Effect of local filtering on Freezing Phenomena of Quantum Correlation

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General quantum correlations measures like quantum discord, one norm geometric quantum discord, exhibit freezing, sudden change, double sudden change behavior in their decay rates under different noisy channels. Therefore, one may attempt to investigate how the freezing behavior and other dynamical features are affected under application of local quantum operations. In this work, we demonstrate the effect of local filtering on the dynamical evolution of quantum correlations. We have found that using local filtering one may remove freezing depending upon the filtering parameter.

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I. INTRODUCTION

Quantification and characterization of quantum correlations that can not be fully captured through entanglement measures has generated lot of interests in recent days. It has been shown that a completely separable mixed state might show a quantum signature to compute information processing tasks [1]. Among several general non-classical correlation measures, Quantum Discord (in short, QD) [2] draws much attention due to its operational significance in various quantum information processing tasks, like quantum metrology [3–5], entanglement activation [6–8], information encoding and distribution [9, 10], etc. In order to reveal quantumness in several composite quantum systems beyond entanglement, many attempts are made to show differences between discord like measures with entanglement. One of these attempts to study dynamics of general quantum correlations in open quantum systems.

Several peculiar properties in the dynamics of classical and quantum correlations have already been established in the presence of Markovian and Non-Markovian noise[11–18]. It has been shown that under dissipative dynamics where entanglement suddenly disappears, known as entanglement sudden death [19], quantum discord vanishes only in the asymptotic limit. In this sense, quantum discord is more robust against decoherence than entanglement [20, 21]. Under Markovian noise quantum discord exhibits some striking phenomena in its decay rate for Bell diagonal states such as freezing [22, 23], single sudden change [24]. This freezing phenomena, not exhibited by any entanglement measure, is very demanding since it indicates that the quantum protocols in which quantum correlations are used as resources, will run with a performance unaffected by specific noisy conditions. Thus more intensive study of the behavior of quantum discord under different noisy channel is highly important.

Alternative to the entropic approach, quantum correlation can be measured in geometric way. Recently, Cianciaruso et.al.[25] have proved that this freezing phenomena occurs for any geometric measure of quantum correlation whenever the distance defining the measure respects a minimal set of physical assumptions, namely dynamical contractivity under quantum channels, invariance under transposition, and convexity. Thus freezing phenomena is revealed as universal property for such geometric measures of quantumness. The examples of such distances are the relative entropy[26, 27], the squared bures distance[23, 28–30], the squared Hellinger distance[31–33] and the trace (or Schatten one-norm) distance[34–36]. The Hilbert-Schmidt distance(or Schatten two-norm)[37] does not respect the contractivity property[38] and as a result the geometric quantum discord(GQD)[39], based on Hilbert-Schmidt distance may increase under local reversible operation on unmeasured party[41]. Thus in-spite of its computational simplicity and operational significance[40] in quantum communication protocol, GQD is not considered as a good measure of non classicality. Another version of geometric discord, based on Schatten one-norm, was introduced by Sarandy et.al.[35]. This measure is more acceptable to us since it does not suffer from the problems like its Schatten two counterpart and also for its computational simplicity. This measure displays freezing[23], single sudden change[42] behaviors for Bell diagonal states under decoherence like quantum discord. It has been shown that under Markovian noise, one-GQD exhibits

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twice transition in its decay rate namely double sudden change[42] for Bell diagonal states. Now, one may ask whether it is possible to maintain freezing (or other effects) or remove it under some local quantum operations or not. In this work, we have addressed this issue under local filtering operations.

Behavior of local filtering operations on entanglement has been studied earlier. A filter can be used for creation as well as purification of entanglement[43, 44]. It is possible to retrieve entanglement by a single local filtering for initially pure W and Cluster states with general amplitude damping channel as noise model[45]. Here, we will explore the effects of single local filtering operation on dynamical evolution of quantum discord and one-GQD in noisy environment. We will analyze the effect of single local filtering against freezing, sudden change and double sudden change. We consider standard Bell diagonal state as initial state and phase flip (PF), bit flip (BF), bit-phase flip (BPF) channels as our noise models for decoherence.

Our paper is organized as follows: in section II we will discuss some measures of quantum correlations beyond entanglement which we will use in our work. In section III we will discuss about behavior of quantum correlations for Bell diagonal states under decoherence and section IV contains our main results. Finally, we conclude in section V.

