Probing the spectrum of the generalized Rabi model by its isomorphism to an atom inside a parametric amplifier cavity

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We show how the generalized Jaynes-Cummings-Rabi model of cavity quantum electrodynamics can be realized via an isomorphism to the Hamiltonian of a qubit inside a parametric amplifier cavity. This realization clears the way to observe the full spectrum of the Rabi model via a probe applied to a parametric amplifier cavity containing a qubit and a parametric oscillator operating below threshold. An important outcome of the isomorphism is that the actual frequencies are replaced by detunings which make it feasible to reach the ultra-strong coupling regime. We find that inside this regime the spectrum is characterized by a narrow resonance peak that is traced back to the transition between the ground and first excited states. The exact form of these states is given at an energy crossing and then extended numerically for nearby points. At the crossing, the eigenstates are entangled states of field and atom where the field is found inside squeezed cat-states.

I. INTRODUCTION

It is well-known that a weak-harmonic field is particularly efficient to induce a transition between two states of an atom when it oscillates at a frequency close to the Bohr frequency separating the states. In this way, the field can be manipulated to probe the atomic energy spectrum and provide a window into the underlying processes that rule the atom dynamics. As the intensity of the field is ramped up, it begins to perturb the atomic response in a way that makes evident that field and atom couple into a single system. The energy diagram of this composite system displays a rich structure with crossings and avoided-crossings whose locations are given by an interplay of the atomic transition frequency, field frequency, and coupling strength [1]. This diagram carries fundamental information on the nature of the light-matter coupling and has been the subject of extensive research ranging from the many-photon [2–6] down to the single-photon limit [7–10].

It has now become possible to increase the single-photon coupling to about 10% of the atomic transition frequency [11,18]. Due to this strong coupling, the description of such systems requires to move beyond the rotating-wave approximation. And, in the idealized case of a two-level system interacting with a single mode of the field under the dipole approximation, the system is accurately described by the Hamiltonian

\[
\hat{H}_0 = \omega_c \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_+ \hat{\sigma}_- + \lambda' (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-) + \eta \lambda' (\hat{a} \hat{\sigma}_- + \hat{a}^\dagger \hat{\sigma}_+). \tag{1}
\]

where \(\hat{a}\) and \(\hat{a}^\dagger\) are the annihilation and creation operators for the cavity mode, \(\hat{\sigma}_+\) and \(\hat{\sigma}_-\) the raising and lowering operators for the atomic levels, and \(\omega_c\) (\(\omega_a\)) the mode (atomic transition) frequency. The parameter \(\lambda'\) represents the single-photon coupling strength and \(\eta\) the ratio between rotating and counter-rotating couplings. This model—referred to as the Jaynes-Cummings-Rabi (JC-Rabi) model—creates a bridge between the Jaynes-Cummings limit (\(\eta = 0\)) and the Rabi limit (\(\eta = 1\)). Its energy spectrum was recently obtained by Tomka and collaborators [19,20] whose analytic results extend on the seminal work by Braak [21] inside the Rabi limit and can be used inside the ultra-strong coupling.

A natural question arises on how to design systems where this spectrum can be probed. Recent approaches have moved to a driven-dissipative setting [22,24] where atomic and mode frequencies are changed to detunings to a driving field and rotating and counter-rotating couplings are generated from interactions between additional levels [22] or degrees of freedom [21]. Here, we propose a different method to study the JC-Rabi model where we present an realizable Hamiltonian that is isomorphic to Eq. (1). In this way, we can use the realizable Hamiltonian to study experimentally the energy spectrum of the JC-Rabi model over a large range of parameters.

In Figure 1 we sketch two possible realizations of the JC-Rabi model that are relevant to this work. On the left frame the model is generated from two metastable-states coupled through a pair of lambda transitions as originally proposed in Ref. [22]. This proposal has found great success on atomic gases where the necessary level structure is encountered [24,26]. On the right frame an isomorphic Hamiltonian is generated from a two-state system coupled to a parametric oscillator. The basic ingredients being available in superconducting circuits architectures [27,28]. This setup is discussed on detail below, where it is shown that the conditions on the level struc-
ture are relaxed in exchange of independent control of the parameters.

The great advantage of using a cavity with parametric amplifier in a cavity was recognized in the context of optomechanics [29] where it was shown how the parametric coupling can result in normal mode splitting and the squeezing of the mechanical oscillator [30]. More recently the use of parametric amplifier has been advocated in enhancing collective effects [31] and in quantum phase transitions [32].

