EXTENDED WEIGHT SEMIGROUPS
OF AFFINE SPHERICAL HOMOGENEOUS SPACES
OF NON-SIMPLE SEMISIMPLE ALGEBRAIC GROUPS

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ABSTRACT. The extended weight semigroup of a homogeneous space \( G/H \) of a connected semisimple algebraic group \( G \) characterizes the spectra of the representations of \( G \) on the spaces of regular sections of homogeneous linear bundles over \( G/H \), including the space of regular functions on \( G/H \). We compute the extended weight semigroups for all strictly irreducible affine spherical homogeneous spaces \( G/H \), where \( G \) is a simply connected non-simple semisimple complex algebraic group and \( H \) a connected closed subgroup of it. In all the cases we also find the highest weight functions corresponding to the indecomposable elements of this semigroup. Among other things, our results complete the computation of the weight semigroups for all strictly irreducible simply connected affine spherical homogeneous spaces of semisimple complex algebraic groups.

1. INTRODUCTION

1.1. Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \) and \( H \) a closed subgroup of it. We consider the homogeneous space \( G/H \) and the algebra \( \mathbb{C}[G/H] = \mathbb{C}[G]^H \) of regular functions on it. This algebra has the structure of a rational \( G \)-module with respect to the action of \( G \) by left multiplication and decomposes into a direct sum of finite-dimensional irreducible \( G \)-modules. A problem of interest is to find all \( \lambda \) for which this decomposition contains the irreducible \( G \)-module \( V_\lambda \) with highest weight \( \lambda \) and to determine the multiplicity \( m_\lambda \) of \( V_\lambda \). Those dominant weights \( \lambda \) of \( G \) satisfying \( m_\lambda \geq 1 \) form a semigroup called the weight semigroup of the homogeneous space \( G/H \). We denote this semigroup by \( \Gamma(G/H) \).

The subgroup \( H \) (resp. the homogeneous space \( G/H \), the pair \( (G,H) \)) is said to be spherical if a Borel subgroup \( B \subset G \) has an open orbit in \( G/H \). The results of the paper [1], which are discussed in §1.3 below, yield that for spherical homogeneous spaces the \( G \)-module \( \mathbb{C}[G/H] \) is multiplicity free. (The converse is also true if \( G/H \) is quasi-affine.) By definition, this means that \( m_\lambda \leq 1 \) for every \( \lambda \). For semisimple \( G \), the classification of affine spherical homogeneous spaces (that is, with reductive \( H \)) up to local isomorphism was obtained in [2]–[4]. Namely, all pairs \( (G,H) \) with \( G \) a simply connected simple algebraic group and \( H \) a connected reductive spherical subgroup of it are found in [2] along with a description of the corresponding weight semigroup for every such pair. Up to local isomorphism, all affine spherical homogeneous spaces of
non-simple semisimple algebraic groups are classified in [3], [4] (see also [5] for a more accurate formulation). Every such space can be obtained by a certain procedure starting from strictly irreducible spherical homogeneous spaces. (Their definition will be given in §1.4.) Up to local isomorphism, these can be of the following three types:

1) spherical spaces $G/H$, where $G$ is simple (classified in [2]);

2) spaces $G/H$, where $G = H \times H$, the subgroup $H$ is embedded in $G$ diagonally, $H$ is simple;

3) spaces $G/H$ corresponding to the pairs $(G, H)$ in Table 1 below (classified in [3], [4]).

The weight semigroups of all spaces of type 1) are computed in [2]. For spaces of type 2), the semigroups $\Gamma(G/H)$ are well known, see §1.4. The weight semigroups for all spaces of type 3) are computed in the present paper.

More precisely, for each spherical homogeneous space in Table 1 we compute its extended weight semigroup (to be precisely defined in §1.2) and the weight functions corresponding to the indecomposable elements of this semigroup. In particular, using these results one can easily compute the weight semigroups for the spaces in Table 1. Yu. V. Dzyadyk reported to E. B. Vinberg that he had computed the extended weight semigroups of all the homogeneous spaces appearing in Table 1 in 1985. He used another method going back to his papers [6], [7] (see also [8]) dealing with the case of symmetric spaces. This method does not require explicit computation of weight functions. Unfortunately, the results of Dzyadyk have not been published hitherto.

1.2. Throughout this paper the base field is the field $\mathbb{C}$ of complex numbers, all topological terms relate to the Zariski topology, all groups are supposed to be algebraic and their subgroups closed. For any group $L$ let $\mathfrak{X}(L)$ denote the group of its characters in additive notation.

In what follows, we keep the notation $G$ for a connected semisimple algebraic group. We suppose that a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$ are fixed. The maximal unipotent subgroup of $G$ contained in $B$ is denoted by $U$. We identify the group $\mathfrak{X}(B)$ with $\mathfrak{X}(T)$ by restricting the characters from $B$ to $T$. The set of dominant weights of $B$ is denoted by $\mathfrak{X}^+(B)$.

The actions of $G$ on itself by left multiplication ($((g, x) \mapsto gx)$ and right multiplication ($((g, x) \mapsto xg^{-1})$) induce its representations on the space $\mathbb{C}[G]$ of all regular functions on $G$ given by $(gf)(x) = f(g^{-1}x)$ and $(gf)(x) = f(xg)$, respectively. For short, we call them the left action and the right action respectively. For any subgroup $L \subset G$, we write $\mathbb{C}[G]^L$ (resp. $\mathbb{C}[G]^L$) for the algebra of functions in $\mathbb{C}[G]$ that are invariant under the left (resp. right) action of $L$.

We now introduce the notion of the extended weight semigroup of a homogeneous space $G/H$.

Let $H \subset G$ be an arbitrary subgroup. For every character $\chi \in \mathfrak{X}(H)$ we denote by $V_\chi$ the subspace of $\mathbb{C}[G]$ consisting of weight functions of weight $\chi$ with respect to the right action of $H$, that is,

$$V_\chi = \{ f \in \mathbb{C}[G] : f(gh) = \chi(h)f(g) \quad \forall g \in G, \forall h \in H \}.$$
Let $H_0 \subset G$ be the intersection of the kernels of all characters in $\mathfrak{X}(H)$: $H_0 = \bigcap_{\chi \in \mathfrak{X}(H)} \ker \chi$. Then $\bigoplus_{\chi \in \mathfrak{X}(H)} V_\chi = \mathbb{C}[G]^{H_0}$. Since the left and right actions of $G$ on $\mathbb{C}[G]$ commute, the subspace $V_\chi$ is invariant under the left action of $G$ for any $\chi \in \mathfrak{X}(H)$.

We note that the quotient group $H/H_0$ is commutative (that is, $H/H_0$ is a quasi-torus) and the natural embedding $\mathfrak{X}(H/H_0) \hookrightarrow \mathfrak{X}(H)$ is an isomorphism.

Suppose that for each subspace $V_\chi$ its decomposition into a direct sum of irreducible $G$-modules is fixed. Then the highest weight vectors of all these $G$-modules taken over all $\chi$ form a basis of the algebra (considered as a vector space over $\mathbb{C}$)

$$A = A(G/H) := U(\mathbb{C}[G]^{H_0}) = U \mathbb{C}[G]^{H_0}.$$ 

Suppose $f \in A \setminus \{0\}$, $\lambda \in \mathfrak{X}_+(T)$, and $\chi \in \mathfrak{X}(H)$. We say that $f$ is a weight function with respect to $B \times H$ (or simply a weight function) of weight $(\lambda, \chi)$ (in the case $\mathfrak{X}(H) = 0$ we simply write $\lambda$ instead of $(\lambda,0)$), if $f$ is the highest weight vector of an irreducible $G$-module $N \subset V_\chi$ with highest weight $\lambda$. We denote by $A_{\lambda, \chi}$ the subspace of $A$ consisting of zero and all weight functions of weight $(\lambda, \chi)$. If $f_1 \in A_{\lambda_1, \chi_1}$ and $f_2 \in A_{\lambda_2, \chi_2}$, then $f_1 f_2 \in A_{\lambda_1 + \lambda_2, \chi_1 + \chi_2}$. Thus the set of pairs $(\lambda, \chi)$ with $\lambda \in \mathfrak{X}_+(B)$, $\chi \in \mathfrak{X}(H)$, and $A_{\lambda, \chi} \neq 0$ is a semigroup. We call this semigroup the extended weight semigroup (the term is suggested by Vinberg) of the homogeneous space $G/H$ and denote it by $\Gamma(G/H)$.

**Remark 1.** We have $\Gamma(G/H) = \{(\lambda, \chi) \in \Gamma(G/H) : \chi = 0\} \subset \Gamma(G/H)$. Clearly, $\mathfrak{X}(H) = 0$ implies that $\Gamma(G/H) = \hat{\Gamma}(G/H)$.

**Remark 2.** The map $\pi : \hat{\Gamma}(G/H) \to \Gamma(G/H_0)$, $(\lambda, \chi) \mapsto \lambda$, is surjective, and $\pi^{-1}(0) = \{(0,0)\}$.

**Remark 3.** There is another interpretation of the semigroup $\hat{\Gamma}(G/H)$ in the case when $H$ is connected. Namely, consider the group $\hat{G} = G \times (H/H_0)$ and its subgroup $\hat{H} \simeq H$ embedded in $\hat{G}$ via $h \mapsto (h, hH_0)$. The algebra $\mathbb{C}[\hat{G}]^{H_0}$ is a $\hat{G}$-module with respect to the left action of $G$ and the right action of $H_0$. Consider the algebra $\mathbb{C}[\hat{G}]^{\hat{H}}$ as a $\hat{G}$-module with respect to the left action of $\hat{G}$. The map $\psi : \mathbb{C}[\hat{G}]^{\hat{H}} \to \mathbb{C}[\hat{G}]^{H_0}$, $(\psi f)(g) = f(g, eH_0)$, is an isomorphism of $\hat{G}$-modules. (The inverse map $F \mapsto \hat{F}$ is given by $\hat{F}(g, hH_0) = F(gh^{-1})$.) Therefore there is a semigroup isomorphism $\hat{\Gamma}(G/H) \simeq \Gamma(\hat{G}/H)$.

1.3. Now suppose that $H \subset G$ is a spherical subgroup. According to Theorem 1 of [1], this implies that the representation of $G$ on $V_\chi$ is multiplicity free for each $\chi \in \mathfrak{X}(H)$. Hence, any pair $(\lambda, \chi) \in \hat{\Gamma}(G/H)$ determines a unique, up to multiplication by a non-zero constant, weight function $f \in A_{\lambda, \chi}$; that is, $\dim A_{\lambda, \chi} = 1$.

Let us prove the following fact. (Cf. [9], Proposition 2.)

**Theorem 1.** Suppose $G$ is simply connected and $H \subset G$ is a connected spherical subgroup. Then the algebra $A$ is factorial and the semigroup $\hat{\Gamma}(G/H)$ is free.

To prove Theorem 1 we need the following

**Theorem 2 ([10], Theorem 3.17).** Suppose a regular action of an algebraic group $L$ on an affine variety $X$ is given. If the algebra $\mathbb{C}[X]$ is factorial and $L$ is connected and has no
non-trivial characters, then the algebra $\mathbb{C}[X]^H$ is also factorial. Moreover, for any element $f \in \mathbb{C}[X]^H$ all its divisors in $\mathbb{C}[X]$ are contained in $\mathbb{C}[X]^H$.

