Landau-Ginzburg Description of D-branes on ALE Spaces

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Abstract

We study the Landau-Ginzburg (LG) mirror theory of the non-linear sigma model on the ALE space \(\mathcal{M}\) obtained by resolving the singularity of the orbifold \(\mathbb{C}^2/\mathbb{Z}_N\). In the LG description, the data of the BPS spectrum and the lines of marginal stability are encoded in the special Lagrangian submanifolds of the mirror manifold \(\mathcal{\hat{M}}\). Our LG description is quite simple as compared with the quiver gauge theory analysis near the orbifold point. Furthermore our mirror analysis can be applied to any point on the moduli space of \(\mathcal{M}\).
1 Introduction

In the Calabi-Yau (CY) compactifications of string theory, a deep understanding of D-branes wrapped around the compact submanifolds is required to determine the physical BPS spectrum. When one moves around the compactification moduli space, the nature of wrapped D-branes is expected to change dramatically. On the moduli space, there are two distinguished points which have been physically well understood: large volume limit and orbifold point. One may have a natural question how one should interpolate these two points to characterize BPS D-branes at arbitrary points on the moduli space. Recently, many analyses concerning this problem have been performed in [1, 2, 3, 4, 5, 6, 7].

In these analyses, mirror symmetry has played a central role. One of the recent developments in this direction is found in [8, 9], where the non-linear sigma model on a toric variety is shown to have an equivalent description given by a Landau-Ginzburg (LG) model with a suitable superpotential. Furthermore, it has been realized there that D-branes wrapping holomorphic cycles in the toric variety are mirror to D-branes wrapped on certain Lagrangian submanifolds on the LG side. The works [10] treat a related subject.

In this paper, we wish to utilize this mirror technique to study the BPS spectrum coming from D-branes on the ALE space \( \mathcal{M} \) given by blowing up the singularity of the orbifold \( \mathbb{C}^2/\mathbb{Z}_N \). In [11], D-branes on \( \mathcal{M} \) near the orbifold point have been investigated in terms of the quiver gauge theory [12]. Moreover the BPS spectrum and the lines of marginal stability were determined. In particular, it has been recognized that all the lines do not lead to the decay of BPS states. However, the analysis has been restricted to the vicinity of the orbifold point in the whole moduli space spanned by complex blowing up parameters. The quiver gauge theory inevitably fixes the value of \( B \) field in the non-linear sigma model on \( \mathcal{M} \).

A distinguished feature of using the LG mirror description is that we can examine the BPS spectrum over the whole moduli space spanned by the blowing up parameters. We map the data of the BPS spectrum into the mirror manifold \( \hat{\mathcal{M}} \). Then, we give the mass formula and the lines of marginal stability in terms of the geometry of the special Lagrangian submanifolds in \( \hat{\mathcal{M}} \).

This paper is organized as follows. In section 2, we formulate the non-linear sigma model on the ALE space \( \mathcal{M} \) using a certain gauged linear sigma model. In section 3, we will explain how BPS spectrum near the orbifold point are studied by the representation of a quiver diagram, following [11]. In section 4, the LG mirror description of the sigma model on \( \mathcal{M} \) will be presented, following [8, 9]. We study BPS spectrum in terms of the special Lagrangian submanifolds in \( \hat{\mathcal{M}} \). Then, we perform explicit analyses for the simplest cases, \( \mathbb{C}^2/\mathbb{Z}_2 \) and \( \mathbb{C}^2/\mathbb{Z}_3 \). The last section is devoted to conclusions and discussion.
2 Non-linear sigma model

The non-linear sigma model on the ALE space $M$, which is given by blowing up the orbifold $\mathbb{C}^2/\mathbb{Z}_N$, is described by the low energy effective theory of a certain $\mathcal{N} = 2$ gauged linear sigma model \[13\] (We use the same notation for $\mathcal{N} = 2$ theories in two-dimensions as \[8\]). The gauged linear sigma model contains $N + 1$ chiral superfields $\Phi_i$ ($i = 1, \ldots, N + 1$) and $N - 1$ vector superfields $V_a$ ($a = 1, \ldots, N - 1$) with field strength $\Sigma_a$, which contribute to the gauge group $U(1)^{N-1} = \prod_{a=1}^{N-1} U(1)_a$. The theory is parameterized by Fayet-Iliopoulos (FI) parameter $r_a$, theta angle $\theta_a$ and gauge coupling $e_a$. We can combine $r_a$ and $\theta_a$ into a complex parameter $t_a = r_a - i\theta_a$. The Lagrangian is given by

$$L = \int d^4\theta \left( \sum_{i=1}^{N+1} \overline{\Phi}_i e^2 \sum_{a=1}^{N-1} Q_{ai} V_a - \sum_{a=1}^{N-1} \frac{1}{2e_a^2} \Sigma_a \right) - \frac{1}{2} \left( \int d^2\tilde{\theta} \sum_{a=1}^{N-1} t_a \Sigma_a + \text{c.c.} \right), \quad (2.1)$$

where the $U(1)_a$ charges $Q_{ai}$ carried by $\Phi_i$ are as follows:

$$Q_{ai} = \begin{cases} -2, & i = a + 1, \\ 1, & i = a, a + 2, \\ 0, & \text{otherwise}. \end{cases} \quad (2.2)$$

These charges are identified with the charge vectors in the toric data for $M$.

After integrating out the auxiliary fields, we have the following scalar potential energy derived from the Lagrangian (2.1)

$$V = \sum_{a=1}^{N-1} \left\{ \left( \frac{e_a^2}{2} \sum_{i=1}^{N+1} Q_{ai} |\phi_i|^2 - r_a \right)^2 + \sum_{i=1}^{N-1} Q_{ai}^2 |\sigma_a|^2 |\phi_i|^2 \right\}, \quad (2.3)$$

where $\phi_i$ and $\sigma_a$ are the scalar components of $\Phi_i$ and $\Sigma_a$, respectively. Let us describe the classical vacuum moduli space which is the space of zeros of $V$ modulo gauge transformations. In the case of $r_a \neq 0$, a solution is given by

$$\sum_{i=1}^{N+1} Q_{ai} |\phi_i|^2 = r_a, \quad a = 1, \ldots, N - 1, \quad (2.4)$$

with the gauge identification $\exp(\sum_{a=1}^{N-1} Q_{ai} \gamma_a) \phi_i \sim \phi_i$ ($\gamma_a \in \mathbb{R}$) and $\sigma_a = 0$. In this region, if we take the limit $e_a^2 \rightarrow \infty$ or the long distance limit, the system reduces to the non-linear sigma model whose target space is the configuration space of classical vacua (2.4) modulo gauge transformation. This target space is nothing but $M$. Quantum mechanically, we would have singularities of the quantum theory which emerge at some specific values of $t_a$. However, two generic smooth points in the moduli space can be smoothly connected
without passing the singularities. Thus, \( M \) can be realized as a vacuum moduli space of the linear sigma model.

