STOCHASTIC QUANTIZATION OF LIOUVILLE CONFORMAL FIELD THEORY

TADAHIRO OH, TRISTAN ROBERT, NIKOLAY TZVETKOV, AND YUZHAO WANG

Abstract. We study a nonlinear stochastic heat equation forced by a space-time white noise on closed surfaces, with nonlinearity $e^{\beta u}$. This equation corresponds to the stochastic quantization of the Liouville quantum gravity (LQG) measure. (i) We first revisit the construction of the LQG measure in Liouville conformal field theory (LCFT) in the $L^2$ regime $0 < \beta < \sqrt{2}$. This uniformizes in this regime the approaches of David-Kupiainen-Rhodes-Vargas (2016), David-Rhodes-Vargas (2016) and Guillarmou-Rhodes-Vargas (2019) which treated the case of a closed surface with genus 0, 1 and $> 1$ respectively. Moreover, our argument shows that this measure is independent of the approximation procedure for a large class of smooth approximations. (ii) We prove almost sure global well-posedness of the parabolic stochastic dynamics, and invariance of the measure under this stochastic flow. In particular, our results improve previous results obtained by Garban (2020) in the cases of the sphere and the torus with their canonical metric, and are new in the case of closed surfaces with higher genus.

CONTENTS

1. Introduction 2
   1.1. Stochastic quantization of LCFT 2
   1.2. Construction of the LQG measure 3
   1.3. Stochastic dynamics and invariance of the measure 7
   1.4. Scheme of the proof 11

2. Background material 14
   2.1. Basic tools from analysis on manifolds 14
   2.2. Schwartz multipliers and Schauder estimate on $M$ 15
   2.3. Properties of the Green’s function 22
   2.4. Conformal change of metric 33
   2.5. Some nonlinear estimates 39

3. GMC theory and the LQG measure 42
   3.1. On punctured Gaussian multiplicative chaos 42
   3.2. Construction of the LQG measure 50

4. Proof of Theorem 1.4 60
   4.1. Construction of the dynamics 60
   4.2. Invariance of the LQG measure 66

2020 Mathematics Subject Classification. 35K15,60H15,58J35.
Key words and phrases. stochastic heat equation on manifolds; exponential nonlinearity; Liouville Quantum Gravity; Gibbs measure.
Appendix A. Remarks on the LQG and $\exp(\phi)^2$ measures with negative cosmological constant

References

1. Introduction

1.1. Stochastic quantization of LCFT. In this work, we discuss the well-posedness of some stochastic dynamics preserving the Liouville quantum gravity (LQG) measure appearing in the Liouville conformal field theory (LCFT) on a general compact surface. Given a connected, closed (compact, boundaryless), orientable, two-dimensional Riemannian manifold $(\mathcal{M}, g)$, the Liouville action $S_L$ is defined on paths $u : \mathcal{M} \to \mathbb{R}$ by

$$S_L(u; g) \overset{\text{def}}{=} \frac{1}{4\pi} \int_{\mathcal{M}} \left\{ |\nabla_g u|^2 + Q R_g u + 4\pi \nu e^{\beta u} \right\} dV_g,$$

(1.1)

where $R_g$ is the Ricci scalar curvature and $dV_g$ is the volume form. There are three positive parameters, namely the cosmological constant $\nu > 0$, the coupling constant $\beta > 0$ and the charge $Q = \frac{2}{\beta} + \frac{\beta}{2}$. The goal of LCFT is then to compute the $L$-points correlation functions

$$\left\langle \prod_{\ell=1}^L \mathcal{V}_{a_\ell}(x_\ell) \right\rangle \overset{\text{def}}{=} \int \prod_{\ell=1}^L \mathcal{V}_{a_\ell}(x_\ell)(u) e^{-S_L(u)} Du,$$

(1.2)

where the so-called vertex operators $\mathcal{V}_{a_\ell}(x_\ell)$ are given by

$$\mathcal{V}_{a_\ell}(x_\ell)(u) \overset{\text{def}}{=} e^{a_\ell u(x_\ell)},$$

for some points $x_\ell \in \mathcal{M}$ and some coefficients $a_\ell \in \mathbb{R}$. More generally, one wants to give a meaning to

$$\int F(u) \prod_{\ell=1}^L \mathcal{V}_{a_\ell}(x_\ell)(u) e^{-S_L(u)} Du$$

(1.3)

for suitable test functions $F$.

The stochastic quantization of LCFT then consists in constructing a parabolic dynamics given by the stochastic nonlinear heat equation

$$\begin{cases}
\partial_t u - \frac{1}{4\pi} \Delta_g u + \mathcal{N}(u) = \xi_g, \quad (t, x) \in \mathbb{R}_+ \times \mathcal{M} \\
u(t=0) = u_0
\end{cases}$$

(1.4)

for some nonlinear interaction $\mathcal{N}(u)$ and a stochastic forcing term $\xi_g : (\Omega, \mathcal{P}) \to \mathcal{S}'(\mathbb{R} \times \mathcal{M})$ given by a space-time white noise (with $\mathcal{S}'(\mathbb{R} \times \mathcal{M})$ being the space of space-time tempered distributions), such that the corresponding stochastic flow map $\Phi : (u_0, \xi_g) \mapsto u$ satisfies for any $t \geq 0$

$$\int E \left[ F(\Phi(t, u, \xi_g)) \prod_{\ell=1}^L \mathcal{V}_{a_\ell}(x_\ell)(\Phi(t, u_0, \xi_g)) e^{-S_L(\Phi(t, u, \xi_g))} \right] Du_0$$

$$= \int F(u_0) \prod_{\ell=1}^L \mathcal{V}_{a_\ell}(x_\ell)(u_0) e^{-S_L(u_0)} Du_0.$$  

(1.5)
LCFT is a special case of Euclidean quantum field theory (QFT), which aims at reconciling quantum mechanics with special relativity. During the 70’s and the 80’s, a systematic investigation of a minimal set of axioms ensuring the existence of such a theory was performed. In particular, a constructive approach to QFT has been developed through the lens of probability theory, which proved to be widely applicable. Namely, provided that the correlation functions (1.2) satisfy some particular properties, there is then a roadmap allowing one to recover a QFT on Minkowski space. This program was particularly successful to treat QFT with polynomial interactions on $\mathbb{R}^{1+1}$ space-time [58, 25].

In this context, the way to build a “uniform” measure on the set of (random) quantum relativistic fields is naturally to look at Gibbs type measures by putting a Boltzmann weight on the contribution of each admissible field, thus yielding to (1.2). Indeed, at the classical level, the functional $S_L$ in (1.1) is well-known for its role in the proof of the uniformization theorem for compact Riemannian 2-manifolds [8]: in the case of a closed surface $M$ of genus greater than 2, it is possible to find a metric with constant negative curvature on $M$ by looking at the critical points of $S_L$ when $Q$ is replaced by $\frac{2}{\beta}$. The formal measure $e^{-S_L(u)} Du$ can then be seen as a natural generalization of the classical Wiener measure on the set of paths $u : [0,1] \to \mathbb{R}$. This latter is indeed formally given by $e^{-E(u)} Du$ for the energy functional $E(u) = \frac{1}{2} \int_0^1 u(x)^2 dx$, and corresponds to the quantum analogue of the classical trajectories $u : [0,1] \to \mathbb{R}$ given by $u =$constant, which are the critical points of $E$. The value of $Q = \frac{2}{\beta} + \frac{\beta}{2}$ in (1.1) can then be seen as a quantum correction of the classical value $\frac{2}{\beta}$; see also the discussion in the introduction of [39].

LCFT is then a Euclidean QFT in $1+1$ dimension, which moreover possesses some extra symmetries, to wit, conformal invariance. It turns out that its importance goes beyond that of only QFT [43]. Although it was introduced by Polyakov in his seminal work [51] as a theory of random surfaces used to describe string theory and quantum gravity, it proved to be also deeply related to probability, geometry and algebra, as it is conjectured to be the scaling limit of random planar maps [41]; it also appears in super-symmetric Yang-Mills [56, 40]. This explains why it has attracted a lot of attention in the past decades in both the physics and mathematics communities.

1.2. Construction of the LQG measure. Although natural, the expression (1.3) for the LQG measure is merely formal, since $Du$ represents a uniform measure on the set of paths $u : \mathcal{M} \to \mathbb{R}$, which is not properly well-defined. The rigorous construction of expressions such as (1.2) has recently attracted a lot of attention [21, 15, 17, 95, 52, 30, 37]. It is now well-known that a way to define properly (1.2) is to start by interpreting $e^{-S_L(u)} Du$ as a measure with density with respect to a base Gaussian measure. Indeed, let us only consider the quadratic part of the action and look at the expression

$$e^{-\frac{1}{4\pi} \int_M |\nabla u|^2 dV} Du = \Xi \prod_{n \geq 1} \frac{\lambda_n}{2\pi} e^{-\lambda_n^2 u_n^2} \, du_n$$

with the normalisation constant formally given by

$$\Xi = \prod_{n \geq 1} \frac{2\pi}{\lambda_n}. \quad (1.6)$$
Here \( 0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots \) are the eigenvalues of \(-\Delta_g\) associated with an orthonormal basis \(\{\varphi_n\}_{n \geq 0}\) of \(L^2(M,g)\) of eigenfunctions, so that we can decompose \(u : M \to \mathbb{R}\) as

\[
u = \sum_{n \geq 0} u_n \varphi_n.
\]

This allows us to interpret for \(n \geq 1\) the one-dimensional measure \(\frac{\lambda_n}{2\pi} e^{-\lambda_n^2 u_n^2} \, du_n\) as the density of a normal distribution with variance \(\frac{\lambda_n}{2\pi}\). However, due to \(\lambda_0 = 0\), we see that the zero-th frequency is distributed uniformly on \(\mathbb{R}\). Thus we can interpret the formal expression

\[
e^{-\frac{1}{4\pi} \int_M |\nabla u|^2_g \, dV_g} \, du = \Xi d\mu_g \otimes d\overline{X}
\]

by decomposing \(u = X_g + \overline{X}\), were \(d\overline{X}\) is the Lebesgue measure on \(\mathbb{R}\), and \(\mu_g\) is the law of the random variable \(X_g\) given by the so-called (mass-less) Gaussian free field (GFF). Namely, \(\mu_g\) is the Gaussian measure on

\[
H^s_g(M,g) = \left\{ u \in H^s(M,g) = (1 - \Delta_g)^{-\frac{s}{2}} L^2(M), \int_M u dV_g = 0 \right\},
\]

\(s < 0\), with covariance \(2\pi(-\Delta_g)^{s-1}\). In particular we can realise \(X_g\) as

\[
X_g(\omega) = \sum_{n \geq 1} \frac{\sqrt{2\pi} h_n(\omega)}{\lambda_n} \varphi_n,
\]

where \(\{h_n\}_{n \geq 1}\) are iid random variables \(\sim \mathcal{N}(0,1)\) on a probability space \((\Omega, \mathbb{P})\).

This allows us to interpret (1.3) as

\[
\int_{H^s_0(M)} \int_{\mathbb{R}} F(X_g + \overline{X}) \, d\rho_{\{a_\ell, x_\ell\}, g}(X_g, \overline{X}),
\]

where the LQG measure \(\rho_{\{a_\ell, x_\ell\}, g}\) is then formally given by

\[
d\rho_{\{a_\ell, x_\ell\}, g}(X_g, \overline{X}) = \Xi \exp \left\{ \sum_{\ell = 1}^L a_\ell (X_g + \overline{X})(x_\ell) - \frac{Q}{4\pi} \int_M R_g(X_g + \overline{X}) \, dV_g - \nu \int_M e^{\beta (X_g + \overline{X})} \, d\mu_g(X_g) \otimes d\overline{X} \right\}.
\]

(1.9)

There are still some issues to deal with in order to make sense of (1.9). The first one comes from the normalisation constant \(\Xi\) in (1.6). Indeed, in view of Weyl’s law (see (2.1) below) the infinite product in (1.6) does not converge. Still, it is possible to interpret it as

\[
\Xi = \frac{\sqrt{V_g(M)}}{\sqrt{\det'(\Delta_g)}},
\]

(1.10)

where \(\det'(\Delta_g)\) is the determinant of the Laplace-Beltrami operator, defined as a suitable expression of the spectral zeta function \(\zeta(s) = \sum_{n \geq 1} \lambda_n^{-s}\); see for example [49] or the discussion in [30] Section 2.3).

\(^{1}\) The law of the Gaussian free field (GFF) [13] is more often referred to as a Gaussian measure on \(L^2_0(M,g)\) with covariance \(2\pi(-\Delta_g)^{-1}\) in the probability literature; here we want to emphasize that this latter operator is not trace class, which makes the support of \(\mu_g\) actually strictly larger than \(L^2_0(M,g)\).

\(^{2}\) which is not to be confused with the random measure \(e^{\beta X_g} \, dV_g\), which is the Gaussian multiplicative chaos associated with the GFF \(X_g\), and is sometimes also referred to as the LQG measure in the probability literature.
The next issue comes from the roughness of the support of \( \mu_g \); we see that the expressions \( X_g(x_\ell) \) and \( e^{\beta X} \) are still not well-defined. This requires a proper renormalization procedure. For any \( N \in \mathbb{N} \) we therefore look at the truncated measure given by
\[
dp_N(g,X) \overset{\text{def}}{=} Z^{-1}_N(g) R_N(g_X + X) d\mu(g_X) \otimes dX,
\]
where \( Z_N \) is the truncated partition function, and the renormalized truncated density in (1.11) is given by
\[
R_N(u) \overset{\text{def}}{=} \exp \left\{ \sum_{\ell=1}^L \left( a_\ell P_N u(x_\ell) - \frac{a_\ell^2}{2} \left( \log N + 2\pi C_P \right) \right) \right. \\
\left. - \frac{Q}{4\pi} \int_{\mathcal{M}} R_g udV_g - \nu \int_{\mathcal{M}} e^{-\beta \beta} C_P N^{-1} e^\beta P_N u dV_g \right\},
\]
where the regularization operator is
\[
P_N = e^{N^{-2} \Delta g},
\]
and \( C_P \) is a constant which only depends on the choice of the regularization procedure; see Lemma 2.10 below.

The difference with (1.9) comes from the introduction of the “ultraviolet cut-off” \( P_N \) in (1.12) as well as the removal of the divergent terms \( \frac{a_\ell^2}{2} \log N \) and \( e^{\frac{a_\ell^2}{2} \log N} \). We can thus hope to have cured all the small scales divergences in the model, and to recover a non trivial limit when letting the cut-off parameter \( N \to \infty \).

Before stating our first result, we need to make several assumptions on the parameters entering the model.

\[
(L^2 \text{ regime}) \quad 0 < \beta < \sqrt{2} \quad (1.14)
\]
\[
(\text{First Seiberg bound}) \quad \chi(\mathcal{M}) Q < \sum_{\ell=1}^L a_\ell \quad (1.15)
\]
\[
(\text{Integrable insertions}) \quad \max_{\ell=1,...,L} a_\ell < \frac{2}{\beta} \quad (1.16)
\]
where \( \chi(\mathcal{M}) \) in (1.15) is the Euler characteristic of \( \mathcal{M} \). We then have the following result.

**Theorem 1.1.** Let \( L \geq 0 \) and \( a_\ell \in \mathbb{R} \) and \( x_\ell \in \mathcal{M} \), \( \ell = 1,...,L \). Let also \( \nu, \beta > 0 \) and \( Q = \frac{2}{\beta} + \frac{\beta}{\mathcal{X}} \) satisfying \( (1.14) \sim (1.15) \sim (1.16) \). Then:

(i) the truncated measure \( \rho_{N,g} \) in (1.11) converges weakly towards a probability measure \( \rho_{\{a_\ell, x_\ell\}, g} \) on \( H_0^1(\mathcal{M}) \otimes \mathbb{R} \), for any \( s < 0 \), which is absolutely continuous with respect to \( dP_g \otimes dX \), where \( P_g \) is the non-centred Gaussian measure on \( H_0^1(\mathcal{M}) \) with covariance \( (-\Delta_g)^{s-1} \) and mean \( \sum_{\ell=1}^L a_\ell G_g(x_\ell, \cdot) \in H_0^1(\mathcal{M}) \), with \( G_g \) being the Green’s function of the Laplace-Beltrami operator on \( \mathcal{M} \);

(ii) the limit measure \( \rho_{\{a_\ell, x_\ell\}, g} \) is independent of the approximation procedure. More precisely, if we replace \( P_N \) in (1.13) by any Schwartz multiplier \( \psi(-N^{-2} \Delta_g) \) with \( \psi \in \mathcal{S}(\mathbb{R}) \) and \( \psi(0) = 1 \), then the same result holds, and the limit \( \rho_{\{a_\ell, x_\ell\}, g} \) obtained this way coincides with the limit \( \rho_{\{a_\ell, x_\ell\}, g} \) obtained by the approximations (1.11);

(iii) the measure \( \rho_{\{a_\ell, x_\ell\}, g} \) is invariant under conformal change of the metric, in the sense

---

3When \( L = 0 \), we simply do not consider any insertion operator in (1.3). See also Remark 1.2 below.
that for any continuous bounded test function \( F \in C_0(H^1_0(\mathcal{M}) \oplus \mathbb{R}) \) and any smooth metrics \( g, g_0 \) on \( \mathcal{M} \) such that \( g = e^{f_0}g_0 \) for some \( f_0 \in C^\infty(\mathcal{M}) \), we have Weyl’s anomaly

\[
\int_{H^1_0(\mathcal{M},g_0)} \int_{\mathbb{R}} F(X_g + \underline{X}) d\rho_{(a, x_1), g}(X_g, \underline{X})
= \exp \left( \frac{c_L}{96\pi} \int_{\mathcal{M}} (|\nabla_0 f_0|^2 + R_0 f_0) dV_0 - \sum_{\ell=1}^L \left( \frac{Q a_\ell}{2} - \frac{a_\ell^2}{4} \right) f_0(x_\ell) \right)
\times \int_{H^1_0(\mathcal{M},g_0)} \int_{\mathbb{R}} F(X_0 + \underline{X} - \frac{Q}{2} f_0) d\rho_{(a, x_1), g_0}(X_0, \underline{X}),
\]

(1.17)

where \( c_L = 1 + 6Q^2 \) is the central charge of LCFT.

Note that the \( \dot{H}^1 \) norm of \( f_0 \) in Weyl’s anomaly does not depend on the metrics \( g \) or \( g_0 \). See Subsection 2.4.

Several comments are in order. First, concerning the assumptions (1.14)-(1.15)-(1.16), note that we only state convergence of the truncated measure in the “\( L^2 \) regime” (1.14) for the coupling constant \( \beta \), whereas it is known \([15, 17, 30]\) that one can define it in the entire sub-critical regime \( 0 < \beta < 2 \), and even at the endpoint \( \beta = 2 \) \([30, 19, 20, 21]\). However, for the construction of the dynamics in (1.5), our argument requires to control the second moment of the Gaussian multiplicative chaos, which gives the restriction (1.14). See also Remark 3.4 below.

As for the constraint (1.16), it is more restrictive than the second Seiberg bound

\[
\max_{\ell=1, \ldots, L} a_\ell < Q = \frac{2}{\beta} + \frac{\beta}{2}
\]

(1.18)

for which the measure is constructed in [15], with even the endpoint case being tractable [16]. Although the full regime \( 0 < \beta < 2 \) and \( \max_{\ell} a_\ell < Q \) can be obtained from the arguments in \([15, 17, 30]\), our argument for the construction of the dynamics (1.20) does not seem to extend beyond (1.14)-(1.15)-(1.16) at this point. See also Remark 3.4 below.

Let us also mention that we stated the uniqueness of the measure only with respect to the class of approximations described in Theorem 1.1 (ii), which includes the natural regularizations by the heat kernel (1.13) or the “smooth” projection on the finite-dimensional subspace \( \text{Vect}\{\varphi_n, \lambda_n \leq N\} \) corresponding to a smooth truncation of the eigenfunctions expansion (1.8). Still our argument also extends to smoothing operators with kernel \( N^2 \chi(Nd_\gamma(x, y)) \), \( \chi \in C_0^\infty(0, \gamma(\mathcal{M})) \) with \( \gamma(\mathcal{M}) \) being the injectivity radius; see Remark 2.11.

**Remark 1.2.** The Seiberg bounds (1.15) and (1.18) (and so a fortiori the assumption (1.19)) imply that the formal measure \( e^{-S_{\xi(\omega)} Du} \) (i.e. without insertions) can only be made sense of in the case of a surface with negative curvature, for which \( L = 0 \) does not violate the condition (1.15). On the other hand, we see that for the sphere or the torus, this latter measure is not finite since the Seiberg bound (1.15) cannot be satisfied for \( L = 0 \) in these cases (see (3.21) below). Combining (1.15) with (1.18), we see that we need at least \( L \geq 3 \) in order to have a non trivial probability measure \( \rho_{(a, x_1), g} \) on \( \mathcal{M} = S^2 \), whereas in the case of the torus, we need \( L \geq 1 \). Namely, the uniform distribution of the zero mode \( d\overline{X} \) raises an issue in defining the probability measure \( e^{-S_{\xi(\omega)} Du} \) in the case of the sphere and the torus, and we fix this issue by inserting punctures in the
definition of the measure (1.9). On the other hand, the negative curvature is favorable for the measure construction, in the sense that this issue does not appear. Interestingly, this deepens the analogy with the classical situation encountered in the uniformization theorem as mentioned in the introduction. Indeed, due to Gauss-Bonnet theorem (see (2.23) below) it is not possible to find a smooth metric with constant negative curvature on \( S^2 \). However, this can be bypassed by considering metrics with conical singularities \([61]\), and the Seiberg bound (1.15) can then be seen at the analogue of the necessary condition on the solid angles of the singularities in order for such a metric to exist.

**Remark 1.3.** From the perspective of Remark 1.2, the LQG measure corresponds to the minimal choice \( L = 3 \) on \( S^2 \), \( L = 1 \) on \( \mathbb{T}^2 \) or \( L = 0 \) on hyperbolic surfaces. In the case of the sphere, fixing three punctures \( x_1, x_2, x_3 \in S^2 \) to define the LQG measure, the expression (1.2) then actually corresponds to the \((L - 3)\)-correlation function of the LQG measure.

Note that the stochastic quantization procedure as introduced in \([50]\) then only corresponds to (1.23) for the minimal choice of \( L \) used to define the LQG measure.

### 1.3. Stochastic dynamics and invariance of the measure.

We now move on to the construction of a stochastic parabolic dynamics leaving the measure \( \rho_{\{a_{\ell}, x_{\ell}\}, g} \) invariant.

Indeed, as mentioned in Subsection 1.1, the measure (1.2) arises in the probabilistic construction of LCFT as a Euclidean QFT. In recent years, we have also seen a rapid development of the (stochastic) PDE approach to constructive Euclidean QFT. Motivated by the stochastic quantization \([50]\) of the QFT models, a lot of attention has then been devoted to the understanding of singular stochastic parabolic PDEs, with the recent breakthroughs of Hairer \([31]\) through the introduction of regularity structures, and Gubinelli and his collaborators \([28]\) through the development of paracontrolled calculus. Let us mention a recent success of this latter approach \([27]\) where the authors follow through the PDE construction of the Euclidean QFT on \( \mathbb{R}^{1+2} \) Minkowski space with quartic interaction potential (the so-called \( \phi_4^4 \) model).

More recently, other stochastic quantizations procedures have been investigated, namely the elliptic \([1, 2]\) and hyperbolic \([29, 47, 46, 48]\) ones, consisting in looking at the elliptic or hyperbolic counterparts of the stochastic parabolic dynamics (1.4). In particular, in \([48]\), three of the authors of the present paper investigated both the parabolic and hyperbolic stochastic quantizations of the \( \exp(\phi) \) model\(^4\) on \( M = \mathbb{T}^2 \), corresponding to (1.2) without insertions and with a mass term \( mu^2 \) in the action, \( m > 0 \). This latter in particular destructs the conformal invariance property of the measure. We were then able to prove invariance of the measure under the parabolic stochastic dynamics in the \( L^2 \) regime (1.14), and under the hyperbolic dynamics for some regime of \( \beta > 0 \). The same result in the parabolic case also appeared in \([34]\).

In \([23]\), Garban studied a parabolic stochastic dynamics which formally preserves (1.9), and discussed its well-posedness on both \( M = \mathbb{T}^2 \) and \( M = S^2 \) (with \( L = 1 \) and \( L = 3 \) respectively; see Remarks 1.2 and 1.7). However the regime of \( \beta > 0 \) covered in \([23]\) is somehow more restrictive, and in particular convergence of the smooth approximations as well as rigorous invariance of the measure under the flow are only established in some smaller regime.

\(^4\)by analogy with the \( P(\phi) \) model \([57]\) dealing with polynomial interactions. This model is also known as the Høegh-Krohn model \([52]\).
More recently, Dubédat and Shen [18] studied the stochastic Ricci flow, which describes the evolution of the conformal factor for the metric $g = e^{t_0}g_0$ with respect to a fixed metric $g_0$ with constant scalar curvature (see also Subsection 2.3.1)

$$\partial_t f_0 = e^{-t_0} \Delta_0 f_0 + \nu e^{-t_0} \xi_0$$ (1.19)

where $\Delta_0$ is the Laplace-Beltrami operator for the metric $g_0$ and $\xi_0$ is a space-time white noise with respect to $g_0$. See also Remark 1.10 for further discussion.

Motivated by these recent developments, and in view of the formal expression (1.3) with the definition of the Liouville action (1.1), we then look at

$$0$$

where $\nu$ is the noise with respect to $g$ for some constant $\nu$. See also Remark 1.10 for further discussion.

In view of the discussion in the previous subsection, in order to make sense of the dynamics (1.20), we look at an approximate one leaving the truncated measure $\rho$ invariant.

Since we have formally $d\rho_{N,g}(u) = e^{-\tilde{E}_N(u)} du$ where the renormalized energy reads

$$\tilde{E}_N(u) \overset{\text{def}}{=} \frac{1}{4\pi} \int_{\mathcal{M}} \left\{ |\nabla_g P_N u|^2 + Q R_g u + 4 \pi \nu e^{-\pi \beta^2 C_P} N^{-\frac{\beta^2}{4}} e^{\beta P_N u} \right\} dV_g$$

$$- \sum_{\ell=1}^L \left( a_{\ell} P_N u(x_\ell) - \frac{a_{\ell}^2}{2} \left( \log N + 2 \pi C_P \right) \right),$$

we thus consider the associated truncated parabolic equation

$$\partial_t \tilde{u}_N - \frac{1}{4\pi} \Delta_g \tilde{u}_N + \frac{Q}{8\pi} R_g + \frac{1}{2} \nu \beta e^{-\pi \beta^2 C_P} N^{-\frac{\beta^2}{4}} e^{\beta P_N \tilde{u}_N} = \frac{1}{2} \sum_{\ell=1}^L a_{\ell} P_N \delta_{x_\ell} + \xi_g,$$ (1.22)

with initial data $\tilde{u}_0$ distributed by the truncated LQG measure $\rho_{N,g}$. As pointed out by Garban [23], the equation (1.22) is difficult to handle as it is because of the rough deterministic term $\sum_{\ell=1}^L a_{\ell} P_N \delta_{x_\ell}$. So, we first do a Girsanov transform in (1.23) to express the $L$-points correlation function $Z_N(g)$ as (see (3.20) below)

$$Z_N(g) = G_N(g) \mathbb{E} \left[ \int_{\mathcal{H}_0^L(\mathcal{M},g)} \exp \left\{ \sum_{\ell=1}^L a_{\ell} \hat{X}_\ell - \frac{Q}{4\pi} \int_{\mathcal{M}} R_g(X_g + \hat{X}) dV_g \right. \right.$$}

$$- \nu e^{-\pi \beta^2 C_P} N^{-\frac{\beta^2}{4}} e^{\beta \hat{X}} \int_{\mathcal{M}} e^{\beta P_N X_g + 2 \pi \beta \sum_{\ell=1}^L a_{\ell} P_N G_g(x_{\ell}, x_\ell)} d\mu_g(X_g) d\hat{X} \left. \right]$$

(1.23)

for some constant $G_N(g)$. Here $G_g$ is the mean zero Green’s function for $(-\Delta_g)$ on $(\mathcal{M}, g)$ (see (2.11) below), and $(P_N \otimes P_N) G_g$ is then the regularization of $G_g$ in both variables.
Thus, in order to remove the deterministic singular part in (1.22), we first do the change of variable

\[ \tilde{u}_N = u_N + 2\pi \sum_{\ell=1}^{L} a_\ell (P_N \otimes \text{Id}) G_g(x_\ell, x), \tag{1.24} \]

so that \( u_N \) now solves the stochastic equation

\[
\partial_t u_N - \frac{1}{4\pi} \Delta_g u_N + \frac{1}{2} \nu \beta e^{-\pi \beta^2 C_P} N^{\frac{\nu^2}{2}} P_N \left\{ e^{\beta P_N u_N + 2\pi \beta \sum_{\ell=1}^{L} a_\ell (P_N \otimes P_N) G_g(x_\ell, x)} \right\} = -\frac{Q}{8\pi} R_g + \frac{1}{2V_g(M)} \sum_{\ell=1}^{L} a_\ell + \xi_g, \tag{1.25} \]

with initial data

\[ u_N |_{t=0} = \tilde{u}_0 - 2\pi \sum_{\ell=1}^{L} a_\ell (P_N \otimes \text{Id}) G_g(x_\ell, x). \]

Note that this change of variable cannot be seen as a Da Prato - Debussche trick, since the remainder \( \tilde{u}_N \) is not smoother than the original unknown \( u_N \). Instead, it can be seen at the equivalent, at the level of the dynamics, of the Girsanov transform (1.23) performed at the level of the LQG measure. In particular, writing

\[ \sigma_N(x) \overset{\text{def}}{=} \int_{H_0^0(M,g)} |P_N X_g(x)|^2 d\mu_g, \tag{1.26} \]

we have under the new measure

\[ \exp \left( \sum_{\ell=1}^{L} a_\ell P_N X_g(x_\ell) - \frac{a_\ell^2}{2} \sigma_N(x_\ell) \right) d\mu_g \otimes d\mathcal{X} \]

that the law of

\[ u_0 \overset{\text{def}}{=} \tilde{u}_0 - 2\pi \sum_{\ell=1}^{L} a_\ell (P_N \otimes \text{Id}) G_g(x_\ell, x) \]

is given by the integrand in (1.23), which is now absolutely continuous with respect to \( d\mu_g \otimes d\mathcal{X} \).

**Theorem 1.4.** Let \( a_\ell \in \mathbb{R}, \ell = 1, \ldots, L \), and \( \nu, \beta > 0 \) and \( Q = \frac{2}{\beta} + \frac{\beta}{2} \) satisfy the assumptions (1.14)-(1.16). Assume also that

\[ 0 < \beta < \sqrt{\frac{a_\ell^2}{\ell_{\text{max}}^2} + 4 - a_{\ell_{\text{max}}}}, \tag{1.27} \]

with \( a_{\ell_{\text{max}}} = \max_{\ell=1,\ldots,L} a_\ell \). Then the equation (1.20) is almost surely globally well-posed and the law of its solution is invariant. More precisely:

(i) for any \( T > 0 \) and all \( N \in \mathbb{N} \), there exists a unique solution \( u_N \in C([0,T];H_0^0(M) \oplus \mathbb{R}) \) to (1.25) for \( d\mu_g(X_g) \otimes d\mathcal{X} \otimes \mathcal{P} \)-almost every \( u_0 = X_g + \mathcal{X} \) and \( \xi_g \), and the solution \( u_N \) converges in measure to some non trivial process \( u = u(t, X_g, \mathcal{X}, \omega) \in C([0,T];H_0^0(M) \oplus \mathbb{R}); \)
(ii) for any test function $F \in C_b(H^s_0(M) \oplus \mathbb{R})$ and any $t \geq 0$ it holds
\[
\int_{H^s_0(M,g)} \int_{\mathbb{R}} \mathbb{E} \left[ F(\bar{u}(t,X_g,X,\omega)) \right] d\rho_{(a_\ell,x_\ell),g}(X_g,X) = \int_{H^s_0(M,g)} \int_{\mathbb{R}} F(X_g + X) d\rho_{(a_\ell,x_\ell),g}(X_g,X),
\]
where
\[
\bar{u}(t,X_g,X,\omega) \overset{\text{def}}{=} u(t,X_g,X,\omega) + 2\pi L \sum_{\ell=1}^L a_\ell G_g(x_\ell,x) \in C(\mathbb{R}_+;H^s_0(M) \oplus \mathbb{R})
\]
is the limit in law of the solution $\bar{u}_N$ to \eqref{eq:1.22}.

**Remark 1.5.** We used the approximate equation \eqref{eq:1.25} with a truncated nonlinearity (but without truncating the noise nor the initial data) in order for the truncated dynamics \eqref{eq:1.25} to preserve the truncated Gibbs measure \eqref{eq:1.11}. However, Theorem 1.4 also holds by replacing \eqref{eq:1.25} with
\[
\partial_t u_N - \frac{1}{4\pi} \Delta_g u_N + \frac{1}{2} \nu \beta e^{-\pi \beta^2 C_N} N^{-\frac{a^2}{\pi}} e^{\beta u_N} + 2\pi \beta \sum_{\ell=1}^L a_\ell (P_N \otimes P_N) G_g(x_\ell,x)
\]
with truncated initial data
\[
u \overset{\text{def}}{=} \frac{Q}{8\pi} R_g + \frac{1}{2V_g(M)} \sum_{\ell=1}^L a_\ell + P_N \xi_g,
\]
with truncated initial data
\[
u \overset{\text{def}}{=} \frac{Q}{8\pi} R_g + \frac{1}{2V_g(M)} \sum_{\ell=1}^L a_\ell + P_N \xi_g.
\]

Let us point out that, apart from the use of the Seiberg bound \eqref{eq:1.15} to ensure that the measure $\rho_{(a_\ell,x_\ell),g}$ is finite, our analysis is completely insensitive to the particular geometry of $\mathcal{M}$. In this aspect, our result unifies the different treatments of \cite{15,17,30} for the measure construction\footnote{However, note that in the case of a hyperbolic surface, our results only treat the case of a fixed conformal class for the metric, and we do not average on the space of Riemannian metrics with negative curvature compared to \cite{30}. See Remark \ref{rem:1.10} below.} and of \cite{23} for the SPDE construction. In order to specialise the regime that is covered by Theorem 1.4 to the different possible geometries, we state the following.

**Corollary 1.6.** Almost sure global well-posedness and invariance of the measure in the sense of Theorem 1.4 hold under the following condition:
(i) For $\mathcal{M}$ a compact hyperbolic surface and $L = 0$, in the whole regime $0 < \beta < \sqrt{2}$;
(ii) For $\mathcal{M} = \mathbb{T}^2$ with $L = 1$, in the regime $a_1 < \frac{2}{\beta}$ and $0 < \beta < \min\left(\sqrt{2}, \sqrt{a_1^2 + 4} - a_1\right)$.
In particular, if $a_1 = \beta$, this regime reduces to $0 < \beta < \sqrt{\frac{4}{3}}$;
(iii) For $\mathcal{M} = \mathbb{S}^2$ with $L = 3$, in the regime $a_{\ell_{\max}} < \frac{2}{\beta}$ and $0 < \beta < \min\left(\sqrt{2}, \sqrt{a_{\ell_{\max}}^2 + 4} - a_{\ell_{\max}}\right)$.
Remark 1.7. Without surprise, the adjunction of punctures (i.e., of singularities in the nonlinearity in (1.25)) reduces the range of admissible $\beta$'s. This was already observed in [23], where Garban obtained uniform (in $N$) well-posedness of (1.25) in the regime

$$\frac{\beta^2}{2} - 2\sqrt{2}\beta + \min(0, \frac{\beta}{2\sqrt{2}} - a_{\ell_{\max}}) > -2,$$

for both $\mathcal{M} = S^2$ with $L = 3$ and $\mathcal{M} = T^2$ with $L = 1$. In this latter case, and for the particular choice $a_1 = \beta$ (related to random planar maps), his result gives the range $0 < \beta < \sqrt{2} - 2 \approx 0.707$, which we modestly improve to $0 < \beta < \sqrt{4 - 2a_1} \approx 1.15$. In particular, this shows that the conjectured threshold $\gamma_{\text{pos}}$ given in [23, Theorem 1.11] ($\gamma_{\text{pos}} = 2\sqrt{2} - 2$ in the case $L = 0$) does not correspond to the actual critical threshold for (1.22); see also the discussion after Theorem 1.1 in [48]. The main difference in our approach comes from the use of the “sign-definite structure” as in [48] (see Section 4) and of $L^p$ based spaces for controlling the solution, whereas Garban used (parabolic) Hölder spaces. In view of the regularity of the main stochastic objects in Proposition 1.8 below, which is very sensitive to their integrability properties, we see that working with the more flexible scale of $L^p$ based spaces allows to improve the admissible range. We believe that it is even possible to cover the full sub-critical regime $0 < \beta < 2$ and $\max_\ell a_\ell < Q$ without requiring a heavy machinery such as regularity structures or higher-order paracontrolled calculus.

