Extended WKB method, resonances and supersymmetric radial barriers

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Abstract. Semiclassical approximations are implemented in the calculation of position and width of low energy resonances for radial barriers. The numerical integrations are delimited by $\tau/\tau_{\text{life}} \ll 8$, with $\tau$ the period of a classical particle in the barrier trap and $\tau_{\text{life}}$ the resonance lifetime. These energies are used in the construction of ‘haired’ short range potentials as the supersymmetric partners of a given radial barrier. The new potentials could be useful in the study of the transient phenomena which give rise to the Moshinsky’s diffraction in time.

1 Introduction

The study of resonant scattering processes has received considerable attention in contemporary physics. Resonances are experimentally observed in atomic, nuclear, and particle physics so that diverse theoretical models have been proposed for their analysis over the years (see, e.g. [1,2]). In a simple picture, a resonance is a special case of scattering state for which the ‘capture’ of the incident wave produces delays in the scattered wave. The ‘time of capture’ can be connected with the lifetime of a decaying system composed by the scatterer and the incident wave. Then, the resonance state is represented by a solution of the Schrödinger equation associated to a complex eigenvalue $\epsilon$ and satisfying purely outgoing conditions (Siegert functions) [3]. These functions are not finite at $r \to \infty$, so that they are not admissible as physical solutions into the mathematical structure of quantum mechanics. Some approaches extend the formalism of quantum theory to the wider background where the Siegert functions are more than a convenient model to solve scattering equations [4–6]. Notwithstanding, the ‘unphysical’ behavior of the Siegert functions has been relevant in the construction of complex supersymmetric partners of a given potential [7–9] (see also [10,11]). The main problem is to evaluate the complex point $\epsilon$ up to a reasonable precision; the binding energy $E = \text{Re}(\epsilon)$ and the lifetime $\tau_{\text{life}} = -1/(2\text{Im}(\epsilon))$ of the decaying composite are then automatically determined. Below, the derivation of analytical expressions for $E$ and $\tau_{\text{life}}$ is discussed such that integrations are achievable up to the precision delimited by $\tau/\tau_{\text{life}} \ll 8$, with $\tau$ the period of a classical particle in a barrier trap. The results are used to get Darboux-deformations of radial barriers presenting ‘hair’ over the top. A stronger resonant phenomenon is expected to be associated with these new interactions.
The avoiding of the asymptotic divergence of the Siegert functions is usually faced with $r$-complex coordinates $r = \rho e^{i\lambda}$, $\lambda > 0$. The Siegert functions are then easily obtained by numerical integration, though they become eigenfunctions associated to complex eigenvalues of a non-Hermitian Hamiltonian, as discussed in Sect. [2]. To improve the numerical approximations it is feasible to include variations in the complex $r$-plane [12], as is briefly outlined in Sect. 2.2. The latter approach, however, depends on one’s ability to guess a trial function as a reasonable approximation to the actual wave function [13]. Variants of the WKB method are also available to get a more direct mechanism of integration [14–16]. Indeed, in most cases, bound and resonance energies correspond to poles of the $S$-matrix, since they are zeros of the Jost function involved. Such connection suggests that the bound energy techniques can be extended to the resonance case. Section 3 deals with the application of semiclassical approximations to calculate $E$ and $\Gamma$ in the one-channel, $s$-wave situation. The embedding of the energies into the complex $\epsilon$-plane due to small variations of the related wave-function is necessary to ensure the analytical continuation to the lower half-plane. As a test, the approach is applied to the simple scattering problem reported in [12]. Our results are in good agreement with those reported in e.g. [12, 17]. Applications to more elaborated potentials (as those reported in [16]) are straightforward. The implementation of simple and double complex Darboux transformations is discussed in Section 4. Interestingly, ‘haired’ barriers are found to be the supersymmetric partners of the conventional radial ones. Connections to the time delay problem in quantum mechanics are transparent, so that the haired barriers could be of interest in the studying of the Moshinsky’s diffraction in time. The paper is closed with some concluding remarks.

