On degree sum conditions for 2-factors with a prescribed number of cycles

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Abstract

For a vertex subset $X$ of a graph $G$, let $\Delta_t(X)$ be the maximum value of the degree sums of the subsets of $X$ of size $t$. In this paper, we prove the following result: Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n \geq 5k - 2$. If $\Delta_2(X) \geq n$ for every independent set $X$ of size $\lceil m/k \rceil + 1$ in $G$, then $G$ has a 2-factor with exactly $k$ cycles. This is a common generalization of the results obtained by Brandt et al. [Degree conditions for 2-factors, J. Graph Theory 24 (1997) 165–173] and Yamashita [On degree sum conditions for long cycles and cycles through specified vertices, Discrete Math. 308 (2008) 6584–6587], respectively.

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. The independence number and the connectivity of a graph $G$ are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. For a vertex $x$ of a graph $G$, we denote by $d_G(x)$ and $N_G(x)$ the degree and the neighborhood of $x$ in $G$. Let $\sigma_m(G)$ be the minimum degree sum of an independent set of $m$ vertices in a graph $G$, i.e., if $\alpha(G) \geq m$, then

$$\sigma_m(G) = \min \left\{ \sum_{x \in X} d_G(x) : X \text{ is an independent set of } G \text{ with } |X| = m \right\};$$

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otherwise, \( \sigma_m(G) = +\infty \). If the graph \( G \) is clear from the context, we often omit the graph parameter \( G \) in the graph invariant. In this paper, “disjoint” always means “vertex-disjoint”.

A graph having a hamilton cycle, i.e., a cycle containing all the vertices of the graph, is said to be hamiltonian. The hamiltonian problem has long been fundamental in graph theory. But, it is NP-complete, and so no easily verifiable necessary and sufficient condition seems to exist. Therefore, many researchers have focused on “better” sufficient conditions for graphs to be hamiltonian (see a survey [14]). In particular, the following degree sum condition, due to Ore (1960), is classical and well known.

**Theorem A (Ore [15])** Let \( G \) be a graph of order \( n \geq 3 \). If \( \sigma_2 \geq n \), then \( G \) is hamiltonian.

Chvátal and Erdős (1972) discovered the relationship between the connectivity, the independence number and the hamiltonicity.

**Theorem B (Chvátal, Erdős [8])** Let \( G \) be a graph of order at least 3. If \( \alpha \leq \kappa \), then \( G \) is hamiltonian.

Bondy [2] pointed out that the graph satisfying the Ore condition also satisfies the Chvátal-Erdős condition, that is, Theorem B implies Theorem A.

By Theorem B we should consider the degree condition for the existence of a hamilton cycle in graphs \( G \) with \( \alpha(G) \geq \kappa(G) + 1 \). In fact, Bondy (1980) gave the following degree sum condition by extending Theorem B.

**Theorem C (Bondy [3])** Let \( G \) be an \( m \)-connected graph of order \( n \geq 3 \). If \( \sigma_{m+1} > \frac{1}{2}(m+1)(n-1) \), then \( G \) is hamiltonian.

In 2008, Yamashita [17] introduced a new graph invariant and further generalized Theorem C as follows. For a vertex subset \( X \) of a graph \( G \) with \( |X| \geq t \), we define

\[
\Delta_t(X) = \max \left\{ \sum_{x \in Y} d_G(x) : Y \subseteq X, \ |Y| = t \right\},
\]

Let \( m \geq t \), and if \( \alpha(G) \geq m \), then let

\[
\sigma_t^m(G) = \min \left\{ \Delta_t(X) : X \text{ is an independent set of } G \text{ with } |X| = m \right\};
\]

otherwise, \( \sigma_t^m(G) = +\infty \). Note that \( \sigma_t^m(G) \geq \frac{m}{t} \cdot \sigma_m(G) \).

**Theorem D (Yamashita [17])** Let \( G \) be an \( m \)-connected graph of order \( n \geq 3 \). If \( \sigma_2^{m+1} \geq n \), then \( G \) is hamiltonian.
This result suggests that the degree sum of non-adjacent “two” vertices is important for hamilton cycles.

