GENERALIZATIONS OF KAPLANSKY THEOREM FOR SOME $(p, k)$-QUASI-HYPONORMAL OPERATORS

Abdelkader Benali [1] and Ould Ahmed Mahmoud Sid Ahmed [2]

[1] Mathematics Department, Faculty of science, University of Hassiba Benbouali, Chlef Algeria. B.P. 151 Hay Essalem, chlef 02000, Algeria.
   benali4848@gmail.com

[2] Mathematics Department, College of Science. Aljouf University
   Aljouf 2014. Saudi Arabia
   sidahmed@ju.edu.sa

January 14, 2018

Abstract

In the present paper, we generalized some notions of bounded operators to unbounded operators on Hilbert space such as $k$-quasihyponormal and $k$-paranormal unbounded operators. Furthermore, we extend the Kaplansky theorem for normal operators to some $(p, k)$-quasihyponormal operators. Namely the $(p, k)$-quasihyponormality of the product $AB$ and $BA$ for two operators.

Keywords. Unbounded operator, normal operator, $(k, p)$-quasihyponormal operator, $k$-paranormal operator.

Mathematics Subject Classification (2010). Primary 47B15, Secondary 46L10.

1 INTRODUCTION

Through out the paper we denote Hilbert space over the field of complex numbers $\mathbb{C}$ by $\mathcal{H}$ and the usual inner product and the corresponding norm of $\mathcal{H}$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let us fix some more notations. We write $\mathcal{B}(\mathcal{H})$ for the set of all bounded linear operators in $\mathcal{H}$ whose domain are equal to $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, the range, the kernel and the adjoint of $A$ are denoted by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $A^*$ respectively. If $\mathcal{M}$ is a...
space of \( \mathcal{H} \), \( \overline{\mathcal{M}} \) and \( \mathcal{M}^\perp \) denote its closure and its orthogonal complement, respectively and \( A|\mathcal{M} \) denotes the restriction of \( A \) to \( \mathcal{M} \).

In this section we introduce basic notations and recall some well-know concepts of some classes of bounded and unbounded operators in a Hilbert space.

An operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be: normal if \( A^*A = AA^* \) (equivalently \( \| Ax \| = \| A^*x \| \) for all \( x \in \mathcal{H} \)), hyponormal if \( A^*A \geq AA^* \) (equivalently \( \| Ax \| \geq \| A^*x \| \) for all \( x \in \mathcal{H} \)), Co-hyponormal if \( AA^* \geq A^*A \) (equivalently \( \| A^*x \| \geq \| Ax \| \) for all \( x \in \mathcal{H} \)), quasinormal if \( AA^*A = A^*A^2 \) (equivalently \( (A^*A)^2 = A^*A^2 \)) and Quasi-hyponormal if \( A^2A^2 \geq (A^*A)^2 \) (equivalently \( \| A^2x \| \geq \| A^*Ax \| \) for all \( x \in \mathcal{H} \)).

An operator \( A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) is said to be densely defined if \( \mathcal{D}(A) \) (the domaine of \( A \)) is dense in \( \mathcal{H} \) and it is said to be closed if its graph is closed. The (Hilbert) adjoint of \( A \) is denoted by \( A^* \) and it is known to be unique if \( A \) is densely defined.

We denote by \( \text{Op}(\mathcal{H}) \) the set of unbounded densely defined linear operators on \( \mathcal{H} \).

Let \( A, B \in \text{Op}(\mathcal{H}) \), the product \( AB \) of two unbounded operators is defined by

\[
(AB)x = A(Bx) \quad \text{on} \quad \mathcal{D}(AB) = \{ x \in \mathcal{D}(B) : Bx \in \mathcal{D}(A) \}.
\]

Let \( A, B \in \text{Op}(\mathcal{H}) \), we recall that \( B \) is called an extension of \( A \), denoted by \( A \subseteq B \), if \( \mathcal{D}(A) \subset \mathcal{D}(B) \) and \( Ax = Bx \) for all \( x \in \mathcal{D}(A) \). An closed operator \( A \in \text{Op}(\mathcal{H}) \) is said to commute with \( B \in \mathcal{B}(\mathcal{H}) \) if \( BA \subseteq AB \), that is, if for \( x \in \mathcal{D}(A) \), we have \( Bx \in \mathcal{D}(A) \) and \( BAx = ABx \).

Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \text{Op}(\mathcal{H}) \) we say that \( A \) commutes with \( B \) if \( BA \subseteq AB \).

Recall also that if \( A, B \) and \( AB \) are all densely defined, then we have \( A^*B^* \subseteq (BA)^* \).

There are cases where equality holds in the previous inclusion, namely if \( B \) is bounded. For other notions and results about bounded and unbounded operators, the reader may consult [19].

A generalization of normal, quasinormal, hyponormal and paranormal operators to unbounded normal quasinormal, hyponormal and paranormal operators has been presented by several authors in the last years. Some important references are [7, 9, 10, 11, 20, 21].

An operator \( A \in \text{Op}(\mathcal{H}) \), we said to be hyponormal if \( \mathcal{D}(A) \subset \mathcal{D}(A^*) \) and \( \| A^*x \| \leq \| Ax \| \) for all \( x \in \mathcal{D}(A) \). We refer to [10] for basic facts concerning unbounded hyponormal operators.

An operator \( A \in \text{Op}(\mathcal{H}) \) is said to be normal if \( A^*A = AA^* \). A closed operator \( A \in \text{Op}(\mathcal{H}) \) is normal if and only if \( \mathcal{D}(A) = \mathcal{D}(A^*) \) and \( \| Ax \| = \| A^*x \| \) for all \( x \in \mathcal{D}(A) \).

A densely defined operator \( A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \) is said to be paranormal if

\[
\| Ax \|^2 \leq \| A^2x \| \| x \| \quad \text{for all} \quad x \in \mathcal{D}(A^2)
\]

or equivalently

\[
\| Ax \|^2 \leq \| A^2x \| \quad \text{for every unit vector} \quad x \in \mathcal{D}(A^2). \quad \text{(See [7]).}
\]
Let $A$ and $B$ be normal operators on a complex separable Hilbert space $\mathcal{H}$. The equation $AX = XB$ implies $A^*X = XB^*$ for some operator $X \in B(\mathcal{H})$ is known as the familiar Fuglede-Putnam theorem. (See [14]).

