Slow-light solitons: Influence of relaxation

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received 6 August 2007; accepted 15 December 2007
published online 28 January 2008

PACS 05.45.Yv – Solitons
PACS 42.50.Gy – Effects of atomic coherence on propagation, absorption, and amplification of light; electromagnetically induced transparency and absorption
PACS 03.75.Lm – Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations

Abstract – We have applied the transformation of the slow-light equations to the Liouville theory that we developed in our previous work, to study the influence of relaxation on the soliton dynamics. We solved the problem of the soliton dynamics in the presence of relaxation and found that the spontaneous emission from the upper atomic level is strongly suppressed. Our solution proves that the spatial shape of the soliton is well preserved even if the relaxation time is much shorter than the soliton time length. This fact is of great importance for applications of the slow-light soliton concept in optical information processing. We also demonstrate that relaxation plays a role of resistance to the soliton motion and slows the soliton down even if the controlling field is constant.

The development of modern methods of optical signal manipulation and control opens wide perspectives for the application of the light in classical and quantum computation. One of such methods of optical signal processing is based on the effect of electromagnetically induced transparency (EIT) [1], which allows for slowing the light down by many orders of magnitude and even bringing it to a complete halt [2–4]. The major advantage of this method is that the velocity of the optical signal is effectively controlled by an auxiliary laser [5–7]. In the linear regime of operation when the intensity of the controlling field is significantly larger than the intensity of the signal, there are some important constraints imposed on the parameters of the signal [1]. These constraints result from the presence of strong optical relaxation in the medium. The medium is typically a gas of alkali atoms whose electronic structure relevant to the EIT effect can be schematically described by the three-level Λ-model (see fig. 1). Relaxation results from spontaneous transitions of the atoms

from the excited upper energy level $|3\rangle$. Usually, such transitions may occur not only at the levels $|1\rangle$, $|2\rangle$ but also to other lower levels, which are not taken into consideration for the sake of simplicity. In any case, relaxation destroys the optical coherence of the signal and therefore must be accounted for.

The atom-field interaction in the system with EIT is substantially nonlinear. In our previous works we provided...
a nonlinear solution of the EIT problem called the slow-light soliton [8]. We also demonstrated that the dynamics of slow-light solitons strongly depends on the form of the controlling field and the solitons can be effectively manipulated concerning their space-time dynamics [9,10]. In our present work we study the dynamics of the solitons in the presence of relaxation. We note that in the framework of the nonlinear theory there might exist different approaches to effectively suppress the influence of relaxation. Here, we demonstrate that by appropriately choosing the parameters of the controlling field we can reduce the influence of relaxation on the slow-light solitons by several orders of magnitude. Our results are found to be in a good agreement with the experiments on EIT.

Within the slowly varying amplitude and phase approximation (SVEPA) the Maxwell equations for the Rabi frequencies are reduced to the the well-known equations governing the dynamics of the atom-field system, viz.

$$\partial_\tau \Omega_a = i\nu_0 \psi_3 \psi_1^*, \quad \partial_\tau \Omega_b = i\nu_0 \psi_3 \psi_2^*. \quad (1)$$

Here $\zeta = z/c, \tau = t - z/c$. Hereafter we set $c = 1$. The Schrödinger equation for the amplitudes $\psi_{1,2,3}$ of atomic wave function reads

$$\begin{align*}
\partial_\tau \psi_1 &= i\frac{1}{2} \Omega_a \psi_3, \\
\partial_\tau \psi_2 &= i\frac{1}{2} \Omega_b \psi_3, \\
\partial_\tau \psi_3 &= -\left(\frac{i}{2} \Delta + \frac{1}{2}\right) \psi_3 + i\frac{1}{2} (\Omega_a \psi_1 + \Omega_b \psi_2).
\end{align*} \quad (2)$$

