VARIABLE SPEED BRANCHING BROWNIAN MOTION 1. 
EXTREMAL PROCESSES IN THE WEAK CORRELATION REGIME

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ABSTRACT. We prove the convergence of the extremal processes for variable speed branching Brownian motions with speed functions that lie below their convex hull and satisfy a certain weak regularity condition. These limiting objects are universal in the sense that they only depend on the slope of the speed function at 0 and the final time $t$. The proof is based on previous results for two-speed BBM obtained in [9] and uses Gaussian comparison arguments to extend these to the general case.

1. INTRODUCTION

Gaussian processes indexed by trees is a topic that received a lot of attention, in particular in the context of spin glass theory (see e.g. [7, 34, 35, 31]) through the so-called Generalised Random Energy Models (GREM), introduced by Derrida and Gardner [22]. One of the issues of interest in this context is the understanding of the structure of the extremal processes that arise in these models in the limit when the size of these trees tends to infinity. A Gaussian process on a tree is characterised fully by the tree and by its covariance, which in the models we are interested in is a function of the genealogical distance on the tree. In the context of the GREM, the tree is just a binary tree of $n$ levels; another popular tree is a supercritical Galton-Watson tree (see, e.g. [3]). These models were first introduced, to our knowledge, by Derrida and Spohn [17].

In this paper we focus on this class of models. They can be constructed as follows. We begin with the construction of the tree. On some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define a supercritical Galton-Watson (GW) tree. The offspring distribution, $\{p_k\}_{k \in \mathbb{N}}$, is normalised for convenience such that $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$, and the second moment, $K = \sum_{k=1}^{\infty} k(k-1)p_k$ is assumed finite. We fix a time horizon $t > 0$. We denote the number of individuals (“leaves”) of the tree at time $t$ by $n(t)$ and label the leaves at time $t$ by $i_1(t), i_2(t), \ldots, i_{n(t)}(t)$. For given $t$ and for $s \leq t$, it will be convenient to let $i_k(s)$ denote the ancestor of particle $i_k(t)$ at time $s$. Of course, in general there will be several indices $k, \ell$ such that $i_k(s) = i_\ell(s)$. The time of the most recent common ancestor of $i_k(t)$ and $i_\ell(t)$ will be given, for $s, r \leq t$, by

$$d(i_k(r), i_\ell(s)) \equiv \sup\{u \leq s \wedge r : i_k(u) = i_\ell(u)\}.$$ 

(1.1)
We denote by $\mathcal{F}_t^{\text{tree}}$ the $\sigma$-algebra generated by the Galton-Watson process up to time $t$. On the same probability space we will now construct, for given $t$, and for any realisation of the GW tree, a Gaussian process.

Let $A : [0,1] \rightarrow [0,1]$ be a right-continuous non-decreasing function. We define a Gaussian process, $x$, labelled by the tree (up to time $t$) with covariance, for $0 \leq s, r \leq t$ and $k, \ell \leq n(t)$

$$E[x_k(s)x_\ell(r)] = tA\left(t^{-1}d(i_k(r), i_\ell(s))\right). \tag{1.2}$$

The existence of such a process is shown easily through a construction as branching Brownian motion. Note first that, in the case when $A(x) = x$, this process is standard branching Brownian motion \cite{30, 33}. For general functions $A$, the models can be constructed from time changed Brownian motion as follows. Let

$$\Sigma^2(s) = tA(s/t). \tag{1.3}$$

Branching Brownian motion with speed function $\Sigma^2$ is constructed like ordinary Brownian motion, except that if a particle splits at some time $s < t$, then the offspring particles perform variable speed Brownian motions with speed function $\Sigma^2$, i.e. they are independent copies of $\{B^\Sigma_t - B^\Sigma_s\}_{t \geq r \geq s}$, all starting at the position of the parent particle at time $s$. We refer to these processes as variable speed branching Brownian motion. The class of processes, labelled by the different choices of functions $A$ provides an interesting set of examples to study the possible limiting extremal processes for correlated random variables. The ultimate goal will be to describe the extremal processes in dependence on the function $A$.

Remark. Strictly speaking, we are not talking about a single stochastic process, but about a family, $\{x^t_k(s), k \leq n(s)\}_{t \leq s \leq t}$, of processes with finite time horizon, indexed by that horizon, $t$. That dependence on $t$ is usually not made explicit in order not to overburden the notation.

Branching Brownian motion has received a lot of attention over the last decades, with a strong focus on the properties of extremal particles. We mention the seminal contributions of McKean \cite{29}, Bramson \cite{13, 12}, Lalley and Sellke \cite{25}, and Chauvin and Rouault \cite{14, 15} on the connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation \cite{21, 24} and on the distribution of the rescaled maximum. In recent years, their has been a revival of interest in BBM with numerous contributions, including the construction of the full extremal process \cite{2, 1}. For a review of these developments see, e.g., the recent survey by Guéré \cite{23} or the lecture notes \cite{8}. Variable speed branching Brownian motion (as well as random walk) has recently been investigated by Fang and Zeitouni \cite{18, 19}, see also \cite{9, 27, 28}.

Naturally, the same construction can be done for any other family of trees. It is widely believed (see \cite{36}) that the resulting structures are very similar, with only details depending on the underlying tree model. More importantly, it is believed that the extremal structure in more general Gaussian processes, such as mean field spin glasses \cite{5, 6} or the Gaussian free field \cite{36} are of the same type; considerable progress in this direction has been made recently by Biskup and Louidor \cite{4}. 

We are interested to understand the nature of the extremes of our processes in dependence on the properties of the covariance functions \( A \). The case when \( A \) is a step function with finitely many steps corresponds to Derrida’s GREMs \([22, 10]\), the only difference being that the deterministic binary tree of the GREM is replaced by a Galton-Watson tree. It is very easy to treat this case.

The case when \( A \) is arbitrary has been dubbed CREM in \([11]\) (and treated for binary regular trees). In that case the leading order of the maximum was obtained, as well as the genealogical description of the Gibbs measures; this analysis carries over mutando mutandis to the analogous BBM situation. The finer analysis of the extremes is, however, much more subtle and in general still open. Fang and Zeitouni \([19]\) have obtained the order of the corrections (namely \( t^{1/3} \)) in the case when \( A \) is strictly concave and continuous. These corrections come naturally from the probability of a Brownian bridge to stay away from a curved line, which was earlier analysed by Ferrari and Spohn \([20]\). There are, however, no results on the extremal process or the law of the maximum.

Another rather tractable situation occurs when \( A \) is a piecewise linear function. The simplest case here corresponds to choosing a speed that takes just two values, i.e.

\[
\sigma^2(s) = \begin{cases} 
\sigma^2_1, & \text{for } 0 \leq s < t \beta, \\
\sigma^2_2, & \text{for } t \beta \leq s \leq t, 
\end{cases}
\]  

(1.5)

with \( \sigma^2_1 b + \sigma^2_2 (1 - b) = 1 \). In this case, Fang and Zeitouni \([19]\) have obtained the correct order of the logarithmic corrections. This case was fully analysed in a recent paper of ours \([9]\), where we provide the construction of the extremal processes.

In the present paper, we present the full picture in the case where \( A(x) < x \) for all \( x \in (0, 1) \), and the slopes of \( A \) at 0 and at 1 are different from 1. We show that there is a large degree of universality in that the limiting extremal processes are those that emerged in the two-speed case and that they depend only on the slopes of \( A \) at 0 and at 1.

The critical cases, \( A(x) \leq x \), involve, besides the well-understood standard BBM, a number of different situations arise that can be quite tricky, and we postpone this analysis to a forthcoming publication.

1.1. Results. We need some mild technical assumptions on the covariance function. Let \( A : [0, 1] \to [0, 1] \) be a right-continuous, non-decreasing function that satisfies the following three conditions:

(A1) For all \( x \in (0, 1) \): \( A(x) < x \), \( A(0) = 0 \) and \( A(1) = 1 \).

(A2) There exists \( \delta_b > 0 \) and functions \( \overline{B}(x), \underline{B}(x) : [0, 1] \to [0, 1] \) that are twice differentiable in \([0, \delta_b]\) with bounded second derivatives, such that

\[
\underline{B}(x) \leq A(x) \leq \overline{B}(x), \quad \forall x \in [0, \delta_b]
\]  

(1.6)

with \( \overline{B}'(0) = \overline{B}'(1) \equiv A'(0) \).

(A3) There exists \( \delta_e > 0 \) and functions \( \overline{C}(x), \underline{C}(x) : [0, 1] \to [0, 1] \) that are twice differentiable in \([1 - \delta_e, 1]\) with bounded second derivatives, such that

\[
\underline{C}(x) \leq A(x) \leq \overline{C}(x), \quad \forall x \in [1 - \delta_e, 1]
\]  

(1.7)

with \( \overline{C}'(1) = \overline{C}'(1) \equiv A'(1) \). The case \( A'(1) = +\infty \) is allowed. This is to be interpreted to mean that, for all \( \rho < \infty \), there exists \( \varepsilon > 0 \) such that, for all \( x \in [1 - \varepsilon, 1] \), \( A(x) \leq 1 - \rho(1 - x) \).
For standard BBM, \( \bar{x}(t) \), recall that Bramson [13] and Lalley and Sellke [25] have shown that
\[
\lim_{t \uparrow \infty} \mathbb{P} \left( \max_{k \leq n(t)} x_k(t) - m(t) \leq y \right) = \omega(x) = \mathbb{E} \left[ e^{-CZ e^{-\sqrt{2y} t}} \right],
\]
(1.8)
where \( m(t) \equiv \sqrt{2t - \frac{3}{2\sqrt{2}} \log t} \), \( Z \) is a random variable, the limit of the so called derivative martingale, and \( C \) is a constant.

In [2] (see also [1] for a different proof) it was shown that the extremal process,
\[
\lim_{t \uparrow \infty} \bar{E}_t \equiv \lim_{t \uparrow \infty} \sum_{k=1}^{n(t)} \delta_{x_k(t) - m(t)} = \bar{E},
\]
exists in law, and \( \bar{E} \) is of the form
\[
\bar{E} = \sum_{k,j} \delta_{\eta_k + \Delta_i^{(k)}},
\]
(1.10)
where \( \eta_k \) is the \( k \)-th atom of a mixture of Poisson point process (Cox process [16]) with intensity measure \( CZe^{-\sqrt{2y} dy} \), with \( C \) and \( Z \) as before, and \( \Delta_i^{(k)} \) are the atoms of independent and identically distributed point processes \( \Delta^{(k)} \), which are the limits in law of
\[
\sum_{j \leq n(t)} \delta_{x_j(t) - \max_{j \leq n(t)} x_j(t)},
\]
(1.11)
where \( \bar{x}(t) \) is BBM conditioned on the event \( \max_{j \leq n(t)} x_j(t) \geq \sqrt{2t} \).