2 Resonances and radial barriers

Consider a spinless particle in a spherical, suitably short range potential $U(r)$. In dimensionless form, the reduced radial Schrödinger equation is given by

$$\left[ \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} - V(r) + k^2 \right] \psi(r) = 0 \quad (1)$$

where $R(r) = \psi(r)/r$ is the radial part of the complete solution $\phi(\vec{r})$. When $r \to \infty$, the effective potential vanishes and $\psi$ behaves like combinations of $e^{\pm ikr}$. Let us take $k = \kappa e^{i\alpha} \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $\kappa := |k|$ and $r \geq 0$ to write $ikr = i\kappa r \cos \alpha - \kappa r \sin \alpha$. Henceforth the plane wave functions

$$\varphi_{\pm}(r, \kappa, \alpha) = e^{\pm ikr} = e^{\pm i\kappa r \cos \alpha} e^{\mp \kappa r \sin \alpha}$$

can be analyzed in terms of $\alpha$. Three general cases are distinguishable: Scattering energies. If $\alpha = 0$ then $k = \kappa$ and the complex functions $\varphi_{\pm} = e^{\pm ikr}$ oscillate with finite amplitude at large distances. Bound energies. If $\alpha = \pi/2$ one gets $k = i\kappa$, and the real function $\varphi_-$ is divergent, while $\varphi_+ = e^{-\kappa r}$ becomes zero for large values of $r$. Resonance energies. If $\alpha = -\beta$, with $0 < \beta < \frac{\pi}{2}$, then $k$ is in the fourth quadrant of the complex $k$-plane and the oscillation amplitude of the complex function $\varphi_-$ ($\varphi_+$) decreases (increases) exponentially.
as \( r \to +\infty \). In any case, the solutions of (1) that actually behave like \( \phi_{\pm} \) are the Hankel functions \( h_{\ell}^{\pm}(z) \), for which \( h_{\ell}^{\pm}(z) \to e^{\pm i(z-\ell\pi/2)} \) as \( z \to \infty \) (see, e.g., [13]). Therefore, solutions corresponding to bound or to resonance states are picked out from the linear combinations of \( h_{\ell}^{\pm}(z) \) such that only outgoing waves exist. That is, these solutions satisfy the Siegert condition,

\[
\lim_{r \to +\infty} \frac{1}{\psi(r)} \frac{d}{dr} \psi(r) = ik, \quad k \in \mathbb{C}. \tag{2}
\]

Thereby both bounded and resonance functions behave as \( \phi_+ \) at large distances. Clearly, these solutions must also be regular at the origin and smooth in between.

### 2.1 Complex-scaling method revisited

Let us take the transformation \( H \to UHU^{-1} = H_\theta \), with \( \theta \) a dimensionless parameter and the operator \( U \) such that \( r \to re^{i\theta} \). Then \( ikr \to ikre^{i\theta} \) and the bound wave-functions \( \varphi_+(r, i\kappa, \pi/2) \) become exponential decreasing complex functions for large values of \( r \) provided that \( -\pi/2 < \theta < \pi/2 \). In turn, the Siegert functions are transformed as follows

\[
\varphi_+(r, k, -\beta) \to e^{i\kappa r \cos(\theta-\beta)} e^{-\kappa r \sin(\theta-\beta)}, \quad 0 < \beta < \frac{\pi}{2}. \tag{3}
\]

Thereby, \( \varphi_+(r, k, -\beta) \) is mapped into a bounded function if \( \theta - \beta > 0 \), i.e. for \( \theta \in (0, \pi/2) \). As regards the scattering states, we have:

\[
e^{\pm ikr} \to e^{\pm i\kappa r \cos(\theta)} e^{\mp \kappa r \sin(\theta)}.
\]

To preserve the original form of scattering wave-functions the kinetic parameter \( k = \kappa \) must be also modified, thus \( \kappa \to \kappa e^{-i\theta} \). This last transformation induces a rotation of the positive real axis of energies in the clockwise direction by the angle \( 2\theta \) : \( E \propto k^2 \to \kappa^2 e^{-2\theta} \propto E e^{-2\theta} \). That is, the rotated scattering energy is complex, \( \epsilon = \epsilon_R - i\epsilon_I \), with \( \epsilon_R = E \cos(2\theta) \) and \( \frac{\epsilon_I}{2} = E \sin(2\theta) \). In summary, the scaling operator \( U \) produces an embedding of the energies \( E \in \mathbb{R} \) into the complex \( \epsilon \)-plane by a rotation of \( E \geq 0 \) such that:

1. The bound state energies are preserved
2. The cut is rotated downward making an angle \( 2\theta \) with the real axis, and
3. The resonances are exposed by the cut.

