_i \mid f_k).$$  
(A.2)

Then, using

$$b_i \mid V_K = V_K \hat{\epsilon}_i, \quad \mathcal{H}^k (b_i \mid f_k) = \frac{1}{2} f_{ki} (f^{kn} - f \eta^{kn}) \hat{\epsilon}_n,$$

we find:

$$\Theta_i = T^n_i \hat{\epsilon}_n, \quad T^n_i := \delta_{ki} V_K - \frac{1}{2} f_{ki} (f^{kn} - f \eta^{kn}),$$  
(A.3)

where $T^n_i$ is the dynamic energy-momentum tensor:

$$\frac{\partial}{\partial b^{i\mu}} (b V_K) = b T^n_i.$$
Using the relations

\[ \epsilon_{ij}^{mn} \nabla_m C_{ni} = \nabla^m \nabla_j L_{mi} - \nabla^m \nabla_m L_{ji} = \hat{R}_{ik} \hat{R}_{kj} - R_{imjn} \hat{R}_{mn} + \frac{1}{4} \nabla_j \nabla_i R - \nabla^2 L_{ij}, \]

\[ R_{imjn} \hat{R}_{mn} = \eta_{ij} \left( \hat{R}_{mn} \hat{R}_{mn} - \frac{1}{2} R^2 \right) + \frac{3}{2} \hat{R}_{ij} R - 2 \hat{R}_{im} \hat{R}_{nj}, \]

\[ L_{ik} G^k_j = \hat{R}_{ik} \hat{R}_{kj} - \frac{3}{4} R \hat{R}_{ij} + \frac{1}{8} \eta_{ij} R^2, \]

one can rederive the forms of \( K_{ij} \) found in [1, 5]:

\[ K_{ij} = -\frac{1}{2} \nabla_j \nabla_i R + 2 \nabla^2 L_{ij} - \frac{3}{2} \hat{R}_{ij} R - \eta_{ij} K + 4 R_{imjn} \hat{R}_{mn} + \frac{9}{2} \hat{R}_{ij} R - 8 \hat{R}_{im} \hat{R}_{nj} + \eta_{ij} \left( 3 \hat{R}_{mn} \hat{R}_{mn} - \frac{13}{8} R^2 \right). \]

## B Reduced phase space formalism

Starting from the basic Poisson brackets (PB) \( \{ b^i_\mu, \pi^j_\nu \} = \delta^i_\mu \delta^j_\nu (x - x') \) etc., one finds that the nontrivial piece of the PB algebra for the primary constraints \( X_A = (\phi_i^\alpha, \Phi_i^\alpha, p_i^\alpha, P_i^\alpha) \) has the form:

\[ \{ \phi_i^\alpha, \Phi_j^\beta \} = -2a \sigma \epsilon^{0 \alpha \beta} \eta_{ij} \delta, \quad \{ \phi_i^\alpha, p_j^\beta \} = -\epsilon^{0 \alpha \beta} \eta_{ij} \delta, \]

\[ \{ \Phi_i^\alpha, P_j^\beta \} = -\frac{a}{m^2} \epsilon^{0 \alpha \beta} \eta_{ij} \delta. \]

The consistency conditions of \( X_A \) determine the corresponding multipliers, which are conveniently written in the form:

\[ u_i^\alpha = \hat{b}_i^\alpha = -\epsilon^{ijk} \omega_j b_{ka} + \nabla_a b_i^0, \]

\[ v_i^\alpha = \omega_i^\alpha = \nabla_a \omega_i^a + \frac{1}{2} b \epsilon^{0 \alpha \beta} (f_i^{\beta} - f h^{ij}_i), \]

\[ w_i^\alpha = \lambda_i^\alpha = -\epsilon^{ijk} \omega_j \lambda_k a + \nabla_a \lambda_i^a + 2a \Lambda \epsilon^{ijk} b_j b_{ka} - a \sigma b \epsilon^{0 \alpha \beta} (f_i^{\beta} - f h^{ij}_i) + \frac{a}{m^2} b \epsilon^{0 \alpha \beta} T_i^\beta, \]

\[ z_i^\alpha = \dot{f}_i^\alpha = -\epsilon^{ijk} \omega_j f_{ka} + \nabla_a f_i^a - \frac{m^2}{a} \epsilon^{ijk} (b_{j0} \lambda_k a - b_{ja} \lambda_k a). \]

