Minimizing GCD sums and applications to non-vanishing of theta functions and to Burgess’ inequality

Régis de la Bretèche          Marc Munsch

January 10, 2019

Abstract

In recent years the question of maximizing GCD sums regained interest due to its firm link with large values of $L$-functions. In the present paper we initiate the study of minimizing for positive weights $w$ of normalized $L^1$-norm the sum $\sum_{m_1,m_2 \leq N} w(m_1)w(m_2)\frac{m_1m_2}{\sqrt{m_1m_2}}$. We consider as well the intertwined question of minimizing a weighted version of the usual multiplicative energy. We give three applications of our results. Firstly we obtain a logarithmic refinement of Burgess’ bound on character sums $\sum_{M<n \leq M+N} \chi(n)$ improving previous results of Kerr, Shparlinski and Yau. Secondly let us denote by $\theta(x, \chi)$ the theta series associated to a Dirichlet character $\chi$ modulo $p$. Constructing a suitable mollifier, we improve a result of Louboutin and the second author and show that, for any $x > 0$, there exists at least $\gg p/(\log p)^{1+o(1)}$ (with $\delta = 1 - \frac{1+\log 2}{\log 2} \approx 0.08607$) even characters such that $\theta(x, \chi) \neq 0$. Lastly, we obtain lower bounds on small moments of character sums.

1 Introduction

1.1 GCD sums

Let $S_\alpha(\mathcal{M})$ be the Gál’s sum associated with a set $\mathcal{M}$ and defined by

$$S_\alpha(\mathcal{M}) := \sum_{m_1,m_2 \in \mathcal{M}} \frac{(m_1,m_2)^{2\alpha}}{(m_1m_2)\alpha} \quad (0 < \alpha \leq 1)$$

0 2010 Mathematics Subject Classification. Primary: 11L40, 11N37. Secondary: 05D05, 11F27.
Key words and phrases. GCD sums, multiplicative energy, character sums, Burgess’ bound, theta functions, mollifiers.

1 After a preprint by the second author [Mun18] was released, the authors worked together and obtained several improvements as well as other results which are now contained in this new version.
Minimizing GCD sums and applications

where as usual \((m_1, m_2)\) denotes the greatest common divisor of \(m_1\) and \(m_2\). Bounding these sums had originally interesting applications in metric Diophantine approximation (see [Har90, Har98]). Recently, further study was carried out due to the connection with large values of the Riemann zeta function (see for instance [ABS15, Hil09, LRT17, Sou08]). In [BS17], [BT18], they were used to prove a lower bound of

\[
\max_{t \in [0,T]} \left| \zeta \left( \frac{1}{2} + it \right) \right| \quad \text{and} \quad \max_{\chi \in \mathcal{X}_p^+} |L(\frac{1}{2}, \chi)|
\]

where \(\mathcal{X}_p^+\) is the set of even characters modulo \(p\) and \(L(s, \chi)\) is the \(L\)-Dirichlet series associated to a character \(\chi\). In [BT18], La Bretèche and Tenenbaum proved that

\[
\max_{|M| = N} \frac{S_{1/2}(M)}{|M|} = \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\log N \log_2 N} \right\},
\]

where \(\log_k\) is the \(k\)-th iterative of the logarithm. In this result, the cardinality of \(M\) is fixed while the size of its elements is not. Moreover this estimate was also satisfied by

\[
Q(M) := \sup_{w \in \mathbb{C}^N, ||w||_2 = 1} \left| \sum_{m_1, m_2 \leq N} w(m_1)w(m_2) \frac{(m_1, m_2)}{\sqrt{m_1m_2}} \right|
\]

where \(||w||_p\) denotes the \(p\)-norm of the \(N\)-tuple \(w \in \mathbb{C}^N\).

In this article, we study the minimal value of the ratio

\[
T_0(N) := \inf_{w \in (\mathbb{R}_+)^N} \left( \frac{N}{||w||_1^2} \sum_{m_1, m_2 \leq N} w(m_1)w(m_2) \frac{(m_1, m_2)}{m_1 + m_2} \right),
\]

and

\[
T_1(N) := \inf_{w \in (\mathbb{R}_+)^N} \left( \frac{N}{||w||_1^2} \sum_{m_1, m_2 \leq N} w(m_1)w(m_2) \frac{(m_1, m_2)}{\sqrt{m_1m_2}} \right).
\]

Our main result is the following.

**Theorem 1.** There exists \(\delta_0 \approx 0.16656 < 1/6\) such that when \(N\) tends to \(+\infty\)

\[
\frac{1}{8} \log_2 N + O(1) \leq T_0(N) \leq T_1(N) \ll (\log N)^{\delta_0 + o(1)}.
\]

We show that this minimization question arises naturally in three different problems. The first application involves logarithmic improvements of the famous Burgess’ bound on multiplicative character sums while the second application is concerned with non-vanishing of theta functions. Though we show in the latter case (cf. Section 1.2 and 1.3) that a related minimization problem gives better results. As an application of this second minimization problem, we obtain lower bounds on small moments of character sums. We believe that this minimization problem might also have applications in metric Diophantine approximation.
1.2 Multiplicative energy

For two sets $A, B \subset [1, N]$, let us consider the multiplicative energy (as defined for instance in [Gow98, Tao08, TV06])

$$E(A, B) := | \{ m_1, n_1 \in A, m_2, n_2 \in B : m_1m_2 = n_1n_2 \}|.$$ 

This quantity appears to be of great importance in additive combinatorics. Under the additional restriction $w(m) \in \{0, 1\}$, the weights $w$ introduced in Section 1.1 can be viewed as the characteristic indicator $w = 1_B$ of a set $B \subset [1, N]$ of integers. In this setting, the problem of minimizing $\mathcal{T}_0(N)$ amounts to minimize the quantity $S(B)/|B|^2$ where

$$S(B) := \sum_{m_1, m_2 \in B} \frac{(m_1, m_2)}{m_1 + m_2}. \quad (3)$$

It is not hard to see that this sum is intimately connected to the quantity

$$E_x(N, B) := | \{ 1 \leq m_1, n_1 \leq N, m_2, n_2 \in B : m_1m_2 = n_1n_2 \}|.$$ 

In view of our applications, we need to bound the multiplicative energy in a symmetric situation, namely $E_x(B, B)$. To be consistent with our previous problem and to give us some flexibility, we define the weighted version of the multiplicative energy:

$$E(N, w) := \sum_{m_1, m_2, n_1, n_2 \leq N} \frac{w(m_1)w(m_2)w(n_1)w(n_2)}{m_1m_2 = n_1n_2}. \quad (4)$$

We want to minimize this quantity among choices of positive weights and introduce for this purpose

$$E(N) := \inf_{w \in (\mathbb{R}_+)^4 \frac{N^2E(N, w)}{||w||_1^4}. \quad (5)$$

When $w = 1_B$ is the characteristic indicator of a set $B$, this equals to minimize $N^2E_x(B, B)/|B|^4$. Using similar techniques as in the proof of Theorem 1, we prove the following bound.

**Theorem 2.** Let $\delta := 1 - \frac{1+\log 2}{\log 2} \approx 0.08607$. When $N$ tends to $+\infty$, we have

$$E(N) \ll (\log N)^{\delta+o(1)}.$$ 

We observe that the exponent $\delta$ is the one appearing in the famous multiplication table problem of Erdős [For08, Ten84].
1.3 First application: Improvement of Burgess’ bound

Let us consider $S_{\chi}(M, N) := \sum_{M < n \leq M + N} \chi(n)$, where $\chi \mod p$ is a multiplicative character. The classical bound of Pólya and Vinogradov gives

$$|S_\chi(M, N)| \ll \sqrt{p} \log p$$

for any non principal $\chi \mod p$. In particular, this is a non trivial result for $N > p^{1/2+\epsilon}$. A major breakthrough was obtained by Burgess [Bur62, Bur63] implying a saving for intervals of length $N \geq p^{1/4+\epsilon}$. Precisely, for any prime number $p$, non trivial multiplicative character modulo $p$ and integer $r \geq 1$, Burgess proved the following inequality

$$|S_\chi(M, N)| \ll N^{1-1/r} p^{(r+1)/4r^2} \log p,$$

where the constant depends only on $r$. Even though much stronger results are expected, this bound remains nowadays the sharpest that could be obtained unconditionally.

