A CHARACTERIZATION OF THE INCLUSIONS BETWEEN MIXED NORM SPACES

IRINA ARÉVALO

Abstract. We consider the mixed norm spaces of Hardy type studied by Flett and others. We study some properties of these spaces related to mean and pointwise growth and complement some partial results by various authors by giving a complete characterization of the inclusion between \( H(p, q, \alpha) \) and \( H(u, v, \beta) \), depending on the parameters \( p, q, \alpha, u, v \) and \( \beta \).

1. Introduction

For \( p, q, \alpha > 0 \), an analytic function on the unit disk \( f \) is said to belong to the mixed norm space \( H(p, q, \alpha) \) if and only if

\[
\|f\|_{p,q,\alpha}^q = \alpha q \int_0^1 (1 - r)^{\alpha q - 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{q/p} dr < \infty.
\]

This expression first appears in Hardy and Littlewood’s paper on properties of the integral mean [13], but the mixed norm space was not explicitly defined until Flett’s works [9], [10]. Since then, these spaces have been studied by many authors (see [1], [5], [6], [11], [17]). Recently, the mixed norm spaces are mentioned in the works [2], [3], which are closely related to the main topic in this paper, see also the forthcoming monograph [14].

The mixed norm spaces form a family of complete spaces that contains the Hardy and Bergman spaces. In the references given, many properties of these spaces have been studied, such as pointwise growth (that will appear later in this work), duality, relation with coefficient multipliers and partial results on inclusions, but, to the best of our knowledge, a complete characterization of the inclusions between different spaces \( H(p, q, \alpha) \) has not been recorded.

In this paper we complete the table of inclusions between different mixed-norm spaces by finding a bound for the norm of the inclusion operator whenever an inclusion holds, and by giving explicit examples of functions to show that no inclusion takes place in some cases. For that, we prove some preliminary results, of interest by themselves, on mean and pointwise growth, norm of the point-evaluation functional and rate of decrease of the integral means.

From now on, we will understand \( 1/\infty \) as zero, the letters \( A, B, C, C', K, m \) will be positive constants, and we will say that two quantities are comparable, denoted by \( \alpha \approx \beta \), if there exist two positive constants \( C \) and \( C' \) such that

\[
C\alpha \leq \beta \leq C'\alpha.
\]
2. Preliminaries

Let $H(D)$ be the space of analytic functions on the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in H(D)$ and $r \in (0,1)$ let $M_p(r, f)$ be the integral mean

$$M_p(r, f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

if $0 < p < \infty$ and

$$M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})| \quad .$$

We consider the spaces $H(p, q, \alpha)$, $0 < p, q \leq \infty$, $0 < \alpha < \infty$, consisting of analytic functions on $D$ such that

$$\|f\|_{p, q, \alpha}^q = \alpha q \int_0^1 (1 - r)^{\alpha q - 1} M_p^q(r, f) \, dr < \infty,$$

if $q < \infty$, and

$$\|f\|_{p, \infty, \alpha} = \sup_{0 \leq r < 1} (1 - r)^{\alpha} M_p(r, f) < \infty.$$

For any $0 < p, q \leq \infty$, $0 < \alpha < \infty$ the space $H(p, q, \alpha)$ is a complete subspace of the space $L(p, q, \alpha)$ of measurable functions in $D$ (see [4]).

In particular, one can identify the weighted Bergman space $A^p_{\alpha}$, $0 < p < \infty$, $-1 < \alpha < \infty$, of analytic functions on the unit disk such that

$$\int_D |f(z)|^p (1 - |z|^2)^{\alpha} \, dA(z) < \infty$$

with the space $H\left(p, p, \frac{\alpha + 1}{p}\right)$ and the Hardy space $H^p$ of functions in $H(D)$ for which

$$\sup_{0 < r < 1} M_p(r, f) < \infty$$

with $H(p, \infty, 0)$. The mixed norm spaces are also related to other spaces of analytic functions, such as Besov and Lipschitz spaces, via fractional derivatives (see [14, Chapter 7]).

Familiar examples of analytic functions on the unit disk are the functions of type $(1 - z)^{-\gamma}$, with $\gamma$ a real constant. It is well known that such function is in the Hardy space $H^p$ if and only if $\gamma < 1/p$ and in the Bergman space $A^p$ if and only if $\gamma < 2/p$. The following lemma determines when these functions belong to $H(p, q, \alpha)$ (see [2]).

Lemma A. Let $0 < p \leq \infty$, $0 < \alpha < \infty$. The functions $f(z) = \frac{1}{(1-z)^\gamma}$ belong to $H(p, q, \alpha)$, $0 < q < \infty$, if and only if $\gamma < \alpha + 1/p$, and to $H(p, \infty, \alpha)$ if and only if $\gamma \leq \alpha + 1/p$.

