Bounds on the spectral radius of real-valued non-negative Kernels on measurable spaces

Wasiur R. KhudaBukhsh\textsuperscript{*}, Technische Universität Darmstadt
Mark Sinzger\textsuperscript{†}, Technische Universität Darmstadt
Heinz Koeppl\textsuperscript{‡}, Technische Universität Darmstadt

In this short technical note, we extend a recently published result \cite{6} on the Perron root (or the spectral radius) of non-negative matrices to real-valued non-negative kernels on an arbitrary measurable space \((E, \mathcal{E})\). To be precise, for any real-valued non-negative kernel \(K : E \times \mathcal{E} \rightarrow \mathbb{R}_+\), we prove that the spectral radius \(\rho(K)\) of \(K\) satisfies

\[
\inf_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)} \leq \rho(K) \leq \sup_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)},
\]

where \(L\) is an arbitrary Kernel on \((E, \mathcal{E})\), which is integrable with respect to the left eigenmeasure of \(K\) and satisfies \(\mathcal{R}L(x) > 0\) for all \(x \in E\), and the operator \(\mathcal{R}\) is defined by \(\mathcal{R}L(x) := \int_E L(x, dy)\).

1 Introduction

Perron roots or spectral radii of matrices and their infinite-dimensional counterpart, non-negative Kernels are a useful quantity in applied probability literature. For instance, the asymptotic behaviour of a General Branching Process (GBP) can be studied via the eigenvalues and eigenfunctions of the “expectation” operators, defined as the partial derivative at zero of the Moment Generating Function (MGF) of a point-distribution representing the objects in the 1st generation of the GBP (see \cite{2}, Chapter III, p. 59, p. 65]). A second application concerns Large Deviations Principles (LDP) of waiting

\textsuperscript{*}Department of Electrical Engineering and Information Technology, Technische Universität Darmstadt, Germany, Email: wasiur.khudabukhsh@bcs.tu-darmstadt.de, ORCID profile: https://orcid.org/0000-0003-1803-0470

\textsuperscript{†}Department of Electrical Engineering and Information Technology, Technische Universität Darmstadt, Germany

\textsuperscript{‡}Department of Electrical Engineering and Information Technology, Technische Universität Darmstadt, Germany, Email: heinz.koeppl@bcs.tu-darmstadt.de
times in queueing systems under exogenous Markov modulation. Interestingly, the \textit{LDP} rate function is often found to be linear for many queueing systems. In fact, probability bounds of the following type, which are useful for performance evaluation purposes, are often derived using martingale techniques (and also from \textit{LDPs})

\[
\lim_{k \to \infty} P(W_k \geq \sigma) \leq a(\sigma)e^{-\theta\sigma},
\]

where \(W_k\) is the waiting time for the \(k\)-th job, the function \(a\) is called the prefactor, and the scalar \(\theta\) is called the effective decay rate of the queueing system. Here, the effective decay rate \(\theta\) turns out to be the spectral radius of a certain exponentially transformed transition kernel for Markov-modulated systems (see [7, 4, 3, 5]). Since the effective decay rate \(\theta\) depends on the particular scheduling algorithm chosen, bounds of the above form are very useful for performance evaluation purposes. In essence, the spectral radius \(\theta\) determines the performance of the entire system.

It is evident from the two examples discussed in the previous paragraph that studying the spectral radius is often beneficial for asymptotic analysis of many stochastic processes arising in applied probability literature. While the exact value of the spectral radius is desirable and most informative, it might be computationally expensive, or even infeasible to find with high degree of accuracy. Moreover, a bound (upper or lower) often suffices for a qualitative analysis. In this short technical note, we extend a recently published result [6] on the Perron root of non-negative matrices to real-valued non-negative kernels on an arbitrary measurable space \((E, \mathcal{E})\). To be precise, for any real-valued non-negative kernel \(K: E \times E \to \mathbb{R}_+\), we prove that the spectral radius \(\rho(K)\) of the kernel \(K\) satisfies

\[
\inf_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)} \leq \rho(K) \leq \sup_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)},
\]

where \(L\) is an arbitrary Kernel on \((E, \mathcal{E})\), which is integrable with respect to the left eigenmeasure of \(K\) and satisfies \(\mathcal{R}L(x) > 0\) for all \(x \in E\), and the operator \(\mathcal{R}\) is defined by \(\mathcal{R}L(x) := \int_E L(x, dy)\). In the next section, we provide a short proof of this bound. In [1], the authors proved similar bounds in the context of integral operators with non-negative kernels.

