Multi-Objective Decision Making Problems with Variable Domination Structure

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Abstract In this study, we investigate multi-objective decision making problems with respect to a variable domination structure. In order to solve such problems, we introduce two types of solutions from vector optimization problems with respect to a variable domination structure; afterward, we characterize them. These solution concepts are helpful in multi-objective decision making problems where different preferences or restrictions of objective functions for different alternatives are at hand; or where different preferences of objective functions with respect to different objective functions are assumed. These solution concepts are proposed for multi-objective location problems where there are different preferences of objective functions at each location; our results are applied for selecting a location to establish an outlet store.

Keywords: Decision making, multi-objective location problems, variable domination structure, minimal and nondominated solutions

1. Introduction

Decision making is a vital tool in business and life. Making the right decisions ensure the success of activities in business and life. By applying a proper decision making theory, productivity and efficiency can be increased at an individual level and in organizations, institutes, and companies.

Multi-objective decision making (MODM) problems deal with solving problems that involve multiple, often conflicting, objectives that are to be minimized or maximized. The set of solutions defines the best tradeoff between considered objectives.

MODM problems are formulated as:

We denote the objective function by \( f : X \rightarrow \mathbb{R}^p \) where \( X \subseteq \mathbb{R}^n \), and \( p, n \in \mathbb{N} \). Let \( C \subseteq \mathbb{R}^p \) be a proper, convex, closed, and pointed cone in \( \mathbb{R}^p \). An element \( f(x^0) \in f(X) \) is an efficient element of \( f(X) \) with respect to \( C \) if the following equality holds:

\[
(f(x^0) - C \setminus \{0\}) \cap f(X) = \emptyset.
\] (1.1)

Recently, variable domination structure has been instrumental in studying MODM problems. Variable domination structure introduced by Yu [15]; Yu compared two elements in terms of domination structure. Some applications have been presented for MODM problems with variable ordering structures. Bao et al. have provided an application of variable ordering structure in Psychology, see [3, 4]. Eichfelder has utilized the concept of variable ordering structure in medical engineering in [7]. The variable ordering structure has also been exploited by Tam [11] to find the optimal radiation in radiotherapy. We investigated multi-objective location problems with respect to variable domination structure [16].

The MODM problems with respect to fixed ordering cone defined in (1.1) are extended to the case of variable domination structure, and the solution concepts of such problems with
respect to variable domination structure are defined with respect to a set-valued mapping $C : \mathbb{R}^p \to \mathbb{R}^p$ [10]. We say that the element/elements $f(x^0) \in f(X)$ is a minimal element of $f(X)$ with respect to $C(\cdot)$ if
\[
(f(x^0) - C(f(x^0)) \setminus \{0\}) \cap f(X) = \emptyset.
\] (1.2)
However, in many real-world facility location problems, there are different preferences or characterizations in different locations; hence, it is more convenient to formulate the problem as a multi-objective facility location problem with respect to variable domination structure [6]. These variable domination structures represent the preferences for each alternative in terms of the given objective functions.

When one compares different locations to select the proper one or rank them in terms of the given criteria or objective functions, the characterization of each location can affect the final decision. We consider these characterizations and preferences by applying the variable domination structure.

In some MODM problems, there are different importance of criteria that could be related to the alternatives or not. In such problems, the index of weight can represent these different weights; see [5, 17] for more detail.

In this study, we present some examples to show why we utilize a MODM problem with respect to a variable domination structure and how these solutions are characterized.

The MODM problems with respect to a variable domination structure are applied where different preferences of objective functions are at hand; note that these variable preferences are related to the alternatives or are related to the objective functions presented in Sections 5 and 4 respectively. This paper is organized as follows: Section 2 refers to the fundamental concepts of the vector optimization problem. Moreover, we investigate these problems with respect to a variable domination structure. Section 3 presents the concept of nondominated elements and the intention of minimal and nondominated elements in applications. To do this, we recall the theorem [16] to solve MODM problems with variable criteria weights based on alternative preferences. We use the result of Theorems 3.1 and 3.2 to define two types of solutions. The advantage of this model is that alternatives are compared with the corresponding preferences. Moreover, in Section 4, we present an application to study MODM problems in terms of fixed ordering cone and variable domination structure and how minimal and nondominated solutions are utilized. In Section 4, we present a numerical experiment to select a location to establish an outlet store.

