Calculations of efficiency of adaptive optics system for atmospheric turbulence correction

L A Bolbasova¹,², and V P Lukin¹

¹V E Zuev Institute of Atmospheric Optics SB RAS, Academician Zuev sq.1, Tomsk, Russia, 634050
²Tomsk State University, Lenin str. 36, Tomsk, Russia, 634055

e-mail: sla@iao.ru

Abstract.
The random refractive-index fluctuations associated with atmospheric turbulence is induced wavefront distortions of optical wave propagating trough atmosphere. The adaptive optical systems are designed to achieve a diffraction-limited image by compensating phase aberrations in real time. The efficiency of adaptive optics correction is limited by time delay and with taking into account the spatial characteristics are defined by isoplanatic patch size. In this paper we investigate theoretically the efficiency of adaptive optics system for atmospheric turbulence correction. We study isoplanatic patch for X and Y tilt wavefront aberrations in Zernike terms dependent on characteristics of turbulent atmosphere for the estimations of the spatio-temporal characteristics of adaptive optical system applications.

1. Introduction

The turbulent effect from the Earth’s atmosphere degrades the performance of an optical imaging system [1]. Adaptive optics (AO) is a real-time correction system for wave front distortions. AO systems are widely used in astronomical telescopes to compensate the atmospheric perturbation [2] and laser systems for high beam qualities [3]. The adaptive optics correction efficiency is limited by time delay. To solve the problems can in predictive correction methods.

Now a great amount of effort has been invested into developing predictive correction methods to overcome this problem and compensate the time delay error of adaptive optical systems [4-6]. The prediction algorithms that have found application in adaptive optical systems are the autoregressive moving-average (ARMA) model, which is a common tool in forecasting time series [4], artificial neural network (ANN) [5] and model predictive control (MPC) approaches including Linear quadratic regulator (LQG) [6].

In most AO systems, the wavefront correction is divided into two stages: a tip–tilt mirror (TTM) for correcting the tip–tilt (TT) aberrations and a deformable mirror (DM) for correcting the high-order aberrations. The tip-tilt disturbances, which are the main components of wave-front distortions, are very detrimental to the performance of AO systems. A predictive control based on the dynamic characteristics of the atmospheric turbulence is an effective method to compensate the time delay connected with tip-tilt of wavefront [7].

The predictive correction methods using transverse wind information are becoming a hot research area [8]. The method have physical base is the Taylor "frozen turbulence" hypothesis. For this reason, having knowledge of such a wind speed is beneficial in terms of AO control system performance. The temporal fluctuations in the transverse wind speed and variations in the measured Fried parameter
severely affect the optimal design metrics and hence the performance of adaptive optical systems. The vertical distribution of wind speed determines the requirements for the operating frequency band of adaptive optic systems, through such parameter as coherence time \( \tau_0 \), depend directly on the wind speed:

\[
f_G = \frac{1}{\tau_0},
\]

\[
\tau_0 = \left(2.91k^2 \sec^{-3} \int_0^\infty d\xi C^2_n(\xi) V_{wind}(\xi)^{5/3}\right)^{-3/5}
\]

where \( C^2_n(\xi) \) - vertical distributions of structure functions of refractive index of atmosphere, \( V_{wind}(\xi) \) - wind speed, \( f_G \) - Greenwood frequency, \( k \) - wave number, \( \theta \) - zenith angle. Wind near the ground can lead to vibrations in the telescope structures, and as a result to the jitter of the image, that is tilt wavefront aberrations also.

The efficiency for adaptive optical corrections of turbulent fluctuations with taking into account the influence of the spatial and temporal characteristics are defined by the wind speed and the isoplanatic patch size[9]:

\[
\tau = L\left(\frac{2}{c} + \frac{\theta_n}{v}\right),
\]

where \( \tau \) - time delay of AO system, \( v \) - wind speed, \( \theta_n \) is isoplanatic angle, \( L \) - length of optical wave propagations, \( c \) - speed of light in vacuum.

In the paper we have aim a study isoplanatic patch size for X and Y tilt wavefront aberrations depends on characteristics of turbulent atmosphere for time delay estimations of AO system applications. AO systems designed individually for atmospheric turbulent placement conditions, which determine the requirements for both the frequency of the system and its design elements: wavefront sensors and a corrector, as well as performance evaluations in the overall system of the whole system. For it we have been calculated angular correlations for the tilt wavefront in Zernike terms. The tilt anisoplanatism in atmospheric turbulence for adaptive optics has studied in recent years [10-11]. However, influence an outer scale on a tilt isoplanatic part have not been described in the literature.

2. Isoplanatismin optical theory and adaptive optics

We have been attempted to compare the concept of isoplanatism in the optical theory and atmospheric adaptive optics for the formation of an image of an object.