The region \( r_a > 0 \) corresponds to the blow-up phase, where we have the exceptional divisors \( \mathbb{P}^1 \) described by \( \phi_a, \phi_{a+2} \) which cannot vanish simultaneously. On the other hand, in the region \( r_a < 0 \), \( \phi_{a+1} \) cannot vanish and the exceptional divisors are blown down. The \( N-1 \) exceptional divisors \( \alpha_a (a = 1, \ldots, N-1) \) appearing from the singularity of \( \mathbb{C}^2/\mathbb{Z}_N \) are given by the loci

\[
\alpha_a : \{ \phi_{a+1} = 0 \}, \quad a = 1, \ldots, N-1.
\]  

(2.5)

Roughly speaking, the FI parameters \( r_a (a = 1, \ldots, N-1) \) control the volume of the 2-cycles \( \alpha_a \). On the other hand, \( \theta_a \) reduce to \( B \) fields in the sigma model. The reader should note that we can rely on the sigma model description only in the blow-up phase.

In the large volume limit (\( r_a \to +\infty \)), the BPS objects in the bulk theory on \( M \) would be provided by D0-branes on \( M \), the bound states of them, (anti-) D2-branes wrapping the 2-cycles \( \alpha_a \) and their bound states with D0-branes. In the next section, we will take the opposite limit (\( r_a \to -\infty \)), i.e. orbifold point where all \( N-1 \) \( \mathbb{P}^1 \) cycles in \( M \) shrink and the smooth ALE space \( M \) becomes the singular orbifold \( \mathbb{C}^2/\mathbb{Z}_N \). We will see how the BPS states in the large volume and orbifold pictures are related to each other. This will be carried out by making use of quiver gauge theory [12].

### 3 Quiver gauge theory

Consider the gauge theory on the worldvolume of a single D0-brane probe placed at the orbifold singularity of \( \mathbb{C}^2/\mathbb{Z}_N \) [12]. The quiver gauge theory under consideration is a projection of the ordinary gauge theory on \( N \) D0-branes in the covering space \( \mathbb{C}^2 \). The resulting field theory is summarized by a quiver diagram. Its gauge group is the product \( \prod_{i=1}^N U(n_i) \). The matter content of the quiver gauge theory is \( a_{ij} \) bi-fundamental hypermultiplets in the representation \( \oplus_{i,j=1}^N a_{ij}(\bar{n}_i, n_j) \). Here, this \( a_{ij} \) is determined by \( C_{ij} = -2\delta_{ij} + a_{ij} \), where \( C_{ij} \) is the Cartan matrix of the affine Lie algebra \( \tilde{A}_{N-1} \). Also the theory has \( N \) Fayet-Iliopoulos (FI) parameters \( \zeta_i (i = 1, \ldots, N) \). Then, the first \( N-1 \) parameters correspond in the large volume limit to the blowing up parameters \( r_a (a = 1, \ldots, N-1) \) which appeared in the previous section. We can remove the \( N \)-th FI parameter \( \zeta_N \) by the condition \( \sum_{i=1}^N \zeta_i = 0 \).

In order to specify BPS states, we have to solve the F- and D-flatness conditions of the gauge theory. In [11], it was realized that a representation theory of a quiver diagram enables us to give an efficient way to analyze BPS spectrum. We will explain how this quiver gauge theory encodes the data about the BPS spectrum near the orbifold point.
3.1 Representation of a quiver diagram

Let us begin with a configuration of the quiver gauge theory as a representation of the quiver diagram. The set of non-negative integers \( n = (n_1, \ldots, n_N) \), which we call dimension vector, specifies the gauge group. The representation space with the fixed dimension vector \( n \) corresponds to the configuration space of the quiver gauge theory. We can specify a subrepresentation of a generic representation with dimension vector \( n \). The dimension vector of this subrepresentation, \( n' = (n'_1, n'_2, \ldots, n'_N) \), is called a subvector of \( n \).

The dimension vectors are in one-to-one correspondence with the lattice spanned by the simple roots of the affine Lie algebra \( \tilde{A}_{N-1} \). The dimension vector specifies the site in the positive root lattice \( \Gamma_+ \) of \( \tilde{A}_{N-1} \)

\[
\Gamma_+ = \left\{ \sum_{i=1}^{N} n_i \alpha_i \right\},
\]  

(3.1)

where \( \alpha_i \) (\( i = 1, \ldots, N \)) are the simple roots of \( \tilde{A}_{N-1} \). The first \( N - 1 \) simple roots \( \alpha_i \) (\( i = 1, \ldots, N - 1 \)) correspond to the \( N - 1 \) exceptional divisors in \( \mathcal{M} \).

Suppose we have a certain BPS state with dimension vector \( n \). The central charge \( Z \) of this state reads

\[
Z = \sum_{i=1}^{N} \zeta_i n_i + i \sum_{i=1}^{N} \frac{n_i}{N}. \tag{3.2}
\]

The mass \( M \) of BPS states is associated with the central charge \( Z \) by \( M = |Z| / g_s \), where \( g_s \) is string coupling constant. BPS states in \( \mathcal{M} \) would occupy only a subset of the lattice \( \Gamma_+ \). Thus, we have to determine which sites in the lattice \( \Gamma_+ \) are occupied by the BPS states and the degeneracy of each states. This is carried out in the following subsection.

3.2 BPS spectrum

Let us introduce the real \( N \)-dimensional vector \( \theta = (\theta_1, \theta_2, \ldots, \theta_N) \). The representation with dimension vector \( n \) is \( \theta \) stable if, for any subrepresentation with subvector \( n' \) and any \( \theta \) vector satisfying

\[
\sum_{i=1}^{N} n_i \theta_i = 0, \tag{3.3}
\]

\[
\sum_{i=1}^{N} n'_i \theta_i > 0.
\]

The importance of this \( \theta \) stability lies within the following mathematical fact: \( \theta \) stable representations are in one-to-one correspondence with the solutions to the F- and D-flatness conditions. Thus, \( \theta \) stable representations correspond to BPS states.
The vector $\theta$ can be mapped into FI parameters in the quiver gauge theory. The relation reads
\[
\theta_i = \zeta_i - \frac{\sum_{i=1}^{N} n_i \zeta_i}{\sum_{i=1}^{N} n_i}.
\] (3.4)

If we shift the value of FI parameters, the volume of exceptional divisors changes. At a certain value, we would encounter BPS states at threshold. Let us sketch this situation as follows. We denote a collection of BPS states by $S^i$ and $S_1^f, \ldots, S_m^f$, whose central charges are $Z^i$ and $Z_1^f, \ldots, Z_m^f$. If the condition
\[
|Z^i| = |Z_1^f| + |Z_2^f| + \ldots + |Z_m^f|,
\]
\[
\text{arg}Z^i = \text{arg}Z_1^f + \text{arg}Z_2^f + \ldots + \text{arg}Z_m^f,
\] (3.5)
is satisfied in the space of FI parameters, the decay of an initial BPS state $S^i$ into final $m$ BPS states $S_1^f, S_2^f, \ldots, S_m^f$ is allowed. However, if the condition (3.5) is satisfied, we should see $S^i$ and $S_1^f, \ldots, S_m^f$ to be an identical state.

