Quasilocal energy-momentum for tensors $B$ and $V$

in small regions

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Abstract

The Bel-Robinson tensor $B$ and the tensor $V$ have the same quasilocal energy-momentum in a small sphere. Using a pseudotensor approach to evaluate the energy-momentum in a half-cylinder, we find that $B$ and $V$ have different values, not proportional to the “Bel-Robinson energy-momentum”. Furthermore, even if we arrange things so that we do get the same “Bel-Robinson energy-momentum” value, the angular momentum gives different values using $B$ and $V$ in a half cylinder. In addition, we find that $B$ and $V$ have a different number of independent components. The fully trace free property of $B$ and $V$ implies conservation of pure “Bel-Robinson energy-momentum” in small regions, and vice versa. In addition, we also demonstrate the tidal heating, rate of change of momentum and spin angular momentum flux by using these two tensors.

1 Introduction

In attempts to identify a good physical expression for the local distribution of gravitational energy-momentum there have been many different approaches which are similar to Einstein’s [1]. For example, those of Landau-Lifshitz [2], Bergmann-Thomson [3], Papapetrou [4] and Weinberg [5]. Most of them deal with the Einstein equation: $G_{\mu \nu} = \kappa T_{\mu \nu}$, where $\kappa$ is a constant, $G_{\mu \nu}$ and $T_{\mu \nu}$ are the Einstein and stress tensors. One can define a superpotential with a suitable anti-symmetry $U_{\alpha \mu \nu} \equiv U_{\alpha [\mu \nu]}$ and remove a divergence of $U_{\alpha \mu \nu}$ from $G_{\mu \nu}$ to define the gravitational energy-momentum density

$$2\kappa t_{\alpha \mu} := \partial_\nu U_{\alpha [\mu \nu]} - 2\sqrt{-g} G_{\alpha \mu}, \tag{1}$$

Note that $t_{\alpha \mu}$ is a pseudotensor [6]. Using the Einstein equation, we have a total energy-momentum density which satisfies

$$\partial_\nu U_{\alpha [\mu \nu]} = 2\kappa T_{\alpha \mu} = 2\kappa \sqrt{-g}(T_{\alpha \mu} + t_{\alpha \mu}), \tag{2}$$

where $T_{\alpha \mu} = \sqrt{-g} T_{\alpha \mu}$ and hence, due to antisymmetry of $U_{\alpha [\mu \nu]}$, is automatically conserved, i.e., has a vanishing divergence.

The proposed criteria for testing quasilocal expressions included: (i) limit to good weak field values (i.e., linearized gravity). (ii) good asymptotic values both at spatial and null infinity. We here emphasize that the criteria for these two are not very restrictive; they only test the quasilocal expression to linear order. (iii) positivity (i.e., globally) is a strong test but is not easy to achieve, (iv) small region inside of matter: the quasilocal energy-momentum expression should, by the equivalence principle, reduce to the material source terms. Most classical pseudotensors pass this test. (v) small region vacuum: positivity for the first non-vanishing parts of the quasilocal expression. This depends on the gravitational field non-linearly, and hence it can give a discriminating test of the expression, it is quite non-trivial but not impossibly difficult.
Positive quasilocal gravitational energy should hold not only on a large scale but also on the small scale [7]. However it is generally not at all easy to prove that a particular expression enjoys this property. A good test case is the small region limit. This will be our concern in this work. Here we consider specifically the pseudotensor expressions. For a small region, one can expand the energy-momentum density in Riemann normal coordinates (RNC) about the origin:

\[
\mathcal{T}_\alpha^\beta(x) = \mathcal{T}_\alpha^\beta|_0 + \partial_\mu \mathcal{T}_\alpha^\beta|_0 x^\mu + \frac{1}{2} \partial_\mu^2 \mathcal{T}_\alpha^\beta|_0 x^\mu x^\nu + \ldots
\]

\[
= \mathcal{T}_\alpha^\beta|_0 + \partial_\mu \mathcal{T}_\alpha^\beta|_0 x^\mu + \ldots + t_\alpha^\beta|_0 + \partial_\mu t_\alpha^\beta|_0 x^\mu + \frac{1}{2} \partial_\mu^2 t_\alpha^\beta|_0 x^\mu x^\nu + \ldots (3)
\]

By construction \( t_\alpha^\beta|_0 \) and \( \partial_\mu t_\alpha^\beta|_0 \) vanish in vacuum. Consequently, for small \( x^\mu \) inside of matter the \( \mathcal{T}_\alpha^\beta \) and \( \partial_\mu \mathcal{T}_\alpha^\beta \) terms dominate (this is a reflection of the equivalence principle). In vacuum regions all the \( \mathcal{T}_\alpha^\beta \) terms vanish, then the lowest order non-vanishing term is \( \frac{1}{2} \partial_\mu^2 \mathcal{T}_\alpha^\beta|_0 x^\mu x^\nu \). This is the object on which we focus our attention in this work. It turns out that for all proposed pseudotensor and quasilocal energy-momentum expressions this fourth rank tensor is quadratic in the Riemann (equivalent in empty space regions to the Weyl) tensor. That is why the quadratic curvature expressions become interesting and important (i.e., \( \partial_\mu^2 \mathcal{T}_\alpha^\beta \approx R_{\ldots R_{\ldots}} \)).

