Hypercyclicity of Composition Operators on Discrete Weighted Banach Spaces

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Abstract. In this paper, we study the hypercyclic composition operators on weighted Banach spaces of functions defined on discrete metric spaces. We show that the only such composition operators act on the “little” spaces. We characterize the bounded composition operators on the little spaces, as well as provide various necessary conditions for hypercyclicity.

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1. Introduction

Let $X$ be a Banach space of functions on a domain $\Omega$. A self-map $\varphi$ of $\Omega$ induces the composition operator defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in X$. We denote the set of analytic functions on the open unit disk $\mathbb{D}$ of the complex plane by $H(\mathbb{D})$. When $X$ is $H(\mathbb{D})$, the research of composition operators is extensive, beginning with the work of Nordgren [21]. For an excellent treatise on the subject, the reader is directed to [11].

One such function space of importance is the so-called weighted Banach space. For a bounded and continuous function $v : \mathbb{D} \to (0, \infty)$, define the weighted Banach space $H_v^\infty$ by

$$H_v^\infty := \left\{ f \in H(\mathbb{D}) : \|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty \right\}.$$
Under the norm $\| \cdot \|_v$, the set $H^\infty_v$ is a Banach space that arises naturally in the fields of complex analysis, Fourier analysis, and partial differential equations; see [6–8,15].

In recent years, the study of operators on so-called discrete function spaces has increased. Most recently, much work has been conducted on spaces where $\Omega$ is an infinite, rooted tree. In this setting, discrete analogs to some classical function spaces on $\mathbb{D}$ have been developed and operators studied, including the Bloch space [1,9], the Hardy space [19,20], and the weighted Banach spaces [2–4].

The study of linear dynamics is an increasingly active area of research. For a treatise on the subject, see for example [13]. Connected to the study of such dynamics is the question of when a composition operator $C_\phi$ is hypercyclic. Hypercyclic composition operators $C_\phi$ on the weighted Banach space $H^0_v$ were investigated in [18], while hypercyclic scalar multiples of composition operators on $H^0_v$ were studied in [14]. In the environment of directed trees, the hypercyclicity of shifts on weighted $L^p$ spaces has been studied in [16].

For a recent exposition on hypercyclic composition operators on Banach spaces of analytic functions, see [10]. The purpose of this paper is to bring the study of hypercyclic composition operators to the setting of discrete function spaces, specifically weighted Banach spaces. In addition to this, we generalize the domains of the function spaces from an infinite, rooted tree to an unbounded, locally finite metric space with a distinguished point.

1.1. Organization of the Paper

In Sect. 2, we define the weighted Banach space $L_\mu(T)$ and little weighted Banach space $L^0_\mu(T)$ on an unbounded, locally finite metric space $T$ with a distinguished element, with respect to a weight (i.e., a positive function) $\mu$ on $T$. We also collect useful facts about these spaces.

In Sects. 3 and 4, we characterize the bounded composition operators acting on the weighted and little weighted Banach spaces, respectively. Finally in Sect. 5, we show that no composition operator is hypercyclic on $L_\mu(T)$, and provide many necessary conditions for $C_\phi$ to be hypercyclic on $L^0_\mu(T)$. We conclude the section with an example of a hypercyclic composition operator on $L^0_\mu(T)$.

2. Weighted Spaces

We start by defining a locally finite metric space and the two function spaces that we will study throughout this paper.

**Definition 2.1.** Let $(T,d)$ be a metric space with a distinguished element $o$. Define $|v| := d(o,v)$. We say that $T$ is *locally finite* if for each $M > 0$, the set $\{v \in T : |v| \leq M\}$ is finite.

**Definition 2.2.** Let $(T,d)$ be an unbounded, locally finite metric space with a distinguished element $o$ and let $\mu$ be a positive function on $T$. We define the
weighted Banach space on $T$ as the set $L_\mu(T)$ of functions $f : T \to \mathbb{C}$ such that

$$\|f\|_\mu := \sup_{v \in T} \mu(v)|f(v)| < \infty.$$  

The little weighted Banach space is the subset $L_0^\mu(T)$ of $L_\mu(T)$ whose elements $f$ satisfy the condition

$$\lim_{|v| \to \infty} \mu(v)|f(v)| = 0.$$  

First, we observe that the point-evaluation functionals on $L_\mu(T)$ are bounded.

**Lemma 2.3.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$ and let $\mu$ be a positive function on $T$. For $v \in T$ and $f \in L_\mu(T)$, we have

$$|f(v)| \leq \frac{1}{\mu(v)} \|f\|_\mu.$$  

Moreover, the point-evaluation functional $e_v$ is bounded on $L_\mu(T)$.

**Proof.** For $v \in T$, observe

$$|f(v)| \leq \frac{1}{\mu(v)} \sup_{u \in T} \mu(u)|f(u)| = \frac{1}{\mu(v)} \|f\|_\mu.$$  

Thus, for $f \in L_\mu(T)$ with $\|f\|_\mu \leq 1$, it follows that

$$\|e_v\| = \sup_{\|f\|_\mu = 1} \|e_v(f)\| = \sup_{\|f\|_\mu = 1} |f(v)| \leq \frac{1}{\mu(v)},$$

and thus the evaluation functionals are bounded on $L_\mu(T)$.  

As expected, the weighted Banach space and little weighted Banach spaces are, in fact, Banach spaces under the norm $\|\cdot\|_\mu$. The following proof is included for completeness. A version of this result when $T$ is an infinite tree and $d$ is the edge-counting metric can be found in [2].

**Theorem 2.4.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$ and let $\mu$ be a positive function on $T$. Then, under the norm $\|\cdot\|_\mu$, $L_\mu(T)$ is a Banach space.

**Proof.** The set $L_\mu(T)$ is clearly a normed linear space. To show completeness, let $\{f_n\}$ be a Cauchy sequence in $L_\mu(T)$. By Lemma 2.3, for each $v \in T$, the sequence $\{f_n(v)\}$ is also Cauchy, and hence converges to a complex number $f(v)$. Now, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for $n, m \geq N$ we have

$$\mu(v)|f_n(v) - f_m(v)| \leq \|f_n - f_m\|_\mu < \varepsilon/2,$$

for all $v \in T$. Letting $m$ go to infinity, we obtain

$$\mu(v)|f_n(v) - f(v)| \leq \varepsilon/2,$$

for each $v \in T$. Taking the supremum over $v \in T$ we obtain $\|f_n - f\|_\mu < \varepsilon$ for all $n \geq N$. In particular, $f_N - f$ is in $L_\mu(T)$ and hence $f \in L_\mu(T)$.  

Recall that a functional Banach space is a Banach space of complex-valued functions on a set such that all point evaluations are bounded linear functionals, no point-evaluation functional is identically 0 and the point-evaluation functionals separate points.

