Photoinduced topologically trivial magnons with finite thermal Hall effect

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In two-dimensional (2D) insulating magnets, the thermal Hall effect of magnons is believed to be a consequence of topological magnon insulator with separated magnon bands and a well-defined Chern number. Due to broken time-reversal symmetry the thermal Hall effect vanishes in Dirac magnons. In this paper, we show that periodically driven semi-Dirac magnon in 2D insulating honeycomb ferromagnet results in a photoinduced Dirac magnon at the topological phase transition between a photoinduced topological and trivial magnon insulator. Remarkably, the photoinduced Dirac magnon and the photoinduced trivial magnon insulator possess a nonzero Berry curvature and exhibit a finite thermal Hall effect. These intriguing properties of periodically driven 2D insulating magnets originate from the bosonic nature of magnons. Hence, they are not expected to exist in 2D electronic Floquet systems.

I. INTRODUCTION

Magnons in 2D insulating honeycomb ferromagnets have recently garnered considerable attention [1–8], due to their similarity to electrons in graphene [9–11]. Moreover, the recent experimental realization of 2D honeycomb ferromagnet in CrI$_3$ and other materials [12, 13] has provided a promising possibility for realizing the magnonic analog of graphene. The Dzyaloshinskii-Moriya (DM) interaction [14–15], which can be allowed in materials with no inversion center is forbidden by symmetry in most 2D insulating honeycomb ferromagnets. Therefore, the magnon dispersions of 2D insulating honeycomb ferromagnet essentially possess Dirac magnon points, reminiscent of Dirac fermions in graphene. However, as magnons are charge-neutral spin-1 bosonic quasiparticles, the Dirac magnons, although very similar to Dirac fermions, are expected to exhibit different transport properties due to their quantum statistics. This discrepancy has been overlooked in previous studies [1–8].

Indeed, magnons in the insulating magnets are simply magnetic dipole moment hopping on the lattice. Therefore, in the presence of an electromagnetic field they accumulate the Aharonov-Casher phase [10], similar to the Aharonov-Bohm phase [17] accumulated by charge particles in a perpendicular magnetic field. In recent years, the physics of magnons in the presence of a time-independent electric field have been explored [18–21], and was recently shown to realize magnonic Landau levels [22]. Quite distinctly, magnons hopping in the presence of a time-dependent oscillating electric field also lead to intriguing features. In particular, the present author has proposed that hopping magnons in the background of a time-dependent oscillating electric field generate the magnonic Floquet topological insulator [23], with similar properties to electronic Floquet topological insulator [24–34]. In the magnonic Floquet topological insulator, an explicit DM interaction or scalar spin chirality is induced by circular-polarized electric field, which breaks the time-reversal symmetry of Dirac magnons.

In this paper, we study another intriguing property of periodically driven 2D insulating honeycomb ferromagnets. We study the driven semi-Dirac magnons with linear dispersion along the $k_y$ momentum direction and quadratic dispersion along the $k_x$ momentum direction. We show that the driven semi-Dirac magnon results in a photoinduced Dirac magnon at the topological phase boundary between the magnonic Floquet topological and trivial insulators.

Usually, the thermal Hall effect in 2D insulating magnets is solely associated with topological magnon insulators with well-separated magnon bands and well-defined Chern numbers [38–45]. Thus, it is expected to vanish in the undriven gapless Dirac magnon due to the presence of time-reversal symmetry. In contrast to this general belief, we find that the photoinduced Dirac magnon at the topological critical point possesses a nonzero Berry curvature, which generates a finite thermal Hall effect even without a well-defined Chern number. We also find that the thermal Hall effect persists in the magnonic Floquet trivial insulator phase. These results are not expected to manifest in the electronic Floquet counterparts. Therefore, they are as a consequence of the bosonic nature of magnons, which can only be thermally excited at low temperatures.

II. MODEL

A. Spin model

We consider the Heisenberg spin Hamiltonian for 2D insulating honeycomb ferromagnets governed by

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j - g \mu_B B \sum_i S_i^z, \quad (1)$$

where $J_{ij} = J$ along the diagonal bonds and $J_{ij} = J'$ along the vertical bonds. The last term is an external magnetic field with $g$ and $\mu_B$ being the spin $g$ factor and the Bohr magneton respectively. The spin Hamiltonian
in Eq. 1 is believed to describe many monolayer 2D insulating honeycomb ferromagnets \cite{12, 40}.