**Proof of Theorem 1.** Since $G$ is simply connected, the algebra $\mathbb{C}[G]$ is factorial (see [11], Corollary from Proposition 1). Let $H = H_u \times R$ be a Levi decomposition of $H$, where $H_u$ (resp. $R$) is the unipotent radical (resp. a maximal reductive subgroup) of $H$. Then $H_0 = H_u \times [R,R]$, where $[R,R]$ is the derived subgroup of $R$. Since $H$ is connected, it follows that $R$ is also connected, whence the group $[R,R]$ is either connected and semisimple or trivial. Therefore $H_0$ is connected and has no non-trivial characters, so the group $U \times H_0$ also possesses these properties. Hence one can apply Theorem 2 to the action of $U \times H_0$ on $G$, where $U$ and $H_0$ act on $G$ by left and right multiplication, respectively. Thus the algebra $A$ is factorial. Let $\{\mu_i\}$ be the set of all indecomposable elements of $\hat{\Gamma}(G/H)$. As mentioned above, the sphericity of $H$ implies that for every element $\mu_i$ there is a unique, up to multiplication by a non-zero constant, weight function $f_i \in A$ of weight $\mu_i$. Moreover, $f_i$ is an irreducible element of $A$. Assume that there is a non-trivial relation $\sum_i k_i \mu_i = \sum_j l_j \mu_j$ for some integers $k_i, l_j > 0$. It implies the relation $c \prod_i f_i^{k_i} = \prod_j f_j^{l_j}$ for some $c \in \mathbb{C}^\times$, which contradicts the factoriality of $A$. \hfill \Box

### 1.4. A direct product of spherical homogeneous spaces

$$(G_1/H_1) \times (G_2/H_2) = (G_1 \times G_2)/(H_1 \times H_2)$$

is again a spherical homogeneous space. Spaces of this kind, as well as spaces locally isomorphic to them are called *reducible*, all others are said to be *irreducible*. A spherical space $G/H$ is said to be *strictly irreducible* if the spherical space $G/N(H)^0$ is irreducible, see [12], 1.3.6. (Here $N(H)^0$ is the connected component of the identity of the normalizer of $H$ in $G$.)

We now formulate the main results of this paper. First, we compute the semigroups $\hat{\Gamma}(G/H)$ for all strictly irreducible spherical pairs $(G, H)$, where $G$ is a simply connected non-simple semisimple algebraic group, $H$ its connected reductive subgroup. All such pairs except for symmetric pairs, which are to be discussed later, are listed in Table II. The indecomposable elements of the corresponding extended weight semigroups are indicated in the column $\hat{\Gamma}(G/H)$’ of Table II. Second, for each space $G/H$ in Table II we find the weight functions in $A$ that correspond to the indecomposable elements of $\hat{\Gamma}(G/H)$. These functions freely generate the algebra $A$.

Having known the semigroups $\hat{\Gamma}(G/H)$ for the spaces in Table II and taking into account Remark IV one can obtain a description of the semigroups $\Gamma(G/H)$ for these spaces. In particular, $\hat{\Gamma}(G/H) = \Gamma(G/H)$ for spaces 2, 4–8 in Table I.

Every simply connected strictly irreducible spherical homogeneous space of a non-simple semisimple algebraic group $G$ that is symmetric is isomorphic to one of the spaces of the form $X(H) = (H \times H)/H$ (the subgroup $H$ is embedded diagonally), where $H$ is simple and simply connected. For spherical homogeneous spaces of this kind, the structure of the semigroup $\hat{\Gamma}(X(H)) = \Gamma(X(H))$ is well known and follows, for instance, from Theorem 5 (see § 2, 1 below) with $L = K = H$. Namely, this semigroup is freely generated by the elements $\pi_i + \varphi^*_i$, $i = 1, \ldots, \text{rk} H$, where $\pi_i$ and $\varphi_i$ are the fundamental weights of the first and second factors of $H \times H$, respectively. The asterisk denotes the highest weight of the dual representation.
Table 1.

| No. | $G \supset H$                                                                 | Embedding diagram | $\text{rk}\hat{f}(G/H)$ | $\hat{f}(G/H)$ | Note |
|-----|------------------------------------------------------------------------------|-------------------|-------------------------|----------------|------|
| 1   | $\text{SL}_n \times \text{SL}_{n+1} \supset \text{SL}_n \times \mathbb{C}^\times$ | ![Diagram](image) | $2n$                    | $(\varphi_1, n\chi_0), (\pi_{n-1}+\varphi_2, (n-1)\chi_0), \ldots, (\pi_1+\varphi_n, \chi_0)$, $(\pi_{n-1}+\varphi_1, -\chi_0), \ldots, (\pi_1+\varphi_{n-1}, -(n-1)\chi_0), (\varphi_n, -n\chi_0)$ | $n \geq 2$ |
| 2   | $\text{Spin}_n \times \text{Spin}_{n+1} \supset \text{Spin}_n$               | ![Diagram](image) | $n$                     | $\varphi_1+\varphi_2, \pi_1+\varphi_1, \pi_1+\varphi_2$ for $n = 3$ | $n \geq 3$ |
|     |                                                                               |                   |                         | $\varphi_1, \pi_1+\varphi_1, \pi_1+\varphi_2, \pi_2+\varphi_2, \ldots, \pi_{k-2}+\varphi_{k-1}, \pi_{k-1}+\varphi_k, \pi_k+\varphi_k$ for $n = 2k$ |              |
|     |                                                                               |                   |                         | $\varphi_1, \pi_1+\varphi_1, \pi_1+\varphi_2, \pi_2+\varphi_2, \ldots, \pi_{k-1}+\varphi_{k-1}, \pi_{k-1}+\varphi_k, \pi_k+\varphi_k+1$ for $n = 2k+1\geq 5$ |              |
| 3   | $\text{SL}_n \times \text{Sp}_{2m} \supset \mathbb{C}^\times, \text{SL}_{n-2} \times \text{SL}_2 \times \text{Sp}_{2m-2}$ | ![Diagram](image) | $6^*$                   | $(\varphi_2, 0, (m \geq 2), (\pi_{n-1}+\varphi_1, \chi_0), (\pi_1+\varphi_{n-1}, 0), (\pi_1+\varphi_1, -\chi_0), (\pi_2, -2\chi_0)$ | $n \geq 3, m \geq 1$ |
| 4   | $\text{SL}_n \times \text{Sp}_{2m} \supset \text{SL}_{n-2} \times \text{SL}_2 \times \text{Sp}_{2m-2}$ | ![Diagram](image) | $6^*$                   | $\pi_{n-2}, \pi_{n-1}+\varphi_1, \pi_1+\varphi_{n-1}, \pi_1+\varphi_1, \pi_2$ | $n \geq 5, m \geq 1$ |
| 5   | $\text{Sp}_{2n} \times \text{Sp}_{2m} \supset \text{Sp}_{2n-2} \times \text{Sp}_2 \times \text{Sp}_{2m-2}$ | ![Diagram](image) | $3^*$                   | $\pi_2 (n \geq 2), \varphi_2 (m \geq 2), \pi_1+\varphi_1$ | $n \geq 1, m \geq 1$ |
| 6   | $\text{Sp}_{2n} \times \text{Sp}_4 \supset \text{Sp}_{2n-4} \times \text{Sp}_4$ | ![Diagram](image) | $6^*$                   | $\pi_1+\varphi_1, \pi_2+\varphi_2, \pi_3+\varphi_1, \pi_4 (n \geq 4), \pi_2, \pi_1+\varphi_4+\varphi_2$ | $n \geq 3$ |
| 7   | $\text{Sp}_{2n} \times \text{Sp}_{2m} \times \text{Sp}_{2l} \supset \text{Sp}_{2n-2} \times \text{Sp}_{2m-2} \times \text{Sp}_{2l-2} \times \text{Sp}_2$ | ![Diagram](image) | $6^*$                   | $\pi_2 (n \geq 2), \varphi_2 (m \geq 2), \psi_2 (l \geq 2), \pi_1+\varphi_1, \varphi_1+\psi_1, \pi_1+\psi_1$ | $n \geq 1, m \geq 1, l \geq 1$ |
| 8   | $\text{Sp}_{2n} \times \text{Sp}_4 \times \text{Sp}_{2m} \supset \text{Sp}_{2n-2} \times \text{Sp}_2 \times \text{Sp}_2 \times \text{Sp}_{2m-2}$ | ![Diagram](image) | $6^*$                   | $\varphi_2, \pi_1+\varphi_1, \varphi_1+\psi_1, \pi_2 (n \geq 2), \psi_4 (m \geq 2), \pi_1+\varphi_2+\psi_1$ | $n \geq 1, m \geq 1$ |

For computation of the extended weight semigroups, two different approaches are used in this paper. The first one is applied to spaces 1 and 2 in Table 1 and the second to all the others.
The paper is organized as follows. In §2 we formulate and prove all the statements needed for our approaches for computation of the extended weight semigroups. In §3 we compute the semigroups \( \Gamma(G/H) \) for each of the homogeneous spaces \( G/H \) appearing in Table 1. We also find the weight functions in \( A \) corresponding to the indecomposable elements of the respective semigroups \( \Gamma(G/H) \).

Let us explain the notation used in Table 1. The first two columns of this table are taken from Table 2 in [12] but are represented in a form which is more convenient for our purpose. The dot between the first and the second factor of \( H \) in row 3 denotes their almost direct product, that is, these two factors have a non-trivial (but finite) intersection.

The embedding diagram describes the embedding of \( H \) in \( G \) and ought to be interpreted as follows. The white nodes correspond to the factors of the group \( G \) and the black ones to those of \( H \). The order of nodes is the same as that of the corresponding factors, except for no. 7 where the upper black node corresponds to the last factor of \( H \). The factor of \( H \) corresponding to a black node \( v \) is diagonally embedded in the product of the factors of \( G \) that correspond to the white nodes joined with \( v \).

The column ‘\( \text{rk} \Gamma(G/H) \)’ shows the rank of the semigroup \( \Gamma(G/H) \), that is, the number of indecomposable elements of \( \Gamma(G/H) \). In this column, an asterisk stands for the cases when, for several small values of the parameters \( n, m, l \), the rank of the extended weight semigroup of the corresponding homogeneous space is less than the value indicated in the table. The exact value of the rank for given values of the parameters can be determined using the information in the column ‘\( \Gamma(G/H) \)’.

The column ‘\( \Gamma(G/H) \)’ contains a list of all indecomposable elements \( (\lambda, \chi) \) of the semigroup \( \Gamma(G/H) \). (If \( \mathfrak{X}(H) = 0 \), then we write simply \( \lambda \) instead of \( (\lambda, 0) \).) These elements freely generate it. If \( (G, H) \) is a pair in Table 1 with \( \mathfrak{X}(H) \neq 0 \), then \( H/H_0 \) is a one-dimensional torus. Each character \( \chi \in \mathfrak{X}(H) \) is identified with some character of this torus and, therefore, the characters of \( H \) are written in the column ‘\( \Gamma(G/H) \)’ as multiples of the character \( \chi_0 \), where \( \chi_0 \) is a fixed basis character of \( H/H_0 \). In each case, an explicit expression for \( \chi_0 \) can be found in §3. Whenever an element \( (\lambda, \chi) \) is followed by parenthesis containing an inequality for one of the parameters \( n, m, l \), the weight \( (\lambda, \chi) \) is contained in the set of indecomposable elements of \( \Gamma(G/H) \) if and only if the corresponding parameter satisfies that inequality.

Some notation and conventions

If the group \( G \) (resp. \( H_0 \)) is a product of several factors, then the \( i \)-th factor is denoted by \( G_i \) (resp. \( H_i \)). We write \( B_i, U_i, \) and \( T_i \) respectively for the Borel subgroup, the maximal unipotent subgroup, and the maximal torus of \( G_i \) such that \( B = \prod B_i, \ U = \prod U_i, \) and \( T = \prod T_i \). By \( \pi_i, \varphi_i, \) and \( \psi_i \) we denote the \( i \)-th fundamental weight of the first, the second, and the third factor of \( G \), respectively.

Our numeration of fundamental weights of simply connected simple algebraic groups is the same as in the book [13].

For every semisimple group \( L \), we denote by \( V_\lambda(L) \) the irreducible \( L \)-module with highest weight \( \lambda \). The weight dual to the weight \( \lambda \) is denoted by \( \lambda^* \).

The identity element of any group is denoted by \( e \).

By \( \mathbb{C}^* \) we denote the multiplicative group of the field \( \mathbb{C} \).
If \( P \) is a matrix, then the equation \( P = (p_{ij}) \) means that \( p_{ij} \) is the element in the \( i \)-th row and the \( j \)-th column of \( P \).