Consider two normal (resp. hyponormal) operators $A$ and $B$ on a Hilbert space. It is known that, in general, $AB$ is not normal (resp. not hyponormal). Kaplansky showed that it may be possible that $AB$ is normal while $BA$ is not. Indeed, he showed that if $A$ and $AB$ are normal, then $BA$ is normal if and only if $B$ commutes with $AA^*$, (see [12]).

In [18, Theorem 3], Patel and Ramanujan proved that if $A$ and $B \in B(\mathcal{H})$ are hyponormal such that $A$ commutes with $|B|$ and $B$ commutes with $|A^*|$ then $AB$ and $BA$ are hyponormal.

The study of operators satisfying Kaplansky theorem is of significant interest and is currently being done by a number of mathematicians around the world. Some developments toward this subject have been done in [6, 10, 12, 15, 16, 17] and the references therein.

The aim of this paper is to give sufficient conditions on two some $(p, k)$-quasihyponormal operators (bounded or not), defined on a Hilbert space, which make their product $(p, k)$-quasihyponormal. The inspiration for our investigation comes from [1], [15] and [17].

The outline of the paper is as follows. First of all, we introduce notations and consider a few preliminary results which are useful to prove the main result. In the second section we discussed conditions which ensure hyponormality, $k$-quasihyponormality or $(p, k)$-quasihyponormality of the product of hyponormal, $k$-quasihyponormal or $(p, k)$-quasihyponormality of operators. In Section three, the concepts of $k$-quasihyponormal and $k$-paranormal unbounded operators are introduced. We give sufficient conditions which ensure $k$-quasihyponormality ($k$-parnormality or $k$-$*$-parnormality) of the product of $k$-quasihyponormal ($k$-parnotmal or $k-*$-paranormal) of unbounded operators.

### 2 KAPLANSKY LIKE THEOREM FOR BOUNDED $(p, k)$-QUASIHYPONORMAL OPERATORS

The next definitions and lemmas give a brief description for the background on which the paper will build on.

**Definition 2.1.** An operator $A \in B(\mathcal{H})$ is said to be

1. $p$-hyponormal if $(A^*A)^p - (AA^*)^p \geq 0$ for $0 < p \leq 1$ ([3]).

2. $p$-quasihyponormal if $A^*\left((A^*A)^p - (AA^*)^p\right)A \geq 0$, $0 < p \leq 1$ ([4]).

3. $k$-quasihyponormal operator if $A^k(A^*A - AA^*)A^k \geq 0$ for positive integer $k$ ([5]).

4. $(p,k)$-quasihyponormal if $A^{*k}\left((A^*A)^p - (AA^*)^p\right)A^k \geq 0$, $0 < p \leq 1$ and $k$ is positive integer ([13]).

A $(p, k)$-quasihyponormal is an extension of $p$-hyponormal, $p$-quasihyponormal and $k$-quasihyponormal.
Remark 2.1. Let $N, hN, p h, Q(p)$ and $Q(p, k)$ denote the classes consisting of normal, hyponormal, $p$-hyponormal, $p$-quasihyponormal and $(p, k)$-quasihyponormal operators. These classes are related by the proper inclusion. (See [14]).

$$N \subset hN \subset p h \subset Q(p) \subset Q(p, k).$$

We need the following lemma which is important for the sequel.

Lemma 2.1. Let $A, B \in B(H)$. Then the following properties hold

1. If $A \geq B$ then $C^* A C \geq C^* B C$, for all $C \in B(H)$.

2. If range of $C$ is dense in $H$ then $A \geq B \iff C^* A C \geq C^* B C$.

Proof. This proof will be left to the reader.

The following famous inequality is needful.

Lemma 2.2. (Hansen’s inequality)

Let $A, B \in B(H)$ such that $A \geq 0$ and $\| B \| \leq 1$ then

$$\left( B^* A B \right)^\alpha \geq B^* A^\alpha B \quad 0 < \alpha \leq 1.$$ 

Kaplansky showed that it may be possible that $AB$ is normal while $BA$ is not. Indeed, he showed that if $A$ and $AB$ are normal, then $BA$ is normal if and only if $B$ commutes with $|A|$.

Kaplansky theorem’s has been extended form normal operators to hyponormal operators and unbounded hyponormal operators by the authors in [1]. We collect some of their results in the following theorem.

Theorem 2.1. (1) Let $A, B \in B(H)$. The following statements hold:

1. If $A$ is normal and $AB$ is hyponormal then

$$B A A^* = A A^* B \implies BA \text{ is hyponormal.}$$

2. If $A$ is normal and $AB$ is co-hyponormal then

$$B A A^* = A A^* B \implies BA \text{ is co-hyponormal.}$$

3. If $A$ is normal, $AB$ is hyponormal and $BA$ is co-hyponormal then

$$B A A^* = A A^* B \iff AB \text{ and } BA \text{ are normal.}$$

We give another proof of Kaplansky theorem.
Theorem 2.2. [Kaplansky, [12]] Let $A$ and $B \in \mathcal{B}(\mathcal{H})$ be two bounded operators such that $AB$ and $A$ are normal. Then

$$A^*AB = BA^*A \iff (BA) \text{ is normal.}$$

Proof. "$\implies$"

Assume that $A^*AB = BA^*A$ and we need to prove that $BA$ is normal.

It is well known that $A$ is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in \mathcal{H}$.