2. Preliminaries

In this part, we recall basic concepts on MODM problems. A decision-making process contains the following steps:
1. Determining the goal of the decision making process;
2. Selecting the set of candidate alternatives, possible actions, or candidate locations which is known as the feasible set in mathematics given by $X = \{x^j | j = 1, \ldots, m\}$, $x^j \in X \subset \mathbb{R}^n$, where $1, m = 1, \ldots, m$;
3. Selecting the set of criteria or objective functions will be given by $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$;
4. Considering a cone in order to compare objective function values;
5. Assigning a weight for each criterion to represent their importance;
6. Selecting a method to solve the MODM problem.

The spaces $\mathbb{R}^n$ and $\mathbb{R}^p$ are called preimage space or decision space and image space or objective space, respectively. For an overview and more detailed results, we refer the
reader to [1, 12]. Since in many MODM problems, we have a finite set of alternatives and objective functions, which are collected in a decision matrix, in the following, we present the decision matrix. To derive a decision matrix, a table with columns and rows is created representing the alternatives and objective function values, respectively, as shown in Table 1. $f_i(x^j)$ represents the performance of the $j^{th}$ alternative with respect to the $i^{th}$ objective function, which are the decision matrix elements, see Table 1. Each row of the decision matrix is an element of the objective space. Usual MODM problems are corresponding to the case of $C := \mathbb{R}^p_+$. In this study, we applied variable domination structures representing the preferences of objective functions in terms of alternatives or some other restrictions related to the objective functions.

To study the application of multi-objective location problems with respect to a variable domination structure, we present some basic concepts of minimal and nondominated elements in vector optimization with respect to variable domination structures.

We consider a variable domination structure defined by a set-valued mapping $C : \mathbb{R}^p \to \mathbb{R}^p$ such that $C(y)$ is a pointed convex cone for all $y \in \mathbb{R}^p$. To give the definition of a variable domination structure, Eichfelder [8] presents these two relations $\leq_1$, $\leq_2$.

Let $y, z \in \mathbb{R}^p$,

$$y \leq_1 z :\iff z - y \in C(y),$$

and

$$y \leq_2 z :\iff z - y \in C(z).$$

**Definition 2.1.** If elements in the $\mathbb{R}^p$ are compared using relation (2.1) or (2.2), then the set-valued mapping $C$ is called variable domination structure on $\mathbb{R}^n$ [8].

In this section, we deal with vector optimization problem with respect to a variable domination structure generated by $C$:

\[
\text{Determine nondominated (or minimal) solutions of } f \text{ w.r.t. } C(\cdot),
\]

where $X \subset \mathbb{R}^n$, $f(X) := \bigcup_{x \in X} f(x)$, $f(X) \subset \mathbb{R}^p$.

Moreover, we present optimality concepts for variable domination structure. Let $f(X) \subset \mathbb{R}^p$, $f(X) \neq \emptyset$.

**Definition 2.2.** ([8], Definition 2.7) Corresponding to the problem (2.3), suppose that $f(\bar{x}) \in f(X)$, minimal and nondominated elements defined by

- According to the relation $\leq_1$ defined in (2.1) a candidate element is called a **nondominated element** (see (2.3)) if it is not dominated by other elements with respect to their corresponding cone. It is formally shown that $f(\bar{x})$ is a nondominated element of $f(X)$ with respect to the domination map $C$, if $f(x) \ngeq_1 f(\bar{x})$ for all $f(x) \in f(X) \setminus \{f(\bar{x})\}$, i.e. if

\[ f \not\in f(X) : f(\bar{x}) \in f(x) + C(f(x)) \setminus \{0_{\mathbb{R}^p}\}, \]

---

**Table 1: Alternatives $x^j$, $j = 1, m$ and objective functions $f_i$, $i = 1, p$.**

|   | $f_1(x)$ | $f_2(x)$ | $\cdots$ | $f_p(x)$ |
|---|-----------|-----------|-----------|-----------|
| $x^1$ | $f_1(x^1)$ | $f_2(x^1)$ | $\cdots$ | $f_p(x^1)$ |
| $\vdots$ |           |           |           |           |
| $x^m$ | $f_1(x^m)$ | $f_2(x^m)$ | $\cdots$ | $f_p(x^m)$ |

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or

\[ f(\bar{x}) \notin \bigcup_{f(x) \in f(X)} f(x) + (C(f(x)) \setminus \{0_{RP}\}). \]

Figure 1: \( f(\bar{x}) \) is not a nondominated element of \( f(X) \)

Moreover, \( f(\bar{x}) \) is a weakly nondominated element of \( f(X) \) with respect to the domination map \( C \), if

\[ \# f(x) \in f(X) : f(\bar{x}) \in f(x) + \text{int}C(f(x)) \setminus \{0_{RP}\}. \]

The set of all weakly nondominated elements is denoted by \( WN(f(X), C(\cdot)) \).

