Varieties swept out by grassmannians of lines

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Dedicated to Andrew J. Sommese in his 60th birthday.

Abstract. We classify complex projective varieties of dimension $2r \geq 8$ swept out by a family of codimension two grassmannians of lines $G(1, r)$. They are either fibrations onto normal surfaces such that the general fibers are isomorphic to $G(1, r)$ or the grassmannian $G(1, r+1)$. The cases $r = 2$ and $r = 3$ are also considered in the more general context of varieties swept out by codimension two linear spaces or quadrics.

1. Introduction

Let $X$ be a smooth complex projective variety. It is a classical question in algebraic geometry to understand to which extent the geometry of $X$ is determined by a particular family of subvarieties of $X$.

Perhaps the first subvarieties that algebraic geometers have considered in that sense are lines in projective varieties $X \subset \mathbb{P}^N$. Examples of the use of this idea can be found all throughout the literature, evolving into the study of rational curves in algebraic varieties that has become a central part of algebraic geometry since Mori’s landmark work in 1980’s. In this paper we will make use of the work of Beltrametti, Sommese and Wiśniewski [BSW], where they study polarized manifolds $(X, H)$ swept out by lines, i.e. rational curves of $H$-degree one.

A naturally related goal is the classification of projective varieties $X \subset \mathbb{P}^N$ dominated by families of linear subspaces $L = \mathbb{P}^t$, see for instance [E], [BSW], [ABW]. The general philosophy here is that these varieties may be classified if the codimension $\dim(X) - t$ is small. In fact, Sato classified them for $\dim(X) \leq 2t$ (cf. [S, Main Thm.]) and recently Novelli and Occhetta have completed the case $\dim(X) = 2t + 1$ (cf. [NO, Thm 1.1]).

One could also study varieties swept out by other types of subvarieties. We would like to point out two different directions. On one side we have the extendability problem, i.e. study which algebraic varieties may appear as an ample divisor on a smooth variety. For instance, it is well known that the quadric $\mathbb{Q}^n$ can only appear as an ample divisor in $\mathbb{P}^{n+1}$ or $\mathbb{Q}^{n+1}$ and that the grassmannian of lines

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\( \mathbb{G}(1, r) \) is not extendable for \( r \geq 4 \) (cf. \([F^2]\)). Extendability has been studied for many other varieties, see for instance \([B]\) and the references therein. On the other side one could consider subvarieties of codimension bigger than one. The case of quadrics has been treated by several authors, see for instance \([KS]\), \([Fu]\) and \([BI]\).

In the three cases they study embedded projective varieties \( X \subset \mathbb{P}^N \) swept out by quadrics of small codimension.

Putting together the previous considerations we find of interest the problem of classificating varieties swept out by codimension two grassmannians of lines. Our main result is the following:

**Theorem 1.1.** Let \((X, H)\) be a polarized variety of dimension \(2r\), \(r \geq 4\). Suppose that \(X\) is dominated by deformations of a subvariety \(G \subset X\) isomorphic to \(\mathbb{G}(1, r)\), such that \(H|_G\) is the ample generator of \(\text{Pic}(G)\). Assume further that \(H\) is very ample and \(H^1(X, \mathcal{I}_{G/X}(H)) = 0\). Then either:

1.1.1. there exists a morphism \( \Phi : X \to Y \) onto a normal surface such that the general fiber is isomorphic to \(G\), or

1.1.2. \(X = \mathbb{G}(1, r + 1)\) and \(H\) is the ample generator of \(\text{Pic}(X)\).

For the sake of completeness we have also dealt with the cases \(r = 2\) and \(r = 3\) which are special since \(\mathbb{G}(1, 2)\) is linear and \(\mathbb{G}(1, 3)\) is a quadric. In fact, our methods allow us to classify \(n\)-dimensional polarized varieties \((X, H)\) swept out by codimension two linear spaces or quadrics (see Propositions 1.1 and 4.3). Observe that the very ampleness of the polarization is not needed in our approach, whereas it was necessary in the results of Sato, \([S]\), Kachi-Sato, \([KS]\), and Beltrametti-Ionescu, \([BI]\), quoted above.

The structure of the paper is the following. In Section 2 we expose some background material, including a result by Beltrametti, Sommese and Wiśniewski on the nef value morphism of polarized varieties swept out by lines, that will be the starting point of our classification. In Section 3 we obtain a structure result on polarized varieties \((X, H)\) swept out by grassmannians of lines, based on their nef value morphisms. We also study the normal bundle to those grassmannians in \(X\). Section 4 deals with the classification of polarized varieties swept out by codimension two linear spaces and quadrics. In the case of quadrics the problem of finding out which Del Pezzo varieties contain quadrics appears. Our solution goes through computing the possible normal bundles to quadrics embedded in certain weighted projective spaces. Finally, we finish the proof of Theorem 1.1 in Section 5. Note that this is the only place where we need very ampleness of the polarization. The proof involves a study of the normal bundle in \(X\) to a linear subspace of \(G\) of maximal dimension, as well as the result by Novelli and Occhetta cited above. With this ingredients at hand we study the variety of tangents to lines in \(X\) passing through a general point, and the proof boils down to using Sato’s Theorem \([S]\) Main Thm.].

