The Complexity of Recognizing Facets for the Knapsack Polytope

Rui Chen\textsuperscript{1}, Haoran Zhu\textsuperscript{2}

\textsuperscript{1}Cornell Tech (rui.chen@cornell.edu)
\textsuperscript{2}Microsoft (haoranzhu@microsoft.com)

Abstract. The complexity class \(D^p\) is the class of all languages that are the intersection of a language in \(NP\) and a language in \(co-NP\), as coined by Papadimitriou and Yannakakis (1982). Hartvigsen and Zemel (1992) conjectured that recognizing a facet for the knapsack polytope is \(D^p\)-complete. While it has been known that the recognition problems of facets for polytopes associated with other well-known combinatorial optimization problems, e.g., traveling salesman, node/set packing/covering, are \(D^p\)-complete, this conjecture on recognizing facets for the knapsack polytope remains open. We provide a positive answer to this conjecture. Moreover, despite the \(D^p\)-hardness of the recognition problem, we give a polynomial time algorithm for deciding if an inequality with a fixed number of distinct positive coefficients defines a facet of a knapsack polytope, generalizing a result of Balas (1975).

Keywords: Knapsack polytope · Facet · \(D^p\)-completeness · Parameterized complexity

1 Introduction

The polyhedral approach has been crucial for the success of solving combinatorial optimization (CO) problems of practical sizes over the last few decades. Many important CO problems can be reformulated as a linear optimization problem over certain discrete sets of vectors. The study of the convex hulls of such discrete sets is a central topic in polyhedral combinatorics as it leads to linear programming reformulations of these CO problems. The convex hulls of such discrete sets associated with some of the well-studied CO problems such as the travelling salesman problem (TSP), the clique problem and the knapsack problem are called the TSP polytope, the clique polytope and the knapsack polytope (KP). The characterization of the facets (i.e., non-redundant linear inequalities) of these polytopes is one of the main subjects in polyhedral combinatorics (see, e.g., [21]). However, a complete list of the non-redundant linear inequalities describing the combinatorial polytope is generally very hard to obtain. Karp and Papadimitriou [15] show that, unless \(NP=co-NP\), there does not exist a computational tractable description by linear inequalities of the polyhedron associated with any \(NP\)-complete CO problem.
Although it is hard to obtain all facet-defining inequalities, from the mixed-integer programming (MIP) perspective, obtaining strong valid inequalities can be critical for reducing the number of nodes required in the branch-and-cut procedure. There has been a very large body of literature aimed at generating valid inequalities for certain combinatorial polytopes, see, e.g., [7][11][16]. In particular, there has been significant interest in studying valid inequalities for the knapsack polytope ([1][2][9][22]), given that knapsack constraints are building blocks of general binary integer programs. Empirically, Crowder et al. [8] and Boyd [3] have shown that the feasible region of many binary integer programming programs can be well-approximated by inequalities valid for individual knapsack polytopes.

As it is only computationally feasible to enumerate some valid inequalities for a CO problem, a natural question to ask next is regarding the strength of these inequalities: Given an inequality and an instance of a CO problem, is this inequality facet-defining for the associated combinatorial polytope? We denote this decision problem by CO FACETS where CO is the specific CO problem. Karp, Papadimitriou and Yannakakis are the first ones taking a theoretical perspective to this problem, and studying its computational complexity. For the decision problem TSP FACETS, some partial results concerning the complexity of this problem are obtained by Karp and Papadimitriou [15]. They show that if TSP FACETS ∈ NP, then NP=co-NP. To provide a more general complexity class for the decision problem of recognizing whether an inequality is a facet of a particular polytope, in another seminal paper by Papadimitriou and Yannakakis [19], they introduce a new complexity class, D^p, defined as the class of all languages that are the intersection of a language in NP and a language in co-NP. It is important to note that NP∩co-NP is a proper sub-class of D^p, as is NP∪co-NP. The complexity class D^p is a natural niche for many important classes of problems. For instance, as the motivation problem in [19], TSP FACETS is in D^p. This is because, a facet-defining inequality of a polytope P is essentially a valid inequality that holds at equality at dim(P) affinely independent points in P. So determining whether an inequality is facet-defining for a polytope is equivalent to deciding (i) if this inequality is valid to the polytope (co-NP problem), and (ii) if there exist dim(P) affinely independent points in P that satisfy the inequality with equality (NP problem). Papadimitriou and Yannakakis [19] show that some other interesting combinatorial problems, including critical problems, exact problems and unique solution problems, are naturally in D^p. Some problems were later shown to be complete for D^p. In particular, Cai and Meyer [5] show that the graph minimal 3-colorability problem is D^p-complete. Rothe [20] show that the exact-4-colorability problem is D^p-complete. Recently, Bulut and Ralphs [4] show that the optimal value verification problem for inverse MIP is D^p-complete. Regarding CO FACETS, in the original paper of Papadimitriou and Yannakakis [19], they show that CLIQUE FACETS is D^p-complete, and conjecture the same hardness for TSP FACETS. This conjecture was later proved by Papadimitriou and Wolfe [18]. When studying the complexity of lifted inequalities for the knapsack problem, along with some other interesting results, Hartvigsen and Zemel [13] show that recognizing valid inequalities for the knapsack polytope is co-NP-
complete, and conjectured that \textit{KNAPSACK FACETS} is $\mathcal{D}^p$-complete. One of the main contributions of this paper is that we give a positive answer to this conjecture.

Despite the $\mathcal{D}^p$-completeness of the facet-recognition problem associated with the KP, one can recognize specific facets of the KP in polynomial time. In Theorem 1 of the paper [1], Balas shows that for an inequality with only binary coefficients on the left-hand side, whether this inequality is facet-defining for a knapsack polytope can be determined in polynomial time. In this paper, we further extend this result to a more general scenario: as long as the inequality has a constant number of distinct positive coefficients, the corresponding \textit{KNAPSACK FACETS} can be solved in polynomial time. In fact, we will show that, the \textit{KNAPSACK FACETS} can be solved in time $n^{K+O(1)}$, where $K$ is the number of distinct positive coefficients of the inequality and $n$ is the dimension.

The remainder of the paper is organized as follows. In Section 2, along with a few auxiliary $\mathcal{D}^p$-complete results, we establish that the recognition problem of a supporting hyperplane for the knapsack polytope is $\mathcal{D}^p$-complete. In Section 3, we prove the main result of this paper, which is that recognizing facets for knapsack polytope is also $\mathcal{D}^p$-complete. In Section 4, we give a polynomial time algorithm for \textit{KNAPSACK FACETS} on inequalities with fixed number of distinct coefficients. Lastly, in Section 5, we show that the problem of recognizing if a given point is in a given knapsack polytope is $\mathcal{NP}$-complete.

\textbf{Notation.} For an integer $n$ we set $[n] := \{1, 2, \ldots, n\}$. We let $\mathbb{N}$ denote the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \ldots\}$. For a vector $w \in \mathbb{R}^n$ and $S \subseteq [n]$, we set $w(S) := \sum_{i \in S} w_i$. For a sequence $f \in \mathbb{R}^\mathbb{N}$ and $S \subseteq \mathbb{N}$ with $|S| < \infty$, we set $f(S) := \sum_{i \in S} f_i$. We let $e_i$ denote the $i$-th unit vector (in the Euclidean vector space with the appropriate dimension).

2 Critical Subset Sum and KP Supporting Hyperplane Problems

In [19], Papadimitriou and Yannakakis show that the \textit{TSP supporting hyperplane problem}, which is the problem of deciding if a given inequality with integer coefficients provides a supporting hyperplane to the given TSP polytope, is $\mathcal{D}^p$-complete. In this section, we extend the same completeness result to the following \textit{KP supporting hyperplane problem}: Given an inequality $\alpha^T x \leq \beta$ with $\alpha \in \mathbb{Z}^n$ and a KP conv \( \{ x \in \{0, 1\}^n : a^T x \leq b \} \), is it true that this inequality is valid for the KP and the corresponding hyperplane has a nonempty intersection with the KP?

