Width Distributions for Convex Regular Polyhedra

STEVEN R. FINCH

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Abstract. The mean width is a measure on three-dimensional convex bodies that enjoys equal status with volume and surface area [1]. As the phrase suggests, it is the mean of a probability density \( f \). We verify formulas for mean widths of the regular tetrahedron and the cube. Higher-order moments of \( f_{\text{tetra}} \) and \( f_{\text{cube}} \) have not been examined until now. Assume that each polyhedron has edges of unit length. We deduce that the mean square width of the regular tetrahedron is \( \frac{1}{3} + \frac{3 + \sqrt{3}}{3\pi} \) and the mean square width of the cube is \( 1 + \frac{4}{\pi} \).

Let \( C \) be a convex body in \( \mathbb{R}^3 \). A width is the distance between a pair of parallel \( C \)-supporting planes. Every unit vector \( u \in \mathbb{R}^3 \) determines a unique such pair of planes orthogonal to \( u \) and hence a width \( w(u) \). Let \( u \) be uniformly distributed on the unit sphere \( S^2 \subset \mathbb{R}^3 \). Then \( w \) is a random variable and

\[
E\left( w_{\text{tetra}} \right) = \frac{3}{2\pi} \arccos \left( \frac{-1}{3} \right)
\]

for \( C = \) the regular tetrahedron with edges of unit length and

\[
E\left( w_{\text{cube}} \right) = \frac{3}{2}
\]

for \( C = \) the cube with edges of unit length [1][2][3]. The probability density of \( w \) is not known. Our humble contribution is to verify the preceding mean width results and to deduce the following mean square width results:

\[
E\left( w_{\text{tetra}}^2 \right) = \frac{1}{3} \left( 1 + \frac{3 + \sqrt{3}}{\pi} \right),
\]

\[
E\left( w_{\text{cube}}^2 \right) = 1 + \frac{4}{\pi}
\]

which appear to be new.

We start with two-dimensional analogs of these results in Sections 1 and 2. The tetrahedral case (Section 3) is more difficult than the cubic case (Section 4); details of

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the evaluation of a certain definite integral are relegated to Section 5. We note that
the phrases **mean breadth** and **mean caliper diameter** are synonymous with
mean width \[2\], and hope that this paper might inspire relevant computations for
other convex bodies.

1. **Equilateral Triangle**

Given a convex region \(C\) in \(\mathbb{R}^2\), a width is the distance between a pair of parallel
\(C\)-supporting lines. Every unit vector \(u \in \mathbb{R}^2\) determines a unique such pair of lines
orthogonal to \(u\) and hence a width \(w(u)\). Let \(u\) be uniformly distributed on the unit
circle \(S^1 \subset \mathbb{R}^2\). We wish to study the distribution of the random variable \(w\) in the
case \(C = \) the equilateral triangle with sides of unit length.

For simplicity, let \(\triangle\) be the equilateral triangle with vertices

\[ v_1 = (0, 1), \quad v_2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right). \]

At the end, it will be necessary to normalize by \(\sqrt{3}\), the side-length of \(\triangle\).

Also let \(\tilde{\triangle}\) be the union of three overlapping disks of radius \(1/2\) centered at \(v_1/2, v_2/2, v_3/2\).
Clearly \(\triangle \subset \tilde{\triangle}\) and \(\tilde{\triangle}\) has centroid \((0, 0)\). A **diameter** of \(\tilde{\triangle}\) is the
length of the intersection between \(\tilde{\triangle}\) and a line passing through the origin.

Computing all widths of \(\triangle\) is equivalent to computing all diameters of \(\tilde{\triangle}\). The
latter is achieved as follows. Fix a point \((a, b)\) on the unit circle. The line \(L\) passing
through \((0, 0)\) and \((a, b)\) has parametric representation

\[ x = ta, \quad y = tb, \quad t \in \mathbb{R} \]

and hence \(y = (b/a)x\) assuming \(a \neq 0\). The nontrivial intersection between top circle
and \(L\) satisfies

\[ x^2 + \left(\frac{b}{a}x - \frac{1}{2}\right)^2 = \frac{1}{4} \]

thus \(x_1 = ab\) since \(a^2 + b^2 = 1\). The nontrivial intersection between right circle and
\(L\) satisfies

