Precise deviations for Hawkes processes

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Abstract

Hawkes process is a class of simple point processes with self-exciting and clustering properties. Hawkes process has been widely applied in finance, neuroscience, social networks, criminology, seismology, and many other fields. In this paper, we study precise deviations for Hawkes processes for large time asymptotics, that strictly extends and improves the existing results in the literature. Numerical illustrations will also be provided.

1 Introduction

We consider the Hawkes process, a simple point process \( N_t \), with the stochastic intensity at time \( t \) given by:

\[
\lambda_t = \nu + \int_0^t h(t-s)dN_s,
\]

where \( \nu > 0 \) and \( h: \mathbb{R}_+ \to \mathbb{R}_+ \) being locally bounded. We assume that \( N_0 = 0 \), that is, the Hawkes process starts at time zero with empty history. The Hawkes process is named after Alan Hawkes [23]. In the literature \( \nu \) is called the baseline intensity, and \( h \) is called the exciting function, or kernel function, encoding the influence of past events on the intensity. Brémaud and Massoulié [9] generalized the dynamics (1.1) and the formula for the intensity (1.1) by a nonlinear function of \( \int_0^t h(t-s)dN_s \), and hence came the name nonlinear Hawkes processes. The original model (1.1) proposed by Hawkes [23] is thus sometimes referred to as the linear Hawkes process.

For the Hawkes process (1.1), the occurrence of a jump increases the intensity of the point process, and thus increases the likelihood of more future jumps. On the other hand, the intensity declines when there is no occurrence of new jumps. The self-exciting and clustering property makes the Hawkes process very appealing in applications in finance and many other fields. The Hawkes process is widely used in the modeling of the limit order books in high frequency trading, see e.g. Alfonsi and Blanc [1] for optimal execution, and Abergel and Jedidi [1] for ergodicity in Hawkes based limit order books models, and also the modeling of the duration between trades, see e.g. Bauwens and Hautsch [7] or the arrival process of buy and sell orders, see e.g. Bacry et al. [6]. The Hawkes process

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also finds applications in dark pool trading [19]. In the context of credit risk modeling, Errais et al. [14] used a top down approach using the affine point process, which includes Hawkes process as a special case. Giot [20], Chavez-Demoulin et al. [10] tested Hawkes processes in the risk management context. A¨ıt-Sahalia et al. [2, 3] used Hawkes processes to model two key aspects of asset prices: clustering in time and cross sectional contamination between regions. The Hawkes process has also been used to explain the supply and demand microstructure in an interest rate model in Hainaut [22]. The applications other than finance include: neuroscience, see e.g. [33, 34, 36, 37], genome analysis, see e.g. [21, 37], networks and sociology, see e.g. [12, 29, 44], queueing theory, see e.g. [18], insurance, see e.g. [38, 45], criminology, see e.g. [28, 30, 35], seismology, see e.g. [31, 32, 41] and many other fields.

In this paper, we consider the linear Hawkes process $N_t$ with $N_0 = 0$ with the intensity 

$$h(t) = \int_0^\infty h(t) \, dt < 1. \quad (1.1)$$

Let us first review the limit theorems for linear Hawkes processes in the literature. It is well known that under the assumption $\|h\|_{L^1} < 1$, there exists a unique stationary Hawkes process, and we have the law of large numbers

$$N_t \to \nu \quad \text{as} \quad t \to \infty.$$

Bacry et al. [5] obtained a functional central limit theorem for multivariate Hawkes process and as a special case of their result, we have

$$N_t - \frac{\nu}{1 - \|h\|_{L^1}} t \to N \left( 0, \frac{\nu}{1 - \|h\|_{L^1}} \right), \quad (1.2)$$

in distribution as $t \to \infty$ under the assumption that $\int_0^\infty t^{1/2} h(t) \, dt < \infty$. Bordenave and Torrisi [8] proved that $\mathbb{P}(N_t \in \cdot)$ satisfies a large deviation principle, with the rate function:

$$I(x) = \begin{cases} x \log \left( \frac{x}{x + x\|h\|_{L^1}} \right) - x + x\|h\|_{L^1} + \nu & \text{if } x \in [0, \infty) \\ +\infty & \text{otherwise} \end{cases}. \quad (1.3)$$

Note that the rate function in the paper [8] is written as the Legendre transform expression, and the formula (1.3) is first mentioned in [47]. Also notice that in [8], the assumption $\int_0^\infty th(t) \, dt < \infty$ is needed, and indeed this assumption is not necessary, see e.g. [27]. A moderate deviation principle is obtained in [47] that fills in the gap between the central limit theorem and the large deviation principle. Other works on the asymptotics of linear Hawkes processes, including the nearly unstable Hawkes processes, that is, when the $\|h\|_{L^1}$ is close to 1, see e.g. [25, 26], and the large initial intensity asymptotics for the Markovian case [16, 17], and the large baseline intensity asymptotics [18].
For nonlinear Hawkes processes, \[46\] studies the central limit theorem, and \[48\] obtains a process-level, i.e., level-3 large deviation principle, and hence has the scalar large deviations as a by-product. An alternative expression for the rate function when the system is Markovian is obtained in \[49\]. Recently, Torrisi \[39, 40\] studies the rate of convergence in the Gaussian and Poisson approximations of the simple point processes with stochastic intensity, which includes as a special case, the nonlinear Hawkes process.

The large deviations \[8\] and moderate deviations \[47\] for linear Hawkes processes are of the Donsker-Varadhan type, which only gives the leading order term, but not the higher order expansion. In many applications in finance, insurance, and other fields, more precise deviations are desired, which motivates us to study the precise deviations for linear Hawkes processes. In this paper, we will derive the precise deviations for linear Hawkes processes, using the recent mod-φ convergence theory developed in \[15\]. The moment generating function for linear Hawkes processes has semi-explicit form due to the immigration-birth representation of linear Hawkes processes, and then the precise deviations results follow from the mod-φ convergence theory after careful analysis and a series of propositions and lemmas. The paper is organized as follows. We will state the main results of our paper in Section 2. In particular, we will give precise large deviations results in Section 2.1 and precise moderate deviations and fluctuation results in Section 2.2. Numerical illustrations will be given in Section 3. All the proofs will be provided in Section 4.

### 2 Main Results

In this paper, we apply the recently developed mod-φ convergence method to obtain precise large deviations for linear Hawkes processes for the large time asymptotic regime. We will also obtain the precise moderate deviations and some fluctuations results.

Let us first recall the definition of mod-φ convergence, see e.g. Definition 1.1. \[15\]. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of real-valued random variables and \(\mathbb{E}[e^{zX_n}]\) exist in a strip \(S_{(c,d)} := \{z \in \mathbb{C} : c < \mathcal{R}(z) < d\}\), with \(c < d\) extended real numbers, i.e. we allow \(c = -\infty\) and \(d = +\infty\) and \(\mathcal{R}(z)\) denotes the real part of \(z \in \mathbb{C}\) throughout this paper. We assume that there exists a non-constant infinitely divisible distribution \(\phi\) with \(\int_{\mathbb{R}} e^{zx} \phi(dx) = e^{\eta(z)}\), which is well defined on \(S_{(c,d)}\), and an analytic function \(\psi(z)\) that does not vanish on the real part of \(S_{(c,d)}\) such that locally uniformly in \(z \in S_{(c,d)}\),

\[
e^{-t_n\eta(z)}\mathbb{E}[e^{zX_n}] \to \psi(z),
\]

where \(t_n \to +\infty\) as \(n \to \infty\). Then we say that \(X_n\) converges mod-φ on \(S_{(c,d)}\) with parameters \((t_n)_{n \in \mathbb{N}}\) and limiting function \(\psi\).

Assume that \(\phi\) is a lattice distribution. Also assume that the sequence of random variables \((X_n)_{n \in \mathbb{N}}\) converges mod-φ at speed \(O((t_n)^{-v})\) (Definition 2.1. in \[15\]), that is,

\[
\sup_{z \in K} \left| e^{-t_n\eta(z)}\mathbb{E}[e^{zX_n}] - \psi(z) \right| \leq C_K(t_n)^{-v},
\]

where \(t_n \to +\infty\) as \(n \to \infty\). Then we say that \(X_n\) converges mod-φ on \(S_{(c,d)}\) with parameters \((t_n)_{n \in \mathbb{N}}\) and limiting function \(\psi\).
where \( C_K > 0 \) is some constant, for any compact set \( K \).

Then Theorem 3.4. [15] says that for any \( x \in \mathbb{R} \) in the interval \((\eta(c), \eta(d))\) and \( \theta^* \)
defined as \( \eta(\theta^*) = x \), assume that \( t_nx \in \mathbb{N} \), then,

\[
\mathbb{P}(X_n = t_nx) = \frac{e^{-t_nF(x)}}{2\pi t_n\eta''(\theta^*)} \left( \psi(\theta^*) + \frac{a_1}{t_n} + \frac{a_2}{t_n^2} + \cdots + \frac{a_{v-1}}{t_n^{v-1}} + O\left(\frac{1}{t_n^v}\right) \right),
\]

as \( n \to \infty \), where \( F(x) := \sup_{\theta \in \mathbb{R}}\{\theta x - \eta(\theta)\} \) is the Legendre transform of \( \eta(\cdot) \), and similarly, if \( x \in \mathbb{R} \) is in the range of \((\eta'(0), \eta'(d))\), then,

\[
\mathbb{P}(X_n \geq t_nx) = \frac{e^{-t_nF(x)}}{2\pi t_n\eta''(\theta^*)} \frac{1}{1 - e^{-\theta^*}} \left( \psi(\theta^*) + \frac{b_1}{t_n} + \frac{b_2}{t_n^2} + \cdots + \frac{b_{v-1}}{t_n^{v-1}} + O\left(\frac{1}{t_n^v}\right) \right),
\]

as \( n \to \infty \), where \((a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty \) are rational fractions in the derivatives of \( \eta \) and \( \psi \) at \( \theta^* \).

For the sake of applications and implementations, we will compute out the sequences \((a_k)_{k=1}^\infty \) and \((b_k)_{k=1}^\infty \) and the proof will be provided in the Appendix.