However, some logarithmic refinements were obtained unconditionally (see [IK04, Chapter 14] following ideas from [FI93]). The last in date is due to Kerr, Shparlinski and Yau who proved for $r \geq 2$

$$|S_\chi(M, N)| \ll N^{1-1/r} p^{(r+1)/4r^2} (\log p)^{1/4r}.$$ 

These improvements rely on an averaging argument which leads to count the number of solutions of certain congruences modulo $p$. Initially, the averaging was carried out over the full interval while in [KSY17], the authors restricted it over numbers without small prime factors. Theorem 3 allows us to perform a similar argument with a set of higher density than the one considered in [KSY17]. We use this in order to prove the following result.

**Theorem 3.** Let $p$ be prime, $r \geq 2$, $M$ and $N$ integers with

$$N \leq p^{1/2+1/4r}.$$ 

For any nontrivial multiplicative character $\chi$ modulo $p$,

$$|S_{\chi}(M, N)| \ll N^{1-1/r} p^{(r+1)/4r^2} \max_{1 \leq x \leq p} T_0(x)^{1/2r} \ll N^{1-1/r} p^{(r+1)/4r^2} (\log p)^{(\delta_0 + o(1))/2r}$$

where $\delta_0 \approx 0.16656$ as in Theorem 7.
1.4 Second application: Non vanishing of theta functions

The distribution of values of $L$-functions is a deep question in number theory which has various important repercussions for the related attached arithmetic, algebraic and geometric objects. The main reason comes from the fact that these values and particularly the central ones hold a lot of fundamental arithmetical information, as illustrated for example by the famous Birch and Swinnerton-Dyer Conjecture \cite{BSD63, BSD65}. It is widely believed that they should not vanish unless there is an underlying arithmetic reason forcing it. Consider the Dirichlet $L$-functions associated to Dirichlet characters $L(s, \chi) := \sum_{n \geq 1} \chi(n) \frac{n}{s} (\Re(s) > 1)$.

In this case there exists no algebraic reason forcing the $L$-function to vanish at $s = \frac{1}{2}$. Therefore it is certainly expected that $L(\frac{1}{2}, \chi) \neq 0$ as firstly conjectured by Chowla \cite{Cho65} for quadratic characters. In the last century the notion of family of $L$-functions has been important as heuristic guide to understand or guess many important statistical properties of $L$-functions. One of the main analytic tools is the study of moments and various authors have obtained results on the mean value of these $L$-series at their central point $s = \frac{1}{2}$.

Using the method of mollification, it was first proved by Balasubramanian and Murty \cite{BM92} that there exists a positive proportion of characters such that the $L$-function does not vanish at $s = \frac{1}{2}$. Their result was improved and greatly simplified by Iwaniec and Sarnak \cite{IS99} enabling them to derive similar results for families of automorphic $L$-functions \cite{IS00}. Since then, a lot of technical improvements and generalizations have been carried out, see for instance \cite{Bui12, KN16, Sou00}.

As initiated in previous works \cite{LM13a, LM13b, MS16, Mun17}, we would like to obtain similar results for moments of theta functions $\theta(x, \chi)$ associated with Dirichlet $L$-functions and defined by

$$\theta(x, \chi) = \sum_{n \geq 1} \chi(n)e^{-\pi n^2 x/p} \quad (\chi \in X_p^+),$$

where $X_p^+$ denotes the subgroup of order $\frac{1}{2}(p - 1)$ of the even Dirichlet characters mod $p$. It was conjectured in \cite{Lou07} that $\theta(1, \chi) \neq 0$ for every non-trivial character modulo a prime (see \cite{CZ13} for a case of vanishing in the composite case). Using the computation of the first two moments of these theta functions at the central point $x = 1$, Louboutin and the second author \cite{LM13b} obtained that $\theta(1, \chi) \neq 0$ for at least $p/\log p$ even characters modulo $p$ (for odd characters, a similar result was already proven

\footnote{Pascal Molin informed the authors that he performed some computations proving that $\theta(1, \chi) \neq 0$ for $p \leq 10^6$.}
by Louboutin in \cite{Lou99}). Constructing different kind of mollifiers than in the case of $L$-functions, we get the following improvement.

**Theorem 4.** Let $x > 0$. For all sufficiently large prime $p$, there exists at least

\[
\gg \frac{p}{\mathcal{E}(\sqrt{p}/3)} + \frac{p}{T_0(\sqrt{p}/3)} \gg \frac{p}{(\log p)^{\delta+o(1)}}
\]

even characters $\chi$ such that $\theta(x, \chi) \neq 0$, where $\delta = 1 - \frac{1 + \log 2}{\log 2} \approx 0.08607$ as before.

For the moment, we only use the bound of $\mathcal{E}$ in Theorem 2 to get a lower bound. The second term $p/T_0(\sqrt{p}/3)$ could be useful if we could improve enough Theorem 1.

### 1.5 Third application: Lower bounds on small moments of character sums

As in Section 1.3, we consider $S_\chi(N) = \sum_{n \leq N} \chi(n)$ where $\chi$ is a multiplicative character modulo a prime $p$. Using probabilistic techniques, Harper \cite{Har18a} recently proved Helson’s conjecture about the first moment of Steinhaus random multiplicative functions (multiplicative random variables whose values at prime integers are uniformly distributed on the complex unit circle). He also investigated the deterministic case and obtained upper bounds on the first moment of character sums. Obtaining sharp lower bounds from the probabilistic methods used in \cite{Har18a} seems harder\footnote{Private communication with Adam Harper.}. Using Theorem 2, we obtain the following lower bound on the $L^r$-norm of character sums.

**Theorem 5.** Let us fix $4/3 < r < 2$. For $N \geq 1$ and $p$ sufficiently large and $N < \sqrt{p}$, we have

\[
\frac{1}{p-1} \sum_{\chi \neq \chi_0} |S_\chi(N)|^r \gg \frac{N^{r/2}}{\mathcal{E}(N)^{1-r/2}}.
\]

In particular, for $p$ sufficiently large and $N < \sqrt{p}$, we have

\[
\frac{1}{p-1} \sum_{\chi \neq \chi_0} |S_\chi(N)| \gg \frac{\sqrt{N}}{\mathcal{E}(N)^{1/2}} \gg \frac{\sqrt{N}}{(\log N)^{\delta/2+o(1)}}
\]

with $\delta/2 \approx 0.043$ and $\delta$ defined in Theorem 2.

**Remark 6.** This result can be easily generalized to composite moduli, but for the sake of simplicity and coherence, we restricted the presentation to the case of prime moduli.
The same method can be also applied to get a lower bound for
\[ \frac{1}{T} \int_0^T \left| \sum_{n \leq N} n^{it} \right|^r \, dt. \]

The study of the limit \( \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left| \sum_{n \leq N} n^{it} \right|^r \, dt \) was initiated by Helson [Hel06] and further investigated by Bondarenko and Seip in [BS16]. For any \( r \leq 1 \), they proved a lower bound of size \( \sqrt{N} (\log N)^{-0.07672} \) and obtained for \( r = 1 \) the same bound with an exponent \(-0.05616\) using a different method than ours. Their method relies on [BS16, Lemma 3] which does not exist for character sums. We illustrate Theorem 2 by the following estimates.

**Theorem 7.** Let us fix \( \frac{4}{3} < r < 2 \). For \( N \geq 1 \) and \( N^2 < T \), we have

\[ \frac{1}{T} \int_0^T \left| \sum_{n \leq N} n^{it} \right|^r \, dt \gg \frac{N^{r/2}}{\mathcal{E}(N)^{1-r/2}}. \]

(10)

In particular, we have

\[ \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left| \sum_{n \leq N} n^{it} \right|^r \, dt \gg \frac{\sqrt{N}}{\mathcal{E}(N)^{1/2}} \gg \frac{\sqrt{N}}{(\log N)^{\delta/2+o(1)}} \]

with \( \delta/2 \approx 0.043 \) and \( \delta \) defined in Theorem 3.