1.4. Scheme of the proof. The proof of Theorem 1.1 follows along the line of the previous works [15, 17, 30], and is essentially a consequence of the argument in [15] along with the study of the Green’s function and its regularizations performed in Section 2.

As for Theorem 1.4, we first precise some notations. From (1.21), we see that the action of $\xi_g$ can be extended to functions in $L^2(\mathbb{R}; L^2(\mathcal{M}, g))$ and if we define for $n \in \mathbb{N}$, $t \geq 0$ the real-valued process

$$B_{n, g}(t) \overset{\text{def}}{=} \langle \xi_g, 1_{[0,t]} \varphi_n \rangle_{t, g}$$

then we have from (1.21) that $\{B_{n, g}\}_{n \geq 0}$ is a family of independent Brownian motions so that

$$\xi_g = \partial_t W$$

in $D'(\mathbb{R}_+ \times \mathcal{M})$, where

$$W = W_g + \mathcal{W} \overset{\text{def}}{=} \sum_{n \geq 1} B_{n, g} \varphi_n + B_0 \varphi_0.$$

In particular we can define $t_g$ as the solution to the linear stochastic equation

$$\begin{cases}
(\partial_t - \frac{1}{4\pi} \Delta) t_g = \partial_t W_g, \\
t_g(0) = X_g,
\end{cases}$$

which we can also write as

$$t_g(t) = e^{\frac{t}{4\pi} \Delta_X} X_g + \int_0^t e^{\frac{t - t'}{4\pi} \Delta_X} dW_g(t').$$

6 along with convergence of the approximations $u_N$ in the smaller regime $\frac{\beta^2}{2} - 2\sqrt{2}\beta + \min(0, \frac{\beta}{2\sqrt{2}} - a_{\ell_{\max}}) > -1.$
It is well-known that for $X_g$ as in (1.8) and $W_g$ as above, $t_g$ is a stationary process belonging almost surely to $C(\mathbb{R}_+; H_0^s(\mathcal{M}))$ for any $s < 0$; see Lemma 3.2 below. In particular, in view of the roughness of $t_g$, we see that $e^{t_g}$ does not make sense, which also justifies the need for the renormalization in the dynamics (1.25).

As in [23, 48], we then start by using Da Prato -Debussche trick [13] and write\footnote{Strictly speaking, the trick used by Da Prato and Debussche in [13] consists in removing the noise in the right-hand side of (1.25) by doing the change of variable $u_N = t_g + v_N$. Here we remove the whole right-hand side of (1.25) by the decomposition (1.30). This is crucial in order to use the “sign-definite structure” as in [48].}

$$u_N = t_g + v_N,$$  \hspace{1cm} (1.30)

where\footnote{Also note that here $z$ is still random as it depends on $X$ and $B_0$. This is different from [48] where $z$ essentially only contained the last two terms in (1.31).}

$$z(t, x, \overline{X}, \omega) = \overline{X} + \frac{B_0(t, \omega)}{V_g(\mathcal{M})} - \frac{Q}{8\pi} \int_{\mathcal{M}} P_g(t, x, y) R_g(y) dV_g(y) + \frac{t}{2V_g(\mathcal{M})} \sum_{\ell=1}^L a_\ell,$$  \hspace{1cm} (1.31)

with $P_g$ being the heat kernel on $(\mathcal{M}, g)$.

The remainder $v_N$ in (1.30) now solves

$$\begin{cases}
\partial_t v_N - \frac{1}{4\pi} \Delta_g v_N + \frac{1}{2} \nu \beta P_N(e^{\beta t} v_N) \Theta_N = 0, \\
v_N|_{t=0} = 0.
\end{cases}$$  \hspace{1cm} (1.32)

where the “punctured” Gaussian multiplicative chaos (GMC) $\Theta_N$ is defined by

$$\Theta_N(t, x) \overset{\text{def}}{=} e^{-\pi \beta^2 C_{\mathbb{P}} N - \frac{\beta^2}{4} N^2 e^{\beta P_N} a(t, x) + 2\nu \beta \sum_{\ell=1}^L a_\ell (P_N \otimes P_N) G(y, x, \ell, \ell)},$$  \hspace{1cm} (1.33)

This type of stochastic object has been largely investigated in the probability literature after the seminal work of Kahane [36] (see also [51, 53]). The following proposition gives the convergence properties of the process $\Theta_N$.

**Proposition 1.8.** Given $1 < p < \infty$, let $m \geq 0$ be an integer such that $2m < p \leq 2(m + 1)$, and let $0 < \beta^2 < 2\min(1, (2m + 1)^{-1})$ and $a_\ell < \frac{2}{\beta}$ for all $\ell = 1, ..., L$ be such that $\alpha = \alpha(p) \in (0, 2)$, where

$$\alpha(p) = \begin{cases}
\max \left\{ \frac{(p-1) \beta^2}{p}, \frac{(p-1) \beta^2}{p} + 2(p-1) \frac{\beta a_\ell}{p} - 2 + \frac{3}{p} \right\} & \text{for } 1 < p \leq 2, \\
\max \left\{ \beta a_\ell - \frac{4}{p}, (p-1) \frac{\beta^2}{2} + 2(p-1) \frac{\beta a_\ell}{p} - \frac{2}{p} \right\} & \text{for } p \geq 2.
\end{cases}$$  \hspace{1cm} (1.34)

Then, given any $T > 0$, the sequence of stochastic processes $\Theta_N$ is a Cauchy sequence in $L^p(\mu_g \otimes \mathbb{P}; L^p([0, T]; B_{p,p}^{\alpha}(\mathcal{M})))$ and hence converges to some limit $\Theta$ in the same class. In particular, $\Theta_N$ converges in probability to $\Theta$ in $L^p([0, T]; B_{p,p}^{\alpha}(\mathcal{M}))$. Finally, the limit $\Theta$ is independent of the approximation procedure in the sense of Theorem 1.1 (ii).

**Remark 1.9.** Here convergence of $\Theta_N$ is only established in probability. However, if $N$ runs over dyadic integers as in [21] instead of $\mathbb{N}$, then we can also prove convergence almost surely.
In Proposition 1.8, the function space $B^{-\alpha}_{p,p}(\mathcal{M})$ denotes the Besov space; see (2.5) below. Then for any positive distribution $\Theta_N \in L^2([0,T];H^{(-1)+}(\mathcal{M}))$ we can solve locally the Cauchy problem (1.32) by a fixed-point argument, and the continuous dependence of the flow of (1.32) in $\Theta_N$ gives the convergence $v_N \to v$ where $v$ solves
\[
\begin{cases}
\partial_t v - \frac{1}{4\pi} \Delta_g v + \frac{1}{2} \nu \beta (e^{\beta(z+v)} \Theta) = 0 \\
v|_{t=0} = 0.
\end{cases}
\] (1.35)
with $\Theta$ given by Proposition 1.8. Global well-posedness of (1.32) and uniform a priori bounds are established by the same argument as in [48, Theorem 1.4] by exploiting the “sign-definite structure” of the equation. Note that this is in order to preserve this structure that we consider regularization by using $P_N$ (1.13) in Theorem 1.4. A compactness argument then provides existence of a limit which satisfies (1.35), and an energy estimate yields the uniqueness property. In particular, the process $u$ in Theorem 1.4 (i) is unique in the class

\[ l_g + z + X_0^T, \]

where $X_0^T$ is the energy space defined in (1.5) below. The invariance of the measure then follows from standard arguments. Finally, observe that the condition $\Theta \in L^2([0,T];H^{(-1)+}(\mathcal{M}))$ implies the condition (1.27) in view of Proposition 1.8 with $\alpha = 1-\frac{1}{4\pi}$ and $p = 2$.

**Remark 1.10.** As pointed out above, another way to look at the stochastic quantization of the Liouville action (1.1) is to consider the stochastic Ricci flow (1.19) as in [18]. Indeed, $S_L$ in (1.1) depends on both the path $u$ and the metric $g$, and the stochastic quantization for $u$ that we considered in (1.20) corresponds to “freezing” $g$. On the contrary, (1.19) takes into account the coupling between $u$ and $g = e^u g_0$ when the metric $g$ belongs to the conformal class of a fixed reference metric $g_0$. Notice that this makes the dynamics (1.19) much more nonlinear.

Interestingly, in the case of a surface with genus strictly greater than one, the probabilistic approach to the full Liouville quantum gravity (LQG) amounts to couple the matter field given by LCFT with gravity by also averaging over all possible metrics, replacing (1.2) with
\[
\int \int \prod_{\ell=1}^L \mathcal{V}_{\alpha}(x_\ell)(u) e^{-S_L(u,g)} D u e^{-S_{EH}(g)} D g
\] (1.36)
where again $Dg$ stands for a “uniform measure” on the moduli space of $\mathcal{M}$, and $S_{EH}$ is the Einstein-Hilbert action
\[
S_{EH}(g) = \frac{1}{2\lambda} \int_{\mathcal{M}} R_g dV_g;
\]
see [30]. It would be interesting to study the stochastic quantization of the full measure (1.36) as a coupled stochastic dynamics on $(u,g)$ without restricting $g$ to a particular conformal class. See also Section 5 in [18].

The rest of the manuscript is organised as follows. In Section 2, we recall the necessary tools to perform the construction of the measure and analyse the stochastic equation; in particular we prove a Schauder estimate as well as pointwise bounds on a large class of regularizations of the Green’s function. Section 3 contains the probabilistic part of the analysis, where we establish Proposition 1.8 in Subsection 3.1 and prove Theorem 1.1 in...
In Section 4 we analyse the stochastic PDE (1.25) and give the proof of Theorem 1.4. Finally, in Appendix A we briefly discuss how the measure construction fails in the case of a negative cosmological constant $\nu < 0$, for both the LQG measure (1.9) and the $\exp(\Phi)^2$ measure $d\rho_{\exp} = \exp\{ -\|u\|^2_{H^1} - \nu \int_M :e^{\beta u}: \, dV_g \} Du$.

2. Background material

2.1. Basic tools from analysis on manifolds. Let $(M, g)$ be a two-dimensional closed (compact, boundaryless), connected, orientable smooth Riemannian manifold, where we fix the metric $g$ once and for all. In local coordinates, the metric $g$ is given by a smooth function $x \mapsto (g_{j,k}(x))_{j,k=1,2}$ taking value in the set of positive symmetric definite matrices. In particular $(g_{j,k})$ is invertible and its inverse is denoted by $(g_{j,k}(x))^{-1}$. We also write $|g(x)| = \det (g_{j,k}(x)) > 0$.

The volume form $V_g$ can then be written locally as

$$dV_g(x) = |g(x)|^\frac{1}{2} \, dx.$$  

The Laplace-Beltrami operator is given in local coordinates by

$$\Delta_g f = |g|^{-\frac{1}{2}} \partial_j (|g|^\frac{1}{2} g^{j,k} \partial_k f)$$

for any smooth function $f \in C^\infty(M)$. Here and in the following we use Einstein’s convention for the summation on repeated indices.

We set $\{\varphi_n\}_{n \geq 0} \subset C^\infty(M)$ to be a basis of $L^2(M, g)$ consisting of eigenfunctions of $\Delta_g$ associated with the eigenvalue $-\lambda_n^2$, assumed to be arranged in increasing order: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$. In particular $\varphi_0 \equiv V_g(M)^{-\frac{1}{2}}$ is constant. As for $\lambda_n$, $n \to \infty$, we have the following asymptotic behaviour given by Weyl’s law:

$$\frac{\lambda_n^2}{n} \to \frac{V_g(M)}{4\pi} \text{ as } n \to \infty. \quad (2.1)$$

Indeed, this is a consequence of the following result of Hörmander [33, Theorem 17.5.7] for the spectral function of $\Delta_g$: there exists $C > 0$ such that for any $\Lambda > 0$ and any $x \in M$, it holds

$$\left| \sum_{\lambda_n \leq \Lambda} \varphi_n(x)^2 - \frac{1}{4\pi} \Lambda^2 \right| \leq C \Lambda. \quad (2.2)$$

In particular we also get that for any $n \geq 1$,

$$\|\varphi_n\|_{L^\infty(M)} \leq \lambda_n^\frac{1}{2} \lesssim \langle n \rangle^\frac{1}{2}. \quad (2.3)$$

On the other hand, the following lemma shows the uniform boundedness in average of the eigenfunctions.

9Compared to the notations of [33, Theorem 17.5.7], we have in our case that the density of the weighted manifold $M$ is given by $|g|^\frac{1}{2} = \det(g^{i,j,k})^{-\frac{1}{2}}$, and the $\lambda_n$’s are the square-roots of the eigenvalues of $-\Delta_g$. 

---

14 T. OH, T. ROBERT, N. TZVETKOV AND Y. WANG
Lemma 2.1. For any $A \in \mathbb{R}$, there exists $C > 0$ such that for any $\Lambda > 0$ and any $x \in \mathcal{M}$, it holds
\[
\sum_{\lambda_n \in [\Lambda, \Lambda+1]} \frac{\varphi_n(x)^2}{\langle \lambda_n \rangle^A} \leq C \sum_{\lambda_n \in [\Lambda, \Lambda+1]} \frac{1}{\langle \lambda_n \rangle^A}.
\]

Proof. This follows from (2.1)-(2.2)-(2.3). See also [12] Proposition 8.3. \qed

Using the above basis of $L^2(\mathcal{M}, g)$, we can expand any $u \in \mathcal{D}'(\mathcal{M})$ as
\[
u = \sum_{n \geq 0} \langle u, \varphi_n \rangle_g \varphi_n,
\]
where $\langle \cdot , \cdot \rangle_g$ denotes the distributional pairing $\mathcal{D}'(\mathcal{M}) \times \mathcal{D}(\mathcal{M}) \to \mathbb{R}$ which coincides with the usual inner product in $L^2(\mathcal{M}, g)$ for distributions which are regular enough. For any $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we thus define the Sobolev spaces
\[
W^{s,p}(\mathcal{M}, g) = \left\{ u \in \mathcal{D}'(\mathcal{M}), \| u \|_{W^{s,p}(\mathcal{M})} < \infty \right\}
\]
where
\[
\| u \|_{W^{s,p}(\mathcal{M})} \overset{\text{def}}{=} \left\| (1 - \Delta_g)^{\frac{s}{2}} u \right\|_{L^p(\mathcal{M})} = \left\| \sum_{n \geq 0} \langle \lambda_n \rangle^{s} \langle u, \varphi_n \rangle_g \varphi_n \right\|_{L^p(\mathcal{M})}.
\]

When $p = 2$ we write $H^s(\mathcal{M}) \overset{\text{def}}{=} W^{s,2}(\mathcal{M})$.

2.2. Schwartz multipliers and Schauder estimate on $\mathcal{M}$. Next, recall that for any self-adjoint elliptic operator $A$ on $L^2(\mathcal{M}, g)$ with discrete spectrum $\{\lambda_n\}$ and orthonormal basis of eigenfunctions $\{\varphi_n\}$, the functional calculus of $A$ is defined for any $\psi \in L^\infty(\mathbb{R})$ by
\[
\psi(A)u = \sum \psi(\lambda_n) \langle u, \varphi_n \rangle_g \varphi_n,
\]
for all $u \in C^\infty(\mathcal{M})$. This in particular allows us to define the more general class of Besov spaces. First, using the functional calculus, we can define the Littlewood-Paley projectors $Q_M$ for a dyadic integer $M \in 2^{\mathbb{Z}_{\geq -1}} \overset{\text{def}}{=} \{0, 1, 2, 4, \ldots\}$ as
\[
Q_M = \left\{ \begin{array}{ll}
\psi_0(- (2M)^{-2} \Delta_g) - \psi_0(- M^{-2} \Delta_g), & M \geq 1, \\
\psi_0(- \Delta_g), & M = 0,
\end{array} \right.
\]
where $\psi_0 \in C^\infty_0(\mathbb{R})$ is non-negative and such that supp $\psi_0$ $\subset$ $[-1, 1]$ and $\psi_0 \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. With the inhomogeneous dyadic partition of unity $\{Q_M\}_{M \in 2^{\mathbb{Z}_{\geq -1}}}$, we can then define the Besov spaces
\[
B^{s}_{p,r}(\mathcal{M}) = \left\{ u \in \mathcal{D}'(\mathcal{M}), \| u \|_{B^{s}_{p,r}(\mathcal{M})} \overset{\text{def}}{=} \left( \sum_{M \in 2^{\mathbb{Z}_{\geq -1}}} (\langle M \rangle^{r} \| Q_M u \|_{L^p(\mathcal{M})}^r \right)^{\frac{1}{r}} < \infty \right\}
\]
for any $p, r \in [1, \infty]$. These function spaces are the natural generalization of the usual Besov spaces on $\mathbb{R}^d$ to the context of closed manifolds. In particular, we proved in [47] Proposition 2.5 the following characterization of these spaces.

Lemma 2.2. Let $(U, V, \kappa)$ be a coordinate patch and $\chi \in C^\infty_0(V)$. For any $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, there exist $c, C > 0$ such that for any $u \in C^\infty(\mathcal{M})$,
\[
c\| \chi u \|_{B^{s}_{p,r}(\mathcal{M})} \leq \| \kappa^s(\chi u) \|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C \| u \|_{B^{s}_{p,r}(\mathcal{M})}.
\]
In order to describe more precisely the operators given above by the functional calculus, we start by looking at the local description of multipliers with smooth symbol. For \( \psi \in C_0^\infty(\mathbb{R}) \), we define through the functional calculus the smoothing operators
\[
\psi(-N^{-2}\Delta_g) : v \in \mathcal{D}'(\mathcal{M}) \mapsto \sum_{n \geq 0} \psi(N^{-2}\lambda_n^2) \langle v, \varphi_n \rangle \varphi_n \in C^\infty(\mathcal{M}),
\]
for any \( N \in \mathbb{N} \). We can see these operators as semi-classical pseudo-differential operators on \( \mathcal{M} \). Indeed, for a semi-classical parameter \( h \in (0, 1] \), we use the quantization rule
\[
a(x, hD) : u \in \mathcal{S}(\mathbb{R}^2) \mapsto a(x, hD)u = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\xi} a(x, h\xi) \hat{u}(\xi) d\xi,
\]
with the convention
\[
\hat{u}(\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} u(x) dx
\]
for the Fourier transform. We then have the following result from \cite{[11]} Proposition 2.1.

**Proposition 2.3.** Let \( A \) be an elliptic self-adjoint operator of order \( m \) on \( L^2(\mathcal{M}, g) \). Let \( \psi \in C_0^\infty(\mathbb{R}) \), \( \kappa : U \subset \mathbb{R}^2 \to V \subset \mathcal{M} \) be a coordinate patch, and \( \chi, \tilde{\chi} \in C_0^\infty(V) \) with \( \tilde{\chi} \equiv 1 \) on \( \text{supp} \chi \). Then there exists a sequence of symbols \( (a_k)_{k \geq 0} \) in \( C_0^\infty(U \times \mathbb{R}^2) \) with the following properties:

(i) for any \( v \in C^\infty(\mathcal{M}) \) and \( K \geq 1 \), we have the expansion
\[
\left\| \kappa^*(\chi\psi(h^m A)v) - \sum_{k=0}^{K-1} h^k a_k(x, hD) \kappa^*(\tilde{\chi}v) \right\|_{H^{s+\sigma}(\mathbb{R}^2)} \lesssim h^{K-s-\sigma} \|v\|_{H^s(\mathcal{M})},
\]
uniformly in \( h \in (0, 1] \);

(ii) for any \( x \in U \) the principal symbol is given by
\[
a_0(x, \xi) = \chi(\kappa(x)) \psi(p_m(x, \xi)),
\]
where \( p_m \) is the principal symbol of \( A \) in \( \kappa \);

(iii) for all \( k \geq 0 \), \( a_k \) is supported in
\[
\{(x, \xi) \in U \times \mathbb{R}^2, \kappa(x) \in \text{supp} \chi, -p_m(x, \xi) \in \text{supp} \psi^{(k)}\}.
\]

**Remark 2.4.** Actually, one can check that the proof of \cite{[11]} Proposition 2.1 also works for multipliers \( \psi \) in the class \( S^{-\gamma}(\mathbb{R}) \) for any \( \gamma > 0 \), yielding a similar decomposition as in \cite{[2,3]} but with symbols \( a_k \in S^{-m\gamma-k}(U \times \mathbb{R}^2) \) instead of \( (2.7) \) (see also \cite{[62]} Theorem 41.9]). This in particular applies to multipliers \( \psi \in S(\mathbb{R}) \) such that \( \psi(x) = e^{-x} \) for \( x \geq 0 \), for which \( \psi(-N^{-2}\Delta_g) = P_N \) defined in \cite{[11]} above.

To pursue our investigation of multipliers on \( \mathcal{M} \) with symbol in \( S(\mathbb{R}) \), using Proposition 2.3 with Remark 2.4, we have the following classical estimate on their kernel.

**Lemma 2.5.** Let \( \psi \in S(\mathbb{R}) \), and for any \( h \in (0, 1] \) define the kernel
\[
\mathcal{K}_h(x, y) \overset{\text{def}}{=} \sum_{n \geq 0} \psi(h^2\lambda_n^2) \varphi_n(x) \varphi_n(y),
\]
for any \( (x, y) \in \mathcal{M} \times \mathcal{M} \).

(2.8)
Then for any $A > 0$, there exists $C > 0$ such that for any $h \in (0, 1]$ and $x, y \in M$ it holds
\[
|K_h(x, y)| \leq C h^{-2} (h^{-1} d_g(x, y))^{-A},
\] (2.9)
where $d_g$ is the geodesic distance on $M$. In particular, it holds
\[
\left\| \psi(-h^2 \Delta_g) \right\|_{L^p(M) \to L^q(M)} \lesssim h^{-2(\frac{1}{p} - \frac{1}{q})}
\] (2.10)
for any $h \in (0, 1]$ and $1 \leq p \leq q \leq \infty$.

**Proof.** Let $h \in (0, 1]$ and $K_h$ be given by (2.8) for some $\psi \in \mathcal{S} (\mathbb{R})$. Fix $(x, y) \in M$.

**Case 1.** If $d_g(x, y) \lesssim h$: in this case, we use that $\psi \in \mathcal{S} (\mathbb{R})$ with Cauchy-Schwarz inequality to bound
\[
|K_h(x, y)| \lesssim \sum_{n \geq 0} |\langle h \lambda_n \rangle|^{-3} |\psi_n(x)\psi_n(y)|
\]
\[
\lesssim \left( \sum_{n \geq 0} \frac{\psi_n(x)^2}{\langle h \lambda_n \rangle^3} \right) \frac{1}{2} \left( \sum_{n \geq 0} \frac{\psi_n(y)^2}{\langle h \lambda_n \rangle^3} \right)^{\frac{1}{2}}.
\]
Now, for any $x \in M$, we can use Hörmander’s bound (2.2) to estimate
\[
\sum_{n \geq 0} \frac{\psi_n(x)^2}{\langle h \lambda_n \rangle^3} \sim \sum_{\lambda_n \leq h^{-1}} \psi_n(x)^2 + h^{-3} \sum_{\lambda_n \geq h^{-1}} \frac{\psi_n(x)^2}{\lambda_n^3}
\]
\[
\lesssim h^{-2} + O(h^{-1}) + h^{-3} \sum_{j=|h^{-1}|}^{\infty} j^{-3} \sum_{\lambda_n \in (j, j+1]} \psi_n(x)^2
\]
\[
\lesssim h^{-2} + O(h^{-1}) + h^{-3} \sum_{j=|h^{-1}|}^{\infty} j^{-2} \lesssim h^{-2}.
\]
This proves that $|K_h(x, y)| \lesssim h^{-2}$ in this case.

**Case 2.** If $h \ll d_g(x, y) \leq \iota(M)$, where $\iota(M)$ is the injectivity radius of $M$: we let $A, K \in \mathbb{N}$, and we also take geodesic normal coordinates $(U, \kappa)$ centred at $x$ and containing $y$. Then we can use Proposition 2.3 with Remark 2.4 to get that
\[
K_h(x, y) = \chi(x) \chi(y) \sum_{k=0}^{K-1} h^k \int_{\mathbb{R}^2} a_k(\kappa^{-1}(x), h \xi) e^{i(\kappa^{-1}(x) - \kappa^{-1}(y)) \cdot \xi} d\xi
\]
\[
+ \chi(x) K_{-K,h}(\kappa^{-1}(x), y),
\]
for some symbols $a_k \in S^{-m-k}(U \times \mathbb{R}^2)$ for any $m > 0$, and where $K_{-K,h}$ is the kernel of
\[
R_{-K,h} : v \in H^s(M) \mapsto \kappa^1_1(\chi_1 \psi(-h^2 \Delta_g) \chi_2 v) - \sum_{k=0}^{K-1} h^k a_k(x, hD) \kappa^*(\overline{\chi} v) \in H^{s+\sigma}(\mathbb{R}^2)
\]
which satisfies
\[
\left\| R_{-K,h} \right\|_{H^s(M) \to H^{s+\sigma}(\mathbb{R}^2)} \lesssim h^{K-\sigma-s}.
\]
For this last term, the previous bound implies that
\[
\left\| \chi(x) K_{-K,h}(\kappa(x), y) \right\|_{L^\infty(M \times M)} \lesssim \left\| R_{-K,h} \right\|_{H^{-1+\varepsilon}(M) \to H^{1+\varepsilon}(\mathbb{R}^2)} \lesssim h^{K-2-2\varepsilon} \lesssim h^{K-2-2\varepsilon} d_g(x, y)^{-A}
\]
for any \( \varepsilon > 0 \), where in the last step we used the compactness of \( M \). This is enough for (2.9) by taking \( K \) large enough.

As for the contribution of the symbols \( a_k \), changing variable and using integration by parts, we can bound it by
\[
\left| h^k \int_{\mathbb{R}^2} a_k(\kappa^{-1}(x), h\xi) e^{i(\kappa^{-1}(x) - \kappa^{-1}(y)) \cdot \xi} d\xi \right| \\
= \left| h^{k-2} \int_{\mathbb{R}^2} a_k(\kappa^{-1}(x), \eta) e^{i(\kappa^{-1}(x) - \kappa^{-1}(y)) \cdot \eta} d\eta \right| \\
\lesssim h^{k-2} \left| h^{-1}(\kappa^{-1}(x) - \kappa^{-1}(y)) \right|^{-A} \int_{\mathbb{R}^2} \left| \langle \eta \rangle^A a_k(\kappa^{-1}(x), \eta) \right| d\eta \\
\lesssim h^{k-2} \left| h^{-1}(\kappa^{-1}(x) - \kappa^{-1}(y)) \right|^{-A} \int_{\mathbb{R}^2} \left| \langle \eta \rangle^{A-K} d\eta \right| \lesssim h^{k-2} \left| h^{-1} d_g(x, y) \right|^{-A},
\]
where we used that \( \left| \partial_x^\alpha a_k(x, \eta) \right| \lesssim \langle \eta \rangle^{[\alpha]-m-k} \) for any \( \alpha \in \mathbb{N}^2 \) and \( m \in \mathbb{R} \) since by Remark 2.4 \( a_k \in \cap_{m \in \mathbb{R}} S^{-m-k}(U \times \mathbb{R}^2) \) since \( \psi \in S(\mathbb{R}) \). Note that the last integral converges by taking \( K \) large enough.

**Case 3:** if \( d_g(x, y) > \frac{\langle M \rangle}{2} \): then we can repeat the same argument as above, and use that the pseudo-differential operators \( \chi \kappa \ast a_k(z, D) \chi^* \) are properly supported (as can be checked on (2.6)), which implies in this case that their contribution to \( K_h \) vanishes. Indeed, take charts \((U_j, V_j, \kappa_j)\), \( j = 1, 2 \), then since \( d_g(x, y) > \frac{\langle M \rangle}{2} \), we can find \( \chi_j \in C_0^\infty(V_j) \) such that \( \text{supp} \chi_1 \cap \text{supp} \chi_2 = \emptyset \), then using (2.6) with \( v = \chi_2 v \) we see that all the term in the sum on \( k = 0, ..., K - 1 \) vanish, so that
\[
|K_h(x, y)| = |\chi_1(x) \chi_2(y) K_h(x, y)| = |\chi_1(x) \chi_2(y) K_{-K, h}(\kappa^{-1}(x), y)| \\
\lesssim h^{K-2-2\varepsilon} \lesssim h^{K-2-2\varepsilon} d_g(x, y)^{-A}
\]
for any \( \varepsilon > 0 \). This proves (2.9) by taking \( K \) large enough.

The bound (2.10) then follows from
\[
\left\| K_h(x, y) \right\|_{L^r_x L^s_y} \lesssim \left\| h^{-2} \left( h^{-1} d_g(x, y) \right)^{-A} \right\|_{L^2_x L^1_y} \lesssim h^{\frac{s}{2}-2}
\]
for any \( r \geq 1 \), the symmetry of \( K_h \) and Schur’s lemma.

As a corollary of the estimates above, we can then investigate the behaviour of the solutions to the linear heat equation on \( M \). Indeed, we can use the eigenfunctions expansion to represent the solution of the heat equation
\[
\begin{cases}
\partial_t u - \frac{1}{4\pi} \Delta_g u = 0, \\
u_{t=0} = u_0 \in \mathcal{D}'(M) \\
(t, x) \in \mathbb{R}_+ \times M,
\end{cases}
\]
as the distribution
\[ u(t) = e^{\frac{t}{4\pi}\Delta_g} u_0 = \sum_{n \geq 0} e^{-\frac{t}{4\pi}\lambda_n^2} \langle u_0, \varphi_n \rangle \varphi_n. \]

The well-known heat kernel is then the kernel of the above propagator, defined as
\[
P_g(t, x, y) \overset{\text{def}}{=} \sum_{n \geq 0} e^{-\frac{t}{4\pi}\lambda_n^2} \varphi_n(x) \varphi_n(y).
\]

The convergence of the sum above is a priori in the sense of distributions, but we have the following properties of \( P_g \).

**Lemma 2.6.** Let \( P_g \) be the heat kernel defined in (2.11) above.

(i) For any \( t > 0 \), \( P_g(t, \cdot, \cdot) \) is a smooth, symmetric, non-negative function on \( \mathcal{M} \times \mathcal{M} \), which satisfies the semigroup property
\[
P_g(t+s, x, y) = \int_{\mathcal{M}} P_g(t, x, z) P_g(s, y, z) dV_g(z)
\]
for any \( t, s > 0 \). Moreover for any \( f \in C(\mathcal{M}) \) it holds
\[
\int_{\mathcal{M}} P_g(t, x, y) f(y) dV_g(y) \to f(x)
\]
uniformly as \( t \to 0^+ \).

(ii) For any \( 1 \leq p \leq q \leq \infty \), any \( 1 \leq r \leq \infty \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \) with \( \alpha_1 \leq \alpha_2 \), there exists \( C > 0 \) such that for any \( 0 < t \leq 1 \) and any \( u \in B_{p,r}^{\alpha_1, q, r}(\mathcal{M}) \), we have Schauder’s estimate
\[
\left\| e^{\frac{t}{4\pi}\Delta_g} u \right\|_{B_{q,r}^{\alpha_2}(\mathcal{M})} \leq C t^{-\frac{\alpha_2 - \alpha_1}{2} - \left( \frac{1}{p} - \frac{1}{q} \right)} \left\| u \right\|_{B_{p,r}^{\alpha_1}(\mathcal{M})}.
\]

Using the aforementioned smoothing properties of the heat kernel, we can then define the smoothing operators \( P_N, N \in \mathbb{N} \) using \( P_g \) as in (1.13) above.

**Proof.** The properties (i) are well-known, and can be found e.g. in [26].

As for (ii), since we can write \( e^{t\Delta_g} = \psi(-h^2\Delta_g) \) with \( \psi \in \mathcal{S}(\mathbb{R}) \) such that \( \psi(x) = e^{-x} \) for \( x \geq 0 \) and with the semiclassical parameter \( h = \sqrt{t} \), we then have that the heat kernel \( P_g \) satisfies the assumptions of Lemma 2.5. In particular, (2.10) gives (ii) in the case \( \alpha_1 = \alpha_2 \); indeed, using the definition of \( B_{q,r}^{\alpha_2}(\mathcal{M}) \) and (2.10) we have
\[
\left\| e^{\frac{t}{4\pi}\Delta_g} u \right\|_{B_{q,r}^{\alpha_2}(\mathcal{M})} = \left( \sum_{M \in \mathbb{Z}^{d \geq -1}} \langle M \rangle^{r\alpha_2} \left\| e^{\frac{t}{4\pi}Q_M} u \right\|_{L^r(\mathcal{M})} \right)^{\frac{1}{r}}
\]
\[
\lesssim t^{-\left( \frac{1}{p} - \frac{1}{r} \right)} \left( \sum_{M \in \mathbb{Z}^{d \geq -1}} \langle M \rangle^{r\alpha_2} \left\| Q_M u \right\|_{L^p(\mathcal{M})} \right)^{\frac{1}{r}}
\]
\[
\lesssim t^{-\left( \frac{1}{p} - \frac{1}{2} \right)} \left\| u \right\|_{B_{p,r}^{\alpha_2}(\mathcal{M})}.
\]
As for the case \( \alpha_1 < \alpha_2 \), writing \( e^{\frac{t}{\pi} \Delta_k} = e^{\frac{t}{\pi} \Delta_k} e^{\frac{t}{\pi} \Delta_k} \) and using again (2.10) we have

\[
\left\| e^{\frac{t}{\pi} \Delta_k} u \right\|_{B^2_{p,2}} \leq \left( \sum_{M \in 2^{\geq -1}} \langle M \rangle^{\alpha_2} \left\| e^{\frac{t}{\pi} \Delta_k} Q_M u \right\|_{L^p(M)}^r \right)^{\frac{1}{r}} \lesssim t^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \left( \sum_{M \in 2^{\geq -1}} \langle M \rangle^{\alpha_2} \left\| e^{\frac{t}{\pi} \Delta_k} Q_M u \right\|_{L^p(M)}^r \right)^{\frac{1}{r}},
\]

so it suffices to prove

\[
\left\| e^{t \Delta_k} Q_M \right\|_{L^p(M) \to L^p(M)} \lesssim (\sqrt{t} \langle M \rangle)^{-\alpha}
\]

for any \( t \in (0, 1] \), \( p \in [1, \infty] \) and \( M \in 2^{\geq -1} \), where \( \alpha = \alpha_2 - \alpha_1 > 0 \). Note that \( e^{t \Delta_k} \) and \( Q_M \) both satisfy the assumptions of Lemma 2.5 with respective semiclassical parameter \( h = \sqrt{t} \) and \( h = M^{-1} \), so that using (2.10) again we have

\[
\left\| e^{t \Delta_k} Q_M \right\|_{L^p(M) \to L^p(M)} \lesssim \left\| Q_M \right\|_{L^p(M) \to L^p(M)} \lesssim 1,
\]

which is enough for (2.13) when \( M \leq t^{-\frac{1}{2}} \) or \( M \lesssim 1 \). Thus in the following we assume \( M \geq t^{-\frac{1}{2}} \) and \( M \gg 1 \).

