Unitarity of
The Realization of Conformal Symmetry
in The Quantum Hall Effect

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Abstract

We study the realization of conformal symmetry in the QHE as part of the $W_{\infty}$ algebra. Conformal symmetry can be realized already at the classical level and implies the complexification of coordinate space. Its quantum version is not unitary. Nevertheless, it can be rendered unitary by a suitable modification of its definition which amounts to taking proper care of the quantum measure. The consequences of unitarity for the Chern-Simons theory of the QHE are also studied, showing the connection of non-unitarity with anomalies. Finally, we discuss the geometrical paradox of realizing conformal transformations as area preserving diffeomorphisms.

1 Introduction

The Quantum Hall Effect (QHE) is related to the existence of energy levels with infinite (macroscopic) degeneracy for a system of two-dimensional electrons in a perpendicular uniform magnetic field. The infinite degeneracy immediately suggests the presence of an infinite group of symmetry, which must have an important role in the understanding of this effect. Both the Integral and Fractional QHE are related to the incompressibility of filled energy levels \[\text{[4]}\]. Therefore, it was natural to associate the symmetry group

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to the group of area preserving diffeomorphisms, as was formally done in [3, 2]. They showed that the symmetry of the classical system is indeed that of area preserving diffeomorphisms, while at the quantum level it becomes deformed to a well-known algebra, namely $W_\infty$.

On the other hand, alternative infinite symmetries, namely, conformal or Kac-Moody, have also been shown to have a role in the physics of the QHE. The 2d Coulomb gas, the simplest 2d CFT, has been utilized since the original work of Laughlin [1]. Later, there were introduced vertex operator techniques [4]. Independently, the study of edge waves led to a Kac-Moody symmetry and, consequently, conformal symmetry [5].

While the importance of these symmetries is well established, it remains to clearly understand their interrelations, in particular, that between $W_\infty$ symmetry and conformal symmetry. Cappelli et al observed that the $W_\infty$ algebra contains a sub-algebra isomorphic to the algebra of conformal transformations and that this sub-algebra is indeed realized as conformal transformations on analytic wave functions. However, they point out that there are important differences with the usual realization of this symmetry in 2d CFT [1], for example, the absence of negative modes. Besides, they leave unexplained the appearance of conformal transformations as a subgroup of area preserving diffeomorphisms, rather contradictory from the geometrical point of view.

Our intention here is to clarify the nature of the realization of conformal symmetry in the Hall effect. We consider in section 2 the classical system, namely, one electron (or several non-interacting electrons) moving in a plane under a perpendicular uniform magnetic field. For zero energy there is symmetry under area preserving diffeomorphisms [3]. In the Hamiltonians formalism, the symmetry is realized as a group of canonical transformations. If we admit complex canonical transformations, the conformal transformations appear as a subgroup. Upon quantization (section 3), the algebra of canonical transformations becomes an algebra of differential operators; the conformal subgroup corresponds to those of first degree in derivatives. However, they are not self adjoint and therefore do not generate unitary transformations in the Hilbert space of holomorphic functions. The reason is that the transformation of the integration measure has not been accounted for. We will show that making the generator self adjoint (taking its real part) is equivalent to adding the missing piece that takes care of the change of the measure under conformal transformations. In section 4 are some pertinent remarks on the peculiarities of the realization of conformal symmetry in the QHE. Section 5 is dedicated to Chern-Simons theory. This theory describes the gauge fluctuations in the Hall system and is mathematically a functional version of the formalism introduced in section 3. The realization of conformal symmetry is again related through unitarity to the larger topological symmetry of the theory. Furthermore, in this case a clear connection with anomalies arises. The paper ends with a discussion on the relation between conformal transformations and area-preserving diffeomorphisms in the QHE.
2 Symmetry of the classical system.

Let us briefly recall the constrained Hamiltonians formulation of one electron with zero energy in a plane under a perpendicular uniform magnetic field [7, 3]. The Hamiltonians

\[ H = \frac{1}{2} m v^2 \]  

is constrained to be zero by imposing that the velocity \( \mathbf{v} \),

\[ \mathbf{v}^a = \frac{1}{m} (p_a + \frac{B}{2} \epsilon_{ab} x^b) \]

vanish. One usually defines complex phase space variables,

\[ a = \frac{1}{2} (-v_2 + i v_1) \]

and \( b \), which results from \( a \) by reversing the sign of the magnetic field \( B \). The constraints are then expressed as

\[ a = a^* = 0, \]

which imply that

\[ p_y = x, \quad p_x = -y. \]

The four-dimensional phase space effectively reduces to a two-dimensional phase space. This fact is the first sign of the dynamics being reduced to the edge of the sample. In the reduced phase space, the value of \( b \) only depends on the coordinates \( x \) and \( y \):

\[ b = x - i y \equiv z^*, \quad b^* = x + i y \equiv z. \]

An arbitrary function of \( x \) and \( y \) generates canonical transformations that leave the (null) Hamiltonians unchanged and are therefore symmetries. Furthermore, they preserve the element of area \( dx \wedge dy \), which is the symplectic form, and constitute the algebra of area preserving diffeomorphisms, called \( w_\infty \).

