Is there a physical meaning of the Breit-Wigner parameters?

S. Ceci, M. Vukšić, and B. Zauner

Rudjer Bošković Institute, Bijenička 54, HR-10000 Zagreb, Croatia
University of Zagreb, Bijenička 34, HR-10000 Zagreb, Croatia

Using the first order expansion of a resonant scattering amplitude we demonstrate that major objections against the Breit-Wigner parameters: their model dependence, unphysical background dependence, and field-transformation dependence, are not valid. We show that the Breit-Wigner parameters connect the shape of the resonant peak to the phase of the resonant pole residue.

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All particles recently discovered in the large experimental facilities, elementary and composite alike, are resonances. Resonances appear when there is enough energy crammed in a tiny volume, live for an extremely short period of time, and then decay into two or more lighter particles that are allowed by conservation laws of particular interactions. Due to the short lifetime of resonances, there is no direct way to measure their masses. Instead, the masses are inferred from the analysis of their decay products. When total c.m. energy of incident particles is close to the resonance mass $m$, the decay products are more frequently observed in particle detectors. That is manifested as a peak in the cross section data

$$\sigma = \frac{4\pi}{q^2} \frac{2J+1}{(2s_1+1)(2s_2+1)} |A_{\text{R}}|^2 + \sigma_{\text{bg}},$$

where $q$ is center of mass momentum of incident particles, $J$ is the spin of the resonance, $s_1$ and $s_2$ are spins of incident particles, and $\sigma_{\text{bg}}$ is background. The key object in this relation is a resonant amplitude $A_{\text{R}}$, originally written in a well known Breit-Wigner formula [1,2] as

$$A_{\text{R}}^{\text{original}} = \frac{\Gamma_{\text{par}}/2}{m-W-i\Gamma_{\text{tot}}/2}.$$  

The partial decay width $\Gamma_{\text{par}}$ is the inverse of resonance mean lifetime and gives the rate at which a resonance decays into the observed channel. In natural units, it has the same unit as the mass $m$. The total decay width $\Gamma_{\text{tot}}$ is the sum of all partial widths, and $W$ is c.m. energy.

In more realistic situations, the parameters in Eq. (2) will generally be energy dependent

$$A_{\text{R}}^{\text{realistic}} = \frac{\Gamma_{\text{par}}(W)/2}{m(W)-W-i\Gamma_{\text{tot}}(W)/2}.$$  

If resonant parameters become functions, it is not clear what should we use, if anything, for the mass and width of the resonance. There is a long-standing debate on that matter (see e.g. Ref. [3]). One side strongly advocates using well-defined mathematical quantities, such as the poles of the scattering amplitude (the signatures of short-living states in the complex-energy plane) and their residues. If, for simplicity, we write the amplitude from Eq. (3) in a less specific form

$$A_{\text{R}} = \frac{N(W)}{D(W)},$$

the definition of pole parameters will be given by

$$D(W_{\text{pole}}) = 0 \Rightarrow W_{\text{pole}} = M - i\Gamma/2,$$

$$|r| e^{i\theta} = \lim_{W \to W_{\text{pole}}} \frac{[F(W) - W] A_{\text{R}}(W)],}{r}$$

where the real part of the pole position is the mass $M$, the imaginary part is related to the total decay width $\Gamma$, while $|r|$ is the magnitude and $\theta$ is the phase of the pole residue. (Due to convention, the sign of the residue is opposite from its mathematical definition.)

The other side prefers staying on the real-valued physical energies, and using the so called Breit-Wigner or conventional resonant parameters (see e.g. Ref. [3,5])

$$\text{Re } D(W_{\text{BW}}) = 0 \Rightarrow W_{\text{BW}} = M_{\text{BW}},$$

$$\Gamma_{\text{BW}} = -2 \text{Im } D(W_{\text{BW}}),$$

where $M_{\text{BW}}$ is the mass, and $\Gamma_{\text{BW}}$ is the decay width.

In the Review of Particle Properties [2], some resonances are described by their Breit-Wigner parameters, e.g. the Z boson and the $\rho$ meson, while others are given by their pole parameters, such are the light spinless mesons. Nucleon resonances, such is the $\Delta(1232)$, have two kinds of tables in Ref. [2]: one kind with the Breit-Wigner, and the other with the pole parameters.

Three strong objections are commonly raised against the Breit-Wigner parameters: that they are model dependent [3], that they depend on the non-physical background contribution [3], and that they are ill-defined on a fundamental, quantum-field-theoretical level [5,6].

In this Letter we demonstrate that these objections are not valid, and show that the Breit-Wigner parameters are the link between the shape of the resonant peak, and the phase of the resonant pole residue $\theta$.

It turns out that we can address the first objection immediately. It is evident from the definitions for pole and Breit-Wigner parameters in Eqs. (5) and (7) that...
they depend on the particular choice of functions \( N \) and \( D \). Thus, both are in fact model dependent.

To address the second objection, the unphysical background dependence, we build a simplified model of a resonant scattering amplitude. We do it in the following way: close to a resonance, we keep only the linear terms in Taylor expansion of functions \( N \) and \( D \), similarly to Refs. [7, 8]. We rewrite the approximate amplitude explicitly giving the pole parameters

\[
A_R^{\text{linear}} = \frac{|r| e^{i\theta}}{M - W - i\Gamma/2} + A_B e^{i\beta},
\tag{9}
\]

where we have two additional real parameters: the background \( A_B \), and its phase \( \beta \). Assuming that the numerator \( N \) is a real-valued function, which is the prevailing form in the literature, we get the following relation

\[
A_B = \frac{2 |r|}{\Gamma} \sin(\theta - \beta).
\tag{10}
\]

To simplify the notation, we introduce the branching fraction \( x = 2 |r|/\Gamma \), an overall phase \( \eta = 2 \beta - \theta \), and the deformation phase

\[
\delta = \theta - \beta.
\tag{11}
\]

Putting it all together, the amplitude in Eq. (9) becomes

\[
A_R = x e^{i\eta} \left( \frac{\Gamma/2 e^{2i\delta}}{M - W - i\Gamma/2} + e^{i\delta} \sin(\delta) \right).
\tag{12}
\]

This relation was obtained empirically in Ref. [7]. Here we show that it emerges naturally if \( N \) is a real function.

