SPECTRA OF EXPANDING MAPS ON BESOV SPACES

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Abstract. A typical approach to analysing statistical properties of expanding maps is to show spectral gaps of associated transfer operators in adapted function spaces. The classical function spaces for this purpose are Hölder spaces and Sobolev spaces. Natural generalisations of these spaces are Besov spaces, on which we show a spectral gap of transfer operators.

1. Introduction. Let $M$ be a compact smooth Riemannian manifold endowed with the normalised Lebesgue measure $\text{Leb}_M$, and $f : M \to M$ a $C^r$ expanding map with $r > 1$. It is well known that individual trajectories of expanding maps tend to have “chaotic behaviour”. Therefore, to analyse statistical properties of the expanding map $f$, it is typical to instead study how densities of points evolve under a so-called transfer operator $L_{f,g}$ induced by $f$ with a given $C^{\tilde{r}}$ weight function $g : M \to \mathbb{C}$ with $0 < \tilde{r} \leq r$ (the precise definition will be given below). In his celebrated paper [23], Ruelle first showed a spectral gap of the transfer operator of expanding maps on the usual Hölder space $C^s(M)$ with $0 < s \leq \tilde{r}$ (when $g$ is real-valued and strictly positive), resulting in the demonstration of the existence of a unique equilibrium state $\mu_g$ of $f$ at $g$ and exponential decay of correlation functions of any $C^s$ observables with respect to $\mu_g$. (The existence and uniqueness of equilibrium states for $C^r$ expanding maps had been proved earlier in his monograph [22] through a thermodynamic approach.) Furthermore, the spectral gap of the transfer operator was used to investigate the dynamical zeta function [24, 25], several limit theorems [1, 16] and strong stochastic stability [13, 14]. As another (deep) development, Gundlach and Latushkin [17] obtained an exact formula of the essential spectral radius of the transfer operator on $C^s(M)$ in thermodynamic expression (see also Remark 4).

Recently, the transfer operator was shown to also have a spectral gap on the Sobolev space $W^{s,p}(M)$ in Baillif and Baladi [3] (see also Faure [15] for the spectral...
gap via a semiclassical approach, and Thomine [28] for a spectral gap of the transfer operator of piecewise expanding maps on \( \mathcal{C}^s(M) \) and the “little Hölder space” \( \mathcal{C}^r(M) \) in Baladi and Tsujii [12]. Our goal in this paper is to show a spectral gap of the transfer operator on Besov spaces \( B_{pq}^s(M) \), which are closely related with the previously-studied function spaces (see Remark 2). We also refer to [5, 7, 6] and references therein for recent development of Banach spaces adapted to (hyperbolic) dynamical systems.

Our method in the proof is a natural generalisation of the best technology developed in Tsujii and Baladi [11]. Our result gives an answer to Problem 2.40 in the monograph by Baladi [8].

1.1. Definitions and results. Before precisely stating our main result, we introduce some notation. Recall that \( M \) is a compact smooth Riemannian manifold and \( f : M \to M \) is of class \( \mathcal{C}^r \) with \( r > 1 \). Let \( f \) be an expanding map, i.e., there exist constants \( C > 0 \) and \( \lambda_0 > 1 \) such that \( |Df^n(x)| \geq C\lambda_0^n |v| \) for each \( x \in M \) and \( v \in T_xM \). Let us set the minimal Lyapunov exponent

\[
\chi_{\min} = \lim_{n \to \infty} \frac{1}{n} \log \inf_{x \in M} \inf_{|v|=1} |Df^n(x)v|.
\]

Then \( \chi_{\min} > 0 \) when \( f \) is an expanding map. (For the properties of expanding maps, the reader is referred to [22, 19, 21] e.g.)

Let \( g \) be a complex-valued \( \mathcal{C}^r \) function on \( M \) with \( 0 < \tilde{r} \leq r \). We assume a technical condition

\[
\tilde{r} \geq 1 \quad \text{or} \quad \tilde{r} \leq r - 1.
\]

Note that (1) is satisfied for an important application \( g = |\det Df|^{-1} \), and that if \( r \geq 2 \), then the condition (1) always holds (see also Remark 4 for the condition).

With the notation \( g^{(n)}(x) = \prod_{j=1}^{n} g(f^{j}(x)) \), we set

\[
R(g) = \lim_{n \to \infty} R_n(g)^{1/n}, \quad R_n(g) = \|g^{(n)}\|_{L^\infty} \det Df^n\|_{L^\infty} \quad (n \geq 1).
\]

For each \( 0 \leq s \leq \tilde{r} \), the transfer operator \( L_{f,g} : \mathcal{C}^s(M) \to \mathcal{C}^s(M) \) of \( f \) with a \( \mathcal{C}^\tilde{r} \) weight function \( g \) is defined by

\[
L_{f,g}u(x) = \sum_{f(y)=x} g(y) \cdot u(y), \quad u \in \mathcal{C}^s(M).
\]

Standard references for transfer operators are [4, 8].

Remark 1. Denote by \( r(A|E) \) and \( r_{ess}(A|E) \) the spectral radius and the essential spectral radius of a bounded operator \( A : E \to E \) on a Banach space \( E \), respectively. Due to Ruelle [23], \( r(L_{f,g}|\mathcal{C}^s(M)) \) is bounded by \( \exp P_{\text{top}}(\log |g|) \), and is equal to \( \exp P_{\text{top}}(\log g) \) when \( g \) is real-valued and strictly positive (i.e., \( \inf_{x \in M} g > 0 \)), where \( P_{\text{top}}(\phi) \) is the topological pressure of a continuous function \( \phi : M \to \mathbb{R} \) (refer to [32] for the definition of topological pressure). It also follows from [8, Lemma 2.16] that \( R(g) \geq \exp P_{\text{top}}(\log |g|) \).

Furthermore, in view of [11, Lemma A.1] regarding coincidence of eigenvalues of abstract linear operators in different Banach spaces outside of the essential spectral radii, we get the following: If \( r_{ess}(L_{f,g}|B_{pq}^s(M)) < \exp P_{\text{top}}(\log g) \) with real-valued and strictly positive \( g \), then \( r(L_{f,g}|B_{pq}^s(M)) = \exp P_{\text{top}}(\log g) \), where \( B_{pq}^s(M) \) is the Besov space on \( M \) (see below for definition). In particular, the transfer operator on \( B_{pq}^s(M) \) is quasi-compact (has a spectral gap), which is our goal in this paper.
\( \mathcal{L}_{f,g} \) can be extended to a bounded operator on \( L^p(M) \) with \( p \in [1, \infty] \). (Since some Besov spaces do not coincide with the completion of \( \mathscr{C}^\infty(M) \) with respect to its Besov norm, this extension would be necessary; see [27, §A.1].) Indeed, a change of variables shows
\[
\int \mathcal{L}_{f,g} u \cdot \varphi \, d\text{Leb}_M = \int u \cdot \varphi \circ f \cdot g \cdot |\det Df| \, d\text{Leb}_M, \tag{3}
\]
for any \( u \in L^\infty(M) \) and \( \varphi \in L^1(M) \). On the other hand, the operator \( \varphi \mapsto \varphi \circ f \cdot g \cdot |\det Df| \) is bounded on \( L^{p'}(M) \) with \( 1/p' + 1/p = 1 \): notice that \( ||\varphi \circ f \cdot g \cdot |\det Df||_{L^{p'}} \leq R_1(g)||\varphi \circ f||_{L^{p'}} \) and that in the case \( 1 \leq p' < \infty \),
\[
||\varphi \circ f||_{L^{p'}} = \left( \int \mathcal{L}_{f,|\det Df|^{-1}} \cdot |\varphi|^p \, d\text{Leb}_M \right)^{1/p'} \leq ||\mathcal{L}_{f,|\det Df|^{-1}}||_{L^{\infty}} ||\varphi||_{L^{p'}}
\]
by virtue of (3), while \( ||\varphi \circ f||_{L^\infty} \leq ||\varphi||_{L^\infty} \). Therefore, the transfer operator has a continuous extension to \( L^p(M) \) by the duality (3). The extension will also be denoted by \( \mathcal{L}_{f,g} \).

Below we define Besov spaces associated with a partition of unity of \( M \). We first recall the definition of the Besov spaces on \( \mathbb{R}^d \), where \( d \) is the dimension of \( M \). Let \( \rho : \mathbb{R} \to \mathbb{R} \) be a \( \mathscr{C}^\infty \) function such that
\[
0 \leq \rho \leq 1, \quad \rho(t) = \begin{cases} 1 & (t \leq 1), \\ 0 & (t \geq 2). \end{cases}
\tag{4}
\]
For each nonnegative integer \( n \), define radial functions \( \psi_n \in \mathscr{C}^\infty_0(\mathbb{R}^d, \mathbb{R}) \) by
\[
\psi_n(\xi) = \begin{cases} \rho(|\xi|) & (n = 0), \\ \rho(2^{-n}|\xi|) - \rho(2^{-n+1}|\xi|) & (n \geq 1). \end{cases}
\tag{5}
\]
For a tempered distribution \( u \) (that is, \( u \) is in the dual space of the set of rapidly decreasing test functions), the operator \( \Delta_n \) is given by \( \Delta_n u = \mathcal{F}^{-1}[\psi_n \mathcal{F} u] \) with \( n \geq 0 \), where \( \mathcal{F} \) is the Fourier transform. Then we have \( \sum_{n \geq 0} \Delta_n u = u \), called the Littlewood-Paley (dyadic) decomposition. This decomposition was first employed in context of dynamical systems theory to analyse the spectra of transfer operators of Anosov diffeomorphisms by Baladi and Tsujii [10]. They also applied the decomposition to spectral analysis of transfer operators of expanding maps in the little Hölder space \( \mathscr{C}^\alpha_0(M) \) in a survey [12, Subsection 3.2], see Remark 2 for the definition of \( \mathscr{C}^\alpha_0(M) \).

