LEARNING IN RANDOM UTILITY MODELS VIA ONLINE DECISION PROBLEMS

EMERSON MELO

ABSTRACT. This paper studies the Random Utility Model (RUM) in a repeated stochastic choice situation, in which the decision maker is imperfectly informed about the payoffs of each available alternative. We develop a gradient-based learning algorithm by embedding the RUM into an online decision problem. We show that a large class of RUMs are Hannan consistent \cite{Hannan1957}; that is, the average difference between the expected payoffs generated by a RUM and that of the best-fixed policy in hindsight goes to zero as the number of periods increase. In addition, we show that our gradient-based algorithm is equivalent to the Follow the Regularized Leader (FTRL) algorithm, which is widely used in the machine learning literature to model learning in repeated stochastic choice problems. Thus, we provide an economically grounded optimization framework to the FTRL algorithm. Finally, we apply our framework to study recency bias, no-regret learning in normal form games, and prediction markets.

Keywords: Random utility models, Multinomial Logit Model, Generalized Nested Logit model, GEV class, Online optimization, Online learning, Hannan consistency, No-regret learning, Recency bias, Prediction markets.

JEL classification: D83; C25; D81

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Department of Economics, Indiana University, Bloomington, IN 47408, USA. Email: emelo@iu.edu. I am very grateful to Bob Becker, Austin Knies, Jorge Lorca, and Pablo Pincheira for their valuable comments and suggestions that have greatly improved the paper.
1. Introduction

The random utility model (RUM) introduced by Marschak [1959], Block and Marschak [1959], and Becker et al. [1963] has become the standard approach to model stochastic choice problems. The seminal work by McFadden [1978a,b, 1981] takes the RUM approach to a whole new level by making this theory empirically tractable. In particular, he provides an economic foundation and econometric framework which connects observables to stochastic choice behavior. This latter feature makes the RUM suitable to deal with complex choice environments and welfare analysis (McFadden [2001] and Train [2009]).

In a RUM, a decision maker (DM) faces a discrete choice set of alternatives in which each option is associated with a random utility. Then the DM chooses a particular option with a probability equal to the event that such alternative yields the highest utility among all available alternatives. Most of the applied literature models the random utility associated with each alternative as the sum of an observable and deterministic component and a random preference shock. Under this additive specification, different distributional assumptions on the random preference term will generate different stochastic choice rules. Thus, all the effort is to provide conditions on the distribution of the random preference shock such that the choice probabilities are consistent with the random utility maximization hypothesis (McFadden [1981]). From the description above, we remark that a RUM relies on two fundamental assumptions in addition to the distributional requirements. First, the RUM represents a static choice situation, ruling out dynamic environments where the DM may face a repeated stochastic choice problem. Second, the RUM assumes that the utilities (deterministic plus random components) are known to the DM. However, it is not difficult to find situations where informational frictions prevent the DM from learning the utilities associated to the different alternatives. In particular, the DM may be imperfectly informed about the actual value of the deterministic component associated with each option. These informational frictions may be caused by attentional limits, personal inclinations or biases, or just by the inherent complexity of the alternatives presented to the DM.

1 The seminal paper by Tversky [1969] reports early evidence on stochastic choice behavior. Recently, Agranov and Ortoleva [2017] provided experimental evidence supporting that decision-makers exhibit stochastic choice behavior in repeated choice situations.

2 The recent literature in stochastic choice incorporates informational frictions by explicitly modeling the sources of information. For instance, the papers by Matějka and Mckay [2015], Caplin and Dean [2015], Caplin et al. [2018], Fosgerau et al. [2020], and Natenzon [2019] study static stochastic choice problems with costly information frictions by incorporating the mechanism by which the DM acquires information and learns about the utilities associated to the alternatives. Webb [2019] derives a RUM using a bounded accumulation model, which can capture the dynamic of evidence accumulation. Webber’s framework provides an alternative mechanism to information acquisition in stochastic choice models. In a
This paper studies a RUM which relaxes the assumptions of static choice and perfect information. In doing so, we embed the RUM into an online decision problem (ODP) in which the DM must choose a probability distribution over a set of discrete actions at each point in time. At each period, there is a random utility vector describing the utilities associated with each option. However, and different from the traditional RUM, the deterministic component in the random utility vector is unknown to the DM at the moment of making a choice. The realization of this random vector depends on a probability distribution unknown to the DM. In this environment, when making a decision (selecting a probability distribution), the DM uses the accumulated information up to the previous period, which is subject to random preference shocks. Thus, at each point of time, the DM’s probability distribution can be considered generated by a RUM in which she imperfectly estimates the performance (utilities) associated to each alternative using the accumulated information until the previous period. To make the connection between both approaches explicit, we denote the resulting model as RUM-ODP.

In the RUM-ODP model, the DM wants to choose a sequence of probability distributions to suffer as little regret as possible, where regret is defined as the difference between the DM’s cumulative expected payoffs and that of the best-fixed action in hindsight (Bell [1982], Loomes and Sugden [1982], and Fishburn [1982]). More importantly, when the DM can find a sequence of probability vectors such that the average regret associated to it becomes arbitrarily small as $T$ grows, we say that such a sequence is Hannan consistent (Hannan [1957]). Similarly, when a sequence of probability distributions is Hannan consistent, we say such a sequence enjoys the no-regret learning property (Cesa-Bianchi and Lugosi [2003] and Roughgarden [2016]).

Examples that fit into the RUM-ODP framework include prediction from expert advice, repeated consumer choice, adversarial learning, and online shortest path problems, among many others (Hazan [2017]). Furthermore, from a strategic point of view, the RUM-ODP problem can be cast as a structured repeated game between the DM and the environment (nature), where the performance metric (or equilibrium concept) is the regret suffered by the DM.

A fundamental property of the RUM-ODP is the fact that we do not need to specify the DM’s priors over the set of possible utility vectors. Similarly, in the RUM-ODP, we do not need to specify the mechanism by which

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3This strategic interpretation in the ODP literature has been analyzed by Littlestone and Warmuth [1994], Freund and Schapire [1997], Freund and Schapire [1999], Foster and Vohra [1999], and Cesa-Bianchi and Lugosi [2006] among many others.

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recent paper, Cerreia-Vioglio et al. [2021] axiomatize the multinomial logit (MNL) model, where stochastic choice behavior is caused by time-constrained information processing. They provide a neural and behavioral foundation for the MNL in environments where the DM faces a pressing deadline that affects her choices.
the DM acquires information. This is in stark contrast with other models that connect RUM with stochastic choice under informational frictions (e.g. Matějka and McKay [2015] and Fosgerau et al [2020]).

This paper studies the RUM-ODP model from an algorithmic standpoint, making at least five general contributions. First, we show that the RUM-ODP model can be studied in terms of an algorithm which we call the Social Surplus Algorithm (SSA). This algorithm provides a simple economic and behavioral foundation to study learning in the RUM-ODP. In doing so, we use the result that, at each point in time, the DM’s stochastic choice rule is given by the gradient of the Social Surplus function. This latter relationship characterizes the entire class of RUMs.\footnote{In a RUM, the Social Surplus function is defined as $\varphi(v) = E_\varepsilon(\max_{i \in A} \{v_i + \varepsilon_i\})$, where $A = \{1, \ldots, N\}$ is the choice set, $v_i$ is a deterministic term and $\varepsilon_i$ is a random preference shock. McFadden [1981] and Rust [1994] show that $\frac{\partial \varphi(v)}{\partial v_i} = P(i = \arg \max_{j \in A} \{v_j + \varepsilon_j\})$ for $j = 1, \ldots, N$.}

Then, by exploiting convex analytic results, we show that the SSA is Hannan consistent for a large class of RUMs. A fundamental property of the SSA is that it allows for environments with arbitrary degrees of correlation (similarity) between the alternatives. Thus, we can study the RUM-ODP in situations where the alternatives in the choice set may be correlated, have complex substitution patterns, have an ordered structure, or be grouped into several different classes (nests).

In our second contribution, we focus on a particular instance of the RUM-ODP model. In modeling the random preference shock, we focus on the class of RUMs known as Generalized Extreme Value (GEV) models. Examples within this class include the multinomial logit (MNL), the Nested Logit (NL), the Paired Combinatorial Logit (PCL), the Generalized Nested Logit (GNL), and the Ordered GEV (OGEV) models. We show that the SSA is Hannan consistent for all of these models. From an applied perspective, this result is useful in modeling dynamic discrete choice demand and learning in complex environments. To the best of our knowledge, the GEV class has not been studied previously in the context of a model like the RUM-ODP.

Third, we show that the SSA approach is equivalent to the Follow the Regularized Leader (FTRL) algorithm, widely used in machine learning problems involving repeated stochastic choice situations. We show that the SSA and the FTRL are dual to each other in a convex duality sense. The relevance of this equivalence result comes from the fact we can avoid the specification of a regularization function which is fundamental to implementing the FTRL algorithm in applied settings.\footnote{The FTRL algorithm models the DM as solving an explicit strictly concave optimization problem combining past information (cumulative payoffs) along with a deterministic regularization term. The idea of regularization comes from the machine learning literature (Shalev-Schwartz [2012]). Intuitively, this idea can be interpreted as hedging against bad} In fact, the only regularization function available in closed form is the Shannon entropic term which yields the MNL (cf. Hazan}
More importantly, from an applied standpoint, our equivalence result allows us to provide a behavioral interpretation of the FTRL in terms of a RUM-ODP model. In addition, we show that for a large class of RUMs, the choice probability vector generated by the FTRL algorithm can be written in a general recursive way. This latter result generalizes in a nontrivial way the popular *Exponential Weights Algorithm* (EWA) approach, which heavily relies on the closed-form expression of the MNL model (cf. Hazan [2017] and Roughgarden [2016]). Furthermore, we discuss how to incorporate the phenomenon of recency bias, which refers to a situation in which a DM reacts more heavily to recent observations than she does to old ones (Erev and Haruvy [2016]). In Proposition 6, we show how the RUM-ODP is Hannan consistent under different forms of recency bias. This result provides an economic foundation to the Optimistic FTRL algorithm proposed by Rakhlin and Sridharan [2013].

In our fourth contribution, we provide a complete analysis of the NL model in terms of the FTRL algorithm. McFadden [1978a] introduces the NL model as a particular instance of the GEV class. The fundamental property of the NL is that the choice set is divided into a collection of *mutually* exclusive nests. The utilities between alternatives within the same nest are correlated, while the utilities of alternatives in different nests are independent. In this case, we show that the FTRL can be implemented using a new regularization function, which naturally generalizes the Shannon Entropic term. From a behavioral standpoint, our regularization function captures substitution patterns *within* and *between* nests. Furthermore, we show that in the case of the NL model, the choice probability vector (the gradient of the social surplus function) can be written in an explicit recursive form, highlighting the role of past information and the nesting structure in the DM’s learning.

In our last contribution, we focus on two concrete problems. First, we apply our framework and results to study no-regret learning in normal form games. In particular, we discuss how our approach differs from the *Potential-Based dynamics* framework introduced by Cesa-Bianchi and Lugosi [2003], Hart and Mas-Colell [2001], and Hart and Mas-Colell [2003]. Second, use our approach to study predictions markets. A prediction market is a future market in which prices aggregate information and predict future events (Hanson [2002]). Applications of these markets include electoral markets, science and technology events, sports events, the success of movies, etc. (Wolfers and Zitzewitz [2004]). We show how the RUM-ODP and the SSA approach are useful for studying this class of markets. In Proposition 9, we exploit the mathematical structure of the social surplus function to connect the SSA with a large class of future events or as avoiding overfitting the observed data. Thus, the solution to this regularized optimization problem at each point yields the DM’s optimal probability distribution as a function of cumulative payoffs.
of prediction markets. In economic terms, this result establishes a formal relationship between machine learning, the RUM-ODP model, and prediction markets. Our results extend in a nontrivial way the findings in Chen and Vaughan [2010] and Abernethy et al. [2013].

The rest of the paper is organized as follows. §2 describes the model and studies the SSA. §3 studies the connection between the SSA using the GEV class. §4 analyzes the FTRL algorithm and NL model. In §5 discusses no-regret learning in normal form games. §6 discusses the connection between our results and prediction markets. §7 provides an in-detail discussion of the related literature to this paper. Finally, §8 concludes. Proofs and technical lemmas are gathered in Appendix A.

Notation. Let ⟨·,·⟩ denote the inner product between two vectors. For a convex function \(f : K \subseteq \mathbb{R}^n \rightarrow \mathbb{R}\), \(\partial f(x)\) denotes the subgradient of \(f\) at \(x\). Let \(\nabla f(x)\) denote the gradient of a function \(f : K \subseteq \mathbb{R}^N \rightarrow \mathbb{R}\) evaluated at point \(x\). The \(i\)th element of \(\nabla f(x)\) is denoted by \(\nabla_i f(x)\). The Bregman divergence associated to a function \(f\) is given by

\[
D_f(x||y) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle.
\]

The Hessian of \(f\) at point \(x\) is denoted by \(\nabla^2 f(x)\) with entries \(\nabla^2_{ij} f(x)\) for \(i,j = 1,\ldots,N\). Let \(\| \cdot \|\) denote a norm in \(\mathbb{R}^N\) where \(\| \cdot \|_*\) is its dual norm. Let \(A \in \mathbb{R}^{N \times N}\) denote a \(N\)-square matrix. We define the norm \(\| \cdot \|_{\infty,1}\) associated to the matrix \(A\) as \(\|A\|_{\infty,1} = \max_{\|v\|_1 \leq 1} \|Av\|_1\). Finally, the trace of a matrix \(A \in \mathbb{R}^N\) is denoted by \(Tr(A)\).

2. Online decision problems and the Social Surplus Algorithm

Let \(A = \{1,\ldots,N\}\) be a finite set of alternatives. Let \(\Delta_N\) denote the \(N\)-dimensional simplex over the set \(A\). Let \(T \geq 2\) denote the (exogenous) number of periods. Let \(u_t = (u_{1t},\ldots,u_{Nt})\) be a random vector, where \(u_{it}\) denotes the stochastic payoff associated to option \(i \in A\) for \(t = 1,\ldots,T\). The realizations of the vector \(u_t\) are determined by the environment (nature), which in principle can be adversarial. We assume that the vector \(u_t\) takes values on a compact set \(U \subseteq \mathbb{R}^N\). In particular, we assume that \(\|u_t\|_{\infty} \leq u_{\text{max}}\) for all \(t\).

In this paper, we study the following ODP. At each period of time \(t = 1,\ldots,T\), the DM chooses \(x_t \in \Delta_N\) based on the data received up to the previous period. After committing to \(x_t\), the DM observes the realization of the payoff vector \(u_t\). Then the DM experiences the expected payoff \(\langle u_t, x_t \rangle\). The DM’s goal is to choose a sequence \(x_1,\ldots,x_T\) that minimizes her \textit{regret} between the total expected payoffs she has incurred and that of the best choice in \textit{hindsight}. Formally, the regret associated with a sequence of choices is defined as follows.
**Definition 1.** Consider $T$ periods and a sequence of choices $A = \{x_1, \ldots, x_T\}$. The regret associated with the sequence $A$ is defined as:

\[
R^T_A = \max_{x \in \Delta_N} \langle \theta_T, x \rangle - \sum_{t=1}^{T} \langle u_t, x_t \rangle,
\]

where $\theta_T \triangleq \sum_{t=1}^{T} u_t$ is the cumulative payoff vector until period $T$.

In the previous definition it is easy to see that $R^T_A$ can be interpreted as the comparison between the sequence $x_1, \ldots, x_T$ against the best choice in hindsight $x^* \in \arg \max \langle \theta_T, x \rangle$. Note that $x^*$ is computed in the ideal situation where the DM knows in advance the sequence of payoffs $u_1, \ldots, u_T$. In particular, in an ODP, the sequence $A$ describes how the DM learns over time using past information. Thus, $A$ can be associated with particular learning algorithms.

To formalize the notion that the DM implements learning algorithms that minimize the regret associated with her sequence of choices, we introduce the notion of Hannan Consistency (Hannan [1957]).

**Definition 2** (Hannan [1957]). A sequence $A$ is Hannan Consistent if the regret in Eq. (1) is small, i.e.,

\[
R^T_A = o(T).
\]

Intuitively, Definition 2 establishes that the sequence $A$ is Hannan consistent if the averaged regret associated to $A$ goes to zero as $T \to \infty$. In formal terms, this is equivalent to say that $A$ is Hannan consistent if the $R^T_A$ is sublinear in $T$, i.e., if Eq. (2) holds. Thus, Hannan consistency is equivalent to saying that a sequence of choices performs as well as the best-fixed strategy in hindsight $x^*$. Alternatively, when a sequence of choices $A$ satisfies the condition (2), we say that the algorithm $A$ satisfies the no-regret property.

Coupling the ODP model with the notion of regret, we can develop learning algorithms that satisfy Definition 2. In doing so, most of the Game Theory and the Online Convex Optimization (OCO) literature has focused on the study of the FTRL algorithm using the entropic penalty (see Hazan [2017] and references therein).

This paper shows that a large class of discrete choice models are Hannan consistent. Our results extend the scope of no-regret learning analysis far

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6We note that Definition 1 can be equivalently written as

\[
R^T_A = \max_{i \in A} \{\theta_{iT}\} - \sum_{t=1}^{T} \langle u_t, x_t \rangle,
\]

where $\theta_{iT} \triangleq \sum_{t=1}^{T} u_{it}$ is the cumulative payoff for alternative $i$ until period $T$.

7Throughout this paper, we use the terms Hannan consistency and no-regret learning interchangeably.
beyond the traditional MNL model. As a consequence, no-regret learning can be studied with a richer class of behavioral models.