Before proceeding to the proof of the main result in this section, we first introduce a $\mathcal{D}^p$ problems that were previously defined in literature.

\textit{Exact vertex cover (EVC)}: Given graph $G = (V, E)$ and a positive integer $k$, is it true that the minimum vertex cover of $G$ has size exactly $k$, i.e., there exists $V'$ of size $k$ but no $V'$ of size $k - 1$ such that $V' \cap e \neq \emptyset$ for all $e \in E$? We will use $(V, E, k)$ to denote one particular instance of EVC.
It has been shown that a class of exact problems, including the exact vertex cover problem, are \( \mathsf{D^p}\)-complete.

**Theorem 1** \([19]\). \( \mathsf{EVC} \) is \( \mathsf{D^p}\)-complete.

In this section, we will first define an auxiliary problem, which we call the critical subset sum (CSS) problem, and show that \( \mathsf{EVC} \) is reducible to CSS, and then show that CSS is reducible to the KP supporting hyperplane problem, thus establishing \( \mathsf{D^p}\)-completeness of the \( \mathsf{KP} \) supporting hyperplane problem. Here we remark that all reductions we mention in this paper refer to the polynomial time many-one reduction, or Karp reduction \([14]\).

Now we define the CSS problem, which is a slight variant of the subset sum problem.

**Critical subset sum:** Given \( w \in \mathbb{Z}_+^n \) and a target-sum \( t \), is it true that there exists a subset \( S \subseteq \{1, \ldots, n\} \) such that \( w(S) = t - 1 \), but does not exist subset \( T \subseteq \{1, \ldots, n\} \) such that \( w(T) = t \)? We will use \((w,t)\) to denote one particular instance of CSS.

Next, using the standard reduction from **vertex cover** to **subset sum**, we prove the following result, which will play a crucial rule in the next section.

**Theorem 2.** CSS is \( \mathsf{D^p}\)-complete.

**Proof.** By Theorem \([1]\) it suffices to show that is reducible to CSS. Given instance \((V,E,k)\) of the exact vertex cover problem, define the following CSS instance. Assume \( V = \{v_1, \ldots, v_n\} \) and \( E = \{e_1, \ldots, e_m\} \). For \( i = 1, \ldots, n \), define \( w_i := 1 + \sum_{j=1}^{m} (n+1)^j \mathbb{I}(v_i \in e_j) \). For \( j = 1, \ldots, m \), define \( w_{n+j} := (n+1)^j \). Define \( t := n - k + 1 + \sum_{j=1}^{m} (n+1)^j \). Then the CSS instance \((w,t)\) has polynomial encoding size with respect to the input size of the EVC instance \((V,E,k)\).

Let \( \tilde{I} := \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\} \) and \( \tilde{V} := \{v_{i_1}, \ldots, v_{i_p}\} \subseteq V \). Note that \( \tilde{V} \) is a vertex cover of \( G \) if and only if the \((q+1)\)-th digit of \( w(\tilde{I}) \) (in base \( n+1 \)) is at least 1 for \( q = 1, \ldots, n \). Define \( \tilde{I} := \{1, \ldots, n\} \setminus \tilde{I} \). Then \( \tilde{V} \) is a vertex cover of \( G \) if and only if the \((q+1)\)-th digit of \( w(\tilde{I}) \) is at most 1 for \( q = 1, \ldots, n \). Also note that the first digit of \( w(\tilde{I}) \) is \( n - p \). Define \( \tilde{J} := \{n+j : j \in [m], \ e_j \cap \tilde{V} = \emptyset\} \). Then \( \tilde{V} \) being a vertex cover of \( G \) implies \( w(\tilde{I} \cup \tilde{J}) = n - p + \sum_{j=1}^{m} (n+1)^j \). On the other hand, \( w(S) = n - p + \sum_{j=1}^{m} (n+1)^j \) implies \( |\{v_i \in \tilde{V} \setminus S\} = \emptyset \) and \( \{v_i \in [n] \setminus S\} \) being a vertex cover of \( G \). Then by definitions of the EVC instance \((V,E,k)\) and the CSS instance \((w,t)\), we have the EVC instance has a “yes” answer if and only if the CSS instance has a “yes” answer.

The above theorem has an immediate corollary on the **exact knapsack** (EK) problem which asks: Given a \( n \)-dimensional vector \( c \), a knapsack constraint \( a^T x \leq b \) and an integer \( L \), is it true that \( \max \{c^T x : a^T x \leq b, x \in \{0,1\}^n \} = L \)?

**Corollary 1.** EK is \( \mathsf{D^p}\)-complete.
Proof. It is easy to see that $E \in \mathbb{D}$. We next show a reduction from $CSS$ to $E$. Let $(w, t)$ be a $CSS$ instance. Then this instance has “yes” answer if and only if $\max\{w^T x : w^T x \leq t, x \in \{0,1\}^n\} = t - 1$, which is a “yes” answer to a particular $E$ instance. ⊓ ⊔

We are finally in the position to prove the main result of this section.

Theorem 3. The KP supporting hyperplane problem is $\mathbb{D}$-complete.

Proof. By Theorem 2, it suffices to establish that $CSS$ is reducible to the KP supporting hyperplane problem. Given a $CSS$ instance $(w, t)$, consider the following instance of the KP supporting hyperplane problem: Given an inequality $\sum_{i=1}^n w_i x_i \leq t - 1$ and a KP conv ($\{x \in \{0,1\}^n : \sum_{i=1}^n w_i x_i \leq t\}$), is it true that this inequality is valid for the KP and the corresponding hyperplane has a nonempty intersection with the KP? It is easy to see that this KP supporting hyperplane instance has a “yes” answer if and only if $\sum_{i=1}^n w_i x_i = t - 1$. This is equivalent to saying that the $CSS$ instance $(w, t)$ has a “yes” answer. ⊓ ⊔

3 KNAPSACK FACETS

As one of the main results of this paper, in this section, we are going to resolve the conjecture raised by Hartvigsen and Zemel: $KNAPSACK$ FACETS is $\mathbb{D}$-complete. This result can also be seen as a stronger version of the KP supporting hyperplane problem in the previous section.

Before proving the main result of this section, we first present some results regarding an elegant sequence $(f_i)_{i=1}^\infty$ constructed by Gu in defined by:

$$f_1 = f_2 = f_3 = 1, \text{ and } f_i = f_{i-2} + f_{i-1} \text{ for } i \geq 4. \quad (1)$$

Note that this sequence is also used in [6] to construct a hard instance for sequentially lifting a cover inequality. The idea of incorporating the sequence $f$ into the reduction technique that we will use later to prove the main result, is also motivated by the constructive example in [6].

For this particular sequence $f$, we have the following observations, which can be easily verified by induction.

Observation 1 For $j \geq 3$, $f_j = \sum_{i=1}^{j-2} f_i$.

Observation 2 For $j \geq 3$, $\frac{\sqrt{2^j - 1}}{4} \leq f_j \leq 2^j$.

The sequence $f$ also has the following nice property.

Lemma 1 (Lemma 4.1 [6]). Let $f$ be defined as in (1) and $r \geq 1$ be a given integer. For any $\tau \in \mathbb{Z}^+$ satisfying $0 \leq \tau \leq \sum_{i=1}^{2r+1} f_i$, there exists a subset $S \subseteq [2r+1]$ such that $f(S) = \tau$.

We are now ready to prove one of the main results of this paper.
Theorem 4. **KNAPSACK FACETS** is \( \text{D}^\text{p} \)-complete.

Proof. It suffices to show that **CSS** is reducible to **KNAPSACK FACETS**, as **CSS** is \( \text{D}^\text{p} \)-complete according to Theorem 2. Consider any **CSS** instance \((w, t)\), i.e., “is it true that there exists \( S \subseteq [n] \) such that \( w(S) = t - 1 \), but there does not exist \( T \subseteq [n] \) such that \( w(T) = t \)?” Without loss of generality, here we assume that \( w_i \leq t - 1 \) for all \( i \in [n] \) and \( t \geq 2 \).

