In how many ways can quantum information be split?

Sreraman Muralidharan†
Loyola College, Nungambakkam, Chennai - 600 034, India

Siddharth Karumanchi
Birla Institute of Technology and Science, Pilani, Rajasthan- 333031, India

R. Srikanth
Poornaprajna Institute of Scientific Research, Devanahalli, Bangalore 562 110, India and
Raman Research Institute, Sadashiva Nagar, Bangalore 5600012, India

Prasanta K. Panigrahi‡
Indian Institute of Science Education and Research - Kolkata,
Mohanpur, BCKV Campus Main Oce, Mohanpur - 741252, India and
Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India

We quantify the different ways in which an arbitrary \(n\) qubit state can be split among a set of \(k\) participants using a \(N\) qubit entangled channel, such that the original information can be completely reconstructed only if all the participants cooperate. After proving that the first party needs to possess at least \(n\) qubits for this purpose, we show that the maximum number of protocols that one can construct for the splitting of an arbitrary \(n\) qubit state among two parties using an \(N\) qubit entangled channel is \((N - 2n)\). Then we generalize this result to \(k\) parties and illustrate its usefulness by providing an explicit physical example for the same.

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I. INTRODUCTION

Splitting and sharing of secret information among a group of parties such that none of them can completely reconstruct the secret information by themselves is a common requirement in financial and defence sectors. Classical information theory offers a solution to this problem through “secret sharing” †, where the secret message is encrypted and split among various parties. However, the security of all these classical schemes are conditional and undeterministic.

The laws of quantum mechanics ‡ enable one to carry out a number of tasks which would otherwise be impossible in classical world §. For instance, the no cloning theorem ‡ protects the quantum information from being copied. The amalgamation of principles of quantum mechanics and classical secret sharing has given rise to “quantum secret sharing” ‡ which provides unconditional security for the splitting and sharing of secret information. Infact, quantum secret sharing can be used for protecting both classical and quantum information ‡.

Sharing of quantum information among a group of parties such that none of them can reconstruct the unknown information completely by operating on their own share is usually referred to as “Quantum Information Splitting” (QIS). In a landmark paper, Hillery et al. §, proposed the first scheme for the QIS of an arbitrary single qubit state \(|\psi_1\rangle = \alpha |0\rangle + \beta |1\rangle\) (\(\alpha, \beta \in C\) and \(|\alpha|^2 + |\beta|^2 = 1\)) using a three qubit GHZ state, given by

\[ |\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC}, \]

as a shared entangled resource using controlled teleportation §. In their scheme, three parties namely, Alice, Bob and Charlie possess one qubit each from the \(|\text{GHZ}_3\rangle\). Alice possesses \(|\psi_1\rangle\) which she wants Bob and Charlie to share, such that neither of them should be able to reconstruct \(|\psi_1\rangle\) by operating on their own qubits. To achieve this, Alice performs a Bell measurement on her qubits so that the Bob-Charlie system collapses to an entangled state given by,

\[ (U_x \otimes I)(|00\rangle + |11\rangle)_{BC}, \]

where \(U_x \in \{I, \sigma_x, i\sigma_y, \sigma_z\}\). At this point, both the subsystems of Bob and Charlie are maximally mixed. Alice can designate any one of them (Bob or Charlie) to retrieve the final state and convey the outcome of her measurement to the other using two classical bits. After, Alice conveys the outcome of her measurement to Bob or Charlie, the reduced density matrices of Bob and Charlie have amplitude, but no phase information about the initial state \(|\psi_1\rangle\).

Hence, neither Bob nor Charlie can reconstruct the information by themselves by operating on their own qubits. Now, one of the parties (say Bob) needs to perform a Hadamard measurement in the basis \(|\frac{1}{\sqrt{2}}(0 \pm 1)|\)

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*Electronic address: sreramanm@yahoo.co.in
†Electronic address: prasanta@prl.res.in
and convey the outcome of his measurement to Charlie via one classical bit. Having known the outcomes of both their measurements, Charlie can obtain the unknown qubit state \(|\psi_1\rangle\) by local unitary operations. This completes the protocol for QIS of \(|\psi_1\rangle\) using \(|GHZ\rangle_4\) as a shared entangled resource. This protocol has also been experimentally realized \[8\].

As is evident, in a realistic situation, for the splitting of an arbitrary multiqubit state, many parties need to be in an entangled quantum network. Keeping in mind the complexity of the multipartite entangled system, there will arise more than one way of carrying out QIS in a quantum network, given a fixed number of parties. This needs to be quantified for the fundamental understanding of QIS.