In the optics the definition of strict isoplanatism for optical systems of image formation is the invariance to the displacement of the imaging operator \( L \):

\[
I(\eta', \eta) = \int \int I(\eta, \eta') L(\eta', -\eta, \eta') d\eta d\eta',
\]

where \( \eta \) - canonical coordinates on the subject and image, respectively.

The strict isoplanatism is very rarely observed in practice, therefore the operation of the optical systems is considered in separate isoplanatic zones [12], into which the field of view is divided, where it is assumed that the point spread function (PSF) has the same shape throughout the field of view. However, this is not always true for images formatting trough atmosphere, where the PSF varies over the field of view.

The imaging operator for linear instruments, as is known, can be expressed through a function describing the image of a point object \( h(x', y') \) is point spread function (PSF), which in the case of strict isoplanatism does not depend on generalized coordinates on the subject and image [13]:

\[
I(\eta', \eta) = \int \int I(x, y) h(x', -Vx, y', -Vy) dx dy
\]

where \( V \) - magnification.
The condition of isoplanatism in optical theory is in the equality to zero of the derivative of the PSF with respect to the displacement of the object \( \frac{dh(x-Vx)}{dx} \), and the norm of this derivative can serve as a measure of non-isoplanatism, and the task of the theory of isoplanatism in classical optical systems is to estimate precisely this quantity. The PSF of optical systems is completely determined by aberrations, as well as by the shape, size, and transmission of the pupil [14]. If we assume that the last factors are constant up to second-order smallness values when the point of the object is displaced, in this case the non-isoplanatism is determined by the aberrations.

This leads to the use of the term isoplanatic angle in the case of optical systems to isolate the angular region of the field of view in which the constancy of the aberrations of the optical system and, consequently, the optical transfer function (OTF) is constant [13]. Outside this limited area, the optical system is considered to be non-isoplanatic. However, when observing an image of objects through atmospheric turbulence, optical rays from different areas of the object have different optical paths in a turbulent atmosphere, each of which has its own OTF, it is obvious that the adaptive optical system cannot correct this set simultaneously [15].

In AO the definitions of the isoplanatic angle of atmospheric turbulence in adaptive optics is

\[
\theta_i = 2.91 k^3 \int_0^\infty d\xi \xi^{3/5} C_n^2(\xi)^{-3/5},
\]

where \( k = \frac{2\pi}{\lambda} \), \( \lambda \)- wavelength, \( \alpha \)- zenith angle. Or through \( r_0 \)- Fried parameter [16]:

\[
\theta_i = 0.31 \frac{r_0^\alpha}{h^\gamma},
\]

where

\[
h^\gamma = \int_0^\infty d\xi \xi^{3/5} C_n^2(\xi) \int_0^\infty d\xi C_n^2(\xi) = \mu_1/\mu_0
\]

Fried parameter:

\[
r_0 = 0.423 k^2 \sec \alpha \int_0^\infty d\xi C_n^2(\xi)^{-3/5}
\]

\( C_n^2(\xi) \)- is vertical profile of the structural parameter of fluctuations in the refractive index of the turbulent atmosphere.

The isoplanatic patch in atmosphere is presented in figure 1.

![Figure 1. Isoplanatic part.](image-url)

Angle for tilt named isokinetic for image motion and defined as

\[
\theta_i \approx 0.3 \frac{D}{h^\gamma}
\]

where \( D \)- diameter aperture, \( h^\gamma \)- defined by equation (9).

We note that the classical theory of isoplanatism [12-14,16] is developed for the axial zone of centered optical systems with an image at a finite distance. In adaptive optical systems operated in atmosphere
aberrations due to the action of atmospheric turbulence. Thus the definition of isoplanatism in the classical theory as uniformity of aberrations of an optical system over the field leads to the necessity to describe it with the aid of the aberration function as a function of ray coordinates. In geometric optics, it is either wavefront aberration or eikonal [16]. In atmospheric adaptive optical systems, this role is played by wave aberrations caused by the atmospheric turbulence.

3. Angular correlations for Y-Tilt and X-Tilt
The wave aberration function can be represented in a series in polynomials Zernike, where each term of the series characterizes aberrations by radial and azimuthal degrees. The tilt is a deviation in the direction a beam of light propagates and quantifies the average slope in both the X and Y directions of a wavefront (Y-Tilt, vertical tilt and X-Tilt, horizontal tilt). Rapid optical tilts in both X and Y directions are jitter of image of object [18].

