Angular momentum
and long-range gravitational interactions
in Matrix theory

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Abstract

We consider subleading terms in the one-loop Matrix theory potential between a
classical membrane state and a supergraviton. Nontrivial terms arise at order \( v/r^8 \) and
\( v^3/r^8 \) which are proportional to the angular momentum of the membrane state. The
effective potential for a graviton moving in a boosted Kerr-type metric is computed and
shown to agree precisely with the Matrix theory calculation at leading order in the long-
distance expansion for each power of the graviton velocity. This result generalizes to
arbitrary order; we show that terms in the membrane-graviton potential corresponding
to \( n \)th moments of the membrane stress-energy tensor are reproduced correctly to
to all orders in the long-distance expansion by terms of the form \( F^4 X^n \) in the one-loop
Matrix theory calculation.

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1 Introduction

There is by now abundant evidence that the Matrix theory proposal of Banks, Fischler, Shenker and Susskind [1] has exposed an extraordinarily close relationship between supersymmetric matrix quantum mechanics and 11-dimensional supergravity. For recent reviews of the subject, see [2, 3]. Numerous calculations have shown that in various particular cases the leading term in the one-loop Matrix theory potential reproduces correctly the leading term in the long-distance supergravity potential (the original examples of this calculation appeared in [4, 1]; a general proof of this result appears in [5]). Subleading terms in the Matrix theory potential are less well understood, although some progress has been made towards understanding both subleading terms in the one-loop potential and terms arising from higher loop effects [6, 7, 8, 9, 10, 11]. Processes involving longitudinal momentum transfer have been studied in Matrix theory and found to agree with supergravity [12]. Scattering of more than two gravitons has also been considered [13], giving an apparent discrepancy between Matrix theory and supergravity; a possible resolution of this discrepancy appears in [14]. Spin-dependent effects in graviton scattering were considered in [15, 16, 17].

To date, most studies of subleading effects in Matrix theory have focused on supergraviton interactions. The first nonvanishing subleading term in the effective potential between a pair of gravitons (without spin) is of order $v^6/r^{14}$ and arises as a two-loop effect [9]. In this paper, we consider subleading terms in the potential between a classical membrane configuration of finite size and a single graviton. The leading term in the long-distance membrane-graviton potential was calculated for infinite flat membranes in [18, 19], and for compact membranes in [20]. In the case of the compact membrane it was shown that the potential between the membrane and a static graviton contains a time-dependent part related to gravitational radiation from the classically oscillating membrane as well as a constant part which is proportional to the square of the energy of the state. The stationary component of the potential precisely reproduces the static supergravity potential around a massive object, so that this result was interpreted as a demonstration of the equivalence principle in Matrix theory. Because the membrane is an extended object, subleading terms arise in this potential at order $v^k/r^8$ for $k = 0, \ldots, 4$. In this paper we calculate these terms, as well as the velocity-dependent terms at order $1/r^7$, for an arbitrary membrane configuration and analyze them in detail. Just as for the static membrane-graviton potential, we find that each term can be expressed as a sum of a stationary component and a time-dependent component related to radiation effects. The stationary components at order $v/r^8$ and $v^3/r^8$ are proportional to the angular momentum of the membrane state. We compare the effective potential calculated in this way to supergravity using several methods. We compute the effective potential for a graviton moving in the metric produced by a heavily boosted rotating source in a space which has been compactified in a timelike direction. This potential agrees perfectly with the results of our Matrix theory calculation. We also compare our results directly with the potential arising from single-graviton exchange in 11D supergravity. We find agreement in this case also. From the graviton exchange calculation, it is clear that there are terms of order $v^k/r^{n+7}$
for every \( n \geq 0 \) in the membrane-graviton potential which arise from \( n \)th moments of the stress-energy tensor of the membrane. We show that these terms are reproduced precisely to all orders by certain terms in the one-loop Matrix theory calculation.

The paper is organized in the following fashion: In Section 2 we carry out the one-loop Matrix theory calculation of the membrane-graviton potential to order \( 1/r^8 \). We rewrite this potential in the membrane language of de Wit, Hoppe and Nicolai [21] and analyze the structure of the time-independent components of the potential. In Section 3 we calculate the appropriate metric for a boosted rotating object in lightlike compactified supergravity; we calculate the effective graviton potential in this background and demonstrate agreement with large \( N \) Matrix theory. In Section 4 we calculate the potential from single-graviton exchange processes in supergravity between a membrane with a given stress-energy tensor and a graviton with given momentum. We analyze the effects of higher moments in the exchange process and show that the resulting terms are reproduced to all orders in Matrix theory. Section 5 describes the simple example of a rotating spherical membrane. It is found that the usual relation between angular momentum and the \( 1/r^8 \) term in the potential is modified by finite \( N \) effects. Section 6 contains concluding remarks. Throughout the paper we use the notation and conventions of [20].

2 Matrix theory potential at order \( 1/r^8 \)

In this section we calculate the one-loop effective potential governing the long-distance interaction of a graviton with an arbitrary localized state in Matrix theory. We arrive at a general expression for the potential at leading and subleading order in the inverse separation. Specializing to states which describe classical membranes in the large \( N \) limit, we simplify the resulting expressions and show that at each order in \( v \) the potential is described by a time-independent term depending only on conserved charges of the membrane plus a time-dependent term which averages to zero. We arrive at a general formula for the time-averaged potential containing the leading term at each power of the graviton velocity up to fourth order. For \( v \) and \( v^3 \) the leading contributions come in at order \( 1/r^8 \) and are proportional to the transverse angular momentum of the Matrix theory object.

2.1 One-Loop Matrix theory potential

The one-loop Matrix theory calculation describing the effective potential between a pair of separated states was used in [4] to describe the scattering of a pair of branes in type IIA string theory. Since then this calculation has been performed in a wide variety of contexts, describing interactions between many Matrix theory objects. In this subsection we carry out this calculation to order \( 1/r^8 \) for the potential between a graviton and an arbitrary Matrix theory object. We follow the quasi-static approach to this calculation described in [20]. It was shown in [22] that there are discrepancies in some subleading terms between potentials calculated using this method and those calculated using the phase shift method.
of [4]. However, this method is known to be valid for the leading order terms, and we find in this paper that it is also accurate for a particular infinite series of subleading corrections.

