Log Canonical Thresholds on Burniat Surfaces with $K^2 = 6$ via Pluricanonical Divisors

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Abstract. Let $S$ be a Burniat surface with $K_S^2 = 6$ and $\varphi$ be the bicanonical map of $S$. In this paper we show optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems of $S$ via Klein group $G$ induced by $\varphi$. Indeed, for a positive even integer $m$, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m)$ (resp. $1/(2m - 2)$). For a positive odd integer $m$, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m - 5)$ (resp. $1/(2m)$). The inequalities are all optimal.

1. Introduction

Let $X$ be a variety and $p \in X$ be a smooth point. And let $D$ be an effective Cartier divisor on $X$. The log canonical threshold or the complex singularity exponent of $D$ at $p$ is the number

$$
lct_p(X, D) := \sup \{ c \in \mathbb{Q} \mid ||f||^{-c} \text{ is locally } L^2 \text{ near } p \},$$

where $f$ is a local defining equation of $D$ at $p$. In [7] we have the following inequalities

$$
\frac{1}{\text{mult}_p(D)} \leq \lct_p(X, D) \leq \frac{\dim X}{\text{mult}_p(D)},
$$

and the log canonical threshold of $D$ at $p$ is equal to the absolute value of the largest root of the Bernstein–Sato polynomial of $f$.

The log canonical threshold can be formally defined for log pairs (cf. [7, 8.2 Proposition]). Let $X$ be a normal variety with at worst log canonical singularities, $Z$ be a closed subvariety of $X$ and $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. The log canonical threshold of $D$ along $Z$ on $X$ is the number

$$
lct_Z(X, D) := \sup \{ c \in \mathbb{Q} \mid (X, cD) \text{ is log canonical in an open neighborhood of } Z \}.$$ 

For simplicity, we put $\lct(X, D) = \lct_X(X, D)$.

We have the following invariant for every polarised pair $(X, \mathcal{L})$.

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Definition 1.1. Let $X$ be a normal variety with at worst log canonical singularities, and $\mathcal{L}$ be an ample $\mathbb{Q}$-Cartier divisor on $X$. The global log canonical threshold of a pair $(X, \mathcal{L})$ is the number

$$\text{glct}(X, \mathcal{L}) := \inf \{\text{lct}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-Cartier divisor on } X, \mathbb{Q}\text{-linearly equivalent to } \mathcal{L}\}.$$ 

Chen, Chen and Jiang [5] proved the Noether inequality for projective 3-folds of general type. They use the global log canonical threshold of a surface of general type with $p_g = 2$ and $K^2 = 1$ via its ample canonical divisor (see the appendix by Kollár in [5]).

The authors in [6] showed that the global log canonical threshold of a Burniat surface with $K^2 = 6$ via its ample canonical divisor is $1/2$, where the Burniat surface is a minimal surface of general surface with $p_g = 0$ and $K^2 = 6$.

In this paper, we give optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems via Klein group induced by the bicanonical map of a Burniat surface with $K^2 = 6$.

Let $S$ be a Burniat surface with $K_S^2 = 6$ (see [1, 2, 8–10]). The bicanonical map $\varphi$ of $S$ has an image, a del Pezzo surface $\Sigma$ of degree 6 in $\mathbb{P}^6$ which is a blow-up $\rho: \Sigma \to \mathbb{P}^2$ at three point $p_1, p_2, p_3$ in general position. Denote by $e_i$ the $(-1)$-curve corresponding to $p_i$, by $e'_i$ the strict transform of the line passing through the two points $p_j$ and $p_k$ by $\rho$, and by $m^i_l$ the strict transform of a general line passing through the point $p_i$ by $\rho$ for each $\{i, j, k\} = \{1, 2, 3\}$ and $l = 1, 2$. Then $\varphi$ is a bidouble covering map over $\Sigma$ with a branch divisor $B := B_1 + B_2 + B_3$ satisfying $2L_i \sim B_j + B_k$ for a line bundle $L_i$ on $\Sigma$ and $\{i, j, k\} = \{1, 2, 3\}$, where

$$B_1 = e_1 + e'_1 + m_1^2 + m_2^2,$$
$$B_2 = e_2 + e'_2 + m_1^3 + m_3^2,$$
$$B_3 = e_3 + e'_3 + m_1^1 + m_2^1,$$

and $\sim$ means the linearly equivalent relation between divisors.