With this kept in mind, we can proceed by introducing $\theta$ semi-stability. The representation with dimension vector $n$ is $\theta$ semi-stable if, for any subrepresentation with subvector $n'$ and any real $\theta$ vector satisfying $\sum_{i=1}^{N} n_i \theta_i = 0$,
\[
\sum_{i=1}^{N} n_i' \theta_i \geq 0.
\] (3.6)

In general, a $\theta$ semi-stable representation cannot correspond to any solution to the F- and D-flatness conditions. However, there is a distinguished $\theta$ semi-stable representation which corresponds to a solution to the F- and D-flatness conditions. Actually, this representation is a direct sum of $\theta$ stable representations, which was called a graded representation in [11]. This representation is none other than what we want. Furthermore, the notion of $S$-equivalence enables us to identify $S^i$ and $S_1^f, \ldots, S_m^f$ if the condition (3.3) is satisfied. We should note that $S$-equivalence excludes all the $\theta$ semi-stable representations which do not correspond to solutions to the F- and D-flatness conditions.

Now, we define the moduli space of BPS states as the space of $\theta$ semi-stable representations with $S$-equivalence. We should assure that this moduli space have a non-negative dimension. Then, the correspondence between the subset of $\Gamma_+$ and the BPS spectrum in the large volume limit is determined as follows [11].

(I) The positive root of $A_{N-1}$ given by $\alpha_+$ corresponds to D2-brane wrapping the 2-cycle $\alpha_+$ in $\mathcal{M}$.

(II) The null roots $n\delta$ ($n \geq 1$) correspond to the bound states of $n$ D0-branes on $\mathcal{M}$. 5
(III) The roots of the form $\alpha_+ + n\delta$ ($n \geq 1$) correspond to the $n$-th KK excitation of D2-brane wrapping the 2-cycle $\alpha_+$ in $\mathcal{M}$.

(IV) The roots of the form $-\alpha_+ + n\delta$ ($n \geq 1$) correspond to the $(n-1)$-th KK excitation of anti-D2-brane wrapping the 2-cycle $\alpha_+$ in $\mathcal{M}$.

We have introduced above the notations $\alpha_+$ and $\delta$:

$$\alpha_+ \in \{ \alpha_{ij} \equiv \alpha_i + \alpha_{i+1} + \cdots + \alpha_j ; 1 \leq i \leq j \leq N-1 \},$$

$$\delta = \sum_{i=1}^{N} \alpha_i,$$

which stand for a positive root of the Lie algebra $A_{N-1}$ and the null root of $\hat{A}_{N-1}$, respectively. Then, the positive root $\alpha_+ = \alpha_i + \cdots + \alpha_j$ can be interpreted as the homological sum of the 2-cycles $\alpha_i, \ldots, \alpha_j$ in (2.5). We use the same symbol $\alpha_+$ to specify the corresponding 2-cycle in $\mathcal{M}$. The dimension vector corresponding to the simple root $\alpha_i$ has entry one in the $i$-th component with all other entries being zero, and one associated with the null vector is $(1, 1, \ldots, 1)$.

### 3.3 Marginal stability

We give now general remarks on the moduli space of BPS states. If the condition (3.5) is satisfied, the decay of the initial BPS state $S_i$ into other final $m$ BPS states $S_1, S_2, \ldots, S_m$ is allowed. In terms of the representation of a quiver diagram, $\theta$ stable representation becomes $\theta$ semi-stable representation, which is $S$-equivalent to a direct sum of $\theta$ stable subrepresentations. In general, the condition (3.5) is satisfied in a certain submanifold of the moduli space. This submanifold is usually called a “line of marginal stability”. However, the reader should not confuse the term “line” with a line in the moduli space. Actually, the locus of the marginal stability is a codimension one submanifold in the moduli space. On the other hand, due to the celebrated McKay correspondence, we have a one-to-one correspondence between BPS states in $\mathcal{M}$ and root vectors of $\hat{A}_{N-1}$ throughout the moduli space. This fact means that we cannot miss any BPS states in wandering the moduli space. Thus, we always have the lines of marginal stability, which do not lead to the decay of BPS states.

Instead of making a general analysis on lines of marginal stability, we show an example only for the simplest case, $\mathbb{C}^2/\mathbb{Z}_2$ orbifold. Consider the $\theta$ stable representation with $n = (1, 1)$. This corresponds to a D0-brane in $\mathcal{M}$. We choose $(\theta_1, \theta_2)$ to satisfy the relation $\sum_{i=1}^{2} n_i \theta_i = \theta_1 + \theta_2 = 0$. At $\theta_1 = 0$, $\theta$ stable representation $n$ becomes $\theta$ semi-stable, and $S$-equivalent to the direct sum of two subrepresentation with $n' = (1, 0)$ and
The corresponding marginal stability line $\theta_1 = 0$ is rewritten in terms of the FI parameter as $\zeta_1 = 0$ using (3.4). In other words, we have the following BPS state at threshold on the marginal stability line

$$D0 \leftrightarrow D2 + \overline{D2}.$$ \hspace{1cm} (3.8)

Note that, on both sides of marginal stability line, D0-brane is $\theta$ stable. Thus, D0-brane cannot decay anywhere in the moduli space. In this way, the line of marginal stability can be determined by examining the $\theta$ stability, not by solving the condition (3.3).

One might wonder that the analysis in this section cannot be applied to the large volume region in the moduli space of $\mathcal{M}$. Our motivation for using mirror symmetry [8, 9] is to remedy this deficiency. In the next section, the LG mirror of the sigma model on $\mathcal{M}$ will be studied. We will often use there the terms “D2-brane” or “D0-brane” in the sigma model sense to specify BPS states, although they are meaningful notations only in the large volume limit of the sigma model. Furthermore, we remind the reader not to confuse D2-branes in the sigma model with D2-branes in the LG theory which will appear in the next section.