II. MEASURES OF QUANTUM CORRELATIONS

a. Quantum Discord: Olliver and Zurek [2] introduced the concept of Quantum Discord as a measure of genuine quantum correlation. The total correlation, i.e., the total amount of classical and quantum correlations of a bipartite state ρ_{AB} is given by its quantum mutual information

I(ρ_{AB}) = S(ρ_A) + S(ρ_B) − S(ρ_{AB})

(1)

where S(ρ) = −Tr(ρ log_2 ρ) is the Von Neumann entropy of a state ρ and ρ_A, ρ_B are the reduced density matrices of the state ρ_{AB}. On the other hand, its classical correlation is captured by

C(ρ_{AB}) = max_{\{Π_j^A\}} [S(ρ_{AB}[Π_j^A])] = max_{\{Π_j^A\}} [S(ρ_B) − \sum_j p_j S(ρ_{B/j})]

(2)

where maximum is taken over all complete set of orthogonal projectors \{Π_j^A\} on subsystem A and μ_{B/j} = Tr[Π_j^A ⊗ I ρ_{AB}[Π_j^A ⊗ I]] is the reduced density matrix of the subsystem B after obtaining measurement outcome j with probability p_j = Tr[Π_j^A ⊗ I ρ_{AB}[Π_j^A ⊗ I]]. Then Quantum Discord (QD) of a bipartite state ρ_{AB} is defined as the difference between its total correlation and classical correlation and is given by

Q(ρ_{AB}) = I(ρ_{AB}) − C(ρ_{AB})

(3)

This is possibly the most important quantifier of quantum correlations beyond entanglement. There are separable states with non-zero quantum discord. The zero discord states have the form ρ_{AB} = \sum_i p_i |i⟩⟨i|^A ⊗ ρ_B^i, where p_i is a probability distribution, \{|i⟩^A\} denotes an orthonormal basis for subsystem A and ρ^i_B is an ensemble of states of subsystem B. Zero-discord states are usually known as classical-quantum states and the set of all zero-discord state is not convex unlike set of all separable states. For this reason in general it is really hard to calculate quantum discord for most of the states. Only few results are available. For pure bipartite states quantum discord coincides with entropy of entanglement.

b. One-norm Geometric Quantum Discord: Let us consider the geometric quantum discord based on more general norm

D_p = \min_{\tilde{ρ}_{AB}} \|ρ_{AB} − \tilde{ρ}_{AB}\|_p^p

(4)

where \|X\|_p = Tr[(X^†X)^{\frac{p}{2}}]^{\frac{2}{p}} is the Schatten p-norm with p as positive integer and Ω_0 is the set of classical quantum states. In this notation the geometric quantum discord (GQD) introduced by Dakic et.al.[39] is simply obtained by taking p = 2. In spite of its computation simplicity it fails to establish as a good quantifier of quantum correlation since it may increase under local reversible operation on unmeasured party [41]. Sarandy et.al. [35] have shown that only One-norm Geometric Quantum Discord (1-GQD) is the only possible Schatten p-norm which does not suffer from this local ancillary problem. The one-norm Geometric Quantum Discord of a bipartite state ρ_{AB} is defined as [35]

D_1 = D_1 = \min_{\tilde{ρ}_{AB}} \|ρ_{AB} − \tilde{ρ}_{AB}\|_1

(5)

where \|X\|_1 = Tr[√(X^†X)] is the trace norm.

Like Quantum Discord, 1-GQD is zero if and only if ρ_{AB} is classical-quantum state. It is invariant under local unitary operations. For pure bipartite states One-norm Geometric Quantum Discord is an entanglement monotone.
III. DYNAMICS OF QUANTUM CORRELATION FOR BELL DIAGONAL STATES UNDER DECOHERENCE:

Any standard Bell diagonal state can be written as

\[
\rho = \frac{1}{4} [I \otimes I + \sum_{i=1}^{3} c_i \sigma^i \otimes \sigma^i] = \sum_{i=1}^{4} \lambda_i |\phi_i\rangle \langle \phi_i|
\]  

(6)

where \(|\phi_{1,3}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |\phi_{2,4}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)\) and \(\lambda_i\)'s \((\geq 0)\) are eigenvalues with \(\lambda_{1,3} = \frac{1}{4}(1 \pm c_1 \mp c_2 + c_3), \lambda_{2,4} = \frac{1}{4}(1 \pm c_1 \pm c_2 - c_3)\). We will consider the system environment interaction through operator sum representation formalism and both qubits are under similar noise. So, the dynamical evolution of the state \(\rho\) under decoherence, described by a completely positive trace preserving channel \(\$\), is given by,

\[
\$\rho\) = \sum_{i,j} (E_i \otimes E_j) \rho (E_i \otimes E_j)\]

