On Abelian Closures of Infinite Non-binary Words

Juhani Karhumäki, Svetlana Puzynina, Markus A. Whiteland

Abstract

Two finite words $u$ and $v$ are called abelian equivalent if each letter occurs equally many times in both $u$ and $v$. The abelian closure $\mathcal{A}(x)$ of an infinite word $x$ is the set of infinite words $y$ such that, for each factor $u$ of $y$, there exists a factor $v$ of $x$ which is abelian equivalent to $u$. The notion of an abelian closure gives a characterization of Sturmian words: among uniformly recurrent binary words, periodic and aperiodic Sturmian words are exactly those words for which $\mathcal{A}(x)$ equals the shift orbit closure $\Omega(x)$. Furthermore, for an aperiodic binary word that is not Sturmian, its abelian closure contains infinitely many minimal subshifts. In this paper we consider the abelian closures of well-known families of non-binary words, such as balanced words and minimal complexity words. We also consider abelian closures of general subshifts and make some initial observations of their abelian closures and pose some related open questions.

1. Introduction

Let $x \in \Sigma^\mathbb{N}$ be an infinite word over an alphabet $\Sigma$. We define the language of $x$, denoted by $\mathcal{L}(x)$, as the set of factors of $x$, i.e., blocks of consecutive letters of $x$. A subshift $\Omega(x)$ generated by an infinite word $x$ can be defined as the set of infinite words whose languages are included in $\mathcal{L}(x)$: $\Omega(x) = \{ y \in \Sigma^\mathbb{N} : \mathcal{L}(y) \subseteq \mathcal{L}(x) \}$. In this paper, we consider an abelian version of the notion of a subshift. Two finite words $u$ and $v$ are called abelian equivalent, denoted by $u \sim_{ab} v$, if each letter occurs equally many times in both $u$ and $v$. Various abelian properties of words have been actively studied recently, e.g., abelian complexity, abelian powers, abelian periods, etc. \cite{4, 21, 22}. We define the abelian closure $\mathcal{A}(x)$ of an infinite word $x$ as the set of infinite words $y$ such that, for each factor $u$ of $y$, there exists a factor $v$ of $x$ with $u \sim_{ab} v$. Clearly, $\Omega(x) \subseteq \mathcal{A}(x)$ for any word $x$.

We start with two examples showing completely different structure of abelian closures: Sturmian words and the Thue–Morse word. Sturmian words can be defined as infinite aperiodic words which have $n+1$ distinct factors for each length $n$. They admit various characterizations; in particular, they are exactly the aperiodic balanced words (i.e., the numbers of occurrences of 1 in factors of the same length differ by at most 1). It is not hard to see that, for a Sturmian word $x$, $\Omega(x) = \mathcal{A}(x)$ (so, the abelian closure is small, contains only its subshift). Indeed, due to balance there are exactly two abelian classes of factors of each
length. Therefore, any word \( y \in A(x) \) must be balanced. Further, the frequencies of letters of \( y \) are uniquely defined by \( A(x) \). Thus \( y \) is Sturmian with the same letter frequencies as \( x \), i.e., \( y \in \Omega(x) \). In fact, the property \( \Omega(x) = A(x) \) characterizes Sturmian words among uniformly recurrent binary words (see Theorem 2.6).

The Thue–Morse word \( \text{TM} = 011010011001 \cdots \) can be defined as the fixed point starting with 0 of the morphism \( \mu : 0 \mapsto 01, 1 \mapsto 10 \). For odd lengths \( \text{TM} \) has two abelian factors, and for even lengths three. Further, the number of occurrences of 1 in each factor differs by at most 1 from half of its length [22]. It is easy to see that any factor of any word in \( \{\varepsilon, 0, 1\} \cdot \{0, 1\}^N \) has the same property, i.e., \( \{\varepsilon, 0, 1\} \cdot \{0, 1\}^N \subseteq A(\text{TM}) \). In fact, equality holds: \( A(\text{TM}) = \{\varepsilon, 0, 1\} \cdot \{0, 1\}^N \) (so, the abelian closure of \( \text{TM} \) is huge compared to its shift orbit closure). Indeed, let \( x \in A(\text{TM}) \). Then \( x \) has blocks of each letter of length at most 2 (since there are no factors 000 and 111). Moreover, between two consecutive occurrences of 00 there must occur 11, and vice versa (otherwise we have a factor \( 00(10)^n0 \), where the number of occurrences of 1 differs by more than 1 from half of its length). Clearly, such a word is in \( \{\varepsilon, 0, 1\} \cdot \{0, 1\}^N \). In [20], we show that this fact can be generalized to all binary words: In fact, each binary aperiodic uniformly recurrent word which is not Sturmian, admits infinitely many minimal subshifts in its abelian closure. Moreover, in the case of rational letter frequency, the abelian closure always contains a morphic image of the full shift.

In general, the abelian closure of an infinite word might have a pretty complicated structure. T. Hejda, W. Steiner, and L.Q. Zamboni studied the abelian shift of the Tribonacci word \( T \) defined as the fixed point of \( \tau : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0 \). They have announced that \( A(T) \) contains only one minimal subshift, namely \( \Omega(T) \) itself, but that there exist other words in it as well [12, 24].

The study of abelian closures is motivated by the question of, given an infinite word \( x \), how strong is the bond between its abelian factors and its language. We quantify this bond by the size of the abelian closure. By size we do not mean the usual cardinality of a subshift, rather, we mean the number of disjoint minimal subshifts contained in \( A(x) \). A shift orbit closure \( \Omega(x) \) is minimal if does not properly contain another shift orbit closure. If \( A(x) \) is huge (it contains infinitely many minimal subshifts), then this bond is quite weak. On the other hand, if \( A(x) \) is small (finitely many minimal subshifts), then the bond is quite strong. The strongest bond is attained when \( A(x) \) is a minimal subshift itself. In this case we necessarily have \( A(x) = \Omega(x) \). It is not hard to see that for purely periodic words, their abelian closure is finite (see Proposition 2.3). On the other hand, the abelian closure of an ultimately periodic word can be huge (see Example 2.4). For reasons stemming from this observation, when dealing with abelian closures of individual words, we shall assume the words to define minimal shift orbit closures.

There is indeed no particular reason to restrict the definition of abelian closures to just individual words; the abelian closure of a set of words \( X \) comprises those infinite words \( y \) whose each factor is abelian equivalent to some factor of one of the words in \( X \). For example, the abelian closure of \( \Omega(x) \) coincides with \( A(x) \). As mentioned previously, the shift orbit closure of an infinite word is a certain type of a subshift. In general, a non-empty set \( X \subseteq \Sigma^N \) of infinite words is called a subshift if it is closed (as a subset of the compact metric space \( \Sigma^N \) equipped with the usual product topology defined by the discrete topology on \( \Sigma \)) and that \( \sigma(X) \subseteq X \), where the shift map \( \sigma \), is defined as \( \sigma(x)_i = x_{i+1}. \) A subshift is called minimal

1Subshifts are often defined over bi-infinite words, in which case we require \( \sigma(X) = X \) in the definition.
if it does not contain a proper subshift. Hence a minimal subshift is always the shift orbit closure of some word \( x \). It is routinely checked that \( \mathcal{A}(X) \) is a subshift for any set \( X \) of words.

In this paper, we study how the characterization of Sturmian words as aperiodic uniformly recurrent words with \( \mathcal{A}(x) = \Omega(x) \) extends to non-binary alphabets. We then study the abelian closures of certain generalizations of Sturmian words, and some preliminary results have been reported at DLT 2018 conference [15]. Besides that, we discuss abelian closures of subshifts in general. In Section 3 we characterize the abelian subshifts of aperiodic recurrent balanced words; they are a finite union of minimal subshifts. In Section 4 we consider abelian closures of words over a \( k \)-letter alphabet with factor complexity \( n + k - 1 \) for each \( n \), which are aperiodic words of minimal complexity involving \( k \) letters. The behavior is different depending on \( k \). For \( k = 2 \), we are in the case of Sturmian words, so we have \( \mathcal{A}(x) = \Omega(x) \). Surprisingly, the most complicated behaviour is exhibited in the ternary alphabet. We show that for \( k = 3 \), depending on the word \( x \), its abelian closure \( \mathcal{A}(x) \) contains either exactly one, or uncountably many minimal subshifts. For alphabets of size greater than 3, \( \mathcal{A}(x) \) equals the union of exactly two minimal subshifts, \( \Omega(x) \) and its "reversal". Further, in Section 5, we show that for Arnoux–Rauzy words, their abelian closures contain non-recurrent words, and hence \( \mathcal{A}(x) \neq \Omega(x) \). We then extend our interest to general subshifts in Section 6. Our focus is on subshifts defined using notions from formal language theory. We show that the abelian closure of a subshift of finite type (resp., sofic subshift) is not necessarily a subshift of finite type (resp., a sofic subshift) (see Section 6 for definitions). We then conclude with open problems.

2. Notation and first observations

We recall some notation and basic terminology from the literature of combinatorics on words. We refer the reader to [17, 18] for more on the subject.

The set of finite words over an alphabet \( \Sigma \) is denoted by \( \Sigma^* \) and the set of non-empty words is denoted by \( \Sigma^+ \). The empty word is denoted by \( \varepsilon \). We let \( |w| \) denote the length of a word \( w \in \Sigma^* \). By convention, \( |\varepsilon| = 0 \). A factor of a word \( x \) is any block of its consecutive letters, and we let \( \mathcal{L}(x) \) denote the set of factors of \( x \). The length \( n \) factors of \( x \) is denoted by \( \mathcal{L}_n(x) \). The length \( n \) prefix of the word \( x \) is denoted by \( \text{pref}_n(x) \). The factor complexity function \( \mathcal{P}_x \) is defined by \( \mathcal{P}_x(n) = |\mathcal{L}_n(x)| \). An infinite word \( x \) is called recurrent if each factor of \( x \) occurs infinitely many times in \( x \). Further, \( x \) is uniformly recurrent if for each factor \( u \in \mathcal{L}(x) \) there exists \( N \in \mathbb{N} \) such that \( u \) occurs as a factor in each factor of length \( N \) of \( x \). For a finite word \( u \in \Sigma^* \), we let \( |u|_a \) denote the number of occurrences of the letter \( a \in \Sigma \) in \( u \). For a finite word \( v \), we let \( v^\omega \) denote the infinite word obtained by repeating \( v \) infinitely many times.