**Proposition 1.** Let \( S_n \) be the set consisting of all the \( n \)-tuples of non-negative integers \((m_1, \ldots, m_n)\) satisfying the following constraint:

\[
1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 + \cdots + n \cdot m_n = n.
\]

(i) For every \( k \geq 1 \),

\[
a_k = \sum_{\ell=0}^{2k} \psi(2k-\ell)(\theta^*) \sum_{S_\ell} \frac{(-1)^{m_1+\cdots+m_\ell}}{m_1!1!m_2!2!m_2!\cdots m_\ell!\ell!m_\ell!}
\]

\[
\cdot \prod_{j=1}^\ell \left( \frac{1}{\eta''(\theta^*)} \frac{(j+2)(j+1)}{(j+2)(j+1)} \right)^{m_j} \frac{(-1)^{k}(2(k+m_1+\cdots+m_\ell)-1)!!}{(\eta''(\theta^*))^k}.
\]

(ii) For every \( k \geq 1 \),

\[
b_k = \sum_{n=0}^{2k} \psi^{(2k-n)}(\theta^*) \sum_{S_n} \frac{e^{-(m_1+\cdots+m_n)\theta^*} (m_1 + \cdots + m_n)!(1 - e^{-\theta^*})^{-(m_1+\cdots+m_n)-1}}{m_1!1!m_2!2!m_2!\cdots m_n!n!m_n!} \prod_{j=1}^n (-1)^{j \cdot m_j}
\]

\[
\cdot \sum_{\ell=0}^{2k-n} \psi(2k-\ell-n)(\theta^*) \sum_{S_\ell} \frac{(-1)^{m_1+\cdots+m_\ell}}{m_1!1!m_2!2!m_2!\cdots m_\ell!\ell!m_\ell!}
\]

\[
\cdot \prod_{j=1}^\ell \left( \frac{1}{\eta''(\theta^*)} \frac{(j+2)(j+1)}{(j+2)(j+1)} \right)^{m_j} \frac{(-1)^{k}(2(k+m_1+\cdots+m_\ell)-1)!!}{(\eta''(\theta^*))^k}.
\]
2.1 Precise Large Deviations

Our main results for the precise large deviations for the Hawkes process is stated as follows. It provides the full expansion to arbitrary order in the large time asymptotic regime, which generalizes and the large deviations result in [3].

**Theorem 2.** Given \( v \in \mathbb{N} \). Let us assume that for any \( \alpha \in (0, \frac{1}{\|h\|_{L^1}}) \),

\[
\int_0^\infty s^v H(s) ds < \infty, \quad \int_0^\infty s^v \sum_{k=1}^\infty \alpha^k h^{*k} * H(s) ds < \infty,
\]

where \( H(s) := \int_s^\infty h(t) dt \). Then, we have:

(i) For any \( x > 0 \), and \( tx \in \mathbb{N} \), as \( t \to \infty \),

\[
\mathbb{P}(N_t = tx) = e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} \left( \psi(\theta^*) + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right)\right),
\]

where for any \( R(z) \leq \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \),

\[
\psi(z) := e^{\psi(z)}, \quad \text{and} \quad \varphi(z) := \int_0^\infty [F(s; z) - x(z)] ds,
\]

which is analytic in \( z \) for any \( R(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \), and \( F \) is the unique solution that satisfies

\[
F(t; z) = e^{z + \int_0^t (F(t-s; z)-1) h(s) ds},
\]

with the constraint \( |F(t; z)| \leq \frac{1}{\|h\|_{L^1}} \), and it is analytic in \( z \) for any \( R(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \), and \( x(z) := F(\infty; z) \) exists and it satisfies the equation

\[
x(z) = e^{z + \|h\|_{L^1}(x(z)-1)},
\]

and it is analytic in \( z \) for any \( R(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \). And \( I(x) \) is defined in (1.3), \( I''(x) = \frac{\nu^2}{x(\nu + \|h\|_{L^1} x)^2} \), and

\[
\theta^* = \log \left( \frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1},
\]

where \( (a_k)_{k=1}^\infty \) are rational fractions in the derivatives of \( \eta \) and \( \psi \) at \( \theta^* \) given in (2.14), where \( \eta(z) := \nu(x(z) - 1) \).

(ii) For any \( x > \frac{\nu}{1 - \|h\|_{L^1}} \), as \( t \to \infty \),

\[
\mathbb{P}(N_t \geq tx) = e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} \frac{1}{1 - e^{-\theta^*}} \left( \psi(\theta^*) + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right)\right),
\]

where \( (b_k)_{k=1}^\infty \) are rational fractions in the derivatives of \( \eta \) and \( \psi \) at \( \theta^* \) given in (2.14).
In order to apply and implement Theorem 2, we need to compute the sequences \((a_k)_{k=1}^\infty\) and \((b_k)_{k=1}^\infty\) whose formulas are provided in Proposition 1 which rely on the derivatives of \(\eta\) and \(\psi\) at \(\theta^\ast\). Next, we provide recursive formulas to compute the derivatives of \(\eta\) and \(\psi\) at \(\theta^\ast\) of any order.

**Proposition 3.** (i) \(\eta(\theta^\ast) = \nu(x(\theta^\ast) - 1)\), and for \(k \geq 1\), \(\eta^{(k)}(\theta^\ast) = \nu x^{(k)}(\theta^\ast)\). For \(k \geq 1\), \(x^{(k)}(\theta^\ast)\) can be computed recursively as:

\[
x^{(k)}(\theta^\ast) = \frac{x(\theta^\ast)}{1 - \|h\|_{L^1} x(\theta^\ast)} \sum_{k,m_k=0}^k \frac{k! \cdot \|h\|_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! \cdots 2^{m_2} \cdots m_k! k!^{m_k}} \cdot \prod_{j=1}^k (x^{(j)}(\theta^\ast))^{m_j} + \frac{x(\theta^\ast)}{1 - \|h\|_{L^1} x(\theta^\ast)} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{\ell=1}^\infty \frac{\ell! \cdot \|h\|_{L^1}^{m_1 + \cdots + m_\ell}}{m_1! m_2! \cdots 2^{m_2} \cdots m_\ell! \ell!^{m_\ell}} \cdot \prod_{j=1}^\ell (x^{(j)}(\theta^\ast))^{m_j},
\]

where the right hand side depends on \(x^{(j)}(\theta^\ast), j = 0, 1, \ldots, k - 1\).

(ii) For every \(k \geq 1\),

\[
\psi^{(k)}(\theta^\ast) = \sum_{S_k} \frac{k! \cdot \nu^{m_1 + \cdots + m_k} \cdot \psi(\theta^\ast)}{m_1! m_2! 2^{m_2} \cdots m_k! k!^{m_k}} \cdot \prod_{j=1}^k \left( \int_0^\infty \left[ F^{(j)}(s; \theta^\ast) - x^{(j)}(\theta^\ast) \right] ds \right)^{m_j},
\]

where \(F^{(k)}(\cdot; \theta^\ast), k \geq 1\), can be computed recursively as

\[
F^{(k)}(t; \theta^\ast)
= F(t; \theta^\ast) \cdot \int_0^t F^{(k)}(t - s; \theta^\ast) h(s) ds
+ F(t; \theta^\ast) \cdot \sum_{S_k,m_k=0}^k \frac{k!}{m_1! m_2! 2^{m_2} \cdots m_k! k!^{m_k}} \cdot \prod_{j=1}^k \left( \int_0^t F^{(j)}(t - s; \theta^\ast) h(s) ds \right)^{m_j}
+ F(t; \theta^\ast) \cdot \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{S_\ell} \frac{\ell!}{m_1! m_2! 2^{m_2} \cdots m_\ell! \ell!^{m_\ell}} \cdot \prod_{j=1}^\ell \left( \int_0^t F^{(j)}(t - s; \theta^\ast) h(s) ds \right)^{m_j},
\]

where the right hand side depends on \(F^{(j)}(\cdot; \theta^\ast), j = 0, 1, \ldots, k - 1\).

The main strategy of the proof of Theorem 2 is by showing the mod-\(\phi\) convergence as defined in [15] and apply their Theorem 3.4. The proof of Theorem 2 relies on a series of lemmas and propositions that we will state later. We will first recall and discuss some well-known properties for the linear Hawkes process that will be used extensively in our proofs later.

Hawkes and Oakes [24] first discovered that a linear Hawkes process has an immigration-birth representation. The immigrants (roots) arrive according to a standard Poisson process.
with intensity $\nu > 0$ at time $t$. Each immigrant generates children according to a Galton-Watson tree, that is, the number of children of each immigrant follows a Poisson distribution with parameter $\|h\|_{L^1}$, and each child will independently generate children according to the same Poisson distribution, and so on and so forth. In addition, when the children are born, they are born at the same time, with the probability density function $h$, for being born at time $t$. Then, the Hawkes process $N_t$ is the number of all the immigrants and their descendants that arrive on the time interval $[0,t]$.

By the immigration-birth representation for linear Hawkes processes, it is well-known that one can compute that, see e.g. \cite{47,27,19}, for any $z \in \mathbb{C}$, such that

$$\mathcal{R}(z) < \theta_c := \|h\|_{L^1} - 1 - \log \|h\|_{L^1},$$

we have

$$\mathbb{E}[e^{zN_t}] = e^{\nu \int_0^t (F(s;z)-1)ds},$$

where $F$ satisfies the equation:

$$F(t;z) = e^{z \int_0^t (F(s;z)-1)h(s)ds},$$

for any $t \geq 0$.

Note that by the immigration-birth representation, we can interpret $F(t;z)$ as:

$$F(t;z) = \mathbb{E}[e^{zS_t}],$$

where $S_t$ is the number of all the descendants of an immigrant that arrives at time 0, on the time interval $[0,t]$ including the immigrant. Moreover, let us define:

$$x(z) = \mathbb{E}[e^{zS_\infty}].$$

It is well known that $x(z)$ satisfies the algebraic equation, see e.g. \cite{27}:

$$x(z) = e^{z \int \|h\|_{L^1} (x(z)-1)},$$

This algebraic equation may have more than one solution. It is known that for $z \in \mathbb{R}$, there are at most two solutions of this algebraic equation and $\mathbb{E}[e^{zS_\infty}]$ is the smaller solution, see e.g. \cite{27}.

By dominated convergence theorem, for $\mathcal{R}(z) < \theta_c$, where $\theta_c$ is defined in \eqref{2.17},

$$F(t,z) \rightarrow x(z), \quad \text{as } t \rightarrow \infty.$$  

The limit $x(\cdot)$ has the following properties:

**Proposition 4.** For any $\theta \in \mathbb{R}$, and $\theta \leq \theta_c$, where $\theta_c$ is defined in \eqref{2.17}, we have

(i) $x(\theta)\|h\|_{L^1} \leq 1.$

(ii) $x'(\theta) \rightarrow \infty$ as $\theta \uparrow \theta_c$. 

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We know that \( E[e^{zS_t}] \) satisfies the equation (2.11). Thus, as a by-product of the immigration-birth representation for linear Hawkes processes, we get the existence of the solution of (2.11).

Let us notice that for any \( R(z) \leq \theta_c \),

\[
|F(t; z)| = |E[e^{zS_t}]| \leq E[|e^{zS_t}|] = E[e^{R(z)S_t}] \leq \frac{1}{\|h\|_{L^1}}.
\]  

(2.24)

Therefore, it suffices to consider the solution of the equation (2.11) that satisfies the constraint \( |F(t; z)| \leq \frac{1}{\|h\|_{L^1}} \).

With this additional constraint, the equation (2.11) has a unique solution:

**Proposition 5.** Let \( z \in \mathbb{C} \) and \( R(z) \leq \theta_c \). The equation (2.11) with the constraint \( |F(t; z)| \leq \frac{1}{\|h\|_{L^1}} \) has a unique solution.

