TRANSLATION HYPERSURFACES WITH CONSTANT CURVATURE IN 4-DIMENSIONAL ISOTROPIC SPACE

MUHITTIN EVREN AYDIN, ALPER OSMAN OGRENMIS

Abstract. There exist four non-equivalent types of the translation hypersurfaces in the 4-dimensional isotropic space \( I^4 \) generated by translating the curves lying in perpendicular \( k \)-planes \((k = 2, 3)\), due to its absolute figure. In arbitrary dimensional case; constant Gauss-Kronecker and mean curvature translation hypersurfaces of type 1, i.e. the hypersurfaces whose the translating curves lie in perpendicular isotropic 2-planes, were investigated by the same authors in [1]. The present study concerns such hypersurfaces in \( I^4 \) of other three types.

1. Introduction

Dillen et al. [7] introduced a translation hypersurface \( M^{n-1} \) in a \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) as the graph of the form

(1.1) \( y_n = f_1(y_1) + \ldots + f_{n-1}(y_{n-1}) \),

where \((y_1, \ldots, y_n)\) denote orthogonal coordinates in \( \mathbb{R}^n \) and \( f_1, \ldots, f_n \) smooth functions of single variable. They proved that if \( M^{n-1} \) is minimal, it is either a hyperplane or \( M^{n-1} = M^2 \times \mathbb{R}^{n-3} \), where \( M^2 \) is the Scherk’s minimal surface ([31]) given in explicit form

\[ y_1 = c^{-1} \left( \ln |\cos(cy_2)| - \ln |\cos(cy_1)| \right), \quad c \in \mathbb{R}, \ c \neq 0. \]

In many different ambient spaces, one was tried to generalize the Scherk’s result as defining the translation (hyper)surfaces, see [6, 8, 11, 12, 15, 18, 19, 20, 34, 35, 37].

In addition, Seo [32] extended the above result to the translation hypersurfaces with arbitrary constant mean and Gauss-Kronecker curvature.

Most recently, Munteanu et al. [25] initiated a different notion on this framework, so-called translation graph. Obviously, they defined that a translation graph in \( \mathbb{R}^{p+q} \) is given in explicit form

\[ y_{p+q}(y_1, y_2, \ldots, y_{p+q-1}) = f_1(y_1, \ldots, y_p) + f_2(y_{p+1}, \ldots, y_{p+q-1}), \]

providing certain minimality results. In addition, Moruz and Munteanu [24] concerned the minimal graphs of the form

\[ y_4(y_1, y_2, y_3) = f_1(x_1) + f_2(x_2, x_3), \]

which can be expressed as the sum of a curve in \( y_1y_4 \)-plane and a surface in \( y_2y_3y_4 \)-space.

Note that the graph of the form (1.1) is formed by translating \( n-1 \) curves (called generating curves) lying in mutually perpendicular 2-planes. As the restrictions on

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generating curves are removed, the different kinds of the translation hypersurfaces arise. For example; in the particular case \( n = 3 \), Liu and Yu [16] introduced the notion of affine translation surface, i.e., the surface whose the generating curves lie in non-perpendicular planes. They obtained minimal affine translation surfaces, so called affine Scherk surfaces. Furthermore, arbitrary constant mean curvature and Weingarten affine translation surfaces were given in [13, 17].

This notion was generalized to arbitrary dimension by the first author in [3], defining that an affine translation hypersurface in \( \mathbb{R}^n \) is the graph of the form

\[
y_n(y_1,y_2,\ldots,y_{n-1}) = f_1(\zeta_1) + \ldots + f_{n-1}(\zeta_{n-1}),
\]

where \( \zeta_i = \sum_{j=1}^{n-1} a_{ij} y_j, i = 1,\ldots,n-1 \), \( det(a_{ij}) \neq 0 \) and \( [a_{ij}] \) is non-orthogonal matrix. He proved that such a hypersurface with constant Gauss-Kronecker curvature in \( \mathbb{R}^n \) is congruent to a cylinder.

In this study, we are interested in the counterparts of translation hypersurfaces in isotropic geometry, i.e., a particular Cayley-Klein geometry. For details, see [14, 26, 36]. In 3-dimensional isotropic space \( \mathbb{I}^3 \), when the generating curves lie in perpendicular planes, three types of translation surfaces exist up to the absolute figure:

Type 1. Both generating curves lie in isotropic planes; that is, a graph of \( x_3(x_1,x_2) = f(x_1) + g(x_2) \), where \( (x_1, x_2, x_3) \) denote the isotropically orthogonal coordinates in \( \mathbb{I}^3 \).

Type 2. One generating curve lies in non-isotropic plane and other in isotropic plane; that is, a graph of \( x_2(x_1,x_3) = f(x_1) + g(x_3) \).

Type 3. Both generating curves lie in non-isotropic planes; that is, a graph of \( x_1(x_2,x_3) = \frac{1}{2}(f(x_2 + x_3 - \pi/2) + g(\pi/2 - x_2 + x_3)) \).

As well as the non-isotropic planes, Strubecker [33] obtained the minimal translation surfaces in \( \mathbb{I}^3 \), so called isotropic Scherk's surfaces of type 1,2,3 and respectively given in explicit form \( x_3 = c\sqrt{x_1^2 - x_2^2} \), \( x_2 = c^{-1} \ln|x_3| - \ln|\cos cx_1| \) and

\[
x_1 = (2c)^{-1} \ln|\cos c(x_2 + x_3 - \pi/2)| - \ln|\cos c(\pi/2 - x_2 + x_3)|, c \in \mathbb{R}, c \neq 0.
\]

Afterwards, his results were generalized by Sipus [21] to the translation surfaces in \( \mathbb{I}^3 \) with arbitrary constant Gaussian and mean curvature. The situation that the generating curves in \( \mathbb{I}^3 \) are non-planar or lie in non-perpendicular planes extends the above categorization and the results, see [2].

In \( \mathbb{I}^4 \), there are four types of translation hypersurfaces whose the generating curves lie in mutually perpendicular \( k \)-planes \( (k = 2,3) \), see Section 3. In more general case, i.e. in arbitrary dimensional isotropic spaces, the translation hypersurfaces whose the generating curves lie in isotropic \( 2 \)-planes, said to be of type 1, were investigated in [12]. The present study deals with other three types of translation hypersurfaces in \( \mathbb{I}^4 \) with constant Gauss-Kronecker and mean curvature.

In addition, due to the absolute figure of \( \mathbb{I}^n \ n \geq 3 \), the graph hypersurfaces associated with the form \( x_n = f(x_1,\ldots,x_{n-1}) \) differ from other hypersurfaces, for a smooth real-valued function \( f \). For example; the Gauss-Kronecker and mean curvature for such a graph hypersurface in \( \mathbb{I}^n \) correspond to the determinant and trace of the Hessian of \( f \), respectively. The formulas of these curvatures were initiated by Chen et al. [5], besides obtaining flat and minimal graphs associated with most famous production models in microeconomics.
As far as we know, this is first study formulating such fundamental curvatures of a generic hypersurface in $\mathbb{I}^n$.

