THE SUPERSINGULAR LOCI AND MASS FORMULAS ON
SIEGEL MODULAR VARIETIES

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Abstract. We describe the supersingular locus of the Siegel 3-fold with a
parahoric level structure. We also study its higher dimensional generalization.
Using this correspondence and a deep result of Li and Oort, we evaluate
the number of irreducible components of the supersingular locus of the Siegel
moduli space \( \mathcal{A}_{g,1,N} \) for arbitrary \( g \).

1. INTRODUCTION

In this paper we discuss some extensions of works of Katsura and Oort [5], and
of Li and Oort [8] on the supersingular locus of a mod \( p \) Siegel modular variety.

Let \( p \) be a rational prime number, \( N \geq 3 \) a prime-to-\( p \) positive integer. We
choose a primitive \( N \)-th root of unity \( \zeta_N \) in \( \mathbb{Q} \subset \mathbb{C} \) and an embedding \( \mathbb{Q} \hookrightarrow \mathbb{Q}_p \).

Let \( \mathcal{A}_{g,1,N} \) be the moduli space over \( \mathbb{Z}/p \mathbb{Z} \) of \( g \)-dimensional principally polarized
abelian varieties \((A, \lambda, \eta)\) with a symplectic level-\( N \) structure (See Subsection 2.1).

Let \( \mathcal{A}_{2,1,N,(p)} \) be the cover of \( \mathcal{A}_{2,1,N} \) which parametrizes isomorphism classes of
objects \((A, \lambda, \eta, H)\), where \((A, \lambda, \eta)\) is an object in \( \mathcal{A}_{2,1,N} \) and \( H \subset \mathcal{A}[p] \) is a finite
flat subgroup scheme of rank \( p \). It is known that the moduli scheme \( \mathcal{A}_{2,1,N,(p)} \) has
semi-stable reduction and the reduction \( \mathcal{A}_{2,1,N,(p)} \otimes \mathbb{F}_p \) modulo \( p \) has two irreducible
components. Let \( \mathcal{S}_{2,1,N,(p)} \) (resp. \( \mathcal{S}_{2,1,N} \)) denote the supersingular locus of the
moduli space \( \mathcal{A}_{2,1,N,(p)} \otimes \mathbb{F}_p \) (resp. \( \mathcal{A}_{2,1,N} \otimes \mathbb{F}_p \)). Recall that an abelian variety \( A \) in
characteristic \( p \) is called supersingular if it is isogenous to a product of supersingular
elliptic curves over an algebraically closed field \( k \); it is called superspecial if it is
isomorphic to a product of supersingular elliptic curves over \( k \).

The supersingular locus \( \mathcal{S}_{2,1,N} \) of the Siegel 3-fold is studied in Katsura and Oort
[5]. We summarize the main results for \( \mathcal{S}_{2,1,N} \) (the local results obtained earlier in
Koblitz [7]):

Theorem 1.1.

(1) The scheme \( \mathcal{S}_{2,1,N} \) is equi-dimensional and each irreducible component is
isomorphic to \( \mathbb{P}^1 \).

(2) The scheme \( \mathcal{S}_{2,1,N} \) has

\[
|\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \left( \frac{p^2 - 1}{5760} \right)
\]

irreducible components.

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level structure, mass formula.\]
(3) The singular points of \( S_{2,1,N} \) are exactly the superspecial points and there are
\[
|\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \frac{(p-1)(p^2+1)}{5760}
\]
of them. Moreover, at each singular point there are \( p+1 \) irreducible components
passing through and intersecting transversely.

**Proof.** See Koblitz [7, p.193] and Katsura-Oort [5, Section 2, Theorem 5.1, Theorem 5.3].

In this paper we extend their results to \( S_{2,1,N,(p)} \). We show

**Theorem 1.2.**

1. The scheme \( S_{2,1,N,(p)} \) is equi-dimensional and each irreducible component is
isomorphic to \( \mathbb{P}^1 \).
2. The scheme \( S_{2,1,N,(p)} \) has
\[
|\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4} \left( [p^2-1] + (p-1)(p^2+1) \right)
\]
irreducible components, where \( \zeta(s) \) is the Riemann zeta function.
3. The scheme \( S_{2,1,N,(p)} \) has only ordinary double singular points and there are
\[
|\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4} (p-1)(p^2+1)(p+1)
\]
of them.
4. The natural morphism \( S_{2,1,N,(p)} \to S_{2,1,N} \) contracts
\[
|\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)\zeta(-1)\zeta(-3)}{4} (p-1)(p^2+1)
\]
projective lines onto the superspecial points of \( S_{2,1,N} \).

**Remark 1.3.**

1. By the basic fact that
\[
\zeta(-1) = -\frac{1}{12} \quad \text{and} \quad \zeta(-3) = \frac{1}{120},
\]
the number (1.3) (of the vertical components) equals the number (1.2) (of superspecial points), and the number (1.3) (sum of vertical and horizontal components) equals the sum of the numbers (1.1) (of irreducible components) and (1.2) (of superspecial points). Thus, the set of horizontal irreducible components of \( S_{2,1,N,(p)} \) is in bijection with the set of irreducible components of \( S_{2,1,N} \).
2. Theorem (1.2) (4) says that \( S_{2,1,N,(p)} \) is a “desingularization” or a “blow-up” of \( S_{2,1,N} \) at the singular points. Strictly speaking, the desingularization of \( S_{2,1,N} \) is its normalization, which is the (disjoint) union of horizontal components of \( S_{2,1,N,(p)} \). The vertical components of \( S_{2,1,N,(p)} \) should be the exceptional divisors of the blowing up of a suitable ambient surface of \( S_{2,1,N} \) at superspecial points.

In the proof of Theorem (1.2) (Section 4) we see that

- the set of certain superspecial points (the set \( \Lambda \) in Subsection 4.1) in \( S_{2,p,N} \) (classifying degree \( p^2 \) polarized supersingular abelian surfaces) is in bijection with the set of irreducible components of \( S_{2,1,N} \), and
- the set of superspecial points in \( S_{2,1,N} \) is in bijection with the set of irreducible components of \( S_{2,p,N} \), furthermore
the supersingular locus $S_{2,1,N,(p)}$ provides the explicit link of this duality as a correspondence that performs simply through the “blowing-ups” and “blowing-downs”.

In the second part of this paper we attempt to generalize a similar picture to higher (even) dimensions.

Let $g = 2D$ be an even positive integer. Let $H$ be the moduli space over $\mathbb{Z}/(p)\{ζ\}$ which parametrizes equivalence classes of objects $(ϕ : A_1 \to A_2)$, where
- $A_1 = (A_1, λ_1, η_1)$ is an object in $A_{g,1,N}$,
- $A_2 = (A_2, λ_2, η_2)$ is an object in $A_{g,pD,N}$, and
- $ϕ : A_1 \to A_2$ is an isogeny of degree $pD$ satisfying $ϕ^*λ_2 = pλ_1$ and $ϕ^*η_1 = η_2$.

The moduli space $H$ with natural projections gives the following correspondence:

Let $S$ be the supersingular locus of $H \otimes \overline{F}_p$, which is the reduced closed subscheme consisting of supersingular points (either $A_1$ or $A_2$ is supersingular, or equivalently both are so).

In the special case where $g = 2$, $H$ is isomorphic to $A_{2,1,N,(p)}$, and $S \simeq S_{2,1,N,(p)}$ under this isomorphism (See Subsection 4.5).

As the second main result of this paper, we obtain

**Theorem 1.4.** Let $C$ be the number of irreducible components of $S_{g,1,N}$. Then

$$C = |Sp_{2g}(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{g(g+1)/2}}{2g} \prod_{i=1}^{g} ζ(1 - 2i) \cdot L_p,$$

where

$$L_p = \begin{cases} \prod_{i=1}^{g} \left( p^i + (-1)^i \right) & \text{if } g \text{ is odd;} \\ \prod_{i=1}^{D} (p^{4i-2} - 1) & \text{if } g = 2D \text{ is even.} \end{cases}$$

In the special case where $g = 2$, Theorem 1.4 recovers Theorem 1.1 (2).

