Relatively bounded operators and the operator E-norms (addition to arXiv:1806.05668)

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Abstract

In this brief note we describe relations between the well known notion of a relatively bounded operator and the operator E-norms considered in arXiv:1806.05668.

We show that the set of all \( \sqrt{G} \)-bounded operators equipped with the E-norm induced by a positive operator \( G \) is the Banach space of all operators with finite E-norm and that the \( \sqrt{G} \)-bound is a continuous seminorm on this space.

We also show that the set of all \( \sqrt{G} \)-infinitesimal operators (operators with zero \( \sqrt{G} \)-bound) equipped with the E-norm induced by a positive operator \( G \) is the completion of the algebra \( \mathfrak{B}(\mathcal{H}) \) of bounded operators w.r.t. this norm.

Some properties of \( \sqrt{G} \)-infinitesimal operators are considered.

1 Introduction

A linear operator \( A \) on a Hilbert space \( \mathcal{H} \) is called relatively bounded w.r.t. a linear operator \( B \) (briefly, \( B \)-bounded) if \( D(B) \subseteq D(A) \) and

\[
\| A \varphi \|^2 \leq a^2 \| \varphi \|^2 + b^2 \| B \varphi \|^2 \quad \forall \varphi \in D(B)
\]  

for some nonnegative numbers \( a \) and \( b \). The infimum over \( b \) for which (1) holds (with some \( a \)) is called the \( B \)-bound for \( A \). If this \( B \)-bound is equal to zero then \( A \) is called \( B \)-infinitesimal operator (infinitesimally bounded w.r.t. \( B \)). These notions are widely used in the modern operator theory, in particular, in analysis of perturbations of unbounded operators in a Hilbert space [3, 8].

The operator E-norm \( \| A \|_G^E \) of a bounded operator \( A \) on a Hilbert space \( \mathcal{H} \) induced by a positive unbounded operator \( G \) is introduced\(^1\) in [5] as the maximum of \( \sqrt{\text{Tr} \rho A^*} \) over all states (positive operators with unit trace) \( \rho \) such that \( \text{Tr} G \rho \leq E \). This norm was used in [5] to obtain the modification of the Kretschmann-Schlingemann-Werner

\(^{1}\)As far as I know. I would be grateful for any references.

\(^{2}\)The value of \( \text{Tr} G \rho \) (finite or infinite) is defined as \( \sup_n \text{Tr} P_n G \rho \), where \( P_n \) is the spectral projector of \( G \) corresponding to the interval \( [0, n] \).
Theorem\footnote{The original Kretschmann-Schlingemann-Werner theorem obtained in [4] quantifies continuity of the Stinespring dilation of CP linear maps w.r.t. the diamond norm (\(cb\)-norm) topology on the set of CP linear maps and the operator norm topology on the set of Stinespring operators.} which quantifies continuity of the Stinespring dilation of a quantum channel w.r.t. the strong convergence topology on the set of channels and the strong operator topology on the set of Stinepring isometries.\footnote{If \(G\) is a unbounded operator with discrete spectrum of finite multiplicity then any of the \(E\)-norms \(\| \cdot \|_E^G, E > 0\), generates the strong operator topology on the unit ball of \(B(\mathcal{H})\) [6, Proposition 2].} These norms are studied in detail in [6], where they are extended to unbounded operators. The extended operator \(E\)-norms and the corresponding Banach spaces of unbounded operators have different applications described in [6, Section 5], [7].

In this note we describe relations between the notion of a relatively bounded operator and the \(E\)-norms extended to unbounded operators.

We show that the set of all \(\sqrt{G}\)-bounded operators equipped with the \(E\)-norm induced by a positive operator \(G\) coincides with the Banach space of all operators with finite \(E\)-norm denoted by \(B^G_G(H)\) in [6] and that the \(\sqrt{G}\)-bound of an operator is a continuous seminorm on \(B^G_G(H)\). We obtain an explicit formula for the \(E\)-norm \(\|A\|_E^G\) in terms of the set of coefficients \((a,b)\) for which (1) holds with \(B = \sqrt{G}\) and the expression for the \(\sqrt{G}\)-bound of an operator \(A\) via \(\|A\|_E^G\).