2.1. The Social Surplus Algorithm. In this section, we develop a simple algorithm combining the ODP approach with the theory of RUMs (McFadden [1981, Ch. 5]). The combination of the ODP and the RUM approaches allows us to extend Hannan [1957]'s approach. Formally, Hannan studies an ODP model in which the DM’s choice is given by:

\[ \hat{x}_{t+1} \in \arg \max_{x \in \Delta_N} (\theta_t + \eta \epsilon_{t+1}, x) \quad \text{for} \quad t = 1, \ldots, T, \]

where \( \theta_t \) is the cumulative payoff vector until period \( t \), \( \epsilon_{t+1} = (\epsilon_{1t+1}, \ldots, \epsilon_{Nt+1}) \) is a random preference shock vector, and \( \eta \) is a strictly positive parameter, i.e., \( \eta > 0 \).

The expression (3) establishes that the choice \( \hat{x}_{t+1} \) is the solution of a recursive problem that depends on the past information contained in the cumulative payoff vector \( \theta_t \), the realization of \( \epsilon_{t+1} \), and \( \eta \), which is interpreted as a learning parameter. Alternatively, the parameter \( \eta \) is a measure of accuracy on the DM’s choices.

Intuitively, the interpretation of (3) is that the DM samples a random realization of \( \epsilon_{t+1} \) to smooth out her optimization problem. In the OCO literature, this approach is called Follow the Perturbed Leader (FTPL). A common assumption in this framework is to assume that \( \epsilon_{t+1} \) is sampled from a \( N \)-dimensional uniform distribution. The main advantage of the FTPL idea is the possibility of inducing stability in the DM’s choices. In fact, Hannan [1957] shows that when \( \epsilon_t \) is i.i.d. following a uniform distribution, the sequence generated by problem (3) satisfies Definition 2. However, the regret analysis of this approach relies on probabilistic arguments about the stochastic structure of \( \epsilon_{t+1} \) rather than having a general framework (Abernethy et al. [2017]). More importantly, from an economic standpoint, the interpretation of the FTPL approach is unclear, which makes it difficult to provide a behavioral foundation for the repeated stochastic choice problem.

In this section, we propose that instead of focusing on particular realizations of \( \epsilon_{t+1} \), we can exploit the entire distribution of it. In particular, under reasonably general distributional assumptions on \( \epsilon_{t+1} \), we can use the theory of RUMs to generate an alternative approach to Hannan’s original FTPL algorithm idea. In doing so, throughout the paper, we use the following assumption.

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8The interpretation of \( \eta \) as a measure of accuracy is also considered in Cerreia-Vioglio et al. [2021] in a different framework.

9In the Game Theory literature, this approach is known as fictitious stochastic play. For details, we refer the reader to Fudenberg and Levine [1998].
Assumption 1. For all $t \geq 1$ the random vector $\epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{N_t})$ follows a joint distribution $F = (F_1, \ldots, F_N)$ with zero mean that is absolutely continuous with respect to the Lebesgue measure, independent of $t$ and $\theta_t$, and fully supported on $\mathbb{R}^N$.

Assumption 1 is standard in random utility and discrete choice models (McFadden [1981, Ch, 5]). The full support and absolute continuity conditions imply that (3) can be rewritten as:

$$
\max_{x \in \Delta_N} \langle \theta_t + \eta \epsilon_{t+1}, x \rangle = \max_{j \in A} \{\theta_{jt} + \eta \epsilon_{jt+1}\} \quad \text{for } t = 1, \ldots, T.
$$

From (4), it follows that Assumption 1 implies that $\tilde{x}_{t+1}$ corresponds to a corner solution, i.e., the DM chooses one of the alternatives with probability one. Furthermore, noting that $\max \{\cdot\}$ is a convex function and defining $\tilde{\varphi}(\theta_t + \eta \epsilon_{t+1}) \triangleq \max_{j \in A} \{\theta_{jt} + \eta \epsilon_{jt+1}\}$, $\tilde{x}_{t+1}$ is characterized as

$$
\tilde{x}_{t+1} \in \partial \tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})
$$

Thus an optimal solution $\tilde{x}_{t+1}$ is given by a subgradient of $\tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})$ (Rockafellar [1970]).

Characterization (5) is derived under the assumption of a single realization of $\epsilon_{t+1}$. However, a natural extension of this result is to consider the entire distribution of $\epsilon_{t+1}$ by considering the expectation of $\tilde{\varphi}(\theta_t + \epsilon_{t+1})$. Formally, define the function $\varphi : \mathbb{R}^N \mapsto \mathbb{R}$ as

$$
\varphi(\theta_t) \triangleq \mathbb{E}_t \left( \max_{i \in A} \{\theta_{it} + \eta \epsilon_{it+1}\} \right).
$$

In the RUM literature $\varphi(\theta_t)$ is known as the social surplus function, which summarizes the effect of $\epsilon_t$. More importantly, $\varphi(\theta_t)$ is convex and differentiable in $\mathbb{R}^N$. This latter property implies that the choice probability vector $x_{t+1}$ can be characterized as:

$$
\nabla \varphi(\theta_t) = x_{t+1} \quad \text{for } t = 1, \ldots, T - 1.
$$

The previous result follows from the well-known Williams-Daly-Zachary theorem (see Rust [1994, p. 3104]). From an economic standpoint, the social surplus function allows us to interpret the DM’s choice in terms of the theory of RUMs. To see this, the cumulative payoff vector $\theta_t$ can be naturally interpreted as an estimate of the unknown utility $u_t$ at time $t$. In particular, $\theta_{jt}$ provides cumulative information about alternative $j$’s past performance. Accordingly, the random variable $\epsilon_{jt+1}$ is interpreted as a preference shock that affects how the DM perceives the cumulative payoff associated with this particular alternative. The same logic applies to all alternatives in $A$. Thus, the DM’s stochastic choice is consistent with RUMs. More importantly, this connection clarifies that different distributional assumptions on $\epsilon_{t+1}$ will imply stochastic choice rules capturing different behavioral aspects. This feature
allows one to study choice models in which the alternatives exhibit high similarity and correlation. For instance, we can study models like probit and NL. Given the connection between RUMs and the ODP described above, we denote the resulting model as the RUM-ODP model.

A second important implication of expression (7) is that $x_{t+1}$ can be interpreted as the expected value of $\tilde{x}_{t+1}$. The next proposition formalizes this fact.

**Proposition 1.** Let Assumption 1 hold. Then, for $t = 1, \ldots, T$

$$E(\tilde{x}_{t+1}) = x_{t+1}.$$  

From the previous result, it follows that

$$\nabla \varphi(\theta_t) = E \left( \arg \max_{x \in \Delta_n} \langle \theta_t + \eta \epsilon_{t+1}, x \rangle \right),$$  

$$= \left( P \left( i = \arg \max_{j \in A} \left\{ \theta_{jt} / \eta + \epsilon_{t+1} \right\} \right) \right)_{i \in A},$$  

$$= (P(\theta_{it} / \eta + \epsilon_{it} \geq \theta_{jt} / \eta + \epsilon_{jt} \forall j \neq i))_{i \in A}.$$  

Furthermore, from the definition of the social surplus, it is easy to see that $\varphi(\theta) = \eta \varphi(\theta / \eta)$ and $\nabla \varphi(\theta) = \nabla \varphi(\theta / \eta)$. We shall use this relationship in deriving our results.

More importantly, the convex structure of the function $\varphi$ enables us to develop a learning algorithm called the Social Surplus Algorithm (SSA). In doing so, we assume that the Hessian of $\varphi(\theta_t)$ satisfies the following technical condition.

**Assumption 2.** For all $t \geq 1$ the Hessian of the Social Surplus function $\nabla^2 \varphi(\theta_t)$ satisfies the following condition:

$$2 \text{Tr}(\nabla^2 \varphi(\theta_t)) \leq \frac{L}{\eta},$$

with $L > 0$.

The previous assumption is less standard as it imposes a condition on the trace of Hessian of $\varphi(\theta_t)$. This requirement allows us to establish the Lipschitz continuity of $\nabla \varphi(\theta_t)$.

**Lemma 1.** Let Assumptions 1 and 2 hold. Then $\varphi(\theta)$ has a gradient-mapping that is Lipschitz continuous with constant $\frac{L}{\eta}$:

$$\| \nabla \varphi(\theta) - \nabla \varphi(\tilde{\theta}) \|_1 \leq \frac{L}{\eta} \| \theta - \tilde{\theta} \|_1, \quad \forall \theta, \tilde{\theta} \in \mathbb{R}^N,$$

with $L > 0$. 
Two remarks are in order. First, as we noted before, the social surplus function \( \varphi(\theta) \) can be equivalently written as \( \eta \varphi(\theta/\eta) \). Using this equivalence, Lemma \[2\] can be equivalently stated as: the social surplus function \( \eta \varphi(\theta/\eta) \) has a gradient-mapping that is \( L \)-Lipschitz continuous, i.e.,

\[
\|\nabla \varphi(\theta/\eta) - \nabla \varphi(\tilde{\theta}/\eta)\|_1 \leq L \|\theta/\eta - \tilde{\theta}/\eta\|_1, \quad \forall \theta, \tilde{\theta} \in \mathbb{R}^N,
\]

with \( L > 0 \).

Our second observation is related to the fact that despite being a technical requirement, Assumption \[2\] is satisfied by many well-known RUMs, like the MNL, NL, and the GEV class, as we show in \$3\.

Now we are ready to introduce the SSA as follows:

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**Algorithm 1 Social Surplus Algorithm**

1: Input: \( \eta > 0, \) \( F \) a distribution on \( \mathbb{R}^N \), and \( \Delta_N \).
2: Let \( \theta_0 = 0 \) and choose \( x_1 = \nabla \varphi(0) \)
3: for \( t = 1 \) to \( T \) do
4: The DM chooses \( x_t = \nabla \varphi(\theta_{t-1}) \)
5: The environment reveals \( u_t \)
6: The DM receives the payoff \( \langle u_t, \nabla \varphi(\theta_{t-1}) \rangle \)
7: Update \( \theta_t = u_t + \theta_{t-1} \) and choose
\[
x_{t+1} = \nabla \varphi(\theta_t)
\]
8: end for

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The previous algorithm exploits the social surplus function and its gradient. In this sense, our algorithm is similar to the *Potential Based Algorithm* proposed in \cite{Cesa-Bianchi and Lugosi 2003, 2006} and to the *Gradient Based Algorithm* studied by \cite{Abernethy et al. 2017}. The idea of applying a gradient descent-like approach to study repeated stochastic choice problems dates back to \cite{Blackwell 1956}. \cite{Hart and Mas-Colell 2001} and \cite{Hart and Mas-Colell 2003} apply Blackwell’s approachability theorem to sequential decision problems. They introduce the notion of *Lambda—* strategies, which relies on the notion of potential functions.

A natural question is whether \cite{Cesa-Bianchi and Lugosi 2003}’s results apply to the study of the RUM-ODP model. To use their results, we must assume that the social surplus function \( \varphi \) has an additive structure. In terms of our approach, this is equivalent to assume that the random variables \( \epsilon_{it} \) are i.i.d. for all \( t \). For instance, the MNL model satisfies this condition. However, most of the RUMs considered in the discrete choice literature (and this paper) do not satisfy this additivity requirement. Similarly, in applying the results in \cite{Hart and Mas-Colell 2001} and \cite{Hart and Mas-Colell 2003}, we need to impose restrictions on the domain of \( \nabla \varphi \). In particular, their approach requires that
the gradient of $\varphi$ vanishes over the approachable set $E$. This latter condition assumes that the random shock vector $\epsilon_t$ has bounded support, which rules out the whole class of RUMs. Thus, while related, Cesa-Bianchi and Lugosi [2003], Hart and Mas-Colell [2001], and Hart and Mas-Colell [2003] results do not apply to the RUM-ODP model.

Another important difference between Cesa-Bianchi and Lugosi [2003], Hart and Mas-Colell [2001], and Hart and Mas-Colell [2003] and our approach is related to the interpretation of the potential function. In their work, the role of a potential function is to provide a way to measure the size of the regret associated with the DM’s choice at a given time. In our framework, we can interpret the social surplus function $\varphi$ in this way. But in addition, $\varphi$ measures the expected utility received at each time point associated with the probability vector $x_t$. In order to formalize this observation, let us define $e_{jt}(\theta_{t-1}) \triangleq \mathbb{E}(\epsilon_{jt} | j = \text{arg max}_k \{\theta_{kt} - 1 + \epsilon_{kt}\})$ for all $j \in A$. Then using the law of iterated expectations, it follows that $\varphi(\theta_{t-1})$ can be expressed as a weighted average:

$$\varphi(\theta_{t-1}) = \sum_{i=1}^{N} x_{it} (\theta_{it} - 1 + \eta e_{jt}(\theta_{t-1}))$$

The expression (9) makes explicit the role of the random shocks in determining the value of $\varphi(\theta_{t-1})$. Different distributions for the random shock $\epsilon_t$ will lead to different social surplus functional forms. This fact is relevant at least for two reasons. First, the SSA allows us to study discrete choice models with arbitrary degrees of correlation (similarity) between the different alternatives. In particular, the SSA allows us to study repeated stochastic problems in environments where the DM’s choices are represented by preference trees (e.g. NL and GNL models). Second, by exploiting the structure of discrete choice models, the SSA can be implemented using closed-form expressions for $\nabla \varphi(\theta_t)$. In particular, we will show that the SSA can be implemented with the GEV class, which contains the MNL as a particular case.

Without further delay, we establish the main result of this section.

**Theorem 1.** Let Assumptions 1 and 2 hold. Then in the SSA we have:

$$R_{SSA}^T \leq \eta \varphi(0) + \frac{L}{2\eta} T u_{\text{max}}^2.$$  

Furthermore, setting $\eta = \sqrt{\frac{LT u_{\text{max}}^2}{2\varphi(0)}}$ we get

$$R_{SSA}^T \leq u_{\text{max}} \sqrt{2\varphi(0)LT}.$$  

Some remarks are in order. First, Theorem 1 establishes that by implementing the SSA, a large class of RUMs are Hannan consistent. The result

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10 Formally, this is equivalent to assume that $\nabla \varphi(z) = 0$ for all $z \in \mathbb{R}^N$. 

highlights the role of $L$, $\eta$, and the social surplus function evaluated at $u = 0$.\[^{11}\]

This result enables one to implement the SSA exploiting different functional forms for the stochastic choice rule given by $\nabla \varphi(\theta_t)$. In §3 we will show how the result in Theorem 1 applies to a large class of RUMs.

Second, from a technical point of view, the proof of Theorem 1 relies on the convex structure of $\varphi(\theta_t)$. Formally, we exploit convex duality arguments to bound the regret of the SSA. In §4 we shall further exploit these results to analyze the FTRL algorithm.

Third, Theorem 1 is related to the results in Cesa-Bianchi and Lugosi [2003]. Their analysis focuses on several potential functions that generate the choice probability vector $x_{t+1}$. Our analysis differs from theirs in that $\varphi(\theta_t)$ has an economic meaning in terms of RUMs. Similarly, it is worth pointing out that Theorem 1 generalizes the analysis in Abernethy et al. [2014, 2017]. They focus mainly on the MNL model, which corresponds to the case of $\epsilon_t$ following an extreme value type 1 distribution. The following result formalizes this observation.

**Corollary 1.** Assume that for all $t$ the random shock vector $\epsilon_t$ follows an Extreme Value type 1 distribution, given by

$$F(\epsilon_{1t}, \ldots, \epsilon_{Nt}) = \exp \left(- \sum_{j=1}^{N} e^{-\epsilon_{jt}} \right) \quad \forall t.$$  

Then, in the SSA, setting $\eta = \sqrt{\frac{\text{Regret}_T}{2 \log N}}$ yields:

$$\text{Regret}_T \leq u_{\max} \sqrt{2 \log N}.$$  

The previous corollary follows from the fact that for the MNL it is well known that (cf. McFadden [1981, Ch. 5] and Train [2009, Ch. 3])

$$\varphi(\theta_t) = \eta \log \left( \sum_{j=1}^{N} e^{\theta_{jt}/\eta} \right).$$  

Noting that $\eta \varphi(0) = \eta \log \sum_{i=1}^{N} e^{0} = \eta \log N$ and using the fact that $\nabla \varphi(\theta) = \left( \frac{e^{\theta_{jt}/\eta}}{\sum_{j=1}^{N} e^{\theta_{jt}/\eta}} \right)_{t \in N}$ is $\frac{1}{\eta}$-Lipschitz continuous, it follows that Corollary 1 is a direct application of Theorem 1.

From a behavioral standpoint, the MNL imposes that the stochastic shocks $\{\epsilon_{jt+1}\}_{j \in A}$ are independent across alternatives. This assumption can be strong, especially in environments where the payoffs associated with the options in $A$ can be correlated. The following section discusses how to relax the independence assumption to implement the SSA with more flexible RUMs.

\[^{11}\text{Note that in this case } \eta \varphi(0) = \eta \mathbb{E}(\max_{i=1,\ldots,N} \{\epsilon_i\}). \]
3. The SSA and GEV models

This section aims to connect the RUM-ODP model and the SSA approach with the class of GEV models. To the best of our knowledge, the GEV class has not been studied in the context of no-regret learning models.

The GEV class was introduced by McFadden [1978a] to generalize the MNL model by allowing general patterns of dependence among the unobserved components of the options while yielding analytical and tractable analytical closed forms for the choice probabilities under consideration.

In developing the GEV class, McFadden introduces the notion of a generator function, which we define now.

