Abstract

Recent work on the spectrum of the Euclidean Dirac operator spectrum show that the exact microscopic spectral density can be computed in both random matrix theory, and directly from field theory. Exact relations to effective Lagrangians with additional quark species form the bridge between the two formulations. Taken together with explicit computations in the chGUE random matrix ensemble, a series of universality theorems are used to prove that the finite-volume QCD partition function coincides exactly with the universal double-microscopic limit of chUE random matrix partition functions. In the limit where $N_f$ and $N_c$ both go to infinity with the ratio $N_f/N_c$ fixed, the relevant effective Lagrangian undergoes a third order phase transition of Gross-Witten type.
1 Introduction

Over the last five years it has gradually become clear that the eigenvalue spectrum of the Dirac operator in Euclidean QCD (and other gauge theories) can be computed exactly in a particular finite-volume scaling region. The origin of these developments dates back to work in the 1980’s on QCD in a finite volume (see e.g. ref. [1]), but the main breakthrough came with two very influential papers in 1992 by Leutwyler and Smilga [2], and by Shuryak and Verbaarschot [3]. In the latter paper an intriguing relation to random matrix theory was pointed out for the first time, and this led quickly to a series of theoretical developments that clarified the connection between Dirac eigenvalue spectra in gauge theories and random matrix theory [4, 5, 6] (see also the reviews of ref. [7]). Here we shall discuss some very recent developments.

The central object of study is the spectral density \( \rho(\lambda; m_1, \ldots, m_{N_f}) = \sum_n \langle \delta(\lambda - \lambda_n) \rangle_{\nu} \), where the average is taken over all gluon configurations with fixed topological charge \( \nu \), and where the Dirac eigenvalues \( \lambda_n \) are solutions to \( D\phi_n = \lambda_n \phi_n \). In a finite volume it is convenient to introduce instead a rescaled, double-microscopic, spectral density [3]

\[
\rho_S(\zeta; \mu_1, \ldots, \mu_{N_f}) \equiv \frac{1}{V \Sigma} \rho \left( \frac{\zeta}{V \Sigma}; \frac{\mu_1}{V \Sigma}, \ldots, \frac{\mu_{N_f}}{V \Sigma} \right), \quad V \to \infty ,
\]

which computes the local density near \( \lambda \sim 0 \), on a scale set by the chiral condensate \( \Sigma \), as computed in the massless theory. In the infinite-volume limit \( \Sigma \) will be proportional to \( \rho(0) \), but the microscopic spectral density will generically vanish at \( \zeta = 0 \), in accordance with the fact that in any finite volume \( V \) there is no spontaneous chiral symmetry breaking.

Consider now a finite-volume range \( 1/\Lambda_{QCD} << L << 1/m_\pi \), where \( L \sim V^{1/4} \). The essential observation of Leutwyler and Smilga [2] was that in a sector of fixed topological charge \( \nu \) this actually defines a finite-size scaling region. What this means becomes clear when one considers the finite-volume partition function (really the generating function for the chiral condensate). In the above limit it equals

\[
Z_\nu^{(N_f)}(\mu_1, \ldots, \mu_{N_f}) = \int dU (\det U)^\nu \exp \left[ V \Sigma \Re \Tr[\mathcal{M}U] \right] , \tag{2}
\]

where the integral is taken over \( U(N_f) \), and \( \mathcal{M} \) is the quark mass matrix, which we take to be diagonal in the masses \( m_i \). This generating function depends only on one very particular combination, \( \mu_i \equiv m_i V \Sigma \), and is in this sense a scaling function. The only needed ingredient is the existence of a non-vanishing chiral condensate \( \Sigma \). Although the effective Lagrangian is consistent with the Gell-Mann–Oakes–Renner relation, the partition function does not depend on other dimensionful parameters, such as \( f_\pi \). Of course, the above representation becomes exact only in the limit, but this is precisely what we mean by having an exact finite-size scaling function: the universal result can be recovered to any required accuracy by tuning \( V \). Corrections are suppressed by powers of \( 1/V \); we will comment on the nature of such corrections below.

