Notes on linearly H-closed spaces and od-selection principles

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Abstract

A space is called linearly H-closed iff any chain cover possesses a dense member. This property lies strictly between feeble compactness and H-closedness. While regular H-closed spaces are compact, there are linearly H-closed spaces which are even collectionwise normal and Fréchet-Urysohn. We give examples in other classes, and ask whether there is a first countable normal linearly H-closed non-compact space in ZFC. We show that PFA implies a negative answer if the space is moreover either locally separable or locally compact and locally ccc. Ostaszewski space (built with ♦) is an example which is even perfectly normal. We also investigate Menger-like properties for the class of od-covers, that is, covers whose members are open and dense.

1 Introduction

This note is mainly about a property (to our knowledge not investigated before) we decided to call linear H-closeness, which lies strictly between H-closeness and feeble compactness. Since it came up while investigating simple instances of od-selection properties (see below), and all have a common ‘density of open sets’ flavor, we included a section about this latter topic, although they are not related more than on a superficial level.

By ‘space’ we mean ‘topological space’. A cover of a space always means a cover by open sets, and a cover is a chain cover if it is linearly ordered by the inclusion. In any Hausdorff space (of cardinality at least 2), each point has a non-dense neighborhood, and thus the space has the property of possessing a cover by open non-dense sets. But the chain-generalization of this property may fail.

Definition 1.1. A space $X$ is linearly H-closed iff any chain cover has a member which is dense in $X$ (or equivalently iff any chain cover has a finite subfamily with a dense union).

Recall that a space any of whose covers has a finite subfamily with a dense union is called $H$-closed, whence the name ‘linearly H-closed’. While H-closed regular spaces are compact (see [18, Corollary 4.8(c)] for a proof), there are plenty of Tychonoff linearly H-closed non-compact spaces, the simplest being maybe the deleted Tychonoff
plank (see Example 2.6). We will give examples in various classes such as first countable, normal, collectionwise normal, etc, but while there are consistent examples of perfectly normal first countable linearly H-closed spaces, we were unable to determine whether a first countable normal linearly H-closed space exists in ZFC alone. A partial result is that PFA prevents such a space to exist if it is moreover either locally separable or locally compact and locally ccc (see Theorem 2.11). These results are contained in Section 2.

In Section 3, we investigate Menger-like properties for od-covers of topological spaces, that is, covers whose members are open and dense. In our short study, we show in particular that the class of spaces satisfying $U_{\text{fin}}(\mathcal{O}, \Delta)$ does contain some Hausdorff spaces but no regular space, and that a separable space satisfies $U_{\text{fin}}(\Delta, \mathcal{O})$ iff it satisfies $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$, where $\Delta$ is the class of od-covers. We defer the definitions of $U_{\text{fin}}(\mathcal{A}, \mathcal{B})$ until Section 3. Research on selection principles (such as Menger-like properties) currently flourishes and sees an impressive flow of new results (see for instance [19, 20] for surveys about recent activity in the field). The author is not an expert of the subject and admits to feeling kind of lost in its numerous subtleties, we shall thus content ourselves with an humble introduction to the class of od-covers, and to derive some basic properties.

For convenience, we now give a grouped definition: the (od-)[linear-]Lin{\ss}dl"of number $L(X)$ (od$L(X)$) ($tL(X)$) of a space $X$ is the smallest cardinal $\kappa$ such that any (od-)[chain] cover of $X$ has a subcover of cardinality $\leq \kappa$. A space is od-compact iff any od-cover has a finite subcover, and we define similarly od-Lindel"of, linearly-Lindel"of, etc. We do not assume separation axioms in any of these properties. Some remarks about od-compact and od-Lindelöf spaces (and the ignorance of past results) are given in Section 3.

A note on separation axioms: in this paper, ‘regular’ and ’normal’ imply ‘Hausdorff’.

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2 Linearly H-closed spaces

In this section, each space is assumed to be Hausdorff, even though it is not needed for every assertion, and we will repeat the assumption often (for emphasis). Any chain cover possesses a subcover indexed by a regular cardinal, for simplicity we will always use such indexing. The first lemma is almost immediate.

**Lemma 2.1.** A space is linearly H-closed iff any infinite cover of it has a subfamily of strictly smaller cardinality with a dense union.