We also show that the set of all \(\sqrt{G}\)-infinitesimal operators (operators with zero \(\sqrt{G}\)-bound) equipped with the \(E\)-norm \(\| \cdot \|_E^G\) coincides with the completion of the algebra \(\mathcal{B}(\mathcal{H})\) of bounded operators w.r.t. this norm denoted by \(B^0_G(H)\) in [6].

\section{Definitions and the main result}

Let \(\mathcal{H}\) be a separable infinite-dimensional Hilbert space, \(\mathcal{B}(\mathcal{H})\) – the algebra of all bounded operators on \(\mathcal{H}\) with the operator norm \(\| \cdot \|\) and \(\mathfrak{T}(\mathcal{H})\) – the Banach space of all trace-class operators on \(\mathcal{H}\) with the trace norm \(\| \cdot \|_1\) (the Schatten class of order 1) [3, 8]. Let \(\mathfrak{S}(\mathcal{H})\) be the set of quantum states – positive operators in \(\mathfrak{T}(\mathcal{H})\) with unit trace [2].

Let \(G\) be a positive (semidefinite) operator on \(\mathcal{H}\) with a dense domain \(\mathcal{D}(G)\) such that
\[
\inf \{ \|G\varphi\| \mid \varphi \in \mathcal{D}(G), \|\varphi\| = 1\} = 0.
\]
For a given linear (bounded or unbounded) operator \(A\) such that \(\mathcal{D}(\sqrt{G}) \subseteq \mathcal{D}(A)\) the operator \(E\)-norm induced by \(G\) is defined in [6] as
\[
\|A\|_E^G = \sup \left\{ \sqrt{\text{Tr}A\rho A^*} \mid \rho \in \mathfrak{S}(\mathcal{H}), \text{Tr}G\rho \leq E, \text{rank}\rho < +\infty \right\}
\]
where we assume that\footnote{This assumption is made to avoid the notion of adjoint operator.}
\[
A\rho A^* = \sum_i |\alpha_i\rangle\langle\alpha_i|, \quad |\alpha_i\rangle = A|\varphi_i\rangle,
\]
provided that ρ = ∑n ∣ψn⟩⟨ψn⟩ (by using Schrodinger’s mixture theorem (see [1] Ch. 8))

it is easy to show that the r.h.s. of (11) does not depend on this decomposition of ρ)

By using purification of a state it is easy to see that

\[ \|A\|_E^G = \sup_n \{ \|A \otimes I_{H_n} \varphi\| : \varphi \in \mathcal{H} \otimes H_n, \|\varphi\| = 1, \|\sqrt{G} \otimes I_{H_n} \varphi\| \leq \sqrt{E} \} , \quad (5) \]

where \( \mathcal{H}_n \) and \( I_{H_n} \) denote, respectively, a \( n \)-dimensional Hilbert space and the unit operator in this space.

For any given operator \( A \) the nonnegative nondecreasing function \( E \mapsto [\|A\|_E^G]^2 \) is concave on \( \mathbb{R}_+ \) and tends to \( \|A\| \leq +\infty \) as \( E \to +\infty \) [6]. This implies that

\[ \|A\|_{E_1}^G \leq \|A\|_{E_2}^G \leq \sqrt{E_2/E_1} \|A\|_{E_1}^G \quad \text{for any} \quad E_2 > E_1 > 0. \quad (6) \]

So, for given \( G \) all the norms \( \|A\|_E^G \) are equivalent. In particular, if \( \|A\|_E^G \) is finite for some \( E > 0 \) then \( \|A\|_E^G \) is finite for all \( E > 0 \).

