Mathematical Model and Numerical Simulation of hydrothermal Coupling for Unsaturated Soil Subgrade in the Seasonal Frozen Zone

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Abstract. The subgrade in the seasonal frozen zone is easy to induce uneven settlement, frost heave and thaw settlement under the action of environmental factors such as precipitation, evaporation, radiation, etc. The subgrade is mostly unsaturated soil. In non-saline soil areas, its diseases are mostly related to moisture migration and phase change in the soil. In order to study the moisture migration and temperature distribution of the unsaturated soil subgrade in the seasonal frozen zone, a hydrothermal coupling mathematical model was established. Based on the mathematical model, the hydrothermal coupling numerical simulation was carried out by using the COMSOL Multiphysics finite element program. Given the basic parameters and variables, the model was solved by assigning the same initial and boundary conditions as the field test, and the simulation results were compared with the field test results to obtain very satisfactory temperature simulation results and a good fit between the moisture migration results and the field test values, proving the reliability of the mathematical model.

1. Introduction
In China, about 70% of the ground is covered by permafrost and seasonal frozen soil in cold regions \cite{1, 2}. The subgrade in the seasonal frozen zone is easy to induce uneven settlement, frost heave and thaw settlement under the action of environmental factors such as precipitation, evaporation, radiation, etc. \cite{3, 4}. The subgrade is mostly unsaturated soil. In non-saline soil areas, its diseases are mostly related to moisture migration and phase change in the soil. Therefore, the coupling relationship between the two fields of water and heat needs to be considered. In 1957, Philip and De Vries \cite{5} based on the viscous flow of liquid water in porous media and the principle of heat balance, proposed a hydrothermal coupling migration model, which pioneered the study of hydrothermal coupling in soil. Aboustit et al. \cite{6} successively studied the coupled model of hydrothermal in frozen soil. Harlan \cite{7} constructed a mathematical model to describe the hydrothermal coupling process of the freezing and thawing process of porous media. Subsequently, many scholars have done extended research on this basis, such as Taylor \cite{8} and others used predecessor test data to modify the Harlan model, and constructed a new mathematical model to analyze the law of water and heat migration in frozen soil. Li \cite{9} divided the frozen soil model into different levels. Kurylyk \cite{10} investigated the various forms of the Clapeyron equation, the relationship between the soil moisture curve and the soil freezing curve, and the development process of the soil freezing curve and hydraulic conduction model of partial frozen soils. Vitek \cite{11} describes a thermo-liquid coupling numerical model that is completely consistent with thermodynamics and is designed to simulate artificial ground freezing of saturated
non-deformable porous media under seepage flow conditions. At present, it is mainly focused on the study of hydrothermal coupling in frozen soil, especially saturated frozen soil [12-15]. Few studies have been conducted on hydrothermal coupling in unsaturated seasonal frozen soil.

In this paper, a hydrothermal coupling mathematical model [16] for the subgrade of unsaturated soil in the seasonal freezing zone is established, and field experiment is carried out. COMSOL Multiphysics is used to carry out the numerical simulation of hydrothermal coupling, and the simulated results are compared with field experiment to verify the reliability of the model.

2. Mathematical Model and Numerical Simulation

2.1. Mathematical Model

The governing equation of temperature field is the differential equation of heat conduction (1) as follows:

$$\rho c_v \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right) + q_v$$

In the temperature field control equation (1), the specific heat capacity \(c_v\) and thermal conductivity \(\lambda\) are curve functions of moisture content \(\theta\). Combined with the phase change conditions of the seasonal frozen soil, the volumetric thermal melting in the micro-element body can be described as the phase-change heat generated or absorbed by the ice-water phase change of the micro-element body. Thus it can be obtained.

$$q_v = L \cdot \rho_i \frac{\partial \theta}{\partial t}$$

Thus, Equation (1) can be rewritten as:

$$\rho c_v(\theta) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda(\theta) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda(\theta) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda(\theta) \frac{\partial T}{\partial z} \right) + L \cdot \rho_i \frac{\partial \theta}{\partial t}$$

among them,

$$c_v(\theta) = \frac{\Theta_i \rho_s C_s + \Theta_i \rho_l C_l + \Theta_l \rho_l C_l}{\Theta_i + \Theta_l + \Theta_l}$$

$$\lambda(\theta) = (\lambda_s)^{\Theta_i} (\lambda_l)^{\Theta_l}$$

Equation (3) is the temperature field control equation of hydrothermal coupling, and the thermal conductivity parameters related to water content realize the coupling of moisture content in the temperature field.

