Gaussian field theories, random Cantor sets and multifractality

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The computation of multifractal scaling properties associated with a critical field theory involves non-local operators and remains an open problem using conventional techniques of field theory. We propose a new description of Gaussian field theories in terms of random Cantor sets and show how universal multifractal scaling exponents can be calculated. We use this approach to characterize the multifractal critical wave function of Dirac fermions interacting with a random vector potential in two spatial dimensions.

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Studies of critical theories are usually concentrated on scaling properties of local operators and their correlations. However, there are objects in critical systems (at least in critical theories of disordered systems) that have very complex scaling properties, namely multifractal. They are characterized not by a single scaling exponent, but by infinitely many scaling dimensions. An open question of field theory is how to describe such complex scaling behavior, in particular how to calculate the multifractal scaling functions $\tau(q)$ or $f(\alpha)$.

Multifractality is a property of a distribution in a critical system or some other deterministic systems, such as dynamical systems. The key feature of the phenomenon of multifractality is the property of selfsimilarity of the distribution at different length scales. A more refined description of the property of selfsimilarity relies on the concept of box distribution. Let $\Omega$ be a large box of volume $L^d$, and let $\mu(x)$ be a normalized distribution (assumed to be strictly positive) defined on $\Omega$. Imagine dividing $\Omega$ into $M$ boxes $\Omega_i$, each of volume $a^d$. The box distribution is then defined by $\langle p_i \rangle = \int_{\Omega_i} d^d x \mu(x)$, $i = 1, \ldots, M$, and the distribution $\mu(x)$ is selfsimilar if the scaling law

$$\tau(q) \equiv D(q)(q-1) = \lim_{\xi \to 0} \frac{1}{\ln(\xi)} \ln \left( \sum_{i=1}^M p_i^q \right)$$

holds for real $q$. The function $\tau(q)$ can be shown to be concave and strictly increasing. If $D(q)$ is constant, the distribution $\mu(x)$ is said to be simple fractal of $D$ (Hausdorff) dimensional support. If $D(q)$ is not constant ($\tau(q)$ nonlinear), the distribution $\mu(x)$ is multifractal.

The scaling function $\tau(q)$ is known for many examples of deterministic multifractals. However, much less is known for distributions in critical statistical systems. Statistical multifractals can be found in the problem of localization in two spatial dimensions where the distribution $\mu(x,V)$ is constructed from the squared amplitude of the wave function solving Schrödinger equation with an impurity potential $V(x)$. In this context, a very important example of multifractality is that of the critical wave function at the plateau transition in the integer quantum Hall effect (IQHE). Numerical calculations of $\tau_{\text{iqhe}}(q)$ for the IQHE have been performed for different realizations of the impurity potential. Within numerical errors, $\tau_{\text{iqhe}}(q)$ is independent of the disorder. Moreover, the exponent $\tau_{\text{iqhe}}(2)$, which governs anomalous diffusion, is consistent with the value $\nu \approx 7/3$ for the localization length exponent. Very little is known on the analytical structure of the critical theory in the IQHE. Much more is known, however, on that for Dirac fermions in two spatial dimensions interacting with the random vector potential $A_v(x) = \epsilon_{\nu\rho} \partial_\rho \Theta(x) + \partial_\nu \chi(x)$, distributed according to $\exp[-\frac{1}{2\gamma_A} \int d^2 x A_v^2]$. In this problem of localization, one can construct a distribution

$$\mu(x,\Theta) \propto e^{2\Theta(x)} \int d^2 x e^{2\Theta(x)} P[\Theta] \propto e^{-\frac{1}{2\gamma_A} \int d^2 x (\nabla \Theta)^2}$$

from the zero-energy eigenfunctions $\psi(x) \propto \exp[i \chi(x)]$. Note that in Eq. (3) we are dealing with a statistical ensemble of distributions with weights $P[\Theta]$. Although the disordered critical point is exactly solvable, we still lack a valid calculation of $\tau_{\text{Dirac}}(q)$ for the distributions in Eq. (3). This is so due to the failure of the crucial assumption commonly used in field theoretical calculations of multifractal scaling exponents, whereby it is assumed that averaging $e^{2\Theta(x)}$ over space is equivalent to averaging over the disorder. Under this assumption, one may replace the inverse of the normalization factor $\int d^2 x e^{2\Theta(x)}$ by a number, and use conventional methods from field theory. But, the resulting $\tau_{\text{Dirac}}(q)$ does not satisfy the analytical properties of Eq. (3). When $e^{2\Theta(x)}$ is not selfaveraging, conventional methods can not treat random variables such as $[\int d^2 x e^{2\Theta(x)}]^{-1}$ anymore.

