Cosmological Evolution of a Statistical System of Degenerate Scalarly Charged Fermions with an Asymmetric Scalar Doublet.

II. One-Component System of Doubly Charged Fermions

Yu. G. Ignat’ev1*, A. A. Agathonov2, and D. Yu. Ignatyev1

1Institute of Physics, Kazan Federal University, Kremlyovskaya St. 18, Kazan, 420008 Russia
2Lobachevsky Institute of Mathematics and Mechanics, Kazan Federal University, Kremlyovskaya St. 18, Kazan, 420008 Russia

Received July 18, 2021; revised October 13, 2021; accepted October 22, 2021

Abstract—Based on the previously formulated mathematical model of a statistical system with scalar interaction of fermions, a cosmological model is studied, based on a one-component statistical system of doubly scalarly charged degenerate fermions interacting with an asymmetric scalar doublet of canonical and phantom scalar fields. A relationship of the presented model with previously studied models based on one-component and two-component fermion systems is investigated. The asymptotic and limiting properties of the cosmological model are investigated, it is shown that among all models there is a class of models with finite lifetime. The asymptotic behavior of models near the corresponding singularities is investigated, a qualitative analysis of the corresponding dynamical system is carried out, and numerical implementations of such models are constructed. Based on numerical integration, it is shown that in the present model there can be transitions from a stable asymptotically vacuum state with a zero canonical field and a constant phantom field, corresponding to the phase of cosmological compression, to a symmetric state corresponding to an expansion phase. The time interval of the transition between these phases is accompanied by oscillations of the canonical scalar field.

DOI: 10.1134/S0202289322010066

1. INTRODUCTION

In the recent years, the simultaneous direct experimental detection of gravitational waves and BHs (BHs) in 2016 [1, 2] and their intensive study, in particular, confirmed the earlier indirect observations of the orbits of stars near the supermassive BH at the center of our Galaxy with a mass of about $4\times10^6$ $M_{\odot}$ [see, for example, [3, 4]], as well as the existence of supermassive BHs at the centers of galaxies with masses in the range of $10^9 - 10^{10}$ $M_{\odot}$.

It is believed that supermassive BHs with a mass of $\sim10^9 M_{\odot}$ are the central objects of luminous quasars observed at $z > 6$, but their astrophysical origin remains not fully understood. Currently, more than 200 quasars with $z > 6$ and several objects with $z > 7$ have been discovered. The quasar with the largest redshift at $z = 7.5$, which corresponds to the age of the Universe of 650 million years, has an absolute luminosity of $1.4 \times 10^{47}$ erg/s, while the mass estimate from the gas velocity in the quasar gives a value of $(1.6 \pm 0.4) \times 10^9 M_{\odot}$ [5]. Other detected quasars at $z > 7$ have supermassive BHs of similar mass. These observational data raise the question of the mechanism of formation and rapid growth of such objects in the early Universe.

The results of numerical simulation impose a number of restrictions on the parameters of the formation of supermassive BHs. It was shown by [6] that light embryos with a mass of $M \lesssim 10^8 M_{\odot}$ cannot grow to masses of the order of $10^8 M_{\odot}$ by $z = 6$ even with supercritical accretion. The formation of the first supermassive BHs with masses of $10^8 - 10^9 M_{\odot}$ requires heavier nuclei of $M \sim 10^4 - 10^6 M_{\odot}$ and gas-rich galaxies containing quasars. However, there are currently no convincing models for the emergence of such heavy embryos. In addition, it was found that the spatial density of luminous quasars decreases rapidly with increasing redshift, and this trend increases beyond $z = 5 - 6$ [7], so that such quasars are the rarest objects detected at a large redshift.

The interest in the formation mechanisms of supermassive BHs, taking into account the fact of their

*E-mail: ignatev-yuri@mail.ru, yuri.ignatev.1947@yandex.ru
dominant presence in the composition of quasars, is caused, in particular, by the fact that such BHs are formed in the composition of quasars at fairly early stages of the evolution of the Universe, before the formation of stars. This circumstance, in particular, opens up the possibility of the formation of supermassive BHs under conditions where scalar fields and baryonic dark matter can have a significant influence on this process. In this regard, we note the papers [8], which consider the possible existence of scalar halos and scalar hair in the vicinity of supermassive BHs.

In the papers [11] and [12], the following assumption was formulated, based on a study of systems of scalarly charged degenerate fermions with phantom interaction: (1). a cold completely degenerate Fermi system with very large effective masses of scalarly charged fermions can become a good model of dark matter; (2). at a certain stage of cosmological evolution, gravitational instabilities in nonrelativistic matter can lead to the emergence of isolated regions with dark matter; (3) standard Cooper mechanisms in Fermi systems with particle attraction can lead to the formation of bosons from pairs of fermions and thereby to superfluidity of dark matter regions; (4) with the growth of the effective masses of fermions in the growing scalar field above the Planck value, massive fermions can form stable primary BHs, taking into account the Hawking theorems about BHs, in the scenario with superfluid quasibosons with zero spin.

In [13], the assumption about the instability of short-wave perturbations was confirmed, but for Fermi systems with a canonical scalar interaction. These preliminary studies have shown the need for a comprehensive and more in-depth study of statistical systems of scalar charged particles.

Further, in [14], two simplest models of the interaction of fermions with an asymmetric scalar doublet were proposed: in the first model, such an interaction is carried out by two types of different-grade fermions, one of which is the source of a canonical scalar field, and the second is a phantom field (model $\mathcal{M}_1$); in the second model, there is one kind of fermions with a pair charge—canonical and phantom (model $\mathcal{M}_2$). A qualitative analysis of the dynamic systems of the $\mathcal{M}_1$ model was also carried out there. In [15], a numerical simulation of the $\mathcal{M}_1$ cosmological model was carried out, on the basis of which the features of this model were revealed, in particular, the possible existence of phases of cosmological compression, oscillations of the Hubble parameter, as well as the possibility of universes with finite lifetime.

In this paper, we investigate the $\mathcal{M}_2$ model of a one-component statistical system with an asymmetric scalar interaction of fermions and compare it with the results of previous studies. Due to a large number of parameters of the $\mathcal{M}_2$ model, as well as on the basis of aesthetic considerations, we will restrict ourselves to the case of zero seed mass of $\zeta$-fermions and identify the most typical cases of the cosmological model behavior.

2. MATHEMATICAL MODEL OF A COSMOLOGICAL SYSTEM OF SCALARLY CHARGED FERMIONS WITH AN ASYMMETRIC SCALAR HIGGS DOUBLET

Consider the spatially flat model of the Friedmann universe:\(^2\)

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).$$

(1)

Let us introduce the Hubble parameter $H(t)$,

$$H = \frac{\dot{a}}{a},$$

(2)

and the invariant cosmological acceleration $\Omega(t)$,

$$\Omega = \frac{\ddot{a}a}{a^2} = 1 + \frac{\dddot{H}}{H^2} = -\frac{1}{2}(1 + 3w),$$

(3)

where $w$ is the barotropic coefficient

$$w = p/\varepsilon,$$

(4)

$p$ and $\varepsilon$ being the total pressure and energy density of cosmological matter.

