K-THEORY OF THE LEAF SPACE OF FOLIATIONS
FORMED BY THE GENERIC K-ORBITS OF SOME INDECOMPOSABLE $MD_5$-GROUPS

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Abstract
The paper is a continuation of the authors’ work [18]. In [18], we consider foliations formed by the maximal dimensional K-orbits ($MD_5$-foliations) of connected $MD_5$-groups such that their Lie algebras have 4-dimensional commutative derived ideals and give a topological classification of the considered foliations. In this paper, we study K-theory of the leaf space of some of these $MD_5$-foliations and characterize the Connes’ C*-algebras of the considered foliations by the method of K-functors.

INTRODUCTION

In the decades 1970s − 1980s, works of D.N. Diep [4], J. Rosenberg [10], G. G. Kasparov [7], V. M. Son and H. H. Viet [12],... have seen that K-functors are well adapted to characterize a large class of group C*-algebras. Kirillov’s method of orbits allows to find out the class of Lie groups MD, for which the group C*-algebras can be characterized by means of suitable K-functors (see [5]). In terms of D. N. Diep, an MD-group of dimension n (for short, an $MD_n$-group) is an n-dimensional solvable real Lie group whose orbits in the co-adjoint representation (i.e., the K-representation) are the orbits of zero or maximal dimension. The Lie algebra of an $MD_n$-group is called an $MD_n$-algebra (see [5, Section 4.1]).

In 1982, studying foliated manifolds, A. Connes [3] introduced the notion of C*-algebra associated to a measured foliation. In the case of Reeb foliations (see A. M. Torpe [14]), the method of K-functors has been proved to be very effective in describing the structure of Connes’ C*-algebras. For every MD-group G, the family of K-orbits of maximal dimension forms a measured foliation in terms of Connes [3]. This foliation is called MD-foliation associated to G.

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6Key words: Lie group, Lie algebra, $MD_5$-group, $MD_5$-algebra, K-orbit, Foliation, Measured foliation, C*-algebra, Connes’ C*-algebra associated to a measured foliation.

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Combining methods of Kirillov (see [8, Section 15]) and Connes (see [3, Section 2, 5]), the first author had studied $MD_4$-foliations associated with all indecomposable connected $MD_4$-groups and characterized Connes’ $C^*$-algebras of these foliations in [16]. Recently, Vu and Shum [17] have classified, up to isomorphism, all the $MD_5$-algebras having commutative derived ideals.

In [18], we have given a topological classification of $MD_5$-foliations associated to the indecomposable connected and simply connected $MD_5$-groups, such that $MD_5$-algebras of them have 4-dimensional commutative derived ideals. There are exactly 3 topological types of the considered $MD_5$-foliations, denoted by $F_1, F_2, F_3$. All $MD_5$-foliations of type $F_1$ are the trivial fibrations with connected fibre on 3-dimesional sphere $S^3$, so Connes’ $C^*$-algebras of them are isomorphic to the $C^*$-algebra $C(S^3) \otimes K$ following [3, Section 5], where $K$ denotes the $C^*$-algebra of compact operators on an (infinite dimensional separable) Hilbert space.

The purpose of this paper is to study K-theory of the leaf space and to characterize the structure of Connes’ $C^*$-algebras $C^*(V,F)$ of all $MD_5$-foliations $(V,F)$ of type $F_2$ by the method of K-functors. Namely, we will express $C^*(V,F)$ by two repeated extensions of the form

$$0 \longrightarrow C_0(X_1) \otimes K \longrightarrow C^*(V,F) \longrightarrow B_1 \longrightarrow 0,$$

$$0 \longrightarrow C_0(X_2) \otimes K \longrightarrow B_1 \longrightarrow C_0(Y_2) \otimes K \longrightarrow 0,$$

then we will compute the invariant system of $C^*(V,F)$ with respect to these extensions. If the given $C^*$-algebras are isomorphic to the reduced crossed products of the form $C_0(V) \rtimes H$, where $H$ is a Lie group, we can use the Thom-Connes isomorphism to compute the connecting map $\delta_0, \delta_1$.

In another paper, we will study the similar problem for all $MD_5$-foliations of type $F_3$.

1 THE $MD_5$—FOLIATIONS OF TYPE $F_2$

Originally, we will recall geometry of K-orbit of $MD_5$-groups which associate with $MD_5$-foliations of type $F_2$ (see [18]).

