1. Introduction

In the last over thirty years, following the discovery of the quantum Hall effect [1,2], physicists have established a new classification of fundamental states of quantum matter, called topological quantum phases [3,4], which are beyond the scope of the celebrated Landau-Ginzburg-Wilson framework that characterizes quantum matter by local symmetry-breaking orders [5]. A topological quantum phase can be classified by bulk topological invariant defined in the ground state at equilibrium, and hosts protected boundary modes through the bulk-boundary correspondence [6,7]. This characterization crucially influences the strategies of identifying topological states in experiment, which is challenging in general since a non-local topological invariant may not have direct physical measurements. Despite the great success achieved with various strategies in discovering new topological matter, such as topological insulators [8–11] and semimetals [12,13] whose boundary modes can be probed by transport measurement or angle resolved photoemission spectroscopy, by far only a small portion of topological states predicted in theory have been uncovered in experiment [14]. In some circumstances the measurements are not fully unambiguous for being not direct observations of topological numbers, for which there are topological states, e.g., the topological superconductors [15–19], even well explored in theory, necessitating further experimental confirmation.

As yet, the topological phases have been primarily characterized with classification theories developed for equilibrium systems, ranging from symmetry-protected topological phases [20,21] to intrinsic topological orders [4]. A fundamental question arises that, for a generic topological quantum phase defined for the equilibrium ground state, is there a non-equilibrium classification of such phase? We address this fundamental issue in the present work, and establish a dynamical classification theory for topological quantum phases characterized by integers of multiband and all dimensions. Not only being of the clear fundamental significance, the dynamical classification is also experimentally important in probing new topological quantum physics, particularly relevant for ultracold atom systems where, due to heating, the equilibrium ground states are usually hard to achieve, but the quantum dynamics can be readily engineered as having been considered in the recent studies.

Keywords:
Quench dynamics
Topological quantum phase
Band inversion surface
Bulk-surface duality
Dynamical topological invariant
Ultracold atoms
Synthetic gauge field

Abstract

Topological phase of matter is now a mainstream of research in condensed matter physics, of which the classification, synthesis, and detection of topological states have brought excitements over the recent decade while remain incomplete with ongoing challenges in both theory and experiment. Here we propose to establish a universal non-equilibrium characterization of the equilibrium topological quantum phases classified by integers, and further propose the high-precision dynamical schemes to detect such phases. The framework of the dynamical classification theory consists of basic theorems. First, we uncover that classifying a d-dimensional (dD) gapped topological phase of generic multibands can reduce to a (d − 1)D invariant defined on so-called band inversion surfaces (BISs), rendering a bulk-surface duality which simplifies the topological characterization. Further, we show in quenching across phase boundary the (pseudo) spin dynamics to exhibit unique topological patterns on BISs, which are attributed to the post-quench bulk topology and manifest a dynamical bulk-surface correspondence. For this the topological phase is classified by a dynamical topological invariant measured from an emergent dynamical spin-texture field on the BISs. Applications to quenching experiments on feasible models are proposed and studied, demonstrating the new experimental strategies to detect topological phases with high feasibility. This work opens a broad new direction to classify and detect topological phases by non-equilibrium quantum dynamics.

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of topological systems [22–26]. The cold atom experiments have demonstrated the feasibility of simulating novel topological systems with quantum gases, including the 1D Su–Schrieffer–Heeger (SSH) chain [27,28], 1D chiral topological phase [29,30], and 2D Chern insulators [31–34]. In particular, characterizing the band topology of an equilibrium Hamiltonian from quantum dynamics was proposed and explored very recently in theory and experiment [30,35,36]. Nevertheless, these studies are not universal and valid for two-band models in specific dimensions. No dynamical classification theory was proposed until the present work.

The framework of the dynamical classification theory is made up of several basic theorems uncovered in this work. First, for a generic d-dimensional (dD) gapped topological phase characterized by integer invariants, we show a bulk-surface duality that its classification reduces to a (d − 1)-D invariant defined on the so-called band inversion surfaces (BISs). Further, in quenching from trivial to topological phases, the BISs are captured dynamically that on BISs the time-averaged (pseudo) spin-polarizations vanish in the unitary evolution. Finally, the bulk topology of the post-quench phase is classified by the dynamical topological invariant determined via an emergent dynamical spin-texture field on BISs. The results manifest a dynamical bulk-surface correspondence which can be directly measured with high feasibility in quenching experiments. With the dynamical classification theory we propose various high-precision dynamical schemes to detect topological quantum states, of which the application to measuring Chern insulators has been achieved in a latest follow-up experiment [37].

