Dispatching Parallel Jobs to Achieve Zero Queuing Delay

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Abstract
Zero queueing delay is highly desirable in large-scale computing systems. Existing work has shown that it can be asymptotically achieved by using the celebrated Power-of-d-choices (Pod) policy with a probe overhead \( d = \omega \left( \frac{\log N}{1 - \lambda} \right) \), and it is impossible when \( d = O \left( \frac{1}{\log k} \right) \), where \( N \) is the number of servers and \( \lambda \) is the load of the system. However, these results are based on the model where each job is an indivisible unit, which does not capture the parallel structure of jobs in today’s predominant parallel computing paradigm.

This paper thus considers a model where each job consists of a batch of parallel tasks. In this model, we say a policy leads to zero (asymptotic) queueing delay if the job delay under the policy approaches the delay given by the max of service times of its tasks, as if tasks entered service right upon arrival. We show that zero queueing delay for such parallel jobs can be achieved using a variant of the Pod policy, the batch-filling policy, with a probe overhead \( d = \omega \left( \frac{1}{(1-x) \log k} \right) \), where \( k \) is the number of tasks in each job. This result demonstrates that for parallel jobs, zero queueing delay can be achieved with a smaller probe overhead. We also establish a lower bound on the minimum \( d \) needed: we show that zero queueing delay cannot be achieved if \( d = e^{O \left( \frac{\log N}{\log k^2} \right)} \).

1 Introduction

In view of the rise in the amount of latency-critical workloads in today’s datacenters [26, 22], load-balancing policies with ultra-low latency have attracted great attention (see, e.g., [21, 16, 7, 17, 15]). In particular, it is highly desirable to have a policy under which the delay due to queueing is minimal.

In a classical setting of load-balancing, the celebrated greedy policy, Join-the-Shortest-Queue (JSQ), achieves a minimal queueing delay in the sense that the queueing delay is diminishing as the system becomes large, even in heavy-traffic regimes [31, 30, 21]. Therefore, we say that JSQ achieves a zero (asymptotic) queueing delay. Specifically, consider a system with \( N \) servers where jobs arrive into the system following a Poisson process. Each server has its own queue and serves jobs in the queue in a First-Come-First-Serve manner. Under JSQ, each incoming job will be assigned to a server with the shortest queue length. Then the expected time (in steady state) a job spends in the queue before entering service goes to zero as \( N \) goes to infinity.

However, a drawback of JSQ is that it has a high communication overhead, which can cancel out its advantage of achieving zero queueing delay. For assigning each job, JSQ requires the knowledge of the queue-length information of all the \( N \) servers, which will be referred to as having a probe overhead of \( N \). In a typical cluster of servers, \( N \) is in the tens of thousands range, resulting in intolerable delay due to communication [26, 22].

A load-balancing algorithm that provides tradeoffs between queueing delay and communication overhead is the Power-of-d-choices (Pod) policy [27, 20]. For each incoming job, Pod selects \( d \) queues out of \( N \) queues uniformly at random, and assigns the job to a shortest queue among the \( d \) selected queues. Therefore, Pod has a probe overhead of \( d \). It is
easy to see that when $d = N$, Pod coincides with JSQ, thus achieving a zero queueing delay. However, a fundamental question is: Can zero queueing delay be achieved by Pod with a $d$ smaller than $N$? Or, what is the smallest $d$ for achieving zero queueing delay?

This question has been recently answered in a line of research [21][16][17][15]. In particular, the following results are the most relevant to our paper. Suppose the job arrival rate is $N\lambda$ and job service times are exponentially distributed with rate 1. Then the load of the system is $\lambda$. Consider a heavy-traffic regime with $\lambda = 1 - \beta N^{-\alpha}$, where $\alpha$ and $\beta$ are constants with $0 < \beta < 1$ and $0 < \alpha < 1$. It has been shown that Pod achieves zero queueing delay when $d = \Omega\left(\frac{\log N}{1-\beta}\right)$, and does not have zero queueing delay when $d = O\left(\frac{1}{1-\beta}\right)$.

Although these prior results provide great insights into achieving zero queueing delay, they are all for the classical setting where each job is an indivisible unit. In today’s applications, parallel computing has emerged as a dominant paradigm to support the rapidly growing data volume and computation demands. A job with a parallel structure is no longer a single unit, but consists of multiple components that can run in parallel, resulting in a system dynamics that is very different from the non-parallel model. Therefore, it is of great importance to revisit the fundamental question on the minimum probe overhead needed for achieving zero queueing delay, and answer it under the new parallel paradigm.

In this paper, to capture the parallel structure, we consider a model where each job consists of $k$ tasks. Tasks can run on different servers in parallel, and a job is completed when all its tasks are completed. We assume that task service times are independent and exponentially distributed with rate 1. Recall that $N$ denotes the number of servers in the system. We assume that $k$ grows with $N$ (with exact assumption specified later on), but we suppress this dependency in notation for conciseness.

**Zero queueing delay for parallel jobs**

We are interested in achieving zero queueing delay since this is the regime where the delay due to queueing is minimal and jobs are only subject to delay due to their inherent sizes. In the non-parallel model, it is clear that the delay due to queueing for a job is just the time a job spends waiting in the queue. However, when a job consists of multiple tasks, quantifying the delay due to queueing is more complicated since different tasks experience different queueing times.

In this paper, we propose the following notion of zero queueing delay for parallel jobs. Let $X_1, X_2, \ldots, X_k$ denote the service times of a job’s $k$ tasks. Then if a job does not experience any queueing, its delay is given by $T^* = \max\{X_1, X_2, \ldots, X_k\}$. This is the job delay when all the tasks of the job enter service immediately, so we call it the inherent delay. Let $T$ denote the delay of a job in steady state. Then the delay due to queueing is characterized by the difference $\mathbb{E}[T - T^*]$. We say jobs have zero queueing delay if

$$\frac{\mathbb{E}[T - T^*]}{\mathbb{E}[T^*]} \to 0 \quad \text{as } N \to \infty,$$

i.e., the queueing delay takes a diminishing fraction of the inherent delay. Interestingly, under this notion, zero queueing delay allows tasks in a job to wait in queues for non-negligible times.

**Probe overhead and batch-filling policy**

When a job arrives into the system, a task-assigning policy samples some queues to obtain their queue length information, and then decides how to assign the $k$ tasks to the sampled servers. If the policy samples $kd$ queues, then we say its probe overhead $\mathbb{E}[\mathbb{T}]$ is $d$ since $d$ is the average number of samples per task.

In this paper, we focus on a policy called batch-filling, which has been shown to outperform the naive implementation of Pod and also another policy called batch-sampling for parallel jobs [24][22]. Batch-filling assigns the tasks one by one to the shortest queue, where the queue length is updated after every task assignment.

**Challenges and our results**

For the non-parallel model, to show a zero queueing delay, it suffices to characterize the fraction of non-idle servers since a job can only land in one single queue. However, for parallel jobs, crucially, zero queueing delay of jobs can be achieved even when tasks have non-zero queueing delays. As a result, the analysis becomes much harder – we need to characterize the fractions of servers with queue lengths ranging from zero to a certain threshold. More specifically, the threshold here is $o(\log k)$. A key in our analysis is an interesting state-space collapse result that we discover. This result enables us to use the powerful framework of Stein’s method [4][5].

We consider a system with a job arrival rate of $N\lambda/k$. Then $\lambda$ is the load of the system. We focus on a heavy-traffic regime where $\lambda = 1 - \beta N^{-\alpha}$ with $0 < \beta < 1$ and $0 < \alpha < 0.5$, i.e., the sub-Halfin-Whitt regime. Note that the
larger $\alpha$ is, the faster the load approaches 1 as $N \to \infty$. All the order notation and asymptotic results in this paper are with respect to the regime that $N \to \infty$.

Our main result is that zero queueing delay is achieved when the probe overhead $d$ satisfies

$$d = \omega \left( \frac{1}{(1 - \lambda) \log k} \right),$$

where the number of tasks $k$ satisfies $k = o \left( \frac{N^{0.5} \log^2 N}{\log^2 N} \right)$ and $\frac{1}{\log k} = \Omega(\log N)$. For example, this includes $k = \log^2 N$, $k = N^{0.1}$ when $\alpha < 0.4$, and so on.

Recall that for the non-parallel model, a lower bound result is that zero queueing cannot be achieved when the probe overhead is $O \left( \frac{1}{1 - \lambda} \right)$. In contrast, we can see that for parallel jobs, the probe overhead in (2) can be orderly smaller than $\frac{1}{1 - \lambda}$.

We also prove a lower bound result on the minimum $d$ needed: zero queueing delay is not achievable if

$$d = e^{o \left( \frac{\log N}{\log^2 \log N} \right)},$$

where $k$ satisfies that $k = e^{o(\sqrt{\log N})}$ and $k = \omega(1)$. To establish this lower bound, we utilize the tail bound given by a Lyapunov function in a novel way. This proof technique we develop may be of separate interest itself.

Related works

Load-balancing systems for non-parallel jobs have been extensively studied in the literature. It is well-known that JSQ is delay-optimal under a wide range of assumptions [31, 30]. Although getting exact-form stationary distributions is typically not feasible for most load-balancing policies, many results and approximations are known for various asymptotic regimes.

For JSQ in heavy-traffic regimes, Eschenfeldt and Gamarnik [6] obtain a diffusion approximation in the Halfin-Whitt regime ($\alpha = 0.5$), which has a zero queueing delay in the diffusion limit. The convergence result in [6] is on the process level. Braverman [3] later establish steady-state results and their results imply the convergence of the stationary distributions to the diffusion limit. JSQ has also been studied in the nondegenerate slowdown (NDS) regime ($\alpha = 1$) [10].

The problem of achieving zero queueing delay with Pod has been studied in [21, 16, 17, 15]. Mukherjee et al. [21] show through stochastic coupling that the diffusion limit of Pod with $d = \omega(\log^2 N)$ converges to that of JSQ in the Halfin-Whitt regime, thus resulting in a zero queueing delay. The convergence to the diffusion limit in [21] is on the process level. Zero queueing delay for Pod in steady state is first studied by Liu and Ying [17] for the regime where $\alpha < \frac{1}{2}$, where they show that the waiting probability goes to 0 as $N \to \infty$ when $d = \omega \left( \frac{1}{1 - \lambda} \right)$. The results are later extended to the sub-Halfin-Whitt regime ($0 < \alpha < 0.5$) for both exponential and Coxian-2 service times [16, 15] and beyond-Halfin-Whitt regime ($0.5 \leq \alpha < 1$) [15], where it is shown that zero queueing delay is achieved when $d = \Omega \left( \frac{\log N}{1 - \alpha} \right)$. The paper [17] also provides a lower bound result: the waiting probability is bounded away from 0 when $d = O \left( \frac{1}{1 - \alpha} \right)$ for $0 \leq \alpha < 1$.

Pod has also been analyzed in the regime with a constant load ($\alpha = 0$) as $N \to \infty$. Mean-field analysis has been derived for a constant $d$ in [20, 27], and Mukherjee et al. [21] show $d = \omega(1)$ leads to zero queueing delay. We remark that mean-field analysis results are also available for other policies such as Join-the-Idle-Queue (JIQ) [18, 24], and also for delay-resource tradeoffs [7].

