Dynamical solitons and boson fractionalization in cold-atom topological insulators

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We study the $\mathbb{Z}_2$ Bose-Hubbard model at incommensurate densities, which describes a one-dimensional system of interacting bosons whose tunneling is dressed by a dynamical $\mathbb{Z}_2$ field. At commensurate densities, the model is known to host intertwined topological phases, where long-range order coexists with non-trivial topological properties. This interplay between spontaneous symmetry breaking (SSB) and topological symmetry protection gives rise to interesting fractional topological phenomena when the system is doped to certain incommensurate fillings. In particular, we hereby show how topological defects in the $\mathbb{Z}_2$ field can appear in the ground state, connecting different SSB sectors. These defects are dynamical and can travel through the lattice carrying both a topological charge and a fractional particle number. In the hardcore limit, this phenomenon can be understood through a bulk-defect correspondence. Using a pumping argument, we show that it survives also for finite interactions, demonstrating how boson fractionalization can occur in strongly-correlated bosonic systems, the main ingredients of which have already been realized in cold-atom experiments.

The recent progress in the field of atomic, molecular and optical (AMO) physics has established a new paradigm in our understanding of quantum matter: it is nowadays possible to experiment with systems of many particles in a pristine and highly-controllable environment. In contrast to solid-state materials, the properties of AMO quantum matter are controlled and probed at the single-particle level, yielding a unique toolbox to explore collective quantum-mechanical effects [1]. Historically, systems of ultracold bosonic atoms have played a fundamental role in this regard, allowing for the first experimental demonstration of the condensation predicted by S. N. Bose and A. Einstein [2, 3]. Additionally, R. Feynman’s dream of a quantum simulator [4], i.e. a controllable quantum device that can be used as an efficient alternative to numerical simulations of quantum many-body problems with classical computers, was also accomplished for the first time in systems of ultra-cold bosons [5]. This trend has continued in subsequent years with, among other things, the experimental demonstration of static gauge fields and topological phases of matter [6–10].

To exploit these quantum simulators in condensed matter or high-energy physics, one typically focuses on AMO systems of fermionic atoms or Bose-Fermi mixtures [11] and targets specific models of those disciplines [1, 12]. A broader perspective, however, would be to exploit this unique quantum technology to synthesize new forms of quantum matter which, while capturing the essence of exotic phenomena in these two disciplines, do so in an entirely different context and via distinct microscopic models that are genuine to AMO setups. Clearly, experiments with ultracold bosonic atoms are the key to this endeavor, as no other setup with many interacting bosons can rival with the control and flexibility characteristic of AMO setups. Specifically, ultracold bosons have been used as quantum simulators of topological insulators, such as the Su-Schrieffer-Heeger model [13–15], or the Hofstadter model [7, 8], which captures the essence [16] of the integer quantum Hall effect [17] despite consisting of interacting bosons. A current challenge is the quantum simulation of strongly-correlated topological insulators and lattice field theories of matter coupled to dynamical gauge fields.

The former can lead to fascinating physics such as anyons or charge fractionalization [18]. Specifically, the fractionalization of electric charge is a groundbreaking phenomenon in condensed matter, which plays a crucial role in quantum Hall samples. Historically, however, charge fractionalization was first considered in high-energy physics through a relativistic quantum field theory of fermions coupled to a solitonic background [19]. The soliton, which is a topological excitation that cannot be deformed into the groundstate at a finite energy cost, polarizes the fermionic vacuum in such a way that quasi-particles with fractional charge appear in the spectrum. This general phenomenon also takes place in systems where fermions are coupled to phonons [20] or to an optical waveguide [21]. In this work, we address the following questions: (i) could one observe soliton-induced fractionalization in models genuine to cold-atomic setups?, and (ii) do solitons lead to charge fractionalization of not only fermionic particles, but also bosonic ones?. A priori, since bosons tend to condense in the lowest energy level, one would naively expect them to be insensitive to the solitonic excitations at higher energies. Nonetheless, as shown below, the interplay of strong correlations, topology, and spontaneous symmetry breaking, may cooperate to allow for this exotic effect to occur.  