For \( x \in \Sigma^\mathbb{N} \) and \( a \in \Sigma \), the limits

\[
\overline{\text{freq}}_x(a) := \lim_{n \to \infty} \sup_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n} \quad \text{and} \quad \underline{\text{freq}}_x(a) := \lim_{n \to \infty} \inf_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n}
\]

exist. and, moreover,

\[
\overline{\text{freq}}_x(a) = \inf_{n \in \mathbb{N}} \sup_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n} \quad \text{and} \quad \underline{\text{freq}}_x(a) = \sup_{n \in \mathbb{N}} \inf_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n}.
\]
This follows from Fekete’s lemma as sup\(_{v \in L_\alpha(x)} |v|_\alpha\) (resp., inf\(_{v \in L_\alpha(x)} |v|_\alpha\)) is subadditive (resp., superadditive) with respect to \(n\). It immediately follows that

\[
\sup_{v \in L_\alpha(x)} |v|_\alpha \geq n \overline{\text{freq}}_x(a) \quad \text{and} \quad \inf_{v \in L_\alpha(x)} |v|_\alpha \leq n \underline{\text{freq}}_x(a)
\]

for all \(n \in \mathbb{N}\). If \(\overline{\text{freq}}_x(a) = \underline{\text{freq}}_x(a)\), we denote the common limit by \(\text{freq}_x(a)\) and we say that \(x\) has uniform frequency of \(a\).

A subshift \(X \subseteq \Sigma^\mathbb{N}\), \(X \neq \emptyset\), is a closed set (with respect to the product topology on \(\Sigma^\mathbb{N}\)) satisfying \(\sigma(X) \subseteq X\), where \(\sigma\) is the shift operator (defined by \(\sigma(a_0a_1a_2\cdots) = a_1a_2\cdots\)). For a subshift \(X \subseteq \Sigma^\mathbb{N}\) we let \(\mathcal{L}(X) = \cup_{y \in X} \mathcal{L}(y)\). A subshift \(X \subseteq \Sigma^\mathbb{N}\) is called minimal if \(X\) does not contain any proper subshifts. Observe that two minimal subshifts \(X\) and \(Y\) are either equal or disjoint. Let \(x \in \Sigma^\mathbb{N}\). We let \(\Omega(x)\) denote the shift orbit closure of \(x\), which may be defined as the subshift \(\{y \in \Sigma^\mathbb{N} : \mathcal{L}(y) \subseteq \mathcal{L}(x)\}\). Thus \(\mathcal{L}(\Omega(x)) = \mathcal{L}(x)\) for any word \(x \in \Sigma^\mathbb{N}\).

It is known that \(\Omega(x)\) is minimal if and only if \(x\) is uniformly recurrent. For a morphism \(\varphi : \Sigma \to \Delta^*\) (that is, \(\varphi(uv) = \varphi(u)\varphi(v)\) for all \(u, v \in \Sigma^*\)) and a subshift \(X \subseteq \Sigma^\mathbb{N}\), we define \(\varphi(X) = \cup_{x \in X} \Omega(\varphi(x))\). When using erasing morphisms, that is, some letter maps to \(\varepsilon\), we make sure that no point in \(X\) gets mapped to a finite word. For more on this topic we refer the reader to [16].

We recall definitions and properties related to Sturmian words from [18, Chapter 2]. We identify the interval \([0, 1]\) with the unit circle \(T\) (the point 1 is identified with point 0). For points \(x, y \in T\), we let \(I(x, y)\) (resp., \(I^+(x, y)\)) denote the half-open interval on \(T\) containing \(x\) (resp., \(y\)) and starting from \(x\) and ending at \(y\) in counter-clockwise direction. We omit the bar whenever it does not matter which endpoint is in the interval. Let \(\alpha \in T\) be irrational or rational and let \(\rho \in T\). The map \(R_\alpha : T \to T\), \(x \mapsto \{x + \alpha\}\), \(\{x \in T\} = x - \lfloor x \rfloor\) is the fractional part of \(x \in \mathbb{R}\), defines a (counter-clockwise) rotation on \(T\). Divide \(T\) into two half-open intervals \(I_0 = I^+(0, 1 - \alpha)\) and \(I_1 = I^+(1 - \alpha, 1)\) (resp., \(I_0 = I(0, 1 - \alpha)\) and \(I_1 = I(1 - \alpha, 1)\)) and define the coding \(\nu : T \to \{0, 1\}, x \mapsto i\) if \(x \in I_i\), \(i = 0, 1\). The rotation word \(\mathbf{s}_{\alpha, \rho}\) (resp., \(\mathbf{s}_{\alpha, \rho}\)) of slope \(\alpha\) and intercept \(\rho\) is the word \(a_0a_1\cdots \in \{0, 1\}^\mathbb{N}\) defined by \(a_n = \nu(R_\alpha^n(\rho))\) for all \(n \in \mathbb{N}\). Note that 00 occurs in \(\mathbf{s}_{\alpha, \rho}\) if and only if \(\alpha < 1/2\). Clearly, \(\mathbf{s}_{\alpha, \rho}\) is aperiodic if and only if \(\alpha\) is irrational. Each aperiodic rotation word is a Sturmian word and vice versa. We call periodic rotation words periodic Sturmian. Observe the special role played by \(\mathbf{s}_{\alpha, 0} = \mathbf{s}\): both 01s and 10s \(\in \Omega(\mathbf{s})\) for \(\alpha \in (0, 1)\).

Every Sturmian word \(\mathbf{s}\) is uniformly recurrent so that \(\Omega(\mathbf{s})\) is minimal. Further, \(\mathbf{s}' \in \Omega(\mathbf{s})\) if and only if \(\mathbf{s}\) and \(\mathbf{s}'\) are of the same slope. In particular, the intercepts or the endpoints of \(I_0\) and \(I_1\) do not play any role when speaking of the shift orbit closure of a Sturmian word. (In Section 4 we pay attention to these choices.)

An infinite word \(x \in \Sigma^\mathbb{N}\) is called balanced if, for all \(v, v' \in \Sigma^\mathbb{N}\) with \(|v| = |v'|\) and for all \(\alpha \in \Sigma\), we have \(||v|_\alpha - |v'|_\alpha| \leq 1\). Periodic and aperiodic Sturmian words are exactly the recurrent balanced binary words [19]. It follows that, for each (aperiodic or periodic) Sturmian word \(\mathbf{s}\), the set \(\{|v|_1 : v \in L_n(\mathbf{s})\}\) consists of at most two values \(k\) and \(k + 1\) for some \(k\) depending on \(\mathbf{s}\) and \(n\). Observe now that \(\text{freq}_x(1) = \alpha\) for any Sturmian word \(\mathbf{s}\) of slope \(\alpha\). By (1), \(k\) above equals \(|n\alpha|\).

We turn to the main notion of this paper.

**Definition 2.1.** For \(x \in \Sigma^\mathbb{N}\) we define the abelian closure of \(x\) as

\[
\mathcal{A}(x) = \{y \in \Sigma^\mathbb{N} | \forall u \in \mathcal{L}(y) \exists v \in \mathcal{L}(x) : u \sim_{ab} v\}.
\]
Now, for any $x \in \Sigma^\mathbb{N}$, the abelian closure $A(x)$ is indeed a subshift. We make preliminary observations on abelian closures of infinite words.

**Lemma 2.2.** Assume $x \in \Sigma^\mathbb{N}$ has uniform frequency of a letter $a \in \Sigma$. Then any word $y \in A(x)$ has uniform frequency of $a$ and $\text{freq}_y(a) = \text{freq}_x(a)$.

**Proof.** For all $y \in A(x)$ and for all $n \in \mathbb{N}$, we have immediately from the definition of $A(x)$ that

$$\sup_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n} \geq \sup_{v \in \mathcal{L}_n(y)} \frac{|v|_a}{n} \geq \inf_{v \in \mathcal{L}_n(y)} \frac{|v|_a}{n} \geq \inf_{v \in \mathcal{L}_n(x)} \frac{|v|_a}{n}.$$

Letting $n \to \infty$ gives our claims. \qed

We immediately have that if $x$ has an irrational uniform frequency of some letter $a$, then $A(x)$ contains only aperiodic words. We continue by observing how the abelian closures of periodic and ultimately periodic words can differ.

**Proposition 2.3.** For any periodic word $x$, the abelian closure $A(x)$ is finite.

**Proof.** A word $y$ is periodic if and only if all factors of length $n$ are abelian equivalent for some $n \geq 1$ [5]. Let $n$ be the least such integer for $x$. It follows that all factors of length $n$ of any word $y \in A(x)$ are abelian equivalent. Thus $y = v^\omega$ with $|v|$ dividing $n$. There are finitely many such words. \qed

In general, the abelian closure of an ultimately periodic word can be huge.

**Example 2.4.** Let $x = 0011(001101)^\omega$. It is readily verified that for odd lengths $x$ has two abelian factors, and for even lengths three. Further, for each factor of $x$, the number of occurrences of 1 differs by at most one from half of its length. Thus, by the discussion in the introduction, we have $TM \in A(x)$ so that $A(x) = A(TM) = \{\varepsilon, 0, 1\} \cdot \{01, 10\}^\mathbb{N}$.

A family of such examples are given in [15, Ex. 2].

In the end of this section we show that, for a uniformly recurrent binary word $x$, $A(x)$ contains exactly one minimal subshift if and only if $x$ is a Sturmian word. We start with a crucial observation, which is characteristic only for binary words.

**Lemma 2.5** (Corridor Lemma). Let $x$ be a binary word. Then $y \in A(x)$ if and only if, for all $n \in \mathbb{N}$, we have

$$\inf_{u \in \mathcal{L}_n(y)} |u|_1 \geq \inf_{u \in \mathcal{L}_n(x)} |u|_1 \text{ and } \sup_{u \in \mathcal{L}_n(y)} |u|_1 \leq \sup_{u \in \mathcal{L}_n(x)} |u|_1.$$

**Proof.** It is easy to see (e.g., by a sliding window argument) that, for any $n \geq 1$, there exists a word $u \in \mathcal{L}_n(z)$ with $|u|_1 = m$ if and only if $\inf_{v \in \mathcal{L}_n(x)} |v|_1 \leq m \leq \sup_{v \in \mathcal{L}_n(x)} |v|_1$. Applying this observation to $x$ and $y$ we have that for each $v \in \mathcal{L}_n(y)$ there exists $u \in \mathcal{L}_n(x)$ such that $v \sim_{ab} u$ if and only if $\inf_{u \in \mathcal{L}_n(x)} |u|_1 \leq \inf_{u \in \mathcal{L}_n(y)} |u|_1$ and $\sup_{u \in \mathcal{L}_n(y)} |u|_1 \leq \sup_{u \in \mathcal{L}_n(x)} |u|_1$. \qed

We now characterize Sturmian words in terms of abelian closures.

**Theorem 2.6.** Let $x \in \{0, 1\}^\mathbb{N}$ be uniformly recurrent. Then $A(x)$ contains exactly one minimal subshift if and only if $x$ is Sturmian.
Another version of this is the following:

**Theorem 2.7.** Let \( x \in \{0, 1\}^\mathbb{N} \) be aperiodic and uniformly recurrent. Then \( \mathcal{A}(x) = \Omega(x) \) if and only if \( x \) is Sturmian.

**Proof.** First we show that \( \mathcal{A}(x) = \Omega(x) \) for Sturmian words. Let \( x \) be a Sturmian word of slope \( \alpha \) and \( y \in \mathcal{A}(x) \). By the Corridor Lemma, \( y \) is balanced and, by Lemma 2.2, has uniform frequencies of letters equal to those of \( x \). Thus \( y \in \Omega(x) \).

