REINHARDT DOMAINS WITH A CUSP AT THE ORIGIN

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Abstract. Let $\Omega$ be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with many strictly pseudoconvex points and logarithmic image $\omega$. It was known that the maximal ideal in $H^\infty(\Omega)$ consisting of all functions vanishing at $(p_1, p_2) \in \Omega$ is generated by the coordinate functions $z_1 - p_1, z_2 - p_2$ (meaning that one can solve the Gleason problem for $H^\infty(\Omega)$) if $\omega$ is bounded. We show that one can solve Gleason’s problem for $H^\infty(\Omega)$ as well if there are positive numbers $a, b$ and a positive rational number $\frac{k}{l}$ such that $\Omega$ looks like $\{(z_1, z_2) \in \mathbb{C}^2 : a|z_2|^l \leq |z_1|^k \leq b|z_2|^l\}$ for small $z$.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, let $p = (p_1, \ldots, p_n)$ a point in $\Omega$. Recall the Gleason problem, cf. [4] : is the maximal ideal in $A(\Omega)$ (or $H^\infty(\Omega)$) consisting of functions vanishing at $p$ generated by the (translated) coordinate functions $z_1 - p_1, \ldots, z_n - p_n$? We say that one can solve the Gleason problem if this is indeed the case for every $p \in \Omega$. Gleason mentioned the difficulty of solving this problem even for such a simple domain as the unit ball $B(0, 1)$ in $\mathbb{C}^2, p = (0, 0)$. This case was solved by Leibenzon ([4]), who gave a solution tot the Gleason problem for every convex domain in $\mathbb{C}^n$ with $C^2$ boundary.

Kerzman and Nagel ([3]) used sheaf-theoretic methods and estimates on the solutions of $\bar{\partial}$-problems to solve the Gleason problem for $A(\Omega)$, where $\Omega$ is a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with $C^4$ boundary. Lieb ([2]) independently solved the Gleason problem for $A(\Omega)$ on bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with $C^5$ boundary; Øvrelid improved this in [4] to $C^2$ boundary. See also Henkin ([3]) and Jakobczak ([3]).

In $\mathbb{C}^2$ the Gleason problem was also solved for domains of finite type ([4], [5]). Backlund and Fällström showed ([1]) that there exists an $H^\infty$-domain of holomorphy on which the Gleason problem is not solvable. In [1] Backlund and Fällström used ideas similar to those of Beatrous ([5]), to solve the Gleason problem for $A(\Omega)$ if $\Omega$ is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with $C^2$ boundary that contains the origin. These ideas were expanded by the authors ([1]), who solved the Gleason problem for both $A(\Omega)$ and $H^\infty(\Omega)$ if $\Omega$ is a bounded Reinhardt domain in $\mathbb{C}^2$ with $C^2$ boundary. Thus the domain does not need to be pseudoconvex, and the condition that it contains the origin could also be dropped. The condition of $C^2$ boundary could be weakened quite a lot, since it was only the behavior of the domain at the origin that was important. In this paper, we consider bounded boundaries.

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pseudoconvex Reinhardt domains $\Omega$ in $\mathbb{C}^2$ that for small $z$ look like
\[
\{(z_1, z_2) : a < \left| \frac{z_k}{z_l} \right| < b\}, \quad k, l \in \mathbb{N}^+, a, b \in \mathbb{R}^+,
\]
and are rounded of strictly pseudoconvexily. Thus, $\partial \Omega$ is non-smooth near the origin. We solve the Gleason problem for $H^\infty(\Omega)$ in a way like [11]. More detailed, we divide the domain in two parts. On one part the problem is solved by splitting $f$ into functions for which an explicit solution is constructed. Adding these explicit solutions then gives a solution to the Gleason problem for $f$ on this part of $\Omega$. On the other part, the problem is solved using the $\partial$-methods of [11]. Then we patch the two local solutions together to a global solution, using a new $\partial$-result.

We conclude by solving the Gleason problem for $H^\infty(\Omega)$ on the Hartogs triangle and related domains.

2. Definitions

We let $\mathbb{C}^*$ stand for $\mathbb{C} \setminus \{0\}$. Let
\[
L : (\mathbb{C}^*)^2 \to \mathbb{R}^2, \quad L(z_1, z_2) := (\log |z_1|, \log |z_2|).
\]
Throughout this paper $\Omega$ will be a bounded Reinhardt domain in $\mathbb{C}^2$. We denote its logarithmic image $L(\Omega \cap (\mathbb{C}^*)^2)$ by $\omega$. The boundary of $\Omega$ and $\omega$ will be denoted by $\partial \Omega$ and $\partial \omega$ respectively, while $S(\Omega)$ shall stand for the strictly pseudoconvex boundary points of $\Omega$ that are $C^5$.

We denote the derivative of a function $g$ with respect to the $j$'th coordinate with $D_j g$. The interior and the closure of a set $V$ are denoted by $V^\circ$ and $\overline{V}$ respectively.

We denote the set
\[
\{(z_1, z_2) \in \mathbb{C}^2 : g(z_1, z_2) = c\}
\]
by $[g(z_1, z_2) = c]$, and use a similar notation with e.g. $\leq$ instead of $=$.

Definition. We say that $\Omega$ is an $A$-domain, if $\Omega$ is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ such that

- There exist $a, b, \epsilon \in \mathbb{R}^+, k, l \in \mathbb{N}^+$, with
  \[
  \Omega \cap B(0, \epsilon) = \{(z_1, z_2) \in B(0, \epsilon) : a < \left| \frac{z_k}{z_l} \right| < b\}
  \]
- The boundary points of $\Omega$ outside $\overline{B(0, \epsilon)}$ are all $C^5$ and strictly pseudoconvex.

Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. For $0 < \alpha < 1$ we define
\[
\Lambda_\alpha(U) = \{f \in C(U) : \sup_{x, x+h \in U} |f(x+h) - f(x)|/|h|^\alpha + \|f\|_{L^\infty(U)}
= \|f\|_{\Lambda_\alpha(U)} < \infty\}.
\]

3. Solving a Cauchy-Riemann equation

The goal of this section is to prove the following theorem.

Theorem 1. Let $\Omega$ be an $A$-domain. Suppose that $f$ is a $\overline{\partial}$-closed $(0, 1)$-form with coefficients that are smooth and bounded on $\Omega$, and that $\text{supp} f \cap \overline{B(0, \epsilon)} = \emptyset$. Then there exists a $u \in \Lambda^{1/2}(\Omega)$ with $\overline{\partial} u = f$. 

From this follows immediately that this \( u \) is bounded on \( \Omega \). Note that under the assumptions of the theorem, the support of \( f \) near the boundary lies only near the strictly pseudoconvex points. The setup of the proof is very similar to that of the standard result on strictly pseudoconvex domains with \( C^5 \) boundary. We will follow the book of Krantz (\cite{10}), sections 5.2 and 9.1-9.3 (10.1-10.3 in the new edition). The proof is subdivided in a series of lemmas. Proofs are given or indicated if there is a difference with the standard situation, otherwise we refer to \cite{10}. We do realize that the reader who is not that familiar with \( \overline{\partial} \)-problems will not be very happy about this decision. In our opinion the alternative, copying over 25 pages word by word, would be even worse.

Both in our case and the standard case, one has to construct holomorphic support functions \( \Phi(\cdot, P) \). Estimates on it are derived by solving a \( \overline{\partial} \)-problem using the \( L^2 \)-technique with weights of Hörmander (\cite{6}). In our case, we use that the \( A \)-domain \( \Omega \) is contained in a slightly larger \( A \)-domain \( \Omega_1/n \). The necessary estimate on a ball \( B \) around the origin is derived by a smart choice of the weight function \( \phi \). The estimate on \( \Omega \setminus B \) is derived using that \( \Omega \setminus B \) is compactly contained in \( \Omega_1/n \setminus B \).

