On the large $N$ limit of the Schwinger-Dyson equation of tensor field theory

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Abstract

We analyze in this paper the large $N$ limit of the Schwinger-Dyson equations in tensor quantum field theory, which are derived with the help of Ward-Takahashi identities. In order to have a well-defined large $N$ limit, appropriate scalings in powers of $N$ for the various terms present in the action are explicitly found. A perturbative check of our results is done, up to second order in the coupling constant.

1 Introduction

Tensor models are witnessing a considerable regain of interest since the implementation of their large $N$ limit (see [1], [2, 3, 4, 5, 6] or the review articles [7] and [8]). Recently, tensor models have been related to the celebrated Sachdev-Ye-Kitaev AdS/CFT toy-model [9, 10] in [11] and [12] - see also [13, 14, 15, 16] and the lectures [17].

In [18] and [19], the Ward-Takahashi identity (WTI) has been extensively used in order to derive the tower of exact Schwinger-Dyson equations (SDE) for an $U(N)$-invariant tensor models whose kinetic part is modified to include a Laplacian-like operator (more exactly, this operator is a discrete Laplacian in the Fourier transformed space of the tensor index space). Let us emphasize here that this type of tensor model has been used as a test-bed for applying renormalization techniques to tensor models - see [20], the thesis [21] and references within.

Let us also mention here that the WTI has been already successfully used to study the SDE in the context of non-commutative quantum field theory - see [22] and [23]. In tensor models, a WTI appeared for the first time in [24], whose consequences are still under investigation [24].

This paper is a follow up of [18] and [19] in the sense that we study in detail the large $N$ limit of the SDE obtained via the use of WTI. We thus find appropriate scalings in powers of $N$ for the various terms present in the action of the model. Moreover, we analyse in detail a case where the boundary graph is disconnected (as explained in the following section, in tensor models, boundary graphs index the expansion of the free energy). This case has not been treated in [18] and [19].

Let us also mention here that in [26], scaling dimension for interactions in Abelian tensorial group field theories with a closure constraint have been obtained. However, the mathematical
Let us first consider a complex rank-3 bosonic tensor field theory with an action of the form

\[ S(\varphi, \bar{\varphi}) = S_0(\varphi, \bar{\varphi}) + S_{\text{int}}(\varphi, \bar{\varphi}) \]

\[ = \sum_{x} \varphi^3(x) |x|^2 \varphi^3(x) + \lambda \sum_{c=1}^{N} \varphi^{a_c} \varphi^{b_c} \varphi^{c_a}, \]

with \( x = (x_1, x_2, x_3) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^3 \), \( |x|^2 = x_1^2 + x_2^2 + x_3^2 \), \( \lambda = N^\delta \lambda \), \( a, b, c = (a_1, \ldots, a_{c-1}, b, a_{c+1}, \ldots, a_D) \) for \( D \)-tuple. Note that the interaction terms in the action, called pillow interaction terms, are invariant under the action of the group \( U(N)^3 \). These terms are also sometimes referred, in the tensor model literature, as melonic bubbles.

Let us emphasise here that the kinetic term above represents the discrete Laplacian in the Fourier transformed space of the tensor index space.

The generating functional of the model writes:

\[ Z[J, \bar{J}] = \int D\varphi D\bar{\varphi} \exp \left( -N^\gamma S(\varphi, \bar{\varphi}) + N^\beta \sum_{x} (J_{x} \varphi^x + \bar{J}_{x} \bar{\varphi}^x) \right). \]

Note that we have introduced here the scaling \( \delta, \gamma \) and \( \beta \) for the action and the source terms; these scalings will be determined in the sequel, using the SDE.

Our paper is organized as follows. In the following section we give the action of the tensor model we work with, and we recall tensor model tools used in the sequel, such as the boundary graph expansion of the free energy and the WTI. The third section is dedicated to the analysis of the scalings in powers of \( N \) of the various terms present in the action. Having a well-defined large \( N \) limit of the SDE imposes a series of constraints on these scalings. The following section treats in detail the case of the 4-point function with disconnected boundary graph. In the last section we find appropriate scalings in order to have a coherent large \( N \) limit of the SDE. In the appendix a perturbative expansion check of these results is performed up to second order.

## 2 The model and the tools

Let us first consider a complex rank-3 bosonic tensor field theory with an action of the form

\[ S(\varphi, \bar{\varphi}) = S_0(\varphi, \bar{\varphi}) + S_{\text{int}}(\varphi, \bar{\varphi}) \]

\[ = \sum_{x} \varphi^3(x) |x|^2 \varphi^3(x) + \lambda \sum_{c=1}^{N} \varphi^{a_c} \varphi^{b_c} \varphi^{c_a}, \]

with \( x = (x_1, x_2, x_3) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^3 \), \( |x|^2 = x_1^2 + x_2^2 + x_3^2 \), \( \lambda = N^\delta \lambda \), \( a, b, c = (a_1, \ldots, a_{c-1}, b, a_{c+1}, \ldots, a_D) \) for \( D \)-tuple. Note that the interaction terms in the action, called pillow interaction terms, are invariant under the action of the group \( U(N)^3 \). These terms are also sometimes referred, in the tensor model literature, as melonic bubbles.

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Note that we have introduced here the scaling \( \delta, \gamma \) and \( \beta \) for the action and the source terms; these scalings will be determined in the sequel, using the SDE.

The construction of boundary graphs is largely explained in [18] and [19]. Here we briefly recall this in Fig. 1 and Table 1. The free energy is written as an expansion over boundary graphs (see again [18] for more details):

\[ W[J, \bar{J}] = \sum_{k=1}^{\infty} \sum_{B \in \partial_{\text{int}} S_{\text{int}}} \sum_{\mathbb{X}(B)} \frac{N^\alpha(B)}{|\text{Aut}(B)|} G^{(2k)}(\mathbb{X}) \cdot J(B)(\mathbb{X}), \]

where \( \partial_{\text{int}} S_{\text{int}} \) is the set of boundary graphs associated to the interaction terms, \( V(B) \) is the number of vertices of \( B \), \( \mathbb{X} = (x_1, \ldots, x_n) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^3 \), \( \text{Aut}(B) \) is the symmetry group of the graph \( B \), and \( J(B)(\mathbb{X}) = J_{x_1} \ldots J_{x_n} \lambda_{1} \ldots \lambda_{n} \). Here \( \mathbb{X} = \mathbb{X}(B) \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}^3 \) is a momentum triplet determined by the boundary graph \( B \). For instance, for the boundary graph \( V_1 \) (see figure [1]), \( J(V_1)(x, y) = J_{x_1} J_{y_1} J_{x_1 y_2 y_3} J_{y_1 x_2 x_3} \).

Let us mention here that the coefficient \( \alpha(B) \) does not depend on the choice of colouring that can be made for the respective pillow term. For example, \( \alpha(V_1) = \alpha(V_2) = \alpha(V_3) \).
Figure 1: Two connected Feynman graphs and the associated boundary graphs. To each external leg of a Feynman diagram is associated an external vertex so that the open graph is bipartite. These vertices are exactly the vertices of the boundary graph. An edge of color \( c \) in the boundary graph, corresponds to a path between two external leg in the Feynman graph, which alternates between dotted lines and lines of colour \( c \). White and black vertex in a boundary graph \( B \) correspond in \( J(B) \) to the sources \( J \) and \( \bar{J} \) respectively. In the figure a) the boundary graph \( \partial V_1 \) is connected. In fig. b) the boundary graph \( \partial B \) is disconnected.

A \( 2k \)-point function for a connected boundary graph \( B \) is taken to be

\[
G^{(2k)}_B(X) = \frac{N^{-\alpha(B)}}{Z_0} \prod_{i=1}^k \left( \frac{\delta}{\delta J_{p_i}} \frac{\delta}{\delta J_{x^i}} \right) Z[J, \bar{J}] \bigg|_{J=J=0},
\]

where for all \( c \in \{1, 2, 3\} \) and \( (i, j) \in \{1, \ldots, k\}^2 \), \( x_i^c \neq x_j^c \).