On the other hand, it is known that a 2-factor is one of the important generalizations of a hamilton cycle. A 2-factor of a graph is a spanning subgraph in which every component is a cycle, and thus a hamilton cycle is a 2-factor with “exactly 1 cycle”. As one of the studies concerning the difference between hamilton cycles and 2-factors, in this paper, we focus on 2-factors with “exactly k cycles”. Similar to the situation for hamilton cycles, deciding whether a graph has a 2-factor with k (≥ 2) cycles is also NP-complete. Therefore, the sufficient conditions for the existence of such a 2-factor also have been extensively studied in graph theory (see a survey [11]). In particular, the following theorem, due to Brandt, Chen, Faudree, Gould and Lesniak (1997), is interesting. (In the paper [5], the order condition is not “n ≥ 4k − 1” but “n ≥ 4k”. However, by using a theorem of Enomoto [9] and Wang [16] (“every graph G of order at least 3k with σ2(G) ≥ 4k − 1 contains k disjoint cycles”) for the cycles packing problem, we can obtain the following. See the proof in [5, Lemma 1].)

**Theorem E (Brandt et al. [5])** Let k be a positive integer, and let G be a graph of order n ≥ 4k − 1. If σ2 ≥ n, then G has a 2-factor with exactly k cycles.

This theorem shows that the Ore condition guarantees the existence of a hamilton cycle but also the existence of a 2-factor with a prescribed number of cycles.

By considering the relation between Theorem A and Theorem E, Chen, Gould, Kawarabayashi, Ota, Saito and Schiermeyer [6] conjectured that the Chvátal-Erdős condition in Theorem E also guarantees the existence of a 2-factor with exactly k cycles (see [6, Conjecture 1]). Chen et al. also proved that if the order of a 2-connected graph G with α(G) = α ≤ κ(G) is sufficiently large compared with k and with the Ramsey number r(α + 4, α + 1), then the graph G has a 2-factor with k cycles. In [12], Kaneko and Yoshimoto “almost” solved the above conjecture for k = 2 (see the comment after Theorem E in Chen et al. [6] for more details). Another related result can be found in [7]. But, the above conjecture is still open in general. In this sense, there is a big gap between hamilton cycles and 2-factors with exactly k (≥ 2) cycles.

In this paper, by combining the techniques of the proof for hamiltonicity and the proof for 2-factors with a prescribed number of cycles, we give the following Yamashita-type condition for 2-factors with k cycles.

**Theorem 1** Let k be a positive integer, and let G be an m-connected graph of order n ≥ 5k − 2. If σ2[m/k]+1 ≥ n, then G has a 2-factor with exactly k cycles.

This theorem implies the following:

**Remark 2**

- Theorem 1 is a generalization of Theorem D.
Theorem 1 leads to the Bondy-type condition: If $G$ is an $m$-connected graph of order $n \geq 5k - 2$ with $\sigma_{[m/k]+1}(G) > \frac{1}{2}([m/k] + 1)(n - 1)$, then $G$ has a 2-factor with exactly $k$ cycles. Therefore, Theorem 1 is also a generalization of Theorem 2 for sufficiently large graphs. (Recall that $\sigma^m_t(G) \geq \frac{1}{m} \cdot \sigma_m(G)$ and $\sigma_m(G) \geq \frac{m}{2} \cdot \sigma_2(G)$ for $m \geq t \geq 2$.)

Theorem 1 leads to the Chvátal-Erdős-type condition: If $G$ is a graph of order at least $5k - 2$ with $\alpha(G) \leq \lceil \kappa(G)/k \rceil$, then $G$ has a 2-factor with exactly $k$ cycles.

The complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ ($n$ is odd) does not contain a 2-factor, and hence the degree condition in Theorem 1 is best possible in this sense. The order condition in Theorem 1 comes from our proof techniques. Similar to the situation for the proof of Theorem 2, we will use the order condition only for the cycles packing problem (see Lemma 2 and the proof of Theorem 1 in Section 3). The complete bipartite graph $K_{2k-1,2k-1}$ shows that $n \geq 4k - 1$ is necessary. In the last section (Section 4), we note that “$n \geq 5k - 2$” can be replaced with “$n \geq 4k - 1$” for the Bondy-type condition (and the Chvátal-Erdős-type condition) in Remark 2.