Consider a statistical system consisting of $n$ kinds of degenerate scalarly charged fermions with scalar charges $q_{(a)}$ with respect to $N$ scalar fields $\Phi_r$. Let then

$$m_{(a)} = m_{(a)}^0 + \sum_r q_{(a)}^r \Phi_r$$

(5)

be the dynamic masses of these fermions [14], and $L_s$ is the Lagrange function of noninteracting scalar Higgs fields, $L_s = \sum_r L_{(r)}$, with

$$L_{(r)} = \frac{1}{16\pi} \sum_r \left( e_r g^{ik} \Phi_{r,i} \Phi_{r,k} - 2 V_r(\Phi_r) \right);$$

(6)

$$V_r(\Phi_r) = -\frac{\alpha_r}{4} \left( \Phi_r^2 - \frac{m_r^2}{\alpha_m} \right)^2$$

(7)

is the potential energy of the corresponding scalar fields, $\alpha_r$ are constants of their self-action, $m_r$ their quantum masses, and $e_r = \pm 1$ their indicators (the plus sign corresponds to canonical scalar fields, and the minus sign to phantom ones).

\(^2\)In this article, we use the metric signature $(- + + +)$, and the Ricci tensor is determined by convolution of the first and third indices of the curvature tensor (see, for example, [16]).
Further, the energy-momentum tensor of the scalar fields corresponding to the Lagrange function (30) is

\[ T_{(s)k}^i = \frac{1}{16\pi} \sum_r \left( 2e_r \Phi_r \Phi_{r,k} - e_r \delta_{ij} \Phi_{r,j} \Phi_{r}^j + 2V_r(\Phi_r) \delta_{ik} \right), \]

and the energy-momentum tensor of an equilibrium statistical system is

\[ T_{(p)k}^i = (\varepsilon_p + p_p)u_i u_k - \delta_{ik} p_p, \]

where \( u^i \) is the macroscopic velocity vector of the statistical system, \( \varepsilon_p \) and \( p_p \) are its energy density and pressure. Einstein’s equations for the system “scalar fields + particles” have the form

\[ R_k^i - \frac{1}{2} \delta_k^i R = 8\pi T_{(s)k}^i + \delta_k^i \Lambda_0, \]

where

\[ T_{(s)k}^i = T_{(s)k}^i + T_{(p)k}^i. \]

\( \Lambda_0 \) is the seed value of the cosmological constant associated with its observed value \( \Lambda \) by the relation

\[ \Lambda = \Lambda_0 - \frac{1}{2} \sum_r m_r^4. \]

2.1. General Relations for Models of Degenerate Fermions with Scalar Interaction

One can prove\(^3\) that a strict consequence of the general-relativistic kinetic theory for statistical systems of completely degenerate fermions is the Fermi momentum conservation law \( \pi_{(a)} \) for each component,

\[ a(t)\pi_{(a)}(t) = \text{const}. \]

Assuming further for certainty \( a(0) = 1 \) (see [15]) and

\[ \xi = \ln a, \quad \xi \in (-\infty, +\infty), \quad \xi(0) = 0, \]

we introduce the dimensionless functions

\[ \psi_{(a)} = \frac{\pi_{(a)}^{0} e^{-\xi}}{|m_{(a)}|} \left( \pi_{(a)} = \pi_{(a)}(0) \right), \]

equal to the ratio of the Fermi momentum \( \pi_{(a)} \) to the total energy of the fermion, as well as the functions \( F_1(\psi) \) and \( F_2(\psi) \):

\[ F_1(\psi) = \psi \sqrt{1 + \psi^2} - \ln(\psi + \sqrt{1 + \psi^2}) \]

\[ F_2(\psi) = \psi \sqrt{1 + \psi^2} + 2\psi \ln(\psi + \sqrt{1 + \psi^2}), \]

with the help of which we will determine the macroscopic scalars of the statistical system: the particle number density of the \( a \)-th component

\[ n_{(a)} = \frac{1}{\pi^2} \pi_{(a)}^3, \]

the pressure of the \( a \)-th component

\[ p_p = \sum_a m_{(a)}^4 \frac{1}{2\pi^2} F_2(\psi_{(a)}), \]

the energy density of the fermion system

\[ p_p = \sum_a m_{(a)}^4 \frac{1}{2\pi^2} F_2(\psi_{(a)}), \]

and the scalar charge density of the fermion system with respect to the scalar field \( \Phi_r \)

\[ \sigma_r = \sum_a q_{(a)}^r \frac{m_{(a)}^3}{2\pi^2} F_1(\psi_{(a)}). \]

We will also write out useful expressions for further derived functions \( F_1(x) \) and \( F_2(x) \):

\[ F_1(x) = \frac{2 \psi^2}{\sqrt{1 + \psi^2}}, \quad F_2(x) = 8 \psi^2 \sqrt{1 + \psi^2}. \]

and the useful relation

\[ \varepsilon_p + p_p = \frac{1}{3\pi^2} \sum_a m_{(a)}^4 \psi_{(a)}^3 \sqrt{1 + \psi_{(a)}^2}. \]

Thus in the cosmological model consisting of a system of degenerate scalarly charged fermions and scalar fields, all macroscopic scalars are determined by explicit algebraic functions of the scalar potentials \( \Phi_1(t), \ldots, \Phi_N(t) \).

A complete closed normal autonomous system of ordinary differential equations describing a cosmological model based on a statistical system of completely degenerate scalarly charged fermions has the form (see [14, 18])

\[ \dot{\xi} = H, \]

\[ \dot{H} = -\sum_r e_r Z_r^2 - \sum_a \frac{4m_{(a)}^2 \psi_{(a)}^3}{3\pi} \sqrt{1 + \psi_{(a)}^2}, \]

\[ \dot{\Phi_r} = Z_r, \quad (r = 1, N), \]

\[ e_r \dot{Z}_r = -e_r 3HZ_r - m_{(a)}^2 \Phi_r + \alpha_r \Phi_r^3 - 8\pi \sigma_r(t). \]

A strict consequence of the dynamical system (23)–(26) is the total energy integral

\[ 3H^2 - 8\pi E_{\text{eff}} = \]
const, the partial zero value of which corresponds to the Einstein equation (23)
\[3H^2 - \Lambda - \sum_r \left( \frac{e_r Z^2}{2} - \frac{m_r^2 \Phi_r^2}{2} + \alpha_r \Phi_r^4 \right)\]
\[-\frac{1}{\pi} \sum_a \frac{m_{(a)}^4 F_2(\psi_{(a)})}{2} + \sum_r \left( \frac{e_r Z^2}{2} - \frac{m_r^2 \Phi_r^2}{2} - \alpha_r \Phi_r^4 \right) = 0. \quad (27)\]

