ON A GENERALIZED UNIFORM ZERO-TWO LAW FOR POSITIVE CONTRACTIONS OF NON-COMMUTATIVE $L_1$-SPACES AND ITS VECTOR-VALUED EXTENSION

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Abstract. First, Ornstein and Sucheston proved that for a given positive contraction $T : L_1 \to L_1$ there exists $m \in \mathbb{N}$ such that $\|T^{m+1} - T^m\| < 2$ then

$$\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$ 

Such a result was labeled as "zero-two" law. In the present paper, we prove a generalized uniform "zero-two" law for multi-parametric family of positive contractions of the non-commutative $L_1$-spaces. Moreover, we also establish a vector-valued analogous of the uniform "zero-two" law for positive contractions of $L_1(M, \Phi)$-- the non-commutative $L_1$-spaces associated with center valued trace.

Keywords: zero-two law, positive contraction, bundle; non-commutative

AMS Subject Classification: 47A35, 17C65, 46L70, 46L52, 28D05.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive $\sigma$-additive measure $\mu$ and let $L_1(X, \mathcal{F}, \mu)$ be the usual associated real $L_1$-space. A linear operator $T : L_1(X, \mathcal{F}, \mu) \to L_1(X, \mathcal{F}, \mu)$ is called a positive contraction if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$.

Jamison and Orey [14] proved that if $P$ is a Markov operator recurrent in the sense of Harris, with $\sigma$-finite invariant measure $\mu$, then $\|P^ng\|_1 \to 0$ for every $g \in L^1$ with $\int g \, d\mu = 0$ if (and only if) the chain is aperiodic. Clearly, when the chain is not aperiodic, taking $f$ with positive and negative parts supported in different sets of the cyclic decomposition, we have $\lim_{n \to \infty} \|P^nf\|_1 = 2\|f\|_1$.

Ornstein and Sucheston [38] obtained an analytic proof of the Jamison-Orey result, and in their work they proved the following theorem [38, Theorem 1.1].

Theorem 1.1. Let $T : L_1 \to L_1$ be a positive contraction. Then either

$$\sup_{\|f\|_1 \leq 1} \lim_{n \to \infty} \|T^{n+1}f - T^nf\| = 2.$$ 

or $\|T^{n+1}f - T^nf\| \to 0$ for every $f \in L^1$.

This result was later called a strong zero-two law. Consequently, [38, Theorem 1.3], if $T$ is ergodic with $T^*1 = 1$ (e.g. $T$ is ergodic and conservative), then either (1.1) holds, or $\|T^ng\|_1 \to 0$ for every $g \in L^1$ with $\int g \, d\mu = 0$. Some extensions of the strong zero-two law can be found in [2, 42].

Interchanging "sup" and "lim" in the strong zero-two law we have the following uniform zero-two law, proved by Foguel [12] using ideas of [38] and [11].
Theorem 1.2. Let $T : L_1 \to L_1$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then 
\[
\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.
\]

Zahoropol [44] has provided another proof of Theorem 1.2 which is given in the following theorem.

Theorem 1.3. [44] Let $T : L_1 \to L_1$ be a positive contraction. Then for the following statements:

(i) there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - T^m\| < 2$;
(ii) there is some $m \in \mathbb{N}$ such that $\|T^{m+1} - (T^{m+1} \wedge T^m)\| < 1$;
(iii) one has 
\[
\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.
\]

the implications hold: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

To establish the implication (ii) $\Rightarrow$ (iii) in [44] the following auxiliary fact was established.

Theorem 1.4. [44] Let $T, S : L_1 \to L_1$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$.

In [23, 34, 30] we have extended the last result for several kind of Banach spaces. Therefore, it is natural step is to find analogous of Theorem 1.3 in a non-commutative setting. Note that a strong version of the uniform "zero-two" law was proved in [32], but it was not a desirable result.

The aim of this paper is to prove a non-commutative version of a generalized uniform "zero-two" law for multi-parametric family of positive contractions of $L_1$-spaces associated with von Neumann algebras. As a particular case (when the algebra is commutative), we recover the results of [30, 31]. Moreover, we emphasize that Theorem 1.2 will be included in the main result as a particular case.

On the other hand, development of the theory of integration for measures $\mu$ with the values in Dedekind complete Riesz spaces has inspired the study of (bo)-complete lattice-normed $L_p$-spaces (see, for example, [29]). The existence of center-valued traces on finite von Neumann algebras naturally leads to develop the theory of integration for this kind of traces. In [16] non-commutative $L_p$-spaces associated with with central-valued traces have been investigated. Furthermore, in [5] more general $L_p$-spaces associated with Maharam traces have been studied.

Therefore, another main aim of this paper is to establish the uniform "zero-two" law for non-commutative $L_1$-spaces associated with with central-valued traces. It is known [16] that $L_p$-spaces associated with central-valued traces are Banach-Kantorovich spaces. The theory of Banach-Kantorovich spaces is now sufficiently well-developed (for instance, see [29]). One of the important approach to study Banach-Kantorovich spaces is provided by the theory of continuous and measurable Banach bundles [25, 26]. In this approach, the representation of a Banach-Kantorovich lattice as a space of measurable sections of a measurable Banach bundle makes it possible to obtain the needed properties of the lattice by means of the corresponding stalkwise verification of the properties.
As an application of this approach, in [16] noncommutative $L_p(M, \Phi)$-spaces associated with center-valued traces are represented as bundle of noncommutative $L_p$-spaces associated with numerical traces. For other applications of the mentioned method, we refer the reader to [1, 4], [19]-[23].

In the second part of this paper, we are going to prove a vector-valued analogous of the main result for positive contractions of non-commutative $L_1$-spaces associated with central-valued trace. To establish the result, we mainly employ the last mentioned approach for the existence of vector-valued lifting, which allowed us to prove the required result. If the algebra is commutative, then the obtained result extends the main result of [23].

Let us outline of the organization of the present paper. In section 2, we collect some necessary well-know facts about non-commutative $L_1$-spaces. In section 3, we prove an auxiliary result (a non-commutative analogous of Theorem 1.4) about dominant operators. Section 4 is devoted to the proof of a generalized uniform "zero-two" law for multi-parametric family of positive contractions of the non-commutative $L_1$-spaces. In section 5, we recall necessary definitions about $L_1(M, \Phi)$ – the non-commutative $L_1$-spaces associated with center valued traces. Section 6 contains an auxiliary result about the existence of the non-commutative vector-valued lifting. Finally, in section 7, by means of the result of section 6, we first prove that every positive contraction of $L_1(M, \Phi)$ can be represented as a measurable bundle of positive contractions of non-commutative $L_1$-spaces, and this allowed us to establish a vector-valued analogous of the uniform "zero-two" law for positive contractions of $L_1(M, \Phi)$.

2. Preliminaries

Throughout the paper $M$ would be a von Neumann algebra with the unit $I$ and let $\tau$ be a faithful normal semi finite trace on $M$. Therefore we omit this condition from the formulation of theorems. Recall that an element $x \in M$ is called self-adjoint if $x = x^*$. The set of all self-adjoint elements is denoted by $M_{sa}$. By $M_*$ we denote a pre-dual space to $M$ (see for definitions [3], [41]).