We give the idea of the proof. Let $Λ_{g,1,N}$ denote the set of superspecial (geometric) points in $A_{g,1,N} \otimes \overline{F}_p$. For $g = 2D$ is even, let $Λ_{g,pD,N}^*$ denote the set of superspecial (geometric) points $(A, λ, η)$ in $A_{g,pD,N} \otimes \overline{F}_p$ satisfying $\ker λ = A[F]$, where $F : A \to A(p)$ is the relative Frobenius morphism on $A$. These are finite sets and each member is defined over $\overline{F}_p$. By a result of Li and Oort [8] (also see Section 5), we know

$$C = \begin{cases} |Λ_{g,1,N}| & \text{if } g \text{ is odd;} \\ |Λ_{g,pD,N}^*| & \text{if } g \text{ is even.} \end{cases}$$

One can use the geometric mass formula due to Ekedahl [2] and some others (see Section 3) to compute $|Λ_{g,1,N}|$. Therefore, it remains to compute $|Λ_{g,pD,N}^*|$ when $g$ is even. We restrict the correspondence $S$ to the product $Λ_{g,1,N} \times Λ_{g,pD,N}^*$ of superspecial points, and compute certain special points in $S$. This gives us relation between $Λ_{g,pD,N}^*$ and $Λ_{g,1,N}$. See Section 6 for details.
Theorem 1.4 tells us how the number $C = C(g, N, p)$ varies when $p$ varies. For another application, one can use this result to compute the dimension of the space of Siegel cusp forms of certain level at $p$ by the expected Jacquet-Langlands correspondence for symplectic groups. As far as the author knows, the latter for general $g$ is not available yet in the literature.

The paper is organized as follows. In Section 2, we recall the basic definitions and properties of the Siegel moduli spaces and supersingular abelian varieties. In Section 3, we state the mass formula for superspecial principally polarized abelian varieties due to Ekedahl (and some others). The proof of Theorem 1.2 is given in Section 4. In Section 5, we describe the results of Li and Oort on irreducible components of the supersingular locus. In Section 6, we introduce a correspondence and use this to evaluate the number of irreducible components of the supersingular locus.

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2. Notation and preliminaries

2.1. Throughout this paper we fix a rational prime $p$ and a prime-to-$p$ positive integer $N \geq 3$. Let $d$ be a positive integer with $(d, N) = 1$. We choose a primitive $N$-th root of unity $\zeta_N$ in $\mathbb{Q} \subset \mathbb{C}$ and an embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$. The element $\zeta_N$ gives rise to a trivialization $\mathbb{Z}/N\mathbb{Z} \simeq \mu_N$ over any $\mathbb{Z}_p[\zeta_N]$-scheme. For a polarized abelian variety $(A, \lambda)$ of degree $d^2$, a full symplectic level-$N$ structure with respect to the choice $\zeta_N$ is an isomorphism $\eta : (\mathbb{Z}/N\mathbb{Z})^{2g} \simeq A[N]$ such that the following diagram commutes:

\[
\begin{array}{ccc}
(Z/N\mathbb{Z})^{2g} \times (Z/N\mathbb{Z})^{2g} & \xrightarrow{(\eta, \eta)} & A[N] \times A[N] \\
\downarrow \langle \cdot, \cdot \rangle & & \downarrow \epsilon_\lambda \\
\mathbb{Z}/N\mathbb{Z} & \xrightarrow{\zeta_N} & \mu_N,
\end{array}
\]

where $\langle \cdot, \cdot \rangle$ is the standard non-degenerate alternating form on $(\mathbb{Z}/N\mathbb{Z})^{2g}$ and $\epsilon_\lambda$ is the Weil pairing induced by the polarization $\lambda$.

Let $A_{g,d,N}$ denote the moduli space over $\mathbb{Z}_p[\zeta_N]$ of $g$-dimensional polarized abelian varieties $(A, \lambda, \eta)$ of degree $d^2$ with a full symplectic level $N$ structure with respect to $\zeta_N$. Let $S_{g,d,N}$ denote the supersingular locus of the reduction $A_{g,d,N} \otimes \overline{\mathbb{F}}_p$ modulo $p$, which is the closed reduced subscheme of $A_{g,d,N} \otimes \overline{\mathbb{F}}_p$ consisting of supersingular points in $A_{g,d,N} \otimes \overline{\mathbb{F}}_p$. Let $\Lambda_{g,d,N}$ denote the set of superspecial (geometric) points in $S_{g,d,N}$; this is a finite set and every member is defined over $\overline{\mathbb{F}}_p$.

For a scheme $X$ of finite type over a field $K$, we denote by $\Pi_0(X)$ the set of geometrically irreducible components of $X$.

Let $k$ be an algebraically closed field of characteristic $p$. 

2.2. Over a ground field $K$ of characteristic $p$, denote by $\alpha_p$ the finite group scheme of rank $p$ that is the kernel of the Frobenius endomorphism from the additive group $\mathbb{G}_a$ to itself. One has
\[
\alpha_p = \text{Spec}K[X]/X^p, \quad m(X) = X \otimes 1 + 1 \otimes X,
\]
where $m$ is the group law.

By definition, an elliptic curve $E$ over $K$ is called supersingular if $E[p](K) = 0$. An abelian variety $A$ over $K$ is called supersingular if it is isogenous to a product of supersingular elliptic curves over $K$; $A$ is called superspecial if it is isomorphic to a product of supersingular elliptic curves over $K$.

For any abelian variety $A$ over $K$ where $K$ is perfect, the $a$-number of $A$ is defined by
\[
a(A) := \dim_K \text{Hom}(\alpha_p, A).
\]
The following interesting results are well-known; see Subsection 1.6 of [8] for a detail discussion.

**Theorem 2.1** (Oort). If $a(A) = g$, then $A$ is superspecial.

**Theorem 2.2** (Deligne, Ogus, Shioda). For $g \geq 2$, there is only one $g$-dimensional superspecial abelian variety up to isomorphism over $k$.

### 3. THE MASS FORMULA

Let $\Lambda_g$ denote the set of isomorphism classes of $g$-dimensional principally polarized superspecial abelian varieties over $\mathbb{F}_p$. Write
\[
M_g := \sum_{(A, \lambda) \in \Lambda_g} \frac{1}{|\text{Aut}(A, \lambda)|}
\]
for the mass attached to $\Lambda_g$. The following mass formula is due to Ekedahl [2, p.159] and Hashimoto-Ibukiyama [3, Proposition 9].

**Theorem 3.1.** Notation as above. One has
\[
M_g = \left(\frac{-1)^{g(g+1)/2}}{2^g} \prod_{k=1}^{g} \zeta(1 - 2k) \right) \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k) \right\}.
\]

Similarly, we set
\[
M_{g,1,N} := \sum_{(A,\lambda,\eta) \in \Lambda_{g,1,N}} \frac{1}{|\text{Aut}(A,\lambda,\eta)|}.
\]

**Lemma 3.2.** We have $M_{g,1,N} = |\Lambda_{g,1,N}| = |\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})| \cdot M_g$.

**Proof.** The first equality follows from a basic fact that $(A, \lambda, \eta)$ has no non-trivial automorphism. The proof of the second equality is elementary; see Subsection 4.6 of [11].

**Corollary 3.3.** One has
\[
|\Lambda_{2,1,N}| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{\zeta(-1)}\zeta(-3)}{4} (p - 1)(p^2 + 1).
\]
4. Proof of Theorem 1.2

4.1. In this section we consider the case where \( g = 2 \). Let

\[ \Lambda := \{(A, \lambda, \eta) \in S_{2,p,N}; \quad \ker \lambda \cong \alpha_p \times \alpha_p \} \]

Note that every member \( A \) of \( \Lambda \) is superspecial (because \( A \supset \alpha_p \times \alpha_p \)), that is, \( \Lambda \subset \Lambda_{2,p,N} \). For a point \( \xi \) in \( \Lambda \), consider the space \( S_\xi \) which parametrizes the isogenies of degree \( p \)

\[ \varphi : (A_\xi, \lambda_\xi, \eta_\xi) \to A = (A, \lambda, \eta) \]

which preserve the polarizations and level structures. Let

\[ \psi_\xi : S_\xi \to S_{2,1,N} \]

be the morphism which sends \( (\varphi : \xi \to A) \) to \( A \). Let \( V_\xi \subset S_{2,1,N} \) be the image of \( S_\xi \) under \( \psi_\xi \).

