A discrete time relativistic Toda lattice

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Abstract. Four integrable symplectic maps approximating two Hamiltonian flows from the relativistic Toda hierarchy are introduced. They are demonstrated to belong to the same hierarchy and to exemplify the general scheme for symplectic maps on groups equipped with quadratic Poisson brackets. The initial value problem for the difference equations is solved in terms of a factorization problem in a group. Interpolating Hamiltonian flows are found for all the maps.
1 Introduction

Although the subject of integrable symplectic maps received in the recent years a considerable attention, the order in this area seems still to lack. Given an integrable system of ordinary differential equations with such attributes as Lax pair, $r$–matrix and so on, one would like to construct its difference approximation, desirably also with a (discrete–time analog of) Lax pair, $r$–matrix etc. Recent years brought us several successful examples of such a construction [1–8], but still not the general tools and recipes, not to say about algorithms.

Recently there appeared for the first time examples where the Lax matrix of the discrete–time approximation coincides with the Lax matrix of the continuous–time system, so that the discrete–time system belongs to the same integrable hierarchy as the underlying continuous–time one (systems of Calogero–Moser type [7,8]). We want to present here one more example of this type, which can be studied in full (and beautiful) details, – the discrete–time analog of the relativistic Toda lattice [9], see also [10–12].

The paper is organized as follows. In sect. 2,3 we recall some facts about the continuous–time relativistic Toda lattice, its $r$–matrix structure and the solution in terms of a factorization problem in a matrix group. The most part of these facts is by now well known, but it turned out to be rather difficult or even impossible to find them in literature in the form suitable for our present purposes. In sect. 4 we introduce the equations of motion of the discrete–time relativistic Toda lattice and discuss their symplectic structure. Sect. 5 contains the Lax pair representation for our system, and in sect. 6 we give the solution of the initial value problem for our system in terms of a factorization problem in a matrix group.

2 Relativistic Toda lattice

The relativistic Toda lattice with the coupling constant $g^2 \in \mathbb{R}$ is described by the Newtonian equations of motion

$$\ddot{x}_k = \dot{x}_{k+1}\dot{x}_k \frac{g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)} - \dot{x}_k \dot{x}_{k-1} \frac{g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - x_{k-1})}, \quad 1 \leq k \leq N. \tag{2.1}$$
with one of the two types of boundary conditions: open-end,

\[ x_0 \equiv \infty, \quad x_{N+1} \equiv -\infty, \]

or periodic,

\[ x_0 \equiv x_N, \quad x_{N+1} \equiv x_1. \]

It is a known fact (although usually not stressed in the literature) that the equation (2.1) may be put into the Hamiltonian form in two different ways, which lead to two different Hamiltonian functions belonging, remarkably, to one and the same integrable hierarchy.

The first way to introduce the variables \( p_k \) canonically conjugated to \( x_k \) is:

\[ \exp(p_k) = \frac{\dot{x}_k}{1 + g^2 \exp(x_{k+1} - x_k)}, \quad (2.2) \]

which leads to the system

\[ \dot{x}_k = \exp(p_k)(1 + g^2 \exp(x_{k+1} - x_k)), \]
\[ \dot{p}_k = g^2 \exp(x_{k+1} - x_k + p_k) - g^2 \exp(x_k - x_{k-1} + p_{k-1}), \]

a Hamiltonian system with the Hamiltonian function

\[ J_+ = \sum_{k=1}^{N} \exp(p_k)(1 + g^2 \exp(x_{k+1} - x_k)). \quad (2.3) \]

The second way to introduce the momenta \( p_k \) is:

\[ \exp(p_k) = \frac{-1 + g^2 \exp(x_k - x_{k-1})}{\dot{x}_k}, \quad (2.4) \]

which leads to the system

\[ \dot{x}_k = -\exp(-p_k)(1 + g^2 \exp(x_k - x_{k-1})), \]
\[ \dot{p}_k = -g^2 \exp(x_{k+1} - x_k - p_{k+1}) + g^2 \exp(x_k - x_{k-1} - p_k), \]

a Hamiltonian system with the Hamiltonian function

\[ J_- = \sum_{k=1}^{N} \exp(-p_k)(1 + g^2 \exp(x_k - x_{k-1})). \quad (2.5) \]
The Lax representation and the integrability for the flows with the Hamiltonians (2.3),(2.5) are dealt with in the following statement. Introduce two $N \times N$ matrices depending on the phase space coordinates $x_k, p_k$ and (in the periodic case) on the additional parameter $\lambda$:

$$L = \sum_{k=1}^{N} \exp(p_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},$$

$$U = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} g^2 \exp(x_{k+1} - x_k + p_k) E_{k,k+1}.$$

Here $E_{jk}$ stands for the matrix whose only nonzero entry on the intersection of the $j$th row and the $k$th column is equal to 1. In the periodic case we set $E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1};$ in the open–end case we set $\lambda = 1,$ and $E_{N+1,N} = E_{N,N+1} = 0.$ Consider also following two matrices:

$$T_+ = LU^{-1}, \quad T_- = U^{-1}L.$$ \hspace{1cm} (2.6)

**Theorem 1.** The flow with the Hamiltonian (2.3) is equivalent to the following matrix differential equations:

$$\dot{L} = LB - AL, \quad \dot{U} = UB - AU,$$

which imply also

$$\dot{T}_+ = [T_+, A], \quad \dot{T}_- = [T_-, B],$$

where

$$A = \sum_{k=1}^{N} (\exp(p_k) + g^2 \exp(x_k - x_{k-1} + p_{k-1})) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},$$

$$B = \sum_{k=1}^{N} (\exp(p_k) + g^2 \exp(x_{k+1} - x_k + p_k)) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}.$$
where
\[
C = -\lambda^{-1} \sum_{k=1}^{N} g^2 \exp(x_{k+1} - x_k - p_{k+1} + p_k) E_{k,k+1},
\]
\[
D = -\lambda^{-1} \sum_{k=1}^{N} g^2 \exp(x_{k+1} - x_k) E_{k,k+1}.
\]

So we see that either of the matrices \(T_\pm\) (they are in fact connected by means of a similarity transformation) serves as the Lax matrix for both the flows (2.3), (2.5). Note also that the Hamiltonians \(J_\pm\) belong to the set of invariant functions of \(T_\pm\), as it is easy to check that
\[
J_+ = \text{tr}(T_\pm), \quad J_- = \text{tr}(T_\pm^{-1}).
\]

It is often convenient to use instead of the canonically conjugated variables \(x_k, p_k\) another set of variables \(c_k, d_k\) defined as
\[
d_k = \exp(p_k), \quad c_k = g^2 \exp(x_{k+1} - x_k + p_k), \quad (2.7)
\]
which satisfy the Poisson brackets
\[
\{c_k, c_{k+1}\} = -c_k c_{k+1}, \quad \{c_k, d_{k+1}\} = -c_k d_{k+1}, \quad \{c_k, d_k\} = c_k d_k
\]
(only the non-vanishing brackets are written down). In terms of these variables the Hamiltonians \(J_\pm\) are expressed as
\[
J_+ = \sum_{k=1}^{N} (c_k + d_k), \quad J_- = \sum_{k=1}^{N} \frac{c_k + d_k}{d_k d_{k+1}},
\]
and the corresponding Hamiltonian flows read:
\[
\dot{c}_k = c_k (d_{k+1} + c_{k+1} - d_k - c_{k-1}), \quad \dot{d}_k = d_k (c_k - c_{k-1}) \quad (2.8)
\]
for the \(J_+\) Hamiltonian, and
\[
\dot{c}_k = c_k \left( \frac{1}{d_k} - \frac{1}{d_{k+1}} \right), \quad \dot{d}_k = d_k \left( \frac{c_k}{d_k d_{k+1}} - \frac{c_{k-1}}{d_{k-1} d_k} \right) \quad (2.9)
\]
for the \(J_-\) Hamiltonian.
The fundamental matrices $L, U$ have in new coordinates the form

$$L = \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (2.10)$$

$$U = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1} \quad (2.11)$$

For the further reference we give here also the expressions in the variables $c_k, d_k$ for the matrices involved in the theorem 1:

$$A(c,d,\lambda) = \sum_{k=1}^{N} (d_k + c_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (2.12)$$

$$B(c,d,\lambda) = \sum_{k=1}^{N} (d_k + c_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (2.13)$$

$$C(c,d,\lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k+1} E_{k,k+1}, \quad (2.14)$$

$$D(c,d,\lambda) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k} E_{k,k+1}. \quad (2.15)$$

