POSITIVE PERIODIC SOLUTIONS OF THE WEIGHTED $p$-LAPLACIAN WITH NONLINEAR SOURCES

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ABSTRACT. In this paper we study the existence of time periodic solutions for the evolutionary weighted $p$-Laplacian with a nonlinear periodic source in a bounded domain containing the origin. We show that there is a critical exponent $q_c = q_c(\alpha, \beta) = \frac{(N+\beta)p}{N+\alpha} - 1$ and a singular exponent $q_s = p - 1$: there exists a positive periodic solution when $0 < q < q_c$ and $q \neq q_s$; while there is no positive periodic solution when $q \geq q_c$. The case when $q = q_s$ is completely different from the remaining case $q \neq q_s$, the problem may or may not have solutions depending on the coefficients of the equation.

1. Introduction. We are concerned with the existence of positive periodic solutions of the following weighted $p$-Laplacian with a nonlinear periodic source

$$\frac{\partial u}{\partial t} = \text{div}(\|x\|^\alpha|\nabla u|^{p-2}\nabla u) + m(x,t)|x|^\beta u^q, \quad (x,t) \in \Omega \times \mathbb{R},$$

subject to the homogeneous boundary condition and the periodic condition

$$u(x,t) = 0, \quad x \in \partial \Omega, \ t \in \mathbb{R},$$

$$u(x,t) = u(x,t + \omega), \quad x \in \Omega, \ t \in \mathbb{R},$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $0 \in \Omega$, $p > 1$, $\alpha \geq 0$, $\beta \geq 0$, $q \geq 0$ are constants, $m(x,t) \in C^1(\bar{\Omega} \times \mathbb{R})$ is a positive function which is $\omega$-periodic in time with $\omega > 0$.

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Periodic parabolic problems of non-weighted case have been extensively studied since the last century. Among the earliest works of this aspect, Seidman [22] (1975) studied the related special case of (1), namely
\[
\frac{\partial u}{\partial t} = \text{div}(|\nabla u|^{p-2}\nabla u) + m(x,t), \quad (x,t) \in \Omega \times \mathbb{R}.
\]
He established the existence of nontrivial periodic solutions for \(p \geq 2\) and \(m(x,t) \not\equiv 0\). It is worth mentioning the work of Beltramo and Hess [3] (1984), who considered the linear case
\[
\frac{\partial u}{\partial t} = \Delta u + m(x,t)u, \quad (x,t) \in \Omega \times \mathbb{R}.
\]
It was shown that only for some special \(m(x,t)\) can the equation have nontrivial periodic solutions. The nonlinear case
\[
\frac{\partial u}{\partial t} = \Delta u + m(x,t)u^q, \quad (x,t) \in \Omega \times \mathbb{R},
\]
was verified by Esteban [6] (1986) (see also [7] (1988)). Her results imply the existence of positive periodic solutions for \(q > 1\) with \(N \leq 2\) or \(1 < q < \frac{N}{N-2}\) with \(N > 2\) and star-shaped domains. The gap \(\frac{N}{N-2} \leq q < \frac{N+2}{N-2}\) with \(N > 2\) was filled by Quittner [20], in which he proved the existence with some restrictions on the structure of \(m(x,t)\). For the evolution \(p\)-Laplacian, Wang et al. [25] established the existence of positive periodic solutions for the case \(p-1 < q < p-1 + \frac{\beta}{N}\) and \(p \geq 2\).

A rather complete characterization, in terms of the parameters \(p > 1\) and \(q > 0\), of whether or not the positive periodic solutions exist was given by Yin et al. [27].

A special case of the steady state of equation (1) was studied by Song et al. [24] recently, where \(m \equiv 1\) and \(\Omega\) is the unit \(N\)-ball without the origin. They considered the unbounded, nonnegative, singular solutions and gave a classification of the behavior of these solutions at the isolated singularity, the origin.

To the best of our knowledge, the existence and nonexistence results of positive periodic solutions for the weighted case are not fully established, especially in the case of weighted flux. The main interest in the past decades lies in the periodic parabolic eigenvalue problem with weights in the source,
\[
u_t - \mathcal{L}u = \lambda m(x,t)u, \quad x \in \Omega, \quad t \in \mathbb{R},
\]
where \(\mathcal{L}\) is a linear uniformly elliptic operator, \(m(x,t)\) is a given weight function, see [5, 9, 10, 13] and the original work of Beltramo and Hess [2, 3].

We are quite interested in the case of weighted flux, in which (1) may have degeneracy or singularity due to the weight \(|x|^{\alpha}\) and the \(p\)-Laplacian when \(p \neq 2\). We shall give a complete characterization of whether or not the positive periodic solutions exist for the parameters \(p > 1\) and \(q > 0\). Indeed, there is a singular exponent \(q_s = p-1\) and a critical exponent \(q_c = q_c(\alpha, \beta) = \left(\frac{N+\beta}{N+\alpha-p}\right)^{\frac{N+\alpha-p}{N+\beta}} - 1\), such that under certain conditions:

(i) the problem (1)–(3) admits positive periodic solutions for \(0 < q < q_c\) with \(q \neq q_s\);
(ii) the problem (1)–(3) admits no positive periodic solutions for \(q \geq q_c\);
(iii) the problem (1)–(3) admits positive periodic solutions for some \(m(x,t)\) and admits no positive periodic solutions for some other \(m(x,t)\) for \(q = q_s\).
This paper is organized as follows. In Section 2, we discuss the sublinear case $0 \leq q < p - 1$ and establish the existence result. In Section 3, we consider the superlinear case $q > p - 1$, in which we will show the existence for $p - 1 < q < q_c$ and nonexistence for $q \geq q_c$. The last section is devoted to the singular case $q = p - 1$.

2. The sublinear case. In this section, we shall show the existence of positive periodic solutions in the case $0 \leq q < p - 1$. Let $\tau \in \mathbb{R}$ be fixed and set

$$ Q = \Omega \times (0, +\infty), \quad Q_\omega = \Omega \times (\tau, \tau + \omega), $$

$$ m = \inf_Q m(x, t), \quad M = \sup_Q m(x, t). $$

The weighted space $L^r(\Omega; |x|\alpha)$ for $r \geq 1$ and $\alpha \geq 0$ is defined as the set of all functions $u = u(x)$ that are measurable on $\Omega$ and satisfy

$$ \int_{\Omega} u^r |x|^\alpha \, dx < +\infty, $$

with the norm

$$ ||u||_{L^r(\Omega; |x|\alpha)} = \left( \int_{\Omega} u^r |x|^\alpha \, dx \right)^{\frac{1}{r}}. $$

Other weighted spaces such as $L^r(Q_\omega; |x|\alpha)$ and $W^{1,p}(\Omega; |x|^\alpha)$ are defined similarly. Denote by $E$ and $E_0$ the reasonable solution spaces, namely

$$ E = \{ u \in L^{r+1}(Q_\omega; |x|^\beta) : \frac{\partial u}{\partial t} \in L^2(Q_\omega), \nabla u \in L^p(Q_\omega; |x|^\alpha) \}, $$

$$ E_0 = \{ u \in E : u(x, t) = 0 \text{ for } x \in \partial \Omega \text{ in the sense of trace} \}. $$

Because of the singularities of $p$-Laplacian and the weight $|x|\alpha$, the problem (1)–(3) might not have classical solutions in general; hence we consider the following weak solutions.

**Definition 2.1.** A function $u \in E$ is called a weak $\omega$-periodic upper solution of the problem (1)–(3) provided that for any nonnegative function $\varphi \in C^1(\overline{Q}_\omega)$ with $\varphi(x, t) = 0$ for $x \in \partial \Omega$, there holds

$$ \int_{\Omega} \frac{\partial u}{\partial t} \varphi \, dx \, dt + \int_{Q_\omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi |x|^{\alpha} \, dx \, dt \geq \int_{Q_\omega} m(x, t) u^q \varphi |x|^\beta \, dx \, dt, $$

$$ u(x, t) \geq 0, \quad x \in \partial \Omega, $$

$$ u(x, t) \geq u(x, t + \omega), \quad x \in \Omega. $$
Lemma 2.3. Let \( \nabla u \)\n\( \nabla u \)

Noticing that \( \nabla u \)\n\( \nabla u \)

Replacing “\( \geq \)“ by “\( \leq \)“ in the above inequalities, it follows the definition of a weak lower solution. Furthermore, if \( u \) is a weak upper solution as well as a weak lower solution, then we call it a weak solution of the problem (1)–(3).

We use the method of upper and lower solutions to show the existence of weak solutions. First we present the following comparison principle.

Lemma 2.2. Let \( u_1, u_2 \in E \) such that

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} - \text{div}(|x|^\alpha |\nabla u_1|^{p-2} \nabla u_1) &\geq \frac{\partial u_2}{\partial t} - \text{div}(|x|^\alpha |\nabla u_2|^{p-2} \nabla u_2), \quad (x, t) \in Q_\omega, \\
u_1(x, t) &\geq u_2(x, t), \quad (x, t) \in \partial \Omega \times (\tau, \tau + \omega), \\
u_1(x, \tau) &\geq u_2(x, \tau), \quad x \in \Omega.
\end{aligned}
\]

Then \( u_1 \geq u_2 \) for all \( x \in \Omega \) and \( t \in [\tau, \tau + \omega] \).

Proof. This is proved by the standard test function method, see for example [11, 26]. For any \( 0 \leq \varphi \in L^2(\tau, \tau + \omega; H^1_0(\Omega)) \), there holds

\[
\iint_{Q_\omega} (u_1 - u_2) \varphi \, dx \, dt + \iint_{Q_\omega} |x|^\alpha (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla \varphi \, dx \, dt \geq 0.
\]

Since \( u_1(x, t) \geq u_2(x, t) \) for \( (x, t) \in \partial \Omega \times (\tau, \tau + \omega) \), we can choose \( \varphi = (u_2 - u_1)_+ \) as the test function and get

\[
\frac{1}{2} \iint_{Q_\omega} \frac{\partial |\varphi|^2}{\partial t} \, dx \, dt + \iint_{Q_\omega} |x|^\alpha (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla (u_2 - u_1)_+ \, dx \, dt \leq 0.
\]

Noticing that \( u_1(x, \tau) \geq u_2(x, \tau) \) for \( x \in \Omega \), we have

\[
\frac{1}{2} \iint_{Q_\omega} \varphi^2(x, \tau + \omega) \, dx \, dt + \iint_{Q_\omega} |x|^\alpha (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla (u_2 - u_1)_+ \, dx \, dt \leq 0.
\]

Further we find

\[
\iint_{Q_\omega} |x|^\alpha \frac{|\nabla \varphi|^{p+(2-p)_+}}{(|\nabla u_1| + |\nabla u_2|)^{(2-p)_+}} \, dx \, dt \\
\leq \iint_{Q_\omega} |x|^\alpha (|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1) \cdot \nabla (u_2 - u_1)_+ \, dx \, dt \leq 0.
\]

Thus, \( \nabla \varphi = 0 \) and \( \varphi = 0 \) a.e. in \( Q_\omega \).

\[\square\]

Lemma 2.3. Let \( \overline{u}, \underline{u} \), with \( \overline{u} \geq u \geq \underline{u} \geq 0 \) be a pair of bounded upper and lower solutions of the problem (1)–(3). Then the problem (1)–(3) admits a bounded weak solution \( u \in E_0 \) with \( \overline{u} \leq u \leq \underline{u} \).