- Based on the relation \( \leq_2 \) defined in (2.2) a candidate element is called a minimal element if it is not dominated by any other element with respect to the cone related with the candidate element. This expression defines that \( f(\bar{x}) \) is a minimal element of the set \( f(X) \) with respect to the domination map \( C \), if \( f(x) \not\leq_2 f(\bar{x}) \) for all \( f(x) \in f(X) \setminus \{f(\bar{x})\} \), i.e. if

\[ \# f(x) \in f(X) : f(\bar{x}) \in f(x) + C(f(x)) \setminus \{0_{RP}\}, \]

or

\[ \left( \{f(\bar{x})\} - C(f(\bar{x})) \right) \cap f(X) = \{f(\bar{x})\}. \]

The set of all minimal elements is denoted by \( M(f(X), C(\cdot)) \).

Figures 1, 2 show both optimality conditions.

Remark 2.1. Note that in Definition 2.2, it is not necessary that the sets \( C(f(x)) \) are convex pointed cones.

Remark 2.2. Note that in Multi-objective location problems w.r.t. variable domination structure, the domination structures are related to \( x \) or \( f(x) \); It means \( C(x) \) or \( C(f(x)) \) respectively. The domination structure \( C(x) \) is defined from preimage space to image space, and \( C(f(x)) \) is defined from image space to image space. Depending on the problem, each domination structure is applied. In Section 4, the domination structure is defined from image space to image space, and in the case study presented in Section 5, the domination structure is defined from preimage space to image space. In [16], the authors derive a new method for decision-making problems using definitions of minimal and weakly nondominated solutions. The following section presents these results and shows how we can apply these results in application.
3. Solution Type I and Type II of Multi-Criteria Decision Making Problem with Variable Domination Structure

The following assumption is used in this study.

**Assumption 3.1.** Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a vector function, $m$ is a positive integer number, $X \subset \mathbb{R}^n$, and $X := \bigcup_{j=1}^m X_j$ where $X_j \subset \mathbb{R}^n$ for $j = \overline{1, m}$, and $X_s \cap X_r = \emptyset$ whenever $s \neq r$. Let $\alpha^j := (\alpha^j_1, \ldots, \alpha^j_p)^T \in \mathbb{R}^p$ for $j = \overline{1, m}$, $C : \mathbb{R}^n \Rightarrow \mathbb{R}^p$ is a convex domination map defined as

$$
C(x) := \begin{cases} 
\{(y_1, y_2, \ldots, y_p)^T \in \mathbb{R}^p \mid \alpha^1_1 y_1 + \alpha^1_2 y_2 + \ldots + \alpha^1_p y_p \geq 0\} & \text{if } x \in X_1, \\
\cdots \cdots \\
\{(y_1, y_2, \ldots, y_p)^T \in \mathbb{R}^p \mid \alpha^m_1 y_1 + \alpha^m_2 y_2 + \ldots + \alpha^m_p y_p \geq 0\} & \text{if } x \in X_m.
\end{cases}
$$

(3.1)

The corresponding vector optimization problem w.r.t. the variable domination structure generated by (3.1) is described as follows:

Determine nondominated (or minimal) solutions of $f$ w.r.t. $C(\cdot)$. \hspace{1cm} (PC(\cdot))

We recall the following theorem which, discusses a result concerning the sufficient conditions of the minimal and weakly nondominated solutions of problem $PC(\cdot)$ \cite{16}. The solutions Type I and Type II are derived from the result of Theorem 3.1.

**Theorem 3.1.** \cite{16} Let Assumption 3 be fulfilled and $x^0 \in X$. We suppose that $X_j = \{x^j\}$, $X = \{x^1, \ldots, x^m\}$. We assume that the following inequality holds for all $x^j \in X$, $j = \overline{1, m}$

$$
\sum_{i=1}^p \alpha^j_i f_i(x^0) \leq \sum_{i=1}^p \alpha^j_i f_i(x^j).
$$

(3.2)

Then, $f(x^0) \in WN(f(X), C(\cdot))$.

This assertion is proved in [Theorem 4.1, \cite{16}].

The same relation holds true for a location problem where distance functions are considered for objective functions.