Assume then that \( \mathcal{A}(x) \) contains exactly one minimal subshift, namely \( \Omega(x) \), and let \( \alpha = \text{freq}_x(1) \). Take a (periodic or aperiodic) Sturmian word \( s \) of slope \( \alpha \). By the Corridor Lemma, we have \( \Omega(s) \subseteq \mathcal{A}(x) \) using (1). Since \( \Omega(s) \) is also minimal, we have \( \Omega(x) = \Omega(s) \). Thus \( x \) is Sturmian, and \( \mathcal{A}(x) = \Omega(x) \).

The following example shows that we cannot omit the assumption of uniform recurrence from the statement of the above theorem.

**Example 2.8.** Take the Champernowne word \( C_2 \) (over the binary alphabet), which is obtained by concatenating all finite words ordered by length and lexicographic order for the same length:

\[
C_2 = 0 1 00 01 10 11 000 001 010 011 100 101 110 111 \cdots
\]

Clearly, both \( \Omega(C_2) \) and \( \mathcal{A}(C_2) \) are equal to the full shift, i.e., contain all binary words.

Note that the property \( \mathcal{A}(x) = \Omega(x) \) or \( \mathcal{A}(x) \) containing exactly one minimal subshift does not characterize Sturmian words among uniformly recurrent words over arbitrary alphabets. Let \( f \) be the Fibonacci word, which is a Sturmian word defined as the fixed point of the morphism \( 0 \mapsto 01, 1 \mapsto 0 \). Let then \( \varphi : 0 \mapsto 02, 1 \mapsto 12 \). Then \( \mathcal{A}(\varphi(f)) = \Omega(\varphi(f)) \) (see Theorem 4.4).

We investigate possible generalizations of the property \( \mathcal{A}(x) = \Omega(x) \) to nonbinary alphabets in the next sections.

3. Abelian closures of balanced words

In this section we study the abelian closures of non-binary aperiodic balanced words. We prove that the abelian closure of such a word is a finite union of minimal subshifts:

**Theorem 3.1.** Let \( u \) be aperiodic recurrent and balanced. Then \( \mathcal{A}(u) \) is the union of finitely many minimal subshifts.

As we will show below, abelian closure of a recurrent balanced word can contain one or more (yet a finite number) of minimal subshifts, depending on its structure, and we can in fact compute this number. Our results rely heavily on the characterization of aperiodic recurrent balanced words by R. Graham [11] and P. Hubert [13]. In fact, the characterization allows us to characterize the abelian closures of slightly more general words. In particular, the the techniques used in the proof of the above theorem give us, for each \( k \), an aperiodic word \( x_k \) over a four-letter alphabet such that \( \mathcal{A}(x_k) \) equals the union of \( k \) distinct minimal subshifts (see Proposition 3.10).

We need some notation to give a characterization of aperiodic recurrent words.

**Definition 3.2.** A word is called **constant gap** if each letter occurs with a constant gap.
For example, \((abac)^\infty\) is a constant gap word.

**Definition 3.3.** Let \(x\) be a finite or infinite binary word, \(z_0 \in A^N\) and \(z_1 \in B^N\), where \(A\) and \(B\) are some alphabets. Let \(S(x, z_0, z_1)\) denote the word obtained from \(x\) by substituting the \(n\)th occurrence of 0 (resp., 1) in \(x\) by the \(n\)th letter of \(z_0\) (resp., \(z_1\)).

We illustrate the above operation with an example.

**Example 3.4.** Let \(f = 0100101001001001\) be the Fibonacci word, \(z_0 = (0102)^\omega\), and \(z_1 = (ab)^\omega\). Then \(S(f, z_0, z_1) = 0a1b2a01b2a0b10a2b\) is balanced.

The following theorem characterises recurrent balanced words using constant gap words and the operation \(S\).

**Theorem 3.5 ([13, Thm. 1], [11]).** An aperiodic word \(u \in \Sigma^N\) is recurrent and balanced if and only if there exist a partition \(\{A, B\}\) of \(\Sigma\), two constant gap words \(z_0 \in A^N\) and \(z_1 \in B^N\), and a Sturmian word \(s\), such that \(u = S(s, z_0, z_1)\).

We remark that although the structure of aperiodic balanced words is clear, the structure of periodic balanced words is a mystery: the following conjecture by Fraenkel, 1973, remains open despite efforts of different scientists: The unique (up to a permutation of letters) balanced word on \(k \geq 3\) letters with all distinct frequencies of letters is \((F_k)^\omega = (F_{k-1}kF_{k-1})^\omega\) where \(F_2 = 121\) [10]. The conjecture has been verified for \(k \leq 7\) (see [2] and references therein).

In fact, throughout this section we consider slightly more general words, namely, we relax the condition of \(z_0\) and \(z_1\) being constant gap. We obtain a characterization of the abelian closures of such words, from which the characterization of abelian closures of recurrent balanced words follows.

We need the following lemma:

**Lemma 3.6.** Let \(u = S(s, z_0, z_1)\) with \(s\) Sturmian and \(z_i\) periodic words. Then

\[
\mathcal{L}(u) = \{S(x, z'_0, z'_1) : x \in \mathcal{L}(s), z'_0, z'_1 \in \Omega(z_i)\}.
\]

Further, \(u\) is uniformly recurrent.

**Proof.** In fact, this result is implicitly contained in [13], only stated in slightly weaker form. Theorem 2 and Proposition 3.1 of [13] essentially state the following. Let \(v = S(s, y_0, y_1)\), where \(s\) is a Sturmian word and \(y_0\) and \(y_1\) are constant gap words with periods \(l_0\) and \(l_1\) respectively. Then the factor complexity function of \(v\) satisfies \(P_v(n) = l_0l_1(n+1)\) for all large enough \(n\). Further [13, Prop. 5.1] states that \(v\) is uniformly recurrent.

Observe that \(\mathcal{L}(v) \subseteq \{S(x, \sigma^i(y_0), \sigma^j(y_1)) : x \in \mathcal{L}(s), i, j \in N\}\) and the cardinality of the latter set equals \(l_0l_1(n+1)\) for large enough \(n\). This coincides with the factor complexity \(P_v(n)\) by Hubert’s result. We deduce that for all \(x \in \mathcal{L}(s), i, j \in N\), we have \(S(x, \sigma^i(y_0), \sigma^j(y_1)) \in \mathcal{L}(v)\).

Now we can take the constant gap words \(y_0 = (a_1 \ldots a_{l_0})^\omega\) and \(y_1 = (b_1 \ldots b_{l_1})^\omega\), where \(l_0\) and \(l_1\) are the lengths of the periods of \(z_0\) and \(z_1\): \(z_i = u_i^\omega, |u_i| = l_i\). Define a coding \(\tau\) so that \(a_1 \ldots a_{l_0} \mapsto u_0\) and \(b_1 \ldots b_{l_1} \mapsto u_1\). Then \(\tau(v) = u\) so that \(u\) is uniformly recurrent. Further, \(\tau(S(x, \sigma^i(y_0), \sigma^j(y_1))) = S(x, \sigma^i(z_0), \sigma^j(z_1))\). The claim follows now immediately. \(\square\)
In the above lemma we allow the periodic words \( z_0 \) and \( z_1 \) contain common letters. On the other hand, in the following proposition we assume that they do not share common letters. This puts us in the position of characterizing the abelian closures of such words.

**Proposition 3.7.** Let \( u = S(s, z_0, z_1) \) for some Sturmian word \( s \) and periodic words \( z_0 \in A^N \) and \( z_1 \in B^N \), where \( A \) and \( B \) are disjoint alphabets. Then

\[
A(u) = \bigcup_{t_j \in A(z_i)} \Omega(S(s, t_0, t_1)),
\]

and \( A(u) \) is a finite union of minimal subshifts.

**Proof.** Let first \( x \in \Omega(S(s, t_0, t_1)) \) for some \( t_i \in A(z_i) \). For any factor \( x \in L(x) \), we have \( x = S(y, \sigma^i(t_0), \sigma^j(t_1)) \) for some \( y \in L(s) \) and \( i, j \geq 0 \) by the above lemma. Let \( u = \text{pref}_n(\sigma^i(t_0)) \) and \( v = \text{pref}_m(\sigma^j(t_1)) \), where \( n = |y|_0 \), \( m = |y|_1 \). By assumption, there exist \( u' \in L(z_0) \) and \( v' \in L(z_1) \) such that \( u \sim_{ab} u' \) and \( v \sim_{ab} v' \). Choose \( r, s \) such that \( \sigma^r(z_0) \) begins with \( u' \) and \( \sigma^s(z_1) \) begins with \( v' \). It follows that \( x \sim_{ab} S(y, \sigma^r(z_0), \sigma^s(z_1)) \in L(u) \). We thus have \( x \in A(u) \).

Let then \( x \in A(u) \). Take \( \varphi: \Sigma \to \{0, 1\} \) such that \( \varphi(a) = 0 \) if and only if \( a \in A \). It follows that \( \varphi(x) \in \Omega(\varphi(u)) = \Omega(s) \) since the alphabets \( A \) and \( B \) are disjoint. Take then the morphism \( \varphi_A: \Sigma \to A^* \) such that \( \varphi_A(a) = a \) for \( a \in A \), otherwise \( \varphi_A(a) = \varepsilon \). Define the morphism \( \varphi_B: \Sigma \to B^* \) analogously. We again have that \( \varphi_A(x) \in \Omega(\varphi_A(u)) \) where \( \varphi_A(u) = z_0 \). Similarly \( \varphi_B(x) \in A(z_1) \). It is now evident that \( x = S(s', t_0, t_1) \) for some \( t_i \in A(z_i) \), \( i = 0, 1 \), and \( s' \in \Omega(s) \). The above lemma implies that \( x \in \Omega(S(s, t_0, t_1)) \).

As \( \Omega(z_i) \) is finite by **Proposition 2.3**, \( A(u) \) is a finite union of minimal subshifts. This concludes the proof. \( \boxdot \)

The above proposition has **Theorem 3.1** as an immediate corollary.

Let us consider what the above proposition says. The number of distinct minimal subshifts is bounded above by the product of the number of minimal subshifts in \( A(z_0) \) and the number of minimal subshifts in \( A(z_1) \).

**Example 3.8.** Let \( z_0 = (0102)^\omega \) and \( z_1 = (34)^\omega \). Now \( A(z_i) = \Omega(z_i) \) as is readily verified. Thus \( A(u) = \Omega(u) \) for \( u = S(s, z_1, z_2) \), \( s \) Sturmian, by the above proposition.

For the case of recurrent balanced words, the periodic words in the construction are constant gap words. It is natural to ask whether the extra property of constant gaps restricts the cardinality of the number of minimal subshifts in its abelian closure. We give a negative answer to this:

**Example 3.9.** Let \( z_1 = (a_0a_1a_2a_3a_4A_0 \cdot a_0a_1a_2a_3a_4A_1 \cdot a_0a_1a_2a_3a_4A_2)^\omega \). Here the letters \( a_i \) have constant gaps of length 6, and \( A_i \) have constant gaps of length 18. Take any constant gap sequence \( z_2 \) which is not closed under reversal (e.g., \( (abc)^\omega \)), and \( u = S(s, z_1, z_2) \). Then we can independently take reversal inside \( z_1 \), inside \( z_0 \), and inside the arithmetic progression given by \( A_0A_1A_2 \) in \( z_0 \). We thus get eight minimal subshifts. Note that this construction can be generalized to produce \( 2^k \) minimal subshifts for any \( k \).