4 Landau-Ginzburg theory

The non-linear sigma model in the previous section has a mirror counterpart which is described in terms of a LG theory [8, 9]. The field contents of the mirror LG theory consist of $N + 1$ twisted chiral superfields $Y_i$ ($i = 1, \ldots, N+1$) and the $N-1$ twisted chiral superfields $\Sigma_a$ ($a = 1, \ldots, N-1$) which have already appeared in the previous section as the superfield strengths for $V_a$. The twisted superpotential reads [8]

$$\tilde{W} = \sum_{a=1}^{N-1} \Sigma_a \left( \sum_{i=1}^{N+1} Q_{ai} Y_i - t_a \right) + \mu \sum_{i=1}^{N+1} e^{-Y_i},$$ \hspace{1cm} (4.1)

where $\mu$ is a cut-off parameter.

Our aim in this section is to study the BPS states and their stability in the bulk theory. To this end, it suffices to compute the period integral of the form [12, 8]

$$\Pi = \int \prod_a d\Sigma_a \prod_i dY_i \exp(-\tilde{W})$$ \hspace{1cm} (4.2)

which, when integrated over appropriate regions in the field configuration space, yields the BPS masses. The expression of the period (4.2) can be simplified following the manipulation used in [9]. Integrating out $\Sigma_a$ leads to the constraints among the superfields
Let us apply the constraints (4.3) to the twisted superpotential (4.1). If we define two twisted chiral superfields by

\[ v = \mu \exp \left[ -Y_2 - \frac{1}{N} \sum_{i=1}^{N-1} (N-i) t_i \right] \]
\[ w = \exp \left[ Y_1 - Y_2 - \frac{1}{N} \sum_{i=1}^{N-1} (N-i) t_i \right] \]

which take their values in \( \mathbb{C}^* \), then all other superfields \( Y_i \) \( (i = 3, \ldots, N+1) \) can be expressed by \( v \) and \( w \) with the help of the charges (2.2). The resulting period is

\[ \Pi = \int \frac{dv dw}{vw} \exp(-\tilde{W}), \quad \tilde{W} = v \left( w^{-1} + \sum_{i=1}^{N-1} c_i w^{i-1} + w^{N-1} \right), \]

where the complex parameters \( c_i \) \( (i = 1, \ldots, N-1) \) are given by

\[ c_i = \exp \left[ \left( 1 - \frac{i}{N} \right) \sum_{j=1}^{i-1} j t_j + \frac{i}{N} \sum_{j=i}^{N-1} (N-j) t_j \right]. \]

In (4.3) and hereafter, we will omit unimportant numerical factors in front of the period. The variable \( v \) cannot be simply integrated out since it takes values in \( \mathbb{C}^* \). In other words, the integral measure in (4.3) contains the factor \( dv/v \). To appropriately eliminate \( v \), we introduce two additional twisted chiral superfields \( x \) and \( y \) which take values in \( \mathbb{C} \). Then, it is easy to verify that the period (4.3) is equivalent to

\[ \Pi = \int \frac{dv dw dx dy}{w} \exp(-\tilde{W}), \quad \tilde{W} = v \left( w^{-1} + \sum_{i=1}^{N-1} c_i w^{i-1} + w^{N-1} + xy \right), \]

where \( v \) can be viewed as a \( \mathbb{C} \) variable. In fact, integrating \( x \) and \( y \) out reproduces (4.3) and especially the measure \( dv/v \), which ensures that \( v \) was originally a \( \mathbb{C}^* \) variable.

The variable \( v \) can now be safely integrated out, leading to the following expression of the period,

\[ \Pi = \int \frac{dw dx dy}{w} \delta \left( w^{-1} + \sum_{i=1}^{N-1} c_i w^{i-1} + w^{N-1} + xy \right). \]

This result implies that we can analyze the BPS spectrum by identifying the LG theory with the sigma model on the mirror CY \( \tilde{\mathcal{M}} \) given by

\[ \tilde{\mathcal{M}} : \begin{cases} 
 f(w, z) \equiv 1 + zw + \sum_{i=2}^{N} c_i w^i + w^N = 0, \\
 xy = z - c_1, 
\end{cases} \]
where we have introduced an extra $C$ variable $z$. The final expression of the period is obtained by integrating out $y$ in (4.8) as

$$
\Pi = \int \Omega, \quad \Omega = \frac{dwdx}{wx}.
$$

(4.10)

The 2-form $\Omega$ defined above is none other than the holomorphic 2-form on $\hat{M}$.

The geometry of $\hat{M}$ takes the form of the product of two fibrations over the base $B \simeq C$ parameterized by $z$. One of the two fibers is the set of $N$ points $F_1 = \{w_1(z), \ldots, w_N(z)\}$ in the $w$ plane, which solve the equation $f(w, z) = 0$. Let us define the discriminant of the polynomial $f(w, z)$ by $\Delta(z) \equiv \prod_{i=1}^{N}(z - z_i)$. Then, some two points in $F_1$ coalesce when $z$ meets one of the $N$ points $P = \{z_1, \ldots, z_N\}$ on $B$. The other fiber is the algebraic torus $F_2 \simeq C^*$ given by $xy = z - c_1$, which degenerates at $\{z = c_1\}$.

4.1 Special Lagrangian submanifolds

We have to look for all the special Lagrangian submanifolds in $\hat{M}$, which contribute to the BPS states in the bulk theory. Before identifying the special Lagrangian submanifolds, let us consider for an exercise how middle dimensional compact submanifolds, which are not necessarily special Lagrangian, are embedded in $\hat{M}$. A simple example would be 2-sphere, as we will explain now. First consider a line interval $L$ on $B$ connecting $\{z = c_1\}$ and one of the $N$ points in $P$, say $\{z = z_1\}$. Then, a 2-sphere $S$ can be constructed as a submanifold fibered over $L$. The real one dimensional fiber $f$ of $S$ lies in the fiber of $\hat{M}$, $F_1 \times F_2$. The projection of $f$ on $F_1$ consists of the two points in $F_1$, which meet each other at the endpoint $\{z = z_1\}$ of $L$. On the other hand, the projection of $f$ on $F_2 \simeq C^*$, which can be interpreted as an infinitely elongated cylinder, is the compact circle in the cylinder which shrinks at $\{z = c_1\}$. It is evident that $S$ obtained this way forms a 2-sphere in $\hat{M}$. It is shown in Figure 1 how the 2-sphere $S$ is embedded in $\hat{M}$.

Let us turn to the problem of finding the special Lagrangian submanifolds in $\hat{M}$. For a submanifold $C$ to be special Lagrangian, the following conditions must be satisfied [15, 16, 17]:

$$\omega|_C = 0,$$  \hspace{1cm} (4.11)

$$\text{Im}(e^{-i\theta}\Omega)|_C = 0.$$  \hspace{1cm} (4.12)

Here, $\omega$ is the Kähler form on $\hat{M}$,

$$\omega = \frac{i}{2}(dwd\bar{w} + dxd\bar{x} + dyd\bar{y}),$$  \hspace{1cm} (4.13)
Figure 1: Non-compact mirror Calabi-Yau $\hat{M}$ and a 2-sphere $S$ embedded in it.

where $f(w, xy + c_1) = 0$ is implicitly assumed to eliminate $y$. The phase $\theta \in [0, 2\pi)$ should be constant over all points on $C$. We often call $C$ a special Lagrangian submanifold with phase $\theta$.