The parameter \( \delta \) can be obtained by fitting the total cross section, using this \( A_R \) in the original Breit-Wigner formula in Eq. (1). In Fig. 1 we see how the shape of \( |A_R|^2 \) depends on \( \delta \).

![FIG. 1. The value of \( \delta \) changes the resonant shape \( |A_R|^2 \)](image)

To determine the Breit-Wigner parameters, we rewrite the amplitude in Eq. (12) as

\[
A_R = x e^{i(\rho + \beta)} \sin(\rho + \delta),
\tag{13}
\]

where the new function \( \rho \) is defined by

\[
\tan \rho = \frac{\Gamma/2}{M - W}.
\tag{14}
\]

Assuming again that \( N \) is real, the definition of Breit-Wigner mass from Eq. (7) becomes

\[
\text{Re } A_R (M_{BW}) = 0 \Rightarrow \cos [\rho (M_{BW}) + \beta] = 0.
\tag{15}
\]

The tan \( \rho \) at the energy equal to \( M_{BW} \) is then

\[
\tan \rho (M_{BW}) = \cot \beta.
\tag{16}
\]

We rewrite it using Eq. (14) to get the Breit-Wigner mass

\[
M_{BW} = M - \Gamma/2 \tan (\theta - \delta),
\tag{17}
\]

The Breit-Wigner width is then obtained using Eq. (8)

\[
\Gamma_{BW} = \Gamma / \cos^2(\theta - \delta).
\tag{18}
\]

Eqs. (17) and (18) are consistent with Ref. [8]. However, neither the residue phase \( \theta \), nor the shape parameter \( \delta \) were considered there.

With all this known, we can address the second objection against the Breit-Wigner parameters: that they depend on the nonphysical background. We see that the Breit-Wigner parameters, apart from the fundamental pole parameters \( M \) and \( \Gamma \), indeed depend on the background phase \( \beta \). However, this \( \beta \) is fixed by two fundamental and observable resonance parameters: the pole residue phase \( \theta \), and the deformation phase \( \delta \)

\[
M_{BW} = M - \Gamma/2 \tan (\theta - \delta),
\tag{19}
\]

\[
\Gamma_{BW} = \Gamma / \cos^2(\theta - \delta).
\tag{20}
\]

Thus, the Breit-Wigner parameters are the link between the fundamental pole parameters \( M \), \( \Gamma \), and \( \theta \), and the observable shape of the resonant peak \( \delta \).

From Eq. (13), we see that \( \beta \) and \( \delta \) are constant background phase shifts. If they are both zero, we get the original Breit-Wigner formula where the amplitude depends solely on the pole parameters \( M \) and \( \Gamma \). Otherwise, each background shift changes the different phase. If the amplitude is written as \( A_R = |A_R| e^{i\alpha} \), we see that \( \beta \) shifts the complex phase of the amplitude

\[
\alpha = \rho + \beta,
\tag{21}
\]

while \( \delta \) shifts the real phase of the amplitude magnitude

\[
|A_R|^2 = x^2 \sin^2(\rho + \delta).
\tag{22}
\]

This means that we can independently extract \( \delta \) from \( |A_R|^2 \), as was done in Ref. [7], and \( \beta \) from the phase shift \( \alpha \). Once the \( \beta \) and \( \delta \) are extracted, the pole residue phase \( \theta \) is given by their sum

\[
\theta = \delta + \beta.
\tag{23}
\]

Consequently, even though the claim that the Breit-Wigner parameters depend on the background is true, we see that the fundamental pole parameter \( \theta \) also depends on the same background phase \( \beta \) and, in addition, on another background phase \( \delta \).

Before we draw any strong conclusions on this matter, we remind ourselves that this is a relatively crude approximation which needs to be tested on some realistic
examples. We begin by fitting the Breit-Wigner formula from Eq. (1), using $A_R$ from Eq. (12), to the $\pi N$ elastic total cross section data collected in Ref. [2]. We use the fitting strategy with non-coherent background as described in Ref. [7], and show the results in Fig. 2. The obtained mass $M$ is $1211 \pm 1$ MeV, and the total width $\Gamma$ is $101 \pm 2$ MeV, which is in excellent agreement with estimates from Ref. [2], where $M_{PDG}$ is $1210 \pm 1$ MeV, and $\Gamma_{PDG}$ is $100 \pm 2$ MeV.

![Graph](image)

**FIG. 2.** Fit of Eq. (12) to the $|A_R|^2$ data obtained from cross section $\pi$ for the process $\pi N \to \pi N$ using Eq. (1). The data is taken from the PDG database [2]. The fitting strategy and background are the same as in Ref. [7].

In the same reference, the residue phase $\theta_{PDG}$ for the $\pi N$ elastic reaction is $-47 \pm 1^\circ$. From the fit, we obtained the deformation phase $\delta = -23 \pm 1^\circ$. Subtracting $\delta$ from $\theta_{PDG}$ gives us $\beta = -24 \pm 1.4^\circ$. Using our pole parameters $M$ and $\Gamma$ with this $\beta$, we get the Breit-Wigner mass $M_{BW} = 1233 \pm 2$ MeV and width $\Gamma_{BW} = 121 \pm 4$ MeV. Checking Ref. [2] again, we see that $M_{BW}^{PDG} = 1232 \pm 2$ MeV and $\Gamma_{BW}^{PDG} = 117 \pm 3$ MeV. Thus, our Breit-Wigner parameters agree very well to the PDG estimates.