2. Preliminaries

Some differential geometric approaches on the curves and the hypersurfaces in isotropic geometry can be found in [9, 10, 22, 23, 27, 28, 29, 30].

Let $\mathbb{P}^n$ denote the $n$–dimensional real projective space, $\omega$ a hyperplane in $\mathbb{P}^n$ and $\mathbb{I}^n = \mathbb{P}^n \setminus \omega$ the obtained affine space. We call $\mathbb{I}^n$ $n$–dimensional isotropic space if $\omega$ contains a hypersphere $S$ with null radius. Then the pair $\{\omega, S\}$ is called absolute figure of $\mathbb{I}^n$ determined in the homogeneous coordinates by

$\omega : u_0 = 0, \ S : u_0 = u_1^2 + ... + u_{n-1}^2 = 0.$

The vertex of $S$ is $F(0 : 0 : ... : 1)$ which we call absolute point.

Denote the affine coordinates $x_1 = \frac{u_1}{u_0}, ..., x_n = \frac{u_n}{u_0}, u_0 \neq 0.$ Then the group of motions of $\mathbb{I}^n$ which preserves the absolute figure is given in terms of affine coordinates by

$$\begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix},$$

where $A$ is an orthonogal $(n - 1, n - 1)$–matrix, $B$ a real $(1, n - 1)$–matrix.

Let $p = (p_1, ..., p_n), \ q = (q_1, ..., q_n)$ be two points in $\mathbb{I}^n$. The isotropic distance between $p$ and $q$ is defined by

$$d_i(p, q) = \sqrt{\sum_{i=1}^{n-1} (p_i - q_i)^2}.$$ 

If $d_i = 0$, then the so-called range between $p$ and $q$ is defined as $d_r = |p_n - q_n|.$

A line is said to be isotropic if its point at infinity is absolute. Other lines are non-isotropic. We call a $k$–plane isotropic (non-isotropic) if it contains (does not) an isotropic line. In the affine model of $\mathbb{I}^n$, the isotropic lines and the isotropic $k$–planes are parallel to the $x_n$–axis. For example; the following

$$a_1 x_1 + + a_n x_n = b, \ a_i, b \in \mathbb{R},$$

determines an isotropic (non-isotropic) hyperplane if $a_n \neq 0 \ (\neq 0).$

Note that the hyperplane $x_n = 0$, so-called basic hyperplane, is Euclidean and therefore the Euclidean metric is used in it.

As distinct from the Euclidean case, the orthogonality in $\mathbb{I}^n$ does not mean the perpendicularity. Obviously, two non-isotropic lines are orthogonal if their projections onto the basic hyperplane are perpendicular up to the Euclidean metric. Nevertheless, an isotropic line is orthogonal to some non-isotropic line. As a consequence, each non-isotropic hyperplane is orthogonal to the isotropic one. In addition, two isotropic hyperplanes are orthogonal if their projections onto the basic hyperplane are perpendicular.

We call a curve isotropic (non-isotropic) $k$–planar if it lies in an isotropic (non-isotropic) $k$–plane.
2.1. Curvature theory of hypersurfaces. This part of isotropic geometry is close to the Euclidean case.

Let $M^{n-1}$, $n \geq 3$, be a hypersurface in $\mathbb{I}^n$ whose the tangent hyperplane at each point is non-isotropic. Then the coefficients $g_{ij}$ of the first fundamental form are calculated by the induced metric from $\mathbb{I}^n$. The normal vector field $U$ is completely isotropic, i.e. $(0,0,\ldots,1)$.

For the second fundamental form, let us consider a curve $r$ on $M^{n-1}$ with isotropic arclength $s$ and the tangent vector $t(s) = r'(s) = \frac{dr}{ds}$. Denote $S$ the projection of $r''(s) = \frac{d^2r}{ds^2}$ onto the tangent hyperplane of $M^{n-1}$. Then, the following decomposition occurs:

$$ r''(s) = \kappa_g \sigma + \kappa_n U, $$

where $\kappa_g$ and $\kappa_n$ are geodesic and normal curvatures of $r$, respectively. Hence, it follows $\kappa_g = \|r''(s)\|_i$, where we mean the induced norm by $\|\cdot\|$. In addition, by a direct computation, we have

$$ \kappa_n = \frac{1}{\sqrt{\det g_{ij}}} \sum_{i,j=1}^{n-1} \det \left( r_{x_1}, \ldots, r_{x_{n-1}}, r_{x_i x_j} \right) \frac{dx_i}{ds} \frac{dx_j}{ds}, $$

where $r_{x_i} = \frac{\partial r}{\partial x_i}$ and $r_{x_i x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j}$, $1 \leq i, j \leq n-1$. If we put

$$ h_{ij} = \frac{\det \left( r_{x_1}, \ldots, r_{x_{n-1}}, r_{x_i x_j} \right)}{\sqrt{\det g_{ij}}}, $$

into (2.1) then one can be written in the matrix form as

$$ \kappa_n = \hat{t}^T \cdot [h_{ij}] \cdot \hat{r}, \quad \hat{t} = \left( \frac{dx_1}{ds}, \ldots, \frac{dx_{n-1}}{ds} \right)^T, $$

where "$\cdot$" denotes the matrix multiplication. If $r$ is a curve with arbitrary parameter, then (2.2) turns to

$$ \kappa_n = \frac{\hat{t}^T \cdot [h_{ij}] \cdot \hat{r}}{\hat{t}^T \cdot [g_{ij}] \cdot \hat{r}}. $$

The extreme values of $\kappa_n$ which we call principal curvatures correspond to the eigenvalues of the matrix $[h_{ij}] \cdot [g_{ij}]^{-1}$. Denote the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ and $[a_{ij}] = [h_{ij}] \cdot [g_{ij}]^{-1}$. Therefore, the characteristic equation of $[a_{ij}]$ follows

$$ \det ([a_{ij}] - \lambda I_{n-1}) = \lambda^{n-1} - tr [a_{ij}] \lambda^{n-2} + \ldots + (-1)^{n-1} \det [a_{ij}] = 0, $$

which provides the fundamental curvatures, called isotropic Gauss-Kronecker curvature (or relative curvature) and isotropic mean curvature. We shortly call them Gauss-Kronecker ($K$) and mean curvature ($H$). Obviously, one obtains

$$ K = \kappa_1 \ldots \kappa_{n-1} = \det [a_{ij}] \quad \text{or} \quad K = \frac{\det [h_{ij}]}{\det [g_{ij}]}, $$

and

$$ (n-1)H = \kappa_1 + \ldots + \kappa_{n-1} = tr [a_{ij}], $$

where $tr$ denotes the trace of a matrix.