Compare this to the strictly pseudoconvex case, where one uses that the domain is compactly contained in a strictly pseudoconvex domain that is strictly larger.

We fix an \( A \)-domain \( \Omega \). Let \( \epsilon \) be the smallest number such that \( \partial \Omega \setminus B(0, \epsilon) \) contains only strictly pseudoconvex points. We set \( V := \{ w \in \partial \Omega : |w| > \epsilon \} \); then \( V \) contains only strictly pseudoconvex points. Let \( \rho : \mathbb{C}^2 \to \mathbb{R} \) be a defining function for \( \Omega \) that is \( C^5 \) and strictly plurisubharmonic on a neighborhood of \( V \). The function \( L : \mathbb{C}^2 \times \mathbb{C}^2 \setminus B(0, \epsilon) \to \mathbb{C} \) given by

\[
L_P(z) = L(z, P) := \rho(P) + \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(P)(z_j - P_j)
\]

\[
+ \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 \rho(P)}{\partial z_j \partial z_k}(P)(z_j - P_j)(z_k - P_k)
\]

is known as the Levi polynomial at \( P \). It has the following properties:

1. For all \( P \in \mathbb{C}^2 \setminus B(0, \epsilon) \), the function \( z \mapsto L(z, P) \) is holomorphic (it is even a polynomial).
2. For all \( P \in V \), there is a neighborhood \( U_P \) such that if \( z \in \overline{\Omega} \cap \{ w \in U_P : L_P(w) = 0 \} \) then \( z = P \).

The goal is to construct for every \( P \in V \) a holomorphic support function \( \Phi(\cdot, P) \). This is a smooth function on \( \Omega \times V \) that is holomorphic in the first variable, such that \( \Phi(z, P) = 0 \Leftrightarrow z = P \). Thus, this function should have the first property of the Levi polynomial at \( P \). The difference is that one does not have to restrict in (2) to a small neighborhood of \( P \in V \). The construction of these functions \( \Phi(\cdot, P) \) will be done via some lemmas.

Choose \( \gamma, \delta > 0 \) such that

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \rho(P)}{\partial z_j \partial z_k}(P)|v_j v_k| \geq \gamma |v|^2 \quad \forall P \in \{ z \in \mathbb{C}^n \setminus B(0, \epsilon) : |\rho(z)| < \delta \}, v \in \mathbb{C}^n.
\]
Lemma 2. There is a $\lambda > 0$ such that if $P \in V$ and $|z - P| < \lambda$, then $2\Re L_P(z) \leq \rho(z) - \frac{2|z - P|^2}{z}$. For every $n \in \mathbb{N}$, we shall now define $A$-domains $\Omega_{1/n}$ that are close to $\Omega$. That is:

$$\Omega_{1/n} \cap B(0, \epsilon) = \{(z_1, z_2) \in B(0, \epsilon) : (1 - 1/n)a < \left| \frac{z_1}{z_2} \right| < (1 + 1/n)b \},$$

and $\Omega_{1/n}$ is rounded off strictly pseudoconvexily, having a $C^5$ defining function $\rho_{1/n}$ on a neighborhood $U$ of $\Omega_{1/n} \setminus B(0, \epsilon)$ such that

- $\Omega \subset \Omega_{1/n}$, $\partial \Omega \cap \partial \Omega_{1/n} = \{0\}$
- $\Omega_{1/(n+1)} \subset \Omega_{1/n} \quad \forall n \in \mathbb{N}, \partial \Omega_{1/(n+1)} \cap \partial \Omega_{1/n} = \{0\}$
- $\lim_{n \to \infty} ||\rho_{1/n} - \rho||_{C^5(U)} = 0$.

We also construct $A$-domains $\Omega_{-1/n}$ that are close to $\Omega$. That is:

$$\Omega_{-1/n} \cap B(0, \epsilon) = \{(z_1, z_2) \in B(0, \epsilon) : (1 + 1/n)a < \left| \frac{z_1}{z_2} \right| < (1 - 1/n)b \},$$

and $\Omega_{-1/n}$ is rounded off strictly pseudoconvexily, having a $C^5$ defining function $\rho_{-1/n}$ on a neighborhood $U$ of $\Omega \setminus B(0, \epsilon)$ such that

- $\Omega \subset \Omega_{-1/n}$, $\partial \Omega \cap \partial \Omega_{-1/n} = \{0\}$
- $\Omega_{-1/(n+1)} \subset \Omega_{-1/n} \quad \forall n \in \mathbb{N}, \partial \Omega_{-1/(n+1)} \cap \partial \Omega_{-1/n} = \{0\}$
- $\lim_{n \to \infty} ||\rho_{-1/n} - \rho||_{C^5(U)} = 0$.

This is possible, cf. the setup in [1]: we only need to consider convex domains in $\mathbb{R}^2$ instead of pseudoconvex Reinhardt domains in $\mathbb{C}^2$.

We choose $n \in \mathbb{N}$ such that $||\rho_{1/n} - \rho||_{C^5(U)} \leq \frac{20}{n}$ (where $\lambda$ is the constant of lemma 3). We may assume that $||\rho_{1/n} - \rho||_{C^5(U)} < \lambda < \delta < 1$.

Lemma 3. If $P \in V$, $z \in \Omega_{1/n}$, $\lambda/3 \leq |z - P| \leq 2\lambda/3$, then $\Re L_P(z) < 0$.

Let $\eta : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function that satisfies $\eta(x) = 1$ for $x \leq \lambda/3$, $\eta(x) = 0$ for $x \geq 2\lambda/3$.

Lemma 4. Let $P \in V$. The $(0, 1)$-form

$$f_P(z) = \begin{cases} -\overline{\partial} \left( \eta((z - P)) \cdot \log L_P(z) \right), & \text{if } |z - P| < \lambda, z \in \Omega_{1/n} \\ 0, & \text{if } |z - P| \geq \lambda, z \in \Omega_{1/n} \end{cases}$$

is well defined (if we take the principal branch for the logarithm) and has $C^\infty$ coefficients for $z \in \Omega_{1/n}$. Furthermore, $\overline{\partial} f_P(z) = 0$ on $\Omega_{1/n}$.

Lemma 5. Let $f$ be a $\overline{\partial}$-closed $(0, 1)$-form on $\Omega_{1/n}$ with $C^1$ coefficients that are bounded. Suppose that $\overline{B(0, \epsilon) \cap \supp f} = \emptyset$. Then there exist a $C_c$ (that does not depend on $f$) and a function $u$ with $\overline{\partial} u = f$ such that

$$||u||_{L^\infty(\Omega_{1/2n})} \leq C_c ||f||_{L^\infty(\Omega_{1/n})}.$$
The first inequality is the estimate of Hörmander, the second one holds because \( f \) has bounded coefficients. We start by showing that the assumption that there is a sequence \( \{z_n\}_{n=1}^{\infty} \) in \( \Omega_{1/2n} \) that converges to 0 such that \( |u(z_n)| \geq \|f\|_{L^{\infty}(\Omega_{1/n})} \) leads to a contradiction. This yields an estimate for \( \|f\|_{\infty} \) near the origin.

There are constants \( R, \beta > 0 \) such that

\[
z \in \Omega_{1/(2n)} \cap B(0, \epsilon) \Rightarrow B(z, R|z|^\beta) \subset \Omega_{1/n}.
\]

Thus for large \( n \), one has that \( B(z_n, R|z_n|^\beta) \) is contained completely in \( B(0, \epsilon) \cap \Omega_{1/n} \). We choose \( k > 4\beta \). We assumed that \( f \) has no support on \( \overline{B(0, \epsilon)} \cap \Omega_{1/n} \), thus \( u \) is holomorphic there. We now apply the mean value inequality on \( u \).

\[
\infty > \int_{\Omega_{1/n}} |f(z)|^2 e^{-\phi(z)} d\lambda \geq \int_{\Omega_{1/n}} \frac{|u(z)|^2 e^{-\phi(z)}}{(1 + |z|^2)^2} d\lambda > \int_{B(z_n, R|z_n|^\beta)} \frac{|u(z_n)|^2 R^4|z_n|^{4\beta}}{|z_n|^k} > C'|z_n|^{4\beta - k} \to \infty
\]

if \( n \to \infty \). Thus there is a \( \delta \) with \( 0 < \delta < \epsilon \) such that \( |u(z)| \leq \|f\|_{L^{\infty}(\Omega_{1/n})} \) for \( z \in \Omega_{1/(2n)} \cap B(0, \delta) \).

