WELL-POSEDNESS AND DIRECT INTERNAL STABILITY OF COUPLED NON-DEGENRATE KIRCHHOFF SYSTEM VIA HEAT CONDUCTION

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Abstract. In the paper under study, we consider the following coupled non-degenerate Kirchhoff system

\[
\begin{align*}
  &y_{tt} - \varphi\left(\int_{\Omega} |\nabla y|^{2} \, dx\right) \Delta y + \alpha \Delta \theta = 0, \quad \text{in } \Omega \times (0, +\infty) \\
  &\theta_t - \Delta \theta - \beta \Delta y_t = 0, \quad \text{in } \Omega \times (0, +\infty) \\
  &y = \theta = 0, \quad \text{on } \partial \Omega \times (0, +\infty) \\
  &y(\cdot, 0) = y_0, \ y_t(\cdot, 0) = y_1, \ \theta(\cdot, 0) = \theta_0, \quad \text{in } \Omega
\end{align*}
\]

where \(\Omega\) is a bounded open subset of \(\mathbb{R}^n\), \(\alpha\) and \(\beta\) be two nonzero real numbers with the same sign and \(\varphi\) is given by \(\varphi(s) = m_0 + m_1 s\) with some positive constants \(m_0\) and \(m_1\). So we prove existence of solution and establish its exponential decay. The method used is based on multiplier technique and some integral inequalities due to Haraux and Komornik [5, 6].

1. Introduction

In these last few years, Kirchhoff-type equations with non-linear or linear internal feedback and source term have been studied by many authors. For instance, the primary equation due to Kirchhoff is

\[
\rho u_{tt} - \left(p_0 + \frac{\varepsilon h}{2L} \int_{0}^{L} |u_x|^2 \, dx\right) u_{xx} + \delta u_t + f(x, u) = 0
\]

for \(t \geq 0\) and \(0 < x < L\), where \(u = u(t, x)\) is the lateral displacement at the time \(t\) and at the space coordinate \(x\), \(\varepsilon\) the Young modulus, \(\rho\) the mass density, \(h\) the crosssectional area, \(L\) the length of the string, \(p_0\) the initial axial tension, \(\delta\) the resistance modulus, and \(f\) the external force. When \(\delta = f = 0\), Eq. (2) was introduced by Kirchhoff in [7]. Further details and physical phenomena described by Kirchhoff’s classical theory can be found in [17]. Let us first review some known results on analogous problems. So, following Nishihara and Yamada [12, Ono 13], they established existence of global solutions for small data, decay property of the energy and blow-up of solutions. In addition, degenerate or nondegenerate Kirchhoff equation with weak dissipation is in the

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following form
\[ y_{tt} - \varphi \left( \int_{\Omega} |\nabla y|^2 \, dx \right) \Delta y + \sigma(t)g(y_t) = 0. \]

Benaissa et al. [4] used the multiplier method and general weighted integral inequalities to estimate the whole energy of such system.

Lasiecka et al. [10] studied the existence and exponential stability of solutions to a quasilinear system arising in the modeling of nonlinear thermoelastic plates.

Also, Lasiecka et al. [11] considered the thermoelastic Kirchhoff-Love plate, they studied the local well-posedness, so they proved that unique classical local solution is extended globally, provided the initial data are sufficiently small at the lowest energy level and an exponential decay rate is further stated.

Tebou [10] considered
\[
\begin{align*}
y_{tt} - c^2 \Delta y + \alpha (\Delta) \mu \theta &= 0, \\
\theta_t - \nu \Delta \theta - \beta \Delta y_t &= 0.
\end{align*}
\] (3)

He showed that the associated semigroup is not stable (uniformly) for the values of \( \mu \in [0, 1] \). Hence, he proposed an explicit non-uniform decay rate. Afterwards, for \( \mu = 1 \), system [3] was discussed by Lebeau and Zuazua [15] and subsequently by Albano and Tataru [2]. So in the same paper [10], Tebou showed that the corresponding semigroup is exponentially stable but not analytic.

In other context, Tebou et al. [16] investigated a thermoelastic plate with rotational forces as
\[
\begin{align*}
y_{tt} + (\Delta) \mu y_{tt} + \Delta^2 y + \alpha \Delta \theta &= 0, \\
\theta_t - \nu \Delta \theta - \beta \Delta y_t &= 0.
\end{align*}
\]

They showed that, for every \( \delta > (2 - \mu)/(2 - 4 \mu) \) and for both clamped and hinged boundary conditions, the corresponding semigroup is of Gevrey class when the parameter \( \delta \) lies in the interval \((0, 1/2)\). Then, they obtained exponential decay for the associated semigroup for hinged boundary conditions, when \( \mu \) lies in \((0, 1]\). At the end, they ensured, by constructing a counterexample, that, under hinged boundary conditions, the semigroup is not analytic, for all \( \mu \) in \((0, 1]\).