Equations (27) define a certain hypersurface \(\Sigma_E\) in the \(2N + 2\)-dimensional phase space of the dynamical system (23)–(26). In the future, we will call this hypersurface the Einstein hypersurface. The points of the phase space \(\mathbb{R}_{2N+2}\) in which the effective energy \(E_{\text{eff}}\) is negative are inaccessible to the dynamical system. The inaccessible region is separated from the accessible region of phase space by a hypersurface of zero effective energy \(S_E \subset \mathbb{R}_{2N+2}\), which is a cylinder with the axis \(OH\):
\[8\pi E_{\text{eff}} \equiv \Lambda + \frac{1}{\pi} \sum_{(a)} m_{(a)}^4 F_2(\psi_{(a)}) + \sum_r \left( \frac{e_r Z^2}{2} - \frac{m_r^2 \Phi_r^2}{2} - \alpha_r \Phi_r^4 \right) = 0, \quad (28)\]
moreover, the hypersurface of zero effective energy (28) is tangent to the Einstein hypersurface (27) in the hyperplane \(H = 0\):
\[\Sigma_E \cap S_E = H = 0. \quad (29)\]

The fact that the Einstein equation (27) is a special first integral of the dynamical system (23)–(26) allows one to use it to determine the initial value of the Hubble parameter \(H_0 = H(0)\), which we will do in the future. Equation (27) is quadratic with respect to \(H(t)\), so it has two symmetric roots \(\pm H(t)\), and the positive root corresponds to the expansion of the universe while the negative root corresponds to compression.

2.2. Cosmological Model \(\mathfrak{M}_2\) with Scalar Interaction of a One-Component Fermion System

Further, \(L_s\) is the Lagrange function of interacting canonical (\(\Phi\)) and phantom (\(\varphi\)) scalar fields:
\[L_s = \frac{1}{16\pi} \left( g^{ik} \Phi_{ik,\Phi} - 2V(\Phi) \right) + \frac{1}{16\pi} \left( -g^{ik} \varphi_{ik,\varphi} - 2V(\varphi) \right), \quad (30)\]
where
\[V(\Phi) = -\frac{\alpha}{4} \left( \Phi^2 - \frac{m^2}{\alpha} \right)^2, \quad \varphi(\varphi) = -\frac{\beta}{4} \left( \varphi^2 - \frac{m^2}{\beta} \right)^2\]
are the potential energies of the corresponding scalar fields, \(\alpha\) and \(\beta\) are the constants of their interaction, \(m\) and \(m\) are masses of their quanta. As a carrier of scalar charges, we consider a one-component degenerate fermion system in which the fermions \(\zeta\) simultaneously have two charges: the canonical charge \(e\) with respect to the canonical scalar field \(\Phi\) and the phantom charge \(e\) with respect to the phantom field \(\varphi\). In this case, the fermions can have some seed mass \(m_0^4\) and the initial Fermi momentum \(\pi_0\). In this case, Eqs. (14) and (20) take the form
\[\dot{\psi} = \frac{\pi_0}{|e\Phi + e\varphi|} e^{-\xi}, \quad (31)\]
\[\sigma_c = \frac{e}{2\pi^2} (e\Phi + e\varphi)^3 F_1(\psi), \quad \sigma_f = \frac{e}{2\pi^2} (e\Phi + e\varphi)^3 F_1(\psi). \quad (32)\]
Thus, in the \(\mathfrak{M}_2\) model, the following relation takes place:
\[\epsilon \sigma_c = \epsilon \sigma_f. \quad (33)\]

Let us write out a complete normal set of Einstein’s equations and those of the scalar fields \(\Phi(t)\) and \(\varphi(t)\) for this one-component system of scalarly charged degenerate fermions. In an obviously non-singular form, the normal set of ordinary differential equations of the model under study (23), (24), (25)–(26), taking into account (32), takes the form
\[\dot{Z} = -3HZ - m^2 \Phi + \alpha \Phi^3 - 4\epsilon \pi_0 (e\Phi + e\varphi) e^{-\xi}\]
\[\times \left( \pi_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2 + 4\epsilon \pi_0 (e\Phi + e\varphi)^3 \right) \times \ln \left( \frac{\pi_0 e^{-\xi} + \sqrt{\pi_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2}}{\sqrt{(e\Phi + e\varphi)^2}} \right); \quad (37)\]
\[\dot{\zeta} = -3HZ - m^2 \varphi - \beta \varphi^3 + 4\epsilon \pi_0 (e\Phi + e\varphi) e^{-\xi}\]
\[\times \left( \pi_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2 + 4\epsilon \pi_0 (e\Phi + e\varphi)^3 \right) \times \ln \left( \frac{\pi_0 e^{-\xi} + \sqrt{\pi_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2}}{\sqrt{(e\Phi + e\varphi)^2}} \right). \quad (38)\]

\^See a discussion on this issue in [15].
The first integral of the set of equations (27) for the $\mathcal{M}_2$ model takes the form

$$3H^2 - \frac{\alpha}{2} \Phi^2 - \frac{\beta}{2} \Phi^2 - \frac{\alpha \Phi^4}{4} + \frac{\mathbf{m}^2}{2} \varphi^2 - \frac{\frac{\beta}{4} \varphi^2}{\pi} e^{-\frac{\xi}{\pi}} \rho_0 \sqrt{\rho_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2} 
\times (2\pi_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2) + \frac{(e\Phi + e\varphi)^4}{\pi} \ln \left( \frac{(\rho_0 e^{-\xi} + \sqrt{\rho_0^2 e^{-2\xi} + (e\Phi + e\varphi)^2})}{(e\Phi + e\varphi)^2} \right) = 0.$$  

(39)

Thus, the complete normal autonomous system of ordinary differential equations of the $\mathcal{M}_2$ model consists of Eqs. (34)–(37) and (38). Moreover, we have the total energy integral (39) as the first integral of this system, which represents the Einstein hypersurface equation $\Sigma_E$ in the 6-dimensional phase space of the dynamical system

$$\mathbb{R}_6 = \{\xi, H, \Phi, Z, \varphi, z\},$$

on which all the phase trajectories of this system lie.

