COMBINATORIAL IDENTITIES INVOLVING MERTENS FUNCTION THROUGH RELATIVELY PRIME SUBSETS

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Abstract. In this note we give some identities which involve the Mertens function $M(n)$. Our proofs are combinatorial with relatively prime subsets as a main tool.

1. Introduction

Mertens function given by

$$M(n) = \sum_{d=1}^{n} \mu(d),$$

where $\mu$ denotes the Möbius mu function, is an important function in (analytic) number theory. Most of mathematical identities where $M(n)$ appears are either recursive formulas for $M(n)$ or formulas with an analytic flavor relating $M(n)$ to other functions. For a survey on identities involving $M(n)$ we refer to [1, 4] and for a survey on combinatorial identities involving other arithmetical functions we refer to [7, 10] and their references. In this work we will give some other identities involving the function $M(n)$. Our proofs are combinatorial and based on relatively prime subsets of sets of positive integers. We list two of the identities which we intend to prove. Let $\lfloor x \rfloor$ denote the floor of $x$.

1) If $n > 3$, then

$$\sum_{d=1}^{n} \mu(d) 2^{\lfloor \frac{n}{d} \rfloor - \lfloor \frac{n-3}{d} \rfloor} = \begin{cases} 3 + M(n), & \text{if } n \text{ is even} \\ 4 + M(n), & \text{if } n \text{ is odd} \end{cases}$$

2) If $1 < m < n$, then

$$\sum_{d=1}^{n+1} \mu(d) 2^{\lfloor \frac{n+1}{d} \rfloor - \lfloor \frac{n+1}{d} \rfloor + \lfloor \frac{m}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor} = \begin{cases} 2 + M(n+1), & \text{if } (m,n) > 1 \text{ and } (m,n+1) > 1 \\ 3 + M(n+1), & \text{if } (m,n) = 1 \text{ and } (m,n+1) > 1 \text{ or } (m,n) > 1 \text{ and } (m,n+1) = 1 \\ 4 + M(n+1), & \text{if } (m,n) = (m,n+1) = 1. \end{cases}$$
2. RELATIVELY PRIME SUBSETS OF $[l_1, m_1] \cup [l_2, m_2]$

Throughout this section let $k$, $l$, $m$, $l_1$, $l_2$, $m_1$, and $m_2$ be positive integers such that $l \leq m$, $l_1 \leq m_1$ and $l_2 \leq m_2$, let $[l, m] = \{l, l + 1, \ldots, m\}$, and let $A$ be a nonempty finite set of positive integers. The set $A$ is called relatively prime if $\gcd(A) = 1$.

**Definition 1.** Let

$$f(A) = \#\{X \subseteq A : X \neq \emptyset \text{ and } \gcd(X) = 1\}$$

and

$$f_k(A) = \#\{X \subseteq A : \#X = k \text{ and } \gcd(X) = 1\}.$$  

Nathanson in [8] introduced among other functions $f(n)$ and $f_k(n)$ (in our terminology $f([1, n])$ and $f_k([1, n])$ respectively) and found

$$f([1, n]) = \sum_{d=1}^{n} \mu(d)(2^{\left\lfloor \frac{n}{d} \right\rfloor} - 1) \text{ and } f_k([1, n]) = \sum_{d=1}^{n} \mu(d)\left(\left\lfloor \frac{n}{d} \right\rfloor \right).$$

Formulas for $f([m, n])$ and $f_k([m, n])$ are found in [3][9]. Recently Ayad and Kihel in [3] considered relatively prime subsets of sets which are in arithmetic progression and obtained formulas for $f([l, m])$ and $f_k([l, m])$ as consequences since the integer interval $[l, m]$ is in arithmetic progression. However the authors' argument seems not to extend to unions of integer intervals. In this section we will give formulas for $f([l_1, m_1] \cup [l_2, m_2])$ and for $f_k([l_1, m_1] \cup [l_2, m_2])$. For the sake of completeness we include the following result which is a natural extension of [3 Theorem 2 (a)] on Möbius inversion for arithmetical functions of several variables. For simplicity of notation we let

$$(\overline{m}_a, \overline{m}_b) = (m_1, m_2, \ldots, m_a, n_1, n_2, \ldots, n_b)$$

and

$$(\overline{m}_a \div \overline{m}_b) = \left(\frac{m_1}{d}, \frac{m_2}{d}, \ldots, \frac{m_a}{d}, \left\lfloor \frac{n_1}{d} \right\rfloor, \frac{n_2}{d}, \ldots, \left\lfloor \frac{n_b}{d} \right\rfloor\right).$$