Following \[18\], we use the WTI, which for rank-\( D \) tensors, writes:

\[
\sum_{q_a} \frac{\delta Z[J, \bar{J}]}{\delta J_{q_a m_a}} \delta_{m_a n_a} Y^{(a)}_{m_a}[J, \bar{J}]; Z[J, \bar{J}] = \frac{N^{\beta_{-2}}}{m_a^2 - n_a^2} \sum_{q_a} \left( J_{q_a m_a} \frac{\delta}{\delta J_{q_a n_a}} - J_{q_a n_a} \frac{\delta}{\delta J_{q_a m_a}} \right) Z[J, \bar{J}],
\]

Note that we have used here the notation: \( q_a = (q_1, \ldots, q_{a-1}, q_{a+1}, \ldots, q_D) \). The Y-term above is a functional given by

\[
Y^{(a)}_{m_a}[J, \bar{J}] = \delta_{m_a n_a} \sum_{q_a} \frac{\delta^2 W[J, \bar{J}]}{\delta J_{q_1 \ldots q_{a-1} m_a q_{a+1} \ldots q_D} \delta J_{q_1 \ldots q_{a-1} n_a q_{a+1} \ldots q_D}}
= \sum_{B \in \partial S_{\text{int}}} \sum_X f_a(X; m_a; B; J(B)(X)),
\]

with

\[
f_a(X; m_a; B) = \sum_{\hat{\pi} \in \text{Aut}_c(B)} (\pi^* f^{(a)}_{\hat{B}, m_a})(X),
\]

where \( \pi \) is the restriction of the automorphism \( \hat{\pi} \) to the white vertices of \( B \), \( (\pi^* f)(x^1, \ldots, x^k) = f(\pi^{-1}(x^1), \ldots, \pi^{-1}(x^k)) \) and \( f_a(X; m_a; B) \) is the function-coefficient of \( J(B)(X) \) in the graph expansion of the Y-term. For the pillow graphs \( V_a \), equation (7) states that

\[
f_a(x, y; m_a; V_a) = \sum_{\hat{\pi} \in \mathbb{Z}_2} (\pi^* f^{(a)}_{V_a, m_a})(x, y)
= f^{(a)}_{V_a, m_a}(x, y) + f^{(a)}_{V_a, m_a}(y, x).
\]

Here we are only interested in the explicit coefficients up to order four. In the following equations, \( \{a, b, c\} = \{1, 2, 3\} \), an automatic reordering of the entries by ascending sub-index is implied, and
we omit the powers in $N$ associated to each Green’s function. One thus has:

\[
\begin{align*}
\hat{f}_{m,s}^{(a)}(x) &= G_a^{(4)}(x, s_a, x_b, x_c) + \sum_{c \neq a} \sum_{q_b \in \mathbb{Z}} G_c^{(4)}(x; s_a, q_b, x_c) + \sum_{q_b, q_c} G_m^{(4)}(x, s_a, q_b, q_c), \\
\hat{f}_{V_a,s}^{(a)}(x, y) &= \frac{1}{3} \left( G_a^{(6)}(s_a, x_b, x_c, x, y) + \text{cyclic perm.} \right) + \frac{1}{3} \left( G_b^{(6)}(s_a, x_b, y_c; x, y) + \text{cyclic perm.} \right) + \frac{1}{3} \left( G_c^{(6)}(s_a, x_b, x_c; x, y) + \text{cyclic perm.} \right) + \frac{1}{2} \sum_{q_b, q_c} G_{F_{a,b}}^{(6)}(s_a, q_b, q_c; x; y), \\
\hat{f}_{V_a,s}^{(a)}(x, y) &= \frac{1}{3} \left( \sum_{q_b} G_{Q_a}^{(6)}(s_a, q_b, y_c; x, y) + \text{cyclic perm.} \right) + \frac{1}{2} \sum_{q_b, q_c} G_{F_{a,b}}^{(6)}(s_a, q_b, q_c; x; y), \\
\hat{f}_{V_a,s}^{(a)}(x, y) &= \frac{1}{3} \left( \sum_{q_b} G_{Q_b}^{(6)}(s_a, q_b, y_c; x, y) + \text{cyclic perm.} \right) + \frac{1}{2} \sum_{q_b, q_c} G_{F_{a,b}}^{(6)}(s_a, q_b, q_c; x; y), \\
\hat{f}_{V_a,s}^{(a)}(x, y) &= \left( \sum_{q_b, q_c} G_{Q_a}^{(6)}(s_a, q_b, q_c, x, y) + \text{cyclic perm.} \right) + G_{F_{a,b}}^{(6)}(x, s_a, x_b, y_c, y), \\
\hat{f}_{V_a,s}^{(a)}(x, y) &= \left( \sum_{q_b} G_{Q_b}^{(6)}(s_a, q_b, q_c, x, y) + \text{cyclic perm.} \right) + G_{F_{a,b}}^{(6)}(x, s_a, x_b, y_c, y), \\
\hat{f}_{m[s]}^{(a)}(x, y) &= \left( \sum_{q_b, q_c} G_{m[V_a]}^{(6)}(s_a, q_b, q_c, x, y) + \text{cyclic perm.} \right) + G_{F_{a,b}}^{(6)}(x, s_a, x_b, y_c, y) \\
&+ \sum_{q_b} G_{m[V_b]}^{(6)}(x, s_a, y_b, y_c) + \sum_{q_c} G_{m[V_c]}^{(6)}(x, s_a, y_b, y_c) + \sum_{q_b} G_{m[V_b]}^{(6)}(x, s_a, y_b, y_c) + \frac{1}{2} \sum_{q_b, q_c} G_{m[V_a]}^{(6)}(x, y, s_a, y_b, y_c). \tag{14}
\end{align*}
\]

Note that, except for equation (14), the other equations were already present in [19].
3 Constraints on the scalings in \( N \)

3.1 2-point function SDE

In this subsection, we start with the explicit definition of the 2-point function, and using the WTI to obtain SDE, we finally get a set of inequalities between the scaling coefficients \( \alpha, \beta, \gamma \) and \( \delta \).

The 2-point function explicitly writes

\[
G^{(2)}(x) = \frac{N^{-\alpha} \delta^2 Z[J, \bar{J}]}{Z_0} \exp \left( -N^\gamma S_{\text{int}} \left[ \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta J} \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta \bar{J}} \right] \right) \exp \left( N^{2\beta-\gamma} \sum_x \frac{J_x \bar{J}_x}{|x|^2} \right) \bigg|_{J=\bar{J}=0}
\]

\[
= \frac{N^{-\alpha} \delta^2}{Z_0} \frac{\delta}{\delta J} \frac{\delta}{\delta \bar{J}} \exp \left( -N^\gamma S_{\text{int}}^\theta \right) \left[ \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta J} \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta \bar{J}} \right] \exp \left( N^{2\beta-\gamma} \sum_x \frac{J_x \bar{J}_x}{|x|^2} \right) \bigg|_{J=\bar{J}=0}
\]

\[
= \frac{N^{2\beta-\gamma-\alpha}}{|p|^2} + \frac{N^{2\beta-\gamma-\alpha}}{Z_0} \exp \left( -N^\gamma S_{\text{int}}^\theta \right) \left[ \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta J} \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta \bar{J}} \right] \exp \left( N^{2\beta-\gamma} \sum_x \frac{J_x \bar{J}_x}{|x|^2} \right) \bigg|_{J=\bar{J}=0},
\]

where we note \( F^\theta = F \left[ \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta J}, \frac{1}{N^{2\beta-\gamma}} \frac{\delta}{\delta \bar{J}} \right] \). In order for the free propagator to be dominant in the large \( N \) limit, one has:

\[
\alpha = 2\beta - \gamma.
\]

To simplify the equations, we consider first the contribution of the pillow interaction \( V_1 \) and we then add the analogous contributions coming from the contributions of the pillow interactions \( V_2 \) and \( V_3 \). One has:

\[
N^\gamma \left( \frac{\partial S_{\text{int}}}{\partial \phi^x} \right)^\theta = 2\lambda N^{5\gamma+\delta} \sum_a \frac{\delta}{\delta J_{x_1x_2x_3}} \frac{\delta}{\delta J_{a_1a_2a_3}} \frac{\delta}{\delta J_{x_1a_2a_3}} \frac{\delta}{\delta J_{x_1a_2a_3}}.
\]

Using the WTI for the two rightmost derivatives in the expression (17) enables us to write:

\[
N^\gamma \left( \frac{\partial S_{\text{int}}}{\partial \phi^x} \right)^\theta \bigg|_{J=\bar{J}=0} = \frac{2\lambda N^{5\gamma+\delta}}{N^{8\beta}} \sum_a \frac{\delta}{\delta J_{a_1a_2a_3}} \frac{\delta}{\delta J_{x_1a_2a_3}} \left\{ \left( \frac{\delta_{x_1a_1} Y_{a_1}^{(1)}[J, \bar{J}]}{\delta J_{a_1a_2a_3}} \right) \cdot Z[J, \bar{J}] \right\} \bigg|_{J=\bar{J}=0}
\]

\[
+ \sum_{a_2, a_3} \frac{N^{3\beta-2\gamma}}{a_1^2 - x_1^2} \left( J_{a_1a_2a_3} \frac{\delta}{\delta J_{x_1a_2a_3}} - J_{x_1a_2a_3} \frac{\delta}{\delta J_{a_1a_2a_3}} \right) Z[J, \bar{J}] \bigg|_{J=\bar{J}=0}.
\]