For \( s \in \mathbb{R}, p, q \in [1, \infty] \), we define the Besov space \( B^{s,p}_q(\mathbb{R}^d) \) as a set of tempered distributions \( u \) on \( \mathbb{R}^d \) whose norm
\[
||u||_{B^{s,p}_q} = \begin{cases} \left( \sum_{n \geq 0} 2^{snp} ||\Delta_n u||^q_{L^p} \right)^{1/q} & (q < \infty), \\ \sup_{n \geq 0} 2^{sn} ||\Delta_n u||_{L^p} & (q = \infty) \end{cases}
\tag{6}
\]
is finite. We remark that \( u \in B^{s}_p(\mathbb{R}^d) \) if and only if there are a constant \( C > 0 \) and a nonnegative sequence \( \{c_n\}_{n \geq 0} \in \ell^q \) with \( ||\{c_n\}_{n \geq 0}||_{\ell^q} \leq 1 \) such that
\[
||\Delta_n u||_{L^p} \leq C 2^{-ns} c_n
\tag{7}
\]
for all \( n \geq 0 \). Once we know \( u \in B^{s}_p(\mathbb{R}^d) \), we can take \( C = ||u||_{B^{s,p}_q} \).

Since \( M \) is compact, there are a finite open covering \( \{V_i\}_{i=1}^I \) and a system of local charts \( \{\kappa_i\}_{i=1}^I \) such that \( \kappa_i : V_i \to U_i \) is of class \( \mathscr{C}^\infty \) with an open set \( U_i \subset \mathbb{R}^d \) for each \( 1 \leq i \leq I \). Let \( \{\phi_i\}_{i=1}^I \) be a family of \( \mathscr{C}^\infty \) functions such that \( \{\phi_i \circ \kappa_i\}_{i=1}^I \)

is a partition of unity of $M$ subordinate to the covering $\{V_i\}_{i=1}^I$, that is, $\phi_i \circ \kappa_i$ is a $C^\infty$ function on $M$ with values in $[0, 1] \subset \mathbb{R}$ whose support is contained in $V_i$ for each $1 \leq i \leq I$ and $\sum_{i=1}^I \phi_i \circ \kappa_i(x) = 1$ for all $x \in M$. In this paper, the support of a continuous function $\phi : M \to \mathbb{R}$ is defined as the closure of $\{x \in M \mid \phi(x) \neq 0\}$.

**Definition 1.1.** The Besov space $B^{s}_{pq}(M)$ on $M$ is the space of tempered distributions $u$ on $M$ whose norm

$$
\|u\|_{B^{s}_{pq}} = \sum_{1 \leq i \leq I} \|\phi_i \cdot u \circ \kappa_i^{-1}\|_{B^{s}_{pq}}
$$

is finite. This definition does not depend on the choice of charts or the partition of unity, see [30].

Let $U \subset \mathbb{R}^d$ be a nonempty open set. We write $\mathcal{C}^s(U)$, $L^p(U)$ and $B^s_{pq}(U)$ for the subspace of $\mathcal{C}^s(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$ and $B^s_{pq}(\mathbb{R}^d)$, respectively, such that the support of each element in these spaces is included in $U$. Then we have $B^s_{pq}(U) \subset L^p(U)$ for $s > 0$ and $p, q \in [1, \infty]$.\(^1\) Hence, from the argument following (3) it holds that $\mathcal{L}_{f,g}u$ is in $L^p(M)$ for each $u \in B^s_{pq}(M)$ with $s > 0$ and $p, q \in [1, \infty]$. In particular, $\mathcal{L}_{f,g}u$ is a tempered distribution and $\|\mathcal{L}_{f,g}u\|_{B^{s}_{pq}}$ is well-defined (but possibly takes $+\infty$).

Here we provide our main theorem for an upper bound of the essential spectral radius of the transfer operator on $B^s_{pq}(M)$, which concludes (due to Remark 1) a spectral gap of the transfer operator on $B^s_{pq}(M)$ when $g$ is real-valued and strictly positive and $s$ is sufficiently large:

**Theorem 1.2.** Let $f : M \to M$ be a $C^r$ expanding map with $r > 1$, and $g : M \to \mathbb{C}$ a $C^\tilde{r}$ function with $0 < \tilde{r} \leq r$ satisfying (1). Let $s \in (0, \tilde{r}]$ and $p, q \in [1, \infty]$. In addition, when $q < \infty$, assume that $s$ is strictly smaller than $\tilde{r}$. Then $\mathcal{L}_{f,g}$ can be extended to a bounded operator on $B^s_{pq}(M)$, and it holds

$$
r_{\text{ess}}(\mathcal{L}_{f,g})_{B^s_{pq}(M)} \leq \exp(-s \chi_{\min}) \cdot R(g),
$$

where $R(g)$ is defined in (2).

**Remark 2.** The Besov spaces can represent many other function spaces. As an important example, $B^{s}_{\infty\infty}(\mathbb{R}^d)$ is known to coincide with the Hölder space $C^s(\mathbb{R}^d)$ when $s$ is not an integer (whereas $B^{s}_{\infty\infty}(\mathbb{R}^d)$ strictly includes $C^s(\mathbb{R}^d)$ when $s$ is an integer), see [27, §A.1]. Hence Theorem 1.2 is a generalisation of a well-known result by Ruelle [23, Theorem 3.2]. We note that the *little Hölder space* $C^s_{\text{little}}(M)$ given in [12] is defined as the completion of $\mathcal{C}^\infty(M)$ for the norm $\|\cdot\|_{B^{s}_{\infty\infty}}$ (which is smaller than $\mathcal{C}^s(M)$, see [27, §A.1]), so that one can see that the proof of Theorem 1.2 can be translated literally to the case when the functional space where the transfer operator acts is $\mathcal{C}^s_{\text{little}}(M)$ (this was essentially done in [12, Theorem 3.1] with $\tilde{r} = r - 1$).

Furthermore, for the Sobolev space $W^{s,p}(\mathbb{R}^d)$ with $s \in \mathbb{R}$ and $p \in (1, 2]$, it holds $B^{s}_{pp}(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d) \subset B^{s}_{pq}(\mathbb{R}^d)$ and $B^{s}_{p/2}(\mathbb{R}^d) \subset W^{s,p/2}(\mathbb{R}^d) \subset B^{s}_{p,p}(\mathbb{R}^d)$, where $1/p + 1/p' = 1$ (if $p \neq 2$, these inclusions are strict; see [26, p.161] for the cases

\(^1\) Indeed, by Hölder’s inequality, for $1 \leq q \leq \infty$ it holds

$$
\|u\|_{L^p} = \left\| \sum_{n \geq 0} \Delta_n u \right\|_{L^p} \leq \sum_{n \geq 0} \|\Delta_n u\|_{L^p} \leq \left( \sum_{n \geq 0} 2^{nq} \|\Delta_n u\|_{L^p} \right)^{1/q} \left( \sum_{n \geq 0} 2^{-sn} \right)^{1/q'} = (1 - 2^{-s})^{-1/q'} \|u\|_{B^{s}_{pq}},
$$

where $1/q + 1/q' = 1$ with a usual modification for $q = \infty$. 
when $s \in \mathbb{N}$). These inclusions imply $\mathcal{W}^{s,2}(M) = \mathcal{B}^s_{22}(M)$, thus Theorem 1.2 recovers a part of the result by Baillif and Baladi [3].

Other famous functional spaces constructed by using the Littlewood-Paley decomposition are Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^d)$. It seems possible to prove Theorem 1.2 with $\mathcal{B}^s_{pq}(\mathbb{R}^d)$ replaced by $F^s_{pq}(\mathbb{R}^d)$ in the same manner as in the proof of Theorem 1.2. We note that $W^{s,p}(\mathbb{R}^d) = F^{s,2}_{22}(\mathbb{R}^d)$ for each $1 < p < \infty$, see [29].

**Remark 3.** Theorem 1.2 is far from optimal: Gundlach and Latushkin [17] showed that $\text{ess}\left(\mathcal{L}_{f,g}\big|\mathcal{C}^s(M)\right)$ is equal to

$$\sup_{\mu \in \text{Erg}(f)} \exp\left(-s\chi^-_\mu + h_\mu + \int \log |g| d\mu\right),$$

(10)

where $\text{Erg}(f)$ is the set of ergodic $f$-invariant probability measures, $\chi^-_\mu$ is the smallest Lyapunov exponent of $\mu$ (for $Df$ over $f$), and $h_\mu$ is the Kolmogorov-Sinai entropy of $\mu$ (over $f$). We refer to [32, 31] for the definitions of Lyapunov exponents and Kolmogorov-Sinai entropy. By the variation principle for topological pressure (see [32, §9] e.g.), we have

$$P_{\text{top}}(\phi) = \sup_{\mu \in \text{Erg}(f)} \left(h_\mu + \int \phi \, d\mu\right),$$

so that, since $\chi^-_\mu \geq \chi_{\text{min}}$ for any $\mu \in \text{Erg}(f)$, (10) is bounded by $\exp(-s\chi_{\text{min}} + P_{\text{top}}(\log |g|))$. Therefore, when $g$ is real-valued and strictly positive, the transfer operator on $\mathcal{C}^s(M)$ is quasi-compact for any $0 < s \leq \tilde{r}$, while our $s$ should be large to deduce the quasi-compactness from Theorem 1.2.

We also give a remark on an exceptional case $g = |\det Df|^{-1}$: If $g = |\det Df|^{-1}$, then $R(g) = 1$ and it follows from the Ruelle inequality and the Pesin identity (see e.g. [20]) that $\exp P_{\text{top}}(\log |g|) = 1$. Therefore, together with Remark 1, Theorem 1.2 implies the quasi-compactness of $\mathcal{L}_{f,g}$ for any $0 < s < \tilde{r}$.