The techniques used in the proof of the above theorem give us the following proposition. We remark that the words in question are not necessarily balanced:
Proposition 3.10. For each \( k \geq 1 \) there exists an aperiodic word \( x_k \) over a three-letter alphabet such that \( A(x_k) \) equals the union of \( k \) distinct minimal subshifts.

Proof. For \( k = 1 \) we may take any Sturmian word. Let thus \( k \geq 2 \). Consider first the abelian closure of the periodic word \( z = (0^{2k-1}11)^\omega \). It is readily verified that \( A(x) \) contains the minimal subshifts generated by the words \( (0^{2k-1-i}10^i1)^\omega, \) \( i = 0, \ldots, k - 1 \). To see that there is nothing else in \( A(z) \), we observe the following. By Proposition 2.3, any word in \( A(z) \) is periodic with period dividing \( 2^{k+1} \), an odd number. The period cannot be less than \( 2^{k+1} \), as the number of 1s in factors of length \( 2^{k+1} \) is only 2. On the other hand, all words of length \( 2^{k+1} \) containing two occurrences of 1 (and hence the periodic words they generate) occur already in the subshifts above.

For the claim we set \( x_k = S(s, z, a^\omega) \) where \( s \) is an aperiodic Sturmian word. By Proposition 3.7,

\[
A(x_k) = \bigcup_{i=0}^{k-1} \Omega(S(s, (0^{2k-1-i}10^i1)^\omega, a^\omega)),
\]
a union of \( k \) distinct minimal subshifts. \( \square \)

4. Abelian closures of words of minimal complexity

First we study the abelian closures of aperiodic nonbinary words of minimal factor complexity. Over an alphabet \( \Sigma \) with at least two letters, the minimal complexity is \( n + |\Sigma| - 1 \). The structure of words of complexity \( n + C \) is related to the structure of Sturmian words and is well understood (\([7, 9, 14]\)). The main goal of this subsection is to prove that for aperiodic ternary words of minimal complexity their abelian closure consists of either one or uncountably many minimal subshifts (Theorem 4.4); for alphabets of size greater than 3 the abelian closure contains exactly two minimal subshifts (when the words are assumed to be recurrent, Theorem 4.13). Recall that for binary alphabet, we have exactly one minimal subshift (Theorem 2.6).

A proof of the following is explained in \([9]\) in the discussion following Lemma 1.

Lemma 4.1. A minimal complexity word \( u \) over the alphabet \( A \) is of the form \( a_0 \cdots a_t u' \), where \( u' \) is a recurrent minimal complexity word over an alphabet \( A' \subseteq A \), and \( a_0, \ldots, a_t \) are distinct letters of \( A \setminus A' \).

We shall consider the abelian subshifts of recurrent aperiodic minimal complexity words. The following lemma then extends those results to handle non-recurrent ones as well.

Lemma 4.2. Let \( u = a_0 \cdots a_t u' \) be an aperiodic minimal complexity word as in the above lemma. Then \( A(u) = \Omega(u) \cup A(u') \).

Proof. Let \( y \in A(u) \), so it contains at most one occurrence of each of the letters \( a_i \). If it contains none of them, there is nothing to prove.

So let us write \( y = p a_i y' \), where \( y' \) contains none of the letters \( a_j, j = 1, \ldots, t \). First of all, \( i = t \), as \( a_t \) must be followed by \( a_{t+1} \) (or \( a_{t-1} \), in which case we reach \( a_0 \) at some point, after which we cannot add anything). We show that \( y' = u' \). Assume that \( y' \neq u' \), so \( y \) contains the factor \( a_t w a \), while \( u \) contains \( a_t w b \), for some \( a \neq b \). This is impossible, as \( a_t w b \).
is the only factor of length \(|w| + 2\) of \(u\) that contains only letters from \(A \setminus A'\) (apart from \(a_t\)). Hence \(y = pa_tu'\).

Now either \(p\) ends with \(a_{t-1}\) or with a letter \(a \in A'\). In the latter case, we must have that \(a\) is the first letter of \(u'\), and so \(y\) contains \(aa\). It follows that \(u'\) must begin with \(aa\), so \(y\) contains \(aa\). By continuing with this line of reasoning, we see that \(u' = a^\omega\), which is contrary to the assumption that \(u\) is aperiodic. We deduce that \(p\) must end with \(a_{t-1}\). Now \(p\) cannot contain any letter from \(A'\) (it would be followed by a letter \(a_i\) with \(i < t\), a contradiction). The only option is that \(y = a_ja_{j+1}\cdots a_tu'\) for some \(j \geq 0\), which suffices for the proof.

\[\square\]

### 4.1. Ternary minimal complexity words

We start with infinite words for which \(p(n) = n + 2\) for all \(n \geq 1\). Observe that this implies that we deal with ternary words.

In [3], J. Cassaigne characterizes words having factor complexity \(n + C\) for all \(n \geq n_0\), \(C\) a constant. Here we consider the case of \(C = 2\) and \(n_0 = 1\). We first recall their characterization, which can be deduced from [14] (see also [7]).

**Theorem 4.3.** A word \(u \in \{0, 1, 2\}\) has factor complexity \(p(n) = n + 2\) for all \(n \geq 1\) if and only if \(u\) is of the form (up to permuting the letters)

1. \(u = 2s\) for some Sturmian word \(s \in \{0, 1\}^N\), or
2. \(u \in \Omega(\varphi(s))\), where \(s\) is a Sturmian word and \(\varphi\) is defined by
   - \(0 \mapsto 02, 1 \mapsto 12;\)
3. \(0 \mapsto 0, 1 \mapsto 12.\)

In this subsection we study the abelian closures of these words. The main result is the following theorem.

**Theorem 4.4.** Let \(u\) be a word of factor complexity \(n + 2\) for all \(n \geq 1\). If \(u\) is as in Theorem 4.3 (1) or (2), then \(A(u) = \Omega(u)\). If \(u\) is as in (3), then \(A(u)\) contains uncountably many minimal subshifts.

In fact we are able to characterize the abelian closures of these words. We do this in parts, the first two cases are straightforward and we prove them first. For the last case we need some further notions.

**Remark 4.5.** In the following, we often use the following argument. Assume that each letter in \(x \in \Sigma^N\) occurs with bounded gaps. Let \(\varphi\) be a morphism such that \(|\varphi(a)| \leq 1\). Then \(\varphi(A(x)) \subseteq A(\varphi(x))\). Indeed, since in \(x\) each letter occurs with bounded gaps, the same holds for any \(y \in A(x)\). Consequently \(\varphi(y)\) is infinite. Letting \(v\) be a factor of \(\varphi(y)\), there exists a factor \(v'\) of \(y\) such that \(\varphi(v') = v\) due to the length assumption on \(\varphi\). As \(y \in A(x)\) there exists a factor \(w'\) of \(x\) abelian equivalent to \(v'\). It thus follows that \(\varphi(x)\) contains the factor \(\varphi(w')\) abelian equivalent to \(v\). As \(y\) and \(v\) were arbitrary, we conclude that \(\varphi(y) \in A(\varphi(x))\).

In particular, if \(\varphi(x)\) is Sturmian, then \(\varphi(A(x)) \subseteq \Omega(\varphi(x))\) by Theorem 2.6.

**Proposition 4.6.** Let \(u\) as in Theorem 4.3 (1) or (2). Then \(A(u) = \Omega(u)\).

**Proof.** Assume first that \(u = 2s\). The claim follows immediately from Lemma 4.2 together with Theorem 2.7.
Assume then that \( u \) is as in (2). Let \( x \in \mathcal{A}(u) \). By applying the morphism \( 2 \mapsto 2, 1 \mapsto 0, 0 \mapsto 0 \), we see that every second letter of \( x \) is 2. Further, by mapping \( 2 \mapsto \varepsilon, 1 \mapsto i \) for \( i = 0, 1 \), we see that \( x \) maps into a word in \( \Omega(s) \). It is straightforward to see that now \( x \in \Omega(u) \).

The rest of this subsection is devoted to the case where \( u \) is as in Theorem 4.3 (3). This case is more intricate as shown by the following example: \( \mathcal{A}(u) \) contains non-recurrent words, similar to the Tribonacci word.

**Example 4.7.** Let \( \alpha \in T \), let \( \varphi \) be as in Theorem 4.3 (3), and \( u = \varphi(s_{0,\alpha}) \). The words \( u_1 = \varphi(1s_{0,\alpha}) = 012u \) and \( u_2 = \varphi(10s_{0,\alpha}) = 120u \) are both in \( \Omega(u) \). We claim that the non-recurrent word \( x = 02u \in \mathcal{A}(u) \). Indeed, \( \sigma(x) = \sigma^2(u_1) \in \Omega(u) \) (recall \( \sigma \) is the shift map). Further, any prefix of \( x \) of length at least 2 is abelian equivalent to the prefix of \( \sigma(u_2) \).

Thus \( x \in \mathcal{A}(u) \).

We now analyze the structure of \( \mathcal{A}(u) \). Without loss of generality we may take \( u = \varphi(s) \), since \( \varphi(s) \) is uniformly recurrent. Consider the images of \( u \) under the morphisms \( \varphi_1 : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0 \) and \( \varphi_2 : 0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1 \). We have \( \varphi_1(u) = s_1 \), a Sturmian word of some slope \( \alpha \) and intercept \( \rho \). (Indeed, \( \varphi(u) = G \circ E(s) \) using the notation of [18, Chap. 2, p. 72].) Symmetrically, \( \varphi_2(u) = s_2 \) is a Sturmian word of slope \( \alpha \) and intercept \( \rho' \). (Again, \( \varphi_2(u) = D \circ E(s) \), see again the above reference.) In fact we can say more: \( \rho' = \rho - \alpha \) or, equivalently, \( s_1 = \sigma(s_2) \). Observe now that \( s_1 \) contains the factor 00 meaning that \( \alpha < 1/2 \).

By mapping any word \( x \in \mathcal{A}(u) \) with the morphisms \( \varphi_1 \) and \( \varphi_2 \), we obtain two Sturmian words with the same slope \( \alpha \). Further, by applying \( 0 \mapsto \varepsilon \) on \( u \), we see that \( x \in \Omega((12)^\omega) \). This implies that all words in \( \mathcal{A}(u) \) are obtained by somehow "interleaving" two Sturmian words of the same slope (\( s_2 \) is encoded by \( 1 \mapsto 2 \)). In the following we define dynamical systems, called *ternary codings of rotations*, which capture this phenomenon.

