STOCHASTIC HOMOGENIZATION FOR VARIATIONAL SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

C. VITERBO

Wednesday 12th May, 2021, 02:40

CONTENTS

1. Introduction 2
2. Notations and abbreviations 8
3. Acknowledgments and general remarks. 9
4. Non-compact supported Hamiltonians 10
5. Spectral invariants in cotangent bundles of non-compact manifolds 13
5.1. The case of Lagrangians 13
5.2. The case of Hamiltonians in $T^*\mathbb{R}^n$ 22
6. Compactness and ergodicity 24
7. Some results on compact abelian metric groups 28
8. Regularization of the Hamiltonians in $\text{Ham}_f^c$ 29
9. Homogenization in the almost periodic case 33
10. Proof of the Main Theorem 43
11. The coercive case 45
12. The discrete case (Proof of Corollary 1.6) 45
13. Extending the Main Theorem 45

Appendix 1: Generating Function Quadratic at Infinity (GFQI) for non-compact Lagrangians: Proof of Theorem 4.5 46
Appendix 2: Proof of Lemma 7.3 and Lemma 9.17 47
Appendix 3: Approximation of Generating functions and symplectic integrators 49
Appendix 4: Proof of Proposition 9.3 50
References 54

DMA, École Normale Supérieure, 45 Rue d’Ulm, 75230 Cedex 05, FRANCE. On leave from Department of Mathematics, Université de Paris-Sud, Orsay. We also acknowledge support from ANR MICROLOCAL (ANR-15-CE40-0007) and NSF grant DMS- 1440140.
1. **Introduction**

Let \((\Omega, \mu)\) be a probability space endowed with an ergodic action \(\tau\) of \((\mathbb{R}^n, +)\). This means that if \(X \subset \Omega\) satisfies \(\tau_a X \subset X\) for all \(a \in \mathbb{R}^n\), then \(\mu(X) = 0\) or \(1\). Let \(H(x, p; \omega) = H_\omega(x, p)\) be a smooth Hamiltonian on \(T^*\mathbb{R}^n\) parametrized by \(\omega \in \Omega\) and such that

\[
H(a + x, p; \tau_a \omega) = H(x, p; \omega)
\]

We shall specify later the assumptions satisfied by \(H\). We now consider for an initial condition \(f \in C^0(\mathbb{R}^n)\), the family of stochastic Hamilton-Jacobi equations

\[
(HJS_\varepsilon) \quad \begin{cases}
\frac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\left(\frac{x}{\varepsilon}, \frac{\partial u^\varepsilon}{\partial x}(t, x; \omega)\right) = 0 \\
u^\varepsilon(0, x; \omega) = f(x)
\end{cases}
\]

Fixing \(\omega\), we can consider different type of generalized solutions (there is generally no smooth solution) for this equation. The most interesting ones are either the viscosity solution of Crandall-Lions (see \([C-L]\) and also \([Ba, B-C]\)), or the variational solutions defined in \([Sik2, Ch, V3, V4]\), both requiring some assumptions on \(f\) and \(H\) that will be specified later. The problem of stochastic homogenization for the above equation is to determine whether for \(\mu\)-a.e. \(\omega\), the sequence \(u^\varepsilon(t, x; \omega)\) \(C^0\)-converges on compact sets to \(\overline{u}(t, x)\), solution of

\[
(H\overline{H}) \quad \begin{cases}
\frac{\partial v}{\partial t}(x) + \overline{H} \left(\frac{\partial v}{\partial x}(x)\right) = 0 \\
v(0, x) = f(x)
\end{cases}
\]

where \(\overline{H}\) is to be determined (and in general cannot be defined explicitly). Note that \(\overline{H}\) does not depend on \(\omega\) by the ergodicity hypothesis.

A classical case is the so-called (non-stochastic) periodic case, corresponding to the case where \(\Omega = \mathbb{T}^n\), \(\tau_a\) is the translation on the torus, and \(H(x, p; \omega) = K(x - \omega, p)\) where \(K : T^*\mathbb{T}^n \rightarrow \mathbb{R}\) is a Hamiltonian on the torus.\(^1\) For viscosity solutions, homogenization in the periodic non-stochastic case has been settled by \([L-P-V]\) in 1988, and for variational solutions by \([V5]\) in 2008.

For the general stochastic case, this problem has been solved for viscosity solutions by Rezakhanlou-Tarver and Souganidis in \([R-T, Soug]\),

\(^{1}\)Indeed, if \(u^\varepsilon(t, x)\) is the solution (either viscosity or variational) of \(\frac{\partial v^\varepsilon}{\partial t}(t, x) + K(\frac{x}{\varepsilon} - \omega, \frac{\partial v^\varepsilon}{\partial x}(t, x)) = 0\) then \(v^\varepsilon(t, y) = u^\varepsilon(t, y + \varepsilon \omega) = 0\) satisfies \(\frac{\partial v^\varepsilon}{\partial t}(t, x) + K(\frac{x}{\varepsilon}, \frac{\partial v^\varepsilon}{\partial x}(t, y)) = 0\). Thus \(u^\varepsilon(t, y) = v^\varepsilon(t, y - \varepsilon \omega)\), and convergence of \(v^\varepsilon\) to \(\overline{v}\) as \(\varepsilon\) goes to 0 is equivalent to convergence of \(u^\varepsilon\) to \(\overline{u} = \overline{v}\). See the proof of Corollary 1.6 for another method of reducing to the periodic case.
assuming $H$ is convex in $p$. Beyond the quasi-convex case and some very special cases (see for instance [A-T-Y]), nothing is known for viscosity solutions in the general (i.e. for $H$ non-convex in $p$) case, and counterexamples have been found, first by Ziliotto and then by Feldman-Souganidis ([Zi], [F-S]).

We settle here the case of variational solutions without any convexity assumption. Note that the construction of a variational solution relies on the choice of a field of coefficients for the homology theory we use, but once the field is chosen, the variational solution is uniquely defined. We shall here fix once and for all a coefficient field (the reader can think of $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{R}$ for example). As in [V5], our results hold when $H$ is either compact supported or coercive in the $p$ direction. Note that fixing $\omega$, if $V_t(H)f = u(t,x)$ is the variational solution operator of the Hamilton-Jacobi equation, and $S_t(H)f$ the viscosity semigroup, we know that for $H$ convex in $p$ we have $S_t(H) = V_t(H)$. Our result thus implies the stochastic homogenization for viscosity solutions in the convex case as in [R-T] and [Soug]. In the general case it has been proved in [WQ1, WQ2] (see also [R], theorem 1.19) that

$$S_t(H) = \lim_{n \to +\infty} (V_t/(H))^n$$

However, in the non-convex case, stochastic homogenization of the viscosity solutions is not a consequence of stochastic homogenization for variational solutions, unless the above convergence is uniform in $\varepsilon \in [0,1]$, for $H_\varepsilon(x,p) = H(x/\varepsilon, p)$.

Of course, as in [V5], the equation $(HJS_\varepsilon)$ is related to the Hamiltonian flow of $H(x/\varepsilon, p; \omega)$ given by

$$\phi^{t}_{\varepsilon,\omega} = \rho_{\varepsilon}^{-1} \phi^{t}_{\omega} \rho_{\varepsilon}$$

where $\phi^{t}_{\omega}$ is the flow of $H(x, p; \omega)$ and $\rho_{\varepsilon}(x, p) = (x/\varepsilon, p)$.

$^2$See for example in [C-V] and more explicitly in [WQ2] and Appendix B in [R2]

$^3$This means that it sends $f$ to the variational solution of

$$(HJS) \begin{cases} \frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0 \\ u(0,x) = f(x) \end{cases}$$

Note that the operator is neither linear, not a semigroup (since variational solutions do not have the Markov property).

$^4$However in that case our method is much more complicated.

$^5$Indeed, if we knew that $V_t(\varepsilon) = V_t(H_\varepsilon) = \overline{V}_t + R_t(\varepsilon)$ where $\|R_t(\varepsilon)\| \leq C \varepsilon$, and $\overline{V}_t = V_t(\overline{H})$ is the homogenized operator, we would get that $\|V_t/(\varepsilon))^n - (\overline{V}_t/n)^n\| \leq C \varepsilon$ hence, setting $\overline{S}_t = \lim_n (\overline{V}_t/n)^n$ we would have $\|S_t(\varepsilon) - \overline{S}_t\| \leq C \varepsilon$ hence $\lim_{\varepsilon \to 0} S_t(\varepsilon) = \overline{S}_0$. Since there are counterexamples to stochastic homogenization for viscosity solutions (see [Zi], [F-S]), the above inequality cannot hold in general.
We shall prove analogously to [V5] that for almost all \( \omega \), we have
\[
\varphi_{t,\omega}^\epsilon \xrightarrow{\gamma_c} \varphi_{t,\omega}^c
\]
but since we are on a non-compact base we have to redefine the \( \gamma \)-distance, that we shall denote by \( \gamma_c \).

Our main result is

**Theorem 1.1** (Main theorem). Let \( H(x, p; \omega) \) be a stochastic Hamiltonian on \( T^*\mathbb{R}^n \times \Omega \), where \((\Omega, \mu)\) is a probability space endowed with an action \( \tau \) of \( \mathbb{R} \). We assume the following conditions are satisfied:

1. For all \( a \in \mathbb{R}^n \), the map \( \tau_a \) is measure preserving and the action \( \tau \) is ergodic for the measure \( \mu \) (i.e. invariant sets have measure 0 or 1).
2. We have for all \( a \in \mathbb{R}^n \), \((x, p) \in T^*\mathbb{R}^n \) and almost all \( \omega \in \Omega \) the identity \( H(x + a, p, \tau_a \omega) = H(x, p, \omega) \).
3. The map \((x, p) \mapsto H(x, p, \omega) \) is \( C^{1,1} \) for \( \mu \)-almost all \( \omega \).
4. For almost all \( \omega \), \( H \) is compact supported in the \( p \) direction i.e. the set \( \{ p \mid \exists x \in \mathbb{R}^n, H(x, p; \omega) \neq 0 \} \) is bounded.
5. There exists \( C \) such that for almost all \( \omega \) and for all \((x, p) \in T^*\mathbb{R}^n \) we have \( \left| \frac{\partial H}{\partial p}(x, p; \omega) \right| \leq C \).
6. There exists \( C \) such that for almost all \( \omega \) we have \( \sup_{(x, p) \in T^*\mathbb{R}^n} |H(x, p; \omega)| \leq C \).

Then if \( \varphi_{t,\omega}^\epsilon \) is the flow of \( H_{\epsilon,\omega}(x, p) = H(\frac{x}{\epsilon}, p) \) there is a function \( \overline{H} \) in \( C^0(\mathbb{R}^n, \mathbb{R}) \) such that
\[
\varphi_{t,\omega}^\epsilon \xrightarrow{\gamma_c} \varphi_{t,\omega}^c
\]
for the topology \( \gamma_c \) that will be defined in section 5. Here \( \varphi_{t,\omega}^\epsilon \) denotes the flow of \( \overline{H} \) in \( \mathcal{D} \mathcal{H} \mathcal{A} \text{m}(T^*\mathbb{R}^n) \) the \( \gamma_c \)-completion of \( \mathcal{D} \mathcal{H} \mathcal{A} \text{m}(T^*\mathbb{R}^n) \). As a consequence if \( f \) is uniformly continuous on \( \mathbb{R}^n \), then a.s. in \( \omega \in \Omega \) the variational solution \( u^\epsilon(t, x; \omega) \) of \( (HJS^\epsilon) \) converges to the variational solution \( \overline{u}(t, x) \) of \( (HJH) \).

**Remark 1.2.**
1. Existence and uniqueness of the variational solution for \( (HJS^\epsilon) \) follows from [C-V], pp. 266-276 (since we are in the case of a non-compact base). The bounded propagation speed condition in [C-V] are more general than the ones in the present paper and are obviously satisfied in the fiberwise compact supported case.
2. Set \( \Omega_c = \{ \omega \in \Omega \mid \sup_{(x, p) \in T^*\mathbb{R}^n} |H(x, p; \omega)| \geq c \} \) is \( \tau \) invariant. If it has measure 0, then \( \sup_{(x, p) \in T^*\mathbb{R}^n} |H(x, p; \omega)| = +\infty \) for a.e. \( \omega \).
3. The set \( \Omega^c_R = \{ \omega \in \Omega \mid \text{supp}(H) \subset \mathbb{R}^n \times B(R) \} \) is also invariant by \( \tau \). It thus either has measure 1 for some \( R \), and then the bound in [4],...
is independent from $\omega$ in set of full measure, or it has measure 0 for all $R$ and then for a.e. $\omega$, condition (4) is not satisfied. In the first case, we shall say that the $H_\omega$ have uniform fiber compact support.

The compact supported case is usually not the most interesting in applications. However the above theorem implies

**Corollary 1.3 (Main corollary).** Let $H(x, p; \omega)$ be a stochastic Hamiltonian on $T^*\mathbb{R}^n \times \Omega$, where $(\Omega, \mu)$ is a probability space endowed with an action $\tau$ of $\mathbb{R}^n$. We assume the following conditions are satisfied:

1. Conditions (1)-(3) as in the Main Theorem
2. For all $(x, p; \omega)$ we have $|\frac{\partial H}{\partial p}(x, p; \omega)| \leq h(|p|)$ for almost all $\omega$ for some continuous function $h: \mathbb{R} \rightarrow \mathbb{R}$.
3. for almost all $\omega$ $H$ is coercive, that is $\lim_{|p| \rightarrow +\infty} |H(x, p; \omega)| = +\infty$ uniformly in $x$

If $H$ satisfies the above assumptions and $f$ is uniformly continuous on $\mathbb{R}^n$, there is a coercive function $H$ in $C^0(\mathbb{R}^n, \mathbb{R})$ such that a.e. in $\omega$, the variational solution $u^\varepsilon(t, x; \omega)$ of

$$
\frac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\left(\frac{x}{\varepsilon}, \frac{\partial u^\varepsilon}{\partial x}(t, x; \omega); \omega\right) = 0
$$

converges to the variational solution $\overline{u}(t, x)$ of

$$
\frac{\partial v}{\partial t}(t, x) + \overline{H}\left(\frac{\partial v}{\partial x}(t, x)\right) = 0
$$

with

$$
v(0, x) = f(x)
$$

**Examples 1.4.**

1. Let $\Omega$ be the space of $C^1$ functions on $\mathbb{R}^n$, $(\tau_a f)(x) = f(x+a)$ and $\mu$ some measure on $\Omega$ invariant by $\tau_a$ and ergodic. Let $V$ be a bounded function. Set $H(x, p; \omega) = \frac{1}{2} h(p) - V(\omega(x))$, where $h$ is coercive. This satisfies the assumptions of the Corollary and corresponds to a random potential, with probability $\mu$.

2. (P-R, example 2.4(ii)) Let $H_0(q, p)$ be a Hamiltonian and $H(q, p; \omega) = \sum_{j \in \mathbb{Z}} H_0(q - \omega_j, p)$ where $\omega = \omega_j$ is a stationary point process, that is a probability on $\mathbb{R}^2$ invariant by translation. This makes sense provided $H_0$ decreases fast enough as $q$ goes to infinity. Then $H$ satisfies the assumption of the above Corollary.

**Remark 1.5.** Here are a few comments

1. We could of course also state a convergence result in the coercive case for the sequence $\varphi_{\varepsilon, \omega}$, it is just that the statement of convergence would be a little more complicated to state.
(2) Note that ergodicity implies that
\[ \Omega(m, p) = \left\{ \omega \mid \sup_{x \in \mathbb{R}^n} H(x, p; \omega) \leq m \right\} \]
is an invariant set for the \( \tau \)-action, hence has measure either 0 or 1. As a result by ergodicity property [3a] or [4] hold either for a set of \( \omega \) of zero measure or for a.e. \( \omega \). We set \( h_+(p) \) to be the smallest \( m \) such that \( \Omega(m, p) \) has measure 1, where \( h_+(p) \in \mathbb{R} \cup \{+\infty\} \). Thus we have \( \sup_{x \in \mathbb{R}^n} H(x, p; \omega) = h_+(p) \) a.e. in \( \Omega \) and similarly \( \inf_{x \in \mathbb{R}^n} H(x, p; \omega) = h_-(p) \) a.e. in \( \Omega \), where \( h_-(p) \in (-\infty, +\infty) \). Notice that assumption [3a] implies that \( h_+(p) \) is finite, and that \( \lim_{|p| \to +\infty} h_+(p) = +\infty \). This condition is more or less explicit in both [R-T] (see their conditions (Aii)-(Aiii)) and [Soug] (see his Condition 0.2). Similarly
\[ h_+(p, \omega) = \sup_{x \in \mathbb{R}^n} \left| \frac{\partial H}{\partial p}(x, p; \omega) \right| \]
is invariant by \( \tau \) hence independent from \( \omega \) a.e. in \( \Omega \), and equal to \( h_+(p) \), so [2a] and [5] either hold a.s. or do not hold a.s in \( \Omega \).

(3) Setting \( h_+(r) = \sup \{h_+(p) \mid |p| \leq r \} \) and \( h_-(r) = \inf \{h_-(p) \mid |p| \geq r \} \) we have that a.s. in \( \omega \), the set \( \{(q, p) \mid H_\omega(q, p) \leq c \} \) contains \( \mathbb{R}^n \times B_r \) for \( c \geq h_+(r) \) and is contained in \( \mathbb{R}^n \times B_r \) for \( c \leq h_-(r) \). In particular in case [3a] the coercivity is automatically uniform in \( x \) (i.e. we can bound \( \{p \mid H(x, p; \omega) \leq c \} \) independently from \( x \)).

(4) We shall reduce the case [3a] where \( H \) is coercive to the uniformly fiberwise compact supported case by replacing \( H \) by \( \chi_R(H) \) that is compact supported where \( \chi_R : \mathbb{R} \to \mathbb{R} \) is a function supported in \( \left]-\infty, R + 1\right] \) such that \( \chi'(t) = 1 \) for \( t \leq R \) (see [C-V], Appendix B). Then \( H_{\chi_R} = \chi_R(H) \), also satisfies \( H_{\chi_R}(x + a, p; \tau_a \omega) = H_{\chi_R}(x, p; \omega) \).

(5) Let us consider a Hamiltonian \( H \) convex in \( p \) and uniformly coercive (i.e. there exists a function \( h_+(p) \) going to infinity, such that \( h_-(p) \leq H(x, p; \omega) \leq h_+(p) \)) as follows form the assumptions in both [R-T] (2.5) (ii) and (2.8) p. 280 and [Soug] (see Condition 0.2). Then we claim that its truncation \( H_{\chi_R} = \chi_R(H) \) satisfies assumption [5] of the Main Theorem (condition [4] is obvious). This is because \( \frac{\partial H_{\chi_R}}{\partial p} = \chi_R'(H)\frac{\partial H}{\partial p} \), so it is enough to prove that \( \frac{\partial H}{\partial p} \) is bounded on a set \( |p| \leq C \). But if \( |\frac{\partial H}{\partial p}(x_0, p_0)| \geq A \), we can find \( p_1 \) with \( |p_1| \leq 2C \) such that \( p_0 - p_1 \) is collinear with \( \frac{\partial H}{\partial p}(x_0, p_0) \) and...
\[ |p_0 - p_1| = C, \text{ so that} \]
\[ \sup_{|p| \leq 2C} h_+(p) - \inf_{|p| \leq 2C} h_-(p) \geq H(x, p_1) - H(x, p_0) \geq 0 \]
\[ \left\langle \frac{\partial H}{\partial p}, p_0 - p_1 \right\rangle \geq C \left| \frac{\partial H}{\partial p} \right| = CA \]

hence \( A \) is bounded.

Our result can be easily extended, since we do not need the full action of \( \mathbb{R}^n \). For example if we have an action of \( \mathbb{Z}^n \) we get the following

**Corollary 1.6.** With the same assumptions as in the Main Theorem except that we have an action of \( \mathbb{Z}^n \) (instead of \( \mathbb{R}^n \)) on \( \Omega \), still denoted by \( \tau \), and the first two assumptions are replaced by

1. (1b) For all \( z \in \mathbb{Z}^n \), the map \( \tau_z \) is measure preserving and ergodic
2. (2b) We have for all \( z \in \mathbb{Z}^n \), \((x, p) \in T^* \mathbb{R}^n \) and almost all \( \omega \in \Omega \) the identity

\[ H(x + z, p, \tau_z \omega) = H(x, p, \omega) \]

while conditions (3)-(6) are unchanged. We then have the same conclusion as in the Main Theorem.

Finally, note that ergodicity of \( \tau \) on \( \Omega \) is not required, since we can use the ergodic decomposition theorem (cf. [Gr-Sch]), which holds for Borel spaces\(^6\) and obtain the

**Corollary 1.7.** With the same assumptions as in the main theorem (resp. Corollary 1.6) except that the action \( \tau \) is not supposed to be ergodic but we

\(^6\)That is isomorphic (as a measured space) to a complete separable metric space with a measure defined on its Borel algebra.
assume \((\Omega, \mu)\) is a Borel space, we have the same conclusion, except that \(\overline{H}(p; \omega)\) now depends on \(\omega \in \Omega\) and is constant on each ergodic component of \(\tau\).

Our proof will require the following steps, starting from the uniformly fiber compact supported case:

1. On \(\mathcal{Ham}_{fc}(T^*\mathbb{R}^n)\) the set of uniformly fiberwise compact supported Hamiltonians on \(T^*\mathbb{R}^n\) we define a metric \(\gamma_c\) (see Section 4 and 5).
2. We identify \(\Omega\) to \(\widehat{\mathcal{H}}_\Omega\) the set of \(H_\omega\) for \(\omega \in \Omega\), and \(\widehat{\mathcal{H}}_\Omega\) its completion for \(\gamma_c\). We then prove that ergodicity implies compactness of the metric space \((\widehat{\mathcal{H}}_\Omega, \gamma_c)\) (see Section 6 and 7). The action of \(\mathbb{R}^n\) on \(\mathcal{H}_\Omega\) given by \((\tau_a H)(x, p; \omega) = H(x - a, p; \omega) = H(x, p; \tau_a \omega)\) extends to an action of a compact connected metric abelian group \(\mathbb{A}_\Omega\) on \((\widehat{\mathcal{H}}_\Omega, \gamma_c)\), and \(\mathbb{R}^n\), through the action \(\tau\), is identified to a dense subgroup of \(\mathbb{A}_\Omega\). Moreover we prove that for \(\mu\)-almost all \(H\) in \(\mathcal{H}_\Omega\), the \(\mathbb{A}_\Omega\) orbit of \(H\) is equal to \(\widehat{\mathcal{H}}_\Omega\).
3. In Section 8 we prove a regularization theorem showing that the action of \(\mathbb{A}_\Omega\) on \(\widehat{\mathcal{H}}_\Omega\) can be approximated by an action of a finite-dimensional torus (note that \(\mathbb{A}_\Omega\) is not in general a finite dimensional torus, but is a projective limit of finite dimensional tori).
4. We prove in Section 9 that homogenization holds when \(\mathbb{A}_\Omega\) is a finite dimensional torus (quasi-periodic case) and \(\omega \mapsto H_\omega\) is continuous for the \(C^0\)-topology instead of the \(\gamma_c\)-topology.
5. In Section 10 we conclude the proof in the fiberwise compact case, and in Section 11 for the coercive case and in Section 12 for the discrete case.