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Theorem 3.2. Let Assumption 3 be hold. For $x^0 \in X$, $\alpha^0 := (\alpha_1^0, \ldots, \alpha_p^0)$, if the following assertion holds true for all $x^j \in X$

$$
\sum_{i=1}^{p} \alpha_i^0 f_i(x^0) \leq \sum_{i=1}^{p} \alpha_i^0 f_i(x^j), \quad j = \overline{1,m}.
$$

(3.3)

Then, $f(x^0) \in M(f(X), C(\cdot))$.

Proof. Assume that the inequality $\alpha_i^0 f_i(x^0) \leq \alpha_i^0 f_i(x^j)$ holds for $j = \overline{1,m}$. To prove that $x^0$ is a minimal solution of problem $(P_{C^\cdot})$, we assume by contradiction that there is $\bar{x} \in X$ such that $f(\bar{x}) \in f(x^0) - C(x^0) \setminus \{0\}$. It follows that $f(x^0) - f(\bar{x}) \in C(x^0) \setminus \{0\}$. Therefore, $(f(x^0) - f(\bar{x}))\alpha^0 > 0$. This is equivalent to $\alpha^0 f(x^0) > \alpha^0 f(\bar{x})$, which contradicts the assumption. \hfill \Box

Remark 3.1. Observe that the converse implication of the results in Theorem 3.1 can be found in [Theorem 3.2 and 3.3 [16]].

Since, in many MODM problems, we have a finite set of alternatives and objective functions, we apply Theorems 3.1 and 3.2 to use in such problems. Consider the decision matrix given by

$$
D_{m \times p} := \begin{pmatrix}
  f_1(x^1) & \ldots & f_p(x^1) \\
  \vdots & \ddots & \vdots \\
  f_1(x^m) & \ldots & f_p(x^m)
\end{pmatrix},
$$

(3.4)

moreover, we assume the weight matrix as

$$
W_{m \times p} := \begin{pmatrix}
  \alpha_1^1 & \ldots & \alpha_p^1 \\
  \vdots & \ddots & \vdots \\
  \alpha_1^m & \ldots & \alpha_p^m
\end{pmatrix},
$$

(3.5)

where $\alpha^j_i \in \mathbb{R}$ is the weight of the $i^{th}$ criteria with respect to the $j^{th}$ alternative. There are many methods to calculate the criteria’s weight, see [13, 19].

The matrix $\Phi$ is defined to find minimal and weakly nondominated solutions presented in Theorem 3.1, $\Phi_{m \times m} := D_{m \times p} \times W_{p \times m}^T$, where $W_{p \times m}$ denotes the transpose matrix to $W_{m \times p}$, and $D_{m \times p}$ is a normalized decision matrix, see [14]. More explicitly:

$$
\Phi_{m \times m} = \begin{pmatrix}
  \sum_{i=1}^{p} \alpha_i^1 f_i(x^1) & \ldots & \sum_{i=1}^{p} \alpha_i^m f_i(x^1) \\
  \vdots & \ddots & \vdots \\
  \sum_{i=1}^{p} \alpha_i^1 f_i(x^m) & \ldots & \sum_{i=1}^{p} \alpha_i^m f_i(x^m)
\end{pmatrix}.
$$

(3.6)

Then, $\Phi_{kl}$ is the performance of alternative $x^k$ corresponding to the weight vector $\alpha^l = (\alpha_1^l, \ldots, \alpha_p^l)$, i.e., $\Phi_{kl} = \alpha_1^l f_1(x^k) + \ldots + \alpha_p^l f_p(x^k)$.

For problem $(P_{C^\cdot})$, we define two types of solutions as follows:

**Definition 3.1.** Consider problem $(P_{C^\cdot})$ and $x^k$ is a given alternative, $k = \overline{1,m}$.

(a) **Solution Type I** If an element in the main diagonal of the matrix $\Phi$ is the smallest element in its corresponding column, it is regarded that this element is a solution Type I. This is equivalent that $x^k$ is a solution Type I of $(P_{C^\cdot})$ if

$$
\forall l = \overline{1,m}: \Phi_{kk} \leq \Phi_{lk}.
$$

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(b) **Solution Type II** If an element in the main diagonal of the matrix $\Phi$ is the greatest element in its corresponding row, it is regarded that this element is a solution Type II. This is equivalent to $x^k$ is a solution Type II of $(P_{C(\cdot)})$ if

$$\forall l = 1, m : \Phi_{kl} \leq \Phi_{ll}.$$ 

**Remark 3.2.** From Theorem 3.1, if we consider the problem $(P_{C(\cdot)})$ and the set-valued mapping $C(\cdot)$ given by (3.1), where $X = \{x^1, \ldots, x^m\}$. $X_j = \{x^j\}$ and $a^j$, $j = 1, m$ are weight vectors, then $x^k$ is a solution of Type II for the problem $(P_{C(\cdot)})$ if and only if $f(x^k)$ is a weakly non-dominated element of $f(X)$ with respect to $C(\cdot)$, i.e., $x^k$ is a weakly nondominated solution of $(P_{C(\cdot)})$. Similarly, from Theorem 3.2, $x^k$ is a solution of Type I for the problem $(P_{C(\cdot)})$ if and only if $f(x^k)$ is a minimal element of $f(X)$ with respect to $C(\cdot)$, i.e., $x^k$ is a minimal solution of $(P_{C(\cdot)})$.

