On Itō’s formula for convoluted Lévy processes

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Abstract

We derive an Itō formula for stochastic processes which are constructed by a convolution of a deterministic kernel with a Lévy process. Compared to related change of variable formulas in the literature we can treat a larger class of kernels, which includes the case of a fractional Lévy process (in the Mandelbrot-Van Ness sense). Moreover, we show that the growth conditions in the Itō formula can be weakened in the presence of a nontrivial Gaussian part. We also provide results on the path regularity and on the existence of moments of the running maximum for a large class of convoluted Lévy processes.

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1 Introduction

In this paper we study a class of stochastic processes which is constructed by a convolution of a deterministic kernel with a Lévy process. An example of these ‘convoluted’ Lévy processes is the class of fractional Lévy processes introduced in [24]. In this setting the kernel is the one of the Mandelbrot-Van Ness representation for fractional Brownian motion, but the driving Brownian motion is replaced by a more general Lévy process. In this way a fractional Lévy process inherits the correlation structure of a Brownian motion, but the choice of the driving Lévy process provides additional flexibility to incorporate distributional properties such as heavier tails than the ones of a normal distribution. More generally, the fact that the second order structure (and hence the memory) of a convoluted Lévy process is encoded in the choice of the kernel is an intriguing feature from a modelling point of view. Applications to financial mathematics include volatility modelling ([17], [9]), modelling of trading time ([7], [15]), and credit risk models [13]. Moreover, [2] apply the related class of Lévy semistationary processes as a model for energy spot prices.

Depending on the choice of the kernel and the driving Lévy process, convoluted Lévy processes may fail to be semimartingales, see e.g. [3] and [5]. Hence, the classical Itō calculus is not at our disposal. [6] introduced a Skorokhod integral with respect to convoluted Lévy processes driven by a pure jump Lévy process and derived an Itō formula under rather restrictive assumptions on the kernel which do not cover the fractional Lévy case. Their definition of the Skorokhod integral is motivated

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by the $S$-transform characterisation of the Skorokhod integral with respect to a Lévy process and
generalises the $S$-transform approach to integration with respect to a fractional Brownian motion
introduced in [4].

In the present paper we extend the results of [6] in several respects. The Skorokhod integral with
respect to convoluted Lévy processes is introduced in the presence of a nontrivial Gaussian part of
the driving Lévy process. Moreover, in the case of a predictable integrand we express this integral
in terms of a classical Itô integral with respect to the driving Lévy process and a Skorokhod integral
with respect to the Gaussian part and the compensated jump measure. As a main result we are able
to obtain a generalisation of the change of variable formula in [6]. On the one hand we significantly
enlarge the class of kernels for which this formula is valid. In particular, this larger class of kernels
now includes fractional Lévy processes, the prime example of a convoluted Lévy process. On the
other hand we derive the Itô formula also in the presence of a Gaussian part and show that the
growth conditions in the Itô formula can be significantly relaxed when the Gaussian component is
nonzero. This is due to the fact that the characteristic function of the convoluted Lévy process
under a sufficiently large class of measures is rapidly decreasing, when the Gaussian component is
present.

The paper is organised as follows: In Section 2 we recall the definition of a convoluted Lévy process,
introduce the class of kernels which we consider throughout the paper, and illustrate this class of
kernels by several examples. In Section 3 we state the main results of the paper. Besides the
Itô formula, these include results on the path regularity of convoluted Lévy processes and the
existence of moments of their running maximum. The proofs of the latter two results are provided
in Section 4. Section 5 is devoted to the introduction of Skorokhod integration for convoluted Lévy
processes via the $S$-transform, while the proof of the Itô formula is presented in Section 6.

2 Convoluted Lévy processes

In this section we start by describing the underlying Lévy process and we then proceed to define
the convolution of this process with some integration kernel, which results in the main object under
consideration in the present paper. We define convoluted Lévy processes for a class of integration
kernels that fulfil an $L^2$-condition and later on restrict ourselves to a smaller class of kernel
functions for which we prove our main results. In this section we also provide some examples of
that latter class of kernel functions.

Throughout this paper we fix some $T \geq 0$.

2.1 Set-up

Regarding some background on the theory of Lévy processes we refer e.g. to [8] and [18]. Stochastic
analysis with respect to Lévy processes is treated in [1].

Let $(\gamma, \sigma, \nu)$ be a triplet consisting of constants $\gamma \in \mathbb{R}$ and $\sigma \geq 0$ as well as a measure $\nu$ on $\mathbb{R}_0$ that satisfy

$$\int_{[-1,1]} x^2 \nu(dx) + \int_{\mathbb{R}\setminus[-1,1]} |x|^k \nu(dx) < \infty \quad (1)$$

for all $k \in \mathbb{N}$ and

$$\gamma = -\int_{\mathbb{R}\setminus[-1,1]} x \nu(dx). \quad (2)$$
Observe that \((\gamma, \sigma, \nu)\) is a so-called characteristic triplet that determines the distribution of a Lévy process on \((\Omega, \mathcal{F}, \mathbb{P})\). Hence, let \(\hat{L} := (\hat{L}(t))_{t \geq 0}\) be a Lévy process on \((\Omega, \mathcal{F}, \mathbb{P})\) with characteristic triplet \((\gamma, \sigma, \nu)\) and càdlàg paths, whose jump measure we denote by \(N(dx, ds)\). Furthermore, let \(\hat{L} := (\hat{L}(t))_{t \geq 0}\) be a Lévy process on \((\Omega, \mathcal{F}, \mathbb{P})\) with characteristic triplet \((\gamma, \sigma, \nu)\) and càdlàg paths, whose jump measure we denote by \(N(dx, ds)\). Furthermore, let \(\Psi(u) := \ln \left( \mathbb{E} \left( e^{iu\hat{L}(1)} \right) \right) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}_0} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(dx) \) be the characteristic exponent of \(\hat{L}\) which is given by the Lévy-Khintchine formula. Note that (1) implies that \(\hat{L}(t) \in L^k(\mathbb{P})\) holds for all \(k \in \mathbb{N}\) and (2) is equivalent to \(\mathbb{E}(\hat{L}(1)) = 0\). The latter assertion holds, since \(\mathbb{E}(\hat{L}(1)) = \Psi'(0^+) = \int_{\mathbb{R}_0} x \tilde{N}(dx, ds)\), where \(\Psi'\) denotes the derivative of \(\Psi\). The process \(\hat{L}(t)\) can be represented as

\[
\hat{L}(t) = \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} x \tilde{N}(dx, ds)
\]

where \(W\) is a standard Brownian motion, \(\sigma\) is the standard deviation of the Gaussian component of \(\hat{L}(1)\), \(\nu(dx)\) is the Lévy measure and \(\tilde{N}(dx, ds) = N(dx, ds) - \nu(dx)ds\) is the compensated jump measure of the Lévy process \(\hat{L}\).

Later on we shall need the \(n\)-th order cumulants \(c^n_t\) of \(\hat{L}(t)\) for \(n \geq 3\) which are given by

\[
c^n_t := \frac{d^n}{du^n} \ln \left( \mathbb{E} \left( e^{iu\hat{L}(t)} \right) \right) \bigg|_{u=0} = t \cdot \frac{d^n}{du^n} \Psi(u) \bigg|_{u=0},
\]

where we used the stationary and independent increments of \(\hat{L}\). Differentiating (3) thus results in

\[
c^n_t = t \int_{\mathbb{R}_0} x^n \nu(dx)
\]

for all \(n \geq 3\). We construct a two-sided Lévy process \(L := (L(t))_{t \in \mathbb{R}}\) by taking two independent copies \((L_1(t))_{t \geq 0}\) and \((L_2(t))_{t \geq 0}\) of \(\hat{L}\) and defining

\[
L(t) := \begin{cases} L_1(t), & t \geq 0 \\ -L_2(-t-), & t < 0. \end{cases}
\]

We now introduce the main object under consideration in the present paper, which is the following class of stochastic integrals with respect to \(L\).

**Definition 1** For any function \(f : \mathbb{R}^2 \to \mathbb{R}\) such that \(f(t, \cdot) \in L^2(\lambda)\) for every \(t \in \mathbb{R}\) we define a stochastic process \(M := (M(t))_{t \geq 0}\) by

\[
M(t) = \int_{\mathbb{R}} f(t, s) L(ds)
\]

for every \(t \geq 0\). The process \(M\) is referred to as **convoluted Lévy process**.  

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Note that the integrability condition on $f$ implies that such integrals exist since $L$ is a càdlàg square integrable martingale.

In the literature, processes as defined in Definition 1 are occasionally also referred to as filtered Lévy processes (see e.g. [11]).

Let us introduce the following class of Volterra type kernels:

**Definition 2** We denote by $K$ the class of measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ with $\text{supp} f \subset [\alpha, \infty)^2$ for some $\alpha \in [-\infty, 0]$ such that

(i) $\forall s > t \geq 0 : f(t, s) = 0$,

(ii) $f(0, \cdot) = 0$ $\lambda$-a.e.,

(iii) the function $f$ is continuous on the set $\{(t, s) \in \mathbb{R}^2 : \alpha \leq s \leq t\}$,

(iv) $\lambda (\{s \in \mathbb{R} : f(t, s) \neq 0\}) \neq 0$ for all $0 < t \leq T$,

(v) for all $s > -\infty$ the mapping $t \mapsto f(t, s)$ is differentiable on the set $(s, \infty)$ and there exist some $C > 0$, $\beta \in \mathbb{R}$ and $\gamma < 1$ such that

$$\left| \frac{\partial}{\partial t} f(t, s) \right| \leq C|s|^\beta|t - s|^{-\gamma}$$

for all $t > s > -\infty$, and for any fixed $t > 0$ there are $\epsilon > 0$ and $\theta > (\gamma - \beta - 1) \wedge (-\frac{1}{2} - \epsilon)$ such that

$$\sup_{r \in (t-\epsilon, t+\epsilon)} |f(r, s)| = O\left(|s|^\theta\right) \quad \text{as } s \to -\infty.$$  

(vi) for every fixed $t > 0$ the function $s \mapsto f(t, s)$ is absolutely continuous on $[\alpha, t]$ with density $\frac{\partial}{\partial s} f(t, \cdot)$, i.e.

$$f(t, s) = f(t, \alpha) + \int_\alpha^s \frac{\partial}{\partial s} f(t, s) \, ds, \quad \alpha \leq s \leq t,$$

where $f(t, -\infty) := \lim_{x \to -\infty} f(t, x) = 0$, such that

(a) the function $t \mapsto \frac{\partial}{\partial s} f(t, s)$ is continuous for $\lambda$-a.e. $s \in [\alpha, \infty)$,

(b) there exist $a < 0$ and $\epsilon > 0$ such that

$$\sup_{t \in [0, T]} \left| \frac{\partial}{\partial s} f(t, s) \right| \leq |s|^{-\left(\frac{3}{2} + \epsilon\right)}$$

holds for all $s \leq a$,

(c) for every $a < 0$ there exists some $\epsilon > 0$ such that

$$\sup_{t \in [0, T]} \int_a^t \left| \frac{\partial}{\partial s} f(t, s) \right|^{1+\epsilon} \, ds < \infty.$$

**Remark 3** Let us provide some interpretation of the conditions in Definition 2

- The condition concerned with the support is used to treat kernels with compact support and unbounded support simultaneously.
• Condition (i) implies that

\[ M(t) = \int_{\mathbb{R}} f(t, s) \, L(ds) = \int_{-\infty}^{t} f(t, s) \, L(ds). \]

Roughly speaking this means that the convoluted Lévy process does not look into the future. That is to say, \( M \) is adapted to the filtration generated by \( L \).

• Condition (ii) ensures that \( M(0) = 0 \).

• Note that condition (iii) and equation (6) in condition (v) imply that \( f(t, \cdot) \in L^p(\lambda) \) for every \( t > 0 \) and \( p \geq 2 \). This fact is used later on to prove that all moments of \( M(t) \) exist.

• Condition (iv) is equivalent to \( M(t) \) not being identical to zero for \( t > 0 \).