B. Bosonic tight-binding model

We describe the magnetic excitations of the spin Hamiltonian Eq. 1 by the Holstein Primakoff (HP) transformation:

\[ S_i^z = S - a_i ^\dagger a_i, \quad S_i^\pm \approx \sqrt{2S} a_i = (S_i^-) ^\dagger, \]

where \( a_i ^\dagger \) are the bosonic creation (annihilation) operators, and \( S_i^z \pm S_i^y = S_i^y \pm iS_i^x \) denote the spin raising and lowering operators. By substituting the HP transformation into Eq. 1 and Fourier transforming into momentum space, we arrive at the free magnon Hamiltonian

\[ \mathcal{H} = \sum_{\vec{k}} \psi_\vec{k} ^\dagger \mathcal{H}(\vec{k}) \psi_{\vec{k}} , \]

with \( \psi_{\vec{k}} ^\dagger = (a_{\vec{k},A} ^\dagger, a_{\vec{k},B} ^\dagger) \).

\[ \mathcal{H}(\vec{k}) = h_0 \mathbf{1} + h_x(\vec{k}) \sigma_x + h_y(\vec{k}) \sigma_y , \]

where \( \sigma_i \) are Pauli matrices with identity \( \mathbf{1} \), \( h_0 = t' + 2t + t_B \), with \( t(t')(t_B) = JS(J'S)(g\mu_B B/S) \). \( h_x = \text{Re}[\rho(\vec{k})], \quad h_y = -\text{Im}[\rho(\vec{k})] \).

\[ \rho(\vec{k}) = \sum_\ell \bar{t}_\ell e^{i\bar{k}\ell} , \]

where \( \bar{t}_1 = t_2 = t, \bar{t}_3 = t' \). The momentum is \( \bar{k} = \vec{k} \cdot \bar{a}_\ell \) and the primitive vectors of the honeycomb lattice in Fig. 1a are \( \bar{a}_{1,2} = a(\mp \sqrt{3}/2 + 3g/2), \) and \( \bar{a}_3 = (0,0) \). Since the magnetic field simply rescales \( h_0 \), we will set \( t_B = 0 \) in the following.

C. Dirac and Semi-Dirac magnons

For \( t' = t \), there are two Dirac magnon points at \( \pm \vec{k} = (\pm 2\pi/\sqrt{3}, 2\pi/3) \) in the Brillouin zone (BZ) of Fig. 1b.) The Brillouin zone (BZ) of the honeycomb lattice. Points with the same colour are related by symmetry.

For \( t' = 2t \), there are two semi-Dirac magnon points at \( \pm \vec{k} = (\pm 2\pi/\sqrt{3}, 2\pi/3) \) in the Brillouin zone (BZ) of Fig. 1b.) The Brillouin zone (BZ) of the honeycomb lattice. Points with the same colour are related by symmetry.

Fig. 1b. They remain intact for \( t' < 2t \), but move away from the \( \pm \vec{K} \) point to \( \pm \bar{K} = (\pm k_0^0, 2\pi/3) \), where \( k_0^0 = \frac{2}{\sqrt{3}} \arccos(t'/2t) \) as shown in Fig. 2(a). Eventually, they emerge at \( M = (0, 2\pi/3) \) for \( t' = 2t \) as shown in Fig. 2(b). In this case, however, the magnon bands are linear along the \( k_y \) direction (see inset of (b)) and quadratic along the \( k_x \) direction and form a semi-Dirac magnon band. Subsequently, a gap reopens for \( t' > 2t \) and the system becomes a trivial magnon insulator. We now obtain the effective Hamiltonian in the vicinity of the magnon band crossing points.

Case (i): \( t' < 2t \), the effective Hamiltonian near \( \vec{k} = \pm \vec{K} \) is given by

\[ \mathcal{H}(\vec{q} + \eta \vec{K}) = v_0 \mathbf{1} - (\eta v_x \sigma_x q_x + v_y \sigma_y q_y) , \]

where \( \eta = \pm, \quad v_0 = h_0, \quad v_x = \sqrt{3t} \sin(\sqrt{3}k_0^0/2) \) and \( v_y = 3t \cos(\sqrt{3}k_0^0/2) \).

Case (ii): \( t' = 2t \), the effective Hamiltonian near \( \vec{k} = \vec{M} \) is given by

\[ \mathcal{H}(\vec{q} + \vec{M}) = v_0 \mathbf{1} - \frac{q_x^2}{2m} \sigma_x - \tilde{v}_1 \sigma_y q_y , \]

where \( m = 1/v_D \) and \( \tilde{v}_1 = 2v_D \) with \( v_D = 3t/2 \). This is the semi-Dirac magnon Hamiltonian which is our focus in this paper.