The form of $\Omega = d(\ln w)d(\ln x)$ and the condition (4.12) suggest that a compact special Lagrangian submanifold projected on both the $(\ln w)$- and $(\ln x)$-plane must be a straight line. To specify how the straight line is lying on the $(\ln w)$-plane, it is convenient to introduce an infinite number of points,

$$u_i(n) = \ln w_i(c_1) + 2\pi in, \quad i = 1, \ldots, N, \quad n \in \mathbb{Z}, \quad (4.14)$$

where $0 \leq \text{Im} \ln w_i(c_1) < 2\pi$ is assumed. There are two possibilities for the form of the straight line on the $(\ln w)$-plane. If the straight line is the line interval connecting $u_i(n)$ and $u_j(m)$ with $i \neq j$, the associated special Lagrangian submanifold has the topology of 2-sphere. In fact, we can parameterize the submanifold by parameters $s, t \in [0, 1]$ as

$$\ln w = u_i(n)(1 - t) + u_j(m)t,$$

$$\ln x = \frac{1}{2} \ln \left| \frac{f(w, c_1)}{w} \right| + 2\pi is, \quad (4.15)$$

1 In [4], it has been argued that this phase should be extended to any real number $\mathbb{R}$. This leads us to introduce the notion “graded Lagrangian submanifold” as in [18].
where we have obtained the second line by examining the condition (4.11). Keep in mind that \( w \) in the second line must be replaced by a function of \( t \) using the first line. The variables \( t \) and \( s \) parameterize a line interval and a circle which is fibered over the interval and shrinks at the endpoints of it, respectively. The special Lagrangian submanifolds of this type belong to the same homology class as that of the 2-sphere \( S^2 \) mentioned above.

On the other hand, we can imagine that the straight line on the \((\ln w)\)-plane starts from an arbitrary real number \( a \in \mathbb{R} \) and ends on \( a + 2\pi i \). Similarly to the above case, the form of the straight line on the \((\ln x)\)-plane can be determined from (4.11). The result is

\[
\begin{align*}
\ln w &= a(1 - t) + (a + 2\pi i)t, \\
\ln x &= \frac{1}{2} \ln \left[ b + \sqrt{b^2 + \left| \frac{f(w, c_1)}{w} \right|^2} \right] + 2\pi is,
\end{align*}
\tag{4.16}
\]

where \( b \) is an arbitrary real constant. Again we have to use the first line to eliminate \( w \) in the second line. In this case, the associated special Lagrangian submanifold is a torus. Notice that this special Lagrangian torus has two real dimensional moduli \( a \) and \( b \), while special Lagrangian 2-spheres have no moduli. This reflects the fact that, in general, a compact special Lagrangian submanifold \( C \) in a CY manifold has a deformation moduli space of real dimension \( b^1(C) \), the first Betti number \([16]\).

We can choose \( N \) special Lagrangian 2-spheres \( C_i \) \((i = 1, \ldots, N)\), so that their homology classes \([C_i]\) form a basis of the group of the compact 2-cycles in \( \hat{\mathcal{M}} \), \( H_2(\hat{\mathcal{M}}, \mathbb{Z}) \). The straight line intervals, which are the projection of \( C_i \) on the \((\ln w)\)-plane, are given by

\[
\begin{align*}
C_i &: \quad \ln w = u_i(0)(1 - t) + u_{i+1}(0)t, \quad i = 1, \ldots, N - 1, \\
C_N &: \quad \ln w = u_N(0)(1 - t) + u_1(1)t.
\end{align*}
\tag{4.17}
\]

We show in Figure 2 the intervals associated with \( C_i \). The adjacent two submanifolds \( C_i \) and \( C_{i+1} \) intersect only at a single point \( \{w = w_{i+1}(c_1), z = c_1, x = y = 0\} \) for \( i = 1, \ldots, N - 1 \), and the unique intersection point between \( C_1 \) and \( C_N \) is given by \( \{w = w_1(c_1), z = c_1, x = y = 0\} \). Then we can give the homology group \( H_2(\hat{\mathcal{M}}, \mathbb{Z}) \) a structure of lattice by the intersection numbers,

\[
(C_i \cdot C_j) = \begin{pmatrix}
-2, & 1, & 0, & \cdots & 0, & 1 \\
1, & -2, & 1, & \cdots & \cdots & 0 \\
0, & 1, & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0, & \cdots & \cdots & 1, & -2, & 1 \\
1, & 0, & \cdots & 0, & 1, & -2
\end{pmatrix}.
\tag{4.18}
\]
Here we have a special 2-cycle given by the homological sum of all the \( N \) 2-cycles,

\[
C_0 = \sum_{i=1}^{N} C_i. 
\]  

(4.19)

As the intersection matrix \((4.18)\) indicates, \((C_0 \cdot C_i) = 0\) for \( i = 1, \ldots, N \). This can be geometrically understood by identifying \( C_0 \) with a submanifold which admits a fibration over a sufficiently large circle on \( B \) surrounding all \( N \) points in \( P \). The fiber of \( C_0 \) is a compact circle wrapped on the cylinder \( F_2 \simeq \mathbb{C}^* \). Therefore, the submanifold \( C_0 \) has the topology of torus, whereas \( C_i \) \( (i = 1, \ldots, N) \) has the shape of 2-sphere. The special Lagrangian torus given by \((4.16)\) is in the homology class \( C_0 \).

### 4.2 BPS spectrum

In the original sigma model, the BPS spectrum is provided by D0-branes, (anti- ) D2-branes wrapped on holomorphic rational curves in \( \mathcal{M} \) and their KK excitations. These BPS objects are mapped, in the LG mirror, to the D2-branes wrapped on special Lagrangian submanifolds in \( \hat{\mathcal{M}} \). The special Lagrangian submanifolds can be represented by linear combinations of \( N \) submanifolds \( C_i \) given above. In fact the root lattice of the affine Lie algebra \( \hat{A}_{N-1} \) is identified with the homology lattice generated by \( C_i \). The correspondence between the roots and the submanifolds \( C_i \) is simply given by

\[
\alpha_i \leftrightarrow C_i, \quad i = 1, \ldots, N,
\]
\[ \alpha_+ \leftrightarrow C_+ \in \{ C_{ij} = C_i + C_{i+1} + \cdots + C_j ; \ 1 \leq i \leq j \leq N - 1 \}, \]
\[ \delta \leftrightarrow C_0. \]  
(4.20)

The BPS objects in the sigma model on \( \mathcal{M} \), which are classified by the positive roots of \( \hat{A}_{N-1} \), are therefore in one-to-one correspondence with D2-branes wrapped on special Lagrangian submanifolds in \( \hat{\mathcal{M}} \) as follows.