Let $\mathfrak{N} = \{x \in M : \tau(|x|) < \infty\}$, here $|x|$ denotes the modules of an element $x$, i.e. $|x| = \sqrt{x^*x}$. Define the map $\| \cdot \|_1 : \mathfrak{N} \rightarrow [0, \infty)$ defined by the formula $\|x\|_1 = \tau(|x|)$ is a norm (see [36]). The completion of $\mathfrak{N}$ with respect to the norm $\| \cdot \|_1$ is denoted by $L_1(M, \tau)$. It is known [36] that the spaces $L_1(M, \tau)$ and $M_*$ are isometrically isomorphic, therefore they can be identified. Further we will use this fact without noting.

Theorem 2.1. [36] The space $L_1(M, \tau)$ coincides with the set

$$L_1 = \{x = \int_{-\infty}^{\infty} \lambda e_\lambda : \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda) < \infty\}.$$ 

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda).$$

It is known [36] that the equality

$$(2.1) \quad L_1(M, \tau) = L_1(M_{sa}, \tau) + iL_1(M_{sa}, \tau)$$
is valid. Note that \( L_1(Msa, \tau) \) is a pre-dual to \( M_{sa} \).

Let \( T : L_1(M, \tau) \to L_1(M, \tau) \) be any bounded linear operator, by \( \dot{T} \) we denote its restriction to \( L_1(M_{sa}, \tau) \). Then due to (2.1) we have \( T(x + iy) = \dot{T}(x) + i\dot{T}(y) \), where \( x, y \in L_1(M_{sa}, \tau) \). This means that any linear bounded operator is uniquely defined by its restriction to \( L_1(M_{sa}, \tau) \). Therefore, in what follows, we only consider linear operators on \( L_1(M_{sa}, \tau) \) over real numbers.

Recall that a linear operator \( T \) is called positive if \( Tx \geq 0 \) whenever \( x \geq 0 \). A linear operator \( T \) is said to be a contraction if \( \|T(x)\|_1 \leq \|x\|_1 \) for all \( x \in L_1(M_{sa}, \tau) \). Denote

\[
\|T\| = \sup\{\|Tx\|_1 : \|x\|_1 = 1, x \in L_1(M_{sa}, \tau)\}.
\]

Let \( T, S : L_1 \to L_1 \) be two positive contractions. In what follows, we write \( T \leq S \) if \( S - T \) is a positive operator.

The following auxiliary facts are well known (see for example [34]).

**Lemma 2.2.** Let \( T : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau) \) be a positive operator. Then

\[
\|T\| = \sup_{\|x\|_1 = 1} \|Tx\| = \sup_{\|x\|_1 = 1, x \geq 0} \|Tx\|.
\]

**Lemma 2.3.** Let \( T, S : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau) \) be two positive contraction such that \( T \leq S \). Then for every \( x \in L_1(M_{sa}, \tau) \), \( x \geq 0 \) the equality holds

\[
\|Sx - Tx\| = \|Sx\| - \|Tx\|.
\]

3. Dominant operators

In this section we are going to prove an auxiliary result related to dominant operators.

**Theorem 3.1.** Let \( Z, T, S : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau) \) be positive contractions such that \( T \leq S \) and \( ZS = SZ \). If there is an \( n_0 \in \mathbb{N} \) such that \( \|Z(S^{n_0} - T^{n_0})\| < 1 \). Then \( \|Z(S^n - T^n)\| < 1 \) for every \( n \geq n_0 \).

**Proof.** Assume the contrary, i.e. \( \|Z(S^n - T^n)\| = 1 \) for some \( n > n_0 \). Denote

\[
m = \min\{n \in \mathbb{N} : \|Z(S^{n_0+n} - T^{n_0+n})\| = 1\}.
\]

It is clear that \( m \geq 1 \). The positivity of \( Z \) with \( T \leq S \) implies that \( Z(S^{n_0+n} - T^{n_0+n}) \) is a positive operator. Then according to Lemma 2.2 there exists a sequence \( \{x_n\} \in L_1(M_{sa}, \tau) \) such that \( x_n \geq 0, \|x_n\| = 1, \forall n \in \mathbb{N} \) and

\[
\lim_{n \to \infty} \|Z(S^{n_0+n} - T^{n_0+n})x_n\| = 1.
\]

The positivity of \( Z(S^{n_0+n} - T^{n_0+n}) \) and \( x_n \geq 0 \) together with Lemma 2.3 yield that

\[
\|Z(S^{n_0+n} - T^{n_0+n})x_n\| = \|ZS^{n_0+n}x_n\| - \|ZT^{n_0+n}x_n\|
\]

for every \( n \in \mathbb{N} \). It then follows from (3.1), (3.2) that

\[
\lim_{n \to \infty} \|ZS^{n_0+n}x_n\| = 1,
\]

\[
\lim_{n \to \infty} \|ZT^{n_0+n}x_n\| = 0.
\]
Thanks to the contractivity of $S$, $Z$ together with $ZS = SZ$ we obtain

$$\|ZS^{m_0+m}x_n\| = \|S(ZS^{m_0+m-1}x_n)\| \leq \|ZS^{m_0+m-1}x_n\| \leq \|S^m x_n\|.$$ 

Hence, the last ones with (3.3) imply

$$\lim_{n \to \infty} \|ZS^{m_0+m-1}x_n\| = 1, \quad \lim_{n \to \infty} \|S^m x_n\| = 1. \quad (3.5)$$

Moreover, the contractivity of $Z, S$ and $T$ ($i = 1, 2$) implies that $\|ZT^{m_0+m-1}x_n\| \leq 1$, $\|T^m x_n\| \leq 1$ and $\|ZS^{m_0}T^m x_n\| \leq 1$ for every $n \in \mathbb{N}$. Therefore, we may choose a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequences $\{\|ZT^{m_0+m-1}y_k\|\}, \{\|T^m y_k\|\}$, $\{\|ZS^{m_0}T^m y_k\|\}$ converge. Hence, denote their limits as follows

$$\alpha = \lim_{k \to \infty} \|ZT^{m_0+m-1}y_k\|, \quad (3.6)$$

$$\beta = \lim_{k \to \infty} \|ZS^{m_0}T^m y_k\|, \quad (3.7)$$

$$\gamma = \lim_{k \to \infty} \|T^m y_k\|. \quad (3.8)$$

The inequality $\|Z(S^{m_0+m-1} - T^{m_0+m-1})\| < 1$ with (3.5) implies that $\alpha > 0$. Hence we may choose a subsequence $\{z_k\}$ of $\{y_k\}$ such that $\|ZT^{m_0+m-1}z_k\| \neq 0$ for all $k \in \mathbb{N}$. From $\|ZT^{m_0+m-1}z_k\| \leq \|T^m z_k\|$ together with (3.6), (3.8) we find $\alpha \leq \gamma$, and hence $\gamma > 0$.

Now using Lemma 2.3 one gets

$$\|ZS^{m_0}T^m z_k\| = \|ZS^{m_0+m}z_k - Z(S^{m_0+m} - S^{m_0}T^m)z_k)\|
= \|ZS^{m_0+m}z_k\| - \|ZS^{m_0}(S^m - T^m)z_k\|
\geq \|ZS^{m_0+m}z_k\| - \|S^m z_k - T^m z_k\|
= \|ZS^{m_0+m}z_k\| - \|S^m z_k\| + \|T^m z_k\| \quad (3.9)$$

Due to (3.3) and (3.5) we have

$$\lim_{k \to \infty} \left(\|ZS^{m_0+m} z_k\| - \|S^m z_k\|\right) = 0;$$

which with (3.9) implies that

$$\lim_{k \to \infty} \|ZS^{m_0}T^m z_k\| \geq \lim_{k \to \infty} \|T^m z_k\|,$$

this means $\beta \geq \gamma$.