**Definition 3.** A function \( G : \mathbb{R}^N_+ \rightarrow \mathbb{R}_+ \) is a generator if the following conditions hold:

i) For all \( y = (y_1, \ldots, y_N) \in \mathbb{R}^N_+ \), \( G(y) \geq 0 \).

ii) The function \( G \) is homogeneous of degree 1: \( G(\lambda y) = \lambda G(y) \) for all \( y \in \mathbb{R}^N_+ \) and \( \lambda > 0 \).

iii) For \( i = 1, \ldots, N \), \( G(y) \rightarrow \infty \) as \( y_i \rightarrow \infty \).

iv) If \( i_1, \ldots, i_k \) are distinct from each other \( k \) distinct components, then \( i_1, \ldots, i_k \), the \( k \)-th order partial derivative \( \frac{\partial G(y_1, \ldots, y_N)}{\partial y_{i_1} \cdots \partial y_{i_k}} \geq 0 \) when \( k \) is odd, whereas \( \frac{\partial G(y_1, \ldots, y_N)}{\partial y_{i_1} \cdots \partial y_{i_k}} \leq 0 \) when \( k \) is even.

McFadden [1978a, 1981] show that when the generator function \( G \) satisfies i)-iv), then the random vector \( \epsilon = (\epsilon_1, \ldots, \epsilon_N) \) follows a Multivariate Extreme Value distribution:

\[
F(\epsilon_1, \ldots, \epsilon_N) = \exp \left( -G(e^{-\epsilon_1}, \ldots, e^{-\epsilon_N}) \right).
\]

More importantly, McFadden shows that when \( \epsilon \) follows distribution (13), a random utility maximization model is consistent with the RUM hypothesis. Furthermore, McFadden [1978a] establishes that for a deterministic utility vector \( \theta \) with \( y = (e^{\theta_1}, \ldots, e^{\theta_N}) \), the social surplus function can be expressed in “closed” form expression as:

\[
\varphi(\theta) = \log G(y) + \gamma,
\]

where \( \gamma = 0.57721 \) is the Euler’s constant.

Using the fact \( \nabla \varphi(u) \) yields the choice probabilities, we get

\[
\nabla_i \varphi(u) = \frac{y_i G_i(y)}{\sum_{j=1}^N y_j G_j(y)} \quad \text{for} \quad i = 1, \ldots, N,
\]

\[
= \frac{e^{\theta_i + \log G_i(e^\theta)}}{\sum_{j=1}^N e^{\theta_j + \log G_j(e^\theta)}}.
\]

Eqs. (14) and (15) highlight two fundamental properties of the GEV class. First, the choice probability vector has a logit-like form, which makes the
analysis tractable. Second, the closed form expression for $\varphi$ helps us to study the SSA in a large class of RUMs.

3.1. The RUM-ODP model and the GEV class. Now we connect the GEV class with our RUM-ODP approach. In doing so, we note that given the cumulative payoff vector $\theta_t$ and the learning parameter $\eta$, the social surplus function can be written as:

$$\varphi(\theta_t) = \mathbb{E} \left( \max_{j \in A} \{ \theta_{jt} + \eta \epsilon_{jt+1} \} \right) = \eta \varphi(\theta_t / \eta).$$

Then, Eqs. (14) and (15) can be written as:

$$\begin{align*}
(16) \quad & \varphi(\theta_t) = \eta \left( \log G(e^{\theta_t/\eta}) + \gamma \right) \\
(17) \quad & \nabla_i \varphi(\theta_t) = \frac{e^{\theta_{it}/\eta} G_i(e^{\theta_t/\eta})}{\sum_{j=1}^N e^{\theta_{jt}/\eta} G_j(e^{\theta_t/\eta})} \quad \text{for } i = 1, \ldots, N.
\end{align*}$$

It is worth remarking that expression (17) is derived using the fact that for all $i$, $\frac{\partial \varphi(\theta_t)}{\partial \theta_{it}} = \eta \frac{\partial \varphi(\theta_t / \eta)}{\partial \theta_{it}}$. In deriving this relationship, we define the GEV model in terms of the scaled utility vector $\theta_t / \eta = (\theta_{it} / \eta)_{i \in A}$.

**Example 2.** Consider the linear aggregator $G(e^{\theta_t/\eta}) = \sum_{j=1}^N e^{\theta_{jt}/\eta}$. Then it follows that

$$\begin{align*}
\varphi(\theta_t) &= \eta \varphi(\theta_t / \eta) \\
&= \eta \log G(e^{\theta_t/\eta}) + \eta \gamma \\
&= \eta \log \sum_{j=1}^N e^{\theta_{jt}/\eta} + \eta \gamma \\
\text{and} \\
\nabla_i \varphi(\theta_t) &= \frac{e^{\theta_{it}/\eta}}{\sum_{j=1}^N e^{\theta_{jt}/\eta}} \quad \text{for } i = 1, \ldots, N.
\end{align*}$$

By exploiting the properties of the GEV class, we can establish that the SSA achieves Hannan consistency. In doing so, we use the following technical result from Muller et al. [2022, Thm. 3].

**Lemma 2.** Let $y_t = (e^{\theta_{1t}/\eta}, \ldots, e^{\theta_{Nt}/\eta}) \in \mathbb{R}_+^N$ and let $G$ be a generator function satisfying the following inequality

$$\begin{align*}
(18) \quad & \sum_{i=1}^N \frac{\partial^2 G(y_t)}{\partial y_{it}^2} \cdot (y_{it})^2 \leq MG(y_{1t}, \ldots, y_{Nt}), \quad \text{for all } t = 1, \ldots, T,
\end{align*}$$

for some $M \in \mathbb{R}_{++}$. Then the social surplus function $\varphi(\theta_t) = \eta (\log G(y_t) + \gamma)$ has a Lipschitz continuous gradient with constant $L = \frac{2M+1}{\eta}$. 

The previous lemma establishes that the Lipschitz constant \( L \) depends on the parameters \( M \) and \( \eta \). With this result in place, we can show the following bound on the regret for the GEV class:

**Theorem 3.** Let Assumption 1 hold. In addition, assume that there exists a generator function \( G \) satisfying Eq. (18). Then, in the SSA

\[
R_{SSA}^T \leq \eta \log G(1) + \frac{L}{2\eta} T u_{max}^2,
\]

where \( L = 2M + 1 \). Furthermore, setting \( \eta = \sqrt{\frac{LT u_{max}^2}{2\log G(1)}} \), we get:

\[
R_{SSA}^T \leq u_{max} \sqrt{2 \log G(1)(2M + 1) T}.
\]

The previous result provides a bound to the regret associated with the SSA when the GEV is considered. In other words, when the GEV class is combined with the RUM-ODP model, the sequence of choices generated by the SSA achieves Hannan consistent. To the best of our knowledge, Theorem 3 is the first regret analysis using the GEV class. In the next sections we discuss several widely used GEV models that satisfy condition (18).

### 3.2. The Generalized Nested Logit (GNL) model.

The most widely used model in the GEV class is the Generalized Nested Logit (GNL) model ([McFadden 1978a] and [Wen and Koppelman 2001]). This approach generalizes several GEV models like NL and Ordered GEV (OGEV). Given its flexibility, the GNL has been applied to energy, transportation, housing, telecommunications, demand estimation, etc.\(^{12}\) The main advantage of using a GNL is the possibility of incorporating correlation between the elements of the random shock vector \( \epsilon_{i+1} \) in a relatively simple and tractable manner.

Let \( A \) be the set of options partitioned into \( K \) nests labeled \( N_1, \ldots, N_K \). Let \( \mathcal{N} \) be the set of all nests. In defining the set \( \mathcal{N} \), we allow for overlapping between the nests. In particular, an option \( i \) may be an element of more than one nest. For instance, an option \( i \) may be an element of nests \( N_k, N_{k'}, N_{k''} \) simultaneously.

For \( k = 1, \ldots, K \), let \( 0 < \lambda_k \leq 1 \) be nest-specific parameters. From an economic standpoint, the parameter \( \lambda_k \) is a measure of the degree of independence between each random shock \( \epsilon_i \) in nest \( k \). In particular, the statistic \( 1 - \lambda_k \) is a measure of correlation ([Train 2009, Ch. 5]). Thus, as the value of \( \lambda_k \) increases, the value of this statistic decreases, indicating less correlation. Given the cumulative payoff vector \( \theta_i \), the generator function \( G \) for the GNL

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\(^{12}\)For a discussion about several applications of GEV models, we refer the reader to [Train 2009] and the references therein.
is the following:

\[ G(e^{\theta_t/\eta}) = \sum_{k=1}^{K} \left( \sum_{i=1}^{N} (\alpha_{ik} \cdot e^{\theta_{it}/\eta})^{1/\lambda_k} \right)^{\lambda_k} \] 

The parameter \( \alpha_{ik} \geq 0 \) characterizes the “portion” of the alternative \( i \) assigned to nest \( k \). Thus for each \( i \in A \) the allocation parameters must satisfy:

\[ \sum_{k=1}^{K} \alpha_{ik} = 1. \]

Using the previous condition, the set of alternatives within the \( k \)-th nest is defined as:

\[ \mathcal{N}_k = \{ i \in A \mid \alpha_{ik} > 0 \}, \]

where \( A = \bigcup_{k=1}^{K} \mathcal{N}_k \).

Based on the previous description, it is easy to show that the function \( G \) in Eq. (21) defines a GEV model (cf. Wen and Koppelman [2001]).

To describe the choice probability vector \( x_{t+1} \) to be used in the SSA, we note that the GNL is a model in which the underlying choice process comprises two stages. In the first stage, the DM chooses nest \( k \) with probability

\[ p_{kt} = \frac{e^{v_{kt}}}{\sum_{\ell=1}^{K} e^{v_{\ell t}}} \]

where

\[ v_{kt} = \lambda_k \log \left( \sum_{i=1}^{N} (\alpha_{ik} \cdot e^{\theta_{it}/\eta})^{1/\lambda_k} \right) \]

stands for the inclusive value contained in nest \( k \).

In the second stage, the probability of choosing alternative \( i \) within nest \( k \) is given by:

\[ p_{ikt} = \frac{(\alpha_{ik} \cdot e^{\theta_{it}/\eta})^{1/\lambda_k}}{\sum_{j=1}^{N} (\alpha_{jk} \cdot e^{\theta_{jt}/\eta})^{1/\lambda_k}}. \]

Thus, according to the GNL, the probability of choosing alternative \( i \) is expressed as \( \nabla_i \varphi(\theta_t) = x_{it+1} \), where

\[ x_{it+1} = \sum_{k=1}^{K} p_{kt} \cdot p_{ikt} \quad \text{for } i = 1, \ldots, n, \ t = 1, \ldots, T - 1. \]

Lemma 9 in Appendix A establishes two important properties of the GNL model. First, it shows that \( \nabla \varphi(\theta_t) \) is Lipschitz continuous with constant \( \frac{2}{\min_{k} \lambda_k - 1} \). The second important property is that \( \log G(1) \leq \log N \). These two observations allows us to establish the following result.
**Proposition 2.** Let $0 < \lambda_k \leq 1$ for $k = 1, \ldots, K$. In addition, set $\eta = \sqrt{\frac{(2\min_k \lambda_k - 1)T u_{max}^2}{2 \log N}}$. Then in the GNL we have

$$R_{SSA}^T \leq u_{max} \sqrt{2 \log N \left( \frac{2}{\min_k \lambda_k} - 1 \right) T}. \tag{22}$$

Some remarks are in order. First, Proposition 2 is a direct corollary of Theorem 3. It establishes that in the RUM-ODP model, the sequence of choices generated by the GNL model is Hannan consistent.

Second, Proposition 2 expands the scope of the SSA to more general environments than those considered by the traditional MNL model. The fundamental object in this result is the specification of the generator function $G$. In Appendix B.3 we discuss how different specifications of the GNL model yield the Paired Combinatorial logit (PCL), the OGEV model, and the Principles of Differentiation GEV model.

Finally, it is worth pointing out that Proposition 2 is related to the Potential-Based Algorithm introduced by Cesa-Bianchi and Lugosi [2003] and further explored by Abernethy et al. [2014, 2017]. In particular, the generator $G(e^{\theta_i/\eta})$ can be seen as a potential function that generates the choice prediction vector $x_{t+1}$. Thus Proposition 2 can be seen as a way to extend Cesa-Bianchi and Lugosi [2003] and Abernethy et al. [2014, 2017] approach to RUMs far beyond the MNL model.

### 3.3. The Nested Logit (NL) model.

The NL model proposed by McFadden [1978a] as a particular case of the GNL in the sense that each alternative $i \in A$ belongs to a unique nest $N_k$. In other words, the NL is a model in which the nests are mutually exclusive. In particular, each allocation parameter $\alpha_{ik}$ is:

$$\alpha_{ik} = \begin{cases} 1 & \text{if alternative } i \in N_k \\ 0 & \text{otherwise} \end{cases}.$$ 

Accordingly, the generator $G$ corresponds to:

$$G(e^{\theta_i/\eta}) = \sum_{k=1}^{K} \left( \sum_{i \in N_k} e^{\theta_{ia}/\eta} \right)^{1/\lambda_k} \lambda_k.$$ 

In this model the gradient $\nabla \varphi(\theta)$ is Lipschitz continuous with constant $\left( \frac{2}{\min \lambda_k} - 1 \right) / \eta$. Thus, by Proposition 2 we conclude that the NL model is Hannan consistent.

\[13] In addition, in Appendix B.3 we discuss the CNL, OGEV, and the PDGEV models.
It is worth mentioning that when $\lambda_k = 1$ for $k = 1, \ldots, K$, the NL boils down to the MNL. Under this parametrization, the generator $G$ can be written as
\[
G(e^{\theta_t/\eta}) = \sum_{k=1}^{K} \sum_{i \in N_k} e^{\theta_{ti}/\eta},
\]
\[
= \sum_{i=1}^{N} e^{\theta_{ti}/\eta}.
\]
Thus, as a direct result of Proposition 2, we find that the MNL is Hannan consistent.

4. The RUM-ODP model and the FTRL algorithm

As described in the introduction section, the FTRL algorithm models the DM as solving a strictly concave optimization problem combining past information (cumulative payoffs) and a deterministic regularization term. Intuitively, a regularization function can be seen as a hedging mechanism against bad future events or as a way of avoiding overfitting the observed data (Shalev-Shwartz [2012]). Thus, at each point in time, the solution to this regularized optimization problem yields the DM’s optimal probability distribution as a function of cumulative payoffs.

In this section, we provide an economic foundation for the FTRL algorithm. Formally, we establish the connection between the RUM-ODP, the SSA, and the FTRL algorithm. We make three contributions. First, we exploit the convex structure of the social surplus function to derive a strongly convex regularization function denoted by $\mathcal{R}(x)$. Formally, $\mathcal{R}(x)$ corresponds to the convex conjugate of $\varphi(\theta)$. As a direct consequence of this convex conjugacy, we show that the SSA and the FTRL algorithms are equivalent from a convex analysis point of view. In addition, we show that the choice probability vector generated by the FTRL algorithm can be expressed in a general recursive form. This latter fact implies that our approach generalizes in a nontrivial way the EWA, which only applies to the MNL model. Second, we show that in the case of the NL model, $\mathcal{R}(x)$ can be expressed in closed form. Finally, we show how by using the FTRL approach, the RUM-ODP model can incorporate the recency bias effect.

4.1. Convex conjugate and regularization. From we know that $\varphi(\theta)$ is convex and differentiable. Following Rockafellar [1970], we define the convex conjugate of $\varphi(\theta)$, denoted by $\varphi^*(x)$ as:
\[
\varphi^*(x) = \sup_{\theta \in \mathbb{R}^N} \left\{ \langle \theta, x \rangle - \varphi(\theta) \right\}.
\]

\(^{14}\)We recall that when $\lambda_k = 1$ for $k = 1, \ldots, N$ the elements of the random vector $\epsilon_t$ are independent.
Using the fact that \( \varphi(\theta) = \eta \varphi(\theta/\eta) \) combined with Beck [2017, Thm. 4.14(b)] we can define the function \( R(x) = \eta \varphi^*(x; \eta) \), where \( \varphi^*(x; \eta) \) is the convex conjugate of the parametrized social surplus function \( \varphi(\theta/\eta) \). The next result summarizes some fundamental properties of \( R(x) \).

**Proposition 3.** Let Assumptions 1 and 2 hold. Then:

i) \( R(x) \) is \( \frac{\eta}{L} \)-strongly convex.

ii) \( R(x) \) is differentiable for all \( x \in \text{int} \Delta_N \).

iii) The optimization problem

\[
\max_{x \in \Delta_N} \{ \langle \theta, x \rangle - R(x) \}
\]

has a unique solution. Furthermore:

\[
\nabla \varphi(\theta) = \arg \max_{x \in \Delta_N} \{ \langle \theta, x \rangle - R(x) \}.
\]

Some remarks are in order. First, part i) follows from a fundamental equivalence between the Lipschitz continuity of \( \varphi(\theta) \) and the strong convexity of its convex conjugate \( \varphi^*(x) \). This equivalence is known as the Baillon-Haddad Theorem; see, e.g., Rockafellar and Wets [1997, 12, Section H] and Bauschke and Combettes [2010]. Second, Proposition 3ii) establishes that \( R(x) \) is differentiable so that we can exploit the first order conditions to find the optimal \( x^* \). Third, Proposition 3iii) follows from the Fenchel equality. The main implication of this part is that given the cumulative payoff \( \theta_t \), the choice probability vector \( x_{t+1} \) can be equivalently characterized as the unique solution of a strongly concave optimization program. This latter fact allows us to provide a simple interpretation to \( R(x) \). In particular, for \( x_t = \nabla \varphi(\theta_{t-1}) \), and using Eq. (9) combined with the Fenchel equality, we find that

\[
R(x_t) = -\eta \sum_{j=1}^{N} x_{jt} e_{jt}(\theta_{t-1}) \quad \text{for } t \geq 1.
\]

The expression (25) makes explicit the role of the distribution of \( \epsilon_t \) in determining the shape of the regularizer. For instance, for the the MNL model, it is well known that \( e_{jt}(\theta_{t-1}) = -\log x_{jt} \) for all \( j \in A \). In this case we obtain \( R(x_t) = \eta \sum_{j=1}^{N} x_{jt} \log x_{jt} \), which is just the familiar entropic regularization. As we shall see in §4.4, in applying similar arguments to the case of the NL model, we can also provide a closed form expression for \( R(x_t) \).