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**1.1. Conventions and definitions.** We will work over the complex numbers and we will freely use the notation and conventions appearing in \([Ha]\). When there is no ambiguity we will denote a line bundle \(\mathcal{O}_X(M)\) on a variety \(X\) by \(\mathcal{O}(M)\).
Along the paper a polarized variety will be a pair \((X, H)\) where \(X\) is a smooth irreducible projective variety and \(H\) is an ample line bundle on \(X\). The \textit{nef value} of \((X, H)\) is the minimum number \(\tau\) such that \(K_X + \tau H\) is nef but not ample. If \(K_X\) is not nef then \(\tau\) is rational and the \(\mathbb{Q}\)-divisor \(K_X + \tau H\) is semiample. We will denote by \(\Phi : X \to Y\) the morphism with connected fibers determined by \(m(K_X + \tau H)\), \(m >> 0\), and we will call it the \textit{nef value morphism} of \((X, H)\).

We will denote by \((G(k, n), \mathcal{O}(1))\) the grassmannian of linear subspaces of dimension \(k\) in \(\mathbb{P}^n\) polarized by the ample generator of its Picard group, and by \((\mathbb{Q}^n, \mathcal{O}(1))\) the smooth quadric of dimension \(n\) polarized by the very ample divisor defining the embedding \(\mathbb{Q}^n \subset \mathbb{P}^{n+1}\) as a hypersurface of degree two.

We will say that \((X, H)\) is a \textit{scroll} over a smooth projective variety \(B\) if there exists a vector bundle \(E\) on \(B\) such that \(X = \mathbb{P}(E)\) and \(H\) is the tautological line bundle.

Given an irreducible family \(C\) of rational curves in \(X\) we will say that \(X\) is \textit{rationally chain connected by the family} \(C\) if two general points of \(X\) can be connected by a chain of curves of \(C\). We refer to [Hw] and [KeSo] for notation and generalities on rational curves and the variety of minimal rational tangents.

A vector bundle on a projective variety \(X\) is called \textit{generically globally generated} (g.g.g. for brevity) if it is globally generated at the general point.

### 2. Preliminars

We begin by recalling the following well known features of families of subschemes:

**Remark 2.1.** Let \((X, H)\) be a polarized variety. Set \(G \subset X\) an irreducible smooth subvariety. The universal family parametrized by an irreducible component \(\mathcal{H}\) containing \([G]\) of the Hilbert scheme \(\text{Hilb}(X)\) dominates \(X\) if and only if the normal bundle of a general deformation of \(G\) in \(\mathcal{H}\) is g.g.g.. For simplicity we will say that \(\mathcal{H}\) dominates \(X\), or that \(X\) is dominated by a family of deformations of \(G\).

Given a family \(\mathcal{H}\) of smooth subschemes of \(X\), one may wish to study semicontinuity properties on the normal bundles. In order to do that we introduce the following notation:

**Notation 2.2.** Let \((X, H)\) be a polarized variety and \([G]\) be a smooth point of a dominating component \(\mathcal{H} \subset \text{Hilb}(X)\). Let \(I = \{(p, [G']): p \in G', [G'] \in \mathcal{H}\}\) be the universal family, \(\pi_1\) and \(\pi_2\) the corresponding projections, \(\mathcal{H}_0\) the open set of smooth points of \(\mathcal{H}\) and \(I_0 = \pi_2^{-1}(\mathcal{H}_0)\). Shrinking \(\mathcal{H}_0\) if necessary, we get a diagram of sheaves over \(I_0\) with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & T_{I_0}\mathcal{H}_0 & \to & T_{I_0} & \to & \pi_2^*T_{\mathcal{H}_0} & \to & 0 \\
\text{d} & & \text{d} & & \text{d} & & \text{d} & & \text{d} \\
0 & \to & T_{I_0}\mathcal{H}_0 & \to & \pi_1^*T_X & \to & N_{\mathcal{H}_0} & \to & 0
\end{array}
\]

where \(N_{\mathcal{H}} := \text{coker}(\pi_1)\) verifies \(N_{\mathcal{H}|G'} \cong N_{G'/X}\), for \([G'] \in \mathcal{H}_0\).

Varieties swept out by lines have been extensively studied. We will make use of a particular result in this direction, extracted from a more detailed exposition due to Beltrametti, Sommese and Wiśniewski (cf. [BSW]).
Theorem 2.3 ([BSW] Thms. 2.1-2.5). Let \((X, H)\) be a polarized variety such that for each point \(x \in X\) there exists a rational curve \(\ell \subset X\) with \(x \in \ell\) and \(H \cdot \ell = 1\). With the notation of \([\mathcal{L}]\) we get:

2.3.1. Either \(\Phi\) contracts \(\ell\) (equivalently, \(\tau = -K_X \cdot \ell\)), or \(-K_X \cdot \ell + 1 \leq \tau \leq n + K_X \cdot \ell + 2\) and, in particular, \(-K_X \cdot \ell \leq (n+1)/2\).