Proof. Define a function sequence \( \{u_n\}_{n=0}^\infty \) by the following iteration scheme

\[
\begin{aligned}
\frac{\partial u_n}{\partial t} &\geq \text{div}(|x|^\alpha |\nabla u_n|^{p-2} \nabla u_n) + m(x, t)|x|^\beta u_n^q, \quad (x, t) \in Q_\omega, \\
u_n(x, t) &\geq 0, \quad (x, t) \in \partial \Omega \times (\tau, \tau + \omega), \\
u_n(x, \tau) &\geq u_{n-1}(x, \tau + \omega), \quad x \in \Omega.
\end{aligned}
\]
where \( u_0 = \underline{u} \). The existence and uniqueness of solutions for the above problem is classical, so \( u_n \) is well defined. Then we have
\[
\underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \leq \cdots \leq \overline{u}.
\]
In fact, by the iteration scheme and the definition of upper and lower solutions, we have
\[
\begin{cases}
\frac{\partial u_1}{\partial t} - \text{div}(|x|^\alpha \nabla u_1)|^{p-2} \nabla u_1 = m(x, t)|x|^{\beta} u_0^q \\
\geq \frac{\partial u_0}{\partial t} - \text{div}(|x|^\alpha \nabla u_0)|^{p-2} \nabla u_0), \quad (x, t) \in Q_\omega, \\
u_1(x, t) = 0 \geq u_0(x, t), \quad (x, t) \in \partial \Omega \times (\tau, \tau + \omega), \\
u_1(x, \tau) = u_0(x, \tau + \omega) \geq u_0(x, \tau), \quad x \in \Omega.
\end{cases}
\]
The comparison principle Lemma 2.2 shows that \( u_1 \geq u_0 \). Other order relations can be verified similarly. By the monotonicity of \( u_n \) with respect to \( n \), there exists a function \( u \) such that \( u_n(x, t) \) tends to \( u \) a.e. in \( Q_\omega \), and \( u(x, \tau) = u(x, \tau + \omega) \), \( \underline{u} \leq u \leq \overline{u} \). Moreover, multiplying the first equation of (4) by \( u_n \), and integrating over \( Q_\omega \), we have
\[
\int_{Q_\omega} |\nabla u_n|^p |x|^\alpha \, dx \, dt \leq \int_{Q_\omega} m(x, t) u_n^{q-1} u_n |x|^\beta \, dx \, dt \leq \int_{Q_\omega} m \Omega^{2+1} |x|^\beta \, dx \, dt,
\]
since
\[
\int_{Q_\omega} \frac{\partial u_n}{\partial t} \, dx \, dt = \frac{1}{2} \int_{\Omega} u_n^{q-1} u_n \, dx \, dt - \frac{1}{2} \int_{\Omega} u_n^{q-1} \, dx \, dt \geq 0.
\]
Furthermore, multiplying the first equation of (4) by \( \frac{\partial u_n}{\partial t} \) yields
\[
\int_{Q_\omega} \left| \frac{\partial u_n}{\partial t} \right|^2 \, dx \, dt + \int_{Q_\omega} \frac{1}{p} \frac{\partial}{\partial t} (|\nabla u_n|^p |x|^\alpha) \, dx \, dt = \int_{Q_\omega} m(x, t) u_n^{q-1} \frac{\partial u_n}{\partial t} |x|^\beta \, dx \, dt
\]
\[
\leq \frac{1}{2} \int_{Q_\omega} \left| \frac{\partial u_n}{\partial t} \right|^2 \, dx \, dt + \frac{1}{2} \int_{Q_\omega} m \Omega^{2q} |x|^{2\beta} \, dx \, dt.
\]
By the periodicity, we obtain
\[
\|u_n\|_{L^\infty(Q_\omega)} \leq C, \quad \|\nabla u_n\|_{L^p(Q_\omega; |x|^{\alpha})} \leq C, \quad \left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(Q_\omega)} \leq C,
\]
for some constant \( C \) independent of \( n \). It follows that for any \( r \geq 1 \),
\[
u_n \to u \text{ in } L^r(Q_\omega; |x|^{\alpha}), \quad \nabla u_n \to \nabla u \text{ in } L^p(Q_\omega; |x|^{\alpha}), \quad \frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(Q_\omega),
\]
as \( n \to \infty \), which implies that \( u \in E_0 \) with \( \underline{u} \leq u \leq \overline{u} \) is a bounded weak periodic solution of the problem (1)–(3).

We need the following weighted Sobolev inequality, see for example [4, 21]. Throughout this paper, we define
\[
p^* = p^*(\alpha, \beta) = \begin{cases}
\frac{(N + \beta)p}{N + \alpha - p}, & N + \alpha > p, \\
\infty, & N + \alpha \leq p.
\end{cases}
\]

\textbf{Lemma 2.4.} Suppose that \( 0 \leq \alpha < N(p - 1) \), \( \beta \geq 0 \), \( (N - p)\beta \leq N\alpha \), \( \alpha \leq \beta + p \). Then for any \( \varphi \in W_0^1, p(\Omega; |x|^{\alpha}) \), the following inequality holds
\[
\left( \int_{\Omega} |\varphi|^r |x|^{\beta} \, dx \right)^{\frac{1}{r}} \leq C_0 \left( \int_{\Omega} |\nabla \varphi|^p |x|^{\alpha} \, dx \right)^{\frac{1}{p}},
\]
where \( p < r < p^* \) and \( C_0 = C_0(N, r, p, \alpha, \beta, \Omega) \) is independent of \( \varphi \).
Proof. Since $\beta \geq 0$ and $0 \leq \alpha < N(p - 1)$, the weights $|x|^{\beta}$ and $|x|^{-\frac{\beta}{p+r}}$ satisfy the reverse doubling condition defined in [21]. For any $p < r < p^*$, the inequality $(N-p)\beta \leq N\alpha$ implies that $-\frac{\beta}{r} \leq 1 - \frac{\alpha}{p}$. The condition $\alpha < \beta + p$ ensures the existence of $r$ such that $p < r < p^*$. This Lemma is a consequence of Theorem 1 in [21].

In what follows, we shall show that all periodic solutions of the problem (1)–(3) are uniformly bounded in the case $0 < q < p-1$.

**Theorem 2.5.** Assume that $0 \leq q < p-1$ and the hypotheses of Lemma 2.4 hold. If $m(x,t)$ depends on $t$, we further assume that $\beta = 0$. Then there exists a constant $M > 0$, such that for any periodic solution $u \in E_0$ of the problem (1)–(3), there holds $\|u\|_{L^\infty(Q_{\omega})} \leq M$.

**Proof.** We shall give the proof in four steps.

**Step 1.** Let $u \in E_0$ be a solution of the problem (1)–(3). Multiplying (1) by $u^r$ with $r > 0$ being arbitrary, then integrating the resulting equation over $\Omega$ yields

$$
\frac{1}{r+1} \frac{d}{dt} \int_{\Omega} u^{r+1}(x,t) \, dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla u^r |x|^\alpha \, dx = \int_{\Omega} m(x,t) u^{q+r} |x|^\beta \, dx.
$$

(5)

Here we have assumed the integrability of $u$ in $L^{q+r}(\Omega; |x|^\beta)$; otherwise we can pass through a regularization approach. In fact, we can multiply (1) by $u_M^r$ instead of $u^r$, and derive uniform estimates independent of $M > 1$, where $u_M$ is a cut-off of $u$ such that $|u_M| \leq M$, $u_M(x) = u(x)$ if $|u(x)| \leq M-1$ and $\|u_M\|_{W^{1,r}(\Omega; |x|^{\alpha})} \leq C \|u\|_{W^{1,r}(\Omega; |x|^\alpha)}$. Integrating the above inequality over $(\tau, \tau+\omega)$, using the periodicity of $u$ and Lemma 2.4, we have

$$
\int_{Q_{\omega}} |\nabla u|^{|r+1-\frac{p}{r+1}} |x|^\alpha \, dx dt \leq C \int_{Q_{\omega}} u^{q+r} |x|^\beta \, dx dt
$$

since $-\frac{Np+\alpha}{p} < 1 - \frac{N+q}{p}$ as $q < p-1$ and $\alpha < \beta + p$. Here and in what follows we continue to use the letter $C$ as a generic positive constant, whose particular value depends on the context and may change within the proof. Noticing that $\frac{q+r}{p+r-1} < 1$, we have

$$
\int_{Q_{\omega}} |\nabla u|^{|r+1-\frac{p}{r+1}} |x|^\alpha \, dx dt \leq C.
$$

Applying Lemma 2.4 again, we obtain that

$$
\int_{Q_{\omega}} u^{\frac{N(q+r-1)}{N-r+q}} \, dx dt \leq C \int_{Q_{\omega}} |\nabla u|^{|r+1-\frac{p}{r+1}} |x|^\alpha \, dx dt \leq C,
$$

(6)

since $-\frac{Np+\alpha}{p+r-1} < 1 - \frac{N+q}{p}$ as $\alpha < \beta + p$. Recalling (5), we have

$$
\frac{1}{r+1} \frac{d}{dt} \int_{\Omega} u^{r+1} \, dx + r \left(\frac{p}{p+r-1}\right)^p \int_{\Omega} |\nabla u|^{|r+1-\frac{p}{r+1}} |x|^\alpha \, dx
$$

$$
\leq C \left(\int_{\Omega} |\nabla u|^{|r+1-\frac{p}{r+1}} |x|^\alpha \, dx\right)^{\frac{p+r-1}{p}}.
$$

It follows

$$
\frac{d}{dt} \int_{\Omega} u^{r+1}(x,t) \, dx \leq C,
$$

(7)
Applying Lemma 2.4 and Hölder’s inequality to the above inequality yields
\[
\int_\Omega u^{r+1}(x,t_0) \, dx < C.
\]
Combining the last inequality with (7) and using the periodicity of \(u(x,t)\), we obtain that
\[
\sup_{t \in (\tau, \tau+\omega)} \int_\Omega u^{r+1}(x,t) \, dx \leq C.
\]
We further have
\[
\sup_{t \in (\tau, \tau+\omega)} \int_\Omega u^{r+1}(x,t)|x|^\beta \, dx \leq C. \tag{8}
\]

**Step 2.** Multiplying (1) by \((u - k)^{r+1}_+\chi_{[t_1,t_2]}(t)\), with \(k > 0\) being arbitrary and \(\chi_{[t_1,t_2]}(t)\) the characteristic function of the interval \([t_1,t_2]\), then integrating over \(\Omega \times (t_1,t_2)\) yields
\[
\frac{1}{r+1} \int_{t_1}^{t_2} \frac{d}{dt} \int_\Omega (u - k)^{r+1}_+ \, dx \, dt + r \int_{t_1}^{t_2} \int_\Omega (u - k)^{r+1}_+ |\nabla u|^p \, dx \, dt 
\leq \frac{m}{r+1} \int_{t_1}^{t_2} \int_\Omega u^q (u - k)^{r+1}_+ |x|^\beta \, dx \, dt.
\]
Let
\[
I_k(t) = \int_\Omega (u - k)^{r+1}_+ \, dx.
\]
Assume that the absolutely continuous function \(I_k(t)\) attains its maximum at \(\rho \in [\tau, \tau + \omega]\). Since \(I_k(\tau) = I_k(\tau + \omega)\), we can take \(\rho > \tau\). Set \(t_1 = \rho - \varepsilon\), \(t_2 = \rho\), with \(\varepsilon > 0\) small enough such that \(t_1 > \tau\). Therefore, we have \(I_k(\rho) \geq I_k(\rho - \varepsilon)\), which implies
\[
r\left(\frac{p}{p + r - 1}\right) \frac{1}{\varepsilon} \int_{\rho - \varepsilon}^{\rho} \int_\Omega |\nabla (u - k)^{r+1}_+|^p |x|^\alpha \, dx \, dt 
\leq \frac{m}{\varepsilon} \int_{\rho - \varepsilon}^{\rho} \int_\Omega u^q (u - k)^{r+1}_+ |x|^\beta \, dx \, dt.
\]
Letting \(\varepsilon \to 0^+\) and recalling (8), we have
\[
\int_\Omega |\nabla (u(x,\rho) - k)^{r+1}_+|^p |x|^\alpha \, dx 
\leq C \int_\Omega u^q (u(x,\rho) - k)^{r+1}_+ |x|^\beta \, dx
\leq C \left( \int_\Omega u^{r(N + \beta + p) + \frac{\beta + r - 1}{N + \beta + p}} (u(x,\rho) - k)^{r+1}_+ \, dx \right)^{\frac{N + \alpha}{N + \beta + p}}
\leq C \left( \int_\Omega (u(x,\rho) - k)^{r(N + \beta + p) + \frac{\beta + r - 1}{N + \beta + p}} |x|^\beta \, dx \right)^{\frac{N + \alpha}{N + \beta + p}}.
\]
Let
\[
A_k(t) = \{ x \in \Omega : u(x,t) > k \}, \quad \mu_k = \sup_{t \in (\tau, \tau+\omega)} |A_k(t)|,
\]
\[
|A_k(\rho)| |x|^\beta = \int_{A_k(\rho)} |x|^\beta \, dx.
\]
Applying Lemma 2.4 and Hölder’s inequality to the above inequality yields
\[
\left( \int_{A_k(\rho)} (u(x,\rho) - k)^{r+1}_+ |x|^\beta \, dx \right)^{\frac{r + p - 1}{p}}
\leq C \int_{A_k(\rho)} |\nabla (u(x,\rho) - k)^{r+1}_+|^p |x|^\alpha \, dx
\]
\[
\leq C \left( \int_{A_k(\rho)} (u(x, \rho) - k)_+^{r(\frac{N+\beta+p}{N+\alpha})} |x|^{\beta} \, dx \right)^{\frac{N+\alpha}{r+\beta+p}} 
\leq C |A_k(\rho)|^{(\frac{N+\alpha}{r+\beta+p})} \left( \int_{A_k(\rho)} (u(x, \rho) - k)_+^{r'} |x|^{\beta} \, dx \right)^{-\frac{r'}{r}} ,
\]

provided that \(-\frac{N+\beta}{r+\beta+p} \leq 1 - \frac{N+\alpha}{p}\) and \(r' > \frac{(N+\beta+p)r}{N+\alpha}\). We further have
\[
\left( \int_{A_k(\rho)} (u(x, \rho) - k)_+^{r'} |x|^{\beta} \, dx \right)^{\frac{p-1}{r'}} \leq C |A_k(\rho)|^{(\frac{N+\alpha}{r+\beta+p})} \left( \frac{N+\alpha}{r+\beta+p} \right)^{\frac{p-1}{r'}} .
\]

**Step 3.** If \(m(x,t)\) is independent of \(t\), then the periodic solution of the problem \((1)-(3)\) must be a steady state. In fact, multiplying \((1)\) by \(u_t\) and integrating over \(Q_\omega\) yield
\[
\int_{Q_\omega} u_t^2 \, dx dt + \int_{Q_\omega} \frac{1}{p} \frac{\partial}{\partial t} (|\nabla u|^p |x|^\alpha) \, dx dt = \int_{Q_\omega} \frac{1}{q+1} \frac{\partial}{\partial t} (m(x)u^{q+1} |x|^\beta) \, dx dt ,
\]

which, together with the periodicity of \(u(x,t)\), implies that
\[
\int_{Q_\omega} u_t^2 \, dx dt = 0 .
\]