Theorem 2.4 shows that $L_\mu(T)$ is a Banach space, Lemma 2.3 shows that all point evaluations are bounded; clearly, no point-evaluation is identically zero and point evaluations separate points. Hence, $L_\mu(T)$ is a functional Banach space.

Next, we show $L_\mu^0(T)$ to be a closed and separable subspace of $L_\mu(T)$.

**Proposition 2.5.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$ and let $\mu$ be a positive function on $T$. The set $L_\mu^0(T)$ is a closed subspace of $L_\mu(T)$.

**Proof.** Let $\{f_n\}$ be a sequence in $L_\mu^0(T)$ and assume $f_n \to f$ for some $f \in L_\mu(T)$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|f_N - f\|_\mu < \varepsilon/2$. Let $M \in \mathbb{N}$ be chosen so that

$$\mu(v)|f_N(v)| < \varepsilon/2$$

for all $v \in T$ with $|v| \geq M$. Then, for all such $v$ we have

$$\mu(v)|f(v)| \leq \mu(v)|f(v) - f_N(v)| + \mu(v)|f_N(v)|$$

$$\leq \|f_N - f\|_\mu + \mu(v)|f_N(v)| < \varepsilon,$$

which shows $f \in L_\mu^0(T)$. \qed

The space $L_\mu^0(T)$ is separable, as the next theorem shows.

**Theorem 2.6.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$ and let $\mu$ be a positive function on $T$. The set

$$\{f : T \to \mathbb{C} : f \text{ has finite support}\}$$

is dense in $L_\mu^0(T)$ and hence $L_\mu^0(T)$ is separable.

**Proof.** Let $f \in L_\mu^0(T)$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\mu(v)|f(v)| < \frac{\varepsilon}{2},$$

for all $|v| \geq N$.

For $n \in \mathbb{N}$, let $U_n := \{v \in T : |v| < n\}$, which is a finite set since $T$ is locally finite. Let $\chi_{U_n}$ be the characteristic function of $U_n$ and define $g_n := f\chi_{U_n}$. Clearly, $g_n$ has finite support. Then, for each $n \in \mathbb{N}$ we have

$$\|f - g_n\|_\mu = \sup_{v \in T} \mu(v)|(f - g_n)(v)| = \sup_{v \in T} \mu(v)|f(v)| |1 - \chi_{U_n}(v)| = \sup_{|v| \geq n} \mu(v)|f(v)|,$$

since $1 - \chi_{U_n}(v) = 0$ whenever $|v| < n$.

Now, let $n \geq N$. Since if $|v| \geq N$ then $\mu(v)|f(v)| < \frac{\varepsilon}{2}$, we have

$$\|f - g_n\|_\mu = \sup_{|v| \geq n} \mu(v)|f(v)| < \varepsilon,$$

which proves that $\lim_{n \to \infty} g_n = f$, as desired.
Clearly the set \( \{ f : T \to C : f \) has finite support\} can be approximated by the set \( \{ f : T \to \mathbb{Q} + i\mathbb{Q} : f \) has finite support\}, which is countable since \( T \) is countable. Hence \( L^0_\mu(T) \) is separable.

On the other hand, \( L^\mu_\mu(T) \) is never separable: indeed, define
\[
\mathcal{F} := \{ f \in L^\mu_\mu(T) : f(v) \in \{0, 1/\mu(v)\} \text{ for each } v \in T\}.
\]
Clearly, if \( f, g \in \mathcal{F} \) and \( f \neq g \), then \( \|f - g\| = 1 \). Since \( \mathcal{F} \) is uncountable, it follows (for example, see [17, Proposition 1.12.1]) that \( L^\mu_\mu(T) \) is not separable.

3. Boundedness of Composition Operators on the Big Space

In this section, we study when composition operators are bounded on \( L^\mu_\mu(T) \) and we obtain a complete characterization of such operators.

**Theorem 3.1.** Let \( (T, d) \) be an unbounded, locally finite metric space with a distinguished element \( o \), let \( \mu \) be a positive function on \( T \) and let \( \varphi \) be a self-map of \( T \). The operator \( C_\varphi \) is bounded on \( L^\mu_\mu(T) \) if and only if
\[
\sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))} < \infty.
\]
Furthermore, \( \|C_\varphi\| = \sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))} \).

**Proof.** Let \( C := \sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))} \).

First, assume that \( C < \infty \). Let \( f \in L^\mu_\mu(T) \). Then
\[
\|C_\varphi f\|_\mu = \sup_{v \in T} \mu(v)|f(\varphi(v))| = \sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))}\mu(\varphi(v))|f(\varphi(v))| \leq C \sup_{v \in T} \mu(\varphi(v))|f(\varphi(v))| \leq C \sup_{w \in T} \mu(w)|f(w)| = C\|f\|_\mu,
\]
which shows that \( C_\varphi \) is bounded and \( \|C_\varphi\| \leq C \).

Now, for \( w \in T \) define \( g_w(v) := \frac{1}{\mu(\varphi(w))} \chi_{\varphi(w)}(v) \). Clearly \( \|g_w\|_\mu = 1 \). Then
\[
\|C_\varphi\| = \sup_{\|f\|_\mu = 1} \|C_\varphi f\|_\mu \geq \|C_\varphi g_w\|_\mu = \sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))} \chi_{\varphi(w)}(\varphi(v)) \geq \frac{\mu(w)}{\mu(\varphi(w))}.
\]
Hence, if \( C_\varphi \) is bounded, then
\[
\sup_{w \in T} \frac{\mu(w)}{\mu(\varphi(w))} < \infty.
\]
and
\[ \sup_{w \in T} \frac{\mu(w)}{\mu(\varphi(w))} \leq \|C_\varphi\|, \]
which finishes the proof. \(\square\)

In the remainder of the paper, we shall restrict our attention to composition operators that are bounded on the space \(L_\mu(T)\). For easier reference, we adopt the following definition.

**Definition 3.2.** Let \((T, d)\) be an unbounded, locally finite metric space with a distinguished element \(o\), \(\mu\) a positive function on \(T\), and \(\varphi\) a self-map of \(T\). We say \(\varphi\) is an *admissible symbol* for a composition operator (or simply *admissible*) if the composition operator \(C_\varphi\) is bounded on \(L_\mu(T)\).

Note by Theorem 3.1, \(\varphi\) is admissible if and only if there exists a positive constant \(C\) such that
\[ \mu(w) \leq C\mu(\varphi(w)) \]
for all \(w \in T\).

The following proposition shows that, with an additional hypothesis on \(\varphi\), admissibility is characterized by boundedness of the weight.

**Proposition 3.3.** Let \((T, d)\) be an unbounded, locally finite metric space with a distinguished element \(o\), let \(\mu\) be a positive function on \(T\) and let \(\varphi\) be a self-map of \(T\). If \(\varphi\) has finite range, then \(\varphi\) is admissible if and only if \(\mu\) is bounded.

