Dissipation induced quantum synchronization in few-body spin systems

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We explore the synchronization phenomenon in quantum few-body spin system with the non-local dissipation by quantum trajectory approach. We find that even without external drive, the system can achieve spontaneous synchronization due to the interaction of non-local dissipation, and the time-dependent non-stationarity oscillations can be confirmed through the purely imaginary eigenvalues of the Liouvillian. We theoretically analyze the dissipative dynamics of the system and consider the measure to quantify synchronization through the stochastic quantum trajectories. In each quantum trajectory, it is also possible for the system to ignore dissipative process and build up oscillations in the long-time evolution. We finally investigate the robustness of the oscillations to perturbations, and determine the so-call the largest Lyapunov exponent to confirm the stability of oscillations.

I. INTRODUCTION

The classical synchronization is a fascinating and multidisciplinary topic that is found in abundance in both the natural and social sciences, e.g., applause, traffic lights, heart cells, etc.[1–3]. The phenomenon of synchronization was first noticed by Huygens in the 17th century [4]. In general, synchronization can be classified into two different types: forced synchronization [5–8] and spontaneous synchronization [9, 10]. In contrast to the forced case, spontaneous synchronization is not driven by any externally driven forces, only as a consequence of the interaction between the subsystems, of which the subsystems have identical or similar time-dependent properties. Due to the widespread nature of synchronization phenomena, over the past few centuries, classical synchronization has been extensively studied in various fields such as physics, chemistry and biology [11–14].

Quantum synchronization, as the counterpart and extension of classical synchronization in the quantum regime, has also received considerable attention in recent decades. The studies in the field of quantum synchronization, on the one hand, mainly focus on the issue: how to measure the degree of quantum synchronization. Unlike the technologies, which are well-established and have been widely used in the classical case for quantification of synchronization, the absence of a clear notion of phase-space trajectories prevents the straightforward extension of classical synchronization measures to the quantum regime [15]. In addition, quantum correlation also plays a significant role in quantum synchronization and which must be considered in the measures [10, 15–18]. To this aim, plenty of temporal correlations has been addressed to quantify quantum synchronization. For instance, the Pearson correlation coefficient, which is based on the local observables [16]. The error function, a combination of synchronization error and the nonlocal term, first proposed by Mari, et al. [15]. Until now, certain desirable methods have been designed and developed for different systems and conditions to quantify the quantum synchronization [10, 15–22].

Finding out the quantum systems which may have the existence of synchronization phenomenon is another fascinating direction of the investigation for quantum synchronization. Previously, quantum synchronization has been explored in plenty of different open quantum systems, inducing the discrete variables systems and continuous variables systems. For instance, the van der Pol (VdP) oscillators [23–25], atomic ensembles [26], superconducting circuit systems [27, 28], optomechanical systems [29, 30], and so on. Lots of theoretical and experimental explorations have been realized [31–33]. Moreover, the relevant studies even have been explored in the collision model. In the collision model framework, G. Karpat, et al. proposed a scheme to study the environment induced spontaneous synchronization [34]. Based on the setting of stroboscopic collisions, they realized the mutual synchronization between two spin-1/2 particles and investigated the relevant properties by the Person correlation coefficient, concurrence, and mutual information, the work provides a novel path to explore the continuous oscillations.

Under usual circumstances, the dynamics of the open quantum system, which is governed in Lindblad form, will always point to a time-independent steady state and this characterization is only determined by the structure of the corresponding Liouvillian [35–38]. However, the non-stationarity state can be realized in some specific dissipative quantum systems.

To our best knowledge, B. Buca, et. al. originally proposed a novel mechanism, which shows that coupling to some specific dissipations can induce the non-stationarity, especially the oscillation in quantum many-body systems [39, 40], the oscillation is stabilized in the long-time limit and directly induced by dark states of the so-called dark Hamiltonian. Therefore, under this condition, the dissipation splits the whole state space into decaying state space and non-decaying state space. Due to the dissipative dynamics caused by the external environment, states in the decaying state space will always eventually disappear, like being erased during evolution; but dark states can be preserved during the dissipative process, and they construct the different disjoint sectors of non-decaying state space. In this case, the system can oscillate between disjoint sectors under the driven of the dark Hamiltonian [39, 40].