One can consider as generators arbitrary functions \( F(z, z^*) \), that is not necessarily real. This implies complexifying \( x \) and \( y \). Hence, according to the usual convention we use \( \bar{z} \) instead of \( z^* \).

The transformations of \( z \) and \( \bar{z} \) are still given by

\[ \delta_F z = i \frac{\partial F}{\partial \bar{z}} \]

\[ \delta_F \bar{z} = -i \frac{\partial F}{\partial z} \]

\(^1\)To simplify, we will use units such that \( c = \hbar = 1 \), \( m = 1 \) and \( B = 2 \), implying that the magnetic length is also one.

\(^2\)Note that \( \bar{z} = x - i y \) whereas \( z^* = x^* - i y^* \).
but now $z$ and $\bar{z}$ transform independently. An interesting example is given by

$$F(z, \bar{z}) = -i \bar{z} \epsilon(z),$$

which yields

$$\delta_F z = \epsilon(z),$$

$$\delta_F \bar{z} = -\partial_z \epsilon(z) \bar{z}. \quad (10)$$

Recalling that $\bar{z}$ plays the role of the conjugate momentum of $z$, since their Poisson bracket is

$$\{z, \bar{z}\}_{P.B.} = i,$$

this transformation is the complex analogue of a point transformation: $z$ undergoes an arbitrary (holomorphic) diffeomorphism and $\bar{z}$ transforms as a co-vector.

Given the dual role of $z$ and $\bar{z}$, we can also consider

$$F(z, \bar{z}) = -i z \bar{\epsilon}(\bar{z}),$$

$$\delta_F z = -\partial_{\bar{z}} \bar{\epsilon}(\bar{z}) z,$$

$$\delta_F \bar{z} = \bar{\epsilon}(\bar{z}). \quad (15)$$

Thus, we have holomorphic and anti-holomorphic transformations as subgroups of the complex area preserving diffeomorphisms. However, holomorphic transformations of $z$ affect $\bar{z}$ as well and vice-versa anti-holomorphic transformations, thence they do not commute, unlike those standard in 2dCFT.

3 Realization of conformal symmetry in the quantum system

In the quantum system one has operators, $a, a^\dagger, b$ and $b^\dagger$, in place of phase space variables. The Hamiltonians

$$H = 2 a^\dagger a + 1$$

commutes with any power of $b$ and $b^\dagger$. The constraints are implemented a la Gupta-Bleuler,

$$a \Psi = 0,$$

meaning that the state $\Psi$ belongs to the lowest Landau level. This condition implies that the wave function adopt a particular form,

$$\Psi(x, y) = \psi(z) e^{-\frac{1}{2} z \bar{z}}, \quad (18)$$

4
which is called the holomorphic or coherent state representation. The operators $b$ and $b^\dagger$ act on holomorphic wave functions as:

\begin{align}
    b \psi(z) &= \partial_z \psi(z), \\
    b^\dagger \psi(z) &= z \psi(z).
\end{align}

Hence, the infinite symmetry is generated by arbitrary differential operators in $z$. A convenient basis is given by

\[ \mathcal{L}_{n,m} = z^{n+1} \partial_z^{m+1}, \quad n, m \geq -1. \]

They satisfy the commutation relations of the celebrated $W_\infty$ algebra, a deformation of $w_\infty$.

A notable sub-algebra occurs when $m = 0$, namely, the algebra of holomorphic transformations

\[ \delta \psi(z) = z^{n+1} \partial_z \psi(z). \]

This algebra can be regarded as the quantum representation of $\mathfrak{g}$. We remarked there that the function $F$ in (9) is complex and does not generate transformations in the real $(x, y)$ plane. Its quantum version, a linear combination of operators $\mathcal{L}_{n,0}$, is not self adjoint, as one can easily see. Let us expound this point. We know after Dirac that canonical transformations are represented in Quantum Theory as unitary transformations of the Hilbert space. The generator of a canonical transformation becomes a self-adjoint operator. In particular, $w_\infty$ comprises all the canonical transformations of the reduced phase space; its quantum counterpart, $W_\infty$, is the algebra of unitary transformations on the lowest Landau level [2]. However, the complexification necessary to include holomorphic transformations spoils unitarity and the operators $\mathcal{L}_{n,0}$ do not provide a unitary representation of conformal symmetry.