Table 1 summarizes the conditions mentioned in the above.

|                        | hamilton cycle | 2-factor with $k$ cycles |
|------------------------|----------------|-------------------------|
| **Ore-type**           | $\sigma_2 \geq n$ | $\sigma_2 \geq n$       |
| **Chvátal-Erdős-type** | $\alpha \leq \kappa$ | $\alpha \leq \lceil \kappa/k \rceil$ |
| **Bondy-type**         | $\sigma_{n+1} > \frac{1}{2}(\kappa + 1)(n - 1)$ | $\sigma_{[\kappa/k]+1} > \frac{1}{2}(\lceil \kappa/k \rceil + 1)(n - 1)$ |
| **Yamashita-type**     | $\sigma_{\frac{\kappa}{k}+1} \geq n$ | $\sigma_{\frac{\kappa}{k}+1} \geq n$ |

Table 1: Comparison of the degree conditions

To prove Theorem 1 in the next section, we extend the concept of insertible vertices which was introduced by Ainouche [1], and we prove Theorem 1 in Section 3 by using it.

2 The concept of insertible vertices

In this section, we prepare terminology and notations and give some lemmas.

Let $G$ be a graph. For $v \in V(G)$ and $X \subseteq V(G)$, we let $N_G(v; X) = N_G(v) \cap X$ and $d_G(v; X) = |N_G(v; X)|$. For $V, X \subseteq V(G)$, let $N_G(V; X) = \bigcup_{v \in V} N_G(v; X)$. For
$X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. An $(x, y)$-path in $G$ is a path from a vertex $x$ to a vertex $y$ in $G$. We write a cycle (or a path) $C$ with a given orientation by $\overrightarrow{C}$. If there exists no fear of confusion, we abbreviate $\overrightarrow{C}$ by $C$. Let $C$ be an oriented cycle (or path). We denote by $\overleftarrow{C}$ the cycle $C$ with a reverse orientation. For $x \in V(C)$, we denote the successor and the predecessor of $x$ on $\overrightarrow{C}$ by $x^+$ and $x^-$. For $x, y \in V(C)$, we denote by $\overrightarrow{xy}$ the $(x, y)$-path on $\overrightarrow{C}$. The reverse sequence of $\overrightarrow{xy}$ is denoted by $\overleftarrow{xy}$. In the rest of this paper, we consider that every cycle (path) has a fixed orientation, unless stated otherwise, and we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$.

The following lemma is obtained by using the standard crossing argument, and so we omit the proof.

**Lemma 1** Let $G$ be a graph of order $n$, and let $P$ be an $(x, y)$-path of order at least 3 in $G$. If $d_G(x) + d_G(y) \geq n$, then $G$ contains a cycle of order at least $|P|$.

In [1], Ainouche introduced the concept of insertible vertices, which has been used for the proofs of the results on hamilton cycles. In this paper, we modify it for 2-factors with $k$ cycles, and it also plays a crucial role in our proof. Let $G$ be a graph, and let $\mathcal{D} = \{D_1, \ldots, D_{r+s}\}$ $(r + s \geq 1)$ be the set of $r$ cycles and $s$ paths in $G$ which are pairwise disjoint. For a vertex $x$ in $G - \bigcup_{1 \leq p \leq r+s} D_p$, the vertex $x$ is **insertible** for $\mathcal{D}$ if there is an edge $uv$ in $E(D_p)$ such that $xu, xv \in E(G)$ for some $p$ with $1 \leq p \leq r+s$. In the following lemma, “partition” of a graph means a partition of the vertex set.

**Lemma 2** Let $G$ be a graph, and let $\mathcal{D} = \{D_1, \ldots, D_{r+s}\}$ $(r + s \geq 1)$ be the set of $r$ cycles and $s$ paths in $G$ which are pairwise disjoint, and $P$ be a path in $G - \bigcup_{1 \leq p \leq r+s} D_p$. If every vertex of $P$ is insertible for $\mathcal{D}$, then $G[\bigcup_{1 \leq p \leq r+s} V(D_p) \cup V(P)]$ can be partitioned into $r$ cycles and $s$ paths.