We next construct a **KNAPSACK FACETS** instance. Let \( L = w([n]) \), \( r = \lceil \log_2(30L + 20) - 1 \rceil \), and

\[
a_i = \begin{cases} 
  tf_i, & \text{for } i = 1, \ldots, 2r + 1, \\
  t(2L + 1) + 1, & \text{for } i = 2r + 2, \\
  (t + 1)w_{i-2r-2}, & \text{for } i = 2r + 3, \ldots, 2r + n + 2, \\
  t_{2r+1} + t^2 + t(2L + 2) + 1, & \text{for } i = 2r + n + 3, \\
  t + 1, & \text{for } i = 2r + n + 4.
\end{cases}
\]

(2)

\[
b = t \sum_{i=1}^{2r+1} f_i + t^2 + t(2L + 2) + 1,
\]

(3)

\[
\alpha_i = \begin{cases} 
  f_i, & \text{for } i = 1, \ldots, 2r + 1, \\
  2L + 2, & \text{for } i = 2r + 2, \\
  w_{i-2r-2}, & \text{for } i = 2r + 3, \ldots, 2r + n + 2, \\
  f_{2r+1} + t + 2L + 1, & \text{for } i = 2r + n + 3, \\
  0, & \text{for } i = 2r + n + 4.
\end{cases}
\]

(4)

\[
\beta = \sum_{i=1}^{2r+1} f_i + t + 2L + 1.
\]

(5)

Here \( N := 2r + n + 4 \) is the dimension of the vectors \( a \) and \( \alpha \). Consider the following instance of **KNAPSACK FACETS**: Given an inequality \( a^T x \leq \beta \) and a KP \( \text{conv}\{x \in \{0, 1\}^N : a^T x \leq b\} \), is this inequality facet-defining to the KP? It is easy to verify that the input size of this **KNAPSACK FACETS** instance is polynomial of that of the **CSS** instance \((w, t)\). To complete the proof of this theorem, we are going to show: there is a “yes” answer to the **CSS** problem \((w, t)\) if and only if \( a^T x \leq \beta \) is a facet-defining inequality to the KP \( \text{conv}\{x \in \{0, 1\}^N : a^T x \leq b\} \).

Given the **CSS** instance and the **KNAPSACK FACETS** instance, we have the following claims.

Claim 1. \( \sum_{i=1}^{2r} f_i > 3L + 2 \).

Proof of claim. The claim follows from \( \sum_{i=1}^{2r} f_i = f_{2r+2} \geq \frac{\sqrt{2} - 1}{4} \sqrt{2^{2r+2}} > 2^{r+1}/10 \geq \frac{\sqrt{2} + \sqrt{20}}{10} = 3L + 2 \), where the first equality is from Observation 1, the second inequality is from Observation 2 and the last inequality is from the definition of \( r \).
Claim 2. Inequality \( \sum_{i=1}^{2^r+1} \alpha_i x_i \leq \sum_{i=1}^{2^r} f_i \) is a facet-defining inequality for KP \( \text{conv}\{ x \in \{0,1\}^{2^r+1} : \sum_{i=1}^{2^{r+1}} a_i x_i \leq \sum_{i=1}^{2^r} t f_i \} \).

**Proof of claim.** Note that for \( \gamma = 1, \ldots, r \), by our definition in (2) and (3), both inequalities \( \sum_{i=1}^{2^\gamma+1} \alpha_i x_i \leq \sum_{i=1}^{2^\gamma} f_i \) and \( \sum_{i=1}^{2^{\gamma+1}} \alpha_i x_i \leq \sum_{i=1}^{2^\gamma} t f_i \) are identical to \( \sum_{i=1}^{2^\gamma+1} f_i x_i \leq \sum_{i=1}^{2^\gamma} f_i \). We prove a stronger version of the claim: For \( \gamma = 1, \ldots, r \), inequality \( \sum_{i=1}^{2^\gamma+1} \alpha_i x_i \leq \sum_{i=1}^{2^\gamma} f_i \) is a facet-defining inequality for KP \( \text{conv}\{ x \in \{0,1\}^{2^{\gamma+1}} : \sum_{i=1}^{2^{\gamma+1}} a_i x_i \leq \sum_{i=1}^{2^\gamma} t f_i \} \). We proceed by induction on \( \gamma \). When \( \gamma = 1 \), the claim is: \( x_1 + x_2 + x_3 \leq 2 \) is facet-defining for \( \text{conv}\{ x \in \{0,1\}^3 : x_1 + x_2 + x_3 \leq 2 \} \), which is obviously true. Assume that this claim is true when \( \gamma = R - 1 \) for some integer \( R \in [2, r - 1] \); \( \sum_{i=1}^{2^{R-1}} f_i x_i \leq \sum_{i=1}^{2^{R-2}} f_i \) is a facet-defining inequality for \( \text{conv}\{ x \in \{0,1\}^{2^{R-1}} : \sum_{i=1}^{2^{R-1}} f_i x_i \leq \sum_{i=1}^{2^{R-2}} f_i \} \). So there exists affinely independent points \( v_1, \ldots, v_{2R-1} \in \{0,1\}^{2^{R-1}} \), satisfying \( \sum_{i=1}^{2^{R-1}} f_i x_i \leq \sum_{i=1}^{2^{R-2}} f_i \) at equality. For \( v \in \{0,1\}^{2^{R-1}} \), let \( (v, 0, 1) \) denote the binary point in \( \{0,1\}^{2^{R+1}} \) obtained by appending to \( v \) two new components with values 0 and 1. It is then easy to verify that, for any \( j \in [2R-1], x = (v_j, 0, 1) \) satisfies \( \sum_{i=1}^{2^{R+1}} f_i x_i \leq \sum_{i=1}^{2^R} f_i \) at equality as \( f_{2R+1} = f_2 + f_{2r-1} \). Define \( p := (1, \ldots, 1, 0) \in \{0,1\}^{2^{R+1}} \), then \( \sum_{i=1}^{2^{R+1}} f_i p_i = \sum_{i=1}^{2^R} f_i \). Define \( q := (0, \ldots, 0, 1, 1) \in \{0,1\}^{2^{R+1}} \), then \( \sum_{i=1}^{2^{R+1}} f_i q_i = f_2 + f_{2R+1} = \sum_{i=1}^{2^R} f_i \). Here the last equality is from Observation 1. Therefore, we have obtained the following \( 2^R + 1 \) binary points in \( \{0,1\}^{2^{R+1}} : (v_1, 0, 1), \ldots, (v_{2R-1}, 0, 1), p, q \), where \( v_1, \ldots, v_{2R-1} \) are affinely independent in \( \{0,1\}^{2^{R-1}} \). It is easy to see that these \( 2^R + 1 \) binary points are affinely independent in \( \{0,1\}^{2^{R+1}} \), and satisfy \( \sum_{i=1}^{2^{R+1}} f_i x_i \leq \sum_{i=1}^{2^R} f_i \) at equality. \( \square \)

Claim 3. Inequality \( \sum_{i=1}^{2^{r+2}} \alpha_i x_i \leq \sum_{i=1}^{2^r} f_i \) is a facet-defining inequality for KP \( \text{conv}\{ x \in \{0,1\}^{2^{r+2}} : \sum_{i=1}^{2^{r+2}} a_i x_i \leq \sum_{i=1}^{2^r} t f_i \} \).