In this section, $G$ will be always an connected and simply connected $MD_5$-group such that its Lie algebras $G$ is an indecomposable $MD_5$-algebra generated by $\{X_1, X_2, X_3, X_4, X_5\}$ with $G^1 := [G, G] = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \cong \mathbb{R}^4$, $ad_{X_1} \in End(G) \equiv Mat_4(\mathbb{R})$. Namely, $G$ will be one of the following Lie algebras which are studied in [17] and [18].

1. $G_{5,4,11}(\lambda_1, \lambda_2, \varphi)$

$$ad_{X_1} = \begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 & 0 \\
\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & \lambda_2
\end{bmatrix}; \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi).$$
2. $G_{5,4,12}(\lambda,\varphi)$

$$ad_{x_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

3. $G_{5,4,12}(\lambda,\varphi)$

$$ad_{x_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

The connected and simply connected Lie groups corresponding to these algebras are denoted by $G_{5,4,11}(\lambda_1,\lambda_2,\varphi)$, $G_{5,4,12}(\lambda,\varphi)$, $G_{5,4,13}(\lambda,\varphi)$. All of these Lie groups are $MD_5$-groups (see [17]) and $G$ is one of them. We now recall the geometric description of the K-orbits of $G$ including $F = \{\alpha, \beta \in \mathbb{R}^\ast \}$. The family $\mathcal{F}$ of maximal-dimensional K-orbits of $G$ forms measured foliation in terms of Connes on the open submanifold

$$V = \{(x,y,z,t,s) \in G^\ast : y^2 + z^2 + t^2 + s^2 \neq 0\} \cong \mathbb{R} \times (\mathbb{R}^4)^\ast \subset G^\ast \cong \mathbb{R}^5$$

Furthermore, all foliations $(V, \mathcal{F}_{5,4,11}(\lambda,\varphi)), (V, \mathcal{F}_{5,4,12}(\lambda,\varphi)), (V, \mathcal{F}_{5,4,13}(\lambda,\varphi))$ are topologically equivalent to each other $(\lambda_1, \lambda_2, \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi))$. Thus, we need only choose an envoy among them to describe the structure of the $C^\ast$-algebra. In this case, we choose the foliation $(V, \mathcal{F}_{4,12}(1,\pi))$.

In [18], we have described the foliation $(V, \mathcal{F}_{4,12}(1,\pi))$ by a suitable action of $\mathbb{R}^2$. Namely, we have the following proposition.
PROPOSITION 1. The foliation $(V, F_{4,12}(1, \frac{\pi}{2}))$ can be given by an action of the commutative Lie group $\mathbb{R}^2$ on the manifold $V$.

Proof. One needs only to verify that the following action $\lambda$ of $\mathbb{R}^2$ on $V$ gives the foliation $(V, F_{4,12}(1, \frac{\pi}{2}))$

$$\lambda: \mathbb{R}^2 \times V \to V$$

$$(r, a) \cdot (x, y + iz, t, s) = (x + r, (y + iz).e^{-ia}, t.e^a, s.e^a)$$

where $(r, a) \in \mathbb{R}^2, (x, y + iz, t, s) \in V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{R}^2)^* \cong \mathbb{R} \times (\mathbb{R}^4)^*$. Hereafter, for simplicity of notation, we write $(V, F)$ instead of $(V, F_{4,12}(1, \frac{\pi}{2}))$.

\qed

It is easy to see that the graph of $(V, F)$ is identical with $V \times \mathbb{R}^2$, so by [3, Section 5], it follows from Proposition 1 that:

COROLLARY 1 (analytical description of $C^*(V, F)$). The Connes $C^*$-algebra $C^*(V, F)$ can be analytically described the reduced crossed of $C_0(V)$ by $\mathbb{R}^2$ as follows

$$C^*(V, F) \cong C_0(V) \rtimes_\lambda \mathbb{R}^2.$$ 

\qed

2 \quad $C^*(V, F)$ AS TWO REPEATED EXTENSIONS

2.1. Let $V_1, W_1, V_2, W_2$ be the following submanifolds of $V$

$$V_1 = \{(x, y, z, t, s) \in V : s \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^*,$$

$$W_1 = V \setminus V_1 = \{(x, y, z, t, s) \in V : s = 0\} \cong \mathbb{R} \times (\mathbb{R}^3)^* \times \{0\} \cong \mathbb{R} \times (\mathbb{R}^3)^*,$$

$$V_2 = \{(x, y, z, t, 0) \in W_1 : t \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*,$$

$$W_2 = W_1 \setminus V_2 = \{(x, y, z, t, 0) \in W_1 : t = 0\} \cong \mathbb{R} \times (\mathbb{R}^2)^*.$$ 