This point can be more clearly seen through the following example: For instance, the QIS of \(|\psi_1\rangle\) can be achieved using a four qubit entangled channel in two distinct ways. Let Alice, Bob and Charlie share a four qubit GHZ state:

\[
|GHZ\rangle_4 = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)_234. \tag{3}
\]

In the first protocol, Alice possesses qubits 1, Bob 2, 3 and Charlie 4. In this scenario, Alice performs a Bell measurement on qubits \(|\psi_1\rangle\) and 1. Now the Bob-Charlie system collapses into an entangled state given by

\[
(U_x \otimes I \otimes I)(|000\rangle + |111\rangle)_234. \tag{4}
\]

Subsequently, Bob performs Bell measurement on his qubits 2 and 3, so that Charlie’s system collapses into \(U_x|\psi_1\rangle\). Charlie then applies a suitable unitary operator \(U_x^\dagger\) and obtains \(|\psi_1\rangle\).

Another protocol for the QIS of \(|\psi_1\rangle\) using \(|GHZ\rangle_4\) can be constructed by redistributing the qubits of the entangled channel among the parties. Suppose Alice possesses qubit 1, 2, Bob 3, and Charlie 4. Then, Alice can perform a three particle measurement in the GHZ basis and convey the outcome of her measurement to Charlie via two classical bits. Now, the Bob-Charlie system collapses into \((U_x \otimes I)(|00\rangle + |11\rangle)_24\). Bob performs a measurement in Hadamard basis and conveys the outcome of his measurement to Charlie via one classical bit. Having known the outcomes of both their measurements, Charlie can obtain \(|\psi_1\rangle\) by LOCC. This completes the second protocol for QIS of \(|\psi_1\rangle\) using \(|GHZ\rangle_4\) as an entangled channel. Though, the classical communication and the entanglement cost remains the same in both these protocols, one protocol might be preferred over the other in terms of feasibility and security. In a generalized scenario, involving \(N\) parties, one would think of many combinations for distributing the qubits, however all such combinations will not work out for QIS.

QIS of \(|\psi_1\rangle\) has also been carried out using an asymmetric W state \[9\] and multipartite cluster states \[10, 11\]. It has been found that while the five qubit Brown states \[10\] and the cluster states \[10\] are useful for QIS of \(|\psi_2\rangle = \sum_{i} \alpha_{i1_2...i_n} |i_1i_2...i_n\rangle\) among two parties, a four qubit cluster state cannot be used for the same in the scenario where they need not meet. It has also been shown that it is possible to devise two protocols for the QIS of an arbitrary two qubit state \(|\psi_2\rangle\) using six qubit cluster and the Borras et al. \[13\] states as entangled resources. Further, it was conjectured recently that one can devise \((N - 2n)\) protocols for the QIS of an arbitrary \(n\) qubit state \(\sum_{i} \alpha_{i1_2...i_n} |i_1i_2...i_n\rangle\) among two parties in the case where they need not meet up \[10\]. These points naturally lead to questions regarding the number of ways in which quantum information can be split among many parties. However, to the best of authors’ knowledge, the general quantification of the different ways in which an arbitrary \(n\) qubit state can be split among a set of \(k\) participants using a general \(N\) qubit entangled channel, has not been established. In this paper, we hope to answer this question by proving two theorems which throws light on the structure of global entanglement that is required for the study of QIS.

\[\text{II. THEOREMS}\]

**Theorem 1:** If Alice, Bob and Charlie share an \(N\) qubit entangled state and Alice has an arbitrary \(n\) qubit state that she wants Bob and Charlie to share, then Alice needs to possess a minimum of \(n\) qubits for this purpose.

**Proof:** If we let Alice possesses \(m\) qubits in the entangled quantum network then it can be proved that \(m \geq n\) from a quantum encryption perspective as follows: After Alice’s joint measurement in \(\mathcal{H}_x \otimes \mathcal{H}_A\), but before her classical communication to Bob, the no-signaling theorem demands that Bob’s density operator should not have changed, i.e., it must remain maximally mixed. Here, \(\mathcal{H}_x\) and \(\mathcal{H}_A\) refer to the respective Hilbert spaces of the unknown secret information and Alice’s part of the entangled network respectively. On the other hand, we know by the way teleportation works that Bob’s state has become:

\[
T : |\psi\rangle \longrightarrow \sum_{j=1}^{P} U_j |\psi\rangle \langle \psi | U_j^\dagger \tag{5}
\]