**Proof of claim.** First, let’s verify that \( \sum_{i=1}^{2^{r+2}} \alpha_i x_i \leq \sum_{i=1}^{2^r} f_i \) is valid for such KP. When \( x_{2r+2} = 0 \), it is trivially valid. When \( x_{2r+2} = 1 \), the knapsack constraint implies that \( \sum_{i=1}^{2^{r+2}} f_i x_i \leq \sum_{i=1}^{2^r} f_i - (2L + 1) - 1 \). In this case \( \sum_{i=1}^{2^{r+1}} f_i x_i \leq \sum_{i=1}^{2^r} f_i - 2L - 2 \), which means that \( \sum_{i=1}^{2^{r+1}} f_i x_i + (2L + 2)x_{2r+2} \leq \sum_{i=1}^{2^r} f_i \) is a valid inequality. Second, from the last Claim, it suffices to show that there exists a binary point \( x^* \in \{0,1\}^{2^{r+2}} \) with \( x_{2r+2} = 1 \), such that \( \sum_{i=1}^{2^{r+1}} \alpha_i x_i + a_{2r+2} \leq \sum_{i=1}^{2^r} f_i \) while \( \sum_{i=1}^{2^{r+1}} \alpha_i x_i + a_{2r+2} = \sum_{i=1}^{2^r} f_i \). By definition of \( a \) in (2) and \( \alpha \) in (4), it suffices to find a binary point \( x^* \), such that \( \sum_{i=1}^{2^{r+1}} f_i x_i^* = \sum_{i=1}^{2^r} f_i - 2L - 2 \). From the above Claim, \( \sum_{i=1}^{2^r} f_i - 2L - 2 \geq L \). By Lemma 1, we know such binary point \( x^* \) must exist. \( \square \)

Claim 4. Inequality \( \sum_{i=1}^{2^{r+n+2}} \alpha_i x_i \leq \sum_{i=1}^{2^r} f_i \) is a facet-defining inequality for KP \( \text{conv}\{ x \in \{0,1\}^{2^{r+n+2}} : \sum_{i=1}^{2^{r+n+2}} a_i x_i \leq \sum_{i=1}^{2^r} t f_i \} \).
Proof of claim. By definition of \( a \) in (2) and \( \alpha \) in (4), we need to show that inequality

\[
\sum_{i=1}^{2r+1} f_i x_i + (2L + 2) x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_i x_i \leq \sum_{i=1}^{2r} f_i
\]

is facet-defining for the KP defined by the following knapsack constraint:

\[
\sum_{i=1}^{2r+1} tf_i x_i + (t(2L + 1) + 1) x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} (t+1) w_i x_i \leq \sum_{i=1}^{2r} tf_i.
\]

First of all, we verify that inequality (6) is indeed valid for the KP defined by constraint (7). For any binary point \( x \in \{0, 1\}^{2r+n+2} \), knapsack constraint (7) implies that

\[
\sum_{i=1}^{2r+1} f_i x_i \leq \sum_{i=1}^{2r} f_i - (2L + 1) x_{2r+2} - \sum_{i=2r+3}^{2r+n+2} w_i x_i - \left( \frac{x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_i x_i}{t} \right).
\]

Therefore, we have

\[
\sum_{i=1}^{2r+1} f_i x_i + (2L + 2) x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_i x_i \leq \sum_{i=1}^{2r} f_i - x_{2r+2} - \left( \frac{x_{2r+2} + \sum_{i=2r+3}^{2r+n+2} w_i x_i}{t} \right)\leq \sum_{i=1}^{2r} f_i,
\]

i.e., inequality (6) is valid. To complete the proof, it suffices to show that there exist \( 2r + n + 2 \) affinely independent binary points satisfying the knapsack constraint (7), on which (6) holds at equality. From the last Claim 3, we can find \( 2r + 2 \) affinely independent binary points \( v_1, \ldots, v_{2r+2} \) in \( \{0, 1\}^{2r+n+2} \) satisfying

\[
\sum_{i=1}^{2r+1} f_i x_i + (2L + 2) x_{2r+2} = \sum_{i=1}^{2r+1} f_i \quad \text{and} \quad \sum_{i=1}^{2r+1} tf_i x_i + (t(2L + 1) + 1) x_{2r+2} \leq \sum_{i=1}^{2r+1} tf_i.
\]

It implies that \( (v_1, 0, \ldots, 0), \ldots, (v_{2r+2}, 0, \ldots, 0) \) are affinely independent in \( \{0, 1\}^{2r+n+2} \), satisfying the knapsack constraint (7), and satisfying (6) at equality. Now, for each \( i \in [n] \), consider \( \sum_{j=1}^{2r} f_i - 2L - 2 - w_i \). By Claim 1, we know that \( \sum_{i=1}^{2r+1} f_i - 2L - 2 - w_i \geq 0 \). So from Lemma 4, we can find \( x^* \) with \( x^*_{2r+2} = x^*_{2r+2+j} = 1 \), and \( x^*_i = 0 \) for all \( j \in [n] \setminus \{i\} \), and

\[
\sum_{i=1}^{2r+1} f_i x^*_i = \sum_{i=1}^{2r} f_i - 2L - 2 - w_i.
\]

Also note that \( w_i \leq t - 1 \). It is then easy to verify that \( x^* \) satisfies the knapsack constraint (7), and satisfies (6) at equality. Therefore, we have found in total \( 2r + n + 2 \) binary points that satisfy knapsack constraint (7), and satisfy (6) at equality. Moreover, these \( 2r + n + 2 \) points are obviously affinely independent.
Now, we are ready to prove the validity of the reduction: there is a “yes” answer to the CSS instance \((w, t)\) if and only if \(\alpha^T x \leq \beta\) is a facet-defining inequality for the KP \(\text{conv}\{x \in \{0, 1\}^N : a^T x \leq b\}\).

We first verify that \(w(S) \neq t\) for all \(S \subseteq [n]\) if and only if inequality \(\alpha^T x \leq \beta\) is valid for the KP defined by \(a^T x \leq b\). In other words, we would like to show that \(w(S) \neq t\) for all \(S \subseteq [n]\) if and only if for all \(\bar{x} \in \{0, 1\}^N\) with \(a^T \bar{x} \leq b\), we have \(\alpha^T \bar{x} \leq \beta\). Consider an arbitrary \(\bar{x} \in \{0, 1\}^N\) with \(a^T \bar{x} \leq b\). Depending on the values of \(\bar{x}_{N-1}\) and \(\bar{x}_N\), we consider the following four cases.

1. \(\bar{x}_{N-1} = 1, \bar{x}_N = 0\). In this case, \(a^T x \leq b\) reduces to \(\sum_{i=1}^{2r+1} a_i x_i \leq 2t f_i\), and \(\alpha^T x \leq \beta\) is the same as \(\sum_{i=1}^{2r+1} \alpha_i x_i \leq \sum_{i=1}^{2r} f_i\). From Claim 4, we have that \(\alpha^T \bar{x} \leq \beta\) is always satisfied in this case.

2. \(\bar{x}_{N-1} = 1, \bar{x}_N = 1\). From \(a^T \bar{x} \leq b\), we have

\[
\sum_{i=1}^{2r+1} t f_i \bar{x}_i + (t(2L + 1) + 1) \bar{x}_{2r+2} + (t + 1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r} t f_i - t - 1.
\]

Since \(\bar{x} \in \{0, 1\}^N\), we have \(\sum_{i=1}^{2r+1} f_i \bar{x}_i \in \mathbb{Z}\). It implies that

\[
\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r} f_i - (2L + 1) \bar{x}_{2r+2} - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left[ \frac{1 + \bar{x}_{2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right].
\]

Hence, in this case we always have

\[
\alpha^T \bar{x} = \sum_{i=1}^{2r+1} f_i \bar{x}_i + (2L + 2) \bar{x}_{i+2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} + f_{2r+1} + t + 2L + 1
\leq \sum_{i=1}^{2r+1} f_i + \bar{x}_{2r+2} + t + 2L - \left[ \frac{1 + \bar{x}_{2r+2} + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right]
\leq \sum_{i=1}^{2r+1} f_i + t + 2L = \beta - 1.
\]

3. \(\bar{x}_{N-1} = 0, \bar{x}_N = 0\). In this case, if \(\bar{x}_{2r+2} = 0\), then \(\alpha^T \bar{x} \leq \sum_{i=1}^{2r+1} f_i + \sum_{i=1}^{n} w_i \bar{x}_i < \beta\). So we assume \(\bar{x}_{2r+2} = 1\). Then from \(a^T \bar{x} \leq b\), we have

\[
\sum_{i=1}^{2r+1} t f_i \bar{x}_i + (t + 1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+1} t f_i + t^2 + t.
\]

This implies

\[
\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t + 1 - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left[ \frac{\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t} \right]. \quad (8)
\]
Depending on the value of $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}$, we further consider the following three cases.