III. FLOQUET MAGNON

A. Time-dependent Aharonov-Casher phase

Magnons in 2D insulating magnets are simply hopping magnetic dipole moment \( g\mu_B \hat{z} \). Hence, in the presence of a time-periodic electric field \( \vec{E}(\tau) \), they accumulate the time-dependent Aharonov-Casher phase (see Appendix)

\[ \theta_{ij}(\tau) = \mu_m \int_{\tau_i}^{\tau_f} \vec{E}(\tau) \times \hat{z} \cdot d\vec{r}, \]
where $\mu_m = g\mu_B/\hbar c^2$, $h = h/2\pi$ and $c$ are the reduced Planck’s constant and the speed of light respectively, and $\vec{r}_i$ is the coordinate of the lattice at site $i$. The electric field obey the relation: $\vec{E}(\tau) = -\partial \vec{A}(\tau)/\partial \tau$, where $\vec{A}(\tau)$ is the time-periodic vector potential given by

$$\vec{A}(\tau) = [A_x \sin(\omega \tau + \phi), A_y \cos(\omega \tau), 0], \quad (9)$$

where $A_i = E_i/\omega \ (i = x, y)$ are the amplitudes of the time-dependent vector potential. Here $\omega$ is the frequency of the light and $\phi$ is the phase difference. Hence, the electric field is given by

$$\vec{E}(\tau) \times \hat{z} = [E_y \sin(\omega \tau), E_x \sin(\omega \tau + \phi), 0] \quad (10)$$

where $E_x$ and $E_y$ are the amplitudes of the time-periodic electric field along the $x$ and $y$ directions. Note that the time-periodicity is given by $\vec{E}(\tau) = \vec{E}(\tau + T)$ where $T = 2\pi/\omega$ is the period. The resulting time-dependent Hamiltonian $\mathcal{H}(\vec{k}, \tau)$ is obtained by making the time-dependent Peierls substitution $\vec{k} \to \vec{k} + \mu_m(\vec{E}(\tau) \times \hat{z})$ into Eq. (10). Therefore, we can now study periodically driven magnon systems in a same manner to driven electronic systems.

### B. Floquet-Bloch formalism

The Hamiltonian can now be expanded in Fourier space as

$$\mathcal{H}(\vec{k}, \tau) = \sum_{n=-\infty}^{\infty} e^{in\omega \tau} \mathcal{H}_n(\vec{k}), \quad (11)$$

where the Fourier component is given by $\mathcal{H}_n(\vec{k}) = \frac{1}{\pi} \int_0^\pi e^{-in\omega \tau} \mathcal{H}(\vec{k}, \tau) d\tau = \mathcal{H}_n^{(1)}(\vec{k})$. Its eigenvectors has the form $|\psi_n(\vec{k}, \tau)\rangle = e^{-i\epsilon_n(\vec{k}) \tau} |\xi_n(\vec{k})\rangle$, where $|\xi_n(\vec{k})\rangle = \sum_n e^{in\omega \tau} |\xi_n(\vec{k})\rangle$ is the time-periodic Floquet-Bloch wave function of magnons and $\epsilon_n(\vec{k})$ are the magnon quasienergies. The Floquet operator is defined as $\mathcal{F}(\vec{k}, \tau) = \mathcal{H}(\vec{k}, \tau) - i\partial_\tau$, which leads to the Floquet eigenvalue equation

$$\sum_{n} [\mathcal{H}_{n-m}(\vec{k}) + m\omega \delta_{nm} \mathcal{F}_n(\vec{k})] = \epsilon_n(\vec{k}) \xi_n(\vec{k}). \quad (12)$$

