What can one reconstruct from the representation ring of a compact group?

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Abstract

It is well known that there exist non-isomorphic compact groups with isomorphic representation rings (fusion rules). Nevertheless, considerable structural information about the group can be reconstructed from its representation ring. We review these types of partial reconstruction theorems, including some recent results. In the Appendix a derivation of the Clebsch-Gordan series of $SU(2)$ based only on information about the dimensions of the irreducible representations is presented.

1 Introduction

Although the representation theory of compact groups is far from being completely understood, it is a very well established field of mathematics. One of the basic theorems in this field is that every continuous unitary representation of a compact group on a complex Hilbert space is equivalent to a direct sum of irreducible finite-dimensional unitary representations (shortly: irreps). In particular, the tensor product of any two irreps can be decomposed to the so called Clebsch-Gordan series:

$$D_p \otimes D_q \cong \bigoplus_r N^r_{p,q} D_r.$$  

The multiplicities $N^r_{p,q}$ are often referred to as fusion rules. In terms of irreducible characters the above decomposition can be written in the form

$$\chi_p \cdot \chi_q = \sum_r N^r_{p,q} \chi_r, \quad \chi_r \in I_G, \quad (1)$$

here and throughout the paper $I_G$ will denote the set of characters belonging to the irreps of a compact group $G$.

From equation (1) the pointwise product of characters of arbitrary finite-dimensional continuous unitary representations can be deduced (provided that
their decompositions to irreducibles are known), and one can also extend this product to generalized characters, i.e. to differences of characters. In this way a ring $\mathcal{R}(G)$ is obtained. This ring admits a natural partial ordering making it into a partially ordered ring with positive cone $\mathcal{R}(G)^+$ consisting of characters, i.e., $\chi_2 \prec \chi_1$ holds, if $(\chi_1 - \chi_2) \in \mathcal{R}(G)^+$. The ordered ring $\mathcal{R}(G)$ is called the representation ring of the compact group $G$. The irreducible characters are the minimal positive elements of $\mathcal{R}(G)$ and form a $\mathbb{Z}$-basis for the ring. The representation ring encodes exactly the same information as the fusion rules.

Considerable amount of work has been devoted to the derivation of fusion rules of various compact groups. In this contribution we will consider the less frequently investigated dual question: Provided that the fusion rules, or representation ring, of a (possibly unknown) compact group is known, what attributes of the group can one reconstruct from this information? Our intention is to give a short and informal review on this subject.

## 2 Complete reconstructions

In general it is not possible to recover a compact group completely from its representation ring, since there exist non-isomorphic compact groups - even finite groups - with isomorphic representation rings (see Fig. 1.).

![Multiplication tables of the dihedral group $D_4$ and the quaternion group $Q_8$.](image_url)

Figure 1. Multiplication tables of the dihedral group $D_4$ and the quaternion group $Q_8$. in both tables black squares denote the unit elements. These two groups are not isomorphic, since the number of elements of order two differs, but the groups have equivalent character tables, thus isomorphic fusion rules.

It is quite remarkable, that the phenomenon illustrated in Fig. 1. cannot occur in the case of connected compact groups. According to a beautiful theorem of Handelman [1], two connected compact groups can only have order isomorphic representation rings (equivalent fusion rules), if the groups themselves are isomorphic. However, Handelman’s theorem is not a reconstruction theorem - it only states this unicity, but does not provide a general reconstruction method.

There are, however, other types of representation theoretical data that can be used to completely recover any compact group. Namely, a compact group can be
reconstructed from its Krein-algebra (Tannaka-Krein reconstruction theorem [2, chapter 7]), or from its representation category, viewed as a rigid, monoidal, symmetric C*-category (Doplicher-Roberts reconstruction theorem [3]). This latter theorem provides also the basis for the theory of superselection sectors in Algebraic Quantum Field Theory [4].