The rest of the paper is structured as follows. Besides the present introduction, section 2 is devoted to state our main results concerning global well-posedness as well as exponential stability of Eqs. (1). In, Section 3 and 4, we prove our main results.

**Conceptualization and Methodology:**

The purpose of this study is to construct a stability theory under suitable conditions for system (1) and apply it to specific physical and mechanical engineering models. In certain instances, exponential stability with respect to the state space energy can be readily derived using Lyapunov, energy and spectral methods. The energy method is a popular strategy in showing stability of systems defined in the entire space. However, employing the energy method to some physical systems on bounded domains necessitates additional regularity and compatibility conditions on the data. We emphasize here that significant feature and difficulty of the problem fits in the quasilinearity appearing in Eq. (1) which is topologically hard since the nonlinear coefficient depends only on the time component.
2. Well-posedness and energy decay

The main results of the paper reads as follows.

**Theorem 2.1.** (Well-posedness) Let \((y_0, y_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega), \quad \theta_0 \in H_0^1(\Omega)\) and assume that \(\{y_0, y_1, \theta_0\}\) is small enough. Then the problem \((\text{I})\) has a unique weak solution \((y, y_t, \theta)\) such that for any \(T > 0\), we have

\[(y, y_t) \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \times L^\infty(0, T; H_0^1(\Omega)), \quad \theta \in L^\infty(0, T; H_0^1(\Omega)).\]

**Theorem 2.2.** (Exponential stability.) Let \((y, y_t, \theta)\) be the solution of \((\text{I})\). Then the energy functional \((\text{IV})\) satisfies

\[E(t) \leq C E(0) e^{-\omega t}, \quad \forall t \geq 0\]

where \(C\) and \(\omega\) are positive constants independent of the initial data.

Let us now introduce the energy functional associated to \((\text{I})\) which is given by

\[E(t) = \frac{1}{2} \int_\Omega |y_t|^2 dx + \frac{m_0}{2} \int_\Omega |\nabla y|^2 dx + \frac{m_1}{4} \left( \int_\Omega |\nabla y|^2 dx \right)^2 + \frac{\alpha}{2|\beta|} \int_\Omega |\theta|^2 dx, \quad \forall t \geq 0.\]

(4)

So, as a first result of this paper, we have the following.

**Lemma 2.1.** Let \((y, y_t, \theta)\) be a solution to the problem \((\text{I})\). Then, the energy functional defined by \((\text{IV})\) satisfies

\[E'(t) = -\frac{\alpha}{|\beta|} \int_\Omega |\nabla \theta|^2 dx \leq 0, \quad \forall t \geq 0.\]

(5)

That is, the energy functional is a nonincreasing function.

**Proof.** Integrating by parts the first equation of \((\text{I})\) after multiplying it by \(y_t\), yielding

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |y_t|^2 dx + \frac{m_0}{2} \frac{d}{dt} \int_\Omega |\nabla y|^2 dx + \frac{m_1}{4} \frac{d}{dt} \left( \int_\Omega |\nabla y|^2 dx \right)^2 + \alpha \int_\Omega \nabla \theta \nabla y_t dx = 0.
\]

(6)

Afterwards, as previous, integrating the second equation of \((\text{I})\) over \(\Omega\) after multiplying it by \(\theta\), we obtain

\[
\frac{1}{|\beta|} \frac{d}{dt} \int_\Omega |\theta|^2 dx + \frac{1}{|\beta|} \int_\Omega |\nabla \theta|^2 dx = \int_\Omega \nabla \theta \nabla y_t dx.
\]

(7)

Inserting \((\text{IV})\) into \((\text{VI})\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |y_t|^2 dx + \frac{m_0}{2} \frac{d}{dt} \int_\Omega |\nabla y|^2 dx + \frac{m_1}{4} \frac{d}{dt} \left( \int_\Omega |\nabla y|^2 dx \right)^2 + \frac{\alpha}{|\beta|} \frac{d}{dt} \int_\Omega |\theta|^2 dx = -\frac{\alpha}{|\beta|} \int_\Omega |\nabla \theta|^2 dx.
\]

