PIEZOELECTRIC STACKS AS AN EFFECTIVE GAUGE OF SMALL DYNAMIC STRAINS

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Abstract. The theory of a new piezoelectric gauge of small dynamic strains is developed. The first previously proposed piezoelectric gauge is a thin piezoelectric element with preliminary polarization in the direction of its thickness. As a new gauge, piezoelectric stacks are used. Its direction of preliminary polarization coincides with the direction of the measured strains. As shown in the paper this kind of piezoelectric composite gauge can significantly increase the value of the measured electrical signal and the accuracy of the measured strains. Piezoelectric laminated stacks has a periodic structure, therefore, the homogenization method is used to correctly determine its effective characteristics (physical constants).

1. Introduction

The author of the first gauges of small tangential strains is Lord Kelvin [1, 2], who owns the theory and experiments that confirm the theory. The idea of Kelvin gauge is as follows: a conductor (gauge) through which electric current flows is on the surface of the deformable body. It is assumed that, in the contact area, the conditions of perfect contact between the conductor and the body are fulfilled. When the body is deformed, then the conductor is also deformed. As a result of the deformation of the gauge, its cross-sectional area is changed. The change in electrical resistance is proportional to the change in the cross-sectional area of the conductor.

The theory and experiments of piezoelectric gauge of small dynamic strains was first described in [3, 4]. Thin piezoelectric ceramic elements in the form of a rod and a plate, preliminary polarized in the direction of the thick coordinate, were used as a gauge. For the body under study, this direction is normal to the body surface at the point under investigation. A piezoelectric element placed on the surface of a deformable body is deformed together with it. As a result of the inverse piezoelectric effect, an electric charge appears on the electrodes of the piezoelectric gauge. Deformations are calculated by the measured difference in electrical potentials at the electrodes. The efficiency of the gauge as an energy converter is more than 30%. This means that more than 30% of the mechanical energy of the deformation of the gauge is converted into electrical energy. The direction of the preliminary polarization vector and the electric field strength vector is orthogonal to the tangential displacement vector. In order to increase the efficiency of the gauge, it is necessary to use a piezoelectric element with preliminary polarization of the parallel surface of the deformable body. In this case, the efficiency of the gauge increases to 70%. The problem is that it is not possible to obtain uniform longitudinal pre-polarization in the rod or plate. To obtain a piezoelectric element with longitudinal polarization, proceed as follows: a large number of thin plates with a thickness polarization are glued together and piezoelectric stacks are obtained. It is a piezoelectric rod of a periodic structure with longitudinal polarization (Figure 1 a). Piezoelectric composite rods (stacks)
are manufactured industrially and are widely used in modern technology as highly efficient converters of mechanical energy into electrical energy and vice versa. [5-9].

Note the fundamental difference between piezoelectric gauge and Kelvin gauge. Piezoelectric gauge measures the main value — the difference in electric potentials, which is proportional to the magnitude of the tangential deformation. Kelvin gauge measures a small value - a change in the resistance of the conductor, which requires additional equipment and introduces additional error in the determination of deformations.

2. Basic equations

One of the methods for manufacturing composite active elements (stacks) is that several tens of thin piezoelectric plates with a thickness of 0.3 to 1 mm, pre-polarized in thickness, with electrodes on the front surfaces are glued together to form a rod.

A composite rod with a periodic structure, referred to the Cartesian coordinate system, is schematically depicted in figure 1a. The piezoelectric layers are shaded, the metal electrodes are shown in black, the adhesive layers in gray. One cell, consisting of a piezoelectric layer, the front surfaces of which are coated with metal electrodes, and an adhesive layer are depicted in figure 1b. Electrodes, as a rule, are applied to piezoelectric layers by spraying.

![Figure 1](image_url)

Figure 1. Schematic structure of stacks a) and its periodicity cell b).