Let then \( Q_M = \psi(-M^{-2} \Delta_k) \) where \( \psi \) is the symbol of \( Q_M \) in (2.21), and let \((U, V, \kappa)\) be a chart on \( M \) with \( \chi \in C_0^\infty(V) \). Then using Proposition 2.3 (twice) we can expand locally

\[
\kappa^*(\chi Q_M e^{t \Delta_k}) = \chi \left( \sum_{l=0}^{L-1} M^{-\ell} a_l(x, M^{-1} D) \kappa^* \tilde{\chi}^* + R_{-L,M} \right) \kappa^*(\tilde{\chi} e^{t \Delta_k})
\]

\[
= \chi \sum_{l=0}^{L-1} M^{-\ell} a_l(x, M^{-1} D) \left( \sum_{k=0}^{K-1} t^{\frac{1}{2}} a_k(x, \sqrt{t} D) \kappa^* \tilde{\chi}^* + R_{-K,l} \right) + R_{-L,M} e^{t \Delta_k},
\]

for some \( \tilde{\chi} \in C_0^\infty(V) \) and some symbols \( a_k, a_l \in S^{-\infty} \) as in Proposition 2.3 and the remainders satisfy

\[
\left\| R_{-K,l} \right\|_{H^{-s_1}(M) \to H^{s_2}(\mathbb{R}^2)} \lesssim t^{\frac{K-s_1-s_2}{2}}
\]

and

\[
\left\| R_{-L,M} \right\|_{H^{-s_1}(M) \to H^{s_2}(\mathbb{R}^2)} \lesssim M^{s_1+s_2-L}
\]

for any \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 \leq \min\{K, L\} \). In particular, taking \( s_1, s_2 \) large enough so that \( H^{s_2}(M) \subset L^\infty(M) \) and \( L^1(M) \subset H^{-s_1}(M) \) and using (2.10) again, we see that the contribution of \( R_{-L,M} e^{t \Delta_k} \) satisfies (2.13) provided that we take \( K, L \) large enough. We also have for any \( \ell = 0, \ldots, L-1 \)

\[
\left\| a_l(x, M^{-1} D) R_{-K,l} \right\|_{L^p(M) \to L^p(\mathbb{R}^2)} \lesssim \left\| a_l(x, M^{-1} D) R_{-K,l} \right\|_{H^{-s_1}(M) \to H^{s_2}(\mathbb{R}^2)} \lesssim M^{-\alpha} \left\| R_{-K,l} \right\|_{H^{-s_1}(M) \to H^{s_2+\alpha}(\mathbb{R}^2)} \lesssim M^{-\alpha} t^{\frac{K-s_1-s_2-\alpha}{2}}
\]

by taking \( s_1, s_2 \) and then \( K \) large enough, and using that the symbols \( a_l \) are supported on an annulus in view of Proposition 2.3 with \( M \gg 1 \).
As for the products $a_l(x, M^{-1}D)a_k(x, \sqrt{t}D)$, similarly as for the usual composition rule for (semiclassical) pseudo-differential operators (see e.g. [62]), their symbol is given by

$$b_{\ell,k,M,t}(x, \xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\cdot a_{\ell}(x, M^{-1}(\xi + \eta))} a_k(x + z, \sqrt{t}\xi) dzd\eta$$

$$= \sum_{|\gamma|=0}^{A-1} \frac{M^{-|\gamma|}}{\gamma!} \partial^\gamma_{\xi} a_{\ell}(x, M^{-1}\xi) \partial^\gamma_{\eta} a_k(x, \sqrt{t}\xi) + \tilde{R}_{\ell,k,A,M,t}(x, \xi)$$

with

$$\tilde{R}_{\ell,k,A,M,t}(x, \xi) = \frac{1}{(2\pi)^2} \sum_{|\gamma|=A} \frac{AM^{-A}}{\gamma!} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\cdot a_{\ell}(x, M^{-1}(\xi + \eta))}$$

$$\times \int_0^1 (1 - \theta)^A \partial^\gamma_{\xi} a_k(x + \theta z, \sqrt{t}\xi) d\theta dzd\eta.$$ Integrating by parts in $\eta$, we find that the corresponding kernel is bounded by

$$|\tilde{R}_{\ell,k,A,M,t}(x, y)| \lesssim M^{-A} \sum_{|\gamma|=A} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle z \rangle^{-3} \langle D_\eta \rangle^3 \partial^\gamma_{\eta} a_{\ell}(x, M^{-1}(\xi + \eta)) \bigg| 
\times \bigg| \int_0^1 (1 - \theta)^A \partial^\gamma_{\xi} a_k(x + \theta z, \sqrt{t}\xi) d\theta \bigg| dzd\eta d\xi$$

Using next the properties of the symbols $a_k, a_\ell$ given by Proposition 2.3 and Remark 2.4, we can continue with

$$\lesssim M^{-A} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle z \rangle^{-3} \big( \langle z \rangle^0 \lesssim M \langle \sqrt{t}\xi \rangle^{-B} \big) dzd\eta d\xi$$

for any $B > 0$. This yields the bound

$$\lesssim t^{-1}M^{2-A}$$

which is enough for (2.13) by taking $A$ large enough since we are in the case $M \geq t^{-\frac{1}{2}}$.

At last, the contribution of the leading symbols are then

$$|K_{k,\ell,\gamma,M,t}(x, y)| \sim \bigg| \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} \partial^\gamma_{\xi} a_{\ell}(x, M^{-1}\xi) \partial^\gamma_{\eta} a_k(x, \sqrt{t}\xi) d\xi \bigg|$$

$$= M^2 \bigg| \int_{\mathbb{R}^2} e^{iM(x-y) \cdot \xi} \partial^\gamma_{\xi} a_{\ell}(x, \xi) \partial^\gamma_{\eta} a_k(x, \sqrt{t}M\xi) d\xi \bigg|$$

Then we integrate by parts in $\xi$ to get

$$\lesssim M^2 \langle M(x - y) \rangle^{-B} \bigg| \int_{\mathbb{R}^2} e^{iM(x-y) \cdot \xi} \langle D_\xi \rangle^B \big[ \partial^\gamma_{\xi} a_{\ell}(x, \xi) \partial^\gamma_{\eta} a_k(x, \sqrt{t}M\xi) \big] d\xi \bigg|$$

for any $B > 0$. We can then use again the properties of the symbols $a_\ell, a_k$ in Proposition 2.3 to get

$$\lesssim M^2 \langle M(x - y) \rangle^{-B} \int_{|\xi| \sim 1} \langle \sqrt{t}M \rangle^B \langle \sqrt{t}M\xi \rangle^{-D} d\xi$$

$$\lesssim M^2 \langle M(x - y) \rangle^{-B} \langle \sqrt{t}M \rangle^{B-D}$$
for any $D > 0$. We thus get

$$\left\| K_{k,\ell,\gamma,M,t}(x,y) \right\|_{L^\infty L^1_y} \lesssim \langle \sqrt{t}M \rangle^{B-D},$$

by taking $B > 2$, which along with Schur’s lemma shows (2.13) by taking $D$ large enough. This completes the proof of Lemma 2.6.

2.3. Properties of the Green's function. We now investigate the properties of the fundamental solution for the Laplace equation. It is defined as

$$G_g(x,y) \defeq \sum_{n \geq 1} \frac{\varphi_n(x,g)\varphi_n(y,g)}{\lambda_n(g)^2},$$

(2.14)

where the convergence of the sum holds again in the sense of distributions. Here we wrote $\varphi_n(x,g)$ and $\lambda_n(g)$ to emphasize the dependence of the eigenfunctions and eigenvalues (and hence of $G_g$) on the metric.

For a distribution $f \in \mathcal{D}'(M)$, we also abuse notations and define

$$\langle f \rangle_g \defeq \langle f, \varphi_0 \rangle_g \varphi_0 = \frac{1}{V_g(M)} \int_M f(x) dV_g(x).$$

Then we can check on (2.14) that for any $f \in \mathcal{D}'(M)$, it holds

$$u(x) = \int_M G_g(x,y) f(y) dV_g(y) \iff \begin{cases} -\Delta_g u = f - \langle f \rangle_g \\ \langle u \rangle_g = 0, \end{cases}$$

(2.16)

where the equalities are again in distributional sense.

We now list some further properties of the Green’s function.

**Lemma 2.7.** Let $G_g$ be given by (2.14). (i) $G_g$ is symmetric and smooth on $M \times M \setminus \Delta$, where $\Delta = \{(x,x) : x \in M\}$ is the diagonal. (ii) There exists a continuous function $\tilde{G}_g$ on $M \times M$ such that for any $(x,y) \in M \times M \setminus \Delta$ it holds

$$G_g(x,y) = -\frac{1}{2\pi} \log (d_g(x,y)) + \tilde{G}_g(x,y).$$

In particular, $G_g$ is bounded from below. (iii) There exists a constant $C > 0$ such that for any $(x,y) \in M \times M \setminus \Delta$ it holds

$$|\nabla_x G_g(x,y)|_g \leq Cd_g(x,y)^{-1}.$$

**Proof.** This is classical, and can be found e.g. in [4, Section 4.2].

Next, we investigate the behaviour of the approximations

$$(P_N \otimes P_N)G_g(x,y) \defeq \int_M \int_M P_g(4\pi N^{-2}, x,z)P_g(4\pi N^{-2}, y,z')G_g(z,z')dV_g(z)dV_g(z')$$

of the Green’s function. Recall indeed that the smoothing operator $P_N$ has been defined in (1.13) above, so that its kernel is given by $P_g(4\pi N^{-2}, \cdot, \cdot)$.

**Lemma 2.8.** There exists $C > 0$ such that for any $N \in \mathbb{N}$ and $(x,y) \in M \times M \setminus \Delta$, it holds

$$-C \leq (P_N \otimes P_N)G_g(x,y) \leq G_g(x,y) + \frac{1}{N^2 V_g(M)}.$$
Moreover, for any \( N_1 \leq N_2 \in \mathbb{N} \) and \((x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta\) we have for any \( j = 1, 2 \)

\[
\left| (\mathbf{P}_{N_j} \otimes \mathbf{P}_{N_j}) G_g(x, y) - (\mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2}) G_g(x, y) \right|
\leq \min \left\{ 1 + \log \frac{1}{d_g(x, y)}, N_1^{-1} d_g(x, y)^{-1} \right\}. \tag{2.18}
\]

Proof. We begin by proving the first claim. For the lower bound, we have from Lemma 2.7(ii) that there exists \( C > 0 \) such that \( G_g(x, y) \geq -C \) for any \((x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta\). In particular, since the kernel of \( \mathbf{P}_N \) is non-negative in view of (1.13) and Lemma 2.6(i), we get that

\[
(\mathbf{P}_N \otimes \mathbf{P}_N)[G_g + C](x, y) \geq 0,
\]

i.e.,

\[
(\mathbf{P}_N \otimes \mathbf{P}_N)G_g(x, y) \geq -(\mathbf{P}_N \otimes \mathbf{P}_N)C = -C.
\]

For the upper bound, note that since \( G_g \) has its eigenfunctions expansion only on the diagonal and since the heat kernel \( P_g \) is symmetric, we actually have

\[
(\mathbf{P}_N \otimes \mathbf{P}_N)G_g(x, y) = \sum_{n \geq 1} e^{-2N^{-2} \Delta^g n} \varphi_n(x) \varphi_n(y) = \int_{\mathcal{M}} P_g(8\pi N^{-2} x, z) G_g(z, y) dV_g(z) \]

\[
= (e^{2N^{-2} \Delta^g} G_g)(x, y) = (\mathbf{P}_N^2 \otimes \mathbf{Id})G_g(x, y),
\]

where the notation \( e^{t \Delta^g} \) means that the Laplace-Beltrami operator only acts on the \( x \) variable.

Fix then \( y \in \mathcal{M} \) and for \( t \geq 0 \) and \( x \in \mathcal{M} \) with \( y \neq x \), define

\[
u(t, x) \overset{\text{def}}{=} G_g(x, y) + \frac{t}{4\pi V_g(\mathcal{M})} - e^{t \Delta^g} G_g(x, y).
\]

Then we have \( u \in C(\mathbb{R}_+; L^2(\mathcal{M})) \) and it satisfies

\[
u(t) = \frac{1}{4\pi} \int_0^t P_g(t - t', x, y) dt'.
\]

This indeed follows from (2.11), (2.14) and a straightforward computation. The first claim then follows from the previous identity with Lemma 2.6, which ensures that \( u(t) \geq 0 \) for any \( t \geq 0 \), and taking \( t = 8\pi N^{-2} \) with the definition of \( \mathbf{P}_N(1.13) \).

As for the convergence property, for \( N_1 \leq N_2 \) and \((x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta\), we use the remark above with the semigroup property of \( P_g \) to write for \( j = 1, 2 \)

\[
\left| (\mathbf{P}_{N_j} \otimes \mathbf{P}_{N_j}) G_g(x, y) - (\mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2}) G_g(x, y) \right|
\leq \min \left\{ 1 + \log \frac{1}{d_g(x, y)}, N_1^{-1} d_g(x, y)^{-1} \right\}. \tag{2.18}
\]

We first deal with the inner integral. For \( z', z \in \mathcal{M} \), let \( \gamma : t \in [0, d_g(z', z)] \mapsto \gamma(t) \in \mathcal{M} \) be a unit speed geodesic between \( z \) and \( z' \), then we can use the mean value theorem and
Lemma 2.5 for $h = \sqrt{4\pi (N_1^{-2} - N_2^{-2})}$, to bound for any $A > 0$

$$\left| \int_{\mathcal{M}} P_{\gamma}(h^2, z', z) \left[ G_{\gamma}(z, y) - G_{\gamma}(z', y) \right] dV_{\gamma}(z) \right|$$

$$\lesssim \int_{\mathcal{M}} h^{-2}(h^{-1} d_{\gamma}(z', z))^{-A} d_{\gamma}(z', z) dV_{\gamma}(z)$$

for some $t_z \in (0, d_{\gamma}(z', z))$. Using then Lemma 2.7 (iii) we can continue with

$$\lesssim \int_{\mathcal{M}} h^{-2}(h^{-1} d_{\gamma}(z', z))^{-A} d_{\gamma}(z', z) dV_{\gamma}(z)$$

$$\lesssim \int_{\mathcal{M}} h^{-2}(h^{-1} d_{\gamma}(z', z))^{-A} d_{\gamma}(z', z) [d_{\gamma}(z, y) \wedge d_{\gamma}(z', y)]^{-1} dV_{\gamma}(z).$$

We then distinguish several cases to estimate the integral above, depending on which side is the smallest in the triangle made by $z, z'$ and $y$.

**Case 1:** If $d_{\gamma}(z', z) \lesssim d_{\gamma}(z, y) \sim d_{\gamma}(z', y)$. In this case we can bound the integral with

$$d_{\gamma}(z', y)^{-1} \int_{\mathcal{M}} h^{-2}(h^{-1} d_{\gamma}(z', z))^{-A} d_{\gamma}(z', z) dV_{\gamma}(z) \lesssim d_{\gamma}(z', y)^{-1} h,$$

where the last bound comes from integrating separately on the regions $d_{\gamma}(z', z) \lesssim h$ and $d_{\gamma}(z', z) \gtrsim h$ and taking $A$ large enough.

**Case 2:** If $d_{\gamma}(z', y) \lesssim d_{\gamma}(z', z) \sim d_{\gamma}(z, y)$. We can proceed as in the previous case to get the same bound as above.

**Case 3:** If $d_{\gamma}(z, y) \lesssim d_{\gamma}(z', z) \sim d_{\gamma}(z', y)$. In this case, we get the bound

$$\int_{\mathcal{M}} h^{-2}(h^{-1} d_{\gamma}(z', z))^{-A} d_{\gamma}(z', z) dV_{\gamma}(z).$$

In the region $d_{\gamma}(z', z) \lesssim h$ we can estimate it with

$$h^{-2} d_{\gamma}(z', y) \int_{\mathcal{M}} d_{\gamma}(z, y)^{-1} dV_{\gamma}(z) \lesssim h^{-2} d_{\gamma}(z', y)^2 \lesssim 1 \lesssim d_{\gamma}(z', y)^{-1} h,$$

where we used that in this case the integral runs over $d_{\gamma}(z, y) \lesssim d_{\gamma}(z', y) \sim d_{\gamma}(z', z) \lesssim h$. In the other region $d_{\gamma}(z', z) \gtrsim h$ we have the bound

$$h^{A-2} d_{\gamma}(z', y)^{1-A} \int_{\mathcal{M}} d_{\gamma}(z, y)^{-1} dV_{\gamma}(z) \lesssim h^{A-2} d_{\gamma}(z', y)^{2-A} \lesssim d_{\gamma}(z', y)^{-1} h$$

by using again that the integral runs over $d_{\gamma}(z, y) \lesssim d_{\gamma}(z', y)$ and by choosing $A = 3$.

Plugging this bound in the double integral above and using Lemma 2.5 with Remark 2.4 again, we finally estimate for any $A > 0$

$$\left| (P_{N_j} \otimes P_{N_j}) G_{\gamma}(x, y) - (P_{N_1} \otimes P_{N_2}) G_{\gamma}(x, y) \right| \lesssim \int_{\mathcal{M}} N_0^2 (N_0 d_{\gamma}(x, z'))^{-A} d_{\gamma}(z', y)^{-1} h dV_{\gamma}(z'),$$

where $N_0 = (N_j^{-2} + N_2^{-2})^{-\frac{1}{4}}$. To bound this last integral we divide again the argument into three cases.
where in the second to last step we chose $A = 2$.

**Case 1:** If $\mathbf{d}_g(x, y) \lesssim \mathbf{d}_g(x, z') \sim \mathbf{d}_g(z', y)$. In this case we have

$$
\left| \left( \mathbf{P}_{N_j} \otimes \mathbf{P}_{N_j} \right) G_g(x, y) - \left( \mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2} \right) G_g(x, y) \right| \\
\lesssim h \int_M N_0^2 \langle N_0 \mathbf{d}_g(x, z') \rangle^{-A} \mathbf{d}_g(x, z')^{-1} dV_g(z') \\
\lesssim hN_0^2 \int \mathbf{d}_g(x, y) \lesssim N_0^{-1} \mathbf{d}_g(x, z')^{-1} dV_g(z') \\
+ hN_0^{-2} \int \mathbf{d}_g(x, y) \lesssim N_0^{-1} \mathbf{d}_g(x, z')^{-1} - A dV_g(z') \\
\lesssim hN_01(\mathbf{d}_g(x, y) \lesssim N_0^{-1}) + h\left( N_0^{-1} \mathbf{d}_g(x, y) \right)^{-1} \\
\lesssim \mathbf{d}_g(x, y)^{-1} h
$$

where in the second to last step we chose $A = 2$.

**Case 2:** If $\mathbf{d}_g(x, z') \lesssim \mathbf{d}_g(x, y) \sim \mathbf{d}_g(z', y)$. In this case we have the bound

$$
\left| \left( \mathbf{P}_{N_j} \otimes \mathbf{P}_{N_j} \right) G_g(x, y) - \left( \mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2} \right) G_g(x, y) \right| \lesssim h \mathbf{d}_g(x, y)^{-1} \int_M N_0^2 \langle N_0 \mathbf{d}_g(x, z') \rangle^{-A} dV_g(z') \\
\lesssim \mathbf{d}_g(x, y)^{-1} h
$$

by separating again the regions $\mathbf{d}_g(x, z') \lesssim N_0^{-1}$ and $\mathbf{d}_g(x, z') \lesssim N_0^{-1}$ and taking for example $A = 3$.

**Case 3:** If $\mathbf{d}_g(z', y) \lesssim \mathbf{d}_g(x, y) \sim \mathbf{d}_g(x, z')$. The same argument as in the previous Case 3 gives the final bound

$$
\left| \left( \mathbf{P}_{N_j} \otimes \mathbf{P}_{N_j} \right) G_g(x, y) - \left( \mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2} \right) G_g(x, y) \right| \lesssim \mathbf{d}_g(x, y)^{-1} h.
$$

The second bound in the right-hand of the claim \eqref{eq:2.18} then follows from the previous bound with the definition of $h = \sqrt{4\pi(N_1^{-2} - N_2^{-2})}$. The first one is a consequence of the triangle inequality with \eqref{eq:2.17} and Lemma \ref{lem:2.7} (ii).

This completes the proof of Lemma \ref{lem:2.8}.

The upper bound \eqref{eq:2.17} in Lemma \ref{lem:2.8} is enough to bound the (punctured) Gaussian multiplicative chaos $\Theta_N$ in \eqref{eq:1.33}, uniformly in $N \in \mathbb{N}$. However, in order to prove the convergence of the truncated LQG measure \eqref{eq:1.11}, and prove the uniform boundedness of the negative moments of $\Theta_N$, for which we need a two-sided bound on $\left( \mathbf{P}_N \otimes \mathbf{P}_N \right) G_g$ in order to use Kahane’s inequality (Lemma \ref{lem:3.5}) and the argument of \ref{lem:4.2}. Thus we also establish the following lemma, which gives a two-sided bound for a more general regularization of the Green’s function.

**Lemma 2.9.** Let $\psi \in \mathcal{S}(\mathbb{R})$. There exists $C > 0$ such that for any $N \in \mathbb{N}$ and $(x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta$ we have

(i) if $\mathbf{d}_g(x, y) \geq N^{-1}$ then

$$
\left| (\psi \otimes \psi)(-N^{-2} \Delta_g) G_g(x, y) - G_g(x, y) \right| \leq C;
$$
(ii) if \( d_g(x, y) \leq N^{-1} \) then

\[
\left| (\psi \otimes \psi)( - N^{-2} \Delta_g) G_g(x, y) - (\psi \otimes \psi)( - N^{-2} \Delta_g) G_g(x, x) \right| \leq C.
\]

**Proof.** As in the proof of Lemma 2.5, note that it is enough to treat the case \( d_g(x, y) \leq \frac{i(M)}{2} \).

**Case 1:** if \( d_g(x, y) \geq N^{-1} \). In this case, with the same observation as in the proof of Lemma 2.8, we have

\[
(\psi \otimes \psi)( - N^{-2} \Delta_g) G_g(x, y) = \int_{\mathcal{M}} K_N(x, z) G_g(z, y) dV_g(z).
\]

where we wrote \( K_N \) for the kernel of \( \psi^2(-N^{-2} \Delta_g) \). Thus we have

\[
\left| (\psi \otimes \psi)( - N^{-2} \Delta_g) G_g(x, y) - G_g(x, y) \right| = \left| \int_{\mathcal{M}} K_N(x, z) \left[ \log \left( \frac{d_g(z, y)}{d_g(x, y)} \right) + \tilde{G}_g(z, y) - \tilde{G}_g(x, y) \right] dV_g(z) \right| \lesssim N^2 \int_{\mathcal{M}} \langle N d_g(x, z) \rangle^{-10} \left| \log \left( \frac{d_g(z, y)}{d_g(x, y)} \right) \right| dV_g(z) + O(1),
\]

where in the first step we used Lemma 2.7 and in the second one we used Lemma 2.5 with \( K_N \) being the kernel of the multiplier with symbol \( \psi^2 \in \mathcal{S}(\mathbb{R}) \). Here \( O(1) \) stands for a term bounded uniformly in both \( N \in \mathbb{N} \) and \( (x, y) \in \mathcal{M} \times \mathcal{M} \).

**Subcase 1.1:** If \( d_g(x, z) \lesssim d_g(x, y) \sim d_g(x, y) \). Then the log term in the previous integral is bounded above and below and the integral is \( O(1) \).

**Subcase 1.2:** If \( d_g(z, y) \ll d_g(x, z) \sim d_g(x, y) \). Then using polar geodesic coordinates around \( y \) and integrating by parts, we can estimate the integral above by

\[
N^{-8} d_g(x, y)^{-10} \int_0^{d_g(x,y)} \log \left( \frac{d_g(x, y)}{r} \right) r \, dr = N^{-8} d_g(x, y)^{-10} \left[ \frac{r^2}{2} \log \left( \frac{d_g(x, y)}{r} \right) \right]_0^{d_g(x,y)}
\]

\[
+ N^{-8} d_g(x, y)^{-10} \int_0^{d_g(x,y)} \frac{r}{2} dr \lesssim N^{-8} d_g(x, y)^{-8} \lesssim 1.
\]

**Subcase 1.3:** If \( N^{-1} \leq d_g(x, y) \ll d_g(x, z) \sim d_g(z, y) \). In this case, using again polar coordinates around \( y \) and noting \( \delta(\mathcal{M}) < \infty \) for the diameter of \( \mathcal{M} \), we get the bound

\[
N^{-8} \int_{d_g(x,y)}^{\delta(\mathcal{M})} r^{-10} \log \left( \frac{r}{d_g(x, y)} \right) r \, dr \lesssim N^{-8} \left[ r^{-8} \log \left( \frac{d_g(x, y)}{r} \right) \right]_{d_g(x,y)}^{\delta(\mathcal{M})} d_g(x, y)
\]

\[
+ N^{-8} \int_{d_g(x,y)}^{\delta(\mathcal{M})} r^{1-10} dr \lesssim N^{-8} \log \left( d_g(x, y) \right) + N^{-8} d_g(x, y)^{-8} \lesssim 1.
\]
Case 2: If \( d_g(x, y) \leq N^{-1} \). In this case we have similarly

\[
\left| (\psi \otimes \psi)(-N^{-2}\Delta_g)G_g(x, y) - (\psi \otimes \psi)(-N^{-2}\Delta_g)G_g(x, x) \right| \\
\lesssim N^2 \int_{\mathcal{M}} |N d_g(x, z)|^{-10} \left| \log \left( \frac{d_g(z, y)}{d_g(z, x)} \right) \right| dV_g(z) + O(1).
\]

Subcase 2.1: If \( d_g(x, y) \lesssim d_g(z, x) \sim d_g(z, y) \). In this case the log term is bounded above and below, so the integral above is \( O(1) \).

Subcase 2.2: If \( d_g(z, x) \lesssim d_g(z, y) \sim d_g(x, y) \leq N^{-1} \). In this case, using polar geodesic coordinates around \( x \) and integrating by parts we can bound the previous integral with

\[
N^2 \int_0^{d_g(x, y)} \log \left( \frac{d_g(x, y)}{r} \right) r dr = N^2 \left[ \frac{r^2}{2} \log \left( \frac{d_g(x, y)}{r} \right) \right]_0^{d_g(x, y)} + \frac{N^2}{2} \int_0^{d_g(x, y)} r dr \\
= \frac{N^2}{4} d_g(x, y)^2 \lesssim 1.
\]

Subcase 2.3: If \( d_g(z, y) \lesssim d_g(z, x) \sim d_g(x, y) \leq N^{-1} \). This case follows from the same computation as in Subcase 2.2.

This completes the proof of Lemma 2.9 \( \Box \)

In view of Lemma 2.7, the bound of Lemma 2.9 (i) is enough for our purpose. In order to precise the one of Lemma 2.9 (ii), we prove the following.

Lemma 2.10. Let \( \psi \in \mathcal{S}(\mathbb{R}) \) such that \( \psi(0) = 1 \). Then there exists a constant \( C_\psi \in \mathbb{R} \) such that

\[
\left\| (\psi \otimes \psi)(-N^{-2}\Delta_g)G_g(x, x) - \frac{1}{2\pi} \log N - \widetilde{G}_g(x, x) - C_\psi \right\|_{L^\infty(\mathcal{M})} \longrightarrow 0
\]

as \( N \rightarrow \infty \), where \( \widetilde{G}_g \) is given in Lemma 2.7.

Proof. Let \( \mathcal{K}_N \) be as in the proof of Lemma 2.9. Note that we have again the identity (2.19). Using Lemma 2.7, we first decompose

\[
(\psi \otimes \psi)(-N^{-2}\Delta_g)G_g(x, x) = -\frac{1}{2\pi} \int_{\mathcal{M}} \mathcal{K}_N(x, y) \log \left( d_g(y, x) \right) dV_g(y) \\
+ \int_{\mathcal{M}} \mathcal{K}_N(x, z) \widetilde{G}_g(y, x) dV_g(y).
\]

Since \( \widetilde{G}_g \) is continuous on \( \mathcal{M} \times \mathcal{M} \), the use of Lemma 2.5 shows that the second term converges to \( \widetilde{G}_g(x, x) \) uniformly.

It remains to treat the first term. We first take a finite partition of unity \( \{ \chi_j \} \) such that \( \chi_j \) are supported in balls of radii \( \ll \iota(\mathcal{M}) \), so that the exponential chart centred at some point in \( \text{supp} \chi_j \) entirely covers \( \text{supp} \chi_j \). Then, using Proposition 2.3 with Remark 2.4 we have

\[
\mathcal{K}_N(x, y) \log \left( d_g(y, x) \right) \\
= \sum_j \left[ (a_j(0, N^{-1}D) + N^{-1}a_j(0, N^{-1}D))(\exp_x)_* \chi_j + R_{j, N} \right] \log \left( d_g(\cdot, x) \right),
\]
where \( \exp_x : T_x \mathcal{M} \simeq \mathbb{R}^2 \to \mathcal{M} \) is the exponential map at \( x \in \mathcal{M} \), \( \bar{\chi}_j \in C_0^\infty(\mathcal{M}) \) with \( \bar{\chi}_j \equiv 1 \) on \( \text{supp} \chi_j \), and \( \| R_{j,N} \|_{H^{-s_2}(\mathcal{M}) \to H^{s_2} (\mathbb{R}^2)} \lesssim N^{s_1 + s_2 - 2} \) for any \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 \leq 2 \).

The principal symbol of \( \psi^2 \left( -N^{-2} \Delta_g \right) \) is
\[
a_j(z, \xi) = \chi_j(\exp_x(z)) \psi^2 \left( N^{-2} g^{j,k}(\exp_x(z)) \xi_j \xi_k \right),
\]
with \( \exp_x(0) = x \) and \( g^{j,k}(0) = \delta_{j,k} \), and \( \bar{a}_j \in \mathcal{S}^{-\infty}(\mathbb{R}^2 \times \mathbb{R}^2) \) and is compactly supported in \( z \).

This gives the decomposition
\[
K_N(x, y) = \sum_j \chi_j(x) \bar{\chi}_j(y) \left( \frac{1}{(2\pi)^2} \right) \int_{\mathbb{R}^2} e^{-i \exp_x^{-1}(y) \cdot \xi} \psi^2 \left( N^{-2} |\xi|^2 \right) d\xi + N^{-1} \bar{K}_{j,N} + K_{j,N}
\]
where \( \bar{K}_{j,N} \) is the kernel of \( \bar{a}_j(z, N^{-1} D) \) and \( K_{j,N} \) the one to \( R_{j,N} \).

From Proposition 2.3, we have
\[
\| R_{j,N} \|_{L^2(\mathcal{M}) \to H^{1+\delta}(\mathcal{M})} \lesssim N^{\delta - 1}.
\]
In particular this shows that
\[
\left\| \int_{\mathcal{M}} K_{j,N}(x, y) \log (d_g(x, y)) dV_g(y) \right\|_{L^\infty(\mathcal{M})} \lesssim \left\| R_{j,N} \log (d_g(\cdot, x)) \right\|_{L^\infty_x H^{1+\delta}_y} \lesssim N^{\delta - 1} \left\| \log (d_g(x, y)) \right\|_{L^\infty_x L^2_y} \lesssim N^{\delta - 1},
\]
for any \( 0 < \delta \ll 1 \).

As for \( \bar{K}_{j,N} \), we have by integrations by parts
\[
|\bar{K}_{j,N}(x, y)| = \left| \chi_j(x) \bar{\chi}_j(y) \left( \frac{1}{(2\pi)^2} \right) \int_{\mathbb{R}^2} e^{-i \exp_x^{-1}(y) \cdot \xi} \bar{a}_j(0, N^{-1} \xi) d\xi \right|
\leq \chi_j(x) \bar{\chi}_j(y) N^2 |N \exp_x^{-1}(y)|^{-2} \int_{\mathbb{R}^2} |\Delta_\xi \bar{a}_j(0, \xi)| d\xi
\lesssim \chi_j(x) \bar{\chi}_j(y) (d_g(x, y))^{-2}.
\]
Interpolating with the trivial bound \( |\bar{K}_{j,N}(x, y)| \lesssim \chi_j(x) \bar{\chi}_j(y) N^2 \), we get
\[
|\bar{K}_{j,N}(x, y)| \lesssim \chi_j(x) \bar{\chi}_j(y) N^\delta (d_g(x, y))^{\delta - 2}
\]
for any \( 0 < \delta \ll 1 \). This shows that
\[
N^{-1} \left\| \int_{\mathcal{M}} \bar{K}_{j,N}(x, y) \log (d_g(x, y)) dV_g(y) \right\|_{L^\infty(\mathcal{M})} \lesssim N^{\delta - 1} \int_{\mathcal{M}} (d_g(x, y))^{\delta - 2} \log (d_g(x, y)) dV_g(y)
\lesssim N^{\delta - 1}.
\]

Finally, to deal with the leading term, we have
\[
- \frac{\chi_j(x)}{2\pi} \int_{\mathcal{M}} \bar{\chi}_j(y) \left( \frac{1}{(2\pi)^2} \right) \int_{\mathbb{R}^2} e^{-i \exp_x^{-1}(y) \cdot \xi} \psi^2 \left( N^{-2} |\xi|^2 \right) d\xi \log (d_g(x, y)) dV_g(y)
= - \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \bar{\chi}_j(\exp_x(z)) \psi^2 \left( N^{-2} |\xi|^2 \right) \log (|z|) |g(z)\right|^\frac{1}{2} dz d\xi.
\]
Changing variables in $\xi$ then in $z$, we continue with

\[
= -\frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \tilde{\chi}_j(\exp_x(N^{-1}z)) \psi^2(|\xi|^2) \log(N^{-1}|z|) |g(N^{-1}z)|^{1/2} dz d\xi
\]

\[
= \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \tilde{\chi}_j(\exp_x(N^{-1}z)) \psi^2(|\xi|^2) \left[-\log N + \log(|z|)\right] |g(N^{-1}z)|^{1/2} dz d\xi
\]

\[
= I_j + II_j.
\]

The first term is then given by

\[
I_j = \frac{1}{2\pi} \log N \left\{ \frac{\chi_j(x)}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \tilde{\chi}_j(\exp_x(N^{-1}z)) \psi^2(|\xi|^2) |g(N^{-1}z)|^{1/2} dz d\xi \right\}
\]

\[
= \frac{1}{2\pi} \log N \left\{ \frac{\chi_j(x)}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F} \left[ \psi^2(|\xi|^2) \right](z) \tilde{\chi}_j(\exp_x(N^{-1}z)) |g(N^{-1}z)|^{1/2} dz \right\}.
\]

Note that this last integral converges to $\tilde{\chi}_j(x)$. Indeed, since $\psi \in \mathcal{S}(\mathbb{R})$ and $\exp_x g$ are smooth with $D \exp_x g$ depending continuously on $x \in \mathcal{M}$, we have by the mean value theorem

\[
\left\| \int_{\mathbb{R}^2} \mathcal{F} \left[ \psi^2(|\xi|^2) \right](z) \tilde{\chi}_j(\exp_x(N^{-1}z)) |g|^{1/2}(N^{-1}z) - \tilde{\chi}_j(\exp_x(0)) |g(0)|^{1/2} dz \right\|_{L^\infty(\mathcal{M})}
\]

\[
\lesssim \sup_{x \in \mathcal{M}} \int_{\mathbb{R}^2} (z)^{-10} N^{-1} |z| \left\| \int_0^1 \nabla_z [\tilde{\chi}_j \circ \exp_x |g|^{1/2}] (\theta N^{-1}z) d\theta dz \right\|
\]

\[
\lesssim N^{-1} \int_{\mathbb{R}^2} |z|^2 (z)^{-10} dz \lesssim N^{-1}.
\]

Thus, using that $g(0) = \text{Id}$ with $\exp_x(0) = x$, that $\{\chi_j\}$ is a partition of unity with $\tilde{\chi}_j \equiv 1$ on supp $\chi_j$, and that $\psi(0) = 1$, we have

\[
\sum_j I_j = \sum_j \frac{\chi_j(x)}{2\pi} \log N \left\{ \frac{\chi_j(x)}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F} \left[ \psi^2(|\xi|^2) \right](z) dz \right\} + o(1)
\]

\[
= \sum_j \frac{\chi_j(x) \psi^2(0)}{2\pi} \log N + o(1) = \frac{1}{2\pi} \log N + o(1),
\]

where $o(1) \to 0$ as $N \to \infty$, uniformly on $\mathcal{M}$.

