ON THE VAN EST ANALOGY IN HOPF CYCLIC COHOMOLOGY

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INTRODUCTION

Hopf cyclic cohomology has emerged from the attempt to unravel the local formula [7] for the cyclic cohomological Chern character [5] of the hypoelliptic spectral triple [8] modeling the space of leaves of a foliation. A close inspection of that formula brought to light a Hopf algebra of ‘moving frames’ $\mathcal{H}(n, \mathbb{R})$ (abbreviated as $\mathcal{H}_n$), which appeared to assume in the transverse geometry of foliations a role similar to that of $\text{GL}(n, \mathbb{R})$ as structure group of the frame bundle of a manifold. Further examination revealed that the index formula in question was in fact the expression of a characteristic cocycle in the range of a certain cohomology specific to Hopf algebras. In the particular case of $\mathcal{H}_n$, this Hopf cyclic cohomology turned out to be isomorphic to the Gelfand-Fuchs cohomology of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$ which, together with its relative versions, is well-known to encode the geometric characteristic classes of foliations. The benefit of having the same classes recaptured in Hopf cyclic cohomology is that the latter affords a direct characteristic map to the cyclic cohomology of étale foliations groupoids. Ultimately, the main outcome of the aforementioned investigation was the realization that the Hopf cyclic cohomology $HP^\bullet(\mathcal{H}_n; \mathbb{C}_\delta)$ of $\mathcal{H}(n, \mathbb{R})$ together with its relative versions $HP^\bullet(\mathcal{H}_n, \text{O}_n; \mathbb{C}_\delta)$ and $HP^\bullet(\mathcal{H}_n, \text{GL}(n, \mathbb{R}); \mathbb{C}_\delta)$ provide the appropriate repository for all geometric characteristic classes of foliations in the framework of noncommutative geometry.

On the other hand in the classical theory it was known that the characteristic classes can be constructed not just in terms of the usual connection and curvature procedure, but also by simplicial cohomological methods involving $\text{GL}(n, \mathbb{R})$ as structure group (cf. [1, 3, 20, 28, 12]). A. Gorokhovsky figured out how to convert that kind of representation into the formalism of Hopf cyclic cohomology by extending it to differential graded (DG) Hopf algebras. He showed [17] that the suitably
truncated Hopf cyclic cohomology $HP^*((\Omega^* \text{GL}(n, \mathbb{R}))_m)$ of the DG Hopf algebra of differential forms on $\text{GL}(n, \mathbb{R})$ captures the characteristic classes of flat $\Gamma$-equivariant vector bundles of rank $n$, where $\Gamma$ is a discrete group of diffeomorphisms of the base manifold of dimension $m$. In particular his results yield an isomorphism

\[(0.1)\quad HP^*(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \cong HP^*((\Omega^* \text{GL}(n, \mathbb{R}))_n),\]

which bears a conspicuous resemblance with the van Est isomorphism for the continuous cohomology of $\text{GL}(n, \mathbb{R})$.

The aim of this note is to show that this analogy is not coincidental, and it in fact extends to other avatars of the van Est isomorphism, thus providing the connection with the absolute cohomology $HP^*(H_n; \mathbb{C}_\delta)$, which accounts for the characteristic classes of framed foliations, and with the relative cohomology $HP^*(\mathcal{H}_n, \text{GL}(n, \mathbb{R}); \mathbb{C}_\delta)$, which only stores the Chern classes. More specifically, we prove that in addition to (0.1) there are canonical isomorphisms

\[(0.2)\quad HP^*(\mathcal{H}_n, \text{GL}(n, \mathbb{R}); \mathbb{C}_\delta) \cong HP^*((\Omega^* \text{GL}_{\text{alg}}(n, \mathbb{R}))_n),\]

where $\Omega^* \text{GL}_{\text{alg}}(n, \mathbb{R})$ is the Hopf algebra of forms on $\text{GL}(n, \mathbb{R})$ viewed as an algebraic group, and

\[(0.3)\quad HP^*(\mathcal{H}_n; \mathbb{C}_\delta) \cong HP^*((\bar{\Omega}^* \text{GL}_{\text{germ}}(n, \mathbb{R}))_n),\]

where $\bar{\Omega}^* \text{GL}_{\text{germ}}(n, \mathbb{R})$ is the Hopf algebra of germs of forms on the group germ $\text{GL}_{\text{germ}}(n, \mathbb{R})$ defined by $\text{GL}(n, \mathbb{R})$.

The material is organized as follows. After recalling the basic definitions related to $\mathcal{H}_n$ and Hopf cyclic cohomology in §1 we outline Gorokhovsky’s computation of the cyclic cohomology of $\Omega^* \text{GL}(n, \mathbb{R})$ in §2.1, explain the ‘behind-the-scenes’ role of the van Est isomorphism in §2.2, and then prove the above mentioned results in §2.3 (Theorem 2.1) and §2.4 (Theorem 2.2). Finally, in §2.5 we illustrate the convenience of using $HP^*((\Omega^* \text{GL}(n, \mathbb{R}))_n)$ as a repository of characteristic classes by giving examples of representative cocycles.

1. Hopf algebra of moving frames and its cyclic cohomology

We recall in this section the definition of the Hopf algebra of moving frames in $\mathbb{R}^n$, its Hopf cyclic cohomology, and how the relationship with the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$. 

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1.1. **Hopf algebra \( \mathcal{H}_n \).** Consider the groupoid \( \Gamma_n = \text{Diff}_n \ltimes \mathbb{R}^n \), where \( \text{Diff}_n \) denotes the group of diffeomorphisms of \( \mathbb{R}^n \) equipped with the discrete topology. Let \( F \Gamma_n = \text{Diff}_n \ltimes F \mathbb{R}^n \), where \( F \mathbb{R}^n = \mathbb{R}^n \times \text{GL}(n, \mathbb{R}) \) denote the 'moving frames' groupoid, with \( \text{Diff}_n \) acting by prolongation:

\[
\varphi(x, y) := (\varphi(x), \varphi'(x)y), \quad \text{where } \varphi'(x) = (\partial_j \varphi^i(x)) \in \text{GL}(n, \mathbb{R}).
\]

The convolution algebra \( C_c^\infty(F \Gamma_n) \) is spanned by monomials \( f U_\varphi \) with \( f \in C_c^\infty(F \mathbb{R}^n) \) and \( \varphi \in \text{Diff}_n \), with the product given by the rule

\[
f_1 U_{\varphi_1} \cdot f_2 U_{\varphi_2} = f_1 (f_2 \circ \varphi_1^{-1}) U_{\varphi_1 \varphi_2}, \quad \text{where } U_\varphi(f) = f \circ \varphi^{-1}.
\]