\section{Main result}

\subsection{Notational conventions}

The following notational conventions are adhered to throughout this technical report. We denote the set of natural numbers and the set of real numbers by \(\mathbb{N}\) and \(\mathbb{R}\) respectively. Let \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). For \(N \in \mathbb{N}\), let \([N] := \{1, 2, \ldots, N\}\). The set of non-negative real numbers is denoted by \(\mathbb{R}_+\). For \(A \subseteq \mathbb{R}\), we denote the Borel \(\sigma\)-field of subsets of \(A\) by \(\mathcal{B}(A)\). For any \(f: \mathbb{R} \to \mathbb{R}\), we denote the effective domain of \(f\) by \(\mathcal{D}(f)\), i.e., \(\mathcal{D}(f) := \{x \in \mathbb{R} | |f(x)| < \infty\}\). For an event \(A\), we denote the indicator function of \(A\) by \(1(A)\), taking value unity when \(A\) is true and zero otherwise.
2.2 Bounds on the Perron root

Let \((E, E)\) be a given measurable space. The set \(E\) need not be countable, but we assume \(E\) is a complete and separable metric space. Denote the space of real-valued non-negative Kernels on \((E, E)\) by \(K_E\), i.e., \(K_E := \{f : E \times E \to \mathbb{R}_+\}\). Given \(K \in K_E\), define the operator \(R\) as

\[
R(K)(x) := \int_E K(x, dy).
\] (2.1)

The operator \(R\) is analogous to the row sum in the finite case, i.e., when \(E\) is finite and \(K\) is a real non-negative matrix. We are interested in getting upper and lower bounds on the spectral radius \(\rho(K)\) of the kernel \(K\). Before presenting our main result, we have the following two lemmas.

**Lemma 1.** Suppose \(f : E \to \mathbb{R}_+\) and \(g : E \to \mathbb{R}_+\) are such that \(\mathcal{D}(f) = \mathcal{D}(g) = E\) and \(0 < \int_E f \, d\mu < \infty, 0 < \int_E g \, d\mu < \infty\). Then, given a measure \(\mu\) on \(E\),

\[
\inf_{x \in E} \frac{f(x)}{g(x)} \leq \frac{\int_{E} f \, d\mu}{\int_{E} g \, d\mu} \leq \sup_{x \in E} \frac{f(x)}{g(x)}.
\] (2.2)

**Proof of Lemma 1.** Note that

\[
\frac{\int_{E} f \, d\mu}{\int_{E} g \, d\mu} = \frac{1}{\int_{E} g \, d\mu} \int_{E} f \, d\mu = \int_{E} \frac{f}{g} \, d\nu,
\]

where the probability measure \(\nu\) on \((E, E)\) is defined by

\[
\nu(A) := \frac{1}{\int_{E} g \, d\mu} \int_{A} g \, d\mu, \text{ for } A \in \mathcal{E}.
\]

Note that,

\[
\int_{E} \frac{f}{g} \, d\nu \geq \inf_{x \in E} \frac{f(x)}{g(x)} \int_{E} d\nu = \inf_{x \in E} \frac{f(x)}{g(x)} \nu(E) = \inf_{x \in E} \frac{f(x)}{g(x)},
\]

and

\[
\int_{E} \frac{f}{g} \, d\nu \leq \sup_{x \in E} \frac{f(x)}{g(x)} \int_{E} d\nu = \sup_{x \in E} \frac{f(x)}{g(x)} \nu(E) = \sup_{x \in E} \frac{f(x)}{g(x)}.
\]

Therefore, we get

\[
\inf_{x \in E} \frac{f(x)}{g(x)} \leq \frac{\int_{E} f \, d\mu}{\int_{E} g \, d\mu} \leq \sup_{x \in E} \frac{f(x)}{g(x)}.
\] (2.3)
Next, we define the product of two kernels on the measurable space \((E, \mathcal{E})\). Let \(F\) and \(G\) be two kernels on \((E, \mathcal{E})\). Analogous to matrix multiplication, define the new kernel \(F \cdot G : E \times \mathcal{E} \to \mathbb{R} \cup \{\infty, -\infty\}\) as
\[
F \cdot G(x, A) := \int_E F(x, dy)G(y, A).
\] (2.4)

The next lemma concerns the row sums of the product of two kernels.

Lemma 2. Let \(F\) and \(G\) be two kernels on \((E, \mathcal{E})\) such that \(D(\mathcal{F} \cdot \mathcal{G}) = E\). And suppose, that \(\int_E \int_E |F(x, dy)G(y, dz)| < \infty\) holds. Then,
\[
\mathcal{R}F \cdot \mathcal{G}(x) = \int_E F(x, dy)\mathcal{G}(y).
\] (2.5)

Proof of Lemma. Note that by Fubini’s theorem,
\[
\mathcal{R}F \cdot \mathcal{G}(x) = \int_E \int_E F(x, dy)G(y, dz) \\
= \int_E \int_E F(x, dy)G(y, dz) \\
= \int_E F(x, dy) \int_E G(y, dz) \\
= \int_E F(x, dy)\mathcal{G}(y).
\]

This completes the proof.

Now we present our main result on the Perron root \(\rho(K)\) of \(K\).