For $i = 1, 2, 3$, we note $\varphi^*(B_i) = 2R_i$ for some divisor $R_i$ ramified by $\varphi$, and denote by $G$ the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\text{Id}_S, \sigma_1, \sigma_2, \sigma_3\}$ induced by $\varphi$ such that $R_i$ is the divisorial fixed part of $\sigma_i$.

For a positive integer $m$, the natural action of the group $G$ splits the set of global sections of the pluricanonical divisor $mK_S$ of $S$ into eigen spaces via the characters of $G$:

$$H^0(S, mK_S) = H^0(S, mK_S)^{\text{inv}} \oplus \bigoplus_{i=1}^{3} H^0(S, mK_S)\chi_i,$$

where $\chi_i$ is a character of $G$ such that $\chi_i(\sigma_j) = \delta_{ij}$ for $i, j \in \{1, 2, 3\}$. Then the pluricanonical linear system $|mK_S|$ for a positive integer $m$ contains an invariant part $|mK_S|_0$.
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(resp. an anti-invariant part \( |mK_S|_i \)) that consists of zeros of sections of \( H^0(S, mK_S)^{\text{inv}} \) (resp. \( H^0(S, mK_S)^{\chi_i} \)) for \( i = 1, 2, 3 \), that is,

\[
|mK_S| \supseteq |mK_S|_0 \cup \bigcup_{i=1}^{3} |mK_S|_i.
\]

We consider the log canonical threshold of members of the invariant and anti-invariant parts of the complete linear system \( |mK_S| \), where \( m \) is a positive integer. To calculate the log canonical threshold, we use the following representation of pluricanonical linear systems for a bidouble covering map \( \varphi: S \to \Sigma \). Denote by \( R \) the ramification divisor \( R_1 + R_2 + R_3 \) of \( \varphi \).

**Proposition 1.2.** (cf. [10, Proposition 1.6]) For a positive integer \( n \) and each \( i = 1, 2, 3 \) with \( \{i, j, k\} = \{1, 2, 3\} \),

(i) \( |2nK_S|_0 = \varphi^*|n(2K_\Sigma + B)| \) and \( |2nK_S|_i = R_j + R_k + |\varphi^*(n(2K_\Sigma + B) - L_i)| \);

(ii) \( |(2n + 1)K_S|_0 = R + |\varphi^*((2n + 1)K_\Sigma + nB)| \) and \( |(2n + 1)K_S|_i = R_i + |\varphi^*((2n + 1)K_\Sigma + nB + L_i)| \).

We apply \( B \sim -3K_\Sigma \) to Proposition 1.2 and obtain log canonical thresholds of members of the pluricanonical sublinear systems of Burniat surfaces \( S \) with \( K_S^2 = 6 \) via the Klein group induced by the bicanonical map of \( \varphi \) as follows.

**Theorem 1.3** (Main theorem). Let \( S \) be a Burniat surface with \( K_S^2 = 6 \). Then for a positive integer \( n \) and each \( i = 1, 2, 3 \),

(i) if \( D_0 \in |2nK_S|_0 \) and \( D_i \in |2nK_S|_i \),

\[
lct(S, D_0) \geq \frac{1}{4n} \quad \text{and} \quad lct(S, D_i) \geq \frac{1}{4n - 2};
\]

(ii) if \( D_0' \in |(2n + 1)K_S|_0 \) and \( D_i' \in |(2n + 1)K_S|_i \),

\[
lct(S, D_0') \geq \frac{1}{4n} \quad \text{and} \quad lct(S, D_i') \geq \frac{1}{4n + 2}.
\]

Moreover the inequalities are optimal.