**Remark 8.** Studying the proofs of [BS16], it is not difficult to see that their method gives also the same exponent \( \frac{\delta}{2} \). In our result, one can expect new improvements of Theorem 3 to get a better lower bound. As the proof of our Theorem 7 is similar to Theorem 5, we do not give any details.

### 2 Proof of Theorem 1

We choose weights defined by the sequence \( w \) in order to minimize

\[ T_0(N, w) := \frac{N}{\|w\|_1^2} \sum_{m_1, m_2 \leq N} w(m_1) w(m_2) \frac{(m_1, m_2)}{m_1 + m_2}. \]

\[ \frac{4}{4} \] This was pointed out by “Lucia” without detailing the proof on mathoverflow https://mathoverflow.net/questions/129264/short-character-sums-averaged-on-the-character-in-may-of-2017. As it was quoted by “Lucia”, the method of [BS16] relies on some input from analysis (lemma 3 of [BS16]) which permits to restrict the sum over the set of integers \( n \) such that \( \Omega(n) \) is constant whereas we avoid this part using some weights.
One can prove $T_0(N, w) \geq \frac{1}{2}$ by taking the contribution of the case $m_1 = m_2$. Indeed, by Cauchy-Schwarz inequality, we get

$$T_0(N, w) \geq \frac{N||w||^2}{2||w||^2} \geq \frac{1}{2}. \quad (12)$$

By the large sieve, for any sequence $w$, we have

$$\sum_{p \leq N^{1/3}} p \left| \sum_{m \leq N, \gcd(p, m) = 1} w(m) - \frac{1}{p} \sum_{m \leq N} w(m) \right|^2 \leq (1 + N^{-1/3})N||w||^2. \quad (13)$$

Using the identity $n = (1 \ast \varphi)(n)$, we have

$$T_0(N, w) = \frac{N||w||^2}{2||w||^2} \sum_{d \leq N} \varphi(d) \sum_{m_1, m_2 \leq N, d|\gcd(m_1, m_2)} \frac{w(m_1)w(m_2)}{m_1 + m_2} \geq \frac{1}{2||w||^2} \sum_{p \leq N^{1/3}} (p - 1) \left( \sum_{m \leq N, p|m} w(m) \right)^2. \quad (14)$$

By (13), we get

$$T_0(N, w) \geq \frac{1}{4||w||^2} \left( \sum_{p \leq N^{1/3}} \frac{p - 1}{p^2} \sum_{m \leq N, \gcd(p, m) = 1} w(m) - \frac{||w||}{p} \right)^2 \geq \sum_{p \leq N^{1/3}} \frac{1 - 1/p}{4p} - (1 + N^{-1/3}) \frac{N||w||^2}{2||w||^2}. \quad (15)$$

By (12), we deduce that

$$(2 + N^{-1/3})T_0(N, w) \geq \sum_{p \leq N^{1/3}} \frac{1 - 1/p}{4p} \geq \frac{1}{4} \log_2 N + O(1)$$

which concludes the proof of the lower bound.

As the proof works for $T_1(N)$, to get an upper bound, we study

$$T_1(N, w) := \frac{N||w||^2}{||w||^2} \sum_{m_1, m_2 \leq N} w(m_1)w(m_2)\frac{(m_1, m_2)}{\sqrt{m_1m_2}}. \quad (16)$$

We consider the case where the weights are supported on the set of integers with a fixed number of prime factors. Precisely, we choose

$$w(m) = w_k(m) := \begin{cases} 1 & \text{if } \Omega(m) = k, \\ 0 & \text{otherwise}. \end{cases} \quad (17)$$
where \( k = \kappa \log_2 N \in \mathbb{N} \), \( \kappa \in ]0,1[ \), and \( \Omega(n) \) the number of prime factors of \( n \) with multiplicity. We introduce the function \( Q \) defined by

\[
Q(\lambda) := \lambda \log \lambda - \lambda + 1.
\]

It is decreasing in the range \([0,1]\) and increasing in \([1,\infty]\). Assuming \( \kappa \in [\kappa_0,2-\kappa_0] \) with \( \kappa_0 \) fixed in \([0,1]\), we have uniformly (see for instance [Ten15, Chapter II.6, Theorem 6.5]) for large \( N \)

\[
||w_k||_1 = \sum_{m \leq N} w_k(m) \asymp \frac{N}{(\log N)Q(\kappa)\sqrt{\log_2 N}}. \tag{16}
\]

Moreover, when \( k = \kappa \log_2 N \in \mathbb{N} \) and \( \kappa \in [0,2-\kappa_0] \), we have uniformly

\[
\sum_{m \leq N} w_k(m) \ll \frac{N}{(\log N)Q(\kappa)}. \tag{17}
\]

But we have also an uniform upper bound without restriction on \( k \)

\[
\sum_{m \leq N} w_k(m) \ll \frac{N}{(\log N)\min\{Q(\kappa),3/8\}}. \tag{18}
\]

since \( \frac{3}{8} < Q(2) \).

In order to bound \( T_1(N,w_k) \), we write

\[
T_1(N,w_k) \leq 2\frac{N}{||w||_1^2} (S_1 + S_2)
\]

with

\[
S_1 := \sum_{d \leq \sqrt{N}} \sum_{m_1 \leq m_2 \leq N/d} \frac{w_k(dm_1)w_k(dm_2)}{\sqrt{m_1m_2}},
\]

\[
S_2 := \sum_{m_1 \leq m_2 \leq \sqrt{N}} \sum_{d \leq N/m_2} \frac{w_k(dm_1)w_k(dm_2)}{\sqrt{m_1m_2}}.
\]

Let \( S_1(j) \) and \( S_2(j) \) be the contribution in each sum corresponding with \( d \) such that \( \Omega(d) = j = \lambda \log_2 N \leq k \). We have, using \([18]\) once and \([17]\) twice

\[
S_1(j) = \sum_{d \leq \sqrt{N}} w_j(d) \sum_{m_1 \leq m_2 \leq N/d} \frac{w_{k-j}(m_1)w_{k-j}(m_2)}{\sqrt{m_1m_2}}
\]

\[
\ll \sum_{d \leq \sqrt{N}} w_j(d) \sum_{m_2 \leq N/d} \frac{w_{k-j}(m_2)}{(\log 2m_2)^{\min\{Q((\kappa-\lambda)\log_2 N/\log_2 m_2),3/8\}}}.
\]

\[
\ll N((\log N)^{-2Q(\kappa-\lambda)} + (\log N)^{-Q(\kappa-\lambda)-3/8}) \sum_{d \leq \sqrt{N}} \frac{w_j(d)}{d}
\]

\[
\ll N((\log N)^{-2Q(\kappa-\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}}(1 + (\log N)^{1-Q(\lambda)+o(1)})
\]

\[
\ll N((\log N)^{-2Q(\kappa-\lambda)+1-Q(\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}+o(1)}.
\]
The sums $S_2(j)$ satisfy the same kind of bound. We have

$$S_2(j) = \sum_{m_1 \leq m_2 \leq \sqrt{N}} \frac{w_{k-j}(m_1)w_{k-j}(m_2)}{\sqrt{m_1m_2}} \sum_{d \leq N/m_2} w_j(d)$$

$$\ll N(\log N)^{-\Omega(\kappa)} \sum_{m_1 \leq m_2 \leq \sqrt{N}} \frac{w_{k-j}(m_1)w_{k-j}(m_2)}{m_1^{1/2}m_2^{3/2}}$$

$$\ll N(\log N)^{-\Omega(\kappa)} \sum_{m_2 \leq \sqrt{N}} \frac{w_{k-j}(m_2)}{m_2(\log 2m_2)^{\min\{Q(\kappa-\lambda), \log_2 N/\log_2 m_2, 3/8\}}}$$

$$\ll N(\log N)^{-\Omega(\kappa)} + N(\log N)^{1-2Q(\kappa-\lambda)-Q(\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}+o(1)}.$$  