The proof of Lemma 2.1 is thus complete. \(\square\)
3. Proof of Theorem 2.1

As a powerful tool to prove the existence of a global solutions for problem (11) is the Faedo-Galerkin method. In fact, let \((e_k)_{k \in \mathbb{N}}\) be normalized eigenfunctions of the negative Laplacian with Dirichlet boundary conditions
\[
\begin{aligned}
- \Delta e_k &= \lambda_k e_k, \quad \text{in } \Omega \\
e_k &= 0, \quad \text{in } \partial \Omega.
\end{aligned}
\]

Then, the family \(\{e_k|k \in \mathbb{N}\}\) forms an orthonormal basis of \(L^2(\Omega)\). Furthermore, we consider \(V^n = \text{span}\{e_m|m = 1, 2, \ldots, n\}\). So here, several steps are involved.

**Step 1:** We construct approximate solutions \((y^n, y^n_t, \theta^n), n = 1, 2, 3, \ldots,\) in the form
\[
y^n(x, t) = \sum_{m=1}^{n} h^n_m(t)e_m(x),
\]
and
\[
\theta^n(x, t) = \sum_{m=1}^{n} c^n_m(t)e_m(x)
\]
where \(h^n_m, c^n_m (m = 1, 2, \ldots, n)\) are determined by the following ordinary differential equations
\[
\begin{aligned}
(y^n_{tt} - \varphi \left( \int_{\Omega} |\nabla y^n|^2 \right) \Delta y^n + \alpha \Delta \theta^n, e_m) &= 0 \quad \forall e_m \in V^n \\
(\theta^n_t - \Delta \theta^n - \beta \Delta y^n_t, e_m) &= 0 \quad \forall e_m \in V^n
\end{aligned}
\]
with initial conditions
\[
\begin{aligned}
y^n(x, 0) &= y^n_0 = \sum_{m=1}^{n} (f, e_m)e_m \rightarrow y_0, \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega) \text{ as } n \rightarrow \infty, \\
y^n_t(x, 0) &= y^n_1 = \sum_{m=1}^{n} (f_t, e_m)e_m \rightarrow y_1, \quad \text{in } H^1_0(\Omega) \text{ as } n \rightarrow \infty, \\
\theta^n(x, 0) &= \theta^n_0 = \sum_{m=1}^{n} (g, e_m)e_m \rightarrow \theta_0, \quad \text{in } H^1_0(\Omega) \text{ as } n \rightarrow \infty.
\end{aligned}
\]

The system (9)-(11) of ordinary differential equation of variable \(t\) admits a solution \((y^n, y^n_t, \theta^n)\) on the interval \([0, t_n]\). At the beginning, we will start to identify some a priori estimates in order to prove that \(t_n = \infty\). After that, we will show that the sequence of solutions to (9) converges to a solution of (11) with the claimed smoothness.

**Step 2:** If we multiply the first and the second equations of (9) by \(h^n_m(t)\) and \(c^n_m(t)\) respectively and sum over \(m\) from 1 to \(n\), we get
\[
\begin{aligned}
\int_{\Omega} |y^n_t|^2 \, dx &+ \left( m_0 + \frac{m_1}{2} \right) \int_{\Omega} |\nabla y^n|^2 \, dx \int_{\Omega} |\nabla y^n|^2 \, dx + \frac{\alpha}{\beta} \int_{\Omega} |\theta^n|^2 \, dx + \frac{2\alpha}{\beta} \int_{0}^{t} \int_{\Omega} |\nabla \theta^n(s)|^2 \, dx \, ds \\
&\leq \int_{\Omega} |y^n_0|^2 \, dx + \left( m_0 + \frac{m_1}{2} \right) \int_{\Omega} |\nabla y^n_0|^2 \, dx \int_{\Omega} |\nabla y^n_0|^2 \, dx + \frac{\alpha}{\beta} \int_{\Omega} |\theta^n_0|^2 \, dx \\
&\leq 2E(0), \quad \forall t \in [0, t_n).
\end{aligned}
\]

Therefore, we deduce that \(t_n = \infty\), and that
\[
\begin{aligned}
y^n &\text{ is bounded in } L^\infty(0, T; H^1_0(\Omega)) \\
y^n_t &\text{ is bounded in } L^\infty(0, T; L^2(\Omega))
\end{aligned}
\]
\( \theta^n \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \). (15)