(7)

where \(\{E_k\}\) is the set of Kraus operators associated with the decohering process of a single qubit with trace preserving condition \(\sum_k E_k^\dagger E_k = I\). Here we will consider Phase Flip (PF), Bit Flip (BF), Bit-phase Flip (BPF) channels as our noise models for decoherence and corresponding Kraus operators are summarized in Table(1). As under every such decoherence \(\rho\) preserves its Bell diagonal form, the state after evolution is given by

\[
\rho(p) = \$\rho) = \frac{1}{4} [I + \sum_{i=1}^{3} c_i(p) \sigma^i \otimes \sigma^i] = \sum_{i=1}^{4} \lambda_i(p) |\phi_i\rangle \langle \phi_i|
\]

(8)

where \(\lambda_i(p)\)'s are eigenvalues of the state \(\rho(p)\) with

\[
\begin{align*}
\lambda_{1,3}(p) &= \frac{1}{4}(1 \pm c_1(p) \mp c_2(p) + c_3(p)) \\
\lambda_{2,4}(p) &= \frac{1}{4}(1 \pm c_1(p) \pm c_2(p) - c_3(p))
\end{align*}
\]

(9)

and \(\rho(0) \equiv \rho\). Time dependent correlation functions \(c_i(p)\)'s are given in the Table(1) in terms of parameters \(c_i(0) \equiv c_i\) of initial state and parameterized time \(p = 1 - d(t)\) \((0 \leq p \leq 1)\) of channel \(\$\) where \(d(t)\) depicts the degradation of coherence w.r.t. time \(t\) and it often takes an exponential form for Markovian decoherence process.

| Channel          | Kraus operators | \(c_1(p)\)     | \(c_2(p)\)     | \(c_3(p)\)     |
|------------------|-----------------|-----------------|-----------------|-----------------|
| Phase flip       | \(E_1 = \sqrt{1 - p/2} I, E_2 = \sqrt{p/2} \sigma_1\) | \((1 - p)^2 c_1\) | \((1 - p)^2 c_2\) | \(c_3\)        |
| Bit flip         | \(E_1 = \sqrt{1 - p/2} I, E_2 = \sqrt{p/2} \sigma_1\) | \(c_1\)          | \((1 - p)^2 c_2\) | \((1 - p)^2 c_3\) |
| Bit-phase flip   | \(E_1 = \sqrt{1 - p/2} I, E_2 = \sqrt{p/2} \sigma_1\) | \((1 - p)^2 c_1\) | \(c_2\)          | \((1 - p)^2 c_3\) |

TABLE I: Kraus operators \(E_i\) and correlation functions \(c_i(p)\) for phase flip, bit flip and bit-phase flip channels in terms of \(p\) and \(c_i\).

Since the evolutions of the state \(\rho(p)\) under bit flip and bit phase flip channel are symmetric with that of phase flip channel, we will consider only phase flip channel as noise model. Now we rename \(c_1(p)\) and \(c_2(p)\) by \(c_+(p)\) and \(c_-(p)\) in such a way that \(|c_+(p)|\) and \(|c_-(p)|\) are the maximum and minimum of \(|c_1(p)|, |c_2(p)|\) respectively and \(c_i \equiv c_i(0)\) \(i = +, -, 1, 2, 3\).

Quantum Discord under phase flip channel: As the state \(\rho(p)\) given in Eq.(8) is in Bell diagonal form, its total correlation \(I(\rho(p))\) and classical correlation \(C(\rho(p))\) are given by \[17\]

\[
I(\rho(p)) = 2 + \sum_{i=1}^{4} \lambda_i(p) \log_2 \lambda_i(p)
\]

(10)

\[
C(\rho(p)) = 1 + \sum_{i=1}^{2} \frac{1 + (-1)^i \theta}{2} \log_2 \frac{1 + (-1)^i \theta}{2}
\]

(11)
respectively where $\theta \equiv \max \{|c_1(p)|, |c_2(p)|, |c_3(p)|\} = \max \{|c_+(p)|, |c_3|\}$.

The necessary and sufficient condition \cite{48} (in terms of correlation functions) for freezing phenomena of quantum discord for Bell diagonal state in Eq. (6) under this channel can be obtained as

$$|c_+| \geq |c_3|, \quad c_- = -c_+ c_3$$  \quad (12)

Now we focus on the class of initial states $\rho$ satisfying the condition in Eq. (12). From Eq.(9) and (10) it is easy to observe that total correlation of this class of states takes the form

$$I(\rho(p)) = \sum_{i=1}^{2} \frac{1 + (-1)^i c_3}{2} \log_2[1 + (-1)^i c_3] + \sum_{i=1}^{2} \frac{1 + (-1)^i |c_+(p)|}{2} \log_2[1 + (-1)^i |c_+(p)|]$$  \quad (13)

As under this channel $c_1(p)$ and $c_2(p)$ display the same decay rate w.r.t. $p$, $\theta = |c_+(p)|$ until a parameterized time $p_{\text{sc}} = 1 - \sqrt{\frac{|c_3|}{c_+}}$ and after that $\theta = |c_3|$. From (11) and Eq.(13) it is clear that when $p \leq p_{\text{sc}}$, classical correlation $C(\rho(p))$ decays monotonically and coincides with the 2nd term of total correlation $I(\rho(p))$.