(I) D2-brane wrapped on \( \alpha_{ij} \) in \( \mathcal{M} \) is mapped to D2-brane wrapped on \( C_{ij} \) in \( \hat{\mathcal{M}} \).

(II) The bound state of \( n (\geq 1) \) D0-branes on \( \mathcal{M} \) is mapped to D2-brane wrapped on \( nC_0 \) in \( \hat{\mathcal{M}} \).

(III) The \( n (\geq 1) \)-th KK excitation of D2-brane wrapped on \( \alpha_{ij} \) in \( \mathcal{M} \) corresponds to D2-brane wrapped on \( C_{ij} + nC_0 \) in \( \hat{\mathcal{M}} \).

(IV) The \( n (\geq 0) \)-th KK excitation of anti-D2-brane wrapped on \( \alpha_{ij} \) in \( \mathcal{M} \) is identified with D2-brane wrapped on \( -C_{ij} + (n + 1)C_0 \) in \( \hat{\mathcal{M}} \).

The mass of the BPS state, which corresponds to the D2-brane wrapped on a special Lagrangian submanifold \( C \) of \( \hat{\mathcal{M}} \), is given by
\[ M = \frac{1}{4\pi^2 g_s} \left| \int_C \Omega \right|. \]  
(4.21)

In this formula, the normalization factor is included to reproduce the results in the large volume limit which will be presented below. It is rather difficult for an arbitrary \( N \) to obtain the BPS mass formula in a closed form by computing (4.21). We will perform a detailed analysis only for the simplest cases \( N = 2, 3 \) in the following subsections. Before doing it, however, it would be worthwhile to confirm that our mirror model reproduces some known results for general \( N \) in the two extremal situations: the large volume and orbifold limits. Specifically, we will focus on \( C_i (i = 1, \ldots, N - 1) \) and evaluate the complexified Kähler class of their mirror
\[ B_{\alpha_i} + iJ_{\alpha_i} = \frac{1}{4\pi^2} \int_{C_i} \Omega, \]  
(4.22)
in these limits. Here, \( J_{\alpha_i} \) and \( B_{\alpha_i} \) denote respectively the Kähler form and \( B \) field on \( \mathcal{M} \) integrated over the 2-cycle \( \alpha_i \).

**Large volume limit**

The large volume limit is to take \( \text{Re} \ t_a = r_a \to +\infty \). Let us study how the Kähler class (4.22) behaves in this limit. For \( \text{Re} \ t_a \gg 1 \), the \( N \) solutions \( \{ w_1(c_1), \ldots, w_N(c_1) \} \) to
the equation \( f(w, c_1) = 0 \) are approximately given by

\[

t_{1}(c_1) \sim 1/c_1, \\
\hat{t}_{i}(c_1) \sim c_{i-1}/c_i, \quad i = 2, \ldots, N - 1, \\
\hat{t}_N(c_1) \sim c_{N-1}.
\]

(4.23)

This estimation was derived by assuming \( |\hat{w}_1(c_1)| \ll |\hat{w}_2(c_1)| \ll \cdots \ll |\hat{w}_N(c_1)| \). Then, the Kähler class of the holomorphic 2-cycles in \( \mathcal{M} \) mirror to \( C_i \) can be read off by combining (4.10), (4.14), (4.17), (4.21) and (4.23) as

\[
B_{\alpha_i} + iJ_{\alpha_i} = \frac{i}{2\pi} \log \frac{\hat{w}_{i+1}(c_1)}{\hat{w}_i(c_1)}, \\
\sim \frac{i}{2\pi} \hat{t}_i.
\]

(4.24)

This reproduces the classical picture that \( it_i = \theta_i + ir_i \) can be interpreted as the complexified Kähler class of the exceptional divisor \( \alpha_i \).

**Orbifold limit**

The orbifold limit is realized by taking Re \( t_a = r_a \to -\infty \). It is interesting to take this limit and see the fate of the Kähler class. The \( N \) solutions \( \{\hat{w}_1(c_1), \ldots, \hat{w}_N(c_1)\} \) in the limit Re \( t_a \to -\infty \) are given by

\[
\hat{w}_i(c_1) = \exp \left[ \frac{(2i - 1)\pi i}{N} \right], \quad i = 1, \ldots, N.
\]

(4.25)

As in the large volume limit, let us confine ourselves to the D2-brane wrapping \( C_i \). The resultant Kähler class is

\[
B_{\alpha_i} + iJ_{\alpha_i} = \frac{i}{2\pi} \log \frac{\hat{w}_{i+1}(c_1)}{\hat{w}_i(c_1)}, \\
= -\frac{1}{N}.
\]

(4.26)

We have again succeeded to rederive well known facts at the orbifold point. Namely, it was verified in (4.26) that the \( B \) field takes the value \(-1/N\) \[19, 20\] and consequently the BPS mass is smaller than the ordinary unit \( 1/g_s \) for D0-brane by the factor \( 1/N \). Therefore, we can conclude that the D2-brane wrapping \( C_i \) \( i = 1, \ldots, N - 1 \) becomes a fractional brane \[21, 2\] in the orbifold limit.
4.3 $C^2/Z_2$