In this paper, we introduce the non-local dissipations into a four-body spin-1/2 system. With the help of the environment-induced driven, we find that the evolution of the given system induces relaxation to a non-stationarity oscillator, and the synchronization phenomenon with continuous oscillation occurs.

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We start our discussions by considering two types of Hamiltonian: the XXZ-type case and XYZ-type case. After showing the eigenvalue spectrum of Liouvillian in the complex plane, we confirm the existence of the purely imaginary eigenvalues in the XXZ-type case, which implies that the sustaining oscillations may be found in this system. We employ the fast Fourier transform (FFT) to numerically determine the dominant frequency of oscillation, then we study the synchronization properties through quantum trajectory. In order to explore the robustness of oscillation to perturbations, we calculate the first and the non-local dissipations are considered, and we identify two dissipative operators in Eq:(3): \( \hat{O}_1 = \sqrt{\gamma}(\hat{\sigma}_-^x + \hat{\sigma}_+^x) \), \( \hat{O}_2 = \sqrt{\gamma}(\hat{\sigma}_+^x + \hat{\sigma}_-^x) \), with \( \hat{\sigma}_-^x = (\hat{\sigma}_j^x - i\hat{\sigma}_j^z)/2 \) is the lowering operator for the \( j \)-th spin and describes the incoherent spin-flip process, which implies that the operator tends to align the spin toward the negative direction of the z axis. Especially, the decay rate \( \gamma \) has already been introduced inside the definition of dissipative operators and we will always work in units of \( \gamma \) for simplicity.

In general, the Liouvillian superoperator can always be diagonalized and has a complex eigenvalue spectrum,

\[ \mathcal{L}\hat{\rho}_j = \lambda_j \hat{\rho}_j, \]

where \( \lambda_j \) (\( j = 0, 1, 2, \ldots \)) are the eigenvalues of \( \mathcal{L} \), and can be sorted by the real part, i.e. \( 0 \geq \text{Re}[\lambda_0] > \text{Re}[\lambda_1] \cdots > \text{Re}[\lambda_j] \).

In general, the real part of these eigenvalues are always semi-negative definite and exists at least one eigenvalue whose real part is zero. One can see from the combination of Eq.(2) with Eq.(4), the spectral properties of \( \mathcal{L} \) can directly determine the basic characteristics of dynamical process of system \([42, 43]\),

\[ \dot{\hat{\rho}}(t) = \hat{\rho}_{ss} + \sum_{j=0}^{\infty} c_j e^{i\lambda_j t} \hat{\rho}_j, \]

where \( \hat{\rho}_{ss} \) indicates the steady state, and \( c_j \) is derived from the unique decompositions of initial state \( \hat{\rho}(0) \) by the eigenstates of Liouvillian. In the long-time limit, the system will eventually reach an asymptotic steady state, \( \hat{\rho}_{ss} = \lim_{t\to\infty} e^{t\mathcal{L}}\hat{\rho}(0) \), and the steady state corresponds to the state \( \hat{\rho}_{ss} = \hat{\rho}_0/\text{Tr}[\hat{\rho}_0] \) with \( \text{Re}[\lambda_0] = 0 \). In addition, the eigenvalue with the largest nonzero real part is also defined as the asymptotic decay rate, which determines the slowest relaxation dynamics \([42, 43]\).

In the general case, the master equation invariably points to a stationary steady state, which satisfies the formula \( \mathcal{L}\hat{\rho}_{ss} = 0 \).

However, according to the specific form of Eq.(5), we can find that when Liouvillian has purely imaginary eigenvalues, then the system obtains a non-stationary steady state, and the dynamics governed as an oscillation. In Ref.[39], B. Buča, et. al. have introduced the non-stationary coherent quantum many-body dynamics, and here we briefly review the previous work.

