ON A PROBLEM OF KAHANE IN HIGHER DIMENSIONS

JAMES WRIGHT

Dedicated to Mike Christ on his 60th birthday

Abstract. We characterise those real analytic mappings from $T^k$ to $T^d$ which map absolutely convergent Fourier series on $T^d$ to uniformly convergent Fourier series via composition. We do this with respect to rectangular summation on $T^k$ (more precisely, unrestricted rectangular summation). We also investigate uniform convergence with respect to square sums and highlight the differences which arise.

1. Introduction

The famous Beurling-Helson theorem [2] from the early 1950’s states that the only mappings of the circle $\Phi : T \to T$ which preserve the space $A(T)$ of absolutely convergent Fourier series are affine maps (see also [9] and [6]). More precisely, if $\Phi : T \to T$ has the property that $f \circ \Phi \in A(T)$ whenever $f \in A(T)$, then $\Phi(e^{2\pi it}) = e^{2\pi i(c+kt)}$ for some constant $c \in \mathbb{R}$ and integer $k \in \mathbb{Z}$. In a remarkable paper [4], P. Cohen extended this to mappings between any two locally compact abelian groups $G_1$ and $G_2$ with the same rigidity outcome: only piecewise affine maps $\Phi : G_1 \to G_2$ transport $A(G_2)$ to $A(G_1)$ via composition with $\Phi$.

Kahane then asked what happens when we relax the condition of absolute convergence to uniform convergence; see for example, [7]. Are there nonlinear mappings $\Phi : T \to T$ (and if so, which ones) that carry the space $A(T)$ to the space of uniformly convergent Fourier series $U(T)$ via composition with $\Phi$? In this direction, we have the following nice result of L. Alpár [1].

Alpár’s Theorem For any real-analytic mapping $\Phi : T \to T$, we have $f \circ \Phi \in U(T)$ whenever $f \in A(T)$. Furthermore this property can fail for smooth $\Phi$; more precisely, there exists $\Phi \in C^\infty(T)$ and $f \in A(T)$ such that $f \circ \Phi \notin U(T)$.

Alpár’s original proof was very complicated but along the way (as people asked what happens between the real-analytic and $C^\infty$ categories of mapping of the circle; see for example, [8], [12] and [18]) simplifications were made. Most notable is the work of R. Kaufman [8] where some interesting connections were established between Kahane’s problem and the theory of thin sets in harmonic analysis.

1991 Mathematics Subject Classification. 42B05, 42B08, 42B20.
In this paper, we investigate what happens with mappings \( \Phi : \mathbb{T}^k \to \mathbb{T}^d \) between higher dimensional tori. Since \( \mathbb{T}^k \) and \( \mathbb{T}^d \) are two examples of compact abelian groups, Cohen’s result applies and we see that the only mappings \( \Phi \) which preserve the space of absolutely convergent Fourier series are affine ones. As Kahane proposed, we ask here what happens when we relax the condition of absolute convergence to uniform convergence. We begin by stating a simple extension of Alpár’s Theorem.

**Proposition 1.1.** Let \( \Phi : \mathbb{T} \to \mathbb{T}^d \) be any real-analytic mapping. Then \( f \circ \Phi \in U(\mathbb{T}) \) whenever \( f \in A(\mathbb{T}^d) \).

The proof of Proposition 1.1 is an adaptation of Kaufman’s proof of Alpár’s theorem. We give the proof in the next section. Much more interesting is what happens when the domain \( \mathbb{T}^k \) is higher dimensional; that is, when \( k \geq 2 \). In this paper, we will restrict ourselves to \( k = 2 \). Already in this case, we will see most of the new phenomena and challenges. The situation for larger values of \( k \) is similar but matters become much more technical. As we will see, the analysis is already quite involved for \( k = 2 \) but the statements of the main results in this setting are simple. At the end of the paper we will discuss what happens when \( k \geq 3 \).

When \( k = 2 \), we want to understand when \( f \circ \Phi \in U(\mathbb{T}^2) \) whenever \( f \in A(\mathbb{T}^d) \) but now we need to stipulate how we are summing the Fourier series

\[
 f \circ \Phi(x, y) = \sum_{(k, \ell) \in \mathbb{Z}^2} c_{k, \ell} e^{2\pi i [kx + \ell y]};
\]

here \( c_{k, \ell} \) denote the Fourier coefficients of \( f \circ \Phi \). That is, the definition of the subspace \( U(\mathbb{T}^2) \) of continuous functions \( C(\mathbb{T}^2) \) on the 2-torus \( \mathbb{T}^2 \) which have a uniformly convergent Fourier series depends on how we sum the series (this of course is not needed when we consider the subspace \( A(\mathbb{T}^2) \subset C(\mathbb{T}^2) \) of absolutely convergent Fourier series).

We will be mainly interested in rectangular summation or what is more precisely referred to as **unrestricted rectangular summation**. This is the space of continuous functions \( g \in C(\mathbb{T}^2) \) which have the property that the rectangular Fourier partial sums

\[
 S_{M,N}g(x, y) = \sum_{|k| \leq M, |\ell| \leq N} \hat{g}(k, \ell) e^{2\pi i [kx + \ell y]} = \iint_{\mathbb{T}^2} g(x-s, y-t) D_M(s) D_N(t) \, ds \, dt
\]

converges uniformly to \( g \): \( \|S_{M,N}g - g\|_{L^\infty(\mathbb{T}^2)} \to 0 \) as \( \min(M, N) \to \infty \). Here \( D_N(y) = \sin(2\pi(N + 1/2)y) / \sin(\pi y) \) is the \( N \)th Dirichlet kernel. We will denote this subspace as \( U_{rect}(\mathbb{T}^2) \) which in fact is a Banach space in its own right with respect to the norm

\[
 \|g\|_{U_{rect}} := \sup_{M,N} \|S_{M,N}g\|_{L^\infty(\mathbb{T}^2)}
\]

of uniform rectangular convergence. We will also consider the subspace \( U_{sq}(\mathbb{T}^2) \) of continuous functions \( g \) whose Fourier series converges uniformly with respect to **square sums**, \( \|S_{N}g - g\|_{L^\infty} \to 0 \) as \( N \to \infty \). This is also a Banach space with the norm

\[
 \|g\|_{U_{sq}} := \sup_{N} \|S_{N}g\|_{L^\infty(\mathbb{T}^2)}
\]
of uniform square convergence. Of course we could also consider spherical summation (of various orders) $U_{sp}(T^2)$ but this problem has a different character and we will investigate this setting elsewhere.

We will only consider real-analytic mappings $\Phi: T^2 \to T^d$ and in this category, it turns out that it is sometimes the case and sometimes not the case that $f \circ \Phi \in U_{\text{rect}}(T^2)$ whenever $f \in A(T^d)$. The same holds for square sums $U_{\text{sq}}(T^2)$. One of our main goals will be to give a simple characterisation of those real-analytic mappings $\Phi: T^2 \to T^d$ which have the property

$$f \circ \Phi \in U_{\text{rect}}(T^2) \quad \text{whenever} \quad f \in A(T^d). \quad (\Phi)_{\text{rect}}$$

We will denote by $(\Phi)_{\text{sq}}$ the analogous property but with respect to square sums.

It is easy to see that any continuous mapping $\Phi: T^k \to T^d$ can be described as follows: if $\vec{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ parameterises a point $P(\vec{t}) = (e^{2\pi it_1}, \ldots, e^{2\pi it_k}) \in T^k$ on the $k$-torus, then

$$\Phi(P(\vec{t})) = (e^{2\pi i[\phi_1(\vec{t})+L_1\cdot \vec{t}]}, \ldots, e^{2\pi i[\phi_d(\vec{t})+L_d\cdot \vec{t}]})$$

for some $d$-tuple $\vec{\phi}(\vec{t}) = (\phi_1(\vec{t}), \ldots, \phi_d(\vec{t}))$ of 1-periodic functions on $\mathbb{R}^k$ and some $d$-tuple $\vec{L} = (L_1, \ldots, L_d)$ of lattice points in $\mathbb{Z}^k$. Hence every map

$$\Phi = \Phi_{\vec{\phi}, \vec{L}}: T^k \to T^d$$

has a periodic part $\vec{\phi}$ and a linear part $\vec{L}$. The mapping $\Phi$ is said to be affine precisely when the periodic part $\vec{\phi} = (c_1, \ldots, c_d)$ is constant.

Now let us restrict our attention to $k = 2$ so that the periodic part $\vec{\phi}(s, t)$ is a $d$-tuple of analytic, 1-periodic functions of two variables. We put these $d$ periodic functions together as a single periodic function $\psi(s, t, \omega) := \omega \cdot \vec{\phi}(s, t)$ where $\omega \in S^{d-1}$ is a point on the $(d-1)$-dimensional unit sphere $S^{d-1}$. Hence $\psi$ defines a real-analytic map on the compact manifold $T^2 \times S^{d-1}$. For any compact, analytic manifold $M$ we say that an analytic function $\psi$ on $T^2 \times M$ satisfies the factorisation hypothesis (FH) on $M$ if at every point $P = (s, t, \omega) \in T^2 \times M$, the 2nd order partial derivatives

$$\psi_{ss} = \frac{\partial^2 \psi}{\partial s^2}, \quad \psi_{tt} = \frac{\partial^2 \psi}{\partial t^2}, \quad \psi_{st} = \frac{\partial^2 \psi}{\partial s \partial t}$$

of $\psi(s, t, \omega)$ satisfy the factorisation identities

$$\psi_{st}(s, t, \omega) = K(s, t, \omega) \psi_{ss}(s, t, \omega) \quad \text{and} \quad \psi_{st}(s, t, \omega) = L(s, t, \omega) \psi_{tt}(s, t, \omega)$$

for analytic functions $K$ and $L$ in some neighbourhood of $P$. This condition can be compactly expressed in terms of germs of analytic functions.

**Definition 1.2.** A real-analytic map $\psi: T^2 \times M \to \mathbb{R}$ satisfies the factorisation hypothesis if at every point in $T^2 \times M$, the divisibility conditions

$$\psi_{ss}, \ psi_{tt}, \ psi_{st} \quad (FH)$$

hold as germs of analytic functions.

Our main result is the following.
Theorem 1.3. Let $\Phi : \mathbb{T}^2 \to \mathbb{T}^d$ be a real-analytic map. Suppose that $\overline{\phi}$ is the periodic part of $\Phi$ and set $\psi(s,t,\omega) = \omega \cdot \overline{\phi}(s,t)$. Then $(\Phi)_{\text{rect}}$ holds if and only if $\psi$ satisfies (FH) on $M = \mathbb{S}^{d-1}$.

The hypothesis (FH) on $M = \mathbb{S}^{d-1}$ with $\psi(s,t,\omega) = \omega \cdot \overline{\phi}(s,t)$ is satisfied if $\overline{\phi}(s,t) = \mathcal{F}(s) + \mathcal{Y}(t)$ or if $\overline{\phi}(s,t) = \mathcal{F}(ks + \ell t)$ where $\mathcal{F}, \mathcal{Y}$ are both $d$-tuples of real-analytic, 1-periodic functions of one variable and $k, \ell$ are integers. Furthermore, this factorisation hypothesis remains true under certain small analytic perturbations; namely, if $\psi$ satisfies (FH), then for any real-analytic function $K$, there is an $\epsilon > 0$ such that $\zeta = \psi + \epsilon K(\psi_{ss})^3(\psi_{tt})^3$ also satisfies (FH). On the other hand, for any two real-analytic, 1-periodic, nonconstant functions $f, g$ of one variable, the function $\psi(s,t) = f(s)g(t)$ will never satisfy (FH).

Note that in Theorem 1.3, the necessary and sufficient condition for $(\Phi)_{\text{rect}}$ to hold only depends on the periodic part $\overline{\phi}$ and does not depend on the linear part $L$ of the mapping $\Phi : \mathbb{T}^2 \to \mathbb{T}^d$. In particular, if (FH) fails for $\psi(s,t,\omega) = \omega \cdot \overline{\phi}(s,t)$, then $(\Phi)_{\text{rect}}$ fails for all $\Phi = \Phi_{\overline{\phi},L}$ with periodic part $\overline{\phi}$. The same is not true for the square sum $(\Phi)_{sq}$ problem, at least when $d = 1$.

Theorem 1.4. Let $\psi(s,t)$ be any real-analytic, 1-periodic function of two variables. Then there exists infinitely many lattice points $L = (k,\ell) \in \mathbb{Z}^2$ such that $(\Phi)_{sq}$ holds where $\Phi = \Phi_{\phi,L} : \mathbb{T}^2 \to \mathbb{T}$. Furthermore, there exist real-analytic maps $\Phi = \Phi_{\phi,L} : \mathbb{T}^2 \to \mathbb{T}$ such that $(\Phi)_{sq}$ fails for some lattice point $L = (k,\ell)$.

Hence we see that there exist real-analytic, 1-periodic functions $\phi(s,t)$ of two variables such that $(\Phi)_{sq}$ holds for some maps $\Phi = \Phi_{\phi,L}$ and $(\Phi)_{sq}$ fails for other maps $\Phi = \Phi_{\phi,L'}$, both maps having the same periodic part. Thus the linear part $L = (k,\ell)$ of the map $\Phi(e^{2\pi is}, e^{2\pi it}) = e^{2\pi i[\phi(s,t) + ks + \ell t]}$ can determine whether or not the property $(\Phi)_{sq}$ holds. This is not the case for $(\Phi)_{\text{rect}}$.

1.5. Outline of the paper. In the next section, we give some preliminary reductions and we give the proof of Proposition 1.1. In Section 3 we discuss some consequences of the factorisation hypothesis (FH). Sections 4 and 5 develop a more robust one-dimensional theory which will be needed in, as well as giving a prelude to, the proof of the main results. We prove Theorem 1.3 in Sections 6 and 7 and the proof of Theorem 1.4 is given in Section 8. In the final section we discuss the situation for mappings $\Phi : \mathbb{T}^k \to \mathbb{T}^d$ from higher dimensional tori.

1.6. Notation. Uniform bounds for oscillatory integrals lie at the heart of this paper. Keeping track of constants and how they depend on the various parameters will be important for us. For the most part, constants $C$ appearing in inequalities $A \leq CB$ between positive quantities $A$ and $B$ will be absolute or uniform in that they can be taken to be independent of the parameters of the underlying problem. We will use $A \lesssim B$ to denote $A \leq CB$. When we allow the constant $C$ to depend on a parameter (or parameters), say $\omega \in \mathbb{S}^{d-1}$, we will write $A \lesssim_{\omega} B$ or $A \lesssim_{\omega} B$. For

---

1 We thank Sandy Davie for pointing this out to us.
quantities $Q$ and $B$ where $B$ is nonnegative, we write $Q = O(B)$ (or $Q = O_{\omega}(B)$) to denote $|Q| \leq CB$ (or $|Q| \leq C_{\omega}B$). Also $P \sim Q$ or $P \sim_{\omega} Q$ will denote $C^{-1}|P| \leq |Q| \leq C|P|$ or $C_{\omega}^{-1}|P| \leq |Q| \leq C_{\omega}|P|$.

We will use multi-index notation

$$\frac{\partial^{k+\ell} \phi}{\partial s^k \partial t^\ell}(s,t) = \partial^{k,\ell} \phi(s,t)$$

for partial derivatives but also when $k$ or $\ell = 0$, we adopt the shorthand notation $\partial^s \phi = \partial^n \phi$ and $\partial^m \phi = \partial^{0,m} \phi$. Finally we will also use $\phi_{ss}, \phi_{tt}$ and $\phi_{st}$ to denote the second order derivatives which play a special role.

2. Preliminary reductions and the proof of Proposition 1.1

For the proof of Proposition 1.1, or to establish $(\Phi)_{\text{rect}}$, or $(\Phi)_{\text{sq}}$, we need to show that whenever $f \in A(T^d)$, we have $f \circ \Phi \in U(T), f \circ \Phi \in U_{\text{rect}}(T^2)$ or $f \circ \Phi \in U_{\text{sq}}(T^2)$, respectively. A simple application of the closed graph theorem shows that this is equivalent to proving that $f \to f \circ \Phi$ is a continuous map from $A(T^d)$ to $U(T), U_{\text{rect}}(T^2)$ or $U_{\text{sq}}(T^2)$; that is, these three problems are equivalent to establishing the apriori bounds

$$\|f \circ \Phi\|_{U(T)}, \|f \circ \Phi\|_{U_{\text{rect}}(T^2)} \text{ or } \|f \circ \Phi\|_{U(T^2)} \leq C\|f\|_{A(T^d)}. \quad (1)$$

Let $U_\ast$ denote either $U(T), U_{\text{rect}}(T^2)$ or $U_{\text{sq}}(T^2)$. If

$$f(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}^d} \hat{f}(\vec{m}) e^{i \pi \vec{m} \cdot \vec{x}} \in A(T^d), \text{ then } f \circ \Phi = \sum_{\vec{m} \in \mathbb{Z}^d} \hat{f}(\vec{m}) e^{i \pi \vec{m} \cdot \vec{y}}$$

where $\vec{y}(\vec{t}) = \vec{\phi}(\vec{t}) + \vec{T} \cdot \vec{t}$ is the vector-valued function on $\mathbb{R}^k$ parameterising the mapping $\Phi : T^k \to T^d$. By the triangle inequality, we have

$$\|f \circ \Phi\|_{U_\ast} \leq \sum_{\vec{m} \in \mathbb{Z}^d} |\hat{f}(\vec{m})| \|e^{i \pi \vec{m} \cdot \vec{y}}\|_{U_\ast}$$

and so (1) holds if and only if the $U_\ast$ norms of the oscillations $e^{i \pi \vec{m} \cdot \vec{y}}$ are uniformly bounded in $\vec{m} \in \mathbb{Z}^d$; that is,

$$\sup_{\vec{m} \in \mathbb{Z}^d} \|e^{i \pi \vec{m} \cdot \vec{y}}\|_{U_\ast} < \infty. \quad (2)$$

A similar reduction can be made for the Beurling-Helson theorem but now the $U_\ast$ norms are replaced with the Weiner norm $\|f\| = \sum_n |\hat{f}(n)|$ on $A(T)$.

2.1. Reduction when $k = 1$; for mappings $\Phi : T \to T^d$. In the case $k = 1$, we have $\vec{y}(t) = \vec{\phi}(t) + \vec{T}t$ where $\vec{\phi}$ is a $d$-tuple of real-analytic, 1-periodic real-valued functions of a single variable and $\vec{T} \in \mathbb{Z}^d$. Hence if $S_N f(x) = \sum_{|n| \leq N} \hat{f}(n) e^{i \pi nx}$ denotes the classical $N$th Fourier partial sum of a function $f \in C(T)$, we have

$$\|e^{i \pi \vec{m} \cdot \vec{y}}\|_{U(T)} = \sup_N \|S_N (e^{i \pi \vec{m} \cdot \vec{y}})\|_{L^\infty(T)} = \sup_{N,x} \left| \int_T e^{i \pi \vec{m} \cdot [\vec{\phi}(t) + \vec{T}(x-t)]} D_N(t) \, dt \right|.$$
Since the Dirichlet kernel
\[ D_N(t) = \frac{\sin(2\pi(N+1/2)t)}{\sin(\pi t)} = \frac{e^{2\pi i(N+1/2)t} - e^{-2\pi i(N+1/2)t}}{\pi t} \]
for \( t \in [-1/2, 1/2) \), we see that to establish (2) in the case of Proposition 1.1, it suffices to obtain (writing \( \overline{\Phi} = \lambda \omega \) for some \( \omega \in \mathbb{S}^{d-1} \)) the uniform bound
\[ \sup_{\lambda, \omega \in \mathbb{R}, \omega \in \mathbb{S}^{d-1}} \left| p.v. \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i[\lambda \omega \cdot x + \rho t]} \frac{dt}{t} \right| \leq \sup_{\lambda, \rho \in \mathbb{R}, (x, t) \in \mathbb{T}^2} \left| e^{2\pi i \overline{\Phi}(x, t)} \right| < \infty. \]  

2.2. Reduction when \( k = 2 \); for mappings \( \Phi : \mathbb{T}^2 \to \mathbb{T}^d \). In the case \( k = 2 \), we have \( \overline{\Phi}(s, t) = \overline{\phi}(s, t) + \overline{L}_1 s + \overline{L}_2 t \) where \( \overline{\phi} \) is a \( d \)-tuple of real-analytic, 1-periodic real-valued functions of two variables and \( \overline{L} = (\overline{L}_1, \overline{L}_2) \) is a pair of lattice points in \( \mathbb{Z}^d \). Hence if \( S_{M,N} f(x, y) = \sum_{|k| \leq M, |\ell| \leq N} \hat{f}(k, \ell) e^{2\pi i kx + \ell y} \) denotes an rectangular Fourier partial sum of a function \( f \in C(\mathbb{T}^2) \), we have \( \|e^{2\pi i \overline{\Phi}}\|_{U_{rec}(\mathbb{T}^2)} = \sup_{M,N} \|S_{M,N}(e^{2\pi i \overline{\Phi}})\|_{L^\infty(\mathbb{T})} \leq \sup_{M,N} \int_{\mathbb{T}^2} e^{2\pi i \overline{\Phi}(x, y) + \overline{L}(s, t)} D_M(s) D_N(t) ds dt \). From the formula and properties of the Dirichlet kernel displayed in (3) we see that the integral above underpinning the norms \( \|e^{2\pi i \overline{\Phi}}\|_{U_{rec}(\mathbb{T}^2)} \) splits into a sum of eight integrals, four of these have the form
\[ \int_{\mathbb{T}^2} e^{2\pi i \overline{\Phi}(x, y) + \overline{L}(s, t)} D_M(s) D_N(t) ds dt \]
where \( A_\pm = \overline{\Phi} \cdot \overline{L}_1 \pm (M + 1/2) \) and \( B_\pm = \overline{\Phi} \cdot \overline{L}_2 \pm (N + 1/2) \). The other four integrals have the form
\[ \int_{\mathbb{T}^2} e^{2\pi i \overline{\Phi}(x, y) + \overline{L}(s, t)} D_M(s) D_N(t) ds dt, \]
each multiplied by an \( O(1) \) term. Here \( C = \overline{\Phi} \cdot \overline{L}_2 \) and \( D = \overline{\Phi} \cdot \overline{L}_1 \). These last four integrals will be uniformly bounded if we can establish a more robust estimate than (4); namely,
\[ S(\omega) := \sup_{\lambda, \rho \in \mathbb{R}, (x, t) \in \mathbb{T}^2} \left| p.v. \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i[\lambda \omega \cdot x + \rho t]} \frac{ds}{s} \right| < \infty \]
is a bounded function on \( \mathbb{S}^{d-1} \), together with an similar bound for the oscillatory integral with the roles of the variables \( s \) and \( t \) interchanged. Indeed we will show that \( \sup_{\omega \in \mathbb{S}^{d-1}} S(\omega) < \infty \) holds when the factorisation hypothesis (FH) holds on \( \mathbb{S}^{d-1} \) for \( \psi(s, t, \omega) = \omega \cdot \overline{\phi}(s, t) \). Furthermore, even when (FH) does not hold, we still have an estimate for the oscillatory integral in (6) which is uniform in the parameters \( \lambda, x, \tau \) and \( \rho \) but not necessarily in \( \omega \); that is, \( S(\omega) < \infty \) for all \( \omega \in \mathbb{S}^{d-1} \).
2.3. The work of C. Fefferman. In [5], Fefferman constructed a continuous function \( f \in C(\mathbb{T}^2) \) such that the Fourier series of \( f \) with respect to unrestricted rectangular summation diverges everywhere; more precisely, the limit of the rectangular Fourier partial sums \( S_{M,N} f(x,y) \) of \( f \) does not exist for any point \((x,y) \in \mathbb{T}^2\).

At the heart of the construction is the observation that if \( g_\lambda(s,t) = e^{2\pi i \lambda st}, \) then for large \( \lambda > 0 \) and every \((x,y) \in \mathbb{T}^2\) with \( x,y \neq 0, \)

\[
\sup_{M,N} |S_{M,N} g_\lambda(x,y)| = \sup_{M,N} \left| \int_{\mathbb{T}^2} e^{2\pi i \lambda(st-s-yt)} D_M(s) D_N(t) \, ds \, dt \right| \geq C \log \lambda
\]

for some \( C > 0. \) To see this, the integral above splits into four integrals plus some \( O(1) \) terms (as in (5) and (6)). The four integrals have the form

\[
\int_{\mathbb{T}^2} e^{2\pi i \lambda st + E_\pm s + F_\pm t} \frac{ds \, dt}{s \, t}
\]

where \( E_\pm = -\lambda y \pm (M + 1/2) \) and \( F_\pm = -\lambda x \pm (N + 1/2). \) Depending on the signs of \( \lambda y \) and \( \lambda x, \) we can choose positive integers \( M, N \) such that for one out of the four choices of \((\pm, \pm)\) we have \( E_\pm, F_\pm = O(1) \) and for the other three choices, either \( |E_\pm| \sim |\lambda| \) or \( |F_\pm| \sim |\lambda| \) (or both). For the integral with both \( E_\pm, F_\pm = O(1), \)

\[
\int_{\mathbb{T}^2} e^{2\pi i \lambda st + E_\pm s + F_\pm t} \frac{ds \, dt}{s \, t} = \int_0^{1/2} \int_0^{1/2} \sin(\lambda st) \frac{ds \, dt}{s \, t} + O(1).
\]

Changing variables \( \sigma = \lambda st \) in the \( s \) integral and interchanging the order of integration shows that the sine integral above is equal to \( c \log \lambda + O(1) \) as \( \lambda \to \infty. \) The other three integrals are easily seen to be \( O(1) \) for large \( \lambda. \)

Although the function \( g_\lambda(s,t) = e^{2\pi i \lambda st} \) can never be of the form \( e^{i \lambda \varphi(s,t)} \) since the phase \( st \) is not a periodic function of \((s,t), \) the above observation of Fefferman does give a strong indication that the estimates (2) may fail for the norms \( U_{rec}(\mathbb{T}^2). \)

Let us analyse a little further the relationship between the hyperbolic phase \( \lambda st \) appearing in Fefferman’s work and the phase \( \lambda \varphi(x-s,y-t) =: \lambda \psi(x-s,y-t,\omega) \)
arising from the norms \( U_{rec}(\mathbb{T}^2). \) Expanding \( \psi \) by its Taylor series at \((x,y), \)

\[
\psi(x+s,y+t,\omega) = \sum_{k,\ell=0}^{\infty} \frac{1}{k!\ell!} \partial^{k,\ell} \psi(x,y,\omega) s^k t^\ell.
\]

We see that the hyperbolic oscillation \( st \) arises when \( \psi_{st}(x,y,\omega) = \partial^{1,1} \psi(x,y,\omega) \) is nonzero. The only way that this term in the Taylor expansion vanishes for all points \((x,y,\omega) \in \mathbb{T}^2 \times S^{d-1} \) is when \( \psi(s,t,\omega) = \omega \cdot \overrightarrow{f}(s) + \omega \cdot \overrightarrow{h}(t) \) for a pair of \( d \)-tuple real-analytic functions \( \overrightarrow{f} \) and \( \overrightarrow{h} \) implying that the norm

\[
\| e^{2\pi i \overrightarrow{m} \overrightarrow{n}} \|_{U_{rec}(\mathbb{T}^2)} = \| e^{2\pi i \overrightarrow{m} \overrightarrow{n}} \|_{U(\mathbb{T})} \cdot \| e^{2\pi i \overrightarrow{m} \overrightarrow{n}} \|_{U(\mathbb{T})}
\]
factors into a product of two norms on \( U(\mathbb{T}) \) whose uniform boundedness is the content of Proposition 1.1. Here \( \overrightarrow{m}(s) = \overrightarrow{f}(s) + \tilde{L}_1 s \) and \( \overrightarrow{n}(t) = \overrightarrow{h}(t) + \tilde{L}_2 t. \) The example \( \overrightarrow{\varphi}(s,t) = \overrightarrow{f}(s) + \overrightarrow{h}(t) \) satisfies the factorisation hypothesis (FH) on \( S^{d-1} \) and so the fact that the associated mapping \( \Phi : \mathbb{T}^2 \to \mathbb{T}^d \) transports \( A(\mathbb{T}^d) \) to \( U_{rec}(\mathbb{T}^2) \) is covered by Theorem 1.3. Of course this also follows from two applications of Proposition 1.1.
Even if the hyperbolic term $\psi_{st} st$ arises, the other two quadratic terms $\psi_{a, s} s^2$ and $\psi_{tt} t^2$ can conspire against it to ameliorate its effect. For instance, while the oscillatory integral

$$p.v. \int_{T^2} e^{2\pi i \lambda t} \frac{ds dt}{s t} = c \log(\lambda) + O(1)$$

for large $\lambda > 0$ and some $c \neq 0$, we have the uniform bound

$$\sup_{\lambda \in \mathbb{R}} \left| p.v. \int_{T^2} e^{2\pi i \lambda (a x + b t)^2} \frac{ds dt}{s t} \right| < \infty$$

for any choice of $a, b$. Such conspiracy can happen for mappings $\Phi : \mathbb{T}^2 \to \mathbb{T}^d$ where the periodic part $\Phi(s, t) = \mathcal{F}(k s + \ell t)$ and $\mathcal{F}$ is a $d$-tuple of real-analytic, periodic functions of one variable and $k, \ell \in \mathbb{Z}$. Of course this family of examples satisfies the factorisation hypothesis (FH) and so Theorem 1.3 implies that $\Phi$ transports $A(\mathbb{T}^d)$ to $U_{rec}(\mathbb{T}^2)$ via composition. It is no longer the case that the positive result for this family of mappings follows from Proposition 1.1. In fact, its study was the inspiration to find the argument to establish the part of Theorem 1.3 when the factorisation hypothesis (FH) holds.

2.4. Proof of Proposition 1.1. Let us set $\psi(t, \omega) := \omega \cdot \phi(t)$ which is a real-valued, real-analytic function on the compact manifold $T \times S^{d-1}$. First suppose there exists a point $(t_0, \omega_0) \in T \times S^{d-1}$ such that $\partial_k^\omega \psi(t_0, \omega_0) = 0$ for all $k \geq 2$. Hence $\psi(t, \omega_0) = \psi(t, \omega_0) = \partial_t \psi(t_0, \omega_0)(t - t_0)$ for $t$ near $t_0$ and hence for all $t \in T$ since both sides of the equality are analytic functions on the connected manifold $T$. But $\psi(t, \omega_0)$ is periodic in $t$ and this forces $\partial_t \psi(t_0, \omega_0) = 0$ since otherwise the function $g(t) := \psi(t, \omega_0) + \partial_t \psi(t_0, \omega_0)(t - t_0)$ is not periodic in $t$. This implies that $\omega_0 \cdot \partial_t (t) \equiv \text{constant}$ and so one of the functions $\phi_j(t)$ in the $d$-tuple of analytic functions $\phi(t) = (\phi_1(t), \ldots, \phi_d(t))$ can be written as a linear combination of the other $d - 1$ functions which (after a simple re-writing of the oscillatory integral appearing in (4)) reduces matters to establishing (4) with $d$ replaced by $d - 1$.

