Gravitational Lensing in the Universe

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This work reviews the basic theoretical aspects, the main observational evidences and the recent applications of gravitational lensing in the Universe. The article is aimed particularly at providing the readers who don’t work on gravitational lensing a relatively easy introduction to this active research field in today’s astrophysics.

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INTRODUCTION

One of the most important tasks for astronomers and physicists is to study the matter distribution in the Universe. Based on the assumption of “light-traces-mass”, the map of the Universe can be directly drawn
from the measurements of the apparent positions of various luminous objects on the sky (two dimensions) and their distances from the earth (one dimension). This method has been widely used nowadays and has revealed the existence of large-scale structures, such as the “Great Wall”, voids, filaments, etc. However, the dark matter puzzle in today’s physics and astrophysics casts doubt on the hypothesis of using luminous objects as the tracers of the total matter distribution of the Universe. Indeed, astronomical observations can only give rise to the distributions of those celestial objects that have electromagnetic radiation strong enough to be captured by telescopes, which might not reflect at all the real matter distribution in the Universe.

Dynamical methods are traditionally used for the determination of the masses gravitationally bound in the celestial bodies, which have successfully led to the discoveries of the excess of dynamical masses in galaxies and clusters of galaxies as compared with their luminous masses. Nevertheless, the employment of dynamical analysis requires that the systems are in the state of dynamical equilibrium, while it has remained unclear to date whether the large gravitationally-dominated systems like clusters of galaxies have reached the virialized cosmo-dynamical state, especially at high redshifts. Furthermore, the mass determination using dynamical method relies on the detailed knowledge of the celestial bodies: The rotational velocities and/or velocity dispersions should be well measured in galaxies and galaxy clusters in order to estimate their dynamical masses. Yet, this turns out to be quite difficult for the distant galaxies and galaxy clusters.

A new method of mapping the total matter distribution (luminous+dark) in the Universe stems from the effect of gravitational lensing, which has been available only for about 16 years since the discovery of the first gravitationally lensed double quasar 0957+561A,B (Walsh, Carswell and Weymann, 1979). The masses derived from gravitational lensing reflect the total matter contained in the lensing objects, independent of whether or not the lensing systems have reached the virial equilibrium. Therefore, gravitational lensing provides an independent way to test the evolution and to set constraints on the possible form of matter distributions of the lensing objects. Moreover, studies of gravitational lensing open a possibility of weighing the unseen matter in the Universe by comparing the gravitational masses deduced from lensing with the luminous masses estimated from optical observations. Furthermore, the effect of gravitational lensing magnifies the apparent luminosities of background objects, making the intrinsically faint sources enter into the detection thresholds of telescopes, which acts in fact like a “gravitational telescope”.

Recall that modern cosmology is based on the so-called “cosmological principle”, which assumes a spatially isotropic and homogeneous matter distribution of the Universe. The isotropy of the Universe has been well demonstrated by the measurement of the 3K microwave background radiation (Smoot et al., 1992), whilst the homogeneity turns to be somewhat hard to describe quantitatively. Actually, it is not so clear on what scales the Universe can be treated as homogeneous, though the largest coherent structure seems to have scale of ~ 100 Mpc. Matter condensations occur on scales up to a few tens of megaparsecs: planets, stars, galaxies, galaxy clusters, superclusters, voids, Great Walls, etc. All these matter clumps may affect the propagation of light through the effect of gravitational lensing, according to the prediction of the theory of general relativity. Therefore, gravitational lensing is a common phenomenon in astronomical observations and reasonable caution should be exercised in the identification of various celestial bodies. For instance, some close double or multiple images may be due to single sources (e.g. quasars) gravitationally split by the intervening objects (e.g. galaxies), and the arc-like images could be the result of the gravitationally distorted background galaxies by the foreground galaxy clusters.

The history of studies of gravitational lensing can be divided into three periods: 1704 – 1964, 1964 – 1979 and 1979 – present. Newton (1704) addressed the question nearly three hundreds years ago if the celestial bodies could bend light rays. The deflection of light by a spherical body of mass $M$ based on the Newtonian mechanics was computed to be $\alpha = 2GM/c^2\xi$, assuming an impact distance of $\xi$, i.e., the shortest distance from $M$ to the light path. $c$ and $G$ are the speed of light and the gravitational constant, respectively. This formula was found to underestimate the deflection angle by a factor of 2 by Einstein in 1915 in terms of his gravitational theory. In particular, utilizing the result of general relativity to the Sun predicts a deflection angle of 1.75" for a light ray passing near the solar limb. This prediction was very soon confirmed during the solar eclipse in 1919 by a team led by Eddington (Dyson, Eddington and Davidson, 1920). Although some progress had been made since then on the theoretical aspects of light bending, not much interest had been really drawn on this field until 1963–1964 when the geometry of lensing was studied independently by Klimov (1963), Liebes (1964) andRefsdal (1964a). Their work can be considered pioneering in the sense that
they set up the foundation of the modern theoretical research on gravitational lensing. During the period of 1964 – 1979 some important progress was made in the computations of deflection angle (e.g., Bourassa and Kantowski, 1975; 1976) and the light propagation in the model of an inhomogeneous Universe (Press and Gunn, 1973; Dyer and Roeder, 1972; 1973; 1974). These achievements have played an important role in the lensing studies after the detection of the first multiple images of quasar 0957+561A,B by Walsh, Carswell and Weymann in 1979. In the decade following this landmark, the detections of new lensing phenomena have increased dramatically, including multiply-imaged quasars, giant luminous arcs and arclets, radio Einstein rings, microlensing events, whilst the theoretical investigation of gravitational lensing has concerned many interesting subjects of modern cosmic physics, such as determinations of total masses of the lensing systems, determinations of $H_0$ and $\Omega_0$, explanation of the associations of background sources with foreground objects, searches for dark matter candidates, etc. Today, gravitational lensing has become one of the most active fields in cosmological research.

It is surely impossible to cover all the topics of lensing research in this review as it has expanded very rapidly in the past years. The purpose of this work is to concentrate on the basic theories and the new progress of gravitational lensing as well as their applications in astrophysics. Alternatively, this review will not trace the history of development of gravitational lensing but follow the scales of lenses from compact objects to large-scale inhomogeneities.

1 BASIC THEORY

1.1 Deflection of light

Gravitational lensing is based on the theory of general relativity, which predicts that light rays would be bent when they pass near a massive body. For a pointlike mass $M$ the deflection angle of light rays is

$$\alpha = \frac{4GM}{c^2\xi},$$

(1)

where $\xi$ represents the impact distance of light rays. This deflection angle is twice as large as the value predicted from Newtonian mechanics.

For an extended mass distribution, the deflection of light passing through the mass system cannot be simply obtained in the frame of general relativity. Only for some special matter distributions can the metric and the solution to the photon geodesic equations, hence the angle of light bending, be found exactly. For instance, in a spherical uniform matter distribution the deflection of light, to first order in the Newtonian gravitational potential, is (Wu, 1989a)

$$\alpha \approx \frac{4GM}{c^2\xi} \left[1 - \left(1 - \frac{\xi^2}{R^2}\right)^{3/2}\right].$$

(2)

Here $M$ and $R$ are the total mass and the radius of the massive sphere, respectively. Another matter profile that has been commonly used for galaxies and clusters of galaxies is the singular isothermal sphere model having mass density

$$\rho(r) = \frac{\sigma_v^2}{2\pi Gr^2},$$

(3)

where $\sigma_v$ measures the line-of-sight velocity dispersion. The deflection angle of light rays can be found to first order in $(\sigma_v/c)^2$ to be (Wu, 1989a)

$$\alpha = 4\pi \frac{\sigma_v^2}{c^2},$$

(4)

i.e., a constant deflection independent of the impact distance.
For photons traveling in the gravitational field of an irregular matter distribution, the deflection angle of light rays is often computed in a linearized Einstein approximation

\[ \alpha = \frac{2}{c^2} \int_{-\infty}^{\infty} \nabla \phi dt, \]  

in which \( \phi \) represents the Newtonian gravitational potential of the matter distribution. An equivalent but simple formula is obtained by dividing the system into a number of pointlike masses \( M_i \) and then summing up the contributions from each small mass piece

\[ \alpha = \sum_i \frac{4GM_i}{c^2} \frac{r - r_i}{|r - r_i|^2}, \]  

where \( r - r_i \) is the impact vector of light rays from the mass unit \( M_i \). The integral form is

\[ \alpha = \frac{4G}{c^2} \int \frac{\xi - \xi'}{|\xi - \xi'|^2} \Sigma(\xi')d^2\xi'. \]  

Here \( \Sigma(\xi') \) is the surface mass density at position \( \xi' \) obtained by projecting all the mass along the line of sight onto the “lens plane”. In particular, for a spherical matter distribution \( \rho(r) \) the above equation can be simplified to be

\[ \alpha = \frac{4Gm(\xi)}{c^2\xi}, \]  

where \( m(\xi) \) is the total projected mass along the light-of-sight enclosed within the impact distance \( \xi \). If \( \rho(r) \) is confined within the radius \( R \), then

\[ m(\xi) = M - 4\pi \int_\xi^R \sqrt{r^2 - \xi^2} r \rho(r)dr, \quad \xi < R; \]
\[ m(\xi) = M, \quad \xi \geq R, \]  

in which \( M \) is the total mass of the spherical system. Therefore, if the deflector has spherical symmetry, the light bending can be obtained simply by replacing the pointlike mass \( M \) of eq.(1) by the projected mass \( m(\xi) \). For example, replacing the mass density in eq.(9) by a constant and inserting \( m(\xi) \) for \( \xi < R \) into eq.(8) recover the result of eq.(2). It should be pointed out that eq.(9) holds true even if the deflector has no boundary or infinite mass, in which

\[ m(\xi) = M(\xi) + 4\pi \int_{\xi}^{\infty} (r - \sqrt{r^2 - \xi^2})r \rho(r)dr. \]  

So, the deflection angle [eq.(4)] of a singular isothermal sphere can be easily obtained by inserting the density profile eq.(3) into eqs.(8) and (10).

1.2 Lensing geometry and various lenses

Suppose that a light-ray from a distant source at redshift \( z_s \) passes through or near a massive system at redshift \( z_d \) \( (z_d < z_s) \) and then reaches the observer at redshift \( z = 0 \) (Figure 1). If there were no intervening massive system, the light from the source would have arrived at the observer along a straight line. The gravitational field of the massive system now bends the light ray, causing the direction of light to be changed by an angle of \( \alpha \). If we use \( \eta \) (\( \beta \) in angle) to denote the true position of the source or the alignment parameter in the source plane and \( \xi \) (\( \theta \) in angle), the observed positions of the images in the lens plane, we have the following geometrical relation, namely, the “lensing equation”:

\[ \frac{D_s}{D_d} \xi - \eta = D_{ds} \alpha, \]  

where \( D_d \) is
Figure 1: Scheme of basic geometry of gravitational lensing. The true position of the distant source is \( \beta \) with an angular diameter distance \( D_s \) from the observer. The light rays with an impact distance \( \theta \) would be deflected by an angle \( \alpha \) by the gravitational field of an intervening massive object, resulting in multiple and distorted images.

or

\[ \theta - \beta = \frac{D_{ds}}{D_s} \alpha \]  

(12)

where \( D_d, D_s \) and \( D_{ds} \) are the angular diameter distances to the deflector, to the source and from the deflector to the source, respectively. As seen from eq.(7), the deflection angle \( \alpha \) can be written as the gradient of a two-dimensional potential \( \psi \)

\[ \frac{D_{ds}}{D_s} \alpha = \nabla \psi \]  

(13)

and

\[ \psi(\theta) = \int \frac{\Sigma(\theta')}{\pi \Sigma_c} \ln |\theta - \theta'| \, d^2 \theta', \]  

(14)

in which the quantity \( \Sigma_c \) is called the critical surface mass density

\[ \Sigma_c = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}, \]  

(15)

and the physical surface mass density term \( \Sigma \) should satisfy the two dimensional Poisson’s equation (Blandford and Narayan, 1986)

\[ \nabla^2 \psi = \frac{2\Sigma}{\Sigma_c} \equiv 2\kappa. \]  

(16)

For a given source position \( \beta \), the lensing equation may have several solutions for the image position \( \theta \). As a consequence, we can observe multiple images of a single source on the sky. Furthermore, the lensing equation (12) describes the distortion of the surface brightness of background sources according to the mapping from source plane to lens plane:

\[ I'(\theta) = I(\beta) = I(\theta - \nabla \psi), \]  

(17)

where \( I' \) and \( I \) are, respectively, the observed and intrinsic surface brightness patterns. The magnification factor \( \mu \) describes the change of apparent luminosity of the source, which is characterized by the Jacobian for the mapping \( \beta \rightarrow \theta \) (Schneider, Ehlers and Falco, 1992):

\[ \mu \equiv |\det \frac{\partial \beta}{\partial \theta}|^{-1} = \left[ (1 - \kappa)^2 - \gamma^2 \right]^{-1}, \]  

(18)

here

\[ \kappa = \frac{1}{2}(\psi_{11} + \psi_{22}); \]

\[ \gamma = \sqrt{\gamma_1^2 + \gamma_2^2}; \]

\[ \gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}); \]

\[ \gamma_2 = \psi_{12} = \psi_{21}, \]  

(19)

in which \( \psi_{ij} \equiv \partial^2 \psi/\partial \theta_i \partial \theta_j \). When the line of sight completely misses the deflector, \( \Sigma(\theta) = 0 \) and the \( \kappa \) term vanishes in terms of eq.(16). So, \( \kappa \) represents the amplitude of the convergence due to the matter within
the light-ray (also referred to as Ricci focusing), while the \( \gamma \) term is the amplitude of the shear due to the matter outside the beam (also referred to as Weyl focusing). The latter can be easily calculated in the case of spherical deflector, which reads
\[
\gamma = \frac{4G[m(\theta) - \overline{m}(\theta)]}{c^2 \theta^2} \frac{D_{ds}}{D_d D_s},
\]
and \( \overline{m}(\theta) = \pi \theta^2 D_s^2 \Sigma(\theta) \) is the mass of the cylinder of radius \( \theta \) with a uniform surface mass density equal to \( \Sigma(\theta) \). When a beam of light-rays pass through the center of deflector, \( m'(0) = m''(0) = 0 \) and the \( \gamma \) term then becomes zero.

(1) Pointlike mass as deflector. This model can be considered to be a good approximation for many celestial bodies like “Jupiters”, stars, black holes, and even galaxies, when the light rays from background sources pass outside the deflectors. Solving the lensing equation, using the deflection of eq.(1), yields
\[
\theta_{\pm} = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right),
\]
where
\[
\theta_E = \left( \frac{4GM}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{1/2}
\]
is the critical radius (or \( a_E = \theta_E D_d \) in linear size) corresponding to a ring-like image of a background source when \( \beta = 0 \), which is often called the “Einstein ring”, named after the pioneering work by Einstein (1936). \( \theta_+ \) and \( \theta_- \) describe the positions of the two images produced by a point mass \( M \). Their magnifications are
\[
\mu_{\pm} = \frac{1}{1 - (\theta_E/\theta_{\pm})^2}.
\]
If the separation between the two images is too small to be resolved by modern telescopes, as for the microlensing events (section 2), the total magnification is often used for their combined effect
\[
\mu = \mu_+ + \mu_- = \frac{u^2 + 2}{u \sqrt{u^2 + 4}}.
\]
Here \( u \) is defined as \( u = \beta/\theta_E \). \( u = 1 \) or \( \beta = \theta_E \) is often taken to be a typical case that characterizes the efficiency of the lens, which corresponds to \( (\mu_+ - \mu_-) = (1.17, 0.17) \) and \( \mu = 1.34 \) or \( \Delta m = 0.32 \) in apparent magnitude. Pointlike masses as lenses play an important role in the study of microlensing (section 2.1) and in searches for dark matter candidates in the Galactic halo using microlensing effect (section 2.2).

The image configurations by a pointlike mass is well illustrated in Figure 2 by changing the relative positions between the source which is chosen to be the Einstein portrait and the lens which is assumed to be a massive black hole. When the source approaches the lens, the two images are elongated, showing two arclike structures. Finally, two arc images merge into an “Einstein ring” when the lens is completely coincident with the source.

(2) A singular isothermal sphere. The studies of the flat rotation curves of galaxies and the galaxy/gas distributions in clusters of galaxies suggest that the total matter profiles in these systems follow very well the singular isothermal sphere (SIS) model [see eq.(3)]. The constant deflection of \( \alpha = 4\pi(\sigma_v/c)^2 \) in the lensing equation gives
\[
\theta_{\pm} = \theta_E \pm \beta
\]
Figure 3: Solution to lensing equation by ISC. The solid curves represent the lensing equation $\beta_0 = \theta_0 - D[(1 + \theta_0^2)^{1/2} - 1]/\theta_0$. The intersecting points with the line $\beta_0 =$constant give the number and positions of the lensed images.

for $\beta < \theta_E$, where the critical radius is

$$\theta_E = 4\pi \frac{\sigma_v^2 D_{ds}}{c^2}.$$  (26)

In particular, the image separation is just the diameter of the Einstein ring: $\Delta \theta = \theta_+ + \theta_- = 2\theta_E$. Note that if the alignment parameter $\beta$ is larger than the Einstein ring, only one image appears instead of two. The magnifications of the images are simply

$$\mu_\pm = \left| 1 \pm \frac{\theta_E}{\theta_\pm} \right|^{-1} = \left| 1 \pm 4\pi \frac{\sigma_v^2 D_{ds}}{c^2 \beta D_s} \right|.$$  (27)

(3) A softened singular isothermal sphere. SIS, though simple, is an unphysical model because the mass density reaches infinity at the center. Instead, a SIS with a finite core radius of $r_c$ (or $\theta_c$ in angle) (ISC) seems to be more reasonable for the matter distributions of galaxies and galaxy clusters (Hinshaw and Krauss, 1987):

$$\rho(r) = \frac{\sigma_v^2}{2\pi G} \frac{1}{r^2 + r_c^2}.$$  (28)

This density profile reduces to SIS when $r_c = 0$ or $r \gg r_c$. The surface mass density and the total projected mass within $\xi$ are

$$\Sigma(\xi) = \frac{\sigma_v^2}{2\pi G} \frac{1}{\sqrt{\xi^2 + r_c^2}};$$

$$m(\xi) = \pi \frac{\sigma_v^2}{c^2} \left( \sqrt{\xi^2 + r_c^2} - r_c \right),$$  (29)

respectively. The deflection of light can be directly obtained from eq.(9) and then, the lensing equation reads

$$\theta_0 = \beta_0 + D \sqrt{1 + \theta_0^2} - 1/\theta_0,$$  (30)

in which $\theta_0$ and $\beta_0$ are in unit of $\theta_c$. The lensing parameter, defined as $D \equiv (4\pi \sigma_v^2/c^2)(D_d D_{ds}/r_c D_s)$, determines the number of the solutions (Figure 3) (Wu, 1989b). In the case of $D \leq 2$, ISC always produces a single image, while it may result in three images for $D > 2$ if $\beta_0$ is sufficiently small. The intersections of the lines $\beta_0 =$constant with the curves $\beta_0 = \theta_0 - D(\sqrt{1 + \theta_0^2} - 1)/\theta_0$ give the solutions to the lensing equation. The magnification is found to be

$$\mu = \left| \left( 1 - D \frac{\sqrt{1 + \theta_0^2} - 1}{\theta_0^2} \right) \left( 1 + D \frac{\sqrt{1 + \theta_0^2} - 1}{\theta_0^2} - D \frac{1}{\sqrt{1 + \theta_0^2}} \right) \right|^{-1}.$$  (31)

(4) Other spherical models. Two other models which are also frequently adopted for the matter distributions in galaxies and in galaxy clusters are the King model (or the modified Hubble model) and the de Vaucouleur model (or the $r^{1/4}$ law). Their lensing properties are very similar to those of ISC and are summarized in Table 1.

(5) Asymmetric lenses. In principle, the lensing geometry can be established for any kind of geometrically-thin matter inhomogeneity in the Universe through eqs.(7) and (12). Among them, the properties of elliptical
Figure 4: Image configurations by an elliptical lens. The source planes are on the left and the corresponding images are on the right. The solid lines are the caustics and the dashed lines are the corresponding critical lines. (from Blandford and Narayan, 1992)

lenses have been thoroughly studied (e.g., Bourassa and Kantowski, 1975; 1976; Blandford and Kochanek, 1987; Narasimha, Subramanian and Chitre, 1987; Schramm, 1990; Wallington and Narayan, 1993; Kassiola and Kovner, 1993, etc.). A simple elliptical lens assumes the following “non-singular pseudo-elliptical isothermal potential”

$$\psi(\theta_1, \theta_1) = 4\pi \sigma_c^2 D_d D_s \frac{D_s}{D_s} \sqrt{\theta_c^2 + (1 - \epsilon)\theta_1^2 + (1 + \epsilon)\theta_2^2}, \quad (32)$$

where $\epsilon$ is the ellipticity and $\theta_c$ is the core radius of the lens. Solving the lens equation

$$\beta = \theta - \nabla \psi, \quad (33)$$

one has (Blandford and Kochanek, 1987; Wallington and Narayan, 1993)

$$\beta_1 = \theta_1 - \theta_E \frac{(1 - \epsilon)\theta_1}{\sqrt{\theta_c^2 + (1 - \epsilon)\theta_1^2 + (1 + \epsilon)\theta_2^2}},$$

$$\beta_2 = \theta_2 - \theta_E \frac{(1 - \epsilon)\theta_2}{\sqrt{\theta_c^2 + (1 - \epsilon)\theta_1^2 + (1 + \epsilon)\theta_2^2}}, \quad (34)$$

where $\theta_E$ is the Einstein radius in SIS [eq.(26)]. Magnifications of the images can be calculated using eq.(18)

$$\mu = \left| \frac{\partial \beta_1}{\partial \theta_1} \frac{\partial \beta_2}{\partial \theta_2} - \frac{\partial \beta_1}{\partial \theta_2} \frac{\partial \beta_2}{\partial \theta_1} \right|^{-1}. \quad (35)$$

Inverting eq.(34) gives the image positions in the lens plane for pointlike sources. An extended source can be considered as a set of pointlike sources and its image configuration produced by an elliptical lens can be drawn through the above equation for each source element. In principle, the images with arbitrary shapes can be produced by an elliptical lens as long as the position and the shape of the extended source are properly chosen. Some image configurations produced in this procedure for a circular source are shown in Figure 4.