In the viewpoint of Liouvillian, Eq.(4) can be rewritten as \( \mathcal{L}\hat{\rho}_j = -i\mathcal{H}\hat{\rho}_j = -i\hat{H}\hat{\rho}_j \), and \( \mathcal{H} \) is the dark Hamiltonian. When the system exist a subset of states \( \{|\psi_1\rangle, |\psi_2\rangle, \ldots , |\psi_j\rangle\} \), which all are the eigenstates of the system Hamiltonian but are the dark states of arbitrary jump operators \( \hat{O}_k \), i.e. \( \hat{O}_k|\psi_j\rangle = 0 \), we can call those states are the dark states of \( \mathcal{H} \), and the collection of eigenstates can span a subspace.

In this subspace, we can define a set of pseudo-density matrices, which satisfies \( \mathcal{L}|\psi\rangle\langle\psi| = -i(\omega_j - \omega)|\psi\rangle\langle\psi| \), and hints that the pseudo-density matrices will undergo the coherent oscillation. In fact, the pseudo-density matrices can construct a valid density matrix whose diagonal elements will not change with time but the off-diagonal elements are time-dependent. Thus, driven by the dark Hamiltonian, the dynamics can be regarded as a state erasing process, and finally, only those dark states are left. Then the subspace is decomposed into different disjoint sectors depending on the eigenvalues, the state of the system varies in different sectors, and resulting in a time-dependent steady state finally, and the corresponding properties can be determined by the initial conditions and constructions of the subspaces \([39, 40]\). Thanks to the special properties of the Liouvillian, the steady state of system can be lead to the oscillatory phenomena, so we can focus on the synchronization properties of the subsystems within this special system.

As the counterpart idea of classical synchronization, we put our investigation in the framework of quantum trajectories to explore the dynamics properties of this specific model. Thus we introduce the quantum trajectory method and the quantum synchronization measure in follows.

We investigate the stochastic evolution of quantum trajectories through the diffusion form of the stochastic Schrödinger equation, and classify the fluctuation of quantum dynamics. In this viewpoint, we are interested in the quantum synchronization measurements summarized in Sec.IV.
equation, which can be obtained as [44],
\[ d|\psi(t)\rangle = D^1[|\psi(t)\rangle]dt + \sum_k D^2_k[|\psi(t)\rangle]dW_k(t), \]
(6)
with the drift term,
\[ D^1[|\psi(t)\rangle] = -i\hat{H}_{\text{eff}} + \sum_k \frac{s_k(t)}{2}\langle\hat{O}_k - \frac{s_k(t)}{4}\rangle|\psi(t)\rangle, \]
(7)
and the diffusion part,
\[ D^2_k[|\psi(t)\rangle] = \langle\hat{O}_k - \frac{s_k(t)}{2}\rangle|\psi(t)\rangle, \]
(8)
where the effective Hamiltonian can be obtained as \( \hat{H}_{\text{eff}} = \hat{H} - i\sum_k \hat{O}_k \hat{O}_k/2 \), and the scalar quantity \( s_k(t) = \langle|\psi(t)\rangle\hat{O}_k + \hat{O}_k^\dagger|\psi(t)\rangle \) is the expectation value of a linear combination of dissipation operators. The random variable \( dW_k(t) \) which with the standard normal distribution is the stochastic Wiener increment, obeys the Itô rule \( dW_k^2 = dt \).

As for the quantum synchronization measure, following the specific definition given by Ref.[45], we consider a complex-valued measure to characterize the degree of synchronization, this temporal measure is defined by the nonlocal correlation,
\[ C_{jl}(t) = \frac{\langle\hat{\sigma}_j^x\hat{\sigma}_l^x\rangle}{\sqrt{\langle\hat{\sigma}_j^x\hat{\sigma}_j^x\rangle\langle\hat{\sigma}_l^x\hat{\sigma}_l^x\rangle}}, \]
(9)
the complex-valued correlation can be expressed as following \( C_{jl}(t) = |C_{jl}|e^{i\phi_{jl}} \). The angle \( \phi_{jl} \) can determine the phase difference between the \( j \)th and \( l \)th spin and which is restricted in \([-\pi, \pi]\). The modulus of \( C_{jl}(t) \) indicates the degree of correlation between two systems, \( |C_{jl}| = 1 \) implies the phase-locking, and \( |C_{jl}| = 0 \) means those two operators are completely uncorrelated.