3 Pontryagin duality and the abelianization of a compact group

Let \( G \) be a locally compact abelian group, and let \( \hat{G} \) denote the set of continuous unitary one-dimensional characters. The product of two such characters will again be in \( \hat{G} \), hence \( \hat{G} \) has a natural abelian group structure. Furthermore, it can also naturally be endowed with a topology [2, chapter 6], which is compatible with its group structure. This topology of \( \hat{G} \) is simply the discrete topology, when \( G \) is compact. \( \hat{G} \), regarded as a topological group, is called the dual of \( G \). Also the dual of \( \hat{G} \) can be constructed, the following well-known theorem shows the importance of this "double dual" construction:

**Theorem** (Pontryagin-van Kampen duality theorem)

Let \( G \) be a locally compact abelian group, and let \( \hat{\hat{G}} \) be the dual of its dual. \( \hat{\hat{G}} \) and \( G \) are topologically isomorphic.

This theorem gives a method to recover an abelian compact group \( G \) from its representation ring. The fusion rules in this case simply encode the multiplication laws of \( \hat{G} \), and as mentioned earlier, \( \hat{G} \) has a trivial discrete topology. Now, according to the Pontryagin-van Kampen theorem, \( G \) can be recovered (up to topological isomorphism) by finding the one-dimensional unitary characters of \( \hat{G} \).

If \( G \) is a nonabelian compact group, then the group of continuous, unitary one-dimensional characters of \( G \) will be isomorphic to the dual of the abelianization of \( G \), i.e. to the dual of \( G/[G,G] \), where \( [G,G] \) denotes, as usual, the commutator subgroup of \( G \) [2, Theorem 23.8]. Hence, the abelianization of \( G \) can (in principle) be recovered from \( \mathcal{R}(G) \).

4 The center of a compact group and the chain group construction

The other abelian compact group, beside the abelianization, that can canonically be associated to every compact group \( G \) is its center. It is natural to ask whether, similarly to the abelianization, also the center can be reconstructed from \( \mathcal{R}(G) \). This question has only recently been answered by Baumgärtel and Lledó [5], and Müger [6].

In [5] the so called chain group construction was introduced. The construction is based on the following equivalence relation \( \sim \) on the set of irreducible characters: let \( \chi, \eta \in I_G \), we define \( \chi \sim \eta \) if there exists \( \psi_1, \psi_2, \ldots, \psi_n \in I_G \) (a "chain" of irreducible characters), such that \( \chi \prec \psi_1 \cdot \psi_2 \cdot \ldots \cdot \psi_n \), i.e., both \( \chi \) and \( \eta \) is contained in the irreducible decomposition of the above product.
The \( \sim \)-equivalence class of a character \( \chi \) will be denoted by \( \langle \chi \rangle \). The quotient \( C(G) := I_G / \sim \) is an abelian group, with respect to the operations \( \langle \chi \rangle \cdot \langle \psi \rangle = \langle \eta \rangle \), where \( \eta \) is any irreducible character with the property \( \eta \prec \chi \cdot \psi \). The group \( C(G) \) is called the \textit{chain group} of \( G \).

As an example, let us derive the chain group of \( SU(2) \). Let \( \chi_{1/2} \in I_{SU(2)} \) denote the unique two-dimensional irreducible character of \( SU(2) \). For any two odd-dimensional irreducible characters \( \psi, \eta \in I_{SU(2)} \), there is an appropriately large even number \( n \), such that \( \psi, \eta \prec \chi_{1/2}^n \). Similarly, for any two even-dimensional irreducible characters, there is an appropriately large odd number \( m \), such that both even-dimensional characters are contained in \( \chi_{1/2}^m \). However, no products of irreducible characters contains simultaneously an even- and an odd-dimensional element of \( I_{SU(2)} \). Thus, \( SU(2) \) has exactly two \( \sim \)-equivalence classes, and \( C(SU(2)) \cong \mathbb{Z}_2 \).

The meaning of the chain group is revealed by the following theorem, which was conjectured by Baumgärtel and Lledó, and was proved by Müger [6]:

\textbf{Theorem}

The chain group \( C(G) \) of a compact group \( G \) and \( \hat{Z}(G) \), the dual of the center of \( G \), are isomorphic.

This means that one can reconstruct the dual of the center of \( G \) from \( R(G) \), and thus in principle also \( Z(G) \).