(6) Cosmic strings as lenses. Cosmic strings are topological defects that are believed to be created in the very early Universe and could act as the seeds of formations of galaxies, galaxy clusters and even large-scale structures (Vilenkin, 1981), although there has been no observational evidence to date for the existence of such cosmic strings in the Universe. Cosmic strings, if real, may essentially exist in two forms: straight and loop strings. Thus, the long-lived cosmic strings are able to cause gravitational lensing effect on background sources.

The exterior gravitational field of a straight string is ($G/c^2 = 1$) (Gott, 1985)

$$ds^2 = -dt^2 + dr^2 + (1 - 4\mu_s)^2 r^2 d\phi^2 + dz^2. \quad (36)$$

Thus, a cosmic string is completely described by its linear mass density $\mu_s$ ($\sim 10^{-6}$). Consider that a light ray propagates in the plane perpendicular to the string (Figure 5), one can easily solve the null geodesic equation and find the light deflection in $\phi$ direction to be
The light rays that propagate along $r$ and $z$ directions are unaffected by the gravitational field of a straight string, according to the metric of eq.(36). Two images of a background source with the separation of $8\pi\mu_s$ but with the same apparent luminosity appear around the string. Note that eq.(36) adopts a cylindrical coordinate rather than the spherical one as in SIS although both lenses produce the impact parameter-independent deflections. So, the double images of a background source by a straight string can be identified relatively easily from their equal luminosities.

The metric of a long-lived loop string is unknown today. Its lensing properties can be studied only in the linearized gravitational approximation. For the simple case of a “face-on” loop string with radius of $a$ in the sky, the light from the source behind the string is bent by (Wu, 1989c)

$$\alpha = \begin{cases} 
0, & \xi < a; \\
8\pi\mu_sa/\xi, & \xi > a,
\end{cases}$$

(38)
i.e., the light rays which pass through the loop remain unaffected while the light rays outside the loop behave in the same way as those by a point mass of $M = 2\pi\mu_sa$ at the center. Therefore, if the alignment parameter $\eta$ of a background source is smaller than the loop radius $a$, we would always expect to see the original source through the loop. In particular, when $\eta$ satisfies $a - 8\pi\mu_s\!(D_zD_d/D_s) < \eta < a$, the arclike image of the lensed source appears outside the loop. Figure 6 shows such an example, in which the source and loop string are 10 kpc and 100 kpc in radii and located at $z_s = 1$ and $z_d = 0.25$, respectively. The equivalent mass of the string is $\sim 10^{14}M_\odot$. Therefore, a single giant arclike image, instead of multiple arclike images in the case of massive spherical deflectors, can be produced by the loop string.

### 1.3 Lensing efficiency

In the above discussion, it appears that lensing magnification of the apparent luminosities of background sources may tend towards infinity in some cases. For example, the impact parameters that satisfy the condition $(1 - \kappa)^2 - \gamma^2 = 0$ in eq.(18) correspond to images with an infinitely large magnification, whilst the same situation occurs for a pointlike lens when the alignment parameter is $\eta = 0$. These apparent unphysical results arise from the hypothesis that the background source is pointlike.

In general, the total magnification of an extended source with surface brightness $I(\eta_1, \eta_2)$ can be obtained by summing up the contribution $\mu(\eta_1, \eta_2)$ of each source element $I(\eta_1, \eta_2)d\eta_1d\eta_2$ (Bontz, 1979):

$$\mu = \frac{\iint \mu(\eta_1, \eta_2)I(\eta_1, \eta_2)d\eta_1d\eta_2}{\iint I(\eta_1, \eta_2)d\eta_1d\eta_2}.$$  

(39)

For simplicity, we consider a circular disk source with uniform surface brightness and radius $R_s$ instead of the point source approximation, and we further assume a spherical matter distribution for the deflector. In this situation, the maximum magnification $\mu_{max}$ corresponds to the case where the centers of the source and of the lens and the observer lie perfectly on a straight line, i.e, the Einstein ring shows up. So, eq.(39) reduces to the ratio of the area of the Einstein ring to the luminous area of the original source. This ratio or
Figure 7: Maximum magnification and source radius. The background source is assumed to be a luminous circular disk with radius $R_s$ at $z_s = 2$ while the lens is modeled by a point mass for star [eq.(40)] and SIS for galaxy and cluster of galaxy [eq.(41)] at $z_d = 0.5$.

maximum magnification is not infinite any longer. Actually, $\mu_{\text{max}}$ measures the efficiencies of gravitational lensing by various lenses for the same source. The larger the $\mu_{\text{max}}$ is, the stronger the lens would be. For a pointlike lens $M$,

$$\mu_{\text{max}} = \sqrt{1 + \frac{16GM}{c^2} \frac{D_d D_s}{R_s^2 D_d}}, \quad (40)$$

and for SIS,

$$\mu_{\text{max}} = \begin{cases} 
16\pi \frac{\sigma^2}{c^2} \frac{D_d}{R_s} & \theta_s < \theta_E \\
1 + 8\pi \frac{\sigma^2}{c^2} \frac{D_d}{R_s} & \theta_s > \theta_E
\end{cases}, \quad (41)$$

where $\theta_s = R_s/D_s$ is the angular radius of the source.

To show how efficiently the various lenses act on the background source, the maximum magnification of a background circular disk source is illustrated in Figure 7 for three typical lenses in the Universe: stars, galaxies and clusters of galaxies (Wu, 1992a). The star is modeled by a point mass with a solar mass $M_\odot$, while SIS is adopted for galaxies and galaxy clusters whose velocity dispersions are taken to be 200 km/s and 1500 km/s, respectively. It is concluded from Figure 7: (1) Compact objects like stars as lenses are capable of producing significant lensing effect on sources with sizes smaller than $\sim 0.01$ pc. Therefore, they can affect AGNs ($\sim 10^{-3}$ pc), quasars and normal stars; (2) Lenses on scale of galaxies can magnify any sources with size smaller than galaxies themselves; (3) Finally, clusters of galaxies are efficient lenses for nearly all luminous objects including galaxies of sizes of $\sim 10$ kpc.

## 2 COMPACT OBJECTS AND MICRO-LENSING

### 2.1 Microlensing

For a pointlike lens $M$ and a background source, both at their typical cosmological distances of $z_d = 0.5$ and $z_s = 1$, the angle subtended by the Einstein radius is [eq.(22)]

$$\theta_E \approx 1.4 \times 10^{-6} \left( \frac{M}{M_\odot} \right)^{1/2} \text{arcseconds.} \quad (42)$$

Thus, a compact star-like lens at cosmological distance gives rise to an image splitting of a background source of the order of “microarcseconds”. Hence, the terms “microlens” and “microlensing” are used for the compact object and its lensing phenomena so as to distinguish it from the “macrolens” like a galaxy which results in the image separation of the order of $\sim 1$ arcsecond according to eq.(42). Nevertheless, it appears to be hopeless to resolve the images microlensed by star-like compact objects in the Universe even with the present advanced telescopes.

Because of the too small separation between the multiply microlensed images, the combined magnification of the images turns to be the unique feature that arises from microlensing. Unfortunately, this feature cannot be straightforwardly used to detect microlensing events since it is impossible to separate the magnification effect from the intrinsic luminosity of the source. However, if the source and/or the lens have a transverse motion with respect to the line of sight, the lensing magnification would vary with time, leading to the
variation of apparent luminosity of the source. This effect was firstly predicted by Chang and Refsdal in 1979. Since then, a great number of papers have appeared, trying to apply this property for the explanation of QSO variabilities and the searches of dark matter candidates of the Galactic halo (e.g., Young, 1981; Canizares, 1982; Ostriker and Vietri, 1985; Nottale, 1986; Paczyński, 1986; Schramm et al., 1994; etc.).

The timescale of luminosity variability of a source by a microlens $M$ can be estimated using the time of the source crossing the Einstein radius $a_E$ with a relative velocity $v$: $T = 2a_E/v$. (1) For a local Galactic lens, e.g., an object in the Galactic halo ($D_d \sim 10$ kpc) acting as lens and a star of the Large Magellanic Cloud (LMC) ($D_s = 50$ kpc) being the target source:

$$T \approx 0.2 \text{yr} \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{v}{200 \text{ km s}^{-1}} \right)^{-1};$$

and (2) for a lens at cosmological distance ($z_d = 0.5$ and $z_s = 1$)

$$T \approx 10 \text{yr} \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{v}{1000 \text{ km s}^{-1}} \right)^{-1} h_{50}^{-1/2},$$

where (also hereafter) $H_0 = 50h_{50}$ km s$^{-1}$ Mpc$^{-1}$ is the Hubble constant. The timescale of luminosity variability of a source due to microlensing depends on the mass of the microlens as $\sim (M/M_\odot)^{1/2}$. It turns out that the masses of the lensing objects can be determined by monitoring the variations of the apparent luminosities of some sources if the distances of lenses and sources as well as the transverse speed of the lens $v$ are known. Note, however, that this procedure is actually restricted by the sensitivities of the microlensing observations, which cannot include the events with timescales lasting both shorter than the observing periods of sampling and longer than the observing coverage. The present observations are then sensitive to those events with timescales ranging from $\sim 10$ minutes to $\sim 10$ years.

The convincing evidence that a distant source is microlensed has been found so far only in QSO 2237+0305 ($z_s = 1.695$) associated with a foreground spiral galaxy ($z_d = 0.0394$). This lens system was discovered by Huchra et al. in 1985, in which the quasar appears to be coincident with the nucleus of the galaxy. The subsequent observations (De Robertis and Yee, 1988; Yee, 1988; Schneider et al., 1988) show that this quasar actually consists of four components with the maximum separation of 1.8′′ (see Figure 8), which is also referred to as the Einstein cross. The photometric monitoring of QSO 2237+0305 has been made for several years and the brightness variations of each component are shown in Figure 9. Overall, the light curves of four components show no apparent correlations, indicating that their variations must be due to the microlensing magnification by the compact objects and their relative motions in the spiral galaxy. Note that the time delay between the components from the microlensing is estimated to be about one day. To be specific, a sharp variation of at least 0.2 magnitude in the component A was detected within 26 days during 1988-1989 (Irwin et al., 1989; Corrigan et al., 1991), while the other components didn’t exhibit a similar feature. Another strong microlensing occurred in component B in 1991: the brightness of B is magnified by a factor of about 0.5 magnitude (Yee and De Robertis, 1992; Racine, 1992). As a comparison, components C and D appear to be relatively stable.
2.2 MACHO searches

The inferred masses from the rotation curves of galaxies, including our Galaxy, are an order of magnitude larger than their luminous masses, implying that galaxies are embedded in invisible massive halos (see Ashman 1992 for a recent review). The nature of the dark matter in the halos of galaxies is still unknown today. Basically, two kinds of candidates have been proposed: WIMPs (weakly interacting massive particles such as axions and supersymmetric neutralinos) and MACHOs (massive astrophysical compact halo objects such as brown dwarfs, low mass stars and black holes). WIMPs are some kinds of unknown non-baryonic matter which may dominate the Universe according to the standard inflation cosmological model and the Big Bang primordial nucleosynthesis, especially if the new measurement of the deuterium abundance in a high redshift primordial hydrogen cloud (Songalia et al., 1994; Carswell et al., 1994) is confirmed, whilst astronomers might favour MACHOs in the sense that MACHOs may cause the observational effect – microlensing.

If the relative velocity of the microlens $M$ (or the source) transverse to the line of sight is $v$, the variability of the total magnification of a pointlike source follows eq.(24). Figure 10 shows the light curve of a background source passing near a microlens with impact parameter of $0.25a_E$. The main characteristic features of the microlensing light curve are the achromaticity and time-symmetry around the point of maximum magnification, which can then be distinguished from the known variable star phenomena.

The probability that a source is gravitationally lensed is described by the so-called optical depth ($\tau$). The optical depth to microlensing without involving the lenses at cosmological distance is simply the number of microlenses or MACHOs inside the “microlensing tube” which has a cross-section of $\pi a_E^2$:

$$\tau = \int_0^{D_s} \pi a_E^2 n(D_d) dD_d,$$

where $n(D_d)$ is the number density of the MACHOs at distance $D_d$. Note that within the microlensing tube, $\mu > 1.34$ or $|\Delta m| > 0.32$. Suppose that the Galactic halo is composed of MACHOs, their density profile can be estimated from the rotational velocity $v_G$ which is about 220 km/s at the position of the Sun. The original work by Paczyński (1986) used a SIS for the massive halo, and the subsequent work by Griest (1991) modified SIS by introducing a core radius (ISC) which actually does not provide any significant difference from SIS in the calculation of $\tau$. Employing ISC for the MACHO distribution of the Galactic halo in eq.(45) yields

$$\tau_G = \tau_0 \int_{x_s}^{x_h} \frac{x(x_s-x)dx}{x_s(1+x_c^2-2x \cos \alpha + x^2)},$$

and

$$\tau_0 = \left( \frac{v_G}{c} \right)^2 \frac{1}{1-\frac{D_h}{r_c} \arctan \frac{D_h}{r_c}},$$

where $r_c$ is the core radius of ISC, $\alpha$ is the angle between the line of sight to the source and the direction to the Galactic center, and $D_h$ is the extent of the Galactic halo. All the distances are measured in unit of $R_{GC}$, the distance to the Galactic center ($\approx 8.5$ kpc), so that $x = D_d/R_{GC}$, $x_s = D_s/R_{GC}$ and $x_c = r_c/D_{GC}$.

Now consider a star in our neighbour galaxy, the LMC, as the target. The light rays from a star of LMC reach the observer by passing through the LMC halo, the LMC disk and finally, the Galactic halo. MACHOs of both the LMC halo/disk and the Galactic halo are able to gravitationally magnify the apparent luminosity of the star in the LMC. The contribution of the LMC halo to the microlensing optical depth can be computed in a similar way to that of the Galactic halo

$$\tau_{LMC, \text{halo}} = \tau_0 \int_{x_h}^{x_s} \frac{x(x_s-x)dx}{x_s(1+x_c^2-2x \cos \beta + x^2)},$$

where $\beta$ is the angle between the line of sight to the source and the direction to the Galactic center, and $D_h$ is the extent of the Galactic halo. All the distances are measured in unit of $R_{GC}$, the distance to the Galactic center ($\approx 8.5$ kpc), so that $x = D_d/R_{GC}$, $x_s = D_s/R_{GC}$ and $x_c = r_c/D_{GC}$.
Furthermore, the inclination of the LMC disk can be simply taken to be 0° (Schommer et al., 1992) and the center of the LMC, and Rn is the extent of the LMC halo. The distances in eq.(47) are measured in unit of Dc, the distance to the LMC center (≈ 50.6 kpc), so that x = Dd/Dc, xs = Ds/Dc and xc = xc/Dc. The LMC disk can be modeled by an isothermal self-gravitating disk having density (van der Kruit and Searle, 1981) n = n0 exp(−R/h)sech²(z/z0), with the scale length h in the radial direction and the scale height z0 in the z-direction. The central number density n0 relates with the maximum rotational velocity (v_m) through (Freeman, 1970) n0 = (1/4πMGhz0)(v_m/0.62)², in which M is the mass of the MACHO. Furthermore, the inclination of the LMC disk can be simply taken to be 0°, i.e., a face-on distribution of the disk matter. The optical depth to microlensing by the disk is then

$$\tau_{\text{LMC,disk}} = \tau_0 \int_0^{x_s} \frac{x(x_s - x)}{x_s} \left( \frac{v_m}{0.62} \right)^2 \left( \frac{1 - x \cos \beta}{z_0/D_c} \right) dx,$$  

and

$$\tau_0 = \left( \frac{v_{\text{LMC}}}{c} \right)^2 \frac{1}{1 - \frac{R_n}{h \times \arctan \frac{R_h}{r_c}}}. $$

Here rc is the core radius of the LMC halo described by ISC, β is the angle between the line of sight to the star and the center of the LMC, and Rh is the extent of the LMC halo. The optical depth to microlensing by the Galactic halo, the LMC halo and the LMC disk to the microlensing optical depth for the stars of the LMC are plotted in Figure 11. The halo of our Galaxy gives rise to a nearly constant optical depth crossing the LMC disk, τ_G = 5 × 10⁻⁷, while the LMC halo/disk provide an optical depth depending sharply on the positions of the stars, which arises from the fact that only the foreground MACHOs of the LMC are able to act as lenses for the stars of the LMC itself. For the stars near the LMC center (β ≤ 0°.5), τ_{LMC,halo} ≥ 2 × 10⁻⁷ and τ_{LMC,disk} ≥ 3 × 10⁻⁸. Therefore, if the halo of our Galaxy and of the LMC are composed of MACHOs, several million stars should be monitored for the discovery of the microlensing events in the LMC.

Following the proposal of Paczyński (1986), the EROS (Expérience de Recherche d’Objets Sombres) and the MACHO collaboration commenced in 1990 their searches for microlensing events of the LMC by monitoring the brightness of a few million stars in the LMC, and the OGLE (Optical Gravitational lensing Experiment) began in 1992 to conduct a similar search in the direction of the Galactic bulge. Three groups announced their discoveries at almost the same time in the autumn of 1993: Three events were detected in the LMC (Alcock et al. 1993; Aubourg et al. 1993) and six were seen in the Galactic bulge (Udalski et al., 1993). During the writing process of this article, the total “local” microlensing events have grown to ~ 70. These include more than 10 events found by the OGLE collaboration (Udalski et al., 1994a; 1994b; 1995) and more than 40 events by the MACHO collaboration (Bennett et al., 1994; Alcock et al., 1995a) in the Galactic bulge, and 3 events by the MACHO collaboration (Alcock et al., 1995b) and 2 events by the EROS team in the LMC [Note that the EROS n°2 candidate may be an eclipsing binary system rather than a microlensing event (Ansari et al., 1995)]. Figure 12 shows the light curves of, and the lensing model fits to the MACHO microlensing events and Table 2 summarizes the properties of the 5 microlensing candidates of the LMC, which can be regarded as the representatives of all the reported microlensing candidates.

If these events are indeed generated by the microlensing of the MACHOs along the light of sight rather than a new kind of variable stars, one can estimate the mass of the MACHOs using the event duration T (i.e., the Einstein ring crossing time) and the maximum magnification µ_{max}. At the point where the background
star enters into the microlensing tube (denoted by subscript “min”), eq.(24) reads
\[
\mu_{\text{min}} = \frac{u_{\text{min}}^2 + 2}{u_{\text{min}} \sqrt{u_{\text{min}}^2 + 4}} = 1.34,
\]  
(49)
in which \(u_{\text{min}} = 1\), while at the maximum magnification (denoted by subscript “T”)
\[
\mu_{T} = \frac{u_{T}^2 + 2}{u_{T} \sqrt{u_{T}^2 + 4}}.
\]  
(50)
Alternatively, \(u_{T}\) is related with \(u_{\text{min}}\) through geometrical relation
\[
u_{\text{min}}^2 - u_{T}^2 = \left(\frac{vT}{2a_E}\right)^2,
\]  
(51)
where \(v\) is the relative velocity of the star or the microlens. These three equations give rise to the mass of microlens
\[
M = \frac{c^2}{32G} \frac{D_s}{D_d D_{ds}} \frac{\mu_{\text{min}}^2 - 1}{\sqrt{\mu_{\text{min}}^2 - 1}} \frac{v^2 T^2}{\sqrt{u_{\text{min}}^2 + 1}}.
\]  
(52)
If the MACHOs of Galactic halo are responsible for the observed events, we can take \(D_d = 10\) kpc and \(v = 220\) km/s for a numerical estimate. Utilizing the observed microlensing event duration of typically \(T \sim 30\) days and the maximum magnification of a few leads to \(M \sim 0.1 M_\odot\), i.e., sub-solar objects in the Galactic halo are likely to be the deflectors for the microlensing events of the LMC.

However, one cannot conclude from the presently detected microlensing events in the LMC that the halos of the galaxies are dominated by \(\sim 0.1\) solar mass objects. In fact, the positions \(D_d\) and relative velocities \(v\) of traverse motion of the lensing objects are two unknown factors in the determination of masses of MACHOs. Unless a statistical sample of microlensing events in the LMC is completed, it is in principle impossible to draw a decisive conclusion about the masses of the MACHOs in the Galactic halo. Gould (1994) argued that the observed optical depth toward the LMC center in the MACHO collaboration is only \(7 - 9 \times 10^{-8}\), much less than that expected from the Galactic halo made of MACHOs. Using the fact that 3 events were detected among 9.5 million monitored stars in LMC for 1.1 years, the MACHO Collaboration (Alcock et al., 1995a,c) has recently reached a similar microlensing optical depth of \(8.8^{+0.7}_{-0.5} \times 10^{-8}\), nearly an order of magnitude lower than the expected optical depth of \(\sim 5 \times 10^{-7}\). So, the halo of our Galaxy may have not been detected at all. If so, the microlensing events seen by EROS and MACHO collaboration may have arisen from the stars of the LMC disk (Wu, 1994b; Sahu, 1994) and self-lensing by a stellar disk remains to be an interesting model for further investigation (Gould, 1995).
2.3 Simulations and observations

Cosmological compact objects either bounded in galaxies or distributed randomly in the Universe are capable of magnifying temporarily the background sources like quasars, AGNs, etc., resulting in variations of their apparent luminosities. Besides the significant difference of timescales between the cosmological microlenses and the local ones like those in the Galactic halo and in the LMC [see eqs.(43) and (44)] which we have discussed in the above subsection, the optical depth to microlensing arising from the lenses at cosmological distance may be a few orders of magnitude larger than the local optical depth, depending on the content of compact objects of the Universe. For example, the optical depth to microlensing for a distant source at redshift \( z_s = 3 \sim 4 \) can be of order of unity if the Universe is composed of compact objects (see Figure 17). So, one now needs to deal with the problem that a background source is simultaneously microlensed by \( n \)-pointlike masses.