Thus we have \(u(x,t) \equiv u(x)\), \(A_k(t) \equiv A_k\) and \((9)\) can be interpreted as
\[
\int_{A_k} (u(x) - k)_+^{r'} |x|^{\beta} \, dx \leq C |A_k|^{(\frac{N+\alpha}{r+\beta+p})} \left( \frac{N+\alpha}{r+\beta+p} \right)^{\frac{p-1}{r'}} .
\]

In addition, we note that
\[
\int_{A_k} (u(x) - k)_+^{r'} |x|^{\beta} \, dx \geq (h - k)_+^{r'} |Ah|_{|x|^{\beta}} .
\]

Combining the last inequality with \((10)\) yields
\[
|Ah|_{|x|^{\beta}} \leq \left( \frac{C}{h - k} \right)^{\frac{r'}{(\frac{N+\alpha}{r+\beta+p})}} .
\]

Take
\[
r' = \left\{ \begin{array}{ll}
\frac{(N+\beta)(r+p-1)}{N+\alpha} & \text{if } p < N + \alpha, \\
\frac{(N+\beta+p)(r+p)}{N+\alpha} & \text{if } p \geq N + \alpha.
\end{array} \right.
\]

Then we have \(-\frac{N+\beta}{r+\beta+p} \leq 1 - \frac{N+\alpha}{p}\), \(r' > \frac{(N+\beta+p)r}{N+\alpha}\) and \((\frac{N+\alpha}{r+\beta+p})^{\frac{p-1}{r'}} > 1\). By Lemma 4.1.1 of \([26]\) and \((11)\), we conclude that
\[
|A_C|_{|x|^{\beta}} = 0 ,
\]

which means that
\[
\|u(x)\|_{L^\infty(\Omega)} = \|u(x)\|_{L^\infty(\Omega; |x|^{\beta})} \leq C .
\]

**Step 4.** If \(m(x,t)\) depends on \(t\), we have assumed that \(\beta = 0\). Then \((9)\) and Hölder’s inequality yield
\[
I_k(\rho) = \int_\Omega (u(x, \rho) - k)_+^{r+1} \, dx \leq \left( \int_{A_k(\rho)} (u(x, \rho) - k)_+^{r'} \, dx \right)^{\frac{r+1}{r'}} |A_k(\rho)|^{1 - \frac{r+1}{r'}} 
\leq C |A_k(\rho)|^{(\frac{N+\alpha}{r+\beta+p})} \left( \frac{N+\alpha}{r+\beta+p} \right)^{\frac{p-1}{r'}} ,
\]
provided that \( r' \geq r + 1 \). We also have

\[
I_k(p) \geq I_k(t) \geq \int_{A_k(t)} (u(x, t) - k)^{r+1} dx \geq (h - k)^{r+1} |A_h(t)|.
\]

Combining the above two inequalities yields

\[
\mu_k \leq \left( \frac{C}{h - k} \right)^{r+1} \mu_k^{(N+\alpha)(N+p)r - \frac{r+1}{p}}.
\]

Take \( r' \) as defined in (12) for \( \beta = 0 \), then \( r' > r + 1 \) if we choose \( r \) appropriately large since \( \alpha < p \), \( \frac{N}{N+\alpha-p} > 1 \) for \( p < N + \alpha \), and \( \frac{N+p}{N+\alpha} > 1 \) for \( p \geq N + \alpha \). Further we can verify that \( -\frac{N+\beta}{r'+p} \leq 1 - \frac{N+\alpha}{p} \), \( r' > \frac{(N+p)(1+\alpha)_{p}}{N+\alpha} > 1 \) and \( \frac{(N+\alpha)(N+p)r - \frac{r+1}{p}}{r'} + 1 - \frac{r+1}{p} > 1 \). Similar to Step 3, we conclude that \( \mu_C = 0 \) and

\[
\| u(x, t) \|_{L^\infty(Q)} \leq C.
\]

This completes the proof. \( \Box \)

Next we consider the eigenvalue problem

\[
\begin{cases}
-\text{div}(|x|^\alpha |\nabla \varphi|^{p-2} \nabla \varphi) = \lambda |x|^\beta |\varphi|^{p-2} \varphi, & x \in \Omega, \\
\varphi(x)|_{\partial \Omega} = 0,
\end{cases}
\]

which is closely related to the problem (1)–(3).

We note that the eigenvalue problem (13) might not admit its principal eigenvalue nor the principal eigenfunction. In fact, the weighted embedding inequality

\[
\int_{\Omega} |\varphi|^p |x|^\beta dx \leq C_0 \int_{\Omega} |\nabla \varphi|^p \varphi |x|^\alpha dx, \quad \forall \varphi \in C_0^1(\Omega),
\]

(14) is not valid for any \( C_0 \) independent of \( \varphi \) if \( \alpha > \beta + p \).

**Lemma 2.6.** Assume that the hypotheses of Lemma 2.4 hold. Then the eigenvalue problem (13) admits its first eigenvalue \( \lambda_1 \) and the first eigenfunction \( \varphi_1(x) \) that satisfy

\[
\lambda_1 > 0, \quad \varphi_1 > 0, \quad x \in \Omega, \quad \varphi_1 \in L^\infty(\Omega).
\]

**Proof.** By the weighted Sobolev inequality Lemma 2.4, we see that (14) holds for some \( C_0 = C_0(N, p, \alpha, \beta, \Omega) \) independent of \( \varphi \). Thus

\[
\mu = \inf_{\varphi \in W_0^1(\Omega; |x|^\alpha)} \frac{\int_{\Omega} |\nabla \varphi|^p \varphi |x|^\alpha dx}{\int_{\Omega} |\varphi|^p |x|^\beta dx} \geq \frac{1}{C_0} > 0.
\]

Similar to the well known results of non-weighted \( p \)-Laplacian eigenvalue problems, see for example [15], using the same variational approach, we can verify that the infimum can be taken as a minimum. It follows that \( \mu \) is actually the first eigenvalue \( \lambda_1 \) of the problem (13) and there exists an eigenfunction \( 0 < \varphi_1 \in W_0^1(\Omega; |x|^\alpha) \).

Since the eigenvalue problem (13) is homogeneous with respect to \( \varphi \), for any given \( r > 0 \), we can take \( \varphi_1 \) such that

\[
\int_{\Omega} |\varphi_1(x)|^{r+1} |x|^\beta dx \leq 1.
\]

Replacing (8) by the above inequality in the proof of Theorem 2.5, then repeating Step 2 and Step 3 therein, we obtain \( \varphi_1 \in L^\infty(\Omega) \). \( \Box \)
Theorem 2.7. Assume that $0 \leq q < p - 1$ and the hypotheses of Lemma 2.4 hold. Then the problem (1)–(3) admits at least one bounded positive periodic solution $u \in E_0$.

Proof. Choose $\tilde{R}$ to be appropriately large such that $\Omega \subset B_{\tilde{R}}/2$. Let $\lambda_1$, $\tilde{\lambda}_1$ be the first eigenvalues of the weighted $p$-Laplacian with homogeneous Dirichlet boundary conditions on $\Omega$ and $B_{\tilde{R}}$ respectively, and let $\varphi$, $\tilde{\varphi}$ with $\|\varphi\|_{L^q(\Omega)} = \|\tilde{\varphi}\|_{L^q(B_{\tilde{R}})} = 1$ be the eigenfunctions corresponding to $\lambda_1$ and $\tilde{\lambda}_1$ respectively. Precisely speaking, $\varphi$ and $\tilde{\varphi}$ satisfy

$$
\begin{cases}
-\text{div}(|x|^{\alpha} |\nabla \varphi|^{p-2} \nabla \varphi) = \lambda_1 |x|^{\beta} |\varphi|^{p-2} \varphi, & x \in \Omega, \\
\varphi(x)|_{\partial \Omega} = 0,
\end{cases}
$$

and

$$
\begin{cases}
-\text{div}(|x|^{\alpha} |\nabla \tilde{\varphi}|^{p-2} \nabla \tilde{\varphi}) = \tilde{\lambda}_1 |x|^{\beta} |\tilde{\varphi}|^{p-2} \tilde{\varphi}, & x \in B_{\tilde{R}}, \\
\tilde{\varphi}(x)|_{\partial B_{\tilde{R}}} = 0.
\end{cases}
$$

Lemma 2.6 provides the existence of the first eigenvalues and the first eigenfunctions. Furthermore, there exists a constant $\delta > 0$ such that $\tilde{\varphi} \geq \delta$ for $x \in \Omega$. Then $\Phi = \varepsilon \varphi$ with $\varepsilon = (m/\lambda_1)^{1/(p-1-q)}$ is a periodic lower solution of the problem (1)–(3), and $\Psi = \kappa \tilde{\varphi}$ with $\kappa = (m/\tilde{\lambda}_1)^{1/(p-1-q)}/\delta$ is a periodic upper solution of the same problem. The monotonic dependence of the first eigenvalue with respect to the domain shows that $\tilde{\lambda}_1 \leq \lambda_1$. Thus we have $\Psi \geq \Phi$ for $x \in \Omega$. Lemma 2.3 implies that the problem (1)–(3) admits a periodic solution $u \in E_0$ with $\Phi(x) \leq u(x,t) \leq \Psi(x)$. \qed

3. The superlinear case. In this section, we prove the existence and nonexistence results of positive periodic solutions. We begin with some Liouville type results, which will be used to show the existence of periodic solutions of the problem (1)–(3).

First, we show the asymptotic behavior of solutions of the following inequality

$$
-\text{div}(|x|^{\alpha} |\nabla u|^{p-2} \nabla u) \geq 0,
$$

which is a simple modification of Lemma 2.3 in [23].

Lemma 3.1. Suppose that $N + \alpha > p$ and $\{x \in \mathbb{R}^N : |x| > R\} \subset \Omega$ with given $R > 0$. Let $u$ be a positive weak solution of the inequality (15) in the domain $\Omega$. Then there exists a constant $C = C(N, p, \alpha, R, u) > 0$ such that

$$
u(x) \geq C|x|^{-\frac{\alpha+\alpha-p}{p-1}}, \quad |x| > 2R.
$$

Proof. Define

$$K = (2R)^{\frac{N+\alpha-p}{p-1}} \min_{|x|=2R} u(x) > 0
$$

and

$$v(x) = K|x|^{-\frac{\alpha+\alpha-p}{p-1}}.
$$

Then $v$ is a fundamental solution of $-\text{div}(|x|^{\alpha} |\nabla u|^{p-2} \nabla u) = 0$, so by applying the comparison lemma (a similar version as Lemma 2.2) in the domain $|x| > 2R$ we get (16). \qed

Next, we prove the following nonexistence result of the weighted differential inequality

$$
-\text{div}(|x|^{\alpha} |\nabla u|^{p-2} \nabla u) \geq m(x)|x|^{\beta} u^q,
$$

the proof is similar to the proof of non-weighted cases in [16, 27].
Lemma 3.2. Assume that \( p - 1 < q \leq \frac{(N + \beta)(p - 1)}{N + \alpha - \beta} \) (which means \( p - 1 < q < +\infty \) if \( p \geq N + \alpha \)) and \( m(x) \) is an appropriately smooth function with \( m(x) \geq m_0 > 0 \). Then the problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi |x|^\alpha \, dx 
\geq \int_{\mathbb{R}^N} m(x) u^{q} \varphi |x|^\beta \, dx, & \forall \varphi \in C_0^1(\mathbb{R}^N), \\
u(x) > 0, & \forall x \in \mathbb{R}^N
\end{array} \right.
\end{align*}
\]

(17)

has no solution in the space \( W^{1,p}_0(\mathbb{R}^N; |x|^\alpha) \cap L^q_0(\mathbb{R}^N; |x|^\beta) \).