The following theorem is due to Katsura and Oort \[ 5, \text{Theorem 2.1 and Theorem 5.1}\]:

**Theorem 4.1** (Katsura-Oort). Notation as above. The map \( \xi \mapsto V_\xi \) gives rise to a bijection \( \Lambda \cong \Pi_0(S_{2,1,N}) \) and one has

\[ |\Lambda| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})|(p^2 - 1)/5760. \]

We will give a different way of evaluating \( |\Lambda| \) that is based on the geometric mass formula (see Corollary 4.6).

4.2. Dually we can consider the space \( S'_\xi \) for each \( \xi \in \Lambda \) that parametrizes the isogenies of degree \( p \)

\[ \varphi' : A = (A, \lambda, \eta) \to \xi = (A_\xi, \lambda_\xi, \eta_\xi), \]

with \( A \in A_{2,1,N} \otimes \mathbb{F}_p \), such that \( \varphi' \eta = \eta_\xi \) and \( p \varphi' \lambda = \lambda_\xi \). Let

\[ \psi'_\xi : S'_\xi \to S_{2,1,N} \]

be the morphism which sends \( (\varphi' : A \to \xi) \) to \( A \). Let \( V'_\xi \subset S_{2,1,N} \) be the image of \( S'_\xi \) under \( \psi'_\xi \).

For a degree \( p \) isogeny \( (\varphi : A_1 \to A_2) \) with \( A_2 \in A_{2,1,N} \), \( p \varphi \lambda_2 = \lambda_1 \) and \( \varphi \eta_1 = \eta_2 \), we define

\[ (\varphi : A_1 \to A_2)^* = (\varphi' : A'_2 \to A'_1), \]

where \( \varphi' = \varphi^t \) and

\[ A'_2 = (A'_2, \lambda_2^{-1}, \lambda_2 \circ \eta_2), \]

\[ A'_1 = (A'_1, p \lambda_1^{-1}, \lambda_1 \circ \eta_1). \]

Note that \( \varphi' \eta_2 = \eta'_1 \) as we have the commutative diagram:

\[
\begin{array}{ccc}
(Z/N\mathbb{Z})^4 & \overset{\eta_2}{\longrightarrow} & A_2[N] \overset{\lambda_2}{\longrightarrow} A'_2[N] \\
\downarrow & & \uparrow \varphi \\
(Z/N\mathbb{Z})^4 & \overset{\eta_1}{\longrightarrow} & A_1[N] \overset{\lambda_1}{\longrightarrow} A'_1[N].
\end{array}
\]

If \( A_1 \in \Lambda \), then \( A'_1 \) is also in \( \Lambda \). Therefore, the map \( \xi \mapsto V'_\xi \) also gives rise to a bijection \( \Lambda \cong \Pi_0(S_{2,1,N}) \).
4.3. We use the classical contravariant Dieudonné theory. We refer the reader to Demazure [14] for a basic account of this theory. Let $K$ be a perfect field of characteristic $p$, $W := W(K)$ the ring of Witt vectors over $K$, $B(K)$ the fraction field of $W(K)$. Let $\sigma$ be the Frobenius map on $B(K)$. A quasi-polarization on a Dieudonné module $M$ here is a non-degenerate (meaning of non-zero discriminant) alternating pairing

$$\langle \cdot, \cdot \rangle : M \times M \to B(K),$$

such that $\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma$ for $x, y \in M$ and $(M^t, M') \subset W$. Here we regard the dual $M^t$ of $M$ as a Dieudonné submodule in $M \otimes B(K)$ using the pairing. A quasi-polarization is called separable if $M^t = M$. Any polarized abelian variety $(A, \lambda)$ over $K$ naturally gives rise to a quasi-polarized Dieudonné module. The induced quasi-polarization is separable if and only if $(p, \deg \lambda) = 1$.

Recall (Subsection 2.4) that $k$ denotes an algebraically closed field of characteristic $p$.

**Lemma 4.2.**

1. Let $M$ be a separably quasi-polarized superspecial Dieudonné module over $k$ of rank 4. Then there exists a basis $f_1, f_2, f_3, f_4$ for $M$ over $W := W(k)$ such that

$$F f_1 = f_3, F f_3 = p f_1, \quad F f_2 = f_4, F f_4 = p f_2$$

and the non-zero pairings are

$$\langle f_1, f_3 \rangle = -(f_3, f_1) = \beta_1, \quad \langle f_2, f_4 \rangle = -(f_4, f_2) = \beta_1,$$

where $\beta_1 \in W(\mathbb{F}_{p^2})^\times$ with $\beta_1^p = -\beta_1$.

2. Let $\xi$ be a point in $\Lambda$, and let $M_\xi$ be the Dieudonné module of $\xi$. Then there is a $W$-basis $e_1, e_2, e_3, e_4$ for $M_\xi$ such that

$$F e_1 = e_3, \quad F e_2 = e_4, \quad F e_3 = p e_1, \quad F e_4 = p e_2,$$

and the non-zero pairings are

$$\langle e_1, e_2 \rangle = -\langle e_2, e_1 \rangle = \frac{1}{p}, \quad \langle e_3, e_4 \rangle = -\langle e_4, e_3 \rangle = 1.$$

**Proof.** (1) This is a special case of Proposition 6.1 of [8].

(2) By Proposition 6.1 of [8], $(M_\xi, \langle \cdot, \cdot \rangle)$ either is indecomposable or decomposes into a product of two quasi-polarized supersingular Dieudonné modules of rank 2. In the indecomposable case, one can choose such a basis $e_i$ for $M_\xi$. Hence it remains to show that $(M_\xi, \langle \cdot, \cdot \rangle)$ is indecomposable. Let $(A_\xi[p^\infty], \lambda_\xi)$ be the associated polarized $p$-divisible group. Suppose it decomposes into $(H_1, \lambda_1) \times (H_2, \lambda_2)$. Then the kernel of $\lambda$ is isomorphic to $E[p]$ for a supersingular elliptic curve $E$. Since $E[p]$ is a nontrivial extension of $\alpha_p$ by $\alpha_p$, one gets contradiction. This completes the proof.

4.4. Let $(A_0, \lambda_0)$ be a superspecial principally polarized abelian surface and $(M_0, \langle \cdot, \cdot \rangle_0)$ be the associated Dieudonné module. Let $\varphi' : (A_0, \lambda_0) \to (A, \lambda)$ be an isogeny of degree $p$ with $\varphi'^* \lambda = p^\lambda_0$. Write $(M, \langle \cdot, \cdot \rangle)$ for the Dieudonné module of $(A, \lambda)$. Choose a basis $f_1, f_2, f_3, f_4$ for $M_0$ as in Lemma 4.2. We have the inclusions

$$(F, V) M_0 \subset M \subset M_0.$$
Modulo \((F,V)M_0\), a module \(M\) corresponds a one-dimensional subspace \(\overline{M}\) in \(\overline{M}_0 := M_0/(F,V)M_0\). As \(\overline{M}_0 = k < f_1, f_2 >, \overline{M}\) is of the form 
\[
\overline{M} = k < af_1 + bf_2 >, \quad [a : b] \in \mathbb{P}^1(k).
\]

The following result is due to Moret-Bailly \cite{9} p.138-9). We include a proof for the reader’s convenience.

**Lemma 4.3.** Notation as above, ker \(\lambda \simeq \alpha_p \times \alpha_p\) if and only if the corresponding point \([a : b]\) satisfies \(a^{p+1} + b^{p+1} = 0\). Consequently, there are \(p+1\) isogenies \(\phi'\) so that ker \(\lambda \simeq \alpha_p \times \alpha_p\).

**Proof.** As \(\phi^*\lambda = p\lambda_0\), we have \([\ast, \ast] = \frac{1}{p}([\ast, \ast])_0\). The Dieudonné module \(M(\ker \lambda)\) of the subgroup ker \(\lambda\) is equal to \(M/M^t\). Hence the condition ker \(\lambda \simeq \alpha_p \times \alpha_p\) is equivalent to that \(F\) and \(V\) vanish on \(M(\ker \lambda) = M/M^t\).