2. Generic model

We start with the generic dD gapped topological phases, including insulators and superconductors, which are classified by integer invariants in the Altland–Zirnbauer (AZ) symmetry classes [38–41]. The basic Hamiltonian can be written in the elementary representation matrices of the Clifford algebra [42,43] (see also the Electronic Supplementary Material)

\[
\tau\tau(k) = \hat{h}(k) \cdot \vec{\tau} = \sum_{i=0}^{d} h_i(k) \gamma_i, \tag{1}
\]

where the \( \gamma_i \) matrices define a (pseudo) spin obeying anticommutation relation \( \{ \gamma_i, \gamma_j \} = 2 \delta_{ij} \), with \( i, j = 0, 1, \ldots, d \), and \( \hat{h}(k) \) mimics a (d + 1)D Zeeman field depending on the Bloch momentum \( \mathbf{k} \) in BZ. The dimensionality of \( \gamma_i \) matrices in the elementary representation reads \( n_d = 2^{d/2} \) (or \( 2^{(d+1)/2} \)) if \( d \) is even (or odd), which is the minimal requirement to open a topological gap for the dD topological phase [43]. In the 1D/2D regimes, for instance, the \( \gamma_i \) matrices simply reduce to the Pauli ones, and \( \tau\tau(k) \) describes a two-band model for the topological states, e.g., the well-known SSH model [27,41] for 1D BDI class insulator and the Haldane model [44] for 2D Chern insulator. Similarly, for 3D/4D phases, the \( \gamma_i \) matrices take the Dirac forms, and a fully gapped topological phase has to involve at least four bands, such as the 3D AII class topological insulator (also DIII class superconductors) [45,39] and 4D quantum Hall effect [46]. While for convenience the dynamical classification theory is formulated with the basic Hamiltonian written in the above elementary representation, the theory applies to any generic multiband model of the dD phase (see proof in the Electronic Supplementary Material). Therefore, all the main results presented based on Eq. (1) directly apply to the generic dD gapped topological phases characterized by integer invariants.

To facilitate the description, we pick up an arbitrary component, say \( h_{\mathbf{n}_j}(k) \), from the vector field \( \hat{h}(k) \) to characterize the “dispersion” of the \( n_j \) decoupled bands. Accordingly, we denote the remaining components of \( \hat{h}(k) \) as the “spin-orbit” (SO) field \( h_{\mathbf{n}_j}(k) = (h_1, \ldots, h_d) \). Without the SO field, the band crossing occurs for the \( n_j \) decoupled bands if \( h_0(k) = 0 \) results at certain momentum points of the BZ. All such momentum points form closed (d − 1)D surface, dubbed band inversion surface (BIS), which is a key concept in this work. The BIS is related to, but conceptually different from the familiar phenomenon of band inversion in topological insulators, and can be defined for broad classes of phases including topological superconductors. Across the BIS the energy difference between half of the \( n_j \) bands and the remaining half switches sign. In general one can have multiple BISs in the first BZ. The gap opens when the nonzero SO field \( h_{\mathbf{n}_j}(k) \) is switched on in the BISs. As shown below, the dD gapped phase being topologically nontrivial necessitates that the BZ includes at least one BIS which hosts a nonzero (d − 1)D invariant.

3. Bulk-surface duality

The topology of the Hamiltonian \( \tau\tau(k) \) is classified by dD winding number for odd dimensionality \( d = 2n - 1 \), or the \( n \)-th Chern number if \( d = 2n \). The topological characterization of the equilibrium phase is formulated by a mapping from the BZ, i.e., a dD torus \( T^d \), to the dD spherical surface \( S^d \) through the unit vector field \( \mathbf{n}(k) = \hat{h}(k) / |\hat{h}(k)| \). The topological number counts the times that the mapping covers \( S^d \). Interestingly, as detailed in the Electronic Supplementary Material, we show that the topological characterization by dD winding number (or Chern number) reduces to a (d − 1)D Chern number (or winding number) \( w_{d-1} \) defined on the BISs

\[
v_d(h_0, h_0) = w_{d-1}(h_0), \tag{2}
\]