![FIG. 1: Fig(1) describes the evolution of QD (Q) of the state with $c_1 = 0.9$, $c_2 = -0.36$, $c_3 = 0.4$ under phase flip channel. Freezing of QD (magenta solid line) appears for $0 \leq p \leq 0.333333(= p_{\text{sc}})$ with a sudden transition in its decay rate at $p = 0.333333$.](image)

Therefore, for $0 \leq p \leq p_{\text{sc}}$ quantum discord $Q(\rho(p))$ equals to the first term in Eq(13) which is constant, i.e., decay rate of quantum discord is zero for finite period of time. This behavior of QD is known as freezing phenomena. On the other hand when $p \geq p_{\text{sc}}$, $C(\rho(p))$ in Eq(11) is constant since then $\theta = |c_3|$ and hence QD $Q(\rho(p))$ decays monotonically. Such evolution of QD indicates abrupt transition in its decay rate w.r.t. parameterized time $p$ at a specific point $p = p_{\text{sc}}$. Thus QD exhibits freezing behavior for $0 \leq p \leq p_{\text{sc}}$ with a sudden change at $p = p_{\text{sc}}$. This dynamics is illustrated in Fig(1) where we have chosen $c_1 = 0.9$, $c_2 = -0.36$, $c_3 = 0.4$. We have observed that freezing is obtained for $0 \leq p \leq 0.333333$ with a sudden change at $p = 0.333333$.

**One-norm Geometric Quantum Discord under phase flip channel:** As $\rho(p)$ given in Eq(8) is a Bell diagonal state, Eq.(5) reduces to \cite{49}

$$D_G = \frac{1}{2} \times \{\text{intermediate value of } |c_1(p)|, |c_2(p)|, |c_3|\}$$  \quad (14)

As $c_1(p)$ and $c_2(p)$ display the same decay rate w.r.t. $p$, they do not cross each other. Therefore sudden change (or double sudden change) in decay rate of $D_G$ occurs due to the allowed crossing of $|c_3|$ with either $|c_1(p)|$ or $|c_2(p)|$ (or both). Now we consider two classes of Bell diagonal states $\rho(p)$ depending upon two types of initial conditions as follows: **Type 1:** if $|c_+| > |c_3| > |c_-|$ and **Type 2:** if $|c_-| > |c_3|$.
For type 1 states Eq. (14) reduces to

\[
D_G = \frac{1}{2} \begin{cases} 
|c_3| & \text{if } 0 \leq p \leq p_{sc} \\
|c_+ (p)| & \text{if } p_{sc} \leq p \leq 1.
\end{cases}
\]  

(15)

It is clear from Eq. (15) that \( D_G \) exhibits freezing phenomena for \( 0 \leq p \leq p_{sc} \) and decays monotonically for \( p_{sc} \leq p \leq 1 \). Thus a sudden change in its decay rate occurs at \( p = p_{sc} = 1 - \sqrt{|c_3|/|c_+|} \), caused by the only allowed crossing \( |c_3| = |c_+ (p)| \).

For type 2 states Eq. (14) takes the form

\[
D_G = \frac{1}{2} \begin{cases} 
|c_- (p)| & \text{if } 0 \leq p \leq p_{sc1} \\
|c_3| & \text{if } p_{sc1} \leq p \leq p_{sc2} \\
|c_+ (p)| & \text{if } p_{sc2} \leq p \leq 1.
\end{cases}
\]  

(16)

Here the crossings \( |c_3| = |c_- (p)| = (1 - p)^2 |c_-| \) and \( |c_3| = |c_+ (p)| = (1 - p)^2 |c_+| \) are allowed and these imply sudden transitions in decay rates of 1-GQD at two parameterized times \( p_{sc1} = 1 - \sqrt{|c_3|/|c_+|} \) and \( p_{sc2} = 1 - \sqrt{|c_3|/|c_+|} \) which is known as double sudden change behavior. Hence for these type of states 1-GQD exhibits freezing for \( p_{sc1} \leq p \leq p_{sc2} \) with double sudden changes at \( p_{sc1} \) and \( p_{sc2} \).