Therefore by a simple induction on dimension argument, we may assume without loss of generality that for every point $(t, \omega) \in T \times S^{d-1}$, there is an integer $k = k(t, \omega) \geq 2$ such that $\partial_k^t \psi(t, \omega) \neq 0$. Furthermore there exists an $\epsilon = \epsilon(t, \omega) > 0$ such that

$$\left| \partial_k^t \psi(t', \omega') \right| \geq \frac{1}{2} \left| \partial_k^t \psi(t, \omega) \right|$$

for every $(t', \omega') \in B_\epsilon(t, \omega)$ in a ball of radius $2\epsilon$ centred at $(t, \omega)$. We cover $T \times S^{d-1}$ with balls $B_\epsilon$ of radius $\epsilon(t, \omega) > 0$ and use the compactness of $T \times S^{d-1}$ to extract a finite subcover $\{B_{\jmath} \}_{\jmath=0}^M$ of $T \times S^{d-1}$ and reduce matters to establishing a uniform bound for the oscillatory integral in (4) for parameters $\lambda, \rho \in \mathbb{R}$ and $(x, \omega) \in B_{\jmath}$ for some $0 \leq \jmath \leq M$. For such parameters, we write the integral in (4) as

$$p.v. \int_{\mathbb{T}} e^{i(\lambda \omega - \phi(x - t) + \rho t)} \frac{dt}{t} = p.v. \int_{|t| < \epsilon_\jmath} e^{i(\lambda \omega - \phi(x - t) + \rho t)} \frac{dt}{t} + O(\log(1/\epsilon_\jmath)).$$

For convenience we take $\jmath = 0$. The ball $B_0$ has radius $\epsilon_0$ with centre $(x_0, \omega_0)$, say, so that the parameters $(x, \omega)$ lie in the ball $B_{\epsilon_0}(x_0, \omega_0)$. Furthermore by (8), the
phase of the oscillatory integral \( \varphi(t) := \lambda \cdot \overline{\varphi}(x - t) + pt \) satisfies

\[
|\varphi^{(k_0)}(t)| \geq |\lambda| |\partial_{t_0}^{k_0} \psi(x_0, \omega_0)| / 2 =: c_0|\lambda|
\]

for all \(|t| < \epsilon_0\). Here \(k_0 = k_0(x_0, \omega_0) \geq 2\) is the integer which makes \(c_0\) above nonzero. By van der Corput’s lemma (see page 332 in [13]), together with an integration by parts argument, we see that

\[
\left| \int_{|\lambda|^{-1/k_0} \leq |t| \leq \epsilon_0} e^{i[\lambda \omega \cdot \overline{\varphi}(x-t) + pt]} \frac{dt}{t} \right| \leq C(k_0, c_0).
\]

We remark that ensuring \(k_0 \geq 2\) has two advantages. First the linear term \(pt\) in the phase \(\varphi(t)\) is killed off when we compute the \(k_0\) derivative (thus making it possible to obtain a bound from below on \(\partial_{t_0}^{k_0} \varphi(t)\)). And second, when employing van der Corput’s lemma we do not need to ensure the additional condition that \(\partial_t \varphi(t)\) is monotone (a task which could be difficult to achieve) which is needed when \(k_0 = 1\).

In the complimentary range \(|t| \leq |\lambda|^{-1/k_0}\), we compare the phase \(\varphi(t)\) with its \(k_0\) Taylor polynomial

\[
P_{k_0, x, \omega}^{\varphi}(t) = \sum_{\ell=0}^{k_0-1} \frac{1}{\ell!} \varphi^{(\ell)}(0) t^{\ell} = \varphi(0) + \varphi'(0) t + \lambda \sum_{\ell=2}^{k_0-1} \frac{(-1)^\ell}{\ell!} (\omega \cdot \overline{\varphi})^{(\ell)}(x) t^{\ell}
\]

so that

\[
|\varphi(t) - P_{k_0, x, \omega}^{\varphi}(t)| \leq \|\overline{\varphi}\|_{C^{k_0}(T)} |\lambda||t|^{k_0}
\]

and therefore

\[
\left| \int_{|t| \leq |\lambda|^{-1/k_0}} e^{i[\lambda \omega \cdot \overline{\varphi}(x-t) + pt]} \frac{dt}{t} \int_{|t| \leq |\lambda|^{-1/k_0}} e^{iP_{k_0, x, \omega}^{\varphi}(t)} \frac{dt}{t} \right| \leq C|\lambda| \int_0^{1/|\lambda|^{-1/k_0}} t^{k_0-1} dt \leq C
\]

where \(C = C(k_0, \overline{\varphi})\). Hence we are left with estimating the oscillatory integral

\[
\int_{|t| \leq |\lambda|^{-1/k_0}} e^{iP_{k_0, x, \omega}^{\varphi}(t)} \frac{dt}{t}
\]

(9)

where \(P_{k_0, x, \omega}^{\varphi}\) is a polynomial of degree at most \(k_0\). Here we can appeal to a result of Stein and Wainger [16] which states that for any \(d \geq 0\), there is a universal constant \(C_d\) such that for any real polynomial \(P(t) = a_d t^d + \cdots + a_1 t + a_0\) of degree at most \(d\), and for any \(0 \leq a < b\),

\[
\left| \int_{a \leq |t| \leq b} e^{i[a_d t^d + \cdots + a_1 t + a_0]} \frac{dt}{t} \right| \leq C_d.
\]

(10)

In particular the bound is uniform in the coefficients \(\{a_0, \ldots, a_d\}\) and the interval \(a \leq |t| \leq b\) of integration. This completes the proof (4) and therefore Proposition 1.1.

2.5. Developing a perspective. Before we move on to truly higher dimensional variants of Kahane’s problem, we pause here to make a few comments on the argument presented above and indicate some of the issues which lie ahead when we turn to mappings from \(T^2\) to \(T^d\).
First of all, matters are quickly reduced to proving a uniform bound for a certain integral, namely (4). This integral contains an oscillatory factor $e^{i\varphi(t)}$ with a real-valued phase $\varphi(t) = \lambda \omega \cdot \overline{\varphi}(x - t) + pt$ (depending on a number of parameters $\lambda$, $\omega$, $x$ and $p$) which is integrated against a Calderón-Zygmund kernel, the classical Hilbert transform kernel $1/t$. The task is to find enough cancellation in the integral to overcome the logarithmic divergence of the integral $\int 1/|t|$ and do so obtaining a bound which is uniform in all the parameters. Cancellation arises in two ways: (1) from rapid oscillation of $e^{i\varphi(t)}$ which is detected by some derivative of the phase $\varphi$ being large and (2) from the singular kernel $1/t$ which is detected by the fact that the Fourier transform of the principal-valued distribution $p.v.1/t$,

$$
(p.v.1/t)(\xi) = \text{p.v.} \int e^{-2\pi i \xi t} \frac{dt}{t} \quad (11)
$$

defines a bounded function in $\xi \in \mathbb{R}$. Equivalently, convolution with $p.v.1/t$, the classical Hilbert transform, is bounded on $L^2(\mathbb{R})$.

These two types of cancellation often compliment each other perfectly giving us a successful bound on an oscillatory integral. For instance, for the oscillatory integral in (4), we now give a slightly altered argument, using only the types of cancellation described above (and in particular, avoiding the robust result of Stein and Wainger (10)). We begin as before and first take care of the degenerate case to reduce ourselves to the situation where there is a nonzero derivative $\partial^k_x \psi$ of $\psi$ at every point $(x, \omega) \in \mathbb{T} \times S^{d-1}$.

Now suppose we chose the integer $k = k(x, \omega) \geq 2$ to be minimal with respect to the property that $\partial^k_x \psi(x, \omega) \neq 0$. Then the $k$th Taylor polynomial

$$
P^k_{x,\omega}(t) = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \varphi^{(\ell)}(0) t^\ell = \varphi(0) + \varphi'(0) t
$$

of $\varphi(t) = \lambda \omega \cdot \overline{\varphi}(x - t) + pt$ is a linear polynomial. For this fixed point $(x, \omega)$ we could choose $\delta = \delta(x, \omega) > 0$ such that $|\partial^k_x \varphi(t)| \geq (1/2)|\lambda||\partial^k_x \psi(x, \omega)|$ for all $|t| < \delta$. Hence as above, using the fact that the $k$th derivative of the phase $\varphi$ is large, we have the bound

$$
\left| \int_{|t| \leq |\lambda|^{-1/k}} e^{i|\lambda| \omega \cdot \overline{\varphi}(x-t) + pt} \frac{dt}{t} \right| \leq C(k, c) + O(\log(1/\delta))
$$

where $c = |\partial^k_x \varphi(x, \omega)| > 0$. For the complimentary integral, we have

$$
\int_{|t| \leq |\lambda|^{-1/k}} e^{i|\lambda| \omega \cdot \overline{\varphi}(x-t) + pt} \frac{dt}{t} = 
$$

$$
\int_{|t| \leq |\lambda|^{-1/k}} \left[ e^{i|\lambda| \omega \cdot \overline{\varphi}(x-t) + pt} - e^{iP^k_{x,\omega}(t)} \right] \frac{dt}{t} + A \int_{|t| \leq |\lambda|^{-1/k}} e^{i\xi t} \frac{dt}{t} =: I + II
$$

where $A = e^{i\varphi(0)} = e^{i\lambda \omega \cdot \overline{\varphi}(x)}$ and $\xi = \varphi'(0)$. The first term satisfies $|I| \leq 2 \|\overline{\varphi}\|_{C^k(\mathbb{T})}$ as before and $II$ is uniformly bounded by a slightly more robust version of (11). So we have managed to estimate (4) without appealing to the result (10) of Stein and Wainger.
Although we obtain in this way a bound which is uniform in the parameters \( \lambda, \rho \in \mathbb{R} \), the bound depends on the point \((x, \omega) \in \mathbb{T} \times \mathbb{S}^{d-1}\). In fact the bound depends on \(k, \delta^{-1}\) and \(e^{-1}\) which all depend on \((x, \omega)\). In particular, if we slightly perturb \((x, \omega)\) to \((x', \omega')\), the associated quantities \( \delta \) and \( c \) could vanish; we have made no provisions to prevent this from happening. This is why in the original argument, we employed a compactness argument to stabilise the integer \( k \) and most importantly, we stabilised the derivative bound of the phase from below. This is the content of (8). However we are unable to stabilise the minimality property of \( k \) used in the above, alternative argument. The minimality property of \( k \) is highly unstable. If we perturb slightly the point \((x, \omega)\) to \((x', \omega')\), we can still achieve a bound from below on \( \partial_{t}^{k}(\omega' \cdot \vec{\phi})(x' - t) \) with \( k = k(x, \omega) \) BUT it is no longer the case that all lower order derivatives \( \partial_{t}^{\ell}(\omega' \cdot \vec{\phi})(x') \) vanish for \( 2 \leq \ell \leq k - 1 \). Generically, these derivatives will become nonzero.

For these reasons we are led to employ the result of Stein and Wainger in (10). One way to view (10) is to observe that the \( L^2 \) boundedness of convolution with the distribution \( p.v. 1/t \) (this is the content of (11)) remains true if we multiply the Hilbert transform kernel \( 1/t \) by any polynomial oscillation; that is, convolution with \( p.v. e^{itP(t)}/t \) is bounded on \( L^2 \) and moreover, the \( L^2 \) operator norm is independent of the coefficients of \( P \), depending only on the degree of \( P \). This is precisely the content of (10).\(^2\) This observation has been extended to higher dimensions by Stein in [14] and by Ricci and Stein [11] to the nontranslation-invariant setting. This extension of Calderón-Zygmund theory, the stability of \( L^p \) mapping properties of classical Calderón-Zygmund singular integral operators by multiplication of the CZ kernel by a general polynomial oscillation is intimately connected to the theory of Singular Radon Transforms, see [11]. Hence there is an interesting connection between Kahane’s problem and the theory of Singular Radon Transforms – these matters are discussed in a nice survey paper by Wainger, [18]. The connections remain when we move to higher dimensions but in our case, we will need to move from one parameter to multiparameter Calderón-Zygmund theory.

When we move to higher dimensions \((k = 2)\), the \( U_{rec}(\mathbb{T}^2) \) and \( U_{sq}(\mathbb{T}^2) \) norms involve a product of Dirichlet kernels \( D_{M}(s)D_{N}(t) \) which will lead to two dimensional variants of integrals in (4) but now we will be integrating an oscillation against a product-type Calderón-Zygmund kernel which is in fact the quintessential product-type CZ kernel, the double Hilbert transform kernel \( 1/st \). We will be confronted with oscillatory integrals of the form

\[
p.v. \int_{\mathbb{T}^2} \frac{e^{i\psi(s,t)}}{s} \frac{ds \ dt}{t}
\]

where \( \psi \) is some real-valued, real-analytic phase depending on several parameters and we will need to obtain bounds for this integral which are uniform with respect to these parameters. Again one of these parameters will be a point \( \omega \in \mathbb{S}^{d-1} \) on the unit sphere and so \( \psi(s,t) = \psi_{\omega}(s,t) \). Here the nondegenerate case is when for all points \((s,t,\omega) \in \mathbb{T}^2 \times \mathbb{S}^{d-1}\), there are integers \( m = m(s,t,\omega) \geq 2 \) and \( n = n(s,t,\omega) \geq 2 \) such that both derivatives \( \partial_{t}^{(m)} \psi_{\omega}(s,t) \) and \( \partial_{s}^{(n)} \psi_{\omega}(s,t) \) are

\(^2\)The bound (10) in fact says a little more - this \( L^2 \) boundedness remains true after truncation of the kernel \( e^{itP(t)}/t \) to an interval \( a \leq |t| \leq b \) with bounds uniform in the truncation.
nonzero. This will be much more difficult to achieve (and in fact we will not be able to get ourselves exactly in this situation).

Once in the nondegenerate case (or some variant), we can use a compactness argument to stabilise the integers \( m \) and \( n \) and corresponding derivative bounds from below. The minimality of \( k = k(x, \omega) \) from the discussion above is replaced by the so-called *Newton diagram* of the phase function \( \psi \), based at some point \((x, y, \omega) \in \mathbb{T}^2 \times S^{d-1}\). Newton diagrams are highly unstable and can dramatically change under small perturbations of the phase function. Proceeding in some analogous way to the proof of Proposition 1.1 given above, we will be led to enquire whether multiparameter Calderón-Zygmund theory (at least the part of the theory concerned with \( L^2 \) mapping properties of convolution with product-type Calderón-Zygmund kernels) is stable under multiplication of the kernel by general polynomial oscillations. Fortunately this has already been considered in the context of singular Radon transforms (see [3] and [17]) but unfortunately the answer is NO in general. The theory is not stable under multiplication by general polynomial oscillations and part of our endeavour here is to understand to what extent (or for which polynomial oscillations) it is stable. This explains in part why there will be many analytic mappings \( \Phi : \mathbb{T}^2 \to \mathbb{T}^d \) which do NOT carry \( A(\mathbb{T}^d) \) to \( U_{\text{rect}}(\mathbb{T}^2) \).

3. Consequences of the factorisation hypothesis (FH)

Let \( M \) be an real-analytic manifold. Suppose that \( \psi \) is an analytic function on \( \mathbb{T}^2 \times M \) which satisfies the factorisation hypothesis (FH) on \( M \); that is every point of \( \mathbb{T}^2 \times M \) has a neighbourhood \( U \) on which

\[
\partial_{s,t,1}^1 \psi(s, t, \omega) = L(s, t, \omega) \partial_{s}^1 \psi(s, t, \omega) \quad \text{and} \quad \partial_{s,t,1}^1 \psi(s, t, \omega) = K(s, t, \omega) \partial_{s}^1 \psi(s, t, \omega)
\]

for some real-analytic functions \( L \) and \( K \) on \( U \). From these two relations on \( U \), it is a simple matter to see that every partial derivative \( \partial^{k,\ell}_s \psi(s, t) \) with \( k, \ell \geq 1 \), can be written as some linear combination of the pure derivatives \( \partial_s^1 \psi \) and \( \partial_t^1 \psi \) with coefficients depending on \( L \) and \( K \).

**Lemma 3.1.** For every \( k, \ell \geq 1 \), we have

\[
\partial^{k,\ell}_s \psi = \sum_{j=2}^{k+\ell} Q^{k,\ell}_j \partial^j_t \psi
\]

on \( U \). Furthermore we have the following relationships.

\[
\begin{align*}
\text{j} = 2 & \quad Q^{k,1}_2 = \partial_s Q^{k-1,1}_2 + \sum_{j=2}^{k} Q^{k-1,1}_j \partial^j_t L \\
3 \leq j \leq k & \quad Q^{k,1}_j = \partial_s Q^{k-1,1}_j + \sum_{r=j-1}^{k} \binom{r-1}{j-2} Q^{k-1,1}_r \partial^{j-1}_t L \\
\text{j} = k+1 & \quad Q^{k,1}_{k+1} = L^k.
\end{align*}
\]
When \( k = 1 \), the three formulae above collapse to a single formula; namely \( Q_2^{1,1} = L \). When \( \ell \geq 2 \), we have

\[
\begin{align*}
\dot{j} &= 2 & Q_2^{k,\ell} &= \partial_t Q_2^{k,\ell-1} \\
3 \leq j \leq k + \ell - 1 & \quad Q_j^{k,\ell} = \partial_t Q_j^{k,\ell-1} + Q_j^{k,\ell-1} \\
\dot{j} = k + \ell & \quad Q_{k+\ell}^{k,\ell} = L^k.
\end{align*}
\]

We also have \( \partial_t^{k,\ell} \psi = \sum_{j=2}^{k+\ell} R_j^{k,\ell} \partial_t^j \psi \) where the coefficients \( R_j^{k,\ell} \) satisfy similar relations.

**Proof** First we observe that by differentiating \( j - 1 \) times the factorisation hypothesis \( \partial^{1,1} \psi = L \partial_t^2 \psi \) with respect to \( t \), we have the useful formula

\[
\partial^{1,j} \psi = \sum_{r=2}^{j+1} \binom{j-1}{r-2} \partial_t^{j+1-r} L \partial_t^r \psi.
\]  

(14)

This can be verified by a simple induction argument on \( j \).

Next we will establish (12) and we will do so by induction on \( k \). The case \( k = 1 \) follows immediately from the factorisation hypothesis \( \partial^{1,1} \psi = L \partial_t^2 \psi \) and so \( Q_2^{1,1} = L \). Suppose now that (12) holds for some \( k - 1 \geq 1 \). In particular \( \partial_t^{k-1,1} \psi = \sum_{j=2}^{k-1} Q_j^{k-1,1} \partial_t^j \psi \) and by differentiating this relation with respect to \( s \), we see that

\[
\partial_t^{k,1} \psi = \sum_{j=2}^{k} \left[ \partial_s Q_j^{k-1,1} \partial_t^1 \psi + Q_j^{k-1,1} \partial_t^{1,j} \psi \right]
\]

Applying the useful formula (14) to the second term, we have

\[
\sum_{j=2}^{k} Q_j^{k-1,1} \partial_t^{1,j} \psi = \sum_{j=2}^{k} Q_j^{k-1,1} \sum_{r=2}^{j+1} \binom{j-1}{r-2} \partial_t^{j+1-r} L \partial_t^r \psi
\]

\[
= \partial_t^2 \psi \sum_{j=2}^{k} Q_j^{j-1,1} \partial_t^{-1} L + \sum_{r=3}^{k+1} \left[ \sum_{j=r-1}^{k} \binom{j-1}{r-2} Q_j^{k-1,1} \partial_t^{j+1-r} L \right] \partial_t^r \psi
\]

and therefore \( \partial_t^{k,1} \psi = \sum_{j=2}^{k+1} Q_j^{k,1} \partial_t^j \psi \) where the \( \{Q_j^{k,1}\}_{j=1}^{k+1} \) satisfy the three relations given in (12). Note that in particular, the above gives \( Q_{k+1}^{k,1} = Q_k^{k-1,1} L \) and iterating this \( k \) times, we obtain \( Q_{k+1}^{k,1} = L^k \).

Let us now turn to (13) which we will establish by induction on \( \ell \). Suppose that

\[
\partial_t^{k,\ell-1} \psi = \sum_{j=2}^{k+\ell-1} Q_j^{k,\ell-1} \partial_t^j \psi
\]

holds for some \( \ell - 1 \geq 1 \) and all \( k \geq 1 \). Differentiating this identity with respect to \( t \) gives us

\[
\partial_t^{k,\ell} \psi = \sum_{j=2}^{k+\ell-1} \left[ \partial_t Q_j^{k,\ell-1} \partial_t^j \psi + Q_j^{k,\ell-1} \partial_t^{j+1} \psi \right] = \partial_t Q_2^{k,\ell-1} \partial_t^2 \psi
\]
\[ + \sum_{j=3}^{k+\ell-1} \left[ \partial_j Q_{j-1}^{k,\ell-1} \partial_j^2 \psi + Q_{j-1}^{k,\ell-1} \partial_j^3 \psi \right] + Q_{k+\ell-1}^{k,\ell} \partial_{k+\ell}^2 \psi. \]

Therefore \( \partial^{k,\ell} \psi = \sum_{j=1}^{k+\ell} Q_j^{k,\ell} \partial_j^2 \psi \) where \( Q_j^{k,\ell} \) satisfy the relations (13).

To be clear, for the final relation in (13), we have \( Q_k^{k,\ell} = Q_{k+\ell-1}^{k,\ell-1} \) and upon iterating we obtain \( Q_k^{k,\ell} = Q_{k+1}^{k,1} = L_k \) by (12).

As an immediate consequence of Lemma 3.1, we have the following observation which will be useful to us in the next section when we turn to developing a more robust one dimensional theory.

**Corollary 3.2.** Let \( \psi \) be a real-analytic function on \( \mathbb{T}^2 \times M \) satisfying the factorisation hypothesis (FH) on \( M \). Suppose that there exists a point \( (x_0, y_0, \omega_0) \in \mathbb{T}^2 \times M \) such that

\[ \partial_t^m \psi(x_0, y_0, \omega_0) = 0 \quad \text{for all } m \geq 2. \]

Then \( \psi(x, y, \omega_0) \equiv \psi(x, y_0, \omega_0) \).

**Proof** Lemma 3.1 implies \( \partial^{k,\ell} \psi(x_0, y_0, \omega_0) = 0 \) for all \( k, \ell \geq 1 \). Hence

\[ \psi(x, y, \omega_0) = \sum_{k, \ell \geq 0} \frac{1}{k! \ell!} \partial^{k,\ell} \psi(x_0, y_0, \omega_0)(x-x_0)^k (y-y_0)^\ell \]

\[ = \psi(x, y_0, \omega_0) + \partial_t \psi(x, y_0, \omega_0)(y-y_0) \]

for \( (x, y) \) near \( (x_0, y_0) \) and hence for all \( (x, y) \) by analyticity. However we also have

\[ \psi(x_0, y, \omega_0) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \partial^{\ell} \psi(x_0, y_0, \omega_0)(y-y_0)^\ell = \psi(x_0, y_0, \omega_0) + \partial_t \psi(x_0, y_0, \omega_0)(y-y_0) \]

for \( y \) near \( y_0 \) and hence for all \( y \) by analyticity. This forces \( \partial_t \psi(x_0, y_0, \omega_0) = 0 \) since \( \psi(x_0, y, \omega_0) \) is a periodic function in \( y \). Plugging this back into (15) shows \( \psi(x, y, \omega_0) = \psi(x, y_0, \omega_0) \), completing the proof of the corollary.

Another immediate consequence of Lemma 3.1 which is needed in the next key proposition is the following.

**Corollary 3.3.** Let \( \psi \) be a real-analytic function on \( \mathbb{T}^2 \times M \) satisfying the factorisation hypothesis (FH) on \( M \). Suppose that there is a point \( (x, y, \omega) \in \mathbb{T}^2 \times M \) and a pair of integers \( m, n \geq 2 \) such that

\[ \partial_k^m \psi(x, y, \omega) = 0 \quad \text{for } 2 \leq k < n \quad \text{and} \quad \partial_\ell^m \psi(x, y, \omega) = 0 \quad \text{for } 2 \leq \ell < m. \]

If \( m \) or \( n \) is equal to 2, the corresponding condition above is vacuous. Then \( \partial^{k,\ell} \psi(x, y, \omega) = 0 \) for all \( k, \ell \geq 1 \) satisfying \( k + \ell < \max(m, n) \).

Before we give the proof, let us see how to interpret this corollary if \( m = n = 2 \). In this case, \( \max(m, n) = 2 \) and there are no pairs of integers \( k, \ell \geq 1 \) satisfying the condition \( k + \ell < 2 \). Hence the conclusion is vacuous and there is nothing to prove in this case.
Proof If \( \max(m, n) = m \) and \( k, \ell \geq 1 \) satisfies \( k + \ell < m \), then
\[
\partial^{k,\ell} \psi(x, y, \omega) = \sum_{j=2}^{k+\ell} Q_{j}^{k,\ell} \partial_{j} \psi(x, y, \omega) = 0.
\]
Also if \( \max(m, n) = n \) and \( k, \ell \geq 1 \) satisfies \( k + \ell < n \), then
\[
\partial^{k,\ell} \psi(x, y, \omega) = \sum_{j=2}^{k+\ell} R_{j}^{k,\ell} \partial_{j} \psi(x, y, \omega) = 0.
\]
This completes the proof of the corollary.

At the heart of the proof of Theorem 1.3 when we are assuming that the factorisation hypothesis holds is the following proposition.

Proposition 3.4. Let \( \psi \) be a real-analytic function on \( \mathbb{T}^2 \times M \) satisfying the factorisation hypothesis (FH) on \( M \). Suppose at a point \( (x, y, \omega) \in \mathbb{T}^2 \times M \), there exist integers \( m, n \geq 2 \) such that \( \partial_{m} \psi(x, y, \omega) \neq 0 \), \( \partial_{n} \psi(x, y, \omega) \neq 0 \) but \( \partial_{k} \psi(x, y, \omega) = 0 \) for \( 2 \leq k < n \) and \( \partial_{\ell} \psi(x, y, \omega) = 0 \) for \( 2 \leq \ell < m \).

Fix \( r \geq 0 \) and suppose \( m + r < n \). Then \( \partial^{k,\ell} L(x, y, \omega) = 0 \) for all nonnegative \( k, \ell \) satisfying \( k + \ell \leq r \).

Let us try to get some feeling for what this proposition says in the case \( r = 0 \). In this case the condition \( m + r < n \) simply says \( m < n \). The only pair of nonnegative integers \( k, \ell \) satisfying \( k + \ell \leq 0 \) is \( k = \ell = 0 \) and so we would like to conclude that \( L(x, y, \omega) = 0 \) under our conditions on \( \partial_{m} \psi \) and \( \partial_{n} \psi \) at the point \( (x, y, \omega) \).

Corollary 3.3 implies that \( \partial^{k,\ell} \psi(x, y, \omega) = 0 \) for all integers \( k, \ell \geq 1 \) satisfying \( k + \ell < \max(m, n) = n \). In particular if we choose \( k = 1 \) and \( \ell = m - 1 \geq 1 \) (since \( m \geq 2 \)), then \( \partial^{1,m-1} \psi(x, y, \omega) = 0 \) since \( k + \ell = 1 + m - 1 = m < n = \max(m, n) \).

But on the other hand, by Lemma 3.1, we have
\[
\partial^{1,m-1} \psi(x, y, \omega) = \sum_{j=2}^{m} Q_{j}^{1,m-1} \partial_{j} \psi(x, y, \omega) = Q_{m}^{1,m-1} \partial_{m} \psi(x, y, \omega)
\]
since we are assuming \( \partial_{j} \psi(x, y, \omega) = 0 \) for all \( 2 \leq j < m \). Since \( Q_{m}^{1,m-1} = L \) by (12) and (13), we have
\[
0 = \partial^{1,m-1} \psi(x, y, \omega) = L(x, y, \omega) \partial_{m} \psi(x, y, \omega)
\]
and this forces \( L(x, y, \omega) = 0 \) since we are assuming \( \partial_{m} \psi(x, y, \omega) \neq 0 \).

Unfortunately matters become more complicated for larger values of \( r \) and it will be necessary to have some knowledge of the coefficients \( Q_{j}^{k,\ell} \) which are clearly certain polynomials in \( L \) and the various derivatives \( D^{\beta} L = \partial^{\beta_{1},\beta_{2}} L \) of \( L \). In fact we will need to have a fairly good understanding about the structure of these polynomials.
We organise the polynomial $Q_j^{k,\ell}$ in terms of powers of $L$; it is clear that the largest power of $L$ which arises is $L^k$. We write

$$Q_j^{k,\ell} = \sum_{r=0}^{k} P_{r}^{k,\ell,j} L^{k-r}$$

where $P_{r}^{k,\ell,j}$ is a polynomial whose variables are the derivatives of $L$:

$$Y_{\beta} = D^{\beta} L = \partial^{\beta_1,\beta_2} L, \quad \beta = (\beta_1, \beta_2) \text{ with } |\beta| := \beta_1 + \beta_2 \geq 1.$$  

For convenience we enumerate the variables $\{Y_{\beta}\} = \{X_j\}_{j \geq 1}$ sequentially; namely, $X_j = Y_{\beta_j}$ where

$\beta_1 = (1, 0), \beta_2 = (0, 1), \beta_3 = (2, 0), \beta_4 = (1, 1), \beta_5 = (0, 2), \beta_6 = (3, 0),$ etc...

Precisely, for each $n \geq 2$ and $n(n - 1)/2 \leq j < n(n + 1)/2$, we define

$$\beta_j = \left(\frac{(n-1)(n+2)}{2} - j, j - \frac{n(n-1)}{2}\right).$$

Then $P_{r}^{k,\ell,j} \in \mathbb{Z}[X_1, X_2, X_3, \ldots]$. We will use the multi-index notation $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of length $n$ and size $|\alpha| := \alpha_1 + \cdots + \alpha_n$. It turns out that the polynomials

$$P_{r}^{k,\ell,j} = \sum_{\alpha} c_{\alpha}^{k,\ell,j,r} X^\alpha$$

arising as the coefficients of $L^{k-r}$ in the expression for $Q_j^{k,\ell}$ possess two different kinds of homogeneity. To formulate this we introduce two weights defined on each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$; namely,

$$h(\alpha) := \sum_{u=1}^{n} \alpha_u |\beta_u| \quad \text{and} \quad w(\alpha) := \sum_{u=1}^{n} \alpha_u (\beta_u^1 + 1).$$

Here $\beta_u = (\beta_u^1, \beta_u^2) \in \mathbb{N}^2$ with $|\beta_u| = \beta_u^1 + \beta_u^2$.

It will be helpful to use the notation $e^q = (0, 0, \ldots, 1, 0, \ldots)$ to denote the multi-index of any length with all zeros except for 1 in the $q$th place.

The following proposition contains the structural information we need about the polynomials $Q_j^{k,\ell}$ in order to carry out the proof of Proposition 3.4.

**Proposition 3.5.** Each coefficient $Q_j^{k,\ell}$ arising in Lemma 3.1 can be written as

$$\sum_{r=0}^{k} P_{r}^{k,\ell,j} L^{k-r} \quad \text{where} \quad P_{r}^{k,\ell,j} = \sum_{\alpha: w(\alpha) = r} c_{\alpha}^{k,\ell,j,r} X^\alpha \in \mathbb{Z}[X_1, X_2, \ldots].$$

Furthermore for any $k, \ell \geq 1$ and $2 \leq j \leq \ell + 1$, the coefficient $c_{\alpha}^{k,\ell,j,k}$ of $X^\alpha = \partial^{k-1,\ell+1-j} L$ is strictly positive.

**Proof** The proof will proceed by induction utilising the relations among the $Q_j^{k,\ell}$ described in (12) and (13). With this in mind we will need to understand the effect of certain operations on any particular monomial

$$X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} = Y_{\beta_1}^{\alpha_1} Y_{\beta_2}^{\alpha_2} \cdots Y_{\beta_n}^{\alpha_n},$$

where
namely, $\partial_{s}X^{\alpha}, \partial_{t}X^{\alpha}, (\partial_{t}L)X^{\alpha}$ and $(\partial{s}L)X^{\alpha}$.