where \( w_{d-1} \) characterizes the integer times that the unit SO field \( h_{\mathbf{n}}(k) = h_0(k) / |h_0(k)| \) covers over the (d − 1)D spherical surface \( S^{d-1} \) when \( \mathbf{k} \) runs over all the BISs of the BZ. Without giving details, we show the above invariant \( w_{d-1} \) intuitively by clarifying the key ingredients of the theorem, while the rigorous proof can be found in the Electronic Supplementary Material. As for topological phase, the mapping \( T^d \rightarrow S^d \) covers nonzero integer times of the dD spherical surface, any component of the vector field \( \hat{h}(k) \), including \( h_0(k) \), spans both positive and negative values in the BZ. Thus the BIS with \( h_0(k) = 0 \) must be obtained. It is convenient to denote by \( \Psi_{\text{BIS}} \) the vector volume of the BZ with \( h_0(k) < 0 \), and \( \Psi_{\text{BIS}} \) the volume with \( h_0(k) > 0 \). Note that \( w_{d-1} \) in the right hand side of Eq. (2) is given by a (d − 1)D surface integral enclosing the open vector volume \( \Psi_{\text{BIS}} \) (see the Electronic Supplementary Material), which is gauge-independent and counts the total topological charges located at the singularities with \( h_{\mathbf{n}}(k) = 0 \) in this volume (see Fig. 1a). This result can be intuitively understood in the following way, for which we rewrite that \( h_0(k) = m_0 + \hat{h}_0(k) \), with \( \hat{h}_0(k) \) being momentum-dependent and \( m_0 \) mimicking a constant “magnetization” which tunes the topology of the phase. It is ready to know that when \( m_0 > \max ||\hat{h}_0(k)|| \) for all \( \mathbf{k} \), the phase is trivial since no BIS can be obtained in the BZ. The BIS starts to emerge in the BZ by reducing the magnetization to \( m_0 \leq \max ||\hat{h}_0(k)|| \), while the gap of the system keeps open if no topological charges (at \( h_0 = 0 \)) enter the vector volume \( \Psi_{\text{BIS}} \). Further reducing \( m_0 \) eventually enables that the topological charges pass through the BIS and enter the volume \( \Psi_{\text{BIS}} \). Note that when a topological charge crosses the BIS, the bulk gap closes one time since \( h_0 = h_0 = 0 \) occurs at the crossing momentum on the BIS. This leads to a topological phase transition with the topological invariant varying by one. Then the number of topological charges with \( h_0(k) = 0 \) enclosed in the volume \( \Psi_{\text{BIS}} \) reflects the
Numerical results for the 1D model. Time evolution (upper panel) and the counts the total topological charges \( \mathcal{C}_i \) of the SO field in the region \( \mathcal{V}_{\text{BIS}} \) (gray) enclosed by BISs with \( h_0 < 0 \). Here the momentum \( \mathbf{k} \) is decomposed into \( (k_x, k_z) \) (black arrows) and \( k_z \) perpendicular to the BISs and pointing to the complementary region of \( \mathcal{V}_{\text{BIS}} \), i.e., \( \mathcal{V}_{\text{BIS}} \) with \( h_0 > 0 \). (b), (c) Detecting topological patterns on the BISs by quench dynamics. A quench process is employed, where a trivial system with highly polarized (pseudo) spins in the negative direction of the axes \( h_0 \) is suddenly quenched to a topological band with non-trivial spin textures (b). Subsequently, dramatic spin procession occurs near the BISs with the rotation axis being the post-quench vector field \( \vec{h}(\mathbf{k}) \) (c). The SO field is encoded by the variation of time-averaged spin polarizations \( \vec{\sigma}_{\mathbf{k}} \) with respect to \( \mathbf{k} \). Here the blue and red dashed lines in (b) represent the \( h_0 \) component of the post-quench Hamiltonian with the band crossing being the \( (d-1) \)-D BIS, and \( \phi \) in (c) denotes the relative angle between the inital spin (blue) and the post-quench vector field \( \vec{h}(\mathbf{k}) \). (d), (e) Numerical results for the 1D model. Time evolution (upper panel) and the corresponding averages (lower panel) of spin textures are shown after a sudden quench from \( m_0 = 8 \sigma_{z} \) (trivial) to \( 0 \) (topological) with \( t_0 = 0.2 \tau_0 \). The BIS is determined by momenta with vanishing time-averaged spin-polarizations \( \langle \vec{\sigma}_{\mathbf{k}} \rangle \) (dashed lines), and the new dynamical spin-texture field \( \vec{g}(\mathbf{k}) = -i \partial_t \vec{\sigma}_{\mathbf{k}} \) (purple arrows in (e)) characterizes the nontrivial topology.