The main result of the present paper is the following theorem.

**Theorem 1.1.** Assume that \( A : [0, 1] \rightarrow [0, 1] \) satisfies (A1)-(A3). Let \( A'(0) = \sigma_e^2 < 1 \) and \( A'(1) = \sigma_b^2 \geq 1 \). Let \( \bar{m}(t) = \sqrt{2t - \frac{1}{2\sqrt{2}} \log t} \). Then there is a constant \( \bar{C}(\sigma_e) \) depending only on \( \sigma_e \) and a random variable \( Y_{\sigma_b} \) depending only on \( \sigma_b \) such that

(i) \[
\lim_{t \uparrow \infty} \mathbb{P} \left( \max_{1 \leq i \leq n(t)} x_i(t) - \bar{m}(t) \leq x \right) = \mathbb{E} \left[ e^{-\bar{C}(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2x} t}} \right].
\]
(1.12)

(ii) The point process
\[
\sum_{k \leq n(t)} \delta_{x_k(t) - \bar{m}(t)} \rightarrow \mathcal{E}_{\sigma_b, \sigma_e} = \sum_{i,j} \delta_{p_i + \sigma_e \Lambda_i^{(i)}},
\]
(1.13)
as \( t \uparrow \infty \), in law, where the \( p_i \) are the atoms of a Poisson point process on \( \mathbb{R} \) with intensity measure \( \bar{C}(\sigma_e) Y_{\sigma_b} e^{-\sqrt{2x} dx} \), and the \( \Lambda_i^{(i)} \) are the limits of the processes as in (1.11), but conditioned on the event \( \{ \max_k x_k(t) \geq \sqrt{2\sigma_b} t \} \).

(iii) If \( A'(1) = \infty \), then \( \bar{C}(\infty) = 1/\sqrt{4\pi} \), and \( \Lambda_i^{(i)} = \delta_0 \), i.e. the limiting process is a Cox process.

The random variable \( Y_{\sigma_b} \) is the limit of the uniformly integrable martingale
\[
Y_{\sigma_b}(s) = \sum_{i=1}^{n(s)} e^{-s(1+\sigma_b^2)/2+2\sigma_b \bar{x}_i(s)},
\]
(1.14)
where \( \bar{x}_i(s) \) is standard branching Brownian motion.
Remark. The special case of Theorem 1.1 when \( A \) consists of two linear segments was obtained in [9]. Theorem 1.1 shows that the limiting objects under conditions \((A1) - (A3)\) are universal and depend only on the slopes of the covariance function \( A \) at 0 and at 1. This could have been guessed, but the rigorous proof turns out to be quite involved. Note that \( \sigma_e = \infty \) is allowed. In that case the extremal process is just a mixture of Poisson point processes. If \( \sigma_b = 0 \), then \( Y_{\sigma_b} \) is just an exponential random variable of mean 1.

1.2. Outline of the proof. The proof of Theorem 1.1 is based on the corresponding result obtained in [9] for the case of two speeds, and on a Gaussian comparison method. We start by showing the localisation of paths, namely that the paths of all particles that reach a hight of order \( \tilde{m}(t) \) at time \( t \) has to lie within a certain tube. Then we show tightness of the extremal process.

The remainder of the paper is then concerned with proving the convergence of the finite dimensional distributions. To this end we use Laplace transforms. We introduce auxiliary two speed BBM’s whose covariance functions approximate \( \Sigma^2(s) \) well around 0 and \( t \). Moreover we choose them in such a way that their covariance functions lie above respectively below \( \Sigma^2(s) \) in a neighbourhood of 0 and \( t \) (see Figure 1).

We then use Gaussian comparison methods to compare the Laplace transforms. The Gaussian comparisons comes in three main steps. In a first step we introduce the usual interpolating process and introduce a localisation condition on its paths. In a second step we justify a certain integration by parts formula, that is adapted to our setting. Finally the resulting quantities are decomposed in a part that has a sign and a part that converges to zero.

![Figure 1. Gaussian Comparison: The extremal process of BBM with covariance \( A \) (black curve) is compared to process with covariances functions \( \overline{A} \) (red curve), respectively \( \underline{A} \) (blue, curve).](image-url)
2. Localization of paths

In this section we show where the paths of particles that are extreme at time $t$ are localised. This is essentially inherited from properties of the standard Brownian bridge. For a given covariance function $\Sigma^2$, and a subinterval $I \subset [0, t]$, define the following events on the space of paths, $X : \mathbb{R}_+ \to \mathbb{R}$,

$$T_{t,I,\Sigma^2} = \left\{ X \bigg| \forall s : s \in I : \left| X(s) - \frac{\Sigma^2(s)}{t} X(t) \right| < (\Sigma^2(s) \wedge (t - \Sigma^2(s)))^\gamma \right\}. \quad (2.1)$$

**Proposition 2.1.** Let $x$ denote the variable speed BBM with covariance function $\Sigma^2$. For any $\frac{1}{2} \leq \gamma < 1$ and for all $d \in \mathbb{R}$, there exists $r$ sufficiently large such that, for all $t > 3r$,

$$\mathbb{P} \left( \exists k \leq n(t) : \{x_k(t) > \bar{m}(t) + d\} \wedge \left\{ x_k \not\in T_{t,I,\Sigma^2} \right\} \right) < \epsilon, \quad (2.2)$$

where $I_r \equiv \{ s : \Sigma^2(s) \in [r, t - r] \}$.

To prove Proposition 2.1, we need the following basic lemma on Brownian bridges (see [12]).

**Lemma 2.2.** Let $\frac{1}{2} < \gamma < 1$. Let $\xi$ be a Brownian bridge from 0 to 0 in time $t$. Then, for all $\epsilon > 0$, there exists $r$ large enough such that, for all $t > 3r$,

$$\mathbb{P} \left( \exists s \in [r, t - r] : |\xi(s)| > (s \wedge (t - s))^\gamma \right) < \epsilon. \quad (2.3)$$

More precisely,

$$\mathbb{P} \left( \exists s \in [r, t - r] : |\xi(s)| > (s \wedge (t - s))^\gamma \right) < 8 \sum_{k=\lfloor r \rfloor}^{\lfloor t/2 \rfloor} k^{\frac{3}{2}\gamma - 1} e^{-k^{2\gamma - 1}/2}. \quad (2.4)$$

**Proof.** The probability in (2.3) is bounded from above by

$$\sum_{k=\lfloor r \rfloor}^{\lfloor t/2 \rfloor} \mathbb{P} \left( \exists s \in [k - 1, k] : |\xi(s)| > (s \wedge (t - s))^\gamma \right)$$

$$\leq 2 \sum_{k=\lfloor r \rfloor}^{\lfloor t/2 \rfloor} \mathbb{P} \left( \exists s \in [k - 1, k] : |\xi(s)| > (s \wedge (t - s))^\gamma \right) \quad (2.5)$$

by the reflection principle for the Brownian bridge. This is now bounded from above by

$$2 \sum_{k=\lfloor r \rfloor}^{\lfloor t/2 \rfloor} \mathbb{P} \left( \exists s \in [0, k] : |\xi(s)| > (k - 1)^\gamma \right) \quad (2.6)$$

Using the bound of Lemma 2.2 (b) of [12] we have

$$P \left( \exists s \in [0, k] : |\xi(s)| > (k - 1)^\gamma \right) \leq 4(k - 1)^{\frac{3}{2}\gamma - 1} e^{-(k - 1)^{2\gamma - 1}/2}. \quad (2.7)$$

Using this bound for each summand in (2.6) we obtain (2.4). Since the sum on the right-hand side of (2.4) is finite (2.3) follows. \hfill \Box

**Proof of Proposition 2.7.** Using a first moment method, the probability in (2.2) is bounded from above by

$$e^t \mathbb{P} \left( B_{\Sigma^2(t)} > \bar{m}(t) + d, B_{\Sigma^2(t)} \not\in T_{t,I,\Sigma^2} \right), \quad (2.8)$$

...
where $B_{\Sigma(t)}$ is a time change of an ordinary Brownian motion. Using that $\Sigma^2(s)$ is a non-decreasing function on $[0, t]$ with $\Sigma^2(t) = t$, we bound (2.8) from above by
\[ e^t P\left( \{ B_t > \tilde{m}(t) + d \} \land \left\{ \exists s \in [r, t-r] : \left| B_s - \frac{s}{t} B_t \right| > (s \land (t-s))^{\gamma} \right\} \right). \] (2.9)
Now, $\xi(s) \equiv B_s - \frac{s}{t} B_t$ is a Brownian bridge from 0 to 0 in time $t$, and it is well known that $\xi(s)$ is independent of $B_t$. Therefore, it holds that (2.9) is equal to
\[ e^t P(B_t > \tilde{m}(t) + d) P(\exists s \in [r, t-r] : |\xi(s)| > (s \land (t-s))^{\gamma}). \] (2.10)
Using the standard Gaussian tail bound,
\[ \int_u^\infty e^{-x^2/2} dx \leq u^{-1} e^{-u^2/2}, \quad \text{for } u > 0, \] (2.11)
we have
\[ e^t P(B_t > \tilde{m}(t) + d) \leq e^t \frac{\sqrt{t}}{\sqrt{2\pi(\tilde{m}(t) + d)}} e^{-(\tilde{m}(t) + d)^2/2t} \]
\[ = \frac{t}{\sqrt{2\pi(\tilde{m}(t) + d)}} e^{-\sqrt{2d}} \leq M, \] (2.12)
for some constant $M > 0$. By Lemma 2.2 we can find by $r$ large enough such that
\[ P(\exists s \in [r, t-r] : |\xi(s)| > (s \land (t-s))^{\gamma}) < \epsilon/M. \] (2.13)
Using the bounds of (2.12) and (2.12) we can bound (2.10) from above by $\epsilon$. □

3. PROOF OF THEOREM 1.1

In this section we present the proof of our main theorem assuming Proposition 3.2 below, whose proof will be postponed to the following two sections.

**Proof of Theorem 1.1** We show the convergence of the extremal process
\[ E_t = \sum_{k \leq n(t)} \delta_{\tilde{x}_k(t) - \tilde{m}(t)} \] (3.1)
by showing the convergence of the finite dimensional distributions and tightness. Tightness of $(E_t)_{t \geq 0}$ is implied by the following bound on the number of particles above a level $d$ (see [32], Lemma 3.20).