The vertical vector fields \( Y^j_i \) and the horizontal vector fields \( X_k = \sum_\mu y^\mu_k \frac{\partial}{\partial y^\mu_j} \), are made to act on \( C_c^\infty(F \Gamma_n) \) as follows: if \( Z \) is one of these, then \( Z(f U_\varphi) := Z(f) U_\varphi \). Since the right action of \( \text{GL}(n, \mathbb{R}) \) on \( F \mathbb{R}^n \) commutes with the action of \( \text{Diff}_n \), the vertical operators so extended remain derivations:

\[
Y^j_i(a b) = Y^j_i(a) b + a Y^j_i(b), \quad a, b \in C_c^\infty(F \Gamma_n).
\]

By contrast, the horizontal vector fields are not \( \text{Diff}_n \)-invariant, and satisfy instead

\[
\varphi_*(X_k) \equiv U_\varphi X_k U_\varphi^{-1} = X_k - \gamma^j_{jk}(\varphi^{-1}) Y^j_i, \quad \text{where}
\]

\[
\gamma^j_{jk}(\varphi)(x, y) = (y^{-1} \cdot \varphi'(x)^{-1} \cdot \partial_\mu \varphi^i(x) \cdot y)^j_i y^\mu_k.
\]

Consequently the extended operators \( X_k \) are no longer derivations but satisfy the generalized Leibniz rule

\[
X_k(a b) = X_k(a) b + a X_k(b) + \delta^j_{jk}(a) Y^j_i(b), \quad a, b \in C_c^\infty(F \Gamma_n).
\]

Here \( \delta^j_{jk} \) are multiplication operators given by

\[
\delta^j_{jk}(f U_\varphi) = \gamma^j_{jk}(\varphi^{-1}) f U_\varphi = -\gamma^j_{jk}(\varphi) \circ \varphi^{-1} f U_\varphi
\]

and they satisfy the usual Leibniz rule

\[
\delta^j_{jk}(a b) = \delta^j_{jk}(a) b + a \delta^j_{jk}(b).
\]

The operators \( \text{Id}, \{X_k, Y^j_i, \delta^j_{jk}\} \) generate a unital subalgebra of the linear operators on \( C_c^\infty(F \Gamma_n) \), which defines \( \mathcal{H}_n \) as an algebra. Automatically, \( \mathcal{H}_n \) is also a Lie algebra with respect to the usual commutator, and the commutator relations between its generators give more insight into its nature. The vector fields satisfy the standard commutation relations of the affine group, and the iterated commutators of the operators \( \delta^j_{jk} \)'s with the \( X^i \)'s yield further operators

\[
\delta^j_{lk \ell_1 \ldots \ell_r} := [X_{\ell_r}, \ldots [X_{\ell_1}, \delta^j_{lk} \ldots]].
\]
which act by multiplication
\[ \delta^i_{jk\ell_1\cdots\ell_r} (f U_\varphi) := \gamma^i_{jk\ell_1\cdots\ell_r}(\varphi^{-1}) f U_\varphi, \]
where
\[ \gamma^i_{jk\ell_1\cdots\ell_r}(\varphi) := X_{\ell_r} \cdots X_{\ell_1}\gamma^i_{jk}(\varphi), \quad \varphi \in \mathcal{C}_n, \]
and therefore commute with each other. They are no longer derivations but do satisfy (generalized) Leibniz rules.

The Leibniz rules obeyed by the generators extend by multiplicativity to all \( h \in \mathcal{H}_n \). In turn, these rules uniquely determine a coproduct \( \Delta : \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_n \) by the stipulation expressed in the Sweedler notation as follows:
\[ (1.1) \quad \Delta(h) = \sum h(1) \otimes h(2) \quad \text{iff} \quad h(ab) = \sum h(1)(a)h(2)(b). \]
The counit \( \epsilon : \mathcal{H}_n \to \mathbb{C}, \epsilon(h) := h(1) \) and the anti-automorphism \( S : \mathcal{H}_n \to \mathcal{H}_n \) defined on generators by
\[ S(Y^j_i) = -Y^j_i, \quad S(X_k) = -X_k + \delta^i_{jk} Y^j_i, \quad S(\delta^i_{jk}) = -\delta^i_{jk}, \]
complete the properties which make \( \mathcal{H}_n \) a Hopf algebra.

The antipode \( S \) is not involutive but it can be twisted into an involution by means of the character \( \delta : \mathcal{H}_n \to \mathbb{C} \) which trivially extends the trace map on \( \mathfrak{gl}_n(\mathbb{C}) \), i.e. defined by
\[ \delta(Y^j_i) = \delta^i_j, \quad \delta(X_k) = 0, \quad \text{and} \quad \delta(\delta^i_{jk}) = 0. \]
Then \( S_\delta(h) := \sum \delta(h(1))S(h(2)), h \in \mathcal{H}_n \) satisfies \( S_\delta^2 = \text{Id} \).

1.2. Hopf cyclic cohomology. By its very construction, the Hopf algebra \( \mathcal{H}_n \) acts on the algebra \( C_c^{\infty}(F\Gamma_n) \), and \( \square \) ensures that \( C_c^{\infty}(F\Gamma_n) \) is actually a left \( \mathcal{H}_n \)-module algebra. \( C_c^{\infty}(F\Gamma_n) \) has a canonical trace, namely
\[ (1.2) \quad \tau(f U_\varphi) = \int_{F\Gamma_n} f \varpi, \quad \text{if} \ \varphi = \text{Id}, \quad \tau(f U_\varphi) = 0, \quad \text{if} \ \varphi \neq \text{Id}, \]
where \( \varpi \) is the volume form dual to \( X_1 \wedge \cdots \wedge X_n \wedge Y^1 \wedge \cdots \wedge Y^n \) (lexicographically ordered); \( \tau \) is tracial since \( \varpi \) is \( \text{Diff}_n \)-invariant, and in addition it is \( \delta \)-invariant with respect to the action of \( \mathcal{H}_n \), i.e. satisfies
\[ \tau(h(a)) = \delta(h) \tau(a), \quad \forall \ h \in \mathcal{H}, \ a \in C_c^{\infty}(F\Gamma_n). \]