Recall the definition of Sturmian words as codings of rotations on the torus \( T \) with the half-open intervals \( I_0 = I(0, 1 - \alpha) \) and \( I_1 = I(1 - \alpha, 1) \). We assume here that \( \alpha < 1/2 \). Take \( \zeta \in I(\alpha, 1 - \alpha) \) and split torus \( T \) into four (three if \( \zeta = \alpha \) or \( \zeta = 1 - \alpha \)) intervals defined by the points \( 0, \zeta, \alpha, \zeta, 1 \) in increasing order: Define the disjoint intervals \( J_0 = I(\zeta - \alpha, \zeta) \) (resp., \( J_2 = I(\zeta - \alpha, \zeta) \)) and \( J_1 = I_1 \) and \( J_0 = I_0 \setminus J_2 \). We must be careful with the value \( \zeta = \alpha \) (resp., \( \zeta = 1 - \alpha \)): If \( 1 \in I_1 \) (resp., \( 1 - \alpha \in I_1 \)) then \( J_2 = I(0, \alpha) \) (resp., \( J_2 = I(1 - 2\alpha, 1 - \alpha) \)). Take the rotation \( R_\alpha \) and the encoding \( \nu : T \rightarrow \{0, 1, 2\} \), \( x \mapsto i \) if and only if \( x \in I_i \). The word \( t_{\alpha, \zeta, x} = (\nu(R_\alpha(x)) \) is called the rotation word of slope \( \alpha \), offset \( \zeta \), and intercept \( x \). See Figure 1a for an illustration. When indicate the choices of endpoints of \( J_1 \) as follows. If \( 1 \in J_1 \) and \( \zeta \in J_2 \) (resp., \( 1 \notin J_1 \), \( \zeta \notin J_2 \)) we denote the obtained word by \( t_{\pi^-_{\alpha, \zeta, \rho}} \) (resp., \( t_{\pi^+_{\alpha, \zeta, \rho}} \)). If \( 1 \in J_1 \) and \( \zeta \notin J_2 \) (resp., \( 1 \notin J_1 \), \( \zeta \in J_2 \)), we denote this by \( t_{\pi^-_{\alpha, \zeta, \rho}} \) (resp., \( t_{\pi^+_{\alpha, \zeta, \rho}} \)). Notice that \( t_{\pi^-_{\alpha, \zeta, \rho}} \) and \( t_{\pi^+_{\alpha, \zeta, \rho}} \) are not defined: this would imply that the intervals \( J_1 \) and \( J_2 \) overlap.

Observe now that, by the discussion following Example 4.7, for \( u \) of factor complexity \( n + 2 \) as in Theorem 4.3(3), we have \( u = t_{\alpha, \zeta, \rho} \) for some \( \rho \in T \) (see Figure 1b). Further, any word \( x \in \mathcal{A}(u) \) is of form \( t_{\alpha, \zeta, \rho'} \) for some \( \zeta \in I(\alpha, 1 - \alpha) \), \( \rho' \in T \). Our main goal is to show that \( t_{\alpha, \zeta, \rho} \in \mathcal{A}(u) \) for all possible \( \zeta \in [\alpha, 1 - \alpha] \).

Recall that Sturmian words are balanced, so that for each \( n \in N \) and for each \( i = 1, 2 \), the set \( \{v|_i : v \in \mathcal{L}(u)\} \) comprises two values (depending on \( n \) and \( \alpha \)). We say that a factor \( v \) is 1-heavy (resp., 2-heavy) if \( |v|_1 \) (resp., \( |v|_2 \)) attains the larger of the two possible values. Otherwise we say that \( v \) is 1-light (resp., 2-light). If \( v \) is 1-heavy and 2-heavy, we say that
$v$ is 1-2-heavy. Similarly, $v$ is called 1-2-light if $v$ is 1-light and 2-light. We make use of the following result appearing in [23, part of Thm. 19].

**Proposition 4.8.** Let $s$ be a Sturmian word of slope $\alpha$ and intercept $\rho$ and let $m \geq 1$. Then the prefix of length $m$ of $s$ is heavy if and only if $\rho \in I(R_{\alpha}^{-m}(0), 1)$. Here $I(R_{\alpha}^{-m}(0), 1)$ contains the point $R_{\alpha}^{-m}(0)$ if and only if $1 \notin I_1$.

We may apply **Proposition 4.8** to determine whether a point starts with a heavy factor of length $m$ or not. Indeed, for the letter 1, the proposition stands as is: $t_{\alpha, \zeta, \rho}$ begins with a 1-heavy factor of length $m$ if and only if $\rho \in I(\{-m\alpha\}, 1)$. For 2-heavy factors we take into account the rotation induced by the offset $\zeta$. Thus the word $t_{\alpha, \zeta, \rho}$ begins with a 2-heavy factor if and only if $\rho \in I(\{-m\alpha + \zeta\}, \zeta)$. Here the interval $I(\{-m\alpha + \zeta\}, \zeta)$ contains the point $\{-m\alpha + \zeta\}$ (resp., $\zeta$) if and only if $\zeta \notin J_2$ (resp., $\zeta \in J_2$). We define the following distance on the torus: $||x|| = \min\{x, 1-x\}$. Thus, e.g., $\max\{x, 1-x\} = 1 - ||x||$.

**Lemma 4.9.** Let $x = t_{\alpha, \zeta, \rho}$. Then

1. $x$ contains a 1-heavy–2-light and a 2-heavy–1-light factor for each length.

2. There exists a 1-2-heavy factor $v \in \mathcal{L}(x)$ of length $m$ if and only if $\{-m\alpha\} < 1 - ||\zeta||$, or $\{-m\alpha\} = 1 - ||\zeta||$ and $x = t_{\alpha, \{m\alpha\}, \{-m\alpha\}}$ or $x = t_{\alpha, \{m\alpha\}, \{-m+n\alpha\}}$ for some $n \geq 0$.

3. There exists a 1-2-light factor $v \in \mathcal{L}(x)$ of length $m$ if and only if $\{-m\alpha\} > ||\zeta||$, or $\{-m\alpha\} = ||\zeta||$ and $x = t_{\alpha, \{m\alpha\}, \{-m+n\alpha\}}$ or $x = t_{\alpha, \{m\alpha\}, \{-m\alpha\}}$.

**Proof.** We give a proof case by case. Consider factors of length $m$ and write $\mu = \{-m\alpha\}$ for short.

1. We first consider 1-heavy–2-light factors. By the preceding observations on 1-heavy and 2-heavy factors, $x$ has a 1-heavy–2-light factor if and only if $I(\mu, 1) \cap I(\zeta, \{\zeta + \mu\}) \neq \emptyset$. The interval $I(\max\{\zeta, \mu\}, \min\{\zeta + \mu, 1\})$ is always in the intersection, since $\zeta, \mu < 1$ and $\zeta, \mu < \zeta + \mu$. Since $\{-m\alpha\}_n$ is dense in $[0,1)$, some shift of $x$ corresponds to a coding of a point in this interval.

Similarly, $x$ has a 2-heavy–1-light factor if and only if $I(0, \mu) \cap I(\{\zeta + \mu\}, \zeta) \neq \emptyset$. The interval $I(\max\{0, \mu + \zeta - 1\}, \min\{\zeta, \mu\})$ is always in the intersection, since $\zeta, \mu > 0$ and $\zeta, \mu > \mu + \zeta - 1$.

2. We then consider 1-2-heavy factors. Similar to above, $x$ has a 1-2-heavy factor if and only if $I(\mu, 1) \cap I(\{\mu + \zeta\}, \zeta) \neq \emptyset$. Assume first that $\mu < 1 - ||\zeta||$. If $\mu < \zeta$ then $I(\mu, \zeta)$ is in the
intersection. If $\zeta \leq \mu < 1 - \|\zeta\|$, then $\mu + \zeta < 1$ so that $\{\mu + \zeta\} = \mu + \zeta$ and $I(\mu + \zeta, 1)$ is in the intersection. The denseness of $\{-n\alpha\}_{n \in \mathbb{N}}$ in $\mathbb{T}$ again implies that some shift of $x$ corresponds to a point in this interval.

Assume then that $\mu = 1 - \|\zeta\|$. If $\zeta = \|\zeta\|$, then $\{\mu + \zeta\} = 0$. Now $I(\{\mu + \zeta\}, \zeta)$ and $I(\mu, 1)$ can share at most one point in common, namely the point 1. Now the intersection is non-empty if and only if $1 \in J_1$ and $\zeta \notin J_2$. Further, $t_{\sigma, \zeta, 0}$ is the only word starting with a 1-2-heavy factor. To hit the point 0 in the orbit starting from $\rho \in \mathbb{T}$, we must have $\rho = \{-n\alpha\}$ for some $n \geq 0$. So, we have $x$ contains a 1-2-heavy factor of length $m$ if and only if $x = t_{\sigma, \zeta, 0}(\{-n\alpha\})$, where $\zeta = 1 - \mu = 1 - \{-m\alpha\} = \{m\alpha\}$.

Similarly, if $\zeta = 1 - \|\zeta\|$, then $I(\{\mu + \zeta\}, \zeta)$ and $I(\mu, 1)$ can share at most one point in common, namely $\zeta$. The intersection is not empty if and only if $1 \notin J_1$ and $\zeta \in J_2$. In this case $t_{\sigma, \zeta, 0}$ is the only point starting with a 1-2-heavy factor. To hit the point $\zeta$ in the orbit of $\rho$, we must have $\rho = \{\zeta - n\alpha\}$ for some $n \geq 1$. Hence $x$ contains a 1-2-heavy factor if and only if $x = t_{\sigma, \zeta, 0}(\{-n\alpha\})$, where $\zeta = 1 - \|\zeta\| = \mu = \{-m\alpha\}$.

Assume finally that $\mu > 1 - \|\zeta\|$. It follows that $\mu > \zeta$ and $\mu + \zeta > 1$ and thus $0 < \{\mu + \zeta\} < \zeta$. Therefore $I(\mu, 1) \cap I(\{\mu + \zeta\}, \zeta) = \emptyset$. This concludes the case of 1-2-heavy factors.

3. Let us then finally consider 1-2-light factors. We proceed analogous to the previous case. The word $x$ has a 1-2-light factor of length $m$ exists if and only if $I(0, \mu) \cap I(\zeta, \{\zeta + \mu\}) \neq \emptyset$.

Assume first that $\mu > \|\zeta\|$. If $\mu > \zeta$, then the interval $I(\zeta, \mu)$ is in the intersection. If $\zeta > \mu \geq 1 - \|\zeta\|$, then $\mu + \zeta > 1$ so that $\{\mu + \zeta\} > 0$. Now $I(0, \{\mu + \zeta\})$ is in the intersection.

Assume then that $\mu = \|\zeta\|$. If $\zeta = \|\zeta\|$, then $\mu + \zeta < 1$. Now $I(\zeta, \mu + \zeta)$ and $I(0, \mu)$ can share at most one point in common, namely the point $\zeta$. By the observations preceding the lemma, the intersection is not empty if and only if $1 \in J_1$ and $\zeta \notin J_2$. Now $t_{\sigma, \zeta, 0}$ is the only point starting with a 1-2-light factor. The only way to hit the point $\zeta$ in the orbit of $\rho$ is that $\rho = \{\zeta - \alpha\}$ for some $n \geq 0$. It follows that $x$ contains a 1-2-light factor if and only if $x = t_{\sigma, \zeta, 0}(\{-n\alpha\})$, where $\zeta = \{-m\alpha\}$.

