IN SDP RELAXATIONS, INACCURATE SOLVERS DO ROBUST OPTIMIZATION

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ABSTRACT. We interpret some wrong results (due to numerical inaccuracies) already observed when solving SDP-relaxations for polynomial optimization on a double precision floating point SDP solver. It turns out that this behavior can be explained and justified satisfactorily by a relatively simple paradigm. In such a situation, the SDP solver (and not the user) performs some “robust optimization” without being told to do so. Instead of solving the original optimization problem with nominal criterion \( f \), it uses a new criterion \( \tilde{f} \) which belongs to a ball \( B_\infty(f, \varepsilon) \) of small radius \( \varepsilon > 0 \), centered at the nominal criterion \( f \) in the parameter space. In other words the resulting procedure can be viewed as a “max – min” robust optimization problem with two players (the solver which maximizes on \( B_\infty(f, \varepsilon) \) and the user who minimizes over the original decision variables). A mathematical rationale behind this “autonomous” behavior is described.

1. INTRODUCTION

Certified optimization algorithms provide a way to ensure the safety of several systems in engineering sciences, program analysis as well as cyber-physical critical components. Since these systems often involve nonlinear functions, such as polynomials, it is highly desirable to design certified polynomial optimization schemes and to be able to interpret the behaviors of numerical solvers implementing these schemes. Wrong results (due to numerical inaccuracies) in some output results from semidefinite programming (SDP) solvers have been observed in quite different applications, and notably in recent applications of the Moment-SOS hierarchy for solving polynomial optimization problems, see e.g., [20, 19]. In fact this particular application has even become a source of illustrating examples for potential pathological behavior of SDP solvers [16]. An intuitive mathematical rationale for the wrong results has been already provided informally in [5] and [13], but not a satisfactory picture for the whole process.

An immediate and irrefutable negative conclusion is that double precision floating point SDP solvers are not robust and cannot be trusted as they sometimes provide wrong results in these so-called “pathological” cases. The present paper (with a voluntarily provocative title) is an attempt to provide a different and more positive viewpoint around the interpretation of such inaccuracies in SDP solvers, at least when applying the Moment-SOS hierarchy of semidefinite relaxations in polynomial optimization as described in [7, 9].

We claim that in such a situation, in fact the floating point SDP solver (and not the user) is precisely doing some robust optimization, without being told to

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1
do so. It solves a “max − min” problem in a two-player zero-sum game where the solver is the leader who maximizes (over some ball of radius $\varepsilon > 0$) in the parameter space of the criterion, and the user is a “follower” who minimizes over the original decision variables. In traditional robust optimization, one solves the “min − max” problem where the user (now the leader) minimizes to find a “robust decision variable”, whereas the SDP solver (now the follower) maximizes in the same ball of the parameter space. In this convex relaxation case, both min − max and max − min problems give the same solution. So it is fair to say that the solver is doing what the optimizer should have done in robust optimization.

As an active (and even leader) player of this game, the floating point SDP solver can also play with its two parameters which are (a) the threshold level for eigenvalues to declare a matrix positive semidefinite, and (b) the tolerance level at which to declare a linear equality constraint to be satisfied. Indeed, the result of the “max − min” game strongly depends on the absolute value of both levels, as well as on their relative values.

Of course and so far, this viewpoint which provides a more positive view of inaccurate results from semidefinite solvers, is proper to the context of semidefinite relaxations for polynomial optimization. Indeed in such a context we can exploit a mathematical rationale to explain and support this view. However a topic of further investigation is to try to extend this viewpoint to a larger class of semidefinite programs and perhaps the canonical form of SDPs:

$$\min_{X} \{ \langle F_0, X \rangle : \langle F_\alpha, X \rangle = c_\alpha; \ X \succeq 0 \},$$

in which case the SDP solver would solve the robust optimization problem

$$\max_{\tilde{c} \in B_{\infty}(c, \varepsilon)} \min_{X} \{ \langle F_0, X \rangle : \langle F_\alpha, X \rangle = \tilde{c}_\alpha; \ X \succeq 0 \},$$

where $\langle \cdot \rangle$ stands for the matrix trace and “$\succeq 0$” means positive semidefinite.

2. SDP solvers and the Moment-SOS hierarchy

**Notation.** For a fixed $j \in \mathbb{N}$, let us note $\mathbb{R}[x]_{2j}$ the set of polynomials of degree at most $2j$ and $S_{n,j}$ the set of real symmetric matrices of size $\binom{n+j}{n}$. For any real symmetric matrix $M$, denote by $\|M\|_*$ its nuclear norm and recall that if $M \succeq 0$ then $\|M\|_* = \langle I, M \rangle$. We also note $\Sigma[x]_j$ for the convex cone of SOS polynomials of degree at most $2j$. In the sequel, we will use a generalization of Von Neumann’s minimax theorem, namely the following Sion’s minimax theorem [17]:

**Theorem 2.1.** Let $B$ be a compact convex subset of a linear topological space and $Y$ be a convex subset of a linear topological space. If $h$ is a real-valued function on $B \times Y$ with $h(b, \cdot)$ lower semi-continuous and quasi-convex on $Y$, for all $b \in B$ and $h(\cdot, y)$ upper semi-continuous and quasi-concave on $B$, for all $y \in Y$, then

$$\max_{b \in B} \inf_{y \in Y} h(b, y) = \inf_{y \in Y} \max_{b \in B} h(b, y).$$

The Moment-SOS hierarchy was introduced in [7] to solve the global polynomial optimization problem

$$P : \ f^* = \min_{x} \{ f(x) : x \in K \},$$
where \( f \) is a polynomial and \( K := \{ x \in \mathbb{R}^n : g_l(x) \geq 0, l = 1, \ldots, m \} \) is a basic closed semi-algebraic set with \( (g_l) \subset \mathbb{R}[x] \). Let us note \( g_0 := 1 \) and \( d_\ell := \deg g_\ell \), for each \( \ell = 0, \ldots, m \).

A systematic numerical scheme consists of solving a hierarchy of convex relaxations:

\[
P^j : \quad \rho^j = \min_y \{ L_\gamma(f) : y_0 = 1; y \in C_j(g_1, \ldots, g_m) \},
\]

whose dual reads

\[
D^j : \quad \delta^j = \max_\lambda \{ \lambda : f - \lambda \in C_j(g_1, \ldots, g_m)^* \},
\]

where \( (C_j(g_1, \ldots, g_m)^*)_{d \in \mathbb{N}} \subset \mathbb{R}[x] \) is a nested family of convex cones contained in \( C(K) \), the convex cone of polynomials nonnegative on \( K \), and \( L_\gamma : \mathbb{R}[x] \rightarrow \mathbb{R} \) is the Riesz Linear functional:

\[
f \quad (= \sum_\alpha f_\alpha x^\alpha) \mapsto L_\gamma(f) = \sum_\alpha f_\alpha y_\alpha.
\]

When \( C_j(g_1, \ldots, g_m)^* \) comes from an appropriate SOS-based (Putinar) representation of polynomials positive on \( K \), both \( P^j \) and \( D^j \) are \textit{semidefinite programs} (SDP).