It is shown in [6] that the set of all operators \( A \) with finite \( \|A\|_E^G \) equipped with the norm \( \| \cdot \|_E^G \) and naturally defined linear operations is a nonseparable Banach space denoted therein by \( \mathcal{B}_G(\mathcal{H}) \). The completion \( \mathcal{B}_G(\mathcal{H})^0 \) of the algebra \( \mathcal{B}(\mathcal{H}) \) w.r.t. to the norm \( \| \cdot \|_E^G \) is a proper subspace of \( \mathcal{B}_G(\mathcal{H}) \) determined by the condition

\[ \|A\|_E^G = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty. \quad (7) \]

If \( G \) is a unbounded operator with discrete spectrum of finite multiplicity then the Banach space \( \mathcal{B}_G^0(\mathcal{H}) \) is separable and for any \( A \in \mathcal{B}_G^0(\mathcal{H}) \) its \( E \)-norm can be defined by the simple formula

\[ \|A\|_E^G = \sup \{ \|A\varphi\| : \varphi \in \mathcal{H}, \|\varphi\| = 1, \|\sqrt{G} \varphi\| \leq \sqrt{E} \} , \quad (8) \]

which means that the first supremum in (5) is achieved at \( n = 1 \) and the supremum in (3) can be taken over pure states [6, Theorem 3F]. Due to the assumption (2) the condition \( \|\varphi\| = 1 \) in (5) and (8) can be replaced by \( \|\varphi\| \leq 1 \) [6, Proposition 3A] [8].

According to the general definition mentioned in the Introduction an operator \( A \) is called relatively bounded w.r.t. the operator \( \sqrt{G} \) (briefly, \( \sqrt{G} \)-bounded) if \( D(\sqrt{G}) \subseteq D(A) \) and

\[ \|A\varphi\|^2 \leq a^2 \|\varphi\|^2 + b^2 \|\sqrt{G} \varphi\|^2, \quad \forall \varphi \in D(\sqrt{G}) \quad (9) \]

for some nonnegative numbers \( a \) and \( b \). Denote by \( \Gamma_{\sqrt{G}}(A) \) the set of all pairs \( (a, b) \) for which (9) holds. It is easy to see that \( \Gamma_{\sqrt{G}}(A) \) is a closed subset of \( \mathbb{R}_+^2 \). The \( \sqrt{G} \)-bound of \( A \) (denoted by \( b_{\sqrt{G}}(A) \) in what follows) is defined as

\[ b_{\sqrt{G}}(A) = \inf \{ b \mid (a, b) \in \Gamma_{\sqrt{G}}(A) \} . \]

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6For any vector \( |\alpha\rangle \) the symbol \( |\alpha\rangle\langle\alpha| \) denotes the 1-rank operator mapping a vector \( |\beta\rangle \) to \( \langle\alpha|\beta\rangle|\alpha\rangle \).

7We identify operators coinciding on the set \( D(\sqrt{G}) \).

8The question about coincidence of (5) and (6) for any positive operator \( G \) is open. It is easy to show that this coincidence is equivalent to concavity of the r.h.s. of (8) as a function of \( E \).
Lemma 1. A pair \((a, b)\) belongs to the set \(\Gamma_{\sqrt{G}}(A)\) if and only if \(\|A\|_E^{G} \leq \sqrt{a^2 + b^2 E}\) for all \(E > 0\).

Proof. If \(\|A\|_E^{G} \leq \sqrt{a^2 + b^2 E}\) then definition (5) implies that

\[
\|A\varphi\| \leq \|A\|_E^{G} \|\varphi\|^2 \leq \sqrt{a^2 + b^2} \|\varphi\|^2
\]

for any unit vector \(\varphi\) in \(\mathcal{D}(\sqrt{G})\). Hence \((a, b) \in \Gamma_{\sqrt{G}}(A)\).