The time-dependent Hamiltonian $\mathcal{H}(\vec{k}, \tau)$ is given by

$$\mathcal{H}(\vec{k}, \tau) = \rho_0 \mathbf{1} + \left( \begin{array}{cc} 0 & \rho(\vec{k}, \tau) \\ \rho^* (\vec{k}, \tau) & 0 \end{array} \right), \quad (13)$$

where $\rho_0 = h_0$ and $\rho(\vec{k}, \tau) = -\sum_{\ell} \epsilon_{\ell} e^{i\epsilon_{\ell} \epsilon_{\mu_m} \vec{E}(\tau) \cdot \vec{d}_\ell}$. Here $\vec{d}_1 = a(\sqrt{3}/2, -1/2)$, $\vec{d}_2 = -a(\sqrt{3}/2, 1/2)$, and $\vec{d}_3 = a(0, 1)$ are the vectors of the nearest-neighbour bonds. The Fourier components of the Hamiltonian are given by

$$\mathcal{H}_n(\vec{k}) = \rho_0 \delta_{q, 0} \mathbf{1} + \left( \begin{array}{cc} 0 & \rho_n(\vec{k}) \\ \rho_n^* (\vec{k}) & 0 \end{array} \right), \quad (14)$$

where $\rho_n(\vec{k}) = -\sum_{\ell \in q} \epsilon_{\ell} e^{i\epsilon_{\ell} \epsilon_{\mu_m} \vec{E}(\tau) \cdot \vec{d}_\ell}$, and $q = n - m = -1, 0, 1$. The renormalized interactions are given by

$$t_{1, q} = t_1 \mathcal{J}_{-q}(\mathcal{E}_- e^{-i\epsilon_n \Phi_0}), \quad t_{2, q} = t_1 \mathcal{J}_q(\mathcal{E}_+ e^{i\epsilon_n \Phi_0}), \quad (15)$$

where $\mathcal{J}_n(x)$ is the Bessel function of order $n$.

$$\mathcal{E}_\pm = \frac{1}{2} \sqrt{3E_y^2 + E_x^2 \pm 2\sqrt{3}E_y E_x \cos(\phi)}, \quad (16)$$

$$\Phi_\pm = \arctan \left( \frac{E_x \sin(\phi)}{\sqrt{3}E_y \pm E_x \cos(\phi)} \right). \quad (17)$$

In the magnonic Floquet formalism, the light intensity is characterized by the dimensionless quantity

$$\mathcal{E}_i = \frac{g\mu_B E_i a}{\hbar c^2}, \quad (18)$$

where $i = x, y$ and $a$ is the lattice constant. Since there is no frequency denominator in Eq. (18), the magnonic Floquet formalism is different from that of electronic Floquet formalism [24, 25].

### C. Magnonic Floquet quasienergy bands

We consider the off-resonant limit $\omega \gg t, t'$. In this limit, the system can be described by an effective time-independent Hamiltonian [28, 29], which can be obtained perturbatively in $1/\omega$ expansion as

$$\mathcal{H}_{\text{eff}}(\vec{k}) = \mathcal{H}_0(\vec{k}) - \frac{1}{\omega} \left( [\mathcal{H}_0(\vec{k}), \mathcal{H}_{-1}(\vec{k})] - [\mathcal{H}_0(\vec{k}), \mathcal{H}_1(\vec{k})] + [\mathcal{H}_{-1}(\vec{k}), \mathcal{H}_1(\vec{k})] \right). \quad (19)$$

In Fig. 3, we have shown the Floquet magnon quasienergy bands. Fig. 3(a) shows the magnonic Floquet topological insulator for $t' < 2t$ and Fig. 3(b) shows the photoinduced Dirac magnon for $t' = 2t$. We can see that driven semi-Dirac magnon remain gapless at $t' = 2t$, due to the competition between $k_x$ and $k_y$ (cf. Eq. 7). Note that the Floquet magnon quasienergy bands are gapped in the regime $t' > 2t$ (not shown), which corresponds to a Floquet trivial magnon insulator. As we will show later, the Floquet trivial magnon insulator for $t' > 2t$ and $\phi = \pi/2$ is different from the one induced by linearly-polarized light for $\phi = 0$. It is crucial to note that the photoinduced Dirac magnon in Fig. 3(b) is reminiscent of 2D Dirac semimetals with SOC [17]. In fact,
for \( \phi = \pi/2 \) and \( \mathcal{E}_x = \mathcal{E}_y = \mathcal{E} \), the effective Hamiltonian of the photoinduced Dirac magnons near \( \tilde{k} = M \) is given by

\[
\mathcal{H}(\tilde{q} + M) = v_0 \mathbf{1} + v_x(\mathcal{E}, \omega)q_x \sigma_z - v_y(\mathcal{E})\sigma_y q_y,
\]

where \( v_x(\mathcal{E}, \omega) \) and \( v_y(\mathcal{E}) \) are the group velocities along the \( x \) and \( y \) directions respectively.