To the best of our knowledge, very limited work has been done on achieving zero queueing delay for parallel jobs, or on analyzing delay for parallel jobs in general. Only the regime with a constant load as $N \to \infty$ has been studied. Mukherjee et al. [21] briefly touch upon this topic and show that fluid-level optimality can be achieved with probe overhead $d \geq \frac{1}{1 - \alpha}$ under the so-called batch-sampling policy [22]. Ying et al. [34] provide limiting distributions for the stationary distributions under (batch-version) Pod, batching-sampling, and batch-filling, but have not analyzed delay of jobs. Wang et al. [29] analyze job delay under a (batch-version) random-routing policy, which does not achieve zero queueing delay. There have been no results for heavy-traffic regimes.

Finally, the techniques we use in this paper are based on Stein’s method and drift-based state-space collapse. Proposed in [23], Stein’s method has been an effective tool for bounding the distance between two distributions. The seminal papers [4, 5, 11] build an analytical framework for Stein’s method in queueing theory that consists of generator
2 Model

We consider a system with \( N \) identical servers, illustrated in Figure 1. Each server has its own queue and serves tasks in its queue in a First-Come-First-Serve manner. Since each queue is associated with a server, we will refer to queues and servers interchangeably. Jobs arrive into the system following a Poisson process. To capture the parallel structure of jobs, we assume that each job consists of \( k \) tasks that can run on different servers in parallel. A job finishes when all of its tasks finish. We study the large-system regime where the number of servers, \( N \), becomes large, and we will let \( k \) increase to infinity with \( N \) to capture the trend of growing job sizes.

We denote the job arrival rate by \( N \lambda/k \) and assume that the service times of tasks are independent and exponentially distributed with rate 1. Then \( \lambda \) is the load of the system. We consider a heavy-traffic regime where \( \lambda = 1 - \beta N^{-\alpha} \) with \( 0 < \beta < 1 \) and \( 0 < \alpha < 0.5 \), i.e., the so-called sub-Halfin-Whitt regime [16, 12].

When a job arrives into the system, we sample \( kd \) queues and obtain their queue length information. Since the average overhead is \( d \) samples per task, the probe overhead is \( d \). We then assign the \( k \) tasks of the job to the \( kd \) selected queues using the batch-filling policy proposed in [34]. Batch-filling assigns the tasks one by one to the shortest queue, where the queue length is updated after each task assignment. Specifically, the task assignment process runs in \( k \) rounds.

For each round, we put a task into the shortest queue among sampled queues. We then update the queue length, and continue to the next round.

Now we give an equivalent description of batch-filling, which is useful in our analysis. For each queue and a positive integer \( \ell \), we use the number of spaces below threshold \( \ell \) to refer to the quantity \( \max\{\ell - \text{queue length}, 0\} \), i.e., the number of tasks we can put in the queue such that the queue length after receiving the tasks is no larger than \( \ell \). For a set of queues \( \mathcal{A} \), we use \( N_\ell(\mathcal{A}) \) (or just \( N_\ell \) when it is clear from the context) to denote the total number of spaces below \( \ell \) in \( \mathcal{A} \). Figure 2 gives an example of \( N_\ell(\mathcal{A}) \). We say a task is at a queueing position \( p \) if there are \( p - 1 \) tasks ahead of it in the queue. With the above terminology, the batch-filling policy can be described in the following way: it finds a minimum threshold \( \ell \) such that the total number of spaces below \( \ell \) in the sampled queues is at least \( k \). Then it fills the \( k \) tasks into these spaces from low positions to high positions.

To define zero queueing delay for parallel jobs, let \( X_1, X_2, \ldots, X_k \) be the service times of the tasks of a job. When a job does not experience any queueing, its delay is given by \( T^* = \max\{X_1, \ldots, X_k\} \), which we call the inherent delay of this job. If the actual delay of the job is very close to its inherent delay, it is as if the job immediately gets
service when it arrives to the system. Therefore, we say a job experiences zero queueing delay if the steady state delay of the job, $T$, satisfies that

$$\frac{E[T - T^*]}{E[T^*]} \to 0 \quad \text{as} \quad N \to \infty.$$ 

We note that as the service time of each task is exponentially distributed with mean 1, it holds that

$$E[T^*] = H_k = \ln k + o(\ln k),$$

where $H_k$ is the $k$-th harmonic number \[19\].

We make the following interesting observation, which provides a basis for our delay analysis of parallel jobs: a job can have zero queueing delay even when its tasks are assigned to non-idle servers. In fact, we establish a necessary and sufficient condition: a job has zero queueing delay if and only if all of its tasks are at queueing positions below a threshold $h$ with $h = o(\log k)$ after assigned to servers, noting that the inherent delay is $\ln k + o(\ln k)$. The formal proof is based on Lemma \[9\]. This phenomenon allows us to have a zero queueing delay with low probe overhead. But it also makes the analysis hard since it implies that there are many situations that can lead to zero queueing delay.

We assume that every queue has a finite buffer size of $b$ including the task in service. If the dispatcher routes a task to a queue with length equal to $b$, we simply discard this task and all the other tasks of the same job. In this case, we say the job is dropped; otherwise, we say the job is admitted. We remark that this assumption is not restrictive for the following two reasons: (1) our results hold for a very large range of $b$ (see Theorem \[1\]), and (2) the probability of discarding a job is very small (see Theorem \[2\]).

To represent the state of the system, let $S_i(t)$ denote the fraction of servers that have at least $i$ jobs at time $t$, where $0 \leq i \leq b$. Note that it always holds $S_0(t) = 1$. Then $S(t) = (S_0(t), S_1(t), \ldots, S_b(t))$ forms a continuous-time Markov chain (CTMC) since batch-filling is oblivious to labels of servers. The state space is as follows:

$$S = \{s = (s_0, s_1, s_2, \ldots, s_b) : 1 \leq s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_b, \quad Ns_i \in \mathbb{N}, \forall 1 \leq i \leq b\}.$$

It can be verified that $\{S(t) : t \geq 0\}$ is irreducible and positive recurrent, thus having a unique stationary distribution. Let $\pi_S$ denote this stationary distribution, and let $S = (S_1, \ldots, S_b)$ be a random element with distribution $\pi_S$.

### 3 Main Results

Our main results provide bounds on queue lengths and delay, which lead to corresponding bounds on the probe overhead for achieving zero queueing delay. We divide our results into upper-bound and lower-bound results. Again, all the asymptotics are with respect to the regime that the number of servers, $N$, goes to infinity.

#### Upper-Bound Results

We first give an upper bound on $E\left[\sum_{i=1}^{b} S_i\right]$, the expected number of tasks in each server, in Theorem \[1\]. This upper bound underpins our analysis of job delay.

**Theorem 1.** Consider a system with $N$ servers where each job consists of $k$ tasks. Let the load be $\lambda = 1 - \beta N^{-\alpha}$ with $0 < \beta < 1$ and $0 < \alpha < 0.5$. Under the batch-filling policy with a probe overhead of $d$ such that $d \geq \frac{8}{(1-\lambda)h}$ for some $h = o(\log k)$ and $h = \omega(1)$, it holds that

$$E\left[\max\left\{\sum_{i=1}^{b} S_i - h \left(1 - \frac{1}{2} \beta N^{-\alpha}\right), 0\right\}\right] \leq \frac{5}{\sqrt{N \log N}},$$

(4)

where $k$ satisfies that $k = o\left(\frac{N^{0.5-\alpha}}{\log N}\right)$ and $\frac{k}{\log k} = \Omega(\log N)$, the buffer size $b = \min\left\{N^\alpha, \frac{N^{0.5-\alpha}}{k}\right\}$, and $N$ is sufficiently large.

We remark that the $h = o(\log k)$ in this theorem represents the threshold position we pointed out for zero queueing delay, i.e., a job has zero queueing delay if all of its tasks are at queueing positions below $h$ after assigned to servers.

The upper bound on $E\left[\sum_{i=1}^{b} S_i\right]$ in Theorem \[1\] indicates how full the queues are. This enables us to analyze the probability that all the tasks of an incoming job end up in positions below $h$ under batch-filling, which further leads to the zero queueing delay result below in Theorem \[2\]. Recall that the buffer size $b$ of each queue is finite, so a job will get dropped if at least one of its tasks is assigned to a queue with a full buffer. We denote the probability of dropping an incoming job in steady state by $p_d$. 

5
**Theorem 2.** Under the assumptions of Theorem 1, the dropping probability under batch-filling, \( p_d \), can be upper bounded as follows when \( N \) is sufficiently large:

\[
p_d \leq \frac{11}{b \sqrt{N \log N}}.
\]

The steady-state delay of jobs that are admitted satisfies that

\[
E[T | \text{admitted}] = \ln k + o(\ln k).
\]  

(5)

Therefore, the batch-filling policy achieves zero queueing delay for parallel jobs.

Theorems 1 and 2 imply that zero queueing delay for parallel jobs can be achieved with a probe overhead \( d = \omega \left( \frac{1}{1 - \frac{1}{4} b N^{-\alpha}} \right) \). This breaks the lower bound of \( \omega \left( \frac{1}{1 - \frac{1}{4} b N^{-\alpha}} \right) \) for achieving zero queueing delay for non-parallel jobs, i.e., single-task jobs [17]. Therefore, the parallel structure helps reduce communication overhead.

**Lower-Bound Results**

To complement the upper-bound results, below we investigate when zero queueing delay cannot be achieved. In Theorem 3, we find conditions under which \( \sum_{i=1}^{h} S_i \) is lower bounded with a constant probability.

**Theorem 3.** Consider a system with \( N \) servers where each job consists of \( k \) tasks. Let the load be \( \lambda = 1 - \beta N^{-\alpha} \) with \( 0 < \beta < 1 \) and \( 0 < \alpha < 0.5 \). Assume that \( b = \infty \) and \( k \) satisfies that \( k = e^{\Theta(\sqrt{\log N})} \) and \( k = \omega(1) \). For any stable task-assigning policy with a probe overhead of \( d \) such that \( d = e^{\Theta(\frac{1}{\log k})} \) and any \( h \) with \( h = O(\log k) \), it holds that when \( N \) is sufficiently large,

\[
P \left\{ \sum_{i=1}^{h} S_i \geq h - \frac{1}{3d} \right\} \geq \frac{1}{4e^2}.
\]  

(6)

The lower bound on \( \sum_{i=1}^{h} S_i \) in Theorem 3 guarantees that an incoming job will have a significant delay in addition to its inherent delay, and thus fails to have zero queueing delay. This result is formally stated in Theorem 4 below.

**Theorem 4.** Under the assumptions of Theorem 3, the steady-state job delay, \( T \), satisfies that

\[
E[T] \geq 2 \ln k
\]  

when \( N \) is sufficiently large. Therefore, to achieve zero queueing delay, the probe overhead \( d \) needs to be at least \( e^{\Omega\left(\frac{1}{\log k} \right)} \).

### 4 Proofs of Upper-Bound Results

In this section, we prove the upper-bound results in Theorems 1 and 2. We first give a proof sketch that provides an overview of the structure of the proofs. We then present the formal proofs of Theorems 1 and 2 in Sections 4.1 and 4.2, respectively. These two proofs rely on lemmas that are presented in Section 4.3, followed by their proofs in Section 4.4. Throughout this section, we assume that the assumptions in Theorem 1 hold.