The following section compares the solution set of a vector optimization problem with respect to a fixed ordering cone and the solution set of a corresponding problem with respect to a variable domination structure. It shows why the decision maker intended to apply a variable domination structure instead of a fixed cone in MODM problems.

### 4. Selecting a Location for a New Market Using Multi-Objective Location Problem

To present a practical model, we are looking to select a location for a new market; note that these concepts for all location problems hold true. Consider an enterprise with $p$ existing distributing warehouses localized in $a^1, \ldots, a^p \in \mathbb{R}^2$. Each of these warehouses provides different products, for instance, snacks, sandwiches, drinks, etc.

A decision maker is looking for a location $x \in \mathbb{R}^2$ to establish a new market. The $p$ given warehouses provide the goods for this market. The decision maker may desire to select a new facility to establish a market by minimizing the distances to these $p$ existing warehouses.

The distance from the new market $x \in \mathbb{R}^2$ to a given existing facility $a^i \in \mathbb{R}^2$, $i = 1, p$ is measured by the distance function induced by the Manhattan norm $|| \cdot ||_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ with

$$||x - a^i||_1 := \sum_{j=1}^{2} |x^j - a^j_i|, \ i = 1, p.$$ 

In this case, we use the Manhattan norm to calculate the distance; while Euclidean distance gives the shortest distance between two points, Manhattan distance has specific performance. It is related to the urban structure. For instance, in a city like Manhattan, applying Manhattan distance is proper when traveling from one point to another.

The objective function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^p$ for the location problem is given by

$$f(x) := \left( \begin{array}{c} ||x - a^1||_1 \\ \ldots \\ ||x - a^p||_1 \end{array} \right). \quad (4.1)$$

#### 4.1. Multi-objective location problems with fixed ordering structure

Consider the case that the decision maker is looking for a new facility to establish a market by considering the same preference of minimizing the distance to the existing warehouses for all feasible points. It is evident that if there is a location with a minimum distance to
all warehouses, this location is selected as an optimal solution, and the other locations are dominated by it. However, in real world problems, usually, such location does not exist, and each feasible location $x$ is close to some warehouses while other feasible locations are close to some other warehouses. So, if there are no other preferences or restrictions, the concept of Pareto efficiency w.r.t. a fixed ordering cone (in this case $R^n_+$) is applied to determine the efficient solution set. Pareto efficient solution set collects the locations that are not dominated by the other locations. This means that in the case of using $R^n_+$ as a fixed cone, it is only possible to improve an objective function by worsening another objective function.

Let $X \subseteq \mathbb{R}^2$ be the feasible set. In this case study, $X$ is a finite set of possible locations to establish a new market.

The constrained multi-objective location problem w.r.t. fixed order involving the Manhattan norm is given by

$$\begin{cases} f(x) = (||x - a^1||_1, \ldots, ||x - a^p||_1) \to \min \text{ w.r.t. } \mathbb{R}_+^p \\ x \in X. \end{cases}$$

(POLP$_X$)

A point $x \in X$ is called Pareto efficient solution for (POLP$_X$) if

$$\not\exists x' \in X \text{ s.t. } \begin{cases} \forall i \in \{1, 2, \ldots, p\} : ||x' - a^i||_1 \leq ||x - a^i||_1, \\ \exists j \in \{1, 2, \ldots, p\} : ||x' - a^j||_1 < ||x - a^j||_1, \end{cases}$$

or equivalently,

$$f(X) \cap (f(x) - \mathbb{R}_+^p \setminus \{0\}) = \emptyset.$$ 

$f(x)$ is a vector with $p$ components, and the location $x$ with all components $f_i(x)$ greater than the others are dominated by the other locations. It means that the other locations dominate the locations, which have longer distances to all warehouses. The decision maker collects the set of Pareto efficient solutions, which has at least the distance to a warehouse less than the other elements compared. If the decision maker does not assume the other restrictions or preferences, then there is no other decision that can improve the outcome given by Pareto optimal solution.

