The First Physics Picture of Contractions from a Fundamental Quantum Relativity Symmetry Including all Known Relativity Symmetries, Classical and Quantum

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Abstract

In this article, we utilize the insights gleaned from our recent formulation of space(-time), as well as dynamical picture of quantum mechanics and its classical approximation, from the relativity symmetry perspective in order to push further into the realm of the proposed fundamental relativity symmetry \( SO(2, 4) \) of our quantum relativity project. We explicitly trace how the diverse actors in this story change through various contraction limits, paying careful attention to the relevant physical units, in order to place all known relativity theories – quantum and classical – within a single framework. More specifically, we explore both of the possible contractions of \( SO(2, 4) \) and its coset spaces in order to determine how best to recover the lower-level theories. These include both new models and all familiar theories, as well as quantum and classical dynamics with and without Einsteinian special relativity. Along the way, we also find connections with covariant quantum mechanics. The emphasis of this article rests on the ability of this language to not only encompass all known physical theories, but to also provide a path for extensions. It will serve as the basic background for more detailed formulations of the dynamical theories at each level, as well as the exact connections amongst them.

Keywords: Relativity Symmetry, Quantum Relativity, Lie Algebra Contractions, Quantum and Classical Dynamics, Quantum Nonrelativistic

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1. Introduction

This article is a study within our group’s Quantum Relativity Project, the key idea of which is to formulate pictures of quantum spacetime and the related dynamics from a (relativity) symmetry perspective. We expect the models of quantum spacetime to be of a noncommutative nature, seeing them as intrinsically quantum; hence, they may not be fully described by any real number geometric picture of finite dimension. The latter, as classical/commutative geometry, is of course applicable to modeling classical spacetime, as in the Newtonian, Minkowskian, as well as the dynamical Einstein general-relativistic spacetime. What we should bear in mind is that all of these are only theoretical models of the notion of spacetime, and as such are only as good as the corresponding model of physical dynamics. The pursuit of better models of fundamental physics should go hand-in-hand with the pursuit of better models of spacetime. Real number geometry may not maintain its role as a successful, not to mention convenient, tool in this endeavor. The point of view underlying our project highlights the difficulty one encounters in appreciating traditional quantum mechanics. Quantum mechanics indeed sounds strange, or even counter-intuitive, when thinking about it as a theory of mechanics on classical spacetime. What we hope to convince people of, however, is that it is no less intuitive than classical mechanics when one thinks about it with the proper model of a quantum (physical) space in hand \[1, 2\] (a conceptual discussion has been presented in an article in Chinese \[3\]). A quantum particle, for example, always has a definite position within the quantum model of physical space, though it notably cannot be modeled or represented by a finite number of real-valued coordinates. This is ultimately the source of the widespread confusion regarding the probabilistic nature of quantum mechanics, which should really only arise when discussing von Neumann measurements.

Since Einstein, physicists have learned to appreciate how intimately connected the notion of spacetime is with its relativity symmetry – the symmetry of admissible reference frame /coordinate transformations. With the development of the mathematics of group theory and their representations, as well as their applications in physics (mostly in quantum mechanics), we have also
learned to appreciate the perspective of taking the symmetry as the starting point for the construction of a theory. In particular, both Newtonian space-time and Minkowski spacetime can be thought of as a (coset) representation space of the Galilean and Poincaré symmetry groups, respectively. The Newtonian model, as an approximation of the Einstein/Minkowski one, can be retrieved as a symmetry contraction limit [4, 5, 6]. The latter is just the mathematical way to implement, from the relativity symmetry perspective, the physical notion of taking the $c \to \infty$ limit, when all of the involved velocities are small compared to that of light ($c$). More interestingly, one can reason the other way around. The symmetry (algebra) of the Galilean boosts plus rotations is unstable against perturbations. When the vanishing structure constants in the commutator of two boost generators are made nonzero, one is forced to retrieve the (algebra of) the Lorentz symmetry as the unique (up to isomorphism) stabilized symmetry, with $\frac{1}{c}$ as the deformation parameter corresponding to an invariant speed $c$. Even if $c$ as never been measured to be finite, honest physicists can do nothing but give $\frac{1}{c}$ a lower bound. Whatever nonzero value for $\frac{1}{c}$ we admit, Lorentz symmetry provides one with a correct description of the relevant physics, while the Newtonian model can never be confirmed to be (exactly) correct – merely correct up to some limitation in our measurements. Our proposed fundamental quantum relativity symmetry of $SO(2, 4)$ [7] comes from the idea of a fully stabilized symmetry, incorporating all the known fundamental constants $G$, $\hbar$, and $c$ into the algebra structure, with and the Poincaré symmetry as (part of) a contraction limit. For the sake of convenience, we note first the $SO(2, 4)$ algebra is defined by

$$[J_{RS}, J_{MN}] = -i(\eta_{SM}J_{RN} - \eta_{RM}J_{SN} + \eta_{RN}J_{SM} - \eta_{SN}J_{RM}),$$

where $R$, $S$, $M$, and $N$ range from 0 to 5, and we choose the metric convention $\eta_{MN} = (-1, 1, 1, 1, 1, -1)$. All $SO(m, n)$ symmetries involved in this article are (preserved) subgroups/subalgebras of $SO(2, 4)$ for which we typically skip repeating the commutator structures among the $J$-generator sets.

In Ref.[8], we looked into various contraction limits of the $SO(2, 4)$ symmetry, as well as the corresponding contractions of the relevant coset space representations. The focus there was on symmetries that maintain a $SO(m, n)$ subgroup. $(m + n) \leq (2 + 4)$ is called the dimension of the relativity symmetry and the most interesting symmetries are obtained by contractions which take that dimension down one at a time. The first step of the contraction is fixed as $SO(2, 4) \to ISO(1, 4)$ [7]. Options for the further contractions
from ISO(1, 4) were explored mathematically, while focusing on physically plausible pictures that result in, and hence go beyond, $G(1, 3)$ – a (1 + 3)D relativity symmetry of Galilean type. The present analysis, however, focuses on the contraction sequences passing through $H_{\ell}(1, 3)$ instead. By this we mean a (1 + 3)D relativity symmetry of the Heisenberg-Weyl type. More specifically, it is a symmetry with generators $X_\mu$ and $P_\mu$, each transforming as components a four-vector under the $SO(1, 3)$ subgroup generated by $J_{\mu\nu}$, together with a central charge generator giving the $X$-$P$ commutator a Heisenberg-type commutation relation. In total, the full $H_{\ell}(1, 3)$ group therefore has 15 generators.

In the more recent studies [1, 2], we have given a successful formulation of quantum dynamics on a quantum space from a representation of what is essentially an $H_{\ell}(3)$ symmetry. The latter is an invariant subgroup of the $U(1)$ central extension $\tilde{G}(3)$ of the Galilei group $G(3)$ with the Hamiltonian, or time symmetry generator, taken out. A sketch of the story is as follows: owing to the semidirect product structures

$$\tilde{G}(3) = H_{\ell}(3) \rtimes T_t = H(3) \rtimes (SO(3) \times T_t),$$

(2)

where $T_t$ denotes the one parameter group of time translations, the standard Hilbert space representation of the Heisenberg-Weyl group $H(3)$ serves as a (spin zero) representation of $H_{\ell}(3)$, or $\tilde{G}(3)$, in which the extra generators are simply represented by combinations of $\hat{X}_i$ and $\hat{P}_i$, i.e. the operators representing $X_i$ and $P_i$. The other relativity transformations act on $H(3)$ as outer automorphisms, as well as on its group algebra as inner automorphisms. The optimum framework for conceptual clarity is provided by formulating the Hilbert space as being spanned by the set of canonical coherent states $e^{i\theta}|p^i, x^i\rangle$. Such states can themselves be identified with a coset space of $H_{\ell}(3)$ or $\tilde{G}(3)$, and moreover in a way, as the group manifold of $H(3)$. The latter admits the coordinates $(p^i, x^i, \theta)$, with each group element given in the form $e^{i(p^i x_i - x^i p_i + \theta I)}$, where $I$ is the central charge. On the Hilbert space $\mathcal{K}$ of wavefunctions $\phi(p^i, x^i) = \langle p^i, x^i|\phi\rangle$, $\hat{X}_i$ and $\hat{P}_i$ are given by

$$x_i^{\star} = x_i + i\partial_{p^i},$$
$$p_i^{\star} = p_i - i\partial_{x_i},$$

(3)

where $\star$ is the Moyal star product (in the $\hbar = 2$ units). The full algebra of observables is essentially the group $C^\ast$-algebra $C^\ast(H(3))$, represented as $(L^\ast)$
functions of the two basic operators listed above, \( C(p_i, x_i\star) = C(p_i, x_i)\star \); or, equivalently, the multiplier algebra \( \mathcal{M}' = \{ \beta \in S' : \beta \star \alpha \in L^2, \forall \alpha \in L^2 \} \) represented as \( \mathbb{B}(\mathcal{K}) \). Observe that \( \mathcal{K} \) actually sits inside the group algebra as the collection of partial isometries \( \phi(p^i, x^i)\star \). Moreover, each real function \( \alpha(p^i, x^i) \) gives rise to a Hermitian operator \( \alpha(p^i, x^i)\star \equiv \alpha(p_i\star, x_i\star) \). Such an operator generates a one parameter group of unitary transformations on \( \mathcal{K} \). The Schrödinger picture, envisaged as corresponding to a group of automorphisms on \( C(p_i\star, x_i\star) \), can then be matched to the Heisenberg picture given by the latter description. The case in which \( \alpha(p^i, x^i)\star \) is the energy operator yields time translation/evolution, and when the energy operator is furthermore given by \( \frac{p_i^2}{2m} \star \), corresponding to a free particle, one obtains the Hamiltonian among the \( \tilde{G}(3) \) generators.

The above description of quantum mechanics does not seem to offer any notion of the quantum configuration space, to say nothing of a quantum model of physical space. The Hilbert space \( \mathcal{K} \), as a quantum phase space, is essentially the only variety of irreducible unitary representation of \( H(3) \). Ref.\[1\], however, gives a clear justification for also interpreting \( \mathcal{K} \) as a configuration space, from the perspective provided by the relevant coset space structures, as well as the relativity contraction limit trivializing the Heisenberg commutation relation yielding the classical/Newtonian approximation. The configuration space of a free particle is the only model of physical space one can have from any theory of particle dynamics; hence, from the quantum relativity perspective, the projective Hilbert space \( \mathcal{P}(\mathcal{K}) \), as an infinite-dimensional manifold, should be taken as the quantum model of physical space. Unlike the classical phase space, the quantum phase space as a representation of the its relativity symmetry is irreducible. The classical phase space is a sum of the configuration space and the momentum space, which is a splitting that cannot be made at the quantum level. As an irreducible representation of the quantum symmetry and the observable algebra, \( \mathcal{K} \) becomes reducible upon the contraction. This can be seen as reducing the representation to the simple sum of the one-dimensional rays associated with the coherent state or the position eigenstate basis. Only such rays survive as pure (classical) states in the contraction limit. That is to say, the corresponding projective Hilbert spaces are exactly the phase space and (configuration) space cosets of the classical Galilean symmetry.

This picture works well from the dynamical point of view, as well \[2\]. Implementing the contraction on the observable algebra as the extension
of the unitary representation of \( H(3) \) to \( C^*(H(3)) \) results in the classical Poisson algebra. In other words, the deformation of \( C(p^i, x^i) \) to \( C(p^i\star, x^i\star) \), as in deformation quantization, is really a deformation of the corresponding relativity symmetry implemented on a representation of the relevant group \( C^*-\text{algebra} \), which can be matched with a coset space representation and the corresponding unitary group representation (essentially on the Koopman-von Neumann Hilbert space of mixed states). The contraction is the ‘inverse’ of the deformation, hence our formulation is one of dequantization.

