Continuous-Time Quantum Walks on Trees
in Quantum Probability Theory

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Abstract

A quantum central limit theorem for a continuous-time quantum walk on a homogeneous tree is derived from quantum probability theory. As a consequence, a new type of limit theorems for another continuous-time walk introduced by the walk is presented. The limit density is similar to that given by a continuous-time quantum walk on the one-dimensional lattice.

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I. INTRODUCTION

Two types of quantum walks, discrete-time or continuous-time, were introduced as the quantum mechanical extension of the corresponding classical random walks and have been extensively studied over the last few years, see [1, 2] for recent reviews. In this paper we consider a continuous-time quantum walk on a homogeneous tree in quantum probability theory. The walk is defined by identifying the Hamiltonian of the system with a matrix related to the adjacency matrix of the tree. Concerning continuous-time quantum walks, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for examples.

Let \( T^{(p)}_M \) denote a homogeneous tree of degree \( p \) with \( M \)-generation. After we fix a root \( o \in T^{(p)}_M \), a stratification (distance partition) is introduced by the natural distance function in the following way:

\[
T^{(p)}_M = \bigcup_{k=0}^{M} V_k^{(p)}, \quad V_k^{(p)} = \{ x \in T^{(p)}_M : \partial(o, x) = k \}.
\]

Here \( \partial(x, y) \) stands for the length of the shortest path connecting \( x \) and \( y \). Then

\[
|V_0^{(p)}| = 1, \quad |V_1^{(p)}| = p, \quad |V_2^{(p)}| = p(p-1), \ldots, \quad |V_k^{(p)}| = p(p-1)^{k-1}, \ldots,
\]

where \( |A| \) is the number of elements in a set \( A \). The total number of points in \( M \)-generation, \( |T^{(p)}_M| \), is \( p(p-1)^M - (p-1) \).

Let \( H^{(p)}_M \) be a \( |T^{(p)}_M| \times |T^{(p)}_M| \) symmetric matrix given by the adjacency matrix of the tree \( T^{(p)}_M \). The matrix is treated as the Hamiltonian of the quantum system. The \((i, j)\) component of \( H^{(p)}_M \) denotes \( H^{(p)}_M(i, j) \) for \( i, j \in \{0, 1, \ldots, |T^{(p)}_M|-1\} \). In our case, the diagonal component of \( H^{(p)}_M \) is always zero, i.e., \( H^{(p)}_M(i, i) = 0 \) for any \( i \). On the other hand, the diagonal component of corresponding matrix \( H^{(p)}_{M,MB} \) investigated in [15] for \( p = 3 \) case is not zero. For example, \( H^{(3)}_{1,MB}(0, 0) = -3, \quad H^{(3)}_{1,MB}(1, 1) = H^{(3)}_{1,MB}(2, 2) = H^{(3)}_{1,MB}(3, 3) = -1 \).

The evolution of continuous-time quantum walk on the tree of \( M \)-generation, \( T^{(p)}_M \), is governed by the following unitary matrix:

\[
U^{(p)}_M(t) = e^{itH^{(p)}_M}.
\]

The amplitude wave function at time \( t \), \( |\Psi^{(p)}_M(t)\rangle \), is defined by

\[
|\Psi^{(p)}_M(t)\rangle = U^{(p)}_M(t)|\Psi^{(p)}_M(0)\rangle.
\]
In this paper we take $|\Psi_M^{(p)}(0)\rangle = [1, 0, 0, \ldots, 0]^T$ as an initial state, where $T$ denotes the transposed operator.

