On cosmology in nonlinear multidimensional gravity with multiple factor spaces

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Within the scope of multidimensional Kaluza–Klein gravity with nonlinear curvature terms and two spherical extra spaces $S^m$ and $S^n$, we study the properties of an effective action for the scale factors of the extra dimensions. Dimensional reduction leads to an effective 4D multiscalar-tensor theory. Based on qualitative estimates of the Casimir energy contribution on a physically reasonable length scale, we demonstrate the existence of such sets of initial parameters of the theory in the case $m = n$ that provide a minimum of the effective potential that yield a fine-tuned value of the effective 4D cosmological constant. The corresponding size of extra dimensions depends of which conformal frame is interpreted as the observational one: it is about three orders of magnitude larger than the standard Planck length if we adhere to the Einstein frame, but it is $n$-dependent in the Jordan frame, and its invisibility requirement restricts the total dimension to values $D = 4 + 2n \leq 20$.

1 Introduction

In modern physics, the concept of extra dimensions appears in various theoretical contexts such as Kaluza–Klein models [1–3], brane-world models [4, 5], superstring/M-theories [6], multidimensional cosmology, etc. Despite the unobservable nature of extra dimensions, their possible existence provides a very elegant theoretical background for a number of key physical problems (geometrization of the interactions, possible variations of fundamental constants, etc., see, e.g., [7–10]), and it is also an essential and inevitable feature of many theories such as superstring theory [6]. At present, quite a general class of multidimensional theories is under consideration, including nonlinear functions of the Ricci scalar ($F(R)$ theories) and high-order curvature invariants (e.g., the Gauss—Bonnet term) motivated by various inflationary scenarios and low-energy limits of superstring models.

One of possible explanations of the unobservable nature of extra dimensions is the concept of spontaneous compactification [2, 3], according to which the geometry of the entire multidimensional manifold (bulk) can be understood as a topological product $M^4 \times V^n$ of the observable 4D space-time $M^4$ and a compact extra space $V^n$ of appropriate dimension and topology with a very small characteristic length scale, in many cases close to Planckian. This kind of explanation is widely used in Kaluza–Klein-like theories. A realistic model of such a type should include (at least in principle) some explanatory mechanism for stabilization of the radii of compact extra dimensions to keep them unobservable on a sufficiently large time scale of the Universe evolution. This stability can be global or local, and may be achieved on the (quasi)classical or/and quantum level.

In the simplest cases, the stability conditions of ground state manifolds of the type $M^4 \times V^n$ can be related to the existence of minima of an effective potential of scalar fields appearing from the extra-dimensional metric tensor components at dimensional reduction. It is clear that, beyond a classical level, there can also be some quantum vacuum contributions to the effective potential such as the Casimir energy of scalar, vector gauge, and spinor fields due to the compact topology of the extra factor space [11–14].

In this paper we continue our study of cosmological models in the framework of nonlinear multidimensional gravity, taking into account possible contributions of quantum vacuum effects. In [15, 16], the space-time geometry was chosen in the extended Kaluza-Klein form, $M^4 \times S^n$, where $M^4$ is the observed weakly curved 4D space-time, and $S^n$ is an $n$-dimensional sphere of sufficiently small size to be invisible by modern instruments. Now we
extend the study to more complex geometries containing a number (at least two) of internal factor spaces.

As compared to the Casimir energy on $\mathbb{M}^4 \times S^n$ manifolds [13, 14], calculations of this energy on $\mathbb{M}^4 \times S^p \times S^m$ make a much more complicated problem which has been studied in a number of papers [17, 18]. In the case of an even number of extra dimensions, there is an additional difficulty in renormalizing the logarithmically divergent terms in the Casimir energy [19].

The analysis of a possible stability of extra dimensions in multidimensional gravity with high-order curvature invariants (up to $R^m$) on $\mathbb{M}^4 \times S^D$ geometries under a number of assumptions such as neglecting high-order curvature contributions to the kinetic terms in $1/D$-approximation and a vanishing effective cosmological constant. For geometries $\mathbb{M}^4 \times S^m \times S^n$ it was argued that the effective potential may not be bound from below.