In general, the new ‘complex eigenvalues’ are \( \theta \)-independent, so that the resonance phenomenon is associated to the discrete part of the complex-scaled Hamiltonian \( H_\theta \). Finally, considering the operators \( R \) and \( \Pi \), with \([R, \Pi] = iI\), one realizes that \( U = e^{-i\theta \Pi} \) is the scaling operator we are looking for. Indeed, since \( H_\theta = e^{-2i\theta \Pi} + V(e^{i\theta} R) \) is such that \( H_\theta^1 \neq H_\theta \), the ‘regularized’ Siegert functions (3) are square-integrable eigenfunctions associated to complex eigenvalues \( \epsilon \) of the non-Hermitian Hamiltonian \( H_\theta \).
2.2 BBJS complex-coordinate analysis

The repulsive potential

\[ V(r) = V_0 r^2 e^{-\lambda r} \] (4)

represents a radial barrier of maximum height \( V_{\text{max}} = \frac{4V_0}{(\lambda e)^2} \) at \( r_0 = 2\lambda^{-1} \), which admits narrow shape resonances in low-energy scattering. Introduced in 1974 by R. A. Bain, J. N. Bardsley, B.R. Junker and C. V. Sukumar (BBJS), this potential is widely used in the testing of diverse approaches developed to calculate resonances (see e.g. [12,17–20]). In their paper, Bain and coworkers present a combination of the complex-scaling method with variational principles to analyze the resonance phenomenon. Indeed, they found a single resonance \( \epsilon_0 = 6.8722 - 0.025549 \) (a factor 2 must be considered in [12]) for the potential (4) with \( \lambda = 1 \) and \( V_0 = 15 \). To depict the BBJS procedure consider equation (1). After a complex scaling transformation, we get

\[ M_{\theta} \psi \equiv e^{2i\theta} [H_{\theta} - k^2] \psi = 0. \]

The next step is to assume that the integral

\[ I = \int_0^{\infty} \psi(r)M_{\theta}\psi(r) dr \]

is preserved for diverse values of \( \theta \) and small variations of \( \psi(r) \). Introducing a trial function \( f_\epsilon \), parameterized by a set of variable numbers \( c_1, \ldots, c_n \), the approximation \( f_\epsilon(r; c_1, \ldots, c_n) \approx \psi(r) \) is better for small variations of \( I \) with respect to each of the parameters. That is, every one of the expressions \( \frac{\partial I}{\partial c_i} \approx 0 \) becomes an equality if \( f_\epsilon = \psi \). To state precisely: the BBJS method allows for the determination of approximate values of \( k \), leading to points \( \tilde{\epsilon} = k^2 \in \mathbb{C} \) which are in the vicinity of the resonances \( \epsilon = E - i\frac{\Gamma}{2} \) we are interested in.

3 The WKB method for resonances

In this section we formulate an extension of the WKB method to include the calculation of complex-valued energies \( \epsilon = E - i\frac{\Gamma}{2} \). For simplicity, we shall focus on low energy s-wave shape resonances. We use the abbreviation \( p(r) = \sqrt{|T|} \), with \( T = E - V(r) \), the kinetic parameter for a given energy \( E \). The WKB wave function in a classically accessible region \((T > 0)\) and in the nonclassical domain \((T < 0)\) is, respectively, written as follows

\[ \psi(r) = \frac{1}{\sqrt{p(r)}} e^{\pm iW(c,r)}, \quad \psi(r) = \frac{1}{\sqrt{p(r)}} e^{\pm \Omega(c,r)}, \] (5)

where

\[ W(c,r) = \int_c^r p(r) dr, \quad \Omega(c,r) = iW(c,r). \] (6)

