A rational trigonometric relationship between the dihedral angles of a tetrahedron and its circumradius

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Abstract

This paper will extend a known relationship between the circumradius and dihedral angles of a tetrahedron in three-dimensional Euclidean space to three-dimensional affine space over a general field not of characteristic two, using only the framework of rational trigonometry devised by Wildberger. In this framework, a linear algebraic view of trigonometry is presented, which allows the associated three-dimensional vector space of such a three-dimensional affine space to be equipped with a non-degenerate symmetric bilinear form; this will also generalise the results presented to arbitrary geometries parameterised by such a non-degenerate symmetric bilinear form.

Keywords: tetrahedron, circumradius, dihedral angle, rational trigonometry, symmetric bilinear form

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1 Introduction

The following result, from Cho (2000) [3], gives the relationship between the dihedral angles of a tetrahedron in three-dimensional Euclidean space over the real number field and its circumradius.

**Theorem 1** Let $A_0, A_1, A_2$ and $A_3$ be the points of a tetrahedron in three-dimensional Euclidean space over the real number field. Let $R$ be its circumradius, and for distinct $i$ and $j$ in the set $\{0 1 2 3\}$, let $\theta_{ij}$ be the interior dihedral angle between $A_i$ and $A_j$. Then, the volume $V$ of the tetrahedron is

$$V = \frac{32}{3} \left( \frac{N_0 N_1 N_2 N_3}{M^2} \right) R^3$$

where, for $l$ in the set $\{0 1 2 3\}$,

$$N_l \equiv \det \begin{pmatrix} 1 & \cos \theta_{ij} & \cos \theta_{ik} \\ \cos \theta_{ji} & 1 & \cos \theta_{jk} \\ \cos \theta_{ki} & \cos \theta_{kj} & 1 \end{pmatrix}$$

1
and

\[
M \equiv -\det \begin{pmatrix}
0 & \sin^2 \theta_{01} & \sin^2 \theta_{02} & \sin^2 \theta_{03} \\
\sin^2 \theta_{10} & 0 & \sin^2 \theta_{12} & \sin^2 \theta_{13} \\
\sin^2 \theta_{20} & \sin^2 \theta_{21} & 0 & \sin^2 \theta_{23} \\
\sin^2 \theta_{30} & \sin^2 \theta_{31} & \sin^2 \theta_{32} & 0
\end{pmatrix}.
\]

In this paper, we aim to derive a similar result using only the framework of rational trigonometry, as devised by Wildberger (2005) [8]. We will also draw on a result from Crelle (1821) [4] with regards to the circumradius of a tetrahedron and its volume and side lengths to obtain such a result, and ultimately this gives rise to an interesting set of implications. Proving this result allows us to obtain a result pertaining to the ratio of the product of opposing dihedral angles to the product of opposing side lengths, as well as allowing us to express the circumradius explicitly in terms of the dihedral angles, volume and areas of the tetrahedron.

In rational trigonometry, the metrical notions of distance and angle are replaced respectively by quadrance and spread, which have purely linear algebraic definitions. This allows us to define the metrical notions of rational trigonometry in three-dimensional affine space; we can thus equip to its associated three-dimensional vector space a non-degenerate symmetric bilinear form that can be represented by a $3 \times 3$ invertible symmetric matrix, for which we then have a generalised definition of perpendicularity. This gives us a general metrical structure by which the results seen in this paper can be generalised to arbitrary geometries. We should also note that these results can also be generalised to arbitrary fields not of characteristic 2, so that the Zero denominator convention in [8, p. 28] is adopted without constant reference. The notions of area, volume and dihedral angle from classical trigonometry will also be replaced by the rational trigonometric notions of quadrea, quadrume and dihedral spread.

We will also adopt a novel approach to proving the results in this paper, which involves taking an affine map from a general tetrahedron to a special tetrahedron whose points are

\[
X_0 \equiv [0,0,0], \quad X_1 \equiv [1,0,0], \quad X_2 \equiv [0,1,0] \quad \text{and} \quad X_3 \equiv [0,0,1].
\]

This special tetrahedron is named the Standard tetrahedron, and it allows to analyse a specific tetrahedron over a general symmetric bilinear form rather than a general tetrahedron over a specific symmetric bilinear form. Using this tool, a key property of the affine map implies that any result we prove for the Standard tetrahedron can be generalised to a general tetrahedron, by way of the inverse affine map; thus, it is sufficient that any result presented in this paper is proven for the Standard tetrahedron.