2. Notations and abbreviations

- \(\Omega\) a probability space with measure \(\mu\)
- a.s. or a.e.: almost surely or almost everywhere in \((\Omega, \mu)\)
- GFQI: Generating Function Quadratic at Infinity
- \(H^*, H_*\) cohomology and homology with coefficients in some field \(\mathbb{K}\).
- \(\mu_N\) the fundamental class in \(H^d(N)\) (for a closed manifold) or \(H^d(N, \partial N)\) (for a manifold with boundary) or \(H^d_c(N)\) (for a non-compact manifold) where \(d = \dim(N)\)
- \(1_N\) the generator of \(H^0(N)\)
- \(T^*N\) the cotangent bundle of \(N\) with the standard symplectic form \(\omega = d\lambda\), where \(\lambda = pdq\)
- \(\overline{T^*N}\) the cotangent bundle of \(N\) with the opposite of the standard symplectic form \(\omega = -d\lambda\), where \(\lambda = pdq\)
• $C^0_{f_c}([0, 1] \times T^*\mathbb{R}^n)$ set of continuous functions on $[0, 1] \times T^*\mathbb{R}^n$ (viewed as “continuous Hamiltonians” which are fiberwise compact)
• $\mathcal{H}am_{f_c}(T^*N)$ the set of uniformly fiberwise compact supported autonomous Hamiltonians
• $\mathcal{H}am_{f_c}([0, 1] \times T^*\mathbb{R}^n)$ the set of uniformly fiberwise compact supported time-dependent Hamiltonians
• For a Hamiltonian $H$ on $T^*N$, $\phi^t_H$ is the solution of $\frac{d}{dt}\phi^t_H(z) = X_H(t, \phi^t_H(z))$ such that $\phi^0_H(z) = z$. We set $\phi_H = \phi^1_H$
• $\mathcal{D}\mathcal{H}am_{FP}(T^*N)$ is the image by $H \mapsto \phi_H$ of $\mathcal{H}am_{f_c}([0, 1] \times T^*\mathbb{R}^n)$
• $\mathcal{D}\mathcal{H}am_{BP}(T^*N)$ (resp. $\mathcal{H}am_{BP}(T^*N)$, $\mathcal{H}am_{BP}([0, 1] \times T^*N)$) elements in $\mathcal{D}\mathcal{H}am(T^*N)$ (resp. $\mathcal{H}am(T^*N)$, $\mathcal{H}am([0, 1] \times T^*N)$) having FPS.
• $\gamma_c$ the uniform topology on $L(T^*N)$ (see Definition 5.15)
• $\hat{L}(T^*N)$ the completion for $\gamma_c$ of $L(T^*N)$ (see Definition 5.15)
• $\mathcal{D}\mathcal{H}am_{FP}(T^*N)$ (resp. $\mathcal{D}\mathcal{H}am_{BP}(T^*N)$, $\mathcal{D}\mathcal{H}am_{f_c}(T^*N)$) the completion for $\gamma_c$ of $\mathcal{D}\mathcal{H}am_{FP}(T^*N)$ (see Definition 5.21)
• $G_f$ the graph of $df$ in $T^*N$
• $L$: for $L \in \mathcal{L}(T^*N)$ we define $\overline{L} = \{(x, -p) \mid (x, p) \in L\}$ where $f_{\overline{L}} = -f_L$.

3. ACKNOWLEDGMENTS AND GENERAL REMARKS.

I would like to acknowledge the hospitality of M.S.R.I. during the semester "Hamiltonian systems, from topology to applications through analysis" in the fall of 2018. The idea of this work came through attending talks and several discussions with Fraydoun Rezakhanlou, that I want to thank wholeheartedly here for generously sharing his ideas.

As the reader will check, and analogously to [V5], the methods used here are drawn from symplectic topology. This paper can be considered as part of a program to study symplectic topology in a random framework (or random phenomena having a symplectic structure) of which a foundational example is the random version of Poincaré-Birkhoff theorem from [P-R].

\footnote{i.e. the support is contained in $\mathbb{R}^n \times B(R)$ for some $R$}
\footnote{supported by NSF grant DMS-1440140}
4. NON-COMPACT SUPPORTED HAMILTONIANS

Let \( N \) be a non-compact manifold. We shall assume that \( N \) is homeomorph to the interior of a manifold with smooth boundary. \(^9\)

**Definition 4.1.** Let \( \varphi \in \mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}(T^*N) \). We say that \( \varphi \) has **finite propagation speed** (FPS, for short), if for each bounded set \( U \), there is a bounded set \( V \) such that \( \varphi(T^*U) \subset T^*V \). A subset in \( \mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}(T^*N) \) has uniformly finite propagation speed if each element has finite propagation speed, and moreover given \( U \), the set \( V \) can be chosen to be the same for all the elements in the subset. We write \( \mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}_{FP}(T^*N) \) for the set of Hamiltonians maps with finite propagation speed. By abuse of language, we use the same terminology in \( \mathcal{H}\mathcal{a}\mathcal{m}(T^*N) : H \) has **finite propagation speed** if \( \varphi_H \) has finite propagation speed, etc. We use the notation \( \mathcal{H}\mathcal{a}\mathcal{m}_{FP}(T^*N) \) for this set.

Note that for instance if \( \left| \frac{\partial \varphi}{\partial p}(t,q,p) \right| \leq C_U \) for all \((q,p) \in T^*U \) then \( H \) has FPS.

The following lemma will prove useful.

**Lemma 4.2.** Let \( U \subset V \) be relatively compact open sets in \( N \) such that for any compact set \( K \) in \( N \) there exists an isotopy of \( N \) sending \( K \) in \( V \). Let \( \varphi \in \mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}(T^*N) \) be such that \( \varphi(T^*U) \subset T^*V \). Then we can find a Hamiltonian isotopy \((\varphi^t)_{t \in [0,1]}\) from the identity to \( \varphi \) such that for all \( t \in [0,1] \) we have \( \varphi^t(T^*U) \subset T^*V \).

**Proof.** Let \( \psi^t \) be an isotopy from id to \( \psi^1 = \varphi \). Let \( X \) be a vector field corresponding to the isotopy for a compact set containing the projection of \( \bigcup_{t \in [0,1]} \psi^t(U) = K \) and pointing inwards on \( \partial V \). Let \( \rho^t \) be the Hamiltonian vector field of \( H(t,x,p) = \langle p, X(t,x) \rangle \) which projects on the flow of \( X \). Possibly replacing \( \rho^t \) by a \( \rho^{\alpha(t)} \), we may assume that for all \( t \in [0,1] \) we have \( \rho^t \circ \psi^t(T^*U) \subset T^*V \). Then \( \rho^1 \psi^1(T^*U) \subset T^*V \) and since \( \psi^1(T^*U) \subset T^*V \) the set of \( t \) such that \( \rho^t \psi^1(T^*U) \subset T^*V \) is an interval, it must contain \([0,1]\) hence concatenating the Hamiltonian isotopy \( t \mapsto \rho^t \psi^t \) with \( t \mapsto \rho^{1-t} \psi^1 \), we get a new Hamiltonian isotopy that we denote \( \varphi^t \) such that \( \varphi^t(T^*U) \subset T^*V \) for all \( t \in [0,1] \). \( \square \)

Note that our hypothesis on \( N \) implies that we can find an exhausting sequence \((U_j)_{j \geq 1}\) of \( N \) satisfying the assumptions of Lemma 4.2.

We shall now prove that \( \mathcal{D}\mathcal{H}\mathcal{a}\mathcal{m}_{fc} \) the set of Hamiltonians which are uniformly fiberwise compact supported is contained in \( \mathcal{H}\mathcal{a}\mathcal{m}_{FP} \).

---

\(^9\)We eventually only use the case \( N = \mathbb{R}^n \). For this section we actually only need that there is an exhausting sequence of open bounded sets \((U_j)_{j \in \mathbb{N}}\) such that \( U_j \subset U_{j+1} \) and for \( j \) large enough, \( U_j \) is ambient isotopic to \( U_{j+1} \).
Proposition 4.3. If $H \in \mathcal{Ham}_{f_c}(T^*\mathbb{R}^n)$ is uniformly fiberwise compact supported, then $H$ has FPS.

Proof. Indeed, if for some $C$, $\varphi$ is the identity outside of $DT^*_C(N) = \{(q, p) \mid |p| \leq C\}$, then $\varphi(T^*U) \subset T^*U \cup \varphi(T^*U \cap T^*_C N)$, but since $T^*U \cap T^*_C N$ is compact its image is contained in some $T^*V$ for $V$ bounded, and we get $\varphi(T^*U) \subset T^*(U \cup V)$. □

The usefulness of this notion will be clear on several occasions. Remember that a generating function quadratic at infinity for $(L, f_L)$ where $L$ is a smooth Lagrangian, and $f_L$ a function such that $d f_L = \Lambda_L$, is a smooth function $S : E = N \times F \rightarrow \mathbb{R}$ where $E$ is a finite dimensional vector space\(^{10}\) such that

1. $S$ coincides with a non-degenerate quadratic form $Q$ on the vector space $F$ for $\xi$ large enough
2. $(x, \xi) \mapsto \frac{\partial S}{\partial \xi}(x, \xi)$ is transverse to $0$
3. setting $\Sigma_S = \{(x, \xi) \mid \frac{\partial S}{\partial \xi}(x, \xi)\}$ the image of this submanifold by $i_S : (x, \xi) \mapsto \frac{\partial S}{\partial \xi}(x, \xi)$ has image $L$
4. $f_L \circ i_S = S$

Let $S_1, S_2$ be two GFQI. They are said to be equivalent if they are fiberwise diffeomorphic after stabilization, that is there are two non-degenerate quadratic forms $q_1, q_2$ such that if

$$\tilde{S}_j(x, \xi_j, \eta_j) = S_j(x, \xi_j) + q_j(\eta_j)$$

there is fiber preserving diffeomorphism

$$(x, \xi_1, \eta_1) \mapsto (x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1))$$

such that

$$S_2(x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1)) = S_1(x, \xi_1, \eta_1)$$

We shall say that $S_1, S_2$ are equivalent over $U$ if the fiber-preserving diffeomorphism is defined for $x \in U$. Note that the customary “addition of a constant” for the equivalence of generating functions is not needed here, since generating functions are normalized so that $S|_L = f_L$.

We cannot expect a non-compact Lagrangian to have a GFQI in this sense, since the number of variables required could go to infinity. We can either assume $F$ is a Hilbert space, but then positive and negative

\(^{10}\)All this discussion also works if we replace $N \times F$ by a general finite-dimensional vector bundle. Then we must replace in the sequel the Künneth isomorphism by the Thom isomorphism.
eigenspaces will generally be infinite dimensional so that $H^* (S^b, S^a) = 0$ which is a notorious drawback.\footnote{That we could avoid by using Floer homology everywhere, but would make reading this paper even harder for the Hamilton-Jacobi community!} Here we have

**Definition 4.4.** We say that $L$ has a GFQI if for each bounded set, $U$, there is a GFQI defined over $U \times F$ (where $F$ depends on $U$), $S_U$, and there is a set $V \supset U$ such that the $S_W$ are all equivalent over $U$ for $W \supset V$. Two GFQI are equivalent if they are equivalent over each bounded set.

**Theorem 4.5.** Let $\varphi$ be an element in $\mathcal{D}\mathcal{H}\mathcal{A}m_{FP}(T^* N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique up to equivalence.

**Proof.** See Appendix 1.\hfill \square

**Remarks 4.6.** Notice that

1. If $\varphi$ does not have FPS, $\varphi(0_N)$ does not even need to have surjective projection on $N$: for example take on $T^* \mathbb{R}$ the Hamiltonian $\frac{\pi}{4} (x^2 + p^2)$, then $\varphi(0_\mathbb{R}) = [0] \times \mathbb{R}$!

2. Using Lemma 4.2 we may assume we have a sequence $U_\nu$ of domains such that for all $t \in [0, 1]$ we have $\varphi^t(U_\nu) \subset T^* U_{\nu+1}$. We denote by $S_\nu = S_{U_\nu}$ and notice that we may assume that the restriction of $S_\mu$ over $U_\nu$ is exactly $S_\nu \oplus q_{\nu, \mu}$ by composing $S_\mu$ with an extension of the fiber preserving diffeomorphism realizing the equivalence.\footnote{The existence of the extension follows from the fact that we may assume that for $\mu, \nu$ large enough, the inclusion $U_\nu \subset U_{\mu}$ is a homotopy equivalence.} We shall always make this assumption in the sequel.

3. We will use the expression “$S$ is a GFQI for $L$” meaning there is a sequence $(S_\nu)_{\nu \geq 1}$ of GFQI for $L$ over $U_\nu$ to avoid cumbersome indexes. Most of the time this means we consider $S_\nu$ for $\nu$ large enough.

**Definition 4.7.** We denote by $\mathcal{L}(T^* N)$ the set of Lagrangians of the type $\varphi(0_N)$ where $\varphi \in \mathcal{D}\mathcal{H}\mathcal{A}m_{FP}(T^* N)$.

On a Riemannian manifold, there is a more precise notion than FPS.

**Definition 4.8.** Let $N$ be a manifold with a distance $d$ and $\varphi \in \mathcal{D}\mathcal{H}\mathcal{A}m(T^* N)$. We say that $\varphi$ has **bounded propagation speed** (BPS for short), if there is a constant $r_0$ such that for any ball $B(x_0, r)$ we have $\varphi(T^* B(x_0, r)) \subset T^* B(x_0, r + r_0)$. A subset in $\mathcal{D}\mathcal{H}\mathcal{A}m(T^* N)$ has **uniformly bounded propagation speed** if each has bounded propagation speed, and moreover the constant $r_0$ can be chosen to be the same for all the elements in the subset. We write $\mathcal{D}\mathcal{H}\mathcal{A}m_{BPS}(T^* N)$ for the set of Hamiltonians maps with bounded...
propagation speed. By abuse of language, we use the same terminology in $\Ham(T^* N) : H$ has \textbf{bounded propagation speed} if $\varphi_H$ has bounded propagation speed.

Example 4.9. If $\left| \frac{\partial H(t, q, p)}{\partial p} \right| \leq C$ for all $(q, p) \in T^* \mathbb{R}^n$ then $H$ has BPS. In particular Assumption (5) implies B.P.S.

Remark 4.10. (1) Of course Bounded Propagation Speed implies Finite Propagation Speed.

(2) Our definition of finite propagation speed does not exactly coincide with the terminology of [C-V] Definition B.5 p. 271. Our definition was more involved and the notion of finite propagation speed defined there is weaker than the present one, but would still be sufficient to prove our theorems. However this would have made an already long paper even longer.

5. Spectral Invariants in Cotangent Bundles of Non-Compact Manifolds

The goal of this section is to define and state the main properties of the metric $\gamma$ that occurs in the statement of the Main Theorem. This has been done in [V1] in the case of a compact base, the present situation, for a non-compact base, is unfortunately slightly more involved.

5.1. The case of Lagrangians. Let $L$ be an exact Lagrangian in $T^* N$ with $N$ not necessarily compact (but assumed, for simplicity, to be connected). We assume a primitive of $\lambda|_L$, $f_L$ is given. We shall assume that $L$ has a unique GFQI, $S$ such that $f_L = S$ on $L$ (through the identification $i_S(x, \xi) = (x, \frac{\partial S}{\partial x}(x, \xi))$). For example according to Theorem 4.5, this is the case if $L = \varphi_H(0_N)$ with $\varphi \in \DHam_F(P(T^* N)).$

We denote by $T_F$ the generator of $H^i(D(F^{-}), S(F^{-}))$ where $F^{-}$ is the negative eigenspace of $Q$, $i = \dim(\text{F}^{-})$ and $D(\text{F}^{-}), S(\text{F}^{-})$ are respectively the disc and sphere in $\text{F}^{-}$, so that $\alpha \mapsto \alpha \otimes T_F$ is an isomorphism (the Künneth isomorphism) from $H^*(U)$ to $H^{*+i}(U \times D(F^{-}), U \times S(F^{-})) = H^*(U) \otimes H^*(D(F^{-}), S(F^{-}))$ for $U \subset N$. By abuse of language we again denote by $T_F$ its homological counterpart in $H_i(D(F^{-}), S(F^{-}))$. We denote by $S'_{U} = \{ (x, \xi) \in U \times E \mid S(x, \xi) \leq t \}$ (we omit the subscript for $U = N$) and $S_{-\infty}^c$ (resp. $S_{+\infty}^c$) any of the $S_{-\infty}^c$ (resp. $S_c$) for $c$ large enough (by Morse's lemma they are all isotopic). Classically we have a homotopy equivalence between $(S_{-\infty}^c, S_{-\infty}^c)$ and $U \times (D(F^{-}), S(F^{-)))$.

\textsuperscript{13} Even though we write $L$, we always mean the pair $(L, f_L)$.

\textsuperscript{14} Remember cf Remark 4.6.3 that this means there is a sequence $S_{v}$ of GFQI over $U_{v}$ such that for $v \leq \mu$, the function $S_{\mu}$ restricts to the stabilization of $S_{v}$ over $U_{v}$. 

Definition 5.1. Let $S$ be a GFQI for $L \in \Sigma(T^*N)$ and $U$ a bounded open set with smooth boundary. We define

1. For $\alpha \in H^*(U)$
   \[ c(\alpha, S) = \inf \{ t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{U}^f, S_{U}^\infty) \} \]

2. For $\alpha \in H_*(U, \partial U)$
   \[ c(\alpha, S) = \inf \{ t \mid T \otimes \alpha \text{ is in the image of } H_*(S_{U}^f, S_{U}^\infty \cup S_{\partial U}^f) \} \]

3. For $\alpha \in H^*_*(U) = H^*(U, \partial U)$
   \[ c(\alpha, S) = \inf \{ t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{U}^f, S_{U}^\infty \cup S_{\partial U}^f) \} \]

4. For $\alpha \in H_*(U)$
   \[ c(\alpha, S) = \inf \{ t \mid T \otimes \alpha \text{ is in the image of } H_*(S_{U}^f, S_{U}^\infty) \} \]

5. For $L_1, L_2 \in \Sigma(T^*N)$, having unique GFQI, $S_1, S_2$, we set $(S_1 \otimes S_2)(x, \xi, \eta) = S_1(x, \xi) - S_2(x, \eta)$ and $\alpha \in H^*(U)$ or $H^*(U, \partial U)$
   \[ c(\alpha, L_1, L_2) = c(\alpha, (S_1 \otimes S_2)) \]
   We set $c(\alpha, L) = c(\alpha, L, 0_N)$.

6. We set $\mu_U \in H^n(U, \partial U), 1_U \in H^0(U)$ and $\gamma_U(L_1, L_2) = c(\mu_U, L_1, L_2) - c(1_U, L_1, L_2)$. If $c(1_U, L_1, L_2) = 0$ we shall write $L_2 \leq_U L_1$ and if this holds for all bounded sets $U$, we write $L_2 \preceq L_1$.

7. We set $GH^*(L_1, L_2; a, b) = H^{* - i}((S_1 \otimes S_2)^b, (S_1 \otimes S_2)^a)$

Remark 5.2. We notice that

1. As we said, $S$ is shorthand for $S_\gamma$ defined on $U_\gamma$. As long as $U \subset U_\gamma$ it is easy to see that for $\alpha \in H^*(U)$ (resp. $H^*(U, \partial U)$) the $c(\alpha, S_\gamma)$ do not depend on $\gamma$.

2. The function $(S_1 \otimes S_2)$ is not quadratic at infinity, but a standard trick allows us to deform it to a function quadratic at infinity (see [V4], prop. 1.6). The $GH^*$ functor is called Generating function homology (see [R]) and coincides with Floer homology (see [V2] for the equivalence of the two homologies) that we shall not introduce here.

3. Note that if $S$ has no fiber variables, $c(1_U, S) = \inf_{x \in U} S(x)$ and $c(\mu_U, S) = \sup_{x \in U} S(x)$.

It is often convenient to express the cohomological critical values in terms of their homology counterpart. Note that $H^*(U)$ is dual to $H_{n-*}(U, \partial U)$ and $H^*(U, \partial U)$ is dual to $H_{n-*}(U)$ by Lefschetz duality (cf [Hat], p. 254). We have a fundamental class $\mu_U \in H^n(U, \partial U)$ dual to $[pt_U] \in H_0(U)$ and $1_U \in H^0(U)$ dual to $[U] \in H_n(U, \partial U)$. The following Lemma will be useful
Lemma 5.3. We have for $S$ a GFQI

1. $c(1_U, S) = c([pt_U], S)$
2. $c(\mu_U, S) = c([U], S)$

We also have the duality identity

$$c(1_U, L) = -c(\mu_U, L)$$

Proof. The first two properties follow from Proposition B.3 in [V5]. The duality identity is a consequence of the identity $c(1_U, -S) = -c(\mu_U, S)$. Both are easily adapted from the case $U = N$ closed to the present situation. This follows from the following argument (see [V1], prop. 2.7, p. 692). We give here details for the proof of duality. First notice that $(-S)^f = E \setminus S^{-1}$, so we look for the smallest $t$ such that $1_U \neq 0$ in $H^*(E_t \setminus S^{-1}_U, E_t \setminus S^{-0}_U)$. We then apply Alexander duality, noting that even though $E_t = U \times \mathbb{R}^k$ is non-compact, it is contained in the compact manifold $U \times S^k$. Then get the diagram where vertical maps correspond to long exact sequences of triples, and horizontal to Alexander isomorphisms (omitting the subscript $U$)

$$
\begin{align*}
H_*(S^{-t}, S^{-\infty}) & \xrightarrow{\sim} H^*(E \setminus S^{-\infty}, E \setminus S^{-t}) = H^*((-S)^t, (-S)^{-\infty}) \\
H_*(S^{+\infty}, S^{-\infty}) & \xrightarrow{\sim} H^*(E \setminus S^{-\infty}, E \setminus S^{+\infty}) = H^*((-S)^{+\infty}, (-S)^{-\infty}) \\
H_*(S^{+\infty}, S^{-t}) & \xrightarrow{\sim} H^*(E \setminus S^{-t}, E \setminus S^{\infty}) = H^*((-S)^{+\infty}, (-S)^{-\infty})
\end{align*}
$$

Using the universal coefficient theorem (remember, our coefficient ring is a field) we see that $H_*(S^{+\infty}_U, S^{-\infty}_U)$ is a vector space dual to $H^*(S^{+\infty}_U, S^{-\infty}_U)$. We denote again $1_U$ the element in $H_*(S^{+\infty}_U, S^{-\infty}_U)$ sent to $1_U \in H^*(S^{+\infty}_U, S^{-\infty}_U)$ to 1, and we have $c(1_U, S)$ is the same whether we consider $1_U$ in homology or cohomology. On the other hand the second line of the diagram sends $1_U$ to $\mu_U$, since in this case Alexander duality corresponds to Poincaré duality. Now saying that $1_U$ is in the image of $H_*(S^{-t}, S^{+\infty})$ is equivalent to saying that $\mu_U$ is in the image of $H^*((-S)^t, (-S)^{-\infty})$. In other words, $-t \leq c(1_U, S)$ is equivalent to $t \leq c(\mu_U, S)$ and this means $c(1_U, -S) = -c(\mu_U, S)$. □

Definition 5.4. Let $U$ be a bounded domain with smooth boundary, $\partial U$.
We say that the sequence of smooth functions $(f_k)_{k \geq 1}$ defines $U$ if

1. there is a decreasing family $F_k$ of closed subset of $U$ such that $\cap_k F_k = \overline{U}$
2. $f_k = 0$ on $F_k$
(3) $f_k$ is a decreasing sequence converging to $-\infty$ on $N \setminus U$.