(3a) If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq t - 1$, then $\alpha^T \bar{x} = \sum_{i=1}^{2r+2} \alpha_i + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+2} f_i + 2L + 2 + t - 1 = \beta$.

(3b) If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} = t$, then consider the new point $\hat{x} \in \{0, 1\}^N$ with $\hat{x}_i = 1$ for $i = 1, \ldots, 2r+2$, $\hat{x}_i = \bar{x}_i$ for $i = 2r+3, \ldots, 2r+n+2$, $\hat{x}_{N-1} = \hat{x}_N = 0$. We have $\alpha^T \hat{x} = \sum_{i=1}^{2r+1} t f_i + t(2L+1) + 1 + t(t+1) = \beta$ while $\alpha^T \bar{x} = \sum_{i=1}^{2r+1} f_i + 2L + 2 + t = \beta + 1$. So here $\hat{x}$ is in the KP but it does not satisfy the inequality $\alpha^T x \leq \beta$.

(3c) If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \geq t+1$, then (3b) implies that $\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - 1$. It implies that $\alpha^T \bar{x} = \sum_{i=1}^{2r+1} f_i \bar{x}_i + 2L + 2 + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \beta$.

Cases (3a)-(3c) imply that $\alpha^T x \leq \beta$ is valid for any binary point $\bar{x}$ satisfying $\alpha^T \bar{x} \leq b$ and $\bar{x}_{N-1} = \bar{x}_N = 0$ if there does not exist $S \subseteq [n]$ such that $w(S) = t$, in which case (3b) does not happen. On the other hand, if there exists $S \subseteq [n]$ such that $w(S) = t$, then one can construct $\hat{x}$ like in case (3b) to show that $\alpha^T x \leq \beta$ is not valid. Therefore, $\alpha^T x \leq \beta$ is valid for any binary point $\bar{x}$ satisfying $\alpha^T \bar{x} \leq b$ and $\bar{x}_{N-1} = \bar{x}_N = 0$ if and only if there does not exist $S \subseteq [n]$ such that $w(S) = t$.

(4) $\bar{x}_{N-1} = 0, \bar{x}_N = 1$. In this case, if $\bar{x}_{2r+2} = 0$, then $\alpha^T \bar{x} \leq \sum_{i=1}^{2r+1} f_i + \sum_{i=1}^{n} w_i < \beta$. So we assume $\bar{x}_{2r+2} = 1$. Then $\alpha^T \bar{x} \leq b$ reduces to

$$\sum_{i=1}^{2r+1} f_i \bar{x}_i + (t+1) \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+1} t f_i + t^2 - 1.$$ 

This implies

$$\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - \left[1 + \frac{\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2}}{t}\right].$$

(9)

If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq t - 1$, then $\alpha^T \bar{x} = \sum_{i=1}^{2r+2} \alpha_i + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \sum_{i=1}^{2r+1} f_i + 2L + 2 + t - 1 = \beta$. If $\sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \geq t$, then (9) yields that $\sum_{i=1}^{2r+1} f_i \bar{x}_i \leq \sum_{i=1}^{2r+1} f_i + t - \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} - 2$. Therefore, $\alpha^T \bar{x} = \sum_{i=1}^{2r+1} f_i \bar{x}_i + 2L + 2 + \sum_{i=1}^{n} w_i \bar{x}_{i+2r+2} \leq \beta - 1$.

From the discussion of the above four cases, we have known that for any binary point $\bar{x} \in \{x \in \{0, 1\}^N : \alpha^T x \leq b\}$ with $\bar{x}_{N-1} + \bar{x}_N \geq 1$, inequality $\alpha^T \bar{x} \leq \beta$ always holds. At the same time, inequality $\alpha^T x \leq \beta$ is valid for any binary point $\bar{x}$ satisfying $\alpha^T \bar{x} \leq b$ and $\bar{x}_{N-1} = \bar{x}_N = 0$ if and only if there does not exist $S \subseteq [n]$ such that $w(S) = t$. We have thus concluded that $w(S) \neq t$ for any subset $S \subseteq [n]$ if and only if inequality $\alpha^T x \leq \beta$ is valid for $\{x \in \{0, 1\}^N : \alpha^T x \leq b\}$.

Lastly, we want to show that there exist $N$ affinely independent points in $\{x \in \{0, 1\}^N : \alpha^T x \leq b, \alpha^T x = \beta\}$ if and only if there exists subset $S \subseteq [n]$ such that $w(S) = t - 1$. 


From Claim 4, there exist \( v_1, \ldots, v_{N-2} \in \{0,1\}^{N-2} \) that are affinely independent, and they satisfy \( \sum_{i=1}^{N-2} a_i x_i \leq \sum_{i=1}^{2r} t f_i \) and \( \sum_{i=1}^{N-2} a_i x_i = \sum_{i=1}^{2r} f_i \). Therefore, points \((v_1, 1, 0), \ldots, (v_{N-2}, 1, 0) \in \{0,1\}^N\) are affinely independent, and they satisfy

\[
\sum_{i=1}^{N} a_i x_i \leq \sum_{i=1}^{2r} t f_i + a_{N-1} = \sum_{i=1}^{2r+1} t f_i + t^2 + t(2L + 2) + 1 = b
\]

and

\[
\sum_{i=1}^{N} \alpha_i x_i = \sum_{i=1}^{2r} f_i + \alpha_{N-1} = \sum_{i=1}^{2r+1} f_i + t + 2L + 1 = \beta.
\]

If there exists \( x^* \in \{0,1\}^n \) such that \( \sum_{i=1}^{n} w_i x^*_i = t-1 \), then we can construct another two points \( p, q \) as follows:

\[
p_i = \begin{cases} 
1, & \text{for } i = 1, \ldots, 2r + 1, \\
1, & \text{for } i = 2r + 2, \\
x^*_{i-2r-2}, & \text{for } i = 2r + 3, \ldots, 2r + n + 2, \\
0, & \text{for } i = 2r + n + 3, \\
0, & \text{for } i = 2r + n + 4.
\end{cases}
\]

Notice that

\[
a^T p \leq a^T q = \sum_{i=1}^{2r+1} t f_i + t(2L + 1) + 1 + \sum_{i=1}^{n} (t + 1) w_i x^*_i + t + 1 = b,
\]

\[
\alpha^T p = \alpha^T q = \sum_{i=1}^{2r+1} f_i + 2L + 2 + \sum_{i=1}^{n} w_i x^*_i = \sum_{i=1}^{2r+1} f_i + t + 2L + 1 = \beta.
\]

So both points \( p \) and \( q \) satisfy \( a^T x \leq b \) and \( \alpha^T x = \beta \). It is easy to check that these \( N \) points \((v_1, 1, 0), \ldots, (v_{N-2}, 1, 0)\), \( p, q \) are affinely independent points in \( \{x \in \{0,1\}^N : a^T x \leq b, \alpha^T x = \beta \} \).

On the other hand, assume that there exist \( N \) affinely independent points in \( \{x \in \{0,1\}^N : a^T x \leq b, \alpha^T x = \beta \} \). Then there must exist a point \( p^* \in \{x \in \{0,1\}^N : a^T x \leq b, \alpha^T x = \beta \} \) with \( p^*_N = 1 \), since otherwise \( \{x \in \{0,1\}^N : a^T x \leq b, \alpha^T x = \beta \} \) will be contained in the hyperplane given by \( x_N = 0 \), which violates the assumption. Furthermore, here \( p^*_{N-1} = 0 \), because if \( p^*_{N-1} = 1 \), then point \( p^* \) falls into the case (2) above, in which case we have \( \alpha^T p^* \leq \beta - 1 \), contradicting the assumption that \( \alpha^T p^* = \beta \). Hence, we have \( p^*_{N-1} = 0, p^*_N = 1 \), and \( p^* \) falls into the case (4) above. Following the same argument there, in order to have \( \alpha^T p^* = \beta \), we must have \( p^*_{2r+2} = 1 \) and \( \sum_{i=1}^{n} w_i p^*_{i+2r+2} \leq t - 1 \). If \( \sum_{i=1}^{n} w_i p^*_{i+2r+2} \leq t - 2 \), then \( \alpha^T p^* \leq \sum_{i=1}^{2r+1} f_i + 2L + 2 + \sum_{i=1}^{n} w_i p^*_{i+2r+2} \leq \sum_{i=1}^{2r+1} f_i + t + 2L = \beta - 1 \). Therefore, we must have \( \sum_{i=1}^{n} w_i p^*_{i+2r+2} = t - 1 \), which means that there exists subset \( S \subseteq [n] \) such that \( w(S) = t - 1 \).

