Groups with normal restriction property

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Abstract. Let $G$ be a finite group. A subgroup $M$ of $G$ is said to be an $NR$-subgroup if, whenever $K \trianglelefteq M$, then $K^G \cap M = K$ where $K^G$ is the normal closure of $K$ in $G$. Using the Classification of Finite Simple Groups, we prove that if every maximal subgroup of $G$ is an $NR$-subgroup then $G$ is solvable. This gives a positive answer to a conjecture posed in [2].

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1. Introduction

All groups considered are finite. Let $G$ be a group. Following Berkovich in [2], a triple $(G, H, K)$ is said to be special in $G$ if $K \trianglelefteq H \leq G$ and $H \cap K^G = K$, where $K^G$ is the normal closure of $K$ in $G$. A subgroup $H$ is called an $NR$-subgroup (Normal Restriction) if, whenever $K \trianglelefteq H$, then $(G, H, K)$ is special in $G$. The main result of this paper is a proof of Conjecture 2 raised in [2].

Theorem 1.1. ([2] Conjecture 2) If all maximal subgroups of $G$ are $NR$-subgroups then $G$ is solvable.

In order to prove Theorem 1.1 we need a result on the factorization of almost simple groups. Unfortunately, we cannot avoid using the Classification of Finite Simple Groups in the proof of that result (see Theorem 1.2). Recall that a group $G$ is said to be almost simple if $S \leq G \leq Aut(S)$ for some non-abelian simple group $S$. If $K$ is a proper subgroup of $G$ and $H$ is a subgroup of $G$ with $K \leq H < G$, then $H$ is called a proper over-group of $K$ in $G$. Moreover, a subgroup $K$ of $G$ is said to be $p$-local in $G$ if $K = N_G(P)$ for some non-trivial $p$-subgroup $P$ of $G$, $p$ prime. We also say that $K$ is local in $G$ if $K$ is $p$-local in $G$ for some prime $p$.

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Finally, a subgroup $K$ of $G$ is said to be \textit{local maximal} if it is both maximal and local in $G$.

\textbf{Theorem 1.2.} Let $S$ be a non-abelian simple group and $S \leq G \leq \text{Aut}(S)$. Then there exists a non-trivial subgroup $K$ of $S$ such that all proper over-groups of $K$ in $S$ are local in $S$ and $G = N_G(K)S$.

The following corollary is used to show that the minimal counter-example to Theorem 1.1 is not simple.

\textbf{Corollary 1.3.} Let $S$ be a non-abelian simple group. Then $S$ contains a local maximal subgroup.

\textit{Proof.} Let $G = \text{Aut}(S)$ and $K$ be the subgroup of $S$ obtained from Theorem 1.2. Consider the set $\mathcal{A}$ of all proper over-groups of $K$ in $S$. Clearly, $\mathcal{A}$ is non-empty and every element of $\mathcal{A}$ is a local subgroup of $S$ containing $K$. The maximum element of $\mathcal{A}$ is a maximal subgroup of $S$ and is local. \hfill $\square$

\section{Preliminaries}

In this section, we collect some results that we need for the proofs of the theorems above.

\textbf{Lemma 2.1.} Let $K \trianglelefteq H \leq G$. If $H$ is an NR-subgroup of $G$ then $HK^G/K^G$ is an NR-subgroup of $G/K^G$. In particular, if $K \trianglelefteq G$ and all maximal subgroups of $G$ are NR-subgroups, then all maximal subgroups of $G/K$ are also NR-subgroups.

\textit{Proof.} The first statement is Lemma 4(c) in [2]. The second statement follows easily. \hfill $\square$

\textbf{Theorem 2.2.} ([1] Theorem 4.3) Let $P$ be a $p$-Sylow subgroup of a group $G$. If $P$ lies in the center of $N_G(P)$ then $G$ has a normal $p$-complement.

\textbf{Theorem 2.3.} ([2] Proposition 7) Let $H$ be a maximal solvable subgroup of $G$. If $H$ is an NR-subgroup of $G$ then $H = G$.