In Fig. (2) we describe these dynamical evolutions of 1-GQD(\( D_G \)) under phase flip channel for both type 1 and type 2 states by taking initial state parameters as \( c_1 = 0.8 \), \( c_2 = 0.3 \), \( c_3 = -0.45 \) and \( c_1 = 0.8 \), \( c_2 = -0.45 \), \( c_3 = 0.3 \) respectively. We have observed that for both the type 1 and type 2 states QD shows freezing for finite period of time but for type 1 state it exhibits a single sudden transition at \( p = 0.25 \) where as double sudden changes appear at \( p = 0.183503 \) and \( p = 0.387628 \) for type 2 state.

![Fig. 2:](image)

**FIG. 2:** Fig. (2a) shows the evolution of 1-GQD(\( D_G \)) for a type 1 state with \( c_1 = 0.8 \), \( c_2 = 0.3 \), \( c_3 = -0.45 \). In this dynamics \( D_G \) (magenta solid line) becomes frozen for \( 0 \leq p \leq p_{sc} \) with single sudden change at \( p = p_{sc} = 0.25 \) where as Fig. (2b) describes the time evolution of \( D_G \) for type 2 state with \( c_1 = 0.8 \), \( c_2 = -0.45 \), \( c_3 = 0.3 \). Under this evolution \( D_G \) (magenta solid line) exhibits freezing phenomena for \( p_{sc1} \leq p \leq p_{sc2} \) with double sudden changes at two parameterized times \( p_{sc1} = 0.183503 \) and \( p_{sc2} = 0.387628 \).

**IV. DYNAMICAL EVOLUTION OF QUANTUM CORRELATIONS AFTER SINGLE LOCAL FILTERING**

In this section, we will discuss the effect of using local filtering on the dynamical evolution of quantum correlations. Local filtering has been seen to be capable of revealing hidden nonlocality for certain classes of states [50, 51] and useful for creation as well as purification of entanglement [43–45]. Practically this map can be realized as a null-result weak measurement [52].

Let us now perform a single local filtering operation on the subsystem \( A \) of state (8) and the state after filtering is given by

\[
\rho^K (p) = (F \otimes I) \rho (p) (F \otimes I)
\]  

(17)
where filtering operator $F$ is a non-trace preserving map and can be written as

$$F = \sqrt{1 - k}|0\rangle\langle 0| + \sqrt{k}|1\rangle\langle 1|, \quad 0 < k < 1.$$  \hfill (18)

Eigen values of the state in Eq.(17) are $\lambda_i^k(p)'s (\geq 0)$ with

$$
\lambda_{1,3}^k(p) = \frac{1}{4}(1 + c_3(p)) \pm \sqrt{(1 - 2k)^2(1 - c_3(p))^2 + 4k(1 - k)(c_1(p) - c_2(p))^2} \\
\lambda_{2,4}^k(p) = \frac{1}{4}(1 + c_3(p)) \pm \sqrt{(1 - 2k)^2(1 - c_3(p))^2 + 4k(1 - k)(c_1(p) - c_2(p))^2} 
$$

We now discuss the effects case by case.

**Quantum Discord after filtering:** The total correlation and classical correlation of the states $\rho^k(p)$ are given by [53]

$$I(\rho^k(p)) = S(\rho_A^k(p)) + S(\rho_B^k(p)) + \sum_{i=1}^{4} \lambda_i^k(p) \log_2 \lambda_i^k(p)$$

$$C(\rho^k(p)) = S(\rho_B^k(p)) + \sum_{i=1}^{2} \frac{1 + (-1)^i \beta}{2} \log_2 \frac{1 + (-1)^i \beta}{2}$$

respectively where

$$\beta = \sqrt{c_3^2 (1 - 2k)^2 + 4k(1 - k)\theta^2},$$

$$S(\rho_A^k(p)) = -(1 - k) \log_2 (1 - k) - k \log_2 k,$$

$$S(\rho_B^k(p)) = \sum_{i=1}^{2} \frac{1 + (-1)^i (1 - 2k)c_3(p)}{2} \log_2 \frac{1 + (-1)^i (1 - 2k)c_3(p)}{2}.$$