We say that $(f_k)_{k \geq 1}$ is a standard defining sequence if there is a function $r \in C^\infty(\mathbb{R})$ such that

1. $r(t) = 0$ for $t \leq 0$
2. $r'(t) < 0$ for $0 < t < 1,$
3. $r(t) = -1$ for $t \geq 1$

and for some increasing sequence $a_k$ converging to $+\infty$ we have

$$f_k(x) = a_k r_k(a_k \cdot d(x, U))$$

Notice that given a sequence $f_k$ defining $U$, we can find standard sequences $g_k, h_k$ such that $g_k \leq f_k \leq h_k$.

We define for a smooth function $f$ the graph of its differential, $G_f = \{(x, df(x)) \mid x \in N\}$. This is an exact Lagrangian, with primitive $f$. If $L$ is a Lagrangian with GFQI $S$, we define $L + G_f$ the Lagrangian generated by $S + f$.

We notice that

**Lemma 5.5.** Let $(f_k)_{k \geq 1}$ be a sequence defining $U$, and $V$ be any bounded open set such that $V \supset \overline{U}$. Then for $L_1, L_2 \in \mathcal{L}(T^* N)$ we have

$$c(1_U, L_1, L_2) = \lim_k c(1_V, L_1 - G_{f_k}, L_2) = \lim_k c(1_V, L_1, L_2 + G_{f_k})$$

**Proof.** Let $S_j$ be G.F.Q.I. for $L_j$ and $S = S_1 \cap S_2$. We have $S_j^c = \lim_k (S - f_k)^c$, therefore for Čech cohomology, according to theorem 5 in [L-R] we have

$$\lim_k H^*((S - f_k)_j^c, (S - f_k)^b) = H^*(S_j^c, S_j^b)$$

and from the definition of $c(1_U, S)$ the Proposition follows. $\square$

Let $U$ be an open set with smooth boundary and set $\nu(x) \in T^*_x U$ to be the exterior conormal to $\partial U$ at $x \in \partial U$, i.e. $\nu(x) = 0$ on $T \partial U$ and $\langle \nu(x), n(x) \rangle > 0$ where $n(x)$ is the exterior normal to $U$ at $x$. The conormal of $U$ is then defined as

$$\nu^* U = \{ (x, p) \in T^* N \mid x \in U, p = 0, \text{ or } x \in \partial U, p = c \nu(x), c \leq 0 \}\$$

We now prove that the values of $c(\alpha, L)$ correspond to intersection points of $L$ and $\nu^* U$ (or $L$ and $\overline{\nu^* U}$)

**Proposition 5.6** (Representation theorem). Let $U$ be a bounded open set with smooth boundary, we then have

1. For $\alpha \in H^*(U)$, $c(\alpha; L_1, L_2)$ is given by $f_1(x_\alpha, p_{1, \alpha}) - f_2(x_\alpha, p_{2, \alpha})$ where $(x_\alpha, p_{1, \alpha}) \in L_1$ and $(x_\alpha, p_{2, \alpha}) \in L_2$ and $(x_\alpha, p_{1, \alpha} - p_{2, \alpha}) \in \nu^* U$.
2. The same holds for $\alpha \in H^*(U, \partial U)$ but with $\overline{\nu^* U}$.
Proof. This is the representation theorem (Proposition 2.4 in [VI]), using a standard defining sequence for $U$ and the fact that $c(1_U; L_1, L_2) = \lim_k c(1_V; L_1 - G_{f_k}, L_2)$. Indeed, a converging sequence of points in $G_{f_k}$ will converge to a point in $v^* U$. Then compactness of $L_1 \cap T^* U$ and $L_2 \cap T^* U$ imply the result. \hfill \Box

For $(f_k)_{k \geq 1}$ a defining sequence of $U$, then $v^* U$ is the “limit” of the $G_{f_k}$ for $k \geq 1$. We will formally write $c(\alpha, L, v_* U)$ for $c(\alpha_U, L)$.

Remarks 5.7. Here are some comments:

1. The same will hold for $U \subset V$ and any $\alpha_V \in H^* (V)$ having restriction $\alpha_U \in H^* (U)$:
   \[
   c(\alpha_U, L_1, L_2) = \lim_k c(\alpha_V, L_1 - G_{f_k}, L_2) = \lim_k c(\alpha_V, L_1, L_2 + G_{f_k})
   \]
   In particular, if $M$ is a closed manifold containing $N$, we have
   \[
   c(1_U, L_1, L_2) = \lim_k c(1_M, L - G_{f_k}) = \lim_k c(1_M, L_1, L_2 + G_{f_k})
   \]

2. Let $\overline{U} \subset V$, then with obvious abuse of notations $c(1_V, v^* U) = -\infty, c(\mu_V, v^* U) = 0$ and of course $c(1_U, v^* V) = 0, c(\mu_U, v^* V) = +\infty$. This means that for $(f_k)_{k \geq 1}$ and $(g_k)_{k \geq 1}$ defining $U$ and $V$, we have $\lim_k c(1_M, G_{f_k}, G_{g_k}) = -\infty$ and $\lim_k c(\mu_M, G_{f_k}, G_{g_k}) = 0$.

3. Note also that symbolically we have for $\overline{U} \subset V$ that $v^* U + v^* V = v^* U$, meaning that if $(f_k)_{k \geq 1}$ defines $U$ and $(g_k)_{k \geq 1}$ defines $V$ then $(f_k + g_k)_{k \geq 1}$ defines $U$. More generally if $U \cap V \subset W$ we have $v^* U + v^* V \leq v^* W$ where this means that if $(f_k)_{k \geq 1}$ defines $U$ and $(g_k)_{k \geq 1}$ defines $V$, there is a sequence $(h_k)_{k \geq 1}$ defining $W$ such that $f_k + g_k \leq h_k$.

We will now prove some of the properties of these invariants.

**Proposition 5.8.** Let $\varphi \in \mathcal{D}\mathcal{Y} \mathcal{A}\mathcal{M}_{FP}(T^* N)$ and $L = \varphi^1 (0_N)$ be a Lagrangian submanifold. We have
\[
\gamma_U (L) = c(\mu_U, L) - c(1_U, L) \geq 0
\]
and equality implies that $L \cap T^* U \ni 0_U$.

Proof. The proof follows from the triangle inequality (see [VI], prop 3.3 p.693) $c(\alpha \cup \beta, L) \geq c(\alpha, L)$ applied to the product
\[
H^* (U) \otimes H^*_c (U) \longrightarrow H^*_c (U)
\]
Thus we have $c(\mu_U, L) = c(1_U \cup \mu_U, L) \geq c(1_U, L)$ and equality implies that $\mu_U$ is non zero in $K_c \cong L \cap v^* U$. But this implies $\pi (L \cap v^* U) \ni U$, hence $L$ contains $0_U$. Note that in general, contrary to the case where $N = U$ is compact $L \cap T^* U$ may contain other connected components than $0_U$. \hfill \Box
Proposition 5.9. The following holds for \( L_i \in \mathcal{L}(T^*N) \)

1. We have \( c(\mu_U, L_1, L_2) = -c(1_U, L_2, L_1) = -c(1_U, L_1, L_2) \).

2. For \( U \subset V \) and \( L_1, L_2 \) Lagrangian submanifolds we have
   
   \[ \begin{aligned}
   &\text{c}(\mu_U, L_1, L_2) \leq \text{c}(\mu_V, L_1, L_2) \\
   &\text{c}(1_U, L_1, L_2) \geq \text{c}(1_V, L_1, L_2) \\
   &\gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2)
   \end{aligned} \]

3. We have \( \gamma_U(L_1, L_3) \leq \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3) \)

4. If \( \gamma_U(L_1, L_2) = 0 \) then \( L_1 \cap L_2 \) has a connected component with projection on \( N \) containing \( U \).

5. If \( L_1 \leq L_2 \) then \( c(\alpha, L_1) \leq c(\alpha, L_2) \) for all \( \alpha \).

Proof. (1) The proof is the same as in Lemma 5.3, since \( S_{L_1} \oplus S_{L_2} = -(S_{L_2} \oplus S_{L_1}) \).

(2) If \( U \subset V \) note that

\[ c(1_U; L_1, L_2) = \lim_{k} c(1_N, L_1 - G_{f_k}, L_2) \]

Since we may choose defining sequences \( (f_k)_{k \geq 1}, (g_k)_{k \geq 1} \) for \( U, V \) such that \( f_k \leq g_k \) we have for \( S_1 \) a GFQI of \( L_1 \) that \( S_1 - f_k \geq S_1 - g_k \) hence \( c(1_N, L_1 - G_{f_k}) \geq c(1_N, L_1 - G_{g_k}) \), and going to the limit we get \( c(1_U, L_1, L_2) \geq c(1_V, L_1, L_2) \). By the duality formula, we get \( c(\mu_U; L_1, L_2) \leq c(\mu_V; L_1, L_2) \), hence \( \gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2) \).

(3) We have \( S_1 \oplus G_{2, f} \oplus S_3 = (S_1 \oplus G_f) \oplus (S_3 \oplus G_f) \) and \( (S_1 \oplus G_f) \oplus S_2 = S_1 \oplus (G_f \oplus S_2) \). Now remarking that if \( (f_k)_{k \geq 1} \) defines \( U \), then so does \( (2 \cdot f_k)_{k \geq 1} \) we have

\[ \gamma_U(L_1, L_3) = \lim_{k} \gamma_V(S_1 \oplus G_{2 \cdot f_k} \oplus S_3) = \lim_{k} \gamma_V((S_1 \oplus G_{f_k}) \oplus (S_3 \oplus G_{f_k})) \leq \lim_{k} \gamma_V(S_1 \oplus G_{f_k} \oplus S_2) + \lim_{k} \gamma_V(S_2 \oplus (G_{f_k} \oplus S_3)) = \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3) \]

(4) This follows from Lusternik-Schnirelmann theory as in the proof of Prop. 2.2 page 691 of [VII] (see also Proposition 5.8).

(5) \( L_1 \leq L_2 \) implies \( c(\mu_U, L_1, L_2) = 0 \) for all \( U \). By the triangle inequality, if \( \beta \cup \alpha = \mu_U \) we have

\[ 0 = c(\mu_U, L_1, L_2) \geq c(\alpha, L_1, 0_N) + c(\beta, 0_N, L_2) \geq c(\alpha, L_1) - c(\alpha, L_2) \]

since \( c(\beta, 0_N, L) = -c(\alpha, L, 0_N) \) according to the proof of proposition B.3 in [V].

\[ \square \]
We must now see what happens when we make a coordinates change in $T^* N$. We start with three lemmata

**Lemma 5.10.** Let $S$ be a G.F.Q.I. defined on $E = Y \times F$ and for $f : X \longrightarrow Y$ a smooth map a map $\tilde{f} : X \times F \longrightarrow Y \times F$ living over $f$, i.e. the following diagram

$$
\begin{array}{ccc}
X \times F & \xrightarrow{\tilde{f}} & Y \times F \\
\downarrow f & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

is commutative. We then have for $\alpha \in H^*(Y)$ and $(f)^*(\alpha) \in H^*(X)$

$$c(\alpha, S) \leq c(f^*(\alpha), S \circ \tilde{f})$$

**Proof.** Indeed, if $T \in H^*(D(F^-), S(F^-))$ is the Thom class for $F^-$, then $(\tilde{f})^*(T) = \tilde{T}$ is the Thom class for $\tilde{f}^*(F^-)$ and we have, denoting by $\pi, \tilde{\pi}$ the projections on $Y$ and $X$,

$$(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \tilde{T}$$

Now if $c \leq c(\alpha, S)$ then $\pi^*(\alpha) \cup T$ vanishes in $H^*(S'^c, S'^\infty)$ and this implies that $(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \tilde{T}$ vanishes in $H^*((S \circ \tilde{f})^c, (S \circ \tilde{f})^\infty)$, i.e. $c \leq c(f^*(\alpha), S \circ \tilde{f})$. This implies the lemma. \qed

**Lemma 5.11.** We have

$$c(1_U \boxtimes 1_U; L_1 \times L_2, \nu^* \Delta_N) = c(1_U; L_1, L_2)$$

**Proof.** Let $d^c : N \times N \longrightarrow \mathbb{R}$ be smooth function vanishing on $\Delta_N$ and converging as $\epsilon$ goes to 0, to $-\infty \cdot (1 - \chi_{\Delta_N})$ where $\chi_{\Delta_N}$ is the characteristic function of $\Delta_N$. For example we can choose

$$d^c(x, y) = \frac{-1}{\epsilon} d(x, y)$$

Similarly define $d^c_U(x, y) = d^c(x, y) + f^c_U(x) + f^c_U(y)$ where $f^c_U$ converges to $-\infty (1 - \chi_U)$ as $\epsilon$ goes to 0.

Setting $[S_1 \boxtimes (-S_2)] (x_1, x_2, \xi_1, \xi_2) = S_1(x_1, \xi_1) + S_2(x_2, \xi_2)$, and

$$[S_1 \oplus (-S_2)] (x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2)$$

we may write

$$c(1_U \times U; L_1 \times L_2, \Delta_N) = \lim_{\epsilon \to 0} c(1_{N \times N}; (L_1 - d f^c_U) \times (L_2 - d f^c_U), \nu^* \Delta_N) = \lim_{\epsilon \to 0} c(1_{N \times N}; (S_1 - f^c_U) \boxtimes (-S_2 - f^c_U), d^c) = c(1_{N \times N}; ((S_1 - f^c_U) \boxtimes (-S_2 - f^c_U)) - d^c$$

Now $\lim_{\epsilon \to 0} (S_1 \boxtimes (-S_2) - d^c) = (S_1 \oplus (-S_2))^c$ and if $\delta : \Delta_N \longrightarrow N \times N$ is the diagonal map, $\delta^*(1_N \boxtimes 1_N) = 1_{\Delta_N}$, so from Lemma 5.10 we get

$$c(1_U \boxtimes 1_U; L_1 \times L_2, \nu^* \Delta_N) \leq c(1_N, (S_1 - f^c_U) \oplus (S_2 - f^c_U)) \leq c(1_U; L_1, L_2)$$
Conversely we notice that given $c$, for $\varepsilon$ small enough, $(S_1 \boxminus (-S_2) - d^c)^c$ is contained in a neighborhood of $\Delta_N$. Thus if $1_U \otimes 1_U$ does not vanish in
\[ H_c(((S_1 - f_U^c) \boxminus (-S_2 - f_U^c)) - d^c)^c, \left\{ ((S_1 - f_U^c) \boxminus (-S_2 - f_U^c)) - d^c \right\}^{-\infty} \]
i.e. $c \geq c(1_U \boxminus 1_U; L_1 \times L_2, v^* \Delta N)$ then its restriction to $\Delta_N$, that is $1_U$ does not vanish either, and $c \geq c(1_U/L_1, L_2)$, so
\[ c(1_U \boxminus 1_U; L_1 \times L_2, v^* \Delta N) \geq c(1_U/L_1, L_2) \]
and we have equality.

**Lemma 5.12.** Let us consider a bounded open set with boundary $U \subset N$ and $v^* \Delta U \subset T^* N \times T^* \overline{N}$, where $\Delta_U$ is the diagonal in $U$ Then if $\varphi^1(T^* U) \subset T^* V$ we have
\[ (\varphi \times \varphi)(v^* \Delta U) \preceq v^* \Delta V \]

**Proof.** Let $(q, p, q, p') \in v^* \Delta_U$ and notice that unless $q \in \partial U$ we have $p = p'$. Then according to Lemma 4.2 we may assume $\varphi^1(T^* U) \subset T^* V$ for all $t \in [0, 1]$, so setting $(\varphi^1 \times \varphi^1)(q, p, q, p') = (Q_t, P_t, Q'_t, P'_t)$ we know that when $(q, p, q, p') \in v^* \Delta_U \subset T^* \overline{U}$, we have $Q_t, Q'_t \notin \partial V$. So if $(Q_t, P_t, Q'_t, P'_t) \in v^* V$ we must have $Q_t = Q'_t, P_t = P'_t$, but then $p = p'$. In other words
\[ (\varphi^1 \times \varphi^1)(v^* \Delta_U) \cap v^* \Delta V = (\varphi^1 \times \varphi^1)(\Delta T^* U) \cap \Delta T^* V = (\varphi^1 \times \varphi^1)(\Delta T^* U) \]
So the intersection $(\varphi^1 \times \varphi^1)(v^* \Delta_U) \cap v^* \Delta V$ is constant and by a classical argument, this implies that as a function of $t$ $c(\alpha, (\varphi^1 \times \varphi^1)(v^* \Delta_U), v^* \Delta V)$ is constant. Since $v^* \Delta_U \preceq v^* \Delta V$ we have for all $t$ $(\varphi \times \varphi)(v^* \Delta_U) \preceq v^* \Delta V$. 

Using Proposition 5.9, we may conclude that the limits in the following proposition are well-defined in $\mathbb{R} \cup \{\pm \infty\}$.

**Definition 5.13.** When $U$ is an unbounded set we define $\mathcal{B}(U)$ to be the set of bounded subsets in $U$ and
\[ c(\mu_U/L_1, L_2) = \lim_{V \in \mathcal{B}(U)} c(\mu_V/L_1, L_2) \]
\[ c(1_U/L_1, L_2) = \lim_{V \in \mathcal{B}(U)} c(1_V/L_1, L_2) \]

We may now prove

**Proposition 5.14.** We have for $\varphi \in \mathcal{D}(\text{Ham}(T^* N))$ such that $\varphi(T^* U) \subset T^* V$ and $L_1, L_2 \in \mathcal{L}(T^* N)$
\[ \gamma_U(\varphi(L_1), \varphi(L_2)) \leq \gamma_V(L_1, L_2) \]
Proof. We use Lemma 5.10 so we replace $c(1_U, \varphi(L_1), \varphi(L_2))$ by

$$c(1_U \otimes 1_U, \varphi(L_1) \times \varphi(L_2), \nu^* \Delta_N)$$

and this in turn equals

$$c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^* \Delta_N + \nu^*(U \times U))$$

Since

$$\nu^* \Delta_N + \nu^*(U \times U) \preceq \nu^*(\Delta_N \cap (U \times U)) = \nu^* \Delta_U$$

As a result

$$c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^* \Delta_N + \nu^*(U \times U)) \geq c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^* \Delta_U) = c(1_N \otimes 1_N, (\varphi \times \varphi)^{-1}(\nu^* \Delta_U))$$

and using Lemma 5.12 we get that the last term is greater than

$$c(1_{N \times N}, L_1 \times L_2, \nu^* \Delta_V) = c(1_V, L_1, L_2)$$

We may thus conclude that

$$c(1_V, L_1, L_2) \leq c(1_U, \varphi(L_1), \varphi(L_2))$$

By duality, we get

$$c(\mu_V, L_1, L_2) \geq c(\mu_U, \varphi(L_1), \varphi(L_2))$$

and our result follows. \qed

**Definition 5.15.** A sequence $(L_k)_{k \geq 1} \in \Sigma(T^* N)$ $\gamma_c$-converges to $L \in \Sigma(T^* N)$ if for all bounded domains $U$ the sequence $\gamma_U(L_k, L)$ converges to 0. We shall write $L_k \xrightarrow{\gamma_c} L$. The completion of $\Sigma(T^* N)$ for $\gamma_c$, is the set of equivalence classes of $\gamma - c$-Cauchy sequences $(L_k)_{k \geq 1}$ for the following relation:

$$(L_k)_{k \geq 1} \simeq (L'_k)_{k \geq 1}$$

if for all bounded domains $U$, $\gamma_U(L_k, L'_k)$ converges to 0.

We denote by $\hat{\Sigma}(T^* N)$ this completion.

**Remark 5.16.** Of course we may take a cofinal sequence $U_k$ of bounded open sets in $N$ and define

$$d(L_1, L_2) = \sum_{j=1}^{+\infty} 2^{-j} \max \left\{ 1, \gamma_{U_j}(L_1, L_2) \right\}$$

and then take the completion with respect to this metric. It is easy to see that the completion coincides with the above, hence does not depend on the choice of the sequence $U_k$ (this is just rephrasing the fact that the $\gamma_U$ define a uniform structure, see [We1], or [Bou], chap. II).
Example 5.17. Let \( f_k \) be a sequence of smooth functions. Then \( \gamma \)-convergence of the \( L_k = \text{gr}(df_k) \) is equivalent to uniform convergence on compact sets of the \( f_k \).

We shall need the following Proposition

**Proposition 5.18.** We have for \( L = \varphi_H^1(0_N) \in L(T^* N) \) the inequalities

\[
\begin{align*}
\gamma_U(L) &\leq \|H\|_{C^0(T^* U)} \\
c(\mu_H, L) &\leq \sup_{(q,p) \in T^* U} H(q,p) \\
c(1_U, L) &\geq \inf_{(q,p) \in T^* U} H(q,p)
\end{align*}
\]

Proof. Let \( H(q,p) = h(q) \) and \( L_h = \varphi_H(0_N) \). Then according to Remark 5.2 (3) we have

\[
\begin{align*}
c(\mu_H, L_h) &\leq \sup_{q \in U} h(q), \quad c(1_U, L_h) \geq \inf_{q \in U} h(q)
\end{align*}
\]

because \( \mathcal{L} = \{(q,dh(q)) \mid q \in N\} \). Now since for \( H \leq h(q) = \sup_{p \in \mathcal{U}_q N} \) we have \( H \leq H \) we get \( L \leq L_h \), so \( c(\mu_H, L) \leq c(\mu, L_h) = \sup_{q \in U} h(q) = \sup_{(q,p) \in T^* U} H(q,p) \) we get the first equality. The other two equalities follow immediately from this one.

5.2. **The case of Hamiltonians in** \( T^* \mathbb{R}^n \). Let \( H \in \mathcal{H}am_{fc}([0,1] \times T^* \mathbb{R}^n) \) and \( \varphi_H^f \) be its flow. Let \( s_1,s_2 \) be the symplectomorphisms

\[
T^* \mathbb{R}^n \times T^* \mathbb{R}^n \to T^* \Delta T^* \mathbb{R}^n
\]

defined respectively by

\[
\begin{align*}
s_1(q,p,Q,P) &= (q,P,p-P,Q-q) \\
s_2(q,p,Q,P) &= (Q,P,p-P,Q-q)
\end{align*}
\]

Denoting by \((x,y,X,Y)\) the coordinates in \( T^* \Delta T^* \mathbb{R}^n\), we have \( s_i^*(dY \wedge dy + dX \wedge dx) = dp \wedge dq - dP \wedge dQ\), so the \( s_i \) are symplectic.