Starting with these examples we can search for functions with faster growth for $z \in \mathbb{R}$, $0 < z < 1$. The following lemma gives us examples of functions which attain the critical exponent shown in the last lemma, but still belong to the space (see [2]).
Lemma B. Let $0 < p \leq \infty$, $0 < \alpha < \infty$. The functions
\[
f(z) = \frac{1}{(1-z)^{\alpha+1/p}} \left( \log \frac{e}{1-z} \right)^{-c}
\]
belong to $H(p,q,\alpha)$ if and only if $c > 1/q$ for $q < \infty$, and $c \geq 0$ for $q = \infty$.

Another well-known class of analytic functions is the class of lacunary series. Such series belongs to the Hardy space $H^p$ if and only if the sequence formed with its coefficients belongs to the $l^2$ space. In that case (and only then) the function has radial limits almost everywhere, and otherwise, has radial limits almost nowhere. The following result appears in [14, Thm. 8.1.1], based on [16].

Lemma C. Let $f(z) = \sum_{n=0}^{\infty} a_n z^{2^n-1}$ and $0 < p, q \leq \infty$, $0 < \alpha < \infty$. Then $f \in H(p,q,\alpha)$ if and only if \( \{ 2^{-n\alpha}a_n \} \in l^q \).

In particular, there are functions with radial limits almost nowhere in each $H(p,q,\alpha)$ with $\alpha > 0$ (for instance, the lacunary series with coefficients equal to 1 satisfies \( \sum_{n=0}^{\infty} 2^{-n\alpha}|a_n|^q < \infty \) for every $0 < p, q \leq \infty$, $0 < \alpha < \infty$, but \( \sum_{n=0}^{\infty} |a_n|^2 = \infty \)). Therefore, the Hardy space does not contain any $H(p,q,\alpha)$ with $\alpha > 0$ or $q > 0$.

3. Pointwise and mean estimates

If $f$ is a function in $H(p,q,\alpha)$, we have the following estimate for its integral means.

Lemma 1. If $f \in H(p,q,\alpha)$, $0 < p \leq \infty$, $0 < q, \alpha < \infty$, then
\[
M_p(r,f) = o \left( (1-r)^{-\alpha} \right)
\]
as $r \to 1$.

Proof. Since the integral
\[
\alpha q \int_0^r (1-\rho)^{\alpha q-1} M_p^q(\rho, f) \, d\rho
\]
converges to $\|f\|_{p,q,\alpha}^q$ as $r \to 1$, then for every $\varepsilon > 0$ there exists $r_0$ such that
\[
(3.1) \quad \alpha q \int_r^1 (1-\rho)^{\alpha q-1} M_p^q(\rho, f) \, d\rho < \varepsilon
\]
for every $r > r_0$. Therefore, since the integral means are increasing as functions of $r$, we get
\[
(1-r)^{\alpha q} M_p^q(r, f) = \alpha q \int_r^1 (1-\rho)^{\alpha q-1} M_p^q(r, f) \, d\rho 
\leq \alpha q \int_r^1 (1-\rho)^{\alpha q-1} M_p^q(\rho, f) \, d\rho < \varepsilon.
\]

Moreover, it follows from the proof that if $f \in H(p,q,\alpha)$, then
\[
(3.2) \quad M_p(r,f) \leq \frac{\|f\|_{p,q,\alpha}}{(1-r)^{\alpha}}
\]
since we can bound the integral in (3.1) by the norm of $f$ instead of $\varepsilon$.  

Although the result in Lemma 1 fails for \( q = \infty \) as the function \( f(z) = (1 - z)^{-\alpha - 1/p} \) shows, the above bound for the integral mean still holds since

\[
\|f\|_{p,\infty,\alpha} = \sup_{0 \leq \rho < 1} (1 - \rho)^{\alpha} M_p(\rho, f) \geq (1 - r)^{\alpha} M_p(r, f)
\]

for any \( r, 0 < r < 1 \), and therefore

\[
M_p(r, f) \leq \|f\|_{p,\infty,\alpha} (1 - r)^{\alpha}
\]

for \( f \in H(p, \infty, \alpha) \).

For the Bergman spaces \( A^p \), besides the well-known big-Oh growth inequality, we have the estimate \( |f(z)| = o((1 - |z|)^{q+1/p}) \) as \( |z| \to 1 \) for every \( f \in A^p \). This is a consequence of the subharmonicity of \( |f|^p \) and the inequality:

\[
\int_{D(a,r)} |f(z)|^p \, dA(z) \leq \int_D |f(z)|^p \, dA(z) = \|f\|_{A^p}^p
\]

for \( a \in \mathbb{D} \) and \( r < 1 \) (see [8, Page 7]). We can obtain an analogous result for the Hardy spaces with similar techniques that cannot be used in the mixed norm spaces. However, the result still holds, as we shall show next.

**Proposition 1.** If \( f \in H(p, q, \alpha) \), \( 0 < p \leq \infty \), \( 0 < q, \alpha < \infty \), then

\[
|f(z)| = o((1 - |z|)^{\alpha + 1/p})
\]

as \( |z| \to 1 \).

In the proof we will use the following identity.