To look further into this problem, we shall first consider the unitarity of canonical transformations and, in particular, point transformations in a more general setting; namely, an arbitrary quantum system, which we take one dimensional for simplicity. There is one coordinate $x$ and one momentum $p$; states are given by their wave functions $\psi(x)$. Point transformations are generated by

\[ F(x, p) = f(x) p, \]

with

\begin{align}
    \delta_F x &= f(x), \\
    \delta_F p &= -f'(x) p.
\end{align}

\[ ^3 \text{This representation can alternatively be obtained by quantizing the constrained system with Poisson bracket (12).} \]

\[ ^4 \text{True at least locally.} \]
Since in the coordinate representation \( p = -i \partial_x \), one may think that
\[
\delta_F \psi(x) \equiv i F \psi(x) = f(x) \partial_x \psi(x).
\]  
(26)

Hence, \( \psi(x) \) transforms as a scalar. To see the change of the norm
\[
\langle \psi | \psi \rangle = \int dx \psi^*(x) \psi(x),
\]  
(27)
we calculate
\[
\delta_F (\psi^*(x) \psi(x)) = f(x) \partial_x (\psi^*(x) \psi(x))
\]  
(28)
and find a non-null value.

It is natural because we have not taken into account the change of the element of volume in (27). On the other hand, as the alert reader has probably noticed, the operator \( F \) in (26) is not self-adjoint. We must consider instead
\[
\hat{F} = \frac{1}{2} (F + F^\dagger) = \frac{1}{2} (f(x)(-i \partial_x) + (-i \partial_x)f(x)),
\]  
(29)
and
\[
\delta_{\hat{F}} \psi(x) = f(x) \partial_x \psi(x) + \frac{1}{2} \partial_x f(x) \psi(x).
\]  
(30)
Now,
\[
\delta_{\hat{F}} (\psi^*(x) \psi(x)) = f(x) \partial_x (\psi^*(x) \psi(x)) + \partial_x f(x) (\psi^*(x) \psi(x)),
\]  
(31)
which accounts for the change of the element of volume, so that
\[
\delta_{\hat{F}} \langle \psi | \psi \rangle = 0.
\]  
(32)
This extends to any canonical transformation: One cannot just take the action of \( F \) on wave functions. One must also take care of the element of volume and this is automatically done by taking the self-adjoint part of \( F \).

Let us return to transformations of holomorphic wave functions. Comparing (22) with (26), one may think by analogy that what is missing in the former to be self adjoint is the change of the corresponding element of volume. Let us recall the form of the norm in the holomorphic representation,
\[
\langle \psi | \psi \rangle = \int dz \, d\bar{z} \frac{1}{2\pi i} e^{-z\bar{z}} \bar{\psi}(\bar{z}) \psi(z).
\]  
(33)
The element of volume or holomorphic measure is
\[
d\mu = \frac{dz \, d\bar{z}}{2\pi i} e^{-z\bar{z}}.
\]  
(34)
Every operator has a holomorphic-antiholomorphic kernel \[8\]. A classical function and the kernel of its corresponding quantum operator coincide when the anti-normal
Hence we have full confirmation. In particular, the kernel of a holomorphic transformation of wave functions, (linear combination of (32)), is precisely (3). Since this operator is not self adjoint, we define

$$\hat{F} = F + \bar{F} = -i (\bar{z} \epsilon(z) - z \bar{\epsilon}(\bar{z})).$$

When $x$ and $y$ are real, $\hat{F}$ is a real function and is as well the kernel of a self-adjoint operator. The coordinate transformation produced by this function according to (7, 8) is

$$\delta_F z = \epsilon(z) - z \partial_z \bar{\epsilon}(\bar{z}),$$
$$\delta_F \bar{z} = \epsilon(z) - \bar{z} \partial_{\bar{z}} \epsilon(z),$$

which is a perfect area-preserving diffeomorphism of real $x$ and $y$. The quantum transformation is given by

$$\delta_{\hat{F}} \psi(z) = \delta_F \psi(z) + \delta_F \bar{\psi}(z),$$

with

$$\delta_F \psi(z) \equiv i F \psi(z) = \bar{z} \epsilon(z) \psi(z) = \partial_z (\epsilon(z) \psi(z)) = \partial_z \epsilon(z) \psi(z) + \epsilon(z) \partial_z \psi(z).$$

Similarly,

$$\delta_{\hat{F}} \bar{\psi}(\bar{z}) = \partial_{\bar{z}} \bar{\psi}(\bar{z}) \bar{\psi}(\bar{z}) + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} \bar{\psi}(\bar{z}).$$