**Proof.** Assume \(\varphi\) is admissible. Then, there exists \(C > 0\) such that \(\mu(v) \leq C\mu(\varphi(v))\) for all \(v \in T\). Since \(\varphi\) has finite range, the set \(\{\mu(\varphi(v)) : v \in T\}\) is finite and hence bounded. Hence \(\mu\) is bounded as well.

Conversely, suppose \(\mu\) is bounded with upper bound \(M > 0\). Since \(\varphi\) has finite range, \(\delta := \min_{v \in T} \mu(\varphi(v)) > 0\). Thus, by Theorem 3.1,
\[ \|C_\varphi\| = \sup_{v \in T} \frac{\mu(v)}{\mu(\varphi(v))} \leq \frac{M}{\delta}, \]
proving that \(\varphi\) is admissible. \(\square\)

### 4. Boundedness of Composition Operators on the Little Space

The first result in this section, which highlights the connection to Proposition 3.3 for the big space case, gives a necessary and sufficient condition for \(C_\varphi\) to be bounded on \(L^0_\mu(T)\) whenever \(\varphi\) has finite range.

**Proposition 4.1.** Let \((T, d)\) be an unbounded, locally finite metric space with a distinguished element \(o\), let \(\mu\) be a positive function on \(T\) and let \(\varphi\) be a self-map of \(T\). If \(\varphi\) has finite range, then \(C_\varphi\) is bounded on \(L^0_\mu(T)\) if and only if \(\lim_{|v| \to \infty} \mu(v) = 0\).
Proof. Let \( S = \varphi(T) \) be a finite set. Suppose first that \( C_\varphi \) is bounded on \( L^0_\mu(T) \), and for purposes of contradiction assume \( \lim_{|v| \to \infty} \mu(v) \) does not equal zero. Then there exist \( c > 0 \) and a sequence \( \{v_n\} \) in \( T \) such that \( |v_n| \to \infty \) and \( \mu(v_n) \geq c \). Since \( \chi_S \) is in \( L^0_\mu(T) \), then \( C_\varphi(\chi_S) = \chi_S \circ \varphi \) is in \( L^0_\mu(T) \). But

\[
\mu(v_n)|\chi_S(\varphi(v_n))| = \mu(v_n) \neq 0,
\]

which is a contradiction. Hence \( \lim_{|v| \to \infty} \mu(v) = 0 \).

Conversely, suppose \( \mu(v) \to 0 \) as \( |v| \to \infty \). To show that \( C_\varphi \) is bounded on \( L^0_\mu(T) \) it suffices to show that \( f \circ \varphi \) is in \( L^0_\mu(T) \) for each \( f \in L^0_\mu(T) \) by the Closed Graph Theorem (see [11, Exercise 1.1.1]). Fixing such an \( f \), let \( M := \max_{v \in T} |f(\varphi(v))| \). Then

\[
0 \leq \lim_{|v| \to \infty} \mu(v)|f(\varphi(v))| \leq \lim_{|v| \to \infty} M\mu(v) = 0,
\]

and thus \( f \circ \varphi \in L^0_\mu(T) \), as desired. \( \square \)

Observe that, unlike the case for \( L_\mu(T) \) (see Theorem 3.1) admissibility itself cannot be enough to guarantee boundedness of \( C_\varphi \) on \( L^0_\mu(T) \). For example, Proposition 4.1 guarantees that if \( \varphi \) has finite range and \( \mu \) is does not go to zero, then \( C_\varphi \) does not map \( L^0_\mu(T) \) into \( L^0_\mu(T) \) even in the event that \( \varphi \) is admissible. Nevertheless, admissibility of \( \varphi \) is a necessary condition for the boundedness of the composition operator \( C_\varphi \) on \( L^0_\mu(T) \).

**Proposition 4.2.** Let \( (T,d) \) be an unbounded, locally finite metric space with a distinguished element \( o \), let \( \mu \) be a positive function on \( T \) and let \( \varphi \) be a self-map of \( T \). If \( C_\varphi \) is bounded on \( L^0_\mu(T) \), then \( \varphi \) is admissible.

This follows from the proof of Theorem 3.1 since the test function \( g_w \) used is an element of \( L^0_\mu(T) \).

By the previous result, to show that \( C_\varphi \) is bounded on \( L^0_\mu(T) \) it is permissible to assume \( C_\varphi \) is bounded on \( L_\mu(T) \). This additional hypothesis allows us to replace the assumption of finite range with admissibility.

**Proposition 4.3.** Let \( (T,d) \) be an unbounded, locally finite metric space with a distinguished element \( o \), \( \mu \) be a positive function on \( T \) and \( \varphi \) admissible. If \( \lim_{|v| \to \infty} \mu(v) = 0 \), then \( C_\varphi \) is bounded on \( L^0_\mu(T) \).

**Proof.** Let \( C > 0 \) be a constant such that \( \mu(w) \leq C\mu(\varphi(w)) \) for all \( w \in T \). Fix \( \varepsilon > 0 \) and suppose \( f \in L^0_\mu(T) \). We can then choose \( N_1 \in \mathbb{N} \) such that

\[
\mu(v)|f(v)| < \frac{\varepsilon}{C}, \quad \text{for all } |v| \geq N_1.
\]

(1)

Let \( S := \{v \in T : |v| \leq N_1\} \), which is a finite set since \( T \) is locally finite, and let \( m := \max_{v \in S} |f(v)| \). Also, since \( \mu(v) \to 0 \) as \( |v| \to \infty \), we can choose \( N_2 \in \mathbb{N} \) such that

\[
\mu(v) < \frac{\varepsilon}{m+1}, \quad \text{for all } |v| \geq N_2.
\]

(2)

Now, if \( |\varphi(v)| \geq N_1 \), then, by (1), we have

\[
\mu(v)|f(\varphi(v))| \leq C\mu(\varphi(v))|f(\varphi(v))| < C\frac{\varepsilon}{C} = \varepsilon.
\]
Also, if $|v| \geq N_2$ and $|\varphi(v)| \leq N_1$, then, by (2) and the definition of $m$, we have
\[ \mu(v)|f(\varphi(v))| \leq \mu(v)m \leq \frac{\varepsilon}{m+1}m < \varepsilon. \]
Therefore, if $|v| \geq N_2$, we have $\mu(v)|f(\varphi(v))| < \varepsilon$, which shows that $f \circ \varphi \in L^0_\mu(T)$. Since $C\varphi$ is bounded on $L_\mu(T)$, this implies that $C\varphi$ is bounded on $L^0_\mu(T)$. \hfill \Box

The previous result gave a condition on $\mu$ that yields the boundedness of the composition operator. The following proposition gives instead a condition on the symbol $\varphi$.

**Proposition 4.4.** Let $(T,d)$ be an unbounded, locally finite metric space with a distinguished element $o$, $\mu$ be a positive function on $T$ and $\varphi$ admissible. If $\varphi^{-1}(v)$ is a finite set for every $v \in T$, then $C\varphi$ is bounded on $L^0_\mu(T)$.