Next, we test our approximation on another resonance that has a significant difference between the pole and Breit-Wigner mass, the Z boson. We use the Breit-Wigner parametrization of $A_R$ from Ref. [2], with the PDG values of the Breit-Wigner mass $M_{BW}^{PDG} = 91188 \pm 2$ MeV, and the width $\Gamma_{BW}^{PDG} = 2495 \pm 2$ MeV in the parametrization. Here, the goal is to see whether our formula reproduces these Breit-Wigner parameters.

We extract the pole parameters from the parametrization to get mass $M = 91162$ MeV, total width $\Gamma = 2494$ MeV, and residue phase $\theta = -2.4^\circ$. In Ref. [7], the phase $\delta$ was estimated from the $e^+e^-$ scattering data to be $-1.1 \pm 0.1^\circ$. Here, we determine $\delta$, without fitting, from the peak position of $|A_R|^2$. From Eq. (13), it is easy to show that the relation between the peak position $M_{peak}$ and $\delta$ is given by

$$M_{peak} = M - \Gamma/2 \tan \delta.$$  \hspace{1cm} (24)

The same relation was obtained in Ref. [7]. From the peak of $|A_R|^2$ we estimate $\delta = -1.2^\circ$. With $\theta$ and $\delta$ known, we easily calculate $\beta = -1.2 \pm 0.1^\circ$, which gives us $M_{BW} = 91188 \pm 2$ MeV, and the width $\Gamma_{BW} = 2495 \pm 2$ MeV. These are exactly the values of the Breit-Wigner parameters in Ref. [2].

In order to test the amplitude approximation in Eq. (13), we build it using the PDG parameters $M$, $\Gamma$, $M_{BW}$, and $\theta$ for some resonances with known amplitudes. From these parameters we first calculate $\beta$ using Eq. (17), and then $\delta$ using Eq. (11). Finally, the amplitude is compared to the data from the SAID $\pi N$ elastic partial waves [9]. The comparison is shown in Fig. 3.

![Graph](image)

**FIG. 3.** The thick black line represents $|A_R|^2$ (left column) and the phase shift $\alpha$ (right column) in the energy range $M \pm \Gamma$. They are calculated (not fitted!) using Eqs. (21) and (22). Resonant parameters are PDG estimates from Ref. [2]. The data points are constructed using the real and imaginary parts of $\pi N \to \pi N$ partial waves from SAID database [9].

We have shown here that the proposed approximation is valid for more than a few resonances. Naturally, there
are exceptions, but since the approximation works rather well, it seems reasonable to use it as a starting point or consistency test for future analyses.

The final objection against the Breit-Wigner parameters is that they are not invariant under particular field transformations that do not change any observables, including pole parameters. For the Z boson case see Ref. [5], and for \( \Delta(1232) \) see Ref. [6].

Going into the details of particular field transformations would take us beyond the scope of this Letter. However, it is clear that a field transformation generally affects the whole amplitude. The denominator \( D \) changes due to the modified self energy term in the propagator, but the numerator \( N \) may be different as well because the vertices are also changed by the transformation. However, in Refs. [3][4] only the change in \( D \) was considered, since the definition of the Breit-Wigner parameters in Eqs. (7) and (8) uses only \( D \). There is a serious problem with this approach, which we illustrate by constructing a simple amplitude transformation. In it, both \( N \) and \( D \) from Eq. (4) are multiplied by the same complex number, e.g., \( e^{i\varphi} \), where \( \varphi \) is some real constant. The modified amplitude \( \tilde{A}_R \) is then given by

\[
\tilde{A}_R = \frac{e^{i\varphi} N(W)}{e^{i\varphi} D(W)} = \frac{\tilde{N}(W)}{\tilde{D}(W)}. \tag{25}
\]

This transformation, evidently, does not change anything, since the \( \tilde{A}_R \) is absolutely indistinguishable from \( A_R \). Indeed, the pole mass and width, and even the residue phase extracted from \( \tilde{D}(W) \) are invariant to this transformation. Yet, the Breit-Wigner parameters depend on the unobservable parameter \( \varphi \).

This example may seem to be somewhat oversimplified, but that is exactly what happens in our linear approximation. In \( N \), we end up with a term \( (W_{\text{zero}} - W) \) multiplied by a complex constant. In \( D \), we have \( (W_{\text{pole}} - W) \) times another complex constant. Multiplying, or dividing, both constants by any complex number would change the Breit-Wigner mass and width defined by Eqs. (7) and (8).

Since in our approach \( N \) is a real function, the Breit-Wigner mass is given by the energy where the phase shift \( \alpha \) crosses 90°, which is invariant under the transformation in Eq. (25). Therefore, even though we cannot claim that this definition will be invariant under elaborate field transformations from [5][6], we clearly see that any definition using only \( D \) function is bound to fail even under our simple transformation.

Generally, a resonance participates in more than one process. It is interesting to see what is the value of \( \beta \) in different processes. The amplitude \( A_R \) is a matrix in channel indices \( ab \), and each process from channel \( a \) to channel \( b \) (e.g., from \( e^+e^- \) to \( \pi^+\pi^- \)) is described by a corresponding matrix element of the amplitude

\[
[A_R]_{ab} = x_{ab} e^{i(\rho + \beta_{ab})} \sin (\rho + \delta_{ab}). \tag{26}
\]

When the unitarity constraint \( A_R^\dagger A_R = \text{Im} A_R \) is imposed on the amplitude, we see that it generally cannot be satisfied, unless all parameters \( \beta_{ab} \) are the same for all processes. Thus, the unitarity constraint demands that the phase \( \beta \) is same in all processes. If there is more than one decay channel, \( \beta \) will be the same in all channels.

