Fermion perturbations in string theory black holes

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Abstract

In this paper we study fermion perturbations in four-dimensional black holes of string theory, obtained either from a non-extreme configuration of three intersecting five-branes with a boost along the common string or from a non-extreme intersecting system of two two-branes and two five-branes. The Dirac equation for the massless neutrino field, after conformal re-scaling of the metric, is written as a wave equation suitable to study the time evolution of the perturbation. We perform a numerical integration of the evolution equation, and with the aid of Prony fitting of the time-domain profile, we calculate the complex frequencies that dominate the quasinormal ringing stage, and also determine these quantities by the semi-analytical sixth-order WKB method. We also find numerically the decay factor of fermion fields at very late times, and show that the falloff is identical to those showing for massless fields in other four-dimensional black hole spacetimes.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

String theory and its generalization in terms of extended objects named M/Dp-branes is a promising candidate for a fundamental quantum theory of all interactions [1], including a description of black holes in a quantum gravity framework [2].

The scenario of string theory is consistently formulated only in higher dimensions, which are usually compactified, so a full description of a black hole whose event horizon is comparable to the size of extra dimensions has to be in terms of such higher dimensional description.

A unified treatment of the proprieties of black holes which comes from the string theory can be obtained through the compactified configurations of intersecting two-branes and five-branes of 11-dimensional M-theory. For instance, ten-dimensional solutions with NS-NS and
R-R charges appear on the same footing when viewed in the framework of 11-dimensional M-theory.

One class of solutions which encompass the features of string theory and also can be interpreted as black holes are those non-extremal solutions which came from intersections of branes in string theory or in M-theory [3–6]. The parameters of such black holes can be deduced by corresponding compactifications of the higher dimensions.

The central idea of the compactified configurations is to relate non-extreme/extreme static black hole solutions to non-extreme/extreme versions of intersecting M-brane solutions. This approach allows us to learn more about the structure of the non-extremal black hole solutions, shedding some light on the origin of the black hole entropy as pointed out in [3].

The study of test field perturbations in black hole backgrounds gives us the opportunity to consider such questions as the stability of the compact object under such fluctuations, as well as the form in which the field relaxes at very late times. We can also find the characteristic oscillation frequencies (quasinormal frequencies) of the black hole solution that carry information about the particular properties of the spacetime.

Quasinormal modes (QNM) of black holes have been extensively studied since the pioneering work on the stability of Schwarzschild singularity performed by Regge and Wheeler [7]. In studying the quasinormal spectrum, we can gain some valuable information about the parameters which characterize the solution, because there are definite relations between the QNM and the parameters of solution, see [8] and references therein. Such QNMs are characterized by a well-defined set of complex frequencies and encode the response of black hole to external perturbation; so we can study the stability of a black hole against small perturbations due to probe fields or the geometry itself through its quasinormal spectrum.

Besides, further developments in QNM research in the context of AdS/CFT and holography allow us to calculate the location of the poles of the retarded correlators of certain gauge theories [9] and reveals some connection between the dynamics of black hole horizons and the hydrodynamics [10, 11].

Regarding the stability of extended objects from the low energy limit of string theory, some previous work has been done [12], where it was considered a probe scalar field evolving in a $p$-brane geometry and its quasinormal spectrum in order to analyze the stability and the relation between the parameters of $p$-brane and its quasinormal ringing.

This work is the first of a series of papers devoted to the study of perturbations of a family of four-dimensional black holes coming from string theory. In this work we aim at study specifically the evolution of fermion perturbations in the form of massless uncharged Dirac fields.

Fermionic perturbations are important for several reasons. Fermions are universally describing matter fields, neutrinos, and most of perturbations (especially those coming from matter) are essential for the structure of the solution.

We have several motivations to carry out this work. In the first place, we believe it is important to know if the solutions are stable against small perturbations in order to make the solutions more reliable. Also the analysis of the quasinormal spectrum of the black hole in string/M theory provides a way to fix the parameters of black hole and consequently of string/M theory.

A further point concerns the AdS/CFT analysis. Fermionic perturbations of the gravitational bulk field equations turn out to be fundamental matter fields in the boundary conformal field theory, providing an important class of objects relevant for condensed matter physics.

The paper is organized as follows: section 2 gives a review about the causal structure of a four-dimensional black hole obtained from intersections of branes. In section 3 we obtain the
wave equation suitable to analyze the propagation of a Dirac field in the background geometry of the four-dimensional stringy black hole. Section 4 is devoted to numerically solving the evolution equation of such fields in the considered background, and in section 5 we are concerned with the quasinormal stage, and compute the complex quasinormal frequencies using two methods: Prony fitting of time domain data and the sixth-order WKB approach. The results of a numerical investigation of the relaxation of Dirac perturbations in stringy black holes at very long times are presented in section 6. In section 7 we present an analytical expression for the quasinormal frequencies in the limit of large angular momentum, followed by the last section of the paper, which contains some concluding remarks.

2. (3+1)-dimensional black hole solution from intersecting branes

The metric of the non-extremal four-dimensional black hole obtained from intersections of five-branes read as \[1\]
\[
ds^2 = - f^{-1/2} \left( 1 - \frac{r_H}{r} \right) dt^2 + f^{1/2} \left[ \left( 1 - \frac{r_H}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2 \right],
\]
where
\[
f = \left( 1 + \frac{r_H Q_1}{r} \right) \left( 1 + \frac{r_H Q_2}{r} \right) \left( 1 + \frac{r_H Q_3}{r} \right) \left( 1 + \frac{r_H Q_4}{r} \right).
\]

This metric represents a family of black hole solutions parametrized by \(r_H\), which gives the location of the event horizon, and four charges \(Q_1, Q_2, Q_3, Q_4\) given by
\[
Q_i = \sinh^2(\delta_i), \quad i = 1, 2, 3, 4,
\]
that are written in terms of five-brane parameters \(\delta_i\), resulting from the compactification of higher dimensions. This solution describes a four-dimensional Schwarzschild black hole in the case in which the charges \(Q_i\) are all zero. Also the metric is asymptotically flat and, as we shall see below, has a regular event horizon located at \(r = r_H\) even with all charges different from zero. Besides, the redshift goes to infinity in such surface.

Infinity redshift surfaces can be found, for a given metric, through the following relation:
\[
\nu = \nu_0 \sqrt{g_{00}(x^a_{\text{source}})} / g_{00}(x^a),
\]
which relates the frequency \(\nu\) measured by an observer at rest away from the source, whose emission frequency of, say, light pulses is \(\nu_0\).

In order to have an infinity redshift surface, the frequency \(\nu\) has to be zero, it means that the frequency \(\nu\) was infinitely delayed due to gravitational effects. So, equation (3) implies
\[
g_{00}(x^a_{\text{source}}) = 0,
\]
given the location of the infinity redshift surfaces. In the case of the spacetime of four-dimensional stringy black holes (1) it is easy to see that \(r = r_H\) and \(r = 0\) are the surfaces where the redshift of in-falling objects goes to infinity.