Similarly, if $\zeta = 1 - \|\zeta\|$, then $\mu + \zeta = 1$. Now $I(\zeta, \{\mu + \zeta\})$ and $I(0, \mu)$ can share at most one point in common, namely the point 1. By the observations preceding the lemma, the intersection is not empty if and only if $1 \notin J_1$ and $\zeta \in J_2$. Now $t_{\sigma, \zeta, 0}$ is the only factor starting with a 1-2-light factor. Again, we have $x$ contains a 1-2-light factor if and only if $x = t_{\sigma, \zeta, 0}(\{-n\alpha\})$, where $\zeta = \{\mu\} = \{m\alpha\}$.

Finally, if $\mu < \|\zeta\|$, then $\mu < \zeta$ and $\mu + \zeta \leq \mu + 1 - \|\zeta\| < 1$. Thus $I(0, \mu)$ and $I(\zeta, \mu + \zeta)$ do not intersect. This concludes the proof.

As is evident from Lemma 4.9(2), the existence of a 1-2-heavy factor of a certain length depends not only on $\zeta$, but also on $\rho$ and how the endpoints of the intervals are defined. For example, the word $t_{\sigma, \{\mu\}, 0}$ begins with a 1-2-heavy factor of length $m$, while $t_{\sigma, \{\mu\}, \{-n\alpha\}}$ does not contain such a factor. Note further that $t_{\sigma, \{\mu\}, 0}$ contains only one occurrence of such a factor, and hence is non-recurrent. In fact, any word $t_{\sigma, \{\mu\}, \{-n\alpha\}}$, $n \geq 0$, contains exactly one such factor of length $m$, namely at position $n$.

**Lemma 4.10.** If $\|\zeta\| > \|\zeta'\|$ then $t_{\alpha, \zeta, \rho} \in A(t_{\alpha, \zeta', \rho'})$ but $t_{\alpha, \zeta', \rho'} \notin A(t_{\alpha, \zeta, \rho})$. 

13
Proof. Let \( x = t_{\alpha, \zeta, \rho} \) and \( u = t_{\alpha, \zeta', \rho} \) for short. Observe that for any \( w, w' \), where \( w \in \mathcal{L}_m(u) \) and \( w' \in \mathcal{L}_m(x) \), we have \(|w|_1 - |w'|_1| \leq 1 \) and \(|w|_2 - |w'|_2| \leq 1 \).

Let us first show that \( x \in \mathcal{A}(u) \). By Lemma 4.9(1), both words contain both 1-heavy-2-light and 2-heavy-1-light factors of each length. We show that whenever \( x \) contains a 1-2-heavy factor or a 1-2-light factor length \( m \), then \( u \) contains such a factor as well, which suffices for the claim. To this end, let \( w \in \mathcal{L}_m(x) \). If \( w \) is a 1-2-heavy factor, then by Lemma 4.9(2), we have \( \{-ma\} \leq 1 - ||\zeta|| < 1 - ||\zeta'|| \) so that \( u \) contains a 1-2-heavy factor by the same lemma. If \( w \) is a 1-2-light factor, then by Lemma 4.9(3), \( \{-ma\} \geq ||\zeta|| > ||\zeta'|| \) so that \( u \) again contains a 1-2-light factor of length \( m \).

We then show that \( u \not\in \mathcal{A}(x) \). Since \( \{\{-ma\}_{m \geq 1} \) is dense in \([0,1)\), there must exist \( m \in \mathbb{N} \) for which \( ||\zeta|| > \{-ma\} > ||\zeta'|| \). By Lemma 4.9(3), \( u \) contains a 1-2-light factor of length \( m \), while \( x \) does not. It follows that \( u \not\in \mathcal{A}(x) \).

We may now characterize the abelian closure of \( n + 2 \) factor complexity words via ternary codings of rotations.

**Proposition 4.11.** Let \( u = t_{\alpha, \zeta, \rho} \) for some \( \rho \in \mathbb{T} \). Then \( \mathcal{A}(u) = \bigcup_{\zeta \in [\alpha, 1-\alpha]} \Omega(t_{\alpha, \zeta, \rho}) \).

**Proof.** By the above lemma we have \( t_{\alpha, \zeta, \rho} \in \mathcal{A}(u) \) for all \( \zeta \in (\alpha, 1 - \alpha) \). For \( \zeta = \alpha \) or \( \zeta = 1 - \alpha \), all words \( t_{\alpha, \zeta, \rho} \) either contain or do not contain a 1-2-light (resp. 1-2-heavy) factor regardless of \( \rho \). (Recall that the words \( t_{\alpha, \zeta, \rho} \) and \( t_{\alpha, 1-\alpha, \rho} \) are not defined.) As there are no other words in \( \mathcal{A}(u) \), this concludes the proof. \( \square \)

In fact, utilising Lemma 4.10 we can characterize the abelian closure any word \( t_{\alpha, \zeta, \rho} \). The proof above applied to the setting \( ||\zeta|| = ||\alpha|| \), i.e., when \( u \) is a minimal complexity word, carries over to arbitrary \( \zeta \) with minor modifications:

**Proposition 4.12.** Let \( u = t_{\alpha, \zeta, \rho} \) with \( ||\zeta|| > ||\alpha|| \). Then

\[
\mathcal{A}(u) = \bigcup_{||\zeta|| > ||\alpha||} \Omega(t_{\alpha, \zeta, \rho}) \setminus S,
\]

where \( S \) is a countable set of words depending on \( \zeta \) and \( \rho \) as follows.

1. If \( 1 - ||\zeta||, ||\zeta|| \notin \{-ma\} : m \in \mathbb{N} \), then \( S = \emptyset \).

2. Assume that \( 1 - ||\zeta|| = \{-ma\} \) for some \( m \geq 1 \). If \( u = t_{\alpha, \{ma\}, \{-na\}} \) or \( u = t_{\alpha, \{na\}, \{-ma\}} \) for some \( n \geq 0 \), then \( S = \emptyset \). Otherwise

\[
S = \{ t_{\alpha, \{na\}, \{-ma\}} : n \geq 0 \} \cup \{ t_{\alpha, \{ma\}, \{-na\}} : n \geq 0 \}.
\]

3. Assume that \( ||\zeta|| = \{-ma\} \) for some \( m \geq 1 \). If \( u = t_{\alpha, \{-ma\}, \{-na\}} \) or \( u = t_{\alpha, \{ma\}, \{-na\}} \) for some \( n \geq 0 \), then \( S = \emptyset \). Otherwise

\[
S = \{ t_{\alpha, \{-ma\}, \{-na\}} : n \geq 0 \} \cup \{ t_{\alpha, \{ma\}, \{-na\}} : n \geq 0 \}.
\]

**Proof.** Notice that any word \( y \in \mathcal{A}(u) \), we have that \( y \) is of the form \( t_{\alpha, \zeta', \rho} \). Indeed, using the mappings \( \varphi_1 \) and \( \varphi_2 \) as in the discussion following Example 4.7, \( \varphi_1(y) \) and \( \varphi_2(y) \) are
Sturmian words with slope $\alpha$. We deduce that they are interleavings of Sturmian words giving rise to the claimed form of $y$.

Lemma 4.10 then gives that $A(u)$ is a subset of $\bigcup\{\|\xi\| \geq \|\zeta\| : \Omega(t_{\alpha,\xi,\rho})\}$, but it is possibly a proper subset. The same lemma shows that $A(u)$ is a superset of $\bigcup\{\|\xi\| > \|\zeta\| : \Omega(t_{\alpha,\xi,\rho})\}$ in any case.

Therefore, we may focus on words $y = t_{\alpha,\xi,\rho}$ with offset $\xi$ having $\|\xi\| = \|\zeta\|$. Notice that the three points are disjoint. Indeed, in 2., we assume that $1 - \|\xi\| = \{-ma\}$, which gives $\|\xi\| = \{ma\}$. Hence $\|\xi\| \neq \{-m'\alpha\}$ for any $m' \geq 1$, as otherwise $\{(m + m')\alpha\} = 0$ which would leave $\alpha$ rational.

To identify the set of words $S$ not in the abelian closure of $u$, we employ Lemma 4.9.

1. Assume that $1 - \|\xi\|, \|\zeta\| \notin \{-ma\}$: $m \in \mathbb{N}$. Lemma 4.9(1) then states that the existence of a 1-2-heavy or light factor does not depend on the point whose orbit we encode, nor the choices of the endpoints of the intervals. That is to say, all words with offset $\xi$, $\|\xi\| = \|\zeta\|$, simultaneously either have or do not have a 1-2-heavy (resp., light) factor of length $m$ independent to the choice of starting point $\rho'$ of the orbit. This suffices to show that $S = \emptyset$ in this case.

2. Assume that $1 - \|\xi\| = \{-ma\}$ for some $m \geq 1$. There is only one length of factors in which the existence of a 1-2-heavy factor depends on the starting point $\rho$ and the choice of the endpoints of the intervals. This length is $m$. By Lemma 4.9(2), if $u = t_{\pi_{\{ma\}}\{-na\}}$ or $u = t_{\pi_{\{ma\}}\{-ma\}},\{-ma\}$ for some $n \geq 0$, then the word contains such a factor. In this case $S = \emptyset$. If $u$ is not of this form, then it does not contain such a factor, while all the words $t_{\pi_{\{ma\}}\{-na\}}$ and $t_{\pi_{\{ma\}}\{-ma\}},\{-ma\}$, $n \geq 0$, do. The claim then follows.

3. This is analogous to the one above.

4.2. Recurrent minimal complexity words with at least four letters

Surprisingly, for alphabet of size greater than 3 there are always only finitely many subshifts:

Theorem 4.13. Let $u$ be a recurrent word of factor complexity $n + C$ for all $n \geq 1$, where $C > 2$. Then $A(u)$ contains exactly two minimal subshifts.

The proof is based on the characterization of words of factor complexity $n + C$ for all $n \geq 1$ from [9].

Lemma 4.14 ([9, Lem. 4]). Let $u$ be a recurrent word of minimal complexity an alphabet $A$; then there exist distinct elements $e_1, \ldots, e_b$, $f_1, \ldots, f_c$, $g_1, \ldots, g_d$ in $A$ such that the sets $E = \{e_1, \ldots, e_b\}$, $F = \{f_1, \ldots, f_c\}$, and $G = \{g_1, \ldots, g_d\}$ are pairwise disjoint, $E \cup F \cup G = A$, with $G \neq \emptyset$, and $E \cup F \neq \emptyset$, and there exists a Sturmian word $s$ on $\{0, 1\}$ such that, if $\sigma$ is the substitution

\[
\begin{align*}
0 & \mapsto g_1 \cdots g_d e_1 \cdots e_b \\
1 & \mapsto g_1 \cdots g_d f_1 \cdots f_c
\end{align*}
\]

then $\sigma(s) = W u$, where $W$ is a (possibly empty) prefix of $\sigma(0)$ or $\sigma(1)$.
Lemma 4.15. Let \( w \) be a recurrent word of minimal complexity an alphabet \( A \) of cardinality at least 3. Then each \( w' \in \mathcal{A}(w) \) is a concatenation of blocks of the form \( \sigma(0) \) and \( \sigma(1) \) (or their reversals), where \( \sigma \) is as in the previous lemma, preceded by a possibly empty suffix of \( \sigma(0) \) or \( \sigma(1) \) (or a reversal of a prefix).