**Proof.** Given a chain cover indexed by a regular cardinal, a subcover of strictly smaller cardinality is contained in some member, so the latter implies the former. If
X is linearly H-closed, given a cover \( \{ U_\alpha : \alpha \in \kappa \} \), then the \( V_\alpha = \bigcup_{\beta < \alpha} U_\beta \) form a chain cover and some \( V_\alpha \) is dense.

Linear H-closeness is linked to other generalized compactness properties, as seen on Figure 1 below. Plain straight arrows denote implications that hold for Hausdorff space (and most of them for any space) while additional properties (for instance those written on their side) are needed for those denoted by dotted curved arrows. Recall that a space is \emph{feebly compact} iff every locally finite cover is finite, or equivalently (for Hausdorff spaces) iff every countable cover of \( X \) has a finite subfamily with a dense union (see [18, Theorem 1.11(b)]); and a space is \emph{pseudocompact} iff any real valued function on it is bounded. All implications in Figure 1 are classical except linearly H-closed \( \rightarrow \) feebly compact and its converses for some (*) whose proofs are given in Lemma 2.2.

\[ \begin{align*}
\text{Compact} & \quad \leftrightarrow \quad \text{Lindelöf} \\
\text{Regular} & \quad \leftrightarrow \quad \text{Lindelöf} \\
\text{Countably compact} & \quad \leftrightarrow \quad \text{Pseudo-compact} \\
\text{H-closed} & \quad \leftrightarrow \quad \text{Feebly compact} \\
\text{Lindelöf} & \quad \leftrightarrow \quad \text{Tychonoff} \\
\text{Linearly H-closed} & \quad (\ast) \\
\end{align*} \]

Figure 1: Some implications for Hausdorff spaces.

We decided to state this lemma in an almost absurd amount of generality, so we need some definitions. The good news is that more readable corollaries do follow quite easily. Given an infinite cardinal \( \kappa \), a space is \emph{initially \( \kappa \)-[linearly] Lindelöf} iff any open [chain] cover of cardinality \( \leq \kappa \) has a countable subcover. Notice that any space is initially \( \omega \)-[linearly] Lindelöf. The \emph{weak Lindelöf number} \( wL(X) \) of a space \( X \) is the least cardinal such that any open cover of \( X \) has a subfamily of cardinality \( \leq \kappa \) whose union is dense. Notice that if \( Y \subset X \) is dense, then \( wL(X) \leq wL(Y) \).

**Lemma 2.2.**

1. A linearly H-closed space is feebly compact.
2. Let \( X \) be a Hausdorff space, \( Y \subset X \) be dense in \( X \), and \( \kappa \) be an infinite cardinal. Assume that \( wL(X) \leq \kappa \) and that either (i) \( X \) or (ii) \( Y \) is both initially \( \kappa \)-linearly Lindelöf and feebly compact. Then \( X \) is linearly H-closed.

**Proof.**

1. Given a countable cover \( U = \{ U_n : n \in \omega \} \) of a linearly H-closed \( X \), set \( V_n = \bigcup_{m \leq n} U_n \), then \( V_n \) is dense for some \( n \), and the result follows.
2. Let \( U = \{ U_\alpha : \alpha \in \lambda \} \) be an infinite chain cover of \( X \), with \( \lambda \) a regular cardinal. Assume first that \( \lambda \leq \kappa \). If (i) holds, there is a countable subfamily that covers \( X \), and then some \( U_\alpha \) is dense in it by feebly compactness. The same is true for \( Y \) if (ii) holds. Then \( X \) is linearly H-closed.
holds, and then \( U_\alpha \) is dense in \( X \) as well. Now, suppose that \( \lambda > \kappa \), since \( wL(X) \leq \kappa \) there is some subfamily of cardinality \( \leq \kappa < \lambda \) whose union is dense in \( X \), and by regularity of \( \lambda \) its union is contained in some \( U_\alpha \).

A case not covered by this lemma is the following triviality.

**Lemma 2.3.** Let \( X \) be an Hausdorff space containing a dense feebly compact linearly Lindelöf subspace \( Y \). Then \( X \) is linearly H-closed.

**Proof.** Given a chain cover of \( Y \), linear Lindelöfness gives a countable subcover and feebly compactness a finite one.

For a cardinal \( \kappa \), a space is \( \kappa \)-cc (or ccc if \( \kappa = \omega \)) iff any disjoint collection of open sets has cardinality at most \( \kappa \). A space with a dense subset of cardinality \( \kappa \) is obviously \( \kappa \)-cc.