It is important to note, as illustrated in Ref.\(^2\), that our picture of (quantum) relativity symmetries within the Lie group/algebra setting is a lot more powerful and generic than it may seem to be. The group \( C^*-\text{algebra} \) provides one with a noncommutative algebra to be taken as the observable algebra, on which the Lie group acts via automorphisms. It appears there is little reason to expect that the collection of all noncommutative algebras obtainable as the group \( C^*-\text{algebra} \) of some Lie group is not enough to describe the observable algebra of any fundamental physical theory we might have in mind. Moreover, to the extent that we would like to be able to retrieve some (quantum/noncommutative) spacetime picture out of it, we do expect a notion of relativity symmetry underlying everything. In physics, it sure looks as though we need little beyond the basic set of phase space coordinate observables in order to describe all observables, though there may be a generalized notion of the latter beyond the classical position and momentum observables. This basic set is to be found among the generators of the relativity (Lie) algebra, while the full set is offered by the corresponding group \( C^*-\text{algebra} \), interpreted as functions on this basic set. Thus, this basic set should be enough to fully illustrate the spacetime picture one is after. Specifically, the noncommutative geometry \(^9\) of the \( C(p\star, x\star) \) algebra should be some manner of geometry equipped with the noncommutative coordinates \( p\star \) and \( x\star \).

The success of the 3D quantum relativity picture naturally leads to the question of whether or not the analogous \((1+3)\)D picture works as well. This question will be addressed below, together with the related question of how the 3D picture is to be retrieved from the \((1+3)\)D picture as a relativity symmetry contraction limit. It is important to note that the “dimension” in both 3D and \((1+3)\)D here is merely the dimension of the relativity symmetry, which corresponds to the dimension of the corresponding space(time) in only the classical cases. It can also likely be thought of as some variety of noncommutative dimension – for instance, the 3D quantum relativity has
three (noncommutative) $\tilde{X}_i$ coordinate observables. Our first quantum space model in this setting is infinite-dimensional when thought of as a (commutative) manifold. Models for the higher levels, yet to be constructed, may even go completely beyond such real number geometric pictures.

The primary goal of the recent articles [1, 2] was to present a detailed picture of the feasibility of this whole scheme at the first level, namely that of the coset space representations. We will illustrate here not only that a (1 + 3)D picture of what we have done in Refs. [1, 2] can be formulated, but also that the relevant coset space representations (of the $H_R(1,3)$ relativity symmetry) can be incorporated in sequences of representations starting from $SO(2,4)$, pass through $H_R(1,3)$, as well as including the extended symmetries of $H_R(3)$ or $\tilde{G}(3)$, before eventually arriving at those relevant for Newtonian physics. We will, however, only briefly discuss the key notions relating to the full formulation of the associated dynamical theories, leaving such detailed investigations to be reported on in future publications. Moreover, we mostly leave such discussions until the end of the present article.

Let us elaborate a bit more on the basic framework of the program, especially in regards to one of the more challenging aspects that is crucial to the formulation and interpretation of the physical pictures at the various levels. The $SO(2,4)$ symmetry can be seen as arising from a stabilization of the algebra containing the Poincaré symmetry and the 3D Heisenberg algebra. Both from the perspective of our relativity symmetry stabilization, and that of requiring a consistent physical account for the relevant structures, we need to supplement these fourteen generators with an additional generator $X_0$, promoting the Heisenberg structure to that of the (1 + 3)D version. While it looks like we can essentially use the Galilean boost $K_i$ generators as the position observable $X_i (= \frac{1}{m}K_i)$ at the 3D relativity symmetry level, we need the full $\tilde{G}(3)$ symmetry (as the $U(1)$ central extension of the quantum relativity symmetry), which cannot be obtained from the Poincaré symmetry $ISO(1,3)$. In fact, the latter has one less generator. One can contract $ISO(1,3)$ to $G(3) [8]$ or to $H_R(3) \equiv C(3) [8]$, but that is not enough. Similarly, in order to have a (1 + 3)D picture of both quantum and classical physics from this perspective, we need two different relativity symmetries connected by a contraction trivializing the Heisenberg commutation relation. As the Poincaré symmetry does not even admit a nontrivial $U(1)$ central extension, this ten generator framework is certainly not enough. The $SO(2,4)$ symmetry is supposed to possess some manner of invariant length and in-
variant momentum, characterizing the noncommutativity among momentum and position observables, respectively, eventually. The $X_\mu$ generate what are called “momentum boosts,” which would be contracted to commuting momentum translations at the lower levels. We have looked somewhat into the physics of such momentum boosts/translations in various settings \cite{7, 8, 10} (see also \cite{11, 12}). The basic feature that will likely be applicable in all cases is that $p^\mu$ has to be generally defined as the derivative of $x^\mu$ with respect to a new invariant parameter $\sigma$, i.e. $p^\mu = \frac{\partial x^\mu}{\partial \sigma}$, and this is independent of the Newtonian idea of mass times velocity ($p^i = mv^i$). In the case of an Einsteinian particle of mass $m$, we need to have $\sigma = \frac{\tau}{m}$, where $\tau$ is the particle’s proper time, in order to retrieve the desired Einsteinian momentum expressions. Note that from the point of view of Hamiltonian mechanics, which is definitely the preferred setting for classical physics within the contraction formulation \cite{2}, even $p^i = mv^i$ is to be interpreted as an equation of motion, rather than a definition of momentum.

From the phase space geometry point of view, at both classical and quantum levels, having momentum translations is as natural as having position translations, and they are, indeed, strongly suggested from the canonical coherent state picture. The crucial challenge that still needs to be fully appreciated is that such momentum translations would change the Einsteinian invariant particle mass. Moreover, we should be able to see this fact as a variety of reference frame transformation, though not necessarily one that is practically implementable. It is particularly interesting to note that the invariant parameter $\sigma$ has essentially been introduced in the theory of covariant quantum mechanics \cite{13}, which has a (1 + 3)D version of the Schrödinger equation. The latter theory is somewhat aligned with the basic spirit of the (1 + 3)D quantum relativity presented here. It is not, therefore, much of a surprise that it is more or less the theory of quantum mechanics we obtained. Admitting such kinds of physical pictures as plausible theories – only the limiting cases of which have been explored in the presently established cases – seems to be very reasonable. Given that, we will illustrate below how our whole framework of quantum relativity looks quite promising, with step-by-step contractions, at least at the kinematical level. Furthermore, a first look at the dynamical setting will be discussed here. As we mentioned above, a fully dynamical analysis along the lines of Ref. \cite{2}, and the explicit contractions giving the lower levels as approximations, has to be left to future publications.

We begin, in the next section, with the picture of the $H_{\alpha}(1, 3)$ symme-
try as a quantum relativity symmetry, and then briefly discuss the picture of covariant quantum mechanics that would result from the coherent state representations. This is based on the relevant cosets, in a fashion paralleling the case of the $H_R(3)$ symmetry. In Sec.III, analysis of the potential contractions from $SO(2,4)$ illuminates which sequences of cosets give rise to those for the desired $H_R(1,3)$ symmetry. Particular attention is paid to the notion of physical dimensions, or the introduction of physical units, which can be seen as a consequence of the contractions. Sec.IV deals with contractions applied to the Lorentz symmetry sitting within the $H_R(1,3)$ symmetry. The focus is on determining the 3D relativity symmetry pictures coming out of the quantum level first, and then taking them further down to the classical level. The quantum picture seems to be somewhat richer than that of the $H_R(3)$ or $\tilde{G}(3)$ relativity symmetry – some key physical issues related to this will be addressed in Sec.VI, after the analysis of the contractions of $H_R(1,3)$ to classical symmetries – before further contracting the Lorentz symmetry – in Sec.V. Finally, we provide some discussion and concluding remarks in Sec.VI. Note that we skip citations of the background references directly involved in analysis of the kind presented in Ref.[2], and leave it to interested readers to check the discussion and references contained therein.

2. (1 + 3)D Quantum Relativity Symmetry, Covariant Quantum Mechanics, and Classical Limits

The quantum relativity symmetry perspective takes the Heisenberg commutation relation as a part of the relativity symmetry algebra. In order to have a similar formulation with the Lorentz symmetry of $SO(1,3)$ incorporated, the natural candidate to consider is that of the $H_R(1,3)$ symmetry. We will first highlight the particularly relevant coset space representations, in view of the analysis of Ref.[1]. The nonzero commutators among the generators outside of the pure $SO(1,3)$ portion are taken to be

$$[J_{\mu\nu}, X_\sigma] = -i(\eta_{\nu\sigma}X_\mu - \eta_{\mu\sigma}X_\nu), \quad [J_{\mu\nu}, E_\sigma] = -i(\eta_{\nu\sigma}E_\mu - \eta_{\mu\sigma}E_\nu), \quad [X_\mu, E_\nu] = 2i\eta_{\mu\nu}I.$$ (4)

Perhaps we should explain here our not-so-conventional notation. The last commutator is the (1 + 3)D Heisenberg commutation relation in which we use $E_\mu$, rather than $P_\mu$, and have an additional factor of 2. The former is actually a natural feature of the relativity symmetry contraction picture, in
which the introduction of \( P_\mu = \frac{1}{c}E_\mu \) with a nontrivial \( c \) is really appropriate for taking the \( c \to \infty \) contraction limit described below\(^1\). The factor of 2 arises from taking \( \hbar = 2 \) units, which is the preferred choice of units for quantum mechanics. Note that the algebra would be the same under any independent changes of units for \( X_\mu \) and \( E_\mu \), as such variations can be absorbed into a redefined \( I \). As \( I \) commutes with everything else, it has to be represented by a scalar multiple of the identity operator in any irreducible unitary representation. It is most convenient to choose a system of physical units such that \( I \) in the above algebra can be taken as exactly the identity, and therefore \( I \) is dimensionless. The physical dimension of \( X_\mu \) and \( E_\mu \) would then be reciprocals of one another, and an even better choice would be to take all of them as dimensionless.

The first coset of interest is obtained by factoring out the copy of \( ISO(1,3) \) generated by the \( J_{\mu\nu} \) and \( X_\mu \) generators. The infinitesimal transformations of \( H_R(1,3) \), or equivalently the action of the algebra element\(^2\) \( \frac{1}{2}\omega^{\mu\nu}J_{\mu\nu} + \bar{p}^\mu X_\mu - \lambda^\mu E_\mu + \bar{\theta}I \) on the coset space coordinates \( (\lambda^\mu, \theta) \), is given as follows:

\[ \begin{pmatrix} d\lambda^\mu \\ d\theta \end{pmatrix} = \begin{pmatrix} \omega^\mu_\nu & 0 & \bar{\lambda}^\mu \\ 2\bar{p}_\nu & 0 & \bar{\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda^\nu \\ \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \omega^\mu_\nu \lambda^\nu + \bar{\lambda}^\mu \\ 2\bar{p}_\nu \lambda^\nu + \bar{\theta} \\ 0 \end{pmatrix}. \] (5)

Another coset space of interest is \( H_R(1,3)/SO(1,3) \), given by

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\(^1\) One can introduce \( P_\mu = \frac{1}{c}E_\mu \) and \( I = \frac{1}{c}F \), with \( c \) being the speed of light, to write the algebra in terms of \( J_{\mu\nu} \), \( X_\mu \), \( P_\mu \), and \( I \) as generators. The latter form would be more familiar looking. Physics at that level would be better described in \( c = 1 \) units anyway. Even a simple contraction picture of the Poincaré to Galilean symmetry has the same feature. Interested readers can see Ref.\(^4\), which gives a detailed pedagogical description of the story.