For instance, Figure 3 illustrates Pareto efficient solutions of the Problem (POLP$_X$) where $p = 2$ (However, in location problems w.r.t. fixed ordering cone, it makes sense where $p > 2$; here we assume $p = 2$ to illustrate it on a figure and to comparison with location problems w.r.t. variable domination structure). It shows that $x^1$, $x^2$, $x^3$, $x^4$ belong to the Pareto efficient solution set. If we compare the locations $x^2$, $x^4$, it is evident that $x^2$ is closer to warehouse $a^1$ than $x^4$ while $x^4$ is close to warehouse $a^2$ than $x^2$. It indicates that both these locations are appropriate to establish a new market if there are no other preferences or restrictions involved.

4.2. Multi-objective location problems with variable domination structure

This section considers the concept of variable domination structures by Yu (1974) and Chen et al. (2005)

Suppose that the decision-maker may desire to select a location for a new market by considering that minimizing the distance to some warehouses is more preferred than minimizing the distance to the other warehouses at each location. This situation can arise when customer requirements are not the same at each candidate location, and these requirements should be met by different warehouses. For instance, we assume that we select a location $x^j$ for our market close to cinema; the demand for popcorn is more than sandwiches, while if we select the other location for our market, it is another way around. Consequently,
minimizing the distance to the warehouse/warehouses that provide the popcorn and snacks is more preferred than minimizing the distance to the warehouse/warehouses that provide the sandwiches if we select the location \( x_j \) for our market.

It indicates that the preferences of objective functions are not the same at each location; therefore, this condition leads the decision maker to formulate these preferences by applying a variable domination structure \( C(\cdot) \) instead of the fixed ordering cone \( \mathbb{R}^p_+ \).

Here, we assume \( p = 2 \) in order to illustrate it in a figure. Then, \( f_1(x) = ||x - a^1||_1, f_2(x) = ||x - a^2||_1 \). \( a^1 \) and \( a^2 \) are two warehouses to provide sandwiches and snacks/drinks, respectively. The decision-maker considers a stronger preference relation for \( f_1(x) \geq 1 \).

The decision maker assumes that there are many cinemas and many attractive places in \( x \) with \( f_1(x) \geq 1 \). It implies that the markets that are located in \( x \) with \( f_1(x) \geq 1 \) at each period should provide the requirements three times by warehouse \( a^2 \) and once by warehouse \( a^1 \). It indicates that in \( x \) with \( f_1(x) \geq 1 \), we want to minimize the sum of distances by the following relations, and the goal is to find \( x^0 \in \mathbb{R}^2 \) s.t.

\[
f_1(x^0) + 3f_2(x^0) < f_1(x) + 3f_2(x),
\]

this means that the decision maker compares the outcome values by

\[
C(f(x)) := \begin{cases} D & \text{if } f_1(x) \geq 1, \\ \mathbb{R}^2_+ & \text{otherwise}, \end{cases}
\]

where \( D := \{(y_1, y_2) \mid y_1 + 3y_2 \geq 0\} \).

Note that, in this case, the domination map is defined from image space to image space.

The corresponding multi-objective location problem w.r.t. this variable domination structure is given by

\[
\begin{align*}
\min & \quad C(f(x)) \\
\text{s.t.} & \quad f(x) = (||x - a^1||_1, ||x - a^2||_1) \\
& \quad x \in X.
\end{align*}
\]

The Figure 3 shows the minimal and nondominated solutions of the problem \((P_{C(f(x))})\), see Definition 2.2. The figure indicates that, \( f(x^1) \) and \( f(x^2) \) are minimal elements of \((P_{C(f(x))})\) w.r.t. the domination structure \( C(f(x)) \); and \( f(x^4) \) is a minimal and nondominated element of \((P_{C(f(x))})\) w.r.t. the domination structure \( C(f(x)) \). The element \( f(x^3) \) is not an efficient element of the problem \((P_{C(f(x))})\) w.r.t. the domination structure defined for this area; while \( f(x^3) \) is an efficient element of \((POLP_X)\).

Therefore, when the decision maker attempts to select an appropriate place for a new market by minimizing the distance to the warehouses, all these locations could be selected (all the locations which are belonging to the Pareto efficient solution set). While, if the decision maker assumes additional restrictions or preferences presented in 4.2, \( x^3 \) is not an optimal solution considering the decision maker’s preferences. In this case, \( x^1, x^2, x^4 \) are efficient solutions by applying their preferences.