The outline of the paper is as follows. In Section II we present the equivalence between the JC-Rabi Hamiltonian and the Hamiltonian of a qubit inside a parametric amplifier operating below threshold. Our focus is centered on the symmetries shared by both models and the general structure of their eigenvalues and eigenstates. In Section III the eigenvalues are found numerically and then matched to a simulated resonance spectrum obtained by applying a weak-coherent field with tunable frequency. The resonance spectrum is characterized by a sharp peak whose breath decreases as we dwelve deeper into the strong coupling regime. Section IV is devoted to describe the structure of the eigenstates that lead to this peak. The states are shown to form a long-living pair described by two-photon cat-states conditioned to the state of the atom. These entangled states minimize the two-photon fluctuations induced by the Jaynes-Cummings coupling and parametric amplifier. Section V is left for discussion.

II. ISOMORPHISM OF THE GENERALIZED JC-RABI MODEL TO THE QUBIT INSIDE A PARAMETRIC AMPLIFIER

We consider a two-state system coupled to the subharmonic mode of a parametric oscillator under the dipole and rotating-wave approximations as sketched in Fig. 1. For simplicity, we adopt the physical picture of an atom placed inside a parametric cavity which contains a second-order nonlinear material pumped by an external field. The subharmonic mode is driven by a two-photon pump which is assumed to be highly populated and is treated as a classical field of amplitude $G/2$ and frequency $\omega_p$. The system Hamiltonian reads

$$\hat{H}_G = \omega_c \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_+ \hat{\sigma}_- + \lambda (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-) + \frac{G}{2} (e^{2i\omega_p t} \hat{a}^2 + e^{-2i\omega_p t} \hat{a}^2)$$

(2)

and we will refer to it as the parametric model throughout this work. In a frame rotating at the pump frequency the Hamiltonian takes the form

$$\hat{H}_G = \Delta_G \hat{a}^\dagger \hat{a} + \Delta_a \hat{\sigma}_+ \hat{\sigma}_- + \lambda (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-) + \frac{G}{2} (\hat{a}^2 + \hat{a}^2) .$$

(3)

with detunings

$$\Delta_G = \omega_c - \omega_p ,$$

(4)

$$\Delta_a = \omega_a - \omega_p .$$

(5)

The parametric Hamiltonian description is supplemented by dissipative process. Losses appear in the form of photons leaving the subharmonic mode at a rate $\kappa$ and spontaneous emission of the atom at a rate $\gamma$. The master equation for the density operator of the system $\rho$ reads

$$\dot{\rho} = -i [\hat{H}_G, \rho] + \kappa \mathcal{L} [\hat{a}] \rho + \frac{1}{2} \gamma \mathcal{L} [\hat{\sigma}_-] \rho ,$$

(6)

with Lindblad superoperators $\mathcal{L} [\xi] \equiv 2 \xi \cdot \xi^\dagger - \xi^\dagger \xi - \xi \cdot \xi^\dagger$.

The two-photon pump is responsible for a process of optical amplification that has been been studied on detail in the absence of the qubit [33, 34]. Its effect on the squeezing of the mechanical oscillator [30]. More recently the use of parametric amplifier in a cavity was recognized in the context of optomechanics [29] where it was shown how the parametric coupling can result in normal mode splitting and the squeezing transformation [35].

$$S(z) = \exp \frac{1}{2} (z^* \hat{a}^2 - z \hat{a}^2)$$

with parameter

$$z = \frac{1}{4} \ln \frac{\Delta_G - G}{\Delta_G + G} .$$

(7)

This amplification is responsible of connecting the generalized JC-Rabi model Hamiltonian with the atom inside the parametric amplifier,

$$S(z) \hat{H}_G S^\dagger (z) = \hat{H}_G + \frac{1}{2} (\Delta_c - \omega_G) ,$$

(8)

as the atom—now coupled to a squeezed mode—probes the amplified quadratures of the field through rotating and counter-rotating terms. The strength of these cou-
plings is given by
\[ \cosh z = \frac{\lambda'}{\lambda}, \]
\[ \sinh z = \frac{\eta\lambda'}{\lambda}, \]
while the mode frequencies are related by the hyperbolic relation
\[ \omega_c = \sqrt{\Delta_G^2 - G^2}. \]