**Corollary 2.4.** If \( X \) is Hausdorff and possesses a dense feebly compact ccc subspace \( Y \), then \( X \) is linearly H-closed.

**Proof.** It is well known that a \( \kappa \)-cc space has weak Lindelöf number \( \leq \kappa \), hence \( wL(Y) \leq \omega \). Invoking the vacuousness of the definition, \( Y \) is also initially \( \omega \)-Lindelöf, and the conditions of Lemma 2.2 (2) (ii) are thus fulfilled.

Recall that a space is perfect iff any closed subset is a \( G_\delta \).

**Corollary 2.5.** Let \( X \) be a feebly compact regular perfect space. Then \( X \) is first countable and linearly H-closed.

**Proof.** Lemmas 2.2 and 2.3 in [17] show that if \( \{ p \} \) is a \( G_\delta \) in a feebly compact regular space \( X \), then \( X \) is first countable at \( p \), and moreover, if each closed set in \( X \) is a \( G_\delta \), then \( X \) is ccc.

We can use Lemma 2.2 to obtain simple examples:

**Example 2.6.** There are linearly H-closed Tychonoff spaces of arbitrarily high weak Lindelöf number and cellularity.

**Construction.** A very classical example: the deleted Tychonoff plank of cardinality \( \kappa \). Let us recall the construction and its properties for convenience. Fix a cardinal \( \kappa \). Let \( X \) be the subspace of the product \((\kappa^+ + 1) \times (\kappa + 1)\) given by \( \kappa^+ \times (\kappa + 1) \cup \{ \kappa^+ \} \times \kappa \), where each ordinal is given the order topology.

As a subspace of a compact space, \( X \) is Tychonoff. The cellularity of \( X \) is at least \( \kappa^+ \) since \( \{ \alpha \} \times (\kappa + 1) \) for successor \( \alpha \in \kappa^+ \) is a disjoint collection of open subsets. The cover \( \{ \alpha \times (\kappa + 1) : \alpha \in \kappa^+ \} \cup \{ (\kappa^+ + 1) \times \beta : \beta \in \kappa \} \) shows that \( wL(X) \geq \kappa \).

Since \( (\kappa^+ + 1) \times \kappa \) is the union of \( \kappa \) compact sets and is dense in \( X \), \( wL(X) \leq \kappa \).

Recall that \( \kappa^+ \) with the order topology is initially \( \kappa \)-compact, and so is its product with the compact space \((\kappa + 1)\) (see Theorem 2.2 in [11]). Thus \( Y = \kappa^+ \times (\kappa + 1) \) is in particular feebly compact and initially \( \kappa \)-Lindelöf. This implies that \( X \) is linearly H-closed by Lemma 2.2 (b) (ii).
Of course, these spaces are not first countable. Let us give more elaborate examples. All are ‘classical’ spaces which happen to be linearly H-closed. In the following, we refer to [21] for the definitions of the ‘small’ uncountable cardinals $p, b$, but recall that $\omega_1 \leq p \leq b \leq 2^{\aleph_0}$ and that each inequality may be strict. The diamond axiom ♦ implies the continuum hypothesis $\text{CH}$ and is defined in any book on set theory.

**Examples 2.7.** There are linearly H-closed non-compact spaces with the following additional properties:

(a) (Bell) First countable, Tychonoff, Lindelöf number $\omega_1$.
(b) (Isbell) First countable, locally compact (and thus Tychonoff), perfect.
(c) (Franklin and Rajagopalan, in effect) $(p = \omega_1)$ First countable, locally compact, normal.
(d) (Ostaszewski) ♦ First countable, locally compact, perfectly normal.
(e) Frechet-Urysohn, collectionwise normal.

**Details.** Linear H-closedness follows from Corollary 2.4 in each case except (b) where Corollary 2.5 is used.

(a) M.G. Bell [3, Example 1] constructed a first countable countably compact ccc (non-separable) Tychonoff space $X$. Since $X$ is an increasing union of $\aleph_1$-many compact spaces, it has Lindelöf number $\omega_1$.

(b) Isbell’s space $\Psi$ (see for instance [8, Exercise 5I]), is first countable, perfect, Tychonoff and feebly compact. This space is not countably compact, and thus non-normal.