III. RESULTS

As a preliminary exploration, we first focus on the spectral properties of Liouvillians in different types of interaction, and the spectrums are shown in Fig.1. In Fig.1(a), we show the eigenvalues spectrum of Liouvillian of XXZ-type Hamiltonian in the complex plane. The total number of eigenvalues is determined by the dimension of Liouvillian, which is \( \dim(\mathcal{L}) = 2^8 = 256 \). One can see that the eigenvalues exhibit a concentrated distribution in the complex plane, which means that the eigenvalue spectrum has highly degenerated.

We mainly focus on the eigenvalues with the real parts close to zero value, and highlight them in the inset of Fig.1(a). Different from the general case, the spectrum has purely imaginary eigenvalues in addition to the degenerate eigenvalues in which both real and imaginary parts are nearly zero. The purely imaginary eigenvalues can be regarded as the steady-state global phases, which also indicates that in the long-time limit, the system will reach a non-stationary state and persistently oscillate, while the period is related to the purely imaginary parts.

Despite having the identical dissipative operators and decay rates, the spectrum of Liouvillians of XYZ-type Hamiltonian is relatively trivial. We present its eigenvalue spectrum in Fig.1(b). The inset of Fig.1(b) implies that the eigenvalues are degenerate when the real part is zero, but the purely imaginary case absents, the system will eventually asymptotically evolves into a time-independent steady state.

Next we investigate the dynamical properties of both cases. In Fig.2(a), we concentrate on the expectations of given operators of XXZ-type Hamiltonian, e.g. \( \langle\hat{\sigma}_j^x\rangle, (j = A, B, C, D) \), which are considered as the indicators to describes the dynamical properties of dissipative system. After exhibiting irregular variations at the initial moment, the time-varying expectations rapidly form oscillations that may have well-defined periods.

In order to distinguish periodic oscillation from other non-stationary states, we confirm the stability of the oscillation’s period by means of the fast Fourier transform, and determine the dominant frequency \( f_d \) of the oscillation.

As shown in Fig.2(b), the time-dependent expectations are converted to the frequency domain through fast Fourier transform, and all curves exhibit the peak around the identical frequency. We show the concrete details in the inset of Fig.2(b), one can see that the dominant frequency can be numerically determined as \( f_d \approx 0.57 \). We can theoretically determine \( f_d \) by Eq.(5). In the long-time limit, the components of the right-hand-side of Eq.(5) that correspond to the eigenvalues with nonzero real part will decay and eventually lost, and the com-
components of which the real parts are zero, inducing the purely imaginary part can survive. With the parameters are given as: $J_x = 1, J_y = 1, J_z = 0.9$, we can obtain the purely imaginary eigenvalues are $\lambda_1 = 3.6i$ and $\lambda_2 = -3.6i$. Then the theoretical dominant frequency $f_d^* = 3.6/2\pi \approx 0.5730$, and the theoretical results are in good agreement with the numerical results when the computational errors are ignored.