\(^2\) Strictly speaking, we should write the algebra elements with a factor of \(-i\), which we leave out here and below. The true generators of the real Lie algebra are really of the form \(-iJ\), rather than simply \( J \) itself. The conventional \(-i\) is, of course, to have the ‘generators’ \( J \) represented by Hermitian operators in a unitary representation.
\[ H_r(1,3)/SO(1,3) : \]
\[
\begin{pmatrix}
  dp^\mu \\
  d\lambda^\mu \\
  d\theta \\
  0
\end{pmatrix}
\begin{pmatrix}
  \omega^\mu _\nu & 0 & 0 & \tilde{p}^\mu \\
  0 & \omega^\mu _\nu & 0 & \tilde{\lambda}^\mu \\
  -\tilde{\lambda}_\nu & \tilde{p}_\nu & 0 & \tilde{\theta} \\
  0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  p^\nu \\
  \lambda^\nu \\
  \theta \\
  1
\end{pmatrix}
= \begin{pmatrix}
  \omega^\mu _\nu p^\nu + \tilde{p}^\mu \\
  \omega^\mu _\nu \lambda^\nu + \tilde{\lambda}^\mu \\
  \tilde{p}_\nu \lambda^\nu - \tilde{\lambda}_\nu p^\nu + \tilde{\theta} \\
  0
\end{pmatrix}. \tag{6}
\]

These two cosets can be matched with their own associated coherent states, defined in terms of the unitary representation of operators on \( H(3) \) given by
\[ e^{i\theta |\lambda^\mu \rangle} = e^{i(\theta I - \lambda^\mu \hat{E}_\mu)} |0\rangle \]
and
\[ e^{i\theta |p^\mu , \lambda^\mu \rangle} = e^{i(\theta I + \nu^\mu \hat{X}_\mu - \lambda^\mu \hat{E}_\mu)} |0, 0\rangle, \]
respectively. Each set of such states (without the phase factor) can be taken as a basis spanning a Hilbert space. Just as \( \hat{E}_\mu \) effectively stands in for \( \hat{P}_\mu \), \( \lambda^\mu \) stands in for \( x^\mu \) – this means no more than expressing the same quantities in different physical units (see footnote\[^1\]). Taking \( c = 1 \) units, one can simply identify each of these pairs. The two corresponding Hilbert space representations are, of course, equivalent. On each of the Hilbert spaces, as constructed above, a transformation (of the relativity group) takes a coherent state to another coherent state, exactly in accordance with the transformation between the corresponding coset space points. The \( |\lambda^\mu \rangle \) states are therefore eigenstates of \( \hat{X}_\mu \) with eigenvalue \( \lambda^\mu \). As such, we have a wavefunction representation of a generic state \( |\phi \rangle \) given by \( \phi(\lambda^\mu) \equiv \langle \lambda^\mu | \phi \rangle \), which are essentially the same as the wavefunctions found in covariant quantum mechanics \[^{13}\]. Wavefunctions on the canonical coherent state basis of \( |p^\mu , \lambda^\mu \rangle \), as given by \( \phi(p^\mu , \lambda^\mu) \equiv \langle p^\mu , \lambda^\mu | \phi \rangle \), can be acted on by \( \hat{X}_\mu \) and \( \hat{E}_\mu \) via the star product action
\[
\hat{X}^\mu_\mu = \lambda^\mu \ast = \lambda^\mu + i\partial_{p^\mu} , \\
\hat{E}^\mu_\mu = p^\mu \ast = p^\mu - i\partial_{\lambda^\mu} . \tag{7}
\]
with a generic operator from \( C^*(H(1,3)) \) considered as the operator function \( \alpha(p^\mu \ast , \lambda^\mu \ast) = \alpha(p^\mu , \lambda^\mu) \ast \). The latter is a great setting for the description of the dynamics in the Heisenberg picture and its contraction to the classical limit \[^2\]. The observables in the classical limit would be \( \alpha(p^\mu_\mu , \lambda^\mu_\mu) \) functions of the classical phase space variables, the coset space picture of which will be obtained below. The Hilbert space becomes completely reducible as the sum of the one dimensional rays of the basis states. In other words, it effectively becomes the classical coset space in the contraction limit. In terms of the wavefunctions, only the coordinate delta functions \( \delta(p^\mu_\mu , \lambda^\mu_\mu) \) survive as pure states \[^{1,2}\].
The contraction to the classical limit is morally about decoupling $I$ – removing it from any consideration of kinematics or dynamics. Fortunately, this is straightforward, as shown in Ref. [1]. One simply introduces $X^c_\mu = \frac{1}{k} X_\mu$ and $E^c_\mu = \frac{1}{k} E_\mu$, and then takes the resulting commutation relations to the $k \to \infty$ limit. Apart from having $X_\mu$ and $E_\mu$ replaced by $X^c_\mu$ and $E^c_\mu$, the algebra resulting from this further contraction only differs from the original by the now vanishing $[X^c_\mu, E^c_\nu]$. The resulting symmetry is that of $S(1,3)$. \[8\]

If the $H_R(1,3)$ symmetry can be taken as the relativity symmetry for quantum physics on Minkowski spacetime, $S(1,3)$ would be the appropriate one for corresponding classical theory. As an example, let us illustrate the contracted result of the first coset above as

\[ S(1,3)/ISO(1,3) : \]

\[
\begin{pmatrix}
\frac{d\lambda^\mu_c}{d\theta} \\
\omega^\mu_\nu \\
\lambda^\nu \\
\theta
\end{pmatrix} = \begin{pmatrix}
\omega^\mu_\nu \\
2k \tilde{p}_\nu \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
\tilde{X}^\mu_c \\
\tilde{X}^\mu_c \\
1
\end{pmatrix} = \begin{pmatrix}
\omega^\mu_\nu \lambda^\nu_c + \tilde{\lambda}^\mu_c \\
\theta \\
0
\end{pmatrix}. \tag{8}
\]

Apart from the completely decoupled $\theta$ coordinate, we have Minkowski space coordinated by the finite four-vector $\lambda^\mu_c = k \lambda^\mu$, which obey the relativity symmetry consisting of translations and Lorentz transformations. Note that for the other coset, $S(1,3)/SO(1,3)$, we also need $p^\mu_c = k p^\mu$: these are from the relations $p^\mu_c X^\mu_c = p^\mu X^\mu$ and $\lambda^\mu_c E^\mu_c = \lambda^\mu E^\mu$.

As was mentioned in the introductory section, the physical picture with $H_R(1,3)$ and its classical limit is somewhat different from, or rather more general than, the limiting case as described by Einsteinian (special) relativity, and it is not our plan to give the full formulation and analysis of that case in this article. We will only give a brief sketch, drawing on references from the literature, to illustrate how such theories look very plausible as sensible generalizations of those based on Einsteinian relativity. In fact, some earlier efforts of our group developed a formulation of quantum and classical physics in a very similar setting \[11, 12\], most parts of which are expected to still be feasible within the current framework. The setting presented in this earlier work is that of a $G(1,3)$ relativity – obtained from an alternative path of contractions from $ISO(1,4)$ – and its $U(1)$ central extension, \emph{i.e.} a $\tilde{G}(1,3)$ relativity. The quantum mechanics based on the latter can be seen as a geometric group quantization \[14, 15\] of the classical $G(1,3)$ theory. Then, based on the lessons of Ref. \[2\], we can infer that the basic quantum theory
would mainly be a story of the $H(1,3)$ invariant subgroup and its group $C^*$-algebra. To the extent that the rest of the generators are to be represented as some Hermitian function $\alpha(p_\mu\star, \lambda_\mu\star)$ generating inner automorphisms of the observable algebra, they have no significance in the formulation of the quantum dynamics. Their role is no different from any generic Hermitian observable, which also generates an automorphism, and the corresponding unitary transformation on the Hilbert space of $\phi(\lambda^\mu)$. In the case of the 3D version of $\tilde{G}(3)$ and $G(3)$, the only special importance of the time translation generator is in the classical coset space representation that depicts Newtonian space-time, and the corresponding description of the Galilean boosts as space-time reference frame transformations. The analog of the latter here in our $(1+3)$D case is the parameter $\sigma$ mentioned in the introductory section and the generator giving its translations. We surely can proceed happily without it and formulate the $\sigma$-translation only through an automorphism arising from the right generator.

The naive quantum dynamical picture for states, as described by the wavefunctions $\phi(\lambda^\mu)$ (or more conventionally as $\phi(x^\mu)$ in the literature), is given by a proper time evolution of the form of the Schrödinger equation with $-\frac{\hbar^2}{2} \hat{P}_\mu \hat{P}^\mu$ as the Hamiltonian operator. Here, of course, we use $\hat{P}_\mu = -i\hbar \frac{\partial}{\partial \lambda_\mu}$. The lesson of the previous paragraph suggests that so long as we take the operator as the generic generator of a unitary flow, we get an associated ‘equation of motion’ with the flow parameter as the ‘time’. The operator is to be taken as determining a version of free particle Hamiltonian $\tilde{G}(1,3)$ dynamics, the corresponding classical case of which retrieves exactly the particle dynamics as described by Einsteinian special relativity. The Klein-Gordon equation would be obtained as the ‘time’ independent equation of motion where the particle rest mass (squared) has the role of an eigenvalue, rather than being an intrinsic characteristic of the particle. The ‘time’ parameter in this case is $\sigma$, from which the Einstein proper time is given by dividing it by the mass. On the other hand, a formulation of Heisenberg picture dynamics as automorphism flow on $\alpha(p_\mu\star, \lambda_\mu\star) = \alpha(p_\mu, \lambda_\mu)\star$ would have the classical

---

3 Constructions of $ISO(m,n)$ to $G(m,n)$ or $H\delta(m,n)$ (also commonly denoted by $C(m,n)$) differ only for one generator of the algebra, which plays the role of the time translation generator in $G(3)$ and the central charge needed for the Heisenberg commutation relation as in $H\delta(3)$; for $G(1,3)$, it gives translations of an absolute (proper) time-like coordinate $\sigma$ in a five dimensional ‘spacetime’ coset picture, besides the then relative coordinate time $t$, as in the $(1+3)$D Minkowski spacetime.
limit from the contraction given above given by Poisson/Hamilton dynamics, in agreement with that corresponding to $G(1, 3)$. Therefore, we consider the $H_R(1, 3)$ to $S(1, 3)$ picture presented here to be convincingly acceptable at this stage.

3. The $H_R(1, 3)$ Cosets from a Contraction of $SO(2, 4)$

Following Ref.[7] closely would require the contraction to go through an intermediate $ISO(1, 4)$; however, it is of interest to explore alternatives.

3.1. $SO(2, 4) \rightarrow ISO(1, 4) \rightarrow H_R(1, 3)$

Let us first trace the contraction sequence explicitly, giving due attention to the important physical notion of the physical dimensions of quantities. From the Lie algebra of $SO(2, 4)$, as given by Eq.(1), we introduce the rescaled generators $E_A = -\frac{1}{\lambda} J_{A5}$, $A$ from 0 to 4, and proceed to take them to the $\lambda \rightarrow \infty$ limit. This results in the following commutators

$$[J_{AB}, E_c] = -i(\eta_{BC} E_A - \eta_{AC} E_B),$$
$$[E_A, E_B] = -\frac{i}{\lambda^2} J_{AB} \rightarrow 0. \quad (9)$$

A generic element of the algebra can be written as

$$\frac{1}{2} \omega^{MN} J_{MN} = \frac{1}{2} \omega^{AB} J_{AB} - \lambda \omega^{A5} E_A \longrightarrow \frac{1}{2} \omega^{AB} J_{AB} - \lambda^A E_A,$$

where the $\lambda^A \equiv \lambda \omega^{A5}$ are taken to be finite in the $\lambda \rightarrow \infty$ limit. These $\lambda^A$ are the new parameters for the contracted algebra, which are matched to generators $E_A$, and they have the physical dimensions $[\lambda]^{-1}$ (while $\lambda^A$ has dimension $[\lambda]$). The resulting symmetry is $ISO(1, 4)$. Note that the natural choice of physical units in the $SO(2, 4)$ case is no unit at all, i.e. all $J_{MN}$ and $\omega^{MN}$ are dimensionless. The physical meaning of the $\lambda \rightarrow \infty$ contraction as an approximation [3, 8], however, says that observables corresponding to the $E_A$ generators appear to be different kinds of physical quantities than the $J_{AB}$. Reflecting on the physical meaning of this statement about the $ISO(1, 4)$ symmetry, physicists would introduce a physical unit for $E_A$ (different from that of $J_{AB}$) – the actual finite numerical value of $\lambda$ with respect to this unit would be taken as a fundamental constant. That is, for example, the nature of the speed of light $c$ [6].
For the second contraction, we want to separate $E_4$ from the remaining $E_{\mu}$, and rescale either one along with the $J_{\mu\nu}$. Again, the idea is to reduce the dimension of the relativity symmetry by one, specifically from $(1 + 4)$ to $(1 + 3)$. Summarizing the results in Ref.[8]: taking $E_{\mu}$ through this contraction yields the aforementioned $G(1,3)$, while rescaling $E_4$ instead results in the $C(1,3)$ symmetry; performing neither or both gives $S(1,3)$, which itself can be obtained by a further contraction from $G(1,3)$ or $C(1,3)$. The $C(1,3)$ notation comes from $C(3)$ – the so-called Carroll symmetry that has been around in relativity symmetry discussions for some time (see for example Ref.[16]), but apparently without realizing it has anything to do with quantum mechanics. Mathematically, the Carroll symmetry is really just what we have denoted by $H_R$, and as such we will drop the $C(m,n)$ notation in favor of $H_R(m,n)$ for the remainder of this paper. An explicit illustration of the contraction $ISO(1,4) \rightarrow H_R(1,3)$ can be given as the $p \rightarrow \infty$ limit applied to $X_\mu = \frac{1}{p} J_{\mu\nu}$ and $F = \frac{1}{p} E_4$; we then obtain (skipping the $[J,J]$ part)

\[
\begin{align*}
[J_{\mu\nu}, X_{\sigma}] &= -i(\eta_{\sigma\sigma}X_\mu - \eta_{\mu\sigma}X_\nu) , \\
[J_{\mu\nu}, E_{\sigma}] &= -i(\eta_{\sigma\sigma}E_\mu - \eta_{\mu\sigma}E_\nu) , \\
[X_\mu, E_\nu] &= i\eta_{\mu\nu}F , \\
[X_\mu, F] &= -i\frac{p}{p^2}E_\mu \rightarrow 0 , \\
[J_{\mu\nu}, F] &= 0 , \tag{10}
\end{align*}
\]

which, upon identifying $F$ with $2I$, is exactly the algebra of Eq.(4).