(c) Franklin and Rajagopalan introduced a class of spaces called $\gamma N$ spaces, which consist of a dense discrete countable set to which is ‘attached’ a copy of $\omega_1$ in such a way that the space is locally compact and normal, with various additional properties depending on the way the attachment is done. The constructions were later simplified and generalized by van Douwen, Nyikos and Vaughan, and a version of $\gamma N$ which is countably compact and first countable can be built iff $p = \omega_1$ (see for instance [13], Theorem 2.1 and Example 3.4, or [15]).

(d) The celebrated Ostaszewski’s space [16]: a first countable, perfectly normal, hereditarily separable, countably compact, locally compact, non-compact space built with ♦.

(e) The sigma-product of $2^{\omega_1}$, i.e. the subspace of the compact space $2^{\omega_1}$ where at most countably many coordinates have value 1, is collectionwise normal, Frechet-Ursohn, countably compact and ccc (see for instance H. Brandsma’s answer here [5]).

More than ZFC is necessary for the construction in (c), see Theorem 2.12 below. Bell’s space in (a) cannot be shown to be locally compact in ZFC by Theorem 2.13 below. It is also not separable, and no separable regular example with Lindelöf number $\omega_1$ can be found in ZFC, as the next lemma shows.

**Lemma 2.8.** A first countable separable linearly H-closed Hausdorff space of Lindelöf number $p$ is H-closed (and thus compact if regular).
Notice the similarity with the fact (proved in [10]) that a regular separable countably compact space of Lindelöf number $< p$ is compact.

**Proof.** A first countable separable space has countable $\pi$-weight, as easily seen. Since $X$ is linearly H-closed, it is feebly compact. A feebly compact space with countable $\pi$-weight and Lindelöf number $< p$ is H-closed (Lemma 3.1 in [17]).

Likewise, Example 2.7 (d) cannot be constructed in $\text{ZFC} + \text{CH}$ alone.

**Lemma 2.9.** It consistent with $\text{ZFC}$ (and even with $\text{ZFC} + \text{CH}$) that a perfectly normal linearly H-closed space is compact. In particular, it follows from $\text{MA} + \neg \text{CH}$.

**Proof.** A linearly H-closed normal space is countably compact, Weiss [22] showed that $\text{MA} + \neg \text{CH}$ implies that a countably compact regular perfect space is compact, and Eisworth [6] showed that this latter result is compatible with $\text{CH}$.

**Question 2.10.** Is there a normal first countable linearly H-closed non-compact space in $\text{ZFC}$?

The following theorem is a partial answer.

**Theorem 2.11.** (PFA) Let $X$ be normal, linearly H-closed and non-compact. If either (a) $X$ is countably tight and locally separable, or (b) $X$ is first countable, locally compact and locally ccc, then $X$ is compact.

We will use the following results in our proof.

**Theorem 2.12.** ([2], Corollary 2)

(PFA) Every separable, normal, countably tight, countably compact space is compact.

**Theorem 2.13.** (Hajnal-Juhász [9])

(MA + ¬CH) Every first countable, locally compact, ccc space is separable.

Recall that (PFA) implies (MA + ¬CH).

**Proof of Theorem 2.11.**

(a) For $\alpha < \omega_1$ we will define open subsets $U_\alpha \subset X$ such that $\overline{U_\beta} \subset U_\alpha$ whenever $\beta < \alpha$. Then $Y = \bigcup_{\alpha < \omega_1} U_\alpha$ is a clopen subset of $X$: openness is immediate, to see that it is closed, notice that given a point $x \in Y$ there is a countable subset of $Y$ having $x$ in its closure by countable tightness. But a countable subset of $Y$ is contained in some $U_\alpha$, so $x \in U_{\alpha + 1} \subset Y$. It follows that $X$ is not linearly H-closed, since no member of the chain cover $\{(X - Y) \cup U_\alpha : \alpha \in \omega_1\}$ is dense in $X$.

To find $U_\alpha$, we proceed by induction, each will be a separable open subset of $X$. Let $U_0$ be any such open separable subset. Assume that $U_\beta$ is defined for each $\beta < \alpha$. Recall that by normality and linear H-closedness $X$ is countably compact. Thus, $Z = \bigcup_{\beta < \alpha} U_\beta$, being separable, is compact by Theorem 2.12. If $Z = X$, then $X$ is compact. Otherwise choose a point $x \notin Z$, cover $\{x\} \cup Z$ by open separable sets and take the union of a finite subcover to obtain a separable $U_\alpha$ properly containing $Z$, in particular $U_\beta \subset U_{\alpha + 1}$ for all $\beta < \alpha$. This defines $U_\alpha$ for each $\alpha < \omega_1$ with the required properties.
(b) We proceed as in (a), defining $U_\alpha$ to be ccc with compact closure. The successor stages are the same, if $\alpha$ is limit then $\bigcup_{\beta<\alpha} U_\beta$, having a dense ccc subspace, is ccc. By Theorem 2.13 it is separable under $\text{MA} + \neg \text{CH}$, and thus compact under $\text{PFA}$.