Next we calculate Loschmidt echo $L(t)$ as another probe to indicate the persistent oscillation, which expressed as [46, 47]

$$L(t) = \text{Tr}[\hat{\rho}(t)\hat{\rho}(0)],$$

and $L(t)$ signals the overlap between the initial state $\hat{\rho}(0)$ and the state at the time $t$. As shown in Fig. 2(c), Loschmidt echo varies over time and governs an oscillation with a well-defined period, which means that the system reaches a stable oscillation, and it eventually dissipates. The behaviors of numeric in the frequency domain exhibit that almost all imaginary parts of Liouvillian have the approximate weight in evolution, except for the dominant frequency, and the stable and sustaining oscillations are governed rapidly. (b). The fast Fourier transform of the time-varying expectations of the observables shown in Fig. 2(a). The fast Fourier transform of the time-varying expectations of the observables shown in Fig. 2(b) in the frequency region [0, 0.7], the black dash line denotes the theoretical predicted value of the dominant oscillation frequency. (c). The red line denotes the Loschmidt echo as a indicator to probe the persistent oscillation, and the blue line indicates the purity of the steady-state density matrix. The parameters of Hamiltonian are: $J_x = 1, J_y = 1, J_z = 0.9$. In addition, we present the purity of the density matrix during the evolution, $P(t) = \text{Tr}[\hat{\rho}(t)^2]$. Different from Loschmidt echo, the purity $P(t)$ reaches an asymptotical stabilization when the system establishes a cyclic evolution, and remains unchanged in the following dynamical process. The behavior of the purity indicates that the diagonal elements of the density matrix will not change after the oscillation is stabilized, and the oscillation originates from the off-diagonal part of the density matrix. Besides, the stabilization value of the purity reveals the oscillator is a mixed state, which hints that the current dynamical process of system is different the decoherence-free subspace.

We proceed to make a brief discussion on the other Hamiltonian. Compared with the $XXZ$-type case, the results of the $XYZ$-type Hamiltonian are much more trivial. Although the oscillation can also be established during the initial evolution, and it eventually dissipates. The behaviors of numeric in the frequency domain exhibit that almost all imaginary parts of Liouvillian have the approximate weight in evolution, ex-

FIG. 2: (a). In the $XXZ$-type interaction, the expectations of the observables as a function of time. After a short-term irregular evolution, the stable and sustaining oscillations are governed rapidly. (b). The fast Fourier transform of the time-varying expectations of the observables shown in Fig. 2(a). The inset: partial detailed display of the fast Fourier transforms of the time-varying expectations of the observables shown in Fig. 2(b) in the frequency region [0.5, 0.7], the black dash line denotes the theoretical predicted value of the dominant oscillation frequency. (c). The red line denotes the Loschmidt echo as a indicator to probe the persistent oscillation, and the blue line indicates the purity of the steady-state density matrix. The parameters of Hamiltonian are: $J_x = 1, J_y = 1, J_z = 0.9$.

FIG. 3: (a). In the $XYZ$-type interaction, the expectations of the observables as a function of time. (b). The fast Fourier transforms of the time-varying expectations of the observables shown in Fig. 3(a). (c). Rapid decay of the Loschmidt echo indicates that the system will eventually evolve to a trivial state. The parameters of Hamiltonian are: $J_x = 0.8, J_y = 1, J_z = 0.9$. 

$P(t) = \text{Tr}[\hat{\rho}(t)^2]$. Different from Loschmidt echo, the purity $P(t)$ reaches an asymptotical stabilization when the system establishes a cyclic evolution, and remains unchanged in the following dynamical process. The behavior of the purity indicates that the diagonal elements of the density matrix will not change after the oscillation is stabilized, and the oscillation originates from the off-diagonal part of the density matrix. Besides, the stabilization value of the purity reveals the oscillator is a mixed state, which hints that the current dynamical process of system is different the decoherence-free subspace.
FIG. 4: (a). The expectations of the given observable $\langle \hat{\sigma}_A^2 \rangle$ obtained by the quantum trajectory method vary with time. The light red lines represent ten randomly selected single quantum trajectories, and the red line denotes the statistical average of all the quantum trajectories. (b). The modulus $|C_{\text{AB}}|$ and phase difference $\phi_{\text{AB}}$ of the steady-state quantum synchronization measure, which has been employed on a set of single quantum trajectories. The set contains all the quantum trajectories used for our simulation with $N = 500$ and $dt = 10^{-3}$.

cept for the frequency corresponding to the peak, which is the imaginary part of the eigenvalue corresponding to the asymptotic dissipative rate.

