A diagrammatic calculus of $n$-term syllogisms

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Abstract
We extend the diagrammatic calculus of syllogisms introduced in [9] to the general case of $n$-term syllogisms, showing that the valid ones are exactly those whose conclusion follows by calculation. Moreover, by pointing out the existing connections with the theory of rewriting systems we will also single out a suitable category theoretic framework for the calculus.

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1 Introduction
The main aims of the present paper are on one hand that of extending to the $n$-term case the diagrammatic calculus of syllogisms introduced in [9], where we dealt with the basic 3-term case and on the other hand that of single out a suitable category theoretic framework for the calculus itself. To the author’s knowledge, another diagrammatic approach based on directed graphs already exists, see [11], whereas for a category theoretic point of view, the reader may consult [5]. In section 2 we briefly recall the basics on syllogisms and the the diagrammatic calculus we hinted at above. In section 3, we will deal with $n$-term syllogisms and prove that the calculus extends to them, by showing in turn that the valid $n$-term syllogisms are exactly those whose conclusion follows from their premisses by calculation. Moreover, we will also retrieve the well-known result that the valid $n$-term syllogisms are $3n^2 − n$.

In section 4, we will point out the existing connections with the theory of rewriting systems, by approaching them through polygraphs, mainly referring to [2].

2 Preliminaries on syllogisms
We will refer to nouns, adjectives or more complicated expressions of the natural language as to terms, generically, and denote them by using upper case letters which we will also call term-variables. The first systematization of syllogistic is due to Aristotle. Since him the following four kinds of propositions were recognized as fundamental throughout the research in logic:

- $\mathbf{A}_{AB}$: All A is B (universal affirmative proposition)
- $\mathbf{E}_{AB}$: No A is B (universal negative proposition)
- $\mathbf{I}_{AB}$: Some A is B (particular affirmative proposition)
O_{AB}: Some A is not B (particular negative proposition)

Following the tradition that is, loosely, the medieval systematization of syllogistic, we will henceforth refer them to as **categorical propositions**. In each of them, A denotes the subject whereas B denotes the predicate of the corresponding proposition.

A *syllogism* is a rule of inference that involves three categorical propositions that are distinguished by referring them to as *first premise*, *second premise* and *conclusion*. Moreover, a syllogism involves exactly three term-variables S, P and M in the following precise way: M does not occur in the conclusion whereas, according to the tradition, P occurs in the first premise and S occurs in the second premise. The term-variables S and P occur as the subject and predicate of the conclusion, respectively, and are also referred to as *minor term* and *major term* of the syllogism, whereas M is also referred to as *middle term*.

The *mood* of a syllogism is the sequence of the kinds of categorical propositions by which it is formed, whereas its *figure* is the position of the term-variables S, P and M in it. There are four possible figures, as shown in the table

| first premise | fig. 1 | fig. 2 | fig. 3 | fig. 4 |
|---------------|--------|--------|--------|--------|
| MP           |        |        |        |        |
| SM           |        |        |        |        |
| SP           |        |        |        |        |

and a syllogism is completely determined by its mood and by its figure together. We write syllogisms so that their mood and figure can be promptly retrieved and let the symbol |= separate the premisses from the conclusion. For example, in the syllogism

$$ A_{MP}, A_{SM} \models A_{SP} $$

it is possible to recognize from left to right the first premise, the second premise and the conclusion, moreover the mood, which is AAA, and the figure which is the first one. The combination of the moods and figures gives rise to 256 syllogisms in total, of which only 24 are *valid*, that is such that the verification of the premisses necessarily entails the verification of the conclusion. Venn diagrams can be used to verify the validity of syllogisms, see [10] for example. We hasten to say that of the 24 valid syllogisms, 9 are valid under suitable additional assumptions and will be henceforth referred to as *syllogisms with assumption of existence* for reasons that will be cleared later on, whereas the remaining 15 are valid without any further assumption and in the present section we continue refer them to as syllogisms, simply. These are the ones listed in the table

| fig. 1 | fig. 2 | fig. 3 | fig. 4 |
|--------|--------|--------|--------|
| AAA    | EAE    | IAI    | AEE    |
| EAE    | AEE    | AII    | IAI    |
| AII    | EIO    | OAO    | EIO    |
| EIO    | AOO    | EIO    | EIO    |

Now, for the previously cited diagrammatic calculus of syllogisms, graphical representations of the categorical propositions are correspondingly given, that is

$$ A \xrightarrow{A_{AB}} B \quad A \xrightarrow{E_{AB}} B $$

$$ A \xleftarrow{I_{AB}} B \quad A \xleftarrow{O_{AB}} B $$

2
which will be henceforth referred to as Aristotelian diagrams. Each of them has a corresponding dual, namely

\[
\begin{align*}
\text{A} & \quad \text{A} \\
\text{A} & \quad \text{B} \\
\text{A} & \quad \text{B}
\end{align*}
\]

Two or more Aristotelian diagrams, and their duals, can be concatenated and reduced, if possible. In such a concatenation, a reduction applies by formally composing two or more consecutive and accordingly oriented arrow symbols separated by a single term-variable, thus deleting it. Such a reduction will be henceforth referred to as syllogistic inference. By means of syllogistic inferences, Aristotelian diagrams can be used to verify the validity of syllogisms. This is obtained by using three Aristotelian diagrams, as the first premise, the second premise, and the conclusion of the syllogism. Moreover, these involve three distinguished term-variables, denoted \( S \), \( P \) and \( M \), in such a way that \( M \) occurs in both the Aristotelian diagrams in the premisses and does not in the conclusion, whereas \( S \) and \( P \) occur in the conclusion as well as in the premisses. Following the tradition, \( P \) will occur in the first premise whereas \( S \) in the second. Syllogistic inferences will be represented by diagrams filled in with the symbol \( | \) upside down, so to explicitly underline the fact that the notion of syllogistic inference is a directed one but also written in line, by letting \( \# \) denote the concatenation of Aristotelian diagrams. Thus, for example, the syllogistic inference associated with the valid syllogisms (2) can be either diagrammatically represented as

\[
\begin{align*}
\text{S} & \quad \text{A} \\
\text{A} & \quad \text{M} \\
\text{A} & \quad \text{P}
\end{align*}
\]

or written as

\[
(A_{MP}) \# (A_{SM}) \vdash (A_{SP}).
\]