The above theorem shows a bulk-surface duality which maps the classification of bulk topology to the characterization on BISs. This bulk-surface duality is valid for the generic systems with arbitrary multiple bands and multiple band crossings in the BZ. The key observation is that for a generic \( d \) D topological phase, the gap must be opened through each group of \( n_b \) bands \( (n_b = 2^{d/2} \text{ or } 2^{d-1}/2) \) with hybridization at BISs of these bands. Accordingly, by transforming the system into a new trivial bases, which block diagonalize the Hamiltonian \( \mathcal{H}(\mathbf{k}) = \mathcal{H}_1(\mathbf{k}) \oplus \mathcal{H}_2(\mathbf{k}) \oplus \cdots \) at each momentum \( \mathbf{k} \), one can show directly that only those blocks involving \( n_b \) bands with BISs have nontrivial contribution to the topology.

The topology of the bulk bands can then be reduced to the topological invariants defined on the corresponding BISs of each block. More details of the generalization can be found in the Electronic Supplementary Material.

The bulk-surface duality provides a simplified characterization of the topological phases with lower-dimensional invariants, while there are two important issues having to be addressed. First, the BISs are not unique but depends on the choice of \( h_0 \) axis. Then how to precisely identify BISs in a real experiment? Further, how to measure the topology defined on the BISs? As we show below that, the uncovered bulk-surface duality is further mapped to a novel dynamical form in the quench dynamics, which directly measures all the topological features on BISs with high feasibility.

4. Non-equilibrium classification: dynamical bulk-surface correspondence

We proceed to consider the non-equilibrium characterization of topological phases based on the bulk-surface duality shown above. The quantum dynamics can be induced by quenching the phase from a deep initially trivial phase at time \( t = 0 \), which is given by setting \( m_0|_{t=0} \gg \max \left[ |\vec{h}(\mathbf{k})| \right] \), to the final topological state which is obtained in proper regime \( |m_0|_{t=0} < \max[|h_0(\mathbf{k})|] \) (Fig. 1b). The unitary evolution of the (pseudo) spin \( \vec{q} \) is then governed by the post-quench Hamiltonian \( \mathcal{H}(m_0, \mathbf{k}) \). The quenching dynamics can be quantified by introducing the dynamical averaging of the spin-polarization, which is obtained by

\[
\langle \vec{q}(\mathbf{k}) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \text{d}t \text{Tr}[\rho_0 e^{i\mathcal{H}t} e^{-i\mathcal{H}T}],
\]

where \( \rho_0 \) is density of matrix of the initial state. The peculiar effects are obtained on the BISs. Note that the system is initialized in the fully spin-polarized phase with the (pseudo) spin of all Bloch states pointing along negative \( h_0 \) axis (for positive \( m_0 \)). The spin processes with respect to \( \vec{h}(\mathbf{k}) \) after quenching. On the BISs, the vector \( \vec{h}(\mathbf{k}) = h_{0z}(\mathbf{k}) \) is perpendicular to initial spin polarization, which leads to the process of the spin vector \( \vec{q} \) within the plane perpendicular to \( h_{0z} \) while incorporating \( h_{0x}(\mathbf{k}) \) axis (Fig. 1c). As a result, the dynamical averaging \( \langle \vec{q}(\mathbf{k}) \rangle \) vanishes right on the BISs for all \( i \)-components. On the other hand, at the momentum \( \mathbf{k} \) away from the BISs the \( h_{0z} \) component is nonzero, so \( \vec{h} \) is not perpendicular to the initial spin polarization. Then the spin procession leads to a nonzero \( \langle \vec{q}(\mathbf{k}) \rangle \) along the \( \vec{h}(\mathbf{k}) \)-direction (Fig. 1c). With these results we conclude a dynamical characterization of the BISs that

\[
\langle \vec{q}(\mathbf{k}) \rangle = 0, \quad \text{for } \mathbf{k} \in \text{BISs}, \quad i = 0, 1, 2, \ldots, d.
\]

The above characterization can also be understood that on BISs, the resonant spin-reversing transitions \( \vec{q} \rightarrow -\vec{q} \) are induced by the SO field \( h_{0z}(\mathbf{k}) \) due the vanishing gap (for band-crossing surfaces). Thus the time-averaged spin-polarizations vanish at BISs. In contrast, away from the BISs, the nonzero \( h_{0x}(\mathbf{k}) \) severs as a detuning for the spin-reversing transitions, so the time-averaged spin-polarizations do not vanish.