A hypersurface is said to be flat (minimal) if $K (H)$ is identically zero.

Notice that the isotropic counterpart for the notion of shape operator in the Euclidean (or Riemannian) sense of a hypersurface is indeed a zero map. In $\mathbb{I}^n$, the matrix $[a_{ij}]$ however plays its role.


3. Categorization of translation hypersurfaces

Let $M^3$ be a translation hypersurface in $\mathbb{I}^4$ generated by translating three curves lying in perpendicular $k$–planes, $k = 2, 3$. Denote the generating curves $\alpha, \beta, \gamma$. Up to the absolute figure of $\mathbb{I}^4$, there are four types of such hypersurfaces listed as below:

Type 1. Three of $\alpha, \beta, \gamma$ are isotropic 2-planar. Then $M^3$ is parameterized by

$$r(u, v, w) = (u, v, w, f(u) + g(u) + h(w)),$$

where $\alpha, \beta$ and $\gamma$ lie in $x_1x_4$–plane, $x_2x_4$–plane and $x_3x_4$–plane, respectively.

Type 2. $\alpha$ is non-isotropic 2-planar and $\beta, \gamma$ isotropic 2-planar. Then $M^3$ is parameterized by

$$r(u, v, w) = (u + v, w, f(u), g(u) + h(w)),$$

where $\alpha, \beta$ and $\gamma$ lie in $x_1x_3$–plane, $x_1x_4$–plane and $x_2x_4$–plane, respectively. The regularity implies that $f$ is a non-constant function.

Type 3. $\alpha, \beta$ are non-isotropic 2-planar and $\gamma$ isotropic 2-planar. Then $M^3$ is parameterized by

$$r(u, v, w) = (u + v + w, f(u), g(v), h(w)),$$

where $\alpha, \beta$ and $\gamma$ lie in $x_1x_2$–plane, $x_1x_3$–plane and $x_1x_4$–plane, respectively. The regularity implies that neither $f$ nor $g$ is a constant function.

Type 4. Three of $\alpha, \beta, \gamma$ are non-isotropic hyperplanar. The curves $\alpha, \beta, \gamma$ and the hyperplanes $P_\alpha, P_\beta, P_\gamma$ containing them can be chosen as

$$\alpha(u) = (f(u), u, u, u + \pi), \quad P_\alpha: -2x_2 + x_3 + x_4 = \pi;$$
$$\beta(v) = (g(v), v, v, -v + \frac{\pi}{2}), \quad P_\beta: 2x_2 + x_3 + 3x_4 = \pi;$$
$$\gamma(w) = (h(w), 6w, -w, w - \frac{\pi}{2}), \quad P_\gamma: x_2 + 4x_3 - 2x_4 = \pi.$$

Then $M^3$ is parameterized by

$$r(u, v, w) = (f(u) + g(v) + h(w), u + v + 6w, u + v + w, u - v - w + \frac{\pi}{w}).$$

The regularity implies that $\frac{df}{du} - \frac{dg}{dv} \neq 0$.

A translation hypersurface of above one type is no equivalent to that of other type due to the absolute figure of $\mathbb{I}^4$.

We hereinafter denote the derivatives of $f, g, h$ with respect to the given variable by a prime and so.

4. Translation hypersurfaces of type 2

The Gauss-Kronecker and the mean curvature respectively follows

(4.1) \[ K = \frac{g'f''g'h''}{(f')^3}, \]

and

(4.2) \[ 3H = \frac{f''g'}{(f')^3} + \frac{g'1 + (f')^2}{(f')^2} + h''. \]
Theorem 4.1. A flat translation hypersurface of type 2 in $\mathbb{H}^4$ is a cylindrical hypersurface with non-isotropic rulings. Furthermore, if one has nonzero constant Gauss-Kronecker curvature, then the following occurs:

$$f(u) = \lambda u^2, \quad g(v) = \mu v^2, \quad h(w) = \xi w^2,$$

where $\lambda, \mu, \xi \in \mathbb{R}$ and $\lambda \mu \xi \neq 0$.

Proof. (4.1) follows that $K$ vanishes if at least one of $f, g, h$ is a linear function with respect to the given variable, that is, at least one of the generating curves turns to a non-isotropic line. Now, let assume that the Gauss-Kronecker curvature is a nonzero constant $K_0$. So, (4.1) leads to

$$f' f'' = \lambda, \quad g' g'' = \mu, \quad h'' = \xi,$$

for $\lambda, \mu, \xi \in \mathbb{R}$ and $K_0 = \lambda \mu \xi \neq 0$. After solving (4.3), we obtain

$$f(u) = \pm \frac{1}{\lambda} \sqrt{-2\lambda u + c_1 + c_2}, \quad g(v) = \pm \frac{1}{3\mu} (2\mu v + c_3)^2 + c_4$$

and

$$h(w) = \frac{\xi}{2} w^2 + c_5 w + c_6,$$

where $c_1, \ldots, c_6 \in \mathbb{R}$. This concludes the proof. □

Theorem 4.2. Let $M^3$ be a minimal translation hypersurface of type 2 in $\mathbb{H}^4$. Then it is either a non-isotropic hyperplane or $M^3 = S^2 \times \mathbb{R}$, where $S^2$ is the isotropic Scherk’s surface of type 2 in $\mathbb{H}^3$.

Proof. (4.2) leads to

$$f'' g' (f')^3 + g'' (f')^2 + h'' = 0,$$

which implies $h'' = h_0, h_0 \in \mathbb{R}$. To solve (4.4) we distinguish two cases depending on $h_0 = 0$ or not.

1. $h_0 = 0$. (4.4) can be rewritten as

$$f'' f' [1 + (f')^2] + g'' g' = 0.$$

The situation that $f'' = g'' = 0$ is a solution to (4.5), which leads $M^3$ to be a non-isotropic hyperplane. If $f'' g'' \neq 0$, (4.5) implies

$$f'' f' [1 + (f')^2] = \lambda = -\frac{g''}{g'},$$

for $\lambda \in \mathbb{R}, \lambda \neq 0$. By solving (4.6), we derive

$$f(u) = \pm \frac{1}{\lambda} \arccos \left( c_1 e^{\lambda u} \right), \quad g(v) = -\frac{c_2}{\lambda} e^{-\lambda v},$$

for $c_1, c_2 \in \mathbb{R}, c_1 c_2 \neq 0$. Up to suitable translations and constants, $M^3$ can be written by a change of parameter in (4.7) as

$$r(\tilde{u}, \tilde{v}, w) = \left( \frac{1}{\lambda} \ln \frac{\cos \lambda \tilde{u}}{\lambda \tilde{v}}, 0, \tilde{u}, \tilde{v} \right) + w(0, 1, 0, 0).$$

