Moduli of flat bundles on open Kähler manifolds.

Jean-Luc Brylinski¹, Philip A. Foth

March 19, 2022

Abstract

We consider the moduli space $M_N$ of flat unitary connections on an open Kähler manifold $U$ (complement of a divisor with normal crossings) with restrictions on their monodromy transformations. Using intersection cohomology with degenerating coefficients we construct a natural symplectic form $F$ on $M_N$. When $U$ is quasi-projective we prove that $F$ is actually a Kähler form.

Contents

1 Introduction 1
2 Description of the moduli space 3
3 Deformations of representations of a discrete group 5
4 Intersection cohomology and the construction of the 2-form 8
5 The universal bundle 11
6 Gauge group and $L_2$ bundles 13
7 The 2-form is symplectic 15
8 Projective case 17

1 Introduction

Let $X$ be a compact Kähler manifold and $D$ a divisor on $X$ with normal crossings. There exists a moduli space $M_N$ of flat irreducible unitary bundles on $U = X \setminus D$ such that the monodromy transformation around each smooth irreducible component of $D$ lies in a prescribed conjugacy class in $U(N)$. We

¹The first author was supported in part by NSF grant DMS-9504522.
develop a theory of deformations of representations of a discrete group relative to our situation. We obtain a condition (similar to Goldman-Millson \[15\]) when \( \mathcal{M}_N \) has a manifold structure and further we work under this assumption. If we pick a representation \( \rho \) of \( \pi_1(U) \) satisfying these conditions, then we get a local system \( \tilde{g} \) on \( U \) associated with the representation \( \rho \). Since \( \tilde{g} \) has singularities on \( X \) (along \( D \)) it is quite natural to express the tangent space \( T_\rho M_N \) in terms of the intersection cohomology groups of \( X \) with coefficients in \( \tilde{g} \). Lemma 4.2 shows that this tangent space identifies with the group \( IH^1(X, \tilde{g}) \). We introduce a natural 2-form on \( M_N \) as a pairing \( IH^1(X, \tilde{g}) \times IH^1(X, \tilde{g}) \to \mathbb{R} \).

To see that our form is actually closed, we map our manifold \( M_N \) to an infinite-dimensional affine space such that its tangent space at any point consists of \( L_2 \) forms on \( X \) having certain additional properties. This affine space admits a constant coefficient 2-form such that its pull-back to \( M_N \) is given by a pairing in \( L_2 \) cohomology. We further use the isomorphism between intersection cohomology and \( L_2 \) cohomology constructed by Cattani-Kaplan-Schmid \[10\], and Kashiwara-Kawai \[21\]. As an auxiliary tool we introduce the notion of \( L_2 \) vector bundle, which seems to be of independent interest.

We prove

**THEOREM 1.1** The moduli space \( M_N \) is symplectic.

Intersection cohomology enjoys such properties as Poincaré duality and the Hard Lefschetz theorem. This follows from an unpublished work of Deligne and is partially explained by Zucker in \[28\]. The point is that the Hard Lefschetz theorem holds true for the \( L_2 \) cohomology groups, because they are finite-dimensional, which again is a consequence of the isomorphism between \( L_2 \) and intersection cohomologies. The Hard Lefschetz theorem provides us with an isomorphism \( IH^{d-j}(X, \tilde{g}) \simeq IH^{d+j}(X, \tilde{g}) \), where \( d = \dim_\mathbb{C} X \). This allows us to see that our 2-form is non-degenerate.

We also prove

**THEOREM 1.2** When \( X \) is projective, the moduli space \( M_N \) is Kähler.

When \( X \) is a curve we use the identification of \( M_N \) with the moduli space of stable parabolic vector bundles given by Mehta-Seshadri \[23\] to get a Kähler structure on the space \( M_N \). Then we reduce the case of general projective manifold \( X \) to the case of a curve by taking appropriate number of hyperplane sections of \( X \).

The result was known in some special cases. For example, in the compact case (when \( D \) is empty), the symplectic structure on the moduli space appears in the works of Atiyah-Bott \[6\], Goldman \[16\], and Karshon \[17\]. In the case of a Riemann surface with punctures the symplectic structure on the moduli space was described by Atiyah \[3\], Biquard \[5\] (see also \[4\]), and Witten \[27\]. It was also the subject of a paper by Biswas-Guruprasad \[7\]; a proof in this case using group cohomology is due to Guruprasad-Huebschmann-Jeffrey-Weinstein \[18\].

We would like to thank P. Deligne for his useful comments.
2 Description of the moduli space

Let \( X \) be a compact Kähler manifold of complex dimension \( d \) endowed with a Kähler form \( \lambda \), and let \( D \) be a divisor on \( X \) with normal crossings such that \( D = \bigcup_{i=1}^{r} D_i \) is a decomposition of \( D \) into the union of smooth irreducible complex analytic subvarieties. Let \( G \) be a compact connected Lie group and \( \mathfrak{g} \) its Lie algebra with fixed non-degenerate invariant bilinear form \( \langle \cdot, \cdot \rangle \). From now on we fix the set \( N = (C_1, C_2, \ldots, C_r) \) of \( r \) conjugacy classes in \( G \). On the complement of \( D \) we also have a Kähler form - the restriction of the form \( \lambda \) so that one has an open Kähler manifold \( U := X \setminus D \). We notice that any quasi-projective smooth algebraic variety can be obtained this way.

Take a base point \( b \in U \) and denote by \( \pi_1(U) = \pi_1(U, b) \) the fundamental group of \( U \) with base point \( b \) and by \( \pi_1(X) = \pi_1(X, b) \) the fundamental group of \( X \) with the same base point. Let us define \( \bar{M}_N \) - the moduli space of flat \( G \)-bundles on \( U \) such that the monodromy transformation around \( D_i \) lies in \( C_i \). Later on we will provide the reader with some examples when \( \bar{M}_N \) is actually smooth.

The moduli space \( M \) of flat \( G \)-bundles over \( X \) identifies with the space \( \text{Hom}(\pi_1(X), G)/G \), where the group \( G \) acts by conjugation. As it is well-known \cite{10, 24} the space \( M \) admits a natural symplectic structure. Let \( \rho \) be a smooth point on \( M \). Then the Zariski tangent space at the class of \( \rho \) is \( T_{[\rho]} M = H^1(\pi_1(X), \mathfrak{g}) \), where \( \mathfrak{g} \) is considered as a \( \pi_1(X) \)-module via the adjoint representation followed by \( \rho \). Let \( \tilde{\mathfrak{g}} \) stand for the local system associated with the representation \( \rho \). Due to the isomorphism \( H^1(\pi_1(X), \mathfrak{g}) \simeq H^1(X, \tilde{\mathfrak{g}}) \) and the natural map \( H^2(\pi_1(X), \mathfrak{g}) \to H^2(X, \tilde{\mathfrak{g}}) \) we get the operation

\[
\begin{align*}
H^1(\pi_1(X), \mathfrak{g}) \cup H^1(\pi_1(X), \mathfrak{g}) & \to H^2(\pi_1(X), \mathfrak{g}) \\
H^1(X, \tilde{\mathfrak{g}}) \cup H^1(X, \tilde{\mathfrak{g}}) & \to H^2(X, \tilde{\mathfrak{g}})
\end{align*}
\]

Then we compose

\[
B(\cdot \cup [\lambda]^{d-1}) : H^1(X, \tilde{\mathfrak{g}}) \times H^1(X, \tilde{\mathfrak{g}}) \to \mathbb{R}
\]

to complete the pairing.