Proof. For any \( 0 \leq \varphi \in C_0^1(\mathbb{R}^N) \), taking \( u^{-r} \varphi \) as the test function in (17), where \( r \) with \( 0 < r < \min\{\frac{q-(p-1)}{p}, p-1\} \) is to be determined, we have

\[
\int_{\mathbb{R}^N} u^{-r} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi |x|^\alpha \, dx - r \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx
\]

\[
\geq \int_{\mathbb{R}^N} m(x) u^{q-r} \varphi |x|^\beta \, dx,
\]

that is

\[
r \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx + \int_{\mathbb{R}^N} m(x) u^{q-r} \varphi |x|^\beta \, dx \leq \int_{\mathbb{R}^N} u^{-r} |\nabla u|^{p-1} |\nabla \varphi| |x|^\alpha \, dx
\]

\[
\leq \frac{r}{2} \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx + C_r \int_{\mathbb{R}^N} u^{p-1-r} |\nabla \varphi|^p \varphi^{p-1} |x|^\alpha \, dx
\]

\[
\leq \frac{r}{2} \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx + \frac{1}{2} \int_{\mathbb{R}^N} m(x) u^{-r'} \varphi |x|^\beta \, dx
\]

\[
+ \tilde{C}_r \int_{\mathbb{R}^N} \frac{|\nabla \varphi|^{\frac{p(q-r)}{q-r}}}{\varphi^{\frac{p(q-r)}{q-r}-1}} |x|^\alpha \, dx
\]

with \( \delta = \frac{(q-r)\alpha - (p-1-r)\beta}{q-(p-1)} \) since \( r < p - 1 < q \), which implies that

\[
r \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx + \int_{\mathbb{R}^N} m(x) u^{-r'} \varphi |x|^\beta \, dx \leq 2\tilde{C}_r \int_{\mathbb{R}^N} |\nabla \varphi|^{\frac{p(q-r)}{q-r}} \varphi^{\frac{p(q-r)}{q-r}-1} |x|^\alpha \, dx.
\]

(18)

Furthermore, using (17) again, we see that

\[
\int_{\mathbb{R}^N} m(x) u^{q} \varphi |x|^\beta \, dx
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla u|^{p-1} |\nabla \varphi| |x|^\alpha \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} u^{(r+1)(p-1)} |\nabla \varphi|^p \varphi^{p-1} |x|^\alpha \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_{\mathbb{R}^N} u^{-r-1} |\nabla u|^p \varphi |x|^\alpha \, dx \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} m(x) u^{-r'} \varphi |x|^\beta \, dx \right)^{\frac{(r+1)(p-1)}{p(q-r)}}
\]

\[
\times \left( \int_{\mathbb{R}^N} |\nabla \varphi|^{\frac{p(q-r)}{q-r}} \varphi^{\frac{p(q-r)}{q-r}-1} |x|^\alpha \, dx \right)^{\frac{q-r-(r+1)(p-1)}{p(q-r)}}
\]

(19)
where \( \mu = \frac{(q-r)(r+1)(p-1)}{q-r-(r+1)(p-1)} \) since \( r < \frac{q-(p-1)}{p} \). Combining the last inequality with (18), we obtain that
\[
\int_{\mathbb{R}^N} m(x)u^q |x|^\beta \, dx \leq M \left( \int_{\mathbb{R}^N} \left| \nabla \varphi \right|^{\frac{p(q-r)}{q-(r+1)(p-1)}} |x|^\delta \, dx \right)^{\frac{p-1}{q}} + \frac{(r+1)(p-1)}{p(q-r)} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \left| \nabla \varphi \right|^{\frac{p(q-r)}{q-r-(r+1)(p-1)}} |x|^\mu \, dx \right)^{\frac{q-r-(r+1)(p-1)}{p(q-r)}} dx. \tag{20}
\]

Let
\[ \xi(x) = \xi_0 \left( \frac{|x|}{R} \right), \]
where \( \xi_0 \in C^1(\mathbb{R}^+) \) with \( 0 \leq \xi_0 \leq 1 \) satisfying
\[ \xi_0(\eta) = \begin{cases} 1, & 0 \leq \eta \leq 1, \\ 0, & \eta \geq 2. \end{cases} \]

Take \( \varphi = \xi^\kappa \) with \( \kappa \) chosen appropriately large. For example,
\[ \kappa \geq \max \left\{ \frac{p(q-r)}{q-(p-1)}, \frac{p(q-r)}{q-r-(r+1)(p-1)} \right\}. \]

A straightforward calculation shows that \( \left| \nabla \varphi \right|^\sigma / \varphi^{\sigma-1} \) is integrable for any \( 1 < \sigma < \kappa \). Noticing that \( \nabla \varphi \equiv 0 \) for \( |x| < R \) or \( |x| > 2R \), we further obtain that \( |x|^{\nu} |\nabla \varphi|^{\sigma} / \varphi^{\sigma-1} \) is integrable for any \( 1 < \sigma < \kappa \) and \( \nu \in \mathbb{R} \). Thus we have
\[
\int_{\mathbb{R}^N} \left| \nabla \varphi \right|^\sigma / |x|^{\nu} \, dx \leq C_{\sigma, \nu} R^{N+\nu-\sigma}.
\]

Recalling (20), we have
\[
\int_{B_R} u^q |x|^\beta \, dx \leq \frac{1}{m_0} \int_{\mathbb{R}^N} m(x)u^q |x|^\beta \, dx \leq CR^\rho,
\]
where
\[
\rho = \left( N + \delta - \frac{p(q-r)}{q-(p-1)} \right) \left( \frac{p-1}{p} + \frac{(r+1)(p-1)}{p(q-r)} \right) + \left( N + \mu - \frac{p(q-r)}{q-r-(r+1)(p-1)} \right) \frac{q-r-(r+1)(p-1)}{p(q-r)}
\]
\[
= \left( N + \frac{(q-r)\alpha-(p-1)\beta}{q-(p-1)} \right) \left( \frac{p-1}{p} + \frac{(r+1)(p-1)}{p(q-r)} \right) + \left( N + \frac{(q-r)\alpha-(p-1)\beta}{q-(p-1)} \right) \frac{q-r-(r+1)(p-1)}{p(q-r)}
\]
\[
= N + \frac{q\alpha}{q-(p-1)} - \frac{(p-1)\beta}{q-(p-1)} - \frac{pq}{q-(p-1)}. \]

If \( p-1 < q < \frac{(N+\beta)(p-1)}{N+\alpha-p} \), we can verify that \( \rho < 0 \). Letting \( R \to \infty \), we obtain that
\[
\int_{B_R} u^q |x|^\beta \, dx = 0, \quad \forall R > 0,
\]
which means that \( u \equiv 0 \). If \( q = \frac{(N+\beta)(p-1)}{N+\alpha-p} \), then \( \rho = 0 \) and \( u^q |x|^\beta \in L^1(\mathbb{R}^N) \). Taking \( R = 1 \) in Lemma 3.1 and using (16), we have
\[
\int_{\mathbb{R}^N} u^q |x|^\beta \, dx \geq C \int_{\mathbb{R}^N \setminus B_2} |x|^{-\frac{N+\beta}{p-1}q} |x|^\beta \, dx = C \int_{\mathbb{R}^N \setminus B_2} |x|^{-N} \, dx,
\]
which contradicts to the integrability. \( \square \)
Now we consider the Liouville type results of the weighted $p$-Laplacian

$$-\text{div}(|x|^\alpha |\nabla u|^{p-2} \nabla u) = m(x)|x|^\beta u^q.$$ 

Rather than the weighted differential inequality in Lemma 3.2, we shall show that the nonexistence result can be extended to a larger range of exponents. For the Liouville type results of non-weighted differential equations or inequalities associated with the $p$-Laplacian, we refer the readers to [18, 23] and references therein. We only prove the one dimensional case and the radially symmetric case below.

**Lemma 3.3.** Assume that $q > p - 1$ for $p \leq N + \alpha$, or $p - 1 < q < \frac{(N+\beta)p}{N+\alpha} - 1$ and $\alpha < \beta + p$ for $p < N + \alpha$, and $m_0 > 0$. Then the problem

$$\left\{ \begin{array}{l}
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi |x|^{\alpha} dx = \int_{\mathbb{R}^N} m_0 u^q \varphi |x|^{\beta} dx, \quad \forall \varphi \in C_0^1(\mathbb{R}^N), \\
0 < u \in W^{1,p}_\text{loc}(\mathbb{R}^N; |x|^\alpha) \cap L^\infty_\text{loc}(\mathbb{R}^N),
\end{array} \right.$$ 

has no solution for $N = 1$ and has no radially symmetric solution for $N \geq 2$.

**Proof.** According to Lemma 3.2, we only need to consider the case $p < N + \alpha$, $\alpha < \beta + p$, and $p - 1 < \frac{(N+\beta)(p-1)}{N+\alpha} - 1 < q < \frac{(N+\beta)p}{N+\alpha} - 1$. Suppose that $u > 0$ is a weak solution for $N = 1$ or a radially symmetric solution for $N \geq 2$ of (21). For the case when $N = 1$, we set $r = x$; while for the radially symmetric case, we may assume that $u(x) = u(r)$ with $r = |x|$. Then we have

$$\left\{ \begin{array}{l}
-(r^{N+\alpha+1} |u'|^{p-2} u')' = m_0 r^{N+\beta-1} u^q, \quad r > 0, \\
u(r) > 0, \quad r > 0,
\end{array} \right.$$ 

in the weak sense. Define

$$s = r \frac{N+\alpha-p}{p-1}, \quad w(s) = u(r).$$

The following ordinary differential equation arises

$$\left\{ \begin{array}{l}
-(|u'|^{p-2} w')' = \mu \frac{w^q}{s^d}, \quad s > 0, \\
w > 0, \quad s > 0,
\end{array} \right.$$ 

with $d = 1 + \frac{(N+\beta)(p-1)}{N+\alpha} - 1$ and $\mu = \left(\frac{p-1}{N+\alpha} - 1\right)^p m_0$. This generalized homogeneous equation can be transformed into an autonomous equation by setting

$$w(s) = s^b v(t), \quad t = \ln s,$$

with $b = \frac{d-p}{q-(p-1)}$. That is,

$$\left( \frac{1}{v'} + bv \right)^{p-2} (v' + bv) + (b-1)(p-1) \left( |v'| + bv \right)^{p-2} (v' + bv) + \mu v^q = 0, \quad t \in \mathbb{R}.$$ 

The condition $\frac{(N+\beta)(p-1)}{N+\alpha} - 1 < q < \frac{(N+\beta)p}{N+\alpha} - 1$ is equivalent to $\frac{p-1}{p} < b < 1$. Since $0 < u \in L^\infty_\text{loc}(\mathbb{R}^N)$, we obtain that $u$ is bounded in $0 < r < 1$, and $w$ is bounded in a neighborhood of $s = +\infty$. Thus $v > 0$ for $t \in \mathbb{R}$ and $\lim_{t \to +\infty} v(t) = \lim_{s \to +\infty} w(s)/s^b = 0$. Define

$$y(t) = |v'| + bv \frac{p-2}{p} (v' + bv).$$

It follows that the pair $(v, y)$ solves the ordinary differential system

$$\left\{ \begin{array}{l}
v' = |v|^{\frac{p-2}{p-1}} y - bv, \\
y' = (1-b)(p-1)y - \mu v^q.
\end{array} \right.$$ 

(22)
We may assume that $\mu = 1$, since a simple rescaling of $\tilde{u} = \kappa u$ in (21) can make change of the coefficient $\tilde{m}_0 = \kappa^{-q}(p-1)\tilde{m}_0$.

We claim that the singular dynamical system (22) admits no solution $(v, y)$ such that $v(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t \rightarrow +\infty} v(t) = 0$. We only need to consider the half plane $\{(v, y); v \geq 0\}$. Denote the vector field of the system (22) by $(\Phi_1, \Phi_2)$ and define two curves $\Gamma_1 = \{(v, y); y = \frac{2-v}{p-1} y = b v, v > 0\}$ and $\Gamma_2 = \{(v, y); (1-b)(p-1)y = \mu v^q, v > 0\}$. Bendixson’s negative criterion shows that there exists no periodic orbit since $\text{div}(\Phi_1, \Phi_2) = -b + (1-b)(p-1) < 0$. There are two equilibrium points:

$P_0 = (0, 0)$ and $P_1 = (v_0, y_0)$ with $y_0 = b^\frac{2-v}{p-1} = b^\alpha(1-b)(p-1) > 0$ and $y_0 = b v_0$.

The linearization matrix at $P_1$ is

$$A = \begin{pmatrix}
-b & \frac{1}{p-1}y_0^\frac{2-v}{p-1} \\
-qv_0^\frac{2-v}{p-1} & (1-b)(p-1)
\end{pmatrix}.$$

We can verify that $\text{trace} A = -b + (1-b)(p-1) = p - 1 - bp < 0$ and $\text{det} A = -b(1-b)(p-1) + \frac{q}{p-1}y_0^\frac{2-v}{p-1} = b(1-b)(q - (p-1)) > 0$, since $\frac{p-1}{p} < b < 1$. It follows that $P_1 = (v_0, y_0)$ is a sink.

A standard analysis of the system (22) shows that $v' < 0$ and $y' < 0$ in the fourth quadrant. Thus, the trajectory $(v(t), y(t))$ denoted by $\Gamma$ that satisfies $v(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t \rightarrow +\infty} v(t) = 0$ can not run through the $y$-axis or leave from the sink $P_1$. Then the Poincaré–Bendixson theorem implies that $\Gamma$ must enter the equilibrium point $P_0$ from the first quadrant and can only behave like one of the following cases: (i) $\lim_{t \rightarrow -\infty} (v(t), y(t)) = (0, 0)$; (ii) $\lim_{t \rightarrow +\infty} v(t) = +\infty$. The case (i) can be excluded by an argument similar to the proof of Lemma 2.2 in [12]. In the case (ii), we see that $\Gamma$ does not intersect $\Gamma_1$; otherwise, suppose the last point that $\Gamma$ intersects $\Gamma_1$ before entering $(0,0)$ is $(v(t_0), y(t_0))$, then $v'$ turns to positive for $t < t_0$ and $v(t) \leq v(t_0)$, which is a contradiction. Therefore, $\Gamma$ is contained in the region $G = \{(v, y); v > 0, y > 0, |y|^{2-p)/(p-1)y < b v\}$.

Similar to the proof of Lemma 2.5 in [12], we complete this proof by constructing a curve denoted by $\Gamma_0$ that does not intersect $\Gamma$, and then $\Gamma$ is contained in a bounded domain, which contradicts to the case (ii). Define $\Gamma_0 = \{(v, y); y = k(v^a - c), v > 0, y > 0\}$, where $p - 1 < a < q$ and $k, c$ are positive constants. The condition $a > p - 1$ ensures that $\Gamma_0$ intersects $\Gamma_1$, and then $\Gamma_0$ divides $G$ into a bounded part and an unbounded part. The normal vector of curve $\Gamma_0$ is $(kav^{a-1}, -1)$. We only need to show that $(kav^{a-1}, -1) \cdot (\Phi_1, \Phi_2) > 0$ for $v > c^\frac{1}{a}$. That is, $kav^{a-1}k^\frac{1}{a}(v^a - c)^\frac{1}{a} - kav^a + v^q - (1-d)(p-1)k(v^a - c) > 0, \quad v > c^\frac{1}{a}$.