2.3.2. If \(-K_X \cdot \ell \geq (n+1)/2\) then \(-K_X \cdot \ell = (n+1)/2\) unless \(\Phi\) is the contraction of the extremal ray \(\mathbb{R}_+ [\ell]\).

As an application we get the following lemma.

Lemma 2.4. Let \((X, H)\) be a polarized variety. Assume that \(X\) is dominated by deformations of a smooth subvariety \(G \subset X\). Assume further that \(G\) is rationally connected by a family \(\mathcal{C}\) of rational curves of \(H\)-degree one. Set \(c := \det(N_{G/X}) \cdot \ell\) for \([\ell]\) \(\in \mathcal{C}\). If \(c > K_G \cdot \ell + (n+1)/2\) then, with the notation of \([\mathcal{L}]\), \(\tau = -K_G \cdot \ell + c\), \(\Phi\) is the contraction of the extremal ray \(\mathbb{R}_+ [\ell]\) and \((\Phi(G))\) is a point.

Proof. Being \(f : \mathbb{P}^1 \to \ell\) the normalization morphism, the hypotheses imply that \(f^* N_{G/X}\) is g.g.g. and hence it is nef. But \(G\) is dominated by \(\mathcal{C}\), hence \(f^* T_G\) is nef. It follows that \(f^* T_X\) is nef too and, equivalently, \(X\) is swept out by rational curves of \(H\)-degree one. Since \(-K_X \cdot \ell = -K_G \cdot \ell + c > (n+1)/2\), Theorem 2.3 implies that \(\tau = -K_G \cdot \ell + c\) and \(\Phi : X \to Y\) is the (fiber-type) contraction of the extremal ray \(\mathbb{R}_+ [\ell]\). Finally, since \(G\) is rationally chain connected by the family \(\mathcal{C}\), its image by \(\Phi\) is a point.

3. Varieties swept out by grassmannians

Let us start this section by fixing the setup:

Setup 3.1. Let \((X, H)\) be a polarized variety of dimension \(n = 2r\). We assume that \(X\) is dominated by a family of deformations of \(G \cong G(1, r), r \geq 2\). Assume further that \(O(1) = H|_G\) generates \(\text{Pic}(G)\) and write \(\det(N_{G/X}) \cong cH|_G, c \in \mathbb{Z}\).

Remark 3.2. Note that the vanishing \(H^1(G(1, r), T_{G(1, r)}) = 0\) (obtained by Littlewood-Richardson formula, for instance) implies that the general deformation of \(G\) inside \(X\) is isomorphic to \(G(1, r)\).

We begin by applying the results of the previous section to a polarized variety \((X, H)\) verifying the hypotheses we have just imposed.

Proposition 3.3. Let \((X, H)\) be as in \([\mathcal{L}]\). Then either

3.3.1. \(Y\) is a normal surface, the general fiber of \(\Phi\) is isomorphic to \(G\) and \(N_{G/X} \cong \mathcal{O}^{\oplus 2}\), or

3.3.2. \(Y\) is a smooth curve, the general fiber of \(\Phi\) is either \(\mathbb{P}^3\) or a smooth 5-dimensional quadric and \(N_{G/X} \cong \mathcal{O} \oplus \mathcal{O}(1)\), or

3.3.3. \(Y\) is a smooth curve, the general fiber of \(\Phi\) is \(\mathbb{P}^5\) and \(N_{G/X} \cong \mathcal{O} \oplus \mathcal{O}(2)\), or

3.3.4. \(Y\) is a point, \(\text{Pic}(X) = \mathbb{Z}H\) and \(-K_X = (r+1+c)H\).

Proof. Since \(N_{G/X}\) is g.g.g., then \(c \geq 0\). Hence, by Lemma 2.4 \(\tau = r+1+c\) and \(\Phi\) contracts \(G\) to a point, which in particular gives \(\dim(Y) \leq 2\).

If \(\dim(Y) = 2\), since the fibers of \(\Phi\) are connected, the general deformation of \(G\) coincides with the fiber containing it, hence 3.3.1 holds.
If $Y$ is a point, then a multiple of $K_X + \tau H$ is trivial. Since $\Phi$ is an elementary contraction, it follows that $X$ is a Fano manifold of Picard number 1. In particular $\text{Pic}(X)$ has no torsion, thus $K_X + \tau H$ is trivial too and [3.3]4 follows.

Thus we are left with the case $\dim(Y) = 1$. Let us denote by $F$ the general fiber of $\Phi$, that contains a grassmannian $G$. Applying Lemma [2.3] to $(F, H|_F)$ and using that the nef value morphism $\Phi_F$ coincides with $\Phi|_F$ we obtain that $F$ is a Fano manifold whose Picard group is generated by $H|_F$. But $G$ appears as an effective, and hence ample, divisor on $F$. In particular $c \geq 1$. On the other side it is classically known that this is only possible (cf. [F1] Theorem 5.2) if $r = 2, 3$. It follows that $G$ is either $\mathbb{P}^2$ or a smooth quadric of dimension 4. If the former holds then $F = \mathbb{P}^3$, the exact sequence

$$0 \to N_{G/F} \to N_{G/X} \to \mathcal{O} \to 0,$$

splits and we get [3.3]2. If the later holds then $-K_G = 4H|_G$. This implies that $-K_F = (4 + c)H|_F$, and applying Kobayashi-Ochiai characterization of quadrics and projective spaces, [KO], either $c = 1$ and $F$ is a 5-dimensional quadric, or $c = 2$ and $F$ is isomorphic to $\mathbb{P}^5$. The splitting of the exact sequence [1] concludes the proof.