Once again, we want to trace the (relative) physical dimensions of the quantities represented by the generators through the contraction. One can see that $E_\mu$ possesses the dimensions $[\lambda]^{-1}$, while $X^\mu$ has $[p]^{-1}$, and $F$ carries the dimensions of $[\lambda]^{-1}[p]^{-1}$. Obviously, the dimension of $F$ is that of $\hbar$.

Generic elements of the algebras are related as follows:

\[
\frac{1}{2} \omega^{AB} J_{AB} - \lambda^A E_A = \frac{1}{2} \omega^{J_{\mu\nu}} J_{\mu\nu} + p \omega^{\mu\nu} X_\mu - \lambda^\mu E_\mu - p \lambda^4 F \\
\rightarrow \frac{1}{2} \omega^{\mu\nu} J_{\mu\nu} + p^\mu X_\mu - \lambda^\mu E_\mu + f F . \tag{11}
\]

Note that $p^\mu \equiv p \omega^{\mu\mu}$ and $f \equiv -p \lambda^4$ are the finite parameters at the corresponding contraction limits, with physical dimensions of $[p]$ and $[\lambda][p]$, respectively.

The sequence of symmetry contractions can moreover be implemented on the coset spaces $SO(2,4)/SO(1,4)$ and $SO(2,4)/SO(1,3)$, which are in turn
contracted as
\[ SO(2, 4)/SO(1, 4) \rightarrow ISO(1, 4)/SO(1, 4) \rightarrow H_r(1, 3)/ISO(1, 3), \]
and
\[ SO(2, 4)/SO(1, 3) \rightarrow ISO(1, 4)/SO(1, 3) \rightarrow H_r(1, 3)/SO(1, 3). \]
The coset space \( SO(2, 4)/SO(1, 4) \) can be considered as the hyperbolic space, sitting inside of the \((2 + 4)\)D pseudo-Euclidean space with coordinates \( Z^M \), defined by the condition
\[ \eta_{MN} Z^M Z^N = -1. \]
The infinitesimal action of \( SO(2, 4) \) on these coordinates is simply given by
\[ dZ^M = \omega^M_N Z^N. \] (12)

In order to see how the contraction can be implemented more explicitly, let us first rewrite the above as

- \( SO(2, 4)/SO(1, 4): \)
  \[
  \begin{pmatrix}
  dZ^A \\
  dZ^5 
  \end{pmatrix} = \begin{pmatrix}
  \omega^A_B \\
  \omega^5_A 
  \end{pmatrix} \begin{pmatrix}
  Z^B \\
  Z^5 
  \end{pmatrix} = \begin{pmatrix}
  \omega^A_B - \frac{1}{\lambda} \bar{\lambda}^A \\
  \frac{1}{\lambda} \eta_{BA} \bar{\lambda}^A \\
  0 \\
  0 
  \end{pmatrix} \begin{pmatrix}
  Z^B \\
  Z^5 
  \end{pmatrix},
  \] (13)
where following the above, we use \( \bar{\lambda}^A = \lambda \omega^{A5} \) in place of \( \lambda^A \) in the transformation, mostly to distinguish it from the latter in the coset space coordinates below. One merely needs to use the new coordinates \( \lambda^A = \lambda Z^A \), focused on the region around the point \( Z^5 = -1 \), and then take the picture to the \( \lambda \to \infty \) limit. \( \bar{\lambda}^A \) becomes what is essentially a translation of \( \lambda^A \). The following would be the result of this.

- \( ISO(1, 4)/SO(1, 4): \)
  \[
  \begin{pmatrix}
  d\lambda_A \\
  d(1) 
  \end{pmatrix} = \begin{pmatrix}
  \frac{1}{\lambda} \bar{\lambda}^A \\
  \frac{1}{\lambda} \bar{\lambda}^A 
  \end{pmatrix} \begin{pmatrix}
  \lambda^B \\
  1 
  \end{pmatrix} = \begin{pmatrix}
  \omega^A_B \lambda^B + \bar{\lambda}^A \\
  \frac{1}{\lambda^2} \bar{\lambda}^A \lambda^B \\
  0 \\
  0 
  \end{pmatrix} \rightarrow \begin{pmatrix}
  \omega^A_B \lambda^B + \bar{\lambda}^A \\
  \frac{1}{\lambda^2} \bar{\lambda}^A \lambda^B \\
  0 \\
  0 
  \end{pmatrix}.
  \] (14)

This is exactly the structure we have for the \( ISO(1, 4)/SO(1, 4) \) coset, akin to a five dimensional Minkowski space. Contraction pictures of this kind are quite standard, and are well-described with a geometric language.
Another account based on more physical reasoning can be given as follows: for large values of $|Z^i|$, probably with all $|Z^i|$ large as well, we can take approximation of the $SO(2, 4)$ coset space as satisfying $\eta_{MN}Z^MZ^N = 0$; hence, it is no longer curved. This is equivalent to $\eta_{AB}Z^AZ^B = |Z^5|^2$, the latter of which can be taken as a free (positive) parameter, and we can forget about $Z^5$ as a coordinate. One would then describe all of the large numerical values of $Z^A$ by some $\lambda^A$, with a convenient choice of physical units, which reduces the size of their numerical values. This is in much the same vein as how the Planck length provides a natural length scale for nature compared to which the usual scale of laboratory physics is essentially infinite. Planck length is expected to be a notion of minimal length, close to the scale of which we expect very nontrivial structure of spacetime. In the ‘normal’ setting of laboratory physics, we can neglect this and see that the notion of metric distance has no lower bound. This is precisely the spirit behind introducing a relativity deformation to $SO(2, 4)$, and the above contraction is the ‘inverse’ of this deformation.

Going from $ISO(1, 4)/SO(1, 4)$ to $H_R(1, 3)/ISO(1, 3)$ is straightforward, we have:

- $ISO(1, 4)/SO(1, 4) \rightarrow H_R(1, 3)/ISO(1, 3)$:

$$\begin{vmatrix}
\frac{d\lambda^\mu}{df} \\
0
\end{vmatrix} = \begin{pmatrix}
\omega^\mu_\nu - \frac{1}{p^2}\bar{p}^\mu & \bar{\lambda}^\mu \\
\bar{p}_\nu & 0 & f & 0
\end{pmatrix} \begin{pmatrix}
\lambda^\nu \\
f
1
\end{pmatrix} = \begin{pmatrix}
\omega^\mu_\nu \lambda^\nu + \bar{\lambda}^\mu \\
\bar{p}_\nu \lambda^\nu + f
\end{pmatrix}, \quad (15)$$

in which we have used $f$ and $\bar{p}^\mu$ instead of $\lambda^4$ and $\omega^{\mu 4}$. The result is to be read at the $p \rightarrow \infty$ limit, though we show the terms vanishing in the limit in the calculation so that readers can easily trace how the result is obtained (a presentational feature that we will maintain below). The coset is, of course, the same as the somewhat differently written $\text{Eq}(15)$.

Similar considerations to those given above lead us to write:

- $SO(2, 4)/SO(1, 3) \rightarrow ISO(1, 4)/SO(1, 3)$ —

$$\begin{vmatrix}
\frac{d\lambda^A}{d(1)} \\
\frac{dY^\mu}{dY^4}
\end{vmatrix} = \begin{pmatrix}
\omega^A_B & \bar{\lambda}^A & 0 & 0 \\
\frac{1}{\bar{\lambda}^A\lambda_B} & 0 & 0 & 0 \\
0 & 0 & \omega^\mu_\nu & \omega^\mu_4 \\
0 & 0 & \omega^4_\nu & 0
\end{pmatrix} \begin{pmatrix}
\lambda^B \\
1 \\
Y^\nu \\
Y^4
\end{pmatrix} = \begin{pmatrix}
\omega^A_B \lambda^B + \bar{\lambda}^A \\
0 \\
\omega^\mu_\nu Y^\nu + \omega^\mu_4 Y^4 \\
\omega^4_\nu Y^4
\end{pmatrix}, \quad (16)$$

which is simply the sum of two parts, namely $ISO(1, 4)/SO(1, 4)$, as described above in $\text{Eq}(14)$, and $SO(1, 4)/SO(1, 3)$, described along the lines of
Eq. (15) in terms of the $Y^A$ coordinates. It is then straightforward to take a further contraction, essentially along the lines of taking $SO(1, 4)/SO(1, 3)$ to $ISO(1, 3)/SO(1, 3)$, and similar to the discussion above (here with $p^\mu = p Y^\mu$ and $Y^A \sim 1$). The resulting coset space is given by

$$\bullet \ ISO(1, 4)/SO(1, 3) \rightarrow H_{R}(1, 3)/SO(1, 3)$$

$$\left( \begin{array}{c} dp^\mu \\ d\lambda^\mu \\ df \\ dp \end{array} \right) = \left( \begin{array}{cccc} \omega^\mu_{\nu} & 0 & 0 & \bar{p}^\mu \\ 0 & \omega^\mu_{\nu} & \bar{\lambda}^\mu & 0 \\ 0 & \bar{p}^\nu & 0 & \bar{f} \\ \frac{1}{p^\nu} \bar{p}^\nu & 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} p^\nu \\ \lambda^\nu \\ f \\ 1 \end{array} \right) = \left( \begin{array}{c} \omega^\mu_{\nu} p^\nu + \bar{p}^\mu \\ \omega^\mu_{\nu} \lambda^\nu + \bar{\lambda}^\mu \\ \bar{p}^\nu \lambda^\nu + \bar{f} \\ 0 \end{array} \right), \quad (17)$$

which can be seen as giving essentially the same physical picture as Eq. (10). We leave details of the various issues one might be concerned with to be discussed below.

### 3.2. $SO(2, 4) \rightarrow ISO(2, 3) \rightarrow H_{R}(1, 3)$ and $SO(2, 4) \rightarrow H_{R}(1, 3)$

Directly

$ISO(2, 3)$ is an obvious alternative symmetry between $SO(2, 4)$ and $H_{R}(1, 3)$, the contraction sequence of which should not be expected to be much different from the one given above passing through $ISO(1, 4)$. We sketch it briefly here to address any differences that may be worth some attention.