All in all, we have established that:
There exists $N$ affinely independent points in $\{ x \in \{0,1\}^N : a^T x \leq b, a^T x = \beta \}$ if and only if if there exists subset $S \subseteq [n]$ such that $w(S) = t - 1$.

Combining (i) and (ii), we have shown that there is a “yes” answer to the CSS problem $(w, t)$ if and only if $a^T x \leq \beta$ is a facet-defining inequality to the KP $\text{conv}(\{ x \in \{0,1\}^N : a^T x \leq b \})$. \hfill $\Box$

4 KNAPSACK FACETS on Inequalities With Fixed Number of Distinct Coefficients

In the previous section, we have shown that for a general inequality and a knapsack polytope, it is $\mathcal{D}^{\mathcal{P}}$-complete to recognize whether this inequality is facet-defining for the knapsack polytope. However, for a canonical inequality, i.e., an inequality with binary left-hand-side coefficients, there exists a simple characterization of such inequalities that are facet-defining, provided by Balas \cite{Balas1977}.

**Theorem 5 (\cite{Balas1977}).** The inequality $\sum_{j \in M} x_j \leq k$, where $|M| \geq 2$, defines a facet of the knapsack polytope $\text{conv}(\{ x \in \{0,1\}^n : a^T x \leq b \})$ if and only if $M$ is the extension of a strong cover $C$ for $a^T x \leq b$, such that $|C| = k + 1$, and $\sum_{j \in T} a_j \leq b$, where $T = (S - \{ j_1, j_2 \}) \cup \text{argmax}_{i \in [n]} a_i$, with $j_1 = \text{argmax}_{j \in C} a_j$, and $j_2 = \text{argmax}_{j \in C - \{ j_1 \}} a_j$.

This theorem immediately implies that, determining whether a canonical inequality is facet-defining for a knapsack polytope $\text{conv}(\{ x \in \{0,1\}^n : a^T x \leq b \})$ can be done in polynomial time. In this section, we show a more general result, that is, assuming that each arithmetic operation takes $O(1)$ time, determining whether an inequality $a^T x \leq \beta$ with $(a, \beta) \in \mathbb{Z}_+^{n+1}$ is facet-defining for the knapsack polytope is slice-wise polynomial (XP) with respect to $|a|_+$, where $|a|_+$ denotes the cardinality of the set $\{ \alpha_i : \alpha_i > 0, i \in [n] \}$, i.e., the number of distinct positive values that coefficients $\alpha_i$ are taking. In other words, when $|a|_+$ is a fixed constant, the problem of determining if $a^T x \leq \beta$ is facet-defining for a knapsack polytope can be solved in polynomial time.

Throughout this section, let $S$ denote the knapsack set $\{ x \in \{0,1\}^n : a^T x \leq b \}$ and $P$ denote the knapsack polytope $\text{conv}(S)$. We assume $\|a\|_\infty \leq b$, in which case both $S$ and $P$ are full-dimensional. Then up to a positive scaling, any facet-defining inequality $a^T x \leq \beta$ of $P$ that is different from the non-negativity constraints $x \geq 0$ must have $(a, \beta) \in \mathbb{Z}_+^{n+1}$. Without loss of generality, we assume that $a^T x = \sum_{k=1}^{K} \gamma_k \sum_{i \in I_k} x_i$ where $K := |a|_+, \gamma \in \mathbb{N}^K$, $I_k = \{ i_{k-1} + 1, i_{k-1} + 2, \ldots, i_k \}$ with $0 = i_0 \leq i_1 \leq \ldots \leq i_K \leq n$, vector $a$ satisfies $a_{i_{k-1}+1} \leq a_{i_{k-1}+2} \leq \ldots \leq a_{i_k}$ for $k = 1, \ldots, K$, and $a_{i_k+1} \leq a_{i_k+2} \leq \ldots \leq a_n$.

A useful tool we use for proving our XP result is the concept of minimal basic knapsack solutions.

**Definition 1.** Given vector $z \in \mathbb{Z}_+^K$ with $0 \leq z_k \leq |I_k|$ for $k \in [K]$, we call the vector $x[z] \in \{0,1\}^n$, satisfying the following, the minimal $z$-basic knapsack solution (with respect to $\alpha$):
Proposition 1. Validity of the inequality $\alpha^T x \leq \beta$ with $(\alpha, \beta) \in \mathbb{Z}_+^{n+1}$ for the knapsack polytope $P$ can be verified in time $n^{K+O(1)}$.

Proof. Assume that $\bar{x} \in \{0,1\}^n$ satisfies $\alpha^T \bar{x} \leq b$ while $\alpha^T \bar{x} > \beta$. Define $\bar{z} \in \mathbb{Z}_+^K$ such that $\bar{z}_k = \sum_{i \in I_k} \bar{x}_i$ for $k = 1, \ldots, K$. Consider the $\bar{z}$-basic knapsack solution $x[\bar{z}]$. Since $a_{i_k-1+1} \leq a_{i_k+2} \leq \ldots \leq a_{i_k}$ for $k = 1, \ldots, K$, we have $\alpha^T x[\bar{z}] \leq \alpha^T \bar{x} \leq b$ and $\alpha^T x[\bar{z}] = \alpha^T \bar{x} > \beta$. Therefore, if $\alpha^T x \leq \beta$ is not valid for $P$, then there exists a minimal $\bar{z}$-basic knapsack solution satisfying $\alpha^T x[\bar{z}] \leq b$ and $\alpha^T x[\bar{z}] > \beta$. However, if $\alpha^T x \leq \beta$ is valid for $P$, then such solution does not exist. Therefore, verifying validity of the inequality $\alpha^T x \leq \beta$ amounts to checking through all minimal basic knapsack solutions, which takes time $O(n \prod_{k=1}^K (|I_k| + 1)) \leq n^{K+O(1)}$.

Proposition 2. Assume inequality $\alpha^T x \leq \beta$ with $(\alpha, \beta) \in \mathbb{Z}_+^{n+1}$ is valid for the knapsack polytope $P$. Then inequality $\alpha^T x \leq \beta$ defines a facet of $P$ if and only if the following two conditions hold:

1. There exists $\bar{x} \in S$ such that $\alpha^T \bar{x} = \beta$ and $\bar{x}_n = 1$.
2. Inequality $\sum_{k=1}^K \gamma_k \sum_{i \in I_k} x_i \leq \beta$ is facet-defining for the knapsack polytope $P' = \text{conv}\{x \in \{0,1\}^K : \sum_{i=1}^K a_i x_i \leq b\}$.

Moreover, the first condition can be checked in time $n^{K+O(1)}$.

Proof. We first show that both conditions are necessary for inequality $\alpha^T x \leq \beta$ to be facet-defining for $P$. Assume inequality $\alpha^T x \leq \beta$ is facet-defining for $P$. Then the first condition must hold. Otherwise, by full dimensionality of $P$, $\alpha^T x \leq \beta$ can only be a multiple of $x_n \geq 0$, which contradicts the fact that $(\alpha, \beta) \in \mathbb{Z}_+^{n+1}$. Inequality $\sum_{k=1}^K \gamma_k \sum_{i \in I_k} x_i \leq \beta$ is valid for $P'$ as $\alpha^T x \leq \beta$ is valid for $P$. Note that the face of $P'$ defined by $\sum_{k=1}^K \gamma_k \sum_{i \in I_k} x_i \leq \beta$ is the orthogonal projection of the face of $P$ defined by $\alpha^T x \leq b$ to its first $n_K$ coordinates. It then follows that the second condition must hold as inequality $\alpha^T x \leq \beta$ is facet-defining for $P$.