Our purpose is to construct a symplectic structure (and, moreover a Kähler structure) on the moduli space \( \mathcal{M}_N \) (at least on its smooth locus) satisfying the following condition: if \( X \) is smooth and \( C_i = 1 \) for each \( i \) (under this assumption of course \( M = \mathcal{M}_N \)) then this form is the one described above. The moduli space \( \mathcal{M}_N \) is the same as \( \text{Hom}_N(\pi_1(U), G)/G \), where the subscript \( N \) means that the prescribed generator of \( \pi_1(U) \) corresponding to a given loop \( \gamma_i \) around \( D_i \) goes to \( C_i \in G \). We will denote by \( \mathcal{M}_N \) the locus of \( \mathcal{M}_N \) corresponding to the irreducible representations of \( \pi_1(U) \).
Remark. The conjugacy classes $C_i$ may be chosen in such a fashion that the moduli space is empty (on $\mathbb{CP}^1$ with one puncture one can take $C \neq \text{Id}$, see also [26]). Also we notice that the obvious map

$$\pi_1(U) \to \pi_1(X)$$

is surjective.

**Proposition 2.1** The Zariski tangent space to the moduli space $M_N$ at the point $\rho$ is

$$T_{[\rho]}M_N = \text{Ker}(H^1(\pi_1(U), \mathfrak{g}) \to \prod_i H^1(\Gamma_i, \mathfrak{g})), $$

where $\Gamma_i \simeq \mathbb{Z}$ is generated by the class of a loop encircling $D_i$.

**Proof.** We shall repeatedly use the fact that for every connected manifold $Z$ the two groups $H^1(Z, \tilde{\mathfrak{g}})$ and $H^1(\pi_1(Z), \mathfrak{g})$ are canonically isomorphic. To understand the tangent space $T_{[\rho]}M_N$ we recall the well-known fact that it is a subspace of the tangent space $T_{[\rho]}(\text{Hom}(\pi_1(U), G)/G)$. Also we note that the tangent space to the conjugacy class $C_g$ of $g \in G$ is the subspace of $g$ given as the range of $\text{Ad}(g) - 1$ if we identify as usual the tangent space $T_gG$ with $g$ via the action of the left translation by $g$. Now if $\gamma_i$ is the class of a loop encircling $D_i$ and if $\Gamma_i \subset \pi_1(U)$ is the cyclic subgroup generated by $\gamma_i$ then the cohomology group $H^1(\Gamma_i, \mathfrak{g})$ is the cokernel of the map $\text{Ad}(g) - 1 : \mathfrak{g} \to \mathfrak{g}$, so the equality follows.

The space $\text{Ker}(H^1(\pi_1(U), \mathfrak{g}) \to \prod_i H^1(\Gamma_i, \mathfrak{g}))$ is called the parabolic cohomology of $X$ with coefficients in the local system $\tilde{\mathfrak{g}}$ on $U$ associated to the action of $\pi_1(U)$ on $\mathfrak{g}$.

We will briefly discuss some of the known results for punctured Riemann surfaces. Here $D_1, ..., D_r$ are just distinct points on $\Sigma$. Let $G_C = SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, or $Sp(2n, \mathbb{C})$ and let $G$ be its standard maximal compact subgroup. Let $\mathcal{C}^r$ be the standard representation space of $G_C$ (so that $r = n$ or $2n$) and consider $V = \oplus_{i=1}^r \wedge^i \mathcal{C}^r$ which is naturally a representation space of $G_C$ too. We say that an element $A \in G_C$ (or its conjugacy class) satisfies property P if two stabilizers have the same dimension: $\dim(V^{A_s}) = \dim(V^T)$, where $T$ is a maximal torus in $G_C$ (or in $G$) and $A = A_s A_n$ is a Jordan decomposition of $A$ into the product of commuting unipotent and semisimple elements. In terms of the eigenvalues $\lambda_1, ..., \lambda_n$ of $A$ property P means that the product $\lambda_{i_1} \cdots \lambda_{i_k} \neq 1$ for any $i_1 < \cdots < i_k$ and $k < n$. For more details we refer to a paper of the second author [14] where it was shown that in the case of a Riemann surface with one puncture the moduli space of flat $SL(n, \mathbb{C})$- (or $SU(n)$-) bundles is smooth if and only if the monodromy transformation around the puncture has
property P. It was also shown that if $G_C = SO(n, \mathbb{C})$, or $Sp(2n, \mathbb{C})$, then the moduli space has at worst quotient singularities (i.e. it is a quotient of a smooth manifold by an action of a finite group).

3 Deformations of representations of a discrete group

We have to make an additional assumption in order to guarantee that each point $[\rho] \in \mathcal{M}_{\mathcal{X}}$ (corresponding to an irreducible representation $\rho$) is a smooth point and the tangent space $T_{[\rho]} \mathcal{M}_{\mathcal{X}}$ to the manifold is given by Proposition 2.1. For this it is enough to show that every infinitesimal deformation (an element of the Zariski tangent space) is tangent to an analytic path in $\mathcal{M}_{\mathcal{X}}$. We recall a theorem of M. Artin [1], which asserts that an infinitesimal deformation is tangent to an analytic path if and only if there exists a formal power series deformation with the infinitesimal deformation as its leading term. In the compact Kähler case, the criterion is established by Goldman-Millson [15]. For example, it is enough to have $H^2(\pi_1(X), \mathfrak{g}) = 0$. Analogously to their methods we will show that in our situation a sufficient condition is the vanishing of the second relative cohomology group $H^2(\pi_1(U), \{\Gamma_i\}, \mathfrak{g})$.

Therefore, for the rest of the paper we will assume that we are dealing with the situation when $H^2(\pi_1(U), \{\Gamma_i\}, \mathfrak{g}) = 0$ and thus we have no obstruction for representing the actual tangent space as the kernel of the map $H^1(\pi_1(U), \mathfrak{g}) \to \prod_i H^1(\Gamma_i, \mathfrak{g})$. In all other situations we only get "formal symplectic structure" and it is a separate problem which will be treated elsewhere to understand its actual meaning.

In the present section we will establish all the results in the following generality. Let $\pi$ be a discrete finitely generated group and let $(\{\Gamma_i\})_{i=1}^r$ be a system of its subgroups such that $\Gamma_i \simeq \mathbb{Z}$ is generated by an element $\gamma_i$. We consider the representation variety $\mathcal{M}_{\mathcal{X}} = \text{Hom}(\pi, G)_{\mathcal{X}}/G$ ($G = U(N)$), consisting of classes of such group homomorphisms that $\text{Im}(\pi)$ is not contained in any proper parabolic subgroup of $G$ (irreducibility condition) and $\text{Im}(\gamma_i) \in \mathcal{C}_i$. We saw that the Zariski tangent space to a point $[\rho_0]$ is given by

$$T_{[\rho_0]} \mathcal{M}_{\mathcal{X}} = \text{Ker}[H^1(\pi, \mathfrak{g}) \to \oplus_i H^1(\Gamma_i, \mathfrak{g})].$$

Our task is to prove

PROPOSITION 3.1 If $H^2(\pi, (\Gamma_i), \mathfrak{g}) = 0$ then for any $\eta \in T_{[\rho_0]} \mathcal{M}_{\mathcal{X}}$ there exists a formal power series $\rho_t$ of $\rho_0$ representing $\eta$.

We devote the rest of the section to the proof of this statement.

Let us have

$$\rho_t(g) = \rho_0(g) \exp(\sum_{i=1}^{\infty} f_i(g) t^i),$$

5
where \( f_i : \pi \to \mathfrak{g} \) is a group 1-cochain and \( t \) is a formal parameter. We recall that \( \mathfrak{g} \) is a \( \pi \) - module via the adjoint representation followed by \( \rho_0 \). We have the following two conditions to satisfy. First, \( \rho_i \) is a group homomorphism, therefore \( \rho_i(g_1 g_2) = \rho_i(g_1) \rho_i(g_2) \). Secondly, the condition of mapping \( \gamma_i \) to fixed conjugacy classes is written as

\[
\rho_i(\gamma_j) = \exp(\sum_{i=1}^{\infty} c^j_i t^i) \rho_0(\gamma_j) \exp(-\sum_{i=1}^{\infty} c^j_i t^i), 
\]

where \( c^j_i \in \mathfrak{g} \). Next we make a change of notation \( h_i(\gamma) = f_i(\gamma^{-1}) \) and using the Campbell-Hausdorff formula spell out these conditions. To the first order of \( t \) we have

\[
\partial h_1(g_1, g_2) = \text{Ad} \rho_0(g_1) h_1(g_2) - h_1(g_1 g_2) + h_1(g_1) = 0, \\
h_1(\gamma_j) = \text{Ad} \rho_0(\gamma_j) c^j_1 - c^j_1 = \partial c^j_1(\gamma_j),
\]

where \( c^j_1 \) is considered as a zero-cochain. As one sees, these equations are equivalent to the fact that \( h_1 \in \text{Ker}[H^1(\pi, \mathfrak{g}) \to \oplus_i H^1(\Gamma_i, \mathfrak{g})] \).