3 Algebraic structure

Here we recall some of the results of [11,12] on the algebraic interpretation of the relativistic Toda lattice as a Hamiltonian system on a particular orbit of a certain Poisson bracket on a matrix group ([11] deals with a gauge transformed Lax matrix, which results in a different Poisson bracket on a group). The results concerning the difference equations (part c of the Theorem 2 below) are, to my knowledge, new; however the similar results for less general Poisson brackets can be found in [13,14].

First of all, we define the relevant algebras, groups and decompositions.

1) For the open–end case we set $\mathfrak{g} = gl(N)$. As a linear space, $\mathfrak{g}$ may be represented as a direct sum of two subspaces, which serve also as subalgebras: $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ ($\mathfrak{g}_-$) is a space of all lower triangular (resp. strictly upper triangular) $N$ by $N$ matrices. The corresponding groups are: $G = GL(N)$; $G_+$ ($G_-$) is a group of all nondegenerate lower triangular $N$ by
N matrices (resp. upper triangular N by N matrices with unities on the diagonal).

2) For the periodic case \( g \) is a certain twisted loop algebra over \( gl(N) \):

\[
g = \{ \tau(\lambda) \in gl(N)[\lambda, \lambda^{-1}] : \Omega \tau(\lambda) \Omega^{-1} = \tau(\omega\lambda) \},
\]

where \( \Omega = \text{diag}(1, \omega, \ldots, \omega^{N-1}) \), \( \omega = \exp(2\pi i/N) \). Again, as a linear space \( g = g_+ \oplus g_- \), where \( g_+ \) (\( g_- \)) is a subspace and subalgebra consisting of \( \tau(\lambda) \) containing only non–negative (resp. only negative) powers of \( \lambda \). The corresponding groups are:

\( G \), the twisted loop group, i.e. the group of \( GL(N) \)–valued functions \( T(\lambda) \) of the complex parameter \( \lambda \), regular in \( \mathbb{C}P^1 \setminus \{0, \infty\} \) and satisfying \( \Omega T(\lambda) \Omega^{-1} = T(\omega\lambda) \); \( G_+ \) (\( G_- \)) is the subgroup consisting of \( T(\lambda) \) regular in the neighbourhood of \( \lambda = 0 \) (resp. regular in the neighbourhood of \( \lambda = \infty \) and taking the value \( I \) in \( \lambda = \infty \)).

For both the open–end and periodic cases every \( \tau \in g \) admits a unique decomposition \( \tau = l - u \), where \( l \in g_+ \), \( u \in g_- \). We denote \( l = \pi_+(\tau) \), \( u = \pi_-(\tau) \). Analogously, for the both cases every \( T \in G \) from some neighbourhood of the group unity admits a unique factorization \( T = L U^{-1} \), where \( L \in G_+, U \in G_- \). We denote the factors as \( L = \Pi_+(T) \), \( U = \Pi_-(T) \).

Recall also that the derivative \( d\varphi(T) \) of the conjugation invariant function \( \varphi : G \mapsto \mathbb{C} \) is defined by the relation

\[
\text{tr}(d\varphi(T)u) = \left. \frac{d}{d\varepsilon} \varphi(Te^{\varepsilon u}) \right|_{\varepsilon=0} = \left. \frac{d}{d\varepsilon} \varphi(e^{\varepsilon u}T) \right|_{\varepsilon=0}, \forall u \in g.
\]

**Theorem 2.**

a) Equip \( G \times G \) with the quadratic Poisson bracket (38)–(41) from Ref. [12], and \( G \) with the quadratic Poisson bracket (33) from Ref. [12]. Then the set of pairs of matrices \( \{(L(c,d,\lambda), U(c,d,\lambda))\} \) forms a Poisson submanifold in \( G \times G \), the set of matrices \( \{T_\pm(c,d,\lambda)\} \) forms a Poisson submanifold in \( G \), and the maps \( (L, U) \mapsto T_+ = LU^{-1} \) and \( (L, U) \mapsto T_- = U^{-1}L \) are Poisson maps from \( G \times G \) into \( G \).

b) Let \( \varphi : G \mapsto \mathbb{C} \) be an invariant function on \( G \). Then the Hamiltonian flow on \( G \times G \) with the Hamiltonian function \( \varphi(LU^{-1}) = \varphi(U^{-1}L) \) has the form

\[
\dot{L} = L\pi_+(d\varphi(T_-)) - \pi_-(d\varphi(T_+))L,
\]

\[
\dot{U} = U\pi_+(d\varphi(T_-)) - \pi_-(d\varphi(T_+))U,
\]

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and the Hamiltonian flow on $G$ with the Hamiltonian function $\varphi(T)$ has the form

$$\dot{T} = [T, \pi_{\pm}(d\varphi(T))], \quad T = T_+ \quad \text{or} \quad T_-.$$ 

These flows admit the following solution in terms of the factorization problem

$$e^{td\varphi(T_{\pm}(0))} = L_{\pm}(t)U_{\pm}^{-1}(t), \quad L_{\pm}(t) \in G_+, \quad U_{\pm}(t) \in G_-$$

(this problem has solutions at least for sufficiently small $t$):

$$L(t) = L_{\pm}^{-1}(t)L(0)L_{\pm}(t) = U_{\pm}^{-1}(t)L(0)U_{\pm}(t),$$

$$U(t) = L_{\pm}^{-1}(t)U(0)L_{\pm}(t) = U_{\pm}^{-1}(t)U(0)U_{\pm}(t),$$

so that

$$T_{\pm}(t) = L_{\pm}^{-1}(t)T_{\pm}(0)L_{\pm}(t) = U_{\pm}^{-1}(t)T_{\pm}(0)U_{\pm}(t).$$

(c) If $f : G \mapsto G$ is the derivative of an invariant function on $G$, then the system of difference equations ($t \in h\mathbb{Z}$)

$$L(t + h) = \Pi_{\pm}^{-1}(f(T_{\pm}(t)))L(t)\Pi_{\pm}(f(T_{\pm}(t))),$$

$$U(t + h) = \Pi_{\pm}^{-1}(f(T_{\pm}(t)))U(t)\Pi_{\pm}(f(T_{\pm}(t)))$$

defines a Poisson map $G \times G \mapsto G \times G$, and the difference equation

$$T(t + h) = \Pi_{\pm}^{-1}(f(T(t)))T(t)\Pi_{\pm}(f(T(t))), \quad T = T_+ \quad \text{or} \quad T_-$$

defines a Poisson map $G \mapsto G$. These difference equations admit following solution in terms of the factorization problem

$$f^n(T_{\pm}(0)) = L_{\pm}(nh)U_{\pm}^{-1}(nh), \quad L_{\pm}(nh) \in G_+, \quad U_{\pm}(nh) \in G_-$$