• In Section 4 equation (6) and condition (vi) are used to obtain an integration by parts formula and to show that convoluted Lévy processes with kernels \( f \in \mathcal{K} \) have càdlàg paths and moments of all orders.

\[ \diamond \]

### 2.2 Examples

As we prove our main results for functions \( f \in \mathcal{K} \) we restrict ourselves from now on to this case. Examples of such convoluted Lévy processes \( M \) include:

**Example 4**

(i) The Lévy process \( L_1 \) itself that is recovered by choosing

\[ f(t, s) = \begin{cases} 1, & 0 \leq s \leq t < \infty \\ 0, & \text{otherwise} \end{cases} \]

(ii) *One-sided shot noise processes* with kernel

\[ f(t, s) = \begin{cases} r(t - s), & 0 \leq s \leq t < \infty \\ 0, & \text{otherwise} \end{cases} \]

for some \( r > 0 \).

(iii) *One-sided Ornstein-Uhlenbeck type processes* with kernel

\[ f(t, s) = \begin{cases} e^{-r(t-s)}, & 0 \leq s \leq t < \infty \\ 0, & \text{otherwise} \end{cases} \]

for some \( r \geq 0 \).

(iv) *Fractional Lévy processes* with the Mandelbrot-van Ness kernel (cf. Lemma 3), defined by

\[ f_d(t, s) = \frac{1}{\Gamma(d+1)} \left( (t-s)^d_+ - (-s)^d_+ \right) \]

for all \( s, t \in \mathbb{R} \) and a fractional integration parameter \( d \in (0, \frac{1}{2}) \) and where \( \Gamma \) denotes the Gamma function. The parameter \( d \) is related to the well-known Hurst parameter by \( d = H - \frac{1}{2} \). Below, \( M^d \) denotes the convoluted Lévy process with kernel \( f_d \).
The following Lemma shows that the kernel $f_d$ defined in Example 4(iv) is indeed in $K$.

**Lemma 5** The function $f_d : \mathbb{R}^2 \to \mathbb{R}$, defined in (7), satisfies the assumptions in Definition 2 for $\alpha = -\infty$.

**Proof** It is easy to see that $f_d$ satisfies (i)-(iv) in Definition 2. Condition (v) follows from the following expression containing the derivative with respect to the first argument of $f_d$, i.e.

$$\frac{\partial}{\partial t} f_d(t, s) = \frac{1}{\Gamma(d + 1)} d(t - s)^{d-1}.$$

If we choose $\beta = 0$ and $\gamma = 1 - d$ we deduce that equation (5) in condition (v) is satisfied. Moreover, we infer by means of the Mean Value Theorem that

$$\sup_{r \in (t - \epsilon, t + \epsilon)} |f(r, s)| \leq \sup_{r \in (t - \epsilon, t + \epsilon)} \sup_{u \in (-s, r - s)} \frac{1}{\Gamma(d + 1)} dr |u|^{d-1} \leq \frac{1}{\Gamma(d + 1)} d(t + \epsilon)|s|^{d-1}.$$

Hence, equation (6) in condition (v) holds with $\theta = d - 1$.

In order to check condition (vi) of the above definition, observe that $f_d(t, \cdot)$ is absolutely continuous with density given by

$$\frac{\partial}{\partial s} f_d(t, s) = \begin{cases} \Gamma(d + 1)^{-1} \left( (d(t - s))_{+}^{d-1} - (d(-s))_{+}^{d-1} \right), & t > s \\ 0, & t \leq s. \end{cases}$$

Clearly, $f_d$ satisfies (vi)(a) and (vi)(c). In addition, (vi)(b) holds, since an application of the Mean Value Theorem results in

$$\left| (d(t - s))_{+}^{d-1} - (d(-s))_{+}^{d-1} \right| \leq d(1 - d) t |s|^{d-2}$$

for all $t > 0$ and $s < 0$. \hfill \Box

We are mainly interested in the fractional kernel $f_d$ as defined in Example 4(iv). As it is standard in the literature we call the convoluted Lévy process $M_d$ a *fractional Lévy process*. The motivation for this name is that it generalises the Mandelbrot-Van Ness definition of a fractional Brownian motion as an integral of the same kernel with respect to Brownian motion. However, let us emphasise that one needs to be a bit careful with the name *fractional Lévy process* as there are various definitions of a fractional Brownian motion (e.g. also via Molchan-Golosov kernels) that may be extended by replacing the Brownian motion by a general Lévy process. In contrast to the Gaussian case, this does not generally result in stochastic processes which are equal in distribution. Here we do not go into detail regarding this issue and refer instead to [5] and [28] to name just two references concerned with different approaches to define fractional Lévy processes.

Let us mention that the fractional Lévy processes as defined above have a modification which is pathwise defined as an improper Riemann integral and have continuous paths, see Theorem 3.4 in [24] for $\sigma = 0$ and Proposition 1.3 in [10] for the Gaussian part. In the forthcoming Proposition 6 we show that a related integration by parts formula also holds for any $M$ with $f \in K$.

Fractional Lévy processes aim at combining infinitely divisible distributions and the nice covariance structure of a fractional Brownian motion $(B^H(t))_{t \geq 0}$, which is given by

$$\text{cov} \left( B^H(t), B^H(s) \right) = \mathbb{E} \left( B^H(t)B^H(s) \right) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right)$$

(8)
for a Hurst parameter $H \in (0, 1)$ and any $s, t \geq 0$. Indeed, it is known that $M^d(t), t \geq 0$, is infinitely divisible (see also Lemma 25 below for a simple way to calculate its characteristic function) and it follows from Theorem 4.1 in [24] combined with (8) that the covariance structure of a fractional Lévy process $(M^d(t))_{t \geq 0}$ is essentially the same as for a fractional Brownian motion. More precisely,

$$\text{cov} \left( M^d(t), M^d(s) \right) = \left( \frac{1}{2} + \frac{\mathbb{E}(L_j(1)^2)}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} \right) \left( \left| t \right|^{2d+1} + \left| s \right|^{2d+1} - \left| t-s \right|^{2d+1} \right)$$

holds for $d \in (0, 1/2)$ and all $s, t \geq 0$, where $L_j$ is a Lévy process with characteristic triplet $(\gamma, 0, \nu)$. Let us mention (cf. [5]) that $M^d$ is not a martingale and that for a large class of Lévy processes it is neither a semimartingale. Moreover, like a fractional Brownian motion also $M^d$ has stationary increments, since the substitution $v := u - s$ and the stationary increments of $L$ result in

$$M^d(t) - M^d(s) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left( (t-u)^d_+ - (-u)^d_+ - ((s-u)^d_+ - (-u)^d_+) \right) L(du)$$

To conclude this brief digression into the general theory of fractional Lévy processes, let us mention that, in contrast to fractional Brownian motion, such processes are not self-similar as long as they are not a fractional Brownian motion.

### 3 Main results

In this section we present the main results of this paper. We start with some results regarding certain properties that $L$ possesses and which we show to hold true also for convoluted processes with a kernel $f \in K$. These properties are interesting in their own right, but they are also used for the Itô formula that constitutes the main result of this paper and is stated below. Our first result provides an integration by parts formula for the process $M$.

**Proposition 6** For each $f \in K$ the process $M$ satisfies

$$M(t) = f(t,t)L(t) - f(t,\alpha)L(\alpha) - \int_{\alpha}^{t} L(s) \frac{\partial}{\partial s} f(t,s) \, ds \quad (9)$$

$\mathbb{P}$-a.s., where $f(t,-\infty)L(-\infty) := \lim_{N \to \infty} f(t,-N)L(-N) = 0$ holds.

Throughout the rest of this paper we work with the modification of $M$ given by the right hand side of the above integration by parts formula. For notational convenience we denote this modification again by $M$.

One question one could ask is whether convoluted Lévy processes have good path properties, similar to Lévy processes. This question is answered affirmatively by the following result, which yields that $M$ inherits the càdlàg paths of the underlying Lévy process $L$.

**Theorem 7** For any $f \in K$ the process $M$ has càdlàg paths.

The second theorem of this paper is concerned with the $p^{th}$ moment of $M$.  

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Theorem 8 For each \( f \in \mathcal{K} \) we have \( \sup_{t \in [0,T]} |M(t)| \in \mathcal{L}^p(\mathbb{P}) \) for every \( p \geq 1 \).

That is to say also the nice integrability properties of \( L \) are carried over to the convoluted process \( M \) for suitable kernel functions.

Throughout this paper we use the following definition:

\[
\mathcal{A}(\mathbb{R}) := \{ \xi \in \mathcal{L}^1(\lambda) : \mathcal{F}\xi \in \mathcal{L}^1(\lambda) \},
\]

where \( \mathcal{F}\xi \) denotes the Fourier transform of \( \xi \). Recall that the functions in \( \mathcal{A}(\mathbb{R}) \) are continuous and bounded.

The main result in the present paper is the following Itô formula. The diamonds indicate that integration is understood in the Hitsuda-Skorokhod sense, and the precise definitions of the these integrals are given in Section 5.2 below.

Theorem 9 Let \( f \in \mathcal{K} \) as well \( G \in C^2(\mathbb{R}) \) such that one of the following assumptions holds:

(i) \( \sigma > 0 \) and \( G, G', G'' \) are of polynomial growth,

(ii) \( G, G', G'' \in \mathcal{A}(\mathbb{R}) \).

Then the Itô formula

\[
G(M(T)) = G(0) + \frac{\sigma^2}{2} \int_0^T G''(M(t)) \left( \frac{d}{dt} \int_{-\infty}^t f(t,s)^2 ds \right) dt \\
+ \sum_{0 < t \leq T} \left[ G(M(t)) - G(M(t-)) - G'(M(t-)) \Delta M(t) \right] \\
+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}_0} \left( G'(M(t)) + x f(t,s) \right) - G'(M(t)) \frac{\partial}{\partial t} f(t,s) \ N^\diamond(dx, ds) \ dt \\
+ \int_0^T G'(M(t)) \ M^\diamond(dt)
\]

holds \( \mathbb{P} \)-a.s. if all the terms exist in \( \mathcal{L}^2(\mathbb{P}) \).

Remark 10 1. For an interpretation of some of the terms occurring in Theorem 9 we refer e.g. to p. 512 in [6].

2. Due to the polynomial growth of \( G, G' \) and \( G'' \) as well as Theorem 8 all the integrands in Theorem 9 are elements of \( \mathcal{L}^2(\mathbb{P}) \). Moreover, the integrals can be rearranged in a way that all terms exist in \( \mathcal{L}^2(\mathbb{P}) \), see Theorem 23 below.

\( \diamond \)

4 Path properties and moments

This section is concerned with the proofs of Proposition 6, Theorem 7 and Theorem 8. We start with the proof of the integration by parts formula which is an important ingredient for the other two proofs.
Proof of Proposition 6
For every \( N \in \mathbb{N} \) we can decompose the process \( M \) as
\[
M(t) = \int_{-\infty}^{\alpha \vee -N} f(t, s) \, L(ds) + \int_{\alpha \vee -N}^{t} f(t, s) \, L(ds).
\]
(11)
The first term clearly tends to 0 as \( N \) goes to infinity. For the second term the (standard) integration by parts formula for fixed \( t \) yields thanks to the absolute continuity of \( f \) in \( s \) (Definition 2(vi))
\[
\int_{\alpha \vee -N}^{t} f(t, s) \, L(ds) = f(t, t)L(t) - f(t, \alpha \vee -N)L(\alpha \vee -N) - \int_{\alpha \vee -N}^{t} L(s) \frac{\partial}{\partial s} f(t, s) \, ds.
\]
(12)
In the case \( \alpha > -\infty \) we have \( \alpha \vee -N = \alpha \) for \( N \) sufficiently large, which proves \( \| \). If instead \( \alpha = -\infty \) we have according to (11) that for \( s \) sufficiently small the estimate
\[
|f(t, s)| \leq |s|^{-\left(\frac{1}{2} + \varepsilon\right)}
\]
holds for some \( \varepsilon > 0 \). Furthermore, note that by means of Gnedenko’s law of the iterated logarithm for Lévy processes, see Proposition 48.9 in [26], there exists some \( \tau : \Omega \to (-\infty, a] \), where \( a \) is given by Definition 2(vi)(b), such that
\[
\lim_{N \to \infty} |f(t, -N)L(-N)| = 0 \quad \mathbb{P}\text{-a.s.}
\]
In the light of (12), taking the limit as \( N \to \infty \) in (11) thus proves the assertion. \( \square \)

