On the Construction of Polar Codes in the Middleton Class-A Channels

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Abstract—Although power line communication (PLC) systems are available everywhere, unfortunately these systems are not suitable for information transmission due to the effects of the impulsive noise. Therefore, many previous studies on channel codes have been carried out for the purpose of reducing the impulsive noise in such channels. This paper investigates some methods for the construction of polar codes under PLC systems in the presence of Middleton class-A noise. We discuss here the most feasible construction methods which already have been adopted with other channels. In addition, we present an illustrative example for the construction in these methods and also we discuss a comparison between the methods in terms of performance.

Index Terms—Impulsive noise, polar codes, power line communications (PLC), Middleton Class-A channel.

I. INTRODUCTION

PLC systems have drawn a huge interest as they can be easily accessed over the existing power line networks, which are available in almost every building on the planet. Hence, PLC technology is preferred in some wireless environments which have a significant propagation loss. Moreover, the huge demand for communication and internet networks makes PLC one of the most important competing technologies. However, PLC networks are not suitable for the communication signals because the connected electrical appliances in these networks can be considered as abundant sources of interference and noise [1]. Thus, the noise over the PLC channels could be divided into two types, namely, colored background noise and impulsive noise. In fact the latter has the worst impact on the PLC network reliability due to its large power spectral density (PSD) which is normally 10 – 15 dB higher than the PSD of the colored background noise. For the PLC noise, Middleton class-A model is adopted in this paper because it is the most accepted model amid PLC channel models [2].

Many studies have been proposed to mitigate the impact of impulsive noise; this includes the use of nonlinear pre-processing and channel coding [3]–[7]. As for channel coding, binary turbo codes were proposed over the PLC channels in order to reduce the impact of impulsive noise [5] [6]. In this regard, the authors in [7] have discussed the performance of polar codes over the PLC channels by using signal limiter. The popular code family, which is the low density parity check (LDPC), was investigated with both single-carrier and multi-carriers techniques over the PLC channels and resulted in further enhancement in the reduction of impulsive noise impact [8]–[10]. Furthermore, comparisons between the performance of LDPC and turbo codes in the context of the PLC channels were explored in [11]. In the other hand, previous research has presented a comparison between polar codes with LDPC codes in both single-carrier and multi-carriers modulations over PLC channels [12]. Also, a recent work showed a comparison between these two codes for smart grid (SG) using orthogonal frequency modulation multiplexing (OFDM) in the presence of impulsive noise [13].

In the last decade, polar codes family has attracted an attention since it can achieve the maximum capacity of the channel by using a low complexity decoder in comparison to some other family codes [14]. The underlying key of polar codes is the channel polarization phenomenon in which a number of the binary discrete memoryless channels (B-DMCs) are combined to create one comprehensive channel. Then, this single channel splits into two types of bit-channels namely, information, and frozen bit-channels. The frozen bits are fixed to zero and the decoder has the full knowledge about their positions in the codeword [15].

Unlike other channel codes, polar codes are not universal codes which mean that the code construction is dependent on the signal to noise ratio (SNR) of the channel. It is worth mentioning that the heuristic method was applied as the earliest method of construction and it was utilized by the help of the binary erasure channels (BEC) [16]. The key element of the heuristic method is the Bhattacharyya parameter which can be denoted by $Z(W)$. The definition of $Z(W)$ is the upper bound of the error rate for a channel in binary transmission. The initial value $Z(W_i)$ for of this method is usually considered as 0.5. On the other hand, the heuristic based capacity method was adopted in polar codes in [17]. This method differs from the normal heuristic method in its initial value where $Z(W_i)$ is evaluated according to the capacity of the using channel. Moreover, Monte Carlo method which based on repeated random sampling for achieving results [18]. It should be noted that there are other construction methods have been developed such as quantization approximations method [19], and a method based on linear complexity convolution [20]. Furthermore, a practical construction method was proposed for the general binary input channels by introducing a new algorithm for finding the exact values of the Bhattacharyya...
parameter methods is presented for the purpose of finding the best design-SNR for the different methods in additive white Gaussian noise (AWGN) channel [22].

