Commuting Hamiltonians from Seiberg-Witten Θ-Functions

A.Mironov†, A.Morozov‡

Elementary MAPLE calculations are used to support the claim of hep-th/9906240 that the ratios of theta-functions, associated with the Seiberg-Witten complex curves, provide Poisson-commuting Hamiltonians which describe the dual of the original Seiberg-Witten integrable system.

The goal of this note is to suggest a numerical evidence in favor of the manifest construction proposed in [1] for the dual integrable systems [2, 3]. This construction allows one to build systems dual to the elliptic Calogero and Ruijsenaars models with elliptic dependence on momenta and ultimately leads to the (self-dual) double elliptic system describing 6d SUSY gauge theories with the adjoint matter hypermultiplet. We start with repeating and discussing the main steps of the construction [1].

1. Standard approach [4] to the Seiberg-Witten anzatz [5] is to associate with every theory an integrable system $\mathcal{S}$ given by a Lax operator $L(\vec{p}, \vec{q}|\xi)$ that naturally gives rise to a family of the spectral curves of genus $g$,

$$C : \quad \det (L(\vec{p}, \vec{q}|\xi) - \lambda) = 0$$  \hspace{1cm} (1)

parametrized by the Poisson-commuting Hamiltonians (moduli) $h_n(\vec{p}, \vec{q})$, $n = 1, \ldots, g$,

$$\{h_m, h_n\} = \frac{\partial h_m}{\partial \vec{p}} \frac{\partial h_n}{\partial \vec{q}} - \frac{\partial h_n}{\partial \vec{p}} \frac{\partial h_m}{\partial \vec{q}} = 0$$\hspace{1cm} (2)

A distinguished set of the Poisson-commuting quantities is provided by the “flat” moduli (action variables of the system $\mathcal{S}$)

$$\vec{a} \equiv \oint_{\vec{a}} d\mathcal{S}$$  \hspace{1cm} (3)

which are the $A$-periods of the presymplectic 1-form $d\mathcal{S}$ characterized by the property that its infinitesimal variations with respect to moduli is holomorphic on the curve $C$,

$$\bar{\partial} \frac{\partial d\mathcal{S}}{\partial \text{ moduli}} = 0$$  \hspace{1cm} (4)

See refs. [1, 2] for further details and references.

†Theory Dept., Lebedev Physical Inst. and ITEP, Moscow, Russia
‡ITEP, Moscow, Russia
2. In [1] it was claimed that another interesting set of Poisson-commuting quantities is provided by the ratios of Riemannian theta-functions (KP $\tau$-functions) associated with the curve $C$ – and they were interpreted as the Hamiltonians of the dual integrable system.

In the case of $GL(N)$, $N = g + 1$ an explicit construction looks as follows. The spectral curve of the original integrable system (Toda chain, Calogero, Ruijsenaars or the most interesting double elliptic system) has a period matrix $T_{ij}(\vec{a})$, $i, j = 1, \ldots, N$ with the special property:

$$ \sum_{j=1}^{N} T_{ij}(a) = \tau, \quad \forall i $$

(5)

where $\tau$ does not depend on $a$. As a corollary, the genus-$N$ theta-function is naturally decomposed into a linear combination of genus-$g$ theta-functions:

$$ \Theta^{(N)}(p_i|T_{ij}) = \sum_{n_i \in \mathbb{Z}} \exp \left( i\pi \sum_{i,j=1}^{N} T_{ij} n_i n_j + 2\pi i \sum_{i=1}^{N} n_i p_i \right) $$

(6)

where

$$ \Theta_k \equiv \Theta^{(g)}(\vec{p}_i|\vec{T}_{ij}) $$

(7)

and $p_i = \zeta + \vec{p}_i$, $\sum_{i=1}^{N} \vec{p}_i = 0$; $T_{ij} = \vec{T}_{ij} + \vec{\tau}$, $\sum_{j=1}^{N} \vec{T}_{ij} = 0$, $\forall i$.

