A shifted binomial theorem and trigonometric series

Stéphane Ouvry* and Alexios P. Polychronakos†

October 23, 2020

Abstract

We introduce a shifted version of the binomial theorem, and use it to study some remarkable trigonometric integrals and their explicit rewriting in terms of binomial multiple sums. Motivated by the expressions of area generating functions arising in the counting of closed walks on various lattices, we propose similar sums involving fractional values of the area and show that they are closely related to their integer counterparts and lead to rational sequences converging to powers of π. Our results, other than their mathematical interest, could be relevant to generalizations of statistical mechanical models of the Heisenberg chain type involving higher spins or $SU(N)$ degrees of freedom.

* LPTMS, CNRS, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay Cedex, France; stephane.ouvry@u-psud.fr

† Department of Physics, City College of New York, NY10031 and the Graduate Center of CUNY, New York, NY 10016, USA; apolychronakos@ccny.cuny.edu

1 Introduction

The study of the statistics of lattice random walks is a subject rich with challenges and rewards. Closely related to the celebrated Hofstadter model [1] and its “butterfly” spectrum, it has led to exact expressions for the algebraic area counting of walks involving remarkable trigonometric sums [2] and to intriguing connections with exclusion statistics [3]. The mathematical and physical content of these results seems to be well beyond what has been explored so far.

In [4] we focused on a particular class of objects, namely trigonometric integrals of the type

$$\int_0^1 dt \prod_{i=1}^j \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{r_i}$$

(1)
where \( l_1, l_2, \ldots, l_j \) is a set of positive or null integers and \( r \) a positive integer. They are important building blocks of the Hofstadter model and play a key role in the algebraic area enumeration of various types of lattice walks, such as, in particular, the triangular lattice chiral walks introduced in [3]. The integers \( l_i \) are a generalized composition (called \( g \)-composition [1]) of an integer \( n \) representing the length of the walk, and a summation over their possible values is part of the process, although they have no immediate impact in the analysis of the above sums.

A key ingredient for obtaining the algebraic area combinatorics of such walks was the rewriting of the trigonometric integrals (1) in terms of the binomial multiple sums

\[
\sum_{k_3=-r l_3/2}^{r l_3/2} \cdots \sum_{k_j=-r l_j/2}^{r l_j/2} \left( r l_1/2 + A/2 + \sum_{i=3}^{j} (i-2)k_i \right) \left( r l_2/2 - A/2 - \sum_{i=3}^{j} (i-1)k_i \right) \prod_{i=3}^{j} \left( r l_i/2 + k_i \right)
\]

summed with weight \( e^{i\pi A p/q} \) over a variable \( A \), which ends up being the algebraic area; namely

\[
\int_0^1 dt \prod_{i=1}^{j} \left( 2 \cos \left( \pi t - \pi (i-1) p/q \right) \right)^{r l_i} = \sum_{A} e^{i\pi A p/q} \\
\sum_{k_3=-r l_3/2}^{r l_3/2} \cdots \sum_{k_j=-r l_j/2}^{r l_j/2} \left( r l_1/2 + A/2 + \sum_{i=3}^{j} (i-2)k_i \right) \left( r l_2/2 - A/2 - \sum_{i=3}^{j} (i-1)k_i \right) \prod_{i=3}^{j} \left( r l_i/2 + k_i \right)
\]

The allowed values of the summed variable \( A \) in (3) are dictated by the condition that the binomial entries \( r l_1/2 + A/2 + \sum_{i=3}^{j} (i-2)k_i \) and \( r l_2/2 - A/2 - \sum_{i=3}^{j} (i-1)k_i \) be non-negative integers for all \( k_i \in \{-r l_i/2, r l_i/2\} \), with \( i = 3, \ldots, j \). It follows that when \( r \) is even, since then the \( k_i \)'s are all integers, \( A \) has to be even; when \( r \) is odd, since then the \( k_i \)'s can be either integers or half-integers, \( l_1 + l_2 + \ldots + l_j \) has to be even and \( A \) of the same parity as \( l_1 + l_3 + \ldots \) (or equivalently \( l_2 + l_4 + \ldots \) since \( l_1 + l_2 + \ldots + l_j \) is even). In both cases the summation range is \( A \in \left[-(g-1)r[(l_1+\ldots+l_j)^2/4], (g-1)r[(l_1+\ldots+l_j)^2/4]\right] \).

Note that in the limit \( q \to \infty \) the integral becomes trivial

\[
\int_0^1 dt \left( 2 \cos(\pi t) \right)^{r(l_1+l_2+\ldots+l_j)} = \left( \frac{r(l_1+l_2+\ldots+l_j)}{r(l_1+l_2+\ldots+l_j)/2} \right)
\]

with an overall binomial counting as output. Note also that the multiple binomial sums in (2) are \( A \to -A \) symmetric (changing \( A \) for \(-A \) does not affect it since one can harmlessly replace the \( k_i \)'s by \(-k_i \)) meaning that the \( e^{i\pi A p/q} \) expansion in (3) is actually a \( \cos(\pi A p/q) \) expansion.

From now on we focus on the trigonometric integral (1) and the identity (3), and for simplicity we consider the \( r \) even case. A question naturally arising is whether the sum in
the RHS of (3) generalizes to encompass different types of summands and, correspondingly, new types of walks or physical models. As stressed above, the sum has to be over even values of $A$ ($A/2$ integer). An obvious generalization, then, is to sums over \textit{fractional} values $A/2$, potentially of infinite range. The most obvious such generalization involves summing over \textit{odd} values of $A$ ($A/2$ half-integer), and this will be the focus of our work. In the process, we will encounter binomial sums of the type (4) but shifted, i.e., where $k_i$ is now a half-integer with infinite range instead of an integer. This will lead us to formulate a \textit{shifted} version of Newton’s binomial theorem (5) where the summation index $k$ can be shifted by any real parameter $s$ off the integers. Further, as already alluded to in [4], this procedure will allow for rational sequences converging to powers of $\pi$, when $s = 1/2$, and to other quantities of interest when $s$ has other rational values.