The key to prove the main result Theorem 2 is to verify the mod-\( \phi \) convergence. More precisely, we need to show that

\[
e^{-t \eta(z)} E[e^{zN_t}] \to \psi(z) := e^{\nu \varphi(z)},
\]  

(2.25)

as \( t \to \infty \) locally uniformly in \( z \) for \( R(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \), where

\[
\varphi(z) = \int_0^\infty [F(s; z) - x(z)] ds,
\]  

(2.26)

is analytic in \( z \) and

\[
\eta(z) = \nu(x(z) - 1),
\]  

(2.27)

and

\[
e^{\eta(z)} = e^{\nu(x(z) - 1)} = E[e^{zY}],
\]  

(2.28)

for some random variable \( Y \), where \( Y \) has an infinitely divisible distribution.

We will show the mod-\( \phi \) convergence below via a series of lemmas.

First, we show that \( Y \) is infinitely divisible. The infinite divisibility is a limitation of the method of mod-\( \phi \) convergence. Fortunately, the limiting distribution in the case of the linear Hawkes process is indeed infinitely divisible.

**Lemma 6.** \( Y \) has an infinitely divisible distribution.

To show the mod-\( \phi \) convergence, the main technical lemma is given as follows:

**Lemma 7.** For any \( R(z) < \theta_c \), where \( \theta_c \) is defined in (2.17),

\[
\varphi(z) = \int_0^\infty [F(s; z) - x(z)] ds
\]  

(2.29)

is well-defined and analytic, and as \( t \to \infty \),

\[
e^{-t(\nu(x(z) - 1))} E[e^{zN_t}] \to e^{\nu \varphi(z)},
\]  

(2.30)

locally uniformly in \( z \).
To this end, we have established the mod-\(\phi\) convergence for the linear Hawkes process for the large time limit. The proofs of all the propositions, lemmas and Theorem 2.1 will be given in Section 4.

2.2 Precise Moderate Deviations and Fluctuations

The mod-\(\phi\) convergence implies also the precise moderate deviations and central limit theorem, see Theorem 3.9. [15].

By Theorem 3.9. [15], we have the following central limit theorem result:

**Theorem 8.** For any \(y = o(t^{1/6})\), as \(t \to \infty\),

\[
\mathbb{P}
\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \frac{\sqrt{\nu}}{1 - \|h\|_{L^1}^{3/2}} \frac{y}{\sqrt{t}}\right) \approx \Phi(y)(1 + o(1)),
\]

where \(\Phi(y) := \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\).

**Remark 9.** Note that Theorem 8 generalizes the univariate case of the Hawkes process central limit theorem considered in [5] since here we allow \(y = o(t^{1/6})\) for \(t \to \infty\).

By Theorem 3.9. [15], we can also study the moderate deviations result. If \(1 \ll y \ll \sqrt{t}\) for \(t \to \infty\), i.e., the moderate deviations regime, and if we let:

\[
s_t := \frac{\nu}{1 - \|h\|_{L^1}} t + \frac{\sqrt{\nu}}{1 - \|h\|_{L^1}^{3/2}} \frac{y}{\sqrt{t}},
\]

then, as \(t \to \infty\),

\[
\mathbb{P}
\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \frac{\sqrt{\nu}}{1 - \|h\|_{L^1}^{3/2}} \frac{y}{\sqrt{t}}\right) \approx e^{-tI(s_t)} \frac{\theta^*}{\sqrt{2\pi t} \eta''(\theta^*)} (1 + o(1)),
\]

where \(\eta'(\theta^*) = s_t\).

Corollary 3.13. [15] gives a more explicit form of Theorem 3.9. [15]. By using Corollary 3.13. [15], we have the following result:

**Theorem 10.** (i) If \(y = o(t^{1/4})\), then as \(t \to \infty\),

\[
\mathbb{P}
\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \frac{\sqrt{\nu}}{1 - \|h\|_{L^1}^{3/2}} \frac{y}{\sqrt{t}}\right) \approx (1 + o(1)) e^{-\frac{\nu^2}{2y\sqrt{2\pi}}} e^{-\frac{\eta'''(0)}{6(\eta''(0))^{3/2}}} \frac{y^3}{t!}.
\]

(ii) If \(y = o(t^{1/2-1/m})\), where \(m \geq 3\), then as \(t \to \infty\),

\[
\mathbb{P}
\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \frac{\sqrt{\nu}}{1 - \|h\|_{L^1}^{3/2}} \frac{y}{\sqrt{t}}\right) \approx (1 + o(1)) e^{-\frac{\nu^2}{2y\sqrt{2\pi}}} e^{-\frac{\eta'''(0)}{6(\eta''(0))^{3/2}}} \frac{y^3}{t!}.
\]
where $I(\cdot)$ is defined in (1.3) and for any $i \geq 2$,

$$I^{(i)}(x) = (i - 2)!(-1)^{i-2}x^{1-i-1} \left((i-1) \left( \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} \right)^i - i \left( \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} \right)^{i-1} + 1 \right),$$  \hspace{1cm} (2.36)

and

$$\eta'(0) = \frac{\nu}{1 - \|h\|_{L^1}}, \quad \eta''(0) = \frac{\nu}{(1 - \|h\|_{L^1})^3}, \quad \eta'''(0) = \nu \frac{1 + 2\|h\|_{L^1}}{(1 - \|h\|_{L^1})^5}. \hspace{1cm} (2.37)$$

Remark 11. Note that Theorem 11 (i) gives a precise moderate deviation result, and it provides a more precise tail estimate than [47]. To see this, let $y = \frac{(1-\|h\|_{L^1})^{3/2} a(t)}{\sqrt{t}} x$, where $x$ is a constant independent of $t$. Then for $t^{1/2} \ll a(t) \ll t^{3/4}$, as $t \to \infty$, we have

$$\mathbb{P} \left( N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + a(t)x \right) = \frac{\sqrt{\nu(1 + o(1))}}{(1 - \|h\|_{L^1})^{3/2} \sqrt{2\pi} a(t)x} e^{(1-\|h\|_{L^1})^{3/2} a(t)^2 x^2} e^{6(\eta''(0))^{3/2} a(t)^3 x^3},$$

where $\eta'(0)$, $\eta''(0)$, and $\eta'''(0)$ are given in (2.37).

3 Numerical Illustrations

In this section, we illustrate our precise deviations results by comparing the approximation of the tail probability $\mathbb{P}(N_t \geq xt)$ by using our formulas and by using Monte Carlo simulations. Since the event $\{N_t \geq xt\}$ we are interested in is a rare event, we will first develop the importance sampling. Rare event simulations using importance sampling have been studied for affine point processes, a generalization of the linear Hawkes process when the exciting function is exponential, see [43, 42]. We are interested in the importance sampling for linear Hawkes processes with general exciting functions, which are non-Markovian in general, and is not covered in [43, 42]. The importance sampling has also been used to estimate the ruin probability in a risk model where the arrival process of the claims follows a Hawkes process [38].

We are interested to estimate right tail probability $\mathbb{P}(N_t \geq xt)$, where $x > \frac{\nu}{1 - \|h\|_{L^1}}$. Note that $\{N_t \geq xt\}$ is a rare event for $x > \frac{\nu}{1 - \|h\|_{L^1}}$. The idea of the importance sampling is to change the measure from $\mathbb{P}$ to a new measure $\hat{\mathbb{P}}$ under which the event $\{N_t \geq xt\}$ becomes a typical event.

Let us define a new probability measure $\hat{\mathbb{P}}$ under which the $N_t$ process is again a linear Hawkes process, but with baseline intensity $\gamma \nu$ and the exciting function $\gamma h(\cdot)$, where $\gamma$
is a positive constant to be chosen later. Under the new measure, the \( N_t \) process has the intensity \( \lambda_t = \gamma \lambda_t \).

We have the following result whose proof will be presented in the Appendix.

**Proposition 12.**

\[
\mathbb{P}(N_t \geq xt) = e^{-tI(x)} \hat{\mathbb{E}} \left[ 1_{N_t \geq xt} \cdot e^{((\gamma - 1)\|h\|_{L^1} - \log \gamma)(N_t - xt) - (\gamma - 1) \int_0^t H(t-u)dN_u} \right],
\]

where \( H(t) := \int_t^\infty h(s)ds \) denotes the right tail of the exciting function, and \( I(x) \) is given in (1.3).

Our numerical illustrations include three different methods: (1) importance sampling; (2) precise deviations up to the first-order approximation; (3) precise deviations up to the second-order approximation.

1. **Importance sampling.** We simulate under \( \hat{\mathbb{E}} \) a Hawkes process \( N_t \) with intensity:

\[
\hat{\lambda}_t = \gamma \nu + \gamma \int_0^t h(t-s)dN_s,
\]

where

\[
\gamma = \frac{x}{\nu + \|h\|_{L^1} x}.
\]

Note that under the new measure \( \hat{\mathbb{P}} \), we have

\[
\hat{\mathbb{P}} \left( \frac{N_t}{t} \to \frac{\gamma \nu}{1 - \gamma \|h\|_{L^1}} \right) = 1.
\]

By choosing \( \gamma \) as in Eqn. (3.3), so that \( \frac{\gamma \nu}{1 - \gamma \|h\|_{L^1}} = x \), it follows that \( 1_{N_t \geq xt} \) is a typical event under the new measure \( \hat{\mathbb{P}} \).

Using importance sampling Monte Carlo method (Proposition 12), we estimate

\[
e^{-tI(x)} \hat{\mathbb{E}} \left[ 1_{N_t \geq xt} \cdot e^{((\gamma - 1)\|h\|_{L^1} - \log \gamma)(N_t - xt) - (\gamma - 1) \int_0^t H(t-u)dN_u} \right],
\]

where \( H(t) := \int_t^\infty h(s)ds \) denotes the right tail of the exciting function, and \( I(x) = x \log \left( \frac{x}{\nu + \|h\|_{L^1} x} \right) - x + x\|h\|_{L^1} + \nu \) as is given in (1.3).

2. **Precise deviations up to the first-order approximation.** We approximate \( \mathbb{P}(N_t \geq xt) \) by the formula:

\[
e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} c_0,
\]

where \( c_0 = \frac{1}{1 - e^{-\theta^*}} \psi(\theta^*) \), and \( I''(x) = \frac{x^2}{\nu^2 + x\|h\|_{L^1}^2} \) is the second derivative of \( I(x) \) defined in (1.3), and \( \theta^* = \log \left( \frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{x\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1} \), where

\[
\psi(\theta) = e^{\nu \int_0^\infty [F(s;\theta) - x(\theta)] ds},
\]
where $F$ is the unique solution that satisfies

$$F(t; \theta) = e^{\theta + \int_0^t (F(t-s; \theta) - 1) h(s) ds}, \quad (3.8)$$

with the constraint $|F(t; \theta)| \leq \frac{1}{\|h\|_{L^1}}$, and $x(\theta) = F(\infty; \theta)$ is the unique solution that satisfies

$$x(\theta) = e^{\theta + (x(\theta) - 1)\|h\|_{L^1}}, \quad (3.9)$$

with the constraint $|x(\theta)| \leq \frac{1}{\|h\|_{L^1}}$. Note that we need to solve (3.8) and (3.9) numerically.

