Sequential static-dynamic hedging for long-term derivatives

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Abstract

This paper presents a new methodology for hedging long-term financial derivatives written on an illiquid asset. The proposed hedging strategy combines dynamic trading of a correlated liquid asset (e.g. the market index) and static positions in market-traded options such as European puts and calls. Moreover, since most market-traded options are relatively short-term, it is necessary to conduct the static hedge sequentially over time till the long-term derivative expires. This sequential static-dynamic hedging strategy leads to the study of a stochastic control problem and the associated Hamilton-Jacobi-Bellman PDEs and variational inequalities. A series of transformations allow us to simplify the problem and compute the optimal hedging strategy.

Keywords: portfolio optimization, employee stock options, optimal stopping, Hamilton-Jacobi-Bellman PDE, variational inequality
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1. Introduction

In the standard no-arbitrage pricing theory, option positions are assumed to be hedged perfectly by continuously trading the underlying asset. The option price is computed from the conditional expectation of discounted payoff under a unique risk-neutral pricing measure. However, in many financial applications, the underlying asset is non-traded. Some examples include weather derivatives [1], employee stock options [2, 3, 4], options on illiquid assets [5, 6]. Instead, derivatives holders manage their risk exposure by trading some liquid assets correlated with the underlying. One candidate hedging instrument is the market index, whose liquidity and relatively low transaction cost permit dynamic trading. Also, standard market-traded options can be used as additional hedging instruments. However, high transaction costs discourage frequent option trades, so recent work (for example [7], [8], and [9]) has focused on static hedging with options, which involves purchasing a portfolio of standard options at initiation and no trades afterwards.

This paper presents a new methodology for hedging long-term options written on a non-traded asset. Specifically, the hedging strategy proposed herein combines dynamic trading of a correlated liquid asset (e.g. the market index) and static positions in market-traded options. Moreover, since most market-traded options are relatively short-term, it is sensible to conduct static hedging sequentially over time till the long-term option expires. For this reason, the proposed strategy is called sequential static-dynamic hedging.

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The sequential static-dynamic hedging mechanism is applicable to both European and American options. One practical financial example is the hedging of employee stock options (ESOs). ESOs considered here are European or American options written on the firm’s stock. The ESO holder (employee) cannot trade the firm’s stock. Instead, he/she dynamically invests in the market index, and buy-and-hold market-traded put options over time. Since the market is incomplete, we will adopt a utility maximization approach to determine the optimal static positions at different times, along with the optimal dynamic trading strategy.

2. Model Formulation

The financial market consists of a riskless bank account, a market index $S$, and the firm’s stock $Y$. The prices of $S$ and $Y$ are modeled as lognormal processes

\[(\text{traded}) \quad dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \]
\[(\text{non-traded}) \quad dY_t = (\nu - q)Y_t \, dt + \eta Y_t (\rho dW_t + \rho' \, d\hat{W}_t), \]

where the processes $W$ and $\hat{W}$ are two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t$ is the augmented $\sigma$-algebra generated by $(W, \hat{W})$. Also, $\rho \in (-1, 1)$ and $\rho' = \sqrt{1 - \rho^2}$. We denote the Sharpe ratios of the market index and the firm’s stock respectively by

$$\lambda = \frac{\mu - r}{\sigma}, \quad \xi = \frac{\nu - q - r}{\eta}.\]

The ESO is a call option written on the firm’s stock with finite maturity $T$. We shall consider European ESOs and American ESOs separately in Sections 3 and 4 respectively.

The employee cannot trade the firm’s stock but dynamically invests in the market index $S$ and the bank account to partially hedge his position. With a dynamic investment strategy $\theta$, the employee’s trading wealth evolves according to the process

$$dX^\theta_t = [\theta_t (\mu - r) + r X^\theta_t] \, dt + \theta_t \sigma \, dB_t, \quad X_0 = x, \quad X_t = x,$$

where $\theta$ represents the cash amount invested in $S$. The set of all admissible strategies, $\Theta_t$, consists of all self-financing $\mathcal{F}_t$-progressively measurable processes $(\theta_t)_{t \geq 0}$ such that the integrability condition $E \{ \int_0^T \theta^2_t \, dt \} < \infty$. For $0 \leq s \leq t \leq T$, we denote by $\Theta_{s,t}$ the set of admissible strategies over the period $[s, t]$.