Our purpose is to find such \( h_2, h_3, ... \) and \( c^j_2, c^j_3, ... \) which satisfy those two conditions. We will do this by the induction process. Let us first show explicitly the existence of such \( h_2 \) and \( c^j_2 \). To the second order of \( t \) we have

\[
\partial h_2(g_1, g_2) = -\frac{1}{2}[\text{Ad} \rho_0(g_1) h_1(g_2), h_1(g_1)], \\
h_2(\gamma_j) - \partial c^j_2 = -\frac{1}{2}[\text{Ad} \rho_0(\gamma_j) c^j_1, c^j_1].
\]

We recall that given a group \( \pi \) and a system of its subgroups \( \Gamma_j \) together with the restriction maps \( \text{Map}(\pi, \mathfrak{g}) \to \text{Map}(\Gamma_j, \mathfrak{g}) \) the relative cochain complex is defined as the cone of the system of maps of complexes \( R_j : C^*(\pi, \mathfrak{g}) \to C^*(\Gamma_j, \mathfrak{g}) \). By definition,

\[
\text{Cone}_R^{k}(\Gamma_1, ..., \Gamma_r) = C^k(\pi, \mathfrak{g}) \oplus \bigoplus_j C^{k-1}(\Gamma_j, \mathfrak{g})
\]

with the differential \((-\partial, R_j + \partial)\).

Therefore, we would be able to find such \( h_2 \) and \( c^j_2 \) if the relative group \( H^2(\pi, (\mathfrak{g}_i), \mathfrak{g}) \) vanishes and the right hand side of (3.2) is a relative 2-cocycle. One easily checks that the cocycle condition is satisfied:

\[
\partial[\text{Ad} \rho_0(g_1) h_1(g_2), h_1(g_3)](g_1, g_2, g_3 = 0, \ g_1, g_2, g_3 \in \pi, \ \text{and}
\]

\[
[\text{Ad} \rho_0(\gamma_1) h_1(\gamma_2), h_1(\gamma_3)] = \partial[\text{Ad} \rho_0(g_1) c^j_1, c^j_1](\gamma_1, \gamma_2), \ \gamma_1, \gamma_2 \in \Gamma_j.
\]

(To verify the equalities we use that \( \partial h_1(g_1, g_2) = 0 \) and \( h_1(\gamma_j) = \partial c^j_1(\gamma_j) \).)

This procedure serves several purposes. First, we get an idea that the obstruction for an element of Zariski tangent space to be tangent to an analytic
path lies in the second relative group cohomology. Besides, we notice that in all successive steps we will deal with the system of the form

\[
\begin{align*}
\partial h_{k+1}(g_1, g_2, g_3) &= F(h_1, ..., h_k) \\
h_{k+1}(\gamma_j) - \partial c_{k+1}^j(\gamma_j) &= H(c_1^j, ..., c_k^j)
\end{align*}
\]  

(3.3)

Also, we see that in order to show that the right hand side of these equations is a relative cocycle, we can restrict ourselves to the case of just one subgroup \(\Gamma \subset \pi\) generated by an element \(\gamma\).

We need the following simple

**Lemma 3.2** Let \(\pi\) be a finitely generated discrete group and let \(\gamma \in \pi\). There exists a free group \(F\) with \(g \in F\) and a surjective homomorphism \(\phi : F \to \pi\) such that \(\phi(g) = \gamma\) and \(g\) is an element of a basis of \(F\).

**Proof.** Let \(F'\) be any free group such that there is a surjective map \(\phi' : F' \to \pi\). If \((g_2, ..., g_l)\) is a basis for \(F'\) then we consider a free group \(F\) obtained from \(F'\) by adding one generator \(g\). Then \((g, g_2, ..., g_l)\) is a basis of \(F\) and we let \(\phi(g) = \gamma\) and \(\phi(g_i) = \phi'(g_i)\) for \(2 \leq i \leq l\). \(\square\)

Now we make the inductive step. Let \(F\) be such a free group that satisfies the condition of the above lemma for our group \(\pi\) and its subgroup \(\Gamma\) generated by \(\gamma\). Let \(\Gamma' \simeq \mathbb{Z}\) be the subgroup of \(F\) generated by the element \(g\) from the above lemma. (We notice that \(H^2(F, \Gamma', g) = 0\).) Let us lift the system of equations (3.3) to the free group \(F\). We were able to find a solution of this system up to order \(k\). The character variety of a free group in \(l\) generators \((g, g_2, ..., g_l)\) such that the image of \(g\) goes to \(C \subset G\) is just

\[
\mathcal{C} \times G \times G \times \cdots \times G
\]

and it is non-singular.

Thus there is no obstruction to finding \(\rho_k : F \to G\) up to order \(k+1\), inducing the given \(k\)-th order formal homomorphism, as well as \(C^l_{k+1}\) such that (3.1) holds up to \(T^{k+1}\). We conclude that in the free group \(F\) it is possible to find a solution of the lift of (3.3). This means that the right hand side not only of the lift of (3.3) but also of (3.3) itself is a relative cocycle. It is a relative coboundary as well, because we assumed the vanishing of the group \(H^2(\pi, (\Gamma_i), g)\). Therefore there exist \(h_{k+1}\) and \(c_{k+1}^j\) satisfying the equations (3.3), which completes the inductive step. This finishes the proof of Proposition 3.1.

We refer to results [19] of Kapovich-Millson for another approach to the relative deformation theory, where they work with differential graded algebras of differential forms on a manifold.
4 Intersection cohomology and the construction of the 2-form

We will construct a non-degenerate 2-form on the space $\mathcal{M}_N$. We first consider the dimension 1 case. The case of a Riemann surface $\Sigma$ with punctures is quite simple because we know the explicit structure of the fundamental group and $\Sigma$ is a $K(\pi, 1)$. The latter property allows us to conclude that $H^1(U, \tilde{g}) = H^1(\pi_1(U), g)$. Consider the exact sequence

$$\cdots \to H^1(\text{Cone}, g) \to H^1(\pi_1(U), g) \to \bigoplus_i H^1(\Gamma_i, g) \to H^2(\text{Cone}_g) \to 0,$$

where $\text{Cone}_g$ is the mapping cone for the morphism of complexes

$$C^*(\pi_1(U), g) \to \bigoplus_i C^*(\Gamma_i, g).$$

Similarly one can define $\text{Cone}_{\mathbb{Z}}, \text{Cone}_{\mathbb{R}}$, etc. If we apply the bilinear form $B(\cdot, \cdot)$ together with the pairing in cohomology then we get a map

$$H^i(\text{Cone}_g) \times H^j(\text{Cone}_g) \to H^{i+j}(\text{Cone}_g).$$

**Lemma 4.1** $H^2(\text{Cone}_{\mathbb{Z}}) \simeq \mathbb{Z}$.

**Proof.** Let $\Delta_i$ be a small disk centered at $i$-th marked point. Then $\Delta_i^*$ be obtained from $\Delta_i$ by removing the marked point. We have the class $\gamma'_i \in \mathbb{H}^1(\Gamma_i, \mathbb{Z}) \simeq H^1(\Delta_i^*, \mathbb{Z})$, which can be defined as follows. In terms of group cohomology it corresponds to the map $\phi : \Gamma_i \to \mathbb{Z}$ sending $\gamma_i$ to 1. Under the above isomorphism this class goes to the class defined in $H^1(\Delta_i^*, \mathbb{Z})$ by the loop $\gamma_i$.