Since \([\ast, \ast]\) is a perfect pairing on \(FM_0\), that is, \((FM_0)^t = FM_0\), we have
\[
P_0 \subset M^t \subset F_0 \subset M \subset M_0.
\]
Changing the notation, put \(M \overline{M}_0 := M_0/pM_0\) and let 
\[
\langle \ast, \ast \rangle_0 : M \overline{M}_0 \times \overline{M}_0 \rightarrow k.
\]
be the induced perfect pairing. In \(M_0\), the subspace \(\overline{M}\) is equal to \(\overline{M}^t\). Indeed,
\[
M^t = \{m \in M_0; (m, x)_0 \in pW \ \forall x \in M\},
\]
\[
\overline{M}^t = \{m \in \overline{M}_0; (m, x)_0 = 0 \ \forall x \in \overline{M}\} = \overline{M}^t.
\]
From this we see that the condition ker \(\lambda \simeq \alpha_p \times \alpha_p\) is equivalent to 
\[
\langle \overline{M}, FM \rangle = \langle \overline{M}, VM \rangle = 0.
\]
Since \(FM_0 = k < f_3, f_4 >\), one has \(M = k < f_1', f_2, f_4 >\) where \(f_1' = af_1 + bf_2\). The condition \(\langle \overline{M}, FM \rangle = \langle \overline{M}, VM \rangle = 0\), same as \(\langle f_1', Ff_1' \rangle = \langle f_1', VF_1' \rangle = 0\), gives the equation \(a^{p+1} + b^{p+1} = 0\). This completes the proof.

Conversely, fix a polarized superspecial abelian surface \((A, \lambda)\) such that ker \(\lambda \simeq \alpha_p \times \alpha_p\). Then there are \(p^2 + 1\) degree-\(p\) isogenies \(\phi' : (A_0, \lambda_0) \rightarrow (A, \lambda)\) such that \(A_0\) is superspecial and \(\phi^*\lambda = p\lambda_0\). Indeed, each isogeny \(\phi'\) always has the property \(\phi^*\lambda = p\lambda_0\) for a principal polarization \(\lambda_0\), and there are \([\mathbb{P}^1(\overline{F}_p)]\) isogenies with \(A_0\) superspecial.

4.5. We denote by \(A'_{2,1,N,(p)}\) the moduli space which parametrizes equivalence classes of isogenies \((\phi' : A_0 \rightarrow A_1)\) of degree \(p\), where \(A_0\) is an object in \(A_{2,p,N}\) and \(A_1\) is an object in \(A_{2,1,N}\), such that \(\phi'^*\lambda_1 = p\lambda_0\) and \(\phi'^*\eta_0 = \eta_1\).

There is a natural isomorphism from \(A_{2,1,N,(p)}\) to \(A'_{2,1,N,(p)}\). Given an object \((A, \lambda, \eta, H)\) in \(A_{2,1,N,(p)}\), let \(A_0 := A, A_1 := A/H\) and \(\phi' : A_0 \rightarrow A_1\) be the natural projection. The polarization \(p\lambda_0\) descends to one, denoted by \(\lambda_1\), on \(A_1\). Put \(\eta_1 := \phi'^*\eta_0\) and \(A_1 := (A_1, \lambda_1, \eta_1)\). Then \((\phi' : A_0 \rightarrow A_1)\) lies in \(A'_{2,1,N,(p)}\) and the morphism
\[
q : A_{2,1,N,(p)} \rightarrow A'_{2,1,N,(p)}, \quad (A, \lambda, \eta, H) \mapsto \phi' : A_0 \rightarrow A_1
\]
is an isomorphism.

We denote by \(S'_{2,1,N,(p)}\) the supersingular locus of \(A'_{2,1,N,(p)} \otimes \overline{F}_p\). Thus we have \(S_{2,1,N,(p)} \simeq S'_{2,1,N,(p)}\). It is clear that \(S_{\xi} \cap S'_{\xi'} = \emptyset\) for each \(\xi \in \Lambda\) and \(S_{\xi} \cap S'_{\xi'} = \emptyset\) if \(\xi \neq \xi'\). For each \(\gamma \in A_{2,1,N}\), let \(S''_{\gamma}\) be the subspace of \(S'_{2,1,N,(p)}\) that consists of objects \((\phi' : A_0 \rightarrow A_1)\) with \(A_0 = A_3\). One also has \(S''_{\gamma} \cap S''_{\gamma'} = \emptyset\) if \(\gamma \neq \gamma'\).
Lemma 4.4.

(1) Let \((M_0, \langle \cdot, \cdot \rangle_0)\) be a separably quasi-polarized supersingular Dieudonné module of rank 4 and suppose \(a(M_0) = 1\). Let \(M_1 := (F, V)M_0\) and \(N\) be the unique Dieudonné module containing \(M_0\) with \(N/M_0 = k\). Let \(\langle \cdot, \cdot \rangle_1 := \frac{1}{k} \langle \cdot, \cdot \rangle_0\) be the quasi-polarization for \(M_1\). Then one has \(a(N) = a(M_1) = 2\), \(VN = M_1\), and \(M_1/M_1' \cong k \oplus k\) as Dieudonné modules.

(2) Let \((M_1, \langle \cdot, \cdot \rangle_1)\) be a quasi-polarized supersingular Dieudonné module of rank 4. Suppose that \(M_1/M_1'\) is of length 2, that is, the quasi-polarization has degree \(p^2\).

(i) If \(a(M_1) = 1\), then letting \(M_2 := (F, V)M_1\), one has that \(a(M_2) = 2\) and \(\langle \cdot, \cdot \rangle_1\) is a separable quasi-polarization on \(M_2\).

(ii) Suppose \((M_1, \langle \cdot, \cdot \rangle_1)\) decomposes as the product of two quasi-polarized Dieudonné submodules of rank 2. Then there are a unique Dieudonné submodule \(M_2\) of \(M_1\) with \(M_1/M_2 = k\) and a unique Dieudonné module \(M_0\) containing \(M_1\) with \(M_0/M_1 = k\) so that \(\langle \cdot, \cdot \rangle_1\) (resp. \(p(\langle \cdot, \cdot \rangle)_1\)) is a separable quasi-polarization on \(M_2\) (resp. \(M_0\)).

(iii) Suppose \(M_1/M_1' \cong k \oplus k\) as Dieudonné modules. Let \(M_2 \subseteq M_1\) be any Dieudonné submodule with \(M_1/M_2 = k\), and \(M_0 \supset M_1\) be any Dieudonné overmodule with \(M_0/M_1 = k\). Then \(\langle \cdot, \cdot \rangle_1\) (resp. \(p(\langle \cdot, \cdot \rangle)_1\)) is a separable quasi-polarization on \(M_2\) (resp. \(M_0\)).

This is well-known; the proof is elementary and omitted.

Proposition 4.5. Notation as above.

(1) One has

\[
S_{2,1,N,(p)}' = \left( \bigcap_{\xi \in \Lambda} S'_\xi \right) \cup \left( \bigcap_{\gamma \in A_{2,1,N}} S''_\gamma \right).
\]

(2) The scheme \(S_{2,1,N,(p)}'\) has ordinary double singular points and

\[
(S_{2,1,N,(p)}')^{\text{sing}} = \left( \bigcap_{\xi \in \Lambda} S'_\xi \right) \cap \left( \bigcap_{\gamma \in A_{2,1,N}} S''_\gamma \right).
\]

Moreover, one has

\[
|S_{2,1,N,(p)}'^{\text{sing}}| = |A_{2,1,N}|(p + 1) = |\Lambda|(p^2 + 1).
\]

Proof. (1) Let \((\varphi' : A_0 \to A_1)\) be a point of \(S_{2,1,N,(p)}'\). If \(a(A_0) = 1\), then \(\ker \varphi'\) is the unique \(\alpha\)-subgroup of \(A_0[p]\) and thus \(A_1 \subseteq \Lambda\). Hence this point lies in \(S'_{\xi}\) for some \(\xi\). Suppose that \(A_1\) is not in \(\Lambda\), then there is a unique lifting \((\varphi' : A_0' \to A_1)\) in \(S_{2,1,N,(p)}'\) and the source \(A_0'\) is superspecial. Hence \(A_0' = A_0'\) is superspecial and the point \((\varphi' : A_0 \to A_1)\) lies in \(S''_{\gamma}\) for some \(\gamma\).