In addition to dynamic trading, the employee also purchases from the market some put options written on the firm’s stock. For the moment, let us consider only European puts all with the same strike $K$, though in reality there is a wide array of options to choose from, and the choices can be path dependent\(^1\). To avoid arbitrage opportunities, we assume that the market price of the market-traded puts lie within the no-arbitrage bounds. Since the underlying asset $Y$ follows a geometric Brownian motion, it makes sense to set the market price to be the Black-Scholes put option price, denoted by $\pi(t, y)$.

After every purchase, the employee will hold the put options till expiration. Typically the market-traded options have short maturities, so the employee will repeat this buy-and-hold strategy several times till the ESO expires. To this end, let the maturity of the market-traded puts be $\Delta t = T/N$ where $N$ is some positive integer. Denote $t_n = n \Delta t$, for $n = 0, 1, \ldots, N - 1$. Therefore, the employee make option purchases at times $[0, t_1, t_2, \ldots, t_{N-1}]$.

The market is incomplete, so we adopt a utility maximization approach to determine the optimal hedges. In particular, we represent the employee’s risk preference by an exponential utility function

$$U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R},$$

with a constant absolute risk aversion $\gamma > 0$. We interpret $U(x)$ as the employee’s utility of having wealth $x$ time $T$.

\(^1\)For example, the strike can be a $\mathcal{F}_t$-measurable random variable, such as $g(Y_t)$, instead of a constant $K'$.\]
3. Sequential Static-Dynamic Hedging for European ESOs

In this section, we discuss the static-dynamic hedging of a European ESO with payoff \( C(Y_T) = (Y_T - K)^+ \). We shall define the ESO holder’s value function recursively backward in time. To start with, suppose at time \( t_{N-1} \), the employee is holding an ESO, along with \( b_{N-1} \) units of put options which will expire at time \( T \) with payoff \( b_{N-1}^*D(Y_T) = b_{N-1}(K^* - Y_T)^+ \). Therefore, the employee’s value function at time \( t_{N-1} \) is given by

\[
V^{(N-1)}(t_{N-1}, x, y; b_{N-1}) = \sup_{\theta_{t_{N-1}, T}} E \left[ U(X_T) + C(Y_T) + b_{N-1}D(Y_T) \mid X_{t_{N-1}} = x, Y_{t_{N-1}} = y \right].
\]  

(2)

Now, accounting for the market price \( \pi \) of the puts at time \( t_{N-1} \), the employee chooses \( b_{N-1} \) so as to maximize the value function:

\[
b^*_n(x, y) = \arg \max_{0 \leq b < \infty} V^{(N-1)}(t_{N-1}, x - b\pi(t_{N-1}, y), y ; b).
\]  

(3)

For convenience, we write the value function and indifference price corresponding to this optimal static hedge as

\[
V^{(N-1)}(t_{N-1}, x, y) = V^{(N-1)}(t_{N-1}, x, y - b^*_n\pi(t_{N-1}, y), y ; b^*_n),
\]  

(4)

\[
p^{(N-1)}_n(t_{N-1}, y) = p^{(N-1)}_n(t_{N-1}, x, y ; b^*_n) - b^*_n\pi(t_{N-1}, y).
\]  

(5)

It is often better for intuitive purposes to work with indifference prices, which we will define next. To do so, we first consider the investment problem in which the risk-averse employee dynamically trades in the market index and bank account without any options till \( T \). This well-studied problem is first introduced by [10]. The employee’s maximal expected utility, called the Merton function, is given by

\[
M(t, x) = \sup_{\theta_{t, T}} E[U(X_T) \mid X_t = x] = -e^{-\gamma r}e^{-\frac{r}{2}(T-t)}.
\]

(6)

At anytime \( t \in [t_n, t_{n+1}) \), the employee’s indifference price \( p^{(n)}(t, x, y; b_n) \) for holding the ESO and \( b_n \) puts is defined by the equation

\[
V^{(n)}(t, x, y; b_n) = M \left( t, x + p^{(n)}(t, x, y; b_n) \right),
\]  

(7)

where \( V^{(n)}(t, x, y; b_n) \) is the value function for time interval \([t_n, t_{n+1})\) (to be defined in (10)). This defining equation allows us to express the optimal static position \( b^*_n \) in (3) in terms of the indifference price.