These solutions are appropriate locations to establish a new market by assuming the customer requirements at each location. The location \( x^4 \) is a nondominated solution too. It indicates that this location is an optimal solution not only by considering the preferences of its location but also by considering the preferences of other alternatives. Hence, \( x^4 \) is an appropriate location for a new market even if the nearby cinemas or attractive places are dissolved. Moreover, assuming that \( f(x^2) \) is a nondominated element, it means that the
Figure 3: Efficient solutions of \((P_{C(f(x))})\)

location \(x^2\) is an appropriate location to establish a new market even if new cinemas are built nearby.

The decision maker selects a proper optimal solution (minimal/nondominated solution) up to the problem. It is clear that, in terms of different preferences of alternatives, it is impossible to find these optimal solutions (minimal and nondominated) by using any other multiobjective decision making solution techniques.

Remark 4.1. Note that, in this case study, the preimage space is a subset of \(\mathbb{R}^2\). To compare this with the preimage space \(\mathbb{R}^2\), we refer to Figure 4.

The bold line shows the set of minimal elements, and \(f(x^5)\) is a minimal and nondominated element of \((P_{C(f(x))})\) w.r.t. the domination structure \(C(f(x))\).

Remark 4.2. It is clear that there are many other optimal solutions for the problems \((POLP_X), (P_{C(f(x))})\), and we consider the elements \(f(x^1), f(x^2), f(x^3), f(x^4)\) to represent and compare the different solution concepts.

5. Selecting a Location to Establish an Outlet Store

This section presents a numerical experiment considering a finite set of alternatives and objective functions; then investigates how the solution Type I and Type II can be helpful in multi-objective decision making problems where different preferences are assumed. We assume that a decision maker wants to establish an outlet store in one of five feasible locations by minimizing the distance to six suppliers.

Consider problem \((P_{C(i)})\) under Assumption 3, where \(X_j = \{x^j\}, j = 1, 5, i = 1, 6\), and the objective function \(f : \mathbb{R}^2 \to \mathbb{R}^6\) is given by

\[
f(x) := \begin{pmatrix} d(x, a^1) \\ \vdots \\ d(x, a^6) \end{pmatrix}
\] (5.1)
Note that, in real-world applications, many other objective functions can involve such as land cost, competitors’ stores’ locations, human resources costs, etc.

Figure 5 illustrates the position of feasible locations and suppliers, Google (n.d.) [9]. The following geographical coordinates for suppliers and feasible locations are at hand from the Google (n.d.), see [9].

\[ a_1 = (54.069760, 11.957710), \quad a_2 = (53.529593, 9.721993), \quad a_3 = (52.662122, 13.666084), \]
\[ a_4 = (51.459898, 6.327217), \quad a_5 = (50.277834, 8.606880), a_6 = (48.733452, 9.233101), \]
\[ x_1 = (51.362233, 9.211128), \quad x_2 = (52.643781, 9.639595), \quad x_3 = (51.103201, 6.683097), \]
\[ x_4 = (51.401094, 11.854868), \quad x_5 = (53.886237, 9.984488). \]

Where the flag points show the location of suppliers and the other signs represent the feasible locations.

Here, we calculate the distance between outlet stores and suppliers by Google Maps (great-circle distance formula) as the following decision matrix, for instance, \( f_1(x^1) = d(x^1, a^1) = 0.17 \); however, one can calculate the distances using Manhattan or Euclidean norms up to the decision case, see [16, 18]. The normalized decision matrix is given by

\[
D^n = \begin{pmatrix}
0.17 & 0.21 & 0.25 & 0.17 & 0.09 & 0.11 \\
0.18 & 0.27 & 0.26 & 0.17 & 0.09 & 0.02 \\
0.19 & 0.12 & 0.18 & 0.17 & 0.15 & 0.19 \\
0.27 & 0.17 & 0.15 & 0.07 & 0.18 & 0.16 \\
0.32 & 0.15 & 0.08 & 0.03 & 0.23 & 0.2
\end{pmatrix}
\]

Assume that at each feasible location \( x^j \), customer demands for the products and brands are not the same; these customer requirements should be met by different suppliers. Therefore, minimizing the distance to some suppliers is more preferred than minimizing the distance to some other suppliers at each feasible location. These preferences are collected in Table 2. For instance, \( \alpha_{1}^{1} = 0.25 \). The matrix \( \Phi \) is calculated using (3.6) as
Figure 5: The position of feasible locations and suppliers, Google (n.d.) [9]