Note: Theorem 5.4 in [13] seems to indicate that there are models of $\text{MA} + \neg \text{CH}$ or even $\text{PFA}^-$ with separable, locally compact, locally countable, countably compact, countably tight normal spaces, but we do not know to which spaces this assertion refers.

We close this section with some remarks about how far a first countable linearly $H$-closed space is from being sequentially compact. Lemma 2.14 below shows that there are restrictions on the Lindelöf number. (The result seems well known, see the remarks before Problem 359 in [21], but we include the proof for completeness.) We first need some vocabulary. A collection of subsets of $X$ is a discrete collection if each point of $X$ possesses a neighborhood intersecting at most one member of the collection. This implies that given any subcollection, the union of the closures of its members is closed. A space satisfies the condition $wD$ if given any infinite closed discrete subspace $D$ of $X$, there is an infinite $D' \subset D$ which expands to a discrete collection of open sets, that is, for each $x \in D'$ there is an open $U_x \ni x$ such that $\{U_x : x \in D'\}$ is a discrete collection.

**Lemma 2.14.** A regular, first countable, feebly compact, non-compact space is either countably compact or has Lindelöf number $\geq b$.

**Proof.** Let $X$ be regular, non-compact, first countable, and suppose that it is non-countably compact and has Lindelöf number $< b$. Let thus $\{x_n \in X : n \in \omega\}$ be a sequence without accumulation point. Since $X$ is first countable, this subset is closed discrete in $X$. A regular first countable space with Lindelöf number $< b$ satisfies $wD$ (see 3.6 & 3.7 in [14]). Let thus $E$ be infinite and $U_n \ni x_n (n \in E)$ be open such that $\{U_x : x \in D'\}$ is a discrete collection. Then

$$V_n = X - \bigcup_{m \leq n, m \in E} U_m$$

is a countable cover of $X$ without any dense member, and thus $X$ is not feebly compact.

3 Od-selection properties

No separation axiom is assumed in this section. Allow us first a remark about the od-Lindelöf number. The author proved in [1] that a $T_1$ space is od-compact iff the subspace of non-isolated points is compact, and that a $T_1$ space with od-Lindelöf number $\leq \kappa$ either has a closed discrete subset of cardinality $> \kappa$, or $\ell L(X) \leq \kappa$ whenever $\kappa$ is regular. We made the remark that since the methods were elementary, it would not be a surprise if similar results we were unaware of had appeared.
elsewhere. It was indeed the case: Mills and Wattel [12] had shown that a $T_1$ space without isolated points with $\text{odL}(X) \leq \kappa$ satisfies $L(X) \leq \kappa$ as well, which is much stronger (the compact case is actually due to Katětov in 1947 [7]). Blair [4] later improved their proof. (Both papers actually deal with $[\kappa, \lambda]$-compactness.) In fact, Blair’s proof shows (after a very small modification) the following:

**Theorem 3.1.** (Mills–Wattel and Blair) Let $\kappa$ be an infinite cardinal. Let $X$ be a $T_1$ space with $\text{odL}(X) \leq \kappa$. Then either $X$ contains a clopen discrete subset of cardinality $> \kappa$, or $L(X) \leq \kappa$. Moreover, the subspace of non-isolated points of $X$ has Lindelöf number $\leq \kappa$.

The ‘moreover’ part is not really contained in Blair’s proof but follows easily (see Lemma 2.4 in [1]).

Let us now turn to selections properties. In what follows, $\mathcal{O}, \Delta$ respectively mean the collection of covers and od-covers of some topological space which will be clear from the context. Recall that a cover is an od-cover iff every member is dense. Given collections $\mathcal{A}, \mathcal{B}$ of covers of a space $X$, we define the following property:

$$U_{\text{fin}}(\mathcal{A}, \mathcal{B}) : \text{For each sequence } \langle U_n : n \in \omega \rangle \text{ of members of } \mathcal{A} \text{ which do not have a finite subcover, there are finite } F_n \subset U_n \text{ such that } \{ \cup F_n : n \in \omega \} \in \mathcal{B}. $$

Recall that the classical Menger property is (equivalent to) $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$, and that

$$\sigma\text{-compact } \rightarrow U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \rightarrow \text{Lindelöf}. $$

### 3.1 $U_{\text{fin}}(\mathcal{O}, \Delta)$

Let us first show the following simple lemma.