\[ \left(x - \frac{\sqrt{3}}{4}\right)^2 + \left(\frac{b}{a}x + \frac{1}{4}\right)^2 = \frac{1}{4} \]

thus \(x_2 = a\left(\sqrt{3}a - b\right)/2\). The nontrivial intersection between left circle and \(L\)
satisfies

\[ \left(x + \frac{\sqrt{3}}{4}\right)^2 + \left(\frac{b}{a}x + \frac{1}{4}\right)^2 = \frac{1}{4} \]

thus \(x_3 = -a\left(\sqrt{3}a + b\right)/2\).
We now examine all pairwise distances, squared, between the three intersection points:

\[(x_i - x_j)^2 + \left(\frac{b}{a} x_i - \frac{b}{a} x_j\right)^2 = \begin{cases} \frac{1}{4} \left(3 - 6\sqrt{3}ab + 6b^2\right) & \text{if } i = 1, j = 2 \\ \frac{3}{4} \left(1 + 2\sqrt{3}ab + 2b^2\right) & \text{if } i = 1, j = 3 \\ 3a^2 & \text{if } i = 2, j = 3 \end{cases}\]

and define

\[g(a, b) = \max \left\{\frac{1}{4} \left(3 - 6\sqrt{3}ab + 6b^2\right), \frac{3}{4} \left(1 + 2\sqrt{3}ab + 2b^2\right), 3a^2\right\}.\]

Therefore the mean width for \(C\) is

\[\frac{1}{\sqrt{3}} \int_0^{2\pi} g(\cos \theta, \sin \theta) \, d\theta = \frac{3}{2\pi} \text{ perimeter of } C\]

and the mean square width is

\[\frac{1}{3} \int_0^{2\pi} g(\cos \theta, \sin \theta) \, d\theta = \frac{1}{2} \left(1 + \frac{3\sqrt{3}}{2\pi}\right).\]

2. Square

We turn to the case \(C = \) the square with sides of unit length. For simplicity, let \(\Box\) be the square with vertices

\[v_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad v_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \quad v_3 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad v_4 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).\]

At the end, it will be necessary to normalize by \(\sqrt{2}\), the side-length of \(\Box\).

Also let \(\widetilde{\Box}\) be the union of four overlapping disks of radius 1/2 centered at \(v_1/2, v_2/2, v_3/2, v_4/2\). A diameter of \(\widetilde{\Box}\) is the length of the intersection between \(\widetilde{\Box}\) and a line passing through the origin.

Computing all widths of \(\Box\) is equivalent to computing all diameters of \(\widetilde{\Box}\). Fix a point \((a, b)\) on the unit circle. The line \(L\) passing through \((0, 0)\) and \((a, b)\) can be represented as \(y = (b/a)x\) assuming \(a \neq 0\). The nontrivial intersection between northeast circle and \(L\) satisfies

\[\left(x - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{b}{a} x - \frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4}\]

thus \(x_1 = \sqrt{2}a(a + b)/2\) since \(a^2 + b^2 = 1\). The nontrivial intersection between southeast circle and \(L\) satisfies

\[\left(x - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{b}{a} x + \frac{\sqrt{2}}{4}\right)^2 = \frac{1}{4}\]
thus $x_2 = \sqrt{2}a(a - b)/2$. The nontrivial intersection between northwest circle and $L$ satisfies
\[
\left( x + \frac{\sqrt{2}}{a} \right)^2 + \left( \frac{b}{a} x - \frac{\sqrt{2}}{a} \right)^2 = \frac{1}{4}
\]
thus $x_3 = -\sqrt{2}a(a - b)/2$. The nontrivial intersection between southwest circle and $L$ satisfies
\[
\left( x + \frac{\sqrt{2}}{a} \right)^2 + \left( \frac{b}{a} x + \frac{\sqrt{2}}{a} \right)^2 = \frac{1}{4}
\]
thus $x_4 = -\sqrt{2}a(a + b)/2$.

