BOHR’S INEQUALITY REVISITED

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Dedicated to Professor Themistocles M. Rassias on his 60th birthday with respect

Abstract. We survey several significant results on the Bohr inequality and presented its generalizations in some new approaches. These are some Bohr type inequalities of Hilbert space operators related to the matrix order and the Jensen inequality. An eigenvalue extension of Bohr’s inequality is discussed as well.

1. Bohr inequalities for operators

The classical Bohr inequality says that

$$|a + b|^2 \leq p|a|^2 + q|b|^2$$

holds for all scalars $a, b$ and $p, q > 0$ with $1/p + 1/q = 1$ and the equality holds if and only if $(p - 1)a = b$, cf. [2]. There have been established many interesting extensions of this inequality in various settings by several mathematicians. Some interesting extensions of the classical Bohr inequality were given by Th.M. Rassias in [17]. In 1993, Th.M. Rassias and Pečarić [16] generalized the Bohr inequality by showing that if $(X, \| \cdot \|)$ is a normed linear space, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing convex function, $p_1 > 0$, $p_j \leq 0$ ($j = 2, \ldots, n$) and $\sum_{j=1}^{n} p_j > 0$, then

$$f \left( \frac{\sum_{j=1}^{n} p_j x_j}{\sum_{j=1}^{n} p_j} \right) \geq \sum_{j=1}^{n} p_j f(\|x_j\|)/\sum_{j=1}^{n} p_j$$

holds for every $x_j \in X, j = 1, \ldots, n$.

In 2003, Hirzallah [10] showed that if $A, B$ belong to the algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on a complex (separable) Hilbert space $\mathcal{H}$ and $q \geq p > 1$ with $1/p + 1/q = 1$, then

$$|A - B|^2 + |(p - 1)A + B|^2 \leq p|A|^2 + q|B|^2,$$

(1.1)

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where $|C| := (C^*C)^{1/2}$ denotes the absolute value of $C \in \mathcal{B}(\mathcal{H})$. He also showed that if $X \in \mathcal{B}(\mathcal{H})$ such that $X \geq \gamma I$ for some positive number $\gamma$, then
\[ \gamma ||| |A - B|^2 ||| \leq |||p|A|^2X + q|B|^2||| \]
holds for every unitarily invariant norm $||| \cdot |||$. Recall that a unitarily invariant norm $||| \cdot |||$ is defined on a norm ideal $\mathcal{C}_{||| \cdot |||}$ of $\mathcal{B}(\mathcal{H})$ associated with it and has the property $|||UAV||| = |||A|||$ for all unitary $U$ and $V$ and $A \in \mathcal{C}_{||| \cdot |||}$.

In 2006, Cheung and Pečarić [4] extended inequality (1.1) for all positive conjugate exponents $p, q \in \mathbb{R}$. Also the authors of [5] generalized the Bohr inequality to the setting of $n$-inner product spaces.

In 2007, Zhang [19] generalized inequality (1.1) by removing the condition $q \geq p$ and presented the identity
\[ |A - B|^2 + |\sqrt{p/q}A + \sqrt{q/p}B|^2 = p|A|^2 + q|B|^2 \quad \text{for } A, B \in \mathcal{B}(\mathcal{H}). \]
In addition, he proved that for any positive integer $k$ and $A_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \ldots, n$
\[ |t_1A_1 + \cdots + t_kA_k|^2 \leq t_1|A_1|^2 + \cdots + t_k|A_k|^2, \quad (1.2) \]
holds for every $t_i > 0$, $i = 1, \ldots, k$ such that $\sum_{i=1}^k t_i = 1$.