In particular, we study a one-dimensional bosonic system described by the $\mathbb{Z}_2$ Bose-Hubbard model [22–24]. We show how, for incommensurate densities, the ground state presents configurations with topological defects. In particular, a soliton-antisoliton pair with $\mathbb{Z}_2$ charges appear for each extra boson doped above half filling. The bosons get thus fractionalized, and each topological defect carries half a boson. Such composite quasiparticles can travel through the system unless externally pinned. Using a local topological invariant, such as $\chi$ (or $\Theta$ for out-of-equilibrium phenomena), one can quantify the fractionalization of quasiparticles.

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FIG. 1. Bosonic Peierls transition and topological defects: In the figure, blue and red spheres represent, respectively, bosonic particles tunneling between sites and $Z_2$ fields located on the bonds. (a) At half filling, the $Z_2$ fields are polarized for weak Hubbard interactions, but order anti-ferromagnetically if the interactions are strong enough, according to two degenerate patterns (b). The SSB drives the bosons from a quasi-superfluid to a BOW phase. Additionally, one of the SSB sectors (B) hosts a SPT phase. (c) Extra bosons create pairs of topological defects in the ground state, interpolating between the different SSB sectors and hosting fractionalized particles.

we show how the defects interpolate between regions with different bulk topology, linking it to the presence of fractional particles through a bulk-defect correspondence in the hardcore limit. Finally, we generalize the connection to softcore bosons through a pumping argument, where quantized transport takes place between topological defects.

The model.– We consider a lattice field theory of bosons sitting on the vertices of a 1D chain $b_i, b_i^\dagger$, where they interact locally with strength $U$, and get coupled to Ising spins residing at the lattice links $\sigma_i^x, \sigma_i^z$. The spins dress the bosonic bare tunnelling $t$ with a strength $\alpha$, and are subjected to local transverse $\beta$ and longitudinal $\Delta$ fields. This spin-boson lattice field theory is governed by the Hamiltonian

$$\begin{align}
H &= \sum_i \left( b_i^\dagger (t + \alpha \sigma_i^{x,i+1}) b_i + \text{H.c.} \right) + \frac{U}{2} \sum_i b_i^\dagger b_i \sigma_i^{z,i+1} + \frac{\Delta}{2} \sum_i \sigma_i^{z,i+1} + \beta \sum_i \sigma_i^{x,i+1},
\end{align}$$

which is directly motivated by recent experimental progress on Floquet-engineered ultra-cold bosons in optical lattices [25, 26]. This model reminds of a $Z_2$ lattice gauge theory [27], where matter not only interacts through the $Z_2$ Ising fields, but also locally through a 4-body term. The main differences are that the local $Z_2$ gauge symmetry is explicitly broken in our model, and that we deal with bosonic rather than fermionic matter. We note, however, that working with fermionic ions and implementing this gauge symmetry pose a number of constraints [28–30] that complicate the experiments [25, 26]. Therefore, it is interesting to dispense with them, trying to elucidate if this spin-boson lattice field theory can lead to genuine quantum matter, hosting topological fractionalized bosons.

Groundstate Ising solitons.– The above Hamiltonian, coined the $Z_2$ Bose-Hubbard model, has been studied at commensurate fillings [23, 24]. Despite the lack of a Fermi surface, when the Hubbard interactions $U$ are sufficiently strong, this model displays a spin-boson version of the Peierls instability of 1D electron-phonon systems [31]. The Ising spins spontaneously break the global $Z_2$ symmetry adopting various magnetic orderings while the bosons, instead of condensing (Fig. 1(a)), form a bond order wave (BOW) (Fig. 1(b)). The later can be understood as an intertwined topological phase, which simultaneously displays both spontaneous symmetry breaking (SSB) with a local order parameter, and symmetry-protected topological (SPT) features characterized by topological invariants.