In this paper, we investigate methods for the construction of polar codes over PLC channels under Middleton class-A model. Therefore, we focus on three methods which are heuristic, heuristic based capacity, and Monte Carlo as they are feasible in the PLC channels. In addition, we present an illustrative example to show how the change in SNR affects on the construction according to each method. For a practical comparison, we show the performance in terms of error rate ratio versus SNR.

The rest of this paper is organized as follows. In Section II, briefly review the preliminaries. In Section III, the Bhattacharyya parameter was discussed in the Middleton class-A channel. The three methods of the construction are discussed in Section IV. The simulation results with discussion are given in Section V. Finally, conclusions are drawn in Section VI.

II. PRELIMINARIES

A. Middleton’s Class-A Noise Model

Middleton class-A has been extensively used as a proper model for the PLC systems due to its suitability for the impulsive noise characterization over PLC. The model contains background noise and impulsive noise. In general, the probability density function (PDF) for this model is expressed as [2]

\[ P_z(z) = \sum_{m=0}^{\infty} \exp(-A) \frac{A^m}{m!} \exp \left( -\frac{z^2}{2\sigma_m^2} \right), \]

(1)

where

\[ \sigma_m = \sqrt{\frac{A^2 + 1}{1 + \Gamma}}, \]

(2)

where \( A \) is the impulsive index, \( \Gamma \) is the Gaussian-to-impulsive power ratio given as \( \Gamma = \sigma_e^2/\sigma_n^2 \), \( \sigma_e^2 \) is the Gaussian noise variance, \( \sigma_n^2 \) is the impulsive noise variance, and the variance of the total noise is given by \( \sigma^2 = \sqrt{\sigma_e^2 + \sigma_n^2} \). Note that the impulsive index identifies the average number of impulses over the signal period, and \( \Gamma \) indicates the strength of impulsive noise compared to the background noise. It was found in [23] that the probability of errors for Middleton’s class-A channel in the binary phase shift keying (BPSK) modulation scheme is

\[ P_e = \frac{\exp(-A)}{2} \sum_{m=0}^{\infty} \frac{A^m}{m!} \text{erfc} \left( \sqrt{\frac{1}{A^2 + \Gamma}} \frac{E_b}{N_0} \right), \]

(3)

where \text{erfc}(.) denotes the complementary error function.

It should be pointed out that the capacity of the Middleton class-A channel is given as [24]

\[ C = \sum_{m=0}^{M} \sigma_m C_m, \]

(4)

where \( C_m \) is the capacity of the AWGN channel capacity with state \( m \) and it can be given by

\[ C_m = B \log_2 \left( 1 + \frac{s}{n} \right), \]

(5)

where \( B \) is the bandwidth, \( s \) is the total signal power, and \( n \) is the power of the noise. It is worth mentioning that the capacity of the Middleton class-A channel approaches the AWGN capacity with low impulsive noise.

B. Polar Codes

In polar codes, \( N \) number, usually power of 2, of identical copies of a channel \( W \) are combined to become one channel \( W_N : X_N \rightarrow Y_N \), where \( X_N = (x_0, x_1, \ldots, x_{N-1}) \) and \( Y_N = (y_0, y_1, \ldots, y_{N-1}) \) are the corresponding channel inputs and channel outputs vectors, respectively. The channel \( W_N \) can be divided into two different sets by the polarization phenomenon. The information set includes the noiseless bit-channels which are useful for transmission, while the bit-channels in the frozen set are fixed to zero values by the encoder of polar codes which utilized by

\[ X = U G_N, \]

(6)

where \( G_N \) denotes the generator matrix. It should be mentioned that \( U \) vector includes the entirely information and frozen bits.

III. BHATTACHARYYA PARAMETER OF MIDDLETON CLASS-A

As we mentioned above that the Bhattacharyya parameter is an important tool for measuring the reliability of channels and it can be given as

\[ Z(W) = \sum_{y \in Y} \sqrt{p(y|0)p(y|1)}. \]

(7)

For the binary erasure channel (BEC), applying (7) resulted in \( Z(w) = \epsilon \), where \( \epsilon \) is the erasure probability. In a similar way, the Bhattacharyya parameter for the binary symmetric channel (BSC) is equivalent to \( \sqrt{p(1-p)} \), where \( p \) is the crossover probability.