(3) Precise deviations up to the second-order approximation. We approximate $P(N_t \geq x(t))$ by adding a higher-order term to Eqn. (3.6) as follows:

$$e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} \left(c_0 + \frac{c_1}{t}\right), \quad (3.10)$$

where $c_0 = \frac{\psi(\theta^*)}{1 - e^{-\theta^*}}$, and $c_1 = \frac{b_1}{1 - e^{-\theta^*}}$, where the formula for $b_1$ can be computed by applying Proposition 1 and Proposition 3. The details for the computations of $b_1$ can be found in the Appendix.

In our numerical illustrations, we take baseline intensity $\nu = 1$, and consider two different exciting functions: $h(t) = e^{-2t}$, $h(t) = \frac{1}{(1+t)^2}$. In both cases, $\|h\|_{L^1} = \frac{1}{2}$, and in the first case, $H(t) = \int_t^\infty h(s) ds = \frac{1}{2}e^{-2t}$, and in the second case, $H(t) = \int_t^\infty h(s) ds = \frac{1}{2(1+t)^2}$.

We then compare the three different methods (1), (2) and (3) by comparing (3.5), (3.6) and (3.10). We summarize the results in Table 1 when the exciting function has exponential decay $h(t) = e^{-2t}$ and Table 2 when the exciting function has polynomial decay $h(t) = \frac{1}{(1+t)^2}$. In Table 1, we take $x = 4$ and $x = 5$, and consider the times $t = 5, 10, 25, 40, 50$. Numerically, we compute that $c_0 = 4.8$ and $c_1 = -22$. In Table 2, we take $x = 4$ and $x = 5$, and consider the times $t = 5, 10, 25, 40, 50$. Numerically, we compute that $c_0 = 3.51$ and $c_1 = -24$. In both tables, the column IS provides the numerical results using the importance sampling; the column 1st Order provides the numerical results using the precise deviations formula up to the first-order approximation; the column 2nd Order provides the numerical results using the precise deviations formula up to the second-order approximation. In both tables, we observe that as time $t$ gets larger, the approximations get better. First-order approximation tend to overestimate the tail probability while second-order approximation tend to underestimate the tail probability in Table 1 while that is not the case in Table 2. This is due to the positivity of $c_0$ and negativity of $c_1$. As a result, when $t$ is small, second-order approximation could give negative values which are unrealistic, e.g. when $t = 5$ in Table 1 and Table 2. In both tables, as $t$ becomes larger, second-order approximation provides better approximation than the first-order approximation.
Table 1: Numerical illustration of $\mathbb{P}(N_t \geq xt)$ for $h(t) = e^{-2t}$, $\nu = 1$ and $x = 4, 5$.

| $t$ (s) | $x = 4$ IS 1st Order | $x = 4$ 2nd Order | $x = 5$ IS 1st Order | $x = 5$ 2nd Order |
|--------|----------------|----------------|----------------|----------------|
| 5      | 4.71E-02 9.19E-02 -2.08E-02 | 1.37E-02 2.68E-02 1.40E-03 |
| 10     | 2.06E-02 3.06E-02 1.18E-02 | 3.19E-03 4.59E-03 2.41E-03 |
| 25     | 1.62E-03 2.02E-03 1.52E-03 | 3.49E-05 4.14E-05 3.35E-05 |
| 40     | 1.46E-04 1.66E-04 1.41E-04 | 4.20E-03 4.66E-03 4.11E-03 |
| 50     | 2.93E-05 3.29E-05 2.89E-05 | 2.14E-08 2.45E-08 2.22E-08 |

Table 2: Numerical illustration of $\mathbb{P}(N_t \geq xt)$ for $h(t) = 1/(1+t)^3$, $\nu = 1$ and $x = 4, 5$.

| $t$ (s) | $x = 4$ IS 1st Order | $x = 4$ 2nd Order | $x = 5$ IS 1st Order | $x = 5$ 2nd Order |
|--------|----------------|----------------|----------------|----------------|
| 5      | 3.04E-02 7.39E-02 -4.37E-02 | 7.50E-03 1.94E-02 -7.60E-03 |
| 10     | 1.33E-02 2.46E-02 5.02E-02 | 1.63E-03 3.33E-03 1.01E-03 |
| 25     | 1.12E-03 1.62E-03 1.11E-03 | 1.95E-05 2.99E-05 2.16E-05 |
| 40     | 1.03E-04 1.34E-04 1.07E-04 | 2.39E-03 3.38E-03 2.79E-03 |
| 50     | 2.28E-05 2.65E-05 2.23E-05 | 1.28E-08 1.78E-08 1.53E-08 |

4 Appendix: Proofs

4.1 Proofs of the Results in Section 2.1

Proof of Proposition 4. Let us prove this first. Consider $\theta \in \mathbb{R}$, and

$$x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta)-1)}.$$  (4.1)

Then $x(\theta)$ is increasing in $\theta$ by $x(\theta) = \mathbb{E}[e^{\theta Y}]$ and the definition of $Y$. Moreover, for $\theta = \theta_c = \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, we have

$$x(\theta_c) = \frac{1}{\|h\|_{L^1}} e^{\|h\|_{L^1} - 1 + \|h\|_{L^1}(x(\theta_c)-1)},$$  (4.2)

which implies that $x(\theta_c)\|h\|_{L^1} = 1$. Thus, for any $\theta \leq \theta_c$, $x(\theta)\|h\|_{L^1} \leq 1$.

We also notice that

$$x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) x(\theta),$$  (4.3)

and therefore

$$x'(\theta) = \frac{x(\theta)}{1 - \|h\|_{L^1} x(\theta)} \to \infty,$$  (4.4)

as $\theta \uparrow \theta_c$. \hfill \Box
Proof of Proposition\[1\] Suppose $F_1$ and $F_2$ are two solutions that satisfy (2.11) with $|F_j| \leq \frac{1}{\|h\|_{L^1}}$, for $j = 1, 2$. Note that for any $z_1, z_2 \in \mathbb{C}$,

$$|e^{z_1} - e^{z_2}| = \left| e^{\sum_{n=1}^{\infty} \frac{z_1^n - z_2^n}{n!}} \right| \leq |z_1 - z_2| \sum_{n=1}^{\infty} \frac{|z_1|^{n-1} |z_2| + \cdots + |z_1||z_2|^{n-1}}{n!} \leq |z_1 - z_2| \sum_{n=1}^{\infty} \frac{n(|z_1| + |z_2|)^{n-1}}{n!} = |z_1 - z_2| |e^{z_1}| + |z_2|.$$  

Hence, for any $T > 0$ and for any $0 \leq t \leq T$,

$$|F_1(t; z) - F_2(t; z)| = |e^z| \left| e^{\int_0^t (F_1(t-s;z) - F_2(t-s;z))h(s)ds} - e^{\int_0^t (F_2(t-s;z) - F_2(t-s;z))h(s)ds} \right| \leq |e^z| \left| \int_0^t (F_1(t-s;z) - F_2(t-s;z))h(s)ds \right| e^{\int_0^t |F_1(t-s;z) - F_2(t-s;z)|h(s)ds} \leq |e^z||h|_{L^\infty[0,T]} e^{2 \left( \frac{1}{\|h\|_{L^1}} + 1 \right)} \int_0^t |F_1(s; z) - F_2(s; z)| ds.$$  

Note that $F_1(0; z) = F_2(0; z) = e^z$ from (2.11). By Gronwall’s inequality, we conclude that $F_1 \equiv F_2$. \hfill \Box

Proof of Lemma\[2\] where $Y$ can be interpreted as $Y = \sum_{i=1}^{K} Z_i$, where $K$ is Poisson random variable with parameter $\nu$ and $Z_i$ are i.i.d. random variable interpreted as the total number of the nodes in a Galton-Watson tree with the number of children being born in each generation Poisson distributed with parameter $\|h\|_{L^1}$. Then, it is clear that we can write $Y = \sum_{j=1}^{n} Y_j$ in distribution, where $Y_j$ are i.i.d. $Y_j = \sum_{i=1}^{K_j} Z_{ij}$, where $Z_{ij}$ are i.i.d. copies of $Z_1$ and $K_j$ are i.i.d. Poisson distributed with parameter $\frac{\nu}{n}$. Therefore, $Y$ has an infinitely divisible distribution. \hfill \Box

Proof of Lemma\[3\] Firstly, it is obvious that for any $s > 0$, and $t > 0$,

$$F(s, z), \ x(z), \ \int_0^t (F(s, z) - x(z)) ds,$$

are analytic in $z$ for $\mathcal{R}(z) < \theta_c$.

Since $\|h\|_{L^1} < 1$ and $x(0) = 1$, for $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, we have $|x(z)|\|h\|_{L^1} \leq x(\mathcal{R}(z))\|h\|_{L^1} < 1$. Therefore, for any compact subset $\mathbb{K} \subset \{ z \in \mathbb{C}; \mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1} \}$, we have

$$\sup_{z \in \mathbb{K}} |x(z)|\|h\|_{L^1} \leq x(\mathcal{R}(z))\|h\|_{L^1} < 1.$$
Since $\int_s^\infty h(u)du \to 0$ as $s \to \infty$ and $F(s; z) = \mathbb{E}[e^{zS_s}] \to x(z) = \mathbb{E}[e^{zS_\infty}]$, we have that
\[
\int_0^s \sup_{z \in K} [F(s - u; z) - x(z)]h(u)du \to 0, \quad \text{as } s \to \infty.
\]
Note that
\[
F(s; z) - x(z) = x(z) \left[ e^{\int_u^s [F(s - u; z) - x(z)]h(u)du} - (x(z) - 1) \int_s^\infty h(u)du - 1 \right],
\]
and for any fixed $\delta > 0$ such that $(1 + \delta)\sup_{z \in K} |x(z)||\|L^1 < 1$, there exists $M > 0$, so that for any $s \geq M$ and $z \in K$, we have
\[
|F(s; z) - x(z)| \leq (1 + \delta)|x(z)| \left[ \int_0^s |F(s - u; z) - x(z)|h(u)du + |x(z) - 1| \int_s^\infty h(u)du \right].
\]
Therefore, we get that for any $T > M$,
\[
\int_0^T \sup_{z \in K} |F(s; z) - x(z)|ds \leq (1 + \delta)\sup_{z \in K} |x(z)| \int_0^T \int_0^s \sup_{z \in K} |F(s - u; z) - x(z)|h(u)du ds + (1 + \delta)\sup_{z \in K} |x(z)||x(z) - 1| \int_0^T \int_s^\infty h(u)du ds
\]
\[
\leq (1 + \delta)\sup_{z \in K} |x(z)| \int_0^T \int_0^s \sup_{z \in K} |F(s - u; z) - x(z)|h(u)du ds + (1 + \delta)\sup_{z \in K} |x(z)||x(z) - 1| \int_0^\infty \int_s^\infty h(u)du ds,
\]
which implies that
\[
\int_0^T \sup_{z \in K} |F(s; z) - x(z)|ds
\]
\[
\leq \int_0^M \sup_{z \in K} |F(s; z) - x(z)|ds + (1 + \delta)\sup_{z \in K} |x(z)| \int_0^T \int_0^s \sup_{z \in K} |F(s - u; z) - x(z)|h(u)du ds + (1 + \delta)\sup_{z \in K} |x(z)||x(z) - 1| \int_0^\infty \int_s^\infty h(u)du ds
\]
\[
= \int_0^M \sup_{z \in K} |F(s; z) - x(z)|ds + (1 + \delta)\sup_{z \in K} |x(z)| \left[ \int_0^T \sup_{z \in K} |F(s - u; z) - x(z)|ds \right] h(u)du + (1 + \delta)\sup_{z \in K} |x(z)||x(z) - 1| \int_0^\infty \int_s^\infty h(u)du ds
\]
\[
\leq \int_0^M \sup_{z \in K} |F(s; z) - x(z)|ds + (1 + \delta)\sup_{z \in K} |x(z)||\|L^1 \int_0^T |F(s; z) - x(z)|ds + (1 + \delta)\sup_{z \in K} |x(z)| \left[ \sup_{z \in K} |x(z)| + 1 \right] \int_0^\infty sh(s)ds,
\]