Since there are two different SSB configurations for the Ising spins, called A and B in Fig. 1(b), one can envisage situations where the spins interpolate between them forming a soliton (Fig. 1(c)). These finite-energy excitations can be created dynamically by crossing the Peierls transition in a finite time. In this work, we show that such Ising solitons may also appear in the groundstate for incommensurate fillings where the system is doped with extra bosons. To analyze this situation, we perform DMRG simulations based on matrix product states (MPS) [32]. In the following, we fix the bond dimension to $D = 100$ and the maximum number of bosons per site to $n_0 = 2$, which is sufficient for strong interactions and low densities [22]. We also fix the parameter $\alpha = 0.5t$.

In this case, the SSB is characterized by a Néel ordering $\langle \sigma_i^{z,i+1} \rangle = (-1)^i \varphi(i)$, where $\varphi(i)$ is a slowly-varying field. The order parameter, defined by the average $\varphi = \sum_i \varphi(i)/\mathcal{N}$, characterizes the two possible SSB sectors $\varphi = \pm 1$, whereas solitons correspond to a scalar field $\varphi(i)$ interpolating between these sectors. As shown in Fig. 2, solitons present the same profile as kinks in $\lambda \varphi^4$ relativistic field theories [33], namely

$$\varphi(i) = \tanh \left( \frac{i - i_0}{\xi} \right),$$

FIG. 2. Ising topological defects: We show the ground state configuration for a chain of $L = 31$ sites and $N = 16$ particles, for $U = 10t$, $\Delta = 0.8t$ and different values of $\beta$, where a topological defect appear for the dimmerized pattern of the $Z_2$ fields. (a) and (b) show the magnetization $\langle \sigma_i^{z,i+1} \rangle$ and the order parameter $\varphi_i$ for $\beta = 0$, respectively, where the defect is a domain wall with topological charge $Q = 1$ (3). (c)-(d) Analogous topological defect for $\beta = 0.03t$, where quantum fluctuations broaden the defect, leading to a soliton of finite width $\xi$, that can be accurately fitted to Eq. (2).
where \( i_p \) is the soliton centre and \( \xi \) its width. By analogy with
the scalar quantum field theory [34], the topological charge
\( Q = \frac{1}{2} \int \mathrm{d}x \partial_x \varphi(x) \) can be evaluated by a finite difference
\begin{equation}
Q = \frac{1}{2} \left( \varphi(i_p + r) - \varphi(i_p - r) \right),
\end{equation}
(3)
at points well separated from the soliton center \( r/\xi \sim O(N) \).

Figures 2(a)-(b) show the results for the \( \beta = 0 \) limit, where
the Ising spins have no quantum fluctuations, and the solitons reduce to static domain walls, the center of which can be found anywhere within the lattice. A topological charge of \( Q = \pm 1 \) can be directly read from the soliton of Fig. 2(b).

We remark two differences with respect to relativistic field-theory solitons: our solitons (i) appear directly in the ground-state, and (ii) are not free to move due to Peierls-Nabarro barriers caused by the lack of a continuous translational invariance [35, 36].

As \( \beta > 0 \) is switched on, the Ising spins are no longer classical discrete variables, but become dynamical fields with quantum-mechanical fluctuations. A direct consequence is that they can tunnel through the barriers and delocalize over the chain. To benchmark the prediction (2), we introduce a pinning mechanism by raising the transverse field \( \beta \to \beta_0 = \beta (1 + \epsilon) \) at two consecutive bonds surrounding the desired pinning centre \( i_p \). In Figs. 2(c)-(d), we show that the effect of quantum fluctuations is to widen the extent of the soliton, such that the interpolating region has now a non-zero width \( \xi > 0 \) that can be extracted by a fit of the corresponding scalar field \( \varphi(i) \) to Eq. (2). In Ref. [37], we present a quantitative study of the aforementioned Peierls-Nabarro barriers, and the quantum-induced widening of the soliton, paying special attention to the role of the repulsive Hubbard interactions that controls the back-action of the bosonic matter on the Ising solitons. Moreover, we explore other fillings that lead to solitons with higher topological charges which, nonetheless, still display clear analogies with the \( \lambda \varphi^4 \) kinks [33].