**Remark 3.4.** Let us remark that in the case [3.3]1 when $r = 3$ we get smoothness of $Y$ when $H$ is very ample or $\Phi$ is equidimensional, see [ABW2, Thm. B] and [BS1 (2.3)]. In fact, $Y$ is conjectured to be smooth, see [BS2 Conj. 14.2.10]. In [3.3]3 or in the first case of [3.3]2, if $H$ is very ample, all fibers of $\Phi$ are isomorphic. However this is not true in general, as one can see by considering certain quadric sections of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^5 \subset \mathbb{P}^{11}$.

At this point, one would like to determine $N_{G/X}$ also in the case [3.3]4, but this task is not as simple as in the other cases. In fact certain restrictions might be imposed in order to get our classification. However the following lemmas allow us to claim that $c$ is different from zero.

**Lemma 3.5.** Let $(X, H)$ be as in [3.3]. If $c = 0$ then $N_{G/X}$ is trivial.

**Proof.** Since $N_{G/X}$ is g.g.g., then $\dim(H^0(X, N_{G/X})) \geq 2$. Taking two independent sections we get a generically injective morphism $\mathcal{O}^{\oplus 2} \to N_{G/X}$. This produces a nonzero global section of the det($N_{G/X}$). Since $c = 0$, this gives det($N_{G/X}$) $\cong \mathcal{O}$ and $N_{G/X} \cong \mathcal{O}^{\oplus 2}$.

**Lemma 3.6.** Let $X$ be an irreducible smooth projective variety of dimension bigger than or equal to 4. Assume that $X$ contains a codimension two smooth subvariety $G \subset X$ such that $b_2(G) = 1$ and $H^1(G, \mathcal{O}) = 0$. If $N_{G/X} \cong \mathcal{O}^{\oplus 2}$ then $\rho(X) > 1$.

**Proof.** The hypotheses imply that $\dim(H^0(G, N_{G/X})) = 2$ and $H^1(G, N_{G/X}) = 0$, hence $[G]$ is a smooth point of a 2-dimensional component $\mathcal{H} \subset \text{Hilb}(X)$. Furthermore, with the notation introduced in [2.2] we may assume that there exists an open subset $\mathcal{H}_0 \subset \mathcal{H}$ such that any element $[G_i] \in \mathcal{H}_0$ corresponds to a smooth projective subvariety $G_i \subset X$ for which $N_{G_i/X} \cong \mathcal{O}^{\oplus 2}$, $d\pi_1$ is an isomorphism and $b_2(G_i) = 1$. This provides a finite morphism $f = \pi_1|_{\mathcal{H}_0} : \mathcal{H}_0 \to X_0$ from the universal family over $\mathcal{H}_0$ onto an open subset $X_0 \subset X$.

If $f$ is generically one to one, then the Picard number of $X$ cannot be one. Thus we may assume that $\deg(f) > 1$. 
Take a general $[G_t] \in \mathcal{H}_0$ and define: $C_t = \{[G_u] \in \mathcal{H}_0 : G_u \cap G_t \neq \emptyset\}$. The subscheme $C_t \subset \mathcal{H}_0$ is nonempty by the hypothesis on $\deg(f)$ and it is different from $\mathcal{H}_0$ since $f$ is a local isomorphism at every point. Given $[G_u] \in C_t$ we claim that $C_t \subseteq C_u$. In fact since $c_2(N_{G_t/X}) = 0$, the self intersection formula tells us that $G_u \cap G_t$ is a divisor on $G_t \cong \mathbb{G}(1, r)$. Hence, given any $[G_v] \in C_t$ we get $G_v \cap G_u \neq \emptyset$. The same argument leads to the equality $C_t = C_u$.

As an abuse of notation let $C_t$ stand now for the union of the one dimensional components of $C_t$. Now define $D_0 = \cup_{u \in C_t} G_u$ which is a divisor on $X_0$ and observe that for the general $[G_s] \in \mathcal{H}_0$ we have $D_0 \cap G_s = \emptyset$. If $\rho(X) = 1$ is one then the closure $D$ of $D_0$ in $X$ would meet $G_s$ so that $(D \setminus D_0) \cap G_s \neq \emptyset$ for any $[G_s] \in \mathcal{H}_0$. But $D \setminus D_0$ is a finite union of codimension two subvarieties of $X$, any of them algebraically equivalent to $G$. Then, the general $G_s$ would meet an irreducible component of $D \setminus D_0$ in a divisor of $G_s$, contradicting the fact that $G_s \cap G_{s'} = \emptyset$ for general $s, s' \in \mathcal{H}_0$.