Acting with the two remaining derivatives in (17) on the second term on the RHS of (18), and using (16), we get:

\[
\frac{2\lambda}{N} \sum_{a_1} \frac{N^{2\gamma+\delta+1-3\beta}}{a_1^2 - x_1^2} \left( G^{(2)}(x) - G^{(2)}(a_1, x_2, x_3) \right).
\]

For this term to give a well defined large \( N \) limit we need the following relation:

\[
3\beta \geq 2\gamma + \delta + 1.
\]

Note that if the inequality (20) is taken to be an equality, then the term (19) is a leading order term in the large \( N \) limit.
Acting with the remaining derivative on the factor $Z[J, \bar{J}]$ of the first term of the RHS of (18) gives:

\[
\frac{2\lambda}{N^{8\beta - 5\gamma - \delta}} \delta_{x_1a_1} Y_{a_1}^{(1)}[0, 0] G^{(2)}(x) = \frac{2\lambda N^{3\gamma + 2 + \delta - 4\beta}}{N^2} \sum_{a_2, a_3} G^{(2)}(x_1, a_2, a_3) G^{(2)}(x).
\]

(21)

This term implies a new inequality on the exponents:

\[4\beta \geq 3\gamma + \delta + 2.\]

(22)

Acting now with these remaining derivatives on the factor $Y_{a_1}^{(1)}[J, \bar{J}]$ of the first term of the RHS of (18) gives:

\[
\delta_{x_1a_1} \frac{\delta Y_{a_1}^{(1)}[J, \bar{J}]}{\delta J x \delta J a_1 x_2 x_3} \bigg|_{J = \bar{J} = 0} = N^{\alpha(V_1)} G_1^{(4)}(x, x) + \frac{N^{\alpha(m|m) + 2}}{N^2} \sum_{a_2, a_3} G_1^{(4)}(x, x_1, a_2, a_3).
\]

(23)

Putting these terms together, we obtain the SDE for the 2-point function:

\[
G^{(2)}(x) = \frac{1}{|x|^2} - \frac{2\lambda}{|x|^2} \left( \frac{N^{3\gamma + 2 + \delta - 4\beta}}{N^2} \sum_{a_2, a_3} G^{(2)}(p_1, a_2, a_3) G^{(2)}(x) + \frac{N^{\alpha(V_1)}}{N^{8\beta - 5\gamma - \delta}} G_1^{(4)}(x, x) \right) + \frac{N^{\alpha(m|m) + 2}}{N^2} \sum_{a_2, a_3} \frac{G_1^{(4)}(x, x_1, a_2, a_3)}{x_1^2 - a_1^2} \left( G^{(2)}(a_1, x_2, x_3) - G^{(2)}(x) \right).
\]

(24)

For the 4-point function to be sub-leading in the large $N$ limit taken in (24) above, we need to impose the following two inequalities on the exponents:

\[\alpha(V_1) < 8\beta - 5\gamma - \delta,\]

(25)

\[\alpha(m|m) < 8\beta - 5\gamma - \delta - 2.\]

(26)

As announced above, we now add the contributions coming from the 2nd and 3rd pillow interaction terms, $V_2$ and $V_3$, of the action. We then get:

\[
G^{(2)}(x) = \frac{1}{|x|^2} - \frac{2\lambda}{|x|^2} \sum_{a_1=1}^3 \left( \frac{N^{3\gamma + 2 + \delta - 4\beta}}{N^2} \sum_{q_a} G^{(2)}(q_a x_a) G^{(2)}(x) + \frac{N^{\alpha(V_1)}}{N^{8\beta - 5\gamma - \delta}} G_a^{(4)}(x, x) \right) + \frac{N^{\alpha(m|m) + 2}}{N^2} \sum_{q_a} \frac{G_a^{(4)}(x, x_1, q_a, x_2, x_3)}{x_1^2 - q_a^2} \left( G^{(2)}(q_a x_2 q_a) - G^{(2)}(x) \right) + \frac{N^{\alpha(V_1) + 1}}{N} \sum_{c \neq a} \sum_{q_b} \frac{G_c^{(4)}(x, x_b, q_b)}{N^2},
\]

(27)

where in the last term $b \neq c$ and $b \neq a$. This last term leads to a stronger condition than (25).

This condition writes:

\[\alpha(V_1) < 8\beta - 5\gamma - \delta - 1.\]

(28)

Moreover for $a_1 \neq x_1$ and using (16), the WTI implies

\[
N^{5\beta - 3\gamma} \frac{G^{(2)}(a_1, x_2, x_3) - G^{(2)}(x)}{x_1^2 - a_1^2} = N^{4\beta - 2\gamma} G^{(2)}(a_1, x_2, x_3) G^{(2)}(x) + \frac{N^{\alpha(V_1) + 2}}{N^2} \sum_{a_2, a_3} G^{(4)}_1(a, x) + \frac{N^{\alpha(V_1) + 1}}{N} \left( \sum_{a_3} G_2^{(4)}(x, a_1, x_2, a_3) + \sum_{a_2} G_3^{(4)}(x, a_1, a_2, x_3) \right).
\]

(29)
This identity rewrites as
\[
\frac{G^{(2)}(a_1, x_2, x_3) - G^{(2)}(x)}{x_1^2 - a_1^2} = \frac{1}{N^{\beta - \gamma}} G^{(2)}(a_1, x_2, x_3) G^{(2)}(x) + \frac{N^{\alpha(V_1) + 2}}{N^{5\beta - 3\gamma}} \frac{1}{N^2} \sum_{a_2, a_3} G^{(4)}_{a_2, a_3} (a, x)
\]
+ \frac{N^{\alpha(V_1) + 1}}{N^{5\beta - 3\gamma}} \left( \frac{1}{N} \sum_{a_3} C^{(4)}_{x_2}(x, a_1, x_2, a_3) + \frac{1}{N} \sum_{a_2} C^{(4)}_{x_3}(x, a_1, a_2, x_3) \right),
\]
(30)

which implies the following two inequalities:
\[
\beta \geq \gamma, \quad \alpha(V_1) \leq 5\beta - 3\gamma - 2.
\]

### 3.2 2k-point function SDE for connected boundary graphs

In this subsection we start with the definition of the 2k-point function, and as above, we use WTI to obtain the SDE. This finally leads to a new inequality between the scaling coefficients.

From now on we consider altogether the contributions coming from the three pillow interactions $V_1$, $V_2$ and $V_3$. Let us recall that the 2k-point function for a connected boundary graph $B$ is:
\[
G^{(2k)}_B(X) = \frac{N^{-\alpha(B)}}{Z_0} \prod_{i=1}^k \left( \frac{\delta}{\delta J_{p_i} \delta J_{\bar{p}_i}} \right) Z[J, \bar{J}] \Big|_{J=\bar{J}=0}.
\]
(33)

Following [19], in order to obtain then SDE, we first consider the term:
\[
\frac{\delta W[J, \bar{J}]}{\delta J_s} = \frac{N^{2\beta - \gamma}}{Z[J, \bar{J}]} \exp \left( -N^\gamma S^{\partial}_{\text{int}} \right) \frac{J_s}{|s|^2} \exp \left( N^{2\beta - \gamma} \sum_a \bar{J}_a J_a \right) - \frac{1}{|s|^2} \frac{N^{2\beta}}{Z[J, \bar{J}]} \frac{\delta S^{\partial}_{\text{int}}}{\delta \bar{p}_s} \frac{\partial}{\partial \bar{p}_s} Z[J, \bar{J}].
\]
(34)

Note that here $s$ is an unspecified vector of indices. The WTI enables us to write:
\[
N^{2\beta} \left( \frac{\delta S^{\partial}_{\text{int}}}{\delta \bar{p}_s} \right) \frac{\partial}{\partial \bar{p}_s} Z[J, \bar{J}] = \frac{2\lambda N^{4\gamma + \delta}}{N^{6\beta}} \sum_{a=1}^3 \sum_{b_a} \frac{\delta}{\delta J_s} \sum_{b_a} \frac{\delta}{\delta J_{b_a}} \sum_{b_a} \frac{\delta}{\delta J_{b_a}} \delta Z[J, \bar{J}]
\]
\[
= \frac{2\lambda N^{4\gamma + \delta}}{N^{6\beta}} \sum_{a=1}^3 \left( \frac{\delta}{\delta J_s} \sum_{b_a} \frac{J_s}{b_a^2 - s_a^2} \delta^2 Z[J, \bar{J}] \right) + \sum_{b_a} \frac{\delta}{\delta J_{b_a}} \sum_{b_a} \frac{\delta}{\delta J_{b_a}} \delta Z[J, \bar{J}]
\]
\[
- N^{3\beta - 2\gamma} \sum_{b} \frac{J_{b_a}}{b_a^2 - s_a^2} \delta^2 Z[J, \bar{J}] + N^{3\beta - 2\gamma} \sum_{b} \frac{\bar{J}_b}{b_a^2 - s_a^2} \delta^2 Z[J, \bar{J}]
\]
(35)