which holds for any $T > M$, and thus we have
\[
\int_0^\infty \sup_{z \in \mathcal{K}} |F(s; z) - x(z)| ds \\
\leq \int_0^M \sup_{z \in \mathcal{K}} |F(s; z) - x(z)| ds + (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \left[ \sup_{z \in \mathcal{K}} |x(z)| + 1 \right] \int_0^\infty sh(s) ds \over 1 - (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \|h\|_{L^1}.
\]
Hence, we conclude that $\int_0^\infty \sup_{z \in \mathcal{K}} |F(s; z) - x(z)| ds \to 0$ as $t \to \infty$, and so
\[
\int_0^t (F(s, z) - x(z)) ds \to \int_0^\infty (F(s, z) - x(z)) ds,
\]
as $t \to \infty$, locally uniformly in $z$ for $\mathcal{R}(z) < \theta_c$. Hence, $\varphi(z)$ is well-defined and is analytic in $z$ for $\mathcal{R}(z) < \theta_c$.

By equation (2.10), we have proved that locally uniformly in $z$ for $\mathcal{R}(z) < \theta_c$,
\[
e^{-t(t(x(z) - 1))} \mathbb{E}[e^{zN_1}] = e^{\nu \int_0^t (F(s, z) - x(z)) ds} \to e^{\nu \varphi(z)}, \quad \text{as } t \to \infty.
\]

To show the mod-$\phi$ convergence at speed $O(t^{-v})$, it suffices to show that
\[
\int_0^\infty t^v \sup_{z \in \mathcal{K}} |F(t; z) - x(z)| dt < \infty. \tag{4.6}
\]
Notice that there exists $M > 0$ so that for any $s \geq M$,
\[
\sup_{z \in \mathcal{Z}} |F(s; z) - x(z)| \leq (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \left[ \sup_{z \in \mathcal{K}} |F(s - u; z) - x(z)| h(u) du \right.
\]
\[
+ (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \sup_{z \in \mathcal{K}} |x(z) - 1| \left. \int_s^\infty h(u) du. \right.
\]
Therefore, there exists some function $g(s)$ so that for every $s \geq 0$,
\[
\sup_{z \in \mathcal{K}} |F(s; z) - x(z)| \leq (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \int_0^s \sup_{z \in \mathcal{K}} |F(s - u; z) - x(z)| h(u) du + g(s), \tag{4.7}
\]
and $g(s)$ is a non-negative continuous function with $g(s) = C \int_s^\infty h(u) du$ for sufficiently large $s$, where $C := (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \sup_{z \in \mathcal{K}} |x(z) - 1|$. Hence, by generalized Gronwall’s inequality [14], we have
\[
\sup_{z \in \mathcal{K}} |F(s; z) - x(z)| \leq g(s) + \int_0^s G(s - u) g(u) du, \tag{4.8}
\]
where
\[
G(t) := \sum_{k=1}^\infty \left( (1 + \delta) \sup_{z \in \mathcal{K}} |x(z)| \right)^k h^k(t). \tag{4.9}
\]
\[16\]
By our assumption on $h(\cdot)$, we have
\[
\int_0^\infty s^v g(s) ds < \infty, \quad \int_0^\infty s^v \int_0^s G(s-u) g(u) du ds < \infty.
\] (4.10)

Hence we have proved the desired result.

Proof of Theorem 2. By Lemma 6 and Lemma 7, we have established the mod-$\phi$ convergence. Hence, by Theorem 3.4. [15], for any $x > 0$, and $tx \in \mathbb{N},$
\[
\mathbb{P}(N_t = tx) = \frac{e^{-t I(x)}}{\sqrt{2\pi t \eta''(\theta^*)}} \left( \psi(\theta^*) + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_{v-1}}{t^{v-1}} + O \left( \frac{1}{t^{v}} \right) \right),
\] (4.11)
and for any $x > \eta'(0) = \frac{\nu}{1 - \|h\|_{L^1}}$,
\[
\mathbb{P}(N_t \geq tx) = \frac{e^{-t I(x)}}{\sqrt{2\pi t \eta''(\theta^*)}} \frac{1}{1 - e^{-\theta^*}} \left( \psi(\theta^*) + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_{v-1}}{t^{v-1}} + O \left( \frac{1}{t^{v}} \right) \right).
\] (4.12)
where $I(x)$ is defined in [13] and $\eta'(\theta^*) = x$.

Note that $\eta(\theta) = \nu (x(\theta) - 1)$, where $x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta) - 1)}$. Thus, $\eta'(\theta) = \nu x'(\theta)$, and
\[
x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) e^{\theta + \|h\|_{L^1}(x(\theta) - 1)} = (1 + \|h\|_{L^1} x'(\theta)) x(\theta),
\] (4.13)
which implies that
\[
x(\theta) = \frac{x'(\theta)}{1 + \|h\|_{L^1} x'(\theta)}.
\] (4.14)

Notice that $\eta'(\theta^*) = x$, and thus
\[
x'(\theta^*) = \frac{\eta'(\theta^*)}{\nu} = \frac{x}{\nu},
\] (4.15)
and
\[
x(\theta^*) = \frac{x'(\theta^*)}{1 + \|h\|_{L^1} x'(\theta^*)} = \frac{x}{\nu + \|h\|_{L^1} x}.
\] (4.16)

Hence,
\[
\frac{x}{\nu} = x'(\theta^*) = (1 + \|h\|_{L^1} x'(\theta^*)) e^{\theta^* + \|h\|_{L^1}(x(\theta^*) - 1)}\]
\[
= \left( 1 + \frac{\|h\|_{L^1}}{\nu + \|h\|_{L^1} x} \right) e^{\theta^* + \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} - \|h\|_{L^1}},
\] which implies that
\[
\theta^* = \log \left( \frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1}.
\] (4.17)
Hence, we have verified that
\[ I(x) = \theta^* - \eta(\theta^*) = x \log \left( \frac{x}{\nu + x \|h\|_{L^1}} \right) - x + x \|h\|_{L^1} + \nu. \] (4.18)

Moreover, by the property of Legendre transform,
\[ \eta''(\theta^*) = \frac{1}{I''(x)}, \] (4.19)
and
\[ I'(x) = \theta^* = \log \left( \frac{x}{\nu + \|h\|_{L^1} x} \right) - \|h\|_{L^1}, \] (4.20)
which implies that
\[ I''(x) = \frac{\nu^2}{x(\nu + \|h\|_{L^1} x)^2}. \] (4.21)

4.2 Proofs of the Results in Section 2.2

Proof of Theorem 10. Since we have established mod-\(\phi\) convergence, and \(N_t\) is lattice distributed, the result follows from Corollary 3.13. [15].

Let us show that (2.36) holds. Recall from (4.21) that
\[ I''(x) = \frac{\nu^2}{x(\nu + \|h\|_{L^1} x)^2}. \] Therefore, for any \(i \geq 2\), by Leibniz formula,
\[ I^{(i)}(x) = \nu^2 \frac{d^{i-2}}{dx^{i-2}} x^{-1} \cdot (\nu + \|h\|_{L^1} x)^{-2} \]
\[ = \nu^2 \sum_{j=0}^{i-2} \frac{(i-2)!}{(i-2-j)!} (-1)^{i-2(j)} j! \|h\|_{L^1}^j (j+1)! x^{-1-(i-2-j)} (\nu + \|h\|_{L^1} x)^{-j-2} \]
\[ = \nu^2 (i-2)! (-1)^{i-2} \frac{x^{1-i}}{(\nu + \|h\|_{L^1} x)^2} \sum_{j=0}^{i-2} j! (j+1) \left( \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} \right)^j \]
\[ = \nu^2 (i-2)! (-1)^{i-2} \frac{x^{1-i}}{(\nu + \|h\|_{L^1} x)^2} \sum_{j=0}^{i-2} \frac{y^{j+1}}{y^{j+1}} \bigg|_{y = \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x}} \]
\[ = \nu^2 (i-2)! (-1)^{i-2} \frac{x^{1-i}}{(\nu + \|h\|_{L^1} x)^2} \left( \frac{(i-1)y^i - iy^{i-1} + 1}{(y-1)^2} \right) \bigg|_{y = \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x}}, \]
which implies (2.36).

Let us show that (2.37) holds. Let us recall that \(\eta(\theta) = \nu(x(\theta) - 1)\), where \(x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta) - 1)}\). Thus, we can compute that
\[ x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) x(\theta), \] (4.22)
and
\[ x''(\theta) = \left[ \| h \|_{L^1} x''(\theta) + (1 + \| h \|_{L^1} x'(\theta))^2 \right] x(\theta), \] (4.23)

and
\[ x'''(\theta) = \left[ \| h \|_{L^1} x'''(\theta) + 2(1 + \| h \|_{L^1} x'(\theta)) \right. \]
\[ \left. + \left[ (1 + \| h \|_{L^1} x'(\theta))^2 \right] (1 + \| h \|_{L^1} x'(\theta)) x(\theta) \right. \] (4.24)

Note that \( x(0) = 1 \). By letting \( \theta = 0 \), we get (2.37).

4.3 Proof of Proposition 1 (i): Derivations of \((a_k)_{k=1}^\infty\)

Before we proceed, let us first introduce the Faà di Bruno’s formula that will be used repeatedly in our proofs.