To describe widely separated objects in Matrix theory, one chooses a background matrix configuration which is block diagonal, with the trace of each block (divided by the rank) giving the center of mass coordinates of the subsystem. We consider a system containing a compact object with center of mass fixed at the origin and a graviton having an arbitrary position and velocity. The appropriate background is given by

\[ X_i(t) = \begin{bmatrix} Y_i(t) & 0 \\ 0 & r_i(t) \end{bmatrix}. \]

Here, \( Y_i(t) \) for \( i = 1, \ldots, 9 \) are \( N \times N \) matrices solving the classical equations of motion of Matrix theory. We assume that \( \text{Tr} (Y_i(t)) = \text{Tr} (\dot{Y}_i(t)) = 0 \) and that the eigenvalues of \( Y_i(t) \) have a finite spread so that we are dealing with a compact object whose center of mass remains at the origin. The lower-right entry is a scalar corresponding to a graviton probe with \( p_- = 1/R \). We assume that the separation distance \( r \) is much greater than the spread of eigenvalues of the matrices \( Y_i(t) \), so that our probe is very distant compared with the extent of our compact object. We also assume that \( v \) is very small, so that the graviton has moved a short distance compared to \( r \) in the natural time scale associated with the classical dynamics of the object at the origin.

Using the quasi-static approach of [20], we expand about this background in the Matrix theory action and compute the one loop effective potential. At quadratic order in the fluctuations of the off-diagonal degrees of freedom, the action describes a system of harmonic oscillators with frequencies determined by the background fields and their time derivatives. There are \( 10N \) complex bosonic oscillators with (frequency)\(^2\) matrix

\[
(\Omega_b)^2 = M_{0b} + M_{1b}
\]

\[
M_{0b} = \sum_i K_i^2 \otimes 1_{10 \times 10}
\]

\[
M_{1b} = \begin{bmatrix} 0 & -2\partial_t K_j \\ 2\partial_t K_i & 2[K_i, K_j] \end{bmatrix}
\]

where

\[ K_i \equiv Y_i - r_i 1_{N \times N}. \]

There are also \( 16N\tilde{N} \) complex fermionic oscillators with (frequency)\(^2\) matrix

\[
(\Omega_f)^2 = M_{0f} + M_{1f}
\]

\[
M_{0f} = \sum_i K_i^2 \otimes 1_{16 \times 16}
\]

\[
M_{1f} = i\partial_t K_i \otimes \gamma^i + \frac{1}{2}[K_i, K_j] \otimes \gamma^{ij}
\]

\[ 3 \]
and (two identical sets of) $N\tilde{N}$ complex scalar ghost oscillators with (frequency)$^2$ matrix

$$(\Omega_g)^2 = \sum_i K_i^2.$$ 

The one loop Matrix theory potential is given by

$$V_{\text{matrix}} = \text{Tr} (\Omega_b) - \frac{1}{2} \text{Tr} (\Omega_f) - 2 \text{Tr} (\Omega_g).$$ (2)

For large $r$, $M_0$ and $M_1$ have eigenvalues of order $r^2$ and 1 respectively, so the $M_1$’s can be treated as perturbations when computing the traces. For each of the traces, the standard Dyson perturbation series gives

$$\text{Tr} \sqrt{M_0 + M_1} = \frac{1}{2} \text{Tr} \int_0^\infty \frac{d\tau}{\tau^{3/2}} e^{-\tau(M_0 + M_1)}$$ (3)

$$= \frac{1}{2} \sqrt{\pi} \sum_n \left\{ \int_0^\infty \frac{d\tau_i}{(\sum \tau_i)^{3/2}} e^{-(\tau_1 + \cdots + \tau_{n+1})r^2} e^{-\tau_1 \hat{M}_0 M_1 e^{-\tau_2 \hat{M}_0} \cdots M_1 e^{-\tau_{n+1} \hat{M}_0}} \right\}$$

$$= \frac{1}{2} \sqrt{\pi} \sum_n \left\{ \int_0^\infty \frac{d\sigma_i}{(\sum \sigma_i)^{3/2}} e^{-(\sigma_1 + \cdots + \sigma_{n+1})} e^{-\sigma_1 \hat{M}_0 M_1 e^{-\sigma_2 \hat{M}_0} \cdots M_1 e^{-\sigma_{n+1} \hat{M}_0}} \right\}.$$ 

Here, we have taken $\sigma_i = r^2 \tau_i$ and $M_0 = r^2 (1 \otimes 1) + \hat{M}_0 = r^2 (1 \otimes 1) - \tilde{M}_0 + \tilde{\tilde{M}}_0$ where

$$\tilde{M}_0 = \sum_i 2r_i Y_i \otimes 1, \quad \tilde{\tilde{M}}_0 = \sum_i Y_i^2 \otimes 1.$$ 

Expanding

$$e^{-\sigma_i \hat{M}_0 / r^2} = 1 - \sigma_i \hat{M}_0 / r^2 + \cdots,$$

we see that a term in the expansion with $n$ powers of $M_1$ and $m$ powers of $\tilde{M}_0$ contains terms with powers of $1/r$ ranging from $1/r^{2n+m-1}$ to $1/r^{2n+2m-1}$. All terms for $n = 0$, $n = 1$, $n = 2$ and $n = 3$ vanish in (2) by Lorentz invariance [20], so for a potential calculation to order $1/r^8$, we need only consider the terms with $n = 4$, $m = 0$ and $n = 4$, $m = 1$. Keeping only these terms we find, to order $1/r^8$

$$V_{\text{matrix}} = -\frac{5}{128} \frac{1}{r^7} W_0 - \frac{35}{256} \frac{1}{r^9} W_1$$ (4)

where

$$W_0 = \text{Tr} (M_{1b}^4) - \frac{1}{2} \text{Tr} (M_{1f}^4)$$

$$W_1 = \text{Tr} (\tilde{M}_{0b} M_{1b}^4) - \frac{1}{2} \text{Tr} (\tilde{M}_{0f} M_{1f}^4).$$