It is easy to see that the action $\lambda$ in Proposition 1 preserves the subsets $V_1, W_1, V_2, W_2$. Let $i_1, i_2, \mu_1, \mu_2$ be the inclusions and the restrictions

$$i_1: C_0(V_1) \to C_0(V), \quad i_2: C_0(V_2) \to C_0(W_1),$$

$$\mu_1: C_0(V) \to C_0(W_1), \quad \mu_2: C_0(W_1) \to C_0(W_2)$$

where each function of $C_0(V_1)$ (resp. $C_0(V_2)$) is extended to the one of $C_0(V)$ (resp. $C_0(W_1)$) by taking the value of zero outside $V_1$ (resp. $V_2$).

It is known a fact that $i_1, i_2, \mu_1, \mu_2$ are $\lambda$-equivariant and the following sequences are equivariantly exact:

$$(2.1.1) \quad 0 \longrightarrow C_0(V_1) \overset{i_1}{\longrightarrow} C_0(V) \overset{\mu_1}{\longrightarrow} C_0(W_1) \longrightarrow 0$$

$$(2.1.2) \quad 0 \longrightarrow C_0(V_2) \overset{i_2}{\longrightarrow} C_0(W_1) \overset{\mu_2}{\longrightarrow} C_0(W_2) \longrightarrow 0.$$
2.2. Now we denote by $(V_1, F_1), (W_1, F_1), (V_2, F_2), (W_2, F_2)$ the foliations-restrictions of $(V, F)$ on $V_1, W_1, V_2, W_2$ respectively.

**THEOREM 1.** $C^*(V, F)$ admits the following canonical repeated extensions

$$(\gamma_1) \quad 0 \to J_1 \xrightarrow{\hat{i}_1} C^*(V, F) \xrightarrow{\hat{\mu}_1} B_1 \to 0,$$

$$(\gamma_2) \quad 0 \to J_2 \xrightarrow{i_2} B_1 \xrightarrow{\mu_2} B_2 \to 0,$$

where

$$J_1 = C^*(V_1, F_1) \cong C_0(V_1) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{R}^3 \cup \mathbb{R}^3) \otimes K,$$

$$J_2 = C^*(V_2, F_2) \cong C_0(V_2) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{R}^2 \cup \mathbb{R}^2) \otimes K,$$

$$B_2 = C^*(W_2, F_2) \cong C_0(W_2) \rtimes_\lambda \mathbb{R}^2 \cong C_0(\mathbb{R}_+^3) \otimes K,$$

$$B_1 = C^*(W_1, F_1) \cong C_0(W_1) \rtimes_\lambda \mathbb{R}^2, \text{ and the homomorphisms } \hat{i}_1, \hat{i}_2, \hat{\mu}_1, \hat{\mu}_2 \text{ are defined by}$$

$$(\hat{i}_k f)(r, s) = i_k f(r, s), \ k = 1, 2$$

$$(\hat{\mu}_k f)(r, s) = \mu_k f(r, s), \ k = 1, 2$$

**Proof.** We note that the graph of $(V_1, F_1)$ is identical with $V_1 \times \mathbb{R}^2$, so by [3, Section 5], $J_1 = C^*(V_1, F_1) \cong C_0(V_1) \rtimes_\lambda \mathbb{R}^2$. Similarly, we have

$$B_1 \cong C_0(W_1) \rtimes_\lambda \mathbb{R}^2,$$

$$J_2 \cong C_0(V_2) \rtimes_\lambda \mathbb{R}^2,$$

$$B_2 \cong C_0(W_2) \rtimes_\lambda \mathbb{R}^2,$$

From the equivariantly exact sequences in 2.1 and by [2, Lemma 1.1] we obtain the repeated extensions $(\gamma_1)$ and $(\gamma_2)$.