**Proposition 3.1.** For any $d \in \mathbb{R}$ and $\epsilon > 0$, there exists $N = N(\epsilon, d)$ such that, for all $t > 0$,
\[ P(E_t[d, \infty) \geq N) < \epsilon. \] (3.2)

**Proof.** By a first order Chebyshev inequality the probability in (3.2) is bounded by
\[ \frac{1}{N} e^t P(B_t > \tilde{m}(t) + d) \leq \frac{M}{N} \] (3.3)
by (2.12), where $M > 0$ is a constant that depends on $d$. Choosing $N > M/\epsilon$ yields Proposition 3.1. □

To show the convergence of the finite dimensional distributions define, for $u \in \mathbb{R}$,
\[ N_u(t) = \sum_{i=1}^{n(t)} \mathbb{1}_{x_i(t) - \tilde{m}(t) > u}, \] (3.4)

that counts the number of points that lie above \( u \). Moreover, we define the corresponding
quantity for the process \( E_{\sigma_e, \sigma_e} \) (defined in (1.13)),
\[
\mathcal{N}_u = \sum_{i,j} 1_{p_{i} + \sigma_e \Lambda_{\sigma_e}^{(i)} > u}.
\] (3.5)

Observe that, in particular,
\[
P \left( \max_{1 \leq k \leq m(t)} x_i(t) - \dot{m}(t) \leq u \right) = P (\mathcal{N}_u(t) = 0).
\] (3.6)

The key step in the proof of Theorem 1.1 is the following proposition, that asserts the convergence of the finite dimensional distributions of the process \( E_t \).

**Proposition 3.2.** For all \( k \in \mathbb{N} \) and \( u_1, \ldots, u_k \in \mathbb{R} \)
\[
\{ \mathcal{N}_{u_1}(t), \ldots, \mathcal{N}_{u_k}(t) \} \overset{d}{\to} \{ \mathcal{N}_{u_1}, \ldots, \mathcal{N}_{u_k} \}
\] (3.7)
as \( t \uparrow \infty \).

The proof of this proposition will be postponed to the following sections.

Assuming the proposition, we can now conclude the proof of the theorem.

The distribution of \( \{ \mathcal{N}_{u_1}(t), \ldots, \mathcal{N}_{u_k}(t) \} \) for all \( k \in \mathbb{N}, u_1, \ldots, u_k \in \mathbb{R} \) characterise the finite dimensional distributions of the point process \( E_t \) since the class of sets \( \{(u, \infty), u \in \mathbb{R}\} \) form a \( \Pi \)-system that generates \( B(\mathbb{R}) \). Hence (3.7) implies the convergence of the finite dimensional distributions of \( E_t \) (see, e.g., Proposition 3.4 in [32]).

Combining this observation with Propositions 3.1, we obtain Assertion (ii) of Theorem 1.1. Assertion (i) follows immediately from Eq. (3.6).

To prove Assertion (iii), we need to show that, as \( \sigma_e^2 \uparrow \infty \), it holds that \( \tilde{C}(\sigma_e) \uparrow 1/4\pi \) and the processes \( \Lambda^{(i)} \) converge to the trivial process \( \delta_0 \). Then,
\[
E_{\sigma_e, \infty} = \sum_{i} \delta_{p_i},
\] (3.8)
where \( (p_i, i \in \mathbb{N}) \) are the points of a PPP with random intensity measure \( \frac{1}{\sqrt{4\pi}} \sigma_e e^{-\sqrt{2\pi} x} dx \).

**Lemma 3.3.** The point process \( E_{\sigma_e, \sigma_e} \) converges in law, as \( \sigma_e \uparrow \infty \), to the point process \( E_{\sigma_e, \infty} \).

**Proof.** The proof of Lemma 3.3 is based on a result concerning the cluster processes \( \Lambda^{(i)} \) as given by Chauvin and Rouault [14], Theorem 5, respectively its backward description, Theorem 2.3 in [1]. To make the dependence on the parameter \( \sigma_e \) explicit, let us denote a copy of these processes by \( \Lambda_{\sigma_e} \).

Let \( Y \) be the path of a Brownian motion starting in 0 with drift \( \sqrt{-2\sigma_e} \). Conditionally on that path, let \( \pi \) be a Poisson point process with intensity measure \( 2P_{Y(r)}(\max_{0 \leq k \leq n(r)} \tilde{x}(r) < 0) dr \), where \( \tilde{x}(r) \) is standard BBM. For each point \( p \in \pi \) start a random number \( \nu_p \) of independent branching Brownian motions \( \{ B^{Y(p), i}, i \leq \nu_p \} \) starting at \( Y(p) \) conditioned on \( \max B^{Y(p), i} < 0 \). The law of \( \nu \) given by the size biased distribution, \( P(\nu_p = k - 1) = \frac{kp}{2} \). Then
\[
\Lambda_{\sigma_e} \overset{\text{law}}{=} \delta_0 + \sum_{p \in \pi} \sum_{i \leq \nu_p} \delta_{B^{Y(p), i}(p)}.
\] (3.9)
Since
\[
\lim_{\sigma_e \uparrow \infty} \mathbb{P} \left( \forall s \geq \sigma_e^{-1/2} : Y(s) + \sqrt{2\sigma_e} s \in [(\sigma_e s)^{\gamma}, (\sigma_e s)^{\gamma}] \right) = 1,
\] (3.10)
it suffices to show that, in the limit as \( \sigma_e \uparrow \infty \), for all \( R \in \mathbb{R}_+ \), for all paths \( Y \) such that
\[
\forall s \geq \sigma_e^{-1/2} : Y(s) + \sqrt{2}\sigma_e s \in \left[ -(\sigma_e s)^\gamma, (\sigma_e s)^\gamma \right],
\]
(3.11)

\[
\lim_{\sigma_e \uparrow \infty} \mathbb{P} \left( \exists p \in \pi, i < \nu_p, j : B_j^{N(p),i} \in [0, -R] \right) = 0.
\]
(3.12)

This probability is bounded by
\[
2K \int_0^\infty \mathbb{P} \left( \max_{s \geq \sigma_e^{-1/2}} B_j^{N(s),1} > -R \right) ds
\]
\[
\leq 2K \int_{\sigma_e^{-1/2}}^\infty \mathbb{P} \left( \max_{s \geq \sigma_e^{-1/2}} B_j^{N(s),1} > -R \right) ds
\]
\[
+ 2K \int_{\sigma_e^{-1/2}}^\infty e^s \mathbb{P}(B(s) > -R + \sqrt{2}\sigma_e s - (\sigma_e s)^\gamma) ds,
\]
(3.13)

where \( K = \sum_{k=1}^\infty k^2 p_k / 2 \). (3.13) is by (2.11) bounded from above by
\[
2K \sigma_e^{-1/2} + 2K \int_{\sigma_e^{-1/2}}^\infty e^{(1-\sigma_e^2)s+O((\sigma_e^2 s)^\gamma)} ds.
\]
(3.14)

From this it follows that (3.14) converges to zero, as \( \sigma_e \uparrow \infty \). Hence we see that as \( \sigma_e \uparrow \infty \)
\( \Lambda_{\sigma_e} \) converges to \( \delta_0 \).

Theorem 2 of [14] gives the following characterisation of the constant \( \widetilde{C}(\sigma_e) \):
\[
\frac{1}{\sqrt{4\pi \widetilde{C}(\sigma_e)}} = \lim_{s \uparrow \infty} \mathbb{E} \left[ \sum_k \mathbf{1}_{\hat{x}_k(s) > \sqrt{2}\sigma_e s} \right]
\]
\[
= \lim_{s \uparrow \infty} \mathbb{E} \left[ \max_k \mathbf{1}_{\hat{x}_k(s) > \sqrt{2}\sigma_e s} \right]
\]
\[
= \Lambda_{\sigma_e}((-E, 0]),
\]
(3.15)

where, by Theorem 7.5 in [9], \( E \) is an exponentially distributed random variable with parameter \( \sqrt{2}\sigma_e \), independent of \( \Lambda_{\sigma_e} \). Combined with the observation that As we have just shown that \( \Lambda_{\sigma_e} \to \delta_0 \), it follows that the right-hand side tends to one, as \( \sigma_e \uparrow \infty \), and hence \( \widetilde{C}(\sigma_e) \uparrow 1 / \sqrt{4\pi} \). Hence the intensity measure of the PPP appearing in \( \mathcal{E}_{\sigma_e, \sigma_e} \) converges to the desired intensity measure \( -\frac{1}{\sqrt{4\pi}} Y_{\sigma_e} e^{-\sqrt{2}x} dx \).

This proves Assertion (iii) of the theorem.

\[ \square \]

4. PROOF OF PROPOSITION 3.2

We prove Proposition 3.2 via convergence of Laplace transforms. Define the Laplace transform of \( \{ N_{u_1}(t), \ldots, N_{u_k}(t) \} \),
\[
\mathcal{L}_{u_1, \ldots, u_k}(t, c) = \mathbb{E} \left( \exp \left( -\sum_{l=1}^k c_l N_{u_l}(t) \right) \right), \quad c = (c_1, \ldots, c_k)^t \in \mathbb{R}_+^k,
\]
(4.1)

and the Laplace transform \( \mathcal{L}_{u_1, \ldots, u_k}(c) \) of \( \{ N_{u_1}, \ldots, N_{u_k} \} \). Proposition 3.2 is then a consequence of the next proposition.

\[ \square \]
Proposition 4.1. For any \( k \in \mathbb{N}, u_1, \ldots, u_k \in \mathbb{R} \) and \( c_1, \ldots, c_k \in \mathbb{R}_+ \)
\[
\lim_{t \to \infty} \mathcal{L}_{u_1, \ldots, u_k}(t, c) = \mathcal{L}_{u_1, \ldots, u_k}(c).
\] (4.2)

The proof of Proposition 4.1 requires two main steps. First, we prove the result for the case of two speed BBM. This was done in our previous paper [9]. In fact, we will need a slight extension of that result where we allow a slight dependence of the speeds on \( t \). This will be given in the next subsection.

The second step is to show that the Laplace transforms in the general case can be well approximated by those of two speed BBM. This uses the usual Gaussian comparison argument in a slightly subtle way.