Computing the traces, we find

$$W_0 = \text{Tr} (F)$$

$$W_1 = \text{Tr} (2r_i Y_i F)$$ (5)
where

\[ F = 8F_{\mu\nu}F^{\nu}\lambda F^\sigma_\lambda F^\mu_\mu + 8(F_{\mu\nu}F^{\lambda}_\mu F^{\nu}_\sigma F^\sigma_\lambda + F^{\mu}_\nu F^{\lambda}_\sigma F^\sigma_\mu F^{\nu}_\lambda) 
- 2(F_{\mu\nu}F^{\lambda}_\rho F^\rho_\lambda + F_{\mu\nu}F^{\lambda}_\sigma F^\rho_\rho F^{\nu}_\lambda) - 2F_{\mu\nu}F^\rho_\sigma F^{\nu}_\mu F^{\rho}_\lambda. \]

The field strength components are given by \( F_{0i} = -F_{i0} = \partial_1 Y_i - v_i, \) \( F_{ij} = i[Y_i, Y_j] \) and indices are raised and lowered with a Minkowski metric \( \eta_{\mu\nu} = \text{diag}(-+\cdots+) \).

Note that the only place in the potential that \( v_i \) appears is through \( F_{0i} \). Since the expansion of \( F \) contains terms with zero, two and four \( F_{0i} \)'s, we will get terms with up to four powers of \( v \) for both \( 1/r^7 \) and \( 1/r^8 \).

### 2.2 Matrix-membrane correspondence

We will now restrict attention to systems where the object at the origin is a membrane. The correspondence between the Matrix theory description of a membrane and the world-volume description of a supergravity membrane in light-front coordinates was developed in \[23, 24, 25, 21\]. We now review some details of this correspondence, using the conventions of \[20\].

The essential feature of the matrix-membrane correspondence is that matrices are taken to correspond to functions on the membrane world-volume. The trace becomes an integral, and commutators become Poisson brackets through

\[ \text{Tr} \rightarrow \frac{N}{4\pi} \int d^2\sigma \quad [\cdot, \cdot] \rightarrow \frac{2i}{N} \{\cdot, \cdot\}. \]

We can translate the field strength components appearing in the Matrix theory action and one-loop potential into membrane language via the correspondence

\[ F_{0i} \rightarrow \dot{Y}_i \quad F_{ij} \rightarrow -\frac{2}{N} \{Y_i, Y_j\}. \]

One notable aspect of this correspondence is that while the algebraic manipulations available for matrices with commutators are mirrored by those of functions with Poisson brackets, translating a given matrix expression into the membrane formalism in this way can drop subleading terms in the \( 1/N \) expansion, such as those arising from commutators of field strength components. For this reason, we expect results derived using this formalism to only be valid at leading order in \( 1/N \).

We now list a number of properties of the membrane variables which will be used to simplify our expressions. The transverse coordinates satisfy the equations of motion (equivalent to the matrix equations of motion)

\[ \ddot{Y}_i = \frac{4}{N^2} \partial_a \left( \gamma^{ab} \partial_b Y_i \right) \]

which follow from the Hamiltonian (equivalent to the matrix Hamiltonian)

\[ H = \frac{N}{4\pi R} \int d^2\sigma \left( \frac{1}{2} \dot{\gamma}_i \dot{Y}_i + \frac{2\gamma}{N^2} \right). \]
Here, $\gamma_{ab} \equiv \partial_a Y_i \partial_b Y_i$ and $\gamma \equiv \det \gamma_{ab}$. This light-front gauge membrane Hamiltonian is related to the auxiliary $Y^{-}$ membrane coordinate by

$$H = \frac{N}{4\pi R} \int d^2\sigma \left( \dot{Y}^- \right).$$

Since $H$ has a constant value $E$ for any solution of the equations of motion, we may write

$$Y^{-}(t, \sigma^a) = \frac{R}{N} E t + \xi(t, \sigma^a)$$

where $E$ is the light front energy of the membrane and $\xi$ is a fluctuation satisfying $\int d^2\sigma \xi = 0$. Constraints imposed by the light front gauge choice imply that $\dot{Y}^-$ satisfies

$$\dot{Y}^- = \frac{1}{2} \ddot{Y}_i \dot{Y}_i + \frac{2\gamma}{N^2}$$

$$\partial_a Y^- = \dot{Y}_i \partial_a Y_i$$

$$\ddot{Y}^- = \frac{4}{N^2} \partial_a (\gamma \gamma_{ab} \partial_b Y^-)$$

Finally, we list a few identities used below which can be easily checked

$$\{ Y_i, Y_j \} \{ Y_j, Y_k \} = -\gamma \gamma_{ab} \partial_a Y_i \partial_b Y_k$$

$$\{ Y_i, Y_j \} \{ Y_i, Y_j \} = 2\gamma$$

$$\dot{Y}_j \dot{Y}_k - \frac{4}{N^2} \gamma \gamma_{ab} \partial_a Y_j \partial_b Y_k = \frac{\partial}{\partial t} (Y_j \dot{Y}_k) - \frac{4}{N^2} \partial_a (\gamma \gamma_{ab} Y_j \partial_b Y_k)$$

$$\int d^2\sigma \left( Y_i \dot{Y}_j \dot{Y}_k - 4 \frac{\gamma \gamma_{ab} \partial_a Y_j \partial_b Y_k}{N^2} \right) = \int d^2\sigma \left( \frac{1}{2} \frac{\partial}{\partial t} (Y_i \dot{Y}_j + Y_j \dot{Y}_k - \dot{Y}_i Y_j Y_k) \right)$$

The third and fourth relations also hold if we substitute $Y^{-}$ for any of the $Y$’s.