Basically, one has to take the two contractions in reverse order, namely taking $X_\mu = \frac{1}{p^\nu} J_{\mu\nu}$ and $X_5 = -\frac{1}{p^\nu} J_{5\nu}$, for the first step, and then $E_\mu = -\frac{1}{\bar{\lambda}} J_{\mu 5}$ and $F = \frac{1}{\bar{\lambda}} X_5$ for the second. The cosets of $ISO(2, 3)/SO(2, 3)$ and $ISO(2, 3)/SO(1, 3)$, obtainable from the contraction of $SO(2, 4)/SO(2, 3)$ and $SO(2, 4)/SO(1, 3)$, respectively, have essentially the same basic structure as $ISO(1, 4)/SO(1, 4)$ and $ISO(1, 4)/SO(1, 3)$. Note, however, that $ISO(2, 3)/SO(2, 3)$ is a pseudo-Euclidean space with signature $\{-1, 1, 1, 1, -1\}$, and the five vectors have as coordinates $p^\mu$ and $p^\nu = p^\nu \omega^\mu$, instead of $\lambda^\mu$. For $H_{R}(1, 3)/ISO(1, 3)$, obtained from the latter, we have

$$\bullet \ H_{R}(1, 3)/ISO(1, 3)$ from $ISO(2, 3)/SO(2, 3)$ : —

$$\left( \begin{array}{c} dp^\mu \\ df \\ 0 \end{array} \right) = \left( \begin{array}{ccc} \omega^\mu_{\nu} & \frac{1}{p^\nu} \bar{\lambda}^\mu & \bar{p}^\mu \\ -\bar{\lambda}^\nu & 0 & \bar{f} \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} p^\nu \\ f \\ 1 \end{array} \right) = \left( \begin{array}{c} \omega^\mu_{\nu} p^\nu + \bar{p}^\mu \\ -\bar{\lambda}^\nu p^\nu + \bar{f} \\ 0 \end{array} \right). \quad (18)$$

One should observe that we have the same expression $df = -\bar{\lambda}^\nu p^\nu + \bar{f}$ for the $H_{R}(1, 3)/ISO(1, 3)$ coset coming from $ISO(2, 3)/SO(1, 3)$, which is however
different from those given in the above presentation of the $H_R(1, 3)/SO(1, 3)$ coset. The coset itself should obviously be the same as the one obtained from $ISO(1, 4)/SO(1, 4)$ as we have the same $H_R(1, 3)$ group and $SO(1, 3)$ subgroup. The difference in the explicit transformation of $f$, which corresponds to the phase of a state in the associated quantum mechanics being described, can in fact be appreciated via a $U(1)$ central extension analysis \[13\]. The group generated by the Heisenberg algebra can be written with formally different group products that are related by some cocycle. This issue also explains the different forms given in the coset presentations of the previous section compared with those in this section. The $H_R(1, 3)/ISO(1, 3)$ coset is, however, not really the same as the other ones above, as the $ISO(1, 3)$ subgroups we are concerned with here are really different from the ones with generators originating as $J_{\mu
u}$ and $J_{\mu 5}$, and $J_{\mu 4}$ and $J_{\mu 5}$ in the other cases. We will not attempt to formulate any dynamical models for physics above the $H_R(1, 3)$ level at this point, and will leave the issues concerning dynamics for future investigation. At the $H_R(1, 3)$ level, it is obvious that we have something like the configuration space coset in one case, and something like the momentum space coset in the other, which are also precisely what we get upon further contractions to decouple the quantum central charge $F$.

One other alternative that is actually more interesting is to contract $SO(2, 4)$ to $H_R(1, 3)$ directly, which can be achieved by taking $E_\mu = -\frac{1}{\lambda} J_{\mu 5}$, $X_\mu = \frac{1}{p} J_{\mu 4}$, and $F = -\frac{1}{p \lambda} J_{45}$ simultaneously to the $\lambda, p \to \infty$ limit. This is more naturally done by simply identifying $\lambda$ and $p$. We keep them separate here mostly for easy comparison with the two-step contraction pictures. One particularly noteworthy point is that for the phase space coset of $H_R(1, 3)/SO(1, 3)$ the two alternative sequences of contractions described above give different expressions for $df$: $f$ from the contraction of $\lambda^4$ yields a $\bar{p}_\nu \lambda^{\nu}$ contribution while $F$ from $p^5$ results in $-\bar{\lambda}_\nu p^{\nu}$, both obviously a consequence of the nontrivial $X_\mu E_\nu$ commutation relation. It is easy to appreciate the fact that taking the single step contraction from $SO(2, 4)$ should not show any preference for one of the two expressions over the other; hence a more symmetric form of Eq.\[16\] is to be expected, \textit{i.e.}, with $df = \frac{\lambda}{2}(\bar{p}_\nu \lambda^{\nu} - \bar{\lambda}_\nu p^{\nu}) + \bar{f}$.

In order to formulate the picture of the passage of $SO(2, 4)/SO(1, 3)$ to $H_R(1, 3)/SO(1, 3)$ along the contraction, one can use a description of the first coset space by a set of eleven coordinates: $Z^\mu$, $Z^5$, $Y^A$ and $W$, with $(Z^\mu, W, Z^5)$ and $(Y^A, W)$ transforming as six-vectors under $SO(2, 4)$, \textit{i.e.} $W = -Z^A Y^5$. We have
This coset description is really just putting together the $SO(2, 4)/SO(1, 4)$
coset picture of $Z^M$ with $\eta_{MN}Z^M Z^N = -1$, and the $SO(2, 4)/SO(2, 3)$
coset picture of $Y^M$ with $\eta_{MN}Y^M Y^N = +1$. The overlapping coordinate $W$
allows for the description of the two pairs to be put into a single framework as the
full $SO(2, 4)/SO(1, 3)$ coset. Complementary cosets of $SO(1, 4)/SO(1, 3)$
and $SO(2, 3)/SO(1, 3)$ in

$$SO(2, 4)/SO(1, 4) \times SO(1, 4)/SO(1, 3) = SO(2, 4)/SO(1, 3)$$
$$= SO(2, 4)/SO(2, 3) \times SO(2, 3)/SO(1, 3)$$

are described by $(Y^\mu, Y^\nu)$ with $(Y^\nu)^2 = (Y^\nu)^2 - (Y^5)^2$ giving
$\eta_{\mu\nu} Y^\mu Y^\nu + (Y^\nu)^2 = +1$, and $(Z^\mu, Z^\nu)$ with $(Z^\nu)^2 = (Z^\nu)^2 - (Z^4)^2$
giving $\eta_{\mu\nu} Z^\mu Z^\nu - (Z^4)^2 = -1$, respectively. Following the above analysis, this
contraction is to be implemented with new parameters $\lambda^\mu = \lambda \omega^\mu 5$,
$p^\mu = p \omega^\mu 5$, and $\bar{f} = -\lambda p \omega^4 5$ in the $\lambda, p \to \infty$ limit, using
the new coordinates $\lambda^\mu = \lambda Z^\mu$, $p^\mu = p Y^\mu$, and
$r = \lambda p W$, under the conditions $Z^5 \sim -1$ and $Y^4 \sim 1$. We have

- $SO(2, 4)/SO(1, 3) \to H_{R}(1, 3)/SO(1, 3)$ —

$$\left(\begin{array}{c}
d\lambda^\mu \\
d(1) \\
dr \\
dp^\mu
\end{array}\right) = \left(\begin{array}{cccc}
\omega^\mu_\nu & \bar{\lambda}_\nu & 0 & 0 \\
\frac{1}{\lambda} \bar{\tilde{f}} & 0 & 0 & 0 \\
\frac{1}{\lambda} \bar{f} & 0 & 0 & 0 \\
\frac{1}{\lambda} \bar{\lambda} & \frac{1}{\lambda} \bar{p} & 0 & 0
\end{array}\right) \left(\begin{array}{c}
\lambda^\nu \\
r \\
p^\nu \\
\omega^\nu_\nu
\end{array}\right) = \left(\begin{array}{cccc}
\omega^\mu_\nu \lambda^\nu + \bar{\lambda}_\nu & 0 & 0 & 0 \\
\bar{p}_\nu \lambda^\nu - \bar{\lambda}_\nu p^\nu + 2\bar{f} & 0 & 0 & 0 \\
\lambda^\nu & 0 & 0 & 0 \\
\omega^\nu_\nu & 0 & 0 & 0
\end{array}\right)$$

assuming $\lambda^4 \to -\lambda$ and $p^5 \to p$. Identifying the $r$ coordinate as $2f$, or
taking $r$ as $\theta$ and $\bar{\theta} = 2\bar{f}$ instead [cf. Eq.(10)], we have obtained exactly
the symmetric description of the $H_{R}(1, 3)/SO(1, 3)$ coset. Alternatively, we
can think of taking $-\frac{1}{p \lambda} J_{45}$ as $2F$ instead of $F$, which therefore naturally
yields $\bar{\theta} I = \omega^5 J_{45}$, giving us $\bar{\theta} = -2p \lambda \omega^5 = 2\bar{f}$. The $W$ (or $r$) coordinate

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is especially introduced to have $r$ bearing the dimensions of $p\lambda$; thereby fitting the contracted symmetry with the $F$, or $I$, generator. This analysis actually indicates that using the generator $I$ provides a more natural picture [cf. the $\hbar = 2$ units for quantum mechanics]. It would be good to have an understanding of the $\lambda^4 \to -\lambda$ and $p^5 \to p$ assumption. Thinking about the two contraction parameters as one, this assumption is firstly the statement that the magnitude of $\lambda^4 (= \lambda Z^4)$ and $p^5 (= p Y^5)$ have to go to $\infty$ with the contraction parameter, i.e. $|Z^4| \sim |Y^5| \to 1$. Otherwise, if they go as any other power of the contraction parameter, one would have either $dr = 0$, for the two staying finite, or $dr \to \infty$. Neither case can be thought of as a sensible result. The signs are a bit more tricky. Explicitly, we have $dr = \bar{p}_\mu \lambda^\nu Y^5 + \bar{f}(Y^5 - Z^4) + \lambda_\nu p^\nu Z^4$. Switching both signs hence only changes $r$ to $-r$, which does not change the actual physical picture being described. Taking both going to $\infty$ with the same sign kills the $\bar{f}$ term, which also seems unreasonable. We are not able to, however, say more about this aspect of the coset contraction picture.

3.3. Remarks About Physical Dimensions

We explained how the physics at the level before and after the first $\lambda \to \infty$ contraction would lead one to seeing $\lambda$ as a fundamental constant with physical dimensions. This comes with a pairing of one’s quantities, like $\lambda_A$ and $E^A$. Similarly, the $p \to \infty$ contraction, whether taken before or after the $\lambda \to \infty$ limit would lead to the introduction of another fundamental physical unit: $[p]$. The parameters $p^\mu$ and $p^5$ would have the dimensions of $[p]$, while $X_\mu$ and $X_5$ have that of $[p]^{-1}$. This means that $f$ would have the dimensions of $[p][\lambda]$, and $F$ would have that of $[p]^{-1}[\lambda]^{-1}$. The latter is obviously essentially that of $h(c^{-1})$; however, the single step contraction $SO(2, 4) \to H_R(1, 3)$ would suggest identifying $p$ and $\lambda$—both would then be $h^4(c^{-2})$. $J$ and $\omega$ remain dimensionless. These are the pictures of physical dimensions suggested by the contraction analysis at the algebra and coset levels. With that said, however, the dynamical picture will usually have $J$ given by the orbital angular momentum; hence having the units of $\hbar$. A more proper way of expressing this fact should actually be that $J_{\mu\nu}$ is represented by $\frac{\hbar}{c} (\hat{X}_\mu \hat{E}_\nu - \hat{E}_\mu \hat{X}_\nu)$. From the perspective of the unitary representations, however, it is more natural to take $h(c^{-1})$ as not having units, and especially with the canonical coherent states, actually taking all quantities without physical dimensions.
4. Contracting the Lorentz Symmetry at the Quantum Level Before Going to the Classical Limit

We can also consider first contracting $H_R(1, 3)$ to the relativity symmetry of Schrödinger quantum physics before going to the classical limit. With $K_i = \frac{1}{c} J_{0i}$, we must also set $P_i = \frac{1}{c} E_i$ while keeping $E_0$ untouched, in order to maintain the Galilean commutation relations between $K_i$ and $P_i$. Maintaining the 3D Heisenberg commutation relation requires putting $G = \frac{1}{c} F$ and keeping $X_i$ unchanged, which forces us to set $T = \frac{1}{c} X_0$. Taking these to the $c \to \infty$ limit, we obtain

\[
\begin{align*}
[J_{ij}, X_k] &= -i(\delta_{jk} X_i - \delta_{ik} X_j), & [J_{ij}, P_k] &= -i(\delta_{jk} P_i - \delta_{ik} P_j), \\
[J_{ij}, K_k] &= -i(\delta_{jk} K_i - \delta_{ik} K_j), & [K_i, K_j] &= -i \frac{i}{c^2} J_{ij} \to 0, \\
[K_i, H] &= -i P_i, & [K_i, P_j] &= -i \frac{i}{c^2} \delta_{ij} H \to 0, & [X_i, P_j] &= i \delta_{ij} G, \\
[T, H] &= -i G, & [K_i, T] &= -i \frac{i}{c^2} X_i \to 0, & [K_i, X_j] &= -i \delta_{ij} T, \tag{21}
\end{align*}
\]

where $H \equiv E_0$. The set of generators $\{J_{ij}, K_i, P_i, H\}$ provides us with the Newtonian/Galilean symmetry of $G(3)$ as a subalgebra. The $\{J_{ij}, X_i, P_i, G\}$ set supplies us with a copy of $H_R(3)$. The generators $\{J_{ij}, X_i, K_i, T\}$ yields another copy of $H_R(3)$. As such, we will henceforth denote the full symmetry by $H_G H_R(3)$. Note also that there is an important difference between the way the two $H_R(3)$ subalgebras sit inside of $H_G H_R(3)$. While the $X-P$ commutator is a central charge for the full algebra, the $X-K$ commutator is central only within the $H_R(3)$ subalgebra it belongs to. Furthermore, observe that the $G(3)$ symmetry considered in Refs. [1, 2] is more akin to the subgroup generated by $\{J_{ij}, X_i, P_i, H, G\}$, though within the framework presented there the subgroup generated by $\{J_{ij}, K_i, P_i, H, G\}$ could serve equally well, assuming $K_i = m X_i$ (as one has for a classical particle within the Newtonian framework). The story is somewhat more complicated here as we have a nonzero $K-X$ commutator. That in and of itself actually causes no harm in the context of the coset representations we are concerned with here. Actually, that suspicious looking commutator will be killed in the classical limit, as shown below.