Finally, the second term is

\[
II_j = -\frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \tilde{\chi}_j(\exp_x(N^{-1}z)) \psi^2(|\xi|^2) \log(|z|) |g(N^{-1}z)|^{1/2} dz d\xi,
\]

and the same argument as above gives

\[
\sum_j II_j = C_\psi + o(1)
\]

uniformly on $\mathcal{M}$, with

\[
C_\psi \overset{\text{def}}{=} -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \mathcal{F} \left[ \psi^2(|\xi|^2) \right](z) \log(|z|) dz < \infty.
\]

This completes the proof of Lemma 2.10. □
Remark 2.11.
(i) The condition \( \psi(0) = 1 \) is necessary to have proper approximations of the identity, i.e. the corresponding family of smooth kernels \( K_N \) satisfy \( K_N(x, y) \to \delta_x(y) \) in the sense of distributions as \( N \to \infty \), and \( \int_{\mathcal{M}} K_N(x, y) dV(x) = 1 \) for any \( x \in \mathcal{M} \). In particular, note that Lemma 2.10 holds for any kernel \( K_N(x, y) = \chi(N^2 d(x, y)^2) \) with \( \chi \in C_0^\infty((0, r)) \) and \( 0 < r \ll \nu(\mathcal{M})^2 \).

(ii) On the other hand, using an approximation by averaging on geodesic circles as in \([15, 17, 30] \) formally corresponds to taking \( \chi = 1_{\{|z| = 1\}} \), which is not covered by our result. Note also that in this case \( F[\psi^2(|\xi|^2)] = 1_{\{|z| = 1\}} \), which explains why \( C_\psi = 0 \) in these works.

Collecting the previous results, we finally obtain the following key bound.

Corollary 2.12. Let \( \psi \in \mathcal{S}(\mathbb{R}) \) such that \( \psi(0) = 1 \). Then there exists \( C > 0 \) such that for any \( N \in \mathbb{N} \) and \( (x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta \) it holds
\[
\left| (\psi \ast \psi)(-N^{-2} \Delta_g) G_g(x, y) + \frac{1}{2\pi} \log (d_g(x, y) + N^{-1}) \right| \leq C. \tag{2.20}
\]

Proof. Since
\[
-\frac{1}{2\pi} \log (d_g(x, y) + N^{-1}) = -\frac{1}{2\pi} \log \left( \max (d_g(x, y), N^{-1}) \right) + O(1),
\]
the bound (2.20) follows from: (a) Lemma 2.9(i) with Lemma 2.7(ii) in the case \( d_g(x, y) \geq N^{-1} \); and (b) Lemma 2.9(ii) with Lemma 2.10 in the case \( d_g(x, y) \leq N^{-1} \). \( \Box \)

At last, we look at the behaviour of different regularizations of \( G_g \).

Lemma 2.13. Let \( \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}) \) be such that \( \psi_1(0) = \psi_2(0) \). Then for any \( 0 < \delta \ll 1 \), there exists \( C > 0 \) such that for any \( N \geq 1 \) and any \( (x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta \) it holds
\[
\left| (\psi_1 \ast \psi_1)(-N^{-2} \Delta_g) G_g(x, y) - (\psi_2 \ast \psi_2)(-N^{-2} \Delta_g) G_g(x, y) \right| \leq C \min \left\{ -\log (d_g(x, y) + N^{-1}) + 1; N^{-1} d_g(x, y)^{-1} \right\}. \tag{2.21}
\]

Proof. This is a straightforward adaptation of the proofs of Lemmas 2.8 and 2.10. First, the use of Corollary 2.12 with the triangle inequality gives the first term in the right-hand side of (2.21).

Next, defining \( \psi(x) = \psi_1^2(x) - \psi_2^2(x) \) we have \( \psi \in \mathcal{S}(\mathbb{R}) \) with \( \psi(0) = 0 \), and
\[
(\psi_1 \ast \psi_1)(-N^{-2} \Delta_g) G_g(x, y) - (\psi_2 \ast \psi_2)(-N^{-2} \Delta_g) G_g(x, y) = \sum_{n \geq 1} \psi(N^{-2} \lambda_n^2) \frac{\varphi_n(x) \varphi_n(y)}{\lambda_n^2}.
\]

First, we replace \( \lambda_n^{-2} \) by \( \langle \lambda_n \rangle^{-2} \), since using (2.3) and that \( \psi(0) = 0 \) with \( \psi \in \mathcal{S}(\mathbb{R}) \), we have
\[
\left\| \sum_{n \geq 1} \psi(N^{-2} \lambda_n^2) \frac{\varphi_n(x) \varphi_n(y)}{\lambda_n^2} - \sum_{n \geq 0} \psi(N^{-2} \lambda_n^2) \frac{\varphi_n(x) \varphi_n(y)}{\langle \lambda_n \rangle^2} \right\|_{L^\infty(\mathcal{M} \times \mathcal{M})} \\
\lesssim \sum_{n \geq 1} \psi(N^{-2} \lambda_n^2) \frac{1}{\lambda_n \langle \lambda_n \rangle^2} \lesssim \sum_{n \geq 1} (1 \wedge N^{-2} \lambda_n^2) \langle N^{-1} \lambda_n \rangle^{-10} \frac{1}{\lambda_n \langle \lambda_n \rangle^2} \\
\lesssim N^{-1} \sum_{n \geq 1} \langle N^{-1} \lambda_n \rangle^{-10} \frac{1}{\langle \lambda_n \rangle^2} \sim N^{-1} \log N.
\]
As for the sum with \( \langle \ldots \rangle \), it can be expressed as the kernel of \((1 - \Delta_g)^{-1} \psi(-N^{-2} \Delta_g)\). This last operator can be expanded locally as
\[
\kappa^*(\chi(1 - \Delta_g)^{-1} \psi(-N^{-2} \Delta_g)) = [a_{-2}(z, D) \kappa^* \chi + R_{-3}] \psi(-N^{-2} \Delta_g)
\]
\[
= a_{-2}(z, D)[a_0(z, N^{-2} D) \kappa^* \chi + R_{-1,N}] + R_{-3} \psi(-N^{-2} \Delta_g),
\]
where \((U, V, \kappa)\) is some chart on \( M \), \( \chi, \tilde{\chi} \in C_0^\infty(V) \) with \( \tilde{\chi} \equiv 1 \) on \( \text{supp} \chi \). Moreover
\[
a_{-2}(z, \xi) = \chi(\kappa^{-1}(z))(1 + g^{ij}(z) \xi_i \xi_j)^{-1}
\]
is the principal symbol of \((1 - \Delta_g)^{-1}\) in \( \kappa \), and
\[
a_0(z, \xi) = \tilde{\chi}(\kappa^{-1}(z)) \psi(g^{ij}(z) \xi_i \xi_j).
\]
The remainders satisfy the bounds
\[
\|R_{-3}\|_{H^s(M) \rightarrow H^{s+2}(\mathbb{R}^2)} \lesssim 1
\]
for any \( s \in \mathbb{R} \), and
\[
\|R_{-1,N}\|_{H^{-s_1}(M) \rightarrow H^{-s_2}(\mathbb{R}^2)} \lesssim N^{s_1 + s_2 - 1},
\]
for any \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 \leq 1 \).

Proceeding as for the classical composition rule for pseudo-differential operators, we can use Taylor’s formula at order one to write the symbol of \( a_{-2}(\kappa^{-1}(x), D) a_0(y, N^{-1} D) \) as
\[
b(\kappa^{-1}(x), \xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iy \eta} a_{-2}(\kappa^{-1}(x), \xi + \eta) a_0(\kappa^{-1}(x) + y, N^{-1} \xi) \, d\eta \, dy
\]
\[
= a_{-2}(\kappa^{-1}(x), \xi) a_0(\kappa^{-1}(x), N^{-1} \xi) + \tilde{R}_{-3,N}(\kappa^{-1}(x), \xi)
\]
with
\[
\tilde{R}_{-3,N}(\kappa^{-1}(x), \xi)
\]
\[
= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \eta} a_{-2}(\kappa^{-1}(x), \xi + \eta) \int_0^1 z \cdot \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1} \xi) \, d\theta \, d\eta \, dz
\]
\[
= \frac{1}{i(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \eta} \nabla_x a_{-2}(\kappa^{-1}(x), \xi + \eta) \cdot \int_0^1 \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1} \xi) \, d\theta \, d\eta \, dz
\]
after integrating by parts in \( \eta \).

Let us then take some charts \((U_1 \times U_2, V_1 \times V_2, \kappa_1 \otimes \kappa_2)\) in \( M \times M \) containing \((x, y)\), and non negative \( \chi_1 \chi_2, \chi_1 \chi_2 \in C_0^\infty(V_1 \times V_2) \) with \( \chi_1(x) = 1 = \chi_2(y) \) and \( \tilde{\chi}_j \equiv 1 \) on \( \text{supp} \chi_j \). We thus have
\[
\chi_1(x) \chi_2(y) \sum_{n \geq 0} \psi(n \lambda_n^2) \frac{\varphi_n(x) \varphi_n(y)}{\lambda_n^2}
\]
\[
= \frac{\chi_1(x) \chi_2(y)}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\kappa_1^{-1}(x) - \kappa_2^{-1}(y)) \xi} a_{-2}(\kappa_1^{-1}(x), \xi) a_0(\kappa_1^{-1}(x), N^{-1} \xi) \, d\xi
\]
\[
+ K_{-1,N}(x, y) + K_{-3,N}(x, y) + \bar{K}_{-3,N}(x, y),
\]
where \( K_{-1,N} \) is the kernel of \( \chi_1(\kappa_1) a_{-2}(z, D) R_{-1,N} \chi_2 \), \( K_{-3,N} \) the one to \( \chi_1(\kappa_1) R_{-3} \psi(-N^{-2} \Delta_g) \chi_2 \) and \( \bar{K}_{-3,N} \) the one to \( \chi_1(\kappa_1) \bar{R}_{-3,N} \chi_2 \).
In particular, we have
\[
\overline{K}_{-3,N}(x,y) = \chi_1(x)\chi_2(y) \frac{1}{i(2\pi)^4} \int_{\mathbb{R}^2} e^{i(\kappa_1^{-1}(x) - \kappa_2^{-1}(y)) \xi} \overline{R}_{-3,N}(\kappa^{-1}(x), \xi) d\xi \\
= \frac{1}{i(2\pi)^4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i[\kappa_1^{-1}(x) - \kappa_2^{-1}(y)] \xi} \nabla_\eta a_{-2}(\kappa^{-1}(x), \xi + \eta) \\
\cdot \int_0^1 \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1}\xi) d\theta d\eta dz d\xi.
\]
We first integrate by parts in \( \eta \) to get some decay in \( z \): for any \( A \in \mathbb{N} \) we then have
\[
\overline{K}_{-3,N}(x,y) = \chi_1(x)\chi_2(y) \frac{1}{i(2\pi)^4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i[\kappa_1^{-1}(x) - \kappa_2^{-1}(y)] \xi} \eta^{-A} \nabla_\eta (D_\eta)^A a_{-2}(\kappa^{-1}(x), \xi + \eta) \\
\cdot \int_0^1 \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1}\xi) d\theta d\eta dz d\xi.
\]
Next, we integrate by parts in \( z \) to get for any \( B \in \mathbb{N} \)
\[
\overline{K}_{-3,N}(x,y) = \chi_1(x)\chi_2(y) \frac{1}{i(2\pi)^4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i[\kappa_1^{-1}(x) - \kappa_2^{-1}(y)] \xi} \eta^{-B} \nabla_\eta (D_\eta)^A a_{-2}(\kappa^{-1}(x), \xi + \eta) \\
\cdot \int_0^1 \langle D_z \rangle^B \{ (z)^{-A} \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1}\xi) \} d\theta d\eta dz d\xi.
\]
Then, using the definition of \( a_{-2} \) and \( a_0 \) with the smoothness of \( g \) and the compactness of \( \mathcal{M} \), we get that
\[
|\nabla_\eta (D_\eta)^A a_{-2}(\kappa^{-1}(x), \xi + \eta)| \lesssim \langle \xi + \eta \rangle^{-3}
\]
and
\[
|\langle D_z \rangle^B \{ (z)^{-A} \nabla_x a_0(\kappa^{-1}(x) + \theta z, N^{-1}\xi) \}| \lesssim \langle z \rangle^{-A N^2} \langle \xi \rangle^2 \langle N^{-1} \xi \rangle^{-C}
\]
for any \( C > 0 \), where we also used that \( \psi(0) = 0 \). Taking \( A > 2 \) we deduce that
\[
|\overline{K}_{-3,N}(x,y)| \lesssim \chi_1(x)\chi_2(y) \int_{\mathbb{R}^2} \langle \eta \rangle^{-B} \langle \xi + \eta \rangle^{-3} \langle \xi \rangle^2 \langle N^{-1} \xi \rangle^{-C} d\eta d\xi \\
\lesssim \chi_1(x)\chi_2(y) \int_{\mathbb{R}^2} \langle \xi \rangle^{-3} \langle \xi \rangle^2 \langle N^{-1} \xi \rangle^{-C} d\xi \\
\lesssim \chi_1(x)\chi_2(y) \left\{ N^{-2} \int_{|\xi| \leq N} \langle \xi \rangle^{-1} d\xi + N^{C-2} \int_{|\xi| \geq N} |\xi|^{-1-C} d\xi \right\} \\
\lesssim N^{-1} \chi_1(x)\chi_2(y).
\]
To deal with the other remainder terms, we have from the properties of \( a_{-2} \) and \( R_{-1,N} \) that for \( 0 < \delta \ll 1 \),
\[
\| a_{-2}(\kappa^{-1}(x), D) R_{-1,N} \|_{H^{-1-\delta}(\mathcal{M}) \rightarrow H^{1+\delta}(\mathcal{M})} \lesssim \| R_{-1,N} \|_{H^{1-\delta}(\mathcal{M}) \rightarrow H^{1+\delta}(\mathbb{R}^2)} \lesssim N^{2\delta - 1},
\]
which implies that
\[
\| \chi_1(x)\chi_2(y) K_{-1,N} \|_{L^\infty(\mathcal{M} \times \mathcal{M})} \lesssim N^{2\delta - 1}.
\]
Similarly,
\[ \| R_{-\delta} \psi ( - N^{-2} \Delta_g ) \|_{H^{1,-\delta} ( \mathcal{M} ) \to H^{1,\delta} ( \mathbb{R}^2 )} \lesssim \| \psi ( - N^{-2} \Delta_g ) \|_{H^{2,-\delta} ( \mathcal{M} ) \to H^{1,\delta} ( \mathcal{M} )} \lesssim N^{2 \delta - 1} \]
for \( 0 < \delta \ll 1 \), where we used that \( \psi (0) = 0 \) in the last step. Along with the previous bounds, this implies that
\[
\chi_1 (x) \chi_2 (y) \sum_{n \geq 0} \psi (N^{-2} \lambda_n^2) \varphi_n (x) \varphi_n (y) / (\lambda_n)^2
\]
\[= \chi_1 (x) \chi_2 (y) \frac{(2 \pi)^2}{\chi_1 (y)} \int_{\mathbb{R}^2} e^{i (\kappa_1^{-1} (x) - \kappa_2^{-1} (y)) \cdot \xi} a_{-2} (\kappa_1^{-1} (x), \xi) a_0 (\kappa_1^{-1} (x), N^{-1} \xi) d \xi + O (N^{2 \delta - 1}).\]

Finally, to deal with the leading term, we see that its contribution is non trivial only if \( V_1 \cap V_2 \neq \emptyset \), and in this case by taking \( V_1 \times V_2 \) to be sufficiently small around \((x, y)\) we can choose \( \kappa_1 = \kappa_2 = \exp_x \), so that \( g^{j,k} (\kappa_1^{-1} (x)) = \delta_{j,k} \) and \( \kappa_1^{-1} (x) = 0 \). From the expression of \( a_{-2} \) and \( a_0 \) and integrating by parts this gives
\[
\left| \int_{\mathbb{R}^2} e^{-i \exp_x^{-1} (y) \cdot \xi} \frac{1}{1 + |\xi|^2} \psi (N^{-2} |\xi|^2) d \xi \right|
\]
\[\lesssim \int_{\mathbb{R}^2} \left| \Delta \left[ \frac{\psi (N^{-2} |\xi|^2)}{1 + |\xi|^2} \right] \right| d \xi \]
\[\lesssim d_x (x, y)^{-2} \int_{\mathbb{R}^2} (\xi)^{-4} N^{-2} |\xi|^2 (N^{-1} \xi)^{-10} + (\xi)^{-2} N^{-2} (N^{-1} \xi)^{-10} \right] d \xi \]
\[\lesssim d_x (x, y)^{-2} N^{-2} \log (N),\]
where we used that \( \psi \in \mathcal{S} (\mathbb{R}) \) with \( \psi (0) = 0 \). Interpolating with the trivial bound
\[
\left| \int_{\mathbb{R}^2} e^{-i \exp_x^{-1} (y) \cdot \xi} \frac{1}{1 + |\xi|^2} \psi (N^{-2} |\xi|^2) d \xi \right| \lesssim \int_{\mathbb{R}^2} N^{-2} (N^{-1} \xi)^{-10} d \xi \lesssim 1
\]
yields (2.21). This completes the proof of Lemma 2.13.

2.4. Conformal change of metric. Recall that we fixed a smooth metric \( g \) on \( \mathcal{M} \) at the beginning of the section. We now invoke the uniformization theorem (see e.g. [4, Section 8.8]) to get that there exists a smooth metric \( g_0 \) with constant curvature in the conformal class of \( g \), i.e.
\[ g (x) = e^{f_0 (x)} g_0 (x) \]
for some \( f_0 \in C^\infty (\mathcal{M}) \) and all \( x \in \mathcal{M} \). Then the Laplace-Beltrami operator \( \Delta_0 \) and the (constant) Ricci scalar curvature \( R_0 \) associated with \( g_0 \) satisfy
\[
\Delta_g u = e^{-f_0} \Delta_0 u \quad \text{and} \quad R_g = e^{-f_0} (R_0 - \Delta_0 f_0)
\]
for any \( u \in C^\infty (\mathcal{M}) \). In particular, the Sobolev space \( H^1_0 (\mathcal{M}) \) defined in (1.7) is invariant under conformal change of the metric. Indeed, it holds for any \( u \in C^\infty (\mathcal{M}) \)
\[
\int_{\mathcal{M}} |\nabla_g u |^2 g dV_g = - \int_{\mathcal{M}} u \Delta_g u dV_g = - \int_{\mathcal{M}} u e^{-f_0} \Delta_0 u e^{f_0} dV_0 = \int_{\mathcal{M}} |\nabla_0 u |^2 dV_0.
\]

Recall also that for a two-dimensional Riemannian manifold, the Ricci scalar curvature is twice the Gaussian curvature, so that Gauss-Bonnet theorem reads
\[
\int_{\mathcal{M}} R_g dV_g = 4 \pi \chi (\mathcal{M}) = \int_{\mathcal{M}} R_0 dV_0 = R_0 V_0 (\mathcal{M}),
\]
(2.23)
where $\chi(M)$ is the Euler characteristic of $M$ and $V_0$ is the volume form associated with $g_0$. The last two equalities indeed follow again from $dV_g = |g|^\frac{1}{2} dx = e^{f_0} dV_0$ with (2.22) and that $\mathcal{R}_0$ is constant.

We also consider the variation of the constant $\Xi = \Xi(g)$ in (1.10). It satisfies the so-called Polyakov formula (see [49] or (2.10) in [30])

$$\Xi(g) = \exp \left( \frac{1}{96\pi} \int_M (|\nabla_0 f_0|^2 + \mathcal{R}_0 f_0) dV_0 \right) \Xi(g_0). \quad (2.24)$$

Finally, up to replacing $g_0$ with $\tilde{g}_0 = \frac{V_g(M)}{V_0(M)} g_0$, whose curvature is also constant, we can then assume that

$$V_0(M) = V_g(M). \quad (2.25)$$

We now investigate the relation between the Green’s function $G_0$ associated with $g_0$ and $G_g$.

**Lemma 2.14.** For all $(x, y) \in M \times M \setminus \Delta$,

$$G_0(x, y) = G_g(x, y) - \langle G_g(x, \cdot) \rangle_0 - \langle G_g(\cdot, y) \rangle_0 + \langle G_g \rangle_0. \quad (2.26)$$

In particular, we have the equality in law

$$X_0 = X_g - \langle X_g \rangle_0. \quad (2.27)$$

**Proof.** Take any $\chi \in C^\infty(M)$ and define

$$u(x) = \int_M \left[ G_0(x, y) + \langle G_g(x, \cdot) \rangle_0 \right] \chi(y) dV_g(y)$$

and $v = u - \langle u \rangle_g$. Then using (2.15)-(2.16)-(2.25) we can compute

$$-\Delta_g v = -e^{-f_0} \Delta_0 \int_M G_0(x, y) \chi(y) e^{f_0} dV_0(y)$$

$$- \frac{1}{V_0(M)} \int_M G_g(x, z) e^{-f_0(z)} dV_g(z) \int_M \chi(y) dV_g(y)$$

$$= e^{-f_0} \left[ \chi e^{f_0} - \langle \chi e^{f_0} \rangle_0 \right](x) + \left[ e^{-f_0} - \langle e^{-f_0} \rangle_g \right](x) \langle \chi \rangle_g$$

$$= \chi(x) - \langle \chi \rangle_g.$$

Using (2.16), the previous computation then shows that

$$v(x) = \int_M G_g(x, y) \chi(y) dV_g(y)$$

for any $\chi \in C^\infty(M)$, which in turn yields

$$G_g(x, y) = G_0(x, y) + \langle G_g(x, \cdot) \rangle_0 - \langle G_0(\cdot, y) \rangle_g. \quad (2.28)$$

Integrating in $x$ with respect to $dV_0$ the previous identity, we find

$$\langle G_g(\cdot, y) \rangle_0 = \langle \langle G_g \rangle_0 \rangle_0 - \langle G_0(\cdot, y) \rangle_g. \quad (2.29)$$

Plugging (2.29) in (2.28) gives (2.26).
As for (2.27), it then follows from (2.26) and the fact that the covariance function completely characterizes the Gaussian processes \(X_g - \langle X_g \rangle_0\) and \(X_0\), and for any \((x, y) \in \mathcal{M} \times \mathcal{M} \setminus \triangle\) this latter is

\[
\mathbb{E} \left[ \left( X_g(x) - \langle X_g \rangle_0 \right) \left( X_g(y) - \langle X_g \rangle_0 \right) \right] = G_g(x, y) - \langle G_g(x, \cdot) \rangle_0 - \langle G_g(\cdot, y) \rangle_0 + \langle \langle G_g \rangle_0 \rangle_0 = G_0(x, y).
\]

This completes the proof of Lemma 2.14

Next, we look at the corresponding relation for the truncated version \((P_N \otimes P_N)G_g\). Indeed, recall that \(P_N = e^{N^{-2} \Delta_g}\) implicitly depends on the metric as well. We then have the following result concerning regularizations with the more general class of multipliers \(\psi \in \mathcal{S}(\mathbb{R})\).

**Lemma 2.15.** Let \(\psi \in \mathcal{S}(\mathbb{R})\) with \(\psi(0) = 1\). Then we have the following.

(i) We have the uniform convergence

\[
\left\| \left( \psi \otimes \psi \right) (-N^{-2} \Delta_g) G_0(x, x) - \left( \psi \otimes \psi \right) (-N^{-2} \Delta_0) G_0(x, x) - \frac{1}{4\pi} f_0(x) \right\|_{L^\infty(\mathcal{M})} \to 0 \quad (2.30)
\]

as \(N \to \infty\);

(ii) There exists \(C > 0\) such that for any \(N \in \mathbb{N}\) and \((x, y) \in \mathcal{M} \times \mathcal{M} \setminus \triangle\),

\[
\left| \left( \psi \otimes \psi \right) (-N^{-2} \Delta_g) G_0(x, y) + \frac{1}{2\pi} \log (d_0(x, y) + N^{-1}) \right| \leq C. \quad (2.31)
\]

**Proof.** We only prove the first point, since (2.31) then follows by combining the arguments for (2.30) with a straightforward adaptation of the proofs of Lemma 2.8

First, using Lemma 2.7, we have

\[
\left( \psi \otimes \psi \right) (-N^{-2} \Delta_g) G_0(x, x) - \left( \psi \otimes \psi \right) (-N^{-2} \Delta_0) G_0(x, x) = -\frac{1}{2\pi} \left( \psi \otimes \psi \right) (-N^{-2} \Delta_g) \log(d_0)(x, x) + \frac{1}{2\pi} \left( \psi \otimes \psi \right) (-N^{-2} \Delta_0) \log (d_0)(x, x)
\]

\[
+ \left( \psi \otimes \psi \right) (-N^{-2} \Delta_g) \tilde{G}_0(x, x) - \left( \psi \otimes \psi \right) (-N^{-2} \Delta_0) \tilde{G}_0(x, x).
\]

Since \(\tilde{G}_0\) is continuous on \(\mathcal{M} \times \mathcal{M}\), we have that the last two terms above both converge uniformly to \(\tilde{G}_0(x, x)\). Hence their contribution cancel each other, and we are left with estimating the contribution of the log terms.

Next, we observe that we can write for any \(u \in \mathcal{D}'(\mathcal{M})\)

\[
\psi(-N^{-2} \Delta_g) u(x) = e^{-\frac{1}{2} f_0(x)} \psi(-N^{-2} \Delta_0)[e^{\frac{1}{2} f_0} u](x).
\]

Here \(\tilde{\Delta}_0 = e^{\frac{1}{2} f_0} \Delta_0 e^{-\frac{1}{2} f_0}\). Indeed, the previous identity can be seen from (2.22) by solving the eigenvalue problem for \(\tilde{\Delta}_0\):

\[
\tilde{\Delta}_0 \varphi = -\lambda^2 \varphi \iff e^{\frac{1}{2} f_0} \Delta_g [e^{-\frac{1}{2} f_0} \varphi] = -\lambda^2 \varphi \iff \lambda = \lambda_n(g) \text{ and } \varphi = e^{\frac{1}{2} f_0} \varphi_n(g),
\]
from which we deduce with the definition of the functional calculus that

\[ \psi(-N^{-2} \Delta_g) u = \sum_{n \geq 1} \psi(N^{-2} \lambda_n) \langle u, \varphi_n(g) \rangle_n \varphi_n(g) = \sum_{n \geq 1} \psi(N^{-2} \lambda_n) \langle e^{\frac{1}{2} \tau_0}, e^{\frac{1}{2} \tau_0} \varphi_n(g) \rangle_{\varphi_n(g)} \]

\[ = e^{-\frac{1}{2} f_0} \sum_{n \geq 1} \psi(N^{-2} \lambda_n) \langle e^{\frac{1}{2} \tau_0}, e^{\frac{1}{2} \tau_0} \varphi_n(g) \rangle_{\varphi_n(g)} e^{\frac{1}{2} \tau_0} \varphi_n(g) \]

\[ = e^{-\frac{1}{2} f_0} \psi(-N^{-2} \Delta_0) \left[ e^{\frac{1}{2} \tau_0} u \right]. \]

Next, using that \(-\Delta_0\) and \(-\Delta_0\) are both elliptic and self-adjoint on \(L^2(\mathcal{M}, g_0)\), we can use Proposition \[2.3\] to write the smoothing operators as

\[ \psi(-N^{-2} \Delta_0) = \sum_j (\kappa_j) \left\{ \left[ \sum_{k=0}^{2} N^{-k} a_{j,k}(z, N^{-1} D) \right] \kappa_j^* \chi_j + (\kappa_j^* \chi_j) R_{j, N} \right\} \]

and

\[ \psi(-N^{-2} \Delta_0) = \sum_j (\kappa_j) \left\{ \left[ \sum_{k=0}^{2} N^{-k} a_{j,k}(z, N^{-1} D) \right] \kappa_j^* \chi_j + (\kappa_j^* \chi_j) \tilde{R}_{j, N} \right\}, \]

where \(\{\chi_j\}\) is a suitable partition of unity adapted to some charts \((U_j, V_j, \kappa_j)\).

With the previous remarks we can write

\[ (\psi \otimes \psi) (-N^{-2} \Delta_g) \log \left( d_0 \right) (x, x) \]

\[ = \int_{\mathcal{M}} \int_{\mathcal{M}} e^{-f_0(x)} \tilde{K}_N(x, y) \tilde{K}_N(x, y') e^{\frac{1}{2} f_0(y)} e^{\frac{1}{2} f_0(y')} \log \left( d_0(y, y') \right) dV_0(y) dV_0(y'), \]

where \(\tilde{K}_N\) is the kernel of \(\psi(-N^{-2} \Delta_0)\). The previous expansion then allows us to decompose this term as

\[ = \sum_j \chi_j(x) e^{-f_0(x)} \int_{\mathcal{M}} \int_{\mathcal{M}} \left[ \sum_{k=0}^{2} N^{-k} \tilde{K}_{j,k,N}(x, y) + \tilde{K}_{j,N}(x, y) \right] \]

\[ \times \left[ \sum_{k=0}^{2} N^{-k} \tilde{K}_{j,k',N}(x, y') + \tilde{K}_{j,N}(x, y') \right] e^{\frac{1}{2} f_0(y)} e^{\frac{1}{2} f_0(y')} \log \left( d_0(y, y') \right) dV_0(y) dV_0(y'), \]

where \(\tilde{K}_{j,k,N}\) is the kernel of \(\chi_j(\kappa_j) a_{j,k}(z, N^{-1} D) \kappa_j^* \chi_j\) and \(\tilde{K}_{j,N}\) the one to \(\chi_j(\kappa_j) R_{j, N}\).

From the property of the remainders in Proposition \[2.3\] we have

\[ \| \tilde{R}_{j, N} \|_{H^{1+\delta}(\mathcal{M}) \to H^{1+\delta}(\mathbb{R}^2)} \lesssim N^{2\delta-1}. \]

Hence we also have that \(\tilde{K}_{j,N}\) is in \(L^\infty(\mathcal{M} \times \mathcal{M})\) with norm \(O(N^{-1+\delta})\). In particular, using also Lemma \[2.5\] we have that

\[ \left| \int_{\mathcal{M}} \int_{\mathcal{M}} \tilde{K}_N(x, y) \tilde{K}_{j,N}(x, y') e^{\frac{1}{2} f_0(y)} e^{\frac{1}{2} f_0(y')} \log \left( d_0(y, y') \right) dV_0(y) dV_0(y') \right| \]

\[ \lesssim N^{-1+\delta} \int_{\mathcal{M}} \int_{\mathcal{M}} \left( N d_0(x, y) \right)^{-10} \left| \log \left( d_0(y, y') \right) \right| dV_0(y) dV_0(y') \lesssim N^{-1+\delta}. \]

Thus it is enough to consider the contribution of the terms

\[ \int_{\mathcal{M}} \int_{\mathcal{M}} \tilde{K}_{j,k,N}(x, y) \tilde{K}_{j,k',N}(x, y') e^{\frac{1}{2} f_0(y)} e^{\frac{1}{2} f_0(y')} \log \left( d_0(y, y') \right) dV_0(y) dV_0(y'). \]
Note also that, repeating the argument as in the proof of Lemma 2.5, these latter kernels also satisfy the bound
\[ |\tilde{K}_{j,k,N}(x,y)| \lesssim N^2 \langle Nd_0(x,y) \rangle^{-10}. \tag{2.32} \]
Thus using the mean value theorem we can control for any \( j, k, k' \)
\[
\left| \int_{\mathcal{M}} \int_{\mathcal{M}} \tilde{K}_{j,k,N}(x,y)\tilde{K}_{j,k',N}(x,y') \left[ e^{-f_0(x)}e^{\frac{i}{2}f_0(y)+\frac{i}{2}f_0(y')} - 1 \right] \log \left( d_0(y,y') \right) dV_0(y) dV_0(y') \right|
\]
\[
\lesssim \int_{\mathcal{M}} \int_{\mathcal{M}} N^4 \langle Nd_0(x,y) \rangle^{-10} \langle Nd_0(x,y') \rangle^{-10} [d_0(x,y) + d_0(x,y')] 
\times [1 \vee \log(\frac{1}{d_0(y,y')})] dV_0(y) dV_0(y').
\]
To compute this last integral, we can assume by symmetry that \( d_0(x,y') \leq d_0(x,y) \). In the case \( d_0(x,y') \lesssim d_0(x,y) \sim d_0(x,y) \) we then have
\[
N^4 \int_{\mathcal{M}} \int_{\mathcal{M}} \langle Nd_0(x,y) \rangle^{-10} \langle Nd_0(x,y') \rangle^{-10} d_0(x,y)^{1-} dV_0(y) dV_0(y') \lesssim N^{(-1)+}.
\]
In the other case \( d_0(x,y') \ll d_0(x,y) \sim d_0(x,y) \), we have
\[
N^4 \int_{\mathcal{M}} \int_{\mathcal{M}} \langle Nd_0(x,y) \rangle^{-10} d_0(x,y)^{0-} dV_0(y) dV_0(y') \lesssim N^{(-1)+}.
\]
Thus we are left with estimating
\[
-\frac{1}{2\pi} \sum_j \sum_{k,k'=0}^2 \int_{\mathcal{M}} \int_{\mathcal{M}} N^{-k-k'} \tilde{K}_{j,k,N}(x,y)\tilde{K}_{j,k',N}(x,y') \log(\frac{d_0(y,y')}{d_0(x,y)}) dV_0(y) dV_0(y').
\]
In the case \( k+k' \geq 1 \), it is enough to use the rough bound (2.32) to estimate the integrals above with
\[
N^{4-k-k'} \int_{\mathcal{M}} \int_{\mathcal{M}} \langle Nd_0(x,y) \rangle^{-10} \langle Nd_0(x,y') \rangle^{-10} d_0(y,y')^{0-} dV_0(y) dV_0(y') \lesssim N^{(-k-k')+}
\]
by proceeding similarly as for the previous integrals.