A sufficient condition is $v^q > kav^a + (1-d)(p-1)k(v^a - c), \quad v^a > c$.

Since $a < q$, this condition is valid for appropriate constants $k$ and $c$. \hfill $\Box$

Lemma 3.3 is sharp in the following sense: there exist bounded positive radially symmetric solutions if $q = (N+\beta)p - 1$ and $\alpha < \beta + p$ for $p < N+\alpha$. A straightforward calculation shows that (21) has solutions of the form

$$u(x) = C(x + |x|^{\frac{\beta+p-\alpha}{p-1}})^{-\frac{N+\alpha-p}{N+p-\alpha}},$$

with $(\frac{N+\alpha-p}{p-1}C)^{p-1} = \frac{m_0}{(\alpha+\beta)p}$ and $\kappa > 0$. 
The Liouville type results for the non-weighted case are needed in the following proofs. Here we recall the following Lemma.

**Lemma 3.4 ([23])**. Assume that $q > p - 1$ for $p \geq N$, or $p - 1 < q < Np - p - 1$ for $p < N$, and $m(x)$ is an appropriately smooth function with $0 < m_0 \leq m(x) \leq M_0$. Then the problem

\[
\begin{align*}
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx &= \int_{\mathbb{R}^N} m(x) u^q \varphi \, dx, \quad \forall 0 \leq \varphi \in C^1_0(\mathbb{R}^N), \\
0 < u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N),
\end{align*}
\]  

(23)

has no solution.

To prove the existence of periodic solutions, we reduce the problem of finding nontrivial solutions to the problem of establishing a priori estimates by the following lemma, which can be found in [1].

**Lemma 3.5.** Let $\mathbb{R}^+ := [0, +\infty)$ and $(E, \| \cdot \|)$ be a real Banach space, and let $G : \mathbb{R}^+ \times E \to E$ be completely continuous. Suppose that $G(0,0) = 0$ and there exists an $R > 0$ such that (i) $u \in E, \|u\| \leq R$ and $u = G(0,u)$ implies $u = 0$; (ii) $\deg(id - G(\cdot, \cdot), B(0, R), 0) = 1$. Let $J$ denote the set of solutions to the problem $u = G(k, u)$ in $\mathbb{R}^+ \times E$, and let $\mathcal{L}$ denote the component (closed connected subset maximal with respect to inclusion) of $J$ to which $(0,0)$ belongs. If

\[ \mathcal{L} \cap \{0\} \times E = \{0,0\}, \]

then $\mathcal{L}$ is unbounded in $\mathbb{R}^+ \times E$.

Here we would like to give an intuitive explanation of the reason why $k$ is introduced in $G$. We are concerned with the operator $G(0,\cdot)$ that admits an isolated zero fixed point and we try to prove the existence of another non-zero fixed point. The idea is to connect those two points together in the $(k, u)$ space by introducing a parameter $k$. Now the zero fixed point of $G(0, \cdot)$ is no longer isolated in the $(k, u)$ space and the continuous component of fixed points $(k, u)$ must intersect with the $\{0\} \times E$ with some positive fixed point provided that this component is a priori bounded in the $(k, u)$ space.

Define an operator

\[ G : \mathbb{R}^+ \times L^\infty_\omega((\tau, \tau + \omega), L^\bar{q}(\Omega; |x|^{\beta})) \to L^\infty_\omega((\tau, \tau + \omega), L^\bar{q}(\Omega; |x|^{\beta})) \]

by $G(k,v) = u$, where $\bar{q} > \max\{2q, \frac{(N+\beta q)^2}{\beta + p - \alpha} + 1\}$ is a constant, $u$ is the solution of the following problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(|x|^\alpha |\nabla u|^{p-2} \nabla u) + m(x,t) |x|^{\beta} (|v| + k)^q, \quad (x,t) \in Q_\omega, \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (\tau, \tau + \omega), \\
u(x, \tau) &= u(x, \tau + \omega), \quad x \in \Omega.
\end{align*}
\]  

(24)

It is clear that $u \geq 0$. First, we verify the compactness and continuity of the operator $G$.

**Lemma 3.6.** Assume that the hypotheses of Lemma 2.4 hold. Then the operator $G$ is completely continuous.

**Proof.** To verify the compactness, we first make some a priori estimates. Multiplying (24) by $u^r$, integrating over $\Omega$, and using Lemma 2.4 yield

\[
\frac{1}{r+1} \int_\Omega u^{r+1} \, dx + r \left( \frac{p}{p + r - 1} \right)^p \int_\Omega |\nabla u|^{p - \frac{r+1}{r}} |x|^\alpha \, dx
\]
Using (28) and a similar argument as in establishing (27), we further obtain that
\[
G \text{ obtain the continuity of }
\]
\[
\text{The compactness of the operator } G \text{ follows from (27), (29) and (30). It is easy to obtain the continuity of } G \text{ by a similar procedure. This completes the proof.} \]

Next, we check the conditions of \( G \) in Lemma 3.5.
Lemma 3.7. Assume that the hypotheses of Lemma 2.4 hold. Then $G(0, 0) = 0$ and there exists an $R > 0$ such that (i) $u \in E$, $\|u\| \leq R$ and $u = G(0, u)$ implies $u = 0$; (ii) $\deg(\text{id} - G(0, \cdot), B(0, R), 0) = 1$.

Proof. Let $u = G(0, 0)$. Multiplying (24) with $v = 0$ and $k = 0$ by $u$, integrating over $Q_\omega$ yields

$$\iint_{Q_\omega} |\nabla u|^p |x|^\alpha \, dx \, dt = 0.$$

By the weighted Sobolev inequality Lemma 2.4, we see that

$$\iint_{Q_\omega} u^p |x|^\beta \, dx \, dt \leq C \iint_{Q_\omega} |\nabla u|^p |x|^\alpha \, dx \, dt = 0,$$

since $\alpha < \beta + p$, which implies that $u = 0$ a.e. in $Q_\omega$. Next, we shall show that there exists an $R > 0$ such that if $u = G(0, u)$ and

$$\sup_t \|u(\cdot, t)\|_{L^q(\Omega; |x|^{\beta})} < R,$$

then $u \equiv 0$. Taking $k = 0$ and $v = u$ in (24), then multiplying it by $u^{\tilde{q} - q}$ and integrating over $\Omega$ yield

$$\frac{1}{q - q + 1} \frac{d}{dt} \int_{\Omega} u^{\tilde{q} - q + 1} \, dx + (\tilde{q} - q) \left( \frac{p}{q - q + p - 1} \right)^p \int_{\Omega} |\nabla u|^{\tilde{q} + p - 1} |x|^\alpha \, dx$$

$$\leq \frac{m}{\alpha} \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx.$$

By virtue of Lemma 2.4, we get

$$\frac{1}{q - q + 1} \frac{d}{dt} \int_{\Omega} u^{\tilde{q} - q + 1} \, dx + \frac{1}{C} (\tilde{q} - q) \left( \frac{p}{q - q + p - 1} \right)^p \left( \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx \right)^{\tilde{q} + p - 1}$$

$$\leq \frac{m}{\alpha} \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx,$$

provided $-\frac{N + \beta}{q - q + p - 1} < 1 - \frac{N + \alpha}{p}$, i.e. $\tilde{q} > \frac{N + \beta}{\beta + p - \alpha} (q - (p - 1))$, which is valid. Thus, we have

$$\frac{1}{q - q + 1} \frac{d}{dt} \int_{\Omega} u^{\tilde{q} - q + 1} \, dx$$

$$\leq \left( \frac{m}{\alpha} - \frac{1}{C} (\tilde{q} - q) \left( \frac{p}{q - q + p - 1} \right)^p \left( \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx \right)^{-\tilde{q} - p + 1} \right) \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx.$$

If

$$\sup_t \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx \leq R_0,$$

with

$$R_0 = \left( \frac{\tilde{q} - q \left( \frac{p}{q - q + p - 1} \right)^p}{2Cm} \right)^{-\tilde{q} + p - 1},$$

then

$$\frac{1}{q - q + 1} \frac{d}{dt} \int_{\Omega} u^{\tilde{q} - q + 1} \, dx \leq -\frac{m}{\alpha} \int_{\Omega} u^{\tilde{q}} |x|^\beta \, dx,$$

which means that $u = 0$ a.e. in $Q_\omega$. It remains to show that there exists an $R < R_0$ such that

$$\deg(\text{id} - G(0, \cdot), B(0, R), 0) = 1.$$
Consider the following problem

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = \text{div}(|x|^\alpha|\nabla u|^{p-2}\nabla u) + \sigma m(x,t)|x|^\beta |v|^{q}, \quad (x,t) \in Q_\omega, \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (\tau, \tau + \omega), \\
u(x,\tau) = u(x,\tau + \omega), \quad x \in \Omega,
\end{aligned}
\]

where \( \sigma \in [0,1] \). Construct a homotopic mapping

\[ T : [0, 1] \times L^\infty_0((\tau, \tau + \omega), L^\tilde{\alpha}(\Omega; |x|^\beta)) \to L^\infty_0((\tau, \tau + \omega), L^{\tilde{\alpha}}(\Omega; |x|^\beta)), \]

by \( T(\sigma, v) = u \). Similar to Lemma 3.6, we see that \( T \) is completely continuous. Assume that

\[ \sup_{t \in (\tau, \tau + \omega)} \|v(\cdot, t)\|_{L^\tilde{\alpha}(\Omega; |x|^\beta)} \leq R, \]

where \( R \leq R_0 \) is to be determined. Multiplying the first equation of (31) by \( u^r \), integrating over \( \Omega \), we get (25) with \( k = 0 \) and \( m(x,t) \) replaced by \( \sigma m(x,t) \), that is

\[
\begin{aligned}
&\frac{1}{r+1} \frac{d}{dt} \int_\Omega u^{r+1} dx + r\left(\frac{p}{p+r-1}\right)^p \int_\Omega |\nabla u|^{\frac{p+1}{p+r-1}} |x|^\alpha dx \\
&\leq CR^\alpha \left(\int_\Omega |\nabla u|^{\frac{p+1}{p+r-1}} |x|^\alpha dx\right)^{\frac{r+1}{r+p-1}}.
\end{aligned}
\]

Similar to the proof of (26), we have

\[
\int_\Omega \int_{Q_\omega} |\nabla u|^{\frac{p+1}{p+r-1}} |x|^\alpha dx dt \leq CR^{\frac{q(r+p-1)}{r-p+1}}.
\]

where \( C \) is independent of \( \sigma, v, u \) and \( R \). Furthermore, we can take \( r \) in the above inequality large as \( \tilde{r} \), such that \( -\frac{N}{r+p-1} < 1 - \frac{N+\alpha}{p} \) and

\[
\int_\Omega \int_{Q_\omega} u^{r+1} dx dt \leq C \left(\int_\Omega \int_{Q_\omega} |\nabla u|^{\frac{p+1}{p+r-1}} |x|^\alpha dx dt\right)^{\frac{r+1}{r+p-1}}
\]

\[
\leq C(R^{\frac{q(r+p-1)}{p+r-1}})^{\frac{r+1}{r+p-1}} \leq CR^{\frac{q(r+1)}{p-1}}.
\]

By the integral mean value theorem, we see that there exists a \( t_\sigma \in [\tau, \tau + \omega) \) such that

\[ \int_\Omega u^{r+1}(x, t_\sigma) dx \leq CR^{\frac{q(r+1)}{p-1}}. \]

Then the periodicity of \( u \) and a similar argument as Theorem 2.5 (Step 1) lead to

\[ \sup_{t \in (\tau, \tau + \omega)} \int_\Omega u^{r+1}(x, t) dx \leq CR^{\frac{q(r+1)}{r-1}} + CR^{\frac{q(r+p-1)}{r-1}}. \]

For appropriately large \( R \), Hölder’s inequality gives

\[ \sup_{t \in (\tau, \tau + \omega)} \left(\int_\Omega |v(x,t)|^q dx\right)^{\frac{1}{q}} \leq CR^{\frac{q}{r-1}} + CR^{\frac{q(r+p-1)}{r-1}} < R, \]

if \( R \) with \( R \leq R_0 \) is appropriately small, since \( q > p - 1 > 1 \). Therefore,

\[ \deg(\text{id} - G(0, \cdot), B(0, R), 0) = \deg(\text{id} - T(1, \cdot), B(0, R), 0) = \deg(\text{id} - T(0, \cdot), B(0, R), 0) = 1. \]

This completes the proof. \( \square \)

Now we can apply Lemma 3.5 to show the following existence result.
Theorem 3.8. Assume that the hypotheses of Lemma 2.4 hold, $\Omega$ is a convex domain, $0 < m(x, t) \in C^1(\Omega \times [\tau, \tau + \omega])$.

(i) If
\[ p - 1 < q \begin{cases} \leq \frac{(N + \beta)(p - 1)}{N + \alpha - p}, & 1 < p < N + \alpha, \\ < +\infty, & p \geq N + \alpha, \end{cases} \]
then the problem (1)–(3) admits at least one positive periodic solution.