We further show the Floquet magnon quasienergy bands for a 1D strip geometry in Fig. 4. In the magnonic Floquet topological insulator phase (a) and (b), there are chiral gapless magnon edge modes traversing the bulk magnon gap. For the photoinduced Dirac magnons in (c), we can see that the bulk gap closes — an indication of a topological phase transition to a magnonic Floquet trivial insulator in (d), with no chiral magnon edge modes.

### D. Topological phase transition

To compute the Berry curvature and the Chern number of the magnon quasienergy bands, we implement the discrete Brillouin zone method proposed by T. Fukui et al. [37]. In this method, the Berry curvature is of the form \( \Omega(\tilde{k}_\ell) \equiv F_{12}(\tilde{k}_\ell)/i \), where \( F_{12}(\tilde{k}_\ell) \) is the lattice field strength, which is defined by \( U(1) \) links [37]. The associated Chern number is defined as,

\[
\mathcal{C} = \frac{1}{2\pi i} \sum_{\ell} F_{12}(\tilde{k}_\ell).
\]

In Fig. 5 we have shown the photoinduced Berry curvature distribution of the lowest magnon band. As shown in Fig. 5(a), the photoinduced Berry curvature has two peaks in the magnonic Floquet topological insulator phase. They correspond to the photoinduced gaps at the Dirac points. In Fig. 5(b) we can also see that the photoinduced Berry curvature vanishes everywhere except at the photoinduced Dirac magnon, where it develops a large peak. Interestingly, the magnonic Floquet trivial insulator induced by circularly-polarized light at \( \phi = \pi/2 \) and \( t' > 2t \) has a different photoinduced Berry curvature distribution from the one induced by linearly-polarized light at \( \phi = 0 \). This can be seen in Figs. 5(c) and (d) respectively. We find that the Chern number changes from \( \mathcal{C} = \pm 1 \) in the magnonic Floquet topological insulator phase \( (t' < 2t) \) to \( \mathcal{C} = 0 \) in the magnonic Floquet trivial insulator phase \( (t' > 2t) \) for \( \phi = \pi/2 \) or \( \phi = 0 \). Therefore, the photoinduced Dirac magnon phase at \( t' = 2t \) is a topological critical point.

### E. Thermal Hall effect

The most interesting aspects of ferromagnetic topological magnons is that they exhibit the thermal Hall effect [38–45]. In ferromagnetic insulators, the thermal Hall effect has only been studied in the topological magnon insulator phase, when the lowest (acoustic) magnon band is well separated and carry a well-defined Chern number. For periodically driven magnon systems, we focus on the regime where the Bose distribution function is close to thermal equilibrium. In this regime, the same theoretical
FIG. 5: Color online. Photoinduced Berry curvature distribution of the lowest Floquet magnon band for the (a) magnonic Floquet topological insulator induced by circularly-polarized light $\phi = \pi/2$ for $t' = 1.5t$, (b) photoinduced Dirac magnon induced by circularly-polarized light $\phi = \pi/2$ for $t' = 2t$, (c) magnonic Floquet trivial insulator induced by circularly-polarized light $\phi = \pi/2$ for $t' = 2.5t$, (d) magnonic Floquet trivial insulator induced by linearly-polarized light $\phi = 0$ for $t' = 1.5t$. The parameters in all plots are $E_x = E_y = 2$ in units of $g \mu_B a / \hbar c^2$, and $\omega = 10t$.

FIG. 6: Color online. Photoinduced thermal Hall conductivity for different light amplitudes in units of $g \mu_B a / \hbar c^2$. (a) Photoinduced Dirac magnon phase for $t' = 2t$. (b) Photoinduced trivial magnon insulator phase for $t' = 2.5t$. We use circularly-polarized electric field $\phi = \pi/2$ and $\omega = 10t$.

The concept of the thermal Hall effect in undriven topological magnon systems \cite{38, 45} can be applied to the driven magnon systems. The transverse component of the thermal Hall conductivity is given explicitly by \cite{10}

$$\kappa_{xy} = -\frac{k_B T}{V} \sum_{\vec{k}} \sum_{\pm} c_2(n_\alpha) \Omega_{\pm}(\vec{k}),$$

(22)

where $V$ is the volume of the system, $T$ and $k_B$ are the temperature and the Boltzmann constant respectively.

$$n_\pm = n[\epsilon_\pm(\vec{k})] = \frac{1}{\exp(\epsilon_\pm(\vec{k})/k_B T) - 1},$$

(23)

is the Bose distribution function close to thermal equilibrium. Here $c_2(x) = (1 + x) \left( \ln \left( \frac{1+x}{x} \right) ^2 - \ln x \right)^2 - 2 \text{Li}_2(-x)$, with $\text{Li}_2(x)$ being the dilogarithm. The plus or minus sign refers to the two middle magnon quasienergies.