The cyclic cohomology of the Hopf algebra \( \mathcal{H}_n \) was introduced in \( \square \) by importing the standard cyclic structure of the algebra \( C_c^{\infty}(F\Gamma_n) \), via the morphism \( \chi_q = \{ \chi_q : \mathcal{H}_n^\otimes q \to CC^q(C_c^{\infty}(F\Gamma_n)) \}_{q \geq 0} \), where
\[ (1.3) \quad \chi_q(h^1 \otimes \cdots \otimes h^q)(a_0, a_1, \ldots, a_q) := \tau(a_0 h^1(a_1) \cdots h^q(a_q)). \]
The stipulation that $\chi_*$ is a morphism of $\Delta C$-objects, where $\Delta C$ is
Connes’ cyclic category, endows $\mathcal{H}^n := \sum_{q \geq 0} \mathcal{H}_{n}^{\otimes q}$ with the basic $\Delta C$-morphisms
\begin{align}
\partial_0 (h^1 \otimes \ldots \otimes h^{q-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{q-1}, \\
\partial_j (h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{q-1} \\
\partial_q (h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes h^{q-1} \otimes 1 \\
\sigma_i (h^1 \otimes \ldots \otimes h^{q+1}) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{q+1} \\
\tau_q (h^1 \otimes \ldots \otimes h^q) &= S_\delta(h^1) \cdot (h^2 \otimes \ldots \otimes h^q \otimes 1).
\end{align}
(1.4)

The involutive property $S_\delta^2 = \text{Id}$ evidently implies $\tau_1^2 = \text{Id}$, and in fact ensures that $\tau_{q+1} = \text{Id}$ for any $q \in \mathbb{N}$ (cf. [3]). The corresponding cyclic cohomology groups $HC^*(\mathcal{H}; \mathbb{C}_\delta)$ are computed from the associated bi-complex $CC^*\bigl(\mathcal{H}_n; \mathbb{C}_\delta\bigr)$ with boundary operators
\begin{align}
b = \sum_{i=0}^{q+1} (-1)^i \partial_i, \quad B = \left( \sum_{i=0}^{q} (-1)^{(q-1)i} \tau_i \right) \sigma_{q-1} (1 - (-1)^q \tau_q),
\end{align}
(1.5)
and the periodic Hopf cyclic cohomology $HP^\ast(\mathcal{H}_n; \mathbb{C}_\delta)$ is the $\mathbb{Z}_2$-graded cohomology of the corresponding total complex.

More generally (cf. [3] Theorem 1), to any Hopf algebra $\mathcal{H}$ endowed with a modular pair in involution $(\sigma, \delta)$, i.e. $\sigma$ is a group element, $\delta$ a character and $\delta(\sigma) = 1$, one associates a $(b, B)$ bicomplex in a similar way, after modifying the operators in (1.4) as follows:
\begin{align}
\partial_q (h^1 \otimes \ldots \otimes h^{q-1}) &= h^1 \otimes \ldots \otimes h^{q-1} \otimes \sigma \\
\tau_q (h^1 \otimes \ldots \otimes h^q) &= S_\delta(h^1) \cdot (h^2 \otimes \ldots \otimes h^q \otimes \sigma).
\end{align}
(1.6)

The definition of Hopf cyclic cohomology was subsequently extended in [18] to a large class of coefficients.

For applications to characteristic classes of foliations, the relative versions of the Hopf cyclic cohomology of $\mathcal{H}_n$ of particular interest are those relative to $O_n$, resp. to $SO_n$ in the orientable case, and to a lesser extent that relative to $GL(n, \mathbb{R})$.

To simplify the notation, in the remainder of this section we abbreviate $GL(n, \mathbb{R})$ by $GL_n$ and $\mathfrak{gl}_n(n, \mathbb{R})$ by $\mathfrak{g}_n$. Letting $H$ denote one of closed subgroups of $GL_n$ enumerated above, we recall that the cohomology of $\mathcal{H}_n$ relative to $H$ is defined as follows (cf. [10]).

First note that the adjoint action of $GL_n$ on the Lie algebra $\mathfrak{gl}_n \ltimes \mathbb{R}^n$ of the affine group $GL_n \ltimes \mathbb{R}^n$, with which we identify the frame bundle $F\mathbb{R}^n$, extends to a linear action of $GL_n$ on $\mathcal{H}_n$. With $\mathfrak{h}$ denoting the Lie algebra of $H$, its universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ is a sub-Hopf algebra of $\mathcal{H}_n$. The quotient $\mathcal{H}_n \otimes \mathcal{U}(\mathfrak{h}) \mathbb{C} \equiv \mathcal{H}_n/\mathcal{H}_n\mathcal{U}^+(\mathfrak{h})$ (where $\mathcal{U}^+(\mathfrak{h})$
denotes the ideal of \( \mathcal{U}(\mathfrak{h}) \) generated by \( \mathfrak{h} \) is an \( \mathcal{H}_n \)-module coalgebra with respect to the coproduct and counit induced from \( \mathcal{H}_n \). We denote by \( \mathcal{Q}_H \) its subspace of invariants under the action of \( H \) on \( \mathcal{H}_n \). The cyclic object defining the relative cohomology consists of \( \sum_{q \geq 0} \mathcal{Q}_H^{\otimes q} \) endowed with the basic \( \Delta C \)-morphisms defined by
\[
\begin{align*}
\partial_0(c^1 \otimes \ldots \otimes c^{q-1}) &= \mathbf{1} \otimes c^1 \otimes \ldots \otimes c^{q-1}, \\
\partial_i(c^1 \otimes \ldots \otimes c^{q-1}) &= c^1 \otimes \ldots \otimes \Delta c^i \otimes \ldots \otimes c^{q-1}, \\
\sigma_i(c^1 \otimes \ldots \otimes c^{q+1}) &= c^1 \otimes \ldots \otimes \epsilon(c^{i+1}) \otimes \ldots \otimes c^{q+1}, \\
\tau_q(h^1 \otimes c^2 \otimes \ldots \otimes c^q) &= S_\delta(h^1) \cdot (c^2 \otimes \ldots \otimes c^q) \mathbf{1}.
\end{align*}
\]

The corresponding bicomplex is denoted \( CC^*(\mathcal{H}_n, H; \mathbb{C}_d) \) and the \( \mathbb{Z}_2 \)-graded cohomology group of the total complex is \( HP^*(\mathcal{H}_n, H; \mathbb{C}_d) \).

The above cohomology groups have been computed in [7] (and by alternative means in [25, 26]). They are isomorphic to the Gelfand-Fuchs cohomology of the Lie algebra \( a \) by \( \mathbb{Q} \) with respect to the coproduct and counit induced from \( H \).

