Doubles of (quasi) Hopf algebras
and some examples of quantum groupoids and vertex groups related to them

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1 Introduction

Let $A$ be a finite dimensional Hopf algebra, $D(A) = A^{\text{cop}} \otimes A$ its Drinfel’d double and $\mathcal{H}(A) = A \# A^*$ its Heisenberg double. The relation between $D(A)$ and $\mathcal{H}(A)$ has been found by J.-H. Lu in [24] (see also [33], p. 196): the multiplication of $\mathcal{H}(A)$ may be obtained by twisting the multiplication of $D(A)^*$ by a certain left 2-cocycle which in turn is obtained from the $R$-matrix of $D(A)$. It was also obtained in [24] that $\mathcal{H}(A)$ becomes a left $D(A)$-module algebra under a certain action of $D(A)$ on $\mathcal{H}(A)$ (formula (35) in [24]).

All these may be obtained alternatively using a more direct approach, which also shows that the above mentioned action of $D(A)$ on $\mathcal{H}(A)$ is manifestly the left regular action of $D(A)$ on $D(A)^*$ (by identifying $\mathcal{H}(A)$ and $D(A)^*$ as linear spaces).

The general setting is the following: if $(H, R)$ is a quasitriangular bialgebra and we define a new multiplication on $H^*$ by $f \cdot g = \sum (R^2 \mapsto g)(R^1 \mapsto f)$, then $H^*$ with this new multiplication (denoted in what follows by $H^*_R$) becomes a left $H$-module algebra under the left regular action $\mapsto$ (this is well-known, see also [3], [29], [18], [10] for some more general versions in terms of Drinfel’d twists). Although very simple, this construction may have some nice applications, for instance $H^*_R$ may be noncommutative even if $H$ was cocommutative—it was discovered recently in [49], [50] that an important algebra arising in noncommutative string theory is an example of this type; this discovery has also been applied to noncommutative quantum field theory in [80]. And, if $A$ is a finite dimensional Hopf algebra and $H = D(A)$, then $H^*_R$ is just $\mathcal{H}(A)$.

For reasons to be discussed below, we were not satisfied with the description of $H^*_R$ as a left $H$-module algebra and we were led to consider also the right regular action of $H$ on $H^*_R$. It turns out that $H^*_R$ is a right $H^{\text{cop}}$-module algebra (that is, the right action satisfies a “reversed Leibniz rule”), so $H^*_R$ is an algebra in the tensor category of $H - H^{\text{cop}}$-bimodules (we say that it is an $H - H^{\text{cop}}$-bimodule algebra). If we endow this category with the braiding given by multiplying to the left by $R_{21}$ (as
usual) and from the right by $R^{-1}$, we shall prove that $H_R^*$ is quantum commutative as an algebra in this braided tensor category (which is also equivalent to saying that $H_R^*$ is a quantum commutative left $H \otimes H^{op \cop}$-module algebra). In particular $\mathcal{H}(A)$ is a quantum commutative $D(A) - D(A)^{cop}$-bimodule algebra and we like to think of this as the most natural “tensor categorical” interpretation of the Heisenberg double.

In section 4 we discuss some more facts about $H_R^*$, Drinfel’d doubles and Heisenberg doubles. For instance, we discuss the relation between the multiplication of $H_R^*$ and Majid’s “covariantised product” and, for a finite dimensional Hopf algebra $A$, we give a formula for the canonical element $W \in \mathcal{H}(A) \otimes \mathcal{H}(A)$, solution to the pentagon equation, in terms of the $R$-matrix of $D(A)$ and the map $Q$ expressing the factorizability of $D(A)$ (the formula is: $W = (Q^{-1} \otimes Q^{-1})(R_{21})$; unfortunately this formula does not seem to offer an answer to the following natural question: is there an “explanation”, in terms of the structure of the Drinfel’d double only, for the fact that $W$ is a solution to the pentagon equation on the Heisenberg double?).

In section 5 we speak about quantum groupoids. The general concepts of bialgebroid and Hopf algebroid (=quantum groupoid) have been introduced by J.-H. Lu in [25], with inspiration and motivation coming from Poisson geometry and by generalizing previous ones ([20], [21], [24], [26]); for discussions concerning the relation between these concepts and other objects known as “quantum groupoids”, such as weak Hopf algebras [2] and Takeuchi’s $\times_R$-bialgebras [47]). In [25] Lu proved that if $A$ is a finite dimensional Hopf algebra and $V$ is a quantum commutative left $D(A)$-module algebra then $V \# A$ is a Hopf algebroid over $V$, and that $A^*$ is a quantum commutative left $D(A)$-module algebra, so that $\mathcal{H}(A^*)$ is a quantum groupoid over $A^*$. We would like to obtain a quantum groupoid having $\mathcal{H}(A)$ as base, and we proceed as follows: first we generalize Lu’s theorem, by proving that if $(H, R)$ is any quasitriangular Hopf algebra and $V$ is a quantum commutative left $H$-module algebra then $V \# R_{(1)}$ is a quantum groupoid over $V$, where $R_{(1)}$ is a certain finite dimensional Hopf subalgebra of $H$ (Radford’s notation). Then, if $A$ is a finite dimensional Hopf algebra, $\mathcal{H}(A)$ is a quantum commutative $D(A) - D(A)^{cop}$-bimodule algebra, hence it is a quantum commutative left $D(A) \otimes D(A)^{op \cop}$-module algebra so that the above result may be applied and we obtain a Hopf algebroid with $\mathcal{H}(A)$ as base and $\mathcal{H}(A) \# (A \otimes A^{*op})$ as total algebra.

Let us mention that a generalization of Lu’s theorem has been independently obtained also very recently in [5].

In section 6 we give a slight generalization of the concept of “vertex group” introduced by Richard Borcherds in his recent (partly Hopf-algebraic) approach to vertex algebras (see [3], [4]), by allowing the “ring of singular functions” to be non-commutative. With this terminology, we prove that if $A$ is a finite dimensional cocommutative Hopf algebra, then $\mathcal{H}(A)$ has some properties making it a vertex group over $A$. These properties are natural from the Hopf-algebraic point of view, but we do not know whether this example may be relevant for the theory of vertex algebras.

Now, we come back to our starting point, namely the construction of $H^*_R$. We have tried to perform it for quasi-bialgebras $H$ instead of bialgebras, the multipli-
cation on $H^*$ being defined by the same formula. Naturally, this multiplication is not associative in general, but surprisingly $H^*_R$ is also not, in general, an algebra in the tensor category of left $H$-modules, as for bialgebras. We have tried to find a tensor category in which $H^*_R$ lives as an algebra, and we found the category of $H - H^{cop}$-bimodules. So, in section 2, the construction and properties of $H^*_R$ are given directly for quasi-bialgebras. This greater generality not only indicates what is the most natural tensor-categorical interpretation for the case of bialgebras, but, in view of the fact that $H(A) = D(A)^*_R$ for Hopf algebras, suggests a possible definition for the Heisenberg double of a finite dimensional quasi-Hopf algebra $A$, as $D(A)^*_R$, where $D(A)$ is the quantum double of $A$ introduced in [28], [13], [20]. In section 7 we compute explicitly this $D(A)^*_R$ for the case when $D(A)$ is a slight generalization of the Dijkgraaf-Pasquier-Roche quasi-Hopf algebra $D^{\omega}(G)$ introduced in [13].

This definition of the Heisenberg double as a non-associative algebra (but which is an algebra in a certain tensor category) may seem rather strange, so let us mention that such non-associative algebras occur naturally in the literature on quasi-Hopf algebras, in various contexts such as smash products ([10]), cohomology and deformation theory ([31], [44]) and algebraic quantum field theory ([26]). A somehow dual situation appears in [1], [2], where it was proposed, as a general philosophy, to try to study non-associative algebras by expressing them, when possible, as algebras in certain tensor categories (especially ones associated to quasi-Hopf algebras). For instance, the octonions and higher Cayley algebras may be studied in this framework, see [1].

2 Preliminaries

In this section, we recall some definitions and fix the notation that will be used in the rest of the paper. Throughout, $k$ will be a fixed field and all algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. For coalgebras and Hopf algebras, we shall use the framework of [13]; in particular, for coalgebras, we shall use $\Sigma$-notation: $\Delta(h) = \sum h_1 \otimes h_2, (I \otimes \Delta)(\Delta(h)) = (\Delta \otimes I)(\Delta(h)) = \sum h_1 \otimes h_2 \otimes h_3$, etc.

**Definition 2.1** ([13]) Let $H$ be a $k$-algebra, $\Delta : H \rightarrow H \otimes H$, $\varepsilon : H \rightarrow k$ two algebra homomorphisms. $H$ is called a quasi-bialgebra if there exists an invertible element $\Phi \in H \otimes H \otimes H$ such that, for all elements $h \in H$, we have:

$$(I \otimes \Delta)(\Delta(h)) = \Phi((\Delta \otimes I)(\Delta(h)))\Phi^{-1}$$

$$(\varepsilon \otimes I)(\Delta(h)) = h$$

$$(I \otimes I \otimes \Delta)(\Phi)(\Delta \otimes I \otimes I)(\Phi) = (1 \otimes \Phi)(I \otimes \Delta \otimes I)(\Phi)(\Phi \otimes 1)$$

$$(I \otimes \varepsilon \otimes I)(\Phi) = 1 \otimes 1$$

where $I = id_H$. The map $\Delta$ is called the coproduct or the comultiplication, $\varepsilon$ the counit and $\Phi$ the associator.
$H$ is called a quasi-Hopf algebra if, moreover, there exist an anti-automorphism $S$ of the algebra $H$ and elements $\alpha$ and $\beta$ of $H$ such that, for all $h \in H$, we have:
\[
\sum S(h_1) h_2 = \varepsilon(h) \alpha \text{ and } \sum h_1 \beta S(h_2) = \varepsilon(h) \beta
\]
\[
\sum X_1^1 \beta S(X^2) \alpha X^3 = 1 \text{ and } \sum S(x_1^1) x_2^2 \beta S(x_3^1) = 1
\]
where $\Phi = \sum X_1^1 \otimes X^2 \otimes X^3$, $\Phi^{-1} = \sum x_1^1 \otimes x^2 \otimes x^3$ (formal notation), and we used also the $\Sigma$-notation: $\Delta(h) = \sum h_1 \otimes h_2$. In this case, $S$ is called the antipode of $H$.

Let us note that every Hopf algebra with bijective antipode is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$.

We note the following two consequences of the definitions of $S, \alpha, \beta$: $\varepsilon(\alpha) \varepsilon(\beta) = 1$, $\varepsilon \circ S = \varepsilon$. Moreover, the axioms imply that $(\varepsilon \otimes I \otimes I)(\Phi) = (I \otimes I \otimes \varepsilon)(\Phi) = 1$.

**Definition 2.2 ([13])** A quasi-bialgebra or a quasi-Hopf algebra $H$ is termed quasi-triangular if there exists an invertible element $R \in H \otimes H$ such that:
\[
(\Delta \otimes I)(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi
\]
\[
(I \otimes \Delta)(R) = \Phi_{231}^{-1} R_{12} \Phi_{213} R_{13} \Phi^{-1}
\]
\[
\Delta^{\cop}(h) = R \Delta(h) R^{-1} \text{ for all } h \in H
\]
where, if $t$ denotes a permutation of $\{1, 2, 3\}$, then we set $\Phi_{t(1)t(2)t(3)} = \sum X^{t^{-1}(1)} \otimes X^{t^{-1}(2)} \otimes X^{t^{-1}(3)}$ and $R_{ij}$ means $R$ acting non-trivially in the $i^{th}$ and $j^{th}$ positions of $H \otimes H \otimes H$.