Proof of Theorem 7
Since \( f \) is continuous and \( L \) is càdlàg, we only have to deal with the third term of (11) which can be decomposed as follows:
\[
\int_{\alpha}^{t} \frac{\partial}{\partial s} f(t, s) \, ds = \int_{\alpha}^{\tau} L(s) \frac{\partial}{\partial s} f(t, s) \, ds + \int_{\tau}^{t} L(s) \frac{\partial}{\partial s} f(t, s) \, ds =: I(t) + II(t)
\]
with \( \tau \) defined as in the proof of Proposition 6. Fix some \( \varepsilon \in (0, (\sqrt{17} - 3)/4) \) complying with Definition 2(vi)(b). To deal with \( I(t) \) we first observe that Definition 2(vi)(b) yields
\[
sup_{u \in [0, T]} \int_{\alpha}^{\tau} |L(s)|^{1+\varepsilon} \left| \frac{\partial}{\partial s} f(u, s) \right|^{1+\varepsilon} |s|^{(1+\varepsilon)^2} |s|^{-\left(1+\varepsilon\right)} \, ds
\]
\[
= \sup_{u \in [0, T]} \int_{\alpha}^{\tau} |L(s)|^{1+\varepsilon} |s|^{-\left(\frac{1}{2} + \varepsilon\right)(1+\varepsilon)} |s|^{\left(\frac{3}{2} + \varepsilon\right)(1+\varepsilon)^2} \left| \frac{\partial}{\partial s} f(u, s) \right|^{1+\varepsilon} \, ds
\]
\[
\leq C^{1+\varepsilon} \sup_{u \in [0, T]} \int_{\alpha}^{a} |s|^{\left(\frac{3}{2} + \varepsilon\right)(1+\varepsilon)^2} \left| \frac{\partial}{\partial s} f(u, s) \right|^{1+\varepsilon} \, ds
\]
\[
\leq C^{1+\varepsilon} \int_{\alpha}^{a} |s|^{\left(\frac{3}{2} + \varepsilon\right)(1+\varepsilon)^2 - \left(\frac{1}{2} + \varepsilon\right)(1+\varepsilon)} \, ds
\]
\[< \infty,
\]
where $C$ is given by (13) and the finiteness follows from
\[
\left(\frac{1}{2} + \epsilon\right)(1 + \epsilon)^2 - \left(\frac{3}{2} + \epsilon\right)(1 + \epsilon) < -1
\]
which holds, since $\epsilon < (\sqrt{7} - 3)/4$. Set
\[
\Upsilon(t, s) := L(s) \frac{\partial}{\partial s} f(t, s)|s|^{1+\epsilon}.
\]
Consequently, we can use the de La Vallée Poussin theorem to deduce that $(\Upsilon(t, \cdot))_{t \in [0, T]}$ is uniformly integrable with respect to the finite measure $|s|^{-(1+\epsilon)} ds$. Now let $t > 0$ and choose an arbitrary sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to t$ as $n \to \infty$. The convergence $\Upsilon(t_n, s) \to \Upsilon(t, s)$ for $\lambda$-a.a. $s \in (-\infty, \tau]$, cf. Definition 2(vi)(a), together with the uniform integrability results in
\[
\lim_{n \to \infty} \int_0^\tau L(s) \frac{\partial}{\partial s} f(t_n, s) \, ds = \lim_{n \to \infty} \int_0^\tau \Upsilon(t_n, s) \, ds = \int_0^\tau L(s) \frac{\partial}{\partial s} f(t, s) \, ds.
\]
This implies that the mapping $t \mapsto I(t)$ is continuous.

Let us now tackle $I(t)$. To this end, observe that
\[
\sup_{u \in [\rho, t]} |L(u)| < \infty
\]
holds $P$-a.s. for any random variable $\rho : \Omega \to \mathbb{R}$. Indeed, we infer from Doob’s maximal inequality that
\[
E\left( \sup_{u \in [s, t]} |L(u)| \right) < \infty,
\]
and consequently $\sup_{u \in [s, t]} |L(u)| < \infty$ $P$-a.s., for every $s \in \mathbb{R}$. For each $n \in \mathbb{Z}$ we set
\[
\mathcal{N}_n := \left\{ \omega \in \Omega : \sup_{u \in [n, t]} |L(u, \omega)| = \infty \right\},
\]
which is a $P$-null set. Therefore,
\[
\mathcal{N} := \bigcup_{n \in \mathbb{Z}} \mathcal{N}_n
\]
satisfies $P(\mathcal{N}) = 0$. For any $\omega \in \Omega \setminus \mathcal{N}$ we thus have
\[
\sup_{u \in [n, t]} |L(u, \omega)| < \infty
\]
for all $n \in \mathbb{Z}$. Hence, we conclude that
\[
\sup_{u \in [\rho(\omega), t]} |L(u, \omega)| \leq \sup_{u \in [\rho(\omega), t]} |L(u, \omega)| < \infty
\]
holds for all $\omega \in \Omega \setminus \mathcal{N}$ and every random variable $\rho : \Omega \to \mathbb{R}$, which proves (13).

We now deduce that
\[
\sup_{u \in [0, T]} \int_{\tau}^t \left| L(s) \frac{\partial}{\partial s} f(u, s) \right|^{1+\epsilon} \, ds \leq \sup_{s \in [\tau, t]} |L(s)|^{1+\epsilon} \sup_{u \in [0, T]} \int_{\tau}^t \left| \frac{\partial}{\partial s} f(u, s) \right|^{1+\epsilon} \, ds < \infty
\]
for all $\omega \in \Omega \setminus \mathcal{N}$ and every random variable $\rho : \Omega \to \mathbb{R}$, which proves (13).
Applying the Hölder inequality and Fubini’s theorem we obtain
\[ L \text{The moment condition on } f \text{ with the continuity of } f \text{ for all } k \]
continuity of the mapping \( t \mapsto \Pi(t) \), which completes the proof. □

As an immediate consequence of the above proof we have the following corollary that specifies the connection between the jumps of \( M \) and those of \( L \).

**Corollary 11** The jumps of the convoluted Lévy process \((M(t))_{t>0}\) fulfil
\[ \Delta M(t) = f(t) \Delta L(t). \]

**Proof** The statement follows easily from Proposition 6 and the proof of Theorem 7, which states that the third term in \((9)\) has continuous paths. □

**Remark 12** Since the fractional kernel \( f_d \) (cf. Definition 4(iv)) vanishes on the diagonal, we see that the fractional Lévy process \( M^d \) has continuous paths. ◊

**Proof of Theorem 8**
Due to the càdlàg paths of \( M \) (cf. Theorem 7) the process \( M \) is separable and thus \( \sup_{t \in [0,T]} |M(t)|^p \) is measurable. To check the integrability of the above random variable we use Jensen’s inequality and write
\[ |v + w|^p \leq 2^{p-1}(|v|^p + |w|^p) \]
for all \( v, w \in \mathbb{R} \) and every \( p \geq 1 \) and. By means of Proposition 6 we therefore deduce that
\[
\sup_{t \in [0,T]} |M(t)|^p \leq C_p \cdot \left( \sup_{t \in [0,T]} |f(t)|^p \sup_{t \in [0,T]} |L(t)|^p + \mathbb{1}_{\mathbb{R}}(\alpha) \sup_{t \in [0,T]} |f(t, \alpha)|^p |L(\alpha)|^p \right)
\[ \quad + \left( \sup_{t \in [0,T]} \int_a^t |L(s)| \left| \frac{\partial}{\partial s} f(t, s) \right| ds \right)^p + \left( \sup_{t \in [0,T]} \int_a^t |L(s)| \left| \frac{\partial}{\partial s} f(t, s) \right| ds \right)^p \]
\[ =: C_p \cdot (I + II + III + IV) \]
holds with a constant \( C_p > 0 \) and \( a \) chosen according to Definition 2(vi)(b). From Doob’s maximal inequality we infer that
\[ \mathbb{E} \left( \sup_{t \in [0,T]} L(t)^{2k} \right) \leq \left( \frac{2k}{2k-1} \right)^{2k} \mathbb{E}(L(T)^{2k}) < \infty \]
for all \( k \in \mathbb{N} \), where we used that \( L \) is a right-continuous martingale as well as \((\Pi)\). This combined with the continuity of \( f \), cf. Definition 2(ii), yields that \( I \) has a finite expectation.

The moment condition on \( L \) together with the continuity of \( f \) implies the finite expectation of \( II \). Applying the Hölder inequality and Fubini’s theorem we obtain
\[
\mathbb{E}(III) = \mathbb{E} \left( \sup_{t \in [0,T]} \left( \int_a^t |L(s)||s|^{-(\frac{1}{2}+2\varepsilon)} |s|^\frac{1}{2}+2\varepsilon \left| \frac{\partial}{\partial s} f(t, s) \right| ds \right)^p \right)
\[ \leq \left( \int_a^t \mathbb{E}(|L(s)|^p)|s|^{-(\frac{1}{2}+2\varepsilon)p} ds \right) \left( \sup_{t \in [0,T]} \int_a^t |s|^{\frac{1}{2}+2\varepsilon} |s|^q \left| \frac{\partial}{\partial s} f(t, s) \right| ds \right)^\frac{p}{q} \]
(15)
for all $\epsilon > 0$ and $p, q > 1$ such that $p^{-1} + q^{-1} = 1$. By Definition 2(vi)(b) we can choose $\epsilon > 0$ such that

$$
\sup_{t \in [0, T]} \int_{a}^{a} |s|^{(\frac{1}{2} + 2\epsilon)} \left| \frac{\partial}{\partial s} f(t, s) \right|^{q} ds \leq \int_{a}^{a} |s|^{(\frac{1}{2} + 2\epsilon)(\frac{1}{2} + 3\epsilon)} ds = \int_{a}^{a} |s|^{-(1+\epsilon)} ds < \infty,
$$

which shows that the second factor on the right hand side of (15) is finite. To prove the finiteness of the first factor we deduce from (3.33) in [16] and the expressions for the cumulants in (4) that for any even $p \geq 4$ the estimate

$$
\mathbb{E}(L(s)^p) \leq K_p |s|^p
$$

holds for some constant $K_p > 0$. Therefore, for even $p > \frac{1}{2\epsilon} \lor 2$ we conclude

$$
\int_{a}^{a} \mathbb{E}(|L(s)|^p) |s|^{-(\frac{1}{2} + 2\epsilon)p} ds \leq K_p \int_{a}^{a} |s|^{-2p} ds < \infty.
$$

In view of (15) this shows that III has finite expectation.

Resorting to Definition 2(vi)(c) and Doob’s maximal inequality we deduce

$$
\mathbb{E}(IV) \leq \mathbb{E} \left( \sup_{s \in [a, T]} |L(s)|^p \right) \left( \sup_{t \in [0, T]} \int_{a}^{t} \left| \frac{\partial}{\partial s} f(t, s) \right| ds \right)^p < \infty,
$$

which completes the proof. $\square$

5 S-transform and Hitsuda-Skorokhod integrals

In this section we make precise the definition of the Hitsuda-Skorokhod integrals which appear in our Itô formula. This definition builds on the injectivity of the Segal-Bargman transform (in short, S-transform)

5.1 The Segal-Bargmann transform

We first recall the definition of the Segal-Bargmann transform, which is a tool from white noise analysis. We shall heavily make use of the injectivity property, which implies that a square-integrable random variable is uniquely determined by its S-transform.

We first introduce a set $\Xi$ by

$$
\Xi := \text{span} \{ g : \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R} : g(x, t) = g_1(x)g_2(t) \text{ for two functions such that } \}
\quad g_1^* \in \mathcal{L}^1(\nu) \cap \mathcal{L}^2(\nu) \text{ and } g_2 \in \mathcal{S} \}.
$$

Here $\mathcal{S}$ is the Schwartz space of rapidly decreasing smooth functions and for any function $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ we define $h^*$ by

$$
h^*(x, t) := xh(x, t)
$$

for all $x, t \in \mathbb{R}$. 

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Remark 13 Let $g \in \Xi$ be given by $g(x,t) = \sum_{j=1}^{N} \mu_j g_{1,j}(x) g_{2,j}(t)$. Using the abbreviations

$$g_1(x) := \sum_{j=1}^{N} |\mu_j g_{1,j}(x)| \quad \text{and} \quad g_2(t) := \sum_{j=1}^{N} |g_{2,j}(t)|$$

we see easily that $g_1^* \in \mathcal{L}^1(\nu) \cap \mathcal{L}^2(\nu)$ and that $\sup_{t \in \mathbb{R}} |g_2(t)p(t)|$ is finite for every polynomial $p$ as well as

$$|g(x,t)| \leq g_1(x) g_2(t)$$

for every $x \in \mathbb{R}_0$ and all $t \in \mathbb{R}$. We will make use of this simple estimate in our subsequent calculations.