In 2009, Chansangiam, Hemchote and Pantaragphong [3] prove that if $A_i \in \mathcal{B}(\mathcal{H})$, $\alpha_{ik}$ and $p_i$ are real numbers, $i = 1, \ldots, n$, $k = 1, \ldots, m$, such that the $n \times n$ matrix $X := (x_{ij})$, defined by $x_{ii} = \sum_{k=1}^m \alpha_{ik}^2 - p_i$ and $x_{ij} = \sum_{k=1}^m \alpha_{ik}\alpha_{jk}$ for $i \neq j$, is positive semidefinite, then
\[ \sum_{k=1}^m \left| \sum_{i=1}^n \alpha_{ik}A_i \right|^2 \geq \sum_{i=1}^n p_i |A_i|^2. \]

In 2010, the first author and Zuo [6] had an approach to the Bohr inequality via a generalized parallelogram law for absolute value of operators, i.e.,
\[ |A - B|^2 + \frac{1}{t} |tA + B|^2 = (1 + t)|A|^2 + \left(1 + \frac{1}{t}\right)|B|^2 \]
holds for every $A, B \in \mathcal{B}(\mathcal{H})$ and a real scalar $t \neq 0$.

In 2010, Abramovich, J. Barić and J. Pečarić [1] established new generalizations of Bohr’s inequality by applying superquadratically.

In 2010, the second author and Rajić [15] presented a new operator equality in the framework of Hilbert $C^*$-modules. Recall that the notion of Hilbert $C^*$-module $\mathcal{X}$ is
a generalization of that of Hilbert space in which the filed of scalars \( \mathbb{C} \) is replaced by a \( C^* \)-algebra \( \mathcal{A} \). For every \( x \in \mathcal{X} \) the absolute value of \( x \) is defined as the unique positive square root of \( \langle x, x \rangle \in \mathcal{A} \), that is, \( |x| = \langle x, x \rangle^{\frac{1}{2}} \). The authors of [15] extended the operator Bohr inequalities of [4] and [10]. One of their results extending (1.2) of Zhang [19] as follows. Suppose that \( n \geq 2 \) is a positive integer, \( T_1, \ldots, T_n \) are adjointable operators on \( \mathcal{X} \), \( T_1^*T_2 \) is self-adjoint, \( t_1, \ldots, t_n \) are positive real numbers such that \( \sum_{i=1}^{n} t_i = 1 \) and \( \sum_{i=1}^{n} t_i|T_i|^2 = I_{\mathcal{X}} \). Assume that for \( n \geq 3 \), \( T_1 \) or \( T_2 \) is invertible in algebra of all adjointable operators on \( \mathcal{X} \), operators \( T_3, \ldots, T_n \) are self-adjoint and \( T_i|T_j| = |T_j|T_i \) for all \( 1 \leq i < j \leq n \). Then

\[
|t_1T_1x_1 + \cdots + t_nT_nx_n|^2 \leq t_1|x_1|^2 + \cdots + t_n|x_n|^2
\]

holds for all \( x_1, \ldots, x_n \in \mathcal{X} \).

Vasić and Kečkić [18] obtained a multiple version of the Bohr inequality, which follows from the Hölder inequality. In [14], the second author, Perić and Pečarić established an operator version of the inequality of Vasić–Kečkić. In 2011, Matharu, the second author and Aujla [12] gave an eigenvalue extension of Bohr’s inequality. In the last section, we present a new approach to the main result of [14]. The interested reader is referred to [7] for many interesting results on the operator inequalities.

## 2. Matrix approach to Bohr inequalities

In this section, by utilizing the matrix order we present some Bohr type inequalities. For this, we introduce two notations as follows. For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we define \( n \times n \) matrices \( \Lambda(x) = x^*x = (x_ix_j) \) and \( D(x) = \text{diag}(x_1, \ldots, x_n) \).

**Theorem 2.1.** If \( \Lambda(a) + \Lambda(b) \leq D(c) \) for \( a, b, c \in \mathbb{R}^n \), then

\[
\left| \sum_{i=1}^{n} a_iA_i \right|^2 + \left| \sum_{i=1}^{n} b_iA_i \right|^2 \leq \sum_{i=1}^{n} c_i|A_i|^2
\]

for arbitrary \( n \)-tuple \((A_i)\) in \( \mathcal{B}(\mathcal{H}) \).

Incidentally, the statement is correct even if the order is replaced by the reverse.