If \((a, b) \in \Gamma_{\sqrt{G}}(A)\) then it is easy to show that \((a, b) \in \Gamma_{\sqrt{G} \otimes I_{\mathcal{H}_n}}(A \otimes I_{\mathcal{H}_n})\), where \(\mathcal{H}_n\) is a \(n\)-dimensional Hilbert space, for any \(n\) [8, Theorem 7.1.20]. Hence

\[
\sup \left\{ \|A \otimes I_{\mathcal{H}_n}\varphi\| \mid \varphi \in \mathcal{H} \otimes \mathcal{H}_n, \|\varphi\| = 1, \|\sqrt{G} \otimes I_{\mathcal{H}_n}\varphi\| \leq \sqrt{E} \right\} \leq \sqrt{a^2 + b^2 E}
\]

for any \(n\) and \(E > 0\). So, definition (5) implies that \(\|A\|_E^{G} \leq \sqrt{a^2 + b^2 E}\). \(\Box\)

Proposition 1. A) The Banach space \(\mathfrak{B}_G(\mathcal{H})\) coincides (as a set) with the set of all \(\sqrt{G}\)-bounded operators. If \(A \in \mathfrak{B}_G(\mathcal{H})\) and \(E > 0\) then

\[
\|A\|_E^{G} = \inf \left\{ \sqrt{a^2 + b^2 E} \mid (a, b) \in \Gamma_{\sqrt{G}}(A) \right\} \quad \text{and} \quad b_{\sqrt{G}}(A) = \lim_{E \to +\infty} \|A\|_E^{G}/\sqrt{E}.
\]

The limit in the last formula can be replaced by the infimum over all \(E > 0\).

B) The completion \(\mathfrak{B}_{G}^0(\mathcal{H})\) of \(\mathfrak{B}(\mathcal{H})\) w.r.t. the norm \(\| \cdot \|_E^{G}\) coincides (as a set) with the set of all \(\sqrt{G}\)-infinitesimal operators, i.e. operators with the \(\sqrt{G}\)-bound equal to 0.

C) The function \(b_{\sqrt{G}}(\cdot)\) is a continuous seminorm on \(\mathfrak{B}_G(\mathcal{H})\) s.t. \(b_{\sqrt{G}}^{-1}(0) = \mathfrak{B}_{G}^0(\mathcal{H})\).

Quantitatively,

\[
|b_{\sqrt{G}}(A) - b_{\sqrt{G}}(B)| \leq b_{\sqrt{G}}(A - B) \leq \|A - B\|_E^{G}/\sqrt{E} \tag{10}
\]

for arbitrary \(A, B\) in \(\mathfrak{B}_G(\mathcal{H})\) and any \(E > 0\).

Proof. Since \(E \mapsto \|A\|_E^{G}/2\) is a concave nonnegative function on \(\mathbb{R}_+\), it coincides with the infimum of all linear functions \(E \mapsto a^2 + b^2 E\) such that \(\|A\|_E^{G}/2 \leq a^2 + b^2 E\) for all \(E > 0\) and the function \(E \mapsto \|A\|_E^{G}/2\) is non-increasing. So, the assertions A and B can be easily derived from Lemma 1.

To prove C note first that the seminorm properites of \(b_{\sqrt{G}}(\cdot)\) follow from the second formula in A, while B implies \(b_{\sqrt{G}}^{-1}(0) = \mathfrak{B}_{G}^0(\mathcal{H})\). So, since the function \(E \mapsto \|A\|_E^{G}/2\) is non-increasing for any given \(A\), the inequality (10) follows from the triangle inequality for \(b_{\sqrt{G}}(\cdot)\). \(\Box\)

Due to Proposition 1 one can reformulate the results in Section 4 in [6] using the notions of \(\sqrt{G}\)-bounded and \(\sqrt{G}\)-infinitesimal operators. In particular, Theorem 3 in [6] implies the following characterization of \(\sqrt{G}\)-infinitesimal operators.

Corollary 1. An operator \(A\) defined on \(\mathcal{D}(\sqrt{G})\) is \(\sqrt{G}\)-infinitesimal if and only if for any separable Hilbert space \(\mathcal{K}\) the operator \(A \otimes I_{\mathcal{K}}\) (naturally defined on the set \(\mathcal{D}(\sqrt{G}) \otimes \mathcal{K}\)) has a continuous linear extension to the set