\section{Proofs of the Theorems}

\textit{Proof of Theorem 1.2.} Without loss of generality, we can assume that $G = \text{Aut}(S)$. By the Classification of Finite Simple Groups, if $S$ is a non-abelian simple group then $S$ is a finite simple group of Lie type, an alternating group of degree at least 5 or one of 26 sporadic groups. In this proof, we treat the Tits group, $^{2}F_4(2)'$ as sporadic group rather than a group of Lie type, and in view of the isomorphisms $A_6 \simeq L_2(9)$, and $A_5 \simeq L_2(5)$, we consider $A_5, A_6$ to be groups of Lie type.

(i) $S$ is a finite simple group of Lie type in characteristic $p$, $S \neq ^{2}F_4(2)'$. By Proposition 8.2.1 and Theorem 13.5.4 in [3], $S$ has a (B,N)-pair. Let $B$ be a Borel subgroup of $S$. Then $B = N_S(U)$, where $U$ is a $p$-Sylow subgroup of $S$. For any
Thus of \( G \), \( G/N \) satisfies the hypothesis of the Theorem and has smaller order than that of \( G \), by the minimality of \( G \), \( G/N \) is solvable. Thus \( N \) is the unique minimal normal subgroup of \( G \), and it coincides with the last term of the derived series of \( G \). If \( N \) is solvable then \( G \) is also solvable and we are done. Thus we assume that \( N \) is not solvable.

\[
\theta \in G, \text{ as } S \leq G, U^\theta \leq S^\theta = S, \text{ and hence } U^\theta \text{ is a } p\text{-Sylow subgroup of } S. \text{ By Sylow’s Theorem } U^\theta = U^g \text{ for some } g \in S. \text{ Observe that }
\]

\[
B^\theta = N_S(U^\theta) = N_S(U^g) = B^g.
\]

Thus \( \theta g^{-1} \in N_G(B) \), so that \( \theta \in N_G(B)S \), and hence \( G = N_G(B)S \). Moreover, if \( H \) is any proper over-group of \( B \) in \( S \), then \( H \) is a parabolic subgroup of \( S \) and \( H < S \), so that \( H \) is \( p\)-local in \( S \). Therefore we can choose \( K \) to be a Borel subgroup of \( S \).

(ii) \( S \) is an alternating group of degree \( n \geq 7 \). In this case \( G = S_n \). Let \( H = S_{n-3} \times S_3 \) and \( K = H \cap S \). Since \( n - 3 > 3 \), it follows from \([5]\) that \( K \) is a maximal subgroup of \( S \), \( H \) is a maximal subgroup of \( G \), and hence \( G = HS \). As \( [H : K] = 2 \), we have \( H = N_G(K) \), so that \( G = N_G(K)S \). The subgroup \( K \) satisfies the Theorem since it is 3-local and maximal in \( S \).

(iii) \( S \) is sporadic or \( S = {}^2F_4(2)' \).

By \([1]\), \( |G : S| = 1 \) or 2. If \( G = S \) then we can choose \( K \) to be any local maximal subgroup of \( S \). The pairs \((S, K)\) are given in Table 1. Otherwise, as in (ii), choose \( H \) to be a maximal subgroup of \( G \) such that \( K = H \cap S \) is a local maximal subgroup of \( S \). Then \( K \) will satisfy the conclusion of the Theorem. The triple \((S, K, H)\) are given in Table 2. The proof is now completed. □

**Proof of Theorem 1.1.** Let \( G \) be a minimal counter-example to Theorem 1.1. We first show that \( G \) is not simple. By contradiction, suppose that \( G \) is simple. By Corollary 1.3, \( G \) contains a \( p\)-local maximal subgroup \( M \). Let \( P \) be a \( p\)-subgroup of \( G \) such that \( M = N_G(P) \). Then \( 1 \neq P \leq M \) and since \( M \) is an \( NR\)-subgroup of \( G \), we have \( P^G \cap M = P \). However as \( G \) is simple and \( P \leq P^G \leq G \), \( P^G = G \). Hence \( P = G \cap M = M \). Let \( P_1 \) be a cyclic subgroup of order \( p \) in the center of \( M \). Then \( P_1 \) is normal in \( M \). Apply the same argument as above, we have \( P_1^G = G \), and so \( P_1 = P_1^G \cap M = M \). Thus \( M \) is a cyclic group of order \( p \).