Our approach to dimensional reduction in theories with nonlinear curvature terms has a number of features different from others in some aspects, such as: using a slow-change approximation; a transition from the Jordan conformal frame to the Einstein one; a dynamic treatment of scale factors of extra dimensions as scalar fields on $\mathbb{M}^4$ with their own kinetic terms (which need to be checked for positive definiteness); a physically reasonable ("semiclassical") choice of the characteristic length scales of extra dimensions; approximate estimation of Casimir contributions at these ranges.

As in [15, 16], we begin with a sufficiently general $D$-dimensional gravitational action, then, under suitable assumptions on the space-time geometry, follow a dimensional reduction and a transition to the Einstein conformal frame. After that, we demonstrate the existence of such sets of the initial parameters that provide a minimum of the effective potential at a physically reasonable length scale, and it is also shown that the kinetic term of the effective scalar fields is positive-definite, hence these minima really describe stable stationary configurations.

2 Basic equations, reduction to 4 dimensions

In our description of multidimensional gravity, we follow [7, 21], where a more detailed derivation can be found. We are considering a $D$-dimensional space-time $\mathbb{M}$ with the structure

$$\mathbb{M} = \mathbb{M}_0 \times \mathbb{M}_1 \times \cdots \times \mathbb{M}_N,$$

where $\dim \mathbb{M}_i = d_i$, and the metric

$$ds^2_D = g_{ab}(x)dx^a dx^b + \sum_{i=1}^N e^{2\beta_i(x)} g^{(i)},$$

where $(x)$ denotes the dependence on the first $d_0$ coordinates $x^\theta$; $g_{ab} = g_{ab}(x)$ is the metric in $\mathbb{M}_0$, and $g^{(i)}$ are $x$-independent $d_i$-dimensional metrics of factor spaces $\mathbb{M}_i$, $i = 1, N$.

We are dealing with a theory of gravity with the action

$$S = \frac{1}{2} m_D^{D-2} \int \sqrt{-g} d^D x \left[ F(R) + c_1 R_{AB} R^{AB} + c_2 K + L_m \right],$$

where $D g = |\det(g_{AB})|$, $F(R)$ is an arbitrary function of the scalar curvature $R$ of $\mathbb{M}$; $c_1$, $c_2$ are constants; $R_{AB}$ and $K = R_{ABCD} R^{ABCD}$ are the Ricci tensor and Kretschmann scalar of $\mathbb{M}$, respectively; $L_m$ is the matter Lagrangian (it can formally include the quantum vacuum contribution). Capital Latin indices cover all $D$ coordinates, small Latin ones ($a, b, \ldots$) the coordinates of the factor space $\mathbb{M}_0$, and $a_i, b_i, \ldots$ the coordinates of $\mathbb{M}_i$.

Let us assume that the factor spaces $\mathbb{M}_i$ are $d_i$-dimensional compact spaces of constant nonzero curvature $K_i = \pm 1$, i.e., spheres ($K_i = 1$) or compact $d_i$-dimensional hyperbolic spaces ($K_i = -1$) with a fixed curvature radius $r_0$, normalized to the $D$-dimensional analogue $m_D$ of the Planck mass, i.e., $r_0 = 1/m_D$ (we are using the natural units $\hbar = c = 1$). Then we have

$$\overline{R}^{ab}_{\phantom{ab}cd} = r_i K_i m_D^2 \delta^{ab}_{\phantom{ab}cd},$$

$$\overline{R}^a_{\phantom{a}ai} = K_i m_D^2 (d_i - 1) \delta^a_{\phantom{a}ai},$$