**Lemma 3.2.** The following equivalences hold for any space $X$.

- (a) Lindelöf & linearly H-closed $\iff$ Lindelöf & H-closed,
- (b) $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$ & linearly H-closed $\iff U_{\text{fin}}(\mathcal{O}, \mathcal{O})$ & H-closed $\iff U_{\text{fin}}(\mathcal{O}, \Delta)$.

Moreover, the properties in (b) imply those in (a).

**Proof.** The moreover part is immediate since $U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \rightarrow \text{Lindelöf}$. Let $\mathcal{U} = \{ U_n : n \in \omega \}$ be a cover of $X$ without a finite subcover. Set $V_n = \cup_{m \leq n} U_m$. Then the $V_n$ form a countable chain cover without a finite subcover and some $V_n$ must be dense.

(b) The leftmost equivalence follows from (a) by Lindelöfness. Let us prove the rightmost equivalence. For the direct implication, let $\mathcal{U}_n$ be a sequence of covers, and let $\mathcal{F}_n \subset \mathcal{U}_n$ be finite such that $\{ \cup F_n : n \in \omega \}$ is a cover of $X$. By H-closedness, we can choose finite $\mathcal{G}_n \subset \mathcal{U}_n$ such that $\cup \mathcal{G}_n$ is dense. Taking $\mathcal{F}_n \cup \mathcal{G}_n$ yields the result. For the converse implication, $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$ trivially holds, so suppose that there is a chain cover $\mathcal{U} = \{ U_n : n \in \omega \}$ without dense member. A finite union of members of $\mathcal{U}_n$ being contained in a member of $\mathcal{U}$, is therefore not dense, taking $\mathcal{U}_n = \mathcal{U}$ for all $n \in \omega$ gives a sequence of open covers violating $U_{\text{fin}}(\mathcal{O}, \Delta)$.

The situation is then very simple for regular spaces:
Proposition 3.3. The following properties are equivalent for regular spaces.
(a) Lindelöf & linearly H-closed,
(b) $\bigcup_{\text{fin}} (\mathcal{O}, \Delta)$,
(c) Compact.

Proof. (b) $\rightarrow$ (a) by Lemma 3.2 and (c) $\rightarrow$ (b) is trivial. Since a regular H-closed space is compact, (a) $\rightarrow$ (c) follows again by Lemma 3.2.

We will show that both (a) $\rightarrow$ (b) and (b) $\rightarrow$ (c) may fail for Hausdorff spaces, that is, we shall exhibit Hausdorff examples of non-compact spaces satisfying $\bigcup_{\text{fin}} (\mathcal{O}, \Delta)$, and Lindelöf linearly H-closed spaces which do not satisfy $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$. H-closed extensions of Hausdorff spaces are well studied (see for instance [18], in particular Proposition 4.5(e)), so it is quite probable that a Lindelöf example can be found in some corner of the literature. But since we found a simple modification of the half disk topology that does the job, we decided to present it here.

Let $X$ be a space equipped with two topologies $\tau, \rho$. Denote by $\hat{X}(\tau, \rho)$ the space whose underlying set is $X \times [0, 1]$ topologized as follows. The topology on $X \times (0, 1]$ is the product topology of $\tau$ and the usual metric topology on $(0, 1]$. Neighborhoods of $\{x, 0\}$ are then defined to be $U \times \{0\} \cup V \times (0, a)$ for $U \in \rho$, $V \in \tau$ with $x \in U \cap V$, and $0 < a \leq 1$.

Lemma 3.4. Assume $\tau \subset \rho$, that is, $\rho$ is finer than $\tau$.
(1) If $X$ is Hausdorff for $\tau$ (and thus for $\rho$), then so is $\hat{X}(\tau, \rho)$.
(2) If $X$ is H-closed for $\tau$, then $\hat{X}(\tau, \rho)$ is H-closed.
(3) If $X$ is compact for $\tau$ and Lindelöf for $\rho$, then $\hat{X}(\tau, \rho)$ is H-closed and Lindelöf.
(4) If $X$ is first countable for both $\tau$ and $\rho$, then so is $\hat{X}(\tau, \rho)$.