Proof. The proof is quite direct. Since \(|A| \geq 4\), at least one of the sets \( G, E, F \) contains at least two letters. Let it be \( E \) (for other sets it is similar). First we show that for any \( w' \in \mathcal{A}(w) \) the letters from \( E \) must occur in blocks \( e_1 \cdots e_b \) (or in \( e_b \cdots e_1 \) – this case is symmetric, all the blocks are reversed). For this, it is enough to consider only factors of length 2 and 3. Indeed, if \( e_1 \) occurs in \( w' \), then the only factors containing \( e_1 \) in \( w \) are \( e_1e_2 \) and \( gd^e_1g \). With the exception of the case \( F = \emptyset \) and \(|G| = 1\), we have that \( gd^e_1g \) is not an abelian factor of \( w \). Since \( gd^e_1g \) is not an abelian factor of \( w \), in \( w' \) we must have \( gd^e_1g \) or \( e_2e_1^g \). Continuing this line of reasoning with \( e_2, e_3 \) instead of \( e_1, g, e_2 \) etc., we get that \( w' \) the letters from \( E \) must occur in blocks \( e_1 \cdots e_b \) (or in \( e_b \cdots e_1 \) ). If \( F = \emptyset \) and \(|G| = 1\), then \(|E| \geq 3\) (the cardinality of the alphabet is at least 4), and we can start by \( e_2 \): by considering factors of length 2 and 3, we see that it can occur only in factors \( e_1e_2e_3 \) and \( e_3e_2e_1 \). The rest of the proof is the same.

In the same way we prove that each such block \( e_1 \cdots e_b \) must be surrounded by \( g_1 \cdots g_d \) from both sides, so we have \( \sigma(0)g_1 \cdots g_d \). Now we show that this block \( \sigma(0) \) must be followed by either \( \sigma(0) \) or \( \sigma(1) \). We already have the beginning of the block \( g_1 \cdots g_d \). After it, one must have either \( e_1 \), or \( f_1 \), or \( g_1 \) in the case if \( F = \emptyset \) (again, it is enough to consider factors of length 2 and 3 containing \( g_d \)). In the cases of \( e_1 \) or \( f_1 \) it must continue with \( e_2 \cdots e_b \) or \( f_2 \cdots f_c \), respectively, thus finishing the block \( \sigma(0) \) or \( \sigma(1) \). In the case when \( F \) is empty, we already have \( \sigma(1) \). In the same way one can show that after the block \( \sigma(1) \) one must also have a full block \( \sigma(0) \) or \( \sigma(1) \).

We remark that the cardinality at least 4 of the alphabet is essential for the above lemma. In the case \( F = \emptyset \), \(|G| = 1\) and \(|E| = 2\) the letters \( e_1 \) and \( e_2 \) can be separated by \( gd \), which corresponds to Theorem 4.4 (3), when we have uncountably many minimal subshifts. The fourth letter blocks this possibility: either we have \(|E| \geq 3\), in which case \( e_2 \) “glues” letters \( e_1 \) and \( e_2 \), or \(|G| \geq 2\), so the two letters \( g_1 \) and \( g_2 \) prevent mixing.

Let \( u \) be uniformly recurrent. We define a word \( u^R \) for which \( \mathcal{L}(u^R) = \mathcal{L}(u)^R \), i.e., the set of reversals of the factors of \( u \). Indeed, take the sequence \( (p_n)_n \) of prefixes of \( u \), and consider the sequence \( (p_n^R) \) of their reversals. There is a subsequence which converges to an infinite word \( u^R \). We claim that \( \mathcal{L}(u^R) = \mathcal{L}(u)^R \). As \( u^R \) is constructed using reversals of factors of \( u \), we have that \( \mathcal{L}(u^R) \subseteq \mathcal{L}(u) \). Let \( x \in \mathcal{L}(u) \). Since \( u \) is uniformly recurrent, \( x \) must occur in \( p_n \) for \( n \) large enough. As \( x \) occurs within bounded gaps, we conclude that the words in the converging subsequence of \( (p_n)_n \) must have \( x^R \) occurring for all \( n \) large enough, the first occurrence occurring with a uniform bound. Hence \( x^R \in \mathcal{L}(u^R) \).

Notice that we immediately have that \( u^R \in \mathcal{A}(u) \).

Proof of Theorem 4.13. The two subshifts are \( \Omega(u) \) and \( \Omega(u^R) \). Suppose that there exists a word \( w \in \mathcal{A}(u) \) such that it is not from \( \Omega(u) \). Due to Lemma 4.15, cutting a short prefix of \( w \), we get a word \( w' \) such that \( w' = \sigma(v') \), \( v' \in \{0,1\}^N \), and \( v' \) is not in the shift orbit closure of \( v \), where \( v \) is a Sturmian word as in Lemma 4.14. So, \( v' \) contains a factor \( w' \) which is not abelian equivalent to any factor of \( v \). It is straightforward to see that then \( \sigma(w') \) is not abelian equivalent to a factor.
consisting of full blocks. And if it happens to be abelian equivalent to a factor which does not consist of full blocks, then it is also equivalent to a shift of this factor that consists of full block, which is not possible. So, \( \sigma(w') \) is not an abelian factor of \( u \), hence \( u' \) is not in \( A(u) \), a contradiction.

5. Abelian closures of Arnoux–Rauzy words

In this section we discuss Arnoux–Rauzy words, which are another generalization of Sturmian words to larger alphabet. One of the ways to define Arnoux–Rauzy words is via palindromic closures. The following basics on Arnoux–Rauzy words are well-known and mostly taken from \([1, 8]\). In fact, this is a generalization of the facts about Sturmian words given for binary words in \([6]\).

A finite word \( v = v_0 \cdots v_{n-1} \) is a palindrome if it is equal to its reversal, i.e., \( v = v_{n-1} \cdots v_0 \). The right palindromic closure of a finite word \( u \), denoted by \( u^+(+) \), is the shortest palindrome that has \( u \) as a prefix. The iterated (right) palindromic closure operator \( \psi \) is defined recursively by the following rules:

\[
\psi(\varepsilon) = \varepsilon, \quad \psi(va) = (\psi(v)a)^+(+)
\]

for all \( v \in \Sigma^* \) and \( a \in \Sigma \). The definition of \( \psi \) may be extended to infinite words \( u \) over \( \Sigma \) as \( \psi(u) = \lim_n \psi(\text{pref}_n(u)) \), i.e., \( \psi(u) \) is the infinite word having \( \psi(\text{pref}_n(u)) \) as its prefix for every \( n \in \mathbb{N} \).

Let \( \Delta \) be an infinite word on the alphabet \( \Sigma \) such that every letter occurs infinitely often in \( \Delta \). The word \( c = \psi(\Delta) \) is then called a characteristic (or standard) Arnoux–Rauzy word and \( \Delta \) is called the directive sequence of \( c \). An infinite word \( u \) is called an Arnoux–Rauzy word if it has the same set of factors as a (unique) characteristic Arnoux–Rauzy word, which is called the characteristic word of \( u \). The directive sequence of an Arnoux–Rauzy word is the directive sequence of its characteristic word. An example of Arnoux–Rauzy word is given by the Tribonacci word \( T \), which can be defined as the fixed point of the morphism \( 0 \to 01, 1 \to 02, 2 \to 0 \). It is not hard to see that the Tribonacci word is an Arnoux–Rauzy word with the directive sequence \( (012)\omega \).

T. Hejda, W. Steiner, and L.Q. Zamboni studied the abelian shift of the Tribonacci word \( T \). They announced that \( A_T \setminus \Omega(T) \neq \emptyset \) but that \( \Omega(T) \) is the only minimal subshift contained in \( A(T) \) \([12, 24]\).

An interesting open question is to understand the general structure of Arnoux–Rauzy words (see Problem 7.2).

6. Abelian closures of general subshifts

In this paper and the previous works on the topic, the focus has been on abelian closures of infinite words. It would be interesting to investigate properties of abelian closures of general subshifts.
We recall some definitions from \cite{18, §1.5}, but slightly modify the terminology. A bi-infinite word \( x \in \Sigma^\mathbb{Z} \) is said to avoid a set of words \( \mathcal{F} \subseteq \Sigma^* \) if \( \mathcal{L}(x) \cap \mathcal{F} = \emptyset \). Let \( X_\mathcal{F} \) denote the set of bi-infinite words avoiding \( \mathcal{F} \). A subshift is a set \( X_\mathcal{F} \) for some \( \mathcal{F} \). The shift operator is defined similar to the case of infinite words. Now a set \( X \subseteq \Sigma^\mathbb{Z} \) is a subshift if and only if \( \sigma(X) = X \) and is closed in the usual topology on bi-infinite words.

Let \( X \) be a subshift and let \( I(X) = \Sigma^* \setminus \mathcal{L}(X) \). Define the set \( \mathcal{F}(X) \) as the set of elements of \( I(X) \) which are minimal for the factor ordering, i.e., have no proper factor in \( I(X) \). Then \( X = X_{\mathcal{F}(X)} \).

**Definition 6.1.** If a subshift \( X = X_\mathcal{F} \) for some finite set \( \mathcal{F} \subseteq \Sigma^* \), then \( X \) is called a subshift of finite type (SFT). If, on the other hand, \( \mathcal{F} \) can be taken regular, then \( X \) is called sofic.

A set \( X \) is a SFT if and only if \( \mathcal{F}(X) \) is finite. Similarly, \( X \) is sofic if and only if \( \mathcal{F}(X) \) is regular.

We may define the abelian closure of a subshift straightforwardly.

**Definition 6.2.** Let \( X \) be a subshift. Then its abelian closure \( \mathcal{A}(X) \) is defined as \( \cup_{x \in X} \mathcal{A}(x) \).

We remark that in the previous text we considered one-way infinite words, as more customary in combinatorics on words, whereas here for general subshifts it is more natural to consider bi-infinite words. Actually, there is no principal difference for our considerations, as all the results can easily be reformulated for one-way or two-way infinite words.

We conclude this paper with a couple of examples of abelian closures of subshifts.

**Example 6.3.** Clearly \( \mathcal{A}(\Sigma^\mathbb{Z}) = \Sigma^\mathbb{Z} \). Let \( \mathcal{F} = \{11\} \subseteq \{0,1\}^* \) and set \( X = X_\mathcal{F} \). The subshift \( X \subseteq \{0,1\}^\mathbb{Z} \) is called the golden mean subshift. Consider the abelian closure of \( X_\mathcal{F} \): it comprises those words for which all 1s are isolated. But this is just \( X_\mathcal{F} \) itself. Thus \( \mathcal{A}(X_\mathcal{F}) = X_\mathcal{F} \).