*Proof.* We claim first that $\lim_{|v| \to \infty} |\varphi(v)| = \infty$. If not, there would exist $M_0 > 0$ and a sequence $\{v_n\}$ in $T$ such that $|v_n| \geq n$ and $|\varphi(v_n)| < M_0$ for each $n \in \mathbb{N}$. But this implies that there exist $w \in T$ with $|w| < M_0$ and an infinite subset $V$ of $T$ such that $\varphi(v) = w$ for each $v \in V$, which contradicts the hypothesis.

Let $f \in L^0_\mu(T)$. Since $\varphi$ is admissible, there exists $C > 0$ such that $\mu(v) \leq C\mu(\varphi(z))$ for all $v \in T$. Then,
\[ \lim_{|v| \to \infty} \mu(v)|(f \circ \varphi)(v)| \leq \lim_{|v| \to \infty} C\mu(\varphi(v))|f(\varphi(v))| = C \lim_{|\varphi(v)| \to \infty} \mu(\varphi(v))|f(\varphi(v))| = 0. \]
Thus, $C\varphi$ maps $L^0_\mu(T)$ into itself. The boundedness of the operator on $L^0_\mu(T)$ follows at once from the admissibility of $\varphi$. \hfill \Box

Observe that the previous proposition includes the case where the function $\varphi$ is injective. For example, $T$ might be an infinite and locally finite graph with the edge-counting metric $d$, $\mu$ any positive function on $T$ and $\varphi$ any admissible automorphism of the graph.

The following proposition allows $\varphi^{-1}(v)$ to be infinite for finitely many $v \in T$, but $\mu$ must tend to zero on the inverse image of such vertices.

**Proposition 4.5.** Let $(T,d)$ be an unbounded, locally finite metric space with a distinguished element $o$, $\mu$ be a positive function on $T$ and $\varphi$ an admissible function. Consider the set $\mathcal{S} := \{w \in T : \varphi^{-1}(w) \text{ is finite}\}$ and assume $\mathcal{S}$ is nonempty. If $\mathcal{S}$ is finite and
\[ \lim_{|v| \to \infty} \mu(v) = 0, \]
then $C\varphi$ is bounded on $L^0_\mu(T)$.

*Proof.* Let $f \in L^0_\mu(T)$. Since $\varphi$ is admissible, we only need to show that $f \circ \varphi$ is in $L^0_\mu(T)$. Let $m := \max\{|f(w)| : w \in \mathcal{S}\}$ and let $C > 0$ such that $\mu(v) \leq C\mu(\varphi(v))$ for all $v \in T$. 
Let $\varepsilon > 0$. Since

$$\lim_{|v| \to \infty \atop v \in \varphi^{-1}(S)} \mu(v) = 0,$$

there exists $N_1 \in \mathbb{N}$ such that

$$\mu(v) < \frac{\varepsilon}{m + 1}, \quad \text{for all } v \in \varphi^{-1}(S) \text{ with } |v| \geq N_1.$$  

We consider the two cases when the complement of $\varphi^{-1}(S)$ is finite or infinite.

Assume first $T \setminus \varphi^{-1}(S)$ is finite. Let $N_2 \in \mathbb{N}$ such that if $|v| \geq N_2$ then $v \notin T \setminus \varphi^{-1}(S)$.

Define $N = \max\{N_1, N_2\}$. Then, if $|v| \geq N$, it follows that $v \in \varphi^{-1}(S)$.

Hence,

$$\mu(v)|f(\varphi(v))| \leq \mu(v)m < \varepsilon.$$  

Hence $f \circ \varphi \in L^0_\mu(T)$.

Next, assume $T \setminus \varphi^{-1}(S)$ is infinite. Since $f \in L^0_\mu(T)$, there exists $N_3 \in \mathbb{N}$ such that

$$\mu(v)|f(v)| < \frac{\varepsilon}{C}, \quad \text{for all } v \in T \text{ with } |v| \geq N_3.$$  

Claim: There exists $N_4 \in \mathbb{N}$ such that for all $|v| \geq N_4$ with $v \notin \varphi^{-1}(S)$ we have $|\varphi(v)| \geq N_3$.

If not, there exists a sequence $\{v_n\}$ of distinct points such that $|v_n| \geq n$, such that $v_n \notin \varphi^{-1}(S)$ and $|\varphi(v_n)| < N_3$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary, it follows that there exists $w \in T$ with $|w| < N_3$ and $\varphi(v_n) = w$ for all $n \in \mathbb{N}$. But this implies that $w \in S$ and hence that $v_n \in \varphi^{-1}(S)$ for all $n \in \mathbb{N}$, contradicting the choice of the sequence and proving the claim.

Define $N = \max\{N_1, N_4\}$. Let $v \in T$ with $|v| \geq N$. If $v \in \varphi^{-1}(S)$, then

$$\mu(v)|f(\varphi(v))| \leq \mu(v)m < \varepsilon.$$  

If $v \notin \varphi^{-1}(S)$, then

$$\mu(v)|f(\varphi(v))| \leq C\mu(\varphi(v))|f(\varphi(v))| < \varepsilon,$$

since $|\varphi(v)| \geq N_3$. In either case,

$$\mu(v)|f(\varphi(v))| < \varepsilon,$$

and hence $f \circ \varphi \in L^0_\mu(T)$. □

Proposition 4.6 below generalizes the conditions of Propositions 4.3, 4.4 and 4.5. Indeed, in its statement we assume a condition that uses an arbitrary positive increasing function $g$ on the nonnegative integers. If $g$ is bounded, the result reduces to Proposition 4.3. Moreover, in the proof of Proposition 4.4 we showed that $|\varphi(v)| \to \infty$ as $|v| \to \infty$. Thus, letting $g$ be the identity, the result follows from Proposition 4.6. The same argument applies to Proposition 4.5, where the assumption combines elements of the previous two propositions.
Proposition 4.6. Let \((T, d)\) be an unbounded, locally finite metric space with distinguished element \(o, g : N_0 \to \mathbb{R}_+\) an increasing function, \(\mu\) a positive function on \(T\) and \(\varphi\) an admissible function. If
\[
\lim_{|v| \to \infty} \min\{(g(|\varphi(v)|))^{-1}, \mu(v)\} = 0, \tag{3}
\]
then \(C_{\varphi}\) is bounded on \(L^0_{\mu}(T)\).