The vanishing \( \langle \vec{q}(\mathbf{k}) \rangle \) on BIS implies that the topological number of the \( d \) D gapped phase cannot be obtained via the time-averaged spin-texture \( \langle \vec{q}(\mathbf{k}) \rangle \). Interestingly, more nontrivial features of the quench dynamics are captured by the variation of the time-averaged spin polarizations across the BISs. Note that the component \( h_0 < 0 \) in the vector volume \( \mathcal{V}_{\text{BIS}} \) and \( h_0 > 0 \) in the volume \( \mathcal{V}_{\text{BIS}} \). When passing through a BIS, the relative angle \( \phi \) between the initial spin-polarization and the vector \( \vec{h}(\mathbf{k}) \) varies from \( \phi < \pi/2 \) in \( \mathcal{V}_{\text{BIS}} \), with a nonzero dynamical averaging \( \langle \vec{q}(\mathbf{k}) \rangle \) pointing to \( \vec{h}(\mathbf{k}) \)-direction, to \( \phi > \pi/2 \) in \( \mathcal{V}_{\text{BIS}} \), with nonzero \( \langle \vec{q}(\mathbf{k}) \rangle \) pointing oppositely to \( \vec{h}(\mathbf{k}) \)-direction (Fig. 1c). This follows that the variation of \( \langle \vec{q}(\mathbf{k}) \rangle \) across the BIS follows the direction of the spin–orbit field \( h_{0x} \).

To quantify this picture, we define a new dynamical spin-texture field \( \vec{g}(\mathbf{k}) \), whose components are given by \( g_i(\mathbf{k}) \equiv -\frac{i}{\mathcal{N}_k} \partial_t \vec{q}_i(\mathbf{k}) \), with \( \mathcal{N}_k \) being the normalization factor. Here \( k_i \) denotes the times gap closing by tuning \( m_0 \) from trivial to the final topological regime, and the total topological charge gives the bulk topological number of the phase, as characterized by the theorem in Eq. (2).

The above theorem shows a bulk-surface duality which maps the classification of bulk topology to the characterization on BISs. This bulk-surface duality is valid for the generic systems with arbitrary multiple bands and multiple band crossings in the BZ. The key observation is that for a generic \( d \) D topological phase, the gap must be opened through each group of \( n_b \) bands \( (n_b = 2^{d/2} \text{ or } 2^{d-1}/2) \) with hybridization at BISs of these bands. Accordingly, by transforming the system into a new trivial bases, which block diagonalize the Hamiltonian \( \mathcal{H}(\mathbf{k}) = \mathcal{H}_1(\mathbf{k}) \oplus \mathcal{H}_2(\mathbf{k}) \oplus \cdots \) at each momentum \( \mathbf{k} \), one can show directly that only those blocks involving \( n_b \) bands with BISs have nontrivial contribution to the topology. The topology of the bulk bands can then be reduced to the topological invariants defined on the corresponding BISs of each block. More details of the generalization can be found in the Electronic Supplementary Material.

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magnitudes of the directional derivatives away from the BISs to zero. The spin-flip oscillation is well suppressed by the unitary evolution operator. The numerical results are shown in Fig. 1d, e, with the parameters taken as $m_z = 0$, $t'_{0x} = 0.2t_0$, and $m_y = m_x = 0$. The time-averaged spin-polarizations exhibit line-shape structures (Fig. 2e), which correspond to the two open BISs at $k_y = 0$ and $\pi$, respectively. Despite a trivial pattern along $k_x$ with $k_y = -\pi$, the winding of the dynamical spin-texture $\mathbf{g}(\mathbf{k})$ (in $x$-$z$ plane) along $k_y = 0$ measures the Chern number $C_1 = -1$ (Fig. 2d). Applying the dynamical classification to measuring topological states with high Chern numbers is straightforward and is found in the Electronic Supplementary Material.