\[
\mathcal{V}_{\sqrt{G} \otimes I_{\mathcal{K}}, E} = \left\{ \varphi \in \mathcal{H} \otimes \mathcal{K} \mid \|\sqrt{G} \otimes I_{\mathcal{K}}\varphi\|^2 \leq E \right\}
\]
for any $E > 0$. If $A$ is a $\sqrt{G}$-infinitesimal operator then
\[ \| A \otimes I_K(\varphi - \psi) \| \leq \varepsilon \| A \|_{G_{4E/\varepsilon^2}} \]  
(11)
for any $\varphi$ and $\psi$ in $\mathcal{V}_{G \otimes I_K} \subseteq \mathcal{V}$ such that $\| \varphi - \psi \| \leq \varepsilon$. The r.h.s. of (11) tends to zero as $\varepsilon \to 0^+$ by condition (7).

Proposition 6 in [6] and Proposition 1B imply that any 2-positive linear map $\Phi : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ such that $\Phi(I_{\mathcal{H}}) \leq I_{\mathcal{H}}$ having the predual map $\Phi^* : \mathfrak{T}(\mathcal{H}) \to \mathfrak{T}(\mathcal{H})$ with finite
\[ Y_{\Phi}(E) \doteq \sup \{ \text{Tr} G\Phi_*(\rho) \mid \rho \in \mathfrak{S}(\mathcal{H}), \text{Tr} G\rho \leq E \} \]
is uniquely extended to a linear transformation of the set of all $\sqrt{G}$-infinitesimal operators bounded w.r.t. the norm $\| \cdot \|_{G_{E}}$.

Finally, consider application of the formula for the $\sqrt{G}$-bound in Proposition 1A.

Example. Let $\mathcal{H} = L^2(\mathbb{R})$ and $S(\mathbb{R})$ be the set of infinitely differentiable rapidly decreasing functions with all the derivatives tending to zero quicker than any degree of $|x|$ when $|x| \to +\infty$. Consider the operators $q$ and $p$ defined on the set $S(\mathbb{R})$ by setting
\[ (q\varphi)(x) = x\varphi(x) \quad \text{and} \quad (p\varphi)(x) = \frac{1}{i} \frac{d}{dx}\varphi(x). \]
These operators are essentially self-adjoint. They represent (sharp) real observables of position and momentum of a quantum particle in the system of units where Planck’s constant $\hbar$ is equal to 1 [2, Ch.12]. For given $\omega > 0$ consider the operators
\[ a = (\omega q + ip)/\sqrt{2\omega} \quad \text{and} \quad a^\dagger = (\omega q - ip)/\sqrt{2\omega} \]  
(12)
defined on $S(\mathbb{R})$. The operator $N = a^\dagger a = aa^\dagger - I_{\mathcal{H}}$ is positive and essentially self-adjoint. It represents (sharp) real observable of the number of quanta of the harmonic oscillator with frequency $\omega$. In [6, Section 5] the following estimates are obtained
\[ \sqrt{2E + 1/2} \omega^{-1} < \| q \|_E^N \leq \sqrt{2E + 1} \omega^{-1}, \quad \sqrt{2E + 1/2} \omega < \| p \|_E^N \leq \sqrt{2E + 1} \omega \]  
(13)
(the $E$-norms of $q$ and $p$ depend on $\omega$, since the operator $N$ depends on $\omega$). Thus, the second formula in Proposition 1A implies that $b_{\sqrt{\mathfrak{T}}}(p) = \sqrt{2/\omega}$ and $b_{\sqrt{\mathfrak{T}}}(q) = \sqrt{2\omega}$. □

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9 It is easy to see that $E \mapsto Y_{\Phi}(E)$ is a concave function. So, finiteness of $Y_{\Phi}(E)$ for some $E > 0$ implies finiteness of $Y_{\Phi}(E)$ for all $E > 0$ and boundedness of the function $E \mapsto Y_{\Phi}(E)/E$.

10 If $G$ is a Hamiltonian of a quantum system then the quantity $Y_{\Phi}(E)/E$ can be treated as an energy amplification factor of $\Phi$. Quantum channels $\Phi$ with finite $Y_{\Phi}(E)$ called energy-limited in [9] naturally appear as realistic quantum dynamical maps.
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