The finite-volume partition function (3) has been evaluated exactly for an arbitrary mass matrix \( \mathcal{M} \) and for any \( N_f \) and \( \nu \) [3]. It turns out that complete spectral information about the Dirac operator in the double-microscopic limit can be obtained from this effective partition function alone [3, 10]. Curiously, the simplest starting point is the random matrix theory formulation of the finite-volume partition function (3):

\[
Z_\nu^{(N_f)}(m_1, \ldots, m_{N_f}) = \int dW \prod_{f=1}^{N_f} \det (M + m_f) \exp \left[ -\frac{N}{2} \Tr V(M^2) \right] , \tag{3}
\]
Here $W$ is a rectangular complex matrix of size $N \times (N + |\nu|)$. In the large-$N$ limit the space-time volume $V$ of QCD is identified with $2N$. The potential $V(M^2)$ can be parametrized in a general way by $V(M^2) = \sum (g_k/k)M^{2k}$. It was proven in refs. \cite{8, 11} that all double-microscopic spectral correlators (including, as the most simple case, the double-microscopic spectral density itself) in fact are \textit{universal}, i.e. independent of the choice of $V(M^2)$ up to a rescaling of the local macroscopic spectral density, here $\rho(0)$. The class of potentials that fall into this specific universality class is huge, its boundary given by potentials for which one has $\rho(0) = 0$, but $\rho^{(2n)}(0) \neq 0$ for some integer $n$ \cite{12}.

To begin, the random matrix theory computations were always performed in the specific case of a Gaussian potential. Rewriting that particular random matrix theory partition it was shown already in refs. \cite{8, 13} that in the double-microscopic limit in which $\zeta \equiv \lambda N^2 \pi \rho(0)$ and $\mu_i \equiv m_i N^2 \pi \rho(0)$ are kept fixed as $N \to \infty$ this \textit{Gaussian} random matrix partition function equals the Leutwyler-Smilga chiral Lagrangian \cite{2} provided one makes the identification $\Sigma = 2 \pi \rho(0)$. This exact equivalence between the random matrix partition function and the QCD effective Lagrangian in this regime, eq. \cite{2}, is crucial for the understanding of why random matrix theory can be used to compute Dirac operator spectra. It is therefore important to notice that also this equivalence holds \textit{universally}, independent of the choice of the random matrix potential (up to the restrictions specified in refs. \cite{6, 12}). We shall now prove this.

### 2 Universality of the partition functions

There are at least two ways to demonstrate universality of the random matrix partition function, and, subsequently, the identity (up to an irrelevant $\mu_i$-independent constant):

$$Z_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f}) = \tilde{Z}_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f}),$$  

(5)

Here the l.h.s. is the finite-volume QCD partition function \cite{8}, and the r.h.s. is the random matrix partition function \cite{8} evaluated in the double-microscopic limit. The most direct way is to use the orthogonal polynomial representation of the latter, \textit{i.e} define monic orthogonal polynomials $P_n(\tilde{\lambda}; m_1, \ldots, m_{N_f})$ so that (expressed in terms of $\tilde{\lambda} = \lambda^2$, which are more convenient variables in the random matrix context):

$$\int_0^\infty d\tilde{\lambda} \tilde{\lambda}^\nu \prod_f (\tilde{\lambda} + m_f^2) e^{-N\tilde{\lambda}} P_k(\tilde{\lambda}; m_1, \ldots, m_{N_f}) P_{\ell}(\tilde{\lambda}; m_1, \ldots, m_{N_f}) = h_k(m_1, \ldots, m_{N_f}) \delta_{k\ell}. \quad (6)$$

The random matrix partition function is directly related to the normalization constants $h_k$:

$$\tilde{Z}_{\nu}^{(N_f)} = N! \prod_{k=0}^{N-1} h_k(m_1, \ldots, m_{N_f}) = N! h_0(m_1, \ldots, m_{N_f}) \prod_{k=1}^{N-1} r_k(m_1, \ldots, m_{N_f}), \quad (7)$$

where $r_k \equiv h_k/h_{k-1}$. It is now possible to use the universality proof of ref. \cite{8}, extended to the case of finite masses \cite{11}, to prove the universal relation \cite{8} in the double-microscopic limit. This is most easily done by taking the logarithm of \cite{8}, and turning the resulting sum into an integral in the large-$N$ limit.
A much simpler way to prove universality of \( \langle 5 \rangle \) is to make use of an interesting relation between the orthogonal polynomials of eq. \( \langle 3 \rangle \) and the random matrix partition functions \( \langle 4 \rangle \):

\[
P_N(-\mu^2_{N_f+1}; \mu_1, \ldots, \mu_{N_f}) = C(-1)^N (\mu_{N_f+1} - \nu) \frac{Z^{(N_f+1)}_{\nu}}{Z^{(N_f)}_{\nu}} \quad (8)
\]