Our point is that

$$H^2(\text{Cone}_{\mathbb{Z}}) = \text{Coker}(H^1(\pi_1(U), \mathbb{Z}) \to \bigoplus_i H^1(\Gamma_i, \mathbb{Z}));$$

the cokernel of the map $\alpha$ is $\mathbb{Z}$ and is generated by the class of the element

$$(\gamma'_1, \ldots, \gamma'_r) \in \bigoplus_i H^1(\Delta_i^*, \mathbb{Z}) \simeq \oplus_i H^1(\Gamma_i, \mathbb{Z}).$$

This lemma gives us the idea how to construct a symplectic form on $\mathcal{M}_N$: $T_{[\rho]} \mathcal{M}_N \times T_{[\rho]} \mathcal{M}_N \to \mathbb{R}$, because

$$T_{[\rho]} \mathcal{M}_N = \text{Ker}(H^1(\pi_1(U), g) \to \prod_i H^1(\Gamma_i, g)) = \text{Im}(H^1(\text{Cone}_g) \to H^1(\pi_1(U), g))$$
and we have a natural pairing

$$H^1(\text{Cone}_g) \times H^1(\text{Cone}_g) \xrightarrow{B(\cdot)} H^2(\text{Cone}_R) \to \mathbb{R}. $$

This pairing is clearly non-degenerate.

Now we move on to arbitrary dimensions. Unfortunately, the usual cohomology does not allow us to construct a non-degenerate pairing on the tangent space of $M_N$ when the local system $\tilde{g}$ cannot be extended from $U$ to $X$, i.e. when at least one of the conjugacy classes $C_i$ is not the class of the identity of $G$. That is the main reason why we need to use intersection cohomology instead.

We introduce the following (canonical) filtration on $X$:

$$X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where $X_j$ consists of all points that belong to at least $n - j$ smooth irreducible components $D_i$ of $D$. We notice that $X \setminus X_{n-1} = U$. We will always work with intersection cohomology for the middle perversity.

We define the truncated complex $\tau_j C^\bullet$ for a complex of sheaves $C^\bullet$. It is the complex which in degree $i$ is

- $C^i$ if $i < j$,
- $\text{Ker}(C^j \to C^{j+1})$ if $i = j$,
- $0$ if $i > j$.

**LEMMA 4.2** $\text{Ker}(H^1(\pi_1(U), g) \to \prod_i H^1(\Gamma_i, g)) = IH^1(X, \tilde{g})$.

**Proof.** An important thing is that the codimension of $D$ in $X$ is 1. That is why in the complex $i^! IC^\bullet_X(\tilde{g})$ has its cohomology sheaves only in degree $\geq 2$. If $j : U \hookrightarrow X$ and $i : D \hookrightarrow X$ are the inclusions then one has an exact (distinguished) triangle of complexes (see [8], p.109)

$$i^! IC^\bullet_X(\tilde{g}) \to IC^\bullet_X(\tilde{g}) \xrightarrow{Rj_* j^!} IC^\bullet_X(\tilde{g}) \to i^! IC^\bullet_X(\tilde{g})[1],$$

(4.1)

which gives rise to the long exact sequence in cohomology (we need only a small part of it):

$$\cdots \to H^1(X, i^! IC^\bullet_X(\tilde{g})) \to H^1(X, IC^\bullet_X(\tilde{g})) \to H^1(U, IC^\bullet_U(\tilde{g})) \to H^2(X, i^! IC^\bullet_X(\tilde{g})) \to H^2(X, IC^\bullet_X(\tilde{g})) \to \cdots$$

The vanishing mentioned above implies that $H^1(X, i^! IC^\bullet_X(\tilde{g})) = 0$. Besides, from definition it follows that $H^1(X, IC^\bullet_X(\tilde{g})) = IH^1(X, \tilde{g})$, and since $U$ is non-singular and the local system $\tilde{g}$ is defined on $U$ one has $H^1(U, IC^\bullet_U(\tilde{g})) = H^1(U, \tilde{g})$. All this means that

$$IH^1(X, \tilde{g}) = \text{Ker}(H^1(U, \tilde{g}) \to H^2(X, i^! IC^\bullet_X(\tilde{g}))).$$
We will show that there is a natural injection
\[ H^2(X, i_! i^! IC^*_X(\mathfrak{g})) \hookrightarrow \prod_i H^1(\Gamma_i, \mathfrak{g}), \]
and in view of Proposition [2.1] it is enough to prove the lemma.

First, we consider the case \( r = 1 \), when the divisor \( D \) is smooth and irreducible. We will use Deligne’s construction of the intersection sheaf complex as in [17]. Let us consider a point \( x \in D \subset X \) and a small neighbourhood \( V \) of \( x \). The group \( \Gamma = \pi_1(V \setminus (V \cap D)) \) is isomorphic to \( \mathbb{Z} \) and is generated by a loop \( \gamma \) encircling \( D \). If \( k : V \hookrightarrow X \) is the inclusion of this neighbourhood, then the sheaf \( \mathfrak{k}_x \mathfrak{g} \) has a stalk \( (\mathfrak{k}_x \mathfrak{g})_x = H^0(\pi_1(V \setminus (V \cap D), G) \). Passing to the derived functor and taking the sheaf cohomology we see that a stalk \( H^1(Rk_! \mathfrak{g})_x = H^1(\Gamma, \mathfrak{g}) \), which is the cokernel of \( \rho(\gamma) - Id \) in \( \mathfrak{g} \). (We recall that \( \mathfrak{g} \) is a \( \pi_1(U) \)-module via the adjoint representation followed by \( \rho : \pi_1(U) \to G \).) For \( i > 1 \) the sheaf cohomology groups vanish: \( H^i(Rk_! \mathfrak{g}) = 0 \). We also notice that \( H^1(V \setminus (V \cap D), \mathfrak{g}) = H^1(\pi_1(V \setminus (V \cap D)), \mathfrak{g}) \).

Let \( S \) stand for a local system on \( D \) with a stalk \( S_x = H^1(\Gamma, \mathfrak{g}) \). Now we have \( IC^*_X(\mathfrak{g}) = \tau_0 Rk_! \mathfrak{g} \), since the canonical filtration simply amounts to \( D \subset X \). Thus we get an exact triangle
\[ \cdots \to IC^*_X \to Rk_! \mathfrak{g} \to i_* S[-1] \to IC^*_X[1] \to \cdots, \]
because the mapping cone for \( IC^*_X \to Rk_! \mathfrak{g} \) is homotopic to the complex having in degree 1 the sheaf \( i_* S \) and nothing else. Comparing this triangle with (4.1) we see that \( i_! IC^*_X(\mathfrak{g}) \simeq S[-2] \) and it means that
\[ H^2(D, i^! IC^*_X(\mathfrak{g})) = H^0(D, S), \]
and it injects into \( H^1(\Gamma, \mathfrak{g}) \) by definition of \( S \).