$(\partial_{s}L)X^{\alpha}$: Since $\partial_{s}L = Y_{\beta_{1}}$, we have $(\partial_{s}L)X^{\alpha} = X^{\tilde{\alpha}}$ where
\[ \tilde{\alpha} = \alpha + e^{1} \text{ and } h(\tilde{\alpha}) = h(\alpha) + 1, \quad w(\tilde{\alpha}) = w(\alpha) + 2. \] (16)

$(\partial_{t}L)X^{\alpha}$: Since $\partial_{t}L = Y_{\beta_{u}(v)}$ where $q(v) = (v+1)(v+2)/2 - 1$ so that $\beta_{q} = (0, v)$, we have $(\partial_{t}L)X^{\alpha} = X^{\alpha^{*}}$ where
\[ \alpha^{*} = \alpha + e^{q(v)} \text{ and } h(\alpha^{*}) = h(\alpha) + v, \quad w(\alpha^{*}) = w(\alpha) + 1. \] (17)

$\partial_{t}X^{\alpha}$: We have
\[ \partial_{t}X^{\alpha} = \sum_{u=1}^{n} \alpha_{u}X^{\alpha_{u}} \text{ and } X^{\alpha^{*}} = Y_{\beta_{1}}^{\alpha_{1}} \cdots (Y_{\beta_{u}}^{\alpha_{u}-1}Y_{\beta_{u+1}}^{1}) \cdots Y_{\beta_{n}}^{\alpha_{n}} \] (18)
where $\alpha^{u} = (\alpha_{1}, \ldots, \alpha_{u}-1, \ldots, \alpha_{n}) + e^{q^{u}}$ and $q^{u} = q_{u}(u)$ satisfies $\beta_{q_{u}} = (\beta_{u}, \beta_{u}^{2} + 1)$. Hence
\[ h(\alpha^{u}) = \sum_{v \neq u} \alpha_{v}|\beta_{v}| + (\alpha_{u} - 1)|\beta_{u}| + |\beta_{u}| + 1 = h(\alpha) + 1 \]
and
\[ w(\alpha^{u}) = \sum_{v \neq u} \alpha_{v}(\beta_{v}^{1} + 1) + (\alpha_{u} - 1)(\beta_{u}^{1} + 1) + \beta_{u}^{1} + 1 = w(\alpha). \]

$\partial_{s}X^{\alpha}$: We have
\[ \partial_{s}X^{\alpha} = \sum_{u=1}^{n} \alpha_{u}X^{\tilde{\alpha}_{u}} \text{ and } X^{\tilde{\alpha}} = Y_{\beta_{1}}^{\tilde{\alpha}_{1}} \cdots (Y_{\beta_{u}}^{\alpha_{u}-1}Y_{\beta_{u+1}}^{1}) \cdots Y_{\beta_{n}}^{\alpha_{n}} \] (19)
where $\tilde{\alpha}^{u} = (\alpha_{1}, \ldots, \alpha_{u}-1, \ldots, \alpha_{n}) + e^{q^{u}}$ and $q^{u}$ satisfies $\beta_{q^{u}} = (\beta_{u}^{1} + 1, \beta_{u}^{2})$. Hence
\[ h(\tilde{\alpha}^{u}) = \sum_{v \neq u} \alpha_{v}|\beta_{v}| + (\alpha_{u} - 1)|\beta_{u}| + |\beta_{u}| + 1 = h(\alpha) + 1 \]
and
\[ w(\tilde{\alpha}^{u}) = \sum_{v \neq u} \alpha_{v}(\beta_{v}^{1} + 1) + (\alpha_{u} - 1)(\beta_{u}^{1} + 1) + \beta_{u}^{1} + 2 = w(\alpha) + 1. \]

Let us now begin the induction argument and we start with $\ell = 1$.

The case $\ell = 1$: Here we want to show that for $2 \leq j \leq k + 1$,
\[ Q_{j}^{k,1} = \sum_{r=0}^{k} P_{r}^{k,1,j} L^{k-r} \text{ where } P_{r}^{k,1,j} = \sum_{\alpha: w(\alpha) = r} c_{\alpha}^{k,1,j,r} X^{\alpha} \] (20)
and when $j = 2$, the coefficient $c_{\alpha}^{k,1,2,k}$ of $X^{\alpha} = \partial_{s}^{k-1} L$ is strictly positive. We will do this by induction on $k$.

$k = 1$: Since $Q_{2}^{1,1} = L$, we see (20) holds and that the coefficient $c_{\alpha}^{1,1,2,1}$ of $X^{\alpha} = L$ is equal to 1.
Suppose now that (20) holds for some \( k' = k - 1 \geq 1 \) and any \( 2 \leq j \leq k' + 1 \).

From (12), see see that when \( 2 \leq j \leq k \), \( Q_j^{k,1} \) is equal to \( \partial_s Q_j^{k-1,1} \) plus a sum of terms of the form \( Q_h^{k-1,1} (\partial_s^i L) \) for \( 2 \leq h \leq k \). Since by induction we are assuming (20) holds for \( Q_j^{k-1,1} \) when \( 2 \leq j \leq k' + 1 = k \), we have

\[
\partial_s Q_j^{k-1,1} = \sum_{r=0}^{k-1} \left[ \partial_s (P_r^{k-1,1,j}) L^{k-1-r} + (k - 1 - r) P_r^{k-1,1,j} L^{k-2-r} (\partial_s L) \right].
\]

This sum splits into two parts. Using (19), we see that the first part is equal to

\[
\sum_{r=0}^{k-1} \sum_{\alpha: w(\alpha^*) = r+1} \alpha_u c^{k-1,1,j,r}_\alpha X^{\tilde{\alpha}^*} \sum_{r=0}^{k} \sum_{\alpha: w(\alpha) = r} \alpha_u c^{k-1,1,j,r-1}_\alpha X^{\tilde{\alpha}^*} L^{k-r}
\]

which is of the form expressed in (20). Furthermore when \( j = 2 \) and \( r = k \), the coefficient of \( \partial_s^{k-1} L \) in the square bracket expression is \( c^{k-1,1,2,k-1}_\alpha \) where \( q = q(k-1,1,2) \) is the integer satisfying \( \beta_q = (k - 2, 0) \). In this case, \( k \geq 3 \) necessarily and this part of \( \partial_s Q_j^{k-1,1} \) does not arise when \( k = j = 2 \) and \( \ell = 1 \). The coefficient \( c^{k-1,1,2,k-1}_\alpha \) is strictly positive by our induction hypothesis.

Using (16), we see that the second part is equal to

\[
\sum_{r=0}^{k-1} (k-1-r) \sum_{\alpha: w(\alpha) = r+1} c^{k-1,1,j,r}_\alpha \sum_{r=0}^{k} \sum_{\alpha: w(\alpha^*) = r} c^{k-1,1,j,r-2}_\alpha X^{\tilde{\alpha}^*} L^{k-r}
\]

which is of the form expressed in (20). When \( j = 2 \) and \( r = k \), the only way \( \tilde{\alpha}^* = \alpha + e^1 \) can equal \( e^{q(k,1,2)} \) is when \( \alpha = 0 \) and \( k = 2 \) and in this case, since \( \partial_s Q_2^{1,1} = \partial_s L \), the coefficient of \( \partial_s L \) in the above expression is \( c_0^{1,1,2,0} = 1 \) which is strictly positive.

Altogether we see that \( \partial_s Q_j^{k-1,1} \) is of the form in (20). Furthermore when \( j = 2 \) and \( r = k \), the coefficient of \( \partial_s^{k-1} L \) is strictly positive.

Using (17) we have \( Q_h^{k-1,1} (\partial_s^i L) = \)

\[
\sum_{r=0}^{k-2} \left[ \sum_{\alpha: w(\alpha^*) = r+1} c^{k-1,1,j,r}_\alpha \sum_{r=0}^{k} \sum_{\alpha: w(\alpha) = r} c^{k-1,1,j,r-1}_\alpha X^{\tilde{\alpha}^*} \right] L^{k-r}
\]

and we see that the term \( X^{\alpha^*} = Y_{(k-1,0)} = \partial_t^{k-1} L \) does not arise since each \( \alpha^* = \alpha + e^{q(\alpha)} \) is not of the form \( e^q \) with \( \beta_q = (k - 1, 0) \).

Since \( Q_h^{k-1,1} \) for \( 2 \leq j \leq k \) is equal to \( \partial_s Q_j^{k-1,1} \) plus a sum of terms of the form \( Q_h^{k-1,1} (\partial_s^i L) \) we see that \( Q_j^{k,1} \) satisfies (20) and the coefficient \( c^{k,1,2,k}_\alpha \) of \( X^{\alpha} = \partial_t^{k-1} L \) is strictly positive.
It remains to consider the case \( j = k + 1 \) but here \( Q_{k+1,1}^j = L^k \) and so (20) clearly holds and there is no coefficient of \( X^\alpha = \partial_{\ell}^{k-1} L \) to consider.

**The cases \( \ell \geq 2 \):** Here we will show that for any \( \ell \geq 2, k \geq 1 \) and \( 2 \leq j \leq k + \ell \),

\[
Q_j^{k,\ell} = \sum_{r=0}^{k} P_r^{k,\ell,j} L^{k-r} \quad \text{where} \quad P_r^{k,\ell,j} = \sum_{\alpha: w(\alpha) = r} \sum_{h(\alpha) = k+\ell-j} c_{\alpha}^{k,\ell,j,r} X^\alpha
\]

(21)

and when \( j \leq \ell + 1 \), the coefficient \( c_{\alpha}^{k,\ell,j,k} \) of \( X^\alpha = \partial_{k-1,\ell+1-j} L \) is strictly positive.

We will do this by induction on \( \ell \) and assume that (21) holds for all \( 1 \leq \ell' \leq \ell - 1, k \geq 1 \) and \( 2 \leq j \leq k + \ell' \). Also we assume that the coefficient of \( X^\alpha = \partial_{k-1,\ell+1-j} L \) is strictly positive.

When \( 2 \leq j \leq k + \ell - 1 \), the relations in (13) express \( Q_j^{k,\ell} \) in terms of \( \partial_{\ell} Q_j^{k,\ell-1} \) which puts us in a position to use the induction hypothesis and write

\[
\partial_{\ell} Q_j^{k,\ell-1} = \sum_{r=0}^{k} \left[ \partial_{\ell}(P_r^{k,\ell-1,j}) L^{k-r} + (k-r)P_r^{k,\ell-1,j} L^{k-1-r} (\partial_{\ell} L) \right].
\]

This sum breaks into two parts \( I + II \). By (18),

\[
I = \sum_{r=0}^{k} \partial_{\ell} P_r^{k,\ell-1,j} L^{k-r} = \sum_{r=0}^{k} \sum_{u=1}^{n} \sum_{\alpha: w(\alpha^*) = r} \sum_{h(\alpha^*) = k+\ell-j} \alpha \kappa^{k,\ell-1,j,r} X^\alpha L^{k-r}
\]

which is of the form (21).

We now examine the coefficient of \( \partial_{k-1,\ell+1-j} L \) arising in the square bracket above. The term \( \partial_{\ell} Q_j^{k,\ell-1} \) clearly does not arise (in either \( I \) or \( II \)) and so we may assume \( 2 \leq j \leq \ell \). In this case, the term \( X^\alpha = \partial_{k-1,\ell+1-j} L \) is given by \( \alpha = e^u \) where \( q_u(u) \) satisfies \( \beta_u(u) = (\beta_u^1, \beta_u^2) = (k-1, \ell+1-j) \). Therefore \( \beta_u = (k-1, \ell-1+j) \) and so with \( \alpha = e^u \), the coefficient of \( \partial_{k-1,\ell+1-j} L \) above is \( c_{\alpha}^{k,\ell-1,j,k} \), which is strictly positive by the induction hypothesis. Here we are implicitly assuming \( k \geq 2 \) when \( j = \ell \) since otherwise the integer \( u \) satisfying \( \beta_u = (k-1, \ell-j) = (0,0) \) does not exist. In fact the term \( \partial_{k-1,\ell+1-j} L \) does not arise in \( I \) when \( k = 1 \) and \( j = \ell \) since \( Q_j^{k,\ell-1} = L^k \).

We now turn our attention to the sum \( II \). By (17), we have

\[
II = \sum_{r=0}^{k} (k-r) \left[ \sum_{\alpha: w(\alpha^*) = r+1} \sum_{h(\alpha^*) = k+\ell-j} c_{\alpha}^{k,\ell-1,j,r} X^\alpha \right] L^{k-1-r}
\]

\[
= \sum_{r=1}^{k} (k-r+1) \left[ \sum_{\alpha: w(\alpha^*) = r} \sum_{h(\alpha^*) = k+\ell-j} c_{\alpha}^{k,\ell-1,j,r-1} X^\alpha \right] L^{k-r}
\]
which again is of the form (21). The term \(\partial^{k-1,\ell+1-j}L\) arises in the sum \(II\) precisely when \(\alpha = (\alpha_1, \alpha_2 + 1, \alpha_3, \ldots) = e^q\) with \(\beta_q = (k - 1, \ell + 1 - j)\); that is, when \(\alpha = 0, k = 1\) and \(j = \ell\). Hence in this case, the coefficient of \(\partial^{k-1,\ell+1-j}L = \partial_t L\) is \(c_0^{1,\ell,0} = 1\) since \(Q_t^{1,\ell-1} = L\) and so \(\partial_t Q_t^{1,\ell-1} = \partial_t L\).

Putting \(I\) and \(II\) together so that \(\partial_t Q_t^{k,\ell-1} = I + II\), we see that \(\partial_t Q_t^{k,\ell-1}\) is of the form (21) and the coefficient of \(\partial^{k-1,\ell+1-j}L\) is strictly positive.

By (13), we have \(Q_2^{k,\ell} = \partial_t Q_2^{k,\ell-1}\) and so we have the desired conclusion (21) in this case. Furthermore when \(3 \leq j \leq k + \ell - 1\), we have \(Q_j^{k,\ell} = \partial_t Q_j^{k,\ell-1} + Q_j^{k,\ell-1}\) and so by the induction hypothesis, we see that (21) holds for \(Q_j^{k,\ell}\). Furthermore (excluding the case \(k = 1\) and \(j = \ell\)) the coefficient of \(\partial^{k-1,\ell+1-j}L\) is equal to \(c_c^{k-1,j,k} + c_c^{k-1,j-1,k}\) where \(u\) and \(q\) are the integers such that \(\beta_u = (k - 1, \ell - j)\) and \(\beta_q = (k - 1, \ell + 1 - j)\). Being the sum of two strictly positive integers, this coefficient is also strictly positive. In the case \(k = 1\) and \(j = \ell\), the coefficient is \(1 + c_c^{1,\ell-1,\ell-1,1}\) which is strictly positive.

Finally since \(Q_{k+\ell}^{k,\ell} = L^k\), we see that (21) holds in all cases and this completes the proof of the proposition.

We can now give the proof of Proposition 3.4.

**Proof of Proposition 3.4** Recall that we are assuming \(2 \leq m\) and \(m + r < n\) for some \(r \geq 0\). Furthermore, we suppose \(\partial_t^{m+n} \psi(x, y, \omega) \neq 0\), \(\partial_t^n \psi(x, y, \omega) \neq 0\) but

\[
\partial_t^k \psi(x, y, \omega) = 0 \quad \text{for} \ 2 \leq k < n \quad \text{and} \quad \partial_t^\ell \psi(x, y, \omega) = 0 \quad \text{for} \ 2 \leq \ell < m.
\]

Under these assumptions we are trying to conclude that \(\partial^{k,\ell} L(x, y, \omega) = 0\) for any nonnegative \(k, \ell\) satisfying \(k + \ell \leq r\).

We proceed by induction on \(r\). The case \(r = 0\) was already treated immediately after the statement of Proposition 3.4. We assume \(m + r < n\) and suppose that the conclusion of the proposition holds for some \(r - 1 \geq 0\); that is, \(\partial^{k,\ell} L(x, y, \omega) = 0\) for all \(k, \ell\) satisfying \(k + \ell \leq r - 1\).

It suffices to show that \(\partial^{k-1,r-k+1} L(x, y, \omega) = 0\) for any fixed \(1 \leq k \leq r + 1\). We first note that \(\partial_t^{m+r-k} \psi(x, y, \omega) = 0\) by Corollary 3.3 and so by Lemma 3.1,

\[
0 = \partial_t^{m+r-k} \psi = \sum_{j=2}^{m+r} Q_j^{k,m+r-k} \partial_t^j \psi = Q_m^{k,m+r-k} \partial_t^m \psi + \sum_{j=m+1}^{m+r} Q_j^{k,m+r-k} \partial_t^j \psi.
\]

Here we used our assumption that \(\partial_t^j \psi = \partial_t^j \psi(x, y, \omega) = 0\) for \(2 \leq j \leq m - 1\).

We now claim that \(Q_j^{k,m+r-k} (x, y, \omega) = 0\) for every \(j \geq m + 1\). To see this, consider the monomial

\[
X^\alpha = Y_{\beta_1}^{\alpha_1} \cdots Y_{\beta_n}^{\alpha_n}
\]

for any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_n)\) satisfying \(h(\alpha) = m + r - j \leq r - 1\). For any \(1 \leq u \leq n\), this implies that \(|\beta_u| \leq h(\alpha) \leq r - 1\). However for any \(\beta_u\) with
and so we conclude that every monomial $X^\alpha(x, y, \omega) = 0$ for any $\alpha$ with $h(\alpha) = m + r - j$. Therefore when $j \geq m + 1,$

$$Q_j^{k, m+r-k} = \sum_{s=0}^k P_s^{k, m+r-k, j} L^{k-s} = P_k^{k, m+r-k, j} = \sum_{\alpha: w(\alpha)=k} c_{\alpha}^{k, m+r-k, j, k} X^\alpha = 0.$$  

Hence

$$0 = \partial^{k, m+r-k} \psi(x, y, \omega) = Q_m^{k, m+r-k}(x, y, \omega) \partial^{m} \psi(x, y, \omega)$$

and so $Q_m^{k, m+r-k} = 0$ as well since we are assuming $\partial^{m} \psi(x, y, \omega) \neq 0.$ However

$$Q_m^{k, m+r-k} = \sum_{s=0}^k P_s^{k, m+r-k, m} L^{k-s} = P_k^{k, m+r-k, m} = \sum_{\alpha: w(\alpha)=k} c_{\alpha}^{k, m+r-k, m, k} X^\alpha.$$  

We saw before that $X^\alpha$ vanishes if $\alpha = (\alpha_1, \ldots, \alpha_n)$ contains any component $u$ with $|\beta_u| \leq r - 1.$ Hence for $\alpha$ satisfying $h(\alpha) = r,$ then $X^\alpha$ being nonzero forces $\alpha = e^i$ for some $q$ so that $X^\alpha = Y_{\beta_q}$ with $|\beta_q| = r.$ If also $k = w(\alpha),$ then

$$k = w(\alpha) = w(e^i) = \beta^1_q + 1 \Rightarrow \beta^1_q = k - 1 \text{ and } \beta^2_q = r - k + 1.$$  

In other words, there is only one nonzero monomial $X^\alpha$ in the sum above for $Q_m^{k, m+r-k}$ and in fact, we have $Q_m^{k, m+r-k} = c_{\alpha}^{k, m+r-k, m, k} \beta^{k-1, r-k+1} L.$

Since we saw above that $Q_m^{k, m+r-k}(x, y, \omega) = 0$ and since the coefficient $c_{\alpha}^{k, m+r-k, m, k}$ of $\partial^{k-1, r-k+1} L$ is strictly positive by Proposition 3.5, we see that necessarily $\partial^{k-1, r-k+1} L(x, y, \omega)$ must be zero. The completes the proof of Proposition 3.4.

Before leaving this section, we record one consequence of Proposition 3.4.

**Corollary 3.6.** Let $\psi$ be a real-analytic function on $\mathbb{T}^2 \times M$ satisfying the factorisation hypothesis (FH) on $M.$ Suppose at a point $(x, y, \omega) \in \mathbb{T}^2 \times M,$ there exist integers $m, n \geq 2$ such that $\partial^m \psi(x, y, \omega) \neq 0,$ $\partial^n \psi(x, y, \omega) \neq 0$ but

$$\partial^k \psi(x, y, \omega) = 0 \text{ for } 2 \leq k < n \text{ and } \partial^\ell \psi(x, y, \omega) = 0 \text{ for } 2 \leq \ell < m.$$  

Suppose $2^m m < n$ for some $\mu \geq 0.$ Then $Q_j^{k, \ell}(x, y, \omega) = 0$ for $\ell + 2^{\mu-1}k < j.$

**Proof** Since $m \geq 2,$ we have

$$m + 2(2^m - 1) \leq m + m(2^m - 1) = 2^m m < n$$

and so we can apply Proposition 3.4 with $r = 2(2^m - 1)$ to conclude that $Y_{\beta} = \partial^{\beta} L = 0$ whenever $|\beta| = \beta_1 + \beta_2 \leq r.$ In particular we have $L(x, y, \omega) = 0$ and so by Proposition 3.5, we have

$$Q_j^{k, \ell} = \sum_{r=0}^k P_r^{k, \ell, j} L^{k-r} = P_k^{k, \ell, j} = \sum_{\alpha: w(\alpha)=k} c_{\alpha}^{k, \ell, j, k} X^\alpha.$$  

ON A PROBLEM OF KAHANE 21
If a multi-index $\alpha = (\alpha_1, \ldots)$ has a component $\alpha_u$ such that $|\beta_u| \leq r = 2(2^u - 1)$, then the monomial $X^\alpha = 0$. When $(n-1)/2 \leq u < n(n+1)/2$, $|\beta_u| = n-1$ and $|\beta_u| = n-1 \leq 2(2^u - 1)$ precisely when $n \leq 2^{u+1} - 1$. Hence, in order for $X^\alpha$ to be nonzero, then necessarily the components $\alpha_u$ of $\alpha$ with $(n-1)/2 \leq u < n(n+1)/2$ and $n \leq 2^{u+1} - 1$ must vanish.

Recall that $h(\alpha) = \sum \alpha_u |\beta_u|$ and $w(\alpha) = \sum \alpha_u (\beta_u^1 + 1)$. For any multi-index $\alpha$ and $n \geq 2$, we will denote

$$N_n := \sum_{u=n(n-1)/2}^{n(n+1)/2} \alpha_u.$$ 

Thus for any multi-index $\alpha$ with $X^\alpha \neq 0$, we have

$$w(\alpha) = \sum_u \alpha_u (\beta_u^1 + 1) = \sum_{n \geq 2^{u+1} u+n(n-1)/2}^{n(n+1)/2} \alpha_u (\beta_u^1 + 1) \leq \sum_{n \geq 2^{u+1}}^{2^{u+1}} n N_n$$

$$= \frac{2^{u+1}}{2u+1} - 1 \sum_{n \geq 2^{u+1}} \frac{2^{u+1} - 1}{2^{u+1}} n N_n \leq \frac{2^{u+1}}{2u+1} - 1 \sum_{n \geq 2} (n-1) N_n = \frac{2^{u+1}}{2u+1} - 1 \sum_u \alpha_u |\beta_u|$$

and the sum in this last quantity is equal to $h(\alpha)$. Hence if $w(\alpha) = k$ and $h(\alpha) = k + \ell - j$,

$$(2^{u+1} - 1)k = (2^{u+1} - 1)w(\alpha) \leq 2^{u+1} h(\alpha) = 2^{u+1} (k + \ell - j)$$

or $j \leq \ell + 2^{-u+1} k$ and this implies $Q_{j,k}^{x,y} \neq 0 \Rightarrow j \leq \ell + 2^{-u-1} k$. This is the desired conclusion.

4. A more robust one dimensional theory

At the heart of Proposition 1.1 is the one dimensional oscillatory integral estimate (4). When we move to the more interesting higher dimensional situations, the heart of the matter will be higher dimensional oscillatory integrals such as the one appearing in (5) from Section 2.2,

$$\int \int_{\mathbb{T}^2} e^{2\pi i [\lambda \omega \cdot \phi(x+s,y+t)+\rho s + \eta t]} \frac{ds}{s} \frac{dt}{t}. \quad (22)$$

In Section 2.2 we saw quickly the need to understand more robust versions of (4); namely (6) where the one dimensional phase $\varphi(s) = \lambda \omega \cdot \phi(x+s, \tau) + \rho s$ is a certain analytic pertubation of the phase $\lambda \omega \cdot \phi(x+s) + \rho s$ appearing in (4). There are other reasons where a more robust estimate such as (6) is needed. For instance, as in the proof of Proposition 1.1, we will want to localise the integral above to small $s$ and $t$,

$$\int \int_{|s|,|t| < \delta} e^{2\pi i [\lambda \omega \cdot \phi(x+s,y+t)+\rho s + \eta t]} \frac{ds}{s} \frac{dt}{t}. \quad (23)$$

for some $\delta > 0$ depending on properties of the analytic function $\psi(s, t, \omega) = \omega \cdot \phi(s, t)$ (for instance, perhaps certain derivative bounds of $\psi(x+s, y+t, \omega)$ are satisfied for $(x, y, \omega)$ near some fixed point $(x_0, y_0, \omega_0)$ and all $s, t$ with $|s|, |t| < \delta$). In order to pass from (22) to (23), two applications of (6) are needed with $O(1/\delta)$ error terms.
Also after we reduced to the integral (23), we may run into a degenerate situation where \( \omega \cdot \overline{\varphi}(x + s, y + t) = \omega \cdot \overline{\varphi}(x + s, y) \). In this case the integral in (23) splits into a product of two oscillatory integrals, one trivial (in \( t \)) to bound and the other is an integral of the form appearing in (6). There may (and will) be other degenerate situations which arise where the oscillation splits into a more complicated product of an \( s \) oscillation and \( t \) oscillation BUT the region of integration may not be a product such as \( |s|, |t| < \delta \), and so the integral no longer decomposes into a product of two 1 dimensional oscillatory integrals.

Specifically we will find ourselves examining an integral of the form (23) but where \( \lambda \omega \cdot \overline{\varphi}(x + s, y + t) = \lambda_1 \mu \cdot \overline{\mathcal{F}}(x + s, \tau) + \lambda_2 \nu \cdot \overline{\mathcal{G}}(\sigma, y + t) \) and the region of integration is \( \mathcal{R} = \{ |s|, |t| < \delta : A|s| < |t| < B|s| \} \); hence

\[
\int_{\mathcal{R}} e^{2\pi i \lambda_1 \mu \cdot \overline{\mathcal{F}}(x + s, \tau) + \rho s + \eta t} \, ds \, dt = \]

\[
\int_{|s| < \delta} e^{2\pi i \lambda_1 \mu \cdot \overline{\mathcal{F}}(x + s, \tau) + \rho s \overline{\tau}} \frac{1}{\overline{s}} \left[ \int_{\mathcal{R}(s)} e^{2\pi i \lambda_2 \nu \cdot \overline{\mathcal{G}}(\sigma, y + t) + \eta \sigma + \overline{\eta} \overline{t}} \, d\sigma \right] \, ds
\]

where \( \mathcal{R}(s) = \{ |t| < \delta : (s, t) \in \mathcal{R} \} \). If we denote by \( F(s) \) the inner integral above, we see that \( F \) is an even function and for \( s > 0 \), \( |F'(s)| \leq C s^{-1} \) where \( C \) depends on \( A, B, a \) and \( b \). Of course if we hope that (6) holds, then we would hope that \( F(s) \) defines a bounded function of \( s \). This motivates the following extension of Proposition 1.1 as encapsulated in (4).

**Proposition 4.1.** Let \( \overline{\varphi}(s, t) \) be a \( d \)-tuple of real-analytic, 1-periodic functions of two variables and set \( \psi(s, t, \omega) = \omega \cdot \overline{\varphi}(s, t) \). Suppose that \( F(s) \) is an even function on \([-1,1], |F(s)| \leq A \) and \( \psi \in C^1 \) on \((0,1)\) such that \( |F'(s)| \leq B s^{-1} \) for \( s \in (0,1) \).

Then for every \( \omega \in S^{d-1} \),

\[
C(\omega) := \sup_{a, b, c, \lambda, \tau, \rho, x} \left\{ \int_{a < |s| < b} e^{2\pi i \lambda \psi(x + s, \tau, \omega) + \rho s} F(s) \, ds \right\} < \infty
\]

where \( C(\omega) \) depends on \( A, B \) and \( \overline{\varphi} \) as well. Furthermore, if \( \psi \) satisfies the factorisation hypothesis (FH) on \( S^{d-1} \), then \( \sup_{\omega \in S^{d-1}} C(\omega) < \infty \).

**Proof** We will give the proof assuming \( F \equiv 1 \) so that we replace the amplitude \( \mathcal{A}(s) = F(s)/s \) with \( \mathcal{A}(s) = 1/s \). As will be evident from the argument below, the only properties of the amplitude \( \mathcal{A}(s) \) we will be using is that it is an odd function, \( |\mathcal{A}(s)| \lesssim |s|^{-1} \) and \( |\mathcal{A}'(s)| \lesssim |s|^{-2} \) which clearly holds for \( F(s)/s \) by our assumptions. Hence, mainly for notational ease, we assume \( F \equiv 1 \).

We first fix \( \omega \in S^{d-1} \) and allow our estimates to depend on \( \omega \). Initially we put no restrictions on the \( d \)-tuple of analytic functions \( \overline{\varphi} \) and in particular, the factorisation hypothesis is not assumed to hold. For notational convenience, we suppress \( \omega \) in what follows, writing \( \psi(s, t, \omega) = \psi(s, t) \), etc...

**Case 1:** There exists a point \((x_0, \tau_0) \in T^2 \) such that

\[
\partial^{k, \ell} \psi(x_0, \tau_0) := \frac{\partial^{k+\ell} \psi}{\partial s^k \partial \tau^\ell}(x_0, \tau_0) = 0 \quad \text{for all} \quad k \geq 2, \ell \geq 0.
\]
Then
\[ \psi(x_0 + s, \tau) = \psi(x_0, \tau) + \partial_s \psi(x_0, \tau)s + \sum_{k=2}^{\infty} \frac{1}{k!} \partial_s^k \psi(x_0, \tau)s^k \quad \text{for small } s. \]

For \( k \geq 2, \)
\[ \partial_s^k \psi(x_0, \tau) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \partial_s^{k-\ell} \psi(x_0, \tau_0)(\tau - \tau_0)^\ell \equiv 0 \quad \text{for } \tau \text{ near } \tau_0. \]

Hence \( \partial_s^k \psi(x_0, \tau) \equiv 0 \) for all \( \tau \) by analyticity. Therefore \( \psi(x_0 + s, \tau) = \psi(x_0, \tau) + \partial_s \psi(x_0, \tau)s \) for small \( s \) and hence all \( s \) by analyticity once again. This forces \( \partial_s \psi(x_0, \tau) = 0 \) since otherwise we would contradict the periodicity of \( \psi(x_0 + s, \tau) \) in \( s \), arriving at the conclusion that for any \( k, \tau \in \mathbb{T}, \psi(x + s, \tau) \equiv \psi(x_0, \tau). \)

\[ \int_{a<s<b} e^{2\pi i [\lambda \psi(x+s,\tau)+\rho s]} \frac{ds}{s} = e^{2\pi i \lambda \psi(x_0,\tau)} \int_{a<s<b} e^{2\pi i \rho s} \frac{ds}{s} \]
and this last integral is easily seen to be uniformly bounded in \( a, b \) and \( \rho \).