Then, integrating over $j$,

$$S_1 + S_2 \ll (\log_2 N) \max_{j \leq k} (S_1(j) + S_2(j)) \ll N(\log N)^{-g_0(\kappa)+o(1)}$$

with

$$g_0(\kappa) := \min \left\{ \min_{\lambda \in [0,\kappa]} \left\{ 2Q(\kappa-\lambda) + Q(\lambda) - 1, Q(\kappa-\lambda) + Q(\lambda) - \frac{5}{8}, Q(\kappa) \right\} \right\}.$$

The minimum on $\lambda$ of the first expression is obtained when $\lambda$ is the solution $\leq \kappa$ of $2\log(\kappa-\lambda) = \log \lambda$, id est

$$\lambda = \lambda_{\kappa,1} := \frac{1}{2}(2\kappa + 1 - \sqrt{4\kappa + 1}).$$

Moreover, the minimum on $\lambda$ of the second term is obtained when $\lambda$ is the solution $\leq \kappa$ of $\log(\kappa-\lambda) = \log \lambda$, id est

$$\lambda = \lambda_{\kappa,2} := \frac{1}{2}\kappa.$$

So we have

$$\min \left\{ 2Q(\kappa - \lambda_{\kappa,1}) + Q(\lambda_{\kappa,1}) - 1, 2Q(\frac{1}{2}\kappa) - \frac{5}{8}, Q(\kappa) \right\} \leq g_0(\kappa)$$

and by (16) we deduce

$$T_1(N, w_k) \ll (\log N)^{f_0(\kappa)+o(1)}$$

with

$$f_0(\kappa) := \max \left\{ 1 + 2Q(\kappa) - 2Q(\kappa - \lambda_{\kappa,1}) - Q(\lambda_{\kappa,1}), 2Q(\kappa) - 2Q(\frac{1}{2}\kappa) + \frac{5}{8}, Q(\kappa) \right\}.$$

It remains to choose $\kappa$ to minimize $f_0(\kappa)$. We choose $\kappa^*$ verifying

$$1 + Q(\kappa^*) - 2Q(\kappa^* - \lambda_{\kappa^*,1}) - Q(\lambda_{\kappa^*,1}) = 0.$$ 

In this case,

$$\delta_0 = \min_{\kappa \in [0,1]} \left\{ f_0(\kappa) \right\} = f_0(\kappa^*) = \max \left\{ Q(\kappa^*), 2Q(\kappa^*) - 2Q(\frac{1}{2}\kappa^*) + \frac{5}{8} \right\}.$$

Solving numerically this equation, we see that $\kappa^* \approx 0.48154$ and $Q(\kappa^*) \approx 0.16656$. We verify numerically that $2Q(\kappa^*) - 2Q(\frac{1}{2}\kappa^*) + \frac{5}{8} \approx 0.1253$. This implies that $T_1(N) \leq T_1(N, w_k) \ll (\log N)^{f_0+o(1)}$ which concludes the proof.

The bound (12) and (16) imply $T_1(N, w_k) = (\log N)^{\delta_0+o(1)}$. 

3 Proof of Theorem \(2\)

Similarly as before, we set \(w = w_k\) as defined in (15) where \(k = \kappa \log_2 N \in \mathbb{N}, \kappa \in [0,1]\). We remark that if \(m_1 n_1 = m_2 n_2\) then \(n_1\) has to be a multiple of \(\frac{m}{(m_1, m_2)}\) and similarly \(n_2\) has to be a multiple of \(\frac{m}{(m_1, m_2)}\). Then we can parametrize the solution of \(m_1 n_1 = m_2 n_2\) by

\[
m_1 = hd_1, \quad m_2 = hd_2, \quad n_1 = \ell d_2, \quad n_2 = \ell d_1,
\]

with \((d_1, d_2) = 1\) so that

\[
\mathcal{E}(N, w) = \sum_{m_1, m_2 \leq N} \left( \sum_{h \leq N/\max \{m_1, m_2\}} w(hm_1)w(hm_2) \right)^2.
\] (19)

This immediately implies

\[
\mathcal{E}(N, w) \leq 2E_1 + 2E_2
\]

where

\[
E_1 := \sum_{h \leq \sqrt{N}} \sum_{m_1 \leq m_2 \leq N/h} \sum_{\ell \leq N/m_2} w_k(hm_1)w_k(hm_2)w_k(\ell m_2),
\]

\[
E_2 := \sum_{m_1 \leq m_2 \leq \sqrt{N}} \left( \sum_{h \leq N/m_2} w_k(hm_1)w_k(hm_2) \right)^2.
\]

Let \(E_1(j)\) and \(E_2(j)\) be the contribution in each sum corresponding with \(h\) such that \(\Omega(h) = j = \lambda \log_2 N \leq k\). As for \(h \leq \sqrt{N}\), we have

\[
\sum_{m_1 \leq m_2 \leq N/h} w_{k-j}(m_1)w_{k-j}(m_2)
\ll \sum_{m_2 \leq N/h} w_{k-j}(m_2)(\log 2m_2)^{-\min\{Q((\kappa - \lambda) \log_2 N/\log_2 m_2), 3/8\}}
\ll \frac{N}{h} (\log N)^{-2Q(\kappa - \lambda) - \min\{0, 3/8 - Q(\kappa - \lambda)\}}
\]
we get, using (17)

\[
E_1(j) = \sum_{h \leq \sqrt{N}} w_j(h) \sum_{m_1 \leq m_2 \leq N/h} w_{k-j}(m_1) w_{k-j}(m_2) \sum_{\ell \leq N/m_2} w_j(\ell)
\]

\[
\ll N \sum_{h \leq \sqrt{N}} w_j(h) \sum_{m_1 \leq m_2 \leq N/h} \frac{w_{k-j}(m_1) w_{k-j}(m_2)}{m_2} \left(\log 2h\right)^{-Q(\lambda)}
\]

\[
\ll N^2 (\log N)^{-2Q(\kappa-\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}} \sum_{h \leq \sqrt{N}} \frac{w_j(h)}{h} \left(\log 2h\right)^{-Q(\lambda)}
\]

\[
\ll N^2 (\log N)^{-2Q(\kappa-\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}} + N^2 (\log N)^{-2Q(\kappa-\lambda)+1-2Q(\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}+o(1)}.
\]

The sums \( E_2(j) \) satisfy the same kind of bound. We have

\[
E_2(j) = \sum_{m_1 \leq m_2 \leq \sqrt{N}} w_{k-j}(m_1) w_{k-j}(m_2) \sum_{h \leq N/m_2} w_j(h) \sum_{\ell \leq N/m_2} w_j(\ell)
\]

\[
\ll N^2 (\log N)^{-2Q(\lambda)} \sum_{m_1 \leq m_2 \leq \sqrt{N}} \frac{w_{k-j}(m_1) w_{k-j}(m_2)}{m_2^{\min\{Q((\kappa-\lambda) \log_2 N/\log_2 m_2); 3/8\}}}
\]

\[
\ll N^2 (\log N)^{-2Q(\lambda)} + N^2 (\log N)^{1-2Q(\kappa-\lambda)-2Q(\lambda)-\min\{0,3/8-Q(\kappa-\lambda)\}+o(1)}.
\]

Then, integrating over \( j \),

\[
E_1 + E_2 \ll (\log_2 N) \max_{j \leq k} (E_1(j) + E_2(j)) \ll N^2 (\log N)^{-g(\kappa)+o(1)}
\]

with

\[
g(\kappa) := \min \left\{ \min_{\lambda \leq [0,\kappa]} \left(2Q(\kappa-\lambda) + 2Q(\lambda) - 1, Q(\kappa-\lambda) + 2Q(\lambda) - \frac{5}{8}, 2Q(\kappa) \right) \right\}.
\]