Similarly we can consider the case \( r > 1 \). Here the canonical filtration is \( \cdots \subset M \subset D \subset X \), where the subvariety \( M \) consists of all points that belong to at least 2 irreducible components of \( D \). For any \( x \in M \) we take its small enough neighbourhood \( V \); the fundamental group \( \pi_1(V \setminus (D \cap V)) \simeq \mathbb{Z}^m \), where \( m \) is the number of intersecting components of \( D \) at \( x \). If \( l : M \to X \) is the inclusion of \( M \) into \( X \) then the complex \( l_! l^! IC^*_X(\mathfrak{g}) \) has its cohomology sheaves only in degree \( \geq 3 \), since the codimension of \( M \) in \( X \) is at least 2. Thus from the exact sequence
\[ \cdots \to H^2(X, l_! l^! IC^*_X(\mathfrak{g})) \to IH^2(X, \mathfrak{g}) \to IH^2(X \setminus M, \mathfrak{g}) \to \cdots \]
one concludes that we have an injection
\[ IH^2(X, \mathfrak{g}) \hookrightarrow IH^2(X \setminus M, \mathfrak{g}). \]
Now the same arguments as above show that the group \( H^2(X, i_! i^! IC^*_X(\mathfrak{g})) \) naturally injects into the product \( \prod_i H^1(\Gamma_i, \mathfrak{g}) \). \( \Box \)
The above lemma proves that \( T_\rho M \cong IH^1(X, \tilde{g}) \), and now we are ready to construct a 2-form \( F \) on the space \( M_N \). The idea is as follows. We have the class \([\lambda] \in H^2(X, \mathbb{R})\) of the Kähler form (when \( X \) is a projective variety the class \([\lambda]\) is just the class of a hyperplane section). Now the form \( F \) is given by the pairing

\[
IH^1(X, \tilde{g}) \times IH^1(X, \tilde{g}) \to \mathbb{R}: \quad \langle x, y \rangle = B(x, y \cup \lambda^{d-1}). \tag{4.2}
\]

Here we used the intersection pairing described in [17].

Now we shall make a choice of Riemannian metric on \( U \) with singularities along \( D \). Locally the description of our metric is as follows. Let \( \Delta \) stand for the standard open disc in \( \mathbb{C} \) given by \(|z| < \varepsilon\) for some positive number \( \varepsilon \) and let \( \Delta^* = \Delta \setminus 0 \). The intersection of a small neighbourhood of \( x \in X \) with \( U \) looks like \((\Delta^*)^r \times \Delta^{d-r}\) if \( r \) components of \( D \) meet in \( x \). In local coordinates \( z_1, ..., z_d \) these components are defined by equations \( z_1 = 0, z_2 = 0, ..., z_r = 0 \). On \( \Delta \) we take the metric \( dz d\bar{z} \) and on \( \Delta^* \) there is the Poincaré metric given in polar coordinates \( r, \theta \) by

\[
dr^2 + (r d\theta)^2 \over (r \ln r)^2.
\]

So we assume that on \( U \) we have a metric that is quasi-isometric to this one over any such open set \((\Delta^*)^r \times \Delta^{d-r}\).

It is important to notice that the local system \( \tilde{g} \) is unitary, since the representation \( g \) of \( G \) is unitary. By a well-known theorem ([1], [2]) one has the isomorphism \( IH^1(X, \tilde{g}) \cong H^1_L(U, \tilde{g}) \) between the intersection cohomology and the \( L_2 \)-cohomology with coefficients in \( \tilde{g} \). Therefore when \( i = 1 \) this space carries a pure Hodge structure of weight 1, with only Hodge types \((0,1)\) and \((1,0)\). This induces a complex structure on the tangent space to \( M_N \).

We refer the reader to [9] for a survey of Hodge theory and its relation with \( L_2 \) cohomology.

5 The universal bundle

From now on we shall concentrate on the case when \( G = U(N) \) and \( G_C = GL(N, \mathbb{C}) \). Let us consider a principal flat \( G \)-bundle \( P \) over \( U \); then there exists canonically a holomorphic vector bundle \( \bar{V} \) over \( X \) such that

\[
\bar{V} | U = V := P \times_G \mathbb{C}^N,
\]

where \( \mathbb{C}^N \) is the standard \( G \)-module. The holomorphic bundle \( \bar{V} \) is called the Deligne extension of \( V \) (see [4]). It has the property that the corresponding connection \( \nabla_0 \) has at worst logarithmic singularities along \( D \). The canonical Deligne extension relies on fixing a unit interval: it is assumed that if \( \mu \) is an eigenvalue of \( Res_D(\nabla) \) then \( 0 \leq \text{Re}(\mu) < 1 \).
Over $U$ we have the Lie algebra bundle $\text{End}(V)$ and it extends to the holomorphic bundle $\text{End}(V)$ over $X$.

Let $Z$ be the subvariety of $\text{Hom}(\pi_1(U), G)$ consisting of all irreducible representations with the monodromy transformation around $D_i$ lying in the fixed conjugacy class $C_i$, so that $\mathcal{M}_N = Z/G$. Let us consider the vector bundle $\mathbb{U}$ over $\tilde{Z} \times U$ given by
\[ \mathbb{U} = (\mathbb{C}^N \times \tilde{U} \times Z)/\pi_1(U), \]
where $\tilde{U}$ is the universal covering of $U$ and the action of an element $a \in \pi_1(U)$ is given by
\[ a(x, \tilde{y}, \rho) = (\rho(a)(x), a(\tilde{y}), \rho), \quad x \in \mathbb{C}^N, \quad \tilde{y} \in \tilde{U}. \]

Also one can construct the universal bundle $E$ over the product $\mathcal{M}_N \times U$:
\[ E = (\mathbb{C}^N \times \tilde{U} \times Z)/\pi_1(U) \times G \]
\[ (a, g)(x, \tilde{y}, \rho) = (g \rho(a) g^{-1}(x), a(\tilde{y}), g \rho g^{-1}), \quad g \in G. \]

We will need the following

**Lemma 5.1** For any $x \in D$ there exists a neighbourhood $V$ of $x$ such that as $\rho$ varies in a connected component of $Z$ the local monodromy representation $Z^k \cong \pi_1(V \setminus (V \cap D)) \to G$ does not change (up to conjugacy).

**Proof.** Clearly one only has to show that the map $\xi$:
\[ \{ k - \text{tuples of commuting elements of } G \}/\text{conjugacy} \]
\[ (G \text{ mod conjugacy})^k \]
is a finite map. It is also important to notice that the source of this map is Hausdorff. Let us exhibit this for $k = 2$, because for $k > 2$ the arguments are the same.

Assume that for $a, b, c, d \in G$ such that $[a, b] = [c, d] = 1$ one has $\xi(a, b) = \xi(c, d)$. Due to the fact that we consider pairs up to conjugacy we may assume that $a = c$. Moreover, $b$ is conjugate to $d$. We identify pairs $(a, b)$ and $(a, d)$ if $b$ is conjugate to $d$ by means of an element from $Z(a)$ - the centralizer of $a$. Next, we observe that each $G$-conjugacy class intersects $Z(a)$ by only finitely many $Z(a)$-conjugacy classes. In fact, the number of these classes is bounded from above by the cardinality of the Weyl group of $G$. \( \Box \)

This lemma allows us to prove the next important result.

**Proposition 5.2** There exists a vector bundle $\tilde{\mathbb{U}}$ over $X \times Z$ extending $\mathbb{U} \to U \times Z$ such that for every smooth point $\rho \in Z$ the restriction $\tilde{\mathbb{U}}|_{X \times \rho}$ is the holomorphic Deligne extension.
Proof. The above lemma allows one to construct the bundle $\tilde{U}$ locally near $D \subset X$, because if one picks a point $x \in D$ then in a neighbourhood $V \ni x$ the local representation of $\pi_1(V \setminus (V \cap D))$ does not vary. So one can take the bundle over $V \times Z$ as a pullback of the bundle over $V$ under the projection onto the first coordinate. Now one notices that the Deligne construction is compatible with restrictions to smaller open sets. It means that for neighbourhoods $V_1$ and $V_2$ of two points $x_1, x_2 \in D$ respectively the restrictions of $\tilde{U}$ onto $V_1$ and $V_2$ agree on the intersection $V_1 \cap V_2$. 

Repeating the above arguments verbatim one can prove the existence of the extension $\tilde{E} \to M$ of the universal bundle $E$. In fact, the bundle $\tilde{E}$ is the one we use in the rest of the paper. The bundle $\tilde{E}$ can also be obtained using the quotient of $\tilde{U}$ by the action of $G$. We also notice that if the monodromy representation of $\pi_1(U)$ is irreducible then the corresponding logarithmic connection is stable in the sense of [24].