For $s = p^1$, recalling that
\[
Y^{(a)}_{p^1} [0, 0] = N^{\alpha} \sum_{q_b} G^{(2)}(q_b p^1_b),
\]
we apply the remaining $2k - 1$ derivatives of (33) to (35). This leads to the SDE for a 2k-point function.
with a connected boundary graph:

\[
G_{\mathcal{B}}^{(2k)}(X) = -\frac{2\lambda}{|p|^2} \sum_{a} \left\{ \frac{N^{3\gamma+2+\delta-4\beta}}{N^2} \sum_{q_a} G^{(2)}(q_a p_a^1) G_{\mathcal{B}}^{(2k)}(X) + \frac{N^{4\gamma+6-6\beta}}{N^2 f_a} \right\}
\]

where \(\mathcal{X}\) corresponds to the only white vertex such that \(x_a^\gamma = s_a\) and \(\mathcal{Z}_{\alpha}(B; 1, \rho)\) is the graph obtained by swapping the \(a\)-coloured lines between \(p_1^1\) and \(p_1^\rho\) (see figure 2 and figure 3).

The first term of the RHS of (37) gives a well-defined large \(N\) limit if (22) is satisfied. The terms of the second line of (37) require (20). The terms contributing to \(f_a(X; p_a^1; B)\) for \(V(B) = 2k\) are \(2k+1\)-point functions with at most two sums on dummy variables. Hence to get a well defined large \(N\) limit we need:

\[
\alpha(B) \geq \alpha(B') + 2 + 4\gamma + \delta - 6, \tag{38}
\]

with \(V(B') = 2k + 2\). If the inequality is strict, the \(2k+1\)-point function terms in the SDE for the \(2k\)-point function are sub-leading and the tower of SDE decouples at leading order.

For the 4-point function and for \(s = (x_1, y_2, y_3)\), the general equation (37) gives

\[
G^{(4)}_{\mathcal{B}}(x, y) = -\frac{2\lambda}{|x|^2} \left\{ \sum_{a=1}^{3} f_a(x, y; s_a; V_a) + \sum_{a=1}^{3} \sum_{q_a} G^{(2)}(q_a s_a) G^{(4)}_{\mathcal{B}}(x, y) \right\}
\]

\[
+ \frac{1}{N} \sum_{b_1} N^{2\gamma+1-3\beta} \frac{1}{b_1^2 - x_1^2} \left( G^{(4)}_{\mathcal{B}}(x, y) - G^{(4)}_{\mathcal{B}}(b_1, x_2, x_3, y) \right)
\]

\[
+ \frac{1}{N} \sum_{b_2} N^{2\gamma+1-3\beta} \frac{1}{b_2^2 - y_2^2} \left( G^{(4)}_{\mathcal{B}}(x, y) - G^{(4)}_{\mathcal{B}}(x, y_1, b_2, y_3) \right)
\]

\[
+ \frac{1}{N} \sum_{b_3} N^{2\gamma+1-3\beta} \frac{1}{b_3^2 - y_3^2} \left( G^{(4)}_{\mathcal{B}}(x, y) - G^{(4)}_{\mathcal{B}}(x, y_1, y_2, b_3) \right)
\]

\[
+ \frac{N^{2\gamma+1-3\beta} N^{2a}}{y_1^2 - x_1^2} \left( G^{(2)}(y) - G^{(2)}(y_1, x_2, x_3) \right)
\]

\[
+ \frac{N^{2\gamma+1-3\beta} N^{2a}}{y_1^2 - x_1^2} \left( G^{(2)}(y) - G^{(2)}(x, y_1, x_2, x_3) \right).
\tag{39}
\]
Using (16), the eighth term in (39) leads to

\[ \alpha(V_1) \geq \beta + \delta, \quad (40) \]

The last term of (39) leads to

\[ \alpha(V_1) \geq \alpha(m|m) + 2\gamma + \delta - 3\beta. \quad (41) \]

Moreover, the fourth and sixth terms imply

\[ 3\beta \geq 2\gamma + \delta, \quad (42) \]

which must be a strict inequality to be consistent with (20). Hence these two terms are sub-leading in the large \( N \) limit.

4 The 4-point function SDE with disconnected boundary graph

In this section, we apply the same approach for the 4-point function SDE with disconnected boundary graph. As already mentioned in the introduction, this case was not considered in [19].

The 4-point function with a disconnected boundary graph writes

\[ G_{m|m}^{(4)}(x, y) = \frac{1}{N^{\alpha(m|m)}} \frac{\delta^4 W[J, \bar{J}]}{\delta J_y \delta J_y \delta J_x \delta J_x} \bigg|_{J=\bar{J}=0}, \quad (43) \]

Let us start from (34) with \( s = x \) and where we applied the three remaining derivatives

\[ \frac{\delta^4 W[J, \bar{J}]}{\delta J_y \delta J_y \delta J_x \delta J_x} = -N^{2\beta} \frac{\delta^2}{|x|^2} \frac{1}{Z[J, \bar{J}]} \frac{\delta}{\delta J_y} \left( \frac{\delta S_{\text{int}}}{\delta \varphi^x} \right)^\partial Z[J, \bar{J}]. \quad (44) \]

For a connected boundary graph, all the derivatives give a vanishing contribution when applied to \( \frac{1}{Z[J, \bar{J}]} \). For the disconnected boundary graph case we treat here, one has:

\[ \frac{\delta^4 W[J, \bar{J}]}{\delta J_y \delta J_y \delta J_x \delta J_x} = -N^{2\beta} \frac{1}{|x|^2} \frac{\delta^3}{Z[J, \bar{J}]} \frac{\delta}{\delta J_y} \left( \frac{\delta S_{\text{int}}}{\delta \varphi^x} \right)^\partial Z[J, \bar{J}] \]

\[ + \frac{N^{2\beta}}{|x|^2} \frac{\delta Z[J, \bar{J}]}{Z[J, \bar{J}]} \frac{\delta}{\delta J_x} \left( \frac{\delta S_{\text{int}}}{\delta \varphi^x} \right)^\partial Z[J, \bar{J}]. \quad (45) \]
The first line is the same as in the case of a connected boundary graph, the second line is a new type of term. As above, the WTI leads to the first term below:

\[
\frac{1}{Z_0} \left. \frac{\delta^4 \left( Y^{(a)}_{x\bar{a}} [J, \bar{J}] \cdot Z[J, \bar{J}] \right)}{\delta J_y \delta J_y \delta J_x \delta J_x} \right|_{J=\bar{J}=0} = N^\alpha \sum_{q_a} G^{(2)}(q_a x \bar{a}) \left( N^{\alpha(m|m)} G^{(4)}_{m|m}(x, y) + N^{2\alpha} G^{(2)}(x) G^{(2)}(y) \right) + N^\alpha G^{(2)}(x) \left. \frac{\delta^2 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta J_y \delta J_y} \right|_{J=\bar{J}=0} + \frac{\delta^4 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta (m|m)} \left|_{J=\bar{J}=0} \right.,
\]

(46)

where

\[
\frac{\delta^4 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta (m|m)} \bigg|_{J=\bar{J}=0} = \delta^{(a)}_{m|m, x\bar{a}} (x, y) + \overline{\delta^{(a)}_{m|m, x\bar{a}} (y, x)},
\]

(47)

\[
\frac{\delta^2 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta J_x \delta J_x} \bigg|_{J=\bar{J}=0} = \delta^{(a)}_{m, x\bar{a}} (x),
\]

(48)

\[
\frac{\delta^2 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta J_y \delta J_y} \bigg|_{J=\bar{J}=0} = \delta^{(a)}_{m, x\bar{a}} (y).
\]

(49)

This term corresponds to the first term in (35). We also need to compute the contribution from the swapping (the term corresponding to the last term of (35)). This writes

\[
\frac{1}{Z_0} \left. \frac{\partial Z[J, \bar{J}]}{\partial \zeta[m|m, 1, 2]} \right|_{J=\bar{J}=0} = N^{\alpha(V_1)} G^{(4)}_{a}(x, y),
\]

(50)