When \( K \) is compact then (under a weak archimedean condition), \( \rho^j = \delta^j \uparrow f^* \) as \( j \rightarrow \infty \), and generically the convergence is even finite \([15]\), i.e., \( f^* = \rho^j \) for some \( j \in \mathbb{N} \). In such case, one may also extract global minimizers from an optimal solution of the corresponding semidefinite relaxation \( P^j \) \([14]\). For more details on the Moment-SOS hierarchy, the interested reader is referred to \([9]\).

At step \( j \) in the hierarchy, one has to solve the SDP-relaxation \( P_j \), for which efficient modern softwares are available. These numerical solvers all rely on interior-point methods, and are implemented either in double precision arithmetics, e.g., SeDuMi \([18]\], SDPA \([21]\), Mosek \([1]\), or with arbitrary precision arithmetics, e.g., SDPA-GMP \([12]\). When relying on such numerical frameworks, the input data considered by solvers might differ from the ones given by the user. Thus the input data, consisting of the cost vector and matrices, are subject to uncertainties. In \([4]\) the authors study semidefinite programs whose input data depend on some unknown but bounded perturbation parameters. For the reader interested in robust optimization in general, we refer to \([2]\).

2.1. \textbf{Two examples of surprising phenomenons.} In general, when applied for solving \( P \), the Moment-SOS hierarchy \([7]\) is quite efficient, modulo its scalability (indeed for large size problems one has to exploit sparsity often encountered in the description of \( P \)). However, in some cases, some quite surprising phenomena have been observed and provided additional support to the pessimistic and irrefutable conclusion that: \textit{Results returned by double precision floating point SDP solvers cannot be trusted as they are sometimes completely wrong}.

Let us briefly describe two such phenomena, already analyzed and commented in \([20, 13]\).

\textbf{Case 1:} When \( K = \mathbb{R}^n \) (unconstrained optimization) then the Moment-SOS hierarchy collapses to the single SDP \( \rho^d = \max_\lambda \{ \lambda : f - \lambda \in \Sigma[x]_d \} \) (with \( 2d \) being the degree of \( f \)). Equivalently, one solves the semidefinite program:

\[
D^d : \quad \rho^d = \max_{X \geq 0, \lambda} \{ \lambda : f_\alpha - \lambda 1_{\alpha = 0} = (X, B_\alpha), \quad \alpha \in \mathbb{N}_0^d \}
\]

for some appropriate real symmetric matrices \((B_\alpha)_{\alpha \in \mathbb{N}_0^d} \); see e.g. \([7]\).
Only two cases can happen: if $f - f^* \in \Sigma[x]_d$ then $\rho^d = f^*$ and $\rho^d < f^*$ otherwise (with possibly $\rho^d = -\infty$). Solving $\rho^j = \max \{ \lambda : f - \lambda \in \Sigma[x]_j \}$ for $j > d$ is useless as it would yield $\rho^j = \rho^d$ because if $f - f^*$ is SOS, then it has to be in $\Sigma[x]_d \subset \Sigma[x]_j$ anyway.

The Motzkin-like polynomial $x \mapsto f(x) = x^2y^2(x^2 + y^2 - 1) + 1/27$ is nonnegative (with $d = 3$ and $f^* = 0$) and has 4 global minimizers, but the polynomial $x \mapsto f(x) - f^*(= f)$ is not an SOS and $\rho^j = -\infty$, which also implies $\rho^j = -\infty$ for all $j$. However, as already observed in [5], by solving (2.3) with $j = 8$ and a double precision floating point SDP solver, we obtain $\rho_8 \approx -10^{-4}$. In addition, one may extract 4 global minimizers close to the global minimizers of $f$ up to four digits of precision! The same occurs with $j > 8$ and the higher is $j$ the better is the result. So undoubtly the SDP solver is returning a wrong solution as $f - \rho^j$ cannot be an SOS, no matter the value of $\rho^j$.

In this case, a rationale for this behavior is that $\tilde{f} = f + \varepsilon(1 + x^{16} + y^{16})$ is an SOS for small $\varepsilon > 0$, provided that $\varepsilon$ is not too small (in [8] it is shown that every nonnegative polynomial can be approximated as closely as desired by a sequence of polynomials that are sums of squares). After inspection of the returned optimal solution, the equality constraints

(2.4) \[ f_\alpha - \lambda 1_{\alpha=0} = \langle X, B_\alpha \rangle, \quad \alpha \in \mathbb{N}^n \]

when solving $P^*_\varepsilon$ in (2.3), are not satisfied accurately and the result can be interpreted as if the SDP solver has replaced $f$ with the perturbed criterion $\tilde{f} = f + \varepsilon$, with $\varepsilon(x) = \sum_\alpha \varepsilon_\alpha X_\alpha \in \mathbb{R}[x]_d$, so that

\[ f_\alpha + \varepsilon_\alpha - \lambda 1_{\alpha=0} = \langle X, B_\alpha \rangle, \quad \alpha \in \mathbb{N}^n \]

and in fact it has done so. A similar “mathematical paradox” has also been investigated in a non-commutative (NC) context [13]. NC polynomials can also be analyzed thanks to an NC variant of the Moment-SOS hierarchy (see [3] for a recent survey). As in the above commutative case, it is explained in [13] how numerical inaccuracies allow to obtain converging lower bounds for positive Weyl polynomials that do not admit SOS decompositions.

Case 2: Another surprising phenomenon occurred when minimizing a high-degree univariate polynomial $f$ with a global minimizer at $x = 100$ and a local minimizer at $x = 1$ with value $f(1) > f^*$ but very close to $f^* = f(100)$. The double precision floating point SDP solver returns a single minimizer $\tilde{x} \approx 1$ with value very close to $f^*$, providing another irrefutable proof that the double precision floating point SDP solver has returned a wrong solution. It turns out that again the result can be interpreted as if the SDP solver has replaced $f$ with a perturbed criterion $\tilde{f}$, as in Case 1.

When solving (2.3) in Case 1, one has voluntarily embedded $f \in \mathbb{R}[x]_6$ into $\mathbb{R}[x]_j$ (with $j > 3$) to obtain a perturbation $\tilde{f} \in \mathbb{R}[x]_j$, whose minimizers are close enough to those of $f$. Of course the precision is in accordance with the solver parameters involved in controlling the semidefiniteness of the moment matrix $X$ and the accuracy of the linear equations (2.4). Indeed, if one tunes these parameters to a much stronger threshold, then the solver returns a more accurate answer with a much higher precision.
In both contexts, we can interpret what the SDP solver does as perturbing the coefficients of the input polynomial data. One approach to get rid of numerical uncertainties consists of solving SDP problems in an exact way \[6\], while using symbolic computation algorithms. However, such exact algorithms only scale up to moderate size instances. For situations when one has to rely on more efficient, yet inexact numerical algorithms, there is a need to understand the behavior of the associated numerical solvers. In \[20\], the authors investigate strange behaviors of double-precision SDP solvers for semidefinite relaxations in polynomial optimization. They compute the optimal values of the SDP relaxations of a simple one-dimensional polynomial optimization problem. The sequence of SDP values practically converges to the optimal value of the initial problem while they should converge to a strict lower bound of this value. One possible remedy, used in \[20\], is to rely on an arbitrary-precision SDP solver, such as SDPA-GMP \[12\] in order to make this paradoxal phenomenon disappear. Relying on such arbitrary-precision solvers comes together with a more expensive cost but paves a way towards exact certification of nonnegativity. In \[10\], the authors present a hybrid numeric-symbolic algorithm computing exact SOS certificates for a polynomial lying in the interior of the SOS cone. This algorithm uses SDP solvers to compute an approximate SOS decomposition after additional perturbation of the coefficients of the input polynomial. The idea is to benefit from the perturbation terms added by the user to compensate the numerical uncertainties added by the solver. The present note focuses on analyzing specifically how the solver modifies the input and perturbs the polynomials of the initial optimization problem.