(2) It is clear that the singularities only occur at the intersection of \(S'_{\xi}\)’s and \(S''_{\gamma}\)’s, as \(S'_{\xi}\) and \(S''_{\gamma}\) are smooth. Let \(x = (\varphi' : A_0 \to A_1) \in S'_{\xi} \cap S''_{\gamma}\). We know that the projection \(p_{0} : S_{2,1,N,(p)}' \to S_{2,1,N}\) induces an isomorphism from \(S'_{\xi}\) to \(V'_{\xi}\). Therefore, \(p_{0}\) maps the one-dimensional subspace \(T_{x}(S'_{\xi})\) of \(T_{x}(A_{2,1,N,(p)}' \otimes \mathbb{F}_{p})\) onto the one-dimensional subspace \(T_{p_{0}(x)}(V'_{\xi})\) of \(T_{p_{0}(x)}(A_{2,1,N} \otimes \mathbb{F}_{p})\), where \(T_{x}(X)\) denotes the tangent space of a variety \(X\) at a point \(x\). On the other hand, \(p_{0}\) maps the subspace \(T_{x}(S''_{\gamma})\) to zero. This shows \(T_{x}(S'_{\xi}) \neq T_{x}(S''_{\gamma})\) in \(T_{x}(A_{2,1,N,(p)}' \otimes \mathbb{F}_{p})\); particularly \(S'_{2,1,N,(p)}\) has ordinary double singularity at \(x\).
Since every singular point lies in each $S'_\xi$ and $S'_\gamma$ for some $\xi, \gamma$, by Subsection 4.3, each $S'_\xi$ has $p^2 + 1$ singular points and each $S'_\gamma$ has $p + 1$ singular points. We get
\[ |(S'_{2,1,N,(p)})^{\text{sing}}| = |S_{2,1,N}(p) + 1|, \quad \text{and} \quad |(S'_{2,1,N,(p)})^{\text{sing}}| = |\Lambda|(p^2 + 1). \]
This completes the proof.

**Corollary 4.6.** We have
\[ |(S'_{2,1,N,(p)})^{\text{sing}}| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{\xi(-1)^{\xi(-3)}}}{4} (p - 1)(p^2 + 1)(p + 1) \]
and
\[ |\Lambda| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \cdot \frac{(-1)^{\xi(-1)^{\xi(-3)}}}{4} (p^2 - 1). \]

**Proof.** This follows from Corollary 3.3 and (2) of Proposition 4.5.

Note that the evaluation of $|\Lambda|$ here is different from that given in Katsura-Oort [3]. Their method does not rely on the mass formula but the computation is more complicated.

Since $S_{2,1,N,(p)} \simeq S'_{2,1,N,(p)}$ (Subsection 4.5), Theorem 4.2 follows from Proposition 4.3 and Corollary 4.6.

As a byproduct, we obtain the description of the supersingular locus $S_{2,p,N}$.

**Theorem 4.7.**
1. The scheme $S_{2,p,N}$ is equi-dimensional and each irreducible component is isomorphic to $\mathbb{P}^1$.
2. The scheme $S_{2,p,N}$ has $|\Lambda_{2,1,N}|$ irreducible components.
3. The singular locus of $S_{2,p,N}$ consists of superspecial points $(A, \lambda, \eta)$ with $\ker \lambda \simeq \alpha_p \otimes \alpha_p$, and thus $|S_{2,p,N}^{\text{sing}}| = |\Lambda|$. Moreover, at each singular point there are $p^2 + 1$ irreducible components passing through and intersecting transversely.
4. The natural morphism $\text{pr}_1 : S_{2,1,N,(p)} \to S_{2,p,N}$ contracts $|\Lambda|$ projective lines onto the singular locus of $S_{2,p,N}$.

5. The class numbers $H_n(p, 1)$ and $H_n(1, p)$

In this section we describe the arithmetic part of the results in Li and Oort [3]. Our references are Ibukiyama-Katsura-Oort [3] Section 2] and Li-Oort [3] Section 4].

Let $B$ be the definite quaternion algebra over $\mathbb{Q}$ with discriminant $p$, and $\mathcal{O}$ be a maximal order of $B$. Let $V = B^\oplus n$, regarded as a left $B$-module of row vectors, and let $\psi(x, y) = \sum_{i=1}^n x_i y_i$ be the standard hermitian form on $V$, where $y_i \mapsto \bar{y}_i$ is the canonical involution on $B$. Let $G$ be the group of $\psi$-similitudes over $\mathbb{Q}$; its group of $\mathbb{Q}$-points is
\[ G(\mathbb{Q}) := \{ h \in M_n(B) \mid hh^t = rI_n \text{ for some } r \in \mathbb{Q}^\times \}. \]

Two $\mathcal{O}$-lattices $L$ and $L'$ in $B^\oplus n$ are called **globally equivalent** (denoted by $L \sim L'$) if $L' = Lh$ for some $h \in G(\mathbb{Q})$. For a finite place $v$ of $\mathbb{Q}$, we write $B_v := B \otimes \mathbb{Q}_v$, $\mathcal{O}_v := \mathcal{O} \otimes \mathbb{Q}_v$ and $L_v := L \otimes \mathbb{Q}_v$. Two $\mathcal{O}$-lattices $L$ and $L'$ in $B^\oplus n$ are called **locally equivalent at $v$** (denoted by $L_v \sim L'_v$) if $L'_v = L_v h_v$ for some $h_v \in G(\mathbb{Q}_v)$. A **genus** of $\mathcal{O}$-lattices is a set of (global) $\mathcal{O}$-lattices in $B^\oplus n$ which are equivalent to each other locally at every finite place $v$. 
Let 
\[ N_p = \mathcal{O}_p^{\oplus n} \cdot \left( \begin{array}{cc} I_r & 0 \\ 0 & \pi I_{n-r} \end{array} \right) \cdot \xi \subset B_p^{\oplus n}, \]
where \( r \) is the integer \([n/2] \), \( \pi \) is a uniformizer in \( \mathcal{O}_p \), and \( \xi \) is an element in \( \text{GL}_n(B_p) \) such that
\[ \xi \mathbf{e}_i = \text{anti-diag}(1,1,\ldots,1). \]

**Definition 5.1.** (1) Let \( \mathcal{L}_n(p,1) \) denote the set of global equivalence classes of \( \mathcal{O} \)-lattices \( L \) in \( B^{\oplus n} \) such that \( L_v \sim \mathcal{O}_v^{\oplus n} \) at every finite place \( v \). The genus \( \mathcal{L}_n(p,1) \) is called the principal genus, and let \( H_n(p,1) := |\mathcal{L}_n(p,1)| \).
(2) Let \( \mathcal{L}_n(1,p) \) denote the set of global equivalence classes of \( \mathcal{O} \)-lattices \( L \) in \( B^{\oplus n} \) such that \( L_p \sim N_p \) and \( L_v \sim \mathcal{O}_v^{\oplus n} \) at every finite place \( v \neq p \). The genus \( \mathcal{L}_n(p,1) \) is called the non-principal genus, and let \( H_n(1,p) := |\mathcal{L}_n(1,p)| \).

Recall (Section 3) that \( \Lambda_g \) is the set of isomorphism classes of \( g \)-dimensional principally polarized superspecial abelian varieties over \( \mathbb{F}_p \). When \( g = 2D > 0 \) is even, we denote by \( \Lambda_{g,p}^* \) the set of isomorphism classes of \( g \)-dimensional polarized superspecial abelian varieties \( (A,\lambda) \) of degree \( p^{2D} \) over \( \mathbb{F}_p \) satisfying \( \ker \lambda = A[F] \).

Let \( \mathcal{A}_{g,1} \) be the coarse moduli scheme of \( g \)-dimensional principally polarized abelian varieties, and let \( \mathcal{S}_{g,1} \) be the supersingular locus of \( \mathcal{A}_{g,1} \otimes \mathbb{F}_p \). Recall (Subsection 2.1) that \( \Pi_0(\mathcal{S}_{g,1}) \) denotes the set of irreducible components of \( \mathcal{S}_{g,1} \).

**Theorem 5.2** (Li-Oort). We have
\[ |\Pi_0(\mathcal{S}_{g,1})| = \begin{cases} |\Lambda_g| & \text{if } g \text{ is odd;} \\ |\Lambda_{g,p}^*| & \text{if } g = 2D \text{ is even.} \end{cases} \]

The arithmetic part for \( \Pi_0(\mathcal{S}_{g,1}) \) is given by the following

**Proposition 5.3.**
1. For any positive integer \( g \), one has \( |\Lambda_g| = H_g(p,1) \).
2. For any even positive integer \( g = 2D \), one has \( |\Lambda_{g,p}^*| = H_g(1,p) \).