**Proof of Lemma 2.** By choosing the following two vertices $u, v \in V(P)$ and the edge $uv^+ \in \bigcup_{1 \leq p \leq r+s} E(D_p)$ inductively, we can get the desired partition of $G[\bigcup_{1 \leq p \leq r+s} V(D_p) \cup V(P)]$. Let $u$ be the first vertex along $\overrightarrow{P}$, and take an edge $uv^+ \in E(D_i)$ ( $\subseteq \bigcup_{1 \leq p \leq r+s} E(D_p)$) such that $uw, uv^+ \in E(G)$ for some $i$ with $1 \leq i \leq r+s$ (since $u$ is insertible for $\mathcal{D}$, we can take such an edge). We let $v$ be the last vertex along $\overrightarrow{P}$ such that $vw, vw^+ \in E(G)$ (may be $u = v$). Then, we can insert all vertices of $u \overrightarrow{P} v$ into $D_i$. In fact, by replacing the edge $uv^+$ by the path $uw \overrightarrow{P} vw^+$, we can obtain a spanning subgraph $D'_i$ of $G[V(D_i \cup u \overrightarrow{P} v)]$ such that $D'_i$ is a cycle if $D_i$ is a cycle; otherwise, $D'_i$ is a path. By the choice of $u$ and $v$, we have $zw \notin E(G)$ or $zw^+ \notin E(G)$ for each vertex $z$ of $P' := P - u \overrightarrow{P} v$, and hence every vertex of $P'$ is insertible for $\mathcal{D}' = \{D_1, \ldots, D_{i-1}, D'_i, D_{i+1}, \ldots, D_{r+s}\}$. Thus, we can repeat this argument for the path $P'$ and the set $\mathcal{D}'$, and we get then the desired
partition. □

In the rest of this section, we fix the following. Let \( C_1, \ldots, C_k \) be \( k \) disjoint cycles in a graph \( G \), and let \( C^* = \bigcup_{1 \leq p \leq k} C_p \). Choose \( C_1, \ldots, C_k \) so that

\[
|C^*| \left( = \sum_{1 \leq p \leq k} |C_i| \right) \text{ is as large as possible.}
\]

Suppose that \( C^* \) does not form a 2-factor of \( G \). Let \( H = G - C^* \), and let \( H_0 \) be a component of \( H \) and \( x_0 \in V(H_0) \). Let

\[ u_1, u_2, \ldots, u_l \] be \( l \) distinct vertices in \( N_G(H_0; C_1) \), where \( l \geq 2 \).

We assume that \( u_1, u_2, \ldots, u_l \) appear in this order on \( C_1 \), and let \( u_{i+1} = u_1 \). Note that by the maximality of \( |C^*| \), \( u_i \neq u_{i+1} \) for \( 1 \leq i \leq l \). We denote by \( Q_i \) and \( Q_{i,j} \) a \((u_i, x_0)\)-path in \( G[V(H_0) \cup \{u_i\}] \) and a \((u_i, u_j)\)-path passing through a vertex of \( H_0 \) in \( G[V(H_0) \cup \{u_i, u_j\}] \), respectively.

**Lemma 3** For \( 1 \leq i \leq l \), \( u_i^+ \overrightarrow{C_1} u_{i+1}^- \) contains a non-insertible vertex for \( \{C_2, \ldots, C_k\} \).

**Proof of Lemma 3** Suppose that every vertex of \( u_i^+ \overrightarrow{C_1} u_{i+1}^- \) is insertible for \( \{C_2, \ldots, C_k\} \). Then, by Lemma 2 \( G \left[ \bigcup_{2 \leq p \leq k} V(C_p) \cup V(u_i^+ \overrightarrow{C_1} u_{i+1}^-) \right] \) has a 2-factor with exactly \( k - 1 \) cycles. With the cycle \( u_{i+1} \overrightarrow{C_1} u_i Q_{i+1,i} u_{i+1} \), we can get \( k \) disjoint cycles in \( G \) such that the sum of the orders is larger than \( |C^*| \), a contradiction. □

For \( 1 \leq i \leq l \), let \( x_i \) be the first non-insertible vertex for \( \{C_2, \ldots, C_k\} \) in \( V(u_i^+ \overrightarrow{C_1} u_{i+1}^-) \) on \( C_1 \), i.e., every vertex of \( u_i^+ \overrightarrow{C_1} x_i^- \) is insertible for \( \{C_2, \ldots, C_k\} \), but \( x_i \) is not insertible (Lemma 3 guarantees the existence of such a vertex \( x_i \)).