On the other hand, from $\|ZS^{m_0}T^m z_k\| \leq \|T^2 z_k\|$ it follows that $\gamma \geq \beta$, so $\gamma = \beta$.

Let us denote

$$u_k = \frac{T^m z_k}{\|T^m z_k\|}, \quad k \in \mathbb{N}.$$ 

Then from $\gamma = \beta$ together with (3.4) we obtain

$$\lim_{k \to \infty} \|ZS^{m_0}u_k\| = \lim_{k \to \infty} \frac{\|ZS^{m_0}T^m z_k\|}{\|T^m z_k\|} = 1,$$

$$\lim_{k \to \infty} \|ZT^{m_0}u_k\| = \lim_{k \to \infty} \frac{\|ZT^{m_0+m} z_k\|}{\|T^m z_k\|} = 0.$$ 

So, keeping in mind Lemma 2.3 and the positivity of $Z(S^{m_0} - T^{m_0})$, one finds that

$$\lim_{k \to \infty} \|Z(S^{m_0} - T^{m_0})u_k\| = 1.$$
Due to \( \|u_k\| = 1, u_k \geq 0 \) (for all \( k \in \mathbb{N} \)) from Lemma \( \text{(2.2)} \) we infer that \( \|Z(S^{n_0} - T^{n_0})\| = 1 \), which is a contradiction. This completes the proof. \( \square \)

We note that the proved theorem extends a main result of the paper \( \text{(33)} \), which can be seen in the following corollary.

**Corollary 3.2.** Let \( T, S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau) \) be positive contractions such that \( T \leq S \). If there is an \( n_0 \in \mathbb{N} \) such that \( \|S^{n_0} - T^{n_0}\| < 1 \). Then \( \|S^n - T^n\| < 1 \) for every \( n \geq n_0 \).

The proof immediately follows if one takes \( Z = Id \). Note that if \( n_0 = 1 \) and \( M \) is a commutative von Neumann algebra, then from Corollary 3.2 we immediately get the Zaharopol’s result (see Theorem \( \text{(1.4)} \)).

4. A MULTI-PARAMETRIC GENERALIZATION OF THE ZERO-TWO LAW

In this section we are going to prove a multi-parametric generalization of the zero-two law for positive contractions of non-commutative \( L_1 \)-space.

Let \( T : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau) \) be a positive contraction. Then its conjugate \( T^* \) acts on \( M_{sa} \), and it is also positive and enjoys \( T^* \mathbf{1} \leq \mathbf{1} \). If one has \( T^* \mathbf{1} = \mathbf{1} \), then \( T \) is called unital positive contraction.

Let us first introduce some notations. Denote \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For any \( m = (m_1, \ldots, m_d), n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d \) \((d \geq 1)\) by the usual way, we define \( m + n = (m_1 + n_1, \ldots, m_d + n_d), \ell n = (\ell n_1, \ldots, \ell n_d) \), where \( \ell \in \mathbb{N}_0 \). We write \( n \leq k \) if and only if \( n_i \leq k_i \) \((i = 1, 2, \ldots, d)\). We denote \( |n| := n_1 + \cdots + n_d \).

Let us formulate our main result.

**Theorem 4.1.** Let \( Z : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau) \) be a unital positive contraction. Assume that \( T_k : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau) \), \((k = 1, \ldots, d)\) be unital positive contractions such that \( ZT_i = T_i^* Z, T_i T_j = T_j T_i \), for every \( i, j \in \{1, \ldots, d\} \). If there are \( m \in \mathbb{N}_0^d, k \in \mathbb{N}_0 \) and a positive contraction \( S : L_1(M_{sa}, \tau) \rightarrow L_1(M_{sa}, \tau) \) such that \( SZ = ZS \)

\begin{align}
ZT^{m+k} &\geq ZS, \quad ZT^m \geq ZS \quad \text{with} \\
\|Z(T^{m+k} - S)\| &< 1, \quad \|Z(T^m - S)\| < 1.
\end{align}

then for any \( \varepsilon > 0 \) there are \( M \in \mathbb{N} \) and \( n_0 \in \mathbb{N}_0^d \) such that

\[ \|Z^M(T^{m+k} - T^n)\| < \varepsilon \quad \text{for all} \quad n \geq n_0. \]

Here \( T^n := T_1^{n_1} \cdots T_d^{n_d}, n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d \).

**Proof.** First we note that for any positive contraction \( T \) on \( L_1 \)-spaces [45, p. 310] there is \( \gamma > 0 \) such that

\[ \left\| \frac{(I + T^k)}{2} - T^k \left( \frac{I + T^k}{2} \right)^\ell \right\| \leq \frac{\gamma}{\sqrt{\ell}}. \]

Now take any \( \varepsilon > 0 \) and fix \( \ell \in \mathbb{N} \) such that \( \gamma/\sqrt{\ell} < \varepsilon/2. \)

Define

\[ Q_1 = \frac{1}{2}(T^{m+k} - S) + \frac{1}{2}T^k(T^m - S). \]
It then follows from (4.8), (4.10) that $ZQ_1$ is positive and $\|ZQ_1\| < 1$. Moreover, one has

$$T^{m+k} = \left( \frac{I + T^k}{2} \right) S + Q_1$$

where $I$ stands for the identity mapping.

For each $\ell \in \mathbb{N}$ let us define

$$Q_{\ell+1} = \left( \frac{I + T^k}{2} \right)^\ell Q_1 S^\ell + T^{m+k} Q_\ell, \ \ell \in \mathbb{N}.$$

Taking into account the positivity of $S$ and $Q_1$, one can see that $Q_\ell$ is a positive operator on $L_1(M_{sa}, \tau)$ and $ZQ_\ell = Q_\ell Z$. Moreover, one has

$$T^{\ell(m+k)} = \left( \frac{I + T^k}{2} \right)^\ell S^\ell + Q_\ell, \ \ell \in \mathbb{N}.$$

Let us prove (4.4) by induction. Clearly, it is valid for $\ell = 1$. Assume that (4.4) is true for $\ell$, and we will prove it for $\ell + 1$. Indeed, one finds

\[
T^{(\ell+1)(m+k)} = T^{m+k} T^{\ell(m+k)} = \left( \frac{I + T^k}{2} \right)^\ell T^{m+k} S^\ell + T^{m+k} Q_\ell
\]
\[
= \left( \frac{I + T^k}{2} \right)^\ell \left( \left( \frac{I + T^k}{2} \right) S + Q_1 \right) S^\ell + T^{m+k} Q_\ell
\]
\[
= \left( \frac{I + T^k}{2} \right)^{\ell+1} S^{\ell+1} + \left( \frac{I + T^k}{2} \right)^\ell Q_1 S^\ell + T^{m+k} Q_\ell
\]
\[
= \left( \frac{I + T^k}{2} \right)^{\ell+1} S^{\ell+1} + Q_{\ell+1}
\]

which proves the required equality.

