A COMBINED SCALARIZATION METHOD FOR
MULTI-OBJECTIVE OPTIMIZATION PROBLEMS

YUAN-MEI XIA
Department of Mathematics, College of Sciences, Shanghai University
Shanghai 200444, China

XIN-MIN YANG* and KE-QUAN ZHAO
School of Mathematical Sciences, Chongqing Normal University
Chongqing 401331, China

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Abstract. In this paper, we propose a new combined scalarization method
of multi-objective optimization problems by using the surplus variables and
the generalized Tchebycheff norm and then use it to obtain some equivalent
scalarization characterizations of (weakly, strictly, properly) efficient solutions
by adjusting the range of parameters. These scalarization results do not need
any convexity assumption conditions of objective functions. Furthermore, we
establish some scalarization results of approximate solutions by means of the
method. Moreover, we also present some examples to illustrate the main re-

1. Introduction. As is known to all, the theory and methods of multi-objective
optimization have been playing an important role in many research fields and real-
life problems (see [2, 4, 6, 7, 9, 15, 21, 26, 29]). Multi-objective optimization
problems are to maximize or minimize multiple and competing objective functions
over a feasible set of decisions, but it is usually not possible to optimize the func-
tions simultaneously. Consequently, efficiency, weak efficiency, strict efficiency and
proper efficiency are commonly used to characterize the relative optimal decisions.
Generally speaking, since it is very difficult to solve these solutions directly, then
various kinds of scalarization methods are presented. Scalarization methods have
been become an important and popular methods to study multi-objective optimization
problems. Steuer and Choo [24] proposed an interactive weighted Tchebycheff
procedure to find efficient solutions. By using $k$th-objective $\varepsilon$-constraint method,
Ruiz-Canales and Rufián-Lizana [23] established a characterization of weakly effi-
cient solutions for multi-objective optimization problems. In order to generate all
weakly efficient solutions of a convex multi-objective optimization problem, Luc et
al. [20] constructed a sequence of scalarizing functions. Based on a Tchebychev-type scalarization method, Dutta and Kaya [8] proposed a new scalarization method and established some scalarization results for weakly efficient solutions. Moreover, Burachik et al. [3] proposed a new scalarization technique to generate the Pareto front for nonconvex problems with disconnected Pareto fronts and domains. Ghanekanafi and Khorram [13] also proposed a new scalarization method for finding the efficient frontier of nonconvex multi-objective optimization problems, and they established the theoretical properties of this scalarization method.

Simultaneously, there are also many significant scalarization results for proper efficient solutions of multi-objective optimization problems. For example, Geoffrion [12] first proposed the proper efficiency for multi-objective optimization problems and established its scalarization results with convexity condition for this solution by using the weighted scalar optimization problem. Choo and Atkins [5] proposed a scalarization method by using an extended form of the generalized Tchebycheff norm to characterize the proper efficient solution for nonconvex multi-objective optimization problems. Huang and Yang [16] established a characterization of properly efficient solutions. Recently, Ehrgott and Ruzika [10] proposed the improved $\varepsilon$-constraint method by using slack variables and surplus variables, established some scalarization results for (weakly, properly) efficient solutions of multi-objective optimization problems. Moreover, Rastegar and Khorram [22] presented a general scalarization technique for solving (weakly, strictly, properly) efficient solutions of multi-objective optimization problems.

In recent years, studying on approximate solutions of multi-objective optimization problems also has drawn more and more researchers’ attention. Kutateladze firstly proposed the notion of approximation solution in [18]. Loridan extended the notion of this solution to vector optimization in [19] and there are many other works for approximate solutions of vector optimization problems (see [25, 27, 28]). For multi-objective optimization problems, Engau and Wieck [11] investigated the scalarization results for $\varepsilon$-efficient solutions by some well-known scalarization methods. Moreover, by utilizing several scalarization approaches, Ghaznavi-ghosoni et al. [14] provided some necessary and sufficient conditions for $\varepsilon$-(strong, weak, proper) efficiency. Rastegar and Khorram also established some relationships between $\varepsilon$-(weakly, properly) efficient points of a general (without any convexity assumption) multi-objective optimization problem and $\epsilon$-optimal solutions of the introduced scalarized problem by using the scalarization method proposed by them in [22].

Although there have been a lot of scalarization methods and scalarization results of (weakly, strictly, properly) efficient solutions of multi-objective optimization problems, we notice that these scalarization methods can not equivalently characterize various kinds of solutions of multi-objective optimization problems. Motivated by [5, 10, 22], in this paper, a new combined scalarization method by using the generalized Tchebycheff norm and surplus variables is first proposed, which can be used to characterize completely the (weakly, strictly, properly) efficient solutions.
Furthermore, some scalarization results for $\varepsilon$-(weakly) efficient solutions are also established.

2. Preliminaries. In this paper, let $\mathbb{R}^p$ be the $p$-dimensional Euclidean space, $\mathbb{R}$ be the set of all real numbers, $\mathbf{0}$ represents a $p$-dimensional zero vector and $\mathbf{e}$ represents a $p$-dimensional vector with $e_i = 1$ for all $i$. We adopt the following order relation for any given $x, y \in \mathbb{R}^p$,

$$ x < y \Leftrightarrow x_i < y_i, \forall i = 1, \ldots, p, $$
$$ x \leq y \Leftrightarrow x_i \leq y_i, \forall i = 1, \ldots, p, $$
$$ x \leq y \Leftrightarrow x \leq y, x \neq y. $$

Moreover, we let $\mathbb{R}^p_{\geq}$ denote the non-negative orthant of $\mathbb{R}^p$.

Consider the following multi-objective optimization problem:

\begin{align*}
\text{(MOP)} \quad \min & \quad f(x) = (f_1(x), f_2(x), \ldots, f_p(x)) \\
\text{s.t.} & \quad x \in X = \{ x \in \mathbb{R}^n | g_j(x) \leq 0, j = 1, \ldots, m; h_k(x) = 0, k = 1, \ldots, l \},
\end{align*}

where $f_i, g_j, h_k : X \subset \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, p, j = 1, 2, \ldots, m, k = 1, 2, \ldots, l$, and the functions $f_i$ are bounded on the constraint set. Moreover, the efficient solution, weakly efficient solution, strictly efficient solution and properly efficient solution for (MOP) are given as follows:

**Definition 2.1** ([9, 12]). Let $\hat{x} \in X$ be a feasible solution of (MOP).

(i) $\hat{x}$ is said to be an efficient solution of (MOP) if there is no $x \in X$ such that $f(x) \leq f(\hat{x})$;

(ii) $\hat{x}$ is said to be a weakly efficient solution of (MOP) if there is no $x \in X$ such that $f(x) < f(\hat{x})$;

(iii) $\hat{x}$ is said to be a strictly efficient solution of (MOP) if there is no $x \in X \setminus \{ \hat{x} \}$ such that $f(x) \leq f(\hat{x})$;