The generalizations considered in this paper, given the connection of the summed variable $A$ to the algebraic area of lattice walks, would correspond to walks with non-integer area in plaquette units. The nature of such nonstandard walks could in principle be unearthed by the properties of the corresponding area enumeration formulæ. In the sequel, we will focus on the mathematical aspects of such sums, leaving their physics for future work. In the Conclusions section we speculate about the possible correspondence of these models with interacting spin systems.

2 Mapping sums over $A$ odd to integrals

We start by reviewing the transition from the LHS trigonometric integral of (3) to its RHS binomial multiple sums. In a nutshell, the basic steps are [4]:

-Use Newton’s binomial theorem for an integer $l$

\[
(1 + x)^l = \sum_{k=0}^{l} \binom{l}{k} x^k \quad \text{or} \quad (x^{1/2} + x^{-1/2})^l = \sum_{k=-l/2}^{l/2} \binom{l}{l/2 + k} x^k \tag{5}
\]

where in the last expression $k$ is integer or half-integer depending on $l$ being even or odd, to binomial-expand each $i = 1, \ldots, j$ cosine in (1) as

\[
\left(2 \cos \left(\pi t - \pi (i - 1) \frac{p}{q}\right)\right)^{rl_i} = \sum_{k_i = -rl_i/2}^{rl_i/2} \binom{rl_i}{rl_i/2 + k_i} e^{2\pi k_i t - 2i\pi k_i (i - 1) \frac{p}{q}} \tag{6}
\]

-Note that $\sum_{i=1}^{j} k_i$ multiplying $2i\pi t$ in the phase is an integer to perform the $t$ integration, obtaining a Kronecker delta on $\sum_{i=1}^{j} k_i = 0$

-Define the new summation variable $A = -2 \sum_{i=1}^{j} k_i (i - 1)$

-Use the two relations arising from the previous two steps, namely $\sum_{i=1}^{j} k_i = 0$ and $A = -2 \sum_{i=1}^{j} k_i (i - 1)$, to eliminate $k_1$ and $k_2$ as summation variables in favor of $A$.  

3 Mapping sums over $A$ odd to integrals
In order to be able to perform the A odd summation instead of the required A even summation, it is useful to reverse the above steps and move from the RHS of (3) to its LHS. This will, then, allow us to express the A odd summation as a generalization of the trigonometric integral (1).

Starting from the A even summation in the RHS of (3), we re-introduce two additional summation variables $k_1, k_2$ and two Kronecker deltas

$$\sum_{k_1, k_2 \text{ integers}} \delta \left( A + 2 \sum_{i=1}^{j} (i - 1)k_i \right) \delta \left( \sum_{i=1}^{j} k_j \right)$$

(7)

Since $k_1, k_2$ do not appear in the summand, summing over them simply enforces the Kronecker deltas and does not change the expression. Summing over $A$ first, however, enforces the first Kronecker delta yielding $A = -2 \sum_{i=1}^{j} (i - 1)k_i$, and substituting this in the sum gives

$$\sum_{k_1, k_2 \text{ integers}} \frac{rl_3}{2} \sum_{k_3=-rl_3/2}^{rl_3/2} \ldots \sum_{k_j=-rl_j/2}^{rl_j/2} \delta \left( \sum_{i=1}^{j} k_i \right) \prod_{i=1}^{j} \left( \frac{rl_i}{2} + k_i \right) e^{-2\pi ik_i p/q}$$

Finally, implementing the Kronecker delta through an integration

$$\delta \left( \sum_{i=1}^{j} k_i \right) = \int_{0}^{1} dt \ e^{2\pi t \sum_{i=1}^{j} k_i} = \int_{0}^{1} dt \ \prod_{i=1}^{j} e^{2\pi t k_i}$$

leads to a sum over binomial terms, one for each $k_i$, of the form

$$\int_{0}^{1} dt \ \prod_{i=1}^{j} \sum_{k_i} \left( \frac{rl_i}{2} + k_i \right) e^{2\pi k_i (t - (i - 1)p/q)}$$

(8)

(a change of variable $t \to t + p/q$ was also performed to bring the exponentials to the form involving $(i - 1)k_i$ rather than $ik_i$).

In the above expression we deliberately left the summations over $k_i$ unspecified. In fact, all $k_1, \ldots, k_j$ will take integer values: $k_3, \ldots, k_j$ by assumption, and $k_1, k_2$ by necessity as explicitly indicated in (7): $A$ is even so $k_2$ has to be integer because of the first Kronecker delta; then $k_1$ also has to be integer because of the second Kronecker delta. This also fixes their range to $[-rl_i/2, rl_i/2]$, in which the binomials do not vanish. An application of the binomial theorem (5) for each term involving $k_i$ then recovers the LHS of (3). So by summing the binomial multiple sum (2) over $A$ even with weight $e^{i\pi Ap/q}$ we obtained the trigonometric integral (1), that is, we recovered the identity (3).

Let us now perform the same summation of the same binomial multiple sum (2) with the same weight $e^{i\pi Ap/q}$ but over A odd instead of A even. We note that, contrary to the A even case, where the summation range is obviously finite, in the A odd case the summation range is by construction infinite. Indeed, since $A$ is odd, both binomial entries $rl_1/2 + A/2 + \sum_{i=3}^{j} (i - 2)k_i$ and $rl_2/2 - A/2 - \sum_{i=3}^{j} (i - 1)k_i$ are now half-integers.
We can proceed in the same way as in the $A$ even case: we introduce two Kronecker deltas and two summation variables $k_1, k_2$ that now need to be half-integers to satisfy the Kronecker deltas with odd $A$,

$$\delta \left( A + 2 \sum_{i=1}^{j} (i - 1) k_i \right) \delta \left( \sum_{i=1}^{j} k_i \right)$$

which again does not change the overall sum. Performing the $A$-summation first and proceeding as before, we end up with almost an identical expression as in (8) with the only difference that the summations $k_1, k_2$ are over half-integer values. The Newton binomial theorem can then be applied only to variables $k_3, \ldots, k_j$ and we end up with the relation

$$\begin{align*}
\sum_{A=-\infty}^{\infty} e^{i \pi A p/q} \
\sum_{k_1=-rl_1/2}^{rl_1/2} \cdots \sum_{k_j=-rl_j/2}^{rl_j/2} \left( r l_1 / 2 + A / 2 + \sum_{i=3}^{j} (i - 2) k_i \right) \left( r l_2 / 2 - A / 2 - \sum_{i=3}^{j} (i - 1) k_i \right) \prod_{i=3}^{j} \left( r l_i / 2 + k_i \right)
\end{align*}$$

which is again $A \to -A$ symmetric, meaning that the $e^{i \pi A p/q}$ expansion is a $\cos(\pi A p/q)$ expansion.