4.1. Approximating two speed BBM. The case \( A'(1) < \infty \). As we will see later, it is enough to compare the process with covariance function \( A \) with processes whose covariance function is piecewise linear with single change in slope. We will produce approximate upper and lower bounds by choosing these in such a way that the covariances near zero and near one are below, resp. above that of the original process. We define
\[
\delta^<(t) = \sup\{ x \in [0, 1] : A(x) \leq t^{-2/3} \}
\]
\[
\delta^>(t) = 1 - \inf\{ x \in [0, 1] : A(x) \geq 1 - t^{-2/3} \}
\] (4.3)

(a) Case 1) \( \sigma_b = 0 \) but \( \lim_{t \uparrow \infty} \delta^<(t) = 0 \)
(b) Case 2) \( \sigma_b = 0 \) but \( \lim_{t \uparrow \infty} \delta^<(t) = \delta^< \neq 0 \)

Figure 2. Different cases for \( \delta^<(t) \) and \( \delta^>(t) \). In Case 1)

By Assumption (A1) it follows that \( \lim_{t \uparrow \infty} \delta^>(t) = 0 \).

Remark. If \( \lim_{t \uparrow \infty} \delta^<(t) = \delta^< \neq 0 \), then it follows by definition of \( \delta^<(t) \) that \( A(x) = 0 \) on \( [0, \delta^<] \).

In the following formulas, we choose the parameter \( n \in \mathbb{N}_{\geq 2} \) as follows. If in Assumption (A2) the functions \( \overline{B}, B \) can be chosen such that there exists \( m \geq 2 \), such that \( \overline{B}^{(k)}(0) = B^{(k)}(0) = 0 \), for all \( 1 \leq k < m \), and in some finite interval \( [0, \delta_b] \), both \( |\overline{B}^{(m)}(x)| \) and \( |B^{(m)}(x)| \) are bounded by some constants \( K_1 \), respectively \( \overline{K}_1 \), then we
choose $n$ as the largest of these integers. Otherwise, we choose $n = 2$. Moreover, let $|C''(x)| \leq K_2$ and $|C''(x)| \leq K_3$ for all $x \in [1 - \delta, 1]$. We define
\[
\Sigma^2(s) = t A(s/t) \tag{4.4}
\]
and
\[
\Sigma^2(s) = t A(s/t). \tag{4.5}
\]
Here
\[
A(x) = \begin{cases} 
(\sigma_e^2 + \frac{K_1}{2} (\delta^< t)^{n-1}) x, & 0 \leq x \leq \bar{b} \\
1 + (\sigma_e^2 - \frac{K_2}{2} \delta^>(x - 1), & \bar{b} < x \leq 1
\end{cases}, \tag{4.6}
\]
with
\[
\bar{b} = \frac{1 - \sigma_e^2 + \frac{K_2}{2} \delta^>(t)}{\sigma_e^2 + \frac{K_2}{2} \delta^>(x)^{n-1} - \sigma_e^2 + \frac{K_2}{2} \delta^>(t)}. \tag{4.7}
\]
If $\sigma_e^2 < \infty$,
\[
A(x) = \begin{cases} 
\left\{ (\sigma_e^2 - \frac{K_1}{2} (\delta^< t)^{n-1}) x \right\} \lor 0, & 0 \leq x \leq \bar{b} \\
1 + (\sigma_e^2 + \frac{K_2}{2} \delta^>(t))(x - 1), & \bar{b} < x \leq 1
\end{cases}, \tag{4.8}
\]
with
\[
\bar{b} = \frac{1 - \sigma_e^2 - \frac{K_2}{2} \delta^>(t)}{\sigma_e^2 - \sigma_e^2 - \frac{K_2}{2} \delta^>(t) - \frac{K_2}{2} (\delta^< t)^{n-1}}. \tag{4.9}
\]
Remark. If $\sigma_e^2 = 0$, $A(x) = 0$ for $0 \leq x \leq \bar{b}$. If $\lim_{t \to \infty} \delta^<(t) = \delta^< \neq 0$ (which implies that all derivatives in zero are 0), we take
\[
A(x) = \begin{cases} 
0, & 0 \leq x \leq \bar{b} \\
1 + (\sigma_e^2 - \frac{K_2}{2} \delta^>(t))(x - 1), & \bar{b} < x \leq 1
\end{cases}, \tag{4.10}
\]
and
\[
\bar{b} = \frac{1 - \sigma_e^2 + \frac{K_2}{2} \delta^>(t)}{-\sigma_e^2 + \frac{K_2}{2} \delta^>(t)} \tag{4.11}
\]
If $A'(1) = \sigma_e^2 = +\infty$, then $\bar{b} = 1$. And $\bar{A} \equiv \bar{A}_p$ is defined by
\[
\bar{A}(x) = \begin{cases} 
(\sigma_e^2 + \frac{K_1}{2} (\delta^< t)^{n-1}) x, & 0 \leq x \leq \bar{b} \\
1 + \rho(x - 1), & \bar{b} < x \leq 1
\end{cases}, \tag{4.12}
\]
and $\bar{b} \equiv \bar{b}_p = \frac{1 - \rho}{\sigma_e^2 + \frac{K_1}{2} (\delta^< t)^{n-1}}$. 

The choice of $\Sigma^2$ and $\Sigma^2$ is motivated by the following properties.

**Lemma 4.2.** $\bar{A}$ and $A$ are piecewise linear, continuous functions with $\bar{A}(0) = A(0) = 0$ and $\bar{A}(1) = A(1) = 1$. Moreover,

(i) If $\lim_{t \to \infty} \delta^<(t) = 0$, then it holds that, for all $s$ with $\Sigma^2(s) \in [0, t^{1/3}]$ and $\Sigma^2(s) \in [t - t^{1/3}, t]$,
\[
\Sigma^2(s) \geq \Sigma^2(s) \geq \Sigma^2(s). \tag{4.13}
\]

(ii) If $\lim_{t \to \infty} \delta^<(t) = \delta^> > 0$, then Eq. (4.13) only holds for all $s$ with $\Sigma^2(s) \in [t - t^{1/3}, t]$ while, for $s \in [0, (\delta \wedge \bar{b}) t)$, it holds that
\[
\Sigma^2(s) = \Sigma^2(s) = \Sigma^2(s) = 0. \tag{4.14}
\]
\begin{proof}
\(\mathcal{A}\) and \(\mathcal{A}\) are obviously piecewise linear. The fact that they are continuous is easily verified. Note that, by definition, \(A'(0) = \sigma_b^2\) and \(A'(1) = \sigma_c^2\).

For all \(s\) such that \(\Sigma^2(s) \in [0, t^{1/3}]\), a \(n\)th-order Taylor expansion of \(\overline{B}\) at \(0\) together with the fact that for \(k < n\), by assumption, \(\overline{B}(0) = \overline{B}_{\leq k}(0) = 0\) shows that

\[
\Sigma^2(s) \leq \overline{B}(s) = t \left[ \overline{B}(0) \frac{s}{t} + \frac{\overline{B}^{(n)}(\xi)}{n!} \left( \frac{s}{t} \right)^n \right], \quad \text{for some } \xi \in (0, s). \tag{4.15}
\]

The reverse inequality holds when \(\overline{B}\) is replaced by \(\underline{B}\). Eq. (4.13) follows then from Assumption (A2). Using a second order Taylor expansion of \(\overline{C}\) and \(\underline{C}\) at \(1\), we obtain Eq. (4.13) for \(\Sigma^2(s) \in [t - t^{1/3}, t]\). Equality (4.14) holds trivially in the specified interval. This concludes the proof of the lemma.
\end{proof}

Let \(\{\overline{y}_i, i \leq n(t)\}\) be the particles of a BBM with speed function \(\Sigma^2\) and let \(\{\underline{y}_i, i \leq n(t)\}\) be particles of a BBM with speed function \(\Sigma^2\). We want to show that the limiting extremal processes of these processes coincide. Set

\[
\overline{N}_u(t) \equiv \sum_{i=1}^{n(t)} 1_{\overline{y}_i(t) - \overline{m}(t) > u}, \tag{4.16}
\]

\[
\underline{N}_u(t) \equiv \sum_{i=1}^{n(t)} 1_{\underline{y}_i(t) - \underline{m}(t) > u}. \tag{4.17}
\]

\begin{lemma}
For all \(u_1, \ldots, u_k\) and all \(c_1, \ldots, c_k \in \mathbb{R}_+\), the limits

\[
\lim_{t \uparrow \infty} \mathbb{E} \left( \exp \left( - \sum_{i=1}^{k} c_i \overline{N}_{u_i}(t) \right) \right) \tag{4.18}
\]

and

\[
\lim_{t \uparrow \infty} \mathbb{E} \left( \exp \left( - \sum_{i=1}^{k} c_i \underline{N}_{u_i}(t) \right) \right) \tag{4.19}
\]

exist. If \(A'(1) < \infty\), the two limits coincide with \(\mathcal{L}_{u_1, \ldots, u_k}(c)\).

If \(A'(1) = \sigma_c^2 = \infty\), then the two limits in (4.18) converges to that in (4.19), as \(\rho \uparrow \infty\).
\end{lemma}

\begin{proof}
We first consider the case when \(A'(1) < \infty\). To prove Lemma 4.3, we show that the extremal processes

\[
\overline{E}_t = \sum_{i=1}^{n(t)} \delta_{\overline{y}_i - \overline{m}(t)} \quad \text{and} \quad \underline{E}_t = \sum_{i=1}^{n(t)} \delta_{\underline{y}_i - \underline{m}(t)} \tag{4.20}
\]

both converge to \(\mathcal{E}_{\sigma_b, \sigma_c}\), that was defined in (1.13). Note that this implies first convergence of Laplace functionals with functions \(\phi\) with compact support, while the \(\mathcal{N}_u(t)\) have support that is unbounded from above. Convergence for these, however, carries over due to the tightness established in Proposition 3.1.

To do so, observe that the slopes at \(0\) of \(\overline{\Sigma}^2\) and \(\underline{\Sigma}^2\) are equal to \(\sigma_b^2\) up to an error of order \(\delta^<(t)\). Moreover, the slope at \(1\) is in both cases, up to an error of order \(\delta^>(t)\), equal to \(\sigma_c^2\). The time of speed change \(\overline{b}(t)\), respectively \(\underline{b}(t)\), is equal to \(\frac{1 - \sigma_c^2}{\sigma_b^2 - \sigma_c^2}\) up to an error of order
δ^>(t) ∨ δ^<(t). For the two-speed BBM with speed

\[
\sigma^2(s) = \begin{cases} 
\sigma_b^2, & \text{for } 0 < s \leq \frac{1-\sigma_b^2}{\sigma_c^2-\sigma_b^2}, \\
\sigma_c^2, & \text{for } \frac{1-\sigma_c^2}{\sigma_c^2-\sigma_b^2} < s < t,
\end{cases}
\]

(4.21)

it was shown in [9] that the maximal displacement is equal to \(\tilde{m}(t)\) and that the extremal process converges to \(E_{\sigma_0,\sigma_c}\) as \(t \uparrow \infty\). The method used to show this is to show the convergence of the Laplace functionals, \(\mathbb{E}(\exp(-\int \phi(x)E_t(dx)))\), \(\phi \in C_c(\mathbb{R}, \mathbb{R}_+)\). The other difference is that the function \(A\) we have to consider now depend (weakly) on \(t\). We need to show that this has no effect.