For every $n \in \mathbb{N}$ let $I_n$ be the $n$-th order multiple Lévy-Itô integral (with respect to $L$), see e.g. page 665 in [27]. For any $g \in \mathcal{L}^2(x^2 \nu(dx) \times \lambda(dt))$ let $g^{\otimes n}$, $n \in \mathbb{N} \cup \{0\}$, be the $n$-fold tensor product of $g$ and define a measure $Q_g$ on $(\Omega, \mathcal{F})$ by the change of measure

$$dQ_g = \exp^{\diamond}(I_1(g)) dP,$$

where the Radon-Nikodým derivative is the Wick exponential of the random variable $I_1(g)$:

$$\exp^{\diamond}(I_1(g)) := \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!}.$$ (16)

However, in this paper we shall not be concerned with Wick calculus. Let us point out that it follows from

$$\mathbb{E} \left( \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!} \right) = 1$$

that $Q_g$ is a signed probability measure. In the following, $\mathbb{E}^{Q_g}$ denotes the expectation under $Q_g$.

Let us mention that according to [19], Theorem 5.8, we have for $g \in \mathcal{L}^2(x^2 \nu(dx) \times \lambda(dt))$ with $g^* \in \mathcal{L}^1(\nu(dx) \times \lambda(dt))$ that

$$\exp^{\diamond}(I_1(g)) = \exp \left\{ \sigma \int_{\mathbb{R}} g(0,t) W(dt) - \frac{\sigma^2}{2} \int_{\mathbb{R}} g(0,t)^2 dt - \int_{\mathbb{R}} \int_{\mathbb{R}_0} g^*(x,t) \nu(dx) dt \right\} \cdot \prod_{t: \Delta L(t) \neq 0} (1 + g^*(\Delta L(t), t)),$$

which equals the Doléans-Dade exponential of $I_1(g)$ at infinity.

Using Proposition 1.4 and formula (10.3) in [12] and the fact that the Brownian part and the jump part are independent, we infer by applying the Cauchy-Schwarz inequality that there exists a constant $e_g > 0$ (only depending on $g$) such that

$$\mathbb{E}^{Q_g}(\|X\|) \leq e_g \cdot \mathbb{E}(\|X\|^2)^{1/2} \cdot \mathbb{E} \left( \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!} \right)^{1/2} \leq e_g \cdot \mathbb{E}(\|X\|^2)^{1/2}$$ (17)
holds for every $X \in \mathcal{L}^2(P)$.

In order to define the Hitsuda-Skorokhod integrals that appear in Theorem 23, we first define the Segal-Bargmann transform on $\mathcal{L}^2(P)$.

**Definition 14** For every $\varphi \in \mathcal{L}^2(P)$ its Segal-Bargmann transform (subsequently referred to as $S$-transform) $S\varphi$ is given as an integral transform on the set $\mathcal{L}^2(x^2\nu(dx) \times \lambda(dt))$ by

$$S\varphi(g) := \mathbb{E}^Q g(\varphi).$$

The following injectivity result for the $S$-transform provides us with a key property for both the definition of Hitsuda-Skorokhod integrals and the proof of Theorem 23.

**Proposition 15** Let $\varphi, \psi \in \mathcal{L}^2(P)$. If $S\varphi(g) = S\psi(g)$ for all $g \in \Xi$, then we have $\varphi = \psi$ $P$-almost surely.

**Proof** Let $g \in \mathcal{L}^2(x^2\nu(dx) \times \lambda(dt))$. In view of $\Xi$ being dense in $\mathcal{L}^2(x^2\nu(dx) \times \lambda(dt))$ we choose a sequence $(g_m)_{m \in \mathbb{N}}$ in $\Xi$ with $g_m \to g$ as $m \to \infty$. Using Remark 5.9 in [19] we obtain from $S\varphi = S\psi$ on $\Xi$ that

$$S\varphi(g) = \lim_{m \to \infty} S\varphi(g_m) = \lim_{m \to \infty} S\psi(g_m) = S\psi(g).$$

Consequently, the two $S$-transforms $S\varphi$ and $S\psi$ coincide on $\mathcal{L}^2(x^2\nu(dx) \times \lambda(dt))$. By [21], p. 16, we deduce that $\varphi$ and $\psi$ coincide $P$-almost surely. Let us point out that the results of [19] and [21], to which we resorted above, were obtained for the so-called white noise probability space but also hold true for a general probability space as in our setting. □

### 5.2 Hitsuda-Skorokhod integrals

Due to the above injectivity property of the $S$-transform we can use it to define integration with respect to convoluted Lévy processes generalising the approach in [4] and [6].

The motivation for our approach of defining Hitsuda-Skorokhod integrals lies in the fact that under suitable integrability and predictability assumptions on the integrand they reduce to the well known stochastic integrals with respect to semimartingales and random measures, respectively.

**Definition 16** Suppose the mapping $t \mapsto S(M(t))(g)$ is differentiable for every $g \in \Xi$, let $\mathcal{B} \subset [0, \infty)$ be a Borel set and let $X : \mathcal{B} \times \Omega \to \mathbb{R}$ be a stochastic process such that $X(t)$ is square-integrable for each $t \in \mathcal{B}$. The process $X$ is said to have a Hitsuda-Skorokhod integral with respect to $M$ if there is a $\Phi \in \mathcal{L}^2(P)$ such that

$$S\Phi(g) = \int_{\mathcal{B}} S(X(t))(g) \frac{d}{dt} S(M(t))(g) \, dt$$

holds for all $g \in \Xi$. As the $S$-transform is injective, $\Phi$ is unique and we write

$$\Phi = \int_{\mathcal{B}} X(t) \, M^c(dt).$$

**Remark 17** A different approach of defining Skorokhod integrals with respect to fractional Lévy processes via white noise analysis can be found in [23]. If the fractional Lévy process is of finite $p$-variation, stochastic integrals with respect to this process can be defined pathwise as an improper
Riemann-Stieltjes integral and have been considered in [14]. In the special case that \( M \) is a Lévy process, it can be shown with the techniques in [12] that our definition of Hitsuda-Skorokhod integrals coincides with the definition of Skorokhod integrals via the chaos decomposition. An approach to Skorokhod integrals with respect to convoluted Poisson processes via Malliavin calculus is provided in [11].

The following explicit formula for the derivative of the \( S \)-transform of \( M(t) \) will be of use later on in the proof of Theorem 23. This formula particularly yields that the differentiability condition on the mapping \( t \mapsto S(M(t))(g) \) in Definition 16 is fulfilled for kernel functions \( f \in K \).

**Lemma 18** For all \( f \in K \) and \( g \in \Xi \) the mapping \( t \mapsto S(M(t))(g) \) is continuously differentiable on the set \((0, \infty)\) with derivative

\[
\frac{d}{dt} S(M(t))(g) = \sigma \left( f(t,t)g(0,t) + \int_{-\infty}^{t} \frac{\partial}{\partial t} f(t,s)g(0,s) \, ds \right) + f(t,t) \int_{\mathbb{R}_0} xg^*(x,t) \nu(dx) + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t,s)xg^*(x,s) \nu(dx) \, ds.
\]

Note that the finiteness of the integrals appearing in the above lemma is guaranteed by Definition 2(v).

**Proof** By the isometry of Wiener-Itô integrals we obtain

\[
S(M(t))(g) = \sigma \int_{-\infty}^{t} f(t,s)g(0,s) \, ds + \int_{-\infty}^{t} \int_{\mathbb{R}_0} f(t,s)xg^*(x,s) \nu(dx) \, ds
\]

(cf. Section 3.1 of [4] and Example 3.6 in [6]). To simplify the notation we prove the assertion for the right derivative. The left derivative can be handled analogously. In order to differentiate the first term of (18) let \( h > 0 \) and consider

\[
\frac{1}{h} \left( \int_{-\infty}^{t+h} \sigma f(t+h,s)g(0,s) \, ds - \int_{-\infty}^{t} \sigma f(t,s)g(0,s) \, ds \right)
\]

\[
= \frac{1}{h} \int_{t}^{t+h} \sigma f(t+h,s)g(0,s) \, ds + \int_{-\infty}^{t} \sigma f(t+h,s) - f(t,s) \frac{h}{h}g(0,s) \, ds
\]

\[=: I + II.
\]

By means of the continuity of \( f \) and \( g(0, \cdot) \) the term I is easy to handle. Indeed,

\[
\sigma \inf_{s \in [t,t+h]} (f(t+h,s)g(0,s)) \leq I \leq \sigma \sup_{s \in [t,t+h]} (f(t+h,s)g(0,s)),
\]

where both sides of the inequality converge to

\[
\sigma f(t,t)g(0,t)
\]

as \( h \) tends to zero from above.

Let us now deal with II. The Mean Value Theorem together with Definition 2(v) yields
\[
\int_{-\infty}^{t} \sup_{h \in (0,1)} \left| \frac{f(t+h,s) - f(t,s)}{h} g(0, s) \right| \, ds \leq \int_{-\infty}^{t} \sup_{u \in [t,t+1]} \left| \frac{\partial}{\partial u} f(u,s) g(0,s) \right| \, ds \\
\leq \int_{-\infty}^{t} C |s|^\beta |t-s|^{-\gamma} |g(0,s)| \, ds < \infty,
\]

and similarly for the limit \( h \uparrow 0 \). By means of the differentiability of \( f \) we can apply the Dominated Convergence Theorem (DCT for short) and thus get

\[
\lim_{h \to 0} \Pi = \int_{-\infty}^{t} \sigma \frac{\partial}{\partial t} f(t,s) g(0,s) \, ds.
\]

Regarding the second term of (18) we apply the same techniques as for the first term and make use of Remark 13 to obtain the expression for the derivative of \( S(M(t))(g) \). Furthermore, by using similar arguments as above, we see that the mapping \( t \mapsto S(M(t))(g) \) is continuous. \( \square \)

We proceed by introducing a Hitsuda-Skorokhod integral that will enable us to establish a connection between the Hitsuda-Skorokhod integral with respect to the convoluted Lévy process \( M \) and the classical integral with respect to the underlying Lévy process \( L \) (see Theorem 21 below).

**Definition 19** Let \( B \subset \mathbb{R} \) be a Borel set and \( X : \mathbb{R} \times B \times \Omega \to \mathbb{R} \) be a square integrable random field.

The **Hitsuda-Skorokhod integral** of \( X \) with respect to the random measure

\[
\Lambda(dx,dt) = x\tilde{N}(dx,d\tau) + \sigma \delta_0(dx) \otimes W(dt),
\]

where \( \delta_0 \) denotes the Dirac measure in 0, is said to exist in \( L^2(\mathbb{P}) \), if there is a random variable \( \Phi \in L^2(\mathbb{P}) \) that satisfies

\[
S\Phi(g) = \int_{B} \int_{\mathbb{R}} S(X(x,t))(g) g^*(x,t) x \cdot \nu(dx) \, dt + \sigma \int_{B} S(X(0,t))(g) g(0,t) \, dt
\]

for all \( g \in \Xi \). In this case \( \Phi \) is uniquely determined by Proposition 15 and we write

\[
\Phi = \int_{B} \int_{\mathbb{R}} X(x,t) \Lambda^\circ(dx,d\tau).
\]

**Remark 20** Let \( X : \mathbb{R} \times [0,T] \times \Omega \to \mathbb{R} \) be a predictable, square integrable random field.

1. Assume that \( \sigma > 0 \) and let \( X \) be given by

\[
X(x,\cdot) = \begin{cases} \frac{1}{\sigma} Y(\cdot), & x = 0 \\ 0, & x \neq 0 \end{cases}
\]

for some stochastic process \( Y : [0,T] \times \Omega \to \mathbb{R} \). Since \( X \) is predictable, we infer from Theorem 3.1 in [1] that the Hitsuda-Skorokhod integral \( \int_{0}^{T} \int_{\mathbb{R}} X(x,t) \Lambda^\circ(dx,dt) \) exists and satisfies

\[
\int_{0}^{T} \int_{\mathbb{R}} X(x,t) \Lambda^\circ(dx,dt) = \int_{0}^{T} Y(t) \, W(dt),
\]
where the last integral is the classical stochastic integral with respect to the Brownian motion $W$. Note that it follows from the calculations in Section 3.1 of [4] that

$$g(0, t) = \frac{d}{dt} \int_0^t g(0, s) \, ds = \frac{d}{dt} S(W(t))(g)$$

(19)

and hence we have

$$S \left( \int_0^T \int_\mathbb{R} X(x, t) \Lambda^\circ(dx, dt) \right)(g) = \int_0^T S(Y(t))(g) \frac{d}{dt} S(W(t))(g) \, dt.$$  

(20)

2. If $X$ fulfills $X(0, \cdot) \equiv 0$, then it follows from Theorem 3.5 in [6] that

$$\int_0^T \int_\mathbb{R} X(x, t) \Lambda^\circ(dx, dt) = \int_0^T \int_{\mathbb{R}_0} xX(x, t) \tilde{N}(dx, dt),$$

where the last integral is the classical stochastic integral with respect to the compensated Poisson jump measure $\tilde{N}$.