### 2.3. Relation of the Mathematical Models $\mathcal{M}_2$, $\mathcal{M}_1$, and $\mathcal{M}_{00}$ for Scalar Singlets

Following [15], we will call the cosmological model with a one-component system of scalarly charged fermions with a scalar singlet with a quadratic interaction potential, a nonnegative dynamic mass and a nonnegative Hubble parameter $H \geq 0$, studied in [12, 20], the $\mathcal{M}_{00}$ model, a model with a one-component system of scalarly charged fermions with a scalar Higgs singlet and an arbitrary sign of the Hubble parameter, investigated in [21], the $\mathcal{M}_0$ model, and finally, a model with a two-component system of single scalarly charged fermions, studied in [15], the $\mathcal{M}_1$ model.5

The dynamic equations of the model $\mathcal{M}_2$ (34)–(39) in the case of one of the scalar singlets, canonical, $\mathbf{C}$, or phantom, $\mathbf{F}$, transform into the corresponding equations of the two-component model $\mathcal{M}_1$, studied in the previous part of the work [15], for the following parameter values and initial conditions:

$$\mathcal{M}_2 \leftrightarrow \mathcal{M}_1 :$$

$\mathbf{C} : \quad \epsilon = 0, \quad \varphi(0) = 0, \quad z(0) = 0,$
$$\pi_f = 0, \quad \pi_c = \pi_0; \quad (40)$$

$\mathbf{F} : \quad e = 0, \quad \Phi(0) = 0, \quad Z(0) = 0,$
$$\pi_c = 0; \quad \pi_f = \pi_0. \quad (41)$$

Further, the dynamic equations of the model $\mathcal{M}_2$, (34)–(39), in the case of one of the scalar singlets, canonical, $\mathbf{C}$, or phantom, $\mathbf{F}$, transform into the corresponding equations of the one-component model $\mathcal{M}_0$, studied in [21], for the following parameter values and initial conditions:

$$\mathcal{M}_2 \leftrightarrow \mathcal{M}_0 :$$

$\mathbf{C} : \quad \epsilon = 0, \quad \varphi(0) = 0, \quad z(0) = 0,$
$$\pi_f = 0, \quad \pi_c = \pi_0; \quad (42)$$

$\mathbf{F} : \quad e = 0, \quad \Phi(0) = 0, \quad Z(0) = 0,$
$$\pi_c = 0; \quad \pi_f = \pi_0. \quad (43)$$

Finally, the dynamic equations of the $\mathcal{M}_2$ (34)–(39) model in the case of one of the scalar singlets, canonical, $\mathbf{C}$, or phantom, $\mathbf{F}$, are equivalent to the dynamic equations of the $\mathcal{M}_{00}$ model studied in [20], only under the following parameter values and conditions:

$$\mathcal{M}_2 \leftrightarrow \mathcal{M}_{00} :$$

$$H \geq 0, \quad \Phi \geq 0, \quad \varphi \geq 0, \quad \alpha = 0, \quad \beta = 0; \quad (44)$$

$\mathbf{C} : \quad \epsilon = 0, \quad \varphi(0) = 0, \quad z(0) = 0,$
$$\pi_f = 0, \quad \pi_c = \pi_0; \quad (45)$$

$\mathbf{F} : \quad e = 0, \quad \Phi(0) = 0, \quad Z(0) = 0,$
$$\pi_c = 0, \quad \pi_f = \pi_0. \quad (46)$$

### 3. Qualitative Analysis and Asymptotic Behavior of the $\mathcal{M}_2$ Model

#### 3.1. Singular Points of the Dynamic System of the $\mathcal{M}_2$ Model

The singular points of a dynamical system are determined by the equality of the right-hand sides of the normal set of equations to zero. Thus we obtain from (34)–(38) a set of algebraic equations for finding the coordinates of singular points:

$$Z = 0; \quad z = 0, \quad (47)$$
$$H = 0, \quad (48)$$

$$-m^2 \Phi + \alpha \Phi^3 - \frac{4\epsilon}{\pi} (e\Phi + e\varphi)^3 F_1(\psi) = 0, \quad (49)$$
$$m^2 \varphi - \beta \varphi^3 + \frac{4\epsilon}{\pi} (e\Phi + e\varphi)^3 F_1(\psi) = 0, \quad (50)$$

$$-\frac{4}{3\pi} (\rho_0^2)^3 e^{-3\xi} \sqrt{(e\Phi + e\varphi)^2 + \rho_0^2 e^{-2\xi}} = 0. \quad (51)$$

In addition, one must take into account the total energy integral (39),—the coordinates of a singular

---

5 Singlets with both canonical and phantom scalar fields were studied in the $\mathcal{M}_{00}$ and $\mathcal{M}_0$ models.
Thus the cosmological constant has singular points only at special values of the scalar potentials; in equation (39) can have either 2 or 4 singular points in the cosmological constant, the dynamical system (34)–(39) continuously transforms to that for the vacuum Higgs doublet (see [17]):

\[
\hat{H} = -\frac{Z^2}{2} + \frac{z^2}{2},
\]

\[
\hat{Z} = -3HZ - m^2\Phi + \alpha\Phi^3,
\]

\[
\dot{\phi} = -3Hz + m^2\varphi - \beta\varphi^3,
\]

\[
3H^2 - \Lambda - \frac{Z^2}{2} + \frac{z^2}{2} - \frac{m^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4}
- \frac{m^2\varphi^2}{2} + \frac{\beta\varphi^4}{4} = 0.
\]

Secondly, we note that the beginning of the Universe (the cosmological singularity \(a = 0\)) corresponds to \(\xi \to -\infty\), and the infinite future \(a \to \infty\) (if the model allows for such a state). It can be easily seen that for \(\xi \to +\infty\), the set of equations (34)–(39) also asymptotically tends to that for the vacuum Higgs doublet (57)–(60). Therefore, if the cosmological model admits the state \(\xi \to +\infty\), then the results of qualitative and numerical analysis of the vacuum model of an asymmetric scalar doublet [17] can be used to study the evolution of the cosmological model at late stages.

Third, for \(e = 0\), \(\Phi = 0\) or \(\epsilon = 0\), \(\varphi = 0\), the model transforms into the model of a single-component scalar Fermi system with the corresponding scalar singlet [18].

Fourth, at both zero charges of \(\zeta\)-fermions, the set of equations (34)–(39) continuously transforms to that for a cosmological model based on a vacuum asymmetric scalar Higgs doublet and a neutral one-component Fermi liquid (it is ultrarelativistic at \(m_0 = 0\)).

3.2.2. Asymptotic properties of the model near the singularity \(a \to 0\). We now investigate the model behavior near the cosmological singularity \(\xi \to -\infty\) (\(a \to 0\)). It follows from the Eqs. (34), (35)–(38) that such a state is always possible for \(\pi_0 \neq 0\). In this case, as \(\xi \to -\infty\), \(H \to \pm\infty\),

\[
|e\Phi + \epsilon\varphi| \ll \pi_0 e^{-\xi},
\]

and the set of equations (34)–(39) reduces to

\[
\dot{\xi} = H; \quad \dot{\Phi} = Z; \quad \dot{\varphi} = z,
\]

\[
\dot{H} = -\frac{4e^{-4\xi}}{3\pi}\pi_0^4,
\]

\[
\dot{Z} = -3HZ - \frac{4e\pi_0^2 e^{-2\xi}}{\pi}(e\Phi + \epsilon\varphi),
\]

\[
\dot{z} = -3Hz + \frac{4e\pi_0^2 e^{-2\xi}}{\pi}(e\Phi + \epsilon\varphi).
\]