\[
b^*_n(x, y) = \arg \max_{0 \leq b < \infty} p^{(n)}(t_n, x, y ; b) - b\pi(t_n, y).
\]  

(8)

Therefore, the optimal static position is found from the Fenchel-Legendre transform of the employees indifference price as a function of the number of puts, evaluated at the market price. As in (4) and (5), we denote the value function and indifference price corresponding to this optimal static hedge by

\[
V^{(n)}(t, x, y; b_n) = V^{(n)} \left( t, x - b^*_n\pi(t, y), y ; b^*_n \right),
\]  

(9)

\[
p^{(n)}_n(t, y) = p^{(n)}_n(t, x, y ; b^*_n) - b^*_n\pi(t, y).
\]

(10)

Starting with the value function \( V^{(N-1)} \), we move backward in time to derive the value functions \( V^{(N-2)}, V^{(N-3)}, \ldots, V^{(0)} \). Indeed, the employee’s value function at time \( t \in [t_n, t_{n+1}) \), for \( n = N-2, N-3, \ldots, 0 \), is given by

\[
V^{(n)}(t, x, y; b_n) = \sup_{\theta_{t_n t_{n+1}}} E \left[ V^{(n+1)}_n(t_{n+1}, X_{t_{n+1}} + b_nD(Y_{t_{n+1}}), Y_{t_{n+1}}) \mid X_t = x, Y_t = y \right]
\]

(11)

\[
= \sup_{\theta_{t_n t_{n+1}}} E \left[ M \left( t_{n+1}, X_{t_{n+1}} + p^{(n+1)}_n(t_{n+1}, Y_{t_{n+1}}) + b_nD(Y_{t_{n+1}}) \right) \mid X_t = x, Y_t = y \right],
\]

where \( V^{(n+1)}_n \) and \( p^{(n+1)}_n \) are defined in (8) and (9) respectively.
3.1. A Recursive System of PDEs

The value functions \( \{ V^{(N-1)}, V^{(N-2)}, \ldots, V^{(0)} \} \) lead us to study a system of PDEs. To do so, let us introduce the differential operators:

\[
L = \frac{\eta^2 y^2}{2} \frac{\partial^2}{\partial y^2} + \rho \theta x \eta y \frac{\partial^2}{\partial x \partial y} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + (v - q)y \frac{\partial}{\partial y} + [\theta(\mu - r) + r x] \frac{\partial}{\partial x},
\]

\[
L^E u = \frac{\eta^2 y^2}{2} \frac{\partial^2 u}{\partial y^2} + (v - q - \rho \mu \eta) y \frac{\partial u}{\partial y},
\]

\[
\mathcal{A}^q u = \frac{\partial u}{\partial t} + L^E u - ru - \frac{1}{2} \gamma (1 - \rho^2) y^2 e^{r(T-t)} (\frac{\partial u}{\partial y})^2.
\]

The operator \( L \) is the infinitesimal generator of \((X, Y)\), \( L^E \) is the infinitesimal generator of \( Y \) under the minimal entropy martingale measure, \( Q^E \), and the last operator \( \mathcal{A}^q \) is \textit{quasilinear}.

First, the value function \( V^{(N-1)}(t, x, y; b) \) is conjectured to solve the following HJB PDE

\[
V^{(N-1)}_t + \sup_{\theta} L V^{(N-1)} = 0,
\]

for \( (t, x, y) \in [t_{N-1}, T) \times \mathbb{R} \times (0, +\infty) \). The boundary conditions are

\[
V^{(N-1)}(T, x, y; b) = -e^{-\gamma(x+C(y)+bD(y))},
\]

\[
V^{(N-1)}(t_0, x, y; b) = -e^{-\gamma(x+C(y)+bD(y))} e^{-\frac{\rho \theta \sigma}{2 r} (T-t)}.
\]

Due to the exponential utility function, the value function has a separation of variables (see [5]):

\[
V^{(N-1)}(t, x, y; b) = M(t, x) \cdot H^{(N-1)}(t, y; b) e^{\frac{1}{2} \rho \theta \sigma (T-t)}.
\]