The minimum on \( \lambda \) of the first expression is obtained when \( \lambda := \frac{1}{2} \kappa \) as before. Moreover, the minimum of the second term is obtained when \( \lambda \) is the solution \( \lesssim \kappa \) of \( \log(\kappa - \lambda) = 2 \log \lambda \), id est

\[
\lambda = \lambda_{\kappa,3} := \frac{1}{2} \left( \sqrt{4\kappa + 1} - 1 \right).
\]

So we have

\[
g(\kappa) \geq \min \left\{ 4Q\left(\frac{1}{2} \kappa\right) - 1, Q(\kappa - \lambda_{\kappa,3}) + 2Q(\lambda_{\kappa,3}) - \frac{5}{8}, 2Q(\kappa) \right\}.
\]

Inserting (16) in (5)

\[
\mathcal{E}(N) \ll (\log N)^{f(\kappa)+o(1)}
\]
with

\[ f(\kappa) := \max \left\{ 4Q(\kappa) - 4Q(\frac{1}{2}\kappa) + 1, 4Q(\kappa) - Q(\kappa - \lambda_{\kappa,3}) - 2Q(\lambda_{\kappa,3}) + \frac{5}{8}, 2Q(\kappa) \right\}. \]

It remains to choose \( \kappa \) to minimize \( f(\kappa) \). This occurs for \( \kappa^* = 1/\log 4 \) verifying

\[ 1 + 2Q(\kappa^*) - 4Q(\frac{1}{2}\kappa^*) = 0. \]

In this case,

\[ \delta = \min_{\kappa \in [0, 1]} \{ f(\kappa) \} = f(1/\log 4) = \max\{ 2Q(1/\log 4), \alpha \} \]

where

\[ \alpha := 4Q(1/\log 4) - Q(1/\log 4 - \lambda_{1/\log 4,3}) - 2Q(\lambda_{1/\log 4,3}) + \frac{5}{8} \approx 0.046. \]

This implies that \( E(N) \ll (\log N)^{\delta + o(1)} \) with \( \delta = 1 - \frac{1+\log 2}{\log 2} \approx 0.08607 \) which concludes the proof.

The exponent \( \delta \) is optimal for the choice of \( w_k \) since we have for any sequence

\[ \frac{N^2 E(N, w)}{||w||_1^4} \geq \frac{N^2 ||w||_2^4}{||w||_1^4}. \]

Moreover if \( w \in \{0, 1\} \), we have

\[ \frac{N^2 E(N, w)}{||w||_1^4} \geq \frac{N^2}{||w||_1^4}. \]

By (16), this gives for \( w = w_k \) the lower bound

\[ \frac{N^2 E(N, w_k)}{||w_k||_1^4} \gg (\log N)^\delta \log_2 N. \]

4 Logarithmic improvement of Burgess’ bound

4.1 Preliminary results

The following result is a consequence of the Weil bounds for complete character sums, see for instance [IK04, Lemma 12.8].

Lemma 9. Let \( r \geq 2 \) be an integer, \( B \geq 1 \), \( p \) a prime and \( \chi \) a nontrivial multiplicative character modulo \( p \). Then we have

\[ \sum_{u=1}^p \left| \sum_{1 \leq b \leq B} \chi(u + b) \right|^{2r} \leq (2r)^r B^r p + 2r B^{2r} p^{1/2}. \]
For any fixed couple \((a_1, a_2)\), we denote by \(T(a_1, a_2; M, N)\) the number of solutions \(M < n_1, n_2 \leq M + N\) of the congruence
\[ n_1 a_1 \equiv n_2 a_2 \pmod{p} \tag{20} \]
and
\[ T_w(M, N, A) := \sum_{a_1, a_2 \leq A} w(a_1)w(a_2)T(a_1, a_2; M, N) \]

Following the lines of the proof of [KSY17, Lemma 4.1], we can prove the following upper bound.

**Lemma 10.** Let \(p\) be a prime and \(M, N, A\) integers such that
\[ A \leq N, \quad AN \leq p. \tag{21} \]
For any sequence \(w \in \mathbb{C}^N\), we have the following upper bound
\[ T_w(M, N, A) \ll \left( \sum_{a \leq A} |w(a)| \right)^2 + N \sum_{a_1, a_2 \leq A} |w(a_1)w(a_2)| \frac{(a_1, a_2)}{a_1 + a_2}. \]

**Proof.** Assume \(T(a_1, a_2; M, N) \neq 0\) and \((n'_1, n'_2)\) to be one fixed solution. For any \((n_1, n_2)\) solution of (20), \((n_1 - n'_1, n_2 - n'_2)\) is counted by \(E_{a_1, a_2}(8N^2; p)\) where
\[ E_{a_1, a_2}(n; p) := \sum_{a_1n_1 \equiv a_2n_2 \pmod{p}} 1. \]
Taking initial intervals in [ACZ96, Lemma 1], we deduce immediately, when \((a_1a_2, p) = 1\), the bound
\[ E_{a_1, a_2}(n; p) \ll 1 + \frac{n}{p} + \frac{\sqrt{n}(a_1 + a_2)}{p(a_1, a_2)} + \sqrt{n} \frac{(a_1, a_2)}{a_1 + a_2}. \tag{22} \]
The majorant is dominated by \(O(1 + N(a_1, a_2)/(a_1 + a_2))\). Summing over \(a_1, a_2 \leq A\), we get the result. \(\square\)

### 4.2 Proof of Theorem 3

We keep the notations of [KSY17] and follow closely their argument. We set
\[ \mathcal{T}_0 := \max_{1 \leq x \leq p} \mathcal{T}_0(x) \]
and proceed by induction on \(N\). Our induction hypothesis is the following. There exists some constant \(c\) such that for any integer \(M\) and any integer \(K < N\) we have
\[ \left| \sum_{M < n \leq M + K} \chi(n) \right| \leq cK^{1-1/r}p^{r+1}/4r^2\mathcal{T}_0^{-1/2r}, \]
and we want to prove that

\[
\left| \sum_{M<n\leq M+N} \chi(n) \right| \leq cN^{1-1/r}p^{(r+1)/4r^2}T_0^{1/2r}.
\]

As in \[\text{KSY17}\], \( N < p^{1/4} \) forms the basis of our induction. We define similarly the integers \( A \) and \( B \) by

\[
A = \left\lfloor \frac{N}{16r^2p/4} \right\rfloor \quad \text{and} \quad B = \left\lfloor rp^{1/2r} \right\rfloor.
\]

For any integers \( 1 \leq a \leq A \) and \( 1 \leq b \leq B \), we have

\[
\sum_{M<n\leq M+N} \chi(n) = \sum_{M<n\leq M+N} \chi(n+ab) + \sum_{M-ab<n\leq M} \chi(n+ab) - \sum_{M+N-ab<n\leq M+N} \chi(n+ab).
\]

By our induction hypothesis, we have

\[
\left| \sum_{M-ab<n\leq M} \chi(n+ab) \right| \leq \frac{c}{4} N^{1-1/r}p^{(r+1)/4r^2}T_0^{1/2r},
\]

and

\[
\left| \sum_{M+N-ab<n\leq M+N} \chi(n+ab) \right| \leq \frac{c}{4} N^{1-1/r}p^{(r+1)/4r^2}T_0^{1/2r},
\]

which combined with the above implies that

\[
\left| \sum_{M<n\leq M+N} \chi(n) - \sum_{M<n\leq M+N} \chi(n+ab) \right| \leq \frac{c}{2} N^{1-1/r}p^{(r+1)/4r^2}T_0^{1/2r}.
\]

The main difference with the method of \[\text{KSY17}\] comes from our choice of the subset used to average. We sum \( w(a) \) over \( a \leq A \) and \( 1 \leq b \leq B \) and obtain

\[
\left| \sum_{M<n\leq M+N} \chi(n) \right| \leq \frac{S}{B||w||_1} + \frac{c}{2} N^{1-1/r}p^{(r+1)/4r^2}T_0^{1/2r},
\]

where

\[
S := \sum_{M<n\leq M+N} \sum_{a\leq A} w(a) \left| \sum_{1\leq b\leq B} \chi(n+ab) \right|.
\]
By multiplying the innermost summation in (25) by $\chi(a^{-1})$ and collecting the values of $na^{-1}(\mod q)$, we arrive at

$$S = \sum_{1 \leq u \leq p} T(u) \left| \sum_{1 \leq b \leq B} \chi(u + b) \right|, \quad (26)$$

where

$$T(u) := \sum_{a \leq A} w(a) \sum_{\substack{M < n \leq M + N \leq M + N \leq N \equiv \mu(a) \mod p}} 1.$$}