### 3 A shifted binomial theorem

In (9), by insisting on summing the binomial multiple sum (2) weighted by $e^{i \pi A p/q}$ over $A$ odd instead of $A$ even as in (3), we ended up trading in the trigonometric integral (1) the first two cosines for the shifted binomial sums

$$\begin{align*}
\sum_{k_1=-\infty}^{\infty} \left( r l_1 / 2 + k_1 \right) e^{2i \pi k_1 t} , \quad \sum_{k_2=-\infty}^{\infty} \left( r l_2 / 2 + k_2 \right) e^{2i \pi k_2 (t - p/q)}
\end{align*}$$

where $k_1$ and $k_2$ are shifted by 1/2 from their usual integer values in (3).

It is natural to wonder if some generalization of the Newton binomial theorem (3) or (6) could hold when, instead of a finite sum over integers, one has an infinite sum over half-integers as in (10).
Indeed, such a generalization exists and plays a central role in rewriting the sums appearing in this paper. Specifically, what we shall call the shifted binomial theorem states that for an integer \( l \) and any real number \( s \)

\[
(1 + x)^l = \sum_{k=-\infty}^{\infty} \binom{l}{k + s} x^{k + s}, \quad x = e^{i\varphi}, \quad -\pi \leq \varphi \leq \pi
\]  

(11)

where the fractional binomial coefficients are expressed in terms of \( \Gamma \)-functions

\[
\binom{l}{k + s} = \frac{l!}{\Gamma(k + s + 1)\Gamma(l - k - s + 1)}
\]

It is a deformation by a shift \( s \) of the standard Newton theorem (5) with the caveat that \( x \) is now restricted to be a phase \( e^{i\varphi} \) with \( -\pi \leq \varphi \leq \pi \). Indeed we note that the left hand side of (11) depends only on \( e^{i\varphi} \), while the right hand side involves the fractional power \((k + s)\varphi\), for which the range of \( \varphi \) is relevant.

We have not found any statement or proof of this theorem in the mathematics literature, so we provide below our own proof. To this end, define the function

\[
S(\varphi) = e^{-i\varphi} \quad \text{for} \quad -\pi \leq \varphi \leq \pi, \quad S(\varphi + 2\pi) = S(\varphi)
\]

\( S(\varphi) \) is by definition a 2\( \pi \)-periodic function (discontinuous at \( \varphi = (2n+1)\pi \)) with discrete Fourier modes for integer \( k \)

\[
S_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\varphi) e^{-ik\varphi} d\varphi = \frac{\sin \left( \pi (k + s) \right)}{\pi (k + s)}, \quad S(\varphi) = \sum_{k=-\infty}^{\infty} S_k e^{ik\varphi}
\]

Then, for \( -\pi < \varphi < \pi \)

\[
S(\varphi) (1 + e^{i\varphi})^l = \sum_{n=0}^{l} \binom{l}{n} \sum_{k=-\infty}^{\infty} \frac{\sin \pi (k + s)}{\pi (k + s)} e^{i(k+n)\varphi}
\]

(shift \( k \rightarrow k - n \))

\[
= \frac{\sin(\pi s)}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k e^{ik\varphi} \sum_{n=0}^{l} \binom{l}{n} \frac{(-1)^n}{k + s - n}
\]

\[
= \frac{\sin(\pi s)}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k e^{ik\varphi} (-1)^l \frac{l!}{\Gamma(k + s + 1)\Gamma(k + s + l)}
\]

where the finite sum over \( n \) was explicitly performed. Finally, using

\[
\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}
\]

for \( z = k + s - l \) we obtain

\[
S(\varphi) (1 + e^{i\varphi})^l = \sum_{k=-\infty}^{\infty} e^{ik\varphi} \frac{l!}{\Gamma(k + s + 1)\Gamma(l - k - s + 1)}
\]
and since $S(\varphi) = e^{-i\varphi}$ when $-\pi \leq \varphi \leq \pi$, we recover the shifted binomial theorem (11) which holds for general fractional $s$. In particular, a generalized Chu-Vandermonde identity automatically follows as

$$\binom{l_1 + l_2}{l_1' + l_2'} = \sum_{k=-\infty}^{\infty} \binom{l_1}{l_1' + k + s} \binom{l_2}{l_2' - k - s}$$

The shifted binomial theorem (11) can be equivalently restated as

$$(2 \cos(\varphi/2))^l = \sum_{k=-\infty}^{\infty} \binom{l}{l/2 + k + s} e^{i(k+s)\varphi}, \quad -\pi \leq \varphi \leq \pi$$

where $k$ is integer or half-integer depending on $l$ being even or odd so that $l/2 + k$ is always an integer. Eq. (13) is the shifted version of the RHS of (5) with $x$ replaced by $e^{i\varphi}$. Note that here and in what follows we take $k$ to be integer or half-integer such that (13) reduces to (5) when $s = 0$. However as far as the validity of (13) per se is concerned, this qualification on $k$ is not needed, i.e., it can be taken integer regardless of the parity of $l$.

The shift $s = 1/2$ is the case of interest in view of the shifted binomial sums in (10). In this case (13) can be subsumed as (absorbing the 1/2 in the summation index $k$ and replacing $\varphi \to 2\pi t$)

$$\sum_{k=-\infty}^{\infty} \binom{l}{l/2 + k} e^{2\pi ikt} = (2 \cos(\pi t))^l, \quad -1/2 \leq t \leq 1/2$$

with the values of $k$ chosen such that $l/2 + k$ be half-integer. Both sums in (10) are precisely of this type for $l$ even, with in the second sum $t \to t - p/q$ with an integration range $-1/2 \leq t - p/q \leq 1/2$ for (14) to be valid.