Phase fluctuations in term Zernike at the angular distance $\theta$ is [19]:

$$ S(\rho, \theta) = \sum_{j=1}^{\infty} a_j(\theta) Z_j(\frac{2\rho}{D}) $$

We calculated the angular correlation functions:

$$ B = \left\{ a_j(\rho_1, \theta) a_j(\rho_2, \theta) \right\} = \int d^2 \rho d^2 \rho' W(\rho) W(\rho') \left\{ S(\rho_1, \theta) S(\rho_2, \theta) \right\} Z_j(\frac{2\rho_1}{D}) Z_j(\frac{2\rho_2}{D}) = $$

$$ = \left( \frac{1}{\pi R^2} \right)^2 \int d^2 \rho d^2 \rho' \left\{ S(\rho_1, \theta) S(\rho_2, \theta) \right\} Z_j(\frac{2\rho_1}{D}) Z_j(\frac{2\rho_2}{D}) $$

We used the Fourier-Stieltjes integral of random process:

$$ S(\rho) = \int \int \exp(i\kappa \rho) d^2 H(\kappa) $$

Then we changed the order of integration over the spatial and frequency coordinates, denoting by the Fourier transform for the Zernike polynomials:

$$ Q(\kappa) = \frac{1}{\pi R^2} \int d^2 \rho Z_j(\frac{2\rho}{D}) \exp(i\kappa \rho) $$

After we obtain the following expression for the correlation function:

$$ B = \int \int \int \left\{ d^2 H(\kappa) d^2 H(\kappa') \right\} $$

Taking into account the delta - correlation of the spectral components of H, the spectral density:

$$ \left\{ d^2 H(\kappa) d^2 H(\kappa') \right\} = F(\kappa) \delta(\kappa - \kappa') d^2 \kappa d^2 \kappa' $$

Then

$$ B = \int \int \left\{ Q(\kappa) \right\}^2 F(\kappa) \exp(i\kappa(\rho_2 - \rho_1)) d^2 \kappa $$

We used the following model of the spectral density of fluctuations in the refractive index:

$$ \Phi_n(\kappa) = 0.033 C_n^2(\frac{\xi}{h_z}) \kappa^{-11/3} $$

$\kappa$ - is the wave number for turbulent inhomogeneities.

We used the following model of the refractive index structural parameter:

$$ C_n^2(\xi) = C_n^2 \exp\left[-\frac{\xi}{h_z}\right] $$

Then, passing to the polar coordinate system and integrating over the angular coordinate and using the recurrence formula, in results we obtain the following expression for the correlation function:

$$ B(\theta) = 8\pi \int_0^\infty \kappa d\kappa F(\kappa) \frac{J_n^2(\kappa)}{\kappa^2} \left[ J_0 \left( \frac{2\theta h_z}{D} \right) J_0 \left( \frac{2\theta h_z}{D} \right) \right] $$

where $J$ - are Bessel functions, $n$ is the order of aberrations.

We used the correlation coefficient:
As a result, from (20) we obtain the following expression for tilt wavefront (n = 1):

$$b_{x,y}(\theta) = \frac{B(\theta)}{B(0)}.$$  \tag{21}

In this formula, the negative sign corresponds to the longitudinal slope along the X-tilt, and the plus sign corresponds to transverse, that is, the Y-tilt.

In results after integration, and using (7) for $h_\kappa \approx 0.31 r_0 / \theta_0$, we obtain the following two analytical expressions.

For X-tilt

$$b_x(\theta) = -0.1023 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-2} + 0.7211 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-1/3} P_{\frac{5}{6}} \left( \frac{5}{6} \theta_0 \right) \left( \frac{5}{6} \right)^{3/2}, \left( -\frac{1}{3} \right), \frac{1}{4} \left( \frac{0.31 D \theta_0}{\theta_0} \right)^2$$

$$-0.0092 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-3} P_{\frac{1}{2}} \left( \frac{23}{6} \right), \left( \frac{13}{3}, \frac{19}{3}, \frac{1}{4} \right) \left( \frac{D \theta_0}{\theta_0} \right)^2$$ \tag{23}

For Y-tilt

$$b_y(\theta) = -0.1023 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-2} + 1.0810 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-1/3} P_{\frac{5}{6}} \left( \frac{5}{6} \theta_0 \right) \left( \frac{5}{6} \right)^{3/2}, \left( -\frac{1}{3} \right), \frac{1}{4} \left( \frac{0.31 D \theta_0}{\theta_0} \right)^2$$

$$-0.0363 \left( \frac{0.31 r_0 \theta}{D \theta_0} \right)^{-1} P_{\frac{1}{2}} \left( \frac{127}{6} \right), \left( \frac{10}{3}, \frac{16}{3}, \frac{1}{4} \right) \left( \frac{D \theta_0}{\theta_0} \right)^2$$ \tag{24}

where $P_{\alpha} [...]$ - generalized hypergeometric function.