**Fractionalization of bosons.**—The fact that the Ising solitons are not restricted to finite-energy excitations, as typically occurs in relativistic quantum field theories, but appear instead in the groundstate is crucial if one aims at finding a bosonic version of charge fractionalization. The bosons, which tend to condense in the lowest-possible energy level, will do so forming a bond-ordered wave that can lead to fractionally-charged quasi-particles. An unambiguous manifestation can be found by doping the half-filled system with a single particle. To accommodate for this particle, an Ising soliton/anti-soliton pair is created, each hosting a bound quasi-particle with a fractionalized number of bosons, i.e. the boson splits into two halves.

This fractionalization mechanism is confirmed by the numerical results presented in Fig. 3. The scalar field associated to the Ising spins displays the aforementioned soliton-antisoliton pair for a chain of \( N = 90 \) sites and filled with \( N_b = 46 \) bosons (see Fig. 3(a) and (d)). One clearly sees that there is a density build-up around the topological defects that follows the superposition of two profiles
\begin{equation}
\langle \gamma \rangle = \langle n_j \rangle - \frac{1}{2} = \frac{1}{4\xi} \text{sech}^2 \left( \frac{j - j_p}{\xi} \right),
\end{equation}
(4)
for the even \( j = 2i \) or odd \( j = 2i + 1 \) sub-lattices, with their corresponding centers \( j_p \) being fixed by the soliton-antisoliton positions. We note that this expression coincides with the profile of fermionic zero-modes for the relativistic Jackiw-Rebbi model [38] where fractionalization was first predicted [19].

In order to test if the bosons fractionalize, we represent in Fig. 3(b) and (e) the integrated density \( N_i = \sum_{j \leq i} \langle n_j \rangle \), which shows two plateaux where the boson change jumps by steps of 1/2. One can thus interpret that the soliton and antisoliton bind half a boson each, forming two fractionalized quasi-particles.

In the companion article [37], we present an in-depth analysis of this fractionalization phenomenon, ruling out the existence of polaron quasi-particles, and showing that the soliton dynamics is crucial to understand the appearance of self-assembled soliton lattices with a periodic arrangement of the fractional charges. Moreover, we extend the analysis to situations where particles are doped above other commensurate fillings, and argue that the larger topological charges of the soliton allows for other fractional densities.

**Many-body topological invariants.**—In this part of the...
In this quantity (6) is associated to the Chern numbers of an extended calculation at the middle of a chain with how the Berry phase FIG. 4.

\[ A = \int_{-\infty}^{\infty} d\phi \beta(\phi) = \frac{1}{\Delta} + \frac{1}{\Delta} \]

\[ B = \int_{-\infty}^{\infty} d\phi \partial_\phi \gamma(\phi), \]

which can be calculated in the many-body case using the approach of the previous section. As shown in Fig. 4, this pumping mechanism can also be applied to bosons in the presence of dynamical solitons. In this case, as shown in Figs. 4(a)-(b), the change of the local Berry phase at the middle of the chain yields the Chern numbers \( \nu = \pm 1 \) depending on the SSB sector A/B of the initial state. By calculating the evolution of the boson density (Fig. 4(c)), one clearly observes that a single bosonic charge is transferred between the soliton andantisoliton, and between each of them and the closest edge. Comparing with Figs. 4(a)-(b), it becomes clear that this pumping is directly associated to the different Chern numbers.

As announced above, the fractional bosons bound to the defects can be understood as remnants of the higher-dimensional conducting states that are localized at the interfaces separating the 2D regions of different Chern number. In the companion paper [37], we present a thorough analysis of this pumping mechanism, and show that it is crucial to understand the topological properties of the soliton quasi-particles that appear at other fractional fillings, unveiling a generalized bulk-defect correspondence.
Conclusions and outlook. — In this work we showed how boson fractionalization can take place in cold-atomic systems. In particular, we study the ground state of the $Z_2$ Bose-Hubbard model for incommensurate densities around half filling, where we found composite quasiparticles consisting on dynamical solitons on the $Z_2$ field together with particles with a fractional bosonic occupation number. We also characterized their properties, including the topological and the fractional charge. Finally, we connected these properties to the topological character of the underlying bulk through a generalized bulk-defect correspondence, where we demonstrated the quantization of inter-soliton transport.

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