As in the Schwarzschild solution, such surface \(r = r_H\) acts as a one-way membrane for physical objects whose trajectories lie in or on the forward light cone. To see this, suppose a smooth hypersurface \(S\) defined by the equation
\[
u(x^a) = \text{const}.
\]
The vector \(N_a = \partial_a u\) is normal to \(S\), so if \(S\) is a one-way membrane it has to be a null hypersurface; therefore, the norm of \(N_a\) has to be null as well. So,
\[
g^{ab} N_a N_b = 0
\]
determine the one-way membranes, which in our case are located at \( r = r_H \) and \( r = 0 \). However, for one of the four charges \( Q_i \) equal to zero, only \( r = r_H \) is a null and the infinity redshift hypersurface \( r = 0 \) is a genuine spacetime singularity, as we will see below in more detail.

Let us consider the Kretschmann invariant for the spacetime (1), where we take for simplicity \( Q_i = q \) (the result holds as well for four different charges):

\[
R^{abcd} R_{abcd} = \frac{4}{(r_H q + r)^8} P(r),
\]

(5)

where

\[
P(r) = r_H^4 q^2 (5 + 4q + q^2) - 2r_H^3 q r (3 + 7q + 2q^2) + (r_H r)^2 (3 + 8q + 8q^2) - 2r_H r^3 + r^4.
\]

As we see from the above expressions, for \( r \to r_H \) we have

\[
R^{abcd} R_{abcd} (r \to r_H) \to \text{const},
\]

and for \( r \to 0 \)

\[
R^{abcd} R_{abcd} (r \to 0) \to \frac{4}{r_H^8 q^8} \left[ r_H^4 q^2 (5 + 4q + q^2) \right].
\]

(6)

We see clearly that \( r = 0 \) is not a spacetime singularity but, as previously pointed out by Horowitz et al [13], it is actually an inner horizon. However, if at least one of the charges is zero, the surface \( r = 0 \) becomes singular as we see explicitly in expression (6).

We conclude that for at least one charge \( Q_i \) zero, we have a four-dimensional black hole with a spacetime singularity located at \( r = 0 \) covered by an event horizon at \( r = r_H \) which is also an infinity redshift surface, as expected for a spherically symmetric space time. On the other hand, if all charges are non-zero, we have a regular black hole with an event horizon at \( r = r_H \) and an inner horizon at \( r = 0 \).

3. Fundamental equations

In curved spacetime the massless Dirac equation is written as

\[
\nabla \Psi = 0,
\]

(7)

where \( \nabla = \Gamma^\mu \nabla_\mu \) is the Dirac operator that acts on the four-spinor \( \Psi \), \( \Gamma_\mu \) are the curved space Gamma matrices and the covariant derivative is defined as \( \nabla_\mu = \partial_\mu - \frac{1}{4} \omega^{ab}_\mu \gamma_a \gamma_b \), with \( a \) and \( \mu \) being tangent and spacetime indices respectively, related by the basis of orthonormal one forms \( \tilde{e}^a = e^\mu_a \). The associated connection one-forms \( \omega^{\mu}_{ab} \equiv \omega^{ab}_\mu \) obey \( d \tilde{e}^a + \omega^{\mu}_{ab} \wedge \tilde{e}^b = 0 \), and the \( \gamma^a \) are flat spacetime gamma matrices related with curved-space ones by \( \Gamma^\mu = e^\mu_a \gamma^a \). They form a Clifford algebra in \( d \) dimensions, i.e. they satisfy the anti-commutation relations \( \{ \gamma^a, \gamma^b \} = -2 \eta^{ab} \), with \( \eta^{00} = -1 \).

First consider some properties of the Dirac operator that will be used in future calculations [14, 15]. Under a conformal transformation of the metric of the form

\[
g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu},
\]

(8)

the spinor \( \psi \) and the Dirac operator transform as

\[
\psi = \Omega^{-\frac{1}{2}} \tilde{\psi},
\]

(9)

\[
\nabla \psi = \Omega^{-\frac{1}{2}} \tilde{\nabla} \tilde{\psi}.
\]

(10)
If the line element of a spacetime metric takes the form of a sum of independent components
\[ ds^2 = dx^2 + dy^2, \]
where \( ds_1^2 = g_{ab}(x) \, dx^a \, dx^b \) and \( ds_2^2 = g_{mn}(y) \, dy^m \, dy^n \), then the Dirac operator \( \nabla \) satisfies a direct sum decomposition
\[ \nabla = \nabla_x + \nabla_y. \]  
(11)

The abovementioned properties of spinors and the Dirac operator in direct sum metrics and conformal related ones allow a simple treatment of the massless Dirac equations in curved spacetimes. For two conformally related metrics, the validity of the massless Dirac equation in one implies the validity of the same equation in the other. Then, the idea is to solve the Dirac equation in the curved space described by an initial metric tensor performing successive conformal transformations that isolate the metric components that depend on a given variable, and apply successively the direct sum decomposition of the Dirac operator, until an equivalent problem in a spacetime of the form \( M \times \Sigma^2 \) is obtained, where \( M \) is a two-dimensional Minkowski spacetime and \( \Sigma^2 \) is a two-dimensional maximally symmetric one, in which the spectrum of the massless Dirac operator is known.

The method works as follows. First, in the following we suppose that our four-dimensional metric is spherically symmetric, and given by
\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -A(r) \, dr^2 + B(r) \, dr^2 + C(r) \, d\Omega^2, \]
(12)
where \( d\Omega^2 \) denotes the metric for the (2)-sphere \( S^2 \).

It is easy to show that with the identification \( (t, r, \theta, \phi) \rightarrow (0, 1, 2, 3) \) if we choose the basis one-forms as \( \bar{e}^0 = e^0(t, r) = f_{(2)}(t, r) \, dt + f_{(2)}(t, r) \, dr \), \( \bar{e}^1 = e^1(t, r) = g_{(2)}(t, r) \, dt + g_{(2)}(t, r) \, dr \) and \( \bar{e}^k = e^k(\theta, \phi) = h_{(k)}(\theta, \phi) \, d\theta + h_{(k)}(\theta, \phi) \, d\phi \), \( k = 2, 3 \), then we have some connection one-forms that are equal to zero, and only \( \omega^0_0, \omega^1_1, \omega^0_1, \omega^1_0 \), \( i, j = 2, 3 \), are different from zero. This fact allows us to write \( \nabla_a = \nabla_a^{(2)} \otimes I_2 \), \( a = 0, \ldots, 3 \), where \( I_2 \) is the unit matrix in two dimensions \([16]\).