Table 2: Preferences of suppliers

|   | $a^1$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ |
|---|-------|-------|-------|-------|-------|-------|
| $x^1$ | 0.25 | 0.16 | 0.16 | 0.08 | 0.25 | 0.08 |
| $x^2$ | 0.08 | 0.25 | 0.08 | 0.33 | 0.08 | 0.16 |
| $x^3$ | 0.3  | 0.15 | 0.15 | 0.08 | 0.23 | 0.08 |
| $x^4$ | 0.15 | 0.23 | 0.15 | 0.08 | 0.3  | 0.08 |
| $x^5$ | 0.16 | 0.16 | 0.25 | 0.08 | 0.25 | 0.08 |

\[ \Phi = \begin{pmatrix} 
0.1706 & 0.182 & 0.1721 & 0.1745 & 0.1778 \\
0.1675 & 0.1692 & 0.1694 & 0.1703 & 0.1747 \\
0.1618 & 0.1581 & 0.1653 & 0.1569 & 0.1609 \\
0.1821 & 0.1392 & 0.1888 & 0.1745 & 0.1713 \\
0.1927 & 0.1298 & 0.2018 & 0.1819 & 0.1711 
\end{pmatrix}, \]

where \( \sum_{i=1}^{6} \alpha_i d(x^1, a^i) = 0.1706 \). Applying Theorems 3.1 and 3.2 the following data indicates that \( x^3 \) is a Type I and Type II solution.

\[
\sum_{i=1}^{6} \alpha_i d(x^3, a^i) \leq \sum_{i=1}^{6} \alpha_i d(x^1, a^i); \quad \sum_{i=1}^{6} \alpha_i d(x^3, a^i) \leq \sum_{i=1}^{6} \alpha_i d(x^1, a^i);
\]

\[
\sum_{i=1}^{6} \alpha_i d(x^3, a^i) \leq \sum_{i=1}^{6} \alpha_i d(x^2, a^i); \quad \sum_{i=1}^{6} \alpha_i d(x^3, a^i) \leq \sum_{i=1}^{6} \alpha_i d(x^2, a^i);
\]
It means that this feasible location is not only a proper location by considering customer demands in this location, but also a proper location by considering customer demands at the other feasible locations. In other words, the nondominated solution could be a proper location to establish an outlet store even if the customer demands change to the condition of other feasible locations. For instance, if the customer demands are related to a nearby city size (large/small), a nondominated solution could be a proper location to establish an outlet store even if the city becomes larger over time.

In many multi-objective decision making problems with respect to variable domination structure, the decision maker is looking to find a solution by considering the preferences of each alternative/location; this refers to solution Type I. In some other multi-objective decision making problems with respect to variable domination structure, the decision maker is looking to find a solution that is an efficient solution by considering the preferences of the other alternatives/locations; it leads the decision maker to find a solution Type II.

For instance, assume that a decision maker is looking to find a place to establish a superstore by minimizing the distances to warehouses. The superstore includes a supermarket plus a large general merchandise section, including apparel, cosmetics, health and beauty care, housewares and home decor, and other non-food items. We assume that the customers’ demands are not the same at each feasible location; these requirements should be met by different warehouses. Therefore, minimizing the distances to some warehouses is more preferred to some others at each feasible location. For instance, assume that nearby a feasible location, there are many other health care and beauty shops; therefore, at this location, providing these items from the corresponding warehouses is less than the other feasible location; whereas nearby some other feasible locations, there are many homewares and home decor shops; therefore, proximity to the warehouses that provide these products is not significant as providing the other products with higher demand at these locations. These different customer demands lead the decision maker to formulate these preferences of proximities to different warehouses by defining a variable domination structure corresponding to each feasible location. If we find a minimal solution, it indicates that this location is appropriate by considering corresponding customer demands. Moreover, suppose that there is a nondominated solution, it means that this location is an appropriate location to establish a superstore, even if nearby this location will be a health and beauty care or home decor shop (even if the preferences of location changed to the preferences of the other feasible location).

Therefore, a decision maker is looking to find minimal/nondominated or both solutions up to the condition of a decision making problem.

6. Conclusions
Since in many real-world MODM problems, there are variable preferences of the criteria (i.e., the objective functions) concerning the alternatives (i.e., locations), utilizing these problems with respect to the variable domination structure has many applications, especially in multi-objective location problems.
The presented solutions (solution Type I and solution Type II) help us select an optimal solution by considering the related preferences of the alternatives.

The domination map used in this study is a hyperplane corresponding to each alternative. The decision maker defines these domination structures up to the preferences of alternatives and sometimes the location’s characterization.

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