If $R$ satisfies these conditions it is called an $R$-matrix. From these relations one can deduce the quasi-Yang-Baxter equation:
\[
R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi = \Phi_{321} R_{23} \Phi_{231}^{-1} R_{13} \Phi_{213} \Phi_{12}
\]

Also, it is easy to see that:
\[
(\varepsilon \otimes I)(R) = (I \otimes \varepsilon)(R) = 1
\]

As a general rule, the tensor components of the associator $\Phi$ of a quasi-bialgebra will be denoted using big letters, for instance
\[
\Phi = \sum X_1^1 \otimes X^2 \otimes X^3 = \sum Y_1^1 \otimes Y^2 \otimes Y^3 \text{ etc}
\]
and the ones of $\Phi^{-1}$ with small letters, for instance
\[
\Phi^{-1} = \sum x_1^1 \otimes x^2 \otimes x^3 = \sum y_1^1 \otimes y^2 \otimes y^3 \text{ etc}
\]

An $R$-matrix will be usually denoted by
\[
R = \sum R_1^1 \otimes R_2^2 = \sum r_1^1 \otimes r_2^2
\]

If $(H, R)$ is a quasitriangular Hopf algebra and $B$ is a left $H$-module algebra (i.e. an algebra in the tensor category $H - \text{mod}$) then $B$ is called quantum commutative in [12] if it is commutative as algebra in the braided tensor category of left $H$-modules, i.e. if $bb' = \sum (R_2^1 \cdot b')(R_1^1 \cdot b)$ for all $b, b' \in B$. We extend this terminology and we call quantum commutative any algebra $B$ in a braided tensor category which is commutative with respect to the braiding $c$ of the category, namely $m_B \circ c_{B, B} = m_B$ where $m_B$ is the multiplication of $B$. 

3 The main result for quasi-bialgebras

Let $H$ be a quasi-bialgebra and denote by $H_l$, $H_r$ and $H_{lr}$ the categories of left $H$-modules, right $H$-modules and $H$-bimodules respectively. In these categories we introduce tensor products, as follows. If $V, W \in H_l$ then $V \otimes W \in H_l$ with $h \cdot (v \otimes w) = \Delta(h) \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$. If $V, W \in H_r$ then $V \otimes W \in H_r$ with $(v \otimes w) \cdot h = (v \otimes w) \cdot \Delta^{cop}(h) = \sum v \cdot h_2 \otimes w \cdot h_1$. If $V, W \in H_{lr}$ then $V \otimes W \in H_{lr}$ with $h \cdot (v \otimes w) \cdot h' = \Delta(h) \cdot (v \otimes w) \cdot \Delta^{cop}(h') = \sum h_1 \cdot v \cdot h'_2 \otimes h_2 \cdot w \cdot h'_1$.

It is well-known (see [23], Chapter XV) that the tensor category $H$ becomes a tensor category, with associativity constraints given by

$$\Phi_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$$

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)) = \sum X^1 \cdot u \otimes (X^2 \cdot v \otimes X^3 \cdot w)$$

for all $U, V, W \in H_l$. Similarly one can prove that $H_r$ and $H_{lr}$ become also tensor categories, with associativity constraints given by

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi_{321} = \sum u \cdot Y^3 \otimes (v \cdot Y^2 \otimes w \cdot Y^1)$$

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)) \cdot \Phi_{321} = \sum X^1 \cdot u \cdot Y^3 \otimes (X^2 \cdot v \cdot Y^2 \otimes X^3 \cdot w \cdot Y^1)$$

for $U, V, W \in H_r$ and $U, V, W \in H_{lr}$ respectively (the unit constraints are the usual ones).

Suppose now that $(H, R)$ is a quasitriangular quasi-bialgebra. It is well-known (see [23]) that the tensor category $H_l$ is braided, the braiding being given by

$$c_{V,W} : V \otimes W \rightarrow W \otimes V$$

$$c_{V,W}(v \otimes w) = \sum R^2 \cdot w \otimes R^1 \cdot v$$

for $V, W \in H_l$. Similarly one can prove that $H_r$ and $H_{lr}$ become also braided tensor categories, the braiding given by

$$c_{V,W}(v \otimes w) = \sum w \cdot U^1 \otimes v \cdot U^2$$

$$c_{V,W}(v \otimes w) = \sum R^2 \cdot w \cdot U^1 \otimes R^1 \cdot v \cdot U^2$$

for $V, W \in H_r$ and $V, W \in H_{lr}$ respectively, where $U = \sum U^1 \otimes U^2$ is the inverse of $R$.

Suppose again that $(H, R)$ is a quasitriangular quasi-bialgebra and consider the left and right regular actions of $H$ on $H^*$, that is $(h \rightarrow p)(h') = p(h'h)$, $(p \leftarrow h)(h') = p(h'h')$, for $p \in H^*$ and $h, h' \in H$. Obviously $H^*$ is an $H$-bimodule with these actions.

On $H^*$ we can consider the convolution product, given by $(fg)(h) = \sum f(h_1)g(h_2)$ (which is not associative in general). We introduce another product in $H^*$, given by

$$f \cdot g = \sum (R^2 \rightarrow g)(R^1 \rightarrow f)$$

which is also not associative in general. Denote by $H^*_R$ the pair $(H^*, \cdot)$. Then we have the following
Theorem 3.1  a) $H^*_R$ is an algebra in the tensor category $H_{tr}$ (we shall say that it is an $H-H^{\text{cop}}$-bimodule algebra), that is, for all $f, g, l \in H^*$ and $h, h' \in H$ we have:

$$h \mapsto (f \cdot g) \leftarrow h' = \sum (h_1 \mapsto f \leftarrow h'_2) \cdot (h_2 \mapsto g \leftarrow h'_1)$$

$$(f \cdot g) \cdot l = \sum (X^1 \mapsto f \leftarrow Y^2) \cdot ((X^2 \mapsto g \leftarrow Y^2) \cdot (X^3 \mapsto l \leftarrow Y^1))$$

$$\varepsilon \cdot f = f \cdot \varepsilon = f$$

$$h \mapsto \varepsilon \leftarrow h' = \varepsilon(h)\varepsilon(h')\varepsilon$$

b) $H^*_R$ is quantum commutative as an algebra in the braided tensor category $H_{tr}$, that is, for all $f, g \in H^*$, we have

$$f \cdot g = \sum (R^2 \mapsto g \leftarrow U^1) \cdot (R^1 \mapsto f \leftarrow U^2)$$

Proof: for $f, g \in H^*$, the product $f \cdot g$ is given by:

$$(f \cdot g)(h) = \sum g(h_1R^2)f(h_2R^1)$$

for all $h \in H$. We shall prove first b). We calculate:

$$((R^2 \mapsto g \leftarrow U^1) \cdot (R^1 \mapsto f \leftarrow U^2))(h) =$$

$$= \sum (R^1 \mapsto f \leftarrow U^2)(h_1r_2^2)(R^2 \mapsto g \leftarrow U^1)(h_2r^1)$$

$$= \sum f(U^2h_1r^2R^1)g(U^1h_2r^1R^2)$$

$$= \sum f(U^2r^2h_2R^1)g(U^1r^1h_1R^2)$$

(using the relation $\Delta^{\text{cop}}(h)R = R\Delta(h)$)

$$= \sum f(h_2R^1)g(h_1R^2)$$

$$= (f \cdot g)(h)$$

Now we prove a). We have:

$$(h \mapsto f \cdot g \leftarrow h')(h'') =$$

$$= (f \cdot g)(h''h)$$

$$= \sum g(h_1''^2h_1h_2R^2)f(h_2''h_2h_2R^1)$$

$$(\sum(h_1 \mapsto f \leftarrow h'_2) \cdot (h_2 \mapsto g \leftarrow h'_1))(h'') =$$

$$= \sum (h_2 \mapsto g \leftarrow h'_2)(h_1''R^2)(h_1 \mapsto f \leftarrow h'_1)(h''_2R^1)$$

$$= \sum g(h_1''h_1''R^2h_2)f(h_2''^2R^1h_1)$$

$$= \sum g(h_1''h_1''h_1R^2)f(h_2''^2h_2R^1)$$

using again the relation $\Delta^{\text{cop}}(h)R = R\Delta(h)$.

For the second relation, we calculate:

$$((f \cdot g) \cdot l)(h) = \sum l(h_1R^2)(f \cdot g)(h_2R^1) =$$

$$= \sum l(h_1R^2)g((h_2)_1(R^1)_2r^2)f((h_2)_2(R^1)_2r^1)$$

$$= \sum l(h_1X^1R^2x^2\rho^2Y^3)g((h_2)_1X^2R^1x^1Y^1r^2)f((h_2)_2X^3x^3\rho^1Y^2r^1)$$

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(using the relation \((\Delta \otimes id)(R) = \sum X^2 R^1 x^1 Y^1 \otimes X^3 x^3 \rho^1 Y^2 \otimes X^1 R^2 x^2 \rho^2 Y^3\), where \(R = \rho = \sum \rho^1 \otimes \rho^2\))

\[
\sum l((h_1 X^1 r^2 x^2 \rho^3 Y^3) g((h_2)_1) R^2 X^3 x^3 \rho^1 Y^2) f((h_2)_2) R^1 X^2 r^1 x^1 Y^1)
\]

by using the quasi-Yang-Baxter equation. We compute now the right hand side evaluated in \(h\):  

\[
\left(\sum (X^1 \to f \leftarrow Y^3) \cdot ((X^2 \to g \leftarrow Y^2) \cdot (X^3 \to l \leftarrow Y^1)\right)\right)(h) = \\
\sum ((X^2 \to g \leftarrow Y^2) \cdot (X^3 \to l \leftarrow Y^1)(h_1) R^2) (X^1 \to f \leftarrow Y^3)(h_2) R^1)
\]

\[
= \sum l(Y^1(h_1)_1 (R^2)_1 r^2 X^3) g(Y^2(h_1)_2 (R^2)_2 r^1 X^2) f(Y^3(h_2) R^1 X^1)
\]

\[
= \sum l(h_1 Y^1 (R^2)_1 r^2 X^3) g((h_2)_1 Y^2 (R^2)_2 r^1 X^2) f((h_2)_2 Y^3 R^1 X^1)
\]

(Using the relation \((id \otimes \Delta)(\Delta(a)) = \Phi(\Delta \otimes id)(\Delta(a)) \Phi^{-1}\))

\[
= \sum l((h_1) Y^1 T^1 R^2 x^2 r^2 X^3) g((h_2)_1 Y^2 g^2 \rho^2 T^3 x^3 r^1 X^2) f((h_2)_2 Y^3 \rho^1 T^2 R^1 x^1 X^1)
\]

(Using the relation \((id \otimes \Delta)(R) = \sum Y^3 \rho^1 T^2 R^1 x^1 \otimes Y^1 T^1 R^2 x^2 \otimes Y^3 \rho^2 T^3 x^3\), where \(\rho = R, \Phi = \sum T^1 \otimes T^2 \otimes T^3\))

\[
= \sum l(h_1 T^1 R^2 x^2 r^2 X^3) g((h_2)_1 \rho^2 T^3 x^3 r^1 X^2) f((h_2)_2 \rho^1 T^2 R^1 x^1 X^1)
\]

and this is obviously equal to the expression obtained for \((f \cdot g) \circ l(h)\).