**Remark 4.** It is a natural question whether one can get a Gundlach-Latushkin type formula of the essential spectral radius on $\mathcal{B}^s_{pq}(M)$, but that we think our method does not work for this purpose. Roughly speaking, we avoided the problem of exponential growth of the number of inverse branches of $f^n$ by a duality argument (see e.g. (14)), and this made our proof a little simpler. Its main shortcoming is that it does not seem that one can use thermodynamic techniques in estimating the essential spectral radius at the final step of the proof, which might be essential to obtain a Gundlach-Latushkin type formula. Baladi [8, Theorem 2.15] recently showed a Gundlach-Latushkin type upper bound of the essential spectral radius of the transfer operator of expanding maps on the Sobolev space $\mathcal{W}^{s,p}(M)$ by using the Littlewood-Paley decomposition, which may be helpful to answering this question.

Another shortcoming of our approach is the technical condition (1) for the regularity of weight functions. This problem is also solved in the Sobolev case in the approach of [8, Theorem 2.15] (see Remark 5 also), and it is highly likely that one can remove the condition (1) by a similar argument.

2. Proof of Theorem 1.2.

2.1. Transfer operators on $\mathbb{R}^d$. We shall deduce Theorem 1.2 from a theorem for transfer operators on Besov spaces of compactly supported functions on $\mathbb{R}^d$. Let
Let $U$ and $U'$ be nonempty bounded open subsets of $\mathbb{R}^d$. Let $F$ be a $C^r$ mapping on $\mathbb{R}^d$ with $r > 1$ such that $U' \cap F^{-1}(U) \neq \emptyset$ and

$$
\lambda = \min_{x \in U' \cap F^{-1}(U)} \min_{v \in \mathbb{R}^d} |DF(x)v|
$$

(11) is larger than 1. Let $G$ be a $C^r$ function whose support is included in $U' \cap F^{-1}(U)$. Furthermore, we assume that there are finitely many compact subsets $\{K_j\}_{j=1}^{N(F)}$ of $\mathbb{R}^d$ such that

$$
\text{int}(K_i) \cap \text{int}(K_j) = \emptyset \quad \text{when} \quad i \neq j, \quad \text{and} \quad \text{supp} \ G = \bigcup_{j=1}^{N(F)} K_j,
$$

(12) and for each $1 \leq j \leq N(F)$ one can find a small neighbourhood $V_j$ of $K_j$ satisfying that $F : V_j \to F(V_j)$ is a $C^r$ diffeomorphism.

Define a bounded operator $\mathbb{I}_{F,G} : C^s(\mathbb{R}^d) \to C^s(U)$ by

$$
\mathbb{I}_{F,G} u(x) = \sum_{F(y)=x} G(y) \cdot u(y).
$$

Then, in a manner similar to one below (3), for each $p \in [1, \infty]$ we can extend $\mathbb{I}_{F,G}$ to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(U)$. The only essential difference from the argument below (3) is that in the case $1 < p \leq \infty$, we need to notice

$$
\|\varphi \circ F \cdot G \det DF\|_{L^p} = \|1_{U' \cap F^{-1}(U)} \cdot \varphi \circ F \cdot G \det DF\|_{L^p}
$$

(14)

$$
\leq \|G \det DF\|_{L^\infty} \|1_{U' \cap F^{-1}(U)} \cdot \varphi \circ F\|_{L^p}
$$

with $1/p' + 1/p = 1$, where $1_{U' \cap F^{-1}(U)}$ is the indicator function of $U' \cap F^{-1}(U)$. Furthermore, it follows from this argument that the operator norm of $\mathbb{I}_{F,G} : L^p(\mathbb{R}^d) \to L^p(U)$ is bounded by a nonnegative number $\alpha \equiv \alpha(F,G,p)$, given by

$$
\alpha = \begin{cases} 
\sup_{x \in F(U') \cap U} \frac{1}{|\det DF(y)|} & \frac{1}{p'} \text{ for } 1 < p \leq \infty, \\
\|G \det DF\|_{L^\infty} & p = 1
\end{cases}
$$

(15)

with $1/p' + 1/p = 1$.

We define $C_1$ as the operator norm of $\Delta_n$ on $L^p(\mathbb{R}^d)$,

$$
C_1 = \sup_{\|u\|_{L^p}=1} \|\Delta_n u\|_{L^p}
$$

(16)

(such $C_1$ is bounded and independent of $u$: see [2, Remark 2.11]). Notice also that if we let $\varphi_0 \in C^\infty(U)$ such that $\varphi_0 \circ F \equiv 1$ on the support of $G$, then $M : u \mapsto u \cdot \varphi_0$ is a bounded operator from $B^s_{pq}(\mathbb{R}^d)$ to $B^s_{pq}(U)$. Denote the operator norm of $M$ by $\tilde{C} = \tilde{C}(s,p,q,\varphi_0) > 0$. For $s > 0$, let

$$
\tilde{\gamma}_s = C_1 2^{5s} (1 - 2^{-s})^{-1} \quad \text{and} \quad \gamma_s = \tilde{C} \tilde{\gamma}_s.
$$

(17)

Theorem 1.2 will follow from the next Lasota-Yorke type inequality.

**Theorem 2.1.** Assume that $(s,p,q)$ satisfies the condition in Theorem 1.2, and let $F$ and $G$ be as above. Then, $\mathbb{I}_{F,G}$ can be extended to a bounded operator from $B^s_{pq}(U')$ to $B^s_{pq}(U)$. Furthermore, there are a constant $\tilde{C}_{F,G} > 0$ and $\sigma \in (0,s)$ such that

$$
\|\mathbb{I}_{F,G} u\|_{B^s_{pq}} \leq \gamma_s \lambda^{-s} \|u\|_{B^s_{pq}} + \tilde{C}_{F,G} \|u\|_{B^s_{pq}}
$$

for all $u \in B^s_{pq}(U')$, where the constants are defined in (11), (15), and (17).
Remark 5. The proof of Theorem 2.1 works even when \( \lambda \leq 1 \), although it may give no spectral information for the original dynamical system \( f \) (in our setting (1) about \( \hat{r} \)): the role of \( F \) will played by \( f^n \) in local chart and \( \lambda \leq 1 \) corresponds to \( \chi_{\text{min}} \leq 0 \), so that Theorem 1.2 only means \( r_{\text{ess}}(L_{f,g})\lesssim (M_f) \) is bounded by \( \exp(-s\chi_{\text{min}})\cdot R(g) \) with \( \exp(-s\chi_{\text{min}}) \geq 1 \) and \( R(g) \geq \exp P_{\text{top}}(\log |g|) \) (see Remark 1). However yet, Theorem 2.1 with \( F = id \) (\( \lambda = 1 \)) seems to be actually helpful to demonstrate a spectral gap with \( \hat{r} \) in full generality (that is, with \( \hat{r} \) being not in the range of (1)), see [8, Theorem 2.15] for the Sobolev case.

Proof. We follow [12] and [8]. Let \( u \in B_{pq}^r(U') \). It follows from (7) that both \( \sum_{\ell,\ell \in n} \Delta_{\ell} u \) and \( \sum_{\ell,\ell \not\in n} \Delta_{\ell} u \) are in \( L^p(\mathbb{R}^d) \) for any \( n \geq 0 \). Thus, if we define \( L_{0,n} u \) and \( L_{1,n} u \) with \( n \geq 0 \) by

\[
L_{0,n} u = \sum_{\ell,\ell \in n} \Delta_{\ell} \|L_{F,G} \Delta_{\ell} u\|, \quad L_{0,n} u = M \circ L_{0,n} u, \\
L_{1,n} u = \sum_{\ell,\ell \not\in n} \Delta_{\ell} \|L_{F,G} \Delta_{\ell} u\|, \quad L_{1,n} u = M \circ L_{1,n} u,
\]

then both \( L_{0,n} u \) and \( L_{1,n} u \) are in \( L^p(U) \) (due to (15) and (16)), where the summations are taken over nonnegative integers \( \ell \) and we write \( \ell \not\in n \) if \( 2^n \leq \lambda^{-1} 2^{s+n} \) and \( \ell \not\not n \) if \( 2^n > \lambda^{-1} 2^{s+n} \). It is obvious from the fact \( \sum_{\ell \geq 0} \Delta_{\ell} u = u \) that

\[
\Delta_n \|L_{F,G} u\| = L_{0,n} u + L_{1,n} u,
\]

so we have by Minkowski’s inequality that

\[
\|L_{F,G} u\|_{B_{pq}^r} \leq \left( \sum_{n \geq 0} 2^{sqn} \|L_{0,n} u\|_{L^p}^q \right)^{1/q} + \left( \sum_{n \geq 0} 2^{sqn} \|L_{1,n} u\|_{L^p}^q \right)^{1/q}
\]

in the case \( q < \infty \) (a corresponding inequality also holds for \( q = \infty \)). Therefore, it suffices to show the following two inequalities for each \( q < \infty \) (and corresponding two inequalities for \( q = \infty \):

\[
\left( \sum_{n \geq 0} 2^{sqn} \|L_{0,n} u\|_{L^p}^q \right)^{1/q} \leq \hat{C}_s \alpha \lambda^{-s} \|u\|_{B_{pq}}, \quad (18)
\]

and

\[
\left( \sum_{n \geq 0} 2^{sqn} \|L_{1,n} u\|_{L^p}^q \right)^{1/q} \leq \tilde{C}_{F,G} \|u\|_{B_{pq}}, \quad (19)
\]

with some constant \( \tilde{C}_{F,G} > 0 \) and \( \sigma \in (0, s) \).

First we show (18). We focus on the case when \( q \) is finite, but the other case is analogous. Using (7), (15) and (16), we can find a nonnegative \( \ell \)-sequence \( \{c_\ell\}_{\ell \geq 0} \) satisfying \( \|\{c_\ell\}_{\ell \geq 0}\|_{\ell^1} \leq 1 \) such that

\[
\left( \sum_{n \geq 0} 2^{sqn} \|L_{0,n} u\|_{L^p}^q \right)^{1/q} \leq C_1 \|u\|_{B_{pq}} \left( \sum_{n \geq 0} 2^{sqn} \left( \sum_{\ell \in n} 2^{-\ell} c_\ell \right)^q \right)^{1/q}
\]

\[
\leq C_1 \|u\|_{B_{pq}} \left( \sum_{n \geq 0} \left( \sum_{\ell \in n} 2^{s(n-\ell)} 1_{\{0 \not\in \ell\}} (n - \ell) c_\ell \right)^q \right)^{1/q}, \quad (20)
\]

where \( 1_{\{0 \not\in \ell\}} \) is the indicator function of \( \{0 \not\in \ell\} = \{ \ell \in \mathbb{Z} | 0 \not\in \ell \} \).