We will explore the properties of these sums in the upcoming sections. But for now, we can take advantage of the shifted binomial expansion (14) to retrieve rational sequences that converge to powers of $\pi$ (see e.g., [6] for a catalogue of $\pi$ formulae).

As a first example, let us fix $t = 0$ in (14) for $l$ even: this would yield

$$\sum_{k=-\infty}^{\infty} \binom{l}{l/2 + k} = 2^l$$

---

In the $s = 1/2$ case it becomes

$$\binom{l_1 + l_2}{l_1' + l_2'} = \sum_{k=-\infty}^{\infty} \binom{l_1}{l_1' + k} \binom{l_2}{l_2' - k}$$

(12)
which, when rewritten as

\[ 2^{-l} \lim_{m \to \infty} \sum_{\substack{k = -m + 1/2 \atop k \text{ half-integer}}}^{m + 1/2} \pi \left( \frac{l}{1/2 + k} \right) = \pi \]

allows for sequences of rational numbers converging to \(\pi\) when \(m \to \infty\), since for \(l\) integer and \(l'\) half-integer

\[ \pi \left( \frac{l}{l'} \right) = (-1)^{l'+1/2} \left( \frac{(-1)^{k-1/2} \pi k}{k} \right) = \left( \frac{l}{l'} \right) \]

is a rational number.

As a second example, we can integrate both sides of (14) over \(t\) in the interval of validity \([-1/2, 1/2]\) of the shifted binomial theorem to obtain, when \(l\) is even

\[ \sum_{k = -\infty}^{\infty} \left( \frac{l}{1/2 + k} \right) \frac{\sin(\pi k)}{\pi k} = \int_{-1/2}^{1/2} (2 \cos(\pi t))^l \]

i.e.,

\[ \sum_{k = -\infty}^{\infty} \left( \frac{l}{1/2 + k} \right) \frac{(-1)^{k-1/2} \pi k}{\pi k} = \left( \frac{l}{1/2} \right) \]

This, when rewritten as

\[ \frac{(l/2)!^2}{l!} \lim_{m \to \infty} \sum_{\substack{k = -m + 1/2 \atop k \text{ half-integer}}}^{m + 1/2} \pi \left( \frac{l}{1/2 + k} \right) \left( \frac{(-1)^{k-1/2}}{k} \right) = \pi^2 \]

allows for a sequence of rational numbers converging now to \(\pi^2\) when \(m \to \infty\).

This construction can be generalized by considering an arbitrary rational \(s\) in (13) to obtain rational sequences converging for example to \(\pi / \sin(\pi s)\) or \(\pi^2 / \sin(\pi s)^2\) instead of \(\pi\) or \(\pi^2\) sequences just obtained for \(s = 1/2\) (see the Appendix for details).

\[ ^{2}\text{The } l\text{ odd case would give } 2^{-l} \lim_{m \to \infty} \sum_{k = -m}^{m} \pi \left( \frac{l}{1/2 + k} \right) = \pi \]

which happens to be identical to the \(l - 1\) even sequence above since

\[ \left( \frac{l}{1/2 + k} \right) = \left( \frac{l - 1}{1/2 + k} \right) + \left( \frac{l - 1}{1/2 - k} \right) \]

\[ ^{3}\text{The } l\text{ odd case would trivially yield } \left( \frac{l}{l/2} \right) = \left( \frac{l}{l/2} \right) \text{ since only the } k = 0 \text{ term in the } k \text{ even summation would contribute.} \]
4 Exploring (9)

Since the trigonometric integral (11) is clearly unaffected by changing the integration range from $[0, 1]$ to any interval of length 1, in particular to the interval $[-1/2, 1/2]$, it can be viewed as a generalization of the integral in (15) to a product of $i = 1, \ldots, j$ cosines with $\pi(i - 1)p/q$ phase shifts. It is therefore tempting to expect that by following the same line of reasoning as in section 2, i.e., trading in (11) some cosines for their shifted binomial sums, we might be able to construct more general sequences for other powers of $\pi$.

There are, however, two caveats to this: Firstly, the freedom to change the integration range of the variable $t$ exists only when the sum of the shifts in all binomials is an integer. This is the only case in which the sum of all $k_i$ will be an integer and the integration will simply pick the term where this sum is zero, all other terms vanishing. In the more general case, all terms will contribute in a way depending on the range of integration.

Secondly, the shifted binomial theorem only holds when the argument $\varphi$ is confined in the interval $[-\pi, \pi]$, while an obvious generalization applies for $\varphi$ outside this range. Therefore it can hold for at most one of the cosines traded for a shifted binomial sum, but not for the remaining ones, because of the varying $-\pi(i - 1)p/q$ phase shifts. E.g., it will hold for the first $i = 1$ traded cosine, provided that the integration range is $[-1/2, 1/2]$ such that $2\pi t$ be in the interval $[-\pi, \pi]$, but not for the $i = 2, \ldots, j$ cosines in which their $-\pi(i - 1)p/q$ phase shifts will drive their arguments outside the range $[-\pi, \pi]$.

There are some notable cases where these caveats are not relevant: if an even number of cosines are traded with shifts $1/2$, as in (9), then, as already stressed above, the integration range can freely be changed. (When an odd number of cosines are traded the results do depend on the range, as in section 5 below). Further, in the $q \to \infty$ limit the $-\pi(i - 1)p/q$ phase shifts vanish and the shifted binomial theorem holds for all traded cosines, thus recovering (11) in the $q \to \infty$ limit, i.e., (1).