### 2.3 Analysis of $1/r^7$ terms

We begin by analyzing the terms of order $1/r^7$ in the potential (5). Our goal is to express each term in terms of a time-independent part plus the world-volume integral of a time-derivative of a bounded fluctuation. The term of order $v^0/r^7$ was shown in [20] to be given by

$$W_0[v^0] = \frac{N}{4\pi} \int d^2\sigma \left[ 24 \dot{Y}_i \dot{Y}_j \dot{Y}_k + \frac{192}{N^2} \gamma \dot{Y}_i \dot{Y}_j + \frac{384}{N^2} \gamma^2 - \frac{384}{N^2} \gamma \gamma_{ab} \dot{Y}_i (\partial_a Y_i) \dot{Y}_j (\partial_b Y_j) \right]$$

$$= \frac{N}{4\pi} \int d^2\sigma \left[ 96 \left( \dot{Y}^- \right)^2 - \frac{384}{N^2} \gamma \gamma_{ab} \partial_a Y^- \partial_b Y^- \right]$$

$$= 96 \frac{R^2}{N} E^2 + \frac{N}{4\pi} \int d^2\sigma \left[ 96 \frac{\partial}{\partial t} (\xi \dot{\xi}) \right].$$
This consists of a constant contribution proportional to the square of the matrix energy, plus a total derivative which vanishes in a time average, since the fluctuation $\xi$ is a function of bounded variation. The linear term in $v$ is

$$W_0[v^1] = \frac{N}{4\pi} \int d^2\sigma \left[ -96 v_i \dot{Y}_i \dot{Y}_j - \frac{384}{N^2} \gamma v_i \dot{Y}_i + \frac{768}{N^2} v_i \gamma^{ab} (\partial_a Y_i)(\partial_b Y_j) \right]$$

$$= \frac{N}{4\pi} \int d^2\sigma \left[ -192 v_i \left( \dot{Y}_i \dot{Y}_j - \frac{4}{N^2} \gamma^{ab} \partial_a Y_i \partial_b Y_j \right) \right]$$

This term vanishes in a time average, so the stationary part of the potential has no linear term in the velocity at order $1/r^7$. The $v^2$ term at order $r^7$ is

$$W_0[v^2] = \frac{N}{4\pi} \int d^2\sigma \left[ 48 v^2 \dot{Y}_i \dot{Y}_j + \frac{192}{N^2} v^2 \gamma + 96 v_i v_j \dot{Y}_i \dot{Y}_j - \frac{384}{N^2} v_i v_j \gamma^{ab} \partial_a Y_i \partial_b Y_j \right]$$

$$= \frac{N}{4\pi} \int d^2\sigma \left[ 96 v^2 \dot{Y}_i \dot{Y}_j + 96 v_i v_j (\dot{Y}_i \dot{Y}_j - \frac{4}{N^2} \gamma^{ab} \partial_a Y_i \partial_b Y_j) \right]$$

$$= 96 v^2 RE + \frac{N}{4\pi} \int d^2\sigma \left[ 96 v_i v_j \frac{\partial}{\partial t}(Y_i \dot{Y}_j) \right].$$

Here we get a nonvanishing constant term proportional to the matrix energy of our object plus a term which vanishes in the time-averaged potential.

Though we will be mainly interested in the constant piece, it is interesting to note that the time-varying portion of the potential for the $v^0$, $v^1$ and $v^2$ terms may be rewritten as the second time derivative of a quadratic term in the membrane coordinates (e.g., $\partial_t^2 \int Y_i Y_j$ for the $v^2$ term). This is exactly the form of the contribution to the potential that we expect from the graviton’s interaction with the object’s quadrupole radiation. We will not discuss this further here.

At third order in $v$, we have

$$W_0[v^3] = -\frac{N}{4\pi} \int d^2\sigma \left[ 96 v^2 \dot{Y}_i \right] = 0,$$

since $\int \dot{Y}_i$ is proportional to the transverse momentum which we have assumed is zero. Finally, at order $v^4$, we have a non-vanishing constant contribution

$$W_0[v^4] = 24 v^4 N.$$

Note that this term is independent of all properties of the object except its longitudinal momentum $p_-.$

Although we have carried out these calculations in the membrane language, the $v^2$, $v^3$ and $v^4$ terms are simple enough that analogous relations can be shown without using the membrane correspondence. Mimicking the manipulations above with matrices, we can derive analogous expressions directly in matrix language. However, the same is not true of the $v^0$
and \( v^1 \) terms. As noted in [20] for the \( v^0 \) term, the analogous relations in Matrix theory break down at subleading order in \( 1/N \), so that the stationary terms in the potential are not simply related to the conserved quantities of the Matrix theory Hamiltonian.

To summarize the results of this subsection, we find that at order \( 1/r^7 \) the stationary part of the potential contains non-vanishing terms at zeroeth, second and fourth orders in \( v \).

### 2.4 Analysis of \( 1/r^8 \) terms

To find the leading terms linear and cubic in the velocity, we look to the order \( 1/r^8 \). Starting with the expression for an arbitrary matrix configuration at any \( N \), it is simple to mimic with a matrix calculation so that the analogous result holds for \( 1/N \) subject to \( 1/r \) corrections. The result of the preceding calculation is that to leading order we now consider only the time-independent part of the potential, performing a time average

\[
W_1[v^1] = \frac{N}{4\pi} \int d^2\sigma \left[ -192 r_l v_i Y_l \dot{Y}_i \dot{\dot{Y}}_i - \frac{768}{N^2} r_l v_i Y_l \dot{Y}_i \gamma + \frac{1536}{N^2} r_l v_i Y_l \gamma_{\alpha\beta} (\partial_a Y_i) \dot{\dot{Y}}_j (\partial_b Y_j) \right]
\]

\[
= \frac{N}{4\pi} \int d^2\sigma \left[ -384 r_l v_i \left( Y_l \dot{Y}_i \dot{Y}^j - \frac{4}{N^2} Y_l \gamma_{\alpha\beta} \partial_a Y_j \partial_b Y^j \right) \right] \tag{7}
\]

We find a non-vanishing contribution to the time-averaged potential proportional to the energy \( E \) and the transverse angular momentum tensor of the object,

\[
a_{ij} = \frac{N}{4\pi R} \int d^2\sigma \left( Y_i \dot{Y}_j - Y_j \dot{Y}_i \right) \sim \frac{1}{R} \text{Tr} \left[ Y_i \dot{Y}_j - Y_j \dot{Y}_i \right].
\]

Note that this angular momentum is a conserved quantity both in the membrane theory and in Matrix theory.