To estimate it, we recall Young’s inequality for the locally compact group \( \mathbb{Z} \). We define \( \ell^q(\mathbb{Z}) \) as the set of (two-sided) nonnegative sequences \( a_\ell = \{a_\ell\}_{\ell \in \mathbb{Z}} \) such that \( \|a_\ell\|_{\ell^q(\mathbb{Z})} := \left( \sum_{\ell \in \mathbb{Z}} q_\ell^q \right)^{1/q} \) is finite. Then, it follows from [2, Lemma 1.4] that
\[ \|a(\cdot)*b(\cdot)\|_{\ell^r(Z)} \leq \|a(\cdot)\|_{\ell^r(Z)} \|b(\cdot)\|_{\ell^r(Z)} \] for all \( a(\cdot) \in \ell^1(Z) \) and \( b(\cdot) \in \ell^q(Z) \), where \( a(\cdot)*b(\cdot) \) is the convolution of \( a(\cdot) \) and \( b(\cdot) \) given by \( a(\cdot)*b(\cdot)(n) = \sum_{\ell \in Z} a(n-\ell)b_\ell \) for \( n \in Z \). Thus, if we let \( \{ \tilde{c}_\ell \}_{\ell \in Z} \) be as \( \tilde{c}_\ell = c_\ell \) for \( \ell \geq 0 \) and \( = 0 \) for \( \ell < 0 \), then we have
\[
\sum_{n \geq 0} \left( \sum_{\ell \geq 0} 2^{\ell(n-\ell)} 1_{\{0 \to (\cdot)\}}(n-\ell)\tilde{c}_\ell \right)^q \leq \sum_{n \in Z} \left( \sum_{\ell \in Z} 2^{\ell(n-\ell)} 1_{\{0 \to (\cdot)\}}(n-\ell)\tilde{c}_\ell \right)^q = \left\| \left( 2^{s(\cdot)} \cdot 1_{\{0 \to (\cdot)\}} \right) \ast \tilde{c}(\cdot) \right\|_{\ell^r(Z)}^q \leq \left( \sum_{\ell \in \{0 \to (\cdot)\}} 2^{s(\cdot)} \right)^q. \tag{21} \]

On the other hand, since
\[
\{ 0 \to (\cdot) \} = \{ \ell \in Z \mid \ell \leq 4 - \log_2 \lambda \}, \tag{22} \]
we have that
\[
\sum_{\ell \in \{0 \to (\cdot)\}} 2^{s(\cdot)} \leq 2^{5(1 - \log_2 \lambda)} \sum_{\ell \geq 0} 2^{-s(\cdot)} = \frac{2^{5s}}{1 - 2^{-s}} \lambda^{-s}. \tag{23} \]

By (20), (21) and (23), we obtain (18).

Next, we shall show (19). Let \( \{ \psi_\ell \}_{\ell \geq 1} \) be a family of smooth functions on \( \mathbb{R}^d \) given by
\[
\tilde{\psi}_n(\xi) = \begin{cases} 
\rho(2^{-1}|\xi|) & (n = 0), \\
\rho(2^{-n-1}|\xi|) - \rho(2^{-n+2}|\xi|) & (n \geq 1)
\end{cases}
\]
for \( \xi \in \mathbb{R}^d \), where \( \rho \) is defined in (4). Notice that \( \tilde{\psi}_n \equiv 1 \) on the support of \( \tilde{\psi}_n \) for each \( n \geq 0 \). For a tempered distribution \( u \), we define \( \tilde{\Delta}_n u \) by \( \tilde{\Delta}_n u = F^{-1}[\tilde{\psi}_n F u] \) for each \( n \geq 0 \). Then it is straightforward to see that
\[
L_{1,\ell}^p u = \sum_{\ell \in \ell \neq n} \Delta_n L_{F,G} \tilde{\Delta}_n \Delta_\ell u \quad \text{for } u \in L^p(U').
\]

We borrow the following crucial lemma by Baladi [8] (from the proof of Lemma 2.21 of [8]), which has been proven in the special case \( \tilde{r} = r-1 \) and \( p = \infty \) in the survey [12] by Baladi and Tsujii (inequality (15) and a comment following inequality (19) of [12]). Since it is a key estimate in the proof of Theorem 2.1, we give a full proof in Appendix A.

**Lemma 2.2.** There exists a constant \( C_{F,G} > 0 \) such that for any \( p \in [1, \infty] \), \( u \in L^p(\mathbb{R}^d) \), \( n \geq 0 \) and \( \ell \neq n \), the following holds: If \( \tilde{r} \leq r - 1 \), then
\[
\| \Delta_n L_{F,G} \tilde{\Delta}_n \Delta_\ell u \|_{L^p} \leq C_{F,G} 2^{-\tilde{r} \max\{n,\ell\}} \| u \|_{L^p}. \tag{24} \]
If \( \tilde{r} \geq 1 \), then
\[
\| \Delta_n L_{F,G} \tilde{\Delta}_n \Delta_\ell u \|_{L^p} \leq C_{F,G} 2^{\min\{n,\ell\}-\tilde{r} \max\{n,\ell\}} \| u \|_{L^p}. \tag{25} \]

Fix \( \sigma \in (0, s) \) satisfying that
\[
\sigma > s - \tilde{r} + 1 + \delta \quad \text{when } \tilde{r} > 1 \tag{26} \]
with some $\delta \in (0, \tilde{r} - 1)$. Let $\gamma_{n, \ell} = 1$ when $\tilde{r} \leq 1$ and $\gamma_{n, \ell} = 2^{\min\{n, \ell\}}$ when $\tilde{r} > 1$. Then, applying Lemma 2.2 together with (7), we can find a nonnegative $\ell^q$ sequence $\{c_\ell\}_{\ell \geq 0}$ satisfying $\|\{c_\ell\}_{\ell \geq 0}\|_{\ell^q} \leq 1$ such that

$$\|L_{1,n}^\ell u\|_{L^p} \leq C_{F,G} \sum_{\ell \neq n} \gamma_{n, \ell} 2^{-\tilde{r} \max\{n, \ell\}} \|\Delta u\|_{L^p} \leq C_{F,G} \sum_{\ell \neq n} \gamma_{n, \ell} 2^{-\ell^{-\tilde{r}} \max\{n, \ell\}} \|u\|_{B^0_{p,q}}.$$  

Therefore, in the case $q = \infty$,

$$\sup_{n \geq 0} 2^{sn} \|L_{1,n}^\ell u\|_{L^p} \leq C_{F,G} \|u\|_{B^0_{p,q}} \sup_{n \geq 0} \sum_{\ell \neq n} \gamma_{n, \ell} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}}.$$  

(27)

Since $-\tilde{r} \max\{n, \ell\} \leq -\tilde{r} n$, and we assumed $\tilde{r} \geq s$ and $\sigma > 0$,

$$\sum_{\ell \neq n} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}} \leq (1 - 2^{-\sigma})^{-1} < \infty.$$  

Furthermore, (26) implies that when $\tilde{r} > 1$, we have

$$\min\{n, \ell\} + sn - \ell - \tilde{r} \max\{n, \ell\} < \begin{cases} -(\tilde{r} - s)(n - \ell) - \delta \ell & (\ell < n) \\ -(s + 1)(n - \ell) - \delta \ell & (\ell \geq n) \end{cases},$$  

(28)

which is bounded by $-\delta \ell$, and thus

$$\sum_{\ell \neq n} 2^{\min\{n, \ell\} + sn - \ell - \tilde{r} \max\{n, \ell\}} < (1 - 2^{-\delta})^{-1} < \infty.$$  

Hence, we obtain (19) with $\tilde{C}_{F,G} = C_{F,G}(1 - 2^{-\sigma})^{-1}$.

When $q < \infty$, we can similarly have

$$\left( \sum_{n \geq 0} 2^{sn} \|L_{1,n}^\ell u\|_{L^p}^q \right)^{1/q} \leq C_{F,G} \|u\|_{B^0_{p,q}} \left( \sum_{n \geq 0} \left( \sum_{\ell \neq n} \gamma_{n, \ell} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}} c_\ell \right)^q \right)^{1/q}$$  

for some $\{c_\ell\}_{\ell \geq 0} \in \ell^q$ satisfying $\|\{c_\ell\}_{\ell \geq 0}\|_{\ell^q} \leq 1$. Let $\tau = \tilde{r} - s$. Then, $\tau > 0$ by assumption of Theorem 1.2, and $sn - \tilde{r} \max\{n, \ell\} \leq sn - \tilde{r} n \leq -\tau(n - \ell)$. Moreover, it follows from (28) that $\min\{n, \ell\} + sn - \ell - \tilde{r} \max\{n, \ell\} \leq -\tau(n - \ell)$ when $\tilde{r} > 1$.

Hence, this leads to

$$\sum_{\ell \neq n} \gamma_{n, \ell} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}} c_\ell \leq \sum_{\ell \geq 0} 1_{\{0 \not= \ell\}}(n - \ell) 2^{-\tau(n - \ell)} c_\ell.$$  

Therefore, using an argument similar to the one following (21) together with (22), we have

$$\left( \sum_{n \geq 0} 2^{sn} \|L_{1,n}^\ell u\|_{L^p}^q \right)^{1/q} \leq C_{F,G} \|u\|_{B^0_{p,q}} \left( \sum_{\ell \in \mathbb{Z} \setminus \{0 \not= \ell\}} 2^{-\tau \ell} \right)^{1/q} \leq C_{F,G} 2^{-\tau (1 - \log_2 \lambda)} \frac{1}{1 - 2^{-\tau}} \|u\|_{B^0_{p,q}}.$$  

This completes the proof of (19). \hfill $\square$

**Remark 6.** We cannot show (19) for $s = \tilde{r}$ and $q < \infty$ in the same manner as the case $q = \infty$. Indeed, in a manner similar to obtaining (29), we have

$$\sum_{\ell \neq n} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}} \geq \sum_{\ell \neq n, \ell \leq n} 2^{-\sigma \ell} \geq 1$$  

for any sufficiently large $n \geq 0$.