Furthermore, the foliation $(V_1, F_1)$ can be derived from the submersion

$$p_1 : V_1 \approx \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^3 \cup \mathbb{R}^3$$

$$p_1(x, y, z, t, s) = (y, z, t, \text{signs}).$$

Hence, by a result of [3, p.562], we get $J_1 \cong C_0(\mathbb{R}^3 \cup \mathbb{R}^3) \otimes K$. The same argument shows that

$$J_2 \cong C_0(\mathbb{R}^2 \cup \mathbb{R}^2) \otimes K, \ B_2 \cong C_0(\mathbb{R}_+^3) \otimes K.$$
3 COMPUTING THE INVARIANT SYSTEM OF $C^*(V, \mathcal{F})$

**DEFINITION.** The set of elements $\{\gamma_1, \gamma_2\}$ corresponding to the repeated extensions $(\gamma_1)$, $(\gamma_2)$ in the Kasparov groups $\text{Ext}(B_i, J_i)$, $i = 1, 2$ is called the system of invariants of $C^*(V, \mathcal{F})$ and denoted by $\text{Index } C^*(V, \mathcal{F})$.

**REMARK.** $\text{Index } C^*(V, \mathcal{F})$ determines the so-called stable type of $C^*(V, \mathcal{F})$ in the set of all repeated extensions $0 \to J_1 \to E \to B_1 \to 0$, $0 \to J_2 \to B_1 \to B_2 \to 0$.

The main result of the paper is the following.

**THEOREM 2.** $\text{Index } C^*(V, \mathcal{F}) = \{\gamma_1, \gamma_2\}$, where

$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ in the group $\text{Ext}(B_1, J_1) = \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$;
$\gamma_2 = (1, 1)$ in the group $\text{Ext}(B_2, J_2) = \text{Hom}(\mathbb{Z}, \mathbb{Z}^2)$.

To prove this theorem, we need some lemmas as follows.

**LEMMA 1.** Set $I_2 = C_0(\mathbb{R}^2 \times \mathbb{R}^*)$ and $A_2 = C_0((\mathbb{R}^2)^*)$.

The following diagram is commutative

\[
\begin{array}{ccccccccc}
\cdots & \to & K_j(I_2) & \to & K_j(C_0(\mathbb{R}^3)^*) & \to & K_j(A_2) & \to & K_{j+1}(I_2) & \to & \cdots \\
& | & \beta_1 & | & \beta_1 & | & \beta_1 & | & \beta_1 & |
\end{array}
\]

\[
\begin{array}{ccccccccc}
\cdots & \to & K_{j+1}(C_0(V_2)) & \to & K_{j+1}(C_0(W_1)) & \to & K_{j+1}(C_0(W_2)) & \to & K_j(C_0(V_2)) & \to & \cdots \\
& & \beta_1 & & \beta_1 & & \beta_1 & & \beta_1 & 
\end{array}
\]

where $\beta_1$ is the isomorphism defined in [13, Theorem 9.7] or in [2, corollary VI.3], $j \in \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Let

\[ k_2 : I_2 = C_0(\mathbb{R}^2 \times \mathbb{R}^*) \to C_0((\mathbb{R}^3)^*) \]
\[ v_2 : C_0((\mathbb{R}^3)^*) \to A_2 = C_0((\mathbb{R}^2)^*) \]

be the inclusion and restriction defined similarly as in 2.1.

One gets the exact sequence

\[
0 \to I_2 \xrightarrow{k_2} C_0((\mathbb{R}^3)^*) \xrightarrow{v_2} A_2 \to 0
\]

Note that

\[ C_0(V_2) \cong C_0(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*) \cong C_0(\mathbb{R}) \otimes I_2, \]
\[ C_0(W_2) \cong C_0(\mathbb{R} \times (\mathbb{R}^2)^*) \cong C_0(\mathbb{R}) \otimes A_2, \]
\[ C_0(W_1) \cong C_0(\mathbb{R} \times (\mathbb{R}^3)^*) \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^3)^*. \]
The extension (2.1.2) thus can be identified to the following one

\[ 0 \rightarrow C_0(\mathbb{R}) \otimes I_2 \xrightarrow{id \otimes k_2} C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^3)^* \xrightarrow{id \otimes v_2} C_0(\mathbb{R}) \otimes A_2 \rightarrow 0. \]

Now, using [13, Theorem 9.7; Corollary 9.8] we obtain the assertion of Lemma 1.