Inspecting the proof of the convergence of the Laplace functional, respectively convergence of the maximum in [9], one sees that nothing changes (since we keep \(t\) fixed) until Eq. (5.28) in [9]. There, we then have to control, for each \(y \in \mathbb{R}\), (in the case of \(\Sigma_\gamma^B\))

\[
\mathbb{E} \left( \exp \left( -C(a) \left( \frac{\sigma_b^2 \sqrt{\delta^>(t)}}{1-\sigma_b^2+\frac{\sqrt{2}}{2}\delta^<(t)B_t} \right)^{1/2} e^{-\sqrt{2}y\tilde{Y}^B_{\sigma_b,B_t,\gamma}} \right) (1 + o(1)) \right),
\]

(4.22)
as \(t \uparrow \infty\), where \(C(a) > 0\) is a constant depending on \(a = \sqrt{2}(\sigma_c-1)\), and

\[
\tilde{Y}^B_{\sigma_b,B_t,\gamma} = \sum_{i=1}^{n(B\sqrt{t})} e^{-(1+\sigma_b^2+\frac{\sqrt{2}}{2}\delta^<(t))B\sqrt{t}+\sqrt{2}y\sqrt{t}\gamma} \mathbb{1}_{\{B\sqrt{t} \leq \gamma\}} (\tilde{Y}^B_{\sigma_b,B_t,\gamma}) (1 + o(1))
\]

(4.23)
The main task is to ensure the convergence of this object to the limit of the corresponding McKean martingale. In the case where \(\lim_{t \uparrow \infty} \delta^<(t) > 0\), this takes the simple form

\[
\tilde{Y}^B_{0,B\sqrt{t},\gamma} = \sum_{i=1}^{n(B\sqrt{t})} e^{-B\sqrt{t}},
\]

(4.24)
which converges to an exponential random variable of mean one, as desired.

Assume now that \(\lim_{t \uparrow \infty} \delta^<(t) = 0\). Observe that in the proof of Theorem 5.1 in [9], \(b\sqrt{t}\) can be replaced by any sequence \(\Delta(t) \uparrow \infty\) such that \(\lim_{t \uparrow \infty} \Delta(t)/t = 0\). Here we adapt \(\Delta(t)\) to the function \(\Sigma_\gamma^2\) and chose

\[
\Delta(t) = (\delta^<(t))^{-1/2}.
\]

(4.25)

Doing so, one obtains analogously to (4.22),

\[
\mathbb{E} \left( \exp \left( -C(a) \left( \frac{\sigma_b^2 \sqrt{\delta^>(t)}}{1-\sigma_b^2+\frac{\sqrt{2}}{2}\delta^<(t)} \right)^{1/2} e^{-\sqrt{2}y\tilde{Y}^B_{\sigma_b,\Delta(t),\gamma}} \right) (1 + o(1)) \right),
\]

(4.26)
By our choice of \(\Delta(t)\),

\[
\left| e^{-\frac{\sqrt{2}}{2}\delta^<(t)\Delta(t)} e^{\sqrt{2}\frac{\sqrt{2}}{2}\delta^<(t)(\Delta(t)+B\Delta(t)\gamma)} - 1 \right| \leq \text{const.} \sqrt{\delta^<(t)},
\]

(4.27)
which tends to zero, as \(t \uparrow \infty\). Thus

\[
\tilde{Y}^B_{\Delta(t),\gamma} = \tilde{Y}^B_{\Delta(t),\gamma} (1 + o(1)),
\]

(4.28)
where

\[
\tilde{Y}^B_{\Delta(t),\gamma} = \sum_{i=1}^{n(\Delta(t))} e^{-(1+\sigma_b^2)\Delta(t)+\sqrt{2}\sigma_b\pi_i(\Delta(t))} \mathbb{1}_{\sigma_b\pi_i(\Delta(t)) \leq [B\Delta(t)\gamma, B\Delta(t)\gamma]}.
\]

(4.29)
By Lemma 4.3 in [9], it follows that $\tilde{Y}^B_{\Delta(t),\gamma}$ converges in probability and in $L^1$ to the random variable $Y_{\sigma_e}$. Since $\tilde{Y}^B_{\Delta(t),\gamma} \geq 0$ and $C(a) > 0$, and since
\[
\lim_{t\uparrow\infty} \left( \frac{\sigma^2 - \bar{\gamma} \kappa^2 \gamma(t)}{1 - (\sigma_e^2 + \bar{\gamma} \kappa^2 \gamma(t)) \kappa(t)/t} \right)^{1/2} = \sigma_e,
\] (4.30)
it follows that
\[
\lim_{B\uparrow\infty} \liminf_{t\uparrow\infty} \mathbb{E} \left( \exp \left( -C(a)\sigma_e e^{-\sqrt{2}y \tilde{Y}^B_{\Delta(t),\gamma}} \right) (1 + o(1)) \right) = \lim_{B\uparrow\infty} \limsup_{t\uparrow\infty} \mathbb{E} \left( \exp \left( -C(a)\sigma_e e^{-\sqrt{2}y \tilde{Y}^B_{\Delta(t),\gamma}} \right) (1 + o(1)) \right) = \mathbb{E} \left( \exp \left( -\tilde{C}(\sigma_e) Y_{\sigma_e} e^{-\sqrt{2}y} \right) \right),
\] (4.31)
where $\tilde{C}(\sigma_e) = \sigma_e C(a)$. The same arguments work when $\Sigma^2$ is replaced by $\Sigma^2$. The limit in (4.31) coincides with the one obtained in [9] for the two-speed BBM with speed given in (4.21). The assertion in the case when $\sigma_e = \infty$ follows directly from Lemma 3.3. \hfill \Box

4.2. Gaussian comparison. We distinguish from now on the expectation with respect to the underlying tree structure and the one with respect to the Brownian movement of the particles.

- $\mathbb{E}_n$: expectation w.r.t. Galton-Watson process.
- $\mathbb{E}_B$: expectation w.r.t. the Gaussian process conditioned on the $\sigma$-algebra $\mathcal{F}_t^{\text{tree}}$ generated by the Galton Watson process.

The proof of Proposition 4.1 is based on the following Lemma that compares the Laplace transform $L_{u_1,...,u_k}(t,c)$ with the corresponding Laplace transform for the comparison processes.

Lemma 4.4. For any $k \in \mathbb{N}$, $u_1,\ldots,u_k \in \mathbb{R}$ and $c_1,\ldots,c_k \in \mathbb{R}_+$ we have
\[
L_{u_1,...,u_k}(t,c) \leq \mathbb{E} \left( \exp \left( - \sum_{l=1}^k c_l \bar{\mathcal{N}}_{u_l}(t) \right) \right) + o(1)
\] (4.32)
\[
L_{u_1,...,u_k}(t,c) \geq \mathbb{E} \left( \exp \left( - \sum_{l=1}^k c_l \bar{\mathcal{N}}_{u_l}(t) \right) \right) + o(1)
\] (4.33)

Proof. The proofs of (4.32) and (4.33) are very similar. Hence we focus on proving (4.32). We will, however, indicate what has to be changed when proving the lower bound as we go along. For simplicity all overlined names depend on $\bar{\Sigma}^2$. Corresponding quantities where $\Sigma^2$ is replaced by $\bar{\Sigma}^2$ are underlined. Set
\[
f(x(t)) \equiv f(x_1(t),\ldots,x_n(t)) \equiv \exp \left( - \sum_{i=1}^{n(t)} \sum_{l=1}^k c_l \mathbb{1}_{x_i(t) - \bar{m}_i(t) > u_l} \right).
\] (4.34)

We want to control
\[
\mathbb{E}_B \left( \exp \left( - \sum_{l=1}^k c_l \bar{\mathcal{N}}_{u_l}(t) \right) \right) - \mathbb{E}_B \left( \exp \left( - \sum_{l=1}^k c_l \mathcal{N}_{u_l}(t) \right) \right)
\] (4.35)
\[
= \mathbb{E}_B \left( f(x_1(t),\ldots,x_{n(t)}(t)) \right) - \mathbb{E}_B \left( f(\bar{y}_1(t),\ldots,\bar{y}_{n(t)}(t)) \right)
\]
Define for $h \in [0, 1]$ the interpolating process
\begin{equation}
    x^h_i = \sqrt{h}x_i + \sqrt{1-h}y_i. \tag{4.36}
\end{equation}

**Remark.** We understand the interpolating process $\{x^h_i, i \leq n(t)\}$ as a new Gaussian process with the same underlying branching structure and speed function

\begin{equation}
    \Sigma^2_h(s) = h\Sigma^2(s) + (1-h)\Sigma^2(s). \tag{4.37}
\end{equation}

Then, (4.35) is equal to
\begin{equation}
    \mathbb{E}_B \left( \int_0^1 \frac{d}{dh} f(x^h(t)) dh \right), \tag{4.38}
\end{equation}
where
\begin{equation}
    \frac{d}{dh} f(x^h(t)) = \frac{1}{2} \sum_{i=1}^{n(t)} \frac{\partial}{\partial x_j} f(x^h_1(t), \ldots, x^h_{n(t)}(t)) \left[ \frac{1}{\sqrt{h}} x_i(t) - \frac{1}{\sqrt{1-h}} y_i(t) \right] \tag{4.39}
\end{equation}
and $\frac{\partial}{\partial x_j} f(x^h_1(t), \ldots, x^h_{n(t)}(t))$ is the weak partial derivative\footnote{In this proof it always suffices to consider weak derivatives since the only non-differentiable function is $f(x) = \mathbb{1}_{x \geq u}$ which can be monotonously approximated by smooth functions, e.g. replace all indicator functions in $f$ by $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2\sigma^2} dz$, rewrite (4.35) as in (4.38) and then take $\sigma \uparrow \infty$.}.\footnote{In this proof it always suffices to consider weak derivatives since the only non-differentiable function is $f(x) = \mathbb{1}_{x \geq u}$ which can be monotonously approximated by smooth functions, e.g. replace all indicator functions in $f$ by $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2\sigma^2} dz$, rewrite (4.35) as in (4.38) and then take $\sigma \uparrow \infty$.}

\begin{equation}
    \frac{\partial}{\partial x_j} f(x^h_1(t), \ldots, x^h_{n(t)}(t)) = - \left( \sum_{l=1}^{k} c_l \delta(x^h_l(t) - m(t) - u_l) \right) f(x^h_1(t), \ldots, x^h_{n(t)}(t)). \tag{4.40}
\end{equation}

The key idea is to introduce a localisation condition on the path of $x^h_i$ into (4.39) at this stage. To do so, we insert into the right-hand side of (4.39) a one in the form
\begin{equation}
    1 = \mathbb{1}_{x^h_i \in T^\gamma_{t, I, x_h^2}} + \mathbb{1}_{x^h_i \in \bar{T}^\gamma_{t, I, x_h^2}}, \tag{4.41}
\end{equation}
with
\begin{equation}
    \bar{I} \equiv [t(\delta^<_1(t) \land \delta^<_1(t)), t(1 - \delta^>_1(t))], \tag{4.42}
\end{equation}
and $T^\gamma_{t, I, x_h^2}$ was defined in (2.1). Here $\delta^<_{1}(t) \equiv \delta^{<,>}(t)$, while $\delta^{<,>}_0$ is defined in the same way, but with respect to the speed function $\Sigma^2$ instead of $\Sigma^2$. We call the two resulting summands $S^<_\gamma$ and $S^>_\gamma$, respectively.