(iv) $\hat{x}$ is said to be a properly efficient solution of (MOP) if $\hat{x}$ is an efficient solution and there exists $M > 0$ such that for all $i$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$, there exists at least one $j$ such that $f_j(\hat{x}) < f_j(x)$ and

$$ \frac{f_j(\hat{x}) - f_j(x)}{f_j(x) - f_j(\hat{x})} \leq M. $$

It should be noted that a strictly (properly) efficient solution for (MOP) must be an efficient solution for (MOP) and an efficient solution for (MOP) must be a weakly efficient solution for (MOP). Moreover, a vector is said to be an ideal vector if it minimizes each of the objective functions. It is well-known that if an ideal vector is feasible for (MOP), then it is an efficient solution for (MOP). However, an ideal vector may not exist and so we generally consider a reference vector called the utopia vector. A utopia vector $f^* = (f_1^*, f_2^*, \ldots, f_p^*)$ is defined as $f^*_i = \min_{x \in X} f_i(x) - \delta$, where $\delta > 0$ for all $i = 1, 2, \ldots, p$. Moreover, the $\varepsilon$-efficient solution and $\varepsilon$-weakly efficient solution for (MOP) are given as follows:

**Definition 2.2** ([19]). Let $\varepsilon \in \mathbb{R}^p_{\geq}$ and $\hat{x} \in X$ be a feasible solution of (MOP).

(i) $\hat{x}$ is said to be an $\varepsilon$-efficient solution of (MOP) if there is no $x \in X$ such that $f(x) \leq f(\hat{x}) - \varepsilon$;

(ii) $\hat{x}$ is said to be an $\varepsilon$-weakly efficient solution of (MOP) if there is no $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$. 
Remark 1. If $\varepsilon = 0$, then $\varepsilon$-efficiency and $\varepsilon$-weak efficiency reduce to the usual definition of efficiency and weak efficiency in Definition 2.1. Moreover, it is clear that $\varepsilon$-efficiency implies $\varepsilon$-weak efficiency.

Consider the general single objective program as follows:

$$(\text{SOP}) \min_{x \in X} g(x),$$

where $g : X \to \mathbb{R}$. The optimal solution and $\varepsilon$-optimal solution for (SOP) are given as follows:

Definition 2.3. Let $\varepsilon \geq 0$ and $\hat{x} \in X$ be a feasible solution of (SOP).

(i) $\hat{x}$ is said to be an optimal solution of (SOP) if $g(x) \geq g(\hat{x})$ for all $x \in X$;

(ii) $\hat{x}$ is said to be an $\varepsilon$-optimal solution of (SOP) if $g(x) \geq g(\hat{x}) - \varepsilon$ for all $x \in X$.

Definition 2.4 ([5]). For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^p$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_p) > 0$, generalized Tchebycheff norm $\|\cdot\|_\beta$ is a real-valued function on $\mathbb{R}^p$ defined by $\|y\|_\beta = \max_{1 \leq i \leq p} \beta_i |(I - \alpha x)^{-1} y_i|$, where $I$ is a $p \times p$ matrix and

$$(I)_{ij} = \begin{cases} 1, & i = j, \\ \alpha, & i \neq j. \end{cases}$$

Lemma 2.5 ([5]). If $-1/2 \leq \alpha \leq 0$, then $I$ is nonsingular and all the elements of the inverse matrix of $I$ is nonnegative. In particular, when $-1/2 < \alpha < 0$, all the elements of the inverse matrix of $I$ is positive.

3. Equivalent characterizations of solutions. We first propose a new combined scalarization method for (MOP) by using the surplus variables and the generalized Tchebycheff norm. Consider the following combined scalarization problem:

$$(\text{SOP})^+_{\alpha\beta\gamma\mu} \min_{x \in X} \|f(x) - f^*\|_\beta + \sum_{i=1}^{p} \mu_i s_i \quad \text{s.t.} \quad f_i(x) - \mu_i s_i \leq \gamma_i, \quad x \in X, s_i \geq 0, i = 1, 2, \cdots, p,$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, $\mu = (\mu_1, \mu_2, \cdots, \mu_p) \geq 0$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_p) \in \mathbb{R}^p$.

In the following, we use $(\text{SOP})^+_{\alpha\beta\gamma\mu}$ to establish some equivalent characterizations of (weakly, strictly, properly) efficient solutions for (MOP).

Lemma 3.1. Let $\hat{x} \in X$, $\hat{s} \geq 0$, $-1/2 < \alpha \leq 0$, $\beta > 0$, $\mu \geq 0$ and $\gamma \in \mathbb{R}^p$. If $(\hat{x}, \hat{s})$ is an optimal solution of $(\text{SOP})^+_{\alpha\beta\gamma\mu}$, then $\hat{x}$ is a weakly efficient solution of (MOP).

Proof. On the contrary, if $\hat{x} \in X$ is not a weakly efficient solution of (MOP), then there exists $\bar{x} \in X$ such that $f(\bar{x}) < f(\hat{x})$. Since $(\hat{x}, \hat{s})$ is feasible for $(\text{SOP})^+_{\alpha\beta\gamma\mu}$, then

$$f_i(\bar{x}) - \mu_i \hat{s}_i < f_i(\hat{x}) - \mu_i \hat{s}_i \leq \gamma_i, \quad i = 1, 2, \cdots, p,$$
Proof. (Necessity). Let 
\( \frac{\alpha}{\beta} > 1 \) and 
\( \beta > 0 \) and 
\( \gamma > 0 \) and 
\( \mu > 0 \) and 
\( \hat{s} \geq 0 \) such that 
\( (\hat{x}, \hat{s}) \) is an optimal solution of 
\( (\text{SOP})_{\alpha\beta\gamma\mu}^+ \).

\( \begin{align*}
\|f(\hat{x}) - f^*\|_\beta^\alpha + \sum_{i=1}^p \mu_i \hat{s}_i &= \max_{1 \leq i \leq p} \beta_i (I_{\alpha i}^{-1}(f(\hat{x}) - f^*)) + \sum_{i=1}^p \mu_i \hat{s}_i \\
&< \max_{1 \leq i \leq p} \beta_i (I_{\alpha i}^{-1}(f(\hat{x}) - f^*)) + \sum_{i=1}^p \mu_i \hat{s}_i \\
&= \|f(\hat{x}) - f^*\|_\beta^\alpha + \sum_{i=1}^p \mu_i \hat{s}_i,
\end{align*} \)

which contradicts the optimality of 
\( (\hat{x}, \hat{s}) \) for 
\( (\text{SOP})_{\alpha\beta\gamma\mu}^+ \). \( \square \)

**Theorem 3.2.** Let \( \hat{x} \in X \), \( \mu \geq 0 \) and \( \gamma \geq f(\hat{x}) \). Then \( \hat{x} \) is a weakly efficient solution of 
\( (\text{MOP}) \) if and only if there exist \( \frac{1}{\beta p} < \alpha \leq 0 \), \( \beta > 0 \) and 
\( \hat{s} \geq 0 \) such that 
\( (\hat{x}, \hat{s}) \) is an optimal solution of 
\( (\text{SOP})_{\alpha\beta\gamma\mu}^+ \).