The function \( H^{(N-1)} \) solves a linear PDE

\[
H^{(N-1)}_t + L^E H^{(N-1)} = 0,
\]

for \( (t, y) \in [t_{N-1}, T) \times (0, +\infty) \), with boundary conditions

\[
H^{(N-1)}(T, y; b) = e^{-\gamma(1-\rho^2)(C(y)+bD(y))}, \quad \text{and} \quad H^{(N-1)}(t_0, y; b) = e^{-\gamma(1-\rho^2)bK}.
\]

By the definition of indifference price, this function \( H^{(N-1)} \) is connected to the indifference price \( p^{(N-1)} \) in the following way:

\[
p^{(N-1)}(t, y; b) = -\frac{1}{\gamma(1-\rho^2)e^{r(T-t)}} \log H^{(N-1)}(t, y; b), \quad t \in [t_{N-1}, T].
\]

which gives

\[
V^{(N-1)}(t, x, y; b) = M(t, x) \cdot e^{-\gamma p^{(N-1)}(t,y; b)e^{r(T-t)}},
\]

and the following PDE

\[
\mathcal{A}^q p^{(N-1)} = 0,
\]

with boundary conditions

\[
p^{(N-1)}(t, y; b) = C(y) + bD(y), \quad \text{and} \quad p^{(N-1)}(t_0, y; b) = bKe^{-r(T-t)}.
\]

In practice, we numerically solve the PDE for \( H^{(N-1)}(t, y; b) \) and use (12) to derive the indifference price \( p^{(N-1)}(t, y; b) \). Then, we optimize over \( b \) (as in (7)) to obtain the employee’s optimal static hedges \( b^*_t \) at time \( t_{N-1} \), along with the corresponding \( V^{(N-1)}(t, x, y), H^{(N-1)}(t, y) \) and \( p^{(N-1)}(t, y) \), which will appear in the terminal conditions for \( V^{(N-2)}(t, x, y; b) \). All these allow us to iterate backward to derive the PDEs for \( V^{(n)}(t, x, y; b) \) and \( p^{(n)}(t, y; b) \) for \( n = N-2, N-3, \ldots, 0 \). To be precise, at time \( t \in [t_n, t_{n+1}) \), the value function \( V^{(n)}(t, x, y; b) \) satisfies the HJB PDE

\[
V^{(n)}_t + \sup_{\theta} L V^{(n)} = 0,
\]
for \((t, x, y) \in [t_n, t_{n+1}] \times \mathbb{R} \times (0, +\infty)\). The boundary conditions are
\[
V^{(n)}(t_{n+1}, x, y; b) = V^{(n+1)}_s(t_{n+1}, x + bD(y), y), \\
V^{(n)}(t, x, 0; b) = -e^{-\gamma(T-t)}bK e^{-\frac{\mu^2}{2\sigma^2}(T-t)}.
\]
Again, we can simplify the PDE by the transformation
\[
V^{(n)}(t, x, y; b) = M(t, x) \cdot H^{(n)}(t, y; b) e^{\frac{1}{1-\rho^2}(T-t)},
\]
The function \(H^{(n)}(t, y; b)\) satisfies the linear PDE
\[
H^{(n)}_t + \mathcal{L}^E H^{(n)} = 0,
\]
for \((t, y) \in [t_n, t_{n+1}] \times (0, +\infty)\), with boundary conditions
\[
H^{(n)}(t_{n+1}, y; b) = e^{-\gamma(1-\rho^2)bD(y)}H^{(n+1)}_s(t_{n+1}, y) \quad \text{and} \quad H^{(n)}(t, 0; b) = e^{-\gamma(1-\rho^2)bK}.
\]
The indifference price is given by
\[
p^{(n)}(t, y; b) = -\frac{1}{\gamma(1-\rho^2)e^{\rho(T-t)}} \log H^{(n)}(t, y; b), \quad t \in [t_n, t_{n+1}).
\]
which yields the PDE
\[
\mathcal{H}^{q(t)} p^{(n)} = 0,
\]
for \((t, y) \in [t_n, t_{n+1}] \times (0, +\infty)\), with boundary conditions
\[
p^{(n)}(t_{n+1}, y; b) = p^{(n+1)}_s(t_{n+1}, y) + bD(y), \quad \text{and} \quad p^{(n)}(t, 0; b) = bK e^{-\rho(T-t)}.
\]

4. Sequential Static-Dynamic Hedging for American ESOS

In this section, we investigate the sequential hedging problem for an American ESO. Our ultimate objectives are to analyze the optimal static positions over time, and examine the nontrivial effect of static hedges on the optimal ESO exercising strategy. We denote by \(T\) the set of all stopping times with respect to \(\mathbb{F}\) taking values in \([0, T]\). This will be the collection of all admissible exercise times for the ESO. For \(s, u \in T\) with \(s \leq u\), we denote the set of stopping times in between by \(T_{su} := \{\tau \in T : s \leq \tau \leq u\}\).