It remains to compute the leading term
\[
-\frac{1}{2\pi} \sum_j \int_{\mathcal{M}} \int_{\mathcal{M}} \tilde{K}_{j,0,N}(x,y)\tilde{K}_{j,0,N}(x,y') \log(\frac{d_0(y,y')}{d_0(x,y)}) dV_0(y) dV_0(y').
\]

Up to taking the charts with a sufficiently small diameter, we can use \( \kappa_j = \exp_x \) on \( V_j \ni x \), so that \( g_0(\kappa_j^{-1}(x)) = \Id \) and \( \kappa_j^{-1}(x) = 0 \). We also set \( \theta = e^{-\frac{1}{2}f_0(x)} > 0 \). We can then write the kernel of \( (\kappa_j)_* a_0(z,N^{-1}D)\kappa_j^* \tilde{\chi} \) as
\[
\tilde{K}_{j,0,N}(x,y) = \frac{\tilde{\chi}_j(y)}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i \exp_x^{-1}(y) \cdot \xi} g_0(0,N^{-1}\xi) d\xi
\]
\[
= \frac{\chi_j(x)\tilde{\chi}_j(y)}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i \exp_x^{-1}(y) \cdot \xi} \psi(N^{-2}\theta^2|\xi|^2) d\xi,
\]
where we used that the principal symbol of \( -\Delta_0 \) in \( \kappa_j \) is
\[
|\xi|_0^2 \overset{\text{def}}{=} g_0(x)\xi_i\xi_\ell.
\]
so that the one of \( -\tilde{\Delta}_0 \) is
\[
e^{-f_0(x)}|\xi_0^2 = \theta^2|\xi|^2
\]
by definition of \( \theta \) and choice of \( \kappa_j \).

This gives
\[
-\frac{1}{2\pi n} \sum_j \int_M \int_M \tilde{K}_{j,0,N}(x, y) \tilde{K}_{j,0,N}(x, y') \log (d_0(y, y')) dV_0(y) dV_0(y')
\]
\[
= - \sum_j \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \psi(N^{-2}|\xi|^2) e^{-iz' \cdot \xi'} \psi(N^{-2}|\xi'|^2)
\]
\[
\times \log (d_0(\exp_x(z), \exp_x(z'))) \tilde{\chi}_j(\exp_x(z)) \tilde{\chi}_j(\exp_x(z')) |g_0(z)|^{1/2} |g_0(z')|^{1/2} d\xi d\xi' dzdz'
\]
\[
= - \sum_j \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \psi(N^{-2}|\xi|^2) e^{-iz' \cdot \xi'} \psi(N^{-2}|\xi'|^2)
\]
\[
\times \log (d_0(\exp_x(\theta z), \exp_x(\theta z'))) \tilde{\chi}_j(\exp_x(\theta z)) \tilde{\chi}_j(\exp_x(\theta z')) |g_0(\theta z)|^{1/2} |g_0(\theta z')|^{1/2} d\xi d\xi' dzdz'
\]
after changing variables in both \( \xi, \xi' \) and \( z, z' \).

Note that \( d_0^2 \) is smooth on the support of \( \tilde{\chi}_j(y) \tilde{\chi}_j(y') \), and we have the Taylor expansion (see e.g. [11, Appendix A])
\[
d_0^2(\exp_x(z), \exp_x(z')) = |z - z'|^2 + O(|z - z'|^2(|z| + |z'|)^2).
\]

In particular, we get that
\[
\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \psi(N^{-2}|\xi|^2) e^{-iz' \cdot \xi'} \psi(N^{-2}|\xi'|^2) \tilde{\chi}_j(\exp_x(\theta z)) \tilde{\chi}_j(\exp_x(\theta z'))
\]
\[
\times \left[ \log (d_0(\exp_x(\theta z), \exp_x(\theta z'))) - \log (\theta |z - z'|) \right] |g_0(\theta z)|^{1/2} |g_0(\theta z')|^{1/2} d\xi d\xi' dzdz'
\]
\[
\lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} N^2(\exp_{\theta z} - \exp_{\theta z'}) - 10 |z - z'|^2 (|z|^2 + |z'|^2) dzdz' \lesssim N^{-2},
\]
where we used the Taylor expansion above after changing variables in \( \xi, \xi' \) and doing integration by parts similarly as in the proof of Lemma 2.5.

To summarize, we have proved so far
\[
(\psi \otimes \psi)(-N^{-2}\Delta_k) \log (d_0)(x, x)
\]
\[
= - \sum_j \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \psi(N^{-2}|\xi|^2) e^{-iz' \cdot \xi'} \psi(N^{-2}|\xi'|^2)
\]
\[
\times \log (\theta |z - z'|) \tilde{\chi}_j(\exp_x(\theta z)) \tilde{\chi}_j(\exp_x(\theta z')) |g_0(\theta z)|^{1/2} |g_0(\theta z')|^{1/2} d\xi d\xi' dzdz' + O(N^{(-1)^+}),
\]
uniformly on \( M \). We also remark that exactly the same computations give
\[
(\psi \otimes \psi)(-N^{-2}\Delta_0) \log (d_0)(x, x)
\]
\[
= - \sum_j \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz \cdot \xi} \psi(N^{-2}|\xi|^2) e^{-iz' \cdot \xi'} \psi(N^{-2}|\xi'|^2)
\]
\[
\times \log (|z - z'|) \tilde{\chi}_j(\exp_x(z)) \tilde{\chi}_j(\exp_x(z')) |g_0(z)|^{1/2} |g_0(z')|^{1/2} d\xi d\xi' dzdz' + O(N^{(-1)^+}).
\]
Changing variables again, we thus have
\[(\psi \otimes \psi)(-N^{-2}\Delta \xi) \log(d_0)(x,x) - (\psi \otimes \psi)(-N^{-2}\Delta_0 \log(d_0)(x,x)\]
\[= -\log(\theta) \sum_j \frac{\chi_j(x)}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \psi(|\xi|^2) e^{-iz'\xi'} \psi(|\xi'|^2) \]
\[\times \tilde{\chi}_j(\exp_x(N^{-1}\theta z)) \tilde{\chi}_j(\exp_x(N^{-1}\theta z')) |g_0(N^{-1}\theta z)|^{\frac{1}{2}} |g_0(N^{-1}\theta z')|^{\frac{1}{2}} d\xi d\xi' dz dz' \]
\[-\sum_j \frac{\chi_j(x)}{(2\pi)^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-iz\xi} \psi(N^{-2}|\xi|^2) e^{-iz'\xi'} \psi(N^{-2}|\xi'|^2) \log(N^{-1}|z - z'|) \]
\[\times \left[ \tilde{\chi}_j(\exp_x(N^{-1}\theta z)) \tilde{\chi}_j(\exp_x(N^{-1}\theta z')) |g_0(N^{-1}\theta z)|^{\frac{1}{2}} |g_0(N^{-1}\theta z')|^{\frac{1}{2}} - \tilde{\chi}_j(\exp_x(N^{-1}z)) \tilde{\chi}_j(\exp_x(N^{-1}z')) |g_0(N^{-1}z)|^{\frac{1}{2}} |g_0(N^{-1}z')|^{\frac{1}{2}} \right] d\xi d\xi' dz dz' \]
\[+ O(N^{-1}+) \]
\[= I + II + O(N^{-1}+). \]

For the first term, the same argument as in the proof of Lemma 2.10 shows that
\[I \to -\frac{\log(\theta)}{2\pi} = \frac{1}{4\pi} f_0(x) \]
as \(N \to \infty\), uniformly on \(\mathcal{M}\).

Finally, for the second term, integrating by parts in \(\xi, \xi'\) and using the mean value theorem gives the estimate
\[|II| \lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} N^2 \langle Nz \rangle^{-10} N^2 \langle Nz' \rangle^{-10} N^0 |z - z'|^{0+} |\theta - 1| N^{-1} (|z| + |z'|) dz dz' \]
\[\lesssim N^{(-2)+}. \]

This concludes the proof of Lemma 2.15. \(\square\)

2.5. Some nonlinear estimates. To conclude this section, we state some useful estimates needed in the analysis of (1.25). We first recall the following embeddings and product estimates in Besov spaces proved in [17, Corollary 2.7].

Lemma 2.16. Let \(B^s_{p,q}(\mathcal{M})\) be the Besov spaces defined above. Then the following properties hold.
(i) For any \(s \in \mathbb{R}\) we have \(B^s_{2,2}(\mathcal{M}) = H^s(\mathcal{M})\), and more generally for any \(2 \leq p < \infty\) and \(\varepsilon > 0\) we have
\[\|u\|_{B^s_p(\mathcal{M})} \lesssim \|u\|_{W^{s,p}(\mathcal{M})} \lesssim \|u\|_{B^s_{p,\infty}(\mathcal{M})} \lesssim \|u\|_{B^{s+\varepsilon}_{p,\infty}(\mathcal{M})}. \]
(ii) Let \(s \in \mathbb{R}\) and \(1 \leq p_1 \leq p_2 \leq \infty\) and \(q_1, q_2 \in [1, \infty]\). Then for any \(f \in B^s_{p_1,q_1}(\mathcal{M})\) we have
\[\|f\|_{B^s_{p_2,q_2}(\mathcal{M})} \lesssim \|f\|_{B^s_{p_1,q_1}(\mathcal{M})}. \]
(iii) Let \(\alpha, \beta \in \mathbb{R}\) with \(\alpha + \beta > 0\) and \(p_1, p_2, q_1, q_2 \in [1, \infty]\) with
\[\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \]
Then for any \( f \in B^\alpha_{p_1,q_1}(\mathcal{M}) \) and \( g \in B^\beta_{p_2,q_2}(\mathcal{M}) \), we have \( fg \in B^{\alpha+\beta}_{p,q}(\mathcal{M}) \), and moreover it holds

\[
\|fg\|_{B^{\alpha+\beta}_{p,q}(\mathcal{M})} \lesssim \|f\|_{B^\alpha_{p_1,q_1}(\mathcal{M})}\|g\|_{B^\beta_{p_2,q_2}(\mathcal{M})}.
\]

In order to take advantage of the “sign-definite structure” used in [48], we also need the following product estimate.

**Lemma 2.17.** Let \( s \geq 0 \) and \( 1 < p < \infty \). Then it holds

\[
\|fg\|_{W^{-s,p}(\mathcal{M})} \lesssim \|f\|_{L^\infty(\mathcal{M})}\|g\|_{W^{-s,p}(\mathcal{M})}
\]

for any \( f \in C(\mathcal{M}) \) and any positive distribution \( g \in W^{-s,p}(\mathcal{M}) \).

**Proof.** We merely repeat the argument of [48, Lemma 2.14]. We consider \( s > 0 \) since the case \( s = 0 \) follows directly from Hölder’s inequality.

Since \( g \) is a positive distribution, it can be identified with a positive Radon measure on \( \mathcal{M} \); see [22]. If \( f \in C(\mathcal{M}) \), then the product \( fg \) is a well-defined function in \( L^1(\mathcal{M}) \); in particular from Lemma 2.6 (i) we have \( P_N f \to f \) in \( C(\mathcal{M}) \) as \( N \to \infty \), so that \( fg = \lim_{N \to \infty} (P_N f)g \) in \( L^1(\mathcal{M}) \). Moreover, for any \( M \in \mathbb{N} \), \( P_M g \) is also a smooth positive distribution which converges to \( g \) in \( W^{-s,p}(\mathcal{M}) \), so that from Lemma 2.16 (i) and (iii) we have that for each fixed \( N \in \mathbb{N} \), \( (P_N f)(P_M g) \) converges to \( (P_N f)g \) in \( L^1(\mathcal{M}) \). Thus from Fatou’s lemma we get

\[
\|(1 - \Delta_g)^{-\frac{s}{2}}(fg)\|_{L^p(\mathcal{M})} \leq \liminf_{N \to \infty} \lim_{M \to \infty} \|(1 - \Delta_g)^{-\frac{s}{2}}[(P_N f)(P_M g)]\|_{L^p(\mathcal{M})}.
\]

It thus remains to prove the product estimate (2.33) for the right-hand side of (2.34). Note that since \( g \) is a positive distribution we also have by Lemma 2.6 (i) and the definition of \( P_M \) (1.13) that \( P_M g \) is a non-negative function for any \( M \in \mathbb{N} \).

Let then \( K_s \) be the distributional kernel of \( (1 - \Delta_g)^{-\frac{s}{2}} \): we can write

\[
K_s(x,y) = \sum_{n \geq 0} \frac{\varphi_n(x)\varphi_n(y)}{\langle \lambda_n \rangle^s} = \sum_{n \geq 0} \varphi_n(x)\varphi_n(y)\Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty t^{\frac{s}{2}-1}e^{-t\langle \lambda_n \rangle^2} dt
\]

\[
= \Gamma\left(\frac{s}{2}\right)^{-1} \int_0^\infty t^{\frac{s}{2}-1}e^{-t}P_k(4\pi t, x, y) dt,
\]

where \( \Gamma \) is the Gamma function, and the equality holds in the sense of distributions on \( \mathcal{M} \times \mathcal{M} \). Note that the last integral converges since we assumed \( s > 0 \). From Lemma 2.6 (i) we then deduce that \( K_s \) is also a positive distribution. This implies that

\[
\|(1 - \Delta_g)^{-\frac{s}{2}}[(P_N f)(P_M g)]\|_{L^p(\mathcal{M})} = \left\| \int_{\mathcal{M}} K_s(x,y)P_N f(y)P_M g(y) dV_g(y) \right\|_{L^p(\mathcal{M})}
\]

\[
\leq \left\| P_N f \right\|_{L^\infty(\mathcal{M})} \left\| \int_{\mathcal{M}} K_s(x,y)P_M g(y) dV_g(y) \right\|_{L^p(\mathcal{M})}
\]

\[
= \left\| P_N f \right\|_{L^\infty(\mathcal{M})} \left\| P_M g \right\|_{W^{-s,p}(\mathcal{M})}.
\]

This allows to bound the right-hand side of (2.34), and taking the limits \( M \to \infty \) finally proves (2.33). This completes the proof of Lemma 2.17. \( \square \)

We will also need a fractional chain rule for the composition with Lipschitz functions. Note that the fractional chain rule for fractional derivatives defined by the right-hand side of (2.35) for \(-2 < s < 0\) is known to hold on any space of homogeneous type [24].
In our case (see Section 4 below) we need it for a Lipschitz nonlinearity acting on functions in Besov spaces. We thus have the following estimates.

**Lemma 2.18.** Let $A : \mathbb{R} \to \mathbb{R}$ with $A(0) = 0$ and $0 < s < 1$. Then:

(i) if $A$ is Lipschitz, it holds

$$\|A(u)\|_{H^s(M)} \lesssim \|A'\|_{L^\infty} \|u\|_{H^s(M)};$$

(ii) if $A$ is $C^1$ and there exists $a \in L^1([0, 1])$, $A$ positive such that for almost every $\theta \in [0, 1]$ and any $u, v$ it holds

$$|A'((\theta u + (1 - \theta)v)| \leq a(\tau)[A(u) + A(v)],$$

then for any $1 < p < \infty$ and any $r \in [1, \infty]$ it holds for any $0 < \varepsilon < 1 - s$

$$\|A(u)\|_{B^p_{r,\varepsilon}(M)} \lesssim \|A(u)\|_{L^\infty(M)} \|u\|_{B^p_{r,\varepsilon}(M)}.$$

**Proof.** Let $(U_j, V_j, \kappa_j)_j$ be a finite atlas on $M$ and $\{\chi_j\}_j$ be an associated partition of unity. Using Lemma 2.2 we thus have

$$\|A(u)\|_{H^s(M)} \lesssim \max_j \|\kappa_j^s(\chi_j A(u))\|_{H^s(\mathbb{R}^2)} = \max_j \|\kappa_j^s(\chi_j)A(\kappa_j^s \chi_j u)\|_{H^s(\mathbb{R}^2)}$$

where $\chi_j \in C_0^\infty(V_j)$, $\chi_j \equiv 1$ on $\text{supp} \chi_j$. Then we can use first the same product estimate as in Lemma 2.16 (iii), which holds on $\mathbb{R}^2$ [5], and then the fractional chain rule for Lipschitz functions on $\mathbb{R}^2$ (see e.g. [60] Proposition 4.1), to continue with

$$\lesssim \max_j \|\kappa_j^s(\chi_j)\|_{B^p_{s,\varepsilon}(\mathbb{R}^2)} \|A(\kappa_j^s \chi_j u)\|_{H^s(\mathbb{R}^2)} \lesssim \max_j \|A'\|_{L^\infty} \|\kappa_j^s \chi_j u\|_{H^s(\mathbb{R}^2)} \lesssim \|A'\|_{L^\infty} \|u\|_{H^s(M)}$$

where in the last step we used Lemma 2.2 again. This proves (i). The second estimate (ii) follows similarly\(^\text{10}\) by using [60] Proposition 5.1. \(\square\)

We finish this section by stating another nonlinear (multilinear) estimate, namely the (geometric) Brascamp-Lieb inequality [7, Example 1.6].

**Lemma 2.19.** Let $p \in \mathbb{N}$ and $f_{q,r} \in L^1(M \times M)$ for $1 \leq q, r \leq 2p$. Then it holds

$$\int_{M^{2p}} \prod_{1 \leq q < r \leq 2p} |f_{q,r}(\pi_{q,r}(\bar{y}))|^{\frac{1}{Q-q}} dV_g(\bar{y})$$

$$\lesssim \prod_{1 \leq q < r \leq 2p} \left( \int_{M \times M} |f_{q,r}(y_q, y_r)| dV_g(y_q) dV_g(y_r) \right)^{\frac{1}{Q-q}},$$

where $\bar{y} = (y_1, ..., y_{2p}) \in M^{2p}$, $dV_g(\bar{y})$ is the corresponding product measure on $M^{2p}$, and $\pi_{q,r}$ denotes the projection defined by $\pi_{q,r}(\bar{y}) = \pi_{q,r}(y_1, \ldots, y_{2p}) = (y_q, y_r)$.

**Proof.** The corresponding estimate on $M = \mathbb{R}^d$ is proved in greater generality in [7]. The estimate (2.36) then follows from (1) in [7] by using a partition of unity with the compactness of $M$, and that in each chart $V_g$ is equivalent to the Lebesgue measure since $g$ is smooth. \(\square\)

\(^\text{10}\)The $\varepsilon$ loss comes from moving from $B^p_{r,\varepsilon}(\mathbb{R}^2)$ to $W^{s+\varepsilon,p}(\mathbb{R}^2)$ in order to use the fractional chain rule on $\mathbb{R}^2$, and can certainly be avoided, by using for example an argument similar to [59] Lemma 9.3.
3. GMC theory and the LQG measure

In this section, we deal with the construction of the main stochastic objects, namely the linear stochastic evolution $\mathbf{g}$ in (1.29), and the “punctured” Gaussian multiplicative chaos $\Theta$ in (1.33) and the LQG measure $\rho_{\{a, x\}, \mathbf{g}}$ in (1.11). We mainly follow the arguments of [18, 15, 17, 30].

3.1. On punctured Gaussian multiplicative chaos. Before moving to the proof of Proposition 1.8, we first express the covariance function of the processes $\mathbf{P}_N \mathbf{g}$ in (1.29) and $\Theta_N(t)$ (1.33).

Lemma 3.1. The following identities hold:
(i) The covariance function of the truncated linear stochastic evolution $\mathbf{g}$ is given by

$$\Gamma_{N_1, N_2}(t_1, t_2, x_1, x_2) \overset{\text{def}}{=} \int_{\mathcal{H}_0^s(\mathcal{M}, \mathbf{g})} \mathbb{E}[\mathbf{P}_N \mathbf{g}(t_1, x_1) \mathbf{P}_N \mathbf{g}(t_2, x_2)] d\mu_{\mathbf{g}}(X_{\mathbf{g}}) = 2\pi e^{\frac{t_2-t_1}{2\pi}} \Delta_s(\mathbf{P}_N \mathbf{g}) G_{\mathbf{g}}(x_1, x_2),$$

for any $(x_1, x_2) \in \mathcal{M} \times \mathcal{M} \setminus \Delta$, $t_1 \leq t_2$ and $N_1, N_2 \in \mathbb{N}$;
(ii) For any $t \geq 0$ and $p \in \mathbb{N}$, the $2p$ (spatial) covariance function of $\Theta_N(t)$ is given by

$$\int_{H_0^s(\mathcal{M}, \mathbf{g})} \mathbb{E}\left[ \prod_{j=1}^{2p} \Theta_N(t, y_j) \right] d\mu_{\mathbf{g}}(X_{\mathbf{g}}) = e^{\pi \beta^2 \sum_{j=1}^{2p} \tilde{G}_{\mathbf{g}}(y_j, y_j) + o(1)} e^{2\pi \beta \sum_{j<k} (\mathbf{P}_N \mathbf{g}) G_{\mathbf{g}}(y_j, y_k) \times e^{2\pi \beta \sum_{j=1}^{2p} \sum_{\ell=1}^{L} a_{\ell} (\mathbf{P}_N \mathbf{g}) G_{\mathbf{g}}(x_\ell, y_j)},$$

where $o(1)$ is deterministic and uniform on $\mathcal{M}$.

Proof. The identity of Lemma 3.1 (i) follows from a straightforward computation.

As for Lemma 3.1 (ii), note that by Lemma 2.10 we have

$$e^{-\pi \beta^2 C_{\mathbf{N}} - \frac{\beta^2}{2}} = e^{-\frac{\beta^2}{2} \sigma_N(x) + \pi \beta^2 \tilde{G}_{\mathbf{g}}(x, x) + o(1)}$$

for any $x \in \mathcal{M}$, where $\sigma_N$ is as in (1.26).

Thus the claim follows from (1.8)-(1.29)-(1.33) with the definitions (2.14) and (2.11) of $G_{\mathbf{g}}$ and $P_{\mathbf{g}}$ along with straightforward computations. □

For the solution $\mathbf{f}_{\mathbf{g}}$ of the linear stochastic heat equation, we have the following classical result.

Lemma 3.2. The process $\mathbf{f}_{\mathbf{g}}$ defined in (1.29) is stationary and belongs almost surely to $C(\mathbb{R}_+; H^s_{\mathbf{g}}(\mathcal{M}))$ for any $s < 0$.

Proof. Since

$$\mathbf{P}_N \mathbf{f}_{\mathbf{g}} = e^{i \beta s \Delta_{\mathbf{g}}(\mathbf{P}_N X_{\mathbf{g}})} + \int_0^t e^{i \beta s \Delta_{\mathbf{g}}(\mathbf{P}_N W_{\mathbf{g}})(t')} dt'$$

is Gaussian and stationary in view of Lemma 3.1 (i), it is enough to construct $\mathbf{f}_{\mathbf{g}}$ as the limit of $\mathbf{P}_N \mathbf{f}_{\mathbf{g}}$. Take any $s < 0$, $p \geq 2$, $t, \delta > 0$ and $N_1, N_2 \in \mathbb{N}$. Write

$$\mathbf{f}_{N_1-N_2} \overset{\text{def}}{=} (\mathbf{P}_{N_1} - \mathbf{P}_{N_2}) \mathbf{f}_{\mathbf{g}},$$

which in view of Lemma 3.1 (i) has covariance function
\[ \Gamma_{N_1 - N_2}(t_1, t_2, x_1, x_2) = 2\pi e^{\frac{1 - \epsilon}{2}} \Delta_{x_1}(\langle \mathbf{P}_{N_1} - \mathbf{P}_{N_2} \rangle) \mathbf{G}_x(x_1, x_2). \] (3.1)

Then using Minkowski’s inequality with \( p \geq 2 \) we can compute
\[
\left\| \mathbf{1}_{t_1 - N_2}(t + \delta) - \mathbf{1}_{t_1 - N_2}(t) \right\|_{L^p(\mu_\epsilon \otimes \mathbb{P}) H^s_0(\mathcal{M})} \\
\leq \left\| (1 - \Delta_\epsilon) \mathbf{1}_{t_1 - N_2}(t + \delta) - \mathbf{1}_{t_1 - N_2}(t) \right\|_{L^2(\mathcal{M}) L^p(\mu_\epsilon \otimes \mathbb{P})}
\]
Using next that \( \mathbf{1}_{t_1 - N_2} \) is a Gaussian process, we can continue with
\[
\lesssim \left\| (1 - \Delta_\epsilon) \mathbf{1}_{t_1 - N_2}(t + \delta) - \mathbf{1}_{t_1 - N_2}(t) \right\|_{L^2(\mathcal{M}) L^2(\mu_\epsilon \otimes \mathbb{P})} \\
= \left\{ \sum_{n \geq 0} (\lambda_n)^{2s} \int_{H^s_0(\mathcal{M})} E \left[ \langle \varphi_n, [\mathbf{1}_{t_1 - N_2}(t + \delta) - \mathbf{1}_{t_1 - N_2}(t)] \rangle^2 d\mu_\epsilon \right] \right\}^{\frac{1}{2}} \\
= \left\{ \sum_{n \geq 0} (\lambda_n)^{2s} \langle \varphi_n \otimes \varphi_n, \Gamma_{t_1 - N_2}(t + \delta, t + \delta) - 2\Gamma_{N_1 - N_2}(t + \delta, t + \delta) + \Gamma_{N_1 - N_2}(t, t) \rangle \right\}^{\frac{1}{2}}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in either \( L^2(\mathcal{M}) \) or \( L^2(\mathcal{M} \times \mathcal{M}) \). Thus in view of (3.1) and using the mean value theorem, we can continue with
\[
= \left\{ \sum_{n \geq 1} (\lambda_n)^{2s} \frac{4\pi}{\lambda_n^2} (1 - e^{-\frac{\delta}{\pi} \lambda_n^2}) \left( e^{-N_1^{-2} \lambda_n^2} - e^{-N_2^{-2} \lambda_n^2} \right)^2 \right\}^{\frac{1}{2}} \\
\lesssim \left\{ \sum_{n \geq 1} \sqrt{\lambda_n^{2s-2}} \min \left( 1, \delta^{\frac{s}{2}} \right) \min \left( 1, |N_1^{-2} - N_2^{-2}| \lambda_n^2 \right) \right\}^{\frac{1}{2}} \\
\lesssim \delta^{\frac{s}{2}} \min(N_1, N_2)^{-\epsilon} \left\{ \sum_{n \geq 1} \lambda_n^{2s-2+4\epsilon} \right\} \lesssim \delta^{\frac{p}{2}} \min(N_1, N_2)^{-p\epsilon},
\]
for \( 0 < \epsilon \ll 1 \) small enough, since \( s < 0 \). In particular, this shows that
\[
\int_{H^s_0(\mathcal{M})} E \left\| \mathbf{1}_{t_1 - N_2}(t + \delta) - \mathbf{1}_{t_1 - N_2}(t) \right\|_{L^p(\mathcal{M})}^p \lesssim \delta^{p\frac{s}{2}} \min(N_1, N_2)^{-p\epsilon},
\]
so that we can conclude from Kolmogorov’s continuity criterion (see e.g. [6] Theorem 8.2) that for any \( T > 0 \), there exists \( p \geq 1 \) large enough such that \( \{ \mathbf{P}_N t_\xi \}_N \) is a Cauchy sequence in \( L^p(\mu_\epsilon \otimes \mathbb{P}; C([0, T]; H^s_0(\mathcal{M}))) \) and in particular \( t_\xi \in C([0, T]; H^s_0(\mathcal{M})) \) almost surely. This proves Lemma 3.2. \( \square \)

We now move on to the proof of Proposition 1.8. Similarly to [48], it will follow from the following set of estimates.

**Lemma 3.3.** Assume that \( a_\xi \in \mathbb{R}, \beta > 0 \) and \( Q = \frac{3}{2} + \beta \) satisfy the assumptions (1.14) - (1.15) - (1.16). Then, for any \( 0 < \epsilon \ll 1 \) and \( T > 0 \), there exists \( C > 0 \) such that for any \( t \in [0, T] \), any \( x \in \mathcal{M} \setminus \{ x_1, \ldots, x_L \} \) and any \( N, N_1, N_2 \in \mathbb{N} \), \( M \in 2^{\epsilon+1} \), the following statements hold:
(i) We have
\[ \int_{H_0^s(M, g)} \mathbb{E}[|\Theta_N(t, x)|] \, d\mu_g \leq C \prod_{\ell=1}^L d_g(x_\ell, x)^{-\beta \alpha_\ell} \in L^1([0, T] \times \mathcal{M}) \]

(ii) Let \( p \) be an even integer, \( 0 < \alpha < 2 \) and \( 0 < (p-1)\beta^2 < 2 \min(1, \alpha) \), then
\[ \int_{H_0^s(M, g)} \mathbb{E}\left[ |Q_M \Theta_N(t, x)|^p \right] \, d\mu_g \leq CM^{p\alpha - \varepsilon} f_{\alpha - \varepsilon, \{x_\ell\}}(x)^{\frac{2}{p}}, \quad (3.2) \]

where \( f_{\alpha, \{x_\ell\}} \) is given by
\[ f_{\alpha, \{x_\ell\}}(x) = \sum_{\ell_1, \ell_2 = 1}^L \left( 1 + d_g(x_{\ell_1}, x)^{\alpha - \beta \alpha_\ell_1} \right) \left( 1 + d_g(x_{\ell_2}, x)^{\alpha - \beta \alpha_\ell_2} \right) + \sum_{\ell = 1}^L d_g(x_\ell, x)^{2\alpha - (p-1)\beta^2 - 2\beta \alpha_\ell}. \quad (3.3) \]

(iii) Given any \( 0 < \beta^2 < 2 \min(1, \alpha) \) for \( 0 < \alpha < 2 \), then it holds
\[ \int_{H_0^s(M, g)} \mathbb{E}\left[ |Q_M \left( \Theta_{N_1}(t, x) - \Theta_{N_2}(t, x) \right)|^2 \right] \, d\mu_g \leq CM^{p\alpha - \varepsilon} |N_1^{-2} - N_2^{-2}|^{-\varepsilon} f_{\alpha - \varepsilon, \{x_\ell\}}(x) \]

for some small \( 0 < \varepsilon \ll \varepsilon \ll 1 \).

(iv) If \( \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}) \) satisfy \( \psi_1(0) = \psi_2(0) \), then
\[ \int_{H_0^s(M, g)} \mathbb{E}\left[ |Q_M \left( \Theta_{N,1}(t, x) - \Theta_{N,2}(t, x) \right)|^2 \right] \, d\mu_g \leq CM^{p\alpha - \varepsilon} N^{-\varepsilon} f_{\alpha - \varepsilon, \{x_\ell\}}(x), \quad (3.4) \]

where
\[ \Theta_{N,j} = e^{-\pi^2 C \psi_j N^{-\frac{\beta^2}{2}}} e^{\beta \psi_j (-N^{-2} \Delta_g) i_k + \sum_{\ell=1}^L \beta \alpha_\ell 2\pi \psi_j \psi_j (-N^{-2} \Delta_g) G_{g}(x_\ell, x)}. \]

Proof. The proof of Lemma 3.3 follows from the same argument as for [18 Proposition 3.2]. Indeed, (i) is a straightforward computation: using Lemmas [3.1, 2.7, 2.8 and 2.13], we indeed obtain
\[ \int_{H_0^s(M, g)} \mathbb{E}[|\Theta_N(t, x)|] \, d\mu_g \leq C e^{2\pi \beta \sum_{\ell=1}^L a_t(\mathbb{P}_N \otimes \mathbb{P}_N) G_{g}(x_\ell, x)} \]
\[ \leq C \prod_{\ell=1}^L e^{\beta \alpha_\ell 2\pi \left( G_{g}(x_\ell, x) + (N^2 V_{g}(\mathcal{M}))^{-1} \right)} \]
\[ \leq C \prod_{\ell=1}^L e^{-\beta \alpha_\ell \log \left( d_g(x_\ell, x) \right)}, \]

where we used that both \( (N^2 V_{g}(\mathcal{M}))^{-1} \) and \( \widetilde{G}_{g} \) in Lemma 2.7 are bounded (uniformly in \( N \)). We also used that the metric is smooth and that \( \mathcal{M} \) is compact. This shows (i).
As for (ii), recall that the kernel $K_M$ of $Q_M$ has been defined in (2.8) with $\psi$ as in (2.4), so that with Lemma 3.1 we can compute for any $p = 2q \in \mathbb{N}$

$$\int_{H_0^p(\mathcal{M}, g)} \mathbb{E} \left[ |Q_M \Theta_N(t, x)|^{2q} \right] d\mu_g$$

$$= \int_{\mathcal{M}} \cdots \int_{\mathcal{M}} \int_{H_0^p(\mathcal{M}, g)} \mathbb{E} \left[ \prod_{j=1}^{2q} \Theta_N(t, y_j) \right] d\mu_g \prod_{j=1}^{2q} K_M(x, y_j) dV_g(y_j)$$

$$= \int_{\mathcal{M}} \cdots \int_{\mathcal{M}} e^{2\pi \beta^2 \sum_{j<k} (|P_N \otimes P_N| G_N(y_j, y_k))} e^{\pi \beta^2 \sum_{j=1}^{2p} G_N(y_j, y_j) + o(1)} \times \prod_{j=1}^{2q} e^{2\pi \beta \sum_{\ell=1}^L a_\ell (|P_N \otimes P_N| G_N(x, y_j)) K_M(x, y_j) dV_g(y_j)}$$

$$\leq C \int_{\mathcal{M}^{2q}} \prod_{j<k} \left[ e^{2\pi \beta (|P_N \otimes P_N| G_N(y_j, y_k))} e^{2\pi \beta \sum_{\ell=1}^L a_\ell ((|P_N \otimes P_N| G_N(x, y_j)) + (|P_N \otimes P_N| G_N(x, y_k))) \times |K_M(x, y_j) K_M(x, y_k)| dV_g(y_j) dV_g(y_k) \right]^{\frac{1}{2q - t}}$$

where $\tilde{y} = (y_1, ..., y_{2q}) \in \mathcal{M}^{2q}$ and $V_g(\tilde{y})$ is the corresponding product measure on $\mathcal{M}^{2q}$. We can then use the Brascamp-Lieb inequality given by Lemma 2.19 with the smoothness of $g$ and the compactness of $\mathcal{M}$ to continue with

$$\lesssim \left( \int_{\mathcal{M}} \int_{\mathcal{M}} d_g(y, z)^{- (2q - 1) \beta^2} \prod_{\ell=1}^L d_g(x, y)^{- \beta a_\ell} d_g(x, z)^{- \beta a_\ell} \right)^{\frac{1}{2q - t}} \times M^2 d_g(x, y)^{- A} M^2 d_g(x, z)^{- A} dV_g(y) dV_g(z)$$

for any $A > 0$. In particular, using that $p = 2q$ and taking $A = 2 - \alpha + \varepsilon > 0$ for some $0 < \varepsilon \ll 1$ and using that $M^2 d_g(x, y)^{- A} \lesssim M^{\alpha - \varepsilon} d_g(x, y)^{\alpha - 2 - \varepsilon}$, (ii) will be established once we show that the double integrals

$$\int_{\mathcal{M}} \int_{\mathcal{M}} d_g(x, y)^{\alpha - 2 - \varepsilon} d_g(x, z)^{\alpha - 2 - \varepsilon} d_g(y, z)^{-(p-1)\beta^2} \times \prod_{\ell=1}^L d_g(x, y)^{- \beta a_\ell} d_g(x, z)^{- \beta a_\ell} dV_g(y) dV_g(z)$$

are bounded by $f_{\alpha - \varepsilon, \{x_k\}}(x)$ as in (5.3).

To bound this last double integral, first note that we only need to consider one of the singularities $d_g(x, y)^{- \beta a_\ell}$ and similarly for $z$. Indeed, if $0 < r \ll \min_{\ell \neq k} d_g(x, x_k)$ and $B_\ell$
is the ball of radius \( r \) around \( x_\ell \), we can bound the previous integrals with
\[
\sum_{\ell,k=1}^{L} r^{-\sum_{\ell' \neq \ell} \beta_{\alpha_{\ell'}} - \sum_{k' \neq k} \beta_{\alpha_{k'}}} \int_{B_\ell} \int_{B_k} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x,z)^{\alpha - 2 - \varepsilon} \\
\times d_g(x_\ell, y)^{\beta_{\alpha_{\ell}}} d_g(x_\ell, z)^{-\beta_{\alpha_{k}}} d_g(y, z)^{-(p-1)\beta^2} dV_g(y) dV_g(z) \\
+ r^{-2 \sum_{\ell=1}^{L} \beta_{\alpha_{\ell}}} \int_{M \setminus (\cup \ell B_\ell)} \int_{M \setminus (\cup \ell B_\ell)} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x,z)^{\alpha - 2 - \varepsilon} d_g(y, z)^{-(p-1)\beta^2} dV_g(y) dV_g(z).
\]
(3.5)

We first deal with the integrals on the second line of (3.5). We can assume by symmetry that \( d_g(x,y) \leq d_g(x,z) \), in which case \( d_g(y,z) \lesssim d_g(x,z) \) and the integrals are bounded by
\[
\sup_y \int_M d_g(y, z)^{\alpha - 2 - \varepsilon (p-1)\beta^2} dV_g(z) < \infty
\]
for \( (p - 1)\beta^2 < \alpha \) and \( 0 < \varepsilon \ll 1 \).