Here \( C \) is an irrelevant \( \mu \)-independent constant that just fixes the overall normalization of the polynomials. The l.h.s. of eq. (8) was proven to have a universal double-microscopic limit in refs. [3, 11] (and the solution is unambiguously defined also for negative eigenvalue entries). The universality proof now proceeds recursively, or by induction. For \( N_f = 0 \) (the quenched case) universality holds trivially. Universality of the random matrix partition functions for \( N_f = 1 \), and higher, then follows recursively, using eq. (8). Of course, this proves only universality of the result from the random matrix side, but not the identity \( \langle 3 \rangle \). Fortunately, the identity has been established for the special case of Gaussian potentials in refs. [3, 13]. The simple proof given here therefore extends this identity to the full universality class.

### 3 Spectral correlators from partition functions

The very useful connection between the orthogonal polynomials and random matrix partition functions \( \langle 8 \rangle \) is actually only one in a series of such relations \( \langle 3 \rangle \ [4 \]. Taken separately, these relations provide very convenient and compact expressions for all the relevant objects that enter in the random matrix computations. But when used in connection with the identity \( \langle 3 \rangle \) they provide a much more intriguing series of relations – relations that now only refer to the finite-volume QCD partition functions.

Most important in this context is the corresponding partition function representation of the kernel in random matrix theory:

\[
K_N(\lambda, \lambda'; m_1, \ldots, m_{N_f}) = e^{-\frac{N}{2}(V(\lambda^2) + V(\lambda'^2))}(\lambda \lambda')^{\nu+\frac{1}{2}} \prod_{f} \sqrt{(\lambda^2 + m_f^2)(\lambda'^2 + m_f^2)} \sum_{i=0}^{N-1} P_i(\lambda^2) P_i(\lambda'^2) . \quad (9)
\]

As is well known, from this kernel one can derive all spectral correlation functions in the limit \( N \rightarrow \infty \):

\[
\rho(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n; m_1, \ldots, m_{N_f}) = \det_{a,b} K(\tilde{\lambda}_a, \tilde{\lambda}_b; m_1, \ldots, m_{N_f}) . \quad (10)
\]

We can now make use of the following kernel representation:

\[
K_N(z, z'; m_1, \ldots, m_{N_f}) = e^{-\frac{N}{2}(V(z^2) + V(z'^2))}(zz')^{\nu+\frac{1}{2}} \prod_{f} \sqrt{(z^2 + m_f^2)(z'^2 + m_f^2)} \sum_{i=0}^{N-1} P_i(\lambda^2) P_i(\lambda'^2) \prod_{f} \prod_{i=1}^{N_f} d\lambda_i \lambda_i^{N_f} \prod_{i=1}^{N-1} \left| \prod_{f} (\lambda_i + m_f^2) e^{-NV(\lambda_i)} \right| \det_{ij} \lambda_j^{-1} \quad (11)
\]

Except for the fact that the last eigenvalue integral runs up to \( N - 1 \) only, the last factor is simply yet another partition function, now with two additional quark species of imaginary mass! Thus, up to corrections of order \( 1/N \), we have in the large-\( N \) limit:

\[
K^{(N_f, \nu)}_N(z, z'; m_1, \ldots, m_{N_f}) = e^{-\frac{N}{2}(V(z^2) + V(z'^2))} (-1)^\nu \sqrt{zz'} \prod_{f} \sqrt{(z^2 + m_f^2)(z'^2 + m_f^2)}
\]

3
We are now ready to take the double-microscopic limit in which \( \zeta \equiv z N^2 \pi \rho(0) \) and \( \mu_i \equiv m_i N^2 \pi \rho(0) \) are kept fixed as \( N \to \infty \). In this limit the prefactor \( \exp[-(N/2)(V(z^2) + V(z'^2))] \) becomes replaced by unity. By identifying \( \Sigma = 2 \pi \rho(0) \), and using the universal relation (13) we finally arrive at the following master formula (14):

\[
K_S^{(N_f)}(\zeta, \zeta'; \mu_1, \ldots, \mu_{N_f}) = C_2 \zeta \prod_{f=1}^{N_f} \left( \zeta_a^2 + \mu_f^2 \right) \frac{Z_{\nu}^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta, i\zeta')}{Z_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})}.
\]

(13)