Given elements \( a_1, \ldots, a_n \) of a group, we write \( S\langle a_1, \ldots, a_n \rangle \) (resp. \( \langle a_1, \ldots, a_n \rangle \)) for the subsemigroup with identity (resp. the subgroup) generated by \( a_1, \ldots, a_n \).

We denote by \( E_n \) the identity matrix of order \( n \) and by \( F_n \) the matrix of order \( n \) with ones on the antidiagonal and zeros elsewhere.

The basis \( e_1, \ldots, e_n \) of the space of the tautological linear representation of the group \( \text{Sp}_{2m} \) is supposed to be chosen in such a way that the matrix of the invariant non-degenerate skew-symmetric bilinear form \( \omega_{2m} \) is

\[
\Omega_{2m} = \begin{pmatrix} 0 & F_m \\ -F_m & 0 \end{pmatrix}.
\]

With this choice of the basis, the Borel subgroup and the maximal unipotent subgroup of \( \text{Sp}_{2m} \) are represented by upper-triangular matrices.

2. Auxiliary results

2.1. Theorems 3–5 are used to compute the extended weight semigroups of spaces 1 and 2 in Table I. Theorems 3, 4 describe known branching rules for the groups \( \text{SL}_{n+1} \), \( \text{Spin}_{n+1} \), respectively (see original papers [14], [15] or a modern exposition in [16]) stated in a form which is convenient for our purpose.

**Theorem 3** (branching rule for the group \( \text{SL}_{n+1} \)). Suppose \( \lambda = c_1 \pi_1 + \cdots + c_n \pi_n \) is a dominant weight of \( \text{SL}_{n+1} \), where \( c_i \geq 0 \). Then the restriction of the irreducible representation of \( \text{SL}_{n+1} \) with highest weight \( \lambda \) to the subgroup \( \text{SL}_n \subset \text{SL}_{n+1} \) has the form

\[
\bigoplus_{\mu \in M(\lambda)} V_\mu(\text{SL}_n),
\]

where the set \( M(\lambda) \) consists of dominant weights \( \mu \) of \( \text{SL}_n \) (possibly with multiplicities) that can be obtained from \( \lambda \) by simultaneously replacing all summand of the form \( c_i \pi_i \) by \( a_i \pi_{i-1} + b_i \pi_i \), where \( a_i, b_i \geq 0 \) and \( a_i + b_i = c_i \). At that, we put \( \pi_0 = 0 \) and \( \pi_n |_{\text{SL}_n} = 0 \).

Before we formulate the next theorem, let us note that every dominant weight \( \lambda \) of the group \( \text{Spin}_{2k+2} \) is uniquely expressible in the form \( \lambda = c_1 \pi_1 + \cdots + c_{k+1} \pi_{k+1} + d(\pi_k + \pi_{k+1}) \), where \( c_i, d \geq 0 \) and at least one of the numbers \( c_k, c_{k+1} \) is zero.

**Theorem 4** (branching rules for the group \( \text{Spin}_{n+1} \)). a) Suppose \( \lambda = c_1 \pi_1 + \cdots + c_k \pi_k \) is a dominant weight of \( \text{Spin}_{2k+1} \), \( k \geq 2 \), where \( c_i \geq 0 \). Then the restriction of the irreducible representation of \( \text{Spin}_{2k+1} \) with highest weight \( \lambda \) to the subgroup \( \text{Spin}_2 \) has the form

\[
\bigoplus_{\mu \in M(\lambda)} V_\mu(\text{Spin}_{2k}),
\]

that can be obtained from \( \lambda \) by simultaneously replacing all summands of the form \( c_i \pi_i \) (for \( i \neq k-1 \)) by \( a_i \pi_{i-1} + b_i \pi_i \), and the summand \( c_{k-1} \pi_{k-1} \) by \( a_{k-1} \pi_{k-2} + b_{k-1} (\pi_{k-1} + \pi_k) \), where \( a_i, b_i \geq 0 \) and \( a_i + b_i = c_i \). At that, we put \( \pi_0 = 0 \).

b) Suppose \( \lambda = c_1 \pi_1 + \cdots + c_{k+1} \pi_{k+1} + d(\pi_k + \pi_{k+1}) \) is a dominant weight of \( \text{Spin}_{2k+2} \), \( k \geq 1 \), where \( c_i, d \geq 0 \) and at least one of the numbers \( c_k, c_{k+1} \) is zero. Then the restriction of the irreducible representation of \( \text{Spin}_{2k+2} \) with highest weight \( \lambda \) to the subgroup \( \text{Spin}_{2k+1} \) has the form

\[
\bigoplus_{\mu \in M(\lambda)} V_\mu(\text{Spin}_{2k+1}),
\]

that can be obtained from \( \lambda \) by simultaneously replacing all summands of the
form $c_i \pi_i$ (for $i < k$) by $a_i \pi_{i-1} + b_i \pi_i$, where $a_i, b_i \geq 0$ and $a_i + b_i = c_i$, all summands of the form $c_i \pi_i$ (for $i = k, k+1$) by $c_i \pi_k$, and the summand $d(\pi_k + \pi_{k+1})$ by $a \pi_{k-1} + 2b \pi_k$, where $a, b \geq 0$ and $a + b = d$. At that, we put $\pi_0 = 0$.

**Theorem 5.** Suppose $L, K$ are connected semisimple algebraic groups and $L \subset K$. Let $m_{\lambda, \mu}$ be the multiplicity with which the irreducible representation of $L$ with highest weight $\mu$ occurs in the irreducible representation of $K$ with highest weight $\lambda$. Consider the group $G = L \times K$ and its subgroup $H \cong L$ embedded in $G$ diagonally. Then there is an isomorphism of $G$-modules $\mathbb{C}[G]^H \cong \bigoplus_{\lambda, \mu} m_{\lambda, \mu} V_{\mu+\lambda^*}(G)$ ($G$ acts on $\mathbb{C}[G]^H$ by left multiplication), where $\lambda, \mu$ run over all dominant weights of $K, L$, respectively.

**Proof.** Consider the space $\mathbb{C}[K]$ as a $G$-module on which $L$ and $K$ act by left and right multiplication, respectively. There is an isomorphism of algebras and $G$-modules $\varphi : \mathbb{C}[G]^H \cong \mathbb{C}[K]$ given by $(\varphi f)(x) = f(e, x)$. Further, it is well-known that the space $\mathbb{C}[K]$, regarded as a $(K \times K)$-module with respect to the actions by left and right multiplication, is isomorphic to the direct sum $\bigoplus_{\lambda} V_\lambda(K)^* \otimes V_\lambda(K)$, where $\lambda$ runs over all dominant weights of $K$. Restricting the action of $K$ by right multiplication to $L$ and taking into account the relations $V_{\lambda}(K)^* \cong V_{\lambda^*}(K)$ and $V_{\lambda^*}(K) \otimes V_{\mu}(L) \cong V_{\lambda^*+\mu}(K \times L) \cong V_{\mu+\lambda^*}(G)$, we get the desired result.

2.2. The results in this subsection are used for computation of the extended weight semigroups of spaces 3–8 in Table 1.

**Lemma 1.** Suppose a group $L$ acts on an irreducible algebraic variety $X$. Suppose there is a closed subset $Y \subset X$, which is called a section, and an open subset $M \subset X$ such that the orbit of any point in $M$ meets $Y$. Then the restriction homomorphism $\rho : \mathbb{C}[X]^L \to \mathbb{C}[Y]$ is injective.

**Proof.** Assume that $\rho(f) = 0$ for some function $f \in \mathbb{C}[X]^L$ satisfies. Then $f|_M = 0$ because invariant functions are constant along orbits. This implies $f \equiv 0$. □

**Theorem 6.** Suppose the group $\text{Sp}_{2m-2k}, k \geq 1$, is embedded in $\text{Sp}_{2m}$ as the central $(2m - 2k) \times (2m - 2k)$ block and acts on $\text{Sp}_{2m}$ by right multiplication. Then the algebra of invariants of this action coincides with the subalgebra of the algebra $\mathbb{C}[\text{Sp}_{2m}]$ generated by the matrix entries of the first $k$ and last $k$ columns.

**Proof.** Let $V$ be the space of the tautological representation of $\text{Sp}_{2m}$. Consider the natural action of $\text{Sp}_{2m}$ on the space

$$W = V \oplus \cdots \oplus V \oplus \underbrace{V \oplus \cdots \oplus V}_k.$$

The subgroup $\text{Sp}_{2m-2k}$ in the hypothesis of the theorem is the stabilizer of the vector $w = (e_1, e_2, \ldots, e_k, e_{2m-k+1}, e_{2m-k+2}, \ldots, e_{2m})$ under this action. The orbit of $w$ in $W$ is isomorphic to the quotient space $\text{Sp}_{2m} / \text{Sp}_{2m-2k}$ and this isomorphism induces the correspondence between regular functions on this orbit and the required invariants. The orbit is closed since it consists of the sets of vectors $(v_1, v_2, \ldots, v_{2k})$ whose Gram matrix with respect to the form $\omega_{2m}$ is $\Omega_{2k}$. Hence, the algebra of regular functions on the orbit is generated by the restrictions of the coordinates of the ambient space $W$. These
coordinates correspond to the matrix entries of the first \( k \) and the last \( k \) columns of a matrix in \( \text{Sp}_{2m} \).

The next three lemmas are technical. The proofs of the first two of them are obtained by direct computation.

**Lemma 2.** For every non-degenerate matrix \( P \) of order \( m \), the matrix

\[
\begin{pmatrix}
P & 0 \\
0 & (P^{-1})^\# 
\end{pmatrix}
\]

of order \( 2m \) is symplectic. (Here the symbol \(#\) denotes the transpose of a matrix with respect to the antidiagonal.)

**Lemma 3.** The matrix

\[
\begin{pmatrix}
E_m & C \\
0 & E_m
\end{pmatrix}
\]

of order \( 2m \) is symplectic if and only if the matrix \( C \) of order \( m \) is symmetric with respect to the antidiagonal.

**Lemma 4.** Let \( P = (p_{ij}) \) be a \( 2m \times 2 \) matrix, \( m \geq 2 \), and \( P_1, P_2 \) its first and second columns, respectively. Suppose \( p_{2m,1} \neq 0, \Delta = p_{2m-1,1}p_{2m,2} - p_{2m-1,2}p_{2m,1} \neq 0, \) and \( P_1^\top \Omega_{2m} P_2 = 1 \) (the symbol \( \top \) denotes the transposed matrix). Then there are upper unitriangular matrices \( u_1, u_2 \in \text{Sp}_{2m} \) such that

\[
\begin{pmatrix}
0 & -1 \\
p_{2m,1} & 0 \\
\vdots & \vdots \\
p_{2m,1} & \Delta \\
p_{2m,1} & p_{2m,2}
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
p_{2m,1} & 0 \\
\vdots & \vdots \\
p_{2m,1} & \Delta \\
p_{2m,1} & p_{2m,2}
\end{pmatrix}
\]

(The dots stand for zero entries.)

**Proof.** Multiplying \( P \) on the left by an appropriate upper unitriangular matrix of type [1], we obtain a matrix \( P' \) whose lower half contains only three non-zero elements: \( p_{2m,1}, p_{2m,2}, \) and \( -\Delta/p_{2m,1} \) (as in the matrices \( u_1 P \) and \( u_2 P \) appearing in the assertion of the lemma). Then, multiplying \( P' \) on the left by an appropriate matrix of type [2], we obtain one of the required matrices. \( \square \)

The following theorem is used at the final stage of the argument in all cases.

**Theorem 7.** Suppose \( G \) is simply connected and \( H \subset G \) is a connected spherical subgroup. Suppose non-zero functions \( f_1, \ldots, f_n \in A \) satisfy the following conditions:

a) \( f_i \in A_{\lambda_i, \chi_i} \) for \( i = 1, \ldots, n \), where \( \lambda_i \in \mathfrak{X}_+(B) \), \( \chi_i \in \mathfrak{X}(H) \), and the weights \( (\lambda_1, \chi_1), \ldots, (\lambda_n, \chi_n) \) are linearly independent;

b) there is an inclusion \( A \subset \mathbb{C}[f_1, \ldots, f_n, f_1^{-1}, \ldots, f_k^{-1}] \) for some \( k \leq n \).