The values of the effective modules do not depend on the type of load, so for simplicity we assume that the harmonic electric load acts on the stacks, changing in time $t = \omega t$, where $\omega$ is the circular frequency. So all the sought quantities vary in time in the same way, we write the initial equations for the amplitude values.
Let us write out one-dimensional equations for layers made of piezoceramics and elastic layers of glue. We will use the one-dimensional equations obtained using the well-known Kirchhoff hypotheses for mechanical quantities and the hypotheses formulated earlier in [4, 10] for the electric unknown quantities.

Vibration equation
\[ \frac{d\sigma_1}{dx_1} + \rho_c \omega^2 u_1 = 0. \]  
(1)

where \( c \) it should be replaced by \( a \) for the adhesive layer and for \( p \) the piezoceramic layer.

Hooke's law for an elastic layer
\[ \sigma_1 = E_1 e_1. \]  
(2)

Electroelasticity ratio for a piezoceramic layer pre-polarized in thickness,
\[ \sigma_1 = \frac{1}{s_{33}} e_1 - \frac{d_{33}}{s_{33}} E_1. \]  
(3)

Electrostatic equations
\[ D_1 = \varepsilon_{33}^T E_1 + d_{33} \sigma_1, \quad E_1 = -\frac{d\varphi}{dx_1}. \]  
(4)

The electric potential on the electrodes of the piezoceramic layer is constant
\[ \varphi \bigg|_{x_{1i}=x_{1i}} = \pm V. \]  
(5)

The relation between deformation and displacement has the form
\[ e_1 = \frac{du_1}{dx_1}. \]  
(6)

The notations used coincide with those adopted previously [4]. In formulas (1)–(6) \( E_1 \) and \( D_1 \) are the components of the electric field vector and electric induction vector in the direction \( x_1 \) respectively, \( s_{33}^E \) is the elastic compliance at zero electric field, \( d_{33} \) is the piezoelectric constant, \( \varepsilon_{33}^T \) is the dielectric constant at zero voltages, \( \varphi \) is the electric potential.

We apply the homogenization method [11, 12] to find the effective modulus.

For a composite with a periodic structure, the electroelastic state depends on two variables — the macroscopic variable and the microscopic variable \( y \) within the periodicity cell. In the homogenization method, for simplicity of notation, it is customary to use the notation \( x \) for a macroscopic variable.

Scale out a single cell
\[ y = x / \varepsilon, \quad \varepsilon = h / l, \quad h = h_s + h_p \]  
(7)

then
\[ \frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y}. \]  
(8)

The sought quantities that determine the behavior of a one-dimensional layered structure of a periodic structure can be represented in the form of asymptotic expansions in a small parameter \( \varepsilon \)
\[ \Phi(x) = \Phi^0(x, y) + \varepsilon \Phi^1(x, y) + \varepsilon^2 \Phi^2(x, y) + \ldots. \]  
(9)

The derivative of the function \( \Phi(x) \), taking into account formula (9), can be written as
\[ \frac{d\Phi(x)}{dx} = \frac{\partial \Phi^0(x, y)}{\partial x} + \varepsilon \frac{\partial \Phi^0(x, y)}{\partial y} + \varepsilon^2 \frac{\partial \Phi^1(x, y)}{\partial x} + \varepsilon^2 \frac{\partial \Phi^1(x, y)}{\partial y} + \ldots \]  
(10)
Any of the sought quantities is meant by \( \Phi(x) \) - displacement, stress, deformation, electrical quantities. All functions \( \Phi^j(x, y) \) are smooth in and periodic in \( y \), equal in magnitude and opposite in sign on opposite sides of the cell. We represent the desired quantities and their derivatives in the form of expansions (9), (10), substitute these expansions in the original equations (1) - (6), set the coefficients equal to zero for the same powers of a small parameter \( \epsilon \). As a result, we obtain the following equations:

Equations of motion
\[
\frac{\partial \sigma^0}{\partial y} = 0, \quad \frac{d \sigma^0}{dx} + \frac{\partial \sigma^1}{\partial y} + \rho \omega^2 u^0(x, y) = 0, \quad \frac{\partial \sigma^k}{\partial x} + \frac{\partial \sigma^{k-1}}{\partial y} + \rho \omega^2 u^k(x, y) = 0, \quad k = 1, 2, \ldots \tag{11}
\]

From the first formula (11) it follows that \( \sigma^0 = \sigma^0(x) \), and this is taken into account in the second equation (11).