6 Gauge group and $L_2$ bundles

In this section we introduce an "$L_2$ gauge group" $G$, later we will use it to prove Theorem 1.1.

As before we consider a (holomorphic) vector bundle $\tilde{V}$ on $X$ which is the Deligne extension of the vector bundle $V$ on $U$ with a fixed flat connection $\nabla_0$. The connection $\nabla_0$ has logarithmic poles along the divisor $D$ and it corresponds to an irreducible unitary representation $\rho_0$ of $\pi_1(U)$. This representation sends monodromy transformations around irreducible components of $D$ to the prescribed set $\mathcal{N}$ of conjugacy classes in $U(N)$.

Now we are ready to introduce the gauge group $G$ that acts on the space of connections on $\tilde{V}$. It is the set of smooth unitary automorphisms of the bundle $V$ over $U$ such that if $g \in G$ then $g^{-1}\nabla_0 g$ is a unitary $L_2$ 1-form with coefficients in $End(V)$.

The topology on $G$ is the coarsest topology satisfying the following three conditions. First, for any compact subset $K$ of $U$, the map from $G$ to the Fréchet space of $C^\infty$ sections of $End(V)$ over $K$ should be continuous. Second, our topology is such that the distance function $G \times G \to \mathbb{R}$ defined by

$$\text{dist}(g_1, g_2) = \sup_{x \in U, v \in V_x, ||v|| = 1} ||g_1^{-1}g_2(v) - v||,$$

is continuous. The above supremum is well-defined since every $g \in G$ is bounded. Finally, we require that in our topology the function $G \to \mathbb{R}$ given by the $L_2$-norm of the covariant derivative of $g \in G$ is continuous.

To see explicitly the manifold structure, we will give a description of the algebra $\text{Lie}(G)$ and construct a map from a small neighbourhood of zero in $\text{Lie}(G)$ to a small neighbourhood of $1 \in G$. The algebra $\text{Lie}(G)$ consists of
bounded sections $u$ of $u(V)$ - the unitary Lie algebra bundle corresponding to $V$ such that $\nabla_0 u$ is $L_2$. Now the map $\text{Lie}(G) \to G$ in the neighbourhood of 0 is just the Cayley transform $(\sqrt{-1}I - A)(\sqrt{-1}I + A)^{-1}$.

Let $\{Y_i\}$ be a finite open cover of $D$ in $X$ by contractible open polycylinders $\Delta^d$ so that $Y_i \simeq (Y_i \cap D) \times \Delta$. Next we need to define a hermitian $L_2$ vector bundle over $X$. It is a triple $(V, h, \{B_i\})$, where $V$ is a vector bundle over $U$ with hermitian metric $h$, and $B_i$ is a class modulo $G$ of frames of $V$ over $Y_i \cap U$ such that $B_i$ and $B_j$ agree over the intersection $Y_i \cap Y_j \cap U$. If we have a refinement $\{\tilde{Y}_j\}$ of $\{Y_i\}$ with similar properties then we naturally obtain a triple $(V, h, \{\tilde{B}_j\})$, defining the same hermitian $L_2$ vector bundle. Two hermitian $L_2$ vector bundles are isomorphic if they are isomorphic over a common refinement. We call elements of $B_i$ local $L_2$-frames. We define a section of a hermitian $L_2$ bundle to be a section of $V$ which is square integrable near $D$.

Let $x \in D$ be an arbitrary point of the divisor $D$ and let $Y$ be a small open polycylinder with Poincaré metric containing $x$. Let $f$ be a smooth compactly-supported function on $Y$. Then it is easy to see that $||df||$ in Poincaré metric is bounded from above. This allows us to use partitions of unity in the $L_2$-context.

Let $Z$ be a smooth manifold (possibly with boundary). We introduce the $L_2$ gauge group $G'$, which consists of those smooth maps $g \in \text{Map}(Z \times (X \cap D), G)$, which satisfy

$$\sup_{z \in C} ||\nabla_X g(\cdot, z)|| < \infty,$$

where $C$ is a compact subset of $Z$, $\nabla_X$ is the covariant derivative in $X$-direction, and $|| \cdot ||$ is the global $L_2$-norm on $X$. Let $p_2$ be the projection $Z \times X \to X$. Then $p_2^{-1}(D)$ has real codimension 2 in $Z \times X$ (and still has normal crossings), and the notion of hermitian $L_2$ bundle over $Z \times X$ makes perfect sense. It is a triple $(E, h, \{p_2^{-1}B_i\})$, where $E$ is a vector bundle over $Z \times (X \cap D)$ equipped with a hermitian metric $h$. One can think of a hermitian $L_2$ bundle over $Z \times X$ as of a family of hermitian $L_2$ bundles over $X$ varying smoothly with $z \in Z$.

The important step in our construction is the following Lemma which has a well-known prototype in topological K-theory.

**Lemma 6.1** Let $Z_1, Z_2$ be smooth contractible manifolds and let $\Phi$ be a hermitian $L_2$ bundle over $Z_2 \times X$. Let $a_0$ and $a_1$ be two smooth homotopic maps $Z_1 \to Z_2$. Then $(a_0 \times \text{Id})^* \Phi$ and $(a_1 \times \text{Id})^* \Phi$ are isomorphic as hermitian $L_2$ vector bundles over $Z_1 \times X$.

**Proof.** Let $I$ denote the unit interval and let $L$ be a hermitian $L_2$ bundle over $X \times I \times Z$. Let $K$ be a closed subset of $X \times I$ (actually, we use $K = X \times \{t\}$). The restriction $L_{K \times Z}$ is a hermitian $L_2$ bundle over $K \times Z$. Locally, a section $s$ of this bundle is a vector-valued function on $K \times Z$ which by definition satisfies the following estimates on its $L_2$-norms:

$$||s|| < \infty, \quad ||\nabla_X s|| < \infty.$$
Therefore the Tietze extension theorem can be applied and for each \( x \in X \times I \times Z \) we may find an open set \( Y \subset X \times I \) satisfying \( x \in Y \times Z \) and a section \( s' \) of \( \mathcal{L} \) such that \( s \) and \( s' \) agree over \( (Y \cap K) \times Z \). Since \( X \times I \) is compact, we can find a finite cover by such open sets and apply \( L_2 \) partition of unity to see that the section \( s \) can be extended to \( X \times I \times Z \). To finish the proof, we apply the classical argument as given on page 17 of [4].

Now we pick a small contractible neighbourhood \( W \subset M_N \) of \( \nabla_0 \). Let \( a_0, a_1 : W \times X \to W \times X \) be two homotopic maps, where the map \( a_0 \) is just the identity map and \( a_1 \) is the map \( W \times X \to \{ \nabla_0 \} \times X \) which induces identity on the second coordinate. The bundle we are interested in is \( \tilde{E} \) from Section 5. We see that the bundle \( a_1^*(\tilde{E}) \) is naturally isomorphic to the bundle \( p_2^*\bar{V} \) over \( W \times X \), where \( p_2 : W \times X \to X \) be the projection onto the second coordinate. Now we apply the above Lemma to the maps \( a_0 \) and \( a_1 \) to get an isomorphism

\[ \psi : \tilde{E} \simeq p_2^*\bar{V} \]

of hermitian \( L_2 \) bundles over \( W \times X \). For any \( \rho \in W \) we let \( \psi_\rho \) stand for the restriction of this isomorphism to \( \rho \times X \).

### 7 The 2-form is symplectic.

Our next goal is to show that the form \( F \) constructed earlier is actually closed. For this we shall construct another 2-form \( H \) on a bigger space such that its closeness is apparent and then we shall see that our 2-form \( F \) is equal to a pullback of the form \( H \).

Let \( Af \) be the affine space of all unitary connections \( \nabla \) on \( \bar{V} \) such that the difference \( \nabla - \nabla_0 \) is an \( L_2 \) 1-form with coefficients in \( \text{End}(\bar{V}) \).

