Classical and quantum spaces as rough images of the fundamental prespace

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April 1, 2022

Abstract

In spite of all no-go theorems (e.g., von Neumann, Kochen and Specker,..., Bell,...) we constructed a realist basis of quantum mechanics. In our model both classical and quantum spaces b are rough images of the fundamental prespace. Quantum mechanics cannot be reduced to classical one. Both classical and quantum representations induce reductions of prespace information.

1. Introduction. In preprints [1] 1 there was constructed a contextual quantum representation of the Kolmogorovian model. That mathematical construction can be used as a realist basis of quantum mechanics (QM). Existence of such a realist “underground” of QM was the question of the great debate since first days of QM, see, e.g., [2], [3] for detail. It should be reminded that A. Einstein strongly supported the idea that such a realist underground of QM could finally be found and W. Heisenberg and N. Bohr claimed that it would be impossible. In this note we present main ideas of [1] without to go in rather technical mathematical details of contextual representation of the Kolmogorovian model in a Hilbert space.

It should be underlined from the very beginning that we do not discuss a reduction of quantum physics to classical one.

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1Plenary talk and a topic of the round table at the International Conference “Quantum Theory: Reconsideration of Foundations-2”, June-2003, Växjö, Sweden.
2. **Prespace, classical space and quantum space.** In my model both quantum and classical states are rough images of contexts – complexes of physical conditions. In the mathematical model [1], cf. [4], contexts are described as sets of fundamental parameters. We call the space of fundamental parameters prespace and denote it Ω. Contexts are represented by a family of subsets of Ω. The prespace Ω is underground of the classical space \( X_{\text{cl}} = \mathbb{R}^3 \) as well as the quantum (Hilbert) space \( X_q = H \). QM gives essentially richer picture of the prespace Ω: the QM-representation of Ω-contexts generates essentially larger class of images than the classical representation. In particular, it is impossible to reduce quantum picture of the prespace Ω to the classical one.

This is a very delicate point of considerations. Dynamics in the prespace Ω is a deterministic dynamics. But it is not “classical dynamics” since the latter takes place not in the prespace Ω but in the classical space \( X_{\text{cl}} \). This is not a question of the mathematical realization of the prespace Ω and the classical space \( X_{\text{cl}} \). It may be that the prespace Ω can also be described as \( \Omega = \mathbb{R}^m \) (or even as \( \Omega = \mathbb{R}^3 \) – so in the same way as \( X_{\text{cl}} = \mathbb{R}^3 \), cf. section 8). The crucial point is that \( X_{\text{cl}} \) is created via the huge **reduction of information** in the process of transition from the prespace contexts to points of \( X_{\text{cl}} \). Each classical point \( x \in X_{\text{cl}} \) is the image of a domain \( B_x \) of the prespace Ω, see [1], and this domain can contain huge (may be even infinite) number of prepoints \( \omega = \omega_x \). By our model there exist mappings:

\[
\Omega \rightarrow X_{\text{cl}}, \quad \Omega \rightarrow X_q, \quad \text{and} \quad X_q \rightarrow X_{\text{cl}}
\]

Thus we also obtain the map:

\[
\Omega \rightarrow X_q \rightarrow X_{\text{cl}}
\]

But there is no pathway:

\[
\Omega \rightarrow X_{\text{cl}} \rightarrow X_q
\]

We underline (see further considerations) that the correspondence principle is based on the map \( \Omega \rightarrow X_q \) and not at all on the map \( X_q \rightarrow X_{\text{cl}} \).

3. **Fundamental incompatible preobservables.** Contextual probabilistic representation [1] of Ω-contexts in the quantum space \( X_q = H \) is based on a fixed pair of incompatible preobservables (**reference observables**):

\[
b, a : \Omega \rightarrow \mathbb{R}
\]  

(1)
In our model [1] preobservables are functions \(d : \Omega \to \mathbb{R}\). Denote the set of all preobservables \(\mathcal{O}(\Omega)\). We interpret observables \(d \in \mathcal{O}(\Omega)\) as realist observables: by fixing a prepoint \(\omega \in \Omega\) we are able to fix the value \(d = d(\omega)\). We are not able to measure an arbitrary \(d \in \mathcal{O}(\Omega)\). But reference preobservables \(\mathbb{I}\) and functions of those preobservables, \(f(b), f(a)\) can be measured [1]. We denote the space of quantum observables by the symbol \(\mathcal{O}(H)\). In mathematical models \(\mathcal{O}(\Omega)\) and \(\mathcal{O}(H)\) are represented by spaces of (Kolmogorovian) random variables and self-adjoint operators, respectively.

4. Position-momentum picture of prespace. In principle a contextual probabilistic representation of the prespace \(\Omega\) in the quantum space \(X_q = H\) can be based on any pair of incompatible preobservables. However, it seems that we (human beings) can use only the special pair of reference preobservables:

\[(q, p) = \text{(position, momentum)}.\]

Thus modern quantum as well as classical physics are the position-momentum pictures of the prespace. All classical and quantum observables are functions of position and momentum observables. In quantum case we use functions \(\hat{d} = u(\hat{q}, \hat{p})\) of operators of the position \(\hat{q}\) and the momentum \(\hat{p}\).

By choosing another pair of reference preobservables we obtain another quantum picture of the prespace \(\Omega\). However, it seems that at the present time we are not able to measure preobservables \(d \in \mathcal{O}(\Omega)\) distinct from functions \(f(q)\) and \(g(p)\).