The relations \(\varepsilon \cdot f = f \cdot \varepsilon = f\) and \(h \to \varepsilon \leftarrow h' = \varepsilon(h)\varepsilon(h')\varepsilon\) are obvious, using the fact that \(\sum \varepsilon(h_1) h_2 = h = \sum h_1 \varepsilon(h_2)\) and \(\sum \varepsilon(R^1) R^2 = \sum R^1 \varepsilon(R^2) = 1\).

\[\square\]

**Remark 3.2** Although \(H^*_R\) is an algebra in the tensor category \(H_{lr}\), in general it is **not** an algebra in the tensor categories \(H_l\) or \(H_r\).

**Remark 3.3** Suppose that \(F : (H, R) \to (H', R')\) is a morphism of quasitriangular quasi-bialgebras and consider the transpose \(F^* : H^*_R \to H^*_R\). It is easy to see that the map \(F^*\) has the following properties:

\[
F^*(f \cdot g) = F^*(f) \cdot F^*(g)
\]

\[
F^*(f(h) \to f) = h \to F^*(f)
\]

\[
F^*(f \leftarrow F(h)) = F^*(f) \leftarrow h
\]

for all \(f, g \in H^*\) and \(h \in H\). The map \(F\) induces a tensor functor \(F_* : H_{lr}^* \to H_{lr}^*\) (see [23]) in the usual manner, and if \(F\) is an isomorphism then this functor is an isomorphism of tensor categories and in this case the above relations imply that \(H_{lr}^*\) and \(F_* (H_{lr}^*)\) are isomorphic as algebras in the tensor category \(H_{lr}\). So, we can say that \(H_{lr}^*\) “depends only on the isomorphism class of \((H, R)\)”.

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4 The Heisenberg double vs the Drinfel’d double for Hopf algebras

Let \((H, R)\) be a quasitriangular bialgebra. We can consider it as a quasitriangular quasi-bialgebra with trivial associator, so the result in the previous section may be applied to \(H\). Since the associator of \(H\) is trivial, \(H^*\) is an associative algebra and is an algebra also in the tensor categories \(H_l\) and \(H_r\) (that is, a left \(H\)-module algebra and a right \(H^{\text{cop}}\)-module algebra), but in general is not quantum commutative as algebra in \(H_l\) or \(H_r\).

Let us note also the trivial fact that if \((H, R)\) is a quasitriangular bialgebra then \((H^{\text{cop}}, R_{21})\) is also a quasitriangular bialgebra and \((H^{\text{cop}})^*_R = (H^*_R)^{\text{op}}\).

Let now \(A\) be a finite dimensional Hopf algebra with antipode \(S\). The Drinfel’d double of \(A\), denoted by \(D(A)\), is a quasitriangular Hopf algebra realized on the \(k\)-linear space \(A^* \otimes A\); its coalgebra structure is the one of \(A^{\text{cop}} \otimes A\), the algebra structure is given by

\[
(p \otimes a)(p' \otimes a') = \sum p(a_1 \rightarrow p' \leftarrow S^{-1}(a_3)) \otimes a_2 a'
\]

for all \(p, p' \in A^*\) and \(a, a' \in A\) (see [38]), and the \(R\)-matrix is

\[
R = \sum (\varepsilon \otimes e_i) \otimes (e_i \otimes 1)
\]

where \(\{e_i\}\) is a basis of \(A\) and \(\{e^i\}\) its dual basis in \(A^*\).

The Heisenberg double of \(A\), denoted by \(\mathcal{H}(A)\), is the smash product \(A \# A^*\), where \(A^*\) acts on \(A\) via the left regular action \(p \rightarrow a = \sum p(a_2) a_1\) for all \(p \in A^*, a \in A\), so its multiplication is

\[
(a \otimes p)(a' \otimes p') = \sum a(p_1 \rightarrow a') \otimes p_2 p'
\]

for all \(p, p' \in A^*\) and \(a, a' \in A\), where \(\Delta(p) = \sum p_1 \otimes p_2\) is the coalgebra structure of \(A^*\). The present name of \(A \# A^*\) is from [13], while in [33] and [27] it was used under the heading “Weyl algebra of \(A\)”.

We have now the following application of the above construction \(H^*_R\):

**Proposition 4.1** If \(A\) is a finite dimensional Hopf algebra and \(H = D(A)\), then \(H^*_R = \mathcal{H}(A)\), so \(\mathcal{H}(A)\) is an algebra in the tensor categories \(D(A)_l, D(A)_r, D(A)_{lr}\) and is quantum commutative as an algebra in \(D(A)_{lr}\).

**Proof**: the fact that the multiplication of \(\mathcal{H}(A)\) may be obtained from the one of \(D(A)^*\) is due to J.-H. Lu in [24], but with a slightly different approach, so we include here a proof for completeness.

The multiplication in \(D(A)^*_R\) is given by:

\[
(b \otimes g) \cdot (a \otimes f) = \sum (R^2 \rightarrow (a \otimes f))(R^1 \rightarrow (b \otimes g))
\]

for all \(a, b \in A\) and \(f, g \in A^*\). Let \(x \in A\), \(p \in A^*\) and denote by \(<,>\) the evaluation map; we calculate:
\[\langle b \otimes g \rangle \cdot (a \otimes f), p \otimes x \rangle =
\]
\[= \sum (a \otimes f, (p \otimes x) R^2) \langle b \otimes g, (p \otimes x) R^1 \rangle
\]
\[= \sum (a \otimes f, (p_2 \otimes x_1)(e^i \otimes 1)) \langle b \otimes g, (p_1 \otimes x_2)(e \otimes e_i) \rangle
\]
\[= \sum (a \otimes f, p_2(x_1 \rightarrow e^i \leftarrow S^{-1}(x_3)) \otimes x_2) \langle b \otimes g, p_1 \otimes x_4 e_i \rangle
\]
\[= \sum (p_2, a_1) \langle e^i, S^{-1}(x_3) a_2 x_1 \rangle \langle f, x_2 \rangle (p_1, b) \langle g, x_4 e_i \rangle
\]
\[= \sum (p, ba_1) \langle f, x_2 \rangle \langle g, x_4 S^{-1}(x_3) a_2 x_1 \rangle
\]
\[= \sum (p, ba_1) \langle f, x_2 \rangle \langle g, a_2 x_1 \rangle
\]
\[= \sum (p, ba_1) \langle g_1, a_2 \rangle \langle g_2, x_1 \rangle \langle f, x_2 \rangle
\]
\[= \sum (p, b(g_1 \rightarrow a)) \langle g_2 f, x \rangle
\]
\[= \sum (b(g_1 \rightarrow a) \otimes g_2 f, p \otimes x)
\]
hence \( (b \otimes g) \cdot (a \otimes f) = \sum b(g_1 \rightarrow a) \otimes g_2 f \), q.e.d.

**Remark 4.2** We would like to emphasize that the actions of \( D(A) \) on \( \mathcal{H}(A) \) are just the left and right regular actions of \( D(A) \) on \( \mathcal{H}(A) \) identified as vector spaces with \( D(A)^* \). These actions may be described explicitly, using the following formula for the comultiplication of \( D(A)^* \) (which may be found in [24]):

\[\Delta_{D(A)^*}(a \otimes p) = \sum (a_1 \otimes e^i p_1 e^j) \otimes (S^{-1}(e_j) a_2 e_i \otimes p_2)\]

where \( \{e_i\} \) is a basis in \( A \) and \( \{e^i\} \) its dual basis in \( A^* \). They look as follows:

\[(p \otimes b) \rightarrow (a \otimes q) = \sum p_2(a_2) q_2(b)(a_1 \otimes p_3 q_1 S^{*-1}(p_1))\]

\[(a \otimes q) \leftarrow (p \otimes b) = \sum p(a_1) q_1(b_2)(S^{-1}(b_3) a_2 b_1 \otimes q_2)\]

for all \( p, q \in A^* \) and \( a, b \in A \) (the first one is formula (35) in [24]).

**Remark 4.3** The fact that \( H^*_R \) is a left \( H \)-module algebra via the left regular action (for any quasitriangular bialgebra \( (H, R) \)) may be obtained alternatively using the more general framework of twisting the multiplication of a module algebra by a Drinfel’d twist (see [3], [29], [18], [10]). We use the Drinfel’d twist \((R_{21})^{-1}\) and from the above mentioned results it follows that \((H^*_R)^{op}\) is a left module algebra over \( H^{cop}\) (hence \( H^*_R \) is a left module algebra over \( H \), and also using [10], Prop. 2.17 we obtain that the algebras \( H^* \# H \) and \((H^*_R)^{op} \# H^{cop}\) are isomorphic. In particular, for a finite dimensional Hopf algebra \( A \), we obtain that the Heisenberg double of \( D(A)^* \) (which is \( D(A)^* \# D(A) \)) is isomorphic to \( \mathcal{H}(A)^{op} \# D(A)^{cop} \) as algebras.

Let us mention that the Heisenberg double and the Drinfel’d double appear also together in [10] where they join to build the algebra \( X = \mathcal{H}(A) \otimes D(A)^{op} \) having the nice property that there exists a vector space — preserving equivalence of categories between the categories of Hopf bimodules over \( A \) and of left \( X \)-modules.
Remark 4.4 Let \((H, R)\) be a quasitriangular bialgebra. Then the braided tensor category \(H_r\) of right \(H^{\text{cop}}\)-modules may be identified with the braided tensor category of left \(H^{\text{cop}}\)-modules, where the \(R\)-matrix of \(H^{\text{cop}}\) is \((R_{21})^{-1}\). Moreover, if \(V\) is an algebra, then \(V\) is a right \(H^{\text{cop}}\)-module algebra if and only if \(V\) is a left \(H^{\text{cop}}\)-module algebra, and it is left quantum commutative if and only if it is right quantum commutative.

Consequently, since \(H_R^*\) is a quantum commutative \(H - H^{\text{cop}}\)-bimodule algebra, we obtain that \(H_R^*\) is a quantum commutative left \(H \otimes H^{\text{cop}}\)-module algebra, where the \(R\)-matrix of \(H \otimes H^{\text{cop}}\) is \(\sum (R^1 \otimes U^2) \otimes (R^2 \otimes U^1)\) with \(U = R^{-1}\) and the left action of \(H \otimes H^{\text{cop}}\) on \(H_R^*\) is of course given by \((h \otimes h') \cdot p = h \mapsto p \leftarrow h'\) for all \(h, h' \in H\) and \(p \in H^*\). In particular, if \(A\) is a finite dimensional Hopf algebra, then \(\mathcal{H}(A)\) is a quantum commutative left \(D(A) \otimes D(A)^{\text{op} \text{cop}}\)-module algebra. We shall use this in the next section.

Proposition 4.5 Let \((H, R)\) be a quasitriangular bialgebra. Then \(H_R^*\) is quantum commutative as a left \(H\)-module algebra if and only if \(R = 1 \otimes 1\). In particular, if \(A\) is a finite dimensional Hopf algebra then \(\mathcal{H}(A)\) is a quantum commutative left \(D(A)\)-module algebra if and only if \(\text{dim}(A) = 1\).

Proof: if \(f, g \in H^*\) then \(f \cdot g = \sum (R^2 \mapsto g) \cdot (R^1 \mapsto f)\) if and only if

\[
\sum g(h_1 r^2) f(h_2 r^1) = \sum f(h_1 r^2 R^1) g(h_2 r^1 R^2)
\]

for all \(h \in H\), and this holds for all \(f, g \in H^*\) if and only if

\[
\sum h_2 r^1 \otimes h_1 r^2 = \sum h_1 r^2 R^1 \otimes h_2 r^1 R^2
\]

for all \(h \in H\), which, by applying the relation \(\Delta^{\text{cop}}(h) R = R \Delta(h)\) several times is equivalent to

\[
\sum R^1 h_1 \otimes R^2 h_2 = \sum R^2 r^1 h_1 \otimes R^1 r^2 h_2
\]

for all \(h \in H\), and this is of course equivalent to \(R = 1 \otimes 1\). The second statement follows immediately from the first.