One goal of this letter is to calculate $\tau_{\text{Dirac}}(q)$ exactly for the first time. We hope that by doing so, we can gain some familiarity with the unusual properties of critical phenomenon for disordered system, in particular those associated with the plateau transition in the IQHE. Since conventional methods relying on the underlying critical field theory are unsatisfactory, we take a very different approach. We first construct explicitly a model for a statistical ensemble of distributions $\mu(x,\Phi)$ ($\Phi$ labels the...
members of the ensemble) and then show that this model is closely related to the critical theory of Dirac fermions with random vector potentials as described, say, in [13].

Our model is a simple generalization of the deterministic two scale Cantor set multifractal [23]. Hence, we call it a random Cantor set (RCS) construction. It should be said that the model itself has been used, for example, in the study of turbulences [1] and of directed polymers in a random medium [24]. The novelty of our work is to relate the RCS to the Gaussian field theory $L = \frac{1}{2} (\nabla \Phi)^2$, and then use this relationship to calculate $\tau_{\text{Dirac}}(q)$. In this way, we hope to illustrate how a new type of equivalence between critical field theories can be taken advantage of.

Our first result is to show that a particular RCS construction describes a Gaussian field theory on an ultrametric space. The correlation functions share the same form as that in the theory $L = \frac{1}{2} (\nabla \Phi)^2$ with an Euclidean metric. Although the two metrics differ, scaling exponents of correlation functions are the same in the two theories. Thus, we can use the one realization in which the calculation is simpler. In the case of the multifractal scaling exponents $\tau(q)$, the RCS is the realization of choice.

Using the RCS realization, we can then show that almost all members $\mu(x, \Phi)$ of the statistical ensemble yield, with the help of Eq. (3), a scaling exponent $\tau(q, \Phi)$ which is selfaveraging, i.e., is almost surely independent of $\Phi$ in the limit $\frac{q}{s} \to 0$. To derive this result, we borrow an exact solution for the free energy of directed polymers on a random tree showing that the free energy is selfaveraging [13,14].

We provide support for the claim that the scaling exponents $\tau(q)$ should be the same in both theories by comparing exact results from the RCS construction with numerical simulations for the $L = \frac{1}{2} (\nabla \Phi)^2$ theory. We also present analytical arguments which indicate a phase transition in $\tau(q)$ for the $L = \frac{1}{2} (\nabla \Phi)^2$ theory at exactly the same $q_s$ for which a transition occurs in the RCS model.

We begin with the construction of the RCS model. Consider a box $\Omega$ of volume $L^2$ in two dimensional Euclidean space. Imagine dividing $\Omega$ into two boxes of equal volume $L^2/2$ and assign to each half a binary address $s_1 = \pm$, or, equivalently, the address $(1,1)$ and $(1,2)$. The procedure is then iterated $n$ times. At level $n$ the space is divided into $2^n$ small boxes addressed by either $n$ bits $(s_1, \ldots, s_n)$ taking the values $s_i = \pm$, $i = 1, \ldots, n$ or by $(n,j)$ where $j = 1, \ldots, 2^n$. Any arbitrary point $x$ in $\Omega$ is then uniquely addressed according to the box in which it lies, say, by the binary address $(s_1, \ldots, s_n)$. Having constructed a selfsimilar structure, namely a binary tree, we define on it random variables. At each level or generation $i = 1, \ldots, n$ of the binary tree one draws $2^n$ independent random variables $\phi_{s_1 \ldots s_n} \equiv \phi_{ij}$ from a Gaussian probability distribution $P[\phi] = (\frac{1}{2\pi \sigma})^{1/2} \exp(-\frac{\phi^2}{2\sigma})$ (see Figure 1). Finally, for any given binary tree made of $n$ generations, we define the random variable

$$\Phi_n(x) = \sum_{i=1}^n \phi_{s_1 \ldots s_n} = \sum_{j=1}^{2^n} e_{ij}(x_k) \phi_{ij}$$