Let us explore the properties of (9) in the light of what has just been said. As a first step, we can harmlessly change the integration range from $[0, 1]$ to $[-1/2, 1/2]$ and then use the shifted binomial theorem to write the first shifted binomial sum as a cosine. The integral in the RHS of (9) can then be rewritten as

$$
\int_0^1 dt \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left( \frac{r_{l_1}}{r_{l_1}/2 + k_1} \right) ^{r_{l_1}} \left( \frac{r_{l_2}}{r_{l_2}/2 + k_2} \right) ^{r_{l_2}} e^{2i\pi k_1 t} e^{2i\pi k_2(t-p/q)} \prod_{i=3}^{j} \left( 2 \cos (\pi t - \pi(i-1)p/q) \right) ^{r_{l_i}}
$$

$$
= \int_{-\frac{1}{2}}^{\frac{1}{2}} dt \left( 2 \cos (\pi t) \right) ^{r_{l_1}} \sum_{k_2 = -\infty}^{\infty} \left( \frac{r_{l_2}}{r_{l_2}/2 + k_2} \right) ^{r_{l_2}} e^{2i\pi k_2(t-p/q)} \prod_{i=3}^{j} \left( 2 \cos (\pi t - \pi(i-1)p/q) \right) ^{r_{l_i}}
$$

Then, we can proceed as before: binomial-expand as in (9) the first cosine and each of the $i = 3, \ldots, j$ cosines, see that the overall $\sum_{i=1}^{j} k_i = k_1 + k_2 + \sum_{i=3}^{j} k_i$ multiplying $2i\pi t$ in the exponential is a half-integer, define $A = -2 \sum_{i=1}^{j} k_i(i - 1) = -2k_2 - 2 \sum_{i=3}^{j} k_i(i - 1)$,
with $A$ being necessarily odd since $2k_2$ is odd, solve for $k_2 = -A/2 - \sum_{i=3}^{j}(i - 1)k_i$ now expressed in terms of $A$ and the $k_i$’s, $i \neq 2$, and finally integrate over $t$ to obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dt \left( 2 \cos (\pi t) \right)^{r_{l_1}} \sum_{k_2 = -\infty}^{\infty} \frac{r_{l_2}}{r_{l_2}/2 + k_2} e^{2i\pi k_2(t-p/q)} \prod_{i=3}^{j} \left( 2 \cos (\pi t - \pi (i - 1)p/q) \right)^{r_{l_i}}$$

which is again $A \to -A$ symmetric because the ratio $\frac{\sin \left( \pi (A/2 + \sum_{i=1}^{j}(i - 2)k_i) \right)}{\pi (A/2 + \sum_{i=1}^{j}(i - 2)k_i)}$ remains unchanged by simultaneously trading $A$ for $-A$ and the $k_i$’s for $-k_i$. It follows that the $e^{i\pi Ap/q}$ expansion (16) is in fact a $\cos(\pi Ap/q)$ expansion. Both expansions in (9) and (16) are identical, meaning that for a given $A$ their respective multiple binomial sums weighted by $e^{i\pi Ap/q}$, which in both cases have finite summation range and are of the form $1/\pi^2 \times$ a rational number\(^4\), are equal.

To get a $\pi^2$ sequence we have to go a step further and use that in the limit $q \to \infty$ the overall binomial counting is unaffected by the trading of cosines: it follows that in (9), or equivalently in (16), setting $e^{i\pi Ap/q} = 1$, the infinite $A$ odd summation from minus to plus infinity of the binomial multiple sums necessarily converges to

$$\left( \frac{r(l_1 + l_2 + \ldots + l_j)}{r(l_1 + l_2 + \ldots + l_j)/2} \right)$$

which is an integer since $r$ is even\(^5\).

Using the symmetry $A \to -A$, consider now instead the cumulative sum $2 \sum_{A=1}^{2m+1} c_{a_{l_1,\ldots,l_j}(m)}/b_{l_1,\ldots,l_j}(m)$ with $a_{l_1,\ldots,l_j}(m)$ and $b_{l_1,\ldots,l_j}(m)$ becoming larger and larger with $m$ going to infinity. It then follows that

$$\frac{a_{l_1,\ldots,l_j}(m)}{b_{l_1,\ldots,l_j}(m)} \to_{m \to \infty} \pi^2 \left( \frac{r(l_1 + l_2 + \ldots + l_j)}{r(l_1 + l_2 + \ldots + l_j)/2} \right)$$

i.e., for any given set of $l_i$’s and $r$ even, we have constructed a sequence of rational numbers which converges when $m \to \infty$ to $\pi^2$ up to the overall binomial factor.

\(^4\)In (9) the first two binomials have half-integer entries, and in (16) only the second binomial has a half-integer entry but there is an additional factor of $1/\pi$.

\(^5\)This can be directly checked, setting $e^{i\pi Ap/q} = 1$ in (9), by first performing the $A$ summation using the generalized Chu-Vandermonde identity (12), then over the $k_i$’s by redefining them appropriately (see [2]).
We can further sum $a_{l_1,...,l_j}(m)/b_{l_1,...,l_j}(m)$ over all $g$-compositions \([3][5]\) of an integer $n$ (meaning that the sets $l_1,l_2,\ldots,l_j$ are now viewed as the $g$-compositions of $n$, i.e., $l_1 + l_2 + \ldots + l_j = n$ with no more than $g - 2$ zeroes in succession) with weight (see \([3]\) for its genesis)

$$c_g(l_1,l_2,\ldots,l_j) = \frac{(l_1 + \cdots + l_{g-1} - 1)!}{l_1!\cdots l_{g-1}!} \prod_{i=1}^{j-g+1} \binom{l_i + \cdots + l_{i+g-1} - 1}{l_{i+g-1}}$$

$$= \frac{\prod_{i=1}^{j-g+1}(l_i + \cdots + l_{i+g-1} - 1)!}{\prod_{i=1}^{j-g}(l_{i+1} + \cdots + l_{i+g-1} - 1)!} \prod_{i=1}^{j} \frac{1}{l_i!}$$

to get the sequence of rational numbers

$$\frac{a_n(m)}{b_n(m)} = gn \sum_{l_1,l_2,\ldots,l_j} c_g(l_1,l_2,\ldots,l_j) \frac{a_{l_1,...,l_j}(m)}{b_{l_1,...,l_j}(m)}$$

By construction this sequence converges when $m \to \infty$ to

$$\frac{a_n(m)}{b_n(m)} \to_{m \to \infty} \pi^2 \left( \frac{rn}{rn/2} \right) \left( \frac{gn}{n} \right)$$

5 Trading a single cosine

In full generality we could trade in \([1]\) not two cosines as in \([9]\), but any number of them for their \textit{shifted} binomial sums. Let us here consider trading the first cosine only. In this case we cannot freely change the integration range and each choice of range will yield a different results. We examine a couple of cases below.