**Proof.** (Necessity). Let
\[
\alpha = 0, \hat{s}_i = 0, \beta_i = \frac{1}{f_i(\hat{x}) - f^*_i}, \quad i = 1, 2, \ldots, p.
\]

On the contrary, if \( (\hat{x}, \hat{s}) \) is not an optimal solution of 
\( (\text{SOP})_{\alpha\beta\gamma\mu}^+ \), then there exist \( \bar{x} \) and \( \bar{s} \) such that
\[
\|f(\bar{x}) - f^*\|_\beta^\alpha + \sum_{i=1}^p \mu_i \bar{s}_i < \|f(\bar{x}) - f^*\|_\beta^\alpha.
\]

It follows that \( \|f(\bar{x}) - f^*\|_\beta^\alpha < \|f(\bar{x}) - f^*\|_\beta^\alpha \). Then we can obtain that
\( f_i(\bar{x}) - f^*_i < f_i(\bar{x}) - f^*_i, \quad i = 1, 2, \ldots, p, \)
which is in contradiction with the weak efficiency of \( \hat{x} \). Therefore, \( (\hat{x}, \hat{s}) \) is an optimal solution of 
\( (\text{SOP})_{\alpha\beta\gamma\mu}^+ \).

(Sufficiency). We can easily obtain the result by using Lemma 3.1. \( \square \)

**Remark 2.** If the condition \( \gamma \geq f(\hat{x}) \) is relaxed to the case of \( \gamma \in \mathbb{R}^p \) or is replaced by \( \gamma \leq f(\hat{x}) \), then the necessity of Theorem 3.2 may not be valid even if 
\( (\text{MOP}) \) is convex. The following example illustrates it.

**Example 1.** Consider the following multi-objective optimization problem:
\[
\begin{align*}
\min & \quad f(x) = (f_1(x), f_2(x)) = (x_1, x_2) \\
\text{s.t.} & \quad x \in X = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = 1, 1 \leq x_2 \leq 2 \}.
\end{align*}
\]

We can obtain that \( (\text{MOP}) \) is convex and the sets of weakly efficient solutions is 
\( X \). Let \( \delta = 0.1, \hat{x} = (1, 2), \bar{x} = (1, 1) \) and \( \gamma = (1, 1) \). Clearly, \( f(\bar{x}) \leq f(\hat{x}), \)
\( f^*_i = f^*_2 = 0.9 \) and then by using Lemma 2.5, we can obtain that for any 
\( \alpha \in \left( -\frac{1}{\beta p}, 0 \right) \)
and \( \beta > 0, \)
\[
\|f(\hat{x}) - f^*\|_\beta^\alpha \leq \|f(\hat{x}) - f^*\|_\beta^\alpha.
\]

Moreover, let \( \mu \geq 0 \) and 
\( X(\hat{x}) = \{(\hat{x}, s) \in X \times \mathbb{R}^2 | f_i(\hat{x}) - \mu_i \hat{s}_i \leq \gamma_i, \hat{s}_i \geq 0, i = 1, 2 \}, \)
\( X(\hat{x}) = \emptyset \) and so it is clear that there is no 
\( \alpha \in \left( -\frac{1}{\beta p}, 0 \right) \), \( \beta > 0 \) and 
\( \hat{s} \geq 0 \) such
that \((\hat{x}, \hat{s})\) is an optimal solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\). Therefore, we assume \(\mu_2 > 0\) and so \(\hat{s}_2 > 0\) for any \((\hat{x}, \hat{s}) \in X(\hat{x})\). Therefore, for any \((\hat{x}, \hat{s}) \in X(\hat{x})\),
\[
\|f(\hat{x}) - f^*\|_3^0 < \|f(\hat{x}) - f^*\|_3^0 + \mu_1 \hat{s}_1 + \mu_2 \hat{s}_2.
\]
Moreover, \((\hat{x}, 0, 0)\) is a feasible solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\) then \((\hat{x}, \hat{s})\) is not an optimal solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\). From the arbitrariness of \((\hat{x}, \hat{s})\), \(\alpha\) and \(\beta\), we know that there is no \(\alpha \in \left(-\frac{1}{2p}, 0\right]\), \(\beta > 0\) and \((\hat{x}, \hat{s}) \in X(\hat{x})\) such that \((\hat{x}, \hat{s})\) is an optimal solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\) with \(\mu \geq 0\).

**Remark 3.** The condition \(\gamma \geq f(\hat{x})\) also can not be relaxed to the case of \(\gamma \in \mathbb{R}^p\) with \(\gamma_i < f_i(\hat{x})\) for some \(i\) in the necessity of Theorem 3.2 even if \((\text{MOP})\) is convex. The following example illustrates it.

**Example 2.** Consider the following multi-objective optimization problem:
\[
\begin{align*}
\text{min} & \quad f(x) = (f_1(x), f_2(x)) = (x_1, x_2) \\
\text{s.t.} & \quad x \in X = \{(x_1, x_2) \in \mathbb{R}^2 | 1 \leq x_1 \leq 2, 1 \leq x_2 \leq 2\}.
\end{align*}
\]
We can obtain that \((\text{MOP})\) is convex and the set of weakly efficient solutions for \((\text{MOP})\) is
\[
\{(x_1, x_2) \in \mathbb{R}^2 | x_1 = 1, 1 \leq x_2 \leq 2\} \cup \{(x_1, x_2) \in \mathbb{R}^2 | 1 \leq x_1 \leq 2, x_2 = 1\}.
\]
Let \(\delta = 0.1\), \(\hat{x} = (1, 2)\), \(\bar{x} = (1, 1)\) and \(\gamma = (2, 1)\). Similar to the analysis of Example 1, we know that there is no \(\alpha \in \left(-\frac{1}{2p}, 0\right]\), \(\beta > 0\) and \(\hat{s} \geq 0\) such that \((\hat{x}, \hat{s})\) is an optimal solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\) with \(\mu \geq 0\).

**Remark 4.** From Remark 2, Example 1 and Remark 3, Example 2, we know that the condition \(\gamma \geq f(\hat{x})\) is very important in Theorem 3.2. However, this condition is not necessary to Lemma 3.1. For example, consider the multi-objective optimization problem in Example 2 and let \(\delta = 0.1\), \(\hat{x} = (2, 1)\), \(\gamma = (2, 0.5)\), \(\delta = 0.1\), \(\alpha = 0\), \(\beta_i = \frac{1}{f_i(\bar{x}) - f_i(\hat{x})}\), \(\mu = (1, 1)\) and \(\hat{s} = (0, 0.5)\). Then we can verify that \((\hat{x}, \hat{s})\) is an optimal solution for \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\) and \(\hat{x}\) is a weakly efficient solution for \((\text{MOP})\).

Under the assumption condition that there exists an ideal vector \(x^I \in X\), we can obtain another scalarization result for weakly efficient solution of \((\text{MOP})\) as follows.