Put \( Z = \langle (\lambda_1, \chi_1), \ldots, (\lambda_k, \chi_k) \rangle + S((\lambda_{k+1}, \chi_{k+1}), \ldots, (\lambda_n, \chi_n)) \). Then:

1) if for each expression

\[
(\lambda_i, \chi_i) = (\mu_1, \sigma_1) + (\mu_2, \sigma_2), \quad \mu_1, \mu_2 \in \mathfrak{X}_+(B) \setminus \{0\}, \quad \sigma_1, \sigma_2 \in \mathfrak{X}(H),
\]
at least one of the weights \((\mu_1, \sigma_1), (\mu_2, \sigma_2)\) is not contained in \(Z\), then the element \(f_i\) is irreducible in \(A\);

2) if \((\lambda_i, \chi_i) = (\lambda_j, \chi_j) + (\lambda_i - \lambda_j, \chi_i - \chi_j), i \neq j\), is the unique expression of the weight \((\lambda_i, \chi_i)\) in the form \((\Box)\) such that both summands belong to \(Z\), and if \(f_i\) is not divisible by \(f_j\) in \(A\), then the element \(f_i\) is irreducible in \(A\);

3) if \(f_i\) is irreducible in \(A\) for \(i = 1, \ldots, n\), then \(A = \mathbb{C}[f_1, \ldots, f_n]\) and \(\hat{\Gamma}(G/H) = S((\lambda_1, \chi_1), \ldots, (\lambda_n, \chi_n))\).

**Proof.** Let us prove 1), 2). Assume that \(f_i\) is reducible in \(A\). Then \(f_i = F_1 F_2\) where for \(j = 1, 2\) the element \(F_j \in A\) is not invertible and \(F_j \in A_{\xi_j, \eta_j}\) for some \(\xi_j \in \mathfrak{X}(B), \eta_j \in \mathfrak{X}(H)\). We have \(\xi_j \neq 0, j = 1, 2\), because otherwise \(F_j\) would be a constant. It follows from b) that each of the functions \(F_1, F_2\) is expressible as an irreducible fraction of the form

\[
F_j = \frac{h_j(f_1, \ldots, f_n)}{f_j^{\beta_j}}
\]

where \(\beta_{j1}, \ldots, \beta_{jk} \geq 0\) and \(h_j\) are polynomials in \(n\) variables. Using a), we obtain

\[
h_j(f_1, \ldots, f_n) = c_j f_{1}^{\alpha_{j1}} \cdots f_{n}^{\alpha_{jn}},
\]

where \(c_j \neq 0\) and \(\alpha_{j1}, \ldots, \alpha_{jn} \geq 0\). Then \((\mu_j, \sigma_j) \in Z\) for \(j = 1, 2\). In the hypothesis of 1), we have already come to a contradiction. In the hypothesis of 2), we see that one of the functions \(F_1, F_2\) has weight \((\lambda_j, \chi_j)\) and, therefore, is proportional to \(f_j\) since \(\dim A_{\lambda_j, \chi_j} = 1\). Thus, \(f_i\) is divisible by \(f_j\), a contradiction.

We now prove 3). Suppose \(f \in A_{\lambda, \chi}\) for some \(\lambda \in \mathfrak{X}(B), \chi \in \mathfrak{X}(H)\). Arguing as in the proof of 1), we see that \(f\) is expressible as an irreducible fraction of the form \(\frac{\prod_{i=1}^{n} f_{i}^{\alpha_{i}}}{\prod_{k=1}^{m} f_{k}^{\beta_{k}}}\), where \(\beta_{1}, \ldots, \beta_{k} \geq 0\) and \(\alpha_{1}, \ldots, \alpha_{n} \geq 0\). Since \(A\) is factorial (Theorem \(\Box\)) and all the elements \(f_i\) are irreducible, it follows that the numerator of this fraction is divisible by its denominator, whence \(\beta_{1} = \cdots = \beta_{k} = 0\). Therefore, \(f = f_{1}^{\alpha_{1}} \cdots f_{n}^{\alpha_{n}}\) and we obtain the required result. \(\square\)

### 3. Computation of the extended weight semigroups

In this section, the cases are numbered in accordance with the numbers in Table \(\Box\). Except in Case 2 we use the following convention. For each factor \(G_i \subset G\) (all of them are of type \(\text{SL}\) or \(\text{Sp}\)), the subgroups \(B_i, U_i, T_i\) consist of all upper triangular, upper unitriangular, and diagonal matrices, respectively, contained in \(G_i\).

#### 3.1. At first, we compute the semigroups \(\hat{\Gamma}(G/H)\) for spaces 1, 2 in Table \(\Box\)

**Case 1.** \(G = \text{SL}_n \times \text{SL}_{n+1}, H = \mathbb{C}^\times, H\) being embedded in \(G\) in such a way that the image in \(G\) of a pair \((P, t) \in H\) is the pair \((P, P') \subset G\), where \(P' = \varphi(P, t) = \begin{pmatrix} P & 0 \\ 0 & t^{-n} \end{pmatrix}\). Further, \(H_0 = \text{SL}_n \times \{e\} \subset H\). The basis character \(\chi_0 \in \mathfrak{X}(H)\) takes each element \((P, t) \in H\) to \(t\).

Let us present \(2n\) functions in \(A\) that are weight functions with respect to \(B \times H\). Given \((P, Q) \in G\), we put \(R = Q P^{-1}\), where \(P = \varphi(P, e)\). We denote by \(\Delta_i, i = 1, \ldots, n\), the minor of \(R\) corresponding to the last \(i\) rows and first \(i\) columns, and by \(\delta_i, i = 1, \ldots, n\), the minor of \(R\) corresponding to the last \(i\) rows and columns \(n+1, 1, 2, \ldots, i-1\). The \(2n\) functions \(\Delta_1, \ldots, \Delta_n, \delta_1, \ldots, \delta_n\) all belong to \(A\) and are weight functions with respect to
The weights of $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}, \Delta_n$ are

\[(\pi_1 + \varphi_n, \chi_0), (\pi_2 + \varphi_{n-1}, 2\chi_0), \ldots, (\pi_{n-1} + \varphi_2, (n-1)\chi_0), (\varphi_1, n\chi_0),\]

respectively, the weights of $\delta_1, \delta_2, \delta_3, \ldots, \delta_n$ are

\[(\varphi_n, -n\chi_0), (\pi_1 + \varphi_{n-1}, -(n-1)\chi_0), (\pi_2 + \varphi_{n-2}, -(n-2)\chi_0), \ldots, (\pi_{n-1} + \varphi_1, -\chi_0),\]

respectively. Moreover, the $2n$ weights of the functions $\Delta_1, \ldots, \Delta_n, \delta_1, \ldots, \delta_n$ are linearly independent, whence $\text{rk} \widehat{\Gamma}(G/H) \geq 2n$. Since $\text{rk} G = 2n - 1$ and $\text{rk} H/H_0 = 1$, it follows that $\text{rk} \widehat{\Gamma}(G/H) \leq 2n$. Hence $\text{rk} \widehat{\Gamma}(G/H) = 2n$.

Applying Theorem [3] for $L = \text{SL}_n$, $K = \text{SL}_{n+1}$ and using Theorem [3], we determine the spectrum of the representation of $G$ on the space $\mathbb{C}[G]^{H_0}$. (Thereby we determine the semigroup $\Gamma(G/H_0)$.) In particular, we obtain the following two facts. First, this spectrum contains the irreducible $G$-modules with highest weights

\[(4) \pi_1 + \varphi_n, \pi_2 + \varphi_{n-1}, \ldots, \pi_{n-1} + \varphi_2, \varphi_1, \varphi_n, \pi_1 + \varphi_{n-1}, \pi_2 + \varphi_{n-2}, \ldots, \pi_{n-1} + \varphi_1\]

(2n weights in total), each of multiplicity 1. Second, any non-zero weight in $\Gamma(G/H_0)$ contains at least one summand of the form $\varphi_i$. It follows that each of the weights (4) is indecomposable in $\Gamma(G/H_0)$. We note that the set of $2n$ weights (4) is the image under the map $\pi$ (see Remark [2]) of the set of $2n$ weights with respect to $B \times H$ that correspond to the functions $\Delta_1, \ldots, \Delta_n, \delta_1, \ldots, \delta_n$. Hence, by Remark [2] the latter $2n$ weights are indecomposable in $\widehat{\Gamma}(G/H)$. Since $\text{rk} \widehat{\Gamma}(G/H) = 2n$, it follows that the semigroup $\widehat{\Gamma}(G/H)$, which is free, is generated by the weights with respect to $B \times H$ of the functions $\Delta_1, \ldots, \Delta_n, \delta_1, \ldots, \delta_n$. Therefore $A = \mathbb{C}[\Delta_1, \ldots, \Delta_n, \delta_1, \ldots, \delta_n]$.

Case 2. $G = \text{Spin}_n \times \text{Spin}_{n+1}$, $H = H_0 = \text{Spin}_n$. Since $\mathfrak{K}(H) = 0$, we get $\widehat{\Gamma}(G/H) = \Gamma(G/H)$ and a description of the semigroup $\Gamma(G/H)$ follows from Theorems [3][4]. Namely, a direct check shows that, for each $n \geq 3$, the weights in the column ‘$\widehat{\Gamma}(G/H)$’ of Table [1] (there are exactly $n$ of them) lie in the semigroup $\widehat{\Gamma}(G/H)$ and linearly independent, therefore $\text{rk} \Gamma(G/H) \geq n$. Since $\text{rk} \Gamma(G/H) \leq \text{rk} G = n$, we get $\text{rk} \Gamma(G/H) = n$. Furthermore, it is easy to see that every non-zero element of $\Gamma(G/H)$ contains one of the elements $\varphi_i$ as a summand, whence all these weights are indecomposable except for the weight $\pi_{k-1} + \varphi_k + \varphi_{k+1}$ for $n = 2k + 1$. The last weight is indecomposable in $\Gamma(G/H)$ because none of the weights $\varphi_k, \varphi_{k+1}$ is contained in $\Gamma(G/H)$. Thus, for each $n \geq 3$ we have found $n$ indecomposable linearly independent weights in the semigroup $\Gamma(G/H)$. Since $\text{rk} \Gamma(G/H) = n$, these $n$ weights freely generate the semigroup.

We now present weight functions generating the algebra $A(G'/H')$ where $G'/H'$ is a homogeneous space locally isomorphic to $G/H$. Namely, we consider the group $G' = \text{SO}_n \times \text{SO}_{n+1}$ and its subgroup $H' = \text{SO}_n$ embedded in $G'$ diagonally. The covering homomorphism of groups $\psi: G \to G'$ induces the morphism $\psi_H: G/H \to G'/H'$, which is a two-sheeted covering. Therefore there is an embedding of algebras

\[\psi_H^*: \mathbb{C}[G'/H'] \hookrightarrow \mathbb{C}[G/H],\]

at that, $\psi_H^*(A(G'/H')) \subset A(G/H)$.

For each $m$ we choose a basis $\{e_i\}$ in the space $V_m$ of the tautological representation of $\text{SO}_m$ such that the matrix of the invariant non-degenerate symmetric bilinear form is $F_m$. Then all upper-triangular and diagonal matrices in $\text{SO}_m$ form a Borel subgroup $\tilde{B}_m$. 

and a maximal torus $\tilde{T}_m$, respectively. We shall consider weights of irreducible representations of $SO_m$ with respect to $\tilde{B}_m$ and $\tilde{T}_m$. We fix the embedding $\tau_n$: $SO_m \hookrightarrow SO_{m+1}$ such that its image is the stabilizer of the vector $e_{m+1}$ for even $m$ and the vector $e_{m+1} - e_{m+1}$ for odd $m$.

We fix the embedding $H'$ in $G'$ sending every matrix $P \in H'$ to $(P, \tau_n(P))$. Suppose $(P, Q) \in G'$ and put $R = Q\tau_n(P)^{-1}$. Now we present the generators of $A(G'/H')$ for each $n$.