In order to retrieve the symmetry for Galilean/Newtonian classical physics, we can take another further contraction to kill the $X-P$ commutator, or more accurately, to decouple $G$. Besides, $[K_i, X_j] = -i \delta_{ij} T \not{=}$ looks strange, at least
for Newtonian physics, in which one should have \( K_i = mX_i \) for a particle of mass \( m \). This will leave \([K_i, H] = -iP_i \) as the only nonzero commutators not involving \( J_{ij} \). We will denote the symmetry resulting from this by \( S_c(3) \).

One nice way to achieve this is the contraction obtained by taking \( K_i^c = \frac{1}{k}K_i \), \( X_i^c = \frac{1}{k}X_i \), \( P_i^c = \frac{1}{k}P_i \), and \( T^c = \frac{1}{k}T \) to the \( k \to \infty \) limit.

The algebra element transition can be written as

\[
\frac{1}{2} J_{\mu \nu} \omega^{\mu \nu} + X_\mu P^\mu + E_\mu \lambda^\mu + Ff \\
\longrightarrow \frac{1}{2} J_{ij} \omega^{ij} + K_i v^i_c + X_i p^i + T e + P_i x^i + H t + G g \\
\longrightarrow \frac{1}{2} J_{ij} \omega^{ij} + K_i^c v^i + X_i^c p^i + T^c e_c + P_i^c x^i + H t + G g ,
\]

(22)

where \( v^i \equiv c \omega^{a^i} \), \( e \equiv c p^a \), \( x^i \equiv c \lambda^i \), \( t \equiv \lambda^0 \), and \( g \equiv c f \), which are followed by \( v^i_c \equiv k v^i \), \( p^i_c \equiv k p^i \), \( e_c \equiv k e \), and \( x^i_c \equiv k x^i \). With all this understood, it is straightforward to trace the contraction of the \( H_3(1,3) \) cosets discussed above. Note that the \( ISO(1,3) \) subgroup of \( H_3(1,3) \) to be factored out of the first coset [cf. Eq. (15)] is contracted to the copy of \( H_3(3) \) obtained from the set \( \{J_{ij}, X_i, K_i, T\} \). We find that

- \( H_{3\eta}(3)/H_3(3) \) from \( H_3(1,3)/ISO(1,3) \) : 

\[
\begin{pmatrix}
\frac{dt}{dx^i} \\
\frac{dx^i}{dg} \\
0
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{c} v^i_j & 0 & \bar{t} \\
v^i_j & \omega^i_j & 0 & \bar{x}^i \\
\bar{e} & -\bar{p}_j & 0 & \bar{g} \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
t \\
x^j \\
g \\
1
\end{pmatrix} = \begin{pmatrix}
\bar{t} \\
v^i_j + \omega^i_j x^j + \bar{x}^i \\
\bar{e} t - \bar{p}_j x^j + \bar{g} \\
0
\end{pmatrix} ; 
\]

(23)

- \( H_{3\eta}(3)/H_3(3) \to S_c(3)/S(3) \) : 

\[
\begin{pmatrix}
\frac{dt}{dx^i_c} \\
\frac{dx^i_c}{dg} \\
0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & \bar{t} \\
v^i_j & \omega^i_j & 0 & \bar{x}^i \\
\bar{e}_c & -\bar{p}_c j & 0 & \bar{g} \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
t \\
x^j_c \\
g \\
1
\end{pmatrix} = \begin{pmatrix}
\bar{t} \\
v^i_j t + \omega^i_j x^j_c + \bar{x}^i_c \\
\bar{g} \\
0
\end{pmatrix} ;
\]

(24)
• $H_{\text{Gal}}(3)/SO(3)$ from $H_{\text{Gal}}(1, 3)/SO(1, 3)$:

\[
\begin{pmatrix}
d e \\
d p^i \\
d t \\
d x^i \\
d g \\
0
\end{pmatrix} = \begin{pmatrix}
0 & v_j & 0 & 0 & 0 & \bar{e} \\
0 & \frac{1}{c^2} v^i & \omega^i_j & 0 & 0 & \bar{p}^i \\
0 & 0 & 0 & \frac{1}{c^2} v_j & 0 & \bar{t} \\
0 & 0 & v^i & \omega^i_j & 0 & \bar{x}^i \\
0 & 0 & \bar{e} & -\bar{p}_j & 0 & \bar{g} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e \\
p^i \\
t \\
x^i \\
g \\
1
\end{pmatrix} = \begin{pmatrix}
v_j p^j + \bar{e} \\
\omega^i_j p^i + \bar{p}^i \\
\bar{t} \\
v^i t + \omega^i_j x^j + \bar{x}^i \\
\bar{e} t - \bar{p}_j x^j + \bar{g} \\
0
\end{pmatrix};
\]

(25)

• $H_{\text{Gal}}(3)/ISO(3) \rightarrow S_\text{c}(3)/ISO(3)$:

\[
\begin{pmatrix}
d e_c \\
d p^i_c \\
d t \\
dx^i_c \\
d g \\
0
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{k} v^i_{cj} & 0 & 0 & 0 & \bar{e}_c \\
0 & \omega^i_j & 0 & 0 & 0 & \bar{p}^i_c \\
0 & 0 & 0 & 0 & 0 & \bar{t} \\
0 & 0 & v^i_c & \omega^i_j & 0 & \bar{x}^i_c \\
0 & 0 & \bar{e}_c & -\bar{p}_j & 0 & \bar{g} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_c \\
p^i_c \\
t \\
x^i_c \\
g \\
1
\end{pmatrix} = \begin{pmatrix}
\bar{e}_c \\
\omega^i_j p^i_c + \bar{p}^i_c \\
\bar{t} \\
v^i_c t + \omega^i_j x^j_c + \bar{x}^i_c \\
\bar{g} \\
0
\end{pmatrix}.
\]

(26)

The first thing we want to note regarding the above results is that the classical picture of what would be the configuration/physical space [Eq. (24)] and phase space [Eq. (31)] are very good. While we have a relativity symmetry group identification that is bigger than the Galilei group, the corresponding cosets are essentially trivial extensions of those from the latter. The simultaneous existence of $K_i$ and $X_i$ as now commuting generators allows for the standard relation of $K_i = m X_i$, which can be taken as a relation between the representations of interest for the otherwise independent generators of the background symmetry algebra. The (infinitesimal) momentum translations, as given by $\bar{p}_c^i$, can be interpreted as merely a consequence of a Galilean boost with $\bar{p}_c^i = m v^i_c$ is imposed. As said before, one expects the latter equation to be retrieved from a Hamiltonian equation of motion under the proper setting. We have also the extra – but completely decoupled – energy ($e$) and ‘quantum phase’ ($g$) translation symmetries, which are irrelevant to the irreducible representations as given by the standard Newtonian configuration/physical space of $x^i$ and momentum space of $p^i$. The phase space is the simple sum of the two, and consequently a reducible representation. The energy translation picture is even there in Newtonian physics as the arbitrariness in setting a
reference zero point for potential energy. This is an incredibly encouraging result, indicating that the full scheme envisioned here can make good sense from a physical perspective. The symmetry picture obtained here for the quantum level, however, needs to be considered more carefully, and so this will be addressed below.

5. Einsteinian/Minkowski Contracted to Galilean/Newtonian Physics

We have briefly addressed the various classical limits of our (1 + 3)D picture of a quantum relativity symmetry, and in particular the symmetry $S(1, 3)$, in Sec.II. We take up the issue further here, and trace its contraction to 3D classical relativity. Again, the key feature is that $H_{el}(1, 3)$ is quite a bit bigger than the usually considered Poincaré symmetry – essentially a central extension of it, if the naive notion of that is admissible. Therefore, its classical limit is likely to be also somewhat different from the standard Einsteinian relativity. The question is whether or not it gives a sensible physical picture – one which includes the latter in some sort of limit. Again, we first focus on the coset structures.

Firstly, we give the infinitesimal transformation descriptions of the two relevant cosets, utilizing the expressions we obtained earlier based on the $k \to \infty$ limit of $X_{\mu} = \frac{1}{k} X_{\mu}$ and $E_{\mu} = \frac{1}{k} E_{\mu}$, namely from Eqs. (15) and (17). The results are simply given by

- $S(1, 3)/ISO(1, 3)$:

\[
\begin{pmatrix}
\frac{d \lambda_{\mu}^c}{df} \\
0
\end{pmatrix} = \begin{pmatrix}
\omega_{\nu}^{\mu} & 0 & \tilde{\lambda}_{\nu}^{\mu} \\
-\frac{1}{k} f p_{cv} & 0 & \tilde{f} \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\lambda_{\nu}^{c} \\
f \\
1
\end{pmatrix} = \begin{pmatrix}
\frac{\omega_{\nu}^{\mu} \lambda_{\nu}^{c} + \tilde{\lambda}_{\nu}^{\mu}}{f} \\
\tilde{f} \\
0
\end{pmatrix}, \quad (27)
\]

- $S(1, 3)/SO(1, 3)$:

\[
\begin{pmatrix}
\frac{d p_{c}^{\mu}}{df} \\
\frac{d \lambda_{c}^{\mu}}{df}
\end{pmatrix} = \begin{pmatrix}
\omega_{\nu}^{\mu} & 0 & 0 & \tilde{p}_{c}^{\mu} \\
0 & \omega_{\nu}^{\mu} & 0 & \tilde{\lambda}_{\nu}^{\mu} \\
0 & -\frac{1}{k} f p_{cv} & 0 & \tilde{f} \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
p_{c}^{\nu} \\
\lambda_{c}^{\nu} \\
f \\
1
\end{pmatrix} = \begin{pmatrix}
\omega_{\nu}^{\mu} p_{c}^{\nu} + \tilde{p}_{c}^{\mu} \\
\omega_{\nu}^{\mu} \lambda_{c}^{\nu} + \tilde{\lambda}_{\nu}^{\mu} \\
\tilde{f} \\
0
\end{pmatrix}, \quad (28)
\]

where $\lambda_{c}^{\mu} = k \lambda^{\mu}$ and $p_{c}^{\mu} = kp^{\mu}$. Note that these results are exactly the same, apart from a normalization of the decoupled ‘quantum phase’ (as given by $f$ here), as if we had applied the contraction to the forms of the
cosets as given in Sec.II instead. We have Minkowski spacetime arising as
\( S(1,3)/ISO(1,3) \sim ISO(1,3)/SO(1,3) \), here described by four ‘time’ co-
dinates, and a matching phase space with four additional momentum coordi-
nates. As discussed somewhat in the introductory section, these momentum
translations are beyond the standard Einsteinian formulation. Note that
here they are transformations independent from, and in addition to, the
Lorentz boosts, as described infinitesimally by the \( \omega^\mu_\nu \). Thinking about the
\( E_\mu \) generators and the corresponding \( p^\mu \) parameters as describing the energy-
momentum four-vector as observables, having their components transforming
as a Lorentz four-vector is of course an actual necessity, so long as all \( p^\mu \), for
example, are to be included as phase space coordinates.