Proof. Denote by $\tau \times \mu$ the product topology of $\tau$ on $X$ and the usual metric topology $\mu$ on $[0, 1]$. Notice that if $\tau \subset \rho$, the topology on $\hat{X}(\tau, \rho)$ is finer than $\tau \times \mu$.
(1) Immediate since $\tau \times \mu$ is Hausdorff.
(2) Let $W$ be a cover of $\hat{X}(\tau, \rho)$. For each $x \in X$, fix $W_x \in W$ which contains $\{x, 0\}$. Then for some $V_x \in \tau$ containing $x$, $U \in \rho$ and $a_x \in (0, 1)$, $W_x$ contains $U \times \{0\} \cup V_x \times (0, a)$. Then $\{V_x : x \in X\}$ is a cover of $X$, by H-closeness for $\tau$ we may extract a finite subfamily with a dense union. There is thus some finite $E \subset X$ such that $\cup_{x \in E} W_x$ is dense in $X \times (0, 1)$ for $a = \min_{x \in E} a_x$. H-closeness is productive [18] Proposition 4.8(1)], so $X \times [a/2, 1]$ is H-closed, and thus a finite subfamily of $W$ has a dense union in $X \times (0, 1]$ which is dense in $\hat{X}(\tau, \rho)$.
(3) H-closeness follows as in (2). The other claim holds since $\hat{X}(\tau, \rho) = X \times (0, 1] \cup X \times \{0\}$ and both are Lindelöf.
(4) Straightforward: a neighborhood basis for $\{x, 0\}$ is given by $\{U_n \times \{0\} \cup V_m \times (0, 1/\ell) : \ell, m, n \in \omega\}$, where $U_n, V_n$ are local bases for $x$ in the $\rho$ and $\tau$ topologies.

Notice that in most cases $\hat{X}(\tau, \rho)$ is not regular.

Proposition 3.5. The following holds.
(1) There are Hausdorff H-closed spaces of arbitrary high Lindelöf number.
(2) There is a Hausdorff non-compact first countable space satisfying $\bigcup_{\text{fin}} (\mathcal{O}, \Delta)$. 
(3) There is a first countable Lindelöf H-closed Hausdorff space which does not satisfy \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \).

**Proof.** The three examples are of the form \( \hat{X}(\tau, \rho) \): Hausdorffness, H-closedness, Lindelöfness and first countability in (2) and (3) all follow from Lemma 3.2.

(1) This is well known, but let us give an example anyway. Take \( X \) to be the ordinal \( \kappa + 1 \), \( \tau \) the order topology (which makes it compact) and \( \rho \) the discrete topology. Then \( \hat{L}(\hat{X}(\tau, \rho)) = \kappa \).

(2) Take \( \kappa = \omega \) in (1). Then for each \( \alpha \in \omega + 1 \), \( \{\alpha\} \times [0,1] \) is homeomorphic to \([0,1] \), so \( \hat{X}(\tau, \rho) \) is a \( \sigma \)-compact space which thus satisfies \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \), and we apply Lemma 3.2 to obtain \( U_{\text{fin}}(\mathcal{O}, \Delta) \). Notice that \( \omega + 1 \) is first countable in the order topology.

(3) Take \( X \) to be \([0,1] \), \( \tau \) its usual topology, while \( \rho \) is the refining of \( \tau \) that makes \( \mathbb{Q} \cap [0,1] \) clopen and discrete. Thus, a \( \rho \)-open set is the union of some subset of \( \mathbb{Q} \) and of \( U - \mathbb{Q} \), for \( U \) open for the usual topology. Denote as usual the irrational numbers by \( \mathbb{P} \). It is well known that \( \mathbb{P} \cap [0,1] \) does not satisfy \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \), take a sequence \( \mathcal{U}_n \) of covers of \( \mathbb{P} \cap [0,1] \) witnessing this fact. Then the sequence of covers \( \mathcal{W}_n = \{(U \cup \mathbb{Q}) \times \{0\} \sqcup [0,1] \times \{0,1\} : U \in \mathcal{U}_n\} \) shows that \( \hat{X}(\tau, \rho) \) does not satisfy \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \). \( \square \)

### 3.2 \( U_{\text{fin}}(\Delta, \mathcal{O}) \)

For \( n \in \omega \), \( \Delta_n \) denotes the collection of open covers with at least \( n \) dense members. First, some easy facts.