4. Varieties swept out by codimension two linear spaces and quadrics

The main result of this paper, Theorem 1.1, classifies, under certain assumptions, varieties swept out by deformations of $\mathbb{G}(1, r)$, with $r \geq 4$. For the sake of completeness, we have addressed in this section the cases $r = 2, 3$, for which some ad hoc arguments are needed. They allow us to classify $n$-dimensional varieties swept out by codimension two linear spaces and quadrics. These problems have been already addressed by many authors, see for instance [S], [KS] and [BI]. Note that they allow higher codimension but they assume very ampleness of the polarization, which is not necessary in our case.

An analogue of Proposition 3.3 already allow us to study varieties swept out by codimension 2 linear spaces. We have skipped the proof since it follows verbatim [3.3]. Note that we need to use that projective spaces are rigid $(H^1(\mathbb{P}^{n-2}, T_{\mathbb{P}^{n-2}}) = 0)$ and that the only varieties containing linear spaces as ample divisors are linear spaces themselves. Note also that the only quadric containing codimension 2 linear spaces is $\mathbb{Q}^4$.

**Proposition 4.1.** Let $(X, H)$ be a polarized variety of dimension $n \geq 4$. Suppose that $X$ is dominated by a family of deformations of $L \cong \mathbb{P}^{n-2}$, with $H|_L \cong \mathcal{O}(1)$. Then, with the notation of [4.4] either:

4.1.1. $Y$ is a normal surface and the general fiber of $\Phi$ is $\mathbb{P}^{n-2}$, or
4.1.2. $Y$ is a smooth curve and the general fiber of $\Phi$ is $\mathbb{P}^{n-1}$, or
4.1.3. $(X, H) = (\mathbb{P}^n, \mathcal{O}(1))$, or
4.1.4. $(X, H) = (\mathbb{Q}^4, \mathcal{O}(1))$.

**Remark 4.2.** The hypothesis $n \geq 4$ in [4.1] is needed in order to get the bound $K_L \cdot \ell + (n + 1)/2 < 0$ that allows us to apply Lemma 2.4. If $n = 3$ and $c = 0$ these arguments do not work. Nevertheless we can apply basic results of adjunction theory, see [1] Section 1, to describe this case. If $(X, H)$ is not $(\mathbb{P}^3, \mathcal{O}(1))$, $(\mathbb{Q}^3, \mathcal{O})$ or a scroll over a curve, then $K_X + 2H$ is nef and so $\tau = 2$. Hence, either $(X, H)$ is a Del Pezzo threefold, or a quadric fibration over a smooth curve, or a scroll over a surface.

In the case of quadrics, reasoning as above we obtain the following:
Suppose that $H$ meet the singular locus of $N$. We claim that the pair $(X, H)$ is completely analogous. The following argument was suggested to us by M. Reid.

The classification will be completed by determining which Del Pezzo varieties may be swept out by codimension two quadrics:

**Proposition 4.4.** Let $X$ be a Del Pezzo variety of dimension $n \geq 4$ and $Pic(X) = \mathbb{Z}H$. If $X$ contains a $(n - 2)$-dimensional smooth quadric $Q_n - 2$ of $H$-degree 2, then $X$ is isomorphic to a linear section of $G(1, 4)$.

This fact is based on Fujita’s classification of Del Pezzo varieties (cf. [F2] 8.11, p. 72). We are interested in those of Picard number 1, which are:

I. $X \cong X_3 \subset \mathbb{P}^{n+1}$ is a hypersurface of degree three, or

II. $X \cong X_{2.2} \subset \mathbb{P}^{n+2}$ is the complete intersection of two quadrics, or

III. $X \cong X_4$ is a degree four hypersurface in the weighted projective space $\mathbb{P}(1^{n+1}, 2)$, or

IV. $X \cong X_6 \subset \mathbb{P}(1^n, 2, 3)$ is a degree 6 hypersurface, or

V. $X$ is isomorphic to a linear section of $G(1, 4) \subset \mathbb{P}^9$.

**Proof of Proposition 4.4.** For each case denote by $\mathbb{P}$ the corresponding ambient space. We will discard Types I to IV by showing that $Q := Q_n - 2$ does not meet the singular locus of $\mathbb{P}$ and that the normal bundle $N_{Q/P}$ of a quadric $Q$ in $\mathbb{P}$ of $H$-degree 2 does not admit a surjective morphism onto $N_{X/P}|_Q$. More concretely, we claim that the pair $(N_{Q/P}, N_{X/P}|_Q)$ takes the values $(O(H)^2 \oplus O(2H), O(3H))$, $(O(H)^3 \oplus O(2H), O(2H)^2)$, $(O(H) \oplus O(2H)^2, O(4H))$, $(O(2H)^2 \oplus O(3H), O(6H))$ for Types I to IV, respectively. In Types I and II the line bundle $H$ is very ample and the statement is immediate. We will show how to discard Type IV, being Type III completely analogous. The following argument was suggested to us by M. Reid.