The \( \mathcal{O} \) stands for the DG subalgebra of \( \mathcal{H}_n \) on compacta of functions and their derivatives, is a Hopf algebra with the coproduct \( \Delta : \mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{O} \) defined by \( \Delta(f)(x, y) = f(xy) \), the counit given by evaluation at the unit \( \mathbf{1} \in G \) and the antipode \( S \) induced by inversion. As topological vector spaces

2. **Hopf algebras of forms and their cyclic cohomology**

2.1. **DG Hopf algebra of a Lie group.** Let \( G \) be an almost connected Lie group. Its algebra of differential forms \( \Omega^*(G) \) has an natural structure of a topological (Fréchet) DG Hopf algebra, extending that of the function algebra \( C^\infty(G) = \Omega^0(G) \). Indeed \( C^\infty(G) \), equipped with its standard topological vector space structure of uniform convergence on compacta of functions and their derivatives, is a Hopf algebra with the coproduct \( \Delta : C^\infty(G) \rightarrow C^\infty(G) \otimes C^\infty(G) = C^\infty(G \times G) \) defined by \( \Delta(f)(x, y) = f(xy) \), the counit given by evaluation at the unit \( \mathbf{1} \in G \) and the antipode \( S \) induced by inversion. As topological vector spaces
\( \Omega^*(G) \cong C^\infty(G) \otimes \bigwedge g^* \), where \( g \) is the Lie algebra of \( G \), and the Hopf structure of \( C^\infty(G) = \Omega^0(G) \) extends naturally to \( \Omega^*(G) \). Together with the de Rham differential, the couple \((\Omega^*(G), d)\) is a DG (Fréchet) Hopf algebra.

In [17, §3] Gorokhovsky defines a cyclic object associated to an arbitrary DG Hopf algebra \((H^*, d)\) endowed with a modular pair, by a natural graded extension of the cyclic object associated to \( H^0 \). In particular this applies to \((\Omega^*(G), d)\) which, having involutive antipode, carries a trivial modular pair. The cyclic object consists of

\[
(\Omega^*G)^\otimes := \sum_{q \geq 0} \Omega^*(G)^{\otimes q} \equiv \sum_{q \geq 0} \Omega^*(G^{\otimes q})
\]
equipped with the \( \Delta C \)-morphisms similar to those in [14], with the difference that the summands (implicitly present) in the expression of the cyclic operator \( \tau_q \) acquire appropriate signs (cf. [17, §3, (3.3)]). The corresponding \((b, B)\)-bicomplex \( C^{*,*}(\Omega^*(G), d) \) is then upgraded to a tricomplex \( C^{*,*,*}(\Omega^*(G), d) \) by the addition of the differential

\[
d(\alpha_1 \otimes \cdots \otimes \alpha_q) = \sum_{i=1}^q (-1)^{\deg \alpha_1 + \cdots + \deg \alpha_i} \alpha_1 \otimes \cdots \otimes d\alpha_i \cdots \otimes \alpha_q,
\]
The latter is filtered by the subcomplexes

\[
F^m C^{*,*}(\Omega^*(G), d) = \{ \sum \alpha_1 \otimes \cdots \otimes \alpha_q \mid \deg \alpha_1 + \cdots + \deg \alpha_q > m \},
\]
and gives rise, for each \( m \in \mathbb{Z}^+ \), to a truncated complex

\[
C^{*,*,*}(\Omega^*(G)_m, d) = C^{*,*,*}(\Omega^*(G), d) / F^m C^{*,*}(\Omega^*(G), d);
\]
Its Hochschild, cyclic and periodic cyclic cohomology, formed using only finite cochains, will be respectively denoted by \( HH^*( (\Omega^*G)_m) \), \( HC^* ( (\Omega^*G)_m) \) and \( HP^* ( (\Omega^*G)_m) \). Gorokhovsky computed them in terms of the cohomology of truncated Weil algebras as follows.

**Theorem 2.1** ([17], §6). With \( K \) denoting the maximal subgroup of \( G \), there are canonical isomorphisms

\[
(2.1) \quad HH^* ( (\Omega^*G)_m) \cong H^* (W(g, K)_m), \quad \forall \ m \geq 0,
\]
\[
(2.2) \quad HC^q ( (\Omega^*G)_m) \cong \bigoplus_{i \geq 0} H^{q-2i} (W(g, K)_m).
\]

For the clarity of the ensuing discussion we outline Gorokhovsky's proof. The crucial observation is that the Hochschild bicomplex \( C^{*,*,*}(\Omega^*(G), b, d) \) coincides with the Bott simplicial de Rham bicomplex \( \{ \Omega^*(NG), \delta, d \} \) of the nerve \( NG \) of \( G \). Recall that \( NG := \bigsqcup N_p G \), where \( N_p G = G^p \), and \( \Omega^{p,q}(NG) = \Omega^q(G^p) \).
As was shown in \cite{3} for a general simplicial manifold \( X = \{ X_p \} \), the cohomology of \( \{ \Omega^*(X), \delta, d \} \) is isomorphic to the singular cohomology of the geometric realization of \( X \). On the other hand, Dupont \cite{12}) introduced a related de Rham complex, \( \{ \Omega^*(||X||), d \} \), where \( ||X|| \) is the “thick” geometric realization of \( X \), which has the advantage of being graded commutative. A \( q \)-form \( \omega \in \Omega^q(||X||) \) is a collection \( \{ \omega_p \} \) of de Rham \( q \)-forms on \( \bigsqcup_p X_p \), satisfying the compatibility condition

\[
(\varepsilon^i \times \text{Id})^*\omega_p = (\text{Id} \times \varepsilon_i)^*\omega_{p-1},
\]

where \( \varepsilon_i : X_p \to X_{p-1} \) are the face operators corresponding to the inclusions of the faces \( \varepsilon^i : \Delta^{p-1} \to \Delta^p, i = 0, 1, \ldots, p \). Dupont has shown that integration over simplices \( \int_{\Delta^p} \alpha | \Delta^p \times X_p \)

defines a quasi-isomorphism of \( \{ \Omega^*(||X||), d \} \) with with the total complex \( \{ \Omega^*(X), \delta \pm d \} \).

We next recall that the geometric realization \( EG \) of the simplicial manifold \( \bar{N}G \), with \( \bar{N}_p G = G^{p+1} \) on which \( G \) acts diagonally, gives the total space of the universal (right) \( G \)-bundle \( \pi : EG \to BG \). This bundle has a canonical connection \( \theta \) induced by the Maurer-Cartan form on \( G \); for a linear group \( G \) the expression of \( \theta | \Delta^p \times \bar{N}_p G \)

\[
\theta(t_0, \ldots, t_p; g_0, \ldots, g_p) = t_0 g_0^{-1} dg_0 + \cdots + t_p g_p^{-1} dg_p,
\]

and the curvature form is \( \Omega = d\theta + \theta \wedge \theta \).