**Proof.** We define a positive linear mapping \( \Phi \) from \( \mathcal{B}(\mathcal{H})^n \) to \( \mathcal{B}(\mathcal{H}) \) by

\[
\Phi(X) = (A_1^* \cdots A_n^*) X^T (A_1 \cdots A_n),
\]
where \(T\) denotes the transpose operation. Since \(\Lambda(a) = (a_1, \cdots, a_n)^T(a_1, \cdots, a_n)\), we have
\[
\Phi(\Lambda(a)) = \left(\sum_{i=1}^{n} a_i A_i\right)^* \left(\sum_{i=1}^{n} a_i A_i\right) = \left|\sum_{i=1}^{n} a_i A_i\right|^2,
\]
so that
\[
\left|\sum_{i=1}^{n} a_i A_i\right|^2 + \left|\sum_{i=1}^{n} b_i A_i\right|^2 = \Phi(\Lambda(a) + \Lambda(b)) \leq \Phi(D(c)) = \sum_{i=1}^{n} c_i |A_i|^2.
\]
The additional part is easily shown by the same way. □

The meaning of Theorem 2.1 will be well explained in the following theorem.

**Theorem 2.2.** Let \(t \in \mathbb{R}\).

(i) If \(0 < t \leq 1\), then
\[
|A \mp B|^2 + |tA \pm B|^2 \leq (1 + t)|A|^2 + (1 + \frac{1}{t})|B|^2.
\]

(ii) If \(t \geq 1\) or \(t < 0\), then
\[
|A \mp B|^2 + |tA \pm B|^2 \geq (1 + t)|A|^2 + (1 + \frac{1}{t})|B|^2.
\]

**Proof.** We apply Theorem 2.1 to \(a = (1, \mp 1), b = (t, \pm 1)\) and \(c = (1 + t, 1 + 1/t)\). Then we consider the order between corresponding matrices:
\[
T = \begin{pmatrix} 1 + t & 0 \\ 0 & 1 + \frac{1}{t} \end{pmatrix} - \begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix} - \begin{pmatrix} t^2 & \pm t \\ \pm t & 1 \end{pmatrix} = (1 - t) \begin{pmatrix} t & \pm 1 \\ \pm 1 & \frac{1}{t} \end{pmatrix}.
\]
Since \(\det(T) = 0\), \(T\) is positive semidefinite (resp. negative semidefinite) if \(0 < t < 1\) (resp. \(t > 1\) or \(t < 0\)). □

As another application of Theorem 2.1, we give a new proof of [19, Theorem 7] as follows.

**Theorem 2.3.** If \(A_i \in \mathcal{B}(\mathcal{H})\) and \(r_i \geq 1, i = 1, \ldots, n\), with \(\sum_{i=1}^{n} \frac{1}{r_i} = 1\), then
\[
\left|\sum_{i=1}^{n} A_i\right|^2 \leq \sum_{i=1}^{n} r_i |A_i|^2.
\]
In other words, it says that \(K(z) = |z|^2\) satisfies the (operator) Jensen inequality
\[
\left|\sum_{i=1}^{n} t_i A_i\right|^2 \leq \sum_{i=1}^{n} t_i |A_i|^2 \quad \text{for } A_i \in \mathcal{B}(\mathcal{H}) \text{ and } t_i > 0, i = 1, \ldots, n, \text{ with } \sum_{i=1}^{n} t_i = 1.
\]
Proof. We check the order between the corresponding matrices $D = \text{diag}(r_1, \cdots, r_n)$ and $C = (c_{ij})$ where $c_{ij} = 1$. All principal minors of $D - C$ are nonnegative and it follows that $C \leq D$. Really, for natural numbers $k \leq n$, put $D_k = \text{diag}(r_{i_1}, \cdots, r_{i_k})$, $C_k = (c_{ij})$ with $c_{ij} = 1$, $i, j = 1, \ldots, k$ and $R_k = \sum_{j=1}^{k} 1/r_{i_j}$ where $1 \leq r_{i_1} < \cdots < r_{i_k} \leq n$. Then

$$\det(D_k - C_k) = (r_{i_1} \cdots r_{i_k})(1 - R_k) \geq 0 \quad \text{for arbitrary } k \leq n.$$ 

Hence we have the conclusion of our Theorem by Theorem 2.1. \hfill \square

As another application of Theorem 2.1, we give a new proof of [19, Theorem 7] as follows.