The next contraction to consider is again taking the Lorentz boosts to
the Galilean boosts, as in the last section. We take
\( K^c_i = \frac{1}{c} J^c_{0i} \) and \( P^c_i = \frac{1}{c} E^c_i \)
to the \( c \to \infty \) limit.

\[
\begin{align*}
[J_{ij}, K_k] &= -(\delta_{jk} K_i - \delta_{ik} K_j), & [K_i, K_j] &= -\frac{1}{c^2} J_{ij} \to 0, \\
[K_i, H^c] &= -P^c_i, & [K_i, P^c_j] &= -\frac{1}{c^2} \delta_{ij} H \to 0, \\
[J_{ij}, P^c_k] &= -(\delta_{jk} P^c_i - \delta_{ik} P^c_j), & [J_{ij}, H^c] &= 0, & [P^c_i, H^c] &= 0, \\
[K^c_i, X_0] &= -\frac{1}{c} X_i \to 0, & [K^c_i, X_j] &= -\frac{1}{c} \delta_{ij} X_0 \to 0,
\end{align*}
\]

where we have introduced \( H^c \equiv E^c_0 \). This result, apart from some difference
in notation, is really essentially the same as the result given in the last
section. Mathematically, it is the same \( S_G(3) \). The cosets for the Newtonian
configuration/physical space(-time) and phase space, as given in the last
section, are to be obtained from this alternative line of contractions, which
under the present notation are given by

\[ S_G(3)/\overline{S}(3) : \]

\[
\begin{pmatrix}
dt^c \\
dx^c_i \\
df \\
0
\end{pmatrix}
= \begin{pmatrix}
0 & \frac{1}{c^2} v_j & 0 & \bar{t}^c \\
v^i & \omega^i_j & 0 & \bar{x}^i \\
0 & 0 & 0 & \bar{f} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
t^c \\
x^c_i \\
f \\
1
\end{pmatrix}
= \begin{pmatrix}
\bar{t}^c \\
v^i t^c + \omega^i_j x^j + \bar{x}^i \\
f \\
0
\end{pmatrix},
\]

(30)
\[ \begin{pmatrix}
    dc_c \\
    dp^i_c \\
    dt_c \\
    dx^i_c \\
    df
\end{pmatrix} =
\begin{pmatrix}
    0 & \frac{1}{c}v_j & 0 & 0 & 0 & \bar{e}_c \\
    \frac{1}{c}v^i & \omega_j^i & 0 & 0 & 0 & \bar{p}_c^i \\
    0 & 0 & 0 & \frac{1}{c}v_j & 0 & \bar{t}_c \\
    0 & 0 & 0 & 0 & 0 & \bar{x}_c^i \\
    0 & 0 & 0 & 0 & 0 & \bar{f}
\end{pmatrix}
\begin{pmatrix}
    e_c \\
    p_c^i \\
    t_c \\
    x_c^i \\
    f
\end{pmatrix} =
\begin{pmatrix}
    \bar{e}_c \\
    \omega_j^i p_c^i + \bar{p}_c^i \\
    t_c \\
    v^i t_c + \omega_j^i x_c^i + \bar{x}_c^i
\end{pmatrix},
\] (31)

where \( t_c \equiv \lambda^0_c, x_c^i = c \lambda^i_c, \) and \( e_c \equiv p_c^0. \) The exact differences between the above and the presentation of the cosets in last section are: the absolute Newtonian time is \( t_c \) here and \( t \) there; we moreover have \( v^i \) here and \( v_i \) there (and the decoupled phase as \( f \) versus \( g \)); note also that \( v^i t = v^i t_c \). So, the difference is in the ‘natural’ choice of units for measuring velocity and time only. For that matter, the (relative) choice of physical units in the practical setting seems to be more in line with those here in the above equations. It should also be noted, though, that that is the case only so long as the phase space coordinate picture is concerned – looking at the generators directly as describing physical quantities would indicate that the other contraction path having \( T^c \) and \( X^i_c \), instead of \( X^0 \) and \( X^i \), may be more appealing (but that also has \( H \) instead of \( H^c \)). This issue will be addressed further below, together with explicit tracing of the physical dimensions of all quantities.

6. Discussion and Concluding Remarks

In the preceding sections we first put together the relevant relativity symmetry contractions and the resulting contractions of the relevant coset space representations, starting from the \( SO(2, 4) \) symmetry. The coset space representations are what can be called the (configuration) space coset and the phase space coset at each level, which in the classical cases give pictures of space and phase space for a single particle system (or the center of mass of a system of particles). We have considered alternative contraction sequences, such as taking the Lorentz symmetry to the Galilean limit first before going from the quantum case to that of classical physics and the other way round – achieving virtually the same result. We essentially recover Newtonian spacetime and phase space, as well as Minkowski spacetime. The \( (1+3) \)D relativity symmetry has a phase space picture with full Minkowski energy-momentum
four-vector coordinates admitting independent translations in all directions. Earlier analysis indicates that, at least for Hamiltonian evolution as generated by the square of the energy-momentum four-vector, one recovers particle dynamics of Einsteinian special relativity with proper time (or rather proper time/rest mass) as the evolution parameter. All of this can be considered as preliminary success of the scheme being advocated for here, which now includes all known particle dynamical pictures with their identified relativity symmetries. This is the key message of this article.

As discussed in the introduction, the relativity symmetry perspective requires going beyond the Poincaré symmetry and its stabilization. Formulating physics consistent with all available experimental results within this scheme may already be quite a challenge. The more exciting prospect of obtaining new predictions is even more interesting. A key point to be made here is that the proper interpretation of theories of this kind is also likely to require adjustments to our understanding of existing physical concepts beyond our old frameworks. One example we have discussed a bit above is the version of quantum mechanics with the $H_R(1,3)$ symmetry. The basic features of said quantum theory would be like those found in covariant quantum mechanics, which has been studied by various authors before while being mostly neglected by other physicists. Extending a $\phi(x^i)$ wavefunction to one of the $\phi(x^\mu)$ wavefunctions, and considering its proper time evolution, sounds like a very natural way to put quantum mechanics within the framework of Einsteinian special relativity. There has actually been a long history, comprised of many diverse efforts, along this line, which also brought up key notions such as mass indefiniteness, the introduction and interpretation of antiparticles, and the direction of time (etc.), which are to be addressed below. As such, we do not even consider it of much interest to the readers to cite here more references pertaining to these issues. On the other hand, though, a particularly noteworthy reference comes from Feynman's work on quantum electrodynamics [17]. The master went beyond everybody, actually taking the Klein-Gordon equation exactly as the $\sigma$-independent equation (again $\sigma = \frac{\sigma_m}{m}$) of the $\sigma$-evolution Schrödinger equation in covariant quantum mechanics, and discussed the $\frac{d}{d\sigma} < 0$ case in connection with the notion of antiparticles [18]. Readers interested in more details regarding this point are suggested to consult Refs. [11, 12], which present analyses in that direction based on a setting that is somewhat different from, but very compatible with, the one presented here. A key point is that – like what lies behind the wisdom of Stückelberg-Feynman – all theoretical results presented there should
be (re-)interpreted in laboratory terms, based on a coordinate time \( t \) with its forward-increasing direction, at least to the extent that a classical time idea is involved at all. For example, things ‘evolving’ backwards in time is to be interpreted in our forward-increasing time somewhat differently – in exactly the same vein as thinking of a particle moving backward in time as an antiparticle moving forward in time.

We want to leave this subject matter until we have a full dynamical analysis of the \( H_R(1, 3) \) theory and its contraction limits, except for one final point: the question of the interpretation of the \( \phi(x^\mu) \) wavefunction. Obviously, a Born probability interpretation would be very problematic. Our perspective is that quantum mechanics is about quantum models of spacetime to which the classical models provide only an approximation. These quantum models, like the example discussed in Ref.\[1\], are not finite-dimensional, real number geometries. The (projective) Hilbert space of a quantum system provides one with a real-number-geometric description of the otherwise noncommutative geometry. A wavefunction as a description of a vector in the Hilbert space is really the infinite number of coordinates under a fixed choice of coordinate frame, where the basis is being provided by \( |x^\mu\rangle \) in this case. A quantum state has a completely fixed position (in the quantum spacetime) without an uncertainty. However, such a position is to be described by some noncommutative values instead of real number values, or equivalently an infinite number of the latter. How single classical-physics-like real number values of a (repeated) von-Neumann measurement, and the probability distribution of such results, is obtained is only a matter of the kind of measurements being performed. This can be considered quite well-explained by decoherence theory, at least for standard quantum mechanics. In that sense, the problem is completely disposed of. Of course, how to better understand the nature of these noncommutative coordinate values, especially for the case of the physical time variable \( \hat{T} \), is a key challenge.

Among all of the coset results presented here, the one whose details are less obvious – which apparently would provide quite a challenge in formulating a dynamical theory – is actually the case of 3D quantum relativity symmetry \( H_{GH}(3) \) [cf. Eqs.(23) and (25)]. Looking at it from the formulation of quantum mechanics as discussed in Ref.\[2\], however, we see that this does not actually present much of a problem. We still have \( H(3) \) as an invariant subgroup. The cosets still possess an absolute Newtonian time. At least if we write quantum mechanics based on the first coset, \textit{i.e.} with the \( \phi(x^\mu) \) wavefunction. The only somewhat nontrivial issue is the quantum
phase contribution from the energy-time product. So long as we do not consider energy translations, the extra contribution drops out; hence, the usual formulation works perfectly, at least as a special case. While a canonical coherent state formulation from the phase space coset looks somewhat complicated, we know the physical story has to be the same as in the other case, as the irreducible unitary representation on the Hilbert space for the $H(3)$ part is essentially unique. There will be technical challenges for a dynamical formulation from the perspective of Ref. [2], which begins with the canonical coherent states from the full coset. Moreover, a more interesting aspect to analyze would be how one obtains the dynamics from the Lorentz to Galilean contraction of $H_R(1,3)$ discussed somewhat above.

So, to conclude, the analysis presented above of contractions of coset representations indicates no inconsistency with established theories, which should be considered as successfully retrieved from the appropriate limits and special cases. More has to be learned and some technically-detailed challenges remain to be surmounted, from which we may discover new features about Nature.

Finally, we would like to look into the issue of physical dimensions in some more detail. We have discussed how the contraction processes, as well as the studying of theories on the lower levels of the contraction sequences, suggest the introduction of (relative) physical dimensions to various quantities in terms of a ‘natural’ choice of different physical units; units which would not naturally be used in the more fundamental theory. The relationship between the units chosen by humans and the truly natural (numerical) representations of such ‘quantities’ in the mathematical structure lurking beneath gives rise to fundamental constants in physics such as $c$, $\hbar$, and $G$. For example, taking the Galilean approximation of Lorentz symmetry suggests space and time are independent; hence to be measured in different units. With Lorentz symmetry, $\frac{1}{c^2}$ is just a structural constant of the $SO(1,3)$ symmetry algebra, which is stable against deformation $\xi_0$, meaning any nonzero value(s) of the structural constants $\frac{1}{c^2}$ give the same symmetry, and the natural choice is $c$ being unity and dimensionless. This corresponds to a nontrivial, fixed value in, say, meters per second. We have also explicitly traced the physical dimensions of quantities through each of the steps of various contractions in Sec.III. The latter illuminates an idea that can be easily applied to all of the other cases discussed above. Again, how $c$ plays the role of the contraction parameter (which suggests, for example, splitting off the $x^i$ from the $\lambda^\mu$ or $t$) having the new physical dimension of $[c][\lambda]$, can be seen from results of
Sec.IV.

Let us try to see what we can learn from this coupled with our practical usage of physical dimensions. Firstly, we tabulate all of the quantities with physical dimensions obtainable from the above contraction analysis. In Table 1, we present the results for each relativity symmetry level – first for the coset coordinates and the parameters of infinitesimal transformations, followed by the symmetry generators. Let us take a look at the \( H_R(1,3) \) and \( H_{cal}(3) \) cases. As discussed in Sec.III.C, either of the two-step contractions from \( SO(2,4) \) results in \( H_R(1,3) \) with the invariant time \( \lambda \) and momentum \( p \) – the product of which is essentially \( \hbar \), which characterize the Heisenberg commutation relation. Looking at the coset level, therefore, a nontrivial fundamental constant of \( \hbar \) should actually be part of the natural choice of units.