**Lemma 2.** For \( 0 < p, q, \alpha < \infty \) and \( z \in \mathbb{D} \),

\[
\int_{|z|}^1 (1 - \rho)^{\alpha q - 1}(\rho - |z|)^{q/p} \, d\rho = B(\alpha q, q/p + 1)(1 - |z|)^{\alpha q + q/p},
\]

where \( B(a, b) = \int_0^1 (1 - x)^{a-1} x^{b-1} \, dx \), \( a, b > 0 \), is the Beta function.

**Proof.** With the change of variables \( x = \frac{\rho - |z|}{1 - |z|} \),

\[
\int_{|z|}^1 (1 - \rho)^{\alpha q - 1}(\rho - |z|)^{q/p} \, d\rho = \int_0^1 (1 - x)^{\alpha q - 1}(1 - |z|)^{\alpha q - 1} x^{q/p} (1 - |z|)^{q/p} (1 - |z|) \, dx
\]

\[
= (1 - |z|)^{\alpha q + q/p} \int_0^1 (1 - x)^{\alpha q - 1} x^{q/p} \, dx.
\]

Next, we prove Proposition 1.

**Proof of Proposition 1.** If \( p = \infty \), it is easy to see that, for \( r \) close enough to 1 (as in Lemma 1),

\[
|f(re^{i\theta})|^q (1 - r)^{\alpha q} = \alpha q |f(re^{i\theta})|^q \int_r^1 (1 - \rho)^{\alpha q - 1} \, d\rho
\]

\[
\leq \alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_{\infty}^q(\rho, f) \, d\rho < \varepsilon.
\]

If \( 0 < p < \infty \), we estimate the integral mean \( M_p(r, f) \), using the Poisson integral:
Let $\rho \in (0, 1)$ and define $f_\rho(z) = f(\rho z)$, for $f \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Since $f_\rho \in H^\infty$ for any $f \in H(p, q, \alpha)$, we have, as in [11],

$$|f(re^{i\theta})|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \frac{\rho^2 - r^2}{|\rho - re^{i(\theta - t)}|^2} dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \frac{\rho^2 - r^2}{(\rho - r)^2} dt$$

$$\leq \frac{2}{\rho - r} \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p dt = \frac{2}{\rho - r} M_p^p(\rho, f).$$

Hence,

$$|f(re^{i\theta})|(\rho - r)^{1/p} \leq 2^{1/p} M_p(\rho, f).$$

Using the identity in Lemma 2 as in (3.1), for $\varepsilon$ small enough,

$$\frac{\alpha q}{2q/p} B(\alpha q, q/p + 1) |f(re^{i\theta})|^q (1 - r)^{\alpha q + q/p}$$

$$= \frac{\alpha q}{2q/p} |f(re^{i\theta})|^q \int_r^1 (1 - \rho)^{\alpha q - 1} (\rho - r)^{q/p} d\rho$$

$$\leq \alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) d\rho < \varepsilon.$$
Proposition 2. For \( z \in \mathbb{D} \), \( 0 < p, q \leq \infty \), \( 0 < \alpha < \infty \) and \( s > 0 \), the functions

\[
f_z(w) = \frac{(1 - |z|^2)^s}{(1 - \zeta w)^{\alpha + \frac{s}{2} + s}}
\]

satisfy \( |f_z(z)| \approx \|\phi_z\| \) and \( \|f_z\|_{p,q,\alpha} \approx 1 \).

Proof. First we check that \( f_z \) belongs to \( H(p, q, \alpha) \) and estimate its norm:

If \( w = re^{i\theta} \), then (see [7, Page 65])

\[
M^p_p(r, f_z) = \int_0^{2\pi} \frac{(1 - |z|^2)^{ps}}{|1 - zre^{i\theta}|^{p(\alpha + \frac{s}{2} + s)}} \frac{d\theta}{2\pi} \approx \frac{(1 - |z|^2)^{ps}}{(1 - r|z|)^{p(\alpha + \frac{s}{2} + s) - 1}} = \frac{(1 - |z|^2)^{ps}}{(1 - r|z|)^{(\alpha + s)p}}
\]

for \( p < \infty \) and

\[
M_\infty(r, f_z) \approx \frac{(1 - |z|^2)^{s}}{(1 - r|z|)^{\alpha + s}}.
\]

Therefore, if \( q < \infty \) and \( 0 < p \leq \infty \),

\[
\|f_z\|_{q,p,\alpha}^q = \alpha q \int_0^1 (1 - r)^{aq-1} M^p_p(r, f_z) \, dr
\]

\[
\approx \alpha q (1 - |z|)^q \int_0^1 (1 - r)^{aq-1} \frac{1}{(1 - r|z|)^{(\alpha + s)q}} \, dr.
\]