We have shown that \( \beta \) is an important parameter with useful properties. But, is the Breit-Wigner mass a physical, observable property of a resonance, in a sense in which \( M_{\text{peak}} \) is the peak of the squared amplitude, which is related to an observable, the partial cross section \( \sigma \) in Eq. (1)? It turns out that it is, at least within the proposed approximation. From Eq. (15) it is evident that

\[
(\text{Im} A_R)^2/|A_R|^2 = \sin^2(\rho + \beta). \tag{27}
\]

From comparison with Eq. (22), we see that \( \beta \) determines the shape of \( (\text{Im} A_R)^2/|A_R|^2 \) in exactly the same manner in which \( \delta \) does it for \( |A_R|^2 \), which means that \( M_{\text{BW}} \) is the peak of \( (\text{Im} A_R)^2/|A_R|^2 \). And that is interesting because for elastic processes, due to the optical theorem, \( \text{Im} A_R \) is related to another observable: the total cross section \( \sigma_{\text{tot}} \).

In conclusion, we have demonstrated here that the common objections against the Breit-Wigner parameters are not valid. It is clear even from the definition that their model dependence is not worse than the model dependence of the pole parameters. We have used the first order expansion of a resonant amplitude to show that even though the Breit-Wigner mass and width do depend on the background phase \( \beta \), this \( \beta \) is in fact fixed by the physical properties of the resonance. Thus, the Breit-Wigner parameters link the observable shape parameter \( \delta \) to the fundamental pole parameter \( \theta \). Due to the unitarity, \( \beta \) is unique for all processes involving the resonance. Consequently, the Breit-Wigner mass and width are unique as well. Finally, using a simple but realistic example, we have shown that the Breit-Wigner definition used in Refs. [3][4] is inappropriate because it may produce unphysical results. A more appropriate definition of the Breit-Wigner mass is the energy at which the amplitude phase \( \alpha \) crosses 90°.

\*sasa.ceci@irb.hr

[1] G. Breit and E. Wigner, Phys. Rev. 49, 519 (1936).
[2] J. Beringer et al. (PDG), Phys. Rev. D 86, 010001 (2012).
[3] G. Höhler, “Against Breit-Wigner parameters – A pole-emic” in D. E. Groom et al. Eur. Phys. J. C15, 1 (2000).
[4] N. A. Törnqvist, Z. Phys. C 68, 647 (1995).
[5] A. Sirlin, Phys. Rev. Lett. 67, 2127 (1991).
[6] D. Djukanovic, J. Gegelia, and S. Scherer, Phys. Rev. D 76, 037501 (2007).
[7] S. Ceci, M. Korolija, and B. Zauner, Phys. Rev. Lett. 111, 112004 (2013).
[8] D. M. Manley, Phys. Rev. D 51, 4837 (1995).
[9] SAID database, http://gwdac.phys.gwu.edu/
SUPPLEMENTARY MATERIALS

A word on our branching fraction $x$

In the literature, see e.g. Ref. [2], our branching fraction $x$ is commonly written as

$$x = \sqrt{B_{\text{in}}/B_{\text{out}}},$$

(28)

where $B_{\text{in}}$ and $B_{\text{out}}$ are the resonance branching fractions, i.e. the ratios of the partial and total decay widths, to the entrance and the exit channels, respectively.

Derivation of Eq. (12)

We begin by expanding all functions in Eq. (3) and keeping only linear terms

$$A_R = \frac{aW + b}{eW + f - W + i(gW + h)}.$$  

(29)

where all parameters ($a$, $b$, $e$, $f$, $g$, and $h$) are real constants. With this choice, the function in numerator will be real-valued. Rewriting it further, we obtain

$$A_R = \frac{aW + b}{(1 - e - ig)W + (f + ih)},$$

(30)

$$= \frac{a}{1-e-ig} \frac{W + \frac{b}{1-e-ig}}{\frac{1-e-ig}{1-e-ig} - W}.$$  

(31)

To simplify the notation, we write

$$A_R = \frac{kW + l}{n - W},$$

(32)

where

$$k = \frac{a}{1-e-ig},$$

(33)

$$l = \frac{b}{1-e-ig},$$

(34)

$$n = \frac{f + ih}{1-e-ig}. $$

(35)

Note that, since all of the parameters from $a$ to $f$ are real numbers, the complex phases of $k$ and $l$ are the same (up to $\pi$). Now, we rewrite the amplitude as

$$A_R = \frac{kn + l}{n - W} - k,$$

(36)

and immediately see the connection with the resonance pole parameters

$$n = M - i \Gamma/2,$$

(37)

$$kn + l = x \frac{\Gamma/2}{M - W - i \Gamma/2} e^{i\theta},$$

(38)

$$k = -x \kappa e^{i\beta},$$

(39)

where $x$ is $2|\gamma|/\Gamma$.

Combining it all together, we get the form analogous to Eq. [18]

$$A_R = \frac{x \Gamma/2 e^{i\theta}}{M - W - i \Gamma/2} + x \kappa e^{i\beta}.$$  

(40)

This form is still not very practical. However, we have not yet used all available information. Namely, we did not use the fact that $k$ and $l$ have the same complex phase (up to $\pi$).