**Corollary 2.4.** If $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $p = (p_1, p_2)$ satisfy $p_1 \geq a_1^2 + b_1^2$, $p_2 \geq a_2^2 + b_2^2$ and $(p_1 - (a_1^2 + b_1^2))(p_2 - (a_2^2 + b_2^2)) \geq (a_1a_2 + b_1b_2)^2$, then

$$|a_1A + a_2B|^2 + |b_1A + b_2B|^2 \leq p_1|A|^2 + p_2|B|^2$$

for all $A, B \in \mathbb{B}$. \hfill \square

Concluding this section, we observe the monotonicity of the operator function $F(a) = \left| \sum_{i=1}^{n} a_i A_i \right|^2$.

**Corollary 2.5.** For a fixed $n$-tuple $(A_i)$ in $\mathbb{B}$, the operator function $F(a) = \left| \sum_{i=1}^{n} a_i A_i \right|^2$ for $a = (a_1, \cdots, a_n)$ preserves the order operator inequalities, that is,

if $\Lambda(a) \leq \Lambda(b)$, then $F(a) \leq F(b)$.

Proof. We prove this putting $F(a) = \Phi(a^*a)$, where $\Phi$ is a positive linear mapping as in the proof of Theorem 2.1. \hfill \square

The following corollary is 3D version of [19, Lemma 2].

**Corollary 2.6.** If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ satisfy $|a_i| \leq |b_i|$ for $i = 1, 2, 3$ and $a_i b_j = a_j b_i$ for $i \neq j$, then $F(a) \leq F(b)$.
Proof. It follows from assumptions that if \( i \neq l \) and \( j \neq k \), then

\[
\begin{vmatrix}
\begin{array}{cc}
a_ia_j - b_ib_j & a_ia_k - b_ib_k \\
a_la_j - b_lb_j & a_la_k - b_lb_k
\end{array}
\end{vmatrix} = a_kb_j(a_ib_l - b_ia_l) + a_jb_k(b_ia_l - a_ib_l) = 0.
\]

This means that all 2nd principal minors of \( \Lambda(b) - \Lambda(a) \) are zero. It follows that \( \det(\Lambda(b) - \Lambda(a)) = 0 \). Since the diagonal elements satisfy \( |a_i| \leq |b_i| \) for \( i = 1, 2, 3 \), we have the matrix inequality \( \Lambda(a) \leq \Lambda(b) \). Now it is sufficient to apply Corollary 2.5. □

3. A generalization of the operator Bohr inequality via the operator Jensen inequality

As an application of the operator Jensen inequality, in this section we consider a generalization of the operator Bohr inequality. Namely the Jensen inequality implies the Bohr inequality even in the operator case.

For this, we first target the following inequality which is an extension of the Bohr inequality, precisely, it is a multiple version of the Bohr inequality due to Vasić and Kečkić [18] as follows. If \( r > 1 \) and \( a_1, \ldots, a_n > 0 \), then

\[
\left| \sum z_i \right|^r \leq \left( \sum a_i^{\frac{1}{r}} \right)^{r-1} \sum a_i |z_i|^r \quad (3.1)
\]

holds for all \( z_1, \ldots, z_n \in \mathbb{C} \).

We note that it follows from Hölder inequality. Actually, \( p = \frac{r}{r-1} \) and \( q = r \) are conjugate, i.e., \( \frac{1}{p} + \frac{1}{q} = 1 \). We here set

\[
u_i = a_i^{-\frac{1}{q}}; \quad w_i = u_i^{-1}z_i, \quad i = 1, \ldots, n
\]

and apply them to the Hölder inequality. Then we have

\[
\left| \sum_{i=1}^n z_i \right|^r = \left| \sum_{i=1}^n u_i w_i \right|^r \leq \left( \sum_{i=1}^n |u_i|^p \right)^{\frac{r}{p}} \left( \sum_{i=1}^n |w_i|^q \right)^{\frac{r}{q}} = \left( \sum_{i=1}^n a_i^{\frac{1}{r-p}} \right)^{r-1} \sum_{i=1}^n a_i |z_i|^r.
\]

Now we propose its operator extension, see also [14, 12].