The \( v^3 \) term is simply given by

\[
W_1[v^3] = \frac{N}{4\pi} \int d^2\sigma \left[ -192 r_l v^2 v_i Y_l \dot{Y}_i \right]
\]

\[
= -96 R v^2 v_i r_l a_{it} + \frac{N}{4\pi} \int d^2\sigma \left[ -96 v^2 v_i r_l \frac{\partial}{\partial t} (Y_l \dot{Y}_i) \right].
\]

Again, we have a term proportional to the membrane angular momentum plus a time derivative which vanishes in the average. Like the terms of order \( v^2/r^7 \), \( v^3/r^7 \) and \( v^4/r^7 \), this term is simple enough to mimic with a matrix calculation so that the analogous result holds for an arbitrary matrix configuration at any \( N \). The same is not true of the term (7), which is subject to \( 1/N \) corrections.

### 2.5 Time-averaged potential

We now consider only the time-independent part of the potential, performing a time average to eliminate the other terms. The result of the preceding calculations is that to leading order in \( 1/r \) for each power of velocity separately, the membrane-graviton potential is given by

\[
\langle V_{\text{matrix}} \rangle = -\frac{15 R^2 E^2}{4N} \frac{1}{r^7} - \frac{15 R E v^2}{4} \frac{1}{r^7} - \frac{15 N v^4}{16} \frac{1}{r^7} + \frac{105 R^2 E a_{it} r_l v_i}{4N} \frac{1}{r^9} + \frac{105 R a_{it} r_l v_i v^2}{8} \frac{1}{r^9} \tag{8}
\]
Note that the only properties of the object appearing in this expression are the conserved quantities of the classical membrane state: the energy, light-front momentum \( p_- = N/R \) and the transverse angular momentum. As discussed above, the terms of order zero and one in the velocity are subject to correction at finite \( N \); the remaining terms are correct for any compact object to all orders in \( N \).

It is interesting to compare this result to a similar expression which was derived in [17] for the interaction potential between gravitons with spin. In that case, a similar term appeared at order \( v^3/r^8 \), but there was no term of order \( v/r^8 \). This is consistent, since the term of this order in (8) is proportional to the light-front energy \( E \), which vanishes for a graviton with no transverse velocity.

3 Classical supergravity potential

We would like to compare the potential calculation just performed with the corresponding result from a classical supergravity calculation. If the Matrix theory conjecture is correct, we would expect to find the same result, since we are dealing with a process which should be adequately described by the classical theory. In this section we compare the potential computed above to the effective action for a graviton moving in the appropriate metric on a compactified space. We proceed in two steps. First, we calculate the leading terms in the stationary long-distance metric of an arbitrary compact object with fixed momentum in a direction of space-time which has been compactified on a lightlike circle. This is the appropriate frame for comparison with Matrix theory. Then we determine the Lagrangian for a graviton moving in this background; we read off the potential from this Lagrangian, and compare with the matrix expression derived above. A similar procedure was used in [17] to study spin dependence of graviton scattering in Matrix theory.

3.1 Long-distance metric in lightlike compactified spacetime

We will now compute the leading terms in the desired stationary long-distance metric using supergravity. We assume that the object at the origin has no net charges other than those associated with Poincare symmetries, so that we can calculate the metric at large distances using Einstein’s equations in 11 dimensions.

In uncompactified 11 dimensional spacetime in coordinates where the body’s center of mass is fixed at the origin and the metric at infinity is Minkowski the long-distance stationary metric is given by [26, 27]

\[
G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

where

\[
h_{00} = \frac{CM}{r^8} + O(1/r^9)
\]

\[
h_{0I} = \frac{9Ca_{I}x^{J}}{2r^{10}} + O(1/r^{10})
\]

9
Here, $M$ is the total energy of the object in its rest frame, $a_{IJ}$ is the angular momentum tensor, and $C$ is a constant given by $C = 64G/3\pi^4$.

To make a comparison with Matrix theory, we should consider the metric of such an object in a lightlike compactified theory in a frame where the object has $p_- = N/R$. To find this, we follow the kinematics of Seiberg [28]. Starting with the metric (9), we boost along $x^{10}$ to a frame where $p_{10} = N/R_s$ before compactifying $x^{10}$ on a circle of radius $R_s$. The appropriate boost parameter satisfies

$$\gamma M v = \frac{N}{R_s}$$

To find the metric in the compactified space, we note that all of the terms in (9) come from the linearized Einstein equations, so that to this order the compactified metric may be obtained by a “method of images”, taking (tildes refer to boosted quantities)

$$h_{\mu\nu}^{\text{comp}} = \sum_n \tilde{h}_{\mu\nu}(\tilde{x}^0, \tilde{x}^i, \tilde{x}^{10} + 2\pi n R_s).$$

To compute the leading order terms in $1/r$, we need only keep the zeroeth Fourier mode of $h$ on the compact circle, so we may average over $\tilde{x}_{10}$ to get

$$h_{\mu\nu}^{\text{comp}} = \frac{1}{2\pi R_s} \int_{-\infty}^{+\infty} d\tilde{x}_{10} \tilde{h}_{\mu\nu}(\tilde{x}^0, \tilde{x}^i, \tilde{x}^{10}).$$