Now, under a conformal re-scaling of the form \( ds^2 = C(r) \, ds^2 \), where
\[ ds^2 = -\frac{A}{C} \, dr^2 + \frac{B}{C} \, dr^2 + d\Omega^2, \]
(13)
we need to solve the equation \( \nabla \tilde{\psi} = \tilde{\Gamma}^\mu \nabla_\mu \tilde{\psi} = 0 \), with \( \tilde{\psi} = C^2 \psi \). In general, the Dirac matrices \( \tilde{\Gamma}^\mu \) can be chosen as
\[ \tilde{\Gamma}^0 = \Gamma^{(2)}_0 \otimes I_2, \]
(14)
\[ \tilde{\Gamma}^1 = \Gamma^{(2)}_1 \otimes I_2, \]
(15)
\[ \tilde{\Gamma}^2 = \Gamma^{(2)}_5 \otimes \Gamma^{(2)}_0, \]
(16)
\[ \tilde{\Gamma}^3 = \Gamma^{(2)}_5 \otimes \Gamma^{(2)}_1, \]
(17)
where \( \Gamma^{(2)}_0, \Gamma^{(2)}_1 \) and \( \Gamma^{(2)}_5 \) are two-dimensional Gamma matrices, and \( (\Gamma^{(2)}_5)^2 = I_2 \).

Since the orbit-space part and the angular part of the metric are completely separated, one can write the Dirac equation in the form
\[ \left[ \left( \tilde{\Gamma}^0 \tilde{\nabla}_0 + \tilde{\Gamma}^1 \tilde{\nabla}_1 \right) \otimes I_2 \otimes \left( \tilde{\Gamma}^a \tilde{\nabla}_a \right) S_2 \right] \tilde{\psi} = 0, \]
(18)
where \( \tilde{\Gamma}^a \tilde{\nabla}_a \) denotes the Dirac operator in the 2-sphere, whose orthogonal set of eigenspinors \( \xi^{(\ell)}_\pm \) is defined by \([17]\)
\[ (\tilde{\Gamma}^a \tilde{\nabla}_a) S_2 \xi^{(\pm)}_\ell = \pm i(\ell + 1) \xi^{(\pm)}_\ell, \]
(19)
where \( l = 0, 1, 2, \ldots \). Now expanding \( \tilde{\psi} \) as
\[
\tilde{\psi} = \sum_{\ell} \left( \tilde{\psi}_{\ell}^{(+)} \xi_{\ell}^{(+)} + \tilde{\psi}_{\ell}^{(-)} \xi_{\ell}^{(-)} \right),
\] (20)
we can put the Dirac equation in the form
\[
\left\{ \Gamma_{0}^{(2)} \partial_{0} + \Gamma_{1}^{(2)} \partial_{1} + \Gamma_{5}^{(2)} \left[ \pm i(l + 1) \right] \right\} \tilde{\psi}_{\ell}^{(k)} = 0.
\] (21)
In the following we work with the + sign solution, because the \(-\) sign case can be worked in the same form. Denoting as \( d_{s}^{2} = -\frac{A}{C} \, dt^{2} + \frac{B}{C} \, dr^{2} \) the \( t-r \) part of the metric (13) and doing a conformal rescaling of it in the form
\[
d_{\tilde{s}}^{2} = \frac{A}{C} [-dt^{2} + dr^{2}],
\] (22)
where \( dr_{*} = \sqrt{\frac{B}{A}} \, dr \) defines the tortoise coordinate, we have
\[
\left[ \gamma' \partial_{t} + \gamma' \partial_{r_{*}} + i\gamma^{5} \sqrt{\frac{A}{C}} (l + 1) \right] \tilde{\psi}_{\ell}^{(s)} = 0,
\] (23)
\( \gamma' \) and \( \gamma' \) being the Dirac matrices in the two-dimensional Minkowski spacetime with metric \( ds^{2} = -dt^{2} + dr^{2} \).

Now choosing the representation of Dirac matrices given by
\[
\gamma' = -i\sigma^{3}, \quad \gamma' = \sigma^{2}, \quad \gamma^{5} = (\sigma^{3})(\sigma^{2}) = -\sigma^{1},
\] (24)
where \( \sigma^{i} \) are the Pauli matrices:
\[
\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\] (25)
and writing
\[
\tilde{\psi}_{\ell}^{(s)} = \begin{pmatrix} i\zeta_{\ell}(t, r) \\ \chi_{\ell}(t, r) \end{pmatrix},
\] (26)
we obtain the following equations for each component of the Dirac spinor \( \tilde{\psi}_{\ell} \):
\[
i \frac{\partial \zeta_{\ell}}{\partial t} + \frac{\partial \chi_{\ell}}{\partial r_{*}} + \Lambda_{\ell} \chi_{\ell} = 0,
\] (27)
and
\[
i \frac{\partial \chi_{\ell}}{\partial t} - \frac{\partial \zeta_{\ell}}{\partial r_{*}} + \Lambda_{\ell} \zeta_{\ell} = 0,
\] (28)
where
\[
\Lambda_{\ell}(r) = \sqrt{\frac{A}{C}} (l + 1).
\] (29)
The above equations can be separated to obtain
\[
\frac{\partial^{2} \zeta_{\ell}}{\partial t^{2}} - \frac{\partial^{2} \zeta_{\ell}}{\partial r_{*}^{2}} + V_{+}(r) \zeta_{\ell} = 0,
\] (30)
and
\[
\frac{\partial^{2} \chi_{\ell}}{\partial t^{2}} - \frac{\partial^{2} \chi_{\ell}}{\partial r_{*}^{2}} + V_{-}(r) \chi_{\ell} = 0,
\] (31)
where
\[
V_{\pm} = \pm \frac{d \Lambda_{\ell}}{dr_{*}} + \Lambda_{\ell}^{2}.
\] (32)
The above equations give the temporal evolution of Dirac perturbations outside the black hole spacetime [18]. As the potentials $V_+$ and $V_-$ are supersymmetric to each other in the sense considered by Chandrasekhar in [19], $\zeta_\ell(t, r)$ and $\chi_\ell(t, r)$ will have similar time evolutions and then they will have the same spectra, both for scattering and quasi-normal. At this point it should be stressed that for the spinor $\tilde{\phi}_\ell^{(-)}$, we have these two potentials again. In the following we will work with equation (30) and we eliminate the subscript $^+$ for the effective potential, defining $V(r) \equiv V_+(r)$.