**Case 2:** For every point \((x, \tau) \in \mathbb{T}^2, \) there exists \( k = k(x, \tau) \geq 2 \) and \( \ell = \ell(x, \tau) \geq 0 \) such that \( \partial^{k,\ell} \psi(x, \tau) \neq 0. \)

We choose \( \ell = \ell(x, \tau) \) to be minimal with this property; that is, if \( \ell < \ell(x, \tau), \) then \( \partial^{k,\ell} \psi(x, \tau) = 0 \) for all \( k \geq 2. \) There is an \( \epsilon = \epsilon(x, \tau) > 0 \) such that
\[ |\partial^{k,\ell} \psi(x', \tau')| \geq \frac{1}{2} |\partial^{k,\ell} \psi(x, \tau)| \quad \text{for all } (x', \tau') \in B_{2\epsilon}(x, \tau). \]

We cover the compact \( \mathbb{T}^2 \) with the family of balls \( \{B_{\epsilon(x, \tau)}(x, \tau)\}_{(x, \tau) \in \mathbb{T}^2} \) and extract a finite subcover \( \{B_{\epsilon_j}(x_j, \tau_j)\}_{j=0}^{M} \), reducing to establishing a uniform bound for
\[ T := \int_{a<s<b} e^{2\pi i [\lambda \psi(x+s,\tau)+\rho s]} \frac{ds}{s} \]
uniformly for \((x, \tau) \in B_{\epsilon_j}(x_j, \tau_j) \) (and also, \( a, b, \lambda \) and \( \rho \)) for each \( 0 \leq j \leq M. \) This follows from the simple observation that
\[ \int_{a<s<b} e^{2\pi i [\lambda \psi(x+s,\tau)+\rho s]} \frac{ds}{s} \leq \int_{a<s<b} e^{2\pi i [\lambda \psi(x+s,\tau)+\rho s]} \frac{ds}{s} + O(\log(1/\epsilon_j)). \]

For convenience we take \( j = 0 \) and denote \( B_0 = B_{\epsilon_0}(x_0, \tau_0). \) We also write \( k_0 = k(x_0, \tau_0) \geq 2 \) and \( \ell_0 = \ell(x_0, \tau_0) \geq 0 \) and note that from (24), we have
\[ |\partial^{k_0,\ell_0} \psi(x+s, \tau)| \geq \frac{1}{2} |\partial^{k_0,\ell_0} \psi(x_0, \tau_0)| = c_0 \]
for all \((x, \tau) \in B_0 \) and \( |s| < \epsilon_0. \)

**Claim:** For \((x, \tau) \in \mathbb{T}^2 \) and \( |s| < \epsilon_0, \) we have \( \psi(x+s, \tau) = P(\tau) + E(s, \tau) \) where \( P(\tau) = P_{x_0, \tau_0}(\tau) \) is some polynomial in \( \tau \) of degree at most \( \ell_0 - 1 \) and \( E(s, \tau) = E_{x, \tau_0}(s, \tau) \) satisfies the derivative bound
\[ |\partial_s^{\ell_0} E(s, \tau)| \geq c_0 |\tau - \tau_0|^{\ell_0} \]
for all \( |s| < \epsilon_0. \)
When \( \ell_0 = 0 \), we take \( P = 0 \) so that \( E_{x, \tau_0}(s, \tau) = \psi(x + s, \tau) \) and the claim simply states that \( |\partial^{k_0}_x \psi(x + s, \tau)| \geq c_0 \) whenever \( (x, \tau) \in B_0 \) and \( |s| < c_0 \) which of course holds by (26). So in order to prove the claim, it suffice to take \( \ell_0 \geq 1 \). By the minimality of \( \ell_0 \), we see that for any \( \ell < \ell_0, \partial^{k, \ell}_x \psi(x, \tau_0) = 0 \) for all \( k \geq 2 \) and hence

\[
\partial^{\ell}_x \psi(x_0 + s, \tau_0) = \partial^{\ell}_x \psi(x, \tau_0) + \partial^{1, \ell}_x \psi(x, \tau_0)s + \sum_{k=2}^{\infty} \frac{1}{k!} \partial^{k, \ell}_x \psi(x, \tau_0)s^k
\]

and therefore for all \( s \) by analyticity. Again the periodicity of \( \partial^{1, \ell}_x \psi(x, \tau_0) \) in \( s \) forces \( \partial^{1, \ell}_x \psi(x, \tau_0) = 0 \) and so

\[
\partial^{\ell}_x \psi(x + s, \tau_0) = \partial^{\ell}_x \psi(x, \tau_0).
\] (28)

Using (28) with \( \ell = 0 \), we see that for any \( (x, \tau) \in B_0 \) and \( |s| < c_0 \),

\[
\psi(x + s, \tau) = \psi(x, \tau_0) + \psi(x + s, \tau) - \psi(x, \tau_0) = \psi(x, \tau_0) + (\tau - \tau_0) \int_0^1 \partial_1 \psi(x + s, \tau_0 + r(\tau - \tau_0)) dr.
\]

Noting that \( (x, \tau_0 + \tau') := (x, \tau_0 + r(\tau - \tau_0)) \in B_0 \) for any \( 0 < r < 1 \) and using (28) with \( \ell = 1 \), we reason similarly to see

\[
\partial_1 \psi(x + s, \tau_0 + \tau') = \partial_1 \psi(x, \tau_0) + \partial_1 \psi(x + s, \tau_0 + \tau') - \partial_1 \psi(x + s, \tau_0) = \partial_1 \psi(x, \tau_0) + (\tau - \tau_0) \int_0^1 \partial^2_1 \psi(x + s, \tau_0 + ru(\tau - \tau_0)) drdu.
\]

Hence

\[
\psi(x + s, \tau) = \psi(x, \tau_0) + \partial_1 \psi(x, \tau_0)(\tau - \tau_0) + (\tau - \tau_0)^2 \int_0^1 \int_0^1 \partial^2_1 \psi(x + s, \tau_0 + ru(\tau - \tau_0)) drdu.
\]

Iterating we conclude that \( \psi(x + s, \tau) = P(\tau) + E(s, \tau) \) where

\[
P(\tau) := P_{x_0, \tau_0}(\tau) := \sum_{\ell=0}^{\ell_0} \frac{1}{\ell!} \partial^{\ell}_x \psi(x, \tau_0)(\tau - \tau_0)^\ell
\]

and \( E(s, \tau) = E_{x, \tau_0}(s, \tau) = (\tau - \tau_0)^{\ell_0} \int_0^1 \cdots \int_0^1 \partial^{\ell_0}_x \psi(x + s, \tau_0 + r_1 \cdots r_{\ell_0} (\tau - \tau_0)) r_2 r_3^2 \cdots r_{\ell_0 - 1} d r_1 \cdots d r_{\ell_0} \).

Finally the derivative bound (26) implies (27), completing the proof of the claim.

Returning to the oscillatory integral in (25), we have

\[
T = e^{2\pi i \lambda P_{x_0, \tau_0}(\tau)} \int_{a < |s| < c_0} e^{2\pi i [\lambda E(s, \tau) + \rho s]} \frac{ds}{s}
\]

and by van der Corput’s lemma, together with an integration by parts argument, we see that (27) implies that

\[
T^\prec := \int_{|s| < c_0} e^{2\pi i [\lambda E(s, \tau) + \rho s]} \frac{ds}{s} \leq C
\]

and

\[
T^\succ := \int_{|s| < c_0} e^{2\pi i [\lambda E(s, \tau) + \rho s]} \frac{ds}{s} \leq C
\]
where $C$ only depends on $k_0$ and $c_0 = \|\partial^{k_0} F(x_0, \tau_0)\|/2$ (also on the constants $A$ and $B$ if our amplitude is the original $F(s)/s$). In the other region where $|s| < \kappa$ and $\kappa = \min((|\tau - \tau_0|/\lambda)^{-1/k_0}, \epsilon_0)$, we will compare the complimentary integral
\[
T_c := \int_{a<s<\kappa} e^{2\pi i (|\lambda E(s, \tau) + \rho s|)} \frac{ds}{s}
\]
to an integral with a certain polynomial phase
\[
\mathcal{C} = \int_{a<s<\kappa} e^{2\pi i (|\lambda E(s, \tau) + \rho s|)} \frac{ds}{s} - \int_{a<s<\kappa} e^{2\pi i (|\lambda Q(s) + \rho s|)} \frac{ds}{s}.
\]
The polynomial $Q$ is simply the Taylor polynomial of $E(s, \tau)$ of order $k_0$, thought of as a function of $s$:
\[
Q(s) = Q(s, x, x_0, \tau, \tau_0) = \sum_{k=0}^{k_0-1} \frac{1}{k!} \partial_s^k E(0, \tau)s^k.
\]
Hence $E(s, \tau) = Q(s) + R_{k_0}(s)$ where the remainder term can be expressed as
\[
R_{k_0}(s) = s^{k_0} \int_0^1 \cdots \int_0^1 \partial_s^{k_0} E(r_1 \cdots r_{k_0} s, \tau)r_1 r_2 \cdots r_{k_0-1} dr_1 \cdots dr_{k_0}.
\]
Using the formula for $E$ above, we see that the remainder term $R_{k_0}(s)$ satisfies the bound $|R_{k_0}(s)| \lesssim |\lambda^0|s^{k_0}$ and therefore
\[
|\mathcal{C}| \lesssim |\lambda(\tau - \tau_0)^0| \int_{|s|<\kappa} |s|^{k_0-1} ds \lesssim 1
\]
since $\kappa \leq |\lambda(\tau - \tau_0)^0|^{-1/k_0}$. This shows that, up to a uniformly bounded error term, the oscillatory integral
\[
T = e^{2\pi i (\lambda \rho x_0, \tau_0)} \int_{a<s<\kappa} e^{2\pi i (|\lambda Q(s) + \rho s|)} \frac{ds}{s} + O(1)
\]
and this last remaining oscillatory integral is uniformly bounded thanks to (10), the useful one dimensional result of Stein and Wainger.

We now turn to discuss how to achieve uniformity in the parameter $\omega \in S^{d-1}$ when $\psi(s, t, \omega) = \omega \cdot \bar{\phi}(s, t)$ satisfies the factorisation hypothesis (FH) on $S^{d-1}$. Again we divide the analysis into two cases but the cases are different.

First suppose there exists a point $(x_0, \tau_0, \omega_0) \in \mathbb{T}^2 \times S^{d-1}$ such that
\[
\partial_s^k \psi(x_0, \tau_0, \omega_0) = 0 \quad \text{for all } k \geq 2.
\]
Then by Corollary (3.2), we have $\psi(s, \tau, \omega_0) \equiv \psi(x_0, \tau, \omega_0)$ or $\omega_0 \cdot \bar{\phi}(s, \tau) \equiv \omega_0 \cdot \bar{\phi}(x_0, \tau)$. Hence there exists a $\phi_j$ in the $d$-tuple $\bar{\phi}$ which we can solve in terms of the others; namely,
\[
\phi_j(s, \tau) = \sum_{k \neq j} c_k \phi_k(s, \tau) + \omega_0 \cdot \bar{\phi}(x_0, \tau)
\]
and so (after re-writing the integral appearing in the statement of the proposition) we have reduced the dimension by one, reducing matters to examining an oscillatory integral with $d - 1$ real-analytic functions defining the phase.
Therefore by a simple induction on dimension argument, we may assume without loss of generality that for every point \((x, \tau, \omega) \in T^2 \times S^{d-1}\), there is a \(k = k(x, \tau, \omega) \geq 2\) such that \(\partial_k^s \psi(x, \tau, \omega) \neq 0\). Hence there is an \(\epsilon = \epsilon(x, \tau, \omega) > 0\) such that

\[
| \partial_k^s \psi(x', \tau', \omega') | \geq \frac{1}{2} | \partial_k^s \psi(x, \tau, \omega) | \quad \text{for all } (x', \tau', \omega') \in B_{2\epsilon}(x, \tau, \omega). \tag{29}
\]

So we are in a much better position than before, effectively we have reduced ourselves to the Case 2 situation above but with \(\ell = 0\) and even more, the derivative bound in (29) is more robust than we had in (24) since it holds in an entire neighbourhood of \((x, \tau, \omega)\) and not just in a neighbourhood of \((x, \tau)\) for a fixed \(\omega\).

The derivative bound (29) is the analogue of (8) but with one additional parameter \(\tau\) and the argument to show that our oscillatory integral

\[
\int_{a < |s| < b} e^{2\pi i [\lambda \psi(x+s, \tau, \omega) + \rho s]} \frac{ds}{s}
\]

is uniformly bounded now proceeds exactly as in the proof of Proposition 1.1, establishing (4). The reader can readily check the details.

### 5. Prelude to the proof of Theorem 1.3

Theorem 1.3 determines exactly when \((\Phi)_{rect}\) holds for any given real-analytic map \(\Phi : T^2 \to T^d\). In a previous section we saw that \((\Phi)_{rect}\) holds precisely when the oscillatory integral

\[
\int_{T^2} e^{2\pi i [\lambda \psi(x+s, y+t) + \rho (s,t)]} D_M(s) D_N(t) \, ds \, dt
\]

is uniformly bounded in the parameters \(\lambda, x, y, M\) and \(N\); see (2) and the beginning of Section 2.2.

The proof of Proposition 1.1 used two standard types of cancellation, as outlined in Section 2.5, to reduce the estimate of the one dimensional oscillatory integral (4) to the integral (9), an oscillatory integral with a polynomial phase of bounded degree (but we have no control on the coefficients). At this point we appealed to a result of Stein and Wainger (10) which has no satisfactory counterpart in the multiparameter setting, when we move from oscillatory integrals involving a classical CZ kernel \(1/t\) to ones involving a product-type kernel, namely \(1/st\).

In this section, we begin the analysis of the oscillatory integral (30). In a way analogous to what we did in the one dimensional setting, passing from (4) to (9), we will reduce the study of (30) to an oscillatory integral with an almost polynomial phase of bounded degree. By almost polynomial we mean a polynomial modulo a real-analytic part that is a function of \(s\) and \(t\) separately; that is, the phase will be of the form \(P(s,t) + f(s) + g(t)\) where \(P\) is a polynomial of bounded degree and \(f\) and \(g\) are real-analytic functions of a single variable. In this section we will accomplish this under certain derivative bounds of the phase function. This is the main goal of this section and to achieve this goal, we will follow the arguments in

---

The text continues with further details and proofs pertaining to the oscillatory integral and its analysis.
[3] which dealt with a related problem in the theory of multiparameter singular Radon transforms.

As we saw in a previous section, using properties of the Dirichlet kernel, the study of the integral (30) can be reduced to studying oscillatory integrals of the form

$$\int_{\mathbb{T}^2} e^{2\pi i [\lambda \omega \cdot \Phi(x+y,t) + \rho s + \eta t]} \frac{ds \, dt}{s \, t}$$

for some parameters $\lambda, \omega, x, y, \rho$ and $\eta$. Employing estimates for the one dimensional oscillatory integral in Proposition 4.1 twice, matters are reduced to considering the oscillatory integral

$$I := \int_{|s|,|t| < \alpha} e^{[\lambda \psi(x+s,y+t,\omega)] + \rho s + \eta t} \frac{ds \, dt}{s \, t}$$

for some small $0 < \alpha < 1$ depending on the $d$-tuple $\Phi$ of real-analytic functions. Here $\psi(s,t,\omega) := \omega \cdot \Phi(s,t)$ is an analytic function depending on $\omega$. The estimates we derive for $I$ above will be uniform in $\omega \in \mathbb{S}^{d-1}$ as well as $x, y, \lambda, \rho$ and $\eta$. They will depend only on $\Phi$ and certain assumed derivative bounds of $\psi$ from below. For notational convenience, we suppress the dependence on $\omega$ for the rest of this section and write $\psi(s,t)$ instead of $\psi(s,t,\omega)$.

Suppose that there is an integer $m_0 \geq 2$ such that

$$|\partial_{t}^{m_0} \psi(x+s,y+t)| = |\frac{\partial^{m_0} \psi}{\partial t^{m_0}}(x+s,y+t)| \geq A \quad \text{for all } |s|, |t| < \alpha \quad (31)$$

for some $A \lesssim 1$. Under such a condition, we will show how to successfully compare parts of $I$ to an oscillatory integral with an almost polynomial phase of bounded degree. We will consider those parts of $I$ where the monomial $t^{m_0}$ dominates the majority of the monomials arising in the Taylor expansion

$$\psi(x+s,y+t) = \sum_{k,\ell \geq 0} \frac{1}{k! \ell!} \partial^{k,\ell} \psi(x,y) s^k t^\ell.$$ 

In order to do this efficiently, it will be convenient to decompose $I$ dyadically, resolving the singularities arising from the kernel $1/st$. We introduce an appropriate smooth nonnegative function $\zeta$, supported in $[-2, -1] \times [1, 2]$, and write the integral $I$ as

$$\sum_{p,q} \int \int e^{[\lambda \psi(x+s,y+t)] + ps + \eta t} \zeta(2^p s) \zeta(2^q t) \frac{ds \, dt}{s \, t} =: \sum_{p,q} I_{p,q}$$

where the sum is over large positive integers $(p, q)$ such that both $2^{-p}, 2^{-q} < \alpha$.

For any pair $(n_1, m_1)$ with $n_1 > 0$ and $m_1 < m_0$, we will consider the part of $I$ where the monomial $t^{m_0}$ is pointwise larger than $s^{n_1} t^{m_1}$; that is, we consider

$$I_{m_0, m_1} := \int \int_{|s^{n_1} t^{m_1}| < |t^{m_0}|} e^{[\lambda \psi(x+s,y+t)] + ps + \eta t} \frac{ds \, dt}{s \, t} = \sum_{(p,q) \in \mathcal{R}} I_{p,q}$$

where $\mathcal{R} = \{(p, q) \text{ large : } n_1 p - (m_0 - m_1) q \geq 0\}$. In the region $|s^{n_1} t^{m_1}| < |t^{m_0}|$, the only monomials $s^k t^\ell$ which are not pointwise dominated by $t^{m_0}$ are those $(k, \ell)$
satisfying \( k(m_0 - m_1) + \ell n_1 < m_0 n_1 \). In other words the part of the Taylor expansion of \( \psi \) NOT controlled in some weak pointwise sense by the monomial \( t^{m_0} \) is

\[
\sum_{k,\ell \geq 0} \frac{1}{k!\ell!} \partial^{k,\ell} \psi(x, y) s^k t^\ell
\]

which is a polynomial whose degree only depends on \( m_0, m_1 \) and \( n_1 \). It will be necessary to keep all the pure terms in the Taylor expansion (monomials of the form \( s^k \) and \( t^\ell \)) and so we compare \( I_{m_0,m_1,n_1}^p \) to \( II_{m_0,m_1,n_1}^q \) where

\[
II_{p,q} := \int_0^1 e^{i \lambda (P_{x,y}(s,t) + \psi(x+s,t) + \psi(x,y+t) - \psi(0,0) + \rho s + \eta t)} \frac{d\lambda}{\lambda} \frac{ds}{s} \frac{dt}{t}
\]

We set \( D_{p,q} = I_{p,q} - II_{p,q} \) and make the changes of variables \( s' = q^p s \) and \( t' = q^\ell t \) in both integrals. We have the simple comparison bound

\[
|D_{p,q}| \lesssim \sum_{k \geq 1, \ell \geq 1} |\lambda|2^{-pk-q\ell} \left| \frac{1}{k!\ell!} \partial^{k,\ell} \psi(x, y) \right| \leq C|\lambda|2^{-\delta N}2^{-m_0 q}.
\]  

(32)

Here \( C \) depends only on \( \phi \) (and in particular independent of \( \omega \in S^{d-1} \)), \( \delta = 1/n_1 \) and \( N := n_1 p - (m_0 - m_1)q \geq 0 \). In fact for any \( (k, \ell) \) satisfying \( k \geq 1 \), we have

\[
2^{-kp-q\ell} = 2^{-\frac{k}{n_1} + \frac{\ell}{q} - \frac{k}{n_1} - \frac{\ell}{q} + m_0 \frac{k}{m_0 - m_1}} \leq 2^{-\delta N}2^{-\frac{k}{n_1} - \frac{\ell}{q} (n_1 + k(m_0 - m_1))}
\]

and therefore

\[
2^{-kp-q\ell} \leq 2^{-\delta N}2^{-m_0 q}
\]

if in addition, \( k(m_0 - m_1) + \ell n_1 \geq m_0 n_1 \).

The phase function in \( I_{p,q} \), after making the changes of variables \( s' = q^p s \) and \( t' = q^\ell t \), is given by

\[
\varphi(s, t) = \lambda \psi(x + 2^{-p} s, y + 2^{-q} t) + \rho 2^{-p} s + \eta 2^{-q} t
\]

and so the derivative bound in (31) implies that that \( |\partial^{m_0} \varphi(s, t)| \geq A|\lambda|2^{-m_0 q} \) for \( |s|, |t| \sim 1 \). Hence van der Corput’s lemma implies that

\[
|I_{p,q}| \leq C_{m_0} (A|\lambda|2^{-m_0 q})^{-1/m_0}.
\]  

(33)

Similarly, since the largest power of \( t \) appearing in \( P_{x,y}(s,t) \) is strictly less than \( m_0 \), we see that

\[
|II_{p,q}| \leq C_{m_0} (A|\lambda|2^{-m_0 q})^{-1/m_0}.
\]  

(34)

Taking a convex combination of the estimates (32), (33) and (34) (and noting \( A \gtrsim 1 \)) we have

\[
|D_{p,q}| \leq C 2^{-\delta N} A^{-\varepsilon_0} \min(|\lambda|2^{-qm_0}, (|\lambda|2^{-qm_0})^{-\varepsilon_1})
\]
for some exponents \( \delta > 0 \) (possibly different), \( \epsilon_0, \epsilon_1 > 0 \), depending only on \( m_0 \) and \( n_1 \). The constant \( C \) depends on \( m_0, n_1 \) and \( \phi \). Therefore

\[
|I_{m_0,m_1,n_1}^* - I_{m_0,m_1,n_1}^{|}| = \left| \sum_{(p,q) \in R} D_{p,q} \right| \leq CA^{-\epsilon_0} \sum_{N \geq 0} \sum_{p,q: n_1 p - (m_0 - m_1) q = N} |D_{p,q}|
\]

\[
\leq CA^{-\epsilon_0} \sum_{N \geq 0} 2^{-\delta N} \sum_{q: \min(|\lambda| 2^{-q m_0}, (|\lambda| 2^{-q m_0})^{-\epsilon_1})} \leq CA^{-\epsilon_0}.
\]

Summarising, under the derivative bound (31), we have successfully compared \( I_{m_0,m_1,n_1}^* \), the part of the integral \( I \) where the integration is taken over the region \( R = \{|s|, |t| < a : |s^{m_1} t^{m_1}| < |t^{m_0}|\} \) to a corresponding integral with an almost polynomial phase \( \phi(s, t) = P_{x,y}(s, t) + \psi(x + s, y) + \psi(x, y + t) \) of bounded degree;

\[
I_{m_0,m_1,n_1}^* = \int_R e^{i[\lambda \psi(x+s,y+t)+\rho s+\eta t]} \frac{ds \ dt}{s \ t} = \int_R e^{i[\lambda \phi(s,t)+\rho s+\eta t]} \frac{ds \ dt}{s \ t} + O(A^{-\epsilon_0}).
\]

We emphasise here that the constant appearing implicitly in the error bound \( O(A^{-\epsilon_0}) \) is a uniform constant and can be taken to be independent of \( x, y, \lambda, \rho, \eta \) and \( \omega \). Furthermore in many of our applications, the constant \( A \) appearing above and in \( (31) \) is a uniform constant, depending on \( \phi \) but otherwise independent on \( x, y, \lambda, \rho, \eta \) and \( \omega \). However there is one situation where \( A \) is not uniform but we still need our bounds to be uniform; that is, independent of \( A \). We discuss this situation in Section 7 below.

5.1. A small variant. We note that the above argument still works if the phase function \( \lambda \psi(x+s,y+t)+\rho s+\eta t \) in \( I \) or \( I_{m_0,m_1,n_1}^{|} \) is replaced by \( \lambda \psi(x+s,y+t) + H(s) + \rho s + \eta t \) where \( H \) is any real-analytic function. The derivative bound (31) still implies the corresponding bound for this new phase since the function \( H(s) \) is killed off when computing any derivative in \( t \). Hence we still conclude that if

\[
I_{m_0,m_1,n_1}^* = \int_R e^{i[\lambda \psi(x+s,y+t)+H(s)+\rho s+\eta t]} \frac{ds \ dt}{s \ t},
\]

then

\[
I_{m_0,m_1,n_1}^* = \int_R e^{i[\lambda \phi(s,t)+H(s)+\rho s+\eta t]} \frac{ds \ dt}{s \ t} + O(A^{-\epsilon_0}),
\]

where again \( \phi(s, t) = P_{x,y}(s, t) + \psi(x + s, y) + \psi(x, y + t) \) is the almost polynomial from before.

5.2. The case \((n_1,m_1) = (n_0,0)\). Let us apply the discussion above to the case \((n_1,m_1) = (n_0,0)\) to conclude that

\[
I_{m_0,n_0}^* = \int_R e^{i[\lambda \psi(x+s,y+t)+\rho s+\eta t]} \frac{ds \ dt}{s \ t} = \int_R e^{i[\lambda \phi(s,t)+\rho s+\eta t]} \frac{ds \ dt}{s \ t} + O(A^{-\epsilon_0}),
\]
for some \( \epsilon_0 > 0 \) depending on \( m_0 \) and \( n_0 \). Here \( R = \{ |s|, |t| < a : |s|^{m_0} < |t|^{m_0} \} \) and \( \varphi(s, t) = P_{x, y}(s, t) + \psi(x + s, y) + \psi(x, y + t) \) where now

\[
P_{x, y}(s, t) = \sum_{k, \ell \geq 1} \frac{1}{k! \ell!} \partial_s^k \partial_t^\ell \psi(x, y) s^k t^\ell.
\]

Now suppose we knew in addition to the derivative bound (31), that the derivative bound (this time with respect to \( s \))

\[
|\partial_s^m \psi(x + s, y + t)| = |\partial_s^m \psi_s(x + s, y + t)| \geq B \quad \text{for all } |s|, |t| < a
\]

holds for some \( B \leq 1 \) but with the same \( x, y \) (and implicitly for the same \( \omega \) in the definition of \( \psi(s, t) = \psi_\omega(s, t) = \omega \cdot \overline{\sigma}(s, t) \)). For the complimentary integral where we integrate over \( \mathcal{R}' = \{ |s|, |t| < a : |t|^{m_0} \leq |s|^{m_0} \} \), we argue similarly to conclude that

\[
I_{m_0, n_0}^< = \int_{\mathcal{R}'} e^{i[\lambda \psi(x + s, y + t) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t} = \int_{\mathcal{R}'} e^{i[\lambda \psi(s, t) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t} + O(B^{-\epsilon_0})
\]

where \( \varphi(s, t) = P_{x, y}(s, t) + \psi(x + s, y) + \psi(x, y + t) \) is the same as before. Again we emphasise here that the constant appearing implicitly in the error bound \( O(B^{-\epsilon_0}) \) is a uniform constant and can be taken to be independent of \( x, y, \lambda, \rho, \eta \) and \( \omega \).

Since the oscillatory integral

\[
I := \int_{|s|, |t| < a} e^{i[\lambda \psi(x + s, y + t) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t} = I_{m_0, n_0}^< + I_{m_0, n_0}^>,
\]

we see that the above argument successfully reduces the study of \( I \) to an oscillatory integral with an almost polynomial phase of bounded degree \textit{when we have managed to reduce ourselves to certain derivative bounds (31) and (36) holding. This was the main goal of the section.}

6. The proof of Theorem 1.3 – When (FH) holds

In this section we will show that if \( \psi(s, t, \omega) = \omega \cdot \overline{\sigma}(s, t) \) satisfies the factorisation hypothesis (FH) on \( S^{d-1} \), then the oscillatory integral

\[
\int_{|s|, |t| < 1/2} e^{2\pi i[\lambda \psi(x + s, y + t, \omega) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t}
\]

is uniformly bounded in \( \lambda, \rho, \eta \in \mathbb{R}, \omega \in S^{d-1} \) and \( (x, y) \in T^2 \). From our discussion in Section 2, this will give a proof of Theorem 1.3 under the factorisation hypothesis.

First suppose that there exists an \( \omega_0 \in S^{d-1} \) and \( (x_0, y_0) \in T^2 \) such that

\[
\partial_s^k \psi(x_0, y_0, \omega_0) = 0 \quad \text{for all } k \geq 2, \quad \text{and} \quad \partial_t^\ell \psi(x_0, y_0, \omega_0) = 0 \quad \text{for all } \ell \geq 2.
\]

By two applications of Corollary 3.2, we see that \( \psi(s, t, \omega_0) \equiv \psi(x_0, y_0, \omega_0) \) (in fact this follows easily by examining the Taylor expansion of \( \psi \) and using periodicity – one does not need the factorisation hypothesis here). Hence \( \omega_0 \cdot \overline{\sigma}(s, t) \equiv \text{constant} \)
and so there exists a \( \phi_j \) in the \( d \)-tuple \( \overline{\phi} \) which we can solve in terms of the others; namely,
\[
\phi_j(s,t) = \sum_{k \neq j} c_k \phi_k(s,t) + \omega_0 \cdot \overline{\phi}(x_0, y_0)
\]
and so (after re-writing the integral (37)) we have reduced the dimension by one, reducing matters to examining an oscillatory integral with \( d-1 \) real-analytic functions defining the phase. Note that after this reduction, the factorisation hypothesis still holds for the \((d-1)\)-tuple of real-analytic functions.

Therefore by a simple induction on dimension argument, we may assume without loss of generality that for every point \((x, y, \omega) \in \mathbb{T}^2 \times S^{d-1}\), either

1. there exists an \( n \geq 2 \) such that \( \partial_s^n \psi(x, y, \omega) \neq 0 \); or
2. there exists an \( m \geq 2 \) such that \( \partial_t^m \psi(x, y, \omega) \neq 0 \).

For each \((x, y, \omega) \in \mathbb{T}^2 \times S^{d-1}\), we define \( \epsilon = \epsilon(x, y, \omega) > 0 \) as follows: if (1) holds, choose \( \epsilon > 0 \) such that
\[
|\partial_s^n \psi(x', y', \omega')| \geq \frac{1}{2} |\partial_s^n \psi(x, y, \omega)| \quad \text{for all } (x', y', \omega') \in B_{2\epsilon}(x, y, \omega),
\]
and if (1) does not hold (then (2) necessarily holds), we define \( \epsilon > 0 \) analogously using the \( m \)th derivative with respect to \( t \).

This gives us a covering \( \{B_{\epsilon}(x, y, \omega)\} \) of the compact space \( \mathbb{T}^2 \times S^{d-1}\) from which we extract a finite subcover \( \{B_j\}_{j=0}^M \) and reduce matters to estimating the oscillatory integral in (37) for \((x, y, \omega) \in B_j\) for some \( 0 \leq j \leq M \). For convenience, we take \( j = 0 \) and write \( B_0 = B_{\epsilon_0}(x_0, y_0, \omega_0) \). We will assume, without loss of generality, that (1) holds for \((x_0, y_0, \omega_0)\); that is,
\[
|\partial_s^n \psi(x + s, y + t, \omega)| \geq A \quad \text{for all } (x, y, \omega) \in B_0 \quad \text{and } |s|, |t| < \epsilon_0 \quad \text{(38)}
\]
where \( A = (1/2)|\partial_s^n \psi(x_0, y_0, \omega_0)| > 0 \) and \( n_0 = n_0(x_0, y_0, \omega_0) \geq 2 \).

Using Proposition 4.1 twice, we see that the integral in (37) is equal to
\[
\iint_{|s|,|t|<\epsilon_0} e^{2\pi i \lambda \psi(x+s, y+t, \omega)+\rho s+\eta t} \frac{ds\,dt}{s\,t} + O(\log(1/\epsilon_0)).
\]
Here we are using the conclusion of Proposition 4.1 under the factorisation hypothesis (FH) and so the implicit constant appearing in the O term can be taken to be independent of \( \omega \in S^{d-1}\) as well as \( \lambda, \rho, \eta, x \) and \( y \).

For the next lemma we use the notation \( \psi_e(s, t, \nu) := \nu_1 \phi_1(s, t) + \cdots + \nu_e \phi_e(s, t) \) when \( 1 \leq e \leq d \) and \( \nu = (\nu_1, \ldots, \nu_e) \in S^{e-1}\). We note that if \( \psi = \psi_{\psi} \) satisfies (FH) on \( S^{d-1}\), then for each \( 1 \leq e \leq d \), \( \psi_e \) satisfies (FH) on \( S^{e-1}\).

**Lemma 6.1.** After a possible permutation of the \( d \)-tuple \( \overline{\phi} = (\phi_1, \ldots, \phi_d) \), there exists an \( 0 \leq e \leq d \) such that

(a) for every \((x, y, \nu) \in \mathbb{T}^2 \times S^{e-1}\), there is an \( m \geq 2 \) such that \( \partial_t^m \psi_e(x, y, \nu) \neq 0 \), and
(b) there exists $\omega_1 \in S^{d-1}, \omega_2 \in S^{d-2}, \ldots, \omega_{d-e} \in S^e$ and $\{y_1, y_2, \ldots, y_{d-e}\} \subset T\) such that

\begin{align*}
  b_1 & \quad \psi_d(s, t, \omega_1) \equiv \psi_d(s, y_1, \omega_1), \quad \omega_{1,d} \neq 0 \\
  b_2 & \quad \psi_{d-1}(s, t, \omega_2) \equiv \psi_{d-1}(s, y_2, \omega_2), \quad \omega_{2,d-1} \neq 0 \\
  \vdots & \quad \psi_{d-e}(s, t, \omega_{d-e}) \equiv \psi_{d-e}(s, y_{d-e}, \omega_{d-e}), \quad \omega_{d,e,e+1} \neq 0
\end{align*}

We remark that the cases $e = 0$ and $e = d$ are allowed. The case $e = 0$ means that (a) is vacuous and (b) holds with $d$ equations. The case $e = d$ means that (b) is vacuous and (a) holds with the original $d$-tuple, $\psi_d = \psi$.

**Proof** If (a) holds with $e = d$, then we are done and so suppose there exists $(x_1, y_1, \omega_1) \in T^d \times S^{d-1}$ such that $\partial^m_t \psi(x_1, y_1, \omega_1) = 0$ for all $m \geq 2$. Then Corollary 3.2 implies that $\psi(s, t, \omega_1) \equiv \phi(x, y_1, \omega_1)$; that is, $b_1$ holds. Without loss of generality, we may assume $\omega_{1,d} \neq 0$ (otherwise make a permutation). Hence (b) holds with $e = d - 1$. If (a) holds with $e = d - 1$, we are done.

If (a) does not hold with $e = d - 1$, then there exists $(x_2, y_2, \omega_2) \in T^2 \times S^{d-2}$ such that $\partial^m_t \psi_{d-1}(x_2, y_2, \omega_2) = 0$ for all $m \geq 2$. Since $\psi_{d-1}$ satisfies (FH) on $S^{d-2}$, Corollary 3.2 implies that $\psi_{d-1}(s, t, \omega_2) \equiv \psi_{d-1}(s, y_2, \omega_2)$. Without loss of generality, we may assume that $\omega_{2,d-1} \neq 0$. Hence (b) holds with $e = d - 2$. If (a) holds with $e = d - 2$ we are done.