The claim of ref. [1] is that all the ratios $\Theta_k/\Theta_l$ are Poisson-commuting with respect to the Seiberg-Witten symplectic structure

$$ \sum_{i=1}^{N} dp_i \wedge da_i $$

(8)

(where $a_N \equiv -a_1 - \ldots - a_{N-1}$, i.e. $\sum_{i=1}^{N} a_i = 0$):

$$ \left\{ \frac{\Theta_k}{\Theta_l}, \frac{\Theta_m}{\Theta_n} \right\} = 0 \quad \forall k, l, m, n $$

(9)

or

$$ \Theta_i\{\Theta_j, \Theta_k\} + \Theta_j\{\Theta_k, \Theta_i\} + \Theta_k\{\Theta_i, \Theta_j\} = 0 \quad \forall i, j, k $$

(10)

or

$$ \{\log \Theta_i, \log \Theta_j\} = \left\{ \log \frac{\Theta_i}{\Theta_j}, \log \Theta_k \right\} \quad \forall i, j, k $$

(11)

The Hamiltonians of the dual integrable system can be chosen in the form $H_k = \Theta_k/\Theta_0$, $k = 1, \ldots, g$.

3. This claim was partly supported by the old observation [8, 9] that zeroes of the KP (Toda) $\tau$-function (i.e. essentially the Riemannian theta-function), associated with the spectral curve [1] are nothing but the coordinates $q_i$ of the original (Calodero, Ruijsenaars) integrable system $S$. In more detail, due to the property [1], $\Theta^{(N)}(p|T)$ as a function of $\zeta = 1/\mathcal{N} \sum_{i=1}^{N} p_i$ is an elliptic function on the
torus \((1, \tau)\) and, therefore, can be decomposed into an \(N\)-fold product of the genus-one theta-functions. Remarkably, their arguments are just \(\zeta - q_i:\)

\[
\Theta^{(N)}(p|T) = c(p, T, \tau) \prod_{i=1}^{N} \theta(\zeta - q_i(p, T)|\tau)
\]

(In the case of the Toda chain when \(\tau \to i\infty\) this "sum rule" is implied by the standard expression for the individual \(e^{q_i}\) through the KP \(\tau\)-function.) Since one can prove that \(q_i\) form a Poisson-commuting set of variables with respect to the symplectic structure \(\mathfrak{g}\), this observation indirectly justifies the claim of ref.\([1]\).

A deeper argument for the commutativity of such ratios should come from the study of (the quasiclassical limit of) quantum \(\tau\)-functions and their properties \([10]\) (i.e. from group theory), but this is beyond the scope of the present letter.

Postponing discussion of deep theoretical origins of \((9)\), we report here some results of MAPLE calculations, which provide a nontrivial check of those relations.

4.1. In the simplest case of the Seiberg-Witten family

\[
w = P_N(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i); \quad dS = \lambda d\log w
\]

the flat moduli \(a_i = \lambda_i\), the period matrix is singular and only finite number of terms survives in the series for the theta-function:

\[
\Theta^{(N)}(p|T) = \sum_{k=0}^{N-1} e^{2\pi ik\zeta} H_k^{(0)}(p, a),
\]

\[
H_k^{(0)}(p, a) = \sum_{I, |I|=k} \prod_{i \in I} e^{2\pi ip_i} \prod_{j \in \bar{I}} F_{ij}^{(0)}(a)
\]

Here

\[
F_{ij}^{(0)}(a) = \frac{\Lambda}{a_{ij}}
\]

and \(I\) are all possible partitions of \(N\) indices into the sets of \(k = |I|\) and \(N - k = |\bar{I}|\) elements. Parameter \(\Lambda\) becomes significant only when the system is deformed: either non-perturbatively or to more complex systems of the Calogero–Ruijsenaars–double-elliptic family.