From this one single formula all double-microscopic spectral correlators can be computed directly from QCD chiral Lagrangians in the appropriate scaling regime. In particular, for the spectral density itself we find

\[
\rho_S(\zeta; \mu_1, \ldots, \mu_{N_f}) = C_2 \zeta \prod_{f=1}^{N_f} \left( \zeta_a^2 + \mu_f^2 \right) \frac{Z_{\nu}^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta, i\zeta)}{Z_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})}.
\]

(14)

The overall proportionality factor \( C_2 \) can be determined by using the matching condition

\[
\lim_{\zeta \to \infty} \rho_S(\zeta; \mu_1, \ldots, \mu_{N_f}) = 1/\pi,
\]

which fixes \( C_2 = (1)^{N_f}/\pi \).

The higher \( k \)-point double-microscopic spectral correlation functions are conveniently evaluated using the double-microscopic limit of the general relation (11). Curiously, it is also possible to relate these higher \( k \)-point functions to finite-volume QCD partition functions with \( 2k \) additional quark species (12):

\[
\rho_S(\zeta_1, \ldots, \zeta_k; \mu_1, \ldots, \mu_{N_f}) = C^{(k)} \prod_{i=1}^{k} \zeta_i \prod_{f=1}^{N_f} \left( \zeta_a^2 + \mu_f^2 \right) \prod_{j=1}^{k} (\zeta_j^2 - \zeta_i^2)^2 \times \frac{Z_{\nu}^{(N_f+2k)}(\mu_1, \ldots, \mu_{N_f}, \{i\zeta_1\}, \ldots, \{i\zeta_k\})}{Z_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})},
\]

(16)

Each additional imaginary quark mass \( i\zeta_j \) is thus doubly degenerate. The overall proportionality constant \( C^{(k)} \) can again be fixed by a matching condition. For \( k = 1 \) the relation (12) simply coincides with the previous expression for the double-microscopic spectral density. But already for \( k = 2 \) (and all higher values of \( k \)) the expressions are completely different, relating as they do the spectral correlators to finite-volume QCD partition functions with different numbers of flavors. It is quite amazing that the finite-volume QCD partition function (2) has all this structure, which takes on such a simple form in random matrix language, encoded in it. In fact, by combining eqs. (15) and (16) one obtains an infinite sequence of consistency conditions for QCD partitions. The relations become particularly transparent if we first take the additional fermion masses to physical values by replacing \( \zeta_j \to -i\zeta_j \) (inspection of the explicit solution of ref. 3 shows immediately that this can be done unambiguously). We then find the following infinite sequence of consistency conditions (17):

\[
\det_{1 \leq a, b \leq k} \prod_{f=1}^{N_f} \sqrt{\zeta_a \zeta_b} \prod_{f=1}^{N_f} \left( \mu_f^2 - \zeta_a^2 \right) \left( \mu_f^2 - \zeta_b^2 \right) Z_{\nu}^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, \zeta_a, \zeta_b) =
\]

(17)
\[
\tilde{C}^{(k)} = \prod_{i}^{k} \left( \zeta_i - \zeta_j^2 \right) \prod_{j < l} \left| \zeta_j^2 - \zeta_l^2 \right|^{2 \nu} \frac{Z_{\nu}^{(N_f + 2k)}(\mu_1, \ldots, \mu_{N_f}, \{\zeta_1\}, \ldots, \{\zeta_k\})}{Z_{\nu}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})^{1-k}},
\]

where \( \tilde{C}^{(k)} \) is some overall \( \mu \)-independent normalization constant. Precisely these relations encode in the finite-volume QCD partition function the fact that in the random matrix picture the kernel (9) generates all spectral correlation functions through the relation (10).

4 Direct computations from chiral Lagrangians

We have learned that the massless spectral sum rules [2] do not provide the proper starting point for computing the microscopic spectral density. It is the double-microscopic limit [3, 15, 11, 16] that is needed. This was in fact clear already from the first demonstration of the equivalence of the random matrix theory partition function and the finite-volume QCD partition function [3, 13]. Instead of the massless spectral sum rules, one should focus on the “massive spectral sum rules” [3, 17] because these contain the analytical structure that allows one to unravel the spectral correlators from the finite-volume partition function. This fact becomes very clear when one considers the most simple example, the massive spectral sum rule corresponding to quenched QCD. Defining

\[
G(\mu) = 2\mu \int_{0}^{\infty} d\lambda \frac{\rho_S(\lambda)}{\lambda^2 + \mu^2},
\]

this can be written as a Stieltjes transform:

\[
\int_{0}^{\infty} dt \frac{\rho_S(t)/\sqrt{t}}{t + y} = G(\sqrt{y}) \equiv F(y).
\]