We assume that the employee will reinvest the ESO proceeds, if any, into the dynamic trading portfolio. Moreover, after the ESO exercise, there is no need for future static hedges, so the sequential static hedging will terminate by the next expiration date. Precisely, if the ESO has been exercised at time \(t \in (t_n, t_{n+1})\), \(n \in \{0, 1, \ldots, N-1\}\), then the employee, who still holds some \(a\) units of European puts expiring at \(t_{n+1}\), faces the investment problem
\[
u^{(n)}(t, x, y; a) = \sup_{\theta_{t_{n+1}}} E\left[M(t_{n+1}, X_{t_{n+1}} + aD(Y_{t_{n+1}})) \mid X_t = x, Y_t = y\right].
\]
This is a standard utility maximization problem with European puts. We can express it in terms of the indifference price for the puts, denoted by \(h^{(n)}(t, y; a)\):
\[
u^{(n)}(t, x, y; a) = M(t, x + h^{(n)}(t, y; a)).
\]
Now suppose the employee is holding the American ESO at \(t_{N-1}\), along with \(a_{N-1}\) units of put options. Then, the value function is given by
\[
u^{(N-1)}(t_{N-1}, x, y; a_{N-1}) = \sup_{\tau \in T_{t_{N-1}, T}} \sup_{\theta_{t_{N-1}}} E\left[M\left(\tau, X_{\tau} + C(Y_{\tau}) + h^{(N-1)}(\tau, Y_{\tau}; a_{N-1})\right) \mid X_{t_{N-1}} = x, Y_{t_{N-1}} = y\right].
\]
Now, accounting for the market price $\pi$ of the puts at time $t_{N-1}$, the employee chooses $a_{N-1}$ so as to maximize the value function:

$$a_{N-1}^*(x,y) = \arg \max_{0 \leq a < \infty} \hat{V}^{(N-1)}(t_{N-1}, x - a\pi(t_{N-1}, y), y; a).$$  \hfill (19)

For any time $t \in [t_n, t_{n+1})$, the employee’s indifference price $\hat{p}_n(t, x, y; a_n)$ for holding the American ESO and $a_n$ puts is defined by the equation

$$\hat{V}^{(n)}(t, x, y; a_n) = M \left(t, x + \hat{p}_n(t, x, y; a_n)\right),$$  \hfill (20)

where $\hat{V}^{(n)}(t, x, y; a_n)$ is defined in (24) below. We can express the optimal static position $a_n^*$ (decided at time $t_n$) in terms of the indifference price and market price.

$$a_n^*(x,y) = \arg \max_{0 \leq a < \infty} \hat{p}_n^{(n)}(t_n, x, y; a) - a\pi(t_n, y).$$  \hfill (21)

We write the value function and indifference price with this optimal static hedge as

$$\hat{V}_n^{(n)}(t, x, y; a_n) = \hat{V}_n^{(n)}(t, x - a_n^*\pi(t, y), y; a_n^*),$$  \hfill (22)

$$\hat{p}_n^{(n)}(t, y) = \hat{p}_n^{(n)}(t, x, y; a_n^*) - a_n^*\pi(t, y).$$  \hfill (23)

Then, for $n \in \{N-2, N-3, \ldots, 0\}$, the employee’s value function at time $t \in [t_n, t_{n+1})$ with $a_n$ put options is given recursively by

$$\hat{V}^{(n)}(t, x, y; a_n) = \sup_{\tau \in \mathcal{T}_{t_n,t_{n+1}}} \sup_{a_n \in [0,1]} E\left[\hat{V}_n^{(n+1)}(t_{n+1}, X_{t_{n+1}} + a_n D(Y_{t_{n+1}}), Y_{t_{n+1}}) \cdot 1_{\tau = t_{n+1}}\right]$$
$$+ u^{(n)}(\tau, X_{\tau} + C(Y_{\tau}), Y_{\tau}; a_n) \cdot 1_{\tau < t_{n+1}} \mathbb{I}_{X_t = x, Y_t = y},$$  \hfill (24)

where $u^{(n)}$ is defined in (17).