Finally, the contribution from the two remaining terms of (35) writes

\[
- N^{3\beta - 2\gamma} \sum_{b} \frac{\delta^3}{\delta J_y \delta J_y \delta J_x} \left( J_{ba x\bar{a}} \right) \left. \frac{\delta^2 Z[J, \bar{J}]}{\delta J_y \delta J_y \delta J_x} \right|_{J=\bar{J}=0} - \frac{1}{N} \sum_{b} N^{3\beta - 2\gamma + 1} \left( N^{\alpha(m|m)} G^{(4)}_{m|m}(x_a b_{a\bar{a}}, y) + N^{2\alpha} G^{(2)}(x_{a\bar{a}}, y) G^{(2)}(y) + N^{\alpha(a)} \delta^{b_{a\bar{a}}} G^{(4)}_{a}(x_{a\bar{a}}, y) \right),
\]

(51)

\[
\sum_{b} N^{3\beta - 2\gamma + 1} \frac{1}{N} \left. \frac{\delta^4 Z[J, \bar{J}]}{\delta J_y \delta J_y \delta J_y \delta J_x} \right|_{J=\bar{J}=0} = \frac{1}{N} \left( N^{\alpha(m|m)} G^{(4)}_{m|m}(x, y) + N^{2\alpha} G^{(2)}(x) G^{(2)}(y) \right).
\]

(52)

Let us note here that in (51), we obtain not only a contribution coming from the disconnected 4-point function, but also a supplementary contribution as a product of 2-point functions. These products of 2-point functions and the term

\[
G^{(2)}(y) \frac{\delta^2 Y^{(a)}_{x\bar{a}} [J, \bar{J}]}{\delta J_y \delta J_y},
\]

(53)

give rise to disconnected Feynman graphs because the dependence in momenta factorises. They should not appear in a connected Green’s function, hence they need to be compensated.

They will be cancelled by the term coming from the second line of (45). This will give us new relations on the exponents of \( N \). Noting that

\[
\frac{\delta}{\delta J_x} \left( \frac{\delta S_{\text{int}}}{\delta \phi} \right) \frac{\partial}{\partial \phi} Z[J, \bar{J}] = \left( \frac{\delta S_{\text{int}}}{\delta \phi} \right) \frac{\partial}{\partial \phi} Z[J, \bar{J}],
\]

(54)
we already have computed these terms in the SDE for the 2-point function. Indeed, all the terms proportional to $\hat{\lambda}$ in the SDE for the 2-point function are multiplied by

$$-\frac{N^{2\gamma-\gamma} \delta^2 Z[J,\bar{J}]}{N^{\alpha(m|m)} \delta J_x \delta J_y} \tag{55}$$

to obtain the contribution from the second line of (45) in the SDE for the 4-point function with a disconnected boundary graph. This writes:

$$\frac{2\hat{\lambda}}{|x|^2} \frac{G(2)\langle y\rangle}{N^{\alpha(m|m)}} \sum_{a=1}^{3} \left( \frac{N^{3\gamma+2+\delta-4\beta}}{N^2} \sum_{q_a} G(2)\langle q_a x_a \rangle \frac{G(2)}{N^{8\beta-3\gamma-\delta}} + \frac{f^{(a)}_{m,x_a}}{N^{\beta+\delta+1}} \right) \tag{56}$$

Collecting all the terms above and again making use of (16), we get

$$G^{(4)}_{m|m}(x,y) = \frac{2\hat{\lambda}}{|x|^2} \sum_{a=1}^{3} \left( \frac{1}{N^2} \sum_{q_a} G(2)\langle q_a x_a \rangle \left( \frac{G^{(4)}_{m|m}(x,y)}{N^{4\beta-3\gamma-3\delta-2}} + \frac{N^{3\gamma+2+\delta-4\beta}}{N^{\alpha(m|m)}} \frac{G(2)}{N^{2\gamma-\gamma}} \right) \right)$$

$$- \frac{1}{N} \sum_{q_a} \frac{1}{q_a^2 - x_a^2} \left( \frac{N^{3\gamma+\delta-3\beta+1}}{N^{\alpha(m|m)}} G^{(4)}_{m|m}(x_a q_a, y) + \frac{N^{\beta+\delta+1}}{N^{\alpha(m|m)}} G(2)(x_a q_a) G(2)(x) \right)$$

$$+ \frac{1}{N} \sum_{q_a} \frac{1}{q_a^2 - x_a^2} \left( \frac{N^{3\gamma+\delta-3\beta+1}}{N^{\alpha(m|m)}} G^{(4)}_{m|m}(x,y) + \frac{N^{\beta+\delta+1}}{N^{\alpha(m|m)}} G(2)(x) G(2)(y) \right)$$

$$- \frac{N^{\alpha(V_1)+2\gamma+\delta-3\beta}}{N^{\alpha(m|m)}} \frac{1}{y_a^2 - x_a^2} \left( G^{(4)}(x,y) - G^{(4)}(x_a y_a, x^2) \right)$$

$$+ \frac{N^{\beta+\delta+1}}{N^{\alpha(m|m)}} \left( f^{(a)}_{m|m,x_a^1} (x, y) + f^{(a)}_{m|m,x_a^1} (y, x) + \frac{N^{3\gamma+2+\delta-4\beta}}{N^{\alpha(m|m)}} G(2)(x) f^{(a)}_{m,x_a} (x) \right)$$

$$- \frac{N^{\alpha(V_1)+2\gamma+\delta-3\beta}}{N^{\alpha(m|m)}} \frac{1}{y_a^2 - x_a^2} \left( G^{(4)}(x_a q_a, y) - G^{(2)}(x) \right)$$

$$- \frac{N^{3\gamma+\delta+1-3\delta}}{N^{\alpha(m|m)}} \frac{1}{N} \sum_{q_a} \frac{1}{q_a^2 - x_a^2} \left( G^{(2)}(x_a q_a) - G^{(2)}(x) \right) \right) \tag{57}$$

Let us determine the conditions on the exponents for the disconnected term to be cancelled. We have the following three identities:

$$\frac{N^{3\gamma+\delta-4\beta}}{N^{\alpha(m|m)}} G(2)(x^2) f^{(a)}_{m,x_a^1} (x^1) = \frac{N^{5\gamma+\delta-8\beta}}{N^{\alpha(m|m)}} G(2)(x^2) f^{(a)}_{m,x_a^1} (x^1), \tag{58}$$

$$\frac{N^{\beta+\delta+1}}{N^{\alpha(m|m)}} G(2)(x^1) G(2)(x^2) = \frac{N^{3\gamma+2+\delta-4\beta}}{N^{\alpha(m|m)}} G(2)(x^1) G(2)(x^2), \tag{59}$$

$$\frac{N^{\beta+\delta+1}}{N^{\alpha(m|m)}} G(2)(x^2) G(2)(x_a^1 q_a) - G^{(2)}(x^2) \frac{1}{E(x_a^1, q_a)} = \frac{N^{3\gamma+\delta+1-3\delta}}{N^{\alpha(m|m)}} G(2)(x^2) G(2)(x_a^1 q_a) - G^{(2)}(x^1). \tag{60}$$

Each of these identities leads to the condition:

$$2\beta = \gamma. \tag{61}$$
The SDE for the 4-point function with a disconnected boundary graph then writes:

\[
G^{(4)}_{m|m}(x, y) = -\frac{2\lambda}{|x|^2} \sum_{a=1}^{3} \left\{ \frac{1}{N^2} \sum_{\mathbf{q}_a} G^{(2)}(\mathbf{q}_ax) \left( \frac{G^{(4)}_{m|m}(x, y)}{N^{4\beta-3\gamma-\delta-2}} \right) + \frac{N^{2\gamma+\delta-6\beta}}{N^{\alpha(m|m)}} \left( f^{(a)}_{m|m,x_a}(x, y) + f^{(a)}_{m|m,x_a}(y, x) \right) + \frac{1}{N} \sum_{\mathbf{q}_a} N^{2\gamma+\delta-3\beta+1} E(x_a, q_a) \left( G^{(4)}_{m|m}(x,aq_a, y) - G^{(4)}_{m|m}(x, y) \right) + \frac{N^{\alpha(V_1)+2\gamma+\delta-3\beta}}{N^{\alpha(m|m)}} \left( G^{(4)}_a(x, y) - G^{(4)}_a(x, y) \right) + 2 \left( G^{(4)}_a(x, y) - G^{(4)}_a(x, y) \right) \right\}. \]  

(62)

The first term of the RHS requires again \([22]\); the third term gives again \([20]\). Then, the fourth term gives the relation:

\[
\alpha(m|m) \geq \alpha(V_1) + 2\gamma + \delta - 3\beta. \]  

(63)

To obtain relations on the exponents from the last term we need the following expression

\[
f^{(a)}_{m|m,x_a}(y) = \frac{N^{\alpha(V_1)}}{N^{\alpha(m|m)}} G^{(4)}_a(y, x, a, y) + \frac{N^{\alpha(V_1)+1}}{N} \sum_{c\neq a} \sum_{\mathbf{q}_c} G^{(4)}_c(y, x, a, q_c) + \frac{N^{\alpha(m|m)+2}}{N^2} \sum_{\mathbf{q}_a,\mathbf{q}_c} G^{(4)}_{m|m}(y, x, a, q_c). \]  

(64)

From the first term of this equation we recover the same relation between \(\alpha(m|m)\) and \(\alpha(V_1)\) as above, but we also have a stronger condition from the second term. This condition writes:

\[
\alpha(m|m) \geq \alpha(V_1) + 2\gamma + \delta - 3\beta + 1, \]  

(65)

which becomes an equality if one wants the second order graphs in the perturbation expansion (which are the lowest order graphs) to be leading order. The last term requires again \([22]\). Finally, the terms in \(f^{(a)}_{m|m,x_a}\) give the same type of relations as \([38]\).