Since $AB$ is normal we have

$$\|(AB)Ax\| = \|(AB)^*Ax\| \text{ for all } x \in \mathcal{H}$$

and we deduce that

$$\|A(BA)x\| = \|A(BA)^*x\| \text{ for all } x \in \mathcal{H}.$$  

By hypotheses given in the theorem, we have

$$\|A(BA)x\| = \|A(BA)^*x\|$$

$$\iff \langle A(BA)x, A(BA)x \rangle = \langle A(BA)^*x, A(BA)x \rangle$$

$$\iff \langle [A(BA)]^* A(BA)x, x \rangle = \langle [A(BA^*)]^* A(BA)x, x \rangle$$

$$\iff \langle (BA)^* (BA)x, x \rangle = \langle A(A^*BA)x, x \rangle$$

$$\iff \langle (BA)^*(BA), A^*Ax \rangle = \langle (BA)(BA)^*, A^*Ax \rangle$$

$$\iff \langle (BA)^*(BA) - (BA)(BA)^*, A^*Ax \rangle = 0 \text{ for all } x \in \mathcal{H}.$$  

Put $T = \left( (BA)(BA)^* - (BA)^*(BA) \right)$.

Form the identities above we have $\langle Tx, A^*Ax \rangle = 0$ for all $x \in \mathcal{H}$. This implies that

$$Tx \in \mathcal{R}(A^*A)^\perp = \overline{\mathcal{R}(A^*A)}^\perp = \mathcal{N}(A^*) = \mathcal{N}(A) \text{ for all } x \in \mathcal{H}.$$  

Now if $\mathcal{N}(A) = \{0\}$, we have $Tx = 0$ for all $x \in \mathcal{H}$ and $T \equiv 0$ i.e., $BA$ is normal.

If $\mathcal{N}(A) \neq \{0\}$. Suppose that, contrary to our claim, the operator $T \neq 0$. There exists $x_0 \in \mathcal{H}$, $x_0 \neq 0$ such that $Tx_0 \neq 0$. Since $Tx_0 \in \mathcal{N}(A)$ and $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(A)^\perp$, it follows that $Tx_0 \notin \mathcal{N}(A)^\perp$. From this we deduce that there exists $z_0 \in \mathcal{N}(A)$ so that $\langle Tx_0, z_0 \rangle \neq 0$.

As usual, this leads to the statement that

$$0 \neq \langle Tx_0, z_0 \rangle = \langle x_0, Tz_0 \rangle \implies x_0 \notin \mathcal{N}(A)^\perp \text{ (since } Tz_0 \in \mathcal{N}(A)).$$
This means that, \( x_0 \in \mathcal{N}(A) \) and
\[
Tx_0 = ((BA)^*(BA) - (BA)(BA)^*)x_0 = -(BA)(BA)^*x_0 = 0.
\]
This contradicts the assumption that \( Tx_0 \neq 0 \).

(2) "\( \iff \)" The reverse application is even evidence of Kaplansky

We have \( ABA = ABA \Rightarrow (AB)A = A(BA) \), and by the theorem of Fuglede-Putnam
\[
(AB)^*A = A(BA)^* \Rightarrow ((AB)^*A = (A(BA))^*).
\]
So
\[
A^*(AB) = (BA)A^* \Rightarrow A^*AB = BAA^*.
\]

Consider two quasihyponormal operators \( A \) and \( B \) on a Hilbert space. It is known that, in general, \( AB \) is not quasihyponormal.

**Example 2.1.** Let \( \mathcal{H} = l^2(\mathbb{N}) \) with the canonical orthonormal basis \( (e_n)_{n\in\mathbb{N}} \) and consider \( A \) the unilateral right shift operator on \( \mathcal{H} \) defined by \( Ae_n = e_{n+1} \) for all \( n \in \mathbb{N} \) and \( B \) the operator defined on \( \mathcal{H} \) by
\[
B_{en} = \begin{cases} 
  e_n, & \text{if } n \neq 1 \\
  0 & \text{if } n = 1.
\end{cases}
\]

A simple calculation shows that \( A \) and \( B \) are quasihyponormal and \( AB \) does not quasi-hyponormal, since
\[
\|AB^*(AB)e_0\| = 1 \text{ and } \|(AB)^2e_0\| = 0.
\]

Denote by \( \mathbb{C}^{mn} \) the set of all \( m \times n \) complex matrix.

In [8], the authors proved the following results.

**Theorem 2.3.** ([8]) Let \( A = UH \), where \( H \in \mathbb{C}^{nn} \) is positive semidefinite Hermitian and \( U \in \mathbb{C}^{nn} \) is unitary, and let \( B \in \mathbb{C}^{nm} \),

1. if \( BU \) is normal and \( HBU = BUH \), then \( AB \) and \( BA \) are normal,
2. if \( AB \) and \( BA \) are normal, then \( HBU = BUH \).

**Theorem 2.4.** ([8]) Let \( A \in \mathbb{C}^{mn} \) and \( B \in \mathbb{C}^{nm} \). Then \( AB \) and \( BA \) are normal if and only if \( A^*AB = BA^*A \) and \( ABB^* = BB^*A \).

We show here the main results of this paper. Our intention is to study some conditions for which the product of operators will be hyponormal and \( k \)-quasihyponormal or \( (k, p) \)-quasihyponormal.
Proposition 2.1. Let $A = U|A| \in \mathcal{B}(\mathcal{H})$ with $U$ is unitary and let $B \in \mathcal{B}(\mathcal{H})$ such that $|A|BU = BU|A|$. The following properties hold

1. If $UB$ is hyponormal, then $AB$ is hyponormal.
2. If $BU$ is hyponormal, then $BA$ is hyponormal.
3. If $UB$ is quasihyponormal, then $AB$ is quasihyponormal.
4. If $BU$ is quasihyponormal, then $BA$ is quasihyponormal.

Proof. (1) Suppose that $UB$ is hyponormal. Then

\[
\|(AB)^*x\| = \|B^*|A|U^*x\| = \|B^*U^*|A|U^*x\| = \|(UB)^*|A|U^*x\| \\
\leq \|UB|A|U^*x\| = \|U|A|BUU^*x\| = \|ABx\|.
\]

This shows that $AB$ is hyponormal.

(2) Suppose that $BU$ is hyponormal. Then

\[
\|(BA)^*x\| = \||A|U^*B^*x\| = \||A|(BU)^*x\| = \|(BU)^*|A|x\| \\
\leq \|BU|A|x\| = \|BAx\|.
\]

This shows that $BA$ is hyponormal.