(this problem has solutions for a given $n$ at least if $f(T_{\pm}(0))$ is sufficiently close to the group unity $I$):

$$L(nh) = L_{\pm}^{-1}(nh)L(0)L_{\pm}(nh) = U_{\pm}^{-1}(nh)L(0)U_{\pm}(nh),$$

$$U(nh) = L_{\pm}^{-1}(nh)U(0)L_{\pm}(nh) = U_{\pm}^{-1}(nh)U(0)U_{\pm}(nh),$$

so that

$$T_{\pm}(nh) = L_{\pm}^{-1}(nh)T_{\pm}(0)L_{\pm}(nh) = U_{\pm}^{-1}(nh)T_{\pm}(0)U_{\pm}(nh).$$
d) The solutions of the difference equations of the part c) are interpolated by the flows of the part b) with the Hamiltonian function \( \varphi(T) \) defined by
\[
d\varphi(T) = h^{-1} \log(f(T)).
\]

The part b) of the last theorem explains, in particular, the theorem 1, as for \( J_+(T) = \text{tr}(T), \ J_-(T) = \text{tr}(T^{-1}) \) we have
\[
dJ_+(T) = T, \ dJ_-(T) = -T^{-1},
\]
and it is not hard to check that
\[
A = \pi_+(T_-), \; B = \pi_+(T_+), \; C = \pi_-(-T_-^{-1}), \; D = \pi_-(-T_+^{-1}).
\]

4 Discrete equations of motion and symplectic structure

We propose two systems of difference equations as discrete–time versions of the relativistic Toda lattice. The equations of motion of the first system read:
\[
\frac{\exp(x_k(t + h) - x_k(t)) - 1}{\exp(x_k(t) - x_k(t - h)) - 1} = \frac{(1 + g^2 \exp(x_{k+1}(t) - x_k(t)))}{(1 + g^2 \exp(x_{k+1}(t - h) - x_k(t)))} \cdot \frac{(1 + g^2 \exp(x_k(t) - x_{k-1}(t + h)))}{(1 + g^2 \exp(x_k(t) - x_{k-1}(t)))}, \quad 1 \leq k \leq N.
\]
(4.1)

The equations of motion of the second one read:
\[
\frac{\exp(-x_k(t + h) + x_k(t)) - 1}{\exp(-x_k(t) + x_k(t - h)) - 1} = \frac{(1 + g^2 \exp(x_{k+1}(t + h) - x_k(t)))}{(1 + g^2 \exp(x_{k+1}(t) - x_k(t)))} \cdot \frac{(1 + g^2 \exp(x_k(t) - x_{k-1}(t)))}{(1 + g^2 \exp(x_k(t) - x_{k-1}(t - h)))}, \quad 1 \leq k \leq N.
\]
(4.2)
The both systems are considered under one of the two types of boundary conditions, either open–end:

\[ x_0(t) \equiv \infty, \quad x_{N+1}(t) \equiv -\infty \quad \text{for all} \quad t \in h\mathbb{Z}, \]

or periodic:

\[ x_0(t) \equiv x_N(t), \quad x_{N+1}(t) \equiv x_1(t) \quad \text{for all} \quad t \in h\mathbb{Z}. \]

The functions \( x_k(t) \) in (4.1), (4.2) are supposed to be defined for \( t \in h\mathbb{Z}, h > 0 \) (note that the equations themselves do not depend on \( h \) explicitly). However, it is convenient to consider (4.1), (4.2) as finite–difference approximations to (2.1), and in this context \( x_k(t) \) are to be considered as the smooth functions of \( t \in \mathbb{R} \). Then the left–hand sides are expanded in powers of \( h \) as \( 1 + h \frac{\ddot{x}_k}{x_k} + O(h^2) \), and the right–hand sides as

\[
1 + h \left( \frac{g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)} - \frac{g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - x_{k-1})} \right) + O(h^2),
\]

so that we recover in the continuous limit the equation (2.1).

Of course, the both systems are closely related, namely by means of the time reversion operation. (Note that the underlying continuous–time system (2.1) is invariant with respect to this operation).

We would like to remark that the equations (4.1), (4.2) admit a simple non–relativistic limit: set \( x_k(t) = q_k(t) + ct \) in (4.1) (resp. \( x_k(t) = q_k(t) - ct \) in (4.2)) with \( c > 0 \) playing the role of the speed of light; then in the limit \( c \to \infty \) the both equations tend to one and the same system:

\[
\exp(q_k(t+h) - 2q_k(t) + q_k(t-h)) = \frac{1 + g^2 \exp(q_{k+1}(t) - q_k(t))}{1 + g^2 \exp(q_k(t) - q_{k-1}(t))}, \quad 1 \leq k \leq N,
\]

i.e. to the equations of motion of the discrete–time Toda lattice from [3].

In the following we will adopt the notations from [7,8]: if \( x_k = x_k(t) \), then \( \bar{x}_k = x_k(t+h) \), \( \bar{x}_k = x_k(t-h) \), so that (4.1), (4.2) take the form

\[
\frac{\exp(\bar{x}_k - x_k) - 1}{\exp(x_k - \bar{x}_k) - 1} = \frac{1 + g^2 \exp(x_{k+1} - x_k)(1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k))(1 + g^2 \exp(x_k - x_{k-1}))}, \quad 1 \leq k \leq N;
\]

(4.3)
\[
\frac{\exp(-\bar{x}_k + x_k) - 1}{\exp(-x_k + \bar{x}_k) - 1} = \frac{(1 + g^2 \exp(x_{k+1} - x_k)) (1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k)) (1 + g^2 \exp(x_k - x_{k-1}))}, \quad 1 \leq k \leq N,
\]

respectively. Obviously, (4.3), (4.4) are systems of nonlinear algebraic equations for \(\bar{x}_k, 1 \leq k \leq N\) of the form

\[
F_k(\bar{x}, x, \bar{x}) = 0, \quad 1 \leq k \leq N.
\]

It will be convenient to discuss the solvability of these systems simultaneously with the symplectic structure.

The general recipe to derive the invariant symplectic structure for a discrete evolutionary equation of the type (4.5) was given in [1,15]: represent the equations of motion in the Lagrangian form

\[
\partial \left( \Lambda(\bar{x}, x) + \Lambda(x, \bar{x}) \right) / \partial x_k = 0,
\]

then the momenta \(p_k\) canonically conjugate to \(x_k\) are defined as

\[
p_k = \partial \Lambda(x, \bar{x}) / \partial x_k.
\]

The map \((x, p) \mapsto (\bar{x}, \bar{p})\) preserves the standard symplectic 2–form \(\sum_{k=1}^{N} dx_k \wedge dp_k\). Note that for the \(\bar{x}\) we have the equation

\[
p_k = -\partial \Lambda(\bar{x}, x) / \partial x_k,
\]

and then \(\bar{p}\) may be computed as

\[
\bar{p}_k = \partial \Lambda(\bar{x}, x) / \partial \bar{x}_k.
\]

**Remark.** Note that if the system (4.5) may be represented in the Lagrangian form (4.6) with the Lagrangian function \(\Lambda(x, \bar{x})\), then the system

\[
F_k(x, x, \bar{x}) = 0, \quad 1 \leq k \leq N
\]

may be represented in the Lagrangian form with the Lagrangian function \(-\Lambda(x, \bar{x})\). Now it is easy to see from (4.7)–(4.9)
that the symplectic maps \((x, p) \mapsto (\tilde{x}, \tilde{p})\) corresponding to these two systems are mutually inverse. This is just the case for (4.1), (4.2), up to minor modifications due to the "artificial" parameter \(h\) introduced in the definitions of momenta below. We prefer, however, to consider all the maps separately, and will return to the above-mentioned circumstance at the end of the section 6.