Note that, when proving the lower bound, we choose instead of $\bar{I}$, the interval
\begin{equation}
    \bar{I} \equiv [t(\delta^<_0(t) \land \delta^<_0(t)), t(1 - \delta^<_0(t))]. \tag{4.43}
\end{equation}

The next lemma shows that $S^>_\gamma$ does not contribute to the expectation in (4.39), as $t \to \infty$.

**Lemma 4.5.** With the notation above, we have
\begin{equation}
    \lim_{t \to \infty} \mathbb{E}_B \left( \int_0^1 \mathbb{E}_B(|S^>_\gamma|) dh \right) = 0. \tag{4.44}
\end{equation}

The proof of this lemma will be postponed.

We continue with the proof of Lemma 4.4. We are left with controlling
\begin{equation}
    \mathbb{E}_B(S^>_\gamma) = \mathbb{E}_B \left( \frac{1}{2} \sum_{i=1}^{n(t)} \frac{\partial}{\partial x_j} f(x^h_i(t)) \mathbb{1}_{x^h_i \in T^\gamma_{t, I, x_h^2}} \left[ \frac{x_i(t)}{\sqrt{h}} - \frac{y_i(t)}{\sqrt{1-h}} \right] \right). \tag{4.45}
\end{equation}
By the definition of $\mathcal{T}_{t,1,\Sigma^2_h}$,

$$1_{x_j^h \in \mathcal{T}_{t,1,\Sigma^2_h}} = 1_{\forall s \in \bar{I}, |\xi(s) - \Sigma^2_h(t)| \leq (\Sigma^2_h(t) \wedge (t - \Sigma^2_h(t)))^\gamma},$$  \hspace{1em} (4.46)

where $\xi(\cdot)$ is a Brownian bridge from 0 to 0 in time $t$, that is independent of $x_j^h(t)$. We want to apply a Gaussian integration by parts formula to (4.45). However, we need to take care of the fact that each summand in (4.45) depends on the whole path of $\xi$ through the term in (4.46). Therefore, we first approximate that indicator function in (4.46) by a discretised version. Let, for $n \in \mathbb{N}$, $t_1, \ldots, t_{2n}$ be a sequence of equidistant points in $[0, t]$. Define the following sequence of approximations to the indicator function in (4.46),

$$g(\xi(t_1), \ldots, \xi(t_{2n})) = \prod_{j=1}^{2n} 1_{t_j \in \bar{I}} |\xi(t_j)| \leq (\Sigma^2_h(t_j) \wedge (t - \Sigma^2_h(t_j)))^\gamma.$$

By the Gaussian integration by parts formula (see, e.g., [26]), we have, for any given $n \in \mathbb{N},$

$$\mathbb{E}_B \left( x_j^h \frac{\partial}{\partial x_j} f(x_j^h(t)) g(\xi(t_1), \ldots, \xi(t_{2n})) \right) = \sum_{j=1}^{2n} \mathbb{E}_B \left( (x_j^h \xi(\Sigma^2_h(t_j))) \mathbb{E}_B \left( f(x_j^h(t)) \frac{\partial}{\partial x_j} g(\xi(t_1), \ldots, \xi(t_{2n})) \right) \right)$$

$$+ \sum_{2 \leq j \leq n} \mathbb{E}_B (x_j^h(t) x_j^h(t)) \mathbb{E}_B \left( g(\xi(t_1), \ldots, \xi(t_{2n})) \frac{\partial^2}{\partial x_j \partial x_i} f(x_j^h(t)) \right),$$

where the weak partial derivative $\frac{\partial}{\partial x_j} g(\xi(t_1), \ldots, \xi(t_{2n}))$ is given by

$$(\delta(\xi(\Sigma^2_h(t_j))) + (\Sigma^2_h(t_j) \wedge (t - \Sigma^2_h(t_j)))^\gamma) - \delta(\xi(\Sigma^2_h(t_j))) - (\Sigma^2_h(t_j) \wedge (t - \Sigma^2_h(t_j)))^\gamma)$$

$$\times 1_{t_j \in \bar{I}} \prod_{p \neq j} 1_{t_p \in I} |\xi(t_p)| \leq (\Sigma^2_h(t_p) \wedge (t - \Sigma^2_h(t_p)))^\gamma.$$

(4.49)

Since $x_j^h(t)$ is independent of $\xi$, we have

$$\mathbb{E}_B (x_j^h(t) \xi(\Sigma^2_h(t_j))) = 0, \hspace{1em} \forall j \in \{1, \ldots, 2^n\},$$

(4.50)

so that (4.48) is equal to

$$\sum_{2 \leq j \leq n} \mathbb{E}_B (x_j^h(t) x_j^h(t)) \mathbb{E}_B \left( \frac{\partial^2}{\partial x_j \partial x_i} f(x_j^h(t)) \right) 1_{t_j \in \bar{I}} |\xi(t_j)| \leq (\Sigma^2_h(t_j) \wedge (t - \Sigma^2_h(t_j)))^\gamma.$$  \hspace{1em} (4.51)

Using the fact that $\xi(\cdot)$ is a continuous process and that $\Sigma^2_h(\cdot)$ is right-continuous and increasing, by monotone convergence we may pass to the limit as $n \uparrow \infty$ to obtain

$$\mathbb{E}_B \left( x_j^h \frac{\partial}{\partial x_j} f(x_j^h(t), \ldots, x_{n(t)}^h(t)) 1_{\forall s \in \bar{I}, |\xi(s)| \leq (\Sigma^2_h(s) \wedge (t - \Sigma^2_h(s)))^\gamma} \right)$$

$$= \sum_{j=1}^{n(t)} \mathbb{E}_B (x_j^h(t) x_j^h(t)) \mathbb{E}_B \left( \frac{\partial^2}{\partial x_j \partial x_i} f(x_j^h(t), \ldots, x_{n(t)}^h(t)) 1_{x_j^h \in \mathcal{T}_{t,1,\Sigma^2_h}} \right).$$  \hspace{1em} (4.52)
Hence
\[
\mathbb{E}_B(\overline{S}_T) = \sum_{i,j=1}^{n(t)} \left[ \mathbb{E}_B(x_i(t),x_j(t)) - \mathbb{E}_B(\overline{y}_i(t),\overline{y}_j(t)) \right] \mathbb{E}_B \left( 1_{x_i^h(t) \in T_{i,j} \text{ for all } i,j} \frac{\partial^2 f(x^h(t))}{\partial x_i \partial x_j} \right),
\]

(4.53)

where
\[
\frac{\partial^2 f(x^h(t))}{\partial x_i \partial x_j} = \sum_{l,j'=1}^{k} c_l c_{j'} \delta(x_i^h - \tilde{m}(t) - u_l) \delta(x_j^h - \tilde{m}(t) - u_{j'}) f(x_i^h(t), \ldots, x_n^h(t)).
\]

(4.54)

Introducing
\[
1 = \mathbb{1}_{d(x^h(t),x^h(t)) \in I} + \mathbb{1}_{d(x^h(t),x^h(t)) \in \overline{I}},
\]

(4.55)

into (4.53) we rewrite (4.53) as \((\overline{T}1) + (\overline{T}2)\), where
\[
(\overline{T}1) = \sum_{i,j=1}^{n(t)} \left[ \mathbb{E}_B(x_i(t),x_j(t)) - \mathbb{E}_B(\overline{y}_i(t),\overline{y}_j(t)) \right] \mathbb{E}_B \left( 1_{d(x^h(t),x^h(t)) \in I} 1_{x_i^h(t) \in T_{i,j} \text{ for all } i,j} \frac{\partial f(x^h(t))}{\partial x_i \partial x_j} \right),
\]

(4.56)

\[
(\overline{T}2) = \sum_{i,j=1}^{n(t)} \left[ \mathbb{E}_B(x_i(t),x_j(t)) - \mathbb{E}_B(\overline{y}_i(t),\overline{y}_j(t)) \right] \mathbb{E}_B \left( 1_{d(x^h(t),x^h(t)) \in \overline{I}} 1_{x_i^h(t) \in T_{i,j} \text{ for all } i,j} \frac{\partial f(x^h(t))}{\partial x_i \partial x_j} \right).
\]

(4.57)

The term \((\overline{T}1)\) is controlled by the following Lemma.

**Lemma 4.6.** With the notation above, there exists a constant \(\overline{C} < \infty\), such that
\[
\left| \mathbb{E}_n \left( \int_0^1 (\overline{T}1) \, dh \right) \right| \leq \bar{C} \left| \int_{t(1-\delta_\infty^+(t))}^{t(1-\delta_\infty^-(t))} \left( e^{-s+\Sigma^2(s)+O(s^\gamma)} - e^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right) \, ds \right|.
\]

(4.58)

Moreover, we have:

**Lemma 4.7.** If \(\Sigma^2\) satisfies (A1)-(A3), and \(\overline{\Sigma}^2\) is as defined in (4.4), then
\[
\lim_{t \to \infty} \left| \int_t^{(\delta_\infty^-(t))} \left( e^{-s+\Sigma^2(s)+O(s^\gamma)} - e^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right) \, ds \right| = 0.
\]

(4.59)

We postpone the proofs of these lemmata.

Up to this point the proof of (4.33) works exactly as the proof of (4.32) when \(\Sigma^2\) is replaced by \(\overline{\Sigma}^2\). For \((\overline{T}2)\) and \((T2)\) we have:

**Lemma 4.8.** For almost all realisations of the Galton-Watson process, the following statements hold:

(i) If \(\lim_{t \to \infty} \delta_\infty^-(t) = 0\), then
\[
(\overline{T}2) \leq 0,
\]

and
\[
(T2) \geq 0.
\]
(ii) If \( \lim_{t \uparrow \infty} \delta^< (t) = \delta^< > 0, \) then
\[
\lim_{t \uparrow \infty} (T^2) \leq 0, \quad (4.62)
\]
and
\[
\lim_{t \uparrow \infty} (T^2) \geq 0. \quad (4.63)
\]

The proof of this lemma is again postponed.