Basically, for an ensemble of compact objects as microlenses which are often assumed to be on a single lens plane, the total magnification of a pointlike source at the position \( \beta \) is the sum of the magnification \( \mu_i \) of each micro image at position \( \theta_i \) on the lens plane

\[
\mu(\beta) = \sum_i \mu_i = \sum_i \left| \det \frac{\partial \beta}{\partial \theta_i} \right|^{-1}.
\]

However, this straightforward method cannot be efficiently employed for a computation of the total magnification. In practice, one is unable to find analytically all the micro images when the number of lenses is very large, especially for extended sources. Many numerical techniques have been developed to deal with the problem of large number of microlenses including Monte-Carlo simulation (Young, 1981), the ray-shooting method (Kayser, Refsdal and Stabell, 1986; Schneider and Weiss, 1987), the Fourier method (Katz, Balbus and Paczyński, 1986), the Markoff method (Deguchi and Watson, 1988), and the parametric representation of caustics (Witt, 1990). In particular, the inverse ray-shooting method has been widely used in recent years in microlensing simulations: Light rays are traced backwards from the observer to the source plane, on which the magnification pattern is represented by the intersection of the rays. So, the number density of rays is proportional to the magnification. A typical magnification pattern produced by an ensemble of \( N \) star-like microlenses is shown in Figure 13. If a background source traverses the magnification regions due to either the motion of the source itself or the velocity dispersion of the stars associated with the lens galaxies, the apparent luminosity of the source would vary as a function of time. To most observers, this kind of variability is something like a “noise”. However, it should be noticed that the microlensing-induced variations could be very dramatic sometimes, which may explain the unusual features associated with some special objects.

It was noticed that the violently variable objects 0846+51W1 (quasar), AO 0235+164 (BL Lac object) and PKS 0537-441 (blazar) might be the results of microlensing (Nottale, 1986; Stickel, Fried and Kühr, 1988a,b; 1989). It was even speculated that the variability of apparent magnitude of quasars are partially due to microlensing rather than their intrinsic physical processes (Peacock, 1986; Kayser, Refsdal and Stabell, 1986; Schneider and Weiss, 1987). These arguments have recently been strengthened by Hawkins (1993), based on the analysis of a complete sample of \( \sim 300 \) quasars selected from their variability over 17 years (Hawkins and Véron, 1993; hereafter HV). Some typical light curves from their sample are plotted in Figure 14 for two high redshift \( (z_s = 2) \) and two low redshift \( (z_s = 0.2) \) quasars, respectively. To investigate whether these variabilities are intrinsic to quasars or due to gravitational lensing by compact objects along the lines...
of sight, an analysis of the time-varying autocorrelation function was made. It turns out that the timescale of quasar luminosity variations decreases with increasing redshift. This is inconsistent with the theory that the expansion of the Universe should cause observed timescales to increase linearly with $(1 + z)$ due to time dilation. Therefore, Hawkins concluded that the quasar variabilities cannot be intrinsic to quasars themselves, and gravitational lensing is the most possible cause, indicative of the existence of a large number of compact objects up to 10% of the critical mass density of the Universe. Nevertheless, this claim should be taken very cautiously in the sense that the observed feature of quasar variability increasing with their redshift can also be interpreted as the result of cosmic evolution or various observational limitations, e.g. the finite duration of the monitoring campaign, the finite photometric sensitivity (Alexander, 1995) and the observing wavelength dependence arising from the accretion disk model of quasar (Baganoff and Malkan, 1995). Another interesting issue is the number deficit in HV, as compared with the optically selected quasars (Boyle, Shanks and Peterson, 1988; hereafter BSP) (see Figure 15). It remains worth investigating whether the quasar number discrepancy in HV is related to microlensing or to the observational methods. Finally, the most important aspect of studying cosmological microlensing is to set constraints on the fraction of compact objects ($\Omega_c$) in the matter density of the Universe by analyzing complete samples of variability-selected sources, which is quite similar to the purpose of the ongoing MACHO experiments in our Galaxy, although one cannot separate the microlensing-induced variability from the variability intrinsic to sources. For instances, using the HV sample, Schneider (1993) obtained an upper limits of $\Omega_c < 0.1$ for compact objects with masses ranging from $10^{-3} M_\odot$ to $3 \times 10^{-2} M_\odot$. Dalcanton et al. (1994) have recently found $\Omega_c < 0.1$ in the mass range $0.01 M_\odot - 20 M_\odot$, $\Omega_c < 0.2$ for $0.001 M_\odot - 60 M_\odot$ and $\Omega_c < 1$ for $0.001 M_\odot - 300 M_\odot$ by comparing the distributions of the AGN and quasar equivalent widths of emission lines at low and high redshifts. It is expected that observations of cosmological microlensing can set more stringent limits on $\Omega_c$ in the next few years.

### 2.4 Inhomogeneous Universe

Astrophysical observations indicate that the Universe tends to be locally inhomogeneous on scales less than $\sim 100$ Mpc. Since the early 1960's there have been many studies about the influence of matter inhomogeneities on the propagation of light rays from distant sources, especially on the magnitude-redshift ($m \sim z$) relation (Zel'dovich, 1964; Bertotti, 1966; Gunn, 1967; Kantowski, 1969; Dyer and Roeder, 1972; 1973; Canizares, 1982; Nottale, 1982a,b; 1983; Vietri and Ostriker, 1983; Schneider and Weiss, 1988a,b; Isaacson and Canizares; 1989; Wu, 1990b; 1992b; Kantowski, Vaughan and Branch, 1995). Many authors addressed the question if the classical $m \sim z$ relation (Mattig, 1958) should be modified, i.e., if the $m \sim z$ relation in a locally inhomogeneous Universe differs from that in a standard Friedmann-Lemaître Universe.

In an inhomogeneous Universe the propagation of a bundle of light rays is controlled by two different effects: A beam of light traveling outside the mass clump would diverge faster because the matter density in such a region is lower than the mean matter density of the Friedmann-Lemaître Universe, whereas a beam of light passing near the clump would be sheared by the gravity of the clump, leading to the convergence of
light rays. Flux conservation requires that the divergence due to the absence of matter inside the beam be balanced on average by the convergence due to the gravitational effect of the clump, so that the luminosity distances in both the inhomogeneous Universe and the Friedmann-Lemaître Universe remain statistically equal if the size of the inhomogeneities is sufficiently small (typically, the size of galaxy) (Weinberg, 1976).

Suppose that the Universe is uniformly filled by both the intergalactic medium of density of $\tilde{\alpha}\rho$ and matter clumps of density of $(1 - \tilde{\alpha})\rho$ so that the mean mass density is the same as that in the Friedmann-Lemaître Universe. $\tilde{\alpha}$ denotes the fraction of the total mass density that is intergalactic. $\tilde{\alpha} = 1$ corresponds to a completely homogeneous Universe, i.e., the Friedmann-Lemaître model, and $\tilde{\alpha} = 0$ describes a completely inhomogeneous Universe in which all the matter is concentrated into clumps. For a beam of light rays with vertex at the observer ($z = 0$) propagating far away from any matter clumps, the shearing effect of the beam can be neglected and the propagation of light is determined by the optical scalar equation (Sachs, 1961)

$$x(x - 1)\frac{d^2\tilde{D}}{dx^2} + \left(\frac{7}{2}x - 3\right) \frac{d\tilde{D}}{dx} + \frac{3}{2} \tilde{\alpha} \tilde{D} = 0;$$

in which we use the angular diameter distance $\tilde{D}$ (in units of $c/H_0$) as the variable (Dyer and Roeder, 1973),

$$\Omega \equiv \frac{\rho}{\rho_c}$$

is the cosmological mass density parameter and

$$\rho_c \equiv \frac{3H_0^2}{8\pi G}$$

is the critical mass density of the Universe. The initial conditions of eq.(54) can be conveniently chosen to be the values of angular diameter distance and the local expansion of the Universe at the observer:

$$\tilde{D}|_{z=0} = 0; \quad \frac{d\tilde{D}}{dz}|_{z=0} = 1.$$  \hspace{1cm} (57)

Under these initial conditions the solution to eq.(54) with $\tilde{\alpha} = 1$ gives the well-known angular diameter distance in the Friedmann-Lemaître Universe

$$\tilde{D} = \frac{2|z\Omega + (\Omega - 2)(-1 + \sqrt{1 + \Omega z})|}{\Omega^2(1 + z)^2}.$$  \hspace{1cm} (58)

In the case of $\tilde{\alpha} = 0$ eq.(54) reduces to (Dyer and Roeder, 1972)

$$\tilde{D} = \int_0^z \frac{dz}{(1 + z)^3\sqrt{1 + \Omega z}}.$$  \hspace{1cm} (59)

In particular, the solution to eq.(54) for a flat Universe of $\Omega = 1$ is

$$\tilde{D} = \frac{2}{\tilde{\beta}}(1 + z)^{\tilde{\beta}/2} \left[ 1 - (1 + z)^{-\tilde{\beta}/2} \right]$$

in which $\tilde{\beta} = \sqrt{25 - 24\tilde{\alpha}}$. Seitz and Schneider (1994) have recently shown that a general solution to eq.(54) can be found for any values of $\Omega$ and $\tilde{\alpha}$ by transforming eq.(54) into the Legendre differential equation.

Luminosity distances in the Friedmann-Lemaître Universe and in the inhomogeneous Universe can be denoted by $D_{L0}$ and $D_L$, respectively, and relate with angular diameter distances by multiplying a factor of $(1 + z)^2$. Their difference reflects the divergence of the light propagation in the Universe, whilst the convergence can be described by gravitational lensing effect when a beam of light rays passes near the clumps. Hence, the parameter

$$\Delta m = 5 \log \frac{D_L}{D_{L0}} - 2.5 \log \mu$$

(61)
indicates a deviation of the actually observed apparent magnitude of a source from its theoretically expected value in a completely homogeneous Universe. Note that $\Delta m$ may have relatively large variations, i.e., a distant source may dramatically change its apparent luminosity. This arises because the magnification $\mu$ can vary in principle from unity to infinity, depending on the distance of the line of sight to the background source from the mass clumps. However, energy conservation requires that luminosity distances in a homogeneous Universe and a clumpy Universe should be statistically equal, which reads (Ehlers and Schneider, 1986)

$$\langle \mu \rangle = \left(\frac{D_L}{D_{L0}}\right)^2.$$  \hfill (62)

Thus, the mean magnification correction is

$$\langle \Delta m \rangle = 2.5[\log \langle \mu \rangle - \langle \log \mu \rangle].$$  \hfill (63)

The probability that a source at $z_s$ is gravitationally magnified by a factor of $\mu$ due to pointlike lenses within $(z_d, z_d + dz_d)$ is

$$dp = n_d (\pi \theta_d^2 D_d^2) (dr_{\text{prop}}/dz_d) dz_d$$ \hfill (64)

where $n_d$ is the number density of the lenses, $dr_{\text{prop}}$, the differential proper distance around the lens at $z_d$, and $\pi \theta_d^2 D_d^2$ is the lensing cross-section which is given by the lensing equation eq.(24):

$$\pi \theta_d^2 = 2\pi \theta_E^2 \left(\frac{\mu}{\sqrt{\mu^2 - 1}} - 1\right).$$ \hfill (65)

Assuming a uniform distribution of pointlike lenses in the Universe, i.e. $n_d = (1 + z_d)^3 n_d0$, and defining the mass density parameter of the lenses as

$$(1 - \tilde{\alpha})\Omega \equiv \frac{n_{d0}M}{\rho_c} = \frac{8\pi GMn_{d0}}{3H_0^2},$$ \hfill (66)

we have the total probability for a source at $z_s$ to be magnified by a factor of $\mu$ due to foreground compact objects

$$P_1(\mu) = 3\Omega(1 - \tilde{\alpha}) \frac{H_0}{c} \left(\frac{\mu}{\sqrt{\mu^2 - 1}} - 1\right) \int_{z_s}^{z_d} \left[ \frac{D_d}{D_L} \right] \frac{1 + z_d}{\sqrt{1 + \Omega z_d}} dz_d.$$ \hfill (67)

Fortunately, this expression can be separated into two parts

$$P_1(\mu) = f_1(\mu) \tau$$ \hfill (68)

where

$$f_1(\mu) = 2 \left(\frac{\mu}{\sqrt{\mu^2 - 1}} - 1\right);$$ \hfill (69)

$$\tau = \frac{3}{2} \Omega(1 - \tilde{\alpha}) \frac{H_0}{c} \int_{z_s}^{z_d} \left[ \frac{D_d}{D_s} \right] \frac{1 + z_d}{\sqrt{1 + \Omega z_d}} dz_d.$$ \hfill (70)

$\tau$ is the optical depth to gravitational lensing in terms of the definition of eq.(45), which describes the total number of lenses enclosed within the Einstein ring along the line of sight to the source. Figure 16 shows the “maximum” optical depth contributed by pointlike lenses that compose all the matter of the Universe, i.e., $\tilde{\alpha} = 0$. Note that eq.(70) is valid for computation of the optical depth to gravitational lensing by various compact objects with different masses. Meanwhile, $\tau$ is an indicator of the significance of multiple lenses. In the case of $\tilde{\alpha} = 0$ and $\Omega_0 = 1$ (Figure 16), a source with $z_s > 3$ may be lensed by more than one lensing object ($\tau \sim 1$). Therefore, multiple lenses need to be taken into account for the high-redshift sources if a relatively large amount of matter of the Universe is concentrated into compact objects.

Multiple lenses fall into two classes: geometrically-thin multiple lenses located on a single lens plane, as was discussed in the above subsection, and spatially discrete multiple lenses which are distributed on
different lens planes along the line of sight. For the multiple lens plane deflections, the gravitational lensing equation can be formally written as (Blandford and Narayan, 1986; Schneider, Ehlers and Falco, 1992)

\[ \eta = \frac{D_s}{D_1} \xi_1 - \sum_{i=1}^{N} D_{i,s} \alpha_i(\xi_i), \]  
\[ \xi_j = \frac{D_j}{D_i} \xi_1 - \sum_{i=1}^{j-1} D_{ij} \alpha_i(\xi_i), \]

where \( \eta \) is the position of the source in source plane, \( \alpha_i(\xi_i) \) is the deflection angle of the light ray with the impact distance \( \xi_i \) by the deflectors in the \( i \)-th lens plane, \( D_{ij} \) denotes the angular diameter distance from the \( i \)-th lens plane to the \( j \)-th lens plane and \( D_i, \) from the observer to the \( i \)-th lens plane (Note that \( j = s \) refers to the source). The total magnification of the source luminosity is finally given by the inverse of the determinant of the magnification matrix which relates to the Jacobian matrices of the mapping from the \( (i - 1) \)-th lens plane to the \( i \)-th lens plane \( (1 < i < N) \). Both numerical techniques (e.g. Schneider and Weiss, 1988a,b) and analytical methods (Seitz and Schneider, 1992, 1994) have been employed in the determination of the magnification factor as well as the magnification probability of a distant source by the deflectors in multiple lens planes, although the procedures turn to be relatively complex.

In the practical computation of magnification probability by multiple lenses, some approximations are often employed in order to avoid the above complexity. The fact that the probability \( P_1(\mu) \) by a single lens factorizes into two independent parts in eq.(68) leads to the speculation that the probability \( P_n(\mu) \) for a distant source by \( n \) lenses be written as a product (Wu, 1990a)

\[ P_n(\mu) = f_n(\mu) g_n(\tau), \]  

i.e., the \( \mu \) variable separates from the \( \tau \) variable. \( g_n(\tau) \) is a function that describes the probability of a background source being lensed by \( n \) foreground objects. Obviously, \( g_n(\tau) \) follows a Poisson distribution

\[ g_n(\tau) = \frac{\tau^n}{n!} e^{-\tau}. \]

Therefore, the magnification distribution function can be obtained by summing up \( P_n(\mu) \) over all \( n \)

\[ P(\mu) = \sum_{n=0}^{\infty} \frac{\tau^n e^{-\tau}}{n!} f_n(\mu). \]

Unfortunately, no exact expressions have been established for \( P_{\geq 2}(\mu) \). An oversimple assumption is that the total magnification \( \mu \) is the product of the individual magnification \( \mu_i \) so that \( P(\mu) \) is the convolution of \( P_i(\mu) \) of each lens (Canizares, 1982; Vietri and Ostriker, 1983; Peacock, 1986; Isaacson and Canizares, 1989; Schneider, 1993; Pei, 1993). Nevertheless, a justification for this assumption is rather hard, and it seems that this multiplication method may be questionable (Wu, 1990a).

Finally, in a single lensing approximation the mean correction of the apparent magnitude [eq.(63)] to the Mattig’s relation is (Wu, 1990b,1992b)

\[ \langle \Delta m \rangle = 1.16 \tau + O(\tau^2) \approx 0.58(1 - \tilde{\alpha})(\Omega/2)z_s^2 + \cdots, \]

in comparison with the correction found by Kantowski (1969) and Dyer and Roeder (1974) based on the Swiss-cheese model for the inhomogeneities of the Universe:

\[ \langle \Delta m \rangle \approx 1.086(1 - \tilde{\alpha})(\Omega/2)z_s^2 + \cdots. \]
These formulae can be used to statistically estimate the modification to the classical magnitude-redshift relation due to the matter inhomogeneities, and the quantity depends sharply on the matter content of compact objects $(1 - \tilde{\alpha})\Omega$ in the Universe.

3 GALAXIES AND MULTIPLE IMAGES

3.1 Multiply-imaged quasars

The most significant feature of gravitational lensing is the multiple imaging of the lensed source. The first gravitational lens system was discovered in 1979 (Walsh, Carswell and Weymann, 1979) during the identification of the optical counterpart of a radio source. This famous system 0957+561 consists of two quasar images (A and B) at the same redshift of $z_s = 1.41$ and with separation of $6''$.1. A galaxy at $z_d = 0.36$ was soon detected at the position near the image B (Adams and Boroson, 1979; Young et al., 1980; Stockton, 1980), which is believed to be the main deflector with mass of $\sim 10^{12} M_\odot$ (see Figure 17). VLBI observation of 0957+561A,B (e.g. Gorenstein et al., 1988; Garret et al., 1994) strongly confirmed the lensing origin of QSO 0957+561A,B by revealing the same radio morphology (one compact core and three jets) of the two images (Figure 18).

The criteria for the determination of a gravitationally lensed quasar system are (1) multiple images, (2) similar spectra and the same redshifts of the images and (3) the detection of intervening galaxies as lenses. A list of the accepted and the proposed lensed quasar systems compiled by Surdej and Soucail in 1993 is updated in Table 3. The proposed rather than confirmed cases arise mainly from the fact that no corresponding deflectors have been found to be responsible for the multiple images. The absence of the lensing galaxies in the direct imaging centered on some of the multiple quasars even with HST is a well-known puzzle in gravitational lensing. It remains unclear today if one should really reject these lensing candidates in which the “dark” lensing galaxies are apparently missing. The recent detections of a very faint galaxy ($B = 25.0$) and the faint cluster of galaxies at $z > 1$ (Mellier et al., 1994) associated with the double quasar 2345+007 (Fischer et al., 1994) may be promising for the future searches of the deflectors in other lensing candidate systems.

Table 3: Gravitationally-Lensed Multiple Quasars
3.2 Multiply-imaged radio sources and radio rings

Radio observation turns out to be an efficient way of finding multiple images due to its consistently high dynamical range and resolution of the maps. Indeed, the VLBI observation of double quasar 0957+561A,B (Figure 18) played an important role in the confirmation of their lensing origin (Gorenstein et al., 1988). Today, with the completeness of a large lens survey at the Very Large Array, the Cosmic Lens All-Sky Survey would provide a large number of lensing candidates for both modeling of the lensing systems and statistical study. The recent discoveries of a double lens images 1600+434 (Jackson et al., 1995) and a quadruple lens system 1608+656 (Myers et al., 1995; Snellen et al., 1995) have marked the success of this survey.

Another success of radio observation in gravitational lensing is the detection of the Einstein ring. As was illustrated in Figure 2, when the background source lies in a position behind the foreground lensing object, the multiple images merge into a ring-like image, i.e. the Einstein ring. The first radio ring (Figure 19) was discovered by Hewitt et al. in 1988, which is the image of a radio lobe at $z_s = 1.13$ lying perfectly behind a foreground galaxy at $z_d = 0.85$. Six more radio rings have been so far observed: MG 1654+1346 (Langston et al., 1989), PKS 1830−211 (Rao and Subrahmanyan, 1988; Jauncey et al., 1991), MG 1549+3047, MG 0751+2716, B 1938+666 (Lehár et al., 1993), and B 0218+357 A−B (Patnaik et al., 1993). Besides constraining the mass profile in the lensing galaxies from modeling of the radio rings, the measurement of time delay in the radio ring would be of great interest for the determination of the Hubble constant (see section 3.4).

3.3 Galaxies as Lenses

Galaxies are found to be the main deflectors for the ∼10 confirmed gravitationally-lensed multiple quasars in Table 3. Modeling each lens system based on the observational data has reproduced quite well the observed multiple images of quasars. Actually, this procedure is no more than to solve the lensing equation for different gravitational potentials, which has been extensively discussed in section 1.2. The most important issue in the study of multiply-imaged quasars, however, is the statistical properties, which raises the question if there are enough massive galaxies in the Universe to be responsible for the observed events of gravitationally-lensed quasars. Recall that the lensing galaxies have not been detected today in some of the proposed lens candidates. Moreover, statistical lensing of galaxies provides the information on how many multiply-imaged quasars would be expected to observe over the sky. By comparing the theoretical statistical predictions with the observations of multiple quasars, one can also set useful constraints on the mass density of the lensing objects in the Universe (e.g., Hewitt et al., 1986; Claeskens et al., 1993; Surdej et al., 1995) and the cosmological constant $\Lambda$ (e.g., Turner, 1990; Fukugita et al., 1992; Sasaki and Takahara, 1993; Rix et al., 1994; Kochanek, 1992;1993a,c;1995).