5. Application to quenching experiments

We first consider the 1D topological phases of all class obtained by the Hamiltonian $\mathcal{H}(k) = \mathbf{h}(k) \cdot \sigma$, where $\mathbf{h}_x = t_0 \sin k_x \equiv h_{zo}$, $h_y = 0$, and $\mathbf{h}_z = m_z - t_0 \cos k_x \equiv h_{zo}$, with $t_0$ and $h_{zo}$ denoting the nearest-neighbor spin-conserved and spin-flipped hopping coefficients. The model was proposed in Ref. [29] and realized recently in experiment for $^{173}$Yb fermions [30]. The effective magnetization $m_z$ can be precisely tuned by a bias magnetic field. The quench study is performed by tuning the Zeeman term from $m_z \gg t_0$, which corresponds to the deep trivial regime with the system initialized in spin-down, to $|m_z| < t_0$, which lies in topological regime. The quantum dynamics is then governed by the unitary evolution operator. The numerical results are shown in Fig. 1d, e, with the parameters taken as $t_0 = 0.2t_0$. The resonant spin-flip transitions driven by SO field emerge at two momentum points $k_{BIS} = \pm \cos^{-1}(m_z/t_0)$ with vanishing time-averaged spin polarizations $\langle \sigma \rangle$, which manifests the BIS (band inversion points). Away from the BIS the spin-flip oscillation is well suppressed by the energy gap. The right hand side of the above formula describes the mapping $\mathbf{g}(k)_{BIS} = \mathbf{h}(k)$ over the full (d - 1)D spherical surface. The results in Eqs. (4)-(6) manifest a highly nontrivial dynamical bulk-surface correspondence, which enables us to propose high-precision dynamical schemes to detect the bulk topology, as studied below with experimentally feasible models.

We note that the dynamical classification theory is essentially based on the mapping of the dD bulk topology to (d - 1)D BISs. This feature is valid for generic multiband systems, in which case the BISs are defined among every $n_d$ bands in the BZ. The topology only relies on the (d - 1)D invariants “locally” defined on the BISs, which can be measured based on a sequence of quenches by initializing the system in different bands (see the Electronic Supplementary Material). This dimension reduction is the essential difference from the previous dynamical schemes which are applicable to two-band models [30,35,36].

**Fig. 2.** Characterizing 2D Chern insulators in different quench processes. (a), (b) Topological patterns by quenching along $h_z(k)$ axis. Time-averaged spin textures (a) are measured after a sudden quench from $m_z = 0$ to $t_0$ ($m_z = m_y = 0$). The dynamical field $\mathbf{g}(k)$ obtained from a indicates the Chern number $C_1 = -1$. (b), (c) Topological patterns by quenching $h_x(k)$ axis. Time-averaged spin textures (c) and the dynamical field on the BIS (d) are shown after a quench from $m_z = -t_0$ to $0$, with $m_z = -t_0$, $m_y = 0$, and $t'_{0x} = -0.5t_0$. Here the two BISs are at $k_y = 0$. While the winding of $\mathbf{g}(k)$ is trivial along $k_y = 0$, the non-zero winding number along $k_y = 0$ characterizes the topological phase with the Chern number $C_1 = -1$.
We further consider the application to 3D topological phases, whose hosting Hamiltonian reads $\mathcal{H}(\mathbf{k}) = \tilde{h}(\mathbf{k}) \cdot \mathbf{\sigma}$, with $h_0(\mathbf{k}) = m_z - t_0 \sum \cos k_i$ and $h_1 = t_{\text{so}} \sin k_i$ (for $i = x, y, z$). Here we take that $\gamma_0 = \sigma_z \otimes \tau_z$, $\gamma_1 = \sigma_z \otimes 1$, $\gamma_2 = \sigma_z \otimes 1$ and $\gamma_3 = \sigma_z \otimes \tau_z$, where the Pauli matrices $\sigma_i$ and $\tau_i$ may refer to the real- and pseudo-spin degrees of freedom (e.g., sublattices or orbitals). The Hamiltonian has a chiral symmetry defined by $\sigma_z \otimes \tau_z$, so it belongs to AII class and is classified by 3D winding numbers in equilibrium theory [38–40]. The trivial phase corresponds to $|m_z| > 3t_{\text{so}}$, while the topological phases include three regions: (I) $t_0 < m_z < 3t_0$ with winding number $\nu_3 = -1$; (II) $-t_0 < m_z < t_0$ with $\nu_3 = 2$; and (III) $-3t_0 < m_z < -t_0$ with $\nu_3 = -1$. We consider the quench from $m_z$ to $t_0$ to say phase (I) ($m_z = 1.3t_0$ and $t_{\text{so}} = 0.2t_0$). The dynamical results being partially presented in Fig. 3 (more is shown in the Electronic Supplementary Material). The BIS of the post-quench band is given in Fig. 3a, with the vector arrows denoting the dynamical field $\mathbf{g}(\mathbf{k})$. In a practical measurement, by resolving the time-averaged spin polarizations $\langle \mathbf{r} \rangle$ on two closed surfaces slightly inside (with $h_0 = -0.1t_0$) and outside (with $h_0 = 0.1t_0$) the BIS, one can qualitatively determine the vector field $\mathbf{g}(\mathbf{k})$ (without affecting measurement of topology) from the subtraction of $\langle \mathbf{r} \rangle$ between the two surfaces. A few examples for $\langle \mathbf{r} \rangle$ are given in Fig. 3b–d, similar for $\langle \mathbf{r} \rangle$ from the dynamical field $\mathbf{g}(\mathbf{k})$ on the BIS, as illustrated with arrows in Fig. 3a, which gives a nonzero Chern number $C_1 = -1$ on the 2D BIS, corresponding to the 3D winding number $\nu_3 = -1$ of the post-quench 3D system.