(ii) If
\[ \frac{(N + \beta)(p - 1)}{N + \alpha - p} < q < \frac{(N + \beta)p}{N + \alpha - p} - 1, \quad 1 < p < N + \alpha, \]
m(x, t) = m(t), and further $N = 1$ or $\Omega = B_{R_0}(0)$ with $R_0 > 0$, then the problem (1)–(3) admits at least one positive periodic solution.

Proof. According to Lemma 3.5, we only need to verify the boundedness of the set $\mathcal{L}$, which is shown by a contradiction argument. Suppose $\mathcal{L}$ is not bounded, then there exist two sequences, $k_n$ and $u_n$, such that $u_n = G(k_n, u_n)$ and
\[ k_n + \sup_{(r, r + \omega)} \| u_n(r, t) \|_{L^p(\Omega; |x|^q)} \to \infty, \quad n \to \infty, \]
which implies that
\[ k_n + \| u_n \|_{L^\infty(Q_{\omega})} \to \infty, \quad n \to \infty. \] (33)
The rest of the proof is rather complicated and we present an outline as follows:
(i) show that $k_n/\| u_n \|_{L^\infty(Q_{\omega})} \to 0$ and $\| u_n \|_{L^\infty(Q_{\omega})} \to +\infty$ as $k \to \infty$;
(ii) find the accumulation point such that $x_n \to x_0$ and $t_n \to t_0$ as $k \to \infty$;
(iii) if $x_0 \neq 0$, rescale the time and spatial variables together with the solutions similar to the non-weighted case;
(iv) if $x_0 = 0$, the rescaling coincides with the weights;
(v) in both cases (iii) and (iv), derive energy estimates to provide locally uniform convergence to a Liouville type problem;
(vi) finish the contradiction argument by utilizing the Liouville type results in both non-weighted case and weighted case.

We claim that
\[ \| u_n \|_{L^\infty(Q_{\omega})} \to 0, \quad n \to \infty. \] (34)
Suppose to the contrary, there exists a constant $C > 0$ and a subsequence denoted by the same symbol such that $\| u_n \|_{L^\infty(Q_{\omega})}/k_n \leq C$. Note that if $k_n$ is bounded, then (33) shows $\| u_n \|_{L^\infty(Q_{\omega})} \to \infty$, which means (34). Thus, without loss of generality, we may assume that $0 < k_n \to +\infty$. Making change of variable
\[ v_n = \frac{u_n}{k_n}, \]
we have
\[ k_n^{2-p} \frac{\partial v_n}{\partial \tau} - \text{div}(\{ |x|^\alpha |\nabla v_n|^{p-2}\nabla v_n \}) = k_n^{q-(p-1)} m(x, t)|x|^\beta (v_n + 1)^q. \]
For any $\varphi \in C^1_\omega(Q_{\omega})$ with $\varphi|_{\partial \Omega} = 0$, we have
\[ \int_{Q_{\omega}} k_n^{2-p} \int_{Q_{\omega}} \frac{\partial v_n}{\partial \tau} \varphi \, dxdt + \int_{Q_{\omega}} |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \varphi |x|^{\alpha} \, dxdt = k_n^{q-(p-1)} \int_{Q_{\omega}} m(x, t)(v_n + 1)^q \varphi |x|^{\beta} \, dxdt. \]
We may take \( \varphi = v_n \) by the density of \( C_+^1(Q_\omega) \) in \( L^p_\omega((\tau, \tau + \omega), W^{1,p}(\Omega; |x|^\alpha)) \). Then
\[
\iint_{Q_\omega} |\nabla v_n|^p |x|^\alpha \, dx \, dt \leq C m k_n^{p-1}.
\]
In addition, for any \( 0 \leq \varphi(x) \in C_+^1(\Omega) \), we also have
\[
k_n^{p-1} \iint_{Q_\omega} m(x,t)|\varphi^\beta| \, dx \, dt
\leq k_n^{p-1} \iint_{Q_\omega} m(x,t)(v_n + 1)\varphi|x|^{\beta} \, dx \, dt
\]
\[
= \iint_{Q_\omega} |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \varphi |x|^{\alpha} \, dx \, dt
\]
\[
\leq \left( \iint_{Q_\omega} |\nabla v_n|^p |x|^{\alpha} \, dx \, dt \right)^{\frac{p-1}{p}} \left( \iint_{Q_\omega} |\nabla \varphi|^p |x|^{\alpha} \, dx \, dt \right)^{\frac{1}{p}}
\leq \left( C m k_n^{p-1} \right)^{\frac{p-1}{p}} \left( \iint_{Q_\omega} |\nabla \varphi|^p |x|^{\alpha} \, dx \, dt \right)^{\frac{1}{p}},
\]
which is
\[
k_n^{p-1} \iint_{Q_\omega} m(x,t)|\varphi^\beta| \, dx \, dt \leq C \left( \iint_{Q_\omega} |\nabla \varphi|^p |x|^{\alpha} \, dx \, dt \right)^{\frac{1}{p}}.
\]
Clearly, it is a contradiction since \( k_n \to \infty \). Therefore, (34) holds, which also implies that \( \|u_n\|_{L^\infty(Q_\omega)} \to \infty \).

Let \( \rho_n = \|u_n\|_{L^\infty(Q_\omega)} = u_n(\cdot, t_n) \to \infty \). Since the weights \( |x|^\alpha \) and \( |x|^\beta \) only degenerate at the interior point 0 \( \in \Omega \), by the convexity of \( \Omega \) and similar to the non-weighted case, there exists a \( \delta_0 > 0 \) such that \( \text{dist}(x_n, \partial \Omega) \geq \delta_0 \), see for example [8, 17]. Then we can choose a subsequence, for simplicity, denoted by the same symbol, such that \( x_n \to x_0, t_n \to t_0 \), with \( \text{dist}(x_0, \partial \Omega) \geq \delta_0 \).

If \( x_0 \neq 0 \), let
\[
w_{nj}(y,s) = \rho_n^{-q} u_n(y + \frac{x_0}{\rho_n} \cdot y, t_j + j s), \quad \bar{m}_{nj}(y,s) = m(y + \frac{x_0}{\rho_n} \cdot y, t_j + j s),
\]
and
\[
\Omega_n = \{ y; y = \rho_n^{-q} (x - x_0), x \in \Omega \}, \quad Q_{nj} = \Omega_n \times \left( \frac{\tau - t_j}{j}, \frac{\tau + \omega - t_j}{j} \right).
\]
Then \( w_{nj} \) with \( \|w_{nj}\|_{L^\infty(Q_{nj})} = 1 \) satisfies
\[
\rho_n^{1-q} \frac{\partial w_{nj}}{\partial s} - j \text{div} \left( |x_0 + \rho_n^{-q} \frac{x}{y} - \frac{\alpha}{\beta} y | \nabla w_{nj} \right)^{p-2} \nabla w_{nj} = j \bar{m}_{nj}(y,s) |x_0 + \rho_n^{-q} \frac{x}{y} - \frac{\alpha}{\beta} y | \beta (w_{nj} + k_n \rho^{-1})^q.
\]
Therefore for any \( \psi(y,s) \in C_{\omega/j}^1(Q_{nj}) \) with \( \psi = 0 \) on \( \partial \Omega_n \), we have
\[
\iint_{Q_{nj}} \rho_n^{1-q} \frac{\partial w_{nj}}{\partial s} \psi \, dy \, ds + j \iint_{Q_{nj}} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \psi |x_0 + \rho_n^{-q} \frac{x}{y} - \frac{\alpha}{\beta} y | \alpha \, dy \, ds
\]
\[
= j \iint_{Q_{nj}} \bar{m}_{nj}(y,s) (w_{nj} + k_n \rho^{-1})^q \psi |x_0 + \rho_n^{-q} \frac{x}{y} - \frac{\alpha}{\beta} y | \beta \, dy \, ds.
\]
For simplicity, we denote $x(y) = x_0 + \rho_n \frac{y - (p-1)}{p} y$ for $y \in \Omega_n$. Taking $\psi = w_{nj}$, we have

$$j \int_{Q_{nj}} |\nabla w_{nj}|^p |x(y)|^\alpha dy ds = j \int_{Q_{nj}} \tilde{m}_{nj}(y, s) |w_{nj}|^{p-1} q w_{nj} |x(y)|^{\beta} dy ds \leq C \int_{\Omega} |x(y)|^{\beta} \rho_n^{\frac{2-(p-1)}{p}} dx \leq C |\Omega| |x|^{\beta} \rho_n^{\frac{2-(p-1)}{p}} N,$$

which means that there exists $\sigma_j \in \left[ \frac{T_{-t_1}}{2}, \frac{T+\omega-t_1}{2} \right)$ such that

$$\int_{\Omega_n} |\nabla w_{nj}(y, \sigma_j)|^p |x(y)|^\alpha dy \leq C |\Omega| |x|^{\beta} \rho_n^{\frac{2-(p-1)}{p}} N.$$  

Here $|\Omega| |x|^{\beta} = \int_{\Omega} |x|^{\beta} dx$ denotes the weighted measure. For any $s > \sigma_j$, taking $\psi = \chi(\sigma_j, s) - \frac{\partial w_{nj}}{\partial x}$ in (35) yields

$$\int_{\Omega_n} |\nabla w_{nj}(y, s)|^p |x(y)|^\alpha dy \leq \int_{\Omega_n} |\nabla w_{nj}(y, \sigma_j)|^p |x(y)|^\alpha dy + \frac{p}{q+1} \int_{\Omega_n} \tilde{m}_{nj}(y, s) |w_{nj}|^{p-1} q w_{nj} |x(y)|^{\beta} dy \leq C |\Omega| |x|^{\beta} \rho_n^{\frac{2-(p-1)}{p}} N.$$  

By the periodicity of $w_{nj}$, the above inequality holds for all $s \in \left( \frac{T-t_1}{2}, \frac{T+\omega-t_1}{2} \right)$. That is

$$\sup_{s} \int_{\Omega_n} |\nabla w_{nj}(y, s)|^p |x(y)|^\alpha dy \leq C |\Omega| |x|^{\beta} \rho_n^{\frac{2-(p-1)}{p}} N.$$  

In addition, we note that for any $\varphi \in C_0^1(\Omega_n)$,

$$j \int_{Q_{nj}} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \varphi |x(y)|^\alpha dy ds = j \int_{Q_{nj}} \tilde{m}_{nj}(y, s) |w_{nj}|^{p-1} q \varphi |x(y)|^{\beta} dy ds.$$  

By Lebesgue’s differential theorem, there exists $s_j \in \left( \frac{T-t_1}{2}, \frac{T+\omega-t_1}{2} \right)$ such that

$$\int_{\Omega_n} |\nabla w_{nj}|^{p-2} \nabla w_{nj}(y, s_j) \cdot \nabla \varphi |x(y)|^\alpha dy = \int_{\Omega_n} \tilde{m}_{nj}(y, s_j) |w_{nj}(y, s_j) + k_n \rho_n^{-1} q \varphi |x(y)|^{\beta} dy.$$  

Then there exists a function $w_n \in W^{1,p}(\Omega_n; |x(y)|^\alpha)$ with $\|w_n\|_{L^\infty} = 1$ such that as $j \to \infty$ (passing to a subsequence if necessary)

$$\nabla w_{nj} \rightharpoonup \nabla w_n$$

in $L^p(\Omega_n; |x(y)|^\alpha)$, \quad $w_{nj} \to w_n$ in $L^r(\Omega_n)$ for any $r > 1$, \quad \text{and} \quad w_{nj} \to w_n$ in $L^1(\Omega_n; |x(y)|^{-\beta}).$
and

\[ \tilde{m}_{n}(y, s_{j}) \to \tilde{m}(y) \text{ locally uniformly,} \quad \tilde{m}(y) \in C^{\gamma}(\Omega_{n}) \text{ for some } 0 < \gamma < 1. \]

In fact, we note that

\[ \tilde{m}_{n}(y, s_{j}) = \int_{B_{r}} \tilde{m}(y, x_{0} + \rho_{n}^{q-p-1} y) dx \]

for some \( r_{j} \in [\tau, \tau + \omega] \) since \( m(x, t) \) is \( \omega \)-periodic with respect to time. Further, according to the uniform continuous of \( m(x, t) \) on \( \Omega \times [\tau, \tau + \omega] \), we find

\[ \int_{B_{r}} \tilde{m}(y, x_{0} + \rho_{n}^{q-p-1} y) dx \to \tilde{m}(y) \]

locally uniformly on \( \Omega_{n} \) as \( j \to \infty \),

provided that \( r_{j} \to r_{0} \) (passing to a subsequence). Since \( \{r_{j}\} \subset [\tau, \tau + \omega] \), we know that the above convergence of time sequence is valid. Then we obtain that

\[ \int_{\Omega_{n}} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi |x(y)|^{\alpha} dy = \int_{\Omega_{n}} \tilde{m}(w_n + k_n \rho_n^{-1}) \varphi |x(y)|^{\beta} dy. \]

