Stability of the inverse scattering problem for the self-adjoint matrix Schrödinger operator on the half line

Xiao-Chuan Xu¹ | Natalia Pavlovna Bondarenko²,³

¹School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, Jiangsu, China
²Department of Applied Mathematics and Physics, Samara National Research University, Moskovskoye Shosse 34, Samara, Russia
³Department of Mechanics and Mathematics, Saratov State University, Astrakhanskaya 83, Saratov, Russia

Abstract
In this work, we study the inverse scattering problem for the self-adjoint matrix Schrödinger operator on the half line. We estimate the difference of two potentials and the difference of the two unitary matrices in the boundary conditions when the two sets of scattering data are close enough (or coincide) on a finite interval, which implies the stability of the inverse scattering problem.

KEYWORDS
finite data, inverse scattering problem, matrix Schrödinger operator, stability

1 | INTRODUCTION

Consider the matrix Schrödinger equation on the half-line

\[-\Psi''(x) + V(x) \Psi(x) = k^2 \Psi(x), \quad x \in \mathbb{R}^+ := (0, +\infty),\]
where \( \Psi \) is either an \( n \times n \) matrix-valued function or a column vector-valued function with \( n \) components, and the potential \( V(x) \) belongs to \( L_1^1(\mathbb{R}^+) \) and satisfies

\[
V(x)^\dagger = V(x). \tag{2}
\]

Here the dagger “\( ^\dagger \)" denotes the matrix adjoint (complex conjugate and matrix transpose), and \( V \in L_1^1(\mathbb{R}^+) \) means that

\[
\int_0^\infty (1 + x)\|V(x)\|dx < \infty, \tag{3}
\]

where the matrix norm is defined as

\[
\|V(x)\| = \max_i \sum_{s=1}^n |V_{is}(x)|, \tag{4}
\]

here \( V_{is}(x) \) denotes the \((i, s)\)-entry of \( V(x) \).

Let us consider (1) with the most general self-adjoint boundary condition

\[-B^\dagger \Psi(0) + A^\dagger \Psi'(0) = 0_n, \tag{5}\]

where (see Refs. 1, 2)

\[A = \frac{1}{2}(U + I_n) \quad \text{and} \quad B = \frac{i}{2}(U - I_n) \tag{6}\]

for some constant \( n \times n \) unitary matrix \( U \). Here \( I_n \) denotes the \( n \times n \) identity matrix, \( 0_n \) means an \( n \times n \) zero matrix or a zero column vector. It should be pointed out that instead of (6), the following equivalent condition can be considered:

\[B^\dagger A = A^\dagger B, \quad A^\dagger A + B^\dagger B > 0. \tag{7}\]

Indeed, it is obvious that (6) implies (7). Conversely, if \( A \) and \( B \) fulfill (7), then the matrices \( AD \) and \( BD \) satisfy (6) with \( U = (A - iB)(A + iB)^{-1} \) and the invertible matrix \( D = (A + iB)^{-1} \) (see, e.g., Refs. 3–5). Here for convenience of studying the inverse scattering stability, we use the condition (6), and denote by \( L(V, U) \) the problems (1) and (5). Note that (5) turns into the Dirichlet boundary condition \( \Psi(0) = 0 \) if \( U = -I_n \) and into the Neumann one \( \Psi'(0) = 0 \) if \( U = I_n \).

The matrix Schrödinger equation arises in quantum mechanics, electronics, nano science, and other branches of science and engineering,\(^3,4,6–9\) and also has important applications to the matrix-valued integrable system.\(^10–12\) The matrix Schrödinger equation (1) with the self-adjoint boundary condition (5) is connected with scattering in quantum mechanics involving particles of internal structures as spins, scattering on graphs and quantum wires (see Refs. 1, 3, 4, 7–9, 13–15 and the references therein).

The classical inverse scattering problem for the matrix Schrödinger equation is formulated as follows.
Inverse Problem 1. Given the scattering data $S := \{S(k), k_j, C_j\}_{k \in \mathbb{R}, j=1}^{N}$, find the potential $V(x)$ and the boundary condition coefficient $U$.

Here, $S(k)$ is the scattering matrix, $\{-k_j^2\}_j^{N}$ are the bound states (the eigenvalues), and $\{C_j\}_j^{N}$ are the normalization matrices. The rigorous definitions of these characteristics are provided in Section 2.

Inverse scattering problem for the scalar Schrödinger operator on the half line has been studied thoroughly.\textsuperscript{16–20} In particular, in the revised book,\textsuperscript{18} Marchenko provided the stability of the inverse scattering problem with the Dirichlet boundary condition.

Matrix Schrödinger operators are more complicated than the scalar ones. Agranovich and Marchenko\textsuperscript{21} were the first ones who studied the inverse scattering problem for the matrix Schrödinger operator on the half-line, while only for the problem with the Dirichlet boundary condition. In the last 20 years, the matrix problems with the general self-adjoint boundary condition (5) attracted much attention because of their applications to quantum graphs. For the matrix scattering problem (1),(5), Harmer deduced the Marchenko equation \textsuperscript{2} and also applied his results to the scattering problem on the star-shaped graph.\textsuperscript{1} Aktosun and coauthors implemented the small and large energy analysis of the scattering matrix.\textsuperscript{5,22,23} Xu and Yang proved the uniqueness theorems with/without bound states.\textsuperscript{24} The characterization of the scattering data was given in Ref. 25. For more results on the direct and inverse scattering for the matrix Schrödinger operator with the most general self-adjoint boundary condition, we refer readers to the monograph.\textsuperscript{26} It is worth mentioning that the inverse spectral problem by the Weyl matrix for the matrix Schrödinger operator has been investigated in Refs. 27, 28. Without claim to completeness, we also mention some other works on the matrix Schrödinger operators. Inverse spectral problems for such operators on a finite interval were studied in Refs. 29–34, and the inverse scattering on the full line, in Refs. 35–38.

Although there are many interesting results on the inverse problems for the self-adjoint matrix Schrödinger operators on the half-line, as far as we know, the aspect of stability has not been considered before, even in the Dirichlet case. This paper aims to fill this gap. From the physical point of view, it is natural to assume that the scattering matrix $S(k)$ is known only on a finite interval $k \in (0, T)$, $T > 0$. Therefore, instead of Inverse Problem 1, we focus on the following problem.

Inverse Problem 2. Given the finite scattering data: $\{k_j, C_j\}_j^{N}$ and $S(k)$ on $(0, T)$, find $V(x)$ and $U$.

It is important to note that, in contrast to Inverse Problem 1, the solution of Inverse Problem 2 is nonunique. Nevertheless, it is natural to study the difference $(V(x) - \tilde{V}(x))$ of the potential $V(x)$ and of (any) potential $\tilde{V}(x)$ reconstructed by using the finite data of Inverse Problem 2. The goal of this paper is to estimate this difference in the two cases: (i) The finite scattering data corresponding to $V$ and $\tilde{V}$ coincide and (ii) the finite scattering data is given with an error at most $\varepsilon$. In addition, we study the difference of the boundary condition coefficients $(U - \tilde{U})$. The obtained estimates lead to the uniform stability of Inverse Problem 2 as $T \to \infty$ under some restrictions on $V$ and $U$.

Here, we should mention the stability results in the scalar case. Marchenko and Lundina\textsuperscript{18,39,40} were the first ones who studied the stability of the inverse scattering problem with the Dirichlet boundary condition. They estimated the difference $|V(x) - \tilde{V}(x)|$ in terms of the function $\varepsilon(T) := \int_0^{T-1} a(x)dx$, under the additional assumption that the potentials have bounded derivatives, where $a(x)$ satisfies $\int_x^{\infty} |V(t)| dt \leq a(x)$. Recently, Xu\textsuperscript{20} proved the stability for the Dirichlet...
and non-Dirichlet boundary conditions, without the differentiability hypothesis of the potentials. In particular, the estimate for $| \int_x^\infty (V(t) - \tilde{V}(t)) dt |$ was given in terms of $\frac{c_0}{t^p}$ with some constant $c_0 > 0$ and $p \in (0, 1)$. Furthermore, the stability of the boundary condition coefficient was also given in the non-Dirichlet case. In this paper, we generalize the ideas of Refs. 18, 20 to the matrix case, and obtain the stability estimates for the matrix inverse scattering problem.

This paper is organized as follows. In Section 2, we provide some notations and the known results. In Section 3, we obtain the relation for the difference of two potentials, which is used in the further proofs. In Section 4, we estimate difference of the scattering matrix and its limit. In Section 5, we study the stability of the inverse scattering problem by finite scattering data. In Section 6, the stability is studied for the case when the finite scattering data are given with errors. In the last section, we summarize the main results and give some comments.

2 | PRELIMINARIES

In this section, we introduce some notations, and recall the definitions of some relevant physical qualities together with their properties. The results presented in this section can be found in Refs. 5, 21–23, 35.

Let $\mathbb{C}^n$ be the space of complex column vector with $n$ components. By $\bar{k}$ we mean the complex conjugate of the number $k$. Denote $\mathbb{C}^\pm = \{k : \pm \text{Im}k > 0\}$ and $\mathbb{C}^{\pm} = \mathbb{C}^\pm \cup \mathbb{R}$, and $\mathbb{R}^+ = (0, +\infty)$. Let $L^p((a, b); \mathbb{C}^n)$ and $L^p((a, b); \mathbb{C}^{n\times n})$ be the spaces of all column vector- and matrix-valued functions, respectively, with each element belonging to $L^p(a, b)$, where $p = 1, 2$ and $-\infty < a < b \leq +\infty$. If it does not cause misunderstanding, we may just use the notation $L^p(a, b)$ for simplicity.