**Proof Sketch**

We start by setting the goal to be proving the zero queueing delay result in Theorem 2. The need for the fundamental characterizations of the system in Theorem 1 will emerge during the analysis. We first note that the steady-state job delay \( T \) can be upper bounded in the following way:

\[
E[T] \leq E \left[ T \right| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right] + E \left[ T \right| \sum_{i=1}^{h} S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right] \cdot P \left\{ \sum_{i=1}^{h} S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\},
\]  

(8)
where we have used the fact that \( \Pr \{ \sum_{i=1}^{h} S_i \leq h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \} \leq 1. \)

In this upper bound, the conditions in the expectations are based on the threshold value \( h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \) for \( \sum_{i=1}^{h} S_i. \)

We choose this particular threshold value for the following reason. Given the condition \( \sum_{i=1}^{h} S_i \leq h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \), we can show that with high probability, all the tasks of an incoming job will be assigned to queueing positions below \( h \) with \( h = o(\log k) \), thus resulting in a zero queueing delay for the job. Specifically, suppose a job arrives to the system with state \( s \). If we choose one queue uniformly at random from all the queues, then the probability for the chosen queue to have a length of \( i \) is \( s_i - s_{i+1} \). So the expected number of spaces below position \( h \) in the chosen queue is \( \sum_{i=0}^{h}(h-i)(s_i-s_{i+1}) = h - \sum_{i=1}^{h} s_i \). The batch-filling policy samples \( kd \) queues. Thus the total expected number of spaces below position \( h \) in the \( kd \) sampled queues is \( kd \left(h - \sum_{i=1}^{h} s_i\right) \). To fit all the \( k \) tasks of the incoming job to positions below \( h \), we need
\[
kd \left(h - \sum_{i=1}^{h} s_i\right) \geq k,
\]
which becomes
\[
\sum_{i=1}^{h} s_i \leq h \left(1 - \frac{1}{8}\beta N^{-\alpha}\right)
\]
when \( d \geq \frac{8}{(1-\lambda)h} = \frac{8N^{\alpha}}{\beta h} \) as required in Theorem (1). We strengthen this requirement to the condition \( \sum_{i=1}^{h} S_i \leq h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \) to obtain a high-probability guarantee using proper concentration inequalities.

The second summand in the upper bound (8) is based on the condition \( \sum_{i=1}^{h} S_i > h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \). Under this condition, we may not be able to put all the tasks of an incoming job to positions below \( h \). But we show that the probability \( \Pr \{ \sum_{i=1}^{h} S_i > h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \} \) is very small in Theorem (1). To this end, we first upper-bound it using the Markov inequality:
\[
\Pr \left\{ \sum_{i=1}^{h} S_i > h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \right\} \leq \frac{\mathbb{E} \left[ \max \left\{ \sum_{i=1}^{h} S_i - h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right), 0 \right\} \right]}{\frac{1}{4}\beta N^{-\alpha}}.
\]
It then suffices to bound \( \mathbb{E} \left[ \max \left\{ \sum_{i=1}^{h} S_i - h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right), 0 \right\} \right]. \)

We focus on the dynamics of \( \sum_{i=1}^{h} S_i(t) \) in the proof of Theorem (1) which is equal to the total queue length at time \( t \) divided by \( N \). Our proof follows the framework of Stein’s method. The main idea is to couple our Markov chain \( \{S(t) : t \geq 0\} \) with an auxiliary process that is easier to analyze, and bound their difference through generator approximation. Here we consider the following simple fluid model as our auxiliary process:
\[
\dot{x}(t) = (-\delta) \cdot 1_{\{x>0\}},
\]
x(t) is continuous,
\[\text{(9)}\]
where \( \delta = \frac{(k+1)\log N}{\sqrt{N}} \), and we then compare the dynamics of \( \sum_{i=1}^{h} S_i(t) \) with that of \( x(t) \). Based on this coupling, we derive an upper bound on \( \mathbb{E} \left[ \max \left\{ \sum_{i=1}^{h} S_i - h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right), 0 \right\} \right] \) in Section 4.1 below. We reiterate that a key in our analysis is a novel state-space collapse result that we establish.

Combining the arguments above for both the condition \( \sum_{i=1}^{h} S_i \leq h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \) and the condition \( \sum_{i=1}^{h} S_i > h \left(1 - \frac{1}{4}\beta N^{-\alpha}\right) \), we can conclude that the upper bound on \( \mathbb{E}[T] \) in (8) implies zero queueing delay.

4.1 Proof of Theorem (1)

Proof. As we explained in the proof sketch, we consider the fluid model in (9). The generator of this fluid model, denoted as \( \mathcal{G} \), is simply given by
\[
\mathcal{G}g(x) = g'(x) \cdot (-\delta) \cdot 1_{\{x>0\}}
\]
for a differentiable function \( g \). Recall that we will compare the dynamics of \( x(t) \) in this fluid model with that of \( \sum_{i=1}^{b} S_i(t) \).

The quantity of interest in Theorem \([1]\) is \( \mathbb{E} \left[ \max \left\{ \sum_{i=1}^{b} S_i - \eta, 0 \right\} \right] \), where we have used the notation \( \eta = h (1 - \frac{1}{2} \beta N^{-\alpha}) \) for conciseness. Recall that \( S \) follows the stationary distribution of \( \{S(t) : t \geq 0\} \). To couple \( \{S(t) : t \geq 0\} \) with the fluid model, we solve for a function \( g \) such that

\[
\mathcal{G} g(x) = \max \left\{ x - \eta, 0 \right\},
\]

\( g(0) = 0 \). (10)

It is not hard to see that the solution is

\[
g(x) = \frac{(x - \eta)^2}{2(-\delta)} \mathbb{1}(x \geq \eta).
\]

(11)

Now we utilize this function \( g \) to bound \( \mathbb{E} \left[ \max \left\{ \sum_{i=1}^{b} S_i - \eta, 0 \right\} \right] \) through generator approximation. Let \( G \) be the generator of \( \{S(t) : t \geq 0\} \). Then

\[
G \left( \sum_{i=1}^{b} s_i \right) = \sum_{s' \in S} r_{s \rightarrow s'} \left( g \left( \sum_{i=1}^{b} s'_i \right) - g \left( \sum_{i=1}^{b} s_i \right) \right),
\]

where \( r_{s \rightarrow s'} \) is the transition rate from state \( s \) to state \( s' \). Since \( g \left( \sum_{i=1}^{b} s_i \right) \) is bounded on \( S \), it holds that

\[
\mathbb{E} \left[ G \left( \sum_{i=1}^{b} S_i \right) \right] = 0.
\]

(12)

Combining this with the equations in (10) gives,

\[
\mathbb{E} \left[ \max \left\{ \sum_{i=1}^{b} S_i - \eta, 0 \right\} \right] = \mathbb{E} \left[ \mathcal{G} g \left( \sum_{i=1}^{b} S_i \right) \right] = \mathbb{E} \left[ \mathcal{G} g \left( \sum_{i=1}^{b} S_i \right) - G g \left( \sum_{i=1}^{b} S_i \right) \right] = \mathbb{E} \left[ g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - G g \left( \sum_{i=1}^{b} S_i \right) \right]
\]

(13)

This is referred to as the generator approximation since we are approximating the generator \( G \) with \( \mathcal{G} \).

Next we take a closer look at the term \( G g \left( \sum_{i=1}^{b} S_i \right) \) and derive an upper bound for (13). Let \( P_A(s) \) be the probability that a job arrival is admitted into the system given that the system is at state \( s \), i.e., the probability that all the tasks of the job are routed to positions below \( b \). Then

\[
G g \left( \sum_{i=1}^{b} s_i \right) = \frac{N\lambda}{k} P_A(s) \left( g \left( \sum_{i=1}^{b} s_i + \frac{k}{N} \right) - g \left( \sum_{i=1}^{b} s_i \right) \right) + \frac{\eta}{N} \left( g \left( \sum_{i=1}^{b} s_i - \frac{1}{N} \right) - g \left( \sum_{i=1}^{b} s_i \right) \right),
\]

where first term is the drift due to a job arrival and the second term is due to a task departure. To derive an upper bound on (13), we divide the discussion into the three cases below. Recall that \( g(x) = \frac{(x-\eta)^2}{2(-\delta)} \mathbb{1}(x \geq \eta) \) and \( g'(x) = \frac{x-\eta}{-\delta} \mathbb{1}(x \geq \eta) \).

**Case 1:** \( \sum_{i=1}^{b} S_i < \eta - \frac{k}{N} \). In this case, clearly \( g' \left( \sum_{i=1}^{b} S_i \right) = 0 \) and \( G g \left( \sum_{i=1}^{b} S_i \right) = 0 \).
Case 2: \( \sum_{i=1}^{b} S_i \in [\eta - \frac{k}{N}, \eta + \frac{1}{N}] \). By the mean value theorem,
\[
g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - Gg \left( \sum_{i=1}^{b} S_i \right) = g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - \left( \frac{N\lambda}{k} P_A(S) \frac{k}{N} g'(\xi) + \frac{2N^2 g''(\xi)}{2N^2} \right)
\]
\[
\leq g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - \lambda g'(\xi) + S_1 g'(\xi),
\]
where \( \xi \in \left( \sum_{i=1}^{b} S_i, \sum_{i=1}^{b} S_i + \frac{k}{N} \right), \xi \in \left( \sum_{i=1}^{b} S_i - \frac{k}{N}, \sum_{i=1}^{b} S_i \right) \), and (14) is true since \( P_A(S) \leq 1 \) and \( g'(x) \leq 0 \) for all \( x \).

Case 3: \( \sum_{i=1}^{b} S_i \geq \eta + \frac{1}{N} \). Since \( g'(x) \) is continuous for all \( x \), by the second order Taylor expansion in the Lagrange form,
\[
g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - Gg \left( \sum_{i=1}^{b} S_i \right) = g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - \frac{N\lambda}{k} P_A(S) \left( \frac{k}{N} g' \left( \sum_{i=1}^{b} S_i \right) + \frac{k^2}{2N^2} g''(\tilde{\xi}) \right)
\]
\[
- NS_1 \left( \frac{1}{N} g' \left( \sum_{i=1}^{b} S_i \right) + \frac{1}{2N^2} g''(\tilde{\xi}) \right)
\]
\[
\leq g' \left( \sum_{i=1}^{b} S_i \right) (-\delta - \lambda + S_1) - \frac{1}{2N^2} (\lambda k g''(\xi) + S_1 g''(\tilde{\xi})),
\]
where \( \zeta \in \left( \sum_{i=1}^{b} S_i, \sum_{i=1}^{b} S_i + \frac{k}{N} \right), \zeta \in \left( \sum_{i=1}^{b} S_i - \frac{k}{N}, \sum_{i=1}^{b} S_i \right) \).