3. According to 1. and 2. we have

$$\int_0^T \int_\mathbb{R} X(x, t) \Lambda^\circ(dx, dt) = \int_0^T \int_{\mathbb{R}_0} xX(x, t) \tilde{N}(dx, dt) + \sigma \int_0^T X(0, t) \, W(dt).$$

$\diamond$

Now we are in the position to state the connection between the Hitsuda-Skorokhod integral with respect to $M$ and the Itô integral with respect to the driving Lévy process. It is a direct consequence of Lemma 18 and Remark 20.

**Theorem 21** Suppose that $X$ is a predictable and square integrable process. Then,

$$\int_0^T X(t) \, M^\circ(dt)$$

exists, if and only if

$$\int_0^T \int_{\mathbb{R}_0} \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t, s)X(t) \Lambda^\circ(dx, ds) \, dt$$

exists. In this case

$$\int_0^T X(t) \, M^\circ(dt) = \int_0^T f(t, t)X(t) \, L(dt) + \int_0^T \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t, s)X(t) \Lambda^\circ(dx, ds) \, dt.$$  

(21)

Now considering the jump measure $N$ instead of the compensated jump measure $\tilde{N}$ naturally leads to the following definition by adding the $S$-transform of the integral with respect to the compensator. The corresponding Hitsuda-Skorokhod integral appears in Theorem 9.

**Definition 22** Let $B \subset \mathbb{R}$ be a Borel set and $X : \mathbb{R}_0 \times B \times \Omega \rightarrow \mathbb{R}$ be a square integrable random field. The **Hitsuda-Skorokhod integral** of $X$ with respect to the jump measure $N(dx, dt)$ is said to exist in $\mathcal{L}^2(\mathbb{P})$, if there is a (unique) random variable $\Phi \in \mathcal{L}^2(\mathbb{P})$ that satisfies

$$S\Phi(g) = \int_B \int_{\mathbb{R}_0} S(X(x, t))(g)(1 + g^\circ(x, t)) \, \nu(dx) \, dt$$

for all $g \in \Xi$. We write

$$\Phi = \int_B \int_{\mathbb{R}_0} X(x, t) \, N^\circ(dx, dt).$$
6 Itô formulas

This section is devoted to proving the Itô formula in Theorem 9. We first present a different formulation of the Itô formula which has the advantage that all the terms occurring in that formula exist in $L^2(\mathbb{P})$. The proof of Theorem 9 is then based on rearranging the terms in the Itô formula below. The advantage of the formula in Theorem 9 is the more intuitive interpretation of the separate terms.

**Theorem 23** Let $f \in K$ as well $G \in C^2(\mathbb{R})$ such that one of the following assumptions holds:

(i) $\sigma > 0$ and $G, G', G''$ are of polynomial growth,

(ii) $G, G', G'' \in A(\mathbb{R})$.

Then the Itô formula

\[
G(M(T)) = G(0) + \sigma^2 \int_0^T G''(M(t)) \left( \frac{d}{dt} \int_{-\infty}^t f(t,s)^2 ds \right) dt \\
+ \sum_{0 < t \leq T} \left[ G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t) \right] \\
+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} G'(M(t) + xf(t,s)) \frac{\partial}{\partial t} f(t,s) \Lambda^o(dx,ds) \ dt \\
+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left( G'(M(t) + xf(t,s)) - G'(M(t)) \right) x \frac{\partial}{\partial t} f(t,s) \nu(dx) \ ds \ dt \\
+ \int_0^T G'(M(t-))f(t,t) \ L(dt)
\]

(22)

holds $\mathbb{P}$-almost surely. In particular, all the integrals and the sum appearing in (22) exist in $L^2(\mathbb{P})$.

**Remark 24** Note that the choice $f(t,s) = 1_{[0,t]}(s)$ for $s, t \geq 0$ results in the time derivative of the kernel being zero such that the integrals involving $\frac{\partial}{\partial t} f$ vanish. In that case the Itô formula (22) is just the same as the familiar one for the Lévy process $L$. 

In order to prove Theorem 23 we start with a heuristic calculation that gives a rough outline of the steps of our proof and motivates the auxiliary results that we shall prove below. Suppose the Fundamental Theorem of Calculus enables us to write

\[
S(G(M(T)))(g) = G(0) + \int_0^T \frac{d}{dt} S(G(M(t)))(g) \ dt. \tag{23}
\]

Subsequently, by making use of the Fourier inversion theorem we could obtain

\[
S(G(M(t)))(g) = \mathbb{E}^{Q_0}(G(M(t))) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{Q_0}(e^{iuM(t)}) \ du. \tag{24}
\]

Differentiating the right hand side of (24) with respect to $t$, using some standard manipulations of the Fourier transform, plugging the resulting formula for $\frac{d}{dt} S(G(M(t)))(g)$ into (23) and using
the injectivity of the $S$-transform would give an explicit expression for $G(M(T))$ leading to an Itô formula.

Our approach to proving Theorem 23 is based on several auxiliary results. More precisely, following the above motivation we derive explicit expressions for the characteristic functions $\mathbb{E}(e^{iuM(t)})$ and $\mathbb{E}_Q(e^{iuM(t)})$ as well as the derivative $\frac{\partial}{\partial t}\mathbb{E}_Q(e^{iuM(t)})$ in Lemma 25, Proposition 26 and Lemma 27. In (the proof of) Proposition 30 we will show that the integral appearing in (24) is well-defined and that the mapping $t \mapsto S(G(M(t)))(g)$ is indeed differentiable. We then complete the proof of Theorem 23 by using the explicit expression for $\frac{\partial}{\partial t}\mathbb{E}_Q(e^{iuM(t)})$ and the injectivity of the $S$-transform.

Let us now follow our approach by calculating the characteristic function of $M(t)$.

**Lemma 25** For every $f \in K$ and $t \geq 0$ we have

\[
\mathbb{E}(e^{iuM(t)}) = \exp\left(-\frac{\sigma^2 u^2}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right) \nu(dx) \, ds\right).
\]

**Proof** A proof of this kind of result is given in [25], see Propositions 2.6 and 2.7. Nonetheless, in our situation we prefer to give a more elementary proof. Recall the Lévy-Khintchine representation (3). In particular, this implies that $(L_{r}(t))_{t \in \mathbb{R}}$, given by

\[
L_r(t) = L(t) - L(r) = \gamma(t - r) + \sigma(W(t) - W(r)) + \int_{r}^{t} \int_{\mathbb{R}_0} x \left[N(dx, ds) - \nu(dx) \, ds\right]
\]

for any $r \in \mathbb{R}$, is a semimartingale with characteristics

\[
(0, \sigma^2(t - r), 1_{[r,t]}(s) \nu(dx))
\]

Consequently, the characteristic function of $L_r(t)$ is

\[
\mathbb{E}(e^{iuL_r(t)}) = \exp\left(-\frac{\sigma^2 u^2}{2} (t - r) + \int_{r}^{t} \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right) \nu(dx) \, ds\right).
\]

Let $(f_{N,+(t,\cdot)})_{N \in \mathbb{N}}$ and $(f_{N,-(t,\cdot)})_{N \in \mathbb{N}}$ be a sequences of step functions

\[
f_{N,\pm}(t,s) = \sum_{j=1}^{N} \mathbb{1}_{(t_j^N,t_{j+1}^N]}(s) \alpha_{j,\pm}^N(t)
\]

such that $f_{N,\pm}(t,\cdot) \uparrow f_{\pm}(t,\cdot)$ $\lambda$-a.e. as $N \to \infty$ and define

\[
\alpha_j^N(t) = \alpha_{j,+}^N(t) - \alpha_{j,-}^N(t)
\]

for all $N \in \mathbb{N}$, $j = 1, \ldots, N$ and $t \geq 0$. In view of the Monotone Convergence Theorem this implies that $f_+^N(t,\cdot) \uparrow f_{\pm}^2(t,\cdot)$ in $L^2(\lambda)$ as $N \to \infty$. Resorting to the independence of the increments of $L$ this results in

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\[ \mathbb{E} \left( e^{iuM(t)} \right) = \mathbb{E} \left( \exp \left( iu \mathcal{L}^2 \lim_{N \to \infty} \int_{-\infty}^{t} f_N(t,s) \, L(ds) \right) \right) \\
= \lim_{k \to \infty} \prod_{j=1}^{N_k} e^{iu \omega_j(t)}(L(t_{j+1}) - L(t_j)) \\
= \lim_{k \to \infty} \prod_{j=1}^{N_k} \mathbb{E} \left( e^{iu \omega_j(t)} L(t_j(t_{j+1})) \right), \]

for some sequence \((N_k)_{k \in \mathbb{N}}\) in \(\mathbb{N}\), where we used the DCT in the last step. Using \([20]\) we see that the last expression equals

\[
\lim_{k \to \infty} \exp \left( \sum_{j=1}^{N_k} \left( -\frac{\sigma^2 (u \omega_j N_k(t))^2}{2} (t_{j+1} - t_j) + \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}_0} \left( e^{iu \omega_j N_k(t)} - 1 - iu \omega_j N_k(t) \right) \nu(dx) \, ds \right) \right) \\
= \lim_{k \to \infty} \exp \left( -\frac{\sigma^2 u^2}{2} \int_{-\infty}^{t} f_N(t,s)^2 \, ds + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \left( e^{iu f_N(t,s)} - 1 - iu f_N(t,s) \right) \nu(dx) \, ds \right) \\
= \exp \left( -\frac{\sigma^2 u^2}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \left( e^{iu f(t,s)} - 1 - iu f(t,s) \right) (1 + g^*(x,s)) \nu(dx) \, ds \right). 
\]

Note that interchanging the limit and the integrals in the last step is justified by the DCT and the fact that the mapping \(z \mapsto \frac{e^{iu z} - 1 - iuz}{z^2}\) is bounded. \(\square\)

The following proposition is concerned with the characteristic function of \(M(t)\) under the signed measure \(\mathcal{Q}_g\).