In results the dependence of the angular correlation of tilt wavefront on aperture size and Fred parameter is obtained.

4. Numerical results

Figures 2 and 3 shows numerical calculations of correlation coefficient for various ratio $D / r_0$. The ratio of the size of the aperture of the telescope to the Fried parameter are defined for the classification of the telescope size and the influence of atmosphere. If the ratio equal 10 is medium-sized telescope, the telescope can be considered small, if <10 and in the opposite case it is large telescopes in adaptive optics. The argument is relation of the isokinetic angle to the isoplanatic angle. For comparisons we present numerical results for coma and defocus aberrations also.

**Figure 2.** Dependence of the angular correlation coefficient from top to bottom: X-axis tilt, Y- tilt axis, coma, defocus for $D / r_0 = 50$. 

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Figure 3. Dependence of the angular correlation coefficient from top to bottom: X-axis tilt, Y-tilt axis, coma, defocus for a- \( D / r_0 = 20 \).

Figure 2 is typical conditions for large ground-based astronomical telescopes with AO systems for best astronomical seeing. Figure 3 is near to conditions of operations of laser systems with AO in atmosphere.

The size of the isoplanatism patch for the slopes is two orders of magnitude greater than for other wavefront aberrations. And it is larger for X-tilt than for Y-tilt. The results are confirmed the formula (10) for isokinetic angle in general.

5. Influence the outer scale of atmospheric turbulence

In accordance with the theory of turbulence, it can be characterized by two characteristic scales of motion: the outer scale of turbulence and the internal scale. We have aimed investigated influence outer scale of atmospheric turbulence as important parameter optical turbulence.

For it we use following the model of the spectral density of fluctuations in the refractive index:

\[
\Phi_n(\kappa) = 0.033C_n^2(\xi)\kappa^{-11/3} (1 - \exp(-\kappa^2/\kappa_0^2))
\]  

(25)

where \( \kappa_0 = 2\pi / L_0 \), \( L_0 \) - effective outer scale of atmospheric turbulence [18]:

\[
L_0^* = \left( \int_0^\infty d\xi C_n^2(\xi) \int_0^\infty d\xi C_n^2(\xi) \right)^{1/5},
\]

(26)

In results following expression for the coefficient of angular correlation \( b_{\omega,\omega}(\theta) = B(\theta)/B(0) \) for tilt wavefront (n=1) are obtained:

\[
b_{\omega,\omega}(\theta) = \frac{\int_0^\infty d\xi C_n^2(\xi) \int_0^\infty d\kappa \kappa^{-11/3} \{1 - \exp(-\kappa^2/\kappa_0^2)\} J_2(\kappa) \left[ J_0 \left( \frac{2\theta h_z}{D} \kappa \right) + \frac{2\theta h_z}{D} \kappa \right] J_2 \left( \frac{2\theta h_z}{D} \kappa \right) \}}{\int_0^\infty d\xi C_n^2(\xi) \int_0^\infty d\kappa \kappa^{-11/3} \{1 - \exp(-\kappa^2/\kappa_0^2)\} J_2(\kappa) \left[ J_0 \left( \frac{2\theta h_z}{D} \kappa \right) + \frac{2\theta h_z}{D} \kappa \right] J_2 \left( \frac{2\theta h_z}{D} \kappa \right) \}}
\]

(27)

where parameter \( \gamma = \theta h_z / D \).

In Figure 4 present the coefficient of the angular correlation for Y-tilt and X-tilt (a and b) in the case of an infinite external turbulence scale (curve 1) and when the outer turbulence scale of atmospheric turbulence is taken into account (curve 2 and 3). These results of numerical calculations are presented in a logarithmic scale.

The obtained results show that the value of the outer scale of atmospheric turbulence strictly affects the angular correlation of the tilt wavefront, and as a consequence, the size of the isoplanatism patch of the AO system correcting the tip–tilt aberrations. The values obtained with an infinite outer scale are higher than in the case of a model dependent on the outer scale. In results the field of view of the AO system is reduced. The outer scale of atmospheric turbulence will be influences on the spatial and temporal characteristics of AO correcting the tip–tilt aberrations also. Therefore significant increase in frequency band of AO systems of tilt corrections can be required.
6. Conclusions
In the paper we focused on tilt wavefront correlations for determinate of isoplanatic requirements as size of atmosphere where adaptive optics correcting of the tip–tilt aberrations is effectively. We obtained analytical expressions for correlations coefficient for X-tilt and Y-tilt. In results we detected influence the outer scale on for adaptive optics correcting the tip–tilt aberrations in atmosphere. The dependence from various atmospheric turbulent conditions are investigated.

7. References
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Acknowledgments
This work was supported by Russian Science Foundation grant № 17-79-20077.