Combining these three cases yields
\[
E \left[ g' \left( \sum_{i=1}^{b} S_i \right) (-\delta) - Gg \left( \sum_{i=1}^{b} S_i \right) \right]
\]
\[
\leq E \left[ g' \left( \sum_{i=1}^{b} S_i \right) (-\delta - \lambda g'(\xi) + S_1 g'(\tilde{\xi})) \right] \mathbb{I} \{ \sum_{i=1}^{b} s_i \in [\eta - \frac{k}{N}, \eta + \frac{1}{N}] \} \tag{16}
\]
\[
- \frac{1}{2N} E \left[ (\lambda k g''(\xi) + S_1 g''(\tilde{\xi})) \right] \mathbb{I} \{ \sum_{i=1}^{b} s_i \geq \eta + \frac{1}{N} \} \tag{17}
\]
\[
+ E \left[ g' \left( \sum_{i=1}^{b} S_i \right) (-\delta - \lambda + S_1) \right] \mathbb{I} \{ \sum_{i=1}^{b} s_i \geq \eta + \frac{1}{N} \} \tag{18}
\]

The first two terms (16) and (17) are easy to bound once we notice that for any \( x \in [\eta - \frac{k+1}{N}, \eta + \frac{k+1}{N}] \), \(|g'(x)| \leq \frac{|x-\eta|}{\delta} \leq \sqrt{1 \log N} \), and for any \( x \in (\eta, +\infty) \), \(|g''(x)| = \frac{1}{\delta} = \frac{\sqrt{N}}{(k+1) \log N} \). Then when \( N \) is sufficiently large,
\[
|16| \leq \frac{1}{\sqrt{N} \log N} \left( \frac{\sqrt{N}}{k+1} \right) \left( \frac{1}{\sqrt{N}} \right) \leq \frac{3}{\sqrt{N} \log N},
\]
and
\[
|17| \leq \frac{1}{2N} \left( \frac{\sqrt{N}}{(k+1) \log N} \right) \leq \frac{1}{\sqrt{N} \log N}.
\]

The key in this proof is to bound the term (18), for which we need the state-space collapse result in Lemma 3 in Section 4.3. Consider the following Lyapunov function:
\[
V(s) = \min \left\{ \frac{1}{h-1} \sum_{i=h}^{b} s_i, b \left( \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right) - \frac{1}{h-1} \sum_{i=1}^{h-1} s_i \right) \right\}^+, \]

where the superscript $+$ denotes the function $x^+ = \max\{x, 0\}$. Then by Lemma 3 when $N$ is sufficiently large,

$$\Pr \left\{ V(S) > B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}} \right\} \leq e^{-\frac{1}{4} \log^2 N},$$

where $B = \frac{b-h+1}{h-1} \left( \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right)$.

We partition the probability space based on the value of $V(S)$. Note that $g' \left( \sum_{i=1}^{b} S_i \right) (-\delta + \lambda + S_i) \mathbb{1}_{\{ \sum_{i=1}^{b} S_i \geq \eta + \frac{h}{N} \}}$ is always no larger than $\frac{2b}{h}$ for large enough $N$. Then (18) can be upper bounded as

$$\begin{align*}
(18) & \leq E \left[ g' \left( \sum_{i=1}^{b} S_i \right) (-\delta + \lambda + S_i) \mathbb{1}_{\{ \sum_{i=1}^{b} S_i \geq \eta + \frac{h}{N} \}} \mid V(S) \leq B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}} \right] \\
& \quad + \frac{2b}{h} \Pr \left\{ V(S) > B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}} \right\}.
\end{align*} \tag{19}$$

Now we focus on the case where we are given the condition that $V(S) \leq B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}}$. Our goal is to show that $S_1$ is large enough such that $\delta + \lambda - S_1 < 0$. Intuitively, this condition on $V(S)$ implies that we either have a small $\sum_{i=b}^{h} S_i$, which leads to a large $S_1$ when combined with the condition $\sum_{i=1}^{b} S_i \geq \eta + \frac{h}{N}$ in the indicator, or a large $\sum_{i=1}^{h-1} S_i$, which directly gives a large $S_1$ since $S_1 \geq \cdots \geq S_{h-1}$.

If $\frac{1}{h-1} \sum_{i=h}^{b} S_i \leq b \left( (1 - \frac{1}{2} \beta N^{-\alpha}) - \frac{1}{h-1} \sum_{i=1}^{h-1} S_i \right)^+$ in $V(S)$, the condition $V(S) \leq B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}}$ implies that

$$\frac{1}{h-1} \sum_{i=h}^{b} S_i \leq \frac{b-h+1}{h-1} \left( \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right) + \frac{2kb \log^2 N}{(h-1)\sqrt{N}}. \tag{20}$$

Recall that $b = \min \{ N^\alpha, \frac{N^{0.5-\alpha}}{k} \}$ and $h = o(\log k)$. Note that the indicator function in (19) makes it sufficient to consider the case where $\sum_{i=1}^{b} S_i \geq \eta$ and $\sum_{i=h}^{b} S_i \geq \eta$, which implies $(h-1)S_1 + \sum_{i=b}^{h} S_i \geq \eta$. Combining this with (20) gives

$$S_1 \geq \frac{\eta}{h-1} \frac{b-h+1}{h-1} \left( \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right) - \frac{2kb \log^2 N}{(h-1)\sqrt{N}}$$

$$\geq 1 + (1 - \beta) \frac{1}{h-1} \frac{1}{h} \frac{1}{2} \beta N^{-\alpha} + o \left( \frac{1}{h} \right)$$

when $N$ is sufficiently large. Note that $\delta = o \left( \frac{1}{h} \right)$ and $\lambda = 1 - \beta N^{-\alpha}$. Therefore, $\lambda + \delta - S_1 < 0$ when $N$ is sufficiently large.

If $\frac{1}{h-1} \sum_{i=h}^{b} S_i > b \left( (1 - \frac{1}{2} \beta N^{-\alpha}) - \frac{1}{h-1} \sum_{i=1}^{h-1} S_i \right)^+$ in $V(S)$, the condition $V(S) \leq B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}}$ implies that

$$b \left( 1 - \frac{1}{2} \beta N^{-\alpha} - \frac{1}{h-1} \sum_{i=1}^{h-1} S_i \right) \leq B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}}.$$ 

Then

$$S_1 \geq \frac{1}{h-1} \sum_{i=1}^{h-1} S_i$$

$$\geq 1 - \frac{1}{2} \beta N^{-\alpha} - \frac{1}{b} \left( B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}} \right)$$

$$\geq 1 - \frac{1}{2} \beta N^{-\alpha} + o(N^{-\alpha}).$$
As a result, again we have
\[ \lambda + \delta - S_1 \leq -\frac{1}{2} \beta N^{-\alpha} + o(N^{-\alpha}) < 0 \] when \( N \) is sufficiently large.

Inserting these bounds back to (19) gives that when \( N \) is sufficiently large,
\[
(18) \leq 0 + \frac{2b}{\delta} \left\{ V(S) > B + \frac{2kb \log^2 N}{(h - 1)\sqrt{N}} \right\}
\leq \frac{2b}{\delta} e^{-\frac{1}{2} \log^2 N}
\leq \frac{1}{\sqrt{N} \log N}.
\]

Combining the bounds for (16), (17) and (18), we have
\[
E \left[ \max \left\{ \sum_{i=1}^{b} S_i - h \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right), 0 \right\} \right] \leq \frac{5}{\sqrt{N} \log N},
\]
which completes the proof of Theorem 1.

4.2 Proof of Theorem 2

Proof. We first bound the dropping probability \( p_d \) using Lemma 1, which will be presented in Section 4.3. Note that an incoming job does not get dropped if and only if all its \( k \) tasks are routed to queueing positions below threshold \( b \), which is the complement of the event \( \text{FILL}_b \) in Lemma 1.

Thus,
\[
p_d = 1 - \mathbb{P} \{ \text{FILL}_b \}
= 1 - \mathbb{P} \left\{ \text{FILL}_b \mid \sum_{i=1}^{b} S_i \leq b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}
\cdot \mathbb{P} \left\{ \sum_{i=1}^{b} S_i \leq b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}
- \mathbb{P} \left\{ \text{FILL}_b \mid \sum_{i=1}^{b} S_i > b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}
\cdot \mathbb{P} \left\{ \sum_{i=1}^{b} S_i > b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}.
\]

We can easily have that \( \mathbb{P} \left\{ \text{FILL}_b \mid \sum_{i=1}^{b} S_i \leq b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\} \leq \frac{1}{N} \) using Lemma 1.

Now we bound \( \mathbb{P} \left\{ \sum_{i=1}^{b} S_i > b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\} \) using Theorem 1. Note that
\[
\mathbb{P} \left\{ \sum_{i=1}^{b} S_i > b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}
\leq \mathbb{P} \left\{ \max \left\{ \sum_{i=1}^{b} S_i - h \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right), 0 \right\} > b - \frac{b}{4} \beta N^{-\alpha} - h \right\}
\leq \mathbb{P} \left\{ \max \left\{ \sum_{i=1}^{b} S_i - h \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right), 0 \right\} > \frac{b}{2} \right\},
\]
where we have used the fact that \( \frac{b}{2} \beta N^{-\alpha} + h \leq \frac{b}{2} \) when \( N \) is sufficiently large due to our assumptions on \( b \) and \( h \).

Then by Markov’s inequality,

\[
\Pr \left\{ \sum_{i=1}^{b} S_i > b \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\} \\
\leq \frac{\mathbb{E} \left\{ \max \{ \sum_{i=1}^{b} S_i - h \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right), 0 \} \right\}}{\frac{b}{2}} \\
\leq \frac{10}{b \sqrt{N} \log N}.
\]

Combining the arguments above yields

\[
p_d \geq 1 - \frac{10}{N \sqrt{N} \log N} \geq 1 - \frac{11}{b \sqrt{N} \log N}
\]
when \( N \) is sufficiently large.

Next we bound the expected job delay given that a job is admitted, i.e., \( \mathbb{E}[T \mid \text{admitted}] \). We define the delay of a job that is dropped to be zero since it leaves the system immediately after arrival. Then \( \mathbb{E}[T] = \mathbb{E}[T \mid \text{admitted}] \cdot (1 - p_d) + \mathbb{E}[T \mid \text{dropped}] \cdot p_d \), and thus \( \mathbb{E}[T \mid \text{admitted}] = \frac{\mathbb{E}[T]}{1 - p_d} \). So we can focus on bounding \( \mathbb{E}[T] \), following the outline given in the proof sketch.

We bound \( \mathbb{E}[T] \) in the following way

\[
\mathbb{E}[T] \leq \mathbb{E} \left[ T \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right] \\
+ \mathbb{E} \left[ T \left| \sum_{i=1}^{h} S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right] \\
\cdot \Pr \left\{ \sum_{i=1}^{h} S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\}.
\]

For the first term (21) in this upper bound, as described in the proof sketch, we will rely on the fact that with high probability, all the \( k \) tasks are assigned to queueing positions below \( h \). Specifically,

\[
\mathbb{E} \left[ T \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right] \\
= \mathbb{E} \left[ T \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right), \text{FILL}_h \right. \right] \\
\cdot \Pr \left\{ \text{FILL}_h \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right\} \\
+ \mathbb{E} \left[ T \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right), \text{FILL}_h \right. \right] \\
\cdot \Pr \left\{ \text{FILL}_h \left| \sum_{i=1}^{h} S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right\},
\]
where \( \text{FILL}_h \) is the complement of \( \text{FILL}_h \).

Suppose \( \text{FILL}_h \) is true. Suppose that the \( k \) tasks of the incoming job land in \( m \) distinct queues with \( m \leq k \). We call the tasks with the highest positions in these \( m \) queues tasks 1, 2, \ldots, \( m \), and let \( n_1, n_2, \ldots, n_m \) denote these positions. Then the delay of task \( i \) can be written as \( Y_i = \sum_{j=1}^{n_i} X_{i,j} \), where \( X_{i,j} \) is the service time of the task at
position \( j \) in the same queue as task \( i \). Clearly \( X_{i,j} \)'s are i.i.d. with an exponential distribution of rate 1. We know that \( n_i \leq h, i = 1, 2, \ldots, m \) given \( \text{FILL}_h \). Then by Lemma 4,
\[
\mathbb{E}[\max \{ Y_1, \ldots, Y_m \}] \leq \ln k + o(\ln k).
\]
When \( \text{FILL}_h \) is true, \( \mathbb{E} \left[ T \left| \sum_{i=1}^h S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right] \leq bk \) since the highest position for a task is \( b \) and the maximum is upper bounded by the sum. Further, \( \mathbb{P} \left\{ \text{FILL}_h \left| \sum_{i=1}^h S_i \leq h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right\} \leq \frac{1}{N} \) by Lemma 1.