Therefore, we can write

$$\delta_{\hat{F}} \psi(z) = (\partial_z \epsilon(z) + \epsilon(z) \partial_z - z \bar{\epsilon}(\bar{z})) \psi(z),$$
$$\delta_{\hat{F}} \bar{\psi}(\bar{z}) = (\partial_{\bar{z}} \bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} - \bar{z} \epsilon(z)) \bar{\psi}(\bar{z}),$$

and, consequently,

$$\delta_{\hat{F}} \left( \bar{\psi}(\bar{z}) \psi(z) \right) = (\epsilon(z) \partial_z + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}}) \left( \bar{\psi}(\bar{z}) \psi(z) \right) +$$
$$\left( \partial_z \epsilon(z) - z \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} \bar{\epsilon}(\bar{z}) - \bar{z} \epsilon(z) \right) \left( \bar{\psi}(\bar{z}) \psi(z) \right).$$

The latter equation has the form of (31). The first term expresses the variation under the conformal transformation generated by $\epsilon(z)$ and we can expect that the second term accounts for the change of the measure (34). Let us compute it,

$$\delta_{\epsilon} d\mu = \delta_{\epsilon} (dz d\bar{z}) e^{-z \bar{z}} + dz d\bar{z} \delta_{\epsilon} e^{-z \bar{z}} =$$

$$(\partial_z \epsilon(z) dz d\bar{z} + dz \partial_{\bar{z}} \epsilon(\bar{z}) d\bar{z}) e^{-z \bar{z}} + dz d\bar{z} \left( \partial_z \bar{\epsilon}(\bar{z}) - \bar{z} \epsilon(z) \right) e^{-z \bar{z}} =$$

$$(\partial_z \epsilon(z) - z \bar{\epsilon}(\bar{z}) + \partial_{\bar{z}} \epsilon(\bar{z}) - \bar{z} \epsilon(z)) d\mu.$$
In the real case, adding $F^\dagger$ (29), is equivalent to give a definite (symmetrical) order to $x$ and $p$. It introduces in (30) the derivative of the diffeomorphism generator $f(x)$, which expresses the change of $dx$. In the present holomorphic case there are two extra pieces in $\delta F \psi(z)$ (41): The anti-normal ordering produces $\partial_z \epsilon(z)$, expressing the change of $dz$, whereas $\bar{F}$ produces the terms that accounts for the change of the non-holomorphic factor $e^{-z \bar{z}}$ in the measure (34). An identical statement can be made for the anti-holomorphic component (42).

One can perceive here an analogy with the modern philosophy of anomalies or, more generally, with the realization of non-linear transformations in quantum theory. It is not sufficient to translate the classical realization of a symmetry to quantum states or operators; the measure in the path integral must also be taken into account. Failure to do this appears as lack of unitarity. The case of the non-linear $\sigma$-model is paradigmatic. The conformal symmetry of the Classical HE, which is translated as the quantum operator $F$, must be supplemented by terms that take care of the measure, contained in $\bar{F}$, curing at the same time the unitarity problem.

4 A closer look at conformal symmetry

So far, the function $\bar{\epsilon}(\bar{z})$ has been taken to be the complex conjugate of $\epsilon(z)$. However, we can consider in (33) $z$ and $\bar{z}$ as independent complex integration variables. Then it is clear that one can perform on them independent arbitrary holomorphic transformations. Hence, we can enlarge our symmetry to the entire group of holomorphic or anti-holomorphic transformations, as in standard 2d CFT. This might seem to contradict our previous assertion that the classical action of holomorphic and anti-holomorphic transformations do not commute with each other. The cause of this non-commutativity was that holomorphic, say, transformations ($\bar{\epsilon}(\bar{z}) = 0$) affect $\bar{z}$ as well. In contrast, although holomorphic transformations act on $\bar{\psi}(\bar{z})$ as

$$\delta_F \bar{\psi}(\bar{z}) = -\bar{z} \epsilon(z) \bar{\psi}(\bar{z}), \quad (45)$$

it is just what corresponds to the factor $e^{-z \bar{z}}$ in the measure; the function itself does not change.

It seems that we have essentially the same realization that in standard 2d CFT. $\psi(z)$ (respectively, $\bar{\psi}(\bar{z})$) transforms as a scalar function under holomorphic (resp., anti-holomorphic) transformations. However, the unitarity conditions are different: Standard 2d CFT is constructed in 2d Minkowski space; hence, $z = x - t$ and $\bar{z} = x + t$ are assumed real and so are the generators $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$. Upon Fourier mode

\footnote{See the introduction of [10] for a brief review of the $\sigma$-model in this context.}
\footnote{This holds at least for transformations close to the identity, which do not spoil the convergence of the integral.}
expansion,

$$\epsilon(z) = \sum_{-\infty}^{\infty} \epsilon_n e^{inz}, \quad l_n = e^{inz} \partial_z,$$

(46)

$$\bar{\epsilon}(\bar{z}) = \sum_{-\infty}^{\infty} \bar{\epsilon}_n e^{innz}, \quad \bar{l}_n = e^{innz} \partial_{\bar{z}},$$

one obtains the usual conditions,

$$l_n^\dagger = l_{-n}, \quad \bar{l}_n^\dagger = \bar{l}_{-n}. \quad (47)$$

They are commonly expressed in another coordinate system $w = e^{iz}$: Fourier mode expansions become Laurent expansions and $w^* = w^{-1}$.