2.2. Contribution. We claim that there is also another possible and more optimistic conclusion if one looks at the above results with new “robust optimization” glasses, not from the viewpoint of the user but rather from the viewpoint of the solver. More precisely, given a polynomial optimization problem \(f^* = \min_x \{ f(x) : x \in K \}\) and its semidefinite relaxation \(P_j\) defined in (2.1) (with dual \(D_j\) in (2.2)),

We interpret the above behavior as the (double precision floating point) SDP solver doing “Robust Optimization” without being told to so so. In the case of individual trace equality perturbations \(\varepsilon\), it solves the max-min problem:

\[
\rho_j^\varepsilon = \max_{\tilde{f} \in \mathcal{B}_\infty(f, \varepsilon)} \{ \inf_y \{ L_y(\tilde{f}) : y_0 = 1; y \in C_j(g_1, \ldots, g_m) \} \},
\]

where \(\mathcal{B}_\infty(f, \varepsilon) := \{ \tilde{f} \in \mathbb{R}[x]_{2j} : \|\tilde{f} - f\|_\infty < \varepsilon \}\), and we provide some numerical experiments to support this claim. Interestingly, if the user would do robust optimization, then he would solve the min-max problem:

\[
\inf_y \{ \max_{\tilde{f} \in \mathcal{B}_\infty(f, \varepsilon)} \{ L_y(\tilde{f}) \} : y_0 = 1, y \in C_j(g_1, \ldots, g_m) \},
\]

which is (2.5) in which the “max” and “min” operators have been switched. It turns out that in this convex case, by Theorem 2.1, the optimal value of (2.6) is \(\rho_j^\varepsilon\).

So from a robustness viewpoint of the solver (not the user), it is quite reasonable to solve (2.5) rather than the original relaxation \(P_j\) of \(P\) with nominal polynomial \(f\). However since \(\rho_j^\varepsilon\) is equal to the optimal value of (2.6), the result is the same as if the user decided to do “robust optimization”! In other words, solving \(P_j\) with nominal \(f\) and numerical inaccuracies is the same as solving the robust problem (2.5) or (2.6) with infinite precision.
3. A “Noise” Model

Given a finite sequence of matrices \((F_{\alpha})_{\alpha \in \mathbb{N}^2_{2j}} \subset S_{n,j}\), a (primal) cost vector \(c = (c_{\alpha})_{\alpha \in \mathbb{N}^2_{2j}}\), we recall the standard form of primal semidefinite program (SDP) solved by numerical solvers such as SDPA [21]:

\[
\min_y \sum_{\alpha \in \mathbb{N}^2_{2j}} c_{\alpha} y_{\alpha}
\]

\[
\text{s.t.} \sum_{\alpha \in \mathbb{N}^2_{2j}} F_{\alpha} y_{\alpha} \succeq F_0,
\]

whose dual is the following SDP optimization problem:

\[
\max_X \langle F_0, X \rangle
\]

\[
\text{s.t.} \langle F_{\alpha}, X \rangle = c_{\alpha}, \quad \alpha \in \mathbb{N}^2_{2j}, \quad \alpha \neq 0,
\]

\[
X \succeq 0, \quad X \in S_{n,j}.
\]

We are interested in the numerical analysis of the moment-SOS hierarchy [7] to solve

\[
P : \min_{x \in K} f(x),
\]

where \(f \in \mathbb{R}[x]_{2j}\). Given \(\alpha, \beta \in \mathbb{N}^n\), let \(1_{\alpha=\beta}\) stands for the function which returns 1 if \(\alpha = \beta\) and 0 otherwise. At step \(d\) of the hierarchy, one solves the SDP primal program (2.1). For the standard choice of the convex cone \(C_j(g_1, \ldots, g_m)\), it reads

\[
\rho^j = \inf_y \{ L_y(f) : y_0 = 1; \quad \mathbf{M}_{j-d}(g_{\ell} y) \succeq 0, \quad \ell = 0, \ldots, m \},
\]

whose dual is the SDP:

\[
\delta^j = \sup_{X,\lambda} \{ \lambda : f - \lambda 1_{\alpha=0} = \sum_{\ell=0}^m \langle \mathbf{C}^\ell, X_\ell \rangle, \quad \alpha \in \mathbb{N}^2_{2j}, \quad X_\ell \succeq 0, \quad X_\ell \in S_{n,j-d_\ell}, \quad \ell = 0, \ldots, m \}
\]

where we have written \(\mathbf{M}_{j-d}(g_{\ell} y) = \sum_{\alpha \in \mathbb{N}^2_{2j}} \mathbf{C}^\ell_{\alpha} y_{\alpha}\); the matrix \(\mathbf{C}^\ell_{\alpha}\) has rows and columns indexed by \(\mathbb{N}^n_{j-d_\ell}\) with \((\beta, \gamma)\) entry equal to \(\sum_{\alpha+\beta+\gamma=\delta} g_{\ell,\delta}\). In particular for \(m = 0\), one has \(y_0 = 1\) and the matrix \(\mathbf{B}_\alpha := \mathbf{C}^0_{\alpha}\) has \((\beta, \gamma)\) entry equal to \(1_{\beta+\gamma=\alpha}\).

For every \(j \in \mathbb{N}\), let

\[
\mathcal{Q}_j(g) = \left\{ \sum_{\ell=0}^m \sigma_{\ell} g_{\ell} : \deg(\sigma_{\ell} g_{\ell}) \leq 2j, \sigma_{\ell} \in \Sigma[x] \right\}
\]

be the “truncated” quadratic module associated with the \(g_{\ell}\)’s. Then the dual SDP (3.4) can be rewritten as

\[
\delta^j = \sup_{\lambda} \{ \lambda : f - \lambda \in \mathcal{Q}_j(g) \} = \sup_{\lambda, \sigma_{\ell}} \{ \lambda : f - \lambda = \sum_{\ell=0}^m \sigma_{\ell} g_{\ell}, \deg(\sigma_{\ell} g_{\ell}) \leq 2j, \quad \sigma_{\ell} \in \Sigma[x] \}.
\]

In floating point computation, the numerical SDP solver treats all (ideally) equality constraints as the following inequality constraints

\[
\sum_{\ell=0}^m \langle \mathbf{C}^\ell_{\alpha}, X_\ell \rangle + \lambda 1_{\alpha=0} - f_{\alpha} = 0, \quad \alpha \in \mathbb{N}^n_{2j},
\]
of (3.4) with the following inequality constraints

\[
\sum_{\ell = 0}^{m} (C^\ell, X_\ell) + \lambda 1_{\alpha = 0} - f_\alpha \leq \varepsilon, \quad \alpha \in \mathbb{N}_2^m, \tag{3.7}
\]

for some a priori fixed tolerance \(\varepsilon > 0\) (for instance \(\varepsilon = 10^{-8}\)). Similarly, we assume that for each \(\ell = 0, \ldots, m\), the SDP constraint \(X_\ell \succeq 0\) of (3.4) is relaxed to \(X_\ell \succeq -\eta I\) for some prescribed \emph{individual semidefiniteness tolerance} \(\eta > 0\)\(^1\).