There is always unfrozen water in the subgrade under freezing-thawing condition, and its migration changes follow Darcy's law. Considering the retardation effect of pore ice on unfrozen water in the micro-element body, the differential equation of unfrozen moisture migration in unsaturated soil is the coupled temperature control equation of moisture field:

$$\frac{\partial}{\partial x} \left[ D_s(\theta) \frac{\partial \theta}{\partial x} \right] + \frac{\partial}{\partial y} \left[ D_s(\theta) \frac{\partial \theta}{\partial y} \right] + \frac{\partial}{\partial z} \left[ D_s(\theta) \frac{\partial \theta}{\partial z} \right] = \frac{\partial k_s(\theta)}{\partial \theta} \frac{\partial \theta}{\partial t} + \frac{\partial k_s(\theta)}{\partial \theta} \frac{\partial \theta}{\partial t}$$
where: $\theta$ is the volume content of unfrozen water in frozen soil; $k_z$ is the permeability coefficient of unsaturated soil in the direction of gravitational acceleration. The diffusivity of moisture in frozen soil $D(\theta)$ is calculated as follows:

$$D(\theta) = \frac{k(\theta)}{c(\theta)} \cdot I$$  \hspace{1cm} (7)

where: $k(\theta)$ is the permeability of unsaturated soil (m/s); $c(\theta)$ is the specific water capacity (1/m), and it can be determined by the stagnant water model; $I$ is the impedance factor, indicating the blocking effect of pore ice on the migration of unfrozen water, which is calculated by the following equation:

$$I = 10^{10\eta}$$  \hspace{1cm} (8)

In order to truly establish the coupling equation between the moisture field and the temperature field, it is necessary to introduce the link equation between $\theta_i$, $\theta$ and $T$, or the constitutive equation. Based on the dynamic equilibrium relationship of phase transition in frozen soil, this equation is established.

$$w_0/w_u = \left(\frac{T}{T_f}\right)^B, \quad T < T_f$$  \hspace{1cm} (9)

where: $T_f$ is freezing temperature of soil (°C), $w_0$ is initial moisture content of soil (%); $w_u$ is the moisture content of unfrozen water at negative temperature $T$ (%); $B$ is a constant and related to soil type and salt content.

The ratio between the volume of pore ice and the volume of unfrozen water in the frozen soil is denoted as $B_i$:

$$B_i = \frac{\theta}{\theta_u} = \begin{cases} \frac{\rho_u}{\rho_f} \left(\frac{T}{T_f}\right)^B & (T < T_f) \\ 0 & (T \geq T_f) \end{cases}$$  \hspace{1cm} (10)

The solid-liquid ratio $B_i$ is a single value function of temperature. Therefore, the relation equation of pore ice, unfrozen water and temperature in frozen soil is

$$\theta_i = B_i(T) \cdot \theta$$  \hspace{1cm} (11)

The hydrothermal coupling equation (12) can be obtained from the coupled temperature equation (3), water field coupled temperature governing equation (6), phase transition dynamic equilibrium relation equation (11) and the calculated parameter equation. That is mathematical model of hydrothermal coupling.
\[
\frac{\rho c_p(\theta)}{\theta} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda(\theta) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda(\theta) \frac{\partial T}{\partial y} \right) + \frac{\partial q}{\partial z} \left( \lambda(\theta) \frac{\partial T}{\partial z} \right) + L \rho \frac{\partial \theta}{\partial t} 
\]

\[
c_p(\theta) = \frac{\theta \rho_s C_s + \theta \rho_l C_l + \theta \rho_w C_w}{\theta_i + \theta + \theta_i} 
\]

\[
\lambda(\theta) = (\lambda_i) + (\lambda_f) + (\lambda_s) 
\]

\[
\frac{\partial}{\partial x} \left[ D_\lambda(\theta) \frac{\partial \theta}{\partial x} \right] + \frac{\partial}{\partial y} \left[ D_\lambda(\theta) \frac{\partial \theta}{\partial y} \right] + \frac{\partial}{\partial z} \left[ D_\lambda(\theta) \left( \frac{\partial \theta}{\partial z} \right) \right] + \frac{\partial k_\lambda(\theta)}{\partial z} \left( \frac{\partial \theta}{\partial z} \right) = \frac{\partial \theta}{\partial t} + \frac{\rho_s}{\rho_w} \cdot \frac{\partial \theta}{\partial t} 
\]