To calculate the phase of $l$, we rewrite Eq. [38] and express $l$ in terms of the resonance parameters from Eqs. [37] and [39]

$$l = x \frac{\Gamma/2}{M - W - i \Gamma/2} e^{i\beta},$$

(41)

$$= x \frac{\Gamma}{2} \left( e^{i\theta} + \kappa \left( \frac{M}{\Gamma/2} - i \right) e^{i\beta} \right),$$

(42)

$$= x \frac{\Gamma}{2} \left( e^{i\theta} + \kappa \frac{M}{\Gamma/2} e^{i\beta} - i \kappa e^{i\beta} \right).$$

(43)

The phase of $l$, let us call it $\phi_l$, is the same as the $\phi_k$, the phase of $k$ (up to $\pi$). We calculate tangents of the two phases by dividing the imaginary and the real parts of the corresponding parameters

$$\tan \phi_k = \frac{\sin \beta}{\cos \beta},$$

(44)

$$\tan \phi_l = \frac{\sin \theta + \kappa \frac{M}{\Gamma/2} \sin \beta - i \kappa \cos \beta}{\cos \theta + \kappa \frac{M}{\Gamma/2} \cos \beta + \kappa \sin \beta}.$$  

(45)

At the first glance, it may seem that the pole and background phases $\theta$ and $\beta$ depend on the resonance mass $M$ and width $\Gamma$. In fact, they do not. It can be easily shown that it works for any $M$ and $\Gamma$. (Note that for $\Gamma \rightarrow 0$, the equation system becomes trivial.) For simplicity, we choose $M = 0$ and get

$$\frac{\sin \theta - \kappa \cos \beta}{\cos \theta + \kappa \sin \beta} = \frac{\sin \beta}{\cos \beta}.$$  

(46)

Solving this for $\kappa$ yields

$$\kappa = \sin(\theta - \beta).$$

(47)

Now, amplitude $A_R$ becomes

$$A_R = x \left[ \frac{\Gamma/2 e^{i\theta}}{M - W - i \Gamma/2} + \sin(\theta - \beta) e^{i\beta} \right].$$

(48)

Eq. (12) is obtained by rewriting the last equation using phases $\delta$ and $\eta$

$$A_R = x e^{i\eta} \left( \frac{\Gamma/2 e^{i\delta}}{M - W - i \Gamma/2} + e^{i\delta} \sin \delta \right).$$

(49)
Derivation of Eq. (13)

It may not be obvious that Eqs. (12) and (13) are equivalent. Here, we show that they indeed are. First, we define

$$T_R = \frac{\Gamma/2}{M-W-i\Gamma/2},$$

and note that it can be written as

$$T_R = e^{i\rho} \sin \rho,$$

where

$$\rho = \arctan \frac{\Gamma/2}{M-W}.$$  

It can be shown that this is true by using

$$\cos (\arctan x) = \frac{1}{\sqrt{1+x^2}},$$

and

$$\sin (\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

For completeness, we show it here. By rewriting $T_R$ in terms of trigonometric functions, we get

$$T_R = \cos \rho \sin \rho + i \sin^2 \rho,$$

$$= \frac{\Gamma/2}{M-W} + i \left( \frac{\Gamma/2}{M-W} \right)^2,$$

$$= \frac{\Gamma/2}{(M-W)^2 + (\Gamma/2)^2}.$$  

We simplify this relation further by expanding the sum of squares in the denominator

$$T_R = \frac{\Gamma/2 (M-W + i\Gamma/2)}{(M-W - i\Gamma/2) (M-W + i\Gamma/2)},$$

$$= \frac{\Gamma/2}{M-W - i\Gamma/2},$$

which proves Eq. (51).

Knowing this, we can easily show the equivalence between the two amplitude forms. First, we write the amplitude from Eq. (12) in terms of $\rho$,

$$A_R = x e^{i\eta} \left( e^{i\delta} e^{i\rho} \sin \rho + e^{i\delta} \sin \delta \right),$$

$$= x e^{i\eta} e^{i\rho+\delta} \left( e^{i\delta} \sin \rho + e^{-i\rho} \sin \delta \right).$$

Expanding the term in brackets into trigonometric functions gives us

$$e^{i\delta} \sin \rho + e^{-i\rho} \sin \delta = \cos \delta \sin \rho + i \sin \delta \sin \rho$$

$$+ \cos \rho \sin \delta - i \sin \rho \sin \delta.$$  

Imaginary terms cancel out, and we get

$$e^{i\delta} \sin \rho + e^{-i\rho} \sin \delta = \cos \delta \sin \rho + \cos \rho \sin \delta,$$

$$= \sin (\rho + \delta).$$

Thus, the amplitude becomes

$$A_R = x e^{i\eta} e^{i\rho+\delta} \sin (\rho + \delta).$$

Finally, Eq. (13) is obtained by using the definitions of $\eta = 2\beta - \theta$ and $\delta = \theta - \beta$

$$A_R = x e^{i\rho+\beta} \sin (\rho + \delta).$$

The Breit-Wigner parameterization of the Z-boson resonant amplitude

The Z-boson amplitude in Ref. [2] is given as the function of Mandelstam variable $s$

$$A_R(s) = x \frac{s \Gamma/M}{M^2 - s - i s \Gamma/M},$$

where $x$ is branching fraction, $M$ and $\Gamma$ are the Breit-Wigner mass and width of the Z boson, and $s$ is defined as the square of the c.m. energy $W$. In our notation, this equation therefore reads

$$A_R(W) = x \frac{\Gamma_{BW} W^2/M_{BW}}{M_{BW}^2 - W^2 - i \Gamma_{BW} W^2/M_{BW}}.$$  

Derivation of Eq. (24)