In the last row of entries for \( H_R(1,3) \) in the table, we put a conventional choice of units that implies that the (dimensionless) Lie algebra elements are more properly written with a \( \frac{-i(c)}{\hbar} \) factor \([i.e. in the form \frac{-i(c)}{\hbar}p^\mu X_\mu]\), which is indeed the common physicists’ convention (here \( c = 1 \)). The story is essentially the same for \( H_{cal}(3) \), which however has the nontrivial \( c \). At this point, the physical dimensions of all quantities already match with our standard practice, with \([\lambda], [p], \) and \([c]\) fixing the usual space, time, and mass units. The further contractions to the classical limits involve another parameter \( k \), which then has to be taken as dimensionless, and the tracing of physical dimensions essentially ends at the \( H_{cal}(3) \) level. From the very beginning \([7]\), \( SO(2,4) \) was constructed as a kind of triply-deformed (special) relativity \([19]\) (see also Ref.\([20]\)). More importantly, the \( k \to \infty \) limit, as the classical limit, then really has nothing directly to do with the \( \hbar \to 0 \) limit, though both trivialize the Heisenberg commutation relation and decouple the central charge for the quantum phase. This result is in line with our analysis in Ref.\([2]\), \([i.e. naively taking \hbar \to 0 \) as the relativity symmetry contraction limit is the wrong thing to do! Again, \( \hbar \) essentially comes from higher level contraction. Unlike \( c \), which is the contraction parameter introduced for contraction from Lorentz symmetry to Galilean, \( \hbar \) plays no such role – a very interesting issue in and of itself.

Let us look at the physical dimension issue, especially in relation to \( \hbar \), from another point of view, beyond the story of the cosets. A single step contraction from \( SO(2,4) \) yields \( H_R(1,3) \) with one parameter which would
essentially be $\sqrt{\hbar}$. The resulting structure has a more symmetric role for $E_\mu$ and $X_\mu$, and this corresponds to our natural choice of phase space coset, as used in Sec. II [cf. Eq. (6)], and similarly in Refs. [1, 2] where the focus is really just the $H(1, 3)$, or $H(3)$, subgroup. The contraction parameter $\sqrt{\hbar}$ is then introduced to get an approximation to the physics of the otherwise $SO(2, 4)$ relativity symmetry. As such, the approximation is comparable to the $\hbar \to \infty$ limit! This is the true parallel of $\sqrt{\hbar}$ to $c$, based on the results here.

More explicitly, one takes $E_\mu = -\frac{1}{\sqrt{\hbar}} J_{\mu\alpha}$ and $X_\mu = \frac{1}{\sqrt{\hbar}} J_{\mu4}$ to the limit where $[E_\mu, E_\nu] = -[X_\mu, X_\nu] = -\frac{i}{\hbar} J_{\mu,\nu} \to 0$. Following the contraction notion naively, one would expect that the very small quantum $\hbar$ is really to be taken as a big parameter, with dimensions much smaller compared to the $SO(2, 4)$ physics of noncommuting $E_\mu$ and noncommuting $X_\mu$. At the Galilean level, we do not see the invariant speed $c$ among the structural constants of the relativity symmetry (or otherwise), but we have physics with the dimension $[c]$ (or equivalently, different physical dimensions for time and distance); at the usual quantum level, as in $H_\mu(1, 3)$ [or $H_\mu(3)$], we do not see $\hbar$ in the symmetry description, but have a notion of the physical dimension of $[\hbar]$. $c$ being an invariant parameter is an issue of Lorentz symmetry; the physics of the $SO(2, 4)$ symmetry reveal $[\hbar]$ as an invariant under (quantum) reference frame transformations. The latter is actually the starting motivation of Snyder in the relativity-symmetry-deformation line of thinking [7, 20]. Looking at things from this perspective, this does not seem to be unreasonable at all, though it is saying that our earlier thinking about the role of $\hbar$ was quite wrong. This is one key lesson here.

Beyond the coset level, we have to look at the unitary representations arising from coherent state constructions, which again suggests the natural choice of using identical units for $E_\mu$ and $X_\mu$. A further simplification is to take $\hbar$ as dimensionless. Note that the exact nature of such formulations at the $(1 + 3)$D and 3D relativity symmetry levels are not the same. At the 3D level of $H_3(3)$, this means identical units for $P_i$ and $X_i$, or equivalently, for $p^i$ and $x^i$. Identical units for $E_\mu$ and $X_\mu$ would yield $P_i$ and $X_i$ with units differing by a factor of $c$, upon taking the Lorentz to Galilean contraction,

\footnote{The standard $\hbar$ dimension is that of $\lambda pc$; hence giving the correct choice as $\lambda = p = \sqrt{\frac{\hbar}{c}}$. Here, we are neglecting $c$, which should be taken to be trivial at this level. All of the $\hbar$’s here are then $\frac{\hbar}{c}$. The exact dimensions – including the $c$ factors – are given in Table 2.}
as shown explicitly in Table 2.

Lastly, let us take a look at the classical structures $S(1,3)$ and $S_c(3)$, with the two slightly different descriptions obtained from the contraction paths of Sec.IV and V, as presented in the same order in Table 1 and Table 2. Again, the two presentations of $S_c(3)$ have different generator sets with direct one-to-one correspondence. The difference is in ‘normalizations,’ or the choice of units in describing some of the quantities. The case of dimensionless $k$ looks simplest. The only particularly interesting point to note, from Table 1, is that instead of the overall adjustment of generators by a factor that makes the central charge generator dimensionless, the factors are now chosen to enforce certain identifications of observables, such as identifying $P^c_i$ with the infinitesimal parameter and (phase space) coordinate $p^c_i$. Note that in the quantum cases, that is also achieved by making the central charge generator dimensionless. The latter, of course, is unimportant in the classical cases as the generator no longer has any role to play in the physics. The most important difference in the phase space cosets at the quantum and classical level is the fact that the representations at the quantum level are irreducible, while the representations at the classical level are reducible into the (configuration) space(time), (energy-)momentum space, and the ‘quantum’ phase parts. Actually, for $S_c(3)$, the energy and (three-)momentum space are separated as irreducible components. The irreducible nature of the quantum phase space coset, illustrated at least at the $H_R(3)$ level [1], is the key issue which reveals that the true quantum (configuration) space cannot be separated from that of the quantum phase space. The latter, as the (projective) Hilbert space constructed out of coherent state basis based on the phase space coset, is (unitary) equivalent to the one constructed from the (configuration) space coset. Of course the key to the irreducibility is the Heisenberg commutation relation.

The classical pictures, as presented in Table 2, deserve some further attention. We have actually presented the table entries with an important modification to the contraction analysis as presented above in Sec. IV and V. Instead of having only one $k$, we have two. Explicitly, we have $X^c_\mu = \frac{1}{k_x} X^c_\mu$ and $E^c_\mu = \frac{1}{k_p} E_\mu$. For example, for the $S(1,3)$ case and similar relations between other ‘classical’ $c$-generators to ‘quantum’ ones ($K^c_i = \frac{1}{k_p} K_i$), and the contraction limit corresponds to $k_x, k_p \to \infty$. The key is to allow different physical dimensions for the two parameters. By not having different dimensions for the ‘length’ and ‘momentum’ at the quantum level (explicitly
$H_R(1, 3)$ here), we must introduce that splitting at the classical level to retrieve the usual pattern of physical dimensions. Not doing this would have, for example, kept $X^\mu_c$ and $E^\mu_c$ as having the same physical dimensions. In fact, we need $[k_x] = [k_\rho]^{-1}$ to have the exact matching. If such a two parameter contraction sounds odd, one can certainly implement it in two separate steps. Adopting this convention, we again have a story fully consistent with known, practical physics. Note that the presentation here of this issue is not optimal. Our choice of notation of all quantities was set from the beginning with the idea of the matching pattern (as presented in Table 1) to practical physics, which is simply adopted here. It is not the best notation to illustrate the somewhat different story being revealed.

What is promising is that, again, a unitary Hilbert space representation of $H_R(1, 3)$, with the corresponding extension to its group $C^*$-algebra (at least for the case of the canonical coherent states), should give a fully dynamical picture of the theory. The various contractions of which would give the corresponding dynamical descriptions at the $H_{\text{cat}}(3)$, $S(1, 3)$, and $S_5(3)$ levels, which should agree with known physics. The coset-level story appears promising enough, as illustrated here. What else can be learned from such a fully dynamical analysis is the exciting task at hand, on which we hope to be able to report soon. The grand game plan is, of course, to push back up to the highest level of $SO(2, 4)$ relativity and formulate its dynamical picture.

**Acknowledgements** The authors are partially supported by research grants number 105-2112-M-008-017 and 106-2112-M-008-008 of the MOST of Taiwan.

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Table 1: Table on Physical Dimensions of Quantities. ([\lambda]^{-1} c^{-1}) stands for [\lambda]^{-1} [k]^{-1} c^{-1}, etc.)

| $SO(2, 4)$ | $\omega^{\mu\nu}$ | $J_{\mu\nu}$ | $\omega^{\mu\nu}$ | $J_{\mu\nu}$ | $\lambda^4 - [\lambda]$ | $E_4 - [\lambda]^{-1}$ |
|------------|-----------------|----------------|-----------------|----------------|-----------------|-----------------|
| $ISO(1, 4)$ | $\lambda^\mu - [\lambda]$ | $E_\mu - [\lambda]^{-1}$ | $\omega^{\mu\nu}$ | $J_{\mu\nu}$ | $p^\mu - [p]$ | $X_\mu - [p]^{-1}$ |
| $ISO(2, 3)$ | $\lambda^\mu - [\lambda]$ | $J_{\mu5}, J_{\mu\nu}$ | $\omega^{\mu\nu}$ | $J_{\mu\nu}$ | $X_\mu - [p]^{-1}$ | $X_5 - [p]^{-1}$ |
| $H_{\mathfrak{ch}}(3)$ | $t - [\lambda]$ | $x^i - [\lambda][c]$ | $J_{ij}$ | $K_i - [\lambda]^{-1}$ | $e - [p][c]$ | $g - [\lambda][p][c]$ |
| $S_{\mathfrak{c}}(3)$ | $T^c - [pck]^{-1}$ | $X^i - [\lambda][c][k]$ | $J_{ij}$ | $K^c_i - [c][k]^{-1}$ | $P^c_i - [\lambda]c^{-1}$ | $G - [\lambda]pc^{-1}$ |
| $S(1, 3)$ | $\lambda_i^\mu - [\lambda][k]$ | $X_\mu - [p][k]^{-1}$ | $J_{ij}$ | $K_i - [\lambda][p][k]$ | $P^c_i - [p][k]$ | $G - [k]^2$ |
| $S_{\mathfrak{c}}(3)$ | $t^c - [\lambda][k]$ | $x^i_c - [\lambda][c][k]$ | $J_{ij}$ | $K^c_i - [c][k]^{-1}$ | $P^c_i - [\lambda]c^{-1}$ | $F - [k]^2$ |
Table 2: Table on Physical Dimensions of Quantities starting with one-step contraction to $H_n(1,3)$, equivalent to $[\lambda] = [p] = [\hbar\frac{\pi}{2}[c]^{-\frac{3}{2}}]$.

| $SO(2,4)$ | $\mu^{\nu}$ | $J_{\mu\nu}$ |
|------------|---------------|---------------|
| $H_n(1,3)$ × $[h][c]^{-1}$ | $\lambda^\mu - [\hbar\frac{\pi}{2}[c]^{-\frac{3}{2}}]$ | $\omega^{\mu\nu}$ |
| $T - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $X_\mu - [\hbar\frac{\pi}{2}[c]^{-\frac{3}{2}}]$ | $J_{\mu\nu} - [h][c]^{-1}$ |
| $t - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $x^\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $p^\mu - [\hbar\frac{\pi}{2}[c]^{-\frac{3}{2}}]$ | $f - [h][c]^{-1}$ |
| $K_i - [c]^{-1}$ | $K_i - [h][c]^{-1}$ | $E_i - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $F - [h][c]^{-1}$ |
| $S_c(3)$ × $[h][k_p][k_p]$ | $J_{ij} - [h]$ | $p^i - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ |
| $T_c - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $x^\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $v^i - [c]$ |
| $e - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ | $e - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}$ |
| $f - [h][c]^{-1}$ |
| $S(1,3)$ × $[h][c]^{-1}[k_p][k_p]$ | $\lambda^\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $p^\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ |
| $X_\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $J_{\mu\nu} - [h][c]^{-1}$ |
| $e - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $f - [h][c]^{-1}$ |
| $F - [k_p][k_p]$ |
| $S_c(3)$ × $[h][k_p][k_p]$ | $t_c - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $x^\mu - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ |
| $J_{ij} - [h][k_p][k_p]$ | $K_i - [h][c]^{-1}[k_p]$ | $e - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $f - [h][c]^{-1}$ |
| $F - [k_p][k_p]$ | $H_c - [h]\frac{\pi}{2}[c]^{-\frac{3}{2}}[k_p]$ | $F - [k_p][k_p]$ |