This implies the hypothesis of the theorem.
Theorem 4.3. Let
\[ A(u) + B(v) = -h_0 C(v) D(u), \]
where
\[ A(u) = \frac{f''}{f' \left[ 1 + (f')^2 \right]}, \quad B(v) = \frac{g''}{g'}, \quad C(v) = \frac{1}{g'}, \quad D(u) = \frac{(f')^2}{1 + (f')^2}. \]

One deduces from (4.9) that \( A, B, C, D \) are all constant. The fact that \( C, D \) are constant yields \( f'' = 0 \) and \( g'' = 0 \). This however contradicts with \( h_0 \neq 0 \) in (4.8).

\[ \square \]

Theorem 4.3. Let \( M^3 \) be a translation hypersurface of type 2 in \( \mathbb{I}^4 \) with nonzero constant mean curvature \( H_0 \). Then, for \( \lambda, \mu, \xi \in \mathbb{R} \), one of the following occurs:

(i) \( f = f(u), f' \neq 0, g(v) = \lambda, h(w) = \frac{3H_0}{2} w^2 \); 
(ii) \( f(u) = \lambda u, g(v) = \mu v, h(w) = \frac{3H_0}{2} w^2, \lambda \mu \neq 0 \); 
(iii) \( f(u) = \lambda u + \frac{4}{3} g(v) = \mu v, h(w) = \xi w^2, \lambda \mu \neq 0, \xi \neq \frac{3H_0}{2} \); 
(iv) \( f(u) = \lambda u, g(v) = \mu v^2, h(w) = \xi w^2, \lambda \mu \neq 0, \xi \neq \frac{3H_0}{2} \); 
(v) \( M^3 = S^2 \times P \), where \( S^2 \) is the isotropic Scherk’s surface of type 2 in \( \mathbb{I}^3 \) and \( P \) is a parabolic circle in \( \mathbb{I}^2 \) with isotropic curvature \( 3H_0 \).

Proof. Reconsidering (4.2) leads to \( h'' = h_0, h_0 \in \mathbb{R} \) and therefore we get
\[ 3H_0 = \frac{f''}{(f')^3} + g' \frac{1 + (f')^2}{(f')^2} + h_0. \]

To solve (4.9), we have two cases:

(1) \( g' = g_0, g_0 \in \mathbb{R} \). In particular, if \( g_0 = 0 \), then we conclude \( h_0 = 3H_0 \) and
\[ h(w) = \frac{3}{2} H_0 w^2 + c_1 w + c_1, \quad c_1, c_2 \in \mathbb{R}, \]
which implies the statement (i) of the theorem. Nevertheless; if \( g_0 \neq 0 \) then, by (4.9), we get
\[ \frac{3H_0 - h_0}{g_0} = \frac{f''}{(f')^3}. \]

If \( 3H_0 = h_0 \) in (4.10), we immediately achieve the proof of the statement (ii) of the theorem. Otherwise, after solving (4.10), we obtain
\[ f(u) = \pm \frac{g_0}{3H_0 - h_0} \sqrt{-6H_0 + 2h_0} u + c_3 + c_4, \]
where \( 3H_0 \neq h_0 \) and \( c_3, c_4 \in \mathbb{R} \). This gives the proof of the statement (iii) of the theorem.

(2) \( g'' \neq 0 \). We consider two cases:

(a) \( f' = f_0 \neq 0, f_0 \in \mathbb{R} \). (4.9) leads to
\[ 3H_0 = \frac{1 + f_0^2}{f_0^2} g'' + h_0, \]
which implies the proof of the statement (iv) of the theorem.
(b) \( f'' \neq 0 \). (4.9) implies \( h_0 = 3H_0 \) and

\[
\frac{f''}{(f')^3} = \frac{1 + (f')^2}{(f')^3} \quad \text{and} \quad g'' = -\lambda g',
\]

where \( \lambda \in \mathbb{R}, \lambda \neq 0 \). After solving (4.11), we obtain

\[
f(u) = \pm \frac{1}{\lambda} \arccos (c_1 e^{\lambda u}), \quad g(v) = -\frac{c_2}{\lambda} e^{-\lambda v}
\]

for \( c_1, c_2 \in \mathbb{R}, c_1 c_2 \neq 0 \). Up to suitable translations and constants, \( M^3 \) can be written by a change of parameter in (4.12) as

\[
r(\tilde{u}, \tilde{v}, w) = \left( \frac{1}{\lambda} \ln \left| \frac{\cos \lambda \tilde{u}}{\lambda \tilde{v}} \right|, 0, \tilde{u}, \tilde{v} \right) + \left( 0, w, 0, \frac{3}{2} H_0 w^2 \right),
\]

which completes the proof of the theorem.

\[\square\]

5. Translation hypersurfaces of type 3

The Gauss-Kronecker and the mean curvature are respectively

\[
K = \frac{(h')^2 f'' g'' h''}{(f' g')^3}
\]

and

\[
3H = h' \left[ \frac{f''}{(f')^3} + \frac{g''}{(g')^3} \right] + h'' \left[ 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right].
\]

Note that the roles of \( f \) and \( g \) are symmetric in (5.2) and henceforth we only discuss the situations depending on \( f \) while solving it.

**Theorem 5.1.** A flat translation hypersurface of type 3 in \( \mathbb{R}^4 \) is a cylindrical hypersurface with non-isotropic rulings. Furthermore; if one has nonzero constant Gauss-Kronecker curvature, then the following occurs:

\[
f(u) = \lambda u^{\frac{1}{2}}, \quad g(v) = \mu v^{\frac{1}{2}}, \quad h(w) = \xi w^{\frac{1}{2}},
\]

where \( \lambda, \mu, \xi \in \mathbb{R} \) and \( \lambda \mu \xi \neq 0 \).