The Lagrangian functions for the equations (4.3), (4.4) are expressed with the help of two functions \(\phi_1(\xi), \phi_2(\xi)\) which are defined by

\[
\phi'_1(\xi) = \log \left| \frac{\exp(\xi) - 1}{h} \right|, \quad \phi'_2(\xi) = \log(1 + g^2 \exp(\xi)).
\]

It turns out that, just as in the continuous-time case, the Lagrangian functions and hence the momenta \(p_k\) may be chosen in two different ways. They lead to two pairs of different symplectic maps (one pair for each of (4.1), (4.2)) belonging, remarkably, to one and the same integrable hierarchy. Still more remarkable, however, is that this hierarchy is the same as in the continuous-time case (see sections 5, 6)!

4.1 System (4.3), the first choice of momenta

It is easy to see that (4.3) is equivalent to (4.6) with

\[
\Lambda(x, x) = \sum_{k=1}^{N} \phi_1(x_k - x_k) + \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}) - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}).
\]

Then the definition (4.7) of momenta \(p_k\) takes the form

\[
\exp(p_k) = \frac{(\exp(x_k - x_k) - 1)}{h(1 + g^2 \exp(x_{k+1} - x_k))}, \quad (4.10)
\]

and the equation (4.8) for \(\tilde{x}\) takes the form

\[
\exp(\tilde{p}_k) = \exp(p_k) \frac{(\exp(\tilde{x}_k - x_k) - 1)}{h(1 + g^2 \exp(x_{k+1} - x_k))} \frac{(1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_k - \tilde{x}_{k-1}))}, \quad (4.11)
\]

(formulas (4.10), (4.11) are to be compared with (2.2)). The equation (4.9) for \(\tilde{p}\) together with (4.11) implies:

\[
\exp(\tilde{p}_k) = \exp(p_k) \frac{(1 + g^2 \exp(x_{k+1} - x_k))}{(1 + g^2 \exp(x_k - x_{k-1}))} \frac{(1 + g^2 \exp(x_k - \tilde{x}_{k-1}))}{(1 + g^2 \exp(x_k - \tilde{x}_{k-1}))}, \quad (4.12)
\]
To solve (4.3) for $\tilde{x}$ is now equivalent to solving (4.11) for $\tilde{x}$. It may be directly verified that this last equation may be rewritten as:

$$h \exp(p_k) \frac{1 + g^2 \exp(x_{k+1} - \tilde{x}_k)}{1 - \exp(x_k - \tilde{x}_k)} =$$

$$= 1 + h \exp(p_k) + g^2 \exp(x_k - x_{k-1}) \frac{1 - \exp(x_{k-1} - \tilde{x}_{k-1})}{1 + g^2 \exp(x_k - \tilde{x}_{k-1})}.$$ 

In terms of the coordinates $c_k, d_k$ this means: if we denote

$$a_k = h \exp(p_k) \frac{1 + g^2 \exp(x_{k+1} - \tilde{x}_k)}{1 - \exp(x_k - \tilde{x}_k)},$$

(4.13) then

$$a_k = 1 + h d_k + \frac{h c_{k-1}}{a_{k-1}}, \quad 1 \leq k \leq N.$$

(4.14)

Suppose for a moment that these recurrent relations define $a_k$ as certain functions of $c_k, d_k$. Then, according to (4.13), this means that the equations for $\tilde{x}$ are solved. Indeed, it follows from (4.13) that

$$\exp(\tilde{x}_k - x_k) = \frac{a_k + h c_k}{a_k - h d_k}$$

To express now the resulting map in terms of the variables $c_k, d_k$ alone, we derive from (4.13)

$$\frac{1 + g^2 \exp(x_{k} - x_{k-1})}{1 + g^2 \exp(x_{k} - \tilde{x}_{k-1})} = a_k - h d_k,$$

which together with the previous expression and (4.12) implies:

$$\tilde{c}_k = c_k \frac{a_{k+1} + h c_{k+1}}{a_k + h c_k}, \quad \tilde{d}_k = d_k \frac{a_{k+1} - h d_{k+1}}{a_k - h d_k}.$$  

(4.15)

Returning to the relations (4.14), we note that in the open-end case $c_0 = 0$, hence we obtain from (4.11) the following finite continued fractions expressions for $a_k$’s:

$$a_1 = 1 + h d_1;$$

$$a_2 = 1 + h d_2 + \frac{h c_1}{1 + h d_1};$$

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Obviously, we have:
\[ a_k = 1 + h(d_k + c_{k-1}) + O(h^2). \] (4.16)

In the periodic case the recurrent relations (4.14) uniquely define the \( a_k \)'s as the \( N \)-periodic infinite continued fractions. It can be proved that these continued fractions converge and their values satisfy (4.16).

Because of (4.16) it is obvious that the map (4.15) is a difference approximation to the flow (2.8).

### 4.2 System (4.3), the second choice of momenta

The equation (4.3) can be treated also as a Lagrangian equation (4.6) with
\[
\Lambda(x, \bar{x}) = - \sum_{k=1}^{N} \phi_1(x_k - x_k) - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}) + \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}).
\]

Then the definition (4.7) takes the form:
\[
\exp(p_k) = \frac{h(1 + g^2 \exp(x_k - x_{k-1}))}{1 - \exp(x_k - x_{k-1})} \frac{(1 + g^2 \exp(x_{k+1} - x_k))}{(1 + g^2 \exp(x_{k+1} - x_k))}. \] (4.17)

and the equation (4.8) for \( \bar{x} \) takes the form:
\[
\exp(p_k) = \frac{h(1 + g^2 \exp(x_k - \bar{x}_{k-1}))}{1 - \exp(\bar{x}_k - x_k)}, \] (4.18)

(Equations (4.17), (4.18) are to be compared with (2.4)). Equation (4.9) together with (4.18) implies:
\[
\exp(\bar{p}_k) = \exp(p_k) \frac{(1 + g^2 \exp(x_{k+1} - \bar{x}_k))}{(1 + g^2 \exp(x_{k+1} - \bar{x}_k))} \frac{(1 + g^2 \exp(\bar{x}_k - x_{k-1}))}{(1 + g^2 \exp(x_k - \bar{x}_{k-1}))}. \] (4.19)
As before, to solve (4.3) for $\tilde{x}$ is equivalent to solving (4.18) for $\tilde{x}$. The formula (4.18) may be rewritten as:

$$\exp(\tilde{x}_k - x_k + p_k) = \exp(p_k) - h - h^2 \exp(x_k - \tilde{x}_{k-1}).$$

In terms of the coordinates $c_k, d_k$ this means: if we denote

$$d_k = g^2 \exp(x_{k+1} - \tilde{x}_k),$$

then

$$\frac{c_k}{d_k} = d_k - h - h d_{k-1}, \quad 1 \leq k \leq N. \quad (4.21)$$

Supposing that the relations define $d_k$'s as certain functions on $c_k, d_k$, we see that (4.20) allows to solve the equation for $\tilde{x}$. In order to express the resulting map in terms of the variables $c_k, d_k$ alone, we derive from (4.18), (4.20) the relations

$$\exp(x_k - \tilde{x}_k) = \frac{d_k d_k}{c_k}, \quad \frac{1 + g^2 \exp(\tilde{x}_{k+1} - \tilde{x}_k)}{1 + g^2 \exp(x_{k+1} - \tilde{x}_k)} = 1 - \frac{h d_k}{d_{k+1}}.$$

We use them and (4.21) to derive from (4.19):

$$\tilde{c}_k = c_{k+1} \frac{c_k + h d_k}{c_{k+1} + h d_{k+1}}, \quad \tilde{d}_k = d_{k+1} \frac{d_k - h d_{k-1}}{d_{k+1} - h d_k}. \quad (4.22)$$

Returning to (4.21), we immediately obtain in the open–end case $d_0 = 0$, and the finite continued fraction expressions for $d_k$'s follow:

$$\varnothing_1 = \frac{c_1}{d_1 - h},$$

$$\varnothing_2 = \frac{c_2}{d_2 - h - \frac{h c_1}{d_1 - h}},$$

$$\ldots$$

$$\varnothing_{N-1} = \frac{c_{N-1}}{d_{N-1} - h - \frac{h c_{N-2}}{d_{N-2} - h - \ldots - \frac{h c_1}{d_1 - h}}}.$$
Obviously, we have
\[ \vartheta_k = \frac{c_k}{d_k} + O(h), \quad 1 \leq k \leq N. \]  \hfill (4.23)

In the periodic case the recurrent relations (4.21) uniquely define \( \vartheta_k \)'s as the \( N \)-periodic infinite continued fractions, that again converge and whose values satisfy the relation (4.22). In view of (4.22) it is obvious that the map (4.21) is a difference approximation to the flow (2.9).