Evidently, the thermal Hall conductivity is simply the Berry curvature weighed by the $c_2$ function. Therefore, its dominant contribution comes from the peaks of the Berry curvature (see Fig. 5). In addition, the thermal Hall transport is determined mainly by the acoustic (lower) magnon branch due to the bosonic nature of magnons. In Fig. 6(a) and (b), we have shown the trend of $\kappa_{xy}$ for different electric field amplitudes in the (a)
photoinduced Dirac magnon phase for \( t' = 2t \) and (b) in the photoinduced trivial magnon insulator phase for \( t' > 2t \). The finite thermal Hall conductivity in these topologically trivial phases can be attributed to the photoinduced Berry curvatures in Figs. 5(b) and (c). We note that the thermal Hall conductivity vanishes for the magnonic Floquet trivial insulator induced by linearly-polarized light \( \phi = 0 \), because the integration of the Berry curvature in Fig. 5(d) vanishes identically. This is expected as linearly-polarized light does not break time-reversal symmetry.

IV. CONCLUSION

We have shown that photoinduced Dirac magnons at the topological phase transition can exhibit interesting transport property that is not expected in the undriven Dirac magnon systems, as well as in electronic systems. Furthermore, we have established that magnon edge modes and well-defined Chern number of magnon bands do not guarantee the presence or absence of the thermal Hall effect in 2D ferromagnetic insulators. In other words, the thermal Hall effect in 2D ferromagnetic insulators is not necessarily a consequence of topological magnon insulator, but it depends solely on the Berry curvature of the magnon bands. We believe that our result offers the manipulation of the intrinsic property of 2D ferromagnetic insulators without an external magnetic field. This could pave the way for investigating interesting potential practical applications of magnons in magnetic insulators, such as photo-magnonics [48], magnon spintronics [49, 50], and ultrafast optical control of magnetic spin currents [21, 51, 53].

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Appendix. Dirac magnons in an external electromagnetic field

A massless charge-neutral Dirac magnon particle in the presence of external electromagnetic field can be described by the Dirac-Pauli Lagrangian [35, 39, 54]

\[ \mathcal{L} = \bar{\psi} \left( -v_0 \gamma^0 + i v_D \gamma^\mu \partial_\mu - \frac{v_D \mu m \sigma^{\mu\nu} F_{\mu\nu}}{2} \right) \psi, \quad (24) \]

where \( v_0 \) accounts for the finite energy Dirac magnon point and \( v_D \) is the group velocity of the Dirac magnon. The two-component wave function is given by \( \psi = (u_A, u_B) \) with \( \psi = \psi^\dagger \gamma^0 \). Here \( F_{\mu\nu} \) is the electromagnetic field tensor and \( \sigma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] = i \gamma^\mu \gamma^\nu, \quad (\mu \neq \nu) \).

In \((2+1)\) dimensions, the Dirac matrices are simply Pauli matrices given by

\[ \beta = \gamma^0 = \sigma_z, \quad \gamma^1 = i \sigma_y, \quad \gamma^2 = -i \sigma_x. \quad (25) \]

They obey the algebra

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu}, \quad (26) \]

where \( g^{\mu\nu} = \text{diag}(1, -1, -1) \) is the Minkowski metric in \( 2+1 \) dimensions. The corresponding Hamiltonian is given by

\[ H = \int d^2 x \left[ \pi(x) \dot{\Psi}(x) - \mathcal{L} \right] \equiv \int d^2 x \, \Psi^\dagger \mathcal{H}_D \Psi, \quad (27) \]

where \( \pi(x) = \frac{\partial \mathcal{L}}{\partial (\dot{\Psi}^\dagger(x))} \) is the generalized momentum.

We consider an electromagnetic field tensor with only time-varying electric field component \( E(\tau) \). In this case, the corresponding Hamiltonian is given by

\[ \mathcal{H}_D = v_0 + v_D \bar{\sigma} \cdot \left[ -i \vec{\nabla} + \mu_m (\vec{E}(\tau) \times \hat{z}) \right], \quad (28) \]

We can see that the time-dependent Aharonov-Casher phase enters the Dirac magnon Hamiltonian as a minimal coupling.

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