The relationship between displacements and deformations
\[
\frac{\partial u^0}{\partial y} = 0, \quad e^0 = \frac{du^0}{dx} + \frac{\partial u^1}{\partial y}, \quad e^k = \frac{du^k}{dx} + \frac{\partial u^{k-1}}{\partial y}, \quad k = 1, 2, \ldots \tag{12}
\]

From the first formula (12) it follows that \( u^0 = u^0(x) \), and this is taken into account in the second equation (12).

Elasticity ratios for adhesive layers
\( \sigma^k = E_a e^k \quad k = 0, 1, 2, \ldots \tag{13} \)

Relations of electroelasticity for piezoelectric layers
\[
\sigma^k = \frac{1}{s_{33}} E_1^k - \frac{d_{33}}{s_{33}} E_1^k, \quad k = 0, 1, 2, \ldots \tag{14}
\]

Electrostatic equations
\[
\varphi^0 = \text{const}, \quad E_1^0 = -\frac{\partial \varphi^1}{\partial y}, \quad E_1^k = -\frac{\partial \varphi^{k-1}}{\partial y}, \quad k = 1, 2, \ldots \tag{15}
\]

In the piezoelectric layer \( E_1^0 \) is a constant value, while \( \varphi \) is a linear function of argument \( y \), in elastic layers these values are not. After satisfying the conditions (5) specified on the electrodes of the piezoceramic layer, the electric potential \( \varphi = \varphi^0 + \omega \varphi^1 \) and electric field strength \( E_1^0 \) are written as
\[
\varphi = V \left( 1 - \varepsilon \frac{2Y}{h_p} + \varepsilon \frac{2y}{h_p} \right), \quad E_1^0 = -V \frac{2}{h}, \quad Y = \frac{h}{\varepsilon} \tag{16}
\]

3. Effective modules and equations of the macroscopic electroelastic state of stacks
We integrate with respect to local variable \( y \) the equations (13) and (14) in the interval of the periodicity cell, taking into account that the properties of the layer materials are piecewise continuous functions
\[
E_y = \begin{cases} E_a, & 0 \leq y \leq h_a / \varepsilon \\ 1/s_{33}, & h_a / \varepsilon \leq y \leq Y \end{cases}, \quad d_{33y} = \begin{cases} 0, & 0 \leq y \leq h_a / \varepsilon \\ d_{33}, & h_a / \varepsilon \leq y \leq Y \end{cases}
\tag{17}
\]

Then the ratio of electroelasticity for the cell is written as
\[
\sigma^0 = E_y (e^0 - d_{33y} E_1^0) \tag{18}
\]
We divide this relation by $E_y$ and integrate over $y$ within the cell of periodicity from 0 to $Y$. As a result, we obtain the averaged ratio of electroelasticity

$$\sigma^0 = \widetilde{E} (e^0 - \widetilde{d}_{33} E_1^0), \quad D_1^0 = \widetilde{e}_{33} E_1^0 + \widetilde{d}_{33} \sigma^0$$

$$\widetilde{E} = h / (h_p s_{33} + h_u / E_u), \quad \widetilde{d}_{33} = d_{33} h_p / h, \quad \widetilde{e}_{33} = e_{33}^T h_p / h$$

(19)

Here $\widetilde{E}$, $\widetilde{d}_{33}$, and $\widetilde{e}_{33}$ are the effective modules.