5 The SDE in the large \(N\) limit

In this section we find appropriate scalings which allow us to obtain a well defined SDE in the large \(N\) limit.

5.1 2- and 4-point functions

Using \([16]\) and \([61]\) one has:

\[
\alpha = 0. \]  

(66)

In the large \(N\) limit, we need the 2-point function SDE to have the following form:

\[
G^{(2)}(x) = \frac{1}{|x|^2} - \frac{2\lambda}{|x|^2} \sum_{a=1}^{3} \sum_{\mathbf{q}_a} G^{(2)}(\mathbf{q}_ax) G^{(2)}(x). \]  

(67)

We need \(4\beta = 3\gamma + \delta + 2\). Using \([20]\), we get:

\[
\delta = -2 - 2\beta, \]  

(68)

\[
\beta > -1. \]  

(69)
The relations (31) and (61) between $\beta$ and $\gamma$ lead to:

$$0 > \beta > \gamma,$$ or $$\beta = \gamma = 0.$$  

(70)

From the inequalities (32) and (40) on $\alpha(V_1)$, we get:

$$\alpha(V_1) = -2 - \beta.$$  

(71)

From now on we chose $\beta = \gamma = 0$. Note that we could chose $0 > \beta > -1$. However, this would change the value of the exponents $\alpha(B)$ but would give the same SDE. Equations (68) and (71) thus become:

$$\alpha(V_1) = -2 = \delta.$$  

(72)

Assuming that $\alpha(V_1) > \alpha(m|m)$ leads to:

$$-2 > \alpha(m|m) \geq -3,$$  

(73)

When chosing $\alpha(m|m) = -3$, we have a well defined large $N$ limit. Moreover, we can see that in general we need that $\alpha(B)$ decreases strictly with the number of points of the Green function and the number of connected components of $B$. Hence at this point we can conjecture that

$$\alpha(B) = 3 - B - 2k,$$  

(75)

where $2k$ is the number of vertices of $B$, $B$ is its number of connected components.

With the scalings above, the SDE in the large $N$ limit writes

$$G^{(2)}(x) = \left( |x|^2 + 2\lambda \sum_{a=1}^{3} \int dq_\delta G^{(2)}(q_\delta x_a) \right)^{-1},$$  

(76)

$$G^{(4)}_1(x, y) = -2\hat{\lambda}G^{(2)}(x_1, y_2, y_3)G^{(2)}(y) \frac{G^{(2)}(x) - G^{(2)}(y_1, x_2, x_3)}{y_1^2 - x_1^2},$$  

(77)

$$G^{(4)}_{m|m}(x, y) = -2\hat{\lambda}G^{(2)}(x) \sum_{c=1}^{3} \left\{ \sum_{d \neq c} \int dq_d G^{(4)}(x_c, q_d, y) + \int dq_c G^{(4)}_{m|m}(q_c x_c, y) \right\},$$  

(78)

where we used the SDE for the 2-point function to rewrite the SDE for the 4-point functions and where $dq_\delta = dq_b dq_c$ for $a \neq b, c$.

### 5.2 Higher-point functions

Let us now look at the SDE for the higher-point functions with a connected boundary graph in the large $N$ limit, and in particular to the 6-point functions.

From (37), we get

$$G^{(2k)}_B(X) = -\frac{2\hat{\lambda}}{s^2} \sum_{a=1}^{3} \left\{ \int dq_\delta G^{(2)}(q_\delta s_a)G^{(2k)}_B(X) \right\} + N^{-\alpha(B) - 2} \sum_{\rho=2}^{k} \frac{1}{(p^e_\rho)^2 - s^2 \sigma_0} \left[ \frac{\partial Z[J, J]}{\partial \zeta_{\alpha}(B; 1, \rho)}(X) - \frac{\partial Z[J, J]}{\partial \zeta_{\alpha}(B; 1, \rho)}(X|_{x_1 \rightarrow p^e_\rho}) \right].$$  

(79)

Let us analyse the large $N$ limit of this equation. The first term in the RHS is always present in the large $N$ limit, but the terms coming from the swappings can be of leading order or be
This gives the following SDE, for $s$ terms in $G$

However, this equation is trivial. This implies that we need to define $\alpha$ such that the terms in $G_{F,a,b,c}^{(6)}$ are also of leading order. We thus need to have the following scaling:

$$\alpha(K(3,3)) = \alpha(F_{a,b,c}) - 2.$$ (81)

This gives the following SDE, for $s = (x_1, y_2, z_3)$ and where we used equation (76):

$$G_{K(3,3)}^{(6)}(x, y, z) = -2\hat{\lambda}G^{(2)}(x_1, y_2, z_3) \left\{ \begin{array}{c}
G_{F_{1,23}}^{(6)}(x, z, y) - G_{F_{1,23}}^{(6)}(y_1, x_2, x_3, z, y) \frac{y_1^2}{x_1^2 - x_2^2} \\
+ \frac{G_{F_{1,23}}^{(6)}(z, x, y) - G_{F_{1,23}}^{(6)}(z, y_1, x_2, x_3, y)}{z_1^2 - x_1^2} \\
+ \frac{G_{F_{1,23}}^{(6)}(y, z, x) - G_{F_{1,23}}^{(6)}(y, x_2, y_3, z, x)}{x_2^2 - y_2^2} \\
+ \frac{G_{F_{1,23}}^{(6)}(y, x, z) - G_{F_{1,23}}^{(6)}(x_2, x_3, y_1, z_2, y_3)}{y_3^2 - z_3^2} \end{array} \right\}. \tag{82}$$

Note that this could be expected because $K(3,3)$ is the first non-planar graph which appears in our analysis. Moreover, in the large $N$ limit and using (76), the SDE for the other 6-point functions with connected boundary graphs (see table 1) are

$$G_{Q_1}^{(6)}(x, y, z) = -2\hat{\lambda}G^{(2)}(x_1, y_2, y_3) \left\{ \begin{array}{c}
G^{(2)}(y) \frac{y^2}{y_1^2 - y_2^2} \\
+ \frac{G^{(4)}(y, z) - G^{(4)}(z_1, x_2, x_3)}{z_1^2 - x_1^2} \end{array} \right\}, \tag{83}$$

for $s = (x_1, y_2, y_3)$, and

$$G_{F_{1,23}}^{(6)}(x, y, z) = -2\hat{\lambda}G^{(2)}(x_1, y_2, x_3)G^{(2)}(x) \frac{G_{F_{3,12}}^{(6)}(y, z, x) - G_{F_{3,12}}^{(6)}(y, x, z_2, y_3)}{x_2^2 - y_2^2}, \tag{84}$$

for $s = (x_1, y_2, x_3)$. 

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We can see that all these equations are algebraic. For a connected boundary graph of degree zero, the SDE depends only on lower-point functions with a connected boundary graph. However, the $K(3, 3)$ SDE depends only on the other 6-point functions and the 2-point function.

Finally, from the previous discussions, we can conjecture a general formula for the scaling

$$\alpha(B) = 3 - B - 2g - 2k,$$

(85)

where $2k$ is the number of vertices of $B$, $B$ its number of connected components and $g$ its genus. Note that, since we deal in this paper with rank three tensors, for boundary graphs (where one colour is lost) the degree is the genus \([27]\) and \([28]\).

6 Concluding remarks

In this paper we have used the WTI to study the large $N$ limit of SDE of tensor field theory. This allowed us to obtain explicit values for the scalings of the various terms appearing in the action of the model studied here.

The first perspectives of this work are the proof of the conjecture (85) and the generalisation of our results for the case when any boundary graph can be disconnected \([29]\).

A second perspective is to solve the SDE in the large $N$ limit. One could initially tackle this hard task using numerical methods, as it was done in \([30]\) for a $\phi^4_5$ just renormalizable tensor model. Another way to tackle this, is to use the analytic method implemented in \([31]\) for non-commutative quantum field theory - see \([32]\).