### 4.3 System (4.4), the first choice of momenta

Turning now to the system (4.2), we see that it is equivalent to (4.6) with
\[ \Lambda(x, x) = -\sum_{k=1}^{N} \phi_1(-x_k + x_{\bar{k}}) - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}) + \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}). \]

Then the definition (4.7) of momenta \( p_k \) takes the form
\[ \exp(p_k) = \frac{(1 - \exp(-x_k + x_{\bar{k}}))}{h(1 + g^2 \exp(x_{k+1} - x_k)) (1 + g^2 \exp(x_k - x_{k-1}))}, \]  \hfill (4.24)

and the equation (4.8) for \( \bar{x} \) takes the form
\[ \exp(p_k) = \frac{1 - \exp(-\bar{x}_k + x_k)}{h(1 + g^2 \exp(\bar{x}_{k+1} - x_k))}, \]  \hfill (4.25)

(formulas (4.24), (4.25) are to be compared with (2.2)). The equation (4.9) for \( \tilde{p} \) together with (4.25) implies:
\[ \exp(\tilde{p}_k) = \exp(p_k) \frac{(1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1})) (1 + g^2 \exp(\bar{x}_{k+1} - x_k))}{(1 + g^2 \exp(\bar{x}_k - x_{k-1})) (1 + g^2 \exp(\bar{x}_{k+1} - \bar{x}_k))}. \]  \hfill (4.26)

To solve (4.4) for \( \bar{x} \) is now equivalent to solving (4.25) for \( \bar{x} \). This last equation may be rewritten as:
\[ \exp(-\bar{x}_k + x_k) = 1 - h \exp(p_k) - \frac{h g^2 \exp(x_{k+1} - x_k + p_k)}{\exp(-\bar{x}_{k+1} + x_{k+1})}. \]
In terms of the coordinates $c_k, d_k$ this means: if we denote

$$\beta_k = \exp(-\bar{x}_k + x_k)$$

then

$$\beta_k = 1 - hd_k - \frac{hc_k}{\beta_{k+1}}, \quad 1 \leq k \leq N. \quad (4.28)$$

Relations (4.28) imply that $\beta_k$ as certain functions (continued fractions, see below) on $c_k, d_k$. According to (4.27), this means that the equations for $\bar{x}$ are solved. To express now the resulting map in terms of the variable $s$ alone, we derive from (4.25), (4.27), and (4.28) the equality

$$\frac{1 + g^2 \exp(\bar{x}_{k+1} - \bar{x}_k)}{1 + g^2 \exp(\bar{x}_{k+1} - x_k)} = 1 - \frac{hc_k}{\beta_{k+1}} = \beta_k + hd_k,$$

so that (4.26) implies

$$\bar{c}_k = c_k \frac{\beta_k - hc_{k-1}}{\beta_{k+1} - hc_k}; \quad \bar{d}_k = d_k \frac{\beta_{k-1} + hd_{k-1}}{\beta_k + hd_k}. \quad (4.29)$$

As for the continued fractions expressions for $\beta_k$'s, we have in the open-end case: $c_N = 0$, hence

$$\beta_N = 1 - hd_N;$$

$$\beta_{N-1} = 1 - hd_{N-1} - \frac{hc_{N-1}}{1 - hd_N};$$

$$\ldots$$

$$\beta_1 = 1 - hd_1 - \frac{hc_1}{1 - hd_2 - \frac{hc_2}{1 - hd_3 - \ldots - \frac{hc_{N-1}}{1 - hd_N}}}. \quad (4.30)$$

Obviously,

$$\beta_k = 1 - h(c_k + d_k) + O(h^2).$$

In the periodic case the recurrent relations (4.28) uniquely define the $\beta_k$'s as the $N$-periodic infinite continued fractions. It can be proved that these continued fractions converge and their values satisfy (4.30). Because of (4.30) it is obvious that the map (4.29) is a difference approximation to the flow (2.8).
4.4 System (4.4), the second choice of momenta

The equation (4.4) can be treated also as a Lagrangian equation (4.6) with

\[ \Lambda(x, \bar{x}) = \sum_{k=1}^{N} \phi_1(-x_k + x_k) + \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}) - \sum_{k=1}^{N} \phi_2(x_k - x_{k-1}). \]

Then the definition (4.7) takes the form:

\[ \exp(p_k) = \frac{h(1 + g^2 \exp(x_k - x_{k-1}))}{\exp(-x_k + x_{k}) - 1}, \] (4.31)

and the equation (4.8) for \( \bar{x} \) takes the form:

\[ \exp(p_k) = \frac{h(1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k))} \frac{1 + g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)}. \] (4.32)

(Equations (4.32), (4.33) are to be compared with (2.4)). Equation (4.9) together with (4.32) implies:

\[ \exp(\bar{p}_k) = \exp(p_k) \frac{1 + g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_{k+1} - x_k)} \frac{1 + g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)}. \] (4.33)

As before, to solve (4.4) for \( \bar{x} \) is equivalent to solving (4.32) for \( \bar{x} \). The formula (4.32) may be rewritten as:

\[ \frac{h(1 + g^2 \exp(x_k - x_{k-1}))}{1 - \exp(x_k - x_{k})} = h + \exp(p_k) + g^2 \exp(x_{k+1} - x_k + p_k) \frac{1 - \exp(x_{k+1} - x_{k+1})}{1 + g^2 \exp(x_{k+1} - x_k)}. \]

In terms of the coordinates \( c_k, d_k \) this means: if we denote

\[ h\gamma_k = c_k \frac{1 - \exp(x_{k+1} - x_{k+1})}{1 + g^2 \exp(x_{k+1} - x_k)}, \] (4.34)

then

\[ \frac{c_{k-1}}{\gamma_{k-1}} = h + d_k + h\gamma_k, \quad 1 \leq k \leq N. \] (4.35)
This defines $\gamma_k$'s as certain functions on $c_k, d_k$, which, according to (4.34), allows to solve the equation for $\bar{x}$. It is not hard to derive from (4.34) the relations

$$\exp(\bar{x}_{k+1} - x_{k+1}) = \frac{1 - \frac{h\gamma_k}{c_k}}{1 + \frac{h\gamma_k}{d_k}}, \quad \frac{1 + g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(\bar{x}_{k+1} - x_k)} = 1 + \frac{h\gamma_k}{d_k}. $$

We use them and (4.33) to express the resulting map in terms of the variables $c_k, d_k$ alone:

$$\bar{c}_k = c_{k-1} \frac{c_k - h\gamma_k}{c_{k-1} - h\gamma_{k-1}}, \quad \bar{d}_k = d_{k-1} \frac{d_k + h\gamma_k}{d_{k-1} + h\gamma_{k-1}}. \quad (4.36)$$