Normally, the expansion of a pseudotensor expression up to second order can only be some linear combination of three tensors \( \{ B, S, K \} \) or \( \{ B, V, S \} \) \([6, 8, 9]\) which are each certain quadratic expressions in the curvature.

According to a review article (4.2.2 in [7]): “Therefore, in vacuum in the leading \( r^5 \) order any coordinate and Lorentz-covariant quasilocal energy-momentum expression which is non-spacelike and future pointing must be proportional to the Bel-Robinson ‘momentum’: \( B_{\mu\lambda\xi\kappa} t^\lambda t^\xi t^\kappa \).” Note that here \( t^\alpha \) is timelike unit vector and ‘momentum’ means 4-momentum (see (28)). This is a strong test. The Bel-Robinson tensor \( B \) has many nice properties such as fully symmetric, traceless and divergence free \([10]\).

It is known that \( B \) contributes positivity in a small sphere region and perhaps it maybe thought that it is the only one. However, we recently proposed an alternative \( V \) (see (18)) which has the identical “Bel-Robinson momentum” at the same limit, i.e., \( (B_{\mu\lambda\xi\kappa} - V_{\mu\lambda\xi\kappa}) t^\lambda t^\xi t^\kappa \equiv 0 \). Confined to a small spherical or cubical regions \([11]\), \( B \) and \( V \) cannot be distinguished. One may suspect that \( V \) is redundant because \( B \) can manage all the jobs, But we claim not.

As the basic requirement for the quasilocal energy is any closed 2-surface, we examined the energy-momentum and angular momentum in other regions (see Table 1). we find for the energy in a small half-cylinder when \( h \neq \sqrt{3}a \) give different values if substituting \( t \) by \( B \) and \( V \), which means that they are distinguishable. Only for one particular ratio \( h = \sqrt{3}a \), \( B \) and \( V \) both give the same “Bel-Robinson momentum” value, however we lose the distinction between them again. Therefore we turn to examining the angular momentum in a small half-cylinder, and show that when we replacing \( t \) by \( B \) and \( V \) in the angular momentum expression they contribute different values, thereby clarifying that the two tensors are really distinguishable.

Here we remark some components of the angular momentum in a hemi-sphere show that \( B \) contributes a null result while \( V \) gives non-zero values (see section 3.2). The reason comes from the fully symmetric property of \( B \), while \( V \) only has some certain symmetry property (see (19)). Consequently, \( V \) is non-replaceable.

Moreover, we also demonstrate the tidal heating, rate of change of momentum and spin angular momentum flux by using tensors \( B \) and \( V \). Once again, they have the same results.
2 Technical background

Using a Taylor series expansion, the metric tensor can be written as
\[ g_{\alpha\beta}(x^\lambda) = g_{\alpha\beta}|_{x_0^\lambda} + \partial_\mu g_{\alpha\beta}|_{x_0^\lambda}(x^\mu - x_0^\mu) + \frac{1}{2} \partial^2_{\mu\nu} g_{\alpha\beta}|_{x_0^\lambda}(x^\mu - x_0^\mu)(x^\nu - x_0^\nu) + \ldots, \]
where the metric signature is +2. For simplicity, let \( x_0^\lambda = 0 \) and at the origin in RNC
\[ g_{\alpha\beta}|_0 = \delta_{\alpha\beta}, \quad \partial_\mu g_{\alpha\beta}|_0 = 0, \]
\[ -3\partial^2_{\mu\nu} g_{\alpha\beta}|_0 = R_{\alpha\beta\lambda\nu} + R_{\alpha\nu\lambda\beta} - 3\partial_\nu \Gamma^\mu_{\alpha\beta}|_0 = R^\mu_{\alpha\beta\nu} + R^\mu_{\beta\alpha\nu}. \]

Three basic tensors [6 8 9] that commonly occurred in the pseudotensor are:
\[ B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2_{\lambda\sigma\rho\tau}, \]
\[ S_{\alpha\beta\mu\nu} := S_{(\alpha\beta)(\mu\nu)} = R_{\alpha\lambda\mu\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R^2_{\lambda\sigma\rho\tau}, \]
\[ K_{\alpha\beta\mu\nu} := K_{(\alpha\beta)(\mu\nu)} = K_{\mu\nu\alpha\beta} := R_{\alpha\lambda\mu\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} R^2_{\lambda\sigma\rho\tau}, \]
where \( R^2_{\lambda\sigma\rho\tau} = R^{\lambda\sigma\rho\tau} R_{\lambda\sigma\rho\tau} \).