By excluding the amplitudes of the lower levels the system of eqs. (1), (2) can be transformed into the following form:

$$\begin{align*}
\partial_\tau \psi_3 &= -\left(\frac{i}{2} \Delta + \frac{1}{2}\right) \psi_3 + i\frac{1}{2} (\Omega_a \psi_1 + \Omega_b \psi_2), \\
\partial_\tau \psi_1 &= i\frac{1}{2} \Omega_a \psi_3, \\
\partial_\tau \psi_2 &= i\frac{1}{2} \Omega_b \psi_3, \\
\text{eqs. (3)}-\text{(5)} \text{ take the form}
\end{align*} \quad (3)$$

where $\alpha(\tau)$ is a first-order correction to the exact slow-light–soliton solution. It will be shown below that $\alpha(\tau)$ is always negative. In the convenient notations

$$|\Omega_a| \equiv e^{-\rho}, \quad \Omega_b \equiv \eta, \quad \rho \equiv \rho + \alpha, \quad k = \frac{\nu_0}{8(c_0^2 + \Delta^2)}$$

and the amplitude $\Omega_a$ of the excited state $|3\rangle$ is

$$\begin{align*}
\text{eqs. (3)}-\text{(5)} \text{ take the form}
\end{align*} \quad (8)$$

$$\begin{align*}
\partial_\tau \alpha \partial_\tau \rho + \partial_\zeta \rho &= -k e^{-2\rho}, \\
\partial_\zeta \eta + \partial_\tau \rho \partial_\zeta \eta &= k e^{-2\rho} \eta, \\
4k(\partial_\tau + \gamma) e^{-2\rho} &= -\partial_\zeta^2 (\eta^2 + e^{-2\rho}).
\end{align*} \quad (9)$$

Equation (6) for the phase $\varphi_3$ can easily be integrated after we solved the first three.

We now assume that the effective relaxation described by $\alpha$ varies slowly in time $\tau$, i.e.

$$|\partial_\tau \alpha \partial_\zeta \rho| \ll k e^{-2\rho}, \quad (10)$$

and therefore neglect the first term in eq. (9). The range of validity of the approximation eq. (10) will be analyzed below. Within this approximation eq. (9) transforms into the well-known Liouville equation, viz.

$$\partial_\zeta \rho = -k e^{-2\rho}, \quad (11)$$

whose general solution is readily available:

$$\rho = -\frac{1}{2} \log \left[ \frac{k^2 e^{2\alpha} \partial_\zeta A_+(\zeta) \partial_\tau A_-(\tau)}{(1 - A_+ A_-)^2} \right]. \quad (12)$$

Here $A_+ (\zeta), A_-(\tau)$ are arbitrary functions. To obtain the solution of the whole system including eqs. (10) and (11)
we specify these arbitrary functions as follows:

\[ A_+(\zeta) = -\exp[-8\epsilon_0 k \zeta], \quad (15) \]
\[ A_-(\tau) = \exp \left[ 2\epsilon_0 \int \frac{e^{2\alpha(\tau)}}{p^2(\tau) + 1} \, d\tau \right], \quad (16) \]
\[ \eta = -2p \partial_\tau p + 2\partial_\tau p - 2p \partial_\tau \alpha, \quad (17) \]
\[ \partial_\tau \alpha(\tau) = -\frac{\gamma/2}{p^2(\tau) + 1}. \quad (18) \]

Here \( p(\tau) \) is an arbitrary function. We note from eq. (18) that the correction \( \alpha \) to the exact solution without relaxation obtained in our previous papers vanishes for \( \gamma = 0 \) as expected. Employing specifications eqs. (15), (16) we obtain for the fields

\[ \Omega_a(\tau, \zeta) = \frac{2\epsilon_0 e^{2\alpha/\gamma}}{\sqrt{p^2(\tau) + 1}} \, \text{sech}(\varphi), \quad (19) \]
\[ \Omega_b(\tau, \zeta) = -\frac{2\epsilon_0 p(\tau)e^{2\alpha/\gamma}}{p^2(\tau) + 1} \, \tanh(\varphi) + \frac{2\partial_\tau p(\tau) - \gamma p(\tau)}{p^2(\tau) + 1}. \quad (20) \]

with

\[ \varphi = -4k \epsilon_0 (\zeta - \zeta_0) + \frac{1 - e^{2\alpha(\tau)}}{\gamma}. \quad (21) \]