$$\overline{R}_i = K_i m_D^2 d_i (d_i - 1).$$

The scale factors $r_i(x) \equiv e^{\beta_i}$ in (2) are dimensionless. The overbar marks quantities obtained from the factor space metrics $g_{ab}$ and $g^{(i)}$ taken separately, $\beta_a = \partial_a \beta$, and $\delta^{ab}_{\phantom{ab}cd} = \delta^a_{\phantom{a}c} \delta^b_{\phantom{b}d} - \delta^a_{\phantom{a}d} \delta^b_{\phantom{b}c}$ and similarly for other kinds of indices.
To simplify further calculations, we are using the slow-change approximation [22]: we suppose that all derivatives \( \partial_a \) are small as compared to the extra-dimension scale, so that each \( \partial_a \) involves a small parameter \( \varepsilon \), and we neglect all quantities of orders higher than \( O(\varepsilon^2) \). This approximation proves to be valid in almost all thinkable situations. In the descriptions of the modern Universe with small extra dimensions it is valid up to tens of orders of magnitude.

Then, integrating out all subspaces \( M_i \) in (3) and subtracting a full divergence, we obtain the \( d_0 \)-dimensional action

\[
S = \frac{1}{2} \mathcal{V} m_D^{d_0-2} \int \sqrt{g_0} d^{d_0} x \left\{ e^\sigma F'(\phi) \bar{R}_0 + K_J - 2[V_J(\phi_i) + V_{J(Cas)}] \right\},
\]

where \( g_0 = |\det(g_{ab})|, \mathcal{V} \) is a product of volumes of \( n \) compact \( d_i \)-dimensional spaces \( M_i \) of unit curvature;

\[
\phi_i := K_i m_D^2 (d_i - 1) e^{-2\beta_i}, \quad \phi := \sum_i \phi_i;
\]

with the kinetic and potential terms

\[
K_E = \frac{1}{d_0 - 2} \left( (\partial^2 + F'/F' \partial^2 \phi)^2 \right) \left( \frac{F''}{F'} \right)^2 \left( \partial \phi \right)^2 + \sum_i \left[ d_i \left( 1 + \frac{4}{F'}(c_1 + c_2) \phi_i \right) \right] \left( \partial \beta \right)^2, \quad (11)
\]

\[
-2V_E(\phi_i) = e^{-2(\sigma/(d_0 - 2))} \left| F' \right| - d_0/(d_0 - 2) \left[ F(\phi) + \sum_i d_i \phi_i^2 \left( c_1 + \frac{2c_2}{d_i - 1} \right) \right], \quad (12)
\]

where the metric \( \bar{g}_{ab} \) is used, the indices are raised and lowered with \( g_{ab} \) and \( \bar{g}^{ab} \), and \( V_{E(Cas)} \) is the Casimir contribution to the total Einstein-frame potential obtained from \( V_{J(Cas)} \) after the transformation (9). The quantities \( \beta_i \) and \( \sigma \) are expressed in terms of the \( n \) fields \( \phi_i \), whose numbers coincide with the numbers of factor spaces.

### 3 A search for stable extra dimensions

#### 3.1 Equations for \( M_0 \times S^m \times S^n \)

In what follows, we will try to find stable equilibria of the system with the action (12) that correspond to the modern state of the expanding Universe with the metric \( g_{ab} \), naturally putting \( d_0 = 4 \), so that \( M_0 = M_4 \). We will also restrict ourselves to two extra factor spaces \( M_1 \) and \( M_2 \). Furthermore, a final interpretation of the results depends on which conformal frame is chosen as the physical (observational) one [23, 24], and this in turn depends on the way in which fermions enter into the (so far unknown) underlying unification theory of all interactions. From an infinite number of such opportunities, we will consider two most natural ones: the Einstein frame with the action (10), and the Jordan frame with the action (5), obtained directly from the D-dimensional theory.