Now we shall make the appropriate estimate on \( \Omega_{1/(2n)} \setminus B(0, \delta) \). Remember the Hörmander construction ([1], chapter 4), with \( \phi, \phi_1, \phi_2 \) and

\[
T = \mathfrak{T}_{0,0} : L^2(0,0)_{\Omega_{1/n}, \phi_1} \to L^2(0,1)_{\Omega_{1/n}, \phi_2}.
\]

Then

\[
sup_{\Omega_{1/(2n)} \setminus B(0, \delta)} |u| \leq C(\|u\|_{L^2(\Omega_{3/(4n)} \setminus B(0, \delta))} + \|\mathfrak{T}u\|_{L^2(\Omega_{3/(4n)} \setminus B(0, \delta))})
\]

\[
\leq C'(\|u\|_{L^2(\Omega_{1/n} \setminus B(0, \delta), \phi_1)} + \|f\|_{L^2(\Omega_{3/(4n)} \setminus B(0, \delta))})
\]

\[
\leq C''(||f\|_{L^2(\Omega_{1/n} \setminus B(0, \delta), \phi_2)} + \|f\|_{L^2(\Omega_{3/(4n)} \setminus B(0, \delta))})
\]

\[
\leq C''||f\|_{L^2(\Omega_{1/n} \setminus B(0, \delta))}
\]

since \( e^{-\phi(z)} \) tends to zero as \( z \) tends to a boundary point of \( \Omega_{1/n} \) that is non-zero.

For every \( P \in V \), we let \( u_P \) be a solution of \( \mathfrak{T}u_P = f_P \) that satisfies the estimate above. We now define

\[
\Phi(z, P) = \begin{cases} L_P(z) \cdot \exp(u_P(z)), & \text{ if } |z - P| < \lambda/3 \\
\exp(u_P(z) + \eta(|z - P|) \log L_P(z)), & \text{ if } \lambda/3 \leq |z - P| < \lambda \\
\exp(u_P(z)), & \text{ if } \lambda \geq |z - P| \end{cases}
\]

We proceed to show that these functions \( \Phi(\cdot, P) \) are holomorphic support functions.

**Lemma 6.** For every \( P \in V \), the function \( \Phi(\cdot, P) \) is holomorphic on \( \Omega_{1/n} \). For fixed \( z \in \Omega_{1/(2n)} \), \( \Phi(z, \cdot) \) is continuous in \( P \). There is a \( C > 0 \), independent of \( P \), such that for all \( z \in \Omega_{1/(2n)} \) we have

\[
\begin{align*}
\text{if } |z - P| < \lambda/3, & \text{ then } |\Phi(z, P)| \geq C|L_P(z)|, \\
\text{if } |z - P| \geq \lambda/3, & \text{ then } |\Phi(z, P)| \geq C.
\end{align*}
\]
Proof. The function \( f_P \) is bounded on \( \Omega_{1/(2n)} \) uniformly in \( P \), hence \( u_P \) is bounded on \( \Omega_{1/(2n)} \) uniformly in \( P \). Thus there is a \( C > 0 \) such that \( |\exp u_P(z)| \geq C \). Working this out yields the appropriate estimates.

Lemma 7. For every \( P \in V \) there exist functions \( \Phi_1(z, P) \), \( \Phi_2(z, P) \) that are holomorphic in \( z \in \Omega_{1/n} \) and a constant \( C \) that does not depend on \( P \), such that

\[
\Phi(z, P) = \Phi_1(z, P)(z_1 - P_1) + \Phi_2(z, P)(z_2 - P_2) \quad \forall z \in \Omega_{1/n},
\]

\[
|\Phi_j(z, P)| \leq C \quad \forall z \in \Omega_{1/(2n)}, P \in V, j = 1, 2.
\]

Proof. We will follow the approach of Backlund and Fälström in [2]. A line with positive rational slope \( \frac{k}{n} \) in \( \mathbb{R}^2 \) passing through \( L(p) \) can be seen as the logarithmic image of the zero set of the polynomial \( z_1^k p_2^k - z_2^k p_1^k \), while a line with negative rational slope \( \frac{k}{m} \) in \( \mathbb{R}^2 \) passing through \( L(p) \) can be seen as the logarithmic image of the zero set of the polynomial \( z_1^k z_2^m - p_1^k p_2^m \).

Fix \( P \in V \). We choose polynomials \( g \) and \( h \) such that \( L(Z_g) \) and \( L(Z_h) \) are lines in \( \mathbb{R}^2 \) that intersect the boundary of \( \Omega \) only in \( V \), and \( \{ g = 0 \} \cap \{ h = 0 \} \cap \Omega_{1/n} = \{ P \} \).

Now choose a ball \( U_0 \) around \( P \) that lies compactly in \( \Omega_{1/n} \), and choose open sets \( U_1, U_2 \) such that

- \( \bigcup_{i} U_i \subset \Omega_{1/n} \)
- For a certain positive number \( \mu \) one has that \( |g| > \mu \) on \( U_1 \), \( |h| > \mu \) on \( U_2 \).
- \( U_1 \cap U_2 \cap B(0, \epsilon) = \emptyset \).

Now choose functions \( \phi_k \in C_0^\infty(U_k) \) \((k = 0, 1, 2) \) such that \( 0 \leq \phi_k \leq 1 \) and \( \sum_{k=0}^2 \phi_k = 1 \) on \( \Omega_{1/n} \). Recall that \( \Phi(\cdot, P) \) vanishes at \( z = P \). Because \( \Phi(\cdot, P) \) is holomorphic on \( \Omega_{1/n} \), and \( U_0 \subset \subset \Omega_{1/n} \), the lemma of Oka-Hefer implies that there exist functions \( \Phi_0^1(\cdot, P) \), \( \Phi_0^2(\cdot, P) \in H^\infty(U_0) \) such that

\[
\Phi(z, P) = \Phi_1^0(z, P)(z_1 - P_1) + \Phi_2^0(z, P)(z_2 - P_2) \quad \forall z \in U_0.
\]

We set

\[
\tilde{\Phi}_1^1(z, P) := \frac{\Phi(z, P)}{g(z)}, \tilde{\Phi}_2^1(z, P) := 0,
\]

\[
\tilde{\Phi}_1^2(z, P) := 0, \tilde{\Phi}_2^2(z, P) := \frac{\Phi(z, P)}{h(z)}.
\]

Then \( \tilde{\Phi}_j^i \in H^\infty(U_i \cap \Omega_{1/n}) \) and

\[
\Phi(z, P) = \tilde{\Phi}_1^i(z, P)g(z) + \tilde{\Phi}_2^i(z, P)h(z) \quad \forall z \in U_i, i \in \{ 1, 2 \} \quad (*).
\]

Since \( g \) is an analytic polynomial, vanishing at \( P \), there are polynomials \( g_1, g_2 \in H(\mathbb{C}^2) \) such that \( g = g_1(z_1 - P_1) + g_2(z_2 - P_2) \) on \( \mathbb{C}^2 \). A similar formula holds for \( h \). Substituting this in (**), we obtain the existence of functions \( \Phi_j^i \in H^\infty(U_i \cap \Omega_{1/n}) \), \( i = 1, 2 \), such that

\[
\Phi(z, P) = \Phi_1^i(z, P)(z_1 - p_1) + \Phi_2^i(z, P)(z_2 - p_2) \quad \text{on} \quad U_i \cap \Omega_{1/n}, \quad i = 1, 2.
\]