For the sake of convenience, we recall some notations and definitions. Let \( \mathcal{A} \) be a \( C^* \)-algebra of Hilbert space operators and let \( T \) be a locally compact Hausdorff space. A field \( (A_t)_{t \in T} \) of operators in \( \mathcal{A} \) is called a continuous field of operators if the function \( t \mapsto A_t \) is norm continuous on \( T \). If \( \mu \) is a Radon measure on \( T \) and the function
If $t \mapsto \|A_t\|$ is integrable, then one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathcal{A}$ such that 

$$\varphi \left( \int_T A_t d\mu(t) \right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional $\varphi$ in the norm dual $\mathcal{A}^*$ of $\mathcal{A}$; cf. [8, Section 4.1].

Furthermore, a field $(\varphi_t)_{t \in T}$ of positive linear mappings $\varphi_t : \mathcal{A} \to \mathcal{B}$ between $C^*$-algebras of operators is called continuous if the function $t \mapsto \varphi_t(A)$ is continuous for every $A \in \mathcal{A}$. If the $C^*$-algebras include the identity operators (i.e. they are unital $C^*$-algebras), denoted by the same $I$, and the field $t \mapsto \varphi_t(I)$ is integrable with integral equals $I$, then we say that $(\varphi_t)_{t \in T}$ is unital.

Recall that a continuous real function $f$ defined on a real interval $J$ is called operator convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ holds for all $\lambda \in [0,1]$ and all self-adjoint operators $A, B$ acting on a Hilbert space with spectra in $J$.

Now, we cite the Jensen inequality for our use below.

**Theorem 3.1.** [9, Theorem 2.1] Let $f$ be an operator convex function on an interval $J$, let $T$ be a locally compact Hausdorff space with a bounded Radon measure $\mu$, and let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$-algebras. If $(\psi_t)_{t \in T}$ is a unital field of positive linear mappings $\psi_t : \mathcal{A} \to \mathcal{B}$, then

$$f \left( \int_T \psi_t(A_t) d\mu(t) \right) \leq \int_T \psi_t(f(A_t)) d\mu(t)$$

holds for all bounded continuous fields $(A_t)_{t \in T}$ of selfadjoint elements in $\mathcal{A}$ whose spectra are contained in $J$.

**Theorem 3.2.** Let $T$ be a locally compact Hausdorff space with a bounded Radon measure $\mu$, and let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$-algebras. If $1 < r \leq 2$, $a : T \to \mathbb{R}$ is a bounded continuous nonnegative function and $(\phi_t)_{t \in T}$ is a field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ satisfying

$$\int_T a(t)^{\frac{1}{1-r}} \phi_t(I) d\mu(t) \leq \int_T a(t)^{\frac{1}{1-r}} d\mu(t) I,$$

then

$$\left( \int_T \phi_t(A_t) d\mu(t) \right)^r \leq \left( \int_T a(t)^{\frac{1}{1-r}} d\mu(t) \right)^{r-1} \int_T a(t) \phi_t(A_t^r) d\mu(t)$$

holds for all continuous fields $(A_t)_{t \in T}$ of positive elements in $\mathcal{A}$. 
Proof. We set $\psi_t = \frac{1}{M} a(t)^{\frac{1}{1-r}} \phi_t$, where $M = \int_T a(t)^{\frac{1}{1-r}} d\mu(t) > 0$. Then we have $\int_T \psi_t(I) d\mu(t) \leq I$. By a routine way, we may assume that $\int_T \psi_t(I) d\mu(t) = I$. Since $f(t) = t^r$ is operator convex for $1 < r \leq 2$, then we applying Theorem 3.1 we obtain

$$\left( \int_T \frac{1}{M} a(t)^{\frac{1}{1-r}} \phi_t(\tilde{A}_t) d\mu(t) \right)^r \leq \int_T \frac{1}{M} a(t)^{\frac{1}{1-r}} \phi_t(A^r_t) d\mu(t)$$

for every bounded continuous fields $(\tilde{A}_t)_{t \in T}$ of positive elements in $\mathcal{A}$. Replacing $\tilde{A}_t$ by $a(t)^{-1/(1-r)} A_t$, the above inequality can be written as