Furthermore, \( Z(W) \) can be given for any continuous channel by

\[ Z(W) = \int_{-\infty}^{\infty} \sqrt{p(y|0)p(y|1)} |d_y|. \]

(8)

Therefore, in the practical AWGN channel, it can be calculated by [17]

\[ Z(W) = \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\pi\sigma^2} e^{-\frac{(y_e - y_0)^2}{2\sigma^2}}} \frac{1}{2\pi\sigma^2} e^{-\frac{(y_e - y_1)^2}{2\sigma^2}} |d_y|. \]

(9)

\[ = e^{-\frac{1}{2\sigma^2}}, \]

(10)

where \( \sigma^2 \) is the AWGN channel variance. Therefore, by applying (8) in the Middleton class-A channel, the Bhattacharyya parameter can be given in (11). However, there is no direct solution to (11); hence, the numerical integration is a feasible
the Bhattacharyya parameter versus the bit index for the conventional heuristic method proposed by Arikan when $N = 64$, and half code-rate. The information channels are below the red lines while the frozen channels are above the red lines.

Figure 1. Bhattacharyya parameter versus the bit index for three cases of erasure rate $\epsilon = 0.4$, $0.5$, and $0.6$.

solution for finding the $Z(W)$ values in the Middleton class-A channels.

IV. THE CONSTRUCTION OF POLAR CODES

A. Heuristic method

Let $Z(W_{N}^{i})$ denotes the Bhattacharyya parameter of the bit-channel $W_{N}^{i}$. The recursive channel parameters for any B-DMC channel could be calculated as given in [18]

\begin{equation}
Z(W_{N}^{2i-1}) \leq 2Z(W_{N/2}^{i}) - (Z(W_{N/2}^{i}))^{2},
\end{equation}

\begin{equation}
Z(W_{N}^{2i}) = (Z(W_{N/2}^{i}))^{2},
\end{equation}

where $i$ is an integer number $1 \leq i \leq N$. In BEC, (12) has equality status which leads to the fact that this construction could be applied under the BEC only. Consequently, for other channels such as AWGN, the author of [16] proposed the heuristic method in which any channel is treated as an equivalent BEC with an initial $Z(W_{1}) = 0.5$. The criteria of this method is to select the channels with smaller $Z(W)$ for sending information bits, while select the channels with higher $Z(W)$ as frozen bits. Due to its simplicity, this method was adopted by a lot of researchers on polar codes.

To highlight this idea, suppose that $N = 64$ and the code rate is 0.5. Fig. 1 depicts $Z(W)$ of each single channel bits versus the bit index for three cases of erasure rate $\epsilon = 0.4$, 0.5 and 0.6. Hence, 32 channel bits with the higher $Z(W)$ values are selected as frozen bits, which are above the red line. In contrast, 32 channel bits with the lower $Z(W)$ are selected as information bits, which are below the red line. It can be noticed that there is a symmetry between the right side and left side of the figures, especially when $\epsilon = 0.5$. This can be attributed to the fact that for each channel bit $i : i \in N$ with $Z(W_{N}^{i})$, there is another channel bit $j : j \in N$ with a Bhattacharyya parameter $Z(W_{N}^{j}) = 1 - Z(W_{N}^{i})$. Furthermore, it can be seen that when the erasure rate changes, the values of $Z(W)$ are changed. Thus, for $\epsilon = 0.4$, the red line has a lower value which means that the 32 information bits have more reliabilities due to their Bhattacharyya parameters. In contrast, when $\epsilon = 0.6$, it can be noticed that the information bits have higher values of Bhattacharyya parameters, i.e., less reliabilities.

Hence, the heuristic method depends on the properties of the BEC channel only and it is not an optimum method for other channels such as AWGN and Middleton class-A. However, this method can give accepted results when it is used with other channel types. What is interesting in this method is that its simplicity compared to other methods. In addition, it is independent of the channel SNR values. Hence, the code designer needs to apply it according to the codeword length and number of information bits regardless the channel parameters.