**Step 3:** Replacing \( e_m \) by \( -\Delta e_m \), and doing in the same manner as previous step, we get

\[
\frac{d}{dt} \left[ \int\Omega |\nabla y^n|^2 \, dx + \left( m_0 + \frac{m_1}{2} \int\Omega |\nabla y^n|^2 \, dx \right) \int\Omega |\Delta y^n|^2 \, dx + \frac{\alpha}{\beta} \int\Omega |\nabla \theta^n|^2 \, dx \right]
+ \frac{2\alpha}{\beta} \int\Omega |\Delta \theta^n|^2 \, dx
= \int\Omega |\Delta y^n|^2 \, dx \frac{d}{dt} \left[ m_0 + \frac{m_1}{2} \int\Omega |\nabla y^n|^2 \, dx \right]
= m_1 \left( \int\Omega |\nabla y^n \nabla y^n|^2 \, dx \right) \int\Omega |\Delta y^n|^2 \, dx.
\]

Using (12) and Cauchy-Schwarz inequality the following estimate holds

\[
\frac{d}{dt} \left[ \int\Omega |\nabla y^n|^2 \, dx + \left( m_0 + \frac{m_1}{2} \int\Omega |\nabla y^n|^2 \, dx \right) \int\Omega |\Delta y^n|^2 \, dx + \frac{\alpha}{\beta} \int\Omega |\nabla \theta^n|^2 \, dx \right]
+ \frac{2\alpha}{\beta} \int\Omega |\Delta \theta^n|^2 \, dx
\leq C \left( \int\Omega |\nabla y^n|^2 \, dx \right)^{\frac{2}{3}} \int\Omega |\Delta y^n|^2 \, dx.
\]

Integrating the last inequality over \((0, t)\), we get

\[
E_n^*(t) + \frac{2\alpha}{\beta} \int_0^t \int\Omega |\Delta \theta^n(s)|^2 \, dx \, ds \leq E_n^*(0) + C \int_0^t (E_n^*(s))^\frac{2}{3} \, ds
\]  
(16)

where

\[
E_n^*(t) = \int\Omega |\nabla y^n|^2 \, dx + \left( m_0 + \frac{m_1}{2} \int\Omega |\nabla y^n|^2 \, dx \right) \int\Omega |\Delta y^n|^2 \, dx + \frac{\alpha}{\beta} \int\Omega |\nabla \theta^n|^2 \, dx.
\]

To complete this step, we need the following Lemma.

**Lemma 3.1.** (Modified Gronwall inequality) Let \( G \) and \( f \) be non-negative functions on \([0, +\infty)\) satisfying

\[
0 \leq G(t) \leq K + \int_0^t f(s)G(s)^{r+1} \, ds,
\]

with \( K > 0 \) and \( r > 0 \). Then

\[
G(t) \leq \left\{ K^{-r} - r \int_0^t f(s) \, ds \right\}^{-1/r},
\]

as long as the RHS exists.

So, an immediate application of this lemma with

\[
G(t) = E_n^*(t), \quad K(t) = E_n^*(0) \quad \text{and} \quad f(t) = C
\]

gives us

\[
E_n^*(t) \leq \left\{ \left( E_n^*(0) \right)^{-\frac{1}{2}} - C \frac{1}{2} \int_0^t ds \right\}^{-2}.
\]

Therefore, if initial data \( \{u_0, u_1\} \) are sufficiently small, we deduce that

\[
E_n^*(t) \leq \left\{ \left( E_n(0) \right)^{-\frac{1}{2}} - \frac{CT}{2} \right\}^{-2}.
\]
Hence, we conclude that
\[
y^n_t \text{ is bounded in } L^\infty(0, T, H^1_0(\Omega)), \quad (17)
\]
\[
\left(m_0 + \frac{m_1}{2} \int_\Omega |\nabla y^n|^2 \, dx\right) \Delta y^n \text{ is bounded in } L^\infty(0, T, L^2(\Omega)), \quad (18)
\]
and
\[
\Delta y^n \text{ is bounded in } L^\infty(0, T, L^2(\Omega)), \quad (19)
\]
\[
\nabla \theta^n \text{ is bounded in } L^\infty(0, T, L^2(\Omega)). \quad (20)
\]