To get the peak position of $|A_R|^2$, we start with Eq. (13),

$$A_R = x e^{i[\rho(W)+\beta]} \sin [\rho(W) + \delta].$$

Now, we could calculate the first and second derivative of $|A_R|^2$ over energy, but instead we just note that $|A_R|^2$ has a maximum when

$$\sin^2 [\rho(M_{peak}) + \delta] = 1,$$

or

$$\rho(M_{peak}) + \delta = \frac{\pi}{2} + n \pi,$$

where $n$ is any whole number. It means that

$$\tan \rho(M_{peak}) = \cot \delta.$$  

Using the definition of the $\rho$ function, given in Eq. (14), we get

$$\frac{\Gamma/2}{M - M_{peak}} = \cot \delta.$$  

We obtain Eq. (24) by solving the last equation for $M_{peak}$

$$M_{peak} = M - \Gamma/2 \tan \delta.$$
The fitting strategy

In Fig. [2], we show the fit of our amplitude to the the $\pi N \rightarrow \pi N$ data. The parameterization that we use to fit the data is

$$\frac{(2s_1 + 1)(2s_2 + 1)}{2J + 1} \frac{q^2}{4\pi} = |A_R|^2 + \sum_{k=0}^{n} B_k W^{2k},$$  \hspace{1cm} (76)

where $|A_R|^2$ is given by Eq. [12], and $B_n$ are real parameters. For convenience of the reader, we repeat here the fitting strategy from Ref. [7], which is used in this paper as well: “(...) To extract the resonance parameters, we do local fits of this parameterization to a broad range of data points in the vicinity of the resonance peak. To estimate the proper order $n$ of polynomial background, we vary endpoints of the data range and check the convergence of the physical fit parameters: $M$, $\Gamma$, and $x$. Goodness of the convergence is estimated by calculating $c_n, l$ parameters for each data range and for all polynomial orders $n$ and $l$

$$c_{n,l} = \sum_{y=m,G,x} (y_n - y_n)^2/y_n^2.$$  \hspace{1cm} (77)

Smaller $c_{n,l}$ means better convergence. To avoid false positive convergence signals as much as possible, we demand good convergence not just for two, but for three consecutive polynomial orders by using

$$c_n = c_{n,n+1} + c_{n,n+2}.$$  \hspace{1cm} (78)

Final result is the one having the smallest reduced $\chi^2$, among several fits (we use ten) with lowest convergence parameters $c_n$. When statistical errors turn out to be unrealistically small due to the dataset issues, the spread in extracted pole parameter values is used to estimate parameters errors. (…)"

In Fig. [2] black data points are automatically chosen to be fitted by this strategy. Thus, the line passing through the gray data points is the extrapolation.

Error analysis for $M_{BW}$ and $\Gamma_{BW}$

To extract the Breit-Wigner parameters, we use Eqs. (17) and (18), which we repeat here for convenience

$$M_{BW} = M - \frac{\Gamma}{2} \tan \beta,$$  \hspace{1cm} (79)

$$\Gamma_{BW} = \frac{\Gamma}{\cos^2 \beta}.$$  \hspace{1cm} (80)

We want to know how the Breit-Wigner mass and width depend on errors in pole mass $\Delta M$, pole width $\Delta \Gamma$, and background phase $\Delta \beta$. Assuming the errors are independent, we calculate the error propagation for the Breit-Wigner parameters.

First, we calculate all partial derivatives of the Breit-Wigner mass

$$\frac{\partial M_{BW}}{\partial M} = 1,$$  \hspace{1cm} (81)

$$\frac{\partial M_{BW}}{\partial \Gamma} = -\frac{\tan \beta}{2},$$  \hspace{1cm} (82)

$$\frac{\partial M_{BW}}{\partial \beta} = -\frac{\Gamma}{2 \cos^2 \beta},$$  \hspace{1cm} (83)

and get

$$\left(\Delta M_{BW}\right)^2 = \left(\Delta M\right)^2 + \frac{\tan^2 \beta}{4} \left(\Delta \Gamma\right)^2 + \frac{\Gamma^2}{4 \cos^4 \beta} \left(\Delta \beta\right)^2.$$  \hspace{1cm} (84)

Then, we calculate all partial derivatives of the Breit-Wigner width

$$\frac{\partial \Gamma_{BW}}{\partial M} = 0,$$  \hspace{1cm} (85)

$$\frac{\partial \Gamma_{BW}}{\partial \Gamma} = \frac{1}{\cos^2 \beta},$$  \hspace{1cm} (86)

$$\frac{\partial \Gamma_{BW}}{\partial \beta} = \frac{2 \sin \beta}{\cos^3 \beta},$$  \hspace{1cm} (87)

which gives

$$\left(\Delta \Gamma_{BW}\right)^2 = \frac{1}{\cos^4 \beta} \left(\Delta \Gamma\right)^2 + \Gamma^2 \frac{4 \sin^2 \beta}{\cos^6 \beta} \left(\Delta \beta\right)^2$$  \hspace{1cm} (88)

The results are given in Table IV.

| Res | $M$ (MeV) | $\Gamma$ (MeV) | $\beta = \theta - \delta$ | $M_{BW}$ (MeV) | $\Gamma_{BW}$ (MeV) |
|-----|------------|----------------|-----------------------------|----------------|-------------------|
| $\Delta$ | 1211 ± 1 | 101 ± 2 | −24 ± 1.4 | 1233 ± 2 | 121 ± 4 |
| PDG | 1210 ± 1 | 100 ± 2 | 1232 ± 2 | 117 ± 3 |
| Z | 91 162 ± 2 | 2494 ± 2 | −1.2 ± 0.1 | 91 188 ± 3 | 2495 ± 2 |
| PDG | 91 188 ± 3 | 2495 ± 2 |                          | 2495 ± 2 |

Testing the amplitude defined in Eq. (13)

To test the amplitude approximation in Eq. (13), we construct it using the PDG parameters $M$, $\Gamma$, $M_{BW}$, and $\theta$ of some resonances with known amplitudes. From these parameters, we first calculate $\beta$ using Eqs. (17), and then $\delta$, using Eq. (11). The results are given in Table IV.

The amplitude constructed by using PDG parameters in Eq. (13) is compared to the partial waves from the SAID database [9]. The results are show in Fig. 3 and results are quite promising. However, not all resonances
were in a good agreement with the data. We show here a few of those as well, and try to explain the observed discrepancies.