Now, on the one hand,

\[
\|f_z\|_{p,q,\alpha}^q \approx \alpha q (1 - |z|)^q \int_0^1 (1 - r)^{aq-1} \frac{1}{(1 - r|z|)^{(\alpha + s)q}} \, dr
\]

\[
\geq \alpha q (1 - |z|)^q \int_0^1 (1 - r)^{aq-1} \frac{1}{(1 - r|z|)^{(\alpha + s)q}} \, dr
\]

\[
\geq \alpha q \frac{(1 - |z|)^q}{(1 - |z|^2)^{(\alpha + s)q}} \int_0^1 (1 - r)^{aq-1} \, dr
\]

\[
\approx \frac{1}{(1 - |z|)^aq} (1 - |z|)^aq = 1
\]

and, on the other hand, integrating by parts and using \((1 - r)^{aq} \leq (1 - r|z|)^{aq}\),

\[
\|f_z\|_{p,q,\alpha}^q \approx (1 - |z|)^q \int_0^1 \alpha q (1 - r)^{aq-1} \frac{1}{(1 - r|z|)^{(\alpha + s)q}} \, dr
\]

\[
= (1 - |z|)^q \left( 1 - (\alpha + s)q |z| \int_0^1 (1 - r)^{aq} \frac{1}{(1 - r|z|)^{(\alpha + s)q+1}} \, dr \right)
\]

\[
\leq (1 - |z|)^q \left( 1 - (\alpha + s)q |z| \int_0^1 (1 - r|z|)^{-aq+1} \, dr \right)
\]

\[
= (1 - |z|)^q \left( 1 - \frac{\alpha + s}{s} (1 - (1 - |z|)^{-aq} - 1) \right)
\]

\[
= \left( 1 - \frac{\alpha + s}{s} \right) (1 - |z|)^aq - \frac{\alpha + s}{s} \approx 1.
\]

If \( q = \infty \),

\[
\|f_z\|_{\infty,\alpha} = \sup_{0 \leq r < 1} (1 - r)^{\alpha} M_p(r, f_z) \approx \sup_{0 \leq r < 1} (1 - r)^{\alpha} \frac{(1 - |z|^2)^s}{(1 - r|z|)^{\alpha + s}}.
\]
A CHARACTERIZATION OF THE INCLUSIONS BETWEEN MIXED NORM SPACES

Since $1 - r \leq 1 - r|z|$ and $1 - |z| \leq 1 - r|z|$, we have
\[
\|f_z\|_{p,\infty,\alpha} \approx \sup_{0 \leq r < 1} (1 - r)^{\alpha} \frac{(1 - |z|)\alpha}{(1 - r|z|)\alpha + s} \leq \sup_{0 \leq r < 1} (1 - r|z|)\alpha \frac{(1 - r|z|)\alpha}{(1 - r|z|)\alpha + s} = 1
\]

hence
\[
\|f_z\|_{p,\infty,\alpha} \geq \sup_{|z| < r < 1} (1 - r)^{\alpha} \frac{(1 - |z|)\alpha}{(1 - r|z|)\alpha + s} \geq \frac{(1 - |z|)\alpha}{(1 - |z|^2)\alpha + s} \sup_{|z| < r < 1} (1 - r)^{\alpha}
\]
\[
= \frac{(1 - |z|)\alpha(1 - |z|)\alpha}{(1 - |z|^2)\alpha + s} \approx 1.
\]

Now that we know that $f_z \in H(p,q,\alpha)$, we see easily that
\[
|f_z(z)| = \frac{(1 - |z|^2)^s}{(1 - |z|^2)^{\alpha + \frac{1}{p} + s}} = \frac{1}{(1 - |z|^2)^{\alpha + \frac{1}{p}}} \approx \frac{1}{(1 - |z|)^{\alpha + \frac{1}{p}}}
\]
and from here
\[
\|\phi_z\| \approx \|\phi_z\||f_z||_{p,q,\alpha} \geq |f_z(z)| \approx \frac{1}{(1 - |z|)^{\alpha + \frac{1}{p}}}.
\]

With (3.9), we get
\[
|f_z(z)| \approx \|\phi_z\| \approx \frac{1}{(1 - |z|)^{\alpha + \frac{1}{p}}}.
\]

\[]

4. INCLUSIONS BETWEEN MIXED NORM SPACES

The main theorems in this work, which characterize completely the inclusions between mixed norm spaces, are the following. To avoid repetitions, we recall here that we are assuming our parameters to be $0 < \alpha, \beta < \infty$ and $0 < p, q, u, v \leq \infty$.

**Theorem 1.** If $p \geq u$, then
\[
H(p,q,\alpha) \subseteq H(u,v,\beta) \iff \begin{cases} 
\alpha < \beta \\
\alpha = \beta \quad \text{or} \\
q \leq v.
\end{cases}
\]

**Theorem 2.** If $p < u$, then
\[
H(p,q,\alpha) \subseteq H(u,v,\beta) \iff \begin{cases} 
\alpha + \frac{1}{p} < \beta + \frac{1}{u} \\
\alpha + \frac{1}{p} = \beta + \frac{1}{u} \quad \text{and} \\
q \leq v.
\end{cases}
\]

It is worth noticing that we need $\alpha$ to be greater than zero as we stated when these spaces were defined. In the limit case $\alpha = 0$, by a theorem by Hardy and Littlewood (related to the Isoperimetric Inequality, see [15], [18]), we have $H^p \subseteq A^{2p}$. That is, $H(p,\infty,0) \subseteq H(2p,2p,1/2p)$, although these parameters do not satisfy Theorem 2.