$$\left( \int_T \phi_t(A_t) d\mu(t) \right)^r \leq M^{r-1} \int_T a(t) \phi_t(A^r_t) d\mu(t)$$

which is the desired inequality. \hfill \square

Remark 3.3. We note that with the notation as in above and

$$\int_T a(t)^{\frac{1}{1-r}} \phi_t(I) d\mu(t) \leq k \int_T a(t)^{\frac{1}{1-r}} d\mu(t) I,$$

for some $k > 0$,

then

$$\left( \int_T \phi_t(A_t) d\mu(t) \right)^r \leq k^{r-1} \left( \int_T a(t)^{\frac{1}{1-r}} d\mu(t) \right)^{r-1} \int_T a(t) \phi_t(A^r_t) d\mu(t).$$

For a typical positive linear mapping $\phi(A) = X^*AX$ for some $X$, Theorem 3.2 can be written as follows.

Corollary 3.4. Let $T$ be a locally compact Hausdorff space with a bounded Radon measure $\mu$, and let $\mathcal{A}$ be unital $C^*$-algebra. If $1 < r \leq 2$, $a : T \to \mathbb{R}$ is a bounded continuous nonnegative function and $(X_t)_{t \in T}$ is a bounded continuous field of elements in $\mathcal{A}$ satisfying

$$\int_T a(t)^{\frac{1}{1-r}} X^*_t X_t d\mu(t) \leq \int_T a(t)^{\frac{1}{1-r}} d\mu(t) I,$$

then

$$\left( \int_T X^*_t A_t X_t d\mu(t) \right)^r \leq \left( \int_T a(t)^{\frac{1}{1-r}} d\mu(t) \right)^{r-1} \int_T a(t) X^*_t A^r_t X_t d\mu(t)$$

holds for all continuous fields $(A_t)_{t \in T}$ of positive elements in $\mathcal{A}$.

Similarly, putting a positive linear mapping $\phi(A) = \langle Ax, x \rangle$ for some vector $x$ in a Hilbert space in Theorem 3.2 we obtain the following result.
Corollary 3.5. Let \((A_t)_{t \in T}\) be a continuous field of positive operators on a Hilbert space \(\mathcal{H}\) defined on a locally compact Hausdorff space \(T\) equipped with a bounded Radon measure \(\mu\).

If \(1 < r \leq 2\), \(a : T \to \mathbb{R}\) is a bounded continuous nonnegative function and \((x_t)_{t \in T}\) is a continuous field of vectors in \(\mathcal{H}\) satisfying
\[
\int_T a(t) \frac{1}{1-r} \|x_t\|^2 d\mu(t) \leq \int_T a(t) \frac{1}{1-r} d\mu(t) ,
\]
then
\[
\left( \int_T \langle A_t x_t, x_t \rangle d\mu(t) \right)^r \leq \left( \int_T a(t) \frac{1}{1-r} d\mu(t) \right)^{r-1} \int_T a(t) \langle A_t^r x_t, x_t \rangle d\mu(t) .
\]

The following corollary is a discrete version of Theorem 3.2.

Corollary 3.6. If \(1 < r \leq 2\), \(a_1, \ldots, a_n > 0\) and \(\phi_1, \ldots, \phi_n\) are positive linear mappings \(\phi_i : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})\) satisfying
\[
\sum_{i=1}^n a_i^{\frac{1}{1-r}} \phi_i(I) \leq \sum_{i=1}^n a_i^{\frac{1}{1-r}} I ,
\]
then
\[
\left( \sum_{i=1}^n \phi_i(A_i) \right)^r \leq \left( \sum_{i=1}^n a_i^{\frac{1}{1-r}} \right)^{r-1} \sum_{i=1}^n a_i \phi_i(A_i^r)
\]
holds for all bounded continuous fields \((A_t)_{t \in T}\) of positive elements \(A_1, \ldots, A_n \geq 0\) in \(\mathbb{B}(\mathcal{H})\).

We can obtain the above inequality in a broader region for \(r\) under conditions on the spectra. For this result, we cite version of Jensen’s operator inequality without operator convexity.