Now let us put $V^{(1)} = S^{\ell}$, and

$$V^{(d+1)}_{\ell} = T^{\ell(m+k)} V^{(d)}_{\ell} + V^{(1)}_{\ell} Q^d, \ d \in \mathbb{N}.$$  

One can see that for every $d, \ell \in \mathbb{N}$ the operator $ZV^{(d)}_{\ell}$ is positive, since $Z$ and $S$ are commuting. Moreover, one has

$$T^{d\ell(m+k)} = \left( \frac{I + T^k}{2} \right)^\ell V^{(d)}_{\ell} + Q^d_{\ell}, \ d, \ell \in \mathbb{N}.$$
Again let us prove the last equality by induction. Keeping in mind that (4.5) is true for \( d \), it is enough to establish (4.5) for \( d + 1 \). Indeed, we have

\[
T^{(d+1)\ell(m+k)} = T^{\ell(m+k)}T^{d(m+k)} = T^{\ell(m+k)} \left( \left( \frac{I + T^k}{2} \right)^\ell V^{(d)}_\ell + Q^{d}_\ell \right)
\]

\[
= \left( \frac{I + T^k}{2} \right)^\ell T^{d(m+k)}V^{(d)}_\ell + \left( \left( \frac{I + T^k}{2} \right)^\ell S^d + Q^d \right)Q^{d}_\ell
\]

\[
= \left( \frac{I + T^k}{2} \right)^\ell T^{\ell(m+k)}V^{(d)}_\ell + V^{(1)}_\ell Q^{d}_\ell + Q^{d+1}_\ell
\]

which proves (4.5).

From \( Z^*(1) = T^*(1) = 1 \), it follows from (4.5) that

\[
V^{(d)*}_\ell(1) + Q^{*d}_\ell(1) = 1.
\]

Now the positivity of \( ZV^{(d)}_\ell \) and \( ZQ^d_\ell \) imply that \( \|ZV^{(d)}_\ell\| \leq 1 \) and \( \|ZQ^d_\ell\| \leq 1 \).

From (4.3) and (4.10), due to Theorem 3.1 one finds that \( \|Z(T^m - S^\ell)\| < 1 \) for all \( \ell \in \mathbb{N} \). Using this inequality with \( T^*(1) = 1 \) and the positivity of \( Z(T^m - S^\ell) \) we find that

(4.6) \[
\|Z(T^m - S^\ell)\| = \|(T^*)^m - S^*\ell\|Z^* = \|1 - S^*\ell(1)\| < 1.
\]

The equality (4.4) yields that

\[
Q^{*}_\ell(1) = 1 - S^{*\ell}(1).
\]

Hence, from (4.6) with the positivity of \( ZQ^d_\ell \) we obtain

\[
\|ZQ^d_\ell\| = \|Q^{*}_\ell(1)\| = \|1 - S^{*\ell}(1)\| < 1
\]

for all \( \ell \in \mathbb{N} \).

Therefore, there is a number \( d_\varepsilon \in \mathbb{N} \) such that \( \|(ZQ^d_\ell)^{d_\varepsilon}\| < \frac{\varepsilon}{4} \). From the commutativity of \( Z \) and \( Q^d_\ell \) one finds

(4.7) \[
\|Z^{d_\varepsilon}Q^{d_\varepsilon}_\ell\| < \frac{\varepsilon}{4}.
\]
Now putting $n_0 = d_ε(\ell_ε(m) + k)$, from (4.3) with (4.7) we obtain
\[
\|Z^{d_ε}(T^{n_0 + k} - T^{n_0})\| = \|Z^{d_ε}(T^{d_ε(\ell_ε(m + k))} - T^{d_ε(\ell_ε(m + k))})\|
\]
\[
\leq \left\| Z^{d_ε}\left(T^k(I + T^k)^{\ell_ε} - (I + T^k)^{\ell_ε}\right) \right\| + \left\| Z^{d_ε}Q^{d_ε}_e(T^k - I) \right\|
\]
\[
\leq \left\| T^k(I + T^k)^{\ell_ε} - (I + T^k)^{\ell_ε} \right\| + 2\|Z^{d_ε}Q^{d_ε}_e\|
\]
\[
\leq \frac{\gamma}{\sqrt{\ell_ε}} + 2 \cdot \frac{\varepsilon}{4} < \varepsilon.
\]
Take any $n \geq n_0$ then from the last inequality one gets
\[
\|T^{n+k} - T^n\| = \|T^{n-n_0}(T^{n_0+k} - T^{n_0})\| \leq \|T^{n_0+k} - T^{n_0}\| < \varepsilon
\]
which completes the proof.

**Corollary 4.2.** Assume that $T_k : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau)$, $(k = 1, \ldots, d)$ be unital positive contractions such that $T_i T_j = T_j T_i$, for every $i, j \in \{1, \ldots, d\}$. If there are $m \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0^d$ and a positive contraction $S : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau)$ such that
\[
T^{m+k} \geq S, \quad T^m \geq S
\]
with
\[
\|T^{m+k} - S\| < 1, \quad \|T^m - S\| < 1.
\]
then one has
\[
\lim_{n \to \infty} \|T^{n+k} - T^n\| = 0.
\]
The proof immediately follows from Theorem 4.1 if one takes $Z = Id$.

**Remark 4.3.** We note that in [24, 37] a similar kind of result, for a single contractions of $C^*$-algebras, has been proved. Our main result extents it for more general multi-parametric contractions. We point out that if the algebra becomes commutative, then the proved theorems cover main results of [31].

**Corollary 4.4.** Let $T, S : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau)$ be two commuting unital positive contractions. If for some $m_0 \in \mathbb{N}$ and a positive contraction $S : L_1(M_{sa}, \tau) \to L_1(M_{sa}, \tau)$ one has
\[
T^{m_0+k} S^{m_0} \geq S, \quad T^{m_0} S^{m_0} \geq S
\]
with
\[
\|T^{m_0+k} S^{m_0} - S\| < 1, \quad \|T^{m_0} S^{m_0} - S\| < 1.
\]
then
\[
\lim_{m, n \to \infty} \|T^{n+k} S^m - T^n S^m\| = 0.
\]
The proof immediately follows from Corollary 4.2 if one takes $m = (m_0, m_0)$ and $k = (k, 0)$. 
Remark 4.5. Since the dual of $L_1(M_{sa}, \tau)$ is $M_{sa}$ then due to the duality theory the proved Theorem 4.1 holds true if we replace $L_1$-space with $M_{sa}$.

5. Noncommutative $L_1$-space associated with center valued trace

In this section we recall some necessary notions and facts about the noncommutative $L_1$-spaces associated with center valued trace.

Let $M$ be any finite von Neumann algebra, $S(M)$ be the set all measurable operators affiliated to $M$ (see [40] for definitions). Let $Z$ be some subalgebra of the center $Z(M)$. Then one may identify $Z$ with $*$-algebra $L_\infty(\Omega, \Sigma, m)$ and do $S(Z)$ with $L_0(\Omega, \Sigma, m)$. Recall that a center valued (i.e. Z-valued) trace on the von Neumann algebra $M$ is a $Z$-linear mapping $\Phi : M \rightarrow Z$ with $\Phi(x^*x) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_+) \subset Z_+$. A trace $\Phi$ is said to be faithful if the equality $\Phi(x^*x) = 0$ implies $x = 0$, normal if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_{sa}, x_\alpha \uparrow x$. Note that the existence of such kind of traces has been studied in [6].