Take \( \varphi = w_n \eta^{2p}(y) \), where

\[ \eta(y) = \begin{cases} 1, & y \in B_{R}(0), \\ 0, & y \in B_{2R}(0) \end{cases} \]

with \( 0 \leq \eta \leq 1 \) sufficiently smooth and \( |\nabla \eta| \leq \frac{C}{R} \). Then for sufficiently large \( n \), we have \( B_{2R} \subset \Omega_{n}, \frac{1}{2} |x_{0}| \leq |x(y)| = |x_{0} + \rho_{n}^{q-p-1} y| \leq 2 |x_{0}| \) for \( y \in B_{2R} \), and

\[ \int_{B_{2R}} \eta^{2p} |\nabla w_n|^{p} |x(y)|^{\alpha} dy \]

\[ = -2p \int_{B_{2R}} \eta^{2p-1} w_n |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \eta |x(y)|^{\alpha} dy \]

\[ + \int_{B_{2R}} \tilde{m}(w_n + k_n \rho_n^{-1}) \eta^{2p} w_n \eta^{2p} |x(y)|^{\beta} dy \]

\[ \leq \frac{1}{2} \int_{B_{2R}} \eta^{2p} |\nabla w_n|^{p} |x(y)|^{\alpha} dy + \frac{C}{R^{p}} \int_{B_{2R}} \eta^{p} w_n^{p} |x(y)|^{\alpha} dy \]

\[ + m \int_{B_{2R}} (w_n + k_n \rho_n^{-1})^{q} |w_n \eta^{2p} |x(y)|^{\beta} dy \]

\[ \leq \frac{1}{2} \int_{B_{2R}} \eta^{2p} |\nabla w_n|^{p} |x(y)|^{\alpha} dy + CR^{N-p} + CR^{N}. \]

That is

\[ \int_{B_{R}} |\nabla w_n|^{p} dy \leq C \int_{B_{2R}} \eta^{2p} |\nabla w_n|^{p} |x(y)|^{\alpha} dy \leq CR^{N}, \quad (37) \]

for sufficiently large \( R > 0 \), since \( 0 < \frac{1}{2} |x_{0}| \leq |x(y)| \leq 2 |x_{0}| \). Then there exists a function \( \hat{w} \in W_{loc}^{1,p}(\mathbb{R}^{N}) \) such that, passing to a subsequence if necessary, as \( n \to \infty \)

\[ \tilde{m}_{n}(y) \to \tilde{m}(y), \quad |x(y)| \to |x_{0}| \text{ uniformly on } B_{R}, \]

\[ \nabla w_n \to \nabla \hat{w} \text{ in } L^{p}(B_{R}), \quad w_n \to \hat{w} \text{ in } L^{r}(B_{r}) \text{ for any } r > 1. \]

Then we have

\[ \left\{ \begin{array}{l}
\int_{B_{R}} |\nabla \hat{w}|^{p-2} \nabla \hat{w} \cdot \nabla \varphi dy = \int_{B_{R}} \tilde{m}(y) |x_{0}|^{\beta-\alpha} \hat{w}^{q} \varphi dy \text{ for any } \varphi \in C_{c}^{1}(B_{R}), \\
\|\hat{w}\|_{L^{\infty}(B_{R})} = 1, \quad \hat{w} > 0, \quad y \in B_{R}.
\end{array} \right. \]
Taking balls $B_R$ larger and larger, and repeating the argument for the subsequence $u_n$ obtained in the previous step, we get a Cantor diagonal subsequence, still denoted by $w_k$ for convenience, which converges in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ to a function $w \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ satisfying
\[
\begin{cases}
\int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \, dy = \int_{\mathbb{R}^N} \tilde{m}(y) |x|^{\beta-\alpha} w^q \varphi \, dy \text{ for any } \varphi \in C^1_0(\mathbb{R}^N), \\
\|w\|_{L^\infty(\mathbb{R}^N)} = 1, \quad w > 0, \quad y \in \mathbb{R}^N.
\end{cases}
\] (38)

We can now conclude that (38) is a contradiction. Indeed, thanks to the non-weighted Liouville type result Lemma 3.4, for $p - 1 < q < p^*(\alpha, \beta) \leq p^*(0,0)$, we see that (38) has no solution.

It remains to prove the case $x_0 = 0$, in which the rescaling coincides with the weights. If $x_0 = 0$, let
\[w_{nj}(y,s) = \rho_n^{-1} u_n(\rho_n^{\frac{q-(p-1)}{p}} y, t_j + js), \quad \tilde{m}_{nj}(y,s) = m(\rho_n^{\frac{q-(p-1)}{p}}, y, t_j + js),\]
and
\[\Omega_n = \{y; y = \frac{\rho_n^{\frac{q-(p-1)}{p}} x, x \in \Omega\}, \quad Q_{nj} = \Omega_n \times \left(\frac{\tau - t_j}{j}, \frac{\tau + \omega - t_j}{j}\right)\}.
Then $w_{nj}$ with $\|w_{nj}\|_{L^\infty(\Omega_n)} = 1$ satisfies
\[\rho_n^{1-q + \frac{q-(p-1)}{p}} \frac{\partial w_{nj}}{\partial s} - j \text{div}(|y|^\alpha |\nabla w_{nj}|^{p-2} \nabla w_{nj}) = j \tilde{m}_{nj}(y,s)|y|^\beta (w_{nj} + k_n \rho_n^{-1})^q.
Therefore for any $\psi(y,s) \in C^1_{\omega/\omega_j}(Q_{nj})$ with $\psi = 0$ on $\partial \Omega_n$, we have
\[\int_{Q_{nj}} \rho_n^{1-q + \frac{q-(p-1)}{p}} \frac{\partial w_{nj}}{\partial s} \psi \, dyds + j \int_{Q_{nj}} |\nabla w_{nj}|^{p-2} \nabla w_{nj} \cdot \nabla \psi |y|^\alpha \, dyds = j \int_{Q_{nj}} \tilde{m}_{nj}(y,s)(w_{nj} + k_n \rho_n^{-1})^q \psi |y|^\beta \, dyds.
Similar to the case $x_0 \neq 0$, we repeat the argument therein by replacing $x(y) = y$ for $y \in \Omega_n$ and some other modifications. In fact, instead of (36), we have
\[\sup_s \int_{\Omega_n} |\nabla w_{nj}(y,s)|^p |y|^\alpha \, dy \leq C|\Omega_n|^{\frac{(q-(p-1)}{p} N}.
Then (37) can be replaced by
\[\int_{B_R} |\nabla w_n|^p |y|^\alpha \, dy \leq CR^{N+\alpha}.
Then there exists a function $\hat{w} \in W^{1,p}_{\text{loc}}(\mathbb{R}^N; |y|^\alpha)$ such that, passing to a subsequence if necessary, as $n \to \infty$
\[\tilde{m}_n(y) \to \tilde{m}(y), \quad |x(y)| \to |x_0| \text{ uniformly on } B_R,
\nabla w_n \to \nabla \hat{w} \text{ in } L^p(B_R; |y|^\alpha), \quad w_n \to \hat{w} \text{ in } L^r(B_r) \text{ for any } r > 1.
Then we have
\[\begin{cases}
\int_{B_R} |\nabla \hat{w}|^{p-2} \nabla \hat{w} \cdot \nabla \varphi |y|^\alpha \, dy = \int_{B_R} \tilde{m}(y) \hat{w}^q \varphi |y|^\beta \, dy, \quad \text{for any } \varphi \in C^1_0(B_R),
\|\hat{w}\|_{L^\infty(B_R)} = 1, \quad \hat{w} > 0, \quad y \in B_R.
\end{cases}
\]
Similarly, taking balls $B_R$ larger and larger, and repeating the above argument, we get a Cantor diagonal subsequence, still denoted by $w_k$, which converges in $W^{1,p}_0(\mathbb{R}^N; |y|^\alpha)$ to a function $w \in W^{1,p}_0(\mathbb{R}^N; |y|^\alpha)$, such that

$$
\begin{align*}
\left\{ \begin{array}{ll}
\int_{\mathbb{R}^N} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi |y|^\alpha dy = \int_{\mathbb{R}^N} \tilde{m}(y)w^q \varphi |y|^\beta dy & \text{for any } \varphi \in C_0^1(\mathbb{R}^N), \\
\|w\|_{L^\infty(\mathbb{R}^N)} = 1, & w > 0, \quad y \in \mathbb{R}^N.
\end{array} \right.
\end{align*}
$$

(39)

Again, we can now conclude that (39) is a contradiction. Indeed, for the case $q > p - 1$ with $p \geq N + \alpha$ and the case $p - 1 < q \leq \frac{(N+\beta)(p-1)}{N+\alpha-p}$ with $1 < p < N + \alpha$, the weighted Liouville type result Lemma 3.2 implies that (39) has no solution. Next, for the case $\frac{(N+\beta)(p-1)}{N+\alpha-p} < q < \frac{(N+\beta)p}{N+\alpha-p} - 1$ with $1 < p < N + \alpha$ and $N = 1$, the same contradiction follows from Lemma 3.3. Last, for the case $\frac{(N+\beta)(p-1)}{N+\alpha-p} < q < \frac{(N+\beta)p}{N+\alpha-p} - 1$ with $1 < p < N + \alpha$ and $N \geq 2$, $\Omega = B_{R_0}(0)$, we may restrain ourselves in searching for radially symmetric solutions. That is, the operator $G$ is replaced by

$$
G : \mathbb{R}^+ \times L^\infty_{\omega}((\tau, \tau + \omega), \tilde{L}^q(B_{R_0}(0); |x|^\beta)) \to L^\infty_{\omega}((\tau, \tau + \omega), \tilde{L}^q(B_{R_0}(0); |x|^\beta))
$$

as $G(k, \tau) = u$, where $u$ is the solution of the problem (24) and $\tilde{L}^q(B_{R_0}(0); |x|^\beta)$ is the radially symmetric subset of $L^q(B_{R_0}(0); |x|^\beta)$. A simple phase plane analysis similar to the proof of Lemma 3.3 shows that the maximum of $\|u_n\|_{L^\infty(\Omega_S); \mu_n = u_n(x_n, \tau_n)}$ must be attained at $x_n = 0$. Thus the only accumulation point of $\{x_n\}$ is $x_0 = 0$ and the functions obtained by the change of variable are also radially symmetric. Therefore, the weighted Liouville type result Lemma 3.2 in radially symmetric case implies that (39) has no solution. The above contradictions imply that $k_n + \|u_n\|_{L^\infty} = 0$ is uniformly bounded. This completes the proof.

The results of Theorem 3.8 are sharp in the sense of the following nonexistence results.

**Theorem 3.9.** Asssume that the hypotheses of Lemma 2.4 hold, $1 < p < N + \alpha$ and $0 < m(x, t) \in C^1(\overline{\Omega} \times [\tau, \tau + \omega])$. If $m(x, t)$ depends on $t$, we further assume that $x \cdot \nabla m(x, t) \leq 0$ for $(x, t) \in Q_\omega$. If $\Omega$ is star-shaped and $q > \frac{(N+\beta)p}{N+\alpha-p} - 1$, or $\Omega$ is strictly star-shaped and $q \geq \frac{(N+\beta)p}{N+\alpha-p} - 1$, then there is no positive periodic solution to the problem (1)-(3).

**Proof.** If $m$ is independent of $t$, then the periodic solution of the problem (1)-(3) must be a steady state, as proved in the Step 3 of Theorem 2.5. Therefore, there is no positive solution if $\Omega$ is star-shaped.

If $m$ depends on $t$, we assume that $x \cdot \nabla m(x, t) \leq 0$. Suppose that (1)-(3) admits a positive periodic solution $u$. For any fixed $s \in [\tau, \tau + \omega]$, multiplying (1) by $x_i \frac{\partial u(x, s)}{\partial x_i}$, and integrating over $\Omega$, then summing up from 1 to $N$ yields