From a technical point of view, Hofbauer and Sandholm [2002], Abernethy et al. [2014, 2017], Feng et al. [2017], and Fosgerau et al. [2020] establish a similar equivalence as the one in (24). The proof of our result borrows some of their arguments. We contribute to their results by adding the property of strong convexity of \( R(x) \).

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15 Lemma 4 in Appendix A provides the formal statement of this result.
4.2. The FTRL algorithm. The main implication of Proposition 3 is that $R(x)$ can be seen as a regularization function (Hazan [2017]). This result implies that introducing a stochastic perturbation $\epsilon_t$ is equivalent to introducing a deterministic regularization $R(x)$. More importantly, Proposition 3 allows one to write the FTRL algorithm as follows:

**Algorithm 2** Follow the regularized leader

1: Input: $\eta > 0$, $R$, and $\Delta_N$.
2: Let $x_1 = \arg \max_{x \in \Delta_N} \{-R(x)\}$.
3: for $t = 1$ to $T$ do
4: Predict $x_t$.
5: The environment reveals $u_t$.
6: The DM receives the payoff $\langle u_t, x_t \rangle$.
7: Update $\theta_t = u_t + \theta_{t-1}$ and choose
   $$x_{t+1} = \arg \max_{x \in \Delta_N} \{\langle \theta_t, x \rangle - R(x)\}.$$  
8: end for

Using the FTRL approach, we can establish the Hannan consistency of a large class of discrete choice models.

**Theorem 4.** Let Assumptions 1 and 2 hold. Then the FTRL Algorithm satisfies the following bound

$$R^T_{FTRL} \leq \eta \varphi(0) + \frac{L}{2\eta} Tu_{\max}^2.$$  

Furthermore, setting $\eta = \sqrt{\frac{LTu_{\max}^2}{2\varphi(0)}}$ we get

$$R^T_{FTRL} \leq u_{\max}\sqrt{2\varphi(0)LT}.$$  

Three remarks are in order. First, Theorem 4 establishes that the FTRL algorithm achieves the Hannan consistency property. This result is similar to the conclusion we obtained in Theorem 1. Its proof exploits the convex duality structure of the social surplus function. Thus, the SSA and the FTRL algorithm are dual to each other. This result implies any of the two algorithms achieves Hannan consistency.

Second, we point out that in obtaining the result in Theorem 4, no knowledge about the functional form of $R(x)$ is required. This feature is not new in the analysis of the FTRL algorithm. However, thanks to the RUM-ODP model’s convex structure, we can use the information contained in the choice probability vector to perform our regret analysis. Thus, from a behavioral standpoint, the FTRL approach can be interpreted as an algorithm that allows learning in the case of perturbed random utility models (cf. Hofbauer and Sandholm [2002], Fudenberg et al. [2015], and Fosgerau et al. [2020]). More importantly,
our result shows that the FTRL algorithm has an economically grounded optimization interpretation.

We close this section by formalizing the equivalence between the FTRL and SSA approaches.

**Theorem 5.** Let Assumptions 1 and 2 hold. Then the SSA and the FRTL algorithms are equivalent.

The proof of the previous theorem relies on Proposition 3. In simple, Theorem 5 establishes that the FTRL algorithm and the SSA approach are dual to each other. From a technical standpoint, the assumption that $\phi(\theta_t)$ has a Lipschitz continuous gradient is key in deriving this equivalence. As discussed in §3, a large class of GEV satisfies this condition. Thus, the FTRL algorithm is useful for studying no-regret learning algorithms in cases far beyond the MNL case.

### 4.3. A general recursive structure.

In the previous sections we have defined $R(x)$ as the convex conjugate of $\varphi(\theta)$. In particular, we discussed how specific assumptions on the distribution of $\epsilon_t$ lead to different functional forms for $R(x)$. This section aims to show how under Assumptions 1 and 2, a general recursive structure can is available for the choice probability vector $x_t$. The main appeal of this recursive structure is that it does not require knowledge of the functional form associated with $R(x)$.

We begin noticing that the choice probabilities are given by:

\begin{equation}
    x_{it+1} = \frac{H_i(e^{\theta_t/\eta})}{\sum_{j=1}^N H_j(e^{\theta_t/\eta})} \quad \text{for all } i \in A, t \geq 1,
\end{equation}

where the vector-valued function $H(\cdot) = (H_j(\cdot))_{j=1,...,N} : \mathbb{R}_+^n \mapsto \mathbb{R}_+^n$ is defined as the gradient of the exponentiated surplus, i.e.

\begin{equation}
    H(e^{\theta_t/\eta}) = \nabla (e^{\varphi(\theta_t)}).
\end{equation}

Two remarks are in order. First, the derivation in Eqs. (26)-(27) were proposed by Fosgerau et al. [2020] as a way to characterize the probability vector in the case of discrete choice models satisfying Assumptions 1. In particular, Eq. (26) is a straightforward application of their results, showing that the choice probabilities have a logit-like form. Second, from Fosgerau et al. [2020, Prop. 2] it follows that the vector-valued function $H(\cdot)$ is globally invertible. Exploiting this property, we can define $\Phi(\cdot) \equiv H^{-1}(\cdot)$. Combining the previous definition with Fosgerau et al. [2020, Prop. 3ii)], it follows that $R(x_t) = \eta \langle x_t, \log \Phi(x_t) \rangle$. Based on these observations, we establish the following result:
Proposition 4. Let Assumption 1 and 2 hold. Then in the FTRL algorithm we have:

\begin{equation}
\begin{aligned}
x_{it+1} &= H_i(e^{u_t/\eta + \alpha(x_t)}) \\
&= \frac{H_i(e^{u_t/\eta + \alpha(x_t)})}{\sum_{j=1}^{N} H_j(e^{u_t/\eta + \alpha(x_t)})} \quad \forall i \in A, t \geq 1
\end{aligned}
\end{equation}

with \(\alpha(x_t) \triangleq \log \Phi(x_t)\).

Some remarks are in order. First, Proposition 4 provides a recursive expression for the choice probabilities at each period. Intuitively, Eq. (28) shows that in the FTRL algorithm, the DM incorporates the past information to choose the vector \(x_{t+1}\) through the term \(\alpha(x_t)\). In particular, the term \(\alpha(x_t)\) is a weight that increases (decreases) the choice probability of those alternatives that have had better (worse) payoffs in the past. To see this, we note that for each alternative \(i \in A\), the associated payoff can be written as \(u_t + \alpha(x_t)\) for \(t \geq 1\). Thus, the term \(\alpha(x_t)\) contains the past information about the performance of the different alternatives. A second observation is related to the fact that Proposition 4 makes explicit that different assumptions on \(\epsilon_t\) will generate different choice probability vectors. In particular, the functional form of \(\Phi(x_t)\) is determined by the distribution of \(\epsilon_t\). The argument behind this fact comes from [Fosgerau et al., 2020, Prop. 2], which implies that \(-\log \Phi_j(x_t) = \varphi(\theta_t) - \theta_{jt}\) for all \(j \in A, t \geq 1\).

4.4. NL and a closed form expression for \(R(x)\). Formally, when implementing the FTRL algorithm, a fundamental question is how to choose the regularization \(R(x)\). In the previous section we defined \(R(x)\) as the convex conjugate of \(\varphi(\theta)\). However, there is no available closed form for this regularization term for general distributions of the random vector \(\epsilon_t\). This feature is common in the theory of online learning problems, where the Euclidean and entropic penalty terms are the two commonly known and widely used regularization functions. The former yields the popular Online Descent Gradient (ODG) algorithm, while the latter results in the widely used Exponential Weights Algorithm (EWA). Furthermore, [Hazan, 2017, p.101] notices the following: “There are surprisingly few cases of interest besides the Euclidean and Entropic regularizations and their matrix analogues”.

In this section, we show that a “new” regularization function is available in the case of the NL model. As we discussed in §3 for the NL case the generator function \(G\) is given by \(G(e^{\theta_t/\eta}) = \sum_{k=1}^{K} \left(\sum_{i \in N_k} e^{\theta_{it}/\eta \lambda_k}\right)^{\lambda_k}\) and \(\varphi(\theta_t) = \eta \log G(e^{\theta_t/\eta}) + \eta \gamma\). Exploiting this specific functional form, \(R(x)\) can be expressed in closed form.
Lemma 3. In the NL model the following hold:

\[(29) \quad R(x_t) = \eta \sum_{k=1}^{K} \sum_{i \in \mathcal{N}_k} \lambda_k x_{it} \log x_{it} + \eta \sum_{k=1}^{K} (1 - \lambda_k) \left( \sum_{i \in \mathcal{N}_k} x_{it} \right) \log \left( \sum_{i \in \mathcal{N}_k} x_{it} \right) \quad \text{for} \quad t = 1, \ldots, T.\]

The previous lemma provides a simple expression for \(R(x)\), which is a generalization of the Shannon entropic term. To see this, we note that in Eq. (29) the first term captures the Shannon entropy within nests, whereas the second term captures the information between nests. Accordingly, \(R(x)\) can interpreted as an augmented (or generalized) version of Shannon entropy. In the context rational inattention models, the function (29) was introduced by [Fosgerau et al. 2020]. Our proof is a simple adaptation of their arguments. However, to our knowledge, regularization (29) is new to the FTRL algorithm literature.

Based on Lemma 3 in the following proposition, we show that in the case of the NL model, the choice probability vector \(x_t\) has a recursive structure.

Proposition 5. Supposed that the FTRL algorithm is implemented with the NL model. Then the choice probability vector \(x_t\) satisfies:

\[(30) \quad x_{it+1} = \left( \Phi_i(x_t) \right)_{it} \frac{\mathbb{P}_{ikt} \mathbb{P}_{kt}}{\mathbb{P}_{kt}} \quad \forall i, k, t,\]

where:

\[\Phi_i(x_t) \triangleq x_{it}^{\lambda_k} \left( \sum_{j \in \mathcal{N}_k} x_{jt} \right)^{1 - \lambda_k},\]

\[\mathbb{P}_{ikt} \triangleq \frac{e^{u_{it}/\eta \lambda_k}}{\sum_{j \in \mathcal{N}_k} \Phi_j(x_t)e^{u_{jt}/\eta \lambda_k}},\]

and

\[\mathbb{P}_{kt} \triangleq \frac{\left( \sum_{j \in \mathcal{N}_k} \Phi_j(x_t)e^{u_{jt}/\eta \lambda_k} \right)^{\lambda_k}}{\sum_{l=1}^{K} \left( \sum_{j' \in \mathcal{N}_l} \Phi_{j'}(x_t)e^{u_{jt'}/\eta \lambda_l} \right)^{\lambda_l}}.\]

Formally, this result allows one to understand how \(x_t\) evolves as the DM learns about past realizations of the payoff vector \(u_t\).

More importantly, Proposition 5 makes explicit the recursive structure of the FTRL when we use the NL model. In particular, our result is a generalization of the EWA approach. When the values of the nesting parameters converge to one, expression (30) boils down to the recursive MNL model. The following corollary formalizes this observation.
Corollary 2. In Proposition \( \Box \) let \( \lambda_k = 1 \) for all \( k \). Then

\[
x_{it+1} = \frac{x_{it} e^{u_{it}/\eta}}{\sum_{j=1}^{N} x_{jt} e^{u_{jt}/\eta}} \quad \forall i, t.
\]

4.5. Recency bias. So far in our regret analysis, we have assumed that the DM weights all past observations similarly. However, there is plenty of evidence that in repeated choice problems, a DM reacts more heavily to recent observations than she does to old ones. In the learning literature this phenomenon is referred as recency bias (e.g. Erev and Haruvy [2016], Fudenberg and Peysakhovich [2014], and Fudenberg and Levine [2014]).

In this section, we show how to incorporate the recency bias effect into the RUM-ODP model in a simple way. Following Rakhlin and Sridharan [2013] we introduce a sequence of functions \( \beta_t : \mathcal{U}^{t-1} \mapsto \mathcal{U} \) for each \( t = 1, \ldots, T \), which define a predictable sequence

\[
\beta_1(0), \beta_2(u_1), \ldots, \beta_T(u_1, \ldots, u_{T-1}).
\]

Intuitively, the sequence (32) can be seen as a way of incorporating prior knowledge about the sequence \( u_1, \ldots, u_T \). In particular, the sequence of functions (32) allows us to model the magnitude and effect of recency bias by specifying different summary statistics. Accordingly, we introduce the recency bias by incorporating the sequence (32) to modify the FTRL algorithm as follows \( \Box \)

\[
x_{t+1} = \begin{cases} 
\arg \max_{x \in \Delta_N} \{-R(x)\} & \text{for } t = 0 \\
\arg \max_{x \in \Delta_N} \{\langle \theta_t + \beta_t, x \rangle - R(x)\} & \text{for } t = 1, \ldots, T.
\end{cases}
\]

Expression (33) makes explicit the fact that by adding the term \( \beta_t \), we obtain a variant of the FTRL algorithm that incorporates recency bias. Following Rakhlin and Sridharan [2013], we denote the resulting algorithm as Optimistic FTRL (OFTRL).

Following Rakhlin and Sridharan [2013] and Syrgkanis et al. [2015], we define three types of recency biases we use in our regret analysis.

Definition 4. In the OFTRL algorithm we say that the DM exhibits:

a) One step recency bias if \( \beta_t = u_{t-1} \) for \( t = 1, \ldots, T \).

b) S-step recency bias if \( \beta_t = \frac{1}{S} \sum_{\tau=t-S}^{t-1} u_{\tau} \) for \( t = 1, \ldots, T \).

c) Geometrically discounted recency bias if \( \beta_t = \frac{1}{\sum_{\tau=0}^{t-1} \delta^{t-\tau}} \sum_{\tau=0}^{t-1} \delta^{\tau} u_{\tau} \)

where \( \delta \in (0, 1) \) is a discount factor for \( t = 1, \ldots, T \).

In the previous definition, each functional form for \( \beta_t \) captures different ways of using the information in more recent observations. As the following proposition shows, under these three types of recency bias, the OFTRL algorithm is Hannan consistent.

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16For ease of notation, we denote \( \beta_t(u_1, \ldots, u_{t-1}) \triangleq \beta_t \) for \( t = 1, \ldots, T \).
**Proposition 6.** Let Assumptions 1 and 2 hold. In addition, assume that the following condition holds

\begin{equation}
\|u_t - u_{t-1}\|_* \leq B \text{ for } t = 1, \ldots, T.
\end{equation}

Then the following statements hold:

i) In the one-step recency bias with \(\eta = \sqrt{LT^2 \beta^2} \frac{2\phi(0)}{\beta} \) the OFTRL algorithm is Hannan consistent with:

\[ R_T^{OFTRL} \leq B \sqrt{2LT \phi(0)}. \]

ii) In the S-step recency bias with \(\eta = \sqrt{LT^2 \beta^2} \frac{2\phi(0)}{\beta} \) the OFTRL algorithm is Hannan consistent with:

\[ R_T^{OFTRL} \leq SB \sqrt{2LT \phi(0)}. \]

iii) In the geometrically discounted recency bias with \(\eta = \sqrt{LT^2 \beta^2} \frac{2\phi(0)}{\beta(1 - \delta)} \) the OFTRL algorithm is Hannan consistent with:

\[ R_T^{OFTRL} \leq B \sqrt{2LT \phi(0)} \left(\frac{1}{1 - \delta}\right)^3. \]

To understand the intuition behind Proposition 6, we note that Lemma 11 in Appendix A establishes that the regret associated with the OFTRL algorithm satisfies the following bound:

\begin{equation}
R_T^{OFTRL} \leq \eta \phi(0) + \frac{L}{2\eta} \sum_{t=1}^{T} \|u_t - \beta_t\|^2.
\end{equation}

Intuitively, Eq. (35) establishes that when the sequence \((\beta_t)_{t \geq 1}\) predicts \(u_t\) well then the RUM-ODP model can achieve low regret. Thus, when the DM exhibits recency bias, she can achieve low regret by implementing the OFTRL algorithm.

Our second observation is related to the technical details in proving Proposition 6. Formally, our proof is based in adapting the arguments in Rakhlin and Sridharan [2013, Lemma 2] and Syrgkanis et al. [2015, Lemmas 21 and 22]. However, a fundamental difference between Proposition 6 and their results is our use of the convex structure of the RUM. Thus, Proposition 6 provides an economic justification for the OFTRL algorithm.

Finally, we remark that by using the equivalence in Theorem 5, we can derive an SSA version incorporating the recency bias effect. In doing so, we note that under recency bias, the social surplus function is

\[ \varphi(\theta_t + \beta_{t+1}) = \mathbb{E} \left( \max_{i \in A} \{\theta_{it} + \beta_{it+1} + \epsilon_{it+1}\} \right). \]
Then, using Theorem\footnote{In this section, we closely follow the exposition in Syrgkanis et al.\cite{2015}.} we know that
\[ x_{t+1} = \nabla \varphi(\theta_t + \beta_{t+1}) = \arg \max_{x \in \Delta_N} \{ \langle \theta_t + \beta_{t+1}, x \rangle - R(x) \}. \]

Thus, we can naturally define an optimistic SSA which connects the notion of recency bias, the social surplus function, and the class of RUMs.