Alice’s classical communication will be the number \(j\) which will allow Bob to apply operation \(U_j\) that restores his object’s state to \(|\psi\rangle\). We require the minimal number \(P\) in Eq. 5 such that for an arbitrary input state \(|\psi\rangle\), we obtain \(T(|\psi\rangle) = I/D\), where \(I\) is the unit matrix and \(D = 2^n\) is \(\dim(\mathcal{H}_B)\). According to Ref. 15, which provides a protocol for classically encrypting a quantum state, \(P = D^2\). This in turn means that Alice’s classical communication must be \(\log(D^2) = 2n\) bits long. In turn this means that the classical outcome of Alice’s Bell state measurement, which is \((m + n)\) qubits, must satisfy \(m + n \geq 2n\), or, \(m \geq n\), as required.
We let Alice possess the unknown arbitrary two qubit state and perform a measurement to get the above state into another state through LOCC by performing another measurement nor one can transform the above state to another state through LOCC and perform a measurement to get (N−2n). Hence, the total number of protocols that one can construct is (N−2n).

Lemma 1 : The maximum number of protocols one can construct for this purpose is (N−2n).

Proof :

We let the third person (say Charlie) have the last n qubits, on which he will apply a suitable local unitary transformation and reconstruct the unknown n qubit information. Therefore Charlie will possess, (N−n+1)th qubit to the Nth qubit. Now, the first (N−n) qubits need to be distributed among Alice and Bob. This would correspond to (N−n+1) protocols. However, from the above theorem, all the protocols, in which Alice possesses less than n qubits fail. Hence, the total number of protocols that one can construct is (N−2n).

Corollary :

By substituting N = 4 and n = 2 in this formula, we can deduce that four qubit states cannot be used for the QIS of |ψ⟩2. This shall be illustrated below as follows. We let Alice possess the unknown arbitrary two qubit state |ψ⟩2 and qubit 1, Bob possess qubit 2 and Charlie 3,4 in the four qubit cluster state [14],

|C4⟩ = 1/2(|0000⟩ + |0110⟩ + |1001⟩ − |1111⟩)1234. (6)

respectively. Alice can perform a three partite measurement on |ψ⟩2 as in the above protocol. For instance, if Alice performs a measurement in the basis

1/2(|000⟩ + |100⟩ + |011⟩ − |111⟩), (7)

then, the Bob-Charlie system collapses to

α(|000⟩ + |110⟩) + µ(|000⟩ + |110⟩) +
γ(|001⟩ − |111⟩) + β(|001⟩ − |111⟩). (8)

However, from the above state one cannot obtain |ψ⟩2, by performing another measurement nor one can transform the above state into another state through LOCC and perform a measurement to get |ψ⟩2. Hence, this protocol fails illustrating the usefulness of the theorem.

Theorem 2 : If k (k ≥ 3) parties share an N qubit entangled state and the first party has an arbitrary n qubit state that he/she wants the remaining members to share, then the maximum number of protocols he/she can construct for this purpose is N−2nCk−2.

Proof :

The number of ways of distributing n qubits among k participants such that each participant gets at least one qubit is n−1Ck−1. In the first case, when the first and the last participants possesses n qubits each, the remaining (N−2n) qubits can be distributed among (k−2) parties in N−2n−2Ck−3 ways. The process of redistribution can continue until all the (k−2) participants have exactly one qubit each. Therefore, from the case, where the first participant possess (n + p), qubits we obtain (N−2n−p) = k−2 or p = (N−2n−k−2). Since all the protocols are mutually exclusive of each other, explicit calculations on all different arrangements and their summation yields N−2nCk−2 assuming γC0 = 1 for any positive integer y. Hence, the theorem is proved. It is worth noting that this yields Lemma 1, when k = 3.

III. CONCLUSION

In conclusion, we have proved two theorems which is important for the study of QIS. We initially proved that the number of qubits possessed by Alice should be greater than the dimension of the quantum secret in order for QIS to be possible. We showed that this implies that the maximum number of protocols that one can construct for the splitting of an arbitrary n qubit state among two parties using an N qubit entangled channel is (N−2n). We illustrated this point by taking the example of a four qubit cluster state. We later, considered a general scenario and showed that one can construct a maximum of N−2nCk−2 protocols when there are k parties involved in QIS. These theorems illustrate why certain entangled channels cannot be used for QIS of certain quantum states. These will also help in deciding the threshold number of qubits that an entangled channel should possess for the QIS of certain unknown composite quantum systems. The study of the different possible protocols that one can construct for QIS of certain quantum systems is important because certain protocols may be preferred over the other in terms of security and feasibility. For instance, only one protocol for the QIS of |ψ⟩1 using the four qubit cluster state was found to be secure [10]. All theorems are considered in the scenario where the participants need not meet. We hope that these theorems will shed light into the future research on QIS as the feasibility and security of different protocols needs to be considered for the practical implementation of QIS.

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