That is, all iterates \((X_{\ell, k})_{k \in \mathbb{N}}\) of the implemented minimization algorithm satisfy (3.7) and \(X_{\ell, k} \succeq -\eta I\) instead of the idealized (3.6) and \(X_{\ell, k} \succeq 0\).

Therefore we interpret the SDP solver behavior by considering the following “noise” model which is the \((\varepsilon, \eta)\)-perturbed version of SDP (3.4):

\[
\sup_{X_{\ell, \lambda}} \{ \lambda : -\varepsilon \leq \sum_{\ell = 0}^{m} (C^\ell, X_\ell) + \lambda 1_{\alpha = 0} - f_\alpha \leq \varepsilon, \quad \alpha \in \mathbb{N}_2^m, \quad X_\ell \succeq -\eta I, \quad X_\ell \in S_{n,j-d_\ell}, \quad \ell = 0, \ldots, m \}, \tag{3.8}
\]

now assuming exact computations.

**Proposition 3.1.** The dual of Problem (3.8) is the convex optimization problem

\[
\inf_y \{ L_y(f) + \eta \sum_{\ell = 0}^{m} \|M_{j-d_\ell}(gt y)\|_s + \varepsilon \|y\|_1 : \quad y_0 = 1; \quad M_{j-d_\ell}(gt y) \succeq 0, \quad \ell = 0, \ldots, m \} \tag{3.9}
\]

which is an SDP.

\textbf{Proof.} Let \(y_{\alpha}^{+}, y_{\alpha}^{-}\) be the nonnegative dual variables associated with the constraints

\[
\pm \left( \sum_{\ell = 0}^{m} (C^\ell, X_\ell) + \lambda 1_{\alpha = 0} - f_\alpha \right) \leq \varepsilon, \quad \alpha \in \mathbb{N}_2^m,
\]

and let \(S_\ell \succeq 0\) be the dual matrix variable associated with the SDP constraint \(X_\ell \succeq -\eta I, \ell = 0, \ldots, m\). Then the dual of (3.8) is a semidefinite program which reads:

\[
\inf_{S_\ell \succeq 0, y_\alpha^{+}, y_\alpha^{-} \succeq 0} \{ \sum_{\alpha} (f_\alpha (y_{\alpha}^{+} - y_{\alpha}^{-}) + \varepsilon (y_{\alpha}^{+} + y_{\alpha}^{-})) + \eta \sum_{\ell} (I, S_\ell) : \quad S_\ell - \sum_{\alpha} C_\alpha (y_{\alpha}^{+} - y_{\alpha}^{-}) = 0, \quad \ell = 0, \ldots, m, \quad y_0^{+} - y_0^{-} = 1 \}. \tag{3.10}
\]

In view of the nonnegative terms \(\varepsilon \sum_{\alpha} (y_{\alpha}^{+} + y_{\alpha}^{-})\) in the criterion, at an optimal solution we necessarily have \(y_{\alpha}^{+} y_{\alpha}^{-} = 0\), for all \(\alpha\). Therefore letting \(y_\alpha := y_{\alpha}^{+} - y_{\alpha}^{-}\), on obtains \(y_\alpha^2 + y_\alpha = |y_\alpha| \) for all \(\alpha\), and \(\sum_{\alpha} (y_{\alpha}^{+} + y_{\alpha}^{-}) = \|y\|_1\). Similarly as \(S_\ell \succeq 0\), \(\langle I, S_\ell \rangle = \|S_\ell\|_1, \ell = 0, \ldots, m\). This yields the formulation (3.9). \hfill \Box

Notice that the criterion of (3.9) consists of the original criterion \(L_y(f)\) perturbed with a sparsity-inducing norm \(\varepsilon \|y\|_1\) for the variable \(y\), and a low-rank-inducing norm \(\eta \sum_\ell \|M_{j-d_\ell}(gt y)\|_s\) for the localizing matrices.

We now distinguish among two particular cases.

\(^1\)This latter relaxation of \(\succeq 0\) to \(\succeq -\eta I\) is used here as an idealized situation for modeling purpose; in practice it seems to be more complicated.
3.1. Priority to trace equalities. With \( \varepsilon = 0 \) and individual semidefiniteness-tolerance \( \eta \), Problem (3.9) becomes

\[
\rho^\eta = \inf_y \{ L_y(f) + \eta \sum_{\ell=0}^m \| M_{j-d_\ell}(g_\ell y) \|_* \}
\]

subject to \( y_0 = 1; \ M_{j-d_\ell}(g_\ell y) \geq 0, \ \ell = 0, \ldots, m \} \).

Given \( \eta > 0, \ j \in \mathbb{N} \), let us define:

\[
B^\eta_\infty(f, K, \eta) := \{ f + \theta \sum_{\ell=0}^m g_\ell(x) \sum_{\beta \in \mathbb{N}^n_{j-d_\ell}} x^{2\beta} : |\theta| \leq \eta \},
\]

\[
B_\infty(f, K, \eta) := \bigcup_{j \in \mathbb{N}} B^\eta_\infty(f, K, \eta).
\]

Recall that SDP (3.11) is the dual of SDP (3.8) with \( \varepsilon = 0 \), that is,

\[
\sup_{x \in K} \{ \lambda : f_\alpha - \lambda 1_{\alpha=0} = \sum_{\ell=0}^m (C_\alpha, X_\ell), \ \alpha \in \mathbb{N}^n_{2j}, \ X_\ell \succeq -\eta I, \ X_\ell \in S_{n,j-d_\ell}, \ \ell = 0, \ldots, m \}.
\]

Fix \( j \in \mathbb{N} \) and consider the following robust polynomial optimization problem

\[
P^\max_\eta : \max_{\hat{f} \in B_\infty(f, K, \eta)} \{ \min_{x \in K} \{ \hat{f}(x) \} \}.
\]

If in (3.14), we restrict ourselves to \( B^\eta_\infty(f, K, \eta) \) and we replace the inner minimization by its step-\( j \) relaxation, we obtain

\[
P^\max_\eta : \max_{\hat{f} \in B^\eta_\infty(f, K, \eta)} \{ \inf_y \{ L_y(\hat{f}) : y_0 = 1; \ M_{j-d_\ell}(g_\ell y) \geq 0, \ \ell = 0, \ldots, m \} \}.
\]

Observe that Problem \( P^\max_\eta \) is a strenghtening of Problem \( P^\max_\eta \), that is, the optimal value of the former is smaller than the optimal value of the latter.