**Remark 1.4.** Since \( |2K_S|_i = \emptyset \) for all \( i = 1, 2, 3 \) (see [9, Proposition 3.1]), we actually have \( lct(S, D_i) \geq 1/(4n - 2) \) for any \( D_i \in |2nK_S|_i \) when an integer \( n \geq 2 \) in Theorem 1.3(i).
Corollary 1.5. Let $S$ be a Burniat surface with $K_S^2 = 6$. Then for a positive integer $n$ and each $i = 1, 2, 3$,

(i) if $D_i \in |2nK_S|_i$,
\[ \lct(S, D_i) > \frac{1}{4n}; \]

(ii) if $D'_0 \in |(2n + 1)K_S|_0$,
\[ \lct(S, D'_0) > \frac{1}{4n + 2}. \]

Remark 1.6. Corollary 1.5(i) is [6, Proposition 5.2].

Since
\[ \text{glct}(S, K_S) = \frac{1}{2} \]
(see [6, Theorem 1.3]), we obtain

Corollary 1.7. Let $S$ be a Burniat surface with $K_S^2 = 6$. For any positive even (resp. odd) integer $m$, if a divisor $D$ is in the linear system $|mK_S|$ such that $\text{glct}(S, K_S) = \lct(S, \frac{1}{m}D)$, then the divisor $D$ is not in the anti-invariant parts $|mK_S|_i$ (resp. the invariant part $|mK_S|_0$) for $i = 1, 2, 3$.

Proof. We get the result by Corollary 1.5.

2. Preliminaries

Let $X$ be a normal variety with at worst log canonical singularities. Note that $\sim_{\mathbb{Q}}$ means the $\mathbb{Q}$-linearly equivalent relation.

Lemma 2.1. Let $N_0 \sim_{\mathbb{Q}} A$ be an effective $\mathbb{Q}$-Cartier divisor on $X$ such that the log pair $(X, N_0)$ is not log canonical at a point $p$. And let $N \sim_{\mathbb{Q}} A$ be an effective $\mathbb{Q}$-Cartier divisor on $X$ such that the log pair $(X, N)$ is log canonical at the point $p$. Then there is an effective $\mathbb{Q}$-Cartier divisor $N' \sim_{\mathbb{Q}} A$ on $X$ such that at least one component of $N'$ is not contained in the support of $N'$ and the log pair $(X, N')$ is not log canonical at the point $p$.

Proof. See [4, Remark 2.22].

The following is used for a non log canonical pair at some smooth point.

Lemma 2.2. (cf. [7, 8.10 Lemma]) Let $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. If the log pair $(X, D)$ is not log canonical at some smooth point $p$, then the inequality
\[ \text{mult}_p(D) > 1 \]
holds.
3. Proof of the main theorem

We remark that for $i = 1, 2, 3$ and $j = 1, 2$,

\[ E_i^2 = E_i' = -1, \quad K_S \cdot E_i = K_S \cdot E_i' = 1, \quad M_j^2 = 0 \quad \text{and} \quad K_S \cdot M_j = 2, \]

where $\varphi^*(e_i) = 2E_i, \varphi^*(e_i') = 2E_i'$ and $\varphi^*(m_j) = 2M_j$.

3.1. Even pluricanonical linear system

For a positive integer $n$, the complete linear system $|2nK_S|$ contains the invariant part $|2nK_S|_0$ and the anti-invariant parts $|2nK_S|_i$ with $i = 1, 2, 3$, that is,

\[ |2nK_S| \supseteq \bigcup_{i=0}^{3} |2nK_S|_i. \]

3.1.1. Invariant part

In [6] we have

\[ \text{gltc}(S, 2K_S) = \text{lct}(S, D_0) = \frac{1}{4} \]

for some divisor $D_0 \in |2K_S|$. For example, $D_0 := 2E_1 + 4E_3 + 2E_1' + 4E_2'$, then

\[ \text{lct}(S, D_0) \geq \frac{1}{4n} \]

for any $D_0 \in |2nK_S|_0$ and the inequality is optimal.

3.1.2. Anti-invariant parts

To show

\[ \text{lct}(S, D_i) \geq \frac{1}{4n - 2} \]

for any $D_i \in |2nK_S|_i$, we need the following lemma.