In view of the maximality of \( M \) and the simplicity of \( G \), \( M \) is a \( p\)-Sylow subgroup of \( G \) and \( N_G(M) = M \). By Theorem 2.2, \( G \) has a normal \( p\)-complement. This contradicts to our assumption. Thus \( G \) is not simple.

Let \( N \) be any minimal normal subgroup of \( G \). By Lemma 2.2, the group \( G/N \) satisfies the hypothesis of the Theorem and has smaller order than that of \( G \), by the minimality of \( G \), \( G/N \) is solvable. Thus \( N \) is the unique minimal normal subgroup of \( G \), and it coincides with the last term of the derived series of \( G \). If \( N \) is solvable then \( G \) is also solvable and we are done. Thus we assume that \( N \) is not solvable.

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**Table 1.** \(|\text{Out}(S)| = 1\)

| \( S \) | \( M_{11} \) | \( J_1 \) | \( M_{23} \) | \( M_{24} \) | \( Ru \) | \( Co_1 \) | \( Co_2 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|
| \( K \) | 2 \( S_4 \) | 7 : 6 | 23 : 11 | \( 2^4 : A_8 \) | 5 : 4 \( \times A_5 \) | 2 \( \times M_{12} \) | \( 2^6 : M_{22} : 2 \) |
| \( S \) | Ly | Th | \( F_{223} \) | \( Co_1 \) | \( J_4 \) | B | M |
| \( K \) | 37 : 18 | 31 : 15 | 2 \( F_{222} \) | \( S_3 \times A_9 \) | 37 : 12 | 47 : 23 | 2 \( B \) |
Then $N = S_1 \times S_2 \times \cdots \times S_t$, where $S_i = S^{x_i}$, $S$ is a non-abelian simple group, and $x_1, x_2, \cdots, x_t \in G$. Let $K$ be the subgroup of $S$ obtained from Theorem 1.2 and $T = K_1 \times K_2 \times \cdots \times K_t$, where $K_i = K^{x_i}$. Then $T$ is a non-trivial proper subgroup of $N$. Since $N$ is the unique minimal normal subgroup of $G$, $N_{G}(T) < G$. We will show that $G = N_{G}(T)N$. For any $g \in G$, since $N^{g} = N$, there exists a permutation $\pi$ of degree $t$ acting on $\{1, 2, \cdots, t\}$ such that $S^{x_{g}} = S^{x_{\pi}}$. Let $g_{i} = x_{i}g_{x_{i}}^{-1}$. Then $g_{i} \in N_{G}(S)$. We have

$$T^{g} = K^{x_{g}} \times K^{x_{g}2} \times \cdots \times K^{x_{g}t} = K^{g_{1}x_{1}} \times K^{g_{2}x_{2}} \times \cdots \times K^{g_{t}x_{t}} = K^{h_{1}x_{1}} \times K^{h_{2}x_{2}} \times \cdots \times K^{h_{t}x_{t}} = K^{h_{1}x_{1}, s_{1}} \times K^{h_{2}x_{2}, s_{2}} \times \cdots \times K^{h_{t}x_{t}, s_{t}}$$

where $K^{g_{i}x_{i}} = K^{h_{i}}$ with $h_{i} \in S$ by Theorem 1.2 and $s_{i} = h_{i}^{-1}$. Let $s = s_{1}, s_{2}, \cdots, s_{t} \in N$. Since $[S_{i}, S_{j}] = 1$ if $i \neq j \in \{1, 2, \cdots, t\}$, $K_{s} = K_{s_{i}}$. Thus $T^{g} = T^{s}$, where $s \in N$. Therefore $G = N_{G}(T)N$.