Stable points must be found in any case using the action (10) as minima of the potential (12) provided that the kinetic term is positive-definite (which is a priori not at all guaranteed). Other conditions to be fulfilled by such a minimum are:

A. Since we adhere to classical gravity, the size of the extra dimensions should appreciably exceed the fundamental length scale \( r_0 = 1/m_D \), i.e., \( r_i/r_0 = e^{\beta_i} \gg 1 \ (i = 1, 2) \).
B. The extra dimensions should not be observable by modern instruments, hence, \( r_i = r_0 e^{\beta_i} \lesssim 10^{-17} \) cm, which is close to the TeV energy scale.

C. The 4D cosmological constant \( \Lambda_4 \) which corresponds to the minimum value of the potential, should be responsible for the observed dark energy density, so that

\[
0 < \Lambda_4/m_4^2 \sim 10^{-120}, \tag{13}
\]

where \( m_4 \sim 10^{-5} \) g is the 4D Planck mass.

Let us assume that our space-time contains two spherical extra factor spaces with dimensions \( d_1 = m, \ d_2 = n \), so that \( D = 4 + m + n \). Also, for simplicity, we assume that

\[
F(R) = -2\Lambda_D + R, \tag{14}
\]

where \( \Lambda_D \) is the D-dimensional cosmological constant. Then \( F(\phi) = -2\Lambda_D + \phi, \ F' = 1, \ \Lambda^n = 0 \), and the curvature nonlinearity of the theory is only contained in terms with \( c_1 \) and \( c_2 \) in the action (3).

Now we can write for the dimensionless quantity \( W(x, y) = r_0^2 V_E(x, y) \) and the kinetic term:

\[
W(x, y) = \frac{x^m y^n}{2} \left[ k_1 x^4 + k_2 y^4 - m(m-1)x^2 - n(n-1)y^2 + \lambda \right] + W_{\text{Cas}}, \tag{15}
\]

\[
K(x, y) = \frac{m}{2} (\partial x)^2 \left[ \frac{m+2}{x^2} + 8(m-1)(C_1+C_2) \right] + \frac{n}{2} (\partial y)^2 \left[ \frac{n+2}{y^2} + 8(n-1)(C_1+C_2) \right] + \frac{mn(\partial x, \partial y)}{xy}. \tag{16}
\]

where \( x = e^{-\beta_1}, \ y = e^{-\beta_2}, \ C_1 = c_1/r_0^2, \ C_2 = c_2/r_0^2, \ \lambda = r_0^2\Lambda_D \), and

\[
k_1 = -\frac{m(m-1)}{2} \left[ C_1(m-1) + 2C_2 \right],
\]

\[
k_2 = -\frac{n(n-1)}{2} \left[ C_1(n-1) + 2C_2 \right].
\]

We will now seek such combinations of the input parameters \( m, n, C_1, C_2 \) that \( W(x, y) \) has a local minimum at some \( x = x_0 \) and \( y = y_0 \) much smaller than unity in order to satisfy requirement A. On the other hand, \( x_0 \) and \( y_0 \) should not be too small in order to conform to requirement B, but the corresponding estimate crucially depends on our assumption on the value of \( m_D = 1/r_0 \). By requirement C, the value of \( W \) at such a minimum must be positive but extremely small.

### 3.2 The Casimir contribution

Let us now estimate the Casimir term \( W_{\text{Cas}} \) in the potential \( W(x, y) \). The corresponding calculations are quite complicated and have been performed in a number of papers [17–19] assuming that the 4D subspace \( M_0 \) is flat. Quite evidently, we can use this approximation for calculating \( W_{\text{Cas}} \) if we suppose that \( M_0 \) is very weakly curved as compared to the extra factor spaces \( M_1, M_2 \), and it is precisely this approximation that we have been using to obtain all other terms in \( W(x, y) \) and \( K(x, y) \).

Therefore it looks reasonable to use the results obtained with flat \( M_0 \) in our wider context.