For each \( \rho \in W \) we have a pull-back \( \nabla_\rho = \psi_\rho^*\nabla_0 \) of the connection \( \nabla_0 \) in \( V \). The corresponding map \( \beta : W \to Af, \beta(\rho) = \nabla_\rho \) is a smooth embedding.

We define the following 2-form on \( Af \)

\[ H(v_1, v_2) = \int_U Tr(v_1 \cup v_2) \omega^{d-1}, \quad v_1, v_2 \in T_aAf. \tag{7.1} \]

(The integral in question is defined since \( \omega \) is a bounded form on \( U \), and the \( v_j \)'s are \( L_2 \).) The form \( H(.,.) \) is naturally closed, because it is a constant coefficient 1-form over an affine space.

The isomorphism mentioned in Section 4 between \( L_2 \) and intersection cohomologies (with coefficients in \( \bar{g} \)) has the property that corresponding pairings on degree 1 cohomology given by (4.2) and (7.1) correspond to one another. Thus we get

**Lemma 7.1** The pull-back \( \beta^*H(.,.) \) is equal to \( F \).
The pull-back $\beta^*H(\ldots)$ is closed and the point $\nabla_0$ can be chosen to be an arbitrary representation with trivial stabilizer, hence $F$ is closed as well.

**Remark.** Let $\mathcal{A}$ be the space consisting of flat irreducible connections $\nabla$ on $V$ which are compatible with the unitary structure and have the property that the difference $\nabla - \nabla_0 \in A^1(U, \text{End}(\tilde{V}))$ is an $L_2$ 1-form. We also require that $\nabla$ and $\nabla_0$ have the same monodromies (up to conjugation) around the irreducible components of $D$. (It is a mater of simple computations to see that this condition is redundant and follows from the others.) The tangent space $T\nabla_0 \mathcal{A}$ is the space of closed unitary $L^2$-forms on $U$ such that their restriction to a small enough punctured polycylinder $\Delta^r \times (\Delta^*)^{d-r}$ is $\nabla_0$-exact. The smooth structure on $\mathcal{A}$ naturally comes from the fibering

$$G \to \mathcal{A}$$

$$\downarrow \pi$$

$$\mathcal{M}_\mathcal{N}.$$

It comes down to considering the natural smooth map $s : \mathcal{A} \to Af$. The form $H(\ldots)$ on $Af$ is invariant under the action of $G$ and the pull-back $s^*H$ of the form $H(\ldots)$ to $\mathcal{A}$ (which we denote by $H_\mathcal{A}$) is closed too.

The form $H_\mathcal{A}$ is $G$-invariant and vertical, meaning that $H_\mathcal{A}(v_1, v_2) = 0$, where $v_1, v_2 \in T\nabla_0 \mathcal{A}$ and $v_1$ is $\nabla_0$-exact. This shows that the form $H_\mathcal{A}$ descends to a 2-form on $\mathcal{M}_\mathcal{N}$ and it can be seen directly from the definitions that this form coincides with $F$.

To finish the proof of Theorem 1.1 we need

**Lemma 7.2** The form $F$ is non-degenerate.

**Proof.** Let $\rho \in \mathcal{M}_\mathcal{N}$. The 2-form $F$ on $T_{\rho} \mathcal{M}_\mathcal{N} = IH^1(X, \tilde{g})$ is given by

$$IH^1(X, \tilde{g}) \times IH^1(X, \tilde{g}) \to \mathbb{R} : \quad F(x, y) = B(x, y \cup \lambda^{d-1}).$$

It is clearly skew-symmetric; it is also non-degenerate, because the Poincaré duality pairing is non-degenerate and the Hard Lefschetz theorem says that iterated cap-product by $\lambda$ induces an isomorphism

$$IH^{d-k}(X, \tilde{g}) \simeq IH^{d+k}(X, \tilde{g}).$$

Here we use the fact that the Hard Lefschetz theorem holds for $L_2$ cohomology and we recall (Section 4) the isomorphism between $L_2$ and intersection cohomologies. \(\Box\)

Combining this result with our above observation that $F$ is closed, we conclude that $F$ gives a natural symplectic structure on $\mathcal{M}_\mathcal{N}$. This proves Theorem 1.1.
8 Projective case.

In this section we assume that $X$ is a projective manifold and we intend to show that the form $F$ defined in Section 4 is actually a Kähler form. (One can conjecture that this is true under the assumption that $X$ is just a Kähler manifold, but we do not know the proof of this more general fact.)

Let $X \subset \mathbb{C}P^M$ be an embedding of $X$. Let us take a curve $C$ in $X$ obtained by intersection with $d-1$ generic hyperplanes in $\mathbb{C}P^M$. This curve satisfies the following properties. First, $C$ is smooth and, secondly, if $D = \bigcup_{i=1}^r D_i$ is the decomposition of $D$ into the union of irreducible complex analytic subvarieties then $C$ intersects $D_i$ transversally and $C$ does not meet $D_i \cap D_j$ for $i \neq j$. Thus $C \cap D$ is a finite set of points on $C$. Let $S := C \setminus (C \cap D)$ be the open Riemann surface. There is a natural set of conjugacy classes $N'$ in the group $G$ assigned to $S$, which comes from the set $N$. There is also the moduli space $\mathcal{M}_{N'}$ of flat irreducible $G$-bundles on $S$ with monodromies around the punctures defined by the set $N'$. The following result, which is a direct consequence of Lemma 1.4 in [13] comes handy.

**LEMMA 8.1** Let $z \in S$. The natural morphism

$$\pi_1(S, z) \to \pi_1(U, z)$$

is surjective.

This result allows us to get a smooth inclusion:

$$T : \mathcal{M}_N \hookrightarrow \mathcal{M}_{N'}.$$  

If we pick an irreducible representation $\rho$ of $\pi_1(U)$, it gives rise to an irreducible representation $\rho'$ of $\pi_1(S)$. There is an obvious compatibility of images by $\rho$ and $\rho'$ of loops encircling irreducible components of the divisor $D$ and the punctures on $S$ respectively.

The tangent space to $\mathcal{M}_{N'}$ is identified (by Proposition 2.1) with the group $\text{Ker}[H^1(\pi_1(S), g) \to \prod_j H^1(\Gamma'_j, g)]$, where the groups $\Gamma'_j \cong \mathbb{Z}$ are generated by the classes of loops encircling the punctures.

The above Lemma also gives us the existence of injective homomorphisms:

$$H^1(\pi_1(U), g) \to H^1(\pi_1(S), g),$$

$$\prod_i H^1(\Gamma_i, g) \hookrightarrow \prod_j H^1(\Gamma'_j, g)).$$

(Recall that $\Gamma_i$ are generated by the classes of loops encircling irreducible components of the divisor $D$.) Thus we get the following injection

$$\text{Ker}[H^1(\pi_1(U), g) \to \prod_i H^1(\Gamma_i, g)] \hookrightarrow \text{Ker}[H^1(\pi_1(S), g) \to \prod_j H^1(\Gamma'_j, g)].$$
which is equal to \(dT : T_\rho \mathcal{M}_\mathcal{N} \hookrightarrow T_\rho \mathcal{M}_\mathcal{N}'\). (The Lefschetz hyperplane theorem for intersection cohomology \([17]\) gives the same result.) Thus we proved

**PROPOSITION 8.2** The map \(T : \mathcal{M}_\mathcal{N} \rightarrow \mathcal{M}_\mathcal{N}'\) defined above is an embedding.