The Dupont complex being a \( \mathfrak{g} \)-DG algebra, by the universal property of the Weil algebra (see \cite{31}) this connection determines a morphism \( w : W(\mathfrak{g}) \to \Omega^*(EG) \). In turn, the latter descends to \( K \)-invariants yielding a morphism \( w_K : W(\mathfrak{g}, K) \to \Omega^*(EG/K) \). Using the contractibility of \( G/K \) and the canonical construction of geodesic simplices (cf. \cite{12} §5) for metrics of nonpositive curvature, one defines a simplicial cross-section \( s : BG \to EG/K \) (cf. \cite{17} Eqs. (6.10),(6.11)), which moreover is equivariant with respect to the cyclic group action (cf. \cite{17} (6.8)). The composition

\[
\mu = \mathcal{I}_\Delta \circ s^* \circ w_K : W(\mathfrak{g}, K) \to \Omega^*(NG)
\]

preserves the canonical filtrations and descends to the truncated complexes of any level \( m \geq 0 \), inducing in cohomology a morphism

\[
\mu^*_HH : H^*(W(\mathfrak{g}, K)_m) \to H^q((\Omega^*(NG))_m) = HH^q((\Omega^*(G))_m).
\]

In view of results proved in \cite{20} and \cite{28}, it follows that \( \mu^*_HH \) is actually an isomorphism. Finally, since the connection (2.3) and the cross-section \( s \) are invariant under the natural cyclic action, the Hochschild
cocycles in the image of $\mu$ are actually cyclic. This implies that Connes’ periodicity exact sequence splits into short exact sequences

$$0 \to HC^{q-2}((\Omega^* G)_m) \xrightarrow{S} HC^q((\Omega^* G)_m) \xrightarrow{I} HH^q((\Omega^* G)_m) \to 0,$$

which in turn implies the isomorphism (2.2) for cyclic cohomology.

### 2.2. Role of van Est isomorphism

We now revisit the proof of the isomorphism (2.5) and give an alternative argument emphasizing the role of the van Est isomorphism, which will serve as a template for dealing with the counterparts of the isomorphism (2.2) corresponding to $H = \{1\}$ in (1.8) and $H = GL(n, \mathbb{R})$ in (1.9).

A key element of the approach we are about to describe is the linkage made by Bott [1] with the continuous group cohomology. Relying on the Dold-Puppe homology theory for nonadditive functors, Bott proved a basic Decomposition Lemma [1, §2] for the simplicial de Rham complex $\Omega^*(NG)$ and use it show that for any Lie group $G$ the spectral sequence of the bicomplex $\{\Omega^*(NG), d, \delta\}$ filtered by the degree of the forms converges to the cohomology of $BG$. In particular (cf. [1, Theorem 1]) he expressed the $E_1$-term as continuous group cohomology with coefficients:

$$(2.6) \quad E_1^{pq} = H^p_\delta(\Omega^q(NG)) \simeq H^p_{cont}(G; S^q(g^*)) .$$

By the van Est isomorphism [14],

$$(2.7) \quad H^*_{cont}(G; S^*(g^*)) \simeq H^*(g, K; \mathbb{R}) \otimes S^*(g^*)^G ,$$

where $K$ is the maximal compact subgroup. In the case when $G$ itself is compact one has

$$H^0_{cont}(G; S^*(g^*)) \simeq S^*(g^*)^G \quad \text{and} \quad H^k_{cont}(G; S^*(g^*)) , \quad \forall k > 0,$$

and so (2.6) and (2.7) imply that there is no cohomology above the diagonal. This allows Bott to conclude (cf. [1, Remark (a)]) that the edge homomorphism to the cohomology of the total complex

$$S^*(g^*) \to H^*(\Omega^*(NG)) ,$$

which under the identification $H^*(\Omega^*(NG)) \simeq H^*(BG)$ coincides with the Chern-Weil homomorphism, is an isomorphism.

For the remainder of this section, we specialize to the group $GL(n, \mathbb{R})$ in order to remain consistent with the context of §1. To keep the notation convenient though we will simply denote it by $G$ (except for the rare occasion when $G$ is allowed to be an arbitrary Lie group, in which case this will be explicitly stated).
Returning to the Bott spectral sequence, where now for $G = \text{GL}(n, \mathbb{R})$ and $K = O_n$, it is well known that
\begin{align}
H^*(\mathfrak{g}, K; \mathbb{R}) &\cong E(h_1, h_3, \ldots, h_{2\lfloor \frac{n}{2}\rfloor+1}), \\
\text{and} \quad S^*(\mathfrak{g}^*)^G &\equiv P[c_1, c_2, \ldots, c_n];
\end{align}
here $E$ stands for the exterior algebra in generators $h_i$ of degree $\deg h_i = 2i - 1$, and $P[c_1, c_2, \ldots, c_n]$ denotes the polynomial algebra in generators $c_i$ of degree $\deg c_i = 2i$. Since the sequence converges to $H^*(BG) = P[c_2, c_4, \ldots, c_{2\lfloor \frac{n}{2}\rfloor}]$, it can be seen, successively, that $d_{2i-1}$ sends $h_{2i-1}$ to a non-zero multiple of $c_{2i-1}$, and that $c_{2i}$'s survive; also, $d_r = 0$ for $r > 2\frac{n+1}{2}$.

A similar pattern occurs for the spectral sequence of the truncated complex $(\Omega^*(NG))_n$, the main difference being that the polynomial algebra $P[c_1, c_2, \ldots, c_n]$ is replaced by its truncation modulo the ideal of elements of degree $> 2n$, denoted $S^*(\mathfrak{g}^*)^{GL_n} \equiv P_n[c_1, c_2, \ldots, c_n]$. In particular (2.6) and (2.7) become
\begin{equation}
H^p_\delta(\Omega^*(NG)_n) \cong H^p_{\text{cont}}(G; S^q(\mathfrak{g}^*)_n),
\end{equation}
respectively
\begin{equation}
H^*_{\text{cont}}(G; S^*(\mathfrak{g}^*)_n) \cong H^*(\mathfrak{g}, K; \mathbb{R}) \otimes S^*(\mathfrak{g}^*)^G_n.
\end{equation}
Thus, in view of (2.8), the corresponding $E_1$-term is
\begin{equation}
E_1 \cong WO_n := E(u_1, u_3, \ldots, u_{2\lfloor \frac{n}{2}\rfloor+1}) \otimes P_n[c_1, c_2, \ldots, c_n].
\end{equation}

There is a natural inclusion of the DG algebra $WO_n$ into the quotient $W(\mathfrak{g}, K)_n$ of $W(\mathfrak{g}, K)$ by the ideal of elements of degree $> 2n$, and the map $WO_n \hookrightarrow W(\mathfrak{g}, K)_n$ is a quasi-isomorphism. Indeed, this follows from the comparison theorem for the spectral sequences associated to the quotient of the standard filtration (by the ideal generated by the elements of degree $> 2n$), as the $E_1$ term of both spectral sequences is the same as in (2.11). Although the spectral sequences of $W(\mathfrak{g}, K)_n$ and $\Omega^*(NG)_n$ cannot be directly compared, by chasing their differentials one can still infer that $H^*(W(\mathfrak{g}, K)_n)$ and $H^*(\Omega^*(NG)_n)$ are formally isomorphic.