Combining the arguments above, we have the following bound for term (21):
\[
\mathbb{E} \left[ T \left| \sum_{i=1}^h S_i \right. \right] \leq \ln k + o(\ln k) + \frac{bk}{N}.
\]

Now we go back to the term (22). Again, it is easy to see that \( \mathbb{E} \left[ T \left| \sum_{i=1}^h S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right. \right] \leq bk \). Utilizing Theorem 1, we have
\[
\mathbb{P} \left\{ \sum_{i=1}^h S_i > h \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \right\} \leq \mathbb{P} \left\{ \max \left\{ \sum_{i=1}^b S_i - h \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right), 0 \right\} > \frac{1}{4} h \beta N^{-\alpha} \right\}
\]
\[
\leq \frac{20}{h \beta N^{\frac{1}{2} - \alpha} \log N}.
\]

With the bounds above on (21) and (22), we have
\[
\mathbb{E}[T] \leq \ln k + o(\ln k) + \frac{bk}{N} + \frac{20bk}{h \beta N^{\frac{1}{2} - \alpha} \log N}.
\]

Consequently,
\[
\mathbb{E}[T \mid \text{admitted}] = \frac{\mathbb{E}[T]}{1 - p_d} \leq \ln k + o(\ln k) + \frac{bk}{N} + \frac{20bk}{h \beta N^{\frac{1}{2} - \alpha} \log N} \leq \ln k + o(\ln k),
\]
which completes the proof.

### 4.3 Lemmas Needed For the Upper Bound Results

In Lemma 1 below, we consider the event that all the \( k \) tasks of an incoming job are routed to queueing positions below some threshold value \( \ell \). Let this event be denoted by \( \text{FILL}_\ell \), and Lemma 1 lower-bounds its probability \( \mathbb{P}\{\text{FILL}_\ell\} \) for several values of \( \ell \) of interest. Lemma 1 is an essential building block and is needed for establishing the state-space collapse result in Lemma 3 and bounding job delay in Theorem 2.

**Lemma 1 (Filling Probability).** Under the assumptions of Theorem 7 given that the system is in a state \( s \) such that \( \sum_{i=1}^{\ell} s_i \leq \ell \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right) \), the probability of the event \( \text{FILL}_\ell \) for any \( \ell \in \{ h - 1, h, b \} \) can be bounded as follows when \( N \) is sufficiently large:
\[
\mathbb{P}\{\text{FILL}_\ell\} \geq 1 - \frac{1}{N}.
\]

Lemma 2 bounds the distribution tails of a Lyapunov function, which slightly generalizes the tail bounds in [28], [15] and [2].
lemma 2. Consider a continuous time Markov chain \( \{ S(t) \} \) with a unique stationary distribution \( \pi \). Assume it has a finite state space \( S \). For a Lyapunov function \( V : S \to [0, +\infty) \), define the drift of \( V \) at a state \( s \in S \) as

\[
\Delta V(s) = GV(s) = \sum_{s' \in S, s \neq s'} r_{s \to s'} (V(s') - V(s)),
\]

where \( r_{s \to s'} \) is the transition rate from state \( s \) to \( s' \).

Assume that

\[
\nu_{\text{max}} := \sup_{s, s' \in S: r_{s \to s'} > 0} |V(s) - V(s')| < \infty
\]

\[
f_{\text{max}} := \max \left\{ 0, \sup_{s \in S} \sum_{s' : V(s') > V(s)} r_{s \to s'} (V(s') - V(s)) \right\} < \infty.
\]

If there is a set \( \mathcal{E} \) with \( B > 0, \gamma > 0, \delta \geq 0 \) such that

- \( \Delta V(s) \leq -\gamma \) when \( V(s) \geq B \) and \( s \in \mathcal{E} \).
- \( \Delta V(s) \leq \delta \) when \( V(s) \geq B \) and \( s \notin \mathcal{E} \).

Then it holds that for all \( j \in \mathbb{N} \),

\[
\mathbb{P} \{ V(s) \geq B + 2\nu_{\text{max}} j \} \leq \left( \frac{f_{\text{max}}}{f_{\text{max}} + \gamma} \right)^j + \left( \frac{\delta}{\gamma} + 1 \right) \mathbb{P} \{ s \notin \mathcal{E} \}.
\]

This tail bound in Lemma 2 is slightly more general than existing bounds in that it allows different drift bounds based on whether a state \( s \) is in a set \( \mathcal{E} \) or not, which will be needed in the proof of lower-bound results.

We utilize Lemma 2 to establish the state-space collapse result below in Lemma 3. Here we simply let \( \mathcal{E} \) be the whole state space.

lemma 3 (State-Space Collapse). Under the assumption of Theorem 1, consider the following Lyapunov function

\[
V(s) = \min \left\{ \frac{1}{h - 1} \sum_{i=1}^{b} s_{i, b} \left( 1 - \frac{1}{2} \beta N^{-\alpha} \right) - \frac{1}{h - 1} \sum_{i=1}^{h-1} s_{i} \right\} + \frac{b}{\sqrt{N}},
\]

where the superscript \(^+\) denotes the function \( x^+ = \max\{x, 0\} \). Let \( B = \frac{b - h + 1}{h - 1} \left( \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right) \). For any state \( s \) such that \( V(s) > B \), its Lyapunov drift satisfies

\[
\Delta V(s) = GV(s) \leq -\frac{b}{\sqrt{N}}.
\]

Consequently, when \( N \) is sufficiently large,

\[
\mathbb{P} \left\{ V(S) > B + \frac{2kb \log^2 N}{(h - 1)\sqrt{N}} \right\} \leq e^{-\frac{1}{2} \log^2 N},
\]

Lemma 4 below is used in the proof of Theorem 1 to bound job delay based on queueing positions of its tasks. It states that if \( m \) tasks are in queueing positions below \( o(\log m) \), then the maximum delay of these tasks is \( \ln m + o(\ln m) \). Due to space limitations, the proof is given in Appendix A.

lemma 4. Consider \( m \) independent random variables \( Y_1, \ldots, Y_m \) where each \( Y_i \) is the sum of \( n_i \) i.i.d. random variables that follow the exponential distribution with rate 1. In the asymptotic regime that \( m \) goes to infinity, if \( \max\{n_1, \ldots, n_m\} = o(\log m) \), then

\[
\mathbb{E}[\max\{Y_1, \ldots, Y_m\}] \leq \ln m + o(\ln m).
\]

4.4 Proofs of Lemmas 1 and 3

Proof of Lemma 1 (Filling Probability)

Proof. Assume that a job arrival sees a state \( S = s \) that satisfies

\[
\sum_{i=1}^{t} s_i \leq \ell \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right).
\]
We focus on the the number of spaces below the threshold $\ell$ in the sampled queues, denoted by $N_{\ell}$. Then $N_{\ell}$ is the maximum number of tasks that can be put into these queues such that all of these tasks are at queueing positions below $\ell$. Therefore,

$$\Pr \{ \text{FILL}_\ell \} = \Pr \{ N_{\ell} \geq k \} \geq 1 - \Pr \{ N_{\ell} \leq k \}. $$

Now we bound $\Pr \{ N_{\ell} \leq k \}$. We can think of the sampling process of batch-filling as sampling $kd$ queues one by one without replacement. Let $X_1, X_2, \ldots, X_{kd}$ be the numbers of spaces below $\ell$ in the 1st, 2nd, $\ldots$ $kd$th sampled queues, respectively. Then $N_{\ell} = X_1 + \cdots + X_{kd}$. It is not hard to see that for each of the sampled queue and each integer $x$ with $1 \leq x \leq \ell$,

$$\Pr \{ X_i = x \} = s_{\ell-x} - s_{\ell-x+1},$$

and $\Pr \{ X_i = 0 \} = s_{\ell}$.

Note that since we sample without replacement, $X_1, X_2, \ldots, X_{kd}$ are not independent. But we can still derive concentration bounds using a result of Hoeffding [13, Theorem 4]. By this result, we have $E X$ and $P X$ are integer $x$ with $1 \leq x \leq \ell$.

Therefore, the upper bound becomes

\[ E \left[ f \left( \sum_{i=1}^{kd} X_i \right) \right] \leq \exp \left( t \right) \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right), \]

where $Y_1, Y_2, \ldots, Y_{kd}$ are i.i.d. and follow the same distribution as $X_1$. We take the function $f(\cdot)$ to be $f(x) = e^{-tx}$ with $t > 0$. Then

$$\Pr \{ N_{\ell} \leq k \} = \Pr \{ e^{-tN_{\ell}} \geq e^{-tk} \} \leq e^{tk} \prod_{i=1}^{kd} \left( 1 - \frac{1}{t} \sum_{j=1}^{\ell} (s_{\ell-j} - s_{\ell-j+1}) (1 - e^{-tj}) \right).$$

Since $1 - x \leq e^{-x}$ for each $x \geq 0$, this can be further bounded as

$$\Pr \{ N_{\ell} \leq k \} \leq \exp \left( tk + kd \sum_{j=1}^{\ell} (s_{\ell-j} - s_{\ell-j+1}) (1 - e^{-tj}) \right) \leq \exp \left( tk + kd \sum_{j=1}^{\ell} (s_{j-1} - s_{j}) (e^{-t(\ell-j)+1}) - 1 \right) \quad (23)$$

Rearranging the terms in the sum in (23), we get

$$\sum_{j=1}^{\ell} (s_{j-1} - s_{j}) (e^{-t(\ell-j)+1}) - 1 = (e^{-\ell t} - 1) + (e^t - 1) \sum_{j=1}^{\ell} s_j e^{-t(\ell-j)}, \quad (24)$$

Since $1 \geq s_1 \geq \cdots s_\ell$ and we have assumed that $\sum_{j=1}^{\ell} s_j \leq \ell \left( 1 - \frac{1}{4} \beta N^{-\alpha} \right)$, (24) is maximized when $s_1 = s_2 = \cdots = s_\ell = 1 - \frac{1}{4} \beta N^{-\alpha}$.