Together with (1), we consider the following equation:

$$ -\Phi'' + \Phi V(x) = k^2 \Phi, \quad x > 0, \quad (8) $$

where $\Phi$ is either an $n \times n$ matrix-valued function or a row vector-valued function with $n$ components. Let $[\Phi; \Psi] := \Phi \Psi' - \Phi' \Psi$ denote the matrix Wronskian. It is easy to prove that, if $\Phi(k, x)$ satisfies (8) and $\Psi(k, x)$ satisfies (1), then the Wronskian $[\Phi(k, x); \Psi(k, x)]$ does not depend on $x$. In addition, if $\Psi(k, x)$ is a solution of (1) for real $k$, then $\Psi(\pm k, x)$ are solutions of (8). In case $\Psi(k, x)$ and $\Psi(-k, x)$ have analytic extensions from $k \in \mathbb{R}$ to $k \in \mathbb{C}^+$ and $\Psi(k, x)$ solves (1), then $\Psi(-k, x)$ solves (8), and hence $\Psi(-k, x)$ is independent of $x$ and analytic for $k \in \mathbb{C}^+$.

Let $f(k, x)$ be the Jost solution to (1), which is a matrix-valued function satisfying the integral equation

$$ f(k, x) = e^{ikx} I_n + \int_x^\infty \frac{\sin k(t - x)}{k} V(t) f(k, t) dt, \quad k \in \mathbb{C}^+. \quad (9) $$

In Ref. 21, the following representation of the Jost solution in terms of the transformation operator has been obtained:

$$ f(k, x) = e^{ikx} I_n + \int_x^\infty K(x, t) e^{ikt} dt, \quad x \geq 0, \quad k \in \mathbb{C}^+, \quad (10) $$
where $K(x, t)$ is a continuous matrix function of two variables, having first derivatives with respect to $x$ and $t$. Actually, the matrix-valued function $K(x, t)$ here has the same properties as that in the scalar case (see Ref. 18):

$$
||K(x, t)|| \leq e^{K(x) - \frac{K(x) + t}{2}} \frac{1}{2} |V(t)| dt, \quad j = 0, 1,
$$

(11)

$$
K(x, x) = \frac{1}{2} \int_x^\infty V(t) dt.
$$

(12)

Using the estimate (11), we derive

$$
\|f(k, x)\| \leq e^{\frac{1}{2} \int_x^\infty \|K(x, t)\| dt} \leq e^{\frac{1}{2} \int_x^\infty \|V(t)\| dt}, \quad k \in \mathbb{C}^+.
$$

(13)

Note that $f(k, x)$ is analytic for $k \in \mathbb{C}^+$, and

$$
\dot{f}(k, x) := \frac{\partial f(k, x)}{\partial k} = ixe^{ikx}I_n + \int_x^\infty itK(x, t)e^{ikt} dt, \quad k \in \mathbb{C}^+.
$$

(14)

Consequently,

$$
\|\dot{f}(k, x)\| \leq \frac{1}{\text{Im}k} \left( 1 + \int_x^\infty \|K(x, t)\| dt \right) \leq \frac{e^{\sigma_1(x) - \frac{1}{2} \text{Im}k}}{\text{Im}k}, \quad k \in \mathbb{C}^+, \quad x \geq 0.
$$

(15)

Here we have used the inequality

$$
xe^{-\tau x} \leq \frac{1}{\tau e} \quad \text{for} \quad \tau > 0 \quad \text{and} \quad x \geq 0.
$$

(16)

Define the Jost matrix

$$
J(k) := f(-\kappa, 0)^\dagger B - f'(-\kappa, 0)^\dagger A.
$$

(17)

Recall that the eigenvalue of the problem $L(V, U)$ is the $k^2$-value for which (1) has a nonzero column vector solution $\Psi \in L^2(\mathbb{R}^\neq)$ satisfying (5), and the corresponding solution is called an eigenfunction. It was shown in Ref. 23 that $L(V, U)$ has a finite number of negative eigenvalues, whose square roots coincide with the zeros of the function $\det J(k)$, which all lie on the positive imaginary axis $i\mathbb{R}^\neq$. Assume that the function $\det J(k)$ has $N$ different zeros on $i\mathbb{R}^\neq$, denoted by \{ik_j\}_{j=1}^N with $0 < k_1 < k_2 < ... < k_N$. Define the scattering matrix

$$
S(k) = -J(-\kappa)J(k)^{-1}, \quad k \in \mathbb{R},
$$

(18)

and the normalization matrices

$$
C_j := P_j P_j^\dagger P_j \left( P_j \int_0^\infty f(ik_j, x)^\dagger f(ik_j, x) dx P_j + I_n - P_j \right)^{-\frac{1}{2}}, \quad j = 1, 2, ..., N,
$$

(19)

where $P_j$ is the orthogonal projection onto $\ker J(ik_j)^\dagger$, $j = 1, 2, ..., N$. 

**Proposition 1** see Refs. 5, 23. Assume that $V \in L^1_1(\mathbb{R}^+)$ and satisfies (2), and the matrix $U$ in the boundary condition (5) is unitary. Then the scattering matrix $S(k)$ defined in (18) is continuous on $\mathbb{R}$ and satisfies

$$S(-k) = S(k)^\dagger = S(k)^{-1}, \quad k \in \mathbb{R},$$

and

$$S(k) = U_0 + O\left(\frac{1}{k}\right), \quad |k| \to \infty,$$

where $U_0$ is some unitary Hermitian matrix specified uniquely by $U$.

**Definition 1.** The data $S := \{S(k), k^j, C^j\}_{k \in \mathbb{R}, j = 1}^N$ are called the scattering data.

The scattering data are connected with the potential and the unitary matrix in the boundary condition through the following Marchenko equation 2,26:

$$F(x + y) + K(x, y) + \int_x^\infty K(x, t)F(t + y)dt = 0, \quad y > x \geq 0,$$

where

$$F(x) := \sum_{j=1}^N C_j^2 e^{-k_j x} + \frac{1}{2\pi} \int_{-\infty}^\infty [S(k) - U_0]e^{ikx}dk.$$  

(23)

It is known 24,26 that the matrix-valued function $F(x)$ is Hermitian, belongs to $L^1(\mathbb{R}^+)$, and goes to zero as $x \to \infty$. Finding $K(x, y)$ from (22), one can determine the potential $V(x) = 2 \frac{d}{dx} K(x, x)$ from (12).

### 3 | DIFFERENCE OF TWO POTENTIALS

In this section, we obtain the relation for the differences of two potentials, which further plays an important role in the proofs of the main results. Here, we follow the strategy of Ref. [18, Chapter 5].

For any $f(\cdot) \in L^p((b, \infty); \mathbb{C}^n)$ with $b \geq 0$ and $p = 1, 2, \infty$, define the following operators mapping $L^p(b, \infty)$ into itself:

$$(K_{b,f})(x) := \int_b^\infty K(x, t)f(t)dt, \quad (K^\ast_{b,f})(y) := \int_b^\infty K(t, y)^\dagger f(t)dt,$$

$$(F_{b,f})(x) = \int_b^\infty F(x + t)f(t)dt, \quad (I_f)(y) = f(y).$$

(24)

Because $K(x, y) = 0$ for $x > y$, Equation (24) can also be written as

$$(K_{b,f})(y) = (K_{y,f})(y) = \int_y^\infty K(y, t)f(t)dt, \quad (K^\ast_{b,f})(y) = \int_b^y K(t, y)^\dagger f(t)dt.$$  

(25)
Using the arguments similar to Ref. [18, p. 203], it is easy to get the following lemma. For convenience of readers, we provide the proof in the Appendix.

**Lemma 1.** The following relation is valid:

\[
(I + F_b)^{-1} = (I + K_x^e)(I + K_b). \tag{26}
\]

Together with the problem \(L(V, U)\), we consider a problem \(L(\tilde{V}, \tilde{U})\) of the same form but with different coefficients \(\tilde{V}\) and \(\tilde{U}\). We agree that, if a certain symbol \(\delta\) denotes an object related to \(L(V, U)\), then \(\tilde{\delta}\) will denote an analogous object related to \(L(\tilde{V}, \tilde{U})\).

By (22), we have

\[
\tilde{K}(x, y) - K(x, y) + \tilde{F}(x + y) - F(x + y) + \int_x^\infty \tilde{K}(x, t)\tilde{F}(t + y)dt \\
- \int_x^\infty K(x, t)F(t + y)dt = 0_n, \quad y \geq x. \tag{27}
\]

Denote

\[
\tilde{K}(x, t) := \tilde{K}(x, y) - K(x, y), \quad \tilde{F}(x) := \tilde{F}(x) - F(x). \tag{28}
\]

It follows that

\[
\tilde{K}(x, y) + \int_x^\infty \tilde{K}(x, t)\tilde{F}(t + y)dt + \tilde{F}(x + y) + \int_x^\infty K(x, t)\tilde{F}(t + y)dt = 0_n, \quad y \geq x. \tag{29}
\]

Taking the matrix adjoint of the above equation and noting that \(F = F^\dagger\), we have

\[
\tilde{K}(x, y)^\dagger = -(I + \tilde{F}_x)^{-1}\left[\tilde{F}(x + y) + \int_x^\infty \tilde{F}(t + y)\tilde{K}(x, t)^\dagger dt\right]. \tag{30}
\]

Using (23) together with (10), we calculate

\[
\tilde{F}(x + y) + \int_x^\infty \tilde{F}(t + y)\tilde{K}(x, t)^\dagger dt = \sum_{j=1}^N C_j^2 e^{-\bar{k}_j y}f(i\bar{k}_j, x)^\dagger - \sum_{j=1}^N C_j^2 e^{-k_j y}f(i\bar{k}_j, x)^\dagger \\
+ \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{S}(k)e^{ik y}f(-\bar{k}, x)^\dagger dk, \tag{31}
\]

where

\[
\tilde{S}(k) = S(k) - S(k) + U_0 - \bar{U}_0. \tag{32}
\]

Here, we have used the Fourier transform theory to interchange the order of integration.