The expressions for $\gamma_k$'s in the open–end case follow from (4.35) and $\gamma_N = 0$:

$$\gamma_{N-1} = \frac{c_{N-1}}{d_N + h},$$

$$\gamma_{N-2} = \frac{c_{N-2}}{d_{N-1} + h + \frac{hc_{N-1}}{d_N + h}},$$

$$\gamma_1 = \frac{c_1}{d_2 + h + \frac{hc_2}{d_3 + h + \ldots + \frac{hc_{N-1}}{d_N + h}}}. $$

Obviously, we have

$$\gamma_k = \frac{c_k}{d_{k+1}} + O(h), \quad 1 \leq k \leq N. \quad (4.37)$$

In the periodic case the recurrent relations (4.35) uniquely define $\gamma_k$'s as the $N$–periodic infinite continued fractions, that again converge and whose values satisfy the relation (4.37). In view of (4.37) it is obvious that the map (4.36) is a difference approximation to the flow (2.9).
5 Lax representations

We show in this section that our discrete–time systems admit the discrete analogs of Lax representations with the same matrices $L, U, T_{\pm}$ (see (2.10), (2.11), (2.6)) as the continuous–time relativistic Toda lattice. In the following theorems we adopt the conventions described before the formula (2.6). The dependence of all the matrices below on the discrete time $t \in h\mathbb{Z}$ is supposed to appear through the dependence of $c_k, d_k$ on $t$.

**Theorem 3.** The symplectic map defined by (4.14), (4.15) admits a representation in the form of matrix equations

$$A(t)L(t + h) = L(t)B(t), \quad A(t)U(t + h) = U(t)B(t),$$

so that

$$T_+(t + h) = A^{-1}(t)T_+(t)A(t), \quad T_-(t + h) = B^{-1}(t)T_-(t)B(t)$$

with the matrices

$$A(c, d, \lambda) = \sum_{k=1}^{N} a_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k},$$

$$B(c, d, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k},$$

where $a_k$’s are defined by (4.14), and $b_k$’s by (5.7) or (5.8) below.

**Proof.** It is straightforward to check that the matrix equations (5.1) are equivalent to the following ones:

$$a_k \tilde{d}_k = d_k b_k, \quad h\tilde{d}_k + a_{k+1} = hd_{k+1} + b_k,$$

$$a_k \tilde{c}_k = c_k b_{k+1}, \quad h\tilde{c}_k - a_{k+1} = hc_{k+1} - b_{k+1}.$$  

Now it would be not hard to check directly that the equations (5.5), (5.6) are satisfied with the following identifications: (4.11), (4.12) for $d_k, \tilde{d}_k$; $c_k = g^2 \exp(x_{k+1} - x_k)d_k$ and similarly for $\tilde{c}_k$; 

$$a_k = \exp(\bar{x}_k - x_k)\frac{(1 + g^2 \exp(x_{k+1} - \bar{x}_k)) (1 + g^2 \exp(x_k - x_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k)) (1 + g^2 \exp(x_k - \bar{x}_{k-1}))},$$

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as in (4.13), and \( b_k = \exp(\bar{x}_k - x_k) \).

We prefer, however, to work directly with (5.5), (5.6), not using the expressions in terms of the \( x_k \) variables. Namely, (5.5) is equivalent to

\[
\bar{d}_k = d_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k}, \quad b_k = a_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k},
\]

(5.7)

and (5.6) is equivalent to

\[
\bar{c}_k = c_k \frac{a_{k+1} + hc_{k+1}}{a_k + hc_k}, \quad b_{k+1} = a_k \frac{a_{k+1} + hc_{k+1}}{a_k + hc_k}.
\]

(5.8)

The first equations in (5.7), (5.8) coincide with (4.15), and the compatibility of the second ones is equivalent to

\[
\frac{a_k (a_{k+1} - hd_{k+1})}{a_k + hc_k} = \frac{a_{k-1} (a_k - hd_k)}{a_{k-1} + hc_{k-1}},
\]

which is a direct consequence of (4.14). This completes the proof.

We would like to mention here that the matrices (5.3), (5.4) when compared with (2.12), (2.13) satisfy

\[
A(c, d, \lambda) = I + hA(c, d, \lambda) + O(h^2), \quad B(c, d, \lambda) = I + hB(c, d, \lambda) + O(h^2),
\]

as it follows from (4.16).

**Theorem 4.** The symplectic map defined by (4.21), (4.22) admits a representation in the form of matrix equations

\[
L(t + h)D(t) = C(t)L(t), \quad U(t + h)D(t) = C(t)U(t),
\]

(5.9)

so that

\[
T_+(t + h) = C(t)T_+(t)C^{-1}(t), \quad T_-(t + h) = D(t)T_-(t)D^{-1}(t)
\]

(5.10)

with the matrices

\[
C(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} \epsilon_k E_{k,k+1},
\]

(5.11)

\[
D(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} \eta_k E_{k,k+1},
\]

(5.12)
where \( \varphi_k \)'s are defined by (4.21), and \( \epsilon_k \)'s by (5.15) or (5.16) below.

**Proof.** It is straightforward to check that the matrix equations (5.9) are equivalent to the following ones:

\[
\begin{align*}
\tilde{d}_k \varphi_k &= c_k d_{k+1}, \\
\tilde{d}_k + h \varphi_{k-1} &= d_k + h \epsilon_k, \tag{5.13}
\end{align*}
\]

\[
\begin{align*}
\tilde{c}_k \varphi_{k+1} &= c_k c_{k+1}, \\
\tilde{c}_k - h \varphi_k &= c_k - h \epsilon_k. \tag{5.14}
\end{align*}
\]

Now it could be checked by means of direct calculation that the equations (5.13), (5.14) are satisfied with the following identifications: (4.18), (4.19) for \( d_k, \tilde{d}_k; c_k = g^2 \exp(x_{k+1} - x_k) d_k \) and similarly for \( \tilde{c}_k; \tilde{\varphi}_k = g^2 \exp(x_{k+1} - \tilde{x}_k) \) as in (4.2), and

\[
\begin{align*}
\epsilon_k &= g^2 \exp(x_{k+1} - \tilde{x}_k) \frac{(1 - \exp(\tilde{x}_{k+1} - x_{k+1}))}{(1 - \exp(\tilde{x}_k + x_k))} \frac{(1 + g^2 \exp(\tilde{x}_k - x_k - 1))}{(1 + g^2 \exp(\tilde{x}_{k+1} - x_{k+1}))},
\end{align*}
\]

Working, however, again directly with (5.13), (5.14), without turning to the representation through the \( x_k \) variables, we note that (5.13) is equivalent to

\[
\begin{align*}
\tilde{d}_k &= d_{k+1} \frac{d_k - h \varphi_{k-1}}{d_{k+1} - h \varphi_k}, \\
\epsilon_k &= \frac{d_k - h \varphi_{k-1}}{d_{k+1} - h \varphi_k}, \tag{5.15}
\end{align*}
\]

and (5.14) is equivalent to

\[
\begin{align*}
\tilde{c}_k &= c_{k+1} \frac{c_k + h \varphi_k}{c_{k+1} + h \varphi_{k+1}}, \\
\epsilon_k &= \frac{c_k + h \varphi_k}{c_{k+1} + h \varphi_{k+1}}. \tag{5.16}
\end{align*}
\]

The first equations in (5.15), (5.16) coincide with (4.22), and the compatibility of the second ones is equivalent to

\[
\frac{\varphi_{k+1}(d_{k+1} - h \varphi_{k-1})}{c_{k+1} + h \varphi_{k+1}} = \frac{\varphi_k(d_k - h \varphi_{k-1})}{c_k + h \varphi_k},
\]

which is a direct consequence of (4.21). The proof is complete.