(3)

Here, for any generation $i$ there exists one and only one integer $0 \leq j_i(x_k) \leq 2^n$ such that $e_{ij}(x_k) = \delta_{ij, (x_k)}$. By construction, $\Phi_n(x)$ is a random Gaussian variable. We see that random variables $\phi_{s_1 \ldots s_n}$ at different levels $i$ describe fluctuations at different length scales. Thus, $\Phi_n(x)$, containing fluctuations at all length scales, has a selfsimilar structure.

**FIG. 1.** First two levels of RCS model.

Next, we calculate the generating function of $N$-point correlation functions for the RCS model [Eq. (3)]. We show that it is quadratic in the sources, i.e., describes a Gaussian field theory. We then discuss the relationship between the Gaussian field theory for the RCS model and the Euclidean field theory $L = \frac{1}{2} (\nabla \Phi)^2$.

We consider an arbitrary collection of $N$ points $x_1, \ldots, x_N$ in a large box $\Omega$ of volume $L^2$ embedded in two dimensional Euclidean space. We want to calculate the correlation function

$$Z_n(1, \ldots, N) = \exp \left[ i \sum_{k=1}^N q_k \Phi_n(x_k) \right]$$

(4)

where $q_1, \ldots, q_N$ are real and the overline denotes averaging over all random variables $\phi_{ij}$. The Gaussian random variables $\Phi_n(x_k), k = 1, \ldots, N$, are not necessarily independent. Their statistical correlations can be measured by the scalar product $I_{kl} = \sum_{x \in \Omega} \Delta_{x_1} q_1 \Phi_n(x) \phi_{ij}$.

The quantity measures how “close” $x_k$ and $x_l$ are in the binary partition of $\Omega$ by returning the number $I_{kl}$ of common boxes of volume $L^2/2, \ldots, L^2/2^{l-1}$, respectively, that contain both $x_k$ and $x_l$. On the other hand, one can perform the Gaussian integrations over all independent Gaussian random variables $\phi_{ij}$, and one finds

$$Z_n(1, \ldots, N) = e^{-\frac{N}{2} \sum_{k=1}^N \sum_{l=1}^N \Delta_{x_1} \Phi_n(x_k) \phi_{ij} \phi_{ij}}$$

(5)

Eq. (4) shows that the RCS model defined by Eq. (3) describes a Gaussian field theory, since Eq. (3) implies that $N$-point functions are constructed from 2-point functions in the same way as in the $L = \frac{1}{2} (\nabla \Phi)^2$ theory.

Now, let $d(x_k, x_l) = L 2^{-l_{kl}/2}$. One can easily show that $d(x_k, x_m) \leq \max[d(x_k, x_l), d(x_l, x_m)], \forall x_k, x_l, x_m \in \Omega$, i.e., $d(\phi, \phi)$ is an ultrametric (and thus a metric), and that the 2-point function in the binary tree is given by

$$G_{\text{RCS}}(x, y) \equiv \Phi_n(x) \Phi_n(y) = -\frac{2g}{\ln 2} \ln \frac{d(x, y)}{L}$$

(6)
Thus, the RCS 2-point function resembles very much the 2-point function \( G(x,y) = -\frac{2}{\ln q} \ln \frac{\ln q}{\ln Z} \), in the theory \( L = \frac{1}{2gA} (\nabla \Phi)^2 \). Notice that by tuning \( g = \frac{\ln 2}{2gA} \), we can match the coefficients in front of the logarithms.