Finally, we perform another boost in the (opposite) $x^{10}$ direction with boost parameter

$$\tilde{\gamma} = \sqrt{\frac{R^2}{2R_s^2} + 1}$$

which, in the $R_s \to 0$ limit gives the desired form of the metric for compactification on a lightlike circle of radius $R$ with $p_- = N/R$. The resulting metric (keeping only leading terms) has

$$h_{i j} = \frac{5CM^2}{256 N r^8} \delta_{i j}$$

$$h_{+ i} = \frac{315 M^2 R C a_{i j} x^j}{1024 N^2 r^9}$$

$$h_{- i} = \frac{315 C a_{i j} x^j}{512 R r^9}$$

$$h_{++} = \frac{45CM^4 R^2}{1024 N^3 r^7}$$

$$h_{+-} = \frac{35CM^2}{512 N r^7}$$

$$h_{--} = \frac{45CN}{256 R^2 r^7}$$

Note that to this order, the components $a_{10 i}$ do not appear in the expressions for $h$. Both $h_{++}$ and $h_{--}$ have contributions at order $1/r^8$ proportional to $a_{10 i} x^i / r^9$, but these are at subleading order.
3.2 Calculation of graviton potential

We will now compute the effective potential for a graviton moving in a metric of the type just described. We follow the approach used by Becker, Becker, Polchinski and Tseytlin in [9]. The action of a scalar particle in 11-dimensional gravity is

$$S = -m \int d\tau (-G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}$$

where $G_{\mu\nu}$ is the background. For the process we will consider, $p_-$ is to be fixed, so the appropriate Lagrangian in the transverse coordinates is

$$\mathcal{L}'(p_-) = \mathcal{L} - p_- \dot{x}^-(p_-). \quad (11)$$

Now, from the action, we calculate:

$$p_- = m \frac{(G_{++} + h_- \dot{x}^- + h_{-i} \dot{x}^i)}{(-G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}}$$

where

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (12)$$

Solving for $\dot{x}^-(p_-)$ and plugging into (11) we find in the $m \to 0$ limit at fixed $p_-$

$$\mathcal{L}' = p_- \left[ \frac{v^2}{2} + \frac{1}{2} h_{++} + h_{+i} v^i + \frac{1}{2} h_{ij} v^i v^j + \frac{1}{2} h_{+v} v^++ \frac{1}{2} h_{-i} v_i v^+ + \frac{1}{8} h_{-v} v^4 + O(1/r^{14}) \right]. \quad (13)$$

Each component of the metric appears in this expression coupled to a different term in the velocity expansion. Thus, the graviton is a probe of all components of the metric. We now use the values of $h$ calculated in (10) and read off the effective potential. In order to compare with the Matrix theory calculation in Section 2, we recall that the matrix energy of a state is related to its rest frame energy by

$$E_{\text{matrix}} = \frac{R}{2N} M^2$$

in the convention with $H_{\text{matrix}} = R^{-1} \text{Tr} \left( \frac{1}{2} \dot{Y}^i \dot{Y}^i - \frac{1}{4} [Y^i, Y^j]^2 \right)$. Also, in these conventions we have $G = 2\pi^4 R^3$, so $C = 128 R^3/3$. Using these expressions and $p_- = 1/R$, we find

$$V_{\text{gravity}} = -\frac{15 R^2 E^2}{4N} \frac{1}{r^7} - \frac{15 R E v^2}{4} \frac{1}{r^7} - \frac{15 N v^4}{16 \frac{1}{r^7}} + \frac{105 R^2 E a_i r_i v_i}{4N \frac{1}{r^7}} + \frac{105 R a_i r_i v_i v^2}{8 \frac{1}{r^7}}$$
This is exactly the stationary part of the potential calculated in Matrix theory (8). Thus, we have shown that for an arbitrary compact membrane state in Matrix theory (and in complete generality for the $v^2$, $v^3$ and $v^4$ terms) the time-averaged one-loop Matrix theory potential for a distant graviton reproduces the supergravity result for the leading term in $1/r$ at each power of velocity.

4 Graviton exchange and higher order terms

4.1 Graviton exchange and angular momentum

Another way to understand the correspondence between the Matrix theory potential calculation and supergravity is by considering single-graviton exchange processes in supergravity. The effective action from one-graviton exchange between an extended object with stress-energy tensor $T^{\mu\nu}$ and a pointlike object with momentum $p^\mu$ located at $y$ in a flat background metric in light-front coordinates is

$$S_{\text{eff}} = -\frac{1}{8} \int d^{11}x T_{\mu\nu}(x) D^{\mu\nu\lambda\sigma}(x-y) \frac{p_\mu p_\nu}{p^+} \tag{14}$$

where the (harmonic gauge) graviton propagator in 11 spacetime dimensions is

$$D^{\mu\nu\lambda\sigma}(x-y) = 16\pi G \left( \eta^\mu\lambda \eta^\nu\sigma + \eta^\mu\sigma \eta^\nu\lambda - \frac{2}{9} \eta^\mu\nu \eta^{\lambda\sigma} \right) \int \frac{d^{11}k}{(2\pi)^{11}} \frac{e^{ik\cdot(x-y)}}{-k^2}.$$  

The stress tensor for a small object with center of mass at light-front coordinates $(z^-, z^i)$ can be expressed in a moment expansion

$$T^{\mu\nu}(x^+, x^-, x^i) = T^{\mu\nu} \delta(x^--z^-)\delta(x^i-z^i) + T^{\mu\nu(\lambda)} \partial_{x^\lambda} \left( \delta(x^--z^-)\delta(x^i-z^i) \right) + \cdots$$

where the zeroth and first moments of the stress-energy tensor are given by

$$T^{\mu\nu} = \int dz^- d^9 z^i T^{\mu\nu}(z^+, z^-, z^i),$$

$$T^{\mu\nu(\lambda)} = \int dz^- d^9 z^i [z^\lambda T^{\mu\nu}(z^+, z^-, z^i)].$$

The supergravity interaction potential can be computed exactly to order $1/r^8$ from these expressions. For example, the term linear in the graviton velocity appears in the potential with terms of the form

$$V_{\text{gravity}}[v^i] = \frac{15R_v}{2} \left[ T^{-i} + \frac{7T^{-i(j)}r_j}{r^2} + \cdots \right]. \tag{15}$$

To compare this with the Matrix theory calculation in membrane language we note that for the membrane

$$T_{m^i} = \frac{N}{4\pi R} \int d^2\sigma \left( \dot{Y}_i \dot{Y}^- - \frac{4}{N^2} \gamma^{ab} \partial_a Y^- \partial_b Y^i \right). \tag{16}$$
This expression appeared in the formula (3) for $W_0[v^1]$ and appeared with an extra factor of $Y_l$ in (4) for $W_1[v^1]$. Thus, these terms are precisely proportional to the zeroth and first moments of the membrane stress-energy tensor component $T_{-i}^j$. This shows that even before time-averaging, Matrix theory correctly reproduces the expected supergravity potential for these terms. A similar argument can be used to show that the terms proportional to other powers of the velocities are also proportional to moments of the membrane stress-energy tensor.