**Proposition 26** For every \(f \in K\) and \(g \in \Xi\) we have

\[
\mathbb{E} \mathcal{Q}_g \left( e^{iuM(t)} \right) \\
= \exp \left( iu \int_{-\infty}^{t} \sigma^2 f(t,s) g(0,s) \, ds + iu \int_{-\infty}^{t} \int_{\mathbb{R}_0} x f(t,s) g^*(x,s) \nu(dx) \, ds \right) \\
- \frac{\sigma^2 u^2}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \left( e^{iu f(t,s)} - 1 - iu f(t,s) \right) (1 + g^*(x,s)) \nu(dx) \, ds. \tag{27} 
\]

**Proof** Recall that

\[ M(t) = I_1(f(t, \cdot)). \tag{28} \]

Approximating \(f(t, \cdot)\) by the sequence of functions \((f_n(t, \cdot))_{n \in \mathbb{N}}\) in \(\mathcal{L}^1(\lambda) \cap \mathcal{L}^\infty(\lambda)\) defined via \(f_n(t, \cdot) := 1_{[-n,t]}(\cdot) f(t, \cdot)\) and using Theorem 2.9 in \([21]\) we see that \(\mathbb{P}\text{-a.s.}\) the equation

\[ e^{I_1(iuf_n(t, \cdot))} = \mathbb{E} \left( e^{I_1(iuf_n(t, \cdot))} \right) \exp^\circ (I_1(k_{t,n})) \]

holds with

\[ k_{t,n}(x,s) := 1_{[0]}(x) iuf_n(t,s) + 1_{\mathbb{R}_0}(x) \frac{e^{iu f_n(t,s)} - 1}{x} \]

\[ = 1_{[-n,0]}(s) k_t(x,s), \]

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where \( k_t \) is given by \( k_t(x, s) := \mathbb{1}_{\{0\}}(x) iuf(t, s) + \mathbb{1}_{\mathbb{R}_0}(x) \frac{x^{iuM(t)(x)} - 1}{x} \). In view of (16) this yields

\[
\mathbb{E}^g \left( e^{I_1(\mathcal{g}f_n(t, \cdot))} \right) = \mathbb{E} \left( e^{I_1(\mathcal{g}f_n(t, \cdot))} \right) \mathbb{E} \left( \exp^\circ(I_1(g)) \exp^\circ(I_1(k_{t, n})) \right).
\]

To deal with the product of the two Wick exponentials we apply Proposition 3.2 in [22] to obtain

\[
\mathbb{E}^g \left( e^{I_1(\mathcal{g}f_n(t, \cdot))} \right) = \mathbb{E} \left( e^{I_1(\mathcal{g}f_n(t, \cdot))} \right) \exp \left( \int_{-\infty}^t \int_R g(x, s) k_t(x, s) \tilde{\nu}(dx, ds) \right) \mathbb{E} \left( \exp^\circ(I_1(g + k_{t, n} + (gk_{t, n})^*)) \right),
\]

where the measure \( \tilde{\nu} \) is defined as \( \tilde{\nu}(dx, ds) = x^2 \nu(dx) \times ds \) if \( x \neq 0 \) and \( \tilde{\nu}(dx, ds) = \sigma^2 \lambda(ds) \) if \( x = 0 \). Since the expectation of the Wick exponential on the right hand side of (29) equals 1, resorting to (28) and (29), taking the limit as \( n \to \infty \) and using the DCT results in

\[
\mathbb{E}^g \left( e^{iuM(t)} \right) = \mathbb{E} \left( e^{iuM(t)} \right) \exp \left( \int_{-\infty}^t \int_R g(x, s) k_t(x, s) \tilde{\nu}(dx, ds) \right).
\]

Plugging in the formula for \( \mathbb{E}(e^{iuM(t)}) \) from Lemma 25 completes the proof.

In the spirit of Lemma 18 we now derive a formula for the derivative of the \( S \)-transform of \( e^{iuM(t)} \). This result makes use of Proposition 26.

**Lemma 27** For all \( f \in \mathcal{K} \), \( g \in \Xi \) and \( u > 0 \) the map \( t \mapsto \mathbb{E}^g \left( e^{iuM(t)} \right) \) is continuously differentiable with derivative

\[
\frac{\partial}{\partial t} \mathbb{E}^g \left( e^{iuM(t)} \right) = \mathbb{E}^g \left( e^{iuM(t)} \right) \left[ \begin{array}{l}
\frac{iu}{dt} S(M(t))(g) \\
\frac{\sigma^2 u^2}{2} \left( f(t, t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t, s) \cdot f(t, s) \ ds \right) \\
+ \int_{\mathbb{R}_0} \left( e^{iuM(t, t)} - 1 - iuxf(t, t) \right) \left( 1 + g^*(x, t) \right) \nu(dx) \\
+ \int_{-\infty}^t \int_{\mathbb{R}_0} \left( iux \frac{\partial}{\partial t} f(t, s) \left( e^{iuM(t, s)} - 1 \right) \left( 1 + g^*(x, s) \right) \right) \nu(dx) \ ds \end{array} \right].
\]

**Proof** By the differentiability of the exponential function we only have to prove the differentiability of the terms in the exponential of (27). The first two of these summands are easily identified as the terms that occur in (18) and are already treated in Lemma 18. Moreover, the differentiability of the third summand in the exponential of (27) can be proven by techniques similar to those in the proof of Lemma 18 and the details are therefore omitted.

To deal with the fourth summand in the exponential of (27) we consider the right derivative and assume \( u > 0 \). The left derivative and the case \( u < 0 \) (the case \( u = 0 \) is trivial) can be handled analogously. We obtain
To deal with \( I(h) \) for some constant \( C > 0 \) we can apply the DCT to the last expression in (30). This results in

\[
\frac{1}{h} \left( \int_{-\infty}^{t+h} R_0 \left( e^{iuxf(t+h,s)} - 1 - iuxf(t+h,s) \right) (1 + g^*(x,s)) \right) \nu(dx) \ ds
\]

\[
- \int_{t}^{t+h} R_0 \left( e^{iuxf(t,s)} - 1 - iuxf(t,s) \right) (1 + g^*(x,s)) \nu(dx) \ ds
\]

\[
= \frac{1}{h} \int_{t}^{t+h} R_0 \left( e^{iuxf(t+h,s)} - iuxf(t+h,s) - (1 + g^*(x,s)) \nu(dx) \ ds
\]

\[
+ \int_{t}^{t+h} R_0 \left( e^{iuxf(t,s)} - iuxf(t+h,s) - f(t,s) \right) \nu(dx) \ ds
\]

\[
=: I(h) + II(h)
\]

for any \( h > 0 \).

To handle \( I(h) \) we assume without loss of generality \( h \leq 1 \) and consider the estimate

\[
\left| I(h) - \int_{R_0} \left( e^{iuxf(t,t)} - 1 - iuxf(t,t) \right) (1 + g^*(x,t)) \nu(dx) \right|
\]

\[
= \left| \int_{R_0} \frac{1}{h} \left( e^{iuxf(t+h,s)} - 1 - iuxf(t+h,s) \right) \nu(dx) \right|
\]

\[
\leq \int_{R_0} \frac{1}{h} \left( e^{iuxf(t+h,s)} - 1 - iuxf(t+h,s) \right) \nu(dx) \sup_{(t_0,s_0) \in [t,t+h]^2} \left| \left( e^{iuxf(t_0,s_0)} - 1 - iuxf(t_0,s_0) \right) \right|
\]

\[
(1 + g^*(x,s)) \ ds \nu(dx)
\]

\[
= \int_{R_0} \sup_{(t_0,s_0) \in [t,t+h]^2} \left| \left( e^{iuxf(t_0,s_0)} - 1 - iuxf(t_0,s_0) \right) \right|
\]

\[
(1 + g^*(x,s)) \right| \nu(dx)
\]

Using the estimate

\[
e^{iy} - 1 - iy \leq Cy^2
\]

for some constant \( C > 0 \) and all \( y \in \mathbb{R} \) as well as the continuity of \( f \) and Remark 13 we see that we can apply the DCT to the last expression in (30). This results in

\[
\lim_{h \to 0} I(h) = \int_{R_0} \left( e^{iuxf(t,t)} - 1 - iuxf(t,t) \right) (1 + g^*(x,t)) \nu(dx).
\]

To deal with \( II(h) \) we apply the Mean Value Theorem in conjunction with Definition 13 and 14.
as well as Remark 13 to write

\[
\int_{-\infty}^{\ell} \int_{\mathbb{R}} \sup_{h \in (0,1)} \left| e^{iux(t+h,s)} - e^{iux(t,s)} - iux(f(t+h,s) - f(t,s)) \right| \frac{(1 + g^*(x,s))}{h} \nu(dx) \, ds
\]

\[
\leq \int_{-\infty}^{\ell} \int_{\mathbb{R}} \sup_{r \in (t,t+1)} \left| iux \frac{\partial}{\partial r} f(r,s) \left( e^{iux(r,s)} - 1 \right) \right| (1 + g^*(x,s)) \nu(dx) \, ds
\]

\[
\leq 2u^2 \sup_{y \in \mathbb{R}} \frac{|\gamma - 1|}{y} \int_{\mathbb{R}} x^2 (1 \vee g_1^2(x)) \nu(dx) \int_{-\infty}^{t} \sup_{r \in (t,t+1)} |f(r,s)| \cdot C |s|^\beta |t-s|^{-\gamma} (1 \vee g_2(s)) \, ds
\]

\[
\leq 2u^2 \sup_{y \in \mathbb{R}} \frac{|\gamma - 1|}{y} \int_{\mathbb{R}} x^2 (1 \vee g_1^2(x)) \nu(dx) \cdot C \sup_{s \in [-\infty,t]} \sup_{r \in (t,t+1)} \left( |f(r,s)||s|^{-\eta} \right) \cdot \int_{-\infty}^{t} |s|^\theta |t-s|^{-\gamma} (1 \vee g_2(s)) \, ds
\]

\[
< \infty,
\]

where \( C, \beta, \gamma \) and \( \theta \) are given by Definition 2. Consequently, we can apply the DCT and obtain that

\[
\lim_{h \to 0} \Pi(h) = \int_{-\infty}^{\ell} \int_{\mathbb{R}} iux \frac{\partial}{\partial t} f(t,s) \left( e^{iux(t,s)} - 1 \right) (1 + g^*(x,s)) \nu(dx) \, ds.
\]

Using the same considerations for the limit from below we obtain the desired expression for the derivative. Along the lines of the above argument we deduce that the function \( t \mapsto \frac{\partial}{\partial t} \mathbb{E}^{Q_\theta} (e^{iuM(t)}) \) is continuous.

The proof of the following result is based on Theorem 8 and generalises the first part of Proposition 4.2 in [20].

**Proposition 28** Suppose \( L \) has a nontrivial Gaussian part (i.e. \( \sigma > 0 \)). Then, for every \( t > 0 \), the mapping \( u \mapsto \mathbb{E}^{Q_\theta} (e^{iuM(t)}) \) is a Schwartz function on \( \mathbb{R} \).

**Proof** The proof is divided into two parts. The first part deals with the derivative of the map \( u \mapsto \mathbb{E}^{Q_\theta} (e^{iuM(t)}) \), which is then used in the second part to prove the assertion.

**Part 1** At first we show by induction that the above mapping is smooth with \( j \)-th derivative, \( j \in \mathbb{N} \cup \{0\} \), given by

\[
\frac{d^j}{du^j} \mathbb{E}^{Q_\theta} (e^{iuM(t)}) = \mathbb{E}^{Q_\theta} \left( i^j M(t)^j e^{iuM(t)} \right).
\]

For \( j = 0 \) the assertion is trivial. Now let the statement hold for some \( k \in \mathbb{N} \). For the purpose of interchanging differentiation and integration we consider

\[
\sup_{h \in (0,\infty)} \left| \frac{i^k M(t)^k e^{i(u+h)M(t)} - e^{iuM(t)}}{h} \right| = \left| M(t)^k \right| \sup_{h \in (0,\infty)} \left| \frac{M(t) e^{i \theta h M(t)} - 1}{h M(t)} \right| \cdot |e^{iuM(t)}|
\]

\[
\leq \left| M(t)^{k+1} \right| \sup_{x \in \mathbb{R}} \left| \frac{e^{ix} - 1}{x} \right|.
\]

Since the last supremum is finite, the term on the right hand side is bounded by \( C |M(t)^{k+1}| \) for some \( C > 0 \). In the light of (17) this yields

\[
\mathbb{E}^{Q_\theta} \left( \sup_{h \in (0,\infty)} \left| \frac{i^k M(t)^k e^{i(u+h)M(t)} - e^{iuM(t)}}{h} \right| \right) \leq \mathbb{E}^{Q_\theta} \left( C |M(t)^{k+1}| \right) \leq c \mathbb{E} \left( \left| M(t)^{2(k+1)} \right| \right)^{1/2},
\]

23
which is finite according to Theorem 8, where \(\epsilon\) is a constant depending only on \(g\). Therefore, we can apply the DCT in order to interchange differentiation and integration and obtain

\[
\frac{d^{k+1}}{du^{k+1}} \mathbb{E}_g \left( e^{iuM(t)} \right) = \frac{d}{du} \mathbb{E}_g \left( e^{iM(t)k e^{iuM(t)}} \right) = \lim_{h \downarrow 0} \mathbb{E}_g \left( \frac{e^{iM(t)(k+h) e^{iuM(t)}} - e^{iM(t)k e^{iuM(t)}}}{h} \right)
\]

\[
= \mathbb{E}_g \left( e^{iM(t)(k+1) e^{iuM(t)}} \right),
\]

where the first equality follows from the induction hypothesis. This proves (31) and hence finishes the first part of the proof.