The inverse of this is given by the discontinuity:

\[
\frac{\rho_S(\sqrt{t})}{\sqrt{t}} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left[ F(-t - i\epsilon) - F(-t + i\epsilon) \right].
\]

So if one could compute the l.h.s. of (18) directly from a finite-volume partition function, one would have achieved a derivation of the spectral density without having at any intermediate stage to go through the random matrix theory framework at all. The trouble is that the massive spectral sum rule (18) refers to a “quenched quark” of (rescaled) mass \( \mu \). If it were a dynamical quark, one could compute the function \( G(\mu) \) straightforwardly from [17]

\[
G(\mu) = \frac{\partial}{\partial \mu} \ln Z_{\nu}(\mu) - \frac{\nu}{\mu},
\]

where the last term subtracts the contribution from the zero modes. The needed trick, recently discovered by Osborn, Toublan and Verbaarschot [10], is to compute the r.h.s. of eq. (21) in a finite-volume field theory that contains yet another “quark”, now of opposite statistics and of initially different mass (so that, after taking the degenerate mass limit, the two determinants cancel in the partition function itself). The result is (for \( \nu = 0 \)) [10]:

\[
G(\mu) = \Sigma \mu [I_0(\mu) K_0(\mu) + I_1(\mu) K_1(\mu)]
\]

a result that was first derived the other way around, from the random matrix theory result, by Verbaarschot [18]. Substituting this into eq. (20), one finds, straight from the finite-volume partition function,

\[
\rho_S(\lambda) = \frac{1}{2} |\lambda| \left[ J_0(\lambda)^2 + J_1(\lambda)^2 \right]
\]

where \( J_n(\lambda) \) are Bessel functions of the first kind.
which of course agrees with the result obtained from random matrix theory. Not only could those authors compute the function $G(\mu)$ this way, they also managed to rewrite the general expression (20) in precisely the form (14) using the technique of partially quenched chiral perturbation theory [19], here based on the super Lie group $U(N_f + 1|1)$. Their result naturally generalizes to higher $k$-point spectral correlation functions, now given in the form (16). The relevant super Lie group will here be $U(N_f + k|k)$.

5 Finite-volume corrections

We have seen that the microscopic spectral correlators have a natural interpretation as finite-size scaling functions. This makes them ideally suited for lattice gauge theory studies, and in fact there have now been Monte Carlo tests of the spectral densities of QCD in both (3+1) dimensions [21] and (2+1) dimensions [22]. (There have also been interesting studies of the applicability of random matrix techniques beyond the microscopic limit, in the “bulk” [23]). One obvious question in that connection concerns finite-size corrections. In actual computations the volume $V$ is often far from being asymptotically large, and one could ask whether it is also possible to analytically calculate subleading corrections. For example, for the double-microscopic spectral density itself one could envisage an expansion of the kind

$$
\rho_S^{(V)}(\zeta; \mu_1, \ldots, \mu_{N_f}) = \rho_S^{(\infty)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \left[ 1 + \frac{A}{V} f(\zeta; \mu_1, \ldots, \mu_{N_f}) + \ldots \right],
$$

(24)

with $A$ some dimensionful constant, and $f$ a correction-to-scaling function. One could hope that such corrections could be computed analytically using the random matrix theory formulation. In fact, if we go back to the derivation of eq. (2) from (1), we could try to keep the subleading corrections that come from ignoring the difference between $N$ and $N - 1$ in (1). However, on top of these we can also get subleading contributions from the potential $V(\lambda^2)$, even in the microscopic limit. We therefore conclude that such subleading $1/N$ corrections in the random matrix picture will be non-universal, and hence cannot be expected to be related to the Dirac eigenvalue spectrum. This is completely in accord with the field theory picture, in which subleading terms in $1/V$ will involve non-static modes of the pseudo-Goldstone bosons. The kinetic term $\frac{1}{2}f_\pi^2 Tr[\partial_\mu U \partial_\mu U]$ in the effective Lagrangian can therefore not be neglected. A new dimensionful scale (namely $f_\pi$) has entered, and the Dirac spectrum will cease to be a scaling function related to just $V$ and $\Sigma$. Of course, one could try to systematically analyze $1/V$ corrections in this very precise framework of the effective Lagrangian. The most natural starting point will unfortunately not be the usual expansion around the kinetic term, but rather a low-momentum expansion around the mass term $Tr[MU]$.

