Renormalization Group Approach to Matrix Models

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Abstract. Matrix models of 2D quantum gravity are either exactly solvable for matter of central charge $c \leq 1$, or not understood. It would be useful to devise an approximate scheme which would be reasonable for the known cases and could be carried to the unsolved cases in order to achieve at least a qualitative understanding of the properties of the models. The double scaling limit is an indication that a change of the length scale induces a flow in the parameters of the theory, the size of the matrix and the coupling constants for matrix models, at constant long distances physics. We construct here these renormalization group equations at lowest orders in various cases to check that we reproduce qualitatively the properties of $c \leq 1$ models.

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1. Introduction

The matrix model representation of two-dimensional quantum gravity has led to explicit solutions for minimal conformal fields coupled to gravity and to beautiful connections with various branches of mathematical physics such as integrable flows or topological field theories. However this approach has not been of real help for understanding the difficulties which arise when the central charge $c$ of the matter field is larger than one although it is very easy to write matrix models for those cases as well. Indeed if the topological expansion of field theories in dimension larger than one is notoriously difficult to handle, one can write (for instance) a matrix model for $n$ Ising spins coupled to a randomly triangulated surface, with a central charge $c = n/2$, as an integral over $2^n$ matrices. Unfortunately these models are not solvable (up to now) for $n > 1$.

Therefore we are in the unfortunate situation of either solving exactly (for $c \leq 1$) or not understanding at all what is happening. We are not even guaranteed that a matrix model candidate to describe a $c > 1$ model has anything to do with the continuum description, although we should keep in mind that (i) it is not yet clear that there is a continuum theory at present for $c > 1$, (ii) we have invested enough efforts in matrix models to try to find out whether they do have a continuum limit for $c > 1$; is there still a double scaling limit with some new type of exponent, or no continuum limit at all? If there is a continuum limit is it still a string theory in non-critical dimension?

We have no answer to provide to these important questions, but we would like to devise a method which may be only approximate for the soluble cases $c \leq 1$, but which would allow us to understand at least qualitatively $c > 1$. The attempt that we present here is based on simple minded analogies with critical phenomena. We note first that the central result of matrix models is the existence of the “double scaling limit”, i.e. a continuum limit with critical exponents which describe how the two coupling constants of the theory have to be tuned to reach this limit. These simple scaling laws for $c < 1$, with logarithmic deviations at $c = 1$, are reminiscent of the theory of phase transitions for dimensions larger or equal to four. There we know that a non-trivial IR fixed point governs dimensions smaller than four and that we have to rely on approximate methods such as the $\varepsilon$ or $1/N$ expansions. Is there a similar phenomenon for matrix models? This question makes sense only if we understand renormalization group flows. Indeed if we had such flow equations, we know that the scaling laws and the exponents, which characterize the double scaling limit, would arise automatically. Therefore we have to understand how the two coupling constants of the theory, the string coupling constant and the cosmological constant.
(mapped respectively into the size $N$ of the matrices and the matrix coupling constant $g$), evolve under a rescaling of the regularization length introduced in the triangulation of the worldsheet. We thus expect that a change $N \to N + \delta N$ can be compensated by a change $g \to g + \delta g$ with the same continuum physics. We shall see that for matrix models this indeed the case, at the expense of enlarging the space of coupling constants very much as in the Wilson’s scheme of integration over the momenta in the shell $\Lambda - d\Lambda < |p| < \Lambda$.

Let us review first briefly the $c < 1$ scaling laws: the singular part of the string partition function satisfies

$$Z = \Delta^{2-\gamma_0} f\left(\Delta \frac{N^2}{\gamma_1}\right)$$

with

$$\Delta = g_c - g$$

and

$$\gamma_1 = 2 - \gamma_0 = \frac{1}{12} \left[25 - c + \sqrt{(1 - c)(25 - c)}\right].$$

The string susceptibility exponent at genus $h$ is

$$\gamma_h = \gamma_0 + h \gamma_1.$$

The relation

$$\gamma_0 + \gamma_1 = 2$$

independent of the explicit values of $\gamma_0$ and $\gamma_1$, is easily obtained from the consideration of the torus.