We now turn to the terms in the first line of (3.5), and for each \( \ell, k = 1, \ldots, L \) we estimate
\[
\int_{B_\ell} \int_{B_k} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x,z)^{\alpha - 2 - \varepsilon} d_g(x_\ell, y)^{-\beta_{\alpha_{\ell}}} d_g(x_\ell, z)^{-\beta_{\alpha_{k}}} d_g(y, z)^{-(p-1)\beta^2} dV_g(y) dV_g(z).
\]

**Case 1:** if \( \ell \neq k \). In this case the integrals can be bounded by
\[
r^{-(p-1)\beta^2} \left( \int_{B_\ell} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x_\ell, y)^{-\beta_{\alpha_{\ell}}} dV_g(y) \right) \left( \int_{B_k} d_g(x,z)^{\alpha - 2 - \varepsilon} d_g(x_\ell, z)^{-\beta_{\alpha_{k}}} dV_g(z) \right).
\]

From [4, Proposition 4.12], we have that
\[
\int_{B_\ell} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x_\ell, y)^{-\beta_{\alpha_{\ell}}} dV_g(y) \lesssim \begin{cases} 1 & \text{if } \alpha > \beta_{\alpha_{\ell}} \\
1 + | \log (d_g(x, x_\ell)) | & \text{if } \alpha = \beta_{\alpha_{\ell}}, \\
d_g(x, x_\ell)^{\alpha - \beta_{\alpha_{\ell}}} & \text{if } \alpha < \beta_{\alpha_{\ell}}, \end{cases}
\]
and similarly for the integral in \( z \). In any case, these are then bounded by the first term in the right-hand side of (3.3).

**Case 2:** if \( \ell = k \). We now need to bound
\[
\int_{B_\ell} \int_{B_\ell} d_g(x,y)^{\alpha - 2 - \varepsilon} d_g(x,z)^{\alpha - 2 - \varepsilon} d_g(x_\ell, y)^{-\beta_{\alpha_{\ell}}} d_g(x_\ell, z)^{-\beta_{\alpha_{k}}} d_g(y, z)^{-(p-1)\beta^2} dV_g(y) dV_g(z).
\]
(3.6)

To estimate these integrals we look at the following regions:
\[
\mathcal{R}_1 \overset{\text{def}}{=} \{ y \in B_j, \ d_g(x,y) \lesssim d_g(x_\ell, y) \sim d_g(x,x_\ell) \}, \\
\mathcal{R}_2 \overset{\text{def}}{=} \{ y \in B_j, \ d_g(x_\ell, y) \lesssim d_g(x,y) \sim d_g(x,x_\ell) \}, \\
\mathcal{R}_3 \overset{\text{def}}{=} \{ y \in B_j, \ d_g(x,x_\ell) \lesssim d_g(x,y) \sim d_g(x_\ell, y) \},
\]
and for \( i_1, i_2 \in \{1, 2, 3\} \) we define the subregion of \( B_\ell \times B_\ell \) as
\[
\mathcal{R}_{i_1,i_2} \overset{\text{def}}{=} \mathcal{R}_{i_1} \times \mathcal{R}_{i_2}.
\]
In particular note that \( B_\ell \times B_\ell = \cup_{i_1,i_2=1}^{3} \mathcal{R}_{i_1,i_2} \).
Contribution of $\mathcal{R}_{1,1}$: In this region we can bound (3.6) with

$$d_g(x, x')^{-2\beta a_t} \int_{\mathcal{R}_1} \int_{\mathcal{R}_1} d_g(x, y)^{\alpha - 2 - \varepsilon} d_g(x, z)^{\alpha - 2 - \varepsilon} d_g(y, z)^{-(p-1)\beta^2} dV_g(z) dV_g(y). \quad (3.7)$$

Proceeding then as for the second line of (3.5), in the case $d_g(x, y) \lesssim d_g(x, x') \sim d_g(y, z)$ we get the bound

$$d_g(x, x')^{-2\beta a_t} \int_{\mathcal{R}_1} \int_{\mathcal{R}_1} d_g(x, z)^{\alpha - 2 - \varepsilon - (p-1)\beta^2} dV_g(y) dV_g(z)$$

$$\lesssim d_g(x, x')^{-2\beta a_t} \int_{\mathcal{R}_1} \int_{\mathcal{R}_1} d_g(x, z)^{2\alpha - 2 - \varepsilon - (p-1)\beta^2} dV_g(z)$$

where in the last step we used the condition $2\alpha > (p-1)\beta^2$ along with the definition of $\mathcal{R}_1$. The case $d_g(x, z) \lesssim d_g(x, y) \sim d_g(y, z)$ is treated similarly by exchanging the roles of $y$ and $z$, and in the case $d_g(y, z) \lesssim d_g(x, x') \sim d_g(x, z)$ we can bound (3.7) with

$$d_g(x, x')^{-2\beta a_t} \int_{\mathcal{R}_1} \int_{\mathcal{R}_1} d_g(x, z)^{2\alpha - 2 - \varepsilon - (p-1)\beta^2} dV_g(z)$$

$$\lesssim d_g(x, x')^{-2\beta a_t + 2\alpha - 2\varepsilon - (p-1)\beta^2},$$

where in the first step we used the condition $(p-1)\beta^2 < 2$ and in the second step we used that $2\alpha > (p-1)\beta^2$.

Contribution of $\mathcal{R}_{1,2}$: In this region we estimate (3.6) with

$$d_g(x, x')^{\alpha - 2 - \varepsilon - \beta a_t} \int_{\mathcal{R}_1} \int_{\mathcal{R}_2} d_g(x, y)^{\alpha - 2 - \varepsilon} d_g(x', z)^{\alpha - 2 - \varepsilon} d_g(y, z)^{-(p-1)\beta^2} dV_g(z) dV_g(y). \quad (3.8)$$

Similarly as for $\mathcal{R}_{1,1}$, in the case $d_g(x, y) \lesssim d_g(x, x') \sim d_g(y, z)$ we can bound (3.8) with

$$d_g(x, x')^{\alpha - 2 - \varepsilon - \beta a_t} \int_{\mathcal{R}_2} d_g(x, z)^{\alpha - \varepsilon - \beta a_t} dV_g(z) \lesssim d_g(x, x')^{2\alpha - 2 - \varepsilon \beta a_t - (p-1)\beta^2},$$

where in the last estimation we used that $\alpha - \beta a_t - (p-1)\beta^2 > -2$ since both $\alpha > (p-1)\beta^2$ and $\beta a_t < 2$. The case $d_g(x, x') \lesssim d_g(x, y) \sim d_g(y, z)$ gives the bound

$$d_g(x, x')^{\alpha - 2 - \varepsilon - \beta a_t} \int_{\mathcal{R}_1} d_g(x, y)^{\alpha - \varepsilon - \beta a_t} dV_g(y) \lesssim d_g(x, x')^{2\alpha - 2 - \varepsilon \beta a_t - (p-1)\beta^2}$$

where we used in the first step that $\beta a_t < 2$ and again that $\alpha - \beta a_t - (p-1)\beta^2 > -2$ in the last step. Finally, the last case $d_g(y, z) \lesssim d_g(x, y) \sim d_g(x, x')$ with the condition $(p-1)\beta^2 < 2$ leads to

$$d_g(x, x')^{\alpha - 2 - \varepsilon - \beta a_t} \int_{\mathcal{R}_1} d_g(x, y)^{\alpha - \varepsilon - \beta a_t} dV_g(y) \lesssim d_g(x, x')^{2\alpha - 2 - \varepsilon \beta a_t - (p-1)\beta^2}.$$
and we can conclude similarly as for the region $R_{1,1}$, the only difference being that in the case $d_g(y,z) \lesssim d_g(x,y) \sim d_g(x,z)$, we then have $d_g(x,y) \sim d_g(x,z) \sim d_g(x,x_\ell)$ by definition of $R_1$ and $R_3$, and we get the same bound as in the other cases.

**Contribution of $R_{3,3}$:** The contribution of this region can also be bounded by (3.7) and we can conclude as for $R_{1,1}$.

The remaining contributions of $R_{1,i_1,j_2}$ can then be estimated similarly as for $R_{1,1}$, $R_{1,2}$, $R_{1,3}$ or $R_{3,3}$ up to a permutation of the singularities. This leads to (3.3).

Finally, Lemma 3.3 (iii) follows from similar computations as in the proof of Proposition 1.1: indeed, we have

$$
\int_{H^0(M,\Sigma)} \mathbb{E}[Q_M(\Theta_{N_1}(t, x) - \Theta_{N_2}(t, x))]|^2 d\mu_g
= \int_M \int_M K_M(x, y_1)K_M(x, y_2) \left\{ H_{N_1}(y_1)H_{N_1}(y_2)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_1})G_g(y_1, y_2)} - 2H_{N_1}(y_1)H_{N_2}(y_2)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_2})G_g(y_1, y_2)} + H_{N_1}(y_1)H_{N_2}(y_2)e^{2\pi\beta^2(P_{N_2} \otimes P_{N_2})G_g(y_1, y_2)} \right\} dV_g(y_1)dV_g(y_2),
$$

where we wrote

$$
H_N(y) = e^{2\pi\beta \sum_{\ell=1}^L a_\ell (P_{N} \otimes P_{N}) G_g(x_{\ell}, y) + o(1)) \quad \text{(3.10)}.
$$

We deal with

$$
H_{N_1}(y_1)H_{N_1}(y_2)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_1})G_g(y_1, y_2)} - H_{N_1}(y_1)H_{N_2}(y_2)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_2})G_g(y_1, y_2)}
= \left( H_{N_1}(y_2) - H_{N_2}(y_2) \right) H_{N_1}(y_1)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_1})G_g(y_1, y_2)}
+ H_{N_1}(y_1)H_{N_2}(y_2) \left( e^{2\pi\beta^2(P_{N_1} \otimes P_{N_1})G_g(y_1, y_2)} - e^{2\pi\beta^2(P_{N_1} \otimes P_{N_2})G_g(y_1, y_2)} \right)
= I + II.
$$

Using the mean value theorem, Lemma 2.8 by interpolating between the two bounds in (2.18), and Lemma 2.7 with the definition of $H_N$ (3.10), we have

$$
|I| \lesssim \sum_{\ell=1}^L \left| (P_{N_1} \otimes P_{N_1}) G_g(x_{\ell}, y_2) - (P_{N_2} \otimes P_{N_2}) G_g(x_{\ell}, y_2) \right|
\times \left( \prod_{k=1}^L d_g(x_k, y_2)^{-\beta a_k} \right) H_{N_1}(y_1)e^{2\pi\beta^2(P_{N_1} \otimes P_{N_1})G_g(y_1, y_2)}
\lesssim \sum_{\ell=1}^L N_1^{\frac{\varepsilon}{4}} d_g(x_{\ell}, y_2)^{-\varepsilon} \left( \prod_{k=1}^L d_g(x_k, y_2)^{-\beta a_k} d_g(x_k, y_1)^{-\beta a_k} \right) d_g(y_1, y_2)^{-\beta^2}
\lesssim N_1^{-\frac{\varepsilon}{2}} \left( \prod_{\ell=1}^L d_g(x_{\ell}, y_2)^{-\beta a_k} d_g(x_{\ell}, y_1)^{-\beta a_k} \right) d_g(y_1, y_2)^{-\beta^2},
$$
for any $0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$ and $N_1 \leq N_2$. Similarly, we use again Lemmas 2.7 and 2.8 to bound

$$\|P\| \lesssim \left| (P_{N_1} \otimes P_{N_1})G_g(y_1, y_2) - (P_{N_1} \otimes P_{N_2})G_g(y_1, y_2) \right|$$

$$\times \left( \prod_{\ell=1}^L d_g(x_k, y_2)^{-\beta a \ell} d_g(x_k, y_1)^{-\beta a \ell} \right) d_g(y_1, y_2)^{-\beta^2}$$

$$\lesssim \left( \prod_{\ell=1}^L d_g(x_k, y_2)^{-\beta a \ell} d_g(x_k, y_1)^{-\beta a \ell} \right) N_1^{-\frac{\tilde{\varepsilon}}{2}} d_g(y_1, y_2)^{-\tilde{\varepsilon} - \beta^2}$$

$$= N_1^{-\frac{\tilde{\varepsilon}}{2}} \left( \prod_{\ell=1}^L d_g(x_k, y_2)^{-\beta a \ell} d_g(x_k, y_1)^{-\beta a \ell} \right) d_g(y_1, y_2)^{-\tilde{\varepsilon} - \beta^2}$$

for some $0 < \tilde{\varepsilon} \ll \varepsilon \ll 1$.

Plugging these bounds into (3.9) and proceeding as for (ii), we get

$$\int_{H_0^\alpha(M,g)} \mathbb{E} \left[ \left| Q_M(\Theta_{N_1}(t,x) - \Theta_{N_2}(t,x)) \right|^2 \right] d\mu_g$$

$$\lesssim M^{2(\alpha-\varepsilon)} N_1^{-\frac{\tilde{\varepsilon}}{2}} \int_\mathcal{M} \int_\mathcal{M} d_g(x, y_1)^{\alpha - 2 - \tilde{\varepsilon}} d_g(x, y_2)^{\alpha - 2 - \tilde{\varepsilon}}$$

$$\times \left( \prod_{\ell=1}^L d_g(x_k, y_2)^{-\tilde{\varepsilon} - \beta a \ell} d_g(x_k, y_1)^{-\tilde{\varepsilon} - \beta a \ell} \right) d_g(y_1, y_2)^{-\tilde{\varepsilon} - \beta^2} dV_g(y_1) dV_g(y_2)$$

$$\lesssim M^{2(\alpha-\varepsilon)} N_1^{-\frac{\tilde{\varepsilon}}{2}} f_{\alpha-\varepsilon - 2\tilde{\varepsilon}, \{x_\ell\}}(x).$$

This finally establishes Lemma 3.3 (iii). The estimate (3.1) of Lemma 3.3 (iv) then follows from the same computations as above up to using Lemma 2.13 in place of Lemma 2.8. □

**Remark 3.4.**

(i) Note that the constraint (1.31) in Proposition 1.8 ensures that $f_{\alpha-\varepsilon, \{x_\ell\}}$ given by Lemma 3.3 above belongs to $L_2^\varepsilon(\mathcal{M})$, so that using Fubini’s theorem and summing (3.2) in $M \in 2^{\tilde{\varepsilon}}\varepsilon^{-1}$, we have that $\|\Theta_N\|_{L^p(\mu_G \otimes \mathcal{F}) \cap L^p_{\mathcal{M}}(\mathcal{M})}$ is uniformly bounded in $N \in \mathbb{N}$. The convergence claimed in Proposition 1.8 then follows from interpolating between (i) and (iii) when $1 \leq p \leq 2$, and (ii) and (iii) for $p \geq 2$.

(ii) As mentioned in the introduction, the condition (1.16) is more restrictive than the usual second Seiberg bound (1.18) usually assumed for the construction of the LQG measure. However in our situation, the bound (1.16) is even required for the weakest bound

$$\sup_N \int \mathbb{E} \left[ \left| \Theta_N \right|^2 \right] d\mu_G < \infty.$$  

Moreover our argument for the proof of Theorem 1.4 requires $\Theta_N$ to be bounded in $L^2([0, T]; H^{-1}(\mathcal{M}))$, and the computation of the second moment in Lemma 3.3 (ii) explicitly requires the constraint (1.16).

(iii) Finally, note that the construction of $\Theta$ in Proposition 1.8 follows from the estimates in Lemma 3.3 above, and in particular Lemma 3.3 (iv) shows that $\Theta$ is independent of the choice of the approximation by a multiplier $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(0) = 1$. Note that in [15 30], only regularization by circle averaging is considered.
3.2. Construction of the LQG measure. We now turn to the convergence properties of the truncated measure $\rho_{N,g}$ in (1.11). In order to prove Theorem 1.1, we need several technical lemmas. These are mainly a unified adaptation of [15, 17, 30], though our regularization is different from those works.

A first observation is that $\Theta_N$ and its limit $\Theta$ given in Proposition 1.8 are random positive distributions on $[0,T] \times \mathcal{M}$ (for any $T > 0$), and thus $(\Theta_N)_{t=0}$ and $\Theta|_{t=0}$ can be identified with random Radon measures on $\mathcal{M}$. Thus for any Borel set $B \subset \mathcal{M}$, let us define the truncated Liouville measure as the random variable on $(H^0_0(\mathcal{M}), \mu_g)$ given by

$$Y_N(B) = Y_N(B, X_g) \overset{\text{def}}{=} \hat{B} \text{e}^{-\pi \beta^2 C P N - \frac{\beta^2}{2} e^{\beta P N} X_g(x)} H_N(x) dV_g(x),$$

(3.11)

where we redefine the function

$$H_N(x) \overset{\text{def}}{=} \text{e}^{2\pi \beta \sum_{\ell=1}^L a_\ell (P_N \otimes P_N) G_g(x, x)}.$$

(3.12)

We also define the closely related random measure

$$X_N(B) = X_N(B, X_g) \overset{\text{def}}{=} \int_B \Theta_N(0, x) dV_g(x),$$

(3.13)

with $\sigma_N(x, g)$ as in (1.26).

We first recall some basic facts about Gaussian processes, namely Kahane’s convexity inequality.

**Lemma 3.5.** Let $\{X_j\}_{j=1,\ldots,n}$ and $\{Y_j\}_{j=1,\ldots,n}$ be two centred Gaussian vectors such that

$$E[X_j X_k] \leq E[Y_j Y_k]$$

for any $j, k = 1, \ldots, n$. Then for all positive numbers $p_j$ and any convex function $F : \mathbb{R} \to \mathbb{R}$ it holds

$$E \left[ F \left( \sum_{j=1}^n p_j e^{X_j - \frac{1}{2} E[X_j^2]} \right) \right] \leq E \left[ F \left( \sum_{j=1}^n p_j e^{Y_j - \frac{1}{2} E[Y_j^2]} \right) \right].$$

**Proof.** See for example [53, Corollary A.2].

Next we state the existence of negative moments for $X_N$. Note that taking $a_\ell = 0$ for any $\ell$, we have that

$$e^{\beta X_g(x)} : = \lim_{N \to \infty} e^{\beta P_N X_g - \frac{\beta^2}{2} \sigma_N}$$

is well-defined, where the convergence holds in $L^p(\mu_g; B^{(p)}_{\alpha(p)}(\mathcal{M}))$ for any $p \geq 1$, with $\alpha(p)$ as in Proposition 1.8 for $a_\ell = 0$.

**Lemma 3.6.** Let $0 < \beta^2 < 2$, and let $X_N$ be defined as in (3.13). Then for any $a > 0$, there exists $C > 0$ such that for any $y_0 \in \mathcal{M}$, any $0 < r < \iota(\mathcal{M})$ and any $N \in \mathbb{N}$, it holds

$$\int X_N \left( B(y_0, r) \right)^{-a} d\mu_g \leq C.$$
Moreover, we have the convergence
\[
\int \mathcal{X}_N(B(y_0, r))^{-a}d\mu_g \rightarrow \int \mathcal{X}(B(y_0, r))^{-a}d\mu_g
\]
as $N \to \infty$, where
\[
\mathcal{X}(B(y_0, r)) \defeq \int_{B(y_0, r)} : e^{\beta X_g(x)} : dV_g(x).
\]

Proof. First, note that for any $B \subset \mathcal{M}$, $\mathcal{X}(B)$ as in Lemma 3.6 is a well-defined random variable. Indeed, thanks to the positivity of $e^{\beta P_N X_g(x)} - \frac{\beta^2}{2} \sigma_N(x)$ it holds $0 < \mathcal{X}_N(B) \leq \mathcal{X}_N(\mathcal{M})$ and this last term is in $L^2(\mu_g)$, uniformly in $N \in \mathbb{N}$. This follows by taking $a_\ell = 0$ for any $\ell$, so that we have
\[
\int \mathcal{X}_N(\mathcal{M})^2d\mu_g(X_g) \leq \int \mathcal{M} \int \mathcal{M} e^{2\pi \beta^2(P_N \otimes P_N)G_g(x,y)}dV_g(x)dV_g(y)
\leq \int \mathcal{M} \int \mathcal{M} d_g(x, y)^{-\beta^2}dV_g(x)dV_g(y) < \infty
\]
uniformly in $N \in \mathbb{N}$ since $0 < \beta^2 < 2$.

As for the convergence of $\mathcal{X}_N(B)$, we have
\[
\int |\mathcal{X}_{N_1}(B) - \mathcal{X}_{N_2}(B)|^2d\mu_g = \int \mathcal{M} \int \mathcal{M} \left[ e^{2\pi \beta^2(P_{N_1} \otimes P_{N_1})G_g(x,y)} - 2e^{2\pi \beta^2(P_{N_1} \otimes P_{N_2})G_g(x,y)} - e^{2\pi \beta^2(P_{N_2} \otimes P_{N_2})G_g(x,y)} \right]dV_g(x)dV_g(y)
\]
which converges to 0 as $N_1, N_2 \to \infty$ by similar (simpler) computations as for Lemma 3.3 (iii).

To prove Lemma 3.6 we fix a realization of $X_g$ as in (1.8) on $(\Omega, \mathbb{P})$ and we then follow the argument in [42, Proposition 4]. It is of a general nature: provided that one has an inequality in law of the type
\[
\mathcal{X}_N(B) \geq \sum_{k=1}^{K^2} \omega_k \mathcal{X}_{N,k}(B)
\]
for some independent copies $\mathcal{X}_{N,k}(B)$ of $\mathcal{X}_N(B)$ and some independent random variables $\omega_k$ admitting negative moments, then this argument implies that $\mathcal{X}_N(B)$ has in turn negative moments. Here, as in [34], we will rely on Kahane’s inequality in Lemma 3.5 and the main computations from the proof of Proposition 4(a) in [42] will then apply.

Let us then take geodesic normal coordinates $\kappa$ centred at $y_0 \in \mathcal{M}$, and $0 < r \ll \iota(\mathcal{M})$. We define $\mathcal{C} \subset \mathbb{R}^2$ to be the cube of side-length $r$ centred around 0; in particular we have $\mathcal{C} \subset B(0, r) = \kappa^{-1}(B(y_0, r)) \subset 2\mathcal{C}$. Let us take an integer $K \in \mathbb{N}$ and denote by $C_k$, $k = 1, \ldots, K^2$ the partition of $\mathcal{C}$ in cubes $C_k$ of side-length $K^{-1}r$ centred around some $z_k \in \mathcal{C}$. We also define the smaller cubes $C'_k$ centred around $z_k$ with side-length $(4K)^{-1}r$. Since $e^{\beta P_N X_g - \frac{\beta^2}{2} \sigma_N}$ is positive, and since $V_g$ is equivalent to the Lebesgue measure in $\kappa$, we
have

\[ X_N(B(y_0, r)) \geq X_N(\kappa^{-1}(C)) \geq C \int_C e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z) \, dz \]

\[ = \sum_{k=1}^{K^2} \int_{C_k} e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z) + c \, dz \geq \sum_{k=1}^{K^2} \int_{C_k} e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z) + c \, dz \]

\[ = \sum_{k=1}^{K^2} (8K)^{-2} \int_{2C} e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z_k + (8K)^{-1} z) + c \, dz. \] (3.14)

Next, for any \( J \in \mathbb{N} \) we can subdivide again \( 2C = \bigcup_{j=1}^{J^2} C_j \) into cubes of side-length \( J^{-1} r \) centred at some \( \tilde{z}_j \in 2C \). Since for any fixed \( N \in \mathbb{N} \), \( \kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N] \) has almost surely continuous paths, for any \( k = 1, ..., K^2 \) the following convergence holds almost surely:

\[ \int_{2C} e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z_k + (8K)^{-1} z) + c \, dz = \lim_{J \to \infty} \sum_{j=1}^{J^2} J^{-2} e^{\kappa_*[\beta P_N X_k - \frac{\beta^2}{2} \sigma_N]}(z_k + (8K)^{-1} \tilde{z}_j) + c. \] (3.15)

Now from the same computation as in Lemma 3.1 it holds for any \( k_1, k_2 = 1, ..., K^2 \) and \( j_1, j_2 = 1, ..., J^2 \):

\[ \mathbb{E} \left[ (\kappa_* P_N X_k(z_{k_1} + (8K)^{-1} \tilde{z}_{j_1}) + c) (\kappa_* P_N X_k(z_{k_2} + (8K)^{-1} \tilde{z}_{j_2}) + c) \right] \]

\[ = 2\pi(P_N \otimes P_N) G_\beta \left( \kappa(z_{k_1} + (8K)^{-1} \tilde{z}_{j_1}), \kappa(z_{k_2} + (8K)^{-1} \tilde{z}_{j_2}) \right) + c. \]

From the two-sided bound of Corollary 2.12 we can bound this term with

\[ - \log \left( \left| (z_{k_1} - z_{k_2}) + (8K)^{-1} (\tilde{z}_{j_1} - \tilde{z}_{j_2}) \right| + N^{-1} \right) + C_K \]

for some constant \( C_K > 0 \) independent of \( J \) and \( N \).

In the case \( k_1 = k_2 \), we have then (using Corollary 2.12 again)

\[ \mathbb{E} \left[ \kappa_* P_N X_k(z_k + (8K)^{-1} \tilde{z}_{j_1}) \kappa_* P_N X_k(z_k + (8K)^{-1} \tilde{z}_{j_2}) \right] \]

\[ \leq - \log \left( \left| \tilde{z}_{j_1} - \tilde{z}_{j_2} \right| + 8KN^{-1} \right) + \log K + C_K \]

\[ \leq - \log \left( \left| \tilde{z}_{j_1} - \tilde{z}_{j_2} \right| + N^{-1} \right) + C_K \]

\[ \leq \mathbb{E} \left[ \kappa_* P_N X_k(z_k) \kappa_* P_N X_k(z_k) \right] + C_K \]

where \( X_{g, k}, k = 1, ..., K^2 \) are independent copies of \( X_g \), and \( C_K > 0 \) is some constant independent of \( N, J, \tilde{z}_j \).

On the other hand, in the case \( k_1 \neq k_2 \), we have by choice of \( z_k \) and \( \tilde{z}_j \) that \( |z_{k_1} - z_{k_2}| \geq K^{-1} r \) and \( |\tilde{z}_{j_1} - \tilde{z}_{j_2}| \leq 2\sqrt{2r} \), which in turn implies

\[ - \log \left( \left| (z_{k_1} - z_{k_2}) + (8K)^{-1} (\tilde{z}_{j_1} - \tilde{z}_{j_2}) \right| + N^{-1} \right) \]

\[ \leq - \log \left( \left| (z_{k_1} - z_{k_2}) - (8K)^{-1} (\tilde{z}_{j_1} - \tilde{z}_{j_2}) \right| + N^{-1} \right) \]

\[ \leq - \log \left( cK^{-1} + N^{-1} \right) \leq \log K + C. \]
Hence we arrive at

\[
\mathbb{E}\left[ (\kappa_N \mathbf{P} X_{g}^k(z_{k_1}) + (8K)^{-1}z_{j_1}) + \kappa_N \mathbf{P} X_{g}^k(z_{k_2}) + (8K)^{-1}z_{j_2}) + c \right] 
\leq \mathbb{E}[\kappa_N \mathbf{P} X_{g,k_1}(z_{j_1}) \kappa_N \mathbf{P} X_{g,k_2}(z_{j_2})] + C_K
\]  

(3.16)

for some constant $C_K > 0$ independent of $N, J \in \mathbb{N}$. If we then take some independent $h_k \sim \mathcal{N}(0, C_K)$ (and independent of $X_g$ and $(X_{g,k})_{k=1,\ldots,K^2}$), we can then use (3.14)-(3.15)-(3.16) with Kahane’s inequality in Lemma 3.5 to get for any decreasing and convex function $F : \mathbb{R}_+ \to \mathbb{R}$,

\[
\mathbb{E}\left[ F\left(\mathcal{X}_N(B(y_0, r))\right)\right] \leq \mathbb{E}\left[ F\left(\sum_{k=1}^{K^2} (8K)^{-2} \int_c e^{\kappa_N \mathbf{P} X_{g,k} - \frac{\beta^2}{4} \sigma_N} (z_k + (8K)^{-1}z) + \epsilon dz\right)\right]
\]

\[
= \lim_{J \to \infty} \mathbb{E}\left[ F\left(\sum_{k=1}^{K^2} \sum_{j=1}^{J^2} (8KJ)^{-2} e^{\kappa_N \mathbf{P} X_{g,k} - \frac{\beta^2}{4} \sigma_N} (z_k + (8K)^{-1}z_j) + \epsilon\right)\right]
\]

\[
\leq \lim_{J \to \infty} \mathbb{E}\left[ F\left(\sum_{k=1}^{K^2} \sum_{j=1}^{J^2} (8KJ)^{-2} e^{\kappa_N \mathbf{P} X_{g,k} - \frac{\beta^2}{4} \sigma_N} (\bar{z}_k + \beta h_k - \frac{\beta^2}{4} C_K\right)\right]
\]

\[
= \mathbb{E}\left[ F\left(\sum_{k=1}^{K^2} (8K)^{-2} \int_{2\mathcal{C}} e^{\kappa_N \mathbf{P} X_{g,k} - \frac{\beta^2}{4} \sigma_N} + \beta h_k - \frac{\beta^2}{4} C_K dV_g\right)\right]
\]

(3.17)

We can now prove the existence of negative moments for $\mathcal{X}_N(B(y_0, r))$, uniformly in $N \in \mathbb{N}$. First, we will prove that $\mathbb{E}\left[ \mathcal{X}_N(B(y_0, r))^{-\epsilon}\right] < \infty$ uniformly in $N \in \mathbb{N}$ for some $0 < \epsilon \ll 1$. To do so, let us introduce the moment-generating function

$$
\varphi(t) \overset{\text{def}}{=} \mathbb{E}\left[ e^{-t\mathcal{X}_N(B(y_0, r))}\right], \quad t \geq 0.
$$

Note that by Fubini’s theorem

$$
\int_0^\infty t^{\epsilon-1} \varphi(t) dt = \Gamma(\epsilon) \mathbb{E}\left[ \mathcal{X}_N(B(y_0, r))^{-\epsilon}\right],
$$

where $\Gamma$ is the Gamma function. Since $\mathcal{X}_N(B(y_0, r))$ is almost surely positive, we always have $\varphi(t) \leq 1$ and that $\varphi$ is decreasing with $\varphi(t) \to 0$ as $t \to +\infty$. Hence it is enough to prove that $\varphi(t) \leq ct^{-\epsilon}$ for some $c > 0$, $\epsilon' > \epsilon$ and $t \geq t_0$ large enough.
From (3.17) with the decreasing convex function \( x \mapsto e^{-tx} \) for any \( t > 0 \) and the independence of \( h_k \) and \( X_k \) with Young’s and Jensen’s inequalities, we then have

\[
\varphi(t) \leq \prod_{k=1}^{K^2} \mathbb{E} \left[ \exp \left(-t(8K)^{-2}e^{\beta h_k - \frac{\beta^2}{2}C_k} \int_{B(y_0, r)} e^{\beta P_N x_{g,k} - \frac{\beta^2}{2} \sigma_N dV_g} \right) \right] \\
\leq \sum_{k=1}^{K^2} K^{-2} \mathbb{E} \left[ \exp \left(-t(8K)^{-2}e^{\beta h_k - \frac{\beta^2}{2}C_k} \int_{B(y_0, r)} e^{\beta P_N x_{g,k} - \frac{\beta^2}{2} \sigma_N dV_g} | h_k \right) \right]^{K^2} \\
\leq \sum_{k=1}^{K^2} K^{-2} \mathbb{E} \left[ \varphi(t(8K)^{-2}e^{\beta h_k - \frac{\beta^2}{2}C_k})K^2 \right] = \mathbb{E} \left[ \varphi(t(8K)^{-2}e^{\beta h_1 - \frac{\beta^2}{2}C_k})K^2 \right] .
\]

(3.18)

In particular, using that \( \varphi(t) \leq 1 \) for any \( t > 0 \), we deduce from (3.18) that

\[
\varphi(t^2) \leq \mathbb{E} \left[ 1 \{ t(8K)^{-2}e^{\beta h_1 - \frac{\beta^2}{2}C_k} < 1 \} \right] \varphi(t(8K)^{-2}e^{\beta h_1 - \frac{\beta^2}{2}C_k})K^2 \\
+ \mathbb{E} \left[ 1 \{ t(8K)^{-2}e^{\beta h_1 - \frac{\beta^2}{2}C_k} \geq 1 \} \varphi(t(8K)^{-2}e^{\beta h_1 - \frac{\beta^2}{2}C_k})K^2 \right] \\
< \mathbb{P} \left( e^{\beta h_1 - \frac{\beta^2}{2}C_k} < \frac{(8K)^2}{t} \right) + \varphi(t)^2.
\]

Using then Chebychev’s inequality, we can bound for any \( p \geq 1 \)

\[
\mathbb{P} \left( e^{\beta h_1 - \frac{\beta^2}{2}C_k} < \frac{(8K)^2}{t} \right) \leq \frac{(8K)^{2p}}{t^{2p}} \mathbb{E} \left[ e^{-2p(\beta h_1 - \frac{\beta^2}{2}C_k)} \right] = \frac{(8K)^{2p}}{t^{2p}} e^{\frac{\beta^2}{2}C_k 2p(2p+1)} \\
\leq C(K, p)t^{-2p}.
\]

All in all, provided that \( K \geq 2 \) we arrive at

\[
\varphi(t^2) < q(t) \left[ t^{-p} + \varphi(t)^2 \right]
\]

where \( q(t) = C(K, p)(t^{-p} + \varphi(t)^{K^2-2}) \to 0 \) as \( t \to +\infty \). This is precisely the estimate at the bottom of p. 687 in [42], and from there the same computations apply and give

\[
\varphi(t) \leq ct^{-\epsilon'}
\]

for some \( 0 < \epsilon' \ll 1 \) and all \( t \geq t_0 \) large enough. This shows that

\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \mathcal{X}_N(B(y_0, r))^{-\epsilon} \right] < \infty
\]

(3.19)

for some \( 0 < \epsilon < \epsilon' \ll 1 \). For some arbitrary \( a > 0 \), we then use (3.17) again with the decreasing convex function \( x \mapsto x^{-a} \) and Young’s inequality with the independence of \( h_k \) and \( X_k \) to get

\[
\mathbb{E} \left[ \mathcal{X}_N(B(y_0, r))^{-a} \right] \leq \left[ \sum_{k=1}^{K^2} (8K)^{-2}e^{\beta h_k - \frac{\beta^2}{2}C_k} \int_{B(y_0, r)} e^{\beta P_N x_{g,k} - \frac{\beta^2}{2} \sigma_N dV_g} \right]^{-a} \\
\leq \prod_{k=1}^{K^2} \mathbb{E} \left[ (e^{\beta h_k - \frac{\beta^2}{2}C_k} - aK^{-2}) \right] \prod_{k=1}^{K^2} \mathbb{E} \left[ \left( \int_{B(y_0, r)} e^{\beta P_N x_{g,k} - \frac{\beta^2}{2} \sigma_N dV_g} \right)^{-aK^{-2}} \right] \\
\leq \left( \mathbb{E} \left[ \mathcal{X}_N(B(y_0, r))^{-aK^{-2}} \right] \right)^{K^2} .
\]
We can then iterate the inequality above \( n \) times so that \( aK^{-2n} < \varepsilon \) and conclude from (3.19). This shows that all the negative moments are bounded uniformly in \( N \in \mathbb{N} \).