Lemma 3.1. [6 Lemma 4.1] Let $\psi: X \to Y$ be a bidouble covering map between a normal variety $X$ and a smooth variety $Y$ branched along an effective divisor $B$ on $Y$, and $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. Then

\[ (X, D) \text{ is log canonical if } \left( Y, \psi(D) + \frac{1}{2}B \right) \text{ is log canonical.} \]

We deal with an integer $n \geq 2$ by Remark 1.4. Suppose that $\text{lct}(S, D_i) < 1/(4n - 2)$. Then the log pair $(S, \frac{1}{4n-2}D_i)$ is not log canonical at some point $p$. By Lemma 2.2

\[ \text{mult}_p(D_i) > 4n - 2. \]
We put an effective divisor \( d_i := \varphi(D_i) \) on \( \Sigma \). Then
\[
\left( \Sigma, \frac{1}{4n-2} d_i + \frac{1}{2} B \right)
\]
is not log canonical at a point \( \varphi(p) \) on \( \Sigma \) by Lemma 3.1.

We consider the case \( \varphi(p) \notin B_1 \cup B_2 \cup B_3 \). Then \( \left( \Sigma, \frac{1}{4n-2} d_i \right) \) is not log canonical at \( \varphi(p) \) which implies
\[
\text{glct}(\Sigma, d_i) < \frac{1}{4n-2}.
\]
However, it contradicts because \( d_i \sim_Q -nK_\Sigma \) and \( \text{glct}(\Sigma, \Delta) \geq 1/2 \) for any effective \( Q \)-Cartier divisor \( \Delta \sim_Q -K_\Sigma \) since \( \Sigma \) is a nonsingular del Pezzo surface of degree 6 (see [3, Theorem 1.7]). Thus \( \varphi(p) \in B_1 \cup B_2 \cup B_3 \).

By Proposition 1.2, we have an effective \( Q \)-Cartier divisor \( D_i - (R_j + R_k) \) for \( \{i, j, k\} = \{1, 2, 3\} \). We may deal with \( i = 1 \).

The case \( p \in E_1 \cap E_2^\prime \). We have
\[
D_1 = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3' E_3^\prime + \Omega,
\]
where rational numbers \( \alpha_1 \geq 0 \) and \( \alpha_2, \alpha_3' \geq 1 \), and \( E_1, E_2, E_3^\prime \not\subset \text{Supp}(\Omega) \) with an effective \( Q \)-Cartier divisor \( \Omega \) (denote by \( \text{Supp}(\Omega) \) the support of \( \Omega \)). Since \( p \notin E_2 \cup E_3^\prime \), the log pair \( (S, D_1 - \alpha_2 E_2 - \alpha_3' E_3^\prime) \) is not log canonical at the point \( p \).

Suppose \( \alpha_1 = 0 \), and then \( 2n = D_1 \cdot E_1 \geq \text{mult}_p(D_1) \text{mult}_p(E_1) > 4n - 2 \) which is a contradiction. So \( \alpha_1 \neq 0 \).

Since \( D_1 - (R_2 + R_3) \) is effective,
\[
\Omega \cdot M_1^1 \geq (M_1^3 + M_2^3) \cdot M_1^1 = 2.
\]
Thus \( 4n = D_1 \cdot M_1^1 = \alpha_1 + \Omega \cdot M_1^1 \) implies \( 4n - 2 \geq \alpha_1 \), and so
\[
\frac{\alpha_1}{4n - 2} \leq 1.
\]
We have a pair \( (S, E_1 + \frac{1}{4n-2} \Omega) \) is not log canonical at \( p \). By the inversion of adjunction formula,
\[
\text{the pair } \left( E_1, \frac{1}{4n-2} \Omega_{| E_1} \right) \text{ is not log canonical at } p.
\]
This implies that
\[
2n + \alpha_1 - \alpha_3' = (D_1 - \alpha_1 E_1 - \alpha_2 E_2 - \alpha_3' E_3^\prime) \cdot E_1 > 4n - 2.
\]
On the other hand, since \( D_1 - (R_2 + R_3) \) is effective,
\[
2n = D_1 \cdot E_3^\prime = \alpha_1 + \alpha_2 - \alpha_3' + \Omega \cdot E_3^\prime \geq \alpha_1 + \alpha_2 - \alpha_3' + (M_1^3 + M_2^3) \cdot E_3^\prime = \alpha_1 + \alpha_2 - \alpha_3' + 2.
\]
Hence
\[ \alpha_2 < 0 \]
which is a contradiction.