**Lemma 4** Let \( i, j \) be integers with \( 1 \leq i, j \leq l \) and \( i \neq j \). If \( x \in V(u_i^+ \overrightarrow{C_1} x_i) \) and \( x' \in \{x_0, u_j^+\} \), then (i) \( xx' \notin E(G) \), and (ii) \( d_G(x; H \cup C_1) + d_G(x'; H \cup C_1) \leq |H \cup C_1| - 1 \).

**Proof of Lemma 4** Consider the path

\[
P = \begin{cases} 
  x \overrightarrow{C_1} u_i Q_i x_0 & \text{(if } x' = x_0) \\
  x \overrightarrow{C_1} u_i Q_i j u_i^+ & \text{(if } x' = u_j^+) 
\end{cases}
\]

See Figure 1. Then, \( P \) is a path in \( G[V(H \cup x \overrightarrow{C_1} u_i)] \) passing through all vertices of \( x \overrightarrow{C_1} u_i \) and a vertex of \( H_0 \). Recall that every vertex of \( u_i^+ \overrightarrow{C_1} x_i^- \) is insertible for \( \{C_2, \ldots, C_k\} \), and hence \( G \left[ \bigcup_{2 \leq p \leq k} V(C_p) \cup V(u_i^+ \overrightarrow{C_1} x_i^-) \right] \) has a 2-factor with exactly \( k - 1 \) cycles (by Lemma 2). Hence, the maximality of \( |C^*| \) and Lemma 1 yield that \( xx' \notin E(G) \) and \( d_G(x; H \cup x \overrightarrow{C_1} u_i) + d_G(x'; H \cup x \overrightarrow{C_1} u_i) \leq |H \cup x \overrightarrow{C_1} u_i| - 1 \).


In particular, (i) holds. Then, by applying (i) for each vertex in \( u_i^+ C_1 x^- \) and the vertex \( x' \), we have \( N_G(x'; u_i^+ C_1 x^-) = \emptyset \). Combining this with the above inequality, we get,

\[
d_G(x; H \cup C_1) + d_G(x'; H \cup C_1)
\]
\[
= d_G(x; H \cup x C_1 u_i) + d_G(x'; H \cup x C_1 u_i) + d_G(x; u_i^+ C_1 x^-)
\]
\[
\leq (|H \cup x C_1 u_i| - 1) + |u_i^+ C_1 x^-| = |H \cup C_1| - 1.
\]

Thus (ii) also holds. \( \square \)

3 Proof of Theorem \( \text{(1)} \)

Before proving Theorem \( \text{(1)} \) we will give the following lemma for the cycles packing problem.

**Lemma 5** Let \( k, m, n \) and \( G \) be the same ones as in Theorem \( \text{(1)} \). Under the same degree sum condition as Theorem \( \text{(1)} \), \( G \) contains \( k \) disjoint cycles.

**Proof of Lemma 5.** If \( k = 1 \), then it is easy to check that \( G \) contains a cycle. If \( \lfloor m/k \rfloor = 1 \) or \( \lfloor m/k \rfloor \geq 3 \), then by a theorem of Enomoto \[9\], \( G \) contains \( k \) disjoint cycles (note that if \( \lfloor m/k \rfloor \geq 3 \), then \( G \) is \((2k+1)\)-connected, that is, the minimum degree \( \delta(G) \) is at least \( 2k+1 \)). Thus, we may assume that \( k \geq 2 \) and \( \lfloor m/k \rfloor = 2 \). Then, we have \( \delta(G) \geq m \geq k+1 \) and \( \sigma_3(G) = \sigma_2^{\lfloor m/k \rfloor +1}(G) \geq n \geq 5k-2 \). Note that, by the definition of \( \sigma_3(G) \) and \( \sigma_3(G) \), \( \sigma_3(G) \geq \sigma_2^3(G) + \delta(G) \). Note also that \( n \geq 5k-2 \geq 3k+2 \geq 8 \) because \( k \geq 2 \). Hence, by a theorem of Fujita et al. \[10\] (“every graph \( G \) of order at least \( 3k+2 \geq 8 \) with \( \sigma_3(G) \geq 6k-2 \) contains \( k \) disjoint cycles”), we can get the desired conclusion. \( \square \)