It may be worthwhile to emphasize that \( B \) has a very good analog with the electromagnetic energy-momentum tensor \( T^{\mu\nu} \). In Minkowski coordinates \((t, x, y, z)\):
\[ T^{00} = \frac{1}{2} (E_a E^a + B_a B^a), \]
\[ T^{0i} = \delta^{ij} \epsilon_{jab} E^a B^b = (\vec{E} \times \vec{B})^i, \]
\[ T^{ij} = \frac{1}{2} \left[ \delta^{ij} (E_a E^a + B_a B^a) - 2 \left( E^i E^j + B^i B^j \right) \right]. \]

where \( \vec{E} \) and \( \vec{B} \) refer to the electric and magnetic field density. In order to appreciate the nice properties of \( B \), we compare the energy density with \( S \) and \( K \)
\[ B_{0000} = E_{ab}^2 + H_{ab}^2, \quad S_{0000} = 2(E_{ab}^2 - H_{ab}^2), \quad K_{0000} = -E_{ab}^2 + 3H_{ab}^2. \]
where the evaluation has used the electric part \( E_{ab} \) and magnetic part \( H_{ab} \), defined in terms of the Weyl tensor [12]. \( E_{ab} := C_{a000} \) and \( H_{ab} := *C_{a000} \) where \(*C_{a\beta\mu\nu} \) means its dual. Likewise for the momentum density (i.e., Poynting vector)
\[ B_{000i} = 2\epsilon_{ij} k E^j d H^k d, \quad S_{000i} = 0, \quad K_{000i} = 2\epsilon_{ij} k E^j d H^k d. \]

Finally, the stress,
\[ B_{00ij} = \delta_{ij}(E_a E^a + H_a H^a) - 2(E_i d E^d j + H_id H^d j), \]
\[ S_{00ij} = -2 \left[ \delta_{ij}(E_a E^a - H_a H^a) + 2(E_i d E^d j - H_id H^d j) \right]. \]
\[ K_{00ij} = \delta_{ij}(5E_a E^a - 3H_a H^a) - 4E_i d E^d j. \]

We observe that summing up \( S \) and \( K \) has exactly the same energy as \( B \): \((B_{0000} - S_{0000} - K_{0000}) = 0 \equiv (B_{000j} - S_{000j} - K_{000j})\delta^{ij} \). It is natural to define the alternative 4th rank tensor [9] as follows
\[ V := S + K \equiv B + W, \]
where
\[ W_{\alpha\beta\mu\nu} := \frac{3}{2} S_{\alpha\beta\mu\nu} - \frac{1}{8} (5g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) R^2_{\lambda\sigma\rho\tau}. \]  

(19)

This is the tensor that we prefer to focus on. Both \( V \) and \( W \) satisfy the following properties
\[ X_{\alpha\beta\mu\nu} \equiv X_{(\alpha\beta)(\mu\nu)} \equiv X_{\mu\nu\alpha\beta}, \quad X_{\alpha\beta\mu} \equiv 0 \equiv X_{\alpha\beta\mu}. \]  

(20)

It is known \( B \) has the dominant energy property \[ B_{\alpha\beta\mu\nu} w_1^\alpha w_2^\beta w_3^\mu w_4^\nu \geq 0, \]
where \( w_1, w_2, w_3, w_4 \) are any future-pointing causal vectors. Intuitively, referring to \[ \text{(18)}, \]
\( V \) may contain more non-trivial independent components than \( B \) and indeed it is the case (see section (3.3)). While \( V \) only satisfies the weak energy condition and \( W \) fulfills none of them. For \( W \), it does not contribute energy-momentum in small sphere. For a comparison of \( B \) and \( V \), we find that it is more convenient to use \( (B + W) \) instead of \( (S + K) \) for the representation of \( V \).

In our work, we are mainly dealing with expression of the 4th rank which are quadratic in the curvature tensor. There are four tensors which form a basis with appropriate symmetries \[ \text{(14)}, \]
we use
\[ \tilde{B}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\nu} R_{\beta}^{\lambda} \sigma + R_{\alpha\lambda\nu}\sigma R_{\beta}^{\lambda} \mu, \quad \tilde{S}_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\nu}\sigma R_{\beta}^{\mu} \lambda + R_{\alpha\mu\lambda}\sigma R_{\beta}^{\mu} \lambda (21) \]
\[ K_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\mu} R_{\nu}^{\lambda} \sigma + R_{\alpha\lambda\beta}\sigma R_{\nu}^{\lambda} \mu, \quad \tilde{T}_{\alpha\beta\mu\nu} := -\frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2_{\lambda\sigma\rho\tau}. \]  

(22)

They are designed to describe the gravitational energy expression based on the pseudotensor (see \text{(30)}) and are manifestly symmetric in the last two indices, i.e., \( \tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{\alpha\beta}(\mu\nu) \). Then \( \tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{(\alpha\beta)(\mu\nu)} \) and it also naturally turns out that \( \tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{\mu\nu\alpha\beta} \). In addition, here we write down another form of the representation for this Bel-Robinson tensor:
\[ B_{\alpha\beta\mu\nu} \equiv -\frac{1}{2} S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} + \frac{1}{8} (5g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) R^2_{\lambda\sigma\rho\tau}. \]  

(23)

Here come to the situation for applying the tidal heating, rate of change of momentum and time dependent spin angular momentum by using the tensors \( B \) and \( V \). In weak field the metric tensor can be decomposed as \( g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \), and its inverse \( g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \). Here we mainly use the first order and ignore the higher orders. According to Zhang \[ \text{(15)}, \]
the metric components can be written as
\[ h^{00} = \frac{3}{r^5} I_{ab} x^a x^b - E_{ab} x^a x^b, \]  

(24)
\[ h^{0j} = \frac{4}{r^3} \epsilon^{ij} \epsilon^p q B^p q x^a x^b + \frac{2}{3} \epsilon^{ij} \epsilon^p q B^p q x^a x^j + \frac{2}{r^3} \epsilon^{ij} \epsilon^a q x^a x^j + \frac{10}{21} \dot{E}_{ab} x^a x^b x^j - \frac{4}{21} \dot{E}^j q x^a x^j, \]  