5. NO-REGRET LEARNING IN GAMES

In this section, we apply the RUM-ODP model to the study of no-regret learning in games. We consider a static game \( \mathcal{G} \) among a set \( \mathcal{P} \) of \( P \) players\footnote{\cite{2015}}. Each player \( j \) has a strategy space \( S_j \) and a utility function \( u_j : S_1 \times \ldots \times S_P \rightarrow [0, 1] \) that maps a strategy profile \( s = (s_1, \ldots, s_P) \) to a utility \( u_j(s) \). We assume that the strategy space of each player is finite and has cardinality \( N_j \) \( \in \mathbb{N} \) \( j \in \mathcal{P} \). The set of profile of mixed strategies is denoted as \( \Delta^N = \prod_{j \in \mathcal{P}} \Delta_{N_j} \). The set of profile of mixed strategies is denoted as \( \Delta^N = \prod_{j \in \mathcal{P}} \Delta_{N_j} \). Finally let \( U_j(x) = \mathbb{E}_{s \sim \pi} [u_j(s)] \), the expected utility of player \( j \).

We consider a situation where the game \( \mathcal{G} \) is played repeatedly for \( T \) time steps. We denote this repeated as \( \mathcal{G}^T \). At each time step \( t \) each player \( j \) chooses a mixed strategy \( x^j_t \in \Delta_{N_j} \). At the end of the iteration, each player \( j \) observes the expected utility he would have received had he played any possible strategy \( k \in S_j \). More formally, let \( u^j_{tk} = \mathbb{E}_{s \sim \pi_j} [u_j(k, s_{-j})] \), where \( s_{-j} \) is the set of strategies of all but the \( j^{th} \) player, and let \( u^j_t = (u^j_{tk})_{k \in S_j} \). At the end of each iteration, each player \( j \) observes \( u^j_t \). It follows that the expected utility of a player at iteration \( t \) is given by the inner product \( \langle x^j_t, u^j_t \rangle \).

To model no-regret learning, we assume that each player decides her mixed strategy \( x^{j+1}_t \) using the SSA. In doing so, we define player \( j \)'s social surplus function as \( \varphi_j(\theta^j_t) \triangleq \mathbb{E}(\max_{k \in S_j} \{ \theta^j_{tk} + \eta_j \epsilon^j_{tk} \}) \), where the expectation is taken with respect to \( \epsilon^j_{tk} \) and \( \eta_j > 0 \) is the player-specific learning parameter.

In this strategic setting, the SSA is defined as \( x^0_j = \nabla \varphi_j(0) \) and for \( t = 1, \ldots, T \) we get:
\[ x^{j+1}_t = \nabla \varphi_j(\theta^j_t) \quad \forall j \in \mathcal{P}, \]
where \( \theta^j_t \triangleq \sum_{i=1}^t u^j_i \).

In the repeated game \( \mathcal{G}^T \), the regret after \( T \) periods is equal to the maximum gain that player \( j \in \mathcal{P} \) could have achieved by switching to any other fixed strategy in hindsight:
\[ R^T_{SSA_j} \triangleq \max_{x^j_t \in \Delta_{N_j}} \sum_{t=1}^T \langle x^j_t - x^j_t, u^j_t \rangle. \]

\footnote{We note that in this strategic environment, \( \Delta(S_j) \triangleq \Delta_{N_j} \).}
It is straightforward to show that under Assumptions 1 and 2, we can apply Theorem 1 to bound $R_{SSA}^T$. In particular, in the repeated game $G^T$, setting $\eta_j = \sqrt{\frac{L_j^T}{2\varphi_j(0)}}$, we obtain:

$$R_{SSA}^T \leq \sqrt{2\varphi_j(0)L_j^T} \text{ for all } j \in \mathcal{P}.$$  

The main implication of using the RUM-ODP model combined with the SSA is that we can bound the regret associated with each player using a large class of discrete choice models. For instance, we can consider cases where some players may use an MNL model, and others can use GNL. In general, our approach is flexible enough to accommodate players using different discrete choice models to compute $\nabla \varphi_j(\theta_j^t) = x_j^{t+1}$ for each player $j \in \mathcal{P}$.

It is worth mentioning that a significant advantage of using the SSA to study no-regret learning in normal form games is that we do not need to specify the regularization term $R_j(x_j^t)$. This feature is different from most of the literature on no-regret dynamics, which focuses on the idea of regularized learning. In particular, our SSA approach applies even in situations where $R_j(x_j^t)$ may not have a closed form expression. In addition, we mention that from using the results in §4.5, we can combine the SSA with the notion of recency bias. Thus, our analysis can accommodate this type of learning behavior.

5.1. Coarse Correlated Equilibrium. An important implication of using the RUM-ODP model to study no-regret learning in games is that we can expand the class of choice models that allow us to approximate coarse correlated equilibrium (CCE). Formally, a CCE is defined as follows:

**Definition 5** (Coarse Correlated Equilibrium (CCE)). A distribution $\sigma$ on the set $S_1 \times \cdots \times S_P$ of outcomes of the game $G$ is a coarse correlated equilibrium (CCE) if for every agent $i \in \mathcal{P} = \{1, 2, \ldots, P\}$ and every unilateral deviation $s_j' \in S_j$,

$$E_{\sigma \sim \sigma} [u_j(s)] \geq E_{\sigma \sim \sigma} [u_j(s_j', s_{-j})]. \quad (36)$$

The condition (36) is the same as that for a mixed strategy Nash equilibrium, except without the restriction that $\sigma$ is a product distribution. Intuitively, this condition applies to a situation where an agent $i$ contemplating a deviation $s_i'$ knows only the distribution $\sigma$ and not the component $s_i$ of the realization. In other words, a CCE only protects against unconditional unilateral deviations, as opposed to the unilateral deviations conditioned on $s_i$ that are addressed.

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19 We note that in setting $\eta_j = \sqrt{\frac{L_j^T}{2\varphi_j(0)}}$ we have used the fact that $u_{max} = 1$ for all player $j \in \mathcal{P}$. In addition, the parameter $L_j$ corresponds to the Lipschitz constant associated to player $j$’s social surplus function $\varphi_j$.

20 See, for instance, Syrgkanis et al. [2013] and Mertikopoulos and Sandholm [2016].
in the definition of a correlated equilibrium \cite{Aumann1974}. Furthermore, it follows that every CE is a CCE, so a CCE is guaranteed to exist and is computationally tractable.\footnote{For an in-depth discussion of this issue we refer the reader to Roughgarden \citeyear{Roughgarden2016}.} More importantly, it is well-known that no-regret dynamics converge to the set of coarse correlated equilibria \cite{Roughgarden2016,Prop. 17.9}. The following result adapts this observation to the case of the RUM-ODP model.

**Proposition 7.** Let Assumptions 1 and 2 hold. Suppose that at periods \(t = 1, \ldots, T\), players choose their strategy \(x^t_j\) according to the SSA. Let \(\sigma^t = \prod_{j=1}^P x^t_j\) denote the outcome distribution at iteration \(t\) and \(\sigma = \frac{1}{T} \sum_{t=1}^T \sigma^t\) the time-averaged history of these distributions. Then \(\sigma\) is an approximate coarse correlated equilibrium, in the sense that

\[
E_{s \sim \sigma} [u_j(s)] \geq E_{s \sim \sigma} [u_j(s'_j, s_{-j})] - \delta.
\]

for every agent \(i\) and unilateral deviation \(s'_i\) where \(\delta \triangleq \max_{j \in P} \{R^{T}_{\text{SSA},i}\}\).

Some remarks are in order. First, the result in Proposition 7 is well-known in the algorithmic game theory literature.\footnote{\cite{Roughgarden2016, pp. 240-241} provides a proof of this result. For completeness, we provide the proof of this result in Appendix A.} The main innovation on it is the definition of \(\delta\) in terms of the players’ regret bounds. This parameter incorporates the role of the RUM under consideration through the social surplus function \(\varphi\) and the Lipschitz constant \(L\). Thus, our version of this result provides a connection between no-regret dynamics, RUMs, and the set of coarse correlated equilibria of the game \(G\).

Second, we point out that given the flexibility of the SSA approach, a large class of RUMs is available to study no-regret dynamics and CCE points. This feature contrasts with most of the literature on no-regret learning, which primarily focuses on the FTRL using well-known regularization functions. In fact, to the best of our knowledge, most of the literature on no-regret learning has concentrated on the MNL model for which the term \(R_j(x^t_j)\) corresponds to the Shannon entropy. Thus, our result allows us to study no-regret learning and convergence to the set of CCE in normal form games considering a large class of RUMs.

As discussed in \cite{Roughgarden2016}, the social surplus function corresponds to a potential function. This feature implies that the SSA approach is related to the framework of potential-based dynamics developed in Cesa-Bianchi and Lugosi \citeyear{Cesa-Bianchi2003}, Hart and Mas-Colell \citeyear{Hart2001}, and Hart and Mas-Colell \citeyear{Hart2003}. However, we pointed out in \cite{Roughgarden2016} that our framework and theirs differ in at least three important aspects. First, Cesa-Bianchi and Lugosi \citeyear{Cesa-Bianchi2003}’s approach requires the existence of a potential function that satisfies such a requirement. Our use of the social surplus function \(\varphi\) (as a potential) does not require this condition.
As we discussed earlier, this is equivalent to assuming that the random preference shocks are i.i.d., which is very restrictive, ruling out a large class of RUMs. Second, the approach in Hart and Mas-Colell [2001] and Hart and Mas-Colell [2003] requires the existence of a potential function $P : \mathbb{R}^N \mapsto \mathbb{R}_+$ with the property that the gradient $\nabla P$ vanishes over the approachable set $\mathbb{R}_+^N$. In terms of the RUM-ODP model, this is equivalent to assuming that the random preference shock $\epsilon$ has bounded support. This condition rules out the whole family of RUMs discussed in this paper. Finally, a third difference is related to the economic interpretation of the potential function. As we showed in Eq. (9), the social surplus function $\varphi$ has a clear economic interpretation, which highlights the role of the cumulative payoff vector $\theta_j$ and the random preference shock $\epsilon$. Thus, while related, our approach is different from theirs.

5.2. Efficiency. An important application of no-regret learning is the possibility of analyzing the average welfare in a repeated game where players are no-regret learners. In order to formalize this, for a given strategy profile $s$ the social welfare is defined as the sum of the player utilities:

$$W(s) \triangleq \sum_{j \in P} u_i(s).$$

Similarly, given a mixed strategy profile $x$, we define

$$W(x) \triangleq \mathbb{E}_{s \sim x}[W(s)].$$

Our basic goal is to set a lower bound on how far the sequence’s average welfare concerns the static game’s optimal welfare $G$. Formally, we focus on the following measure:

$$\text{OPT} \triangleq \max_{s \in S_1 \times \ldots \times S_n} W(s).$$

From an economic standpoint, the optimal welfare OPT corresponds to a situation where players do not have incentives to be strategic and if a central planner could enforce (or dictate) each player’s strategy. Following the Algorithmic Game Theory literature, we define the class of smooth games [Roughgarden 2015].

**Definition 6** (Roughgarden 2015). A game is $(\lambda, \mu)$-smooth if there exists a strategy profile $s^*$ such that for any strategy profile $s$:

$$\sum_{j \in P} u_j(s_j^*, s_{-j}) \geq \lambda \text{OPT} - \mu W(s).$$

Intuitively, Definition 6 establishes that any player using her optimal strategy continues to do well irrespective of other players’ strategies. This condition implies near-optimality of no-regret dynamics when we apply the RUM-ODP model.
Proposition 8. Let Assumptions 1 and 2 hold. Then in a $(\lambda, \mu)$-smooth game, the following holds:

$$\frac{1}{T} \sum_{t=1}^{T} W(x^t) \geq \frac{\lambda}{1 + \mu} \text{OPT} - \frac{1}{1 + \mu} \frac{1}{T} \sum_{j \in P} R_{SSA_j}^T = \frac{1}{\rho} \text{OPT} - \frac{1}{1 + \mu} \frac{1}{T} \sum_{j \in P} R_{SSA_j}^T,$$

where the factor $\rho = (1 + \mu) / \lambda$ is called the price of anarchy (PoA).

The proof of Proposition 8 follows the arguments in Syrgkanis et al. [2015, Prop. 2] and Roughgarden [2015]. We contribute to this result by providing an explicit expression for the terms $R_{SSA_j}^T$. In particular, given that the PoA is driven by the quantity $\frac{1}{1 + \mu} \frac{1}{T} \sum_{j \in P} R_{SSA_j}^T$, we can connect the no-regret learning behavior with a large class of RUMs. In particular, given Assumptions 1 and 2, we know that $R_{SSA_j}^T = O(\sqrt{\varphi_j(0)} L_j T)$. From §2 we know that this explicit bound captures the structure of the RUM-ODP model through the parameters $\varphi_j(0)$ and $L_j$. Thus, our contribution is to provide economic content to the fact that by employing the SSA, the average welfare converges to the PoA.

6. Prediction markets and the RUM-ODP model

A prediction market is a future market in which prices aggregate information and predict future events (Hanson [2002]). The designer of these markets is called a market maker, and her goal is to incentivize accurate predictions of uncertain outcomes. In these markets, goods correspond to securities with payoffs contingent on uncertain outcomes. Applications of prediction markets include electoral markets, science and technology events, sports events, the success of movies, etc. (Wolfers and Zitzewitz [2004]).

Recently, Chen and Vaughan [2010] and Abernethy et al. [2014] have established the connection between prediction markets and online learning models. They show that a general class of cost-function-based prediction markets corresponds to applying the FTRL algorithm to these specific markets.

This section shows how the RUM-ODP and the SSA can be used to study prediction markets. In doing so, we exploit the mathematical structure of the social surplus function, which enables us to connect the SSA with a large class of prediction markets. In economic terms, our analysis establishes a formal relationship between machine learning, the RUM-ODP model, and prediction markets.

6.1. Cost-function based prediction markets. A popular approach in the study of prediction markets is the idea of cost-function-based prediction markets. In this setting, there is an agent denoted as the market maker, who

\textsuperscript{23} This section closely follows the discussion of cost-function-based markets described in Chen and Vaughan [2010].
trades a set of securities corresponding to each potential outcome of an event. Formally, let \( \Omega = \{1, \ldots, N\} \) be a set of mutually exclusive and exhaustive outcomes of a particular event (for instance, an electoral result or the winner in a sporting event). To price the securities associated with the uncertain events, the market maker uses a differentiable cost function \( C : \mathbb{R}^N \to \mathbb{R} \) to determine the prices. This cost function describes the amount of money currently wagered in the market as the number of shares purchased. Let \( q_i \) be the number of shares of security \( i \) currently held by traders for \( i = 1, \ldots, N \). Accordingly, let \( q = (q_1, \ldots, q_N) \) be the vector of shares. A trader would like to purchase \( r_i \) shares of each security paying \( C(q + r) - C(q) \) to the market maker, where \( r = (r_1, \ldots, r_N) \in \mathbb{R}^N \).

Given an infinitely small number of shares, the instantaneous price of security \( i \) is given by \( p_i(q) = \frac{\partial C(q)}{\partial q_i} \). In other words, the price per share of an infinitely small number of shares is given by the gradient of the cost function \( C \).

Chen and Vaughan [2010] define a cost function \( C \) to be valid if the associated prices satisfy the following two simple conditions:

1. For every \( i \in 1, \ldots, N \) and every \( q \in \mathbb{R}^N, p_i(q) \geq 0 \).
2. For every \( q \in \mathbb{R}^N, \sum_{i=1}^{N} p_i(q) = 1 \).

Condition 1 establishes that the price of a security is non-negative. In the case of a negative price \( p_i(q) \), a trader could purchase shares of this security at a guaranteed profit. Similarly, condition 2 establishes that the prices of all securities add up to 1. If it were the case that \( \sum_{i=1}^{N} p_i(q) < 1 \), then a trader could purchase small equal quantities of each security for a guaranteed profit. Combining these two requirements on prices, we ensure that there are no arbitrage opportunities within the market. More importantly, conditions 1 and 2 allow us to interpret the price vector as a valid probability distribution over the outcome space. In particular, these prices represent the market’s current estimate of the probability that outcome \( i \) will occur (Manski [2006]).

Chen and Vaughan [2010] provide necessary and sufficient conditions for the cost function \( C \) to be valid. Given the relevance to our analysis, we state their result for completeness.

**Theorem 6.** [Chen and Vaughan (2010)] A cost function \( C \) is valid if and only if it satisfies the following three properties:

1. **Differentiability:** The partial derivatives \( \frac{\partial C(q)}{\partial q_i} \) exist for all \( q \in \mathbb{R}^N \) and \( i \in \{1, \ldots, N\} \).

The share \( r_i \) could be zero or even negative, representing the sale of shares. Thus the vector \( r \) can be treated as an element of \( \mathbb{R}^N \).

In the case \( \sum_{i=1}^{N} p_i(q) > 1 \), then a trader could sell small equal quantities of each security for a guaranteed profit.
2. **Increasing Monotonicity:** For any \( q \) and \( q' \), if \( q \geq q' \), then 
\[ C(q) \geq C(q') \]

3. **Positive Translation Invariance:** For any \( q \) and any constant \( k \), 
\[ C(q + k) = C(q) + k \]

The previous theorem characterizes a valid cost function in terms of three natural conditions. Hanson [2003] and Chen and Pennock [2007] show that the cost function

\[
C(q) = b \log \left( \sum_{i=1}^{N} e^{q_i/b} \right) \quad b > 0,
\]

satisfies the requirements of Theorem 6. Moreover, taking the partial derivative with respect to \( q_i \), it is easy to see that the pricing function \( p_i(q) \) is given by:

\[
p_i(q) = \frac{e^{q_i/b}}{\sum_{j=1}^{N} e^{q_j/b}}, \quad \text{for } i = 1, \ldots, N.
\]

In the prediction markets literature, expressions (37) and (38) define a *Logarithmic Market Scoring Rule* (LMSR) which was introduced by Hanson [2003, 2002]. In particular, the prices in (38) allows us to capture a situation in which a trader who changes the market probabilities from \( r \) to \( r' \) obtains the same payoff for every outcome \( i \) as a trader who changes the quantity vectors from any \( q \) to \( q' \) such that \( p(q) = r \) and \( p(q') = r' \) in the cost function formulation (Chen and Vaughan [2010] and Abernethy et al. [2013]).