**Theorem 3.3.** Let \(\hat{x} \in X\). If there exists an ideal vector \(x^I \in X\), then \(\hat{x}\) is a weakly efficient solution of \((\text{MOP})\) if and only if there exist \(-\frac{1}{2p} < \alpha \leq 0\), \(\beta > 0\), \(\mu \geq 0\), \(\gamma \geq f(\hat{x})\) with \(\gamma_i < f_i(\hat{x})\) or \(\gamma_i = 0\) for some \(i\) and \(\hat{s} \geq 0\) such that \((\hat{x}, \hat{s})\) is an optimal solution of \((\text{SOP})_{1\alpha\beta\gamma\mu}^+\).

**Proof.** (Necessity). Assume that \(\hat{x}\) is a weakly efficient solution of \((\text{MOP})\) and let \(I = \{i | f_i(\hat{x}) = f_i(x^I)\}\). From the weak efficiency of \(\hat{x}\), we get \(I \neq \emptyset\). For \(i = 1, 2, \ldots, p\), we take \(\alpha = 0\) and
\[
\beta_i = \frac{1}{f_i(\bar{x}) - f_i(\hat{x})}, \quad \gamma_i = \left\{ \begin{array}{ll}
\min \left\{ \frac{f_i(\bar{x})}{2}, 2f_i(\hat{x}) \right\}, & i \in I, \\
\frac{f_i(\bar{x}) - \gamma_i}{\mu_i}, & i \notin I.
\end{array} \right.
\]
where \(\rho = (\rho_1, \rho_2, \ldots, \rho_p) \geq 0\) with \(\rho_i > 0\) for \(i \in I\). Obviously, \(-\frac{1}{2p} < \alpha \leq 0\), \(\beta > 0\), \(\mu \geq 0\), \(\gamma \geq f(\hat{x})\) with \(\gamma_i < f_i(\hat{x})\) or \(\gamma_i = 0\) for \(i \in I\), \(\hat{s} \geq 0\) and \((\hat{x}, \hat{s})\) is a feasible
solution of $(SOP)_{\alpha \beta \gamma \mu}^+$. If $(\tilde{x}, \tilde{s})$ is not an optimal solution of $(SOP)_{\alpha \beta \gamma \mu}^+$, then there exist $\tilde{x}$ and $\tilde{s}$ such that $(\tilde{x}, \tilde{s})$ is feasible and

$$
\|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i < \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i
$$

$$
= \|f(\tilde{x}) - f^*\|_\beta^2 + \left(\sum_{i \in I} (f_i(\tilde{x}) - \gamma_i) \right).
$$

Since $(\tilde{x}, \tilde{s})$ is feasible, then $f_i(\tilde{x}) - \mu_i \tilde{s}_i \leq \gamma_i$, $i = 1, 2, \cdots, p$. It follows from the definition of $I$ that

$$
\mu_i \tilde{s}_i \geq f_i(\tilde{x}) - \gamma_i \geq f_i(\tilde{x}) - \gamma_i, \quad i \in I.
$$

Hence from $\mu_i \geq 0$ and $\tilde{s}_i \geq 0$, we have

$$
\sum_{i=1}^p \mu_i \tilde{s}_i \geq \sum_{i \in I} (f_i(\tilde{x}) - \gamma_i).
$$

Therefore,

$$
\|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i \in I} (f_i(\tilde{x}) - \gamma_i) \leq \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i
$$

$$
< \|f(\tilde{x}) - f^*\|_\beta^2 + \left(\sum_{i \in I} (f_i(\tilde{x}) - \gamma_i) \right).
$$

Whence, $\|f(\tilde{x}) - f^*\|_\beta^2 < \|f(\tilde{x}) - f^*\|_\beta^2$. Then we can obtain that

$$
f_i(\tilde{x}) - f_i^* < f_i(\bar{x}) - f_i^*, \quad i = 1, 2, \cdots, p,
$$

which is in contradiction with the weak efficiency of $\tilde{x}$. Therefore, $(\tilde{x}, \tilde{s})$ is an optimal solution of $(SOP)_{\alpha \beta \gamma \mu}^+$.

(Sufficiency). We can easily obtain the result by using Lemma 3.1. \qed

**Remark 5.** From the proof of Theorem 3.3, we can easily obtain the more general result as follows:

Let $\tilde{x} \in X$. If there exists an ideal vector $x^I$ in $X$, $I = \{i | f_i(\tilde{x}) = f_i(x^I)\}$, $\gamma \leq f(\tilde{x})$ with $\gamma_i = f_i(\tilde{x})$ for $i \notin I$ and $\gamma_i < f_i(\tilde{x})$ or $\gamma_i = 0$ for $i \in I$, and $\mu \geq 0$ with $\mu_i > 0$ for $i \in I$, then $\tilde{x}$ is a weakly efficient solution of (MOP) if and only if there exist $-\frac{1}{2p} < \alpha \leq 0, \beta > 0$ and $\tilde{s} \geq 0$ such that $(\tilde{x}, \tilde{s})$ is an optimal solution of $(SOP)_{\alpha \beta \gamma \mu}^+$.

**Theorem 3.4.** Let $\tilde{x} \in X, \ -\frac{1}{2p} < \alpha < 0$ and $\beta > 0$. Then $\tilde{x} \in X$ is an efficient solution of (MOP) if and only if there exist $\mu \geq 0$, $\gamma \in \mathbb{R}^p$ and $\tilde{s} \geq 0$ such that $(\tilde{x}, \tilde{s})$ is an optimal solution of $(SOP)_{\alpha \beta \gamma \mu}^+$.

**Proof.** (Necessity). Let

$$
\gamma_i = f_i(\tilde{x}), \mu_i = \tilde{s}_i = 0, \quad i = 1, 2, \cdots, p.
$$

Obviously, $(\tilde{x}, \tilde{s})$ is a feasible solution of $(SOP)_{\alpha \beta \gamma \mu}^+$. Moreover, let $(\tilde{x}, \tilde{s})$ be the feasible solution of $(SOP)_{\alpha \beta \gamma \mu}^+$. Then

$$
f_i(\tilde{x}) \leq \gamma_i = f_i(\tilde{x}), \quad i = 1, 2, \cdots, p.
$$

It follows from the efficiency of $\tilde{x}$ that $f(\tilde{x}) = f(\bar{x})$. Hence for any $\alpha \in (-\frac{1}{2p}, 0)$ and $\beta > 0$,

$$
\|f(\tilde{x}) - f^*\|_\beta^2 = \|f(\bar{x}) - f^*\|_\beta^2.
$$

Thus, $(\tilde{x}, \tilde{s})$ is an optimal solution of $(SOP)_{\alpha \beta \gamma \mu}^+$. \hfill \qed
Proposition 1. Let \( \hat{x} \in X \).

(i) If \( \hat{x} \in X \) is an efficient solution of (MOP), then there exist \( -\frac{1}{2p} < \alpha \leq 0, \beta > 0, \mu \geq 0, \gamma \in \mathbb{R}^p \) and \( \tilde{s} \geq 0 \) with \( \mu_i \tilde{s}_i \neq 0 \) for some \( i \) such that \((\hat{x}, \tilde{s})\) is an optimal solution of \((\text{SOP})^{+\alpha\beta\gamma\mu}\).