Notice also that it is only to be expected that the relation between the spaces would depend on the relation between the parameters $p$ and $u$, since, ultimately, in order to compare the different spaces we need to compare the sizes of the integral means. In turn, the integral means relate in a different fashion according to the parameters $p$ and $u$.

Therefore, in order to prove these theorems we will need the following estimates of the integral means, which can be found in the literature (see [5], [7, Thm. 5.9], [13]).
Lemma D. If $f \in H(p, q, \alpha)$ and $q \leq v < \infty$, then
\[
M_p^v(r, f) \leq \|f\|_{p, q, \alpha} (1 - r)^{-\alpha(v-q)} M_p^q(r, f).
\]

Proof. If $f \in H(p, q, \alpha)$, by the bound on the integral mean (3.2)
\[
M_p(r, f) \leq \|f\|_{p, q, \alpha} (1 - r)^{-\alpha},
\]
and since $q \leq v < \infty$,
\[
M_p^v(r, f) = M_p^v(r, f) M_p^q(r, f) \leq \|f\|_{p, q, \alpha} (1 - r)^{-\alpha(v-q)} M_p^q(r, f).
\]

Lemma E. If $f \in H(p, q, \alpha)$ and $p < u$, then
\[
M_u(r, f) \leq m^{1-\frac{u}{p}} \|f\|_{p, q, \alpha} (1 - r)^{-\alpha + \frac{1}{p} - \frac{u}{p}},
\]
where
\[
m = \frac{2^{1/p}}{(ao B(aq, q/p + 1))^{1/q}}.
\]

Proof. The pointwise inequality (3.7)
\[
M_\infty(r, f) \leq m \|f\|_{p, q, \alpha} (1 - r)^{-\alpha - \frac{u}{p}}
\]
is the case $u = \infty$. Now if $u < \infty$,
\[
(4.1) \quad M_u(r, f) = \left( \int_0^1 |f(re^{i\theta})|^{u-p} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/u} \leq M_\infty^{1-\frac{u}{p}}(r, f) M_p^{\frac{u}{p}}(r, f)
\]
\[
\leq m^{1-\frac{u}{p}} \|f\|_{p, q, \alpha} (1 - r)^{(1-\frac{u}{p})(-\alpha - \frac{1}{p})} \|f\|_{p, q, \alpha} (1 - r)^{-\alpha + \frac{1}{p}}
\]
\[
= m^{1-\frac{u}{p}} \|f\|_{p, q, \alpha} (1 - r)^{-\alpha + \frac{1}{p} - \frac{u}{p}}.
\]

Proposition 3. If $f \in H(p, q, \alpha)$ and $p < u$, then for every constant $C > 1$ there exists $r_0$, $0 < r_0 < 1$, such that for every $r > r_0$ we have
\[
M_u(r, f) \leq K (1 - r)^{\frac{1}{p} - \frac{u}{p}} M_p(r, f),
\]
with $K = m^{1-\frac{u}{p}} C$.

To prove this result we will need the next lemma, which gives us a bound for the norm of $f$ involving its integral mean. We already know the inequality
\[
(1 - r)^{\alpha} M_p(r, f) \leq \|f\|_{p, q, \alpha}
\]
for every $r$, $0 < r < 1$. Notice that, taking supremum over $r$, we get
\[
(4.2) \quad \|f\|_{p, \infty, \alpha} \leq \|f\|_{p, q, \alpha},
\]
and therefore $H(p, q, \alpha) \subseteq H(p, \infty, \alpha)$ for every $0 < p, q \leq \infty$, $0 < \alpha < \infty$. 

Lemma 3. If \( f \in H(p, q, \alpha) \), then for every constant \( C > 1 \) there exists \( r_0 \), \( 0 < r_0 < 1 \), such that
\[
\| f \|_{p, q, \alpha} \leq C(1 - r)^{\alpha} M_p(r, f)
\]
for every \( r > r_0 \).

Proof. If \( q = \infty \), the result is clear. If \( 0 < q < \infty \),
\[
\| f \|_{p, q, \alpha}^q = \alpha q \int_0^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho
\]
\[
= \alpha q \int_0^r (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho + \alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho
\]
\[
\leq (1 - r)^{\alpha q} M_p^q(r, f) + \alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho.
\]
Since the integral \( \alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho \) tends to zero as \( r \) goes to 1, for every constant \( C' < 1 \) there exists \( r_0 \), \( 0 < r_0 < 1 \), such that for every \( r > r_0 \)
\[
\alpha q \int_r^1 (1 - \rho)^{\alpha q - 1} M_p^q(\rho, f) \, d\rho \leq C' \| f \|_{p, q, \alpha}^q.
\]
It follows that
\[
(1 - C')^{1/q} \| f \|_{p, q, \alpha} \leq (1 - r)^{\alpha} M_p(r, f)
\]
and
\[
\| f \|_{p, q, \alpha} \leq C(1 - r)^{\alpha} M_p(r, f)
\]
with \( C = \frac{1}{(1 - C')^{1/q}} \).