Let us begin with the simplest example, $C^2/Z_2$. For this case, the defining equation (4.3) of $\hat{\mathcal{M}}$ is given by
\[
f(w, z) = w^2 + zw + 1 = 0, \quad xy = z - e^{t_1/2}.
\] (4.27)
The discriminant $\Delta(z)$ of the first equation in (4.27) is
\[
\Delta(z) = (z - z_1)(z - z_2), \quad z_1 = -2, \quad z_2 = 2.
\] (4.28)
The special Lagrangian submanifolds in $\hat{\mathcal{M}}$ can be found by the general procedure explained in subsection 4.1. The two solutions of $f(w, e^{t_1/2}) = 0$ are given by
\[
w_1(e^{t_1/2}) = \frac{1}{2} \left( e^{t/2} + \sqrt{e^t - 4} \right), \quad w_2(e^{t_1/2}) = \frac{1}{2} \left( e^{t/2} - \sqrt{e^t - 4} \right).
\] (4.29)
These two points in the $w$-plane are mapped to an infinite number of points aligned on the imaginary axis of the $(\ln w)$-plane,
\[
u_{1,2}(n) = \ln w_{1,2}(e^{t_1/2}) + 2\pi in, \quad n \in \mathbb{Z},
\] (4.30)
where we can assume $0 \leq \text{Im}[\ln w_1(e^{t_1/2})] \leq \text{Im}[\ln w_2(e^{t_1/2})] < 2\pi$ without loss of generality. Then, we associate the special Lagrangian submanifold $C_1$ ($C_2 = -C_1 + C_0$), which corresponds to a (anti-) D2-brane on $\mathcal{M}$, with the straight line segments on the $(\ln w)$-plane,
\[
C_1 : \ln w = u_1(0)(1 - t) + u_2(0)t,
C_2 : \ln w = u_2(0)(1 - t) + u_1(1)t.
\] (4.31)
Other special Lagrangian submanifolds, which give rise to the $n$ ($\geq 1$)-th KK modes, are given by
\[
C_1 + nC_0 : \ln w = u_1(0)(1 - t) + u_2(n)t,
-C_1 + (n + 1)C_0 : \ln w = u_2(0)(1 - t) + u_1(n + 1)t.
\] (4.32)
All other line segments are identical with the ones appearing in (4.31) and (4.32). Finally, the special Lagrangian submanifold $nC_0$, which is identified with the bound state of $n$ D0-branes on $\mathcal{M}$, is parameterized by
\[
nC_0 : \ln w = a + 2\pi int,
\] (4.33)
where $a \in \mathbb{R}$ describes the moduli of deforming $C_0$ in $\hat{\mathcal{M}}$ along the $(\ln w)$-plane. Recall that there exists one more modulus $b$ as in (4.16). When combined with the real two
dimensional moduli space of the Wilson line on a D2-brane wrapping the 2-torus $C_0$, the two parameters $a$ and $b$ account for the real four dimensional moduli space of a D0-brane moving on $M$. In Figure 3, we depict these special Lagrangian submanifolds projected on the $(\ln w)$-plane.

The mass of a (anti-) D2-brane wrapped on the holomorphic curve $\alpha_1$ in $M$ can then be determined by (4.31) as

$$D2 : M = \frac{1}{2\pi g_s} \left| \ln \left( \frac{e^{t_1/2} - \sqrt{e^{t_1} - 4}}{e^{t_1/2} + \sqrt{e^{t_1} - 4}} \right) \right|, \quad \overline{D2} : M = \frac{1}{2\pi g_s} \left| \ln \left( \frac{e^{t_1/2} - \sqrt{e^{t_1} - 4}}{e^{t_1/2} + \sqrt{e^{t_1} - 4}} \right) - 2\pi i \right|. \quad (4.34)$$

This is in agreement, as expected, with the result of [19] in which the GKZ system of differential equations has been used to compute the period (4.10). Similarly, we find from (4.32) that the bound state of $n$ D0-branes with these states has the mass

$$D2/nD0 : M = \frac{1}{2\pi g_s} \left| \ln \left( \frac{e^{t_1/2} - \sqrt{e^{t_1} - 4}}{e^{t_1/2} + \sqrt{e^{t_1} - 4}} \right) + 2\pi n i \right|, \quad \overline{D2}/nD0 : M = \frac{1}{2\pi g_s} \left| \ln \left( \frac{e^{t_1/2} - \sqrt{e^{t_1} - 4}}{e^{t_1/2} + \sqrt{e^{t_1} - 4}} \right) - 2\pi (n + 1) i \right|. \quad (4.35)$$
The mass of the bound state of \( n \) D0-branes can be read off from (4.33),

\[
M = \frac{n}{g_s},
\]

which is a familiar result.

We can use the data of special Lagrangian submanifolds to examine the stability of the BPS states. Let us consider the decay of a BPS state \( S^i \) into other \( n \) BPS states \( S_1^f, S_2^f, \ldots, S_n^f \). We denote the special Lagrangian submanifolds in \( \mathcal{M} \) which correspond to these states by \( C_i \) and \( C_1^f, C_2^f, \ldots, C_n^f \), respectively. Furthermore, let \( \theta_i \) and \( \theta_1^f, \ldots, \theta_n^f \) be the special Lagrangian phases of them. For this decay to be possible, the following geometrical condition is required to hold:

\[
C^i = C_1^f + C_2^f + \cdots + C_n^f, \\
\theta^i = \theta_1^f = \cdots = \theta_n^f.
\]

The first equation guarantees RR charge conservation. The second condition states that this decay process is energetically allowed, since it implies that the mass of the initial state \( S^i \) is equal to the sum of the masses of the final states \( S_1^f, S_2^f, \ldots, S_n^f \). In the present case, there exist submanifolds such that the condition (4.37) is satisfied if and only if \( \ln w_1(e^{t_1/2}) \) and \( \ln w_2(e^{t_1/2}) \) have the same real part, as shown in Figure 4. This condition is rephrased in terms of the parameter \( t_1 \) as

\[
z_1 \leq \text{Re } e^{t_1/2} \leq z_2, \quad \text{Im } e^{t_1/2} = 0.
\]

On this line, the decay of the initial state \( S^i = D0 \) into the two final states \( S_1^f = D2 \)
and $S^f = \overline{D2}$ is allowed. More generally, the bound state of a D2 state and $n$ D0 states can decay into $n + 1$ D2 states and $n \overline{D2}$ states. In summary, the following transition processes can occur on the line of marginal stability given by (4.38),

$$
\begin{align*}
D0 & \leftrightarrow D2 + \overline{D2}, \\
D2/D0 & \leftrightarrow 2D2 + \overline{D2}, \\
& \vdots \\
D2/nD0 & \leftrightarrow (n + 1)D2 + n\overline{D2},
\end{align*}
$$

(4.39)

As stressed in the previous section, on both sides of the line, the states on the l.h.s. of (4.39) are stable. In this regard, our model, which possesses 16 supercharges, is clearly distinguished from theories with 8 unbroken supersymmetries where the BPS spectrum can jump.

4.4 $C^2/Z_3$

Let us proceed to the $C^2/Z_3$ orbifold. The equation (4.9) of $\hat{M}$ reads

$$
f(w, z) = w^3 + e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}w^2 + zw + 1 = 0, \quad xy = z - e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}.
$$

(4.40)

Special Lagrangian submanifolds in $\hat{M}$ are constructed following the general procedure in subsection 4.1. We write three solutions of $f(w, e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}) = 0$ by $w_i(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})$ with $i = 1, 2, 3$. We map these three points in the $w$-plane into an infinite number of points aligned on the imaginary axis of the $(\ln w)$-plane,

$$
u_i(n) = \ln w_i(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}) + 2\pi in, \quad n \in \mathbb{Z},
$$