Assume now that we have indeed a Callan–Symanzik like differential equation for this string partition function which reads

$$[N \partial/N - \beta(g) \partial/\partial g + \gamma(g)]Z(N, g) = r(g)$$

with a fixed point $g^*$ given by $\beta(g^*) = 0$ and $\beta'(g^*) > 0$. It is then elementary to verify that we recover the scaling law with

$$\gamma_1 = \frac{2}{\beta'(g^*)} , \quad \gamma_0 = 2 - \frac{\gamma(g^*)}{\beta'(g^*)}.$$

With several coupling constants $g_k$, the scaling exponents are given by the eigenvalues of the matrix

$$\Omega_{kl} = \frac{\partial \beta_k}{\partial g_l} (g^*_m).$$
Note that the scaling law (5) requires that $\gamma (g^*) = 2$.

These flow equations will be constructed by integrating out one line and one row of an $(N + 1) \times (N + 1)$ matrix, thereby reducing it to an $N \times N$ matrix. In this process we shall prove that the matrix partition function $\zeta_N(g)$ fulfills the equation

$$\zeta_{N+1}(g) = [\lambda(g)]^{N^2} \zeta_N(g')$$

(9)

with

$$g' = g + \frac{1}{N} \beta(g) + O \left( \frac{1}{N^2} \right)$$

(10)

and

$$\lambda(g) = 1 + \frac{1}{N} r(g) + O \left( \frac{1}{N^2} \right).$$

(11)

It follows immediately that the string partition function

$$Z(N, g) = \frac{1}{N^2} \ln \zeta_N(g)$$

(12)

satisfies (6) with

$$\gamma(g) = 2,$$

(13)

and therefore the scaling law (5) does hold.

The set-up of this article is the following: we first discuss at lowest order pure gravity, then multicritical points of one-matrix models, then a more appropriate saddle-point method and the new features which manifest themselves at higher orders.

2. A Simple Perturbative Calculation

We begin with a one-matrix $\phi^4$ model, which near its critical point describes pure gravity ($c = 0$). It consists of an integral over an $N \times N$ hermitian matrix $\phi_N$:

$$\zeta_N(g) = \int d\phi_N \exp \left[ -S_N (\phi_N, g) \right],$$

(14)

with an action

$$S_N [\phi_N, g] = N \text{ tr} \left[ \frac{1}{2} \phi_N^2 + \frac{g}{4} \phi_N^4 \right].$$

(15)

From the exact solution [2] we know that the double scaling limit is reached in the vicinity of

$$g_c = -1/12$$

(16)
with an exponent
\[ \gamma_1 = \frac{5}{2}. \] (17)

The matrix \( \phi_{N+1} \) is parametrized in terms of an \( N \times N \) submatrix \( \phi_N \), a complex \( N \)-component vector \( v_a \), and a number \( \alpha \):

\[ \phi_{N+1} = \begin{pmatrix} \phi_N & v_a \\ v_a^* & \alpha \end{pmatrix}, \] (18)

but one verifies easily that all the terms involving \( \alpha \) are of relative order \( 1/N \) and can be dropped in the continuum limit; in other words we can set \( \alpha = 0 \). We then define

\[ \exp \left[ -S'_N (\phi'_N, g') \right] = \lambda_N(g) \int d^N v \, d^N v^* \, \exp \left[ -S_{N+1} (\phi_N, v, g) \right], \] (19)

in which \( \phi'_N \) is obtained after a rescaling which normalizes the coefficient of \( \phi^2_N \) to \( N/2 \) as in (15).

An easy calculation yields

\[ S_{N+1} (\phi_{N+1}) = (N + 1) \left[ \operatorname{tr} \left( \frac{1}{2} \phi^2_N + \frac{g}{4} \phi^4_N \right) + v^* \cdot v \right] + (N + 1)g \left[ v^* \phi^2_N v + \frac{1}{2} (v^*, v)^2 \right], \] (20)

after which we expand the exponential to first order in \( v \), perform the easy integrations over the \( v_a \)'s and re-exponentiate to get a new effective action

\[ S' (\phi_N) = (N + 1) \left[ \operatorname{tr} \left( \frac{1}{2} \phi^2_N + \frac{g}{4} \phi^4_N \right) \right] + g \operatorname{tr} \phi^2_N. \] (21)

We then rescale \( \phi_N \) to

\[ \phi_N = \rho \phi'_N \] (22)

with

\[ \rho = 1 - \frac{2g + 1}{2N} + O \left( \frac{1}{N^2} \right) \] (23)