**Proof.** (5.1) follows that \( K \) vanishes if at least one of \( f, g, h \) is a linear function with respect to the given variable; that is, at least one of the generating curves turns to be a non-isotropic line. Now, let us assume that it is a nonzero constant. So, (5.1) leads to

\[
\frac{f''}{(f')^3} = \lambda, \quad \frac{g''}{(g')^3} = \mu, \quad (h')^2 h'' = \xi,
\]

for \( \lambda, \mu, \xi \in \mathbb{R} \) and \( \lambda \mu \xi \neq 0 \). After solving (5.3), we obtain

\[
f(u) = \pm \frac{1}{\lambda} \sqrt{-2\lambda u + c_1 + c_2}, \quad g(v) = \pm \frac{1}{\lambda} \sqrt{-2\mu v + c_3 + c_4}
\]

and

\[
h(w) = \frac{1}{4\xi} (3\xi w + c_5)^{\frac{1}{2}} + c_6,
\]

where \( c_1, ..., c_6 \in \mathbb{R} \). This concludes the proof. \( \square \)
Theorem 5.2. Let $M^3$ be a minimal translation hypersurface of type 3 in $\mathbb{R}^4$. Then, it is either a non-isotropic hyperplane or one of the following occurs:

(i) $f = f(u), f' \neq 0, g = g(v), g' \neq 0, h(w) = \lambda$;
(ii) $M^3 = S^2 \times \mathbb{R}$, where $S^2$ is the isotropic Scherk’s surface of type 2 in $\mathbb{R}^3$;
(iii) $f(u) = \lambda(-u) \frac{1}{2}, g(v) = \lambda v^2, h(w) = \mu w, \lambda \mu \neq 0$;
(iv) $f(u) = \eta \ln \left| \frac{1 + \sqrt{1 + \kappa e^{\lambda u}}}{1 - \sqrt{1 + \kappa e^{\lambda u}}} \right|$ or $f(u) = \kappa e^{\lambda u}, g(v) = \mu \ln \left| \frac{1 + \sqrt{1 + \kappa e^{\lambda u}}}{1 - \sqrt{1 + \kappa e^{\lambda u}}} \right|$ or $g(v) = \xi e^{\omega v}, h(w) = \rho e^{\tau w}$, where $\eta, \kappa, \lambda, \mu, \xi, \omega, \rho, \tau$ are nonzero constants.

Proof. Due to $H = 0$, (5.2) implies

$$f''(u) \left( \frac{f''}{(f')^3} + \frac{g''}{(g')^3} \right) + h'' \left[ 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right] = 0. \quad (5.4)$$

The situation that $h' = 0$ is a solution to (5.4), which is the proof of the statement (i) of the theorem. Assume that $h' \neq 0$. In order to solve (5.4), we have to consider the following cases:

1. If $f' = f_0, f_0 \in \mathbb{R}, f_0 \neq 0$. Then, (5.4) reduces to

$$\frac{g''h'}{(g')^3} + h'' \left[ 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right] = 0. \quad (5.5)$$

To solve (5.5), we have two possibilities: The first one is the situation that $M^3$ is a non-isotropic hyperplane. The second one is that $g''h'' \neq 0$. So, (5.5) can be rewritten as

$$\frac{f_0^2 g''}{g' (1 + f_0^2 (g')^2 + f_0^2)} = \lambda = \frac{-h''}{h'}, \quad (5.6)$$

where $\lambda \in \mathbb{R}, \lambda \neq 0$. After solving (5.6), we obtain

$$g(v) = \pm \frac{f_0}{\sqrt{1 + f_0^2}} \arccos \left( c_1 \left[ 1 + f_0^2 \right] e^{\lambda v} \right), \quad h(w) = \frac{-c_2}{\lambda} e^{-\lambda w},$$

where $c_1, c_2 \in \mathbb{R}, c_1c_2 \neq 0$, which is the proof of the statement (ii).

2. $f'' \neq 0$. By symmetry, we have $g'' \neq 0$ and distinguish two cases:

a. If $h'' = h_0, h_0 \in \mathbb{R}, h_0 \neq 0$. (5.4) implies

$$\frac{f''}{(f')^3} = \lambda = -\frac{g''}{(g')^3}. \quad (5.7)$$

Solving (5.7) leads to

$$f(u) = \pm \frac{1}{\lambda} \sqrt{-2\lambda u + c_1} + c_2, \quad g(v) = \pm \frac{1}{\lambda} \sqrt{2\lambda v + c_3} + c_4,$$

where $c_1, ..., c_4 \in \mathbb{R}$, which indicates the proof of the statement (iii) of the theorem.

b. If $h'' \neq 0$. (5.4) yields that $h'' = \mu, \mu \in \mathbb{R}, \mu \neq 0$, or $h(w) = c_1 e^{\mu w}, c_1 \in \mathbb{R}, c_1 \neq 0$. Thereby, (5.4) reduces to

$$\frac{f''}{(f')^3} + \frac{\mu}{(f')^2} + \frac{g''}{(g')^3} + \frac{\mu}{(g')^2} = -\mu, \quad (5.8)$$
which implies

\[(5.9) \quad \frac{f''}{(f')^3} + \frac{\mu}{(f')^2} = \xi,\]

and

\[(5.10) \quad \frac{g''}{(g')^3} + \frac{\mu}{(g')^2} = \rho,\]

where \(\xi \in \mathbb{R}\) and \(\rho = -\mu - \xi\). From (5.9), we have

\[(5.11) \quad f'(u) = \pm \left( \frac{\xi}{\mu} + \frac{c_2}{\mu} e^{2\mu u} \right)^{\frac{1}{2}}, \quad c_2 \in \mathbb{R}, \quad c_2 \neq 0.\]

If \(\xi = 0\) in (5.11), then we can derive

\[f(u) = \pm \left( \frac{c_2}{\mu} e^{2\mu u} \right)^{\frac{1}{2}}.\]

Otherwise, we get

\[f(u) = -\frac{1}{\sqrt{\mu \xi}} \tanh^{-1} \left( \sqrt{1 + \frac{c_2}{\xi} e^{2\mu u}} \right) = -\frac{1}{2\sqrt{\mu \xi}} \ln \left| \frac{1 + \sqrt{1 + \frac{c_2}{\xi} e^{2\mu u}}}{1 - \sqrt{1 + \frac{c_2}{\xi} e^{2\mu u}}} \right|.
\]

Same solutions are also satisfied to (5.10) and therefore we complete the proof.

\[\square\]

**Theorem 5.3.** Let a translation hypersurface of type 3 have nonzero constant mean curvature \(H_0\) in \(\mathbb{R}^4\). Then, for \(\lambda, \mu, \xi \in \mathbb{R}\), one of the following occurs:

(i) \(f(u) = \lambda u, \quad g(v) = \mu v, \quad h(w) = 3H_0(\lambda \mu)^2 w^2, \quad \lambda \mu \neq 0;\)

(ii) \(f(u) = \lambda u, \quad g(v) = \left( -\frac{2\mu}{3H_0} \right)^{\frac{1}{2}}, \quad h(w) = \mu w, \quad \lambda \mu \neq 0;\)

(iii) \(f(u) = \lambda u^{\frac{1}{2}}, \quad g(v) = \mu v^{\frac{1}{2}}, \quad h(w) = \xi w, \quad \lambda \mu \xi \neq 0.\)

**Proof.** Due to \(H_0 \neq 0\), \(h\) cannot be constant in (5.2). Nevertheless, we have to distinguish several cases to solve (5.2):

(1) \(f' = f_0, \quad f_0 \in \mathbb{R}, \quad f_0 \neq 0\). Then (5.2) follows

\[(5.12) \quad 3H_0 = \left( \frac{g''}{(g')^3} \right) + h'' \left[ \lambda + \frac{1}{(g')^2} \right].\]

where \(\lambda = \frac{f_0^2 + 1}{f_0^2}\). In order to solve (5.12) the regularity provides two cases:

(a) \(g' = g_0, \quad g_0 \in \mathbb{R}, \quad g_0 \neq 0.\) (5.12) yields

\[h(w) = \frac{3H_0}{\mu} w^2 + c_1 w + c_2,\]

where \(c_1, c_2, \mu \in \mathbb{R}, \quad \mu = \lambda + \frac{1}{g_0}\). This is the proof of statement (i) of the theorem.