Now we are ready to prove Theorem \( \text{(1)} \).
Proof of Theorem \[1\]. Let \( G \) be an \( m \)-connected graph of order \( n \geq 5k - 2 \) such that \( \sigma_2^{[m/k]+1}(G) \geq n \). We show that \( G \) has a 2-factor with exactly \( k \) cycles. By Theorem \[2\] we may assume that \( \lceil m/k \rceil \geq 2 \). By Lemma \[3\] \( G \) contains \( k \) disjoint cycles. Let \( C_i \) for \( 1 \leq i \leq k \), \( C^* \), \( H \), \( H_0 \), \( x_0 \) and \( u_i \) for \( 1 \leq i \leq l \) be the same graphs and vertices as the ones described in the paragraph preceding Lemma \[3\] in Section \[2\]. In particular, we may assume that \( l = \lceil m/k \rceil \). Because, since \( G \) is \( m \)-connected, it follows that \( |N_G(H_0; C^*)| \geq m \) (note that by the maximality of \( |C^*| \), \( |C^*| > m \)), and hence, without loss of generality, we may assume that \( |N_G(H_0; C_1)| \geq \lceil m/k \rceil (\geq 2) \).

We first consider the set
\[
X = \{x_0\} \cup \{u_i^+: 1 \leq i \leq l\}.
\]

Then, Lemma \[4\] implies the following:

1. \( X \) is an independent set of size \( l + 1 \).
2. \( d_G(x; H \cup C_1) + d_G(x'; H \cup C_1) \leq |H \cup C_1| - 1 \) for \( x, x' \in X \) (\( x \neq x' \)).

On the other hand, by the maximality of \( |C^*| \) and Lemma \[2\] \( x_0 \) is non-insertible for \( \{C_2, \ldots, C_k\} \). This implies the following:

3. \( d_G(x_0; C_p) \leq |C_p|/2 \) for \( 2 \leq p \leq k \), and hence \( d_G(x_0; C^* - C_1) \leq |C^* - C_1|/2 \).

Since \( \sigma_2^{[m/k]}(G) \geq n \), it follows from \[1\] that there exist two distinct vertices \( x \) and \( x' \) in \( X \) such that \( d_G(x) + d_G(x') \geq n \). Then, by \[2\], we get
\[
d_G(x; C^* - C_1) + d_G(x'; C^* - C_1) \geq n - (|H \cup C_1| - 1) = |C^* - C_1| + 1.
\]

Combining this with \[3\] and the definition of \( X \), we may assume that

4. \( d_G(u_i^+; C^* - C_1) > |C^* - C_1|/2 \).

Next, let \( x_1 \) be the first non-insertible vertex for \( \{C_2, \ldots, C_k\} \) in the path \( u_1^- \overset{C_1}{\rightarrow} C_1 u_i^+ \) on \( C_1 \) (we can take such a vertex by Lemma \[3\] and the symmetry of \( C_1 \) and \( C_1 \)), and we consider the set
\[
Y = \{x_0, x_1\} \cup \{u_i^-: 2 \leq i \leq l\}.
\]

Then, by the symmetry of \( C_1 \) and \( C_1 \), Lemma \[4\] and since \( x_1 \) is non-insertible for \( \{C_2, \ldots, C_k\} \), we have the following:

5. \( Y \) is an independent set of size \( l + 1 \).

6. \( d_G(y; H \cup C_1) + d_G(y'; H \cup C_1) \leq |H \cup C_1| - 1 \) for \( y, y' \in Y \) (\( y \neq y' \)).

7. \( d_G(x_1; C_p) \leq |C_p|/2 \) for \( 2 \leq p \leq k \), and hence \( d_G(x_1; C^* - C_1) \leq |C^* - C_1|/2 \).
Since \( \sigma_{l+1}(G) \geq n \), it follows from (5) that there exist two distinct vertices \( y \) and \( y' \) in \( Y \) such that \( d_G(y) + d_G(y') \geq n \). Then, by (6), we get
\[
d_G(y; C^* - C_1) + d_G(y'; C^* - C_1) \geq n - (|H \cup C_1| - 1) = |C^* - C_1| + 1.
\]
Combining this with (3), (7) and the definition of \( Y \), we have the following:

\[
(8) \quad d_G(u^-; C^* - C_1) + d_G(u^+; C^* - C_1) > |C^* - C_1|/2 \text{ for some } i \text{ with } 2 \leq i \leq l.
\]

By (4) and (8), we have
\[
d_G(u^+ + 1; C^* - C_1) + d_G(u^-; C^* - C_1) > |C^* - C_1| = \sum_{2 \leq p \leq k} |C_p|.
\]