Remark 4.6 Let \(A\) be a finite dimensional Hopf algebra. Then the left action of \(D(A)\) on \(\mathcal{H}(A)\) restricts to an action of \(D(A)\) on \(A^* \subseteq \mathcal{H}(A)\), given by

\[
(p \otimes b) \rightarrow q = \sum q_2(b) p_2 q_1 S^{* -1}(p_1)
\]

So, \(A^*\) is a left \(D(A)\)-module algebra under this action, and it was proved in [23] that it is quantum commutative, a fact which allowed Lu to introduce a quantum groupoid structure on \(\mathcal{H}(A^*)\) (we shall come back to the subject of quantum groupoids in the next section).

Let us mention that \(A\) is also a quantum commutative left \(D(A)\)-module algebra under the action:

\[
(p \otimes a) \cdot b = \sum (a_1 b S(a_2)) \leftarrow S^{-1}(p)
\]
for all \(a, b \in A\) and \(p \in A^*\). This action is nicely obtained in \([52]\) by considering the right regular action of \(D(A)\) on \(D(A)^*\), restricting it to an action on \(A^{op}\) and then transforming it via \(S\) into a left action on \(A\).

If \((H, R)\) is a quasitriangular Hopf algebra, we shall denote by \(Q\) the map \(Q : H^* \to H\), \(Q(p) = \sum p(R^2 r^1)R^1 r^2\). With this notation, let us recall the following

**Definition 4.7** ([44]) If \((H, R)\) is a finite dimensional quasitriangular Hopf algebra, it is called factorizable if the map \(Q\) is a linear isomorphism.

This is a natural condition, which was also proved in \([21]\) to be useful in defining certain invariants for 3-manifolds.

It is natural to see how far is the map \(Q\) from being an algebra map, if we consider on \(H^*\) the algebra structure \(H^*_\).

**Proposition 4.8** If \(f, g \in H^*\), then

\[
Q(f \cdot g) = \sum Q(R^1 \to f)Q(g \leftarrow R^2)
\]

**Proof:** let us compute (we denote \(R = r = \rho = T = \alpha = \beta = \gamma = \delta\):

\[
Q(f \cdot g) = \sum (f \cdot g)(R^2 r^1)R^1 r^2
\]

\[
= \sum g((R^2)_1(r^1)_1T^1)f((R^2)_2(r^1)_2T^1)R^1 r^2
\]

\[
= \sum f(\gamma^2 \beta^1 T^1)g(\delta^2 \alpha^1 T^2)\gamma^1 \delta^1 \alpha^2 \beta^2
\]

(using: \((\Delta \otimes id)(R) = \sum \alpha^1 \otimes \beta^1 \otimes \alpha^2 \beta^2\) and \((id \otimes \Delta)(R) = \sum \gamma^1 \delta^1 \otimes \delta^2 \otimes \gamma^2\)

\[
= \sum f(\gamma^2 \beta^1 T^1)g(\delta^2 \alpha^1 T^2)\gamma^1 \delta^1 \beta^2 \alpha^2
\]

(using the Yang-Baxter equation: \(\sum \beta^1 T^1 \otimes \alpha^1 T^2 \otimes \alpha^2 \beta^2 = \sum T^1 \beta^1 \otimes T^2 \alpha^1 \otimes \beta^2 \alpha^2\)

\[
= \sum f(\gamma^2 \beta^1 T^1)g(T^2 \delta^2 \alpha^1)\gamma^1 \beta^2 \alpha^2
\]

(using the Yang-Baxter equation: \(\sum T^1 \beta^1 \otimes \delta^1 \beta^2 \otimes \delta^2 T^2 = \sum \beta^1 T^1 \otimes \beta^2 \delta^1 \otimes T^2 \delta^2\)

\[
= \sum (T^1 \to f)(\gamma^2 \beta^1)\gamma^1 \beta^2(g \leftarrow T^2)(\delta^2 \alpha^1)\delta^1 \alpha^2
\]

\[
= \sum Q(T^1 \to f)Q(g \leftarrow T^2), \text{q.e.d.}
\]

It is natural to try to define an algebra structure on \(H^*\) such that, with respect to this structure, \(Q\) becomes an algebra map. In view of the previous proposition and the fact that \(R^{-1} = \sum R^1 \otimes S^{-1}(R^2)\), it is natural to define

\[
f \cdot g = \sum (R^1 \to f) \cdot (g \leftarrow S(R^2))
\]

and indeed one can now check that this multiplication is associative (due to the associativity of \(\cdot\)) and that \(Q(f \cdot g) = Q(f)Q(g)\) (due to the previous proposition).

This multiplication \(\cdot\) is just Majid’s “covariantised product” (see \([27]\), Th. 7.4.1), as one can easily check. The fact that \(Q(f \cdot g) = Q(f)Q(g)\) is also proved in \([27]\), Prop. 7.4.1.
7.4.3.

Let us say few more words about the relation between the two multiplications \( \cdot \) and \( \circ \) on \( H^* \). Define the following action of \( H \) on \( H^* \):

\[
h \triangleright f = \sum h_2 \to f \leftarrow S(h_1)
\]

Obviously \( H^* \) is a left \( H \)-module via this action; using the description of \( \cdot \) in terms of \( \circ \) and the fact that \((H^*_R, \cdot)\) is an algebra in the category of \( H - H^{cop} \)-bimodules, one can see immediately that \((H^*_R, \cdot)\) is a left \( H \)-module algebra with respect to the action \( \triangleright \). It is not quantum commutative, but satisfies a condition called by Majid “braided commutativity” (see [27], Exp. 9.4.10). It is nice that this condition may be easily proved using the fact that \((H^*_R, \cdot)\) is quantum commutative as an \( H - H^{cop} \)-bimodule algebra.

Of course, if \( H \) is finite dimensional and factorizable, we can express the covariantised product as

\[
f \circ g = Q^{-1}(Q(f)Q(g))
\]

Now let \( A \) be a finite dimensional Hopf algebra. It is well-known that the Drinfel’d double of \( A \) is factorizable. An explicit proof (for the explicit realization of the double we work with) is written down in [39]. It uses the following

**Proposition 4.9** ([39]) Let \((H, R)\) be a finite dimensional quasitriangular Hopf algebra, and denote by \( u = \sum S(R^2)R^1 \) the canonical Drinfel’d element of \( H \). Then \((H, R)\) is factorizable if and only if \( u^{-1} \leftarrow H^* = H \), or equivalently \( H \) is a free right \( H^* \)-module with generator \( u^{-1} \). Moreover, in this case, if \( y \in H \), then \( Q^{-1}(y) = p \leftarrow u^{-1} \), where \( p \in H^* \) is the unique element satisfying the relation \( u^{-1}y = u^{-1} \leftarrow p \).

Another criterion for factorizability may be found in [17].

Actually, one can give a direct proof of the factorizability of \( D(A) \): it is easy to see that in this case the map \( Q \) is given by

\[
Q(a \otimes p) = \sum a_1 \to p \leftarrow S^{-1}(a_3) \otimes a_2
\]

for all \( a \in A, p \in A^* \), and it has an inverse given by

\[
Q^{-1} : D(A) \to D(A)^*
\]

\[
Q^{-1}(p \otimes a) = \sum a_2 \otimes S^{-1}(a_1) \to p \leftarrow a_3
\]

We can also write down explicit formulae for Radford’s criterion applied to a Drinfel’d double:

**Proposition 4.10** Let \( p \otimes a \in D(A) \); then we have

\[
u^{-1}(p \otimes a) = u^{-1} \leftarrow (\sum S^{-2}(a_2) \otimes S^{-1}(a_1) \to p)
\]

Consequently, we have:

\[
Q^{-1}(p \otimes a) = (\sum S^{-2}(a_2) \otimes S^{-1}(a_1) \to p) \leftarrow u^{-1}
\]
Proof: a direct computation, using the formula \( u^{-1} = \sum e^i \otimes S^2(e_i) \).

Now we would like to write down the formula for the covariantised product corresponding to a Drinfel’d double \( D(A) \). It is slightly easier to do this using the explicit formulae obtained above for \( Q \) and \( Q^{-1} \) rather than the definition in terms of \( \cdot \). We have then:

\[
(a \otimes p)_2(a' \otimes p') = Q^{-1}(Q(a \otimes p)Q(a' \otimes p'))
\]

for \( a, a' \in A \) and \( p, p' \in A^* \), and by a direct computation we obtain:

\[
(a \otimes p)_2(a' \otimes p') = \sum aa'_2 \otimes (S^{-1}(a'_1) \rightarrow p \leftarrow a'_3)p'
\]

and this is just the multiplication of the realization of the Drinfel’d double on \( A \otimes A^* \), see [27], p. 290. Moreover, if \( R = \sum (\varepsilon \otimes e_i) \otimes (e^i \otimes 1) \) is the \( R \)-matrix of \( D(A) = A^* \otimes A \), then one can see that \( (Q^{-1} \otimes Q^{-1})(R) = \sum (e_i \otimes \varepsilon) \otimes (1 \otimes e^i) \), which is the \( R \)-matrix of \( D(A) = A \otimes A^* \). Finally, the comultiplication of \( D(A) = A \otimes A^* \) is

\[
\Delta(a \otimes p) = \sum a_1 \otimes p_2 \otimes a_2 \otimes p_1
\]

and one can see that \( Q : D(A) = A \otimes A^* \rightarrow D(A) = A^* \otimes A \) is a coalgebra map.

In conclusion, for a Drinfel’d double, the map \( Q \) in the definition of factorizability gives an isomorphism of quasitriangular Hopf algebras between the two realizations of the double, on \( A^* \otimes A \) and \( A \otimes A^* \). For the general meaning of the map \( Q \) (i.e. for any quasitriangular Hopf algebra) in terms of “braided groups”, we refer to [27], p. 490.

**Remark 4.11** If we take the above realization of the double on \( A \otimes A^* \) (denoted also by \( D(A) \)), then one can check that the multiplication of \( D(A)^*_R \) is given by

\[
(p \otimes a) \cdot (p' \otimes a') = \sum p'(a'_1 \rightarrow p) \otimes a'_2a
\]

so that \( D(A)^*_R = \mathcal{H}(A^*)^op \). So, we can obtain \( \mathcal{H}(A^*) \) from \( D(A) \) by the same method used for obtaining \( \mathcal{H}(A) \) (let us mention that in [24] \( \mathcal{H}(A^*) \) is obtained form \( D(A) \) without using the \( R \)-matrix of \( D(A) \)).