5.1 Integrating from $-1/2$ to $1/2$

Clearly

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dt \sum_{k_1=\text{half-integer}}^{\infty} \left( \frac{rl_1}{rl_1/2 + k_1} \right) e^{2\pi k_1 t} \prod_{i=2}^{j} \left( 2 \cos (\pi t - \pi (i-1)p/q) \right)^{rl_i}$$

is identical to \([1]\) by virtue of the \textit{shifted} binomial theorem. We can again use the usual strategy to extract from \([18]\) the relevant binomial multiple sum: binomial-expand each $i = 2,\ldots,j$ cosine, see that the overall $\sum_{i=1}^{j} k_i$ multiplying $2\pi t$ in the exponential is now half-integer because of $k_1$ being half-integer, define $A = -2 \sum_{i=1}^{j} (i-1)k_i = -2 \sum_{i=2}^{j} (i-$
5.2 Integrating from 0 to 1 \( k_i \), which is thus even, solve for \( k_2 = -A/2 - \sum_{i=3}^{j} (i - 1)k_i \) and finally integrate over \( t \). We obtain

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} dt \prod_{i=1}^{j} \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{rl_i}
\]

\[
= \sum_{A \text{ even}} e^{i\pi Ap/q} \sum_{k_1=\text{half-integer}}^{k_1=-\infty} \sum_{k_3=-rl_3/2}^{rl_3/2} \cdots \sum_{k_j=-rl_j/2}^{rl_j/2} \frac{\sin \left( \pi (A/2 + \sum_{i=1}^{j} (i - 2)k_i) \right)}{\pi (A/2 + \sum_{i=1}^{j} (i - 2)k_i)}
\]

\[
\left( \frac{rl_1}{rl_1/2 + k_1} \right) \left( \frac{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^{j} (i - 1)k_i} \right) \prod_{i=3}^{j} \left( \frac{rl_i}{rl_i/2 + k_i} \right)
\]

which is again \( A \rightarrow -A \) symmetric (this a \( \cos(\pi Ap/q) \) expansion). Note that in (19) the multiple binomial sums weighted by \( e^{i\pi Ap/q} \) have the same form as whose in (16) with the caveat that in (19) \( A \) is even and \( k_1 \) half-integer whereas in (16) \( A \) is odd and \( k_1 \) integer. (19) is yet another rewriting of (3) as again a summation over \( A \) even of binomial multiple sums which are now \( 1/\pi^2 \times \) a rational number (there is an explicit \( 1/\pi \) and another \( 1/\pi \) coming from the half-integer \( k_1 \) binomial). For a given \( A \) the multiple binomial sums in (3) and (19) match one to one, meaning that an integer in (3) is equal to \( 1/\pi^2 \times \) a rational number in (19) whose numerator and denominator become larger and larger with the \( k_1 \) summation range going to infinity. Focusing instead on the cumulative sum \( \sum_{k_1=-m+1/2}^{m+1/2} \) and multiplying it by \( \pi^2 \), it follows that its ratio with the corresponding integer in (3) yield sequences of rational numbers converging when \( m \rightarrow \infty \) to \( \pi^2 \).

5.2 Integrating from 0 to 1

Let us now consider instead of the integration range \(-1/2 \) to \( 1/2 \) in (18), the range 0 to 1. The shifted binomial theorem does hold anymore. The relevant binomial multiple sums can be extracted as usual by binomial-expanding each \( i=2, \ldots, j \) cosine, defining \( A = -2 \sum_{i=1}^{j} (i - 1)k_i \) which is even, extracting \( k_2 = -A/2 - \sum_{i=3}^{j} (i - 1)k_i \) and finally integrating over \( t \)

\[
\int_{0}^{1} dt \sum_{k_1=\text{half-integer}}^{k_1=-\infty} \left( \frac{rl_1}{rl_1/2 + k_1} \right) e^{2i\pi k_1 t} \prod_{i=2}^{j} \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{rl_i}
\]

\[
= \sum_{A \text{ even}} e^{i\pi Ap/q} \sum_{k_1=\text{half-integer}}^{k_1=-\infty} \sum_{k_3=-rl_3/2}^{rl_3/2} \cdots \sum_{k_j=-rl_j/2}^{rl_j/2} e^{-i\pi (A/2 + \sum_{i=1}^{j} (i - 2)k_i)} \frac{\sin \left( \pi (A/2 + \sum_{i=1}^{j} (i - 2)k_i) \right)}{\pi (A/2 + \sum_{i=1}^{j} (i - 2)k_i)}
\]

\[
\left( \frac{rl_1}{rl_1/2 + k_1} \right) \left( \frac{rl_2}{rl_2/2 - A/2 - \sum_{i=3}^{j} (i - 1)k_i} \right) \prod_{i=3}^{j} \left( \frac{rl_i}{rl_i/2 + k_i} \right)
\]
or, since \( A \) being even and \( k_1 \) half-integer implies \( e^{2i\pi(A/2+\sum_{i=1}^{j}(i-2)k_i)} = -1 \) so that

\[
e^{-i\pi(A/2+\sum_{i=1}^{j}(i-2)k_i)} \sin \left(\pi(A/2 + \sum_{i=1}^{j}(i-2)k_i)\right) = -i
\]

\[
\int_0^1 dt \sum_{k_1=-\infty}^{\infty} \sum_{k_1 \text{ half-integer}}^{\infty} \left( rl_1/2 + k_1 \right) e^{2i\pi k_1 t} \prod_{i=2}^{j} \left( 2 \cos \left( \pi t - \pi(i-1)p/q \right) \right)^{rl_i}
\]

\[
= -i \sum_{A \text{ even}} \sum_{k_1=-\infty}^{\infty} \sum_{k_1 \text{ half-integer}}^{\infty} \left( rl_1/2 + k_1 \right) \left( rl_2/2 - A/2 - \sum_{i=3}^{j}(i-1)k_i \right) \prod_{i=3}^{j} \left( rl_i/2 + k_i \right)
\]

(20)

which is now anti-symmetric under \( A \to -A \), since the denominator \( \pi(A/2 + \sum_{i=1}^{j}(i-2)k_i) \) changes sign when simultaneously exchanging \( A \) for \(-A\) and the \( k_i \)'s for \(-k_i \). It follows that this \( e^{i\pi Ap/q} \) expansion is in fact a \( \sin(\pi Ap/q) \) expansion. In (20) the binomial multiple sums weighted by \( e^{i\pi Ap/q} \) are again (up to a factor \( i \)) of the form \( 1/\pi^2 \times \) a rational number (there is an explicit \( 1/\pi \) and another \( 1/\pi \) coming from the half-integer \( k_1 \) binomial) whose numerator and denominator become larger and larger with \( k_1 \) going to infinity.