A third perspective appears to us to be the implementation of the analytic methods used in this paper for the study of SYK-like tensor models such as the ones of \([11]\) and \([12]\). The main difficulty here would come from the fact that one would then need to take into consideration an additional time coordinate (SYK models being (0+1)-dimensional models, and not 0-dimensional models such as the model studied in this paper).

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A Perturbative expansion

In this appendix, we perform a perturbative check of the SDE up to second order of the coupling constant, before and after taking the large $N$ limit. For simplicity, we do not write the powers in $N$ in the equations.
2-point function

The SDE for the 2-point function is

\[ G^{(2)}(x) = \frac{1}{|x|^2} - \frac{2\lambda}{|x|^2} \sum_c \left( \sum_{q_c} G^{(2)}(q_c x_c) G^{(2)}(x) + G^{(4)}_c(x, x) \right) \]

\[ + \sum_{q_c} G^{(4)}_{m|m}(q_c x_c, x) + \sum_{q_c} \frac{1}{x_c^2 - q_c^2} \left( G^{(2)}(x; q_c) - G^{(2)}(x) \right) + \sum_{d \neq c} \sum_{q_c} G^{(4)}_d(x, x; q_c), \] \hspace{1cm} (86)

Let us look at the perturbative equation up to 2nd order in the coupling constant. We can first remark that the term with \( \lambda G^{(4)}_{m|m} \) will only start contributing at order \( \lambda^3 \). The other terms give

\[ - \frac{2\lambda}{|x|^2} \sum_c \sum_{q_c} G^{(2)}(q_c x_c) G^{(2)}(x) = 2 \sum_{c=1}^3 \] \hspace{1cm} (87)

\[ + 4 \sum_{c=1}^3 \sum_{d=1}^3 \left( \right) \]

\[ + 4 \sum_{c=1}^3 \sum_{d=1}^3 \left( \right) + O(\lambda^3), \]

\[ - \frac{2\lambda}{|x|^2} \sum_c G^{(4)}_c(x, x) = 4 \sum_{c=1}^3 \] \hspace{1cm} (88)

\[ - \frac{2\lambda}{|x|^2} \sum_{c=1}^3 \sum_{q_c} \sum_{d \neq d} G^{(4)}_d(x, x; q_c) = 4 \sum_{c=1}^3 \sum_{d \neq d} \] \hspace{1cm} (89)

It is more involved to obtain the perturbative expansion from the difference of 2-point functions. At first order, we have

\[ - \frac{2\lambda}{|x|^2} \sum_{c=1}^3 \sum_{q_c} \frac{1}{x_c^2 - q_c^2} \left( G^{(2)}(x; q_c) - G^{(2)}(x) \right) = 2 \sum_c \] \hspace{1cm} (90)
We are going to take the example of \( c = 1 \) and compute explicitly the diagrams at 2\(^{\text{nd}}\) order in the coupling constant.

\[
- \frac{2\lambda}{|x|^2} \sum_{a_1} \frac{1}{x_1^2 - a_1^2} \left( G^{(2)}(a_1, x_2, x_3) - G^{(2)}(x) \right) \bigg|_{x^2} = 2
\]

\[
\left( \frac{4\lambda^2}{|x|^2} \sum_{b_1} \frac{1}{x_1^2 + x_2^2 + x_3^2} \sum_{a_1} \frac{1}{a_1^2 - x_1^2} \left( \frac{1}{|x|^4} - \frac{1}{(a_1^2 + x_1^2 + x_2^2 + x_3^2)^2} \right) \right)
\]

\[
= \frac{4\lambda^2}{|x|^4} \sum_{a_1, b_1} \frac{1}{(b_1^2 + x_1^2 + x_2^2)(a_1^2 + x_1^2 + x_2^2)} \left( \frac{1}{|x|^2} - \frac{1}{a_1^2 + x_1^2 + x_2^2} \right) \]

\[
= 2 + 2
\]

And combining the terms with sums on \( b_1, b_2 \) and \( b_1, b_3 \) gives

\[
\left( \frac{4\lambda^2}{|x|^2} \sum_{b_1, b_2, b_3} \frac{1}{b_1^2 + b_2^2 + b_3^2} \sum_{a_1} \frac{1}{a_1^2 - x_1^2} \left( \frac{1}{|x|^4} - \frac{1}{(a_1^2 + x_1^2 + x_2^2 + x_3^2)^2} \sum_{b_2, b_3} \frac{1}{a_1^2 + b_2^2 + b_3^2} \right) \right).
\]

Now let us look at the two terms

\[
\frac{4\lambda^2}{|x|^2} \sum_{a_1} \frac{1}{a_1^2 - x_1^2} \left( \frac{1}{|x|^4} \sum_{b_2, b_3} \frac{1}{b_2^2 + b_3^2} - \frac{1}{(a_1^2 + x_1^2 + x_2^2 + x_3^2)^2} \sum_{b_2, b_3} \frac{1}{a_1^2 + b_2^2 + b_3^2} \right).
\]

and compute

\[
(a_1^2 + x_2^2 + x_3^2)^2(a_1^2 + b_2^2 + b_3^2) - |x|^4(x_1^2 + b_2^2 + b_3^2) =
\]

\[
a_1^6 - x_1^6 + 2(a_1^4 - x_1^4)(x_2^2 + x_3^2) + (a_1^2 - x_1^2)(x_2^4 + x_3^4 + 2(b_2^2 + b_3^2)(x_2^2 + x_3^2)).
\]
By writing
\[ a_1^0 - x_1^0 = (a_1^2 - x_1^2)(a_1^4 + x_1^4) + x_1^2 a_1^2 - x_1^2 x_1^2 = (a_1^2 - x_1^2)(a_1^4 + x_1^4 + a_1^2 x_1^2), \]
we get
\[ 2(a_1^4 - x_1^4)(x_2^2 + x_3^2) = 2(a_1^2 - x_1^2)(a_1^4 + x_1^4)(x_2^2 + x_3^2), \]
which gives
\[ 4\lambda^2 \sum_{a_1,b_2,b_3} \frac{a_1^4 + x_1^4 + a_1^2 x_1^2 + 2(x_2^2 + x_3^2)(a_1^2 + x_1^2) + x_2^4 + x_3^4 + 2(b_2^2 + b_3^2)(x_2^2 + x_3^2)}{|x|^4(a_1^2 + b_2^2 + b_3^2)(a_1^2 + x_2^2 + x_3^2)^2(a_1^2 + b_2^2 + b_3^2)}. \]
Now we can factorise
\[ a_1^4 + x_1^4 + a_1^2 x_1^2 + 2(x_2^2 + x_3^2)(a_1^2 + x_1^2) + x_2^4 + x_3^4 + 2(b_2^2 + b_3^2)(x_2^2 + x_3^2) = |x|^2(a_1^2 + x_2^2 + x_3^2) + (a_1^2 + x_2^2 + x_3^2)(a_1^2 + b_2^2 + b_3^2) + |x|^2(x_1^2 + b_2^2 + b_3^2), \]
which gives
\[ 4\lambda^2 \sum_{a_1,b_2,b_3} \left( \frac{1}{|x|^2(a_1^2 + b_2^2 + b_3^2)(a_1^2 + x_2^2 + x_3^2)^2(a_1^2 + b_2^2 + b_3^2)} + \frac{1}{|x|^4(a_1^2 + b_2^2 + b_3^2)(a_1^2 + x_2^2 + x_3^2)^2(a_1^2 + b_2^2 + b_3^2)} \right) \]
\[ = 4 \quad + 4 \quad + 4 \quad + 4. \]
Then by combining the terms with a sum on \( b_2 \) or on \( b_3 \), we get an analogous result which correspond to replace \( b_3 \) by \( x_3 \) or \( b_2 \) by \( x_2 \) in the previous equation. And we obtain the following diagrams

This computation is completely analogous for \( c = 2, 3 \). Collecting all the diagrams we get
\[ G^{(2)}(x) = \ldots + \sum_{c=1}^{3} \left\{ \begin{array}{c} 2 \end{array} \right\} + 2 \]
\[ + \sum_{d=1}^{3} \left\{ \begin{array}{c} 4 \end{array} \right\} \]
\[ = 4 \quad + 4 \quad + 4 \quad + 4. \]