**Proposition 3.2.** Problem \( P^\max_\eta \) is equivalent to SDP (3.11). Therefore, solving primal SDP (3.11) (resp. dual SDP (3.13)) can be interpreted as solving exactly, i.e., with no semidefiniteness-tolerance, the step-\( j \) strengthening \( P^\max_\eta \) associated with Problem \( P^\max_\eta \).

**Proof.** Remind that for every \( \ell = 0, \ldots, m \), one has \( M_{j-d_\ell}(g_\ell y) \succeq 0 \) and

\[
\| M_{j-d_\ell}(g_\ell y) \|_* = \text{Trace} (M_{j-d_\ell}(g_\ell y)) = L_y \left( \sum_{\beta \in \mathbb{N}^n_{j-d_\ell}} x^{2\beta} g_\ell(x) \right).
\]

For \( \hat{f} = f + \eta \sum_{\beta \in \mathbb{N}^n_{j-d_\ell}} x^{2\beta} g_\ell(x) \), one has \( L_y(\hat{f}) = L_y(f) + \eta \sum_{\ell=0}^m \| M_{j-d_\ell}(g_\ell y) \|_* \).

Since \( \hat{f} \) is feasible for Problem \( P^\max_\eta \), the optimal value of Problem \( P^\max_\eta \) is greater than the value of SDP (3.11).

Next, by Theorem 2.1, Problem \( P^\max_\eta \) is equivalent to

\[
\inf_y \max_{\hat{f} \in B^\eta_\infty(f, K, \eta)} \{ L_y(\hat{f}) : y_0 = 1; \ M_{j-d_\ell}(g_\ell y) \geq 0, \ \ell = 0, \ldots, m \}.
\]

For all \( \hat{f} \in B^\eta_\infty(f, K, \eta) \), \( L_y(\hat{f}) \leq L_y(f) + \eta \sum_{\ell=0}^m \| M_{j-d_\ell}(g_\ell y) \|_* \), which proves that the optimal value of (3.15) is less than the value of SDP (3.11).
This yields the equivalence between Problem $P^{\text{max-j}}_\eta$ and SDP (3.11). □

In the unconstrained case, i.e. when $m = 0$, solving $P_\eta$ boils down to minimize the perturbed polynomial $f_{\eta,j}(x) := f(x) + \eta \sum_{|\beta| \leq j} x^{2\beta}$, that is the sum of $f$ and all monomial squares of degree up to $2j$ with coefficient magnitude $\eta$. As a direct consequence from [8], the next result shows that for given nonnegative polynomial $f$ and perturbation $\eta > 0$, the polynomial $f_{\eta,j}$ is SOS for large enough $j$.

**Corollary 3.3.** Let assume that $f \in \mathbb{R}[x]$ is nonnegative over $\mathbb{R}^n$ and let us fix $\eta > 0$. Then $f_{\eta,j} \in \Sigma[x]$, for large enough $j$.

**Proof.** For fixed nonnegative $f \in \mathbb{R}[x]$ and $\eta > 0$, it follows from [8, Theorem 4.2 (ii)] that there exists $j_\eta$ (depending on $f$ and $\eta$) such that the polynomial

$$f + \eta \sum_{k=0}^j \sum_{i=1}^n \frac{x_i^{2k}}{k!},$$

is SOS for any $d \geq d_\eta$. Let us select $j := j_\eta$. Notice that

$$f_{\eta,j} = f + \eta \sum_{|\beta| \leq j} x^{2\beta} = f + \eta \sum_{k=0}^j \sum_{i=1}^n \frac{x_i^{2k}}{k!} + \eta \sum_{k=0}^j \sum_{i=1}^n \left(1 - \frac{1}{k!}\right) x_i^{2k} + \eta q_j,$$

where $q_j$ is a sum of monomial squares. Since $\left(1 - \frac{1}{k!}\right) \geq 0$, the second sum of the right hand side is SOS, yielding the desired claim. □

### 3.2. Priority to semidefiniteness inequalities.

Problem (3.9) with $\eta = 0$ and individual trace equality perturbation $\varepsilon$ becomes

$$\rho^j_\varepsilon = \inf_{y} \left\{ L_y(f) + \varepsilon \|y\|_1 \right\} : \quad \begin{array}{l}
\text{s.t.}\quad y_0 = 1; \quad M_{j-d_\ell}(g_\ell y) \succeq 0, \quad \ell = 0, \ldots, m \}. 
\end{array}$$

Given $\varepsilon > 0$, $j \in \mathbb{N}$, let us define

$$B^j_\infty(K, \eta) := \left\{ \tilde{f} \in \mathbb{R}[x]_{2j} : \|f - \tilde{f}\|_\infty \leq \varepsilon \right\}, \quad B_\infty(f, \varepsilon) := \bigcup_{j \in \mathbb{N}} B^j_\infty(f, \varepsilon).$$

Recall that (3.16) is the dual of (3.8) with $\eta = 0$, that is,

$$\sup_{f, \lambda} \left\{ \lambda : \tilde{f} - \lambda \in \mathcal{Q}(g); \quad |f_{\alpha} - \tilde{f}_{\alpha}| \leq \varepsilon, \quad \alpha \in \mathbb{N}^2 \right\}, \quad \lambda \in \mathbb{R}, \quad \tilde{f} \in \mathbb{R}[x]_{2j} \right\}. $$

Fix $j \in \mathbb{N}$ and consider the following robust polynomial optimization problem:

$$P^{\text{max}}_\varepsilon : \quad \max_{f \in B_\infty(f, \varepsilon)} \left\{ \min_{x \in K} \{ \tilde{f}(x) \} \right\}. $$

If in (3.19), we restrict ourselves to $B^j_\infty(f, \varepsilon)$ in the outer maximization problem and we replace the inner minimization by its step-$j$ relaxation, we obtain

$$P^{\text{max-j}}_\varepsilon : \quad \max_{\tilde{f} \in B^j_\infty(f, \varepsilon)} \left\{ \sup_{\lambda} \left\{ \lambda : \tilde{f} - \lambda \in \mathcal{Q}(g) \right\} \right\}$$

$$= \max_{\tilde{f} \in B^j_\infty(f, \varepsilon)} \left\{ \inf_{y} \left\{ L_y(\tilde{f}) : y_0 = 1; \quad M_{j}(g_\ell y) \succeq 0, \quad \ell = 0, \ldots, m \right\} \right\}$$

which is a strengthening of $P^{\text{max}}_\varepsilon$ and whose dual is exactly (3.16). That is:
Proposition 3.4. Solving (3.16) (equivalently (3.18)) can be interpreted as solving exactly, i.e. with no trace-equality tolerance, the step-\(j\) reinforcement \(P^\text{max},j\) associated with \(P^\text{max}\).