(b) \(g'' \neq 0.\) We have two cases:

(1) \(h' = h_0, \quad h_0 \in \mathbb{R}, \quad h_0 \neq 0.\) By (5.12), we derive

\[(5.13) \quad \frac{3H_0}{h_0} = \left( \frac{g''}{(g')^3} \right).\]
Solving (5.13) leads to
\[ g(v) = \pm \frac{h_0}{3H_0} \sqrt{-6H_0v + c_3}, \]
where \( c_3 \in \mathbb{R} \) and this proves the statement (ii) of the theorem.

(ii) \( h'' \neq 0 \). Dividing (5.12) with \( h' \) and taking partial derivative respect to \( w \) gives the following polynomial equation on \((g')^2\)
\[ \left( \frac{3H_0h''}{(h')^2} + \lambda \left( \frac{h''}{h'} \right) \right)(g')^2 + \left( \frac{h''}{h'} \right)' = 0, \]
which yields a contradiction.

(2) \( f'' \neq 0 \). The symmetry gives \( f''g'' \neq 0 \). We have two cases:
(a) \( h' = h_0, h_0 \in \mathbb{R}, h_0 \neq 0 \). (5.2) reduces to
\[ \frac{3H_0}{h_0} = \frac{f''}{(f')^3} + \frac{g''}{(g')^3}. \]
Solving (5.14) gives
\[ f(u) = \pm \frac{1}{\frac{3H_0}{h_0} - \lambda} \sqrt{-2 \left( \frac{3H_0}{h_0} - \lambda \right) u + c_1 + c_2} \]
and
\[ g(v) = \pm \frac{1}{\lambda} \sqrt{-2\lambda v + c_3 + c_4}, \]
for \( \lambda, c_1, ..., c_4 \in \mathbb{R}, \lambda \neq 0 \). This is the proof of the statement (iii) of the theorem.

(b) \( h'' \neq 0 \). Dividing (5.2) with \( h' \) and taking its partial derivative with respect to \( w \), we deduce
\[ -3H_0 \frac{h''}{(h')^2} = \left( \frac{h''}{h'} \right)' \left[ 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right]. \]
Both-hand side must be nonzero in (5.15) and thus we can rewrite it as follows:
\[ -3H_0 \frac{h''}{(h')^2} \left[ \left( \frac{h''}{h'} \right) \right]^{-1} = 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}. \]
This is a contradiction due to the fact that the right-hand side of (5.16) cannot be a constant.

\[ \square \]

6. Translation Hypersurfaces of Type 4

The Gauss-Kronecker and the mean curvature are respectively
\[ K = \frac{8f''g''h''}{49 (f' - g')^5} \]
and
\[ 3H = \frac{2}{49(f' - g')^5} \left\{ \left[ 37 (g')^2 + 2 (h')^2 - 10g'h' + 49 \right] f'' + \right. \\
\left. + \left[ 37 (f')^2 + 2 (h')^2 - 10f'h' + 49 \right] g'' + 2h'' (f' - g')^2 \right\}. \]
As in previous section, the roles of $f$ and $g$ are symmetric in (6.2) and, while solving it, the situations depending on $f$ are only considered.

**Theorem 6.1.** There does not exist a translation hypersurface of type 4 in $\mathbb{I}^4$ with constant Gauss-Kronecker curvature, except the cylindrical hypersurfaces with non-isotropic rulings.

**Proof.** Assume that $K = K_0 \neq 0$ and thus $f'' g'' h'' \neq 0$. (6.1) follows

\begin{equation}
\frac{49K_0}{8h_0} = \frac{f'' g''}{(f' - g')^2},
\end{equation}

where $h'' = h_0 \neq 0$. The partial derivative of (6.3) with respect to $u$ yields

\begin{equation}
(f'''(f' - g') - 5(f'')^2 = 0.
\end{equation}

The fact that the coefficient of the term $g'$ in (6.4) must be zero leads to the contradiction $f'' = 0$. \qed

**Theorem 6.2.** Let a translation hypersurface of type 4 be minimal in $\mathbb{I}^4$. Then it is either a non-isotropic hyperplane or, for $\lambda, \mu, \xi \in \mathbb{R}$, one of the following occurs:

(i) $f(u) = \lambda u$, $g(v) = \lambda v - \frac{1}{\mu} \ln |\mu v|$, $h(w) = \frac{h}{2} w + \frac{1}{\mu} \ln |\cos \xi w|$, $\mu \xi \neq 0$;

(ii) $f(u) = \lambda u - \frac{1}{\mu} \ln |\cos \xi w|$, $g(v) = \lambda v + \frac{1}{\mu} \ln |\cos \xi w|$, $h(w) = \frac{37h}{5} w$, $\mu \xi \neq 0$.

**Remark 6.1.** If $\lambda = 0$ in the statement (ii) of Theorem 6.2, then $M^3 = S^2 \times \mathbb{R}$, where $S^2$ is isotropic Scherk’s surface of type 3 in $\mathbb{I}^4$ with codimension 2. For details, see Appendix 1.

**Proof.** (6.2) follows

\begin{equation}
0 = \left[ 37(f')^2 + 2(h')^2 - 10g'h' + 49 \right] f'' + \left[ 37(f')^2 + 2(h')^2 - 10f'h' + 49 \right] g'' + 2h'' (f' - g')^2.
\end{equation}

We have two cases to solve (6.5):

1. $f' = f_0$, $f_0 \in \mathbb{R}$. (6.5) reduces to

\begin{equation}
\frac{g''}{(f_0 - g')^2} + \frac{2h''}{2(h')^2 - 10f_0h' + 37f_0^2 + 49} = 0.
\end{equation}

The situation that $g'' = h'' = 0$, $g' \neq f_0$, leads $M^3$ to be a non-isotropic hyperplane. If $g'' h'' \neq 0$, (6.6) implies

\begin{equation}
\frac{g''}{(f_0 - g')^2} = \lambda = \frac{-2h''}{2(h')^2 - 10f_0h' + 37f_0^2 + 49},
\end{equation}

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$. After solving (6.7), we conclude

\begin{equation}
g(v) = f_0v - \frac{1}{\lambda} \ln |\lambda v + c_1| + c_2
\end{equation}

and

\begin{equation}
h(w) = \frac{5f_0}{2} w + \frac{1}{\lambda} \ln \left| \cos \left( -\frac{7\lambda\sqrt{2 + f_0^2} w}{2} + c_3 \right) \right| + c_4.
\end{equation}

where $c_1, ..., c_4 \in \mathbb{R}$. This is the proof of the statement (i) of the theorem.