(25)
\[ h^{ij} = \eta^{ij} h^{00} + \tilde{h}^{ij}, \]  

(26)
where
\[ \tilde{h}^{ij} = \frac{8}{3r^3} \epsilon_{pq} (i) \epsilon^{ij} \epsilon^p q x^q + \frac{5}{21} x^j (i) \epsilon^p q \dot{B}^q x^p x^j - \frac{1}{21} r^2 \epsilon_{pq} (i) \dot{B}^j q x^p. \]  

(27)

Zhang used \( \tilde{h}^{ij} \) for the manipulation while we prefer using \( h^{ij} \), the transformation is as follows
\[ \tilde{h}^{ij} = h^{ij} - \frac{1}{2} \eta^{ij} \dot{h}. \]  

(28)
The corresponding first order harmonic gauge is $\partial_\gamma \tilde{h}^{\alpha\beta} = 0$. Moreover, we will substitute the mass quadrupole moment $I_{ij}$ and current quadrupole moment $J_{ij}$ as determined by Poisson \[16\]:

$$I^{ij} = \frac{32}{45} M^6 \hat{E}^{ij}, \quad J^{ij} = \frac{8}{15} M^6 \hat{B}^{ij},$$  \hspace{1cm} \text{(29)}$$

where $M$ is the mass of the black hole. The value of the tidal heating is something like $I^{ij} \hat{E}_{ij} + J^{ij} \hat{B}_{ij} \approx \hat{E}^2_{ij} + \hat{B}^2_{ij}$, where $\hat{E}^2_{ij}$ means $\hat{E}^{ij} \hat{E}_{ij}$, likewise for $\hat{B}^2_{ij}$. While for the total time derivative of $\partial_\nu (I^{ij} E_{ij})$ and $\partial_\nu (J^{ij} B_{ij})$, they correspond to a change of a state function, this kind of reversible changes do not involve energy dissipation. Similarly, for the quantities $\partial_\nu (I^{ij} B_{ij})$ and $\partial_\nu (J^{ij} E_{ij})$.

3 Energy-momentum tensors of $B$ and $V$

3.1 Alternative gravitational energy-momentum tensor $V$

Let $x^\mu = (t, x, y, z)$ and using a RNC Taylor expansion around any point, consider all the possible combinations of the small region in vacuum, the total energy-momentum density pseudotensor is in general expressed as

$$\mathcal{T}_\alpha^\beta = \kappa^{-1} G_{\alpha}^\beta + (a_1 \bar{B}_\alpha^\beta x_\kappa + a_2 \bar{S}_\alpha^\beta x_\kappa + a_3 \bar{K}_\alpha^\beta x_\kappa + a_4 \bar{T}_\alpha^\beta x_\kappa) x^\xi x^\kappa + \mathcal{O}(\text{Ricci}, x) + \mathcal{O}(x^3),$$  \hspace{1cm} \text{(30)}$$

where $a_1$ to $a_4$ are constants. Since our concern is the vacuum case, so $G_{\alpha\beta} = 0 = T_{\alpha\beta}$. Then the first order linear in Ricci terms $\mathcal{O}(\text{Ricci}, x)$ vanish. The lowest order non-vanishing term is of second order, and compared to this in the small region limit we ignore the third order terms $\mathcal{O}(x^3)$. It should be noted that $\mathcal{T}_\alpha^\beta$ in \[(2)\] or \[(30)\] is a pseudotensor, but in the Taylor expansion on the right hand side in \[(30)\] the coefficients of the various powers of $x$ are tensors. As argued in \[14\], $\partial_\mu^2 \mathcal{T}_\alpha^\beta (0)$ must be some linear combination of 4 tensors, here we use \{$\bar{B}, \bar{S}, \bar{K}, \bar{T}$\}. From now on, we only keep the second order term and drop the others. There are two physical conditions which can constrain the unlimited combinations between \{$\bar{B}, \bar{S}, \bar{K}, \bar{T}$\}:

4-momentum conservation and positivity, both considered in the small region vacuum limit (i.e., not restricted to a 2-sphere)

First condition: energy-momentum conservation. Consider \[(2)\] and \[(30)\] in vacuum

$$0 = \partial_\beta t_{\alpha}^\beta = \frac{1}{4}(a_1 - 2a_2 + 3a_3 - a_4) g_{\alpha\beta} x^\beta R^2_{\lambda\sigma\rho\nu}. \hspace{1cm} \text{(31)}$$

Therefore, the constraint for the conservation of the energy-momentum density is

$$a_4 = a_1 - 2a_2 + 3a_3. \hspace{1cm} \text{(32)}$$

No single element from \{$\bar{B}, \bar{S}, \bar{K}, \bar{T}$\} can satisfy \[(31)\], however certain linear combinations of them can. Eliminate $\bar{T}$ which is absorbed by $\bar{B}, \bar{S}$ or $\bar{K}$, comparing \[(2)\] and using \[(32)\] rewrite \[(30)\]

$$t_{\alpha}^\beta = \left[ a_1 (\bar{B}_{\alpha\beta\xi\kappa} + \bar{T}_{\alpha\beta\xi\kappa}) + a_2 (\bar{S}_{\alpha\beta\xi\kappa} - 2\bar{T}_{\alpha\beta\xi\kappa}) + a_3 (\bar{K}_{\alpha\beta\xi\kappa} + 3\bar{T}_{\alpha\beta\xi\kappa}) \right] x^\xi x^\kappa$$(33) $$= (a_1 B_{\alpha\beta\xi\kappa} + a_2 S_{\alpha\beta\xi\kappa} + a_3 K_{\alpha\beta\xi\kappa}) x^\xi x^\kappa$$