Lemma 13 (Faà di Bruno’s formula).

\[
\frac{d^n}{dx^n} f(g(x)) = \sum_{S_n} \frac{n!}{m_1! m_2! \ldots m_n! n!} \cdot f^{(m_1 + \cdots + m_n)}(g(x)) \cdot \prod_{j=1}^n (g^{(j)}(x))^{m_j}, \tag{4.25}
\]

where the sum is over the set \( S_n \) consisting of all the \( n \)-tuples of non-negative integers \((m_1, \ldots, m_n)\) satisfying the following constraint:
\[
1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 + \cdots + n \cdot m_n = n. \tag{4.26}
\]

We recall from the proof of Theorem 3.4. and Remark 3.7 in [15] that for \( \mathbb{Z} \)-valued random variables \( X_n \),
\[
\mathbb{P}(X_n = t_n x) = \frac{e^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(\theta^*)}} \left( \phi(\theta^*) + \frac{a_1}{t_n} + \cdots + \frac{a_{v-1}}{t_n^{v-1}} + O(t_n^{-v}) \right), \tag{4.27}
\]
as \( n \to \infty \), where
\[
a_k = \int_{-\infty}^{\infty} \alpha_k(w) \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw, \tag{4.28}
\]
where \( \alpha_k \)'s are defined via the expansion:
\[
\sum_{k=0}^{2v-1} \frac{\alpha_k}{t_n^{k/2}} + O(t_n^{-v}) = \sum_{k=0}^{2v-1} \frac{\phi^{(k)}(\theta^*)}{k!} \left( \frac{iw}{\sqrt{t_n \eta''(\theta^*)}} \right)^k e^{-\frac{w^2}{2\eta''(\theta^*)}} \sum_{k=1}^{2v-1} \frac{z^{(k+2)}(\theta^*)}{(k+2)!} \left( \frac{iw}{\sqrt{t_n \eta''(\theta^*)}} \right)^k. \tag{4.29}
\]

Let us define \( f(x) = e^{-x} \) and
\[
g(x) = \frac{w^2}{\eta''(\theta^*)} \sum_{k=1}^{\infty} \eta^{(k+2)}(\theta^*) (k + 2)! x^k. \tag{4.30}
\]
Then

\[ f(g(x)) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(g(0)) \frac{x^n}{n!}, \]  

(4.31)

where we can compute that \( g(0) = 0 \) and for every \( j \in \mathbb{N}, \)

\[ g^{(j)}(0) = \frac{w^2}{\eta''(\theta^*)} \frac{\eta^{(j+2)}(\theta^*)}{(j+2)(j+1)}, \]  

(4.32)

which implies that by Faà di Bruno’s formula

\[ f(g(x)) = \sum_{n=0}^{\infty} \sum_{m_1!m_2!m_3! \ldots m_n!} \frac{n! (-1)^{m_1 + \ldots + m_n}}{m_1!^{m_1} m_2!^{m_2} \ldots m_n!^{m_n}} \prod_{j=1}^{n} \left( \frac{w^2}{\eta''(\theta^*)} \frac{\eta^{(j+2)}(\theta^*)}{(j+2)(j+1)} \right)^{m_j} \frac{x^n}{n!}. \]  

(4.33)

It follows that

\[ \frac{\alpha_k}{t_n^{k/2}} = \sum_{\ell=0}^{k} \frac{\psi^{(k-\ell)}(\theta^*)}{(k-\ell)!} \sum_{S_\ell} \frac{(-1)^{m_1 + \ldots + m_\ell}}{m_1!^{m_1} m_2!^{m_2} \ldots m_\ell!^{m_\ell}} \prod_{j=1}^{\ell} \left( \frac{w^2}{\eta''(\theta^*)} \frac{\eta^{(j+2)}(\theta^*)}{(j+2)(j+1)} \right)^{m_j} \left( \frac{iw}{\sqrt{t_n} \eta''(\theta^*)} \right)^k, \]  

(4.34)

which implies that

\[ \alpha_k(w) = \sum_{\ell=0}^{k} \frac{\psi^{(k-\ell)}(\theta^*)}{(k-\ell)!} \sum_{S_\ell} \frac{(-1)^{m_1 + \ldots + m_\ell}}{m_1!^{m_1} m_2!^{m_2} \ldots m_\ell!^{m_\ell}} \prod_{j=1}^{\ell} \left( \frac{w^2}{\eta''(\theta^*)} \frac{\eta^{(j+2)}(\theta^*)}{(j+2)(j+1)} \right)^{m_j} \left( \frac{iw}{\sqrt{t_n} \eta''(\theta^*)} \right)^k. \]  

(4.35)

By the property of standard normal random variable,

\[ \int_{-\infty}^{\infty} w^m \cdot \frac{e^{-w^2/2}}{\sqrt{2\pi}} dw = (m-1)!!, \]  

(4.36)
if \( m \) is even and 0 if \( m \) is odd. Hence, we conclude that

\[
a_k = \sum_{\ell=0}^{2k} \frac{y_{\ell}^{(2k-\ell)}(\theta^*)}{(2k-\ell)!} \sum_{S_{\ell}} \frac{(-1)^{m_1+\cdots+m_{\ell}}}{m_1!m_2!m_3!\cdots m_{\ell}!} \prod_{j=1}^{\ell} \left( \frac{1}{\eta''((\theta^*)^j(j+2)(j+1))} \right)^{m_j} (-1)^k(2(k+m_1+\cdots+m_{\ell}) - 1)!!
\]

(4.37)

4.4 Proof of Proposition (ii): Derivations of \((b_k)^{\infty}_{k=1}\)

We recall from the proof of Theorem 3.4 and Remark 3.7. in [15] that for \( Z \)-valued random variables \( X_n \),

\[
\mathbb{P}(X_n \geq t_n x) = \frac{e^{-t_n F(x)}}{\sqrt{2\pi t_n} \eta''((\theta^*)^1)} \frac{1}{1 - e^{-\theta^*}} \left( \psi(\theta^*) + \frac{b_1}{t_n} + \frac{b_2}{t_n^2} + \cdots + \frac{b_{v-1}}{t_n^{v-1}} + O\left( \frac{1}{t_n^v} \right) \right),
\]

(4.38)

where

\[
b_k = \int_{\mathbb{R}} \beta_{2k}(w) \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw,
\]

(4.39)

where \( \beta_j(w) \) via the expansion:

\[
g_n(w) = \sum_{k=0}^{2\nu-1} \frac{\beta_k(w)}{(t_n)^{k/2}} + O(t_n^{-v}),
\]

(4.40)

where

\[
g_n(w) := \sum_{k=0}^{\infty} e^{-k\theta^*} e^{-\frac{kw}{\sqrt{t_n} \eta''((\theta^*)^1)}} \cdot \psi \left( \theta^* + \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \right) \cdot t_n \left( \eta\left( \theta^* + \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \right) - \eta(\theta^*) - \eta'(\theta^*) \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \frac{w^2}{2t_n^2} \right)
\]

\[
= \frac{1}{1 - e^{-\frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)}}} \cdot \psi \left( \theta^* + \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \right) \cdot t_n \left( \eta\left( \theta^* + \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \right) - \eta(\theta^*) - \eta'(\theta^*) \frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)} \frac{w^2}{2t_n^2} \right),
\]

Note that

\[
\sum_{k=0}^{2\nu-1} \frac{\beta_k(w)}{(t_n)^{k/2}} + O(t_n^{-v}) = \frac{1}{1 - e^{-\frac{iw}{\sqrt{t_n} \eta''((\theta^*)^1)}}} \sum_{k=0}^{2\nu-1} \frac{\alpha_k(w)}{(t_n)^{k/2}}.
\]

(4.41)

We define

\[
f(x) = \frac{1}{1 - e^{-\theta^* x}} \quad g(x) = e^{-x}.
\]

(4.42)
Then,
\[
f(g(x)) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(g(0)) \frac{x^n}{n!}, \tag{4.43}\]
where we can compute that
\[
f^{(k)}(x) = e^{-k\theta^*} k!(1 - e^{-\theta^*})^{-k-1}, \tag{4.44}\]
and \(g^{(k)}(x) = (-1)^k e^{-x}\), and by Faà di Bruno's formula, we get
\[
\frac{d^n}{dx^n} f(g(0)) = \sum_{k=0}^{n} \frac{n!e^{-(m_1+\cdots+m_n)\theta^*}}{m_1!m_1!m_2!m_2!\cdots m_n!m_n!} \prod_{j=1}^{n} (-1)^{j-m_j}. \tag{4.45}\]
By the formula for \(\alpha_k(w)\), we have
\[
\beta_k(w) = \sum_{n=0}^{k} \sum_{S_n} \frac{e^{-(m_1+\cdots+m_n)\theta^*}}{m_1!m_1!m_2!m_2!\cdots m_n!m_n!} \prod_{j=1}^{n} (-1)^{j-m_j} \cdot \prod_{\ell=0}^{k-n} \frac{\psi^{(k-n-\ell)}(\theta^*)}{(k-n-\ell)!} \sum_{S_\ell} \frac{(-1)^{m_1+\cdots+m_\ell}}{m_1!m_1!m_2!m_2!\cdots m_\ell!m_\ell!} \prod_{j=1}^{\ell} \left( \frac{1}{\eta''(\theta^*)} \left( \frac{j+2}{(j+2)(j+1)} \eta_j(\theta^*) \right) \right)^{m_j} \frac{(i)^k w^{k+2(m_1+\cdots+m_\ell)}}{(\eta''(\theta^*))^{k/2}}. \tag{4.46}\]
Hence, we conclude that
\[
b_k = \sum_{n=0}^{2k} \sum_{S_n} \frac{e^{-(m_1+\cdots+m_n)\theta^*}}{m_1!m_1!m_2!m_2!\cdots m_n!m_n!} \prod_{j=1}^{n} (-1)^{j-m_j} \cdot \prod_{\ell=0}^{2k-n} \frac{\psi^{(2k-n-\ell)}(\theta^*)}{(2k-n-\ell)!} \sum_{S_\ell} \frac{(-1)^{m_1+\cdots+m_\ell}}{m_1!m_1!m_2!m_2!\cdots m_\ell!m_\ell!} \prod_{j=1}^{\ell} \left( \frac{1}{\eta''(\theta^*)} \left( \frac{j+2}{(j+2)(j+1)} \eta_j(\theta^*) \right) \right)^{m_j} \frac{(-1)^{k}(2(k+m_1+\cdots+m_\ell) - 1)!}{(\eta''(\theta^*))^{k}}. \tag{4.47}\]

4.5 Proof of Proposition 3 (i): Computations of \(\eta^{(k)}(\theta^*)\)
We recall that \(\eta(\theta) = \nu(x(\theta) - 1)\) so that
\[
\eta^{(k)}(\theta^*) = \nu_x^{(k)}(\theta^*), \quad k \in \mathbb{N}. \tag{4.48}\]
Note that

\[ x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta) - 1)} = e^{\theta - \|h\|_{L^1}} e^{\|h\|_{L^1} x(\theta)}. \]  

(4.49)

Therefore, by Leibniz formula,

\[
x^{(k)}(\theta^*) = \sum_{\ell=0}^{k} \binom{k}{\ell} e^{\theta^* - \|h\|_{L^1}} \frac{d^\ell}{d\theta^\ell} e^{\|h\|_{L^1} x(\theta)} \bigg|_{\theta=\theta^*}.
\]

(4.50)

By Faà di Bruno’s formula, we get

\[
\frac{d^\ell}{d\theta^\ell} e^{\|h\|_{L^1} x(\theta)} \bigg|_{\theta=\theta^*} = \sum_{S_\ell} \ell! \cdot \frac{||h||_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! m_3! \cdots m_k! k!} \cdot e^{\|h\|_{L^1} x(\theta^*)} \cdot \prod_{j=1}^{k} (x^{(j)}(\theta^*))^{m_j}.
\]