## 2 Preliminaries

### 2.1 Tetrahedron in three-dimensional affine space

We start with the three-dimensional affine space over a general field $\mathbb{F}$ not of characteristic 2, which will be denoted by $\mathbb{A}^3$. The associated three-dimensional vector space $\mathbb{V}^3$ is equipped with a non-
degenerate symmetric bilinear form \( b \) represented by a \( 3 \times 3 \) invertible symmetric matrix
\[
B \equiv \begin{pmatrix}
a & b & b \\
b & a & b \\
b & b & a
\end{pmatrix}
\]
and defined by
\[
b(u, v) \equiv u \cdot_B v \equiv uBv^T
\]
for vectors \( u \) and \( v \) in \( \mathbb{V}^3 \), which will be called a \textit{B-scalar product}. We will use the notation \( u \cdot_B v \) throughout the paper to denote the \( B \)-scalar product. From this, we also define a \textit{B-quadratic form} by
\[
Q_B(v) \equiv v \cdot_B v.
\]
If \( u \cdot_B v = 0 \) then \( u \) and \( v \) are \textit{B-perpendicular}.

The primary objects in \( \mathbb{A}^3 \) are called \textit{points} and are denoted in this paper by a triple enclosed in rectangular brackets, and the primary objects in \( \mathbb{V}^3 \) are called \textit{vectors} and are typically denoted as a three-dimensional row matrix. The association mentioned above is described by the operation
\[
X + v = Y
\]
for points \( X \) and \( Y \) in \( \mathbb{A}^3 \), and \( v \) a vector in \( \mathbb{V}^3 \), which then allows us to define a vector between \( X \) and \( Y \) by
\[
\overrightarrow{XY} \equiv v = Y - X
\]

A \textit{tetrahedron} in \( \mathbb{A}^3 \) is an unordered collection of four points in \( \mathbb{A}^3 \); for points \( A_0, A_1, A_2 \) and \( A_3 \) in \( \mathbb{A}^3 \), a tetrahedron containing these points will be denoted by \( A_0A_1A_2A_3 \). An \textit{edge} of a tetrahedron \( A_0A_1A_2A_3 \) is defined to be an unordered collection of any two points of \( A_0A_1A_2A_3 \) and is denoted by \( A_iA_j \) for integers \( i \) and \( j \) satisfying \( 0 \leq i < j \leq 3 \). Furthermore, a \textit{triangle} of a tetrahedron \( A_0A_1A_2A_3 \) is defined to be an unordered collection of any three points of \( A_0A_1A_2A_3 \) and is denoted by \( A_iA_jA_k \) for integers \( i, j \) and \( k \) satisfying \( 0 \leq i < j < k \leq 3 \). Note that there are six edges and four triangles associated to any tetrahedron in \( \mathbb{A}^3 \), and that there are three edges associated to each triangle of such a tetrahedron. We will also define the \textit{midpoint} of the edge \( A_iA_j \) to be the point \( M_{ij} \) satisfying
\[
\overrightarrow{A_iM_{ij}} = \frac{1}{2} \overrightarrow{A_iA_j}.
\]

Associated to each edge of a tetrahedron \( A_0A_1A_2A_3 \) is a \textit{B-quadrance}, which is the number
\[
Q_B(A_iA_j) \equiv Q_B \left( A_iA_j^\leftrightarrow \right) = A_iA_j^\leftrightarrow \cdot_B A_iA_j^\leftrightarrow
\]
for integers \( i \) and \( j \) satisfying \( 0 \leq i < j \leq 3 \). This will be denoted for the rest of this paper by \( Q_{ij} \).