In the QHE, we are in a genuinely euclidean 2d space. Hence, $\bar{z}$ is the complex conjugate of $z$ and $\bar{\epsilon}(\bar{z})$ the complex conjugate of $\epsilon(z)$. Therefore, we directly use a Taylor expansion and unitarity requires

$$l_n^\dagger = \bar{l}_n \quad (48)$$

instead. This unitarity condition mixes the holomorphic and anti-holomorphic sectors and is therefore unrelated to the inner product of standard 2d CFT, where those sectors are independent. However, that mixing was to be expected for the inner product provided by the norm in the holomorphic representation $33$.

As regards to the comparison with standard 2d CFT, there is another point that also deserves attention: The absence of negative modes, according to $31$. In contrast to their conclusion, our opinion is that that absence makes no essential difference. The argument relies on the previous paragraph. Although in standard 2d CFT the generator of transformations is allowed to be singular at the origin, this is a coordinate dependent statement: It is singular in $w$ but not in $z$. We can further Taylor expand in $33$,

$$l_n = e^{inz} \partial_z = \sum_{k=0}^{\infty} \frac{1}{k!} (inz)^k \partial_z, \quad (49)$$

showing that the positive modes suffice in this coordinate system.$10$

5 Gauge fluctuations and Chern-Simons theory

To treat fluctuations of the gauge field we shall now include it as a dynamical field in the system of many interacting electrons. Then it fluctuates around the value of the

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8 Valid for periodic $x$, that is, the topology of a cylinder.
9 Corresponding to mapping the cylinder onto the punctured plane.
10 We recall that the type of expansion is motivated by global considerations and should not affect local objects.
external magnetic field $B$. The Chern-Simons interaction is known to be dominant at long distance in electrodynamics in $2+1$ dimensions \cite{11}. Its topological character implies that the dynamics is confined to the boundary of the manifold (usually a cylinder). Therefore it directly relates to the edge-current picture of the QHE. It is also noteworthy that these theories are equivalent to theories of chiral bosons living at the edge of the sample \cite{13}, thus providing a concrete Lagrangian that realizes 2d conformal symmetry. From a more mathematical standpoint, non-abelian Chern-Simons theory was formulated by Witten as a covariant 3d theory of knots. In his paper \cite{15} the connection with 2d CFT is already made. This connection was explained in subsequent papers by other authors. In particular, the application to the QHE, based on the group $U(1)$, was studied in \cite{12}. Here I intend to sketch how conformal symmetry arises in Chern-Simons theory and to show that the issue of unitarity in Chern-Simons theory is related not to unitarity in 2d CFT but to modular invariance. Besides, it is also worthwhile to point out the relation between unitarity and the anomaly of Chern-Simons theory in the presence of a boundary.

The observables of Chern-Simons theory are given by functional integration of the appropriate classical functions weighted with the exponential of the action. The connection with 2d CFT is made through canonical quantization (or operator formalism) and elimination of the gauge degrees of freedom \cite{15}. It is carried through by first cutting the 3d manifold $M$ along a Riemann surface $\Sigma$ to establish the classical phase space, which produces the Hilbert space. The operation is best made in the holomorphic representation for there are constraints (the Gauss law). In this representation the partition function adopts the form of a scalar product which is a functional version of \cite{15, 16, 17}

$$Z = \langle \Psi | \Psi \rangle = \int \frac{DA_z DA_{\bar{z}}}{2\pi i} \exp \left( - \int d^2 z A_z A_{\bar{z}} \right) \Psi [A_{\bar{z}}] \Psi [A_z]. \quad (50)$$

This functional integral is over configurations on $\Sigma$. The wave functionals can be obtained by functional integration of the Chern-Simons action with a boundary term \cite{17},

$$\Psi [A_z] = \int DA \exp \left( i \int_M A \wedge dA + \frac{1}{2} \int_\Sigma d^2 z A_z A_{\bar{z}} \right). \quad (51)$$

After removing gauge equivalent configurations in (50) one is left with an integral over the moduli space of $U(1)$ flat connections. On general grounds, this integral represents the corresponding WZW model and has conformal symmetry \cite{15, 16, 17}.