Now we shall speak about the pentagon equation, which, as the Yang-Baxter equation, appears in various contexts in mathematics and physics (see for instance [3]). If \( A \) is a finite dimensional Hopf algebra, there exist two canonical procedures to construct (invertible) solutions for the pentagon equation: on the one hand, the map \( w \in \text{End}(A \otimes A) \) given by \( w(a \otimes b) = \sum a_1 \otimes a_2b \) satisfies the pentagon equation \( w_{12}w_{13}w_{23} = w_{23}w_{12} \), and its inverse is given by \( w^{-1}(a \otimes b) = a_1 \otimes S(a_2)b \) (see [27], p. 29). On the other hand, if we consider the element \( W = \sum (1 \otimes e^i) \otimes (e_i \otimes \varepsilon) \) in \( \mathcal{H}(A) \otimes \mathcal{H}(A) \), where \( \{e_i\} \) is a basis of \( A \) and \( \{e^i\} \) its dual basis in \( A^* \), then \( W \) is also a solution to the pentagon equation (see [18], [22]). These two approaches are actually equivalent, because if we consider the algebra isomorphism \( \lambda : A^\# A^* \rightarrow \text{End}(A) \) given by \( \lambda(a \otimes f)(b) = af(b_2)b_1 \), for all \( a, b \in A \) and \( f \in A^* \) (see [23], p. 162) then we have \( (\lambda \otimes \lambda)(W) = w \).

Since the Heisenberg double \( \mathcal{H}(A) \) may be obtained from the Drinfel’d double \( D(A) \), it is natural to see whether we can obtain the element \( W \) from the \( R \)-matrix of \( D(A) \).
Proposition 4.12 If $A$ is a finite dimensional Hopf algebra, $R$ is the $R$-matrix of $D(A)$, $W$ is the canonical element of $H(A)$ and $Q : D(A)^* \to D(A)$ is the map expressing the factorizability of $D(A)$, then we have

$$W = (Q^{-1} \otimes Q^{-1})(R_{21})$$

where we identified $D(A)^*$ and $H(A)$ as linear spaces.

Proof: recall that the map $Q$ is given by $Q(a \otimes f) = \sum a_1 \to f \leftarrow S^{-1}(a_3) \otimes a_2$, so we shall compute:

$$(Q \otimes Q)(W) = \sum Q(1 \otimes e^i) \otimes Q(e_i \otimes \varepsilon)$$

$$= \sum (e^i \otimes 1) \otimes ((e_i)_1 \to \varepsilon \leftarrow S^{-1}((e_i)_3) \otimes (e_i)_2))$$

$$= \sum (e^i \otimes 1) \otimes (\varepsilon \otimes e_i) = R_{21}, \text{ q.e.d.}$$

Let us note that although this proposition expresses $W$ in terms of the structure of $D(A)$, it does not give an explanation, in terms of $D(A)$, for why is $W$ a solution to the pentagon equation. This would have been the case if we could have this proposition as a particular case of a more general statement, for instance, it was tempting to conjecture the following: “If $(H, R)$ is a finite dimensional quasitriangular factorizable Hopf algebra, then the element $W = (Q^{-1} \otimes Q^{-1})(R_{21})$ is a solution to the pentagon equation with respect to the algebra structure of $H_R^*$. Unfortunately, at least in this generality, this is false. A counterexample may be obtained as follows. Suppose $\text{char}(k) = 0$ and $k$ contains a primitive root of unity of order 3, $\omega$ say. Take $G = \{1, a, a^2\}$ a group of order 3, and take $H = kG$, with the following $R$-matrix (see [39]):

$$R = \frac{1}{3}(1 \otimes 1 + 1 \otimes a + 1 \otimes a^2 + a \otimes a^2 + a^2 \otimes 1 + \omega^2(a \otimes a) + \omega(a \otimes a^2) + a^2 \otimes 1 + \omega(a^2 \otimes a) + \omega^2(a^2 \otimes a^2))$$

It is known from [39] that $(H, R)$ is factorizable, a fact which may be proved also directly: if $\{e^0, e^1, e^2\}$ is the basis of $H^*$ dual to $\{1, a, a^2\}$, then one can check that the map $Q : H^* \to H$ is given by:

$$Q(e^0) = \frac{1}{3}(1 + a + a^2)$$

$$Q(e^1) = \frac{1}{3}(1 + \omega a + \omega^2 a^2)$$

$$Q(e^2) = \frac{1}{3}(1 + \omega^2 a + \omega a^2)$$

and that it is bijective, its inverse being the map $Q^{-1} : H \to H^*$:

$$Q^{-1}(1) = e^0 + e^1 + e^2$$

$$Q^{-1}(a) = e^0 + \omega^2 e^1 + \omega e^2$$

$$Q^{-1}(a^2) = e^0 + \omega e^1 + \omega^2 e^2$$
The multiplication of \( H^*_R \) is given by:

\[
\begin{align*}
e^0 \cdot e^0 &= \frac{1}{3}(e^0 + \omega^2 e^1 + \omega e^2) \\
e^1 \cdot e^1 &= \frac{1}{3}(e^1 + \omega^2 e^0 + \omega^2 e^2) \\
e^2 \cdot e^2 &= \frac{1}{3}(e^2 + \omega^2 e^0 + \omega^2 e^1) \\
e^0 \cdot e^1 &= e^1 \cdot e^0 = \frac{1}{3}(e^0 + e^1 + \omega e^2) \\
e^0 \cdot e^2 &= e^2 \cdot e^0 = \frac{1}{3}(e^0 + e^2 + \omega e^1) \\
e^1 \cdot e^2 &= e^2 \cdot e^1 = \frac{1}{3}(e^1 + e^2 + \omega e^0)
\end{align*}
\]

The element \( W \) is given by

\[
W = (Q^{-1} \otimes Q^{-1})(R_{21}) = e^0 \otimes e^0 + e^0 \otimes e^1 + e^0 \otimes e^2 + e^1 \otimes e^0 + \omega(e^1 \otimes e^1) + \omega^2(e^1 \otimes e^2) + e^2 \otimes e^0 + \omega^2(e^2 \otimes e^1) + \omega(e^2 \otimes e^2)
\]

and one can prove by a direct (but tedious) computation that \( W \) is not a solution to the pentagon equation.

However, one can ask for what classes of Hopf algebras the conjecture is true. A useful result may be the following:

**Proposition 4.13** Let \((H, R)\) and \((H', R')\) be two quasitriangular Hopf algebras and \(F : H \to H'\) a map of quasitriangular Hopf algebras. Then:

a) \(F^* : H^*_R \to H'^*_R\) is an algebra map

b) if \(H\) and \(H'\) are factorizable and \(F\) is injective, then

\[
(F^* \otimes F^*)(Q^{-1} \otimes Q^{-1})(R'_{21}) = (Q^{-1} \otimes Q^{-1})(R_{21})
\]

Consequently, if \((Q^{-1} \otimes Q^{-1})(R'_{21})\) is a solution to the pentagon equation, so is \((Q^{-1} \otimes Q^{-1})(R_{21})\).

**Proof:** a) has been noticed before, so we prove b). By applying \(Q \otimes Q\), the relation we have to prove is equivalent to

\[
\sum Q(F^*(Q^{-1}(R'^{1}))) \otimes Q(F^*(Q'^{-1}(R'^{2}))) = R
\]

which is equivalent to

\[
\sum Q'^{-1}(R'^{0}))(F(\alpha^{2})F(\beta^{1}))(\alpha^{1}\beta^{2} \otimes Q'^{-1}(R'^{2}))(F(\gamma^{2})F(\delta^{1}))(\gamma^{1}\delta^{2}) = R
\]

where \(\alpha = \beta = \gamma = \delta = R\), which, by applying \(F \otimes F\) (which is injective) and taking into account that \((F \otimes F)(R) = R'\), is equivalent to

\[
\sum Q'(Q'^{-1}(R'^{1})) \otimes Q'(Q'^{-1}(R'^{2})) = R'
\]

which is obviously true. \(\square\)
Remark 4.14 Let $A$ be a finite dimensional Hopf algebra and define the map $\varphi : D(A) \to \mathcal{H}(A^*) \otimes \mathcal{H}(A)$ by

$$\varphi(p \otimes a) = \sum (p_2 \# a_1) \otimes (a_2 \# p_1 \leftarrow a_3) = \sum (p_3 \# a_1) \otimes (p_1 \rightarrow a_2 \# p_2)$$

for all $p \in A^*$ and $a \in A$. Then one can check, by a direct computation, that $\varphi$ is an algebra map (this is a “coordinate-free” version of a result in \[22\]) and that

$$(id \otimes Q) \varphi = (id \otimes T \otimes id) \Delta_{D(A)}$$

where $Q : A \otimes A^* \to A^* \otimes A, Q(a \otimes p) = \sum a_1 \rightarrow p \leftarrow S^{-1}(a_3) \otimes a_2$ and $T : A \otimes A^* \to A \otimes A^*, T(a \otimes p) = \sum a_1 \otimes a_2 \rightarrow p$.

Then, by identifying $\mathcal{H}(A^*) \simeq \text{End}(A^*)$ and $\mathcal{H}(A) \simeq \text{End}(A)$ via the algebra map $\lambda$ mentioned before and then by identifying $\text{End}(A^*) \otimes \text{End}(A) \simeq \text{End}(A^* \otimes A)$, and finally by composing with $\varphi$, we obtain an algebra map $\alpha : D(A) \to \text{End}(A^* \otimes A)$

$$\alpha(p \otimes a)(q \otimes b) = \sum p_2(a_1 \rightarrow q) \otimes p_1 \rightarrow (a_2b)$$

hence a $D(A)$-module structure on $A^* \otimes A$, which seems to be noncanonical.

5 A quantum groupoid with Heisenberg double as base

We recall first the following definitions from \[25\]:

Definition 5.1 A bialgebroid consists of the following data:
1) an algebra $T$ called the total algebra
2) an algebra $B$ called the base algebra
3) the source map: an algebra homomorphism $\alpha : B \to T$
4) the target map: an algebra anti-homomorphism $\beta : B \to T$

such that $\alpha(b)\beta(b') = \beta(b')\alpha(b)$ for all $b, b' \in B$.

4) the coproduct: a $B - B$-bimodule map

$$\Delta : T \to T \otimes_B T, \ h \mapsto \sum h_1 \otimes h_2$$

with $\Delta(1) = 1 \otimes 1$, satisfying the coassociativity condition

$$(\Delta \otimes_B id_T) \Delta = (id_T \otimes_B \Delta) \Delta : T \to T \otimes_B T \otimes_B T$$
where \( T \) is a \( B - B \)-bimodule with structure maps
\[
\lambda : B \otimes T \to T, \quad b \otimes t \mapsto \alpha(b)t
\]
\[
\rho : T \otimes B \to T, \quad t \otimes b \mapsto \beta(b)t
\]
The coproduct \( \Delta \) and the algebra structure of \( T \) are required to be compatible in the sense that the kernel of the map
\[
\phi : T \otimes T \otimes T \to T \otimes_B T, \quad t \otimes t' \otimes t'' \mapsto (\Delta(t))(t' \otimes t'')
\]
is a left ideal of \( T \otimes T \otimes_B T \) (we have used the fact that \( T \otimes T \) acts on \( T \otimes_B T \) from the right by right multiplication).
5) the counit map: a \( B - B \)-bimodule map
\[
\varepsilon : T \to B
\]
satisfying \( \varepsilon(1) = 1 \) (it follows then that \( \varepsilon \beta = \varepsilon \alpha = id_B \)) and
\[
\lambda(\varepsilon \otimes id_T)\Delta = \rho(id_T \otimes \varepsilon)\Delta = id_T : T \to T
\]
where \( \lambda \) and \( \rho \) are the maps defined above.
It is also required that \( \varepsilon \) is compatible with the algebra structure of \( T \) in the sense that the kernel of \( \varepsilon \) is a left ideal of \( T \).

**Remark 5.2** It was proved in [25] that in the case \( B = k \) the definition of a bialgebroid reduces to the one of a bialgebra over \( k \).