Therefore

\[
j_1 := \sum_{k=0}^2 \phi_k \Phi_1^k \quad \text{and} \quad j_2 := \sum_{k=0}^2 \phi_k \Phi_2^k.
\]
give a smooth solution of our problem. We want to find \( u \) such that
\[
\Phi_1 = j_1 + u(z_2 - P_2) \quad \text{and} \quad \Phi_2 = j_2 - u(z_1 - P_1)
\]
are in \( H(\Omega_{1/n}) \cap L^\infty(\Omega_{1/(2n)}) \). Define a form \( \lambda \) as follows:
\[
\lambda := \frac{-\overline{\partial} j_1}{z_2 - P_2} - \frac{\overline{\partial} j_2}{z_1 - P_1}
\]
This form \( \lambda \) is a bounded \( \overline{\partial} \)-closed \((0,1)\)-form on \( \Omega_{1/n} \). The support of \( \lambda \) is contained in \( \overline{U_i \cap U_j}, \ i \neq j \). These sets all lie outside \( \overline{B}(0, \epsilon) \). Lemma 3 gives the existence of a function \( u \in L^\infty(\Omega_{1/(2n)}) \) such that \( \overline{\partial}u = \lambda \). With this \( u, \Phi_1, \Phi_2 \) as defined at (**),
\[
\Phi = \Phi_1(z_1 - P_1) + \Phi_2(z_2 - P_2)
\]
on \( \Omega_{1/n} \), and \( \Phi_1(\cdot, P), \Phi_2(\cdot, P) \) both belong to \( H(\Omega_{1/n}) \cap L^\infty(\Omega_{1/(2n)}) \).
For fixed \( z \in \Omega_{1/(2n)} \), the function \( \Phi(z, \cdot) \) depends continuously on \( P \). Studying the construction above carefully, we see that we can choose \( \Phi_1(z, \cdot) \) and \( \Phi_2(z, \cdot) \) continuously in \( P \) as well. Thus, because \( \text{supp} \lambda \cap \partial \Omega \) is compact, there exists a uniform bound on \( ||\Phi_i(z, P)||_{\Omega_{1/(2n)}} \).

**Theorem 8.** Let \( \Omega \) be an A-domain. Let \( f \) be a \( \overline{\partial} \)-closed \((0,1)\)-form on an A-domain that contains \( \Omega_1 \setminus \{0\} \) with \( C^1 \) coefficients. Suppose that \( \text{supp} f \cap B(0, \epsilon) = \emptyset \). Then there is a function \( u \) such that \( \overline{\partial}u = f \), and
\[
||u||_{L^\infty(\Omega)} \leq C ||f||_{L^\infty(\Omega)}.
\]

**Proof.** Let \( H_\Omega(f)(z) \) be the Khenkin solution to the \( \overline{\partial} \) equation; then \( \overline{\partial}H_\Omega(f) = f \).
To prove the necessary estimates, we start by writing \( f = f_1 d\overline{\zeta}_1 + f_2 d\overline{\zeta}_2 \). Then the Khenkin solution can be rewritten to
\[
H_\Omega(f)(z) = \frac{1}{4\pi^2} \left\{ \int_\Omega \frac{f_1(\zeta) \cdot (\overline{\zeta}_1 - \overline{z}_1) + f_2(\zeta) \cdot (\overline{\zeta}_2 - \overline{z}_2)}{|\zeta - z|^4} \times d\overline{\zeta}_1 \wedge d\overline{\zeta}_2 \wedge d\zeta_1 \wedge d\zeta_2 \\
- \int_{\partial \Omega} \frac{\Phi_1(z, \zeta) (\overline{\zeta}_2 - \overline{z}_2) - \Phi_2(z, \zeta) (\overline{\zeta}_1 - \overline{z}_1)}{\Phi(z, \zeta)|\zeta - z|^2} \times (f_1(\zeta) d\overline{\zeta}_1 + f_2(\zeta) d\overline{\zeta}_2) \wedge d\zeta_1 \wedge \zeta_2 \right\}
\]
\[
= \int_\Omega f_1(\zeta) K_1(z, \zeta) d\zeta + \int_\Omega f_2(\zeta) K_2(z, \zeta) d\zeta + \int_{\partial \Omega} f_1(\zeta) L_1(z, \zeta) d\zeta + \int_{\partial \Omega} f_2(\zeta) L_2(z, \zeta) d\zeta
\]
where the identity defines the kernels. Now let \( T \) be so large that \( \Omega \subseteq B(z, T) \) for every \( z \in \Omega \). Then
\[
\int_\Omega |K_j(z, \zeta) d\zeta| \leq \int_{B(z, T)} |z - \zeta|^{-3} d\zeta = C \int_0^T r^{-3} r^3 dr = C' \quad j = 1, 2.
\]
Because \( f \) has no support on \( \overline{B(0, \epsilon)} \), one has that
\[
\int_{\partial \Omega} f_j(\zeta) L_j(z, \zeta) d\zeta = \int_V f_j(\zeta) L_j(z, \zeta) d\zeta \quad j = 1, 2.
\]
Using lemmas 6 and 7, one can prove that
\[
\int_V |L_j(z, \zeta)|dV(\zeta) \leq D_j, \quad j = 1, 2,
\]
where the bounds are independent of \( z \in \Omega \). This implies that
\[
||H_\Omega(f)||_{L^\infty(\Omega)} \leq (2C' + D_1 + D_2)||f||_{L^\infty(\Omega)}.
\]
Keeping in mind that \( |\Phi_1(z, \zeta)| \) and \( |\Phi_2(z, \zeta)| \) are bounded on \( \Omega_{1/(2n)} \) uniformly in \( \zeta \), one can simply follow [10].

Repeating all the arguments used over there exactly, yields:

**Theorem 9.** Let \( \Omega \) be an A-domain. Let \( f \) be a \( \overline{\partial} \)-closed \((0, 1)\)-form on an A-domain that contains \( \overline{\Omega} \setminus \{0\} \) with \( C^1 \) coefficients. Suppose that \( \text{supp}\cap B(0, \epsilon) = \emptyset \). Then \( H_\Omega(f) \) is well defined, continuous on \( \Omega \) and
\[
||H_\Omega(f)||_{\Lambda_{1/2}(\Omega)} \leq C||f||_{L^\infty(\Omega)}.
\]

**Theorem 10.** Let \( \Omega \) be an A-domain. Then there is an \( N \in \mathbb{N} \) such that if \( n \geq N \), then theorem 8 holds on \( \Omega_{-1/n} \) with \( C_{\Omega_{-1/n}} \leq 2C_\Omega \).

Now we give the proof of theorem 8.

**Proof.** Let \( \Omega \) be an A-domain. For \( n \in \mathbb{N} \) large, the stability result will apply on \( \Omega_{-1/n} \). Now let \( f \) be a \( \overline{\partial} \)-closed \((0, 1)\)-form defined on \( \Omega \) (not necessarily on a neighborhood of \( \overline{\Omega} \)) with bounded \( C^1 \) coefficients. For each sufficiently small \(-1/n < 0\), the form \( f \) satisfies the hypotheses of theorem 8 on \( \Omega_{-1/n} \). Therefore \( H_{\Omega_{-1/n}}(f) \) is well defined and satisfies \( \overline{\partial}H_{\Omega_{-1/n}}(f) = f \) on \( \Omega_{-1/n} \). Moreover,
\[
||H_{\Omega_{-1/n}}(f)||_{\Lambda_{1/2}(\Omega)} \leq C_{\Omega_{-1/n}}||f||_{L^\infty(\Omega)} \leq 2C_\Omega||f||_{L^\infty(\Omega)}.
\]
Thus, given a compact subset \( K \) of \( \Omega \), the functions \( \{H_{\Omega_{-1/n}}(f)\} \) form an equicontinuous family on \( K \) if \( n \) is large. Of course, this family is also equi-bounded. By the Arzelà-Ascoli theorem and diagonalization, we see that there is a subsequence \( H_{\Omega_{-1/j}}(f), j = 1, 2, \ldots, \) such that \( H_{\Omega_{-1/j}}(f) \) converges uniformly on compacta to a \( u \in \Lambda_{1/2}(\Omega) \) with \( \overline{\partial}u = f \) on \( \Omega \).