Given a triangle \( A_iA_jA_k \) of a tetrahedron \( A_0A_1A_2A_3 \), for integers \( i, j \) and \( k \) satisfying \( 0 \leq i < j < k \leq 3 \), we have the three edges \( A_iA_j \), \( A_iA_k \) and \( A_jA_k \) associated to it, with respective \( B \)-quadrances
Consider an affine map which sends a general tetrahedron $\overline{A_0A_1A_2A_3}$ itself is the $B$-\textbf{quadrum}e, which is the number

$$V_B (\overline{A_0A_1A_2A_3}) \equiv \frac{1}{2} \det \begin{pmatrix}
2p_1 & p_1 + p_2 - q_2 & p_1 + p_3 - q_2 \\
p_1 + p_2 - q_2 & 2p_2 & p_2 + p_3 - q_1 \\
p_1 + p_3 - q_2 & p_2 + p_3 - q_1 & 2p_3
\end{pmatrix}$$

is Euler’s four-point function [8] p. 191]. This function is essentially the Cayley-Menger determinant (see [2], [5] and [7]) and it satisfies the properties

$$E (q_1, q_2, q_3, p_1, p_2, p_3) = E (p_1, p_2, p_3, q_1, q_2, q_3)$$

and

$$E (q_1, q_2, q_3, p_1, p_2, p_3) = E (q_i, q_j, q_k, p_i, p_j, p_k)$$

for any permutation $i$, $j$ and $k$ of the integers 1, 2 and 3. For the rest of this paper, this quantity is denoted by $\mathcal{Y}$.

For $0 \leq i < j \leq 3$ and indices $k$ and $l$ distinct from $i$ and $j$, we can associate to a pair of triangles $\overline{A_iA_jA_k}$ and $\overline{A_iA_jA_l}$ of a tetrahedron $\overline{A_0A_1A_2A_3}$ the number

$$E_{ij} \equiv \frac{4Q_{ij} \mathcal{Y}}{A_{ijk}A_{ijl}}.$$ 

This quantity is called the $B$-\textbf{dihedral spread} between the triangles $\overline{A_iA_jA_k}$ and $\overline{A_iA_jA_l}$, with the edge $\overline{A_iA_j}$ being common in these two triangles.

### 2.2 Standard tetrahedron

Consider an affine map which sends a general tetrahedron $\overline{A_0A_1A_2A_3}$ to the tetrahedron $\overline{X_0X_1X_2X_3}$, where

$$X_0 \equiv [0, 0, 0], \quad X_1 \equiv [1, 0, 0], \quad X_2 \equiv [0, 1, 0] \quad \text{and} \quad X_3 \equiv [0, 0, 1].$$

Such a tetrahedron will be called the **Standard tetrahedron**. This affine map can be defined by translating the point $A_0$ of the general tetrahedron $\overline{A_0A_1A_2A_3}$ to $X_0$ and then applying a linear map with matrix representation $L$ to send the other three vertices to $X_1$, $X_2$ and $X_3$. If we equip $\mathcal{V}^3$ with a symmetric bilinear form with matrix representation $C$ then this affine mapping induces a new
symmetric bilinear form, defined by
\[ u \cdot_C v = uCv^T = u(\text{LL}^{-1}) \left( \text{LL}^{-1} \right)^T v^T \]
\[ = (uL) \left[ \left( L^{-1} \right) C \left( L^{-1} \right)^T \right] (vL)^T. \]

For \( M \equiv L^{-1} \), we set the matrix \( MCM^T \) to be the matrix \( B \), so that
\[ u \cdot_C v = (uL) \cdot_B (vL). \]

With this tool we may, without any loss of generality, transform a general tetrahedron to the Standard tetrahedron \( X_0X_1X_2X_3 \) so that any result can be proven by proving it for the Standard tetrahedron, so that the result will automatically generalise to an arbitrary tetrahedron by way of performing the inverse affine map.

2.3 Trigonometric quantities of the Standard tetrahedron

In what follows, we define
\[ r_1 \equiv a_2 + a_3 - 2b_1, \quad r_2 \equiv a_1 + a_3 - 2b_2, \quad r_3 \equiv a_1 + a_2 - 2b_3 \]
and
\[ \Delta \equiv \det B = a_1a_2a_3 + 2b_1b_2b_3 - a_1b_1^2 - a_2b_2^2 - a_3b_3^2. \]