Proof of Proposition 3. First we shall prove the statement for \( u = \infty \) : If \( f \in H(p, q, \alpha) \), by Lemma 3, we have that
\[
|f(re^{i\theta})| \leq m \frac{\| f \|_{p, q, \alpha}}{(1 - r)^{\alpha + \frac{1}{q}}}
\]
According to Lemma 3 for every \( C > 1 \) there exists \( r_0 \) such that for \( r > r_0 \) we have
\[
|f(re^{i\theta})| \leq m \frac{C(1 - r)^{\alpha} M_p(r, f)}{(1 - r)^{\alpha + \frac{1}{q}}} = m C(1 - r)^{-\frac{1}{q}} M_p(r, f),
\]
and hence
\[
M_{\infty}(r, f) \leq m C(1 - r)^{-\frac{1}{q}} M_p(r, f).
\]
Now let \( u \) be an arbitrary parameter such that \( u > p \). For \( r > r_0 \), as in Proposition 3,
\[
M_u(r, f) \leq M_{\infty}^{1 - \frac{u}{p}}(r, f) M_p^{\frac{u}{p}}(r, f) \leq m^{1 - \frac{u}{p}} C^{1 - \frac{u}{p}} (1 - r)^{\frac{1}{p} - \frac{1}{q}} M_p(r, f).
\]

Proof of Theorem 1. Throughout this proof, we will assume that \( p \geq u \). The key to proving the sufficiency is the inequality of the integral means: if \( p \geq u \), then \( M_u(r, f) \leq M_p(r, f) \).
We suppose first that \( \alpha < \beta \). Then, since \( M_p(r, f) \leq \| f \|_{p,q,\alpha}(1 - r)^{-\alpha} \) by (3.2), we have that, if \( v \) is finite,
\[
\| f \|_{u,v,\beta}^v = \beta v \int_0^1 (1 - r)^{\beta v - 1} M_p^v(r, f, v) \, dr \leq \beta v \int_0^1 (1 - r)^{\beta v - 1} M_p^{v\alpha}(r, f, v) \, dr
\]
\[
\leq \beta v \| f \|_{p,q,\alpha}^v \int_0^1 (1 - r)^{\beta v - 1} (1 - r)^{-\alpha v} \, dr = \beta v \| f \|_{p,q,\alpha}^v \int_0^1 (1 - r)^{\alpha(\beta - \alpha) - 1} \, dr = \frac{\beta}{\beta - \alpha} \| f \|_{p,q,\alpha}^v
\]
if \( v < \infty \), and, by (4.2),
\[
\| f \|_{u,\infty,\beta} = \sup_{0 < r < 1} (1 - r)^{\beta} M_u(r, f, \beta) \leq \sup_{0 < r < 1} (1 - r)^{\alpha} M_p(r, f, \alpha) = \| f \|_{p,\infty,\alpha} \leq \| f \|_{p,q,\alpha}.
\]

Therefore, \( f \in H(u, v, \beta) \) for every \( f \in H(p, q, \alpha) \). Now, if \( \alpha = \beta \) and \( q \leq v \), by Lemma [D]
\[
\| f \|_{u,v,\beta}^v = \beta v \int_0^1 (1 - r)^{\beta v - 1} M_p^v(r, f, v) \, dr \leq \beta v \int_0^1 (1 - r)^{\beta v - 1} M_p^{vq}(r, f, v) \, dr
\]
\[
\leq \beta v \| f \|_{p,q,\alpha}^v \int_0^1 (1 - r)^{\beta v - 1} (1 - r)^{-\alpha q} M_p^{q}(r, f, q) \, dr
\]
\[
= \beta v \| f \|_{p,q,\alpha}^v \int_0^1 (1 - r)^{\alpha q - 1} M_p^{q}(r, f, q) \, dr = \frac{\beta v}{\alpha q} \| f \|_{p,q,\alpha}^v \| f \|_{p,q,\alpha}^q = \frac{v}{q} \| f \|_{p,q,\alpha}^v.
\]
if \( v < \infty \), and, again by (4.2),
\[
\| f \|_{u,\infty,\beta} = \sup_{0 < r < 1} (1 - r)^{\beta} M_u(r, f, \beta) \leq \sup_{0 < r < 1} (1 - r)^{\alpha} M_p(r, f, \alpha) = \| f \|_{p,\infty,\alpha} \leq \| f \|_{p,q,\alpha}.
\]

Hence, in both cases \( H(p, q, \alpha) \subseteq H(u, v, \beta) \), and the sufficiency is proven.

For the necessity, we need to see that \( H(p, q, \alpha) \not\subseteq H(u, v, \beta) \) when the parameters do not relate as in the statement of the theorem. For this, consider a function of type \( f(z) = \sum_{n=1}^{\infty} a_n z^{2^{n-1}} \) as in Lemma [C]. Recall that \( f \) belongs to \( H(p, q, \alpha) \) if and only if \( \{2^{-\alpha n} a_n\} \in l^q \).