If \( p(a) = p(b) = 0 \), the roots \( r = a \) and \( r = b \) are the classical turning points for energies below the barrier (see Figure 1). The origin is a fixed turning point so that \((0,a)\) and
\((b, +\infty)\) are the classically allowed regions while \((a, b)\) is the nonclassical domain. The connection formulae can be summarized as follows
\[
\frac{1}{\sqrt{p(r)}} \cos \left( W(c, r) - \frac{\pi}{4} + w \right) \leftrightarrow \psi \rightarrow \frac{\sin w \ e^{\Omega(r,c)}}{\sqrt{p(r)}} + \frac{1}{2} \frac{\cos w \ e^{-\Omega(r,c)}}{\sqrt{p(r)}} \quad (7)
\]
where \(w\) is a real parameter defined by the turning point which is under analysis. We want to get the quantization rule for the resonance energy \(\epsilon = E - i \frac{\Gamma}{2}\) by imposing the Siegert condition and looking for regular solutions at the origin.

Let the solution in the nonclassical domain be written in the form
\[
\varphi_2(r) = \frac{A}{2\sqrt{p(r)}} e^{-\Omega(a,r)} + i \frac{B}{2\sqrt{p(r)}} e^{-\Omega(r,b)}, \quad (8)
\]
with \(A\) and \(B\) constants. The additivity of the \(\Omega\)-functions \(\Omega(a, b) = \Omega(a, r) + \Omega(r, b)\) leads to
\[
\varphi_2(r) = \frac{A}{2\sqrt{p(r)}} e^{-\Omega(a,b)} e^{\omega(r,b)} + i \frac{B}{2\sqrt{p(r)}} e^{-\Omega(r,b)}. \quad (9)
\]
This last solution is connected to that of region 3 via (7) with \(w = \frac{\pi}{2}\):
\[
\varphi_2 \rightarrow \frac{2iB}{\sqrt{p(r)}} \left( \cos \left[ W(b, r) - \frac{\pi}{4} \right] + \frac{iA}{4B} e^{-\Omega(a,b)} \sin \left[ W(b, r) - \frac{\pi}{4} \right] \right). \quad (10)
\]
Now, imposing \(4B = A e^{-\Omega(a,b)}\) we arrive at the plane wave
\[
\varphi_3(r) = i \frac{A}{2\sqrt{p(r)}} \exp \left\{ -\Omega(a, b) + i \left[ W(b, r) - \frac{\pi}{4} \right] \right\}. \quad (11)
\]
In a similar manner, the connection to region 1 \((w = 0)\) leads to
\[
\varphi_1(r) = \frac{A}{\sqrt{p(r)}} \left( \cos \left[ W(r, a) - \frac{\pi}{4} \right] - \frac{i}{4} e^{-2\Omega(a,b)} \sin \left[ W(r, a) - \frac{\pi}{4} \right] \right). \quad (12)
\]
At the origin one must get $\varphi_1(0) = 0$. However, this condition is not trivially fulfilled by (12) so that we first constrain the $W$-function to satisfy

$$W(0,a) = \left(n + \frac{3}{4}\right) \pi,$$

and then we impose the condition

$$\left|\frac{e^{-2\Omega(a,b)}}{4}\right| \ll 1.$$

That is, we look for a function (12) such that for small values of $r$ the real part goes to zero faster than its imaginary counterpart. The imaginary part of $\varphi_1(r)$ is then expected to produce the embedding of the energy eigenvalues into the complex $\epsilon$-plane by the addition of a small (negative) imaginary term to each of the energies:

$$\mathbb{R} \ni E \rightarrow \mathbb{C} \ni \epsilon = E + \delta E, \quad \delta E = -\frac{i}{2} \Gamma, \quad \Gamma > 0.$$

This embedding is relevant since $W$ is displaced to $W + \delta W$, where

$$\delta W(0, a) = \left[\frac{1}{2} \int_0^a p^{-1}(r) dr\right] \delta E \equiv \frac{\tau}{4} \delta E = -i \frac{\tau \Gamma}{8}.$$