If $n = 2k$, then we consider the following functions on $R$: $\Delta_i$, $i = 1, \ldots, k$, is the minor corresponding to the last $i$ rows and first $i$ columns; $\delta_i$, $i = 1, \ldots, k$, is the minor corresponding to the last $i$ rows and columns $k+1, \ldots, i-1$; $\Phi$ is the minor corresponding to the last $k$ rows and columns $k+2, 1, \ldots, k-1$. We have $\delta_i^2 = -2\Delta_i\Phi$. All these functions are weight functions with respect to $(\tilde{B}_n \times \tilde{B}_{n+1}) \times H'$ and generate $A(G'/H')$. The weight of $\Delta_i$ is $\pi_i + \varphi_i$ for $i \leq k-2$, $\pi_{k-1} + \varphi_k + \varphi_{k-1}$ for $i = k-1$, and $2(\pi_k + \varphi_k)$ for $i = k$. The weight of $\delta_i$ is $\pi_{i-1} + \varphi_i$ for $i \leq k-1$ (we put $\pi_0 = 0$) and $\pi_{k-1} + \varphi_k + 2\varphi_k$ for $i = k$. The weight of $\Phi$ is $2(\pi_{k-1} + \varphi_k)$. The algebra $A(G/H)$ also contains functions $\Delta$ and $D$ such that $\Delta^2 = \psi_H^{\ast}(\Delta_k)$, $D^2 = \psi_H^{\ast}(\Phi)$, and $\sqrt{-2}\Delta D = \psi_H^{\ast}(\delta_k)$. Their weights are $\pi_i + \varphi_k$ and $\pi_{i-1} + \varphi_k$, respectively. The functions $\psi_H^{\ast}(\Delta_j)$ and $\psi_H^{\ast}(\delta_j)$, $i, j \leq k-1$, along with the functions $\Delta$ and $D$ correspond to the indecomposable elements of $\hat{G}/H$ and generate $A(G/H)$.

If $n = 2k+1$, then we consider the following functions on $R$: $\Delta_i$, $i = 1, \ldots, k$, is the minor corresponding to the last $i$ rows and first $i$ columns; $\delta_i$, $i = 1, \ldots, k+1$, is the difference of two minors, the first corresponding to the last $i$ rows and columns $k+1, \ldots, i-1$, and the second corresponding to the last $i$ rows and columns $k+2, 1, \ldots, i-1$; $\Phi$ is the minor corresponding to rows $1, \ldots, k, k+2$ (counting from the bottom) and columns $1, \ldots, k, 2\left[\frac{k}{2}\right] + 2$. We have $\Delta_k^2 = (-1)^{k+1}\delta_{k+1}\Phi$. All these functions are weight functions with respect to $(\tilde{B}_n \times \tilde{B}_{n+1}) \times H'$ and generate $A(G'/H')$. The weight of $\Delta_i$ is $\pi_i + \varphi_i$ for $i \leq k-1$ and $2\pi_k + \varphi_k + \varphi_{k+1}$ for $i = k$. The weight of $\delta_i$ is $\pi_{i-1} + \varphi_i$ for $i \leq k-1$, $\pi_{k-1} + \varphi_k + \varphi_{k+1}$ for $i = k$, and $2(\pi_k + \varphi_{k+1})$ for $i = k+1$ (we put $\pi_0 = 0$). The weight of $\Phi$ is $2(\pi_k + \varphi_k)$. The algebra $A(G/H)$ also contains functions $\delta$ and $D$ such that $\delta^2 = \psi_H^{\ast}(\delta_{k+1})$, $D^2 = \psi_H^{\ast}(\Phi)$, and $\sqrt{(-1)^{k+1}}\delta D = \Delta_k$. Their weights are $\pi_k + \varphi_{k+1}$ and $\pi_k + \varphi_k$, respectively. The functions $\psi_H^{\ast}(\Delta_j)$, $i \leq k-1$, $\psi_H^{\ast}(\delta_j)$, $j \leq k$, $\delta$, and $D$ correspond to the indecomposable elements of $\hat{G}/H$ and generate $A(G/H)$.

3.2. We now proceed to computing the extended weight semigroups of spaces 3–8 in Table I. First we describe the general method.

In each of the cases considered below we search for the algebra $A$. The functions in this algebra satisfy $f(g) = f(u^{-1}gh)$ for all $g \in G$, $u \in U$, $h \in H_0$. To find such functions, we multiply an arbitrary element $g$ of some dense open subset $M \subset G$ by appropriate elements in $U$ and $H_0$ so as to obtain an element of ‘canonical’ form. A canonical form for elements of $M$ is specified by the condition that some of the matrix entries equal zero, some others equal one, and some of the remaining entries equal minus one. The set $Y$ of elements of the whole group $G$ (not only in $M$) satisfying these restrictions is closed in $G$ and serves as a section in the sense of Lemma I. By that lemma, $A$ is contained in the algebra $\mathbb{C}[Y]$. 

In all the cases we first present a set of functions \( f_1, \ldots, f_p \in A \) that are weight functions with respect to \( B \times H \) and then we use the canonical form to prove that these functions generate \( A \). (In almost all cases the functions \( f_1, \ldots, f_p \) naturally arise when reducing to canonical form.) The set \( M \) is determined by the condition that some of the functions \( f_1, \ldots, f_p \) do not vanish. 

If one of the factors of \( G \) is \( \text{Sp}_{2m} \) and one of the factors of \( H_0 \) is \( \text{Sp}_{2m-2k} \) embedded only in \( \text{Sp}_{2m} \) (as the central \((2m-2k) \times (2m-2k)\) block), then by Theorem \([6]\) every function in \( A \) is independent of the matrix entries of the factor \( \text{Sp}_{2m} \) located in columns \( k + 1, k + 2, \ldots, 2m - k - 1, 2m - k \).

Therefore, when reducing to canonical form we may care not about transformations of these entries under the actions of \( U \) and \( H_0 \). In this connection, firstly, we do not consider anymore the action of the factor \( \text{Sp}_{2m-2k} \subset H_0 \) since it transforms in a non-trivial way only columns of \( \text{Sp}_{2m} \) mentioned above. Secondly, it is sufficient for our purpose to reduce to canonical form not the whole of a matrix in \( \text{Sp}_{2m} \) but only its first \( k \) and last \( k \) columns. In other words, it suffices to impose restrictions of the form \( g_\alpha = c, \) where \( c \in \{0, 1, -1\} \), defining a canonical form of a matrix in \( \text{Sp}_{2m} \) only on the matrix entries \( g_\alpha \) of the factor \( \text{Sp}_{2m} \) that are located in the \( 2k \) columns indicated above. Therefore, when formally considering matrices \( Q \in \text{Sp}_{2m} \), we actually deal only with their submatrices \( \overline{Q} \) consisting of the first \( k \) and the last \( k \) columns of \( Q \), and it is \( \overline{Q} \) that is reduced to canonical form. This can be interpreted as follows: the factor \( \text{Sp}_{2m} \) of \( G \) is replaced by the quotient space \( \text{Sp}_{2m} / \text{Sp}_{2m-2k} \) on which the actions of \( U \) and the remaining factors of \( H_0 \) are preserved. As we see from the proof of Theorem \([6]\) this quotient space can be thought of as the set of \( 2m \times 2k \) matrices whose columns satisfy the same relations as the first \( k \) and the last \( k \) columns of a matrix in \( \text{Sp}_{2m} \).

It always turns out that the matrix entries of the canonical form of an element \( g \in M \) on which the functions in \( A \) can depend are rational functions (more precisely, Laurent polynomials) in the values of \( f_1, \ldots, f_p \) at the point \( g \). We denote these rational functions, which are obviously invariant with respect to \( U \) and \( H_0 \), by \( r_1(f_1, \ldots, f_p), \ldots, r_q(f_1, \ldots, f_p) \). Since regular functions on the section are generated by the restrictions of the coordinate functions on \( G \), it follows that every function \( f \in A \) is a polynomial in the functions \( r_1, \ldots, r_q \), that is,

\[
f(g) = F(r_1(f_1(g), \ldots, f_p(g)), \ldots, r_q(f_1(g), \ldots, f_p(g)))
\]

for every \( g \in M \), where \( F(x_1, \ldots, x_q) \) is a polynomial. Thus, there is an inclusion \( A \subset \tilde{A} = \mathbb{C}[r_1(f_1, \ldots, f_p), \ldots, r_q(f_1, \ldots, f_p)] \).

Since the algebra \( \tilde{A} \) consists of rational functions that are invariant under \( U \) and \( H_0 \), it follows that \( A = \tilde{A} \cap \mathbb{C}[G] \). Therefore our next step aims at extracting regular functions from \( \tilde{A} \). This is carried out using Theorem \([7]\) Namely, it follows from assertions 1) and 2) (the latter one is used only in Case 3 for \( n = 3 \) and Case 8) of Theorem \([7]\) that each of the functions \( f_1, \ldots, f_p \) is irreducible in \( A \). Then assertion 3) yields that \( A = \mathbb{C}[f_1, \ldots, f_p] \).

A description of the semigroup \( \Gamma(G/H) \) also follows from assertion 3) of Theorem \([7]\).

In all the cases the conditions a) and b) of Theorem \([7]\) are verified directly, therefore we do not even mention that except in Cases 3, 4. We check the hypothesis of assertion 1) of this theorem only in Cases 3, 4 since it is verified similarly in all the remaining cases.
We now proceed to consideration of all the cases. Our argument follows the plan discussed above, which is therefore used without extra explanation.

Cases 3, 4. It is convenient to consider the pairs in rows 3, 4 of Table II together. In both cases, \( G = SL_n \times Sp_{2m} \), in Case 3 we have \( H = C^\times \cdot SL_{n-2} \times SL_2 \times Sp_{2m-2} \), in Case 4 we have \( H = SL_{n-2} \times SL_2 \times Sp_{2m-2} \). Case 4 is considered only for \( n \geq 5 \), otherwise the space \( G/H \) is not spherical. The embedding of \( H \) in \( G \) is as follows. The factor \( SL_{n-2} \) is embedded in the factor \( SL_n \) of \( G \) as the upper left \( (n-2) \times (n-2) \) block. The factor \( SL_2 \) is diagonally embedded in \( G \) as the lower right \( 2 \times 2 \) block in \( SL_n \) and as the \( 2 \times 2 \) block corresponding to the first and last rows and columns of the factor \( Sp_{2m} \). The factor \( Sp_{2m-2} \) is embedded in \( Sp_{2m} \) as the central \( (2m-2) \times (2m-2) \) block. In Case 3 the torus \( C^\times \subset H \) is embedded in \( SL_n \) as \( E_{n-2}t^{n-2} \) for odd \( n \) and as \( E_{n-2}t^{-1} \) for even \( n \). The group \( H_0 \) is the same in both cases and equals \( SL_{n-2} \times SL_2 \times Sp_{2m-2} \). In Case 3 the basis character \( \chi_0 \in \mathcal{X}(H) \) acts on the torus \( C^\times \subset H \) as \( t \mapsto t^{n-2} \) for odd \( n \) and as \( t \mapsto t^{-n-2} \) for even \( n \).

We search for functions \( f \in \mathbb{C}[G] \) satisfying

\[
f(P, Q) = f(u_1^{-1}Pu_2, u_2^{-1}Qh_2h_3)
\]

for all \( P \in G_1, Q \in G_2, u_1 \in U_1, u_2 \in U_2, h_i \in H_i, i = 1, 2, 3 \). Suppose \( P = (p_{ij}) \) and \( Q = (q_{ij}) \). We denote by \( P_{ij} \) the \((i,j)\)-cofactor of \( P \) so that \( P^{-1} = (P_{ji}) \).

Theorem 6 allows us not to consider the action of \( H_3 \) and to reduce to canonical form the first and last columns of \( Q \).