In Fig. 4, we see that the amplitude approximation from Eq. (13) for $N(1535)$ does not work that well. The reason could be that the $N(1535)$ is in a close proximity of the $\eta N$ threshold at 1486 MeV, and strongly overlaps with the neighboring resonance $N(1650)$.

**TABLE II.** The pole and Breit-Wigner PDG \[2\] values, as well as the parameters $\beta$ and $\delta$ obtained from them, which are used to construct the amplitudes in Fig. 3.

| Resonance | $M$ (MeV) | $\Gamma$ (MeV) | $M_{BW}$ (MeV) | $\beta$ | $x$ | $\theta$ | $\delta$ |
|-----------|-----------|----------------|----------------|---------|-----|---------|---------|
| $\Delta(1232)$ | 1210 | 100 | 1232 | -23 | 100 | -47 | -24 |
| $N(1520)$ | 1510 | 110 | 1520 | -10 | 60 | -10 | 0 |
| $N(1675)$ | 1660 | 135 | 1675 | -13 | 40 | -25 | -12 |
| $N(1680)$ | 1675 | 120 | 1685 | -9 | 68 | -10 | -1 |

In the case of the Roper resonance $N(1440)$, we immediately see from Fig. 5 a) and b) that it is less than perfect. It could be that the problem is its large width of almost 200 MeV. It could also be that the PDG estimates are not precise enough. Though, since this is a relatively crude approximation, it could be that it is simply too crude for this resonance. However, in Fig. 5 c) we use the $\delta$ extracted from the SAID $\pi N$ elastic amplitude in Ref. [7], where $\delta = -40^\circ$, and the resemblance to data seems much better.

If $\beta$ is allowed to have a linear energy dependence, we can describe the data in the substantially larger range, as is clear from Fig. 5 d). Though, that would be far beyond the proposed approximation.

In Fig. 6 we can see that the strong discrepancy in the amplitude magnitude $|A_R|^2$ for $N(2190)$ (full line) can be drastically reduced if we assume that the branching fraction $x$ is 25% instead of 15% (dashed line).

In Fig. 7 we see the drastic discrepancy in the case of $\Delta(1620)$ (full lines). Curiously, it is almost removed by simply adding a flat background term, roughly equal to $-0.22 + i 0.33$, to the amplitude $A_R$ (dashed lines). This number is estimated using the eye-ball fit.

It is interesting to note that such a simple modification, adding a flat complex background to the amplitude, changes the result that much. We also tested the linear complex background, which fits the data perfectly. However, that would be way beyond our first order approximation.

Another distinctive feature of $\Delta(1620)$ is a peculiar...
relation between its \( \theta \) and \( \delta \). All other resonances we have analyzed had \( \beta \) either roughly equal to \( \theta \), meaning \( \delta \approx 0^\circ \), or to \( \delta \), which means \( \theta \approx 2\beta \approx 2\delta \). In the case of \( \Delta(1620) \), \( \delta \) is three times larger than \( \beta \).

Why is \( \delta \) roughly equal to \( \beta \) for some resonances?

For (almost) elastic resonances, such as \( \Delta(1232) \), total and partial decay widths are almost the same. Therefore, within our approximation, we can write

\[
A_R = \frac{aW + b}{eW + f - W - i(aW + b)}. \tag{89}
\]

We rearrange it to

\[
A_R = \frac{aW + b}{f - ib - (1 - e + ia)W}, \tag{90}
\]

\[
= \frac{aW + b}{\frac{f - ib}{1 - e + ia} - W}. \tag{91}
\]

This form has the pole \( W_p \) and residue \( r \) given by

\[
W_p = \frac{f - ib}{1 - e + ia}, \tag{92}
\]

\[
r = \frac{aW_p + b}{1 - e + ia}. \tag{93}
\]

Here, we are interested in the pole residue

\[
|r| e^{i\theta} = \frac{a f - ib + b(1 - e + ia)}{1 - e + ia}, \tag{94}
\]

\[
= \frac{a f}{1 - e + ia} + b(1 - e + ia)^2, \tag{95}
\]

\[
= \frac{a f + b(1 - e)(1 - e - ia)^2}{(1 - e + ia)^2}, \tag{96}
\]

\[
= \frac{a f + b(1 - e)}{(1 - e + ia)^2}. \tag{97}
\]

From the last relation we see that the residue phase \( \theta \) is (up to \( \pi \)) equal to the phase of \( (1 - e - ia)^2 \)

\[
\tan \theta = \frac{\text{Im} (1 - e - ia)^2}{\text{Re} (1 - e - ia)^2}, \tag{98}
\]

\[
= -\frac{2a(1 - e)}{(1 - e)^2 - a^2}, \tag{99}
\]

\[
= -\frac{2a}{1 - e} \left( 1 - \frac{a}{1 - e} \right)^2. \tag{100}
\]

To simplify this, we introduce

\[
\tan \psi = \frac{a}{1 - e}, \tag{101}
\]

and immediately see that

\[
\tan \theta = -\tan(2\psi). \tag{102}
\]

To find the connection between \( \theta, \psi, \) and \( \beta \), we need to go back to Eq. (33), which in our case looks like

\[
k = \frac{a}{1 - e - i\gamma} \rightarrow \frac{a}{1 - e + ia}, \tag{103}
\]

\[
= \frac{a(1 - e + ia)}{(1 - e)^2 + a^2}. \tag{104}
\]

The phase of \( k \), the \( \phi_k \), is then given by

\[
\tan \phi_k = -\frac{a}{1 - e}, \tag{105}
\]

and therefore

\[
\tan \phi_k = -\tan \psi. \tag{106}
\]

From Eq. (44), we know that

\[
\tan \phi_k = \tan \beta. \tag{107}
\]

Gathering all relations involving \( \theta, \psi, \phi_k, \) and \( \beta \), we finally get

\[
\tan \beta = \tan(\theta/2). \tag{108}
\]

Since \( \delta = \theta - \beta \), we see that

\[
\tan \delta = \tan \beta, \tag{109}
\]

and that is what we indeed see for elastic \( \Delta(1232) \), but also for many inelastic resonances, e.g. Z boson, \( N(1675) \), and even modified Roper resonance, the \( N(1440) \).