(3) Assume that $UB$ is quasihyponormal, then

\[
\|(AB)^*(AB)x\| = \|B^*|A|^2Bx\| \\
\leq \|B^*|A|^2BUU^*x\| = \|B^*BU|A|^2U^*x\| = \|B^*U^*UBU|A|^2U^*x\| = \|(UB)^*(UB)|A|^2U^*x\| \\
\leq \|(UB)^2|A|^2U^*x\| = \|UBUBU|A|^2U^*x\| = \|UBU|A|^2Bx\| \\
\leq \|U|A|BU|A|Bx\| = \|(AB)^2x\|.
\]

This shows that $AB$ is quasihyponormal.
(4) Assume that $BU$ is quasihyponormal. Then

$$
\|(BA)^*(BA)x\| = \|A^*B^*BAx\| \\
\leq \|A(BU)^*BUA|x\| \\
= \|(BU)^*(BU)|A^2x\| \\
\leq (BU)^2|A|^2|x| \quad \text{(since $BU$ is quasihyponormal)} \\
\leq BU|A|BU|A|x| \\
= \|(BA)^2x\|.
$$

This shows that $BA$ is quasihyponormal.

Proposition 2.2. Let $A$ and $B \in B(\mathcal{H})$ are hyponormal operators. If $BA^* = A^*B$, then $AB$ and $BA$ are $k$-quasihyponormal.

**Proof.** Let $x \in \mathcal{H},$

$$
\|(AB)^*(AB)^kx\| = \|B^*A^*(AB)^kx\| \\
\leq \|BA^*(AB)^kx\| \quad \text{(since $B$ is hyponormal)} \\
\leq \|A^*B(AB)^kx\| \\
\leq \|AB(AB)^kx\| \quad \text{(since $A$ is hyponormal)} \\
\leq \|(AB)^{k+1}x\|
$$

we even have evidence to $k$- quasi-hyponormal.

Proposition 2.3. Let $A$ and $B \in B(\mathcal{H})$ such that $A$ and $B$ are doubly commutative $k$-quasi-hyponormal operators then $AB$ is $k$- quasi-hyponormal.

**Proof.** Let $x \in \mathcal{H},$

$$
\|(AB)^*(AB)^kx\| = \|A^*A^kB^kB^*x\| \leq \|A^{k+1}B^*B^kx\| \leq \|A^{k+1}B^{k+1}x\| = \|(AB)^{k+1}x\|
$$

Proposition 2.4. Let $A$ and $B \in B(\mathcal{H})$ are $k$-quasihyponormal operators for some positive integer $k$. The following statements hold

1. If $A^*A^kB = BA^*A^k$ and $A^jB^j = (AB)^j$ for $j \in \{k, k+1\}$, then $AB$ is $k$- quasihyponormal.

2. If $B^kB^jA = AB^kB^j$ and $B^jA^j = (BA)^j$ for $j \in \{k, k+1\}$, then $BA$ is $k$- quasihyponormal.
Proof. (1)

\[
\|(AB)^*(AB)^kx\| = \|B^*A^kB^kx\| \\
= \|B^*B^kA^kx\| \\
\leq \|B^{k+1}A^kx\| \quad \text{(since } B \text{ is } k-\text{quasihyponormal)} \\
\leq \|A^kA^{k+1}Bx\| \\
\leq \|A^{k+1}B^{k+1}x\| \quad \text{(since } A \text{ is } k-\text{quasihyponormal)} \\
\leq \|(AB)^{k+1}x\| \quad \text{for all } x \in \mathcal{H}.
\]

(2)

\[
\|(BA)^*(BA)^kx\| = \|A^*B^kA^{k+1}x\| \\
= \|A^*A^kB^kx\| \\
\leq \|A^{k+1}B^kA^{k+1}x\| \quad \text{(since } A \text{ is } k-\text{quasihyponormal)} \\
\leq \|B^kA^{k+1}B^{k+1}x\| \\
\leq \|B^{k+1}A^{k+1}x\| \quad \text{(since } B \text{ is } k-\text{quasihyponormal)} \\
= \|(BA)^{k+1}x\| \quad \text{for all } x \in \mathcal{H}.
\]

\[\square\]

**Proposition 2.5.** Let \(A\) and \(B\) \(\in \mathcal{B}(\mathcal{H})\) such that \(A\) is normal and \(AB\) is quasinormal, then

\(A^*AB = BA^*A \implies BA\) is quasinormal.

*Proof.* Let \(A = U|A|\) with \(U\) unitary. Since \(A^*AB = BA^*A\) we have \(|A|B = B|A|\). These facts and the quasinormality of \(AB\) give

\[
(BA)(BA)^*(BA) = U^*(AB)U(U^*(AB)U)^*(U^*(AB)U) \\
= U^*(AB)(AB)^*(AB)U \\
= U^*(AB)^*(AB)^2U \\
= (U^*(AB)U)^*(U^*(AB)U)^2 \\
= (BA)^*(BA)^2.
\]

\[\square\]

**Remark 2.2.** The reverse implication does not hold in the previous result (even if \(A\) is self-adjoint) as shown in the following example.

**Example 2.2.** Let \(A\) and \(B\) be acting on the standard basis \((e_n)\) of \(\ell^2(N)\) by:

\[Ae_n = \alpha_n e_n \quad \text{and} \quad Be_n = e_{n+1}, \quad \forall n \geq 1\]

respectively. Assume further that \(\alpha_n\) is bounded, real-valued and positive, for all \(n\). Hence \(A\) is self-adjoint (hence normal!) and positive. Then

\[ABe_n = \alpha_{n+1} e_{n+1}, \quad \forall n \geq 1.\]
For convenience, let us carry out the calculations as infinite matrices. Then

\[
AB = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 \\
0 & \alpha_3 & 0 & \ddots \\
0 & 0 & \ddots & \ddots \\
\end{pmatrix}
\]

so that \((AB)^*\) =

\[
\begin{pmatrix}
0 & \alpha_1 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 \\
0 & 0 & 0 & \alpha_3 \\
0 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