B. Heuristic based capacity method

The algorithm of this method uses (12) and (13) which are used in the normal heuristic method but the major difference is the initial value which is considered always as 0.5 in the heuristic method; whereas in this method, the initial value is evaluated based on the Bhattacharyya parameters of the channel. Hence, this method could be regarded as a channel selective method. For example, the initial value for AWGN is applied according to (9).

In this framework, we apply the heuristic based capacity method for finding the construction of polar codes under Middleton class-A channel by using (11). Hence, the SNR change affects the construction in this method. On other words, the optimum code construction for a specific value of SNR may not be an optimum construction for other values. Therefore, this method needs to apply on each specific value of SNR.
For more clarity, Fig. 2 depicts the Bhattacharyya parameters for channel bits versus the channel index for some values of SNR values. As expected, the reliabilities of the channel bits are increased with the increasing of the SNR values as it can be seen that the \( Z(W) \) values of the channel bits are reduced with SNR increasing.

C. Monte Carlo method

The metric of choosing the best channels is related to the bit error rate (BER) of the channels as [18]

\[
BER_i = Z(W^i) .
\]  \hspace{1cm} (14)

Hence, the BER for the bit channels can be evaluated by simulations with the help of the statistical techniques. Since the fact that there is no direct algorithm for finding the optimum construction of polar codes under Middleton class-A parameters, we can adopt the Monte Carlo simulation for the channel-bits selection. The Monte Carlo is an approximation statistical method uses samples from \( U, Y \) as inputs and the output is the BER values. The values of BER here play the same role of the \( Z(W) \) values in the previous methods. Hence, we can consider the bits with the highest BER values as frozen bits.

Fig. 3 shows the BER for the channel bits when \( N = 64 \). It is clearly seen that each SNR gives different values of BER which resulted in different construction due to the changes in the information set. It can be also noticed that the bits have different reliabilities; hence, the construction of polar codes selects the bits with the lowest values of BERs to become the information bits. The size of information set depends on the code design and in general when the size increases, it is expected that less reliabilities bits are included in the information set. The disadvantage of this method is the huge complexity compared to other methods.

V. Simulation Results and Discussions

In this section, we investigate the construction of the three methods in different values of SNR. Fig. 4 illustrates the three methods of construction where the code length is 64 and the code-rate is 0.5 for three values of SNR. The construction in this case is responsible for selecting 32 bits from the entire codeword for carrying the information and 32 bits as frozen bits. The first row in the figure represents the heuristic based capacity method, the second row represents the conventional heuristic method and finally the third row represents the Monte Carlo method. It can be seen that the heuristic method is not affected by the change of the SNR values, while the two other methods can be affected by SNR. Although the constructions look similar for the three methods, the slight differences in SNR may affect the performance. It is worth mentioning that
these differences are increased with the increasing of the code length.

For better clarity, we compared the performance of the code under only heuristic and heuristic based capacity methods due to the huge complexity which is needed for the Monte Carlo method. Hence, a comparison between these two methods in terms of BER performance is illustrated in Fig. 5 when $N = 64$ and Fig. 6 when $N = 512$. It can be noticed that with a small size of codeword, the heuristic based capacity method performance is similar to the heuristic method performance. In contrast, when $N = 512$, the heuristic based capacity method outperforms the normal heuristic method with larger $E_b/N_0$. This can be attributed to the positive impact of the polarization phenomenon on the heuristic based capacity method as it is known that the polarization is increased with the increasing in the code size. Hence, the construction with the heuristic based capacity method becomes more useful with large code sizes while for the small and moderate codewords, both methods give similar performance.

VI. CONCLUSION

We investigated in this paper the construction of polar codes under the impact of the impulsive noise in PLC systems. We focused on three methods of code construction which are heuristic, heuristic based capacity and Monte Carlo methods. Furthermore, we showed a comparison of the performance between the heuristic and heuristic based capacity methods in terms of the BER behavior. Note that the successive cancellation decoder is adopted in both hard-decision (HD) and soft-decision (SD) in the comparison. We found that there are slight differences between these methods with moderate and small block length but when the codeword is increased, the heuristic based capacity method outperforms the performance of the conventional heuristic method.

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