(ii) Assume that \( \tilde{s} \geq 0, -\frac{1}{2p} < \alpha \leq 0, \beta > 0, \mu \geq 0, \gamma \in \mathbb{R}^p \) and \( \mu_i \tilde{s}_i \neq 0 \) for all \( i \). If \((\hat{x}, \tilde{s})\) is an optimal solution of \((\text{SOP})^{+\alpha\beta\gamma\mu}\), then \( \hat{x} \) is an efficient solution of \((\text{MOP})\).

Proof. (i) Let \( \alpha = 0, \beta_i = \frac{1}{\hat{f}_i(\hat{x}) - \hat{f}_i}(i = 1, 2, \ldots, p) \) and

\[
\gamma_i = \begin{cases} f_i(\hat{x}) - 1, & i = 1, \\ f_i(\hat{x}), & i \neq 1, \end{cases} \quad \mu_i = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1, \end{cases} \quad \tilde{s}_i = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1. \end{cases}
\]

Obviously, \((\hat{x}, \tilde{s})\) is a feasible solution of \((\text{SOP})^{+\alpha\beta\gamma\mu}\). Moreover, let \((\tilde{x}, \tilde{s})\) be the feasible solution of \((\text{SOP})^{+\alpha\beta\gamma\mu}\). Then \( f_i(\hat{x}) \leq f_i(\tilde{x}) (\forall i \neq 1) \) and \( f_1(\hat{x}) - \tilde{s}_1 \leq f_1(\tilde{x}) - 1 \).

It follows from the efficiency of \( \hat{x} \) that \( f_1(\hat{x}) \geq f_1(\tilde{x}) \) and \( \tilde{s}_1 \geq 1 \). Hence

\[
\|f(\tilde{x}) - f^*\|^\alpha_\beta = \frac{f_1(\tilde{x}) - f^*_1}{f_1(\tilde{x}) - f^*_1} \geq 1 = \|f(\tilde{x}) - f^*\|^\beta_\gamma.
\]

Thus, \((\hat{x}, \tilde{s})\) is an optimal solution of \((\text{SOP})^{+\alpha\beta\gamma\mu}\).
(ii) On the contrary, if \( \hat{x} \) is not an efficient solution of (MOP), then there exists \( \bar{x} \in X \) such that \( f_i(\bar{x}) \leq f_i(\hat{x}) \) for all \( i = 1, 2, \cdots, p \) and \( f_j(\bar{x}) < f_j(\hat{x}) \) for some \( j \). Then it follows from the definition of \( f^* \) and Lemma 2.5 that

\[
\|f(\bar{x}) - f^*\|_\beta^2 \leq \|f(\hat{x}) - f^*\|_\beta^2.
\]

Let \( \bar{s} \) satisfy the following condition:

\[
\bar{s}_i = \begin{cases} 
\max \left\{ 0, \frac{f_i(\bar{x}) - f_i(\hat{x})}{\mu_i} \right\}, & i = j, \\
\bar{s}_i, & i \neq j.
\end{cases}
\]

Then \( 0 \leq \bar{s} \leq \hat{s} \). It follows that

\[
\|f(\bar{x}) - f^*\|_\beta^2 + \sum_{i=1}^{p} \mu_i \bar{s}_i < \|f(\bar{x}) - f^*\|_\beta^2 + \sum_{i=1}^{p} \mu_i \bar{s}_i.
\]

Moreover, by using the definition of \( \bar{s} \), we can obtain that for all \( i = 1, 2, \cdots, p \),

\[
f_i(\bar{x}) - \mu_i \bar{s}_i \leq f_i(\hat{x}) - \mu_i \bar{s}_i \leq \gamma_i.
\]

This implies that \( (\bar{x}, \bar{s}) \) is a feasible solution of \( (SOP)_{\alpha \beta \gamma \mu}^+ \) and the value of the objective function at this point is less than \( (\hat{x}, \hat{s}) \). But this is a contradiction to the optimality of \( (\hat{x}, \hat{s}) \) for \( (SOP)_{\alpha \beta \gamma \mu}^+ \). Hence \( \hat{x} \) is an efficient solution of (MOP). \( \square \)

**Remark 6.** The range of \( \alpha \) in Theorem 3.4 can not be relaxed to the case of \(-\frac{1}{2p} < \alpha \leq 0 \). The following example illustrates it.

**Example 3.** Consider the following multi-objective optimization problem:

\[
\begin{aligned}
\min \quad & f(x) = (f_1(x), f_2(x)) = (x_1, x_2) \\
\text{s.t.} \quad & x \in X = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = 2, 2 \leq x_2 \leq 4\}.
\end{aligned}
\]

We can obtain that the sets of efficient and weakly efficient solutions for (MOP) are \( \{(2, 2)\} \) and \( X \), respectively. Let \( \delta = 0.1, \hat{x} = (2, 4), \gamma = f(\bar{x}) \) and \( \alpha = 0, \beta_i = \frac{1}{f_i(\bar{x}) - f_i^*} > 0, \mu_i = \bar{s}_i = 0, \quad i = 1, 2. \)

Then by using Theorem 3.2, we know that \( (\hat{x}, \hat{s}) \) is an optimal solution for \( (SOP)_{\alpha \beta \gamma \mu}^+ \) but \( \hat{x} \) is not an efficient solution for (MOP).

**Theorem 3.5.** Let \( \hat{x} \in X, -\frac{1}{2p} < \alpha \leq 0 \) and \( \beta > 0 \). Then \( \hat{x} \) is a strictly efficient solution of (MOP) if and only if there exist \( \mu \geq 0, \gamma \in \mathbb{R}^p \) and \( \hat{s} \geq 0 \) such that \( (\hat{x}, \hat{s}) \) is an optimal solution of \( (SOP)_{\alpha \beta \gamma \mu}^+ \) and \( \hat{x} \) is unique to the optimal value of \( (SOP)_{\alpha \beta \gamma \mu}^+ \).

**Proof.** (Necessity). Let

\[
\gamma_i = f_i(\bar{x}), \mu_i = \bar{s}_i = 0, \quad i = 1, 2, \cdots, p.
\]

Obviously, \( (\hat{x}, \hat{s}) \) is a feasible solution of \( (SOP)_{\alpha \beta \gamma \mu}^+ \). Moreover, let \( (\bar{x}, \bar{s}) \) be the feasible solution of \( (SOP)_{\alpha \beta \gamma \mu}^+ \). Then it follows from the strictly efficiency of \( \hat{x} \) that \( \bar{x} = \hat{x} \). Thus \( (\hat{x}, \hat{s}) \) is an optimal solution of \( (SOP)_{\alpha \beta \gamma \mu}^+ \) and \( \hat{x} \) is unique to the optimal value of \( (SOP)_{\alpha \beta \gamma \mu}^+ \).
(Sufficiency). On the contrary, if \( \hat{x} \) is not a strictly efficient solution, then there exists \( \bar{x} \in X \) such that \( \bar{x} \neq \hat{x} \) and for all \( i = 1, 2, \cdots, p \), \( f_i(\bar{x}) \leq f_i(\hat{x}) \). Since \( (\hat{x}, \tilde{s}) \) is a feasible solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \), then
\[
 f_i(\bar{x}) - \mu_i\tilde{s}_i \leq f_i(\hat{x}) - \mu_i\tilde{s}_i \leq \gamma_i, \quad i = 1, 2, \cdots, p,
\]
which implies that \( (\bar{x}, \tilde{s}) \) is a feasible solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \). Moreover, from Lemma 2.5, we can obtain that
\[
 \|f(\bar{x}) - f^*\|_\beta + \sum_{i=1}^{p} \mu_i\tilde{s}_i \leq \|f(\hat{x}) - f^*\|_\beta + \sum_{i=1}^{p} \mu_i\tilde{s}_i,
\]
which contradicts to the fact that \( (\hat{x}, \tilde{s}) \) is the optimal solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \) and \( \hat{x} \) is unique to the optimal value of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \).