This reconstruction theorem can also be used to prove several propositions about the center of various types of compact groups. An example of such a theorem is the following: if the non-zero fusion rules of a compact group are odd numbers (e.g. if the group is simply reducible), then the existence of a pseudo-real (symplectic) representation implies that the group has a non-trivial center. This theorem is a consequence of an interesting fusion property of pseudo-real representations [7]. The detailed proof of this and similar propositions together with a generalization of the chain group construction will be published in [8].

5 Closed normal subgroups and representation subrings

A subring \( R \) of \( R(G) \) will be called a \textit{representation subring}, if it is spanned by irreducible characters, i.e., if \( \exists J \subset I_G \) which is a \( \mathbb{Z} \)-basis of \( R \). There is a one-to-one Galois-correspondence between representation subrings \( R \subset R(G) \) and closed normal subgroups \( H \subset G \) given by

\[
H_R = \{ g \in G \mid g \in \text{Ker}(\chi), \ \forall \chi \in R^+ \},
\]

\[
R_H^+ = \{ \chi \in R(G)^+ \mid g \in \text{Ker}(\chi), \ \forall g \in H \},
\]

where \( R^+ \) denotes the positive elements of a representation subring \( R \), i.e., the characters contained in \( R \). This means that one can reconstruct the number (or cardinality) of closed normal subgroups of a compact group from its representation ring. Furthermore, \( R_H \cong R(G/H) \).
hence also the representation rings of the quotient groups can be obtained.

Moreover, for every $R_H$ the chain group construction can be generalized in a non-trivial way [8], and one can recover also the dual of $Z(G) \cap H$. This means that the number (or cardinality) of closed normal subgroups that contain a given subgroup of the center can be reconstructed, too.

It is noteworthy that the representation subring belonging to the center, i.e. $R_{Z(G)}$, can be canonically characterized for every compact $G$, it is the smallest representation subring that contains all irreducible characters that appear in a decomposition of $\chi \cdot \chi^*$ for some $\chi \in I_G$ [6, 9].

For the sake of completeness let us also mention that if $G$ is a connected compact group, and $R_H$ is a representation subring of $R(G)$, then one can recover from $R(G)$ whether $H$ is invariant with respect to all automorphisms of $G$. Namely, $H$ has this property if and only if all order automorphism of $R(G)$ leaves $R_H$ invariant (this is a direct consequence of Theorem 2.14. in [1]). For non-connected compact groups this does not hold in general.

6 Character tables of finite groups and the dimensions of irreps of compact groups

The way to obtain fusion rules from the character tables of finite groups is a subject which is covered by almost all standard textbooks on finite groups. It is, however, seldom mentioned that also the character table of a finite group can be recovered from the fusion rules. This can be done, because the only non-trivial sets of complex solutions $\{\alpha_p\}$ of the set of equations

$$\alpha_p \cdot \alpha_q = \sum_r N_{p,q}^r \alpha_r$$

are exactly the columns of the character table, i.e. the character values of certain conjugacy classes. This is not true for compact groups in general.

From the character table of a finite group one can directly obtain the abelianization, the center, and the number of normal subgroups, i.e. properties that can also be reconstructed from the fusion rules of a compact group in general. In addition, one can recover the dimensions of the irreps, the number of conjugacy classes that are left invariant by the inverse, the sizes of the conjugacy classes, and "the conjugacy class structure" of the commutator subgroup and other normal subgroups, and much other similar information [10].

In the case of generic compact groups the set of equations equations (2) has usually many other solutions than the character values of a given group element (or conjugacy class). One can not even obtain in general the dimensions of the irreps from (2), since there might be several sets of positive integers that satisfy this equation (this is the case even for $SU(2)$). Despite this, there is a more complicated way of obtaining the dimension of the irreps, which is presented in [1]. It is an open question, at least according to the authors knowledge, whether also other character values can be recovered from the representation ring for general compact groups. It is interesting that in the case of $SU(2)$ one can
reconstruct the fusion rules only from the dimensions of the irreps, we present this short and easy derivation in the Appendix.