It follows that $\left\{ \sum_{\ell \neq n} 2^{sn - \ell^{-\tilde{r}} \max\{n, \ell\}} \right\}_{n \geq 0}$ is not in $\ell^q$ ($q < \infty$).
2.2. **Completion of the proof.** Now we can complete the proof of Theorem 1.2 by reducing the transfer operator \( L_{f,g^{(n)}} = L_{f,g}^{n} \) to a family of transfer operators on the local charts and applying Theorem 2.1 for each \( n \geq 1 \).

**Proof of Theorem 1.2.** For the time being, we shall fix \( n \geq 1 \) and suppress it from the notation. Let \( \iota \) be an isometric embedding from \( \mathcal{B}_{pq}^{s}(M) \) to \( \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}) \) (equipped with the norm \( \| (u_{i})_{i=1}^{I} \|_{\bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i})} = \sum_{i=1}^{I} \| u_{i} \|_{B_{pq}^{s}} \) given by

\[
i(u) = (\phi_{i} \circ u \circ \kappa_{i}^{-1})_{i=1}^{I} \quad u \in \mathcal{B}_{pq}^{s}(M).
\]

For \( 1 \leq i,j \leq I \), we let \( U_{ij} \equiv U_{n,ij} = \kappa_{ij}(V_{j} \cap (f^{n})^{-1}(V_{i})) \) and let \( F_{ij} \equiv F_{n,ij} \) be a \( \mathcal{C}^{r} \) mapping on \( \mathbb{R}^{d} \) such that

\[
F_{ij}(x) = \kappa_{i} \circ f^{n} \circ \kappa_{j}^{-1}(x) \quad \text{for } x \in U_{ij}.
\]

Furthermore, let \( G_{ij} \equiv G_{n,ij} \) be a compactly supported \( \mathcal{C}^{s} \) function on \( \mathbb{R}^{d} \) such that

\[
G_{ij}(x) = \phi_{i} \circ F_{ij}(x) \cdot \phi_{j}(x) \cdot g^{(n)} \circ \kappa_{j}^{-1}(x) \quad \text{for } x \in \mathbb{R}^{d}.
\]

By the existence of Markov partitions for expanding maps [21], one can find finitely many compact sets \( \{ K_{i} \}_{i=1}^{N(f^{n})} \) and their neighbourhoods \( \{ V_{i} \}_{i=1}^{N(f^{n})} \) satisfying (12) and (13). Finally we define \( \mathbf{L} \equiv \mathbf{L}_{n} : \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}) \to \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}) \) by

\[
\mathbf{L} ( (u_{i})_{i=1}^{I} ) = \left( \sum_{i=1}^{I} \mathbb{L}_{ij} u_{j} \right)_{i=1}^{I} \quad \text{for } (u_{i})_{i=1}^{I} \in \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}),
\]

(30)

where \( \mathbb{L}_{ij} \equiv \mathbb{L}_{n,ij} : B_{pq}^{s}(U_{j}) \to B_{pq}^{s}(U_{i}) \) for \( 1 \leq i,j \leq I \) is given by

\[
\mathbb{L}_{ij} u(x) = \sum_{F_{ij}(y) = x} G_{ij}(y) \cdot u(y) \quad \text{for } x \in \mathbb{R}^{d}.
\]

(31)

(More precisely, we define \( \mathbf{L} \) as a bounded operator on \( \bigoplus_{i=1}^{I} \mathcal{C}^{s}(U_{i}) \) by (30) and (31), and it can be extended to a bounded operator on \( \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}) \) by virtue of Theorem 2.1.) Then it is straightforward to see that the following commutative diagram holds:

\[
\begin{array}{ccc}
\mathcal{B}_{pq}^{s}(M) & \xrightarrow{L_{\mathbf{L}}^{s}} & \mathcal{B}_{pq}^{s}(M) \\
\bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}) & \xrightarrow{\mathbf{L}} & \bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i}).
\end{array}
\]

(32)

In particular, \( \| L_{f,g}^{n} u \|_{\mathcal{B}_{pq}^{s}} = \| \mathbf{L} \circ \iota (u) \|_{\bigoplus_{i=1}^{I} B_{pq}^{s}(U_{i})} \) for each \( u \in \mathcal{B}_{pq}^{s}(M) \).

It follows from Theorem 2.1 that one can find a constant \( \hat{C}_{n} > 0 \) and \( \sigma \in (0,s) \) such that for each \( 1 \leq i,j \leq I \) satisfying \( U_{ij} \neq \emptyset \) and each \( u \in B_{pq}^{s}(U_{j}) \), we get the bound of \( \| \mathbb{L}_{ij} u \|_{B_{pq}^{s}} \) by

\[
\gamma_{s} \sup_{x \in F_{ij}(U_{ij})} \left| \sum_{F_{ij}(y) = x} \frac{1}{| \det DF_{ij}(y) |} \right|^{1/p'} \left\| G_{ij} \det DF_{ij} \right\|_{L^{\infty}} \times \left( \min_{x \in U_{ij}} \min_{v \in \mathbb{R}^{d}} |DF_{ij}(x)v| \right)^{-\sigma} \| u \|_{B_{pq}^{s}} \left( \hat{C}_{n} \| u \|_{B_{pq}^{s}} \right)^{-s},
\]

(33)
where the second factor of the first term is replaced with 1 when \( p = 1 \). On the other hand, it is straightforward to see that there is a constant \( C_2 > 0 \) independently of \( n,i,j \) such that the first term of (33) is bounded by

\[
C_2 \gamma_s |\mathcal{L}_{f,i}^{1/p} \det Df|^{-1/2^n} \int_0^1 M^{1/2^n} R_n(g) \left( \min_{x \in \mathcal{M} \cap \mathcal{T}_{\mathcal{M}}} \min_{\frac{|v|}{|w|} = 1} |Df^n(x)v| \right)^{-s} \|u\|_{B_{pq}^s}. \tag{34}
\]

Moreover, it follows from the estimate (4.5) in [9] that \( \|\mathcal{L}_{f,i}^{1/p} \det Df|^{-1/2^n} \int_0^1 M^{1/2^n} \) is bounded by a constant \( C_3 > 1 \) which is independent of \( n \).

Therefore, by (32), we obtain the bound of \( \|\mathcal{L}_{f,i}^{1/p} u\|_{\mathcal{B}_{pq}^s} \) by

\[
IC_2 C_3^{1/p'} \gamma_s \left( \min_{x \in \mathcal{M} \cap \mathcal{T}_{\mathcal{M}}} \min_{\frac{|v|}{|w|} = 1} |Df^n(x)v| \right)^{-s} R_n(g) \|u\|_{\mathcal{B}_{pq}^s} + I\hat{C}_n \|u\|_{\mathcal{B}_{pq}^s}
\]

for any \( n \geq 1 \). Since the embedding \( \mathcal{B}_{pq}^s(M) \hookrightarrow \mathcal{B}_{pq}^s(\mathcal{M}) \) is compact (see [2, Corollary 2.96]), we can apply Hennion’s theorem [18] (see also [8, Appendix A]): we finally have

\[
\rho_{ess}(\mathcal{L}_{f,g}|_{\mathcal{B}_{pq}^s(\mathcal{M})}) \leq \liminf_{n \to \infty} \left( IC_2 C_3^{1/p'} \gamma_s \left( \min_{x \in \mathcal{M} \cap \mathcal{T}_{\mathcal{M}}} \min_{\frac{|v|}{|w|} = 1} |Df^n(x)v| \right)^{-s} R_n(g) \right)^{1/n}.
\]

This completes the proof of Theorem 1.2. \( \square \)

**Appendix A. Proof of Lemma 2.2.** We follow [12, 8]. Recall \( \{V_j\}_{j=1}^{N(F)} \) in (13).

Let \( \{\theta_j\}_{j=1}^{N(F)} \) be a partition of unity subordinate to \( \{V_j\}_{j=1}^{N(F)} \). Given \( 1 \leq j \leq N(F) \), let \( I_j \) be the set of integers \( i \) in \( \{1, \ldots, N(F)\} \) such that \( F(V_i) \cap V_j \neq \emptyset \). For \( 1 \leq j \leq N(F) \), let \( T_j : \mathbb{R}^d \to \mathbb{R}^d \) be a \( \mathcal{C}^r \) mapping such that

\[
T_j(x) = (F|V_j)^{-1}(x) \quad \text{for } x \in F(V_j),
\]

that \( T_j(x) \notin V_j \) if \( x \notin F(V_j) \), and that \( |T_j(x) - T_j(y)| > c_j |x - y| \) for any \( x, y \in \mathbb{R}^d \) with some constant \( c_j > 0 \). Then we have

\[
\mathcal{L}_{F,G} u = \sum_{j=1}^{N(F)} \sum_{i \in I_j} \theta_j \cdot (\theta_i \cdot G \cdot u) \circ T_i. \tag{35}
\]

Indeed, we have

\[
\sum_{j=1}^{N(F)} \sum_{i \in I_j} \theta_j(x) (\theta_i \cdot G \cdot u) \circ T_i(x) = \sum_{j:x \in V_j} \theta_j(x) \sum_{i \in I_j} (\theta_i \cdot G \cdot u) \circ T_i(x).
\]

If \( x \notin F(V_i) \), then \( T_i(x) \notin V_i \) and it yields \( (\theta_i \cdot G \cdot u) \circ T_i(x) = 0 \), thus

\[
\sum_{j:x \in V_j} \theta_j(x) \sum_{i \in I_j} (\theta_i \cdot G \cdot u) \circ T_i(x) = \sum_{j:x \in V_j} \theta_j(x) \sum_{i \in I_j, x \in F(V_i)} (\theta_i \cdot G \cdot u) \circ (F|V_i)^{-1}(x).
\]