If not, there exists $(x_3, y_3, \omega_3) \in T^2 \times S^{d-3}$ such that $\partial^m_t \psi_{d-2}(x_3, y_3, \omega_3) = 0$ for all $m \geq 2$. And so on...

A simple induction argument gives a complete proof of the lemma.

Lemma 6.1 implies that we may write

\[
\psi(s, t, \omega) = \omega \cdot \phi(s, t) = \sum_{j=1}^{e} c_j \phi_j(s, t) + \omega_{e+1} \left( \sum_{j=1}^{d} c_j^{e+1} \phi_j(s, t) + G_{e+1}(s) \right)
\]

\[
+ \omega_{e+2} \left( \sum_{j=1}^{e} c_j^{e+2} \phi_j(s, t) + c_j^{e+1} \sum_{j=1}^{e} c_j^{e+1} \phi_j(s, t) + G_{e+1}(s) \right) + G_{e+2}(s) + \cdots
\]

\[
= \sum_{j=1}^{e} c_j \phi_j(s, t) + \sum_{j=e+1}^{d} \omega_j F_j(s)
\]

(39)

for some real-analytic, 1-periodic functions $\{F_j\}_{j=e+1}^{d}$. If $\tau_e(\omega) := (c_1(\omega), \ldots, c_e(\omega)) \in \mathbb{R}^e \setminus \{0\}$, then

\[
\psi(s, t, \omega) = \lambda_e(\omega) \psi_e(s, t, \nu_e(\omega)) + \sum_{j=e+1}^{d} \omega_j F_j(s)
\]

(40)
where this equation defines \( \lambda_e(\omega) > 0 \) and \( \nu_e(\omega) \in \mathbb{S}^{d-1} \) as functions of \( \omega \in \mathbb{S}^{d-1} \) when \( \tau_\omega(\omega) \neq 0 \).

Lemma 6.1 also implies that for every \( (x, y, \nu) \in \mathbb{T}^2 \times \mathbb{S}^{d-1} \), there is an \( m = m(x, y, \nu) \geq 2 \) such that \( \partial^m_t \psi_e(x, y, \nu) \neq 0 \). Hence for every \( (x, y, \nu) \in \mathbb{T}^2 \times \mathbb{S}^{d-1} \), we can choose a \( \delta = \delta(x, y, \nu) > 0 \) such that

\[
|\partial^m_t \psi_e(x', y', \nu')| \geq \frac{1}{2} |\partial^m_t \psi_e(x, y, \nu)| \quad \text{for all } (x', y', \nu') \in B_{2\delta}(x, y, \nu).
\]

This gives us an open covering \( \{B_{2\delta}(x, y, \nu)\} \) of the compact space \( \mathbb{T}^2 \times \mathbb{S}^{d-1} \) from which we extract a finite subcover \( \{B_j\}_{j=0}^M \) with the property that for every \( (x, y, \nu) \in \mathbb{T}^2 \times \mathbb{S}^{d-1} \), there is a \( j = j(x, y, \nu) \in \{0, 1, \ldots, M\} \) such that \( (x, y, \nu) \in B_j(x_j, y_j, \nu_j) \) and

\[
|\partial^m_s \psi_e(x + s, y + t, \omega)| \geq B_j \quad \text{for all } |s|, |t| < \delta_j
\]

where \( B_j = (1/2)|\partial^m_t \psi_e(x_j, y_j, \nu_j)| > 0 \).

If \( e = 0 \), then

\[
\psi(s, t, \omega) = \omega \cdot \overline{\phi}(s, t) = \sum_{j=1}^{d} \omega_j F_j(s) =: \omega \cdot \overline{F}(s)
\]

and so

\[
\iint_{|s|,|t|<\epsilon_0} e^{2\pi i |\lambda \psi(x + s, y + t, \omega) + \rho s + \nu t|} \frac{ds}{s} \frac{dt}{t} = \int_{|t|<\epsilon_0} \int_{|s|<\epsilon_0} e^{2\pi i \lambda \omega \cdot \overline{F}(s) + \rho s + \nu t} \frac{ds}{s} \frac{dt}{t}
\]

splits as a product of two oscillatory integrals both of which are uniformly bounded by Proposition 4.1.

We may therefore assume \( e \geq 1 \). We again apply Proposition 4.1 to reduce (up to error terms depending only on \( \{\delta_j\}_{j=1}^M \)) our analysis of the oscillatory integral (37) to

\[
I := \iint_{|s|,|t|<\kappa} e^{2\pi i |\lambda \psi(x + s, y + t, \omega) + \rho s + \nu t|} \frac{ds}{s} \frac{dt}{t}
\]

where \( \kappa = \min(\epsilon_0, \delta_0, \ldots, \delta_M) \) and where the derivative bound (38) holds. We fix \( \omega \in \mathbb{S}^{d-1} \) and consider two cases.

**Case 1:** \( \tau(\omega) = 0 \). In this case, we see by (39),

\[
\psi(x + s, y + t, \omega) = \omega \cdot \overline{\phi}(x + s, y + t) = \sum_{j=e+1}^{d} \omega_j F_j(x + s)
\]

and so

\[
I = \int_{|t|<\kappa} e^{2\pi i \omega \cdot \overline{F}(s)} . \int_{|s|<\kappa} e^{2\pi i \lambda \omega_j F_j(x + s) + \rho s + \nu t} \frac{ds}{s}
\]

splits once again into a product of two oscillatory integrals both of which are uniformly bounded by Proposition 4.1.
Case 2: \( \tau(\omega) \neq 0 \). Then \( \nu_c = \nu_c(\omega) \in S^{e-1} \) and we see by (40),

\[
\psi(x + s, y + t, \omega) = \omega \cdot \varphi(x + s, y + t) = \lambda_e \psi_e(x + s, y + t, \nu_c) + \sum_{j=e+1}^{d} \omega_j F_j(x + s)
\]

where \( \lambda_e = \lambda_e(\omega) \). Furthermore \( (x, y, \nu_c(\omega)) \in B_{\delta_j} (x_j, y_j, \nu_j) \) for some \( 0 \leq j \leq M \).

For convenience we assume \( j = 0 \). Hence by (41),

\[
|\partial_t I_{0\omega} \psi_e(x + s, y + t, \nu_c(\omega))| \geq B_0 \text{ for all } |s|, |t| < \delta_0
\]
or

\[
|\partial_t I_{0\omega} \psi(x + s, y + t)| \geq B_0 \lambda_e
\]

holds for all \( |s|, |t| < \delta_0 \). We are now in a position to apply the arguments in Section 5 to successfully compare parts of the integral \( I \) to oscillatory integrals with almost polynomial phases of bounded degrees.

We split \( I = I_{>0} + I_{<0} = I_{>0} + I_{<0} \) as in Section 5 (more specifically, see Section 5.1). The treatment of these two integrals is similar and we choose to concentrate on

\[
I_{0}^{>0} = \iint_{R_0} e^{i[(\lambda \lambda_e) \varphi(s, t) + \sum_{j=e+1}^{d} \lambda \omega_j F_j(x + s) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t}
\]

where \( R_0 = \{|s||t| < \kappa : |s|^{0_m_0} < |t|^{0_m_0} \} \). In fact \( I_{0}^{<0} \) is slightly more involved than the complimentary integral \( I_{0}^{>0} \). We note that

\[
\lambda \psi(x + s, y + t, \omega) = (\lambda \lambda_e) \psi_e(x + s, y + t, \nu_c) + \sum_{j=e+1}^{d} \lambda \omega_j F_j(x + s)
\]

and that (41) holds with \( j = 0 \). This puts us in a position to apply the estimate in Section 5.1 to conclude that

\[
I_{0}^{>0} = \iint_{R_0} e^{i[(\lambda \lambda_e) \varphi(s, t) + \sum_{j=e+1}^{d} \lambda \omega_j F_j(x + s) + \rho s + \eta t]} \frac{ds}{s} \frac{dt}{t} + O(1)
\]

where

\[
\varphi(s, t) = P_{x,y,\omega}^*(s, t) + \psi_e(x + s, y, \nu_c) + \psi_e(x, y + t, \nu_c) - \psi_e(0, 0, \nu_c)
\]

and

\[
P_{x,y,\omega}^*(s, t) = \sum_{k, \ell \geq 1} \frac{1}{k! \ell !} \partial_t^{k, \ell} \psi_e(x, y, \nu_c(\omega)) s^k t^\ell.
\]

Since \( \partial_t^{k, \ell} \psi_e(x, y, \omega) = \lambda_e(\omega) \partial_t^{k, \ell} \psi_e(x, y, \nu_c(\omega)) \) whenever \( \ell \geq 1 \), we see that

\[
P_{x,y,\omega}(s, t) = \lambda_e P_{x,y,\omega}^*(s, t)
\]

where \( P_{x,y,\omega} \) is defined in (35) with respect to \( \psi(s, t) = \psi(s, t, \omega) \). Furthermore,

\[
\psi(x + s, y, \omega) + \psi(x, y + t, \omega) - \psi(0, 0, \omega) = \lambda_e \left[ \psi_e(x + s, y, \omega) + \psi_e(x, y + t, \nu_c) - \psi_e(0, 0, \nu_c) \right] + \sum_{j=e+1}^{d} \omega_j F_j(x + s) + \sum_{j=e+1}^{d} \omega_j (F_j(x) - F_j(0))
\]
and therefore

$$I_0^\prec = e^{-2\pi i \Lambda(x,\omega)} \iint_{R_0} e^{i[(\lambda \varphi(s,t) + \rho s + \eta t)]} \frac{ds}{s} \frac{dt}{t} + O(1)$$

where $\Lambda(x,\omega) = \sum_{j=e+1}^d \omega_j (F_j(x) - F_j(0))$ and

$$\varphi(s,t) = P_{x,y,\omega}(s,t) + \psi(x + s, y + \omega) + \psi(x, y + t, \omega) - \psi(0,0,\omega).$$

(46)

Slightly more straightforward reasoning leads to

$$I_0^\prec = \iint_{|t| < |s| < 0} e^{i[(\lambda \varphi(s,t) + \rho s + \eta t)]} \frac{ds}{s} \frac{dt}{t} + O(1).$$

Therefore we are left with understanding two oscillatory integrals with almost polynomial phases of bounded degree. We choose, without loss of generality, to concentrate on

$$\mathcal{I} := \iint_{R_0} e^{i[(\lambda \varphi(s,t) + \rho s + \eta t)]} \frac{ds}{s} \frac{dt}{t}$$

where $\varphi = \varphi_{x,y,\omega}$ is the almost polynomial phase defined above in (46). Recall that we are fixing $x, y$ and $\omega$ in this discussion and we have deduced various properties of $\psi$ which hold at $(x,y,\omega)$. In particular we know that both $\partial_n^m \psi(x,y,\omega)$ and $\partial_t^m \psi(x,y,\omega)$ are nonzero, see (38) and (42). However the exponents $m_0, n_0 \geq 2$ may not be minimal with respect to this property. Let $m = m(x,y,\omega) \geq 2$ be minimal with respect to $\partial_t^m \psi(x,y,\omega) \neq 0$. In particular we have $\partial_t^m \psi(x,y,\omega) = 0$ for all $2 \leq \ell < m$. Hence $\partial_t^m \psi(x,y,\omega) \neq 0$ and $\partial_t^\ell \psi(x,y,\omega) = 0$ for $2 \leq \ell < m$.

Likewise, let $n = n(x,y,\omega) \geq 2$ be minimal with respect to $\partial_n^m \psi(x,y,\omega) \neq 0$. By Corollary 3.3, we have $\partial^{k+\ell} \psi(x,y,\omega) = 0$ whenever $k + \ell < \max(m,n)$. Hence we may write

$$P_{x,y,\omega}(s,t) = \sum_{\substack{k+\ell \leq \max(m,n) \leq \max(m,n) \leq m_0 + \ell n_0 < m_0 n_0}} 1_{k\ell \geq \max(m,n)} \partial^{k+\ell} \psi(x,y,\omega) s^{k} t^{\ell}.$$ 

We decompose $\mathcal{I} = \mathcal{I}' + \mathcal{I}''$ where

$$\mathcal{I}' := \iint_{R'} e^{i[(\lambda \varphi(s,t) + \rho s + \eta t)]} \frac{ds}{s} \frac{dt}{t}$$

where $R' = \{(s,t) \in R_0 : |s|^n < |t|^m\}$. We will show that $\mathcal{I}'$ is uniformly bounded. The argument for $\mathcal{I}''$ is similar although slightly less notationally cumbersome. The first step is to reduce the region of integration to $R = \{(s,t) \in R_0 : C|s|^n \leq |t|^m\}$ for some large but absolute constant $C$ which may depend on the $d$-tuple of analytic functions $\varphi$ but it will not depend on the other parameters, $x, y, \lambda, \rho$ and $\eta$.

**Proposition 6.2.** The integral

$$\iint_{(s,t) \in R_0, C^{-1}|t|^m < |s|^n < |t|^m} e^{i[(\lambda \varphi_{x,y,\omega}(s,t) + \rho s + \eta t)]} \frac{ds}{s} \frac{dt}{t}$$

is uniformly bounded in $x, y, \omega, \lambda, \rho$ and $\eta$. 
Fixing either \(s\) or \(t\), the integral in the other variable has a uniformly bounded logarithmic measure and so one expects that this double oscillatory integral is effectively a single oscillatory integral and we should then be appealing to Proposition 4.1 or some variant. However it is a slightly technical argument to decouple the integration and reduce to a one dimensional integral. We choose to postpone the proof of this proposition until the end of the section so as not to detract from the main line of argument.

By Proposition 6.2 the study of \(I'\) is reduced to

\[
I_1 := \int_R \int e^{i[\lambda \varphi(s,t) + \rho_s + \eta t]} \frac{ds}{s} \frac{dt}{t}.
\]

As we have done previously, we decompose \(I_1\) dyadically \(I_1 = \sum_{(p,q) \in R_1} I_{p,q}\) where

\[
I_{p,q} = \int_R \int e^{2\pi i[\lambda \varphi(s,t) + \rho_s + \eta t]} (2^{p_s}) \frac{ds}{s} \frac{dt}{t}.
\]

where \(R_1 = \{(p,q) \in R : C 2^{-pm} \leq 2^{-qm}\} = \{(p,q) \in R : pn - qm \geq N_0\}\) for some large absolute constant \(C\), large integer \(N_0\). Here \(R\) is as in Section 5 which is defined as those pairs \((p, q)\) with both \(2^{-p}, 2^{-q} < \kappa\) such that \(2^{-pm_0} \leq 2^{-qm_0}\).

We will compare \(I_1\) to \(I_2\) where

\[
I_2 = \int_R \int e^{2\pi i[\lambda \psi(x,s,t) + \psi(x,s,t) + \rho_s + \eta t]} \frac{ds}{s} \frac{dt}{t}.
\]

In fact we will show that \(I_1 = I_2 + O(1)\). Since we can write

\[
I_2 = \int_{|s| \leq \kappa} e^{2\pi i[\lambda \psi(x,s,t) + \psi(x,s,t) + \rho_s + \eta t]} G(s) \frac{ds}{s} \quad \text{where} \quad G(s) = \int_{R(s)} e^{2\pi i[\lambda \psi(x,s,t) + \psi(x,s,t) + \rho_s + \eta t]} dt,
\]

where \(R(s) = \{t : (s, t) \in R\}\). By Proposition 4.1, we see that \(G(s) = O(1)\). Also since \(|G''(s)| \leq |s|^{-1}\), we can appeal to Proposition 4.1 once again to conclude that \(I_2\) and hence \(I_1\) is \(O(1)\) as desired.

As with \(I_1\) we will write \(I_2 = \sum_{(p,q) \in R_1} II_{p,q}\) where

\[
II_{p,q} = \int_R \int e^{2\pi i[\lambda \psi(x,s,t) + \psi(x,s,t) + \rho_s + \eta t]} (2^{p_s}) \frac{ds}{s} \frac{dt}{t}.
\]

We write \(D_{p,q} = I_{p,q} - II_{p,q}\) and derive various estimates for \(D_{p,q}\) which will add up to a uniform estimate when we sum over \((p, q) \in R_1\).

We make the changes of variables \(s \to 2^{-p} s\) and \(t \to 2^{-q} t\) in both integrals and after making these changes of variables, the difference of the phases in \(I_{p,q}\) and \(II_{p,q}\) is \(2\pi i \lambda P_{x,s,t}\). Hence the difference \(D_{p,q}\) has the bound

\[
|D_{p,q}| \leq 2\pi |\lambda| \sum_{k \ell \geq \max(m, n)} \frac{1}{k \ell !} (2^{p-k} - 2^{-q \ell}) (2^{-p} \psi \psi^\prime) (x, s, t, \omega) 2^{-p-k-q}.\]
Recall that for $k, \ell \geq 1$, we have $\partial^{k, \ell} \psi(x, y, \omega) = \lambda_\omega \partial^{k, \ell} \psi_e(x, y, \nu_e)$ where $\lambda_\omega = \lambda_e(\omega)$ and $\nu_e = \nu_e(\omega)$. Hence

$$|D_{p,q}| \leq 2\pi |\lambda_\omega| \sum_{k+\ell \geq \max(m,n)} \frac{1}{k!\ell!} |\partial^{k,\ell} \psi_e(x, y, \nu_e)| \ 2^{-pk-q\ell}. \quad (47)$$

At the heart of the argument is a comparison of the weighted mixed derivatives $2^{-pk-q\ell} \partial^{k,\ell} \psi_e(x, y, \nu_e)$ to weighted pure derivatives $2^{-q\ell} \partial^{\ell} \psi_e(x, y, \nu_e)$ with an exponential decay gain. This is the analogue of controlling mixed vector fields by pure ones in the work of Stein and Street, see [15]. The following proposition is the main proposition which gives this comparison.

**Proposition 6.3.** For any $k, \ell$ satisfying $\max(m,n) \leq k+\ell$ and $km_0 + \ell n_0 < m_0 n_0$, we have

$$2^{-pk-q\ell} |\partial^{k,\ell} \psi_e(x, y, \nu_e)| \lesssim 2^{-\delta(pm-qm)} \sum_{j=m}^{m_0} 2^{-qj} |\partial^{j} \psi_e(x, y, \nu_e)| \quad (48)$$

where $\delta = 1/n_0 > 0$.

**Proof** The proof relies on the consequences of the factorisation hypothesis (FH) we derived in Section 3. We will apply the results there to $\psi = \psi_e$ and $M = S^{c-1}$, recalling that $\psi_e$ satisfies (FH) on $S^{c-1}$. By Lemma 3.1, we have

$$\partial^{k,\ell} \psi_e(x, y, \nu_e) = \sum_{j=2}^{k+\ell} Q_j^{k,\ell} (x, y, \nu_e) \partial^j \psi_e(x, y, \nu_e) = \sum_{j=m}^{k+\ell} Q_j^{k,\ell} \partial^j \psi_e$$

since $\partial^j \psi_e(x, y, \nu_e) = 0$ for $2 \leq j < m$. Therefore

$$2^{-pk-q\ell} |\partial^{k,\ell} \psi_e(x, y, \nu_e)| \leq \sum_{j=m}^{k+\ell} 2^{-pk-q\ell} |Q_j^{k,\ell} (x, y, \nu_e) \partial^j \psi_e(x, y, \nu_e)|.$$  

To show (48), it suffices to show

$$\sum_{j=m}^{k+\ell} 2^{-pk-q\ell} |Q_j^{k,\ell} \partial^j \psi_e(x, y, \nu_e)| \lesssim 2^{-\delta N} \sum_{j=m}^{k+\ell} 2^{-qj} |\partial^j \psi_e(x, y, \nu_e)| \quad (49)$$

where $N := pm - qm$. This is clearly the case if $k+\ell \leq m_0$. However if $m_0 < k+\ell$, then for any $m_0 < j \leq k+\ell$, we have

$$2^{-qj} |\partial^j \psi_e(x, y, \nu_e)| \lesssim B_0^{-1} 2^{-qj} |\partial^{m_0} \psi_e(x, y, \nu_e)|$$

by (41) with $j = 0$.

Since $k \geq 1$ and $n \leq n_0$, we see that

$$2^{-pk-q\ell} = 2^{-\frac{1}{n}(pm-qm)} 2^{-q(\ell+km/n)} \leq 2^{-\delta N} 2^{-q(\ell+km/n)} \quad (50)$$

and so

$$2^{-pk-q\ell} \leq 2^{-\delta N} 2^{-q(\ell+k)} \quad \text{if } n \leq m. \quad (51)$$

Therefore when $n \leq m$, we have $2^{-pk-q\ell} \leq 2^{-\delta N} 2^{-qj}$ for all $m \leq j \leq k+\ell$ and this shows that (49) and hence (48) holds.
We may therefore assume that \( m < n \). In particular we have \( \max(m, n) = n \). Our aim now is to show that when \( m < n \), certain coefficients \( Q_{j,\ell}^{k,\ell}(x, y, \nu_e) \) necessarily vanish which then gives us some hope in establishing (49).

Let \( \mu \) be the largest integer such that \( 2^\mu m < n \). Since \( m < n \), we see that \( \mu \geq 0 \) and \( n \leq 2^{\mu + 1}m \). We apply Corollary 3.6 with this \( \mu \) to conclude that \( Q_{j,\ell}^{k,\ell}(x, y, \nu_e) = 0 \) for every \( j > \ell + 2^{-\mu - 1}k \). Therefore we see that several terms on the left hand side of (49) vanish and so we can update (49) and reduce matters to showing

\[
\sum_{j=m}^{\ell+2^{-\mu-1}k} 2^{-p_{\ell}} |Q_{j,\ell}^{k,\ell} \partial_\ell^j \psi_e(x, y, \nu_e)| \lesssim 2^{-\frac{\delta N}{2}} \sum_{j=m}^{\ell+2^{-\mu-1}k} 2^{-q_j} |\partial_\ell^j \psi_e(x, y, \nu_e)|
\]

instead. It therefore suffices to establish that \( 2^{-p_{\ell} - q_j} \leq 2^{-\frac{\delta N}{2}} 2^{-q_j} \) for all \( m \leq j \leq \ell + 2^{-\mu-1}k \) and this would follow if we could improve (51) to

\[
2^{-p_{\ell} - q_j} \leq 2^{-\frac{\delta N}{2}} 2^{-q_j(\ell + 2^{-\mu-1}k)}.
\]

From (50), we see that this is true since \( n \leq 2^{\mu + 1}m \).

By (47) and Proposition 6.3, there is an absolute, uniform constant \( C \) such that

\[
|D_{p,q}| \leq C |\lambda \lambda_e| 2^{-\frac{\delta N}{2}} \sum_{j=m}^{m_0} 2^{-q_j} |\partial_\ell^j \psi_e(x, y, \nu_e)|
\]

(52)

where we recall \( N = pm - qm \). Let us denote the sum on the right hand side by \( S_q = S_q(x, y, \nu_e) \); that is, \( S_q := \sum_{j=m}^{m_0} 2^{-q_j} |\partial_\ell^j \psi_e(x, y, \nu_e)| \). Our goal now is to seek a complementary bound to (52); namely, we will show that

\[
|I_{p,q}|, |II_{p,q}| \leq C (|\lambda \lambda_e| S_q)^{-\epsilon}
\]

(53)

for some absolute, uniform constant \( C \) and \( \epsilon > 0 \). If we can do this, then by taking a convex combination of the bounds in (52) and (53) we can conclude that

\[
|D_{p,q}| \leq C 2^{-\frac{\delta N}{2}} \min(|\lambda \lambda_e| S_q, (|\lambda \lambda_e| S_q)^{-1}) \epsilon
\]

(54)

for some \( \delta', \epsilon' > 0 \). Hence

\[
|I_1 - I_2| \leq \sum_{(p,q) \in \mathcal{R}_1} |D_{p,q}| \leq C \sum_{N \geq 0} 2^{-\frac{\delta N}{2}} \sum_{pm - qm = N} \min(|\lambda \lambda_e| S_q, (|\lambda \lambda_e| S_q)^{-1}) \epsilon'
\]

For fixed \( N \geq 0 \), the inner sum above is easily seen to be uniformly bounded. In fact, the sum is a sum only over integers \( q \) since \( p \) is uniquely determined by \( q \) once \( N \) is fixed. Splitting this inner sum in \( q \) further, depending on which term \( 2^{-q_j} |\partial_\ell^j \psi_e(x, y, \nu_e)| \) in the sum \( S_q \) is largest, and using the complementary estimates manifest in (54), we see that the inner sum is uniformly bounded. This implies that \( I_1 = I_2 + O(1) \) as was to be shown.

It remains to show (53). We first concentrate on establishing (53) for \( I_{p,q} \),

\[
I_{p,q} = \int \int e^{2\pi i (\lambda \varphi(s,t) + \rho \varphi_{s,t})} \zeta(2^p s) \zeta(2^q t) \frac{ds}{s} \frac{dt}{t} = \int \int e^{2\pi i \Phi(2^{-p} s, 2^{-q} t)} \zeta(s) \zeta(t) \frac{ds}{s} \frac{dt}{t}.
\]

Here \( \Phi(2^{-p} s, 2^{-q} t) = \lambda \varphi(2^{-p} s, 2^{-q} t) + \rho 2^{-p} s + \eta 2^{-q} t \) and

\[
\varphi(2^{-p} s, 2^{-q} t) = P_{x,y,\omega}(2^{-p} s, 2^{-q} t) + \psi(x + 2^{-p} s, y, \omega) + \psi(x + 2^{-q} s, y, \omega) + \psi(x, y + 2^{-q} s, \omega)
\]
We will think of $\Phi(2^{-p}s, 2^{-q}t)$ as a function of $t$, denoting it by $\Phi(t)$. By (44) and (45), we have

$$|\Phi^{(m_0)}(t)| = 2^{-q_{m_0}}(|(\lambda\lambda_c)\partial_t^{m_0}\psi_e(x, y + 2^{-q}t, \nu_e)| \geq 2^{-q_{m_0}}|\lambda\lambda_c|$$

by (41). Therefore if $2^{-q_{m_0}} \geq \delta \sum_{j=m}^{m_0-1} 2^{-q_j} |\partial_t^j\psi_e(x, y, \nu_e)|$ for some $\delta > 0$, then an application of van der Corput’s lemma establishes (53). Suppose instead that

$$2^{-q_{m_0}} \leq \delta \sum_{j=m}^{m_0-1} 2^{-q_j} |\partial_t^j\psi_e(x, y, \nu_e)|$$

for some small $\delta > 0$ (55)

which we will choose small enough later. We write

$$\psi(x, y + 2^{-q}t, \omega) = \sum_{\ell=m}^{m_0-1} \frac{1}{\ell!} \partial_t^\ell \psi(x, y, \omega)(2^{-q}t)^\ell + E_q(t) =: Q_{x,y,\omega}(t) + E_q(t)$$

where for any $2 \leq m' \leq m_0 - 1$, $|E_q^{(m')}|(t) \leq |\lambda_c|2^{-q_{m_0}}$. A simple equivalence of norms argument shows there exists a $\delta_{m_0} > 0$ and an $m \leq m' \leq m_0 - 1$ such that

$$|Q_{x,y,\omega}(t)| \geq \delta_{m_0}|\lambda_c| \sum_{j=m}^{m_0-1} 2^{-q_j} |\partial_t^j\psi_e(x, y, \nu_e)|$$

where we continue to employ (45). Choosing $\delta > 0$ small enough in (55), we have

$$|\partial_t^m\psi(x, y + t, \omega)| \geq (\delta_{m_0}/2)|\lambda_c| \sum_{j=m}^{m_0-1} 2^{-q_j} |\partial_t^j\psi_e(x, y, \nu_e)| \geq (\delta_{m_0}/4)|\lambda_c|S_q.$$  

By (44), we have

$$P_{x,y,\omega}(2^{-p}s, 2^{-q}t) = \lambda_c \sum_{k,\ell \geq 1 \atop k\delta_{m_0} + \ell\delta_{m_0} < m_0n_0} \frac{1}{k!\ell!} \partial_t^k\partial_\omega^\ell \psi_e(x, y, \nu_e(\omega))(2^{-p}s)^k(2^{-q}t)^\ell$$

and if we think of this as a polynomial in $t$, denoting it by $P(t)$, we see that

$$|P^{(m')}|(t) \leq |\lambda_c| \sum_{k,\ell \geq 1 \atop k\delta_{m_0} + \ell\delta_{m_0} < m_0n_0} \frac{1}{k!\ell!} 2^{-pk-qt} |\partial_t^k\partial_\omega^\ell \psi_e(x, y, \nu_e(\omega))| \leq 2^{-qN}|\lambda_c|S_q$$

by Proposition 6.3. Since $N \geq N_0$, we now choose $N_0$ large enough so that the bound

$$|\Phi^{(m')}|(t) \geq |\lambda|(|\partial_t^m\psi(x, y + t, \omega)| - |P^{(m')}|(t)) \geq (\delta_{m_0}/8)|\lambda_c|S_q$$

holds. Another application of van der Corput’s lemma establishes (53).

When we turn to $II_{p,q}$, we see that the only difference is that we replace $\Phi(t)$ with

$$\Psi(t) = \lambda_0(\psi(x + 2^{-p}s, y, \omega) + \psi(x, y + 2^{-q}t, \omega)) + p2^{-p}s + q2^{-q}t,$$

the difference being that the polynomial $P_{x,y,\omega}(s, t)$ is no longer present. So the first half of the argument above shows that there is an $m \leq m' \leq m_0$ such that $|\Psi^{(m')}|(t) \geq |\lambda\lambda_c|S_q$ and so van der Corput’s lemma establishes (53) in this case as well. This establishes (53) in all cases, completing the proof of Theorem 1.3 when the factorisation hypothesis (FH) holds.
6.4. **Proof of Proposition 6.2.** Here we return to the proof of Proposition 6.2 where we now establish the uniform boundedness of

\[
\mathcal{O} := \iint_{R_2} e^{i[\lambda \varphi(s,t) + ps + qt]} \frac{ds}{s} \frac{dt}{t}.
\]

Here \( \varphi(s,t) = \varphi_{x,y,\omega}(s,t) = P_{x,y,\omega}(s,t) + \psi(x + s, y, \omega) + \psi(x + y + t, \omega) \) and \( R_2 = \{(s,t) \in R : C^{-1}|t|^m < |s|^n < |t|^m\} \). In order to isolate the main one dimensional oscillatory integral in \( \mathcal{O} \) effectively, we will first reduce matters to the following oscillatory integral with polynomial phase \( \mathcal{P}(s,t) \),

\[
\mathcal{T} = \iint_{R_2} e^{2\pi i[\lambda \mathcal{P}(s,t) + ps + qt]} \frac{ds}{s} \frac{dt}{t},
\]

showing that \( \mathcal{O} = \mathcal{T} + O(1) \). Here

\[
\mathcal{P}(s,t) = P_{x,y,\omega}(s,t) + \sum_{k=0}^{n_0-1} \frac{1}{k!} \partial_s^k \psi(x, y, \omega)s^k + \sum_{\ell=m}^{m_0-1} \frac{1}{\ell!} \partial_t^\ell \psi(x, y, \omega)t^\ell
\]

\[
= \sum_{k,\ell \geq 0 \atop km_0 + \ell m_0 < m_0n_0} \frac{1}{k!\ell!} \partial^{k,\ell} \psi(x, y, \omega)s^k t^\ell.
\]

We will accomplish this in two steps. We first compare \( \mathcal{O} \) with

\[
\mathcal{T}_1 = \iint_{R_2} e^{2\pi i[\lambda \mathcal{P}_1(s,t) + ps + qt]} \frac{ds}{s} \frac{dt}{t}
\]

where

\[
\mathcal{P}_1(s,t) = P_{x,y,\omega}(s,t) + \sum_{k=0}^{n_0-1} \frac{1}{k!} \partial_s^k \psi(x, y, \omega)s^k + \psi(x + y + t, \omega),
\]

and show \( \mathcal{O} = \mathcal{T}_1 + O(1) \).