6 This state is not allowed in all cases of model parameters, see below.
Replacing the variable in Eq. (63) by \( d/d t = H d / d \xi \), we find its solution:

\[
H = \pm \sqrt{\frac{2}{3 \pi} \pi_0^2 e^{-2 \xi}}, \quad (66)
\]

where the plus sign corresponds to exit from the singularity, minus to entering it. Substituting \( H \) from (62) to (66), we obtain a differential equation with respect to \( \xi(t) \):

\[
\dot{\xi} = \pm \sqrt{\frac{2}{3} \pi_0^2 e^{-2 \xi}} \Rightarrow d e^{2 \xi} = \pm \sqrt{\frac{8}{3} \pi_0^2} d t,
\]

whence, taking into account the definition of \( \xi(t) \) (13) and the condition for the existence of a singularity at the point \( t_0 \): \( \xi(t_0) = -\infty \Rightarrow a(t_0) = 0 \), we get the asymptotic behavior near the singularity:

\[
e^\xi \equiv a(t) \propto \left( \frac{8}{3} \right)^{1/4} \pi_0 \sqrt{|t-t_0|}, \quad (67)
\]

Thus near the singularity \( t \to t_0 \), the scale factor and the Hubble parameter have the following asymptotics:

\[
a(t) \bigg|_{t \to t_0} \propto \sqrt{|t-t_0|}, \quad H(t) \bigg|_{t \to t_0} \propto \frac{1}{t-t_0}, \quad (68)
\]

Finally, calculating the invariant cosmological acceleration \( \Omega \) (3) using (63), (66), and (67), we obtain near the singularity

\[
\Omega(t) \bigg|_{t \to t_0} \simeq -1, \quad (69)
\]

which corresponds, as is known, to the ultrarelativistic equation of state \( w = 1/3 \).

Multiplying both parts of Eq. (64) by \( e \) and both parts of Eq. (65) by \( e \), adding the equations obtained in this way and introducing the new field variable

\[\chi(t) \equiv e \Phi + e \varphi, \quad (70)\]

we obtain a linear homogeneous differential equation for the function \( \chi(t) \)

\[
\ddot{\chi} + 3H \dot{\chi} + \frac{4 \pi_0^2 e^{-2 \xi}}{\pi} (e^2 - e^2) \chi = 0. \quad (71)
\]

This equation is integrated with Bessel functions of the first kind \( J_2(z) \) and \( Y_2(z) \):

\[
\chi = C_1 \frac{J_2(\sqrt{\nu|t-t_0|})}{t-t_0} + C_2 \frac{Y_2(\sqrt{\nu|t-t_0|})}{t-t_0}, \quad (72)
\]

where

\[
\nu^2 \equiv \frac{2 \sqrt{6} \pi_0^2}{\pi} |e^2 - e^2|, \quad (73)
\]

Substituting this solution to the right-hand sides of (64) and (65), we obtain asymptotic solutions for the derivatives of the potentials and then the potentials themselves. So, for example,

\[
Z \simeq -\frac{\sqrt{6} e}{\pi \nu^2} \int t (C_1 J_2(\sqrt{\nu t}) + C_2 Y_2(\sqrt{\nu t})) dt,
\]

where \( \tau = t-t_0 \). Replacing the variable \( x = \sqrt{\nu t} \) in this integral, we bring it to the form:

\[
Z \simeq -\frac{2 \sqrt{6} e}{\pi \nu^2} \int x^3 (C_1 J_2(x) + C_2 Y_2(x)) dx.
\]

Applying the well-known recurrence relation for Bessel functions \( Z_p(x) \) (see, e.g., [22])

\[
\frac{d}{dx} x^p Z_p(x) = x^p Z_{p-1}(x),
\]

we find the asymptotic behavior of \( Z(x) \):

\[
Z \simeq -\frac{2 \sqrt{6} e}{\pi \nu^2} (C_1 J_3(\sqrt{\nu t}) + C_2 Y_3(\sqrt{\nu t})). \quad (74)
\]

### 3.3. Eigenvalues of the Dynamic System Matrix

Let us proceed with the study of the nature of singular points of the dynamical system in those special cases I–III (p. 6) when these points exist. Calculating the main matrix of the dynamical system (34)–(38) at the singular points (47) and (48), we find:

\[
A(M) = 
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\partial P_1 / \partial \xi & 0 & \partial P_1 / \partial \Phi & 0 & \partial P_1 / \partial \varphi & 0 \\
\partial P_2 / \partial \xi & 0 & \partial P_2 / \partial \Phi & 0 & \partial P_2 / \partial \varphi & 0 \\
\partial P_3 / \partial \xi & 0 & \partial P_3 / \partial \Phi & 0 & \partial P_3 / \partial \varphi & 0 \\
\partial P_4 / \partial \xi & 0 & \partial P_4 / \partial \Phi & 0 & \partial P_4 / \partial \varphi & 0 \\
\partial P_5 / \partial \xi & 0 & \partial P_5 / \partial \Phi & 0 & \partial P_5 / \partial \varphi & 0 \\
\end{pmatrix}, \quad (75)
\]

where \( P_1, P_2, \) and \( P_3 \) are the right-hand sides of Eqs. (35), (37), and (38), respectively. One can make sure that in the general case the matrix (75) is not degenerate.

Using the obvious relations

\[
\frac{\partial \psi}{\partial \xi} = -\psi, \quad \frac{\partial \psi}{\partial \varphi} = -\frac{e^\psi}{e^\Phi + e^\varphi},
\]

as well as the definitions of the functions \( P_1, P_2, P_3 \) and the expressions (21) for the derivatives of the functions \( F_1(x) \) and \( F_2(x) \), we obtain expressions for partial derivatives of the functions \( P_k \) included in the matrix \( A(M) \):

\[
\frac{\partial P_1}{\partial \xi} = \frac{4 \pi_0^2 e^{-2 \xi}}{3 \pi} \sqrt{(e^\Phi + e^\varphi)^2 + 4 \pi_0^2 e^{-2 \xi}}.
\]

---

\(^7\) See, e.g., [22].
These eigenvalues correspond to the fact that as \( \xi \rightarrow \infty \), the dynamical system becomes degenerate: 

\[
\begin{align*}
\frac{\partial P_1}{\partial \Phi} &= -4\epsilon \pi_0^2 e^{-3\xi} \frac{e\Phi + \epsilon \varphi}{\sqrt{e(\varphi + \epsilon \varphi)^2 + \pi_0^2 e^{-2\xi}}}, \\
\frac{\partial P_1}{\partial \varphi} &= -4\epsilon \pi_0^2 e^{-3\xi} \frac{e\Phi + \epsilon \varphi}{\sqrt{e(\varphi + \epsilon \varphi)^2 + \pi_0^2 e^{-2\xi}}}, \\
\frac{\partial P_2}{\partial \Phi} &= \frac{8\psi^3}{\pi \sqrt{1 + \psi^2}} (e\Phi + \epsilon \varphi)^3, \\
\frac{\partial P_2}{\partial \varphi} &= -m^2 + 3\alpha \Phi^2 \frac{12 e^2}{\pi} (e\Phi + \epsilon \varphi)^2 F_1(\psi) + \frac{8 e^2 \psi^2 \pi_0^2 e^{-2\xi}}{\pi \sqrt{1 + \psi^2}}, \\
\frac{\partial P_3}{\partial \Phi} &= -\frac{8 e^2 \psi^2 \pi_0^2 e^{-2\xi}}{\pi \sqrt{1 + \psi^2}}, \\
\frac{\partial P_3}{\partial \varphi} &= \frac{12 e^2}{\pi} (e\Phi + \epsilon \varphi)^2 F_1(\psi) - \frac{8 e^2 \psi^2 \pi_0^2 e^{-2\xi}}{\pi \sqrt{1 + \psi^2}}.
\end{align*}
\]