Now, we go over to the reduced phase space \( R_1 \), defined by eliminating the momentum variables from the second class constraints \( X_A \). Consider the 24 \( \times \) 24 matrix \( \Delta \) with matrix elements \( \Delta_{AB} = \{ X_A, X_B \} \):

\[ \Delta(x, y) = \left( \begin{array}{ccc} 0 & -2a \sigma & -1 & 0 \\ -2a \sigma & 0 & 0 & -\frac{a}{m^2} \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{a}{m^2} & 0 & 0 \end{array} \right) \otimes \epsilon^{0 \alpha \beta} \eta_{ij} \delta(x - y). \]
The matrix $\Delta$ is regular, and its inverse has the form

$$
\Delta^{-1}(y, z) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{m^2}{a} \\
1 & 0 & 0 & -2m^2\sigma \\
0 & \frac{m^2}{a} & -2m^2\sigma & 0
\end{pmatrix} \otimes \epsilon_{0\beta\gamma} \eta^{ik}\delta(y - z).
$$

Thus, the constraints $X_A$ are second class, and $\Delta^{-1}$ defines the DBs in $R_1$:

$$
\{\phi, \psi\}^*_1 = \{\phi, \psi\} - \{\phi, X_A\}(\Delta^{-1})^{AB}\{X_B, \psi\}.
$$

Explicit form of the nontrivial DBs is displayed in (3.5).

C Dirac brackets

In this appendix, we display the set of DBs, needed in the main text.

We start with the DBs of the secondary constraints:

$$
\begin{align*}
\{\hat{\mathcal{H}}_i, \mathcal{H}_j\}^*_1 &= 4a\Lambda_0\varepsilon_{ijk}\mathcal{T}^k\delta \\
&+ a \frac{\delta}{\delta b^0} \left( \nabla_\alpha (bT^\alpha_i) - \frac{b}{2}\varepsilon_{imn}(f^{ma} - fh^{ma}\lambda^n_\alpha) \right) - (i \leftrightarrow j) \\
\{\hat{\mathcal{H}}_i, \mathcal{K}_j\}^*_1 &= -\varepsilon_{ijk}\hat{\mathcal{K}}^k\delta, \\
\{\mathcal{H}_i, \hat{\mathcal{R}}_j\}^*_1 &= \frac{a}{m^2} \left[ \delta \frac{\delta}{\delta f^0} \nabla_\alpha (bT^\alpha_i) - \delta \frac{\delta}{\delta f^0} \nabla_\alpha (bT^0_i) - \frac{b}{2}\varepsilon_{imn}(\delta^m_jg^{0\alpha} - h^0_jh^{ma}\lambda^n_\alpha) \right] \\
\{\hat{\mathcal{H}}_i, \mathcal{T}_j\}^*_1 &= -\frac{m^2}{a}\mathcal{R}^k\delta + \frac{1}{2}b\varepsilon_{jmn}\left[ (h^0_i0^m - f^m_i(h^{n0} - b^0ng^{00}) - f^0_ih^{m0}b^0_0) \right] \delta, \\
\{\mathcal{K}_i, \hat{\mathcal{K}}_j\}^*_1 &= -\varepsilon_{ijk}\hat{\mathcal{K}}^k\delta, \\
\{\mathcal{K}_i, \mathcal{T}_j\}^*_1 &= -\varepsilon_{ijk}\mathcal{T}^k\delta, \\
\{\hat{\mathcal{R}}_i, \hat{\mathcal{R}}_j\}^*_1 &= \left[ -\partial_\alpha (bh^0_ih^0_j) + b\varepsilon_{imn}\omega^m_\alpha(g^{0\alpha}\delta^n_j - h^{n0}h^0_j) \right] \delta - (i \leftrightarrow j), \\
\{\hat{\mathcal{R}}_i, \mathcal{T}_j\}^*_1 &= \frac{1}{2}b\varepsilon_{jmk}(\delta^m_i g^{0\beta} - h^0_i h^{m0})b^k_\beta \delta, \\
\{\mathcal{T}_i, \mathcal{T}_j\}^*_1 &= 0.
\end{align*}
$$