The graph of \( \varphi_H \) is \((id \times \varphi_H)(\Delta T^* \mathbb{R}^n)\), and its image by \( s_1 \) is denoted by \( \Gamma(\varphi_H) \), while its image by \( s_2 \) will be \( \Gamma(\varphi_H^{-1}) \). Let \( S_H \) be a GFQI for \( \Gamma(\varphi_H) \) which exists and is unique if \( H \in \mathcal{H}am_{BP}(T^* \mathbb{R}^n) \) by Theorem 4.5

**Definition 5.19.** We set for \( W \) a domain contained in \( \Delta T^* \mathbb{R}^n\)

\[
\begin{align*}
(1) \quad &c_W^W(\varphi_H) = c(1_W, \Gamma(\varphi_H)) \\
(2) \quad &c_W^W(\varphi_H) = c(\mu_W, \Gamma(\varphi_H)) \\
(3) \quad &\gamma_W(\varphi_H) = c_W^+(\varphi_H) - c_W^-(\varphi_H)
\end{align*}
\]

**Remark 5.20.** In \( T^* N \) we may define for \( U \subset N \) the numbers

\[
\widehat{\gamma}_U(\varphi_H) = \sup_{L \in \Omega(T^* N)} \gamma_U(L,\varphi_H(L))
\]

which corresponds to (even though we do not claim it is equal to) \( \gamma((U \times \mathbb{R}^n)(\varphi_H)) \).
Let then \((H_v)_{v \geq 1}\) be a sequence of Hamiltonians in \(\mathcal{F}am_{FP}(T^*\mathbb{R}^n)\) and \(\varphi_v = \varphi_{H_v}\).

**Definition 5.21.** The sequence \((\varphi_v)_{v \geq 1}\) \(\gamma_c\)-converges to \(\varphi\) if for all bounded domains \(W\) we have \(\lim_{v} \gamma_W(\varphi_v, \varphi) = 0\). The \(\gamma\)-completion \(\overline{\mathcal{F}am_{FP}(T^*\mathbb{R}^n)}\) is defined as the set of Cauchy sequences in \(\mathcal{D}\mathcal{F}am_{FP}(T^*\mathbb{R}^n)\) for the uniform structure defined by the \(\gamma_W\), in other words the set of sequences which are Cauchy for each \(\gamma_W\), modulo the equivalence relation \((\varphi_v)_{v \geq 1} \equiv (\psi_v)_{v \geq 1}\) if for all \(W\) we have \(\lim_{v} \gamma_W(\varphi_v, \psi_v) = 0\). Similarly we define for \(H \in \mathcal{F}am_{FP}(T^*\mathbb{R}^n)\) the metric \(\gamma_W(H) = \sup_{t \in [0,1]} \gamma_W(\varphi'_t)\) and similarly for \(\gamma_W(H, K)\). Then we define the \(\gamma_c\)-convergence of a sequence in \(\mathcal{F}am_{FP}(T^*\mathbb{R}^n)\) and its completion \(\overline{\mathcal{F}am_{FP}(T^*\mathbb{R}^n)}\).

Note that the property of having FPS or being in \(\mathcal{F}am_{fc}\) can be checked in the \(\gamma\)-completion. Indeed \(\varphi(T^*U) \subset T^*V\) is equivalent to

\[ \Gamma(\varphi) \cap \{(x, p_x, y, p_y) \mid x \in U, y \notin V\} = \emptyset \]

and being supported in \(|p| \leq r\) is equivalent to

\[ \Gamma(\varphi) \cap \{(x, p_x, y, p_y) \mid |p_x| \geq r\} \subset \Gamma(id) \]

and both are closed condition, which makes sense in the completion (see [Hum]).

**Proposition 5.22.** We have

\[ \gamma_W(H) \leq \|H\|_{C^0(W')} \]

As a result we have an embedding

\[ \left( C^0_{fc}[0,1] \times T^*\mathbb{R}^n, d_{C^0} \right) \longrightarrow \mathcal{F}am_{fc}(T^*\mathbb{R}^n) \subset \mathcal{F}am_{BP}(T^*\mathbb{R}^n) \]

When dealing with fiberwise compact supported Hamiltonians, we have

**Definition 5.23.** We set for \(\varphi \in \mathcal{F}am_{fc}(T^*\mathbb{R}^n)\)

\[ \gamma^c(\varphi) = \gamma_{\mathbb{R}^n \times B^n(r)}(\varphi) = \lim_{r \to +\infty} \gamma_{B^n(r) \times B^n(r)}(\varphi) \]

and

\[ \gamma^c(\varphi) = \lim_{r \to +\infty} \gamma_r(\varphi) \in \mathbb{R} \cup \{+\infty\} \]

**Proposition 5.24.** We have if \(h_-(p) \leq H(t, q, p) \leq h_+(p)\) the inequality

\[ \gamma_r(\varphi_H) \leq \sup_{|p| \leq r} h_+(p) - \inf_{|p| \leq r} h_-(p) \]

**Remark 5.25.** The quantity \(\gamma^c(\varphi)\) is finite for \(\varphi \in \mathcal{F}am(T^*\mathbb{R}^n)\) such that \(\|H\|_{C^0(T^*\mathbb{R}^n)} < +\infty\).

Our last results in this section will be
Proposition 5.26. We have the following

(1) Assume $\psi, \psi^{-1}$ send $W = U \times V$ into $W' = U' \times V'$, where $U, U' \subset \mathbb{R}^n, V, V' \subset (\mathbb{R}^n)^*$ then we have

$$\gamma_W(\psi^{-1} \circ \varphi \circ \psi) \leq \gamma_{W'}(\varphi)$$

(2) $\gamma_r(\tau_{-a} \circ \varphi \circ \tau_a) = \gamma_r(\varphi)$

(3) $\gamma_r(\rho_k \circ \varphi \circ \rho_k^{-1}) = \varepsilon \gamma_r(\varphi)$

Proof. In $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ we have that $\gamma(\varphi)$ is obtained by $(q, p, Q, P) \mapsto (q, P, P - p, q - Q)$ where $(Q, P) = \varphi(q, p)$, while $\Gamma(\psi \circ \varphi \circ \psi^{-1})$ is obtained by applying $\psi \times \psi$ to $(q, p, Q, P)$. In other words writing $(q', p') = \psi(q, p), (Q', P') = \psi(Q', P')$, $\Gamma(\psi \circ \varphi \circ \psi^{-1})$ is obtained as

$$\{(q', p', p' - p', q' - Q') \mid \varphi(q, p) = (Q, P)\}$$

Now if $q \in U$ and $p \in V$ we have $q' \in U'$ and $p' \in V'$, hence $(\psi \times \psi)(T^*(U \times V)) \subset T^*(U' \times V')$.

As a result, since $\psi \times \psi$ preserves the diagonal (that is the zero section in the new coordinates) we have

$$\gamma_{U \times V}(\psi^{-1} \varphi \psi) = \gamma_{U \times V}(\psi \times \psi) \Gamma(\varphi), (\psi \times \psi)(\Delta)) \leq \gamma_{U' \times V'}(\Gamma(\varphi), \Delta) = \gamma_{U' \times V'}(\varphi)$$

Statement (2) follows from first applying (1) to $\psi = \tau_a$ so that, setting $U_a = \bigcup_{t \in [-a, a]} \tau_t(U)$

$$\gamma_{U \times B(r)}(\tau_{-a} \varphi \tau_a) \leq \gamma_{U \times B(r)}(\varphi)$$

hence taking the limit for $U \subset \mathbb{R}^n$ we get

$$\gamma_r(\tau_{-a} \varphi \tau_a) \leq \gamma_r(\varphi)$$

and changing $a$ to $-a$ we get equality. The second equality is rather obvious since $\rho_k$ is $k$-conformal and $\rho_k(U \times B_r) = (k \cdot U) \times B_r$. \(\square\)

Remark 5.27. One should be careful, in particular $\gamma_U(\varphi_1, \varphi_2)$ is NOT in general, equal to $\gamma_U(\varphi_2^{-1} \circ \varphi_1) = \gamma_U(\varphi_2^{-1} \circ \varphi_1, \text{id})$. We thus have a priori two types of convergence. We could say that $\varphi_\nu$ converges to $\varphi$ if for all bounded sets $U$ either the sequence $\gamma_U(\varphi_\nu, \varphi)$ goes to 0 or if $\gamma_U(\varphi_\nu \varphi^{-1})$ goes to 0. However if the $\varphi_\nu$ have uniformly bounded propagation speed, that is $\varphi_\nu(T^* B_r) \subset T^* B_{r + r_0}$ for all $\nu$ and all $r$, then the two conditions are equivalent. \(\square\)

6. Compactness and Ergodicity

Let $H : T^* \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ be Hamiltonian satisfying properties [1-6]. Then each $H_\omega = H(\bullet, \bullet, \omega)$ is in $\mathfrak{Ham}_c(T^* \mathbb{R}^n)$ and we identify $\Omega$ with its image in $\mathfrak{Ham}_c(T^* \mathbb{R}^n)$, denoted $\mathfrak{H}_\Omega$. Its closure for the $\gamma$-topology in the
completion $\widehat{\text{Ham}}_{BP}(T^*\mathbb{R}^n)$ is denoted by $\widehat{\Omega}$. The action $\tau$ of $\mathbb{R}^n$ on $\Omega$ induces an action on $\widehat{\Omega}$ by

$$(\tau_a H)(x, p; \omega) = H(x + a, p; \omega) = H(x, p; \tau_a \omega)$$

This action translates into $\varphi \mapsto \tau_a \varphi \tau_a$ on $\widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n)$.

We first want to prove

**Proposition 6.1.** The abelian group $\mathbb{R}^n$ acts continuously by isometries on $(\widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n), \gamma_c)$ and $(\text{DHam}_{c}(T^*\mathbb{R}^n), \gamma_c)$ hence on $(\widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n), \gamma_c)$ and $(\text{DHam}_{c}(T^*\mathbb{R}^n), \gamma_c)$. Therefore the action $\tau$ of $\mathbb{R}^n$ on $\widehat{\Omega}$ is a continuous action by isometries for $\gamma_c$ which extends to a continuous action by isometries on $\widehat{\Omega}$.

**Proof.** That $\mathbb{R}^n$ acts by isometries follows from Proposition 5.26. It is enough according to a theorem by Chernoff and Marsden to prove the separate continuity of the map $\mathbb{R}^n \times \widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n) \to \widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n)$ in each variable. In other words - since $\tau_a$ is an isometry it is obviously continuous in the second variable - we must prove that for all $H \in \widehat{\text{Ham}}_{c}(T^*\mathbb{R}^n)$, we have

$$\lim_{a \to 0} \gamma_c(H, \tau_a H) = 0$$

i.e. we want to prove that for all $r > 0$, $\lim_{a \to 0} \gamma_r(\tau_a^{-1} \varphi^{-1} \tau_a, \varphi) = 0$. But

$$\Gamma(\varphi) = \{(q, P; p - P, Q - q) | \varphi(q, p) = (Q, P)\}$$

while

$$\Gamma(\tau_a^{-1} \varphi \tau_a) = \{(q - a, P; p - P, Q - q) | \varphi(q, p) = (Q, P)\}$$

so that if $S(q, P; \xi)$ is a GFQI for $\Gamma(\varphi)$ and $(\tau_a S)(q, P; \xi) = S(q - a, P; \xi)$ is a GFQI for $\Gamma(\tau_a^{-1} \varphi \tau_a)$. Since critical points of $S(q, P, \xi)$ are contained in $|P| \leq R$ and $a \mapsto S(q - a, P; \xi)$ is uniformly continuous on $|P| \leq R$, we get that $c_W(\alpha, S \circ \tau_a S)$ depends continuously on $a$, and for $a = 0$ is equal to 0 (since it is equal to $c_W(\varphi, \varphi) = 0$).

Proposition 6.1 extends the action $\tau$ to a continuous action by isometries of $\widehat{\Omega}$. Since $\text{Isom}(\widehat{\Omega}, \gamma) \subset \text{Isom}(\widehat{\Omega}, \gamma)$, the map $\tau : \mathbb{R}^n \to \text{Isom}(\widehat{\Omega}, \gamma)$ extends to a map, still denoted $\tau$, from $\mathbb{R}^n$ to $\text{Isom}(\widehat{\Omega}, \gamma)$. Since this is obviously a group morphism, its closure in $\text{Isom}(\widehat{\Omega}, \gamma)$ is an abelian connected and complete metric group.

**Proposition 6.2.** Let us denote the closure of $\tau(\mathbb{R}^n)$ in $\text{Isom}(\widehat{\Omega}, \gamma)$ by $\mathbb{A}_\Omega$. Then $\mathbb{A}_\Omega$ is an abelian, connected and complete metric group.

---

15Which claims that under our assumptions, a separately continuous action is jointly continuous. See [C-M], Theorem 1, extending a theorem of Ellis, in [E].
The goal of this section is to prove that our assumptions on $H$ imply that $A_{\Omega}$ is compact. For this it is enough to prove that $\text{Isom}(\hat{H}_{\Omega}, \gamma_c)$ is compact, but this follows immediately by Ascoli-Arzela’s theorem if we prove that $(\hat{H}_{\Omega}, \gamma_c)$ is compact. Because by assumption $(\hat{H}_{\Omega}, \gamma_c)$ is complete, it is enough to show that it is totally bounded, that is for any $\varepsilon > 0$, $(\hat{H}_{\Omega}, \gamma_c)$ can be covered by finitely many $\gamma_c$-balls of radius $\varepsilon$. Since $(\hat{H}_{\Omega}, \gamma_c)$ is dense in $(\hat{H}_{\Omega}, \gamma_c)$ it is enough to prove that $(\hat{H}_{\Omega}, \gamma_c)$ is totally bounded. We shall prove slightly less but it will be good enough for our purposes:

**Proposition 6.3.** Let $\hat{\mu}_{\Omega}$ be the push forward to $\hat{H}_{\Omega}$ of the measure $\mu$ on $\Omega$. Then the support of $\hat{\mu}_{\Omega}$ is totally bounded hence compact.

This will follow from the following general result

**Proposition 6.4.** Let $(X, \mu)$ be a probability space endowed with a distance $d$ such that $(X, d)$ is separable. Let $G$ be a group acting on $X$ by measure-preserving isometries. Then $\text{supp}(\mu)$ is totally bounded.

We shall first prove

**Lemma 6.5.** Let $\tau$ be an ergodic action of a group $G$ on a probability, separable metric space $(X, \mu, d)$. Then for $\mu$-almost all points $x \in X$ the orbit $G \cdot x$ is dense in $\text{supp}(\mu)$.

**Proof.** This is an immediate consequence of Birkhoff’s ergodic theorem, but we shall give a simpler (or at least easier) proof. Let $Y$ be countable and dense in $X$ and set

$$W = \bigcup_{y \in Y, r \in \mathbb{Q}^+} B(y, r)$$

If $\mu(B(x, r)) = 0$ for some $x \in X, r > 0$ then $x \in W$. Indeed, we may assume $r$ is rational, and choose $y \in Y$ such that $d(y, x) < r/2$. Then $x \in B(y, r/2)$ so $B(y, r/2) \subset B(x, r)$ and we get $\mu(B(y, r/2)) = 0$. This argument implies that

$$W = \{x \in X \mid \exists U \text{ open } x \in U, \mu(U) = 0\}$$

and $W$ is $\tau$ invariant since $\tau$ preserves $\mu$ and the open sets. Now because $W$ is a countable union of open sets of measure 0, it is open and has measure 0. We may then replace $X$ by $X \setminus W$, so we are reduced to the situation where all balls have $> 0$ measure i.e. all open sets have positive measure.

Now let $(U_j)_{j \in \mathbb{N}}$ be a countable basis of open sets (since a separable metric space is second countable). Set $\tau_G A = \{\tau_g x \mid x \in A\}$, then the orbit

---

\[16\] A separable topological space is a space having a countable dense subset.
of \( x \) misses \( U_j \) if and only if \( \tau_G x \cap U_j = \emptyset \) i.e. \( x \notin \tau_G(U_j) \). The points with non-dense orbit must miss at least one \( \tau_G(U_j) \) so they belong to
\[
\bigcup_j (X \setminus \tau_G(U_j)) = X \setminus \bigcap_j \tau_G(U_j)
\]
but by ergodicity \( \tau_G(U_j) \) being \( \tau \) invariant has measure 1 (since it cannot be zero, as its measure is at least the measure of \( U_j \) that is positive by assumption). Therefore \( \bigcap_j \tau_G(U_j) \) as a countable intersection of measure one sets has measure one, and its complement has measure zero. \( \square \)

We are now in a position to prove Proposition 6.4.

**Proof of Proposition 6.4.** Let \( \tau_G x \) be a dense orbit in \( \text{supp}(\mu) \). We shall prove that \( \tau_G(x) \) is totally bounded, arguing by contradiction. Let \( a_1, \ldots, a_k \in G \) be a sequence in \( G \) such that

- \( \bigcup_{j=1}^k \overline{B}(\tau_{a_j} x, \varepsilon) \) does not cover \( \tau_G x \) where \( \overline{B}(x, r) \) is the closed ball of radius \( r \)
- For all \( i \neq j \) we have \( B(\tau_{a_i}, \varepsilon/2) \cap B(\tau_{a_j}, \varepsilon/2) = \emptyset \)

We claim that if \( \tau_G x \) cannot be covered by finitely many balls of size \( \varepsilon \) then we may construct such a sequence by induction. Indeed, assume \( a_1, \ldots, a_k \) have been constructed satisfying the above properties. Then by the first property we may find \( a_{k+1} \) such that \( \tau_{a_{k+1}} x \notin \bigcup_{j=1}^k B(\tau_{a_j} x, \varepsilon) \) and this implies \( B(\tau_{a_j} x, \varepsilon/2) \cap B(\tau_{a_{k+1}} x, \varepsilon/2) = \emptyset \). Hence \( a_1, \ldots, a_{k+1} \) satisfy both properties. But now we found infinitely many disjoint balls of radius \( \varepsilon/2 \) in \( \tau_G x \). Since \( \tau_{a_j} x \in \text{supp}(\mu) \) we have \( \mu(B(\tau_{a_j} x, \varepsilon/2)) > 0 \) and since all the balls \( B(\tau_{a_j} x, \varepsilon/2) \) are isometric they have the same measure. But we cannot have infinitely many disjoint balls with the same positive measure, since the total measure of our space is 1. \( \square \)

We may now conclude with

**Proof of Proposition 6.3.** Here \( G = \mathbb{R}^n \) and \( \tau_a \) induces a measure preserving map on \( (\mathcal{H}_\Omega, \gamma, \mu_\Omega) \). This map is an isometry according to Proposition 6.1, so according to Proposition 6.4 the support of \( \hat{\mu}_\Omega \) is totally bounded. \( \square \)

**Remark 6.6.** As we pointed out already in [V5], there are not so many non-trivial examples of compact subset in \( (\mathcal{H}_{\text{Ham}}(T^*\mathbb{R}^n), \gamma) \) or \( (\mathcal{D}\mathcal{H}_{\text{Ham}}(T^*\mathbb{R}^n), \gamma) \) that is sets that are not already compact for the \( C^0 \)-topology (since \( \gamma \) is continuous for the \( C^0 \) topology on \( \mathcal{H}_{\text{Ham}}(T^* N) \)) and in \( \mathcal{D}\mathcal{H}_{\text{Ham}}(T^* N) \) according to [Sey]). In [V5] we proved that in \( T^* T^n \) the sequence \( (H_k)_{k \geq 1} \) where \( H_k(q, p) = H(k \cdot q, p) \) is converging. Here we extend this to certain families of Hamiltonians on \( T^* \mathbb{R}^n \).
We thus proved that $A_\Omega$ the closure of $\mathbb{R}^n$ in $\text{Isom}(\hat{\Omega}, \gamma)$ is a compact, connected, metric abelian group. We are thus in the following situation: we have an action - again denoted by $\tau$- of $A_\Omega$ on $\hat{\Omega}$ continuous for $\gamma$, by isometries preserving $\hat{\mu}_\Omega$. By compactness of $A_\Omega$, we have that $A_\Omega \cdot H$ is closed for all $H \in \hat{\Omega}$. But since for almost all $H \tau_{\mathbb{R}^n} H$ is dense, we conclude that for almost all $H$ we have $A_\Omega \cdot H = \hat{\Omega}$. To conclude we are reduced to the situation where

\begin{enumerate}
  \item $\Omega = A_\Omega$
  \item $\omega \mapsto H_\omega \in \text{Ham}_{BP}(T^* T^n)$ is continuous for the $\gamma$-topology
  \item On the subgroup $\mathbb{R}^n$ in $A_\Omega$ the action of $\mathbb{R}^n$ on $\Omega$ can be identified with the action by translation of $\mathbb{R}^n$ as a subgroup of $A_\Omega$.
\end{enumerate}

7. SOME RESULTS ON COMPACT ABELIAN METRIC GROUPS

Let $A$ be a compact metric abelian group having $\mathbb{R}^n$ as a dense subgroup (in particular $A$ is connected). According to A. Weil ([We2] p. 110 and [H-M] theorem 8.45) $A$ is the projective limit of finite dimensional tori. In other words there are tori $T^{n_j}$ and group morphisms $f_{j,i} : T^{n_j} \rightarrow T^{n_i}$ for $i < j$ integers, such that $f_{k,j} \circ f_{j,i} = f_{k,i}$ and a map $f_{\infty,i} : A \rightarrow T^{n_i}$ such that $A = \lim \limits_{\rightarrow} T^{n_j}$. We denote by $A_j$ the image of $A$ in $T^{n_j}$, which is clearly a compact subgroup of $T^{n_j}$ hence a subtorus, and we may replace $T^{n_j}$ by $A_j$. Denoting by $p_j = f_{j+1,j}$ and $\pi_j = f_{\infty,j}$ we have a sequence

\[ \cdots \rightarrow A_{j+1} \xrightarrow{p_{j+1}} A_j \xrightarrow{p_j} A_{j-1} \xrightarrow{p_{j-1}} \cdots \]

We set $\mathbb{K}_j = \text{Ker}(\pi_j)$. We then have

**Lemma 7.1.** We have

\[ \lim \limits_{j} \text{diam}(\mathbb{K}_j) = 0 \]

**Proof.** The $\mathbb{K}_j$ are a decreasing sequence of closed - hence compact- subgroups such that $\cap \mathbb{K}_j = \{0\}$ by definition of the projective limit. But this implies the lemma by an easy exercise (or [Ru] theorem 3.10). \qed

Now let $(X, \delta)$ be a metric space. We need the

**Lemma 7.2.** Let us consider the embeddings

\[ \pi_j^* : C^0(A_j, X) \rightarrow C^0(A, X) \]

\[ f \mapsto f \circ \pi_j \]

Then the union of the images of the $\pi_j^*$ is dense in $C^0(A, X)$. 
Proof. Let $f \in C^0(\Lambda, X)$. Then $f$ is uniformly continuous by Heine’s theorem (see [Ru], theorem 4.19):

$$\forall \varepsilon > 0, \exists \eta > 0 \forall x, y \in \Lambda, \ d(x, y) < \eta \implies \delta(f(x), f(y)) < \varepsilon$$

For $j$ large enough $\dim(K_j) < \eta$ so setting $f_j(x) = \min\{f(x + u) \mid u \in K_j\}$, we see that by compactness of $K_j$ the function $f_j$ is continuous and $d(f(x), f_j(x)) < \varepsilon$ provided $\dim(K_j) < \eta$. □

Now remember that we have a group morphism $\tau(\mathbb{R}^n) \rightarrow \Lambda$. By definition of a projective limit, the map $\tau$ is defined by a sequence of maps $\tau_j : \mathbb{R}^n \rightarrow \Lambda_j$ such that $p_j \circ \tau_j = \tau_{j-1}$. Of course the density of $\tau(\mathbb{R}^n)$ implies the density of $\tau_j(\mathbb{R}^n)$ because the preimage by $\pi_j$ of a proper closed subset is a proper closed subset (remember $\pi_j$ is onto by assumption). It will be useful to state

**Lemma 7.3.** Let $u : \mathbb{R}^n \rightarrow T^d$ be a group morphism, and $\tilde{u} : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be its lift. Consider the vector space in $\mathbb{R}^d$ defined by $V = \tilde{u}(\mathbb{R}^n)$. Then $\text{Im}(u)$ is dense if and only if

$$V^\bot \cap \mathbb{Z}^d = \{0\}$$

Proof. See Appendix 2. □

Let us now return to our setting and notice that the situation is the following: we have a subgroup $\Lambda_\Omega$ in $\text{Isom}(\mathcal{H}_\omega, \gamma)$ and for almost every $H$ (for the measure $\tilde{\mu}_\Omega$) we have $\Lambda_\Omega \cdot H = \mathcal{H}_\omega$. Now $\Lambda_\Omega \cdot H$ is approximated by $\Lambda_j \cdot H$ for a finite dimensional torus $\Lambda_j$, and the action of $\mathbb{R}^n$ by $\tau$ yields a dense subgroup of $\Lambda_j$. At the cost of an approximation, we have thus replaced $H_\omega$ for $\omega \in \Lambda_\omega$ by the $H_\omega$ for $\omega \in \Lambda_j$, that is we have a continuous map $\Lambda_j \rightarrow (\mathfrak{ham}_{fc}, \gamma)$ and $\Lambda_j$ is a finite dimensional torus.