Calculating the values of these coefficients of the matrix \( A(M) \) at the points \( M_{\pm \infty} \), we obtain the following values of nonzero coefficients of the matrix:

\[
\begin{align*}
\left. \frac{\partial P_2}{\partial \Phi} \right|_{\xi \rightarrow \infty} &= -m^2 + 3\alpha \Phi^2, \\
\left. \frac{\partial P_3}{\partial \varphi} \right|_{\xi \rightarrow \infty} &= m^2 - 3\beta \varphi^2.
\end{align*}
\]

Thus at the points \( M_{\infty, \pm pm} \) (54), the matrix of the dynamical system becomes degenerate:

\[
A(M) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2m^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2m^2 & 0
\end{pmatrix}_{M_{\infty, \pm \pm}},
\]

and has the following eigenvectors for the pairs of coordinates \([\xi, H], [\Phi, Z] \) and \([\varphi, z] \), respectively:

\[
k_{1,2} = 0; \quad k_{3,4} = \pm \sqrt{2}m; \quad k_{5,6} = \pm 1.\ (79)
\]

These eigenvalues correspond to the fact that as \( \xi \rightarrow +\infty \), the phase trajectories in the plane \([\Phi, Z] \) are unstable near the singular point \( \alpha \Phi^2 = m^2 \) (a saddle), but stable in the plane \([\varphi, z] \) near the singular point \( \beta \varphi^2 = m^2 \) (an attracting center).

At the points \( M_{\infty, \pm 0} \) (55), the matrix has the form

\[
A(M) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 2m^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & m^2 & 0
\end{pmatrix}_{M_{\infty, \pm 0}},
\]

and has the following eigenvectors for the pairs of coordinates \([\xi, H], [\Phi, Z], [\varphi, z] \), respectively:

\[
k_{1,2} = 0; \quad k_{3,4} = \pm \sqrt{2}m; \quad k_{5,6} = \pm \sqrt{2}m. \ (80)
\]

Thus the singular points \( M_{\infty, 0, \pm} \) are stable (attracting) centers, both in the plane \([\Phi, Z] \) \( \Phi = 0 \) and in the plane \([\varphi, z] \) \( \varphi = \pm m \). For \( \xi \rightarrow +\infty \), the phase trajectories asymptotically tend to the singular points \( M_{\infty, 0, \pm} \).

We note the following important circumstance. Since, as we found out in Section 3.2.1, the set of equations (34)–(39), as \( \xi \rightarrow +\infty \), asymptotically tends to the set of equations for the vacuum Higgs doublet (57)–(60), then a \( \xi \rightarrow +\infty \), the dynamical system will have singular points in the phase hyperspace

\[
\mathbb{R}_5 = \{ H, \Phi, Z, \varphi, z \} \subset \mathbb{R}_6,
\]

the singular points that completely coincide with those of the vacuum asymmetric scalar doublet [17]. Therefore, as \( \xi \rightarrow +\infty \), all properties of the vacuum asymmetric scalar doublet are valid for the present system.

4. NUMERICAL SIMULATION

Further, to shorten the text, we will specify a set of fundamental parameters of the \( \mathfrak{M}_2 \) model using the ordered list

\[
\mathbf{P}_2 = \{ \alpha, m, e, \beta, m, e, \pi_0, \Lambda \}.
\]
and the initial conditions with the ordered list
\[ I = [\Phi_0, Z_0, \varphi_0, z_0, \rho], \]
where \( \rho = \pm 1 \), and the value of \( \rho = +1 \) corresponds to a nonnegative initial value of the Hubble parameter, \( H_0 = H_+ \geq 0 \), and \( \rho = -1 \) corresponds to a negative initial value of the Hubble parameter, \( H_0 = H_- < 0 \). At the same time, using the autonomy of the dynamical system, we assume \( \xi(0) = 0 \) everywhere. Thus, the \( M_2 \) model is determined by 10 fundamental parameters and 5 initial conditions, the \( M_1 \) model is determined by 11 parameters
\[ P_1 = [\alpha, m, e, m_c, \pi_c], [\beta, \mu, \epsilon, m_f, \pi_f], \Lambda \]
and 5 initial conditions, the \( M_{00} \) model is determined by 5 fundamental parameters
\[ P_{00} = [\mu, e, q, m_0, \pi_0, \Lambda] \]
and one indicator \( e = \pm 1 \), where \( e = +1 \) corresponds to a canonical field, and \( e = -1 \) to a phantom field. The initial conditions for this model are set by a list of two elements \( I_{00} = [\Phi_0, Z_0] \).

4.1. Relation of the \( M_2 \) Model to Previously Studied Models for Scalar Singlets

In the case where \( \xi(t) \) is a monotonically increasing function, i.e., \( H \geq 0 \), the behavior of all models is practically the same. Let us consider the behavior of models in the case of a canonical scalar field under the following parameters and initial conditions:
\[ P_0 = [1, 1, 1], [1, 1, 0], 0.1, 0.02, \]
\[ I_0 = [0.2, 0.1, 0, 0, 1]. \]

The evolution of the function \( \xi(t) = \ln(\alpha(t)) \) in various models with a single canonical field \( \Phi \) and the parameters (81) and the initial conditions (82): the bold line is the \( M_2 \) model, the dashed line is the \( M_1 \) model, the thin line is the \( M_{00} \) model.

The tendency of the Hubble parameter to the horizontal line, shown in Fig. 2, according to Eq. (3), means that in this case \( \Omega(t \to +\infty) \to 1 \) and \( w = -1 \), i.e., as time passes, the models with such parameters tend to inflation.