Cavity and atomic frequencies have been replaced by detuning parameters which can be controlled by an external drive; thus bringing the ultra-strong coupling regime within reach. Equation (10), however, sets up a limiting pump amplitude
\[ G_{\text{thr}} = \Delta_G \]
that divides the eigenstates of the squeezing operator into discrete and normalizable states below \( G_{\text{thr}} \) and continuous and non-normalizable states above it [36]. The non-normalizable states represent an infinitely squeezed subharmonic mode whose photon intensity grows without bounds so depletion of the pump has to be included into the model to counteract this gain [37]. Depletion of the pump breaks the connection between the atom coupled to the optical parametric amplifier and the JC-Rabi model. Throughout this work we remain below threshold where both Hamiltonians are isomorphic and consider a weak coupling between harmonic and subharmonic modes. It is worth noticing that optical amplification is also intimately connected to the JC-Rabi model generated from two-lambda transitions [38–40].

We now present a discussion of the eigenvalues of \( \hat{H}_G \). The eigenvalue problem according to Eq. (8) can now be written as
\[ \hat{H}_G |\psi_\beta\rangle = E_\beta |\psi_\beta\rangle, \]
\[ \hat{H}_\eta |\phi_\beta\rangle = \tilde{E}_\beta |\phi_\beta\rangle, \]
where eigenstates and eigenvalues are connected by
\[ E_\beta = \tilde{E}_\beta + \frac{1}{2}(\omega_c - \Delta_G), \]
\[ |\psi_\beta\rangle = S^\dagger(z)|\phi_\beta\rangle. \]

Since the parametric Hamiltonian commutes with the parity operator
\[ \hat{\Pi} = \exp(i\pi(\hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_-)), \]
the eigenstates are classified into even- and odd-parity branches according to the \( \pm 1 \) eigenvalues of \( \hat{\Pi} \). All the bare states within each branch couple by different orders in the interaction due to the coexistence of light-matter coupling and two-photon pump. This creates avoided-crossings between states of the same parity and crossings between states of different parities.

In Figure 2 we plot the lowest eigenvalues of \( \hat{H}_G \) as a function of the pump amplitude below threshold. The results are obtained for \( \Delta_a = \Delta_G \) and \( \lambda = 0.95\Delta_G \), deep into the strong coupling regime where the effect of counter-rotating terms is readily available for small pump amplitudes. The crossings and avoided crossings can be observed with the first-crossing being found at \[ G = \frac{1}{\Delta_a} \sqrt{\Delta_G^2 \Delta_a^2 - \lambda^4} \]
Notice that the two lowest energy states meet at this point and appear nearly degenerate outside it. We return to this point at the end of the Section.

Figure 2. Eigenvalues of \( \hat{H}_G \) in units of \( \hbar \Delta_G \) for \( \Delta_a = \Delta_G \) and \( \lambda = 0.95\Delta_G \). Red lines denote states of even parity and blue lines states of odd parity.

Knowledge of the energy lines is complemented by the photon number populations. In Figure 3 we plot the photon number populations correlated to the atomic ground state for the first eigenstates at different pump amplitudes \( G \). For weak-pump amplitudes the eigenstates resemble the Jaynes-Cummings doublets, e.g. states two and five. This is not the case as the pump amplitude increases and states with a well-defined photon number become energy costly. Each eigenstate then begin to populate more photon states to reduce fluctuations in the squeezed quadrature.
where the JC-Rabi Hamiltonian takes the simplified form

\[ \hat{H}_\eta = \omega_c \hat{A}^\dagger \hat{A} - \frac{1}{2} \frac{(1 + \eta^2) \lambda^2}{\omega_c} \]

that obey the commutation relation \([\hat{A}, \hat{A}^\dagger] = 1_{2 \times 2}\) and account for the correlations that rise between field and atom. The degenerate ground states of \(\hat{H}_\eta\) are readily obtained from the eigenvalue equation

\[ \hat{A} |\psi\rangle = 0 \]

whose solutions lead to the dressed states

\[ |\psi_{\text{even}}\rangle = -\frac{|g\rangle (C_\alpha^+ - \sqrt{\eta} |e\rangle) (C_\alpha^-)}{\sqrt{N_+}}, \quad (21a) \]
\[ |\psi_{\text{odd}}\rangle = -\frac{|g\rangle (C_\alpha^- - \sqrt{\eta} |e\rangle) (C_\alpha^+)}{\sqrt{N_-}}, \quad (21b) \]

when diagonalized within the parity basis. These states describe a field in a positive or negative cat-states correlated to the atomic state. The cat-states can be written explicitly as

\[ |C_\alpha^\pm\rangle = [\hat{D}(\alpha) \pm \hat{D}(-\alpha)] |0\rangle \]

where \(\hat{D}(\alpha) = \exp(-\alpha \hat{a} \dagger + \alpha^* \hat{a})\) is the displacement operator with amplitude

\[ \alpha = \sqrt{\eta} \lambda / \omega_c. \]