Theorem 1. Let \(K\) be a real-valued non-negative kernel, i.e., \(K \in \mathcal{K}_E\). Let \(L\) be a kernel on \((E, \mathcal{E})\), which is not necessarily in \(\mathcal{K}_E\), but satisfies \(\int_E \int_E |K(x, dy)L(y, dz)| < \infty\), \(\mathcal{R}L(x) > 0\) for all \(x \in E\) and is integrable with respect to the left eigenmeasure of \(K\). Then,
\[
\inf_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)} \leq \rho(K) \leq \sup_{x \in E} \frac{\mathcal{R}K \cdot L(x)}{\mathcal{R}L(x)}.
\]

Proof of Theorem. Let \(r_K : E \to \mathbb{R}\) and \(l_K : \mathcal{E} \to \mathbb{R}_+\) be respectively the right eigenfunction and the left eigenmeasure of \(K\) associated with \(\rho(K)\). That is,
\[
\rho(K)r_K(x) = \int_E K(x, dy)r_K(y), \forall x \in E,
\]
\[
\rho(K)l_K(A) = \int_E l_K(dx)K(x, A), \forall A \in \mathcal{E}.
\]
Note that $\rho(K)$ is the largest simple eigenvalue of $K$. The existence of $\rho(K), l_K,$ and $r_K$ are guaranteed by [2, Theorem III.10.1].

Let $L$ be an arbitrary kernel on $(E, \mathcal{E})$, which is not necessarily in $K_E$, but satisfies $\int_E \int_E |K(x, dy)L(y, dz)| < \infty$, $RL(x) > 0$ for all $x \in E$ and is integrable with respect to $l_K$ as a measure on $E$. Then, by the definition of the left eigenmeasure $l_K$, we have

$$\int_E l_K(dx)K(x, dy) = \rho(K)l_K(dy)$$

$$\Rightarrow \int_E RL(y)\int_E l_K(dx)K(x, dy) = \rho(K)\int_E l_K(dy)RL(y)$$

$$\Rightarrow \int_E l_K(dx)\int_E K(x, dy)RL(y) = \rho(K)\int_E l_K(dy)RL(y)$$

$$\Rightarrow \int_E l_K(dx)RK\cdot L(x) = \rho(K)\int_E l_K(dy)RL(y),$$

because $RK\cdot L(x) = \int_E K(x, dy)L(y, dz)$, by Lemma 2. Therefore we have

$$\rho(K) = \frac{\int_E l_K(dx)RK\cdot L(x)}{\int_E l_K(dx)RL(x)}.$$

Choosing $\mu = l_K$, $f = RK\cdot L$, and $g = RL$ in Lemma 1, we get

$$\inf_{x \in E} \frac{RK\cdot L(x)}{RL(x)} \leq \rho(K) \leq \sup_{x \in E} \frac{RK\cdot L(x)}{RL(x)}.$$

(2.6)

This completes the proof.

The condition $\int_E \int_E |F(x, dy)G(y, dz)| < \infty$ is necessary and hence, must not be omitted, as shown in the following example.

**Example 1.** Consider the complete separable metric space $E = [0, 1]$. Define $F(x, dy) = dy$ to be the Lebesgue measure on $[0, 1]$ independent of $x$. The kernel $F$ is non-negative and allows for the Lebesgue measure as left eigenmeasure. Furthermore, define the function $g: E \times E \to \mathbb{R}$

$$g(y, z) := \begin{cases} \frac{-1}{y^2}, & 0 < y < z \leq 1 \\ \frac{1}{y^2}, & 0 < z \leq y \leq 1 \\ -1, & z = 0, y \neq 0 \\ 1, & y = 0. \end{cases}$$

Note that $g$ is measurable with respect to the Lebesgue measure on $E \times E$ and defines the kernel

$$G(y, dz) = g(y, z)dz.$$

For all $y \in (0, 1)$, calculate

$$\mathcal{R}G(y) = \int_0^1 g(y, z)dz = \int_0^y \frac{1}{y^2}dz - \int_y^1 \frac{1}{z^2}dz = 1 > 0,$$
and for \( y = 0 \),
\[
\mathcal{RG}(0) = \int_0^1 dz = 1 > 0.
\]

Further, \( \mathcal{RG}(y) \) is integrable with respect to the left eigenmeasure of \( F \). The product of the kernels is
\[
F \cdot G(x, dz) = \left( \int_0^1 g(y, z) dy \right) dz = \left( \int_0^z -\frac{1}{z^2} dy + \int_z^1 \frac{1}{y^2} dy \right) dz = -dz.
\]

Hence
\[
\mathcal{RF} \cdot G(x) = \int_0^1 F \cdot G(x, dz) = -1
\]
independent of \( x \). This implies that \( \inf_{x \in E} \frac{\mathcal{RF} \cdot G(x)}{\mathcal{RG}(x)} \) and \( \sup_{x \in E} \frac{\mathcal{RF} \cdot G(x)}{\mathcal{RG}(x)} \) both equal \(-1\), even though the Perron root \( \rho(F) \) is 1.

**Acronyms**

GBP General Branching Process

LDP Large Deviations Principle

MGF Moment Generating Function

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