The $(n + 1)$-th coordinate of $|\Psi_M^{(p)}(t)\rangle$ is denoted by $|\Psi_M^{(p)}(n, t)\rangle$ which is the amplitude wave function at site $n$ at time $t$ for $n = 0, 1, \ldots, p(p - 1)^M - p$. The probability finding the walker is at site $n$ at time $t$ on $T_M^{(p)}$ is given by

$$P_M^{(p)}(n, t) = \langle \Psi_M^{(p)}(n, t) | \Psi_M^{(p)}(n, t) \rangle.$$ 

Then we define the continuous-time quantum walk $X_M^{(p)}(t)$ at time $t$ on $T_M^{(p)}$ by

$$P(X_M^{(p)}(t) = n) = P_M^{(p)}(n, t).$$

In a similar way, let $X_{M,MB}^{(p)}(t)$ be a quantum walk given by $H_{M,MB}^{(p)}$. As we stated before, $H_{M,MB}^{(p)}(i, i)$ depends on $i$ for any finite $M$. However in $M \to \infty$ limit, the $(i, i)$ component of the matrix becomes $-p$ for any $i$. Remark that the probability distribution of the continuous-time walk does not depend on the value of the diagonal component of the scalar matrix. Therefore the definitions of the walks imply that both quantum walks coincide in $M \to \infty$ limit, i.e.,

$$\lim_{M \to \infty} P(X_M^{(p)}(t) = n) = \lim_{M \to \infty} P(X_{M,MB}^{(p)}(t) = n),$$

for any $t$ and $n$.

This paper is organized as follows. In Sec. 2, we review the quantum probabilistic approach and give preliminaries and some examples for the walk $X_M^{(p)}(t)$. A quantum central limit theorem as $p \to \infty$ is derived from quantum probability theory in Sec. 3. Finally we present a limit theorem for another continuous-time walk $Y(t)$ introduced as a $p$-limit walk of $X_{\infty}^{(p)}(t)$.

II. QUANTUM PROBABILISTIC APPROACH

A. Finite $M$ case

Let $\mu_M^{(p)}$ denote the spectral distribution of our adjacency matrix $H_M^{(p)}$. From the general theory of an interacting Fock space (see [14, 17, 18, 19, 20, 21], for examples), the orthogonal polynomials $\{Q_n^{(p)}\}$ and $\{Q_n^{(p,*)}\}$ associated with $\mu_M^{(p)}$ satisfy the following three-term
recurrence relations with a Szegö-Jacobi parameter \( \{ \omega_n \}, \{ \alpha_n \} \) respectively:

\[
Q_0^{(p)}(x) = 1, \quad Q_1^{(p)}(x) = x - \alpha_1,
\]

\[
xQ_n^{(p)}(x) = Q_{n+1}^{(p)}(x) + \alpha_{n+1}Q_n^{(p)}(x) + \omega_nQ_{n-1}^{(p)}(x) \quad (n \geq 1),
\]

and

\[
Q_0^{(p,*)}(x) = 1, \quad Q_1^{(p,*)}(x) = x - \alpha_2,
\]

\[
xQ_n^{(p,*)}(x) = Q_{n+1}^{(p,*)}(x) + \alpha_{n+2}Q_n^{(p,*)}(x) + \omega_{n+1}Q_{n-1}^{(p,*)}(x) \quad (n \geq 1).
\]

In our tree case,

\[
\omega_1 = p, \quad \omega_2 = \omega_3 = \cdots = \omega_M = p - 1, \quad \omega_{M+1} = \omega_{M+2} = \cdots = 0, \quad \alpha_1 = \alpha_2 = \cdots = 0.
\]

Then the Stieltjes transform \( G_{\mu_M^{(p)}}(x) \) of \( \mu_M^{(p)} \) is given by

\[
G_{\mu_M^{(p)}}(x) = \frac{Q_{n-1}^{(p,*)}(x)}{Q_n^{(p)}(x)},
\]

where \( n = |T_M^{(p)}| = p(p - 1)^M - (p - 1) \).