From Lemma 4.6, Lemma 4.7, and Lemma 4.8 together with (4.45), the bound (4.32) follows. As pointed out, using Lemma 4.8, the bound (4.33) also follows. Thus, Lemma 4.4 is proved, once we provide the postponed proofs of the various lemmata above.

We can now conclude the proof of Proposition 4.1.

**Proof of Proposition 4.1.** Taking the limit as \( t \uparrow \infty \) in (4.32) and (4.33) and using Lemma 4.5 gives, in the case \( A'(1) < \infty, \)
\[
\limsup_{t \uparrow \infty} \mathcal{L}_{u_1, \ldots, u_k} (t, c) \leq \mathcal{L}_{u_1, \ldots, u_k} (c)
\]
and
\[
\liminf_{t \uparrow \infty} \mathcal{L}_{u_1, \ldots, u_k} (t, c) \geq \mathcal{L}_{u_1, \ldots, u_k} (c) \quad (4.64)
\]
Hence \( \lim_{t \uparrow \infty} \mathcal{L}_{u_1, \ldots, u_k} (t, c) \) exists and is equal to \( \mathcal{L}_{u_1, \ldots, u_k} (c) \). In the case \( A'(1) = \infty, \) the same result follows if in addition we take \( \rho \uparrow \infty \) after taking \( t \uparrow \infty \). This concludes the proof of Proposition 4.1.

5. PROOFS OF THE AUXILIARY LEMMATA

We now provide the proofs of the lemmata from the last section whose proofs we had postponed.

**Proof of Lemma 4.5.** We have
\[
\mathbb{E}_B (|\mathcal{F}_t|) \leq \frac{1}{2} \mathbb{E}_B \left( \sum_{i=1}^{n(t)} \left( \sum_{l=1}^{k} c_l \delta (x_{i}^h (t) - \tilde{m}(t) - u_l) \right) \mathbbm{1}_{x_{i}^h \not\in \mathcal{T}^r_{t, i, \mathcal{Y}_h, 1}} \right), \quad (5.1)
\]
For simplicity we introduce another 1 into (5.1), namely
\[
1 = \mathbbm{1}_{|x_{i}(t)| > 2 \tilde{m}(t)} + \mathbbm{1}_{|x_{i}(t)| \leq 2 \tilde{m}(t)}, \quad (5.2)
\]
and
\[
1 = \mathbbm{1}_{|\mathcal{I}(t)| > 2 \tilde{m}(t)} + \mathbbm{1}_{|\mathcal{I}(t)| \leq 2 \tilde{m}(t)}, \quad (5.3)
\]
respectively. Then (5.1) is equal to \((R1) + (R2), \) where
\[
(R1) = \frac{1}{2} \mathbb{E}_B \left( \sum_{i=1}^{n(t)} \left( \sum_{l=1}^{k} c_l \delta (x_{i}^h (t) - \tilde{m}(t) - u_l) \right) \mathbbm{1}_{x_{i}^h \not\in \mathcal{T}^r_{t, i, \mathcal{Y}_h, 1}} \right) \times \left[ \frac{|x_{i}(t)|}{\sqrt{h}} \mathbbm{1}_{|x_{i}(t)| > 2 \tilde{m}(t)} + \frac{|\mathcal{I}(t)|}{\sqrt{1 - h}} \mathbbm{1}_{|\mathcal{I}(t)| > 2 \tilde{m}(t)} \right],
\]
\[
(R2) = \frac{1}{2} \mathbb{E}_B \left( \sum_{i=1}^{n(t)} \left( \sum_{l=1}^{k} c_l \delta (x_{i}^h (t) - \tilde{m}(t) - u_l) \right) \mathbbm{1}_{x_{i}^h \not\in \mathcal{T}^r_{t, i, \mathcal{Y}_h, 1}} \right) \times \left[ \frac{|x_{i}(t)|}{\sqrt{h}} \mathbbm{1}_{|x_{i}(t)| \leq 2 \tilde{m}(t)} + \frac{|\mathcal{I}(t)|}{\sqrt{1 - h}} \mathbbm{1}_{|\mathcal{I}(t)| \leq 2 \tilde{m}(t)} \right]. \quad (5.4)
\]
Proof of Lemma 4.8. We first proof (4.60). Observe that

\[ \lim_{t \to \infty} \int_{0}^{l} (t \int_{1-t}^{1} \langle x \rangle_{t} \, dx + \frac{1}{\sqrt{1-h}} \int_{1-t}^{1} \langle x \rangle_{t} \, dx) \, dh = C_k \]

with \( C_k \equiv \sum_{i=1}^{n} c_i \). Clearly, the right-hand side of (4.5) tends to zero, as \( t \uparrow \infty \). Set \( u_0 = \min_{1 \leq i < k} u_i - 1 \). \( E_n(f(t)R_0(t)) \) is bounded from above by

\[ C_k E_n \left( \int_{0}^{l} \left( \frac{1}{\sqrt{h}} + \frac{1}{\sqrt{1-h}} \right) \sum_{i=1}^{n} \mathbb{P} \left( B_i - \tilde{m}(t) > u_0, B_{T_0(t)} \not\in T_i \right) \, dh \right) = C_k \]

Note that by construction, if \( s \in t \), then for all \( h \in [0, 1], \Sigma_h^1 \geq D t^{1/3} \), and \( \Sigma_h^2 \leq t - D t^{1/3} \), for some constant \( 0 < D < \infty \), depending only on the function \( A \). Thus, by Eq. (2.4) of Lemma 2.2 this expression is bounded from above by

\[ 8m(t)C_k \int_{0}^{l} \left( \frac{1}{\sqrt{h}} + \frac{1}{\sqrt{1-h}} \right) \sum_{k=dt^{1/3}}^{\infty} k^{1/2} e^{-k^{2/3} - 1/2} \, dh. \]  

The sum in the last line is of order \( \exp \left( -D^{2/3} t^{(2/3)-1/3} \right) \), where \( 2\gamma > 1 \), and hence \( (5.7) \) tends to zero, as \( t \uparrow \infty \). This proves the assertion of the lemma.

Proof of Lemma 4.8. We first proof (4.60). Observe that

\[ d(x_i(t), x_j(t)) = d(\bar{y}_i(t), \bar{y}_j(t)) = d(x_i(t), x_j(t)). \]  

Moreover, for all \( 1 \leq i, j \leq n(t), i \neq j \),

\[ \mathbb{P} \left( B_i - \tilde{m}(t) > u_0, B_{T_0(t)} \not\in T_i \right) \]

For \( d(x_i(t), x_j(t)) \in [0, t(\delta_{\gamma}^{\alpha}(t) \wedge \delta_{\gamma}^{\alpha}(t))) \), we distinguish the cases \( \lim_{t \to \infty} \delta^{\alpha}(t) = \delta^{\alpha} > 0 \) and \( \lim_{t \to \infty} \delta^{\alpha}(t) = 0 \), respectively. If \( \lim_{t \to \infty} \delta^{\alpha}(t) = 0 \), then \( A(x) = \bar{A}(x) = A(x) = 0 \), for all \( x \in [0, t(\delta_{\gamma}^{\alpha}(t) \wedge \delta_{\gamma}^{\alpha}(t))) \). Thus all the terms in both (T2) and (T2) with \( i, j \) such that \( d(x_i(t), x_j(t)) \in [0, t(\delta_{\gamma}^{\alpha}(t) \wedge \delta_{\gamma}^{\alpha}(t))) \) vanish.

Next consider the case where \( \lim_{t \to \infty} \delta^{\alpha}(t) = 0 \). By Lemma 4.2 we have, for \( \bar{y}_i(t), \bar{y}_j(t) \) with \( d(\bar{y}_i(t), \bar{y}_j(t)) \in [0, t(\delta_{\gamma}^{\alpha}(t) \wedge \delta_{\gamma}^{\alpha}(t))) \), that

\[ \mathbb{E}_B(\bar{y}_i(t), \bar{y}_j(t)) = \bar{y}_i(t), \bar{y}_j(t)) \]

(5.9)
For (4.61) we proceed in the same way but instead of (5.10) we have, for \( d(y_i(t), y_j(t)) \in [0, t(\delta_1(t) \wedge \delta_0(t)) \] 

\[
\mathbb{E}_B (y_i(t), y_j(t)) = \sum^2 (d(y_i(t), y_j(t))) 
\leq \sum^2 (d(y_i(t), y_j(t))) 
= \sum^2 (d(x_i(t), x_j(t))) = \mathbb{E}_B (x_i(t), x_j(t)). \tag{5.11}
\]

If \( d(\bar{y}_i(t), \bar{y}_j(t)) \in [t(1 - \delta_1(t)), t] \), resp. \( d(y_i(t), y_j(t)) \in [t(1 - \delta_0(t)), t] \), we obtain in both cases from Lemma 4.2 that

\[
\mathbb{E}_B (\bar{y}_i(t), \bar{y}_j(t)) \geq \mathbb{E}_B (x_i(t), x_j(t)), \tag{5.12}
\]

and

\[
\mathbb{E}_B (y_i(t), y_j(t)) \leq \mathbb{E}_B (x_i(t), x_j(t)), \tag{5.13}
\]

respectively. This concludes the proof of Lemma 4.8.

\[\square\]

Proof of Lemma 4.6: Using that \( f \) is always bounded from above by 1, we get the upper bound

\[
\left| \mathbb{E}_n \left( \int_0^1 \overline{T(T)} dh \right) \right| \leq C \left| \mathbb{E}_n \left( \sum_{i,j=1}^{n(t)} \left[ \mathbb{E}_B (x_i(t), x_j(t)) - \mathbb{E}_B (\bar{y}_i(t), \bar{y}_j(t)) \right] \right) \right| \tag{5.14}
\]

\[
\times \int_0^1 \mathbb{E}_B \left( \mathbb{I}_{d(x_i(t), x_j(t)) \in T(T)} \mathbb{I}_{x_i(t), x_j(t) > \bar{m}(t) + \bar{u}} \right) dh \right|,
\]

for some constant \( C > 0 \). We introduce the shorthand notation

\[
A_1 = \Sigma^2 (s)/t \quad A_2 = 1 - \Sigma^2 (s)/t. \tag{5.15}
\]