The time-dependence of the membrane-graviton effective potential arises from the fact that components such as $T_{-i}^j$ of the membrane stress-energy tensor are not conserved in light-front time. From the point of view of supergravity, this time-dependent potential can be understood in terms of outgoing gravitational radiation which gives rise to an instantaneous time-dependent potential in light-front coordinates [29, 5]. Thus, in order to compare to a stationary metric of the type considered in Section 3 we must time-average the components of the stress-energy tensor. As we saw, this gives a precise agreement between the Matrix theory calculation and the effective potential in a static supergravity background metric. In fact, the structure of the terms in the potential (13) arises precisely from the structure of the one-graviton exchange term (14). The different components of the metric $h_{\mu\nu}$ are directly related to the components $T_{\mu\nu}$ of the stress-energy tensor of the extended object.

### 4.2 Higher order terms

We have seen that the $1/r^8$ terms in the Matrix theory potential correspond to angular momentum and other first moments of the membrane stress-energy tensor. It is natural to ask whether further subleading terms can be related to higher moments of the stress-energy tensor. Indeed, this is the case. We will now prove that all the higher-order terms proportional to the graviton velocity which arise from single graviton exchange processes can be reproduced by considering terms in (3) with $n = 4$ and arbitrary $m$.

Generalizing (13) to higher moments, we find that

$$V_{\text{gravity}} = \sum_{p=0}^{\infty} \frac{15Rv^i}{2} \left[ (-1)^p \frac{1}{p!} T^{-i(j_1j_2\cdots j_p)} \partial_{j_1} \partial_{j_2} \cdots \partial_{j_p} \left( \frac{1}{r^7} \right) \right]$$

$$= \sum_{p=0}^{\infty} \frac{Rv^i}{2} \frac{2^{7+p}}{r^{7+p}} \left[ \sum_{k=0}^{p/2} (-1)^k T^{-i(j_1j_2\cdots j_p)} \left( \eta_{j_1j_2\cdots j_p} \eta_{j_{p-2}j_{p-3}\cdots j_{p-2k+2}} \right) \right]$$

$$\times \frac{\left(5 + 2p - 2k\right)!!}{2^k k!} \frac{r_{j_1}r_{j_2} \cdots r_{j_{p-2k}}}{(p-2k)! r^{p-2k}}$$

(17)

where $n!! = n(n-2)(n-4)\cdots 1$.

We can compare this with the higher order terms in the Matrix theory potential arising from $n = 4$ contributions to (3). In the matrix membrane language, each power of $M_0/r^2$ which enters contributes a factor of $-r \cdot Y/r^2$ while each factor of $M_0/r^2$ contributes $Y^2/r^2$. Because on the membrane world-volume the functions $Y$ are commuting, it does not matter
at which position in the trace these terms contribute. It is relatively straightforward to
analyze the combinatorial structure of the various possible terms. We can compute

\[- \frac{1}{2\sqrt{\pi r^7}} \left\{ \int_0^\infty \frac{d^5 \sigma_i}{\sigma^{3/2}} e^{-\sigma} \frac{e^{i\vec{p} \cdot (\vec{M}_0 + \tilde{M}_0)}}{\sigma^{3/2}} \right\} \]

\[= - \sum_{p=0}^{\infty} \sum_{k \leq p/2} \left( \frac{(-1)^{p+k}}{2 \sqrt{\pi r^{7+2p-2k}}} \right) \left\{ \int_0^\infty \frac{d^5 \sigma_i}{\sigma^{3/2}} e^{-\sigma \sigma^{p-k}} \frac{1}{(p-k)!} \left( \frac{p-k}{k} \right)^{M_0 - 2k} \tilde{M}_0^k \right\} \]

\[= \sum_{p=0}^{\infty} \sum_{k \leq p/2} \frac{(-1)^{k+1}}{3 \cdot 2^{r+k} k! (p-2k)!} (r \cdot Y)^{p-2k} Y^{2k} \]

where we have abbreviated \(\sigma = \sigma_1 + \cdots + \sigma_5\). From this and the fact that \(W_0[v^1] = 192 R T_m v^i\) it follows that

\[V_{\text{matrix}}[v^1] = \sum_{p=0}^{\infty} \frac{R v^i}{2 p^{r+p}} \sum_{k \leq p/2} \left[ \frac{(-1)^k T_m^{(i,j_1 j_2 \cdots j_p)} (\eta_{j_1 j_p-1} \eta_{j_p-2j_2-3} \cdots \eta_{j_p-2k+2j_p-2k+1})}{2^k k! (p-2k)! r^{p-2k}} \right] \]

where we define the higher moments \(T_m^{(i,j_1 \cdots j_p)}\) of the membrane stress tensor through (16) with the product \(Y^{j_1} \cdots Y^{j_p}\) inserted into the integral. This expression agrees precisely with (17). Thus, we see that there is an exact agreement at all orders between the set of terms in the one-loop Matrix theory potential which are linear in the velocity and cubic in the membrane field strength and the terms in the supergravity potential which are linear in the velocity and arise from single-graviton exchange processes. It is straightforward to generalize this argument to the terms proportional to all powers in the velocity from zero through four.