Theorem 3.7. [13, Theorem 1] Let \(A_1, \ldots, A_n\) be self-adjoint operators \(A_i \in \mathbb{B}(\mathcal{H})\) with bounds \(m_i\) and \(M_i, m_i \leq M_i, i = 1, \ldots, n\). Let \(\psi_1, \ldots, \psi_n\) be positive linear mappings \(\psi_i : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}), i = 1, \ldots, n\), such that \(\sum_{i=1}^n \psi_i(1_H) = 1_K\). If
\[
(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n ,
\]
where \(m_A\) and \(M_A, m_A \leq M_A\), are bounds of the self-adjoint operator \(A = \sum_{i=1}^n \psi_i(A_i)\), then
\[
f \left( \sum_{i=1}^n \psi_i(A_i) \right) \leq \sum_{i=1}^n \psi_i \left( f(A_i) \right)
\]
holds for every continuous convex function \( f : I \to \mathbb{R} \) provided that the interval \( I \) contains all \( m_i, M_i \).

**Theorem 3.8.** Let \( A_1, \ldots, A_n \) be strictly positive operators \( A_i \in \mathbb{B}(\mathcal{H}) \) with bounds \( m_i \) and \( M_i \), \( 0 < m_i \leq M_i, \ i = 1, \ldots, n \). Let \( \phi_1, \ldots, \phi_n \) be positive linear mappings \( \phi_i : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H}), \ i = 1, \ldots, n \).

If \( r \in (-\infty, 0) \cup (1, \infty) \) and \( a_1, \ldots, a_n > 0 \) such that
\[
\sum_{i=1}^{n} a_i^{1-r} \phi_i(I) \leq \sum_{i=1}^{n} a_i^{1-r} I,
\]
and
\[
(m_A, M_A) \cap [a(t)^{-1/(1-r)} m_i, a(t)^{-1/(1-r)} M_i] = \emptyset \quad \text{for} \ i = 1, \ldots, n,
\]
where \( m_A \) and \( M_A \), \( 0 < m_A \leq M_A \), are bounds of the strictly positive operator \( A = \sum_{i=1}^{n} \phi_i(A_i) \), then
\[
\left( \sum_{i=1}^{n} \phi_i(A_i) \right)^r \leq \left( \sum_{i=1}^{n} a_i^{1-r} \right)^{r-1} \sum_{i=1}^{n} a_i \phi_i(A_i^r).
\]

**Proof.** The proof is quite similar to the one of Theorem 3.2. We omit the details. \( \square \)

In the rest, we shall prove a matrix analogue of the inequality (3.1). For this, we introduce some usual notations. Let \( \mathcal{M}_n \) denote the \( C^* \)-algebra of \( n \times n \) complex matrices and let \( \mathcal{H}_n \) be the set of all Hermitian matrices in \( \mathcal{M}_n \). We denote by \( \mathcal{H}_n(J) \) the set of all Hermitian matrices in \( \mathcal{M}_n \) whose spectra are contained in an interval \( J \subseteq \mathbb{R} \). Moreover, we denote by \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \) the eigenvalues of \( A \) arranged in the decreasing order with their multiplicities counted.

Matharu, the second author and Aujla [12] gave a weak majorization inequality and apply it to prove eigenvalue and unitarily invariant norm extensions of (3.1). Their main result reads as follows.

**Theorem 3.9.** [12, Theorem 2.7] Let \( f \) be a convex function on \( J, \ 0 \in J, \ f(0) \leq 0 \) and let \( A \in \mathcal{H}_n(J) \). Then
\[
\sum_{j=1}^{k} \lambda_j \left( f \left( \sum_{i=1}^{\ell} \alpha_i \Phi_i(A) \right) \right) \leq \sum_{j=1}^{k} \lambda_j \left( \sum_{i=1}^{\ell} \alpha_i \Phi_i(f(A)) \right) \quad (1 \leq k \leq m)
\]
holds for positive linear mappings \( \Phi_i, \ i = 1, 2, \ldots, \ell \) from \( \mathcal{M}_n \) to \( \mathcal{M}_m \) such that \( 0 < \sum_{i=1}^{\ell} \alpha_i \Phi_i(I_n) \leq I_m \) and \( \alpha_i \geq 0 \).
The following result is a generalization of [11, Theorem 1].