5. Nonequivalence of quantum pictures of the prespace. It should be underlined that quantum pictures of the prespace \(\Omega\) based on two different pairs of incompatible observables \((b, a)\) and \((v, w)\) are in general nonequivalent. Of course, the same mathematical formalism – the Hilbert space formalism – can be used for any quantum picture of the \(\Omega\). But we should pay attention to physical structures of representations. So we should not forget about the \((q, p)\)-origin of QM (as a physical theory and not as only a mathematical formalism), see also [5], [6].

6. No “no-go?” Existence of a realist underground model of QM looks very surprising in the view of various no-go theorems, e.g., von Neumann [7], Kochen-Specker [8], Bell [9],... But all those no-go theorems suffered of the absence of physical justification for the list of assumptions on the correspondence between a realist prequantum model and QM. J. Bell performed

\[\text{But Hilbert spaces } H_{b/a} \text{ and } H_{v/w} \text{ corresponding to pictures based on } (b, a) \text{ and } (v, w) \text{ can be different.}\]
the brilliant analysis of assumptions on the “real-quantum” correspondence which were assumed (very often indirectly) in previous no-go theorems, see [9]. We should agree with Bell that von Neumann, Kochen and Specker and many others wanted too much for the “real-quantum” correspondence. Thus despite all pre-Bellian no-go theorems J. Bell was sure that it is possible to construct a realist basis of QM. However, J. Bell also wanted too much for the “real-quantum” correspondence, see, e.g., [10], [6] for analysis of Bell’s assumptions. As a consequence, he came to the conclusion that every realist prequantum model should be nonlocal.

7. Correspondence between preobservables and quantum observables. Correspondence-maps

\[ W : \mathcal{O}(\Omega) \to \mathcal{O}(H) \]

between realist preobservables and quantum observables which were considered by von Neumann, Kochen and Specker, ..., Bell,... were too straightforward. Neither von Neumann and Kochen-Specker nor Bell had physical arguments to present a list of “natural features” of such a correspondence \( W \). I neither have physical arguments. But I have strong probabilistic arguments. There exists a unique quantum representation of a Kolmogorovian model and this representation automatically induces a map \( W \) which have very special properties [1]. Neither von Neumann and Kochen-Specker nor Bell maps have such properties.

In our realist model the map \( W \) is defined only on a proper domain \( D_W \) in \( \mathcal{O}(\Omega) \), namely

\[ D_W = \mathcal{O}(b, a) = \{ d(\omega) = f(b(\omega)) + g(a(\omega)) \}, \]

where \((b, a)\) is the pair of reference preobservables determining the quantum picture of the \( \Omega \). And in general the map \( W \) does not preserve conditional probability distributions. Here we consider two conditionings: contextual conditioning in the \( \Omega \) and quantum state conditioning in the \( H \).

But (!) conditional averages are preserved by the map \( W \):

\[ E(d/C) = (\hat{d}\phi_C, \phi_C), \]

where \( \hat{d} = W(d), d \in D_W \), and \( \phi_C \in H \) is the image of a prespace context \( C \).

It is very important that quantum Hamiltonians belong to the \( W \)-image of the set of preobservables. The operator

\[ \hat{H} = \frac{\hat{a}^2}{2} + V(\hat{b}) \]
is the image of the energy preobservable

\[ \mathcal{H}(\omega) = \frac{a(\omega)^2}{2} + V(b(\omega)). \]

But as we have already underlined the \( W \) does not preserve probability distributions. So \( \hat{\mathcal{H}} \) and \( \mathcal{H}(\omega) \) have different probability distributions. But they have the same average.

8. **Quantization and the correspondence principle.** As we noticed, it was proved [1] that quantum Hamiltonian \( \hat{\mathcal{H}} \) has the same contextual average as the prespace Hamiltonian \( \mathcal{H} \):

\[ E(\mathcal{H}/C) = (\hat{\mathcal{H}} \phi_C, \phi_C). \tag{2} \]

We can speculate that this coincidence of averages is the real basis of quantization rules. By (2) to obtain the correct average of the energy preobservable \( \mathcal{H}(\omega) \) we should put quantum images of the reference preobservables into the prespace Hamiltonian:

\[ b \rightarrow \hat{b} = W(b), \quad a \rightarrow \hat{a} = W(a) \tag{3} \]

and, in particular, for the (position, momentum) quantization

\[ q_\Omega \rightarrow \hat{q} = W(q_\Omega), \quad p_\Omega \rightarrow \hat{p} = W(p_\Omega) \tag{4} \]

We should sharply distinguish the prespace position and momentum, \( q_\Omega, p_\Omega \), and classical space position and momentum \( q_{X_{cl}}, p_{X_{cl}} \). In our model (in the opposite to the very common opinion) \( \hat{q} \neq W(q_{X_{cl}}) \) and \( \hat{p} \neq W(p_{X_{cl}}) \).

By our model the root of the quantization rule is the equality (4) and Hamiltonian dynamics in the prespace \( \Omega \) and not Hamiltonian dynamics in the classical space \( X_{cl} \).

**Conclusion.** *In spite of all no-go theorems the realist model of QM exists.*

I would like to thank A. Plotnitsky for numerous discussions on Heisenberg-Bohr interpretation of QM and L. Accardi, L. Ballentine, S. Gudder for discussions on the role of conditional probabilities in QM which were extremely

\[ \text{It may be that dynamics in \( \Omega \) and \( X_{cl} \) are mathematically described in the same way. But we should distinguish Hamiltonian prespace dynamics and classical Hamiltonian dynamics.} \]


important for the creation of a contextual representation of a Kolmogorovian model in a Hilbert space [1], cf. [4].

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