5 Example: Rotating spherical membrane

As an explicit example of how transverse angular momentum appears in the Matrix theory
potential at order \(v/r^8\), we consider a symmetric spherical membrane of radius \(\tilde{R}\) initially
at \(x_1^2 + x_2^2 + x_3^2 = \tilde{R}^2\) and rotating uniformly in the 1 – 4 plane, the 2 – 5 plane and the
3 – 6 plane. Such a spherical membrane can be described by matrices which are linear in
the \(N \times N\) SU(2) generators in \(U(N)\) through

\[Y_i = \frac{2\tilde{R}}{N} J_i \cos(\omega t)\]

\[Y_i+3 = -\frac{2\tilde{R}}{N} J_i \sin(\omega t)\]

\[Y_7, 8, 9 = 0\]

where the \(N \times N\) matrices \(J_i\) satisfy

\([J_i, J_j] = i \epsilon_{ijk} J_k\).
It is easily verified that this configuration solves the equations of motion

\[ \ddot{Y}_i + [[Y_i, Y_j], Y_j] = 0 \]

for \( \omega = 2\sqrt{2}\tilde{R}/N \). At this frequency, the centrifugal forces are sufficient to keep the membrane from collapsing, and the rotating sphere has a constant radius. The angular momentum of this state is

\[ a_{ij} = \frac{1}{R} \text{Tr} [Y_i \dot{Y}_j - Y_j \dot{Y}_i] \]
\[ = \frac{2\sqrt{2}\tilde{R}^3}{3R} (1 - \frac{1}{N^2}) c_{ij} \]

where \( c_{41} = -c_{14} = c_{52} = -c_{25} = c_{63} = -c_{36} = 1 \) and the matrix energy is

\[ E = \frac{1}{R} \text{Tr} \left( \frac{1}{2} \ddot{Y}_i \dot{Y}_i - \frac{1}{4} [Y_i, Y_j] [Y_i, Y_j] \right) \]
\[ = \frac{6\tilde{R}^4}{RN} (1 - \frac{1}{N^2}) . \]

Let us now consider the interaction between this rotating sphere and a graviton with position \( r_i \) and velocity \( v_i = \dot{r}_i \). The term in the Matrix theory potential proportional to \( v/r^8 \) is given by equation (5) as

\[ V_{v/r^8} = \frac{35}{256r^9} \text{Tr} [2r_i Y_i F_{v,i}] \]
\[ = \frac{3360\sqrt{2}\tilde{R}^7}{r^9 N^7} (v_i \sin(\omega t) + v_{i+3} \cos(\omega t)) (r_i \cos(\omega t) - r_{i+3} \sin(\omega t)) \]
\[ \times \text{Tr} [J_i (J_i J_j J_j + J_j J_i J_i + J_j J_j J_i)] . \]

The \( \cos(\omega t) \sin(\omega t) \) terms time-average to zero while the \( \cos^2 \) and \( \sin^2 \) terms average to \( 1/2 \), so, computing the trace, the time averaged potential becomes

\[ \langle V_{v/r^8} \rangle = \frac{105\sqrt{2}\tilde{R}^7}{r^9 N^2} (r_i v_{i+3} - r_{i+3} v_i) \left( 1 - \frac{10}{3N^2} + \frac{7}{3N^4} \right) . \]

Using the explicit expressions for \( E \) and \( a_{ij} \) for this state, we have

\[ -\frac{105R^2 E a_{ii} r_i v_i}{4N} \frac{1}{r^9} = \frac{105\sqrt{2}\tilde{R}^7}{r^9 N^2} (r_i v_{i+3} - r_{i+3} v_i) \left( 1 - \frac{2}{N^2} + \frac{1}{N^4} \right) . \]

Thus, we see explicitly that in the large \( N \) limit,

\[ \langle V_{v/r^8} \rangle = -\frac{105R^2 E a_{ii} r_i v_i}{4N} \frac{1}{r^9} \]

for this state, and so the matrix energy and angular momentum appear as expected. However, this formula does not hold at finite \( N \), since there is agreement only at leading order in \( N \).
6 Conclusions

We have analyzed subleading terms in the one-loop Matrix theory potential between a classical membrane configuration and a moving graviton. We found that the terms of order $v/r^8$ and $v^3/r^8$ contained stationary components proportional to the angular momentum of the membrane state. These terms were shown to correspond precisely to the predictions of lightlike compactified supergravity. We also showed that the one-loop Matrix theory potential contains an infinite series of terms of order $1/r^{7+n}$ related to higher moments of the membrane stress-energy tensor, and that these terms agree precisely with terms in the supergravity effective potential arising from single graviton exchange.

These results provide further evidence that there is a remarkably deep structure hidden in Matrix theory which is capable of reproducing extremely nontrivial properties of 11-dimensional supergravity. The methods used here provide a systematic framework with which to further explore this structure.

We have not discussed finite $N$ effects much in this paper; indeed, most of our Matrix theory calculations were performed in the membrane language where $1/N$ corrections are dropped. There are currently some apparent contradictions between Matrix theory and supergravity [30, 8, 13, 31, 20, 32], most of which seem to relate to the validity of the finite $N$ DLCQ conjecture [33]. Although there are some fairly convincing arguments that finite $N$ Matrix theory reproduces DLCQ M-theory [34, 28, 11, 14], there are subtleties in these arguments indicating that DLCQ M-theory may not be equivalent to DLCQ supergravity [2, 29]. It was shown in [20] that the stationary part of the leading term in the potential between a membrane and a graviton with no transverse velocity is proportional the square of the Matrix theory energy of the membrane. However, it was pointed out that this relationship only holds at leading order in $1/N$. There are subleading corrections which seem to indicate a breakdown of the equivalence principle at finite $N$. The manipulations we used to show that the $v/r^8$ term is proportional to the matrix angular momentum used the membrane language in which subleading terms in $1/N$ are dropped. Analogous to the subleading corrections in the $v^0/r^7$ term, we found subleading corrections in the $v/r^8$ term which break the relationship between the coefficient of this term and the membrane angular momentum. This gives further evidence that at finite $N$ the usual relationships between the long-distance gravitational field around a finite-size object and the conserved quantities of the object break down. It is difficult to see how to reconcile this observation with the finite $N$ DLCQ conjecture.

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