For estimation purposes we can use the analytic expression for \( M_0 \times S^3 \times S^3 \) [18] found for the case of two spheres \( S^3 \) of approximately equal radii, \( r_2/r_1 = 1 + \epsilon \), where \( \epsilon \lesssim 0.25 \), and, in our notations, \( r_1 = r_0 e^{\beta_1}, \ r_2 = r_0 e^{\beta_2} \):

\[
V_{J_{\text{Cas}}} = -\frac{1}{r_1^4 V} \left[ b \left( 3.639 \times 10^{-4} - 6.053 \times 10^{-4} \epsilon \right) + 3.315 \times 10^{-4} (1 - 2\epsilon) \ln(r_1/\bar{\mu}) \right] + f(3.657 \times 10^{-6} - 5.414 \times 10^{-5} \epsilon), \tag{17}
\]

where \( b \) and \( f \) are the numbers of spin-0 and spin-1/2 fields, and \( \bar{\mu} \) is an unknown constant emerging in the renormalization procedure. From other calculations, which are mostly numerical [17, 18], it follows that \( V_{J_{\text{Cas}}} \) has approximately the same order of magnitude as in Eq. (17), that is, \( 10^{-4} \) or less (due to the factor \( 1/V \)) times \( r_1^{-4} \) (or, restoring the symmetry between \( r_1 \) and \( r_2 \), times \( r_1^{-2}r_2^{-2} = r_0^{-4}x^2y^2 \) times the number of field degrees of freedom. The latter may be probably estimated as being of the order of 100. Assuming that the logarithmic term, appearing in the cases of even \( m + n \), is not too large (see different arguments on this subject in [19]), one can write

\[
V_{J_{\text{Cas}}} \lesssim r_0^{-4} x^2 y^2,
\]

and accordingly for the contribution to \( W(x, y) \)

\[
W_{\text{Cas}} \lesssim x^{2m+2} y^{2n+2}. \tag{18}
\]

Comparing this expression with (15), we see that the contribution (18) contains an extra factor

\[\text{A numerical study shows that if the logarithmic term is large enough to make significant the Casimir contribution to } W, \text{ then a possible minimum of } W \text{ happens to be with } x \text{ or } y \text{ close to unity, where our semiclassical approach is no more applicable.} \]
Figure 1: The function $W(x, y)$ close to its minimum for $m = n = 5$, $C_1 = -127.25$, $C_2 = 128$, $\lambda = 0.0197628$. At $x = y$ we have $W = x^{10}(0.0197628 - 20x^2 + 5060x^4)$. Left panel: $W(x, x)$ for the parameters indicated above (solid line), and a bit larger and smaller $\lambda$ (dashed lines). Right panel: a 3D plot.

3.3 Viable minima of $W(x, y)$

In accord with the above-said, we seek a minimum of $W$ ignoring the Casimir contribution. If we additionally assume $m = n$, the expressions (15) and (16) are symmetric with respect to $x$ and $y$, and it makes sense to seek a minimum of $W$ on the line $x = y$, which substantially simplifies the process. It turns out that the case $n = 3$ is degenerate because, instead of two parameters $C_1$ and $C_2$, both $W$ and $K$ depend on the single combination $C_1 + C_2$. It turns out that in this case there is no minimum of $W$ combined with a positive-definite $K$, which confirms a previously obtained result [18].

For other values of $m = n$ it is possible to find a stable minimum of $W$ under the condition $x = y$, for which the kinetic term $K$ can be shown to be positive-definite under the condition $c_1 + c_2 \geq 0$. It turns out that this minimum occurs at proper “semiclassical” values of $x_0 = y_0 \sim 0.01$ under a proper choice of $c_1$ and $c_2$. Moreover, we have obtained analytically such a value of $\lambda$ that $W(x_0) = 0$, which, under proper fine tuning, makes it possible to satisfy Requirement C:

$$\lambda = -\frac{n(n-1)}{4C_1(n-1) + 8C_2}.$$  (19)

Evidently, in fact, we need a minimum with $W = 0$ in rather a rough approximation: it must be corrected for a corresponding value of $W_{\text{Cas}}$ and fine-tuned to obtain a cosmologically relevant $\Lambda_4$, see the next section.