Therefore, the upper bound becomes

$$\Pr \{ N_{\ell} \leq k \} \leq \exp \left( tk + kd \left( e^{-\ell t} - 1 \right) \frac{1}{4} \beta N^{-\alpha} \right).$$
Now we apply the condition that \( d \geq \frac{gN^a}{hN} \) and let \( t = \frac{\ln(2) - \ln h}{\ell} \). Then
\[
\mathbb{P}\{N_t \leq k\} \\
\leq \exp\left( tk + \frac{2k}{h} (e^{-t\ell} - 1) \right) \\
= \exp\left( \frac{k}{h} \left( \ln(2\ell) - \ln h \right) + \frac{h}{\ell} \right).
\]

Recall the we have assumed that \( \frac{k}{h} = \omega(\log N) \) and \( h = \omega(1) \). Then it can be verified that with a sufficiently large \( N \), \( \frac{h}{\ell} (\ln(2\ell) - \ln h) + \frac{h}{\ell} + 2N^{-0.5} - 2 \) is smaller than a negative constant for all \( \ell \in \{h - 1, h, b\} \). Thus
\[
\mathbb{P}\{N_t \leq k\} \leq \exp(-\omega(\log N)) \leq \frac{1}{N}.
\]

As a result,
\[
\mathbb{P}\{\text{FILL}_t\} \geq 1 - \mathbb{P}\{N_t \leq k\} \geq 1 - \frac{1}{N},
\]
which completes the proof.

\[\square\]

**Proof of Lemma 3 (State-Space Collapse)**

Proof. Consider the Lyapunov function in the lemma, i.e.,
\[
V(s) = \min \left\{ \frac{1}{h - 1} \sum_{i=0}^{b} s_i, b \left( 1 - \frac{1}{2} \beta N^a \right) - \frac{1}{h - 1} \sum_{i=1}^{h} s_i \right\}.
\]

We will refer to the first term and second term in the minimum as \( T_1 \) and \( T_2 \), respectively. Let \( B = \frac{b-h+1}{h-1} (\beta N^{-a} + \frac{\log N}{\sqrt{N}}) \) and suppose \( V(s) > B \). Recall that the drift of \( V \) is given by
\[
\Delta V(s) = GV(s) = \sum_{s' \in S, s \neq s'} r_{s \rightarrow s'} (V(s') - V(s)),
\]
where \( r_{s \rightarrow s'} \) is the transition rate from state \( s \) to \( s' \). Let \( e_i = (0, \cdots , 0, \frac{1}{N}, 0, \cdots , 0) \) be a vector of length \( b \) whose \( i \)th entry is \( \frac{1}{N} \) and all the other entries are zero. We divide the discussion into two cases.

**Case 1:** \( T_1 \leq T_2 \). In this case \( V(s) = T_1 \). When the state transition is due to a task departure from a queue of length \( i \), which has a rate of \( N (s_i - s_i-1) \), then
\[
V(s - e_i) = \begin{cases} 
  V(s), & \text{if } 1 \leq i < h, \\
  V(s) - \frac{1}{N(h-1)}, & \text{if } h \leq i \leq b.
\end{cases}
\]

Now consider the state transition due to a job arrival. Let \( a_i \) be the queueing position that task \( i \) is assigned to. Then the next state can be written as
\[
s + e_{a_1} + \cdots + e_{a_k}.
\]

Note that when the event \( \text{FILL}_{h-1} \) happens, the dispatcher puts all \( k \) tasks to positions below threshold \( h - 1 \). Then under \( \text{FILL}_{h-1} \), \( s_i \) does not change for \( i \geq h \), which implies that
\[
V(s + e_{a_1} + \cdots + e_{a_k}) = V(s).
\]

We can show that \( \mathbb{P}\{\text{FILL}_{h-1}\} \geq 1 - \frac{1}{N} \) using Lemma 1 since \( T_2 \geq T_1 > B > 0 \). Otherwise, i.e., when \( \text{FILL}_{h-1} \) is not true, it is easy to see that
\[
V(s + e_{a_1} + \cdots + e_{a_k}) \leq V(s) + \frac{k}{N(h-1)}.
\]
Therefore,
\[
\Delta V(s) \leq \sum_{i=1}^{b} N(s_i - s_{i+1}) (V(s - e_i) - V(s)) + \frac{N \lambda}{k} \frac{1}{N} \frac{k}{N(h-1)}
\]
\[
= \frac{1}{N(h-1)} - \frac{s_h}{h-1}
\]
\[
\leq \frac{1}{N(h-1)} - \frac{1}{h-1} \sum_{i=h}^{b} s_i.
\]
By the assumption that \( T_1 > B \), we have
\[
\frac{1}{b-h+1} \sum_{i=h}^{b} s_i \geq \frac{h-1}{b-h+1} B = \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}}.
\]
Inserting this back to the upper bound on \( \Delta V(s) \) gives
\[
\Delta V(s) \leq -\frac{1}{h-1} \left( -\frac{1}{N} + \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right).
\]
Since \( \frac{\beta N^{-\alpha}}{h-1} \geq \frac{N^{-\alpha}}{k} \geq \frac{b}{\sqrt{N}} \) and \( \frac{\log N}{\sqrt{N}} \geq \frac{1}{N} \) when \( N \) is sufficiently large, this upper bound becomes
\[
\Delta V(s) \leq -\frac{b}{\sqrt{N}}.
\]
**Case 2:** \( T_1 > T_2 \). In this case \( V(s) = T_2 \). Similarly, a task departs from a queue of length \( i \) at a rate of \( N(s_i - s_{i+1}) \).
The change in \( V(s) \) can be bounded as
\[
V(s - e_i) - V(s) \leq \begin{cases} \frac{b}{N(h-1)} - \alpha, & \text{if } 1 \leq i < h, \\ 0, & \text{if } h \leq i \leq b. \end{cases}
\]
When a job arrives, under the event \( \text{FILL}_{h-1} \),
\[
V(s + e_{a_1} + \ldots + e_{a_k}) = V(s) - \frac{kb}{N(h-1)},
\]
where we have used the fact that \( T_2 > B \). Again, \( \mathbb{P} \{ \text{FILL}_{h-1} \} \geq 1 - \frac{1}{N} \) by Lemma 1. Otherwise, i.e., when \( \text{FILL}_{h-1} \) is not true, \( V(s + e_{a_1} + \ldots + e_{a_k}) \leq V(s) \).

Therefore,
\[
\Delta V(s) \leq \sum_{i=1}^{b} N(s_i - s_{i+1}) (V(s - e_i) - V(s))
\]
\[
= \frac{N \lambda}{k} \left( 1 - \frac{1}{N} \right) \left( -\frac{kb}{N(h-1)} \right)
\]
\[
\leq \frac{b}{h-1} \left( s_1 - s_h \right) - \frac{b}{h-1} \left( 1 - \frac{1}{N} \right) (1 - \beta N^{-\alpha})
\]
\[
\leq \frac{b}{h-1} \left( 1 - \left( \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \right) \right) - \left( 1 - \frac{1}{N} \right) (1 - \beta N^{-\alpha})
\]
\[
= \frac{b}{h-1} \left( 1 - \frac{\log N}{\sqrt{N}} + \frac{1}{N} \right) (1 - \beta N^{-\alpha})
\]
\[
\leq -\frac{b}{h-1} \left( \frac{\log N}{\sqrt{N}} - \frac{1}{\sqrt{N}} \right)
\]
where \( (25) \) is due to the fact that \( s_1 \leq 1 \) and the fact that \( s_h \geq \beta N^{-\alpha} + \frac{\log N}{\sqrt{N}} \) following similar arguments as those in Case 1 noting that \( T_1 > T_2 > B \). When \( N \) is sufficiently large, this upper bound becomes
\[
\Delta V(s) \leq -\frac{b}{\sqrt{N}}.
\]
which completes the proof of the drift bound in Lemma 3.

For this Lyapunov function $V$, under the notation in Lemma 2, we have that $\nu_{\text{max}} \leq \frac{kb}{N(h-1)}$ and $f_{\text{max}} \leq \frac{b}{h-1}$. Let $\mathcal{E} = \mathcal{S}$ and $j = \sqrt{N} \log^2 N$. Then by Lemma 2, the drift bound implies that

$$
\mathbb{P}\left\{ V(S) > B + \frac{2kb \log^2 N}{(h-1)\sqrt{N}} \right\}
= \mathbb{P}\left\{ V(S) > B + \frac{2kb}{(h-1)N^j} \right\}
\leq \left( 1 + \frac{h-1}{\sqrt{N}} \right)^{-j}
\leq \left( \left( 1 + \frac{1}{\sqrt{N}} \right)^{\sqrt{N+1}} \right)^{-\frac{1}{\sqrt{N}}} \sqrt{N} \log^2 N
\leq e^{-\frac{1}{2} \log^2 N},
$$

where the last inequality holds when $N$ is sufficiently large. This completes the proof. \qed

5 Proof of the Lower-Bound Results

In this section, we prove the lower-bound results in Theorems 3 and 4. We first present the proofs of Theorems 3 and 4 in Sections 5.1 and 5.2, respectively. Then we give the lemmas needed in Section 5.3. Due to space limitations, the proofs of the lemmas are given in Appendix B. Throughout this section, we assume that the assumptions in Theorem 3 hold.

5.1 Proof Of Theorem 3

Proof. The proof proceeds in an iterative fashion. The base case is that $E[S_1] = \lambda = 1 - \beta N^{-\alpha}$, which can be proved using the Little’s law. We will then bound $S_1 - S_i$ based on properties of $S_1 - S_{i-1}$.

For simplicity, let $u = 2kd$, this is the ratio appearing in Lemma 6. Consider a Lyapunov function $V_1(s) = s_1$.

Let $h = O(\log k)$ and $B_1 = 1 - h\beta N^{-\alpha}$. For some state $s$ such that $V_1(s) > B_1$, it holds

$$
\Delta V_1(s) = \sum_{s': s \to s'} \text{due to an arrival} r_{s \to s'} (V_1(s') - V_1(s)) + \sum_{s': s \to s'} \text{due to a departure} r_{s \to s'} (V_1(s') - V_1(s)) \leq uh\beta N^{-\alpha} - N(s_1 - s_2) \frac{1}{N} = uh\beta N^{-\alpha} - (s_1 - s_2),
$$

where (a) is due to Lemma 6.

Let $p_2 = \mathbb{P}\{ S_1 - S_2 \leq uh^2 \beta N^{-\alpha} \}$ and

$$
\mathcal{E}_1 = \{ s \in \mathcal{S} | s_1 - s_2 > uh^2 \beta N^{-\alpha} \}.
$$

Then $p_2 = \mathbb{P}\{ S \not\in \mathcal{E}_1 \}$. We now use the tail bound in Lemma 2. Assume that we follow the notation in the lemma. Consider the following two cases:

- $s \not\in \mathcal{E}_1$. $\Delta V_1(s) \leq uh\beta N^{-\alpha} =: \delta$.
- $s \in \mathcal{E}_1$. Let $\gamma = -\Delta V_1(s)$. It holds $\gamma \geq uh\beta N^{-\alpha}(h - 1)$.