$\lambda_i^k(p)'s$ are the eigen values of state $\rho^k(p)$. Now for this particular class of states (state $\rho$ with initial condition given in Eq(12)) the mutual information in Eq(20) becomes

$$I(\rho^k(p)) = S(\rho_A^k(p)) + \sum_{i=1}^{2} \frac{1 + (-1)^i c_3}{2} \log_2 [1 + (-1)^i c_3] + S(\rho_B^k(p)) + \sum_{i=1}^{2} \frac{1 + (-1)^i \alpha}{2} \log_2 [1 + (-1)^i \alpha]$$

where $\alpha = \sqrt{(1 - 2k)^2 + 4k(1 - k)c_3(p)^2}$. Keeping in mind the values of $|c_+(p)|$ and $\theta$ it is straightforward to observe from Eq.(22) and Eq.(21) that when $p \leq p_{uc}$, for each value of filtering parameter $k$ mutual information $I(\rho^k(p))$ and classical correlation $C(\rho^k(p))$ become two different monotonically decreasing function of $p$ and decreasing rate of $I$ is greater than that of $C$. Hence quantum discord is a monotonically decreasing function of $p$. On the other hand, when $p \geq p_{uc}$, $|\theta| = |c_3|$ which implies constant classical correlation $C(\rho^k(p))$, quantum discord decays monotonically. Thus a sudden transition in its decay rate appears at $p = p_{uc}$.

Therefore local filtering affects the freezing behavior of quantum discord by removing freezing but the point of sudden change ($p_{uc}$) remains same. In Fig(3) we describe such type of dynamical evolution of quantum discord for a state with $c_1 = 0.9$, $c_2 = -0.36$, $c_3 = 0.4$

**One-norm Geometric Quantum Discord after filtering:** As the state $\rho^k(p)$ is a X-state its 1-GQD is given by[54]

$$D^k_{G} = \frac{1}{2} \sqrt{\frac{a_1 \max\{a_3, a_2\} - (a_2 - q) \min\{a_3, a_1\}}{\max\{a_3, a_2\} - \min\{a_3, a_1\} + a_1 - (a_2 - q)}}$$

where $a_1 = (1 - q)c_3^2(p)$, $a_2 = (1 - q)c_2^2(p) + q$, $a_3 = c_3^2$ and $q = (1 - 2k)^2 (0 < q < 1)$ is the filtering parameter. The explicit expressions of $D^k_{G}$ depending upon parameter $q$ for type 1 and type 2 are given in Table(II).

For type 1 state 1-GQD($D^k_{G}$) in Eq(23) is any one of $q_i(p)'s (i = 1, 2, 3, 4)$ given in Table(II) depending upon values of $q$ and its decay rate exhibits any one of the following dynamics:
FIG. 3: Fig(3) describes the evolution of QD (Q) of the state ρ with c1 = 0.9, c2 = −0.36, c3 = 0.4 after filtering respectively. In this dynamics freezing of QD(Q) disappears totally for all values of filtering parameter k(≠0.5) where as before filtering for the same state QD exhibits freezing for 0 ≤ p ≤ 0.333333(= psc) (see Fig(1)) under phase flip channel. The sudden change in the decay rate of QD occurs at the parameterized time p=0.333333 in both before and after filtering dynamics.

1. If 0 < q ≤ q1(fig(4a)), DkG shows similar type of evolution as DkG, i.e., it is constant for a finite interval [0,psc1] of parameterized time p and then decays monotonically. As psc1 = pksc1, duration of freezing of DkG is less than that of DkG.

2. If q1 < q ≤ q2 (fig(4b)), DkG decays monotonically until a parameterized time psc1. After that it remains constant for the time interval psc1 ≤ p ≤ psc2 and then decays monotonically again. This implies two times abrupt transition in its decay rate where as, before filtering there was a single sudden change. Since psc1 ≤ psc2 ≤ pscc, the duration of freezing (psc2 − psc1) reduces more as compared with that of the previous case 0 < q ≤ q1.

3. If q2 ≤ q ≤ q3 (fig(4c)), only a single sudden transition is seen at p = psc1 in its decay rate and freezing vanishes.

4. If q3 < q < 1 (fig(4d)), DkG decays monotonically for the whole range of p without any freezing or any sudden change.

For type 2 states depending upon the choice of filtering parameter q its evolution can be any of the following types:

1. If 0 < q ≤ q4(fig(4e)), then nature of decay rate of 1-GQD (DkG) is same as before filtering. DkG becomes freezed for psc1 ≤ p ≤ psc2 and sudden changes appears at the parameterized times psc1 and psc2. But duration of freezing of DkG reduces as compared with that of DkG because psc1 ≤ pscc1 ≤ psc2 ≤ psc2.

2. If q4 ≤ q < q5 (fig(4f)), then freezing phenomena disappears and just a single sudden change in its decay rate is seen at p = pkc.