Note that if we take into account the electrodes, then in the denominator of the expression for $\widetilde{E}$ in parentheses the term $h_m / E_m$ appears, ($h_m$ is the thickness of the electrode, $E_m$ is its elastic modulus). We can neglect it in the framework of the error of the theory of rods, since the ratio $h_m / h_p$ is an order of magnitude from $10^{-4}$ to $10^{-3}$.

Similarly, integrating the second equation (11) within the periodicity cell, we obtain the averaged equation of motion

$$\frac{d\sigma^0}{dx} + \omega^2 \rho \tilde{u}^0 = 0, \quad \rho = \frac{h_p \rho_p + h_u \rho_u}{h}$$

(20)

where $\rho$ is the material density averaged within the periodicity cell.

Integrating the second equality (12) within the periodicity cell, we obtain the relation usual in the theory of isotropic rods

$$e^0 = \frac{du^0}{dx}$$

(21)

It is taken into account that the integrals with respect to $y$ of periodic functions $\partial \sigma^1(x, y) / \partial y$, $\partial u^1(x, y) / \partial y$ are equal to zero.

Equations (19) - (21) constitute a complete system of equations describing the macroscopic behavior of the rod. They differ from the corresponding equations of the theory of isotropic rods only in the sense of physical constants.

There is no need to write out a rapidly changing electroelastic state, since it is not required in the theory of the gauge.

4. Determination of deformations from the measured difference in electric potentials

If the gauge electrodes are closed by an electric circuit with a given complex conductivity $Y = Y_0 + iY_1$, then

$$I = \int_{\Omega} \frac{dD^0}{dt} d\Omega = 2VY$$

(22)

where $I$ is the current strength, $\Omega$ is the surface of the electrode.

Substituting in (22) the formulas (16) and (19), we obtain

$$\frac{d}{dt} \int_{\Omega} - \widetilde{e}_{33}^T (1 - \widetilde{k}_{33}^2) \frac{2V}{h} + \widetilde{d}_{33} \widetilde{E} e^0 d\Omega = 2VY$$

(23)

This formula contains only one unknown quantity, it is a strain.

For harmonic vibrations formula (23) is simplified and takes the form

$$e^0 = \frac{1}{d_{33} \widetilde{E}} \left[ \frac{2V}{\omega} - \widetilde{e}_{33}^T (1 - \widetilde{k}_{33}^2) \frac{2V}{h} \right]$$

(24)

A very important characteristic of the performance of piezoelectric elements is the electromechanical coupling coefficient (EMCC). It gives the ratio of the electrical (mechanical) energy stored in the volume of a piezoelectric body and capable of conversion to the total mechanical (electrical) energy supplied to the body. Generally speaking, this is a difficult problem. But in the case under
consideration, a uniform electroelastic state takes place in the gauge and we can calculate EMCC using the tabular formula [13, 14]. But in this formula, we must replace the physical constants with the above obtained effective modules
\[
\tilde{k}_{33}^2 = \frac{\tilde{d}_{31}^2 \tilde{E}}{\tilde{\varepsilon}_{33}^2}.
\]  
(25)

We analyze the averaged elastic module \( \tilde{E} \) of the composite electroelastic rod as a function of ratio of the thickness of the adhesive layer \( h_a \) to the thickness of the piezoceramic layer \( h_p \). We assume that the electroelastic layers made of piezoceramics PZT-5. The module of elasticity for adhesive is taken as \( 2.35 \cdot 10^9 \text{N/m}^2 \). The calculation is performed for the rod with 20 cells of periodicity, the cell size is \( h = h_a + h_p = 1.1 \text{mm} \). The cell size in the calculation is constant.