Let \(F'\) be the 2-form on the moduli space \(\mathcal{M}_\mathcal{N}'\) defined by Equation (4.2) with \(d = 1\). Theorem 1.1 shows that \(F'\) is a symplectic form. We have the following relation between those two 2-forms:

**PROPOSITION 8.3**

\[ F'|_{\mathcal{M}_\mathcal{N}} = F. \]

**Proof.** Let \(Y \hookrightarrow X\) be a generic hyperplane section of \(X\). Then we have a well-defined map

\[ \alpha^* : IH^j(X, \tilde{g}) \rightarrow IH^j(Y, \tilde{g}), \]

(which is the transpose to the map \(\alpha_*\) of 7.1 \([17]\)). Let \(B_X(\langle , \rangle)\) and \(B_Y(\langle , \rangle)\) be the pairings on \(IH^*(X, \tilde{g})\) and \(IH^*(Y, \tilde{g})\) respectively defined as in (4.2). We have

**LEMMA 8.4** Let \(\lambda \in H^2(X)\) be the class of a hyperplane section and let \(a \in IH^j(X, \tilde{g}), b \in IH^{d-j-1}(X, \tilde{g})\). Then

\[ B_X(a, b \cup \langle C \rangle) = B_H(\alpha^*a, \alpha^*b). \]

We notice that when \(k = j = 1\) the pairings in the left and right hand sides are by definition \(F'\) and \(F\). \(\square\)

By a well-known theorem of Mehta and Seshadri \([23]\) the moduli space \(\mathcal{M}_\mathcal{N}'\) identifies with the moduli space of parabolic stable vector bundles and hence it has a natural structure of complex manifold. In view of the isomorphism between intersection and \(L^2\)-cohomologies, the complex structure on \(\mathcal{M}_\mathcal{N}'\) coming from moduli space of stable parabolic bundles coincides with the almost complex structure

\[ J : IH^1(C, \tilde{g}) \rightarrow IH^1(C, \tilde{g}) \]

we have introduced in Section 4. To conclude that \(\mathcal{M}_\mathcal{N}'\) is a Kähler manifold, it remains to notice that if \(v, w \in IH^1(X, \tilde{g})\) then \(F'(Jv, Jw) = F'(v, w)\).

Moreover, since \(C\) is a general curve, the holomorphic \(L^2\) 1-forms on \(U\) restrict to holomorphic \(L^2\) 1-forms on \(S\) (both with coefficients in \(\tilde{g}\)). Thus
\( IH^1(X, \hat{g}) \) is a complex subspace of \( IH^1(C, \hat{g}) \) and Proposition 8.3 follows that \( \mathcal{M}_N \) is a Kähler submanifold of \( \mathcal{M}_N' \). This ends the proof of Theorem 1.2.

Remark. Let us assume that the monodromy around each \( D_i \) is of finite order \( k \). Then the same is true for the corresponding monodromy transformations around the punctures in \( S \). It is well-known (see e.g. [1], [27]) that in this case the form

\[
\omega = \frac{kF'}{4\pi^2}
\]

defines an integral cohomology class \([\omega]\) in \( H^2(\mathcal{M}_N', \mathbb{R}) \). Moreover, there exists a natural holomorphic line bundle \( L' \) on \( \mathcal{M}_N' \) with curvature equal to \(-2\pi\sqrt{-1}[\omega]\). Explicit constructions of \( L' \) can be found, for instance, in [1] and [2].

Using the fact that \( \mathcal{M}_N \) is a Kähler submanifold of \( \mathcal{M}_N' \), we obtain

**Proposition 8.5** Let the monodromy transformation around each \( D_i \) be of finite order \( k \). Then there exists a natural line bundle \( L \) on \( \mathcal{M}_N \) with the curvature equal to \( kF/2\pi\sqrt{-1} \).

The line bundle \( L \) is simply the restriction of \( L' \) to \( \mathcal{M}_N \). The space of \( L_2 \)-sections of \( L \) can serve as a (partial) goal in the geometric quantization program.

References

[1] M. Artin, On the solution of analytic equations, *Inv. Math.*, 5, 1968, 277-291

[2] M. Atiyah and R. Bott, The Yang-Mills Equations over a Riemann Surface, *Phil. Trans. Roy. Soc.*, A308, 523 (1982)

[3] M. Atiyah, *The geometry and physics of knots*, Cambridge U. Press, 1990

[4] M. Atiyah, *K-theory*, Benjamin, New York, 1967

[5] O. Biquard, These, Ecole Polytechnique, 1991

[6] O. Biquard, Fibrés paraboliques stables et connexions singulières plates, *Bull. Soc. math. France*, 119, 1991, 231-257

[7] I. Biswas and K. Guruprasad, Principal bundles on open surfaces and invariant functions on Lie groups, *Int. J. Math.*, 4, 1993, 535-544

[8] A. Borel at al. Intersection Cohomology, Birkhäuser, 1984

[9] J.-L. Brylinski and S. Zucker, An Overview of Recent Advances in Hodge Theory, *Encyclopaedia of Mathematical Sciences*, 69, Several Complex Variables VI, Springer-Verlag, 1990, 39-142
[10] E. Cattani, A. Kaplan and W. Schmid, $L^2$ and intersection cohomologies for a polarizable variation of Hodge structure, *Invent. Math.*, 87, 1987, 217-252

[11] G. Daskalopoulos and R. Wentworth, Geometric Quantization for the Moduli Space of Vector Bundles with Parabolic Structures, preprint, 1992

[12] P. Deligne, Equations Differentielles a Points Singuliers Reguliers, *Lect. Not. in Math.* 163, Springer-Verlag, 1970

[13] P. Deligne, Le Groupe Fundamental du Complément d'une Courbe Plane n'ayant que des Points Doubles Ordinaires Est Abélien (d’après W. Fulton), Sém. Bourbaki 543, Nov. 1979, *Lect. Not. Math.*, 524

[14] P. A. Foth, Geometry of Moduli Spaces of Flat Connections on Punctured Surfaces, preprint, [alg-geom/9703004](http://arxiv.org/abs/alg-geom/9703004), 1996

[15] W. Goldman and J. Millson, Deformations of flat bundles over Kähler manifolds, *Lect. Not. in Pure and Appl. Math.*, 105, Dekker, NY, 1987, 129-145

[16] W. Goldman, The Symplectic Nature of Fundamental Groups of Surfaces, *Adv. in Math.*, 54, 1984, 200-225

[17] M. Goresky and R. MacPherson, Intersection Homology II, *Inventiones Mathematicae* 71, 1983, 77-129

[18] K. Guruprasad, J. Huebschmann, L. Jeffrey and A. Weinstein, Group Systems, Groupoids, and Moduli Spaces of Parabolic Bundles, preprint, [dg-ga/9510006](http://arxiv.org/abs/dg-ga/9510006), 1995

[19] M. Kapovich and J. Millson, The relative deformation theory of representations and flat connections and deformations of linkages in constant curvature spaces, *Compositio Math.*, 103, 1996, 287-317

[20] Y. Karshon, An algebraic proof for the symplectic structure of moduli space, *Proc. AMS*, 116, 3, 1992, 591-605

[21] M. Kashiwara and T. Kawai, The Poincaré lemma for variations of polarized Hodge structures, *Publ. R.I.M.S. Kyoto Univ.*, 23, 1987, 345-407

[22] H. Konno, On the Natural Line Bundle on the Moduli Space of Stable Parabolic Bundles, *Comm. Math. Phys.*, 155, 1993, 311-324

[23] V. Mehta and C. Seshadri, Moduli of Vector Bundles on Curves with Parabolic Structures, *Math. Ann.*, 248, 1980, 205-239

[24] N. Nitsure, Moduli of semistable logarithmic connections, *J. Amer. Math. Soc.*, 6, 1993, 597-609
[25] C. Simpson, Harmonic Bundles on Noncompact Curves, *J. Amer. Math. Soc.*, 3, 1990, 713-770

[26] C. Simpson, Product of Matrices, *Diff. Geom., Global Anal. & Top.*, CMS Conf. Proc., 12, 1992, 157-185

[27] E. Witten, On Quantum Gauge Theories in Two Dimensions, *Comm. Math. Phys.*, 141, 1991, 153-209

[28] S. Zucker, Hodge theory with degenerating coefficients: $L_2$-cohomology in the Poincaré metric, *Ann. Math.*, 109, 1979, 415-476

Department of Mathematics
Penn State University
University Park, PA 16802
jlb@math.psu.edu, foth@math.psu.edu

*AMS subj. class.* primary 32G13