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\[ + \sum_{d=1}^{3} \left\{ \begin{array}{c} 4 \end{array} \right\} \]
\[ = 4 \quad + 4 \quad + 4 \quad + 4. \]
Once we take the large $N$ limit, we get the following expansion and SDE

\[
G^{(2)}(x) = \ldots + \sum_{c=1}^{3} \left\{ 2 \left[ \sum_{d=1}^{3} G^{(2)}(y) \right] + \sum_{c=1}^{3} \sum_{q_{c}} G^{(2)}(q_{c}; y_{c}) \right\} + O(\lambda^{3}).
\]

\[
= \left( |x|^2 + 2\lambda \sum_{c=1}^{3} \int dq_{c} G^{(2)}(q_{c}; x_{c}) \right)^{-1}.
\]

4-point function with connected boundary

The full SDE for the 4-point function is

\[
G^{(4)}_{1}(x, y) = -\frac{2\lambda}{x_{1}^2 + y_{2}^2 + y_{3}^2} \left\{ \sum_{c=1}^{3} f_{c}(x, y; s_{c}; V_{c}) + \sum_{c=1}^{3} \sum_{q_{c}} G^{(2)}(q_{c}; s_{c}) G^{(4)}_{1}(x, y) \right. \\
+ \sum_{b_{1}} \frac{1}{b_{2}^2 - y_{2}^2} \left( G^{(4)}_{1}(x, y) - G^{(4)}_{1}(y_{1}, x_{2}, x_{3}) \right) \\
+ \sum_{b_{2}} \frac{1}{b_{2}^2 - y_{2}^2} \left( G^{(4)}_{1}(x, y) - G^{(4)}_{1}(x_{1}, y_{2}, b_{3}) \right) \\
+ \sum_{b_{3}} \frac{1}{y_{3}^2 - x_{3}^2} \left( G^{(4)}_{1}(x, y_{1}, y_{2}, x_{3}) - G^{(4)}_{1}(x, y) \right) \\
+ \frac{1}{y_{3}^2 - x_{3}^2} \left( G^{(4)}_{1}(x_{1}, y_{1}, y_{2}, x_{3}) - G^{(4)}_{1}(x_{1}, y_{2}, y_{3}) \right) \\
+ \frac{1}{y_{3}^2 - x_{3}^2} \left( G^{(4)}_{1}(x, y_{1}, x_{2}, x_{3}) - G^{(4)}_{1}(y_{1}, x_{2}, x_{3}) \right) \\
+ \frac{1}{y_{3}^2 - x_{3}^2} \left( G^{(4)}_{1}(y_{1}, x_{2}, x_{3}) - G^{(4)}_{1}(y_{1}, y_{2}, y_{3}) \right) \right\} + O(\lambda^{3}).
\]

where \( s = (x_{1}, y_{2}, y_{3}) \). We can remark that the terms in $\lambda f_{c}$ involve only 6-point functions and start to contribute to the perturbative expansion only at order $\lambda^{3}$, and so does the terms in
Hence up to the 2nd order in the coupling constant the other terms give

\[ \sum_{d=1}^{3} \sum_{q} G^{(2)}(q_{d}^{s_{d}}) G^{(4)}(x, y) = 4 \sum_{d=1}^{3} G^{(2)}(y) - G^{(2)}(y_{1}, x_{2}, x_{3}) = 2 + 4 \]

\[ -\frac{2\lambda}{x_{1}^{2} + y_{2}^{2} + y_{3}^{2}} \sum_{b_{1}} G^{(4)}(x, y) - G^{(4)}(b_{1}, x_{2}, x_{3}, y) = 4 + O(\lambda^{3}), \]

\[ -\frac{2\lambda}{x_{1}^{2} + y_{2}^{2} + y_{3}^{2}} \left( \sum_{b_{2}} \frac{G^{(4)}(x, y) - G^{(4)}(x, y_{1}, b_{2}, y_{3})}{b_{2}^{2} - y_{2}^{2}} + \sum_{b_{3}} \frac{G^{(4)}(x, y) - G^{(4)}(x, y_{1}, y_{2}, b_{3})}{b_{3}^{2} - y_{3}^{2}} \right) = 4 + O(\lambda^{3}), \]
\[-\frac{2\lambda}{x_1^2 + y_2^2 + y_3^2} \left( \frac{G_2^{(4)}(x, y_1, y_2, x_3) - G_2^{(4)}(x, y)}{y_2^2 - x_3^2} + \frac{G_3^{(4)}(x, y_1, x_2, y_3) - G_3^{(4)}(x, y)}{y_2^2 - x_2^2} \right) \]

\[= 4 + 4 + x \quad 2.3 \quad y \quad 3.2 \]

\[= 4 + 4 + y + O(\lambda^3). \quad (111)\]

Finally the expansion of the 4-point function is

\[G_1^{(4)}(x, y) = 2 + 4 + 4 + 4 + 4 + 4 \]

\[+ 4 + 4 + 2.3 \quad 2.3 \quad 3.2 \quad 3.2 \quad y + 4 \sum_{d=1}^{3} \left( \right) + O(\lambda^3). \quad (112)\]

After taking the large \(N\) limit, we get the following SDE and perturbative expansion

\[G_1^{(4)}(x, y) = -\frac{2\lambda}{x_1^2 + y_2^2 + y_3^2} \left( \sum_{c=1}^{3} \int dq c G^{(2)}(q, s_c)G_1^{(4)}(x, y) + G^{(2)}(y) \frac{G^{(2)}(x) - G^{(2)}(y_1, x_2, x_3)}{y_1^2 - x_1^2} \right) \]

\[= 2 + 4 + 4 + 4 + \sum_{d=1}^{3} \left( \right) \]

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4-point function with disconnected boundary

The SDE for the 4-point function with a disconnected boundary graph is

\[
G_{m|m}^{(4)}(x, y) = - \frac{2\lambda}{|x|^2} \sum_{c=1}^{3} \left\{ \sum_{q_c} G_c^{(2)}(q_c x_c) G_{m|m}^{(4)}(x, y) + f_{m|m,x_c}^{(c)}(x, y) + f_{m|m,x_c}^{(c)}(y, x) \\
+ \sum_{b_c} \frac{1}{b_c^2 - x_c^2} \left( G_{m|m}^{(4)}(x, y) - G_{m|m}^{(4)}(b_c x_c, y) \right) + \frac{1}{y_c^2 - x_c^2} \left( G_c^{(4)}(x, y) - G_c^{(4)}(x_c y_c, y) \right) \\
+ G_c^{(2)}(x_c y_c x_c, y) + \sum_{d \neq c} \sum_{q_b} G_d^{(4)}(x_c q_b, y_d, y) + \sum_{q_c} G_{m|m}^{(4)}(q_c x_c, y) \right\},
\]

(114)

where \((x_c, q_b, y_d)\) with \(\{b, c, d\} = \{1, 2, 3\}\) is implicitly reordered. Let us again check the perturbative expansion up to 2nd order in the coupling constant. We can note that the first graphs appearing in \(G_{m|m}^{(4)}\) are of order \(\lambda^2\), hence all terms in the SDE involving \(\lambda G_{m|m}^{(4)}\) will start to contribute only at order \(\lambda^3\), and the same goes for the terms \(\lambda f_{m|m}\). The other terms give

\[
- \frac{2\lambda}{|x|^2} G_c^{(2)}(x_c) \sum_{c=1}^{3} G_{c|m}^{(4)}(y_c x_c, y) = 4 \sum_{c=1}^{3} G_c^{(2)}(x_c y_c x_c, y) + O(\lambda^3),
\]

(115)

\[
- \frac{2\lambda}{|x|^2} G_c^{(2)}(x_c) \sum_{c=1}^{3} \sum_{d \neq c} \sum_{q_b} G_d^{(4)}(x_c q_b, y_d, y) = 4 \sum_{c=1}^{3} \sum_{d \neq c} G_d^{(4)}(x_c q_b, y_d, y) + O(\lambda^3),
\]

(116)

\[
- \frac{2\lambda}{|x|^2} \sum_{c=1}^{3} \frac{G_c^{(4)}(x, y) - G_c^{(4)}(x_c y_c, y)}{y_c^2 - x_c^2} = 4 \sum_{c=1}^{3} \frac{G_c^{(4)}(x, y) - G_c^{(4)}(x_c y_c, y)}{y_c^2 - x_c^2} + O(\lambda^3).
\]

(117)

In the large \(N\) limit, only one of these graphs survives and the SDE becomes

\[
G_{m|m}^{(4)}(x, y) = 4 \sum_{c=1}^{3} G_c^{(4)}(x_c y_c x_c, y) + O(\lambda^3)
\]

(118)
\[-\frac{2\lambda}{|x|^2} \sum_{c=1}^{3} \left\{ \int dq_c G^{(2)}(q_c x_c) G^{(4)}_{m|m}(x, y) + \sum_{d \neq c} \int dq_d G^{(4)}_{d}(x_c, q_b, y_d, y) + \int dq_c G^{(4)}_{m|m}(q_c x_c, y) \right\}.\]

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