Examples of the behavior of $W(x, y)$ leading to viable minima of $W(x, y)$ are shown in Figs. 1, 2. It has been directly verified that the kinetic term is positive-definite in a neighborhood of the minimum under the condition $C_1 + C_2 \geq 0$, which holds in both these examples.

4 4D gravity

In multidimensional gravity with constant extra dimensions, the corresponding 4D theory will be evidently the Einstein theory with certain values of the gravitational constant $G_4$ (or the corresponding Planck mass $m_4 = G_4^{-1/2}$) and the effective cosmological constant $\Lambda_4$. However, their values expressed in terms of the initial parameters of the theory (3) depend on which conformal frame is regarded the physical (observational) one. We will consider two most natural opportunities described above, the Einstein frame with the action (10) and the Jordan frame with the action (5). Note that for our present stable extra dimensions we have $m = n$ and $x_0 = y_0$, therefore $W = W(x_0, y_0) \equiv W(x_0)$. 

$x^m y^n$, which, provided that other coefficients in (15) are of the order of unity, makes the term $W_{\text{Cas}}$ insignificant in a search for a minimum of $W$ at which $x \ll 1$ and $y \ll 1$. 

$x(x - 1/4\sqrt{C_1})^2 + C_2(1/4\sqrt{C_1})$.
requirement B that the extra dimensions should be close to the value at which $\lambda$ have roughly $W$ observations that require $\Lambda_4$ terms very far from unity. As a result, $r_0 \sim 10^{-31}$ cm, manifestly satisfying our requirement B that the extra dimensions should be invisible.

The effective cosmological constant is $\Lambda_4 = W(x_0)/r_0^2 = W(x_0)m_4^2/\sqrt{V(n)}$, and to conform to observations that require $\Lambda_4/m_4^2 \sim 10^{-120}$, we must have roughly $W(x_0) \sim 10^{-117}$. Therefore, fine tuning is necessary: the dimensionless parameter $\lambda$ should be close to the value at which $W(x_0) = 0$ with an accuracy depending on $m = n$. More precisely, according to (15), we have (for $m = n$ and $x = y$)

$$W(x, y) = \frac{1}{2} x^{2n} (\lambda + \text{other terms}),$$

hence $\lambda$ should be fine-tuned up to $10^{-117} x_0^{-2n}$, where $x_0$ is also different for different $n$. Thus, for $n = 5$ (Fig. 1), the fine tuning of $\lambda$ must be about $10^{-104}$. For $m = 11$ (Fig. 2), the corresponding accuracy is $\sim 10^{-81}$.

The Casimir contribution to $W(x_0)$ is very large as compared to $10^{-120}$, e.g., with the above parameter values we have $W_{\text{Cas}} \sim 10^{-24}$ for $n = 5$ and $\sim 10^{-48}$ for $n = 11$. This comparatively large value (though decreasing with growing $m = n$) is compensated by fine-tuned values of other parameters of the theory, above all, $\lambda$.

4.2 The Jordan frame

In the Jordan frame, by (5), the 4D Planck mass is related to $m_D$ by

$$m_D^2 = 1/r_0^2 = m_4^2 x_0^{-2n}/\sqrt{V(n)},$$

and the size of both $M_1$ and $M_2$, $r_1 = r_2 = r_0/x_0$. Since $x_0 \ll 1$, $r_0$ is in general a few orders of magnitude larger than the Planck length, which at large enough $n$ may be in tension with the invisibility of extra dimensions. And indeed, we obtain:

- $n = 5$: $r_0 \approx 1.5 \times 10^{-24}$ cm, $r_1 \approx 3.4 \times 10^{-23}$ cm;
- $n = 8$: $r_0 \approx 1.1 \times 10^{-19}$ cm, $r_1 \approx 3 \times 10^{-18}$ cm;
- $n = 11$: $r_0 \approx 5, 4 \times 10^{-14}$ cm, $r_1 \approx 2.14 \times 10^{-12}$ cm.