As with \( \mathcal{T}_1 \) we decompose these integrals dyadically, writing \( \mathcal{O} = \sum_{(p,q) \in R_2} \mathcal{O}_{p,q} \) where

\[
\mathcal{O}_{p,q} = \iint_{R_2} e^{2\pi i[\lambda \varphi(2^{-p} s, 2^{-q} t) + \rho 2^{-p} s + \eta 2^{-q} t]} \zeta(s) \zeta(t) \frac{ds}{s} \frac{dt}{t}.
\]

and \( \mathcal{T}_1 = \sum_{(p,q) \in R_2} \mathcal{T}_{1,p,q} \) where

\[
\mathcal{T}_{1,p,q} = \iint_{R_2} e^{2\pi i[\lambda \mathcal{P}_1(2^{-p} s, 2^{-q} t) + \rho 2^{-p} s + \eta 2^{-q} t]} \zeta(s) \zeta(t) \frac{ds}{s} \frac{dt}{t}.
\]

Here \( R_2 = \{(p,q) \in R : 0 \leq pm - qn \leq N_0\} \). Again we establish various estimates on the difference \( D_{p,q} = \mathcal{O}_{p,q} - \mathcal{T}_{1,p,q} \) but our task is much easier now since the sum over \( (p,q) \in R_2 \) is effectively a sum over \( p \) alone since for each \( 0 \leq N \leq N_0 \), the integer \( q \) is uniquely determined by \( p \) once \( N \) is fixed and there are only boundedly many \( N \) to consider.

Since the difference of phases appearing in \( \mathcal{O}_{p,q} \) and \( \mathcal{T}_{1,p,q} \) is

\[
2\pi i \lambda \left[ \psi(x + 2^{-p} s, y, \omega) - \sum_{k=0}^{n_0-1} \frac{1}{k!} \partial_s^k \psi(x, y, \omega)(2^{-p} s)^k \right] = O(|\lambda|2^{-pn_0}),
\]
we have $|D_{p,q}| \leq C|\lambda|2^{-p_{m_0}}$. But for both phases in $\mathcal{O}_{p,q}$ and $\mathcal{T}_{p,q}$ the $n_0$th derivative with respect to $s$ is bounded below by $|\lambda|2^{-p_{m_0}}$. Hence van der Corput’s lemma gives us the complimentary estimate $|D_{p,q}| \leq C(|\lambda|2^{-p_{m_0}})^{-1/n_0}$ and so for each $0 \leq N \leq N_0$,

$$\sum_{p:p_{m_0} = N} |D_{p,q}| \leq C \sum_{p:p_{m_0} = N} \min(|\lambda|2^{-p_{m_0}},(|\lambda|2^{-p_{m_0}})^{-1/n_0}) \leq C_N.$$ 

This shows that $\mathcal{O} = \mathcal{T}_1 + O(1)$.

We now compare $\mathcal{T}_1$ to $\mathcal{T}$ and show $\mathcal{T}_1 = \mathcal{T} + O(1)$. As with $\mathcal{O}$ and $\mathcal{T}_1$, we decompose $\mathcal{T} = \sum_{(p,q) \in \mathbb{R}_2} \mathcal{T}_{p,q}$ dyadically where

$$\mathcal{T}_{p,q} = \int \int e^{2\pi i [\lambda t^m(2^{-p_{m_0}}) + \rho t^{n_{m_0}} + \eta t]} \zeta(t) \zeta(s) \frac{ds}{s} \frac{dt}{t}.$$ 

In a similar way as above, we have the following bound for the difference $D_{p,q} = \mathcal{T}_{p,q} - \mathcal{T}_{p,q}$;

$$|D_{p,q}| \leq C \min(|\lambda|2^{-q_{m_0}},(|\lambda|2^{-q_{m_0}})^{-1/m_0})$$

and so for each $0 \leq N \leq N_0$,

$$\sum_{p:p_{m_0} = N} |D_{p,q}| \leq C \sum_{q:p_{m_0} = N} \min(|\lambda|2^{-q_{m_0}},(|\lambda|2^{-q_{m_0}})^{-1/m_0}) \leq C_N.$$ 

This shows that $\mathcal{T}_1 = \mathcal{T} + O(1)$ and hence $\mathcal{O} = \mathcal{T} + O(1)$, as claimed.

We make the change of variables $s \to s/\sqrt{|t|}$ in the $s$ integral so that

$$\mathcal{T} = \int_{|t| < \kappa} \frac{1}{t} [\int_{T(t)} e^{2\pi i [\lambda t^m(2^{-p_{m_0}}) + \rho t^{n_{m_0}}]} \frac{ds}{s}] dt$$

where $T(t) = \{ C^{-1} < |s| < C, |s|^{m_0} \leq |t|^{n_{m_0}} \}$. We now interchange the order of integration so that

$$\mathcal{T} = \int_{|t| < \kappa} \frac{1}{t} [\int_{T} e^{2\pi i [\lambda t^m(2^{-p_{m_0}}) + \rho t^{n_{m_0}}]} \frac{dt}{t} \frac{ds}{s}.$$ 

where $T = \{|t| < \kappa : a < |t| < b\}$ for some $a, b$ depending on $s$. It is possible, after a few more changes of variables, to reduce the inner integral to one with a purely polynomial phase and then we can appeal to the result of Stein and Wainger (10). However in their original paper [16], Stein and Wainger in fact prove an oscillatory integral estimate for phases with rational powers and in particular, their result implies that the inner integral above is uniformly bounded in the coefficients of $P$, $\lambda, \rho, \eta$ and $a, b$. The bound only depends on the degree of $P$ which in turn depends only on $n_0$ and $m_0$. Hence $\mathcal{T} = O(1)$ and this in turn implies $\mathcal{O} = O(1)$, completing the proof of Proposition 6.2.

7. THE PROOF OF THEOREM 1.3 – WHEN (FH) FAILS

We begin with a simple lemma about analytic functions whose proof is an application of the Weierstrass Preparation Theorem. We only present a special case which serves our purposes. The general case and the details of the proof can be found in [10].
Lemma 7.1. Let $M$ be a real-analytic manifold and let $\psi$ be a real-analytic function on $T^2 \times M$. Then the factorisation hypothesis (FH) fails for $\psi$ on $M$ if and only if there is a point $(x, y, \omega) \in T^2 \times M$ such that $\psi_{st}(x, y, \omega) \neq 0$ and either $\psi_{xt}(x, y, \omega) = 0$ or $\psi_{tt}(x, y, \omega) = 0$ (or both).

We now turn to the proof of Theorem 1.3 when the factorisation hypothesis (FH) fails. In this case it suffices to show that the norms $\|e^{2\pi in\overline{\phi}}\|_{U_{REC}(T^2)}$ are unbounded in $\overline{\pi} \in Z^d$ where $\overline{\phi}(s, t) = \overline{\phi}(s, t) + \overline{L}_1 s + \overline{L}_2 t$ parametrises our map $\Phi : T^2 \to T^d$. Here $\overline{\phi} = (\phi_1, \ldots, \phi_d)$ is a $d$-tuple of real-analytic, periodic functions and $\overline{L} := (\overline{L}_1, \overline{L}_2)$ is a pair of lattice points in $Z^d$.

Recall that
\[
\|e^{2\pi in\overline{\phi}}\|_{U_{REC}(T^2)} = \sup_{M,N} \|S_{MN}(e^{2\pi in\overline{\phi}})\|_{L^\infty(T)}
\]
\[
= \sup_{M,N,x,y} \left| \int_{T^2} e^{2\pi in\overline{\phi}(s,y,t)+\overline{L}(s,t)} D_M(s) D_N(t) dsdt \right|
\]
where $D_M$ denotes the Dirichlet kernel of order $M$. Here $\overline{L} : (s, t) = \overline{L}_1 s + \overline{L}_2 t$.

As before we set $\psi(x, t, \omega) = \omega \cdot \overline{\phi}(s, t)$ which defines a real-analytic map on $T^2 \times S^{d-1}$. If $\psi$ does not satsify (FH) on $S^{d-1}$, then Lemma 7.1 implies that there is a point $(x_0, y_0, \omega_0) \in T^2 \times S^{d-1}$ such that (without loss of generality)
\[
\psi_{st}(x_0, y_0, \omega_0) \neq 0 \quad \text{and} \quad \psi_{tt}(x_0, y_0, \omega_0) = 0.
\]

7.2. An intuitive approach which does not quite work. It seems natural now to consider the above oscillatory integral
\[
\int_{T^2} e^{2\pi in\overline{\phi}(s,y,t)+\overline{L}(s,t)} D_M(s) D_N(t) dsdt
\]
evaluated for $x = x_0$, $y = y_0$ and for $\overline{\pi} \in Z^d$ with $\overline{\pi} = \|\overline{\pi}\| \omega$ where $\omega$ is as close as possible to $\omega_0$. In general $\omega_0 \neq \overline{\pi}/\|\overline{\pi}\|$ for any $\overline{\pi} \in Z^d$ but for this discussion, let us simply matters and assume we can take $\omega = \omega_0$.

With strategic choices of $\lambda := \|\overline{\pi}\|$, $M$ and $N$, we would then like to show that the above integral is unbounded as $\lambda \to \infty$. Using the property (3) of Dirichlet kernels, together with Proposition 4.1, we see that
\[
\int_{T^2} e^{2\pi i\lambda \psi(x_0+s,y_0+t)+\overline{L}(s,t)} D_M(s) D_N(t) dsdt =
\]
\[
\int_{|s|,|t| \leq 1/2} e^{2\pi i\lambda \psi(x_0+s,y_0+t)+\omega_0 \overline{L}(s,t)} sin(Ms) sin(Nt) \frac{ds dt}{s t} + O_{\omega_0}(1)
\]
where $\omega_0 \cdot \overline{L}(s,t) := \omega_0 \cdot \overline{L}_1 s + \omega_0 \cdot \overline{L}_2 t$.

Let us denote by $S$ the integral above with the double Hilbert transform singularity $1/st$. We scale $S$ by making the change of variables $s \to \delta^2 s$ and $t \to \delta^2 t$. Hence
\[
S = \int_{|s| \leq \delta^{-1}} \int_{|t| \leq \delta^{-2}} e^{2\pi i\lambda \psi_3(s,t)+\omega_0 \overline{L}(s^2,s^2)} sin(M\delta^3 s) sin(N\delta^2 t) \frac{ds dt}{s t}
\]
where
\[\psi(s, t) := \psi(x_0 + \delta^3 s, y_0 + \delta^2 t, \omega_0) = \psi(x_0, y_0, \omega_0) + \nabla \psi(x_0, y_0, \omega_0) \cdot (\delta^3 s, \delta^2 t) + \delta^5 \left[ \psi_{s,t}(x_0, y_0, \omega_0) s t + \delta \psi_{s,s}(x_0, y_0, \omega_0) s^2 t^2 / 2 + \sum_{k+ℓ≥3} \frac{δ^{3k+2ℓ−5}}{k!ℓ!} \partial^{k,ℓ} \psi(x_0, y_0, \omega_0) s^k t^ℓ \right].\]

The gradient \(\nabla = \nabla_{s,t}\) is taken with respect to the variables \((s, t)\) on \(\mathbb{T}^2\).

This scaling is suggested by the Newton diagram of \(\psi(x, y, \omega)\) based at the point \((x_0, y_0)\) whose essential feature is the content of (56). By choosing \(\lambda = \delta^{-5}\), we see that
\[e^{2πι[\psi_x(s,t)+\psi_y(t)−\psi_t(s)]}(\delta^3 s, \delta^2 t)] = Ce^{2πi[\psi_t−\psi_x−\psi_y]+O(δ)}\]
for some \(|C| = 1\) and \(c \neq 0\). Here \(\vec{v} = \vec{v}(x_0, y_0, \omega_0) = ω_0 \cdot \mathcal{T} + \nabla \psi(x_0, y_0, \omega_0)\) (recall \(\nabla ω_0 \cdot \mathcal{T} = (ω_0 \cdot \mathcal{L}_1, ω_0 \cdot \mathcal{L}_2)\)).

Now if we choose \((M, N)\) so that
\[\vec{v}(x_0, y_0, \omega_0) ∙ \left(\delta^{-2}, \delta^{-3}\right) ∼ (M δ^3, N δ^3)\]
and take \(\lambda\) large (or equivalently, \(δ\) small), we might hope that all the terms in the Taylor expansion of \((s, t) \to \psi(s, t, \omega)\) at \((x_0, y_0)\) of order 2 and larger become less and less significant compared to the hyperbolic oscillation \(ψ_{s,t}(x_0, y_0, \omega_0) s t\), putting us in an analogous situation of (7) arising in C. Fefferman’s work [5] so that we can deduce that the oscillatory integral blows up like \(\log λ = \log(1/δ)\). Such scaling arguments were used in [3] and with some effort all this can be carried out here \(IF\) the vector
\[\vec{v}(x_0, y_0, \omega_0) = ω_0 \cdot \mathcal{T} + \nabla \psi(x_0, y_0, \omega_0) = ω_0 \cdot \left[\nabla ψ(x_0, y_0) + \mathcal{T}\right]\]
has the property that both its components are nonzero.

Unfortunately it may happen that \(\vec{v}\) vanishes. For example, consider the case \(d = 1\) and \(ϕ(s, t) = \sin(2πs) \cos(2πt)\) so that \(ψ = ϕ\). At the point \((x_0, y_0) = (0, 1/4)\), we have \(ϕ_{s,t}(0, 1/4) = 0\) yet \(ϕ_{s,s}(0, 1/4) = −4π^2\) and so the factorisation hypothesis (FH) fails for this example. We take \(\mathcal{T} = (L_1, L_2) = 0\) and so in this case, \(\vec{v} = \nabla ϕ(0, 1/4) = 0\). On the other hand, it is not too difficult to show that
\[\sup_{n, M, N} \left| \iint_{|s|, |t| ≤ 1/2} e^{2πi \left[\sin(2πs) \cos(2πt + π/2)\right]} \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t} \right| < ∞.\]

7.3. A rigorous approach. Therefore we need to proceed in a slightly different way. We will perturb the point \((x_0, y_0) \in \mathbb{T}^2\) in an arbitrarily small neighbourhood so that the two components of
\[\vec{v}(x, y, \omega_0) = ω_0 \cdot \left[\nabla ψ(x, y) + \mathcal{T}\right]\]
are nonzero, \(ψ_{s,t}(x, y, \omega) \neq 0\) and \(0 < |ψ_{s,t}(x, y, \omega)| \leq ε\) is as small as we like. We will then show that there exists \(\overline{A} ∈ \mathbb{Z}^d\) and choices for \(M\) and \(N\) such that the oscillatory
\(\iint_{\mathbb{T}^2} e^{2πi \overline{A} \cdot (s, t)} \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t}\)
is bounded below by \(\log(1/\varepsilon)\). This will establish that the mapping property \((\Phi)_{rect}\) fails for \(\Phi = \Phi_{x,T}\) as well as exhibit an inherent discontinuity in these oscillatory integrals we are studying.

First we observe that \(\psi_{tt}(x, y, \omega_0)\) is not identically zero; otherwise \(\partial_t \psi(x, y, \omega_0) \equiv \partial_t \psi(x, y_0, \omega_0)\) which in turn implies that \(\psi_{st}(x_0, y_0, \omega_0) = 0\), contradicting (56). Similarly \(\psi_{ss}(x, y, \omega_0)\) is not identically zero. Hence the zero set

\[
Z = \{ (x, y) \in T^2 : \psi_{tt}(x, y, \omega_0) = 0 \text{ or } \psi_{ss}(x, y, \omega_0) = 0 \}
\]

is a set of measure zero and in particular, we can find points \((x, y)\) arbitrarily close to \((x_0, y_0)\) such that both \(|\psi_{tt}(x, y, \omega_0)|, |\psi_{ss}(x, y, \omega_0)| > 0\) and \(|\psi_{tt}(x, y, \omega_0)|\) is as small as we like.

In the next lemma we will derive derivative bounds for \(\psi\) in a neighbourhood \(V\) of \((x_0, y_0)\).

**Lemma 7.4.** There exists a pair \(m_0, n_0 \geq 2\) such that for any sufficiently small neighbourhood \(V\) of \((x_0, y_0)\), we have

\[
A/2 \leq |\partial_t^{m_0} \psi(x, y, \omega_0)| \leq 2A \quad \text{and} \quad B/2 \leq |\partial_x^{n_0} \psi(x, y, \omega_0)| \leq 2B
\]

for all \((x, y) \in V, \psi(x_0, y, \omega_0) \neq 0\). Here \(A = |\partial_t^{m_0} \psi(x_0, y_0, \omega_0)|\) or \(A = |\partial_1^{m_0} \psi(x_0, y_0, \omega_0)|\) or \(A = |\partial_2^{m_0} \psi(x_0, y_0, \omega_0)|\) or \(A = |\partial^{m_0,1} \psi(x_0, y_0, \omega_0)|\) depending (respectively) on whether or not there is an \(m \geq 3\) such that \(\partial_t^{m_0} \psi(x_0, y_0, \omega_0) \neq 0\). Similarly \(B = |\partial_x^{n_0} \psi(x_0, y_0, \omega_0)|\) or \(B = |\partial^{n_0,1} \psi(x_0, y_0, \omega_0)|\) or \(B = |\partial^{n_0} \psi(x_0, y_0, \omega_0)|\) or \(B = |\partial^{n_0,1} \psi(x_0, y_0, \omega_0)|\) or \(B = |\partial^{n_0} \psi(x_0, y_0, \omega_0)|\).

Furthermore since \(\psi_{st}(x_0, y_0, \omega_0) \neq 0\), we can choose \(V\) so that for all \((x, y) \in V\),

\[
D/2 \leq |\psi_{st}(x, y, \omega_0)| \leq 2D \quad \text{where} \quad D := |\psi_{st}(x_0, y_0, \omega_0)|.
\]

**Proof** We consider two cases. First suppose there exists \(m_0 \geq 3\) such that \(\partial_t^{m_0} \psi(x_0, y_0, \omega_0) \neq 0\). In this case we set \(A = |\partial_t^{m_0} \psi(x_0, y_0, \omega_0)| > 0\) and choose by continuity an open neighbourhood \(V\) of \((x_0, y_0)\) such that \(A/2 \leq |\partial_t^{m_0} \psi(x, y, \omega_0)| \leq 2A\) for all \((x, y) \in V\).

Next suppose that \(\partial_t^{m_0} \psi(x_0, y_0, \omega_0) = 0\) for all \(m \geq 2\). Then

\[
\psi(x_0, y, \omega_0) = \psi(x_0, y_0, \omega_0) + \partial_t \psi(x_0, y_0, \omega_0)(y - y_0)
\]

for all \(y \in T\) by analyticity. This forces \(\partial_t \psi(x_0, y_0, \omega_0) = 0\) since \(y \rightarrow \psi(x_0, y, \omega_0)\) is a periodic function. Hence \(\psi(x_0, y, \omega_0) \equiv constant\) and so \(\partial_t^{m} \psi(x_0, y, \omega) \equiv 0\) for all \(m \geq 1\). Thus for every \(m \geq 1\),

\[
\partial_t^{m} \psi(x, y, \omega_0) = \int_{x_0}^{x} \partial^{1,m} \psi(u, y, \omega_0) \, du
\]

and we claim there exists an \(m_0 \geq 2\) such that \(\partial^{1,m} \psi(x_0, y_0, \omega_0) \neq 0\). If this is not the case, then

\[
\partial_x \psi(x_0, y, \omega_0) = \partial_x \psi(x_0, y_0, \omega_0) + \partial^{1,1} \psi(x_0, y_0, \omega_0)(y - y_0)
\]

\footnote{In fact these bounds hold for all \((z, w) \in W\) in a complex neighbourhood \(W \subset \mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2\) of \((x_0, y_0) + i(0, 0)\). This remark will allow us to use standard Cauchy estimates on the derivatives of \(\partial_t^{m_0} \psi\) and \(\partial_x^{n_0} \psi\).}
for all \( y \in \mathbb{T} \) by analyticity. But \( \psi_{st}(x_0, y_0, \omega_0) = \partial^{1,1}_y \psi(x_0, y_0, \omega_0) \neq 0 \) which implies that the right hand side is a nonconstant, nonperiodic function of \( y \). This contradicts the periodicity in \( y \) of the left hand side. Thus there exists an \( m_0 \geq 2 \) such that \( \partial^{1,m_0}_y \psi(x_0, y_0, \omega_0) \neq 0 \) and hence a neighbourhood \( V \) of \( (x_0, y_0) \) such that for all \( (x, y) \in V \),

\[
\frac{1}{2} |\partial^{1,m_0}_y \psi(x_0, y_0, \omega_0)||x - x_0| \leq |\partial^{m_0}_t \psi(x, y, \omega_0)| \leq 2|\partial^{1,m_0}_y \psi(x_0, y_0, \omega_0)||x - x_0|.
\]

Therefore the first part of (58) holds with \( A = |\partial^{1,m_0}_y \psi(x_0, y_0, \omega_0)||x - x_0| \).

A similar argument shows the existence of an \( n_0 \geq 2 \) such that the second part of (58) holds. 

For convenience, we write the neighbourhood \( V \) of \( (x_0, y_0) \) as a ball \( B_{r_0}(x_0, y_0) \) so that Lemma 7.4 holds for any sufficiently small \( r_0 > 0 \). We will determine later exactly how small we will take \( r_0 > 0 \) but then we will fix \( r_0 \) and allow all our estimates to depend on \( r_0 \).

Given any \( \varepsilon > 0 \), we can certainly find an \( r = r(\varepsilon) > 0 \) such that \( 0 < r < r_0/2 \) and \( |\psi_{tt}(x, y, \omega_0)| \leq \varepsilon \) for all \( (x, y) \in B_{2r}(x_0, y_0) \). This simply follows from the fact that \( \psi_{tt}(x_0, y_0, \omega_0) = 0 \).

Since \( B_{2r}(x_0, y_0) \subset B_{r_0}(x_0, y_0) = V \), we see that (58) holds with \( (x, y) \) replaced by \( (x + s, y + t) \) where \( (x, y) \in B_r(x_0, y_0) \) and \(|s|, |t| < r_0/2\). Furthermore, (59) and \( |\psi_{tt}(x, y, \omega_0)| \leq \varepsilon \) hold for all \( (x, y) \in B_r(x_0, y_0) \). Our goal is to find a point \( (x, y) \in B_r(x_0, y_0) \) with a few additional properties. But first we need some information about certain zero sets.

**Lemma 7.5.** Each component of the analytic function

\[
\bar{v}(x, y, \omega_0) = \nabla \psi(x, y, \omega_0) + \omega_0 \cdot \mathbf{T} = \omega_0 \cdot [\nabla \phi(x, y) + \mathbf{T}]
\]

is not identically equal to zero and so the zero sets

\[
Z_1 := \{(x, y) \in \mathbb{T}^2 : \partial_x \psi(x, y, \omega_0) + \omega_0 \cdot \hat{L}_1 = 0\}
\]

and

\[
Z_2 := \{(x, y) \in \mathbb{T}^2 : \partial_t \psi(x, y, \omega_0) + \omega_0 \cdot \hat{L}_2 = 0\}
\]

are sets of measure zero.

**Proof** We treat each component of \( \bar{v} \) separately. If \( \partial_x \psi(x, y, \omega_0) = -\omega_0 \cdot \hat{L}_2, \) then \( \omega_0 \cdot \hat{L}_2 = 0 \) since derivatives of periodic functions must vanish somewhere. Hence \( \psi_{st}(x_0, y_0, \omega_0) = 0 \) which contradicts (56).

We are in a position to put our various observations together.

**Lemma 7.6.** Given \( \varepsilon > 0 \). There exist a positive \( r = r(\varepsilon) \) with \( 0 < r < r_0/2 \) and a point \( (x, y) \in B_{r}(x_0, y_0) \) such that for all \(|s|, |t| < r_0/2\),

\[
A/2 \leq |\partial^{m_0}_t \psi(x + s, y + t, \omega_0)| \leq 2A, \quad A > 0, \tag{60}
\]

\[
B/2 \leq |\partial^{m_0}_s \psi(x + s, y + t, \omega_0)| \leq 2B, \quad B > 0 \tag{61}
\]
hold. Also $D/2 \leq |\partial_x \psi(x, y, \omega_0)| \leq 2D$ and $|\psi_t(x, y, \omega_0)| \leq \varepsilon$ hold. Finally, $0 < |\psi_t(x, y, \omega_0)|, |\psi_{xx}(x, y, \omega_0)|$ and both components of the vector
\[
\vec{v}(x, y, \omega_0) = \nabla \psi(x, y, \omega_0) + \omega_0 \cdot \vec{I} = \omega_0 \cdot [\nabla \phi(x, y) + \vec{I}]
\]
are nonzero.

**Proof** We take $r = r(\varepsilon)$ as in the discussion immediately prior to the statement of Lemma 7.5. Since $B_{2r}(x_0, y_0) \subset B_{r_0}(x_0, y_0)$, we see that (60) and (61) hold for all $(x, y) \in B_r(x_0, y_0)$ and $|s|, |t| < r_0/2$. As long as we choose $(x, y)$ such that $x \neq x_0$ and $y \neq y_0$, then both $A$ and $B$ will be nonzero. Furthermore, since the union of the zero sets $Z \cup Z_1 \cup Z_2$ is a set of measure zero, we see that $B_r(x_0, y_0) \setminus (Z \cup Z_1 \cup Z_2)$ is nonempty. By Lemma 7.5, we need only to choose any $(x, y) \in B_r(x_0, y_0) \setminus (Z \cup Z_1 \cup Z_2)$. \hfill \blacksquare

The next proposition completes the proof of Theorem 1.3 when the factorisation hypothesis (FH) fails.

**Proposition 7.7.** Suppose that (FH) fails on $\mathbb{S}^{d-1}$ for $\psi(x, y, \omega) = \omega \cdot \phi(x, y)$. Then, for sufficiently small $\varepsilon > 0$, we have
\[
\sup_{\pi, M, N, x, y} \left| \int_\mathbb{T} e^{2\pi i \pi |\phi(x_0 + s, y_0 + t) + \vec{I}(s, t)|} \sin(Ms) \sin(Nt) \frac{ds \, dt}{s \, t} \right| \gtrsim \log(1/\varepsilon).
\]

The proof will be carried out in several stages.

For given $\varepsilon > 0$, we fix an $r = r(\varepsilon) > 0$ and $(x, y) = (x_\varepsilon, y_\varepsilon) \in \mathbb{T}^2$ with the properties given in Lemma 7.6.

With this point choice $(x, y) \in \mathbb{T}^2$, our main task is to establish
\[
\sup_{\lambda, M, N} \left| \int_\mathbb{T} e^{2\pi i \lambda \varphi(s, t)} \sin(Ms) \sin(Nt) \frac{ds \, dt}{s \, t} \right| \gtrsim \log(1/\varepsilon) \tag{62}
\]
where $\varphi(s, t) = \phi_{x_\varepsilon, x_\varepsilon}(s, t) := \omega_0 \cdot [\phi(x + s, y + t) + \vec{I}(s, t)].$

Since $r_0$ is independent of $\varepsilon > 0$, we see that by a couple applications of Proposition 4.1, the oscillatory integral in (62) is equal to
\[
\int_{|s|, |t| < r_0/2} e^{2\pi i \lambda \varphi(s, t)} \sin(Ms) \sin(Nt) \frac{ds \, dt}{s \, t} + O_{r_0, \omega_0}(1),
\]
with the important observation that the bound in the $O(1)$ term is independent of $\varepsilon$. Let us denote by $\mathcal{I}$ the oscillatory integral in the above displayed equation. Hence (62) follows from the existence of a $\Lambda(\varepsilon)$ such that whenever $\lambda > \Lambda(\varepsilon)$, we can find choices for $M$ and $N$ such that
\[
|\mathcal{I}| \gtrsim \log(1/\varepsilon) \tag{63}
\]
for sufficiently small $\varepsilon > 0$. 
show that instead we will go a bit further than the argument developed in Section 5.2 and unfortunately the bounds derived in Section 5 and Section 5.2 depend on which may depend on $\varepsilon$. Derivative bounds (31) and (36) correspond to the above derivative bounds on $x$. In (62), we have by Lemma 7.6 (see (60) and (61)) that for all $|s|, |t| < r_0/2$,

$$A/2 \leq |\partial_x^{m_0}\varphi(s, t)| \leq 2A, \quad \text{and} \quad B/2 \leq |\partial_x^{n_0}\varphi(s, t)| \leq 2B,$$

where $A = |\partial_x^{m_0}\varphi(0, 0)|$ or $A = |\partial_x^{m_0, n_0}\varphi(0, 0)(x - x_0)|$. Similarly for $B$. Note that $A$ and/or $B$ may depend on $x$ or $y$ which in turn depends on $\varepsilon$. We are in a position to employ the analysis in Section 5 and in particular Section 5.2 since the required derivative bounds (31) and (36) correspond to the above derivative bounds on $\varphi$. Unfortunately the bounds derived in Section 5 and Section 5.2 depend on $A$ and $B$ which may depend on $\varepsilon$ and this is not good for us.

Instead we will go a bit further than the argument developed in Section 5.2 and show that

$$\mathcal{J} = \iint_{R_{r_0}} e^{2\pi i \lambda \varphi(s, t) + \rho s + \eta t} \frac{ds \, dt}{s \, t} + O_{r_0, \omega_0}(1) \quad (64)$$

where

$$R_{r_0} = \{(s, t) : |s|, |t| < r_0/2, \max_{2 \leq m \leq m_0} \varepsilon_m |t|^m, \max_{2 \leq n \leq n_0} \sigma_n |s|^n \leq |s| t |\}.$$

Again the important point here is that the bound $O(1)$ in (64) can be taken to be independent of $\varepsilon$ (and therefore independent of $x, y, \lambda, M$ and $N$).

Here $\varepsilon_m := |\partial_t^m \varphi(0, 0)|$ and $\sigma_n := |\partial_x^n \varphi(0, 0)|$, Note that $\varepsilon_m = |\partial_t^m \psi(x, y, \omega)|$ and $\sigma_n = |\partial_x^n \psi(x, y, \omega)|$ depend on $\varepsilon$ since $(x, y)$ depends on $\varepsilon$. In particular $\varepsilon_{m_0} \sim A$ and $\sigma_{n_0} \sim B$ are both nonzero. Also $0 < \varepsilon_2 = |\psi_{tt}(x, y, \omega)| \leq \varepsilon$ and $0 < \sigma_2 = |\psi_{xx}(x, y, \omega)|$ by our choice of the point $(x, y)$ in Lemma 7.6. These properties will be very important towards the end of the analysis.

We will achieve (64) in a couple of steps. First we will obtain a uniform bound for part of the oscillatory integral $\mathcal{J}$; namely, we consider the part where we integrate over the region $R'_{r_0} = \{(s, t) : |s|, |t| \leq r_0/2, |s| t | \leq \max_{2 \leq m \leq m_0} \varepsilon_m |t|^m \}$.
With the notation in Section 5, we write
\[
\int \int_{R_0^*} e^{i[\lambda \varphi(s,t)+\rho s+\eta t]} \frac{ds}{s} \frac{dt}{t} = \sum_{(p,q) \in R_0^*} I_{p,q}
\]
where now
\[R_0^* = \{(p,q) : 2^{-p}, 2^{-q} \leq r_0/2, \ 2^{-p-q} \leq \max_{2 \leq m \leq m_0-1} \varepsilon_m 2^{-q m}\}.
\]

We split \(R_0^*\) into at most \(m_0\) subsets depending on which of the terms \(\varepsilon_m 2^{-q m}\) is maximal among \(2 \leq m \leq m_0\). We will concentrate on
\[
\sum_{(p,q) \in R_0^m} I_{p,q} = \sum_{(p,q) \in R_0^m} \int \int_{R_0^*} e^{i[\lambda \varphi(s,t)+\rho s+\eta t]} \frac{ds}{s} \frac{dt}{t}
\]
where \(R_0^m\) consists those \((p,q)\) in \(R_0^*\) where \(\varepsilon_m 2^{-q m}\) is maximal. We set \(N = p + q(m-1)\) and define \(L\) so that \(2^L = \varepsilon_m^{-1}\). Hence the pairs \((p,q)\) in \(R_0^m\) satisfy \(N \geq L\).

We compare \(I_{p,q}\) with
\[
II_{p,q} = c \int \int e^{i[\lambda \varphi_1(s)+\varphi_2(t)]+\rho s+\eta t} \zeta(2^p s) \zeta(2^q t) \frac{ds}{s} \frac{dt}{t}
\]
where \(c = e^{\lambda \varphi(0,0)}, \varphi_1(s) = \varphi(s,0)\) and \(\varphi_2(t) = \varphi(0,t)\). We make the change of variables \(s \rightarrow 2^{-p} s\) and \(t \rightarrow 2^{-q} t\) and note that
\[
|D_{p,q}| := |I_{p,q} - II_{p,q}| = |\lambda| \sum_{k,t \geq 1} \frac{1}{k!} \partial^{k,t} \varphi(0,0)(2^{-p} s)^k (2^{-q} t)^t \leq C|\lambda| 2^{-p-q}
\]
\[
= C|\lambda| 2^{-p-\left(m-1\right)} \varepsilon_m^{-1} 2^{-q m} = C|\lambda| 2^{-\left(N-L\right)} \varepsilon_m 2^{-m q}. \quad (65)
\]

**Claim:** If \(\Phi(t) := \varphi(2^{-p} s, 2^{-q} t) = \psi(x + 2^{-p} s, y + 2^{-q} t, \omega_0)\), then there is a 2 \(\leq m' \leq m_0\) such that \(|\Phi^{(m')}(t)| \geq \varepsilon_m 2^{-q m}\) for all \(|t| \sim 1\).