3.3. A two-player game interpretation. If we now assume that one can perform computations exactly, we can interpret the whole process in \(P^\text{max},j\) (resp. \(P^\text{max}\)) as a two-player zero-sum game in which:

- Player 1 (the solver) chooses a polynomial \(\hat{f} \in B_{\infty}(f, K, \eta)\) (resp. \(\tilde{f} \in B_{\infty}^{\infty}(f, \varepsilon)\)).
- Player 2 (the optimizer) then selects a minimizer \(y^*(\hat{f})\) in the inner minimization of (3.20), e.g., with an exact interior point method.

As a result, Player 1 (the leader) obtains an optimal polynomial \(\hat{f}^* \in B_{\infty}(f, K, \eta)\) (resp. \(\tilde{f}^* \in B_{\infty}^{\infty}(f, \varepsilon)\)) and Player 2 (the follower) obtains an associated minimizer \(y^*(\hat{f}^*)\). The polynomial \(\hat{f}^*\) is the worst polynomial in \(B_{\infty}(f, K, \eta)\) (resp. \(B_{\infty}^{\infty}(f, \varepsilon)\)) for the step-\(j\) semidefinite relaxation associated with the optimization problem \(
\min_x \{ f(x) : x \in K \}\). This \(\max - \min\) problem is then equivalent to the single \(\min\)-problem (3.11) (resp. (3.16)) which is a convex relaxation and whose convex criterion is not linear as it contains the sum of \(\ell_\infty\)-norm terms \(\sum_{\ell=0}^n \|M_{j-d\ell} (g \ell \gamma)\|_s\) (resp. the \(\ell_1\)-norm term \(\|y\|_1\)).

Notice that in this scenario the optimizer (Player 2) is not active; initially he wanted to solve the convex relaxation associated with \(f\). It is Player 1 (the adversary uncertainty in the solver) who in fact gives the exact algorithm his own choice of the function \(f \in B_{\infty}^{\infty}(f, K, \eta)\) (resp. \(\tilde{f} \in B_{\infty}^{\infty}(f, \varepsilon)\)). But in fact, as we are in the convex case, Theorem 2.1 implies that this \(\max - \min\) game is also equivalent to the \(\min\)–\(\max\) game. Indeed, \(P^\text{max},j\) is equivalent to

\[
\inf_y \max_{\hat{f} \in B_{\infty}(f, K, \eta)} \{ L_y(\hat{f}) : y_0 = 1; M_j(g \ell \gamma) \succeq 0, \ell = 0, \ldots, m \},
\]

and \(P^\text{max}\) is equivalent to

\[
\inf_y \max_{f \in B_{\infty}^{\infty}(f, K, \eta)} \{ L_y(\tilde{f}) : y_0 = 1; M_j(g \ell \gamma) \succeq 0, \ell = 0, \ldots, m \},
\]

So now in this scenario (which assumes exact computations):

- Player 1 (the robust optimizer) chooses a feasible moment sequence \(y\) with \(y_0 = 1\) and \(M_{j-d\ell} (g \ell \gamma) \succeq 0, \ell = 0, \ldots, m\).
- When priority is given to trace equalities, Player 2 (the solver) then selects \(\hat{f}(y) = \arg \max \{ L_y(\hat{f}) : \hat{f} \in B_{\infty}^{\infty}(f, K, \eta) \}\) to obtain the value \(L_y(f) + \eta \sum_{\ell=0}^n \|M_{j-d\ell} (g \ell \gamma)\|_s\).

When priority is given to semidefinitess inequalities, Player 2 selects \(\tilde{f}(y) = \arg \max \{ L_y(\tilde{f}) : \tilde{f} \in B_{\infty}^{\infty}(f, \varepsilon) \}\) to obtain the value \(L_y (f) + \varepsilon \|y\|_1\), that is \(\tilde{f}(y)_\alpha = f_\alpha + \text{sign}(y_\alpha) \varepsilon, \alpha \in \mathbb{N}^n_{d\ell},\)

Here the optimizer (now Player 1) is “active” as he decides to compute a “robust” optimal relaxation \(y\) assuming uncertainty in the function \(f\) in the criterion \(L_y(f)\).

Since both scenarios are equivalent it is fair to say that the SDP solver is indeed solving the robust convex relaxation that the optimizer should have given to a solver with exact arithmetic (if he had wanted to solve robust relaxations).
Relating to robust optimization. Suppose that there is no computation error
but we want to solve a robust version of the optimization problem \( \min \{ f(x) : x \in K \} \) because there is some uncertainty in the coefficients of the nominal polynomial \( f \in \mathbb{R}[x_1, \ldots, x_d] \). So assume that \( f \in \mathbb{R}[x_1, \ldots, x_d] \) can be considered as potentially of degree at most \( 2j \) (after perturbation).

When priority is given to trace equalities, the robust optimization problem reads:
\[
(3.21) \quad P^{\min,j}_\eta : \min_{x \in K} \{ \max_{\tilde{f} \in B^\infty_j(f,K,\eta)} \{ \tilde{f}(x) \} \}.
\]
It is easy to see that (3.21) reduces to
\[
(3.22) \quad P^{\min,j}_\eta : \min_{x \in K} \left[ f(x) + \eta \sum_{\beta \in N^n_{|1-j|d}} x^{2\beta} g_\ell(x) \right].
\]
which is a polynomial optimization problem.

**Theorem 3.5.** Assume that after solving SDP (3.11), one obtains \( y^* \) such that \( M_j(y^*) \) is a rank-one matrix. Then, the optimal value of \( P^{\min,j}_\eta \) is equal to \( \rho^j_\eta \) and \( P^{\min,j}_\eta \) is equivalent to \( P^{\max,j}_\eta \).

**Proof.** Since \( M_j(y^*) \) is a rank-one matrix, the sequence \( y^* \) comes from a Dirac measure supported on \( x^* \in K \). Then one has
\[
L_{y^*}(f) + \eta \sum_{\ell=0}^m \| M_{j-d\ell}(g_\ell y^*) \|_* = f(x^*) + \eta \sum_{\ell=0}^m \sum_{\beta \in N^n_{|1-j|d}} x^{2\beta} g_\ell(x^*),
\]
Let \( P(K) \) be the space of probability measures supported on \( K \). Then, one has
\[
f(x^*) + \eta \sum_{\ell=0}^m \sum_{\beta \in N^n_{|1-j|d}} x^{2\beta} g_\ell(x^*) \geq \min_{x \in K} \left[ f(x) + \eta \sum_{\ell=0}^m \sum_{\beta \in N^n_{|1-j|d}} x^{2\beta} g_\ell(x) \right] = \inf_{\mu \in P(K)} \left[ \int f d\mu + \eta \sum_{\ell=0}^m \sum_{\beta \in N^n_{|1-j|d}} \int x^{2\beta} g_\ell(x) d\mu \right] \geq \rho^j_\eta = L_{y^*}(f) + \eta \sum_{\ell=0}^m \| M_{j-d\ell}(g_\ell y^*) \|_*.
\]
This implies that \( x^* \) is the unique optimal solution of \( P^{\min,j}_\eta \) and that the optimal value of \( P^{\min,j}_\eta \) is equal to \( \rho^j_\eta \). Eventually, Proposition 3.2 yields the desired equivalence. \( \square \)