2. $f'' \neq 0$. By symmetry, we deduce $g'' \neq 0$. We have two cases:
(a) \( h' = h_0, \ h_0 \in \mathbb{R} \). (6.5) can be rewritten as

\[
(6.8) \quad \frac{f''}{37 (f')^2 - 10h_0 f' + 49 + h_0^2} = \frac{-g''}{37 (g')^2 - 10h_0 f' + 49 + h_0^2}.
\]

Solving (6.8), we conclude

\[
f(u) = -\frac{1}{147} \ln |\cos (\mu \lambda u + c_1)| + \frac{5h_0}{37} u + c_2
\]

and

\[
g(v) = \frac{1}{147} \ln |\cos (-\mu \lambda v + c_3)| + \frac{5h_0}{37} v + c_4,
\]

where \( \mu = \sqrt{1813 + 12h_0^2} \), which proves the statement (ii) of the theorem.

(b) \( h'' \neq 0 \). This case yields a contradiction, see Appendix 2.

**Theorem 6.3.** Let a translation hypersurface of type 4 in \( \mathbb{I}^4 \) have nonzero constant mean curvature. Then, for \( \lambda, \mu, \xi \in \mathbb{R} \), one of the following occurs:

(i) \( f(u) = \lambda u, g(v) = \mu v, h(w) = \frac{1}{147} \rho w^2 \), \( \lambda \neq \mu, \xi \neq 0 \);

(ii) \( f(u) = \lambda u, g(v) = \lambda v + \mu v \), \( h(w) = \frac{1}{147} \rho w, \mu \neq 0 \).

**Proof.** We have several cases to solve (6.2):

(1) \( f' = f_0 \in \mathbb{R} \). (6.2) reduces to

\[
(6.9) \quad \lambda (f_0 - g')^3 = \left[ 2 (h')^2 - 10f_0 h' + 37f_0^2 + 49 \right] g'' + 2h'' (f_0 - g')^2,
\]

where \( \lambda = \frac{147H_0}{2} \neq 0 \). If \( g' = g_0 \in \mathbb{R} \), \( f_0 \neq g_0 \) in (6.9), then we immediately have the proof of the statement (i) of the theorem. Next we assume \( g'' \neq 0 \) and consider the following cases:

(a) \( h' = h_0 \in \mathbb{R} \). (6.9) follows

\[
(6.10) \quad \frac{g''}{(f_0 - g')^3} = \frac{\lambda}{37f_0^2 + 2h_0^2 - 10h_0 f_0 + 49},
\]

for \( \mu = \frac{\lambda}{37f_0^2 + 2h_0^2 - 10h_0 f_0 + 49} \). Solving (6.10) leads to

\[
g(v) = f_0 v \pm \frac{1}{\mu} \left( 2\mu v + c_1 \right)^{\frac{2}{3}} + c_2, \ c_1, c_2 \in \mathbb{R},
\]

which is the proof of the statement (ii) of the theorem.

(b) \( h'' \neq 0 \). The partial derivative of (6.9) with respect to \( w \) gives

\[
(6.11) \quad \frac{g''}{(f_0 - g')^2} + \frac{h'''}{h'' (2h' - 5f_0)} = 0.
\]

where \( h''' \neq 0 \) due to \( g'' \neq 0 \). (6.11) implies

\[
(6.12) \quad \frac{g''}{(f_0 - g')^2} = \rho = -\frac{h''}{h'' (2h' - 5f_0)},
\]

where \( \rho \in \mathbb{R}, \rho \neq 0 \). Considering (6.12) into (6.9) leads to

\[
(6.13) \quad \lambda (f_0 - g') = \left[ 2 (h')^2 - 10h' f_0 + 37f_0^2 + 49 \right] \rho + 2h'',
\]
which is no possible since the left-hand side of (6.13) cannot be a constant.

(2) $f'' \neq 0$. The symmetry follows $g'' \neq 0$. We have two cases depending on $h'' = 0$ or not. These cases however imply some contradictions, see Appendix 3 and Appendix 4.

\[ \square \]

7. Appendix

This appendix provides a detailed explanation for some calculations ignored in the proofs of Theorem 6.2 and Theorem 6.3.

Appendix 1. Isotropic Scherk’s surface of type 3 in $\mathbb{I}^4$ with codimension 2.

The formulas of the Gaussian and the mean curvatures for a surface in $\mathbb{I}^4$ with codimension 2 can be found in [4].

The absolute of $\mathbb{I}^4$ gives rise to three types of the translation surfaces whose both generating curves lie in perpendicular hyperplanes:

Type 1. Both generating curves lie in isotropic hyperplanes determined by the equations:

\[ x_1 + x_2 + x_3 = \pi, \quad 2x_1 - x_2 - x_3 = \pi. \]

The obtained translation surface in $\mathbb{I}^4$ is parameterized by

\[ r(u, v) = (u + v, u + v, -2u + v, f(u) + g(v)), \]

where the generating curves are

\[ \alpha(u) = (u, u, -2u + \pi, f(u)), \quad \beta(v) = (v, v, v - \pi, g(v)). \]

Type 2. One generating curve lies in non-isotropic hyperplane and other one in isotropic hyperplane determined by the equations:

\[ x_2 + x_3 + x_4 = \pi, \quad x_1 + x_2 - x_3 = \pi. \]

The obtained translation surface in $\mathbb{I}^4$ is parameterized by

\[ r(u, v) = (f(u) + v + \pi, u + v, u + 2v, g(v) - 2u + \pi), \]

where the generating curves are

\[ \alpha(u) = (f(u), u, u - 2u + \pi), \quad \beta(v) = (v + \pi, v, 2v, g(v)). \]

Type 3. Both generating curves lie in non-isotropic hyperplanes determined by the equations:

\[ -2x_2 + x_3 + x_4 = \pi, \quad 2x_2 + x_3 + 3x_4 = \pi. \]

The obtained translation surface in $\mathbb{I}^4$ is parameterized by

\[ r(u, v) = \left( f(u) + g(v), u + v, u + v, u - v + \frac{4\pi}{3} \right), \]

where the generating curves are

\[ \alpha(u) = (f(u), u, u + \pi), \quad \beta(v) = \left( g(v), v, v, -v + \frac{\pi}{3} \right). \]
If a translation surface of type 3 in \( \mathbb{I}^4 \) is minimal then we can achieve

\[
\frac{f''}{(f')^2 + 2} + \frac{g''}{(g')^2 + 2} = 0.
\]

The solution of (7.1) parameterizes so-called the isotropic Scherk’s surface of type \( \beta \) in \( \mathbb{I}^4 \).