An embedding $Q \subset \mathbb{P}(1^n, 2, 3) = \mathbb{P}$ is given by sections $s_1, \ldots, s_n \in H^0(Q, O(1))$, $t \in H^0(Q, O(2))$ and $u \in H^0(Q, O(3))$.

If $s_1, \ldots, s_n$ are linearly independent then they generate the homogeneous coordinate ring of $Q$. Choosing appropriate weighted homogeneous coordinates $x_1, \ldots, x_n, y, z$ in $\mathbb{P}$ we may assume that $Q \subset \mathbb{P}$ is a complete intersection defined by the following equations:

$$
(x_1^2 = x_2^2 + \cdots + x_n^2, \ y = x_1 f_1 + f_2, \ z = x_1 g_2 + g_3),
$$

being $f_i$ and $g_i$ homogeneous polynomials of degree $i$ in $x_2, \ldots, x_n$. This implies that the normal bundle takes the desired form.

If $s_1, \ldots, s_n$ are not linearly independent then we may assume that $Q$ lies on a subvariety of equations $s_1 = \cdots = s_i = 0$, isomorphic to $\mathbb{P}(1^{n-i}, 2, 3)$. This case may be ruled out by showing that the quadric defined by the equations (2) cannot be projected isomorphically already into $\mathbb{P}(1^{n-1}, y, z)$ (eliminating the variable $x_1$).
In fact the image of $\mathbb{Q}$ by this projection is defined by equations
\[
\text{rank} \left( \begin{array}{ccc}
y - f_2 & z - g_3 & (x_2^3 + \cdots + x_0^2)f_1 \\
f_1 & g_2 & (x_2^3 + \cdots + x_0^2)g_2 \\
z - g_3 & y - f_2 & (x_2^3 + \cdots + x_0^2)g_2 \end{array} \right) \leq 1,
\]
and the set of points having positive dimensional inverse image, defined by equations $f_1 = g_2 = y - f_2 = z - g_3 = 0$, is nonempty.

**Remark 4.5.** Note that if we skip the hypothesis Pic$(X) = \mathbb{Z}$ in the previous proposition, there is just another possibility, namely $X \cong \mathbb{P}^2 \times \mathbb{P}^2$.

**Remark 4.6.** Similarly to 4.2, let us point out that the hypothesis $n \geq 6$ of 4.3 is needed in order to apply Lemma 2.4. We observe that if $n = 5$ and $c \neq 0$ then 2.4 applies and the same conclusion as in 4.3 follows. If $n = 5$, $c = 0$ then, by [BSW, Thm. 2.5], either $X = \mathbb{P}^2 \times \mathbb{P}^3$, or $\tau = 3$, and $\Phi : X \to Y$ contracts $\mathbb{Q}^3$. If dim$(Y) = 2$ then $X$ is as in 4.3.1. If dim$(Y) = 1$ then, for the general fiber $F$, we get $-K_F \cdot \ell = 3$ so that $\Phi_F = |\Phi|_F$ is the contraction of a extremal ray. Hence $X$ is as in 4.3.2. If dim$(Y) = 0$ then $\rho(X) > 1$ by 3.6. Hence $\Phi$ is not the contraction of a extremal ray and [BSW] 2.5.3 together with [W] describe $(X, H)$ precisely.

If $n = 4$ the situation is slightly different since $\mathbb{Q}^2$ contains two different families of lines. Nevertheless adjunction theory arguments (cf. [1] Section 1, [BS2]) and the understanding of the nef morphism of $(X, H)$ and of its first reduction (cf. [BSW]) allow us to give a more explicit description of $(X, H)$. In fact if $X$ is not $\mathbb{P}^4$, $\mathbb{Q}^4$ or a scroll over a curve, then $K_X + 3H$ is nef and in particular $\tau \leq 3$ and $c \leq 1$. Now, with the exception of the cases in which $(X, H)$ is either Del Pezzo (see Remark 4.3), or a quadric fibration onto a curve, or a scroll over a surface, we may take the first reduction $(X', H')$, that verifies that $K_{X'} + 3H'$ is ample. In particular $\tau < 3$. Now, by [BSW] 2.1 and what we have proved, $2 \leq \tau < 3$. Moreover, $\tau = 2$ by [BSW] Thm. 7.3.4. Hence, the nef value morphism $\Phi : X' \to Y'$ of $(X', L')$ contracts $\mathbb{Q}^2$. If dim$(Y') = 2$ then the general fiber is $\mathbb{Q}^2$. If dim$(Y') = 1$ then the general fiber $F$ has $\rho(F) = 1$ and $F$ is one of the list of [W]. If dim$(Y) = 0$ then $X'$ is a Fano variety of index two, classically called Mukai varieties, described in [CLM] and [M1, M2].

5. Proof of the main theorem

We are ready to prove Theorem 5.1. In view of Proposition 5.3 and Lemma 5.6 we may assume that Pic$(X) = \mathbb{Z}H$ and that $\det(N_{G/X}) = cH|G$ with $c > 0$. Since we are assuming that $H$ is very ample, we will consider $X$ as a subvariety of $\mathbb{P}^N := \mathbb{P}(H^0(X, H))$ and study linear subvarieties of $\mathbb{P}^N$ contained in $X$.