When priority is given to semidefinites inequalities, the robust optimization problem reads:
\[
(3.23) \quad P^{\min,j}_\varepsilon : \min_{x \in K} \{ \max_{\tilde{f} \in B^\infty_j(f,K)} \{ \tilde{f}(x) \} \}.
\]
It is easy to see that (3.23) reduces to
\[
(3.24) \quad P^{\min,j}_\varepsilon : \min_{x \in K} \left[ f(x) + \varepsilon \sum_{\alpha \in N^n_{2j}} |x^\alpha| \right].
\]
which is not a polynomial optimization problem (but is still a semi-algebraic optimization problem). As for Theorem 3.5, one proves the following result:
Theorem 3.6. Assume that after solving SDP (3.16), one obtains $y^\ast$ such that $M_j(y^\ast)$ is a rank-one matrix. Then, the optimal value of $P_{\varepsilon, j}^{\min}$ is equal to $\rho_j^\varepsilon$ and $P_{\varepsilon, j}^{\min}$ is equivalent to $P_{\varepsilon, j}^{\max}$.

Notice an important conceptual difference between the two approaches. In the latter one, i.e., when considering $P_{\varepsilon, j}^{\min}$ (resp. $P_{\varepsilon, j}^{\max}$), the user is active. Indeed the user decides to choose some optimal $\hat{f} \in B_{j, \infty}((f, K, \eta))$ (resp. $B_{j, \infty}((f, \varepsilon))$). In the former one, i.e., when considering $P_{\varepsilon, j}^{\max}$ (resp. $P_{\varepsilon, j}^{\max}$), the user is passive, as indeed he imposes $f$ but the solver decides to choose some optimal $f^\ast \in B_{j, \infty}((f, K, \eta))$ (resp. $B_{j, \infty}((f, \varepsilon))$).

If after solving SDP (3.11) (resp. SDP (3.16)), one obtains $y^\ast$ where $M_j(y^\ast)$ is rank-one (which is to be expected), one obtains the same solution: in other words, we can interpret what the solver does as performing robust polynomial optimization.

In the sequel, we show how this interpretation relates with a more general robust SDP framework, when priority is given to semidefiniteness inequalities.

3.4. Relation with robust semidefinite programming. Let us consider a cost vector $c = (c_\alpha)_{\alpha \in \mathbb{N}^n_2}$, a finite sequence of matrices $(F_\alpha)_{\alpha \in \mathbb{N}^n_2} \subset S_{n,j}$, and let us note $F(y) := \sum_{\alpha \neq 0} F_\alpha y_\alpha - F_0$.

Given $\eta > 0$, $j \in \mathbb{N}$, $C \in S_{n,j}$, we define by a slight abuse of notation w.r.t. (3.17)

$$B_j^{\infty}(c, \varepsilon) := \{ \tilde{c} : \| \tilde{c} - c \|_\infty \leq \varepsilon \},$$

Next, we consider the max-min robust SDP program associated to SDP (3.1):

$$\max_{\tilde{c} \in B_{j, \infty}(c, \varepsilon)} \inf_y \tilde{c}^T y \quad \text{s.t.} \quad F(y) \succeq 0,$$

that is, we consider a perturbation of SDP (3.1), where the cost vector $c$ is replaced by $\tilde{c} \in B_{j, \infty}(c, \varepsilon)$.

Proposition 3.7. The robust SDP (3.25) is equivalent to

$$\inf_y c^T y + \varepsilon \| y \|_1 \quad \text{s.t.} \quad F(y) \succeq 0.$$

Proof. The dual of (3.25) reads as follows:

$$\max_{\tilde{c}} \sup_X \langle F_0, X \rangle \quad \text{s.t.} \quad \langle F_\alpha, X \rangle = \tilde{c}_\alpha, \quad | \tilde{c}_\alpha - c_\alpha | \leq \varepsilon, \quad \alpha \in \mathbb{N}^n_2, \quad \alpha \neq 0, \quad X \succeq 0, \quad X \in S_{n,j},$$

which is equivalent to

$$\sup_X \langle F_0, X \rangle \quad \text{s.t.} \quad | \langle F_\alpha, X \rangle - c_\alpha | \leq \varepsilon, \quad \alpha \in \mathbb{N}^n_2, \quad \alpha \neq 0, \quad X \succeq 0, \quad X \in S_{n,j}.$$

As in the proof of Proposition 3.1, we prove that the dual of SDP (3.28) is (3.26). □

SDP (3.27) is obtained by modifying the dual SDP (3.2) after $\varepsilon$-perturbation of each trace equality. In the particular case of SDP relaxations for polynomial
optimization, we retrieve (3.16) as an instance of (3.26) and (3.18) as an instance of (3.27).

Note also that, by convexity, Theorem 2.1 implies that the max-min robust SDP (3.25) is equivalent to the min-max robust problem

\[
\inf_y \max_{\tilde{c} \in B_{\infty}(c,\epsilon)} \tilde{c}^T y \\
\text{s.t. } F(y) \succeq 0.
\]  

The fact that one can interpret what numerical SDP solvers do as performing robust optimization (w.r.t. a given objective cost \(c\)) is not surprising. But in the case of SDP relaxations for polynomial optimization, it is more surprising. Indeed, some instances (e.g. the Motzkin-like polynomial) cannot be theoretically handled by SDP relaxations, yet higher-order relaxations allow to practically solve them. We presume that similar phenomena could appear when handling polynomial optimization problems with alternative convex programming relaxations relying on interior-point algorithms, for instance linear/geometric programming.

4. Examples

All experimental results are obtained by computing the solutions of the primal-dual SDP relaxations (3.3)-(3.4) of Problem \(P\). These SDP relaxations are implemented in the \texttt{RealCertify} [11] library, available within \textsc{Maple}, and interfaced with the SDP solvers SDPA [21] and SDPA-GMP [12].

For the two upcoming examples, we rely on the procedure described in [5] to extract the approximate global minimizer(s) of some given objective polynomial functions. We compare the results obtained with (1) the SDPA solver implemented in double floating-point precision, which corresponds to \(\epsilon = 10^{-7}\) and (2) the arbitrary-precision SDPA-GMP solver, with \(\epsilon = 10^{-30}\). The value of our robust-noise model parameter \(\rho\) roughly matches with the one of the parameter \(\text{epsilonStar}\) of SDPA.

We also noticed that decreasing the value of the SDPA parameter \(\text{lambdaStar}\) seems to boil down to increasing the value of our robust-noise model parameter \(\eta\). An expected justification is that \(\text{lambdaStar}\) is used to determine a starting point \(X^0\) for the interior-point method, i.e., such that \(X^0 = \text{lambdaStar} \times I\) (the default value of \(\text{lambdaStar}\) is equal to \(10^2\) in SDPA and is equal to \(10^4\) in SDPA-GMP). A similar behavior occurs when decreasing the value of the parameter \(\text{betaBar}\), which controls the search direction of the interior-point method when the matrix \(X\) is not positive semidefinite.