**Lemma 2.** Set \( I_1 = C_0(\mathbb{R}^2 \times \mathbb{R}^*) \) and \( A_1 = C(S^2) \)

The following diagram is commutative

\[ \cdots \rightarrow K_j(I_1) \xrightarrow{\beta_2} K_j(C(S^3)) \xrightarrow{\beta_2} K_j(A_1) \xrightarrow{\beta_2} K_j+1(I_1) \xrightarrow{\beta_2} \cdots \]

\[ \cdots \rightarrow K_j(C_0(V_1)) \xrightarrow{\beta_2} K_j(C_0(V)) \xrightarrow{\beta_2} K_j(C_0(W_1)) \xrightarrow{\beta_2} K_j+1(C_0(V_1)) \xrightarrow{\beta_2} \cdots \]

where \( \beta_2 \) is the Bott isomorphism, \( j \in \mathbb{Z}/2\mathbb{Z} \).

**Proof.** The proof is similar to that of lemma 1, by using the exact sequence (2.1.1) and diffeomorphisms: \( V \cong \mathbb{R} \times (\mathbb{R}^4)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^3, W_1 \cong \mathbb{R} \times (\mathbb{R}^3)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^2 \).

Before computing the K-groups, we need the following notations. Let \( u : \mathbb{R} \rightarrow S^1 \) be the map

\[ u(z) = e^{2\pi i(z/\sqrt{1+z^2})}, \quad z \in \mathbb{R}. \]

Denote by \( u_+ \) (resp. \( u_- \)) the restriction of \( u \) on \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_- \)). Note that the class \([u_+]\) (resp. \([u_-]\)) is the canonical generator of \( K_1(C_0(\mathbb{R}_+)) \cong \mathbb{Z} \) (resp. \( K_1(C_0(\mathbb{R}_-)) \cong \mathbb{Z} \)). Let us consider the matrix valued function \( p : \mathbb{R}^2 \times S^1 \rightarrow \mathbb{R}_+ \rightarrow M_2(\mathbb{C}) \) (resp. \( \overline{p} : \mathbb{R}^2 \rightarrow D/S^1 \rightarrow M_2(\mathbb{C}) \)) defined by:

\[ p(x;y) (\text{resp. } \overline{p}(x,y)) = \frac{1}{2} \left( \begin{array}{cc} 1 - \cos \pi \sqrt{x^2 + y^2} & \frac{x+iy}{\sqrt{x^2+y^2}} \sin \pi \sqrt{x^2 + y^2} \\ \frac{x-iy}{\sqrt{x^2+y^2}} \sin \pi \sqrt{x^2 + y^2} & 1 + \cos \pi \sqrt{x^2 + y^2} \end{array} \right). \]

Then \( p \) (resp. \( \overline{p} \)) is an idempotent of rank 1 for each \((x;y) \in \mathbb{R}^2 \times S^1\) (resp. \((x;y) \in D/S^1\)). Let \([b] \in K_0(C_0(\mathbb{R}^2))\) be the Bott element, \([1]\) be the generator of \( K_0(C(S^1)) \cong \mathbb{Z} \).

**Lemma 3** (See [15, p.234]).

(i) \( K_0(B_1) \cong \mathbb{Z}^2 \), \( K_1(B_1) = 0 \),

(ii) \( K_0(J_2) \cong \mathbb{Z}^2 \) is generated by \( \varphi_0 \beta_1([b] \boxtimes [u_+]) \) and \( \varphi_0 \beta_1([b] \boxtimes [u_-]) ; \ K_1(J_2) = 0 \),

(iii) \( K_0(B_2) \cong \mathbb{Z} \) is generated by \( \varphi_0 \beta_1([1] \boxtimes [u_+]) ; \ K_1(B_2) \cong \mathbb{Z} \) is generated by \( \varphi_1 \beta_1([p] - [\varepsilon_1]) \),

where \( \varphi_j, j \in \mathbb{Z}/2\mathbb{Z}, \) is the Thom-Connes isomorphism (see[2]), \( \beta_1 \) is the isomorphism in Lemma 1, \( \varepsilon_1 \) is the constant matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \boxtimes \) is the external tensor product (see, for example, [2, VI.2]).