Validity of syllogism

\[
A_{PM}, E_{MS} \models E_{SP} \tag{4}
\]

is witnessed by the existence of a syllogistic inference reducing the concatenation of the Aristotelian diagrams for its premisses to the Aristotelian diagram for its conclusion. The concatenation of the Aristotelian diagrams for the premisses of (4) is

\[
(A_{PM}) \# (E_{MS}) \tag{5}
\]

that is

\[
\begin{align*}
\text{S} & \quad \text{E} \\
\text{A} & \quad \text{M} \\
\text{A} & \quad \text{P}
\end{align*}
\]

whereas in its entirety the syllogistic inference can be written as

\[
(A_{PM}) \# (E_{MS}) \vdash (E_{SP}) \tag{5}
\]

or represented by the diagram

\[
\begin{align*}
\text{S} & \quad \text{E} \\
\text{A} & \quad \text{M} \\
\text{A} & \quad \text{P}
\end{align*}
\]
to produce evidence for (5), since in (6) the Aristotelian diagram representing the conclusion of (4) has been obtained by reduction through the formal calculation of the composite $\bullet \leftarrow M \leftarrow P$, making the middle term $M$ disappear. We hasten to remark that the unlabelled version of diagram (6), namely diagram

\[
S \quad \bullet \leftarrow M \leftarrow P \quad \text{(7)}
\]

does not uniquely determine the syllogistic inference (5), since it must be taken into account that the same mood can occur in more than one figure, as clearly shown by table (3). In general, the unlabelled diagram of a syllogistic inference determines the mood of a syllogism only up to figure. For instance, diagram (7) produces evidence for the syllogistic inference

$$\mathbf{A}_{PM} \vdash \mathbf{E}_{SM} \vdash \mathbf{E}_{SP}$$

by relabelling it as

\[
S \quad \bullet \leftarrow M \leftarrow P \quad \text{(A)} \quad \mathbf{E}_{SP}
\]

thus validating the mood AEE in the second figure, namely the syllogism

$$\mathbf{A}_{PM}, \mathbf{E}_{SM} \vdash \mathbf{E}_{SP}$$

whereas the syllogistic inference (5) was validating the mood AEE in the fourth figure. Now, we let the reader convince herself that suitable labellings of the diagram

\[
S \quad \bullet \leftarrow M \leftarrow P \quad \text{(E)} \quad \mathbf{E}_{SP}
\]

produce evidence for the syllogistic inferences

$$\mathbf{E}_{PM} \vdash \mathbf{I}_{SM} \vdash \mathbf{O}_{SP}$$

$$\mathbf{E}_{PM} \vdash \mathbf{I}_{SM} \vdash \mathbf{O}_{SP}$$

$$\mathbf{E}_{MP} \vdash \mathbf{I}_{MS} \vdash \mathbf{O}_{SP}$$

$$\mathbf{E}_{MP} \vdash \mathbf{I}_{MS} \vdash \mathbf{O}_{SP}$$

validating the mood EIO in all the figures.

A feature of the calculus at issue is that in a syllogistic inference, no bullet symbol gets deleted. More precisely, for a valid syllogism, the Aristotelian diagram for its conclusion contains as many bullet symbols as in the Aristotelian diagrams for its premisses. This fact turns out to be useful in showing that a syllogism is not valid. For example, the syllogism

$$\mathbf{O}_{PM}, \mathbf{E}_{MS} \vdash \mathbf{I}_{SP}$$

is not valid since if it were such, then the existence of the syllogistic inference

$$\mathbf{O}_{PM} \vdash \mathbf{E}_{MS} \vdash \mathbf{I}_{SP}$$
would be witnessed by a diagram such as
\[
\begin{array}{c}
S \rightarrow \bullet \rightarrow M \rightarrow \bullet \rightarrow P \\
S \rightarrow \bullet \rightarrow M \rightarrow \bullet \rightarrow P
\end{array}
\]
which fact is impossible since a single bullet symbol occurs in the Aristotelian diagram for the conclusion, whereas three of them occur in those for the premisses. This fact could be observed even without drawing the previous diagram but by simply looking at (8). However, this criterion not always apply. It suffices to consider the syllogistic inference
\[
\begin{array}{c}
S \rightarrow \bullet \rightarrow M \rightarrow \bullet \rightarrow P \\
S \rightarrow \bullet \rightarrow M \rightarrow \bullet \rightarrow P
\end{array}
\]
in which as many bullet symbols occur in the premisses as in the conclusion, that we could be tempted to label as \((O_{PS})\), but doing this would mean the interchanging of the rôles played by the term-variables \(S\) and \(P\). On the other hand, syllogism (8) is not valid even because in diagram (9) \(M\) is not erasable.

For every term-variable \(A\), particularly interesting instances of Aristotelian diagrams are the following:
\[
\begin{array}{c}
A \rightarrow \bullet \rightarrow A \\
A \rightarrow \bullet \rightarrow A \\
A \rightarrow \bullet \rightarrow A \\
A \rightarrow \bullet \rightarrow A
\end{array}
\]
which have to be read as
\[
\begin{array}{c}
A_{AA}: \text{All } A \text{ is } A \\
E_{AA}: \text{No } A \text{ is } A \\
I_{AA}: \text{Some } A \text{ is } A \\
O_{AA}: \text{Some } A \text{ is not } A
\end{array}
\]
respectively. The diagrams \(A_{AA}\) and \(I_{AA}\) are referred to as laws of identity, see [6]. In particular, \(I_{AA}\) will be referred to as an assumption of existence, since it affirms the inhabitation of \(A\) whereas, on the contrary, \(E_{AA}\) affirms its emptyness. The diagram \(O_{AA}\) is an expression of the principle of contradiction, which fact has been discussed in [9], to which we refer the interested reader.

A syllogism with assumption of existence is a syllogism that is valid under an additional assumption of existence of the form \(I_{SS}, I_{MM}\) or \(I_{PP}\). The table

| fig. 1 | fig. 2 | fig. 3 | fig. 4 | assumption |
|-------|-------|-------|-------|------------|
| AAI   | AEO   | AEO   | AEO   | \(I_{SS}\) |
| EAO   | EAO   | EAO   | EAO   | \(I_{SS}\) |
| AAI   | EAO   | AAI   | EAO   | \(I_{MM}\) |
| AAI   | EAO   | AAI   | EAO   | \(I_{PP}\) |

lists the valid syllogisms with assumption of existence. For instance, in order to show that the syllogism with assumption of existence
\[
E_{MP}, A_{SM}, I_{SS} \models O_{SP}
\]
is valid, it suffices to consider the syllogistic inference 

\[(E_{MP})\#(A_{SM})\#(I_{SS}) \models (O_{SP})\]

witnessed by diagram

```
S ← I_S ← A_M ← M ← E_MP ← P
S ← P ← A_S ← S ← I_S ← P
```

whose unlabelled version produces evidence for the syllogistic inference 

\[(E_{PM})\#(A_{SM})\#(I_{SS}) \models (O_{SP})\]

too, validating the syllogism with assumption of existence 

\[E_{PM}, A_{SM}, I_{SS} \models O_{SP}.\]

We end the section by citing

**Theorem 2.1.** A syllogism (with assumption of existence) is valid if and only if there is a necessarily unique syllogistic inference from its premisses to its conclusion.