Here $\tau$ corresponds to the period of a classical particle moving harmonically from the origin to the turning point $a$. Thus, $\tau$ is a quadrature depending on the constants of integration $E$ and $a$. Assuming $\delta W(0, a) \ll 1$, the variation of equation (12) shows that $\delta \varphi_1(r)$ is the correction for $\varphi_1(r)$ to be zero at the origin of the complex plane. In other words, $\varphi(0) + \delta \varphi(0) = 0$ provided that

$$\delta W(0, a) + i\frac{e^{-2\Omega(a,b)}}{4} = 0.$$

From (15) we finally get an analytical expression to calculate the resonance width:

$$\Gamma = 2 \frac{e^{-2\Omega(a,b)}}{\tau}.$$

The time spent by a trapped quantum in the tunnelling of the radial barrier is therefore given as follows:

$$\tau_{\text{life}} = \frac{1}{\Gamma} = \frac{\tau}{2} e^{2\Omega(a,b)} = 2 \left(\frac{\partial W(0, a)}{\partial E}\right) e^{2\Omega(a,b)}.$$

Remark that our approach is useful for $0 < \tau \Gamma \ll 8$; otherwise (14) is not satisfied. Notice also that this result is consistent with our assumption that $\delta W(0, a) \ll 1$. Now, the turning points $a$ and $b$ move farther apart as $\tau \rightarrow 0$ and, as a consequence, the factor $e^{-2\Omega(a,b)}$ decreases exponentially. Therefore $\Gamma$ is narrower as $a \rightarrow 0$. On the other hand, $a$ and $b$ move closer as $E \rightarrow V_{\text{max}}$, so that $\Gamma$ increases as $a \rightarrow b$ up to $E \approx V_{\text{max}}$, where
Table 1: The single resonance $\epsilon_0$ for potential (14) with $\lambda = 1$ and the indicated values of $V_0[V_{\text{max}}]$. A factor 2 must be considered in the data originally reported by BBJS in [12] and KLM in [17].

| $V_0$ | BBJS          | KLM            | Extended WKB        |
|------|---------------|----------------|---------------------|
| 15[08.1201] | $6.8722 - i2.5549(-2)$ | $06.994 - i2.787(-2)$ | $07.01129 - i3.71716(-2)$ |
| 30[16.2402] | $11.104 - i1.321(-4)$ | $11.05705 - i1.41354(-4)$ |                     |
| 45[24.3604] | $14.288 - i6.840(-7)$ | $14.21889 - i7.05499(-7)$ |                     |

the simple WKB method does not apply [14]. For energies $E > V_{\text{max}}$, the turning points diverge into complex values and the approach is not directly applicable (see, however, [16] and [17]).

In summary, our method gives better results for energies which lay deep with respect to $V_{\text{max}}$; improvements can be achieved by considering higher-order phase integral methods [21]. Quite remarkably, the approach presented here could be applied in the study of time delay where transient effects are known to be relevant [22]. That is, the method could be useful in giving a quantum definition to the difference between the time to traverse the barrier and the time of going from $a$ to $b$ as a free particle. Particularly if the energy $E$ is below $V_{\text{max}}$, in which case, classically the particle could not arrive at the region 3 [23] (see also [24]).

3.1 Application: BBJS potential scattering

We have derived two expressions to calculate resonances. Equation (13) corresponds to the WKB quantization and localizes the position $E$ of energies fulfilling the Siegert condition. Equation (17), on the other hand, defines the width $\Gamma$ of the resonance up to the precision established by $0 < \tau \Gamma \ll 8$. We can go a step further in our approach by noticing that potential (14) admits a series of low-energy resonances, in correspondence with the strength $V_0$. For getting ‘low-energy’ resonances, we adopt the criterion of selecting those complex eigenvalues $\epsilon$ whose real part is smaller than, or equal to $V_{\text{max}}$. Thus, for $a = r_0$ such that $V(a) = V_{\text{max}}$, we want $W$ in (13) to have a maximum at $a = r_0$. This condition is that the integrand of $W(0, r_0)$ involve the longest distance between $E$ and $V(r)$ for each point in $[0, r_0]$. The solution of this extremum problem enables us to identify the number $n$ of resonances in terms of $V_0$, as requested. Indeed, the straightforward calculation shows that potential (14) admits $n$ resonances, provided $V_0$ is limited as follows:

$$V_0(n - 1) \leq V_0 < V_0(n), \quad n \in \mathbb{N},$$      (18)

where

$$V_0(n) = \beta_0 \left( n + \frac{3}{4} \right)^2, \quad \beta_0 = \left( \frac{\pi \epsilon}{4\gamma} \right)^2, \quad n = 0, 1, 2, \ldots$$ (19)
and
\[ \gamma = \int_0^1 \sqrt{1 - z^2} e^{2(1-z)} \, dz. \]  

(20)

A numerical integration gives \( \gamma \approx 0.55098 \), so that \( \beta_0 \approx 15.014 \) and potential (4) admits a single resonance if \( 8.44539 \leq V_0 < 45.9804 \). The same potential is also analyzed by H.J. Korsch, H. Laurent and R. Möhlenkamp (KLM) in their study of the Milne's differential equation [17]. Table 1 shows the good agreement of our results with those reported in [12] and [17] for a single resonance. Notice that \( \epsilon_0 \) is such that its real part is close to \( V_{\text{max}} = 8.12012 \).

Figure 2 shows the Siegert function obtained as a numerical solution of the corresponding Schrödinger equation for the extended WKB value of \( \epsilon_0 \). The closeness of the single BBJS resonance to \( V_{\text{max}} \) is also observed for the 'highest' resonance in each of our studied cases. Namely, the highest resonance \( \epsilon_N \) is such that \( \text{Re}(\epsilon_n) < \text{Re}(\epsilon_N) \forall n < N \) and \( \text{Re}(\epsilon_N) \approx V_{\text{max}} \). For instance, Table 2 includes the values of the unique five resonances belonging to potential (4) for \( \lambda = 1 \) and \( V_0 = 350 \). The highest one is such that \( \text{Re}(\epsilon_4) \approx V_{\text{max}} = 189.46939 \). For the lowest resonance, on the other hand, the bigger the difference \( V_{\text{max}} - \text{Re}(\epsilon_0) \), the narrower the width \( \Gamma_0 \), as was noticed in the previous section. According to Table 3 for example, \( V_0 = 350 \) is such that \( V_{\text{max}} - \text{Re}(\epsilon_0) = 143.02639 \) and \( \Gamma_0/2 = 4.2321 \times 10^{-36} \). The strength \( V_0 = 15 \) gives in turn \( V_{\text{max}} - \text{Re}(\epsilon_0) = 1.01883 \) and \( \Gamma_0/2 = 0.03717 \). Under these conditions the classical motion of a trapped particle takes place for larger periods \( \tau_0 \) in the former than in the second case.

![Figure 2: The real (solid curve) and imaginary (dotted curve) parts of the numerically integrated Siegert function belonging to the single resonance of \( V(r) = 15r^2 e^{-r} \).](image)

As a final illustrative example of the method we include the anharmonic spherical oscillator \( V(r) = \frac{1}{2}kr^2 - gr^N, \ k > 0, N = 3, 5, \ldots \) discussed by Mur and Popov in [16]. This potential has a barrier of height \( V_{\text{max}} = \frac{(N-2)}{2N} k^{N/2} (gN)^{-2/(N-2)} \) at \( r_0^{N-2} \approx \frac{k}{g^N} \). Considering \( s\)-waves and the parameters \( N = 3, \ k = 800 = 2g \) (\( V_{\text{max}} = 59.25925 \), the potential possesses a single low-energy resonance at \( \epsilon = 47.0105 - i0.861937 \), which could be also tested by using the BBJS or the KLM methods. Hotter resonances for which \( E > V_{\text{max}} \) can be analyzed with either the Mur-Popov approach or the KLM one.
\[ \text{Re}(\epsilon) - \text{Im}(\epsilon) = \Gamma_0 / 2 \]

Table 2: Extended WKB values of the unique five resonances for potential (4) with \( \lambda = 1 \) and \( V_0 = 350 \) (\( V_{\text{max}} = 189.46939 \)).