By (26) and (30), we obtain

\[
\tilde{K}(x, y)^\dagger = -(I + \tilde{K}_x)(I + \tilde{K}_x)\left[\tilde{F}(x + y) + \int_x^\infty \tilde{F}(t + y)\tilde{K}(x, t)^\dagger dt\right]. \tag{33}
\]
Using (31) and (10), and noting \( y \geq x \), we get
\[
\phi(x, y) := (I + K)x \left[ \hat{F}(x + y) + \int_x^\infty \hat{F}(t + y)K(x, t)^{\dagger} \, dt \right]
\]
\[
= \sum_{j=1}^{\hat{N}} \hat{f}(ik_j, y)\hat{C}_j^2 f(i\hat{k}_j, x) - \sum_{j=1}^{\hat{N}} \hat{f}(ik_j, y)\hat{C}_j^2 f(ik_j, x)^{\dagger}
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k, y)\hat{S}(k)f(-\hat{k}, x)^{\dagger} \, dk.
\] (34)

Note that \( \hat{K}^* \) is the zero operator when \( y = x \) from the second equation in (25). Letting \( y = x \) in (33) and (34), taking the matrix adjoint, and using (12), (20), we obtain the following theorem.

**Theorem 1.** For the two problems \( L(V, U) \) and \( L(\hat{V}, \hat{U}) \), the following relation holds:
\[
\frac{1}{2} \int_{-\infty}^{\infty} [\hat{V}(t) - \hat{V}(t)] \, dt = -\sum_{j=1}^{\hat{N}} f(ik_j, x)\hat{C}_j^2 \hat{f}(ik_j, x)^{\dagger}
\]
\[
- \sum_{j=1}^{\hat{N}} f(ik_j, x)\hat{C}_j^2 \hat{f}(ik_j, x)^{\dagger}
\]
\[
- \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k, x)\hat{S}(k)f(-k, x)^{\dagger} \, dk,
\] (35)
where \( \hat{S}(k) \) is defined in (32).

## 4 | DIFFERENCE OF THE SCATTERING MATRIX AND ITS LIMIT

In this section, we estimate the difference \( \|S(k) - U_0\| \) of the scattering matrix and its limit. Let us first consider the problem \( L(U, 0_n) \). In this case, the Jost matrix has the form
\[
J_0(k) = -ikA + B,
\] (36)
and the scattering matrix,
\[
S_0(k) = -J_0(\hat{k})J_0(k)^{-1} = -(ikA + B)(-ikA + B)^{-1}.
\] (37)

Because \( U \) is a unitary matrix, there exists a unitary matrix \( M \) such that
\[
M^{\dagger}UM = \text{diag}\{ -e^{-2i\theta_1}, ..., -e^{-2i\theta_n_M}, -I_{n_D} \},
\] (38)
where \( \theta_j \in (0, \pi) \), and \( n_M + n_D = n \). Substitute (38) into (6), and get
\[
M^{\dagger}AM = \text{diag}\left\{ \frac{1 - e^{-2i\theta_1}}{2}, ..., \frac{1 - e^{-2i\theta_n_M}}{2}, 0_{n_D} \right\},
\] (39)
\[
M^{\dagger}BM = \text{diag}\left\{ \frac{(e^{-2i\theta_1} + 1)}{2i}, ..., \frac{(e^{-2i\theta_n_M} + 1)}{2i}, -iI_{n_D} \right\}.
\] (40)

It follows from (36), (39), and (40) that
\[ AJ_0(k)^{-1} = M \text{diag}\left\{ \frac{-1}{ik + \cot \vartheta_1}, \ldots, \frac{-1}{ik + \cot \vartheta_{nm}}, 0_{nd} \right\} M^\dagger, \quad (41) \]
\[ BJ_0(k)^{-1} = M \text{diag}\left\{ \cot \vartheta_1, \ldots, \cot \vartheta_{nm}, I_{id} \right\} M^\dagger. \quad (42) \]

In view of Proposition 1,
\[ \lim_{k \to \pm\infty} S(k) = \lim_{k \to \pm\infty} S_0(k) = U_0. \quad (43) \]

Using (41) and (42) in (37), we have
\[ M^\dagger S_0(k)M = \text{diag}\left\{ ik - \cot \vartheta_1, \ldots, ik - \cot \vartheta_{nm}, -I_{nd} \right\} \]
\[ = \text{diag}\{ I_{nm}, -I_{id} \} - \text{diag}\left\{ \frac{2 \cot \vartheta_1}{ik + \cot \vartheta_1}, \ldots, \frac{2 \cot \vartheta_{nm}}{ik + \cot \vartheta_{nm}}, 0_{nd} \right\} \]
\[ = M^\dagger U_0 M - M^\dagger U_1(k) M, \quad (44) \]
where \( U_0 \) is the unitary matrix from (21), \( U_0 = M \text{diag}\{ I_{nm}, -I_{id} \} M^\dagger, \)
\[ U_1(k) := M \text{diag}\left\{ \frac{2 \cot \vartheta_1}{ik + \cot \vartheta_1}, \ldots, \frac{2 \cot \vartheta_{nm}}{ik + \cot \vartheta_{nm}}, 0_{nd} \right\} M^\dagger. \quad (45) \]

Let all potentials considered in the following discussion belong to the following class \( \mathcal{V}_a \):
\[ \mathcal{V}_a := \left\{ V \in L^1_1(0, \infty) : V^\dagger = V, \int_x^\infty \| V(t) \| dt \leq a(x) < \infty \right\}. \quad (46) \]

Let all the unitary matrices in the boundary condition (5) be in the following class:
\[ U_{\Theta} := \left\{ U : U \text{ satisfies } (38) \text{ for some fixed unitary matrix } M \text{ and } \max_j |\cot \vartheta_j| \leq \Theta \right\}. \quad (47) \]

Now let us estimate \( \| S(k) - U_0 \| \) for large \(|k|\) under the assumption that \( V \in \mathcal{V}_a \) and \( U \in U_{\Theta} \). Namely, we prove the following theorem.

**Theorem 2.** Assume \( V \in \mathcal{V}_a \) and \( U \in U_{\Theta} \). Then
\[ \| S(k) - U_0 \| \leq \frac{1}{|k|} \left( 2 \sqrt{n} a_0 + \frac{4 \sqrt{2n} \Theta}{1 + \frac{\Theta}{|k|}} \right) \frac{1}{1 - \frac{a_0}{|k|}}, \quad |k| > a_0, \quad k \in \mathbb{R}, \quad (48) \]
where
\[ a_0 := \max \left\{ 1, \frac{a(0)(1 + \sqrt{n} + 3en)}{2} \right\}. \quad (49) \]

**Proof.** One should first note that the norm of an arbitrary \( n \times n \) unitary matrix is bounded by \( \sqrt{n} \), because the matrix norm is defined as (4). Indeed, for an arbitrary unitary matrix \( W = (w_{ij})_{n \times n} \),
the Cauchy inequality yields

\[
\sum_{j=1}^{n} |w_{ij}| \leq \sqrt{\sum_{j=1}^{n} |w_{ij}|^2} = \frac{1}{\sqrt{n}},
\]

which implies \(\|W\| \leq \sqrt{n}\). In particular, \(\|S_0(k)\| \leq \sqrt{n}\).

Because \(S(k) = J(-k)J_0(-k)^{-1}S_0(k)J_0(k)J(k)^{-1}\), we have

\[
S(k) - U_0 = \{J(-k)J_0(-k)^{-1}S_0(k) - U_0J(k)J_0(k)^{-1}\}[J(k)J_0(k)^{-1}]^{-1}.
\]

It is known that the estimate of the Jost solution can be obtained by the method of successive iteration. Rewrite the Jost solution as

\[
f(k, x) = e^{ikx}z(k, x),
\]

where

\[
z(k, x) = I_n - \frac{1}{2ik} \int_{x}^{\infty} (1 - e^{2ik(t-x)})V(t)z(k, t)dt.
\]

Put

\[
z_0(k, x) = I_n, \quad z_{s+1}(k, x) = -\frac{1}{2ik} \int_{x}^{\infty} (1 - e^{2ik(t-x)})V(t)z_s(k, t)dt.
\]

By induction, one can easily obtain the estimate

\[
\|z_s(k, x)\| \leq \frac{a(x)^s}{|k|^s s!}, \quad k \in \mathbb{C} \setminus \{0\}, \quad x \geq 0.
\]

It follows that

\[
z(k, x) = \sum_{s=0}^{\infty} z_s(k, x).
\]

Replacing \(k\) by \(-\bar{k}\), and taking the matrix adjoint of the above relations, we obtain

\[
\|z_s(-\bar{k}, x)\| \leq \frac{a(x)^s}{|k|^s s!}, \quad k \in \mathbb{C} \setminus \{0\}, \quad x \geq 0.
\]