This is our replacement for the derivative bound (31).

To prove the claim, we first set
\[
H(t) = \varphi(2^{-p} s, 2^{-q} t) - \varphi(0, 2^{-q} t) = 2^{-p} s \int_0^1 \frac{\partial \varphi}{\partial s}(r 2^{-p} s, 2^{-q} t) dr
\]
and note that for any 2 \(\leq m' \leq m_0\),
\[
|H^{(m')}(t)| \leq C2^{-p-r} = C2^{-q r} 2^{-p-r} \leq C2^{-q \varepsilon m 2^{-q m}}. \quad (66)
\]
Since we can take \(2^{-q} < r_0/2\) to be small, we will treat \(H(t)\) as an error term.

Next we write \(\varphi(0, 2^{-q} t) = g(t) + h(t)\) where
\[
g(t) = \sum_{t=0}^{m_0-1} \frac{1}{t!} \partial^t \varphi(0,0)(2^{-q} t)^t \quad \text{and} \quad h(t) = (2^{-q} t)^{m_0} \int_0^1 \cdots \int_0^1 \partial^{m_0} \varphi(0, \sigma(r) 2^{-q} t) d\mu
\]
Here $\sigma(r) = r_1 \cdots r_m$ and $d\mu = r_2 r_3 \cdots r_m^{-1}$. By classical Cauchy estimates applied to the analytic function $z \to \partial_t^{m_0} \psi(x + s, z, \omega_0)$, we have for any $|s|, |t| < r_0/2$,

$$|\partial^{t+m_0} \psi(s, t)| = |\partial_t^{m_0} \psi(x + s, y + t, \omega_0)| \leq C A \lesssim A$$

since $x, y \in B_{r_0/2}(x_0, y_0)$. This follows from Lemma 7.4, see (58) (also see the footnote there). See also (60). Hence for any $2 \leq m' \leq m_0$, we have

$$|h^{(m')}(t)| \lesssim A 2^{-q m_0} \lesssim \epsilon_m 2^{-q m_0}.$$  

(67)

A simple equivalence of norms argument (applied to the space of polynomials of degree at most $m_0 - 1$) shows the existence of a positive constant $\delta_{m_0} > 0$ and an $2 \leq m' \leq m_0 - 1$ such that

$$|g^{(m')}(t)| \geq \delta_{m_0} \max_{2 \leq m'' \leq m_0 - 1} |\partial_t^{m''} \psi(0, 0)| 2^{-q m''}$$  

(68)

for all $|t| \sim 1$. We note that the right hand side of (68) is equal to $\delta_{m_0} \epsilon_m 2^{-q m}$ if $m \leq m_0 - 1$.

To complete the proof of the claim, we split the argument into two cases:

**Case 1:** $\epsilon_m 2^{-q m_0} \geq K^{-1} \delta_{m_0} \epsilon_m 2^{-q m}$ where $4K$ is the implicit constant appearing in the bound (67).

In this case, we have by (58) or (60)

$$|\Phi^{(m_0)}(t)| = 2^{-q m_0} |\partial_t^{m_0} \psi(x + 2^{-p} s, y + 2^{-q} t, \omega_0)| \geq A 2^{-q m_0} / 2$$

for all $|t| \sim 1$, completing the proof of the claim in this case since $A \sim \epsilon_m$.

**Case 2:** $\epsilon_m 2^{-q m_0} \leq K^{-1} \delta_{m_0} \epsilon_m 2^{-q m}$.

In this case we see that $2 \leq m \leq m_0 - 1$. We use the existence of an $2 \leq m' \leq m_0 - 1$ such that (68) holds. Hence by (66), (67) and (68), we have

$$|\Phi^{(m')}(t)| \geq |g^{(m')}(t)| - |h^{(m')}(t)| - |H^{(m')}(t)| \geq \epsilon_m 2^{-q m'} - K \epsilon_m 2^{-q m_0} / 4 - C 2^{-q} \epsilon_m 2^{-q m} \geq (\delta_{m_0} / 2) \epsilon_m 2^{-q m}$$

since $C 2^{-q} \leq C r_0$ and we can take $r_0 < C^{-1} \delta_{m_0} / 4$. This completes the proof of the claim.

A similar but easier argument shows that $|\Psi^{(m')}(t)| \geq \epsilon_m 2^{-q m}$ for all $|t| \sim 1$, with the same $m'$ as in the Claim. Here $\Psi(t) = \psi(x, y + 2^{-q} t, \omega_0)$. Hence an application of van der Corput's lemma shows that

$$|D_{p, q}| = |I_{p, q} - H_{p, q}| \leq C (|\lambda| \epsilon_m 2^{-q m})^{-1/m'}.$$  

Taking a convex combination of this estimate with the estimate (65), we have

$$|D_{p, q}| \leq C 2^{-\delta (N - L)} \min(\epsilon_m 2^{-q m}, (|\lambda| \epsilon_m 2^{-q m})^{-1/m'})$$
for some absolute exponents $\delta > 0$ and $\epsilon > 0$. Hence

$$\left| \sum_{(p,q) \in \mathbb{N}_0^d} D_{p,q} \right| \lesssim \sum_{N \geq L} \sum_{p-(m-1)q=N} |D_{p,q}| \lesssim \sum_{N \geq \delta} 2^{-\delta(N-L)} \sum_{q: p=(m-1)q=N} \min(|\lambda| e_m 2^{-\gamma_m}, (|\lambda| e_2^{-\gamma_m})^{-1}) \epsilon_0$$

which implies that

$$\int_{R_0} e^{i(\lambda s(t)+\rho t+\eta t)} \frac{ds dt}{s t} = c \int_{R_0} e^{i(\lambda \varphi_1(s(t)) + \rho t + \eta t)} \frac{ds dt}{s t} + O(1).$$

The integral on the right can be written as

$$\int_{|s|<r_0/2} e^{i(\lambda \varphi_1(s)+\rho s)} G(s) \frac{ds}{s}$$

where $G(s) = \int_{R(s)} e^{i(\lambda \varphi_2(t)+\eta t)} \frac{dt}{t}$

where $R(s) = \{|t| < r_0/2 : (s, t) \in R_0\}$ and $G(s) = G_{x, y, \lambda, \eta}(s)$ depends on the parameters $x, y, \lambda$ and $\eta$ as well as depending on $\omega_0 \in \mathbb{S}^{d-1}$. An application of Proposition 4.1 shows that $|G(s)| \leq C$ for all $s$ with $|s| < r_0/2$ for some constant $C$ which may be taken to be independent of $x, y, \lambda$ and $\eta$ (but may depend on $\omega_0$). Since $G$ is an even function and clearly, $|G'(s)| \lesssim |s|^{-1}$, another application of Proposition 4.1 shows that

$$\int_{|s|<r_0/2} e^{i(\lambda \varphi_1(s)+\rho s)} G(s) \frac{ds}{s} = O_{r_0, \omega_0}(1)$$

and so

$$\mathcal{J} = \int_{R_{r_0, 1}} e^{2\pi i(\lambda \varphi(s,t)+\rho s+\eta t)} \frac{ds dt}{s t} + O_{r_0, \omega_0}(1)$$

where

$$R_{r_0, 1} = \{(s, t) : |s|, |t| < r_0/2, \max_{2 \leq m \leq m_0} \varepsilon_m |t|^m \leq |s t|\}.$$ 

In exactly the same way as above but now using the derivative bound (58) (or (61)) for $\partial_{s}^{m_0} \varphi(s, t) = \partial_{s}^{m_0} \psi(x + s, y + t, \omega_0)$ which is valid for all $(s, t)$ with $|s|, |t| < r_0/2$, we conclude that

$$\int_{R_{r_0, 2}} e^{2\pi i(\lambda \varphi(s,t)+\rho s+\eta t)} \frac{ds dt}{s t} = O_{r_0, \omega_0}(1)$$

where

$$R_{r_0, 2} = \{(s, t) : |s|, |t| \leq r_0/2, \max_{2 \leq m \leq n_0} \sigma_n |s|^n \leq |s t| \leq \max_{2 \leq m \leq m_0} \varepsilon_m |t|^m\}.$$ 

Hence we arrive at (64).

Next we compare the oscillatory integral in (64) to

$$c \int_{R_0} e^{2\pi i(\lambda A s + B t + C s t)+\rho s+\eta t} \frac{ds dt}{s t}$$

where $c = e^{i \lambda \psi(x, y, \omega_0)}$, $A = \partial_s \varphi(0, 0)$, $B = \partial_t \varphi(0, 0)$ and $C = \partial^{1,1} \varphi(0, 0) = \psi_{st}(x, y, \omega_0)$. Recall that

$$\varphi(s, t) = \psi(x + s, y + t, \omega_0) + \bar{L}_1 s + \omega_0 \cdot \bar{L}_2 t = \omega_0 \cdot \bar{\psi}(x + s, y + t) + \bar{L}_1 s + \bar{L}_2 t.$$
Again we write
\[ \int_{R_0} e^{2\pi i \lambda \varphi(s,t) + \rho s + \eta t} \frac{dt}{s} \frac{ds}{t} = \sum_{(p,q) \in R_0} I_{p,q} \]
and
\[ c \int_{R_0} e^{2\pi i \lambda [A s + B t + C t] + \rho s + \eta t} \frac{dt}{s} \frac{ds}{t} = \sum_{(p,q) \in R_0} II_{p,q} \]
where now \((p,q) \in R_0\) satisfies
\[ 2^{-p}, 2^{-q} < r_0/2, \quad \text{and} \quad \max_{2 \leq m \leq m_0} \varepsilon_m 2^{-qm}, \max_{2 \leq n \leq n_0} \sigma_n 2^{-pn} \leq 2^{-p-q} \]

As before we make the change of variables \(s \to 2^{-p} s\) and \(t \to 2^{-q} t\) in both integrals \(I_{p,q}\) and \(II_{p,q}\), consider the difference \(D_{p,q} = I_{p,q} - II_{p,q}\), first observing
\[
\varphi(2^{-p} s, 2^{-q} t) - \varphi(0,0) = \partial_s \varphi(0,0) 2^{-p} s - \partial_t \varphi(0,0) 2^{-q} t - \frac{1}{2} \partial_{st} \varphi(0,0)(2^{-p} s)(2^{-q} t)
\]
\[
= \sum_{m=2}^{m_0-1} \frac{1}{m!} \varepsilon_m (2^{-q} t)^m + \sum_{n=2}^{n_0-1} \frac{1}{n!} \sigma_n (2^{-p} s)^n + \sum_{k,l \geq 2} \frac{1}{k! l!} \partial^{k,l} \varphi(0,0)(2^{-p} s)^k (2^{-q} t)^l
\]
\[ + h_q(t) + f_p(s) \]
where \(|h_q(t)| \lesssim \varepsilon_{m_0} 2^{-q m_0}\) and \(|f_p(s)| \lesssim \sigma_{n_0} 2^{-p n_0}\). Recall the formula for \(h = h_q\), appearing shortly after (66),
\[ h(t) = (2^{-q} t)^m \int_0^1 \cdots \int_0^1 \partial^{m_0}_t \varphi(0,0) 2^{-q} t) d\mu \]
and so indeed the estimate \(|h_q(t)| \lesssim A 2^{-q m_0} \lesssim \varepsilon_{m_0} 2^{-q m_0}\) follows from (58). Similarly for \(f_p(s)\).

Therefore
\[ |D_{p,q}| \lesssim 2^{-p-2q} + 2^{-2p-q} + \sum_{m=2}^{m_0} \varepsilon_m 2^{-qm} + \sum_{n=2}^{n_0} \sigma_n 2^{-pn} \]
where we recall \(\varepsilon_m = \partial^m \varphi(0,0)\) and \(\sigma_n := \partial^n \varphi(0,0)\). We split the sum over \((p,q) \in R_0\) into various subsums, depending on which of these four terms above is largest. The argument is similar in all cases and so we shall only concentrate on one subsum; namely we restrict those \((p,q) \in R_0\) where the maximal term is \(\max_{2 \leq m \leq m_0} \varepsilon_m 2^{-qm}\). As before we divide these \((p,q)\) depending on which term in this maximum is maximal; say \(\varepsilon_m 2^{-qm}\). We will call this subcollection \(R_0^m\). Therefore for \((p,q) \in R_0^m\), we have
\[ |D_{p,q}| \lesssim \varepsilon_m 2^{-qm} = 2^{-(N-L)} 2^{-p-q} \]
\[ (99) \]
where \(N := q(m-1) - p\) and \(2L := \varepsilon_m\). We note that \(L \leq N\) or \(\varepsilon_m \leq 2^N\) since \(\varepsilon_m 2^{-qm} \leq 2^{-p-q}\) for \((p,q) \in R_0\).

We now aim to obtain a coresponding decay bound for \(I_{p,q}\) and \(II_{p,q}\) whenever \((p,q) \in R_0^m\). We write the phase function in \(I_{p,q}\) as
\[
\lambda \varphi(2^{-p} s, 2^{-q} t) + \rho 2^{-p} s + \eta 2^{-q} t = \lambda 2^{-p-q} \Phi(s,t) + E 2^{-p} s + F 2^{-q} t
\]
where \(\Phi(s,t)\)
\[ = 2^{p+q} \left[ \psi(x + 2^{-p} s, y + 2^{-q} t, \omega_0) - \psi(x, y, \omega_0) - \partial_s \psi(x, y, \omega_0) 2^{-p} s - \partial_t \psi(x, y, \omega_0) 2^{-q} t \right] ,\]
\[ E = \lambda \partial_s \psi(x, y, \omega_0) + \rho = \lambda A + \rho \quad \text{and} \quad F = \lambda B + \eta. \]

The mixed second derivative of the function \( \Phi(s, t) \) is equal to

\[ \dot{\psi}_{st}(x, y, \omega_0) + O(2^{-\min(p, q)}) \quad \text{and so} \quad |\Phi_{st}(s, t)| \geq |\dot{\psi}_{st}(x, y, \omega_0)| - Cr_0 \]

since \( 2^{-\min(p, q)} \leq r_0/2 \) for \((p, q) \in \mathcal{R}_0 \). Hence by Lemma 7.6, we have

\[ |\Phi_{st}(s, t)| \geq D/4 \quad \text{(70)} \]

if we take \( r_0 \) small enough. Recall that we chose the parameter \( r_0 \) after we located the point \((x_0, y_0, \omega_0)\) such that \( D = |\dot{\psi}_{st}(x_0, y_0, \omega_0)| > 0 \) and \( \psi_{tt}(x_0, y_0, \omega_0) = 0 \), see (56).

Since

\[ I_{p, q} = \int \int e^{[\lambda 2^{-p-q}\Phi(s, t) + E2^{-p} + F2^{-q}t]} \zeta(s)\zeta(t) \frac{ds \, dt}{s \, t} \]

the derivative bound (70) puts us in a position to invoke a multidimensional version of van der Corput’s lemma. Such a variant can be found in [13] (see Proposition 5 on page 342) and implies

\[ |I_{p, q}| \leq C(|\lambda|2^{-p-q})^{-1/2} \quad \text{(71)} \]

but the constant \( C \) in the bound here depends on the \( C^3 \) norm of \( f(s, t) := \Phi(s, t) + (\lambda 2^{-p-q})^{-1}[E2^{-p} + F2^{-q}t] \). This is in stark contrast to the one dimensional version of van der Corput’s lemma and it begs the question to what extent is there a multidimensional version of van der Corput’s lemma with all the appropriate uniformity. In any case, an examination of the proof of Proposition 5 in [13] shows in fact the bound \( C \) can be taken to depend only on the \( L^\infty \) norms of the third order derivatives of \( f(s, t) \) and in particular, not on the first order derivatives. The third order derivatives of \( f \) are the same as those of \( \Phi \).

From above, we have \( \Phi(s, t) = 2^{p+q}\Psi(s, t) \) where \( \Psi(s, t) = \)

\[ \sum_{m=2}^{m_0-1} \frac{1}{m!} \varepsilon_m (2^{-q}t)^m + \sum_{n=2}^{n_0-1} \frac{1}{n!} \sigma_n (2^{-p}t)^n + O(2^{-p-q}) + O(\varepsilon_{m_0} 2^{-q_{m_0}}) + O(\sigma_{n_0} 2^{-p_{n_0}}) \]

where the three \( O \) terms are stable under differentiation; for example, \( \partial^{k,f}O(2^{-p-q}) = O(2^{-p-q}) \). Therefore we have the following upper bound on the \( C^3 \) norm of \( \Phi \),

\[ \|\Phi\|_{C^3} \lesssim 2^{p+q} \left[ \max_{2 \leq m \leq m_0} \varepsilon_m 2^{-qm} + \max_{2 \leq n \leq n_0} \sigma_n 2^{-pn} \right] + O(1) = O(1) \]

since \((p, q) \in \mathcal{R}_0 \). This implies that the constant \( C \) in (71) can be taken to be an absolute constant, although depending on \( \mathcal{O} \), it can be taken to be independent of \( p, q, x, y, \lambda, \rho \) and \( \eta \).

A more elementary argument shows that \( |II_{p, q}| \leq C(\lambda|2^{-p-q})^{-1}. \) Taking a convex combination of (69) and (71), we have

\[ |D_{p, q}| \leq C 2^{-\delta(N-L)} \min(\lambda|2^{-p-q}, (\lambda|2^{-p-q})^{-1})^\varepsilon \]
for some absolute exponents $\delta > 0$ and $\epsilon > 0$. Hence

$$| \sum_{(p,q) \in R_0} D_{p,q} | \lesssim \sum_{N \geq L} \sum_{p,q: \ (m-1)q-p = N} |D_{p,q}| \lesssim \sum_{N \geq L} 2^{-\delta(N-L)} \sum_{q: \ (m-1)q-p = N} \min(|\lambda|2^{-p-q}, (|\lambda|2^{-p-q})^{-1})$$

which implies that

$$J = c \int \int_{R_{r_0}} e^{2\pi i \lambda [As+Bt+Cst] + \rho s + \eta t} \frac{ds}{s} \frac{dt}{t} + O_{r_0, \omega_0}(1).$$

We have successfully reduced the analysis of $J$ to an oscillatory integral with a quadratic phase. We label the oscillatory integral above with the quadratic phase as $J'$.

**7.9. Analysis of $J'$: reduction to the heart of the matter.** Recall that our immediate main task (62) follows from (63), the logarithmic bound from below on the oscillatory integral $\mathcal{I} = I + II + III + IV$ where each of these terms is of the form $J$ with $\rho = \pm M$ and $\eta = \pm N$. We rewrite the oscillatory integral $J'$ above as

$$J' = \int \int_{R_{r_0}} e^{2\pi i \lambda Cst + [\lambda A \pm M] s + [\lambda B \pm N] t} \frac{ds}{s} \frac{dt}{t}$$

and rename the coefficients $\lambda' := \lambda C, \mathcal{M} := \lambda A \pm M$ and $N' := \lambda B \pm N$. Since $C = \psi_{s}(x, y, \omega_0)$, we have $|\lambda'| \sim |\lambda|$. In fact by Lemma 7.6, we have $(D/2)|\lambda| \leq |\lambda'| \leq 2D|\lambda|$. Furthermore

$$\mathcal{M} = \lambda \left[ \vec{v}_s(x, y, \omega_0) + \omega_0 \cdot \vec{L}_1 \right] \pm M := \lambda v_1(x, y, \omega_0) \pm M$$

where $v_1(x, y, \omega_0)$ is the first component of the vector

$$\vec{v}(x, y, \omega_0) = \nabla \psi(x, y, \omega_0) + \omega_0 \cdot \vec{L} = \omega_0 \cdot [\nabla \phi(x, y) + \vec{L}],$$

both of whose components are nonzero, see Lemma 7.6. Similarly for $N'$.

To understand $J'$ more precisely, we will decompose the region $R_{r_0}$ into various subregions $R_{m,n}$ determined by where the maxima

$$\max_{2 \leq m' \leq m_0} \epsilon_{m'} |t|^{m'} = \epsilon_{m} |t|^m \quad \text{and} \quad \max_{2 \leq n' \leq n_0} \sigma_{n'} |s|^{n'} = \sigma_{n} |s|^n$$

occurs. We do this for every $2 \leq m \leq m_0$ with $\epsilon_{m} > 0$ and $2 \leq n \leq n_0$ with $\sigma_{n} > 0$. Hence

$$R_{m,n} = \{(s, t) \in R_{r_0} : d_{1,m} \leq |t| \leq d_{2,m} \quad \text{and} \quad c_{1,n} \leq |s| \leq c_{2,n} \}$$

for some choice of exponents $d_{1,m}, d_{2,m}, c_{1,n}$ and $c_{2,n}$. In fact $d_{1,2} = c_{1,2} = 0$ and $d_{2,0} = c_{2,20} = 1$. Otherwise when $\epsilon_{m} > 0$,

$$d_{1,m} := \max_{m' < m} \left( \frac{\epsilon_{m'}}{\epsilon_{m}} \right)^{\frac{1}{m-m'}} \quad \text{and} \quad d_{2,m} := \min_{m < m'} \left( \frac{\epsilon_{m}}{\epsilon_{m'}} \right)^{\frac{1}{m-m'}}$$

and when $\sigma_{n} > 0$,

$$c_{1,n} := \max_{n' < n} \left( \frac{\sigma_{n'}}{\sigma_{n}} \right)^{\frac{1}{n-n'}} \quad \text{and} \quad c_{2,n} := \min_{n < n'} \left( \frac{\sigma_{n}}{\sigma_{n'}} \right)^{\frac{1}{n-n'}}.$$
Hence we can decompose $\mathcal{J}'$ into a sum of oscillatory integrals $\mathcal{J}_{m,n}$ where

$$\mathcal{J}_{m,n} := \iint_{R_{m,n}} e^{2\pi i (\lambda's + Mt + Nt)} \frac{ds \, dt}{s \, t}$$

We split $\mathcal{J}_{m,n} = \mathcal{J}^1_{m,n} + \mathcal{J}^2_{m,n}$ where for $\mathcal{J}^1_{m,n}$ the region of integration $R_{m,n}$ is restricted to $R^1_{m,n} = \{(s,t) \in R_{m,n} : |t| \leq |N|^{-1}\}$. We claim that

$$\mathcal{J}^2_{m,n} = \iint_{R^2_{m,n}} e^{2\pi i (\lambda's + Mt + Nt)} \frac{ds \, dt}{s \, t} = O(1)$$

and to achieve this we split $\mathcal{J}^2_{m,n} = \mathcal{J}^2_{m,n}^- + \mathcal{J}^2_{m,n}^+$ further where $R^2_{m,n} = R^2_{m,n}^- \cup R^2_{m,n}^+$ and $R^2_{m,n}^- = \{(s,t) \in R^2_{m,n} : |s| \leq C|N'/\lambda'|\}$ for some large constant $C$. We write

$$\mathcal{J}^2_{m,n}^- = \int_{|t| \geq |N|^{-1}} e^{2\pi i Nt} F(t) \frac{dt}{t} \quad \text{where} \quad F(t) = \int_{s \in R^2_{m,n}^-(t)} e^{2\pi i (\lambda't + M)s} \frac{ds}{s}$$

and $R^2_{m,n}^-(t) = \{s : (s,t) \in R^2_{m,n}^-\}$. Note that $F(t) = O(1)$ and

$$F'(t) = 2\pi i \lambda' \int_{s \in R^2_{m,n}^-(t)} e^{2\pi i (\lambda't + M)s} \frac{ds}{s} + O(|t|^{-1}).$$

Therefore integration by parts shows that

$$\mathcal{J}^2_{m,n}^- = -\frac{N'}{N} \int_{|t| \geq |N|^{-1}} e^{2\pi i Nt} \left[ \int_{s \in R^2_{m,n}^-(t)} e^{2\pi i (\lambda't + M)s} \frac{ds}{s} \right] \frac{dt}{t} + O(1)$$

and

$$\frac{N'}{N} \int_{|s| \leq C|N'/\lambda'|} e^{2\pi i Ms} \left[ \int_{t \in R^2_{m,n}^-(s)} e^{2\pi i (\lambda's + Nt)} \frac{dt}{t} \right] ds + O(1) = O(1)$$

since the last inner integral is easily seen to be $O(1)$. Next we show that $\mathcal{J}^2_{m,n}^+ = O(1)$. We write

$$\mathcal{J}^2_{m,n}^+ = \int_{|s| \geq C|N'/\lambda'|} e^{2\pi i Ms} G(s) \frac{ds}{s} \quad \text{where} \quad G(s) = \int_{t \in R^2_{m,n}^+(s)} e^{2\pi i (\lambda's + Nt)} \frac{dt}{t}$$

and $R^2_{m,n}^+(s) = \{t : (s,t) \in R^2_{m,n}^+\}$. Note that when $|s| \geq C|N'/\lambda'|$, we have $|\lambda's + N| \sim |\lambda's|$ and a simple integration by parts argument shows that $|G(s)| \lesssim |N'/\lambda'|$, implying that

$$|\mathcal{J}^2_{m,n}^+| \lesssim |N'/\lambda'| \int_{|s| \geq C|N'/\lambda'|} \frac{ds}{s^2} = O(1)$$

and so $\mathcal{J}^2_{m,n} = O(1)$ as claimed.

Since

$$\mathcal{J}^1_{m,n} = \int_{|t| \leq |N|^{-1}} e^{2\pi i Nt} \left[ \int_{s \in R^1_{m,n}(t)} e^{2\pi i (\lambda't + M)s} \frac{ds}{s} \right] \frac{dt}{t},$$

we see that

$$\mathcal{J}^1_{m,n} = \int_{R^1_{m,n}} e^{2\pi i (\lambda's + Ms)} \frac{ds \, dt}{s \, t} + O(1)$$
since $|e^{2\pi i t} - 1| \leq 2|\mathcal{N}|$. As we did with $R_{m,n}^2$, we decompose $R_{m,n}^1 = R_{m,n}^{1,-} \cup R_{m,n}^{1,+}$ where $R_{m,n}^{1,-} = \{ (s, t) \in R_{m,n}^1 : |s| \leq |\mathcal{M}|^{-1} \}$. A similar but more straightforward argument as above shows that $\mathcal{J}_{m,n}^{1,+} = O(1)$ and we are left with $\mathcal{J}_{m,n}^{1,-}$.

Again since

$$\mathcal{J}_{m,n}^{1,-} = \int_{|s| \leq |\mathcal{M}|^{-1}} e^{2\pi i \mathcal{M}s} \left[ \int_{t \in R_{m,n}^{1,-}(s)} e^{2\pi i \lambda' st} \frac{dt}{t} \right] \frac{ds}{s},$$

we see that

$$\mathcal{J}_{m,n}^{1,-} = \int \int_{R_{m,n}^{1,-}(s)} e^{2\pi i \lambda' st} \frac{ds}{s} \frac{dt}{t} + O(1) = \int \int_{R_{m,n}^{1,-}(s)} \frac{\sin(\lambda' st)}{st} \frac{ds}{s} \frac{dt}{t} + O(1)$$

where $P_{m,n} = \{ (s, t) \in R_{m,n}^{1,-} : s, t \geq 0 \}$ is the positive part of $R_{m,n}^{1,-}$.

Summarising, we see that

$$\mathcal{J}' = \sum_{m=2}^{m_0} \sum_{n=2}^{n_0} \mathcal{J}_{m,n} = \sum_{m=2}^{m_0} \sum_{n=2}^{n_0} \int \int_{R_{m,n}^{1,-}(s)} \frac{\sin(\lambda' st)}{st} \frac{ds}{s} \frac{dt}{t} + O(1) \quad (74)$$

but let us not forget that each integral above is a sum of 4 integrals, one for each choice of $\pm$ in the coefficients $\mathcal{M}$ and $\mathcal{N}$ which now only appear in the definition of the region of integration $P_{m,n}^{1,-}$.

### 7.10. Getting to the heart of the matter.

Without loss of generality, let us assume that both $\lambda, \lambda' > 0$. Recall that $\lambda \sim \lambda'$.

Given $\lambda > 0$ (which we choose to be large later, depending on $\varepsilon$) and our point choice $(x, y)$, we now choose $M$ and $N$ so that $M - \Lambda \mathcal{A} = O(1)$ and $N - \Lambda \mathcal{B} = O(1)$.

Recall that $(\mathcal{A}, \mathcal{B}) = \vec{v}(x, y, \omega_0)$ where both components

$$\mathcal{A} = \partial_x \psi(x, y, \omega_0) + \omega_0 \cdot \vec{L}_1 \quad \text{and} \quad \mathcal{B} = \partial_y \psi(x, y, \omega_0) + \omega_0 \cdot \vec{L}_2$$

are nonzero, so we can certainly make such choices for $M$ and $N$. We denote by $\mathcal{M}^* = \lambda \mathcal{A} + M$ and $\mathcal{N}^* = \lambda \mathcal{B} + N$ so that both $|\mathcal{M}^*|, |\mathcal{N}^*| \sim \lambda$. We will reserve $\mathcal{M}$ and $\mathcal{N}$ to denote $\Lambda \mathcal{A} - M$ and $\Lambda \mathcal{B} - N$, respectively, so that both $\mathcal{M}, \mathcal{N} = O(1)$. In fact, $|\mathcal{M}|, |\mathcal{N}| \leq 1$.

Let us now examine the integral

$$S := \int \int_{P_{m,n}^{1,-}} \frac{\sin(\lambda' st)}{st} \frac{ds}{s} \frac{dt}{t}$$

in the case where we take $\mathcal{N}^*$ in the definition of $P_{m,n}^{1,-}$ and the other coefficient can either be $\mathcal{M}$ or $\mathcal{M}^*$. In this case

$$S = \int_{d_1,m \leq t \leq d_2,m} \left( \int_{0 \leq s \leq |\mathcal{N}^*|^{-1}} \frac{\sin(\lambda' st)}{st} \frac{ds}{s} \right) \frac{dt}{t}.$$

When $3 \leq m \leq m_0$, we have $d_{1,m} > 0$ from (72) and since $|\mathcal{N}^*| \sim \lambda$, we see that the interval of the $t$ integration is empty if we choose $\lambda > \Lambda(\varepsilon)$ and $\Lambda(\varepsilon) > 0$ large enough. Hence $S = 0$ in this case.
Next we consider the case \( m = 2 \) so that \( d_{1,2} = 0 \) and hence (for large \( \lambda \)),
\[
S = \int_0^{\left| N^* \right|^{-1}} \left[ \int_{s \in P_{m,n}^1(t)} \frac{\sin(\lambda'st)}{s} ds \right] \frac{dt}{t}
\]
but recall that for \( s \in P_{m,n}^1(t) \), we have the restriction \( c_{1,n} \leq s \leq c_{2,n} \) as well as
\[
\varepsilon_2 t \leq s \leq \left( \frac{t}{\sigma_n} \right)^{1/(n-1)}.
\]
But this is an empty interval of integration in \( s \) for every \( 0 < t < \left| N^* \right|^{-1} \) when \( \left| N^* \right|^{-1} \leq c_{1,n}^{-1} \sigma_n \), which will hold for large \( \lambda \) whenever \( n \geq 3 \) since \( c_{1,n} > 0 \) by (72) (recall that the regions \( R_{m,n} \) only arise when both \( \varepsilon_m, \sigma_n > 0 \)). Hence \( S = 0 \) in this case as well.

Finally we consider the case \( m = n = 2 \). In this case, we have (again for large \( \lambda \))
\[
S = \int_0^{\left| N^* \right|^{-1}} \frac{1}{t} \left[ \int_{s,t} \frac{\sin(\lambda'st)}{s} ds \right] dt = \int_0^{\left| N^* \right|^{-1}} \frac{1}{t} \left[ \int_{s,t} \frac{\sin(\lambda'st)}{s} ds \right] dt
\]
\[
= \int_0^{\left| N^* \right|^{-1}} \frac{\sin(s)}{s} \log(B/A) ds + \int_{s,t} \frac{\sin(\lambda'st)}{s} \log(C/A) ds
\]
where
\[
A = \sqrt{\sigma_2^{-1} s}, \quad B = \sqrt{(\varepsilon_2^{-1} t)^{-1} |N^*|^{-2} s}, \quad \text{and} \quad C = |N^*|^{-1}.
\]
Since \( \log(B/A) = \log(\eta |N^*|) \) where \( \eta = 1/\varepsilon_2 \sigma_2 \) and \( |N^*| \sim \lambda \sim \lambda' \), the first term is \( O(1) \) if \( \lambda > \Lambda(\varepsilon) \) and we choose \( \Lambda(\varepsilon) > 0 \) large enough.