We have shown that the RCS model yields a Gaussian field theory, which shares (symbolically) the same correlation functions as the \( L = \frac{1}{2gA} (\nabla \Phi)^2 \) field theory. The difference between the two theories is that the ultrametric \( d(x,y) \) is not the Euclidean metric \( |x - y| \). The scaling exponents for the two theories are nevertheless the same, despite their different metrics. We believe that the multifractal properties of the critical wave function for Dirac fermions with random vector potential can be obtained from the RCS realization. We give support to this claim by comparing exact results for the RCS model with numerical simulations for the \( L = \frac{1}{2gA} (\nabla \Phi)^2 \) theory.

We now turn to the calculation of \( \tau(q) \) within the RCS model. It is possible to define box probabilities on our random binary tree which obey multifractal scaling of the form defined in Eq. (1). To this end, we define local random events at level \( n \) in the binary tree by \( O_n(x) = \exp \{2\Phi_n(x)\} \) [compare with Eq. (2)]. From Eq. (3), we see that \( O_n(x) \) describes a random multiplicative process. Be aware, however, that \( O_n(x) \) and \( O_n(y) \), for two points \( x \) and \( y \), are not independent random variables. From \( O_n(x) \) we can construct another random variable

\[
Z_n(q; \Phi_n) = \sum_{j=1}^{2^n} O_n^n(x^{[j]}),
\]

where the summation extends over all microscopic boxes \( \Omega_j \) of volume \( a^2 = 2^{-n} L^2 \), labelled by any \( x^{[j]} \in \Omega_j \). In analogy to Eq. (4), we then introduce

\[
\tau_n(q; \Phi_n) = -\frac{2}{\ln 2} \left[ \frac{\ln Z_n(q; \Phi_n)}{n} - q \frac{\ln Z_n(1; \Phi_n)}{n} \right],
\]

for any given random event \( O_n(x) = \exp \{2\Phi_n(x)\} \). Here, we had to divide \( O_n(x) \) by \( Z_n(1; \Phi_n) \) to extract properly normalized box probabilities: hence the second term in the bracket of Eq. (5). Also, the prefactor 2 comes from space being two dimensional, whereas the prefactor \( \ln 2 \) results from having chosen a binary partitioning. As a function of \( q \) alone, \( \tau_n(q; \Phi_n) \) shares by construction all the analytical properties of multifractal scaling exponents for box probabilities [2]. We are going to show that the large \( n \) limit of \( \tau_n(q; \Phi_n) \) exists and, most importantly, is independent of the random event \( O_n \). In other words, \( \tau(q) = \lim_{n \to \infty} \tau_n(q; \Phi_n) \) is self-averaging.

To this end, note that we can interpret \( Z_n(q; \Phi_n) \) as the (random) partition function of directed polymers on a binary tree. The inverse temperature \( 1/T \) is the genus, \( A \), and the critical temperature is \( \frac{\ln 2}{2gA} \), which is the critical value of the disorder. In the weak disorder regime defined by \( q_c \), Eq. (8) holds for any disorder such that the annealed average \( Z_n(q; \Phi_n) \) is a well-defined function of \( q \). We have restricted ourselves to Gaussian random variables. The critical value \( q_c \) is defined by the unique minimum of \( q^{-1} \ln Z_n(q; \Phi_n); q > 0 \). The existence of a critical “temperature” \( q_c^{-1} \) was anticipated in [13]. Above the critical temperature, the quenched and annealed averages over the free energy agree. Below the critical temperature, they do not. In Ref. [14] the low temperature phase is identified as a glassy phase.

The Gaussian average on the right hand side of Eq. (8) is easily performed. For example

\[
q_c = \sqrt{\frac{\ln 2}{2g}},
\]