We now examine all pairwise distances, squared, between the four intersection points:
\[
(x_i - x_j)^2 + \left( \frac{b}{a} x_i - \frac{b}{a} x_j \right)^2 = \begin{cases} 
 2b^2 & \text{if } i = 1, j = 2 \\
 2a^2 & \text{if } i = 1, j = 3 \\
 2 + 4ab & \text{if } i = 1, j = 4 \\
 2 - 4ab & \text{if } i = 2, j = 3 \\
 2a^2 & \text{if } i = 2, j = 4 \\
 2b^2 & \text{if } i = 3, j = 4 
\end{cases}
\]
and define
\[
g(a, b) = \max \left\{ 2a^2, 2b^2, 2(1 + 2ab), 2(1 - 2ab) \right\}
\]
Therefore the mean width for $C$ is
\[
\frac{1}{\sqrt{2}} \int_0^{2\pi} \sqrt{g(\cos \theta, \sin \theta)} \, d\theta = \frac{4}{\pi} = \frac{\text{perimeter of } C}{\pi}
\]
and the mean square width is
\[
\frac{1}{2} \int_0^{2\pi} g(\cos \theta, \sin \theta) \, d\theta = 1 + \frac{2}{\pi}.
\]

3. Regular Tetrahedron

Returning to the three-dimensional setting of the introduction, let $C'$ be the regular tetrahedron with edges of unit length.

For simplicity, let $\triangle$ be the tetrahedron with vertices
\[
v_1 = (0, 0, 1), \quad v_2 = \left( \frac{a}{\sqrt{3}}, 0, -\frac{1}{3} \right), \quad v_3 = \left( -\frac{a}{\sqrt{3}}, \sqrt{\frac{2}{3}}, -\frac{1}{3} \right), \quad v_4 = \left( -\frac{a}{\sqrt{3}}, -\sqrt{\frac{2}{3}}, -\frac{1}{3} \right),
\]
At the end, it will be necessary to normalize by $2\sqrt{2}/3$, the edge-length of $\Delta$.

Also let $\tilde{\Delta}$ be the union of four overlapping balls of radius $1/2$ centered at $v_1/2$, $v_2/2$, $v_3/2$, $v_4/2$. Clearly $\Delta \subset \tilde{\Delta}$ and $\tilde{\Delta}$ has centroid $(0,0,0)$. A diameter of $\tilde{\Delta}$ is the length of the intersection between $\tilde{\Delta}$ and a line passing through the origin.

Computing all widths of $\Delta$ is equivalent to computing all diameters of $\tilde{\Delta}$. The latter is achieved as follows. Fix a point $(a,b,c)$ on the unit sphere. The line $L$ passing through $(0,0,0)$ and $(a,b,c)$ has parametric representation

$$x = ta, \quad y = tb, \quad z = tc, \quad t \in \mathbb{R}$$

and hence $y = (b/a)x$, $z = (c/a)x$ assuming $a \neq 0$. The nontrivial intersection between top sphere and $L$ satisfies

$$x^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}x - \frac{1}{3}\right)^2 = \frac{1}{3}$$

thus $x_1 = ac$ since $a^2 + b^2 + c^2 = 1$. The nontrivial intersection between front sphere and $L$ satisfies

$$\left(x - \frac{\sqrt{2}}{3}\right)^2 + \left(\frac{b}{a}x\right)^2 + \left(\frac{c}{a}x + \frac{1}{6}\right)^2 = \frac{1}{3}$$

thus $x_2 = a \left(2\sqrt{a} - c\right)/3$. The nontrivial intersection between left sphere and $L$ satisfies

$$\left(x + \frac{\sqrt{2}}{6}\right)^2 + \left(\frac{b}{a}x + \frac{\sqrt{6}}{6}\right)^2 + \left(\frac{c}{a}x + \frac{1}{6}\right)^2 = \frac{1}{3}$$

thus $x_3 = -a \left(\sqrt{2}a + \sqrt{6}b + c\right)/3$. The nontrivial intersection between right sphere and $L$ satisfies

$$\left(x + \frac{\sqrt{2}}{6}\right)^2 + \left(\frac{b}{a}x - \frac{\sqrt{6}}{6}\right)^2 + \left(\frac{c}{a}x + \frac{1}{6}\right)^2 = \frac{1}{3}$$

thus $x_4 = a \left(-\sqrt{2}a + \sqrt{6}b - c\right)/3$.