Note that the matrices (5.11), (5.12) when compared with (2.14), (2.15) satisfy

\[
C(c, d, \lambda) = I - hC(c, d, \lambda) + O(h^2), \quad D(c, d, \lambda) = I - hD(c, d, \lambda) + O(h^2),
\]

as it follows from (4.23).
Theorem 5. The symplectic map defined by (4.28), (4.29) admits a representation in the form of matrix equations

\[ L(t+h)B(t) = A(t)L(t), \quad U(t+h)B(t) = A(t)U(t), \quad (5.17) \]

so that

\[ T_+(t+h) = A(t)T_+(t)A^{-1}(t), \quad T_-(t+h) = B(t)T_-(t)B^{-1}(t) \quad (5.18) \]

with the matrices

\[ A(c,d,\lambda) = \sum_{k=1}^{N} \alpha_k E_{kk} - h\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (5.19) \]

\[ B(c,d,\lambda) = \sum_{k=1}^{N} \beta_k E_{kk} - h\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (5.20) \]

where \( \beta_k \)'s are defined by (4.28), and \( \alpha_k \)'s by (5.23) or (5.24) below.

Proof. It is straightforward to check that the matrix equations (5.17) are equivalent to the following ones:

\[ \tilde{d}_k \beta_k = \alpha_k d_k, \quad h\tilde{d}_k - \beta_{k-1} = h d_{k-1} - \alpha_k, \quad (5.21) \]

\[ \tilde{c}_k \beta_{k+1} = \alpha_k c_k, \quad h\tilde{c}_k + \beta_k = h c_{k-1} + \alpha_k. \quad (5.22) \]

Now it would be not hard to check directly that the equations (5.5), (5.6) are satisfied with the following identifications: (4.25), (4.26) for \( d_k, \tilde{d}_k; \)
\( c_k = g^2 \exp(x_{k+1} - x_k) d_k \) and similarly for \( \tilde{c}_k; \beta_k = \exp(-\tilde{x}_k + x_k) \) as in (4.27), and

\[ \alpha_k = \exp(-\tilde{x}_k + x_k) \left( \frac{1 + g^2 \exp(\tilde{x}_{k+1} - x_k)}{1 + g^2 \exp(\tilde{x}_k - \tilde{x}_{k+1})} \right) \left( \frac{1 + g^2 \exp(\tilde{x}_k - x_k)}{1 + g^2 \exp(\tilde{x}_k - x_{k-1})} \right). \]

We prefer, however, to work directly with (5.21), (5.22), not using the expressions in terms of the \( x_k \) variables. Namely, (5.21) is equivalent to

\[ \tilde{d}_k = d_k \frac{\beta_{k-1} + h d_{k-1}}{\beta_k + h d_k}, \quad \alpha_k = \frac{\beta_{k-1} + h d_{k-1}}{\beta_k + h d_k}, \quad (5.23) \]

and (5.22) is equivalent to

\[ \tilde{c}_k = c_k \frac{\beta_k - h c_{k-1}}{\beta_{k+1} - h c_k}, \quad \alpha_k = \frac{\beta_k - h c_{k-1}}{\beta_{k+1} - h c_k}. \quad (5.24) \]
The first equations in (5.23), (5.24) coincide with (4.29), and the compatibility of the second ones is equivalent to
\[ \frac{\beta_{k+1}(\beta_k + hd_k)}{\beta_{k+1} - hc_k} = \frac{\beta_k(\beta_{k-1} + hd_{k-1})}{\beta_k - hc_{k-1}}, \]
which is a direct consequence of (4.28). This completes the proof.

We would like to mention here that the matrices (5.19), (5.20) when compared with (2.12), (2.13) satisfy
\[ A(c,d,\lambda) = I - hA(c,d,\lambda) + O(h^2), \quad B(c,d,\lambda) = I - hB(c,d,\lambda) + O(h^2), \]
as it follows from (4.30).

**Theorem 6.** The symplectic map defined by (4.35), (4.36) admits a representation in the form of matrix equations
\[ C(t)L(t + h) = L(t)D(t), \quad C(t)U(t + h) = U(t)D(t), \quad (5.25) \]
so that
\[ T_+(t + h) = C^{-1}(t)T_+(t)C(t), \quad T_-(t + h) = D^{-1}(t)T_-(t)D(t) \quad (5.26) \]
with the matrices
\[ C(c,d,\lambda) = \sum_{k=1}^{N} E_{kk} - h\lambda^{-1} \sum_{k=1}^{N} \gamma_k E_{k,k+1}, \quad (5.27) \]
\[ D(c,d,\lambda) = \sum_{k=1}^{N} E_{kk} - h\lambda^{-1} \sum_{k=1}^{N} \delta_k E_{k,k+1}, \quad (5.28) \]
where \( \gamma_k \)'s are defined by (4.35), and \( \delta_k \)'s by (5.31) or (5.32) below.

**Proof.** It is straightforward to check that the matrix equations (5.25) are equivalent to the following ones:
\[ \gamma_{k-1}\bar{d}_k = d_{k-1}\delta_{k-1}, \quad \bar{d}_k - h\gamma_k = d_k - h\delta_{k-1}, \quad (5.29) \]
\[ \gamma_{k-1}\bar{c}_k = c_{k-1}\delta_k, \quad \bar{c}_k + h\gamma_k = c_k + h\delta_k. \quad (5.30) \]
Now it could be checked by means of direct calculation that the equations (5.29), (5.30) are satisfied with the following identifications: (4.32), (4.33) for
\[ d_k, \tilde{d}_k; c_k = g^2 \exp(x_{k+1} - x_k)d_k \] and similarly for \( \tilde{c}_k; \delta_k = g^2 \exp(\tilde{x}_{k+1} - x_k) \), and
\[ \gamma_k = g^2 \exp(\tilde{x}_{k+1} - x_k) \frac{\exp(-\tilde{x}_{k+1} + x_{k+1}) - 1}{(1 + g^2 \exp(x_k - x_{k-1}))} \frac{(1 + g^2 \exp(x_{k+1} - x_k))}{(1 + g^2 \exp(x_{k+1} - x_k))}. \]

Working, however, again directly with (5.29), (5.30), without turning to
the representation through the \( x_k \) variables, we note that (5.29) is equivalent
to
\[ \tilde{d}_k = d_{k-1} \frac{d_k + h\gamma_k}{d_{k-1} + h\gamma_{k-1}}, \quad \tilde{\delta}_k = \gamma_{k-1} \frac{d_k + h\gamma_k}{d_{k-1} + h\gamma_{k-1}}, \quad (5.31) \]
and (5.30) is equivalent to
\[ \tilde{c}_k = c_{k-1} \frac{c_k - h\gamma_k}{c_{k-1} - h\gamma_{k-1}}, \quad \tilde{\delta}_k = \gamma_{k-1} \frac{c_k - h\gamma_k}{c_{k-1} - h\gamma_{k-1}}. \quad (5.32) \]
The first equations in (5.31), (5.32) coincide with (4.36), and the compat-
ibility of the second ones is equivalent to
\[ \frac{\gamma_k(d_{k+1} + h\gamma_{k+1})}{c_k - h\gamma_k} = \frac{\gamma_{k-1}(d_k + h\gamma_k)}{c_{k-1} - h\gamma_{k-1}}, \]
which is a direct consequence of (4.35). The proof is complete.

Note that the matrices (5.27), (5.28) when compared with (2.14), (2.15)
satisfy
\[ C(c, d, \lambda) = I + hC(c, d, \lambda) + O(h^2), \quad D(c, d, \lambda) = I + hD(c, d, \lambda) + O(h^2), \]
as it follows from (4.37).