**Remark.** Note that theorem 8 also holds for e.g. a Reinhardt domain \( \Omega \) that for small \( z \) looks like
\[
\{(z_1, z_2) : 0 < |z_1^k| < |z_2^l|\},
\]
and is rounded off strictly pseudoconvexly.

4. Auxiliary results

**Lemma 11.** Let \( \Omega \) be a domain in \( \mathbb{C}^2 \), let \((p_1, p_2) \in \Omega \), let \( k, l \in \mathbb{N}^* \). Suppose that \( \frac{z_1^k}{z_2^l} \in H^\infty(\Omega) \). Let
\[
\begin{align*}
R_1(z_1, z_2) &:= \frac{1}{p_2} \frac{z_1^k - p_1}{z_1 - p_1}, \\
R_2(z_1, z_2) &:= \frac{1}{p_2} \frac{z_1^k p_1^l - z_2^l}{z_2 - p_2}.
\end{align*}
\]
Then
\[
\frac{z_1^k}{z_2} - \frac{p_1^k}{p_2^k} = R_1(z_1, z_2)(z_1 - p_1) + R_2(z_1, z_2)(z_2 - p_2),
\]
and \(R_1, R_2 \in H^\infty(\Omega)\).

**Proof.** This can be checked by hand. \(\square\)

**Lemma 12.** Let \(P\) be a polynomial in \(z_1\) and \(z_2\) that vanishes at \((p_1, p_2) \in \mathbb{C}^2\). There exist polynomials \(P_1, P_2\) such that
\[
P(z_1, z_2) = P_1(z_1, z_2)(z_1 - p_1) + P_2(z_1, z_2)(z_2 - p_2).
\]

**Proof.** For \((p_1, p_2) = (0, 0)\), this follows immediately. For other points apply the appropriate coordinate transform. \(\square\)

**Lemma 13.** Suppose there are points \(t, u, v \in \partial \omega\) having neighborhoods \(T, U, V \subset \partial \omega\) consisting only of strictly convex points of \(\partial \omega\) respectively, such that \(L(p) \in \text{Co}(tuv)\). Then one can solve the Gleason problem for \(H^\infty(\Omega)\) at \(p\).

**Proof.** We choose, just as in lemma 7 analytic polynomials \(g, h\), open sets \(U_0, U_1, U_2\) and a constant \(\mu > 0\) such that:
- \([g(z) = 0] \cap [h(z) = 0] \cap \overline{\Omega} = \{p\}\)
- \(U_0\) is strictly pseudoconvex, and \(p \in U_0 \subset \subset \Omega\)
- \([g] > \mu\) on \(U_1\), \([h] > \mu\) on \(U_2\)
- \(\overline{\Omega} \subset \cup_j U_j\)
- \(U_i \cap U_j \cap B(0, \epsilon) = \emptyset, j = 0, 1, 2\).

Now formulate the corresponding \(\overline{\partial}\)-problem, again as in lemma 7. This yields a bounded \((0, 1)\)-form that has only support outside \(B(0, \epsilon)\). Applying theorem 3 yields a bounded solution to the \(\overline{\partial}\)-problem, and this can be used to solve the Gleason problem in the standard way. \(\square\)

### 5. Dividing \(\Omega\) in Two Pieces

Suppose that \(\Omega\) is an \(A\)-domain, and that \(p \in \Omega\). Then the line with slope \(\frac{k}{l}\) through \(L(p)\) intersects \(\partial \omega\) in only one point \(A\). This point is strictly convex. Thus there is a line \(N\) in \(\mathbb{R}^2\) with rational slope \(\neq \frac{k}{l}\) that intersects \(\partial \omega\) only at strictly convex points such that \(A\) and the part of \(\omega\) in the third quadrant lie on different sides of \(N\). Say \(N\) is given by the equation \(y = \frac{m}{n}x + r\), where \(m, n \in \mathbb{N}\). Then \(N\) is the logarithmic image of \([z_1^m z_2^n = e^{rn}]\). There is a \(\delta > 0\) such that
\[
\{(x, y) \in \partial \omega, r \in [r - \delta, r] : y = \frac{-m}{n}x + \tilde{r}\} \subset S(\omega).
\]
Let
\[
\omega_1 := \{(x, y) \in \omega : y \geq \frac{-m}{n}x + r - \delta\},
\]
\[
\omega_2 := \{(x, y) \in \omega : y \leq \frac{-m}{n}x + r\},
\]
and \(\Omega_1, \Omega_2\) be \((L^{-1}(\omega_1))^\omega, (L^{-1}(\omega_2))^\omega\) respectively. If \(p\) lies in \(\Omega_1\), everything is easy : apply lemma 13 to solve the Gleason problem for \(H^\infty(\Omega)\) at \(p\).

In the rest of the article we shall assume that \(p\) does not lie in \(\Omega_1\). We will use that there is an \(\nu > 0\) such that \(|z_1^n z_2^n - p_1^n p_2^n| > \nu\) for \((z_1, z_2) \in \Omega_1\) to obtain a local
solution on $\Omega_1$. The next section consists of the construction of a local solution on $\Omega_2$. Afterwards, the two local solutions will be patched together using the standard arguments.

6. Constructing a Local Solution

We fix $p = (p_1, p_2) \in \Omega_2$ and $f \in H^\infty(\Omega)$ that vanishes at $p$. The main idea of the following construction is to project $(z_1, z_2)$ on the zero set of $\frac{z_1^k}{z_2} - \frac{p_1}{p_2}$, because

$$f(z_1, z_2) - f\left(\left(\frac{p_1}{p_2}\right)^{1/k}, z_2\right) \leq \frac{z_1^k}{z_2} - \frac{p_1}{p_2} + \frac{f\left(\left(\frac{p_1}{p_2}\right)^{1/k}, z_2\right)}{z_2 - p_2}(z_2 - p_2)$$

comes close to being a solution for the Gleason problem. However, as there appear roots in the argument of the function, we lose in general the holomorphy. We decompose $f$ in functions where one can take the appropriate root. Then we solve the Gleason problem for those functions, add all these solutions and end up with a solution of the Gleason problem for $f$.

By $\zeta$ we denote the $(kn + lm)^{th}$ elementary root of unity.

**Lemma 14.** Suppose $f$ is a bounded holomorphic function on $\Omega_2$. Then for every $0 \leq i, j \leq kn + lm - 1$ there exist functions $f_{i,j} \in H(\Omega_2)$ such that:

- $z_1^iz_2^jf_{i,j}$ is bounded for $0 \leq i, j \leq kn + lm - 1$
- $f_{i,j}(z_1, z_2) = f_{i,j}(\zeta z_1, z_2) = f_{i,j}(z_1, \zeta z_2)$ for all $(z_1, z_2) \in \Omega_2$, $0 \leq i, j \leq kn + lm - 1$
- $f(z_1, z_2) = \sum_{i,j=0}^{kn+lm-1} z_1^iz_2^jf_{i,j}(z_1, z_2)$ for all $(z_1, z_2) \in \Omega_2$.