**Proof.** (1) See [4, Theorem 2.10]. (2) See [8, Proposition 4.7].

6. **Correspondence computation**

6.1. Let \( \hat{M}_0 \) be a superspecial Dieudonné module over \( k \) of rank \( 2g \), and call
\[ \hat{M}_0 := \{ x \in M_0; F^2x = px \}, \]
the skeleton of \( M_0 \) (cf. [8, 5.7]). We know that \( \hat{M}_0 \) is a Dieudonné module over \( \mathbb{F}_{p^2} \) and \( \hat{M}_0 \otimes_{\mathbb{F}_p} W(k) = M_0 \). The vector space \( \hat{M}_0/V\hat{M}_0 \) defines an \( \mathbb{F}_{p^2} \)-structure of the \( k \)-vector space \( M_0/VM_0 \).

Let \( \text{Gr}(n,m) \) be the Grassmannian variety of \( n \)-dimensional subspaces in an \( m \)-dimensional vector space. Suppose \( M_1 \) is a Dieudonné submodule of \( M_0 \) such that
\[ VM_0 \subset M_1 \subset M_0, \quad \dim_k M_1/VM_0 = r, \]
for some integer \( 0 \leq r \leq g \). As \( \dim M_0/VM_0 = g \), the subspace \( \overline{M_1} := M_1/VM_0 \) corresponds to a point in \( \text{Gr}(r,g)(\mathbb{F}_{p^2}) \).

**Lemma 6.1.** Notation as above. Then \( M_1 \) is superspecial (i.e. \( F^2M_1 = pM_1 \)) if and only if \( \overline{M_1} \in \text{Gr}(r,g)(\mathbb{F}_{p^2}) \).
Proof. If $M_1$ is generated by $V\tilde{M}_0$ and $x_1, x_2, \ldots, x_r, x_i \in \tilde{M}_0$ over $W$. Then $\tilde{M}_1$ generates $M_1$ and thus $F^2M_1 = pM_1$. Therefore, $M_1$ is superspecial.

Conversely if $M_1$ is superspecial, then we have

$$V\tilde{M}_0 \subset \tilde{M}_1 \subset \tilde{M}_0.$$ 

Therefore, $\tilde{M}_1$ gives rise to an element in $\text{Gr}(r,g)(F_{p^2})$. 

6.2. Let $L(n, 2n) \subset \text{Gr}(n, 2n)$ be the Lagrangian variety of maximal isotropic subspaces in a $2n$-dimensional vector space with a non-degenerate alternating form.

From now on $g = 2D$ is an even positive integer. Recall (in Introduction and Section 5) that $\Lambda^{\ast}_{g,p,D,N}$ denotes the set of superspecial (geometric) points $(A, \lambda, \eta)$ in $A_{g,p,D,N} \otimes \overline{\mathbb{F}}_p$ satisfying $\ker \lambda = A[F]$.

Lemma 6.2. Let $(A_2, \lambda_2, \eta_2) \in \Lambda^{\ast}_{g,p,D,N}$ and $(M_2, \langle, \rangle_2)$ be the associated Dieudonné module. There is a $W$-basis $e_1, \ldots, e_{2g}$ for $M_2$ such that for $1 \leq i \leq g$

$$Fe_i = e_{g+i}, \quad Fe_{g+i} = pe_i,$$

and the non-zero pairings are

$$\langle e_i, e_{D+i} \rangle_2 = -\langle e_{D+i}, e_i \rangle_2 = \frac{1}{p},$$

$$\langle e_{g+i}, e_{g+D+i} \rangle_2 = -\langle e_{g+D+i}, e_{g+i} \rangle_2 = 1,$$

for $1 \leq i \leq D$.

Proof. Use the same argument of Lemma 4.2 (2).

6.3. Let $H$ be the moduli space over $\mathbb{Z}(p)[\zeta_N]$ which parametrizes equivalence classes of objects $(\varphi: A_1 \to A_2)$, where

- $A_1 = (A_1, \lambda_1, \eta_1)$ is an object in $A_{g,1,N}$,
- $A_2 = (A_2, \lambda_2, \eta_2)$ is an object in $A_{g,p,D,N}$, and
- $\varphi: A_1 \to A_2$ is an isogeny of degree $p^D$ satisfying $\varphi^*\lambda_2 = p\lambda_1$ and $\varphi^*\eta_1 = \eta_2$.

The moduli space $H$ with two natural projections gives the following correspondence:

\[
\begin{array}{ccc}
H & \xrightarrow{pr_1} & A_{g,1,N} \\
\downarrow & & \downarrow \\
A_{g,p,D,N} & \xleftarrow{pr_2} & H
\end{array}
\]

Let $S$ be the supersingular locus of $H \otimes \overline{\mathbb{F}}_p$, which is the reduced closed subscheme consisting of supersingular points (either $A_1$ or $A_2$ is supersingular, or equivalently both are supersingular). Restricting the natural projections on $S$, we have the following correspondence

\[
\begin{array}{ccc}
S & \xrightarrow{pr_1} & S_{g,1,N} \\
\downarrow & & \downarrow \\
S_{g,p,D,N} & \xleftarrow{pr_2} & S
\end{array}
\]
Suppose that \( A_2 \in \Lambda^*_{g,p,N} \). Let \((\varphi : A_1 \to A_2)\) be a point in the pre-image \( \text{pr}^{-1}_2(A_2) \), and let \((M_1, \langle , \rangle_1)\) be the Dieudonné module associated to \( A_1 \). We have
\[
M_1 \subset M_2, \quad p\langle , \rangle_2 = \langle , \rangle_1, \quad FM_2 = M_2^1.
\]
Since \( A_2 \) is superspecial and \( \langle , \rangle_1 \) is a perfect pairing on \( M_1 \), we get
\[
FM_2 = VM_2 = M_2^1, \quad M_1^1 \subset M_1 = M_2.
\]
Therefore, we have
\[
FM_2 = VM_2 \subset M_1 \subset M_2, \quad \dim_k M_1/VM_2 = D.
\]
Put \( \langle , \rangle := p\langle , \rangle_2 \). The pairing
\[
\langle , \rangle : M_2 \times M_2 \to W
\]
duces a pairing
\[
\langle , \rangle : M_2/VM_2 \times M_2/VM_2 \to k
\]
which is perfect (by Lemma 6.2). Furthermore, \( M_1/VM_2 \) is a maximal isotropic subspace for the pairing \( \langle , \rangle \). This is because \( \langle , \rangle_1 \) is a perfect pairing on \( M_1 \) and \( \dim M_1/VM_2 = D \) is the maximal dimension of isotropic subspaces. We conclude that the point \((\varphi : A_1 \to A_2)\) lies in \( \text{pr}^{-1}_2(A_2) \) if and only if \( VM_2 \subset M_1 \subset M_2 \) and \( M_1/VM_2 \) is a maximal isotropic subspace of the symplectic space \((M_2/VM_2, \langle , \rangle)\).

By Lemma 6.1, we have proved

**Proposition 6.3.** Let \( A_2 \) be a point in \( \Lambda^*_{g,p,N} \):

1. The pre-image \( \text{pr}^{-1}_2(A_2) \) is naturally isomorphic to the projective variety \( L(D, 2D) \) over \( k \).
2. The set \( \text{pr}^{-1}_1(\Lambda^*_{g,N}) \cap \text{pr}^{-1}_2(A_2) \) is in bijection with \( L(D, 2D)(\mathbb{F}_p^2) \), where the \( W(\mathbb{F}_p) \)-structure of \( M_2 \) is given by the skeleton \( \tilde{M}_2 \).

### 6.4. We compute \( \text{pr}^{-1}_1(\Lambda^*_1) \cap \text{pr}^{-1}_2(A_2) \) for a point \( A_2 \) in \( \Lambda^*_{g,N} \). Let \( T \) be the closed subscheme of \( S \) consisting of the points \((\varphi : A_1 \to A_2)\) such that \( \ker \lambda_2 = A_2[F] \). We compute the closed subvariety \( \text{pr}^{-1}_1(A_1) \cap T \) first.

Let \( A_1 \in \Lambda^*_{g,N} \), and let \((\varphi : A_1 \to A_2) \in S(k)\) be a point in the pre-image \( \text{pr}^{-1}_1(A_1) \). Let \((M_1, \langle , \rangle_1)\) and \((M_2, \langle , \rangle_2)\) be the Dieudonné modules associated to \( A_1 \) and \( A_2 \), respectively.