**Step 4: Passing to the limit:**

Applying Dunford-Pettis and Banach-Alaoglu-Bourbaki theorems, we conclude from (13) and (19) that there exists a subsequence \(\{y^n, \theta^n\}\) of \(\{y^m, \theta^m\}\) such that
\[
y^n \rightharpoonup^* y, \quad \text{in } L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \quad (21)
\]
\[
y^n_t \rightharpoonup^* y_t, \quad \text{in } L^\infty(0, T; H^1_0(\Omega)) \quad (22)
\]
\[
\left(\int_\Omega |\nabla y^n|^2 \, dx\right) \Delta y^n \rightharpoonup^* \chi, \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad (23)
\]
\[
\theta^n \rightharpoonup^* \theta, \quad \text{in } L^\infty(0, T; H^1_0(\Omega)) \quad (24)
\]
\[
\theta^n \rightharpoonup^* \theta, \quad \text{in } L^2(0, T; H^1_0(\Omega)). \quad (25)
\]

By (22) and (24), we have
\[
\Delta^{-1} \theta^n \rightharpoonup^* \Delta^{-1} \theta, \quad \text{in } L^2(0, T; L^2(\Omega)), \quad (26)
\]
where \(\Delta^{-1}\) denotes the inverse of the Laplacian with zero Dirichlet boundary conditions.

We shall prove that, in fact,
\[
\chi = \left(\int_\Omega |\nabla y|^2 \, dx\right) \Delta y, \quad \text{i.e.}
\]
\[
\left(\int_\Omega |\nabla y^n|^2 \, dx\right) \Delta y^n \rightharpoonup^* \left(\int_\Omega |\nabla y|^2 \, dx\right) \Delta y, \quad \text{in } L^\infty(0, T; L^2(\Omega)). \quad (27)
\]

As \((y^n)\) is bounded in \(L^\infty(0, T, H^2(\Omega) \cap H^1_0(\Omega))\) (by (17)) and the embedding of \(H^2(\Omega)\) in \(L^2(\Omega)\) is compact, we have
\[
y^n \longrightarrow y, \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (28)
\]

On the other hand, for \(v \in L^2(0, T; L^2(\Omega))\), we have
\[
\int_0^T \left(\chi - \left(\int_\Omega |\nabla y^n|^2 \, dx\right) \Delta y, v\right) \, dt
\]
\[
= \int_0^T \left(\chi - \left(\int_\Omega |\nabla y^n|^2 \, dx\right) \Delta y^n, v\right) \, dt + \int_0^T \left(\int_\Omega |\nabla y^n|^2 \, dx\right) \left(\Delta y^n - \Delta y, v\right) \, dt
\]
\[
+ \int_0^T \left(\int_\Omega (|\nabla y^n|^2 - |\nabla y|^2) \, dx\right) (\Delta y^n, v) \, dt. \quad (29)
\]

We deduce from (21) and (23) that the first and the second terms in (29) tend to zero as \(m \to \infty\). For the third term, using (13) and (18), we can write (with \(c\) positive constant)
\[
\int_0^T \left(\int_\Omega (|\nabla y^n|^2 - |\nabla y|^2) \, dx\right) (\Delta y^n, v) \, dt
\]
\[
\leq c \int_0^T \left(\int_\Omega |\nabla y^n - \nabla y|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla y^n + \nabla y|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\Delta y^n|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 \, dx\right)^{\frac{1}{2}} \, dt
\]
\[
\leq c \left(\int_\Omega |\nabla y^n - \nabla y|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 \, dx\right)^{\frac{1}{2}} \, dt.
\]
On the other hand, using (22), (24) and (26), we have

Furthermore, using (21), (24) and (27), we have

Hence we deduce (27) from (28).

**Step 5: Proof of uniqueness:**

Let \((y_1, y_{1,t}, \theta_1)\) and \((y_2, y_{2,t}, \theta_2)\) be solutions of (1) with the same initial data, and setting \(Y = y_1 - y_2\) and \(\theta = \theta_1 - \theta_2\). Hence, we have

\[
\begin{aligned}
\begin{cases}
Y_{tt} - (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_1|^2 \, dx \Delta y_1 + (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_2|^2 \, dx \Delta y_2 + \alpha \Delta \theta = 0, \\
\theta_t - \Delta \theta - \beta \Delta Y_t = 0,
\end{cases}
\end{aligned}
\]

with \(Y = \theta = 0\) on \([0, +\infty) \times \partial \Omega\) and \(Y(0) = Y_t(0) = \theta(0) = 0\) in \(\Omega\).

\[
\begin{aligned}
\begin{cases}
Y_{tt} - (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_1|^2 \, dx \Delta Y - (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_2|^2 \, dx \Delta y_2 \\
+ (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_1|^2 \, dx \Delta y_2 + \alpha \Delta \theta = 0, \\
\theta_t - \Delta \theta - \beta \Delta Y_t = 0,
\end{cases}
\end{aligned}
\]  \hfill (30)