**Theorem 3.6.** Let \( \hat{x} \in X \).

(i) If \( \hat{x} \in X \) is a properly efficient solution of \( (MOP) \), then there exist \(-\frac{1}{2p} < \alpha < 0, \beta > 0, \mu \geq 0, \gamma \in \mathbb{R}^p \) and \( \tilde{s} \geq 0 \) with \( \mu_i\tilde{s}_i \neq 0 \) for some \( i \) such that \( (\hat{x}, \tilde{s}) \) is an optimal solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \).

(ii) Let \( \gamma = (U_1, U_2, \cdots, U_p) \). Then \( \hat{x} \in X \) is a properly efficient solution of \( (MOP) \) if and only if there exist \(-\frac{1}{2p} < \alpha < 0, \beta > 0, \mu \geq 0 \) and \( \tilde{s} \geq 0 \) such that \( (\hat{x}, \tilde{s}) \) is an optimal solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \).

**Proof.** (i). Let \( \tilde{\eta}_i = (I^{-1}_{\alpha}(f(\bar{x}) - f^*))_i, \beta_i = 1/\tilde{\eta}_i (i = 1, 2, \cdots, p) \) and
\[
 \gamma_i = \begin{cases} 
 f_i(\bar{x}) - 1, & i = 1, \\
 f_i(\bar{x}), & i \neq 1,
\end{cases} \quad \mu_i = \begin{cases} 
 1, & i = 1, \\
 0, & i \neq 1.
\end{cases} \quad \tilde{s}_i = \begin{cases} 
 1, & i = 1, \\
 0, & i \neq 1.
\end{cases}
\]
Then
\[
 I\tilde{\eta} = f(\bar{x}) - f^* \quad \text{and} \quad \|f(\bar{x}) - f^*\|_\beta = 1.
\]
It follows from \( f(\bar{x}) - f^* > 0 \) and Lemma 2.5 that \( I^{-1}_{\alpha}(f(\bar{x}) - f^*) > 0 \). In the following, we show \( (\hat{x}, \tilde{s}) \) is an optimal solution of \( (SOP)^{+}_{\alpha\beta\gamma\mu} \). On the contrary, assume there exist \( \bar{x} \) and \( \tilde{s} \) such that \( \|f(\bar{x}) - f^*\|_\beta + \tilde{s}_1 < 2 \). Let
\[
 \tilde{n}_i = (I^{-1}_{\alpha}(f(\bar{x}) - f^*))_i, i = 1, 2, \cdots, p.
\]
Then
\[
 I\tilde{n} = f(\bar{x}) - f^*, \quad \max_{1 \leq i \leq p} \beta_i\tilde{n}_i + \tilde{s}_1 < 2
\]
and
\[
 f_i(\bar{x}) \leq f_i(\hat{x}) (\forall i \neq 1), \quad f_1(\bar{x}) - \tilde{s}_1 \leq f_1(\hat{x}) - 1.
\]
It follows from the efficiency of \( \hat{x} \) that \( f_1(\bar{x}) \geq f_1(\hat{x}) \) and \( \tilde{s}_1 \geq 1 \). Hence \( \max_{1 \leq i \leq p} \beta_i\tilde{n}_i < 1 \) and so
\[
 \tilde{n}_i < \tilde{n}_i, \quad i = 1, 2, \cdots, p.
\]
If \( f_1(\bar{x}) = f_1(\hat{x}) \), then it follows from the efficiency of \( \hat{x} \) and (2) that \( f(\bar{x}) = f(\hat{x}) \). So
\[
 \tilde{n}_1 - \tilde{n}_1 = (I^{-1}_{\alpha}(f(\bar{x}) - f(\hat{x})))_1 = 0.
\]
But this contradicts to (3). Hence \( f_1(\bar{x}) > f_1(\hat{x}) \).

Furthermore, if there is no \( i \) such that \( f_i(\bar{x}) < f_i(\hat{x}) \), then \( f(\bar{x}) \geq f(\hat{x}) \) and so
\[
 \tilde{n}_i - \tilde{n}_i = (I^{-1}_{\alpha}(f(\bar{x}) - f(\hat{x})))_i \leq 0.
\]
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But this contradicts to (3). Hence \( f_i(\tilde{x}) \leq f_i(\bar{x}) (\forall i = 2, 3, \ldots, p) \) and \( f_k(\tilde{x}) < f_k(\bar{x}) \) for some \( k \). Let \( q = \tilde{n}_1 - \bar{n}_1 + \cdots + \tilde{n}_p - \bar{n}_p \). Then
\[
  f_1(\bar{x}) - f_1(\tilde{x}) = (I_\alpha(\tilde{n} - \bar{n}))_1 = \alpha q + (1 - \alpha)(\tilde{n}_1 - \bar{n}_1), \forall i = 1, 2, \ldots, p.
\]
Furthermore, let \( \tilde{n}_k - \bar{n}_k = \max_{1 \leq i \leq p} (\tilde{n}_i - \bar{n}_i) \). Then \( k \neq 1 \), \( f_k(\tilde{x}) > f_k(\bar{x}) \) and \( \tilde{n}_k - \bar{n}_k \geq \frac{q}{p} \). Moreover, from (1),
\[
  \frac{f_k(\tilde{x}) - f_k(\bar{x})}{f_1(\tilde{x}) - f_1(\bar{x})} = \frac{\alpha q + (1 - \alpha)(\tilde{n}_k - \bar{n}_k)}{-\alpha q + (1 - \alpha)(\tilde{n}_1 - \bar{n}_1)} = \frac{\alpha q/(1 - \alpha) + \tilde{n}_k - \bar{n}_k}{-\alpha q/(1 - \alpha) - (\tilde{n}_1 - \bar{n}_1)} \geq \frac{\alpha q/(1 - \alpha) + q/p}{-\alpha q/(1 - \alpha)} = -1 + \frac{1 - \alpha}{-\alpha p} > M,
\]
which contradicts with the proper efficiency of \( \hat{x} \). Whence, \((\hat{x}, \hat{s})\) is an optimal solution of \((SOP)^\alpha_\beta_\gamma_\mu\).