*Proof.* See [9]. □

### 3 n-term syllogisms

Whereas syllogisms, either with assumption of existence or not, involve exactly 3 term-variables, n-term syllogisms involve exactly n term-variables \(A_1, \ldots, A_n\), \(n \geq 1\), linked by \(n\) categorical propositions any two contiguous of which have exactly one term in common. We may represent the \(n\) categorical propositions as

\[X_{A_{n-1}A_n}, X_{A_{n-2}A_{n-1}}, \ldots, X_{A_3A_2}, X_{A_2A_1}, X_{A_1A_2}\]

where \(X\) is a symbol between \(A, E, I, O\) and, for every \(i = 1, \ldots, n-1\), \(X_{A_iA_{i+1}}\) stays either for \(X_{A_iA_{i+1}}\) or for \(X_{A_{i+1}A_i}\). We write

\[X_{A_{n-1}A_n}, X_{A_{n-2}A_{n-1}}, \ldots, X_{A_3A_2}, X_{A_2A_1}, X_{A_1A_2} \models X_{A_1A_n}\]

to denote a generic n-term syllogism. We remark that possibly occurring assumptions of existence of the form \(I_{A_{A_i}}\), for some \(i = 1, \ldots, n\), will be explicitly mentioned when needed. In doing this, we let the expression “n-term syllogism” comprise the case in which no assumption of existence occurs as well as the case in which such an assumption occurs.

It is well known that the total number of valid n-term syllogisms is \(3n^2 - n\), see [3], where such a formula was obtained by rejecting the not valid moods on the bases of the traditional rules of syllogism. The same formula was reobtained in [11] by a diagrammatic method allowing a direct calculation.

The aim of the present section is that of generalize theorem [2, 4] to the case of n-term syllogisms and simultaneously that of directly recalculate the previously cited formula by using syllogistic inferences.
Lemma 3.1. For every positive natural number $n$, a syllogistic inference yields an Aristotelian diagram as a conclusion in exactly the following cases:

(i) $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_i \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{n-1} \rightarrow A_n$.

(ii) $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_i \leftarrow \bullet \rightarrow A_{i+1} \leftarrow \cdots \leftarrow A_{n-1} \leftarrow A_n$, with $1 \leq i \leq n - 1$.

(iii) $A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_i \leftarrow \bullet \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{n-1} \rightarrow A_n$, with $1 \leq i \leq n - 1$.

(iv) $A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_i \leftarrow \bullet \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{n-1} \rightarrow A_n$, with $1 \leq i \leq n$.

(v) $A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_i \leftarrow \bullet \rightarrow A_{i+1} \leftarrow \cdots \leftarrow A_{n-1} \leftarrow A_n$, with $1 \leq i \leq n - 1$.

(vi) $A_1 \leftarrow \cdots \leftarrow A_i \leftarrow \bullet \rightarrow A_{i+1} \rightarrow \cdots \rightarrow A_{j-1} \rightarrow A_j \leftarrow \cdots \leftarrow A_n$, with $1 \leq i < j \leq n$.

(vii) $A_1 \leftarrow \cdots \leftarrow A_i \leftarrow \bullet \rightarrow A_{j-1} \rightarrow A_j \leftarrow \cdots \leftarrow A_n$, with $1 \leq i < j \leq n$.

Proof. It is clear that a syllogistic inference applies to each of the diagrams listed in the statement yielding an Aristotelian diagram involving the terms $A_1$ and $A_n$ only. Conversely, we proceed by cases:

(a) the only way to obtain $A_1 \rightarrow A_n$ as a conclusion of a syllogistic inference is by (i), since no bullet symbol is allowed to occur in the conclusion.

(b) the only way to obtain $A_1 \rightarrow \bullet \leftarrow A_n$ as a conclusion of a syllogistic inference is by (ii), since exactly one bullet symbol must occur in the conclusion with two morphisms converging to it.

(c) the only way to obtain $A_1 \leftarrow \bullet \rightarrow A_n$ as a conclusion of a syllogistic inference is by (iii) or (iv), since exactly one bullet symbol must occur in the conclusion with two morphisms diverging from it.

(d) the only way to obtain $A_1 \leftarrow \bullet \rightarrow A_n$ as a conclusion of a syllogistic inference is by (v), (vi) or (vii), since exactly two bullet symbols must occur in the conclusion, with three alternating morphisms.

Lemma 3.2. For every positive natural number $n$, let $\varphi(n)$ and $\psi(n)$ be the number of diagrams like those in points (vi) and (vii) of lemma 3.1, respectively. The following facts hold

(i) $\varphi(n) = \frac{(n-1)(n-2)}{2}$.

(ii) $\psi(n) = \frac{n(n-1)}{2}$.

Proof. (i) For every positive natural number $n$, $\varphi(n + 1) = \varphi(n) + (n - 1)$. Because, passing from $n$ to $n + 1$ is a matter of inserting one more arrow symbol $\rightarrow$ or $\leftarrow$, on the left, on the right or in the middle of the diagrams constructed at $n$, so to extend them with one more term-variable. There are exactly $n - 1$ possibilities of doing this. Finally, by induction on the number of term-variables, the thesis is easily achieved.
(ii) The argument is completely similar to the previous but for the fact that for every positive natural number \( n \), \( \psi(n + 1) = \psi(n) + n \).

\( \square \)

**Theorem 3.3.** For every positive natural number \( n \), an \( n \)-term syllogism is valid if and only if there is a (necessarily unique) syllogistic inference from its premises to its conclusion. Moreover, the number of valid \( n \)-term syllogisms is \( 3n^2 - n \).

**Proof.** Lemma 3.1 and lemma 3.2 permit to conclude that the \( n \)-term syllogisms in the table

| syllogism | quantity |
|----------|----------|
| \( A_{A_{A_{A_1}}}, \ldots, A_{A_1} \triangleq A_{A_1} \) | 1 |
| \( A_{A_{A_{A_{A_1}}}}, \ldots, A_{A_1} \triangleq A_{E_1} \) | \( n-1 \) |
| \( A_{A_{A_{A_{A_1}}}}, \ldots, E_{A_1} \triangleq A_{A_1} \) | \( n-1 \) |
| \( A_{A_{A_{A_{A_1}}}}, \ldots, E_{A_{A_1}} \triangleq A_{A_1} \) | \( n-1 \) |
| \( A_{A_{A_{A_{A_1}}}}, \ldots, A_{A_1}, \ldots, A_{A_1} \triangleq A_{O_1} \) | \( n \) |
| \( A_{A_{A_{A_{A_1}}}}, \ldots, O_{A_{A_1}} \triangleq A_{A_1} \) | \( n-1 \) |

are all valid. Moreover they are \( 3n^2 - n \). Conversely, we use lemma 3.1 and lemma 3.2 to construct a syllogistic inference to a given possible conclusion:

- By lemma 3.1(i), the diagram

\[
\begin{array}{cccccccc}
A_1 & A_2 & \cdots & A_{i-1} & A_i & \cdots & A_{n-1} & A_n \\
\text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} \\
A_1 & A_2 & \cdots & A_i & & & & A_n
\end{array}
\]

represents the only way to produce evidence for the syllogistic inference

\[(A_{A_{A_{A_1}}} \# \cdots \# (A_{A_1} A_2) \triangleq (A_{A_1} A_n))\]

validating the \( n \)-term syllogism

\[A_{A_{A_{A_{A_1}}}, \ldots, A_{A_1}} \triangleq A_{A_1} \]

- By lemma 3.1(ii), the \( n-1 \) diagrams

\[
\begin{array}{cccccccc}
A_1 & A_2 & \cdots & A_i & & & & A_n \\
\text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} & \text{II} \\
A_1 & A_2 & \cdots & A_i & & & & A_n
\end{array}
\]

represent the only way to produce evidence for the syllogistic inference

\[(A_{A_{A_{A_{A_1}}}}) \# \cdots \# (E_{A_{A_{A_{A_1}}}}) \# \cdots \# (E_{A_{A_{A_{A_1}}}}) \triangleq (E_{A_{A_{A_{A_1}}}})\]
as well as for the syllogistic inference
\[(A_{A_nA_{n-1}})^\# \cdots \# (E_{A_{1}A_{2}})^\# \cdots \# (A_{A_2A_1})^\# \vdash (E_{A_1A_n})\]
validating the \(n-1\) syllogisms
\[A_{A_nA_{n-1}}, \ldots, E_{A_1A_2}, \ldots, A_{A_2A_1} \vdash E_{A_1A_n}\]
and the \(n-1\) syllogisms
\[A_{A_nA_{n-1}}, \ldots, E_{A_1A_2}, \ldots, A_{A_2A_1} \vdash E_{A_1A_n}\]
respectively. Thus, in total there are \(2(n-1)\) valid \(n\)-term syllogisms with conclusion \(E_{A_1A_n}\).

- By lemma 3.1(iii), the \(n-1\) diagrams
\[
\begin{array}{c}
A_1 \quad A_2 \quad \cdots \quad A_i \quad \bullet \quad A_{i+1} \quad \cdots \quad A_{n-1} \quad A_n \\
\hline
A_1 \quad \cdots \quad A_i \quad \bullet \quad A_{i+1} \quad \cdots \quad A_{n-1} \quad A_n \\
\end{array}
\]
represent the only way to produce evidence for the \(n-1\) syllogistic inferences
\[(A_{A_nA_{n-1}})^\# \cdots \# (E_{A_{1}A_{2}})^\# \cdots \# (A_{A_2A_1})^\# \vdash (I_{A_1A_n})\]
as well as for the \(n-1\) syllogistic inferences
\[(A_{A_nA_{n-1}})^\# \cdots \# (I_{A_{1}A_{2}})^\# \cdots \# (A_{A_2A_1})^\# \vdash (I_{A_1A_n})\]
that validate the \(n-1\) \(n\)-term syllogisms
\[A_{A_nA_{n-1}}, \ldots, I_{A_{1}A_{2}}, \ldots, A_{A_2A_1} \vdash I_{A_1A_n}\]
and the \(n-1\) \(n\)-term syllogisms
\[A_{A_nA_{n-1}}, \ldots, I_{A_{1}A_{2}}, \ldots, A_{A_2A_1} \vdash I_{A_1A_n}\]
respectively. By lemma 3.1(iv), the \(n\) diagrams
\[
\begin{array}{c}
A_1 \quad A_2 \quad \cdots \quad A_i \quad \bullet \quad A_{i+1} \quad \cdots \quad A_{n-1} \quad A_n \\
\hline
A_1 \quad \cdots \quad A_i \quad \bullet \quad A_{i+1} \quad \cdots \quad A_{n-1} \quad A_n \\
\end{array}
\]
represent the only way to produce evidence for the \(n\) syllogistic inferences
\[(A_{A_nA_{n-1}})^\# \cdots \# (I_{A_{1}A_{2}})^\# \cdots \# (A_{A_2A_1})^\# \vdash (I_{A_1A_n})\]
that validate the \(n\) \(n\)-term syllogisms
\[A_{A_nA_{n-1}}, \ldots, I_{A_{1}A_{2}}, \ldots, A_{A_2A_1} \vdash I_{A_1A_n}\]
so that in total there are \(2(n-1) + n\) valid \(n\)-term syllogisms with conclusion \(I_{A_1A_n}\).
- By lemma 3.1 (v), the \( n - 1 \) diagrams

\[
\begin{align*}
& A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_j \longrightarrow \bullet \longrightarrow A_{i+1} \cdots \longrightarrow A_{n-1} \longrightarrow A_n \\
& \hspace{1cm} \uparrow \\
& A_1 \longrightarrow \bullet \longrightarrow \cdots \longrightarrow A_j \longrightarrow \bullet \longrightarrow A_{i+1} \cdots \longrightarrow A_{n-1} \longrightarrow A_n
\end{align*}
\]

represent the only way to produce evidence for the \( n - 1 \) syllogistic inferences

\[
(A_1 A_{i-1})^\circ \cdots \# (O_1 A_{i-1})^\circ \cdots \# (A_{1, j} A_j) \models (O_{1, A_j})
\]

that validate the \( n - 1 \) \( n \)-term syllogisms

\[
A_{A_{1, A_{i-1}}}, \ldots, O_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{A_{1 A_j}}.
\]

By lemma 3.1 (vi) and lemma 3.2 (i), the \( \frac{\binom{n-1}{2} (n-2)}{2} \) diagrams

\[
\begin{align*}
& A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \bullet \longrightarrow A_{i+1} \cdots \longrightarrow A_{j-1} \longrightarrow \bullet \longrightarrow A_j \cdots \longrightarrow A_n \\
& \hspace{1cm} \uparrow \\
& A_1 \longrightarrow \bullet \longrightarrow \cdots \longrightarrow A_i \longrightarrow \bullet \longrightarrow A_{i+1} \cdots \longrightarrow A_{j-1} \longrightarrow \bullet \longrightarrow A_j \cdots \longrightarrow A_n
\end{align*}
\]

represent the only way to produce evidence for the 4 \( \frac{\binom{n-1}{2} (n-2)}{2} \) syllogistic inferences

\[
\begin{align*}
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{1 A_j}) \\
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{A_{1 A_j}}) \\
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{A_{1 A_j}}) \\
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{A_{1 A_j}})
\end{align*}
\]

that validate the 4 \( \frac{\binom{n-1}{2} (n-2)}{2} \) \( n \)-term syllogisms

\[
A_{A_{1, A_{i-1}}}, \ldots, E_{A_{1, A_{i-1}}}, \ldots, I_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{A_{1 A_j}}.
\]

\[
A_{A_{1, A_{i-1}}}, \ldots, E_{A_{1, A_{i-1}}}, \ldots, I_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{1, A_j}.
\]