**Lemma 4.**

(i) \( K_0(C^*(V, \mathcal{F})) \cong \mathbb{Z} \), \( K_1(C^*(V, \mathcal{F})) \cong \mathbb{Z} \),
(ii) $K_0(J_1) = 0$; $K_1(J_1) \cong \mathbb{Z}^2$ is generated by $\varphi_1\beta_2([b] \boxtimes [u_+])$ and $\varphi_1\beta_2([b] \boxtimes [u_-])$.
(iii) $K_1(B_1) = 0$; $K_0(B_1) \cong \mathbb{Z}^2$ is generated by $\varphi_0\beta_2[1]$ and $\varphi_0\beta_2([\bar{p}] - [\varepsilon_1])$,
where $1$ is unit element in $C(S^2)$, $\varphi_0$ is the Thom-Connes isomorphism, $\beta_2$ is the Bott isomorphism.

Proof.
(i) $K_i(C^*(V, \mathcal{F})) \cong K_i(C(S^3)) \cong \mathbb{Z}$, $i = 0, 1$.
(ii) The proof is similar to (ii) of lemma 3.
(iii) By [9, p.206], we have

$$K_0(C(S^2)) = \mathbb{Z}[1] + \mathbb{Z}[q], \text{ where } q \in P_2(C(S^2)).$$

Otherwise, in [9, p.48,53,56]; [13, p.162], one has shown that the map

$$\text{dim} : K_0(C(S^2)) \to \mathbb{Z}$$

is a surjective group homomorphism which satisfied $\text{dim}[1] = 1$, $\ker(\text{dim}) = \mathbb{Z}$ and non-zero element $q \in P_2(C(S^2))$ in the kernel of the map $\text{dim}$ has the form $[q] = [\bar{p}] - [\varepsilon_1]$. Hence, the result is derived straight away because $\beta_2$ and $\varphi_1$ are isomorphisms.

\[\square\]

**Proof of theorem 2**

1. Computation of $(\gamma_1)$. Recall that the extension $(\gamma_1)$ in theorem 1 gives rise to a six-term exact sequence

$$0 = K_0(J_1) \to K_0(C^*(V, F)) \to K_0(B_1) \to K_1(B_1) \to K_1(C^*(V, F)) \to K_1(J_1)$$

By [11, Theorem 4.14], the isomorphisms

$$\text{Ext}(B_1, J_1) \cong \text{Hom}((K_0(B_1), K_1(J_1))) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$$

associates the invariant $\gamma_1 \in \text{Ext}(B_1, J_1)$ to the connecting map $\delta_0 : K_0(B_1) \to K_1(J_1)$.

Since the Thom-Connes isomorphism commutes with $K$-theoretical exact sequence (see[14, Lemma 3.4.3]), we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$):

$$
\cdots \to K_j(J_1) \to K_j(C^*(V, F)) \to K_j(B_1) \to K_{j+1}(B_1) \to K_{j+1}(J_1) \to \cdots
\downarrow \varphi_j \downarrow \varphi_j \downarrow \varphi_j \downarrow \varphi_{j+1}
\cdots \to K_j(C_0(V_1)) \to K_j(C_0(V)) \to K_j(C_0(W_1)) \to K_{j+1}(C_0(V_1)) \to \cdots
$$

In view of Lemma 2, the following diagram is commutative

$$
\cdots \to K_j(C_0(V_1)) \to K_1(C_0(V)) \to K_j(C_0(W_1)) \to K_{j+1}(C_0(V_1)) \to \cdots
\downarrow \beta_2 \downarrow \beta_2 \downarrow \beta_2 \downarrow \beta_2
\cdots \to K_j(J_1) \to K_j(C(S^3)) \to K_j(A_1) \to K_{j+1}(J_1) \to \cdots
$$
Consequently, instead of computing $\delta_0 : K_0(B_1) \to K_1(J_1)$, it is sufficient to compute $\delta_0 : K_0(B_1) \to K_1(J_1)$. Thus, by the proof of Lemma 4, we have to define $\delta_0([\bar{p}] - [\varepsilon_1]) = \delta_0([\bar{p}])$ (because $\delta_0([\varepsilon_1]) = (0; 0)$ and $\delta_0([1]) = (0; 0)$). By the usual definition (see [13, p.170]), for $[\bar{p}] \in K_0(B_1)$, $\delta_0([\bar{p}]) = [e^{2\pi i \bar{p}}] \in K_1(I_1)$ where $\bar{p}$ is a preimage of $\bar{p}$ in (a matrix algebra over) $C(S^3)$, i.e. $v_1\bar{p} = \bar{p}$.