We next show that conditions 1 and 2 are sufficient for inequality $\alpha^T x \leq \beta$ to be facet-defining for $P$. Assume conditions 1 and 2 hold. Without loss of generality, we can assume $\bar{x}_{i_K+1} = \bar{x}_{i_K+2} = \ldots = \bar{x}_{n-1} = 0$. (Otherwise, replacing those coordinates by 0 yields such $\bar{x}$.) Since condition 2 holds, there exist affinely independent vectors $y^1, \ldots, y^k \in \{0,1\}^{i_K}$ on the facet of $P'$ defined by $\sum_{k=1}^K \gamma_k \sum_{i \in I_k} x_i \leq \beta$. Observe that the following $n$ vectors are affinely independent and lie on the face of $P$ defined by $\alpha^T x \leq \beta$:

$$\bar{x} + e_{i_K+1} - e_n, \bar{x} + e_{i_K+2} - e_n, \ldots, \bar{x} + e_{n-1} - e_n, \bar{x},$$

$$(y^1, 0_{n-i_K}), (y^2, 0_{n-i_K}), \ldots, (y^k, 0_{n-i_K}).$$
Therefore, $a^T x \leq \beta$ is facet-defining for $P$.

Similar to the proof of Proposition 1, condition 1 holds if and only if there exists there exists vector $z$ and a minimal $z$-basic knapsack solution $x[z]$ such that $\bar{x} = x[z] + e_k$ satisfies condition 1. Therefore, checking condition 1 amounts to checking through all minimal basic knapsack solutions, which takes time $n^{K+O(1)}$.

We next extend the definition of minimal basic knapsack solutions to basic knapsack solutions.

**Definition 2.** For each $z \in \mathbb{Z}_+^K$ satisfying $0 \leq z_k \leq |I_k|$ for all $k \in [K]$ and $\sum_{k=1}^K \gamma_k z_k = \beta$, for $k \in [K]$, we call the following $|I_k| - 1$ vectors the block-$k$ $z$-basic knapsack solutions:

$$x^i[z,k] := x[z] - e_{ik-1+i} + e_{ik-1+z_k+1}, \text{ for } i = 1, \ldots, z_k - 1,$$

and

$$x^i[z,k] := x[z] - e_{ik-1+z_k} + e_{ik-1+i}, \text{ for } i = z_k + 1, \ldots, |I_k|.$$

We call vector $x[z]$ and all block-$k$ $z$-basic knapsack solutions for all $k \in [K]$, in total $\sum_{k=1}^K K - K + 1$ points, the $z$-basic knapsack solutions. We call a $z$-basic knapsack solution $x$ feasible if $a^T x \leq b$, and infeasible otherwise.

We next show some basic properties of feasible $z$-basic knapsack solutions.

**Lemma 2.** Assume $z \in \mathbb{Z}_+^K$ satisfies $0 \leq z_k \leq |I_k|$ for all $k \in [K]$ and $\sum_{k=1}^K \gamma_k z_k = \beta$. Let $X[z]$ denote the set of all feasible $z$-basic knapsack solutions. The following holds:

1. All $z$-basic knapsack solutions are affinely independent of each other;
2. If $x^j[z,k] \notin X[z]$ (i.e., $x^j[z,k]$ is infeasible) for some $k \in [K]$ and $j \leq z_k - 1$, then there does not exist $x \in S$ such that $\sum_{i \in I_k} x_i = z_k$ for $k \in [K]$ and $x_{ik-1+j} = 0$, in which case set $X[z]$ is contained in the affine subspace defined by $x_{ik-1+j} = 1$;
3. If $x^j[z,k] \notin X[z]$ (i.e., $x^j[z,k]$ is infeasible) for some $k \in [K]$ and $j \geq z_k + 1$, then there does not exist $x \in S$ such that $\sum_{i \in I_k} x_i = z_k$ for $k \in [K]$ and $x_{ik-1+j} = 1$, in which case set $X[z]$ is contained in the affine subspace defined by $x_{ik-1+j} = 0$.

**Proof.** The first conclusion follows from the definition of $z$-basic knapsack solutions. For $k \in [K]$ and $j \leq z_k - 1$, note that

$$x^j[z,k] \in \arg\min \{a^T x : x \in \{0, 1\}^n; \sum_{i \in I_k} x_i = z_k, \ k \in [K]; x_{ik-1+j} = 0 \}.$$

The second conclusion then follows. The third conclusion follows similarly. □
We are now prepared to show the main result of this section.

**Theorem 6.** Determining whether inequality $\alpha^T x \leq \beta$ with $(\alpha, \beta) \in \mathbb{Z}_+^{n+1}$ is facet-defining for the knapsack polytope $P = \text{conv(} \{x \in \{0, 1\}^n : a^T x \leq b\} \}$ can be done in time $n^{K+O(1)}$.

**Proof.** By Propositions 1 and 2 we can assume without loss of generality that $i_K = n$ and $\alpha^T x \leq \beta$ is valid for $P$. Also by Theorem 5 we only need to consider the case when $K \geq 2$. We will show that determining whether $\alpha^T x \leq \beta$ is facet-defining for $P$ amounts to checking whether there exist $n$ affinely independent vectors among all feasible basic knapsack solutions. Note that there are at most $O((n - K + 1) \prod_{k=1}^K(|I_k| + 1)) \leq n^{K+O(1)}$ feasible basic knapsack solutions. Checking whether there exist $n$ affinely independent vectors among all feasible basic knapsack solutions can be done in time $n^{K+O(1)}$ by computing the rank of a matrix with $(n + 1)$ rows and up to $n^{K+O(1)}$ columns.

First, if there exist $n$ affinely independent vectors among all feasible basic knapsack solutions, then $\alpha^T x \leq \beta$ is facet-defining for $P$ by the definition of feasible basic knapsack solutions. Now, assume for contradiction that there does not exist $n$ affinely independent vectors among all feasible basic knapsack solutions, and $\alpha^T x \leq \beta$ is facet-defining for $P$. Then there exists $\bar{x} \in S$ lying on the face of $P$ defined by $\alpha^T x \leq \beta$ such that $\bar{x}$ is affinely independent of all feasible basic knapsack solutions. Define $\bar{z} \in \mathbb{Z}_+^K$ such that $\bar{z}_k = \sum_{i \in I_k} \bar{x}_i$ for $k = 1, \ldots, K$. Consider the set $X[\bar{z}]$ of all feasible $\bar{z}$-basic knapsack solutions. Note that $X[\bar{z}] = \emptyset$ if and only if $\alpha^T \bar{x} = b$, which implies $\alpha^T \bar{x} \geq \alpha^T x[\bar{z}] > b$, i.e., $\bar{x} \notin S$. Therefore, $X[\bar{z}] \neq \emptyset$. Since $\bar{x}$ is affinely independent of all feasible basic knapsack solutions, $\bar{x}$ is affinely independent of all points in $X[\bar{z}]$. Note that, since all points in $X[\bar{z}]$ are affinely independent of each other by Lemma 2, the affine hull of $X[\bar{z}]$ is defined exactly by the following $n + 1 - |X[\bar{z}]|$ equations:

1. $\sum_{i \in I_k} x_i = \bar{z}_k$, $k = 1, \ldots, K$;
2. $x_{i_k-1+j} = 1$, for all $k \in [K]$ and $j \leq \bar{z}_k - 1$ with $x^j(\bar{z}, k) \notin X[\bar{z}]$;
3. $x_{i_k-1+j} = 0$, for all $k \in [K]$ and $j \geq \bar{z}_k + 1$ with $x^j(\bar{z}, k) \notin X[\bar{z}]$.

Note that $\bar{x}$ already satisfies the equation $\sum_{i \in I_k} x_i = \bar{z}_k$. Since $\bar{x}$ is affinely independent of all points in $X[\bar{z}]$, one the following two must be true:

1. There exist $k \in [K]$ and $j \leq \bar{z}_k - 1$ such that $x^j(\bar{z}, k) \notin X[\bar{z}]$ while $\bar{x}_{i_k-1+j} = 0$;
2. Or there exist $k \in [K]$ and $j \geq \bar{z}_k + 1$ such that $x^j(\bar{z}, k) \notin X[\bar{z}]$ while $\bar{x}_{i_k-1+j} = 1$.