**Corollary 3.10.** [12, Corollary 2.8] Let $A_1, \ldots, A_\ell \in \mathcal{H}_n$ and $X_1, \ldots, X_\ell \in \mathcal{M}_n$ such that

$$0 < \sum_{i=1}^{\ell} \alpha_i X_i^* X_i \leq I_n,$$

where $\alpha_i > 0$ and let $f$ be a convex function on $\mathbb{R}$, $f(0) \leq 0$ and $f(uv) \leq f(u)f(v)$ for all $u, v \in \mathbb{R}$. Then

$$\sum_{j=1}^{k} \lambda_j \left( f \left( \sum_{i=1}^{\ell} X_i^* A_i X_i \right) \right) \leq \sum_{j=1}^{k} \lambda_j \left( \sum_{i=1}^{\ell} \alpha_i f(\alpha_i^{-1}) X_i^* f(A_i) X_i \right) \quad (3.3)$$

holds for $1 \leq k \leq n$.

**Proof.** Let $A \in \mathcal{M}_{\ell n}$ be partitioned as

$$
\begin{pmatrix}
A_{11} & \cdots & A_{1\ell} \\
\vdots & \ddots & \vdots \\
A_{\ell 1} & \cdots & A_{\ell \ell}
\end{pmatrix},
$$

where $A_{ij} \in \mathcal{M}_n$, $1 \leq i, j \leq \ell$, as an $\ell \times \ell$ block matrix. Consider the linear mappings $\Phi_i : \mathcal{M}_{\ell n} \rightarrow \mathcal{M}_n$, $i = 1, \ldots, \ell$, defined by $\Phi_i(A) = X_i^* A_i X_i$, $i = 1, \ldots, \ell$. Then $\Phi_i$’s are positive linear mappings from $\mathcal{M}_{\ell n}$ to $\mathcal{M}_n$ such that

$$0 < \sum_{i=1}^{\ell} \alpha_i \Phi_i(I_{\ell n}) = \sum_{i=1}^{\ell} \alpha_i X_i^* X_i \leq I_n.$$

Using Theorem 3.9 for the diagonal matrix $A = \text{diag}(A_{11}, \ldots, A_{\ell \ell})$, we have

$$\sum_{j=1}^{k} \lambda_j \left( f \left( \sum_{i=1}^{\ell} \alpha_i X_i^* A_i X_i \right) \right) \leq \sum_{j=1}^{k} \lambda_j \left( \sum_{i=1}^{\ell} \alpha_i X_i^* f(A_i) X_i \right) \quad (1 \leq k \leq n).$$

Replacing $A_{ii}$ by $\alpha_i^{-1} A_i$ in the above inequality, we get (3.3). □

Now we obtain the following eigenvalue generalization of inequality (3.1).

**Theorem 3.11.** [12, Theorem 2.9] Let $A_1, \ldots, A_\ell \in \mathcal{H}_n$ and $X_1, \ldots, X_\ell \in \mathcal{M}_n$ be such that

$$0 < \sum_{i=1}^{\ell} p_i^{1/(1-r)} X_i^* X_i \leq \sum_{i=1}^{\ell} p_i^{1/(1-r)} I_n,$$

where $p_1, \ldots, p_\ell > 0$, $r > 1$. Then

$$\sum_{j=1}^{k} \lambda_j \left( \left| \sum_{i=1}^{\ell} X_i^* A_i X_i \right|^r \right) \leq \left( \sum_{i=1}^{\ell} p_i^{1/(1-r)} \right)^{r-1} \sum_{j=1}^{k} \lambda_j \left( \sum_{i=1}^{\ell} p_i X_i^* |A_i|^r X_i \right)$$

for $1 \leq k \leq n$. 
Proof. Apply Corollary 3.10 to the function \( f(t) = |t|^r \) and \( \alpha_i = \frac{P_i^{1/(1-r)}}{\sum_{i=1}^{\ell} P_i^{1/(1-r)}}. \)

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