The following result was shown in \[14\]:

\[
|\Psi_M^{(p)}(V_k^{(p)}, t)\rangle = \frac{1}{\sqrt{|V_k^{(p)}|}} \int_{\mathbb{R}} \exp(itx) Q_k^{(p)} \mu_M^{(p)}(x) \, dx,
\]

for \( k = 0, 1, 2, \ldots \). Remark that \( |V_k^{(p)}| = \omega_1\omega_2\cdots\omega_k = p(p - 1)^{k-1} \) (\( 1 \leq k \leq M \)) and \( |V_0^{(p)}| = 1 \). It is important to note that

\[
|\Psi_M^{(p)}(n, t)\rangle = \frac{1}{|V_k^{(p)}|} \int_{\mathbb{R}} \exp(itx) Q_k^{(p)} \mu_M^{(p)}(x) \, dx \quad \text{if} \quad n \in V_k^{(p)} \quad (k = 0, 1, \ldots, M).
\]

The proof appeared in Appendix A in \[14\].

**B. \( p = 3 \) and \( M = 2 \) case**

Here we consider \( p = 3 \) and \( M = 2 \) case. Then we have \( n = 10, \omega_1 = 3, \omega_2 = 2, \omega_3 = \omega_4 = \cdots = 0, \alpha_1 = \alpha_2 = \cdots = 0 \). The definitions of \( Q_n^{(3)}(x) \) and \( Q_n^{(3,*)}(x) \) imply

\[
Q_0^{(3)}(x) = 1, \quad Q_1^{(3)}(x) = x, \quad Q_2^{(3)}(x) = x^2 - 3, \quad Q_k^{(3)}(x) = x^{k-2}(x^2 - 5) \quad (k \geq 3),
\]
and
\[ Q_0^{(3, \ast)}(x) = 1, \quad Q_1^{(3, \ast)}(x) = x, \quad Q_k^{(3, \ast)}(x) = x^{k-2}(x^2 - 2) \quad (k \geq 2). \]

Therefore we obtain the Stieltjes transform:
\[
G_{\mu_2^{(3)}}(x) = \frac{Q_0^{(3, \ast)}(x)}{Q_{10}^{(3)}(x)} = \frac{2}{5} \cdot \frac{1}{x} + \frac{3}{10} \cdot \frac{1}{x - \sqrt{5}} + \frac{3}{10} \cdot \frac{1}{x + \sqrt{5}}.
\]

From this, we see that
\[
\mu_2^{(3)} = \frac{2}{5} \delta_0(x) + \frac{3}{10} \delta_{-\sqrt{5}}(x) + \frac{3}{10} \delta_{\sqrt{5}}(x).
\]

Then
\[
|\Psi_2^{(3)}(V_0^{(3)}, t)\rangle = \int_{\mathbb{R}} \exp(itx) \mu_2^{(3)}(dx) = \frac{1}{5} (2 + 3 \cos(\sqrt{5}t)),
\]
\[
|\Psi_2^{(3)}(V_1^{(3)}, t)\rangle = \frac{1}{\sqrt{\omega_1}} \int_{\mathbb{R}} \exp(itx)Q_1^{(3)}(x) \mu_2^{(3)}(dx) = \frac{i\sqrt{3}}{\sqrt{5}} \sin(\sqrt{5}t),
\]
\[
|\Psi_2^{(3)}(V_2^{(3)}, t)\rangle = \frac{1}{\sqrt{\omega_1\omega_2}} \int_{\mathbb{R}} \exp(itx)Q_2^{(3)}(x) \mu_2^{(3)}(dx) = \frac{\sqrt{6}}{5} \left(-1 + \cos(\sqrt{5}t)\right).
\]

Noting that \(|\Psi_2^{(3)}(n, t)\rangle = |\Psi_2^{(3)}(V_k^{(3)}, t)\rangle/|V_k^{(3)}|\) for any \(k = 0, 1, 2\), we obtain the same conclusion as the result given by the eigenvalues and the eigenvectors of \(H_2^{(3)}\).