The normalization \(N_\pm\) reflects the overlap between the coherent states forming each cat

\[ N_\pm = 2 \left(1 + \eta \pm e^{-2|\alpha|^2} (1 - \eta)\right). \]

It can already be seen that the dressed states play a central role in the optical response of a system satisfying the JC-Rabi model (Sec. III). These low energy states couple exclusively to one another under one-photon transitions

\[ \hat{a} |\psi_{\text{even}}\rangle = \alpha \sqrt{\frac{N_-}{N_+}} |\psi_{\text{odd}}\rangle, \quad (25a) \]
\[ \hat{a} |\psi_{\text{odd}}\rangle = \alpha \sqrt{\frac{N_+}{N_-}} |\psi_{\text{even}}\rangle. \quad (25b) \]

Thus forming a long–lived pair when we move into a dissipative setting. Since their structure is inherited from the two-excitation processes (cat-states are eigenstates of the parity \(e^{i\theta}\) and two-photon annihilation \(\hat{a}^2\) operators making them ubiquitous in systems driven by two-photon transitions \[11,14\]), it can then be expected for low-energy states outside the crossing to maintain this form deep into the strong coupling regime (Sec. IV).

Having obtained the closed form structure of the eigenstates at the first crossing \[21\], we now write the eigenstates of \(\hat{H}_G\) at the same point. By applying the squeez-
The field is now described by a superposition of two-photon coherent states \[ |\tilde{\psi}_{\text{even}}\rangle = \frac{|g\rangle \hat{C}^+_\alpha(z) - \sqrt{\eta} \langle e| \hat{C}^-_{\alpha,z}\rangle}{\sqrt{N_+}} \]
\[ |\tilde{\psi}_{\text{odd}}\rangle = -\frac{|g\rangle \hat{C}^-_{\alpha,z} - \sqrt{\eta} \langle e| \hat{C}^+_\alpha(z)i\rangle}{\sqrt{N_-}} \]

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\[ |\tilde{\psi}_{\text{odd}}\rangle = -\frac{|g\rangle \hat{C}^-_{\alpha,z} - \sqrt{\eta} \langle e| \hat{C}^+_\alpha(z)i\rangle}{\sqrt{N_-}} \]

\[ (26a) \]
\[ (26b) \]

### III. PROBED SPECTRUM OF THE JC-RABI MODEL VIA THE PARAMETRIC HAMILTONIAN

We now probe the energy diagram shown in Figure 2 using a coherent beam with tunable frequency to excite the subharmonic mode. The coherent probe has an amplitude \( \epsilon \) and is detuned a frequency \( \delta \) from the pump beam. Its effect over the dynamics of the system is given through an additional term,

\[ \hat{H}_{\text{probe}} = \epsilon (\hat{a} e^{i\delta t} + \hat{a}^\dagger e^{-i\delta t}) \]

inside the master equation \[ (3) \].

For weak probe amplitudes \( \epsilon \ll G, \lambda \) the probe creates one-photon channels that couple states of opposite parities separated by an energy difference \( \Delta E = E_0 - E_\epsilon \). The maximum transition probability is reached for the resonance condition

\[ \delta = \Delta E. \]

We plot the steady-state photon-number expectation obtained from a numerical solution of the master equation in Figure 4. The results are shown for different pump intensities probed by a weak coherent field \( \epsilon \approx 0.03 \lambda \) inside the strong coupling regime \( \kappa \approx \lambda / 10 \). The results are first verified using line (a) where we consider a weak-pump amplitude that allows us to resolve the first Jaynes-Cummings doublet signaled by the two peaks separated a distance \( 2\lambda \). As the pump amplitude \( G \) is ramped up additional peaks start to appear. In Line (b), for example, the transition between the lowest-energy state inside the odd branch to the first excited state in the even branch is already visible at \( \delta / \Delta_G \approx 0.5 \). Past this point additional states become populated and transitions between different excited states occur, as suggested by the peak at negative detunings in lines (c) and (d). These results are in accordance with the energy levels of Fig. 2 and populations of Fig. 3.