(4.51)

Hence,

\[
x^{(k)}(\theta^*) = e^{\theta^* - \|h\|_{L^1}} \frac{d^k}{d\theta^k} e^{\|h\|_{L^1} x(\theta)} \bigg|_{\theta=\theta^*} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} e^{\theta^* - \|h\|_{L^1}} \frac{d^\ell}{d\theta^\ell} e^{\|h\|_{L^1} x(\theta)} \bigg|_{\theta=\theta^*} = e^{\theta^* - \|h\|_{L^1}} ||h||_{L^1} \cdot e^{\|h\|_{L^1} x(\theta^*)} \cdot x^{(k)}(\theta^*)
\]

\[
\quad + e^{\theta^* - \|h\|_{L^1}} \sum_{S_\ell} \ell! \cdot \frac{||h||_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! m_3! \cdots m_k! k!} \cdot e^{\|h\|_{L^1} x(\theta^*)} \cdot \prod_{j=1}^{k} (x^{(j)}(\theta^*))^{m_j}
\]

\[
\quad + \sum_{\ell=0}^{k-1} \binom{k}{\ell} e^{\theta^* - \|h\|_{L^1}} \sum_{S_\ell} \ell! \cdot \frac{||h||_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! m_3! \cdots m_k! k!} \cdot e^{\|h\|_{L^1} x(\theta^*)} \cdot \prod_{j=1}^{k} (x^{(j)}(\theta^*))^{m_j}.
\]

Note that \( x(\theta^*) = e^{\theta^* + \|h\|_{L^1}(x(\theta^*) - 1)}. \) Also that \( \theta^* = \arg \max_{\theta \geq 0} \{ \theta x - \nu(x(\theta) - 1) \}. \) Thus, \( x = \nu x'(\theta^*) \), which gives \( x'(\theta^*) = \frac{x}{\nu} \) and

\[
x(\theta^*) = \frac{x}{\nu + ||h||_{L^1} x}.
\]

(4.53)

For \( k \geq 1 \), \( x^{(k)}(\theta^*) \) can be computed recursively as:

\[
x^{(k)}(\theta^*) = \frac{x(\theta^*)}{1 - ||h||_{L^1} x(\theta^*)} \sum_{S_k, m_k = 0} \frac{k! \cdot ||h||_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! \cdots m_k! k! m_k!} \cdot \prod_{j=1}^{k} (x^{(j)}(\theta^*))^{m_j}
\]

\[
+ \frac{x(\theta^*)}{1 - ||h||_{L^1} x(\theta^*)} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{S_\ell} \ell! \cdot \frac{||h||_{L^1}^{m_1 + \cdots + m_k}}{m_1! m_2! \cdots m_k! k!} \cdot \prod_{j=1}^{k} (x^{(j)}(\theta^*))^{m_j}.
\]

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4.6 Proof of Proposition \(3\) (ii): Computations of \(\psi^{(k)}(\theta^*)\)

Let us recall that

\[
\psi(\theta) = e^{\int_{0}^{\infty} [F(s;\theta) - x(\theta)] ds}.
\]

(4.54)

where \(F(\cdot; \theta)\) satisfies:

\[
F(t; \theta) = e^{\theta + \int_{0}^{t} (F(\cdot-s;\theta) -1) h(s) ds}.
\]

(4.55)

Let \(F^{(k)}(\cdot; \theta)\) denote the \(k\)-th partial derivative of \(F(\cdot; \theta)\) w.r.t. \(\theta\). By Faà di Bruno’s formula, we have

\[
\psi^{(k)}(\theta^*) = \sum_{S_k} \frac{k! \cdot \psi^{m_1 + \cdots + m_k} \cdot \psi(\theta^*)}{m_1! m_2! \cdots m_k! k! m_k} \prod_{j=1}^{k} \left( \int_{0}^{\infty} \left[ F^{(j)}(s;\theta^*) - x^{(j)}(\theta^*) \right] ds \right)^{m_j}.
\]

(4.56)

Moreover, by Leibniz formula, we can compute that

\[
F^{(k)}(t; \theta^*) = \sum_{\ell=0}^{k} \binom{k}{\ell} e^{\theta^* - \int_{0}^{t} h(s) ds} \frac{d^\ell}{d\theta^\ell} e^{\int_{0}^{t} F(t-s;\theta) h(s) ds} \bigg|_{\theta = \theta^*}.
\]

(4.57)

By Faà di Bruno’s formula, similar to our derivations for the formulas for \(x^{(k)}(\theta^*)\), we have

\[
F^{(k)}(t; \theta^*) = F(t; \theta^*) \cdot \int_{0}^{t} F^{(k)}(t-s; \theta^*) h(s) ds
\]

\[
+ F(t; \theta^*) \cdot \sum_{S_k: m_k = 0} \frac{k!}{m_1! m_2! \cdots m_k! k! m_k} \prod_{j=1}^{k} \left( \int_{0}^{t} F^{(j)}(t-s; \theta^*) h(s) ds \right)^{m_j}
\]

\[
+ F(t; \theta^*) \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{S_k} \frac{\ell!}{m_1! m_2! \cdots m_\ell! m_\ell} \prod_{j=1}^{\ell} \left( \int_{0}^{t} F^{(j)}(t-s; \theta^*) h(s) ds \right)^{m_j}.
\]

(4.58)

4.7 Computations of \(a_1\) and \(b_1\) for Theorem \(2\)

By Proposition \(1\) we can compute that

\[
a_1 = -\frac{1}{2} \frac{\psi''(\theta^*)}{\eta''(\theta^*)} + \frac{1}{24} \frac{\psi((\theta^*)^4(\theta^*) + 4\psi'(\theta^*)\eta^{(3)}(\theta^*)}{(\eta''(\theta^*))^2} - \frac{15}{72} \frac{\psi(\theta^*)\eta^{(3)}(\theta^*)^2}{(\eta''(\theta^*))^3},
\]

(4.59)
and

\[
b_1 = -\psi(\theta^*) \frac{1}{2} \frac{e^{-\theta^*} + e^{-2\theta^*}}{(1 - e^{-\theta^*})^2} \frac{1}{\eta''(\theta^*)} - \frac{1}{2} \psi''(\theta^*) \frac{1}{\eta''(\theta^*)} + \psi(\theta^*) \left[ \eta^{(4)}(\theta^*) \frac{3}{24 (\eta''(\theta^*))^2} - \frac{1}{2} (\eta^{(3)}(\theta^*))^2 \frac{15}{36 (\eta''(\theta^*))^3} \right] + \frac{e^{-\theta^*}}{1 - e^{-\theta^*}} \frac{\psi'(\theta^*)}{\eta''(\theta^*)} + \psi'(\theta^*) \eta^{(3)}(\theta^*) \frac{1}{6 (\eta''(\theta^*))^2} - \frac{e^{-\theta^*}}{1 - e^{-\theta^*}} \eta^{(3)}(\theta^*) \frac{3}{16 (\eta''(\theta^*))^2}. \tag{4.60}
\]

Thus to obtain \(a_1\) and \(b_1\), we need to compute \(\psi(\theta^*), \psi'(\theta^*), \psi''(\theta^*), \eta^{(3)}(\theta^*)\) and \(\eta^{(4)}(\theta^*)\) as provided in Proposition 3. Let us recall that

\[
\psi(\theta) = e^\nu \int_0^\infty [F(s; \theta) - x(\theta)] ds. \tag{4.61}
\]

Thus, we have

\[
\psi'(\theta^*) = \nu \int_0^\infty [F^{(1)}(s; \theta^*) - x'(\theta^*)] ds \cdot \psi(\theta^*), \tag{4.62}
\]

and

\[
\psi''(\theta^*) = \left[ \nu \int_0^\infty [F^{(1)}(s; \theta^*) - x'(\theta^*)] ds \right]^2 + \nu \int_0^\infty [F^{(2)}(s; \theta^*) - x''(\theta^*)] ds \cdot \psi(\theta^*). \tag{4.63}
\]

We can first solve \(F(\cdot; \theta^*)\) numerically:

\[
F(t; \theta^*) = e^{\theta^*} + \int_0^t (F(t-s; \theta^*) - 1) h(s) ds. \tag{4.64}
\]

Then, we can solve \(F^{(1)}(\cdot; \theta^*)\) numerically:

\[
F^{(1)}(t; \theta^*) = \left[ 1 + \int_0^t F^{(1)}(t-s; \theta^*) h(s) ds \right] F(t; \theta^*), \tag{4.65}
\]

and finally solve \(F^{(2)}(\cdot; \theta^*)\) numerically:

\[
F^{(2)}(t; \theta^*) = \left[ \int_0^t F^{(2)}(t-s; \theta^*) h(s) ds + \left( 1 + \int_0^t F^{(1)}(t-s; \theta^*) h(s) ds \right)^2 \right] F(t; \theta^*). \tag{4.66}
\]

Let us recall that \(\eta(\theta) = \nu (x(\theta) - 1)\) and

\[
\eta''(\theta^*) = \nu x''(\theta^*), \quad \eta^{(3)}(\theta^*) = \nu x^{(3)}(\theta^*), \quad \eta^{(4)}(\theta^*) = \nu x^{(4)}(\theta^*). \tag{4.67}
\]
We can compute from the formulas in Proposition 3 that

\[
\begin{align*}
x'(\theta) &= \frac{x(\theta)}{1 - \|h\|_{L^1} x(\theta)}, \\
x''(\theta) &= \frac{(1 + \|h\|_{L^1} x'(\theta))^2 x(\theta)}{1 - \|h\|_{L^1} x(\theta)}, \\
x^{(3)}(\theta) &= \frac{3(1 + \|h\|_{L^1} x'(\theta))\|h\|_{L^1} x''(\theta) + (1 + \|h\|_{L^1} x'(\theta))^3 x(\theta)}{1 - \|h\|_{L^1} x(\theta)}, \\
x^{(4)}(\theta) &= \frac{3(\|h\|_{L^1} x''(\theta))^2 + 3(1 + \|h\|_{L^1} x'(\theta))\|h\|_{L^1} x^{(3)}(\theta)) x(\theta)}{1 - \|h\|_{L^1} x(\theta)} \frac{(1 + \|h\|_{L^1} x'(\theta))^4 x(\theta)}{\|h\|_{L^1} x(\theta)}.
\end{align*}
\]