Adopting the simplest matter distribution, SIS, for the lensing galaxy, one can write the lensing cross-section from eq.(27) to be

$$\pi\theta^2 = \frac{\pi\theta^2_E}{(\mu - 1)^2}. \quad (78)$$

Hence, the probability that a source at $z_s$ is magnified by a factor of greater than $\mu$ due to an ensemble of galaxies within redshift $dz_d$ of $z_d$ is

$$dp = F \left( \frac{D_d D_{ds}}{D_s} \right) \frac{(1 + z_d)}{\sqrt{1 + \frac{1}{dz_d}} \frac{1}{(\mu - 1)^2}}, \quad (79)$$

where $F \equiv 16\pi^3 n_0 (c/H_0)^3 (\sigma_v/c)^4$, $n_0$ is the present comoving number density of the galaxies assumed to develop as $n = (1 + z_d)^3 n_0$, $D_d$, $D_s$ and $D_{ds}$ are the angular diameter distances in units of $(c/H_0)$,
Figure 20: The differential optical depth to gravitational lensing by an ensemble of SIS galaxies. The cosmological model is chosen to be $\Omega = 1$ and $\bar{\alpha} = 1$. $F \equiv 16\pi^3n_0(c/H_0)^3(\sigma_v/c)^4$. (cf. Turner, Ostriker and Gott, 1984)

corresponding to $D_d$, $D_s$ and $D_{ds}$, respectively. In a way similar to the lensing probability by pointlike masses, eq.(79) can be separated into two parts by utilizing the optical depth $\tau$

$$dp = d\tau \frac{1}{(\mu - 1)^2}.$$  (80)

Figure 20 shows the differential optical depth $d\tau$ to gravitational lensing for a source at different redshifts $z_s = 1, 2, 3$. The significance of the differential optical depth is that it provides a clear view of the most probable lens position for various sources. It then turns out that for quasars at their typical position of $z_s \approx 2$ the lensing galaxies locate most likely at $z_d \approx 0.5$. The total optical depth $\tau$ can be obtained analytically for some special cosmological models (Turner, Ostriker and Gott, 1984). For example, in the case of $\Omega = 1$ and $\bar{\alpha} = 1$ the total optical depth is simply

$$\tau = \frac{4F}{15} \frac{[(1 + z_s)^{1/2} - 1]^3}{(1 + z_s)^{3/2}}.$$  (81)

For a quasar at $z_s = 2$ this reads

$$\tau = 2 \times 10^{-3}(F/0.1)$$  (82)

An extensive analysis of various types of galaxies by Fukugita and Turner (1991) gives

$$F = 0.019 \pm 0.008 \quad \text{for E galaxies}$$

$$F = 0.021 \pm 0.009 \quad \text{for S0 galaxies}$$

$$F = 0.007 \pm 0.003 \quad \text{for S galaxies}$$  (83)

based on a morphological composition E:S0:S=12:19:69. Thus, for all the galaxies as lenses, $F = 0.047$. This results in a total optical depth of 0.001 for a quasar at $z_s = 2$, i.e., about 1/1000 quasars at $z_s \approx 2$ would be found to be significantly lensed by foreground galaxies. Note that the definition of the optical depth utilizes a cross-section of $\pi\theta_E^2$, so that the total optical depth in SIS corresponds to the total probability of a background source being magnified by a factor of $\mu \geq 2$. Using other models (ISC, KING, $r^{1/4}$, etc.) for the matter distribution of galaxies has yielded statistical properties which explain very well the observed frequency of lensed quasars, the observed distributions of image separations and of apparent magnitudes (Dyer, 1984; Hinshaw and Krauss, 1987; Kochanek and Blandford, 1987; Wu, 1989b; Mao, 1991; Fukugita and Turner, 1991; Kochanek, 1993a,b,c;1995; etc.).

It is worth noticing that the Hubble Space Telescope (HST) Snapshot Survey (Bahcall et al., 1992; Maoz et al., 1992; Maoz et al., 1993a,b; Falco, 1993) provides a sample of 502 luminous and high redshift quasars, among which a search for gravitationally lensed events has been made using the HST Planetary Camera. This sample is of great significance for testing the theoretical model of gravitational lensing which predicts that there should be many gravitationally lensed quasars having $\sim 1$ arcsecond (Turner, Ostriker and Gott, 1984) and even sub-arcsecond image separations (Fukugita and Turner, 1991), whilst the HST Snapshot Survey is capable of observing these small separated cases. One new candidate Q1208+1011 was found in the HST Snapshot Survey. Together with the previously known cases in the sample, the observed frequency of lensing is estimated to be between 3 and 6 out of 502 quasars. The theoretically expected frequency of lensing can be obtained by $\tau(z_s)B(m,z)$, where $B(m,z)$ is the factor by which lensed quasars are over-represented among quasars of a given magnitude $m$ because fainter quasars have been magnified to that magnitude, and its value is determined by the quasar luminosity function (Bahcall et al. 1992). As a consequence, this indeed results in a frequency compatible with the result of the HST Snapshot Survey.
but does not meet the prediction by a cosmological constant dominated Universe (Maoz et al., 1993b). The newly completed lens quasar surveys have even limited $\lambda_0 \equiv \Lambda/3H_0^2$ to $\lambda_0 < 0.66$ (Kochanek, 1995).

### 3.4 Determination of $H_0$

The present status of the uncertainty of determination of the Hubble constant $H_0$ by a factor of about two between 40 km/s/Mpc and 80 km/s/Mpc is unfortunate, which has caused the cosmic distance dispute for decades. The main problem arises from the disagreement of the “standard candles” used as the indicator of the absolute distance (see Fukugita, Hogan and Peebles, 1993 for a recent review). Indeed, it is very unlikely that the debate on $H_0$ would be settled in the next few years if the measurements are still based on the conventional “standard candle” methods. However, the recent progress of determination of $H_0$ using other techniques that are independent of the usual distance-ladder arguments may hopefully help to settle down the debate. The time delay of the multiply-imaged quasars is one of these methods which are promising for the measurement of $H_0$.

The time that the photon traverses a proper distance of $\Delta r_{\text{prop}}$ in the Universe is simply

$$\Delta t = \frac{\Delta r_{\text{prop}}}{c} = \frac{\Delta z}{H_0 (1 + z)^2 \sqrt{1 + \Omega_z}} \sim H_0^{-1}. \quad (84)$$

Therefore, we can obtain the Hubble constant by measuring the time difference $\Delta t$ for a given $\Omega$. The different optical paths to earth between the multiple images of a lensed quasar offer then a possibility of observing $\Delta t$ (the time delay) if the quasar has intrinsic luminosity variability.

Actually, even before the discovery of the gravitationally-doubled quasars it was realized that the Hubble constant $H_0$ could be measured from the time delay of two images of a single background source (Refsdal, 1964a,b, 1966). The expression eq.(84) can now be generally written as

$$H_0 \Delta t = T(\Omega, z_d, z_s) f_{\text{lens}}(\theta_{A,B}, \alpha) \quad (85)$$

where $T$ is called the cosmological correction function which is only dependent on the cosmological parameters, and $f_{\text{lens}}$ is the lens model function which is given by the matter distribution of the lens. $\theta$ and $\alpha$ are the positions of the images and the deflection of light, respectively. It has been shown (Kayser, 1986) that such a separation of the cosmological function $T$ from the lens model function $f$ is indeed possible.

There are essentially three approaches developed to compute the time delay, namely, the wavefront method (Kayser and Refsdal, 1983); (2) the integration method (Cooke and Kantowski, 1975) and (3) the scalar formulation (Schneider, 1985). In the approximation of weak gravitational field $|\phi|$, the traveling time $t$ of a photon is (Cooke and Kantowski, 1975; Borgeest, 1983)

$$ct = \int ds - \frac{2}{c^2} \int \phi ds. \quad (86)$$

The first term gives the length of the light path, and the second is the relativistic time dilatation due to the gravitational potential $\phi$ of the deflector. The integrals are performed along the photon orbit $ds$. Correspondingly, the propagation delay for light-rays from the double images of a lensed source can be split into two components: (1) the geometrical delay $\Delta t_g$ that is caused by the difference of light paths between images and (2) the potential delay $\Delta t_p$ that is induced by the difference of the gravitational field of the intervening lensing object at the image positions. A straightforward computation gives the two components (Cooke and Kantowski, 1975; Borgeest, 1983; Borgeest and Refsdal, 1984):

$$c\Delta t_g = (1 + z_d)D_d \frac{\alpha(\theta_1) + \alpha(\theta_2)}{2} \cdot (\theta_1 - \theta_2) \quad (87)$$

$$c\Delta t_p = (1 + z_d)D_d \int_{\theta_1}^{\theta_2} \alpha(\theta) \cdot d\theta. \quad (88)$$

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Finally, extracting $H_0$ from $D_d$ and combining it with $\Delta t = \Delta t_g + \Delta t_p$ yields the form of eq.(85).

A detailed lens model has been developed for the well-known lens system QSO0957+561 in order to determine the Hubble constant $H_0$ from the gradually accumulated data of the light curves of the double quasar images A,B (Young et al., 1980; Dyer and Roeder, 1980; Greenfield, Roberts and Burke, 1985; Falco, Gorenstein and Shapiro, 1985;1991; Gorenstein, Falco and Shapiro, 1988; etc.). A lensing model that is composed of three matter components has been found to reproduce the known properties of QSO0957+561A,B quite well: (1) the bright galaxy (G1) described by a King profile ($\sigma_v, r_c$); (2) a compact nucleus with mass of $M_c \sim 10^{11}h^{-1}M_\odot$ and (3) the cluster characterized by the surface mass density of a smoothly distributed mass screen (Gorenstein, Falco and Shapiro, 1988; Falco, Gorenstein and Shapiro, 1991). This lens model leads to a relatively simple expression of the Hubble constant $H_0$ from eqs.(85), (87) and (88) (see also Roberts et al., 1991)

$$H_0 = \left\{ \begin{array}{l} 97 \pm 20 \\ 90 \pm 21 \end{array} \right\} \left( \frac{\sigma_v}{390 \text{ km/s}} \right)^2 \left( \frac{1 \text{ yr}}{\Delta t} \right) \frac{\text{km/s/Mpc}}{\Omega = 0} \left\{ \begin{array}{l} \Omega = 0 \\ \Omega = 1 \end{array} \right\}. \quad (89)$$

The error estimate includes measurement error between the VLBI, VLA and optical observations, the unknown values of $\Omega$ and of the clumpiness of the Universe, the non-uniqueness of the cluster model, and errors in the detailed model of G1. The result depends weakly on the cosmological density parameter $\Omega$. It appears that two free parameters, $\sigma_v$ and $\Delta t$, control the actual evaluation of $H_0$. Motivated by the significance of determination of $H_0$ from the time delay, Rhee (1991) obtained the line-of-sight velocity dispersion of $303 \pm 50$ km/s for the bright galaxy G1 in the QSO0957+561 lens system, leaving the final work of finding $H_0$ to the measurement of the time delay between the double images. However, it should be mentioned that the simple lens model from which eq.(89) was derived is by no means unique, and more complicated mass distributions are not only possible but also actually well motivated (Bernstein, Tyson and Kochanek, 1993).

Vanderriest et al. (1989) undertook a 8-years optical photometric monitoring of the double quasar 0957+561A,B from 1980 to 1987, which contains totally 131 observations. The light curves of the image A and B are shown in Figure 21. A significant decrease in brightness around Julian Days 2445700 in A and Julian Days 2446100 in B is clearly seen, indicative of a time delay $\sim 400$ days. The original analysis of Vanderriest et al. (1989) from the cross-correlation function for the two light curves gives $\Delta t = 415$ days, while Press, Rybicki and Hewitt (1992) reached a value of $\Delta t = (537 \pm 11)$ days based on a newly developed mathematical methodology for the same data. Using additional optical data of 3.5 years coverage, Schild (1990) obtained a value of $\Delta t = 404$ days, consistent with the result of Vanderriest et al. (1990). The updated optical light curves to 1994 July (Figure 22) seems also to support $\Delta t \approx 1.1$ years (Schild and Thomson, 1995). Radio monitoring of QSO0957+561A,B with the VLA was reported later for a 10 - 11 years coverage (Roberts et al., 1991; Lehár et al., 1992). The flux curves of the two images are shown in Figure 23. In the absence of specific features in the two curves, an analysis of the cross-correlation function of two signals suggests a time delay of $513 \pm 40$ days. The very recent measurement of the time delay using the hybrid maps of QSO0957+561A,B with VLBI spanning the 6-year interval (1987-1993) yields $\Delta t \sim 1$ year (Campbell et al., 1995), while a reanalysis of the Lehár et al. (1992) observation with a refined method has found that their radio data are compatible with the result of $\sim 1$ year obtained from the optical data.
Figure 23: Radio flux curves of the A (uppercase) and B (lowercase) images of the double QSO0957+561. The letters denote the VLA configurations (P for partial configurations). (From Lehár et al., 1992)

(Pelt et al. 1995). Moreover, the updated VLA light curves of QSO0957+561A,B for 16 years coverage show a time delay of $\Delta t = 455 \pm 40$ days (Haarsma et al., 1995). Apparently, the present results of measurement of the time delay in the double images QSO0957+561A,B are controversial, and the acceptable value ranges from 404 days to 537 days.

Adopting the cosmological density parameter of $\Omega = 1$ and the velocity dispersion of $\sigma_v = 303 \pm 50$ km/s for the G1 in QSO0957+561A,B system, one finds from eq.(89)

$$H_0 = \left\{ \begin{array}{l} 48^{+16}_{-10} \\ 39^{+13}_{-6} \end{array} \right\} \text{km/s/Mpc} \quad \left\{ \begin{array}{l} \Delta t = 415 \pm 20 \text{ days (optical)} \\ \Delta t = 513 \pm 40 \text{ days (radio)} \end{array} \right\}. \quad (90)$$

Though these values still contain large uncertainties, they seem to support a low value of the Hubble constant. Further observations will be needed to find a reliable value of time delay in QSO0957+561A,B as well as to establish a reliable lensing model in order to precisely determine the value of $H_0$. The new observation of luminosity variations in the quadruple-lens system B1422+231 (Hjorth et al., 1995) and the recent detection of time delay in the Einstein ring B 0218+367 (Corbett et al., 1995) and PKS 1830-211 (van Ommeren and Preston, 1995) would be also promising for the measurement of $H_0$.

3.5 Quasar-galaxy associations

One of the important consequences of gravitational lensing, as first realized by Gott and Gunn (1974) even before the discovery of the first gravitationally-imaged quasar 0957+561A,B, is that the surface number density of quasars near foreground galaxies would be enhanced (denoted by the quasar enhancement factor $q_Q$) because the distant quasars lying behind galaxies would be magnified by the lensing effect of the galaxies and then enter into the detection limit (see also Canizares, 1981; Vietri and Ostriker, 1983; Schneider, 1986;1987a,b; Kovner, 1989; etc.). Equivalently, an overdensity of foreground galaxies around high-redshift quasars would also exist (described by the galaxy enhancement factor $q_G$) (Schneider, 1989). The statistical evidence on such quasar-galaxy associations was firstly found by Tyson (1986) and later reported by Webster et al. (1988). They all claimed a significant enhancement of galaxy surface density in the vicinity of distant quasars. Since then, the observational evidences for quasar-galaxy associations have been cumulated (Narayan, 1992). Table 4 summarizes the present status on the optically-selected quasar-galaxy associations, including two negative results. Some suggestions have been made to improve the confidence of the different results such as choosing the same objects, cross-calibrating the different observing techniques, using the same criteria, etc. Yet, large samples will be needed to further confirm the existence of quasar-galaxy associations.

The first effect of gravitational lensing is its magnification ($\mu$), which enhances the apparent brightness of background sources by an amount of $2.5 \log \mu$ in magnitude, leading to an increase of the surface number density ($\sigma$) of background sources by picking up the faint sources: $\sigma(\theta) \sim N(< m + 2.5 \log \mu)/S_0(\theta)$, where $S_0(\theta)$ is the observed area at a distance $\theta$ from the deflector. The second effect is the area distortion $S_0(\theta) + \Delta S_0(\theta)$, which arises from the light bending around the deflector (see Figure 24). This reduces the number counts by losing the sources within the dashed-line regions: $\sigma(\theta) \sim N(< m)/(S_0(\theta) + \Delta S_0(\theta))$. As a whole, the surface number density can be written as

$$\sigma(\theta) = \frac{N(< m + 2.5 \log \mu)}{S_0(\theta) + \Delta S_0(\theta)}. \quad (91)$$

Defining the enhancement factor $q_Q(\theta)$ as the ratio of the disturbed surface number density $\sigma(\theta)$ to the undisturbed one $\sigma_0(\theta) = N(< m)/S_0(\theta)$ and noticing that $\mu(\theta) = [S_0(\theta) + \Delta S_0(\theta)]/S_0(\theta)$, one has (Narayan,
Table 4: Foreground galaxy enhancement $q_G$

| authors     | QSO | selections | $\theta$ range(″) | galaxy(R) | $q_G$  |
|-------------|-----|------------|-------------------|-----------|-------|
| Crampton    | 101 | $V < 18.5$ | 0 − 6            | ~ 23      | 1.4 ± 0.5 |
|             |     | $z > 1.5$  |                   |           |       |
| Kedziora−   | 181 | $V < 18.5$ | 6 − 90           | ~ 21.5    | ~ 1   |
| Chudzer     |     | $z > 0.65$ |                   |           |       |
| Magain      | 153 | $V = 17.4$ | 0 − 3            | ~ 21      | ~ 2.8 |
|             |     | (z) = 2.3  |                   |           |       |
| Thomas      | 64  | $V < 18.5$ | 0 − 10           | ~ 22      | ~ 1.7 |
|             |     | 1 < z < 2.5|                   |           |       |
| Van Drom    | 136 | $V = 17.4$ | 3 − 13.7         | ~ 23      | ~ 1.46|
|             |     | (z) = 2.3  |                   |           |       |
| Webster     | 68  | $V < 18$   | 3 − 10           | ~ 22      | ~ 2   |
|             |     | 0.7 < z < 2.3|                 |           |       |
| Yee         | 94  | $V < 19$   | 2 − 6            | ~ 22.5    | 1.0 ± 0.3 |
|             |     | $z > 1.5$  | 2 − 10           | 1.0 ± 0.2 |
|             |     |             | 2 − 15           | 0.9 ± 0.1 |

Figure 24: Scheme of two effects of gravitational lensing on the number counts. Magnification effect enhances the apparent magnitude of the background sources, which helps to pick up the fainter sources and then leads to an increase of the total number of the sources in a flux-limited sample. The area distortion effect due to the light deflection reduces the source volume (the dashed-line region), resulting in a decrease of the total number counts.
The local quasar enhancement $q_Q$ against quasar limiting magnitude $B$ and local lensing magnification $\mu$ for the BSP and the HV counts. The solid lines correspond to the results within the BSP and HV survey limit of $B = 21$, i.e., both $B$ and $B + 2.5\log\mu$ in the computation of $q_Q$ are confined to $B < 21$, and the dotted lines are the extrapolated results by employing $N(< B)$ beyond $B = 21$. Note that in the HV counts, $q_Q \geq 1$, providing always the positive associations, while in the BSP counts $q_Q$ may be smaller than unity, leading to the “negative” associations.

$\mu$ HV survey limit of $\mu \geq 1$ and local lensing magnification around

It appears that the “local” enhancement parameter $q_Q$ at $\theta$ depends on two factors: the intrinsic number-magnitude relation of the background sources and the local lensing magnification around $\theta$. The average quasar enhancement $\bar{q}_Q(\theta_1, \theta_2)$ over an angular distance of $(\theta_1, \theta_2)$ from the foreground deflector is simply

$$\bar{q}_Q(\theta_1, \theta_2) = \frac{2}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} q_Q(\theta) \theta d\theta.$$ 

Two kinds of quasar number-magnitude relations have been thus far suggested from, respectively, the BSP survey and the HV survey. The significant difference in these two relations is that the slope of the BSP counts changes at $B \approx 19.15$ from 0.86 to 0.28 while there is no such a turnover in the range of $B \leq 21$ in the HV counts (see Figure 15). This probably arises from the different selection methods used in the two surveys. The BSP cumulative counts can be fitted by (Narayan, 1989)

$$N(< B) = 4.66 \times 10^{0.86(B-19.15)}, \quad B < 19.15;$$
$$N(< B) = -10.95 + 15.61 \times 10^{0.28(B-19.15)}, \quad B > 19.15. \quad (94)$$

This relation is valid for $z \leq 2.2$ and $B < 21$. Nevertheless, the subsequent observation (Boyle, Jones and Shanks, 1991) indicates that the above relation holds true also to $B \leq 22$. The HV cumulative counts can be fitted by (Wu, 1994a)

$$N(< B) = 6.25 \times 10^{0.51(B-19.15)}. \quad (95)$$

The local enhancements $q_Q$ are computed for these two kinds of quasar number counts and plotted against the local magnification $\mu$ in Figure 25. One should pay a special attention to the case where $B + 2.5\log\mu$ is larger than the limit of validity of the quasar number count relation. BSP and HV surveys were both restricted within $B \leq 21$. Therefore, the number-magnitude relation $N(< B + 2.5\log\mu)$ fails when $B + 2.5\log\mu > 21$, which occurs for a sufficiently large $\mu$. Strictly speaking, one cannot calculate the enhancement $q_Q$ beyond the survey limit, and the extrapolation of the solid lines in Figure 25 requires the knowledge of fainter quasar counts. Furthermore, the application of eqs.(94) and (95) for the evaluation of $q_Q$ in eq.(92) has presumed that the observed $N(< B)$ remains the same as the intrinsic counts, i.e., quasar counts have not been contaminated significantly by lensing.

A power-law number-magnitude relation with index of $\alpha$, $N(< m) \sim 10^{\alpha m}$, would lead to $q_Q = \mu^{2.5\alpha - 1}$, independent of the limiting magnitude $m$. The HV counts have $\log q_Q/\log\mu = 0.3$, which then cannot provide a large enhancement for a moderate magnification. Conversely, the two power-laws of BSP counts give rise to a relatively wide range of $q_Q$, depending on both the limiting magnitude and the magnification. Bright quasars ($B < 18$) appear to be relatively strongly associated with foreground deflectors, with a maximum enhancement at $dq_Q/d\mu = 0$. On the other hand, faint quasars ($B > 19.15$) exhibit a “negative” association with the foreground deflectors, i.e., fewer quasars would be found near the foreground galaxies than in the rest of the sky.