\[
A_{A_{1, A_{i-1}}}, \ldots, E_{A_{1, A_{i-1}}}, \ldots, I_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{A_{1, A_j}}.
\]

respectively. By lemma 3.1 (vii) and lemma 3.2 (ii), the \( \frac{\binom{n-1}{2}}{2} \) diagrams

\[
\begin{align*}
& \hspace{1cm} \uparrow \\
& A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \bullet \longrightarrow A_{i+1} \cdots \longrightarrow A_{j-1} \longrightarrow \bullet \longrightarrow A_j \cdots \longrightarrow A_n
\end{align*}
\]

represent the only way to produce evidence for the 2 \( \frac{\binom{n-1}{2}}{2} \) syllogistic inferences

\[
\begin{align*}
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{A_{1 A_j}}) \\
(A_{A_{1, A_{i-1}}})^\circ \cdots \# (E_{A_{1, A_{i-1}}})^\circ \cdots \# (I_{A_{1, A_{i-1}}})^\circ \cdots \# (A_{A_{j A_j}}) \models (O_{A_{1 A_j}})
\end{align*}
\]

that validate the 2 \( \frac{\binom{n-1}{2}}{2} \) \( n \)-term syllogisms

\[
A_{A_{1, A_{i-1}}}, \ldots, E_{A_{1, A_{i-1}}}, \ldots, I_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{A_{1 A_j}}.
\]

\[
A_{A_{1, A_{i-1}}}, \ldots, E_{A_{1, A_{i-1}}}, \ldots, I_{A_{1, A_{i-1}}}, \ldots, A_{A_{j A_j}} \models O_{A_{1 A_j}}.
\]

respectively. Thus in total there are \( n - 1 + 4 \cdot \frac{\binom{n-1}{2} (n-2)}{2} + 2 \cdot \frac{\binom{n-1}{2}}{2} \) valid \( n \)-term syllogisms with conclusion \( O_{A_{1 A_j}} \).
In total, the valid $n$-term syllogisms are in number of

$$1 + 2(n - 1) + 2(n - 1) + n + (n - 1) + 4 \cdot \frac{(n - 1)(n - 2)}{2} + 2 \cdot \frac{n(n - 1)}{2} = 3n^2 - n$$

We end the section with the explicit description of the valid $n$-term syllogisms for $n = 1$ and $n = 2$, respectively. In the first case, there is only one figure, that is $A_1A_1$ and only two valid moods for it, that is $A$ and $I$ so that, as observed in [6] and [8], the only valid 1-term syllogisms are $A_1A_1$ and only two valid moods for it, that is $A$ and $I$ so that, as observed in [6] and [8], the only valid 1-term syllogisms are $A_1A_1$ and $I_1A_1$, that is the laws of identity to which we hinted at in the previous section. In the second case there are two figures, as shown in the table

| premise | fig. 1 | fig. 2 |
|---------|--------|--------|
| $A_1A_2$ | $I_1A_2$ | $I_1A_2$ |
| $I_1A_2$ | $I_1A_2$ | $O_1A_2$ |

and ten valid 2-term syllogisms, six in the first figure and four in the second, as follows:

**figure 1:** $A_1A_1 \models A_1A_2$, $E_1A_1 \models E_1A_2$, $I_1A_1 \models I_1A_2$, $O_1A_1 \models O_1A_2$, plus the laws of subalternation $A_1A_2$, $I_1A_2$, $E_1A_2$, $I_1A_2$, $O_1A_2$.

**figure 2:** $E_1A_2 \models E_1A_2$, $I_1A_1 \models I_1A_2$, which are the laws of simple conversion, and $I_1A_2$, $A_1A_1 \models I_1A_2$, $E_1A_2$, $I_1A_1$, $I_1A_2$, $O_1A_2$ which are the laws of conversion per accidens.

### 4 Syllogisms as rewrite rules

We here point out the existing connections between the previously introduced calculus of syllogisms and the rewriting of certain terms, on the base of suitable rewrite rules. On term rewriting systems in general, the reader may consult [1]. Following this, we look at rewrite rules as to directed equations, separating a reducible expression on their left-hand side from a reduced one on their right-hand side, and at term rewriting as to a computation mechanism. Applying a rewrite rule gives rise to a reduction. Loosely, from our standpoint the terms and the rewrite rules we are interested in are finite sequences of Aristotelian diagrams and the valid syllogisms, respectively, whereas reductions are finite pastings of syllogistic inferences, so that $\models$ extends to a reduction relation on the set of finite sequences of Aristotelian diagrams. In pursuing this point of view, we will single out a suitable category theoretic framework for the calculus. Because of this, we assume that the reader has already knowledge of the basics in category theory as can be found in [7], for example.

We recall that a term is in normal form or irreducible if no rewrite rule applies to it. Otherwise, it is reducible. If $\models$ denotes a reduction relation, then repeated applications of rewrite rules to a reducible term $t_1$ yield a descending chain of reductions

$$t_1 \models t_2 \models t_3 \models \cdots$$

which may not be finite. A finite chain of reductions will be more briefly denoted $t_1 \models^* t_n$ and a term $t$ is a direct successor of a term $s$ if $s \models t$. A term rewriting system is
- **normalizing** if every term reduces to a normal form.
- **terminating** if there is no infinite descending chain of reductions
  \[ t_1 \Downarrow t_2 \Downarrow t_3 \Downarrow \cdots \]
- **locally confluent** if in reason of the application of two different rewrite rules to a term \( s \), yielding in turn two different terms \( t_1 \) and \( t_2 \), then a term \( t \) and two finite chains of reductions
  \[ t_1 \Downarrow^* t \quad t_2 \Downarrow^* t \]
  exist.
- **confluent** if whenever \( s \Downarrow^* t_1 \) and \( s \Downarrow^* t_2 \), with \( t_1 \) and \( t_2 \) different terms, then there exists a term \( t \) and two finite chains of reductions
  \[ t_1 \Downarrow^* t \quad t_2 \Downarrow^* t \]
- **convergent** if it is both terminating and confluent.

A terminating rewriting system is normalizing but not the converse, see [1], so that in a terminating rewriting system every term has at least one normal form. On the other hand in a confluent rewriting system every term has at most one normal form. Thus, in a convergent rewriting system every term has exactly one normal form.

Now, for future reference we mention the following

**Lemma 4.1** (Newman’s lemma). A terminating and locally confluent term rewriting system is confluent.

**Proof.** See [1].

For every natural number \( n \), \( n \)-polygraphs and more in general \( \infty \)-polygraphs were introduced in [2]. Existing connections between \( n \)-polygraphs, for \( n = 2, 3 \) in particular, and rewriting systems were pointed out in [3] and [4], for example, so that later on we will feel free to extend to them the previously introduced terminology.

Now, the idea is that of looking at the calculus of \( n \)-term syllogisms as to a rewriting system, or better as to a specific 2-polygraph. In this way, it turns out that the calculus takes place in a suitable categorical structure, freely generated by such a 2-polygraph.