Consider all the possible expressions for the pseudotensors (some of which explicitly included the flat metric), there indeed does appear linear combinations of these three
tensors \[6, 8, 9\]. Explicitly one can use either \(\{B, S, K\}\) or \(\{B, S, W\}\). We prefer the latter because one can even define a new 4th rank energy-momentum tensor as follows

\[
T_{\alpha\beta\mu\nu} := B_{\alpha\beta\mu\nu} + \alpha W_{\alpha\beta\mu\nu},
\]

(34)

where \(\alpha\) is a constant. When \(\alpha = 1\), then \(V\) is recovered.

Second condition: non-negative gravitational energy. For simplicity, we use a small sphere. For any quantity at \(t = t_0\) we consider the limiting value for the radius \(r := \sqrt{x^2 + y^2 + z^2}\). The 4-momentum at time \(t = 0\) is

\[
2\kappa P_\mu = \int t^\mu_{\mu\xi} x^\xi x^\kappa d\Sigma_\rho = t^0_{\mu\mu} \int x^i x^j d^3x = t^0_{\mu\mu} \delta_{ij} \frac{4\pi r^5}{15},
\]

(35)

Thus, from (33)

\[
P_\mu = (-E, \vec{P}) = -\frac{r^5}{60G} [(a_1 + a_3)B_{\mu\alpha\beta\gamma} + (a_2 - a_3)S_{\mu\alpha\beta\gamma}] \delta_{ij},
\]

(36)

The energy-momentum values associated with \(\{B, S, W\}\) are

\[
B_{\mu\alpha\beta\gamma} \delta_{ij} = (E^2_{ab} + H^2_{ab}, 2\epsilon_{cab}E^a E^b H^c), S_{\mu\alpha\beta\gamma} \delta_{ij} = -10(E^2_{ab} - H^2_{ab}, 0), W_{\mu\alpha\beta\gamma} \delta_{ij} = 0.
\]

(37)

Here we emphasize that in a small sphere region, the energy-momentum of \(B\) or \(V\) is inside the light-cone, \(-P_0 \geq |\vec{P}| \geq 0\). Observing (36), basically we are considering the positive energy, \(B\) and \(V\) already satisfy this condition and the remaining job is to find \(\{a_2, a_3\}\). Equation (37) shows that \(S_{\mu\alpha\beta\gamma}\delta_{ij}\) cannot ensure positivity, since we should allow for any magnitude of \(|E_{ab}|\) and \(|H_{ab}|\). The only possibility for (36) to guarantee positivity is to require \(a_1 + a_3 \geq 10|a_2 - a_3| \geq 0\). However, if we insist the pure “Bel-Robinson momentum” \[7\], obviously, we only have one choice \(a_2 = a_3\).

### 3.2 Computing energy-momentum and angular momentum

The Papapetrou pseudotensors \[9\] gives a certain linear combination of \(\{B, V\}\) or \(\{B, W\}\): \(2\kappa P_{\alpha\beta} = \frac{1}{5}(4B_{\alpha\beta\kappa\xi} - V_{\alpha\beta\kappa\xi})x^\kappa x^\xi\). The energy using (33) in a small sphere is

\[
P_0 = -\frac{r^5}{540G} (4B_{00ij} - V_{00ij})\delta_{ij}
\]

\[
= -\frac{r^5}{540G} (3B_{00ij} - W_{00ij})\delta_{ij}
\]

\[
= -\frac{r^5}{180G} B_{00ij}\delta_{ij},
\]

(38)

where \((B_{00ij} - V_{00ij})\delta_{ij} \equiv 0 \equiv W_{00ij}\delta_{ij}\). Before we proceed, one might question that perhaps \(V\) is superfluous since \(B\) and \(V\) have so far shown no distinction. We claim that \(B\) and \(V\) are distinct because they are constructed from different basic quadratic curvatures \((\tilde{B}, \tilde{S}, \tilde{K}, \tilde{T})\): \(B = \tilde{B} + \tilde{T}\) and \(V = \tilde{S} + \tilde{K} + \tilde{T}\). Strictly speaking, we claim \(B\) and \(V\) are fundamentally different \[9\]. But this raises a question regarding how to see the distinction clearly. we realize that it is impossible to distinguish \(\tilde{B}\) and \(\tilde{V}\) if we consider 4-momentum or angular momentum in a small sphere. So we change our strategy to evaluating these physical quantities in other quasilocal volume elements (see Table 1).
Perfect-sphere

\[ P_\mu = \frac{4\pi}{15} \delta_{ij} \delta^3 a^5, \quad r \in [0, a], \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi] \]

\[ J^{0\mu} = (0, 0, 0), \quad (J^{12}, J^{13}, J^{23}) = (0, 0, 0) \]