However, in the cases where the nonnegativity condition of the Hubble parameter \( H \geq 0 \) is violated, the behavior of the \( M_2 \) and \( M_{00} \) models is very different. The behavior of the \( M_2 \) and \( M_{00} \) models in the case of a phantom singlet is shown in Fig. 3. It can be seen from this plot that the evolution of the Hubble parameter \( H(t) \) in the \( M_{00} \) model is restricted to the domain \( H \geq 0 \) and clearly reveals the incorrectness of
this model. In the $\mathcal{M}_2$ model the Hubble parameter starts with a constant asymptotic value $H(\infty) = -0.3$ and eventually reaches the symmetric asymptotic value $H(+\infty) = 0.3$. These values $H = \pm 0.3$ just correspond to the attracting singular points in the model of a vacuum phantom singlet (see [17]). Thus, in this case, an inflationary compression $(H < 0, \Omega = 1)$ in the past is replaced by an inflationary expansion $(H > 0, \Omega = 1)$ in the future.

This example, among other things, shows that although from the point of view of qualitative theory, the dynamical system $\mathcal{M}_2$ in this case, strictly speaking, has no singular points in the phase space $\mathcal{R}_6$, still in fact, due to the asymptotic vacuum character at $t \rightarrow \infty$, the dynamical subsystem in $\mathcal{R}_5$ asymptotically becomes autonomous and has singular points.\(^8\)

### 4.2. Oscillatory Regimes of Cosmological Expansion: $\Lambda < 0$

In the case $\Lambda < 0$ in the $\mathcal{M}_2$ model, as in the $\mathcal{M}_1$ model, oscillatory regimes of cosmological expansion are possible. Their implementation requires components of the scalar doublet that are approximately equal in order of magnitude.

Figure 4 shows a typical example of an oscillatory mode of the model $\mathcal{M}_2$ for the parameters of the model

$$\mathbf{P}_1 = [1, 1, 1, 1, 0.5, 1, 0.01, -0.02]$$

and the initial conditions

$$\mathbf{I}_1 = [0.2, 0.1, 0.1, 0.1, 1].$$

Figures 5 and 6 show three-dimensional projections of the phase trajectories of the system.

It should be noted that the oscillatory behavior of the $\mathcal{M}_2$ model is very close to the behavior of the $\mathcal{M}_1$ model [17] as well as models with a vacuum asymmetric scalar doublet [17].

### 4.3. Example of a Cosmological Model with Finite History

As follows from the results of Section 3.2, in systems of scalarly charged fermions, it is possible to achieve a cosmological singularity $a(t_1) \to 0$ in the expansion phase of $H(t_1) \to +\infty$ and in the compression phase $a(t_2) \to 0$, $H(t_2) \to -\infty$. Such a universe seems to exist for a limited time $t_2 - t_1$.

Note that in the incomplete model $\mathcal{M}_{00}$, due to the non-negativity of the Hubble parameter, the universe exists for an infinite time.

Let us consider a numerical model of such a process with the parameters

$$\mathbf{P}_2 = [0, 0, 0.00001, 0, 10^{-8}, 0.00001, 0, 0.1, 0]$$

and the initial conditions

$$\mathbf{I}_2 = [0, 0, 5 \times 10^{-8}, 0, 1].$$

This case corresponds to a scalar phantom singlet.

The behavior of the scale factor with the same parameters is almost the same in the $\mathcal{M}_2$ and $\mathcal{M}_1$ models (see [21]).

---

\(^8\) We talked about that in Subsection 3.2.1.
5. GENERATING SCALAR DOUBLET COMPONENTS IN THE $\mathcal{M}_2$ MODEL

In the examples above, we have demonstrated the coincidence of the behaviors of the $\mathcal{M}_1$ and $\mathcal{M}_2$ models in the cases of scalar singlets. Therefore, it may seem that these models do not actually differ from each other. However, this is not the case. The fundamental difference is as follows. The scalar charge densities, $\sigma_c$ and $\sigma_f$, (Eqs. (32)) contain the same factor $(e\Phi + e\varphi)^3$, so even if one of the scalar fields is absent in the doublet, the scalar charge density of this component turns out to be nonzero, which leads to generation of this field. We will demonstrate this phenomenon with examples.

5.1. Zero Initial Values of a Phantom Scalar Field

Let us set zero initial conditions for the phantom field $\varphi(0) = 0; \ z(0) = 0$:

$$\mathbf{P}_3 = [(1, 1, 1), [1, 0.5, 1], 0.1, 0.02], \quad (83)$$
$$\mathbf{I}_3 := [0.01, 0, 0, 0, 1]. \quad (84)$$

Figures 8 and 9 show the evolution of scalar potentials under zero initial conditions for a phantom scalar field.

In this case, the canonical scalar field is in fact present only on a small time interval of the order of $\Delta \sim 70$ Planck times, performing several oscillations within this interval. Therefore, in this case, we are dealing with generation of a canonical scalar field by the phantom component of the doublet. In this case, the dynamic system starts from a stable singular point of the vacuum asymmetric scalar doublet with a transition to another stable singular point of the vacuum asymmetric scalar doublet.

As we noted above, the choice of the “initial” time value $t = 0$ is relative, and the graphs in Figs. 8 and 9 convincingly demonstrate this fact. It turns out that the potential of the phantom field $\varphi$ was constant in the past as well as in the future $\varphi(\mp \infty) = \mp 0.5$, and again coincides with the stable singular points of the asymmetric vacuum scalar doublet [17]. It turns out that the potential of the canonical field $\Phi$ both in the past and in the future tends to zero $\Phi(\mp \infty) = 0$, i.e.,
it starts from a stable zero singular point and then returns to it.\(^9\)

Figures 10 and 11 show plots of the evolution of the scale function \(\xi(t) = \ln a(t)\) and the Hubble parameter \(H(t)\) for the case under study. These plots clearly show that asymptotically vacuum stable states correspond to the phases of cosmological compression and expansion, a fairly rapid transition between which is accompanied by bursts of the canonical scalar field. We can say that a dynamic system, with the help of a canonical field, sends some characteristic signal about the transition between the compression and expansion stages.

5.2. Zero Initial Values of a Canonical Scalar Field

Let us set zero initial conditions for the canonical field \(\Phi(0) = 0; Z(0) = 0\), while preserving the values

\(^9\)Numerical integration gives the results \(\Phi(\mp \infty) = 10^{-15} - 10^{-14}\).
of the parameters $P_3$:

$$I_4 := [0, 0, 0.00001, 0, 1].$$

Figures 12 and 13 show the evolution of scalar potentials under zero initial conditions for a canonical scalar field.

We see that the situation does not differ qualitatively from the one discussed in the previous section: a dynamic system starts from a stable point of an asymptotically vacuum state of a scalar doublet with a zero canonical scalar field (compression phase) and passes to another symmetric stable point (expansion phase). Thus we can conclude that it is the canonical scalar field that is generated by the phantom field through the effective charge of the Fermi system.

5.3. Energy Picture of the Process of Generating a Canonical Scalar Field

It was noted in [14] that systems of scalarly charged particles with zero seed mass have a unique property: they allow a scalar neutral state in which scalar fields are completely absent in the presence of arbitrary scalar charges of particles. In this case, the scalar charge density also vanishes, and the particles become massless. Thus, the ultrarelativistic state of a plasma with zero scalar fields becomes the main state of such a statistical system. This property of systems of scalarly charged particles radically distinguishes them from systems with electromagnetic interaction.