Let $M$ be an arbitrary finite von Neumann algebra, $\Phi$ be a center-valued trace on $M$. The locally measure topology $t(M)$ on $S(M)$ is the linear (Hausdorff) topology whose fundamental system of neighborhoods of 0 is given by

$$V(B, \varepsilon, \delta) = \{x \in S(M) : \text{there exists } p \in P(M), z \in P(Z(M)) \text{ such that } xp \in M, \|xp\|_M \leq \varepsilon, z^{-1} \in W(B, \varepsilon, \delta), \Phi_M(zp^{-1}) \leq \varepsilon z\},$$

where $\| \cdot \|_M$ is the $C^*$-norm in $M$. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra [43].

From [35] §3.5] we have the following criterion for convergence in the topology $t(M)$.

Proposition 5.1. A net $\{x_\alpha\}_{\alpha \in A} \subset S(M)$ converges to zero in the topology $t(M)$ if and only if $\Phi_M(E_\lambda^\perp(\|x_\alpha\|) \xrightarrow{t(M)} 0$ for any $\lambda > 0$.

Following [5] an operator $x \in S(M)$ is said to be $\Phi$-integrable if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{t(M)} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(Z)} 0$ as $n, m \rightarrow \infty$.

Let $x$ be a $\Phi$-integrable operator from $S(M)$. Then there exists a $\hat{\Phi}(x) \in S(Z)$ such that $\Phi(x_n) \xrightarrow{t(Z)} \hat{\Phi}(x)$. In addition $\hat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{t(M)} x, \Phi(|x_n - x_m|) \xrightarrow{t(Z)} 0$ [5]. It is clear that each operator $x \in M$ is $\Phi$-integrable and $\hat{\Phi}(x) = \Phi(x)$.

Denote by $L_1(M, \Phi)$ the set of all $\Phi$-integrable operators from $S(M)$. If $x \in S(M)$ then $x \in L_1(M, \Phi)$ iff $|x| \in L_1(M, \Phi)$, in addition $|\hat{\Phi}(x)| \leq \hat{\Phi}(|x|)$ [6]. For any $x \in L_1(M, \Phi)$, set $\|x\|_\Phi = \hat{\Phi}(|x|)$. It is known that $L_1(M, \Phi)$ is a linear subspace of $S(M)$, $ML_1(M, \Phi)M \subset L_1(M, \Phi)$, and $x^* \in L_1(M, \Phi)$ for all $x \in L_1(M, \Phi)$ [6].

Now let us recall some facts about Banach–Kantorovich spaces over the ring of measurable functions [26].

Let $X$ be a mapping which maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \| \cdot \|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function $u$ is said to be a section of $X$, if it is defined almost everywhere in $\Omega$ and takes its value $u(\omega) \in X(\omega)$ for $\omega \in \text{dom}(u)$, where $\omega \in \text{dom}(u)$ is the domain of $u$. Let $L$ be some set of sections.
A pair \((X, L)\) is said to be a **measurable bundle of Banach spaces** over \(\Omega\) if

1. \(\lambda_1 c_1 + \lambda_2 c_2 \in L\) for all \(\lambda_1, \lambda_2 \in \mathbb{R}\) and \(c_1, c_2 \in L\), where \(\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \text{dom}(c_1) \cap \text{dom}(c_2) \rightarrow \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)\);
2. the function \(\|c\| : \omega \in \text{dom}(c) \rightarrow \|c(\omega)\|_{X(\omega)}\) is measurable for all \(c \in L\);
3. for every \(\omega \in \Omega\) the set \(\{c(\omega) : c \in L, \omega \in \text{dom}(c)\}\) is dense in \(X(\omega)\).

A section \(s\) is a step-section, if there are pairwise disjoint sets \(A_1, A_2, \ldots, A_n \in \Sigma\) and sections \(c_1, c_2, \ldots, c_n \in L\) such that \(\bigcup_{i=1}^n A_i = \Omega\) \(s(\omega) = \sum_{i=1}^n \chi_{A_i}(\omega)c_i(\omega)\) for almost all \(\omega \in \Omega\).

A section \(u\) is measurable, if for any \(A \in \Sigma\) there is a sequence \(s_n\) of step-sections such that \(s_n(\omega) \rightarrow u(\omega)\) for almost all \(\omega \in A\).

Let \(M(\Omega, X)\) be the set of all measurable sections. By symbol \(L_0(\Omega, X)\) we denote factorization of the \(M(\Omega, X)\) with respect to almost everywhere equality. Usually, by \(\tilde{u}\) we denote a class from \(L_0(\Omega, X)\), containing a section \(u \in M(\Omega, X)\), and by \(\|\tilde{u}\|\) we denote an element of \(L_0(\Omega)\), containing \(\|u(\omega)\|_{X(\omega)}\). Let \(L^\infty(\Omega, X) = \{u \in M(\Omega, X) : \|u(\omega)\|_{X(\omega)} \in L^\infty(\Omega)\}\) and \(L^\infty(\Omega, X) = \{\tilde{u} \in L_0(\Omega, X) : \|\tilde{u}\| \in L^\infty(\Omega)\}\). One can define the spaces \(L^\infty(\Omega, X)\) and \(L^\infty(\Omega, X)\) with real-valued norms

\[\|u\|_{L^\infty(\Omega, X)} = \sup_{\omega \in \Omega} \|u(\omega)\|_{X(\omega)}\]

and \(\|\tilde{u}\|_{L^\infty(\Omega)} = \left\|\tilde{u}\right\|_{L^\infty(\Omega)}\), respectively.

**Definition 5.3.** Let \(X,Y\) be measurable bundles of Banach spaces. A set linear operators \(\{T(\omega) : X(\omega) \rightarrow Y(\omega)\}\) is called **measurable bundle of linear operators** if \(T(\omega)(u(\omega))\) is measurable section for any measurable section \(u\).

Let \((X, L)\) be a measurable bundle of Banach spaces. If each \(X(\omega)\) is a noncommutative \(L_1\)-space, i.e. \(X(\omega) = L_1(M(\omega), \tau_\omega)\), associated with finite von Neumann algebras \(M(\omega)\) and with strictly normal numerical trace \(\tau_\omega\) on \(M(\omega)\), then the measurable bundle \((X, L)\) of Banach spaces is called **measurable bundle of noncommutative \(L_1\)-spaces**.

**Theorem 5.4.** There exists a measurable bundle \((X, L)\) of noncommutative \(L_1\)-spaces \(L_1(M(\omega), \tau_\omega)\), such that \(L_0(\Omega, X)\) is Banach—Kantorovich *-algebroid, which is isometrically and order *-isomorph to \(L_1(M, \Phi)\). Moreover, the isometric and order *-isomorphism \(H : L_1(M, \Phi) \rightarrow L_0(\Omega, X)\) can be chosen with the following properties

(a) \(\Phi(x)(\omega) = \tau_\omega(H(x)(\omega))\) for all \(x \in M\) and for almost all \(\omega \in \Omega\);

(b) \(x \in M\) if and only if \(H(x)(\omega) \in M(\omega)\) a.e. and there exist positive number \(\lambda > 0\), that \(\|H(x)(\omega)\|_{M(\omega)} \leq \lambda\) for almost all \(\omega\);

(c) \(z \in Z\) if and only if \(H(z) = (\widehat{z(\omega)}1_\omega)\) for some \(\widehat{z(\omega)} \in L_\infty(\Omega)\), where \(1_\omega\) — unit of algebra \(M(\omega)\), in particular \(H(1)(\omega) = 1_\omega\) for almost all \(\omega\).