1989)
Adopting a SIS model for the matter distribution of a lensing galaxy, we can express the local magnification at an angular distance $\theta$ from the center of the galaxy as [eq.(27)]

$$\mu = \frac{\theta}{\theta - \theta_E}. \quad (96)$$

Furthermore, we assume the distance parameter $D_{ds}/D_s$ to be very close to unity, which approximately holds true when the foreground galaxies are at relatively low redshift while quasars are at high redshift in order to guarantee that they are not physically associated systems in the searches for quasar-galaxy associations. In this case, the Einstein radius reads

$$\theta_E = 1''33 \left( \frac{\sigma_v}{215 \text{km/s}} \right)^2. \quad (97)$$

Figure 26 shows the average enhancement $\bar{q}_Q$ versus the limiting magnitudes and the search ranges around a typical galaxy of $\sigma = 215 \text{ km/s}$, the average of the E/S0 galaxy velocity dispersions (Kochanek, 1993c), provided that the extrapolation of both BSP and HV counts to the faint magnitude ($B > 21$) is possible. In HV counts $\bar{q}_Q$ depends only on the search areas and the resulting amplitude turns to be too small to explain the reported enhancements of as large as 2 listed in Table 4. Both positive and “negative” associations are provided by BSP counts, separated in the range of $19 < B < 20$. Other important conclusions are:

1) Positive associations between foreground galaxies and background quasars would be found when one chooses the limiting magnitude of the quasar sample to be brighter than $B \approx 19$. (2) When the faint quasars ($B > 19.5$) are involved, one would expect to detect null or “negative” associations. This scenario of existence of positive/negative quasar-galaxy associations can explain the observed results (Table 4) quite well (Wu, 1994a). In fact, the search for “negative” associations between quasars and galaxies at faint magnitude can be used as the ultimate test for whether or not the quasar-galaxy associations stem from the effect of gravitational lensing. There are no other mechanism known thus far that can result in the negative associations, i.e., the surface number density of distant quasars around foreground galaxies is smaller than the mean quasar density in the rest of the sky.

For foreground galaxies with different luminosities which follow the Schechter function $\phi(L)dL = \phi^*(L/L_*)\nu \exp(-L/L_*)d(L/L_*)$, we can find the average quasar enhancement over a search range of $(\theta_1, \theta_2)$ from

$$\langle \bar{q}_Q(\theta_1, \theta_2) \rangle = \int_0^{\infty} 4\pi D_{ds}^2(1+z_d)^3 \langle q_N \rangle \ dr_{prop,z_d}, \quad (99)$$

where $i$ and $\gamma_i$ represent, respectively, the $i$-th type and composition of galaxies, and the minimum galaxy luminosity $L_{min}$ is related to the galaxy limiting magnitude in the observation of quasar-galaxy associations. The galaxy luminosity $L$ can be converted into the velocity dispersion $\sigma_v$ through the empirical formula such as the Faber-Jackson relation for early-type galaxies (E/S0) $L/L_* = (\sigma_v/\sigma_*)^4$ or the Tully-Fisher relation for spiral galaxies (S) $L/L_* = (\sigma_v/\sigma_*)^{2.6}$. The parameters $L*$ (or $\sigma_*$), $\phi^*$ and $\nu$ have been observationally determined for different galaxies (see Fukugita and Turner, 1991). Finally, the spatial distribution of galaxies needs to be taken into account. Assuming a constant comoving number density of galaxies, we have

$$\langle \bar{q}_Q(\theta_1, \theta_2) \rangle = \frac{\int_0^{\infty} 4\pi D_{ds}^2(1+z_d)^3 \langle q_N \rangle \ dr_{prop,z_d}}{\int_0^{\infty} 4\pi D_{ds}^2(1+z_d)^3 \langle N \rangle \ dr_{prop,z_d}}.$$
in which \( dr_{\text{prop},zd} = (c/H_0)dz_0/[(1 + z_0)^2\sqrt{1 + \Omega z_0}] \). This theoretical expectation can be used straightforwardly to compare with the observations (Wu, Zhu & Fang, 1995).

It should be realized that the quasar-galaxy associations are the statistical results and thereby, statistical lensing should be involved. An exact treatment of this question is to convolve the magnification probability \( P(\mu) \) by foreground galaxies with the intrinsic quasar number counts \( N(< m) \) (Schneider, 1989). The difficulty is that \( P(\mu) \) should be artificially truncated at the faint end of magnification in order to perform the integration \( \int N(< m)dP(\mu) \).

### 4 CLUSTERS OF GALAXIES AND ARCLIKE IMAGES

#### 4.1 Giant arcs and arclets

The arclike image associated with Abell cluster 370 was first detected by Hoag in 1981. However, this blue arc had not been recognized to be the image of a distant galaxy gravitationally lensed by the cluster until a few years later when two groups of astronomers independently announced their convincing evidences of the existence of this peculiar feature in the Universe (Soucail et al., 1987a; Lynds and Petrosian, 1986), followed by Paczyński’s (1987) lensing interpretation. Only two giant arcs (A370 and Cl2244-02, see Figure 27) were known at that time and other explanations remained also possible. The crucial point of interpretation of the arcs as gravitationally imaged background sources rather than some peculiar features physically associated with the clusters is the measurements of redshift of the two giant arcs. They do show higher redshifts than those of their associated clusters: The giant arc in Abell 370 has a redshift of 0.725 (Soucail et al., 1987b; Miller and Goodrich, 1988), in comparison with the redshift of 0.374 for Abell 370 itself, and the arc in Cl 2244-02 has probably an even higher redshift of 2.237 (Mellier et al., 1991) while its associated cluster is only at \( z_d = 0.336 \). These measurements have strongly confirmed the lensing origin of arclike images seen in the cores of rich galaxy clusters. Up to now giant arcs and arclets have been detected in about 30 clusters of galaxies (Table 5) and this number is still increasing dramatically. In particular, the high X-ray luminosity clusters in the EMSS sample turn to be the very efficient deflectors of producing arcs and about 14 arcs have thus far been seen in a subsample of 41 EMSS clusters with \( L_x \geq 2 \times 10^{44}\text{erg/s} \) and \( z_d \geq 0.15 \) (Hammer et al., 1993; Le Fèvre et al., 1994; Gioia and Luppino, 1994; Hammer, 1995; Luppino et al., 1995). Note that there are several multiple arc systems.

Wu and Hammer (1993) classified the elongated images using two parameters: axial ratio \( (L/W, \text{i.e., length/width}) \) and apparent magnitude \( (B) \). The “giant” arcs refer to those images whose axial ratios are greater than 10, i.e., \( L/W \geq 10 \), and mini-arcs or arclets have \( L/W \leq 3 \). The rest arclike images in between \( (3 < L/W < 10) \) are called medium arcs. Arc brightness is represented by its \( B \) magnitude so that the “luminous” arcs have \( B \leq 22.5 \). In the updated list of arclike images of Table 5 there are totally \( \sim 10 \) giant luminous arcs, while arclets are numerous but usually very faint \( (B \sim 26) \).
4.2 Clusters as lenses

Although most of the arclike images remain to be spatially unresolved in width today, the colours and spectra of the arcs are compatible with those of local sub-$L_*$ spiral galaxies, indicating that they are probable spirals at relatively high redshift $z_s \sim 1$. The foreground clusters often have high X-ray luminosity ($L_x > 10^{44}$ erg/s) and/or large velocity dispersion ($\sigma_v > 1000$ km/s), which are then massive enough to act as strong lenses for the background galaxies.

To demonstrate how a rich galaxy cluster at intermediate redshift gravitationally distorts background galaxies, a simulation is made based on the current knowledge of dynamical and spatial properties of galaxies and rich clusters of galaxies. SIS is adopted for the matter distribution of a galaxy cluster with $\sigma_v = 1000$ km/s at $z_d = 0.25$. The luminosity function established by Broadhurst, Ellis and Shanks (1989) in deep redshift survey is employed for the distribution of background spiral galaxies. The luminous area of each galaxy is taken to be a circular disk of radius of $R$ which is assumed to follow the relation $R = R_*(L/L_*)^{1/2}$, where $R_*$ is the characteristic radius corresponding to a $L_*$-galaxy. In the actual simulation, $R = 10^{(-17.16-M)/5}$ kpc (Freeman, 1970). Figure 28(a) shows the unperturbed background galaxies projected on the sky in a field of $1' \times 1'$, in which the orientations of the disk galaxies are randomly placed in space and the population of the galaxies is presumed not to evolve cosmologically. Moreover, we truncate the redshift of the galaxies at $z_s = 1.25$ due to the failure of the $K$-correction in the adopted luminosity function (Broadhurst, Ellis and Shanks, 1989). Figure 28(b) illustrates the same field with a galaxy cluster at the center. All the images brighter than $B = 23$ are selected by taking the magnification effect into account. It appears that galaxies in the field have been strongly elongated around the cluster, indicating that distant rich clusters of galaxies can indeed act as strong lenses and produce the images of giant arcs and arclets.

An interesting issue is the total number of mini-arcs and medium arcs with respect to the number of giant arcs seen up to the same flux threshold in a galaxy cluster. A statistical distribution of arc number appearing in Figure 28(b) against axial ratio $L/W$ is plotted in Figure 29, in which the normalization is made at $L/W = 10$. It turns out that for one observed giant arc, there should be another one medium arc of $L/W \approx 6$ and two arclets of $L/W \approx 3$. The more careful statistical investigations have reached essentially similar conclusion (Grossman and Narayan, 1988; Wu and Hammer, 1993). Unfortunately, this theoretically predicted axial ratio distribution has not been seen in the recent arc survey with the subsample of EMSS clusters. Among the 14 arcs found in the 41 EMSS clusters, 9 are giant arcs (Hammer et al., 1995; Luppino et al., 1995). It is believed that the asymmetrical matter distribution of the arc clusters may account for the discrepancy (Bartelmann and Weiss, 1994; Bartelmann, Steinmetz and Weiss, 1995).

Modeling the known giant luminous arcs, even arclets, associated with clusters of galaxies turns to be very successful. A simple elliptical or bimodal potential for the arc cluster can provide the major properties compatible with the observed arcs. Indeed, any type of elongated configurations including the straight arcs and the radial arcs has been well reproduced based on elongated gravitational potentials tracing the luminous matter distributions (Pello et al., 1991; Mellier, Fort and Kneib 1993). An example of modeling the arcs in the well-known Abell cluster 370 is shown in Figure 30. This arc-cluster system has been extensively studied by a number of authors (e.g. Hammer and Rigaut, 1989; Grossman and Narayan, 1989), in particularly by
Figure 30: Modeling the giant luminous arc and arclets in Abell 370. Upper left: The direct CCD image of the core of Abell 370; Upper right: Image construction. Solid lines and dashed lines are, respectively, the caustics and critical lines (the inner one is for $z_s = 0.725$ and the outer one, $z_s = 0.895$). Dot-dashed lines are the core radius, ellipticity and orientation of the two matter profiles; Low left: Contours of the surface density; Low right: Reconstruction of the source positions (see the upper right panel for the corresponding part). (from Kneib et al., 1993)

the Toulouse Group (e.g. Kneib et al., 1993; Soucail and Mellier, 1993).

4.3 Cluster matter distributions from arcs

Arclike images are robust matter estimators of clusters of galaxies, which is actually the most important issue of studying arclike images today. From the general expression of the lensing equation [eq.(12)] for a spherical matter distribution [eq.(8)], we can write out the projected mass along the line of sight within the arc position $\theta$ (in arcseconds) to be

$$m_g(\theta) = 7.37 \times 10^{11} (\theta - \beta)  \frac{\hat{D}_d \hat{D}_s}{D_s} M_\odot h_{50}^{-1},$$

where $\beta$ is the alignment parameter of background galaxy in arcseconds. This parameter, however, is unmeasurable in practice. Nevertheless, if we further assume a SIS model for the matter distribution of cluster of galaxies and a uniform circular disk for background source, the maximum width ($W$) of the arclike image will be the same as the size of the background galaxy and $\beta$ will satisfies the following approximate geometrical relation

$$\beta = \frac{L}{2(L/W) \sin(L/2\theta D_d)}.$$  (101)

Therefore, the gravitational mass contained in the central core ($< \theta$) of an arc cluster is obtained without any assumptions about the dynamical state of the system. For most of the arc-cluster systems, the gravitational mass is typically $\sim 10^{14} M_\odot$ in the core of a galaxy cluster. The significance of this method is that it provides an independent way of calculating the masses of clusters of galaxies, which can be compared directly with the masses estimated from the dynamical analysis based on the virial theorem. Recall that the latter presumes an hydrostatic equilibrium of both the hot intracluster gas and the galaxies with the binding cluster potential (Cowie, Henriksen and Mushotzky, 1987). This hypothesis, unfortunately, has not been verified using other astrophysical means.

Assuming that both the X-ray gas and the galaxies are in hydrostatic equilibrium with the binding gravitational potential of a spherical galaxy cluster, one can obtain the virial mass of the cluster within radius of $r$ through

$$M_v(r) = -\frac{kT r}{G \mu_p m_p} \left( \frac{d \ln n}{d \ln r} + \frac{d \ln T}{d \ln r} \right),$$  (102)

where $\mu_p$ is the mean particle weight in unit of the proton mass $m_p$, $n$ and $T$ are the gas density and temperature, respectively, which can be found by inverting the observed X-ray surface brightness profile, namely, the $\beta$ model

$$S(\theta) = S_0 \left[ 1 + (\theta/\theta_c)^2 \right]^{1/2-3\beta}$$  (103)

with a core radius of $\theta_c$ (or $r_c$ in linear size). The isothermal gas distribution has been found to be consistent with the X-ray observations of galaxy clusters, leading to $d \ln T/d \ln r = 0$. So, the projected virial mass $m_v(\theta)$ from eq.(102) within a radius of $\theta$ on the cluster plane is (Wu, 1994c)

$$m_v(\theta) = 1.14 \times 10^{14} \hat{n}(\theta) \left( \frac{kT}{\text{keV}} \right) \left( \frac{r_c}{\text{Mpc}} \right) M_\odot;$$  (104)
remains nearly unchanged for \( R \) led to \( \Omega_b \) representative of the matter distribution of the Universe. The baryon catastrophe arises because the virial mass \( m_v \) from arclike images would reduce the baryon fraction \( \Omega_b \) (Gioia et al., 1990; Henry et al., 1992) provide a Schechter luminosity function at low redshift (of clusters of galaxies from the Einstein Observatory and EXOSAT (Edge et al., 1990) as well as the EMSS (Huchra et al., 1986)). Unfortunately, all these distributions have not been very well determined from observations. The X-ray observations provide the cluster velocity dispersion distribution, the length scale distribution and the spatial distribution. Unfortunately, all these distributions have not been very well determined from observations. The X-ray observations provide the cluster velocity dispersion distribution, the length scale distribution and the spatial distribution.

Another useful constraint on the matter distribution of galaxy clusters is provided by the study of statistical lensing, which attempts to statistically investigate the possible form of matter distributions of galaxy clusters as a whole using the properties of giant arcs and/or arclets, such as the total number, the width and the axial ratio of arcs (Hammer, 1991; Wu and Hammer, 1993; Miralda-Escudé, 1993a,b; Grossman and Saha, 1994). This procedure is actually a convolution of the magnification probability of galaxy clusters with the differential probability that a source at \( z_s \) is magnified by a factor within \( d\mu \) of \( \mu \) due to a galaxy cluster at \( z_d \) is proportional to the lensing cross-section \( 2\pi D_s^2D_l d\beta \) [eq.(11)]. To find the total magnification probability \( P(z_s, \mu) \) even for a spherical matter distribution of cluster of galaxies, one still needs to know the cluster velocity dispersion distribution, the length scale distribution and the spatial distribution. Unfortunately, all these distributions have not been very well determined from observations. The X-ray observations of clusters of galaxies from the Einstein Observatory and EXOSAT (Edge et al., 1990) as well as the EMSS (Gioia et al., 1990; Henry et al., 1992) provide a Schechter luminosity function at low redshift (\( z_d < 0.2 \)) and a power-law luminosity function \( \phi(L_{44})dL_x = KL_{44}^\alpha dL_x \) [\( L_{44} \equiv (L_x/10^{44} \text{ ergs s}^{-1}) \)] at intermediate redshift ranging from 0.14 to 0.6, showing a significant evolution of X-ray luminous clusters with cosmic epoch, where

\[
\bar{m}(\theta) = \frac{R_0^3}{1 + R_0^3} = \int_{\theta/\theta_c}^{R_0} x \sqrt{x^2 - \frac{\theta^2}{\theta_c^2}} \frac{3 + x^2}{(1 + x^2)^2} dx. \quad (105)
\]

Here \( R_0 = R/r_c \), and \( R \) is the physical size of the cluster. The numerical computations show that \( m(\theta) \) remains nearly unchanged for \( R \) ranging from 3 Mpc to 100 Mpc.

A comparison of the gravitational masses estimated from arclike images [eq.(100)] with the masses derived from hydrostatic equilibrium [eq.(104)] is shown in Table 6 for four clusters of galaxies in which both arclike images are detected and the X-ray data are available (Henry et al., 1982; Gioia and Luppino, 1994): Abell 370 (A5), MS 1006.0+1202 (arc 4), MS 1008.1-1224 (arc 2) and MS 1910.5+6736. It appears that there exists a significant difference between the virial masses and the gravitational masses from arclike images in all the four systems.

Three groups have independently announced similar results for a total of 8 arc-cluster systems (Wu, 1994c; Fahlman et al., 1994; Miralda-Escudé and Babul, 1995), showing that the virial equilibrium has underestimated the total gravitational masses of clusters of galaxies by a factor of at least 2.5 up to the arc positions. There are three main possibilities that may account for the mass discrepancy: (1) the hot gas in clusters of galaxies may be meanwhile supported by a non-thermal pressure such as magnetic field (Loeb and Mao, 1994; Enßlin et al., 1995); (2) cluster matter distributions may be highly prolate with the long axis along the line of sight (Miralda-Escudé and Babul, 1995); And (3) clusters of galaxies cannot be considered to be the well relaxed virialized systems (Wu, 1994c). Nonetheless, the third possibility, if true, may offer an important clue to resolving the “baryon catastrophe” on scale of clusters of galaxies (White et al., 1993).

The cosmic baryon fraction is the ratio of baryonic matter \( M_b \) (X-ray gas + galaxies) to the total mass \( M \) (baryon + non-baryon) of clusters of galaxies: \( \Omega_b \equiv M_b/M \), provided that clusters of galaxies are representative of the matter distribution of the Universe. The baryon catastrophe arises because the virial mass \( M_v \) derived from eq.(102) is used as the measurement of the total mass \( M \) of cluster of galaxies, which has led to \( \Omega_b \) being 3 to 10 times larger than the prediction of the Big Bang Nucleosynthesis and the standard inflation cosmological model. Now, replacing the virial mass \( m_v \) by the gravitational mass \( m_g \) estimated from arclike images would reduce the baryon fraction \( \Omega_b \) by a corresponding factor of at least 2.5 over the region of the typical core radius of cluster. As a consequence, the “\( \Omega_b \) discrepancy problem” might vanish.

### 4.4 Cluster matter distributions from statistical lensing

Another useful constraint on the matter distribution of galaxy clusters is provided by the study of statistical lensing, which attempts to statistically investigate the possible form of matter distributions of galaxy clusters as a whole using the properties of giant arcs and/or arclets, such as the total number, the width and the axial ratio of arcs (Hammer, 1991; Wu and Hammer, 1993; Miralda-Escudé, 1993a,b; Grossman and Saha, 1994). This procedure is actually a convolution of the magnification probability of galaxy clusters with the distribution of background galaxies.

The differential probability that a source at \( z_s \) is magnified by a factor within \( d\mu \) of \( \mu \) due to a galaxy cluster at \( z_d \) is proportional to the lensing cross-section \( 2\pi D_s^2D_l d\beta \) [eq.(11)]. To find the total magnification probability \( P(z_s, \mu) \) even for a spherical matter distribution of cluster of galaxies, one still needs to know the cluster velocity dispersion distribution, the length scale distribution and the spatial distribution. Unfortunately, all these distributions have not been very well determined from observations. The X-ray observations of clusters of galaxies from the Einstein Observatory and EXOSAT (Edge et al., 1990) as well as the EMSS (Gioia et al., 1990; Henry et al., 1992) provide a Schechter luminosity function at low redshift (\( z_d < 0.2 \)) and a power-law luminosity function \( \phi(L_{44})dL_x = KL_{44}^\alpha dL_x \) [\( L_{44} \equiv (L_x/10^{44} \text{ ergs s}^{-1}) \)] at intermediate redshift ranging from 0.14 to 0.6, showing a significant evolution of X-ray luminous clusters with cosmic epoch, where

### Table 6: Four arc-cluster systems and their masses

| Cluster          | Arcs (1) | Arcs (2) | Arcs (3) | Arcs (4) |
|------------------|----------|----------|----------|----------|
| Abell 370 (A5)   |          |          |          |          |
| MS 1006.0+1202   |          |          |          |          |
| MS 1008.1-1224   |          |          |          |          |
| MS 1910.5+6736   |          |          |          |          |

\[ \tilde{m}(\theta) = \frac{R_0^3}{1 + R_0^3} \int_{\theta/\theta_c}^{R_0} x \sqrt{x^2 - \frac{\theta^2}{\theta_c^2}} \frac{3 + x^2}{(1 + x^2)^2} dx. \quad (105) \]
clusters of galaxies using statistical lensing. Convolving of mass density are clearly seen. This property opens then a possibility to test the matter distribution of galaxy clusters. Furthermore, the extension of the background source is neglected.

The variable $r_c$ distribution of Jones and Forman (1984) is adopted for ISC and KING models and a constant $r_c$ of $6h_{50}^{-1}$ Mpc is used for the $r^{1/4}$ law. A no-evolution scenario is assumed for the number distribution of galaxy clusters. $K$ takes different values at different redshift shells and has units of Mpc$^{-3}$[L$_{44}$]$^{-1}$, and $\nu$ is the power-law index. The X-ray luminosity distribution can be converted into the velocity dispersion distribution through the correlation between X-ray luminosity $L_x$ and velocity dispersion $\sigma_v$ (Quintana and Melnick, 1982; Wu and Hammer, 1993):

$$L_x = 10^{32.71} \sigma_v^{3.94} \text{ergs s}^{-1}.$$  \hspace{1cm} (106)

For the distribution of length scale (core radius $r_c$, effective radius $r_e$, etc.), optical observations exhibit a constant core radius of $r_c = 0.25h_{50}^{-1}$ Mpc for the King model (Bahcall, 1975:1977) and a constant effective radius of $r_e = 6h_{50}^{-1}$ Mpc for the $r^{1/4}$ law (Bears and Tonry, 1986), while X-ray observations provide a distribution of core radius significantly different from the constant core (Jones and Forman, 1984):

$$p_c(\log r_c)d\log r_c = \frac{1}{0.32\sqrt{2\pi}} \exp[-0.5(\log r_c + 0.89^2)/0.32^2] d\log r_c,$$  \hspace{1cm} (107)

which leads to an average core radius of $0.17h_{50}^{-1}$ Mpc. Both the constant and variable scale length distributions will be adopted in the computation of $P(z_s, \mu)$. As for the number density of galaxy clusters, both no-evolution and evolution models from the X-ray data will be used and the respective results will be compared.