From [2], with personal notations, we briefly recall that for every natural number \( n \), an \( n \)-**graph** is a diagram of sets and functions

\[
\begin{align*}
G_0 \xrightarrow{t_0} G_1 \xrightarrow{t_1} G_2 & \quad \cdots \quad G_{n-1} \xrightarrow{t_{n-1}} G_n \\
\end{align*}
\]

such that for every positive natural number \( n \), the **globular identities** \( s_{n-1}s_n = s_{n-1}t_n \) and \( t_{n-1}s_n = t_{n-1}t_n \) hold. For every \( 0 \leq i \leq n \), the functions \( s_i \) and \( t_i \) are referred to as **source** and **target**, respectively, whereas the elements of \( G_i \) are referred to as **i-cells**. A 0-graph is just a set, whereas a 1-graph is an ordinary graph. A **morphism of n-graphs**, is a family of \( n \) functions carrying \( i \)-cells to \( i \)-cells, commuting with source and target.
A 2-category is a 2-graph

\[ G_0 \xrightarrow{s_0} G_1 \xrightarrow{s_1} G_2 \]  

(12)
equipped with a category structure on the graph \((G_0, G_1, s_0, t_0)\), a category structure on the graph \((G_0, G_2, s_0 s_1, t_0 t_1)\) and a category structure on the graph \((G_1, G_2, s_1, t_1)\), reciprocally compatible. In explicit and elementary terms, a 2-category consists of objects and morphisms, also respectively called 0-cells and 1-cells, that identify a category and moreover of 2-cells \(\alpha, \beta, \gamma, \ldots\) between parallel pairs of 1-cells such that to every morphism \(f: A \to B\) corresponds a designated identity 2-cell \(\id_f: f \Rightarrow f\), and to every morphisms \(f, g, h: A \to B\) and 2-cells \(\alpha: f \Rightarrow g\) and \(\beta: g \Rightarrow h\) corresponds a designated vertical composite \(\beta \cdot \alpha: f \Rightarrow h\), in such a way that the axioms for a category are satisfied. Often, the identity 2-cell associated to a 1-cell \(f\) is denoted by the same letter \(f\), thus writing \(f: f \Rightarrow f\). For 1-cells \(f, g: A \to B, f': g': B \to C\), to every 2-cells \(\alpha: f \Rightarrow g\) and \(\alpha': f' \Rightarrow g'\) there corresponds a designated horizontal composite \(\alpha' \cdot \alpha: f' \circ f \Rightarrow g' \circ g\), such that \(\alpha' \cdot \id_B = \alpha' = \id_C \cdot \alpha\). Also, \(f' \cdot f = f' \circ f\). Finally, vertical and horizontal composition of 2-cells are required to interact suitably, so that for every further morphisms \(h: A \to B, h': B \to C\) and for every 2-cells \(\beta: g \Rightarrow h\) and \(\beta': g' \Rightarrow h'\), the interchange law \((\beta' \cdot \alpha') \cdot (\beta \cdot \alpha) = (\beta' \cdot \beta) \cdot (\alpha' \cdot \alpha)\) holds. A morphism between two 2-categories is a morphism between the underlying 2-graphs, furthermore preserving horizontal and vertical compositions and identities, and usually referred to as 2-functor.

**Example 4.2.** Categories, functors and natural transformations considered as 0-cells, 1-cells and 2-cells respectively, identify the paradigmatic example of 2-category.

**Example 4.3.** Every ordinary category can be seen as a 2-category with only identical 2-cells, which is also called locally discrete.

**Example 4.4.** Sets and relations form a category which can be seen as a 2-category by letting the 2-cells be inclusions, i.e. for sets \(A, B\) and relations \(R, S \subseteq A \times B\), there is a 2-cell \(R \Rightarrow S\) if and only if \(R \subseteq S\).

Now, following [2] and with personal notations, we give the following

**Definition 4.5.** A 2-polygraph \(\Sigma\) consists of a diagram

\[ \Sigma_0 \xrightarrow{\Sigma_1} \Sigma_2 \]  

(13)
in which, the functions \(\Sigma_0, \Sigma_1\) are the source and target functions of the free category generated by the graph \((\Sigma_0, \Sigma_1, s_0, t_0)\), and where

\[ \Sigma_0 \xrightarrow{\Sigma_1} \Sigma_2 \]  

(14)
is a 2-graph.
As observed in [2], 2-polygraphs are in connection with the study of the word problem in categories, generalizing the one in monoids. The free 2-category generated by a 2-polygraph \( \Sigma \), is the 2-category \( \Sigma^* \) occurring in the lower part of diagram (15)

![Diagram](image)

that is the free 2-category generated by the category \( \Sigma_0 \xrightarrow{\pi} \Sigma_1^* \) with additional 2-cells provided by \( \Sigma_2 \). By following [2] again, a more explicit description of such a free 2-category can be given by looking at the elements of \( \Sigma_2 \) as to diagrams of the form

![Diagram](image)

and at the elements of \( \Sigma_1^* \) as to 2-paths.

There exists a morphism of 2-graphs from the 2-graph (14) to the 2-graph underlying the 2-category which is the lower part of diagram (15), precisely the one which is the inclusion of \( \Sigma_2 \) in \( \Sigma_2^* \) and the identical function on \( \Sigma_0 \) and \( \Sigma_1^* \), so that freeness of the 2-category

![Diagram](image)

amounts to the following: for every 2-category \( \mathcal{C} = (C_0, C_1, C_2) \) and for every morphism of 2-graphs \( F = (F_0, F_1, F_2): (\Sigma_0, \Sigma_1^*, \Sigma_2) \to \mathcal{C} \) there exists a unique 2-functor \( F^* = (F_0, F_1^*, F_2^*): \Sigma^* \to \mathcal{C} \) extending \( F \), namely such that the diagram

![Diagram](image)

commutes.

**Definition 4.6.** Let \( n \) be a positive natural number. The polygraph for the calculus of \( n \)-term syllogisms is the 2-polygraph \( S \) identified by the following data:

- a set \( S_0 = \{A_1, \ldots, A_n\} \) of term-variables.

- a set \( S_1 \) whose elements are the Aristotelian diagrams

  ![Aristotelian Diagrams](image)

  together with the evident source and target functions.
- a set $S_2$ of rewrite rules, which are the following syllogistic inferences for the corresponding valid syllogisms:

$$(A_{iA}) \models (A_{iA}), \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i \leq n$$

$$(A_{iA}) \models (A_{iA}), \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i < j \leq n$$

$$(E_{iA}) \models (E_{iA}), \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i < j \leq n$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq n$$

$$(E_{iA}) \models (A_{iA}) \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i < j \leq n$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i < j \leq n$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (E_{iA}) \quad 1 \leq i < j \leq n$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq n$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq n$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (E_{iA}) \quad 1 \leq i < j \leq n$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq n$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq n$$

Together with the evident source and target functions.