The convergence of the negative moments then follows from their boundedness and the convergence of \( X_N(B(y_0, r)) \) in \( L^2(\mu_g) \): indeed using the mean value theorem and Cauchy-Schwarz inequality, we have

\[
\left| \mathbb{E}[X_{N_1}(B(y_0, r))^{-a}] - \mathbb{E}[X_{N_2}(B(y_0, r))^{-a}] \right| \\
\lesssim \mathbb{E}\left[|X_{N_1}(B(y_0, r)) - X_{N_2}(B(y_0, r))| \left( X_{N_1}(B(y_0, r))^{-a-1} + X_{N_2}(B(y_0, r))^{-a-1} \right) \right] \\
\lesssim \mathbb{E}\left[|X_{N_1}(B(y_0, r)) - X_{N_2}(B(y_0, r))|^2 \left( \mathbb{E}[X_{N_1}(B(y_0, r))^{-2a-2}] + \mathbb{E}[X_{N_2}(B(y_0, r))^{-2a-2}] \right) \right] \\
\lesssim \mathbb{E}|X_{N_1}(B(y_0, r)) - X_{N_2}(B(y_0, r))|^2 \to 0
\]
as \( N_1, N_2 \to \infty \). This proves Lemma 3.6. \( \square \)

We also show the following estimates on the random variables \( Y_N(B) \) defined in (3.11) above.

**Lemma 3.7.** Suppose that the assumption (1.14) and (1.16) hold. Then for any \( a > 0 \) it holds

\[
\lim_{N \to \infty} \int Y_N(M, X_g)^{-a} d\mu_g(X_g) = \int Y(M, X_g)^{-a} d\mu_g(X_g) < \infty,
\]
where

\[
Y(M, X_g) \overset{\text{def}}{=} \int_M \Theta(0, x) dV_g(x)
\]
is independent of the choice of \( \psi \in S(\mathbb{R}) \) with \( \psi(0) = 1 \) used to define \( P_N = \psi(-N^{-2} \Delta_g) \).

**Proof.** This is similar to [15, Lemma 3.3]. Indeed, since the density is almost surely positive, we have that for any subset \( B \subset M, Y_N(M)^{-a} \leq Y_N(B)^{-a} \). In particular if we take \( B \) to be any ball such that \( x_\ell \notin B \) for any \( \ell = 1, ..., L \), we have that \( H_N \) in (3.12) is bounded from below on \( B \) by some positive constant (depending on \( B \)), so we can estimate

\[
Y_N(B)^{-a} \lesssim_B X_N(B)^{-a}.
\]
The finiteness of the negative moments of the Gaussian multiplicative chaos given by Lemma 3.6 then ensures that \( \int Y_N(B)^{-a} d\mu_g \) is uniformly bounded.

Finally, the convergence of \( \int Y_N(M)^{-a} d\mu_g \) follows from the previous step along with the mean value theorem, Cauchy-Schwarz inequality, and the convergence in \( L^2(\mu_g) \) of \( Y_N(M) \) as in the proof of Lemma 3.6. The independence on the choice of multiplier follows similarly. This proves Lemma 3.7. \( \square \)

With Lemma 3.7 at hand, we can finally give the proof of our first main result.

**Proof of Theorem 1.1.** The proof of Theorem 1.1 is a straightforward adaptation of the proofs of [17, Theorem 4.3] and [15, Theorem 3.2]. In order to compare with our approach in Proposition 1.8 we detail the argument nonetheless.
**Case 1:** if \( g = g_0 \). We begin by treating the case where the metric has constant curvature. Recall from (1.11) that \( d\rho_{N,g_0}(X_0, \overline{X}) = R_N(X_0 + \overline{X})d\mu_0(X_0)d\overline{X} \) where, as in (1.12), the density \( R_N \) is given by
\[
R_N(X_0 + \overline{X}) = \Xi \exp \left\{ \sum_{\ell=1}^{L} \left( a_{\ell} P_N(X_0 + \overline{X})(x_{\ell}) - \frac{a_{\ell}^2}{2} \left( \log N + 2\pi \sigma \right) \right) \right\} - \frac{Q}{4\pi} \int_{\mathcal{M}} R_0(X_0 + \overline{X})dV_0 - \nu \int_{\mathcal{M}} e^{-\pi \beta^2 C P N - \frac{\beta^2}{2} e^{\beta P_N(X_0 + \overline{X})}dV_0} \right\}
\]
with \( \sigma_N \) as in (1.26) and where \( o(1) \) is deterministic and uniform on \( \mathcal{M} \). This follows by using Lemma 2.10 as well as Gauss-Bonnet theorem (2.23) with the fact that \( X_0 \) has mean zero.

We first show the convergence of the partition function \( Z_N(g_0) \) in (1.11). Let us define
\[
\mathcal{G}_{N,0} \overset{\text{def}}{=} \int_{H_0^0(\mathcal{M},g_0)} \left( \prod_{\ell=1}^{L} e^{a_{\ell} \sigma_N(x_{\ell}) - \frac{a_{\ell}^2}{2} \left( \sigma_N(x_{\ell}) - 2\pi \Gamma_N(x_{\ell}, x_{\ell}) + o(1) \right)} \right) d\mu_0(X_0)
\]
\[
= \left( \prod_{\ell=1}^{L} e^{2\pi \sigma_N(x_{\ell}) + o(1)} \right) e^{2\pi \sum_{\ell<k} a_{\ell} a_{k} \Gamma_N(x_{\ell}, x_{k}) + o(1)} > 0.
\]
Note that by Girsanov theorem ([14, Theorem 10.14]), the Gaussian process
\[
\tilde{X}_0 = X_0 - 2\pi \sum_{\ell=1}^{L} a_{\ell} (P_N \otimes \text{Id})G_0(x_{\ell}, x)
\]
has the same law as \( X_0 \) under the new probability measure
\[
e^{-2\pi \sum_{\ell<k} a_{\ell} a_{k} (P_N \otimes P_N)G_0(x_{\ell}, x_{k})} \exp \left( \sum_{\ell=1}^{L} a_{\ell} P_N(X_0)(x_{\ell}) - \frac{a_{\ell}^2}{2} \sigma_N(x_{\ell}) \right) d\mu_0.
\]
Therefore the partition function of \( \rho_{N,g_0} \) can be expressed as
\[
Z_N(g_0) = \int_{H_0^0(\mathcal{M},g_0)} \int_{\mathcal{R}} R_N(X_0 + \overline{X})d\mu_0(X_0)d\overline{X}
\]
\[
= \mathcal{G}_{N,0} \Xi \int_{H_0^0(\mathcal{M},g_0)} \int_{\mathcal{R}} \exp \left\{ \left[ \sum_{\ell=1}^{L} a_{\ell} - Q(\mathcal{M}) \right] \overline{X} - \nu e^{\beta \overline{X}} \mathcal{Y}_N(\mathcal{M}) \right\} d\mu_0(X_0)d\overline{X}
\]
with \( \mathcal{Y}_N(\mathcal{M}) \) as in (3.11). With the change of variable
\[
\overline{X} \mapsto \tau \overset{\text{def}}{=} \nu e^{\beta \overline{X}} \mathcal{Y}_N(\mathcal{M}),
\]
we continue with

\[ Z_N(g_0) = G_{N,0} \Xi \beta^{-1} \nu \beta^{-1} [Q_\chi(M) - \pi] \left[ \int_0^{+\infty} \tau \beta^{-1} [\pi - Q_\chi(M)]^{-1} e^{-\tau} d\tau \right] \]

\[ \times \left[ \int_{H_0^g(M,\mathcal{M}_0)} \nu \beta^{-1} [Q_\chi(M) - \pi] d\mu_0 \right], \tag{3.21} \]

where we set \( \pi = \sum_{\ell=1}^L a_\ell \).

The integral in \( \tau \) in (3.21) is \( \Gamma(\beta^{-1} [\pi - Q_\chi(M)]) \) which is finite under the assumption that the first Seiberg bound (1.15) holds, and the integral with respect to \( \mu_0 \) converges by Lemma 3.7. Since the \( x_\ell \)'s are all different, the constant \( G_{N,0} \) also converges to the constant

\[ G_0 \overset{\text{def}}{=} \left( \prod_{\ell=1}^L e^{2\pi/\beta} G_0(x_\ell, x_\ell) \right) e^{2\pi \sum_{\ell<k} a_\ell a_k G_0(x_\ell, x_k)} < \infty, \]

which shows that the whole partition function \( Z_N(g_0) \) converges to a non trivial real number.

Moreover, for any \( F \in C_0(H_0^g(M) \oplus \mathbb{R}) \), we can use the same argument as in the proof of \[ \text{[15] Theorem 3.2} \]: we have

\[ \int_{H_0^g(M,\mathcal{M}_0)} \int_{\mathbb{R}} F(X_0 + \bar{X}) R_N(X_0, \bar{X}) d\mu_0(X_0) d\bar{X} \]

\[ = G_{N,0} \Xi \beta^{-1} \nu \beta^{-1} [Q_\chi(M) - \pi] \left[ \int_0^{+\infty} \tau \beta^{-1} [\pi - Q_\chi(M)]^{-1} e^{-\tau} \right] \]

\[ \times \int_{H_0^g(M,\mathcal{M}_0)} F \left( X_0 + \beta^{-1} \ln \frac{\tau}{\nu(N(M))} + 2\pi \sum_{\ell=1}^L a_\ell (P_N \otimes P_N) G_0(x_\ell, x) \right) \]

\[ \times \nu \beta^{-1} [Q_\chi(M) - \pi] d\mu_0(X_0) d\tau. \]

Then we note that \( (P_N \otimes P_N) G_0(x_\ell, \cdot) \) converges to \( G_0(x_\ell, \cdot) \) in \( H_0^g(M) \) and that \( 0 < \nu N(M) < \infty \) almost surely and \( \nu N(M) \rightarrow \nu(M) \) in probability from the proof of Lemma 3.7. Since the term with \( F \) is then almost surely uniformly bounded and the integral too in view of the previous step, we can use dominated convergence to conclude that the last term above converges.

**Case 2: general metric.** In the case of a general metric \( g = e^f g_0 \) as in (2.22), we proceed as in \[ \text{[15] Subsection 3.5} \]: we first make the change of variable \( \bar{X} = \bar{X} + \langle X_g \rangle_0 \) and then use Lemma 2.14 to get

\[ \int_{H_0^g(M,\mathcal{M})} \int_{\mathbb{R}} F(X_g + \bar{X}) R_N(X_g, \bar{X}) d\mu_g(X_g) d\bar{X} \]

\[ = \Xi(g) \int_{H_0^g(M,\mathcal{M})} \int_{\mathbb{R}} F(X_g - \langle X_g \rangle_0 + \bar{X}) \exp \left\{ -\frac{Q}{4\pi} \int_{\mathcal{M}} R_g(X_g - \langle X_g \rangle_0 + \bar{X}) dV_g \right. \]

\[ + \sum_{\ell=1}^L \left( a_\ell (P_N \otimes P_N) (X_g - \langle X_g \rangle_0 + \bar{X})(x_\ell) - \frac{a_\ell^2}{2} \left( \log N + 2\pi C_P \right) \right) \]

\[ - \nu \int_{\mathcal{M}} e^{-\pi \beta^2 C_P N - \frac{a_\ell^2}{2} e^{\beta P_N}(X_g - \langle X_g \rangle_0 + \bar{X})} dV_g \right\} d\mu_g(X_g) d\bar{X} \]
In view of (2.22), the shift in

\[ X_0(x) + \frac{Q}{2} \int_M R_g(x_0) G_0(x, y) dV_g(y) \]

is still a mass-less GFF under

\[ \exp \left\{ - \frac{Q}{4\pi} \int_M R_g X_0 dV_g - \frac{1}{2} \int \left| - \frac{Q}{4\pi} \int_M R_g X_0 dV_g \right|^2 d\mu_0(X_0) \right\} d\mu_0. \]

In view of (2.22), the shift in \( X_0 \) can be computed as

\[ \frac{Q}{2} \int_M R_g(y) G_0(x, y) dV_g(y) = \frac{Q}{2} \int_M G_0(x, y) (R_0 - \Delta_0 f_0(y)) dV_0(y) = \frac{Q}{2} (f_0(x) - \langle f_0 \rangle_0) \]

by using that \( R_0 \) is constant, that \( G_0(x, \cdot) \) has mean zero, and (2.16). Similarly, we also have

\[ \int_{H_0^1(M,g)} \left| \int_M R_g X_0 dV_g \right|^2 d\mu_0(X_0) = 2\pi \int_M \int_M R_g(x) R_g(y) G_0(x, y) dV_g(x) dV_g(y) \]

\[ = 2\pi \int_M (R_0 - \Delta_0 f_0(x)) (f_0(x) - \langle f_0 \rangle_0) dV_0(x) \]

\[ = 2\pi \int_M |\nabla_0 f_0|^2 dV_0. \]
With the change of variable $\bar{X} \mapsto \bar{X} + \frac{Q}{2}(f_0),$, this yields
\[
\int_{H_0^N(M,g)} \int_{\mathbb{R}} F(X_g + \bar{X}) R_N(X_g, \bar{X}) d\mu_g(X_g) d\bar{X}
\]
\[
= \Xi(g) \int_{H_0^N(M,g)} \int_{\mathbb{R}} F\left( X_0 - \frac{Q}{2} f_0 + \bar{X} \right) e^{\frac{Q^2}{4} \sum_{\ell=0}^L (\nabla_0 f_0)[\mathcal{M}(f_0) - \sum_{\ell=0}^L (\frac{Q a_\ell}{2} - \frac{Q^2}{4})] f_0(x_\ell) + o(1) \}
\times \exp \left\{ \sum_{\ell=1}^L \left[ a_\ell P_{N,0} X_0(x_\ell) - \frac{a_\ell^2}{2} (2\pi (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x_\ell) - 2\pi G_0(x_\ell, x_\ell) + o(1)) \right] \right\}
+ \left[ \pi - Q \chi(\mathcal{M}) \right] \bar{X} - \nu e^{\beta \bar{X}} \int_{\mathcal{M}} e^{-\pi \beta^2 C P N^{-\beta^2 \frac{2}{7}} e^{2 P N,g} x_0 + (1 - \beta^2 \alpha) f_0 + o(1)} dV_0 \right\} d\mu_0(X_0) d\bar{X}.
\]
Proceeding then as in the previous case by doing the Girsanov transform
\[
\bar{X}_0 = X_0 - 2\pi \sum_{\ell=1}^L a_\ell (P_{N,g} \otimes \text{Id}) G_0(x_\ell, x)
\]
with the new measure
\[
e^{-2\pi \sum_{\ell=1}^L a_\ell a_\ell (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x)} \exp \left( \sum_{\ell=1}^L a_\ell P_{N,0} X_0(x_\ell) - \frac{a_\ell^2}{2} 2\pi (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x_\ell) \right) d\mu_0,
\]
we get
\[
\int_{H_0^N(M,g)} \int_{\mathbb{R}} F(X_g + \bar{X}) R_N(X_g, \bar{X}) d\mu_g(X_g) d\bar{X}
\]
\[
= \Xi(g) G_N,0 (1 + o(1)) \int_{H_0^N(M,g)} \int_{\mathbb{R}} F\left( X_0 - \frac{Q}{2} f_0 + \bar{X} \right) + 2\pi \sum_{\ell=1}^L a_\ell (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x)
\times e^{\frac{Q^2}{4} \sum_{\ell=0}^L (\nabla_0 f_0)[\mathcal{M}(f_0) - \sum_{\ell=0}^L (\frac{Q a_\ell}{2} - \frac{Q^2}{4})] f_0(x_\ell) + o(1) \}
\times \nu e^{\beta \bar{X}} \int_{\mathcal{M}} e^{-\pi \beta^2 C P N^{-\beta^2 \frac{2}{7}} e^{2 P N,g} x_0 + \sum_{\ell=1}^L a_\ell (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x)}
\times e^{(1 - \beta^2 \alpha) f_0 + o(1)} dV_0 \right\} d\mu_0(X_0) d\bar{X},
\]
with again $o(1)$ being deterministic.

Using the fact that the estimate of Lemma 3.3 (iii) also holds for
\[
\Theta_{N,g,0} \overset{\text{def}}{=} e^{-\pi \beta^2 C P N^{-\beta^2 \frac{2}{7}} e^{2 P N,g} x_0 + 2\pi \sum_{\ell=1}^L a_\ell (P_{N,g} \otimes P_{N,g}) G_0(x_\ell, x)}
\]
and using also Lemma 2.15, we find that
\[
\Theta_{N,g,0} \rightarrow e^{\frac{\beta^2}{2} f_0} \Theta
\]
as $N \rightarrow \infty$ in the same topology as in Proposition 1.8. With the identity $\frac{\beta^2}{2} + 1 - \beta^2 \alpha = 0$ (by definition of $Q$), and the same argument as in the proof of Lemma 3.7, we deduce that
\[
\int_{\mathcal{M}} \Theta_{N,g,0}(0) e^{(1 - \beta^2 \alpha) f_0 + o(1)} dV_0 \rightarrow \mathcal{Y}(\mathcal{M})
\]
as $N \to \infty$ in $L^2(\mu_0)$. This shows that

$$
\int_{H_0^c(\mathcal{M},g)} \int_{\mathbb{R}} F(X_g + \mathbb{X}) R_N(X_g, \mathbb{X}) d\mu_g(X_g) d\mathbb{X}
$$

converges to

$$
\Xi(g) e^{\frac{Q}{16\pi} \int_\mathcal{M} |\nabla_0 f_0|^2 dV_0 + \frac{Q}{2} \chi(\mathcal{M})(f_0)_0 - \sum_{t=0}^l \left( \frac{Q}{2} f_0 - \frac{Q}{2} \right) f_0(x_t)}
\times \Xi(g_0)^{-1} \int_{H_0^c(\mathcal{M},g_0)} \int_{\mathbb{R}} F(X_0 + \mathbb{X} - \frac{Q}{2} f_0) d\rho_{(\alpha_0,x_0)}(X_0, \mathbb{X}).
$$

Finally, the identity

$$
\frac{Q^2}{2} \chi(\mathcal{M})(f_0)_0 = \frac{Q^2}{16\pi} \int_\mathcal{M} R_0 f_0 dV_0
$$
given by Gauss-Bonnet (2.23), as well as the use of (2.24) to simplify $\Xi(g) \Xi(g_0)^{-1}$ shows (1.17) and concludes the proof of Theorem 1.1.

\section*{4. Proof of Theorem 1.4}

We know turn to the construction of the dynamics preserving the correlations (1.3).

\subsection*{4.1. Construction of the dynamics}
Recall that we look for a solution of (1.25) under the form (1.30) with $z$ as in (1.31) and where $v_N$ solves (1.32). In particular, note that $z \in C(\mathbb{R}_+; C^\infty(\mathcal{M}))$, $\mathbb{P}$-almost surely (and for any $\mathbb{X} \in \mathbb{R}$). In this subsection, we thus focus on the equation:

$$
\begin{align*}
\partial_t v_N - \frac{1}{4\pi} \Delta_g v_N + \frac{1}{2} \nu \beta P_N \left[ e^{\beta P_N z} e^{\beta P_N v_N} \Theta_N \right] &= 0, \\
v_N \big|_{t=0} &= 0,
\end{align*}
$$

(4.1)

where $z$ is a given deterministic function in $C(\mathbb{R}_+; C^\infty(\mathcal{M}))$, $\Theta_N$ is a given deterministic positive space-time distribution which converges to $\Theta$ in $L^2([0,T]; H^{-1+\varepsilon}(\mathcal{M}))$ for any $T > 0$ and $0 < \varepsilon \ll 1$, $P_N$ is as in (1.13), and $\nu > 0$. In this case, recall from [48] that the “sign-definite structure” allows us to rewrite the equation (4.1) as

$$
\begin{align*}
\partial_t v_N - \frac{1}{4\pi} \Delta_g v_N + \frac{1}{2} \nu \beta P_N \left[ e^{\beta P_N z} N(\beta P_N v_N) \Theta_N \right] &= 0, \\
v_N \big|_{t=0} &= 0,
\end{align*}
$$

(4.2)

where $N$ is a smooth bounded and Lipschitz nonlinearity. Indeed, this follows by writing (4.1) in the Duhamel formulation

$$
v_N(t, x) = -\frac{1}{2} \nu \beta \int_0^t \int_{\mathcal{M}} P_g(t - t', x, y) P_N \left[ e^{\beta P_N z} e^{\beta P_N v_N} \Theta_N \right] (t', y) dV_g(y) dt',
$$

(4.3)

and by using that the heat kernel $P_g$ is positive, which also implies that $P_N$ (1.13) is positivity preserving, and that $\Theta_N$ is a positive distribution with $\nu > 0$, from which we infer on (4.3) that $\beta v_N \leq 0$. In particular if $N \in \mathcal{S}(\mathbb{R})$ satisfies $N(x) \equiv e^x$ for $x \leq 0$ we see that (4.3) is then equivalent to (4.4). We also consider the limit equation

$$
\begin{align*}
\partial_t v - \frac{1}{4\pi} \Delta_g v + \frac{1}{2} \nu \beta \left[ e^{\beta z} N(\beta v) \Theta \right] &= 0, \\
v_N \big|_{t=0} &= 0,
\end{align*}
$$

(4.4)

which is again equivalent to (1.35) due to the positivity of $\Theta$ and $\nu$. 

\section*{Acknowledgments}

This work was supported by the European Research Council grant 320788 and the National Science Foundation grant DMS-1101428.
Proposition 4.1. Let $T > 0$, $0 < \varepsilon \ll 1$, $z \in C([0, T]; C^2(\mathcal{M})$) and $\Theta \in L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M}))$ be a positive distribution. Suppose that a sequence $\{\Theta_N\}_{N \in \mathbb{N}}$ of smooth non-negative functions converges to $\Theta$ in $L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M}))$. Then, for $0 < \delta \ll \varepsilon$, the Cauchy problem (1.2) is well-posed in $X_T^s$ for all $N \in \mathbb{N}$, and the corresponding solution $v_N$ converges to a limit $v$ in $X_T^s$. Furthermore, the limit $v$ is the unique solution to (4.4) in the energy class $X_T^0$.

Here the spaces $X_T^s$ are defined for any $s \in \mathbb{R}$ by

$$X_T^s \triangleq C([0, T]; H^s(\mathcal{M})) \cap L^2([0, T]; H^{1+s}(\mathcal{M})).$$

Using Proposition 1.8, we see that the condition $\Theta \in L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M}))$ for some (small) $\varepsilon > 0$ gives the condition (1.27) in Theorem 1.4 and the convergence of $\Theta_N$ to $\Theta$ in $L^2(\mu_0 \otimes \mathbb{P}; L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M})))$ implies the convergence in measure of $v_N$ to $v$ in (1.30). Thus Theorem 1.4 (i) will be established once we prove Proposition 4.1.

Proof of Proposition 4.1. Let us define the nonlinear operator

$$\Phi_N = \Phi_{N, \Theta_N} \triangleq v_N \mapsto -\frac{1}{2} \nu \beta \int_0^t \exp \left( \frac{t-t'}{2} \right) \mathbb{P}_N \left[ e^{\beta P_N z} \mathcal{N}(\beta P_N v_N) \Theta_N \right](t') dt',$n$$

and similarly for $\Phi = \Phi_{z, \Theta}$. In the following we fix $T > 0$, $z \in C([0, T]; C^\infty(\mathcal{M}))$ and smooth positive functions $\Theta_N$ satisfying $\Theta_N \to \Theta$ in $L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M}))$ for some $0 < \varepsilon \ll 1$ and some positive distribution $\Theta \in L^2([0, T]; H^{-1+\varepsilon}(\mathcal{M}))$. We mainly follow the argument in [18, Section 5].

Step 1: global well-posedness of (4.2). For $0 < \tau \leq T$, we estimate for any $N \in \mathbb{N}$ and any $v_N \in C([0, \tau] \times \mathcal{M})$:

$$\|\Phi_N(v_N)\|_{C_{\tau, x}} \lesssim \left\| \int_0^t (t-t')^{-\frac{1}{2}} \| P_N \left[ e^{\beta P_N z} \mathcal{N}(\beta P_N v_N) \Theta_N \right](t') \|_{L^2} \right\|_{L^\infty_{\tau}} \lesssim \tau^{\frac{1}{2}} e^{C\|z\|_{L^\infty_{\tau, x}}} \| \Theta_N \|_{L^\infty_{\tau, x} L^2}$$

where we used Schauder’s estimate (2.12) and Young’s inequality with the boundedness of $\mathcal{N}$ and the uniform boundedness of $P_N : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ for any $p$. Note that for fixed $N \in \mathbb{N}$, $\Theta_N$ is smooth, so that the right-hand side of (4.6) is indeed finite.

We can also estimate similarly for $v_N, w_N \in C([0, \tau] \times \mathcal{M})$

$$\|\Phi_N(v_N) - \Phi_N(w_N)\|_{C_{\tau, x}} \lesssim \left\| \int_0^t (t-t')^{-\frac{1}{2}} \| P_N \left[ e^{\beta P_N z} \left[ \mathcal{N}(\beta P_N v_N) - \mathcal{N}(\beta P_N w_N) \right] \Theta_N \right](t') \|_{L^2} \right\|_{L^\infty_{\tau}} \lesssim \tau^{\frac{1}{2}} e^{C\|z\|_{L^\infty_{\tau, x}}} \| \mathcal{N}(\beta P_N v_N) - \mathcal{N}(\beta P_N w_N) \|_{L^\infty_{\tau, x}} \| \Theta_N \|_{L^\infty_{\tau, x} L^2} \lesssim \tau^{\frac{1}{2}} e^{C\|z\|_{L^\infty_{\tau, x}}} \| v_N - w_N \|_{L^\infty_{\tau, x}} \| \Theta_N \|_{L^\infty_{\tau, x} L^2}$$

(4.7)
where in the last step we used the mean value theorem and wrote

$$N(\beta P_Nv_N) - N(\beta P_Nw_N) = \beta P_N(v_N - w_N) \int_0^1 N'(\theta \beta P_Nv_N + (1 - \theta)\beta P_Nw_N) d\theta$$

(4.8)

with $N'$ bounded. From the estimates (4.6) and (4.7) we deduce that for $\tau_N = \tau_N(\|\cdot\|_{L^\infty_T}, \|\Theta_N\|_{L^\infty_T L^2}) > 0$, $\Phi_N$ is a contraction on a ball of $C([0, \tau_N] \times M)$ and thus admits a unique fixed point $v_N$ in this ball, which is the unique solution of (1.2) on $[0, \tau_N^*]$ where $0 < \tau_N^* \leq T$ is the maximal time of existence of $v_N$. Moreover, the estimate (4.6) shows that $\|v_N\|_{C_{r,s}}$ stays bounded as $\tau \to \tau_N^*$, so that we can iterate the fixed-point argument to obtain $\tau_N^* = T$.

**Step 2: convergence of $v_N$.** Let $0 < \delta \ll \varepsilon$. Then, proceeding as above and using the Schauder estimate (Lemma 2.6), Lemma 2.17 and Young’s inequality with the boundedness of $N$, we have

$$\|v_N\|_{C_T H^{2\delta}} \lesssim \left\| \int_0^t (t - t')^{-\frac{2+2\delta-\varepsilon}{2}} \|P_N(e^{\beta P_N N^2} N(\beta P_Nv_N) \Theta_N)(t')\|_{H^{-1+\varepsilon}} dt' \right\|_{L^\infty_T}$$

$$\lesssim \|e^{\beta P_N N^2} N(\beta P_Nv_N)\|_{L^\infty_T} \left\| \int_0^t (t - t')^{-\frac{2+2\delta-\varepsilon}{2}} \|\Theta_N(t')\|_{H^{-1+\varepsilon}} dt' \right\|_{L^\infty_T}$$

$$\lesssim e^{C\|z\|_{L^\infty_T}} \|\Theta_N\|_{L^2_T H^{-1+\varepsilon}},$$

uniformly in $N \in \mathbb{N}$. We can bound similarly

$$\|v_N\|_{L^2_T H^{1+2\delta}} \lesssim \left\| \int_0^t (t - t')^{-\frac{2+2\delta-\varepsilon}{2}} \|P_N(e^{\beta P_N N^2} N(\beta P_Nv_N) \Theta_N)(t')\|_{H^{-1+\varepsilon}} dt' \right\|_{L^2_T}$$

$$\lesssim \|e^{\beta P_N N^2} N(\beta P_Nv_N)\|_{L^\infty_T} \left\| \int_0^t (t - t')^{-\frac{2+2\delta-\varepsilon}{2}} \|\Theta_N(t')\|_{H^{-1+\varepsilon}} dt' \right\|_{L^2_T}$$

$$\lesssim e^{C\|z\|_{L^\infty_T}} \|\Theta_N\|_{L^2_T H^{-1+\varepsilon}},$$

and

$$\|\partial_t v_N\|_{L^2_T H^{-1+2\delta}} = \left\| \frac{1}{2} \Delta_g v_N - \frac{1}{2} \nu \beta P_N \left[ e^{\beta P_N N^2} N(\beta P_Nv_N) \Theta_N \right] \right\|_{L^2_T H^{-1+2\delta}}$$

$$\lesssim \|v_N\|_{L^2_T H^{1+2\delta}} + \|e^{\beta P_N N^2} N(\beta P_Nv_N) \Theta_N\|_{L^2_T H^{-1+\varepsilon}}$$

$$\lesssim e^{C\|z\|_{L^\infty_T}} \|\Theta_N\|_{L^2_T H^{-1+\varepsilon}},$$

uniformly in $N \in \mathbb{N}$.

For any $s \in \mathbb{R}$, we define $\overline{X}^s_T$ by

$$\overline{X}^s_T = \{ v \in X^s_T : \partial_t v \in L^2([0, T]; H^{-1+s}(M)) \}.$$

Then we deduce from (4.9), (4.10), and (4.11) along with the convergence of $\Theta_N$ to $\Theta$ in $L^2([0, T]; H^{-1+\varepsilon}(M))$, that $\{v_N\}_{N \in \mathbb{N}}$ is bounded in $\overline{X}^{2\delta}_T$. Moreover, it follows from Rellich’s lemma and the Aubin-Lions lemma (see e.g. [58, Corollary 4 on p. 85]) that the embedding
of $\overline{X}^{2\delta}_T \subset X^\delta_T$ is compact. Hence, there exists a subsequence $\{v_{N_k}\}_{k \in \mathbb{N}}$ converging to some limit $v$ in $X^\delta_T$.

Next, we show that the limit $v$ satisfies (4.4). Since $v_N$ satisfies (4.2), it is enough to prove the convergence of $\Phi_{N_k}(v_{N_k})$ to $\Phi(v)$ in $D'(\{0, T\} \times \mathcal{M})$. We thus estimate

$$\|\Phi_{N_k}(v_{N_k}) - \Phi(v)\|_{L^1_t B^{-1+\varepsilon}_x}$$

$$\lesssim \left\| \int_0^t e^{\frac{t-t'}{2}\Delta_x} [P_{N_k} - \text{Id}] e^{\beta P_{N_k} z} \mathcal{N}(\beta P_{N_k} v_{N_k}) \Theta_{N_k}(t') dt' \right\|_{L^1_t H^{-1+\varepsilon}}$$

$$+ \left\| \int_0^t e^{\frac{t-t'}{2}\Delta_x} \left[ (e^{\beta P_{N_k} z} - e^{\beta z}) \mathcal{N}(\beta P_{N_k} v_{N_k}) \Theta_{N_k} - \Theta \right](t') dt' \right\|_{L^1_t H^{-1+\varepsilon}}$$

$$+ \left\| \int_0^t e^{\frac{t-t'}{2}\Delta_x} \left[ e^{\beta z} \mathcal{N}(\beta P_{N_k} v_{N_k})(\Theta_{N_k} - \Theta) \right](t') dt' \right\|_{L^1_t B^{-1+\varepsilon}_x}$$

$$+ \left\| \int_0^t e^{\frac{t-t'}{2}\Delta_x} \left[ e^{\beta z} (P_{N_k} v_{N_k} - v) \mathcal{N}(P_{N_k} v_{N_k}, v) \Theta \right](t') dt' \right\|_{L^1_t H^{-1+\varepsilon}}$$

$$=: I + II + III + IV, \quad (4.12)$$

where we used (4.8) and wrote

$$\mathcal{N}(P_{N_k} v_{N_k}, v) \overset{\text{def}}{=} \int_0^1 \mathcal{N}(\theta \beta P_{N_k} v_{N_k} + (1 - \theta) \beta v) d\theta.$$

For the first term, note that we have for any $f \in C^\infty(\mathcal{M})$ and any $N \in \mathbb{N}$

$$\left\| [P_N - \text{Id}] f \right\|_{H^{-1}}^2 \lesssim \sum_{n \geq 0} e^{-N^{-2}\lambda_n^2} - 1 \| f, \varphi_n \|_g^2 \lesssim N^{-2\varepsilon} \sum_{n \geq 0} \langle \lambda_n \rangle^{-2+2\varepsilon} \langle f, \varphi_n \rangle_g^2$$

$$\lesssim N^{-2\varepsilon} \left\| f \right\|_{H^{-1+\varepsilon}}^2.$$

Therefore the previous remark with Schauder estimate (2.12), Young’s inequality and Lemma 2.17 yield

$$I \lesssim N^{-\varepsilon} \left\| \int_0^t (t - t')^{-\frac{5}{2}} \left\| e^{\beta P_{N_k} z} \mathcal{N}(\beta P_{N_k} v_{N_k}) \Theta_{N_k}(t') \right\|_{H^{-1+\varepsilon}} dt' \right\|_{L^1_t}$$

$$\lesssim N^{-\varepsilon} e^{C \|z\|_{L^\infty_x, \varepsilon}} \left\| \Theta_{N_k} \right\|_{L^2_x H^{-1+\varepsilon}}, \quad (4.13)$$

Similarly, we bound

$$II \lesssim \left\| e^{\beta P_{N_k} z} - e^{\beta z} \mathcal{N}(\beta P_{N_k} v_{N_k}) \Theta_{N_k} \right\|_{L^1_t H^{-1+\varepsilon}}$$

$$\lesssim \left\| e^{\beta P_{N_k} z} - e^{\beta z} \right\|_{L^\infty_x, \varepsilon} \left\| \Theta_{N_k} \right\|_{L^2_x H^{-1+\varepsilon}}$$

$$\lesssim N^{-\varepsilon} e^{C \|z\|_{L^\infty_x, \varepsilon}} \left\| \Theta_{N_k} \right\|_{L^2_x H^{-1+\varepsilon}}. \quad (4.14)$$
where we used the mean value theorem, Sobolev inequality and the same argument as above to bound
\[
\|e^{\beta z} - e^{\beta z}\|_{L_{T,x}^\infty} \lesssim \| [\mathbf{P}_{N_k} - \text{Id} ] z \|_{L_{T,x}^\infty} C\|z\|_{L_{T,x}^\infty}.
\]

\[
\lesssim \| [\mathbf{P}_{N_k} - \text{Id} ] z \|_{L_{T,x}^{H+1+\epsilon}} C\|z\|_{L_{T,x}^\infty}.
\]

\[
\lesssim N_k^{-\epsilon} \|z\|_{L_{T,x}^{H+1+2\epsilon}} C\|z\|_{L_{T,x}^\infty}.
\]

As for \(\text{III}\), we use again the Schauder estimate \((2.12)\) with Young’s inequality, but due to the lack of positivity of \(\Theta_{N_k} - \Theta\) we use the product estimate of Lemma 2.16 (iii) in place of Lemma 2.17 to get
\[
\text{III} \lesssim \| e^{\beta \mathcal{N}(\mathbf{p}_{N_k} u_{N_k})(\Theta_{N_k} - \Theta)} \|_{L_{1,1,1}^1}.
\]

\[
\lesssim \| e^{\beta \mathcal{N}(\mathbf{p}_{N_k} u_{N_k})} \|_{L_{1}^{H+\frac{1}{2}}} \| \Theta_{N_k} - \Theta \|_{L_{1}^{H-1+\epsilon}}
\]

\[
\lesssim \| e^{\beta \mathcal{N}(\mathbf{p}_{N_k} u_{N_k})} \|_{L_{1}^{H+\frac{1}{2}}} \| \Theta_{N_k} - \Theta \|_{L_{1}^{H-1+\epsilon}}.
\]

Using the fractional chain rule of Lemma 2.18 (i) for \(\hat{A}(u) = \mathcal{N}(u) - 1\), we bound
\[
\| \mathcal{N}(\mathbf{p}_{N_k} u_{N_k}) \|_{L_{1}^{H+\epsilon}} \lesssim T_{\frac{1}{2}}^2 + \| A(\mathbf{p}_{N_k} u_{N_k}) \|_{L_{1}^{H-1+\epsilon}} \lesssim T_{\frac{1}{2}}^2 + \| u_{N_k} \|_{L_{1,1,1}^{H-1+\epsilon}}
\]

\[
\lesssim C(T) (1 + \| u_{N_k} \|_{X_T^\frac{1}{2}}).
\]

We also use the fractional chain rule of Lemma 2.18 (ii) to estimate for some large but finite \(p_\varepsilon\)
\[
\| e^{\beta z} \|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} \lesssim \| e^{\beta z} \|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} \lesssim 1 + \| e^{2\beta z} \|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}}.
\]

Combining \((4.15), (4.16), \) and \((4.17)\) we thus obtain
\[
\text{III} \lesssim e^{\|z\|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} (1 + \|z\|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} (1 + \| u_{N_k} \|_{X_T^\frac{1}{2}})) \| \Theta_{N_k} - \Theta \|_{L_{1}^{H-1+\epsilon}}.
\]

For the last term \(\text{IV}\) in \((4.12)\), we use the Schauder estimate of Lemma 2.6 and the product estimate\footnote{Note that \(v(t)\) is indeed continuous in \(x\) for almost every \(t \in [0, T]\) by the Sobolev embedding \(X_T^\frac{1}{2} \subset L^2([0, T]; C(\mathcal{M}))\).} of Lemma 2.17 with Hölder’s inequality and the boundedness of \(\mathcal{N}\) to bound
\[
\text{IV} \lesssim \| e^{\beta z} (\mathbf{p}_{N_k} u_{N_k} - v) \mathcal{N}(\mathbf{p}_{N_k} u_{N_k}, v) \|_{L_{1}^{H+1+\epsilon}}
\]

\[
\lesssim \| e^{\beta z} (\mathbf{p}_{N_k} u_{N_k} - v) \mathcal{N}(\mathbf{p}_{N_k} u_{N_k}, v) \|_{L_{1}^{H+1+\epsilon}} \| \Theta \|_{L_{1}^{H-1+\epsilon}}
\]

\[
\lesssim e^{\|z\|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} (\| u_{N_k} - v \|_{L_{1}^{H+1+\epsilon}} + \| [\mathbf{p}_{N_k} - \text{Id}] u_{N_k} \|_{L_{1}^{H+1+\epsilon}})} \| \mathcal{N}(u_{N_k}, v) \|_{L_{T,x}^\infty} \| \Theta \|_{L_{1}^{H-1+\epsilon}}
\]

\[
\lesssim e^{\|z\|_{L_{T}^{\infty} L_{1,1,1}^{1-\frac{2}{3}}} (\| u_{N_k} - v \|_{X_T^\frac{1}{2}} + N_k^{-\frac{1}{2}} \| u_{N_k} \|_{X_T^\frac{1}{2}})} \| \Theta \|_{L_{1}^{H-1+\epsilon}}.
\]
$L^1([0,T]; H^{-1+\varepsilon}(\mathcal{M}))$, and a fortiori in $\mathcal{D}'([0,T] \times \mathcal{M})$. Since $v_{N_k} = \Phi_N(v_{N_k})$, we can conclude that

$$v = \lim_{k \to \infty} v_{N_k} = \lim_{k \to \infty} \Phi_N(v_{N_k}) = \Phi(v),$$

as equalities between functions in $X^\delta_T$. This proves existence of a solution $v$ to (4.3) in $X^\delta_T \subset X_T^0$.