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Following the definition in \(\text{[2]}\), it is easy to verify that \(\nu_{\max} \leq \frac{k}{N}\) and \(f_{\max} \leq 1\) for \(V_1(s)\). Let \(j_1 = \left(1 + \frac{N^\alpha}{\beta u h (h-1)}\right) \log^2 N\). By Lemma \(\text{[2]}\), it holds that

\[
\mathbb{P} \{ V_1(S) > B_1 + 2\nu_{\max} j_1 \} \leq \left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_1} + \left( \frac{\delta}{\gamma} + 1 \right) \mathbb{P} \{ S \notin E_1 \}
\]

\[
\leq \left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_1} + \frac{h}{h-1} p_2.
\]

Besides, when \(N\) is large enough,

\[
\left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_1} \leq (1 + uh\beta N^{-\alpha}(h-1))^{-\left(1 + \frac{N^\alpha}{\beta u h (h-1)}\right) \log^2 N}
\]

\[
\leq e^{-\log^2 N}.
\]

As a result,

\[
\mathbb{P} \{ V_1(S) > B_1 + 2\nu_{\max} j_1 \} \leq N^{-\log N} + \frac{h}{h-1} p_2.
\]

Since \(0 < \alpha < 0.5\) and \(k = e^{o(\sqrt{\log N})}\),

\[
1 - (h-1)\beta N^{-\alpha} > 1 - h\beta N^{-\alpha} + 2 \frac{k}{N} \left(1 + \frac{N^\alpha}{\beta u h (h-1)}\right) \log^2 N
\]

when \(N\) is large enough. It follows that

\[
\mathbb{P} \{ V_1(S) > 1 - (h-1)\beta N^{-\alpha} \} \leq \mathbb{P} \{ V_1(S) > B_1 + 2\nu_{\max} j_1 \}
\]

\[
\leq N^{-\log N} + \frac{h}{h-1} p_2
\]

However, by Lemma \(\text{[5]}\),

\[
\mathbb{P} \{ V_1(S) > 1 - (h-1)\beta N^{-\alpha} \} \geq 1 - \frac{1}{h-1}.
\]

Therefore,

\[
\frac{h}{h-1} p_2 + N^{-\log N} \geq \frac{h-2}{h-1},
\]

and thus

\[
\mathbb{P} \{ S_1 - S_2 \leq uh^2 \beta N^{-\alpha} \} = p_2 \geq \frac{h-2}{h} - N^{-\log N}.
\]

Let \(b_q = u^{q-1} h q \beta N^{-\alpha}\) for an integer \(q > 0\). Define a sequence \(a_q\), such that \(a_1 = 0, a_2 = 1\) and \(a_q = (q-2)a_{q-1} + 2\) for \(q > 2\). We now have

\[
\mathbb{P} \{ S_1 - S_2 \leq a_2 b_2 \} \geq \frac{h-2}{h} - N^{-\log N}.
\]

We can use Lemma \(\text{[7]}\) successively to establish

\[
\mathbb{P} \{ S_1 - S_q \leq a_q b_q \} \geq \left( \frac{h-2}{h} \right)^{q-1} - (q-1)N^{-\log N}
\]

for all \(2 \leq q \leq h\).

Let us condition on \(S_1 - S_h \leq a_h b_h\). For ease of notation, let \(p_c = \left( \frac{h-2}{h} \right)^{h-1} - (h-1)N^{-\log N}\), which is a lower bound on the probability of the condition. Note that

\[
E[S_1] \leq E[S_1 | S_1 - S_h \leq a_h b_h] \mathbb{P} \{ S_1 - S_h \leq a_h b_h \} + 1 \cdot \mathbb{P} \{ S_1 - S_h > a_h b_h \}.
\]

Thus

\[
E[S_1 | S_1 - S_h \leq a_h b_h] \geq \frac{1 - \beta N^{-\alpha} - (1 - \mathbb{P} \{ S_1 - S_h \leq a_h b_h \})}{\mathbb{P} \{ S_1 - S_h \leq a_h b_h \}}
\]

\[
\geq 1 - \frac{\beta}{p_c} N^{-\alpha}.
\]
We can also see that

\[ P \left\{ \sum_{i=1}^{h} S_i \geq h - \frac{1}{3d} \right\} \]
\[ \geq P \left\{ \sum_{i=1}^{h} S_i \geq h - \frac{1}{3d} \mid S_1 - S_h \leq a_h b_h \right\} P \{ S_1 - s_h \leq a_h b_h \} \]
\[ \geq p_c P \left\{ h S_1 - h(S_1 - S_h) \geq h - \frac{1}{3d} \mid S_1 - S_h \leq a_h b_h \right\} \]
\[ \geq p_c P \left\{ S_1 \geq 1 - \frac{1}{3dh} + a_h b_h \mid S_1 - S_h \leq a_h b_h \right\} \].

Utilizing the Markov inequality gives

\[ (26) \geq p_c \left( 1 - \frac{3dh - 3dhE [S_1 \mid S_1 - S_h \leq a_h b_h]}{1 - 3dha_h b_h} \right) \]
\[ \geq p_c \left( 1 - \frac{\beta}{p_c} \frac{3dh}{1 - 3dha_h b_h} N^{-\alpha} \right) . \]

Recall that \( a_q = (q - 2)a_{q-1} + 2 \) for \( q > 2 \) and \( a_2 = 1 \). We have \( a_h \leq 2h^\epsilon \), and thus \( a_h b_h \leq 2\epsilon h^2 N^{-\alpha} \). As \( d = e^{o(\log N / \log k)} \), \( k = e^{o(\sqrt{\log N})} \), \( h = O(\log k) \), we have \( \ln(a_h b_h) = \Omega(\log N) \). Furthermore, since \( \ln(3dh) = o(\log N / \log k) + O(\log k), \alpha > 0 \), it holds

\[ 1 - \frac{\beta}{p_c} \frac{3dh}{1 - 3dha_h b_h} N^{-\alpha} \geq \frac{1}{2} \]

if \( N \) is sufficiently large. Note that \( p_c \) is equal to \( (\frac{h-2}{h})^{h-1} - (h-1)N^{-\log N} \) which converges to \( \frac{1}{e^2} \). We could conclude that when \( N \) goes to infinity, we have

\[ P \left\{ \sum_{i=1}^{h} S_i \geq h - \frac{1}{3d} \right\} \geq \frac{1}{4e^2} . \]

\[ \square \]

### 5.2 Proof Of Theorem \[4\]

**Proof.** Let \( h = 12\epsilon^2 \log k \). Then \( h = O(\log k) \). Suppose that we have an incoming job. By Theorem \[3\] and the PASTA property of a Poisson arrival process, with probability at least \( \frac{1}{4e^2} \), this job will see a state \( s \) such that \( \sum_{i=1}^{h} s_i \geq h - \frac{1}{3d} \).

By Lemma \[5\], the dispatcher will route at least one task of this job into a queue of length at least \( h + 1 \) with probability \( 1 - o(1) \). Let \( T \) be the delay of the job. Then it holds for a large enough \( N \),

\[ E[T] \geq 3\log k(1 - o(1)) \geq 2\log k, \]

which completes the proof. \[ \square \]

### 5.3 Lemmas Needed for Lower-Bound Results

**Lemma 5.** Assume that the system is stable. Then for any \( x > 0 \),

\[ P \{ S_1 < 1 - x \} \leq \frac{\beta N^{-\alpha}}{x} . \]

**Lemma 6.** Let \( \ell \) be a threshold such that \( 1 \leq \ell \leq h \) with \( h = O(\log k) \). Suppose that an incoming job sees a state \( s \) such that \( \sum_{i=1}^{\ell} s_i \geq \ell - x \), where \( x = \Omega(hN^{-\alpha}) \) and \( x = e^{-\Omega(\log N)} \). Consider a Lyapunov function \( V(\ell) = s_1 + s_2 + \ldots + s_\ell \). It holds that when \( N \) is sufficiently large,

\[ \sum_{s \rightarrow s'} r_{s \rightarrow s'} (V(\ell')) - V(\ell) \leq 2kdx, \]

where \( r_{s \rightarrow s'} \) is the transition rate, and \( s \rightarrow s' \) due to an arrival means that \( s \) will move to state \( s' \) on the Markov chain only if there is an incoming job.
Lemma 7 below is a key in establishing the iterative proof. This lemma relates $S_q$ to $S_{q-1}$ for $3 \leq i \leq h$.

**Lemma 7.** Define $u = 2kd$ and $b_q = u^{-1}h^3\beta N^{-\alpha}$ for $q \in \mathbb{N}$. Define a sequence $a_q$, such that $a_1 = 0$, $a_2 = 1$ and $a_q = (q-2)a_{q-1} + 2$ for $q > 2$. For any $q$ with $3 \leq q \leq h$, if

$$
\mathbb{P} \{ S_1 - S_{q-1} \leq a_{q-1}b_{q-1} \} \geq \left( \frac{h-2}{h} \right)^{q-2} - (q-2)N^{-\log N},
$$

then

$$
\mathbb{P} \{ S_1 - S_q \leq a_qb_q \} \geq \left( \frac{h-2}{h} \right)^{q-1} - (q-1)N^{-\log N}.
$$

Lemma 8 below complements the probability bound in Lemma 1. Recall that $\text{FILL}_h$ denotes the event that all the $k$ tasks of an incoming job are assigned to queueing positions below a threshold $h$. Lemma 8 gives a condition on the total queue length for $\text{FILL}_h$ to happen with low probability.

**Lemma 8.** Suppose an incoming job sees a state $s$ such that $\sum_{i=1}^h s_i > h - \frac{1}{3m}$. Then when $N$ is sufficiently large,

$$
\mathbb{P} \{ \text{FILL}_h \} = o(1).
$$

6 Conclusions and Future Work

We studied a load balancing algorithm, batch-filling, for a system where each job consists of $k$ parallel tasks in the sub-Halfin-Whitt regime of heavy traffic. We showed that to achieve zero queueing delay for such jobs, we only need a probe overhead of $d = \omega \left( \frac{1}{(1-\lambda)\log k} \right)$ under proper conditions. Existing work has shown that $d = \omega \left( \frac{1}{1-\alpha} \right)$ is necessary for achieving zero queueing delay when each job consists of a single task. Therefore, with a parallel structure, we save a factor of $\log k$ communication overhead. We also established a lower-bound result on the probe overhead $d$, where we showed that $d = \Omega \left( \exp \left( \frac{\log N}{\log k} \right) \right)$ is necessary for achieving zero queueing delay. An interesting future direction is to extend our results to general service time distributions, where it is possible to get more savings when the distributions have a heavy tail.

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A Proof of Lemma 4

Proof. The proof idea is similar to that in [14]. Let \( M_X(s) \) be the moment generating function of a random variable \( X \). By assumption, \( Y_i = \sum_{j=1}^{n_i} X_{i,j} \), and \( X_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n_i \), are all independent and exponentially distributed with mean 1. Therefore, for any \( 1 \leq i \leq m, 1 \leq j \leq n_i \) and any \( s < 1 \),

\[
M_{X_{i,j}}(s) = \mathbb{E}[e^{sX_{i,j}}] = \frac{1}{1-s}^{n_i}
\]

\[
M_{Y_i}(s) = \mathbb{E}[e^{sY_i}] = \left( \frac{1}{1-s} \right)^{n_i}.
\]

Let \( q = \max\{n_1, \cdots, n_m\} \). It holds that for any \( s \in (0, 1) \),

\[
\exp \left( s \mathbb{E} \left[ \max_{j=1}^{m} Y_j \right] \right) \leq \mathbb{E} \left[ \exp \left( s \max_{j=1}^{m} Y_j \right) \right] \leq \sum_{j=1}^{m} \mathbb{E} \left[ \exp (sY_j) \right] \leq m \left( \frac{1}{1-s} \right)^q,
\]

where (27) is due to Jensen’s inequality and (29) is true since the maximum is upper bounded by the sum. As a result,

\[
\mathbb{E} \left[ \max_{j=1}^{m} Y_j \right] \leq \frac{\ln m}{s} + \frac{q}{s} \ln (1-s).
\]

Since we assume that \( q = o(\log m) \), we can write \( q = (\ln m) \cdot \ell(m) \) where \( \ell(m) \to 0^+ \) as \( m \to \infty \). Let \( s = 1 - \ell(m) \), then

\[
\mathbb{E} \left[ \max_{j=1}^{m} Y_j \right] \leq \frac{\ln m}{1 - \ell(m)} \left( 1 - \ell(m) \ln (\ell(m)) \right) = (\ln m) \left( 1 + \frac{\ell(m)}{1 - \ell(m)} \right) \left( 1 - \ell(m) \ln (\ell(m)) \right).
\]

Note that \( \lim_{m \to \infty} \ell(m) \ln (\ell(m)) = 0 \). Then as \( m \to \infty \),

\[
\mathbb{E} \left[ \max_{j=1}^{m} Y_j \right] \leq (\ln m) (1 + o(1)),
\]

which completes the proof.