Therefore, we get

\[
f(-\bar{k}, 0)^\dagger = I_n - \frac{Q_0 - Q_1(k)}{ik} + f_0(k),
\]

where

\[
Q_0 := \frac{\int_{0}^{\infty} V(t)dt}{2}, \quad Q_1(k) := \frac{\int_{0}^{\infty} V(t)e^{2ikt}dt}{2}, \quad \|f_0(k)\| \leq \frac{a(0)^2}{2|k|^2} e^{\frac{a(0)}{|k|}}.
\]
Note that
\[
f'(k, x) - ike^{ikx}I_n = -\frac{e^{ikx}}{2} \int_x^\infty (1 + e^{2ik(t-x)})V(t)z(k, t)dt,
\]
which together with (55) and (56) implies that
\[
f'(-\bar{k}, 0)^\dagger = ikI_n - Q_0 - Q_1(k) + f_1(k),
\]
where
\[
\|f_1(k)\| \leq \frac{a(0)^2}{|k|} e^{\frac{a(0)}{|k|}}.
\]
Substituting (58) and (61) into (17) and taking (36) into account, we have
\[
J(k)J_0(k)^{-1} = I_n - \frac{Q_0 + Q_1(k)S_0(k)}{ik} + (f_0(k)B - f_1(k)A)J_0(k)^{-1} := I_n - f_2(k).
\]
Note that \( \left| \frac{1}{ik + \cot \vartheta} \right| \) and \( \left| \frac{\cot \vartheta}{ik + \cot \vartheta} \right| \) are less than 1 for real \( k \). It follows from (41) and (42) that
\[
\|AJ_0(k)^{-1}\| \leq \frac{n}{|k|}, \quad \|BJ_0(k)^{-1}\| \leq n.
\]
Consequently, the function \( f_2(k) \) defined in (63) satisfies
\[
\|f_2(k)\| \leq \frac{a(0)(1 + \sqrt{n})}{2|k|} + \frac{3na(0)^2e^{\frac{a(0)}{|k|}}}{2|k|^2}
\]
\[
\leq \frac{a(0)\left(1 + \sqrt{n} + 3en\right)}{2|k|} := \frac{a_0}{|k|}, \quad |k| \geq \max\{1, a(0)\}, \quad k \in \mathbb{R}.
\]
Therefore, if \( |k| > a_0 \) then \( \|f_2(k)\| < 1 \), which implies
\[
[J(k)J_0(k)^{-1}]^{-1} = I_n + \sum_{j=1}^\infty f_2(k)^j, \quad \|[J(k)J_0(k)^{-1}]^{-1}\| \leq \frac{1}{1 - \|f_2(k)\|} \leq \frac{1}{1 - \frac{a_0}{|k|}}.
\]
Using (63) and (44), we have
\[
J(-k)J_0(-k)^{-1}\left(1 - U_0J(k)J_0(k)^{-1}\right) = U_0f_2(k) - f_2(-k)U_0 - f_2(-k)U_1(k) + U_1(k).
\]
With the help of the inequality \( |a|^2 + |b|^2 \geq (|a| + |b|)^2/2 \), we obtain
\[
\left| \frac{\cot \vartheta_j}{ik + \cot \vartheta_j} \right| = \frac{|\cot \vartheta_j|}{\sqrt{|k|^2 + |\cot \vartheta_j|^2}} \leq \frac{\sqrt{2}|\cot \vartheta_j|}{|k| + |\cot \vartheta_j|} \leq \sqrt{2}\Theta.
\]
Using this inequality in (45), we have
\[
\|U_1(k)\| \leq \frac{2\sqrt{2}n\Theta}{|k| + \Theta}.
\]
Using (69) and (65) in (67), we obtain
\[
\|J(-k)J_0(k)^{-1}S_0(k) - U_0J(k)J_0(k)^{-1}\| \leq \sqrt{n}\|f_2(k)\| + \|f_2(-k)\| + 2\|U_1(k)\|
\]
\[
\leq 2\sqrt{n}a_0\frac{|k|}{|k| + \Theta} + \frac{4\sqrt{2n}\Theta}{|k| + \Theta}.
\] (70)

Substituting (70) and (66) into (51), we arrive at (48). The proof is complete.

5 | STABILITY OF THE INVERSE SCATTERING PROBLEM

In this section, we give the estimates for the difference of two potentials and the difference of two unitary matrices in the boundary conditions in terms of the difference of the corresponding scattering data, which implies the stability of the inverse scattering problem.

**Theorem 3.** Let \( \mathcal{V}_a \) and \( \mathcal{U}_\Theta \) be defined in (46) and (47), respectively. Assume that \( V, \bar{V} \in \mathcal{V}_a \) and \( U, \bar{U} \in \mathcal{U}_\Theta \). If \( V(x) - \bar{V}(x) \) is bounded, and the scattering data \( S \) and \( \bar{S} \) coincide for \( k^2 \in (-\infty, 0) \cup (0, T^2) \), that is,

\[
N = \bar{N}, \quad k_j = \bar{k}_j, \quad C_j = \bar{C}_j, \quad j = 1, \ldots, N, \quad S(k) = \bar{S}(k), \quad k \in (0, T),
\] (71)

where \( T > a_0 \) and \( a_0 \) is defined in (49), then

\[
\left\| \int_x^\infty [V(t) - \bar{V}(t)] dt \right\| \leq \pi h \sup_{t \in [x, x+h\pi]} \|\bar{V}(t) - V(t)\|
\]
\[
+ \frac{24\sqrt{n}}{\pi}\left( a_0 T + 2\sqrt{2n} \ln \left( 1 + \frac{\Theta T}{T} \right) \right)
\] (72)

for all \( h > T^{-1} \) and \( x \geq 0 \).

**Remark 1.** If we take \( h = T^{-\frac{1}{2}} \) and \( T \to \infty \) in the above theorem, then (72) implies the uniqueness of recovering the potential from the scattering data.

**Proof of Theorem 3.** Note that \( U, \bar{U} \in \mathcal{U}_\Theta \) implies \( U_0 = \bar{U}_0 \). Because \( S(k)^\dagger = S(-k) \) and \( S(k) \) is continuous on \( \mathbb{R} \), then \( S(k) = \bar{S}(k) \) on \( (0, T) \) means \( S(k) = \bar{S}(k) \) on \( [-T, T] \). It follows from (35) and (71) that

\[
Q(x) - \bar{Q}(x) = \frac{1}{\pi} \int_{|k| \geq T} f(k, x)\bar{S}(k)\bar{f}(-k, x)^\dagger dk,
\] (73)

where

\[
Q(x) = \int_x^\infty V(t) dt, \quad \bar{Q}(x) = \int_x^\infty \bar{V}(t) dt.
\] (74)
By (10), we have
\[ f(k, x) = e^{ikx} + g_0(k, x), \quad f'(k, x) = ikI_ne^{ikx} + g_1(k, x), \quad k \in \mathbb{C}^+. \] (75)

From the proof of Theorem 2, we see that
\[ \|g_0(k, x)\| \leq \frac{a(x)e^{a(x)|k|}}{|k|}, \quad \|g_1(k, x)\| \leq a(x)e^{a(x)|k|}, \quad |k| > 0. \] (76)

Substituting the first relation in (75) into (73), we have
\[ Q(x) - \tilde{Q}(x) = \frac{1}{\pi} \int_{|k| \geq T} \hat{S}(k)e^{2ikx}dk + g_2(x), \] (77)

where
\[ g_2(x) = \frac{1}{\pi} \int_{|k| \geq T} \left\{ [\hat{S}(k)g_0(-k, x)^\dagger + g_0(k, x)\hat{S}(k)]e^{ikx} + g_0(k, x)\hat{S}(k)g_0(-k, x)^\dagger \right\} dk. \] (78)

It follows from Theorem 2, (76), and $|k| \geq T > a_0$ that
\[ \|g_2(x)\| \leq \frac{1}{\pi} \int_{|k| \geq T} \|\hat{S}(k)\| (\|g_0(k, x)\| + \|g_0(-k, x)\| + \|g_0(k, x)\| \|g_0(-k, x)\|)dk \]
\[ \leq \frac{2}{\pi} \int_{T}^{\infty} \frac{2}{k} \left( 2\sqrt{n}a_0 + \frac{4\sqrt{2n}}{1 + \Theta/k} \right) \frac{1}{1 - \frac{a_0}{k}} \frac{3a(x)e^{a(x)/k}}{k} dk \]
\[ \leq \frac{24\sqrt{n}a(x)e^{a(x)/T}}{T} \left( \int_{T}^{\infty} \frac{a_0}{k^2} dk + \int_{T}^{\infty} \frac{2\sqrt{2n}}{k^2 + \Theta k} dk \right) \]
\[ = \frac{24\sqrt{n}a(x)e^{a(x)/T}}{T} \left( \frac{a_0}{T} + 2\sqrt{2n} \ln \left( 1 + \frac{\Theta}{T} \right) \right). \] (79)

Replace $x$ in (77) by $x + ht$, multiply the both sides of (77) by $I_n \sin t$, integrate on $(0, \pi)$, and so get
\[ \int_{0}^{\pi} (Q - \tilde{Q})(x + ht) \sin t dt = \frac{1}{\pi} \int_{|k| \geq T} \hat{S}(k) \int_{0}^{\pi} I_n e^{2ik(x + ht)} \sin t dt dk + \int_{0}^{\pi} g_2(x + ht) \sin t dt. \] (80)

Here, the integration order is changed according to the Fourier transform theory. Because $\int_{0}^{\pi} \sin t dt = 2$, then
\[ \int_{0}^{\pi} (Q - \tilde{Q})(x + ht) \sin t dt = \int_{0}^{\pi} \tilde{Q}(x, h, t) \sin t dt + 2[Q(x) - \tilde{Q}(x)], \] (81)

where
\[ \tilde{Q}(x, h, t) := Q(x + ht) - Q(x) - \tilde{Q}(x + ht) + \tilde{Q}(x). \] (82)
By the mean value theorem for the differential, we have
\[
\dot{Q}(x, h, t) := ht[\dot{V} - V](x + h\xi), \quad \xi \in [0, t].
\] (83)

It follows that
\[
\left\| \int_0^\pi \dot{Q}(x, h, t) \sin t \, dt \right\| \leq 2\pi h \sup_{t \in [x, x + h\pi]} \|\dot{V}(t) - V(t)\|. \tag{84}
\]

Observe that
\[
\left\| \int_0^\pi I_n e^{2ik(x + h\xi)} \sin t \, dt \right\| = \left\| I_n \frac{e^{2ik\pi} - 1}{4k^2 h^2} \right\| \leq \frac{1}{2k^2 h^2 \left(1 - \frac{1}{4k^2 h^2}\right)} \leq \frac{2}{3k^2 h^2} \tag{85}
\]
for \(kh > 1\).