If \( \alpha > \beta \), let \( f(z) = \sum_{n=0}^{\infty} 2^{\beta n} z^{2^{n-1}} \). Since
\[
\left\{ \frac{a_n}{2^{\alpha n}} \right\} = \left\{ \frac{1}{2^{n(\alpha - \beta)}} \right\} \in l^q,
\]
the function \( f \) belongs to \( H(p, q, \alpha) \), but
\[
\left\{ \frac{a_n}{2^{\beta n}} \right\} = \{1\} \not\in l^v,
\]
so this function does not belong to \( H(u, v, \beta) \), and therefore \( H(p, q, \alpha) \not\subseteq H(u, v, \beta) \) if \( \alpha > \beta \).

If \( \alpha = \beta \) and \( q > v \), we take \( f(z) = \sum_{n=0}^{\infty} 2^{\alpha n} n^{-1/v} z^{2^{n-1}} \). Similarly,
\[
\left\{ \frac{a_n}{2^{\alpha n}} \right\} = \left\{ \frac{2^{\alpha n} n^{-1/v}}{2^{\alpha n}} \right\} = \left\{ \frac{1}{n^{1/v}} \right\} \in l^q,
\]
and \( f \in H(p,q,\alpha) \), but
\[
\left\{ \frac{a_n}{2^{\beta n}} \right\} = \left\{ \frac{2^{\alpha n}}{2^{\beta n}} \right\} = \left\{ \frac{1}{n^{1/v}} \right\} \not\subset I^v,
\]
so it does not belong to \( H(u,v,\beta) \), and hence \( H(p,q,\alpha) \not\subset H(u,v,\beta) \) if \( \alpha = \beta \) and \( q > v \).

\( \square \)

**Proof of Theorem** As in the last proof, from now on we will assume \( p < u \).

Firstly we shall see that if \( f \in H(p,q,\alpha) \) and the parameters are ordered as in the statement, then \( f \in H(u,v,\beta) \).

If \( \alpha < \beta + \frac{1}{u} - \frac{1}{p} \), by Lemma [E]
\[
\|f\|_{u,v,\beta} = \beta v \int_0^1 (1-r)^{\beta v-1} M_u^v(r,f) \, dr
\]
\[
\leq \beta v m_v^{(1-\frac{1}{p})} \|f\|_{p,q,\alpha} \int_0^1 (1-r)^{\beta v-1} (1-r)^{\alpha + \frac{1}{p} - \frac{1}{u}} \, dr
\]
\[
= \beta v m_v^{(1-\frac{1}{p})} \|f\|_{p,q,\alpha} \int_0^1 (1-r)^{v(\beta-\alpha + \frac{1}{p} - \frac{1}{u}) - 1} \, dr = \frac{\beta m_v^{(1-\frac{1}{p})}}{\beta - \alpha + \frac{1}{p} - \frac{1}{u}} \|f\|_{p,q,\alpha}
\]
for \( v < \infty \), and
\[
\|f\|_{u,\infty,\beta} = \sup_{0 \leq r < 1} (1-r)^\beta M_u(r,f)
\]
\[
\leq m_1^{-\frac{1}{p}} \|f\|_{p,q,\alpha} \sup_{0 \leq r < 1} (1-r)^{\beta (1-r)^{-\alpha + \frac{1}{p} - \frac{1}{u}}} = m_1^{-\frac{1}{p}} \|f\|_{p,q,\alpha}.
\]

If \( \alpha = \beta + \frac{1}{u} - \frac{1}{p} \) and \( q \leq v \), we choose an arbitrary \( C > 1 \) and let \( r_0 \) be as in Proposition [E]. Then, for every \( r > r_0 \) we have if \( v < \infty \),
\[
\|f\|_{u,v,\beta} = \beta v \int_0^1 (1-r)^{\beta v-1} M_u^v(r,f) \, dr
\]
\[
= \beta v \int_0^r (1-r)^{\beta v-1} M_u^v(r,f) \, dr + \beta v \int_r^1 (1-r)^{\beta v-1} M_u^v(r,f) \, dr
\]
\[
\leq (1-r)^{\beta v} M_u^v(r,f) + \beta v m_v^{(1-\frac{1}{p})} C^v \int_r^1 (1-r)^{\beta v-1} (1-r)^{\frac{1}{p} - \frac{1}{u}} M_p^v(r,f) \, dr
\]
\[
= A(r) + \beta v m_v^{(1-\frac{1}{p})} C^v \int_r^1 (1-r)^{\alpha v-1} M_p^v(r,f) \, dr,
\]
and by Lemma [D]
\[
\|f\|_{u,v,\beta} \leq A(r) + \beta v m_v^{(1-\frac{1}{p})} C^v \|f\|_{p,q,\alpha} \int_0^1 (1-r)^{\alpha v-1} (1-r)^{-\alpha (v-q)} M_p^v(r,f) \, dr
\]
\[
\leq A(r) + \beta v m_v^{(1-\frac{1}{p})} C^v \|f\|_{p,q,\alpha} \int_0^1 (1-r)^{\alpha v-1} M_p^v(r,f) \, dr
\]
\[
= A(r) + \frac{\beta v}{\alpha q} m_v^{(1-\frac{1}{p})} C^v \|f\|_{p,q,\alpha} = A(r) + \frac{\beta v}{\alpha q} m_v^{(1-\frac{1}{p})} C^v \|f\|_{p,q,\alpha}.
\]