(d) the section \((H(x)(\omega))^*\) is measurable for all \(x \in L_1(M, \Phi)\).
(e) the section \( H(x)(\omega) \cdot H(y)(\omega) \) is measurable for all \( x, y \in M \).

6. The existence of lifting

In this section we establishes the existence of the lifting in a non-commutative setting. Note that in case of \( C^* \)-algebras, the existence of the lifting has been given in \([17]\) (see also \([20]\)).

Let \( M \) be a von Neumann algebra. Then it can be identify with a linear subspace of \( L^\infty(\Omega, X) \) by the isomorphism \( H \), since if \( x \in M \), then one has

\[
\|H(x)\|_{L^0(\Omega, X)} = \|x\|_1 = \Phi(|x|) \in L^\infty(\Omega)
\]

**Theorem 6.1.** There exists a mapping \( \ell : M(\subset L^\infty(\Omega, X)) \rightarrow L^\infty(\Omega, X) \) with following properties

(a) for every \( x \in M \) one has \( \ell(x) \in x \), \( \text{dom} \ell(x) = \Omega \);

(b) if \( x_1, x_2 \in M \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \), then \( \ell(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \ell(x_1) + \lambda_2 \ell(x_2) \);

(c) \( \|\ell(x)(\omega)\|_{L^0(M(\omega), \tau_\omega)} = p(\|x\|_p)(\omega) \) for all \( x \in M \) and for all \( \omega \in \Omega \);

(d) if \( x \in M \), \( \lambda \in L^\infty(\Omega) \), then \( \ell(\lambda x) = \lambda \ell(x) \);

(e) if \( x \in M \), then \( \ell(x^*) = \ell(x)^* \);

(f) if \( x, y \in M \), then \( \ell(xy) = \ell(x)\ell(y) \);

(g) the set \( \{\ell(x)(\omega) : x \in M\} \) is dense in \( L_p(M(\omega), \tau_\omega) \) for all \( \omega \in \Omega \).

**Proof.** Following \([16]\) for every \( x \in M \) we define

\[
\Phi_0(x) = \Phi(x)(1 + \Phi(1))^{-1}.
\]

One can see that \( \Phi_0 \) is an \( L^\infty(\Omega) \)-valued faithful normal trace on \( M \). By \( \rho \) we denote the lifting on \( L^\infty(\Omega) \) (see \([25]\)). Now define a finite trace \( \varphi_\omega \) on \( M \) by \( \varphi_\omega(x) = \rho(\Phi_0(x))(\omega) \), where \( \omega \in \Omega \). Due to \([7\) Lemma 6.4.1] the function \( s_\omega(x, y) = \varphi_\omega(y^*x) \) is a bi-trace on \( M \), and therefore, the equality \( \langle x, y \rangle_\omega = s_\omega(x, y) \) defines a quasi-inner product on \( M \).

Denote \( I_\omega = \{x \in M : s_\omega(x, x) = 0\} \). It is known that \( I_\omega \) is a two-sided ideal in \( M \), therefore, one considers the quotient space \( \Gamma_\omega = M/I_\omega \), by \( \pi_\omega : M \rightarrow \Gamma_\omega \) we denote the canonical mapping. The involution and multiplication are defined on \( \Gamma_\omega \) by the usual way, i.e. \( \pi_\omega(x)^* = \pi_\omega(x^*) \) and \( \pi_\omega(x) \cdot \pi_\omega(y) = \pi_\omega(xy) \). According to \([7\) Proposition 6.2.3] \( \Gamma_\omega \) is a Hilbert algebra. By \( \mathcal{H}(\omega) \) we denote the Hilbert space which is the completion of \( \Gamma_\omega \); the inner product in \( \mathcal{H}(\omega) \) we denote by the same symbol, i.e. \( \langle \cdot, \cdot \rangle_\omega \).

The mapping \( \pi_\omega(x) \rightarrow \pi_\omega(y)\pi_\omega(x), x, y \in M \), can be extended by continuity to a bounded linear operator \( T_\omega(y) \) on \( \mathcal{H}(\omega) \). It is known \([7\) that \( T_\omega(x) \) is a representation of \( M \) in \( \mathcal{H}(\omega) \). Let \( M(\omega) \) be the von Neumann algebra generated by \( T_\omega(M) \), i.e. \( M(\omega) = T_\omega(M)'' \). By \( \mu_\omega \) we denote the natural trace on \( M(\omega) \) which is defined by \( \mu_\omega(\pi_\omega(x)) = \langle \pi_\omega(x) \mathbf{1}_\omega, \mathbf{1}_\omega \rangle_\omega \) for all \( \pi_\omega(x) \in \Gamma_\omega \). One can see that \( \mu_\omega \) is a faithful,
normal and finite trace on $M(\omega)$ (see [7 Proposition 6.8.3]). Now let us consider
a non-commutative $L_1$-space $L_1(M(\omega), \tau_w)$, where $\tau_w(\cdot) = (1 + \Phi(I))(\omega)\mu_{\omega}(\cdot)$. By $i_{\omega} : \Gamma_\omega \to M(\omega)$ one denotes the canonical embedding, and $j_{\omega} : (M(\omega), \tau_w) \to L_1(M(\omega), \tau_w)$
denotes the natural embedding. Then $\gamma_{\omega} = j_{\omega} \circ i_{\omega} \circ \pi_{\omega}$ is a linear mapping from $M$
to $L_1(M(\omega), \tau_w)$.

Let us define

$$\ell(x)(\omega) = \gamma_{\omega}(x)$$

for any $x \in M$.

(a) Since any element $x \in M$ is identified with the element $\gamma_{\omega}(x)$, then one has

$$\ell(x) \in x \text{ (see [16]).}$$

(b) The linearity of $\ell$ is obvious.

(c) Let $x \in M$. Then

$$\|\ell(x)(\omega)\|_{L_1(M(\omega), \tau_w)} = \|\gamma_{\omega}(x)\|_{L_1(M(\omega), \tau_w)} = \tau_{\omega}(\pi_{\omega}(x))$$

$$= \tau_{\omega}(\pi_{\omega}(|x|)) = \rho(\Phi(|x|))(\omega)$$

$$= \rho(\|x\|_1)(\omega)$$

for all $\omega \in \Omega$.