It is seen that \( \tilde{E} \) decreases rapidly with increasing \( h_a \) : \( \tilde{E} = 0.53 \cdot 10^{11} \text{N/m}^2 \) for \( h_a = 0 \) (this is a homogeneous rod without adhesive layers), \( \tilde{E} = 0.438 \cdot 10^{11} \text{N/m}^2 \) for \( h_a/h_p = 0.01 \), \( \tilde{E} = 0.26 \cdot 10^{11} \text{N/m}^2 \) for \( h_a/h_p = 0.05 \), and \( \tilde{E} = 0.18 \cdot 10^{11} \text{N/m}^2 \) for \( h_a/h_p = 0.1 \). The value of CMCC also decreases rapidly with increasing thickness of adhesive layer. So at \( h_a = 0 \)
\( \tilde{k}_{33} = 0.695 \); at \( h_a/h_p = 0.01 \) \( \tilde{k}_{33} = 0.629 \); at \( h_a/h_p = 0.05 \) \( \tilde{k}_{33} = 0.476 \); at \( h_a/h_p = 0.1 \) \( \tilde{k}_{33} = 0.385 \).

The results confirm the need to define and use effective modules.

5. Conclusions
A new piezoelectric gauge based on a piezoelectric stacks was developed in the paper. The proposed gauge is fundamentally different from the Kelvin gauge. Its advantage is ease of use, high efficiency. The gauge can be made of any size and any shape. Note that the material cost of the new gauge is small, the properties are stable over a wide range, due to which piezoelectric elements are widely used in electronics, robotics, emitters and sound receivers, delay lines, etc.

6. References
[1] Tomson W (Lord Kelvin) 1856 On the electrodynamic qualities of metals. Proc. R. Soc. London 146 pp 649 – 751
[2] Handbook on experimental mechanics Soci for Exper Mech. 1990 Inc Ed. by Kobayashi A S Prentice-Hall, Inc. Englewood Cliffs, New-Jersey 07632 ISBN-10: 0133777062
[3] Infimovskaya A A, Rogacheva N N and Chernishov G N 1989 Use of thin piezoelectric strain gauges to measure small dynamic strains Mechanics of Solids 24(2) pp 157-162 (Allerton Press)
[4] Rogacheva N N 1994 The Theory of Piezoelectric Plates and Shells (CRC Press)
[5] Preumont A 2002 An Introduction to Active Vibration Control/Responsive Systems for Active Vibration Control Proc. NATO Sci. Ser. pp 1 – 41 (Dordrecht: Kluwer)
[6] Preumont A 1997 Vibration Control of Active Structures. An Introduction (Dordrecht: Kluwer)
[7] Janoch H and Clephas B 2000 Measurement and simulation of the electromechanical behavior of piezoelectric stack transducers Piezoelectric Materials: Advances in Science, Technology and Applications Eds. C. Galassi at al pp 179 – 190 (The Netherlands: Kluwer)
[8] IEEE Standart on Piezoelectricity 1987 ANSI-IEEE Std. 176 (New York: IEEE)
[9] Uwe Stobener and Lothar Gaul 2002 Piezoelectric stack actuator: FE-modeling and application for vibration isolation. Responsive Systems for Active Vibration Control Proc. NATO Sci. Ser. 85 pp 253 -266 (Dordrecht: Kluwer)
[10] Rogacheva N N 2007 The dynamic behaviour of piezoelectric laminated bars *J. of Applied Mathematics and Mechanics* 71 pp 494-510 (ELSEVIER)

[11] Bakhvalov N S and Panasenko G P 1984 *Averaging of processes in periodic media* (M : Nauka)

[12] Sanches--Palencia E 1980 *Non-Homogeneous Media and Vibration Theory* (New York: Springer)

[13] Chou C C, Rogacheva N N and Chang H Effective electromechanical coupling coefficient for piezoelements *IEEE Trans. Ultrasonics, Ferroelectrics and Frequency control* 42(4) pp 631 - 640

[14] Berlincourt D A, Curran D R and Jaffe H 1964 Piezoelectric and piezomagnetic materials and their function as transducers W.P.Mason (eds.) *Physical Acoustics, 1A*, pp 204-326 (Academic Press, New York)