3. If q5 ≤ q < 1(fig(4g)), then freezing, sudden change both disappear and DkG decays monotonically.

From the above cases of both type 1 and type 2 states, it is clear that for any values of filtering parameter q, duration of freezing reduces and this reduction increases with the increment of q. If we choose q in such a way that q ≥ q4 or q5, freezing and sudden change both disappear.

Remarks on Quantum Discord and 1-GQD Under Bit flip(or Bit-phase flip) channel : Before and after filtering, the dynamical evolution of quantum discord and 1-GQD under BF (or BPF) channel is symmetric with that of phase flip channel just one has to exchange c3 and c1 (or c2).
Without both freezing and any sudden change in its decay rate. We hope our results will help further to understand the quantum correlation changes. We have found a range of filtering parameters, for which 1-GQD decays monotonically as the amount of filtering increases, duration of freezing decreases and thus the points of abrupt transition in the decay rate increase. On the other hand, in case of 1-GQD, any amount of filtering reduces the duration of freezing. We observed that when the value of filtering parameter increases, phase flip, bit flip, bi-phase flip. During evolution under these channels few crucial features like freezing, Quantum Discord and One Norm Geometric Quantum Discord for the Bell diagonal state under different Markovian noise such as phase flip, bit flip, bi-phase flip. During evolution under these channels... 

In conclusion, we have analyzed in detail, the effect of single local filtering operation on dynamical evolution of Quantum Discord and One Norm Geometric Quantum Discord for the Bell diagonal state under different Markovian noise such as phase flip, bit flip, bi-phase flip. During evolution under these channels few crucial features like freezing, sudden change, double sudden change behaviors of these quantum correlations are seen in its decay rate. We have shown that single local filtering is able to remove this freezing and this disappearance of freezing totally depends on the value of filtering parameter \( q \) \((0 < q < 1)\). In case of Quantum discord, any amount of filtering helps to remove its freezing but the sudden change in its decay rate occurs at the same parameterized time as before filtering. On the other hand in case of 1-GQD, any amount of filtering reduces the duration of freezing. We observed that when amount of filtering increases, duration of freezing decreases and thus the points of abrupt transition in the decay rate of quantum correlation changes. We have found a range of filtering parameter, for which 1-GQD decays monotonically without both freezing and any sudden change in its decay rate. We hope our results will help further to understand basic nature of quantum correlations beyond entanglement.

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| State | \(D_G\) | \(D^k_G\), \(q = (1 - 2k)^2, 0 < k < 1\) | \(g_i(k, p), (i = 1, 2, 3, 4)\) |
|-------|--------|---------------------------------|-------------------------------|
| Type 1 | \(D_G = \frac{1}{2} \left\{ \begin{array}{ll} |c_i| & \text{if } 0 \leq p \leq p_{sc} \\ |c_3(p)| & \text{if } p_{sc} < p \leq 1. \end{array} \right.\) | \(D^k_G = \frac{1}{2} \left\{ \begin{array}{ll} g_1(k, p) & \text{if } 0 \leq p \leq q_i \\ g_2(k, p) & \text{if } q_i \leq p \leq q_2 \\ g_3(k, p) & \text{if } q_2 \leq p \leq q_3 \\ g_4(k, p) & \text{if } q_3 \leq p < 1 \end{array} \right.\) | \(g_1(k, p) = \frac{1}{2} \left\{ \begin{array}{ll} |c_i| & \text{if } 0 \leq p \leq p_{sc} \\ \sqrt{1 - |q|c_i(p)} & \text{if } p_{sc} < p \leq 1. \end{array} \right.\) |
| Type 2 | \(D_G = \frac{1}{2} \left\{ \begin{array}{ll} |c_i| & \text{if } 0 \leq p \leq p_{sc} \\ |c_3(p)| & \text{if } p_{sc} < p \leq p_{sc2} \\ |c_4(p)| & \text{if } p_{sc2} \leq p \leq 1. \end{array} \right.\) | \(D^k_G = \frac{1}{2} \left\{ \begin{array}{ll} g_1(k, p) & \text{if } 0 \leq p \leq q_4 \\ g_2(k, p) & \text{if } q_4 \leq p \leq q_5 \\ g_3(k, p) & \text{if } q_5 \leq p < 1 \end{array} \right.\) | \(g_1(k, p) = \frac{1}{2} \left\{ \begin{array}{ll} |c_i| & \text{if } 0 \leq p \leq p_{sc} \\ \sqrt{1 - |q|c_i(p)} & \text{if } p_{sc} < p \leq 1. \end{array} \right.\) |