It is easy to show why this may work for some inelastic resonances. If \( A_R \) is in the form of the elastic amplitude, i.e. with \( N = -\text{Im} D \), multiplied by a real-valued branching fraction \( x \), as in the case of Z boson, we can repeat the same argumentation with modified Eq. (89) and show that \( x \) only affects the magnitude of the residue in Eq. (97), and leaves the residue phase \( \theta \) unchanged. The rest of derivation is exactly the same, and \( \tan \delta \) is again equal to \( \tan \beta \).

Unitarity and the uniqueness of \( \beta \)

We start from Eq. (13)

\[
A_R = x e^{i(\rho + \beta)} \sin (\rho + \delta). \tag{110}
\]

In a multichannel situation, this function is just one element of a mathematical object called the \( T \) matrix (and sometimes also the \( \tau \) matrix). The scattering matrix, or \( S \) matrix in short, is then given by

\[
S_{ab} = I_{ab} + 2iA_{ab}, \tag{111}
\]

where \( a \) and \( b \) are channel indices, and \( I \) is the unit matrix. We do not use the Kronecker delta symbol \( \delta_{ab} \) to
avoid the confusion with the deformation phase \( \delta \), and drop the index \( R \) to avoid the clutter in the notation. The \( S \) matrix is unitary, which means that

\[
S^\dagger S = S S^\dagger = I. \tag{112}
\]

In the terms of our resonant amplitude, it means

\[
A^\dagger A = \text{Im} \ A. \tag{113}
\]

Now, we impose this unitarity constraint to our ampli-

\[
\begin{align*}
[A^\dagger A]_{ab} &= \sum_{c=a,b,\ldots} x_{ca} e^{-i(\rho + \beta_{ca})} \sin (\rho + \delta_{ca}) x_{cb} e^{i(\rho + \beta_{cb})} \sin (\rho + \delta_{cb}), \\
&= \sum_{c=a,b,\ldots} x_{ca} x_{cb} e^{-i(\beta_{ca} - \beta_{cb})} \sin (\rho + \delta_{ca}) \sin (\rho + \delta_{cb}).
\end{align*} \tag{116}
\]

On the other hand, the imaginary part of the \( A \) matrix is given by

\[
\text{Im} \ A_{ab} = x_{ab} \sin (\rho + \beta_{ab}) \sin (\rho + \delta_{ab}), \tag{118}
\]

and it must be a real function. However, unless all \( \beta_{ab} \) are equal, there could always be some imaginary component in \( [A^\dagger A]_{ab} \). It means that, unlike for the \( \delta_{ab} \) and \( \theta_{ab} \) that change from one process to another, there is only one \( \beta \) for each resonance in all channels.

Why is \( \delta \) roughly equal to zero for some resonances?

The other interesting situation, seen for \( N(1520) \) and \( N(1680) \), and even for the slightly problematic \( N(1535) \), is that all of them seem to have \( \delta \) very close to zero. Here is how it could come about. If we assume that the numerator function varies so slowly with energy that close to the resonance we could ignore its energy dependence altogether, the parameter \( a \) in Eq. \ref{eq:39} will be zero. If \( a \) is zero, from Eq. \ref{eq:33} we see that \( k = 0 \) as well. From tude. First we write it as a matrix element, with the channel indices

\[
A_{ab} = x_{ab} e^{i(\rho + \beta_{ab})} \sin (\rho + \delta_{ab}). \tag{114}
\]

The Hermitian conjugate of this matrix is

\[
[A^\dagger]_{ab} = x_{ba} e^{-i(\rho + \beta_{ba})} \sin (\rho + \delta_{ba}). \tag{115}
\]

We multiply the two to get

\[ k = -x \sin(\theta - \beta) e^{i\beta}. \tag{119} \]

The only way \( k \) will be zero (assuming a nonzero branching fraction \( x \)) is that \( \theta = \beta \), which means that \( \delta \) will be zero.

Therefore, if we see a perfectly bell-shaped resonance in the data (after multiplying \( \sigma \) by initial momentum squared \( q^2 \)), we could expect that it has a pole residue phase \( \theta \) equal to \( \beta \), which should be the same for all processes where this resonance emerges.

The physical meaning of resonant parameters

The original Bret-Wigner parameters are shown in the top left of the Fig. 8. The parameter \( M_{\text{peak}} \) is given by the energy at which the amplitude magnitude has maximal value. The Bret-Wigner mass is the energy at which the amplitude phase \( \alpha \) crosses 90°. By observing \( |A_R|^2 \), it is clear that the \( M_{BW} \), unlike \( M_{\text{peak}} \), does not have any particular relation to its minimal or maximal values. It is the peak of the ratio \( (\text{Im} \ A_R)^2/|A_R|^2 \) that is interesting, because the amplitude magnitude \( |A_R|^2 \) is related to the partial cross section, while the imaginary part of the amplitude is connected to total cross section through optical theorem.
The original Breit–Wigner parameters

\[ x = \frac{\Gamma_{\text{tot}}}{\Gamma_{\text{tot}}} \]

The peak and pole parameters

\[ M_{\text{peak}} = M - \frac{\Gamma}{2} \tan \delta \]
\[ x = 2 \frac{|\Gamma|}{\Gamma} \]

Breit–Wigner parameter \( \beta \)

Breit–Wigner parameter \( M_{BW} \)

FIG. 8. The physical meaning of resonant parameters.