While the above study is based on unitary quench dynamics, we expect that quantum dynamics should be nearly unitary at finite temperature. In the presence of dissipation, we expect that quantum dynamics should be nearly unitary within a short-time evolution after quench, while the dissipation effects shall dominate in the long-term evolution. Taking the 2D model as an example, we describe the dynamics by the Lindblad master equation [49,24]

$$\frac{d}{dt} \rho_k = -i[H, \rho_k] + \eta \left( \mathbf{\sigma} \cdot \mathbf{p}_k \mathbf{\sigma} \cdot \mathbf{p}_k \right).$$

with initial state being $\rho_k(0) = f(E_{-T})^{+} \mathbf{k}^{+} f(E_{+T})^{-} \mathbf{k}^{-}$. Here the distribution function $f(E_{\pm T}) = \left[ e^{(E_{\pm T})/\hbar} \right]^{\pm 1}$ if the system is simulated with fermions (+) and bosons (−), $\pm \mathbf{k}$ are the eigenstates of the pre-quench Hamiltonian, the Pauli matrices $\sigma_z \equiv \left( \sigma_x \pm i \sigma_y \right)/2$ are defined in the eigenbasis of the post-quench Hamiltonian $H$, and $\eta$ is the decay rate. A numerical study with typical parameters is shown in Fig. 4. For a short-term evolution, we find that the dissipative effect is negligible (nearly unitary), as demonstrated in Fig. 4af (gray area) and b. In this regime, the previous dynamical classification is well applicable to measure the locations of BISs and the bulk topology (Fig. 4b). Further, after a long-term evolution, the dissipation drives the occupation of Bloch states to approach equilibrium results. In this case we find that the spin-polarization $\langle \sigma_z(\mathbf{k}) \rangle$ gradually points oppositely to the $\mathbf{h}_{\text{so}}(\mathbf{k})$-direction on BISs of the post-quench system (Fig. 4a). Then the Chern number can be directly read out by the long-term spin texture $\langle \sigma_z(\mathbf{k}) \rangle$, without the further adoption of $\mathbf{g}(\mathbf{k})$ for the classification (Fig. 4a, c).

![Fig. 3. 3D chiral topological phases. (a) The BIS (green sphere) defined by $h_0(\mathbf{k}) = 0$ and the dynamical (pseudo) spin-texture field $\mathbf{g}(\mathbf{k})$ (red arrows) are shown for $m_z = 1.3t_0$ and $t_{\text{so}} = 0.2t_0$. (b) In the practical measurement the $\mathbf{g}(\mathbf{k})$ field can be determined through the variation of spin textures at two equal-energy surfaces close to the BIS, e.g. at $h_0 = -0.1t_0$. (c) Time-averaged spin textures $\langle \mathbf{r} \rangle$ in cross sections $k_y = 0$ (S1 in (c)) and $k_y = \frac{\pi}{2}$ (S2 in (c)). The quench is taken from $m_z = 8t_0$ to 1.3$t_0$ and $\langle \mathbf{r} \rangle$ are shown in the Electronic Supplementary Material.](image-url)