The case \( p \in E_1 \setminus (E_2' \cup E_3') \). We have
\[ D_1 = \alpha_1 E_1 + \alpha'_2 E_2' + \alpha'_3 E_3' + \Omega, \]
where rational numbers \( \alpha_1 \geq 0 \) and \( \alpha'_2, \alpha'_3 \geq 1 \), and \( E_1, E_2', E_3' \not\subseteq \text{Supp}(\Omega) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Omega \). Then
\[ 2n = D_1 \cdot E_3' = \alpha_1 - \alpha'_3 + \Omega \cdot E_3' \geq \alpha_1 - \alpha'_3 + (E_2 + M_1^3 + M_2^3) \cdot E_3' = \alpha_1 - \alpha'_3 + 3. \]
And since
\[ 4n = D_1 \cdot M_1^1 = \alpha_1 + \Omega \cdot M_1^1 \geq \alpha_1 + (M_1^3 + M_2^3) \cdot M_1^1 = \alpha_1 + 2, \]
we obtain
\[ 2n + \alpha_1 - \alpha'_2 - \alpha'_3 = (D_1 - \alpha_1 E_1 - \alpha'_2 E_2' - \alpha'_3 E_3') \cdot E_1 > 4n - 2 \]
by the inversion of adjunction formula. Hence
\[ \alpha'_2 < -1 \]
which is a contradiction.

The case \( p \in M_1^1 \setminus (E_1 \cup E_1') \). The log pair
\[ \left( S, \frac{1}{4n - 2}(M_1^1 + M_1^3 + M_2^3 + D) \right) \]
is not log canonical at the point \( p \), where \( D_1 \sim R_2 + R_3 + D \) for some \( D \in |\varphi^*(-nK_\Sigma - L_1)| \) by Proposition 1.2(i). We have
\[ D = \alpha M_1^3 + \Delta \]
where a rational number \( \alpha \geq 0 \) and \( M_1^3 \not\subseteq \text{Supp}(\Delta) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Delta \). By using a general member \( \overline{M} \) of the linear system \( |2M_1^2| \) such that \( \overline{M} \not\subseteq \text{Supp}(D) \),
\[ 8n - 12 = D \cdot \overline{M} \geq \alpha M_1^3 \cdot \overline{M} = 2\alpha. \]
Thus we can use the inversion of adjunction formula. So the log pair
\[ \left( M_1^3, \frac{1}{4n - 2}(M_1^1 + M_2^3 + \Delta) \bigg|_{M_1^3} \right) \]
is not log canonical at \( p \). Then
\[ 1 + (4n - 4) = (M_1^1 + M_2^3 + \Delta) \cdot M_1^3 \geq \text{mult}_p ((M_1^1 + M_2^3 + \Delta) \bigg|_{M_1^3}) > 4n - 2 \]
which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of $R$. Therefore for all cases $i = 1, 2, 3$,

$$\text{lct}(S, D_i) \geq \frac{1}{4n - 2}$$

for any $D_i \in |2nK_S|_i$.

And the inequality is optimal because $\text{lct}_p(S, D) = 1/(4n - 2)$ for

$$D_i := R_{i+1} + R_{i+2} + 2((2n - 1)E'_i + (n - 2)E'_{i+1} + (2n - 3)E_{i+1} + nE_{i+2}) \in |2nK_S|_i$$

and

$$p \in E'_i \setminus (E_{i+1} \cup E_{i+2} \cup M'_i \cup M'_2),$$

where the index $i \in \{1, 2, 3\}$ is considered as modulo 3.