B Proofs of Lemmas 5–8

B.1 Proof of Lemma 5

Proof. By the Little’s law, it holds that \( \mathbb{E}[S_1] = \lambda = 1 - \beta N^{-\alpha} \). Then \( \mathbb{E}[1 - S_1] = \beta N^{-\alpha} \). Therefore, by the Markov inequality, for any \( x > 0 \),

\[
\mathbb{P} \{ S_1 < 1 - x \} = \mathbb{P} \{ 1 - S_1 > x \} \leq \frac{\beta N^{-\alpha}}{x}.
\]

B.2 Proof of Lemma 6

Proof. Suppose that an arrival sees a state \( s \). Given \( \sum_{i=1}^{\ell} s_i \geq \ell - x \), we have \( s_\ell \geq 1 - x \) since \( s_i \leq 1 \) for all \( 1 \leq i \leq \ell \). Without loss of generality, we can think of the batch-filling policy as sampling the \( kd \) queues one by one.
During the sampling, we always choose at most $kd$ servers of length at least $\ell$. The probability that all $kd$ sampled servers have length at least $\ell$ is thus larger or equal to

$$\left( \frac{N(1-x) - kd}{N} \right)^{kd} = \left( 1 - \left( x + \frac{kd}{N} \right) \right)^{kd}.$$  

Recall that by the assumptions in Theorem 3, we have $x = e^{-\Omega(\log N)}$, $kd = O(N^{1-\alpha})$, and thus $x + \frac{kd}{N} > -1$ when $N$ is sufficiently large. Furthermore, applying Bernoulli’s Inequality and the assumption that $x = \Omega(hN^{-\alpha})$, it holds

$$\left( 1 - \left( x + \frac{kd}{N} \right) \right)^{kd} \geq 1 - kd \left( x + \frac{kd}{N} \right) \geq 1 - 2xkd$$  

for a large $N$. Note that if we put all tasks of this arrival into servers of length at least $\ell$, we will not affect the value of $V_i(s)$. As a result,

$$\sum_{\text{arrivals}} r_{s \rightarrow s'} (V_i(s') - V_i(s))$$  

which completes the proof.

\[ \square \]

### B.3 Proof of Lemma 7

**Proof.** The proof is close to that of Theorem 5. Recall that for each $1 \leq \ell \leq h$ and state $s \in \mathcal{S}$, we define the Lyapunov function

$$V_i(s) = \sum_{i=1}^{\ell} s_i.$$  

For $q$ such that $3 \leq q \leq h$, by assumption,

$$\mathbb{P} \{ S_1 - S_{q-1} \leq a_{q-1} b_{q-1} \} \geq \left( \frac{h - 2}{h} \right)^{q-2} - (q - 2)N^{-\log N}.$$  

It holds

$$\mathbb{P} \{ V_{q-1}(S) < q - 1 - ((q-2)a_{q-1} + 1) b_{q-1} \}$$  

$$\leq \mathbb{P} \{ V_{q-1}(S) < q - 1 - ((q-2)a_{q-1} + 1) b_{q-1},$$  

$$\quad S_1 - S_{q-1} \leq a_{q-1} b_{q-1} \}$$  

$$+ \mathbb{P} \{ S_1 - S_{q-1} > a_{q-1} b_{q-1} \}$$  

$$\leq \mathbb{P} \{ (q - 1) S_1 < q - 1 - b_{q-1} \} + 1 - \left( \frac{h - 2}{h} \right)^{q-2} + (q - 2)N^{-\log N}$$  

(31)

The last inequality uses Lemma 5 and $b_{q-1} = u^{q-2}h^{q-1}N^{-\alpha}$.  

Now let $B_{q-1} = q - 1 - ((q-2)a_{q-1} + 2) b_{q-1}$. We can see that $B_{q-1} = q - 1 - a_q b_{q-1}$. For a state $s$ such that $V_{q-1}(s) > B_{q-1}$, it holds

$$\Delta V_{q-1}(s) = \sum_{s' : s \rightarrow s'} r_{s \rightarrow s'} (V_{q-1}(s') - V_{q-1}(s))$$  

$$+ \sum_{s' : s \rightarrow s'} r_{s \rightarrow s'} (V_{q-1}(s') - V_{q-1}(s)).$$  

Recall that we define $u = 2kd$ and $b_q = u^{q-1}h^{q-1}N^{-\alpha}$. As $V_{q-1}(s) > q - 1 - a_q b_{q-1}$, by Lemma 6 it holds

$$\Delta V_{q-1}(s) \leq 2kd a_q b_{q-1} - (s_1 - s_q)$$  

$$= a_q u^{q-1}h^{q-1}\beta N^{-\alpha} - (s_1 - s_q).$$  

Let $P \{ S_1 - S_q \leq a_q b_q \} = p_q$. Then $P \{ S \not\in E_{q-1} \} = p_q$. For a state $s$, consider the following two cases.
\[ s \notin E_{q-1}, \Delta V_{q-1}(s) \leq a_q u q^{-1} h^{q-1} \beta N^{-\alpha} =: \delta. \]

\[ s \in E_{q-1}. \text{ Let } \gamma = -\Delta V_{q-1}(s). \text{ It holds} \]

\[ \gamma \geq a_q u q^{-1} h^{q-1} \beta N^{-\alpha} (h - 1). \]

We then utilize the tail bound, Lemma 2. Following the definition in Lemma 2, it is easy to verify that \( \nu_{\max} \leq k N, f_{\max} \leq 1 \) for the Lyapunov function \( V_{q-1}(s) \). Let \( j_{q-1} = \left( 1 + \frac{N^\alpha}{a_q u q^{-1} h^{q-1} (h - 1) \beta} \right) \log^2 N. \)

Using Lemma 2,

\[
\mathbb{P}\{V_{q-1}(S) > B_{q-1} + 2\nu_{\max} j_{q-1}\} \\
\leq \left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_{q-1}} + \left( \frac{\delta}{\gamma} + 1 \right) \mathbb{P}\{S \notin E_{q-1}\} \\
\leq \left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_{q-1}} + \frac{h}{h - 1} P_q.
\]

Note that when \( N \) is sufficiently large,

\[
\left( \frac{f_{\max}}{f_{\max} + \gamma} \right)^{j_{q-1}} \leq e^{-\log^2 N}.
\]

Besides, we assume that \( 0 < \alpha < 0.5, k = e^{o(\sqrt{\log N})} \) and \( h = O(\log k) \). As a result, for a large \( N \),

\[
\mathbb{P}\{V_{q-1}(S) \geq q - 1 - ((q - 2)a_{q-1} + 1)b_q\} \\
\leq \mathbb{P}\{V_{q-1}(S) > B + 2\nu_{\max} j_{q-1}\} \\
\leq e^{-\log^2 N} + \frac{h}{h - 1} P_q.
\]

Together with Eq. (31), we have

\[
\left( \frac{h - 2}{h} \right)^{q-2} \left( \frac{q - 1}{u^{q-2} h^{q-1}} \right) - (q - 2)N^{-\log N} \\
\leq \mathbb{P}\{V_{q-1}(S) \geq q - 1 - ((q - 2)a_{q-1} + 1)b_q\} \\
\leq e^{-\log^2 N} + \frac{h}{h - 1} P_q
\]

We can conclude that for a large \( N \),

\[
\mathbb{P}\{S_1 - S_q \leq a_q b_q\} = p_q \geq \left( \frac{h - 2}{h} \right)^{q-1} - (q - 1)N^{-\log N},
\]

which completes the proof.

\[ \square \]

**B.4 Proof of Lemma 8**

**Proof.** We use a similar argument as the proof of Lemma 1. Suppose that an arrival sees a state \( s \). By assumption, it holds

\[
\sum_{i=1}^{h} s_i \geq h - \frac{1}{3d}.
\]

Let \( X_1, \cdots, X_{kd} \) be the numbers of places below \( h \) in each sampled server. The goal is to show

\[
\mathbb{P}\{\text{FILL}_h\} = \mathbb{P}\left\{ \sum_{i=1}^{kd} X_i \geq k \right\} = o(1)
\]

when \( N \) is large enough.
We could see that for each integer \( x \) such that \( 1 \leq x \leq h \), \( \mathbb{P}(X_i = x) = s_{h-x} - s_{h-x+1} \), and \( \mathbb{P}(X_i = 0) = s_h \). Since we are sampling without replacement, \( X_1, \ldots, X_{kd} \) are not independent. But still, utilizing a result of Hoeffding [13, Theorem 4], we have \( \mathbb{E}[f(\sum_{i=1}^{kd} X_i)] \leq \mathbb{E}[f(\sum_{i=1}^{kd} Y_i)] \) for any continuous and convex function \( f(\cdot) \), where \( Y_1, \ldots, Y_{kd} \) are i.i.d. and follow the same distribution as \( X_1 \). Take \( f(\cdot) \) to be \( e^{tx} \) where \( t \) is some positive value.

It then holds
\[
\mathbb{P}\{ \text{FILL}_h \} = \mathbb{P}\left\{ \sum_{i=1}^{kd} X_i \geq k \right\} = \mathbb{P}\left\{ e^{t\sum_{i=1}^{kd} X_i} \geq e^{tk} \right\} \leq e^{-tk} \prod_{i=1}^{kd} \mathbb{E}[e^{tY_i}] = e^{-tk} \prod_{i=1}^{kd} \left( 1 + \sum_{j=1}^{h} \left( e^{t(h-j+1)} - 1 \right) \right).
\]

Since for all \( x > 0, 1 + x \leq e^x \), we can further have
\[
\mathbb{P}\{ \text{FILL}_h \} \leq e^{-tk} \exp\left( k \sum_{j=1}^{h} \left( e^{t(h-j+1)} - 1 \right) (s_{j-1} - s_j) \right).
\]  (32)

Rearranging the sum in (32), we get
\[
\sum_{j=1}^{h} \left( e^{t(h-j+1)} - 1 \right) (s_{j-1} - s_j) = e^{th} - \sum_{j=1}^{h} s_j \left( e^{t(h-j+1)} - e^{t(h-j)} \right) = e^{th} - (e^t - 1) \sum_{j=1}^{h} s_j e^{t(h-j)}.
\]  (33)

Recall that \( \sum_{j=1}^{h} s_j \geq h - \frac{1}{3d} \), and \( 1 \geq s_1 \geq s_2 \geq \cdots \geq s_h \geq 0 \). Eq. (33) is maximized when \( s_1 = s_2 = \cdots = s_h = 1 - \frac{1}{3dh} \) and thus,
\[
(33) \leq (e^{th} - 1) \frac{1}{3dh}.
\]

Plug it into Inequality (32),
\[
\mathbb{P}\{ \text{FILL}_h \} \leq \min_{t>0} \exp\left( k \left( -t + \frac{e^{th} - 1}{3h} \right) \right).
\]

Pick \( t = \ln 3 \). It holds
\[
\mathbb{P}\{ \text{FILL}_h \} \leq \exp\left( \frac{k}{3h} (-3 \ln 3 + 2) \right).
\]

By the assumption that \( \frac{k}{h} = \omega(1) \), we could conclude that
\[
\mathbb{P}\{ \text{FILL}_h \} = o(1)
\]
when \( N \) is sufficiently large.