We will also define
\[ \text{adj} B = \begin{pmatrix} a_2a_3 - b_1^2 & b_1b_2 - a_3b_3 & b_1b_3 - a_2b_2 \\ b_1b_2 - a_3b_3 & a_1a_3 - b_2^2 & b_2b_3 - a_1b_1 \\ b_1b_3 - a_2b_2 & b_2b_3 - a_1b_1 & a_1a_2 - b_3^2 \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \]
to be the adjoint matrix \([I]\) of \( B \), so that we may define
\[ D \equiv \alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3. \]

The \( B \)-quadrances of \( X_0X_1X_2X_3 \) are
\[ Q_{01} = a_1, \quad Q_{02} = a_2, \quad Q_{03} = a_3, \]
\[ Q_{23} = r_1, \quad Q_{13} = r_2 \quad \text{and} \quad Q_{12} = r_3. \]

The \( B \)-quadreas of \( X_0X_1X_2X_3 \) are
\[ A_{012} = 4\alpha_3, \quad A_{013} = 4\alpha_2, \quad A_{023} = 4\alpha_1 \quad \text{and} \quad A_{123} = 4D \]
and the \( B \)-quadrum of \( X_0X_1X_2X_3 \) is
\[ \mathcal{V} = 4\Delta. \]
The $B$-dihedral spreads of $X_0,X_1,X_2,X_3$ are

$$E_{01} = \frac{a_1 \Delta}{a_2 a_3}, \quad E_{02} = \frac{a_2 \Delta}{a_1 a_3}, \quad E_{03} = \frac{a_3 \Delta}{a_1 a_2},$$

$$E_{23} = \frac{r_1 \Delta}{a_1 D}, \quad E_{13} = \frac{r_2 \Delta}{a_2 D} \quad \text{and} \quad E_{12} = \frac{r_3 \Delta}{a_3 D}.$$

### 2.4 Circumquadrance of tetrahedron

One of the most important centres of a tetrahedron is its **circumcentre**. This point will be dependent on the symmetric bilinear form chosen, which therefore varies the definition of perpendicularity between two vectors; hence, we will call this point the **$B$-circumcentre**.

We start by defining a **plane** through a point $A$ and $B$-perpendicular to a vector $v$ to be the space of points $X$ satisfying the equation

$$v \cdot \overrightarrow{AX} = 0.$$

Then we may define a **$B$-midplane** associated to an edge $\overrightarrow{A_iA_j}$ of a tetrahedron $A_0A_1A_2A_3$ to be a plane through the midpoint $M_{ij}$ which is $B$-perpendicular to the vector $\overrightarrow{A_iA_j}$, for $0 \leq i < j \leq 3$.

There are six $B$-midplanes in total for a particular tetrahedron. The following result highlights the concurrency of each of the six $B$-midplanes of a tetrahedron.

**Theorem 2 (Tetrahedron circumcentre theorem)** The six $B$-midplanes associated to each edge of a tetrahedron $A_0A_1A_2A_3$ meet at a single point.

**Proof.** Without loss of generality, transform $A_0A_1A_2A_3$ to the Standard tetrahedron $X_0X_1X_2X_3$.

Note that this will induce a new non-degenerate symmetric bilinear form, but since we started with a general symmetric bilinear form we may use the same one without any loss of generality. For $0 \leq i < j \leq 3$, let $M_{ij}$ be the midpoint of $\overrightarrow{X_iX_j}$, so that any point $C \equiv [x, y, z]$ on each $B$-midplane of $X_0X_1X_2X_3$ satisfies the equation

$$\overrightarrow{X_iX_j} \cdot B \overrightarrow{M_{ij}Y} = 0.$$

This yields the following six equations:

$$a_1 x + b_3 y + b_2 z = \frac{1}{2} a_1,$$

$$b_3 x + a_2 y + b_1 z = \frac{1}{2} a_2,$$

$$b_2 x + b_1 y + a_3 z = \frac{1}{2} a_3,$$

$$(b_3 - a_1) x + (a_2 - b_3) y + (b_1 - b_2) z = \frac{1}{2} (a_2 - a_1),$$

$$(b_2 - a_1) x + (b_1 - b_3) y + (a_3 - b_2) z = \frac{1}{2} (a_3 - a_1),$$

$$(b_2 - b_3) x + (b_1 - a_2) y + (a_3 - b_1) z = \frac{1}{2} (a_3 - a_2).$$
The meet of the six $B$-midplanes is obtained through the common solution to all six of the above equations. Consider the first three equations above as a system of equations. Then write it in matrix form as
\[
\begin{pmatrix}
    a_1 & b_3 & b_2 \\
    b_3 & a_2 & b_1 \\
    b_2 & b_1 & a_3
\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= B
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}.
\]