Ellipsoid

\[ P_\mu = \frac{4\pi}{15} t^0_{\mu 11} a^5 + t^0_{\mu 22} a^5 + t^0_{\mu 33} a^5, \quad x \in [-a, a], \quad y \in [-b, b], \quad z \in [-c, c] \]

\[ J^{0\mu} = (0, 0, 0), \quad (J^{12}, J^{13}, J^{23}) = (0, 0, 0) \]

Hemisphere

\[ P_\mu = \frac{4\pi}{15} t^0_{\mu 11} a^5, \quad r \in [0, a], \quad \theta \in [0, \pi/2], \quad \phi \in [0, 2\pi] \]

\[ J^{0\mu} = \frac{\pi}{24} (2t^0_{0123}, 2t^0_{023}, t^0_{0ij} \delta^j + t^0_{033}) r^6 \]

\[ J^{12} = \frac{\pi}{12} (t^1_{023} - t^2_{013}) r^6, \quad J^{13} = \frac{\pi}{24} (t^1_{0ij} \delta^j + t^1_{033} - 2t^3_{013}) r^6, \]

\[ J^{23} = \frac{\pi}{24} (t^2_{0ij} \delta^j + t^2_{033} - 2t^3_{023}) r^6 \]

Box

\[ P_\mu = \frac{4\pi}{15} t^0_{\mu 11} a^5 + t^0_{\mu 22} a^5 + t^0_{\mu 33} a^5, \quad x \in [-\frac{a}{2}, \frac{a}{2}], \quad y \in [-\frac{b}{2}, \frac{b}{2}], \quad z \in [-\frac{c}{2}, \frac{c}{2}] \]

\[ J^{0\mu} = (0, 0, 0), \quad (J^{12}, J^{13}, J^{23}) = (0, 0, 0) \]

Cylinder

\[ P_\mu = \frac{4\pi}{15} t^0_{\mu 11} a^5 + \frac{4\pi}{21} t^0_{\mu 33} (h^2 - 3a^2) a^5, \quad \rho \in [0, a], \quad \varphi \in [0, 2\pi], \quad z \in [-\frac{c}{2}, \frac{c}{2}] \]

\[ J^{0\mu} = (0, 0, 0), \quad (J^{12}, J^{13}, J^{23}) = (0, 0, 0) \]

Half-cylinder

\[ P_\mu = \frac{4\pi}{15} t^0_{\mu 11} a^5 + \frac{4\pi}{21} t^0_{\mu 33} (h^2 - 3a^2) a^5, \quad \rho \in [0, a], \quad \varphi \in [0, \pi], \quad z \in [-\frac{a}{2}, a] \]

\[ J^{01} = \frac{4}{15} t^0_{0123} a^5 h, \quad J^{02} = \frac{1}{18} t^0_{033} a^3 h^3 + \frac{2}{15} (t^0_{011} + 2t^0_{022}) a^5 h, \quad J^{03} = \frac{4}{15} t^0_{033} a^3 h^3 \]

\[ J^{12} = \frac{1}{18} t^1_{033} a^3 h^3 + \frac{2}{15} (t^1_{011} + 2t^1_{022} - 2t^2_{012}) a^5 h, \quad J^{13} = \frac{4}{9} t^1_{023} a^3 h^3 - \frac{4}{15} t^3_{012} a^5 h \]

\[ J^{23} = \frac{1}{18} (t^2_{023} - t^2_{033}) a^3 h^3 - \frac{2}{15} (t^2_{011} + 2t^2_{022}) a^5 h \]

| Table 1: Energy-momentum and angular momentum in different small regions |

We claim \( B \) and \( V \) can have different energy values, for instance, in a small box with different dimensions. Here we give a concrete example: let \( a = b, c = a + \Delta \) and \( |\Delta| << a \). The energy for substituting \( t \) by \( B \) is \( P_0^B \approx \frac{a^5}{12} (B^0_{0ij} \delta^j + \frac{2\Delta}{a} B^0_{033}) \). Similarly for \( V, P_0^V \approx \frac{a^5}{12} (V^0_{0ij} \delta^j + \frac{2\Delta}{a} V^0_{033}) \). Thus, generally, \( B \) and \( V \) are separable: \( P_0^V - P_0^B \approx \frac{a^5}{6} \Delta W^0_{033} \neq 0 \). Following the restriction that the quasilocal energy-momentum must be a multiple of “Bel-Robinson momentum” \([7]\). We can fulfill this requirement using either \( B \) or \( V \) in a small region for a perfect sphere or a box with \( a \equiv b \equiv c \), i.e., a cube \([11]\), for a cylinder or half-cylinder we need \( h \equiv \sqrt{3a} \). These are desirable results, but unfortunately, we lose the distinction between \( B \) and \( V \) again.