Taking, in particular, \( r = (1+r_0)/2 \), it follows that \( A(r) = A((1+r_0)/2) \leq A \) and
\[
\|f\|_{u,v,\beta} \leq A + \frac{\beta v}{\alpha q} m_v^{(1-\frac{1}{p})} C^v \|f\|_{p,q,\alpha}.
\]
If \( v = \infty \), in a similar way, using Proposition 3 and (4.2), for any \( C > 1 \) and for \( r > r_0 \) we obtain

\[
\|f\|_{u, \infty, \beta} = \sup_{0 \leq \rho < 1} (1 - \rho)^{\frac{1}{p} - \frac{1}{q}} M_p(\rho, f) \leq A(r) + m^{1-\frac{1}{q}} C \|f\|_{p, q, \alpha}.
\]

Finally, we need to see that \( H(p, q, \alpha) \not\subseteq H(u, v, \beta) \) when the parameters are not as in the assumptions of the statement. If \( \alpha + \frac{1}{p} > \beta + \frac{1}{u} \), Lemma A tells us that

\[
f(z) = \frac{1}{(1 - z)^{\frac{1}{p} + 1/v}}
\]

belongs to \( H(p, q, \alpha) \) but not \( H(u, v, \beta) \), and this proves that \( H(p, q, \alpha) \not\subseteq H(u, v, \beta) \) when \( \alpha + \frac{1}{p} > \beta + \frac{1}{v} \).

If \( \alpha + \frac{1}{p} = \beta + \frac{1}{u} \) and \( q > v \), by Lemma B the function

\[
f(z) = \frac{1}{(1 - z)^{\alpha + 1/p} \left( \log \frac{e}{1 - z} \right)^{-1/v}}
\]

is an example of a function in \( H(p, q, \alpha) \) which is not in \( H(u, v, \beta) \), and hence \( H(p, q, \alpha) \not\subseteq H(u, v, \beta) \) for \( \alpha + \frac{1}{p} = \beta + \frac{1}{u} \) and \( q > v \).

\[\square\]

References

1. P. Ahern and M. Jevtić, Duality and multipliers for mixed norm spaces, *Michigan Math. J.* 30 (1983), 53-64.
2. K. L. Avetisyan, A note on mixed norm spaces of analytic functions, *Aust. J. Math. Anal. Appl.* 9 (2012), 1-6.
3. K. L. Avetisyan, Sharp inclusions and lacunary series in mixed-norm spaces on the polydisc, *Complex Var. Elliptic Equ.* 58 (2013), 185-195.
4. A. Benedek and R. Panzone, The space \( L^p \), with mixed norm, *Duke Math. J.* 28 (1961), 301-324.
5. O. Blasco, Multipliers on spaces of analytic functions, *Canad. J. Math.* 47 (1995), 44-64.
6. S. M. Buckley, P. Koskela and D. Vukotić, Fractional integration, differentiation, and weighted Bergman spaces, *Math. Proc. Cambridge Philos. Soc.* 126 (1999), 369-385.
7. P. L. Duren, *Theory of Hp Spaces*, American Mathematical Society, Providence, RI 2004.
8. P. L. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, Providence, RI 2004.
9. T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* 38 (1972), 746-765.
10. T. M. Flett, Lipschitz spaces of functions on the circle and the disk, *J. Math. Anal. Appl.* 39 (1972), 125-158.
11. S. Gadbois, Mixed-norm generalizations of Bergman spaces and duality, *Proc. Amer. Math. Soc.* 104 (1988), 1171-1180.
12. G. H. Hardy and J. E. Littlewood, A convergence criterion for Fourier series, *Math. Z.* 28 (1928), 612-634.
13. G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. II., *Math. Z.* 34 (1932), 403-439.
14. M. Jevtić, D. Vukotić and M. Arsenović, *Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-type Spaces*, to appear.

15. M. Mateljević, The isoperimetric inequality and some extremal problems in $H^1$, *Lecture Notes in Math.* 798 (1980), 364-369.

16. M. Mateljević and M. Pavlović, $L^p$-behavior of power series with positive coefficients and Hardy spaces, *Proc. Amer. Math. Soc.* 87 (1983), 309-316.

17. W. T. Sledd, Some results about spaces of analytic functions introduced by Hardy and Littlewood, *J. London Math. Soc.* 9 (1974/75), 328–336.

18. D. Vukotić, The isoperimetric inequality and a theorem of Hardy and Littlewood, *Amer. Math. Monthly* 110 (2003), 532-536.

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: irina.arevalo@uam.es