$$
\begin{align*}
\sum_{i=1}^N \int_{\Omega} u_t(x, t)x_i \frac{\partial u(x, s)}{\partial x_i} dx &= \sum_{i=1}^N \int_{\Omega} \text{div}(\|x|^\alpha |\nabla u|^{p-2} \nabla u(x, t))x_i \frac{\partial u(x, s)}{\partial x_i} dx \\
&= \sum_{i=1}^N \int_{\Omega} m(x, t)|x|^\beta u^q(x, t)x_i \frac{\partial u(x, s)}{\partial x_i} dx.
\end{align*}
$$
Similarly, multiplying (1) by \(u(x,s)\) and integrating over \(\Omega\) yields
\[
\int_{\Omega} u(x,s) u(x,t) \, dx + \int_{\Omega} |x|^\alpha |\nabla u|^{p-2} \nabla u(x,s) \cdot \nabla u(x,t) \, dx
\]
\[
= \int_{\Omega} m(x,t) |x|\beta u^q(x,t) u(x,s) \, dx.
\]
Combining the above two equalities, we obtain that
\[
\int_{\Omega} u_i(x,t) \left( \sum_{i=1}^N x_i \frac{\partial u(x,s)}{\partial x_i} + \frac{N + \alpha - p}{p} u(x,s) \right) \, dx
\]
\[
- \sum_{i=1}^N \int_{\Omega} \text{div}(|x|^\alpha |\nabla u|^{p-2} \nabla u(x,t)) x_i \frac{\partial u(x,s)}{\partial x_i} \, dx
\]
\[
+ \frac{N + \alpha - p}{p} \int_{\Omega} |x|^\alpha |\nabla u|^{p-2} \nabla u(x,t) \cdot \nabla u(x,s) \, dx
\]
\[
= \sum_{i=1}^N \int_{\Omega} m(x,t) |x|\beta u^q(x,t) x_i \frac{\partial u(x,s)}{\partial x_i} \, dx \tag{40}
\]
\[
+ \frac{N + \alpha - p}{p} \int_{\Omega} m(x,t) |x|\beta u^q(x,t) u(x,s) \, dx.
\]
For simplicity, denote the first term of the left-hand side of the above equality by \(J_1(t)\). By the periodicity of \(u\), we get \(\int_{\tau}^{\tau+\omega} J_1(t) \, dt = 0\). By the integral mean value theorem, there exists \(t \in [\tau, \tau + \omega]\) such that \(J_1(t) = 0\). Define a map
\[
\mathcal{T} : [\tau, \tau + \omega] \to [\tau, \tau + \omega],
\]
by \(\mathcal{T}(s) = t_s\). Since \(\mathcal{T}([\tau, \tau + \omega]) = [\tau, \tau + \omega] \leq 0\), there exists \(t^* \in [\tau, \tau + \omega]\) such that \(\mathcal{T}(t^*) = t^*\). Thus, by (40),
\[
- \sum_{i=1}^N \int_{\Omega} \text{div}(|x|^\alpha |\nabla u|^{p-2} \nabla u(x,t^*)) x_i \frac{\partial u(x,t^*)}{\partial x_i} \, dx + \frac{N + \alpha - p}{p} \int_{\Omega} |x|^\alpha |\nabla u(x,t^*)|^p \, dx
\]
\[
= \sum_{i=1}^N \int_{\Omega} m(x,t^*) |x|\beta u^q(x,t^*) x_i \frac{\partial u(x,t^*)}{\partial x_i} \, dx + \frac{N + \alpha - p}{p} \int_{\Omega} m(x,t^*) |x|\beta u^{q+1}(x,t^*) \, dx. \tag{41}
\]
Denote the two terms on the left-hand side of (41) by \(J_2(t^*)\) and \(J_3(t^*)\), the two terms on the right-hand side by \(J_4(t^*)\) and \(J_5(t^*)\). By a straightforward calculation, we obtain (see also [19]),
\[
J_2(t^*) = \sum_{i=1}^N \int_{\Omega} |x|^\alpha |\nabla u|^{p-2} \nabla u(x,t^*) \cdot \nabla \left( x_i \frac{\partial u(x,t^*)}{\partial x_i} \right) \, dx
\]
\[
- \int_{\partial \Omega} |x|^\alpha |\nabla u|^{p-2} \nabla u(x,t^*) \cdot \nu(x) \left( x_i \frac{\partial u(x,t^*)}{\partial x_i} \right) \, d\sigma
\]
\[
= \int_{\Omega} |x|^\alpha |\nabla u(x,t^*)|^p \, dx \tag{42}
\]
\[
- \sum_{i=1}^N \int_{\Omega} |x|^\alpha x_i \frac{\partial}{\partial x_i} |\nabla u(x,t^*)|^p \, dx
\]
\[
- \int_{\partial \Omega} |x|^\alpha |\nabla u(x,t^*)|^p \left( x \cdot \nu(x) \right) \, d\sigma
\]
where $\nu(x)$ is the unit outward normal at the point $x \in \partial \Omega$, and the equality
\[
\sum_{i=1}^{N} |\nabla u|^{p-2} \nabla u(x, t^*) \cdot \nu(x) \left( x, \frac{\partial u(x, t^*)}{\partial x_i} \right) = |\nabla u(x, t^*)|^p \left( x \cdot \nu(x) \right)
\]
is valid since $\nabla u(x, t^*)$ is parallel to $\nu(x)$, which is deduced by the boundary condition (2) $u(x, t^*) = 0$ for $x \in \partial \Omega$. We further obtain that
\[
J_4(t^*) = \sum_{i=1}^{N} \int_{\Omega} m(x, t^*) |x|^\beta x_i \frac{1}{q + 1} \frac{\partial}{\partial x_i} u^{q+1}(x, t^*) \, dx
\]
\[
= - \frac{N + \beta}{q + 1} \int_{\Omega} m(x, t^*) |x|^\beta u^{q+1}(x, t^*) \, dx - \frac{1}{q + 1} \int_{\Omega} |x|^\beta u^{q+1}(x, t^*) x \cdot \nabla m(x, t^*) \, dx.
\]
Summing up, we obtain
\[
\left( \frac{N + \alpha - p}{p} - \frac{N + \beta}{q + 1} \right) \int_{\Omega} m(x, t^*) |x|^\beta u^{q+1}(x, t^*) \, dx
\]
\[
+ \left( 1 - \frac{1}{p} \right) \int_{\partial \Omega} |x|^\alpha |\nabla u(x, t^*)|^p \left( x \cdot \nu(x) \right) \, d\sigma
\]
\[
= \frac{1}{q + 1} \int_{\Omega} |x|^\beta u^{q+1}(x, t^*) x \cdot \nabla m(x, t^*) \, dx \leq 0.
\]
Note that $x \cdot \nu(x) \geq 0$ if $\Omega$ is star-shaped and $x \cdot \nu(x) \geq \delta > 0$ if $\Omega$ is strictly star-shaped. Thus if $q > \frac{(N+\beta)p}{N+\alpha - p}$, we see that $u(x, t^*) = 0$. Or else if $q = \frac{(N+\beta)p}{N+\alpha - p}$ and $x \cdot \nu(x) \geq \delta > 0$, we obtain $\nabla u(x, t^*) = 0$ for $x \in \partial \Omega$, which together with the boundary condition (2) yields $u(x, t^*) = 0$. It contradicts to the positive condition.

4. The singular case. Now we consider the case of $q = p - 1$. Similar to the non-weighted periodic problem (see [27]), we shall show the speciality of this case, namely, the existence of positive periodic solutions depends on the value of $m(x, t)$.

According to Lemma 2.6, the eigenvalue problem (13) admits its first eigenvalue $\lambda_1 > 0$ and the first eigenfunction $0 < \varphi_1(x) \in L^\infty(\Omega)$, provided that the hypotheses of Lemma 2.4 hold.

**Theorem 4.1.** Assume that the hypotheses of Lemma 2.4 hold and $q = p - 1$.

(i) If $m = \sup m(x, t) < \lambda_1$ or $m = \inf m(x, t) > \lambda_1$, then the problem (1)–(3) admits no positive periodic solution;

(ii) If $m(x, t) \equiv \lambda_1$, there is a positive periodic solution to the problem (1)–(3).

**Proof.** If $m(x, t) \equiv \lambda_1$, then clearly the first eigenfunction $\varphi_1(x)$ is a periodic solution of the problem (1)–(3). We only need to prove (i).

Consider the initial and boundary value problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div} \left( |x|^\alpha |\nabla u|^{p-2} \nabla u \right) + m(x, t) |x|^\beta u^{p-1}, \\
&= \frac{\partial u}{\partial t}, \\
u(x, t) &= 0, \\
\end{aligned}
\]
\[
\begin{aligned}
&u(x, t) = 0, \\
&u(x, 0) = u_0(x), \\
&x \in \Omega, \quad t > 0, \\
&x \in \Omega,
\end{aligned}
\]
where $u_0(x) \geq 0$ for $x \in \Omega$ and $u_0$ satisfies the compatibility condition. Suppose that $u(x, t)$ is a positive solution of (42).
If $m < \lambda_1$, there exists $\tilde{\lambda}$ with $m < \tilde{\lambda} < \lambda_1$ and a domain $\hat{\Omega}$ with $\Omega \subset \subset \hat{\Omega}$ such that $\tilde{\lambda}$ is the first eigenvalue of the weighted $p$-Laplacian eigenvalue problem (13) on $\hat{\Omega}$, and correspondingly, $\tilde{\psi}$ is the first eigenfunction with $\|	ilde{\psi}\|_{L^\infty} = 1$. Further, there exists a constant $\delta > 0$ such that $\tilde{\psi} \geq \delta$ for $x \in \hat{\Omega}$. A simple calculation shows that $K\tilde{\psi}$ is an upper solution of the problem (42) for appropriately large $K > 0$. Since the source $m(x, t)|x|^\beta u^{\alpha-1}$ is locally Lipschitz continuous for $u \in (0, +\infty)$ and $K\delta \leq K\tilde{\psi} \leq K$ for $x \in \Omega$, the comparison principle Lemma 2.2 holds. Then we have $u \leq K\psi$. Let $w$ be the solution of the following problem

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(|x|^\alpha|\nabla u|^{p-2}\nabla u) + m(x, t)|x|^\beta u^{\alpha-1}, \\
u(x, t) &= (K\psi)e^{-t}, \\
u(x, 0) &= K\tilde{\psi}(x),
\end{align*}
$$

(43)

where $\psi$ is an upper solution of problem (42) on $\hat{\Omega}$. We note that there exists a function $w$ such that $w(x) = \lim_{t \to \infty} u(x, t)$. It follows that $w(x)$ is a steady state of the first equation of (43) with homogeneous Dirichlet boundary condition. Clearly, we have $w(x) = 0$ since $m(x, t) < \lambda_1$, which means that $u(x, t)$ tends to $0$ uniformly as $t \to \infty$.

Next, we consider the case $m > \lambda_1$. Take $\hat{\Omega} \subset \subset \Omega$ such that the first eigenvalue $\lambda$ of the weighted $p$-Laplacian eigenvalue problem (13) on $\hat{\Omega}$ satisfying $\lambda_1 < \lambda < m$. Let $\phi$ be the first eigenfunction and $\tilde{\phi}$ be its zero-extension to $\hat{\Omega}$.

When $p > 2$, take $\tilde{u}_0(x) \leq u_0(x)$ satisfying that $0 \leq \tilde{u}_0(x) \neq 0$ for $x \in \hat{\Omega}$ and $\tilde{u}_0 = 0$ for $x \in \partial \hat{\Omega}$. Consider the following problem

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(|x|^\alpha|\nabla u|^{p-2}\nabla u) + m_\lambda|x|^\beta u^{\alpha-1}, \\
u(x, t) &= 0, \\
u(x, 0) &= \tilde{u}_0(x),
\end{align*}
$$

(44)

Denote by $\tilde{u}$ the solution of the above problem. Clearly, the solution $u$ of problem (42) is an upper solution of problem (44) on $\hat{\Omega}$. We note that there exists a function $\delta(T) > 0$ such that $u(x, t) \geq \delta(T)$ for $x \in \Omega$ and $t \in [0, T]$. By the comparison principle, we have $u \geq \tilde{u}$. Similar to the non-weighted result of [14], we see that $\tilde{u}$ blows up in finite time if $p > 2$ since $m > \lambda$, which means that $u$ blows up in finite time.

When $1 < p < 2$, let $\tilde{u} = g(t)\phi$, where $g(t)$ satisfies that

$$
\begin{cases}
g'(t) = (m_\lambda - \lambda)g^{p-1}(t), & t > 0, \\
g(t) > 0, & t > 0, \\
g(0) = 0.
\end{cases}
$$

The condition $1 < p < 2$ ensures the existence of a positive solution of the above problem. We see that $\tilde{u}$ is a lower solution of the problem (42). Furthermore, by the fast diffusion property of (42), we see that $u(x, t) > 0$ for $t > 0$. By the comparison principle, we obtain that $u(x, t) \geq g(t)\phi(x)$. Thus, $\|u\|_{L^\infty}$ tends to infinity since $g(t) \to \infty$ as $t \to \infty$.

It remains to show the case of $p = 2$. Suppose that $u(x, t)$ is a periodic solution of the problem (1)–(3) with $m(x, t) \geq m > \lambda > \lambda_1$. Since the weights only degenerate at the interior point 0, similar to the non-weighted case, we have the local regularity $u \in C^{2+\alpha,1+\alpha/2}(\Omega \setminus \{0\} \times [\tau, \tau + \omega])$ and $\varphi_1 \in C^{2,\alpha}(\Omega \setminus \{0\})$. Let $B_\delta(0)$ be a ball such
that $0 \in B_{\delta}(0) \subseteq \Omega$. There exists a constant $\kappa_1 > 0$, such that $u(x,t) \geq \kappa_1 \varphi_1(x)$ for $x \in \Omega \setminus B_{\delta}(0)$ and $t \in [\tau, \tau + \omega]$. As shown by Lemma 2.6, $\varphi_1 \in L^\infty(\Omega)$. Hence, there exists a constant $\kappa_2 > 0$ such that $u(x,t) \geq \kappa_2 \varphi(x)$ for $x \in B_{\delta}(0)$ and $t \in [\tau, \tau + \omega]$. Take $\kappa = \min\{\kappa_1, \kappa_2\}$. Consider the following periodic problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div}(|x|^p |\nabla u|^{p-2} \nabla u) + \hat{\lambda}|x|^2 u^q, \\
|u(x,t)| &= 0, \\
|u(x,t)| &= u(x,t+\omega),
\end{aligned}
\]

(45)

Since $u(x,t) \geq \kappa \varphi(x)$ and $n(x,t) > \hat{\lambda} > \lambda_1$, we see that $u(x,t)$ and $\kappa \varphi(x)$ are upper and lower solutions of the periodic problem (45). By Lemma 2.3, the problem (45) admits a bounded positive periodic solution, which is a steady state since $\hat{\lambda}$ is a constant. It follows that the eigenvalue problem (13) admits a bounded positive eigenfunction and a positive eigenvalue $\hat{\lambda} > \lambda_1$, which is a contradiction. \hfill \Box

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