(C.1a)

The DBs between the secondary first class constraints are given by:

$$
\begin{align*}
\{\hat{\mathcal{H}}_i, \mathcal{H}_j\} &= -\varepsilon_{ijk}(f^{kn} - fh^{kn})\hat{\mathcal{K}}_n\delta, \\
\{\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\hat{\mathcal{H}}^k\delta, \\
\{\mathcal{K}_i, \hat{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\hat{\mathcal{K}}^k\delta.
\end{align*}
$$

(C.1b)
Finally, we display the most important DBs involving the tertiary constraints:

\[
\{ T_i, \varepsilon^{0\alpha\beta} f_{\alpha\beta} \}_1 = -\frac{m^2}{a} \varepsilon^{0\alpha\beta} \varepsilon_{ijk} b^i \alpha b^k \beta \delta = -\frac{m^2}{a} 2b h' i \delta, \\
\{ T_i, \varepsilon^{0\alpha\beta} \lambda_{\alpha\beta} \}_1 = T_i \delta \approx 0, \\
\{ R_i, \varepsilon^{0\alpha\beta} f_{\alpha\beta} \}_1 = T_i \delta \approx 0, \\
\{ R_i, \varepsilon^{0\alpha\beta} \lambda_{\alpha\beta} \}_1 = \frac{d^2}{2m^2} [f_i^0 - fh_i^0 + g^{00}(f_0 + 2m^2 \sigma b_0) - h_i^0 (f_0 - 2m^2 \sigma)] \delta, \\
\{ \varepsilon^{0\alpha\beta} \lambda_{\alpha\beta}, \varepsilon^{0\gamma\delta} f_{\gamma\delta} \}_1 = -\varepsilon^{0\alpha\beta} f_{\alpha\beta} \delta \approx 0. 
\]

(C.2)

D Two identities

In this appendix, we prove that equations (3.10a) and (3.12b) are identities.

1. Equation (3.10a) can be rewritten in the following form:

\[
Z^{\alpha \beta} = f_{\beta}^\alpha U_{\alpha}^\beta + U_{\alpha}^0 f_0^\alpha = b^{-1} b_i^0 \nabla_{\alpha} \left[ b(f_i^\alpha - f h_i^\alpha) \right].
\]

(D.1)

Using the relations:

\[
Z_{\alpha}^\alpha = h_i^\alpha \nabla_{\alpha} f_i^0, \\
Z_{\alpha}^\alpha = h_i^\alpha \nabla_{\alpha} f_0^i - f_0^\alpha \nabla_{\alpha} f_i^0 \approx b_i^0 \nabla_{\alpha} f_i^0 - f_0^\alpha \nabla_{\alpha} h_i^\alpha,
\]

the left-hand side of (D.1) can be transformed into

\[
L = b_i^0 \nabla_{\alpha} f_i^\alpha - f_0^i \nabla_{\alpha} h_i^\alpha - f_{\beta}^\beta b_i^0 \nabla_{\alpha} h_i^\alpha + f_0^\alpha b_0^0 \nabla_{\alpha} b_i^0.
\]

Let us now rewrite the right-hand side of (D.1) as

\[
R = b_i^0 \nabla_{\alpha} f_i^\alpha - f_0^i \nabla_{\alpha} h_i^\alpha + f_0^\alpha b^{-1} \nabla_{\alpha} b.
\]

By noting that

\[
b^{-1} \nabla_{\alpha} b = h_i^0 \nabla_{\alpha} b_i^0 + h_i^\beta \nabla_{\alpha} b^\beta, \\
f_0^\alpha h_{\beta}^\beta \nabla_{\alpha} b_i^\beta \approx - f_0^\alpha b_0^\beta \nabla_{\alpha} h_i^\beta \approx -(f_i^0 - f_0^i b_i^0) \nabla_{\alpha} h_i^\alpha,
\]

we find \( R \equiv L \), which implies that equation (3.10a) is satisfied identically.