8. **Regularization of the Hamiltonians in $\mathfrak{ham}_{fc}$**

Let $H \in \mathfrak{ham}_{FP}(T^*\mathbb{R}^n)$ and its flow $\varphi_H^t$ in $\mathcal{D}\mathfrak{ham}_{FP}(T^*\mathbb{R}^n)$. We set $c(1_{(q, p)}, S)$ to be the critical value corresponding to the unique cohomology class in $S_{q, p}(\xi) = S(q, p; \xi)$. If $L$ is a Lagrangian, we set $c(1_{(q, p)}, L) = c(1_{(q, p)}, S)$ where $S$ is a GFQI for $L$, and $c(1_{(q, p)}, \varphi) = c(1_{(q, p)}, \Gamma(\varphi))$. We now set

**Definition 8.1.** For $\eta > 0$ we set

$$H_\eta^n(q, p) = \frac{1}{\eta} c(1_{(q, p)}, \varphi_H^\eta) = \frac{1}{\eta} c(1_{(q, p)}, \Gamma(\varphi_H^\eta))$$

This defines a map

$$\sigma_\eta : \mathfrak{ham}_{fc}(T^*\mathbb{R}^n) \rightarrow C^{1,0}_{fc}(T^*\mathbb{R}^n)$$
where $C^0_c(T^*\mathbb{R}^n)$ is the set of Lipschitz functions with fiberwise compact support.

Our goal is to prove that $\sigma_\eta$ is a regularizing operator. This is the content of

**Proposition 8.2.** We have for $H \in \widehat{\text{Ham}}_{f_c}(T^*\mathbb{R}^n)$

1. $\gamma_c - \lim_{\eta \to 0} \sigma_\eta(H) = H$
2. For $H$ supported in $\mathbb{R}^n \times B(R)$ and such that $\varphi_H(T^*B(\rho)) \subset T^*B(\rho + r)$ then $\sigma_\eta(H)$ is $C^{(R+r)}_\eta$-Lipschitz.
3. $\sigma_\eta : \widehat{\text{Ham}}_{f_c}(T^*\mathbb{R}^n) \to C^0_c(T^*\mathbb{R}^n, \mathbb{R})$ is continuous for the $\gamma$-topology.
4. $\sigma_\eta \circ \tau = \tau \circ \sigma_\eta$

**Remark 8.3.** One should be careful. The $\gamma_c$-limit in (1) is not a $C^0$ limit, since $H$ is not continuous in general. But even if $H$ is continuous, we do not claim this.

**Proof.** (1) By density we can find $K \in C^\infty_c(T^*\mathbb{R}^n, \mathbb{R})$ such that $\gamma(H, K) \leq \epsilon$. Now for $K \in C^\infty_c(T^*\mathbb{R}^n, \mathbb{R})$ we may find a GFQI, $S_{K,\eta}$ of $\varphi^K_\eta$ such that

$$S_{K,\eta}(q, p) = \eta \cdot K(q, p) + o(\eta)$$

as $\eta$ goes to zero so that $K^n(1)(q, p) = \frac{1}{\eta} c(1(q, p), S_{K,\eta}) = K(q, p) + o(1)$.

We need the following Lemma, that we shall prove in Appendix 3.

**Lemma 8.4.** We can find a GFQI for $\varphi^K_\eta$, $S_{K,\eta}$ such that

$$\|S_{K,\eta}(q, p) - \eta K(q, p)\| \leq C\eta^2 \|\nabla K\|_{C^0}^2$$

Now the formula $c(1(q, p), S_{K,\eta}) = \eta K(q, p) + o(\eta)$ follows immediately by applying on one hand the triangle inequality (see [VI], prop 3.3 p.693)

$$|c(1_x, L) - c(1_x, L')| \leq \gamma(L, L')$$

and on the other hand Proposition 5.18

$$\|K^n(1)(q, p) - K(q, p)\| \leq \eta \cdot \|\nabla K\|_{C^0}^2$$

Now for $\eta$ small enough we have $\gamma(K^n, K) \leq \epsilon$. Remember that for $H, K \in \widehat{\text{Ham}}(T^*\mathbb{R}^n)$ that $H \leq K$ means $c(1_W, \varphi_K, \varphi_H) = 0$ for all $W$. The reduction inequality ([VI], p. 705, proposition 5.1) implies that $H^n(q, p) \leq K^n(q, p)$ for all $(q, p) \in T^*\mathbb{R}^n$.

Now $\gamma(H, K) \leq \epsilon$ implies that $K - \epsilon\chi_R \leq H \leq K + \epsilon\chi_R$ for $R$ large enough: this follows from the formula $c(1_W, \varphi_K + \epsilon\chi_R, \varphi_H) =$
c(1_W, φ_K, φ_H) + ε for W large enough. Now we have \(K^\eta - \varepsilon \chi_R \leq H^\eta \leq K^\eta + \varepsilon \chi_R\) and for \(\eta\) small enough we get \(\|K - K^\eta\| \leq \varepsilon\) so

\[K - 2\varepsilon \leq H^\eta \leq K + 2\varepsilon\]

hence

\[H - 3\varepsilon \leq K - 2\varepsilon \leq H^\eta \leq K + 2\varepsilon \leq H + 3\varepsilon\]

hence \(\gamma(H^\eta, H) \leq 3\varepsilon\).

(2) We have for \(|q_1 - q_2| + |p_1 - p_2| \leq r\)

\[c(1_{(q_1, p_1)} \phi_H^\eta) - c(1_{(q_2, p_2)} \phi_H^\eta) \leq C(r)\]

because for \(L_{(q, p)}\) Hamiltonian isotopic to the vertical and coinciding with \(T_{(q, p)}^* \Delta_\mathbb{R}^{2n}\) in \(\Delta_\mathbb{R}^{2n} \times B_\rho^{2n}\) we have

\[c(1_{(q, p)}, \Gamma(\phi_H^\eta)) = c(\Gamma(\phi_H^\eta), L_{(q, p)})\]

and

\[|c(\Gamma(\phi_H^\eta), L) - c(\Gamma(\phi_H^\eta), \psi(L))| \leq \gamma(L, \psi(L)) \leq \gamma(\psi)\]

As a result, provided there is a Hamiltonian map \(\psi\) with \(\gamma(\psi) \leq C(r)\) such that

\[\psi(T_{(q_1, p_1)}^* \Delta_\mathbb{R}^{2n}) \cap (\Delta_\mathbb{R}^{2n} \times B_\rho^{2n}) = T_{(q_2, p_2)}^* \Delta_\mathbb{R}^{2n} \cap (\Delta_\mathbb{R}^{2n} \times B_\rho^{2n})\]

where \(\rho\) is such that \(\Gamma(\phi_H^\eta) \subset \mathbb{R}^{2n} \times B_\rho^{2n}\). Since we assumed that \(H\) is supported in \(B_R\) we may assume \(\rho = 2R\) and we have \(C(r) = CR \cdot r\). Indeed if \(\psi_1\) is an isotopy such that \(\psi_1\) sends \((q_1, p_1)\) to \((q_2, p_2)\), and \(\Psi_1\) its natural extension to a Hamiltonian isotopy \(T^*\) \(\Delta_\mathbb{R}^{2n}\) we truncate the Hamiltonian generating \(\Psi_1\) to \(\mathbb{R}^{2n} \times B_\rho^{2n}\) where \(\rho\) is an upper bound for \(|Q_H(q, p) - q| + |P_H(q, p) - p|\). Such an upper bound is given by \(r + 2R\) (\(r\) for \(|Q - q|\) and \(2R\) for \(|P - p|\)). This proves the inequality. \(^{18}\)

(3) We have

\[\|\sigma_\eta(H) - \sigma_\eta(K)\|_{C^0} \leq \frac{1}{\eta} \sup_{(q, p)} c(1_{(q, p)}, \phi_K^\eta(\psi_\eta)^{-1}) \leq \frac{1}{\eta} \gamma(\phi_H^\eta, \phi_K^\eta) \leq \frac{1}{\eta} \gamma(H, K)\]

\(^{17}\)Because if \(S\) is a GFQI for \(\phi_K\) then \(S_\varepsilon(q, p; \xi) = S_0(q, p; \xi) + \varepsilon \chi_R(p)\) is a GFQI for \(\phi_K + \varepsilon \chi_R\) and \(c(1_W, S_\varepsilon) = c(1_W, S_0) + \varepsilon\) for \(R\) and \(W\) large enough.

\(^{18}\)We also can take \(R \approx \eta\|H\|_{C^{0,1}}\), and then \(C(r) \approx CR\|H\|_{C^{0,1}}\) but this requires \(H\) to be Lipschitz. But this proves that the maps \(\sigma_\eta\) does increase the Lipschitz norm by a bounded multiplicative constant only.
where the first inequality is just the triangle inequality (see [VI], prop 3.3 p.693) and the second inequality follows by the reduction inequality ([VI], p. 705, proposition 5.1).

(4) We have \( \sigma_\eta(H_\omega)(x + a, p) = \frac{1}{\eta} c(1, x + a, p, \varphi_\eta H_\omega) = c(1, S_\omega(x + a, P; \xi)) \) but \( S_\omega(x + a, P; \xi) \) is the generating function corresponding to \( \tau_a H_\omega \) i.e. \( \Gamma(\tau_a \varphi_\eta H_\omega, \tau_a) \) is the set of \( (q + a, P; P - p, Q - q) \) where \( \varphi_\eta H_\omega(q, p) = (Q, P) \). So we have \( \Gamma(\tau_a \varphi_\eta H_\omega, \tau_a) = \tau_a(\Gamma(\varphi_\eta H_\omega)) \) and

\[
S_{H_\omega \tau_a}(x, P; \tau_a \xi) = S_{\tau_a H_\omega}(x, P; \xi) = S_{H_\omega}(x + a, P; \xi)
\]

We thus proved that

\[
\tau_a \sigma_\eta(H_\omega)(x, p) = \sigma_\eta(H_\omega)(x + a, p) = \sigma_\eta(\tau_a H_\omega)(x, p) = \sigma_\eta(\tau a H_\omega)(x, p)
\]

We are now in the following situation: we started from a continuous map

\[
H : \mathfrak{A}_j \to (\widetilde{\text{Ham}}(T^* \mathbb{R}^n), \gamma)
\]

and have constructed a map

\[
H^0 : \mathfrak{A}_j \to (C^0_{f_c}(T^* \mathbb{R}^n), d_{C_0})
\]

which is continuous, satisfies \( \tau_a H^0 = H^0 \). Note that we may replace if needed \( C^0_{f_c} \) by \( C^k_{f_c} \) by applying convolution since \( \tau_a (H \ast \chi) = (\tau_a H) \ast \chi = H \ast \chi \) (and of course, since \( \| H \ast \chi - H \| \to 0 \) as \( \chi \to \delta_0 \), we also have \( \chi_c \) convergence).

Let us summarize our findings combining the results of Proposition 8.2 and the conclusions of Sections 6 and 7 the following

**Corollary 8.5.** Let \( H : T^* \mathbb{R}^n \times \Omega \to \mathbb{R} \) satisfy assumption (1)-(6) of the Main Theorem. Given \( \epsilon > 0 \) there exists \( d \in \mathbb{N} \) and \( H^\epsilon : T^* \mathbb{R}^n \times T^d \to \mathbb{R} \) such that

1. \( \omega \to H_\omega^\epsilon \) is continuous from \( T^d \) to \( C^\infty_{f_c}(T^* \mathbb{R}^n, \mathbb{R}) \)
2. \( \gamma(H_\omega, H_\omega^\epsilon) \leq \epsilon \) for all \( \omega \in \Omega \).
3. The Hamiltonians \( H_\omega^\epsilon, H_\omega^\epsilon(\pi_d(\omega)) \) satisfy assumptions (1)-(6)

**Proof.** From Section 6 we get \( H \) from \( \mathfrak{A}_\Omega \) to \( \widetilde{\text{Ham}}(T^* \mathbb{R}^n) \). From Section 7 we can approximate \( H \) by a map from \( T^d \) to \( \widetilde{\text{Ham}}(T^* \mathbb{R}^n) \) and from the present Section, we have an approximating map to \( C^\infty_{f_c}(T^* \mathbb{R}^n, \mathbb{R}) \). \( \square \)
9. HOMOGENIZATION IN THE ALMOST PERIODIC CASE

We assume in this section that we have a a map \((q, p; \omega) \rightarrow H(q, p; \omega) = H_\omega(q, p)\) such that

1. \(\omega \in \Omega = T^d\)
2. The map \(\omega \rightarrow H_\omega\) is continuous for the \(C^\infty_f\) topology In particular the \(H_\omega\) have uniformly fiberwise compact support and the \(H_\omega\) are uniformly BPS by Proposition 4.3.

We set \(\varphi_{\omega t}^t\) to be the time \(t\) flow for \(H_\omega\) and \(\varphi_{\epsilon, \omega} = \rho_{\epsilon} \varphi_{\omega} \rho_{\epsilon}^{-1}\). By compactness of \(\Omega\) we also have a map \(\omega \rightarrow S_\omega(q, p; \xi)\) of GFQI for \(\varphi_{\omega} = \varphi_{1/\omega}\), with \(\xi\) living in a vector space independent from \(\omega\) : indeed its dimension is bounded by \(2nN\) such that \(\varphi_{\omega}^{1/N}\) is in a given neighborhood of \(\text{Id}\) for all \(\omega \in \Omega\).

**Definition 9.1.** We set

\[
h_{k, U}^\omega(p) = \lim_{\rho \in \mathcal{V}} c(\mu_{U \times \mathcal{V}}, \varphi_{k, \omega})
\]

and

\[
h_k^\omega = \lim_{U \in \mathcal{R}^n} h_{k, U}^\omega
\]

Our first result is

**Proposition 9.2.** The sequence \(h_k^\omega\) is equicontinuous and equibounded. All its converging subsequences have the same limit \(h_\omega(p)\) which is in fact independent from \(\omega\) and denoted by \(\overline{H}(p)\). We denote by \(\overline{\varphi^t}\) the flow of \(\overline{H}\) in \(\overline{D^{\text{Sym}}}_f c(T^*\mathbb{R}^n)\) which belongs to \(\overline{D^{\text{Sym}}}_F p c(T^*\mathbb{R}^n)\).

**Proof.** For typographical reasons, the \(\omega\) parameter will be omitted in the notation, but of course everything depends on \(\omega \in \Omega\). Set \(\varphi_k(q, p) = (Q_k(q, p), P_k(q, p))\) and \(Q = Q_1, P = P_1\). Then we have \(\frac{\partial S_k}{\partial q}(q, P; \xi) = 0\) and \(\frac{\partial S_k}{\partial p}(q, P; \xi) = Q_k(q, P) - q\). But

\[
h_{k, U}(p) = S_k(q(p), p; \xi(p))
\]

where \(\frac{\partial S_k}{\partial q}(q, P; \xi) = 0\) and either \(\frac{\partial S_k}{\partial q}(q, P; \xi) = 0\) if \(q \in U\) or \(\frac{\partial S_k}{\partial q}(q, P; \xi) = \lambda \cdot v_U(q)\) if \(q \in \partial U\) and \(v_U(q)\) is the exterior normal. Now as \(p\) varies, we can choose \(p \rightarrow (q(p), \xi(p))\) to be piecewise smooth, so that

\[
d h_k(p) = \
\]

\[
\frac{\partial S_k}{\partial p}(q, P; \xi)(q, p; \xi) + \frac{\partial S_k}{\partial q}(q, P; \xi)(q, p; \xi) \cdot \frac{\partial q}{\partial p} + \frac{\partial S_k}{\partial \xi}(q, P; \xi)(q, p; \xi) \cdot \frac{\partial \xi}{\partial p}
\]

but \(\frac{\partial S_k}{\partial q}(q, P; \xi)(q, p; \xi) = 0\) and then either \(q \in U\) and then \(\frac{\partial S_k}{\partial q}(q, P; \xi)(q, p; \xi) = 0\) or \(q \in \partial U\) and then \(\frac{\partial q}{\partial p} \in T(\partial U)\), so that the term \(\frac{\partial S_k}{\partial q}(q, P; \xi)(q, p; \xi) \cdot \frac{\partial q}{\partial p}\)
also vanishes. In the end \( dh_k(p) = \frac{\partial S}{\partial p} (q, P; \xi)(q, p; \xi) = Q_k(q, p) - q \).

But finite propagation speed implies that \( Q_k(q, p) - q = \frac{1}{k} (Q(kq, p) - kq) \) is bounded, so \( |dh_{k,U}(p)| \) is uniformly bounded, independently from \( U, k \). From this we conclude that the sequence \( h_k \) is equicontinuous. Equiboundedness follows from Definition 5.8 in [V5], which states that a GFQI \( S_k \) of \( \varphi_k \) satisfies \( |S_k - B_k| \leq C \) where \( C \) is a bound for a generating function \( S(q, p) \) for \( \varphi^1 \) and \( B_k \) is a quadratic form. This implies that \( |h_k(p)| \leq C \) and since all this estimates are uniform in \( \omega \), this implies (uniform) equiboundedness. We may thus apply Ascoli-Arzela’s theorem, and conclude that \( h_{\omega_k} \) has a converging subsequence. Proving that the limit is unique follows as in [V5], lemma 5.11.

Finally we prove that \( h_{\omega}(p) \) is independent from \( \omega \), using the commutation of \( \tau_a \) and \( \rho_k \).

We define

\[
\widehat{\text{Ham}}_{f_c, BP}(T^* \mathbb{R}^n) = \overline{\widehat{\text{Ham}}}_{f_c, BP}(T^* \mathbb{R}^n) \cap \overline{\text{Ham}}_{f_c}(T^* \mathbb{R}^n)
\]

The next Proposition is the analog of Proposition 5.15 in [V5]

**Proposition 9.3.** Let \( \alpha \in \overline{\text{Ham}}_{f_c, BP}(T^* \mathbb{R}^n) \). There exists a sequence \( k_v \) such that

\[
\lim_{v \to +\infty} \lim_{U \subset \mathbb{R}^n} c(\mu_U \otimes 1(p), \Gamma(\varphi_{k_v, \omega} \alpha)) = \lim_{U \subset \mathbb{R}^n} c(\mu_U \otimes 1(p), \Gamma(\varphi_{k, \omega} \alpha))
\]

**Proof.** The proof is identical to the proof of Proposition 5.15 in section 6 of [V5].

The next proposition is the analogue of Proposition 7.2 in [V5], but requires an adaptation.

**Proposition 9.4.** For each \( \varepsilon > 0 \) there exists \( K \) such that for all \( k \geq K \) and \( U \) large enough, we have

\[
c(\mu_U \otimes 1(p), \varphi_{k, \omega}) \leq c(1_{U} \otimes 1(p), \varphi_{k, \omega}) + \varepsilon
\]

This implies

**Corollary 9.5.** We have \( \overline{\varphi^{-1}} = (\overline{\varphi})^{-1} \), or equivalently \( \overline{H_{\varphi^{-1}}} = -\overline{H_\varphi} \).
Now putting together Proposition 9.3 and Corollary 9.5 we get

**Proposition 9.6.** For almost all $\omega \in \Omega$ the sequence $\varphi_{k,\omega}$ converges to $\overline{\varphi}$.