One has
\[
M_1^1 = M_1 \supset M_2^1 = FM_2,
\]
and thus has
\[
FM_2 \subset M_1 \subset M_2 \subset p^{-1}VM_1.
\]
Since \( M_1 \) is superspecial, \( p^{-1}VM_1 = p^{-1}FM_1 \). Put \( M_0 := p^{-1}VM_1 \) and \( \langle , \rangle := p\langle , \rangle_2 \). We have
\[
pM_0 \subset M_1^0 = FM_2 \subset M_1 = VM_0 \subset M_2 \subset M_0,
\]
and that \( \langle , \rangle \) is a perfect pairing on \( M_0 \). By Proposition 6.1 of [8] (cf. Lemma 4.2 (1)), there is a \( W \)-basis \( f_1, \ldots, f_g \) for \( M_0 \) such that for \( 1 \leq i \leq g \)
\[
Ff_i = f_{g+i}, \quad Ff_{g+i} = pf_i,
\]
and the non-zero pairings are
\[
\langle f_i, f_{g+i} \rangle = -\langle f_{g+i}, f_i \rangle = \beta_i, \quad \forall 1 \leq i \leq g
\]
where \( \beta_1 \in W(\mathbb{F}_{p^2})^\times \) with \( \beta_2 = -\beta_1 \). In the vector space \( M_2 := M_0/pM_0, \) \( M_2 \) is a vector subspace over \( k \) of dimension \( g + D \) with

\[
M_2 \supset \nabla M_0 = k < f_{g+1}, \ldots, f_2 > \quad \text{and} \quad (M_2, F M_2) = 0.
\]

We can write

\[
M_2 = k < v_1, \ldots, v_D > + \nabla M_0, \quad v_i = \sum_{r=1}^{g} a_{ir} f_r.
\]

One computes

\[
\langle v_i, F v_j \rangle = \left( \sum_{r=1}^{g} a_{ir} f_r, F \left( \sum_{q=1}^{g} a_{jq} f_q \right) \right) = \left( \sum_{r=1}^{g} a_{ir} f_r, \sum_{q=1}^{g} a_{jq}^p f_{g+q} \right) = \beta_1 \sum_{r=1}^{g} a_{ir} a_{jr}^p.
\]

This computation leads us to the following definition.

6.5. Let \( V := \mathbb{F}_{p^2}^{2n} \). For any field \( K \supset \mathbb{F}_{p^2} \), we put \( V_K := V \otimes_{\mathbb{F}_{p^2}} K \) and define a pairing on \( V_K \)

\[
\langle \cdot, \cdot \rangle' : V_K \times V_K \to K, \quad \langle (a_i), (b_i) \rangle' := \sum_{i=1}^{2n} a_i b_i^p.
\]

Let \( X(n, 2n) \subset \text{Gr}(n, 2n) \) be the subvariety over \( \mathbb{F}_{p^2} \) which parametrizes \( n \)-dimensional (maximal) isotropic subspaces in \( V \) with respect to the pairing \( \langle \cdot, \cdot \rangle \).

With the computation in Subsection 6.4 and Lemma 6.1 we have proved

**Proposition 6.4.** Let \( A \) be a point in \( \Lambda_{g,1,N} \) and \( g = 2D \).

1. The intersection \( \text{pr}_1^{-1}(A) \cap \mathcal{T} \) is naturally isomorphic to the projective variety \( X(n, 2n) \) over \( k \).
2. The set \( \text{pr}_1^{-1}(A) \cap \text{pr}_2^{-1}(\Lambda_{g,p,N}^*) \) is in bijection with \( X(D, 2D)(\mathbb{F}_{p^2}) \).

When \( g = 2 \), Proposition 6.4 (2) is a result of Moret-Bailly (Lemma 4.3).

6.6. To compute \( X(n, 2n)(\mathbb{F}_{p^2}) \), we show that it is the set of rational points of a homogeneous space under the quasi-split group \( U(n,n) \).

Let \( V = \mathbb{F}_{p^2}^{2n} \) and let \( x \mapsto \bar{x} \) be the involution of \( \mathbb{F}_{p^2} \) over \( \mathbb{F}_p \). Let \( \psi((x_i), (y_i)) = \sum x_i \bar{y}_i \) be the standard hermitian form on \( V \). For any field \( K \supset \mathbb{F}_p \), we put \( V_K := V \otimes_{\mathbb{F}_p} K \) and extend \( \psi \) to a from

\[
\psi : V_K \otimes V_K \to \mathbb{F}_{p^2} \otimes K
\]

by \( K \)-linearity.

Let \( U(n,n) \) be the group of automorphisms of \( V \) that preserve the hermitian form \( \psi \). Let \( LU(n, 2n)(K) \) be the space of \( n \)-dimensional (maximal) isotropic \( K \)-subspaces in \( V_K \) with respect to \( \psi \). We know that \( LU(n, 2n) \) is a projective scheme over \( \mathbb{F}_p \) of finite type, and this is a homogeneous space under \( U(n,n) \). It follows from the definition that

\[
LU(n, 2n)(\mathbb{F}_p) = X(n, 2n)(\mathbb{F}_{p^2}).
\]

However, the space \( LU(n, 2n) \) is not isomorphic to the space \( X(n, 2n) \) over \( k \).
Let $\tilde{\Lambda}$ be the subset of $\mathcal{S}$ consisting of elements $(\varphi : A_1 \to A_2)$ such that $A_2 \in \Lambda_{g,p,D,N}^*$ and $A_1 \in \Lambda_{g,1,N}$. We have natural projections

\[
\begin{array}{c}
\Lambda_{g,1,N} \xrightarrow{\text{pr}_1} \tilde{\Lambda} \xrightarrow{\text{pr}_2} \Lambda_{g,p,D,N}^*.
\end{array}
\]

By Propositions 6.3 and 6.4 and Subsection 6.6, we have proved Proposition 6.5.

Notation as above, one has

\[
|\tilde{\Lambda}| = |L(D, 2D)(\mathbb{F}_p^2)| \cdot |\Lambda_{g,p,D,N}^*| = |LU(D, 2D)(\mathbb{F}_p)| \cdot |\Lambda_{g,1,N}|.
\]

Theorem 6.6. We have

\[
|\Lambda_{g,p,D,N}^*| = \left| \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right| \cdot \frac{(-1)^{g(g+1)/2}}{2^g} \left( \prod_{i=1}^{g} (1 - 2i) \right) \cdot \prod_{i=1}^{D} (p^{4i-2} - 1).
\]

PROOF. We compute in Section 7 that

\[
|L(D, 2D)(\mathbb{F}_p^2)| = \prod_{i=1}^{D} (p^{2i} + 1),
\]

\[
|LU(D, 2D)(\mathbb{F}_p)| = \prod_{i=1}^{D} (p^{2i-1} + 1).
\]

Using Proposition 6.5, Theorem 3.1 and Lemma 3.2, we get the value of $|\Lambda_{g,p,D,N}^*|$. □

6.7. Proof of Theorem 1.4. By a theorem of Li and Oort (Theorem 5.2), we know

\[
|\Pi_0(S_{g,1,N})| = \begin{cases} 
|\Lambda_{g,1,N}| & \text{if } g \text{ is odd;} \\
|\Lambda_{g,p,D,N}^*| & \text{if } g = 2D \text{ is even.}
\end{cases}
\]

Note that the result of Li and Oort is formulated for the coarse moduli space $S_{g,1}$. However, it is clear that adding the level-$N$ structure yields a modification as above. Theorem 1.4 then follows from Theorem 3.1, Lemma 3.2 and Theorem 6.6. □

7. $L(n, 2n)(\mathbb{F}_q)$ and $LU(n, 2n)(\mathbb{F}_q)$

Let $L(n, 2n)$ be the Lagrangian variety of maximal isotropic subspaces in a 2$n$-dimensional vector space $V_0$ with a non-degenerate alternating form $\psi_0$.

Lemma 7.1. $|L(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^{n} (q^i + 1)$.