**Theorem 2.** If $F$ and $G$ are arithmetical of $a + b$ variables, then

$$G(\overline{m}_a, \overline{m}_b) = \sum_{d \mid (m_1, m_2, \ldots, m_a)} F\left(\overline{m}_a \div \overline{m}_b\right)$$

if and only if

$$F(\overline{m}_a, \overline{m}_b) = \sum_{d \mid (m_1, m_2, \ldots, m_a)} \mu(d)G\left(\overline{m}_a \div \overline{m}_b\right).$$

We need the following three lemmas the proofs of which can be obtained using the same sort of idea and therefore we prove only the first one.

**Lemma 3.** Let

$$g(m_1, l_2, m_2) = \#\{X \subseteq [1, m_1] \cup [l_2, m_2] : l_2 \in X \text{ and } \gcd(X) = 1\},$$

$$g_k(m_1, l_2, m_2) = \#\{X \subseteq [l_1, m_1] \cup [l_2, m_2] : l_2 \in X, \ |X| = k, \text{ and } \gcd(X) = 1\}.$$  

Then

(a) $g(m_1, l_2, m_2) = \sum_{d \mid l_2} \mu(d)2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - l_2/d},$
Lemma 5. Then which by Theorem 2 is equivalent to

\[ \text{Lemma 4.} \]

Let \((b)\) Similarly

\[ \gcd(\text{dividing } l_2) \in \mathbb{N} \]

\[ \text{gcd} (\text{gcd}(\text{dividing } l_2)) = 1. \]

Then \[ 2^{m_1 + m_2 - l_2} = \sum_{d|l_2} \#\mathcal{P}(m_1, l_2, m_2, d) = \sum_{d|l_2} g((m_1/d), l_2/d, [m_2/d]), \]

which by Theorem 2 is equivalent to

\[ g(m_1, l_2, m_2) = \sum_{d|l_2} \mu(d)2^{m_1/d + [m_2/d] - l_2/d}. \]

(b) Similarly

\[ \binom{m_1 + m_2 - l_2}{k - 1} = \sum_{d|l_2} g_k([m_1/d], l_2/d, [m_2/d]), \]

which by Theorem 2 is equivalent to

\[ g_k(m_1, l_2, m_2) = \sum_{d|l_2} \mu(d)\left(\left\lceil\frac{m_1/d}{d}\right\rceil + \left\lceil\frac{m_2/d}{d}\right\rceil - l_2/d\right). \]

\[ \square \]

Lemma 4. Let

\[ h^{(1)}(l_1, m_1) = \#\{X \subseteq [l_1, m_1] : l_1 \in X \text{ and } \gcd(X) = 1\}, \]

\[ h^{(1)}_k(l_1, m_1) = \#\{X \subseteq [l_1, m_1] : l_1 \in X, \#X = k, \text{ and } \gcd(X) = 1\}. \]

Then

\[ (a) \quad h^{(1)}(l_1, m_1) = \sum_{d|l_1} \mu(d)2^{m_1/d - l_1/d}, \]

\[ (b) \quad h^{(1)}_k(l_1, m_1) = \sum_{d|l_1} \mu(d)\left\lceil\frac{m_1/d}{d}\right\rceil - l_1/d\right\rceil_k. \]

Lemma 5. Let

\[ h^{(2)}(l_1, m_1, l_2, m_2) = \#\{X \subseteq [l_1, m_1] \cup [l_2, m_2] : l_1, l_2 \in X \text{ and } \gcd(X) = 1\}, \]

\[ h^{(2)}_k(l_1, m_1, l_2, m_2) = \#\{X \subseteq [l_1, m_1] \cup [l_2, m_2] : l_1, l_2 \in X, |X| = k, \text{ and } \gcd(X) = 1\}. \]

Then

\[ (a) \quad h^{(2)}(l_1, m_1, l_2, m_2) = \sum_{d|(l_1, l_2)} \mu(d)2^{\left\lceil\frac{m_1}{d}\right\rceil + \left\lceil\frac{m_2}{d}\rceil - l_1 = l_2}, \]

\[ (b) \quad h^{(2)}_k(l_1, m_1, l_2, m_2) = \sum_{d|(l_1, l_2)} \mu(d)\left(\left\lceil\frac{m_1}{d}\right\rceil + \left\lceil\frac{m_2}{d}\rceil - l_1 + l_2\right\rceil_{k-2}. \]

We further need the following special case to prove the main theorem of this section.
Theorem 6. We have

\((a)\) \( f([1, m_1] \cup [l_2, m_2]) = \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{2l_2-1}{d} \rfloor} - 1 \right) \),

\((b)\) \( f_k([1, m_1] \cup [l_2, m_2]) = \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{2l_2-1}{d} \rfloor \right) \).