Now for (3+1)-string theory black holes with line element (1), making the identifications $A(r) = f^{-1/2}(1 - \frac{a_0}{r})$, $B(r) = f^{1/2}(1 - \frac{a_0}{r})^{-1}$ and $C(r) = r^2 f^{1/2}$, we can calculate $V(r)$ using (32) with $\Lambda_{\ell}$ given by

$$\Lambda_{\ell} = \frac{r^{-1/2}}{r} \sqrt{\frac{1 - \frac{a_0}{r}}{r} (\ell + 1)}.$$

In figure 1 we show the effective potential for massless Dirac perturbations in four-dimensional stringy black holes for various multipole numbers $\ell$. In this figure distances are measured in units of the black hole horizon radius $r_H$. We see that $V(r)$ has the form of a definite positive potential barrier, i.e. it is a well-behaved function that goes to zero at spatial infinity and gets a maximum value at a well-defined peak near the event horizon. Then we can expect that the stringy black holes are stable under massless Dirac perturbations, a fact supported by the numerical results that we will present in the next section.

4. Time evolution of Dirac perturbations

In order to integrate equation (30) numerically we use the technique developed by Gundlach, Price and Pulling [20].

The first step is to rewrite the wave-like equation (30) in terms light-cone coordinates $du = dt - dr$, and $dv = dt + dr$,

$$\left(4 \frac{d^2}{du dv} + V(u, v)\right) \zeta_\ell(u, v) = 0$$

Figure 1. Potential $V(r)$ for Dirac perturbations with $\ell = 0$ (bottom) to $\ell = 5$ (top). The charges parameters of the solution are $Q_1 = 0.5$, $Q_2 = Q_3 = Q_4 = 1$. 

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$$\left(4 \frac{d^2}{du dv} + V(u, v)\right) \zeta_\ell(u, v) = 0$$

7
and use the following discretized version of the above equation:

\[ \zeta_c(N) = \zeta_c(W) + \zeta_c(E) - \zeta_c(S) - \frac{\Delta u \Delta v}{8} V(S)(\zeta_c(W) + \zeta_c(E)) + O(h^4), \tag{35} \]

where the letters S, W, E, N are used to mark the points that limit a particular integration cell of the grid according to S = (u, v), W = (u + \Delta u, v), E = (u, v + \Delta v), N = (u + \Delta u, v + \Delta v).

In figure 2 we show the cell of a given integration step. We see that the field value at point N depends only of the field values at the points S, E, and W. Given a set of initial conditions at the two null surfaces u = u_0 and v = v_0, we can find, using (35), the value of the field \( \zeta_c \) inside the rhombus which is built on these two null surfaces. Then, by iteration of the integration cell, we find the complete data describing the evolution of the fields with time.

The obtained results from the integration in the case of massless Dirac fields in stringy black hole backgrounds can be observed as the time-domain profile shown in figures 3–7. In such profiles \( r = 3r_H \) and the time is measured in units of black hole event horizon radius, because the time coordinate in the evolution equations can be re-scaled to measure it in such units.

We present the results for stringy black hole spacetimes characterized by different values of the charge parameter. Figures 3 and 4 show the results for the numerical integrations in the case of a background with all the four charges equal. Figure 5 is referred to the particular cases in which there are different values of one of the charges and the other three remain equal, whereas (7) contains the result of the temporal evolution of perturbations in the more general background characterized by different values of all the charges.

An interesting case arises in the specific case in which at most one of the charges is zero, because, as we explain in section 2, this case corresponds to a black hole with a singularity at the origin. Figure 7 shows results for numerical integrations in two cases: a stringy black hole with one charge zero and the other three equal, and a black hole with all the four charges different, including one null charge.
As we can see, the temporal evolution of Dirac perturbations in all the cases studied follows the usual dynamics for fields in black hole spacetimes. After a first transient stage strongly dependent on the initial conditions and the point where the wave profile is observed, we see the characteristic exponential damping of the perturbations called quasinormal ringing, followed by a so-called power law tail at asymptotically late times.

It is important to mention that we perform and extensive numerical exploration of the perturbative dynamics for different values of the angular number $\ell$ and charges $Q_i$, and we have not found any instability for massless Dirac perturbations in four-dimensional stringy black holes.

5. QNMs using the sixth-order WKB method and Prony fitting of characteristic data

In the following, we will assume for the function $\zeta_\ell(t, r)$ the time dependence

$$\zeta_\ell(t, r) = Z_\ell(r) \exp(-i\omega_\ell t).$$

Then, the function $Z_\ell(r)$ satisfies the Schrödinger-type equation

$$\frac{d^2 Z_\ell}{dr^2} + [\omega^2 - V(r)]Z_\ell(r) = 0.$$ \hfill (37)

The QNMs are solutions of the wave equation (30) with the specific boundary conditions requiring pure out-going waves at spatial infinity and pure in-coming waves on the event horizon. Thus, no waves come from infinity or the event horizon. The general form of
equations (30) and (37) and the analytical expression of the effective potential allow us to re-scale the frequency to measure it in units of the black hole horizon radius $r_H$. However, due to the dependence of the potential on the charges $Q_i$, we cannot re-scale the frequencies in the above equations. Then, the only parameters upon which the QNMs depend are the charges resulting from the compactification of higher dimensions.

In order to evaluate the QNMs we used two different methods. The first is a semi-analytical method to solve equation (37) with the required boundary conditions, based on a WKB-type approximation, that can give accurate values of the lowest (that is longer lived) quasinormal frequencies, and was used in several papers for the determination of quasinormal frequencies in a variety of systems [12, 18, 21–25].

The WKB technique was applied up to first order to find QNMs for the first time by Shutz and Will [21]. Later this approach was extended to the third order beyond the eikonal approximation by Iyer and Will [22] and to the sixth order by Konoplya [23, 24]. We use in our numerical calculation of QNMs this sixth-order WKB expansion. The sixth-order WKB expansion gives a relative error which is about two orders less than the third WKB order, and allows us to determine the quasinormal frequencies through the formula

$$i(\omega^2 - V_0)\sqrt{-2V_0} - \sum_{j=2}^{6} \Pi_j = n + \frac{1}{2},$$

where $n = 0, 1, 2, \ldots$ if $\text{Re}(\omega) > 0$ or $n = -1, -2, -3, \ldots$ if $\text{Re}(\omega) < 0$ is the overtone number. In (38) $V_0$ is the value of the potential at its maximum as a function of the tortoise coordinate, and $V_0'$ represents the second derivative of the potential with respect to the tortoise coordinate at its peak. The correction terms $\Pi_j$ depend on the value of the effective potential and its derivatives (up to the 2\text{nd} order) in the maximum, see [27] and references therein. It is a
Figure 5. Logarithmic plots of the time-domain evolution of $\ell = 0$ (left) and $\ell = 1$ (right) massless Dirac perturbations in stringy black holes with charges $Q_1 = 0.5$, $Q_2 = Q_3 = Q_4 = 1$ (top) and $Q_1 = 1.4$, $Q_2 = Q_3 = Q_4 = 1$ (bottom).

Figure 6. Normal (left) and logarithmic (right) plots of the time-domain evolution of $\ell = 0$ massless Dirac perturbations in stringy black holes with charges $Q_1 = 0.2$, $Q_2 = 0.5$, $Q_3 = 0.8$, $Q_4 = 1$.

semi-analytic method, because in spite of having one analytical expression for the quasinormal frequencies, in general we need to find the position of the maximum of the effective potential numerically.