**Proof.** Let

$$f_{i,j}(z_1, z_2) := \frac{1}{(kn + lm)^2z_1^iz_2^j} \sum_{s,t=1}^{kn+lm} \zeta^{-is-jt}f(\zeta^s z_1, \zeta^t z_2).$$

The domain $\Omega$ does not contain points with a zero coordinate, hence $f_{i,j}$ is well defined. Since $f$ is bounded, we see immediately that $z_1^iz_2^jf_{i,j}(z_1, z_2)$ is bounded as well.

$$(kn + lm)^2f_{i,j}(\zeta z_1, z_2) = \frac{1}{(\zeta z_1)^iz_2^j} \sum_{s,t=1}^{kn+lm} \zeta^{-is-jt}f(\zeta^s z_1, \zeta^t z_2) =$$

$$= \frac{1}{(\zeta z_1)^iz_2^j} \sum_{t=1}^{kn+lm} \zeta^{-jt} \sum_{s=2}^{kn+lm+1} \zeta^{-i(s-1)}f(\zeta^s z_1, \zeta^t z_2) =$$

$$= \frac{\zeta^i}{(\zeta z_1)^iz_2^j} \sum_{t=1}^{kn+lm} \zeta^{-jt} \left(\zeta^{-i(kn+lm+1)}f(\zeta^{kn+lm+1} z_1, \zeta^t z_2) + \sum_{s=2}^{kn+lm} \zeta^{-is}f(\zeta^s z_1, \zeta^t z_2)\right) =$$

$$= \frac{1}{z_1^iz_2^j} \sum_{s,t=1}^{kn+lm} \zeta^{-is-jt}f(\zeta^s z_1, \zeta^t z_2) = (kn + lm)^2f_{i,j}(z_1, z_2).$$
The equality $f_{i,j}(z_1, \zeta; z_2) = f_{i,j}(z_1, z_2)$ can be proven similarly. Since
\[
\sum_{i,j=0}^{kn+lm-1} \zeta^{-is-jt} = \sum_{i=0}^{kn+lm-1} \zeta^{-is} \sum_{j=0}^{kn+lm-1} \zeta^{-jt} = \begin{cases} 0 & s, t \neq kn + lm \\ (kn + lm)^2 & s, t = kl + mn \end{cases}
\]
we have that
\[
\sum_{i,j=0}^{kn+lm-1} z_i^j z_{i,j} f_{i,j}(z_1, z_2) = \frac{1}{(kn + lm)^2} \sum_{s,t=1}^{kn+lm} \sum_{i,j=0}^{kn+lm-1} \zeta^{-is-jt} f(\zeta^s z_1, \zeta^t z_2) = \frac{1}{(kn + lm)^2} \sum_{s,t=1}^{kn+lm} f(\zeta^s z_1, \zeta^t z_2) \sum_{i,j=0}^{kn+lm-1} \zeta^{-is-jt} = f(z_1, z_2).
\]

\[\square\]

**Remark.** There is a polynomial $P$ such that
\[P(\zeta^s p_1, \zeta^t p_2) = f(\zeta^s p_1, \zeta^t p_2) \quad \forall 1 \leq s, t \leq kn + lm.\]

From Lemma 14 it follows that one can solve the Gleason problem for the function $f$ if and only if one can solve the Gleason problem for $f - P$. The corresponding functions $(f - P)_{i,j}$ all vanish at $p$. Hence we may assume from now on that $f_{i,j}$ vanishes at $p$.

**Lemma 15.** The multi valued map $\pi$ given below, maps a point $(z_1, z_2) \in \Omega_2$ to the set $\left\{(z_1^m z_2^n) = \frac{p_1^k}{p_2^l}\right\} \cap \Omega$. The function $f_{i,j} \circ \pi$ is a holomorphic single valued map on $\Omega_2$, and it can be viewed as a function of $z_1^m z_2^n$.
\[\pi(z_1, z_2) := \left( (z_1^m z_2^n)^{1/(kn+lm)} \right)^{k} \cdot \left( \left( \frac{p_1}{p_2} \right)^{1/(kn+lm)} \right)^{-m},\]
where in both of the coordinates the same branch of the root is taken.

**Proof.** This follows from an easy computation. Since $f_{i,j}$ has a $kn + lm$-symmetry in the two variables, it is well defined and holomorphic. \[\square\]

**Lemma 16.** For every $0 \leq i, j \leq kn + lm - 1$ there exist functions $f_{i,j}^1, f_{i,j}^2 \in H^\infty(\Omega_2)$ such that
\[z_i^j z_{i,j} f_{i,j}(z_1, z_2) = f_{i,j}^1(z_1, z_2) \left( \frac{z_i^k}{z_{i,j}^{1/2}} - \frac{p_i}{p_2} \right) + f_{i,j}^2(z_1, z_2)(z_1^m z_2^n - p_1^m p_2^n).
\]

**Proof.** We start by constructing good holomorphic candidates for $f_{i,j}^1$ and $f_{i,j}^2$. Then we show that these functions are indeed bounded. A meromorphic solution of the problem is
\[z_i^j z_{i,j} f_{i,j}(z_1, z_2) = z_i^j z_{i,j} \frac{f_{i,j}(z_1, z_2)}{z_i^{1/2} - \frac{p_i}{p_2}} \left( \frac{z_i^{k/2}}{z_{i,j}^{1/2}} - \frac{p_i}{p_2} \right) + 0 \cdot (z_1^m z_2^n - p_1^m p_2^n).
\]

We search for a function $h$ such that
\[f_{i,j}(z_1, z_2) = z_i^j z_{i,j} f_{i,j}(z_1, z_2) + h(z_1, z_2)(z_1^m z_2^n - p_1^m p_2^n) \quad (*)
\]
Hence the function is bounded. Since 

\[ h(z_1, z_2) = \frac{-f^2_{i,j}(z_1, z_2)}{\frac{z^k_1}{z^2_2} - \frac{p^k_i}{p^2_2}} \]

are holomorphic. Then 

\[ f^1_{i,j}(z_1, z_2) = \frac{z^i_1 z^j_2 f_{i,j}(z_1, z_2) - f^2_{i,j}(z_1, z_2)(z^{m_1} z^{n_2} - p^m_1 p^n_2)}{\frac{z^i_1}{z^2_2} - \frac{p^i_1}{p^2_2}} \]

We want \( f^1_{i,j} \) to be holomorphic. Then it is necessary and sufficient that 

\[ f^2_{i,j}(z_1, z_2) = \frac{z^i_1 z^j_2 f_{i,j}(z_1, z_2)}{\frac{z^i_1}{z^2_2} - \frac{p^i_1}{p^2_2}} \]

for points on the zero set of \( \frac{z^i_1}{z^2_2} - \frac{p^i_1}{p^2_2} \). Therefore we define \( f^1_{i,j} \) as

\[ f^1_{i,j}(z_1, z_2) := \frac{z^i_1 z^j_2 (f_{i,j}(z_1, z_2) - f_{i,j}(\pi(z_1, z_2)))}{\frac{z^i_1}{z^2_2} - \frac{p^i_1}{p^2_2}} \]

These are holomorphic functions, and we have that

\[ z^i_1 z^j_2 f_{i,j}(z_1, z_2) = f^1_{i,j}(z_1, z_2)\left(\frac{z^k_1}{z^2_2} - \frac{p^k_i}{p^2_2}\right) + f^2_{i,j}(z_1, z_2)(z^{m_1} z^{n_2} - p^m_1 p^n_2). \]

We proceed to show that the functions \( f^1_{i,j} \) and \( f^2_{i,j} \) are bounded on \( \Omega_2 \). We start with the function \( f^2_{i,j} \). We define a function \( F \), similar to \( f^2_{i,j} \), and show that it is bounded on \( \Omega_2 \).