(we are dropping from now on all terms of relative order \( 1/N^2 \) as well as higher orders in \( g \)). This rescaling gives as coefficient of \( \phi^4 \)

\[ g' = g - \frac{1}{N} (g + 4g^2) \] (24)

and for the prefactor in the partition function

\[ \lambda(g) = 1 - \frac{1}{N} (1 + 3g). \] (25)
Therefore we have established at first order the renormalization group equation (6) with

\[ \beta(g) = -g - 4g^2 + O\left(g^3\right) \]  
\[ \gamma(g) = 2 \]  
\[ r(g) = -1 - 3g + O\left(g^2\right). \]  

(26a)  
(26b)  
(26c)

There are two fixed \( g^* = 0 \) or \(-1/4\), but we have to select the repulsive fixed point

\[ g^* = -\frac{1}{4} \]  

(27)

since the pure gravity exponents are obtained only when we “tune” the cosmological constant \( g \) near its critical value \( g_c \); the true \( g_c \) is \(-1/12\) in the exact theory instead of our first approximation \(-1/4\). Since at this order \( \beta'(g_c) \) is equal to one, this calculation gives

\[ \gamma_1 = 2, \]  

(28)

to be compared to the exact value \( 5/2 \).

We shall return to this \( c = 0 \) case in a more elaborate calculation below, but we first consider within the same approximation the multicritical points of one-matrix models.

3. Multicritical Points

In order to study the existence of Kazakov’s multicritical points \([7]\) within one-matrix models, we allow for an arbitrary polynomial in the action

\[ S[N, g_k] = N \sum_1^\infty \frac{g_k}{2k} \text{tr} \phi^{2k} \]  
\[ \]  

(29)

with \( g_1 = 1 \).

The parametrization \([18]\) of \( \phi_{N+1} \) gives a number of terms, since

\[ \text{tr} \phi^{2k}_{N+1} = \text{tr} \phi^{2k}_N + 2kv^* \phi^{2k-2}_N v + O\left(v^4\right). \]  

(30)

The crudest calculation consists in keeping these quadratic terms in \( v \), expanding to first order in the \( g_k \)'s, integrating over the \( v \)'s and reexponentiating. This gives, up to a normalization constant,

\[ \zeta_{N+1} = \int d\phi_N \exp \left\{ -(N + 1) \text{tr} \left[ \left( \frac{1}{2} + \frac{g_2}{N + 1} \right) \phi_N^2 + \sum_2 \left( \frac{g_k}{2k} + \frac{g_{k+1}}{N + 1} \right) \phi^{2k}_N \right] \right\}. \]  
\[ \]  

(31)
Rescaling $\phi$ in order to set the coefficient of $\phi^2_N$ to $N/2$, we obtain
\[ g'_k = g_k + \frac{1}{N} \left[ - (2kg_2 + k - 1) g_k + 2kg_{k+1} \right], \] (32)
and thus
\[ \beta_k = - (2kg_2 + k - 1) g_k + 2kg_{k+1}. \] (33)

It is easy to verify that there is a multicritical point of order $m$ given by
\[ g^*_k = \begin{cases} \left( \frac{-1}{2m} \right)^{k-1} \binom{m-1}{k-1} & \text{for } 2 \leq k \leq m \\ 0 & \text{for } m < k. \end{cases} \] (34)

The corresponding multicritical potential is
\[ S^*_N = N \text{ tr} \left[ 1 - \left( 1 - \frac{\phi^2_N}{2m} \right)^m \right] \] (35)
which is of course approximate; for $m = 2$ it gives $x^2/2 - x^4/16$, instead of the exact $x^2/2 - x^4/16$; for $m = 3$, $x^2/2 - x^4/12 + x^6/12$ instead of $x^2/2 - x^4/12 + x^6/180$. Although it is only approximate, we do find the right sequence of multicritical points with their characteristic features of being alternatively unbounded below for even $m$’s and bounded for odd $m$’s.