6 Factorization problems and interpolating Hamiltonians

It is very remarkable that the matrices \( A, B, C, D \) and \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) from the
previous section may be identified with the certain factors \( \Pi_{\pm}(f(T_{\pm})) \), as in
the Theorem 2.
Theorem 7. There hold following relations:

\[
A(c, d, \lambda) = \Pi_+ (I + hT_+(c, d, \lambda)) , \tag{6.1}
\]
\[
B(c, d, \lambda) = \Pi_+ (I + hT_-(c, d, \lambda)) , \tag{6.2}
\]
\[
C(c, d, \lambda) = \Pi_-^{-1} \left( I - hT_+^{-1}(c, d, \lambda) \right) , \tag{6.3}
\]
\[
D(c, d, \lambda) = \Pi_-^{-1} \left( I - hT_-^{-1}(c, d, \lambda) \right) . \tag{6.4}
\]

Proof. Define following two matrices:

\[
Q_-(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{a_k} E_{k,k+1} \in G_- ,
\]
\[
Q_+(c, d, \lambda) = \sum_{k=1}^{N} \frac{c_k}{a_k} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \in G_+ .
\]

Note now that the recurrent relation (4.14) is just equivalent to the matrix equality

\[
U(c, d, \lambda) + hL(c, d, \lambda) = A(c, d, \lambda)Q_-(c, d, \lambda) , \tag{6.5}
\]
and the recurrent relation (4.21) is just equivalent to the matrix equality

\[
L(c, d, \lambda) - hU(c, d, \lambda) = Q_+(c, d, \lambda)D(c, d, \lambda) . \tag{6.6}
\]

Multiplying (6.5) from the right by \(U^{-1}\), we obtain:

\[
I + hT_+ = AQ_+ U^{-1} ,
\]
and since \(Q_+ U^{-1} \in G_-\), we obtain (6.1). From the previous equation with the help of (5.1) we derive also

\[
I + hT_- = U^{-1} AQ_- = B \tilde{U}^{-1} Q_- ,
\]
which proves (6.2) in view of \(\tilde{U}^{-1} Q_- \in G_-\).

Next, multiplying (6.6) from the left by \(L^{-1}\), we obtain:

\[
I - hT_-^{-1} = L^{-1} Q_+ D ,
\]
which just implies (6.4) because of \(L^{-1} Q_+ \in G_+\). Finally, from the previous equation we derive with the help of (5.9):

\[
I - hT_+^{-1} = Q_+ D L^{-1} = Q_+ \tilde{L}^{-1} C ,
\]
which means the validity of (6.3) because of $Q_+L^{-1} \in G_+$.

The theorem is proved.

**Theorem 8.** There hold following relations:

\[
\begin{align*}
\mathcal{A}(c, d, \lambda) &= \Pi_+^{-1} \left( (I - hT_+(c, d, \lambda))^{-1} \right), \quad (6.7) \\
\mathcal{B}(c, d, \lambda) &= \Pi_+^{-1} \left( (I - hT_-(c, d, \lambda))^{-1} \right), \quad (6.8) \\
\mathcal{C}(c, d, \lambda) &= \Pi_- \left( (I + hT_+(c, d, \lambda))^{-1} \right), \quad (6.9) \\
\mathcal{D}(c, d, \lambda) &= \Pi_- \left( (I + hT_-(c, d, \lambda))^{-1} \right). \quad (6.10)
\end{align*}
\]

**Proof.** Define following two matrices:

\[
\begin{align*}
P_-(c, d, \lambda) &= \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{\gamma_{k+1}} E_{k, k+1} \in G_-, \\
P_+(c, d, \lambda) &= \sum_{k=1}^{N} \frac{c_{k-1}}{\beta_{k+1}} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1, k} \in G_+.
\end{align*}
\]

Note now that the recurrent relation (4.28) is just equivalent to the matrix equality

\[
U(c, d, \lambda) - hL(c, d, \lambda) = P_-(c, d, \lambda)B(c, d, \lambda), \quad (6.11)
\]

and the recurrent relation (4.35) is just equivalent to the matrix equality

\[
L(c, d, \lambda) + hU(c, d, \lambda) = C(c, d, \lambda)P_+(c, d, \lambda). \quad (6.12)
\]

Multiplying (6.11) from the left by $U^{-1}$, we obtain:

\[
I - hT_- = U^{-1}P_-B,
\]

and since $U^{-1}P_- \in G_-$, we obtain (6.8). From the previous equation with the help of (5.17) we derive also

\[
I - hT_+ = P_-BU^{-1} = P_-\tilde{U}^{-1}A,
\]

which proves (6.7) in view of $P_-\tilde{U}^{-1} \in G_-$. Next, multiplying (6.12) from the right by $L^{-1}$, we obtain:

\[
I + hT_+^{-1} = CP_+L^{-1},
\]
which just implies (6.9) because of $P_L^{-1} \in \mathbf{G}_+$. Finally, from the previous equation we derive with the help of (5.25):

$$I + hT^{-1} = L^{-1}CP = \mathcal{D}L^{-1}P_+,$$

which means the validity of (6.10) because of $L^{-1}P_+ \in \mathbf{G}_+$. The theorem is proved.

Substitute now the expressions (6.1)–(6.4) into the discrete Lax equations (5.1), (5.2), (5.10), and (6.7)–(6.10) into (5.17), (5.18), (5.25), (5.26). One recognizes immediately the difference equations from the part c) of the Theorem 2. This gives us the solution of the initial value problem for the dynamical system (4.14), (4.15) in terms of the factorization of the matrices

$$\left((I + hT_{\pm}(0))\right)^n,$$

the solution of the initial value problem for the dynamical system (4.21), (4.22) in terms of the factorization of the matrices

$$\left((I - hT_{\pm}^{-1}(0))\right)^n.$$

, the solution of the initial value problem for the dynamical system (4.28), (4.29) in terms of the factorization of the matrices

$$\left((I - hT_{\pm}(0))^{-1}\right)^n.$$

, and the solution of the initial value problem for the dynamical system (4.35), (4.36) in terms of the factorization of the matrices

$$\left((I + hT_{\pm}^{-1}(0))^{-1}\right)^n.$$

The part d) of the Theorem 2 may be formulated in our case in the following lines.

**Corollary.** The interpolating Hamiltonian for the map ((4.14), (4.15)) is given by

$$\varphi_+(T) = \text{tr}(\Phi_+(T)) = J_+(T) + O(h), \quad \text{where} \quad \Phi_+(\xi) = h^{-1} \int_0^\xi \frac{d\eta}{\eta} \log(1 + h\eta),$$

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for the map (4.21), (4.22) – by

\[ \varphi_-(T) = \text{tr}(\Phi_-(T)) = J_-(T) + O(h), \quad \text{where} \quad \Phi_-(\xi) = h^{-1} \int_{\xi}^{\infty} \frac{d\eta}{\eta} \log\left(\frac{1}{1-h\eta^{-1}}\right), \]

for the map (4.28), (4.29) – by

\[ \psi_+(T) = \text{tr}(\Psi_+(T)) = J_+(T) + O(h), \quad \text{where} \quad \Psi_+(\xi) = h^{-1} \int_{0}^{\xi} \frac{d\eta}{\eta} \log\left(\frac{1}{1-h\eta}\right), \]

and for the map (4.35), (4.36) – by

\[ \psi_-(T) = \text{tr}(\Psi_-(T)) = J_-(T) + O(h), \quad \text{where} \quad \Psi_-(\xi) = h^{-1} \int_{\xi}^{\infty} \frac{d\eta}{\eta} \log\left(1+h\eta^{-1}\right). \]

7 Conclusion

We have introduced the difference approximations for two Hamiltonian flows from the relativistic Toda hierarchy. They turned out to belong to the same hierarchy. The inclusion in the general scheme of symplectic maps on groups equipped with quadratic Poisson brackets allowed to solve the difference equations in terms of factorization problem in the group and to find the interpolating Hamiltonians.
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