Proof. (a) Clearly

\[ \begin{align*}
  f([1, m_1] \cup [l_2, m_2]) &= f([1, m_2]) - \sum_{i=m_1+1}^{l_2-1} g(m_1, i, m_2) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_2}{d} \rfloor} - 1 \right) - \sum_{i=m_1+1}^{l_2-1} \sum_{d|i} \mu(d) \left( 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{2l_2-1}{d}} - 1 \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_2}{d} \rfloor} - 1 \right) - \sum_{d=1}^{l_2-1} \mu(d) \left( 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \sum_{j=\lfloor \frac{m_1}{d} \rfloor + 1}^{\lfloor \frac{2l_2-1}{d} \rfloor} \frac{1}{2^j} \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_2}{d} \rfloor} - 1 \right) - \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \left( 1 - 2^{-\lfloor \frac{2l_2-1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \right) \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( 2^{\lfloor \frac{m_1}{d} \rfloor} + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{2l_2-1}{d} \rfloor - 1 \right),
\end{align*} \]

where the second identity follows by (1) and Lemma 3.

(b) We have

\[ \begin{align*}
  f_k([1, m_1] \cup [l_2, m_2]) &= f_k([1, m_2]) - \sum_{i=m_1+1}^{l_2-1} g(m_1, i, m_2) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_2}{d} \rfloor \right) - \sum_{i=m_1+1}^{l_2-1} \sum_{d|i} \mu(d) \left( \lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i}{d} \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_2}{d} \rfloor \right) - \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{\lfloor \frac{2l_2-1}{d} \rfloor - 1}{d} \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_2}{d} \rfloor \right) - \sum_{d=1}^{m_2} \mu(d) \left( \left( \lfloor \frac{m_2}{d} \rfloor \right) - \left( \lfloor \frac{m_1}{d} \rfloor + \frac{m_2}{d} - \frac{2l_2-1}{d} \right) \right) \\
  &= \sum_{d=1}^{m_2} \mu(d) \left( \lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{2l_2-1}{d} \right),
\end{align*} \]

This completes the proof. \(\square\)
Proof. (a) Clearly

Theorem 7. We have

\[ f([l_1, m_1] \cup [l_2, m_2]) = \sum_{d=1}^{m_2} \mu(d)(2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor) - \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - 1), \]

\[ f_k([l_1, m_1] \cup [l_2, m_2]) = \sum_{d=1}^{m_2} \mu(d) \left( \left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor \right). \]

We have

\[ f([l_1, m_1] \cup [l_2, m_2]) = f([1, m_1] \cup [l_2, m_2]) - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} h^{(2)}(i, m_1, j, m_2) - \sum_{i=1}^{l_1-1} h^{(1)}(i, m_1) \]

\[ = \sum_{d=1}^{m_2} \mu(d)(2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - 1) - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - \sum_{i=1}^{l_1-1} \sum_{d|i} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor - \frac{1}{d}, \]

where the second identity follows by Theorem [3], Lemma [4] and Lemma [5]. Rearranging the triple summation in identity (3), we get

\[ \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \sum_{d|(i,j)} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - 1 = \sum_{d=1}^{m_2} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - 1 - \sum_{d=1}^{m_2} \sum_{d|(i,j)} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor - \frac{1}{d}. \]

Similarly the last double summation in identity (3) gives

\[ \sum_{i=1}^{l_1-1} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor = \sum_{d=1}^{m_2} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - 1. \]

Appealing to identities (3), (4), and (5) we find

\[ f([l_1, m_1] \cup [l_2, m_2]) = \sum_{d|n} \mu(d)2^\left\lfloor \frac{m_1}{d} \right\rfloor + \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor - \left\lfloor \frac{m_1}{d} \right\rfloor. \]

This completes the proof of part (a). Part (b) follows by similar arguments. \( \square \)

Corollary 8. (Ayad-Kihel [3]) We have

\[ f([l, m]) = \sum_{d=1}^{m} (2^\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - 1), \]

\[ f_k([l, m]) = \sum_{d=1}^{m} \left( \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \right). \]