Proof. Let \(C > 0\) such that \(\mu(v) \leq C \mu(\varphi(v))\) for all \(v \in T\). Let \(f \in L^0_{\mu}(T)\) and \(\varepsilon > 0\). There exists \(N_1 \in \mathbb{N}\) such that
\[
\mu(v)|f(v)| < \frac{\varepsilon}{C}, \quad \text{for all } v \text{ with } |v| \geq N_1. \tag{4}
\]
Let \(m := \max\{|f(v)| : |v| \leq N_1\}\). By hypothesis, there exists \(N_2 \in \mathbb{N}\) such that
\[
\min\{(g(|\varphi(v)|))^{-1}, \mu(v)\} < \min\left\{\frac{\varepsilon}{m + 1}, \frac{1}{g(N_1)}\right\}, \quad \text{for all } v \text{ with } |v| \geq N_2.
\]

Now, let \(v\) be such that \(|v| \geq \max\{N_1, N_2\}\).

- Assume \(\min\{(g(|\varphi(v)|))^{-1}, \mu(v)\} = \mu(v)\). If \(|\varphi(v)| \geq N_1\), then, by the admissibility of \(\varphi\) and equation (4), we have
\[
\mu(v)|f(\varphi(v))| \leq C \mu(\varphi(v))|f(\varphi(v))| < \varepsilon.
\]
If \(|\varphi(v)| \leq N_1\), then
\[
\mu(v)|f(\varphi(v))| \leq \mu(v)m < \varepsilon,
\]
by the definition of \(m\) and since \(\min\{(g(|\varphi(v)|))^{-1}, \mu(v)\} = \mu(v) < \frac{\varepsilon}{m + 1}\).

- Assume \(\min\{(g(|\varphi(v)|))^{-1}, \mu(v)\} = (g(|\varphi(v)|))^{-1}\). Then, \(\frac{1}{g(\varphi(v))} < \frac{1}{g(N_1)}\) and hence \(|\varphi(v)| > N_1\). Therefore,
\[
\mu(v)|f(\varphi(v))| \leq C \mu(\varphi(v))|f(\varphi(v))| < \varepsilon,
\]
as before.

In either case, \(\mu(v)|f(\varphi(v))| < \varepsilon\) and hence \(f \circ \varphi \in L^0_{\mu}(T)\).

In the following example, we cannot apply Propositions 4.3, 4.4, or 4.5, but Proposition 4.6 yields the boundedness of the operator.

Example 4.7. Consider the set \(T = N_0 \times N_0\) with the metric \(d\) given by
\[
d((m, n), (m', n')) = |m - m'| + |n - n'|\]
and take as a distinguished element the pair \((0, 0)\). Define \(\mu\) on \(T\) by \(\mu(m, n) = 3^m 2^{-n}\). The function \(\varphi\) given by \(\varphi(m, n) = (m, 0)\) is easily seen to be admissible. Since
\[
\lim_{|m| + |n| \to \infty} \min\{|m| + 1^{-1}, 3^m 2^{-n}\} = 0,
\]
condition (3) holds for \(g(x) = x + 1, x \in N_0\). Hence, \(C_{\varphi}\) is a bounded operator on \(L^0_{\mu}(T)\).
Note that Proposition 4.6 continues to hold if we assume $g$ to be eventually increasing unboundedly. We now show that a converse of Proposition 4.6 holds under a restriction on the growth of the weight $\mu$. In particular, we obtain a full characterization of boundedness for $C_\varphi$ acting on $L^0_\mu(T)$ if the weight $\mu$ is bounded.

**Theorem 4.8.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$, $g : \mathbb{N}_0 \to \mathbb{R}_+$ an arbitrary increasing function, $\mu$ be a positive function on $T$ such that $\mu(v) = o(g(|v|))$ as $|v| \to \infty$, and $\varphi$ a self-map of $T$. Then $C_\varphi$ is bounded on $L^0_\mu(T)$ if and only if $\varphi$ is admissible and condition (3) holds.

**Proof.** If $g$ is bounded, the result is trivial, since the assumption $\mu = o(g)$ yields $\mu(v) \to 0$ as $|v| \to \infty$, and the conclusion follows at once applying Propositions 4.2 and 4.3.

Thus, let us assume $g$ is unbounded. By Propositions 4.2 and 4.6, it suffices to show that if condition (3) fails, then $C_\varphi$ is not bounded on $L^0_\mu(T)$. Assume (3) fails. Then there exist a sequence of vertices $\{v_n\}$ with $|v_n| \to \infty$ as $n \to \infty$, a positive constant $\delta$ and a positive integer $N$ such that

$$\min\{\left(\frac{1}{g(|\varphi(v_n)|)}\right)^{-1}, \mu(v_n)\} \geq \delta$$

whenever $|v_n| \geq N$. Define

$$f(v) = \begin{cases} \frac{1}{g(|v|)}, & \text{if } v \neq o, \\ 0, & \text{if } v = o. \end{cases}$$

On the one hand, since $\mu(v) = o(g(|v|))$ as $|v| \to \infty$, it follows that $f \in L^0_\mu(T)$.

On the other hand, for all $n \in \mathbb{N}$ such that $|v_n| \geq N$,

$$\mu(v_n)|f(\varphi(v_n))| = \mu(v_n)(g(|\varphi(v_n)|))^{-1} \geq \delta^2.$$ 

Therefore, $C_\varphi f \notin L^0_\mu(T)$. 

□

Are there conditions on the symbol $\varphi$ and on the weight $\mu$ that are both sufficient and necessary for boundedness of $C_\varphi$ on $L^0_\mu(T)$? We have not been able to find an answer to this problem, so we leave it open for future research.

### 5. Hypercyclicity

Let $X$ be a Banach space. Recall that an operator $S : X \to X$ is called hypercyclic if there exists $x \in X$ such that $\{x, Sx, S^2x, S^3x, \ldots\}$ is dense in $X$. Such a vector $x$ is also called hypercyclic. Clearly, the existence of a hypercyclic operator on a Banach space $X$ implies that $X$ is separable. Observe that the set of hypercyclic vectors is a dense set in $X$, since every vector in the orbit $\{x, Sx, S^2x, S^3x, \ldots\}$ is itself hypercyclic.

There has been much research on hypercyclicity and the reader is referred to the books [5, 13] for the necessary background on the study of hypercyclicity. In what follows, we concentrate on the problem of determining hypercyclicity of composition operators on $L^0_\mu(T)$. This study somewhat follows the classical studies of hypercyclicity of composition operators on spaces
of analytic functions (see [10] for a review of hypercyclicity of composition operators on several Banach spaces of analytic functions on the disk).

As it is usually the case for composition operators, if \( \varphi : T \to T \) has a fixed point, then \( C_\varphi \) cannot be hypercyclic on \( L_0^0(T) \), as we now show. A more general result can be obtained: a self-map of \( T \) with periodic points cannot induce a hypercyclic composition operator on a functional Banach space.

**Theorem 5.1.** Let \( X \) be a functional Banach space of complex-valued functions defined on a set \( T \). Let \( \varphi \) be a self-map of \( T \) and assume that \( C_\varphi \) is bounded on \( X \). If \( C_\varphi \) is hypercyclic on \( X \), then \( \varphi \) does not have periodic points.