Hence

\[
(AB)^2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \alpha_1 \alpha_2 & 0 \\
0 & \alpha_2 \alpha_3 & 0 \\
0 & 0 & \ddots \\
\end{pmatrix}
\]

and \([(AB)^*]^2\) =

\[
\begin{pmatrix}
0 & 0 & \alpha_1 \alpha_2 & 0 \\
0 & 0 & 0 & \alpha_2 \alpha_3 \\
0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots \\
\end{pmatrix}
\]

so

\[
(AB)^* AB = \begin{pmatrix}
\alpha_1^2 & 0 & 0 & 0 \\
0 & \alpha_2^2 & 0 & 0 \\
0 & 0 & \alpha_3^2 & 0 \\
0 & 0 & 0 & \ddots \\
\end{pmatrix}
\]

this implies

\[
[(AB)^* AB]^2 = \begin{pmatrix}
\alpha_1^4 & 0 & 0 & 0 \\
0 & \alpha_2^4 & 0 & 0 \\
0 & 0 & \alpha_3^4 & 0 \\
0 & 0 & 0 & \ddots \\
\end{pmatrix}
\]

and

\[
[(AB)^*]^2 [AB]^2 = \begin{pmatrix}
(\alpha_1 \alpha_2)^2 & 0 & 0 & 0 \\
0 & (\alpha_2 \alpha_3)^2 & 0 & 0 \\
0 & 0 & (\alpha_3 \alpha_4)^2 & 0 \\
0 & 0 & 0 & \ddots \\
\end{pmatrix}
\]
It thus becomes clear that $AB$ is quasi hyponormal iff $\alpha_n \leq \alpha_{n+1}$.

Similarly

$$BAe_n = \alpha_ne_{n+1}, \ \forall n \geq 1.$$ 

Whence the matrix representing $BA$ is given by:

$$BA = \begin{pmatrix} 0 & 0 & 0 \\
\alpha_2 & 0 & 0 \\
0 & \alpha_3 & 0 \\
0 & \alpha_4 & \ddots \\
0 & \ddots & 0 \\
\end{pmatrix}$$

so that $(BA)^* = \begin{pmatrix} 0 & 0 & 0 \\
0 & \alpha_2 & \alpha_3 \\
0 & \alpha_3 & \alpha_4 \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
\end{pmatrix}.$

Therefore,

$$(BA)^2 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \alpha_2 \alpha_3 \\
0 & \alpha_3 \alpha_4 & \ddots \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
\end{pmatrix}, \quad [(BA)^*]^2 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \alpha_2 \alpha_3 \\
0 & \alpha_3 \alpha_4 & \ddots \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
\end{pmatrix}.$$ 

and

$$(BA)^*BA = \begin{pmatrix} \alpha_2^2 & 0 \\
0 & \alpha_2^2 \\
0 & \alpha_3^2 \\
0 & \alpha_4^2 & \ddots \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
\end{pmatrix}.$$ 

this implies

$$(BA)^*BA = \begin{pmatrix} \alpha_2^4 & 0 \\
0 & \alpha_2^4 \\
0 & \alpha_3^4 \\
0 & \alpha_4^4 & \ddots \\
0 & \ddots & 0 \\
0 & \ddots & \ddots \\
0 & \ddots & \ddots \\
\end{pmatrix}.$$
\[
[(BA)^*]^2 [BA] = \begin{pmatrix}
((\alpha_2\alpha_3)^2 & 0 & 0 \\
0 & (\alpha_3\alpha_4)^2 & 0 \\
0 & 0 & ((\alpha_4\alpha_5))^2 \\
0 & 0 & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Accordingly, \( BA \) is quasi hyponormal if and only if \( \alpha_n \leq \alpha_{n+1} \) (thankfully, this is the same condition for the hyponormality of \( AB \)).

Finally,

\[
BA^2 = \begin{pmatrix}
0 & 0 & \alpha_1^2 & 0 & \cdots & 0 \\
\alpha_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \alpha_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & \alpha_3 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & \ddots & \ddots
\end{pmatrix}
\neq A^2B = \begin{pmatrix}
0 & 0 & \alpha_1 & 0 & \cdots & 0 \\
\alpha_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \alpha_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & \alpha_3 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & \ddots & \ddots
\end{pmatrix}
\]

**Proposition 2.6.** Let \( A = U|A| \) and \( B \in \mathcal{B}(\mathcal{H}) \) such that \( A \) and \( B \) are quasinormal. If \( BU \) is quasinormal and \( |A|BU = BU|A| \), then \( BA \) is quasinormal.

**Proof.** Let \( A = U|A| \) be the polar decomposition of \( A \) with \( U \) partial isometry. Since \( A \) is quasinormal we have \( |A|U = U|A| \). These facts and the quasinormality of \( AB \) give

\[
(BA)(BA)^*(BA) = (B|A|U)(B|A|U)^*(B|A|U)
= BU|A|^2(BU)^*BU|A|
= |A|^2(BU)(BU)^*(BU)|A|
= |A|^2(BU)^*(BU)^2|A|
= (BU|A|)^*(BU|A|)^2
= (BA)^*(BA)^2.
\]

Thus \( BA \) is quasinormal. \( \Box \)

**Proposition 2.7.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( A \) is normal and \( AB \) is paranormal. Then

\[
A^*AB = BA^*A \implies BA \text{ is paranormal.}
\]

**Proof.** Let \( A = U|A| \) with \( U \) is unitary. Since \( A \) is normal we have \( |A|U = U|A| \) and hence \( B|A| = |A|B \). This then gives

\[
BA = U^*ABU.
\]
From this fact we obtain that for all unit vector $x \in \mathcal{H}$,

$$
\|(BA)x\|^2 = \|U^*(AB)Ux\|^2 \\
\leq \|(AB)Ux\|^2 \\
\leq \|(AB)^2Ux\| \quad (\text{since } AB \text{ is paranormal and } \|Ux\| = 1) \\
= \|U(U^*ABU)^2x\| \\
\leq \|(BA)^2x\|.
$$

Hence $BA$ is paranormal operator.

Let $A$ and $B \in B(\mathcal{H})$. The commutator of $A$ and $B$ is defined as $[A, B] = AB - BA$ and the self-commutator of $A$ is $[A^*, A]$.