We can choose $\bar{p}(x, y, z) = \frac{z}{\sqrt{1 + z^2}}\bar{p}(x, y), (x, y, z) \in S^3$.

Let $\bar{p}_+$ (resp. $\bar{p}_-$) be the restriction of $\bar{p}$ on $\mathbb{R}^2 \times \mathbb{R}_+$ (resp. $\mathbb{R}^2 \times \mathbb{R}_-$). Then we have $\delta_0([\bar{p}]) = [e^{2\pi i \bar{p}_+}] = [e^{2\pi i \bar{p}_-}] \in K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_+)) \oplus K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_-)) \cong K_1(I_1)$

By [13, Section 4], for each function $f : \mathbb{R}_+ \to \widehat{Q_n C_0(\mathbb{R}^2)}$ such that $\lim_{x \to \pm \infty} f(t) = \lim_{x \to \pm \infty} f(t)$, where $Q_n C_0(\mathbb{R}^2) = \{a \in M_n C_0(\mathbb{R}^2), e^{2\pi i a} = Id\}$, the class $[f] \in K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_+))$ can be determined by $[f] = W_f, [b] \otimes [u_+]$, where $W_f = \frac{1}{2\pi i} \int_{\mathbb{R}_+} Tr(f'(z)f^{-1}(z))dz$ is the winding number of $f$.

By simple computation, we get $\delta_0([\bar{p}]) = [b] \otimes [u_+] + [b] \otimes [u_-]$. Thus $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \text{Hom}_Z(\mathbb{Z}^2, \mathbb{Z}^2)$.

2. Computation of $(\gamma_2)$. The extension $(\gamma_2)$ gives rise to a six-term exact sequence

$$
\begin{array}{ccccccccc}
K_0(J_2) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(B_2) \\
\downarrow \delta_1 & & \downarrow \delta_0 & & \downarrow \delta_0 \\
K_1(B_2) & \leftarrow & K_1(B_1) & \leftarrow & K_1(J_2) = 0
\end{array}
$$

By [11, Theorem 4.14], $\gamma_2 = \delta_1 \in \text{Hom}(K_1(B_2), K_0(J_2)) = \text{Hom}_Z(\mathbb{Z}, \mathbb{Z}^2)$. Similarly to part 1, taking account of Lemma 1 and 3, we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$)

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & K_j(J_2) & \longrightarrow & K_j(B_1) & \longrightarrow & K_j(B_2) & \longrightarrow & K_{j+1}(J_2) & \longrightarrow & \cdots \\
\uparrow \varphi_j & & \uparrow \varphi & & \uparrow \varphi & & \uparrow \varphi_j & & \uparrow \varphi_j & & \cdots \\
\cdots & \longrightarrow & K_j(C_0(V_2)) & \longrightarrow & K_j(C_0(W_1)) & \longrightarrow & K_j(C_0(W_2)) & \longrightarrow & K_{j+1}(C_0(V_2)) & \longrightarrow & \cdots \\
\downarrow \beta_1 & & \downarrow \beta & & \downarrow \beta_1 & & \downarrow \beta_1 & & \downarrow \beta & & \cdots \\
\cdots & \longrightarrow & K_{j-1}(J_2) & \longrightarrow & K_{j-1}(C_0(\mathbb{R}^2)^*) & \longrightarrow & K_{j-1}(A_2) & \longrightarrow & K_{j-1}(I_2) & \longrightarrow & \cdots
\end{array}
$$

Thus we can compute $\delta_0 : K_0(A_2) \to K_1(I_2)$ instead of $\delta_1 : K_1(B_2) \to K_0(J_2)$. By the proof of Lemma 3, we have to define $\delta_0([\bar{p}] - [\varepsilon_1]) = \delta_0([\bar{p}])$ (because $\delta_0([\varepsilon_1]) = (0; 0)$). The same argument as above, we get $\delta_0([\bar{p}]) = [b] \otimes [u_+] + [b] \otimes [u_-]$. Thus $\gamma_2 = (1, 1) \in \text{Hom}_Z(\mathbb{Z}, \mathbb{Z}^2) \cong \mathbb{Z}^2$. The proof is completed. \(\square\)
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