Concretely, one can express the partition function with $n$ Wilson lines piercing $\Sigma$ at points $z_1, \ldots, z_n$ as a correlator of vertex operators

$$Z(L_1, \ldots, L_n) = \langle V_{\alpha_1}(z_1, \bar{z}_1) \cdots V_{\alpha_n}(z_n, \bar{z}_n) \rangle_{A_{\bar{z}}} = \sum_{KL} h_{KL} F_K(z_1, \ldots, z_n, \bar{\tau}; A_{\bar{z}}^{cl}) F^*_L(\bar{z}_1, \ldots, \bar{z}_n, \bar{\tau}; A_z^{cl}); \quad (52)$$
$z_i$ are (moduli) variables for the punctures and $\tau$ label other possible moduli of the Riemann surface. The right-hand side is the conformal-block decomposition of the correlator. We mean these Wilson lines to represent physical particles, whereby the background classical field $A_{\text{cl}}$ is produced by these particles. The matrix $h_{KL}$ encodes the data that determine a particular 2d CFT.

As is well known, the partition function $Z(L_1, \ldots, L_n)$ can be exactly calculated in the abelian case since the functional integral is gaussian, to give

$$Z(L_1, \ldots, L_n) = \exp \left( -\frac{i}{2} \sum_{k=1}^{n} \int dx_k A_{cl}^k \right),$$  \hspace{1cm} (54)

which for closed paths yields the linking number (once normalized). To quantize the physical particles we consider the path integral over trajectories $x(t)$ (with only one particle for simplicity)

$$Z' = \int Dx \exp \left( -\frac{i}{2} \int dx A_{cl} \right).$$  \hspace{1cm} (55)

In the QHE $A_{cl}^i$ is to be taken as a mean field, which must coincide with the external field,

$$\langle B \rangle = B.$$  \hspace{1cm} (56)

Thus one arrives to the Lagrangian formulation of the QHE, alternative to the Hamiltonian formulation exposed in section 2. The path integral admits a holomorphic representation, analogous to (50):

$$Z' = \int_\Sigma \frac{dz \, d\bar{z}}{2\pi i} e^{-z\bar{z}} \bar{\psi}(\bar{z}) \psi(z),$$  \hspace{1cm} (57)

with

$$\psi(z) = \int Dx \exp \left( i \int x^i dx + \frac{1}{2} z\bar{z} \right),$$  \hspace{1cm} (58)

where the action

$$S = \int x^i dx = \int dt \left( \dot{x}_1 \dot{x}_2 - x_2 \dot{x}_1 \right)$$  \hspace{1cm} (59)

is the magnetic flux or area enclosed by the trajectory projected on $\Sigma$. Note that it is a topological invariant which plays a similar rôle to the Chern-Simons action. The identification of $z\bar{z}$ as a boundary term is readily made. When performing the path integral (58) no boundary condition is imposed on $\bar{z}$. Therefore, the variation of the action has a boundary component:

$$\delta S = 2 \int \delta x^i dx^i - (x^i \delta x^i)_{t=0},$$  \hspace{1cm} (60)

$$x^i(0) \wedge \delta x^i(0) \equiv \frac{1}{2i} (z \delta \bar{z} - \bar{z} \delta z) = \frac{1}{2i} z \delta \bar{z}.$$  \hspace{1cm} (61)

\footnote{For a review on 2d CFT see, for example, [18].}
This component is canceled by the variation of the boundary term. Incidentally, the same term cancels the boundary component in $\tilde{\psi}(\bar{z})$, but it involves two extra minus signs.

In the many-particle case we have in the integrand of (57) Laughlin wave functions. Chern-Simons mean field equations impose that the average particle density is proportional to the external magnetic field,

$$\langle \rho \rangle = B,$$

hence constant. Since $Z'$ is just the path integral over Wilson lines of the partition function (53), we see that the conformal blocks $F_K$ are to be identified with the holomorphic part of Laughlin’s wave functions [19].

In principle, we would like the correlators of Chern-Simons theory to be independent of the complex structure chosen for the Riemann surface, given the topological character of that theory. We should therefore demand that the partition function $Z$ be independent of the moduli parameters $(z_i, \tau)$. This turns out to be impossible to achieve in the quantum theory because of regularization problems. We have to deal with an anomaly. The theory is no more independent of the metric on $\Sigma$ but depends on the conformal factor. This regularization problem also appears as the necessity of “framing” the Wilson lines, introducing a dependence on local coordinates that amounts to the same thing. This is the way by which conformal invariance comes in this picture of the QHE. The topological nature is now manifested as the invariance under those transformations of the moduli parameters that do not alter the Riemann surface. In other words, the partition function (or the correlators) must be modular invariant. This is the crucial condition on the matrix $h_{KL}$ (modular invariance) and is sufficient to determine it. Given that the partition function is the holomorphic norm (50), we can appreciate that modular invariance is a consequence of the Chern-Simons unitarity condition.