For the uniqueness in $X_T^0$, we proceed via an energy estimate: if $v_1, v_2 \in X_T^0$ are two solutions to (4.3) which are limits in $X_T^0$ of solutions $v_{N,1}, v_{N,2}$ to (4.2), noting $w = v_1 - v_2$ we see that $w$ satisfies

$$\partial_t w - \frac{1}{4\pi} \Delta_g w + \frac{1}{2} \nu \beta e^{\beta z} (\mathcal{N}(\beta v_1) - \mathcal{N}(\beta v_2)) \Theta = 0.$$

We consider the energy of $w$:

$$\mathcal{E}(t) \overset{\text{def}}{=} \frac{1}{2} \| w(t) \|^2_{L^2} + \frac{1}{4\pi} \int_0^t \|\nabla_g w(t')\|^2_{L^2} dt' \geq 0,$$

and similarly for the energy $\mathcal{E}_N(t)$ of $w_N = v_{N,1} - v_{N,2}$. In particular, since $v_{N,j} \to v_j$ in $X_T^0$, we have that $\mathcal{E}_N \to \mathcal{E}$ in $C([0,T])$ for any $T > 0$.

Since $v_{N,j} = 1, 2$, solve (4.2) and that $\Theta_N$ and $z$ are smooth, we see that we have for fixed $N \in \mathbb{N}$ that $v_{N,j} \in C^1([0,T], C^\infty(M))$. Thus the energy functional $\mathcal{E}_N(t)$ is a well-defined differentiable function. Moreover, since $\beta v_{N,j} \leq 0$, $j = 1, 2$, we have again $\mathcal{N}(\beta v_{N,j}) = e^{\beta v_{N,j}}$, so that

$$\bar{\mathcal{N}}(v_{N,1}, v_{N,2}) = \int_0^1 \exp (\theta \beta v_{N,1} + (1 - \theta) \beta v_{N,2}) d\theta \geq 0.$$

This implies that

$$\frac{d}{dt} \mathcal{E}_N(t) = \int_M w_N(t) (\partial_t w_N(t) - \frac{1}{4\pi} \Delta_g w_N(t)) dV_g$$

$$= \frac{1}{2} \nu \beta^2 \int_M w_N(t)^2 e^{\beta z} \bar{\mathcal{N}}(v_{N,1}, v_{N,2}) \Theta_N(t) dV_g$$

$$\leq 0.$$

Since $w_N(0) = 0$, we conclude that $\mathcal{E}_N(t) = 0$ for any $t \geq 0$. From the uniform convergence $\mathcal{E}_N \to \mathcal{E}$, we conclude that $\mathcal{E}(t) = 0$ for any $t \geq 0$, and $v_1 \equiv v_2$. This finally proves uniqueness in the energy space $X_T^0$, and also convergence of the whole sequence $\{v_N\}_{N \in \mathbb{N}}$ in $X_T^\delta \subset X_T^0$. \qed

In particular, Proposition 4.1 implies that the solution

$$u_N = t_g + z + v_N$$

to (1.25) converges in measure to $u = t_g + z + v$ in $X_T^\delta$, which proves Theorem 1.3 (i). Note that $u$ is then unique in the class

$$t_g + z + X_T^\delta \subset C([0,T]; H^\delta_0(\mathcal{M}) \oplus \mathbb{R}).$$
4.2. Invariance of the LQG measure. In this subsection, we finally conclude the proof of Theorem 1.4 (ii). We begin by proving the invariance of the truncated Gibbs measure \( \rho_{N,g} \) (1.11) under the flow \( \tilde{u}_N \) of (1.22) given by (1.24), where \( u_N \) denotes the solution of (1.25) constructed in the previous subsection.

**Lemma 4.2.** For any \( N \in \mathbb{N} \), any \( t \geq 0 \) and any \( F \in C_b(H^s_0(M) \oplus \mathbb{R}) \), it holds

\[
\int_{H^s_0(M,g)} \int_{\mathbb{R}} \mathbb{E} \left[ F(\tilde{u}_N(t,X_g,\tilde{X},\omega)) \right] d\rho_{N,g}(X_g,\tilde{X}) = \int_{H^s_0(M,g)} \int_{\mathbb{R}} F(X_g + \tilde{X}) d\rho_{N,g}(X_g,\tilde{X}).
\]

**Proof.** Fix \( N \in \mathbb{N} \). For any \( M \in \mathbb{N} \), let \( \Pi_M \) be the projection on the space \( \text{Vect}\{\varphi_n, n = 0, \ldots, d_M\} \simeq \mathbb{R}^{d_M} \), with \( d_M = \#\{n \geq 0, \lambda_n \leq M\} \). Similarly as in (1.22), we then look at the dynamics

\[
\partial_t \tilde{u}_{N,M} = \frac{1}{4\pi} \Delta_{\bar{g}} \tilde{u}_{N,M} - \frac{Q}{8\pi} \mathcal{R}_g + \frac{1}{2} \nu \beta e^{-\pi \beta^2 C_P} N^{-\frac{d^2}{2}} \Pi_M \mathcal{P}_N e^{\beta \Pi_M \mathcal{P}_N \tilde{u}_{N,M}}
+ \frac{1}{2} \sum_{\ell=1}^L a_\ell \Pi_M \mathcal{P}_N \delta_{x_\ell} + \xi_g,
\]

with initial data \( \tilde{u}_0 \) randomly distributed by \( \rho_{N,M,g} \), where

\[
d\rho_{N,M,g}(X_g,\tilde{X}) = d\rho_{N,M,g}^F \otimes [\text{Id} - \Pi_M]_* \mu_g.
\]

Here the finite dimensional measure is given by

\[
d\rho_{N,M,g}^F \overset{\text{def}}{=} Z^{-1}_{N,M} e^{-E_{N,M}(U_0,\ldots,U_{d_M-1})} \prod_{n=0}^{d_M-1} dU_n
\]

for the truncated energy

\[
E_{N,M}(U_0,\ldots,U_{d_M-1}) = \sum_{n=0}^{d_M-1} \left\{ \frac{1}{4\pi} \lambda_n^2 e^{-2N^{-2}\lambda_n^2 U_n^2} + \frac{Q}{4\pi} U_n(\mathcal{R}_g,\varphi_n)g - \sum_{\ell=1}^L a_\ell U_0 \right.
+ \nu e^{-2N^{-2}\lambda_n^2} \int_{\mathcal{M}} e^{-\pi \beta^2 C_P} N^{-\frac{\beta^2}{2}} e^{\beta \sum_{m=0}^{d_M-1} e^{-2N^{-2}\lambda_m^2 U_m \varphi_m}} dV_g
- \left. \sum_{\ell=1}^L a_\ell e^{-2N^{-2}\lambda_n^2} U_n \varphi_n(x_\ell) - \frac{a^2_\ell}{2} (\log N + 2\pi C_P) \right\}.
\]

For each fixed \( N \in \mathbb{N} \), the dynamics (4.20) is defined in \( C([0,T];H^s_0(M) \oplus \mathbb{R}) \) for any \( T > 0 \) by a standard argument since the nonlinearity is Lipschitz and the deterministic source terms are smooth. Moreover it holds \( \tilde{u}_{N,M}(t) \to \tilde{u}_N(t) \) as \( M \to \infty \) in law in \( H^s_0(M) \oplus \mathbb{R} \) for any \( t \geq 0 \).
We thus have for any $t \geq 0$

$$
\int_{H_0^2(M,g)} \int_{\mathbb{R}} \mathbb{E} \left[ F(\tilde{u}_N(t, X_g, \overline{X}, \omega)) \right] d\rho_{N,g}(X_g, \overline{X})
$$

$$
= \lim_{M \to \infty} \int_{H_0^2(M,g)} \int_{\mathbb{R}} \mathbb{E} \left[ F(\Pi_M \tilde{u}_N(t, X_g, \overline{X}, \omega)) + (1 - \Pi_M)\tilde{u}_N(t, X_g, \overline{X}, \omega) \right] d\rho_{N,M,g}(X_g, \overline{X})
$$

$$
\times d\rho_{N,M,g}(X_g, \overline{X}) \otimes [\text{id} - \Pi_M]_* \mu_g(X_g).
$$

We see that $\tilde{u}_{N,M}^\perp = (\text{id} - \Pi_M)\tilde{u}_{N,M}$ solves the linear stochastic equation

$$
\partial_t \tilde{u}_{N,M}^\perp - \frac{1}{4\pi} \Delta_{\overline{g}} \tilde{u}_{N,M}^\perp = (\text{id} - \Pi_M)\xi_g
$$

with initial data distributed by $[\text{id} - \Pi_M]_* \mu_g$, so that $[\text{id} - \Pi_M]_* \mu_g$ is the unique invariant measure for $\tilde{u}_{N,M}^\perp$ (see e.g. [14, Section 11.3]).

On the other hand, we can write $\Pi_M \tilde{u}_{N,M}(t) = \sum_{n=0}^{d_M-1} U_n(t) \varphi_n$, so that $U_n$ solves the system of SDEs

$$
dU_n = -\frac{1}{2} \frac{\partial}{\partial U_n} E_{N,M}(U_0, ..., U_{d_M-1}) dt + dB_n, \quad n = 0, ..., d_M - 1, \quad (4.21)
$$

with $B_n$ as in (1.28).

The infinitesimal generator for (4.21) is given by

$$
\mathcal{L}_{N,M} f(U_0, ..., U_{d_M-1}) = \sum_{n=0}^{d_M-1} -\frac{1}{2} \frac{\partial}{\partial U_n} E_{N,M}(U_0, ..., U_{d_M-1}) \frac{\partial}{\partial U_n} f(U_0, ..., U_{d_M-1})
$$

$$
+ \frac{1}{2} \frac{\partial^2}{\partial (U_n)^2} f(U_0, ..., U_{d_M-1})
$$

for any test function $f \in C_0^2(\mathbb{R}^{d_M})$. In particular, we have by integrating by parts

$$
\int_{\mathbb{R}^{d_M}} \mathcal{L}_{N,M} f d\rho_{N,M,g} = \int_{\mathbb{R}^{d_M}} \left[ \sum_{n=0}^{d_M-1} -\frac{1}{2} \frac{\partial}{\partial U_n} E_{N,M} \frac{\partial}{\partial U_n} f + \frac{1}{2} \frac{\partial^2}{\partial (U_n)^2} f \right] e^{-E_{N,M}} dU
$$

$$
= \sum_{n=0}^{d_M-1} -\frac{1}{2} \int_{\mathbb{R}^{d_M}} \frac{\partial}{\partial U_n} E_{N,M} \frac{\partial}{\partial U_n} f e^{-E_{N,M}} dU
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^{d_M}} \frac{\partial}{\partial U_n} E_{N,M} \frac{\partial}{\partial U_n} f e^{-E_{N,M}} dU
$$

$$
= 0.
$$

This shows that $\Pi_M \tilde{u}_{N,M}$ also leaves $\rho_{N,M,g}^F$ invariant.
All in all, we deduce that

\[
\int_{H_0^s(M, g)} \int_\mathbb{R} \mathbb{E} \left[ F \left( \tilde{u}_N(t, x, \bar{X}, \omega) \right) \right] d\rho_{N,g}(X_g, \bar{X})
= \lim_{M \to \infty} \int_{H_0^s(M, g)} \int_\mathbb{R} \mathbb{E} \left[ F \left( \Pi_M \tilde{u}_{N,M}(t, x, \bar{X}, \omega) + (1 - \Pi_M) \tilde{u}_{N,M}(t, x, \bar{X}, \omega) \right) \right] 
\times d\rho_{N,M,g}(X_g, \bar{X}) \otimes [\text{Id} - \Pi_M] \mu_g(X_g)
= \lim_{M \to \infty} \int_{H_0^s(M, g)} \int_\mathbb{R} F \left( \Pi_M (X_g + \bar{X}) + (1 - \Pi_M) X_g \right) 
\times d\rho_{N,M,g}(X_g, \bar{X}) \otimes [\text{Id} - \Pi_M] \mu_g(X_g)
= \int_{H_0^s(M, g)} \int_\mathbb{R} F(X_g + \bar{X}) d\rho_{N,g}(X_g, \bar{X}).
\]

\( \square \)

To conclude the proof of Theorem 1.4 (ii), we observe that by convergence in measure of \( v_N \) in \( X_T^0 \) given by Theorem 1.4 (i) and definition of \( \tilde{u}_N \) and \( u_N \), we have the convergence in law \( \tilde{u}_N(t) \to \tilde{u}(t) \) in \( H_0^s(M) \oplus \mathbb{R} \) for any \( t \geq 0 \). With the weak convergence of \( \rho_{N,g} \) to \( \rho_{\{a_t, x_t\}, g} \) given by Theorem 1.1 along with the invariance of \( \rho_{N,g} \) given by Lemma 4.2, it thus holds

\[
\int_{H_0^s(M, g)} \int_\mathbb{R} \mathbb{E} \left[ F \left( \tilde{u}(t, x, \bar{X}, \omega) \right) \right] d\rho_{\{a_t, x_t\}, g}(X_g, \bar{X})
= \lim_{N \to \infty} \int_{H_0^s(M, g)} \int_\mathbb{R} \mathbb{E} \left[ F \left( \tilde{u}_N(t, x, \bar{X}, \omega) \right) \right] d\rho_{N,g}(X_g, \bar{X})
= \lim_{N \to \infty} \int_{H_0^s(M, g)} \int_\mathbb{R} F(X_g + \bar{X}) d\rho_{N,g}(X_g, \bar{X})
= \int_{H_0^s(M, g)} \int_\mathbb{R} F(X_g + \bar{X}) d\rho_{\{a_t, x_t\}, g}(X_g, \bar{X}).
\]

This concludes the proof of Theorem 1.4.

**Remark 4.3.** Strictly speaking, the solution given by Theorem 1.4 is not a strong solution (in the probability sense) of the original SPDE (1.22) since we did a change of probability space by the Girsanov transform (1.24). It would be very interesting though to be able to deal directly with the singular equation (1.22).

**Remark 4.4.** It would be interesting to also establish the analogue of Theorem 1.4 for the canonical stochastic quantization of the LQG measure, namely prove global well-posedness of the stochastic damped wave equation

\[
(\partial_t^2 - \Delta_g + \partial_t) \tilde{u}_N + \frac{Q}{8\pi} R_g + \frac{1}{2} \nu \beta e^{-\pi \beta^2 C_P N^{-\beta^2}} P_N \left\{ e^{\beta P_N \tilde{u}_N} \right\}
= \frac{1}{2} \sum_{\ell=1}^L a_\ell P_N \delta_{x_\ell} + \sqrt{2} \xi_g,
\]

and invariance of the LQG measure (coupled with the white noise measure on \( \partial_t \tilde{u} \)) under this flow. Although the local well-posedness should follow from a straightforward adaptation...
of [48] Theorem 1.6] combined with the arguments of the present paper, it is not clear to us how to apply Bourgain’s invariant measure argument to globalize this dynamics as in [48, Section 6]. Indeed, the argument in [9, 10] seems to require the densities $R_N$ to be in $L^p(d\mu_g \otimes dX)$ uniformly in $N \in \mathbb{N}$ for some finite $p > 1$, which does not hold since the use of Girsanov transform for $(R_N)^p$, $p > 1$, produces a divergent constant $e^{(p-1)\frac{\sigma_N^2}{2} \sum_{l=1}^L a_l}$.

Appendix A. Remarks on the LQG and $\exp(\phi)_2$ measures with negative cosmological constant

Theorem 1.1 discusses the construction of the $L^2$ points correlations in the case where the cosmological constant $\nu$ is positive (the free case $\nu = 0$ just corresponding to $d\rho_{(a_l,x_l),g} = d\mu_g \otimes dX$). As is clear from (3.21), in the case $\nu < 0$, the change of variable $X \mapsto \tau = -\nu e^{\beta X} \int_{\mathcal{M}} e^{\beta \sum_{l=1}^L a_l P_N G_0(x_l,x_l) + \beta P_N X_0(x) - \frac{\beta^2}{2} \sigma_N(x;g_0)} dV_0(x)$

now gives the expression for the truncated partition function

$Z_N = \mathcal{G}_{N,0} \mathbb{E}^{\nu} \left[ 2\pi Q(X(M)) \right] \int_0^\infty \tau^{\beta-1} \left[ 2\pi Q(X(M)) \right]^{-\frac{1}{2}} e^{\tau d\tau}

\times \int \mathcal{Y}_N(M)^{\beta-1} \left[ 2\pi Q(X(M)) \right]^{-\frac{1}{2}} d\mu_0(X_0)

= +\infty,$

with or without the Seiberg bounds. This proves that there are no well-defined correlation functions in this case.

In [48], we looked at the closely related $\exp(\phi)_2$ (or Høegh-Krohn [32]) measure on the flat torus $M = \mathbb{T}^2$ given by

$d\tilde{\rho} = \lim_{N \to \infty} e^{-\nu \int_M e^{\beta P_N \tilde{X}} - \frac{\beta^2}{2} \sigma_N \tilde{X} dV} d\tilde{\mu}(\tilde{X}),$

where now $\tilde{\mu}$ is the massive Gaussian free field, i.e. under $\tilde{\mu}$ we have

$\tilde{X}(x) = \sum_{n \geq 0} \frac{h_n(\omega)}{\sqrt{1 + \lambda_n^2}} \varphi_n(x)$ \hspace{1cm} (A.1)

for $h_n \sim \mathcal{N}(0, 1)$ on $(\Omega, \mathbb{P})$, and $\sigma_N(x) = \mathbb{E} \left[ |P_N X_0(x)|^2 \right]$. In the defocusing case $\nu > 0$ on $M = \mathbb{T}^2$ with the flat metric \[12\] we proved that the measure $\tilde{\rho}$ above is indeed well-defined as a by-product of the construction of the GMC: $e^{\beta \tilde{X}}$; and the integrability of the whole density which follows in this case from the trivial bound $e^{-\nu \int_M e^{\beta \tilde{X}} dV} \leq 1$ when $\nu > 0$. Namely, since the zero-th Fourier mode of $\tilde{X}$ is now normally distributed (compared to (1.8)), the partition function (i.e. with $L = 0$) is always finite. As for the focusing case $\nu < 0$, we prove the following result.

\[12\]Note that in the massive case there is no conformal invariance anymore. However our proof of the construction of $\tilde{\rho}$ (without curvature term compared to (1.1)-(1.9)) readily extends to any compact surface $(\mathcal{M}, g)$ through the same arguments as in this work.
Proposition A.1. Let \( \nu < 0 \). Then for any \( \beta \neq 0 \), the Høegh-Krohn partition function is not finite. More precisely, for any smooth approximations \( \tilde{X}_N \) of \( \tilde{X} \) such that \( \tilde{X}_N \) is still centred and Gaussian with variance \( \sigma_N + O(1) \), it holds
\[
\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \exp \left( - \nu \int_{\mathcal{M}} e^{\beta \tilde{X}_N(x) - \frac{\beta^2}{2} \sigma_N} dV_{g}(x) \right) \right] = +\infty.
\]

Note that here this holds for a large class of approximations \( \tilde{X}_N \), including the one considered in Section 2.

Proof. We begin by recalling that the Cameron-Martin space for \( \tilde{\mu} \) is given by \( H^1(\mathcal{M}, g) \); in particular, Cameron-Martin’s theorem (see e.g. [14, Proposition 2.26]) states that for \( \tilde{X} \) as in (A.1) and any \( f \in H^1(\mathcal{M}, g) \), \( \tilde{X} - f \) is also a massive GFF under \( e^{(f, \tilde{X})_{H^1} - \frac{1}{2} \| f \|_{H^1}^2} d\mathbb{P} \). In particular, for any \( F \in C_0(H^s(\mathcal{M})) \), it holds
\[
\mathbb{E} \left[ F(\tilde{X}) \right] = \mathbb{E}_f \left[ F(\tilde{X}) e^{\frac{1}{2} \| f \|_{H^1}^2} e^{(f, \tilde{X})_{H^1}} \right] = \mathbb{E} \left[ F(\tilde{X} + f) e^{\frac{1}{2} \| f \|_{H^1}^2} e^{(f, \tilde{X} + f)_{H^1}} \right], \tag{A.2}
\]
where \( \mathbb{E}_f \) is the expectation associated with \( \mathbb{P}_f \triangleq e^{(f, \tilde{X})_{H^1} - \frac{1}{2} \| f \|_{H^1}^2} \mathbb{P} \).

We also recall the following Moser-Trudinger’s inequality [4, Theorems 2.46 and 2.50]: for \( \theta > 0 \), there exists \( C > 0 \) such that for all \( f \in H^1(\mathcal{M}) \),
\[
\int_{\mathcal{M}} e^\theta dV_g \leq C e^{\theta \| f \|_{H^1}^2}
\]
if and only if \( \theta \geq \frac{1}{10\pi} \). Thus for any \( \beta \neq 0 \) and \( N \in \mathbb{N} \), there exists \( f_N \in H^1(\mathcal{M}) \) such that
\[
\int_{\mathcal{M}} e^{\beta f_N} dV_g \geq N e^{\frac{\beta^2}{10\pi} \| f_N \|_{H^1}^2}. \tag{A.3}
\]

Thus, using (A.2), we have for any \( N \in \mathbb{N} \)
\[
\mathbb{E} \left[ \exp \left( - \nu \int_{\mathcal{M}} e^{\beta \tilde{X}_N - \frac{\beta^2}{2} \sigma_N} dV_{g} \right) \right] = \mathbb{E} \left[ \exp \left( - \nu \int_{\mathcal{M}} e^{\beta \tilde{X}_N + \beta f_N - \frac{\beta^2}{2} \sigma_N} dV_{g} \right) \right]
\]
Using Jensen’s inequality, that \( \tilde{X}_N \) is centered and that \( \mathbb{E}[e^{\beta \tilde{X}_N - \frac{\beta^2}{2} \sigma_N}] \sim 1 \), along with (A.3) and that \( \nu < 0 \), we then continue with
\[
\geq \exp \left[ - \nu \int_{\mathcal{M}} e^{1} dV_{g} \right] \geq \exp \left[ - \nu C N e^{\frac{\beta^2}{10\pi} \| f_N \|_{H^1}^2} \right]
\]
for some \( C > 0 \).

Note that, since \( \nu < 0 \), the function \( x \in [0, \infty) \mapsto (\nu)C N e^{\frac{\beta^2}{10\pi} x} - \frac{1}{2} x \) is strictly increasing on \( [0, \infty) \) for any \( N \) large enough (depending on \( \beta \)), so that we have for \( N \) large enough
\[
\mathbb{E} \left[ \exp \left( - \nu \int_{\mathcal{M}} e^{\beta \tilde{X}_N - \frac{\beta^2}{2} \sigma_N} dV_{g} \right) \right] \geq \exp \left[ - \nu C N \right] \to \infty
\]
as \( N \to \infty \). This proves Proposition A.1. \( \square \)
Acknowledgements. The authors are very grateful to Rémi Rhodes and Vincent Vargas for pointing out the relevance of studying the conformal equation (1.20) and for interesting discussions on LCFT which motivated the writing of this paper. They are also very grateful to Christophe Garban, Rémi Rhodes, Vincent Vargas and Younes Zine for helpful comments on a previous version of this work.

T.O. was supported by the European Research Council (grant no. 637995 “ProbDynDisEq” and grant no. 864138 “SingStochDispDyn”). T.R. was supported by the DFG through the CRC 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.”

References

[1] S. Albeverio, F.C. De Vecchi, M. Gubinelli, Elliptic stochastic quantization, arXiv:1812.04422 [math.PR].
[2] S. Albeverio, F.C. De Vecchi, M. Gubinelli, The elliptic stochastic quantization of some two dimensional Euclidean QFTs, arXiv:1906.11187 [math.PR].
[3] J. Aru, E. Powell, A. Sepulveda, Critical Liouville measure as a limit of subcritical measures, Electron. Commun. Probab. 24 (2019), Paper No. 18, 16 pp.
[4] T. Aubin, Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp.
[5] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften, 343. Springer, Heidelberg, 2011. xvi+523 pp.
[6] R. Bass, Stochastic processes. Cambridge Series in Statistical and Probabilistic Mathematics, 33. Cambridge University Press, Cambridge, 2011. xvi+390 pp.
[7] J. Bennett, A. Carbery, M. Christ, T. Tao, The BrascampLieb Inequalities: Finiteness, Structure and Extremals, Geom. Funct. Anal. 17 (2008), no 5, 1343–1415.
[8] M. S. Berger, Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds, J. Differential Geometry 5 (1971), 325–332.
[9] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, Comm. Math. Phys. 166 (1994), 1–26.
[10] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, Comm. Math. Phys. 176 (1996), 421–445.
[11] N. Burq, P. Gérard, N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, American Journal of Mathematics 126 (2004), no. 3, 569–605.
[12] N. Burq, L. Thomann, N. Tzvetkov, Remarks on the Gibbs measures for nonlinear dispersive equations, Ann. Fac. Sci. Toulouse Math. 27 (2018), no. 3, 527–597.
[13] G. Da Prato, A. Debussche, Strong solutions to the stochastic quantization equations, Ann. Probab. 31 (2003), no. 4, 1900–1916.
[14] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions. Second edition. Encyclopedia of Mathematics and its Applications, 152. Cambridge University Press, Cambridge, 2014. xviii+493 pp.
[15] F. David, A. Kupiainen, R. Rhodes, V. Vargas, Liouville quantum gravity on the Riemann sphere, Comm. Math. Phys. 342 (2016), no. 3, 869–907.
[16] F. David, A. Kupiainen, R. Rhodes, V. Vargas, Renormalizability of Liouville quantum field theory at the Seiberg bound, Electron. J. Probab. 22 (2017), Paper No. 93, 26 pp.
[17] F. David, R. Rhodes, V. Vargas, Liouville quantum gravity on complex tori, J. Math. Phys. 57 (2016), no. 2, 022302, 25 pp.
[18] J. Dubédat, H. Shen, Stochastic Ricci flow on compact surfaces, arXiv:1904.10099 [math.PR].
[19] B. Duplantier, R. Rhodes, S. Sheffield, V. Vargas, Renormalization of critical Gaussian multiplicative chaos and KPZ relation, Comm. Math. Phys. 330 (2014), no. 1, 283–330.
[20] B. Duplantier, R. Rhodes, S. Sheffield, V. Vargas, Critical Gaussian multiplicative chaos: convergence of the derivative martingale, Ann. Probab. 42 (2014), no. 5, 1769–1808.
[21] B. Duplantier, S. Sheffield, Liouville quantum gravity and KPZ, Invent. Math. 185 (2011), no. 2, 333–393.
[22] G. Folland, Real analysis. Modern techniques and their applications. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. xvi+386 pp.

[23] C. Garban, Dynamical Liouville. J. Funct. Anal. 278 (2020), no. 6, 108351.

[24] A. E. Gatto, Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition, J. Funct. Anal. 188 (2002), no. 1, 27–37.

[25] J. Glimm, A. Jaffe, Quantum physics. A functional integral point of view, Second edition. Springer-Verlag, New York, 1987. xxii+535 pp.

[26] A. Grigor’yan, Heat kernel and analysis on manifolds. AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. xviii+482 pp.

[27] M. Gubinelli, M. Hofmanova, A PDE construction of the Euclidean $\Phi^4_3$ quantum field theory, arXiv:1810.01700 [math-ph].

[28] M. Gubinelli, P. Imkeller, N. Perkowski, Paracontrolled distributions and singular PDEs, Forum Math. Pi 3 (2015), e6, 75 pp.

[29] M. Gubinelli, H. Koch, T. Oh, L. Tolomeo, Global dynamics for the two-dimensional stochastic nonlinear wave equations, preprint.

[30] C. Guillarmou, R. Rhodes, V. Vargas, Polyakov’s formulation of 2d bosonic string theory, Publ. Math. Inst. Hautes Études Sci. 130 (2019), 111–185.

[31] M. Hairer, A theory of regularity structures, Invent. Math. 198 (2014), no. 2, 269–504.

[32] R. Hoech-Krohn, A general class of quantum fields without cut-offs in two space-time dimensions, Comm. Math. Phys. 21 (1971), 244–255.

[33] L. Hörmander, The analysis of linear partial differential operators III : Pseudo-differential operators, Classics in Mathematics (2007), Springer Berlin.

[34] M. Hoshino, H. Kawabi, S. Kusuoka Stochastic quantization associated with the $\exp(\Phi^2_2)$-quantum field model driven by space-time white noise on the torus, arXiv:1907.07921 [math.PR].

[35] Y. Huang, R. Rhodes, V. Vargas, Liouville quantum gravity on the unit disk, Ann. Inst. Henri Poincaré Probab. Stat. 54 (2018), no. 3, 875–916.

[36] Y. Nakayama, Liouville field theory: a decade after the revolution, Internat. J. Modern Phys. A 19 (2004), no. 17-18, 2771–2930.

[37] D. Maulik, A. Okounkov, Quantum groups and quantum cohomology, Astérisque No. 408 (2019), ix+209 pp.

[38] T. Oh, T. Robert, N. Tzvetkov, Stochastic nonlinear wave dynamics on compact surfaces, arXiv:1904.05277 [math.AP].

[39] T. Oh, T. Robert, Y. Wang, On the parabolic and hyperbolic Liouville equations, arXiv:1908.03944 [math.AP].
[53] R. Robert, V. Vargas, *Gaussian multiplicative chaos revisited*, Ann. Probab. 38 (2010), no. 2, 605–631.

[54] R. Rhodes, V. Vargas, *Gaussian multiplicative chaos and applications: a review*, Probab. Surv. 11 (2014), 315–392.

[55] S. Ryang, T. Saito, K. Shigemoto, * Canonical stochastic quantization*, Progr. Theoret. Phys. 73 (1985), no. 5, 1295–1298.

[56] O. Schiffmann, E. Vasserot, *Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $A^2$*, Publ. Math. Inst. Hautes tudes Sci. 118 (2013), 213–342.

[57] B. Simon, *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.

[58] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. 146 (1987), no 1, 65–96.

[59] C. Sun, N. Tzvetkov, *Gibbs measure dynamics for the fractional NLS*, arXiv:1912.07303 [math.AP].

[60] M. Taylor, *Tools for PDE, Pseudodifferential operators, paradifferential operators, and layer potentials*. Mathematical Surveys and Monographs, 81. American Mathematical Society, Providence, RI, 2000. x+257 pp.

[61] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. 324 (1991), no. 2, 793–821.

[62] M. Zworski, *Semiclassical analysis*. Graduate Studies in Mathematics 138, American Mathematical Society, Providence, RI, 2012.

TADAHIRO OH, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, THE KING’S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM

E-mail address: hiro.oh@ed.ac.uk

TRISTAN ROBERT, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, POSTFACH 10 01 31, 33501 BIELEFELD, GERMANY

E-mail address: trobert@math.uni-bielefeld.de

NIKOLAY TZVETKOV, LABORATOIRE AGM, UNIVERSITÉ DE CERGY-PONTOISE, CERGY-PONTOISE, F-95000, UMR 8088 DU CNRS

E-mail address: nikolay.tzvetkov@u-cergy.fr

YUZHAO WANG, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, WATSON BUILDING, EDBASTON, BIRMINGHAM, B15 2TT, UNITED KINGDOM

E-mail address: y.wang.14@bham.ac.uk