On the other hand, for \( j \) such that \( x \in V_j, x \in F(V_i) \) implies \( i \in I_j \), so

\[
\sum_{i \in I_j, x \in F(V_i)} (\theta_i \cdot G \cdot u) \circ (F|V_i)^{-1}(x) = \sum_{i:x \in F(V_i)} (\theta_i \cdot G \cdot u) \circ (F|V_i)^{-1}(x).
\]
Furthermore, for $i$ such that $x \in F(V_i)$, setting $y = (F|_{V_i})^{-1}(x)$ gives $F(y) = x$ and $y \in V_i$, therefore

$$\sum_{i:x \in F(V_i)} (\theta_i \cdot G \cdot u) = \sum_{F(y) = x \cap y \in V_i} (\theta_i \cdot G \cdot u)(y) = \sum_{F(y) = x} (G \cdot u)(y).$$

Hence, we get (35), and it follows from Minkowski's inequality that

$$\|\Delta_n L_{F,G} \Delta \ell u\|_{L^p} \leq \sum_{j=1}^{N(F)} \sum_{i \in I_j} \|\Delta_n L_{ij} \Delta \ell u\|_{L^p},$$

where $L_{ij} = [\theta_j \cdot (\theta_i \cdot G) \circ T_i] \cdot u \circ T_i$ with a $\mathcal{C}^r$ function $\theta_j \cdot (\theta_i \cdot G) \circ T_i$ and a $\mathcal{C}^r$ mapping $T_i$.

Therefore, it suffices to show the following local version of Lemma 2.2: Let $T: \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{C}^r$ mapping such that $|T(x) - T(y)| > \tilde{c}|x - y|$ for any $x, y \in \mathbb{R}^d$ with some constant $\tilde{c} > 0$, and that $T: V \to T(V)$ is a $\mathcal{C}^r$ diffeomorphism on a bounded open set $V$. Let $\tilde{G}$ be a $\mathcal{C}^\tilde{r}$ function whose support is included in $V$. Define (with slight abuse of notation) $L_{T,\tilde{G}} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ by

$$L_{T,\tilde{G}} u = \tilde{G} \cdot u \circ T, \quad u \in L^p(\mathbb{R}^d).$$

Also, we use the notation $\ell \not\to n$ when it holds $2^n > \Lambda 2^{\ell+4}$, with

$$\Lambda = \max_{w \in \text{supp}(\tilde{G})} |\det DT\ell^\ell w(v)|,$$

where $DT\ell^\ell w(v)$ is the transpose matrix of $DT w$.

**Lemma A.1.** There exists a constant $C_{T,\tilde{G}} > 0$ such that for any $p \in [1, \infty]$, $u \in L^p(\mathbb{R}^d)$, $n \geq 1$ and $\ell \not\to n$, it holds

$$\|\Delta_n L_{T,\tilde{G}} \Delta \ell u\|_{L^p} \leq C_{T,\tilde{G}} 2^{-\tilde{r} \max\{n, \ell\}} \|u\|_{L^p}$$

when $\tilde{r} = r - 1$ and

$$\|\Delta_n L_{T,\tilde{G}} \Delta \ell u\|_{L^p} \leq C_{T,\tilde{G}} 2^{\min\{n, \ell\} - \tilde{r} \max\{n, \ell\}} \|u\|_{L^p}$$

when $\tilde{r} \geq 1$.

**Proof.** We start the proof by noticing that $\Delta_n L_{T,\tilde{G}} \Delta \ell : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is the composition $H^\ell_n \circ M_1$ of a bounded operator $M_1 : \Phi \mapsto \Phi \circ T \cdot |\det DT|$ on $L^p(\mathbb{R}^d)$, whose operator norm is bounded by a constant $C_T > 0$ (see (14)), and an integral operator $H^\ell_n : \Phi \mapsto \int_{\mathbb{R}^d} V^\ell_n (\cdot, y) \Phi(y)dy$ on $L^p(\mathbb{R}^d)$ with kernel

$$V^\ell_n (x, y) = \int_{\mathbb{R}^d} e^{i(x-w)\xi + i(T(w)-T(y))\eta} \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi dn.$$ (38)

We will give a bound of $V^\ell_n$ by a convolution kernel together with the factor $2^{-\tilde{r} \max\{n, \ell\}}$: Let $b : \mathbb{R}^d \to (0, 1]$ be the integrable function given by

$$b(x) = \begin{cases} 1 & (|x| \leq 1), \\ |x|^{-d-1} & (|x| > 1), \end{cases}$$

and set

$$b_m : \mathbb{R}^d \to \mathbb{R}, \quad b_m(x) = 2^{dm} \cdot b(2^m x) \quad \text{for } m > 0.$$ (40)

Then, we will show that there is a constant $\tilde{C}_{T,\tilde{G}} > 0$ such that for all $\ell \not\to n$,

$$|V^\ell_n (x, y)| \leq \tilde{C}_{T,\tilde{G}} 2^{-\tilde{r} \max\{n, \ell\}} b_{\min\{n, \ell\}} (x - y)$$

if $\tilde{r} \leq r - 1$, (41)
and
\[ |V_n^\ell(x, y)| \leq \tilde{C}_{T, G} 2^{\min\{n, \ell\} - \tilde{r} \max\{n, \ell\}} b_{\min\{n, \ell\}} (x - y) \quad \text{if } \tilde{r} > 1. \quad (42) \]

It follows from (41), by Young’s inequality, that
\[
\|H_n^\ell \Phi\|_{L^p} \leq \tilde{C}_{T, G} 2^{-\tilde{r} \max\{n, \ell\}} \|b_{\min\{n, \ell\}} \ast \Phi\|_{L^p} \leq \tilde{C}_{T, G} 2^{-\tilde{r} \max\{n, \ell\}} \|b_{\min\{n, \ell\}}\|_{L^1} \|\Phi\|_{L^p},
\]
which is bounded by \( \tilde{C}_{T, G} \|b\|_{L^1} 2^{-\tilde{r} \max\{n, \ell\}} \|\Phi\|_{L^p} \) since \( \|h_m\|_{L^1} = \|h\|_{L^1} \) for any \( m \geq 1 \), and we get (36) with \( C_{T, G} = C_T \tilde{C}_{T, G} \|b\|_{L^1} \). Similarly, (37) follows from (42). In the rest of the proof, we are dedicated to showing (41) and (42).

**Step 1.** We first prove (41). We further divide the proof of (41) according to whether \( \tilde{r} \) is an integer.

**Case (1-a). \( \tilde{r} \) is an integer.** We recall integrations by parts in \( w \): for each \( \mathcal{C}^2 \) function \( \phi : \mathbb{R}^d \to \mathbb{R} \) and a compactly supported \( \mathcal{C}^1 \) function \( h : \mathbb{R}^d \to \mathbb{R} \) with \( \sum_{j=1}^d (\partial_j \phi)^2 \neq 0 \) on the support of \( h \), we set
\[
h_k^\phi(w) = \frac{i \partial_k \phi(w) \cdot h(w)}{\sum_{j=1}^d (\partial_j \phi(w))^2} \quad k = 1, \ldots, d,
\]
belonging to \( C^1(\mathbb{R}^d) \). Then we get
\[
\int e^{i \phi(w)} h(w) dw = -\sum_{k=1}^d \int i \partial_k \phi(w) e^{i \phi(w)} \cdot h_k^\phi(w) dw = \int e^{i \phi(w)} \cdot \sum_{k=1}^d \partial_k h_k^\phi(w) dw,
\]
where \( w = (w_k)_{k=1}^d \in \mathbb{R}^d \) and \( \partial_k \) denotes partial differentiation with respect to \( w_k \).

**We call this operation integration by parts for \( h \) over \( \phi \). Define**
\[
T_\phi h = \sum_{k=1}^d \partial_k h_k^\phi, \quad \phi(x, y) = (x - w) \xi + (T(x) - T(y)) \eta.
\]

**We simply write \( T_\phi \) for \( T_{\phi(\xi, \eta, x, y)} \) (note that \( T_{\phi(\xi, \eta, x, y)} h \) does not depend on \( x \) and \( y \)). Then, we have**
\[
V_n^\ell(x, y) = \int e^{i \phi(\xi, \eta, x, y)}(w) T_{\tilde{r}} \tilde{G}(w) \psi_n(\xi) \dot{\psi}_\ell(\eta) dw d\xi d\eta,
\]
by integrating (38) by parts for \( \tilde{G} \) over \( \phi(\xi, \eta, x, y) \) \( \tilde{r} \) times (that is possible by definition of \( \tilde{r} \)).

Put
\[
\tilde{G}_{n, \ell}(\xi, \eta, w) = \mathcal{G}_{n, \ell}(2^n \xi, 2^\ell \eta, w) \quad \text{with} \quad \mathcal{G}_{n, \ell}(\xi, \eta, w) = T_{\tilde{r}} \tilde{G}(w) \psi_n(\xi) \dot{\psi}_\ell(\eta).
\]

Then, by scaling we have
\[
\mathcal{F}^{-1} \mathcal{G}_{n, \ell}(u, v, w) = 2^{dn+\ell} \mathcal{F}^{-1} \tilde{G}_{n, \ell}(2^n u, 2^\ell v, w),
\]
where \( \mathcal{F}^{-1} \) is the inverse Fourier transform with respect to the variable \( (\xi, \eta) \). Therefore, by (45) and (46), we have
\[
V_n^\ell(x, y) = \int \mathcal{F}^{-1} \mathcal{G}_{n, \ell}(x - w, T(w) - T(y), w) dw
\]
\[
= 2^{dn+\ell} \int \mathcal{F}^{-1} \tilde{G}_{n, \ell}(2^n (x - w), 2^\ell (T(w) - T(y)), w) dw.
\]
Next we will see that for any integers \( k, k' \geq 0 \), there is a constant \( C_{k,k'} > 0 \) such that
\[
|F^{-1}\tilde{G}_{n,\ell}(u,v,w)| \leq |u|^{-k}|v|^{-k'}C_{k,k'}2^{-\bar{\gamma} \max\{n,\ell\}}
\]
for \((u,v,w) \in \mathbb{R}^{3d}\). This immediately follows if one can see that for any multi-indices \( \alpha, \beta \), there is a constant \( C_{\alpha,\beta} > 0 \) such that
\[
|\partial^\alpha_x \partial^\beta_y \tilde{G}_{n,\ell}(\xi,\eta,w)| \leq C_{\alpha,\beta}2^{-\bar{\gamma} \max\{n,\ell\}}
\]
for \((\xi,\eta) \in \mathbb{R}^{3d}\), where \( \partial^\alpha_x = \partial^{\alpha_1}_x \partial^{\alpha_2}_x \cdots \partial^{\alpha_d}_x \) for \( \alpha = (\alpha_1,\ldots,\alpha_d) \in \{0,1,\ldots\}^d \).