3.2. Odd pluricanonical linear system

For a positive integer $n$, the complete linear system $|(2n + 1)K_S|$ contains the invariant part $|(2n + 1)K_S|_0$ and the anti-invariant parts $|(2n + 1)K_S|_i$ with $i = 1, 2, 3$, that is,

$$|(2n + 1)K_S| \supseteq \bigcup_{i=0}^3 |(2n + 1)K_S|_i.$$

3.2.1. Invariant part

We prove that for any $D'_0 \in |(2n + 1)K_S|_0$, the log pair $(S, \frac{1}{4n-3}D'_0)$ is log canonical. To obtain a contradiction, we assume that there is a member $D'_0$ of $|(2n + 1)K_S|_0$ such that the log pair $(S, \frac{1}{4n-3}D'_0)$ is not log canonical at some point $p$. Note that

$$|(2n + 1)K_S|_0 = R + |2(n - 1)K_S|$$

(see Proposition 1.2 and apply $B \sim -3K_S$ and $K_S \sim \varphi^*(K_S + \frac{1}{2}B)$). Thus there is the member $D'$ of the complete linear system $|2(n - 1)K_S|$ such that $D'_0 = R + D'$. Since the global log canonical threshold of the pair $(S, 2(n-1)K_S)$ is $1/(4n-4)$ (see [6, Theorem 1.3]), $p$ is contained in $R$. We consider the following cases.

The case $p \in E_3 \cap E'_1$. The log pair $(S, \frac{1}{4n-3}(E_3 + E'_1 + D'))$ is not log canonical at the point $p$. For the effective divisor

$$N := (4n - 3)E_3 + (4n - 3)E'_1 + (2n - 2)E_2 + (2n - 2)E'_2 \sim E_3 + E'_1 + D',$$

the log canonical threshold of the log pair $(S, N)$ is $1/(4n - 3)$. By Lemma 2.1, there is an effective $\mathbb{Q}$-Cartier divisor $N' \sim \mathbb{Q} N$ such that at least one component of $N$ is not
contained in the support of $N'$ and the log pair $(S, \frac{1}{4n-3} N')$ is not log canonical at $p$. Thus at least one of $E_2$, $E_3$, $E'_1$ and $E'_2$ is not contained in $\text{Supp}(N')$.

We can represent

$$N' = \alpha_3 E_3 + \alpha'_1 E'_1 + \Omega,$$

where rational numbers $\alpha_3, \alpha'_1 \geq 0$ and $E_3, E'_1 \not\subset \text{Supp}(\Omega)$ with an effective $\mathbb{Q}$-Cartier divisor $\Omega$.

Suppose $E_2 \not\subset \text{Supp}(N')$. Then

$$2n - 1 = N' \cdot E_2 \geq \alpha'_1 E'_1 \cdot E_2 = \alpha'_1$$

By the inversion of adjunction formula, the log pair

$$\left( E'_1, \frac{1}{4n-3} (\alpha_3 E_3 + \Omega) \right)$$

is not log canonical at $p$. Thus

$$(2n - 2) + \alpha'_1 = (\alpha_3 E_3 + \Omega) \cdot E'_1 \geq \text{mult}_p ((\alpha_3 E_3 + \Omega) \cdot E'_1) > 4n - 3$$

which is a contradiction.

For each case $E'_2, E_3$ or $E'_1 \not\subset \text{Supp}(N')$, we also get a contradiction by using a similar argument as above. We remark that $E_3 \not\subset \text{Supp}(N')$ (resp. $E'_1 \not\subset \text{Supp}(N')$) means $\alpha_3 = 0$ (resp. $\alpha'_1 = 0$).

The case $p \in E_3 \setminus (E'_1 \cup E'_2)$. The log pair $(S, \frac{1}{4n-3} (E_3 + M^3_1 + M^3_2 + D'))$ is not log canonical at the point $p$. We have

$$D' = \alpha_3 E_3 + \alpha'_1 E'_1 + \alpha'_2 E'_2 + \Delta,$$

where rational numbers $\alpha_3, \alpha'_1, \alpha'_2 \geq 0$ and $E_3, E'_1, E'_2 \not\subset \text{Supp}(\Delta)$ with an effective $\mathbb{Q}$-Cartier divisor $\Delta$. Let $\tilde{M}$ be a general member of the linear system $|2M^3_1|$ such that $\tilde{M} \not\subset \text{Supp}(D')$. Then