(d) Let $\chi_A \in L^\infty(\Omega)$ and $x \in M$, then $\chi_A \cdot x \in M$. By $\Sigma$ we denote a complete
Boolean algebra of equivalent classes w.r.t. a.e. equality, of sets from $\Sigma$. The lifting $\rho : L^\infty(\Omega) \to L^\infty(\Omega)$ induces a lifting $\tilde{\rho} : \tilde{\Sigma} \to \Sigma$ such that $p(\chi_A) = \chi_{\tilde{\rho}(A)}$. Due to

$$\|\pi_{\omega}(\chi_A x)\|_{L_1(M(\omega), \tau_w)} = \rho(\|\chi_A \cdot x\|_1)(\omega) = \rho(\chi_A)(\omega) \cdot p(\|x\|_1)(\omega)$$

$$= \rho(\chi_A)(\omega) \cdot \|\pi_{\omega}(x)\|_{L_1(M(\omega), \tau_w)}$$

$$= \chi_{\tilde{\rho}(A)}(\omega) \cdot \|\pi_{\omega}(x)\|_{L_1(M(\omega), \tau_w)},$$

we obtain $\pi_{\omega}(\chi_A \cdot x) = 0$, if $\omega \in \tilde{\rho}(A)$. Let $\omega \in \tilde{\rho}(A)$, then

$$\|\pi_{\omega}(\chi_A \cdot x) - \pi_{\omega}(x)\|_{L_1(M(\omega), \tau_w)} = \rho(\|\chi_A \cdot x\|_1 - \|x\|_1)(\omega)$$

$$= \rho(\|\chi_A \cdot x\|_1)(\omega) = \rho(\chi_{\tilde{\rho}(A)})(\omega) \cdot p(\|x\|_1)(\omega)$$

$$= \chi_{\tilde{\rho}(\Omega \setminus A)}(\omega) \cdot \|\pi_{\omega}(x)\|_{L_1(M(\omega), \tau_w)} = 0.$$

Therefore, we have $\pi_{\omega}(\chi_A \cdot x) = \chi_{\tilde{\rho}(A)}(\omega) \cdot \pi_{\omega}(x) = p(\chi_A)(\omega) \cdot \pi_{\omega}(x)$
for all $\omega \in \Omega$.

Let $\lambda = \sum_{i=1}^n \alpha_i \chi_{A_i} \in L^\infty(\Omega)$ be a simple function. Then

$$\pi_{\omega}(\lambda x) = \pi_{\omega}\left(\sum_{i=1}^n \alpha_i \chi_{A_i} x\right) = \sum_{i=1}^n \pi_{\omega}(r_i \chi_{A_i} x)$$

$$= \sum_{i=1}^n r_i \rho(\chi_{A_i})(\omega) \pi_{\omega}(x) = \rho(\lambda)(\omega) \pi_{\omega}(u).$$
The density argument implies that for any \( \lambda \in L^\infty(\Omega) \) there exists a sequence of simple functions \( \{\lambda_n\} \) such that \( \|\lambda_n - \lambda\|_{L^\infty(\Omega)} \to 0 \) as \( n \to \infty \). From
\[
\|\pi_\omega(\lambda_n x) - \pi_\omega(\lambda x)\|_{L^1(M(\omega), \tau_\omega)} = \|\pi_\omega((\lambda_n - \lambda)x)\|_{L^1(M(\omega), \tau_\omega)} = \rho(\| (\lambda_n - \lambda)x \|_1)(\omega)
\]
\[
= \rho(|\lambda_n - \lambda|)(\omega) \cdot \rho(\|x\|_1)(\omega)
\]
\[
\leq \|\rho(\lambda_n - \lambda)\|_{L^\infty(\Omega)} : \|\rho(\|x\|_1)\|_{L^\infty(\Omega)}
\]
\[
= \|\lambda_n - \lambda\|_{L^\infty(\Omega)} \cdot \|\|x\|_1\|_{L^\infty(\Omega)}
\]
one gets \( \pi_\omega(\lambda x) = \lim_{n \to \infty} \pi_\omega(\lambda_n x) \) for all \( \omega \in \Omega \). So,
\[
\pi_\omega(\lambda x) = \lim_{n \to \infty} \pi_\omega(\lambda_n x) = \lim_{n \to \infty} \rho(\lambda_n)(\omega)\pi_\omega(x) = \rho(\lambda)(\omega)\pi_\omega(u)
\]
for all \( \omega \in \Omega \).

Hence,
\[
i_\omega(\pi_\omega(\lambda x) = i_\omega(\rho(\lambda)(\omega)\pi_\omega(x)) = \rho(\lambda)(\omega)i_\omega(\pi_\omega(x)) = \rho(\lambda)(\omega)i_\omega(\pi_\omega(x))
\]
and \( j_\omega(\lambda(x)) = \rho(\lambda)(\omega)j_\omega(x) \) for all \( \omega \in \Omega \). These mean that \( \ell(\lambda x) = \rho(\lambda)\ell(x) \) for any \( x \in M \) and \( \lambda \in L^\infty(\Omega) \).

(e) According to \( \gamma_\omega(x^*) = \gamma_\omega(x)^* \) for any \( x \in M \) we get \( \ell(x^*) = \ell(x)^* \).

(f) From \( \gamma_\omega(xy) = \gamma_\omega(x)\gamma_\omega(y) \) for any \( x, y \in M \) it follows that \( \ell(xy) = \ell(x)\ell(y) \) for any \( x, y \in M \).

(g) By the construction of \( \gamma_\omega \) the set \( \{\gamma_\omega(x) : x \in M\} \) is dense in \( L_1(M(\omega), \tau_\omega) \) for any \( \omega \in \Omega \). Therefore, the set \( \{\ell(x)(\omega) : x \in M\} \) is dense in \( L_1(M(\omega), \tau_\omega) \) for all \( \omega \in \Omega \). The proof is complete. \( \square \)

**Definition 6.2.** The defined map \( \ell \) in Theorem 6.1 is called a noncommutative vector-valued lifting associated with the lifting \( \rho \).

7. **Vector valued analogous of the noncommutative zero-two law**

In this section, we are going to prove a vector-valued analogous of Theorem 4.1.

Let \( M \) be any finite von Neumann algebra, and \( L_1(M, \Phi) \) be the noncommutative \( L_1 \)-space, associated with \( M \) and the center valued trace \( \Phi \). Let \( (X, L) \) be a measurable bundle of noncommutative \( L_1 \)-spaces \( L_1(M(\omega), \tau_\omega) \), associated with finite von Neumann algebras \( M(\omega) \) and with strictly normal numerical traces \( \tau_\omega \) on \( M(\omega) \), corresponding to \( L_1(M, \Phi) \).

**Theorem 7.1.** Let \( T : L_1(M, \Phi) \to L_1(M, \Phi) \) be a positive contraction with \( T(1) \leq 1 \). Then there exists a measurable bundle of positive contractions \( T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega) \) such that
\[
T_\omega(x(\omega)) = (Tx)(\omega)
\]
for all \( x \in L_1(M, \Phi) \) and for almost all \( \omega \in \Omega \), and
\[
\|T\|_\omega = \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega))}
\]

**Proof.** Let \( x \in M_{sa} \). Then
\[
|Tx| \leq T(|x|) \leq \|x\|_M T(1) \leq \|x\|_M 1
\]
i.e. $Tx \in M$. If $x \in M$, there exist $y, z \in M_{sa}$ such that $x = y + iz$. Then $Tx = Ty + iTz$. As $Ty, Tz \in M$ we have $Tx \in M$.

Let $\ell : M(\subset L^\infty(\Omega, X)) \to L^\infty(\Omega, X)$ be the noncommutative vector-valued lifting associated with lifting $p$.