**Remark 5.3** It was proved by P. Xu in [51] that the compatibility condition between \( \Delta \) and the algebra structure of \( T \) in the definition of a bialgebroid may be replaced by a (more manageable) equivalent one, namely the following two conditions have to be satisfied:
\[
\Delta(t)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0 \quad \text{in} \quad T \otimes_B T
\]
\[
\Delta(tt') = \Delta(t)\Delta(t') \quad \text{in} \quad T \otimes_B T
\]
for all \( t, t' \in T \) and \( b \in B \).

**Definition 5.4** A Hopf algebroid (=quantum groupoid) is a bialgebroid \( T \) over an algebra \( B \) with structure maps \( \alpha, \beta, \Delta \) and \( \varepsilon \) together with a linear map \( \tau : T \to T \), called the antipode map, having the following properties:
1) \( \tau \) is an algebra anti-isomorphism of \( T \)
2) \( \tau \beta = \alpha \)
3) \( m_T(\tau \otimes id)\Delta = \beta \varepsilon \tau : T \to T \), where \( m_T \) is the multiplication of \( T \)
4) there exists a linear map \( \gamma : T \otimes_B T \to T \otimes T \) with the following properties:
4a) \( \gamma \) is a section for the natural projection \( p : T \otimes T \to T \otimes_B T \)
4b) the following identity holds:
\[
m_T(id \otimes \tau)\gamma \Delta = \alpha \varepsilon : T \to T
\]
Remark 5.5  a) The existence of the section $\gamma$ is required because the map $m_T(id \otimes \tau)\Delta$ is not well-defined.
b) in general $\varepsilon \tau \neq \varepsilon$, see [25].

Recall also from [25] the following results:

Theorem 5.6 ([25]) If $A$ is a finite dimensional Hopf algebra and $V$ is a quantum commutative left $D(A)$-module algebra, then $V#A$ becomes a Hopf algebroid over $A$, the left action of $A$ on $V$ being the one induced from the action of $D(A)$ on $V$ by the inclusion $A \subseteq D(A)$.

Proposition 5.7 ([25]) If $A$ is a finite dimensional Hopf algebra then $A^*$ is a quantum commutative left $D(A)$-module algebra, so the Heisenberg double $A^*#A$ of $A^*$ is a Hopf algebroid over $A^*$.

Now, we want to obtain an example of a Hopf algebroid having the Heisenberg double of a finite dimensional Hopf algebra $A$ as base. We proceed as follows: first we give a generalization of the above theorem and then we use the quantum commutativity of the Heisenberg double of $A$ as a $D(A)-D(A)$-bimodule algebra.

Let us recall first some notation and results from [38]. If $(H, R)$ is a quasitriangular Hopf algebra, define the subspaces $L = R(l)$ and $D = R(r)$ by $R(l) = \{(id \otimes p)(R)/p \in H^*\}$ and $R(r) = \{(p \otimes id)(R)/p \in H^*\}$. If we write $R = \sum_{i=1}^{m} u_i \otimes v_i \in H \otimes H$ where $m$ is as small as possible, then $\{u_1, ..., u_m\}$ is a basis for $L$ and $\{v_1, ..., v_m\}$ is a basis for $D$, in particular $\dim L = \dim D$ and this common dimension is called the rank of $R$ and is denoted by $\text{rank}(R)$. Moreover, $L$ and $D$ are (finite-dimensional) Hopf subalgebras of $H$ and the map $f : L^\text{cop} \to D$ defined by $f(p) = (p \otimes id)(R)$ for $p \in L^*$ is a Hopf algebra isomorphism.

Now we can generalize Lu’s theorem as follows:

Theorem 5.8 Let $(H, R)$ be a quasitriangular Hopf algebra and denote as above $L = R(l)$, $D = R(r)$. Let $V$ be a quantum commutative left $H$-module algebra with action denoted by $h \otimes V \to V$, $h \otimes v \mapsto h \cdot v$. Then $V#L$ is a Hopf algebroid over $V$, the left action of $L$ on $V$ being the one induced from the action of $H$ on $V$ by the inclusion $L \subseteq H$.

Proof: since (with one exception) the proof follows closely the proof in [25], we shall just write down the structure maps and we shall give only those parts of the proof which, in [28], depend on the explicit structure of the Drinfel’d double.

The map $\alpha : V \to V#L$ is given by $\alpha(v) = v#1$.

Then, we know that the map $f : L^\text{cop} \to D$, $f(p) = \sum p(R^1)R^2$ is an isomorphism of Hopf algebras, so $V$ becomes a left $L^\text{cop}$-module algebra via $f$, in particular it becomes a right $L$-comodule; denote by $\beta : V \to V \otimes L$ the structure map. If we denote by $\beta(v) = \sum v_0 \otimes v_1$, for $v \in V$, we know that this is equivalent to...
\[ p \cdot v = \sum p(v_1)v_0 \text{ for all } p \in L^*, \text{ where } p \cdot v = f(p) \cdot v = \sum p(R^1)R^2 \cdot v. \text{ So, } \beta \text{ is given by the formula } \]
\[ \beta(v) = \sum R^2 \cdot v \otimes R^1 \]

Then, exactly as in [25], using the properties of the \( R \)-matrix \( R \) and the quantum commutativity of \( V \), one can prove that \( \beta \) is an algebra anti-homomorphism and that \( \alpha(v)\beta(w) = \beta(w)\alpha(v) \) for all \( v, w \in V \).

Let us note that, for \( v, w \in V \) and \( \ell \in L \), we have in \( V \# L \):
\[ \beta(v)(w \# \ell) = \sum (R^2 \cdot v \# R^1)(w \# \ell) \]
\[ = \sum (R^2 \cdot v)((R^1)_1 \cdot w) \# (R^1)_2 \ell \]
\[ = \sum (R^2 \cdot v)(R^1 \cdot w) \# r^1 \ell \]
\[ = \sum w(r^2 \cdot v) \# r^1 \ell \]

(we have used the quantum commutativity of \( V \) and the fact that \( (\Delta \otimes id)(R) = \sum R^1 \otimes r^1 \otimes R^2 r^2 \))

The \( V - V \)-bimodule structure of \( V \# L \) is given by
\[ v \cdot (w \# \ell) = \alpha(v)(w \# \ell) = (v \# 1)(w \# \ell) = vw \# \ell \]
\[ (w \# \ell) \cdot v = \beta(v)(w \# \ell) = \sum w(r^2 \cdot v) \# r^1 \ell \]

The coproduct is given by
\[ \Delta : V \# L \to (V \# L) \otimes_V (V \# L) \]
\[ \Delta(w \# l) = \sum (v \# l_1) \otimes_V (1 \# l_2) \]

Obviously \( \Delta(1 \# 1) = (1 \# 1) \otimes_V (1 \# 1) \) and \( \Delta \) is a morphism of left \( V \)-modules. Let us prove that it is also a morphism of right \( V \)-modules. We calculate:
\[ \Delta((w \# l) \cdot v) = \Delta(\sum w(r^2 \cdot v) \# r^1 \ell) \]
\[ = \sum (w(r^2 \cdot v) \# (r^1)_1 l_1) \otimes_V (1 \# (r^1)_2 l_2) \]
\[ = \sum (w(R^2 r^2 \cdot v) \# R^1 l_1) \otimes_V (1 \# r^1 l_2) \]
\[ \Delta(w \# l) \cdot v = \sum (w \# l_1) \otimes_V (1 \# l_2) \cdot v \]
\[ = \sum (w \# l_1) \otimes_V (r^2 \cdot v \# r^1 l_2) \]
\[ = \sum (w \# l_1) \otimes_V (r^2 \cdot v \cdot (1 \# r^1 l_2) \]
\[ = \sum (w \# l_1) \cdot (r^2 \cdot v) \otimes_V (1 \# r^1 l_2) \]

(since the tensor product is over \( V \))
\[ = \sum (w(R^2 r^2 \cdot v) \# R^1 l_1) \otimes_V (1 \# r^1 l_2), \text{ q.e.d.} \]

The coassociativity of \( \Delta \) follows immediately from the one of \( \Delta_L \).

The compatibility of \( \Delta \) with the product of \( V \# L \) may be proved as in [25], once we prove the following identity which in [25] is proved using the structure of the Drinfel’d double:
\[ \sum \beta(l_2 \cdot v)(1 \# l_1) = (1 \# l)\beta(v) \]
for all \( v \in V, l \in L \). To prove this we compute:

\[
\sum \beta(l_2 \cdot v)(1 \# l_1) = \sum (R^2 l_2 \cdot v \# R^1)(1 \# l_1) = \sum R^2 l_2 \cdot v \# R^1 l_1
\]

(1 \# l) \beta(v) = \sum (1 \# l)(R^2 \cdot v \# R^1)

= \sum l_1 R^2 \cdot v \# l_2 R^1

= \sum R^2 l_2 \cdot v \# R^1 l_1

using the identity \( \Delta^{cop}(l)R = R \Delta(l) \).

The section \( \gamma : (V \# L) \otimes_V (V \# L) \rightarrow (V \# L) \otimes (V \# L) \) which is needed in this part of the proof (and also at the end) is given as in \cite{25} by

\[
\gamma((v \# l) \otimes_V (v' \# l')) = \beta(v')(v \# l) \otimes (1 \# l')
\]

Since the proof of this part in \cite{23} is rather tricky, let us give a proof using Xu’s equivalent condition. We have to prove first that

\[
\Delta(v \# l)(\beta(w) \otimes 1 - 1 \otimes \alpha(w)) = 0 \quad \text{in} \quad (V \# L) \otimes_V (V \# L)
\]

for all \( v, w \in V \) and \( l \in L \). We calculate:

\[
\Delta(v \# l)(\beta(w) \otimes 1 - 1 \otimes \alpha(w)) =
\]

\[
= \sum((v \# l_1) \otimes_V (1 \# l_2))((R^2 \cdot w \# R^1) \otimes (1 \# 1) - (1 \# 1) \otimes (w \# 1))
\]

\[
= \sum (v \# l_1)((R^2 \cdot w \# R^1) \otimes (1 \# 1) - \sum (v \# l_1) \otimes_V (1 \# l_2)(w \# 1))
\]

\[
= \sum (v(l_1 \cdot (R^2 \cdot w))) \# l_2 R^1 \otimes_V (1 \# l_3) - \sum (v \# l_1) \otimes_V (l_2 \cdot w \# l_3)
\]

\[
= \sum (v(l_1 \cdot (R^2 \cdot w))) \# l_2 R^1 \otimes_V (1 \# l_3) - \sum (v \# l_1) \otimes_V (l_2 \cdot w) \otimes (1 \# l_3)
\]

\[
= \sum (v(l_1 \cdot (R^2 \cdot w))) \# l_2 R^1 \otimes_V (1 \# l_3) - \sum (v \# l_1) \cdot (l_2 \cdot w) \otimes_V (1 \# l_3)
\]

(since the tensor product is over \( V \))

\[
= \sum (v(l_1 \cdot (R^2 \cdot w))) \# l_2 R^1 \otimes_V (1 \# l_3) - \sum (v(R^2 \cdot (l_2 \cdot w))) \# R^1 l_1 \otimes_V (1 \# l_3)
\]

(we used the formula for the right module structure of \( V \# L \))

\[
= \sum (v(l_1 R^2 \cdot w)) \# l_2 R^1 \otimes_V (1 \# l_3) - \sum (v(R^2 l_2 \cdot w)) \# R^1 l_1 \otimes_V (1 \# l_3)
\]

and this is zero because \( \Delta^{cop}(l)R = R \Delta(l) \).