This is the case of perturbative 4d pure \(N=2\) SYM theory with the prepotential

\[
F^{(0)}(a) = \frac{1}{2i\pi} \sum_{i<j}^{N} a_{ij}^2 \log a_{ij}
\]

The corresponding \(\tau\)-function \(\Theta^{(N)}(p|T)\), eq.\((12)\), describes an \(N\)-soliton solution to the KP hierarchy. The Hamiltonians \(H_k^{(0)}\), eq.\((15)\), are those of the degenerated rational Ruijsenaars system, and they are well-known to Poisson-commute with respect to the relevant Seiberg-Witten symplectic structure \(\mathfrak{g}\).
The same construction for the other perturbative Seiberg-Witten systems ends up with the Hamiltonians of the more sophisticated Ruijsenaars systems.

For the spectral curves \([11]\)

\[
w = \frac{P_N(\lambda)}{P_N(\lambda - m)} = \frac{\prod_{i=1}^n(\lambda - \lambda_i)}{\prod_{i=1}^n(\lambda - \lambda_i - m)}; \quad dS = \lambda d\log w
\]  

(perturbative 4d \(N = 4\) SYM with SUSY softly broken down to \(N = 2\) by the mass \(m\)) the Poisson-commuting (with respect to the same (8)) Hamiltonians \(H_k^{(0)}\) are given by (15) with

\[
F_{ij}^{(0)}(a) = \sqrt{a_{ij}^2 - m^2}
\]
i.e. are the Hamiltonians of the rational Ruijsenaars system.

For the spectral curve \([11]\)

\[
w = e^{-2i\epsilon N} \frac{P_N(\lambda)}{P_N(\lambda e^{-2\epsilon})}; \quad dS = \lambda d\log w
\]

(perturbative 5d \(N = 2\) SYM compactified on a circle with an \(\epsilon\) twist as the boundary conditions) the Hamiltonians are given by (15) with

\[
F_{ij}^{(0)}(a) = \sqrt{\sinh(a_{ij} + \epsilon)\sinh(a_{ij} - \epsilon)}
\]

i.e. are the Hamiltonians of the trigonometric Ruijsenaars system.

Finally, for the perturbative limit of the most interesting self-dual double-elliptic system \([1]\) (the explicit form of its spectral curves is yet unknown) the relevant Hamiltonians are those of the elliptic Ruijsenaars system, given by the same (15) with

\[
F_{ij}^{(0)}(a) = \sqrt{1 - \frac{2g^2}{sn^2(a_{ij})}} \sim \sqrt{\frac{\theta(\tilde{a}_{ij} + \epsilon|\tilde{\tau})\theta(\tilde{a}_{ij} - \epsilon|\tilde{\tau})}{\theta(\tilde{a}_{ij}|\tilde{\tau})}}
\]

where \(\tilde{\tau}\) is the modulus of the second torus associated with the double elliptic system.

In all these examples \(H_0^{(0)} = 1\), the theta-functions \(\Theta^{(N)}\) are singular and given by determinant (solitonic) formulas with finite number of items (only terms with \(n_i = 0,1\) survive in the series expansion of the theta function), and Poisson-commutativity of arising Hamiltonians is analytically checked within the theory of Ruijsenaars integrable systems.

4.2. Beyond the perturbative limit, the analytical evaluation of \(\Theta^{(N)}\) becomes less straightforward.

The deformation of the curve \([13]\),

\[
w + \frac{\Lambda^{2N}}{w} = P_N(\lambda), \quad dS = \lambda d\log w
\]

is associated with somewhat sophisticated prepotential of the Toda chain integrable system,