In fact our proof involves describing the normal bundle $N_{L/X}$ of a general $(r-1)$-dimensional linear subspace $L \subset G$. Note that rank$(N_{L/X}) = \dim(L) + 2$, hence, even if we check that $N_{L/X}$ is uniform, we cannot infer that it is homogeneous. In fact, it has been conjectured that uniform vector bundles on $\mathbb{P}^s$ of rank smaller than $2s$ are homogeneous (cf. [BE]), and homogeneous vector bundles on $\mathbb{P}^s$ are classified for rank smaller than or equal to $s + 2$ (cf. [OSS] 3.4, p. 70, [EI]). However the conjecture has been confirmed only for rank smaller than or equal to $s + 1$, and some extra cases for $s$ small. Nevertheless, in our particular case, we may prove the following:

**Lemma 5.1.** In the conditions of Theorem 5.1 assume further that $\rho(X) = 1$, and let $L \subset G$ be a general $(r-1)$-dimensional linear subspace. Then either
5.11. \( c = 1 \) and \( N_{L/X} = T_L(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O} \), or

5.13. \( c = 2 \) and \( N_{L/X} = T_L(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O} \).

**Proof.** Take a general line \( \ell \subset L \subset G \). Since \( N_{G/X} \) is g.g., then \( N_{G/X}|_{\ell} = \mathcal{O}(a_{\ell}) \oplus \mathcal{O}(b_{\ell}) \) with \( 0 \leq a_{\ell} \leq b_{\ell} \) and \( c = a_{\ell} + b_{\ell} \). The vanishing \( h^1(X, \mathcal{I}_{G/X}(H)) = 0 \) implies that the ideal sheaf \( \mathcal{I}_{G/PN} \) is generated by quadrics and, in particular, \( N^*_{G/PN}(2) \) and its quotient \( N^*_{G/X}(2) \) are globally generated. It follows that \( a_{\ell} \leq b_{\ell} \leq 2 \) and \( c \leq 4 \). Moreover [S] Lemma 2.4 tells us that the restriction of \( N_{G/X}|_L \) to any line \( \ell' \subset L \) passing through a general point \( x \in L \). Therefore \( N_{G/X}|_L = \mathcal{O}(1) \oplus \mathcal{O} \) by [S] Thm. 1.1, and the exact sequence

\[
0 \to N_{L/G} \cong T_L(-1) \to N_{L/X} \to N_{G/X}|_L \to 0.
\]

splits: in fact \( H^1(L, T_L(s)) = 0 \) for all \( s \) since \( \dim(L) \geq 3 \). This leads us to the case 5.11.

The same argument applies to \( c = 3 \) and \( c = 4 \). But in both cases we get that \( \mathcal{O}(2) \) is a direct summand of \( N_{L/X} \), contradicting the fact that this is a subsheaf of \( N_{L/PN} \cong \mathcal{O}(1)^{\oplus n-r+1} \).

It remains to deal with the case \( c = 2 \). Let us observe that in this case \( N_{G/X} \cong N^*_{G/X}(2) \) is globally generated, hence in particular \( N_{G/X}|_L \) is nef and Griffiths vanishing theorem [L] Variant 7.3.2] tells us that \( H^i(L, N_{G/X}|_L(-2)) = 0 \) for \( r > 4 \), \( i > 0 \) and \( H^i(L, N_{G/X}|_L(-3)) = 0 \) for \( r > 5 \), \( i > 0 \). In particular taking cohomology on the Euler sequence tensored with \( N^*_{G/X}|_L \) and using the isomorphism \( N_{G/X}|_L \cong N^*_{G/X}(2) \), we obtain \( H^i(L, N_{G/X}|_L \otimes T_L(-1)) = 0 \) and the exact sequence [3] splits for \( r > 5 \). Now observe that \( N_{L/X}|_L \) is nef (as an extension of two nef vector bundles) and injects into \( N_{L/PN} \), thus it is uniform and its splitting type is composed of 0’s and 1’s. The splitting of [3] implies that \( N_{G/X}|_L \) is uniform too and we get 5.13 for \( r > 5 \) by [OSS] Thm. 3.2.3.

If \( r = 5 \) and \( c = 2 \), consider the tautological line bundle \( \xi \) of the projective bundle \( \pi : P(N_{G/X}|_L) \to L \). Since \( N_{G/X}|_L \) is nef then the Chern-Wu relation implies \( \xi^4 \cdot \pi^*c_1(\mathcal{O}(1)) = 8 - 4c_2 \geq 0 \), where \( c_2 \) stands for the degree of the second Chern class of \( N_{G/X}|_L \). Thus \( c_2 \leq 2 \). But nefness also implies that \( \xi^5 = 16 - 12c_2 + c_2^2 \geq 0 \), hence \( c_2 \leq 1 \), \( \xi^5 > 0 \) and \( N_{G/X}|_L \) is big. In particular [L] Variant 7.3.3] leads us to the vanishing \( h^i(L, N_{G/X}|_L(-a)) = 0 \) for \( i > 0 \) and \( a = 2, 3 \), allowing us to conclude as in the case \( r > 5 \).