The elegant argument though, due to Shulman and Stasheff [28], involves the semi-simplicial Weil algebra of Kamber and Tondeur [20]. Relying on a semi-simplicial generalization of the van Est isomorphism, Shulman and Stasheff arrive to the isomorphism $H^*(W(\mathfrak{g}, K)_n) \cong H^*(\Omega^*(NG)_n)$ not by ad hoc calculations but by exploiting the standard comparison theorem for spectral sequences. To explain their line of argument, we recall that the semi-simplicial Weil algebra $W_1(\mathfrak{g})$ (see [20] [21]) is a $\mathfrak{g}$-DG algebra with underlying vector space $\bigoplus_{p\geq 0} W(\mathfrak{g}^{p+1})$.
and faces induced by the projections \( pr_k : \mathfrak{g}^{p+1} \to \mathfrak{g}^p \), \( 0 \leq k \leq p \), that omit the \((k+1)\)th factor. The projection \( \pi : W_1(\mathfrak{g}) \to W(\mathfrak{g}) \) on the first summand \((p = 0)\) is a map of \( \mathfrak{g}\text{-DG} \) algebras, and is shown to induce a chain equivalence. This is but a special case of a core result of Kamber and Tondeur (\cite[Theorem 8.12]{20}, also \cite[§6]{21}), which for \( W_1(\mathfrak{g}) \) can be stated as follows.

**Theorem 2.2** (\cite[Ch.8]{20}). Let \( G \) be an almost connected Lie group and \( H \) a closed subgroup. There is a suitable filtration \( \mathcal{F}_1 \) of \( W_1(\mathfrak{g}) \) such that the canonical projection at the level of truncated algebras

\[
\pi_m : W_1(\mathfrak{g}, H)_m := W_1(\mathfrak{g}, H)/\mathcal{F}_1^{m+1}W_1(\mathfrak{g}, H) \to W(\mathfrak{g}, H)_m
\]

induces an isomorphism of the associated spectral sequence and hence on cohomology:

\[
\pi^*_m : H^*(W_1(\mathfrak{g}, H)_m) \xrightarrow{\sim} H^*(W(\mathfrak{g}, H)_m).
\]

On the other hand, by the universal property of Weil algebras, the canonical connection on the principal \( G \)-bundle \( NG \to NG \) gives rise to a map of simplicial \( \mathfrak{g}\text{-DG} \) algebras

\[
\varphi : W_1(\mathfrak{g}) = \bigoplus_{p \geq 0} W(\mathfrak{g}^{p+1}) \to \bigoplus_{p \geq 0} \Omega^*(G^{p+1}) = \Omega^*(\bar{N}G).
\]

Shulman and Stasheff \cite{28} define a compatible filtration \( \mathcal{F} \) on \( \Omega^*(\bar{N}G) \) which renders \( \varphi \) filtration-preserving, as follows. First they endow \( \Omega^*(NG) \) with the standard filtration \( F \) by the degree of forms. Then they define \( \mathcal{F} \) as the filtration of \( \Omega^*(\bar{N}G) \) induced by \( \psi : \Omega^*(NG) \to \bar{N}G \), where \( \psi = \{ \psi_p \} \) with \( \psi_p \) corresponding to the projection \( G^{p+1} \to G^p \) on the first \( p \) coordinates. In the homogeneous picture \( \Omega^*(NG) \) is identified with the subcomplex \( \Omega^*(\bar{N}G)_G \) of \( G \)-basic forms on \( \bar{N}G \). Note that \( \mathcal{F} \) also restricts to a filtration on the \( K \)-basic forms \( \Omega^*(\bar{N}G)_K \).

Both \( \varphi \) and \( \psi \) descend to maps \( \varphi_m \), resp. \( \psi_m \), at the level of the quotient algebras by the \((m+1)\)th power of the corresponding ideal, which are shown to give rise to equivalences of spectral sequences.

**Theorem 2.3** (\cite[pp. 68-70]{28}). Each of the maps

\[
\varphi_m : W_1(\mathfrak{g}, K)/\mathcal{F}_1^{m+1} \to \Omega^*(\bar{N}G)_K/\mathcal{F}^{m+1}, \text{ resp.}
\]

\[
\psi_m : \Omega^*(NG)_m \equiv \Omega^*(\bar{N}G)_G/F^{m+1} \to \Omega^*(\bar{N}G)_K/\mathcal{F}^{m+1}
\]

is filtration preserving and induces an isomorphism of spectral sequences.

One thus obtains in homology the isomorphisms

\[
\varphi^*_m : H^*(W_1(\mathfrak{g}, K)_m) \xrightarrow{\sim} H^*(\Omega^*(\bar{N}/K)_m), \quad \forall \ m \geq -1
\]

\[
\psi^*_m : H^*(\Omega^*(NG)_m) \xrightarrow{\sim} H^*(\Omega^*(\bar{N}/K)_m)
\]
which combined with (2.12) give rise to a canonical isomorphism

\[ H^* ((\Omega^* (NG)_m) \cong H^* (W(\mathfrak{g}, K)_m). \]

(2.18)

The above isomorphism coincides with that of (2.5), as both are constructed via the universal property of the Weil algebra in Chern-Weil theory. The difference is that the cross-section in (2.4) is replaced by the equivalence (2.15) of spectral sequences, and the passage through the Dupont complex is altogether bypassed.

In order to see that Theorem 2.3 represents a semi-simplicial generalization of the van Est isomorphism \( H^* (\mathfrak{g}, K; R) \cong H^* \text{cont} (G; R) \), note at the level of 0-th filtration, \( W_1 (\mathfrak{g}, K) / F_1 = \bigoplus_{p \geq 0} (\wedge^p \mathfrak{g}^{p+1})_{K \text{-basic}} \) while \( \Omega^0 (NG) = \bigoplus_{p \geq 0} C^\infty (G^p) \) is the complex of differentiable group cohomology, and \( \varphi_0 \) and \( \psi_0 \) are the very same maps used by van Est in his proof [14].