The span of $A$ and $B$ is

$$
\text{span}\{A, B\} := \{ aA + bB, \quad a, b \in \mathbb{C} \}.
$$

**Proposition 2.8.** Let $A$ and $B \in B(\mathcal{H})$ such that $A$ is quasihyponormal and $B$ is hyponormal. If

$$
[B^*, A] = A^*[B^*, B]B = B^*[A^*, A]B = 0.
$$

Then $T = \omega A + B$ is quasihyponormal for all $\omega \in \mathbb{C}$.

**Proof.** Note that if $\omega \in \mathbb{C}$ and $T = \omega A + B$, then

$$
[T^*, T] = |\omega|^2[A^*, A] + [B^*, B] + 2\text{Re}(\omega[B^*, A]).
$$

By hypotheses given in the theorem, we have

$$
T^*[T^*, T]T = |\omega|^4 A^*[A^*, A]A + |\omega|^2 A^*[B^*, B]A + B^*[B^*, B]B.
$$

By Lemma 2.1 we get $T^*[T^*, T]T \geq 0$. This completes the proof.

**Proposition 2.9.** Let $A$ and $B \in B(\mathcal{H})$ such that $A$ and $B$ are quasihyponormal. If

$$
[B^*, A] = A^*[B^*, B]B = B^*[A^*, A]B = A^*[B^*, B]A = 0.
$$

Then $T = \omega A + B$ is quasihyponormal for all $\omega \in \mathbb{C}$.

**Proof.** By the same arguments as in the proof of the proposition above, we have

$$
T^*[T^*, T]T = |\omega|^4 A^*[A^*, A]A + B^*[B^*, B]B.
$$

The next result is a necessary and sufficient condition for $\text{span}\{A, B\}$ to be quasihyponormal.
Theorem 2.5. Let \( A \) and \( B \in \mathcal{B}(H) \) such that
\[
[B^*, A] = A^*[B^*, B]B = B^*[A^*, A]B = A^*[B^*, B]A = 0.
\]
Then \( A \) and \( B \) are quasihyponormal if and only if \( \text{span}\{A, B\} \) is quasihyponormal.

Proof. First assume that \( A \) and \( B \) are quasihyponormal. It is immediate from the preceding proposition that \( \text{span}\{A, B\} \) is quasihyponormal. The converse is immediate from the definition of \( \text{span}\{A, B\} \).

\[\square\]

Theorem 2.6. Let \( A \) and \( B \in \mathcal{B}(H) \) such that \( A \) is normal and \( AB \) is \((p, k)\)-quasihyponormal. If \( B(AA^*) = (AA^*)B \) then \( BA \) is \((p, k)\)-quasihyponormal for \( 0 < p \leq 1 \) and \( k \in \mathbb{Z}_+ \).

Proof. Since \( A \) is normal, we have \( A = PU = UP \) with \( P \geq 0 \) and \( U \) unitary.

\[B(AA^*) = (AA^*)P^2B = BP^2 \implies PB = BP.\]

A simple computation shows that
\[U^*ABU = BA.\]

since \((AB)^*(AB)\) and \((AB)(AB)^*\) are positive and \(||U|| \leq 1\), by using Lemma 2.2, it follows that
\[
(BA)^k \left( (BA)^* (BA) \right)^p (BA)^k = (U^*(AB)^* U)^k \left( U^*(AB)^*(AB)U \right)^p (U^*(AB)U)^k
\]
\[
= U^*(AB)^k U \left( U^*(AB)^*(AB)U \right)^p U^*(AB)^k U
\]
\[
\geq U^*(AB)^k U (AB)^k U
\]
\[
\geq U^*(AB)^k U (AB)^k U \quad \text{(since } AB \text{ is in } Q(p, k))
\]
\[
\geq U^*(AB)^k U U^*(AB)UU^*(AB)^*U^* \geq 0
\]
\[
\geq U^*(AB)^k U \left( U^*(AB)UU^*(AB)^*U \right)^p U^*(AB)^k U \quad \text{(by Lemma 2.2)}
\]
\[
\geq (U^*(AB)^* U)^k \left( U^*(AB)UU^*(AB)^*U \right)^p (U^*(AB)U)^k
\]
\[
= (BA)^k \left( (BA)(BA)^* \right)^p (BA)^k.
\]

This implies that \( BA \) is \((p, k)\)-quasihyponormal operator. The proof of this theorem is finished.
Theorem 2.7. Let $A$ and $B \in \mathcal{B}(\mathcal{H})$ such that $A$ is $(p,k)$-quasihyponormal and $B$ is invertible. If $A$ commute with $B$ and $B^*$ then $AB$ is $(p,k)$-quasihyponormal for $0 < p \leq 1$ and $k \geq 1$.

Proof.

\[
(AB)^k (AB)^k (AB)^k = (AB)^k (B^*A^*AB)^p (AB)^k \\
\geq (AB)^k B^*(AA) (AB)^k \\
\geq (B^*)^{k+1} A^k (A^*)^p A^k B^{k+1} \\
\geq (B^*)^{k+1} A^k (AA)^p A^k B^{k+1} \\
\geq (AB)^k (AB)(AB)^* (AB)^k
\]

\[ \square \]

3. KAPLANSKY LIKE THEOREM FOR UNBOUNDED $k$-QUASI-HYPONORMAL OPERATORS

In this section, we generalized some notions of bounded operators to unbounded operators on a Hilbert space and we give sufficient conditions which ensure $k$-quasihyponormality ($k$-parnormality or $k$-*$-$paranormality) of the product of $k$-quasihyponormal ($k$-paranormal or $k$-*$-$paranormal) of unbounded operators.

For an $A \in Op(\mathcal{H})$, define $A^2$ by

\[ D(A^2) = \{ x \in D(A) / Ax \in D(A) \}, A^2 x = A(Ax) \]

We can define higher powers recursively. Given $A^n$, define

\[ D(A^{n+1}) = \{ x \in D(A) / Ax \in D(A^n) \}, A^{n+1} x = A^n(Ax) \]

Let us begin with the concept of $k$-quasihyponormality.