Suppose \( \Delta = p_{n-1,n-1}p_{n,n} - p_{n-1,n}p_{n,n-1}, W = q_{2m-1,2m}q_{2m-1,2m} - q_{2m-1,2m}q_{2m,1}, D = p_{n,n-1}q_{2m,2m} - p_{n,n}q_{2m,1}, \Phi_1 = p_{n,n-1}P_{1,n-1} + p_{n,n}P_{1,n}, \Phi_2 = q_{2m,1}P_{1,n-1} + q_{2m,2m}P_{1,n}, \delta \) is the minor of \( P \) corresponding to the last \( n-2 \) rows and the first \( n-2 \) columns. We have \( \Delta, W, D, \Phi_1, \Phi_2, \delta \in A \). At that, \( W \equiv 1 \) for \( m = 1 \) and \( \Phi_1 \equiv -\delta \Delta \) for \( n = 3 \). All these functions are weight functions with respect to \( B \times H \). Their weights are listed in Table 2.

| No. | \( \Delta \)          | \( W \)                     | \( D \)         | \( \Phi_1 \)     | \( \Phi_2 \)     | \( \delta \)  |
|-----|------------------------|-----------------------------|----------------|-----------------|----------------|-------------|
| 3   | \( \pi_{n-2,2}\chi_0 \)\( (\varphi_2,0) (m \geq 2) \) | \( \pi_{n-1,1,\varphi_1,0} \) | \( \pi_1 + \varphi_1 \) | \( 0 \)  | \( 0 \)  | \( 0 \)  |
| 4   | \( \pi_{n-2} \)        | \( \varphi_2 (m \geq 2) \)  | \( \pi_{n-1} + \varphi_1 \) | \( \pi_1 + \varphi_1 \) | \( \pi_1 + \varphi_1 \) | \( \pi_2 \)  |

Below we shall apply Theorem 7 to the functions \( \Delta, W \) (the latter is present for \( m \geq 2 \), \( D, \Phi_1, \Phi_2, \delta \) for \( n \geq 4 \) (in Cases 3, 4) and the functions \( \Delta, W \) (the latter is present for \( m \geq 2 \), \( D, \Phi_1, \delta \) for \( n = 3 \) (in Case 3). We note that in each case the set of weights with respect to \( B \times H \) corresponding to these functions is linearly independent.

Let \( M \) be the open subset of \( G \) given by \( \Delta \neq 0, W \neq 0, D \neq 0, \Phi_1 \neq 0, \delta \neq 0 \) and suppose \( (P, Q) \in M \). Acting by \( H_2 \) we transform \( (P, Q) \) to a pair \( (P', Q') \) such that the lower right \( 2 \times 2 \) block of \( P' \) and the lower \( 2 \times 2 \) block of \( Q' \) are

\[
\begin{pmatrix}
1 & * \\
0 & \Delta
\end{pmatrix},
\begin{pmatrix}
* & W\Delta \\
D & D
\end{pmatrix}
\]
Therefore the desired algebra $n$ in each of the cases $m = 2$, \[ \overline{Q'} = \begin{pmatrix} 0 & \Delta & 0 \\ \vec{D} & 0 & 0 \\ -\Delta & 0 & 0 \end{pmatrix}, \quad m = 2, \quad \overline{Q'} = \begin{pmatrix} 0 & \Delta \\ \vec{D} & 0 \end{pmatrix}, \quad m = 1. \]

(The dots stand for zero entries.) We now turn to the matrix $P'$. Firstly, acting by $U_1$ we make all the entries in the last two columns equal to zero except for the two on the diagonal of the lower $2 \times 2$ block. (These entries are 1 and $\Delta$.) Acting by $H_1$ on the obtained matrix, we transform the block corresponding to the last $n - 2$ rows and first $n - 2$ columns to the form $\text{diag}(1, \ldots, 1, \delta)$. After that, again acting by $U_1$, we transform the new matrix to the form (for $n \geq 5$, $n = 4$, $n = 3$, respectively)

\[
\begin{pmatrix}
0 & \cdots & 0 & \pm \frac{\delta}{\Phi_1} & 0 & 0 \\
0 & \cdots & \frac{\Phi_1}{\Delta \delta} & \pm \frac{\delta}{\Phi_2} & 0 & 0 \\
1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 1 & 0 & 1 & 0 \\
0 & \cdots & 0 & \delta & 0 & \Delta
\end{pmatrix},
\begin{pmatrix}
0 & -\frac{\delta}{\Phi_1} & 0 & 0 \\
\frac{\Phi_1}{\Delta \delta} & \frac{\Phi_2}{\Delta \delta} & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & \delta & 0 & \Delta
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\Phi_2}{\Delta \delta} & 0 \\
0 & \delta & 0 & \Delta
\end{pmatrix}.
\]

(The lower left $(2n - 2) \times (2n - 2)$ block of the first matrix is equal to $\text{diag}(1, \ldots, 1, \delta)$ and the remaining dots stand for zero entries.) We denote the resulting matrix by $P''$ in each of the cases $n \geq 5$, $n = 4$, $n = 3$.

The pair $(P'', Q'')$ is the canonical form of the pair $(P, Q)$. Thus the section is obtained. Therefore the desired algebra $A$ is contained in the algebra

\[
\mathcal{A} = \mathbb{C} \left[ \left\{ \Delta, D, \frac{W \Delta}{D}, \Delta, \delta, \frac{1}{\Delta}, \frac{1}{\Phi_1}, \frac{1}{\Phi_2} \right\} \right] \subset \mathbb{C} \left[ \Delta, W, D, \Phi_1, \Phi_2, \delta, \frac{1}{\Delta}, \frac{1}{\Delta}, \frac{1}{\Phi_1}, \frac{1}{\Phi_2} \right]
\]

for $n \geq 4$ and in the algebra

\[
\mathcal{A} = \mathbb{C} \left[ \left\{ \Delta, D, \frac{W \Delta}{D}, \Delta, \delta, \frac{1}{\Delta}, \frac{1}{\Phi_2} \right\} \right] \subset \mathbb{C} \left[ \Delta, W, D, \Phi_2, \delta, \frac{1}{\Delta}, \frac{1}{\Delta} \right]
\]

for $n = 3$.

We now apply Theorem 7 to the set of functions $\Delta, D, \Phi_1, \delta, \Phi_2, W$ (the last is present for $m \geq 2$) for $n \geq 4$ and $\Delta, D, \delta, \Phi_2, W$ (the last is present for $m \geq 2$) for $n = 3$. We have already seen that condition a) holds. Condition b) follows from the inclusions 6 and 7.

Further we use assertion 1) of Theorem 7. First, we note that the weights of $\Delta, W$ (for $m \geq 2$), $\delta$ admit no representation of the form 3. Hence, these three functions are irreducible in $A$. Every representation of the weight of $D$ in the form 3 has the form $(\pi_{n-1}, a\chi_0) + (\varphi_1, b\chi_0)$ where $a, b \in \mathbb{Z}$ and $a + b = 1$. Every representation of the weight of $\Phi_1$ in the form 3 has the form $(\pi_1, a\chi_0) + (\pi_{n-1}, b\chi_0)$ where $a, b \in \mathbb{Z}$ and $a + b = 0$. Every representation of the weight of $\Phi_2$ in the form 3 has the form $(\pi_1, a\chi_0) + (\varphi_1, b\chi_0)$ where $a, b \in \mathbb{Z}$ and $a + b = -1$. We now distinguish two possibilities: $n \geq 4$ and $n = 3$.  

At first, suppose \( n \geq 4 \). Then it is easy to see that none of the weights of the form \((\pi_1, p\chi_0), (\pi_{n-1}, q\chi_0), (\varphi_1, r\chi_0)\), where \( p, q, r \in \mathbb{Z} \), lies in the set
\[
Z = \langle (\pi_{n-2}, 2\chi_0), (\pi_{n-1} + \varphi_1, \chi_0), (\pi_1 + \pi_{n-1}, 0), (\pi_2, -2\chi_0) \rangle
+ S\langle (\pi_1 + \varphi_1, -\chi_0), (\varphi_2, 0) \rangle.
\]
Therefore the functions \( D, \Phi_1, \Phi_2 \) are also irreducible in \( A \). Thus we have checked the hypothesis of assertion 3) of Theorem 7, hence the functions \( \Delta, D, \Phi_1, \delta, \Phi_2, W (m \geq 2) \) generate the algebra \( A \) and their weights with respect to \( B \times H \) generate the semigroup \( \hat{\Gamma}(G/H) \).

Now suppose \( n = 3 \). We shall prove that the function \( D \) is irreducible using assertion 1) of Theorem 7. Assume that there are integers \( a, b \) such that \( a + b = 1 \) and both weights \((\pi_2, a\chi_0), (\varphi_1, b\chi_0)\) lie in the set
\[
Z = \langle (\pi_1, 2\chi_0), (\pi_2 + \varphi_1, \chi_0) \rangle + S\langle (\pi_2, -2\chi_0), (\pi_1 + \varphi_1, -\chi_0), (\varphi_2, 0) \rangle.
\]
Then it is not hard to show that
\[
(\pi_2, a\chi_0) = p(\pi_1, 2\chi_0) + p(\pi_2 + \varphi_1, \chi_0) + (1 - p)(\pi_2, -2\chi_0) - p(\pi_1 + \varphi_1, -\chi_0),
\]
\[
(\varphi_1, b\chi_0) = -q(\pi_1, 2\chi_0) + (1 - q)(\pi_2 + \varphi_1, \chi_0) + (q - 1)(\pi_2, -2\chi_0)
+ q(\pi_1 + \varphi_1, -\chi_0)
\]
for some integers \( p, q \). At that, \( 1 - p \geq 0, -p \geq 0, q - 1 \geq 0, q \geq 0 \), whence \( p \leq 0 \) and \( q \geq 1 \). Next, we have \( a = 6p - 2 \) and \( b = -6q + 3 \). Since \( a + b = 1 \), it follows that \( p = q \). This contradicts the inequalities \( p \leq 0 \) and \( q \geq 1 \), thus the function \( D \) is irreducible in \( A \). Now let us show that the function \( \Phi_2 \) satisfies the hypothesis of assertion 2) of Theorem 7. First, arguing as for the weight of \( D \) we obtain that \( (\pi_1 + \varphi_1, -\chi_0) = (\pi_1, 2\chi_0) + (\varphi_1, -3\chi_0) \) is the unique representation of the weight of \( \Phi_2 \) in the form \( 2 \) such that both summands lie in \( Z \). At that, \((\pi_1, 2\chi_0)\) is the weight of \( \Delta \). Second, we consider the matrices \( P = -F_3 \in \text{SL}_3 \) and \( Q = E_{2m} \in \text{Sp}_{2m} \). We have \( \Delta(P, Q) = 0, \Phi_2(P, Q) = -1 \neq 0 \), whence \( \Phi_2 \) is not divisible by \( \Delta \). Therefore \( \Phi_2 \) is irreducible. Hence by assertion 3) of Theorem 7 the functions \( \Delta, D, \delta, \Phi_2, W (m \geq 2) \) generate the algebra \( A \) and their weights with respect to \( B \times H \) generate the semigroup \( \hat{\Gamma}(G/H) \).

Case 5. \( G = \text{Sp}_{2n} \times \text{Sp}_{2m}, H = H_0 = \text{Sp}_{2n-2} \times \text{Sp}_2 \times \text{Sp}_{2m-2}, \chi(H) = 0 \). The factor \( \text{Sp}_{2n-2} \) of \( H \) is embedded in the factor \( \text{Sp}_{2n} \) of \( G \) as the central \((2n - 2) \times (2n - 2)\) block. Similarly the factor \( \text{Sp}_{2m-2} \) is embedded in the factor \( \text{Sp}_{2m} \) as the central \((2m - 2) \times (2m - 2)\) block. The factor \( \text{Sp}_2 \) of \( H \) is diagonally embedded in \( G \) as the \( 2 \times 2 \) block in the first and last rows and columns in both factors of \( G \).

We are interested in functions \( f(P, Q) \in \mathbb{C}[G] \) such that
\[
f(P, Q) = f(u_1^{-1}P h_1 h_2, u_2^{-1}Q h_2 h_3)
\]
for all matrices \( P \in G_1, Q \in G_2, u_1 \in U_1, u_2 \in U_2, h_i \in H_i, i = 1, 2, 3 \). Suppose \( P = (p_{ij}), Q = (q_{ij}) \).