As $B$ is invertible, there is a unique solution for $x$, $y$ and $z$ which is written in vector form as
\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
= \frac{1}{2\Delta}
\begin{pmatrix}
    \alpha_1 & \beta_3 & \beta_2 \\
    \beta_3 & \alpha_2 & \beta_1 \\
    \beta_2 & \beta_1 & \alpha_3
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}
= \frac{1}{2}\begin{pmatrix}
    \alpha_1 a_1 + \beta_3 a_2 + \beta_2 a_3 \\
    \beta_3 a_1 + \alpha_2 a_2 + \beta_1 a_3 \\
    \beta_2 a_1 + \beta_1 a_2 + \alpha_3 a_3
\end{pmatrix}.
\]

Thus, the three $B$-midplanes corresponding to the edges $X_0 X_1$, $X_0 X_2$ and $X_0 X_3$ meet at the point
\[
\begin{pmatrix}
    \alpha_1 a_1 + \beta_3 a_2 + \beta_2 a_3 \\
    \beta_3 a_1 + \alpha_2 a_2 + \beta_1 a_3 \\
    \beta_2 a_1 + \beta_1 a_2 + \alpha_3 a_3
\end{pmatrix}
= \frac{1}{2\Delta}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}.
\]

It remains to show this point, which we will call $C$, lies on the remaining three $B$-midplanes. We substitute the point above into the equation
\[
(b_3 - a_1) x + (a_2 - b_3) y + (b_1 - b_2) z = \frac{1}{2} (a_2 - a_1)
\]
to obtain
\[
(b_3 - a_1) (\alpha_1 a_1 + \beta_3 a_2 + \beta_2 a_3) + (a_2 - b_3) (\beta_3 a_1 + \alpha_2 a_2 + \beta_1 a_3) + (b_1 - b_2) (\beta_2 a_1 + \beta_1 a_2 + \alpha_3 a_3)
= \frac{(a_2 - a_1) \Delta}{2\Delta} = \frac{1}{2} (a_1 - a_2).
\]

Similarly,
\[
(b_2 - a_1) (\alpha_1 a_1 + \beta_3 a_2 + \beta_2 a_3) + (b_1 - b_3) (\beta_3 a_1 + \alpha_2 a_2 + \beta_1 a_3) + (a_3 - b_2) (\beta_2 a_1 + \beta_1 a_2 + \alpha_3 a_3)
= \frac{(a_3 - a_1) \Delta}{2\Delta} = \frac{1}{2} (a_3 - a_1)
\]
and
\[
(b_2 - b_3) (\alpha_1 a_1 + \beta_3 a_2 + \beta_2 a_3) + (b_1 - a_2) (\beta_3 a_1 + \alpha_2 a_2 + \beta_1 a_3) + (a_3 - b_1) (\beta_2 a_1 + \beta_1 a_2 + \alpha_3 a_3)
= \frac{(a_3 - a_1) \Delta}{2\Delta} = \frac{1}{2} (a_3 - a_1).
\]

We see then that the six planes are indeed concurrent at a point, as required. ■

The intersection point obtained from the proof of the Tetrahedron circumcentre theorem will be called the $B$-circumcentre of the Standard tetrahedron $X_0 X_1 X_2 X_3$; to obtain the $B$-circumcentre of a general tetrahedron, we merely perform the inverse affine map on such a point.

The $B$-circumcentre of a tetrahedron denotes the center of a sphere passing through each of the points of the tetrahedron; the quadrance between the $B$-circumcentre of a tetrahedron and any point of
this sphere is called the \( \textit{B-circumquadrance} \), and is typically denoted by \( K \). The \( \textit{B-circumquadrance} \) of the Standard tetrahedron \( X_0X_1X_2X_3 \) is

\[
K = Q_B \left( \overrightarrow{X_0C} \right) = \frac{A(a_1b_1, a_2b_2, a_3b_3) + a_1a_2a_3(D - 4(b_1 + b_2 + b_3))}{4\Delta}
\]

The following result, which is an extension of Crelle (1821) \[4\] from the Euclidean setting to a more general setting, links the \( \textit{B-circumquadrance} \) of a tetrahedron with its \( \textit{B-quadrume} \) and \( \textit{B-quadrances} \).