We now examine all pairwise distances, squared, between the four intersection points:

$$\begin{align*}
(x_i - x_j)^2 + \left(\frac{b}{a}x_i - \frac{b}{a}x_j\right)^2 + \left(\frac{c}{a}x_i - \frac{c}{a}x_j\right)^2 = \begin{cases}
\frac{2}{3} \left( a^2 - 2\sqrt{2}a c + 2c^2 \right) & \text{if } i = 1, j = 2 \\
\frac{1}{3} \left( \sqrt{2}a + \sqrt{6}b + 4c \right)^2 & \text{if } i = 1, j = 3 \\
\frac{1}{3} \left( \sqrt{2}a - \sqrt{6}b + 4c \right)^2 & \text{if } i = 1, j = 4 \\
\frac{2}{3} \left( 3a^2 + 2\sqrt{3}ab + b^2 \right) & \text{if } i = 2, j = 3 \\
\frac{2}{3} \left( 3a^2 - 2\sqrt{3}ab + b^2 \right) & \text{if } i = 2, j = 4 \\
\frac{2}{3} b^2 & \text{if } i = 3, j = 4
\end{cases}
\end{align*}$$
and define
\[
g(a, b) = \max \left\{ \frac{8}{9} \left( a^2 - 2\sqrt{2}a c + 2c^2 \right), \frac{1}{9} \left( 2\sqrt{2}a + \sqrt{6}b + 4c \right)^2, \frac{1}{9} \left( 2\sqrt{2}a - \sqrt{6}b + 4c \right)^2, \frac{2}{3} \left( 3a^2 + 2\sqrt{3}a b + b^2 \right), \frac{2}{3} \left( 3a^2 - 2\sqrt{3}a b + b^2 \right) \frac{8}{3} b^2 \right\}.
\]

As for the equilateral triangle, no simplification of \( g \) seems possible. The mean width for \( C \) is
\[
\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} g(\cos \varphi \sin \theta, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi \, d\theta = \frac{3}{2\pi} \arccos \left( -\frac{1}{3} \right)
\]
and the mean square width is
\[
\frac{1}{3} \int_0^{2\pi} \int_0^{\pi} g(\cos \varphi \sin \theta, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi \, d\theta = \frac{1}{3} \left( 1 + \frac{3 + \sqrt{3}}{\pi} \right).
\]

Details on the final integral are given in Section 5.

4. Cube

We turn to the case \( C = \) the cube with edges of unit length. For simplicity, let \( \Box \) be the cube with vertices
\[
v_k = \left( \pm \frac{\sqrt{3}}{3}, \pm \frac{\sqrt{3}}{3}, \pm \frac{\sqrt{3}}{3} \right)
\]
for \( 1 \leq k \leq 8 \). At the end, it will be necessary to normalize by \( 2/\sqrt{3} \), the edge-length of \( \Box \).

Also let \( \Boxhat \) be the union of eight overlapping balls of radius \( 1/2 \) centered at \( v_1/2, v_2/2, v_3/2, \ldots, v_8/2 \). A diameter of \( \Boxhat \) is the length of the intersection between \( \Boxhat \) and a line passing through the origin.

Computing all widths of \( \Box \) is equivalent to computing all diameters of \( \Boxhat \). Fix a point \( (a, b, c) \) on the unit sphere. The line \( L \) passing through \( (0, 0, 0) \) and \( (a, b, c) \) can be represented as \( y = (b/a)x, z = (c/a)x \) assuming \( a \neq 0 \). The nontrivial intersection between each of the eight spheres and \( L \) satisfies
\[
\left( x \pm \frac{\sqrt{3}}{6} \right)^2 + \left( \frac{b}{a}x \pm \frac{\sqrt{3}}{6} \right)^2 + \left( \frac{c}{a}x \pm \frac{\sqrt{3}}{6} \right)^2 = \frac{1}{4}
\]
and consequently
\[
g(a, b) = \max \left\{ \frac{1}{9} a^2, \frac{1}{3} b^2, \frac{1}{3} c^2, \frac{1}{9} \left( 1 + 2ab - c^2 \right), \frac{1}{9} \left( 1 + 2ac - b^2 \right), \frac{1}{9} \left( 1 - 2ab - c^2 \right), \frac{1}{3} \left( 1 - 2ac - b^2 \right), \frac{1}{3} \left( 1 - 2ab + 2c b \right), \frac{1}{3} \left( 1 + 2c b \right), \frac{1}{3} \left( 1 - 2ab - 2ac + 2b c \right), \frac{1}{3} \left( 1 + 2c b - 2c b \right) \right\}
\]
after examining all 28 pairwise distances and extracting 13 distinct expressions. As for the square (in which $g$ simplified to a maximum over two terms), here $g$ simplifies to a maximum over four terms:

$$g(a, b) = \max \left\{ \frac{4}{3} (1 + 2a b + 2ac + 2bc), \frac{4}{3} (1 + 2a b - 2ac - 2bc), \frac{4}{3} (1 - 2a b - 2ac + 2bc), \frac{4}{3} (1 - 2a b + 2ac - 2bc) \right\}.$$  

The mean width for $C$ is

$$\frac{1}{2} \frac{1}{\sqrt{3}} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sqrt{g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)} \sin \varphi \, d\varphi \, d\theta = \frac{3}{2}$$

and the mean square width is

$$\frac{1}{4} \frac{1}{3} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sqrt{g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)} \sin \varphi \, d\varphi \, d\theta = 1 + \frac{4}{\pi}.$$

5. **A Definite Integral**

Considerable work is required to evaluate the definite integral at the end of Section 3. A plot of the surface

$$(\theta, \varphi) \mapsto \sqrt{g(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)} \frac{8}{3}$$

appears in Figure 1, where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$. Figure 2 contains the same surface, but viewed from above. Our focus will be on the part of the surface in the lower right corner, specifically $0 \leq \theta \leq \pi/3$ and $2 \leq \varphi \leq \pi$. The volume under this part is $1/24^{th}$ of the volume under the full surface.

We need to find the precise lower bound on $\varphi$ as a function of $\theta$. Recall the formula for $g$ as a maximum over six terms in Section 3; let $g_\ell$ denote the $\ell^{th}$ term, where $1 \leq \ell \leq 6$. Then the lower bound on $\varphi$ is found by solving the equation

$$g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) = g_4(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

for $\varphi$. We obtain $\varphi(\theta) = 2 \arctan(h(\theta))$, where

$$h(\theta) = \frac{\cos \theta + \sqrt{3} \sin \theta + \sqrt{10 - \cos(2\theta) + \sqrt{3} \sin(2\theta)}}{2\sqrt{2}}$$

and, in particular,

$$\varphi(0) = 2 \arctan \left( \sqrt{2} \right) \approx 1.9106.$$
It follows that $g = g_1$ for $0 \leq \theta \leq \pi/3$ and $2 \arctan(h) \leq \varphi \leq \pi$. Now we have

$$
\frac{1}{8/3} \frac{1}{4\pi} \int_{2 \arctan(h)}^{\pi} g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi
$$

$$
= \frac{(-3 + \cos(2\theta)) \cos(3\varphi) - 3 (7 + 3 \cos(2\theta)) \cos \varphi - 16\sqrt{2} \cos \theta \sin^3 \varphi}{288\pi}
$$

and

$$
\cos(3\varphi)|_{2 \arctan(h)}^{\pi} = -2 \frac{(1 - 3h^2)^2}{(1 + h^2)^3},
$$

$$
\cos(\varphi)|_{2 \arctan(h)}^{\pi} = -\frac{2}{1 + h^2}, \quad \sin(\varphi)|_{2 \arctan(h)}^{\pi} = -\frac{2h}{1 + h^2}
$$

therefore

$$
\frac{1}{8/3} \frac{1}{4\pi} \int_{2 \arctan(h)}^{\pi} g_1(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi) \sin \varphi \, d\varphi
$$

$$
= \frac{6h^4 + 8\sqrt{2}h^3 \cos(\theta) + 3h^2 (1 + \cos(2\theta)) + (3 + \cos(2\theta))}{18\pi(1 + h^2)^3}.
$$

Integrating this expression from 0 to $\pi/3$ gives the desired formula for $E(w_{tetra}^2)$.

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Steven R. Finch
Dept. of Statistics
Harvard University
Cambridge, MA, USA
Steven.Finch@inria.fr
Figure 1: Surface plot of $\sqrt{3g/8}$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.

Figure 2: Another view of $\sqrt{3g/8}$, with contours of intersection.
This figure "Figure1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1110.0671v1
This figure "Figure2.jpg" is available in "jpg" format from:

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