2. The consistency condition (3.12b) can be rewritten in the form:

\[
Z_i^{0\beta} = Z_i^{\beta 0}.
\]

(D.2a)

Since \( Z_i^{0\beta} \) is determined from (3.10b), the proof that (D.2a) is an identity is realized by showing that the substitution of (D.2a) into (3.10b) yields an identity. By making use of (D.2a) and \( Z_i^{0\beta} = Z_i^{\beta 0} \), equation (3.10b) takes the following form:

\[
Z_i^{0\beta} + f_{\beta}^0 U^\alpha_{\alpha} - (f_{\beta}^\alpha - f_0^{\delta^\alpha}) U^\alpha_{\alpha} + b^{-1} b_i^\beta \nabla_{\alpha} [b(f_i^\alpha - f h_i^\alpha)] = 0.
\]

(D.2b)
Let us now use the following relations:

\[
Z^0_\beta = h^0_\beta \nabla f_0 - f^0_\beta \nabla b^0_\beta - \frac{m^2}{a} \varepsilon_{ijk} \left( h^0 b^0_\lambda \lambda^k_\beta - h^0 b^0_\beta \lambda^k_0 \right), \\
\begin{align*}
&f^0_\beta U^\alpha_\alpha - (f^\alpha_\beta - f^\delta_\beta) U^0_\alpha = - f^0_\beta b^0_\alpha \nabla h^0_\alpha + (f^\alpha_\beta - f^\delta_\beta) b^0_\alpha \nabla h^0_0, \\
&b^{-1} b^\beta_\beta \nabla_\alpha \left[ b(f^\alpha_\beta - f h^0_\alpha) \right] = (b^{-1} \nabla b)(f_\beta^\alpha - f^\delta_\beta) + b^\beta_\alpha \nabla f^\alpha_\beta - \partial_\beta f - f b^\beta_\beta \nabla h^0_\alpha,
\end{align*}
\]

and, in addition to that,

\[
(b^{-1} \nabla b)(f^\alpha_\beta - f^\delta_\beta) = -(b^0_0 \nabla h^0_\alpha)(f_\beta^\alpha - f^\delta_\beta) - (\nabla h^0_\alpha)(f^\beta_\alpha - b^0_0 f^\beta_0 - f b^\beta_\beta).
\]

Then, the left-hand side of (D.2b) takes the following form:

\[
\begin{align*}
&h^{\mu}_\beta \nabla_\beta f^{\mu}_\mu - \partial_\beta f - f^{\mu}_\beta \nabla_\beta b^{\mu}_\mu + h^\alpha_\beta (\nabla f^{\beta}_\beta - \nabla f^{\beta}_\alpha) - \frac{m^2}{a} \varepsilon_{ijk} \left( h^0 b^0_\lambda \lambda^k_\beta - h^0 b^0_\beta \lambda^k_0 \right) \\
&- f^{\mu}_\beta h^{\mu}_\beta - f^{\mu}_\beta \nabla_\beta b^{\mu}_\mu - \frac{m^2}{a} \varepsilon_{ijk} \left( h^{\mu}_\beta b^\mu_\alpha \lambda^k_\beta - h^{\mu}_\beta b^\mu_\beta \lambda^k_\mu \right) \\
&\approx (f^{\mu}_\beta - f^{\mu}) \nabla_\beta b^{\mu}_\mu \approx 0.
\end{align*}
\]

Hence, relation (D.2b) is satisfied identically, which completes our proof.

**E Second class constraints**

In this appendix, we show that the set of 20 constraints in the second column of Table 1 are second class. Instead of calculating the determinant of the 20 × 20 matrix of the related DBs, the proof is derived iteratively.