In the above example, both subshifts are of finite type. It was concluded that they are, in fact, their own abelian closures. This property is of course not general for SFTs, as is shown by the following example. In fact, the abelian closure of a SFT is not in general a SFT.

**Example 6.4.** Consider the SFT \( X = X_\mathcal{F} \) with \( \mathcal{F} = \{aa, ac, ba, bb, cb\} \). It can be characterized as the set of two-way infinite walks on the following graph.

\[
\begin{array}{c}
\text{b} \\
\downarrow \\
\text{a} \\
\rightleftharpoons \\
\text{c} \\
\end{array}
\]

Assume for a contradiction, that \( \mathcal{A}(X) \) is a SFT, with \( \mathcal{F}(X) = \mathcal{F}' \). There is an integer \( n \) for which each element of \( \mathcal{F}' \) has length at most \( n \). Consider the word \( x = \omega c \cdot ab \cdot c^n \cdot ba \cdot c^\omega \). Here for a finite word \( v \) by \( \omega v \) we mean the left-infinite word obtained by repeating \( v \) infinitely many times. Observe that the factors of length at most \( n \) of this word occur either in \( \omega c \cdot ba \cdot c^\omega \) or in \( \omega c \cdot ab \cdot c^\omega \). Both of these words are in \( \mathcal{A}(X) \) by inspection, so none of the factors can be in \( \mathcal{F}' \). Thus \( \mathcal{A}(X) \) is not a SFT. Indeed, any word in the language \( \mathcal{L}(\mathcal{A}(X)) \) that contains two occurrences of \( b \) must contain at least one occurrence of \( a \).

The next example shows that this is also possible for binary alphabet:
Example 6.5. Consider an SFT giving words of the form
\[
\cdots 001100110001110001110001 \cdots ,
\]
plus \((0011)^\omega\) and \((000111)^\omega\). It is a SFT of order 6, and its Rauzy graph contains two cycles with the same frequencies of letters and a one-way path between them.

It is readily verified that words of the form
\[
\cdots 00110011000110011001 \cdots 
\]
are in the abelian closure, whereas words of the form
\[
\cdots 0011001100011(0011)+0001100110011 \cdots 
\]
are not. Similarly to the previous example this implies that the abelian closure is not a SFT.

Next we show that the abelian closure of a sofic shift is not necessarily sofic.

Example 6.6. Let \(\Sigma = \{a, b, c, d\}\) be the underlying alphabet. Set
\[
F = \{a, b, d\}c \cup d\{a, b, c\} \cup cRd,
\]
where \(R = \{a, b\}^* \setminus (ab)^*\) and let \(X = X_F\). Hence \(X\) is of the form
\[
X = \{a, b\}^\mathbb{Z} \cup \{\omega x, x^Rd^\omega : x \in \{a, b\}^\mathbb{N}\} \cup \{\omega c(ab)^n d^\omega : n \geq 0\} \cup \{\omega c^\omega, \omega d^\omega\}.
\]
(Here \(x^R\) is the left-infinite word defined by \(x\), i.e., the letter at position \(-n\) of \(x^R\) equals the \(n\)th letter of \(x\).)

Let \(F = F(A(X))\). We show that
\[
F' \cap c\{a, b\}^*d = \{cwd : |w|_a \neq |w|_b\}.
\]

It follows that \(F'\) is cannot be regular, as the language above is well-known to be non-regular. Hence \(A(X)\) is not sofic.

Let us show the \(\subseteq\) direction. Let \(w\) have \(|w|_a = |w|_b\). We show that the word \(\text{"}cwd\text{"}\) is in the abelian closure, and thus \(cwd \notin F'\). Now all factors of the form \(c^n x\) or \(y^d m\), for \(x\) a prefix and \(y\) a suffix of \(w\), occur in the words \(\text{"}cwd\text{"}\) or \(\text{"}wd\text{"}\), which are elements of \(X\). We may thus concentrate on factors of the form \(c^n wd^m\). Now \(c^n wd^m\) is abelian equivalent to \(c^n(ab)^{|w|/2}d^m\) which is clearly in the language of \(X\).

Let then \(w \in \{a, b\}^*\) be such that \(|w|_a \neq |w|_b\). Observe that the proper factors of \(cwd\) are in the language of \(X\). This means that either \(cwd \in F'\) or \(cwd \in L(A(X))\). Assume, for a contradiction that \(cwd \in L(A(X))\). Now \(\Psi(cwd) = (|w|_a, |w|_b, 1, 1)\), but, in \(L(X)\), any word with Parikh vector with last two components equal to 1 is of the form \((m, m, 1, 1)\). Since \(|w|_a \neq |w|_b\), there is no word in \(L(X)\) which is abelian equivalent, so \(cwd\) is not an element of \(L(A(X))\). We conclude that \(cwd \in F'\).

An interesting open question is to find out whether the abelian closure of a subshift of finite type always sofic (see Problem 7.3).
7. Conclusions

In this paper, we introduced and studied a notion of abelian subshifts of infinite words. The main open problem we would like to state in this paper is the following:

**Problem 7.1.** Characterize words for which $A(x) = \Omega(x)$.

Among binary uniformly recurrent words, this property gives a characterization of Sturmian words, but the characterization does not extend to usual generalizations of Sturmian words to non-binary alphabets: neither for balanced words, nor for words of minimal complexity, nor for Arnoux–Rauzy words. A modification of this question is to characterize words for which $A(x)$ contains exactly one minimal subshift.

For Arnoux–Rauzy words, we showed that $A(x) \neq \Omega(x)$, but their abelian closure seems to have rather complicated structure, in particular, it always contains non-recurrent words. An interesting open question is to understand the general structure of Arnoux–Rauzy words:

**Problem 7.2.** Characterize abelian closures of Arnoux–Rauzy words.

Finally, we propose the following open question about general abelian subshifts:

**Problem 7.3.** Is the abelian closure of an SFT always sofic?

Acknowledgements

We are grateful to Joonatan Jalonen, Ville Salo, and Luca Zamboni for fruitful discussions and helpful comments.

Svetlana Puzynina is partially supported by Russian Foundation of Basic Research (grant 20-01-00488) and by the Foundation for the Advancement of Theoretical Physics and Mathematics BASIS”. Part of the research was performed while Markus Whiteland was at the Department of Mathematics and Statistics, University of Turku, Finland.

References

[1] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité $2n+1$. *Bulletin de la Société Mathématique de France*, 119:199–215, 1991. doi:10.24033/bsmf.2164.

[2] J. Bark and P. Varjú. Partitioning the positive integers to seven Beatty sequences. *Indag. Math.*, 14(2):149–161, 2003. ISSN 0019-3577. doi:10.1016/S0019-3577(03)90000-0.

[3] J. Cassaigne. Sequences with grouped factors. In *Developments in Language Theory III, Publications of Aristotle University of Thessaloniki*, pages 211–222, 1998.

[4] S. Constantinescu and L. Ilie. Fine and wilf’s theorem for abelian periods. *EATCS Bull.*, 89:167–170, 2006.

[5] E. M. Coven and G. A. Hedlund. Sequences with Minimal Block Growth. *Math. Syst. Theory*, 7(2):138–153, 1973. doi:10.1007/BF01762232.

[6] A. de Luca. Sturmian words: structure, combinatorics, and their arithmetics. *Theoretical Computer Science*, 183:45–82, 1997. doi:10.1016/S0304-3975(96)00310-6.
[7] G. Didier. Caractérisation des $N$-écritures et application à l'étude des suites de complexité ultimement $n + c^{\text{st}}$. Theoret. Comp. Sci., 215(1-2):31–49, 1999. doi:10.1016/S0304-3975(97)00122-9.

[8] X. Droubay, J. Justin, and G. Pirillo. Episturmian words and some constructions by de Luca and Rauzy. Theoretical Computer Science, 255:539–553, 2001.

[9] S. Ferenczi and C. Mauduit. Transcendence of numbers with a low complexity expansion. Journal of Number Theory, 67:146–161, 1997. doi:10.1006/jnth.1997.2175.

[10] A. S. Fraenkel. Complementing and exactly covering sequences. J. Comb. Theory Ser. A, 14(1):8–20, 1973. ISSN 0097-3165. doi:10.1016/0097-3165(73)90059-9.

[11] R. L. Graham. An efficient algorithm for determining the convex hull of a finite planar set. Inf. Process. Lett., 1(4):132–133, 1972. doi:10.1016/0020-0190(72)90045-2. URL https://doi.org/10.1016/0020-0190(72)90045-2.

[12] T. Hejda, W. Steiner, and L. Q. Zamboni. What is the Abelianization of the tribonacci shift?, 2015. Workshop on Automatic Sequences, Liège, May 2015.

[13] P. Hubert. Suites équilibrées. Theor. Comput. Sci., 242(1-2):91–108, 2000. doi:10.1016/S0304-3975(98)00202-3.

[14] I. Kaboré and T. Tapsoba. Combinatoire de mots récurrents de complexité $n + 2$. ITA, 41(4):425–446, 2007. doi:10.1557/ital:2007027.

[15] J. Karhumäki, S. Puzynina, and M. A. Whiteland. On abelian subshifts. In Developments in Language Theory 2018, volume 11088 of Lecture Notes in Computer Science, pages 453–464. Springer, 2018. doi:10.1007/978-3-319-98654-8_37.

[16] D. Lind and B. Marcus. An Introduction to Symbolic Dynamics and Coding. Camb. Univ. Press, New York, NY, USA, 1995. ISBN 0-521-55900-6.

[17] M. Lothaire. Combinatorics on Words, volume 17 of Encycl. Math. Appl. Addison-Wesley, 1983. ISBN 978-0-201-13516-9.

[18] M. Lothaire. Algebraic combinatorics on words, volume 90 of Encycl. Math. Appl. Cambridge University Press, 2002. ISBN 0-521-81220-8. doi:10.1017/CBO978110736019.

[19] M. Morse and G. A. Hedlund. Symbolic Dynamics II. Sturmian Trajectories. Am. J. Math., 62:1–42, 1940. ISSN 00029327, 10806377.

[20] S. Puzynina and M. A. Whiteland. Abelian closures of infinite binary words. CoRR, abs/2008.08125, 2020. URL https://arxiv.org/abs/2008.08125.

[21] S. Puzynina and L. Q. Zamboni. Abelian returns in Sturmian words. J. Comb. Theory Ser. A, 120(2):390–408, 2013. doi:10.1016/j.jcta.2012.09.002.

[22] G. Richomme, K. Saari, and L. Q. Zamboni. Abelian complexity of minimal subshifts. J. Lond. Math. Soc., 83(1):79–95, 2011. doi:10.1112/jlms/jdq063.

[23] M. Rigo, P. Salimov, and É. Vandomme. Some properties of abelian return words. J. Integer Seq., 16(13.2.5), 2013.

[24] L. Q. Zamboni. Personal communication, 2018.