The right hand side of (5.14) is essentially a constrained second moment estimate. We relax the tube condition except at the splitting time of the two particles, to get an explicit upper bound in the form

\[
C \left| \int \left[ \Sigma^2 (s) - \Sigma^2 (s) \right] e^{2t-s} \right| \tag{5.16}
\]

\[
\times \int_0^1 \int_{A_1 \bar{m}(t) + J(s, \gamma)}^{\infty} \mathbb{E}_n \left( \int_{\bar{m}(t) - \bar{u} - \frac{t^2}{\sqrt{2\pi A_2}}}^{\infty} e^{-\frac{x^2}{2\pi A_2}} \mathbb{I}_{x \in \mathcal{L}(\gamma)} \right) e^{-\frac{y^2}{2\pi A_1}} dy dx dh ds \right|,
\]

where

\[
J(s, \gamma) = \left( \Sigma^2 (s) \wedge (t - \Sigma^2 (s)) \right)^\gamma = ((A_1 \wedge A_2) t)^\gamma. \tag{5.17}
\]

We change variables in (5.16)

\[
y = z + A_1 \bar{m}(t) \tag{5.18}
\]

and obtain

\[
C \left| \int \left[ \Sigma^2 (s) - \Sigma^2 (s) \right] e^{2t-s} \right| \tag{5.19}
\]

\[
\times \int_0^1 \int_{-J(s, \gamma)}^{J(s, \gamma)} \left( \int_{A_2 \bar{m}(t) - \bar{u} - z}^{\infty} e^{-\frac{x^2}{2\pi A_2}} \mathbb{I}_{x \in \mathcal{L}(\gamma)} \right) e^{-\frac{y^2}{2\pi A_1}} dy dx dh ds \right|.
\]
Using the Gaussian tail bound (2.11), we have

\[
\left( \int_{A_2 \tilde{m}(t) - u < -z}^{\infty} e^{-\frac{s^2}{2A_2^2} - \frac{ds}{\sqrt{2\pi A_2}}} \right)^2 \leq \frac{A_2 t}{2\pi (A_2 \tilde{m}(t) - z - u <)^2} \exp \left( -\frac{(A_2 \tilde{m}(t) - z - u <)^2}{A_2 t} \right) \leq \frac{C'}{t} \exp \left( -\frac{(A_2 \tilde{m}(t) - z - u <)^2}{A_2 t} \right),
\]

(5.20)
since \( z \in [-J(s, \gamma), J(s, \gamma)] \).

Inserting (5.20) into (5.19) we see that the latter is bounded from from above by

\[
\frac{\hat{C}}{t} \left| \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] e^{2t - s} \int_0^1 \int_J^{J(s, \gamma)} e^{-\frac{(A_2 \tilde{m}(t) - z - u <)^2}{A_2 t} - \frac{s^2}{2A_1} \frac{ds}{\sqrt{2\pi A_1}}} \right| \]

(5.21)

\[
\leq \frac{\hat{C}}{t} \left| \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] e^{2t - s} \int_0^1 \int \tilde{m}(t)^2 \frac{A_1 A_2 \tilde{m}(t)}{A_1 + 1} + J(s, \gamma) e^{-\frac{1}{\sqrt{A_1 A_2}} \frac{1}{\sqrt{A_1 A_2}} \frac{ds}{\sqrt{2\pi A_1}}} \right|
\]

where we just changed variables

\[
z = w + \frac{A_1 A_2 \tilde{m}(t)}{A_1 + 1}.
\]

(5.22)

Since, for each \( h \in (0, 1) \),

\[
\frac{\sqrt{1 + A_1}}{\sqrt{t A_1 A_2}} \left( \frac{A_1 A_2 \tilde{m}(t)}{A_1 + 1} - J(s, \gamma) \right) \geq ((A_1 \land A_2) \tilde{m}(t))^{-1/2} \left( \frac{1}{4} (A_1 \land A_2) \tilde{m}(t) - (A_1 \land A_2) \gamma t \gamma \right),
\]

(5.23)

which tends to +\( \infty \), as \( t \uparrow \infty \), we can use the Gaussian tail bound (2.11) in \( w \)-integral to show that the right-hand side of (5.21) is bounded from above by

\[
\frac{\hat{C}}{t} \left| \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] e^{2t - s} \int_0^1 \frac{1}{e^{-\frac{\tilde{m}(t)^2}{t + A_1 \tilde{m}(t)}}} e^{-\frac{1}{2} \left( \frac{A_1 A_2 \tilde{m}(t)}{A_1 + 1} - J(s, \gamma) \right) \frac{1}{\sqrt{A_1 A_2}}} \right| dhds
\]

(5.24)

By the definition of \( J(s, \gamma) \) this is equal to

\[
\frac{\hat{C}}{t} \left| \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] e^{2t - s} \int_0^1 e^{-\frac{\tilde{m}(t)^2}{t + A_1 \tilde{m}(t)}} + O(s^\gamma) \right| dhds
\]

(5.25)

Since \( \tilde{m}(t)^2/t = 2t - \log t + O(\log(t)^2/t) \) we can rewrite (5.25) as

\[
\left| \hat{C} \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] \int_0^1 e^{-s + A_1 t + O(s^\gamma)} dhds \right|
\]

(5.26)

By the definition of \( A_1 \), this is equal to

\[
\left| \hat{C} \int \left[ \Sigma^2(s) - \Sigma^2(s) \right] \int_0^1 e^{-s + (h \Sigma^2(s) + (1-h) \Sigma^2(s) + O(s^\gamma)} dhds \right|
\]

(5.27)

This proves Lemma 4.6 \( \square \)
Proof of Lemma 4.7: We split the domain into three different parts. First let \( \delta_3 > 0 \) be such that
\[
\sigma_b^2 + \frac{K}{2} \delta_3 < 1 \quad \text{and} \quad \delta_3 < \delta_b.
\] (5.28)

By a Taylor expansion at zero we have
\[
\Sigma^2(s) \leq (\sigma_b^2 + \frac{K}{2} \delta_3)s, \quad \text{for} \quad s \in [0, \delta_3 t].
\] (5.29)

Moreover, if \( \delta_1 > 0 \), then so is \( \delta_0 \), and we then choose \( \delta_3 < \delta_0 \wedge \delta_1 \) (with \( \delta_1 \equiv \lim_{t \to \infty} \delta_1 \)); hence, for \( t \) large enough it then also holds that \( \delta_3 < \delta_0(t) \wedge \delta_1(t) \).

If \( \delta_1 = 0 \), we set (note that, by monotonicity, in this case \( \delta_0(t) \wedge \delta_1(t) = \delta_0(t) \))
\[
(S1) \equiv \int_{t \delta_0(t)}^{\delta_1(t)} \left( e^{-s+\Sigma^2(s)+O(s^2)} - e^{-s+\Sigma^2(s)+O(s^2)} \right) ds
\]
\[
\leq \int_{t \delta_0(t)}^{\delta_1(t)} \left( e^{-s(1-\sigma_b^2-K\delta_3)+O(s^2)} - e^{-s(1-\sigma_b^2-K\delta^\gamma(t)+O(s^2))} \right) ds.
\] (5.30)

By assumption on \( \delta_3, 1 - \sigma_b^2 - \frac{K}{2} \delta_3 > 0 \) and \( 1 - \sigma_b^2 - \frac{K}{2} \delta^\gamma(t) > 0 \), for all \( t \) sufficiently large. Hence
\[
\lim_{t \to \infty} (S1) = 0.
\] (5.31)

If \( \delta_1 > 0 \), we set \( (S1) = 0 \).

Next we choose \( \delta_4 \) such that
\[
\sigma_e^2 - \frac{K}{2} \delta_4 > 1 \quad \text{and} \quad \delta_4 < \delta_e.
\] (5.32)

Again due to a first order Taylor expansion we have
\[
\Sigma^2(t - \bar{s}) \leq t - (\sigma_e^2 - \frac{K}{2} \delta_4) \bar{s}, \quad \text{for} \quad \bar{s} \in [t \delta_1(t), t \delta_4]
\] (5.33)

Hence
\[
(S2) \equiv \int_{t \delta_1(t)}^{(1-\delta_2(t))} \left( e^{-s+\Sigma^2(s)+O(s^2)} - e^{-s+\Sigma^2(s)+O(s^2)} \right) ds
\]
\[
= \int_{t \delta_1(t)}^{\delta_4(t)} \left( e^{\bar{s}-t+\Sigma^2(t-\bar{s})+O(s^2)} - e^{\bar{s}-t+\Sigma^2(t-\bar{s})+O(s^2)} \right) d\bar{s}
\]
\[
\leq \int_{t \delta_1(t)}^{\delta_4(t)} \left( e^{\bar{s}(1-\sigma_e^2+\frac{K}{2} \delta_4)+O(s^2)} - e^{\bar{s}(1-\sigma_e^2+\frac{K}{2} \delta^\gamma(t)+O(s^2))} \right) d\bar{s}.
\] (5.34)

By assumption on \( \delta_4 \) we have \( 1 - \sigma_e^2 + \frac{K}{2} \delta_4 < 0 \) and, for \( t \) large enough, \( 1 - \sigma_e^2 + \frac{K}{2} \delta^\gamma(t) < 0 \). Hence
\[
\lim_{t \to \infty} (S2) = 0.
\] (5.35)

We still have to control
\[
(S3) \equiv \int_{\delta_4(t)}^{t-\delta_4(t)} \left( e^{-s+\Sigma^2(s)+O(s^2)} - e^{-s+\Sigma^2(s)+O(s^2)} \right) ds.
\] (5.36)

Consider the function \( A(x) \) on the interval \([\delta_3, 1 - \delta_4]\). Since \( A(x) \) is right-continuous, increasing and \( A(x) < x \) on \((0, 1)\), we know that
\[
M \equiv \inf_{x \in [\delta_3, 1 - \delta_4]} (x - A(x)) > 0.
\] (5.37)

Then
\[
s - \Sigma^2(s) = t(s/t - A(s/t)) \geq Mt,
\] (5.38)
which implies
\[ \int_{\delta t}^{t-\delta t} e^{-s+\Sigma^2(s)+O(s^{\gamma})} ds \leq e^{-Mt} \int_{\delta t}^{t-\delta t} e^{O(s^{\gamma})} ds, \] (5.39)
which tends to zero, as \( t \uparrow \infty \). By the same argument it follows that
\[ \lim_{t \uparrow \infty} \int_{\delta t}^{t-\delta t} e^{-s+\Sigma^2(s)+O(s^{\gamma})} ds = 0. \] (5.40)
It follows that \( \lim_{t \uparrow \infty} (S^3) = 0 \), which concludes the proof of Lemma 4.7.

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