Due to this property and the above-mentioned proportionality of the scalar charge densities $\sigma_c$ and $\sigma_f$ to the factor $(e\Phi + \epsilon \varphi)$, each of the components of the asymmetric scalar doublet plays the role of a catalyst for generating the second component. As we saw above, it is the phantom field that is a catalyst for generating the classical one. However, as the analysis shows, the influence of the phantom field on the classical generation process is not restricted to the role of a catalyst, but is significantly determined by the energy of this field and the energy of fermions during a transition from the compression phase to the expansion phase.

Figures 14 and 15 show plot of the energy density evolution of individual components of a dynamical system with the parameters (84) and the initial conditions (85) considered in the previous section. It can be seen that the process of generating a canonical scalar field is characterized, firstly, by a simultaneous burst of the negative energy density of the phantom field and the positive energy density of fermions, and, secondly, by the zero effective total energy $\epsilon_{\text{eff}}$.

6. CONCLUSION

Thus, first, we have conducted a qualitative analysis of a cosmological model based on a one-component system of doubly scalarly charged degenerate fermions, and investigated the asymptotic and limiting properties of this model. Secondly, we have built an appropriate numerical model, with the help of which we have conducted a comparative analysis of this model with previously studied models. Third, we have identified the possibility of generating a canonical scalar field in this model by a phantom scalar field.
COSMOLOGICAL EVOLUTION OF A STATISTICAL SYSTEM 23

with the aid of a canonical scalar charge density self-induced by this field.

Summing up the results of this study, we will list its most important points.
• A closed mathematical model of a cosmological system consisting of a one-component set of degenerate doubly scalarly charged fermions and an asymmetric pair of scalar fields, a canonical one (Φ) and a phantom one (ϕ), with Higgs potential energy, is formulated. It describes a dynamical system in a 6-dimensional phase space \( \mathbb{R}_6 = \{ \xi, H, \Phi, Z = \Phi, \varphi, z = \varphi \} \). The dynamical system is completely described by a normal autonomous set of ordinary differential equations with respect to the cosmological time \( t \). Its behavior in a model with a cosmological constant is determined by 11 parameters.
• The connection has been established between the present models and the previously studied models based on a one-component system of degenerate singly scalarly charged fermions and a two-component system of variously scalarly charged degenerate fermions.
• A qualitative analysis of the dynamical system corresponding to the present cosmological model has been carried out, the character of the singular points corresponding to an infinite future asymptotic vacuum asymmetric scalar doublet has been established.
• The asymptotic and limiting properties of the models are investigated. It has been shown that in the cases admitting an infinite expansion of the universe, its asymptotic properties at infinity are completely determined by vacuum scalar fields, i.e., mainly by a late inflation. At the same time, the model also admits finite histories of the universe; in these cases, an exit from the singular state and an entry into it occur according to the asymptotic features of the ultrarelativistic model. Asymptotically exact solutions have been found for the dynamical system close to the singular state.
• The behavior of the present cosmological model is compared by numerical integration methods with the previously studied incomplete models of the cosmological evolution of charged degenerate fermions. The coincidence of the model properties with those of previously studied models at limiting values of the parameters is demonstrated.
• A unique property of the present model is revealed—the possibility of generating a canonical scalar field by a phantom scalar field by self-induction of a scalar charge density. The mechanism of generation of a standard scalar field by self-induction of a scalar charge near the moment of phase change of cosmological compression and cosmological expansion has been investigated.

Taking into account the results of the study of the gravitational stability of a scalarly charged plasma [13], as well as the results of [12] regarding the possible increase in the effective mass of scalarly charged fermions in the presence of a phantom field, the analysis carried out in this article shows prospects of the proposed model of a one-component system of doubly scalarly charged fermions as a possible model for explaining the formation mechanism of supermassive objects in the early Universe. To implement this idea, it is necessary, above all, to study this model for gravitational stability, which we intend to do in the near future.
FUNDING
This work was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities.

CONFLICT OF INTEREST
The authors declare that they have no conflicts of interest.

REFERENCES
1. B. P. Abbott (LIGO Scientific Collaboration and Virgo Collaboration) et al., Phys. Rev. Lett. 116 (6), 102 (2016).
2. B. P. Abbott, Phys. Rev. Lett. 116 (24), 241103 (2016).
3. S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins, and T. Ott, Astrophys. J. 692, 1075 (2009); arXiv:0810.4674.
4. Sheperd Doeleman, Jonathan Weintroub, Alan E.E. Rogers, et al., Nature 455 78 (2008); arXiv:0809.2442.
5. X. Fan, A. Barth, E. Banados, G. D. Rosa, R. Decarli, A.-C. Eilers, et al., Bulletin of the AAS 51 (3) (2019).
6. B. Trakhtenbrot, arXiv:2002.00972.
7. Q. Zhu et al., arXiv:2012.01458.
8. L. Arturo Ureña-Lopez and Andrew R. Liddle, Phys. Rev. D 66, 083005 (2002); astro-ph/0207493.
9. Pedro V. P. Cunha, Carlos A. R. Herdeiro, Eugen Radu, and Helgi F. Rúnarsson, Int. J. Mod. Phys. D 25 (9), 1641021 (2016).
10. Philippe Brax, Jose A. R. Cembranos, and Patrick Valageas, Phys. Rev. D 101, 023521 (2020); arXiv:1909.02614.
11. Yu. G. Ignat’ev and A.A. Agathonov, Space, Times and Fundamental Interactions, issue 3(16), 48 (2016).
12. Yu. G. Ignat’ev, A. A. Agathonov, and D. Yu. Ignatyev, Grav. Cosmol. 24, 1 (2018); arXiv:1608.05020.
13. Yu. G. Ignat’ev, Grav. Cosmol. 27, 36 (2021); arXiv:2103.13867.
14. Yu. G. Ignat’ev and D.Yu. Ignat’ev, Theor. Math. Phys. 209 (1), 144 (2021).
15. Yu. G. Ignat’ev, A. A. Agathonov, and D. Yu. Ignatyev, Grav. Cosmol. 27 (4), 338 (2021).
16. L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford, 1971).
17. Yu. G. Ignat’ev and I. A. Kokh, Theor. Math. Phys. 207, 514 (2021).
18. Yu. G. Ignat’ev, Grav. Cosmol. 26, 297 (2020).
19. Yu. G. Ignatyev, A. A. Agathonov, and D. Yu. Ignatyev, Grav. Cosmol. 20, 304 (2014).
20. Yu. Ignat’ev, A. Agathonov, M. Mikhailov, and D. Ignatyev, Astron. Space Sci. 357, 61 (2015).
21. Yu. G. Ignat’ev, Grav. Cosmol. 26, 249 (2020).
22. N. N. Lebedev, Special Functions and Their Applications (Prentice-Hall, Englewood Cliffs, N.J., 1965).