\[
F(a) = \frac{1}{2} \sum_{i < j} a_{ij}^2 \log \frac{a_{ij}}{\Lambda} + \sum_{k=1}^{\infty} \Lambda^2 F^{(k)}(a)
\]
The period matrix is
\[
T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = \tau \delta_{ij} + \log \frac{a_{ij}}{\Lambda} + \sum_{k=1}^{\infty} \Lambda^2 \frac{\partial^2 F^{(k)}}{\partial a_i \partial a_j}, \quad i \neq j,
\]
and \( \tau \) in this case can be removed by the rescaling of \( \Lambda \). Then,
\[
\Theta^{(N)}(p|T) = \sum_{k=0}^{N-1} e^{2\pi i k \zeta} \Theta_k(p, a) = \sum_{k=0}^{N-1} e^{2\pi i k \zeta} \sum_{n_i, \sum_i n_i = k} e^{-i \pi \sum_{i<j} T_{ij} n_{ij}^2} e^{2\pi i \sum_i n_i p_i} = \left( 1 + \sum_{i \neq j} e^{2\pi i (p_i - p_j)} F_{ij}^4 \prod_{k \neq i,j} F_{ik} F_{jk} + \sum_{i \neq j \neq k \neq l} e^{2\pi i (p_i + p_j - p_k - p_l)} F_{ik} F_{il} F_{jk} F_{jl} \prod_{m \neq i,j,k,l} F_{im} F_{jm} F_{km} F_{lm} + \ldots \right) + \ldots
\]
with
\[
F_{ij} = e^{-i \pi T_{ij}} = F_{ij}^{(0)} \left( 1 - i \pi \frac{\partial^2 F^{(1)}}{\partial a_i \partial a_j} + \ldots \right), \quad i \neq j
\]
The first few corrections \( F^{(k)} \) to the prepotential are explicitly known in the Toda-chain case [12], for example,
\[
F^{(1)} = -\frac{1}{2i \pi} \sum_{i=1}^{N} \prod_{k \neq i} (F_{ik}^{(0)})^2
\]

The coefficients \( \Theta_k \) in [16] are expanded into powers of \( (\Lambda/a)^{2N} \) and the leading (zeroth-order) terms are exactly the perturbative expressions [15]. Thus, the degenerated Ruijsenaars Hamiltonians [13] are just the perturbative approximations to the \( H_k = \Theta_k/\Theta_0 \) – the full Hamiltonians of integrable system, dual to the Toda chain.

With the help of MAPLE we checked that the first corrections to the degenerated Ruijsenaars Hamiltonians [15] indeed preserve Poisson-commutativity. We did it up to the second order in \( \Lambda^{2N} \) for \( N = 3 \) and up to the first order for \( N = 4 \).

*It would be interesting to compare this system with the recently proposed "elliptic Toda" system [13].
Note that there are three sources of deviations of $H_k = \Theta_k/\Theta_0$ from $H_k^{(0)}$:

- the terms with some $n_i > 1$ are taken into account in the expansion \(^2\)
- $F_{ij} \neq F_{ij}^{(0)}$
- $\Theta_0 \neq 1$

We checked that all these deviations are significant for the Poisson-commutativity: it is important to use all the specifics of the Seiberg-Witten Riemannian theta-function to obtain the Poisson-commuting Hamiltonians.

4.3. Similar checks up to the first non-perturbative order were performed for $N = 3$ for the duals of

- Calogero model – i.e. with $F_{ij}^{(0)}$ of the form \(^3\)
- Ruijsenaars model – i.e. with $F_{ij}^{(0)}$ of the form \(^3\)
- and the double-elliptic model of ref.\(^1\) – with $F_{ij}^{(0)}$ of the form \(^2\); in this case the check was made only with the first non-trivial correction in $\tilde{q} = e^{2\pi i \tau}$

Note that the period matrix in these cases is expanded into powers of $q = e^{2\pi i \tau}$:

$$T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = \tau \delta_{ij} + \log \frac{a_{ij}}{\Lambda} + M^2 \sum_{k=1}^{\infty} q^k \frac{\partial^2 F^{(k)}(a)}{\partial a_i \partial a_j}, \quad i \neq j$$

and the dimensional constant $M$ depends on the system. E.g., in the Calogero model $M = im$ etc. The limit to the Toda system corresponds to $q \to 0$, $qM^{2N}=$ fixed.

The result (Poisson-commutativity of $H_1 = \Theta_1/\Theta_0$ and $H_2 = \Theta_2/\Theta_0$ in this approximation) significantly depends on the form \(^2\) of the first (instanton-gas) correction to the prepotential. It is indeed known to be of this form not only for the Toda chain, but also for Calogero system \(^4\). For Ruijsenaars and double-elliptic systems eq.(\(^2\)) is not yet available in the literature. We checked that \(^2\) is true for these two systems for $N = 2$ (while we used \(^2\) in calculations for $N = 3$).