6 Flavor dependence

We finally address the question of the flavor dependence of all these results. The number of flavors enters in a very simple way in both the field theory and random matrix picture. In the former it determines the coset integration for the effective Lagrangian, while in the latter it enters only through the strength of the determinant in the expression (3). Both lead to an extremely mild dependence on the number of flavors. This is because we throughout normalize the chiral condensate to one flavor-independent number $\Sigma$, – a convenient normalization because it puts the different theories with different flavor content on the same common scale. Clearly the whole framework collapses as the number $N_f$ exceeds the value $N_f^*$ above which QCD no longer supports spontaneous chiral symmetry. If we allow ourselves to treat this upper number of flavors $N_f^*$ as a free and tunable parameter, then
a normalizable condensate $\Sigma$ will simply cease to exist precisely at $N_f = N_f^*$. In the random matrix theory context this may correspond to hitting the boundary where $\rho(0) \to 0$ \cite{12}. Beyond this point all results discussed here will no longer be valid. Just at the point where the condensate disappears one can define critical exponents that count the rate at which $\rho(\lambda)$ vanishes as $\lambda \to 0$ \cite{24}; however they will here not correspond to a physical phase transition (appearing instead as one tunes $N_f$ continuously to $N_f^*$). Beyond that point the spectral density of the Dirac operator will develop a gap around $\lambda = 0$, and there is no obvious way to compute it from the finite-volume partition function (mesons with quantum numbers of the pseudo-Goldstone bosons will still be the lightest excitations \cite{25}, but there will be no analogue of the chiral Lagrangian).

There is however a limit of large $N_f$ in which the results here will continue to be valid. This is what is called the “topological $1/N$ expansion”, where $N_f \to \infty$ and $N_c \to \infty$, with the ratio $\eta \equiv N_f/N_c$ kept fixed. The pertinent random matrix theory ensembles (and chiral Lagrangians) continue to be the same, and if $\eta$ is chosen small enough (as in QCD), the theory will still undergo spontaneous chiral symmetry breaking. However, because $N_f \to \infty$, there is now non-trivial “dynamics” even in the very simple chiral Lagrangian \cite{9}. In fact, when all masses are chosen equal the theory becomes identical to large-$N_c$ lattice QCD in (1+1) dimensions will so-called Wilson action. It is known that this theory undergoes a 3rd order phase transition \cite{29}, and the QCD partition function \cite{9} therefore undergoes precisely such a phase transition in the above limit. Because of the connection between partition functions and the microscopic spectral correlators, such a phase transition is expected to surface also in the Dirac spectrum, once the above limit is taken. Because a new limit ($N_f \to \infty$ and $N_c \to \infty$, with $\eta \equiv N_f/N_c$ fixed) is taken on top of the usual double-microscopic limit, we need to redefine our scaling variables. Let $\kappa \equiv N_f/\mu$, and keep this variable fixed as $N_f \to \infty$. If we define the free energy by $\mathcal{F} \equiv -\ln[Z(N_f)]/N_f^2$, then from the Gross-Witten analysis we know that

$$\mathcal{F} = 1/(4\kappa^2) \quad , \quad \kappa \geq 1$$

$$\mathcal{F} = 1/\kappa + (1/2) \ln(\kappa) - 3/4 \quad , \quad \kappa \leq 1 . \quad (25)$$

Taking derivatives, one indeed verifies that only the 3rd derivative of $\mathcal{F}$ is discontinuous at $\kappa = 1$. It is amusing that this 3rd order phase transition of the effective QCD partition function has a simple interpretation as occurring in a two-dimensional “world volume” taken to be the 2-dimensional space of the $N_f \times N_f$ unitary matrices $U$ \cite{27}. One can plot the complex zeros of the partition function, and see that they as expected precisely pinch the real $\kappa$-axis at $\kappa = 1$. From this one can very accurately compute the correlation length critical index $\nu$, which turns out to agree with an analytical result obtained from the so-called double-scaling limit of models of 2-d quantum gravity \cite{28}. More importantly, the critical scaling corresponding to this critical index sets in at very, very low values of $N_f$ – actually already at $N_f = 2$ \cite{27}. Traces of this analog of the Gross-Witten phase transition therefore persist even in real (3+1)-dimensional QCD, in the microscopic scaling regime.

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