Based on our discussion in §3, it is easy to see that functions (37) and (38) can be seen as a particular application of the MNL to prediction markets. More formally, identifying \( \theta = q \) and \( \eta = b \), we can conclude that for the MNL model, the social surplus function \( \varphi(\theta) \) can be interpreted as a cost function. Strikingly, this relationship is far more general, as the next proposition shows.

**Proposition 9.** Let Assumption 7 hold. Then the social surplus function is a valid cost function.

This result directly implies that the RUM is useful for studying prediction markets. More importantly, Proposition 3 allows us to implement the SSA in the context of prediction markets. Chen and Vaughan [2010] and Abernethy et al. [2013] pointed out the connection between cost-function-based markets and online learning algorithms. They show that the FTRL algorithm is useful for constructing pricing mechanisms in a dynamic environment. Intuitively, this equivalence establishes that the DM uses the FTRL algorithm to select a probability distribution \( x \) while the market maker uses a *duality-based* cost function to compute the price vector \( p(q) \). Unfortunately, this connection relies on knowing the convex conjugate of the cost function \( C \).
By combining Theorem 5 with Proposition 9, we can connect the RUM-ODP and the SSA with prediction markets. In doing so, we identify outcomes in \( \Omega \) with alternatives in a discrete choice set \( A \) and trades made in the market with payoffs observed by the SSA. Thus, we can view the market maker as learning a probability distribution over outcomes by treating each observed trade \( r_t \) as a realization of the environment in the same fashion as the SSA allows the DM to learn a distribution \( x_{t+1} \) over the set \( A = \{1, \ldots, N\} \) using observed realizations of \( u_t \). Using this analogy, we can rewrite the SSA in terms of prediction markets:

**Algorithm 3 Prediction Market Algorithm**

1: Input: \( \eta > 0, F \) a distribution on \( \mathbb{R}^N \), and \( \Delta_N \).
2: Let \( q_0 \in \mathbb{R}^N \) and choose \( x_1 = \nabla \varphi(q_0) \)
3: for \( t = 1 \) to \( T \) do
   • The market maker sets prices \( x_t = \nabla \varphi(q_{t-1}) \)
   • The market maker receives security bundle purchase \( r_t \)
   • The market maker obtains the expected payoff \( \langle u_t, \nabla \varphi(q_{t-1}) \rangle \)
   • The market maker updates accordingly to \( q_t = r_t + q_{t-1} \) and chooses \( x_{t+1} = \nabla \varphi(q_t) \)
4: end for

Given our results in GEV models, it is easy to see that the previous algorithm opens the possibility of using several new cost and pricing functions in the context of cost-function-based prediction markets. Moreover, the regret analysis is similar to the arguments behind Theorem 1. We leave for future work a more profound analysis of the connection between prediction markets, the SSA, and the RUM-ODP model.

7. Related literature

Regret theory was introduced in a series of seminal papers by Bell [1982], Loomes and Sugden [1982, 1987] and Fishburn [1982] as an alternative to the expected utility paradigm. In simple terms, regret theory establishes that a DM wants to avoid outcomes in which she will appear to have made the wrong decision, even if, in advance, the decision appeared correct with the information available at the time. In particular, regret theory entails the possibility of non-transitive pairwise choices\(^{26}\).

Recently, Sarver [2008] and Hayashi [2008] provide an axiomatic foundation for regret preferences. Our paper contributes to this literature by studying algorithmically the no-regret concept in the context of the RUM-ODP model. In this sense, our regret analysis is closer to the one in the algorithmic game theory literature (Roughgarden [2016]).

\(^{26}\)For a complete discussion of regret theory and its contributions, we refer the reader to Bleichrodt et al. [2015].
Our paper mainly relates to the literature on no-regret dynamics in repeated games. As we mentioned earlier, Hannan [1957]’s seminal work introduces the idea of consistency as a benchmark when considering a sequence of repeated play. The papers by Littlestone and Warmuth [1994], Fudenberg and Levine [1995], Freund and Schapire [1997], Freund and Schapire [1999], Blum and Mansour [2007], Foster and Vohra [1997], and Hart and Mas-Colell [2001], among many others, extend Hannan’s analysis to different strategic environments. Recently, Syrgkanis et al. [2015] studied the fast convergence of online learning in the context of regularized games using the FTRL algorithm. Our work differs from these papers in at least two aspects. First, we introduce the SSA, which allows us to study the RUM-ODP model, exploiting the theory of discrete choice models. In particular, we show how the RUM-ODP provides closed-form expressions for several discrete choice models. Second, we provide a regret analysis and a generalization of EWA not covered by the papers cited above.

The papers by Cesa-Bianchi and Lugosi [2003], Hart and Mas-Colell [2001], and Hart and Mas-Colell [2003] study no-regret learning in normal form games exploiting the notion of potential functions. In particular, these papers introduce the notion of potential-based learning. As discussed in the main text, our SSA approach is an instance of a potential-based algorithm. However, our framework differs from this line of work in at least three aspects. First, we do not impose the additivity condition used by Cesa-Bianchi and Lugosi [2003]. Second, we do not impose any condition in the domain of the gradient social surplus. Hart and Mas-Colell [2001] and Hart and Mas-Colell [2003] impose the condition that over the approachable set the gradient of their potential function vanishes. This condition is incompatible with the class of RUMs, implying that their results do not apply to the RUM-ODP model.

As we mentioned in §1, the RUM-ODP model is an instance of a two-person game between the DM and the environment. In this sense, our paper is related to the recent work by Gualdani and Sinha [2020]. They study a discrete choice model in which the DM possesses imperfect information about the utility generated by the available options. They model this problem as an incomplete information game between the DM and the environment, exploiting the notion of Bayes Correlated Equilibrium (Bergemann and Morris [2016]). Our paper differs from theirs in at least three crucial aspects. First, Gualdani and Sinha [2020] analyze a static incomplete information game while we study a repeated choice situation. Second, we focus on understanding under which conditions the RUM-ODP model achieves Hannan consistency, while Gualdani and Sinha [2020]’s goal is the econometric identification of the DM’s preferences. Third,

\footnote{For an in-depth analysis of no-regret learning in games, we refer the reader to Cesa-Bianchi and Lugosi [2006].}
our analysis focuses on discrete choice models. Lomys et al. [2021] study no-regret learning in the context of Bayes-coarse correlated equilibrium. Their main goal is the identification and econometric estimation of the structural parameters describing the underlying game. They do not study the RUM.

Our paper naturally connects with the OCO literature. From this literature, the work by Abernethy et al. [2017] is the closest to our paper. They connect the MNL model with the FTRL approach. Our paper differs from theirs in at least three fundamental aspects. First, we show that the entire class of discrete choice models naturally defines a regularization term to implement the FTRL algorithm. Second, we identify a Lipschitz condition on the gradient of the social surplus function, which allows one to characterize the class of discrete choice models that are Hannan consistent. Third, we generalize in a non-trivial way the EWA algorithm. Concretely, we introduce the NL model providing a new closed-form regularization penalty term.

Our paper is also related to the active and increasing literature on stochastic choice and information frictions. The papers by Caplin and Dean [2015], Matějka and McKay [2015], Caplin and Martin [2015], Caplin et al. [2018], and Fosgerau et al. [2020] study (static) stochastic choice and information acquisition under the Rational Inattention (RI) framework. These papers focus on understanding the relationship between how different cost functions determine stochastic choice behavior. Natenzon [2019] develops a Bayesian Probit approach where the DM observes a noisy signal of the utility associated with each alternative in the choice set. A shared feature of these papers is the assumption that the DM has a prior over the set of possible payoff realizations. In a different framework, Lu [2016] studies a RUM in which the DM has private information before deciding. He provides results where observed stochastic choice behavior is useful to recover private information. Our paper differs from this line of work in at least three aspects. First, our approach neither specifies priors over the payoff realizations nor posterior beliefs. Second, we consider a repeated stochastic choice situation, while these papers focus on a static environment. Third, our analysis focuses on Hannan consistency as a performance benchmark, while the above-cited papers focus on the utility maximization paradigm.

The recent contributions by Webb [2019] and Cerreia-Vioglio et al. [2021] are the closest papers to our work. The paper by Webb derives the RUM using a general class of bounded accumulation models. This connection allows him to characterize the resulting distribution of the stochastic component in a RUM based on response times. The paper by Cerreia-Vioglio et al. [2021] provides an axiomatic characterization of the MNL in which time-constrained information processing causes stochastic choice behavior. In addition, they

28For an excellent treatment of the OCO problem, we refer the reader to Shalev-Shwartz [2012] and Hazan [2017].
propose a neural approach that provides a causal analysis of the decision maker choices through a biologically inspired algorithmic decision process. Our paper differs from [Webb 2019] and [Cerreia-Vioglio et al. 2021] in at least three important aspects. First, in the RUM-ODP model, the DM learns through repeated choice, while in [Webb 2019] and [Cerreia-Vioglio et al. 2021] time is used to accumulate evidence before making a choice. Second, we focus on the notion of regret while they study random utility maximization. Third, our approach is algorithmic.

Our paper is also related to the literature on stochastic choice and perturbed utility models. In particular, our paper is related to the work by [Fudenberg et al. 2015]. They show that stochastic choice corresponds to the optimal solution of maximizing the sum of expected utility and a nonlinear perturbation. This latter term is a regularization function in the language of the FTRL algorithm. While related, the work by [Fudenberg et al. 2015] studies perturbed utility from an axiomatic standpoint without considering learning.

Finally, from a technical point of view, our paper is related to the recent contribution by [Muller et al. 2022]. In particular, our results exploit their connection between the social surplus function and the concept of proxy functions. However, [Muller et al. 2022] do not study the problem of no-regret learning.

8. Final Remarks

This paper proposes the RUM-ODP model to study no-regret learning in uncertain environments. Our approach is algorithmic, providing a connection between the theory of RUMs and the gradient(potential)-based learning dynamics. In particular, we introduced the SSA framework, which allows us to apply a large class of discrete choice models to analyze online decision-making and no-regret learning problems. In addition, we showed that the popular FTRL algorithm has a clear and meaningful economic interpretation. Exploiting this fact, we establish a recursive structure to the choice probability vector generated by the FTRL algorithm. This latter fact generalizes in a non-trivial way the exponential weights algorithm to discrete choice models far beyond the MNL case. In terms of applications, we use our framework to study no-regret learning in normal form games and implement prediction markets.

Finally, we mention that several extensions are possible. First, we relax the complete information assumption in ongoing work by studying the RUM-ODP model using a bandit approach [Lattimore and Szepesvári 2020]. Second, to provide a neurophysiological foundation to the MNL model, [Cerreia-Vioglio et al. 2021] propose the Metropolis-DDM algorithm to model how the DM acquires information. Our algorithm uses ideas from the machine learning and OCO literature.
given the structure of the RUM-ODP model, an important implication of the results derived in this paper is the possibility of studying the econometrics of no-regret learning in discrete choice models.

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Appendix A. Proofs

We begin stating two technical results that will be used throughout this appendix.

**Lemma 4** (Baillon-Haddad Theorem). The following statements are equivalent

i) \( h : E \to \mathbb{R} \) is convex and differentiable with gradient \( \nabla h \) which is Lipschitz continuous with respect to \( \| \cdot \|_E \) with constant \( L > 0 \).

ii) The convex conjugate \( h^* : E^* \to (-\infty, \infty] \) is \( \frac{1}{L} \)-strongly convex with respect to the dual norm \( \| \cdot \|_{E^*} \).

**Proof.** Rockafellar and Wets [1997, Thm. 12, Section H]. \( \square \)

The next lemma establishes the differentiability of \( \mathcal{R}(\mathbf{x}) \).

**Lemma 5.** Let Assumption 1 hold. Then \( \mathcal{R} \) is differentiable.

**Proof.** The proof follows from a direct application of Sorensen and Fosgerau [2021, Thm. 5] or Galichon and Salanie [2021, Thm. 2]. \( \square \)

A.1. **Proof of Proposition 1**. Note that by definition \( \varphi(\theta_t) = \mathbb{E}(\tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})) \). Then combining (5) with Bertsekas [1973, Prop. 2.3] it follows that

\[
\begin{align*}
\tilde{x}_{t+1} & \in \partial \tilde{\varphi}(\theta_t + \eta \epsilon_{t+1}), \\
\mathbb{E}(\tilde{x}_{t+1}) & \in \mathbb{E}(\partial \tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})), \\
& = \partial \mathbb{E}(\tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})), \\
& = \nabla \mathbb{E}(\tilde{\varphi}(\theta_t + \eta \epsilon_{t+1})) = \nabla \varphi(\theta_t).
\end{align*}
\]

\( \square \)

A.2. **Proof of Lemma 1**. In proving this lemma we use the fact that

\[
\varphi(\theta) = \mathbb{E}\left( \max_{j=1,\ldots,N} \{ \theta_j + \eta \epsilon_j \} \right) = \eta \mathbb{E}\left( \max_{j=1,\ldots,N} \{ \theta_j / \eta + \epsilon_j \} \right) = \eta \varphi(\theta / \eta).
\]

Thus, it is easy to see that \( \nabla \varphi(\theta) = \nabla \varphi(\theta / \eta) \). In addition, simple algebra shows that \( \frac{\partial^2 \varphi(\theta)}{\partial \theta_i \partial \theta_j} = \frac{1}{\eta} \frac{\partial^2 \varphi(\theta / \eta)}{\partial \theta_i \partial \theta_j} \), for all \( i, j = 1, \ldots, N \). Under this equivalence we note that the condition in Assumption 2 can be rewritten as \( 2Tr(\varphi(\theta / \eta)) \leq L \).

---

30 We remark that in this theorem \( E^* \) denotes the dual space of \( E \) and \( \| \cdot \|_{E^*} \) denotes its corresponding dual norm.
Define the function \( f(t) = \nabla \varphi(\theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta)) \) with \( f'(t) = \langle \nabla^2 \varphi(\theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta)), \theta_2/\eta - \theta_1/\eta \rangle \). Noticing that

\[
\nabla \varphi(\theta_2/\eta) - \nabla \varphi(\theta_1/\eta) = f(1) - f(0) = \int_0^1 f'(t)dt = \int_0^1 \nabla^2 \varphi(\theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta))(\theta_2/\eta - \theta_1/\eta)dt.
\]

\[
\|\nabla \varphi(\theta_2/\eta) - \nabla \varphi(\theta_1/\eta)\|_1 \leq \int_0^1 \|\nabla^2 \varphi(\theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta))(\theta_2/\eta - \theta_1/\eta)\|_1 dt \leq \int_0^1 \|\nabla^2 \varphi(\theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta))\|_1 \|\theta_2/\eta - \theta_1/\eta\|_1
\]

To complete the proof we stress two properties of the Hessian. First, each row (or columns) of \( \nabla^2 \varphi(\theta_i/\eta) \) sums up to 0. To see this we note that \( \sum_{i=1}^N \nabla_i \varphi(\theta_i/\eta) = 1 \). Then simple differentiation yields \( \sum_{i=1}^N \nabla_i \varphi(\theta_i/\eta) = 0 \) for all \( i = 1, \ldots, N \). Second, it is well known that the off-diagonal elements of \( \frac{1}{\eta} \nabla^2 \varphi(\theta_i/\eta) \) are nonnegative [McFadden 1981, Ch. 5)]. To see why this is true, we recall that for alternative \( i \) the choice probability is given by:

\[
\nabla_i \varphi(\theta_i/\eta) = \Pr(i = \arg \max_{j \in A} \{ \theta_{ij} + \eta \epsilon_{ij} \}).
\]

Then increasing the terms \( \theta_{ij} \) for \( j \neq i \) cannot increase the probability of choosing \( i \), which is formalized as \( \frac{1}{\eta} \nabla_{ij} \varphi(\theta_i/\eta) \leq 0 \).