**Proof assuming Corollary 9.5 and Proposition 9.3.** Let us prove the above Proposition is a consequence of Corollary 9.5 and Proposition 9.3. Indeed the proposition implies

$$\lim_{k \to +\infty} \lim_{\omega} c(\mu_U, \varphi_{k,\omega} \overline{\varphi} ) \leq \lim_{\omega} c(\mu_U, \text{Id}) = 0$$

Applying the same inequality for $\varphi^{-1}$ instead of $\varphi$ and using the Corollary, we get

$$\lim_{k \to +\infty} \lim_{\omega} c(\mu_U, \varphi_{k,\omega}^{-1} \overline{\varphi} ) \leq \lim_{\omega} c(\mu_U, \text{Id}) = 0$$

and this implies

$$\lim_{k \to +\infty} \lim_{\omega} \gamma(\mu_U, \varphi_{k,\omega}^{-1} \overline{\varphi} )$$

which proves our claim. □

**Proof of Corollary 9.5 assuming Proposition 9.4.** Set

$$h^+_{k,\omega} (\varphi; p) = \lim_{U \subset \mathbb{R}^n} c(\mu_U \otimes 1(p), \varphi_{k,\omega})$$

$$h^-_{k,\omega} (\varphi; p) = \lim_{U \subset \mathbb{R}^n} c(1_U \otimes 1(p), \varphi_{k,\omega})$$

so that $h^-_{k,\omega} (\varphi; p) \leq h^+_{k,\omega} (\varphi; p)$. Set $\sigma_{p_0}(q, p) = (q, p + p_0)$ then if $S_p(x; \xi) = S(x, p; \xi)$ is a GFQI for $\varphi$, then $S_p'(x; \xi) = S(x, p; \xi)$ is a GFQI for $\sigma_p'(0) - \varphi(\sigma_p'(0))$. If we assume $\varphi$ has FPS we have from Proposition 5.14

$$c(\mu_U, \sigma_p'(0) - \varphi(\sigma_p'(0))) \leq c(\mu_U, \sigma_p \varphi^{-1}(\sigma_p'(0)))$$

for $V$ such that $\varphi(T^* U) \subset T^* V$. Taking the limit for $U \subset \mathbb{R}^n$ we get

$$\lim_{U \subset \mathbb{R}^n} c(\mu_U, S_p) = \lim_{U \subset \mathbb{R}^n} c(\mu_U, \sigma_p \varphi^{-1}(\sigma_p'(0)))$$

and the same holds for $1_U$ instead of $\mu_U$. Now we may write (omitting the $\omega$) using first Proposition 5.9 (1) and then FPS of $\varphi$

$$h^+_{k} (\varphi^{-1}; p) = \lim_{U \subset \mathbb{R}^n} c(\mu_U \otimes 1(p), \sigma_p \varphi_{k,\omega} p(0)) =$$

$$\lim_{U \subset \mathbb{R}^n} c(1_U \otimes 1(p), 0 - \sigma_p \varphi_{k,\omega} p(0)) \leq$$

$$\lim_{V \subset \mathbb{R}^n} c(1_V \otimes 1(p), \sigma_p \varphi_{k,\omega}^{-1} p(0)) = -h^-_{k} (\varphi; p)$$

As a result

(a) $$h^+_{k} (\varphi^{-1}; p) + h^-_{k} (\varphi; p) \leq 0$$

and as $k$ goes to $+\infty$, Proposition 9.4

$$h^+_{k} (\varphi^{-1}; p) - h^-_{k} (\varphi; p) \leq \varepsilon$$
we get so we get

\[ h_k^+(\varphi^{-1}; p) + h_k^-(\varphi; p) \leq \varepsilon \]

On the other hand, we have using again Proposition \( 5.9 \) (1)

\[ -c(1_U, \sigma_p \varphi_k \sigma_p(0_{\mathbb{R}^n})) \leq -c(1_V, 0_{\mathbb{R}^n}, \sigma_p \varphi_k^{-1} \sigma_p(0_{\mathbb{R}^n})) = c(1_V, 0_{\mathbb{R}^n}) \]

so

\[ -h_k^-(\varphi; p) \leq h_k^+(\varphi; p) \]

so using (a) we get

\[ h_k^+(\varphi; p) + h_k^-(\varphi; p) = 0 \]

Using again Proposition \( 9.4 \) we get for \( k \) large enough

\[ h_k^-(\varphi^{-1}; p) + h_k^-(\varphi; p) \geq -\varepsilon \]

Adding (b) and (d) we get

\[ \left( h_k^+(\varphi^{-1}; p) - h_k^-(\varphi^{-1}; p) \right) + \left( h_k^+(\varphi; p) - h_k^-(\varphi; p) \right) \leq 2\varepsilon \]

Since \( \overline{H}_{\varphi^{-1}} = \lim_k h_k^+(\varphi^{-1}; p) \), inequality (b) implies

\[ \overline{H}_{\varphi^{-1}} + \overline{H}_\varphi \leq 0 \]

Using (d) and (e) we get

\[ \overline{H}_{\varphi^{-1}} + \overline{H}_\varphi \geq 0 \]

so we may conclude

\[ \overline{H}_{\varphi^{-1}} + \overline{H}_\varphi = 0 \]

\[ \square \]

We shall interchangeably the notations \( S_{\omega}(q, p; \xi, \zeta; \omega) \) or \( S(q, p; \xi, \zeta; \omega) \) for the GFQI of \( \varphi_\omega \). We shall make repeated use of the iteration formula (see [V5]), defining the GFQI \( S_{k,\omega} \) for \( \varphi_{k,\omega} \) in terms of the GFQI \( S_\omega \) of \( \varphi_\omega \).

**Definition 9.7** (Iteration formula).

\[ S_{k,\omega}(x, y; \xi, \zeta; \omega) = \frac{1}{k} \left[ S_\omega(kx, p_1; \xi_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j; \xi_j) + S_\omega(kq_k, y; \xi_k) \right] + B_k(x, y; \xi) \]

where \( \xi = (\xi_1, ..., \xi_k), \zeta = (p_1, q_2, ..., p_{k-1}, q_k), q_1 = x, p_k = y \) and

\[ B_k(x, y; \xi) = \langle p_1, q_2 - x \rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j \rangle + \langle y, x - q_k \rangle \]

and \( F_{k,\omega} = S_{k,\omega} - B_k \).

The action of \( \mathbb{R}^n \) is given by

\[ \tau_a^{(k)}(x, y; \xi, \zeta; \omega) = (x + \frac{a}{k}, y; \xi; \tau_{a/k} \xi; \tau_{a} \omega) \]

We now prove
Lemma 9.8. Assume $\omega \mapsto \varphi_\omega$ for $\omega \in \Omega = T^d$ to be continuous. Then we may choose $\omega \mapsto S_\omega(q, p; \xi)$ to be continuous and such that

$$S(q + a, p; \tau_a \xi) = S(q, p; \xi; \omega)$$

Proof. It is enough to prove this assuming $\varphi_\omega$ is $C^1$ small, that is for $\varphi_\omega^{1/k}$ with $k$ large enough using then the iteration formula. But then the graph of $\varphi_\omega$ is the graph of a generating function with no fiber variable, which obviously depends continuously on $\omega$ and satisfies the above formula. \ \Box

Now remember that $\tau_a$ is given on $\Omega = T^d$ by $\tau_a(\omega) = \omega + A \cdot a$ where $A : \mathbb{R}^n \to \mathbb{R}^d$ is a linear injective map with dense image in $T^d$. Consider triples $\alpha, \beta, \gamma$ with $\alpha \in H^*(T^d), \beta \in H^*(U)$ or $H^*(U, \partial U), \gamma \in H^*(V)$ or $H^*(V, \partial V)$. We may then define $^{19}c(\alpha \otimes \beta \otimes \gamma, S)$, and we have

Lemma 9.9. We have the inequalities

$$c(\mu_U \otimes 1(p) ; S_\omega) \leq c(\mu_{T^d} \otimes \mu_U \otimes 1(p) ; S)$$

$$c(1_{T^d} \otimes \mu_U \otimes 1(p) ; S) \leq c(1_U \otimes 1(p) ; S_\omega)$$

Proof. This is the reduction inequality (see [VI] prop. 5.1 p; 705). \ \Box

We now compare spectral invariants of $S$ with those of $S^0$, where we define $S^0(p; \xi; \omega) = S(0, p; \xi; \omega)$

Lemma 9.10. We have

$$\lim_{U \subset \mathbb{R}^n} c(\mu_{T^d} \otimes \mu_U \otimes 1(p) ; S) = c(\mu_{T^d} \otimes 1(0) \otimes 1(p) , S) = c(\mu_{T^d} \otimes 1(p), S^0)$$

and

$$\lim_{U \subset \mathbb{R}^n} c(1_{T^d} \otimes \mu_U \otimes 1(p) ; S) = c(1_{T^d} \otimes 1(0) \otimes 1(p) = c(1_{T^d} \otimes 1(p), S^0)$$

Remark 9.11. This is an extension to GFQI of the following obvious identity for continuous functions $f : \mathbb{R}^n \times T^d \to \mathbb{R}$ such that $f(x + a, \tau_a \omega) = f(x, \omega)$: for any $x_0 \in \mathbb{R}^n$ we have

$$\sup_{(x, \omega) \in \mathbb{R}^n \times T^d} f(x, \omega) = \sup_{\omega \in T^d} f(x_0, \omega)$$

Moreover if the action of $\tau$ has dense orbits, this is also equal to $\sup_{x \in \mathbb{R}^n} f(x, \omega_0)$ for any $\omega_0 \in \Omega$. The analog of this last statement will be our main result.

Proof. Clearly if $0 \in U$ we have

$$c(\mu_{T^d} \otimes \mu_U \otimes 1(p) ; S) \geq c(\mu_{T^d} \otimes 1(0) \otimes 1(p) ; S)$$

$^{19}$Caveat: the cohomology class $\alpha$ corresponds to the last variable, $\omega$!
and we need to prove the reverse inequality. Let C be a cycle representing 
\( \mu_T \otimes 1(p) \in H_\ast((S^0_p)^{\ast}, S^0_p)^{\ast} \) with \( c \leq c(\alpha \otimes 1(0) \otimes 1(p), S) + \varepsilon \) and set

\[
\tilde{C}_U = \{(x, p, \tau_x \xi; \tau_x \omega) \mid (0, p; \xi; \omega) \in C\}
\]

Then \( \tilde{C}_U \subset S_p^c \) and clearly \( [\tilde{C}_U] = \mu_T \otimes U \otimes 1(p) \) thus

\[
c(\mu_T \otimes U \otimes 1(p), S) \leq S(\tilde{C}_U) = S^0(C)
\]

because \( S(x, p, \tau_x \xi, \tau_x \omega) = S(0, p; \xi; \omega) \) and \( S^0(C) \leq c \).

This implies

\[
c(\mu_T \otimes U \otimes 1(p); S) \leq c(\mu_T \otimes 1(0) \otimes 1(p); S)
\]

This proves the first equality. The second one is the dual of the first one, 
since \( \mu_T \otimes U \) is dual to \( 1_T \otimes 1(U) \).

\[\square\]

Our Proposition 9.4 then follows from the following 20.

**Proposition 9.12.** For each \( \varepsilon > 0 \) there exists \( K \) such that for \( k \geq K \)

\[
c(\mu_T \otimes 1(p), S_k^0) \leq c(1_T \otimes 1(p), S_k^0) + \varepsilon
\]

**Proof.** The proof will take up the rest of the section. We rewrite the iteration formula

\[
S_{k,\omega}(x, y; \xi, \xi; \omega) = \frac{1}{k} \left[ S_\omega(kx, p_1; \xi_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j; \xi_j) + S_\omega(kq_k, y; \xi_k) \right] + B_k(x, y; \xi)
\]

where \( \xi = (\xi_1, \ldots, \xi_k), \xi = (p_1, q_2, \ldots, p_{k-1}, q_k), q_1 = x, p_k = y \) and

\[
B_k(x, y; \xi) = \langle p_1, q_2 - x \rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j \rangle + \langle y, x - q_k \rangle
\]

and \( F_{k,\omega} = S_{k,\omega} - B_k \). The action of \( \mathbb{R}^n \) is given by

\[
\tau_a^{(k)}(x, y; \xi, \xi; \omega) = (x + \frac{a}{k}, y; \xi; \tau_{a/k} \xi; \tau_a \omega)
\]

and now \( S_{k,\omega} \) is \( \tau_a^{(k)} \)-invariant, i.e.

\[
S_k(x + \frac{a}{k}, y; \xi, \xi; \omega) = S(x, y; \xi, \xi; \omega)
\]

Let \( a \in \mathbb{R}^n \) such that for some \( v \in \mathbb{Z}^d \) we have \( |A \cdot a - v| \leq \delta \) (that is \( d_T(\tau_a(0), 0) \leq \delta \) where \( d_T \) is the distance on the torus). Then for some constant depending on \( H \) and provided \( \delta \) is small enough

20 The point of replacing \( S \) by \( S^0 \) is to avoid the complications related to the non-compactness of \( x \in \mathbb{R}^n \). Our proofs could be adapted to work directly with \( S \), but proving that the cycles we construct are in the right homology class is slightly more involved.
\[(\ast)\] \(\forall t \in [0,1], \forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega,\)
\[|S(kq + ta, p; \xi; \omega) - S(kq, p; \xi; \omega)| \leq C\]

and
\[(\ast\ast)\] \(\forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega,\)
\[|S(kq + a, p; \xi; \omega) - S(kq, p; \xi; \omega)| = |S(kq, p; \xi; t_{\omega} - a) - S(kq, p; \xi; \omega)| \leq \varepsilon\]

Indeed the first inequality holds because
\[|S(q + a, p; \xi; \omega) - S(q, p; \xi; \omega)| = |S(q, p; \xi; t_{\omega} - a) - S(q, p; \xi; \omega)| \leq \sup_{\omega, \omega'}|S(q, p; \xi; \omega) - S(q, p; \xi; \omega')|\]

This follows from the fact that we may assume that we have no \(\xi\) variable and then use the iteration formula. In this case we may assume \(|S(q, p; \omega) - S(q, p; \omega')| \leq \gamma(\varphi_{\omega}, \varphi_{\omega'})\). The second inequality follows from the fact that \(d_{t_{\omega}}(t_{\omega}, \omega) \leq \delta\) and the continuity of \(S\).

Now let \(\gamma\) be the path in \(\mathbb{R}^n\) defined by \(\gamma(t) = t \cdot a\) for \(0 \leq t \leq 1\). Set \(\tilde{\gamma}^{(k)}\) be the path in \((\mathbb{R}^n)^k\) defined as the concatenation of the \(k\) paths

\[
t \mapsto (\gamma(t), 0, \ldots, 0), \quad \text{for } t \in [0, \frac{1}{k}]
\]

\[
t \mapsto (\gamma(\frac{1}{k}), \gamma(t), \ldots, 0), \quad \text{for } t \in [\frac{1}{k}, \frac{2}{k}]
\]

\[
\vdots
\]

\[
t \mapsto (\gamma(\frac{1}{k}), \gamma(\frac{1}{k}), \ldots, \gamma(\frac{1}{k}), \gamma(t)), \quad \text{for } t \in [\frac{k-1}{k}, 1]
\]

The path \(\tilde{\gamma}^{(k)}\) connects \(\tilde{\gamma}^{(k)}(0) = (0, \ldots, 0)\) to \(\tilde{\gamma}^{(k)}(1) = (\frac{a}{k}, \frac{a}{k}, \ldots, \frac{a}{k})\) through the points \(\tilde{\gamma}^{(k)}(\frac{1}{2k}) = (\frac{a}{k}, 0, \ldots, 0), \tilde{\gamma}^{(k)}(\frac{2}{3k}) = (\frac{a}{k}, \frac{a}{k}, 0, \ldots, 0), \ldots, \tilde{\gamma}^{(k)}(\frac{k-1}{k}) = (\frac{a}{k}, \frac{a}{k}, \ldots, \frac{a}{k}, 0)\).

We shall omit the superscript \(k\) and set \(\tilde{\gamma}(t) = \gamma(t) = (\gamma_1(t), \ldots, \gamma_k(t)) = (\gamma_1(t), \tilde{\gamma}(t))\). We then set \(\tau_{\tilde{\gamma}(t)} \xi = \tau_{\tilde{\gamma}(t)}(p_1, q_2, \ldots, p_{k-1}, q_k) = (p_1, q_2 + \gamma_2(t), \ldots, p_{k-1}, q_k + \gamma_k(t))\) and \(\tau_{\tilde{\gamma}(t)}(x, y; \xi) = (x + \gamma_1(t), y; \tau_{\tilde{\gamma}(t)} \xi)\). Now from \((\ast)\) and \((\ast\ast)\) and the formula

\[
F_k(x, y; \xi; \omega) = \frac{1}{k} \left[ S_0(kx, p_1; \xi_1) + \sum_{j=2}^{k-1} S_0(kq_j, p_j; \xi_j) + S_0(kq_k, y; \xi_k) \right]
\]

we infer that on \([\frac{l}{k}, \frac{l+1}{k}]\) for \(1 \leq l \leq k\).
\[
F_k(\tau_\varphi(t)(x, y; \xi; \zeta; \omega)) = \\
F_{k, \omega}(x, y; \xi; \zeta; \omega) + \frac{1}{k} \left[ S(kx + a, p_1; \xi, \zeta; \omega) - S(kx, p_1; \xi_1; \omega) + \sum_{k=2}^{l} \left( S(kq_j + a, p_j; \xi_j; \omega) - S(kq_j, p_j; \xi_j; \omega) \right) + S(kq_{l+1} + a, p_{l+1}; \xi_{l+1}; \omega) - S(kq_{l+1}, p_{l+1}; \xi_{l+1}; \omega) \right]
\]
so we get

\begin{equation}
(9.2) \quad \left| F_k(\tau_\varphi(t)(x, y; \xi; \zeta; \omega)) - F_k(x, y; \xi, \zeta; \omega) \right| \leq \frac{\varepsilon l}{k} + \frac{C}{k} \leq \frac{C}{k} + \varepsilon
\end{equation}

We now want to estimate the variation of \( B_k \) on \( \tau_\varphi(t)(x, y; \zeta) \) as \( t \) varies from 0 to 1. Note that the choice of this path is crucial to our argument: by changing coordinates one at the time, we achieve an increase of \( S \) by 0(\( 1/k \)) instead of 0(1).

**Lemma 9.13.** We have

\[
|B_k(\tau_\varphi(t)(x, y; \zeta)) - B_k(x, y; \zeta)| \leq (|p_{l+2}| + |p_{l+1}|) \frac{|a|}{k}
\]

**Proof.** Indeed,

\[
B_k(x, y; \zeta) = \langle p_1, q_2 - x \rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j \rangle + \langle y, x - q_k \rangle
\]

and replacing \( x, q_2, \ldots , q_l \) by \( x + \frac{a}{k}, q_2 + \frac{a}{k}, \ldots , q_l + \frac{a}{k}, q_{l+1} \) by \( q_{l+1} + \frac{a}{k} \) and leaving \( q_{l+2}, \ldots , q_k \) unchanged, we get

\[
B_k(\tau_\varphi(t)(x, y; \zeta)) = B_k(x, y; \zeta) + \langle p_{l+1}, t \frac{a}{k} - \frac{a}{k} \rangle - \langle p_{l+2}, t \frac{a}{k} \rangle
\]
and this proves the Lemma. \( \square \)

We must then bound the quantity \((|p_{l+2}| + |p_{l+1}|) \frac{|a|}{k}\) and we shall modify the cycle \( C \) representing the class in \( H_k(S_k^c, S_k^-) \) so that the \( |p_l| \) remain bounded. This follows from the following Lemma, already proved in [V5]:

**Lemma 9.14** (See Lemma 7.5 in [V5]). There exists constants \( K, M \) such that, given a cycle \( C \subset S_k^c \) representing a class \([C] \in H_* (S_k^c, S_k^-)\), we have a cycle \( \tilde{C} \subset S_k^c \) such that \([\tilde{C}] = [C]\) in \( H_* (S_k^c, S_k^-) \) and

1. \( \tilde{C} \subset S_k^{-4K} \cup \{(x, y; \xi, \zeta; \omega) \mid \max_j |p_j| \leq M \} \cap S_k^c \)
2. \( \tilde{C} \cap S_k^{-3K} \subset \{(x, y; \xi, \zeta; \omega) \mid \zeta \in E_k^- \} \) where \( E_k^- \) is the negative eigenspace of \( B_k \).
The Lemma means that we can deform $C$ so that below a certain level of $S_k$ it coincides with the negative bundle of $B_k$.

**Proof.** This is as in lemma 7.5 of [V5]. Let $Z$ be the vector field
\[
\dot{q}_j = \chi(|p_j|)(p_j - p_{j-1}) = Z_{q_j}(q, p), \quad \dot{p}_j = 0 = Z_{p_j}(q, p)
\]
where $\chi(r)$ vanishes for $r \leq 1$. Denoting by $\psi^s$ its flow, we have
\[
\frac{d}{ds} S_k(\psi^s(q, p)) = dS_k(q, p) \cdot Z(q, p) = \left\langle \frac{d}{dq} S_k(q, p), Z_{q_j}(q, p) \right\rangle = -\sum_{j=1}^k \chi(|p_j|) \left\langle \frac{d}{dq} S_k(q, p), p_j - p_{j-1} \right\rangle = -\sum_{j=1}^k \chi(|p_j|)|p_j - p_{j-1}|^2
\]
the last equality because $S$ vanishes on the support of $\chi(|p_j|)$.

Now given $y = p_k$, if $\sup_j |p_j| \geq M$, we have that $\sum_{j=1}^k \chi(|p_j|)|p_j - p_{j-1}|^2$ is bounded from below by some positive quantity $c_k$ (which is $O(1/k)$ but it does not matter). Thus outside the region $\{(q, p) \mid |p_j| \leq M\}$, the vector field $Z$ is a pseudo-gradient vector field for $F_k$. Since $Z$ is complete, its flow $\psi^s$ has the following properties

1. It preserves the $p_j$
2. Outside $\{(q, p) \mid |p_j| \leq M\}$, we have $\frac{d}{ds} S_k(\psi^s(q, p)) \leq -c_k$

As a result if $F_k(q, p) \leq c$ we have $\psi^s(q, p) \in \{(q, p) \mid |p_j| \leq M\} \cup F_k^{c}$.

Now to satisfy (2), we use a "cut and paste" as in [V5] lemma 7.5. We obtain the following

**Proposition 9.15.** Given a class $a$ in $H_*(S^c_k, S^{-\infty}_k)$, we can find a cycle $C$ representing $a$ such that
\[
S_k(\tau_{\mathcal{T}(t)}(C)) \leq S_k(C) + \varepsilon + \frac{C}{k} + \frac{2M|a|}{k}
\]

Now let $a \in H_*(T^d)$ be represented by a map $u : C \to T^d$ and $b \in H_1(T^d)$ be represented by a map $\nu : S^1 \to T^d$. Then the Pontryagin product $a \cdot b$ is represented by $u \cdot \nu : S^1 \times C \to T^d$ given by $u \cdot \nu(z, \theta) = u(z) + \nu(\theta)$.

To conclude the proof of Proposition 9.12 (and as a consequence of Proposition 9.4) we need the
Lemma 9.16. Let \( \nu \in \mathbb{Z}^d \) be such that \( |A \cdot a - \nu| \leq \delta \), and let \( \beta_\nu \) be the class in \( H_1(T^d) \) of the loop \( t \mapsto t \cdot \nu \) (for \( t \in [0, 1] \)). Then for \( k \) large enough, we have
\[
c(\alpha \cdot \beta_\nu \otimes 1(p), S_k^0) \leq c(\alpha \otimes 1(p), S_k^0) + \varepsilon
\]

Proof. Let \( C \) be a cycle representing a class in \( H_*(S_k^0, (S_k^0)^{-\infty}) \). We may assume \( C \) satisfies properties (1) and (2). We are going to construct a cycle in the class of \( \alpha \cdot \beta \) as made of three pieces. First set
\[
C_1 = \bigcup_{t \in [0, 1]} C_1(t)
\]
where
\[
C_1(t) = \{(0, p; \xi, \tau_{-\gamma_1(t)} \tau_{t \cdot \nu} \tau_{k \gamma_1(t)} \omega) \mid (0, p; \xi, \omega) \in C\}
\]
According to Proposition 9.15 since
\[
S_k(0, p; \xi, \tau_{-\gamma_1(t)} \tau_{t \cdot \nu} \tau_{k \gamma_1(t)} \omega) = S_k(\gamma_1(t), p; \xi, \tau_{t \cdot \nu} \tau_{k \gamma_1(t)} \omega) \leq S_k(C) + \varepsilon + \frac{C + 2M|a|}{k}
\]
as a result we have for each \( t \in [0, 1] \) we have
\[
S_k^0(C_1(t)) \leq S_k^0(C) + \varepsilon + \frac{C + 2M|a|}{k}
\]
hence
\[
S_k^0(C_1) \leq S_k^0(C) + \varepsilon + \frac{C + 2M|a|}{k}
\]
Note that
\[
C_1(0) = C
\]
\[
C_1(1) = \{(0, p; \xi, \omega) \mid (0, p; \xi, \omega) \in C\}
\]
Now for \( u \in [0, 1] \) define the path \( \eta(u) = (1 - u)A \cdot a + u \nu \) so that \( \eta(0) + \omega = \tau_{a \omega} \). Set
\[
C_1(1 + u) = \{(0, p; \xi, \omega + \eta(u)) \mid (0, p; \xi, \omega) \in C\}
\]
Now \( C_1(2) = C \) and since \( |\eta(u) - A \cdot a| \leq \delta \) we have
\[
S_k^0(C_1(1 + u)) \leq S_k^0(C_1(1)) + \varepsilon
\]
so that the cycle
\[
\hat{C} = \bigcup_{s \in [0, 2]} C_1(s)
\]
satisfies for \( k \) large enough
\[
S_k^0(\hat{C}) \leq S_k^0(C) + 2\varepsilon
\]
Moreover we claim that the cycle \( \hat{C} \) defines a cycle in the homology class of \( \alpha \cdot \beta \). Indeed the lift of the variable \( \omega \) starting from \( \omega_0 \) is given

(1) for \( s \in [0, 1] \) by the path \( s \mapsto \omega_0 + sA \cdot a \)

(2) for \( s \in [1, 2] \) by \( s \mapsto \omega_0 + (2-s)A \cdot a + (s-1)\nu \)

and since it joins \( \omega_0 \) to \( \omega + \nu \) it belongs to the class \( \beta \). As a result \( [\hat{C}] = \alpha \cdot \beta \in H_\ast((S_k^0)^{+\infty}, S_k^{0 \cdot -\infty}) \) and this proves the lemma.