**Remark 7.1.** The study of above three types of translation surfaces in \( \mathbb{I}^4 \) with prescribed curvature could be an interesting problem.

**Appendix 2.** \( h'' \neq 0 \).

The partial derivative of (6.5) with respect to \( w \) gives

\[
2h' h'' (f'' + g'') - 5h'' (f'' g' + f' g'') + h''' (f' - g')^2 = 0.
\]

If \( h''' = 0 \) in (7.2), then it reduces to

\[
2h' (f'' + g'') - 5 (f'' g' + f' g'') = 0,
\]

which implies \( f'' + g'' = 0 \) and \( f'' g' + f' g'' = 0 \) or \( f' - g' = 0 \). That is no possible due to the regularity. Hence, we deduce \( h''' \neq 0 \). Dividing (7.2) with \( h'' \) and then taking its partial derivative with respect to \( w \), we conclude

\[
2h'' (f'' + g'') + \left( \frac{h'''}{h''} \right)' (f' - g')^2 = 0,
\]

which yields two cases:

1. \( f'' = -g'' \). It follows from (7.3) that \( h''' = \lambda h'' \), \( \lambda \neq 0 \). Putting those into (7.2) leads to

\[
5f'' + \lambda (f' - g') = 0,
\]

which is no possible.

2. \( f'' \neq -g'' \). (7.3) can rewritten as

\[
\frac{f'' + g''}{(f' - g')^2} + \left( \frac{h'''}{h''} \right)' = 0,
\]

which leads to

\[
f'' + g'' = \mu (f' - g')^2, \ \mu \neq 0.
\]

The partial derivative of (7.4) with respect to \( u \) yields

\[
f''' = 2\mu f' f'' - 2\mu f'' g',
\]

in which the fact that the coefficient of the term \( g' \) must vanish leads to the contradiction \( f'' = 0 \).

**Appendix 3.** \( f'' g'' \neq 0 \) and \( h' = h_0 \in \mathbb{R} \).

(6.2) can be rearranged as

\[
\frac{\lambda (f' - g')^3}{f'' g''} = A(u) + \frac{B(v)}{g''},
\]

where \( A(u) = 37 (f')^2 - 10h_0 f' + 2h_0^2 + 49 \) and \( B(v) = 37 (g')^2 - 10h_0 g' + 2h_0^2 + 49 \).

We have two cases:
(1) \( f'' = f_0 \in \mathbb{R}, f_0 \neq 0 \). The partial derivative of (7.5) with respect to \( u \) and \( v \) gives
\[
f' g''' - g' g'' + 2 (g'')^2 = 0,
\]
which yields the contradiction \( g'' = 0 \) due to the fact that the coefficient of the term \( f' \) must vanish.

(2) \( f'' g''' \neq 0 \). The partial derivative of (7.5) with respect to \( u \) and \( v \) gives
\[
6 = 3 \left( C(u) - D(v) \right) (f' - g') + C(u) D(v) (f' - g')^2,
\]
where \( C(u) = \frac{f''}{(f')^2} \) and \( D(v) = \frac{g''}{(g')^2} \). If \( C = C_0 \in \mathbb{R}, C_0 \neq 0 \), then, by taking twice partial derivative of (7.5) with respect to \( u \) leads to the contradiction \( g''' = 0 \). Therefore, \( C \) is no constant, neither is \( D \) by symmetry.

Taking partial derivative of (7.6) with respect to \( u \) and \( v \) and then dividing with the product \( C'D' \) gives
\[
\frac{2 (f'C')}{C'} \left( \frac{(g'D')}{D'} \right) = \frac{(f'g')^2}{C'} - 3f'' + \frac{(g')^2}{D'} - 2g'' - 3g''\]
where \( E, F, G, I \) must be constant, i.e. \( E = E_0 \), \( F = F_0 \), \( G = G_0 \), \( I = I_0 \).

This yields
\[
C(u) = \frac{3}{f' - E_0}, \quad D(v) = \frac{3}{g' - F_0}.
\]

Substituting (7.7) into (7.6) yields the following
\[
\frac{2E_0F_0}{3} + \frac{F_0}{3} g' + \left[ -E_0 + \frac{F_0}{3} \right] f' + \frac{2f'g'}{3} = 0,
\]
which leads to the contradiction \( f'' = 0 \) or \( g'' = 0 \).

**Appendix 4.** \( f'' g'' h'' \neq 0 \).
The partial derivative of (6.2) with respect to \( w \) gives
\[
(7.8) \quad [2h' - 5g'] f'' + [2h' - 5f'] g'' + \frac{h''}{h''} (f' - g')^2 = 0.
\]
If \( h'' = h_0 \in \mathbb{R}, h_0 \neq 0 \), (7.8) reduces to
\[
(7.9) \quad [2h' - 5g'] f'' + [2h' - 5f'] g'' = 0.
\]
The partial derivative of (7.9) with respect to \( w \) leads to \( f'' = -g'' \) and therefore the contradiction \( f' - g' = 0 \) is obtained. Henceforth, we assume \( h'' \neq 0 \). There are two more cases:

1. \( h'' = \lambda h'', \lambda \in \mathbb{R}, \lambda \neq 0 \). Then (7.8) reduces to
\[
(7.10) \quad [2h' - 5g'] f'' + [2h' - 5f'] g'' + \lambda (f' - g')^2 = 0.
\]
The partial derivative of (7.10) with respect to \( w \) gives \( f'' = \mu = -g'' \), \( \mu \in \mathbb{R}, \mu \neq 0 \). Thus, (7.10) yields
\[
5g'' - \lambda (f' - g') = 0,
\]
which implies the contradiction \( \lambda = 0 \).
The partial derivative of (7.8) with respect to $w$ gives

$$
(7.11) \quad 2h'' (f'' + g'') + \left( \frac{h'''}{h''} \right)' (f' - g')^2 = 0,
$$

in which $\left( \frac{h'''}{h''} \right)' = \xi h'', \, \xi \in \mathbb{R}, \, \xi \neq 0$. Thereby (7.11) reduces to

$$
(7.12) \quad 2 (f'' + g'') + \xi (f' - g')^2 = 0.
$$

The partial derivative of (7.12) with respect to $u$ and $v$ concludes the contradiction $\xi = 0$.

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Department of Mathematics, Faculty of Science, Firat University, Elazig, 23200, Turkey

E-mail address: meaydin@firat.edu.tr, aogrenmis@firat.edu.tr