Theorem 7 allows us to consider the actions of \( H_1 \) and \( H_3 \) and to reduce to canonical form only the first and last columns of \( P \) and \( Q \).

We introduce the functions \( \Delta = p_{2n-1,1}p_{2n,2n} - p_{2n-1,2n}p_{2n,1}, \delta = q_{2m-1,1}q_{2m,2m} - q_{2m-1,2m}q_{2m,1}, D = p_{2n,1}q_{2m,2m} - p_{2n,2m}q_{2m,1} \). We have \( \Delta \equiv 1 \) for \( n = 1 \) and \( \delta \equiv 1 \) for \( m = 1 \). It is clear that \( \Delta, \delta, D \) lie in \( A \) and are weight functions with respect to
$B \times H$. Their weights are equal to $\pi_2 (n \geq 2)$, $\varphi_2 (m \geq 2)$, $\pi_1 + \varphi_1$, respectively. Below we shall apply Theorem 7 to these functions.

We consider the open subset $M \subset G$ determined by the conditions $\Delta \neq 0$, $\delta \neq 0$, $D \neq 0$. Suppose $(P, Q) \in M$.

Acting by $H_2$ we transform $(P, Q)$ to a pair $(P', Q')$ such that the lower $2 \times 2$ blocks of the matrices $\overline{P}$ and $\overline{Q}$ are

$$
\begin{pmatrix}
1 & * \\
0 & \Delta
\end{pmatrix}, \quad
\begin{pmatrix}
* & \delta \Delta \\
D & D
\end{pmatrix},
$$

respectively. Then, acting by $U_1$ and $U_2$ (Lemma 4) we transform $P'$, $Q'$ to $P''$, $Q''$, respectively, where

$$
\overline{P}'' = \begin{pmatrix}
0 & 0 \\
0 & -1 \\
\cdots & \cdots \\
1 & 0 \\
0 & \Delta
\end{pmatrix}, \quad
\overline{Q}'' = \begin{pmatrix}
0 & \Delta \\
D & D \\
\cdots & \cdots \\
0 & D \\
D & 0
\end{pmatrix}
$$

for $n, m \geq 2$. In these matrices the dots stand for zero entries. If $n = 1$ or $m = 1$ then

$$
P'' = \overline{P}'' = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad
Q'' = \overline{Q}'' = \begin{pmatrix}
0 & \Delta \\
D & D \\
\Delta & 0
\end{pmatrix}
$$

respectively.

The pair $(P'', Q'')$ is the canonical form of the pair $(P, Q)$. Thus the section is obtained, therefore the desired algebra $A$ is contained in the algebra

$$
\tilde{A} = \mathbb{C} \left[ \Delta, \frac{\Delta}{D}, \frac{D}{\Delta}, \delta \frac{\Delta}{D} \right] \subset \mathbb{C} \left[ \Delta, \delta, D, \frac{1}{\Delta}, \frac{1}{D} \right].
$$

By assertions 1), 3) of Theorem 7 the functions $\Delta (n \geq 2)$, $\delta (m \geq 2)$, $D$ are irreducible and generate the algebra $A$ and their weights with respect to $B \times H$ generate the semigroup $\tilde{\Gamma}(G/H)$.

Case 6. $G = \text{Sp}_{2n} \times \text{Sp}_4$, $H = H_0 = \text{Sp}_{2n-4} \times \text{Sp}_4$, $\chi(H) = 0$. The first factor of $H$ is embedded in the first factor of $G$ as the central $(2n - 4) \times (2n - 4)$ block; the second factor of $H$ is diagonally embedded in $G$, as the $4 \times 4$ block in rows and columns nos. 1, 2, 2n - 1, 2n in the first factor.

We are interested in functions $f(P, Q) \in \mathbb{C}[G]$ such that

$$
f(P, Q) = f(u_1^{-1}Ph_1h_2, u_2^{-1}Qh_2)
$$

for all matrices $P \in G_1$, $Q \in G_2$, $u_1 \in U_1$, $u_2 \in U_2$, $h_1 \in H_1$, $h_2 \in H_2$.

Suppose $(P, Q) \in G$ is an arbitrary pair of matrices, the set $M$ will be chosen later. Let us reduce this pair to canonical form. First of all, we put $h_2 = Q^{-1}u_2$ and thereby transform $Q$ to the identity matrix $E_4$. Now the problem is reduced to finding a canonical form for the matrix $PQ^{-1} \in \text{Sp}_{2n}$ with respect to the right action of $U_1$ and left actions of $U_2$, $H_1$. For short, we put $PQ^{-1} = R$. Suppose $R = (r_{ij})$. By Theorem 6 we do not
consider anymore the action of $H_1$ and restrict ourselves to the problem of reducing only the first two and the last two columns of $R$ to canonical form.

We denote by $\Delta_i$, $i = 1, 2, 3, 4$, the minor of $\overline{R}$ corresponding to the last $i$ rows and first $i$ columns. Let $\Phi$ be the minor of order 3 of $\overline{R}$ corresponding to the last three rows and columns $1, 2, 4$. Next, we put $D = r_{2n,1}r_{2n-1,2n} - r_{2n-1,1}r_{2n,2n} + r_{2n,2}r_{2n-1,2n-1} - r_{2n-1,2}r_{2n,2n-1}$, $F = \Delta_1 \Phi + r_{2n,2} \Delta_3$. Below we shall find out that $\Delta_4 = -D$ for $n = 3$.

The functions $\Delta_1, \Delta_2, \Delta_3, \Delta_4, D, F$ lie in $A$ and are weight functions with respect to $B \times H$. Their weights are equal to $\pi_1 + \varphi_1$, $\pi_2 + \varphi_2$, $\pi_3 + \varphi_1$, $\pi_4$ ($n \geq 4$), $\pi_2$, and $\pi_1 + \pi_3 + \varphi_2$, respectively. Below we shall apply Theorem 7 to the functions $\Delta_1, \Delta_2, \Delta_3, \Delta_4, D, F$ for $n \geq 4$ and functions $\Delta_1, \Delta_2, \Delta_3, D, F$ for $n = 3$.

The following argument will first be performed for $n \geq 4$. We shall reduce the matrix $R$ to canonical form on the open subset $M \subset G$ where $\Delta_i \neq 0$ for $i = 1, 2, 3, 4$.

Using the left action of $U_1$ by matrices of type (1) we transform $R$ to a matrix $R'$ such that all the non-zero elements of the lower half of the matrix $\overline{R}'$ are concentrated in its lower $4 \times 4$ block which has the form

\[
\begin{pmatrix}
0 & 0 & 0 & -\frac{\Delta_4}{\Delta_3} \\
0 & 0 & \frac{\Delta_3}{\Delta_2} & r_6 \\
0 & -\frac{\Delta_2}{\Delta_1} & r_4 & r_5 \\
\Delta_1 & r_1 & r_2 & r_3
\end{pmatrix}
\]

where $r_1 = r_{2n,2}$, $r_2 = r_{2n,2n-1}$, $r_3 = r_{2n,2n}$, $r_4 = \frac{r_{2n,1}r_{2n-1,2n} - r_{2n-1,1}r_{2n,2n} - r_{2n-1,2}r_{2n,2n-1}}{\Delta_1}$, $r_5 = \frac{r_{2n,1}r_{2n-1,2n} - r_{2n-1,1}r_{2n,2n}}{\Delta_1}$, $r_6 = \frac{\Phi}{\Delta_2}$.

Multiplying $R'$ on the right by an appropriate matrix in $U_2$ of type (1) and then by an appropriate matrix in $U_2$ of type (2) we successively obtain two matrices $R'_1$ and $R'_2$ such that the lower $4 \times 4$ blocks of $\overline{R}'_1$ and $\overline{R}'_2$ are

\[
\begin{pmatrix}
0 & 0 & 0 & -\frac{\Delta_4}{\Delta_3} \\
0 & 0 & \frac{\Delta_3}{\Delta_2} & r_6 + \frac{r_1 \Delta_3}{\Delta_2} \\
0 & -\frac{\Delta_2}{\Delta_1} & r_4 & r_5 + \frac{r_1 r_4}{\Delta_2} \\
\Delta_1 & r_1 & r_2 & r_3 + \frac{r_1 r_2}{\Delta_1}
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & -\frac{\Delta_4}{\Delta_3} \\
0 & 0 & \frac{\Delta_3}{\Delta_2} & r_6 \frac{F}{\Delta_1} \\
0 & -\frac{\Delta_2}{\Delta_1} & r_4 & r_5 \frac{D}{\Delta_1} \\
\Delta_1 & 0 & 0 & 0
\end{pmatrix},
\]
respectively. Further, acting on the left by matrices in $U_1$ of type (2) we transform $R_2'$ to a matrix $R''$ where

$$
R'' = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{\Delta_1} \\
0 & 0 & \Delta_1 & 0 \\
0 & 0 & 0 & D \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\frac{\Delta_1}{\Delta_3} \\
0 & 0 & \Delta_3 & \frac{\Delta_1}{\Delta_2} \\
0 & 0 & 0 & \frac{D}{\Delta_1} \\
\Delta_1 & 0 & 0 & 0
\end{pmatrix}.
$$

(The dots stand for zero entries.) The pair $(R'', E_4)$ is the canonical form of the original pair $(P, Q)$. Thus the section is obtained. Therefore the desired algebra $A$ is contained in the algebra

$$
\tilde{A} = \mathbb{C}\left[\frac{\Delta_1}{\Delta_1}, \frac{\Delta_2}{\Delta_2}, \frac{\Delta_3}{\Delta_3}, \frac{\Delta_4}{\Delta_4}, \frac{D}{\Delta_1}, \frac{F}{\Delta_2}, \frac{D}{\Delta_3}, \frac{D}{\Delta_4}, \frac{1}{\Delta_1}\right]
$$

$$
\subset \mathbb{C}\left[\Delta_1, \Delta_2, \Delta_3, \Delta_4, D, F, \frac{1}{\Delta_1}, \frac{1}{\Delta_2}, \frac{1}{\Delta_3}\right].
$$

By assertions 1), 3) of Theorem 7 all the functions $\Delta_1, \Delta_2, \Delta_3, \Delta_4, D, F$ are irreducible and generate $A$ and their weights generate the semigroup $\tilde{\Gamma}(G/H)$.

For $n = 3$ a matrix $R$ in the open subset $M = \{\Delta_1 \neq 0, \Delta_2 \neq 0, \Delta_3 \neq 0\} \subset G$ can similarly be transformed to a new matrix $R'$ where

$$
R' = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{\Delta_1} \\
0 & 0 & \Delta_1 & 0 \\
0 & 0 & 0 & D \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -\frac{\Delta_1}{\Delta_3} \\
0 & 0 & \Delta_3 & \frac{\Delta_1}{\Delta_2} \\
0 & 0 & 0 & \frac{D}{\Delta_1} \\
\Delta_1 & 0 & 0 & 0
\end{pmatrix}.
$$
From this it follows that $\Delta_4 = -D$. The pair $(R', E_4)$ is the canonical form of the original pair $(P, Q)$. Thus the section is obtained. Therefore

$$A \subset \tilde{A} = \mathbb{C}\left[\Delta_1, \Delta_2, \Delta_3, D, F, \Delta_{12}, \Delta_{13}, \frac{1}{\Delta_1}, \frac{1}{\Delta_2}, \frac{1}{\Delta_3}\right] \subset \mathbb{C}\left[\Delta_1, \Delta_2, \Delta_3, D, F, \frac{1}{\Delta_1}, \frac{1}{\Delta_2}, \frac{1}{\Delta_3}\right].$$

By assertions 1), 3) of Theorem 7 all the functions $\Delta_1, \Delta_2, \Delta_3, D, F$ are irreducible and generate the algebra $A$ and their weights generate the semigroup $\hat{\Gamma}(G/H)$.