Returning to the relation of non-unitarity and anomalies already remarked upon at the end of section 3, we can see it more clearly in Chern-Simons theory since it has actual gauge invariance. The gauge transformation of wave functionals is given by

$$\delta_X \Psi [A_z] = \int_{\Sigma} \delta_X A_z \ A_z \Psi = \int_{\Sigma} \frac{\delta \Psi}{\delta A_z} \delta_X A_z = \int_{\Sigma} \frac{\delta \Psi}{\delta A_z} \partial_z X$$

and is anomalous, namely, it is such that $\delta_X Z \neq 0$. The reason is again the presence of a non-trivial integration measure in $Z$ (50). A general non-linear transformation of the wave functional, with $\delta A_z$ arbitrary in (63), is non unitary. Similarly to (41), we must add two terms in (63),

$$\delta \Psi [A_z] = \int_{\Sigma} \left( \delta A_z \frac{\delta}{\delta A_z} + \frac{\delta}{\delta A_z} (\delta A_z) - A_z \delta A_z \right) \Psi [A_z].$$

(64)
The term with the derivative of $\delta A_z$ is absent in the particular case of a gauge transformation. The other extra term is the variation of the exponential part of the measure. As already said, this exponential arises as a boundary term in the functional integral (51) for the wave functional to cancel the boundary component in the equation (compare with (51))

$$\delta \int_M A \wedge dA = 2 \int_M \delta A \wedge dA - \int_{\Sigma} A \wedge \delta A.$$  \hspace{1cm} (65)

When the variation is due to a gauge transformation $\delta \chi A_z = \partial_z \chi$ that extra term can be interpreted as the “gauge anomaly” of Chern-Simons in a manifold with boundary [13]. It is possible to see that it is precisely the one required to cancel the chiral anomaly in 2 dimensions [13, 14].

One can observe that the topological anomaly of Chern-Simons theory can be attributed to the same boundary term (or to the quantum measure). The partition function as a whole must be invariant under diffeomorphisms and it indeed yields topological invariants, like the linking number. However, the wave functions (conformal blocks) are only conformal invariant. It is the boundary term what introduces in the wave function (51) a dependence on the conformal factor of the 2d metric. If we account for it as in (64) we recover topological invariance.

### 6 Discussion

We have seen that a sub-algebra of $W_\infty$ can be realized as conformal transformations, despite being also realized as area preserving diffeomorphisms in the classical system. This dual role may seem puzzling, given the very different geometrical nature of these two types of transformations, but there is no contradiction. One can further confirm that the former realization becomes the latter in the classical limit, when

$$\langle \epsilon(z) \rangle = \epsilon(\langle z \rangle).$$

However, this confirmation is not very enlightening. It may be better to resort to particular examples of $\epsilon(z)$ to gain some insight.

The conformal transformation that most prominently modifies the area is a dilation, $\epsilon(z) = \epsilon \bar{z}$ with $\epsilon$ real. Interestingly, $\hat{F}$ vanishes for it (33), as well as the corresponding classical diffeomorphisms (36, 37). Nevertheless, $\hat{\psi}(z)$ transforms as it should. The best way to understand it is within Laughlin’s original philosophy [1], in which the non-holomorphic factor in the wave functions is interpreted as a background potential for the 2d Coulomb gas. In some sense, the dilation of the holomorphic wave function is compensated for by a similar dilation of the neutralizing background charge, represented by the factor $e^{-z \bar{z}}$ in the measure, as can be read from (13) and (44). It is essentially a self-consistency requirement, since the background charge is determined...
by the electron distribution. On the other hand, for a transformation \( \epsilon(z) = \epsilon z \) with \( \epsilon \) imaginary both terms in (33) are equal instead of canceling one another. This agrees with the fact that this transformation preserves the area, since it is just a rotation.

We can in general split a conformal transformation into two parts of different type in the following way. Let us consider the unit circle and an arbitrary line passing through the origin (the real axis, for example). We define the two types of conformal transformations as the ones that preserve either the circle or the line. Thus the former type of transformations consists of the analytic continuation of diffeomorfisms of the circle \(|z| = 1\). Similarly, the latter is the analytic continuation of diffeomorfisms of the line. If we regard the fluid droplet as confined within the unit circle, the first type is such that preserves its boundary whereas the second type deforms it.

Since the dynamical degrees of freedom live on the edge of the sample, it is natural to think of the diffeomorfisms of the circle as the primary symmetry. Then one can analytically continue a diffeomorfism of the circle to the interior (or exterior). One can as well continue it as an area-preserving transformation. Either way of continuing is achieved by considering \( F \) or \( \hat{F} \), which coincide on the circle (as shown in the appendix). On the contrary, the diffeomorfisms of the line have no physical interpretation. For them \( \hat{F} \) vanishes, as happened in the particular case of a dilation. One must notice, however, that this does not imply in general that the corresponding area-preserving transformations vanish everywhere.