**Theorem 3 (Crelle’s circumquadrance formula)** For a tetrahedron \( \overrightarrow{A_0A_1A_2A_3} \) with \( \textit{B-quadrances} \) \( Q_{ij} \), for \( 0 \leq i < j \leq 3 \), \( \textit{B-quadrume} \) \( V \) and \( \textit{B-circumquadrance} \) \( K \), the relation

\[
4VK = A(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12})
\]

is satisfied.

**Proof.** Without loss of generality, transform \( \overrightarrow{A_0A_1A_2A_3} \) to the Standard tetrahedron \( \overrightarrow{X_0X_1X_2X_3} \); it is sufficient to prove the required result for this tetrahedron. So,

\[
\frac{A(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12})}{4\nu} = \frac{A(a_1r_1, a_2r_2, a_3r_3)}{16\Delta} = \frac{A(a_1b_1, a_2b_2, a_3b_3) + a_1a_2a_3(a_1 + a_2 + a_3 - 2b_1 - 2b_2 - 2b_3)}{4\Delta} = K.
\]

The required result follows. \( \blacksquare \)

This result allows us to conveniently write the \( \textit{B-circumquadrance} \) of \( \overrightarrow{X_0X_1X_2X_3} \) as

\[
K = \frac{A(a_1r_1, a_2r_2, a_3r_3)}{16\Delta}.
\]

**3 Main result**

We now present the main result of this paper, which generalises \[3\] for the rational trigonometric setting over an arbitrary symmetric bilinear form.

**Theorem 4 (Circumquadrance dihedral spread theorem)** For a tetrahedron \( \overrightarrow{A_0A_1A_2A_3} \) in \( \mathbb{A}^3 \), let \( \nu \) be its \( \textit{B-quadrume} \), \( A_{ijk} \) be the \( \textit{B-quadaea} \) of the triangle \( \overrightarrow{A_iA_jA_k} \), \( E_{ij} \) be the \( \textit{B-dihedral spread} \) between \( \overrightarrow{A_iA_jA_k} \) and \( \overrightarrow{A_iA_jA_l} \), and \( K \) be its \( \textit{B-circumquadrance} \). Then,

\[
(A_{012}A_{013}A_{023}A_{123})^2 M = 1024\nu^5K,
\]
where

\[ M = -\det \begin{pmatrix} 0 & E_{01} & E_{02} & E_{03} \\ E_{01} & 0 & E_{12} & E_{13} \\ E_{02} & E_{12} & 0 & E_{23} \\ E_{03} & E_{13} & E_{23} & 0 \end{pmatrix} = A(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}). \]

**Proof.** Perform an affine map on \( A_0A_1A_2A_3 \) to the Standard tetrahedron \( X_0X_1X_2X_3 \), so that we only require to prove this result is true on \( X_0X_1X_2X_3 \). We have

\[ E_{01}E_{23} = \frac{a_1r_1\Delta^2}{\alpha_1\alpha_2\alpha_3 D}, \quad E_{02}E_{13} = \frac{a_2r_2\Delta^2}{\alpha_1\alpha_2\alpha_3 D}, \quad E_{03}E_{12} = \frac{a_3r_3\Delta^2}{\alpha_1\alpha_2\alpha_3 D} \]

and

\[ A_{012}A_{013}A_{023}A_{123} = 256\alpha_1\alpha_2\alpha_3 D \]

so that

\[ M = A(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}) = (\Delta^2(a_1r_1 + a_2r_2 + a_3r_3))^2 - 2\left(\frac{\Delta^4(a_1^2r_1^2 + a_2^2r_2^2 + a_3^2r_3^2)}{(\alpha_1\alpha_2\alpha_3 D)^2}\right) \]

\[ = \frac{\Delta^4(a_1r_1 + a_2r_2 + a_3r_3)^2}{(\alpha_1\alpha_2\alpha_3 D)^2} - 2\Delta^4(a_1^2r_1^2 + a_2^2r_2^2 + a_3^2r_3^2) \]