The total magnification probability $P(z_s, \mu)$ for a source at $z_s = 2$ by intervening clusters of galaxies with different matter distributions is shown in Figure 31. Significant differences in $P(z_s, \mu)$ due to different models of mass density are clearly seen. This property opens then a possibility to test the matter distribution of clusters of galaxies using statistical lensing. Convolving $P(z_s, \mu)$ with the distribution of background galaxies would yield the number of lensed galaxies with magnification greater than $\mu$. However, $\mu$ measures only the magnification of the apparent luminosity rather than the geometrical features of the lensed source. The relation between the magnification $\mu$ and the axial ratio $L/W$ of the elongated image of a background circular galaxy with radius of $R_0$ by a spherical lens can be approximately obtained to be (Wu and Hammer, 1993)

$$\frac{L}{W} = \mu \left[1 + D_0 \frac{m(\theta_0)}{\theta_0^2} - 2\pi D_0 \Sigma(\theta_0)\right]^2 \times \frac{\beta_0}{R_0} \sin^{-1} \frac{R_0}{\beta_0},$$  \hspace{1cm} (108)

in which we have presumed that $W/\theta_0 \ll 1$, i.e., the arc width ($W$) is much smaller than the arc distance ($\theta_0$) from the center of its associated cluster (Apparently, many detected arcs can meet this condition), where $\beta_0$ and $\theta_0$ are the alignment parameter and the corresponding image separation for the center of the background galaxy, and $D_0 = (4G/c^2)(D_d D_s/D_s)$. In particular, as giant arcs trace the critical line or the Einstein ring $\theta_E$, $\theta_0$ can be approximately replaced by $\theta_E$. Furthermore, one can expand $\sin^{-1}(R_0/\beta_0)$ in $(R_0/\beta_0)$ if the complete ring images are excluded in the statistics due to their rareness. Then, eq.(108) becomes

$$\frac{L}{W} = 4\mu(1 - K)^2 \left(1 - \frac{1}{6} \frac{R_0^2}{\theta_0^2} + \cdots\right),$$  \hspace{1cm} (109)

where $K = \pi \theta_E^2 D_d^2 \Sigma(\theta_E)/m(\theta_E)$. Numerical computations using the three spherical models in Table 1 show that the term $(1/6)(R_0/\beta_0)^2$ becomes important only if the radius of the circular galaxy is larger than 8 kpc and $\mu > 30$, while most of the known giant arcs have $\mu \sim 10$. Therefore, neglecting the source extension cannot cause too serious problems in the evaluation of $P(z_s, \mu)$. Thus,

$$\mu = \frac{L/W}{4(1 - K)^2},$$  \hspace{1cm} (110)
This expression relates magnification with axial ratio of the distorted image and is also very useful to estimate the image magnification from its geometrical configuration. For example, for the SIS a simple calculation shows \( K = 1/2 \), which then leads to \( \mu = L/W \). That is to say, the magnification remains the same as the axial ratio for the image produced by SIS and thereby, the magnification probability \( P(z_s, \mu) \) is the probability that a source is elongated by a factor of greater than \( \mu = L/W \). For other spherical mass density models one can find the probability \( P(z_s, \mu, L/W) \) from eq.(110) that a source at \( z_s \) is magnified by a factor of \( \mu \) and elongated by a factor of \( L/W \).

The expected number of giant arcs is finally obtained by convolving the probability \( P(z_s, \mu, L/W) \) with the Schechter luminosity function of background galaxies which is taken from the survey of Broadhurst, Ellis and Shanks (1989). Table 7 gives the result of Wu and Hammer (1993) for the totally expected number of giant luminous arcs with \( B \leq 22.5 \) and \( L/W \geq 10 \) within \( z_s \leq 1.25 \) over the whole sky. Four mass density models are used for the matter distribution of clusters of galaxies which are restricted within the range of \( 0.15 \leq z_d \leq 0.6 \). Both the no-evolution [the Edge et al. (1990) local X-ray Schechter luminosity function] and the evolution model [the Henry et al. (1992) X-ray power-law luminosity function] of galaxy clusters are used. As a consequence of the significant difference in \( P(z_s, \mu) \) arising from the various matter distributions (see Figure 31), the resulting number of giant luminous arcs shows remarkable differences. The strong evolutionary scenario of the X-ray luminous clusters yields fewer clusters than does the no-evolution model, giving rise to fewer giant luminous arcs predicted from the evolution model of clusters. Apparently, very rich clusters of \( \sigma_v \geq 1300 \) km/s have a relatively higher frequency of producing giant arcs than the rich clusters of \( \sigma_v \geq 800 \) km/s. A further computation using the clusters of \( \sigma_v \geq 1300 \) km/s within the redshift interval \( 0.15 \leq z_d \leq 0.4 \) gives a frequency of detecting giant luminous arcs to be as high as 30–40%. Nevertheless, the model ISC in Table 7 has underestimated the number of giant luminous arcs by a factor of at least 2 as compared with the known giant luminous arcs unless the core radii of clusters are reduced significantly, indicative of a smaller core for the total matter distribution than the X-ray gas. In the extreme case of the zero-core radius, SIS predicts a number of giant luminous arcs that marginally reconciles with the observed arcs in the strong evolution scenario of clusters. It is then concluded that the well-known form of ISC, which is found to be the best-fit to the X-ray luminosity distribution in clusters of galaxies, cannot account for the giant luminous arcs if dark matter has the same core radius as the observed one in X-ray. This implies a more compact dark matter distribution in the central core of the galaxy cluster.

The statistical study of arc widths has also reached a similar conclusion (Grossman and Saha, 1994) that cluster mass density profiles must have core radii at least as small as \( \theta_c/\theta_E \leq 0.1 \) in ISC. Interestingly,
the dynamical analysis of a set of 12 clusters of galaxies using the X-ray data from the Einstein Imaging Proportional Counter has found that dark matter appears more peaked in the cluster centers than the X-ray gas and has core radii of only $50 - 100$ kpc (Gerbal et al., 1992; Durret et al., 1994). Although this dynamical method has given a result that agrees with the lensing estimate, it is not so clear if the dynamical masses derived from the assumption of hydrostatic equilibrium for galaxy clusters are reliable values in view of the argument in section 4.3.

It must be pointed out that the above conclusions drawn from statistical lensing by clusters of galaxies should be taken to be very preliminary. Wu (1993d) has discussed four parameters in the statistical lensing which may cause some large uncertainties in the predictions of arc properties due to their limited knowledge from today’s observations: length scale, velocity dispersion, extended source and evolutionary effect. (1) Core or effective radii: Both optical and X-ray data are not sufficient to constrain the distribution of core/effective radii of galaxy clusters. The optical core radii claimed by Bahcall (1975, 1977) are two times smaller than those found by Dressler (1978), whilst the distribution of X-ray core radii eq.(107) shows somewhat larger scatters. Alternatively, the constant effective radius of $6h_{50}^{-1}$ Mpc adopted for the $r^{1/4}$ law is apparently not a good value for different rich clusters of galaxies. So, the arc predictions based on the poor data of length scale of clusters of galaxies may have large uncertainties. (2) Velocity dispersion: Magnification probability for lensing is very sensitive to this parameter. But the average line of sight velocity dispersion along the radius of a galaxy cluster is a variable rather than a constant except for SIS (Wu, 1993b; Kochanek, 1993c). The above statistical lensing takes into account only the asymptotic value of $\sigma_v$, either close to the center or at large radius, which may result in a significant variation in lensing probability. (3) Extended sources: The above point source approximation for background galaxies would overestimate the magnification probability $P(z_s, \mu)$. Recall that a maximum magnification $\mu_{\text{max}}$ instead of infinity is reached for an extended source (see section 1.3). Moreover, the consideration of source extensions might solve the puzzle of too many arclets relative to the giant arcs predicted from statistical lensing (Bergmann and Petrosian, 1993). (4) Evolutionary effect: The strong evolution of galaxy clusters with cosmic epoch would lead to a number decrease of clusters with redshift, providing much fewer lenses at high redshift for background galaxies. Conversely, the no-evolution model is able to produce a relatively larger number of giant luminous arcs in the clusters of galaxies at high redshift. The observed giant luminous arcs then open a possibility of testing the evolution model of distant clusters of galaxies (Wu, 1993a; Bartelmann and Weiss, 1994). Nevertheless, the poorly established evolutionary model of clusters of galaxies may affect the present computation of lensing probability.

4.5 Cluster matter distribution from weak lensing

The parametric likelihood method of determining the cluster matter distribution employed in sections 4.3 and 4.4 is suitable for modeling the giant arcs/arclets, for constraining the length and shape of global cluster potential wells and for investigating the relative redshift distribution of the background objects. The method assumes a priori a density profile of the cluster and determines the most likely values of the model parameters using the properties of the observed arcs/arclets. It appears that the current data of the distorted images are still insufficient to discriminate between the various models. It is, therefore, most desirable that one can directly derive the cluster matter distribution with a parameter-free (or non-parametric) method.

Tyson, Valdes and Wenk (1990) firstly detected the coherent distortion (weak lensing) of faint blue galaxies behind two clusters of galaxies (A1689 and CL 1409+52). They used the alignment statistics to extract the lensing signal which is characterized by an ellipticity (Valdes, Tyson and Jarvis, 1983)

\[
e = \frac{a^2 - b^2}{a^2 + b^2},
\]

where $a$ and $b$ are the semimajor and semiminor axes of the principal axis transformed moments. This yields a positive value of the net alignment for a population of background galaxies around a foreground cluster, in comparison with the zero result for a random population of galaxies. Kaiser and Squires (1993) extended this idea to the outer parts of clusters and to statistical cluster samples, and developed a powerful technique to reconstruct the cluster surface mass density $\Sigma(\theta)$ through the measured distortion $e$, which
is totally independent of the presumed density profile for clusters. This non-parametric method has been successfully used in the mapping of two-dimensional mass distributions in several clusters and is now being further developed for its wide applications in both weak and strong lensing situations (Seitz and Schneider, 1994; Kaiser, 1995; Bartelmann, 1995; Bartelmann and Narayan, 1995; etc.).

The image shapes are characterized by the quadrupole moments (Valdes, Tyson and Jarvis, 1983)

\[
Q_{ij} = \frac{\int d^2 \theta \theta_i I(\theta)}{\int d^2 \theta I(\theta)}
\]

\[
Q'_{ij} = \frac{\int d^2 \theta \theta_i I'(\theta)}{\int d^2 \theta I'(\theta)}
\]

where \( I(\theta) \) and \( I'(\theta) \) are the intrinsic and observed surface brightness distributions of a background source [see also eq.(17)], and the angles are measured from the centroid of the image. Assuming that the source is relatively small so that the magnification matrix \( A \) is constant over the images, we have the following transformation between the quadrupole matrix of source and image according to eq.(17)

\[
Q_{ij} = A_{ik} A_{j\ell} Q'_{k\ell},
\]

or

\[
Q = AQ',
\]

where \( A \) is defined by

\[
A = \begin{bmatrix}
1 - \kappa - \gamma_1 & -\gamma_2 \\
-\gamma_2 & 1 - \kappa + \gamma_1
\end{bmatrix}.
\]

By analogy with eq.(111), we define the intrinsic ellipticity parameter \( e \) of the background source and the observed ellipticity parameter \( e' \) of the image as

\[
e = (e_1, e_2) = \left\{ \frac{Q_{11} - Q_{22}}{Q_{11} + Q_{22}}, \frac{2Q_{12}}{Q_{11} + Q_{22}} \right\};
\]

\[
e' = (e'_1, e'_2) = \left\{ \frac{Q'_{11} - Q'_{22}}{Q'_{11} + Q'_{22}}, \frac{2Q'_{12}}{Q'_{11} + Q'_{22}} \right\},
\]

which can also be denoted in their complex forms: \( e = e_1 + ie_2 \) and \( e' = e'_1 + ie'_2 \).

In the limit of weak lensing, one can use the linear approximation. Under the transformation eq.(115), eqs.(117) and (118) reduce to

\[
e'_1 = e_1 + (\psi_{11} - \psi_{22})(1 - e_1^2) - 2\psi_{12}e_1e_2;
\]

\[
e'_2 = e_2 - (\psi_{11} - \psi_{22})e_1e_2 + 2\psi_{12}(1 - e_2^2).
\]

The mean intrinsic ellipticity of an ensemble of background galaxies should be zero: \( \langle e_1 \rangle = \langle e_2 \rangle = \langle e_1 e_2 \rangle = 0 \), while the factors \( 1 - \langle e_1^2 \rangle \) and \( 1 - \langle e_2^2 \rangle \) are close to unity in practice. As a result, the expectation of the image ellipticity is simply

\[
\langle e'_1 \rangle = \psi_{11} - \psi_{22};
\]

\[
\langle e'_2 \rangle = 2\psi_{12};
\]

or

\[
\langle e'_i \rangle = 2\gamma_i,
\]

i.e., the image distortion is uniquely determined by the tidal field \( \gamma \) of the foreground clusters. However, the shear term \( \gamma \) is related to the matter term \( \kappa \) through [eqs.(14), (16) and (19)]

\[
\gamma(\theta) = \frac{1}{\pi} \int d^2 \theta' \chi(\theta - \theta')\kappa(\theta'),
\]

(124)
where the kernel $\chi(\theta)$ is a complex function (Schneider and Seitz, 1995)

$$
\chi(\theta) = \frac{\theta_2^2 - \theta_1^2 - 2i\theta_1\theta_2}{|\theta|^4}.
$$

(125)

Finally, the inversion of eq.(124) yields the cluster surface mass density $\Sigma(\theta)$, which is expressed by the observed ellipticity parameter of the images,

$$
\Sigma(\theta) = \Sigma_c \pi \int d^2\theta' \text{Re}[\chi^*(\theta - \theta')e'(\theta')],
$$

(126)

where $\chi^*$ is the complex conjugation of $\chi$ and $\text{Re}$ takes the real part of the complex variable.

Since the first detection of gravitational weak shear out to a radius of 3.0 $h^{-1}_{50}$ Mpc in CL 0024+1654 (Bonnet, Mellier and Fort, 1994), several measurements of the weak distortion of background galaxies have been made and the applications of the lensing inversion technique to the observed data have turned to be very successful in the reconstruction of the mass distribution of clusters (Fahlman et al., 1994; Smail et al. 1995; Smail and Dickinson, 1995; Tyson and Fischer, 1995; Squires et al., 1995; Kneib et al., 1995; etc.). Although there are some disagreements about the resulting gravitating masses of clusters, for instance, Fahlman et al. (1994) derived a gravitational mass of 2.5 – 3 times larger than its virial mass in cluster MS 1224 while Squires et al. (1995) found accordance between the two masses in Abell 2218, it appears very promising that one can precisely determine the cluster mass from lensing method in the near future by improving the inversion technique (Bartelmann, 1995; Schneider, 1995; Schneider and Seitz, 1995; Seitz and Schneider, 1995a; Kaiser, 1995).

### 4.6 Quasar-cluster associations

The presence of giant arcs and arclets associated with gravitational potentials of clusters of galaxies indicates that clusters of galaxies are very efficient lenses. Actually, it was noticed at the same time when giant luminous arcs were discovered that some of the high-redshift 3CR galaxies are gravitationally magnified by the low-redshift clusters of galaxies lying in their lines of sight (Hammer, Nottale and Le Fèvre, 1986; Le Fèvre, Hammer and Jones, 1988; Le Fèvre et al., 1988; Hammer and Le Fèvre, 1990). Motivated by these observations, Wu and Hammer (1993) even explored the possibility of whether there are radio arcs behind galaxy clusters.

What would happen to background quasars if foreground clusters of galaxies act as lenses? Four recent measurements have answered this question by discovering a significant quasar overdensity behind foreground clusters using different quasar and cluster samples: (1)the Large Bright Quasar Survey and Zwicky clusters (Rodrigues-Williams and Hogan, 1994), (2)the 1 and 2 Jy Radio Source Surveys and Abell clusters (Wu and Han, 1995), (3)the variability selected quasars and clusters (Rodrigues-William and Hawkins, 1995). (4)the 1 Jy Radio Source Catalog and Zwicky clusters (Seitz and Schneider, 1995b). Table 8 summarizes these searches and their resulted quasar overdensity density $q$, in which only the most significant $q$ measured at a fixed position $\theta$ and a limiting magnitude (or flux) is given. Figure 32 shows the variations of $q$ for Abell clusters at $z_d < 0.2$ around 2 Jy radio quasars at $z_s > 0.5$.

In a similar way to the study of quasar-galaxy associations, we can evaluate the enhancement factor $q$ for the quasar-cluster associations. Given a magnification $\mu$, a general expression is available for both optically- and radio-selected quasars

$$
q = \frac{N(< m + 2.5 \log \mu)}{N(< m)} \frac{1}{\mu} = \frac{N(> S/\mu)}{N(> S)} \frac{1}{\mu}
$$

(127)
where $m$ and $S$ denote the limiting magnitude and the flux threshold, respectively. The surface number density of optically-selected quasars has been given by BSP [see eq.(94)]. For the radio source counts $N(> S)$, a least-square fit of a power-law to the observations at 5 GHz (Langston et al., 1990; Fomalont et al., 1991) yields

$$N(> S) = \begin{cases} 1.27 \times 10^6 S^{-1.46}, & S > 10 \text{ mJy}; \\ 2.10 \times 10^5 S^{-1.10}, & S < 10 \text{ mJy}, \end{cases}$$

(128)

where $S$ is the units of mJy. Note that radio source catalogs are composed not only of galaxies but also quasars. The fraction of quasars in radio source surveys varies with flux threshold. Therefore, the employment of $N(> S)$ in the study of quasar-cluster associations provides only an estimate of $q$.

Now we work with the lensing models of clusters of galaxies and/or their associated matter inhomogeneities and test whether one can explain the reported quasar overdensity behind clusters on scale of $\sim 10$ arcminutes in terms of gravitational lensing. We take an average enhancement $\langle q \rangle$ [eq.(93)] over the search range of $\theta$ around clusters instead of the local enhancement $q$ at $\theta$. We further assume a flat cosmological model $\Omega = 1$ and adopt $H_0 = 50$ km/s/Mpc.

Conventionally, clusters of galaxies should be considered to be the lensing objects for the reported quasar-cluster associations. Utilizing SIS as the mass model and its magnification of eq.(27), we can estimate the cluster velocity dispersion $\sigma_c$ that is required to produce the observed $\langle q \rangle$. Figure 32 plots such a fit to the observed data. Surprisingly, the best fitted Einstein radius in this example is $\sim 0.2$", corresponding to a velocity dispersion of $\sigma_c \approx 5000$ km/s if the typical redshifts of Abell clusters and of radio sources are taken to be 0.1 and 1, respectively. The similar results are found for the rest three measurements (Table 8). Apparently, the masses ($\sim \sigma_c^2$) that are needed to produce the four measured enhancement factors are substantially larger than the realistic value for clusters.

It has been known for some years that the weak lensing by large-scale matter inhomogeneities may contribute a significant effect on the background quasars. It may be the cause for the quasar-galaxy associations observed on the similar scale ($\sim 10$') (Fugmann, 1988;1990; Bartelmann and Schneider, 1993a,b;1994). We can now work out how large an additional mass surface density from the large-scale matter clumps that clusters of galaxies trace is need to produce the quasar-cluster associations. To do this, we add a uniform mass sheet $\Sigma$ to clusters of galaxies. It turns out that the Einstein radius $\theta_E$ and the image separation are increased by a factor of $(1-\Sigma/\Sigma_c)^{-1}$ and the lensing magnification becomes

$$\mu = \frac{1}{|1-\theta_E/\theta|(1-\Sigma/\Sigma_c)^2},$$

(129)

where $\Sigma_c$ is the critical surface mass density that any lens must exceed in order to produce multiple images by itself [eq.(15)]. Quantitatively, the minimum $\Sigma_c$ for a source at $z_s = 2$ is 0.41 g/cm$^2$. In Table 8 we give the required surface mass density $\Sigma$ for each of the measurements. Note that $\Sigma$ deduced from the radio selected quasars is a factor of $\sim 2$ larger than the one from the optically selected samples. This is due to the contamination of radio galaxies in the radio source catalog. The result from Seitz and Schneider (1995) illustrates very well this effect.

We can estimate the matter contribution from all the galaxy clusters that follow the cluster spatial two-point correlation function $\xi(r/r_{cc})^{-1.8}$, where the correlation amplitude is $r_{cc} = 40$ Mpc (Postman, Huchra and Geller, 1992). If we assume that each cluster has a SIS mass density profile and a gravitational radius $R_c$ and the mean number density of clusters is constant, the surface mass density of clusters enclosed within $\theta$ around a given cluster at $z_d$ is (Wu and Fang, 1995)

$$\Sigma(\theta) = 4n_0 M_c r_{cc}(1+z_d)^3 F(\theta, r_{cc}, R_c),$$

(130)
where \( n_0 \) is the present cluster number density, \( M_c = 2\sigma_c^2 R_c / G \) is the total cluster mass and \( F \) is a function given by \( \xi(r) \) and SIS model. Numerical computations show that \( F \approx 2 \sim 3 \) for \( R_c = 3-5 \) Mpc over the range of \( \theta = 1' - 80' \). Let \( \Omega_c \) denote the fraction of the total cluster matter in the mass density of the Universe, we have

\[
\Sigma = 0.01\Omega_c \left( \frac{(1 + z_d)^3}{1.15} \right) \left( \frac{F}{3} \right) \text{g/cm}^2.
\]

Unfortunately, this surface mass density provided by all galaxy clusters following \( \xi(r) \) is an order of magnitude lower than that required for the quasar-cluster associations even if \( \Omega_c = 1 \).