2.3. Hopf algebra of algebraic forms. Regarding \( G = \text{GL}(n, \mathbb{R}) \) as a real algebraic group, we denote by \( \Omega^*_{\text{alg}} (G) \) its graded Hopf algebra of algebraic differential forms. As vector spaces \( \Omega^*_{\text{alg}} (G) \cong C_{\text{alg}} (G) \otimes \bigwedge \mathfrak{g}^* \), where \( C_{\text{alg}} (G) = \mathbb{R} [g_{ij}; \det g^{-1}] \) is the ring of regular rational functions on \( G \), which is the classical Hopf algebra of a linear algebraic group. The associated Hochschild complex of the DG Hopf algebra \( \Omega^*_{\text{alg}} (G) \) coincides with the Bott simplicial de Rham \( (\Omega^*_{\text{alg}} (NG), \delta, d) \), where \( \Omega^p (NG) := \Omega^p (G^p) \).

**Theorem 2.1.** There are canonical isomorphisms

\[ HH^q \left( (\Omega^*_{\text{alg}} G)_n \right) \cong H^* (W(\mathfrak{g}, G)_n), \]

(2.19)

\[ HC^q \left( (\Omega^*_{\text{alg}} G)_n \right) \cong \bigoplus_{i \geq 0} H^{q-2i} (W(\mathfrak{g}, G)_n). \]

(2.20)

**Proof.** The short proof is parallel to Bott’s argument in the case of a compact Lie group (cf. [11] Remark (a)]. Indeed, by the analogue of Bott’s Decomposition Lemma, the \( E_1 \)-term of the spectral sequence for the column filtration of the truncated complex \( (\Omega^*_{\text{alg}} (NG)_n, \delta, d) \) is seen to be

\[ E_1^{p,q} = H^p_{\delta} \left( \Omega^q_{\text{alg}} (G^p)_n \right) \cong H^p_{\text{alg}} (G; S^q (\mathfrak{g}^*)_n). \]

(2.21)

The algebraic counterpart of the van Est isomorphism, which is a consequence of Hochschild’s isomorphism [19] Theorem 5.2, then implies:

\[ H^0_{\text{alg}} (G; S^q (\mathfrak{g}^*)_n) \cong S^q (\mathfrak{g}^*)_n^G, \quad \text{and} \]

\[ H^i_{\text{alg}} (G; S^q (\mathfrak{g}^*)_n) = 0, \quad \text{if} \quad i > 0. \]

(2.22)
From (2.21) and (2.22) it follows that
\[
H^*(\Omega^*_{\text{alg}}(NG)_n) \cong S^*(g^*)_n = P_n[c_1, c_2, \ldots, c_n].
\]
The left hand side is the same as \(HH^*((\Omega^*_{\text{alg}} G)_n)\) while the right hand is the same as \(H^*(W(g, G)_n)\).

The Shulman-Stasheff argument in §2.2 can also be adapted. The algebraic counterpart of \(\psi_m\) in Theorem 2.3 is \(\psi^\text{alg}_m = \text{Id}\), and the proof that
\[
\varphi^\text{alg}_m : W_1(g, G)/F_{m+1} \rightarrow \Omega^*_{\text{alg}}(\bar{NG})_G/F_{m+1} \equiv \Omega^*_{\text{alg}}(NG)_m
\]
induces an isomorphism of spectral sequences follows along the same lines as in [28], with the difference that the semi-simplicial generalization of the van Est bicomplex [13, §10] is replaced by its algebraic counterpart, which can be handled as in the proof of [22, §2] for the algebraic version of the van Est isomorphism.

We finally recall that (2.20) automatically follows from (2.19). □

2.4. Hopf algebra of germs of forms. We now consider a Hopf algebra of forms on the group germ determined by \(G\), which is defined as follows. First, a differential form \(\alpha \in \Omega^*(G)\) will be called \(\text{locally trivial}\) if it vanishes identically in a neighborhood of \(1 := \text{Id}\) in \(G\). We denote the set of such forms by \(\Omega^*_{\text{loc}}(G)\). More generally, for any \(p \in \mathbb{N}\), we define in a similar way the set \(\Omega^*_{\text{loc}}(G^p)\) of locally trivial forms for the group \(G^p\). Obviously, \(\Omega^*_{\text{loc}}(G^p)\) is an ideal in \(\Omega^*(G^p)\), and we form the quotient algebra
\[
\bar{\Omega}^*(G^p) := \Omega^*(G^p)/\Omega^*_{\text{loc}}(G^p), \quad \forall p \in \mathbb{N}.
\]
We then define the tensor product \(\bar{\otimes}\) of two such algebras by stipulating
\[
\bar{\Omega}^*(G^p) \bar{\otimes} \bar{\Omega}^*(G^q) := \bar{\Omega}^*(G^{p+q}),
\]
and equip \(\bar{\Omega}^*(G)\) with the coproduct
\[
\bar{\Delta} : \bar{\Omega}^*(G) \rightarrow \bar{\Omega}^*(G) \bar{\otimes} \bar{\Omega}^*(G)
\]
induced by \(\Delta : \Omega^*(G) \rightarrow \Omega^*(G) \otimes \Omega^*(G)\). The latter is well-defined since if \(\alpha \in \Omega^*_{\text{loc}}(G)\) then \(\Delta \alpha \in \Omega^*(G^2)\). With this coproduct and the obvious antipode, unit and counit, it is straightforward to verify that \(\bar{\Omega}^*(G)\) satisfies the axioms of a DG Hopf algebra. Its Hochschild complex coincides with the simplicial de Rham bicomplex \(\{\bar{\Omega}^*(NG), \delta, d\}\), resp. \(\{\bar{\Omega}^*(NG), \delta, d\}\), where \(\bar{\Omega}^{p,q}(NG) := \bar{\Omega}^p(G^p)\) and \(\bar{\Omega}^{p,q}(NG) := \bar{\Omega}^q(G^{p+1})\).
Theorem 2.2. There are canonical isomorphisms

\[ HH^q((\bar{\Omega}^* G)_n) \cong H^*(W(\mathfrak{g})_n), \]
\[ HC^q((\bar{\Omega}^* G)_n) \cong \bigoplus_{i \geq 0} H^{q-2i}(W(\mathfrak{g})_n). \]