Substituting (81) into (80), and using (84), (85), and (79), we obtain
\[
\left\| Q(x) - \tilde{Q}(x) \right\| \leq \pi h \sup_{t \in [x, x + h\pi]} \|\dot{V}(t) - V(t)\| + \frac{1}{3\pi T h^2} \int_{|k| \geq T} \frac{\|\dot{S}(k)\|}{|k|} \, dk 
+ \frac{24\sqrt{n}a(x)e}{\pi \left(1 - \frac{a_0}{T}\right)} \left(\frac{a_0}{T} + 2\sqrt{2n} \ln \left(1 + \frac{\Theta}{T}\right)\right). \tag{86}
\]

Using the estimate (48) in (86), similarly to (79), we obtain (72). The proof is complete. \(\blacksquare\)

**Theorem 4.** Assume that \(V, \tilde{V} \in \mathcal{V}_a\) and \(U, \tilde{U} \in \mathcal{U}_0\). If the scattering matrices \(S(k)\) and \(\tilde{S}(k)\) coincide for \(k \in (0, T)\), \(T > a_0\), then
\[
\left\| U - \tilde{U} \right\| \leq \sqrt{2n} \left\{ \sqrt{n} \left\| \int_0^\infty (V - \tilde{V})(t) \, dt \right\| + \frac{1}{2} \left\| \int_0^\infty (V - \tilde{V})(t)e^{-2iTt} \, dt \right\| 
+ \frac{n}{2} \left\| \int_0^\infty (V - \tilde{V})(t)e^{2iTt} \, dt \right\| + \frac{2b(T)}{T} \right\}, \tag{87}
\]
where
\[
b(T) = a_2 + \frac{\sqrt{n}a(0)(1 + \sqrt{n})(a_1 + a_{0,1,T})}{2T} + \sqrt{n}(a_1 + a_{0,1,T} + a_1a_{0,1,T}), \tag{88}
\]
here \(a_1, a_{0,1,T}, a_2\) are bounded and defined in (91), (93), (102), respectively.

**Remark 2.** Letting \(T \to \infty\) in the above theorem and taking Remark 1 into account, we obtain the uniqueness result of Ref. 24 for the inverse scattering problem.

**Proof of Theorem 4.** Substitute (44) into (63), and rewrite it as
\[
J(k)J_0(k)^{-1} = I_n - \frac{Q_0 + Q_1(k)U_0}{ik} + f_3(k), \tag{89}
\]
where
\[ f_3(k) := (f_0(k)B - f_1(k)A)J_0(k)^{-1} + \frac{Q_1(k)U_1(k)}{ik}, \quad \|f_3(k)\| \leq \frac{a_1}{|k|^2}, \quad (90) \]
\[ a_1 := \frac{a(0)n(3a(0)e + 2\Theta)}{2}. \quad (91) \]

Using (66), we have

\[ [J(k)J_0(k)^{-1}]^{-1} = I_n + \frac{Q_0 + Q_1(k)U_0}{ik} + f_4(k), \quad (92) \]

where

\[ \|f_4(k)\| \leq \frac{a_{0,1,k}}{|k|^2}, \quad a_{0,1,k} := a_1 + a_2 \left( \frac{1}{1 - \frac{a_0}{|k|}} \right). \quad (93) \]

Rewrite (45) as

\[ U_1(k) = U_2(k) + U_3(k), \quad (94) \]

where

\[ U_2(k) := M \text{diag} \left\{ \frac{2 \cot \theta_1}{ik}, \ldots, \frac{2 \cot \theta_{n_M}}{ik}, 0_{n_D} \right\} M^+, \quad (95) \]
\[ U_3(k) := M \text{diag} \left\{ \frac{2 \cot^2 \theta_1}{k^2 - ik \cot \theta_1}, \ldots, \frac{2 \cot^2 \theta_{n_M}}{k^2 - ik \cot \theta_{n_M}}, 0_{n_D} \right\} M^+. \quad (96) \]

Note that

\[ S(k) = J(-k)J_0(-k)^{-1}S_0(k)[J(k)J_0(k)^{-1}]^{-1}. \quad (97) \]

Substituting (89), (92), and (44) with (94) into (97), and noting \( U_0^2 = I_n \), we get

\[ S(k) = U_0 - U_2(k) + \frac{Q_0U_0 + Q_1(-k) + U_0Q_0 + U_0Q_1(k)U_0}{ik} + H(k), \quad (98) \]

where

\[ H(k) := -U_3(k) - \frac{[Q_0 + Q_1(-k)U_0]U_1(k) + U_1(k)[Q_0 + Q_1(k)U_0]}{ik} \]
\[ - \frac{[Q_0 + Q_1(-k)U_0]S_0(k)[Q_0 + Q_1(k)U_0]}{k^2} \]
\[ + \frac{[Q_0 + Q_1(-k)U_0]S_0(k)f_4(k) + f_3(-k)S_0(k)[Q_0 + Q_1(k)U_0]}{ik} \]
\[ + f_3(-k)S_0(k) + S_0(k)f_4(k) + f_3(-k)S_0(k)f_4(k). \quad (99) \]

Using the estimates

\[ \|Q_0 + Q_1(\pm k)U_0\| \leq \frac{a(0)}{2}(1 + \sqrt{n}), \quad \|U_1(k)\| \leq \frac{2n\Theta}{|k|}, \quad \|U_3(k)\| \leq \frac{2n\Theta^2}{|k|^2}, \quad \|S_0(k)\| \leq \sqrt{n}, \quad (100) \]
together with (90) and (93), we obtain

$$
\|H(k)\| \leq \frac{a_2 + \sqrt{n}a(0)(1+\sqrt{n})(a_1+a_{0,1,k})}{2|k|} + \sqrt{n}(a_1 + a_{0,1,k} + a_1a_{0,1,k})|k| \leq \frac{a(0)^2 \sqrt{n}(1+\sqrt{n})^2}{4} + a_2^2 + 2\sqrt{n}a(0)(1+\sqrt{n}) + a(0)^2 \sqrt{n}(1+\sqrt{n})^2 4
$$

where

$$
a_2 = 2n\Theta^2 + 2n\Theta a(0)(1 + \sqrt{n}) + \frac{a(0)^2 \sqrt{n}(1+\sqrt{n})^2}{4}.
$$

It follows from (98) that

$$
\begin{align*}
\imath k [S(k) - \hat{S}(k)] & = \imath k [U_2(k) - U_2(k) + (Q_0 - \hat{Q}_0)U_0 + [Q_1(-k) - \hat{Q}_1(-k)] \\
& + U_0(Q_0 - \hat{Q}_0) + U_0[Q_1(k) - \hat{Q}_1(k)]U_0 + \imath k (H(k) - \hat{H}(k)).
\end{align*}
$$

Note that the left-hand side of (103) is continuous on \(\mathbb{R}\), and vanishes on \((-T, T)\). Hence, \(S(k) - \hat{S}(k)\) is also equal to zero at \(k = T\). It follows from (95) that

$$
2M \text{diag}\{\cot \theta_1 - \cot \hat{\theta}_1, \ldots, \cot \theta_{n_M} - \cot \hat{\theta}_{n_M}, 0_{n_D}\}M^+ = (Q_0 - \hat{Q}_0)U_0 + [Q_1(-T) - \hat{Q}_1(-T)]
$$

$$
+ U_0(Q_0 - \hat{Q}_0) + U_0[Q_1(T) - \hat{Q}_1(T)]U_0 + iT(H(T) - \hat{H}(T)).
$$

By the mean value theorem for the differential, we know

$$
\begin{align*}
\theta_j - \bar{\theta}_j & = \sin^2 \xi_j (\cot \theta_j - \cot \bar{\theta}_j), \quad \xi_j \in [\theta_j, \bar{\theta}_j] \ (\text{or} [\bar{\theta}_j, \theta_j]), \\
e^{-2i\theta_j} - e^{-2i\bar{\theta}_j} & = -2(\sin 2\alpha_j + i \cos 2\beta_j)(\theta_j - \bar{\theta}_j), \quad \alpha_j, \beta_j \in [\theta_j, \bar{\theta}_j] \ (\text{or} [\bar{\theta}_j, \theta_j]).
\end{align*}
$$

It follows from (105) and (106) that

$$
e^{-2i\theta_j} - e^{-2i\bar{\theta}_j} = -2(\sin 2\alpha_j + i \cos 2\beta_j) \sin^2 \xi_j (\cot \theta_j - \cot \bar{\theta}_j),
$$

which together with (38) imply

$$
U - \hat{U} = 2U_4 M \text{diag}\{\cot \theta_1 - \cot \bar{\theta}_1, \ldots, \cot \theta_{n_M} - \cot \bar{\theta}_{n_M}, 0_{n_D}\}M^+,
$$

where

$$
U_4 := -M \text{diag}\{(\sin 2\alpha_1 + i \cos 2\beta_1) \sin^2 \xi_1, \ldots, (\sin 2\alpha_{n_M} + i \cos 2\beta_{n_M}) \sin^2 \xi_{n_M}, I_{n_D}\}M^+.
$$

Clearly, \(\|U_4\| \leq \sqrt{2n}\). Therefore, using (101), (104), (108), and (109), we obtain (87). The proof is complete.

Theorems 3 and 4 together imply the following corollary, which asserts the uniform stability of Inverse Problem 2 as \(T \to \infty\) under certain restrictions on \(V(t)\) and \(U\).