After analyzing the results of different Hamiltonians, then we concrete on the synchronization phenomenon in XXZ-type Hamiltonian case. Quantum stochastic trajectories are a powerful tool for analyzing the synchronization properties of this given few-body system, with additional insight into other intriguing, novel characteristics. We show our results in the following.

In Fig.4, our study is focused primarily on the single quantum stochastic trajectory in this system, which exhibits different properties than other systems in general. We plot the statistical average result of $N = 500$ quantum trajectories in Fig.4(a), which is named the average trajectory, and we randomly sample 10 single quantum trajectories for the comparison. In our system, it is natural for the average trajectory to have periodic oscillations in the long-term evolution. Surprisingly, the set of randomly drawn individual trajectories also exhibits non-stochastic processes in the long-term evolution, although the evolution of individual trajectory is dependent on random variable and corresponding moment expectations.

To further explore possible phenomenon of synchronization in a single trajectory, we employ the synchronization measure given in Eq.(9), and the quantity can be used to detect the non-local correlations. We show the results in Fig.4(b), each circle in the phase-modulus represents the steady-state properties of a single trajectory. One can see that, the moduli of all quantum trajectories are at a large value, about $|C_{\text{AB}}| \in [0.7, 0.9]$. Although the results do not imply a perfect phase locking, this also indicate a very strong correlation between the subsystems $A$ and $B$. In fact, although we did not specifically mention it before, the two next-nearest neighbour spins, e.g. subsystems $A$ and $C$, or $B$ and $D$, can be always in a strongly correlated relationship due to the existence of non-local dissipative interactions. Also from the point of view of symmetry, the presence of spatial symmetry also implies a dual dynamical properties between the next-nearest neighbours. Thus the strong correlation between the nearest neighbour spins further suggest that a correspondence between the dynamical properties of the various subsystems of the system.

Moreover, we can find in Fig.4(b), the phase difference $\phi_{\text{AB}}$ of each single quantum trajectory shows that the phase differences between subsystems $A$ and $B$ are essentially clustered at the value of $\phi_{\text{AB}} \approx 2\pi/3$. Although each trajectory undergoes an independent stochastic process, they all point to similar properties, which means that in the long time evolution, the stochastic process, or dissipative process, becomes increasingly less influential in the dynamical behavior and the system gradually enters a Hilbert subspace, which is decoupled from the dissipative process.

In the last part, we calculate the so-called largest Lyapunov exponent in quantum regime as a probe to analyze the system stabilization to a perturbation, and the original scheme of the largest Lyapunov exponent was proposed by I. I. Yusipov et. al. in Ref.[41]. Following is a brief review of the definition of the largest Lyapunov exponent in quantum regime.

To a similar extent as the classical definition, the quantum trajectory may also be equally viewed as the evolution trajectory in the phase space. Along this line, the largest Lyapunov exponent $\lambda$ can be defined through the “distance” between two trajectories, the fiducial and auxiliary trajectory [41, 48–50]. In the initial time, the wave function of the auxiliary trajectory $|\psi_a(t = 0)\rangle$ is prepared as a normalized perturbed wave function of the fiducial trajectory $|\psi_f(t = 0)\rangle = (|\psi_f(t = 0)\rangle + \delta|\psi\rangle)/|||\psi_f(t = 0)\rangle + \delta|\psi\rangle||$, and $\delta \ll 1$. In this work, the “distance” is defined as the difference of the expectations of observable $\hat{\sigma}_A^2$. The difference of the two initial wave functions is $\Delta_0 = \langle \hat{\sigma}_A^2(0) \rangle_f - \langle \hat{\sigma}_A^2(0) \rangle_a$, and the temporal changes of the “distance” can be obtained by $\Delta(t) = \langle \hat{\sigma}_A^2(t) \rangle_f - \langle \hat{\sigma}_A^2(t) \rangle_a$. The threshold $\Delta_{\text{max}}$ is the benchmark of the “distance”, the auxiliary wave function has to be renormalized as the perturbed wave function of fiducial trajectory when the “distance” exceeds the threshold $|\Delta(t_0)| > \Delta_{\text{max}}$ at the time $t_0$. Then the event is recorded as the growth factor $d(t_0) = |\Delta(t_0)|/\Delta_0$. In the long-time evolution, the largest Lyapunov exponent can be estimated by summing
FIG. 5: (a). The difference of observables $\Delta(t)$ varies with time (red line), and the black dash lines denote threshold $\Delta_{\text{max}} = 0.05$. (b). The largest Lyapunov exponent changes with time. When the absolute value of the difference of observables $\Delta(t)$ exceeds the threshold $\Delta_{\text{max}}$, an iterative update of the largest Lyapunov exponent occurs. The black dash-dot line denotes the benchmark of zero.