Thus $n = 5$ and $n = 8$ lead to acceptable values of $r_0$ and $r_1$, but at $n > 8$ they are too large.

The effective cosmological constant in the Jordan frame is obtained if we present the integrand in (5) as $\sqrt{g_4} e^{\sigma} F'[R_4 - 2\Lambda_4 + \text{kinetic term}]$, which in our case ($F' = 1, m = n, x_0 = y_0$) leads to $\Lambda_4 = x_0^{-2n} W(x_0)/r_0^2$. However, expressing $r_0$ in terms of $m_4$, we arrive again at the expression $\Lambda_4 = W(x_0)m_4^2/\sqrt{V(n)}$. The same fine tuning of $\lambda$ as in the Einstein frame and have the same estimate of the Casimir contribution to $W(x)$, despite another value of the fundamental length $r_0 = 1/m_D$. 
5 Conclusion

Considering multidimensional gravity with the action (3) in space-times $M^4 \times S^m \times S^n$, we have found stable states of the extra dimensions under the assumption $m = n$, and these states are located on the line $x_0 = y_0$, which means equal radii of $S^m$ and $S^n$. Our attempts to find asymmetric stable states (such that $x_0 \neq y_0$) in the case $m = n$, or those corresponding to $m \neq n$, had no success (the corresponding kinetic terms turned out to be not positive-definite), but this is not a strict result, and the existence of such states is not completely excluded.

The resulting 4D theory coincides with general relativity when using both the Einstein and Jordan frames to be compared with observation, and in both case we need the same fine tuning of the initial parameters of the $D$-dimensional theory in order to have an acceptable value of the cosmological constant $\Lambda_4$. In both cases the fine tuning is slightly weaker than in the “usual” cosmological constant problem. However, in these two frames we obtain substantially different estimates of the $D$-dimensional Planck scale $m_D$ and the size of extra dimensions $r_1$: while in the Einstein frame $m_D$ almost coincides with the conventional Planck mass $m_4$ and $r_1$ is small enough for any $n$, in Jordan’s frame the estimated values of $m_D$ and $r_1$ are strongly $n$-dependent, and acceptable results are obtained for only $n \leq 8$, that is, $D \leq 20$.

The dimensional reduction of our models results in classical general relativity well describing the modern state of the Universe and thus far successfully passing all experimental tests. In cosmological applications, our models actually lead to the Einstein equations with nonzero $\Lambda_4$, used in the “concordance” $\Lambda$CDM model, but a more advanced problem of describing the whole history of the Universe is not addressed. It may seem to be a step back as compared to many studies that try to give such a description, in particular, those using the same multidimensional action (3).

Thus, in [21] it was shown that in a model with two extra factor spaces, one can choose the initial parameters of the theory in such a way that the resulting model describes an early inflationary stage (with rolling down along a comparatively steep slope of an effective potential of two scalars $\phi_1$ and $\phi_2$) and that of modern accelerated expansion described by an extremely slow descent along a shallow valley of the same potential, with slow increase of the extra dimensions within observational limits. However, unlike ours, the model of [21] did not take into account quantum vacuum effects, which are comparatively easily included in a description of the modern stage (maybe actually beginning at the end of the early inflation) but are strong and make a serious problem at the earliest stage, where the extra dimensions should be highly nonstationary. This goes beyond the scope of our present study, but a qualitative picture may look as follows. The Universe emerges from a large fluctuation of space-time foam in a state corresponding to a point somewhere above the minimum of $W(x, y)$ (if some analog of such a potential can be built in a more advanced model). Then comparatively rapidly the fields roll down to the minimum, and, before they settle down there, decaying oscillations around the minimum give rise to creation of matter. Such a scenario actually conforms to chaotic inflation. This construction may be a subject of future work.

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