**Lemma 3.6.** Let \( X \) be a space. The items below are equivalent:

(a) \( X \) satisfies \( U_{\text{fin}}(\Delta, \mathcal{O}) \),
(b) \( X \) satisfies \( U_{\text{fin}}(\Delta, \Delta) \),
(c) \( X \) satisfies \( U_{\text{fin}}(\Delta_1, \mathcal{O}) \),
(d) any closed subset of \( X \) satisfies \( U_{\text{fin}}(\Delta, \mathcal{O}) \),
(e) any closed nowhere dense subset of \( X \) satisfies \( U_{\text{fin}}(\Delta, \mathcal{O}) \).

**Proof.** (c) \( \rightarrow \) (a) \( \leftrightarrow \) (b) are immediate, and (d) \( \rightarrow \) (a) as well.

(a) \( \rightarrow \) (c) Given a sequence of covers such that at least one member \( U(n) \) of each \( \Delta_n \) is dense, set \( \mathcal{V}_n = \{V \cup U(n) : V \in \mathcal{U}_n\} \), then each \( \mathcal{V}_n \) is an od-cover. Applying \( U_{\text{fin}}(\Delta, \mathcal{O}) \) yields (c).

(a) \( \rightarrow \) (d) Let \( Y \subset X \) be closed, any od-cover of \( Y \) yields an od-cover of \( X \) by taking the union of the members with \( X - Y \), and the result follows.

(a) \( \rightarrow \) (e) Let \( Y \subset X \) be closed and nowhere dense. If \( Y \) does not satisfy \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \) take the sequence of covers \( \mathcal{U}_n \) witnessing this fact, set \( \mathcal{V}_n = \{U \cup (X-Y) : U \in \mathcal{U}_n\} \), then \( \mathcal{V}_n \) witnesses that \( X \) does not satisfy \( U_{\text{fin}}(\Delta, \mathcal{O}) \).

(e) \( \rightarrow \) (a) Let \( \mathcal{U}_n \) be od-covers of \( X \) (\( n \in \omega \)), set \( \mathcal{B}_n = \{X - U : U \in \mathcal{U}_n\} \) and take \( B \in \mathcal{B}_0 \). Then \( B \) satisfies \( U_{\text{fin}}(\mathcal{O}, \mathcal{O}) \), and thus (looking at complements) for each \( n \) there is some finite \( \mathcal{F}_n \subset \mathcal{B}_n \) such that

\[
B \cap \bigcap \{\cap \mathcal{F}_n : n \in \omega\} = \emptyset.
\]
Set $G_0 = \mathcal{F}_0 \cup \{B\}$ and $G_n = \mathcal{F}_n$ for $n \geq 1$, then $\bigcap \{\cap G_n : n \in \omega\} = \emptyset$, which implies the result when looking at complements again.

The following proposition settles most of the classical cases (such as sets of reals).

**Proposition 3.7.** Let $X$ be a separable space. Then $X$ satisfies $\mathcal{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ iff $X$ satisfies $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$.

**Proof.** One direction is trivial, so let us assume that $X$ satisfies $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$. Let $D = \{d_i : i \in \omega\}$ be dense in $X$. Given a sequence of open covers $\langle \mathcal{U}_i : i \in \omega \rangle$, take $V_i \in \mathcal{U}_i$ containing $d_i$, since $V = \bigcup_{i \in \omega} V_i$ contains $D$, $X - V$ is closed and nowhere dense and satisfies $\mathcal{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ by Lemma 3.6. Hence there are finite $\mathcal{F}_i \subset \mathcal{U}_i$ such that $\bigcup_{i \in \omega} \cup \mathcal{F}_i \supset X - V$. Include $V_i$ in $\mathcal{F}_i$ to conclude.

Of course, od-compact spaces trivially satisfy $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$. Any non-Lindelöf such space (for instance: an uncountable discrete space) is a trivial example of a space satisfying $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$ but not $\mathcal{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$. But we could not decide on the following question:

**Question 3.8.** Is there a Lindelöf non-od-compact space satisfying $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$ but not $\mathcal{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$?

Another question, inspired by Theorem 3.1:

**Question 3.9.** Let $X$ be a space and $D \subset X$ the subspace of its isolated points. Does the following equivalence hold: $X$ satisfies $\mathcal{U}_{\text{fin}}(\Delta, \mathcal{O})$ $\iff$ $X - D$ satisfies $\mathcal{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$?

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