Could we find if any a sequence associated to these rational numbers converging to a power of \( \pi \)? To do so let us perform the integral in (20) in yet another way by relying again on the shifted binomial theorem. To do so, we split the integration range in two intervals \([0, 1/2]\) and \([1/2, 1]\):

- In the first interval the shifted binomial theorem holds, so the integral is

\[
\int_0^{1/2} dt \prod_{i=1}^{j} \left( 2 \cos \left( \pi t - \pi(i-1)p/q \right) \right)^{rl_i}
\]

- In the second interval, we change variable \( t \to t - 1/2 \in [0, 1/2] \):

\[
\int_0^{1/2} dt \sum_{k_1=-\infty}^{\infty} \left( rl_1/2 + k_1 \right) e^{2i\pi k_1 t} \prod_{i=2}^{j} \left( 2 \cos \left( \pi t - \pi(i-1)p/q \right) \right)^{rl_i}
\]

\[
= \int_0^{1/2} dt \sum_{k_1=-\infty}^{\infty} \left( rl_1/2 + k_1 \right) e^{2i\pi k_1 t} e^{-i\pi k_1} \prod_{i=2}^{j} \left( 2 \cos \left( \pi t + \pi/2 - \pi(i-1)p/q \right) \right)^{rl_i}
\]

\[
= - \int_0^{1/2} dt \prod_{i=1}^{j} \left( 2 \sin \left( \pi t - \pi(i-1)p/q \right) \right)^{rl_i}
\]

the last line following from using the shifted binomial theorem and \( k_1 \) being half-integer.
Altogether we obtain

\[
\int_0^1 dt \sum_{k_1 = -\infty}^{\infty} \left( \frac{r_{l_1}}{r_{l_1}/2 + k_1} \right) e^{2i\pi k_1 t} \prod_{i = 2}^{j} \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{r_{l_i}}
\]

\[
= \int_0^{1/2} dt \left( \prod_{i = 1}^{j} \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{r_{l_i}} - \prod_{i = 1}^{j} \left( 2 \sin \left( \pi t - \pi (i - 1)p/q \right) \right)^{r_{l_i}} \right)
\]

from which the relevant binomial multiple sums can be extracted as above (binomial-expand each \(i = 1, \ldots, j\) cosine and sine, ...) to obtain

\[
\int_0^1 dt \sum_{k_1 = -\infty}^{\infty} \left( \frac{r_{l_1}}{r_{l_1}/2 + k_1} \right) e^{2i\pi k_1 t} \prod_{i = 2}^{j} \left( 2 \cos \left( \pi t - \pi (i - 1)p/q \right) \right)^{r_{l_i}}
\]

\[
= -i \sum_{A \text{ even}} \sum_{k_1 = -r_{l_1}/2}^{r_{l_1}/2} \sum_{k_2 = -r_{l_2}/2}^{r_{l_2}/2} \cdots \sum_{k_j = -r_{l_j}/2}^{r_{l_j}/2} \frac{1 - \cos \left( \pi (A/2 + \sum_{i = 1}^{j} (i - 2)k_i) \right)}{\pi (A/2 + \sum_{i = 1}^{j} (i - 2)k_i)} \left( \frac{r_{l_1}}{r_{l_1}/2 + k_1} \right) \left( \frac{r_{l_2}}{r_{l_2}/2 - A/2 - \sum_{i = 3}^{j} (i - 1)k_i} \right) \prod_{i = 3}^{j} \left( \frac{r_{l_i}}{r_{l_i}/2 + k_i} \right)
\]

In (21) the multiple binomial sums weighted by \(e^{i\pi A p/q}\) are now of the form \(1/\pi \times a\) rational number \(\text{up to a factor } i\). Since they match one to one those in (20), one can again construct a sequence of rational numbers converging now to \(\pi\), since the former being when multiplied by \(\pi^2\) a rational number whose numerator and denominator become larger and larger with the \(k_1\) summation range going to infinity –so here one again focuses on the cumulative sum \(\sum_{k_1 = -m+1/2}^{m+1/2}\) and the latter being when multiplied by \(\pi\) a rational number, their ratio is a rational number which necessarily converges when \(m \to \infty\) to \(\pi\).

### 6 Conclusions

We have explored the relation of trigonometric sums with shifted summation variables to corresponding trigonometric integrals. Clearly this is the tip of a big iceberg that depends on how we wish to extend or deform these sums. E.g., we could consider yet other tradings of cosines for shifted binomial sums, as for example with \textbf{four shifted} binomial sums (i.e., an even number of shifted sums, so integrating over [0, 1] is the same as integrating over

\(\text{Since } A \text{ is even and all the } k_i \text{'s are integers, } \pi (A/2 + \sum_{i = 1}^{j} (i - 2)k_i) \text{ in the denominator can vanish so that } 1 - \cos \left( \pi (A/2 + \sum_{i = 1}^{j} (i - 2)k_i) \right) \text{ in the numerator also vanishes: the indeterminate ratio has to be understood as also vanishing.} \)
\[ \int_{-1/2}^{1/2} dt \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \sum_{k_4=-\infty}^{\infty} \left( r_{l_1} \right) \left( r_{l_2} \right) \left( r_{l_3} \right) \left( r_{l_4} \right) \]

\[ e^{2i\pi k_1} e^{2i\pi k_2 (t-p/q)} e^{2i\pi k_3 (t-2p/q)} e^{2i\pi k_4 (t-3p/q)} \prod_{i=5}^{j} \left( 2 \cos (\pi t - \pi (i-1)p/q) \right) \]

which would again yield in the limit \( q \to \infty \) the overall binomial counting \( \left( \frac{r_{l_1} + l_2 + \ldots + l_j}{r_{l_1} + l_2 + \ldots + l_j} \right) / 2 \).