\[ F(z_1, z_2) := (kn + lm)^2 f^2_{i,j}(z_1, z_2) = \frac{(\frac{z^k_1}{z^2_2})^{kn+m} \sum_{s,t=1}^{kn+lm} \zeta^{-is-jt} f(\zeta^s \left(z^{m_1} z^{n_2}\right)^{\left(\frac{p^k_i}{p^2_2}\right)^n} \zeta^t \left(z^{m_1} z^{n_2}\right)^k (\frac{p^k_i}{p^2_2})^{-m} 1/(kn+lm))}{\left(z^{m_1} z^{n_2} - p^m_1 p^n_2\right)^{kn+lm}}. \]

Then \( (\frac{z^k_1}{z^2_2})^{kn+m} F^{kn+lm} \) is equal to

\[ \frac{\sum_{s,t=1}^{kn+lm} \zeta^{-is-jt} f(\zeta^s \left(z^{m_1} z^{n_2}\right)^{\left(\frac{p^k_i}{p^2_2}\right)^n} \zeta^t \left(z^{m_1} z^{n_2}\right)^k (\frac{p^k_i}{p^2_2})^{-m} 1/(kn+lm))}{\left(z^{m_1} z^{n_2} - p^m_1 p^n_2\right)^{kn+lm}}. \]

We substitute \( x = z^{m_1} z^{n_2} \) in the last line, and it becomes

\[ \frac{\sum_{s,t=1}^{kn+lm} \zeta^{-is-jt} f(\zeta^s \left(x^{\left(\frac{p^k_i}{p^2_2}\right)^n} \zeta^t \left(x^{k} (\frac{p^k_i}{p^2_2})^{-m} 1/(kn+lm))\right) 1/(kn+lm))}{\left(x - p^m_1 p^n_2\right)^{kn+lm}}. \]

The numerator is bounded, and we have a removable singularity at \( x = p^m_1 p^n_2 \). Hence the function is bounded. Since \( (\frac{z^k_1}{z^2_2})^{kn+m-jm} \) is bounded, \( F \) is bounded as well.
The same goes for $f^2_{i,j}$.

Now we turn our attention to the function $f^1_{i,j}$. Remember that $\omega_2$ was given by \{(x, y) : y = \frac{-m}{\pi} x + r\}. The line given by $y = \frac{-m}{\pi} x + r$ corresponds to a curve in $\mathbb{C}^2$ given by $z_1^m z_2^n = nr$. For $|K| \leq nr$, let $\Omega_2^K := \Omega_2 \cap [z_1^m z_2^n = K]$. We will estimate $f^1_{i,j}$ on the sets $\Omega_2^K$. Since we have that $z_1^k z_2^k f_{i,j}(z_1, z_2)$ is bounded (by construction) and that $z_1^k z_2^k f_{i,j}(\pi(z_1, z_2))$ is bounded (as shown while proving that $f^2_{i,j}$ is bounded), for every $\mu > 0$ there exists a constant $C$ such that

$$|f^1_{i,j}(z_1, z_2)| \leq C \quad \text{on} \quad \Omega_2 \setminus \{(z_1, z_2) \in \Omega_2 : \left|\frac{z_1^k}{z_2^k} - \frac{p^k_1}{p^k_2}\right| < \mu\}.$$

The construction of $\Omega_2$ implies the existence of an $\mu > 0$ such that for every $(z_1, z_2) \in \Omega_2^K$ (with $|K| \leq nr$), $\Theta \in [0, 2\pi]$, there is a point $(s, t) \in \Omega_2^K$ with $\frac{z_1^k}{z_2^k} - \frac{p^k_1}{p^k_2} = \mu e^{i\Theta}$. Since $\Omega_2^K$ can locally be seen as an open set in $\mathbb{C}$ (after the appropriate biholomorphic mapping), applying the maximum principle yields that

$$|f^1_{i,j}(z_1, z_2)| \leq \max_{s, t} |f^1_{i,j}(s, t)| \leq C.$$ 

So $f^1_{i,j}$ is bounded as well. \qed

**Proposition 17.** Let $f$ be a bounded holomorphic function on $\Omega_2$ that vanishes at $(p_1, p_2)$. There exist functions $\tilde{f}_1, \tilde{f}_2 \in H^\infty(\Omega_2)$ such that

$$f(z_1, z_2) = \tilde{f}_1(z_1, z_2)(z_1 - p_1) + \tilde{f}_2(z_1, z_2)(z_2 - p_2).$$

**Proof.** Since $z_1^m z_2^n - p_1^m p_2^n$ is a polynomial that vanishes at $p$, there are by lemma 12 polynomials $P_1$ and $P_2$ such that

$$z_1^m z_2^n - p_1^m p_2^n = P_1(z_1, z_2)(z_1 - p_1) + P_2(z_1, z_2)(z_2 - p_2) \quad \forall z_1, z_2 \in \mathbb{C}^2.$$

Use lemma 11 to obtain a similar result for $\frac{z_1^k}{z_2^k} - \frac{p_1^k}{p_2^k}$. We substitute this and the solutions obtained for $z_1^k z_2^k f_{i,j}(z_1, z_2)$ ($0 \leq i, j \leq km + lm - 1$; note that we may assume that $f_{i,j}(p) = 0$, as remarked after lemma 14). This yields that

$$f(z_1, z_2) = \sum_{i,j=0}^{kn+lm-1} z_1^i z_2^j f_{i,j}(z_1, z_2) =$$

$$= \sum_{i,j=0}^{kn+lm-1} \left( f^1_{i,j}(z_1, z_2)\left(\frac{z_1^k}{z_2^k} - \frac{p_1^k}{p_2^k}\right) + f^2_{i,j}(z_1, z_2)(z_1^m z_2^n - p_1^m p_2^n) \right) =$$

$$= \left( \sum_{i,j=0}^{kn+lm-1} f^1_{i,j}(z_1, z_2)R_1(z_1, z_2) + f^2_{i,j}P_1(z_1, z_2) \right) (z_1 - p_1) +$$

$$+ \left( \sum_{i,j=0}^{kn+lm-1} f^1_{i,j}(z_1, z_2)R_2(z_1, z_2) + f^2_{i,j}P_2(z_1, z_2) \right) (z_2 - p_2) =$$

$$\tilde{f}_1(z_1, z_2)(z_1 - p_1) + \tilde{f}_2(z_1, z_2)(z_2 - p_2).$$

The functions $\tilde{f}_1$ and $\tilde{f}_2$ are in $H^\infty(\Omega_2)$. \qed
7. Main result

**Theorem 18.** Let \( \Omega \) be an A-domain. Then for every \( f \in H^\infty(\Omega) \) that vanishes at \( p = (p_1, p_2) \in \Omega \) there exist functions \( f_1, f_2 \in H^\infty(\Omega) \) such that

\[
f(z_1, z_2) = f_1(z_1, z_2)(z_1 - p_1) + f_2(z_1, z_2)(z_2 - p_2) \quad \forall z \in \Omega.
\]

Thus one can solve the Gleason problem for \( H^\infty(\Omega) \).

**Proof.** Let \( \Omega_1, \Omega_2 \) be as in section 5. As noted there, one can find such \( f_1, f_2 \) if \( p \in \Omega_1 \). So suppose \( p \in \Omega_2 \). We make the local solutions on \( \Omega_1 \) and \( \Omega_2 \), using theorem 17. The \( \overline{\partial} \)-problem corresponding to the patching of the two local solutions yields a bounded \((0,1)\)-form that has support outside \( B(0, \epsilon) \). Theorem 1 yields a bounded solution to this particular \( \overline{\partial} \)-problem. Now proceed in the standard way (e.g. lemma 7) to obtain the appropriate \( f_1 \) and \( f_2 \in H^\infty(\Omega) \). \( \square \)

8. The Hartogs triangle and related domains

For \( k, l \in \mathbb{N}^+ \) let \( \Omega_{k,l} \) be the domain defined by

\[
\Omega_{k,l} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2|^l < 1\}.
\]

The Hartogs triangle is exactly \( \Omega_{1,1} \). The situation becomes slightly more complicated compared to the previous sections, since \( \Omega_{k,l} \) contains points of the form \((0, a)\). Thus the functions \( f_{i,j} \) as constructed in lemma 14 may no longer be holomorphic. We will show that one can still solve the Gleason problem for \( H^\infty(\Omega_{k,l}) \).