Let $M$ be any maximal subgroup of $G$ containing $N_{G}(T)$. Let $U = M \cap N$. We have $G = MN$, and $U = M \cap N \leq M$. As $G/N = MN/N \simeq M/U$, $M/U$ is solvable. If $U$ is solvable then $M$ is solvable. By Theorem 2.3 $G = M$, a contradiction. Thus $U$ is non-solvable. Let $L$ be any non-trivial normal subgroup of $M$. Since $M$ is maximal in $G$, $M$ is an NR-subgroup of $G$, so that $L = L^{G} \cap M$. It follows from the fact that $N$ is the unique minimal normal subgroup of $G$, $N \leq L^{G}$. We have $U = N \cap M \leq L^{G} \cap M = L$. We conclude that $U$ is a minimal normal subgroup of $M$. Now, since $U$ is a minimal normal subgroup of $M$ and $U$ is non-solvable, $U = W_{1} \times W_{2} \times \cdots \times W_{k}$, where $W_{i} \simeq W$ for all $1 \leq i \leq k$ and $W$ is a non-abelian simple group. Suppose that there exists $j \in \{1, 2, \cdots, t\}$ such that $S_{j} \leq U$. As $S_{j}$

| S   | K              | H   |
|-----|----------------|-----|
| $M_{12}$ | $4^{2} : D_{12}$ | $K : 2$ |
| $M_{22}$ | $2^{7} : A_{6}$ | $2^{4} : S_{6}$ |
| $J_{2}$ | $A_{4} \times A_{5}$ | $K : 2$ |
| $^2F_{4}(2)'$ | $5^{2} : 4A_{4}$ | $5^{2} : 4S_{4}$ |
| $HS$       | $5 : 4 \times A_{5}$ | $5 : 4 \times S_{5}$ |
| $J_{3}$ | $2^{1+3} : A_{8}$ | $2^{1+3} : S_{5}$ |
| $McL$      | $5^{1+2} : 3 : 8$ | $K : 2$ |
| $He$       | $5^{2} : 4A_{4}$ | $5^{2} : 4S_{4}$ |
| $SU_{3}$   | $3^{3} : M_{11}$ | $3^{3} : (M_{11} \times 2)$ |
| $O'N$      | $4^{3} L_{3}(2)$ | $4^{3} (L_{3}(2) \times 2)$ |
| $Fi_{22}$  | $2^{10} : M_{22}$ | $2^{10} : M_{22} : 2$ |
| $HN$       | $3^{1+3} : 4A_{5}$ | $3^{1+3} : 4S_{5}$ |
| $Fi_{24}$  | $3' O_{7}(3)$ | $3' O_{7}(3) : 2$ |

Table 2. $|Out(S)| = 2$
is normal in $N$, 
$$S_j^G = S_j^{NM} = S_j^M \leq M.$$ 
However as $S_j^G = N$, $G = MN = M$, a contradiction. Therefore $S_j \cap U < S_j$ for any $j \in \{1, 2, \ldots, t\}$. Since $K_j \leq S_j \subseteq N$, $K_j \subseteq S_j \cap U \subseteq U$. As $U$ is a direct product of non-abelian simple groups and $S_j \cap U$ is a non-trivial normal subgroup of $U$, there exists a non-empty set $J \subseteq \{1, 2, \ldots, t\}$ such that $S_j \cap U = \prod_{i \in J} W_i$. Hence 
$$K_j \leq \prod_{i \in J} W_i < S_j,$$
and so 
$$K \leq \prod_{i \in J} W_i^{x_j^{-1}} < S,$$
where $W_i^{x_j^{-1}}$ are non-abelian simple for any $i \in J$. However, by Theorem 1.2, \(\prod_{i \in J} W_i^{x_j^{-1}}\) is local in $S$. This final contradiction completes the proof. □

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