**Step 1.** We begin by considering the subset of constraints \(Y_A := (\theta_0, \varphi, P^{00}, P^0_0)\). The 6 × 6 matrix \(\Delta_1\) with matrix elements \(\{Y_A, Y_B\}_1^*\) has the form:

\[
\Delta_1 = \begin{pmatrix}
0_{3\times3} & A_{3\times3} \\
- A_{3\times3}^T & 0_{3\times3}
\end{pmatrix},
\]

\[
A := \begin{pmatrix}
\{\theta_0, P^{00}\}_1^* & \{\theta_0, P^0_0\}_1^* \\
\{\varphi, P^{00}\}_1^* & \{\varphi, P^0_0\}_1^*
\end{pmatrix}.
\]

The explicit form of \(A\) reads:

\[
A = \begin{pmatrix}
- \delta^\beta_\beta & - g_{00} \\
\Omega^0_0 & \Omega^0_0
\end{pmatrix} \delta.
\]

Using the formulas

\[
\begin{align*}
det A &= g_{00} \Omega^0_0, \\
det \Delta_1 &= det(AA^T) = (det A)^2,
\end{align*}
\]

we see that \(\Delta_1\) is regular provided the condition (3.16b) is satisfied.
Step 2. Next, we focus our attention on the subset $Z_A := (\psi_{0\beta}, \chi, p^{\alpha 0}, p_0^0)$. The corresponding $6 \times 6$ matrix $\Delta_2$ reads:

$$\Delta_2 = \begin{pmatrix} B_{3 \times 3} & C_{3 \times 3} \\ -C_{3 \times 3}^T & 0_{3 \times 3} \end{pmatrix},$$

where

$$B := \begin{pmatrix} -2\varepsilon_{0\alpha\beta}\lambda_{00} & -2\varepsilon_{0\alpha\gamma}\lambda_{0}^\gamma \\ 2\varepsilon_{0\beta\gamma}\lambda_{0}^\gamma & 0 \end{pmatrix} \delta,$$

$$C := \begin{pmatrix} -\delta_{0\beta} & -g_{0\alpha} \\ g_{0\beta} & 1 \end{pmatrix} \delta.$$

The matrix $\Delta_2$ is regular:

$$\det(C) = g_{00}^0 g_{00},$$
$$\det \Delta_2 = (\det C)^2. \quad (E.2)$$

Step 3. Finally, we consider the remaining subset $W_A = (T_i, \varepsilon_{0\alpha\beta}\lambda_{0\alpha\beta}, \hat{R}'_i, \varepsilon_{0\alpha\beta} f_{\alpha\beta})$; these constraints do not contain the variables $f_{i0}, \lambda_i$. The $8 \times 8$ matrix $\{W_A, W_B\}^*_1$ takes the form

$$\Delta_3 = \begin{pmatrix} 0_{4 \times 4} & M_{4 \times 4} \\ -M_{4 \times 4}^T & N_{4 \times 4} \end{pmatrix},$$

where

$$M = \begin{pmatrix} D_{3 \times 3} & E_{3 \times 1} \\ -H_{1 \times 3}^T & 0_{1 \times 1} \end{pmatrix}, \quad N = \begin{pmatrix} F_{3 \times 3} & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 1} \end{pmatrix},$$

and the matrices $D, E, F$ and $H$ are given by

$$D_{ij} := \{T_i, \hat{R}'_j\}^*_1 = -b \left[ \varepsilon_{ijn} \left( \frac{1}{2} h_{i0}^n - g_{00} b_{0}^n \right) - h_{j}^0 \varepsilon_{imn} b_{m}^n h_{i0}^n \right] \delta,$$
$$E_i := \{T_i, \varepsilon_{0\alpha\beta} f_{\alpha\beta}\}^*_1 = -\frac{m^2}{a} 2b h_{i0}^0 \delta,$$
$$H_i := \{\hat{R}'_i, \varepsilon_{0\alpha\beta} \lambda_{0\alpha\beta}\}^*_1 = 2ab \Omega_i^0 \delta,$$
$$F_{ij} := \{\hat{R}'_i, \hat{R}'_j\}^*_1.$$

The calculation of $\det \Delta_3$ yields

$$\det M = \frac{1}{2} \varepsilon^{ijk} \varepsilon_{mn} D_{im} D_{jn} E_k H_l = -m^2 b^4 g_{00} \Omega_{00},$$
$$\det \Delta_3 = (\det M)^2.$$

Thus, $\det \Delta_3 \neq 0$ provided $\Omega_{00} \neq 0$. 

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