(Just consider integration by parts with respect to \((\xi,\eta)\) for \( F^{-1}\tilde{G}_{n,\ell}(u,v,w) = (2\pi)^{-2d} \int e^{i(u\xi + v\eta)}\tilde{G}_{n,\ell}(\xi,\eta,w)d\xi d\eta \),

with noticing that the support of \( \tilde{G}_{n,\ell}(\cdot,\cdot,w) \) is included in \([-2,2]^2\). Hence, we will show (50).

We first note that, by induction with respect to \( r \),
\[
T^r_{\xi,\eta}G(w) = \frac{\Theta_{3\bar{r}}(\xi,\eta,w)}{\xi - DT^{3\bar{r}}(w)\eta}|\xi|^{3\bar{r}},
\]
with some function \( \Theta_{3\bar{r}} = \Theta_{3\bar{r},T,G} \), which is a polynomial function of degree \( 3\bar{r} \) in \( \xi \) and \( \eta \) whose coefficients are \( \mathcal{C}^0 \) in \( w \). Therefore, we have (by induction again) that
\[
|\partial^\alpha_x \partial^\beta_y T^r_{\xi,\eta}G(w)| \leq \frac{\Theta_{3\bar{r} + |\alpha| + |\beta|}(\xi,\eta,w)}{|\xi - DT^{3\bar{r}}(w)\eta|^{4\bar{r} + 2|\alpha| + 2|\beta|}}
\]
with some function \( \Theta_{3\bar{r} + |\alpha| + |\beta|}(\xi,\eta,w) \), which is a polynomial function of degree \( 3\bar{r} + |\alpha| + |\beta| \) in \( \xi \) and \( \eta \) whose coefficients are \( \mathcal{C}^0 \) in \( w \), where \( |\alpha| = \sum_j \alpha_j \) for each multi-index \( \alpha = (\alpha_1,\ldots,\alpha_d) \in \{0,1,\ldots\}^d \). Notice also that by definition, there is an integer \( N_1(T) > 0 \) such that for each \( \xi \in \text{supp}(\psi_n), \eta \in \text{supp}(\tilde{\psi}_\ell) \) and \( w \in \mathbb{R}^d \),
\[
|\xi - DT^{3\bar{r}}(w)\eta| \geq 2^{\max\{n,\ell\} - N_1(T)} \quad \text{if } \ell \neq n.
\]

Hence, for each multi-indices \( \alpha, \beta \) and \((\xi,\eta) \in \mathbb{R}^{3d} \) satisfying \( \psi_n(\xi) \cdot \tilde{\psi}_\ell(\eta) \neq 0 \) with \( \ell \neq n \), we have
\[
|\partial^\alpha_x \partial^\beta_y T^r_{\xi,\eta}G(w)| \leq \tilde{C}_{\alpha,\beta}2^{-(r+|\alpha| + |\beta|)\max\{n,\ell\}} \leq \tilde{C}_{\alpha,\beta}2^{-n|\alpha| - \ell|\beta| - r\max\{n,\ell\}},
\]
for some constant \( \tilde{C}_{\alpha,\beta} > 0 \). Since \( \|\partial^\alpha_x \psi_n\|_{L^\infty} \leq C_\alpha \rho 2^{-n|\alpha|} \) with some constant \( C_\alpha(\rho) > 0 \) (recall (5)), it follows that for each multi-indices \( \alpha, \beta \) and \((\xi,\eta, w) \in \mathbb{R}^{3d} \),
\[
|\partial^\alpha_x \partial^\beta_y G_{n,\ell}(w)| \leq C_{\alpha,\beta}2^{-n|\alpha| - \ell|\beta| - r\max\{n,\ell\}},
\]
for some constant \( C_{\alpha,\beta} > 0 \). Thus, we immediately obtain (50).

Combining (47) and (49), we get that \( |V^\ell_n(x,y)| \) is bounded by
\[
C'2^{dn+dt-\bar{\gamma}\max\{n,\ell\}} \int_K b(2^n(x-w))b(2^\ell(T(w)-T(y)))dw,
\]
with some constant \( C' > 0 \) and \( K = \text{supp } \tilde{G} \). If \( \ell \leq n \), we have
\[
2^dn \int_K b(2^n(x-w))b(2^\ell(T(w)-T(y)))dw \leq 2^{dn} \int_K b(2^n(x-w))dw = \int b(u)du
\]
by setting \( u = 2^n(x-w) \). Hence, with \( C'_1 = C'||b||_{L^1} \), we get
\[
|V^\ell_n(x,y)| \leq C'_12^{dn-\bar{\gamma}\max\{n,\ell\}} \leq C'_12^{dn \min\{n,\ell\} - \bar{\gamma}\max\{n,\ell\}}.
\]
In the other case (when $\ell > n$), by setting $\tilde{u} = 2^{\ell}(T(w) - T(y))$, in a similar manner one can see that $|V_n(x, y)| \leq C_2 2^{d(n-\ell) \max\{n, \ell\}} \leq C'_2 2^{d \min\{n, \ell\} - \ell \max\{n, \ell\}}$ with some constant $C'_2 \equiv C_{2, T} > 0$.

Recalling (39), the above estimates imply (41) for $|x - y| \leq 2^{-\min\{n, \ell\}}$. So, we now assume that $|x - y| > 2^{-\min\{n, \ell\}}$. We consider the case $\ell \leq n$ (the other case can be dealt similarly, as above). Let $q_0 = \lfloor -\log_2 |x - y| \rfloor$, so that either $|x - w| \geq 2^{-q_0 - 1}$ or $|w - y| \geq 2^{-q_0 - 1}$ holds for any $w \in \mathbb{R}^d$. Hence with the notation $K_1 = \{w \in K \mid |x - w| \geq 2^{-q_0 - 1}\}$ and $K_2 = \{w \in K \mid |w - y| \geq 2^{-q_0 - 1}\}$, noting that $w \in K_2$ implies $|T(w) - T(y)| \geq 2^{-q_0 - 1}$, we have the bound of $\int_K b(2^n(x - w))b(2^{\ell}(T(w) - T(y)))dw$ by

$$
\begin{aligned}
&\int_{K_1} |2^n(x - w)|^{-d-1}b(2^{\ell}(T(w) - T(y)))dw \\
&+ \int_{K_2} b(2^n(x - w))b(2^{\ell}(w - y))^{-d-1}dw \\
\leq C_4'(2^{-d\ell -(n-q_0-1)(d+1)} + 2^{-d\ell -(\ell-q_0-1)(d+1)}) \leq C_4' 2^{-dn}b(2^{-\ell}(x - y))
\end{aligned}
$$

with some constants $C'_4 \equiv C'_{3, T, d} > 0$ and $C'_4 \equiv C'_{4, T, d} > 0$, which completes the proof of (41) due to (54).

**Case (1-b).** $\tilde{r}$ is not an integer. We recall regularised integration by parts in $w$: for each $C^{1+\delta}$ function $\phi : \mathbb{R}^d \to \mathbb{R}$ and a compactly supported $C^\delta$ function $h : \mathbb{R}^d \to \mathbb{R}$, for $0 < \delta < 1$, with $\sum_{\ell=1}^d (\partial_\ell \phi)^2 \neq 0$ on the support of $h$, each $h^\phi_k$ given in (43) belongs to $C^\delta_0(\mathbb{R}^d)$ for $k = 1, \ldots, d$. Let $h_k^\phi := h_k^\phi \ast v_\epsilon$ for $\epsilon > 0$, where $v_\epsilon(x) = e^{-\epsilon v(x/\epsilon)}$ with a $C^\infty$ function $v : \mathbb{R}^d \to \mathbb{R}_+$ supported in the unit ball and satisfying $\int v(x)dx = 1$. There is a constant $C_v > 0$, independent of $\phi$, such that for each $C^\delta$ function $u : \mathbb{R}^d \to \mathbb{R}$ and each small $\epsilon > 0$,

$$
\|\partial_k(u \ast v_\epsilon)\|_{C^\delta} \leq C_v \epsilon^{1-\delta} \|u\|_{C^\delta}, \quad \|u - u \ast v_\epsilon\|_{C^\delta} \leq C_v \epsilon^{\delta} \|u\|_{C^\delta}.
$$

Finally, for every real number $L \geq 1$, it holds

$$
\begin{aligned}
&\int e^{iL\phi(w)}h(w)dw = -\sum_{k=1}^d \int i\partial_k \phi(w)e^{iL\phi(w)} \cdot h_k^\phi(w)dw \\
= &\int \frac{e^{iL\phi(w)}}{L} \cdot \sum_{k=1}^d \partial_k h_k^\phi dw - \sum_{k=1}^d \int i\partial_k \phi(w)e^{iL\phi(w)} \cdot (h_k^\phi(w) - h_k^\phi(w))dw.
\end{aligned}
$$

(Compare with (44).) We call this operation regularised integration by parts for $h$ over $\phi$. Define

$$
\mathcal{T}_\phi^{(0, L)} h = L^{-1} \sum_{k=1}^d \partial_k h_k^\phi_{L-1}, \quad \mathcal{T}_\phi^{(1, L)} h = -i \sum_{k=1}^d \partial_k \phi \cdot (h_k^\phi - h_k^\phi_{L-1}),
$$

and denote $\mathcal{T}_\phi^{(0, L)}$ and $\mathcal{T}_\phi^{(1, L)}$ by $\mathcal{T}_\phi^{(0, L)}$ and $\mathcal{T}_\phi^{(1, L)}$, respectively. Then, by integration of (38) by parts for $G$ over $\phi_{[\xi, \eta, x, y]}^{[\tilde{r}]}$ times and regularised integration by parts over $\phi_{[\xi, \eta, x, y]}^{[\tilde{r}]}$ with $\delta = \tilde{r} - [\tilde{r}]$ and $\epsilon = L^{-1}$ (note that
Case (2-a).  We show (42), according to whether ˜

\[ V(0,1) = V_n(0,1) \]

\[ \text{and} \quad V(1,1) = V_n(1,1) \]

which we denote by \( V_n(0,1) \) \((x, y) \) and \( V_n(1,1) \) \((x, y) \).