Now, using previous observation, we have that for a convex combination \( \bar{\theta}/\eta = \theta_1/\eta + t(\theta_2/\eta - \theta_1/\eta) \) we have

\[
\frac{1}{\eta} \|\nabla^2 \varphi(\bar{\theta}/\eta)\|_{\infty, 1} = \frac{1}{\eta} \max_{\|v\| \leq 1} \{ \|\nabla^2 \varphi(\bar{\theta}/\eta)v\|_1 \} \\
\leq \frac{1}{\eta} \sum_{i=1}^N \sum_{j=1}^N |\nabla_{ij} \varphi(\bar{\theta})| \\
= \frac{1}{\eta} 2Tr(\nabla^2 \varphi(\bar{\theta})) \leq \frac{L}{\eta},
\]

where the last inequality follows from Assumption 2. Plugging in, we arrive to the conclusion

\[
\|\nabla \varphi(\theta_2/\eta) - \nabla \varphi(\theta_1/\eta)\|_1 \leq L\|\theta_2/\eta - \theta_1/\eta\|_1 \forall \theta_1, \theta_2.
\]

Finally, using the fact \( \nabla \varphi(\theta) = \nabla \varphi(\theta/\eta) \) we get

\[
\|\nabla \varphi(\theta_2) - \nabla \varphi(\theta_1)\|_1 \leq \frac{L}{\eta} \|\theta_2 - \theta_1\|_1 \forall \theta_1, \theta_2.
\]

Lemma 6. Let Assumption 7 hold. Then \( R(x) \leq 0 \) for all \( x \in \Delta_N \).
Proof. First, we note that the Fenchel equality implies $R(x) = \langle \theta, x \rangle - \varphi(\theta) = \langle \theta, x \rangle - \eta \varphi(\theta/\eta)$ with $x = \nabla \varphi(\theta) = \nabla \varphi(\theta/\eta)$. We recall that $\varphi(\theta) = \mathbb{E}(\max_{j=1,\ldots,n}\{\theta_j + \eta \epsilon_j\})$. Given that $\max\{\cdot\}$ is a convex function, by Jensen’s inequality we get $\max_{j=1,\ldots,n}\mathbb{E}(\theta_j + \eta \epsilon_j) \leq \mathbb{E}(\max_{j=1,\ldots,n}\{\theta_j + \eta \epsilon_j\})$. Then it follows that

$$R(x) = \langle \theta, x \rangle - \varphi(\theta),$$

$$\leq \langle \theta, x \rangle - \max_{j=1,\ldots,n} \mathbb{E}(\theta_j + \eta \epsilon_j),$$

$$= \langle \theta, x \rangle - \max_{j=1,\ldots,n} \theta_j,$$

$$R(x) \leq 0$$

where the last inequality follows from the fact that $x \in \Delta_N$. \hfill $\square$

Lemma 7. Let Assumptions 1 and 2 hold. Then

$$D_{\varphi}(\theta_{t+1}||\theta_t) \leq \frac{L}{2\eta} u_{\text{max}}^2.$$ 

Proof. Using a second order Taylor expansion of $\varphi(\theta_t)$ we get:

$$\varphi(\theta_{t+1}) = \varphi(\theta_t) + \langle \nabla \varphi(\bar{\theta}), u_t \rangle + \frac{1}{2} \langle u_t, \nabla^2 \varphi(\bar{\theta}) u_t \rangle,$$

$$\varphi(\theta_{t+1}) - \varphi(\theta_t) - \langle \nabla \varphi(\bar{\theta}), u_t \rangle = \frac{1}{2} \langle u_t, \nabla^2 \varphi(\bar{\theta}) u_t \rangle,$$

(39) $$D_{\varphi}(\theta_{t+1}||\theta_t) = \frac{1}{2} \langle u_t, \nabla^2 \varphi(\bar{\theta}) u_t \rangle,$$

where $\bar{\theta}$ is some convex combination of $\theta_{t+1}$ and $\theta_t$. From Eq. (39) it follows that

(40) $$D_{\varphi}(\theta_{t+1}||\theta_t) \leq \frac{1}{2} \|\nabla^2 \varphi(\theta)\|_{\infty,1} \|u_t\|_{\infty}^2.$$ 

Noting that

$$\|\nabla^2 \varphi(\bar{\theta})\|_{\infty,1} = \max_{\|v\| \leq 1} \{\|\nabla^2 \varphi(\bar{\theta}) v\|_1\} \leq \sum_{i=1}^{N} \sum_{j=1}^{N} |\nabla^2_{ij} \varphi(\bar{\theta})| = 2Tr(\nabla^2 \varphi(\bar{\theta})) \leq \frac{L}{\eta},$$

where the last inequality follows from Assumption 2.

Plugging the previous bound in (40) combined with $\|u_t\|_{\infty}^2 \leq u_{\text{max}}^2$ we find that

$$D_{\varphi}(\theta_{t+1}||\theta_t) \leq \frac{L}{2\eta} u_{\text{max}}^2.$$ 

Lemma 8. Let Assumptions 1 and 2 hold. Then in the SSA

(41) $$\text{Regret}_T \leq \eta \varphi(0) + \frac{L}{2\eta} T u_{\text{max}}^2.$$
Proof. The proof of this lemma exploits the convex duality structure of the RUM-ODP model. By the Fenchel-Young inequality we know that

$$\forall x \in \Delta_N : R(x) \geq \langle \theta_T, x \rangle - \varphi(\theta_T),$$

where the equality holds when x maximizes $\langle \theta_T, x \rangle - R(x)$.

The Fenchel-Young inequality implies

$$R(x) - \langle \theta_T, x \rangle \geq -\varphi(\theta_T).$$

Noting that $-\varphi(\theta_T)$ can be equivalently expressed as:

$$-\varphi(\theta_T) = -\varphi(0) - \sum_{t=1}^{T} (\varphi(\theta_t) - \varphi(\theta_{t-1})).$$

From the definition of Bregman divergence combined with $x_t = \nabla \varphi(\theta_{t-1})$, it follows that

$$\sum_{t=1}^{T} (\varphi(\theta_t) - \varphi(\theta_{t-1})) = \sum_{t=1}^{T} (D_{\varphi}(\theta_t||\theta_{t-1}) - \langle u_t, x_t \rangle).$$

Thus, it follows that

$$R(x) - \langle \theta_T, x \rangle \geq -\eta \varphi(0) - \sum_{t=1}^{T} (D_{\varphi}(\theta_t||\theta_{t-1}) - \langle u_t, x_t \rangle).$$

Combining Lemmas 6 and 7, the previous inequality can be rewritten as

$$\sum_{t=1}^{T} \langle x - x_t, u_t \rangle \leq R(x) + \eta \varphi(0) + \sum_{t=1}^{T} D_{\varphi}(\theta_t||\theta_{t-1}) \leq \eta \varphi(0) + \frac{L}{2\eta} Tu_{\text{max}}^2.$$ 

Because the previous inequality holds for all $x \in \Delta_N$ we conclude:

$$\text{Regret}_T \leq \eta \varphi(0) + \frac{L}{2\eta} Tu_{\text{max}}^2.$$ 

□

A.3. Proof of Theorem 1. The bound (10) follows from Lemma 8. To derive Eq. (11), define the function $\psi(\eta) = \eta \varphi(0) + \frac{L}{2\eta} Tu_{\text{max}}^2$. Given the strict convexity of $\psi(\eta)$, the first order conditions are necessary and sufficient for a minimum. In particular, we get

$$\psi'(\eta) = \varphi(0) - \frac{L}{2\eta^2} Tu_{\text{max}}^2 = 0$$

The optimal $\eta$ is given by $\eta^* = \sqrt{\frac{LT u_{\text{max}}^2}{2\varphi(0)}}$. Then, it follows that $\psi(\eta^*) = u_{\text{max}} \sqrt{2\varphi(0)LT}$. Thus we conclude that

$$\text{Regret}_T \leq \psi(\eta^*) = 2u_{\text{max}} \sqrt{\varphi(0)LT}.$$
A.4. Proof of Lemma 2 This follows from a direct application of Muller et al. [2022, Thm. 3].

A.5. Proof of Theorem 3 Combining Lemma 2 with Lemma 8 we obtain the bound (19). Following the argument used in proving Theorem 1 combined with \( L = \frac{2M+1}{\eta} \), we obtain the optimized regret bound (20).

Lemma 9. In the GNL model the following statements hold:

i) The Social Surplus function has a Lipschitz continuous gradient with constant \( \left( \frac{2 \min_k \lambda_k}{\lambda_k - 1} \right) / \eta \).

ii) \( \log G(1) = \log \sum_{k=1}^{K} \left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda_k} \right)^{\lambda_k} \leq \log N \),

Proof. i) This follows from a direct application of Muller et al. [2022, Cor. 4].

ii) To prove this, we first show that for \( \lambda_k < \lambda'_{k'} \) we have:

\[
\left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda_k} \right)^{\lambda_k} \leq \left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda'_{k'}} \right)^{\lambda'_{k'}}
\]

for \( k = 1, \ldots, K \).

Let \( p_k = \frac{1}{\lambda_k} \) and \( p_{k'} = \frac{1}{\lambda'_{k'}} \), noting that \( p_k < p_{k'} \) whenever \( \lambda_k < \lambda'_{k'} \). Using this change of variable, we can write the following ratio

\[
\left( \sum_{i=1}^{N} \alpha_{ik}^{p_k} \right)^{1/p_k} \left( \sum_{j=1}^{N} \alpha_{jk}^{p_{k'}} \right)^{1/p_{k'}} = \left( \frac{\sum_{i=1}^{N} \alpha_{ik}^{p_k}}{\sum_{j=1}^{N} \alpha_{jk}^{p_{k'}}} \right)^{1/p_k / \lambda_k} \leq \left( \frac{\sum_{i=1}^{N} \alpha_{ik}^{p_k}}{\sum_{j=1}^{N} \alpha_{jk}^{p_{k'}}} \right)^{1/p_k} = 1.
\]

The last inequality implies that

\[
\left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda_k} \right)^{\lambda_k} \leq \left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda'_{k'}} \right)^{\lambda'_{k'}}
\]

for \( k = 1, \ldots, K \).

Then for \( \lambda_k = 1 \) for \( k = 1, \ldots, K \), we get

\[
\log \sum_{k=1}^{K} \left( \sum_{i=1}^{N} \alpha_{ik}^{1/\lambda_k} \right)^{\lambda_k} \leq \log \sum_{k=1}^{K} \sum_{i=1}^{N} \alpha_{ik} = \log N.
\]
A.6. **Proof of Proposition 2.** Combining Lemma 9 with Theorem 1, the conclusion follows at once. □

A.7. **Proof of Proposition 3.**

i) From the definition of $R(x)$ we know that $R(x) = \eta \varphi^*(x; \eta)$ where $\varphi^*(x; \eta)$ is the convex conjugate of the parametrized social surplus function $\varphi(\theta/\eta)$. By Lemma 1 it follows that $\varphi(\theta/\eta)$ is $L$-Lipschitz continuous. Applying Lemma 4 it follows that $\varphi^*(x; \eta)$ is $\frac{1}{L}$-strongly convex. Then it follows that $R(x)$ is $\eta L$-strongly convex.

ii) This is a direct implication of Lemma 5.

iii) It is easy to see that $\langle \theta, x \rangle - R(x)$ is a $\eta L$-strongly concave function on $\Delta_N$. This implies that the optimization problem $\max_{x \in \Delta_N} \{\langle \theta, x \rangle - R(x)\}$ must have a unique solution. Let $x^*$ be the unique optimal solution. Using the differentiability of $R(x)$ combined with the Fenchel equality it follows that $x^* = \nabla \varphi(\theta)$ iff $\nabla \varphi(\theta) = \arg \max_{x \in \Delta_N} \{\langle \theta, x \rangle - R(x)\}$. □

A.8. **Proof of Theorem 4.** From Proposition 3ii) we know $x_{t+1} = \nabla \varphi(\theta_t) = \arg \max_{x \in \Delta_N} \{\langle \theta_t, x \rangle - R(x)\}$. This fact implies that the FRTL algorithm is equivalent to the SSA. Then the argument used in proving Theorem 1 applies. Thus $R_{\text{FTRL}}^T$ is bounded by the same term that bounds $R_{\text{SSA}}^T$. Similarly, the same optimized bound achieved in $R_{\text{SSA}}^T$ applies to $R_{\text{FTRL}}^T$. □

A.9. **Proof of Theorem 5.** From Proposition 3 we know $\nabla \varphi(\theta_t) = x_{t+1} = \arg \max_{x \in \Delta_N} \{\langle \theta_t, x \rangle - R(x)\}$. Plugging in this observation in the FTRL algorithm the equivalence follows at once. □

A.10. **Proof of Proposition 4.** Let us focus at period $t + 1$. Accordingly, the associated Lagrangian is given by:

$$L(x_{t+1}; \lambda) = \sum_{i=1}^N \theta_i x_{it+1} - \eta \sum_{i=1}^N x_{it+1} \log \Phi_i(x_{t+1}) + \lambda \left( \sum_{i=1}^N x_{it+1} - 1 \right).$$

From [Fosgerau et al., 2020, Prop. A1ii)] we know that $\Phi(x_{t+1})$ is differentiable with

$$\sum_{j=1}^N x_{jt+1} \frac{\partial \Phi_j(x_{t+1})}{\partial x_{it+1}} = 1 \quad \forall i \in A.$$

Using this fact, the set of first order conditions can be written as:

$$\frac{\partial L(x_{t+1}; \lambda)}{\partial x_{t+1}} = \theta_t - \eta \log \Phi(x_{t+1}) - \eta + \lambda = 0.$$  \hspace{1cm} (42)

$$\frac{\partial L(x_{t+1}; \lambda)}{\partial \lambda} = \sum_{i=1}^n x_{it+1} - 1 = 0.$$  \hspace{1cm} (43)
Noting that $\theta_t = u_t + \theta_{t-1}$, Eq. (42) can be expressed as

$$e^{u_t/\eta + \theta_{t-1}/\eta} e^{\lambda/\eta-1} = \Phi(x_{t+1})$$

Recalling that $\Phi(\cdot) = H^{-1}(\cdot)$, from (44) we get:

$$H(e^{u_t/\eta + \theta_{t-1}/\eta} e^{\lambda/\eta-1}) = x_{t+1}.$$  

Noting that $H(\cdot)$ is homogeneous of degree 1, we get:

$$H(e^{u_t/\eta + \theta_{t-1}/\eta}) e^{\lambda/\eta-1} = x_{t+1}.$$  

Using (43) we find that

$$e^{\lambda/\eta-1} = \frac{1}{\sum_{j=1}^{N} H_j(e^{u_t/\eta + \theta_{t-1}/\eta})}.$$  

Then it is easy to see that

$$x_{it+1} = \frac{H_i(e^{u_t/\eta + \theta_{t-1}/\eta})}{\sum_{j=1}^{N} H_j(e^{u_t/\eta + \theta_{t-1}/\eta})} \text{ for all } i \in A, t \geq 1.$$  

From (45), it is easy to see that at period $t$ we must have:

$$x_{it} = \frac{H_i(e^{\theta_{t-1}/\eta})}{\sum_{j=1}^{N} H_j(e^{\theta_{t-1}/\eta})} \text{ for all } i \in A.$$  

Once again, using the fact that $\Phi(\cdot)$ is homogeneous of degree 1, in (46) we find:

$$\Phi(x_t) \sum_{j=1}^{N} H_j(e^{\theta_{t-1}/\eta}) = e^{\theta_{t-1}/\eta} \text{ for all } i \in A.$$  

Define $w_{t-1} \triangleq \log \left( \sum_{j=1}^{N} H_j(e^{\theta_{t-1}/\eta}) \right)$. Using this definition, combined with Eq. (45), we get:

$$x_{it+1} = \frac{H_i(\Phi(x_t)e^{u_t/\eta + w_{t-1}})}{\sum_{j=1}^{N} H_j(\Phi(x_t)e^{u_t/\eta + w_{t-1}})}.$$  

Finally, using the homogeneity of $H$, we conclude that

$$x_{it+1} = \frac{H_i(e^{u_t/\eta + \alpha(x_t)})}{\sum_{j=1}^{N} H_j(e^{u_t/\eta + \alpha(x_t)})}, \quad \forall i \in A, t \geq 1.$$  

□
A.11. **Proof of Lemma 3.** This follows from Fosgerau et al. [2020]. □

In proving the Proposition 5 we make use of the following technical lemma.

**Lemma 10.** Consider the NL model. Define the vector valued function \( \Phi : \Delta_N \rightarrow \mathbb{R}^N_+ \) where the \( i \)-th component is defined as:

\[
\Phi_i(x) = x_i^{\lambda_k} \left( \sum_{j \in N_k} x_j \right)^{1-\lambda_k} \quad \text{for all } i \in N_k, k = 1, \ldots, K,
\]

Then \( \Phi(x) \) is invertible and homogeneous of degree 1.