![Fig. 4. Classifying topology via dissipative dynamics. (a) Post-quench dissipative dynamics of spin polarizations $\langle \sigma_z(t) \rangle$ (red and blue lines) for points $P_{11}$ and $P_{12}$ (green line) for $P_0$ at finite temperature. Due to dissipation, the spin $\langle \sigma_z \rangle$ gradually decays to the opposite direction of the procession axis $\mathbf{h}(\mathbf{k})$ (see insets). The BIS and topological patterns can be determined by time-averaged spin textures over a short period (see b) and a long period (see c), respectively. (b) Time-averaged spin textures $\langle \sigma_z(\mathbf{k}) \rangle$ over a period of 50$t_0$, which resemble the results in Fig. 2a. The BIS is determined by $\langle \sigma_z \rangle \approx 0$ (black dashed line). (c) Time-averaged spin textures $\langle \sigma_z(\mathbf{k}) \rangle$ on the BIS over a long term of 500$t_0$ directly reflect the SO field and thus the topological number. Here we take the temperature $k_B T = 10t_0$ and the decay rate $\eta = 0.005$ with other parameters the same as in Fig. 2a.](image-url)
6. Discussion and outlook

The dynamical classification exhibits essential advantages in experimental investigation of topological quantum states. In particular, applying the classification theory proposed in this work to measuring 2D Chern insulators has been achieved in a recent follow-up experiment [37], which confirms that observing topological phases based on the current dynamical classification is of much higher precision, compared with that based on equilibrium classifications. The advantages are rooted in two essential aspects. First, the bulk-surface duality uncovered here maps the classification of bulk topology to lower-dimensional invariants on BISs, and simplifies the topological characterization. Secondly, the spin dynamics is resonant and nontrivial only on the BISs, so the dynamical bulk-surface correspondence can be easily resolved in the quench study. Finally, the dynamical scheme by nature is robust against the non-ideal conditions in both the state initialization and measurement. In particular, the system starts at deep trivial regime, with the initial phase being dependent on only a single parameter, Zeeman term, immune to non-ideal conditions. Quenching it to topological regime induces quantum dynamics which exhibits resonant oscillations on the BISs. The short-term unitary quench dynamics characterizing the topology of the post-quench system is not affected by detrimental effects like the thermal effects, leading to high-precision measurement of full topological phase diagrams, as confirmed in experiment [37]. In comparison, measuring the topology of static phases necessitates the systems to be carefully initialized, with the quantum states being properly occupied. The preparation is intrinsically sensitive to non-ideal conditions including thermal effects, especially unsatisfactory in the regime close to phase boundaries [34]. Our dynamical classification provides new approaches with high feasibility in exploring topological physics. Moreover, the dynamical classification does not require the initial trivial phase to be fully polarized along certain direction. In the generic case with a trivial initial phase which has trivial but momentum-dependent (pseudo) spin texture, one can rotate the initial (pseudo) spin of each Bloch state to the direction of $\hbar$ by a momentum-dependent unitary transformation. Accordingly, under the transformation the new (pseudo) spin bases become momentum-dependent but still topologically trivial. Then, if expressed in the new (pseudo) spin bases, all the results including the characterization of BISs by vanishing time-averaged spin polarization and dynamical bulk-surface correspondence take the same forms as in the case with a fully polarized initial phase. However, taking the fully polarized (deep trivial) phase as the initial state is most convenient for real experiments.

Finally, this work provides important insights into developing non-equilibrium classification for the complete classes of equilibrium topological quantum states, as conceptually different from the known various equilibrium classification theories. The dynamical classification theory established here covers different categories of the topological phases characterized by integer invariants, including the A, AI, BDI, C, CI, and D classes in the AZ ten-fold ways [38]. It is promising and of great interests to generalize the current classification theory to all ten-fold categories [38–40], crystalline topological states, including the insulators and superconductors protected by space groups [51,50,41,52], and further to the correlated topological states of fermions and bosons [20,21]. We note that for the correlated topological phases whose topology can be characterized by quasiparticles or mean-field picture, the concept of BISs can still be defined and the present dynamical classification theory should be applicable. In turn, the correlation physics can have non-trivial effects on the dynamical bulk-surface correspondence and deserve in-depth studies. We therefore expect that the present work can open a broad new direction to classify and detect topological quantum states by non-equilibrium quantum dynamics.

Conflict of interest

The authors declare that they have no conflict of interest.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.scib.2018.09.018.

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