The second method that we used to find the quasinormal frequencies was the Prony method [26, 27] for fitting the time-domain profile data by superposition of damping exponents in the form

$$\psi(t) = \sum_{k=1}^{p} C_k e^{-i\omega_k t}.$$  

(39)
Figure 7. Normal (left) and logarithmic (right) plots of the time-domain evolution of $\ell = 0$ massless Dirac perturbations in stringy black holes with charges $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 1$ (top) and $Q_1 = 0$, $Q_2 = 0.5$, $Q_3 = 1$, $Q_4 = 1.5$ (bottom).

Assuming that the quasinormal ringing stage begins at $t = 0$ and ends at $t = Nh$, where $N \geq 2p - 1$, expression (39) is satisfied for each value in the time profile data

$$x_n \equiv \psi (nh) = \sum_{k=1}^{p} C_k e^{-i \omega_k nh} = \sum_{k=1}^{p} C_k z^n_k. \quad (40)$$

From the above expression, we can determine, as we know $h$, the quasinormal frequencies $\omega_i$ once we have determined $z_i$ as functions of $x_n$. The Prony method allows us to find the $z_i$ as roots of the polynomial function $A(z)$ defined as

$$A(z) = \prod_{k=1}^{p} (z - z_k) = \sum_{m=0}^{p} \alpha_m z^{p-m}, \quad \alpha_0 = 1. \quad (41)$$

It is possible to show that the unknown coefficients $\alpha_m$ of the polynomial function $A(z)$ satisfy

$$\sum_{m=1}^{p} \alpha_m x_{n-m} = -x_n. \quad (42)$$

Solving the $N - p + 1 \geq p$ linear equations (42) for $\alpha_m$ we can determine numerically the roots $z_i$ and then the quasinormal frequencies.

It is important to mention the fact that with the Prony method we can obtain very accurate results for the quasinormal frequencies, but the practical application of the method is limited because we need to know with precision the duration of the quasinormal ringing epoch. As this stage is not a precisely defined time interval, in practice, it is difficult to determine when the quasinormal ringing begins. Therefore, we are able to calculate with high accuracy only two, or sometimes three, dominant frequencies.
Figure 8. Massless Dirac quasinormal frequencies in a stringy black hole with $Q_1 = Q_2 = Q_3 = Q_4 = 1$, for various values of $l$ and $n$.

Figure 9. Massless Dirac quasinormal frequencies in a stringy black hole with $Q_1 = 0, Q_2 = Q_3 = Q_4 = 1$, for various values of $l$ and $n$.

Tables A1–A14 in the appendix show the results for the quasinormal frequencies measured in units of the black hole horizon radius $r_H$ for the first two overtones corresponding to some values of the multipole moment $\ell$. The numerical results presented have been calculated for
Figure 10. Dependence upon charge $Q_1$ of the real part of the massless Dirac quasinormal frequencies of stringy black holes with $Q_2 = Q_3 = Q_4 = 1$. The results for the first overtone (top) for multipole numbers from $\ell = 0$ to $\ell = 2$ and the second overtone (bottom) for $\ell = 2$ modes.

different sets of values of black hole charges. Figures 8 and 9 show some modes for two sets of values for the charges.

As it is observed, in all cases the sixth-order WKB approach gives results in good correspondence with those obtained by fitting the numerical integration data using the Prony technique. Increasing the values of the multipole number $\ell$, the frequencies for the first two overtones results identical as calculated by the two methods employed, as we can expect due to the fact that the WKB formula gives the better results when $\ell > n$. Also as we expected for stability, all quasinormal frequencies calculated in this work have a well-defined negative imaginary part.
From the numerical results and figures 10 and 11, we observe that increasing the charge produces a decrease in the real part of the quasinormal frequency, as well as a decrease in the damping rate of the oscillations.

For fixed values of charges, as usual, the oscillation frequency increases for higher multipole and fixed overtone numbers. The fundamental mode, i.e. with $\ell = 0$ and $n = 0$, is more long lived with respect to the other modes, but it is interesting that the imaginary part of the $n = 0$ quasinormal frequencies reaches, very quickly, a fixed value for higher multipole numbers.

This situation is different for higher overtone numbers, and the imaginary part for a fixed $n$ decreases with $\ell$. We see in table A1, for example, that for the second overtone number $n = 1$ the damping rate reaches a constant value from $\ell = 8$, within numerical accuracy (see...
Table 1. Dirac quasinormal frequencies in the large angular momentum limit with \( n = 0, r_H = 2 \) and \( Q = 1 \).

| \( \ell \) | Dirac quasinormal frequencies |
|---|---|
| 50 | \( 3.96989 - 0.0314673i \) |
| 100 | \( 7.93959 - 0.0314681i \) |
| 150 | \( 11.9093 - 0.0314682i \) |
| 200 | \( 15.8791 - 0.0314683i \) |

The above qualitative behavior is generic, independent of the particular values of the set of charge parameters of the solution.

6. Large angular momentum limit

It is interesting to study the behavior of quasinormal frequencies in the limit of a large angular momentum number \( \ell \), in which the WKB formula gives an exact result. In the following we assume the simple setup in which all the charges are the same but different from zero, i.e. we take \( Q_i = Q \).

Expanding the effective potential in terms of small values of \( 1/\ell \) and using the lowest order of WKB approximation, we obtain for the quasinormal frequencies the simple expression

\[
\omega^2 = \ell^2 \sigma(r_m) - i \left(n + \frac{1}{2}\right) \sqrt{-2 \frac{d^2 \sigma(r_m)}{dr^2}},
\]

(43)

where

\[
\sigma(r) = \left(1 - \frac{r_H}{r}\right) \left(1 + \frac{r_H Q}{r}\right)^{-4},
\]

and \( r_m \) is where the peak of \( V(r) \) occurs, and it is given by

\[
r_m = \frac{r_H}{2} \left\{ \left(Q + \frac{3}{2}\right) + \left[ \left(Q + \frac{3}{2}\right)^2 - 2Q \right]^{\frac{1}{2}} \right\}.
\]

As we observe from the above expressions, for large values of the angular momentum number, the imaginary part of the quasinormal frequencies for the fixed overtone number \( n \) goes to a constant value that depends of the charge parameter of the stringy black hole. On the other hand, the real part increases as \( \ell \) becomes larger. This fact is shown in table 1 that contains the numerical results for some quasinormal frequencies calculated by the above formula.

7. Long time tails

Another important point to study is the relaxation of the perturbing fermion field outside the black hole. It is a known result that in the Schwarzschild black hole neutral massless fields had a late-time behavior for a fixed \( r \) dominated by a factor \( t^{-(2\ell+3)} \) for each multipole moment \( \ell \) [28, 29].