**Proof.** This follows from Fosgerau et al. [2020, Prop. 8]. □

A.12. **Proof of Proposition 5.** Noting that \( R(x_{t+1}) = \langle x_{t+1}, \log \Phi(x_{t+1}) \rangle \), we can write the associated Lagrangian as:

\[
L(x_{t+1}; \theta_t, \mu, \eta) = \langle \theta_t, x_{t+1} \rangle - \eta \langle x_{t+1}, \log \Phi(x_{t+1}) \rangle + \mu \left( \sum_{i=1}^n x_{it+1} - 1 \right).
\]

Given that \( R(x) \) is \( \frac{1}{L} \)-strongly convex, first order conditions are necessary and sufficient for the existence and uniqueness of a maximum. In particular, we find the maximizer \( x_{t+1} \) by solving:

\[
\frac{\partial L(x_{t+1}; \theta_t, \mu, \eta)}{\partial x_{t+1}} = \theta_t - \eta \log \Phi(x_{t+1}) - \eta + \mu = 0.
\]
\[
\frac{\partial L(x_{t+1}; \theta_t, \mu, \eta)}{\partial \lambda} = \sum_{i=1}^n x_{it+1} - 1 = 0.
\]

From the definition of the cumulative payoff vector, it follows that \( \theta_t = u_t + \theta_{t-1} \). Using this fact we get

\[
e^{u_t/\eta + \theta_{t-1}/\eta} e^{\mu/\eta - 1} = \Phi(x_{t+1}).
\]

Using Lemma 10 we obtain:

\[
H(e^{u_t/\eta + \theta_{t-1}/\eta} e^{\mu/\eta - 1}) = x_{t+1},
\]

where \( H(\cdot) \triangleq \Phi^{-1}(\cdot) \). Noting that \( H(\cdot) \) is homogeneous of degree 1, the previous expression can be rewritten as:

\[
H(e^{u_t/\eta + \theta_{t-1}/\eta}) e^{\mu/\eta - 1} = x_{t+1}.
\]

Now using the constraint \( \sum_{i=1}^N x_{it+1} = 1 \) we get

\[
e^{\mu/\eta - 1} = \frac{1}{\sum_{k=1}^K \sum_{j \in N_k} h_j(e^{u_t/\eta + \theta_{t-1}/\eta})}.
\]

Then we find that for all \( t \)

\[
x_{it+1} = \frac{H_j(e^{u_t/\eta + \theta_{t-1}/\eta})}{\sum_{k=1}^K \sum_{j \in N_k} h_j(e^{u_t/\eta + \theta_{t-1}/\eta})} \quad \text{for } i \in N_k, k = 1, \ldots, K.
\]
In the previous expression, we note that

\[ H_i(e^{u_t/\eta + \theta_{t-1}/\eta}) = \left( \sum_{j \in N_k} e^{(u_{jt} + \theta_{jt-1})/\eta \lambda_k} \right)^{\lambda_k - 1} e^{(u_{it} + \theta_{it-1})/\eta \lambda_k}, \]

\[ \sum_{k=1}^K \sum_{j \in N_k} H_j(e^{u_t/\eta + \theta_{t-1}/\eta}) = \sum_{k=1}^K \left( \sum_{j \in N_k} e^{(u_{jt} + \theta_{jt-1})/\eta \lambda_k} \right)^{\lambda_k}. \]

From (49) it is easy to see that for period \( t \) we must have:

\[ x_{it} = \frac{H_i(e^{\theta_{t-1}/\eta})}{\sum_{k=1}^K \sum_{j \in N_k} H_j(e^{\theta_{t-1}/\eta})} \quad \text{for} \quad i \in N_k, k = 1, \ldots, K. \]

Using once again using the fact that \( \Phi(\cdot) \) is homogenous of degree 1, we get

\[ \Phi(x_i) \sum_{k=1}^K \sum_{j \in N_k} H_j(e^{\theta_{t-1}/\eta}) = e^{\theta_{t-1}/\eta} \quad \text{for} \quad i \in N_k, k = 1, \ldots, K. \]

Taking log on both sides in the previous expression we obtain

\[ \log \Phi(x_t) + \log \left( \sum_{k=1}^K \sum_{j \in N_k} H_j(e^{\theta_{t-1}/\eta}) \right) = \theta_{t-1}/\eta \quad \text{for} \quad i \in N_k, k = 1, \ldots, K. \]

Define \( m_{t-1} \triangleq \log \left( \sum_{k=1}^K \sum_{j \in N_k} H_j(e^{\theta_{t-1}/\eta}) \right) \). Using this definition we get:

\[ x_{it+1} = \frac{H_i(\Phi(x_t)) e^{u_t/\eta + m_{t-1}}}{\sum_{k=1}^K \sum_{j \in N_k} H_j(\Phi(x_t)) e^{u_t/\eta + m_{t-1}}}. \]

Using the homogeneity of \( H \) one last time, we obtain:

\[ x_{it+1} = \frac{H_i(\Phi_i(x_t)) e^{u_t/\eta}}{\sum_{k=1}^K \sum_{j \in N_k} H_j(\Phi_j(x_t)) e^{u_t/\eta}} \quad \text{for} \quad i \in N_k, k = 1, \ldots, K. \]

Finally, replacing the expression for \( H_i \) using the NL assumption, the conclusion follows at once.

A.13. **Proof of Corollary.** When \( \lambda_k = 1 \) for all \( k \in K \), we know that the NL boils to the MNL model. Thus the conclusion follows at once.
**Appendix B. Online Material Not for Publication**

**Lemma 11.** Let Assumptions 1 and 2 hold. Then in the OFTRL algorithm the following hold:

\[
\sum_{t=1}^{T} \langle x^* - x_t, u_t \rangle \leq \eta \varphi(0) + \frac{L}{2\eta} \sum_{t=1}^{T} \|u_t - \beta_t\|_*^2.
\]

*Proof.* The proof of this Lemma follows from a simple adaptation of [Rakhlin and Sridharan, 2013, Lemma 2].

**Lemma 12.** In the OFTRL the following statements hold

i) In the $S$-step recency bias:

\[
\sum_{t=1}^{T} \|u_t - \beta_t\|_*^2 \leq S^2 \sum_{t=1}^{T} \|u_t - u_t\|_*^2
\]

ii) In the geometrically discounted recency bias we have:

\[
\sum_{t=1}^{T} \|u_t - \beta_t\|_*^2 \leq \frac{1}{(1 - \delta)^3} \sum_{t=1}^{T} \|u_t - u_{t-1}\|_*^2
\]

*Proof.* In proving parts i) and ii) we follow the proof of Lemma 21 in Syrgkanis et al. [2015]. Concretely, in proving part i) we have the following:

\[
\sum_{t=1}^{T} \|u_t - \beta_t\|_*^2 = \sum_{t=1}^{T} \left\| u_t - \frac{1}{H} \sum_{\tau=1-H}^{t-1} u_\tau \right\|_*^2,
\]

\[
(51) \quad = \sum_{t=1}^{T} \left( \frac{1}{H} \sum_{\tau=1-H}^{t-1} \|u_t - u_\tau\|_* \right)^2.
\]

By the triangle inequality we get

\[
\frac{1}{H} \sum_{\tau=t-H}^{t-1} \|u_t - u_\tau\|_* \leq \frac{1}{H} \sum_{\tau=t-H}^{t-1} \sum_{q=\tau}^{t-1} \|u_{q+1} - u_q\|_*
\]

\[
= \sum_{\tau=t-H}^{t-1} \frac{t - \tau}{H} \|u_{\tau+1} - u_\tau\|_* \leq \sum_{\tau=t-H}^{t-1} \|u_{\tau+1} - u_\tau\|_*
\]

By the Cauchy-Schwarz inequality, we get:

\[
\left( \sum_{\tau=t-H}^{t-1} \|u_{\tau+1} - u_\tau\|_* \right)^2 \leq H \sum_{\tau=t-H}^{t-1} \|u_{\tau+1} - u_\tau\|_*^2.
\]
Thus it follows that:

\[ \sum_{t=1}^T \|u_t - \beta_t\|_*^2 \leq H \sum_{t=1}^T \sum_{\tau=t-H}^{t-1} \|u_{\tau+1} - u_{\tau}\|_*^2 \]

\[ \leq H^2 \sum_{t=1}^T \|u_t - u_{t-1}\|_*^2 \]

In proving part (ii) we follow the proof of Lemma 22 in Syrgkanis et al. [2015]. We begin noting that

\[ \sum_{t=1}^T \|u_t - \beta_t\|_*^2 = \sum_{t=1}^T \left\| u_t - \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} u_{\tau} \right\|_*^2 \]

We want to show that

\[ \sum_{t=1}^T \left\| u_t - \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} u_{\tau} \right\|_*^2 \leq \frac{1}{(1-\delta)^2} \sum_{t=1}^T \|u_t - u_{t-1}\|_*^2 \]

In proving this, we first note that

\[ \left\| u_t - \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} u_{\tau} \right\|_* = \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} \left\| u_{\tau+1} - u_{\tau} \right\|_* \]

\[ \leq \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} \sum_{q=\tau}^{t-1} \left\| u_{q+1} - u_q \right\|_* \]

\[ = \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{q=0}^{t-1} \left\| u_{q+1} - u_q \right\|_* \sum_{\tau=0}^{q} \delta^{-\tau} \]

\[ = \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{q=0}^{t-1} \left\| u_{q+1} - u_q \right\|_* \delta^{-q} \frac{1 - \delta^{q+1}}{1 - \delta} \]

\[ \leq \frac{1}{1-\delta} \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{q=0}^{t-1} \delta^{-q} \left\| u_{q+1} - u_q \right\|_* \]

Second, by Cauchy-Schwartz we have
\[
\left( \frac{1}{1 - \delta} \sum_{\tau=0}^{t-1} \delta^{-\tau} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_* \right)^2 = \frac{1}{(1 - \delta)^2} \left( \sum_{\tau=0}^{t-1} \delta^{-q/2} \cdot \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_* \right)^2 \leq \frac{1}{(1 - \delta)^2} \sum_{\tau=0}^{t-1} \delta^{-\tau} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2 = \frac{1}{(1 - \delta)^2} \sum_{\tau=0}^{t-1} \delta^{-\tau} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2 \leq \frac{1}{\delta(1 - \delta)^2} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2
\]

Combining previous expressions we get:

\[
\left\| u_t - \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} u_\tau \right\|_*^2 \leq \frac{1}{\delta(1 - \delta)^2} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2
\]

Summing over all \( t \) and re-arranging we get:

\[
\sum_{t=1}^{T} \left\| u_t - \frac{1}{\sum_{\tau=0}^{t-1} \delta^{-\tau}} \sum_{\tau=0}^{t-1} \delta^{-\tau} u_\tau \right\|_*^2 \leq \frac{1}{\delta(1 - \delta)^2} \sum_{t=1}^{T} \sum_{q=0}^{t-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2 = \frac{1}{\delta(1 - \delta)^2} \sum_{q=0}^{T-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2 \sum_{t=q+1}^{T} \delta^t = \frac{1}{\delta(1 - \delta)^2} \sum_{q=0}^{T-1} \delta^{-q} \| u_{q+1} - u_q \|_*^2 \delta \left( \delta^t - \delta^T \right) \frac{1}{1 - \delta} = \frac{1}{(1 - \delta)^3} \sum_{q=0}^{T-1} \| u_{q+1} - u_q \|_*^2 (1 - \delta^{T-q}) \leq \frac{1}{(1 - \delta)^3} \sum_{q=0}^{T-1} \| u_{q+1} - u_q \|_*^2.
\]
B.1. **Proof of Proposition 6.** i) Combining Lemma 11 and Assumption (34) we know that
\[ R_{OFTRL}^T \leq \eta \varphi(0) + \frac{L}{2\eta} TB^2. \]

Optimizing over \( \eta \) we find that the regret is minimized at \( \eta = \sqrt{\frac{LTB^2}{2\varphi(0)}} \). Thus we obtain
\[ R_{OFTRL}^T \leq B \sqrt{2LT \varphi(0)}. \]

ii) Combining Lemma 11 and Assumption (34) we know that
\[ R_{OFTRL}^T \leq \eta \varphi(0) + \frac{L}{2\eta} \sum_{t=1}^{T} \| u_t - \beta_t \|^2. \]

Using Lemma 12i) combined with Eq. (34) we obtain that:
\[ R_{OFTRL}^T \leq \eta \varphi(0) + \frac{L}{2\eta} TS^2 B^2. \]

Optimizing over \( \eta \) we find that the regret is minimized at \( \eta = \sqrt{\frac{LS^2B^2}{2\varphi(0)}} \). In particular, we get:
\[ R_{OFTRL}^T \leq SB \sqrt{2LT \varphi(0)}. \]

In showing part iii), we note that combining Lemma 11, Lemma 12i), and condition (34)
\[ R_{OFTRL}^T \leq \eta \varphi(0) + \frac{L}{2\eta(1-\delta)^3} TB^2 \]

Optimizing over \( \eta \) we find that the regret is minimized at \( \eta = \sqrt{\frac{L TB^2}{2(1-\delta)^3 \varphi(0)}} \). Thus, we get:
\[ R_{OFTRL}^T \leq B \sqrt{\frac{2LT \varphi(0)}{(1-\delta)^3}}. \]

\[ \square \]

B.2. **Proof of Proposition 9.** We proof this result using Theorem 6. Differentiability follows from the Williams-Daly-Zachary theorem. Now let \( q \) and \( q' \) with \( q' \geq q \). Note that \( \max_{i=1m,...,N} \{ q'_i + \epsilon_i \} \geq \max_{i=1m,...,N} \{ q_i + \epsilon_i \} \). Taking expectation with respect to \( \epsilon \), it follows that \( \varphi(q') \geq \varphi(q) \), which implies the increasing monotonicity of \( \varphi \). Finally, positive translation invariance follows from the fact that \( \varphi(q + k1) = \mathbb{E}(\max_{i=1,...,N} \{ q_i + k + \epsilon_i \}) = \mathbb{E}(\max_{i=1,...,N} \{ q_i + \epsilon_i \}) + k = \varphi(q) + k \).
\[ \square \]
B.3. Applications of the GNL model.

B.3.1. The Cross Nested Logit (CNL) model. Vosha [1997] introduces the cross nested logit model. The main assumption of this model is that $\lambda_k = \lambda$ for all $k = 1, \ldots, K$. Thus, the generator $G$ boils down to the expression:

$$G(e^{\theta_t/\eta}) = \sum_{k=1}^{K} \left( \sum_{i=1}^{N} (\alpha_{ik} e^{\theta_{it}/\eta})^{1/\lambda_k} \right)^{\lambda_k}.$$  

In addition, in this case the constant $M$ is given by $M = \frac{2}{\lambda} - 1$. Accordingly, the Social Surplus function has Lipschitz continuous gradient with constant $\left( \frac{2}{\lambda} - 1 \right)/\eta$.

B.3.2. The Paired Combinatorial Logit (PCL) model. In this model each pair of alternatives is represented by a nest. Formally, the set of nests is defined as $N = \{(i,j) \in A \times A : i \neq j\}$. Accordingly, we define

$$\alpha_{ik} = \begin{cases} \frac{1}{2(N-1)} & \text{if } k = (i,j), (j,i) \text{ with } j \neq i \\ 0 & \text{otherwise} \end{cases}.$$  

Using the previous expression, the generator $G$ can be written as:

$$G(e^{\theta_t/\eta}) = \sum_{k=(i,j) \in N} (\alpha_{ik} e^{\theta_{it}/\eta})^{1/\lambda_k} + (\alpha_{jk} e^{\theta_{jt}/\eta})^{1/\lambda_k})^{\lambda_k}.$$  

In this case the Lipschitz constant is $\left( \frac{2}{\min \lambda_k} - 1 \right)/\eta$.

In addition, $\log G(1) \leq \log N$. Then the regret analysis follows from Proposition 2.

B.3.3. The Ordered GEV (OGEV) model. Small [1987] studies a GEV model in which the alternatives are allocated to nests based on their proximity in an ordered set. Following Small [1987], we define the set of overlapping nests to be

$$\mathcal{N} = \{1, \ldots, N + N'\},$$  

with $\alpha_{i\ell} > 0$ for all $\ell \in \{i, \ldots, N + N'\}$ and $\alpha_{i\ell} = 0$ for $i \in \mathcal{N} \setminus \{i, \ldots, N + N'\}$, and each alternative lies exactly in $N' + 1$ of these nests. In the model there are $N + N'$ overlapping nests. Each nest $\ell \in \mathcal{N}$ is defined as $\mathcal{N}_\ell = \{i \in A : l - m \leq i \leq \ell\}$ where $i \in \mathcal{N}_\ell$ for $\ell = i, \ldots, i + N'$. Despite this rather complex description, the generator function $G$ takes the familiar form:

$$G(e^{\theta_t/\eta}) = \sum_{k=1}^{N+N'} \left( \sum_{i \in \mathcal{N}_k} (\alpha_{ik} e^{\theta_{it}/\eta})^{1/\lambda_k} \right)^{\lambda_k}.$$  

Thus in this case the Lipschitz constant is given by $\left( \frac{2}{\min_k = 1, \ldots, K \lambda_k} - 1 \right)/\eta$. Moreover, $\log G(1) \leq \log N$. Thus Proposition 2 applies and we conclude that the OGEV model is Hannan consistent.
B.3.4. Principles of Differentiation GEV model (PDGEV). \cite{Bresnahan2000} introduce the PDGEV model. This approach is based on the idea of markets for differentiated products. Using this idea, the set of nests is defined in terms of the attributes that characterize the different products (goods). For instance, in the context of transportation modeling, the attributes can include mode to work, destination, number of cars, and residential location. Accordingly, let $D$ be the set of attributes with $\mathcal{N} = \bigcup_{d \in D} \mathcal{N}_d$ and $\mathcal{N}_d = \{k \in \mathcal{N} : \text{nest } k \text{ contains products with attribute } d\}$ be the nest that contains the alternatives with attribute $d$. Similarly, let $\mathcal{N}_{kd}$ denote the nest $k$ with attribute $d$.

Let $\alpha_{ik} = \begin{cases} \alpha_d & \text{if } i \in \mathcal{N}_{kd} \text{ and } k \in \mathcal{N}_d \\ 0 & \text{otherwise} \end{cases}$

In this case the generator $G$ takes the form:

$$G(e^{\theta_t/\eta}) = \sum_{d \in D} \alpha_d \sum_{k \in \mathcal{N}_d} \left( \sum_{i \in \mathcal{N}_{kd}} e^{\theta_{it}/\eta \lambda_d} \right)^{\lambda_d}.$$

It is easy to see that the previous generator is a particular case of the GNL model. Furthermore, the Lipschitz constant is $\left(\frac{2}{\min_{d=1,...,D} \lambda_d} - 1\right)/\eta$ and $\log G(1) \leq \log N$. Thus Proposition 2 applies in a direct way.

We close this appendix summarizing the regret bounds for the models described in the main text and in this appendix. Table 1 below makes explicit our regret analysis. The table displays how several GEV models shares the same optimized regret bound.
| Model | Optimal $\eta$ | Regret Bound |
|-------|---------------|---------------|
| RUM   | $\sqrt{\frac{LTu_{max}^2}{2\varphi(0)}}$ | $u_{max}\sqrt{2\varphi(0)LT}$ |
| GEV   | $\sqrt{\frac{(2M+1)Tu_{max}^2}{2\log G(1)}}$ | $u_{max}\sqrt{2\log G(1)(2M+1)T}$ |
| GNL   | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| PCL   | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| CNL   | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| OGEV  | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| PDGEV | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| NL    | $\sqrt{\frac{\left(\frac{2}{\min_k \lambda_k}-1\right)Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log N\left(\frac{2}{\min_k \lambda_k}-1\right)T}$ |
| Logit | $\sqrt{\frac{Tu_{max}^2}{2\log N}}$ | $u_{max}\sqrt{2\log NT}$ |

Table 1. Summary of the optimized regret bound for the RUM and several GEV models.