Definition 3.1. A densely defined operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be $k$-quasihyponormal for some positive integer $k$ if $D(A) \subset D(A^*)$ and

\[ \| A^* A^k x \| \leq \| A^{k+1} x \| \text{ for all } x \in D(A^{k+1}). \]

Remark 3.1. (1) Clearly, the class of all $k$-quasihyponormal operators on $\mathcal{H}$ contains all hyponormal operators.

(2) the class of $k$-quasihyponormal operators properly contains the classes of $k'$-quasihyponormal ($k' < k$).
Definition 3.2. A densely defined operator $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be
(1) $k$-paranormal for some positive integer $k$ if
$$\|Ax\|^k \leq \|A^kx\| \|x\|^{k-1} \text{ for all } x \in \mathcal{D}(A^k),$$
or equivalently
$$\|Ax\|^k \leq \|A^kx\| \text{ for every unit vector } x \in \mathcal{D}(A^k).$$
(2) $k - \ast$-paranormal for some positive integer $k$ if $\mathcal{D}(A) \subset \mathcal{D}(A^\ast)$ and
$$\|A^\ast x\|^k \leq \|A^kx\| \|x\|^{k-1} \text{ for all } x \in \mathcal{D}(A^k),$$
or equivalently $\mathcal{D}(A) \subset \mathcal{D}(A^\ast)$ and
$$\|A^\ast x\|^k \leq \|A^kx\| \text{ for every unit vector } x \in \mathcal{D}(A^k).$$

Example 3.1. Consider the Hilbert space $\mathcal{H} = l^2(\mathbb{Z})$ under the inner product $\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x_n\overline{y_n}$, and let $(e_n)_{n \in \mathbb{Z}}$ be any orthonormal basis for $\mathcal{H}$. Let $(\omega_n)_{n \in \mathbb{Z}}$ be an increasing sequence of numbers such that $\omega_n > 0$ for all $n \in \mathbb{Z}$ and $\sup_n (\omega_n) = \infty$. Consider the unilateral forward weighted shift operator $A$ defined in term of the standard basis of $l^2(\mathbb{Z})$ by
$$Ae_n = \omega_n e_{n+1} \text{ for all } n \in \mathbb{Z}.$$ 
A simple calculation shows that the adjoint of unilateral forward weighted shift is given by
$$A^\ast e_n = \omega_n e_{n-1} \text{ for all } n \in \mathbb{Z}.$$ 
By this we have
$$A^\ast Ae_n = \omega_n^2 e_n \text{ and } A^2 e_n = \omega_n \omega_{n+1} e_{n+2}.$$ 
Consequently
$$\|A^\ast Ae_n\| \leq \|A^2 e_n\| \text{ for all } n \in \mathbb{Z}.$$ 
Which implies that the operator $A$ is quasihyponormal.

Lemma 3.1. If $A \in Op(\mathcal{H})$ be a $k$-quasihyponormal, then
$$\|A^kx\|^2 \leq \|A^{k+1}x\| \|A^{k-1}x\| \text{ for } x \in \mathcal{D}(A^{k+1}).$$
Proof. In fact
$$\|A^kx\|^2 = \langle A^kx, A^kx \rangle = \langle A^\ast A^kx, A^{k-1}x \rangle \leq \|A^\ast A^kx\| \|A^{k-1}x\| \leq \|A^{k+1}x\| \|A^{k-1}x\|.$$
Proposition 3.1. Let $A$ be a closed densely defined operator in $\mathcal{H}$. If $A$ is $k$-quasihyponormal then
$$A^k \left( A^2 - a A^* A + a^2 I \right) A^k \geq 0 \quad \text{for all } a \in \mathbb{R}.$$ 

Proof. Let us suppose that $A$ is $k$-quasihyponormal. Then it follows that the following relation holds:
$$\| A^* A^k x \|^2 \leq \| A^{k+1} x \|^2 \leq \| A^{k+2} x \| \| A^k x \|.$$ 

This means that
$$\| A^* A^k x \|^2 \leq \| A^{k+2} x \| \| A^k x \| \iff 4 \| A^* A^k x \|^2 - 4 \| A^{k+2} x \| \| A^k x \| \leq 0 \iff \| A^{k+2} x \|^2 - 2a \| A^{k+2} x \|^2 + a^2 \| A^k x \|^2 \geq 0 \iff A^k \left( A^2 - a A^* A + a^2 I \right) A^k \geq 0.$$ 

This completes the proof of the proposition. 

Proposition 3.2. Let $A \in Op(\mathcal{H})$ be a $k$-quasihyponormal. if $A$ is invertible then $A$ is hyponormal.

Proof. As $A$ is $k$-quasihyponormal, we have by definition that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and
$$\| A^* A^k x \| \leq \| A^{k+1} x \| \quad \text{for all } x \in \mathcal{D}(A^{k+1}).$$ 

Since $A$ is invertible with an everywhere defined bounded inverse, we have for all $x \in \mathcal{D}(A) : A^{-k} x \in \mathcal{D}(A^{k+1})$
$$\| A^* A^k A^{-k} x \| \leq \| A^{k+1} A^{-k} x \| \quad \text{for all } x \in \mathcal{D}(A^{k+1}).$$ 

Hence we may write
$$\| A^* x \| \leq \| A x \|.$$

Proposition 3.3. Let $A \in B(\mathcal{H})$ and $B \in Op(\mathcal{H})$ such that $A$ and $B$ are hyponormal. If $BA^* \subseteq A^* B$ then $AB$ is $k$-quasihyponormal.

Proof. Let $x \in \mathcal{D}((AB)^{k+1}),$
$$\| (AB)^* (AB)^k x \| = \| B^* A^* (AB)^k x \| \leq \| BA^* (AB)^k x \| \quad \text{(since } B \text{ is hyponormal)} \leq \| A^* B (AB)^k x \| \leq \| AB (AB)^k x \| \quad \text{(since } A \text{ is hyponormal)} \leq \| (AB)^{k+1} x \|.$$ 

we even have evidence to quasi-hyponormal.
Proposition 3.4. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \text{Op}(\mathcal{H})$ such that $A$ is normal and $AB$ is $k$-quasihyponormal. If $B(AA^*) \subseteq (AA^*)B$ then $BA$ is $k$-quasihyponormal.