**Corollary 1.** Let \(A_0, \Theta > 0\) be fixed. Suppose that we have two families of the problem coefficients \((V_T, U_T)\) and \((\hat{V}_T, \hat{U}_T)\) parameterized by \(T > 0\), such that the corresponding scattering data \(S_T\) and...
\( \bar{S}_T \) coincide for \( k^2 \in (-\infty, 0) \cup (0, T^2) \). Assume that \( V_T, \bar{V}_T \in L^1(\mathbb{R}^+) \),

\[
\int_0^\infty \|V_T(t)\| \, dt \leq A_0, \quad \int_0^\infty \|\bar{V}_T(t)\| \, dt \leq A_0,
\]

that is, \( V_T, \bar{V}_T \in \mathcal{V}_a \) with \( a(x) \equiv A_0 \), and, additionally, \( \|V_T(t) - \bar{V}_T(t)\| \leq A_0 \), \( t \in \mathbb{R}^+ \), \( U_T, \bar{U}_T \in \mathcal{U}_0 \). Then,

\[ \lim_{T \to \infty} \left\| \int_x^\infty (V_T(t) - \bar{V}_T(t)) \, dt \right\| = 0, \quad x \in [0, \infty). \]

More precisely,

\[ \left\| \int_x^\infty (V_T(t) - \bar{V}_T(t)) \, dt \right\| \leq c_0 T^{-2/3}, \quad T \geq T_0, \quad x \in [0, \infty), \]

where the constant \( c_0 \) is independent of \( T \) and on the choice of \( V_T, \bar{V}_T \). The certain values of \( T_0 \) and \( c_0 \) can be found by using (72) with \( h = T^{-2/3} \):

\[
T_0 := \max\{2a_0, (9A_0 e)^3\}, \quad c_0 := A_0 \pi + \frac{32 \sqrt{n}}{3\pi} (a_0 + 2\sqrt{2n\Theta}).
\]

Suppose that, in addition,

\[
V'_T, \bar{V}'_T \in L^1(\mathbb{R}^+), \quad \int_0^\infty \|V'_T(t) - \bar{V}'_T(t)\| \, dt \leq A_0.
\]

Then, obviously,

\[ \left\| \int_0^\infty (V_T(t) - \bar{V}_T(t)) e^{\pm 2iTT} \, dt \right\| \leq \frac{A_0}{T}, \quad T > 0. \]

Therefore, the estimate (87) together with (112) and (115) imply

\[ \|U_T - \bar{U}_T\| \leq \sqrt{2n} \left( \sqrt{n}c_0 T^{-2/3} + \frac{(n + 1)A_0 + 4b(T)}{2T} \right), \quad T \geq T_0. \]

Thus,

\[ \lim_{T \to \infty} \|U_T - \bar{U}_T\| = 0. \]

In Corollary 1, we have chosen \( h = T^{-2/3} \) in (72) to make both the terms in the right-hand side of (72) contain an equal power of \( T \). One can also use the estimates of Theorems 3 and 4 to obtain the uniform stability estimates under different restrictions on the potentials \( V, \bar{V} \), and with various choices of \( h = h(T) \).
6 | STABILITY WITH ERRORS

In this section, we consider the case when the given scattering data have errors. In general, the eigenvalues may split under a small perturbation. Such splitting is taken into account in our stability analysis.

Thus, we suppose that the quantities \( N \) and \( \tilde{N} \) of the discrete scattering data may not be equal to each other. However, we can divide the eigenvalues into different close groups. Assume that the numbers \( \{k_j\}_{j=1}^N \) and \( \{\tilde{k}_j\}_{j=1}^\tilde{N} \) are divided into groups \( \{G_l\}_{l=1}^r \) and \( \{\tilde{G}_l\}_{l=1}^\tilde{r} \), respectively, with \( r = \tilde{r} \). Namely, let

\[
G_l = \{k_{l,j}\}_{j=1}^{N_l}, \quad \tilde{G}_l = \{\tilde{k}_{l,j}\}_{j=1}^{\tilde{N}_l},
\]

\[
\bigcup_{l=1}^r G_l = \{k_j\}_{j=1}^N, \quad \bigcup_{l=1}^\tilde{r} \tilde{G}_l = \{\tilde{k}_j\}_{j=1}^\tilde{N}, \quad \sum_{l=1}^r N_l = N, \quad \sum_{l=1}^{\tilde{r}} \tilde{N}_l = \tilde{N}.
\]

The normalization matrices \( \{C_{l,j}\}_{j=1}^{N_l} \) and \( \{\tilde{C}_{l,j}\}_{j=1}^{\tilde{N}_l} \) are also divided into similar groups \( \{C_{l,i}\}_{i=1}^{N_l} \) and \( \{\tilde{C}_{l,i}\}_{i=1}^{\tilde{N}_l} \), respectively, \( l = 1, \ldots, r \).

Recall that \( k_j > k_l > 0 \) for \( j > l \). Even if \( k_{l,i} \) is a multiple zero of \( \det J(k) \), then this number occurs in the sequence \( \{k_j\}_{j=1}^N \) only once. Similarly, the numbers \( k_{l,i} \) are all distinct, and the same is valid for \( \tilde{k}_{l,i} \).

Without loss of generality, for each \( l \), let \( k_{l,i} = k_{l,j} < k_{l,j} \) and \( \tilde{k}_{l,i} = \tilde{k}_{l,j} \) for \( i < j \).

Denote

\[
k_{l,0} := \frac{1}{2}[k_{l,1} + k_{l,N_l}], \quad \tilde{k}_{l,0} := \frac{1}{2}[\tilde{k}_{l,1} + \tilde{k}_{l,N_l}],
\]

\[
\xi_l := \max \left\{ \max_{1 \leq j \leq N_l} |k_{l,j} - k_{l,0}|, \max_{1 \leq j \leq \tilde{N}_l} |\tilde{k}_{l,j} - \tilde{k}_{l,0}|, |k_{l,0} - \tilde{k}_{l,0}| \right\},
\]

\[
\eta_l := \left\| \sum_{j=1}^{N_l} C_{l,j}^2 - \sum_{j=1}^{\tilde{N}_l} \tilde{C}_{l,j}^2 \right\|.
\]

Our goal is to estimate the potential difference \( (V - \tilde{V}) \) in terms of \( \{\xi_l\}_{l=1}^r, \{\eta_l\}_{l=1}^\tilde{r} \), and \( \|S(k) - \tilde{S}(k)\| \). Assume that \( a(x) \) in (46) belongs to \( L^1(0, \infty) \), and denote

\[
a_1(x) := \int_x^\infty a(t)dt.
\]

**Theorem 5.** Let \( V_a \) and \( U_\Theta \) be defined in (46) and (47), respectively. Assume that \( V, \tilde{V} \in V_a \) and \( U, \tilde{U} \in U_\Theta \). Suppose that the numbers \( \{k_j\}_{j=1}^N \) and \( \{\tilde{k}_j\}_{j=1}^\tilde{N} \) are divided into groups (118), the function \( (V(x) - \tilde{V}(x)) \) is bounded, and

\[
\max_{k \in [0, T]} \|S(k) - \tilde{S}(k)\| \leq \epsilon, \quad T > a_0.
\]

Then,

\[
\left\| \int_x^\infty [V(t) - \tilde{V}(t)]dt \right\| \leq \pi h \sup_{t \in [x, x+h\pi]} \|\tilde{V}(t) - V(t)\| + \frac{2\pi T e^{2a_1(x)}}{\pi}
\]
\[\frac{24\sqrt{n}a(x)e^{8\sqrt{n}/3T^{2}}}{\pi\left(1 - \frac{a_0}{T}\right)}\left(\frac{a_0}{T} + 2\sqrt{2n \ln \left(1 + \frac{\Theta}{T}\right)}\right) + \sum_{l=1}^{r}(6D_{1,l}(x)\xi_{l} + D_{2,l}(x)\eta_{l})\]

for all \(h > T^{-1}\) and \(x \geq 0\), where

\[D_{1,l}(x) = \frac{e^{2a_1(x)}}{\min(k_{l,1}, k_{l,1})} e^{1 + \min(k_{l,1}, \tilde{k}_{l,1})x} \max\left\{\sum_{j=1}^{N_l} ||C_{l,j}^{2}||, \sum_{j=1}^{\tilde{N}_l} ||\tilde{C}_{l,j}^{2}||\right\},\]

\[D_{2,l}(x) = \frac{e^{2a_1(x)}}{e^{2\min(k_{l,1}, \tilde{k}_{l,1})x}}.\]

Proof. By (35), we have

\[\frac{1}{2} \int_{x}^{\infty} [V(t) - V(t)]dt = \sum_{j=1}^{N} f(ik_j, x)C_j^{2} \bar{f}(ik_j, x) - \sum_{j=1}^{N} f(ik_j, x)C_j^{2} \bar{f}(i\tilde{k}_j, x)^{\dagger} - \frac{1}{2\pi} \left(\int_{-T}^{T} + \int_{|k| \geq T}\right)f(k, x)\tilde{S}(k)\bar{f}(-k, x)^{\dagger}dk.\]

(127)

Following the proof of Theorem 3, let us consider the part of bound states and scattering matrix on \((-T, T)\) in (127). By (13) and \(\int_{0}^{\pi} \sin t dt = 2\), we have

\[\left\|\int_{-T}^{T} \int_{0}^{\pi} f(k, x + ht)\tilde{S}(k)\bar{f}(-k, x + ht)^{\dagger} \sin t dt dk\right\| \leq 4Te^{2a_1(x)}.\]

(128)

On the other hand, note that

\[\sum_{j=1}^{N} f(ik_j, x)C_j^{2} \bar{f}(ik_j, x) - \sum_{j=1}^{\tilde{N}} f(ik_j, x)C_j^{2} \bar{f}(i\tilde{k}_j, x)^{\dagger} = \sum_{l=1}^{r} \left[\sum_{j=1}^{N_l} f(ik_{l,j}, x)C_{l,j}^{2} \bar{f}(ik_{l,j}, x)^{\dagger} - \sum_{j=1}^{\tilde{N}_l} f(i\tilde{k}_{l,j}, x)\tilde{C}_{l,j}^{2} \bar{f}(i\tilde{k}_{l,j}, x)^{\dagger}\right].\]

(129)