*Proof.* Suppose that there exists \( w \in T \) and \( p \in \mathbb{N} \) such that \( \varphi^p(w) = w \). Let \( f \in X \) be a hypercyclic vector. Then the set

\[
\{ f, f \circ \varphi, f \circ \varphi^2, f \circ \varphi^3, \ldots \}
\]

is dense in \( X \). Since the evaluation functional \( e_w \) is continuous and surjective, it follows that

\[
\{ f(w), f(\varphi(w)), f(\varphi^2(w)), f(\varphi^3(w)), \ldots \}
\]

is dense in \( \mathbb{C} \). Since \( \varphi^p(w) = w \) this set equals

\[
\{ f(w), f(\varphi(w)), f(\varphi^2(w)), \ldots, f(\varphi^{p-1}(w)) \}.
\]

But this set cannot be dense in \( \mathbb{C} \). \( \square \)

It follows that if the self-map \( \varphi \) of \( T \) has a fixed point (or a periodic point) and \( C_\varphi \) is bounded, then \( C_\varphi \) is not hypercyclic on the functional Banach space \( L_0^0(T) \).

The following definition was introduced by Bernal-González and Montes-Rodríguez in a different, but similar, context.

**Definition 5.2.** Let \( (X, d) \) be a metric space and let \( \varphi \) be a self-map of \( X \). We say that \( \varphi \) is a run-away function if for every finite set \( I \), there exists \( N \in \mathbb{N} \) such that \( \varphi^n(I) \cap I = \emptyset \) for every \( n \geq N \).

**Proposition 5.3.** Let \( (X, d) \) be a metric space and let \( \varphi \) be a self-map of \( X \). If \( \varphi \) does not have periodic points, then \( \varphi \) is a run-away function.

*Proof.* Assume that \( \varphi \) is not a run-away function. Therefore, there exists a finite set \( I \) and an increasing sequence \( \{n_k\} \) of natural numbers such that

\[
\varphi^{n_k}(I) \cap I \neq \emptyset
\]

for all \( k \). Since \( I \) is finite, there exists \( w \in I \) such that

\[
w \in \varphi^{n_k}(I)
\]

for infinitely many natural numbers \( k \). Now, since \( I \) is finite, it also follows that there exists \( v \in I \) such that

\[
w = \varphi^{n_k}(v)
\]
for infinitely many natural numbers $k$. Assume then that $w = \varphi^s(v)$ and $w = \varphi^t(v)$, for integers $s$ and $t$ with $s > t$. But then

$$\varphi^{s-t}(w) = \varphi^{s-t}(\varphi^t(v)) = \varphi^s(v) = w,$$

which implies that $w$ is a periodic point, finishing the proof. \hfill $\Box$

Let $X$ be a functional Banach space over a set $T$ and assume $C_\varphi$ is a composition operator on $X$. Then the adjoint of the operator $C_\varphi$ maps point-evaluation functionals to point-evaluation functionals. Specifically, for each $h \in X$ and each $v \in T$, we have

$$(C_\varphi^* e_v)(h) = e_v(C_\varphi h) = e_v(h \circ \varphi) = h(\varphi(v)) = e_{\varphi(v)}(h),$$

and hence $C_\varphi^* e_v = e_{\varphi(v)}$ for each $v \in T$.

**Theorem 5.4.** Let $X$ be a functional Banach space of complex-valued functions defined on a set $T$. Let $\varphi$ be a self-map of $T$ and assume that $C_\varphi$ is bounded on $X$. If $\varphi$ is not injective, then $C_\varphi$ is not hypercyclic on $X$.

**Proof.** Assume $\varphi(v) = \varphi(w)$ for two distinct vertices $v$ and $w$. Let $e_v$ and $e_w$ be the evaluation functionals for $v$ and $w$, respectively. Then $C_\varphi^*(e_v - e_w) = 0$ and hence $C_\varphi^*$ has nontrivial kernel. This implies that the range of $C_\varphi$ cannot be dense and hence $C_\varphi$ cannot be hypercyclic. \hfill $\Box$

We should point out that if $\mu(v) \leq \mu(\varphi(v))$ for all $v \in T$ then $\|C_\varphi\| \leq 1$, which implies that $C_\varphi$ cannot be hypercyclic.

The following theorem gives another condition which prevents hypercyclicity.

**Theorem 5.5.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$, let $\mu$ be a positive function on $T$ and let $\varphi$ be a self-map of $T$. Assume that $C_\varphi$ is bounded on $L^0_\mu(T)$. If $\varphi$ is surjective and $\mu(\varphi(v)) \leq \mu(v)$ for every $v \in T$, then $C_\varphi$ is not hypercyclic.

**Proof.** Observe that, for $f \in L^0_\mu(T)$ and for each $v \in T$ we have

$$\mu(v)|\varphi(\varphi(v)| \geq \mu(\varphi(v))|f(\varphi(v))|$$

and hence

$$\|C_\varphi f\|_\mu \geq \mu(\varphi(v))|f(\varphi(v))|.$$ 

Since $\varphi$ is surjective, this implies that $\|C_\varphi f\|_\mu \geq \|f\|_\mu$ and hence that $\|C_\varphi^n f\|_\mu \geq \|f\|_\mu$ for each $n \in \mathbb{N}$. It follows that $C_\varphi$ cannot be hypercyclic. \hfill $\Box$

Let $n \in \mathbb{N}$ and let $\varphi$ be a self-map of $T$. We denote by $T^n$ the image of $\varphi^n$; that is,

$$T^n := \{w \in T : w = \varphi^n(v) \text{ for some } v \in T\}.$$ 

We also denote by $T^\infty$ the set

$$T^\infty := \{w \in T : \text{ for each } n \in \mathbb{N} \text{ there exists } v \in T \text{ with } \varphi^n(v) = w\}.$$ 

Clearly, $T^\infty = \bigcap_{n=1}^{\infty} T^n$. 
For the proof of our next theorem, we will need the so-called Hypercyclicity Criterion, which was obtained independently by Kitai in 1982 and by Gethner and Shapiro, in 1987. A modern proof can be found in [13, Theorem 3.12].

**Theorem (Hypercyclicity Criterion).** Let $L$ be a separable Banach space and $B$ a bounded operator on $L$. Assume that there exists a dense subset $X$ of $L$, a sequence $\{n_k\}$ of increasing positive integers, and a sequence of functions $\{S_{n_k}\}$, with $S_{n_k} : X \to L$ such that, for every $x \in X$,

1. $B^{n_k}x \to 0$,
2. $S_{n_k}x \to 0$, and
3. $B^{n_k}S_{n_k}x \to x$.

Then, $B$ is hypercyclic.

While the following three theorems can be obtained as corollaries to [12, Theorem 3], we include their proofs, which were, partially, obtained independently, and are specific to composition operators on $L^0_\mu(T)$.