Proof. Since $A$ is normal, we have $A = PU = UP$ with $P \geq 0$ and $U$ unitary.

$$B(AA^*) = (AA^*B) \Rightarrow P^2B = BP^2 \Rightarrow PB = BP.$$ 

A simple computation shows that

$$U^*ABU = BA.$$

So

$$\| (BA)^* (BA)^k x \| = \| U^* (AB)^* UU^* (AB)^k Ux \| = \| U^* (AB)^* (AB)^k Ux \| \leq \| U^* (AB)^{k+1} Ux \| \leq \| (U^* ABU)^{k+1} x \| = \| (BA)^{k+1} x \|,$$

This implies that $(BA)$ is $k$-quasi hyponormal.

\[ \Box \]

Proposition 3.5. Let $A \in \mathcal{B}(\mathcal{H})$ and $B : D(B) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be closed densely defined operator such that $A$ and $B$ are $k$-quasihyponormal. If $A^* A^k B \subseteq BA^* A^k$ and $A^j B^j \subseteq (AB)^j$ for $j \in \{k, k+1\}$, then $AB$ is $k$-quasihyponormal.

Proof. Since $A \in \mathcal{B}(\mathcal{H})$ and $B$ is closed densely defined, it is well known that

$$AB)^* = B^* A^*.$$

Hence we may write

$$\|(AB)^*(AB)^k x\| = \|B^* A^k B^k x\| = \|B^* B^k A^k x\| \leq \|B^{k+1} A^k x\| \text{ (since } B \text{ is } k\text{-quasihyponormal)} \leq \|A^k A^k B^{k+1}\| \leq \|A^{k+1} B^{k+1} x\| \text{ (since } A \text{ is } k\text{-quasi hyponormal)} \leq \|(AB)^{k+1} x\|.$$ 

This completes the proof.

\[ \Box \]

Proposition 3.6. Let $A \in \text{Op}(\mathcal{H})$ is normal and $B \in \mathcal{B}(\mathcal{H})$.

1. If $AB$ is $k$-paranormal and $A^* AB \subseteq BA^* A$, then $BA$ is $k$-paranormal.

2. If $AB$ is $k-\ast$-paranormal and $A^* AB \subseteq BA^* A$, then $BA$ is $k-\ast$-paranormal.
Proof. (1) Using the normality of $A = U|A| = |A|U$ ($U$ unitary) and the fact that

$$A^*AB \subseteq BA^*A$$

we see that $BA = U^*ABU$.

Now we have

$$\|BAx\|^k = \|U^*ABUx\|^k \leq \|ABUx\|^k \leq \|(AB)^kUx\| \leq \|(U^*ABU)^kx\| = \|(BA)^kx\|.$$ 

(2) By similar argument.

References

[1] B. Abdelkader and H. Mortad Mohammed, Generalizations of Kaplansky’s Theorem Involving Unbounded Linear Operators. Bulletin of the Polish Academy of Sciences. Mathematics. Vol. 62, no 2, 181–186 (2014).

[2] A. Aluthge and D. Wang, Powers of p-Hyponormal Operators, J. of Inequal. And Appl., (1999), Vol. 3, pp. 279-284.

[3] A. Aluthge, On p-hyponormal operators for $0 < p < 1$; Integral Equations Operator Theory 13 (1990) 307–315.

[4] C. Arora, P. Arora, On p-quasihyponormal operators for $0 < p < 1$, Yokohama Math. J. 41 (1993) 25–29.

[5] S.L. Campbell, B.C. Gupta, On $k$-quasihyponormal operators, Math. Japonoca 23 (1978) 185–189.

[6] J. B. Conway and W. Szymanski, Linear combinations of Hyponormal operators. Rocky Mountain journal of Mathematics. Volume 18, Number 3, Summer 1988.

[7] A. Daniluk, On the closability of paranormal operators. J. Math. Anal. Appl. 376 (2011) 342-348.

[8] E. Deutsch, P. M. Gibson and H. Schneider, The Fuglede-Putnam Theorem and Normal Products of Matrices. Linear Algebra and its Applications Volume 13, Issues 12, 1976, Pages 53-58.

[9] Z. J. Jablonski, I. B. Jung, J. Stochel, Unbounded quasinormal operators revisited, Integr. Equ. Oper. Theory, 79 (2014), 135-149.

[10] J. Janas, On unbounded hyponormal operators. Ark. Math. 27(1989), 273–281.
[11] W. E. Kaufman, Closed operators and pure contractions in Hilbert space, Proc. Amer. Math. Soc. 87 (1983), 83-87.

[12] I. Kaplansky, Products of normal operators, Duke Math. J., 1953, 20/2, 257–260.

[13] I.H. Kim, On $(p; k)$-quasihyponormal operators. Mathematical inequalities and Applications. Vol. 7, Number 4 (2004), 629–638.

[14] M. Y. Lee, An extension of the Fuglede-Putnam theorem to $(p, k)$-quasihyponormal operators. Kyungpook Math. J. 44(2004), No. 4, 593-596.

[15] M. H. Mortad, On the Normality of the Sum of Two Normal Operators, Complex Anal. Oper. Theory, 6/1 (2012), 105–112. 399-408.

[16] M. H. Mortad, An application of the Putnam-Fuglede theorem to normal products of self-adjoint operators, Proc. Amer. Math. Soc. 131 (2003), 3135-3141.

[17] M. H. Mortad, On some product of two unbounded self-adjoint operators, Integral Equations Operator Theory 64 (2009), 399-407.

[18] A.B. Patel and P.B. Ramanujan, On Sum and Product of Normal Operators. Indian J. Pure Appl. Math., 12(100: 1213–1218(1981).

[19] 17. W. Rudin, Functional analysis, McGraw-Hill, 1991 (2nd edition).

[20] Schöichi Òta and Konrad Schmügen, On some classes of unbounded operators. Integral Equations and Operator Theory Vol. 12 (1989).

[21] J. Stochel, F. H. Szafraniec, On normal extensions of unbounded operators. II, Acta Sci. Math. (Szeged) 53 (1989), 153-177.