It is possible to connect states of the same parity through \( 2m \)-photon transitions \[ (48) \] by increasing the probing field amplitude \( \epsilon \). The \( m \)-th photon transition probability is maximized for the resonance condition

\[ m\delta = \Delta E. \]

In Fig. 5, we plot the absorption spectrum probed by a driving field that allows for multi-photon transitions \( \epsilon \approx 0.12 \lambda \). The two-photon transitions for small pump amplitudes \( G \) are in accordance with excitation of the second Jaynes-Cummings pair displaying peaks at the frequencies \( \delta / \Delta_G \approx 3, 1.7 \) \[ (49, 50) \]. For increased pump and probe field amplitudes the resonances are displaced and radiatively broadened which makes it challenging to relate each peak to a particular transition.

The appearance of single and two-photon transitions is overshadowed by the peak near zero frequency that becomes more pronounced as the pump amplitude is increased. The peak corresponds to the transition between the two lowest-energy states and its narrow-width suggests that this near-degenerate pair has a structure similar to that found at the crossing [see Eqs. (21) and Eqs. (28)]. We study the behaviour of the low-energy states outside the crossing in the next section.
lead to a sharp resonance are maintained.

A. Coupled lambda transitions

We begin with equation (31) and propose the state ansatz

$$\psi_{\text{even}} = -\frac{|g\rangle (C^+_{\alpha_{\text{ans}}}) - \sqrt{\eta} |e\rangle (C^-_{\alpha_{\text{ans}}})}{\sqrt{\mathcal{N}}}, \quad (32a)$$

$$\psi_{\text{odd}} = -\frac{|g\rangle (C^-_{\alpha_{\text{ans}}}) - \sqrt{\eta} |e\rangle (C^+_{\alpha_{\text{ans}}})}{\sqrt{\mathcal{N}}}, \quad (32b)$$

following the same form as Eq. (21), but with the corrected field amplitude

$$\alpha_{\text{ans}} = \sqrt{\eta} \Lambda / (\omega_c + i \kappa) \quad (33)$$

The ansatz is built from the stationary value of a driven-damped harmonic oscillator and our observations of the quantum trajectories of the system. In these trajectories the density matrix is unraveled into pure states whose evolution is conditioned to a simulated photo-electron counting record [37].

Figure 6 is used to illustrate the overlap between the wave-function of the system $|\psi(t)\rangle$ and the ansatz inside a sample quantum trajectory. The system is prepared in the state $|g, 0\rangle$ from which it is quickly driven out to an excited state. The overlap between this excited state and the even- or odd-parity dressed states is denoted by red squares and blue triangles, respectively. Sudden changes between even and odd parities occur each time a photo-electron is detected. This change is reflected on the Wigner distribution of the field [52], which switches between positive and negative cat-states displayed in the lower-right and lower-center frames of the Figure. By performing a time average over many of these cycles the interference fringes that characterize the cat-states disappear, giving way to a statistical mixture shown in the lower-left frame. The statistical mixture is equivalent to the steady-state found under the master equation. Due to the low photon-number expectation $\langle a^\dagger a \rangle_{\text{ss}} \simeq 1.42$, the plus and minus coherent states forming the cats overlap significantly. This allows for changes in parity to have a noticeable effect on the atomic state [see Eqs. (32a) and (32b)]. The probability to find the atom on the excited state is displayed by yellow circles on Fig. 6, where we see how the state of the atom fluctuates between two separate values with each photo-electron detection.

The overlap to the ansatz is not perfect. The probability of finding the system inside the dressed states oscillates during intervals between two detection events with a more prominent effect on the even branch. This behaviour is caused by population to other states outside the pair and the stationary ansatz itself. Working in the low-photon number regime, even (odd) excited states overlap significantly with $|g, 0\rangle$ and $|e, 1\rangle$ ($|e, 0\rangle$ and $|g, 1\rangle$), such that, after the detection of a photon,
Figure 6. Sample quantum trajectory evolution for \( \Delta_G = \Delta_a, \lambda = 0.95\Delta_a, \kappa = 0.1\Delta_G, \) and \( G = 0.8\Delta_G \) (equivalent to \( \eta \simeq 0.5 \)). (Upper row) Probability of finding the system inside the dressed states \( |\psi_{\text{even}}\rangle \) (red triangles), \( |\psi_{\text{odd}}\rangle \) (blue squares), and excited state \( |e\rangle \) (yellow circles). (Lower row) Wigner distribution of the field obtained after the system transitions to the odd-parity branch following the detection of a photon. The left (center) frame gives the distribution conditioned to \( |g\rangle \) (\( |e\rangle \)) while the right frame gives the distribution for the steady state obtained from master equation (31).