There are two distinct regimes depending on the strength \( q \) of the disorder. In the weak disorder regime defined by \( q_c > 1 \), Eqs. (12,13) yield almost surely

\[
\tau(q) = \begin{cases} 
2(1 - \frac{q_c}{q})q, & q < q_c, \\
(2 - \frac{q_c}{q})(q - 1), & q_c \geq q,
\end{cases}
\]

in the limit \( n \to \infty \). In the weak disorder regime, the parabolic approximation \( \tau(q) = D^*(q)\) holds for all moments \( q \) satisfying \( q_c > q \). Here, the parabolic approximation (PA) is obtained from the annealed disorder average over Gaussian random variables \( \tau_n^*(q) = -[2/(2\ln 2)] \ln Z_n(q; \Phi_n; q - q \ln Z_n(1; \Phi_n)) \) instead of the quenched average implied by Eq. (8). The PA breaks down for large moments in view of the inequality \( \lim_{n \to \infty} n^{-1} \ln Z_n(q; \Phi_n) < \ln Z_n(1; \Phi_n) \), \( q > q_c \geq 1 \). It should be noted that the field theory approach of [13] to Dirac fermions coupling to the disorder of Eq. (4) relates primary fields with negative scaling dimensions to the multifractal scaling exponents \( \tau^*(q) \) (for integer valued \( q \)) with \( q^2 = 2\pi/gA \). In the strong disorder regime defined by \( q_c \leq 1 \), the quenched and annealed averages of \( Z_n(q; \Phi_n) \) are unequal for all integer moments. The PA completely breaks down for integer moments \( q \):

\[
\tau(q) = \begin{cases} 
\frac{1}{q_c}(q - |q|), & q < |q|, \\
2(1 - \frac{q_c}{q})^2, & q \geq |q|.
\end{cases}
\]
The multifractal analysis of Eqs. \([1][2]\) consists in performing a Legendre transformation from \(\tau(q)\) to \(f(\alpha)\). The Legendre transformation is well defined since \(\tau(q)\) is concave in the weak as well as strong disorder regimes. The interpretation of \(f(\alpha)\) is that it yields almost surely the Hausdorff dimensions of interwoven Cantor sets characterizing a typical distribution \(\mu(x, \Phi_{\text{typ}})\). Indeed, \(f(\alpha)\) is strictly positive on its domain of definition, in contrast to the Legendre transform \(f^*(\alpha)\) of the scaling exponents \(\tau(q)\) which we recall are calculated from annealed disorder averages as in \([6]\). For weak disorder, \(f(\alpha)\) is a parabola defined on \(D_+ \leq \alpha \leq D_-\), where \(D_\pm = 2(1 \mp 1/q_c)^2\). For strong disorder, \(f(\alpha)\) is a parabola defined on \(0 \leq \alpha \leq 8/q_c\). In both cases, the parabola takes the maximum value 2, the dimensionality of space, and vanishes when \(\alpha = D_+\) with finite slopes. The finiteness of the slope of \(f(\alpha)\) at the end points of its domain of definition comes about as a result of the transition to linear behavior of \(\tau(q)\) when \(q \geq q_c\). This transition is interpreted as a phase transition to a glassy phase in the context of directed polymers in random medium. This property of \(f(\alpha)\) in the RCS construction should be contrasted to the deterministic two scale Cantor set characterized by \(D(\alpha/L)\) containing a term scaling as \((-\alpha)^{1/2}\), \(\alpha \geq 2\). The finiteness of the slope of \(f(\alpha)\) at the end points of its domain of definition comes about as a result of the transition to linear behavior of \(\tau(q)\) when \(q \geq q_c\). This transition is interpreted as a phase transition to a glassy phase in the context of directed polymers in random medium.

We have compared numerical calculations of the scaling exponents \(v(q, \Phi)\) obtained from the \(\mathcal{L} = \frac{m^2}{2\sigma^2}(\nabla \Phi)^2\) field theory with the exact results derived from the RCS construction. We generated four Monte Carlo realizations for the disorder exp\([\Phi(x, y, \Phi_{\text{typ}})], \Phi(x, y, \Phi_{\text{typ}})\) obtained from the RCS model \([\text{Eq. } 11]\) agree. We would like to thank M. Kardar and H. Orland for pointing out to us Ref. \([13]\). This work was supported by NSF grants DMR-9411574 (XGW) and DMR-9400334 (CCC). CM acknowledges a fellowship from the Swiss Nationalfonds and XGW acknowledges the support from A.P. Sloan Foundation.

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