Proceeding as above we would obtain that this trigonometric integral rewrites as

\[ \sum_{A=-\infty}^{\infty} e^{i\pi A p/q} \sum_{k_3=-\infty}^{\infty} \sum_{k_4=-\infty}^{\infty} \sum_{k_5=-rl_2/2}^{rl_2/2} \sum_{k_j=-rl_j/2}^{rl_j/2} \]

\[ \left( r_{l_1} \right) \left( r_{l_2} \right) \left( r_{l_3} \right) \left( r_{l_4} \right) \left( r_{l_i} \right) 
\prod_{i=3}^{j} \left( r_{l_i} / 2 + k_i \right) \]

where \( A \) is necessarily even. One could take advantage of the shifted binomial theorem and rewrite the first shifted binomial sum as a cosine to obtain yet another expression of the \( e^{i\pi A p/q} \) expansion (22). We could also use the overall binomial sum rule. These manipulations would lead to rational sequences for \( \pi^4 \)–since the first four binomial entries in (22) are half-integers– albeit here with two infinite summations over the half-integers \( k_3 \) and \( k_4 \).

It is clear that this pattern generalizes to any number of shifted binomial sums. Extensions of sums to other rational values of \( A \) would also be possible, using the shifted binomial theorem, and would lead to similarly generalized results.

An interesting implication of our results would be in the possible connection of the full sums, over both \( A \) and the \( g \)-compositions \( l_i \), and interacting spin systems. The connection arises by mapping each \( g \)-composition of \( n \) with a configuration of \( n-1 \) \( g \)-level systems (e.g., spin-\((g-1)/2\) \( SU(2) \) spins or fundamental \( SU(g) \) spins). The corresponding spin dynamics are implied by the sums, interpreted as spin partition functions. The generalized sums considered in this work would still correspond to spin systems, but with different, nontrivial couplings.

Finally, the shifted binomial theorem itself could be explored and mined for applications, irrespective of any random walk connection. In particular, it could be used to express periodic functions as a sum of shifted Fourier frequencies, rather than the standard Fourier sum. This could be useful, e.g., for signal processing where the shifted frequency expansion may converge faster, or be more revealing of the properties of the signal.

These and similar considerations are left for future work.
References

[1] D. Hofstadter, “Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields”, Phys. Rev. B 14 (1976) 2239.

[2] S. Ouvry and S. Wu, “The algebraic area of closed lattice random walks”, Journal of Physics A: Mathematical and Theoretical, Volume 52 (2019) 255201.

[3] S. Ouvry and A. Polychronakos, “Exclusion statistics and lattice random walks”, NPB[FS] 948 (2019) 114731.

[4] S. Ouvry and A. Polychronakos, “Lattice walk area combinatorics, some remarkable trigonometric sums and Apéry-like numbers”, NPB[FS] 960 (2020) 115174.

[5] B. Hopkins and S. Ouvry, “Combinatorics of Multicompositions”, Proceedings of the Combinatorial and Additive Number Theory conference, CUNY Graduate Center NY NY, Springer (2020).

[6] D. H. Bayley, “A catalogue of mathematical formulae involving \( \pi \), with analysis” (2020).

7 Appendix

Let us start from the shifted binomial theorem (13)

\[
(2 \cos(\pi t))^l = \sum_{k = \text{integer or half-integer}}^{\infty} \binom{l}{l/2 + k + s} t^{2i\pi(k+s)t}, \quad t \in (-1/2, 1/2)
\]

where \( k \) is integer if \( l \) is even and half-integer if \( l \) is odd and focus on \( s \) a rational number.

Again take \( t = 0 \): we obtain

\[
2^l = \sum_{k = \text{integer or half-integer}}^{\infty} \binom{l}{l/2 + k + s}
\]

When \( l \) is even i.e., \( k \) integer, the shifted binomial \( \binom{l}{l/2+k+s} \) is a rational number up to the factor \( 1/(\Gamma[s] \Gamma[1-s]) = \sin(\pi s)/\pi \). It means that necessarily the sequence

\[
2^{-l} \lim_{m \to \infty} \sum_{k = -m}^{m} \frac{\pi}{\sin(\pi s)} \binom{l}{l/2 + k + s} = \frac{\pi}{\sin(\pi s)}
\]

is rational and converges to \( \pi/\sin(\pi s) \).
Likewise, when $l$ is odd, i.e., $k$ half-integer, the sequence

$$2^{-l} \lim_{m \to \infty} \sum_{k=-m-1/2}^{m-1/2} \frac{\pi}{\sin(\pi s)} \left( \frac{l}{l/2 + k + s} \right) = \frac{\pi}{\sin(\pi s)}$$

is rational and converges to $\pi/\sin(\pi s)$.

Upon integrating over $t$ in the interval \([-1/2, 1/2]\) one would get

$$\left( \frac{l}{l/2} \right) = \sum_{k=-\infty}^{\infty} \left( \frac{l}{l/2 + k + s} \right) \frac{\sin(\pi (k + s))}{\pi (k + s)}$$

Altogether,

- when $l$ is even, the sequence

$$\frac{(l/2)!^2}{l!} \lim_{m \to \infty} \sum_{k=-m}^{m} \frac{\pi}{\sin(\pi s)} \left( \frac{l}{l/2 + k + s} \right) \frac{(-1)^k}{k + s} = \left( \frac{\pi}{\sin(\pi s)} \right)^2$$

is rational and converges to $(\pi/\sin(\pi s))^2$.

- when $l$ is odd, the sequence

$$\frac{(l/2)!^2}{l!} \frac{1}{\pi} \lim_{m \to \infty} \sum_{k=-m-1/2}^{m-1/2} \frac{\pi}{\sin(\pi s)} \left( \frac{l}{l/2 + k + s} \right) \frac{(-1)^{k-1/2}}{k + s} = \frac{\pi}{\sin(\pi s) \cos(\pi s)}$$

($l$ being odd there is an additional $1/\pi$ due to $(l/2)!^2/l!$) is rational and converges to $\pi/(\sin(\pi s) \cos(\pi s))$. 