**Part II** It remains to show that for all \(m, n \in \mathbb{N} \cup \{0\}\) the expression

\[
\left| u^n \frac{d^m}{du^m} \mathbb{E}_g \left( e^{iuM(t)} \right) \right| (32)
\]

is bounded in \(u\). In view of Proposition 26 we start by writing

\[
\mathbb{E}_g \left( e^{iuM(t)} \right) = \exp \left( -\frac{\sigma^2}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds \cdot u^2 \right) \cdot R_{g,t}(u) (33)
\]

with \(R_{g,t}(u)\) given by

\[
R_{g,t}(u) = \exp \left( iu \int_{-\infty}^{t} f(t,s)(\sigma^2 g(0,s)) \, ds \right) \cdot \mathbb{E}_g \left( e^{iM_j(t)} \right),
\]

where the process \(M_j\) is constructed analogously to \(M\) by using the characteristic triple \((\gamma,0,\nu)\) instead of \((\gamma,\sigma,\nu)\). Applying the arguments of Part I to the process \((M_j(t))_{t \in \mathbb{R}}\) we infer that \(R_{g,t}\) has bounded derivatives of every order.

Since \(\frac{\sigma}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds > 0\) according to Definition 2(iv), the mapping

\[
u \mapsto \exp \left( -\frac{\sigma^2}{2} \int_{-\infty}^{t} f(t,s)^2 \, ds \cdot u^2 \right)
\]

is a Schwartz function and thus (32) is bounded in \(u\), which completes the proof. \(\square\)

**Remark 29** Let \(t \in \mathbb{R}\). Using the estimates

\[
\sup_{s \in [t-1,t+1]} \left| \exp \left( \frac{\sigma^2}{2} \int_{-\infty}^{s} f(s,r)^2 \, dr \cdot u^2 \right) \right| = \exp \left( -\frac{\sigma^2}{2} \int_{-\infty}^{s_0} f(s_0,r)^2 \, dr \cdot u^2 \right) \leq \exp \left( -Cu^2 \right)
\]

for some \(s_0 \in [t-1,t+1]\) and \(C > 0\), cf. Definition 2(iv), as well as

\[
|R_{g,t}(u)| \leq 1
\]

we see in view of (33) that the function

\[u \mapsto \sup_{s \in [t-1,t+1]} \mathbb{E}_g \left( e^{iuM(s)} \right)\]

is an element of \(L^2(\lambda)\), provided \(L\) has a nontrivial Gaussian part. \(\Diamond\)
Our starting point in the proof of Theorem \[23\] is the equation
\[
S(G(M(T)))(g) = G(0) + \int_0^T \frac{d}{dt} S(G(M(t)))(g) \, dt, \tag{34}
\]
which follows from the Fundamental Theorem of Calculus if the function \( t \mapsto S(G(M(t)))(g) \) is continuously differentiable. Therefore, we proceed by proving the existence of \( \frac{d}{dt} S(G(M(t)))(g) \).

**Proposition 30** Let \( f \in \mathcal{K} \) as well \( G \in C^2(\mathbb{R}) \) such that one of the following assumptions holds:
a) \( \sigma > 0 \) and \( G \) has compact support,
b) \( G, G', G'' \in \mathcal{A}(\mathbb{R}) \).

Then \( S(G(M(\cdot)))(g) \) is continuously differentiable with derivative
\[
\frac{d}{dt} S(G(M(t)))(g) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{\partial}{\partial t} EQ_s \left( e^{iuM(t)} \right) \, du.
\]

**Proof** Recall that the Fourier transform of a function \( f \in L^1(\mathbb{R}) \) is denoted by \( \mathcal{F}f \). By means of Fourier inversion theorem we deduce that
\[
S(G(M(t)))(g) = EQ_s(G(M(t))) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) EQ_s \left( e^{iuM(t)} \right) \, du \tag{35}
\]
holds, if condition \( b) \) holds. For a function \( G \) fulfilling condition \( a) \) we use a standard approximation via convolution with a \( C^\infty \)-function with compact support and deduce by means of the DCT that \( \text{(35)} \) also holds true in that case. Hence, in both cases we have
\[
\sqrt{2\pi} \frac{S(G(M(t + h)))(g) - S(G(M(t)))(g)}{h} = \int_{\mathbb{R}} (\mathcal{F}G)(u) EQ_s \left( e^{iu(M(t+h))} - e^{iuM(t)} \right) \, du. \tag{36}
\]
Moreover, Resorting to Lemma \[27\] we infer from the Mean Value Theorem that
\[
\sup_{h \in [-1,1] \setminus \{0\}} \left| (\mathcal{F}G)(u) EQ_s \left( e^{iu(M(t+h))} - e^{iuM(t)} \right) \right| \leq |(\mathcal{F}G)(u)| \cdot \sup_{s \in [t-1,t+1]} \left| \frac{\partial}{\partial t} EQ_s \left( e^{iuM(s)} \right) \right|
\]
\[
= |(\mathcal{F}G)(u)| \cdot \sup_{s \in [t-1,t+1]} \left| EQ_s \left( e^{iuM(s)} \right) (I(s) + II(s) + III(s) + IV(s)) \right|,
\]
where the terms \( I(s), II(s), III(s) \) and \( IV(s) \) are given by
\[
I(s) = iu \frac{d}{ds} S(M(s))(g),
\]
\[
II(s) = -\frac{\sigma^2 u^2}{2} \left( f(s,s)^2 + 2 \int_{-\infty}^{s} \frac{\partial}{\partial s} f(s,r) \cdot f(s,r) \, dr \right),
\]
\[
III(s) = \int_{\mathcal{R}_0} \left( e^{iuxf(s,s)} - 1 - iuxf(s,s) \right) (1 + g^*(x,s)) \, \nu(dx)
\]
and
\[
IV(s) = \int_{-\infty}^{s} \int_{\mathcal{R}_0} \left( iux \frac{\partial}{\partial s} f(s,r) \left( e^{iuxf(s,r)} - 1 \right) (1 + g^*(x,r)) \right) \, \nu(dx) \, dr.
\]
Hence,

\[
\sup_{h \in [-1,1] \setminus \{0\}} \left| (FG)(u) \frac{\mathbb{E} Q_s(e^{iuM(t+h)}) - \mathbb{E} Q_s(e^{iuM(t)})}{h} \right| \\
\leq \sup_{s \in [t-1,t+1]} \left| \mathbb{E} Q_s(e^{iuM(s)}) \right| \\
\cdot \sup_{s \in [t-1,t+1]} \left( \left| (FG')(u) \right| \cdot \left| \frac{d}{ds} S(M(s))(g) \right| + \left| (FG'')(u) \right| \cdot \sup_{s \in [t-1,t+1]} \mathcal{E}(s) \right),
\]

where \(\mathcal{E}(s)\) denotes the expression in the square brackets. We now want to prove that under assumption a) or b) the left hand side of (37) is in \(L^1(\lambda)\):

a) If \(\sigma > 0\) and \(G\) has compact support, then in particular \(G', G'' \in L^1 \cap L^2(\lambda)\) and \(FG', FG'' \in L^2(\lambda)\). Since \(s \mapsto \mathcal{E}(s)\) is continuous (cf. Proposition 27), we deduce that

\[
\sup_{s \in [t-1,t+1]} \mathcal{E}(s) < \infty.
\]

According to Lemma 18 also \(s \mapsto \frac{d}{ds} S(M(s))(g)\) is continuous and thus

\[
\sup_{s \in [t-1,t+1]} \left| \frac{d}{ds} S(M(s))(g) \right| < \infty.
\]

In view of Remark 29 we thus infer that both factors on the right hand side of (37) are elements of \(L^2(\lambda)\). By means of the Cauchy-Schwarz inequality it follows that

\[
u(t) \mapsto \sup_{h \in [-1,1] \setminus \{0\}} \left| (FG)(u) \frac{\mathbb{E} Q_s(e^{iuM(t+h)}) - \mathbb{E} Q_s(e^{iuM(t)})}{h} \right| \in L^1(\lambda).
\]

b) Note first that

\[
\sup_{u \in \mathbb{R}} \sup_{s \in [t-1,t+1]} \left| \mathbb{E} Q_s(e^{iuM(s)}) \right| \leq \sup_{u \in \mathbb{R}} \sup_{s \in [t-1,t+1]} \left| \mathbb{E} Q_s(e^{iuM(s)}) \right| = 1.
\]

Moreover, if \(G', G'' \in A(\mathbb{R})\), then \(FG', FG'' \in L^1(\lambda)\). Therefore, in the light of (38) and (39) we conclude that (10) holds.

In both cases we are now able to apply the DCT to (36) and thus infer that \(t \mapsto S(G(M(t)))(g)\) is differentiable and
\[
\frac{d}{dt} S(G(M(t)))(g) = \lim_{h \to 0} \frac{S(G(M(t+h)))(g) - S(G(M(t)))(g)}{h}
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \lim_{h \to 0} \frac{\mathbb{E}_q \left( e^{iuM(t+h)} \right) - \mathbb{E}_q \left( e^{iuM(t)} \right) }{h} du
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{\partial}{\partial t} \mathbb{E}_q \left( e^{iuM(t)} \right) du.
\]

The continuity of the derivative can be proved by using Lemma 27 and the estimates in (37). □

We are now in a position to give the proof of our main result.

**Proof of Theorem 23** The proof is divided into two parts. In the first part we show that the desired formula holds if either

a) \( \sigma > 0 \) and \( G \) has compact support, or

b) \( G, G', G'' \in \mathcal{A}(\mathbb{R}) \).

In the second part we are concerned with proving Theorem 23 (i), i.e. we deal with the situation that \( \sigma > 0 \) and \( G, G' \) and \( G'' \) are of polynomial growth. To this end, we approximate such functions \( G \) by functions that satisfy a) and for which the formula is shown in Part I to hold true.

**Part I** Let \( G \) satisfy a) or b) above. By means of Proposition 30 and Lemma 27 we obtain

\[
S(G(M(T)))(g) - S(G(M(0)))(g) = \int_0^T \frac{d}{dt} S(G(M(t)))(g) \, dt
\]

(41)

with the terms \( I(t), II(t), III(t) \) and \( IV(t) \) given by

\[
I(t) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}_q \left( e^{iuM(t)} \right) \cdot iu \frac{d}{dt} S(M(t))(g) \, du,
\]

\[
II(t) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{-\sigma^2 u^2}{2} \mathbb{E}_q \left( e^{iuM(t)} \right) \left( f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial r} f(t,s) f(t,s) \, ds \right) du,
\]

\[
III(t) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}_q \left( e^{iuM(t)} \right) \int_{\mathbb{R}_0} \left( e^{ixf(t,t)} - 1 - iux f(t,t) \right) \left( 1 + g^*(x,t) \right) \nu(dx) \, du
\]

and

\[
IV(t) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}_q \left( e^{iuM(t)} \right) \cdot \int_{-\infty}^t \int_{\mathbb{R}_0} \left( iux \frac{\partial}{\partial r} f(t,s) \left( e^{ixf(t,s)} - 1 \right) \right)
\]

\[
\cdot \left( 1 + g^*(x,s) \right) \nu(dx) \, ds \, du.
\]

We exemplarily give the argument for \( III(t) \). For this term we have

\[
III(t) = \int_{\mathbb{R}_0} \int_{\mathbb{R}} (\mathcal{F}G)(u) \left( e^{ixf(t,t)} - 1 - iux f(t,t) \right) \left( 1 + g^*(x,t) \right) \cdot \mathbb{E}_q \left( e^{iuM(t)} \right) du \, \nu(dx),
\]

where we used Fubini’s theorem. Note that this is possible, since we have the estimate.
\[
\left| e^{iu\xi f(t,t)} - 1 - iu \xi f(t,t) \right| \leq C u^2 x^2 f(t,t)^2
\]
for some constant \( C > 0 \), and every \( u \in \mathbb{R} \) and \( x \in \mathbb{R}_0 \). By using standard manipulations of the Fourier transform we derive

\[
\text{III}(t) = \int_{\mathbb{R}_0} \int_{\mathbb{R}} \mathcal{F} \left( G(\cdot + x f(t,t)) - G(\cdot) - x f(t,t)G'(\cdot) \right)(u) \cdot \mathbb{E}^\mathcal{Q}_g \left( e^{i u M(t)} \right) (1 + g^*(x,t)) \, du \, \nu(dx).
\]

Consequently, by applying (35) we obtain

\[
\text{III}(t) = \int_{\mathbb{R}_0} \sqrt{2\pi} S \left( G(M(t) + x f(t,t)) - G(M(t)) - x f(t,t)G'(M(t)) \right)(g) (1 + g^*(x,t)) \, \nu(dx).
\]