It follows from (55) and (57) that (with \( L^{-1} = \epsilon \))

\[ |\partial^2_\xi \partial^2_\eta T^{(0,1)}(\xi, \eta) h(w) | \leq \epsilon \sum_{k=1}^d \left| \partial_k \left( (\partial^2_\xi \partial^2_\eta h_k(\xi, \eta, x, y) \ast (w) \right) \right| \]

\[ \leq C_\epsilon \delta \sum_{k=1}^d \| \partial^2_\xi \partial^2_\eta h_k(\xi, \eta, x, y) \| \in C^{\alpha} \]  

In a similar manner, we get the bound of \( |\partial^2_\xi \partial^2_\eta T^{(1,1)}(\xi, \eta) h(w) | \) by

\[ \left. C_\epsilon \delta \sum_{k=1}^d \sum_{\{ \alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)} \}} \| \partial^2_\xi \partial^2_\eta h_k(\xi, \eta, x, y) \| \in C^{\alpha} \right. \]  

where the second summation is taken over multi-indices \((\alpha^{(1)}, \beta^{(1)}, \alpha^{(2)}, \beta^{(2)}) \in \{0, 1, \ldots \}^4 \) such that \( \alpha^{(1)} + \alpha^{(2)} = \alpha \) and \( \beta^{(1)} + \beta^{(2)} = \beta \).

On the other hand,

\[ \partial_k \phi(\xi, \eta, x, y) = \epsilon \Theta_1(w, \xi, \eta), \quad h_k(\xi, \eta, x, y) = \frac{ie^{-1} \Theta_1(w, \xi, \eta) h(w)}{\| \xi - DT^{-1}(w) \eta \|} \]

where \( \Theta_1(w, \xi, \eta) \) is the \( k \)-th element of \( -\xi + DT(\eta \eta) \), which is a polynomial function of degree 1 in \( \xi \) and \( \eta \) whose coefficients are \( \mathcal{C}^{-1} \) (at least \( \mathcal{C}^{\epsilon} \)) in \( w \).

Therefore, applying (59) and (62) for \( h = T^{[\eta]} \), together with (52), we get

\[ |\partial^2_\xi \partial^2_\eta T^{(j,1)}(\xi, \eta) T^{[\eta]}(\xi, \eta) h(w) | \leq C_\epsilon C_{\alpha, \beta} \delta^{2-\max\{|\alpha| - |\beta| - |\eta| \}} \]  

for any \( \xi \in \text{supp } \psi_n \) and \( \eta \in \text{supp } \psi_k \) with \( \ell \text{ } \Rightarrow \text{ } n \), implying that, with \( \epsilon = 2^{-\max\{n, \ell \}} \) and \( \delta = \epsilon \text{ } \Rightarrow \text{ } [\eta] \),

\[ |\partial^2_\xi \partial^2_\eta T^{(j,1)}(\xi, \eta) T^{[\eta]}(\xi, \eta) h(w) | \leq C_\epsilon C_{\alpha, \beta} \delta^{2-\max\{|\alpha| - |\beta| - |\eta| \}} \]  

Therefore, in a manner similar to one in the case (1-a) (replacing \( V_n(0, L) \) with \( V_n(1, L) \) and \( V_n(1, L) \) , one can get (41).

**Step 2.** Next we show (42), according to whether \( \tilde{r} \) is an integer.

**Case (2-a). \( \tilde{r} \text{ is an integer.**** Recall \( V_n(0, L) \) and \( b_n \) from (40). For each \((\xi, \eta, x, y)\), define

\[ \phi_1(\xi, \eta, x, y)(w) = (x - w) \xi + DT(y)(w - y) \eta, \quad \mathcal{R}_{\phi_1, \phi_2} h = e^{i(\phi_1 - \phi_2)} h, \]  

for continuous functions \( \phi_1, \phi_2 : \mathbb{R}^d \rightarrow \mathbb{R} \). We simply write \( \mathcal{R}_0 \) and \( \mathcal{R}_0^{-1} \) for \( \mathcal{R}_{\phi_1, \phi_2} \) and \( \mathcal{R}_{\phi_2, \phi_1} \), respectively, with \( \phi_1 = \phi_1(\xi, \eta, x, y) \) and \( \phi_2 = \phi_2(\xi, \eta, x, y) \). Denote \( \mathcal{T}_{\phi_1(\xi, \eta, x, y)} T^{(0, L)}(\xi, \eta, x, y) \) and \( \mathcal{T}_{\phi_1(\xi, \eta, x, y)} T^{(1, L)}(\xi, \eta, x, y) \) by \( \mathcal{T}_{\xi, \eta, y} T^{(0, L)}(\xi, \eta, x, y) \) and \( \mathcal{T}_{\xi, \eta, y} T^{(1, L)}(\xi, \eta, x, y) \), respectively. Then
(38) can be rewritten as
\[ V_n^\ell(x, y) = \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta. \] (64)

Note that \( \tilde{\phi}(\xi, \eta, x, y) : \mathbb{R}^d \to \mathbb{R} \) is of class \( \mathcal{C}^\infty \) and \( R_0 \tilde{G} \) is \( \mathcal{C}^{\tilde{r}} \) because \( \tilde{r} < r \), so that \( \tilde{T}_{\xi,\eta, y} R_0 \tilde{G} : \mathbb{R}^d \to \mathbb{R} \) is well-defined and of class \( \mathcal{C}^{\tilde{r}-1} \).

Integrating (64) by parts once on \( w \) (for \( R_0 \tilde{G} \) over \( \tilde{\phi}(\xi, \eta, x, y) \)), we obtain
\[ V_n^\ell(x, y) = \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta \] (65)
\[ = \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta. \] (66)

Integrating (65) by parts \( \tilde{r} - 1 \) times on \( w \), we obtain
\[ V_n^\ell(x, y) = \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} \tilde{T}_{\xi,0}^{\tilde{r}-1} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta. \]

In a manner similar to the case (1-a), it can be shown that
\[ \left\| \partial_\xi^\alpha \partial_\eta^\beta \tilde{T}_{\xi,\eta}^{\tilde{r}-1} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G} \right\|_{L_\infty} \leq \tilde{C}_{\alpha, \beta} 2^{-n|\alpha| - \ell|\beta| - \tilde{r} \max\{n, \ell\}}, \] (67)

because one can have, by induction, that
\[ \tilde{T}_{\xi,\eta}^{\tilde{r}-1} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G} = \frac{\Theta^{(2)}_{3(\tilde{r}-1)+1}(\xi, \eta, w) + \Theta^{(3)}_{3(\tilde{r}-1)+1}(\xi, \eta, w)\eta}{|\xi - DT^r(w)| \eta \| \xi - DT^r(y) \eta \|_2}, \] (68)

with some functions \( \Theta^{(2)}_{3(\tilde{r}-1)+1} \) and \( \Theta^{(3)}_{3(\tilde{r}-1)+1} \), which are polynomial functions of degree \( 3(\tilde{r} - 1) + 1 \) in \( \xi \) and \( \eta \) whose coefficients are \( \mathcal{C}^0 \).

We considered \( \tilde{T}_{\xi,\eta}^{\tilde{r}-1} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \) instead of \( \tilde{T}_{\xi,\eta}^{\tilde{r}} \) since \( \tilde{T}_{\xi,\eta}^{\tilde{r}} \) is not well-defined when \( \tilde{r} > r - 1 \). The price we have to pay for this change is the factor \( \eta \) in (68), resulting in \( 2^\ell \) in (67). By definition of \( \ell \neq n, 2^\ell \) is dominated by \( 2^{\min\{n, \ell\}} \) multiplied by a constant. Consequently, as in the case (1-a), one can see that (67) implies (42).

**Case (2-b).** \( \tilde{r} \) is not an integer. We perform single integration by parts of (38) for \( \tilde{\phi}(\xi, \eta, x, y) \), \( \tilde{r} \) - 1 integration by parts over \( \phi \), and a single regularisation integration by parts over \( \tilde{\phi}(\xi, \eta, x, y) \) with \( \delta = \tilde{r} - \tilde{r} \), resulting in
\[ V_n^\ell(x, y) = \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} \tilde{T}_{\xi/\delta, \eta/\delta, y}^{(0, L)} R_0 \tilde{T}_{\xi,\eta}^{(\tilde{r}-1)} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta \]
\[ + \int e^{i\tilde{\phi}(\xi, \eta, x, y)(w)} \tilde{T}_{\xi/\delta, \eta/\delta, y}^{(1, L)} R_0 \tilde{T}_{\xi,\eta}^{(\tilde{r}-1)} R_0^{-1} \tilde{T}_{\xi,\eta, y} R_0 \tilde{G}(w) \tilde{\psi}_n(\xi) \tilde{\psi}_\ell(\eta) dw d\xi d\eta. \]

Then in view of the cases (1-b) and (2-a), we obtain (42).

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