Hence, there exists a cycle \( C_p \) (2 \leq p \leq k), say \( p = 2 \), such that
\[
d_G(u^+_1; C^* - C_1) + d_G(u^-_1; C^* - C_1) = |C^* - C_1| + 1.
\]

This implies that there exists an edge \( uv \) in \( E(C_2) \) such that \( u^+_1, u^-_1, v \in E(G) \). By changing the orientation of \( C_2 \) if necessary, we may assume that \( u^+ = v \). Note that \( i \geq 2 \), and consider two cycles
\[
D_1 = u^-_1 \xrightarrow{\gamma_1} u_i \xrightarrow{Q_1} u_i \text{ and } D_2 = u^+_1 \xrightarrow{\gamma_1} u^-_i \xrightarrow{C_2} uu_1^+ \text{ (see Figure 2)}.
\]

Then, \( D_1, D_2, C_3, \ldots, C_k \) are \( k \) disjoint cycles such that the sum of the orders is

larger than \( |C^*| \), a contradiction.

This completes the proof of Theorem 1. \( \square \)

4 Notes on the order condition

As shown in the argument of the previous section, in the proof of Theorem 1, the order condition \( n \geq 5k - 2 \) is required only to show the existence of \( k \) disjoint cycles in a graph \( G \) (recall that the order condition in Theorem 2 is also). Therefore, the proof of Theorem 1 actually implies the following.
Theorem 3 Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n$. Suppose that $G$ contains $k$ disjoint cycles. If $d_2^{\lceil m/k \rceil + 1} \geq n$, then $G$ has a 2-factor with exactly $k$ cycles.

From this theorem, if we can obtain better results on the cycles packing problem, then the order conditions in Theorem 1 and Remark 2 can be improved. In fact, by using the result of Kierstead, Kostochka and Yeager (2017) and modifying the proof of Lemma 5, we can obtain a sharp order condition for the result in Remark 2 (see Corollary 4).

Theorem F (Kierstead et al. [13]) Let $k$ be an integer with $k \geq 2$, and let $G$ be a graph of order $n \geq 3k$ with $\delta(G) \geq 2k - 1$. Then $G$ contains $k$ disjoint cycles if and only if (i) $\alpha(G) \leq n - 2k$, and (ii) if $k$ is odd and $n = 3k$, then $G \not\cong 2K_k \vee \bar{K}_k$ and if $k = 2$, then $G$ is not a wheel.

Lemma 6 Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n \geq 4k - 1$. If $\sigma_{\lceil m/k \rceil + 1}(G) > \frac{1}{2}\left(\lceil m/k \rceil + 1\right)(n - 1)$, then $G$ contains $k$ disjoint cycles.

Proof of Lemma 6 By a similar argument as in the proof of Lemma 5, we have the following: If $k = 1$, then we can easily find a cycle; If $\lceil m/k \rceil = 1$ or $\lceil m/k \rceil \geq 3$, then by a theorem of Enomoto [9], $G$ contains $k$ disjoint cycles; If $\lceil m/k \rceil = 2$, and $k \geq 3$ or $n \geq 4k$, then by a theorem of Fujita et al. [10], $G$ contains $k$ disjoint cycles. Thus, we may assume that $k = 2$, $\lceil m/k \rceil = 2$ and $n = 4k - 1 = 7$. Then, $\delta(G) \geq m \geq k + 1 = 3 = 2k - 1$ and $\sigma_3(G) > \frac{3}{2}(n - 1) = 6k - 3 = 9$. Since $n = 7$ and $\sigma_3(G) > 9$, it follows that $\alpha(G) \leq 3 = n - 2k$ and $G$ is not a wheel. Hence, by Theorem F, $G$ contains two disjoint cycles. Thus, the lemma follows.

Recall that $\sigma_t^m(G) \geq \frac{t}{m} \cdot \sigma_m(G)$ for $m \geq t \geq 2$, and hence Theorem 3 and Lemma 6 lead to the following.

Corollary 4 Let $k$ be a positive integer, and let $G$ be an $m$-connected graph of order $n \geq 4k - 1$. If $\sigma_{\lceil m/k \rceil + 1}(G) > \frac{1}{2}\left(\lceil m/k \rceil + 1\right)(n - 1)$, then $G$ has a 2-factor with exactly $k$ cycles.

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