Proceeding as in [KSY17], the Hölder inequality gives

$$S^{2r} \leq \left( \sum_{u=1}^{p} T(u)^{2r-2} \right)^{\frac{1}{2r}} \left( \left( \sum_{u=1}^{p} (T(u)^{2}) \right)^{\frac{1}{2}} \right)^{2r} \left( \left( \sum_{1 \leq b \leq B} \chi(u + b) \right)^{2r} \right). \quad (27)$$

By Lemma 9, we see that

$$\sum_{u=1}^{p} \left| \sum_{1 \leq b \leq B} \chi(u + b) \right|^{2r} \leq (2r)^{r} B^{2r} p + 2r B^{2r} p^{1/2}. \quad (27)$$

We trivially have

$$\sum_{u=1}^{p} T(u) = \sum_{\substack{M < n \leq M + N \leq M + N \leq N \equiv \mu(a) \mod p}} \sum_{a \leq A} w(a) = N \sum_{a \leq A} w(a) = N \|w\|_{1}. \quad (28)$$

Furthermore, we have

$$\sum_{u=1}^{p} T(u)^{2} = T_{w}(M, N, A)$$

where $T_{w}(M, N, A)$ is as in Lemma 10. We choose $w$ to minimize $T_{0}(A)$. By Lemma 10 and the hypothesis $N \leq p^{1/2+1/4r}$, we have

$$\sum_{u=1}^{p} T(u)^{2} \ll \|w\|_{1}^{2} (1 + N T_{0}/A) \ll \|w\|_{1}^{2} N T_{0}/A. \quad (29)$$

From (27), (28) and (29), we deduce

$$S^{2r} \ll (2r)^{2r-1} \left( p^{3/2} N^{2r-1} T_{0} \right) \|w\|_{1}^{2r} / A,$$

and hence by our choice of parameters, it follows that there exists an absolute constant $c'$ such that

$$\frac{S}{\|w\|_{1} B} \leq c' N^{1-1/r} p^{(r+1)/4r - 1/2r} T_{0}^{1/2r}. \quad (29)$$
Choosing \( c = 2c' \) and inserting in (24) implies (23)

\[
\sum_{M<n\leq M+N} \chi(n) \leq cN^{1-1/r} p^{(r+1)/4r^2} T_0^{1/2r}
\]

which concludes the proof by induction.

5 Previous results and approaches concerning non-vanishing of theta functions

In order to prove that \( \theta(x, \chi) \neq 0 \) for many of the \( \chi \in X^+_p \), one may proceed as usual and study the asymptotic behavior of the moments of these theta values

\[
S^\pm_{2k}(p) := \sum_{\chi \in X^+_p} |\theta(x, \chi)|^{2k} \quad (k > 0).
\]

Using the computation of the second and fourth moment, it was proved in [LM13b] that \( \theta(1, \chi) \neq 0 \) for at least \( \gg p/\log p \) of the \( \chi \in X^+_p \). Lower bounds of good expected order for the moments were obtained in [MS16] as well as nearly optimal upper bounds conditionally on GRH in [Mun17]. This can be related to recent results of [HNR15], where the authors obtain the asymptotic behavior of moments of Steinhaus random multiplicative function (a multiplicative random variable whose values at prime integers are uniformly distributed on the complex unit circle). This can reasonably be viewed as a random model for \( \theta(x, \chi) \). Indeed, the rapidly decaying factor \( e^{-\pi n^2/q} \) is mostly equivalent to restrict the sum over integers \( n \leq n_0(q) \) for some \( n_0(q) \approx \sqrt{q} \) and the averaging behavior of \( \chi(n) \) with \( n \ll q^{1/2} \) is essentially similar to that of a Steinhaus random multiplicative function.

As noticed by Harper, Nikeghbali and Radziwill in [HNR15], an asymptotic formula for the first absolute moment \( S^+_1(p) \) would probably imply the existence of a positive proportion of characters such that \( \theta(x, \chi) \neq 0 \). Though, quite surprisingly, Harper proved recently both in the random and deterministic case that the first moment exhibits unexpectedly more than square-root cancellation [Har18a, Har18b]

\[
\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \chi(n) \right| \ll \sqrt{N} \min \left\{ \frac{1}{(\log \log L)^{1/4}}, \frac{1}{(\log \log q)^{1/4}} \right\}
\]

(30)

where \( L = \min \{N, q/N\} \). Harper’s result shows that this approach would, in any case, fail to provide the existence of a positive proportion of “good” characters. In the next section, we adapt another approach in order to improve existing results. Precisely, we introduce mollifiers chosen as suitable
weighted Dirichlet polynomials which reduce the problem to the minimization problems considered in Section 1.

Moreover, in section 5, we state and prove a lower bound for the first moment (30).

6 Proof of Theorem 4

For any even character $\chi \in X_p^+$, let us define

$$M(\chi) = \sum_{m \leq \sqrt{p/3}} w(m)\chi(m),$$

where $w(m)$ denote some non-negative weights which will be fixed later. We consider the first mollified moment

$$M_1(p) := \sum_{\chi \in X_p^+} M(\chi)\theta(x, \chi).$$

Let us define

$$M_0(p) := \#\{\chi \in X_p^+, \theta(x, \chi) \neq 0\}.$$

By Hölder inequality, we have

$$M_1(p) \leq M_2(p)^{1/2}M_4(p)^{1/4}M_0(p)^{1/4},$$

with

$$M_2(p) := \sum_{\chi \in X_p^+} |\theta(x, \chi)|^2, \quad M_4(p) := \sum_{\chi \in X_p^+} |M(\chi)|^4.$$

In [LM13b], the authors computed an asymptotic formula for the fourth moment of theta functions showing that the main contribution comes from the solutions $m_1n_1 = m_2n_2$ and obtained a precise asymptotic formula for the related counting function

$$|\{m_1n_1 = m_2n_2, m_1^2 + n_1^2 + m_2^2 + n_2^2 \leq x\}| \sim \frac{3}{8}x \log x.$$

If we want to improve this result, we have to reduce the effect of this logarithmic term. By (33), the problem is related to a similar counting problem restricted to a subset of integers supported by the weight $w$. Precisely, from (33), we get the following lower bound.

Lemma 11. For large prime $p$ and any sequence $w \in [0, +\infty[\sqrt{p/3}$, we have

$$M_0(p) \gg \left(\sum_{n \leq \sqrt{p/3}} w(n)\right)^4 E(\sqrt{p/3}, w)^{-1}$$
where $E(N, w)$ is defined by (4). In particular, we have

$$M_0(p) \gg \frac{p}{E(\sqrt{p/3})}.$$  \hfill (34)

**Proof.** Let us recall the classical orthogonality relations for the subgroup of Dirichlet even characters $X_p^+$

$$\sum_{\chi \in X_p^+} \chi(m)\chi(n) = \begin{cases} \frac{1}{2}(p - 1) & \text{if } m \equiv \pm n \mod p \text{ and } \gcd(m, p) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$M_1(p) = \sum_{\chi \in X_p^+} \sum_{m \leq \sqrt{p/3}} \chi(m)w(m)\sum_{n \geq 1} \chi(n)e^{-\pi n^2x/p} \gg \frac{p - 1}{2} \sum_{m \leq \sqrt{p/3}} w(m)e^{-\pi m^2x/p}.$$  

We deduce that

$$M_1(p) \gg p \sum_{m \leq \sqrt{p/3}} w(m) = p||w||_1.$$  \hfill (35)