If \( p_1 \neq 0 \), we return to the construction in section 5. The domain is now cut off with \( |z_2| = 1 \). We still project a point of \( z_1^m z_2^n = c \) onto the zero set of \( z_1^k - p_1^k \), but now \( m = 0, n = 1 \), thus \( z_1^m z_2^n \) is simply \( z_2 \). Now repeat the proof in section 5 to see that there exist \( f_1, f_2 \in H^\infty(\Omega_{k,l}) \) with \( f(z) = f_1(z)(z_1 - p_1) + f_2(z)(z_2 - p_2) \).

There are only two things to point out:

- The functions \( f_{i,j} \) may no longer be holomorphic (in their definitions we divide by \( z_1^i \)), but \( z_1^i z_2^j f_{i,j} \) is still bounded and holomorphic.

- The expression \( \frac{p_1^{k+i} - z_2^j}{p_2^{l+j}} \) becomes \( \frac{p_1^{k+i}}{z_2^{l+j}} \). Thus we only need that \( \frac{z_1^k}{z_2^l} \) is bounded, and not that \( \frac{z_1^k}{z_2^l} \) is bounded.

Now we consider the case that \( p_1 = 0 \). It is tempting to repeat the previous argument, but this is impossible. Namely, in the remark after 5, we assume that \( f_{i,j} \) vanishes at \( p \). Unfortunately, \( f_{i,j} \) is not defined at \( p \). There is another construction however.

**Lemma 19.** Let \( f \in H^\infty(\Omega_{k,l}) \) such that \( f \) vanishes at \((0, p_2)\). Let

\[
f_1(z_1, z_2) := \frac{z_1^j f(z_1, z_2) - f(0, z_2)}{z_1},
\]

\[
f_2(z_1, z_2) := \frac{p_1^k - z_2^j}{p_2^l(z_2 - p_2)}(f(0, z_2) - f(z_1, z_2)) + \frac{f(0, z_2)}{z_2 - p_2}.
\]

Then \( f_1, f_2 \in H^\infty(\Omega_{k,l}) \) and

\[
f(z_1, z_2) = f_1(z_1, z_2)z_1 + f_2(z_1, z_2)(z_2 - p_2) \quad \forall (z_1, z_2) \in \Omega_{k,l}.
\]
Proof. We see immediately that $f_1$ and $f_2$ are holomorphic, that $f_2$ is bounded and that the last equality holds. We rewrite $f_1$:

$$f_1(z_1, z_2) = \frac{z_1^{k-1}}{p_2^k} \frac{z_2^l}{z_1^l} (f(z_1, z_2) - f(0, z_2)).$$

For every $c \in \mathbb{C}$ with $|c| \leq 1$ the set $\Omega_{k,l} \cap [z_2 = c]$ (a disc) contains a circle with radius $|c|^{l/k}$. On this circle, we have that $|z_1| = 2^l$. Applying the maximum principle yields that

$$|f_1(z_1, c)| \leq \frac{2^{l+1}||f||_\infty}{p_2^k} \quad \forall |z_1| \leq |c|^{l/k}.$$ 

It follows that $f_1$ is bounded on $\Omega_{k,l}$.

Thus we have the following theorem:

**Theorem 20.** For $k, l \in \mathbb{N}^+$ let $\Omega_{k,l}$ be the domain defined by

$$\Omega_{k,l} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2|^l < 1\}.$$ 

One can solve the Gleason problem for $H^\infty(\Omega_{k,l})$.

9. **IF THE DOMAIN MEETS ONE OF THE COORDINATE AXES**

In this section, we study domains that are connected both to the $A$-domains and the domains $\Omega_{k,l}$. Namely, let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain, such that for $c$ close to $-\infty$, $\partial \Omega \cap [y < c]$ consists of 2 arcs, one of them being a half line with rational slope. We assume that $0 \notin \Omega$, and that $\Omega$ meets the $z_2$-axis.

(Because of symmetry, everything will hold if $\Omega$ only meets the $z_1$-axis as well.)

Let $K_1, K_2$ be constants such that $\partial \Omega$ is a half line for $|y < K_1|$, and $\omega$ is rounded off strictly convexly above $|y < K_1|$, such that $\{(x, y) \in \partial \Omega : y > K_1\}$ has $y = K_2$ as horizontal asymptote. We fix $p = (p_1, p_2) \in \Omega$, and an $f \in H^\infty(\Omega)$ that vanishes at $p$. We will now solve the Gleason problem for $f$ at $p$. There is a strictly convex point $A = (a_1, a_2) \in \partial \omega$ with $|p_2| < a_2$. This point has a neighborhood in $\partial \omega$ consisting only of strictly convex points. Take a point $B = (b_1, b_2)$ in this neighborhood with $|p_2| < b_2 < a_2$. Then is $f(z_1, z_2)$ bounded on $\Omega \cap [z_2 > \exp(b_2)]$, and on this set we have that $f(z_1, z_2) = \frac{f(z_1, z_2)}{z_2 - p_2} (z_2 - p_2)$. The boundary of $\omega \cap [y < a_2]$ is a straight line for $y$ small. Thus one can solve the Gleason problem for $H^\infty(\Omega \cap [z_2 > \exp(b_2)])$, just as section 5. One can patch the two local solutions together to a global solution using the standard techniques, since $\partial \Omega \cap [z_2 > \exp(b_2) \cap [z_2 < \exp(a_2)] \subset S(\Omega)$.

The case where a part of $\partial \omega$ is described by $|y = c|$ can be dealt with in a similar way. This yields the following theorem:

**Theorem 21.** Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain, that meets exactly one of the axes. Suppose that one part of $\partial \omega$ is a half line, and that the other boundary points of $\partial \omega$ are strictly convex and $C^5$. Then one can solve the Gleason problem for $H^\infty(\Omega)$. 


10. Final remarks

The results in this article all rely on theorem 1. As noted before, one can prove this theorem for Reinhardt domains that for small \( z \) look like
\[
\{(z_1, z_2) : a < \left| \frac{z_1}{z_2} \right| < b \},
\]
and are rounded off pseudoconvexly. Thus one can still solve the Gleason problem if there are “enough” strictly pseudoconvex points in the sense of \([1]\). The condition that the strictly pseudoconvex points have to be \( C^5 \) can, as usual, be relaxed to \( C^2 \), but this would even need more machinery.

Now let \( \Omega \) be a bounded pseudoconvex Reinhardt domain in \( \mathbb{C}^2 \) that has a strictly pseudoconvex \( C^5 \) boundary outside a ball around the origin. If for \( c \) close to \( -\infty \),\(
\partial \omega \cap [y < c]\)
consists of 2 arcs that have parallel asymptotes with rational slope, theorem \([8]\) holds for \( \Omega \) as well. This is because we are either in the situation described in the previous remark, or every point in \( \omega \) lies in a triangle of strictly convex points of \( \omega \), and one can apply lemma \([13]\).

We do not yet know how to solve the Gleason problem for \( H^\infty(\Omega) \) if \( \Omega \) is a Reinhardt domain that for small \( z \) looks like
\[
\{(z_1, z_2) : a < \left| \frac{z_1}{z_2} \right| < b \},
\]
where \( \alpha \notin \mathbb{Q} \), or (with \( r \neq l \))
\[
\{(z_1, z_2) : a|z_l^2| < |z_k^{\alpha}| < b|z_r^2|\}.
\]
The first problem is hard because \( z_\alpha^l \) is not a holomorphic function; the second problem is hard because the function \( |z_k^{\alpha}|^{\frac{\alpha}{\beta}} \) (that appeared in the proof of theorem \([10]\)) is no longer bounded.

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