The case \( c = 2 \) and \( r = 4 \) must be treated in a different manner. In this case \( N_{L/X} \) is a uniform vector bundle of rank 5 and splitting type \((0, 0, 1, 1, 1)\). Using the classification given in [BE] we get that \( N_{L/X} \) is either

- \( T_L(-1) \oplus \mathcal{O}(1)^{\oplus 2} \), or
- \( \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2) \), or
- \( \Omega_L(2) \oplus \mathcal{O}(1) \oplus \mathcal{O} \).

The first case leads us again to 5.13, and the last two cases can be excluded by proving that \( N_{L/X} \) cannot contain \( \mathcal{O} \) as a direct summand. In fact, using the exact sequence [3], and taking into account that \( \text{Hom}(T_L(-1), \mathcal{O}) = 0 \), the cokernel \( N_{G/X}|_L \) would contain \( \mathcal{O} \) as a direct summand, too, and the only possibility would be \( N_{G/X}|_L = \mathcal{O} \oplus \mathcal{O}(2) \). But then the sequence [3] would split, contradicting again the fact that \( N_{L/X} \) is a subsheaf of \( N_{L/PN} \cong \mathcal{O}(1)^{\oplus n-r+1} \).  


Remark 5.2. Let us point out that the hypothesis $H^1(X, I_{G/X}(H)) = 0$ can be substituted by the hypothesis on the ideal sheaf $I_{G/PN}$ to be generated by quadrics.

The following arguments finish the proof of Theorem 1.1.

End of the proof. With the same notation and assumptions as above, note that Lemma 6.1 implies $H^1(L, N_{L/X}(a)) = 0$ for all $a \in \mathbb{Z}$ and, in particular $[L]$ is a smooth point in Hilb$(X)$. Denote by $\mathcal{H}$ the unique component of Hilb$(X)$ containing $[L]$ and, with the notation presented in [22] (where $L$ plays the role of $G$), denote by $N_{\mathcal{H}}$ the vector bundle on the universal family $I_0$ verifying that $N_{\mathcal{H}}|_{L'} \cong N_{L'/X}$ for $[L'] \in \mathcal{H}_0$. It will allow us to use semicontinuity.

We claim that, if $c = 1$ (respectively $c = 2$), the normal bundle $N_{L/X}$ splits again as $T_{L'}(-1) \oplus O(1) \oplus O$ (resp. as $T_{L'}(-1) \oplus O(1)^{\oplus 2}$). If $c = 1$ (resp. $c = 2$) and $0 \leq j \leq r - 1$, then $H^j(L, N^*_L/X((-j))) \neq 0$ if and only if $i = j = 0, 1, r - 1$ (resp. $i = j = 1, r - 1$). Applying semicontinuity to $N_{\mathcal{H}}$ and its twists, the same occurs for the general $L'$ so that we conclude by using the Beilinson spectral sequence [OSS] Thm. 3.1.3.

In any case the normal bundle of a general deformation $L'$ of $L$ contains $O(1)$ as a direct summand, providing a smooth hyperplane section $X' := H \cap X$ containing $L'$ (cf. [ABW], [BS2 Cor. 1.7.5]). Therefore, by Bertini theorem, the general hyperplane containing $L'$ is smooth, too. Since $L'$ is general we may assume that such a section exists passing through the general point $x \in X$. Moreover by construction of $X'$, either $N_{L'/X'} \cong T_{L'}(-1) \oplus O$ if $c = 1$, or $N_{L'/X'} \cong T_{L'}(-1) \oplus O(1)$ if $c = 2$ (cf. [NO] Lem. 4.3]). Note also that Lefschetz theorem provides Pic$(X') = \mathbb{Z}$.

At this point we apply [NO Cor. 6.1.4] to $X'$, obtaining that it is isomorphic to a linear section of the Plücker embedding of $G(1, r + 1)$ and $c$ is necessarily equal to 1.

Let $C_x \subset \mathbb{P}(\mathcal{O}_{X,x})$ be the variety of minimal rational tangents to $X$ at a general point $x$ (cf. [HW]), which in this case is the set of tangent directions to lines in $X$ through $x$. Since $X'$ is a hyperplane section of $G(1, r + 1)$ then the corresponding hyperplane section of $C_x$ is a hyperplane section of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{r-1} \subset \mathbb{P}^{2r-1}$, in particular it is a variety of minimal degree in $\mathbb{P}(\mathcal{O}_{X,x})$. Being $C_x$ smooth by [HW Prop. 1.5], $C_x$ must be the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{r-1} \subset \mathbb{P}^{2r-1}$.

In particular, through a general point $x \in X$ there exists an $r$-dimensional linear space $M \subset X$. Moreover, the restriction of the normal bundle $N_{M/X}$ to a codimension one linear subspace $L' \subset M$ is $N_{M/X}|_{L'} \cong N_{L'/X'} \cong T_{L'}(-1) \oplus O$ and we conclude that $X \cong G(1, r + 1)$ by [S Main Thm.].

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