**Theorem 5.6.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$, let $\mu$ be a positive function on $T$ and let $\varphi$ be a self-map of $T$. Assume that $C_\varphi$ is bounded on $L^0_\mu(T)$ and $\varphi$ is injective. Then, $C_\varphi$ is hypercyclic if and only if there exists an increasing sequence of positive integers $\{n_k\}$ such that

$$\mu(\varphi^{n_k}(v)) \to 0 \quad \text{for all } v \in T,$$

and

$$\mu(\varphi^{-n_k}(v)) \to 0 \quad \text{for all } v \in T^\infty.$$

For clarity, we prove each direction separately in what follows.

**Theorem 5.7.** Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$, let $\mu$ be a positive function on $T$ and let $\varphi$ be a self-map of $T$. Assume that $C_\varphi$ is bounded on $L^0_\mu(T)$ and $\varphi$ is injective. If there exists an increasing sequence of positive integers $\{n_k\}$ such that

$$\mu(\varphi^{n_k}(v)) \to 0 \quad \text{for every } v \in T,$$

and

$$\mu(\varphi^{-n_k}(v)) \to 0 \quad \text{for every } v \in T^\infty,$$

then $C_\varphi$ is hypercyclic.

**Proof.** We use the Hypercyclicity Criterion. Let $X$ be the set of all finitely supported functions on $T$. Let $f \in X$, $f$ not identically 0, and assume that

$$\{v \in T : f(v) \neq 0\} = \{w_1, w_2, \ldots, w_s\}$$

for some $s \in \mathbb{N}$. 
(1) For each $k \in \mathbb{N}$, we have
\[
\|C^n_k f\|_\mu = \|f \circ \varphi^n_k\|_\mu \\
= \sup_{v \in T} \mu(v)|f(\varphi^n_k(v))| \\
= \sup_{w \in T^{\varphi^n_k}} \mu(\varphi^{-n_k}(w))|f(w)| \\
= \max\{-\mu(\varphi^{-n_k}(w_j))|f(w_j)| : w_j \in T^{\varphi^n_k} \text{ and } j = 1, 2, \ldots, s\},
\]
where the maximum is understood to be 0 if the set $\{w_j : w_j \in T^{\varphi^n_k}\}$ is empty. Taking the limit as $k \to \infty$ guarantees that $C^n_k f \to 0$, as desired.

(2) Define the sequence of functions $S_n$ on $L^0_\mu(T)$ as
\[
(S_n f)(v) = \begin{cases} 
  f(\varphi^{-n}(v)), & \text{if } v \in \varphi^n(T), \text{ and} \\
  0, & \text{if } v \notin \varphi^n(T).
\end{cases}
\]
Clearly, if $f$ has finite support, then so does $S_n f$. Thus, for all $k \in \mathbb{N}$
\[
\|S_n f\|_\mu = \sup_{v \in T} \mu(v)|(S_n f)(v)| \\
= \sup_{v \in \varphi^n(T)} \mu(v)|f(\varphi^{-n_k}(v))| \\
= \sup_{w \in T} \mu(\varphi^n_k(w))|f(w)| \\
= \max\{-\mu(\varphi^n_k(w_j))|f(w_j)| : j = 1, 2, \ldots, s\}.
\]
Taking the limit as $k \to \infty$, we see that $S_n f \to 0$, as desired.

(3) Lastly, observe that $(C^n_k S_n f)(v) = (S_n f)(\varphi^n(v)) = f(v)$ for all $f \in X$ and all $v \in T$. Hence $C^n_k S_n f \to f$ as $k \to \infty$.

By the Hypercyclicity Criterion, it follows that $C_\varphi$ is hypercyclic. 

\begin{theorem}
Let $(T, d)$ be an unbounded, locally finite metric space with a distinguished element $o$, let $\mu$ be a positive function on $T$ and let $\varphi$ be a self-map of $T$. Assume that $C_\varphi$ is bounded on $L^0_\mu(T)$ and $\varphi$ is injective. If $C_\varphi$ is hypercyclic, then there exists a sequence $\{n_k\}$ of increasing positive integers such that
\[
\mu(\varphi^n(v)) \to 0 \quad \text{for all } v \in T,
\]
and
\[
\mu(\varphi^{-n_k}(v)) \to 0 \quad \text{for all } v \in T^\infty.
\]
\end{theorem}

\begin{proof}
We first make the following claim.

\begin{claim}
For every $\varepsilon > 0$, every $N \in \mathbb{N}$ and every finite set $I \subseteq T$, there exists a natural number $n \geq N$ such that
\[
\mu(\varphi^n(v)) < \varepsilon \quad \text{for every } v \in I,
\]
and
\[
\mu(\varphi^{-n}(v)) < \varepsilon \quad \text{for every } v \in I \cap T^\infty.
\]
\end{claim}
To prove the claim, first assume, without loss of generality, that 
\[ 0 < \varepsilon < \min\{\mu(v) : v \in I\}. \]
Since \( C_\varphi \) is hypercyclic, there exists \( f \in L_0^0(T) \) and \( n > N \) such that
\[ \left\| f - \sum_{v \in I} \chi_v \right\|_{\mu} < \frac{\varepsilon}{2} \tag{5} \]
and
\[ \left\| C_\varphi^n f - \sum_{v \in I} \chi_v \right\|_{\mu} < \frac{\varepsilon}{2} \tag{6} \]
Furthermore, since \( \varphi \) must be a run-away function (because, since \( C_\varphi \) is hypercyclic, the function \( \varphi \) has no periodic points), we can assume that \( \varphi^n(I) \cap I = \emptyset \).

Inequality (5) implies that
\[ \mu(w)|f(w)| < \frac{\varepsilon}{2}, \quad \text{for all } w \notin I. \]
Since \( \varphi^n(I) \cap I = \emptyset \), we must have that \( \varphi^n(v) \notin I \) for every \( v \in I \) and hence
\[ \mu(\varphi^n(v))|f(\varphi^n(v))| < \frac{\varepsilon}{2}, \quad \text{for all } v \in I. \tag{7} \]

On the other hand, by inequality (6), we have
\[ \mu(v)|f(\varphi^n(v)) - 1| < \frac{\varepsilon}{2}, \quad \text{for all } v \in I. \]
Therefore,
\[ 1 - |f(\varphi^n(v))| < \frac{\varepsilon}{2\mu(v)} \quad \text{for all } v \in I. \tag{8} \]
Recalling the assumption on \( \varepsilon \) and combining inequalities (8) and (7), we obtain
\[ 0 < 1 - \frac{\varepsilon}{2\mu(v)} < |f(\varphi^n(v))| < \frac{\varepsilon}{2\mu(\varphi^n(v))} \]
for every \( v \in I \), and hence
\[ \mu(\varphi^n(v)) < \frac{\varepsilon}{2 - \frac{\varepsilon}{\mu(v)}}. \]
Since \( 2 - \frac{\varepsilon}{\mu(v)} > 1 \), it follows that
\[ \mu(\varphi^n(v)) < \varepsilon, \quad \text{for all } v \in I. \]

Now, inequality (6) implies that
\[ \mu(w)|f(\varphi^n(w))| < \frac{\varepsilon}{2}, \quad \text{for all } w \notin I. \tag{9} \]
Let \( v \in I \cap T^\infty \). Observe that \( \varphi^{-n}(v) \notin I \). Indeed, if \( \varphi^{-n}(v) \) were in \( I \), then \( v \) would be in \( \varphi^n(I) \) and hence \( v \) would be in \( \varphi^n(I) \cap I \cap T^\infty \), which is impossible since \( \varphi^n(I) \cap I \) is empty.