all the growth factors,

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \sum_k \ln d_k(t_k). \quad (11)$$

In addition, the fiducial and auxiliary trajectories need to go through the identical stochastic process.

We show the temporal quantity $\Delta(t)$ in Fig.5(a), and highlight it with red line. One can see that at the initial time, the “distance” shows rapid and drastic variation, and breaks the threshold $\Delta_{\text{max}}$ (shown as the black dash lines) multiple times. After multiple-time resetting of auxiliary trajectory through renormalization, the “distance” trends to be stable and always within the threshold. The process of updating the largest Lyapunov exponent over time is shown in Fig.5(b). Corresponding to the results in Fig.5(a), the largest Lyapunov exponent is updated multiple times iteratively at the initial moment. After experiencing unstable evolution in a short period, the largest Lyapunov exponent exhibits a monotonically decreasing trend. In the long-time evolution ($t = 50$ in the current simulation), the quantity eventually descends into zero and meets the benchmark of zero, which hints that the system is robust to the perturbation and the oscillations are stable.

IV. SUMMARY

In summary, we have investigated the phenomenon of spontaneous synchronization in a quantum spin-1/2 system, which is governed by four subsystems. In the absence of the presence of a driver, the system generates the oscillatory behavior in the long-time limit due to a special structure of Liouvillian, which is induced by the non-local dissipative operators. To be more specific, we found that the spectrum of Liouvillian exists purely imaginary when the Hamiltonian of the system is of XXZ-type, so that the dynamical semigroup leads to the long-time oscillation.

We further analyzed whether the continuous oscillations are disordered chaos or ordered limit cycles by a series of quantities. After applying the fast Fourier transform on the temporal expectation values of local operators, we found that each subsystem has the similar dominant frequency in the frequency domain, and the frequency results are the same as those analytically obtained numerics through the Liouvillian.

Then we focused on the quantum trajectories, which as the counterpart idea to the trajectories in classical synchronization theory. We found through the non-local synchronization measure that the dynamical behavior of each trajectory obtained from the stochastic Schrödinger equation tends to stabilize the oscillations under long-time evolution, despite the fact that it has to undergo a stochastic process. Moreover, each trajectory has considerable synchronization properties between subsystems.

Finally, we investigated the stability of those oscillations of the system subjected to perturbations in the long-time limit by defining the largest Lyapunov exponent, and this quantity suggests that the system is robust to perturbations.

To sum up, it is interesting to find the phenomenon of spontaneous synchronization in open quantum system, in particular the existence of purely imaginary eigenvalues corresponding to the Liouvillian, which allows the system can be decoupled from its environment under the long-time evolution. Our results reveal that specific non-local dissipation may construct a special dynamical semigroup in few-body systems, and lead to a non-stationary steady state.

In recent years, there has been lot of interest in the existence of the novel states of matter in quantum many-body systems, e.g. quantum time crystal [51–54], which also has an oscillatory character and breaks the time translational symmetry. Combined with the results we obtained, it is therefore also an open question to investigate whether oscillations can be maintained in the thermodynamic limit by special forms of dissipation in interacting spin systems, and giving rise to novel states of matter.

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