5. In conclusion, we found new non-trivial evidence in support of the claim \(^1\) that the theta-functions ratios $H_k = \Theta_k/\Theta_0$ in the case of Seiberg-Witten integrable systems provide Poisson-commuting Hamiltonians of the dual integrable systems. The real raison d’etre (and a reasonable proof) of this property remains to be found.

Also, if the universal expressions like \(^2\) in terms of perturbative $F_{ij}^{(0)}$ – the same for all the systems – will be found for higher corrections to the prepotentials\(^1\), this will immediately give an

\(^1\) To avoid possible confusion, the recurrent relation \(^4\) for the Toda-chain prepotential,

$$\frac{\partial^2 F}{\partial \log \Lambda^2} \sim \frac{\partial^2 F}{\partial \log \Lambda \partial a_i} \frac{\partial^2 F}{\partial \log \Lambda \partial a_j} \frac{\partial^2 F}{\partial \partial p_i \partial p_j} \log \Theta_0 \bigg|_{p=0}$$
explicit (although not the most appealing) construction of the Hamiltonians dual to the Calogero and Ruijsenaars models and – especially important – the self-dual double-elliptic system which was explicitly constructed in [1] only for $N = 2$.

We are indebted for useful discussions to H.W.Braden and A.Marshakov. Our work is partly supported by the RFBR grants 98-01-00328 (A.Mir.), 98-02-16575 (A.Mor.), the Russian President’s Grant 96-15-96939, the INTAS grant 97-0103 and the program for support of the scientific schools 96-15-96798. A.Mir. also acknowledges the Royal Society for support under a joint project.

References

[1] H.W.Braden, A.Marshakov, A.Mironov and A.Morozov, [hep-th/9906240]

[2] S.N.M.Ruijsenaars, Comm.Math.Phys., 115 (1988) 127-165

[3] See also:

V.Fock, Three remarks on group invariants related to flat connections, in: Geometry and Integrable Models, World Scientific, (eds. P.Pyatov and S.Solodukhin), 1995, p.20;

V.Fock and A.Rosly, Poisson structure on moduli of flat connections on Riemann surfaces and $r$-matrix, [math/9802054];

N.Nekrasov, [hep-th/9707111];

V.Fock, A.Gorsky, N.Nekrasov and V.Rubtsov, [hep-th/9906235]

[4] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett. B355 (1995) 466, [hep-th/9505037]

[5] N.Seiberg and E.Witten, Nucl.Phys. B426 (1994) 19, [hep-th/9407087]; Nucl.Phys. B431 (1994) 484, [hep-th/9408099];

A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys.Lett. B344 (1995) 169, [hep-th/9411048];

P.Argyres and A.Farragi, Phys.Rev.Lett. 74 (1995) 3931, [hep-th/9411057];

A.Hanany and Y.Oz, Nucl.Phys. B452 (1995) 283, [hep-th/9505074]

does not immediately provide such universal expressions. Already for $F^{(1)}$ this relation gives

$$F^{(1)} \sim \sum_{i<j}^N \left(F^{(0)}_{ij}\right)^2 \prod_{k \neq i,j}^N F^{(0)}_{ik} F^{(0)}_{jk}$$

which coincides with (28) for the Toda-chain $F^{(0)}_{ij}$, eq.(16), but is not true (in variance with (28)) for Calogero $F^{(0)}_{ij}$, eq.(19). Meanwhile, the recurrent relations of ref.[18] for the Calogero system provide more promising expansion.