The essential factor that makes the difference between the symmetry realizing as conformal or as area-preserving is \( e^{-z \bar{z}} \). This factor is not present in the ordinary representation, \( \Psi(x, y) \). For the conformal transformations that preserve the unit circle \( D \) we have that
\[
\delta \epsilon \int_D \bar{\psi}(\bar{z}) \psi(z) \, d\mu = 0 \quad (66)
\]
If the density \( \rho = |\Psi(x, y)|^2 \) is constant over \( D \) (as corresponds to an incompressible fluid), given that the size of the liquid droplet is arbitrary, then
\[
\delta F \int (dx \, dy) = 0 \quad (67)
\]

This is the condition for area-preserving diffeomorphisms. Thus we see that unitarity and preservation of the area are closely related. Of course, one cannot prove with the present methods of first quantization the existence of incompressibility, which is a property of the ground state that involves interactions or, at least, statistics. (A fluid of bosons would certainly not exhibit it.) However, we know that it can be a simple property, essentially entailed by the coupling of electrons to quantum fluxes embodied by the Chern-Simons Lagrangian and patent in the mean-field theory equation (62).

In conclusion, \( \psi(z) \left( \bar{\psi}(\bar{z}) \right) \) transforms as a scalar function under holomorphic (anti-holomorphic) diffeomorphisms. Thus, the first-quantized Hall effect is a classical 2d

\(^{12}\)See the appendix for an algebraic treatment.
CFT. The holomorphic (anti-holomorphic) wave function can be expressed in terms of another scalar field,

\[ \psi(z) = e^{i\alpha \phi(z)} \quad \left( \bar{\psi}(\bar{z}) = e^{-i\alpha \bar{\phi}(\bar{z})} \right). \tag{68} \]

The field \( \phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}) \) satisfies the free field equation

\[ \partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = 0. \tag{69} \]

Hence we make connection with the simplest 2d CFT, namely, the Coulomb gas. The classical Coulomb gas theory is insufficient to provide an adequate description of the fractional QHE. In particular, it cannot account for the existence of incompressible ground states at preferred densities. Second quantization, appropriate for the many interacting electron system, leads to a fully quantum 2d CFT. The realization of conformal symmetry may have a central charge, which in the simplest Coulomb-gas theory is \( c = 1 \). Excitations are described by vertex operators, corresponding to (68), and their statistical properties can change according to the value of \( \alpha \). Some of these matters have already been discussed elsewhere; see, for instance, ref. [20].

It is important to remark that unitarity of the realization of conformal symmetry in the QHE is not related to the usual notion of unitarity in conformal field theory, as was shown in section 4. It is rather related to modular invariance, according to section 5. It is thus possible to consider non-unitary conformal models as candidate states for the fractional QHE.

Finally, let us point out that there is another realization of 2d CFT in the QHE. It originates in the idea of edge waves and is formulated as a 1 + 1 CFT living on the cylinder swept by the boundary of the sample with time. Its connection to the euclidean 2d CFT at fixed time can be made with the methods of section 5.

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A Appendix

Let us formulate the splitting of a conformal transformation into circle and line-preserving parts. The equations of the circle and the line are \( z\bar{z} = 1 \) and \( z/\bar{z} = 1 \), respectively. Under a conformal transformation

\[ \delta z = \epsilon(z), \tag{70} \]
\[ \delta \bar{z} = \bar{\epsilon}(\bar{z}) \tag{71} \]
\[ \delta(z \bar{z}) = \bar{z} \epsilon(z) + z \bar{\epsilon}(\bar{z}), \quad (72) \]
\[ \delta(z/\bar{z}) = \frac{\bar{z} \epsilon(z) - z \bar{\epsilon}(\bar{z})}{z^2}, \quad (73) \]
on the circle and line, respectively. From these formulas follow the properties of \( F \) or \( \hat{F} \) mentioned in the text.

Introducing the Laurent expansion of \( \epsilon(z) \),
\[ \epsilon(z) = \sum_{-\infty}^{\infty} \epsilon_n z^{n+1}, \]
we can express that the variation (72) vanish as
\[ \epsilon_n + \bar{\epsilon}_{-n} = 0. \quad (74) \]
In particular, if we want the transformation to be regular at the origin, namely, \( \epsilon_n = 0 \) for \( n < -1 \), we obtain from (74) that the only non-vanishing \( \epsilon_n \) occur for \( n = -1, 0, 1 \) and they further satisfy two equations. The total number of real parameters is three and they are the well-known projective transformations of the unit circle.

The condition (73) just implies that
\[ \epsilon(z) = \bar{\epsilon}(\bar{z}), \quad (75) \]
that is the transformation is real (all \( \epsilon_n \) real).

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