Is it possible to keep a multiple of “Bel-Robinson momentum” and still able to tell the difference between \( B \) and \( V \) naturally? Yes, it is possible: we turn to examining the angular momentum (see e.g., §20.3 in \([8]\)) which can be defined as follows

\[ J^{\mu\nu} = \int \langle x^\mu t^\nu \xi_\kappa - x^\nu t^\mu \xi_\kappa \rangle x^\xi x^\kappa d^3x. \]  

\[ \text{(39)} \]

where \( t \) can be \( B \) or \( V \). According to Table 1, we observe that the angular momentum vanishes for a perfect sphere, ellipsoid, box or cylinder. Conversely, both hemisphere and half cylinder \( (h \equiv \sqrt{3a}) \) have non-vanishing angular momentum. In these regions, the angular momentum values for \( B \) and \( V \) are distinguishable, i.e., \( V \) is no longer superfluous. Moreover, we remark that for a hemisphere, if we substitute \( t \) by the completely symmetric \( B, J^{12} = \frac{\pi}{12} (B_{0123} - B_{0213}) r^6 \equiv 0 \). However, if consider \( V, J^{12} = \frac{\pi}{12} (V_{0123} - V_{0213}) r^6 \neq 0 \) generally. Thus, the difference between \( B \) and \( V \) becomes sharply manifest, showing that in this case \( V \) is essential, not redundant.

3.3 Counting the independent components of \( \{B, V, W\} \)

Basically \( B, V \), and \( W \), are fourth rank tensor and could have 256 components. However, by symmetry, they only have a relatively small number of independent components. The counting of the number of independent components of \( B \) has already been done, here we claim there is no common term between \( B \) and \( W \), i.e., \( \{B\} \cap \{W\} = \{\emptyset\} \). We verify this statement as follows:
First, we count the components of $B$. In principle, $B$ is fully symmetric, by explicit examination it reduces to 35 components. There is a formula that directly gives this number. A $k$th rank totally symmetric tensor in $n$ dimensional space has $C_k^{n+k-1}$ components. For our case $C_4^{4+4-1} = 35$. Since $B$ is completely tracefreeness, there are 10 additional constraints which reduce the number of components. Therefore, we have left only 25 components for $B$ (another argument see [17]).

Next, we count the number of independent components of $V$. $V$ does not have the totally symmetric property, but as mentioned in [20] that $V_{\alpha\beta\mu\nu} \equiv V_{(\alpha\beta)(\mu\nu)} \equiv V_{\mu\nu\alpha\beta}$. This reduces $V$ to 55 components. However, the completely traceless condition gives two extra constraints indicated in [20] again: $V_{\alpha\mu\nu} \equiv 0 \equiv V_{\mu\alpha
u}$. Consequently we have $55 - 10 - 10 = 35$ for $V$.

Finally, we count the number of independent components of $W$. Observing that $V$ and $W$ are similar. Referring to [20], there should thus be at most 35 components. However, take care an extra constraint $W_{\alpha(\beta\mu\nu)} \equiv 0$ which gives 25 more constraints. Hence we find $35 - 25 = 10$ for $W$.

### 3.4 Physical meaning of the completely traceless property

It is easy to check that $B$ and $V$ are fully trace free. We are going to verify that this mathematical property and the physical conservation laws are in a 1-1 correspondence in the quasilocal limit. Consider a linear combination between $\{\tilde{B}, \tilde{S}, \tilde{K}, T\}$, let

$$A := a_1 \tilde{B} + a_2 \tilde{S} + a_3 \tilde{K} + a_4 \tilde{T}. \quad (40)$$

We observe that there are only two distinct trace because of the symmetry:

$$8A^\alpha_{\mu\nu} \equiv (a_1 - 2a_2 + 3a_3 - a_4)g_{\mu\nu}R^2_{\sigma\rho\tau}; \quad 2A^\alpha_{\alpha\mu\nu} \equiv (a_1 + a_2 - a_4)g_{\mu\nu}R^2_{\sigma\rho\tau}. \quad (41)$$

The totally traceless condition requires that the above two equations vanish simultaneously

$$0 = a_1 - 2a_2 + 3a_3 - a_4, \quad 0 = a_1 + a_2 - a_4. \quad (42)$$

The first equation in (42) is the same as (32), which indicates one of the mathematical traceless conditions identical to the energy-momentum conservation criterion: solving the equations in (42), we obtain $a_2 = a_3$, and this is proportional to the “Bel-Robinson momentum” requirement found from (36); we have noted that the fully tracefreeness property is related to some physical conditions.

### 3.5 The tidal heating, rate of change of momentum and spin angular momentum flux for $B$ and $V$

The angular momentum for a perfect sphere is vanishing. However, a slightly time dependent deformed sphere such as an ellipsoid would be no longer zero. In order to compute the spin angular momentum flux, the shape will be changed from a perfect sphere to an elliptical. Practically, the lowest order is the quadrupole moment [15]. Based on the Bel-Robinson tensor, the tidal heating can be calculated for the Bel-Robinson tensor as follows

$$W_B = \int_{\partial V} B^0_{\rho\mu\nu} x^f x^m x^n r d\Omega$$

$$= 2 \int_{\partial V} (\partial_{\rho} \Gamma^0_{f\lambda} - \partial_{f} \Gamma^0_{\rho\lambda})(\partial_{\rho} \Gamma^\lambda_{f\nu} - \partial_{\nu} \Gamma^\lambda_{f\rho}) x^f x^m x^n r d\Omega$$
\begin{align}
\int_{\partial V} \left[ -\frac{27}{r^4}I_{cd} \hat{E}_{pq} x^c x^d x^p x^q + 10I_{ij} \hat{B}^{ij} \right] \, d\Omega \\
+ \int_{\partial V} \left[ -\frac{68}{3r^4}I_{cd} \hat{B}_{pq} x^c x^d x^p x^q + \frac{32}{3r^2}J^p \hat{E}_{pd} x^c x^d + 8J_{ij} \hat{B}^{ij} \right] \, d\Omega \\
= \frac{2}{G} \left( I_{ij} \hat{E}^{ij} + \frac{4}{3}J_{ij} \hat{B}^{ij} \right) \\
= \frac{64}{45G} M^6 (\hat{E}_{ij}^2 + \hat{B}_{ij}^2). \tag{43}
\end{align}