\[
D(\theta) = \frac{k(\theta)}{c(\theta)} \cdot I 
\]

\[
I = 10^{10} \theta_i 
\]

\[
\theta_i = B(T) \cdot \theta 
\]

\[
k(\theta) = k \cdot S \left( 1 - \left( 1 - S^{\text{slm}} \right) \right)^2 
\]

\[
c(\theta) = a_m l \left( 1 - m \right) \cdot S^{\text{slm}} \cdot \left( 1 - S^{\text{slm}} \right)^m 
\]

\[
n \cdot (c \nabla u + au - \gamma) + qu = g - h \cdot u \quad \text{on} \Gamma 
\]

\[
u = r \quad \text{on} \Gamma 
\]

where, \( S \) is the relative saturation of frozen soil. It is defined as:

\[
S = \frac{\theta_l}{\theta_f - \theta_r} 
\]

where, \( \theta_l \) and \( \theta_f \) respectively represent the residual moisture content and saturated moisture content of soil.

2.2. Numerical Simulation

According to the established mathematical model, a physical field is established by using the user-defined PDE equation interface in COMSOL Multiphysics mathematical module to simulate the field water migration under freezing conditions. Two physical fields are added through the General Form PDE interface, which are moisture content field and temperature field. The equation to control physical field is shown in Equation (12). The height of the geometric model is 150 cm, and the boundary conditions of the geometric model are consistent with the field experiment environment. The boundary conditions are as follows: the upper condition is the field experiment temperature change condition, the bottom temperature condition is the monitored temperature change condition, and the surrounding condition is zero-energy flux exchange boundary condition. The boundary conditions of moisture are: zero-flow boundary in the upper part, constant humidity boundary at the bottom, and zero-water exchange flux boundary around.

After the geometric model is established, the model is discretized. In order to ensure the convergence of the model, the 1.5 m high geometric model is divided into 150 grids with 150 nodes and the size of each grid is 1cm. Finally, the step size is set and the transient solver is used to solve the model.

3. Field Experiment

The site of the field experiment is located in Changchun, which belongs to the seasonal frozen zone. The experiment pit is shown in figure 1 and the experiment soil is shown in figure 2. The side of the experimental pit is protected by compressed board and foam board. The depth of the experimental pit is 1.5 meters. At the bottom of the experimental pit, 50 cm depth of undisturbed soil is preserved. The
The upper part of the soil is filled with 100 cm silty clay. The sensors embedded are moisture sensors and temperature sensors. When embedding sensors, two sensors are in one group, each group consists of one moisture sensor and one temperature sensor. Sensors of the same group are buried on the same horizontal plane, and eight groups of sensors are buried, with the buried depth of 15 cm, 30 cm, 45 cm, 60 cm, 75 cm, 90 cm, 110 cm, 150 cm. The field experiment lasted for 8 months, and the water content and temperature data of monitoring points were collected at the most representative time points.

4. Comparison of Simulation Results and Experiment Results

Figure 3 and figure 4 are a comparison of the distribution of experimental values and simulated values along the buried depth in the field test for 87 and 151 days. It can be seen from the figure: within the buried depth of 150 cm, the test value of the temperature is almost the same as the simulated value. The simulation calculation curve and the test value are close to overlap. It shows that the calculation model is very reliable for the calculation of temperature distribution in time and space of subgrade soil.

![Figure 1. Experiment pit.](image1)

![Figure 2. Silty clay for experiment.](image2)

![Figure 3. The water content curve.](image3)

![Figure 4. Temperature curve.](image4)
The comparison between the simulated value and the experimental value of moisture migration in the interval of 45 cm to 145 cm shows that the simulated calculated value is very close to the experimental value, and the change trend is consistent. Between 15 and 30 cm, the simulated value differs greatly from the experimental value. The reason may be that the mathematical model established in this paper does not consider that the formation of ice bodies such as ice lens causes the upper frozen soil to overcome the suction effect of water gravity on the lower part. This is due to the limitation of the current research results of the international permafrost disciplines. There is no reliable theoretical basis for equationing the suction equation.

5. Conclusions
A mathematical model of hydrothermal coupling of unsaturated soil subgrades in seasonal frozen zone is established. And COMSOL Multiphysics is used for numerical simulation of hydrothermal coupling, a very ideal temperature simulation result is obtained. The moisture migration results are also well fitted to the field experiments values. The reliability of the mathematical model is proved. It can be used to simulate the hydrothermal coupling field of unsaturated subgrade in seasonal frozen zone under the condition of moisture migration.

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