Taking the \(L^2(\Omega)\) inner product of first and second equation of (30) with \(Y_t\) and \(\theta\) respectively, we get

\[
\begin{aligned}
\frac{d}{dt} \left[ \int_{\Omega} |Y_t|^2 \, dx + (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_1|^2 \, dx \right] &\int_{\Omega} |\nabla Y|^2 \, dx + \frac{\alpha}{\beta} \int_{\Omega} |\theta|^2 \, dx \\
&= \frac{d}{dt} (m_0 + \frac{m_1}{2}) \int_{\Omega} |\nabla y_1|^2 \, dx \int_{\Omega} |\nabla Y|^2 \, dx \\
&+ \int_{\Omega} \left\{ (m_0 + m_1) \int_{\Omega} |\nabla y_1|^2 \, dx \Delta y_2 - (m_0 + m_1) \int_{\Omega} |\nabla y_2|^2 \, dx \Delta y_2 \right\} Y_t \, dx.
\end{aligned}
\]
Using Cauchy-Schwarz and Young’s inequalities, we have

\[
\frac{d}{dt} \left[ \int_\Omega |Y|^2 \, dx + (m_0 + \frac{m_1}{2}) \int_\Omega |\nabla y_1|^2 \, dx \right] \int_\Omega |\nabla Y|^2 \, dx + \frac{\alpha}{\beta} \int_\Omega |\varphi|^2 \, dx + \frac{2\alpha}{\beta} \int_\Omega |\nabla \varphi|^2 \, dx \\
\leq \frac{d}{dt} (m_0 + \frac{m_1}{2}) \int_\Omega |\nabla y_1|^2 \, dx \int_\Omega |\nabla Y|^2 \, dx + 2m_1 \int \{ |\nabla y_1|^2 - |\nabla y_2|^2 \} \, dx \int_\Omega \Delta y_2 Y_1 \, dx \\
\leq 2m_1 \int_\Omega |\nabla Y|^2 \, dx \int_\Omega |\nabla y_1| \nabla y_1 \, dx \times \nabla \right) \|
+ 2m_1 \left( \int_\Omega \{ |\nabla y_1 - \nabla y_2|^2 \} \, dx \right)^{\frac{1}{2}} \left( \int \Omega |\Delta y_2|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla Y|^2 \, dx \right)^{\frac{1}{2}} \\
\leq m_1 \int_\Omega |\nabla Y|^2 \, dx \left( \int_\Omega |\nabla y_1|^2 \, dx + \int_\Omega |\nabla y_2|^2 \, dx \right) \\
+ 2m_1 \int_\Omega |\nabla Y|^2 \, dx \int_\Omega |\Delta y_2|^2 \, dx + \int_\Omega |\nabla Y|^2 \, dx.
\]

Integrating it over \((0, t)\), we conclude that

\[
\int_\Omega |Y|^2 \, dx + (m_0 + \frac{m_1}{2}) \int_\Omega |\nabla y_1|^2 \, dx \int_\Omega |\nabla Y|^2 \, dx + \frac{\alpha}{\beta} \int_\Omega |\varphi|^2 \, dx \\
\leq C \int_0^t \left\{ \int_\Omega |Y(s)|^2 \, dx + m_1 \int_\Omega |\nabla y_1(s)|^2 \, dx \int_\Omega |\nabla Y(s)|^2 \, dx + \frac{\alpha}{\beta} \int_\Omega |\varphi(s)|^2 \, dx \right\} \, ds.
\]

which, by Gronwall’s lemma, implies \(Y \equiv 0\) and \(\theta = 0\). The proof of Theorem 2.1 is now completed.

4. Proof of Theorem 2.2

In this section, we prove our stability result for the energy of the solution of system (1), using the multiplier technique. This proof will be established in three steps and needed the following Lemma due to Martinez [8].