PROOF. Let $e_1, \ldots, e_{2n}$ be the standard symplectic basis for $V_0$. The group $\text{Sp}_{2n}(\mathbb{F}_q)$ acts transitively on the space $L(n, 2n)(\mathbb{F}_q)$. For $h \in \text{Sp}_{2n}(\mathbb{F}_q)$, the map $h \mapsto \{he_1, \ldots, he_{2n}\}$ induces a bijection between $\text{Sp}_{2n}(\mathbb{F}_q)$ and the set $B(n)$ of ordered symplectic bases $\{v_1, \ldots, v_{2n}\}$ for $V_0$. The first vector $v_1$ has $q^{2n} - 1$ choices. The first companion vector $v_{n+1}$ has $(q^{2n} - q^{2n-1})/(q - 1)$ choices as it does not lie in the hyperplane $v_1^\perp$ and we require $\psi_0(v_1, v_{n+1}) = 1$. The remaining ordered
symplectic basis can be chosen from the complement \( \mathbb{F}_q < v_1, v_{n+1} > \perp \). Therefore, we have proved the recursive formula

\[
|\text{Sp}_{2n}(\mathbb{F}_q)| = (q^{2n} - 1)q^{2n-1}|\text{Sp}_{2n-2}(\mathbb{F}_q)|.
\]

From this, we get

\[
|\text{Sp}_{2n}(\mathbb{F}_q)| = q^{n^2} \prod_{i=1}^{n}(q^{2i} - 1).
\]

Let \( P \) be the stabilizer of the standard maximal isotropic subspace \( \mathbb{F}_q < e_1, \ldots, e_n > \). It is easy to see that

\[
P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : AD^t = I_n, \ BA^t = AB \right\}.
\]

The matrix \( BA^t \) is symmetric and the space of \( n \times n \) symmetric matrices has dimension \((n^2 + n)/2\). This yields

\[
|P| = q^{\frac{n^2+n}{2}} |\text{GL}_n(\mathbb{F}_q)| = q^{n^2} \prod_{i=1}^{n}(q^i - 1)
\]

as one has

\[
|\text{GL}_n(\mathbb{F}_q)| = q^{\frac{n^2-n}{2}} \prod_{i=1}^{n}(q^i - 1).
\]

Since \( L(n, 2n)(\mathbb{F}_q) \cong \text{Sp}_{2n}(\mathbb{F}_q)/P \), we get \(|L(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^{n}(q^i + 1)\).

7.1. Let \( V = \mathbb{F}_q^{2n} \) and let \( x \mapsto \bar{x} = x^q \) be the involution of \( \mathbb{F}_q \) over \( \mathbb{F}_q \). Let \( \psi((x_i), (y_i)) = \sum_i x_i y_i \) be the standard hermitian form on \( V \). For any field \( K \supset \mathbb{F}_q \), we put \( V_K := V \otimes_{\mathbb{F}_q} K \) and extend \( \psi \) to a from

\[
\psi : V_K \otimes V_K \rightarrow \mathbb{F}_q \otimes_{\mathbb{F}_q} K
\]

by \( K \)-linearity.

Let \( U(n, n) \) be the group of automorphisms of \( V \) that preserve the hermitian form \( \psi \). Let \( LU(n, 2n)(K) \) be the set of \( n \)-dimensional (maximal) isotropic \( K \)-subspaces in \( V_K \) with respect to \( \psi \). We know that \( LU(n, 2n) \) is a homogeneous space under \( U(n, n) \). Let

\[
I_m := \{ \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{F}_q^m : Q(a) = 0 \},
\]

where \( Q(a) = a_1^{q+1} + \cdots + a_m^{q+1} \).

**Lemma 7.2.** We have \(|I_m| = q^{2m-1} + (-1)^m q^m + (-1)^{m-1} q^{m-1} \).

**Proof.** For \( m > 1 \), consider the projection \( p : I_m \rightarrow \mathbb{F}_q^m \) which sends \((a_1, \ldots, a_m)\) to \((a_1, \ldots, a_{m-1})\). Let \( I_{m-1}^c \) be the complement of \( I_{m-1} \) in \( \mathbb{F}_q^m \). If \( x \in I_{m-1} \), then the pre-image \( p^{-1}(x) \) consists of one element. If \( x \in I_{m-1}^c \), then the pre-image \( p^{-1}(x) \) consists of solutions of the equation \( a_m^{q+1} = -Q(x) \in \mathbb{F}_q^\times \) and thus \( p^{-1}(x) \) has \( q + 1 \) elements. Therefore, \(|I_m| = |I_{m-1}| + (q + 1)|I_{m-1}^c| \). From this we get the recursive formula

\[
|I_m| = (q + 1)q^{2(m-1)} - q|I_{m-1}|.
\]
We show the lemma by induction. When \( m = 1 \), \(|I_m| = 1\) and the statement holds. Suppose the statement holds for \( m = k \), i.e. \(|I_k| = q^{2k-1} + (-1)^k q^k + (-1)^{k-1} q^{k-1}\). When \( m = k + 1 \),
\[
|I_{k+1}| = (q + 1)q^{2k} - q[q^{2k-1} + (-1)^k q^k + (-1)^{k-1} q^{k-1}]
= q^{2k+1} + (-1)^{k+1} q^{k+1} + (-1)^k q^k.
\]
This completes the proof. ■

**Proposition 7.3.** \(|LU(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^n(q^{2i-1} + 1)\).

**Proof.** We can choose a new basis \( e_1, \ldots, e_{2n} \) for \( V \) such that the non-zero pairings are
\[
\psi(e_i, e_{n+i}) = \psi(e_{n+i}, e_i) = 1, \quad \forall 1 \leq i \leq n.
\]
The representing matrix for \( \psi \) with respect to \( \{e_1, \ldots, e_{2n}\} \) is
\[
J = \begin{pmatrix}
0 & I_n \\
I_n & 0
\end{pmatrix}.
\]
Let \( P \) be the stabilizer of the standard maximal isotropic subspace \( \mathbb{F}_q^2 < e_1, \ldots, e_n > \).

It is easy to see that
\[
|P| = q^{n^2} |GL_n(\mathbb{F}_q)| = q^{2n^2} - n \prod_{i=1}^n(q^{2i} - 1).
\]

We compute \(|U(n, n)(\mathbb{F}_q)|\). For \( h \in U(n, n)(\mathbb{F}_q) \), the map
\[
h \mapsto \{he_1, \ldots, he_{2n}\}
\]
gives a bijection between \( U(n, n)(\mathbb{F}_q) \) and the set \( \mathcal{B}(n) \) of ordered bases \( \{v_1, \ldots, v_{2n}\} \)
for which the representing matrix of \( \psi \) is \( J \). The first vector \( v_1 \) has
\[
|I_{2n}| - 1 = q^{4n-1} + q^{2n} - q^{2n-1} - 1 = (q^{2n} - 1)(q^{2n-1} + 1).
\]
choices (Lemma 7.2). For the choices of the companion vector \( v_{n+1} \) with \( \psi(v_{n+1}, v_{n+1}) = 0 \) and \( \psi(v_1, v_{n+1}) = 1 \), consider the set
\[
Y := \{v \in V; \psi(v_1, v) = 1\}.
\]
Clearly, \(|Y| = q^{4n-2}\). The additive group \( \mathbb{F}_q^2 \) acts on \( Y \) by \( a \cdot v = v + av_1 \) for \( a \in \mathbb{F}_q^2, v \in Y \). It follows from
\[
\psi(v + av_1, v + av_1) = \psi(v, v) + \bar{a} + a
\]
that every orbit \( O(v) \) contains an isotropic vector \( v_0 \) and any isotropic vector in \( O(v) \) has the form \( v_0 + av_1 \) with \( \bar{a} + a = 0 \). Hence, the vector \( v_{n+1} \) has
\[
\frac{|Y|q}{q^2} = q^{4n-3}.
\]
choices. In conclusion, we have proved the recursive formula

\[ |U(n, n)(\mathbb{F}_q)| = (q^{2n} - 1)(q^{2n-1} + 1)q^{4n-3}|U(n - 1, n - 1)(\mathbb{F}_q)|. \]

It follows that

\[ |U(n, n)(\mathbb{F}_q)| = q^{2n^2 - n} \prod_{i=1}^{n} (q^{2i} - 1)(q^{2i-1} + 1). \]

Since \( LU(n, 2n)(\mathbb{F}_q) \simeq U(n, n)(\mathbb{F}_q)/P \), we get \( |LU(n, 2n)(\mathbb{F}_q)| = \prod_{i=1}^{n} (q^{2i-1} + 1). \)

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