### 4 Susy transformations and concluding remarks

The Darboux transformation
\[ \tilde{V}(r) = V(r) + 2\beta'(r) \]
(21)
is useful in many branches of the mathematical physics (see e.g. [25]). Of particular interest in quantum theories, this transformation supports the mathematical structure of the supersymmetric approach (for recent reviews, see [26–30]). Moreover, the axioms for Hermitian operators and real spectra can be abolished in some situations (cf. [7–11,31–35] and [36,37]). Two potentials \( V \) and \( \tilde{V} \) are said to be supersymmetric (Susy) partners if the \( \beta \)-function in (21) is a nontrivial solution of the Riccati equation,
\[ -\beta'(r) + \beta^2(r) = V(r) - \epsilon. \]
(22)

In such a case, the spectrum of \( \tilde{V} \) is the same as that of \( V \), occasionally extended by an additional point \( \epsilon \) which could be complex. The non-linear equation (22) is linearized to the Schrödinger equation \( H\psi_\epsilon = \epsilon\psi_\epsilon \) by means of the logarithmic transformation \( \beta(r) = -\frac{d}{dr} \ln \psi_\epsilon(r) \), with \( \psi_\epsilon \) not necessarily in \( L^2(\text{Dom} H) \). For \( \psi_\epsilon \) satisfying the Siegert condition [2], one easily verifies that potential \( \tilde{V} \) inherits the behavior of \( V \) at infinity. It is a custom to take \( \psi_\epsilon \)-functions with at most a single zero, provided this root is one of the borders of \( \text{Dom}(H) \). In the present case, \( \psi_\epsilon \) is the Siegert function belonging to one of the previously derived resonance energies. Thereby, \( \tilde{V} \) is complex-valued and singular.

\[
\begin{array}{cccc}
V_0 & n & V_{\text{max}} & \text{Re}(\epsilon_0) - \text{Im}(\epsilon_0) = \Gamma_0 / 2 \\
15 & 1 & 8.12012 & 7.01129 - 0.03717 \\
60 & 2 & 32.48046 & 16.8584 - 4.86902 \times 10^{-9} \\
150 & 3 & 81.20116 & 28.8664 - 2.14549 \times 10^{-19} \\
250 & 4 & 135.33528 & 38.5165 - 1.97227 \times 10^{-28} \\
350 & 5 & 189.46939 & 46.4430 - 4.23210 \times 10^{-36} \\
\end{array}
\]

Table 3: The extended WKB values of the ‘lowest’ resonance \( \epsilon_0 \) for different strengths \( V_0 \) of potential (4) and \( \lambda = 1 \). The number of resonances \( n \) is in correspondence with the condition (15).
at the origin. A second Darboux transformation, using this time \( \tilde{V} \) and the complex conjugate of \( \beta \), gives rise to a new Susy partner of \( V \), which is a real function. Also this new potential inherits the initial spectrum and is a regular function in \( \text{Dom} H \), as shown in Figure 3.

Figure 3: Haired second order supersymmetric partner of potential \( V(r) = 15r^2 e^{-r} \). Each of the twice Darboux-distortions (couple of hairs) is a smooth curve, as shown at the right.

The new radial potentials exhibit ‘hair’ along the negative slope, which induces stronger resonant phenomena. Since the Siegert function oscillates, while its amplitude increases exponentially (see Fig. 1), for large distances the amplitude of the \( \beta \)-function decreases up to \( -ik \). Hence, according to the Siegert condition (2), the complex double Darboux-distortions (couple of hairs) are cancelled as \( r \to \infty \). The same phenomenon is presented in square wells, where analytical expressions for \( E \) and \( \Gamma \) have been obtained and the number of hairs depends on the excitation of the resonance \( \beta^2 \). It is reasonable to assume that these haired potentials induce delays on the scattering states which are longer than the delay associated with their supersymmetric partners. It is then interesting to analyze the transient phenomena in the scattering process of these new potentials. In this way, the supersymmetric quantum mechanics could be connected to the Moshinsky’s diffraction in time through these haired potentials.

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