However, the correlation between the values of \(\text{lambdaStar}\) (resp. \(\text{betaBar}\)) and \(\eta\) appears to be nontrivial. Thus, our robust-noise model would be theoretically valid if one could impose the value of a parameter \(\eta\), ensuring that \(X \succeq -\eta I\) when the interior-point method terminates. From the best of our knowledge, this feature happens to be unavailable in modern SDP solvers. For that reason, our experimental comparisons are performed by changing the value of \(\text{epsilonStar}\) in the parameter file of the SDP solver.

4.1. Univariate polynomial with minimizers of different magnitudes.

We start by considering the following univariate optimization problem:

\[
f^* = \min_{x \in \mathbb{R}} f(x),
\]
with $f(x) = (x - 100)^2 \left( (x - 1)^2 + \frac{2}{3 \gamma} \right)$ and $\gamma \geq 0$.

Note that the minimum of $f$ is $f^* = 0 = f(100)$ and $f(1) = \gamma$.

We first examine the case where $\gamma = 0$. In this case, $f$ has two global minimizers 1 and 100. At relaxation order $j$, with $2 \leq j \leq 5$, we retrieve the following results (rounded to four significant digits):

1. With $\epsilon = 10^{-7}$, we obtain $\hat{x}^{(1)} = 0.9999 \simeq 1$, corresponding to the smallest global minimizer of $f$.
2. With $\epsilon = 10^{-50}$, we obtain $\hat{x} = 50.5000 = \frac{1+100}{2}$, corresponding to the average of the two global minimizers of $f$.

We also used the realroot procedure, available within Maple, to compute the local minimizers of the following function on $[0, \infty)$:

\[ f_{\epsilon,j}(x) = f(x) + \epsilon \sum_{|\alpha| \leq 2j} |x^\alpha| = f(x) + \epsilon \sum_{|\alpha| \leq 2j} x^\alpha, \]

1. With $\epsilon = 10^{-7}$, we obtain $\hat{x}^{(1)} = 0.9961 \simeq \hat{x}^{(1)}$.
2. With $\epsilon = 10^{-50}$, we obtain $\hat{x}^{(1)} = 0.9961 \simeq \hat{x}^{(1)}$ and $\hat{x}^{(2)} = 99.9960 \simeq 100$, the largest global minimizer of $f$. The corresponding values of $q_{\epsilon,j}$ are 0.1496 and 0.1495, respectively.

These experiments confirm our explanations that the solver computes the solution of SDP relaxations associated to the perturbed function $f_{\epsilon,j}$ from (4.1). With double floating-point precision (1), this perturbed function has a single minimizer, retrieved by the extraction procedure. With higher precision (2), this perturbed function has two local minimizers, whose average is retrieved by the extraction procedure.

Next, we examine the case where $\gamma = 10^{-3}$. In this case, $f$ has a single global minimizer, equal to 100 and another local minimizer. At relaxation order $j$, with $2 \leq j \leq 5$, we retrieve the following results (rounded to four significant digits):

1. With $\epsilon = 10^{-7}$, we obtain $\hat{x}^{(1)} = 0.9999 \simeq 1$, corresponding to the smallest global minimizer of $f$ when $\gamma = 0$.
2. With $\epsilon = 10^{-50}$, we obtain $\hat{x}^{(2)} = 99.1593 \simeq 100$, corresponding to the single global minimizer of $f$.

We also compute the local minimizers of $q_{\epsilon,j}$ with realroot:

1. With $\epsilon = 10^{-7}$, we obtain $\hat{x}^{(1)} = 1.0039 \simeq \hat{x}^{(1)}$.
2. With $\epsilon = 10^{-50}$, we obtain $\hat{x}^{(1)} = 1.0039 \simeq \hat{x}^{(1)}$ and $\hat{x}^{(2)} = 99.9961 \simeq 100$, the single global minimizer of $f$. The corresponding values of $\hat{x}_{\epsilon,j}$ are 0.1505 and 0.1495, respectively. This confirms that $\hat{x}^{(2)}$ is the single global minimizer of $f_{\epsilon,j}$, approximately extracted, as $\hat{x}^{(2)}$.

Here again, our robust-noise model, relying on the perturbed polynomial function $f_{\epsilon,j}$, fits with the above experimental observations. This perturbed function has a single global minimizer, whose value depends on the parameter $\epsilon$, and which can be approximately retrieved by the extraction procedure.

4.2. Motzkin polynomial. Here, we consider the Motzkin polynomial $f = \frac{1}{7} + x_1^2 x_2^2(x_1^2 + x_2^2 - 1)$. This polynomial is nonnegative but is not SOS. The minimum $f^*$ of $f$ is 0 and $f$ has four global minimizers with coordinates $x_1 = \pm \frac{\sqrt{3}}{2}$ and $x_2 = \pm \frac{\sqrt{3}}{2}$. As noticed in [5, Section 4], one can retrieve these global minimizers
by solving the primal-dual SDP relaxations (3.3)-(3.4) of Problem $P$ at relaxation order $j = 8$:

(1) With $\epsilon = 10^{-7}$, we obtain an approximate lower bound of $-1.81 \cdot 10^{-4} \leq f^*$, as well as the four global minimizers of $f$ with the extraction procedure. The dual SDP (3.4) allows to retrieve the approximate SOS decomposition $f(x) = \sigma(x) + r(x)$, where $\sigma$ is an SOS polynomial and the corresponding polynomial remainder $r$ has coefficients of approximately equal magnitude, and which is less than $10^{-8}$.

(2) With $\epsilon = 10^{-30}$, we obtain an approximate lower bound of $-1.83 \cdot 10^1 \leq f^*$ and the extraction procedure fails. The corresponding polynomial remainder has coefficients of magnitude less than $10^{-31}$.

We notice that the support of $r$ contains only terms of even degrees, i.e., terms of the form $x^{2\beta}$, with $|\beta| \leq 8$. Hence we consider a perturbation $\tilde{f}_\gamma$ of $f$ defined by $\tilde{f}_\gamma(x) = f(x) + \gamma \sum_{|\beta| \leq j} x^{2\beta}$, with $\gamma = 10^{-8}$. By solving the SDP relaxation (with $j = 8$) associated to $\tilde{f}_\gamma$, with $\epsilon = 10^{-30}$, we retrieve again the four global minimizers of $f$.

5. DISCUSSION

By considering the hierarchy of SDP relaxations associated to a given polynomial optimization problem, we are facing with a dilemma when relying on numerical SDP solvers. On the one hand, we might want to increase the precision of the solver to get rid of the numerical uncertainties and obtain an accurate solution of the SDP relaxations. On the other hand, working with low precision may allow to obtain hints related to the solution of the initial problem. This has already happened in both commutative and non-commutative contexts, to compute the global minimizers of the Motzkin polynomial in [5] or the bosonic energy levels from [13]. Our theoretical robust-noise model could be extended to problems addressed with structured SDP programs (as, for instance, the moment and localizing matrices coming from polynomial optimization problems). We believe that the use of “inaccurate” SDP solvers could also provide hints for the solutions of such problems.

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