Next we have to prove that

\[
\Delta((v \# l)(v' \# l')) = \Delta(v \# l) \Delta(v' \# l')
\]

in \( (V \# L) \otimes_V (V \# L) \) for all \( v, v' \in V \) and \( l, l' \in L \), and this is trivial.

The counit is given by

\[
\varepsilon : V \# L \rightarrow V
\]

\[
\varepsilon(v \# l) = \varepsilon(l)v
\]

and it is easy to see that it satisfies all the required properties.

So, we have obtained so far a bialgebroid structure on \( V \# L \) over \( V \). We shall construct its antipode. Let \( u = \sum S(R^2) R^1 \) be the canonical Drinfel’d element of \( H \) and define \( d_0 = S(u)^{-1} \) (this element is denoted by \( u_2 \) in \cite{14}). Then we know from
that $d_0$ has the following properties:

$$d_0 = \sum S^2(R^1)R^2$$

$$d_0^{-1} = \sum S^{-1}(R^1)R^2 = \sum R^1S(R^2)$$

$$S^2(h) = d_0hd_0^{-1} \text{ for all } h \in H$$

$$\Delta(d_0) = (R_{21}R)(d_0 \otimes d_0) = (d_0 \otimes d_0)(R_{21}R)$$

As in [25] one can prove that $d_0$ acts as an algebra isomorphism on $V$.

We shall prove that $\tau \beta = \alpha$. Let $v \in V$; we calculate:

$$\tau(\beta(v)) = \sum \tau(R^2 \cdot v\#R^1) = \sum (1\#S(R^1))\beta(d_0R^2 \cdot v)$$

$$= \sum (1\#S(R^1))(v^2d_0R^2 \cdot v\#r^1)$$

$$= \sum S((R^1)_2)r^2d_0R^2 \cdot v\#S((R^1)_1)r^1$$

$$= \sum S(\rho^1)r^2d_0R^2\rho^2 \cdot v\#S(R^1)r^1$$

(Here $\rho = R$ and we used the relation $(\Delta \otimes id)(R) = \sum R^1 \otimes \rho^1 \otimes R^2\rho^2$)

$$= \sum S(\rho^1)r^2S^2(R^2)d_0\rho^2 \cdot v\#S(R^1)r^1$$

(using $S^2(x) = d_0xd_0^{-1}$)

$$= \sum S(\rho^1)r^2S(R^2)d_0\rho^2 \cdot v\#R^1r^1$$

(since $(S \otimes S)(R) = R$, see [44])

$$= \sum S(\rho^1)d_0\rho^2 \cdot v\#1$$

(since $R^{-1} = \sum R^1 \otimes S^{-1}(R^2)$, see [44])

$$= \sum S(\rho^1)S^2(\rho^2)d_0 \cdot v\#1$$

(again using $S^2(x) = d_0xd_0^{-1}$)

$$= \sum \rho^1S(\rho^2)d_0 \cdot v\#1$$

(again since $(S \otimes S)(\rho) = \rho$)

$$= v\#1$$

(since $d_0^{-1} = \sum \rho^1S(\rho^2)$)

$$= \alpha(v), \text{ q.e.d.}$$

The inverse of $\tau$ is given by

$$\tau^{-1} : V\#L \to V\#L$$

$$\tau^{-1}(v\#l) = (1\#S^{-1}(l))\beta(v)$$

One can prove by a direct computation that $\tau^{-1}$ is an algebra anti-homomorphism, then using this one can prove that $\tau^{-1} \tau = id$ and finally that $\tau\tau^{-1} = id$ by a
computation similar to the proof of $\tau \beta = \alpha$. Hence $\tau$ is an algebra anti-isomorphism.
The other axioms for $\tau$ are proved exactly as in [24].

**Remark 5.9**

a) the proof shows that the statement in the theorem remains valid if we replace $L$ by any Hopf subalgebra of $H$ containing $L$.
b) if $A$ is a finite dimensional Hopf algebra and $(D(A), R)$ its Drinfel’d double, it is easy to see that $R(l) = \{ \varepsilon \otimes a / a \in A \}$, which is isomorphic as Hopf algebras to $A$, so indeed the above result generalizes Lu’s theorem.
c) one can check that, in the case $H = D(A)$ and $V = A^*$, the formula $\tau^{-1}(v\#l) = (1\#S^{-1}(l))\beta(v)$ for the inverse of the antipode of $\mathcal{H}(A^*)$ may be written as
\[
\tau^{-1}(p\#a) = \sum (S^{-1}(a) \mapsto (e^j p)) e^j \# S^{-1}(e_j) e_i
\]
for all $a \in A$ and $p \in A^*$, where $\{e_i\}$ is a basis in $A$ and $\{e^i\}$ its dual basis in $A^*$ (the formula for $\tau$ was given in [24]).
d) the generalization of Lu’s theorem in [3] (independently obtained, as we mentioned before) is more general than ours: it concerns quantum commutative algebras in categories of Yetter-Drinfel’d modules.

Now, let $A$ be a finite dimensional Hopf algebra. As we have seen before, $\mathcal{H}(A)$ is a quantum commutative left $D(A) \otimes D(A)^{op\, cop}$-module algebra, so the above theorem may be applied. Denote by $\mathcal{R} = \sum (R^1 \otimes U^2) \otimes (R^2 \otimes U^1)$ the $R$-matrix of $D(A) \otimes D(A)^{op\, cop}$, where $R$ is the $R$-matrix of $D(A)$ and $U = R^{-1}$. Since $R^{-1} = \sum S(R^1) \otimes R^2$ (see [4]) and the antipode of $D(A)$ has the property that $S_{D(A)}(\varepsilon \otimes a) = \varepsilon \otimes S(a)$ for all $a \in A$, it follows that $\mathcal{R}$ is given by $\mathcal{R} = \sum (\varepsilon \otimes e_i \otimes e^j \otimes 1) \otimes (e^i \otimes 1 \otimes \varepsilon \otimes S(e_j))$, where $\{e_i\}$ is a basis in $A$ and $\{e^i\}$ its dual basis in $A^*$. Then it is easy to see that the set $\{\varepsilon \otimes e_i \otimes e^j \otimes 1\}$ is a $k$-linear basis in $\mathcal{R}_{(l)}$, so $\mathcal{R}_{(l)}$ may be identified as linear spaces with $A \otimes A^*$, and from the Hopf algebra structure of $D(A) \otimes D(A)^{op\, cop}$ we can see that actually $\mathcal{R}_{(l)}$ may be identified with $A \otimes A^{*op}$ as Hopf subalgebras of $D(A) \otimes D(A)^{op\, cop}$. Some calculations with the explicit formulae given before for the left and right regular actions of $D(A)$ on $\mathcal{H}(A)$ show that the left $A \otimes A^{*op}$-module structure of $\mathcal{H}(A)$ is given by:
\[
(a \otimes p) \cdot (b \# q) = (b \leftarrow p) \# (a \rightarrow q) = \sum p(b_1)q_2(a)(b_2\#q_1)
\]
for all $a, b \in A$ and $p, q \in A^*$. In conclusion, we obtain:

**Corollary 5.10** $\mathcal{H}(A)\#(A \otimes A^{*op})$ is a Hopf algebroid over $\mathcal{H}(A)$.

### 6 Something like a vertex group

We start with the following definition, which is a variation of the one in [3].
Definition 6.1 Let $H$ be a cocommutative Hopf algebra. A vertex group over $H$ is an $H$-bimodule algebra $K$ (with actions denoted by $h \otimes k \mapsto h \cdot k$ and $k \otimes h \mapsto k \cdot h$ for all $h \in H$ and $k \in K$) together with a map $\alpha : H^* \to K$ which is a morphism of $H$-bimodule algebras ($H^*$ is an $H$-bimodule algebra via the left and right regular actions) and an algebra anti-isomorphism $\tau : K \to K$ satisfying the conditions:

$$\tau(\alpha(p)) = \alpha(S(p))$$
$$\tau(h \cdot k) = \tau(k) \cdot S(h)$$
$$\tau(k \cdot h) = S(h) \cdot \tau(k)$$

for all $h \in H, p \in H^*, k \in K$, where we denoted by $S$ the antipode of $H$ and also the antipode of $H^*$. The map $\tau$ will be called the antipode of $K$. If $K$ is commutative and $\tau^2 = id$ we recover Definition 3.2 in [6].

Remark 6.2 Of course, $H^*$ is a vertex group over $H$ with $\alpha = id$ and $\tau = S$.

Proposition 6.3 If $A$ is a finite dimensional cocommutative Hopf algebra, then $H(A)$ is a vertex group over $A$.

Proof: define $\alpha : A^* \to H(A) = A#A^*$, $\alpha(p) = 1#p$, which is of course an algebra map. We have seen that $H(A)$ is a $D(A) - D(A)^{op}$-bimodule algebra; by restricting the actions of $D(A)$ on $H(A)$ to actions of $A$ and since $A$ is cocommutative, we obtain that $H(A)$ is an $A$-bimodule algebra.

Take $\{e_i\}$ a basis in $A$ and $\{e^i\}$ its dual basis in $A^*$ and define $\tau : A#A^* \to A#A^*$ by

$$\tau(a#p) = \sum (S(p) \rightarrow (e_i a)) e_j S(e^j) e^i$$

This is just the map $\tau^{-1}$ in the previous section written for $H(A)$ instead of $H(A^*)$ (we have used also the fact that $S^2 = id$); we already know that this is an algebra anti-isomorphism.

Let us check the other axioms of a vertex group. In order to simplify the calculations we shall use the version of Sweedler’s sigma notation introduced in [32] for dealing with cocommutative Hopf algebras, namely for $a \in A$ we shall denote $\Delta(a) = \sum a \otimes a$, $(id \otimes \Delta) \Delta(a) = \sum a \otimes a \otimes a$ etc. With this notation, the antipode axiom for $A$ is written as $\sum S(a) a = \sum a S(a) = \varepsilon(a) 1$ for all $a \in A$.

Let us prove that $\alpha$ is a bimodule map, namely

$$\alpha(a \rightarrow p \leftarrow b) = a \rightarrow \alpha(p) \leftarrow b$$

for all $a, b \in A$ and $p \in A^*$, that is

$$1#a \rightarrow p \leftarrow b = (\varepsilon \otimes a) \rightarrow (1#p) \leftarrow (\varepsilon \otimes b)$$

where $\varepsilon \otimes a$ and $\varepsilon \otimes b$ are considered as elements in the Drinfel’d double of $A$. By evaluating in an element $\varphi \otimes x \in A^* \otimes A$, the lhs becomes $\varphi(1)p(bxa)$ and the rhs
may be calculated as:

\[
(1\#p)((\varepsilon \otimes b)(\varphi \otimes x)(\varepsilon \otimes a))
= (1\#p)((\varepsilon \otimes b)(\varphi \otimes xa))
= (1\#p)(b \rightarrow \varphi \leftarrow S(b) \otimes bxa)
= \varphi(S(b)b)p(bxa) = \varphi(1)p(bxa), \text{ q.e.d.}
\]

It is very easy to see that \(\tau(1\#p) = 1\#S(p)\), so let us check the last two axioms for \(\tau\). From the general formulae for the actions of \(D(A)\) on \(\mathcal{H}(A)\) given in a previous section we can write down the actions of \(A\) on \(\mathcal{H}(A)\):

\[
a \mapsto (b\#p) = \sum p_2(a)(b\#p_1)
(b\#p) \leftarrow a = \sum p_1(a)(S(a)ba\#p_2)
\]

for all \(a, b \in A\) and \(p \in A^*\). We calculate:

\[
\tau(a \mapsto (b\#p)) = \sum p_2(a)\tau(b\#p_1)
= \sum p_2(a)(S(p_1) \rightarrow (e_i b))e_j#S(e^i)e^j
\]

which evaluated in an element \(\varphi \otimes x \in A^* \otimes A\) gives

\[
\sum p(S(b)S(x)a)\varphi(xbS(x))
\]

On the other hand, we have:

\[
\tau(b\#p) \leftarrow S(a) =
= \sum((S(p) \rightarrow (e_i b))e_j#S(e^i)e^j) \leftarrow S(a)
= \sum S((e^j)_{2})(S(a))(e^i_1(S(a))(a)(S(p) \rightarrow (e_i b))e_jS(a)#S((e^j)_{1})(e^i)_{2})
\]

which evaluated in \(\varphi \otimes x \in A^* \otimes A\) gives:

\[
\sum((e^j)_{2})(S(a))(e^i_1(S(a))p(S(b)S(x))(e^j_1(S(a))(e^j)_{1}(S(x))(e^i)_{2}(x))
= \sum e^j(S(a)x)e^i(S(x)a)p(S(b)S(e_i))(e^j_1(S(a))p(S(b)S(x)e_i))\varphi(ae_i,bS(a))
= \sum p(S(b)S(x)a)\varphi(aS(a)xS(x)aS(a))
= \sum p(S(b)S(x)a)\varphi(xbS(x))
\]

so we have obtained \(\tau(a \mapsto (b\#p)) = \tau(b\#p) \leftarrow S(a)\).