The mass surface density from large-scale structures of the Universe can be estimated through

\[
\Sigma = \int \left[ \rho(r) - \rho_0 \right] dr \sim 1.45 \times 10^{-3} \delta \left( \frac{r}{100 \text{ Mpc}} \right) \text{g/cm}^2,
\]

in which \( \delta \) is the mean density contrast over scale of \( R \). However, the evaluation of \( \Sigma \) is sharply constrained by the measurements of temperature anisotropy \( \Delta T / T \) of the cosmic background radiation on various scales. Numerical computation indicates that it is impossible to attribute the large mass surface density derived from the quasar-cluster associations to any matter clumps on scale of \( R > 20 \) Mpc in the Universe if \( \Delta T / T = 1 \sim 5 \times 10^{-5} \) (Wu and Fang, 1995).

So, if the reported associations between background bright quasars and foreground clusters of galaxies are not due to statistical variations arising from the quasar/cluster selections and patchy Galactic obscuration, we need to consider the following possibilities: (1) There may exist a large amount of unseen matter between clusters of galaxies on scale of \( \sim 10 \) Mpc, because the above calculations did not include the unbound cluster matter. This can be tested using the N-body simulations, as was made by Bartelmann and Schneider (1993b) for quasar-galaxy associations. (2) The working hypothesis may be wrong, i.e., the observed background quasar counts may deviate from their intrinsic ones. On the scale of galaxies, Schneider (1992) has demonstrated that dropping the unaffected background hypothesis does not significantly improve the situation in quasar-galaxy associations. Whether the cluster matter or large-scale structures would contribute a non-negligible effect on the quasar number counts needs to be further investigated.

**FINAL REMARKS**

The past few years have been exciting times for lensing people. In particular, the microlensing experiments have detected a few ten events associated with the compact objects of the Galactic halo/disk and/or the LMC halo/disk, indicative of the success of using gravitational lensing effect for the searches of dark matter, which will have a strong impact on various aspects of cosmology study today. The rapidly increasing number of new lens systems (multiple quasars/galaxies, radio rings/galaxies, arcs/clusters of galaxies, etc.) has made it possible to study the matter distribution of the Universe statistically and to determine the cosmological parameters \( (H_0, \Omega_0 \text{ and } \lambda_0) \). This is of particularly significance since lensing provides not only an independent means to evaluate these important issues in cosmology but also a test for the validity of other astronomical/physical methods. The determination of the Hubble constant \( H_0 \) from the time delay of double quasars and mapping matter distribution in clusters of galaxies with luminous arcs and arclets are the two excellent examples of the lensing applications in cosmology.

Indeed, the study of gravitational lensing has been developed so rapidly in both theory and observation, and it is not practical to summarize every subject in this review destined for the readers who are not the experts in lensing. In the present article it is even impossible to include some new discoveries, new observations and new theories which appeared during the writing of the article. The interested readers are recommended to refer to the recent reviews on the lensing applications in cosmology (Blandford and Narayan, 1992; Schneider, 1996), on the lensing observations (Refsdal and Surdej, 1994), on the arc(let)s in clusters of galaxies (Fort and Mellier, 1994), in particular, the excellent monograph of gravitational lensing by Schneider, Ehlers and Falco (1992).
Two quotes can be used as the final remarks of this review on gravitational lensing:

“An astronomer can use beams of photons to probe a condensation of dark matter in much the same way that a nuclear physicist used beams of electrons to study the structure of an atomic nucleus.” (Blandford and Kochanek, 1987)

“A galactic gravitational lens can be used as the ultimate astronomical telescope.” (McBreen and Metcalfe, 1987)

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References

- Adams, M. T. and Boroson, T. A. (1979). *Nature*, 282, 183.
- Alcock, C. et al. (1993). *Nature*, 365, 621.
- Alcock, C. et al. (1995a). *Astrophys. J.*, 445, 133.
- Alcock, C. et al. (1995b). *Phys. Rev. Lett.*, 74, 2867.
- Alcock, C. et al. (1995c). preprint astro-ph/9506113.
- Alexander, T. (1995). *Mon. Not. Roy. ast. Soc.*, submitted.
- Allen, S. W., Fabian, A. C. and Kneib, J. P. (1995). astro-ph/9506036.
- Ansari, R. et al. (1995). *Astron. Astrophys.*, 299, L21.
- Ashman, K. M. (1992). *Publ. Astron. Soc. Pacific*, 104, 1109.
- Aubourg, E. et al. (1993). *Nature*, 365, 623.
- Baganoff, F. K. B and Malkan, M. A. (1995). *Astrophys. J. Lett.*, 444, 13.
- Bahcall, N. A. (1975). *Astrophys. J.*, 198, 249.
- Bahcall, N. A. (1977). *Ann. Rev. Astron. Astrophys.*, 15, 505.
[] Bahcall, J. N. et al. (1992). *Astrophys. J.*, **387**, 56.
[] Bartelmann, M. (1995). *Astron. Astrophys.*, **303**, 643.
[] Bartelmann, M. and Narayan, R. (1995). *Astrophys. J.*, **451**, 60.
[] Bartelmann, M. and Schneider, P. (1993a). *Astron. Astrophys.*, **268**, 1.
[] Bartelmann, M. and Schneider, P. (1993b). *Astron. Astrophys.*, **271**, 421.
[] Bartelmann, M. and Schneider, P. (1994). *Astron. Astrophys.*, **284**, 1.
[] Bartelmann, M., Steinmetz, M. and Weiss, A. (1995). *Astron. Astrophys.*, **297**, 1.
[] Beers, T. C. and Tonry, J. L. (1986). *Astrophys. J.*, **300**, 557.
[] Bennett, D. P. et al. (1994). preprint.
[] Bergmann, A. G. and Petrosian, V. (1993). *Astrophys. J.*, **413**, 18.
[] Bernstein, G. M., Tyson, J. A. and Kochanek, C. S. (1993). *Astron. J.*, **105**, 816.
[] Bertotti, B. (1966). *Proc. Roy. Soc. London*, A**294**, 195.
[] Blandford, R. D. and Kochanek, C. S. (1987). *Astrophys. J.*, **321**, 658.
[] Blandford, R. D. and Narayan, R. (1986). *Astrophys. J.*, **310**, 568.
[] Blandford, R. D. and Narayan, R. (1992). *Ann. Rev. Astron. Astrophys.*, **30**, 311.
[] Bonnet, H., Mellier, Y. and Fort, B. (1994). *Astrophys. J. Lett.*, **427**, 83.
[] Bontz, R. J. (1979). *Astrophys. J.*, **233**, 402.
[] Borgeest, U. (1983). *Astron. Astrophys.*, **128**, 162.
[] Borgeest, U. and Refsdal, S. (1984). *Astron. Astrophys.*, **141**, 318.
[] Bourassa, R. R. and Kantowski, R. (1975). *Astrophys. J.*, **195**, 13.
[] Bourassa, R. R. and Kantowski, R. (1976). *Astrophys. J.*, **205**, 674.
[] Boyle, B. J., Jones, L. R., and Shanks, T. (1991). *Mon. Not. Roy. ast. Soc.*, **251**, 482.
[] Boyle, B. J., Shanks, T. and Peterson, B. A. (1988). *Mon. Not. Roy. ast. Soc.*, **235**, 935.
[] Broadhurst, T., J., Ellis, R. S. and Shanks, T. (1989). *Mon. Not. Roy. ast. Soc.*, **235**, 827.
[] Campbell, R. M., Lehár, J., Corey, B. E., Shapiro, I. I. and Falco, E. E. (1995). *Astron. J.*, in press
[] Campusano, L. E. and Hardy, E. (1995). In C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.
[] Canizares, C. R. (1981). *Nature*, **291**, 620.
[] Canizares, C. R. (1982). *Astrophys. J.*, **263**, 508.
[] Carswell, R. F., Rauch, M., Weymann, R. J., Cooke, A. J. and Webb, J. K. (1994). *Mon. Not. Roy. ast. Soc.*, **268**, L1.
[] Chang, K. and Refsdal, S. (1979). *Nature*, **282**, 561.
Claeskens, J. F. et al., (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.

Claeskens, J. F., Surdej, J. and Remy, M. (1995). In C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Cooke, J. H. and Kantowski, R. (1975). *Astrophys. J.*, **195**, L11.

Corbett, E. A., Browne, I. W. A., Wilkinson, P. N. and Patnaik, A. R. (1995). In C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Corrigan, R. T. et al. (1991). *Astron. J.*, **102**, 34.

Cowie, L. L., Henriksen, M. and Mushotzky, R. (1987). *Astrophys. J.*, **317**, 593.

Cumming, C. M. and De Robertis, M. M. (1995). *Publ. Astron. Soc. Pacific*, **107**, 469.

Dahle, H., Maddox, S. J. and Lilje, Per B. (1994). *Astrophys. J. Lett.*, **435**, 79.

Dalcanton, J. J., Canizares, C. R., Granados, A., Steidel, C. C. and Stocke, J. T. (1994). *Astrophys. J.*, **424**, 550.

De Robertis, M. M. and Yee, H. K., C. (1988). *Astrophys. J. Lett.*, **332**, 49.

Deguchi, S. and Watson, W. D. (1988). *Astrophys. J.*, **335**, 67.

Djorgovski, S. and Spinrad, H. (1984). *Astrophys. J. Lett.*, **282**, 1.

Dressler, A. (1978). *Astrophys. J.*, **226**, 55.

Durret, F., Gerbal, D., Lachièze-Rey, M., Lima-Neto, G. and Sadat, R. (1994). *Astron. Astrophys.*, **287**, 733.

Dyer, C. C. (1984). *Astrophys. J.*, **287**, 26.

Dyer, C. C. and Roeder, R. C. (1972). *Astrophys. J. Lett.*, **174**, 115.

Dyer, C. C. and Roeder, R. C. (1973). *Astrophys. J. Lett.*, **180**, 31.

Dyer, C. C. and Roeder, R. C. (1974). *Astrophys. J.*, **189**, 167.

Dyer, C. C. and Roeder, R. C. (1980). *Astrophys. J.*, **241**, 133.

Dyson, F. W., Eddington, A. S. and Davidson, C. R. (1920). *Mem. R. Astron. Soc.*, **62**, 291.

Edge, A. C. et al. (1994). *Astron. Astrophys.*, **289**, L34.

Edge, A. C., Stewart, G. C., Fabian, A. C. and Arnaud, K. A. (1990). *Mon. Not. Roy. ast. Soc.*, **245**, 559.

Ehlers, J. and Schneider, P. (1986). *Astron. Astrophys.*, **168**, 57.

Einstein, A. (1911). *Annalen der Physik*, **35**, 898.

Einstein, A. (1936). *Science*, **84**, 506.

Enßlin, T. A., Biermann, P. L., Kronberg, P. P. and Wu, X. P. (1995). *Phys. Rev. Lett.*, submitted

Fahlman, G, Kaiser, N., Squires, G. and Woods, D. (1994). *Astrophys. J.*, **437**, 56.

Fulco, E. E. (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.
Falco, E. E., Gorenstein, M. V. and Shapiro, I. I. (1985), Astrophys. J. Lett., 289, 1.
Falco, E. E., Gorenstein, M. V. and Shapiro, I. I. (1991), Astrophys. J., 372, 364.
Fischer, P., Tyson, J. A., Bernstein, G. M. and Guhathakurta, P. (1994). Astrophys. J. Lett., 431, L71.
Fomalont, E. B., Windhorst, R. A., Kristian, J. A. and Kellermann, K. I. (1991), Astron. J., 102, 1258.
Fort, B., Le Fèvre, O., Hammer, F. and Cailloux, M. (1992). Astrophys. J. Lett., 399, 125.
Fort, B. and Mellier, Y. (1994). Astron. Astrophys. Rev., 5, 239.
Freeman, K. C. (1970). Astrophys. J., 160, 811.
Fugmann, W. (1988). Astrop. Astrophys., 204, 73.
Fugmann, W. (1990). Astrop. Astrophys., 240, 11.
Fukugita, M., Futamase, T., Masai, M. and Turner, E. L. (1992). Astrophys. J., 393, 3.
Fukugita, M., Hogan, C. J. and Peebles, P. J. E. (1993). Nature, 366, 309.
Fukugita, M. and Turner, E. L. (1991). Mon. Not. Roy. ast. Soc., 253, 99.
Garrett, M. A., Calder, R. J., Porcas, R. W., King, L. J., Walsh, D. and Wilkinson, P. N. (1994). Mon. Not. Roy. ast. Soc., 270, 457.
Gerbal, D., Durret, F. Loma-Neto, G. and Lachièze-Rey, M. (1992). Astron. Astrophys., 253, 77.
Gioia, I. M. et al. (1995). Astron. Astrophys., 297, L75.
Gioia, I. M., Henry, J. P., Maccacaro, T., Morris, S. L., Stocke, J. T. and Wolter, A. (1990). Astrophys. J. Lett., 356, 35.
Gioia, I. M. and Luppino, G. A. (1994). Astrophys. J. Suppl., 94, 583.
Giraud, E. (1988). Astrophys. J. Lett., 334, 69.
Gorenstein, M. V. et al. (1988). Astrophys. J., 334, 42.
Gorenstein, M. V., Falco, E. E. and Shapiro, I. I. (1988). Astrophys. J., 327, 693.
Gott, J. R. (1985). Astrophys. J., 288, 422.
Gott, J. R. and Gunn, J. E. (1974). Astrophys. J. Lett., 190, 105.
Gould, A. (1994). Astrophys. J. Lett., 421, 71.
Gould, A. (1995). Astrophys. J., 441, 77.
Greenfield, P. E., Roberts, D. H. and Burke, B. F. (1985). Astrophys. J., 293, 370.
Griest, K. (1991). Astrophys. J., 366, 412.
Grossman, S. A. and Narayan, R. (1988). Astrophys. J. Lett., 324, 37.
Grossman, S. A. and Narayan, R. (1989). Astrophys. J., 344, 637.
Grossman, S. A. and Saha, P. (1994). Astrophys. J., 431, 74.
Gunn, J. E. (1967). Astrophys. J., 150, 737.
Haarsma, D. B., Hewitt, J. N., Burke, B. F. and Lehár, J. (1995). in C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Hammer, F. (1991). *Astrophys. J.*, **383**, 66.

Hammer, F. (1995). In *the XV Moriond Astrophysics Meeting: Clustering in the Universe*, in press.

Hammer, F., Angonin, M. C., Le Fèvre, O., Wu, X. P., Luppino, G. A. and Gioia, I. M. (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.

Hammer, F. and Le Fèvre, O. (1990). *Astrophys. J.*, **357**, 38.

Hammer, F., Nottale, L. and Le Fèvre, O. (1986). *Astron. Astrophys.*, **169**, L1.

Hammer, F. and Rigaut, F. (1989). *Astron. Astrophys.*, **226**, 45.

Hawkins, M. R. S. (1993). *Nature*, **366**, 242.

Hawkins, M. R. S. and Véron, P. (1993). *Mon. Not. Roy. ast. Soc.*, **260**, 202.

Henry, J. P., Gioia, I. M., Maccacaro, T., Morris, S. L., Stocke, J. T. and Wolter, A. (1992). *Astrophys. J.*, **386**, 408.

Henry, J. P., Soltan, A., Briel, U. and Gunn, J. E. (1982). *Astrophys. J.*, **262**, 1.

Hewett, P. C. et al. (1989). *Astrophys. J. Lett.*, **346**, 61.

Hewitt, J. N., Turner, E. L., Schneider, D. P., Burke, B. F., Langston, G. I. and Lawrence, C. R. (1988). *Nature*, **333**, 537.

Hewitt, J. N. et al. (1986). In A. Hewitt et al. (Eds.), *Observational Cosmology*, Dordrecht, Reidel.

Hinshaw, G. and Krauss, L. M. (1987). *Astrophys. J.*, **320**, 468.

Hjorth, J., Jaunsen, A. O., Patnaik, A. R. and Kneib, J.-P. (1995). in C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Hoag, A. (1981). *Bull. Am. Astr. Soc.*, **13**, 799.

Huchra, J., Gorenstein, M., Kent, S., Shapiro, I., Smith, G., Horine, E. and Perley, R. (1985). *Astron. J.*, **90**, 691.

Irwin, M. J., Webster, R. L., Hewett, P. C., Corrigan, R. T. and Jedrzejewski, R. I. (1989). *Astron. J.*, **98**, 1989.

Isaacson, J. A. and Canizares, C. R. (1989). *Astrophys. J.*, **336**, 544.

Jackson, N. et al. (1995). *Mon. Not. Roy. ast. Soc.*, (in press).

Jauncey, D. L. et al. (1991). *Nature*, **352**, 132.

Jones, C. and Forman, W. (1984). *Astrophys. J.*, **276**, 38.

Kaiser, N. and Squires, G. (1993). *Astrophys. J.*, **404**, 441.

Kaiser, N. (1995). *Astrophys. J.*, **439**, L1.

Kantowski, R. (1969). *Astrophys. J.*, **155**, 89.

Kantowski, R., Vaughan, T. and Branch, D. (1995). *Astrophys. J.*, **447**, 35.

Kassiola, A. and Kovner, I. (1993). *Astrophys. J.*, **417**, 450.
Katz, N., Balbus, S. and Paczyński, B. (1986). *Astrophys. J.*, **306**, 2.

Kayser, R. (1986). *Astron. Astrophys.*, **157**, 204.

Kayser, R. and Refsdal, S. (1983). *Astron. Astrophys.*, **128**, 156.

Kayser, R., Refsdal, S. and Stabell, R. (1986). *Astron. Astrophys.*, **166**, 36.

Klimov, Y. G. (1963). *Soviet Phys. Doklady*, **8**, 119.

Kneib, J.-P., Ellis, R. S., Smail, I., Couch, W. J. and Sharples, R. M. (1995). *Astrophys. J.*, submitted.

Kneib, J.-P., Mellier, Y., Fort, B. and Mathez, G. (1993). *Astron. Astrophys.*, **273**, 367.

Kochanek, C. S. (1992). *Astrophys. J.*, **384**, 1.

Kochanek, C. S. (1993a). *Mon. Not. Roy. ast. Soc.*, **261**, 453.

Kochanek, C. S. (1993b). *Astrophys. J.*, **417**, 438.

Kochanek, C. S. (1993c). *Astrophys. J.*, **419**, 12.

Kochanek, C. S. (1995). *Astrophys. J.*, submitted

Kochanek, C. S. and Blandford, R. D. (1987). *Astrophys. J.*, **321**, 676.

Koo, D. C. (1987). In V. G. Rubin and G. V. Coyne (Eds.), *Large Scale Motions in the Universe*, Princeton University Press, Princeton.

Kovner, I. (1989). *Astrophys. J. Lett.*, **341**, 1.

Langston, G. I. et al. (1989). *Astron. J.*, **97**, 1283.

Langston, G. I., Conner, S. R., Heffin, M. B., Lehár, J. and Burke, B. F. (1990). *Astrophys. J.*, **353**, 34.

Lavery, R. J. and Henry, J. P. (1988). *Astrophys. J. Lett.*, **329**, 21.

Lavery, R. J., Pierce, M. J. and McClure, R. D. (1993). *Astrophys. J.*, **418**, 43.

Lawrence C. R. et al. (1984). *Science*, **223**, 46.

Le Fèvre, O., Hammer, F. and Jones, J. (1988). *Astrophys. J. Lett.*, **331**, 73.

Le Fèvre, O., Hammer, F., Nottale, L., Mazure, A. and Christian, C. (1988). *Astrophys. J. Lett.*, **324**, 1.

Le Fèvre, O., Hammer, F., Angonin, M. C., Gioia, I. M. and Luppino, G. A. (1994). *Astrophys. J. Lett.*, **422**, 5.

Lehár, J., Hewitt, J. N., Roberts, D. H. and Burke, B. F. (1992). *Astrophys. J.*, **384**, 453.

Lehár, J., Langston, G. I., Silber, A., Lawrence, C. R. and Burke, B. F. (1993). *Astron. J.*, **105**, 847.

Liebes, S. (1964). *Phys. Rev.*, **B133**, 835.

Loeb, A. and Mao, S. (1994). *Astrophys. J. Lett.*, **435**, 109.

Luppino, G. A., Gioia, I. M., Annis, J., Le Fèvre, O. and Hammer, F. (1993). *Astrophys. J.*, **446**, 444.

Luppino, G. A., Gioia, I. M., Hammer, F., Le Fèvre, O. and Annis, J., (1995). *Astrophys. J.*, in press.

Lynds, R. and Petrosian, V. (1986). *Bull. Am. Astr. Soc.*, **18**, 1014.

Magain, P. et al. (1988). *Nature*, **334**, 325.
[Patnaik, A. R., Browne, I. W. A., King, L. J., Muxlow, T. W. B., Walsh, D. and Wilkinson, P. N. (1993). Mon. Not. Roy. ast. Soc., 261, 435.]

[Patnaik, A. R., Browne, I. W. A., Walsh, D., Chaffee, F. H. and Foltz, C. B. (1992). Mon. Not. Roy. ast. Soc., 259, 1p.]

[Peacock, J. A. (1986). Mon. Not. Roy. ast. Soc., 223, 113.]

[Pei, Y. C. (1993). Astrophys. J., 403, 7.]

[Pello, R., Le Borgne, J. E., Soucail, G., Mellier, Y. and Sanahuja, B. (1991). Astrophys. J., 366, 405.]

[Pello, R., Soucail, G., Sanahuja, B., Mathez, G. and Ojero, E. (1988). Astron. Astrophys., 190, L11.]

[Pelt, J., Kayser, R., Refsdal, S. and Schramm, T. (1995). Astron. Astrophys., submitted.]

[Pierre, M., Soucail, G., Böhringer, H. and Sauvageot, J. L. (1994). Astron. Astrophys., 289, L37.]

[Postman, M., Huchra, J. P. and Geller, M. J. (1992). Astrophys. J., 384, 404.]

[Press, W. H. and Gunn, J. E. (1973). Astrophys. J., 185, 397.]

[Press, W. H., Rybicki, G. B. and Hewitt, J. N. (1992). Astrophys. J., 385, 404.]

[Quintana, H. and Melnick, J. (1982). Astron. J., 87, 972.]

[Racine, R. (1992). Astrophys. J. Lett., 395, 65.]

[Rao, A. P. and Subrahmanyan, R. (1988). Mon. Not Roy. ast. Soc., 231, 229.]

[Refsdal, S. (1964a). Mon. Not. Roy. ast. Soc., 128, 295.]