For the terms corresponding to \( \text{I}(t) \), \( \text{II}(t) \) and \( \text{IV}(t) \) similar techniques apply and result in

\[
\text{I}(t) = \sqrt{2\pi} S(G'(M(t)))(g) \frac{d}{dt} S(M(t))(g)
\]

and

\[
\text{II}(t) = \frac{\sigma^2}{2} \left( f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, ds \right) \int_{\mathbb{R}} (FG'')(u) \mathbb{E}^\mathcal{Q}_g \left( e^{i u M(t)} \right) \, du
\]

\[
= \frac{\sigma^2}{2} \left( f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, ds \right) \sqrt{2\pi} S(G''(M(t)))(g)
\]

as well as

\[
\text{IV}(t) = \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) \sqrt{2\pi} S(G'(M(t) + x f(t,s)) - G'(M(t)))(g) \cdot (1 + g^*(x,s)) \, \nu(dx) \, ds.
\]

Therefore, plugging the above expressions into (31) we obtain

\[
S(G(M(T)))(g) - S(G(M(0)))(g)
= \int_0^T S(G'(M(t)))(g) \frac{d}{dt} S(M(t))(g) \, dt
\]

\[
\quad + \int_0^T \frac{\sigma^2}{2} \left( f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, ds \right) S(G''(M(t)))(g) \, dt
\]

\[
\quad + \int_0^T \int_{\mathbb{R}_0} S \left( G(M(t) + x f(t,t)) - G(M(t)) - x f(t,t)G'(M(t)) \right)(g) \cdot (1 + g^*(x,t)) \, \nu(dx) \, dt
\]

\[
\quad + \int_0^T \int_{-\infty}^t x \frac{\partial}{\partial t} f(t,s) S(G'(M(t) + x f(t,s)) - G'(M(t)))(g) \cdot (1 + g^*(x,s)) \, \nu(dx) \, ds \, dt
\]

\[
=: \text{I}^* + \text{II}^* + \text{III}^* + \text{IV}^*.
\]
Using Lemma 18 yields for $I^*$ the following equality

\[
\int_0^T S(G'(M(t)))(g) \frac{d}{dt} S(M(t))(g) \, dt
= \int_0^T S(G'(M(t)))(g) \sigma \int_{-\infty}^t \frac{\partial}{\partial t} f(t, s) g(0, s) \, ds \, dt
+ \int_0^T S(G'(M(t)))(g) \sigma f(t, t) g(0, t) \, dt
+ \int_0^T S(G'(M(t)))(g) f(t, t) \int_{\mathbb{R}_0} x g^*(x, t) \nu(dx) \, dt
+ \int_0^T S(G'(M(t)))(g) \int_{-\infty}^t \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t, s) x g^*(x, s) \nu(dx) \, ds \, dt
=: I^*_a + I^*_b + I^*_c + I^*_d.
\]

Resorting to Remark 20 we obtain

\[
I^*_b = S \left( \int_0^T G'(M(t-)) f(t, t) W(dt) \right)(g)
\]
as well as

\[
I^*_c = S \left( \int_0^T \int_{\mathbb{R}_0} G'(M(t-)) f(t, t) x \tilde{N}(dx, dt) \right)(g),
\]
where we used that both integrals exist separately and reduce to classical stochastic integrals because $t$ is $Q_g$-a.s. not a jump time and therefore $S(G'(M(t-)))(g) = S(G'(M(t)))(g)$ as well as the predictability of the integrands. In particular, we infer

\[
I^*_b + I^*_c = S \left( \int_0^T G'(M(t-)) f(t, t) L(dt) \right)(g).
\]

In view of the linearity of the $S$-transform the term $II^*$ can be written as

\[
II^* = S \left( \frac{\sigma^2}{2} \int_0^T (f(t, t))^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t, s) \cdot f(t, s) \, ds \right) G''(M(t)) \, dt \right)(g).
\]

For the third term we obtain

\[
III^* = S \left( \int_0^T \int_{\mathbb{R}_0} \left( G(M(t-)) + x f(t, t) \right) - G(M(t-)) - xf(t, t)G'(M(t-)) \right) N^\circ(dx, dt) \right)(g) 
= S \left( \sum_{0 < t \leq T} G(M(t)) - G(M(t-)) - G'(M(t-)) \Delta M(t) \right)(g).
\]

Note that the predictability of the integrand implies that the Hitsuda-Skorokhod integral is an Itô integral and hence the second equality above holds by means of Corollary 11.

Using the linearity of the $S$-transform we deduce that
\[ IV^* = \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) S(G'(M(t) + xf(t,s)) - G'(M(t)))(g) \cdot (1 + g^*(x,s)) \, \nu(dx) \, ds \, dt \\
= \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) S(G'(M(t)) + xf(t,s))(g) \cdot g^*(x,s) \, \nu(dx) \, ds \, dt \\
- \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) S(G'(M(t)))(g) \cdot g^*(x,s) \, \nu(dx) \, ds \, dt \\
+ S \left( \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) \left( G'(M(t) + xf(t,s)) - G'(M(t)) \right) \nu(dx) \, ds \, dt \right)(g) \\
=: IV_{a}^{**} + IV_{b}^{**} + IV_{c}^{**}.
\]

Observe that \( I_{d}^{**} \) equals the negative of \( IV_{i}^{**} \). Therefore we obtain in view of (41) that

\[
I_{a}^{**} + IV_{a}^{**} = S(G(M(T)) - G(0))(g) \\
- S \left( \frac{\sigma^2}{2} \int_0^T \left( f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, ds \right) G''(M(t)) \, dt \right)(g) \\
- S \left( \sum_{0 < j \leq T} G(M(t)) - G(M(-)) - G'(M(-)) \Delta M(t) \right)(g) \\
- S \left( \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t,s) \left( G'(M(t) + xf(t,s)) - G'(M(t)) \right) x \cdot \nu(dx) \, ds \, dt \right)(g) \\
- S \left( \int_0^T G'(M(-)) f(t,t) \, L(dt) \right)(g).
\]

By linearity of the \( S \)-transform \( I_{a}^{**} + IV_{a}^{**} \) equals the \( S \)-transform of some \( \Phi \in L^2(\mathbb{P}) \) and can thus, by Definition 11, be written as

\[
S \left( \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} \frac{\partial}{\partial t} f(t,s) \left( G'(M(t) + xf(t,s)) \right) \Lambda^0(dx,ds) \, dt \right)(g).
\]

Hence, reordering the terms in (41) and resorting to the injectivity property of the \( S \)-transform (see Lemma 12), this results in the change of variable formula (22).

**Part II** We consider now the case that \( \sigma > 0 \) and \( G, G' \) and \( G'' \) are of polynomial growth, that is we have

\[
|G(x)| + |G'(x)| + |G''(x)| \leq |P(x)|
\]

for some polynomial \( P \) that we can assume to be of the form \( P(x) = C(x^j + 1) \) for \( j \in \mathbb{N} \) and \( C > 0 \).

Our approach is to approximate such \( G \) by functions fulfilling condition a) above as we know from Part I of this proof that the Itô formula holds true for these functions. To this end, let \( (G_n)_{n \in \mathbb{N}} \) be
a sequence of functions \( G_n \in C^2(\mathbb{R}) \) such that

1. \( G_n(u) = G(u), \ u \in [-n, n] \),
2. \( G_n \) has compact support and
3. \( |G_n(x)| + |G_n'(x)| + |G_n''(x)| \leq |\mathcal{P}(x)| \) for all \( x \in \mathbb{R} \).

Below we denote by \( (22)_n \) and \( (45)_n \) the formulas \( (22) \) and \( (45) \) with \( G \) replaced by \( G_n \). In Part I we showed that the Itô formula \( (22)_n \) holds for any \( n \in \mathbb{N} \). It remains to show that this formula also holds true in the limit. For this purpose we need to interchange the limits and the integrals on the right hand side of \( (22)_n \). We first show the argument for the last term on the right hand side of \( (22)_n \). Resorting to the Itô isometry for Lévy processes we obtain

\[
\mathbb{E} \left( \int_0^T (G'(M(t)) - G_n'(M(t))) f(t, t') L(dt') \right) = \sigma^2 \int_0^T \mathbb{E} \left( (G'(M(t)) - G_n'(M(t)))^2 \right) f(t, t')^2 dt
+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E} \left( (G'(M(t)) - G_n'(M(t)))^2 \right) f(t, t')^2 x^2 \nu(dx) dt.
\]

Since \( G_n = G \) on \([-n, n]\), it follows that \( |G_n(M(t)) - G(M(t))| \leq \varepsilon \) implies \( M(t) \geq n \). Consequently,

\[
\mathbb{P} \left( \sup_{t \in [0, T]} |G_n(M(t)) - G(M(t))| \geq \varepsilon \right) \leq \mathbb{P} \left( \sup_{t \in [0, T]} |M(t)| \geq n \right).
\]

This shows that

\[
\sup_{t \in [0, T]} |G_n(M(t)) - G(M(t))| \to 0 \quad (47)
\]

in probability as \( n \to \infty \). Applying the DCT in \( (46) \) thus yields

\[
\mathbb{E} \left( \int_0^T (G'(M(t)) - G_n'(M(t))) f(t, t') L(dt') \right) \to 0
\]
as \( n \to \infty \), which proves the desired convergence of the last term on the right hand side of \( (22)_n \).

Let us now deal with the penultimate term on the right hand side of \( (22)_n \). For this purpose we consider a Taylor expansion of zeroth order of \( G_n \), \( n \in \mathbb{N} \), around \( M(t) \) at \( M(t) + x(t, s) \). With \( R_0 \) denoting the corresponding remainder term, this results in

\[
|G_n'(M(t)) + x f(t, s) - G_n'(M(t))| = |R_0(G_n, M(t) + f(t, s)) |
\leq |G_n''(z_0)| \cdot |x f(t, s)|
\leq |\mathcal{P}(z_0)| \cdot |x f(t, s)|
\leq ([\mathcal{P}(M(t))] + [\mathcal{P}(M(t) + x f(t, s))] ) \cdot |x f(t, s)|
\]

for some \( z_0 \in (M(t) \land M(t) + x f(t, s), M(t) \lor M(t) + x f(t, s)) \), where we used the special form of the polynomial \( \mathcal{P} \). Hence, we deduce that

\[
\int_{\mathbb{R}} \int_{\mathbb{R}_0} \sup_{t \in [0, T]} \sup_{n \in \mathbb{N}} \left| G_n'(M(t) + x f(t, s)) - G_n'(M(t)) \right| |x| \left| \frac{\partial}{\partial t} f(t, s) \right| \nu(dx) ds < \infty.
\]

\(^4\) Details of the construction of such functions are available on request.
Analogously to (47) we see

$$\sup_{t \in [0,T]} |G'(M(t) + xf(t,s)) - G'(M(t)) - (G'_n(M(t) + xf(t,s)) - G'_n(M(t)))| \to 0$$

in probability as $n \to \infty$. By means of the DCT we thus conclude

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left( G'(M(t) + xf(t,s)) - G'(M(t)) \right) x \frac{\partial}{\partial t} f(t,s) \nu(dx) ds dt = \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \left( G'(M(t)) - G'(M(t)) \right) x \frac{\partial}{\partial t} f(t,s) \nu(dx) ds dt.$$

Similar arguments also apply to the first integral and the sum on the right hand side of (22), with an integral representation for the sum in the spirit of (43). It remains to consider the $\Lambda^\diamond$-integral. Note in view of (17) and the convergence in $L^2(\mathbb{P})$ shown above that the right hand side of (45)$_n$ converges to the right hand side of (45) as $n \to \infty$. As for the left hand side of (45)$_n$, observe that (17) and the uniform bound of $G'_n$ by $\mathcal{P}$ as well as similar arguments as above, in conjunction with the DCT, show that the left hand side of (45)$_n$ converges to the left hand side of (45) as $n \to \infty$. In view of the linearity of the $S$-transform we thus deduce analogously to Part I that the left hand side of (45) equals

$$S \left( \int_0^T \int_{-\infty}^t \int_{\mathbb{R}} \frac{\partial}{\partial t} f(t,s) \left( G'(M(t)) + xf(t,s) \right) \Lambda^\diamond(dx,ds) dt \right) (g),$$

which completes the proof. □

We finish off this section of proofs with the proof of our second Itô formula.

**Proof of Theorem 9.** We apply the $S$-transform to the right hand side of (22) and (10), respectively, and make use of Theorem 21. After rearranging the resulting terms and using the injectivity of the $S$-transform (cf. Proposition 15) we deduce that both right hand sides coincide, provided all the terms exist in $L^2(\mathbb{P})$. □

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