To study the late-time behavior, we numerically fit the profile data obtained in the appropriate region of the time domain, to extract the power law exponents that describe
the relaxation. As a test of our numerical fitting scheme, we obtained the power law exponents for the massless Dirac field considered in this paper in the spacetime corresponding to the four-dimensional Schwarzschild black hole with unit event horizon. As we expected, the results obtained are consistent with the power law falloff mentioned in the previous paragraph.

The case for the late-time fermion decay in a (3+1)-stringy black hole is similar, as show the results of our numerical fitting presented in figures 12–17. In these figures we
present some representative results for different sets of values for the black hole charge parameters.

In all cases, with independence of the particular values of charges, we found that the modes with $\ell = 0$ have a numerical decay factor proportional to $t^{-3.09}$, and modes with $\ell = 1$ show a decay factor proportional to $t^{-5.03}$. From the results displayed in figures 12–17, and the other obtained fitting the time-domain data for all the values of the charges and of
angular numbers considered in our numerical work, we can determine the power law factor that dominates the falloff of the Dirac field at very late times, in the spacetime of the stringy black hole. If we suppose that the decay in the Dirac case is governed by factors of the form $t^{-\alpha(\ell+\beta)}$ for each multipole moment $\ell$, then using our numerical results we obtain a decay factor for the Dirac perturbations at late times, that is of the form $t^{-(2\ell+3)}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{Tail for $\ell = 2$, $n = 0$. The power-law coefficients were estimated from numerical data represented by the dotted line. The full red line is the possible analytical result. The charges are chosen as $Q_1 = 0.2$, $Q_2 = 0.5$, $Q_3 = 0.8$, $Q_4 = 1.0$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{Tail for $\ell = 3$, $n = 0$. The power-law coefficients were estimated from numerical data represented by the dotted line. The full red line is the possible analytical result. The charges are chosen as $Q_1 = 0.2$, $Q_2 = 0.5$, $Q_3 = 0.8$, $Q_4 = 1.0$.}
\end{figure}
We remark at this point that this dependence is only a result consistent with our numerical data for all values of the multipole moments studied, and remains an open problem to obtain directly from analytical calculations. Then, we can conclude that, outside four-dimensional stringy black holes, as well as the Schwarzschild black hole, the massless Dirac field shows identical decay at late times.

8. Concluding remarks

We have studied the evolution of massless Dirac perturbations in the spacetime of a (3+1)-dimensional black hole solution from string theory. Solving numerically the time evolution equation for this perturbations, we find similar time-domain profiles as in the case of Dirac fields in other four-dimensional black hole backgrounds. At intermediary times the evolution of fermion perturbations is dominated by quasinormal ringing. We determined the quasinormal frequencies by two different approaches, sixth-order WKB and time-domain integration with Prony fitting of the numerical data, obtaining by both methods very close numerical results.

At the quasinormal ringing epoch, we found that an increase of the charge parameters of the solutions produces a decrease of the oscillating frequency in the system, and a decrease in the damping factor. It is a result valid for all the different setups that can be chosen for the charges, including the case in which one or more charges are zero.

At very late times, the evolution of fermion perturbations in four-dimensional stringy black holes shows a power law falloff proportional to $t^{-\left(2\ell+3\right)}$, as functions of the multipole number. This behavior is identical to that showing in the late time evolution of Dirac fields in other four-dimensional black holes.

There are extensions of this work that will be considered in the future. In the first place it would be interesting to study Dirac perturbations in higher dimensional stringy black holes. Another interesting problem is related with the analytical investigation of the late-time behavior of massless Dirac perturbations outside stringy black holes, to obtain a result in correspondence with our numerical estimate for the decay factor. We also find interesting the consideration of the influence of vacuum polarization effects in the quasinormal spectrum for semiclassical stringy black holes. The solutions to some of the above problems will be presented in future reports.

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Appendix A. Some numerical results for quasinormal frequencies

In this appendix we will show the tables containing the numerical results for the calculations of some QNMs of stringy black holes for different sets of chosen charges.
Table A1. First and second overtones of massless Dirac quasinormal frequencies $\omega_{QY}$ from $\ell = 0$ to $\ell = 10$. The charges are chosen as $Q_1 = Q_2 = Q_3 = Q_4 = 1$.

| $\ell$ | $n$ | Sixth-order WKB | Prony |
|-------|-----|------------------|-------|
| 0     | 0   | 0.1527 − 0.0623i | 0.1525 − 0.0620i |
| 1     | 0   | 0.3147 − 0.0628i | 0.3147 − 0.0626i |
| 2     | 0   | 0.4744 − 0.0629i | 0.4744 − 0.0629i |
| 2     | 1   | 0.4663 − 0.1899i | 0.4661 − 0.1896i |
| 3     | 0   | 0.6337 − 0.0629i | 0.6337 − 0.0629i |
| 3     | 1   | 0.6275 − 0.1894i | 0.6271 − 0.1880i |
| 4     | 0   | 0.7928 − 0.0629i | 0.7928 − 0.0629i |
| 4     | 1   | 0.7878 − 0.1892i | 0.7878 − 0.1892i |
| 5     | 0   | 0.9518 − 0.0629i | 0.9518 − 0.0629i |
| 5     | 1   | 0.9476 − 0.1891i | 0.9476 − 0.1891i |
| 6     | 0   | 1.1107 − 0.0629i | 1.1107 − 0.0629i |
| 6     | 1   | 1.1071 − 0.1890i | 1.1071 − 0.1890i |
| 7     | 0   | 1.2696 − 0.0629i | 1.2696 − 0.0629i |
| 7     | 1   | 1.2665 − 0.1890i | 1.2665 − 0.1890i |
| 8     | 0   | 1.4285 − 0.0629i | 1.4285 − 0.0629i |
| 8     | 1   | 1.4257 − 0.1889i | 1.4257 − 0.1888i |
| 9     | 0   | 1.5873 − 0.0629i | 1.5873 − 0.0629i |
| 9     | 1   | 1.5848 − 0.1889i | 1.5850 − 0.1889i |
| 10    | 0   | 1.7462 − 0.0629i | 1.7462 − 0.0629i |
| 10    | 1   | 1.7439 − 0.1889i | 1.7439 − 0.1889i |

Table A2. First and second overtones of massless Dirac quasinormal frequencies $\omega_{rH}$ from $\ell = 0$ to $\ell = 10$. The charges are chosen as $Q_1 = 0, Q_2 = Q_3 = Q_4 = 1$.