$$8n - 8 = D' \cdot \tilde{M} \geq \alpha_3 E_3 \cdot \tilde{M} = 2\alpha_3$$

implies that $4n - 4 \geq \alpha_3$. By the inversion of adjunction formula, the log pair

$$\left( E_3, \frac{1}{4n-3} (M^3_1 + M^3_2 + \Delta) \right)$$

is not log canonical at $p$. Thus

$$(2n - 1) + \alpha_3 - \alpha'_1 - \alpha'_2 \geq ((M^3_1 + M^3_2 + \Delta) \cdot E_3)_p \geq \text{mult}_p ((M^3_1 + M^3_2 + \Delta) \cdot E_3) > 4n - 3$$

which implies $\alpha_3 > (2n - 2) + \alpha'_1 + \alpha'_2$. Meanwhile, the inequality

$$(2n - 2) - \alpha_3 + \alpha'_1 = \Delta \cdot E'_1 \geq 0$$
implies that \((2n - 2) + \alpha'_1 \geq \alpha_3\) which is a contradiction.

The case \(p \in M_1^1 \setminus (E_1 \cup E'_1)\). Set \(M := M_1^2 + M_2^2 + M_3^3 + M_2^3\). Then the log pair

\[
\left( S, \frac{1}{4n-3}(M_1^1 + M + D') \right)
\]

is not log canonical at the point \(p\). We have

\[D' = \alpha M_1^1 + \Delta,\]

where a rational number \(\alpha \geq 0\) and \(M_1^1 \not\subset \text{Supp}(\Delta)\) with an effective \(\mathbb{Q}\)-Cartier divisor \(\Delta\).

By using a general member \(\hat{M}\) of the linear system \(|2M_1^2|\) such that \(\hat{M} \subset \text{Supp}(D')\),

\[8n - 8 = D' \cdot \hat{M} \geq \alpha M_1^1 \cdot \hat{M} = 2\alpha\]

which implies \(4n - 4 \geq \alpha\). By the inversion of adjunction formula, the log pair

\[
\left( M_1^1, \frac{1}{4n-3}(M + \Delta) \bigg|_{M_1^1} \right)
\]

is not log canonical at \(p\). Then

\[1 + (4n - 4) \geq ((M + \Delta) \cdot M_1^1)_p \geq \text{mult}_p((M + \Delta)|_{M_1^1}) > 4n - 3\]

which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of \(R\). Hence

\[\text{lct}(S, D'_0) \geq \frac{1}{4n-3} \quad \text{for any } D'_0 \in |(2n + 1)K_S|_0.\]

And the inequality is optimal because \(\text{lct}_p(S, D'_0) = 1/(4n - 3)\) for

\[D'_0 := R + 2(n - 1)(2E'_2 + E'_3 + 2E_1 + E_3) \in |(2n + 1)K_S|_0\]

and

\[p \in E'_2 \setminus (E_1 \cup E_3 \cup M_1^2 \cup M_2^2).\]

3.2.2. Anti-invariant part

For a positive integer \(n\) and \(i = 1, 2, 3\), \(|(2n + 1)K_S|_i\) is represented by

\[R_i + |\varphi^*((1 - n)K_{\Sigma} + L_i)|\]

(see Proposition 1.2 and apply \(B \sim -3K_{\Sigma}\)).
We may consider for $i = 1$. The divisor

$$D'_1 := E_1 + E'_1 + M^2_1 + M^2_2 + 2((2n + 1)E'_2 + (n-1)E'_1 + nE_1 + 2nE_3)$$

is in $|(2n + 1)K_S|_1$. The log canonical threshold of the log pair $(S, D'_1)$ is $1/(4n + 2)$. Note that the global log canonical threshold of the log pair $(S, K_S)$ is $1/2$ (see [6, Theorem 1.3]).

This means that the infimum of the set

$$\{ \text{lct}(S, D'_1) \mid D'_1 \in |(2n + 1)K_S|_1 \}$$

is $1/(4n + 2)$. Thus

$$\inf \{ \text{lct}(S, D'_i) \mid D'_i \in |(2n + 1)K_S|_i \} = \frac{1}{4n + 2}$$

for each $i = 1, 2, 3$.

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