By (9) applied to \( \varphi^{-n}(v) \), we obtain
\[ \mu(\varphi^{-n}(v))|f(v)| < \frac{\varepsilon}{2}, \quad \text{for all } v \in I \cap T^\infty. \tag{10} \]
On the other hand, by inequality (5), we have
\[ \mu(v)|f(v) - 1| < \frac{\varepsilon}{2}, \quad \text{for all } v \in I. \] (11)

Proceeding as above, from inequalities (10) and (11), it follows that, for every \( v \in I \cap T^\infty \),
\[ 1 - \frac{\varepsilon}{2\mu(v)} < |f(v)| < \frac{\varepsilon}{2\mu(\varphi^{-n}(v))}. \]
Therefore, for each \( v \in I \cap T^\infty \),
\[ \mu(\varphi^{-n}(v)) < \frac{\varepsilon}{2 - \frac{\varepsilon}{\mu(v)}} < \varepsilon, \]
proving the claim.

Now, for each \( k \in \mathbb{N} \), define \( I_k := \{ v \in T : |v| \leq k \} \). Since \( T \) is locally finite, each \( I_k \) is a finite set. By the claim, there exists \( n_1 \in \mathbb{N} \) such that
\[ \mu(\varphi^{n_1}(v)) < 1, \quad \text{for all } v \in I_1, \]
and
\[ \mu(\varphi^{-n_1}(v)) < 1, \quad \text{for all } v \in I_1 \cap T^\infty. \]

Arguing inductively, assume that for some \( s \in \mathbb{N} \) we have chosen integers \( n_2, \ldots, n_{s-1} \) with \( n_1 < n_2 < n_3 < \cdots < n_{s-1} \) such that, for \( k = 1, 2, \ldots, s-1 \), we have
\[ \mu(\varphi^{n_k}(v)) < \frac{1}{k}, \quad \text{for all } v \in I_k, \]
and
\[ \mu(\varphi^{-n_k}(v)) < \frac{1}{k}, \quad \text{for all } v \in I_k \cap T^\infty. \]
Now, by the claim applied to \( \varepsilon = \frac{1}{s} \), \( N = n_{s-1} \) and \( I = I_s \), it follows that there exists an integer \( n_s > n_{s-1} \) with
\[ \mu(\varphi^{n_s}(v)) < \frac{1}{s}, \quad \text{for all } v \in I_s, \]
and
\[ \mu(\varphi^{-n_s}(v)) < \frac{1}{s}, \quad \text{for all } v \in I_s \cap T^\infty. \]

Lastly, we observe that the sequence \( \{n_k\} \) satisfies the conclusion of the theorem.

Indeed, let \( v \in T \) and \( \varepsilon > 0 \). Choose \( N \in \mathbb{N} \) such that \( N \geq \frac{1}{\varepsilon} \) and \( N \geq |v| \). Clearly, if \( k \geq N \), then \( v \in I_k \) and hence
\[ \mu(\varphi^{n_k}(v)) < \frac{1}{k} \leq \varepsilon. \]
Hence, \( \lim_{k \to \infty} \mu(\varphi^{n_k}(v)) = 0 \), as desired.

Analogously, let \( v \in T^\infty \) and \( \varepsilon > 0 \). Then, choose \( N \in \mathbb{N} \) such that \( N \geq \frac{1}{\varepsilon} \) and \( N \geq |v| \). Clearly, if \( k \geq N \), then \( v \in I_k \cap T^\infty \) and hence
\[ \mu(\varphi^{-n_k}(v)) < \frac{1}{k} \leq \varepsilon. \]

Hence, \( \lim_{k \to \infty} \mu(\varphi^{-n_k}(v)) = 0 \), as desired. \qed

We deduce the following result for the unweighted case. Denote by \( L^0(T) \) the space \( L^0_\mu(T) \) when \( \mu \) is the constant function 1.

**Corollary 5.9.** Let \((T, d)\) be an unbounded, locally finite metric space with a distinguished element \( o \), let \( \varphi \) be a self-map of \( T \). If \( \varphi \) is injective and \( C_\varphi \) is bounded on \( L^0(T) \), then \( C_\varphi \) is not hypercyclic.

The following example shows that hypercyclicity can, indeed, occur.

**Example 5.10.** Let \( T \) be an infinite, locally finite tree with root \( o \). Let \( \varphi \) be a bijection of \( T \) with no periodic points (for example, if \( T \) is a homogeneous tree, automorphisms with no periodic points clearly exist). Define a positive function \( \mu \) on \( T \) in such a way that \( \mu(v) \to 0 \) as \( |v| \to \infty \) (for example, \( \mu(v) = \frac{1}{|v|+1} \)). Then \( C_\varphi \) is hypercyclic on \( L^0_\mu(T) \).

**Proof.** We claim that if \( \varphi \) is a bijection with no periodic points, then \( \varphi^n(v) \to \infty \) as \( n \to \infty \). Suppose not. Then, there must exist a vertex \( v \), a constant \( M \) and an increasing sequence of natural numbers \( \{n_k\} \) such that

\[ |\varphi^{n_k}(v)| \leq M, \quad \text{as } k \to \infty. \]

Since the set \( \{v \in T : |v| \leq M\} \) is finite, this implies that there must exist integers \( s \) and \( t \) with \( n_s < n_t \) such that \( \varphi^{n_s}(v) = \varphi^{n_t}(v) \). But this implies that

\[ \varphi^{n_t-n_s}(\varphi^{n_s}(v)) = \varphi^{n_s}(v), \]

and hence that \( \varphi^{n_s}(v) \) is a periodic point, which is a contradiction.

Now, it follows that, for the full sequence \( \{n\} \) we have that

\[ \mu(\varphi^n(v)) \to 0, \quad \text{as } n \to \infty, \]

for every \( v \in T \), since \( \varphi^n(v) \to \infty \).

Since \( \varphi^{-1} \) is also a bijection of \( T \) without periodic points, it follows that

\[ \mu(\varphi^{-n}(v)) \to 0, \quad \text{as } n \to \infty, \]

for every \( v \in T \). Then Theorem 5.6 guarantees that \( C_\varphi \) is hypercyclic. \qed

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