**Lemma 4.1.** Let \(E : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a non-increasing function and assume that there are two constants \(\mu \geq 0\) and \(\omega > 0\) such that

\[
\int_0^{\infty} E(s)^{\mu+1} \, ds \leq \omega E(0)^\mu E(t), \quad \forall t \geq 0.
\]

Then, we have for every \(t > 0\)

\[
\left\{ \begin{array}{ll}
E(t) \leq E(0) \left( \frac{1+\mu}{1+\omega \mu t} \right)^{-\frac{1}{\mu}}, & \text{if } \mu > 0 \\
E(t) \leq E(0) e^{\frac{-1}{\omega t}}, & \text{if } \mu = 0.
\end{array} \right.
\]

**Step 1:** Let \(\mu \geq 0\) be a non-negative constant. Multiplying the first equation of (1) by \(E^\mu y\), and integrating over \(\Omega \times (S, T)\), we find

\[
0 = \int_S^T E^\mu \int_\Omega y \left( y_{tt} - q \left( \int_\Omega |\nabla y|^2 \, dx \right) \Delta y + \alpha \Delta \theta \right) \, dx \, dt.
\]
Using Green’s formula, we derive

\[
0 = \left[ E^{\mu} \int_{\Omega} y y_t \, dx \right]_S^T - \int_S^T E^{\mu} \int_{\Omega} |y_t|^2 \, dx \, dt - \mu \int_S^T E^{\mu-1} E' \int_{\Omega} y y_t \, dx \, dt + \int_S^T E^{\mu} \phi \left( \int_{\Omega} |\nabla y|^2 \, dx \right) \int_{\Omega} |\nabla y|^2 \, dx \, dt - \alpha \int_S^T E^{\mu} \int_{\Omega} \nabla y \nabla \theta \, dx \, dt.
\]

Using (4), we have

\[
2 \int_S^T E^{\mu+1} \, dt = - \left[ E^{\mu} \int_{\Omega} y y_t \, dx \right]_S^T + 2 \int_S^T E^{\mu} \int_{\Omega} y_t^2 \, dx \, dt + \mu \int_S^T E^{\mu-1} E' \int_{\Omega} y y_t \, dx \, dt + \alpha \int_S^T E^{\mu} \int_{\Omega} \nabla y \nabla \theta \, dx \, dt - \frac{\alpha}{\beta} \int_S^T E^{\mu} \int_{\Omega} \theta^2 \, dx \, dt - \frac{m_1}{2} \int_S^T E^{\mu} \left( \int_{\Omega} |\nabla y|^2 \, dx \right)^2 \, dt.
\]

(31)

Since \( E \) is nonincreasing, and using Cauchy-Schwarz and Poincaré inequalities, we have

\[
\left| E^{\mu} \int_{\Omega} y y_t \, dx \right| \leq C_0 E^{\mu+1} (S)
\]

(32)

\[
\left| \mu \int_S^T E^{\mu-1} E' \int_{\Omega} y y_t \, dx \, dt \right| \leq C_0 \int_S^T E^{\mu} (-E') \, dt \leq C_0 E^{\mu+1} (S)
\]

(33)

\[
\frac{\alpha}{\beta} \int_S^T E^{\mu} \int_{\Omega} \theta^2 \, dx \, dt \leq C_0 \int_S^T E^{\mu} (-E') \, dt \leq C_0 E^{\mu+1} (S)
\]

(34)

\[
\alpha \int_S^T E^{\mu} \int_{\Omega} \nabla y \nabla \theta \, dx \, dt \leq C_0 \int_S^T E^{\mu+1} (-E') \, dt.
\]

Now, fix an arbitrarily small \( \epsilon_0 > 0 \), and applying Young’s inequality, we obtain

\[
\left| \alpha \int_S^T E^{\mu} \int_{\Omega} \nabla y \nabla \theta \, dx \, dt \right| \leq \epsilon_0 \int_S^T E^{2\mu+1} (t) \, dt + C(\epsilon_0) \int_S^T (-E') \, dt
\]

(35)

Taking into account (32) – (34) into (31) and Poincaré inequality, we obtain

\[
2 \int_S^T E^{\mu+1} \, dt \leq \epsilon_0 \int_S^T E^{2\mu+1} \, dt + C_0 E^{\mu+1} (S) + C_0 E (S)
\]