Now we prove the last relation.

\[
\tau((b\#p) \leftarrow a) = \sum p_1(a)\tau(S(a)ba\#p_2)
= \sum p_1(a)((S(p_2) \rightarrow (e_i b))e_j#S(e^i)e^j)
= \sum p_1(a)p_2(S(a)S(b)aS(e_i))(e_i S(a)bae_j #S(e^j)e^j)
= \sum p(S(b)aS(e_i))(e_i S(a)bae_j #S(e^j)e^j)
\]

which evaluated in \(\varphi \otimes x \in A^* \otimes A\) gives

\[
\sum p(S(b)aS(x))\varphi(S(a)baS(x))
\]

\[
S(a) \rightarrow \tau(b\#p) = \sum p(S(b)S(e_i))S(a) \rightarrow (e_i be_j #S(e^i)e^j)
= \sum p(S(b)S(e_i))S((e^j)_{1})(S(a))(e^i)_{2}(S(a))(e_i be_j #S((e^j)_{2})(e^i)_{1})
\]
which evaluated in \( \varphi \otimes x \in A^* \otimes A \) gives:

\[
\sum p(S(b)S(e_i))e^i(xS(a))e^j(aS(x))\varphi(e_ibe_j)
\]

\[
= \sum p(S(b)aS(x))\varphi(xS(a)baS(x))
\]

so that \( \tau((b\#p) \leftarrow a) = S(a) \rightarrow \tau(b\#p) \), q.e.d.

7 The Heisenberg double for quasi-Hopf algebras

In this section we return to quasi-Hopf algebras and suggest a possible definition for Heisenberg doubles of them.

If \( A \) is a finite dimensional quasi-Hopf algebra, it was proved in [28] that there exists a unique (up to isomorphism) quasitriangular quasi-Hopf algebra \((D(A), R)\) with the property that the category \( D(A)\text{-mod} \) is braided equivalent to the centre of the tensor category \( A\text{-mod} \). This \( D(A) \) is called the quantum double of \( A \) (it generalizes the Drinfel’d double for Hopf algebras). Some concrete realizations of \( D(A) \) (on \( A^* \otimes A \) and \( A \otimes A^* \)) have been given in [19], [20].

Now we can propose the following

**Definition 7.1** The Heisenberg double of \( A \) is \( D(A^*)_R \), which is, as we know, a quantum commutative \( D(A) - D(A)^{\text{op}} \)-bimodule algebra. It is denoted by \( \mathcal{H}(A) \).

From a previous discussion and the uniqueness of \( D(A) \), it follows that \( \mathcal{H}(A) \) is “unique up to isomorphism”.

Because of the complicated structure of \( D(A) \), it is quite difficult to write down explicitly the formula for the multiplication in \( \mathcal{H}(A) \), and we shall not do this here. Instead, we shall do this in a particular case, related to the Dijkgraaf-Pasquier-Roche quasi-Hopf algebras \( D^\omega(G) \) introduced in [9]. We shall work in the slightly more general situation of [9] and we begin by recalling the relevant parts of the construction in [9].

Let \( H \) be a finite dimensional cocommutative Hopf algebra (for which we shall use the variation of \( \Sigma \)-notation as in the previous section) with antipode \( S \), and \( \omega : H \otimes H \otimes H \rightarrow k \) a normalized 3-cocycle in the Sweedler cohomology [15], that is \( \omega \) is \( k \)-linear, convolution invertible and satisfies the conditions:

\[
\sum \omega(x, y, z)t\omega(xy, z, t) = \sum \omega(x, y, t)\omega(x, yz, t)\omega(x, y, z)
\]

\[
\omega(1, x, y) = \omega(x, 1, y) = \omega(x, y, 1) = \varepsilon(x)\varepsilon(y)
\]

for all \( x, y, z, t \in H \). Introduce also the following notation: \( g \triangleleft x = \sum S(x)gx \) for all \( g, x \in H \). Then, on the \( k \)-linear space \( H^* \otimes H \) may be constructed a quasitriangular quasi-Hopf algebra, denoted by \( D^\omega(H) \), for which the multiplication is given by

\[
(p \otimes h)(p' \otimes h') = \sum p(h \rightarrow p' \leftarrow S(h))\sigma(h, h') \otimes hh'
\]

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for all \( p, p' \in H^* \) and \( h, h' \in H \), where \( \sigma : H \otimes H \to H^* \) is given by

\[
\sigma(x, y)(g) = \theta(g; x, y)
\]

for all \( x, y, g \in H \), where \( \theta : H \otimes H \otimes H \to k \),

\[
\theta(g; x, y) = \sum \omega(g, x, y)\omega(x, y, g \triangleleft (xy))\omega^{-1}(x, g \triangleleft x, y)
\]

for \( g, x, y \in H \), where \( \omega^{-1} \) is the convolution inverse of \( \omega \).

Define the map \( \gamma : H \otimes H \otimes H \to k \) by

\[
\gamma(g, h; x) = \sum \omega(g, h, x)\omega(x, g \triangleleft x, h \triangleleft x)\omega^{-1}(g, x, h \triangleleft x)
\]

for \( x, g, h \in H \), and the map \( \nu : H \to (H \otimes H)^* \), by \( \nu(h)(x \otimes y) = \gamma(x, y; h) \).

Identifying \((H \otimes H)^*\) with \( H^* \otimes H^* \), we shall write, for any \( h \in H \), \( \nu(h) = \sum \nu_1(h) \otimes \nu_2(h) \in H^* \otimes H^* \), and this relation is equivalent to \( \nu(h)(x \otimes y) = \sum \nu_1(h)(x)\nu_2(h)(y) \) for all \( x, y \in H \). Then the comultiplication of \( D^\omega(H) \) is given by

\[
\Delta : D^\omega(H) \to D^\omega(H) \otimes D^\omega(H)
\]

\[
\Delta(p \otimes h) = \sum (\nu_1(h)p_1 \otimes h) \otimes (\nu_2(h)p_2 \otimes h)
\]

The associator of \( D^\omega(H) \) is \( \omega^{-1} \) (regarded in \((H^*)^{\otimes 3} \subseteq D^\omega(H)^{\otimes 3}\) and the \( R \)-matrix is

\[
R = \sum (e^i \otimes 1) \otimes (\varepsilon \otimes e_i)
\]

where \( \{e_i\} \) is a basis in \( H \) and \( \{e^i\} \) its dual basis in \( H^* \). Recall also from [9] the following identity:

\[
\sum \gamma(x, y \triangleleft x; h)\theta(y; x, h) = \sum \theta(y; h, x \triangleleft h)\gamma(y, x; h)
\]

It was proved in [37] that \( D^\omega(H) \) is (isomorphic to) the quantum double of a certain quasi-Hopf algebra denoted by \( H^*_\omega \), which is just \( H^* \) with its usual Hopf algebra structure but with a nontrivial associator, namely \( \omega^{-1} \).

We can recover the DPR quasi-Hopf algebra \( D^\omega(G) \) in [13] by taking \( H \) to be the group algebra of a finite group \( G \).

Let us note that if \( \omega \) is trivial, then \( D^\omega(H) \) is a Hopf algebra, isomorphic to the Drinfel’d double of \( H^* \); but since we worked with the realization of \( D^\omega(H) \) on \( H^* \otimes H \) rather than \( H \otimes H^* \), we can see from the explicit formulae given above that \((D^{\text{triv}}(H), R) = (D(H)^{\text{cop}}, R_{21})\), where \( D(H) \) is the usual Drinfel’d double of \( H \) and \( R \) in the rhs is the \( R \)-matrix of \( D(H) \).

We can compute now the multiplication in \( D^\omega(H)^*_R \) (which is, by our definition, the Heisenberg double of \( H^*_\omega \)). Take \( x \otimes \varphi, x' \otimes \varphi' \in D^\omega(H)^*_R \) and evaluate their product against an element \( p \otimes h \in D^\omega(H) \):

\[
((x \otimes \varphi) \cdot (x' \otimes \varphi'))(p \otimes h) =
\]

\[
= \sum (x' \otimes \varphi')(p \otimes h_1 R^2)(x \otimes \varphi)((p \otimes h_2) R^1)
\]

\[
= \sum (x' \otimes \varphi')((\nu_1(h)p_1 \otimes h)(\varepsilon \otimes e_i))(x \otimes \varphi)((\nu_2(h)p_2 \otimes h)(e^i \otimes 1))
\]

\[
= \sum (x' \otimes \varphi')((\nu_1(h)p_1 \otimes h)(e_i \otimes he_i))(x \otimes \varphi)((\nu_2(h)p_2 \otimes h)(e^i \otimes 1) S(h) \otimes h)
\]
\[ \sum \nu_1(h)(x')p_1(x')\sigma(h,e_i)(x')\varphi'(he_i)\nu_2(h)(x)p_2(x)e^i(S(h)xh)\varphi(h) \]
\[ = \sum p(x'x)\varphi'(xh)\varphi(h)\gamma(x',x;h)\theta(x';h,x \lhd h) \]
\[ = \sum p(x'x)\varphi'(xh)\varphi(h)\gamma(x,x' \lhd x;h)\theta(x';x,h) \]

So, we have obtained the following formula:

\[ (x \otimes \varphi) \cdot (x' \otimes \varphi') = \sum x'(\varphi'_1 \rightarrow x) \otimes \varphi'_2\varphi \gamma(x,x' \lhd x;\cdot)\theta(x';x,\cdot) \]

Let us note that if \( \omega \) is trivial this formula becomes

\[ (x \otimes \varphi) \cdot (x' \otimes \varphi') = \sum x'(\varphi'_1 \rightarrow x) \otimes \varphi'_2\varphi \]

which is the multiplication of \( \mathcal{H}(H)^{op} \) (this is not a surprize, since \( (D^{triv}(H),R) = (\mathcal{D}(H)^{cop},R_{21}) \)), which in turn is naturally isomorphic to \( \mathcal{H}(H^*) \) (see [24]), so that \( \mathcal{D}^\omega(H)_R \) is indeed a generalization of the usual Heisenberg double of \( H^* \).

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