In a similar way, if \( \lambda > \Lambda(\varepsilon) \) and we choose \( \Lambda(\varepsilon) > 0 \) large enough, the integral
\[
S := \int \int_{P_{m,n}^1} \frac{\sin(\lambda'st)}{st} ds dt = O(1)
\]
when we take \( M^* \) in the definition of \( P_{m,n}^1 \) and the other coefficient can either be \( N \) or \( N^* \).

This leaves us with examining \( S \) when we take both \( M \) and \( N \) (where we have \( |M|, |N| \leq 1 \)) as the two coefficients defining \( S \). The restrictions \( 0 \leq s \leq |M|^{-1} \) and \( 0 \leq t \leq |N|^{-1} \) for \((s, t) \in P_{m,n}^1 \) are no longer restrictions since \( 1 \leq |M|^{-1}, |N|^{-1} \). Hence in this case, when both \( \varepsilon_m, \sigma_n > 0 \), we see that \((s, t) \in P_{m,n}^1 \) are precisely those nonnegative numbers such that \( s, t < r_0/2 \) and
\[
c_{1,n} \leq s \leq c_{2,m}, \quad d_{1,m} \leq t \leq d_{2,m}, \quad \text{and} \quad \varepsilon_m t^{m-1} \leq s \leq (\sigma_n^{-1} t)^{1/(n-1)}.
\]
We write
\[
S = \int_{d_{1,m}}^{d_{2,m}} \frac{1}{t} \left[ \int_{I_{m,n}(t)} \frac{\sin(\lambda'st)}{s} ds \right] dt
\]
where \( I_{m,n}(t) = [c_{1,n}, c_{2,n}] \cap [\varepsilon_m t^{m-1}, (\sigma_n^{-1} t)^{1/(n-1)}] \). We claim that
\[
S = O(1/\lambda) \quad \text{whenever} \quad m \geq 3 \quad \text{or} \quad n \geq 3 \quad (75)
\]
and when $m = n = 2$, 

$$S = O_\varepsilon(1/\lambda) + \frac{\pi}{4} \log 1/(\varepsilon_2 \sigma_2).$$  

(76)

If so, we would be able to conclude that 

$$\mathcal{I} = O_\varepsilon(\log \lambda/\lambda) + \frac{\pi}{4} \log(\varepsilon_2^{-1})$$

since $0 < \sigma_2 \lesssim 1$. Therefore, since $0 < \varepsilon_2 \leq \varepsilon$, we can find a $\Lambda(\varepsilon)$ such that whenever $\lambda > \Lambda(\varepsilon)$, $|\mathcal{I}| \gtrsim \log(\varepsilon^{-1})$, establishing (63) and hence (62). We now prove (75) and (76).

For a fixed $t \in [d_{1,m}, d_{2,m}]$, we make the change of variables $s \to (\lambda t)s$ and write 

$$S = \int_{d_{1,m}}^{d_{2,m}} \frac{1}{t} \left[ \int_{\mathcal{I}_{m,n}(t)} \frac{\sin(s)}{s} ds \right] dt$$

where now $\mathcal{I}_{m,n} = ([\varepsilon_m \lambda^t m, \sigma_n^{-1/(n-1)} \lambda^n/(n-1)] \cap [c_{1,n} \lambda^t, c_{2,n} \lambda^t])$.

When $m \geq 3$, we have $d_{1,m} > 0$ by (72) and so when we interchange the order of integration, $S$ has the form 

$$S = \int_{C_{\varepsilon}} \sin(s) \left[ \int_{d_{1,m}}^{d_{2,m}} \frac{1}{t} ds \right] dt$$

where $g = O(d_{2,n})$ and $C_{\varepsilon} > 0$. From this one can deduce that (75) holds in this case.

Suppose now that $m = 2$ and $n \geq 3$. Hence $d_{1,2} = 0$ and $c_{1,n} > 0$ by (72) and so 

$$S = \int_{0}^{d_{2,2}} \frac{1}{t} \left[ \int_{\mathcal{I}_{2,n}(t)} \frac{\sin(s)}{s} ds \right] dt$$

but if $\mathcal{I}_{2,n}(t)$ is nonempty, it must be the case that $\sigma_{n}c_{1,n}^{-1} \leq t$. This puts us in the same position as before but with $d_{1,m}$ replaced with $\sigma_{n}c_{1,n}^{-1} > 0$ and so, as above, we see that (75) holds in this case as well. Hence this establishes (75) whenever $m \geq 3$ or $n \geq 3$, as claimed.

Finally we turn to the case $m = n = 2$ in which case $c_{1,2} = d_{1,2} = 0$. The same argument as above shows that 

$$\int_{\sigma_2 c_{2,2}}^{d_{2,2}} \frac{1}{t} \left[ \int_{\mathcal{I}_{2,n}(t)} \frac{\sin(s)}{s} ds \right] dt = O_\varepsilon(1/\lambda)$$

and so we may assume $t \leq \sigma_2 c_{2,2}$ in which case $\mathcal{I}_{2,2}(t) = [\varepsilon_2 \lambda^t t^2, \sigma_2^{-1} \lambda^t t^2]$. We are left with examining 

$$S' = \int_{0}^{\min(d_{2,2}, \sigma_2 c_{2,2})} \frac{1}{t} \left[ \int_{\sigma_2^{-1} \lambda^t t^2}^{\sigma_2 \lambda^t t^2} \frac{\sin(s)}{s} ds \right] dt.$$  

Let us call the minimum in the outer limit of integration $\mathfrak{m}$. When we interchange the order of integration, $S'$ splits into a sum $I + II = \int_{\varepsilon_2 \lambda^m t^2}^{\sigma_2^{-1} \lambda^m t^2} \frac{\sin(s)}{s} \left[ \int_{0}^{\mathfrak{m}} \frac{1}{t} ds \right] dt + \int_{\varepsilon_2 \lambda^m t^2}^{\sigma_2 \lambda^m t^2} \frac{\sin(s)}{s} \left[ \int_{0}^{\sqrt{(\varepsilon_2 \lambda^m)^{-1}s}} \frac{1}{t} ds \right] dt$.
of two integrals. As before $I = O(1/\varepsilon)$ but now 

$$II = \frac{1}{2} \log(1/\varepsilon^2) \int_{0}^{2\pi} \frac{\sin(s)}{s} \, ds = \frac{\pi}{4} \log(1/\varepsilon^2) + O(1/\varepsilon),$$

establishing (76).

7.11. **The final step.** We now have established (62). In fact we have proved that there exists a $\Lambda(\varepsilon)$ such that whenever $\lambda > \Lambda(\varepsilon)$, we can find $M = M(\lambda)$ and $N = N(\lambda)$ so that

$$\left| \iint_{T^2} e^{2\pi i (\omega_0 + [\omega (s, t), + L (s, t)]) \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t}} \right| \gtrsim \log(1/\varepsilon)$$

where $(s, t) = (x, y)$ is the point we found from Lemma 7.6.

Next, we appeal to the multidimensional Dirichlet principle which gives a sequence of positive integers $q \to \infty$ and lattice points $M = M(q) \in \mathbb{Z}^d$ such that

$$||M - q \omega_0|| \leq q^{-d}$$

for some $\epsilon_d > 0$ depending only on $d$. We write $\pi = ||\pi|| \omega$ in polar coordinates for some $\omega \in S^{d-1}$. From (77), we see that $\omega \to \omega_0$ and $||\pi||/q \to 1$ as $q \to \infty$. By the Lebesgue dominated convergence theorem, fixing $M$ and $N$ and using (77), we have

$$\int \int_{T^2} e^{2\pi i [\omega (s, t), + L (s, t)] \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t}} - \int \int_{T^2} e^{2\pi i q \omega_0 [\omega (s, t), + L (s, t)] \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t}} \to 0 \quad \text{as} \quad q \to \infty.$$

Hence by taking $q$ large enough (and in particular $q > \Lambda(\varepsilon)$), we can then find a lattice point $\pi \in \mathbb{Z}^d$ and a pair $(M, N)$ such that

$$\left| \iint_{T^2} e^{2\pi i [\omega (s, t), + L (s, t)] \sin(M s) \sin(N t) \frac{ds \, dt}{s \, t}} \right| \gtrsim \log(1/\varepsilon),$$

completing the proof of Proposition 7.7 and hence Theorem 1.3.

8. **Proof of Theorem 1.4**

In this section we consider mappings $\Phi = \Phi_{\phi, L} : T^2 \to T$ parameterised by $g(s, t) = \phi(s, t) + L_1 s + L_2 t$ where $\phi$ is a real-analytic, periodic function of two variables and $L = (L_1, L_2)$ is a pair of integers. We will give a preliminary examination when the property $(\Phi)_{sq}$ holds for $\Phi$; that is, when does $f \circ \Phi$ have a uniformly convergent Fourier series with respect to square sums for any $f \in A(T)$? This boils down to showing whether or not the norms $\|e^{2\pi i n g}\|_{U_q(T^2)}$ are uniformly bounded in $n \in \mathbb{Z}$.

Recall that

$$\|e^{2\pi i n g}\|_{U_q(T^2)} = \sup_M \|S_M(M) n g\|_{L^\infty(T)}$$

$$= \sup_{M, x, y} \left| \int \int_{T^2} e^{2\pi i n [\phi(x, y + t)] + L_1 x + L_2 t]} D_M(s, t) \, ds \, dt \right|$$

where $D_M$ denotes the Dirichlet kernel of order $M$. 
Two applications of Proposition 4.1 show that uniform bounds for the above oscillatory integral are reduced to determining whether or not
\[ D := \int_{|s|,|t| \leq 1/2} e^{2\pi i n |\phi(x,y) + L_1 s + L_2 t|} \frac{\sin(M s) \sin(M t)}{s t} \, ds \, dt \]
is uniformly bounded in the parameters \( n, x, y \) and \( M \).

8.1. **Proof of Theorem 1.4 – existence of uniform bounds.** Here we show that there exists infinitely many pairs \((L_1, L_2)\) such that \( D = O(1) \). This follows from the following proposition.

**Proposition 8.2.** For any pair \((L_1, L_2)\) of integers satisfying \( L_2 > 10^3 L_1 \) and \( L_1 \geq 10^2 \|\phi\|_{C^2} \), we have \( D = O(1) \).

**Proof** Set \( A := n(\partial_t \phi(x, y) + L_2) \pm M \) and \( B := n(\partial_x \phi(x, y) + L_1) \pm M \).

**Claim:** One of the following two statements hold. Either
\[
\begin{align*}
(a) \quad & n, |M| \lesssim |A|, \quad 10n\|\partial_t^2 \phi\|_{L^\infty} \leq |A| \\
\text{or} \quad & (b) \quad n, |M| \lesssim |B|, \quad 10n\|\partial_x^2 \phi\|_{L^\infty} \leq |B|.
\end{align*}
\]

We consider three cases:

**Case 1:** Either \( |M + n(\partial_t \phi(x, y) + L_2)| \geq 10n|\partial_t \phi(x, y) + L_2| \) or
\( |M - n(\partial_t \phi(x, y) + L_2)| \geq 10n|\partial_t \phi(x, y) + L_2| \).

Either situation implies that \( |A| \sim |M| \). Furthermore
\[
|B| \sim |M| \geq 10^{-1}nL_2 \geq 10^{-2}n|\partial_t \phi(x, y) + L_2| \leq |A|/10
\]
and this leads to (a).

**Case 2:** Either \( |M + n(\partial_t \phi(x, y) + L_2)| \leq 10^{-1}n|\partial_t \phi(x, y) + L_2| \) or
\( |M - n(\partial_t \phi(x, y) + L_2)| \leq 10^{-1}n|\partial_t \phi(x, y) + L_2| \).

Either situation implies that \( |M| \sim n|\partial_t \phi(x, y) + L_2| \geq 10^2|\partial_t \phi(x, y) + L_1| \). Hence
\[
|B| \sim |M| \geq 10^{-1}nL_2 \geq 10^2n\|\partial_t^2 \phi\|_{L^\infty}
\]
and this leads to (b).

**Case 3:**
\[
10^{-1}n|\partial_t \phi(x, y) + L_2| \leq |M \pm n(\partial_t \phi(x, y) + L_2)| \leq 10n|\partial_t \phi(x, y) + L_2|
\]
In this case, we have
\[
|A| \sim n|\partial_t \phi(x, y) + L_2| \geq 10^{-1}nL_2 \geq 10^2n\|\partial_t^2 \phi\|_{L^\infty}
\]
and so \(|M| =
\]
\[
|M \pm n(\partial_t \phi(x, y) + L_2) \mp n(\partial_t \phi(x, y) + L_2)| \leq |A| + n|\partial_t \phi(x, y) + L_2| \lesssim |A|.
\]
This leads to (a), establishing the claim in all three cases.
Without loss of generality suppose that (b) holds. We split $D = D_1 + D_2$ where

$$D_1 := \int_{|B|^{-1} \leq |s|} e^{2\pi i n[\phi(x+s,y+t)+L_1s+L_2t]} \sin(Ms) \sin(Mt) \frac{ds \, dt}{s \, t}.$$ 

Using $\sin(u) = (e^{iu} - e^{-iu})/2i$, we write $D_1 = I^+ + I^-$ where

$$I^\pm = \int_{|B|^{-1} \leq |s| \leq 1/2} e^{\mp if_\pm(s)} \left[ \int_{|t| \leq 1/2} e^{2\pi i n[\phi(x+s,y+t)-\phi(x+s,y)+L_2t]} \sin(Mt) \frac{dt}{t} \right] ds$$

where $f_\pm(s) = n(\phi(x+s,y)+L_1s) \pm Ms$. Let us denote by $G(s)$ the inner integral of $I^\pm$ above. By Proposition 4.1, $G(s) = O(1)$.

Integrating by parts in the $s$ integral, we have $I^\pm = BT + II^\pm + III^\pm$ where $BT$ denote the boundary terms,

$$II^\pm = i \int_{|B|^{-1} \leq |s| \leq 1/2} e^{if_\pm(s)} \left( \frac{1}{s f_\pm'(s)} \right)' G(s) \, ds$$

and

$$III^\pm = i \int_{|B|^{-1} \leq |s| \leq 1/2} e^{if_\pm(s)} \frac{1}{s f_\pm'(s)} G'(s) \, ds$$

Note that

$$f_\pm'(s) = n(\partial_x \phi(x+s,y)+L_1) \pm M = B + ns \int_0^1 \partial_{ss}^2 \phi(x+rs,y) \, dr$$

so that

$$\frac{1}{sf_\pm'(s)} = \frac{1}{sB} + \frac{B - f_\pm'(s)}{sBf_\pm'(s)} = \frac{1}{sB} + O\left( \frac{n\|\partial_{ss}^2 \phi\|_{L^\infty}}{B^2} \right).$$

From this we see that $BT = O(1)$. Furthermore

$$f_\pm''(s) = n \int_0^1 \partial_{ss}^2 \phi(x+rs,y) \, dr + ns \int_0^1 \partial_{ss}^2 \phi(x+rs,y) s \, dr$$

and so

$$\left( \frac{1}{sf_\pm'(s)} \right)' = -\frac{1}{s^2 f_\pm'(s)} - \frac{f_\pm''(s)}{sf_\pm'(s)} = O\left( \frac{1}{|B|^2} + \frac{n\|\partial_{ss}^2 \phi\|_{L^\infty}}{|B|^2} + \frac{n\|\partial_{ss}^2 \phi\|_{L^\infty}}{|B|^2} \right).$$

From this we see that

$$II^\pm = O(1) + O(\log |B|/|B|) = O(1).$$

Turning to $III^\pm$, we note that

$$\partial_x \phi(x+s,y+t) - \partial_x \phi(x+s,y) = t \int_0^1 \partial_{ss}^2 \phi(x+s,y+ut) \, du$$

and so $G'(s) = \int_0^1 \left[ \int_{|t| \leq 1/2} e^{2\pi i n[\phi(x+s,y+t)-\phi(x+s,y)+L_2t]} \partial_{xx}^2 \phi(x+s,y+ut) \sin(Mt) \, dt \right] du,$
showing that $G'(s) = O(n)$. Our formula for $(sf_L'(s))^{-1}$ shows that

$$III^\pm = O(n^2/|B|^2) + \frac{n}{B} \int_0^1 \int_{|t| \leq 1/2} e^{2\pi i n L_2 t} \sin(Mt) \, H(t) \, dt \, du$$

where

$$H(t) := \int_{|s| \leq |t| \leq 1/2} e^{2\pi i g_\pm(s,t)} \partial^2_{st} \phi(x + s, y + ut) \frac{ds}{s}$$

and $g_\pm(s, t) = n(\phi(x + s, y + t) + L_1 s) \pm Ms$.

We write

$$\partial^2_{st} \phi(x + s, y + ut) = \partial^2_{st} \phi(x, y + ut) + s \int_0^1 \partial^{2,1} \phi(x + rs, y + ut) \, dr$$

and so

$$H(t) = \partial^2_{st} \phi(x, y + ut) \int_{|s| \leq |t| \leq 1/2} e^{2\pi i g_\pm(s,t)} \frac{ds}{s} + O(1) = O(1)$$

by Proposition 4.1. Hence $III^\pm = O(1)$ and so $I^\pm = O(1)$, showing $D_1 = O(1)$.

We are left with

$$D_2 = \int_{|s| \leq |B|^{-1}} e^{2\pi i n [\phi(x+s,y+t)+L_1 s+L_2 t]} \sin(Ms) \sin(Mt) \frac{ds}{s} \frac{dt}{t}$$

which can be written as

$$\int_{|s| \leq |B|^{-1}} e^{2\pi i n L_1 s} \frac{\sin(Ms)}{s} \left[ \int_{|t| \leq 1/2} e^{2\pi i n [\phi(x+s,y+t)+L_2 t]} \frac{\sin(Mt)}{t} \, dt \right] \, ds.$$ 

The inner integral is uniformly bounded by Proposition 4.1 and so $D_2 = O(|M|/|B|) = O(1)$ by (b). This completes the proof of the proposition, showing $D = O(1)$. 

8.3. **Examples when** $(\Phi)_{sq}$ **fails.** From Theorem 1.3, we know that $(\Phi)_{sq}$ can only fail for maps $\Phi = \Phi_{\phi,L}$ whose periodic part $\phi$ does not satisfy the factorisation hypothesis (FH).

As mentioned at the outset of this section, the property $(\Phi_{\phi,L})_{sq}$ fails if we can show that the oscillatory integral

$$D := \int_{|s|, |t| \leq 1/2} e^{2\pi i n [\phi(x+s,y+t)+L_1 s+L_2 t]} \sin(Ms) \sin(Mt) \frac{ds}{s} \frac{dt}{t}$$

is unbounded in the parameters $n, x, y$ and $M$.

Suppose now (FH) fails for $\phi$. Then there exists a point $(x_0, y_0) \in \mathbb{T}^2$ such that (56) holds; that is, $\phi_{st}(x_0, y_0) \neq 0$ yet $\phi_{tt}(x_0, y_0) = 0$, say. In addition we will assume that we can find such a point such that the difference $\partial_s \phi(x_0, y_0) - \partial_t \phi(x_0, y_0)$ is an integer. There are many such examples; for instance $\phi(x, y) = \cos(x)(-1 + \sin(y))$ has these properties for $(x_0, y_0) = (\pi/4, 0)$. We fix the point $(x, y)$ in $D$ above to be $(x_0, y_0)$. 

We choose any pair \( L = (L_1, L_2) \) of integers so that
\[
\partial_s \phi(x_0, y_0) + L_1 = \partial_t \phi(x_0, y_0) + L_2 \neq 0.
\]
Finally for \( n \) large we choose the integer \( M = M(n) \) such that
\[
n(\partial_s \phi(x_0, y_0) + L_1) - M = n(\partial_t \phi(x_0, y_0) + L_2) - M = O(1).
\]

**Proposition 8.4.** With the point \( (x_0, y_0) \in \mathbb{T}^2 \) and pair of integers \( L = (L_1, L_2) \) described above, we can find an \( N \) such that whenever \( n \geq N \), we have (with the above \( M = M(n) \))
\[
|D| \gtrsim \log n.
\]
Hence \( (\Phi)_{sq} \) fails for \( \Phi = \Phi_{\psi, L} \).

The proof of Proposition 8.4 is much easier than the analysis required to show the corresponding unboundedness in Theorem 1.3 when (FH) fails to hold for \( \phi \). Of course there we established unboundedness for all pairs of integers \( L = (L_1, L_2) \) (although we did have two integers \( M \) and \( N \) to choose instead of a single integer \( M \) in this case). We no longer have to perturb the point \( (x_0, y_0) \) and carry out such an intricate argument. Much of the analysis, even though simpler, is the same as in Section 7 and so we will be brief and only outline the steps.

**Step 1:** Let \( m_0 \geq 3 \) be an integer such that \( \partial_t^{m_0} \phi(x_0, y_0) \neq 0 \) and furthermore we take \( m_0 \) to be minimal with these properties. Recall that \( \partial_t^2 \phi(x_0, y_0) = 0 \). Also let \( n_0 \geq 2 \) be an integer such that \( \partial_t^{n_0} \phi(x_0, y_0) \neq 0 \) and furthermore we take \( n_0 \) to be minimal with these properties. It may be the case that \( m_0 \) and/or \( n_0 = \infty \). We allow this possibility which only will exclude certain successive steps from occuring.

**Step 2:** Reduction to the oscillatory integral
\[
\iint_R e^{2\pi i n[\phi(x+s,y+t)+L_1s+L_2t]} \frac{\sin(Ms)}{s} \frac{\sin(Mt)}{t} dsdt
\]
where
\[
R = \{(s,t) : |t|^{m_0}, |s|^{n_0} \leq |st| \}
\]
The reduction is carried out in Section 5 with \((m_1, n_1) = (1, 1)\). Recall in the general \((m_1, n_1)\) case, we reduced matters to an oscillatory integral with *almost polynomial* phase \( P_{x_0, y_0}(x+s, y+t) + \phi(x+s, y) + \phi(x, y+t) \). When \((m_1, n_1) = (1, 1)\), we see that \( P_{x_0, y_0} \equiv 0 \) and so an application of Proposition 4.1 allows us to complete the analysis over the region \( \{|st| \leq |t|^{m_0} \} \cup \{|st| \leq |s|^{n_0} \} \), reducing matters to the above integral.

Here it is essential that \( m_0 \geq 3 \). Otherwise if \( m_0 = n_0 = 2 \), then \( R = \emptyset \).

**Step 3:** Reduction to an oscillatory integral with quadratic phase
\[
\iint_R e^{2\pi i n[(\partial_s \phi(x_0, y_0)+L_1)s+(\partial_t \phi(x_0, y_0)+L_2)t+\partial_{xt} \phi(x_0, y_0)st]} \frac{\sin(Ms)}{s} \frac{\sin(Mt)}{t} dsdt
\]
A more complicated reduction is used in the proof of Theorem 1.3 as carried out in Section 7.
Step 4: Reduction to
\[\int_{R^+} \frac{\sin(cn\theta)}{st} \, ds \, dt\]
where \(c = \partial_{st} \phi(x_0, y_0)\) and \(R^+ = \{s, t \geq 0 : t^{m_0-1} \leq s \leq t^{n_0/(n_0-1)}\} \).

This reduction is similar but much easier than what we did in Section 7; see in particular the analysis leading up to Section 7.10. It is here where our choice of \(L = (L_1, L_2)\) (which depends on \(\nabla \phi(x_0, y_0)\)) and \(M = M(n, L, \nabla \phi(x_0, y_0))\) comes into play.

Step 5:
\[\int_{R^+} \frac{\sin(cn\theta)}{st} \, ds \, dt = \frac{\pi}{2} \log n^{1 - \frac{m_0}{m_0}} + O(1)\]
This computation is similar but easier than what we did in Section 7.10. As we said before, we allow for \(m_0\) and/or \(n_0 = \infty\). This does not prevent the logarithmic blow up in \(n\).

These steps give a proof (78) and therefore Proposition 8.4.

9. Higher dimensional tori \(\mathbb{T}^k\)

Recall that a map \(\Phi : \mathbb{T}^k \to \mathbb{T}^d\) is parametrised by \(d\) periodic functions \(\vec{\phi} = (\phi_1, \ldots, \phi_d)\) of \(k\) variables \(\vec{t} = (t_1, \ldots, t_k)\) and \(d\) lattice points \(\vec{L} = (L_1, \ldots, L_d)\) in \(\mathbb{Z}^d\). Explicitly, if \(P = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})\) is a point in \(\mathbb{T}^k\), then
\[\Phi(P) = (e^{2\pi i \phi_1(\vec{t}) + L_1 \cdot \vec{t}}, \ldots, e^{2\pi i \phi_d(\vec{t}) + L_d \cdot \vec{t}})\].

Freezing one of the \(k\) variables, say \(t_j\), gives a map \(\Phi_j : \mathbb{T}^{k-1} \to \mathbb{T}^d\).

Furthermore if \((\Phi_j)_{rect}\) holds, then \((\Phi_j)_{rect}\) holds for every \(1 \leq j \leq k\). This follows easily from the definition of unrestricted rectangular convergence. We illustrate this when \(k = 2\): if \(f \in C(\mathbb{T}^2)\), then \(f\) has a uniformly convergent Fourier series with respect to rectangular convergence if
\[
\sup_{x,y} \left| \sum_{|k| \leq M, |\ell| \leq N} \hat{f}(k, \ell) e^{2\pi i (kx + \ell y)} - f(x, y) \right| \to 0 \text{ as } M, N \to \infty.
\]

In particular (letting \(N \to \infty\)) this says that for any \(y \in \mathbb{T}\),
\[
\sup_x \left| \sum_{|k| \leq M} \hat{f}_y(k) e^{2\pi i kx} - f_y(x) \right| \to 0 \text{ as } M \to \infty
\]
where \(f_y \in C(\mathbb{T})\) is defined as \(f_y(x) = f(x, y)\). That is, \(f_y\) has a uniformly convergent Fourier series for every \(y \in \mathbb{T}\).

Hence whatever conditions we impose on \(\Phi\) to guarantee that \((\Phi)_{rect}\) holds, then necessarily \((\Phi_j)_{rect}\) holds and in particular any necessary conditions we happen to know regarding \((\Phi_j)_{rect}\) give us necessary conditions on \(\Phi\). For instance for mappings \(\Phi : \mathbb{T}^3 \to \mathbb{T}^d\) such that \((\Phi)_{rect}\) holds, then necessarily
\[\Phi_1(e^{2\pi i t}, e^{2\pi i u}) = \Phi(e^{2\pi i s}, e^{2\pi i t}, e^{2\pi i u}) = \Phi_2(e^{2\pi i s}, e^{2\pi i u}) = \Phi_3(e^{2\pi i s}, e^{2\pi i t})\]
satisfy $(\Phi_1)_{\text{rect}}$, $(\Phi_2)_{\text{rect}}$ and $(\Phi_3)_{\text{rect}}$, respectively. Therefore if $\overline{\phi}(s, t, u)$ parametrises the periodic part of $\Phi$, Theorem 1.3 implies that for every $u \in \mathbb{T}$, $\psi_u(s, t, \omega) := \omega \cdot \overline{\phi}(s, t, u)$ satisfies the factorisation hypothesis (FH) on $\mathbb{S}^{d-1}$. Likewise $\psi_s$ and $\psi_t$ necessarily satisfy (FH). These conditions allow us to control the particular partial derivatives
\[
\frac{\partial^{k+\ell} \psi}{\partial^k s \partial^\ell t}, \frac{\partial^{k+\ell} \psi}{\partial^k s \partial^\ell u}, \frac{\partial^{k+\ell} \psi}{\partial^k t \partial^\ell u}
\]
for any $k, \ell \geq 1$ in terms of pure derivatives. This is the content of Lemma 3.1.

For our argument it is essential that we are able to control all mixed derivatives by pure derivatives. So when $k = 3$, it remains to control the third order mixed derivative $\partial_{stu} \psi(s, t, u, \omega)$. This leads to our characterisation of when $(\Phi)_{\text{rect}}$ holds when $k = 3$.

**Theorem 9.1.** Let $\Phi : \mathbb{T}^3 \to \mathbb{T}^d$ be a real-analytic map and suppose $\overline{\phi}$ is its periodic part. Set $\psi(s, t, u) = \omega \cdot \overline{\phi}(s, t, u)$. Then $(\Phi)_{\text{rect}}$ holds if and only if
\[
\psi_{ss}, \psi_{tt} \mid \psi_{st}, \psi_{ss}, \psi_{uu} \mid \psi_{su}, \psi_{tt}, \psi_{uu} \mid \psi_{tu}
\]
and
\[
\psi_{sss}, \psi_{ttt}, \psi_{uuu} \mid \psi_{stu}
\]
as germs of real-analytic functions of $\mathbb{T}^3 \times \mathbb{S}^{d-1}$.

Proceeding iteratively we arrive at a characterisation of real-analytic mappings $\Phi$ from $\mathbb{T}^k$ to $\mathbb{T}^d$ such that $(\Phi)_{\text{rect}}$ holds for any $k \geq 1$. The proof is much more technical than Theorem 1.3 but essentially all the ideas are present in the $k = 2$ case and this is the reason we have decided to carry out the analysis only in this case. We may present the details in the general case at a later time.

**References**

[1] L. Alpár, *Sur une classe particulière de séries de Fourier ayant de sommes partielles bornées*, Studia Sci. Math. Hungar. 1 (1966), 189-204.
[2] A. Beurling and H. Helson, *Fourier-Stieltjes transforms with bounded powers*, Math. Scand 1 (1953), 120-126.
[3] A. Carbery, S. Wainger and J. Wright, *Double Hilbert transforms along polynomial surfaces in $\mathbb{R}^3$*, Duke Math. J. 101 (2000), no. 3, 499-513.
[4] P. Cohen, *On homomorphisms of group algebras*, Amer. J. Math. 82 (1960), 213-226.
[5] C. Fefferman, *On the divergence of multiple Fourier series*, Bull. Amer. Math. Soc. 77, no. 2 (1971), 191-195.
[6] J.-P. Kahane, *Sur les fonctions sommes de séries trigonométriques absolument convergentes*, C.R. Acad. Sci. Paris 240 (1955), 36-37.
[7] J.-P. Kahane, *Quatre Lecons sur les Homeomorphisms du Cercle et les Series de Fourier*, Topic in Harmonic Analysis, Instituto Nazionale di Alta Matematica, Rome (1983), 955-990.
[8] R. Kaufman, *Uniform convergence of Fourier series in harmonic analysis*, Studia Sci. Math. Hungar. 10 (1975), no. 1-2, 81-83.
[9] Z.L. Leibenzon, *On the ring of functions with absolutely convergent Fourier series*, Uspechi Matem. Nauk (N.S.) 9 (1954), no. 3, 157-162.
[10] W. Osgood, *Factorization of analytic functions of several variables*, Annals of Math. 19 (1917), no. 2, 77-95.
[11] F. Ricci and E.M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I. Oscillatory integrals*, J. Funct. Anal. 73 (1987), no. 1, 179-194.
[12] A. Santos, *Phd thesis*, University of Wisconsin (1987).
[13] E.M. Stein, *Harmonic Analysis*, Princeton University Press (1993).
[14] E.M. Stein, *Beijing Lectures in Harmonic Analysis*, Annal of Math. Studies 112, Princeton University Press (1986).
[15] E.M. Stein and B. Street, *Multi-parameter singular Radon transforms*, Math. Res. Lett. 18 (2011) no. 2, 257-277.
[16] E.M. Stein and S. Wainger, *The estimation of an integral arising in multiplier transforms*, Studia Math. 35 (1970), 101-104.
[17] B. Street, *Multi-parameter singular integrals*, Annals of Math. Studies 189, Princeton University Press (2014).
[18] S. Wainger, *Problems in harmonic analysis related to curves and surfaces with infinitely flat points*, Harmonic analysis and partial differential equations (El Escorial, 1987), Springer Lecture Notes in Math. 1384 (1989).

E-mail address: J.R.Wright@ed.ac.uk