4.1 A Recursive System of Free Boundary Problems

We proceed to derive the variational inequalities for $\{\hat{V}^{(N-1)}, \hat{V}^{(N-2)}, \ldots, \hat{V}^{(0)}\}$, which will lead to a system of free boundary problems for the corresponding indifference prices.

We first consider the variational inequality for $\hat{V}^{(N-1)}(t, x, y; a)$.

$$\hat{V}_t^{(N-1)} + \sup_{\theta} \mathcal{L} \hat{V}^{(N-1)} \leq 0, \quad \hat{V}^{(N-1)} \geq M \left(t, x + C(y) + h^{(N-1)}(t, y; a)\right),$$  \hfill (25)

$$\left(\hat{V}_t^{(N-1)} + \sup_{\theta} \mathcal{L} \hat{V}^{(N-1)}\right) \left(M \left(t, x + C(y) + h^{(N-1)}(t, y; a)\right) - \hat{V}^{(N-1)}\right) = 0,$$  \hfill (26)

for $(t, x, y) \in [t_{N-1}, T) \times \mathbb{R} \times (0, +\infty)$. The boundary conditions are

$$\hat{V}^{(N-1)}(T, x, y; a) = -e^{-\gamma(x+C(y)+aD(y))},$$

$$\hat{V}^{(N-1)}(t, x, 0; a) = -e^{-\gamma(x+\theta aK)e^{-\frac{\theta^2}{2\gamma}}(T-t)}.$$  

The value function has a separation of variables

$$\hat{V}^{(N-1)}(t, x, y; a) = M(t, x) \cdot \hat{H}^{(N-1)}(t, y; a)^{\frac{1}{1-\theta}},$$  \hfill (27)

where $\hat{H}^{(N-1)}$ solves a linear free boundary problem

$$\hat{H}_t^{(N-1)} + \mathcal{L}^E \hat{H}^{(N-1)} \geq 0, \quad \hat{H}^{(N-1)} \leq \kappa_{N-1},$$  \hfill (28)

$$\left(\hat{H}_t^{(N-1)} + \mathcal{L}^E \hat{H}^{(N-1)}\right) \left(\kappa_{N-1} - \hat{H}^{(N-1)}\right) = 0,$$  \hfill (29)
for \((t, y) \in [t_{N-1}, T) \times (0, +\infty)\), where \(\kappa_{N-1}(t, y; a) = e^{-\gamma(1-\rho^2)(C(y) + h^{(N-1)}(t, y; a))e^{(T-t)}}\). The boundary conditions are
\[
\mathcal{H}^{(N)}(T, y; a) = e^{-\gamma(1-\rho^2)(C(y) + (t, y; a))}, \quad \text{and} \quad \mathcal{H}^{(N-1)}(t, 0; a) = e^{-\gamma(1-\rho^2)\Delta K}.
\]