It is useful to exemplify the previous definition in the cases $n = 1, 2, 3$.

If $n = 1$, then

- $S_0 = \{A_i\}$.

- $S_1 = \{[A_{iA}, (A_{iA})^\circ], [E_{iA}, (E_{iA})^\circ], [I_{iA}, (I_{iA})^\circ], [O_{iA}, (O_{iA})^\circ]\}$.

- $S_2 = \{[A_{iA}, (A_{iA})^\circ], (I_{iA}) \models (I_{iA})\}$.

so that $S^*$ is a locally discrete 2-category, see example [3] that is an ordinary category. The calculus of 1-term syllogisms consists of the sole laws of identity $A_{iA} \models A_{iA}$ and $I_{iA} \models I_{iA}$, recovered by the rewrite rules in $S_2$ above.

If $n = 2$, then

- $S_0 = \{A_1, A_2\}$.

- $S_1 = \{(X_{AA}) | i = 1, 2 \} \cup \{(X_{AA}) | i = 1, 2 \} \cup \{(X_{AA}) | 1 \leq i < j \leq 2 \} \cup \{(X_{AA}) | 1 \leq i < j \leq 2 \}$, where $X \in \{A, E, I, O\}$.

- $S_2$ consists of the rewrite rules

$$(A_{iA}) \models (A_{iA}), \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i \leq 2$$

$$(A_{iA}) \models (A_{iA}), \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i < j \leq 2$$

$$(E_{iA}) \models (E_{iA}), \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(E_{iA}) \models (E_{iA}) \quad (I_{iA}) \models (I_{iA}) \quad 1 \leq i < j \leq 2$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(A_{iA}) \models (I_{iA}), \quad (E_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$

$$(E_{iA}) \models (E_{iA}) \quad (O_{iA}) \models (O_{iA}) \quad 1 \leq i \leq j \leq 2$$
The previous data extend those for the calculus of 1-term syllogisms to recover the calculus of 2-term syllogisms. In particular, it is possible to recognize the syllogistic inferences validating the laws of subalternation, simple conversion and conversion per accidens, already encountered at the end of section 3.

If \( n = 3 \), then the data for the calculus of 3-term syllogisms extend the previous and amount to the whole of those listed in definition 4.6. The syllogistic inferences validating the syllogisms with assumption of existence in Table (10) are obtainable. For instance, the syllogistic inference validating \( \text{AAI} \) in the fourth figure and \( \text{AEO} \) in the second figure are given by the two step reductions

\[
(I_{A_3 A_5}) \Rightarrow (A_{A_2 A_4}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5})
\]

and

\[
(A_{A_3 A_5}) \Rightarrow (E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (A_{A_2 A_4}) \Rightarrow (O_{A_3 A_5}) \Rightarrow (O_{A_3 A_5})
\]

respectively, where the evident rewrite rules have been applied. The calculations for the remaining syllogisms with assumption of existence are similar.

The elements of \( S_1^* \) are words in the elements of \( S_1 \) and will be henceforth referred to as terms. We let the length of a term be the number of Aristotelian diagrams by which it is formed, to which we will also refer to as premises. Thus for example \((A_{A_3 A_5})\) is a term of length 1, whereas \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5}) \Rightarrow (O_{A_3 A_5}) \) is a term of length 3. It is intuitively clear what a subterm is: \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5}) \Rightarrow (O_{A_3 A_5})\) are subterms of \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5})\), for example. Moreover, overlapping subterms may occur, namely those that have some of their premises in common, so that for example \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5}) \Rightarrow (O_{A_3 A_5}) \) are overlapping subterms of \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (I_{A_3 A_5}) \Rightarrow (O_{A_3 A_5}) \), as well as \((E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \) and \((I_{A_3 A_5}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (O_{A_3 A_5}) \). Terms undergo reduction by the rewrite rules in \( S_2 \). These are said to be trivial if they have exactly one premise coinciding with their conclusion, otherwise are non-trivial. In the free 2-category \( S^* \), the trivial rewrite rules will correspond to identical 2-cells.

**Example 4.7.**

(i) the term \((E_{A_4 A_2}) \Rightarrow (A_{A_2 A_4})^* \) cannot be rewritten on the base of any of the rewrite rules in \( S_2 \).

(ii) the term \((E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (E_{A_2 A_4}) \Rightarrow (I_{A_3 A_5}) \)

undergoes rewriting by the sequential application of the rewrite rules

\[
(E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (E_{A_2 A_4}) \Rightarrow (O_{A_3 A_5})
\]

thus obtaining

\[
(E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (E_{A_2 A_4}) \Rightarrow (O_{A_3 A_5}) \Rightarrow (O_{A_3 A_5}) \Rightarrow (O_{A_3 A_5})
\]

giving rise to a term which cannot be further rewritten.

(iii) the term \((E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (E_{A_2 A_4}) \)

can be rewritten in the following ways

\[
(E_{A_4 A_2}) \Rightarrow (I_{A_3 A_5}) \Rightarrow (A_{A_2 A_4})^* \Rightarrow (E_{A_2 A_4}) \Rightarrow (E_{A_2 A_4}) \Rightarrow (O_{A_3 A_5}) \Rightarrow (O_{A_3 A_5}) \Rightarrow (O_{A_3 A_5})
\]
rewriting system is terminating if it undergoes reduction by the application of any of the non-trivial rewrite rules, strictly decreasing. With respect to termination, it must be observed that the length of the terms that undergo reduction by the application of two different rewrite rules, strictly decreasing. The sole exception is represented by the laws of simple conversion, whose application on the other hand cannot be indefinitely iterated. In order to prove that a rewriting system is terminating it suffices to embed it in a rewriting system which is already known to be such, see [1], typically the set $\mathbb{N}$ of natural numbers together with the “greater than” relation $>$. In the case of the 2-polygraph for the calculus of $n$-term syllogisms, it suffices to consider the function $f: \Sigma^* \rightarrow \mathbb{N}$ which, to each term, assigns its length and observe that an infinite chain of length-decreasing reductions would induce an infinite descending chain in $\mathbb{N}$.

With respect to confluence, things are more delicate. The idea is that of proving the local confluence of $\mathcal{S}$ and then conclude by applying Newman’s lemma [1]. Once the number $n$ of occurring term-variables have been fixed, proving local confluence of $\mathcal{S}$ amounts to testing the effect of the application of two different rewrite rules on the same subterm of an arbitrary term. In doing this only the non-trivial rewrite rules have to be taken into account, since otherwise no significant rewriting takes place. Moreover, the interesting case is that of the application of such non-trivial rewrite rules to overlapping subterms. In fact, in general, if $x_1y_1z_2$ is a term in which the non-overlapping subterms $x_1$ and $s_2$ occur, then the application of rewrite rules $R_1$ and $R_2$ to them, yielding the different terms $s_1\sigma_1$ and $s_2\sigma_2$ say, can always be made confluent as
illustrated by the following diagram:

\[
\begin{align*}
xS_1yS_2z & \xrightarrow{R_1} x\sigma_1yS_2x \\
R_2 \downarrow & \\
xS_1y\sigma_2z & \xrightarrow{R_3} x\sigma_1y\sigma_2z
\end{align*}
\]

**Theorem 4.8.** For every positive natural number \(n, n \geq 2\), the 2-polygraph for the calculus of \(n\)-term syllogisms is locally confluent up to renaming of term-variables.