Thus \(x'(\theta^*)\), \(x''(\theta^*)\), \(x^{(3)}(\theta^*)\) and \(x^{(4)}(\theta^*)\) can be computed from \(x(\theta^*)\). Note that \(\theta^* = \arg\max_{\theta \geq 0} \{\theta x - \nu(x(\theta) - 1)\}\). Thus, \(x = \nu x'(\theta^*)\), which gives \(x'(\theta^*) = \frac{x}{\nu}\) and \(x(\theta^*) = \frac{x}{\nu + \|h\|_{L^1} x}\). Hence, we get

\[
\begin{align*}
x'(\theta^*) &= \frac{x}{\nu}, \\
x''(\theta^*) &= \left(1 + \|h\|_{L^1} \frac{x}{\nu}\right)^2 \frac{x}{\nu}, \\
x^{(3)}(\theta^*) &= \left(1 + \|h\|_{L^1} \frac{x}{\nu}\right)^3 \left(1 + 3\|h\|_{L^1} \frac{x}{\nu}\right) \frac{x}{\nu}, \\
x^{(4)}(\theta^*) &= 3 \left(1 + \|h\|_{L^1} \frac{x}{\nu}\right)^4 \left(\|h\|_{L^1} \frac{x}{\nu}\right)^2 + \left(1 + 3\|h\|_{L^1} \frac{x}{\nu}\right) \|h\|_{L^1} \frac{x}{\nu}\right) \frac{x}{\nu} \\
&\quad + \left(1 + \|h\|_{L^1} \frac{x}{\nu}\right)^4 \left(1 + 6\|h\|_{L^1} \frac{x}{\nu}\right) \frac{x}{\nu}.
\end{align*}
\]
4.8 Proof of Proposition 12

Proof. By the change of measure for simple point processes, we have

\[ E[1_{N_t \geq x}t] = \hat{\mathbb{E}} \left[ 1_{N_t \geq x} \cdot e^{\int_0^t \log \left( \frac{\lambda_s}{\hat{\lambda}_s} \right) dN_s - \int_0^t (\hat{\lambda}_s - \lambda_s) ds} \right] \]

\[ = \hat{\mathbb{E}} \left[ 1_{N_t \geq x} \cdot e^{-((\log \gamma)N_t + (\gamma - 1) \int_0^t \lambda_s ds)} \right] \]

\[ = \hat{\mathbb{E}} \left[ 1_{N_t \geq x} \cdot e^{-((\log \gamma)N_t + (\gamma - 1) x + (\gamma - 1) \int_0^t \lambda_s ds - \int_0^t \hat{\lambda}_s - \lambda_s) ds} \right] \]

\[ = \hat{\mathbb{E}} \left[ 1_{N_t \geq x} \cdot e^{-((\log \gamma)N_t + (\gamma - 1) x + (\gamma - 1) \int_0^t \lambda_s ds - \int_0^t \hat{\lambda}_s - \lambda_s) ds} \right] \]

\[ = \hat{\mathbb{E}} \left[ 1_{N_t \geq x} \cdot e^{-((\log \gamma)N_t + (\gamma - 1) x + (\gamma - 1) \int_0^t \lambda_s ds - \int_0^t \hat{\lambda}_s - \lambda_s) ds} \right] \]

where we recall that \( H(t) = \int_t^\infty h(s)ds \) denotes the right tail of the exciting function.

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References

[1] Abergel, F. and A. Jedidi. (2015). Long time behaviour of a Hawkes process-based limit order book. SIAM J. Financial Math. 6, 1026-1043.

[2] Aït-Sahalia, Y., Cacho-Diaz, J. and R. J. A. Laeven. (2015). Modeling financial contagion using mutually exciting jump processes. Journal of Financial Economics. 117, 585-606.

[3] Aït-Sahalia Y., Laeven R.J.A, Pelizzon L. (2014). Mutual excitation in Eurozone sovereign CDS. Journal of Econometrics. 183, 151-167.

[4] Alfonsi, A. and P. Blanc. (2016). Dynamic optimal execution in a mixed-market-impact Hawkes price model. Finance and Stochastics. 20, 183-218.
[5] Bacry, E., Delattre, S., Hoffmann, M. and J. F. Muzy. (2013). Scaling limits for Hawkes processes and application to financial statistics. *Stochastic Processes and their Applications* **123**, 2475-2499.

[6] Bacry, E., Delattre S., Hoffmann, M. and J. F. Muzy. (2013). Modelling microstructure noise with mutually exciting point processes. *Quantitative Finance*. **13**, 65-77.

[7] Bauwens, L. and N. Hautsch. (2009). Handbook of financial time series: modelling financial high frequency data using point processes. Springer, Berlin.

[8] Bordenave, C. and Torrisi, G. L. (2007). Large deviations of Poisson cluster processes. *Stochastic Models*, **23**, 593-625.

[9] Brémaud, P. and Massoulié, L. (1996). Stability of nonlinear Hawkes processes. *Ann. Probab.*, **24**, 1563-1588.

[10] Chavez-Demoulin, V., Davison, A. C. and A. J. McNeil. (2005). A point process approach to value-at-risk estimation. *Quantitative Finance*. **5**, 227-234.

[11] Chu, S. and F. Metcalf. (1967). On Gronwall’s Inequality. *Proc. Amer. Math. Soc.* **18**, 439-440.

[12] Crane, R. and D. Sornette. (2008). Robust dynamic classes revealed by measuring the response function of a social system. *Proc. Nat. Acad. Sci. USA* **105**, 15649.

[13] Dembo, A. and O. Zeitouni. *Large Deviations Techniques and Applications*. 2nd Edition, Springer, New York, 1998.

[14] Errais, E., Giesecke, K. and Goldberg, L. (2010). Affine point processes and portfolio credit risk. *SIAM J. Financial Math*. **1**, 642-665.

[15] Feray, V., Meliot, P.-L., and A. Nikeghbali. (2013). Mod-ϕ convergence I: Normality zones and precise deviations. *arXiv:1304.2934*.

[16] Gao, X. and Zhu, L. (2015). Limit theorems for linear Markovian Hawkes processes with large initial intensity. To appear in *Stochastic Processes and their Applications*.

[17] Gao, X. and Zhu, L. (2018). Large deviations and applications for Markovian Hawkes processes with a large initial intensity. *Bernoulli*. **24**, 2875-2905.

[18] Gao, X. and Zhu, L. (2016). A functional central limit theorem for stationary Hawkes processes and its application to infinite-server queues. To appear in *Queueing Systems*.

[19] Gao, X. and Zhu, L. (2018). Transform analysis for Hawkes processes with applications in dark pool trading. *Quantitative Finance*. **18**, 265-282.

[20] Giot, P. (2005). Market risk models for intraday data. *European Journal of Finance*. **11**, 309-324.

[21] Gusto, G. and S. Schbath. (2005). F.A.D.O.: a statistical method to detect favored or avoided distances between occurrences of motifs using the Hawkes model. *Stat. Appl. Genet. Mol. Biol.*, **4**, Article 24.
[22] Hainaut, D. (2016). A bivariate Hawkes process for interest rates modelling. *Economic Modelling*. **57**, 180-196.

[23] Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **58**, 83-90.

[24] Hawkes, A. G. and Oakes, D. (1974). A cluster process representation of a self-exciting process. *J. Appl. Prob.*** **11**, 493-503.

[25] Jaisson, T. and M. Rosenbaum. (2015). Limit theorems for nearly unstable Hawkes processes. *Annals of Applied Probability*. **25**, 600-631.

[26] Jaisson, T. and M. Rosenbaum. (2016). Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *Annals of Applied Probability*. **26**, 2860-2882.

[27] Karabash, D. and L. Zhu. (2015) Limit theorems for marked Hawkes processes with application to a risk model. *Stochastic Models*. **31**, 433-451.

[28] Lewis, E., Mohler, Brantingham, P. J. and A. Bertozzi. (2011). Self-exciting point process of insurgency in Iraq. *Security Journal*. **25**, 0955-1662.

[29] Li, L. and H. Zha. (2013). Dyadic event attribution in social networks with mixtures of Hawkes processes. *Proceedings of the 22nd ACM International Conference on Information & Knowledge Management* pp 1667-1672.

[30] Mohler, G. O., Short, M. B., Brantingham, P. J., Schoenberg F. P. and G. E. Tita. (2011). Self-exciting point process modelling of crime. *Journal of the American Statistical Association*. **106**, 100-108.

[31] Ogata, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *J. Amer. Statist. Assoc*. **83**, 9-27.

[32] Ogata, Y. (1998). Space-time point-process models for earthquake occurrences. *Ann. Inst. Statist. Math*. **50**, 379-402.

[33] Pernice, V., Staude B., Carandobile, S. and S. Rotter. (2012). How structure determines correlations in neuronal networks. *PLoS Computational Biology*. **85**:031916.

[34] Pernice, V., Staude B., Carandobile, S. and S. Rotter. (2011). Recurrent interactions in spiking networks with arbitrary topology. *Physical Review E*. **75**:e1002059.

[35] Porter, M. and G. White. (2012). Self-exciting hurdle models for terrorist activity. *Annals of Applied Statistics*. **6**, 106-124.

[36] Reynaud-Bouret, P., Rivoirard, V. and C. Tuleau-Malot. (2013). Inference of functional connectivity in Neurosciences via Hawkes processes. *1st IEEE Global Conference on Signal and Information Processing*.

[37] Reynaud-Bouret, P. and S. Schbath. (2010). Adaptive estimation for Hawkes processes; application to genome analysis. *Ann. Statist*. **38**, 2781-2822.
[38] Stabile, G. and G. L. Torrisi. (2010). Risk processes with non-stationary Hawkes arrivals. *Methodol. Comput. Appl. Prob.* **12**, 415-429.

[39] Torrisi, G. L. (2016). Gaussian approximation of nonlinear Hawkes processes. *Annals of Applied Probability*. **26**, 2106-2140.

[40] Torrisi, G. L. (2017). Poisson approximation of point processes with stochastic intensity, and application to nonlinear Hawkes processes. *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*. **53**, 679-700.

[41] Vere-Jones, D. (1978). Earthquake prediction: A statistician’s view. *Journal of Physics of the Earth*. **26**, 129-146.

[42] Zhang, X., Blanchet, J., Giesecke, K., and P. W. Glynn. (2009). Rare event simulation for a generalized Hawkes process. Proceedings of the 2009 Winter Simulation Conference.

[43] Zhang, X., Blanchet, J., Giesecke, K., and P. W. Glynn. (2015). Affine point processes: Approximation and efficient simulation. *Mathematics of Operations Research*. **40**, 797-819.

[44] Zhao, Q., Erdogdu, M. A., He, H. Y., Rajaraman, A. and L. Jure. (2015). SEISMIC: A Self-Exciting Point Process Model for Predicting Tweet Popularity. *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 1513–1522.

[45] Zhu, L. (2013). Ruin probabilities for risk processes with non-stationary arrivals and subexponential claims. *Insurance: Mathematics and Economics*. **53** 544-550.

[46] Zhu, L. (2013). Central limit theorem for nonlinear Hawkes processes. *Journal of Applied Probability*. **50** 760-771.

[47] Zhu, L. (2013). Moderate deviations for Hawkes processes. *Statistics & Probability Letters*. **83**, 885-890.

[48] Zhu, L. (2014). Process-level large deviations for nonlinear Hawkes point processes. *Annales de l’Institut Henri Poincaré-Probabilités et Statistiques*. **50**, 845-871.

[49] Zhu, L. (2015). Large deviations for Markovian nonlinear Hawkes Processes. *Annals of Applied Probability*. **25**, 548-581.