We chose \( \alpha(0) = 0 \). The position of the maximum of the slow-light soliton eq. (19) is given by the following function of retarded time \( \tau \):

\[ \zeta_\tau(\tau) = 1 - e^{2\alpha/\gamma} + \zeta_0, \quad (22) \]

where we chose \( \zeta_\tau(0) = \zeta_0 \). Hence, the phase of the soliton can be rewritten as

\[ \varphi = -4k \epsilon_0 (\zeta - \zeta_\tau(\tau)) \equiv -4k \epsilon_0 \Delta \zeta(\tau). \quad (23) \]

From eq. (21) the group velocity of the slow-light soliton can be readily derived, viz.

\[ v_g = \frac{1}{4k} \frac{e^{2\alpha/\gamma}}{p^2(\tau) + 1}. \quad (24) \]

The group velocity approaches zero as \( \tau \) increases. This means that the slow-light soliton slows down simultaneously losing its magnitude at the exponential rate. The total distance that the slow-light soliton travels before it vanishes under the influence of relaxation is

\[ \mathcal{L}[\Omega] \equiv \zeta_\tau(\infty) - \zeta_0 \leq \frac{1}{4k\gamma}. \quad (25) \]

This formula clearly shows that the maximum distance that the slow-light soliton can travel in the medium is limited by the magnitude of the relaxation constant. The stronger relaxation is, the smaller is the distance that the soliton can propagate in the medium. We emphasize the functional dependence of the distance on the controlling field.

Now we will analyze the validity of the approximation of small \( \partial_\tau \alpha \) as described by eq. (12).

\[ |\sinh(-8k \epsilon_0 \Delta \zeta)| \leq \frac{16\epsilon_0}{\gamma} e^{2\alpha}. \quad (26) \]

This condition is fulfilled in the vicinity of the maximum of slow-light soliton, i.e. in the space-time domain in the vicinity of the trajectories \( \zeta = \zeta_\tau(\tau) \). At the initial moment of time, for \( \gamma < 16\epsilon_0 \) we obtain \( k \epsilon_0 \Delta \zeta(0) \approx \ln \left( \frac{2\epsilon_0}{\gamma} \right) \). The spatial width of the slow-light soliton taken at the half-maximum of the signal, i.e. the “half-width” of the soliton, is \( w_s = 2\Delta \zeta(0) \approx 0.66/k[\epsilon_0] \) and is independent of time. We require condition (26) to be fulfilled at least within the half-width of the signal at the initial moment of time. This leads to the requirement

\[ \epsilon_0 \geq 0.7\gamma, \quad (27) \]

which ensures that the validity window for the approximation eq. (26) is largely maintained. Notice, however, that the validity window closes down with time, as \( \alpha \) is a negative monotonically decreasing function of \( \tau \).

It follows from eq. (22) that the maximum of \( \mathcal{L}[\Omega] \) is reached if \( \alpha(\infty) = -\infty \). In the simplest form, this condition is achieved when \( p(\tau) = p_0 \) is a constant. Then in the course of the dynamics, the soliton monotonically loses its magnitude at the exponential rate. The controlling field \( \Omega(\tau) \) can be obtained from the eq. (20) in the limit \( \zeta \to \infty \). This limit corresponds to the space domain, which the slow-light soliton has not reached yet. We therefore obtain an equation connecting the function \( p(\tau) \) and the controlling field \( \Omega(\tau) \):

\[ \partial_\tau p(\tau) - \frac{\gamma}{2} p(\tau) + |\epsilon_0| p(\tau) e^{2\alpha(\tau)} = \frac{1}{2} \Omega(\tau)(p^2(\tau) + 1). \quad (28) \]

As is evident from eq. (28) the dynamics corresponding to the constant function \( p(\tau) = p_0 \) will be supported by the following controlling field, viz.

\[ \Omega(\tau) = p_0 \tilde{\gamma} \left( \frac{2\epsilon_0}{\gamma} e^{-\tilde{\gamma} \tau} - 1 \right), \quad \tilde{\gamma} = \frac{\gamma}{p_0^2 + 1}. \quad (29) \]

This dynamical regime ensures that the total travel distance reaches its maximal value \( \mathcal{L} = 1/4k\gamma \). The physical meaning of the constant \( \tilde{\gamma} \) will be explained below.