[Refsdal, S. (1964b). Mon. Not. Roy. ast. Soc., 128, 307.]

[Refsdal, S. (1966). Mon. Not. Roy. ast. Soc., 132, 101.]

[Refsdal, S. and Surdej, J. (1994). Rep. Prog. Phys., 56, 117.]

[Rhee, G. (1991). Nature, 350, 211.]

[Rix, H.-W., Maoz, D., Turner, E. L. and Fukugita, M. (1994). Astrophys. J., 435, 49.]

[Roberts, D., Lehár, J., Hewitt, J. N. and Burke, B. F. (1991). Nature, 352, 43.]

[Rodrigues-Williams, L. L. and Hogan, C. J. (1994). Astron. J., 107, 451.]

[Rodrigues-Williams, L. L. and Hawkins, M. R. S. (1995). in Proc. of the 5th Annual Astrophysics Conf., Maryland.]

[Sachs, R. K. (1961). Proc. Roy. Soc. London, A 264, 309.]

[Sahu, K. (1994). Nature, 370, 275.]

[Sasaki, S. and Takahara, F. (1993). Mon. Not. Roy. ast. Soc., 262, 681.]

[Schild, R. E. (1990). Astron. J., 100, 1771.]

[Schild, R. E. (1991). Sky & Telescope, 375.]

[Schild, R. E. and Thomson, D. J. (1995). Astron. J., 109, 1970.]

[Schindler, S. et al. (1995). Astron. Astrophys., 299, L9.]
Schneider, D. P., Turner, E. L., Gunn, J. E., Hewitt, J. N., Schmidt, M. and Lawrence, C. R. (1988). *Astron. J.*, **95**, 1619.

Schneider, P. (1985). *Astron. Astrophys.*, **143**, 413.

Schneider, P. (1986). *Astrophys. J. Lett.*, **300**, 31.

Schneider, P. (1987a). *Astron. Astrophys.*, **179**, 71.

Schneider, P. (1987b). *Astron. Astrophys.*, **179**, 80.

Schneider, P. (1989). *Astron. Astrophys.*, **221**, 221.

Schneider, P. (1992). *Astron. Astrophys.*, **254**, 14.

Schneider, P. (1993). *Astron. Astrophys.*, **279**, 1.

Schneider, P. (1995). *Astron. Astrophys.*, **302**, 639.

Schneider, P. (1996). in the Proceedings of the Laredo Advanced Summer School, in press.

Schneider, P., Ehlers, J. and Falco, E. E. (1992). *Gravitational Lensing*, Springer-Verlag, Berlin.

Schneider, P. and Seitz, C. (1995). *Astron. Astrophys.*, **294**, 411.

Schneider, P. and Weiss, A. (1987). *Astrophys. J.*, **171**, 49.

Schneider, P. and Weiss, A. (1988a). *Astrophys. J.*, **330**, 1.

Schneider, P. and Weiss, A. (1988b). *Astrophys. J.*, **327**, 526.

Schommer, R. A., Olczewski, E. W., Suntzeff, N. B. and Harris, H. C. (1992). *Astron. J.*, **103**, 447.

Schramm, T. (1990). *Astron. Astrophys.*, **231**, 19.

Schramm, J. K., Bian, Y., Borgeest, U. and Swings, J. P. (1994). *Astrophysics*, **25**, 1.

Seitz, S. and Schneider, P. (1992). *Astron. Astrophys.*, **265**, 1.

Seitz, S. and Schneider, P. (1994). *Astron. Astrophys.*, **287**, 349.

Seitz, S. and Schneider, P. (1995a). *Astron. Astrophys.*, **297**, 287.

Seitz, S. and Schneider, P. (1995b). *Astron. Astrophys.*, **302**, 9.

Smail, I. et al. (1991). *Mon. Not. Roy. ast. Soc.*, **252**, 19.

Smail, I. and Dickinson, M. (1995). *Astrophys. J.*, in press.

Smail, I., Ellis, R. S., Fitchett, M. J. and Edge, A. C. (1995). *Mon. Not. Roy. ast. Soc.*, **273**, 277.

Smoot, G. F. et al. (1992). *Astrophys. J.*, **371**, L1.

Snellen, I. A. G., De Bruyn, A. G., Schilizzi, R. T., Miley, G. K. and Myers, S. T. (1995). *Astrophys. J. Lett.*, **447**, L9.

Songalia, A., Cowie, L. L., Hogan, C. J. and Rugers, M. (1994). *Nature*, **368**, 599.

Soucail, G., Arnaud, M. and Mathez, G. (1994). in preparation.

Soucail, G., Fort, B., Mellier, Y. and Picat, J. P. (1987a). *Astron. Astrophys.*, **172**, L14.
Soucail, G. and Mellier, Y. (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.

Soucail, G., Mellier, Y., Fort, B., Hammer, F. and Mathez, G. (1987b). *Astron. Astrophys.*, **184**, L7.

Squires, G., Kaiser, N., Babul, A., Fahlman, G., Woods, D., Neumann, D. M. and Böhringer, H. (1995). *Astrophys. J.*, in press.

Stickel, M., Fried, J. W. and Kühr, H. (1988a), *Astron. Astrophys.*, **198**, L13.

Stickel, M., Fried, J. W. and Kühr, H. (1988b), *Astron. Astrophys.*, **206**, L30.

Stickel, M., Fried, J. W. and Kühr, H. (1989), *Astron. Astrophys.*, **224**, L27.

Stockton, A. (1980). *Astrophys. J. Lett.*, **242**, 141.

Surdej, J. et al. (1987). *Nature*, **329**, 695.

Surdej, J. et al. (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.

Surdej, J. et al. (1995). In C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Surdej, J. and Soucail, G. (1993). In J. Surdej et al. (Eds.), *Gravitational Lenses in the Universe*, Université de Liège, Liège.

Tinney, C. G. (1995). In C. S. Kochanek and J. N. Hewitt (Eds.), *Astrophysical Applications of Gravitational Lensing*, Kluwer Academic Publishers.

Turner, E. L. (1990). *Astrophys. J. Lett.*, **365**, 43.

Turner, E. L., Ostriker, J. P. and Gott, J. R. (1984). *Astrophys. J.*, **284**, 1.

Tyson, J. A. (1986). *Astron. J.*, **92**, 691.

Tyson, J. A. and Fischer, P. (1995). *Astrophys. J. Lett.*, **446**, L55.

Tyson, J. A., Valdes, F. and Wenk, R. A. (1990). *Astrophys. J. Lett.*, **349**, 1.

Udalski, A. et al. (1993). *ACTA Astronomica*, **43**, 289.

Udalski, A., Szymański, M., Kaluzny, J., Kubiak, M., Mateo, M. and Krzeminski, W. (1994a). *Astrophys. J. Lett.*, **426**, 69.

Udalski, A. et al., (1994b). *ACTA Astronomica*, **44**, 165.

Udalski, A. et al. (1995). preprint.

Valdes, F., Tyson, J. A. and Jarvis, J. F. (1983), *Astrophys. J.*, **271**, 431.

van der Kruit, P. C. and Searle, L. (1981). *Astron. Astrophys.*, **95**, 105.

van Ommen, T. D., Jones, D. L. and Preston, R. A. (1995). *Astrophys. J.*, **444**, 561.

Vanderriest, C., Schneider, J., Herpe, G., Chêreton, M., Moles, M. and Wlérick, G. (1989). *Astron. Astrophys.*, **215**, 1.

Vietri, M. and Ostriker, J. P. (1983). *Astrophys. J.*, **267**, 488.

Vilenkin, A. (1981). *Phys. Rev. Lett.*, **46**, 1169.

Wallington, S. and Narayan, R. (1993). *Astrophys. J.*, **403**, 517.
[] Walsh, D., Carswell, R. F. and Weymann, R. J. (1979). Nature, 279, 318.
[] Webster, R. L., Hewett, P. C., Harding, M. E. and Wegner, G. A. (1988). Nature, 336, 358.
[] Weedman, D. W., Weymann, R. J., Green, R. F. and Heckman, T. M. (1982). Astrophys. J. Lett., 255, 5.
[] Weinberg, S. (1976). Astrophys. J. Lett., 208, 1.
[] Weymann, R. J. et al. (1980). Nature, 285, 641.
[] White, S. D. M., Navarro, J. F., Evrard, A. E. and Frenk, C. S. (1993). Nature, 366, 429.
[] Wisotzki, L., Köhler, T., Kayser, R. and Reimers, D. (1993). In J. Surdej et al. (Eds.), Gravitational Lenses in the Universe, Université de Liège, Liège.
[] Witt, H. J. (1990). Astron. Astrophys., 236, 311.
[] Wu, X. P. (1989a). Ph.D. Dissertation, Chinese Academy of Sciences.
[] Wu, X. P. (1989b). Astron. Astrophys., 214, 43.
[] Wu, X. P. (1989c). In J. M. Moran et al. (Eds.), Gravitational Lenses, Lecture Notes in Physics, Vol. 330, Springer-Verlag, Berlin.
[] Wu, X. P. (1990a). Astron. Astrophys., 232, 3.
[] Wu, X. P. (1990b). Astron. Astrophys., 239, 29.
[] Wu, X. P. (1992a). In A. Filippenko (Ed.), Relationships between Active Nuclei and Starburst Galaxies, Astro. Soc. Pac. Conf. Ser., Vol. 31, San Francisco.
[] Wu, X. P. (1992b). Astrophys. and Space Sci., 187, 87.
[] Wu, X. P. (1993a). Astron. Astrophys., 270, L1.
[] Wu, X. P. (1993b). Astrophys. J., 411, 513.
[] Wu, X. P. (1993d). In J. Surdej et al. (Eds.), Gravitational Lenses in the Universe, Université de Liège, Liège.
[] Wu, X. P. (1994a). Astron. Astrophys., 286, 748.
[] Wu, X. P. (1994b). Astrophys. J., 453, 66.
[] Wu, X. P. (1994c). Astrophys. J. Lett., 436, L115.
[] Wu, X. P. and Fang, L. Z. (1995). Astrophys. J., submitted.
[] Wu, X. P. and Hammer, F. (1993). Mon. Not. Roy. ast. Soc., 262, 187.
[] Wu, X. P. and Han, J. (1995). Mon. Not. Roy. ast. Soc., 272, 705.
[] Wu, X. P., Zhu, Z. H. and Fang, L. Z. (1995). Astrophys. J., submitted
[] Yee, H. K. C. (1988). Astron. J., 95, 1331.
[] Yee, H. K. C. and De Robertis, M. M. (1992). Astrophys. J. Lett., 398, 21.
[] Young, P. (1981). Astrophys. J., 244, 756.
[] Young, P., Gunn, J. E., Kristian, J., Oke, J. B. and Westphal, J. A. (1980). Astrophys. J., 241, 507.
[] Zel’dovich, Ya. B. (1964). Soviet Astron., 8, 13.
Table 1  Gravitational Lensing Models

| model          | ISC                  | KING                | $r^{1/4}$ law          |
|----------------|----------------------|---------------------|------------------------|
| surface density| $\Sigma_0 \sqrt{1+\theta_0}$ | $\Sigma_0 \frac{1}{1+\theta_0}$ | $\Sigma_0 \exp[-7.669\theta_0^{1/4}]$ |
| length scale   | $r_c$                | $r_c$               | $r_c$                  |
| central density ($\rho_0$) or total mass ($M_0$) & velocity dispersion ($\sigma_v$) | $\rho_0 = \frac{\sigma_v^2}{r_c^2 G r_c^2}$ | $\rho_0 = \frac{9\sigma_v^2}{4\pi G r_c^2}$ | $M_0 = 9.6 \frac{r_c \sigma_v^2}{G}$ |
| lensing equation | $\beta_0 = \theta_0 - D \sqrt{\frac{1+\theta_0^2}{\theta_0}} - 1$ | $\beta_0 = \theta_0 - D \frac{\ln(1+\theta_0^2)}{\theta_0}$ | $\beta_0 = \theta_0 - D \frac{m_0(\theta_0)}{\theta_0}$ (*)) |
| $D$            | $\frac{4\pi \sigma_v^2}{c^2} \frac{D_s D_{ds}}{r_c D_s}$ | $\frac{18\sigma_v^2}{c^2} \frac{D_s D_{ds}}{r_c D_s}$ | $\frac{36\sigma_v^2}{c^2} \frac{D_s D_{ds}}{r_c D_s}$ |
| critical $D$   | 2                    | 1                   | $3.370 \times 10^{-3}$ |

* $m_0(\theta_0) = 1 - \exp\left(-7.669\theta_0^{1/4}\right) \sum_{n=0}^{7} \frac{1}{n!} \left(7.669\theta_0^{1/4}\right)^n$
Table 3  Gravitationally-Lensed Multiple Quasars

| name       | image No. | lens | $z_d$ | $z_s$ | $\Delta \theta_{max}$ | status   | discovers       | year  |
|------------|-----------|------|-------|-------|------------------------|----------|-----------------|-------|
| 0957+561   | 2         | G+C  | 0.36  | 0.5   | 1.41                   | confirmed| Walsh et al.    | 1979  |
| 1115+080   | ≥ 4       | G    | 0.29  |       | 1.72                   | confirmed| Weymann et al.  | 1980  |
| 2345+007   | 2         | G(?) | 1.49  |       | 2.15                   | possible | Weedman et al.  | 1982  |
| 1634+267   | 2         | G(?) | 0.57  |       | 1.96                   | possible | Djorgovski & Spinrad | 1984 |
| 2016+112   | 3         | G    | 1.01  |       | 3.27                   | confirmed| Lawrence et al. | 1984  |
| 2237+0305  | 4         | G    | 0.04  |       | 1.69                   | confirmed| Huchra et al.   | 1985  |
| 0142-100   | 2         | G    | 0.49  |       | 2.72                   | confirmed| Surdej et al.   | 1987  |
| 1413+117   | 4         | G(?) | 1.4   |       | 2.55                   | confirmed| Magain et al.   | 1988  |
| 1120+019   | 2         | G(?)C(?) | 0.6 |       | 1.46                   | possible | Meylan & Djorgovski | 1989 |
| 1429-008   | 2         | (?)  | 1.6   |       | 2.08                   | possible | Hewett et al.   | 1989  |
| 0952-0115  | 2         | (?)  | 4.5   |       | 0.9                    | possible | McMahon et al.  | 1992  |
| 1208+1011  | 2         | (?)  | 3.8   |       | 0.47                   | possible | Magain et al.   | 1992  |
| 1422+231   | 4         | G(?) | 0.64  |       | 3.62                   | confirmed| Patnaik et al.  | 1992  |
| 1009-025   | 2         | G(?) | 1.62  |       | 2.74                   | possible | Surdej et al.   | 1993  |
| 1104-1805  | 2         | (?)  | 1.66  |       | 2.30                   | possible | Wisotzki et al. | 1993  |
| 0240-343   | 2         | G(?) | 0.34  |       | 1.4                    | possible | Tinney          | 1995  |
| J03.13     | 2         | G(?) | 1.09  |       | 2.55                   | possible | Claeskens et al.| 1995  |

* Radio sources are not included.  G=galaxy;  C=cluster;  $\Delta \theta$=separation
Table 5  Arclike Images

| arc cluster | $z_d$ | $\sigma_v$(km/s) | $L_{x,44}$(ergs/s) | arc redshift ($z_s$) | discovers                  | year    |
|-------------|-------|------------------|--------------------|----------------------|----------------------------|---------|
| Abell 222   | 0.213 | 570              | 3.7                |                      | Smail et al.               | 1991    |
| Abell 370   | 0.374 | 1364             | 9.7                | 0.725,1.3?           | Soucail et al.             | 1987a   |
| Abell 963   | 0.206 | 9.1              | 0.771              |                      | Lavery & Henry             | 1989    |
| Abell 1689  | 0.196 | 1989             | 17.                |                      | Tyson et al.               | 1990    |
| Abell 1942  | 0.224 |                 |                    |                      | Smail et al.               | 1991    |
| Abell 2104  | 0.155 | 8.0              |                    |                      | Pierre et al.              | 1994    |
| Abell 2163  | 0.203 | 0.728,0.742      | 0.91?              |                      | Soucail et al.             | 1994    |
| Abell 2218  | 0.176 | 6.5              | 0.702,1.034        |                      | Pello-Descayre et al.      | 1988    |
| Abell 2219  | 0.225 | 18.              | 1                 | ~ 1                  | Smail et al.               | 1995    |
| Abell 2280  | 0.326 | 5.1              | 0.913              |                      | Gioia et al.               | 1995    |
| Abell 2390  | 0.231 | 0.728,0.742      | 0.91?              |                      | Pello-Descayre et al.      | 1991    |
| Abell 2397  | 0.212 |                 |                    |                      | Smail et al.               | 1991    |
| Abell S295  | 0.301 | 900              |                    |                      | Edge et al.                | 1994    |

| Cl 0024+1654 | 0.391 | 1300          | 2.7                 | 1.39?                | Koo                        | 1987    |
| Cl 0302+1658 | 0.426 | 5.0           |                      |                      | Mathez et al.              | 1992    |
| Cl 0500-24   | 0.316 | 1375          | 0.91?               |                      | Giraud                     | 1988    |
| Cl 1409+52   | 0.46  | 3000          | 9.2                 |                      | Tyson et al.               | 1990    |
| Cl 2236-04   | 0.56  |               | 1.116               |                      | Melnick et al.             | 1993    |
| Cl 2244-02   | 0.336 | 1.5           | 2.237               |                      | Soucail et al.             | 1987a   |

| MS 0440+0204 | 0.190 | 4.0           |                      |                      | Luppino et al.             | 1993    |
| MS 0451-0305 | 0.55  | 20.           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 1006+1202 | 0.221 | 4.8           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 1008-1224 | 0.301 | 4.5           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 1455+2232 | 0.259 | 16.           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 1621+2640 | 0.426 | 4.5           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 1910+6736 | 0.246 | 4.4           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 2053-0449 | 0.583 | 5.8           |                      |                      | Le Fèvre et al.            | 1994    |
| MS 2137-2353 | 0.313 | 16.           |                      |                      | Fort et al.                | 1992    |
| MS 2318-2328 | 0.187 | 6.8           |                      |                      | Le Fèvre et al.            | 1994    |

| AC 114       | 0.31  | 1649          | 4.0                 | 0.639                | Smail et al.               | 1991    |
| 0956+561*    | 0.36  | 0.5           |                      |                      | Bernstein et al.           | 1993    |
| GHO 2154+0508| 0.32  |               | 0.721               |                      | Lavery et al.              | 1993    |
| PKS0745-191  | 0.1028| 0.433         |                      |                      | Allen et al.               | 1995    |
| RXJ1347.5-1145| 0.451 | 62.           |                      |                      | Schindler et al.           | 1995    |
| ACO 3408     | 0.042 | 0.073         |                      |                      | Campusano & Hardy          | 1995    |

* The recent observation (Dahle, Maddox and Lilje, 1994) suggests that this “arc system” is the result of chance alignments of three and two different objects, and not gravitationally lensed arcs.
| cluster name | $z_d$ | $L_{x,44}$ | $\theta''$ | $L''$ | $L/W$ | $z_s$ | $m_v(\theta)\ (M_\odot)$ | $m_g(\theta)\ (M_\odot)$ | $m_g(\theta)/m_v(\theta)$ |
|-------------|------|----------|-----------|------|-------|------|----------------|----------------|-----------------|
| Abell 370   | 0.374| 9.7      | 56        | 9    | 18    | 1.3(?) | $2.26 \cdot 10^{14}$ | $8.20 \cdot 10^{14}$ | 3.63            |
| MS1006.0+1202 | 0.221| 4.819    | 62        | 4.9  | 7.0   | 0.6   | $1.36 \cdot 10^{14}$ | $6.91 \cdot 10^{14}$ | 5.08            |
| MS1008.1-1224 | 0.301| 4.493    | 51        | 4.0  | 6.5   | 0.6   | $1.33 \cdot 10^{14}$ | $7.47 \cdot 10^{14}$ | 5.60            |
| MS1910.5+6737 | 0.246| 4.386    | 67        | 6.1  | 10.5  | 0.6   | $1.63 \cdot 10^{14}$ | $9.95 \cdot 10^{14}$ | 6.12            |
Table 8  Quasar-Cluster Associations: Observations and Models

| clusters | quasars | $\langle z_d \rangle^a$ | $\langle z_q \rangle^b$ | $\theta^c$ | $\langle q \rangle_{obs}$ | $(\sigma_c/10^3)^d$ | $(\sigma_c/10^3)^2$ | $\Sigma^e$ | ref$^f$ |
|----------|---------|-------------------------|------------------------|-----------|-------------------------|---------------------|---------------------|----------|--------|
| Zwicky   | $B \leq 18.5$ | 0.2 | 1.8 | 52 | $1.7^{+0.5}_{-0.4}$ | 5.3$^{+1.6}_{-1.6}$ | 28$^{+20}_{-14}$ | 0.10$^{+0.04}_{-0.05}$ | 1 |
| Abell    | $S \geq 2$ Jy | 0.1 | 2.0 | 24 | $1.7^{+0.5}_{-0.5}$ | 4.7$^{+1.2}_{-1.8}$ | 22$^{+13}_{-14}$ | 0.28$^{+0.10}_{-0.18}$ | 2 |
| UKJ287$^g$ | $B \leq 18.5$ | 0.15 | 1.5 | 7.2 | $2.0^{+0.2}_{-0.2}$ | 2.3$^{+0.2}_{-0.2}$ | 5.3$^{+1.0}_{-0.9}$ | 0.12$^{+0.02}_{-0.02}$ | 3 |
| Zwicky   | $B \leq 19$ | 0.2 | 1 | 78 | $\sim 1.3$ | 4.3 | 18 | 0.06 | 4 |
|          | $S \geq 1$ Jy | | | | | 5.6 | 31 | 0.11 | |

$^a$Mean cluster redshift
$^b$Mean quasar redshift
$^c$Search range in arcminutes
$^d$Required cluster velocity dispersion in units of 1000 km/s
$^e$Required surface mass density in g/cm$^2$ for $\Omega = 1$ and $H_0 = 50$ km/s/Mpc.
$^f$References – (1) Rodrigues-Williams and Hogan, 1994; (2) Wu and Han, 1995; (3) Rodrigues and Hawkins, 1995; (4) Seitz and Schneider, 1995.
$^g$Clusters in UKJ287 field