The critical exponents are related to the matrix (8) of derivatives of the $\beta$-functions at the fixed points. For the $m$-th multicritical points this matrix is real, $(m-1) \times (m-1)$ and non-symmetric. Remarkably enough this matrix turns out to have $(m-1)$ real eigenvalues which are $2/m, 3/m, \ldots, m/m$. For an arbitrary direction of approach of the fixed point the leading critical exponent is the largest one, namely one, which corresponds in fact to the $c = 0$ exponent. If we choose a direction in the space spanned by the $(m-2)$ subleading eigenvalues we shall see in general as leading exponent $(m-1)/m$. In order to tune the $m$-th multicritical point we have to suppress the $(m-2)$ largest eigenvalues by choosing a direction in the $g_k - g^*_k$ space along the eigenvector of the smallest eigenvalue, namely $2/m$. This gives for the $m$-th multicritical point
\[ \gamma_1 = m \] (36)
whereas the exact value is $m + 1/2$. This trivial calculation is thus in good qualitative and semi-quantitative agreement with the exact answer.
4. A Better Calculation

An attempt to expand to higher orders in the coupling constant in order to improve the above calculations reveals that we have not done it in a systematic way. For instance if we follow the same procedure to next order and try a “two-loop” calculation, we find that the $\beta$-function at order $g^2$ receives contributions from higher orders. In fact we have to integrate first over the $N$ complex $v$’s without perturbation theory, and then expand in order to reach the large $N$, double-scaling limit. To this effect we write the effective action

$$S_{N+1}^\text{eff} [\phi_N, v_a, \sigma] = (N + 1) \left[ \operatorname{tr} \left( \frac{1}{2} \phi_N^2 + \frac{g}{4} \phi_N^4 \right) + v^* \cdot v \right]$$

$$+ (N + 1) g \left[ \phi_N^2 v + \sigma v^* \cdot v - \frac{\sigma^2}{2} \right],$$

which, upon integration over $\sigma$, is equivalent to (20). We next integrate over the $v$’s and obtain

$$S_N^\text{eff} [\phi_N, \sigma] = (N + 1) \left[ \operatorname{tr} \left( \frac{1}{2} \phi_N^2 + \frac{g}{4} \phi_N^4 \right) \right] - (N + 1) \frac{g\sigma^2}{2} + \operatorname{tr} \ln \left( 1 + g\sigma + g\phi_N^2 \right).$$

In the large $N$ limit, the single $\sigma$-mode is fixed to its saddle-point value given by

$$\sigma = \frac{1}{N} \operatorname{tr} \left( 1 + g\sigma + g\phi_N^2 \right)^{-1}.$$  

This gives $\sigma$ as a functional of the traces of the even powers of $\phi$, involving also non-linear terms as $(\operatorname{tr} \phi_N^2)^2$ and so on. Therefore the original action has to be enlarged to accomodate such terms. However is we expand again in powers of $g$ these new interactions do not yet appear at order $g^2$ since:

$$\sigma = 1 - (1 + t_2) g + (2 + 3t_2 + t_4) + O \left( g^3 \right)$$

in which

$$t_k = \frac{1}{N} \operatorname{tr} \phi_k$$

and thus, dropping all terms of order $g^3$

$$S_N^\text{eff} = \left[ \frac{1}{2} (N + 1) + g - g^2 \right] t_2 + \left[ \frac{g}{4} (N + 1) - \frac{g^2}{2} \right] t_4.$$  

After a rescaling of $\phi_N$ to enforce to $N/2$ the coefficient of $t_2$ we find a new $\phi^4$ coupling constant

$$g' = g - \frac{1}{N} \left( g + 6g^2 \right) + O \left( g^3 \right)$$
\[ \beta(g) = -g - 6g^2 + O(g^3). \] (44)

The fixed point \( g^* = -1/6 \) is a slight improvement over the previous calculation, but at this order the exponent \( \gamma_1 \) does not change.

A calculation at next order requires two additional coupling constants, namely that of \( \text{tr} \phi^6 \) and of \( (\text{tr} \phi^2)^2 \); the renormalization flow takes place in a three dimensional space. The result of this calculation will be reported elsewhere.

5. Conclusion

Simple renormalization group transformations reproduce qualitatively and semi-quantitatively the results of matrix models. In principle we could apply readily the same technique to matrix models with \( c \) larger than one. However manifestly the method has first to pass a few non-trivial tests: can it be systematically improved at higher orders for simple one-matrix models? For \( c = 1 \) we should find that the fixed point is a double zero of the \( \beta \)-function in order to account for the logarithmic deviations to scaling [4]. The situation is still unclear, but we believe that it is worth exploring the possibilities of this approach further.
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