Case 7. $G = \text{Sp}_{2n} \times \text{Sp}_{2m} \times \text{Sp}_{2l}, \ H = H_0 = \text{Sp}_{2n-2} \times \text{Sp}_{2m-2} \times \text{Sp}_{2l-2} \times \text{Sp}_2, \ \mathcal{X}(H) = 0$. The first three factors of $H$ are embedded in the corresponding factors of $G$ as the central blocks of the appropriate size. The factor $\text{Sp}_2$ of $H$ is diagonally embedded in $G$ as the $2 \times 2$ block in the first and last rows and columns of each factor.

We search for functions $f(P, Q, R) \in \mathbb{C}[G]$ such that

$$f(P, Q, R) = f(u_i^{-1}Ph_ih_4, u_2^{-1}Qh_2h_4, u_3^{-1}Rh_3h_4)$$

for all $P \in G_1, \ Q \in G_2, \ R \in G_3, \ u_i \in U_i, \ i = 1, 2, 3, \ h_j \in H_j, \ j = 1, 2, 3, 4$. Based on Theorem 6 we may discard the actions of the first three factors of $H$ and reduce only the first and last columns of $P, Q, R$ to canonical form. Suppose $P = (p_{ij}), \ Q = (q_{ij}), \ R = (r_{ij})$.

Let us introduce the following functions: $\Delta_1 = p_{2n-1,1}p_{2n,2n} - p_{2n-1,2n}p_{2n,1}, \ \Delta_2 = q_{2m-1,1}q_{2m,2m} - q_{2m-1,2m}q_{2m,1}, \ \Delta_3 = r_{2l-1,1}r_{2l,2l} - r_{2l-1,2l}r_{2l,1}, \ \Delta_1 = p_{2n,1}q_{2m,2m} - p_{2n,2n}q_{2m,1}, \ \Delta_2 = q_{2m,1}r_{2l,2l} - q_{2m,2m}r_{2l,1}, \ \Delta_3 = p_{2n,1}r_{2l,2l} - p_{2n,2n}r_{2l,1}$. If $n = 1$ (resp. $m = 1, l = 1$) then $\Delta_1 \equiv 1$ (resp. $\Delta_2 \equiv 1, \ \Delta_3 \equiv 1$). All the functions $\Delta_1, \ \Delta_2, \ \Delta_3, \ D_1, \ D_2, \ D_3$ lie in $\mathbb{A}$ and are weight functions with respect to $B \times H$. Their weights are $\pi_2$ ($n \geq 2$), $\varphi_2$ ($m \geq 2$), $\psi_2$ ($l \geq 2$), $\pi_1 + \varphi_1, \ \varphi_1 + \psi_1, \ \pi_1 + \psi_1$, respectively. Below we shall apply Theorem 7 to these functions.

We consider the open subset $M \subset G$ determined by the conditions $\Delta_1 \neq 0, \ \Delta_2 \neq 0, \ \Delta_3 \neq 0, \ D_1 \neq 0, \ D_2 \neq 0$. Acting on the triple $(P, Q, R) \in M$ by an appropriate matrix in $H_4$ we obtain a new triple $(P', Q', R')$ such that the lower $2 \times 2$ blocks of $\overline{P'}, \ \overline{Q'}, \ \overline{R'}$ have the form

$$\begin{pmatrix} 1 & * \\ 0 & \Delta_1 \end{pmatrix}, \quad \begin{pmatrix} * & \Delta_2 \Delta_1 \\ -D_1/D_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \Delta_1 & D_1 \\ D_3/D_1 & \Delta_1 \end{pmatrix}.$$
respectively. Now acting by $U$ (Lemma \[\text{[8]}\]) we transform the triple $(P', Q', R')$ to a triple $(P'', Q'', R'')$ where for $n, m, l \geq 2$

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & \ldots & -1 \\
1 & 0 & \Delta_1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \Delta_1 \\
\ldots & \ldots \\
0 & \Delta_2 \Delta_1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \Delta_1 \\
0 & 0 \\
\ldots & \ldots \\
-\frac{D_3}{\Delta_1} & -\frac{D_2 \Delta_1}{\Delta_1}
\end{pmatrix}
$$

(The dots stand for zero entries.) In the cases $n = 1, m = 1, l = 1$ we have

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \Delta_1 \\
0 & 0 \\
\Delta_2 \Delta_1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \Delta_1 \\
0 & 0 \\
\frac{D_3}{\Delta_1} & \frac{D_2 \Delta_1}{\Delta_1}
\end{pmatrix}
$$

respectively. The triple $(P'', Q'', R'')$ is the canonical form of the original triple $(P, Q, R)$. Thus the section is obtained. Therefore the algebra $A$ is contained in the algebra

$$
\tilde{A} = \mathbb{C} \left[ \Delta_1, \frac{\Delta_1}{D_1}, \frac{\Delta_2 \Delta_1}{D_1}, \frac{\Delta_1}{D_1}, \frac{\Delta_1 \Delta_3}{D_3}, \frac{D_3}{\Delta_1}, \frac{D_2 \Delta_1}{D_1} \right]
$$

$$
\subset \mathbb{C} \left[ \Delta_1, \Delta_2, \Delta_3, D_1, D_2, D_3, \frac{1}{\Delta_1}, \frac{1}{D_1}, \frac{1}{D_3} \right].
$$

By assertions 1), 3) of Theorem \[\text{[7]}\] the functions $\Delta_1 (n \geq 2), \Delta_2 (m \geq 2), \Delta_3 (l \geq 2), D_1, D_2, D_3$ are irreducible and generate $A$ and their weights generate the semigroup $\tilde{\mathcal{G}}(G/H)$.

Case 8. $G = \text{Sp}_{2n} \times \text{Sp}_4 \times \text{Sp}_{2m}, H = H_0 = \text{Sp}_{2n-2} \times \text{Sp}_2 \times \text{Sp}_2 \times \text{Sp}_{2m-2}, \mathcal{X}(H) = 0$. The subgroup $H$ is embedded in $G$ as follows. The factor $H_1$ is embedded in $G_1$ as the central $(2n-2) \times (2n-2)$ block. The factor $H_4$ is similarly embedded in $G_3$. The factor $H_2$ is diagonally embedded in $G_1$ and $G_2$ as the $2 \times 2$ block in the first and last rows and columns. The factor $H_3$ is diagonally embedded in $G_2$ and $G_3$, as the central $2 \times 2$ block in $G_2$ and as the $2 \times 2$ block in the first and last rows and columns in $G_3$.

We are interested in those functions $f(P, Q, R) \in \mathbb{C}[G]$ that satisfy $f(P, Q, R) = f(u_i^{-1}P h_1 h_2, u_i^{-1}Q h_2 h_3, u_i^{-1}R h_3 h_4)$ for all $P \in G_1, Q \in G_2, R \in G_3, u_i \in U_i, i = 1, 2, 3, h_j \in H_j, j = 1, 2, 3, 4$. Suppose $P = (p_{ij}), Q = (q_{ij}), R = (r_{ij})$. Using Theorem \[\text{[6]}\] we may discard the actions of $H_1$ and $H_4$ and reduce only the first and last columns of $P, R$ to canonical form.

We introduce the following functions: $\Delta_1 = q_{31} q_{44} - q_{34} q_{41}, \Delta_2 = q_{32} q_{43} - q_{33} q_{42}, \delta_1 = p_{2n,1} q_{44} - p_{2n,2} q_{41}, \delta_2 = r_{2m,1} q_{43} - r_{2m,2} q_{42}, D_1 = p_{2n-1,1} P_{2n,2} - p_{2n-1,2} P_{2n,1}, D_2 = r_{2m-1,1} R_{2m,2} - r_{2m-1,2} R_{2m,1}$. We have $D_1 \equiv 1$ for $n = 1, D_2 \equiv 1$ for $m = 1$. Since the last two columns of $Q$ are skew-orthogonal, it follows that $\Delta_2 = -\Delta_1$. The functions $\Delta_1, \delta_1, \delta_2, D_1, D_2, \Delta$ lie in $A$ and are weight functions with respect to $B \times H$, their weights are $\varphi_2, \pi_1 + \varphi_1, \varphi_1 + \psi_1, \pi_2 (n \geq 2), \nu_2 (m \geq 2), \pi_1 + \varphi_2 + \psi_1$, respectively. Below we shall apply Theorem \[\text{[7]}\] to these functions.
Suppose $M \subset G$ is the open subset determined by the conditions $\Delta_1 \neq 0$, $\delta_1 \neq 0$, $\delta_2 \neq 0$, $D_1 \neq 0$, $D_2 \neq 0$. Suppose $(P, Q, R) \in M$. Acting on this triple by appropriate matrices in $H_2$ and $H_3$ we obtain a new triple of matrices $(P', Q', R')$ where the matrix $Q'$ and the lower $2 \times 2$ blocks of the matrices $\overline{P}$, $\overline{R}$ have the form

$$
\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
1 & 1 & \Delta_2 & \Delta_2 \\
0 & 0 & -D_1 \Delta_1 & -D_2 \Delta_2 \\
\end{pmatrix},
$$

respectively. Next, acting by $U$ we transform the triple $(P', Q', R')$ to a triple $(P'', Q'', R'')$ where for $n \geq 2$ and $m \geq 2$

$$
\overline{P}'' = P'' = \begin{pmatrix} 0 & -\Delta_1 \\
\delta_1 & -\Delta_1 \\
\end{pmatrix}, \quad Q'' = \begin{pmatrix} 1 & 0 & \Delta \\
\Delta_1 & 0 & \delta_1 \delta_2 \\
\end{pmatrix}, \quad R'' = \begin{pmatrix} 0 & -\Delta_2 \\
\delta_2 & -\Delta_2 \\
\end{pmatrix}.
$$

(For the matrices $P', R'$ this is possible by Lemma [4]). In these matrices the dots stand for zero entries. If $n = 1$ or $m = 1$, then

$$
\overline{P}'' = P'' = \begin{pmatrix} 0 & -\Delta_1 \\
\delta_1 & -\Delta_1 \\
\end{pmatrix}, \quad \overline{R}'' = R'' = \begin{pmatrix} 0 & -\Delta_2 \\
\delta_2 & -\Delta_2 \\
\end{pmatrix},
$$

respectively. The triple $(P'', Q'', R'')$ is the canonical form of the triple $(P, Q, R)$. Thus the section is obtained. Therefore there is an inclusion

$$
A \subset \tilde{A} = C \left[ \frac{\Delta_1}{\delta_1}, \frac{D_1 \Delta_1}{\delta_1}, \frac{\delta_1}{\Delta_1}, \frac{D_2 \Delta_1}{\delta_2}, \frac{\delta_2}{\Delta_1}, \frac{1}{\Delta_1}, \frac{\Delta_1 \Delta}{\delta_1 \delta_2}, \frac{\Delta_1}{\delta_1 \delta_2}, \frac{1}{\delta_1}, \frac{1}{\delta_2} \right].
$$

By assertion 1) of Theorem [7] the functions $\Delta_1$, $\delta_1$, $\delta_2$, $D_1$ ($n \geq 2$), $D_2$ ($m \geq 2$) are irreducible. Let us prove that $\Delta$ is also irreducible using assertion 2) of Theorem [4]. It is not hard to prove that $\pi_1 + \varphi_2 + \psi_1 = (\pi_1 + \psi_1) + \varphi_2$ is the unique representation of the weight of $\Delta$ in the form [3] such that both summands lie in $Z$. At that, $\varphi_2$ is the weight of $\Delta_1$. Consider the matrices $P = E_{2n}$, $Q = \Omega_1$, $R = E_{2m}$. We have $(P, Q, R) \in G$, $\Delta_1(P, Q, R) = 0$, $\Delta(P, Q, R) = -1 \neq 0$, whence $\Delta$ is not divisible by $\Delta_1$. Thus $\Delta$ is irreducible. By assertion 3) of Theorem [7] the functions $\Delta_1$, $\delta_1$, $\delta_2$, $D_1$ ($n \geq 2$), $D_2$ ($m \geq 2$), $\Delta$ generate the algebra $A$ and their weights generate the semigroup $\tilde{\Gamma}(G/H)$.
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