(ii). (Necessity). The conclusion can be obtained by using Theorem 3.1 in [5] and the equivalence of \((SOP)^+_\alpha_\beta_\gamma_\mu\) and the model proposed by Choo and Atkins in [5] when \( \mu = 0 \).

(Sufficiency). From Theorem 3.4, we can obtain that \( \hat{x} \in X \) is an efficient solution of \((MOP)\). The remaining proof is similar with Theorem 3.1 in [5].

Remark 7. There are many different scalarization results for various kinds of solutions of multi-objective optimization problems up till the present moment. However, most of them are some sufficient conditions, necessary conditions or necessary and sufficient conditions only for a kind of solutions (see e.g. [1, 3, 5, 10]) by using some known scalarization methods. For example, we consider the following multi-objective optimization problem:
\[
  \min_x \quad f(x) = x
\]
\[
  \text{s.t.} \quad x \in X = \{(x_1, x_2) | 1 \leq x_1 \leq 2, x_2 = 1\}.
\]
From the definition of efficient solutions, we can obtain the set of efficient solution is \( \{(1, 1)\} \). Let \( \mu = (0, 1), \hat{x} = (2, 1), a = f(\hat{x}) = (2, 1), r = (1, 1) \). Then \((\hat{x}, 0, 0)\) is an optimal solution of the scalarization model \(FPS(a, r, \mu)\) in [1]. But \( \hat{x} \) is not the efficient solution of the above multi-objective optimization problem. This example illustrates the fact that the inverse of Theorem 3.4 in [1] may not be valid. It is noting that these known scalarization methods can not be applied to obtain the equivalent characterizations of several kinds of classical solutions for multi-objective optimization problems. In this section, we propose a new scalarization model and establish some equivalent results for (weakly, strictly, properly) efficient solutions of multi-objective optimization problems by adjusting the range of the parameters of this kind of scalarization model. Therefore, in some degree, the scalarization model \((SOP)^+_\alpha_\beta_\gamma_\mu\) presented by us and the corresponding results improve and generalize some known results in the literatures.

4. Scalarization of approximate solutions. In this section, we establish some scalarization results for \(\varepsilon\)-(weakly) efficient solutions of \((MOP)\) by means of the combined scalarization problem \((SOP)^+_\alpha_\beta_\gamma_\mu\).

Theorem 4.1. Let \( \hat{x} \in X, \hat{s} \geq 0, -\frac{1}{2p} < \alpha \leq 0, \beta > 0, \mu \geq 0, \gamma \in \mathbb{R}^p, \varepsilon \in \mathbb{R}^p \) and \( 0 \leq \varepsilon \leq \min_{1 \leq i \leq p} \beta_i \left(I_\alpha^{-1}\varepsilon\right)_i \). If \((\hat{x}, \hat{s})\) is an \(\varepsilon\)-optimal solution of \((SOP)^+_\alpha_\beta_\gamma_\mu\), then \( \hat{x} \) is an \(\varepsilon\)-weakly efficient solution of \((MOP)\).
Proof. If \( \varepsilon = 0 \), then the result is valid by using Lemma 3.1. In the following, we only prove that the result holds in case of \( \varepsilon \in \mathbb{R}^p_+ \). On the contrary, if \( \tilde{x} \in X \) is not an \( \varepsilon \)-weakly efficient solution of (MOP), then there exists \( \tilde{x} \in X \) such that \( f(\tilde{x}) < f(\tilde{x}) - \varepsilon \). Since \( (\tilde{x}, \tilde{s}) \) is feasible for \((\text{SOP})_{\alpha\beta\gamma\mu}^\dagger\), then

\[
\begin{align*}
   f_i(\tilde{x}) - \mu_i \tilde{s}_i < f_i(\tilde{x}) - \mu_i \tilde{s}_i - \varepsilon_i \leq f_i(\tilde{x}) - \mu_i \tilde{s}_i \leq \gamma_i, \quad i = 1, 2, \ldots, p,
\end{align*}
\]

which implies that \((\tilde{x}, \tilde{s})\) is feasible. Moreover, by using \( \varepsilon \leq \min_{1 \leq i \leq p} \beta_i \left( I^{-1}_\alpha \varepsilon \right)_i, \beta > 0 \),

\[
f_i^* = \min_{x \in X} f_i(x) - \delta \text{ and Lemma 2.5}, \text{ we can obtain that}
\]

\[
\begin{align*}
   \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i + \varepsilon & = \max_{1 \leq i \leq p} \beta_i (I^{-1}_\alpha f(\tilde{x}) - f^*)_i + \sum_{i=1}^p \mu_i \tilde{s}_i + \varepsilon \\
   & \leq \max_{1 \leq i \leq p} \beta_i (I^{-1}_\alpha f(\tilde{x}) - f^*_i + \varepsilon)_i + \sum_{i=1}^p \mu_i \tilde{s}_i.
\end{align*}
\]

Moreover, from the inequality \( f(\tilde{x}) < f(\tilde{x}) - \varepsilon \) and Lemma 2.5, we have

\[
\begin{align*}
   \max_{1 \leq i \leq p} \beta_i (I^{-1}_\alpha (f(\tilde{x}) - f^*)_i) < \max_{1 \leq i \leq p} \beta_i (I^{-1}_\alpha (f(\tilde{x}) - f^*)_i).
\end{align*}
\]

Therefore,

\[
\begin{align*}
   \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i + \varepsilon & < \max_{1 \leq i \leq p} \beta_i (I^{-1}_\alpha (f(\tilde{x}) - f^*)_i) + \sum_{i=1}^p \mu_i \tilde{s}_i \\
   & = \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i,
\end{align*}
\]

which is in contradiction with the \( \varepsilon \)-optimality of \((\tilde{x}, \tilde{s})\) for \((\text{SOP})_{\alpha\beta\gamma\mu}^+\).

\[\Box\]

**Theorem 4.2.** Let \( \tilde{x} \in X, \mu \geq 0, \gamma \geq f(\tilde{x}) \) and \( \varepsilon \in \mathbb{R}^p_+ \). If \( \tilde{x} \) is an \( \varepsilon \)-weakly efficient solution of (MOP), then there exist \( -\frac{1}{2p} < \alpha \leq 0, \beta > 0 \) and \( \tilde{s} \geq 0 \) such that \((\tilde{x}, \tilde{s})\) is an \( \varepsilon \)-optimal solution of \((\text{SOP})_{\alpha\beta\gamma\mu}^+\), where \( \varepsilon = \max_{1 \leq i \leq p} \beta_i \left( I^{-1}_\alpha \varepsilon \right)_i \).