All this emphasizes once again the need to study extension of the Whitham theory [19, 17] and WDVV-equations [20] from the Toda chains to the Calogero and Ruijsenaars systems.
[6] E.Martinec and N.Warner, Nucl.Phys. **459** (1996) 97-112, hep-th/9509161;
T.Nakatsu and K.Takasaki, Mod.Phys.Lett. **A11** (1996) 157-168, hep-th/9509162;
T.Eguchi and S.Yang, Mod.Phys.Lett. **A11** (1996) 131-138, hep-th/9510183;
R.Donagi and E.Witten, Nucl.Phys. **B460** (1996) 299, hep-th/9510101;
E.Martinec, Phys.Lett., **B367** (1996) 91, hep-th/9510204;
A.Gorsky and A.Marshakov, Phys.Lett. **B375** (1996) 127, hep-th/9512244;
H.Itoyama and A.Morozov, Nucl.Phys. **B477** (1996) 855, hep-th/9511123, hep-th/9601168;
A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett. **B380** (1996) 75, hep-th/9603140, hep-th/9604078;

[7] for the most recent reviews and references see:
R.Donagi, alg-geom/9705010;
A.Klemm, hep-th/9705131;
C.Gomez and R.Hernandez, hep-th/9711102;
A.Morozov, hep-th/9903087;
A.Mironov, hep-th/9903088;
A.Marshakov, *Seiberg-Witten Theory and Integrable Systems*, World Scientific 1999; hep-th/9903252, hep-th/9906029;
R.Carroll, hep-th/9905010;
M.Mariño, hep-th/9905053;
K.Takasaki, hep-th/9905224;

[8] H.Airault, H.McKean and J.Moser, Comm.Pure and Applied Math., **30** (1977) 95;
I.Krichever, Func.Anal. & Appl.**14** (1980) 282

[9] I.Krichever, O.Babelon, E.Billey and M.Talon, hep-th/9411160;
I.Krichever and A.Zabrodin, Uspekhi Mat.Nauk, **50** (1995) 3-56, hep-th/9505039;
T.Shiota, J.Math.Phys., **35** (1994) 5844-5849, hep-th/9402021;

[10] A.Gerasimov, S.Khoroshkin, D.Lebedev, A.Mironov and A.Morozov, Int.J.Mod.Phys., **A10** (1995) 2589-2614, hep-th/9405011;
S.Kharchev, A.Mironov, A.Morozov, Theor.Math.Phys., **104** (1995) 129-143; q-alg/9501013;
for a review see also:
A.Mironov, Theor.Math.Phys., **114** (1998) 127; q-alg/9711006;

[11] H.W.Braden, A.Marshakov, A.Mironov and A.Morozov, Nucl.Phys., **B558** (1999) 371
[12] See, e.g., A.Klemm, W.Lerche and S.Theisen, Int.J.Mod.Phys., A11 (1996) 1929; hep-th/9505150.
   K.Ito, N.Sasakura, Nucl.Phys., B484 (1997) 141; hep-th/9608054

[13] I.Krichever, hep-th/9909224

[14] E.D’Hoker and D.Phong, hep-th/9709053

[15] G.Moore and E.Witten, hep-th/9709193.
   A.Losev, N.Nekrasov and S.Shatashvili, hep-th/9711108; hep-th/9712980; 1061;
   G.Moore and M.Mariño, hep-th/9712062

[16] J.D.Edelstein, M.Mariño and J.Mas, Nucl.Phys., B541 (1999) 671; hep-th/9805172; hep-th/9902161

[17] A.Gorsky, A.Marshakov, A.Mironov, A.Morozov, Nucl.Phys., B527 (1998) 690-716; hep-th/9802007

[18] J.A.Minahan, D.Nemeschansky and N.P.Warner, Nucl.Phys., B528 (1998) 109; hep-th/9710146

[19] I.Krichever, Comm.Math.Phys., 143 (1992) 415, hep-th/9205110;
   I.Krichever, Comm.Pure Appl.Math., 47 (1994) 437;
   B.Dubrovin, Nucl.Phys., B379 (1992) 627; hep-th/9407018
   H.Itoyama and A.Morozov, Nucl.Phys., B477 (1996) 855; hep-th/9512161

[20] A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., B389 (1996) 43, hep-th/9607109; hep-th/9701123;
   A.Marshakov, A.Mironov and A.Morozov, Mod.Phys.Lett., A12 (1997) 773, hep-th/9701014;
   A.Morozov, Phys.Lett., B427 (1998) 93-96, hep-th/971194;
   A.Mironov and A.Morozov, Phys.Lett., B424 (1998) 48-52, hep-th/9712177