(4.41)

where we assume $0 \leq \text{Im}[\ln w_1(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})] \leq \text{Im}[\ln w_2(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})] \leq \text{Im}[\ln w_3(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})] < 2\pi$. Then, the special Lagrangian 2-spheres $C_1, C_2$, which correspond to D2-branes wrapped around holomorphic rational curves $\alpha_1, \alpha_2$ in $\mathcal{M}$, are associated with the straight line segments on the $(\ln w)$-plane,

$$
C_1 : \ln w = u_1(0)(1 - t) + u_2(0)t,
C_2 : \ln w = u_2(0)(1 - t) + u_3(0)t.
$$

(4.42)

2 The notation $S^f = D0$ might be misleading, since the BPS states are in one-to-one correspondence with the elements of the $L^2$ cohomology group $H^*_{L^2}(M)$ of a BPS D-brane moduli space $\mathcal{M}$, not with the D-branes themselves. However, the moduli space of D0-brane is the ALE space $\mathcal{M}$ itself, and $H^*_{L^2}(\mathcal{M})$ has a unique element given by (the Poincaré dual of) the 2-cycle $\alpha_1$ in the case of $C^2/Z_2$. It is thus justified in this subsection to adopt that notation. In an analogous way, we can use D2, D2/D0 and etc. as the names of BPS states.
The special Lagrangian 2-torus $nC_0$ is parameterized exactly in the same way as (4.33). We depict these special Lagrangian submanifolds in Figure 5. Other special Lagrangian 2-spheres can be identified easily in Figure 4.

Let us study the stability of BPS states in $\tilde{\mathcal{M}}$ using the data of special Lagrangian 2-cycles in $\tilde{\mathcal{M}}$. In particular, let us concentrate on the bound state of a D2-brane wrapping $\alpha_1$ and a D0-brane. This state, which we denote by $D2_1/D0$, is associated with special Lagrangian 2-cycle $C_1 + C_0$ in $\tilde{\mathcal{M}}$. Then, let us consider the following two transition processes

(a) : $D2_1/D0 \leftrightarrow 2D2_1 + \overline{D2_1}$,

(b) : $D2_1/D0 \leftrightarrow D2_{1+2} + \overline{D2_2}$. \hspace{1cm} (4.43)

Here, $D2_{1+2}$ denotes a D2-brane wrapping the holomorphic rational curve $\alpha_1 + \alpha_2$. The decay (a) in (4.43) is allowed if $\ln w_1(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})$ and $\ln w_2(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2})$ have the same real part,

$$\text{Re } \ln \frac{w_2}{w_1}(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}) = 0. \hspace{1cm} (4.44)$$

On the other hand, the decay (b) is allowed if $u_3(0)$ sits on the straight line stretched between $u_1(0)$ and $u_2(1)$ in the ($\ln w$)-plane,

$$\ln \frac{w_2}{w_1}(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}) + 2\pi i = r \left[ \ln \frac{w_2}{w_1}(e^{\frac{2}{3}t_1 + \frac{1}{3}t_2}) - 2\pi i \right], \hspace{1cm} r \in \mathbb{R}. \hspace{1cm} (4.45)$$
These conditions give marginal stability lines as real three-dimensional submanifolds in the real four-dimensional moduli space spanned by two complex FI parameters $t_1, t_2$.

Finally, we wish to show an instructive analysis on the marginal stability lines near the orbifold limit $\zeta_1, \zeta_2 \to -\infty$. As we have shown in subsection 4.2, we have $B_{\alpha_1} = B_{\alpha_2} = -\frac{1}{3}$, $J_{\alpha_1} = J_{\alpha_2} = 0$ in this limit. Let us consider the moduli space spanned by $\zeta_i$ with the value of $B_{\alpha_i}$ fixed. Then, we have to notice the following parameterization

$$
\begin{align*}
2\pi = &\, \text{Im} \ln w_2(c_1) - \text{Im} \ln w_1(c_1), \\
2\pi = &\, \text{Im} \ln w_3(c_1) - \text{Im} \ln w_2(c_1), \\
2\pi \zeta_1 = &\, \text{Re} \ln w_2(c_1) - \text{Re} \ln w_1(c_1), \\
2\pi \zeta_2 = &\, \text{Re} \ln w_3(c_1) - \text{Re} \ln w_2(c_1).
\end{align*}
$$

(4.46)

Here, we have abbreviated $e^{\frac{2\pi t_1 + 2\pi t_2}{3}}$ as $c_1$. In order to search the lines of marginal stability, we choose the value of $\zeta_i$ to satisfy the condition (4.37). In other words, we move each alignment $u_i(n)$ ($i = 1, 2, 3$) to satisfy the condition (4.37) keeping the value of $\text{Im} \ln w_i$ fixed. We depict in Figure 6 the situation where we obtain the BPS states at threshold for the process (4.43). Then, we can easily read off the condition for the decay (4.43) to be possible from Figure 6 and (4.46)

$$
\begin{align*}
(a) &\, : \quad \zeta_1 = 0, \\
(b) &\, : \quad \zeta_1 + 2\zeta_2 = 0.
\end{align*}
$$

(4.47)

These expressions are none other than the results obtained in [11].
5 Conclusions and discussion

In this paper, we have used the idea of \[8, 9\] in order to examine the BPS spectrum of D-branes on the ALE space \(\mathcal{M}\). We have found that the mirror LG theory can provide a quite useful framework to study the BPS spectrum. We have explicitly constructed special Lagrangian submanifolds and BPS charge lattice in the mirror manifold \(\tilde{\mathcal{M}}\). Using these ingredients, we have shown how the lines of marginal stability can be determined geometrically. The prescription is much simpler than that of \[11\]. Our work provides the framework to see the fate of BPS states in the whole moduli space of \(\mathcal{M}\). Although these results are successful, it would be further desired to clarify the role of BPS algebra found in \[11\] in our mirror framework.

In the case of the ALE space \(\mathcal{M}\) with 16 supercharges, there is much simplification in describing the BPS D-brane moduli space. In particular, all the lines of marginal stability cannot give the decay of BPS states. This fact is deeply connected to the well-established McKay correspondence. It is curious to note that our work on a mirror LG theory has appeared after the work \[11\] on a quiver gauge theory. In the case of Calabi-Yau threefolds, many works have used mirror symmetry \[1, 2\] at first. This is deeply related to the fact that a precise mathematical framework of “McKay correspondence” for Calabi-Yau threefolds is still out of reach (however, see \[23\] and \[24\]). Also, it is not obvious how we should define a concept of the stability. Then, one may have a natural question how these two notions are related. Recently, the study on this direction is quite active \[25, 26, 27\]. Researches in this direction would provide us with fruitful insights on the moduli space of D-branes in the near future.

At a different point in the moduli space of \(\mathcal{M}\), there exists a certain LG description \[28, 29, 30\]. Recently, the modular invariance for the ALE space has been studied using a free field approximation \[31\]. We believe that it would be very interesting to understand D-branes in this approach in order to fully understand BPS spectrum in the whole moduli space of \(\mathcal{M}\).

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