\[ = \frac{\Delta^4(A(a_1r_1, a_2r_2, a_3r_3))}{(\alpha_1\alpha_2\alpha_3 D)^2} \]

and thus

\[ (A_{012}A_{013}A_{023}A_{123})^2 M = \frac{\Delta^4(A(a_1r_1, a_2r_2, a_3r_3))}{(\alpha_1\alpha_2\alpha_3 D)^2} (256\alpha_1\alpha_2\alpha_3 D)^2 \]

\[ = 65536\Delta^4 A(a_1r_1, a_2r_2, a_3r_3). \]

By Crelle’s circumquadrance formula,

\[ (A_{012}A_{013}A_{023}A_{123})^2 M = 1024K (1024\Delta^5) \]

\[ = 1024 (4\Delta^5) K \]

\[ = 1024V^5 K \]

as required. ■

We can now provide an alternative expression for the Circumquadrance dihedral spread theorem.

**Corollary 5** If \( N \equiv A(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}) \), where, for \( 0 \leq i < j \leq 3 \), \( Q_{ij} \) denotes the B-quadrances of a tetrahedron \( A_0A_1A_2A_3 \), then the Circumquadrance dihedral spread theorem can alternatively be expressed as

\[ (A_{012}A_{013}A_{023}A_{123})^2 M = 256V^4N. \]
Proof. From Crelle’s circumquadrance formula, we have that

\[ K = \frac{N}{4V} \]

Substitute into the Circumquadrance dihedral spread theorem to obtain

\[
\left( A_{012}A_{013}A_{023}A_{123} \right)^2 M = 1024V^5 \left( \frac{N}{4V} \right) = 256V^4 N,
\]

as required. \( \blacksquare \)

We can now derive a relationship between \( M \) and \( N \).

**Theorem 6 (Dihedral spread ratio theorem)** Given a tetrahedron \( \overline{A_0A_1A_2A_3} \) in \( \mathbb{R}^3 \) with \( B \)-dihedral spreads \( E_{ij} \) and \( B \)-quadrances \( Q_{ij} \), where \( 0 \leq i < j \leq 3 \), the relation

\[
\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}}
\]

is satisfied.

**Proof.** We start by showing that

\[ A(\lambda a, \lambda b, \lambda c) = \lambda^2 A(a, b, c). \]

Let \( V \) be the \( B \)-quadrume of \( \overline{A_0A_1A_2A_3} \) and let \( A_{ijk} \) be their \( B \)-quadreas, for \( 0 \leq i < j < k \leq 3 \).

Then rearrange the equation of Corollary 5 to get

\[
M = \frac{256V^4}{\left( A_{012}A_{013}A_{023}A_{123} \right)^2} N = \left( \frac{16V^2}{A_{012}A_{013}A_{023}A_{123}} \right)^2 N
\]

for

\[ M \equiv A(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}) \quad \text{and} \quad N \equiv A(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}). \]

So,

\[ M = A \left( \frac{16V^2Q_{01}Q_{23}}{A_{012}A_{013}A_{023}A_{123}}, \frac{16V^2Q_{02}Q_{13}}{A_{012}A_{013}A_{023}A_{123}}, \frac{16V^2Q_{03}Q_{12}}{A_{012}A_{013}A_{023}A_{123}} \right). \]

and thus by comparison

\[
\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{16V^2}{A_{012}A_{013}A_{023}A_{123}}
\]

as required. \( \blacksquare \)

From the proof of the Dihedral spread ratio theorem, we can define

\[ R \equiv \frac{16V^2}{A_{012}A_{013}A_{023}A_{123}}. \]

This quantity features prominently in Richardson (1902) [6]. From the Circumquadrance dihedral
spread theorem, we use

\[
K = \frac{(A_{012}A_{013}A_{023}A_{123})^2 M}{1024V^5}
\]

\[
= \frac{M}{4V} \left( \frac{A_{012}A_{013}A_{023}A_{123}}{16V^2} \right)^2
\]

\[
= \frac{M}{4VR^2}.
\]

Crelle’s circumquadrance formula is immediate from this, as \( M = R^2N \) from the proof of the Dihedral spread ratio theorem.

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