**Proof.** Let

\[
\alpha = 0, \tilde{s}_i = 0, \beta_i = \frac{1}{f_i(\tilde{x}) - f_i^*}, \quad i = 1, 2, \ldots, p.
\]

Apparently, \((\tilde{x}, \tilde{s})\) is a feasible solution of \((\text{SOP})_{\alpha\beta\gamma\mu}^+\). Suppose that \((\tilde{x}, \tilde{s})\) is not an \( \varepsilon \)-optimal solution of \((\text{SOP})_{\alpha\beta\gamma\mu}^+\). Then there exist \( \tilde{x} \) and \( \tilde{s} \) such that

\[
\begin{align*}
   \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i < \|f(\tilde{x}) - f^*\|_\beta^2 + \sum_{i=1}^p \mu_i \tilde{s}_i - \varepsilon = \|f(\tilde{x}) - f^*\|_\beta^2 - \varepsilon.
\end{align*}
\]

It follows that

\[
\|f(\tilde{x}) - f^*\|_\beta^2 < 1 - \varepsilon \leq 1 - \beta_i \varepsilon_i, \quad i = 1, 2, \ldots, p.
\]

Therefore, \( f_i(\tilde{x}) - f_i^* < f_i(\tilde{x}) - f_i^* \leq \varepsilon_i, \quad i = 1, 2, \ldots, p \), which contradicts the \( \varepsilon \)-weak efficiency of \( \tilde{x} \). Whence, \((\tilde{x}, \tilde{s})\) is an \( \varepsilon \)-optimal solution of \((\text{SOP})_{\alpha\beta\gamma\mu}^+\).

\[\Box\]

**Theorem 4.3.** Let \( \tilde{x} \in X, \varepsilon \in \mathbb{R}^p_+ \) and \( \mu \geq 0 \) with \( \mu_i > 0 \) for \( \varepsilon_i > 0 \). If \( \tilde{x} \) is an \( \varepsilon \)-weakly efficient solution of (MOP), then there exist \( -\frac{1}{2p} < \alpha \leq 0, \beta > 0, \gamma \in \mathbb{R}^p \) and \( \tilde{s} \geq 0 \) such that \((\tilde{x}, \tilde{s})\) is an \( \varepsilon \)-optimal solution of \((\text{SOP})_{\alpha\beta\gamma\mu}^+\), where \( \varepsilon = \max_{1 \leq i \leq p} \beta_i \left( I^{-1}_\alpha \varepsilon \right)_i + \varepsilon^T \varepsilon \).
Proof. For \( i = 1, 2, \cdots, p \), we take
\[
\alpha = 0, \hat{s}_i = \rho_i \varepsilon_i, \gamma_i = f_i(\hat{x}) - \varepsilon_i, \beta_i = \frac{1}{f_i(\hat{x}) - f_i^*},
\]
where \( \rho \geq 0 \) with \( \rho_i = \frac{1}{\mu_i} \) if \( \varepsilon_i > 0 \). Apparently, \((\hat{x}, \hat{s})\) is a feasible solution of \((SOP)^+_{\alpha \beta \gamma \mu}\). Suppose that \((\hat{x}, \hat{s})\) is not an \( \varepsilon \)-optimal solution of \((SOP)^+_{\alpha \beta \gamma \mu}\). Then there exist \( \tilde{x} \) and \( \tilde{s} \) such that for any \( i = 1, 2, \cdots, p \),
\[
\|f(\tilde{x}) - f^*\|_\beta^\beta + \sum_{i=1}^p \mu_i \tilde{s}_i < \|f(\tilde{x}) - f^*\|_\beta^\beta + \sum_{i=1}^p \mu_i \tilde{s}_i - \varepsilon
\]
\[
= \|f(\tilde{x}) - f^*\|_\beta^\beta - \max_{1 \leq i \leq p} \beta_i \varepsilon_i \leq 1 - \beta_i \varepsilon_i.
\]
It follows \( f_i(\tilde{x}) - f_i^* < f_i(\tilde{x}) - f_i^* - \varepsilon_i, \ i = 1, 2, \cdots, p \) which contradicts to the \( \varepsilon \)-weak efficiency of \( \hat{x} \). Whence, \((\hat{x}, \hat{s})\) is an \( \varepsilon \)-optimal solution of \((SOP)^+_{\alpha \beta \gamma \mu}\).

\[ \square \]

**Theorem 4.4.** Let \( \tilde{x} \in X, \hat{s} \geq 0, -\frac{1}{2p} < \alpha < 0, \beta > 0, \mu \geq 0, \gamma \in \mathbb{R}^p, \varepsilon \in \mathbb{R}^p \geq 0 \) and \( 0 \leq \varepsilon \leq \min_{1 \leq i \leq p} \beta_i (I_\alpha^{-1} \varepsilon)_i \). If \((\hat{x}, \hat{s})\) is an \( \varepsilon \)-optimal solution of \((SOP)^+_{\alpha \beta \gamma \mu}\), then \( \hat{x} \) is an \( \varepsilon \)-efficient solution of \((MOP)\).

**Proof.** On the contrary, if \( \hat{x} \) is not an \( \varepsilon \)-efficient solution of \((MOP)\), then there exists \( \tilde{x} \in X \) such that \( f_i(\tilde{x}) \leq f_i(\hat{x}) - \varepsilon_i \) for all \( i = 1, 2, \cdots, p \) and \( f_j(\tilde{x}) < f_j(\hat{x}) - \varepsilon_j \) for some \( j \). Therefore, \((\tilde{x}, \tilde{s})\) is a feasible solution of \((SOP)^+_{\alpha \beta \gamma \mu}\). Moreover, since \( \varepsilon \leq \min_{1 \leq i \leq p} \beta_i (I_\alpha^{-1} \varepsilon)_i, -\frac{1}{2p} < \alpha < 0, \beta > 0, f_i^* = \min_{x \in X} f_i(x) - \delta_i \), then from Lemma 2.5 and the analysis of sufficiency of Theorem 3.4, we can obtain that
\[
\|f(\tilde{x}) - f^*\|_\beta^\beta + \sum_{i=1}^p \mu_i \tilde{s}_i + \varepsilon \leq \max_{1 \leq i \leq p} \beta_i (I_\alpha^{-1} (f(\tilde{x}) - f^*) + \varepsilon)_i + \sum_{i=1}^p \mu_i \tilde{s}_i
\]
\[
< \max_{1 \leq i \leq p} \beta_i (I_\alpha^{-1} (f(\tilde{x}) - f^*)))_i + \sum_{i=1}^p \mu_i \tilde{s}_i
\]
\[
= \|f(\tilde{x}) - f^*\|_\beta^\beta + \sum_{i=1}^p \mu_i \tilde{s}_i,
\]
which is in contradiction with the \( \varepsilon \)-optimality of \((\hat{x}, \hat{s})\) for \((SOP)^+_{\alpha \beta \gamma \mu}\).

\[ \square \]

5. **Concluding remarks.** We propose a new combined scalarization method by using the surplus variables and the generalized Tchebycheff norm and obtain some equivalent characterizations of (weakly, strictly, properly) efficient solutions of \((MOP)\) by adjusting the range of parameters and without any convexity assumption conditions. We also establish scalarization results of approximate solutions of \((MOP)\). It is worth noting that we do not obtain equivalent scalarization characterizations of approximate solutions of \((MOP)\). It remains one interesting open question.

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E-mail address: mathymxia@163.com
E-mail address: xmyang@cqu.edu.cn
E-mail address: kequanz@163.com