It then contradicts with the last two conclusions of Lemma 2.

**Corollary 2.** Given a valid inequality $\alpha^T x \leq \beta$ with $(\alpha, \beta) \in \mathbb{Z}_+^{n+1}$, the dimension of the face of $P$ defined by $\alpha^T x \leq \beta$ can be computed in time $n^{K+O(1)}$.

**Proof.** The proof of Theorem 6 implies that every $\bar{x} \in S$ can be written as an affine combination of several feasible basic knapsack solutions. Therefore,
computing \( \dim(P) \) amounts to computing the maximum number of affinely independent vectors among all feasible basic knapsack solutions, which can be done in time \( n^{K+O(1)} \) by computing the rank of a matrix with \((n+1)\) rows and up to \( n^{K+O(1)} \) columns.

The proofs in this section can also be easily extended to show that determining whether an inequality \( \alpha^T x \leq \beta \) is facet-defining for totally-ordered multidimensional knapsack polytope [9] is XP with respect to \(|\alpha|_+\).

5 KP Membership Problem

In this section, we study the membership problem associated with KP. Generally speaking, the membership problem asks the following question: Given an element \( x \) and a set \( S \), is \( x \) contained in this set? As trivial as this problem might seem to be, in some cases this decision problem can be rather hard to answer. As a selective list of examples in literature, Murty and Kabadi [17] show that the membership problem for the copositive cone, that is deciding whether or not a given matrix is in the copositive cone, is a \( \text{co-NP} \)-complete problem. Dickinson and Gijben [10] also show that the membership problem for the completely positive cone is \( \text{NP} \)-hard. Moreover, it is worth mentioning that, for any separation problem: “Given point \( x^* \), does there exist an inequality from a given cutting-plane family that is violated by \( x^*? \)” it can be essentially seen as a membership problem: Given a point \( x^* \) and the cutting-plane closure defined by intersecting all inequalities from the family, is \( x^* \) contained in this closure? In combinatorial optimization, the membership problem therein normally takes the form of: Given a point \( p \) and a combinatorial polytope, is \( p \) contained in this polytope? Here the polytope is defined as the convex hull of integer (or binary) points satisfying certain properties, since a linear description of the polytope will trivialize the membership problem. For any polytope arising from combinatorial optimization that is defined as the convex hull of certain set of binary points, e.g., TSP polytope, clique polytope, matching polytope etc., the corresponding membership problem is obviously in \( \text{NP} \), since if point \( p \) in this polytope, then by Carathéodory’s theorem there exist at most \( d+1 \) binary points such that \( p \) can be written as the convex combination of there. Here \( d \) is the dimension of the polytope. Papadimitriou and Yannakakis [19] have shown that the membership problem of TSP polytope is in fact \( \text{NP} \)-complete. The next theorem gives an analogous result for KP. Here recall the well-known partition problem: Given \((w_1, \ldots, w_n) \in \mathbb{Z}_+^n\), does there exist a subset \( S \subseteq [n] \), such that \( w(S) = w([n] \setminus S) \)?

Theorem 7. The membership problem of KP is \( \text{NP} \)-complete.

Proof. Let \((a_1, \ldots, a_n)\) be an input to an instance of the partition problem, let \( x^* := (1/2, \ldots, 1/2) \) and the knapsack constraint given by \( a^T x \leq a([n])/2 \). Since partition problem is \( \text{NP} \)-complete, we only need to show that there exists \( S \subseteq [n] \) with \( a(S) = a([n] \setminus S) \) if and only if \( x^* \in \text{conv} \left\{ x \in \{0,1\}^n : a^T x \leq a([n])/2 \right\} \).
If there exists $S \subseteq [n]$ with $a(S) = a([n] \setminus S)$, then $a(S) = a([n])/2$, so $\chi^S, \chi^{[n] \setminus S} \in \{ x \in \{0,1\}^n : a^T x \leq a([n])/2 \}$. Hence,

$$x^* = 1/2 \chi^S + 1/2 \chi^{[n] \setminus S} \in \text{conv} \{ \{ x \in \{0,1\}^n : a^T x \leq a([n])/2 \} \}.$$ 

On the other hand, if $x^* \in \text{conv} \{ \{ x \in \{0,1\}^n : a^T x \leq a([n])/2 \} \}$, then we know $x^*$ can be written as the convex combination of several points in $\{0,1\}^n$ which all satisfy $a^T x = a([n])/2$ since $a^T x^* = a([n])/2$. The support of any one of these binary points will serve as a yes-certificate to the partition problem. □

References

1. Egon Balas. Facets of the knapsack polytope. *Mathematical programming*, 8(1):146–164, 1975.
2. Egon Balas and Eitan Zemel. Facets of the knapsack polytope from minimal covers. *SIAM Journal on Applied Mathematics*, 34(1):119–148, 1978.
3. E Andrew Boyd. Fenchel cutting planes for integer programs. *Operations Research*, 42(1):53–64, 1994.
4. Aykut Bulut and Ted K Ralphs. On the complexity of inverse mixed integer linear optimization. *SIAM Journal on Optimization*, 31(4):3014–3043, 2021.
5. Jin-Yi Cai and Gabriele E. Meyer. Graph minimal uncolorability is $\mathcal{D}_P$-complete. *SIAM Journal on Computing*, 16(2):259–277, 1987.
6. Wei-Kun Chen and Yu-Hong Dai. On the complexity of sequentially lifting cover inequalities for the knapsack polytope. *Science China Mathematics*, 64(1):211–220, 2021.
7. Sunil Chopra. On the spanning tree polyhedron. *Operations Research Letters*, 8(1):25–29, 1989.
8. Harlan Crowder, Ellis L Johnson, and Manfred Padberg. Solving large-scale zero-one linear programming problems. *Operations Research*, 31(5):803–834, 1983.
9. Alberto Del Pia, Jeff Linderoth, and Haoran Zhu. Multi-cover inequalities for totally-ordered multiple knapsack sets: theory and computation. *Mathematical Programming*, 197(2):847–875, 2023.
10. Peter J. C. Dickinson and Luuk Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. *Comput. Optim. Appl.*, 57(2):403–415, 2014.
11. Martin Grötschel and Manfred W Padberg. On the symmetric travelling salesman problem i: inequalities. *Mathematical Programming*, 16:265–280, 1979.
12. Zonghao Gu. Lifted cover inequalities for 0-1 and mixed 0-1 integer programs. PhD thesis, Georgia Institute of Technology, 1995.
13. David Hartvigsen and Eitan Zemel. The complexity of lifted inequalities for the knapsack problem. *Discrete Applied Mathematics*, 39(2):113–123, 1992.
14. Richard M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103, Boston, MA, 1972. Springer US.
15. Richard M Karp and Christos H Papadimitriou. On linear characterizations of combinatorial optimization problems. *SIAM Journal on Computing*, 11(4):620–632, 1982.
16. Hugues Marchand, Alexander Martin, Robert Weismantel, and Laurence Wolsey. Cutting planes in integer and mixed integer programming. *Discrete Applied Mathematics*, 123(1-3):397–446, 2002.
17. Katta G. Murty and Santosh N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Mathematical Programming*, 39(2):117–129, 1987.
18. Christos H Papadimitriou and David Wolfe. The complexity of facets resolved. Technical report, Cornell University, 1985.
19. Christos H Papadimitriou and Mihalis Yannakakis. The complexity of facets (and some facets of complexity). In *Proceedings of the fourteenth annual ACM symposium on Theory of computing*, pages 255–260, 1982.
20. Jörg Rothe. Exact complexity of exact-four-colorability. *Inform. Process. Lett.*, 87(1):7–12, 2003.
21. Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003.
22. Robert Weismantel. On the 0/1 knapsack polytope. *Mathematical Programming*, 77(3):49–68, 1997.