The term in the square brackets can be represented as follows:

\[\sum_{j=1}^{N_l} f(ik_{l,j}, x)C_{l,j}^{2} \bar{f}(ik_{l,j}, x)^{\dagger} - \sum_{j=1}^{\tilde{N}_l} f(i\tilde{k}_{l,j}, x)\tilde{C}_{l,j}^{2} \bar{f}(i\tilde{k}_{l,j}, x)^{\dagger}\]

\[= \sum_{j=1}^{N_l} [f(ik_{l,j}, x) - f(ik_{l,0}, x)]C_{l,j}^{2} \bar{f}(ik_{l,j}, x)^{\dagger} + \sum_{j=1}^{\tilde{N}_l} f(ik_{l,j}, x)\tilde{C}_{l,j}^{2} \bar{f}(i\tilde{k}_{l,j}, x)^{\dagger} - f(ik_{l,0}, x)^{\dagger}].\]
\[
+ \sum_{j=1}^{\tilde{N}_l} \left[ f(i\tilde{k}_{l,j}, x) - f(i\tilde{k}_{l,j}, x) \right] \tilde{C}_{l,j}^2 \tilde{f}(i\tilde{k}_{l,j}, x)^\dagger \leq \sum_{j=1}^{\tilde{N}_l} \|k_{l,j} - k_{l,j,0}\| \| \tilde{f}(i\tilde{k}_{l,j,0}, x) \tilde{C}_{l,j}^2 \tilde{f}(i\tilde{k}_{l,j}, x)^\dagger \|
\]

Let us transform the last two terms:

\[
f(i\tilde{k}_{l,0}, x) \sum_{j=1}^{N_l} C_{l,j}^2 \tilde{f}(i\tilde{k}_{l,0}, x)^\dagger = \left[ f(i\tilde{k}_{l,0}, x) \sum_{j=1}^{N_l} C_{l,j}^2 \tilde{f}(i\tilde{k}_{l,0}, x)^\dagger \right] + f(i\tilde{k}_{l,0}, x) \sum_{j=1}^{N_l} C_{l,j}^2 \tilde{f}(i\tilde{k}_{l,0}, x)^\dagger.
\]

Further, we substitute (130) and (131) into (129), and estimate the resulting terms by using (13), (15), and the mean value theorem. For example,

\[
\left\| \sum_{j=1}^{N_l} \left[ f(i\tilde{k}_{l,j}, x) - f(i\tilde{k}_{l,j}, x) \right] \tilde{C}_{l,j}^2 \tilde{f}(i\tilde{k}_{l,j}, x)^\dagger \right\| \leq \sum_{j=1}^{N_l} |k_{l,j} - k_{l,j,0}| \| \tilde{f}(i\tilde{k}_{l,j,0}, x) \tilde{C}_{l,j}^2 \tilde{f}(i\tilde{k}_{l,j}, x)^\dagger \|
\]

where \(k_{l,j,0} \in [\min(k_{l,0}, k_{l,j}), \max(k_{l,0}, k_{l,j})] \subset [k_{l,1}, \max(k_{l,0}, k_{l,j})]\). Estimating similarly the other terms in (130) and (131), we show that

\[
\left\| \sum_{j=1}^{N_l} f(i\tilde{k}_{j,0}, x) C_{j}^2 \tilde{f}(i\tilde{k}_{j}, x)^\dagger - \sum_{j=1}^{\tilde{N}_l} f(i\tilde{k}_{j}, x) \tilde{C}_{j}^2 \tilde{f}(i\tilde{k}_{j}, x)^\dagger \right\| \leq \sum_{l=1}^{r} \left( 6D_{1,l}(x) \xi_l + D_{2,l}(x) \eta_l \right).
\]

Using the latter estimate together with (128), (127), and the proof of Theorem 3, we obtain (125). The proof is complete.

**Remark 3.** If \(N = \tilde{N}\), then we can let \(N_l = \tilde{N}_l = 1\) for each \(l\). In this case, the last sum on the right-hand side of the estimate (125) could be replaced by

\[
\sum_{j=1}^{N} |k_j - \tilde{k}_j| D_{1,j}(x) + \sum_{j=1}^{\tilde{N}} \|C_j^2 - \tilde{C}_j^2\| D_{2,j}(x),
\]

where either

\[
D_{1,j}(x) = 2\|\tilde{C}_j^2\| \frac{e^{2a_1(x)}}{\min(k_j, \tilde{k}_j)} e^{1+\min(k_j, \tilde{k}_j)} x, \quad D_{2,j}(x) = \frac{e^{2a_1(x)}}{e^{2k_j} x}.
\]
or

\[ D_{1,j}(x) = 2\| C_j \| \frac{e^{2\alpha_1(x)}}{\min(k_j, \tilde{k}_j)e^{1+\min(k_j, \tilde{k}_j)x}}, \quad D_{2,j}(x) = \frac{e^{2\alpha_1(x)}}{e^{2\tilde{k}_j x}}. \] (136)

**Remark 4.** Suppose that \( V, \tilde{V} \in \mathcal{V}_\alpha, U, \tilde{U} \in \mathcal{U}_\Theta \), and the condition (124) holds. Then, following the proof of Theorem 4 with necessary modifications, we obtain the estimate for \( \| U - \tilde{U} \| \) similar to (87) with the additional term \( \sqrt{2nT\varepsilon} \).

**Corollary 2.** Let \( A_0, \Theta > 0 \) be fixed. Suppose that we have two families of the problem coefficients \((V_{T,\varepsilon}, U_{T,\varepsilon})\) and \((\tilde{V}_{T,\varepsilon}, \tilde{U}_{T,\varepsilon})\) parameterized by \( T > 0 \) and \( \varepsilon > 0 \), such that the corresponding scattering data \( S_{T,\varepsilon} = \{ k_j, C_j, S(k) \} \) and \( \tilde{S}_{T,\varepsilon} = \{ \tilde{k}_j, \tilde{C}_j, \tilde{S}(k) \} \) have the difference at most \( \varepsilon \) for \( k^2 \in (-\infty, T^2] \), that is,

\[ \max_{k \in [0,T]} \| S(k) - \tilde{S}(k) \| \leq \varepsilon, \] (137)

\[ |\xi_j| \leq \varepsilon, \quad |\eta_l| \leq \varepsilon \quad j = 1, \ldots, r. \] (138)

The two important special cases of (138) are:

(i) \( N = \tilde{N}, \ |k_j - \tilde{k}_j| \leq \varepsilon, \ |C_j^2 - \tilde{C}_j^2| \leq \varepsilon, \ j = 1, \ldots, N. \)

(ii) \( N_l = 1, \ k_{l,1} = k_l, \ N_l \geq 1, \ l = 1, \ldots, r, \) and

\[ |k_l - \tilde{k}_{l,0}| \leq \varepsilon, \quad \max_{1 \leq j \leq \tilde{N}_l} |\tilde{k}_{l,j} - \tilde{k}_{l,0}| \leq \varepsilon, \quad \left\| \frac{\tilde{C}_l^2 - \sum_{j=1}^{\tilde{N}_l} \tilde{C}_{l,j}^2}{C_l^2 - \sum_{j=1}^{N_l} C_{l,j}^2} \right\| \leq \varepsilon, \quad l = 1, \ldots, r. \] (139)

Roughly speaking, under a small perturbation, each value \( k_l \) splits into several sufficiently close values \( \{\tilde{k}_{l,j}\}_{j=1}^{\tilde{N}_l}. \)

Assume that \( V_{T,\varepsilon}, \tilde{V}_{T,\varepsilon} \in L^1_1(\mathbb{R}^+) \),

\[ \int_0^\infty (1 + x)\| V_{T,\varepsilon}(t) \| dt \leq A_0, \quad \int_0^\infty (1 + x)\| \tilde{V}_{T,\varepsilon}(t) \| dt \leq A_0, \] (140)

that is, \( a(x) \leq A_0, \ a_1(x) \leq A_0, \) and, additionally, \( \| V_{T,\varepsilon}(t) - \tilde{V}_{T,\varepsilon}(t) \| \leq A_0, \ t \in \mathbb{R}^+, \ U_{T,\varepsilon}, \tilde{U}_{T,\varepsilon} \in \mathcal{U}_\Theta, k_j, \tilde{k}_j \geq A_0^{-1}, \ |C_j^2|, \ |\tilde{C}_j^2| \leq A_0, \ j = 1, \ldots, N. \) Then,

\[ \left\| \int_x^\infty (V_{T,\varepsilon}(t) - \tilde{V}_{T,\varepsilon}(t)) dt \right\| \leq c_0 T^{-2/3} + c_1 \varepsilon T + c_2 \varepsilon, \quad T \geq T_0, \quad x \in [0, \infty), \] (141)

where \( T_0 \) and \( c_0 \) are defined in (113), \( c_1 := \frac{2e^{2A_0}}{\pi} \), and \( c_2 \) can be easily found from (125). In particular, the estimate (141) yields

\[ \lim_{T \to \infty, \varepsilon \to 0} \left\| \int_x^\infty (V_{T,\varepsilon}(t) - \tilde{V}_{T,\varepsilon}(t)) dt \right\| = 0. \] (142)

Suppose that, in addition, the condition (114) holds for \( V_{T,\varepsilon}', \tilde{V}_{T,\varepsilon}' \). Then (141) together with (115) and Remark 4 imply

\[ \lim_{T \to \infty, \varepsilon \to 0} \| U_{T,\varepsilon} - \tilde{U}_{T,\varepsilon} \| = 0. \] (143)
Thus, Corollary 2 shows the uniform stability of Inverse Problem 2 under the perturbation of the given scattering data $S$ for $k^2 \in (-\infty, T^2]$, if the error $\varepsilon$ is connected with $T$ by the condition $\varepsilon T \rightarrow 0$.