7 Appendix - Deriving the Clebsch-Gordan series of SU(2) from the dimensions of the irreducible representations

There exist many methods for deriving the Clebsch-Gordan series of $SU(2)$ (for a recent and very nice derivation see [11]). We present a new type of derivation, which is based only on the compactness of $SU(2)$ and the property that it has one and only one equivalence class of irreps for every dimension. It is noteworthy that the only compact group with this latter property is $SU(2)$.

As usually, $D_j$ ($j \in \frac{1}{2}\mathbb{Z}$) will denote the $2j + 1$ dimensional (up to unitary equivalence) unique irreducible representation of $SU(2)$, and $\chi_j$ denotes its character. Due to the "uniqueness property" $D_j$ must be self-dual (equivalent to its dual), thus the decomposition of $D_j \otimes D_{j'}$ contains the trivial representation $D_0$ only if $j = j'$, and even then only once, i.e. $N^0_{j,j} = 1$.

First, the decompositions of tensor products of type $D_1/2 \otimes D_j$ will be derived. Consider the first non-trivial among these, $D_{1/2} \otimes D_{1/2}$. This is a 4-dimensional representation and, as already mentioned, it contains $D_0$ exactly once. This and the fact that the dimensions should match allow only one possible decomposition: $D_0 \oplus D_1$. The more general formula

$$D_{1/2} \otimes D_j \cong D_{j-1/2} \oplus D_{j+1/2},$$

(3)

can now be proved by induction. Suppose that [3] holds for any non-zero $j$ less than a specific $k \in \frac{1}{2}\mathbb{Z}$. We will prove it also for $k$. If some $D_j$ is contained in the decomposition of $D_{1/2} \otimes D_k$ with a non-zero multiplicity, then $D_0$ has to be contained in $D_j \otimes D_{1/2} \otimes D_k$ with the same multiplicity. But [3] was supposed to hold for $0 < j < k$, thus in this case we can rewrite this product as $(D_{j-1/2} \oplus D_{j+1/2}) \otimes D_k$. This representation contains $D_0$ only when $j = k - \frac{1}{2}$ (for $j < k$), i.e., the only representation with $j < k$ that is contained in $D_{1/2} \otimes D_k$ is $D_{k-1/2}$. Hence the only possible decomposition of $D_{1/2} \otimes D_k$, such that the dimensions match and the aboved-mentioned condition holds is $D_{k-1/2} \oplus D_{k+1/2}$, and [3] is proved by induction. This was the most important part of our derivation.

Now, as the decomposition of the tensor product of $D_{1/2}$ and any other irrep of $SU(2)$ is known, one can decompose any power of $D_{1/2}$:

$$D_{1/2} \otimes D_{1/2} = D_{n/2} \oplus \bigoplus_{i=1}^{[n/2-1]} \left[ \binom{n}{i} - \binom{n}{i-1} \right] D_{n/2-i},$$

where $[n/2 - 1]$ denotes the integer part of $n/2 - 1$. The proof is left to the reader. In terms of characters this reads as:

$$\chi_{1/2}^n = \chi_{n/2} + \sum_{i=1}^{[n/2-1]} \left[ \binom{n}{i} - \binom{n}{i-1} \right] \chi_{n/2-i}.$$
From equation (4) one can express any irreducible characters in terms of $\chi_{1/2}^0$ powers (note that $\chi_{1/2}^0 = \chi_0$):

$$\chi_j = \chi_{1/2}^{2j} + \sum_{i=1}^{[j]} (-1)^i \binom{2j - i}{i} \chi_{1/2}^{2j-2i}.$$ (5)

The product $\chi_j \cdot \chi_{j'}$ can now be evaluated using equations (4) and (5), the result is:

$$\chi_j \cdot \chi_{j'} = \chi_{|j-j'|} + \chi_{|j-j'|+1} + \cdots + \chi_{j+j'-1} + \chi_{j+j'}.$$ Converting this back to representations one recovers the usual Clebsch-Gordan series of SU(2):

$$D_j \otimes D_{j'} \cong D_{|j-j'|} \oplus D_{|j-j'|+1} \oplus \cdots \oplus D_{j+j'-1} \oplus D_{j+j'}.$$ 

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