Here come to the laws of motion and precession. The rate of change of momentum

\[ \dot{P}_i = \int_{\partial V} B^i_{\, fmn} x^f x^m x^n r \, d\Omega \]

\[ = \int_{\partial V} \left( 2R^i_{\, \lambda m \sigma} R^j_{\, \sigma n} x^f x^m x^n r - \frac{1}{8} R^2_{\lambda \rho \sigma r} x^i r^3 \right) \, d\Omega \]

\[ = 2 \int_{\partial V} (\partial_\sigma \Gamma^i_{m \lambda} - \partial_m \Gamma^i_{\lambda \sigma}) (\partial_n \Gamma^\lambda_{j n} - \partial^\sigma \Gamma^\lambda_{j f} x^f x^m x^n ) \, d\Omega \]

\[ = \frac{2}{3} \epsilon^i_{ab} \left( I^{a c} \hat{B}^b_c - \frac{4}{3} J^{a c} \hat{E}^b_c \right) \]

\[ = \frac{128}{135G} M^6 \epsilon^i_{ab} \hat{E}^{a c} \hat{B}^b_c. \tag{44} \]

Note that the maximum value for \( \dot{P}_i = 2 \times (256/225)M^6 \epsilon^i_{ab} \hat{E}^{a c} \hat{B}^b_c \) that can satisfy inside the light-cone requirement, i.e., \( \hat{W}_B \geq |\dot{P}_i| \geq 0 \). Consider the spin angular momentum flux for the Bel-Robinson tensor

\[ \dot{J}^i = \int_{\partial V} \epsilon^i_{jk} B^k_{\, fmn} x^j x^k x^m x^n r \, d\Omega \]

\[ = 2 \int_{\partial V} \epsilon^i_{jk} R^k_{\, \lambda m \sigma} R^j_{\, \sigma n} x^j x^m x^n r \, d\Omega \]

\[ = -\frac{4}{G} \epsilon^i_{ab} \left( I^{a c} E^b_c + \frac{4}{3} J^{a c} B^b_c \right). \tag{45} \]

Moreover, we also demonstrate these three physical quantities \( \hat{W}, \dot{P}_i \) and \( \dot{J}^i \) for replacing the Bel-Robinson tensor by tensors \( S \) and \( W \). We found that all of them give null result. As these three quantities are real physical phenomena, we have to acknowledge that the super-energy Bel-Robinson tensor \( B \) and tensor \( V \) \((V = B+W)\) contribute the same values.

### 4 Conclusion

For describing positivity, the Bel-Robinson tensor is the best, and perhaps has been thought to be the only possibility. We recently proposed an alternative \( V \) in such a way that it shares the same energy-momentum as \( B \) does in the small sphere limit. One might think that \( B \) and \( V \) cannot be distinguished, but we claim they can. After examining the energy found from other 2-surface such as in ellipsoid, box, cylinder and half-cylinder \((h \neq \sqrt{3}a)\), we demonstrate that \( V \) is not redundant because \( B \) and \( V \) are distinguishable. However, if we insist to achieve a multiple of pure “Bel-Robinson momentum” from Szabados’s argument in Living Review, the distinction between \( B \) and \( V \) will be lost once more. For a shape such that both \( B \) and \( V \) give a multiple of the pure “Bel-Robinson momentum” we can turn to investigate the angular momentum. Thus when replacing \( t \) by either \( B \) or \( V \), indeed they do lead to
different angular momentum values for a hemi-sphere or half-cylinder with \( h = \sqrt{3}a. \) Moreover, we emphasize that some of the components of the angular momentum give a null result for \( B \) and a non-vanishing result for \( V \). The reason is based on the elegant completely symmetric property of \( B \), while \( V \) is not fully symmetric. Thus \( V \) can play an essential irreplaceable role.

The tensors \( B \) and \( V \) are constructed from different fundamental quadratic curvatures \( \{ \bar{B}, \bar{S}, \bar{K}, \bar{T} \} \). As a double check, we counted the independent components of \( B \) and \( V \) and find that they are not the same. Finally, we discover the necessary and sufficient conditions for \( B \) and \( V \): fully tracefreeness and conservation of future pointing non-spacelike pure “Bel-Robinson momentum” in the small region limit.

Furthermore, we also demonstrate the tidal heating, rate of change of momentum and spin angular momentum flux by using tensors \( B \) and \( V \). Once again, they have the same results.

**Acknowledgment**

The author would like to thank Dr. Peter Dobson, Professor Emeritus, HKUST, for reading the manuscript and providing some helpful comments. This work was supported by NSC 95-2811-M-032-008, NSC 96-2811-M-032-001, NSC 97-2811-M-032-007 and NSC 98-2811-M-008-078.

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