Proof. Use Theorem [7] with \( l_1 = l, m_1 = m - 1, \) and \( l_2 = m_2 = m. \) \( \square \)
3. Combinatorial identities

**Theorem 9.** If \( n > 1 \), then

\[
\sum_{d=1}^{n+1} \mu(d)(2\left\lfloor \frac{n+1}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor) = 1 + M(n+1)
\]

**Proof.** Apply Corollary 8 to the interval \([n, n+1]\) and use the obvious fact that \( f([n, n+1]) = 1 \). \( \square \)

**Theorem 10.** In \( n > 3 \), then

\[
\sum_{d=1}^{n} 2\left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-3}{d} \right\rfloor = \begin{cases} 3 + M(n), & \text{if } n \text{ is even} \\ 4 + M(n), & \text{if } n \text{ is odd.} \end{cases}
\]

**Proof.** Combine Corollary 8 applied to the interval \([n-2, n]\) and the fact that

\( f([n-2, n]) = \begin{cases} 3, & \text{if } n \text{ is even} \\ 4, & \text{if } n \text{ is odd.} \end{cases} \)

\( \square \)

**Theorem 11.** (a) If \( 1 < m < n \), then

\[
\sum_{d=1}^{n+1} \mu(d)2\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor = \begin{cases} 2 + M(n+1), & \text{if } (m, n) > 1 \text{ and } (m, n+1) > 1 \\ 3 + M(n+1), & \text{if } (m, n) = 1 \text{ and } (m, n+1) > 1 \text{ or } (m, n) > 1 \text{ and } (m, n+1) = 1 \\ 4 + M(n+1), & \text{if } (m, n) = (m, n+1) = 1. \end{cases}
\]

(b) If \( 1 < n < m - 1 \), then

\[
\sum_{d=1}^{m} \mu(d)2\left\lfloor \frac{n+1}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor = \begin{cases} 2 + M(m), & \text{if } (m, n) > 1 \text{ and } (m, n+1) > 1 \\ 3 + M(m), & \text{if } (m, n) = 1 \text{ and } (m, n+1) > 1 \text{ or } (m, n) > 1 \text{ and } (m, n+1) = 1 \\ 4 + M(m), & \text{if } (m, n) = (m, n+1) = 1. \end{cases}
\]

**Proof.** (a) Clearly

\( f([m, m] \cup [n, n+1]) = \begin{cases} 2, & \text{if } (m, n) > 1 \text{ and } (m, n+1) > 1 \\ 3, & \text{if } (m, n) = 1 \text{ and } (m, n+1) > 1 \text{ or } (m, n) > 1 \text{ and } (m, n+1) = 1 \\ 4, & \text{if } (m, n) = (m, n+1) = 1. \end{cases} \)

Combine this identity with Theorem 7(a) applied to \([m, m] \cup [n, n+1]\).

(b) Similar to part (a) with an application of Theorem 7(a) to \([n, n+1] \cup [m, m] \). \( \square \)
References

[1] T. M. Apostol, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer, 1 edition, 1976.
[2] Mohamed Ayad and Omar Kihel, On the Number of Subsets Relatively Prime to an Integer, Journal of Integer Sequences, Vol. 11, (2008), Article 08.5.5.
[3] Mohamed Ayad and Omar Kihel, On Relatively Prime Sets, Integers 9, (2009), 343-352.
[4] F. Dress, Fonction sommatoire de la fonction de Möbius. I. Majorations expériminales, Experiment. Math. 2, (1993), 89-98.
[5] Mohamed El Bachraoui, The number of relatively prime subsets and phi functions for sets \( \{m, m+1, \ldots, n\} \), Integers 7 (2007), A43, 8pp.
[6] Mohamed El Bachraoui, On the Number of Subsets of \( \{1, m\} \) Relatively Prime to \( n \) and Asymptotic Estimates, Integers 8 (2008), A41, 5 pp.
[7] M. Hall, Combinatorial Theory, Second edition, Wiley, New York, 1986.
[8] Melvyn B. Nathanson, Affine invariants, relatively prime sets, and a phi function for subsets of \( \{1, 2, \ldots, n\} \), Integers 7 (2007), A1, 7pp.
[9] Melvyn B. Nathanson and Brooke Orosz, Asymptotic estimates for phi functions for subsets of \( \{m+1, m+2, \ldots, n\} \), Integers 7 (2007), A54, 5pp.
[10] J. Riordan, Combinatorial Identities, Wiley, New York, 1968.

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