We now extend the previous analysis for the master equation (30) and show that the system is now predominantly composed of the dressed-state pair described in Eq. (26). The squeezed states are the result of a competition between drive, atom coupling and dissipation connecting different states. Consider the following chain of events for example,

\[
|g, 2\rangle \xrightarrow{\lambda} |e, 1\rangle \xrightarrow{\kappa} |e, 0\rangle \xrightarrow{G} |e, 2\rangle.
\]

(34)

where only parity was changed by action of the external environment.

The dressed-states are now characterized by two-photon coherent states \( |\tilde{C}_{\alpha, z}\rangle \) conditioned to the atomic state. Each photon detection changes the parity of the system which, unlike the previous subsection, provides a small chance to leak outside the pair [see Eq. (28) above]. The description of the dynamics between detection events also carries some differences due to squeezing. Just like the amplitude of a coherent state is reduced in the absence of photon detection, two-photon coherent states are known to evolve into other two-photon coherent states under a quadratic Hamiltonians, but display a change in both squeezing \( z \) and amplitude \( \alpha \). With an interplay between these two free parameters and the uncertainty relation

\[
\langle 0|\hat{D}^\dagger(\alpha)S(z)|\Delta\hat{a}|^2|S^\dagger(z)\hat{D}(\alpha)|0\rangle = -\cosh z \sinh z.
\]

(35)

it becomes more challenging to determine a steady-state ansatz equivalent to Eqs. (32)-(33). Instead we keep to

B. Atom inside a parametric amplifier

The previous results help to explain the narrow resonance portrayed in Figure 4. Dissipation creates channels that couple the dressed states to one another and create long lived pairs that display a high degree of coherence. In addition, we need to account that the ansatz is built from the stationary value of a driven-damped harmonic oscillator and, as such, can not be expected to capture the transient dynamics that accompany the evolution of the system. During intervals of no-photon detection the amplitude of a coherent state is reduced to \( e^{-\kappa t}\alpha \) a consequence of the information gathered from the system during null-measurements. In Figure 7 we illustrate the case of \( \eta \simeq 0.72 \) where the steady-state displays a higher photon-number expectation. The amplitude of the oscillations is reduced in this scenario. Notice, however, that since the probability of finding the systems in either dressed state is not unity, fluctuations will eventually drive the system outside the dressed state pair for relatively short times.
Figure 8. Sample quantum trajectory for the same parameters as Fig. 6 evolving under the parametric amplifier master equation (30).

in a qualitative description of the states found under the stochastic evolution.

In Figure 8 we show the evolution of the system under a sample quantum trajectory for the parametric amplifier. The Wigner distribution illustrates the state of the field after the detection of a photon has driven the system to the odd-parity branch with the lower left (ceter) frame showing the field conditioned to the ground (excited) atomic state. These fields resemble the squeezed cat states described above, with their statistical average shown in the lower-right frame. We emphasize that the phase and amplitude of the field depend on the decay rate, squeezing, and coupling strength in a non-trivial fashion and fluctuates strongly between photon detection events, as reflected on the atomic state in the upper panel.

V. DISCUSSION

We have presented an isomorphism between an atom coupled to a parametric amplifier and the generalized Jaynes-Cummings-Rabi model. Under this equivalence the ratio between transition frequencies and coupling strengths can be tuned externally using a pump beam that allows for exploration of the dynamics inside the strong-coupling regime. We have shown the energy diagram of the system and presented the absorption spectrum obtained from a coherent probing field. Deep into the strong-coupling regime the spectrum is characterized by a sharp peak whose width becomes more narrow as more photons are injected to the subharmonic mode. The peak is attributed to an induced transition between ground and first-excited state of the system. In particular, we have shown that these states form a long-lived pair under one-photon transitions as those induced by the probe and environment.

We then studied the structure of these long-lived pair starting from a crossing point—where both states are degenerate and the eigenstates can be obtained analytically—and extended the results outside the crossing numerically. The eigenstates correspond to dressed-states where the field is found inside positive or negative cat-states conditioned to the state of the atom. We attributed this structure to the two-excitation process that characterize the JC-Rabi model since cat-states are eigenstates of the parity and two photon annihilation operator. In a regime where these processes dominate, as the one studied here, the ground state is expected to settle into cat-states to minimize the energy costly fluctuations. Similar results have also been found in the parametrically driven Kerr-oscillator [43].

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