We define the linear operator $\varphi_\omega$ from $\{\ell(x) : x \in M\}$ into $L_1(M(\omega), \tau_\omega)$ by

$$\varphi_\omega(\ell(x)(\omega)) = \ell(Tx)(\omega)$$

The contractivity of $T$ implies that

$$\|\varphi_\omega(\ell(x)(\omega))\|_{L_1(M(\omega), \tau_\omega)} = \|\ell(Tx)(\omega)\|_{L_1(M(\omega), \tau_\omega)} = \rho(\|Tx\|_1)(\omega) \leq \rho(\|x\|_1)(\omega) = \|\ell(x)(\omega)\|_{L_1(M(\omega), \tau_\omega)}.$$

This means that $\varphi_\omega$ is bounded and well defined. Moreover, one has

$$\|\varphi_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} \leq 1.$$

The positivity of $T$ yields that $\varphi_\omega$ is positive as well.

Since the set $\{\ell(x)(\omega) : x \in M\}$ is dense in $L_1(M(\omega), \tau_\omega)$, then one can extend $\varphi_\omega$ by continuity to a linear positive contraction $T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)$ by

$$T_\omega(x(\omega)) = \lim_{n \to \infty} \varphi_\omega(\ell(x_n)(\omega)).$$

From $\varphi_\omega(\ell(x)(\omega)) \in L^\infty(\Omega, X)$, for any $x \in M$ we obtain $T_\omega(x(\omega)) \in M(\Omega, X)$ for any $x \in M(\Omega, X)$. Therefore, $\{T_\omega\}$ is a measurable bundle of positive operators.

Using the same argument as in the proof Theorem 4.5 [20] one can prove

$$T_\omega(x(\omega)) = (Tx)(\omega)$$

for all $x \in L_1(M, \Phi)$ and for almost all $\omega \in \Omega$.

Now let us establish $\|T\|(\omega) = \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega))}$.

Let $x \in M$. Then

$$\|\varphi_\omega(\ell(x)(\omega))\|_{L_1(M(\omega), \tau_\omega)} = \|\ell(Tx)(\omega)\|_{L_1(M(\omega), \tau_\omega)} = \rho(\|Tx\|_1)(\omega) \leq \rho(\|T\||\|x\|_1)(\omega) = \rho(\|T\|)(\omega)\rho(\|x\|_1)(\omega)$$

$$= \rho(\|T\|)(\omega)\|\ell(x)(\omega)\|_{L_1(M(\omega), \tau_\omega)}.$$}

If $x(\omega) \in L_1(M(\omega), \tau_\omega)$, then one finds

$$\|T_\omega x(\omega)\|_{L_1(M(\omega), \tau_\omega)} = \lim_{n \to \infty} \|\varphi_\omega(\ell(x_n)(\omega))\|_{L_1(M(\omega), \tau_\omega)} \leq \rho(\|T\|)(\omega) \lim_{n \to \infty} \|\ell(x_n)(\omega)\|_{L_1(M(\omega), \tau_\omega)}$$

$$= \rho(\|T\|)(\omega)\|x(\omega)\|_{L_1(M(\omega), \tau_\omega)}.$$}

Hence, $\|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega))} \leq \rho(\|T\|)(\omega)$.

By [13] Proposition 2 for any $\varepsilon > 0$ there exists $x \in L_1(M, \Phi)$ with $\|x\|_1 = 1$ such that

$$\|Tx\|_1 \geq \|T\| - \varepsilon 1.$$
Then
\[
\rho(\|T\|)(\omega) - \varepsilon \leq \rho(\|Tx\|_1)(\omega) = \|\ell(Tx)(\omega)\|_{L_1(M(\omega), \tau_\omega)}
\]
\[
= \|T_\omega \ell(x)(\omega)\|_{L_1(M(\omega), \tau_\omega)}
\]
\[
\leq \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} \|\ell(x)(\omega)\|_{L_1(M(\omega), \tau_\omega)}
\]
\[
= \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} \|x\|_{L_1(\omega)}
\]
\[
= \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)}.
\]

The arbitrariness of \(\varepsilon\) yields
\[
p(\|T\|)(\omega) \leq \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)},
\]
Hence
\[
p(\|T\|)(\omega) = \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)}
\]
for all \(\omega \in \Omega\) or equivalently we have
\[
\|T\|_{\Omega} = \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)}
\]
for almost all \(\omega \in \Omega\). This completes the proof. \(\Box\)

**Corollary 7.2.** Let \(T : L_1(M, \Phi) \to L_1(M, \Phi)\) be a positive contraction with \(T(1) = 1\). Then there exists a measurable bundle of positive contractions \(T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)\) such that
\[
T_\omega(x(\omega)) = (Tx)(\omega)
\]
for all \(x \in L_1(M, \Phi)\) and for almost all \(\omega \in \Omega\) and
\[
\|T\|_{\Omega} = \|T_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega))}
\]

Let \(T : L_1(M, \Phi) \to L_1(M, \Phi)\) be a positive contraction with \(T(1) = 1\) and \(T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)\) be a measurable bundle of positive contractions. Then \(T\) is called unital positive contraction, if one has \(T_\omega^{-1}(1_\omega) = 1_\omega\) for almost all \(\omega \in \Omega\).

**Theorem 7.3.** Assume that \(T : L_1(M, \Phi) \to L_1(M, \Phi)\) be a unital positive contraction. If there are \(m, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\) and a positive contraction \(S : L_1(M, \Phi) \to L_1(M, \Phi)\) with \(S(1) = 1\) such that
\[
T^{m+k} \geq S, T^m \geq S \quad \text{with}
\]
\[
\|T^{m+k} - S\| < 1, \|T^m - S\| < 1
\]
then
\[
(o) - \lim_{n \to \infty} \|T^{m+k} - T^n\| = 0.
\]

**Proof.** By Corollary 7.2, there exists \(T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)\) and \(S_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)\) such that \(T_\omega(x(\omega)) = (Tx)(\omega)\) and \(S_\omega(x(\omega)) = (Sx)(\omega)\) for all \(x \in L_1(M, \Phi)\) and for almost all \(\omega \in \Omega\).

From \(T^{m+k} \geq S, T^m \geq S\) we get \(T^{m+k} \geq S_\omega, T^m \geq S_\omega\) for almost all \(\omega \in \Omega\). Since
\[
\|T^{m+k} - S\| < 1, \|T^m - S\| < 1
\]
one finds
\[
\|T^{m+k} - S_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} < 1, \|T^m - S_\omega\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} < 1
\]
for almost all $\omega \in \Omega$. Then using $T_\omega^\ast(1_\omega) = 1_\omega$ we obtain that the measurable bundle of positive contractions $T_\omega$ satisfies all conditions Corollary 4.2 for almost all $\omega \in \Omega$. Therefore
\[
\lim_{n \to \infty} \|T^{n+k} - T^n\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)} = 0
\]
for almost all $\omega \in \Omega$.

According to
\[
\|T^{n+k} - T^n\|_{L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega), \text{ a.e.}}
\]
we obtain $\lim_{n \to \infty} \|T^{n+k} - T^n\|_{(\omega)} = 0$ for almost all $\omega \in \Omega$, which means
\[
(o) - \lim_{n \to \infty} \|T^{n+k} - T^n\| = 0.
\]

\section*{Acknowledgement}

The first author acknowledges the MOE Grant FRGS13-071-0312.

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