| Values of \(p_{sc}, p^k_{sc}, q_i, (i = 1, 2)\) and \(q_j, (j = 1, 2, 3, 4)\) | \(p_{sc1} = 1 - \sqrt{\frac{|c_3|}{|c_i|}}, p_{sc2} = 1 - \sqrt{\frac{|c_4|}{|c_3|}} = p_{sc}, p^k_{sc1} = 1 - \frac{c_i - q}{1 - |q|c_i}, p^k_{sc2} = 1 - \frac{c_3 - q}{1 - |q|c_3}, q_1 = \min\{c_i^2 - c_i^2, c_2^2 - c_i^2\}, q_2 = \min\{c_i^2 - c_i^2, c_4^2 - c_3^2\}, q_3 = c_3^2 - c_4^2, q_4 = c_2^2 - c_3^2\) | \(q_1 = \min\{c_i^2 - c_i^2, c_2^2 - c_i^2\}, q_2 = \min\{c_i^2 - c_i^2, c_4^2 - c_3^2\}, q_3 = c_3^2 - c_4^2, q_4 = c_2^2 - c_3^2\) | \(q_1 = \min\{c_i^2 - c_i^2, c_2^2 - c_i^2\}, q_2 = \min\{c_i^2 - c_i^2, c_4^2 - c_3^2\}, q_3 = c_3^2 - c_4^2, q_4 = c_2^2 - c_3^2\) |

**TABLE II: Explicit expressions of 1-GQD before and after filtering \((D_G \text{ and } D^k_G \text{ respectively})\) for type 1 and type 2 states.**

V. CONCLUSION

In conclusion, we have analyzed in detail, the effect of single local filtering operation on dynamical evolution of Quantum Discord and One Norm Geometric Quantum Discord for the Bell diagonal state under different Markovian noise such as phase flip, bit flip, bi-phase flip. During evolution under these channels few crucial features like freezing, sudden change, double sudden change behaviors of these quantum correlations are seen in its decay rate. We have shown that single local filtering is able to remove this freezing and this disappearance of freezing totally depends on the value of filtering parameter \( q \) \((0 < q < 1)\). In case of Quantum discord, any amount of filtering helps to remove its freezing but the sudden change in its decay rate occurs at the same parameterized time as before filtering. On the other hand in case of 1-GQD, any amount of filtering reduces the duration of freezing. We observed that when amount of filtering increases, duration of freezing decreases and thus the points of abrupt transition in the decay rate of quantum correlation changes. We have found a range of filtering parameter, for which 1-GQD decays monotonically without both freezing and any sudden change in its decay rate. We hope our results will help further to understand basic nature of quantum correlations beyond entanglement.

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FIG. 4: Here we describe the evolution of $D_G$ (1-GQD before filtering) (blue dotted line) and $D_G^f$ (1-GQD after filtering) (magenta solid line) for both the type 1 and type 2 states. Fig(4a), Fig(4b), Fig(4c) and Fig(4d) are for the examples of type 1 states with $c_1 = 0.8, c_2 = 0.3, c_3 = -0.45$. For this state $D_G$ remains frozen for $0 \leq p \leq 0.25$ ($= p_{sc}$). After filtering for $q = 0.11$ (Fig(4a)) freezing of $D_G^f$ appears for $0 \leq p \leq 0.227829$ (= $p_{sc}$) i.e., duration of freezing of $D_G^f$ is less than that of $D_G$. For $q = 0.16$ (Fig(4b)) this freezing appears for $p_{sc1} \leq p \leq p_{sc2}$ with double sudden changes at $p_{sc1} = 0.134102$ and $p_{sc2} = 0.216586$. When $q = 0.4$ (Fig(4c)) freezing of 1-GQD totally disappears but a single sudden change is seen at $p_{sc} = 0.147835$ and when $q = 0.8$ (Fig(4d)) $D_G^f$ becomes monotonic without any sudden change. Fig(4e), Fig(4f), Fig(4g) shows the evolution of 1-GQD for type 2 states with $c_1 = 0.8, c_2 = -0.45, c_3 = 0.3$. For this state $D_G$ exhibits freezing behavior for finite time interval $[p_{sc1}, p_{sc2}]$ with double sudden changes at $p_{sc1} = 0.183503$ and $p_{sc2} = 0.387628$. After filtering for $q = 0.06$ (Fig(4e)) $D_G^f$ remains constant or frozen for $p_{sc1} \leq p \leq p_{sc2}$ with $p_{sc1} = 0.360925$ and $p_{sc2} = 0.378081$, i.e., time of freezing becomes less than that of before filtering. For $q = 0.4$ (Fig(4f)) a sudden change occurs at $p_{sc} = 0.304211$ but freezing of $D_G^f$ is removed and for $q = 0.9$ (Fig(4g)) sudden change also disappears.

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