**Proof.** We proceed by cases:

\(n = 2\): The interesting cases are those that amount to the application of non-trivial rewrite rules to subterms in which exactly two distinct term-variables occur, because of the condition \(1 \leq i < j \leq 2\). Thus, the only case to test is provided by the term \((E_{A,A})^\circ \| (I_{A,A})\), to be considered as the overlap of \((E_{A,A}) \| (I_{A,A})\) itself with \((E_{A,A})^\circ\). The local confluence on this subterm is shown by the reductions

\[
(E_{A,A})^\circ \| (I_{A,A}) \vdash (O_{A,A})
\]

\[
(E_{A,A})^\circ \| (I_{A,A}) \vdash (E_{A,A}) \| (I_{A,A}) \vdash (O_{A,A})
\]

in which the evident rewrite rules have been applied.

\(n = 3\): The interesting cases are those that amount to the application of non-trivial rewrite rules to subterms in which exactly two or exactly three distinct term-variables occur, because of the conditions \(1 \leq i < j \leq 3\) and \(1 \leq i < j < k \leq 3\). Thus, with respect to the subterms of length two, the cases to test are

\[
\begin{align*}
(E_{A,A})^\circ \| (I_{A,A}) & \quad (A_{A,A})^\circ \| (E_{A,A})^\circ \\
E_{A,A})^\circ \| (I_{A,A}) & \quad (E_{A,A})^\circ \| (I_{A,A})^\circ \\
A_{A,A}) \| (I_{A,A}) & \quad (A_{A,A}) \| (E_{A,A})^\circ \\
I_{A,A}) \| (E_{A,A})^\circ & \quad (I_{A,A}) \| (E_{A,A}) \| (E_{A,A}) \| (E_{A,A}) \| (I_{A,A})
\end{align*}
\]

all of which are easily seen to be locally confluent. With respect to the subterms of length three, the interesting cases are

\[
\begin{align*}
(A_{A,A}) \| (I_{A,A}) \| (I_{A,A}) & \quad (E_{A,A}) \| (A_{A,A}) \| (I_{A,A}) \\
(A_{A,A}) \| (E_{A,A}) \| (I_{A,A}) & \quad (E_{A,A}) \| (A_{A,A}) \| (I_{A,A}) \\
E_{A,A}) \| (I_{A,A}) \| (A_{A,A}) & \quad (A_{A,A}) \| (E_{A,A}) \| (I_{A,A}) \\
(A_{A,A}) \| (E_{A,A}) \| (I_{A,A}) & \quad (I_{A,A}) \| (A_{A,A}) \| (I_{A,A})
\end{align*}
\]

all of which are easily seen to be locally confluent.

\(n = 4\): The interesting cases are those that amount to the application of non-trivial rewrite rules to subterms in which exactly two, three or four distinct term-variables occur, because of the conditions \(1 \leq i < j \leq 4\), \(1 \leq i < j < k \leq 4\) and \(1 \leq i < j < k < l \leq 4\). The interesting subterms of length two and three that satisfy the first and the second set of conditions, respectively, have been previously considered. There remain the subterms of length three satisfying the third set of
conditions. With respect to these, the interesting cases are

\[
\begin{align*}
(A_{A,A})^{\#}(A_{A,A})^{\#}(A_{A,A})^{\#} & \quad (A_{A,A})^{\#}(A_{A,A})^{\#}(I_{A,A})^{\#} \\
(E_{A,A})^{\#}(A_{A,A})^{\#}(A_{A,A})^{\#} & \quad (E_{A,A})^{\#}(A_{A,A})^{\#}(A_{A,A})^{\#} \\
(E_{A,A})^{\#}(E_{A,A})^{\#}(A_{A,A})^{\#}(I_{A,A})^{\#} & \quad (A_{A,A})^{\#}(E_{A,A})^{\#}(A_{A,A})^{\#} \\
(A_{A,A})^{\#}(E_{A,A})^{\#}(A_{A,A})^{\#}(I_{A,A})^{\#} & \quad (I_{A,A})^{\#}(A_{A,A})^{\#}(A_{A,A})^{\#} \\
(I_{A,A})^{\#}(A_{A,A})^{\#}(I_{A,A})^{\#}(O_{A,A})^{\#} & \quad (E_{A,A})^{\#}(I_{A,A})^{\#}(I_{A,A})^{\#} \\
(O_{A,A})^{\#}(A_{A,A})^{\#}(O_{A,A})^{\#}(I_{A,A})^{\#} & \quad (O_{A,A})^{\#}(A_{A,A})^{\#}(E_{A,A})^{\#} \\
(A_{A,A})^{\#}(A_{A,A})^{\#}(O_{A,A})^{\#}(I_{A,A})^{\#} & \quad (I_{A,A})^{\#}(A_{A,A})^{\#}(E_{A,A})^{\#} \\
(A_{A,A})^{\#}(E_{A,A})^{\#}(A_{A,A})^{\#}(I_{A,A})^{\#} & \quad (I_{A,A})^{\#}(A_{A,A})^{\#}(O_{A,A})^{\#} \\
(O_{A,A})^{\#}(A_{A,A})^{\#}(E_{A,A})^{\#}(I_{A,A})^{\#} & \quad (O_{A,A})^{\#}(A_{A,A})^{\#}(E_{A,A})^{\#}
\end{align*}
\]

all of which can be easily seen to be locally confluent up to renaming of term-variables, as for example in the case of \((O_{A,A})^{\#}(A_{A,A})^{\#}(E_{A,A})^{\#}\), which on one hand reduces to \((O_{A,A})^{\#}(E_{A,A})^{\#}(E_{A,A})^{\#}\). The interesting subterms of length four that satisfy the condition \(1 \leq i < j < k < l \leq 4\) must contain a premise of the form \((I_{A,A})^{\#}, (I_{A,A})^{\#}, (I_{A,A})^{\#}\) or \((I_{A,A})^{\#}\). They fall inside the already tested cases, by thinking of them as obtained from the overlapping of terms of length two.

\[n \geq 5:\] The interesting cases fall inside the already tested cases, by thinking of them as obtained from the overlapping of a suitable amount of terms of length two.

\[\square\]

**Corollary 4.9.** For every positive natural number \(n, n \geq 2\), the 2-polygraph for the calculus of \(n\)-term syllogisms is confluent up to renaming of term-variables.

**Proof.** It follows from theorem 4.8 and Newman’s lemma 4.1 \(\square\)

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