Proof. The analogue of Bott’s Decomposition Lemma yields

\[ E_1^{p,q} = H^p(\bar{\Omega}^q(G^p)_n) \cong H^p_{\square}(G; S^q(\mathfrak{g}^*)_n), \]
where \( H^p_{\square} \) refers to the cohomology of group germs, which by its very definition (see [27, §4]) in the case at hand is the cohomology of the complex \( \{ \bar{\Omega}^0(G^p; S^q(\mathfrak{g}^*)_n), \delta \} \). Furthermore, the counterpart of van Est’s isomorphism for this case was proved by Świeczkowski [27, Theorem 2] and it gives the isomorphism

\[ H^i(\mathfrak{g}; S^q(\mathfrak{g}^*)_n) \cong H^i(\mathfrak{g}; S^q(\mathfrak{g}^*)_n). \]

Since \( H^*(\mathfrak{g}; S^q(\mathfrak{g}^*)_n) \cong H^*(\mathfrak{g}; \mathbb{R}) \otimes S^q(\mathfrak{g}^*)_n \), it follows that

\[ H^*(\mathfrak{g}; \mathbb{R}) \otimes S^q(\mathfrak{g}^*)_n \cong E[h_1, h_2, \ldots, h_n] \otimes P_n[c_1, c_2, \ldots, c_n] \]

After identifying the first differential as being \( d_1 u_i = c_i \) for \( 1 \leq i \leq n \), i.e. showing that \( E_1 \cong W_n \), one can continues as in the first proof in §2.2.

Again, the more satisfying argument follows the pattern of the second proof in §2.2, with the maps \( \varphi_m \) and \( \psi_m \) replaced by \( \varphi^{\text{germ}}_m : W_1(\mathfrak{g})/\mathcal{F}_1^{m+1} \rightarrow \Omega^*(\bar{\Omega}^* G)/\mathcal{F}_1^{m+1} \) and \( \psi^{\text{germ}}_m : \Omega^*(\bar{\Omega}^* G)_m \rightarrow \bar{\Omega}^*(\bar{\Omega}^* G)/\mathcal{F}_1^{m+1} \). The proof that these maps induce isomorphisms of spectral sequences goes along the same lines as in [28], with the necessary modifications similar to those used by Świeczkowski to adapt van Est’s proof for the bicomplex in [13, §10] to the bicomplex in [27, §7].

2.5. Examples of representative cocycles. First we point out that, similarly to the realization of \( \mathcal{H}_n = \mathcal{H}(n, \mathbb{R}) \) as a Hopf algebra of ‘moving frames’, \( \Omega^*(\text{GL}(n, \mathbb{R})) \) has a natural representation as a Hopf algebra of ‘moving coframes’. This representation is given by the coaction of \( G = \text{GL}(n, \mathbb{R}) \) on the étale groupoid \( \Gamma_n = \text{Diff}_n \ltimes \mathbb{R}^n \) determined by the first jet map (analogous to the local coaction of \( \text{GL}(n, \mathbb{R}) \) on the \( C^* \)-algebra of a codimension \( n \) foliation, mentioned by Connes in [4, III 7 α]). Specifically, cf. [17, §5], the formula

\[ \rho(\alpha)(\omega U_\varphi) := J^*(\alpha)\omega U_\varphi, \quad \alpha \in \Omega^*(G), \quad \omega \in \Omega^*_c(\Gamma_n), \]

where \( J : \Gamma_n \rightarrow \text{GL}(n, \mathbb{R}) \) is the Jacobian map

\[ J(\varphi, x) = \varphi'(x) = (\partial_j \varphi^i(x)) \in \text{GL}(n, \mathbb{R}), \]

defines an action \( \rho : \Omega^*(G) \times \Omega^*_c(\Gamma_n) \rightarrow \Omega^*_c(\Gamma_n) \).
Parallel to the case of the action of $H_n$ on $\Gamma_n$, there is a closed graded trace $\int : \Omega^*_c(\Gamma_n) \to \mathbb{C}$, given by

$$\int \omega U_\varphi = \int \omega, \quad \text{if } \varphi = 1,$$

$$\int f U_\varphi = 0, \quad \text{if } \varphi \neq 1,$$

which is $\Omega^*(G)$-invariant, i.e. satisfies

$$\int (d\omega) U_\varphi = 0,$$

and

$$\int \rho(\alpha)(\omega U_\varphi) = \epsilon(\alpha) \int \omega U_\varphi.$$

With these at hand, one can define (see [17, (3.11)]) a characteristic map of cyclic modules $\kappa_\ast : \Omega^*_c(G) \mathcal{C} \to \Omega^*_c(\Gamma_n) \mathcal{C}$ by setting

$$\kappa_q(\alpha^1 \otimes \cdots \otimes \alpha^q)(a_0, a_1, \ldots, a_q) := (-1)^d \int a_0 \rho(\alpha^1)(a_1) \cdots \rho(\alpha^q)(a_q),$$

where $\alpha^i \in \Omega^*(G)$, $a_j \in \Omega^*_c(\Gamma_n)$, and $d := \sum_{i>j} \deg \alpha^i \deg a_j$.

In order to write down examples of Hopf cyclic classes in $HC^*((\Omega^*_c(G))_n)$, let $\theta \in \Omega^*_c(G) \otimes g^*$ denote the Maurer-Cartan form $\theta_g = g^{-1}dg$, and define $\log |\det| \in \Omega^0(G)$ by $\log |\det|(g) := \log |\det g|$. Then

$$gv_{1,q} := \log |\det| \otimes \text{Tr} (\theta^{\otimes q}) \in \Omega^q(G^{q+1})$$

is a cocycle representing in $HC^*((\Omega^*_c(G))_n)$ the Godbillon-Vey class corresponding to $u_{1c}^q \in H^{2q+1}(W(g, G)_n)$. More generally, cocycles representing generalized Godbillon-Vey classes in $HC^*((\Omega^*_c(G))_n)$ can be obtained as suitable linear combinations of terms of the form

$$\log |\det| \otimes \text{Tr} (\theta^{\otimes \ell_1}) \otimes \text{Tr} (\theta^{\otimes \ell_2}) \otimes \cdots \otimes \text{Tr} (\theta^{\otimes \ell_k}) \in \Omega^q(G^{q+1})$$

where $\lambda = (\ell_1, \ldots, \ell_k)$ runs over the partitions of $q$. Carried through the characteristic map (2.27), these cocycles acquire expressions reminiscent of those obtained for the same classes by Bott [2] in group cohomology, and by Crainic and Moerdijk [11, §5.1] in their Čech-de Rham theory for leaf spaces.

The Chern classes in both $HC^*((\Omega^*_c(G))_n)$ and $HC^*((\Omega^*_c(G))_n)$ can also be represented by cocycles of a similar form only without the transcendental factor $\log |\det|$.  

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