We shall also need

**Lemma 9.17.** Let \( \varepsilon > 0 \) and \( A : \mathbb{R}^n \rightarrow \mathbb{R}^d \) be a linear map such that \( A(\mathbb{R}^n) \) has dense projection on \( T^d \). Then there is a basis of integral vectors \( \nu_1, \ldots, \nu_d \) in \( \mathbb{Z}^d \) such that there exists vectors \( a_1, \ldots, a_d \) in \( \mathbb{R}^n \) such that

\[
|A \cdot a_j - \nu_j| \leq \varepsilon
\]

**Proof.** See Appendix 2.

**Proof of Propositions 9.12 and 9.4.** Let \( \alpha_j \in H_1(T^d) \) be the homotopy class of the path \( t \mapsto t \cdot \nu_j \) where \( \nu_j \) is a basis of \( \mathbb{R}^d \) given by Lemma 9.17. Then \( \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_d = c_d\mu_{T^d} \) for some \( c_d \neq 0 \). Since \( c(c_d\mu_{T^d}, f) = c(\mu_{T^d}, f) \) we obtain by repeated applications of Lemma 9.16 that \( c(\mu_{T^d} \otimes 1(p), S_k^0) \leq c(1_{T^d} \otimes 1(p), S_k^0) + \varepsilon \) and this proves Proposition 9.12, hence Proposition 9.4.

10. Proof of the Main Theorem

We first prove that under assumptions (1)-(6) we have \( \lim_{\varepsilon \rightarrow 0} \varphi_{\varepsilon, \omega} = \varphi^t \) for almost all \( \omega \in \Omega \). We start from \( H \) satisfying (1)-(6), then, using the results of Section 6 we get a map \( H : \mathcal{A}_\Omega \rightarrow \text{Hamilton}(T^* T^n) \) such that \( \mathcal{A}_\Omega \) is a compact connected metric abelian group. According to Section 7 \( \mathcal{A}_\Omega \) is the projective limit of finite dimensional tori, \( \mathcal{A}_j \), on which \( \tau_a \) is given by \( \tau_a \omega = \omega + A_j \cdot a \) where the projection of \( A_j(\mathbb{R}^n) \) is dense in \( \mathcal{A}_j \) and \( \omega \mapsto H(\bullet, \bullet; \omega) \) is continuous from \( \mathcal{A}_j \) to \( C^\infty(T^* T^n, \mathbb{R}) \) and satisfies (1)-(6).

By Corollary 8.5 we find \( H^n \) in \( C^\infty(T^* T^n \times \mathcal{A}_j, \mathbb{R}) \) such that

\[
\gamma(H^n_{\pi_j(\omega)}, H_{\omega}) \leq \eta
\]

where \( \pi_j : \mathcal{A}_\Omega \rightarrow \mathcal{A}_j \) is the projection map. According to Section 9 we know that

\[\gamma_c - \lim H^n_{k, \pi_j(\omega)} = \overline{T^n}_{\pi_j(\omega)}\]

and since for all \( k, \omega \), we have \( \gamma_c(H^n_{k, \pi_j(\omega)}, H_{k, \omega}) \leq \eta \) we infer for \( k \) large enough

\[\gamma(\overline{T^n}_{\pi_j(\omega)}), H_{k, \omega}) \leq 2\eta\]
Now consider a sequence $\eta_\nu$ converging to 0 so that $H^\eta_\nu_{\pi_{\nu}(\omega)}$ is a $\gamma - c$-Cauchy sequence, $\gamma_c$-converging to $H_\omega$ uniformly in $\omega$, then $H^\eta_\nu_{\pi_{\nu}(\omega)} = \overline{H}^\eta_\nu$ is also a Cauchy sequence, so converges to some $\overline{H} \in \widehat{\text{Ham}}(T^* T^n)$. But then $H_{k,\omega}$ converges a.s. in $\omega$ to $\overline{H}$.

For the second part of the Main Theorem, we must go from $\gamma$-convergence of the flow to $\gamma$-convergence of the solution of the corresponding Hamilton-Jacobi equation. In the case of a compact base this is achieved in [V4], and the extension to a non-compact base was spelled out in [C-V] p. 266-276.

For $L \in \mathcal{L}(T^* N)$ we define $u_L(x) = c(1_x, L)$. Our first claim is that $\gamma$-convergence for $L$ implies $C^0$-convergence of the $u_L$ uniformly on compact sets.

**Lemma 10.1.** Let $U$ be bounded domain in $N$. If $(L_\nu)_{\nu \geq 1}$ is a Cauchy sequence for $\gamma_U$, then the sequence $u_{L_\nu}$ is a Cauchy sequence for the topology of uniform convergence on $U$. As a result if $(L_\nu)_{\nu \geq 1} \gamma$-converges to $L \in \mathcal{L}(T^* N)$ then the sequence $u_{L_\nu}$ converges uniformly on compact sets to $u_L$.

**Proof.** This is an immediate consequence of the reduction inequality ([V1], p. 705, prop 5.1) which implies that for any $x \in U$,

$$|c(1_x, L) - c(1_x, L')| \leq \gamma_U(L, L')$$

□

Using Proposition 5.14 we get

**Proposition 10.2.** Let $(\varphi_\nu)_{\nu \geq 1}$ be a sequence in $\widehat{\text{Ham}}_{c,FP} \gamma$-converging to $\varphi_\infty \in \widehat{\text{Ham}}_{c,FP}$. Then for any $L \in \mathcal{L}(T^* \mathbb{R}^n)$ (or in $\widehat{\mathcal{L}}(T^* \mathbb{R}^n)$) the sequence $\varphi_\nu(L)$ $\gamma$-converges to $\varphi_\infty(L)$.

**Proof.** Indeed, we proved that $\gamma_U(\psi_1(L), \psi_2(L)) \leq \gamma_{U \times V}(\psi_1, \psi_2)$ provided $\psi_j^{\pm 1}$ sends $T^* U$ to $T^* V$. In our case, we get

$$\gamma_U(\varphi_\nu(L), \varphi_\infty(L)) \leq \gamma_{U \times \overline{V}}(\varphi_\nu, \varphi_\infty)$$

and since the right hand side converges to 0, so does the left hand side. □

We may now conclude our proof. Since $\varphi^t_{k,\omega} \gamma$-converges to $\overline{\varphi}^t$ and is uniformly FPS for bounded $t$, we thus have

$$\varphi^t_{k,\omega}(L_f) \gamma \rightarrow \overline{\varphi}^t(L_f)$$

applying Lemma 10.1 to the sequence $(\varphi^t_{k,\omega})_{k \geq 1}$, this implies uniform convergence on compact sets of the sequence $(u_{k,\omega})_{k \geq 1}$ to its limit $\overline{u}$. This concludes the proof of our Main Theorem.
11. The Coercive Case

We now assume $H$ satisfies assumptions [1a] - [3a] of Corollary 1.3. Let $\chi_A$ be a truncation function, that is an increasing function such that $0 \leq \chi'_A(t) \leq 1$ and $\chi_A(t) = t - 3A/2$ for $t \leq A$ and $\chi_A(t) = 0$ for $t \geq 2A$. We set $H_A(x, p; \omega) = \chi_A(H(x, p; \omega))$. Then coercivity implies \(^{21}\) that $H_A$ has a.s. in $\omega \in \Omega$ the same flow as $H$ in $U_R = \{ (x, p) \mid |p| \leq r(A) \}$ where $\lim_{A \to +\infty} r(A) = +\infty$. We apply the Main Theorem to $H_A$ and obtain a Homogenized Hamiltonian $\overline{H}_A$. We claim now that for $B \geq A$ we have $\overline{H}_A = \overline{H}_B$.

12. The Discrete Case (Proof of Corollary 1.6)

If we have a $\mathbb{Z}^n$ action on $\Omega$, and its standard action on $\mathbb{R}^n$, we construct an $\mathbb{R}^n$ action on $\tilde{\Omega} = \Omega \times \mathbb{R}^n$ where

$$(\omega, t_1, \ldots, t_n) = (T_{-z} \omega, z_1 + t_1, \ldots, z_n + t_n)$$

where $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$. Then $\mathbb{R}^n$ acts on $\tilde{\Omega}$ by translation i.e. $\tilde{T}_a(\omega, t_1, \ldots, t_n) = (\omega, t_1 + a_1, \ldots, t_n + a_n)$. Notice that if $z \in \mathbb{Z}^n$ we have $\tilde{T}_a(\omega, t_1, \ldots, t_n) = (T_z \omega, t_1, \ldots, t_n)$.

Now it is easy to see that $T$ is ergodic if and only if $\tilde{T}$ is ergodic, since any $\tilde{T}$-invariant set will be of the form $U \times [0, 1]^n$ with $U$ a $T$-invariant set. Then if $H$ satisfies $H(x + z, p, T_z \omega) = H(x, p; \omega)$ we can consider $K(x, p, [\omega, t]) = H(x - t, p; \omega)$ and this satisfies $K(x + a, p, \tilde{T}_a([\omega, t]) = K(x, p, [\omega, t])$ for all $a \in \mathbb{R}^n$, and we can apply the stochastic homogenization from the Main Theorem.

13. Extending the Main Theorem

Note that one should be able to extend our methods to the case where we have a Hamiltonian satisfying the assumptions of the Main Theorem, but

1. we have a time dependent Hamiltonian, $H(t, x, p; \omega)$ and an action of $\mathbb{R} \times \mathbb{R}^n$ such that $H(t + s, x + a, p; \tau(s, a) \omega)$ and consider the sequence $H(t, z; \frac{\omega}{\tau}, \omega)$ This has been reduced to our case in the non-stochastic situation in \([V5]\) (Section 12.2 The non-autonomous case)
2. We consider partial homogenization. For example if $X = N \times \mathbb{R}^k$, then we should be able to apply the above propositions as in \([V5]\).
3. We consider the homogenization $H_\varepsilon(x, \frac{p}{\varepsilon}; \omega)$ as $\varepsilon$ goes to 0. This has been reduced to our case in the non-stochastic situation in \([V5]\) (Section 13 Homogenization in the $p$ variable).

\(^{21}\)See Remark 1.5 since $h_-(p) \leq H(x, p; \omega) \leq h_+(p)$ a.s. in $\omega$ where $\lim_{|p| \to +\infty} h_+(p) = +\infty$. 
(4) we have a $\mathbb{Z}^n$ action on a manifold $X$ such that the quotient $X/\mathbb{Z}^n$ is compact and the Hamiltonian satisfies $H(T^*_z x, T^*_z p, T_z \omega) = H(x, p; \omega)$ where $T$ is the action of $\mathbb{Z}^n$ on $X$ and we consider again the sequence $H_\varepsilon(x, p, \omega)$ as $\varepsilon$ goes to 0.

The proof in this last case should be the same as the Main Theorem. We just need to replace $\gamma(\varphi)$ (which is not defined on $T^*X$) with $\hat{\gamma}(\varphi)$ and we shall get an embedding of $\mathbb{Z}^n$ into $\text{Isom}(\hat{\gamma}_\Omega, \gamma)$. According to Weil [We2], the closure of the image of $\mathbb{Z}^n$ is the product of an abelian compact connected metric group, $A^0_\Omega$, and a totally disconnected compact metric abelian group $D_\Omega$. Since we have a morphism $c : \mathbb{Z}^n \to D_\Omega$ and the kernel $L$ must be a cocompact free abelian group, hence a lattice, so $L$ is isomorphic to $\mathbb{Z}^n$ and in suitable integral coordinates, we see that $L = a_1 \mathbb{Z} \oplus a_2 \mathbb{Z} \oplus ... \oplus a_n \mathbb{Z}$, so $D_\Omega = \mathbb{Z}^n / L = \mathbb{Z} / a_1 \mathbb{Z} \oplus \mathbb{Z} / a_2 \mathbb{Z} \oplus ... \mathbb{Z} / a_n \mathbb{Z}$. Replacing $\mathbb{Z}^n$ by $L$, we can reduce ourselves to the case of a compact connected abelian group so we get $\overline{K}(p, \omega)$ where $\overline{K}(p, \cdot)$ is constant on the ergodic components of the action of $L$ and the ergodic component are interchanged by an element of $D_\Omega$, thus we get that $\overline{K}(p, \cdot)$ is indeed constant a.e.

It would be also interesting to see what can be done in the framework of more general groups, as explained in [Sor] (see also [C-I-S]). In this setting a discrete group $G$ is a quotient of the $\pi_1(M)$ where $M$ is a compact manifold, and we see a Hamiltonian on $M$ as a $G$-invariant one on $\tilde{M}$ a cover of $M$. Then Sorrentino considers the Hamiltonian $H(x, \frac{1}{\varepsilon} p)$ as $\varepsilon$ goes to zero, and proves that it converges in some weak sense (we would say in the $\gamma$ topology) to a Hamiltonian defined on $G_\infty$ a graded Lie group associated to $G$ (at least if $G$ is nilpotent).

**APPENDIX 1: GFQI FOR NON-COMPACT LAGRANGIANS: PROOF OF THEOREM 4.5**

The goal of this section is to prove Theorem 4.5 that is

**Theorem 4.5.** Let $\varphi$ be an element in $\mathcal{DH}_\text{am}_{FP}(T^*N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique.

First we claim that the fibration theorem of Théret (see [Thé] theorem 4.2) goes through. Here $\mathcal{F}$ is the set of sequence of GFQI $(S_\nu)_{\nu \geq 1}$ satisfying the above property and $\mathcal{L} = \mathcal{L}(T^*\mathbb{R}^n)$ and we have

**Proposition 13.1.** The projection $\pi : \mathcal{F} \to \mathcal{L}$ is a Serre fibration up to equivalence.

The proof is the same as theorem 4.2 in [Thé]. We may reduce ourselves to the case of a single parameter (as in [Thé]). The proof is then
based on Sikorav’s existence theorem, which uses only the fact that for \( t \) small enough, if \( L \) has a GFQI over \( U_\nu \), then so does \( \varphi^t(L) \). Note that we may always assume that \( \varphi^t(T^*U_\nu) \subset T^*U_{\nu+1} \) and by truncating \( \varphi^t \) beyond \( T^*U_{\nu+1} \), we are reduced to the compact situation.

**Proof of Theorem 4.5.** Using Lemma 4.2 we may assume we have a sequence \( U_\nu \) of domains such that \( \varphi^t(T^*U_\nu) \subset T^*U_{\nu+1} \). Applying a sequence of cut-offs to the Hamiltonian defining \( \varphi \) we can then find a sequence \( L_\nu \) of Lagrangians of the type \( \varphi_\nu^1(0_N) \) where

1. \( \varphi_\nu^t(T^*U_\nu) \subset T^*U_{\nu+1} \) for all \( t \in [0,1] \)
2. \( \varphi_\nu^t \) has compact support in \( T^*U_{\nu+1} \)
3. setting \( \varphi_\nu^t(0_N) = L_\nu(t) \) we have for \( \mu \geq \nu \)
   \[
   L_\nu(t) \cap T^*U_\nu = L_\mu(t) \cap T^*U_\nu = \varphi^t(L) \cap T^*U_\nu
   \]

Then each \( L_\nu(t) \) has a GFQI, \( S_\nu(t) : N \times E_\nu \to \mathbb{R} \) and we claim that for \( \mu \geq \nu \), \( S_\nu(t) \) and \( S_\mu(t) \) are equivalent over \( U_\nu \). Indeed, we have a deformation from \( L_\nu \) to \( L_\mu \) that is the identity on \( T^*U_\nu \). If we denote by \( S_s \) a GFQI covering this deformation (the existence of which follows from [Thé], since we are again in the compact supported case), then \( S_s \) generates a Lagrangian \( L_s \) that is constant over \( T^*U_\nu \). Then using lemma 5.3 in [Thé] we can assume that after applying a fiber preserving diffeomorphism that \( \Sigma_s \cap (U \times F) = \Sigma_0 \cap (U \times F) \) where

\[
\Sigma_s = \left\{ (x,\xi) \mid \frac{\partial S_s}{\partial \xi}(x,\xi) = 0 \right\}
\]

But then as in [Thé] p. 259, using Hadamard’s lemma we prove that there is a fiber preserving diffeomorphism such that \( S_I(x,\xi(x,\eta)) = S_0(x,\eta) \).

So may now assume that the restriction of \( S_\mu \) over \( U_\nu \) is exactly \( S_\nu \circ q_{\nu,\mu} \) by composing \( S_\mu \) with an extension of the fiber preserving diffeomorphism realizing the equivalence. \( \square \)

**Appendix 2: Proof of Lemma 7.3 and Lemma 9.17**

Let \( V \) be a closed subgroup in \( \mathbb{R}^d \). Our first goal is to prove the following Lemma, of which Lemma 7.3 will be an immediate consequence

**Lemma 13.2.** The projection of \( V \subset \mathbb{R}^d \) in \( T^d \) is dense if and only if \( V^\perp \cap \mathbb{Z}^d = \{0\} \).

**Proof.** Density in \( T^d \) is equivalent to saying that \( V + \mathbb{Z}^d \) is dense in \( \mathbb{R}^d \).
Now let \( w \in \mathbb{R}^d \) and consider a sequence \( v_j + z_j \) with \( v_j \in V, z_j \in \mathbb{Z}^d \) such that \( v_j + z_j \) converges to \( w \). Let \( \xi \in \mathbb{Z}^d \cap V^\perp \). Then \( \langle \xi, w \rangle = \lim_j \langle \xi, v_j \rangle + \)

\( \text{22} \) The existence of the extension follows from the fact that we may assume that for \( \mu, \nu \) large enough, the inclusion \( U_\nu \subset U_\mu \) is a homotopy equivalence.
\( \langle \xi, z_j \rangle \). But the first term is zero, while the second is an integer. So this would imply \( \langle \xi, w \rangle \in \mathbb{Z} \) for all \( w \in \mathbb{R}^d \), but this is impossible unless \( \xi = 0 \).

Conversely assume \( V^\perp \cap \mathbb{Z}^d = \{0\} \). Note that the closure of \( V + \mathbb{Z}^d \) is a closed subgroup of \( \mathbb{R}^d \), hence of the form \( W \oplus (\mathbb{Z} e_1 \oplus \ldots \oplus \mathbb{Z} e_k) \) where \( W \) is a real vector space: this is a classical result proved by induction on \( d \) (see [Bou] theorem 2, p.72). In our case we of course have \( V \subset W \). We must prove \( k = 0 \) and \( W = \mathbb{R}^d \).

Set \( W' = W \oplus (\mathbb{R} e_1 \oplus \ldots \oplus \mathbb{R} e_{k-1}) \). We can write \( W' \oplus \mathbb{Z} e_k = W' \oplus \mathbb{Z} f_k \) for some vector \( f_k \). Then we have

\( i) \) \( V \subset W' \) since \( V \subset W \subset W' \)

\( ii) \) \( \mathbb{Z}^d \subset W' \perp \oplus \mathbb{Z} f_k \)

Let \( g_k \) be proportional to \( f_k \) and such that \( \langle g_k, f_k \rangle = 1 \). Then \( \langle g_k, x \rangle \) is an integer for any \( x \in W' \perp \oplus \mathbb{Z} f_k \), hence, according to \( ii \), for all \( x \in \mathbb{Z}^d \). Since \( \mathbb{Z}^d \) is a self dual lattice this implies \( g_k \in \mathbb{Z}^d \). But \( g_k \in W' \perp \) which according to \( ii \) is contained in \( V^\perp \) so this implies \( g_k \in V^\perp \cap \mathbb{Z}^d = 0 \), a contradiction.

We now prove:

**Lemma 9.17.** Let \( \varepsilon > 0 \) and \( A : \mathbb{R}^n \longrightarrow \mathbb{R}^d \) be a linear map such that \( A(\mathbb{R}^n) \) has dense projection on \( T^d \). Then there is a basis of integral vectors \( v_1, \ldots, v_d \) in \( \mathbb{Z}^d \) such that there exists vectors \( a_1, \ldots, a_d \) in \( \mathbb{R}^n \) such that

\[
|A \cdot a_j - v_j| \leq \varepsilon
\]

**Remark** 13.3. We do not claim the basis is an integral basis, i.e. it does not necessarily have determinant 1.

**Proof.** Let us consider the vector space generated by integral vectors such that there exist \( a \in \mathbb{R}^n \) such that \( |A \cdot a - v| \leq \varepsilon \). We must show that the set of such vectors is not contained in a proper vector space \( V_\varepsilon \subset \mathbb{R}^d \). Indeed, if \( V_\varepsilon \) is such a subspace it is an integral subspace (i.e. generated by integral vectors) hence projects on a subtorus \( T_\varepsilon \subset T^d \). The \( V_\varepsilon \) are nested subspaces and we set \( V_0 = \cap_{\varepsilon > 0} V_\varepsilon \). Since the decreasing family \( (V_\varepsilon)_{\varepsilon > 0} \) must be stationary we may choose \( \varepsilon_0 \) such that \( V_{\varepsilon_0} = V_0 \). By density of the image of \( A \) in the torus we may find \( a_0 \) such that \( d(A \cdot a_0, T_0) = \varepsilon_1 < \varepsilon_0 \) with \( \varepsilon_1 > 0 \) since \( T_0 \neq T^d \). We rephrase this as

\[
\forall v \in V_0, \forall v \in \mathbb{Z}^d, |A \cdot a_0 - v - v| \geq \varepsilon_1
\]

\[23\]The classical result omits the orthogonality but this is easy to recover: we can always replace \( e_1 \) by \( e_1 - w_1 \) for some \( w_1 \in W \) so that \( e_1 - w_1 \perp W \).
By Diophantine approximation\textsuperscript{24}, we may find \((k_0, \nu_0) \in (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^d\) such that \(|kA \cdot a_0 - \nu_0| < \epsilon_1\). This last inequality implies \(\nu_0 \in V_{\epsilon_1} = V_0\), while the previous one, together with \(|A \cdot a_0 - \nu_0| < \frac{\epsilon_1}{K}\) implies \(\nu_0 \notin V_0\), a contradiction.