In the same way, we have

$$M_4(p) = \sum_{m_1, m_2, n_1, n_2 \leq \sqrt{p/3}} w(m_1)w(m_2)w(n_1)w(n_2) \sum_{\chi \in X_p^+} \chi(m_2n_2)\chi(m_1n_1)$$

$$= \frac{1}{2}(p - 1) \sum_{m_1, m_2, n_1, n_2 \leq \sqrt{p/3}} \chi(m_1)w(m_2)w(n_1)w(n_2)$$

$$= \frac{1}{2}(p - 1)E(\sqrt{p/3}, w),$$

and

$$M_2(p) = \frac{1}{2}(p - 1) \sum_{n_1, n_2 \geq 1 \atop n_1 \equiv \pm n_2 \mod p} e^{-\pi (n_1^2 + n_2^2)x/p} \ll p^{3/2}.$$  

Reporting these estimates in (33), we finish the proof of Lemma 11. \hfill \Box

Cauchy-Schwarz inequality gives immediately

$$M_0(p) \gg M_1^2(p)/M_{2,2}(p),$$  \hfill (36)

with

$$M_{2,2}(p) := \sum_{\chi \in X_p^+} |M(\chi)\theta(x, \chi)|^2.$$  

The first mollified moment was estimated in (33). The evaluation of the second mollified moment $M_{2,2}(p)$ is a bit more intricate.
Lemma 12. We have

$$M_{2,2}(p) \ll p \left( \sum_{m \leq \sqrt{p}/3} w(m) \right)^2 + p^{3/2} \sum_{m_1, m_2 \leq \sqrt{p}/3} w(m_1)w(m_2) \frac{(m_1, m_2)}{m_1 + m_2}. \quad (37)$$

In particular, we have

$$M_0(p) \gg \frac{p}{T_0(\sqrt{p}/3, w)}.$$

Proof. We have

$$M_{2,2}(p) = \sum_{\chi \in X_p^+} \sum_{m_1, m_2 \leq \sqrt{p}/3} \chi(m_1n_1)\chi(m_2n_2)e^{-\pi(n_1^2 + n_2^2)x/p}$$

$$= \frac{p - 1}{2} \sum_{m_1, m_2 \leq \sqrt{p}/3} w(m_1)w(m_2) \sum_{\chi \in X_p^+} \sum_{n_1 n_2 \equiv m_1 n_2 \mod p} e^{-\pi(n_1^2 + n_2^2)x/p}$$

$$\ll \sum_{m_1, m_2 \leq \sqrt{p}/3} w(m_1)w(m_2) \sum_{n \geq 1} E_{m_1, m_2}(n)e^{-\pi nx/p} \quad (38)$$

where we used partial summation and

$$E_{m_1, m_2}(n) := \sum_{\substack{n_1^2 + n_2^2 \leq n \\ m_1 n_1 \equiv \pm m_2 n_2 \mod p}} 1.$$

By (22), we have

$$E_{m_1, m_2}(n) \ll 1 + \frac{n}{p} + \frac{\sqrt{n}(m_1 + m_2)}{p(m_1, m_2)} + \frac{\sqrt{n}(m_1, m_2)}{m_1 + m_2}. \quad (39)$$

Inserting the bound (39) in the series (38) and using a comparison with an integral, the first three terms in the right hand side of (39) contribute at most \(p \left( \sum_{m \leq \sqrt{p}} w(m) \right)^2 \). The contribution of the last term is bounded by the last term of the bound (37). \( \square \)

Lemma 11 and Lemma 12 combined with Theorem 2 finish the proof of Theorem 4.

7 Proof of Theorem 5

We adopt similar techniques as the ones used in Section 6. In a similar way as in (31) we define

$$M_{\chi}(N) = \sum_{m \leq N} w(m)\overline{\chi}(m),$$
where \( w = (w_n)_{1 \leq n \leq N} \in [0, +\infty[^N \). We introduce the parameters \( \alpha = \frac{r}{3 - 2r} \) and \( \beta = \frac{8 - 6r}{8 - 4r} \) such that \( \alpha + \beta = 1 \). We further set \( p = 4 - 2r \) and \( q = \frac{k - 3r}{3 - 2r} \) which verify \( \frac{1}{p} + \frac{1}{q} + \frac{1}{4} = 1 \). Writing \( S_\chi(N) = S_\chi(N)^\alpha S_\chi(N)^\beta \) and applying Hölder’s inequality, we have

\[
\frac{1}{p - 1} \left| \sum_{\chi \neq \chi_0} S_\chi(N) M_\chi(N) \right| \leq \mathcal{G}_r(N)^{\frac{1}{p - 1}} \mathcal{G}_2(N)^{\frac{1}{p - 1}} \mathcal{M}_4(N)^{1/4}, \tag{40}
\]

where, for any \( k > 0 \), we have

\[
\mathcal{G}_k(N) := \frac{1}{p - 1} \sum_{\chi \neq \chi_0} |S_\chi(N)|^k, \quad \mathcal{M}_4(N) := \frac{1}{p - 1} \sum_{\chi \neq \chi_0} |M_\chi(N)|^4.
\]

Using orthogonality relations, it is easy to see that \( \mathcal{G}_2(N) \ll N \). In the same manner as in Section 6, the left hand side of (40) is bounded from below by \( ||w||_1 \). Similarly, we have \( \mathcal{M}_4(N) \ll \mathcal{E}(N, w) \). Combining together these inequalities, we deduce

\[
\mathcal{G}_r(N)^{\frac{1}{p - 1}} \gg \frac{||w||_1^{\frac{4}{p - 1}}}{N^{2 - \varepsilon} \mathcal{E}(N, w)}.
\]

Hence we get

\[
\mathcal{G}_r(N) \gg \frac{N^{r/2}}{\mathcal{E}(N)^{1 - r/2}}.
\]

### 8 Combinatorial open questions and consequences

Under the additional restriction \( w(m) \in \{0, 1\} \), our first problem considered in Section 1 is equivalent to the construction of a set \( B \subset [1, N] \) of high density such that the associated GCD sum is small. In the present note, we showed that the set of integers having exactly \( k \) prime factors with \( k = \kappa^* \log_2 N \) and \( \kappa^* \approx 0.48154 \) (which is a set of density \( (\log N)^{\delta_0 + o(1)} \) with \( \delta_0 \approx 0.16656 \)) verifies

\[
\sum_{m_1, m_2 \in B} \frac{(m_1, m_2)}{m_1 + m_2} \ll |B|(\log |B|)^{o(1)}
\]

or in another words the multiplicative energy verifies

\[
E_\times(N, B) \ll N|B|(\log N)^{o(1)}.
\]

We address the following problem.

**Question 13.** What is the maximal \( 0 < \alpha < 1 \) (in terms of \( N \)) such that there exists a set \( B \subset [1, N] \) of density \( \alpha \) verifying \( E_\times(N, B) \ll N|B|(\log N)^{o(1)} \) ?
Our previous discussion shows that we can take $\alpha \approx (\log N)^{-\delta_0 + o(1)}$ with $\delta_0 \approx 0.16656$. The following relaxed version is of certain interest:

**Question 14.** Can we construct a set $B \subset [1, N]$ of density $\frac{1}{(\log N)^\alpha}$ such that we have $E_x(B, N) \ll N|B|(\log N)^{\beta + o(1)}$ with $\alpha + \beta < \delta_0$?

For any set $B \subset \mathbb{R}$, a beautiful argument of Solymosi [Sol09] implies an upper bound on the multiplicative energy $E_x(B, B) \ll |B + B|^2 \log |B|$. In particular, it indicates that sets with very small sumset are good candidates for the problem of minimizing the energy. We want to ask the question whether we can remove the logarithmic term in the estimate of Solymosi for sufficiently high-density sets of integers:

**Question 15.** What is the maximal $0 < \alpha < 1$ (in terms of $N$) such that there exists a set $B \subset [1, N]$ of density $\alpha$ verifying the upper bound $E_x(B, B) \ll |B|^2 (\log N)^{o(1)}$?

In the present article, we proved in Theorem 2 that the set of integers having exactly $k$ prime factors with $k = \left[ \frac{\log N}{\log 4} \right]$ gives the admissible density $(\log N)^{-\delta + o(1)}$ with $\delta \approx 0.043$.