Again, we can express the indifference price in terms of function \(\mathcal{H}^{(N-1)}\):
\[
\hat{p}^{(N-1)}(t, y; a) = \frac{1}{\gamma(1-\rho^2)e^{(T-t)}} \log \hat{H}^{(N-1)}(t, y; a), \quad t \in [t_{N-1}, T].
\]
which gives the following quasilinear variational inequality
\[
\mathcal{A}^{\hat{p}} \hat{p}^{(N-1)} \leq 0, \quad \hat{p}^{(N-1)} \geq C(y) + h^{(N-1)}(t, y; a),
\]
for \((t, y) \in [t_{N-1}, T) \times (0, +\infty)\). The boundary conditions are
\[
\hat{p}^{(N-1)}(T, y; a) = C(y) + aD(y), \quad \text{and} \quad \hat{p}^{(N-1)}(t, 0; a) = aKe^{(T-t)}.
\]
Similar to Section 3.1, we numerically solve the free boundary problem for \(\hat{H}^{(N-1)}(t, y; a)\) and use (38) to derive the indifference price \(\hat{p}^{(N-1)}(t, y; a)\). Then, we optimize over \(a\) to obtain the employee’s optimal static hedges \(a_{N-1}^*\) at time \(t_{N-1}\), along with the corresponding \(V^{(n)}(t, y; x)\) and \(\hat{p}^{(n)}(t, y; a)\) which will become the terminal conditions for \(\hat{V}^{(n-2)}(t, y; a)\) \(\hat{V}^{(n)}(t, y; a)\) satisfies the HJB variational inequality
\[
\hat{V}^{(n)}(t, x, y; a) = \mathbb{J} t \times \mathbb{R} \\
\hat{V}^{(n)}(t, x, y; a) = \mathbb{J} t \times \mathbb{R} \\
\hat{V}^{(n)}(t, x, y; a) = \mathbb{J} t \times \mathbb{R} \\
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\hat{V}^{(n)}(t, x, y; a) = \mathbb{J} t \times \mathbb{R}
\]
for \((t, y) \in [t_{n+1}, T) \times (0, +\infty)\). The boundary conditions are
\[
\hat{V}^{(n)}(t_{n+1}, y; a) = V^{(n+1)}(t_{n+1}, x + aD(y), y),
\]
\[
\hat{V}^{(n)}(t, x, 0; a) = -e^{-\gamma(xe^{(T-t)} + aK)} e^{-\gamma x(T-t)}.
\]
Again, the value function admits a separation of variables
\[
\hat{V}(t, x, y; a) = M(t, x) \cdot \hat{H}(t, y; a)^{\frac{1}{1-\rho^2}},
\]
where \(\hat{H}(n)\) solves a linear free boundary problem
\[
\hat{H}^{(n)} + L^x \hat{H}^{(n)} \geq 0, \quad \hat{H}^{(n)} \leq \kappa_n,
\]
for \((t, y) \in [t_n, t_{n+1}) \times (0, +\infty)\), where \(\kappa_n(t, y; a) = e^{-\gamma(1-\rho^2)(C(y) + h^{(n)}(t, y; a))e^{(T-t)}}\). The boundary conditions are
\[
\hat{H}^{(n)}(t_{n+1}, y; a) = e^{-\gamma(1-\rho^2)(aD(y))} \hat{H}^{(n+1)}(t_{n+1}, y), \quad \text{and} \quad \hat{H}^{(n)}(t, 0; a) = e^{-\gamma(1-\rho^2)aK}.
\]
The indifference price is given by
\[
\hat{p}^{(n)}(t, y; a) = -\frac{1}{\gamma(1-\rho^2)e^{(T-t)}} \log \hat{H}^{(n)}(t, y; a), \quad t \in [t_n, t_{n-1}).
\]
which gives the following quasilinear variational inequality
\[
\mathcal{A}^{\hat{p}} \hat{p}^{(n)} \leq 0, \quad \hat{p}^{(n)} \geq C(y) + h^{(n)}(t, y; a),
\]
\[
\mathcal{A}^{\hat{p}} \hat{p}^{(n)} \cdot (C(y) + h^{(n)}(t, y; a) - \hat{p}^{(n)}) = 0,
\]
for \((t, y) \in [t_n, t_{n+1}) \times (0, +\infty)\). The boundary conditions are
\[
\hat{p}^{(n)}(t_{n+1}, y; a) = \hat{p}^{(n+1)}(t_{n+1}, y) + aD(y), \quad \text{and} \quad \hat{p}^{(n)}(t, 0; a) = aKe^{-(T-t)}.
5. Conclusions and Extensions

In summary, we have discussed a new utility-based methodology for hedging long-term financial derivatives. The hedging strategy involves dynamic trading of a correlated liquid asset (e.g. the market index) combined with static positions in market-traded options. In view of the relatively short maturities of most liquid market-traded options, we revise the static hedges sequentially over time till the long-term derivative expires. In order to analyze the optimal sequential static-dynamic hedging strategy, we study the associated Hamilton-Jacobi-Bellman PDEs and variational inequalities. Working with exponential utility, we apply a series of transformations that reduce the problem for tractability. For future investigation, one meaningful extension is to allow to roll over the short-term positions at random times, rather than a fixed schedule. The rolling decisions will lead to an optimal multiple stopping problem. This is also related to the optimal timing to purchase options [11, 12]

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