7 | CONCLUSION

In this paper, we obtained the stability estimates of the inverse scattering problem for the problem $L(V, U)$ with $V \in \mathcal{V}_a$ and $U \in \mathcal{U}_\Theta$, which generalize the stability of the inverse scattering problems in the scalar case. We assume that the scattering matrix $S(k)$ is known only on a finite interval $k \in (0, T)$. The differences of the potentials $(V(x) - \tilde{V}(x))$ and of the boundary condition matrices $(U - \tilde{U})$ are estimated in the two cases: (i) the corresponding finite scattering data coincide and (ii) the finite scattering data are sufficiently close. In case (ii), the splitting of multiple eigenvalues under a small perturbation is taken into account. The obtained estimates imply the uniform stability of the inverse scattering problem as $T \rightarrow \infty$ under some restrictions on $V$ and $U$.

The above results are obtained under the condition that the unitary matrices $U$ and $\tilde{U}$ can be diagonalized by the same unitary matrix $M$ and $n_M = \bar{n}_M$ in (38). This condition can be considered from the following point of view. One can initially apply the unitary transform diagonalizing $U$ to Equation (1) and to the boundary condition (5) and study the stability of the inverse scattering problem in this normalized class. The number $n_M$, roughly speaking, is responsible for the “structure” of the problem. For instance, $n_M = 0$ corresponds to the Dirichlet boundary condition $\Psi(0) = 0$ and $n_M = n$, to the Robin one: $\Psi'(0) + H(0) \Psi(0) = 0$, $H = H^\dagger$. In this paper, we confine ourselves to the case of the two scattering problems $L(V, U)$ and $L(\tilde{V}, \tilde{U})$ having the same “structure.” The opposite case is a topic for separate investigation.

ACKNOWLEDGMENTS

The authors would like to thank the referees for their insightful comments and helpful suggestions. Theorem 5 and Corollaries 1 and 2 have been obtained by N. P. Bondarenko, all the other results of the paper have been obtained by X.-C. Xu. The research work of the author X.-C. Xu was supported by the National Natural Science Foundation of China (11901304). The research work of the author N. P. Bondarenko was supported by Grant 21-71-10001 of the Russian Science Foundation.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

REFERENCES

1. Harmer MS. Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions. *ANZIAM J.* 2002;44:161-168.
2. Harmer MS. Inverse scattering on matrices with boundary conditions. *J Phys A: Math Gen.* 2005;38:4875-4885.
3. Kostrykin V, Schrader R. Kirchhoff’s rule for quantum wires. II: the inverse problem with possible applications to quantum computers. *Fortschr Phys.* 2000;48:703-716.
4. Kurasov P, Stenberg F. On the inverse scattering problem on branching graphs. *J Phys A.* 2002;35:101-121.
5. Aktosun T, Klaus M, Weder R. Small-energy analysis for the self-adjoint matrix Schrödinger operator on the half line. *J Math Phys.* 2011;52:102101.
6. Zakhariev BN, Suzko AA. Direct and Inverse Problems: Potentials in Quantum Scattering. Springer; 1990.
7. Kostrykin V, Schrader R. Kirchhoff’s rule for quantum wires. *J Phys A: Math Gen.* 1999;32:595-630.
8. Kuchment P. Quantum graphs. I. Some basic structures. *Waves Random Media.* 2004;14:S107-S128.
9. Kuchment P. Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J Phys A.* 2005;38:4887-4900.
10. Bondarenko N, Freiling G, Urazboev G. Integration of the matrix KdV equation with self-consistent source. *Chaos Solitons & Fractals.* 2013;49:21-27.
11. Calogero F, Degasperis A. Nonlinear evolution equations solvable by the inverse spectral transform II. *Nuovo Cimento B.* 1977;39(1).
12. Olmedilla E. Inverse scattering transform for general matrix Schrödinger operators and the related symplectic structure. *Inv Probl.* 1985;1:219-236.
13. Berkolaiko G, Carlson R, Fulling SA, Kuchment P., eds. *Quantum Graphs and Their Applications, Contemporary Mathematics 415.* American Mathematical Society; 2006.
14. Buterin SA, Freiling G. Inverse spectral-scattering problem for the Sturm-Liouville operator on a noncompact star-type graph. *Tamkang J Math.* 2013;44:327-349.
15. Exner P, Keating JP, Kuchment P, Sunada T, Teplyaev A., eds. *Analysis on Graphs and Its Applications, Proceedings of Symposia in Pure Mathematics Vol. 77.* Amer Math Soc. Providence, RI, 2006.
16. Levitan BM. *Inverse Sturm-Liouville Problems.* VNU Science Press; 1987.
17. Freiling G, Yurko VA. *Inverse Sturm-Liouville Problems and Their Applications.* NOVA Science Publishers; 2001.
18. Marchenko V. *Sturm-Liouville Operators and Applications.* Revised ed. American Mathematical Society; 2011.
19. Ramm AG. Inverse scattering on the half-line revisited. *Rep Math Phys.* 2015;76:159-169.
20. Xu XC. Stability of direct and inverse scattering problems for the self-adjoint Schrödinger operators on the half-line. *J Math Anal Appl.* 2021;501:125217.
21. Agranovich ZS, Marchenko VA. *The Inverse Problem of Scattering Theory* Gordon and Breach; 1963.
22. Aktosun T, Klaus M, Weder R. Small-energy analysis for the self-adjoint matrix Schrödinger operator on the half line. II. *J Math Phys.* 2014;55:032103.
23. Aktosun T, Weder R. High-energy analysis and Levinson’s theorem for the self-adjoint matrix Schrödinger operator on the half line. *J Math Phys.* 2013;54:012108.
24. Xu XC, Yang CF. Determination of the self-adjoint matrix Schrödinger operators without the bound state data. *Inv. Prob.* 2018;34:065002 (20pp).
25. Aktosun T, Weder R. Inverse scattering on the half line for the matrix Schrödinger equation. *J Math Phy Ana Geo.* 2018;14:273-269.
26. Aktosun T, Weder R. *Direct and Inverse Scattering for the Matrix Schrödinger Equation.* Springer; 2021.
27. Bondarenko N. An inverse spectral problem for the matrix Sturm-Liouville operator on the half-line. *Boundary Value Prob.* 2015;15;(22pp).
28. Freiling G, Yurko VA. An inverse problem for the non-selfadjoint matrix Sturm-Liouville equation on the half-line. *J Inverse Ill-Posed Probl.* 2007;15:785-798.
29. Bondarenko N. Direct and inverse problems for the matrix Sturm-Liouville operator with general self-adjoint boundary conditions. *Math Notes.* 2021;109:358-378.
30. Bondarenko N. Recovery of the matrix quadratic differential pencil from the spectral data. *J Inv Ill-Posed Probl.* 2016;24:245-263.
31. Carlson R. An inverse problem for the matrix Schrödinger equation. *J Math Anal Appl.* 2002;267:564-575.
32. Xu XC. Inverse spectral problem for the matrix Sturm-Liouville operator with the general separated self-adjoint boundary conditions. *Tamkang J Math.* 2019;50:321-336.
33. Yang CF. Trace formula for the matrix Sturm-Liouville operator. *Anal Math Phys.* 2016;6:31-41.
34. Yurko VA. Inverse problems for the matrix Sturm-Liouville equation on a finite interval. *Inverse Probl.* 2006;22:1139-1149.
35. Aktosun T, Klaus M, van der Mee C. Small-energy asymptotics of the scattering matrix for the matrix Schrödinger equation on the line. *J Math Phys.* 2001;42:4627.
36. Bondarenko N. Inverse scattering on the line for the matrix Sturm-Liouville equation. *J Differ Equ.* 2017;262:2073-2105.
37. Gesztesy F, Kiselev A, Makarov KA. Uniqueness results for matrix-valued Schrödinger, Jacobi, and Dirac-type operators. *Math Nachr.* 2002;239/240:103-145.
38. Wadati M, Kamijo T. On the extension of inverse scattering method. *Progr Theoret Phys.* 1974;52:397-414.
APPENDIX A

In the Appendix, we provide the proof of Lemma 1. It is sufficient to show that

\[ I = (I + K_b)(I + F_b)(I + K_b^*), \]  

(A.1)

or equivalently,

\[ 0 = K_b + K_b^* + F_b + K_b F_b + F_b K_b^* + K_b K_b^* + K_b F_b K_b^*. \]  

(A.2)

Here 0 denotes the zero operator. The right-hand side of (A.2) is the integral operator with kernel

\[ G(x, y) = K(x, y) + K(y, x)^\dagger + F(x + y) + \int_b^\infty K(x, t)F(t + y)dt \]
\[ + \int_b^\infty F(x + t)K(y, t)^\dagger dt + \int_b^\infty K(x, t)K(y, t)^\dagger dt \]
\[ + \int_b^\infty \int_b^\infty K(x, t)F(s + t)K(y, s)^\dagger dt ds. \]  

(A.3)

Here, we have used Fubini’s theorem to interchange the order of integration. Denote

\[ D(x, y) := F(x + y) + K(x, y) + \int_x^\infty K(x, z)F(z + y)dz. \]  

(A.4)

Because \( K(y, x) = 0 \) for \( y > x \), it follows from (A.3), (A.4), and (25) that

\[ G(x, y) = D(x, y) + \int_y^\infty D(x, s)K(y, s)^\dagger ds, \quad y > x. \]  

(A.5)

Noting that \( D(x, y) = 0_n \) for \( y > x \) from (22), we get \( G(x, y) = 0 \) for \( y > x \). From (A.3) and \( F(x) = F(x)^\dagger \) we see that \( G(x, y) = G(y, x)^\dagger \). Therefore, the integral operator with kernel \( G(x, y) \) is the zero operator. Namely, the relation (A.1) holds. It follows that

\[ I + F_b = (I + K_b)^{-1}(I + K_b^*)^{-1}, \]  

(A.6)

which implies (26).