(36)
get
\[
0 = \int_{\Omega}^{T} \frac{d}{dt} E^\mu y_t (t) dt = \left[ \int_{\Omega}^{T} E^\mu y_t (t) dt \right]_{S}^{T} - \mu \int_{S}^{T} E^{\mu-1} E' \int_{\Omega}^{T} y_t (t) dt - \int_{S}^{T} E^\mu \int_{\Omega}^{T} \theta y_t (t) dt
\]
\[
+ \int_{S}^{T} E^\mu \int_{\Omega}^{T} \nabla y_t \nabla \theta (t) dt + \beta \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla y_t|^2 dt.
\]
Since \( E \) is nonincreasing, and using Cauchy-Schwarz inequalities, we have
\[
\left| \int_{S}^{T} E^\mu \int_{\Omega}^{T} y_t (t) dt \right| \leq C_1 E^{\mu+1} (S),
\]
\[
\left| \mu \int_{S}^{T} E^{\mu-1} E' \int_{\Omega}^{T} y_t (t) dt \right| \leq C \int_{S}^{T} E^{\mu} (-E') dt.
\]
Applying Young’s inequality, we get
\[
\left| \int_{S}^{T} E^\mu \int_{\Omega}^{T} \nabla y_t \nabla \theta (t) dt \right| \leq \beta \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla y_t|^2 dt + \frac{1}{2\beta} \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla \theta|^2 dt.
\]
\[
\leq \frac{\beta}{2} \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla y_t|^2 dt + \frac{1}{2\beta} \int_{S}^{T} E^\mu (-E') dt.
\]
\[
\leq \frac{\beta}{2} \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla y_t|^2 dt + C_1 E^{\mu+1} (S).
\]
Reporting (38)-(40) into (37) yields
\[
\frac{\beta}{2} \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla y_t|^2 dt \leq C_1 E^{\mu+1} (S) + \int_{S}^{T} E^\mu \int_{\Omega}^{T} \theta y_t (t) dt.
\]

**Step 3:** In this step, we are going to estimate the second term in the right hand side of (41).

Multiplying the first equation in (41) by \( E^\mu \theta \) and using Green’s formula over \( \Omega \times (S, T) \), we find
\[
\int_{S}^{T} E^\mu \int_{\Omega}^{T} \theta y_t (t) dt = - \int_{S}^{T} E^\mu \int_{\Omega}^{T} \phi \left( \int_{\Omega}^{T} |\nabla y|^2 dt \right) \int_{\Omega}^{T} \nabla y \nabla \theta (t) dt + \frac{\alpha}{\beta} \int_{S}^{T} E^\mu \int_{\Omega}^{T} |\nabla \theta|^2 dt.
\]

Thanks to Cauchy-Schwarz inequality, and the definition of the energy \( E \), we easily derive
\[
\left| \int_{S}^{T} E^\mu \phi \left( \int_{\Omega}^{T} |\nabla y|^2 dt \right) \int_{\Omega}^{T} \nabla y \nabla \theta (t) dt \right| \leq \sqrt{2 \int_{S}^{T} \phi \left( \int_{\Omega}^{T} |\nabla y|^2 dt \right) E^{\mu+1} \int_{\Omega}^{T} |\nabla \theta|^2 dt},
\]
Now, pick an arbitrarily small \( \varepsilon_1 > 0 \), applying Young’s inequality and using the expression of \( \phi \), we obtain
\[
\left| \int_{S}^{T} E^\mu \phi \left( \int_{\Omega}^{T} |\nabla y|^2 dt \right) \int_{\Omega}^{T} \nabla y \nabla \theta (t) dt \right| \leq 2 \varepsilon_1 \int_{S}^{T} E^{2\mu+1} dt + \frac{4m_0}{m_0^2} \varepsilon_1 \int_{S}^{T} E^{2\mu+2} dt + C(\varepsilon_1) \int_{S}^{T} \phi \left( \int_{\Omega}^{T} |\nabla y|^2 dt \right) (-E') dt.
\]
Taking into account (44) into (42) and using (5), we obtain
\[ \int T \int S \int \Omega \theta \mu \int t dt \leq C^2 \varepsilon^1 \int T \int S \int \Omega \theta \mu \int t dt + C_2 E(0) \int T \int S \int \Omega \theta \mu \int t dt + C_2 E(0) \int T \int S \int \Omega \theta \mu \int t dt + C_2 E(0) \int T \int S \int \Omega \theta \mu \int t dt \] (45)

Combining (39), (41) and (45), we find
\[ \int T \int S \int \Omega \theta \mu \int t dt \leq C \varepsilon \int T \int S \int \Omega \theta \mu \int t dt + C E(0) \int T \int S \int \Omega \theta \mu \int t dt + C E(0) \int T \int S \int \Omega \theta \mu \int t dt \] (46)

Choosing \( \varepsilon \) small enough and \( \mu = 0 \), we obtain
\[ \int T \int S \int \Omega \theta \mu \int t dt \leq C' E(S) + C E(0) \int T \int S \int \Omega \theta \mu \int t dt \] (47)

where \( C' \) is a positive constant independent of \( E(0) \).

Now, the proof is achieved by applying Lemma 4.1.

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