| $\ell$ | $n$ | Sixth-order WKB | Prony |
|-------|-----|------------------|-------|
| 0     | 0   | 0.1857 − 0.0783i | 0.1855 − 0.0780i |
| 1     | 0   | 0.3818 − 0.0786i | 0.3817 − 0.0783i |
| 2     | 0   | 0.5752 − 0.0786i | 0.5752 − 0.0786i |
| 2     | 1   | 0.5660 − 0.2376i | 0.5656 − 0.2375i |
| 3     | 0   | 0.7682 − 0.0786i | 0.7682 − 0.0786i |
| 3     | 1   | 0.7611 − 0.2368i | 0.7606 − 0.2366i |
| 4     | 0   | 0.9610 − 0.0786i | 0.9610 − 0.0786i |
| 4     | 1   | 0.9553 − 0.2364i | 0.9552 − 0.2364i |
| 5     | 0   | 1.1536 − 0.0786i | 1.1536 − 0.0786i |
| 5     | 1   | 1.1489 − 0.2362i | 1.1489 − 0.2362i |
| 6     | 0   | 1.3462 − 0.0786i | 1.3462 − 0.0786i |
| 6     | 1   | 1.3421 − 0.2361i | 1.3421 − 0.2361i |
| 7     | 0   | 1.5388 − 0.0786i | 1.5388 − 0.0786i |
| 7     | 1   | 1.5352 − 0.2360i | 1.5352 − 0.2360i |
| 8     | 0   | 1.7313 − 0.0786i | 1.7313 − 0.0786i |
| 8     | 1   | 1.7281 − 0.2359i | 1.7281 − 0.2359i |
| 9     | 0   | 1.9239 − 0.0786i | 1.9239 − 0.0786i |
| 9     | 1   | 1.9210 − 0.2359i | 1.9210 − 0.2359i |
| 10    | 0   | 2.1164 − 0.0786i | 2.1164 − 0.0786i |
| 10    | 1   | 2.1137 − 0.2358i | 2.1137 − 0.2358i |
The charges are chosen as $Q = 0.2$, $Q_2 = Q_3 = Q_4 = 1$.

| $\ell$ | $n$ | Sixth-order WKB | Prony |
|--------|-----|-----------------|-------|
| 0      | 0   | 0.1769 − 0.0742i| 0.1767 − 0.0740i |
| 1      | 0   | 0.3641 − 0.0744i| 0.3640 − 0.0742i |
| 2      | 0   | 0.5488 − 0.07445i| 0.5488 − 0.07445i |
| 3      | 0   | 0.7329 − 0.0745i| 0.7329 − 0.0745i |
| 4      | 0   | 0.9169 − 0.07445i| 0.9169 − 0.07445i |
| 5      | 0   | 1.1007 − 0.0745i| 1.1007 − 0.0745i |
| 6      | 0   | 1.2845 − 0.0745i| 1.2845 − 0.0745i |
| 7      | 0   | 1.4683 − 0.0745i| 1.4683 − 0.0745i |
| 8      | 0   | 1.6520 − 0.0745i| 1.6520 − 0.0745i |
| 9      | 0   | 1.8357 − 0.0745i| 1.8357 − 0.0745i |
| 10     | 0   | 2.0194 − 0.0745i| 2.0194 − 0.0745i |

The charges are chosen as $Q = 0.4$, $Q_2 = Q_3 = Q_4 = 1$.

| $\ell$ | $n$ | Sixth-order WKB | Prony |
|--------|-----|-----------------|-------|
| 0      | 0   | 0.1695 − 0.0706i| 0.1693 − 0.0704i |
| 1      | 0   | 0.3492 − 0.0709i| 0.3490 − 0.0707i |
| 2      | 0   | 0.5263 − 0.0710i| 0.5263 − 0.0710i |
| 3      | 0   | 0.7030 − 0.0710i| 0.7030 − 0.0710i |
| 4      | 0   | 0.8794 − 0.0710i| 0.8794 − 0.0710i |
| 5      | 0   | 1.0558 − 0.0710i| 1.0558 − 0.0710i |
| 6      | 0   | 1.2321 − 0.0710i| 1.2321 − 0.0710i |
| 7      | 0   | 1.4083 − 0.0710i| 1.4083 − 0.0710i |
| 8      | 0   | 1.5845 − 0.0710i| 1.5845 − 0.0710i |
| 9      | 0   | 1.7607 − 0.0710i| 1.7607 − 0.0710i |
| 10     | 0   | 1.9369 − 0.0710i| 1.9369 − 0.0710i |

The charges are chosen as $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 0.2$.

| $\ell$ | $n$ | Sixth-order WKB | Prony |
|--------|-----|-----------------|-------|
| 0      | 0   | 0.3063 − 0.1499i| 0.3060 − 0.1496i |
| 1      | 0   | 0.6345 − 0.1513i| 0.6343 − 0.1511i |
| 2      | 0   | 0.9576 − 0.1513i| 0.9576 − 0.1513i |
| 2     | 1   | 0.9340 − 0.4591i| 0.9337 − 0.4587i |
| 3      | 0   | 1.2795 − 0.1513i| 1.2795 − 0.1513i |
| 3     | 1   | 1.2615 − 0.4567i| 1.2615 − 0.4567i |
Table A6. First and second overtones of massless Dirac quasinormal frequencies $\omega_{rH}$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 0$.

| $\ell$ | $n$ | Sixth-order WKB | Prony         |
|-------|-----|------------------|--------------|
| 0     | 0   | 0.2636 $-$ 0.1229i | 0.2633 $-$ 0.1226i |
| 1     | 0   | 0.5445 $-$ 0.1238i | 0.5443 $-$ 0.11235i |
| 2     | 0   | 0.8212 $-$ 0.1237i | 0.8212 $-$ 0.1237i  |
| 2     | 1   | 0.8037 $-$ 0.3750i | 0.8035 $-$ 0.3747i  |
| 3     | 0   | 1.0970 $-$ 0.1237i | 1.0970 $-$ 0.1237i  |
| 3     | 1   | 1.0837 $-$ 0.3733i | 1.0837 $-$ 0.3733i  |

Table A7. First and second overtones of massless Dirac quasinormal frequencies $\omega_{rH}$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 0.6$.

| $\ell$ | $n$ | Sixth-order WKB | Prony         |
|-------|-----|------------------|--------------|
| 0     | 0   | 0.2313 $-$ 0.1036i | 0.2310 $-$ 0.1030i |
| 1     | 0   | 0.4768 $-$ 0.1042i | 0.4765 $-$ 0.1038i |
| 2     | 0   | 0.7188 $-$ 0.1042i | 0.7188 $-$ 0.1042i |
| 2     | 1   | 0.7052 $-$ 0.3155i | 0.7050 $-$ 0.3153i |
| 3     | 0   | 0.9601 $-$ 0.1042i | 0.9601 $-$ 0.1042i |
| 3     | 1   | 0.9497 $-$ 0.3142i | 0.9497 $-$ 0.3142i |

Table A8. First and second overtones of massless Dirac quasinormal frequencies $\omega_{rH}$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 1.4$.