To make contact with the existing experimental evidence we will now implement the results given above in the physically relevant range of parameters. The magnitude of the controlling field at the initial moment of time \( \Omega(0) \equiv \Omega_0 \) is an experimentally measurable parameter of the problem.

Assuming that \( \gamma \gg \Omega_0 \) from eq. (29) at \( \tau = 0 \) we find

\[ p_0 = \frac{2|\epsilon_0| - \gamma}{2\Omega_0} + \sqrt{\left( \frac{2|\epsilon_0| - \gamma}{2\Omega_0} \right)^2 - 1} \approx \frac{2|\epsilon_0| - \gamma}{\Omega_0}. \quad (30) \]

We can heuristically connect the experimentally available duration of the incoming probe signal \( t_p \) with the
The optical relaxation time \( \tau \) is greatly suppressed for soliton pulses due to the lead to the following relation:

\[
\alpha(t) \approx 2 \alpha \text{sech} \left( \frac{\tau}{\Delta t} \right) \approx 2 \alpha \text{sech} \left( \frac{\tau}{t_p/2} \right) = 0.5
\]

To make a simple estimate, we choose \( |\varepsilon_0| = \gamma \) and verify that the conditions eqs. (27), (30), (31) are well satisfied for experimentally feasible values of \( t_p \sim 1 \mu\text{s} \) and \( \Omega_0 \sim 10 \text{MHz} \).

In the final part, we compare the attenuation of the soliton with the experimental results reported for sodium atoms [2] (see fig. 2). The parameters have the following values: \( \gamma = 6.3 \times 10^7 \text{rad s}^{-1}, t_p = 2.5 \mu\text{s}, \) and \( \Omega_0 = 0.56 \gamma \). By solving eqs. (30), (31), we find \( p_0 \approx 18.4, \varepsilon_0 \approx 5.7 \gamma \). Thus, the requirements of our approximation are well satisfied.

We can estimate the reduction in the strength of optical relaxation influencing the dynamics of the slow-light soliton as follows. According to eq. (19), the amplitude of the soliton decays with the rate \( 2\alpha \). Therefore, the quantity \( \gamma \) introduced in eq. (29) has the meaning of effective relaxation constant.

In the case considered above, \( \gamma \approx \gamma / 340 \). We note that the value of the effective relaxation constant \( \gamma \) is significantly lower than \( \gamma \), because the spontaneous emission is greatly suppressed for soliton pulses due to the nonlinear interaction with the medium. The corresponding optical relaxation time \( \tau_{rel} \approx 2.2 t_p \) is larger than the pulse length. For the pulse delay \( \Delta t = 7.05 \mu\text{s} \) reported by Hau [2] we calculated the decay of the soliton amplitude at the maximum and compared it with a reference pulse \( \Omega_0 \):

\[
\frac{\Omega_0 (\Delta t + t_p, \varepsilon_0)}{\Omega_0} \approx 0.2.
\]

The reference pulse \( \Omega_0 \) is not subject to relaxation. For the reference pulse moving in the vacuum it would require at least \( t_p \) seconds to propagate through the medium. Therefore, the delayed pulse spent in the medium \( \Delta t + t_p \) seconds. Our estimate agrees very well with the measurements reported by Hau. We can also note that the distance that the slow-light soliton propagated in the medium during the time \( \Delta t + t_p \) is approximately equal to 200 \( \mu\text{m} \), which is of order of the size of the atomic cloud in the experiment [2]. We need to emphasize that in the presence of relaxation the velocity of the soliton is not a constant any more (see eq. (24)). We calculated that the average value of the velocity is approximately 22 m/s as compared to that experimentally measured of 32 m/s for the Gaussian pulse.

Discussion. – We constructed a nonlinear theory of slow-light pulse propagation with the effects of relaxation taken into account. The results of our work remain valid beyond the constraints of the transparency window defined in the linear theory. We demonstrated that due to the strong nonlinear interaction between the probe and controlling fields it is possible to preserve the spatial shape of the optical signals even in the presence of strong optical relaxation. The comparison of our theory and experimental results [2] shows very good agreement. We provided rigorous analytical estimates for the largest distance that the slow-light soliton can propagate in the medium with relaxation and also described the dependence of the soliton velocity on the relaxation constant \( \gamma \).

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