Appendix 3: Approximation of Generating Functions and Symplectic Integrators

Our goal is to prove Lemma\textsuperscript{8.4}. It is a consequence to the more precise

Lemma 13.4. Let \(\varphi_H^t\) have \(S_t(q, p)\) as Generating function. We have

\[
\|S_t(q, p) - tH(q, p)\| \leq \frac{t^2}{2} \left\| \frac{\partial H}{\partial q} \right\| \cdot \left\| \frac{\partial H}{\partial p} \right\|
\]

Proof. Note that \(S_t\) has no fibre variable. It is well known that \(S_t\) satisfies the following Hamilton-Jacobi equation

\[
\begin{aligned}
\partial S_t(q, p) &= H\left(q + \frac{\partial S}{\partial p}(q, p)\right) \\
S_0(q, p) &= 0
\end{aligned}
\]

Set \(S_t(q, p) = t \cdot H(q, p) + R_t(q, p)\) and replace in the equation, using the inequality

\[
|H(q + \xi, p) - H(q, p)| \leq |\xi| \left\| \frac{\partial H}{\partial q} \right\|_{C^0}
\]

\[
\left\| \frac{\partial R_t(q, p)}{\partial t} \right\| \leq \left\| \frac{\partial H}{\partial q} \right\|_{C^0} \left\| \frac{\partial S_t}{\partial p} \right\|_{C^0} + \left\| \frac{\partial H}{\partial q} \right\|_{C^0} \left\| \frac{\partial R_t(q, p)}{\partial p} \right\|_{C^0}
\]

and since \(R_0(q, p) = 0\). Now the relation

\[
\partial_t R_t(q, p) \leq tA + B \left| \frac{\partial R_t}{\partial p} \right|
\]

implies by monotonicity of the solutions of the Hamilton-Jacobi equations\textsuperscript{25} that \(R_t\) is bounded by the solution \(u_t\) of \(\partial_t u = tA + B|\nabla u|\) that is

\[
u(t, x) = \frac{t^2}{2} A,\text{ so}
\]

\[
R_t(q, p) \leq \frac{t^2}{2} \left\| \frac{\partial H}{\partial q} \right\|_{C^0} \cdot \left\| \frac{\partial H}{\partial p} \right\|_{C^0}
\]

\textsuperscript{24}Indeed for any vector \(\nu \in \mathbb{R}^d\), we can find a pair \((k, \nu) \in (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}^d\). This is equivalent to proving that \(d_T(k|\nu|, 0) < \epsilon\) for some \(k \neq 0\). Since the distance is invariant by translation, this is equivalent to finding \(k, \neq l \in \mathbb{Z}\) such that \(d_T(k|\nu|, l|\nu|) < \epsilon\) but if this did not hold, we would get infinitely many points on the torus with a distance bounded from below by \(\epsilon_1\).

\textsuperscript{25}That is \(H \leq K\) implies that the solutions \(u, w\) of \(\partial_t u = H(x, D_x u)\) corresponding to the same initial condition satisfy \(u \leq w\).
The same argument gives an estimate from below.  

\textbf{APPENDIX 4: PROOF OF PROPOSITION 9.3}

The goal of this section is to prove

\textbf{Proposition 9.3.} Given any \( \alpha \), there exists a sequence \( (\ell_v)_v \geq 1 \) such that for almost all \( \omega \in \Omega \)

\[
\lim_{v \to \infty} \lim_{U \subset \mathbb{R}^n} c(\mu_U, \varphi_{\ell_v}^u \alpha) \leq \lim_{U \subset \mathbb{R}^n} c(\mu_U, \varphi_{\infty}^u \alpha)
\]

The proof is essentially the same as in section 6 of [V5]. We reproduce it here adapted to our situation and notations but notice that \( \omega \) just appears as a parameter and so does not change the proof of Proposition 9.3. In particular the cycles we construct in the proof, do not need to depend continuously on \( \omega \). We first need the next lemma. We define a closed cycle in \( X \) to be a cycle for the Borel-Moore homology of \( X : \) any closed submanifold represents a cycle in Borel-Moore homology, while in ordinary homology, this is the case only for compact submanifolds.

\textbf{Lemma 13.5.} Let \( S \) a G.F.Q.I. defined on \( E \) and \( c = \lim_{U \subset \mathbb{R}^n} c(\mu_U, S) \). There exists a closed cycle \( \Gamma \) such that \( \Gamma_U = \Gamma \cap \pi^{-1}(U) \) satisfies \( [\Gamma_U] = \mu_U \) in \( H_*(S^{+\infty}_U, S^{-\infty}_U) \) and \( S(\Gamma_U) \leq c(\mu_U, S) + \varepsilon \) for \( U \) belonging to a sequence of exhausting open sets with smooth boundary.

\textbf{Proof.} Consider an increasing sequence \( U_n \) of open sets with smooth boundary such that \( N = \bigcup_n U_n \). Notice that there is a restriction map for \( U \subset V \) sending \( H_*(V, \partial V) \to H_*(U, \partial U) \). It induces a map that we denote \( \rho_{U, V} \)

\[
H_*(S_V, S^-_{\infty} \cup E_{|\partial V}) \to H_*(S_U, S^-_{\infty} \cup E_{|\partial U})
\]

and a diagram

\[
\begin{array}{ccc}
H_*(S^{+\infty}_V, S^{-\infty}_{U} \cup E_{|\partial V}) & \xrightarrow{\rho_{U, V}} & H_*(S^{+\infty}_U, S^{-\infty}_{U} \cup E_{|\partial U}) \\
\uparrow & & \uparrow \\
H_*(S^{c+\varepsilon}_V, S^{-\infty}_{U} \cup E_{|\partial V}) & \xrightarrow{\rho_{U, V}} & H_*(S^{c+\varepsilon}_U, S^{-\infty}_{U} \cup E_{|\partial U})
\end{array}
\]

Now the upper horizontal map sends \( \mu_V \) to \( \mu_U \), so applying this to the sequence \( U_n \), we get a sequence \( \Gamma_n \in H_*(S^{c+\varepsilon}_{U_n}, S^{-\infty}_{U_n} \cup E_{|\partial U_n}) \) with image \( \mu_{U_n} \), and we have a sequence such that \( \rho_{U_n U_m} [\Gamma_n] = [\tilde{\Gamma}_n \cap \pi^{-1}(U_m)] \) is constant for \( n \geq m \). Then we may glue the \( \tilde{\Gamma}_n \) as follows: since \( [\tilde{\Gamma}_m] = [\tilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m)] \) in \( H_*(S^{c+\varepsilon}_{U_n}, S^{-\infty}_{U_n} \cup E_{|\partial U_n}) \), we have \( D_m \) such that \( \partial D_m \cap \pi^{-1}(U_m) = \tilde{\Gamma}_m - \tilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m) \) and we can assume \( D_m \subset \pi^{-1}(\tilde{U}_m) \). Now we may consider the cycle

\[
\Gamma_m = \tilde{\Gamma}_m \cup (\partial D_m \cap \pi^{-1}(\tilde{U}_m)) \cup \tilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1} \setminus U_m)
\]
We easily check that
(1) \( \Gamma_m \cap \pi^{-1}(U_m) = \tilde{\Gamma}_m \)
(2) \( \Gamma_m \cap \pi^{-1}(U_{m+1} \setminus U_m) = (\tilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1})) \cup (\partial D_m \cap \pi^{-1}(\partial U_m)) \)
(3) \( \partial \Gamma_m \subset E_{3U_m+1} \)
(4) \( \Gamma_m \subset S^{c+\epsilon} \)

By induction we can build a sequence \( \Gamma_m \) and we have \( \Gamma_n \cap \pi^{-1}(U_m) = \Gamma_m \cap \pi^{-1}(U_m) \) for \( n > m \). Therefore \( \bigcup_n \Gamma_n \) is stationary over any compact set and defines a closed cycle \( \Gamma \) such that \( S(\Gamma) \leq c+\epsilon \).

Now as in [V5], section 5, the generating function for \( \varphi^\omega_k \) is given by

\[
S^\omega_k(x, y; \xi, \zeta) = \frac{1}{k} \left( S^\omega(kx, p_1, \xi_1) + \sum_{j=2}^{k-1} S^\omega(kq_j, p_j, \xi_j) + S^\omega(kq_k, y, \xi_k) \right) + B_k(x, y, \xi)
\]

where \( \xi = (\xi_1, ..., \xi_k), \zeta = (p_1, q_2, ..., q_{k-1}, p_{k-1}, q_k) \)

\[
\tau_{\alpha} \xi = (p_1, q_2 + a, ..., q_{k-1} + a, p_{k-1}, q_k + a)
\]

and

\[
B_k(x, y, \xi) = \langle p_1, q_2 - x \rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j \rangle + \langle y, x - q_k \rangle
\]

Now let \( F(q, p; \eta) \) be a G.F.Q.I. for the graph of \( \alpha \), then

\[
G^\omega_k(u, v; x, y; \eta, \xi, \zeta) = S^\omega_k(u, y; \xi) + F(x, v; \eta) + \langle y - v, u - x \rangle
\]

is a G.F.Q.I. of \( \varphi_k \alpha \). We set

\[
\tilde{G}^\omega_k(u, v; x, y; \eta) = h^\omega_k(y) + F(x, v; \eta) + \langle y - v, u - x \rangle
\]

We shall omit the subscripts \( a, \bar{x} \) for the moment, so in the sequel, \( \tilde{G}^\omega_{k, a, \bar{x}} \)

means \( \tilde{G}^\omega_{k, a, \bar{x}} \) Here the variables \( u, v, x, y \) are in \( \mathbb{R}^n \) and we denote by \( E_k \)
the space of the \( \theta = (\zeta, \xi) \) where \( \xi \in E^k, \zeta \in (\mathbb{R}^{2n})^k \) and \( \eta \in V \). By definition

Figure 2. \( \tilde{\Gamma}_m \) in red, \( \tilde{\Gamma}_{m+1} \) in blue and \( D_m \) in green on the left and \( \Gamma_{m+1} \) on the right.
we have a cycle $\Gamma^\omega_j$ in $U \times \mathbb{R}^n_v \times \mathbb{R}^n_x \times \mathbb{R}^n_y \times E \times V$ relative to $(\overline{G^\omega_k})^{-\infty} \cup \partial U \times \mathbb{R}^n_v \times \mathbb{R}^n_x \times \mathbb{R}^n_y \times E \times V$ and homologous (as a closed cycle) to $U \times \mathbb{R}^n \times \Delta_{x,y} \times E^- \times V^-$ (where $\Delta$ is the diagonal) such that

$$\overline{G^\omega_k}(\Gamma^\omega_j) \leq c(\mu, \overline{G^\omega_k}) + \epsilon = c(\mu, \overline{\varphi}_{k,U} \alpha) + \epsilon$$

where $\overline{\varphi}_{k,U}$ is the flow of $h^\omega_{k,U}(y)$.

Moreover according to Lemma [3.5] we can assume there is a closed (i.e. Borel-Moore) cycle $\Gamma^\omega$ such that $\Gamma^\omega_j = \Gamma^\omega \cap \pi^{-1}(U)$ (at least for a cofinal sequence of $U$'s).

Now let $C^\omega_U(y)$ be a cycle in the class of $U \times E^-_k$ in $H_*(((S^\omega_{k,y})^{+\infty} (S^\omega_{k,y})^{-\infty}$, depending continuously on $y$, such that $26$

$$S^\omega_k(y, C^\omega_U(y)) \leq h^\omega_{k,U}(y) + a\mathcal{\chi}(y) + \epsilon$$

Thus we set for $a \in \mathbb{R}^+, \mathcal{\chi} \in C^\infty(\mathbb{R}^n)$

$$\overline{G^\omega_{k,a,\mathcal{\chi}}}(u, v; x, y, \eta) = h^\omega_{k}(y) + F(x, v; \eta) + (y - v, u - x) + a\mathcal{\chi}(y)$$

We shall omit the subscripts $a, \mathcal{\chi}$ for the moment, so in the sequel, $\overline{g^\omega_{k,a,\mathcal{\chi}}}$ means $\overline{G^\omega_{k,a,\mathcal{\chi}}}$

We again invoke Lemma [3.5] in order to obtain a (closed) cycle $C^\omega(y)$ such that for a cofinal sequence of $U$'s we have $C^\omega_U(y) = C^\omega(y) \cap \pi^{-1}(U)$.

As in [V3], section 6, this is possible provided $\mathcal{\chi}$ is the characteristic function of $\Lambda$, the complement of a disjoint union of sets of diameter less than $\delta$. For example, we can take $\Lambda$ to be a neighborhood of $\Lambda(\delta) = \{ (x_1, \ldots, x_n) | \exists j, x_j \in \delta \mathbb{Z} \}$. We now construct a new (Borel-Moore) cycle, symbolically denoted $\Gamma \times Y C$ and defined as follows (everything depends on $\omega$ but for notational convenience we omit it)

$$\Gamma \times Y C = \{(u, v; x, y, \theta, \eta) \mid (u, v, x, y, \eta) \in \Gamma, (u, \theta) \in C(y) \}$$

we have

1. $(\Gamma \times Y C)_U$ is homologous to $U \times \mathbb{R}^n_v \times \Delta_{x,y} \times E^-_k \times V^-$

2. $G^\omega_k((\Gamma \times C)_U) \leq \overline{G^\omega_{k,a,\mathcal{\chi}}}(\Gamma_U) + \epsilon$

Indeed for (1), it follows from the fact that the homology class of $A \times Y B$ only depends on the homology class of $A, B$ and so $\Gamma_U \times Y C_U$ is homologous to

$$(U \times \mathbb{R}^n_v \times \Delta_{x,y} \times V^-) \times Y (U \times E^-_k) = \{(u, v; x, y, \eta, \theta) \mid u \in U, x = y, \eta \in V^-, \theta \in E^-_k \} = U \times \mathbb{R}^n_v \times \Delta_{x,y} \times V^- \times E^-_k$$

We still have a problematic notation with respect to the order of our variables. By $S_k(y, C_U(y))$ we mean the maximum of $S_k(x, y, \xi, \zeta)$ where $(x, \xi, \zeta) \in C(y)$
As for (2), we have \((\Gamma \times Y \ C^-)_U = \Gamma_U \times Y \ C^-_U\) and

\[
G^o_k(\Gamma_U \times Y \ C^-_U) \overset{def}{=} \sup \{ S^o_k(u, y; \theta) + F(x, v; \eta) + \langle y - v, u - x \rangle \mid (u, v; x, y, \eta) \in \Gamma, (u; \theta) \in C(y) \}
\]

but since \(S_k(u, y; \theta) \leq h^+_k(y) + a\chi(y) + \varepsilon\) for \((u; \theta) \in C(y)\) we have

\[
G^o_k(\Gamma_U \times Y \ C^-_U) \leq \sup \{ F(x, v; \eta) + h^+_k(y) + a\chi(y) + \varepsilon + \langle y - v, u - x \rangle \mid (u, v; x, y, \eta) \in \Gamma^o_u, (u, \theta) \in C^o_U(y) \}
\]

\[
\leq \overline{G}^o_{k, a, \chi}(\Gamma_U) + \varepsilon
\]

Now as in [V5] section 6 p.27 let us consider a collection of \(\ell\) open sets \(\Lambda^j_\delta\) for \(1 \leq j \leq \ell\) such that each of them is a translate of \(\Lambda_\delta\) and any \(n + 1\) of them have empty intersection. We denote by \(\chi_j\) (\(1 \leq j \leq \ell\)) the corresponding functions. We set \(\bar{x} = (x_1, ..., x_\ell), \bar{y} = (y_1, ..., y_\ell), \bar{\theta} = (\theta_1, ..., \theta_\ell)\) and define:\(^{27}\)

\[
G_{k, \ell}(u, v, \bar{x}, \bar{y}, \bar{\theta}, \eta) = F(x_1, v; \eta) + \frac{1}{\ell} \sum_{j=1}^\ell S_k(\ell x_j, y_j, \theta_j) + B_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle
\]

This is a G.E.Q.I. for \(\rho^{-1}_\ell \rho^{-1}_k \varphi \rho \varphi \alpha = \rho^{-1}_\ell \rho^{-1}_k \varphi \rho \varphi \alpha = \varphi \rho \varphi \alpha\).

Let

\[
\overline{G}_{k, \ell}(u, v, \bar{x}, \bar{y}, \eta) = F(x_1, v, \eta) + \frac{1}{\ell} \sum_{j=1}^\ell (h^+_k(y_j) + a\chi_j(y_j)) + B_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle
\]

By definition there is a \(\Gamma_{k, \ell}\) such that

\[
\overline{G}_{k, \ell}((\Gamma_{k, \ell})_U) \leq c(\mu_U, \overline{G}_{k, \ell}) + \varepsilon
\]

and using \(C_j(y_j)\) as before for \(1 \leq j \leq \ell\) and setting

\[
\Gamma_{k, \ell} \times Y \ C^-[\ell] = \left\{ (u, v; \bar{x}, \bar{y}, \bar{\theta}, \eta) \mid (u, v, x, y, \eta) \in \Gamma, (\ell x_j, \xi_j) \in C_j^-(y_j) \right\}
\]

we have

\[
c(\mu_U, G_{k, \ell}) \leq G_{k, \ell}((\Gamma_{k, \ell})_U \times Y (C^-)_U[\ell]) \leq \overline{G}_{k, \ell}((\Gamma_{k, \ell})_U) \leq c(\mu_U, \overline{G}_{k, \ell}) + 2\varepsilon
\]

Finally we claim that

\[
c(\mu_U, \overline{G}_{k, \ell}) \leq c(\mu_U, \overline{G}_{k}) + \frac{(n + 1)a}{\ell}
\]

Indeed, \(\overline{G}_{k, \ell}\) is the generating function of \(\psi_{k, \ell} = \rho^{-1}_\ell \psi_1 \circ ... \circ \psi_\ell \rho \rho \) where \(\psi_\ell\) is the time one flow of \(h_k(y) + a\chi_j(y)\). But these flows commute, so

\(^{27}\)Here we omit \(\omega\) from the notation, which would otherwise become unwieldy.
\( \psi_{k, \ell} \) is the time one flow of

\[
K_{k, \ell}(y) = \frac{1}{\ell} \sum_{j=1}^{\ell} (h_{k,U}(y) + a \chi_j(y))
\]

and we have \( |K_{k, \ell}(y) - h_k(y)| \leq \frac{(1+n)a}{\ell} + \varepsilon \) therefore

\[
c(\mu_U, \overline{G}_{k, \ell}) \leq c(\mu_U, \psi_{k, \ell} \alpha) \leq c(\mu_U, \psi_1 k \alpha) + \frac{(1+n)a}{\ell} + \varepsilon \leq c(\mu_U, \overline{\varphi}_k \alpha) + \frac{(1+n)a}{\ell} + \varepsilon
\]

Thus for \( \ell \) large enough \( c(\mu_U, \overline{G}_{k, \ell}) \leq c(\mu_U, \psi_1 \alpha) + 2\varepsilon \). Taking the limit as \( k \) goes to infinity, we get

\[
c(\mu_U, \varphi_{k, \ell} \alpha) = c(\mu_U, G_{k, \ell}) \leq c(\mu_U, \overline{\varphi} \alpha) + 2\varepsilon \leq c(\mu_U, \overline{\varphi} \alpha) + 3\varepsilon
\]

This concludes the proof of Prop. 9.3.

REFERENCES

[A-T-Y] S. N. Armstrong, H. V. Tran, and Yifeng Yu, *Stochastic homogenization of a non-convex Hamilton-Jacobi equation*. Calc. Var. Partial Differential Equations, vol. 54, 1507-1524 (2015). https://doi.org/10.1016/j.jde.2016.05.010

[B-C] M. Bardi, I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Modern Birkhäuser Classics, Birkhäuser Basel, Springer Science+Business Media New York, 1997. https://doi.org/10.1007/978-0-8176-4755-1

[Ba] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*. Springer-Verlag Berlin Heidelberg, 1994.

[Bou] N. Bourbaki, *Topologie Générale*. Chap. 1 à 4. Springer-Verlag, 1995

[C-L] M. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*. Transactions of the Amer. Math. Soc. vol 277(1), 1983. https://doi.org/10.1090/S0002-9947-1983-0690039-8

[C-M] P. Chernoff and J. Marsden, *On continuity and smoothness of group actions*. Bull. Amer. Math. Soc., Volume 76, Number 5 (1970), 1044-1049. https://projecteuclid.org/euclid.bams/1183532211

[C-V] F. Cardin et C. Viterbo, *Commuting Hamiltonians and multi-time Hamilton-Jacobi equations*. Duke Math Journal, vol. 144, (2008), pp. 235-284. https://projecteuclid.org/euclid.dmj/1218716299

[Ch] Marc Chaperon. *Lois de conservation et géométrie symplectique*. C. R. Acad. Sci. Paris Sér. I Math., 312(4):345–348, 1991.

[C-I-S] G. Contreras, R. Iturriaga, A. Siconolfi *Homogenization on arbitrary manifolds*. https://arxiv.org/pdf/1211.1081.pdf

[E] R. Ellis, *Locally compact transformation groups*. Duke Mathematical Journal, 24(2)(1957), pp. 119-125. https://doi.org/10.1215/S0012-7094-57-02417-1

[F-S] W. Feldman and P. E. Souganidis, *Homogenization and non-homogenization of certain non-convex Hamilton-Jacobi equations*. Journal de Mathématiques Pures et Appliquées. Volume 108, Issue 5, November 2017, Pages 751-782. https://doi.org/10.1016/j.matpur.2017.05.016
[Tr] L. Traynor, *Symplectic homology via Generating Functions*. Geometric and Functional Analysis, vol. 4(1994), pp. 718-748. https://doi.org/10.1007/BF01896659

[V1] C. Viterbo, *Symplectic topology as the geometry of generating functions*. *Mathematische Annalen*, vol. 292, (1992), pp. 685-710. https://doi.org/10.1007/BF01444643

[V2] C. Viterbo, *Functors and Computations in Floer cohomology, II*. https://arxiv.org/abs/1805.01316

[V3] C. Viterbo, *Solutions d’équations d’Hamilton-Jacobi et géométrie symplectique*. Séminaire Équations aux dérivées partielles (Polytechnique), Ecole Polytechnique, Centre de Mathématiques, 1995-1996. http://www.numdam.org/item/SEDP_1995-1996____A22_0

[V4] C. Viterbo, *Symplectic topology and Hamilton-Jacobi equations*. In book “Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology”, (2006), pp.439-459. Springer Netherlands

[V5] C. Viterbo, *Symplectic homogenization*. https://arxiv.org/abs/0801.0206

[WQ1] Q. Wei Solutions de viscosité des équations de Hamilton-Jacobi et minimax itérés. PhD thesis, Université de Paris 7 (2013). https://tel.archives-ouvertes.fr/tel-00963780/

[WQ2] Q. Wei *Viscosity solution of the Hamilton-Jacobi equation by a limiting minimax method*. Nonlinearity, vol. 27(1)(2013), pp. 17–41. https://doi.org/10.1088/0951-7715/27/1/17

[We1] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*. Act. Sci. Ind., vol. 551, Hermann, Paris, 1938.

[We2] A. Weil *L’intégration dans les groupes topologiques. 2ème édition*. Hermann, 1965.

[Wi] N. Wiener, *The ergodic theorem*. Duke Math. J. vol. 5 (1939), pp. 1-18. https://doi.org/10.1215/S0012-7094-39-00501-6

[Z1] T. Zhukovskaya, *Singularités de minimax et solutions faibles d’Équations aux dérivées partielles*. Ph.D. dissertation, Université Paris Diderot-Paris 7, Paris, 1993.

[Z2] T. Zhukovskaya, *Metamorphoses of the Chaperon-Sikorav weak solutions of Hamilton-Jacobi equations*. J. Math. Sci. 82 (1996), 3737–3746. https://doi.org/10.1007/BF02362583

[Zi] B. Ziliotto, *Stochastic Homogenization of Nonconvex Hamilton-Jacobi Equations: A Counterexample*. Comm. Pure Appl. Math., 70: 1798-1809. https://doi.org/10.1002/cpa.21674