| $\ell$ | $n$ | Sixth-order WKB | Prony         |
|-------|-----|------------------|--------------|
| 0     | 0   | 0.1551 $-$ 0.0624i | 0.1547 $-$ 0.0621i |
| 1     | 0   | 0.3183 $-$ 0.0627i | 0.3180 $-$ 0.0621i |
| 2     | 0   | 0.4795 $-$ 0.0627i | 0.4795 $-$ 0.0627i |
| 2     | 1   | 0.4725 $-$ 0.1893i | 0.4723 $-$ 0.1891i |
| 3     | 0   | 0.6403 $-$ 0.0627i | 0.6403 $-$ 0.0627i |
| 3     | 1   | 0.6349 $-$ 0.1889i | 0.6349 $-$ 0.1889i |

Table A9. First and second overtones of massless Dirac quasinormal frequencies $\omega_{rH}$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0$, $Q_2 = Q_3 = Q_4 = 1.8$.

| $\ell$ | $n$ | Sixth-order WKB | Prony         |
|-------|-----|------------------|--------------|
| 0     | 0   | 0.1331 $-$ 0.0517i | 0.1328 $-$ 0.0514i |
| 1     | 0   | 0.2729 $-$ 0.0519i | 0.2727 $-$ 0.0517i |
| 2     | 0   | 0.4110 $-$ 0.0519i | 0.4110 $-$ 0.0519i |
| 2     | 1   | 0.4054 $-$ 0.1568i | 0.4053 $-$ 0.1566i |
| 3     | 0   | 0.5488 $-$ 0.0519i | 0.5488 $-$ 0.0519i |
| 3     | 1   | 0.5445 $-$ 0.1565i | 0.5445 $-$ 0.1565i |
Table A10. First and second overtones of massless Dirac quasinormal frequencies $\omega p H$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0.2$, $Q_2 = 0.4$, $Q_3 = 1$, $Q_4 = 1.4$.  

| $\ell$ | $n$ | Sixth-order WKB | Prony            |
|-------|-----|------------------|------------------|
| 0     | 0   | 0.1858 $- 0.0787i$ | 0.1855 $- 0.0785i$ |
| 1     | 0   | 0.3822 $- 0.0789i$  | 0.3820 $- 0.0786i$  |
| 2     | 0   | 0.5759 $- 0.0789i$  | 0.5759 $- 0.0789i$  |
| 2     | 1   | 0.5664 $- 0.2385i$  | 0.5662 $- 0.2383i$  |
| 3     | 0   | 0.7692 $- 0.1568i$  | 0.7692 $- 0.1568i$  |
| 3     | 1   | 0.7619 $- 0.2378i$  | 0.7619 $- 0.2378i$  |

Table A11. First and second overtones of massless Dirac quasinormal frequencies $\omega p H$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0.8$, $Q_2 = 0.4$, $Q_3 = 0.6$, $Q_4 = 1.0$.  

| $\ell$ | $n$ | Sixth-order WKB | Prony            |
|-------|-----|------------------|------------------|
| 0     | 0   | 0.1879 $- 0.0801i$ | 0.1876 $- 0.0798i$ |
| 1     | 0   | 0.3872 $- 0.0805i$  | 0.3870 $- 0.0803i$  |
| 2     | 0   | 0.5837 $- 0.0805i$  | 0.5837 $- 0.0805i$  |
| 2     | 1   | 0.5735 $- 0.2435i$  | 0.5733 $- 0.2433i$  |
| 3     | 0   | 0.7796 $- 0.0805i$  | 0.7796 $- 0.0805i$  |
| 3     | 1   | 0.7718 $- 0.2427i$  | 0.7718 $- 0.2427i$  |

Table A12. First and second overtones of massless Dirac quasinormal frequencies $\omega p H$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0.6$, $Q_2 = 0.8$, $Q_3 = Q_4 = 1.0$.  

| $\ell$ | $n$ | Sixth-order WKB | Prony            |
|-------|-----|------------------|------------------|
| 0     | 0   | 0.1686 $- 0.0702i$ | 0.1684 $- 0.0699i$ |
| 1     | 0   | 0.3474 $- 0.0706i$  | 0.3472 $- 0.0704i$  |
| 2     | 0   | 0.5372 $- 0.0707i$  | 0.5372 $- 0.0707i$  |
| 2     | 1   | 0.5147 $- 0.2135i$  | 0.5146 $- 0.2133i$  |
| 3     | 0   | 0.6994 $- 0.0707i$  | 0.6994 $- 0.0707i$  |
| 3     | 1   | 0.6926 $- 0.2129i$  | 0.6926 $- 0.2129i$  |

Table A13. First and second overtones of massless Dirac quasinormal frequencies $\omega p H$ from $\ell = 0$ to $\ell = 3$. The charges are chosen as $Q_1 = 0.2$, $Q_2 = Q_3 = Q_4 = 1.0$.  

| $\ell$ | $n$ | Sixth-order WKB | Prony            |
|-------|-----|------------------|------------------|
| 0     | 0   | 0.1769 $- 0.0742i$ | 0.1766 $- 0.0740i$ |
| 1     | 0   | 0.3641 $- 0.0744i$  | 0.3639 $- 0.0742i$  |
| 2     | 0   | 0.5488 $- 0.0745i$  | 0.5488 $- 0.0745i$  |
| 2     | 1   | 0.5396 $- 0.2251i$  | 0.5395 $- 0.2250i$  |
| 3     | 0   | 0.7329 $- 0.0745i$  | 0.7329 $- 0.0745i$  |
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**Table A14.** Sixth-order WKB results for some quasinormal frequencies including high overtones for stringy black holes with all charges $Q_i = 1$.

| $L$ | $n$ | $\omega$   |
|-----|-----|----------------|
| 4   | 2   | 0.7781 $-$ 0.3169i |
| 4   | 3   | 0.7641 $-$ 0.4466i |
| 5   | 2   | 0.9394 $-$ 0.3162i |
| 5   | 3   | 0.9275 $-$ 0.4448i |
| 5   | 4   | 0.9121 $-$ 0.5755i |
| 6   | 2   | 1.100 $-$ 0.3158i |
| 6   | 3   | 1.0897 $-$ 0.4437i |
| 6   | 4   | 1.0762 $-$ 0.5732i |
| 7   | 2   | 1.2603 $-$ 0.3155i |
| 7   | 3   | 1.2511 $-$ 0.4430i |
| 7   | 4   | 1.2392 $-$ 0.5716i |
| 7   | 5   | 1.2246 $-$ 0.7018i |
| 7   | 6   | 1.2077 $-$ 0.8336i |
| 8   | 2   | 1.4202 $-$ 0.3154i |
| 8   | 3   | 1.4120 $-$ 0.4425i |
| 8   | 4   | 1.4012 $-$ 0.5706i |
| 8   | 5   | 1.3881 $-$ 0.6999i |
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