**Acknowledgements**

The first author gratefully acknowledges comments from Gérald Tenenbaum. The second author would like to thank Stéphane Louboutin for valuables remarks as well as Igor Shparlinski for pointing him out the reference [KSY17] after a first version of the draft was released. The authors would like to thank Kannan Soundararajan for drawing their attention to [BS16]. The second author acknowledges support of the Austrian Science Fund (FWF), START-project Y-901 “Probabilistic methods in analysis and number theory” headed by Christoph Aistleitner.

**References**

[ABS15] C. Aistleitner, I. Berkes, and K. Seip. GCD sums from Poisson integrals and systems of dilated functions. *J. Eur. Math. Soc. (JEMS)*, 17(6):1517–1546, 2015.

[ACZ96] A. Ayyad, T. Cochrane, and Z. Zheng. The congruence $x_1 x_2 \equiv x_3 x_4 \pmod{p}$, the equation $x_1 x_2 = x_3 x_4$, and mean values of character sums. *J. Number Theory*, 59(2):398–413, 1996.

[BM92] R. Balasubramanian and V. K. Murty. Zeros of Dirichlet $L$-functions. *Ann. Sci. École Norm. Sup. (4)*, 25(5):567–615, 1992.
Minimizing GCD sums and applications

[BSD63] B. J. Birch and H. P. F. Swinnerton-Dyer. Notes on elliptic curves. I. *J. Reine Angew. Math.*, 212:7–25, 1963.

[BSD65] B. J. Birch and H. P. F. Swinnerton-Dyer. Notes on elliptic curves. II. *J. Reine Angew. Math.*, 218:79–108, 1965.

[BS16] A. Bondarenko and K. Seip. Helson’s problem for sums of a random multiplicative function. *Mathematika*, 62 (2016), no. 1, 101–110.

[BS17] A. Bondarenko and K. Seip. Large greatest common divisor sums and extreme values of the Riemann zeta function. *Duke Math. J.*, 166(9):1685–1701, 2017.

[BT18] R. de la Bretèche and G. Tenenbaum. Sommes de Gâl et applications. *To appear in Proc. London Math. Soc.*, Available at https://arxiv.org/abs/1804.01629.

[Bui12] H. M. Bui. Non-vanishing of Dirichlet $L$-functions at the central point. *Int. J. Number Theory*, 8(8):1855–1881, 2012.

[Bur62] D. A. Burgess. On character sums and $L$-series. *Proc. London Math. Soc. (3)*, 12:193–206, 1962.

[Bur63] D. A. Burgess. On character sums and $L$-series. II. *Proc. London Math. Soc. (3)*, 13:524–536, 1963.

[Cho65] S. Chowla. *The Riemann hypothesis and Hilbert’s tenth problem*. Mathematics and Its Applications, Vol. 4. Gordon and Breach Science Publishers, New York, 1965.

[CZ13] H. Cohen and D. Zagier. Vanishing and non-vanishing theta values. *Ann. Math. Qué.*, 37(1):45–61, 2013.

[FI93] J. Friedlander and H. Iwaniec. Estimates for character sums. *Proc. Amer. Math. Soc.*, 119(2):365–372, 1993.

[For08] K. Ford. The distribution of integers with a divisor in a given interval. *Ann. of Math. (2)*, 168(2):367–433, 2008.

[Gow98] W. T. Gowers. A new proof of Szemerédi’s theorem for arithmetic progressions of length four. *Geom. Funct. Anal.*, 8(3):529–551, 1998.

[Har90] G. Harman. Some cases of the Duffin and Schaeffer conjecture. *Quart. J. Math. Oxford Ser. (2)*, 41(164):395–404, 1990.

[Har98] G. Harman. *Metric number theory*, volume 18 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1998.
Minimizing GCD sums and applications

[Har18a] A. Harper. Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. Preprint, https://arxiv.org/abs/1703.06654.

[Har18b] A. Harper. Moments of random multiplicative functions, II: High moments. Preprint, https://arxiv.org/abs/1804.04114.

[HNR15] A. J. Harper, A. Nikeghbali, and M. Radziwill. A note on Helson’s conjecture on moments of random multiplicative functions. In Analytic number theory, pages 145–169. Springer, Cham, 2015.

[Hel06] H. Helson, Hankel forms and sums of random variables. Studia Math. 176 (2006), 85–92.

[Hil09] T. Hilberdink. An arithmetical mapping and applications to Ω-results for the Riemann zeta function. Acta Arith., 139(4):341–367, 2009.

[IK04] H. Iwaniec and E. Kowalski. Analytic number theory, volume 53 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[IS99] H. Iwaniec and P. Sarnak. Dirichlet $L$-functions at the central point. In Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), pages 941–952. de Gruyter, Berlin, 1999.

[IS00] H. Iwaniec and P. Sarnak. The non-vanishing of central values of automorphic $L$-functions and Landau-Siegel zeros. Israel J. Math., 120(part A):155–177, 2000.

[KSY17] B. Kerr, I. E. Shparlinski, and K. H. Yau. A refinement of the Burgess bound for character sums. Preprint, https://arxiv.org/abs/1711.10582.

[KN16] R. Khan and H. T. Ngo. Nonvanishing of Dirichlet $L$-functions. Algebra Number Theory, 10(10):2081–2091, 2016.

[LR17] M. Lewko and M. Radziwill. Refinements of Gál’s theorem and applications. Adv. Math., 305:280–297, 2017.

[Lou99] S. R. Louboutin. Sur le calcul numérique des constantes des équations fonctionnelles des fonctions $L$ associées aux caractères impairs. C. R. Acad. Sci. Paris Sér. I Math., 329(5):347–350, 1999.

[Lou07] S. R. Louboutin. Efficient computation of root numbers and class numbers of parametrized families of real abelian number fields. Math. Comp., 76(257):455–473, 2007.
Minimizing GCD sums and applications

[LM13a] S. R. Louboutin and M. Munsch. On positive real zeros of theta and L-functions associated with real, even and primitive characters. *Publ. Math. Debrecen*, 83(4):643–665, 2013.

[LM13b] S. R. Louboutin and M. Munsch. The second and fourth moments of theta functions at their central point. *J. Number Theory*, 133(4):1186–1193, 2013.

[Mun17] M. Munsch. Shifted moments of L-functions and moments of theta functions. *Mathematika*, 63(1):196–212, 2017.

[Mun18] M. Munsch. Non vanishing of theta functions and sets of small multiplicative energy. Preprint, https://arxiv.org/abs/1810.05684.

[MS16] M. Munsch and I. E. Shparlinski. Upper and lower bounds for higher moments of theta functions. *Quart. J. Math.*, 67(1):53–73, 2016.

[Sol09] J. Solymosi. Bounding multiplicative energy by the sumset. *Adv. Math.*, 222(2):402–408, 2009.

[Sou00] K. Soundararajan. Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$. *Ann. of Math. (2)*, 152(2):447–488, 2000.

[Sou08] K. Soundararajan. Extreme values of zeta and L-functions. *Math. Ann.*, 342(2):467–486, 2008.

[Tao08] T. Tao. Product set estimates for non-commutative groups. *Combinatorica*, 28(5):547–594, 2008.

[TV06] T. Tao and V. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[Ten84] G. Tenenbaum. Sur la probabilité qu’un entier possède un diviseur dans un intervalle donné. *Compositio Math.*, 51(2):243–263, 1984.

[Ten15] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015.

RÉGIS DE LA BRETÊCHE, INSTITUT DE MATHÉMATIQUES DE JUSSIEU–PRG UMR 7586, UNIVERSITÉ PARIS DIDEROT–PARIS 7, SORBONNE PARIS CITÉ, CASE 7012, F-75013 PARIS, FRANCE

E-mail address: regis.de-la-breteche@imj-prg.fr

MARC MUNSCH, INSTITUT FÜR ANALYSIS UND ZAHLENTHEORIE 8010 GRAZ, STEYRERGASSE 30, GRAZ, AUSTRIA

E-mail address: munsch@math.tugraz.at