C. \(M \to \infty\) case

The quantum probabilistic approach [14, 20, 21] implies that
\[
|\Psi_\infty^{(p)}(V_k^{(p)}, t)\rangle = \lim_{M \to \infty} |\Psi_M^{(p)}(V_k, t)\rangle = \frac{1}{\sqrt{|V_k^{(p)}|}} \int_{\mathbb{R}} \exp(itx) Q_k^{(p)}(x) \mu_\infty^{(p)}(x) \, dx,
\]
for \(k = 0, 1, 2, \ldots\), where the limit spectral distribution \(\mu_\infty^{(p)}(x)\) is given by
\[
I_{(-2\sqrt{p-1}, 2\sqrt{p-1})}(x) \frac{p\sqrt{4(p-1) - x^2}}{2\pi(p^2 - x^2)}.
\]
Here \(I_A\) is the indicator function of \(A\), i.e., \(I_A(x) = 1\), if \(x \in A\), \(= 0\), if \(x \notin A\). This type of measure was first obtained by Kesten [22] in a classical random walk with a different method. An immediate consequence is
\[
P_\infty^{(p)}(V_k^{(p)}, t) = \frac{1}{|V_k^{(p)}|} \left\{ \int_{\mathbb{R}} \cos(tx) Q_k^{(p)}(x) \mu_\infty^{(p)}(x) \, dx \right\}^2 + \left\{ \int_{\mathbb{R}} \sin(tx) Q_k^{(p)}(x) \mu_\infty^{(p)}(x) \, dx \right\}^2,
\]
for \( k = 0, 1, 2, \ldots \). Furthermore, as in the case of finite \( M \), we see that

\[
|\Psi_{\infty}^{(p)}(n, t)\rangle = \frac{1}{|V_{k}^{(p)}|} \int_{\mathbb{R}} \exp(itx) Q_{k}^{(p)}(x) \mu_{\infty}^{(p)}(x) \, dx,
\]

(1)

if \( n \in V_{k}^{(p)} \) \( (k = 0, 1, 2, \ldots) \). From (1) and the Riemann-Lebesgue lemma, we have

\[
\lim_{t \to \infty} |\Psi_{\infty}^{(p)}(n, t)\rangle = 0,
\]

for any \( n \), since \( Q_{k}^{(p)}(x) \mu_{\infty}^{(p)}(x) \in L^{1}(\mathbb{R}) \). Therefore we see that \( \lim_{t \to \infty} P_{\infty}^{(p)}(n, t) = 0 \). So we conclude that \( \bar{P}_{\infty}^{(p)}(n) = 0 \), where \( \bar{P}_{\infty}^{(p)}(n) \) is the time-averaged distribution of \( P_{\infty}^{(p)}(n, t) \).

D. \( p = 2 \) and \( M \to \infty \) case

In this subsection, we consider \( p = 2 \) and \( M \to \infty \), i.e., \( \mathbb{Z}^{1} \) case. Then we have

**Proposition 1.**

\[
|\Psi_{\infty}^{(2)}(V_{0}^{(2)}, t)\rangle = J_{0}(2t), \quad |\Psi_{\infty}^{(2)}(V_{k}^{(2)}, t)\rangle = \sqrt{2} i^{k} J_{k}(2t), \quad (k = 0, 1, 2, \ldots),
\]

where \( J_{n}(x) \) is the Bessel function of the first kind of order \( n \).

**Proof.** We induct on \( k \). For \( k = 0 \) case, we use the following result (see (4) in page 48 in [23]):

\[
\int_{-1}^{1} \exp(isx) \left(1 - x^{2}\right)^{\nu-1/2} \, dx = \frac{\Gamma(1/2)\Gamma(\nu + 1/2)}{(s/2)^{\nu}} J_{\nu}(s),
\]

(2)

where \( \Gamma(x) \) is the Gamma function. Combining \( \Gamma(3/2) = \sqrt{\pi}/2, \Gamma(1/2) = \sqrt{\pi} \) with \( Q_{0}^{(2)}(x) = 1 \) and \( \nu = 0 \) gives

\[
|\Psi_{\infty}^{(2)}(V_{0}^{(2)}, t)\rangle = \int_{-2}^{2} \exp(itx) \frac{1}{\pi\sqrt{4 - x^{2}}} \, dx = J_{0}(2t).
\]

In a similar fashion, we verify that the result holds for \( k = 1, 2 \).

Next we suppose that the result is true for all values up to \( k \), where \( k \geq 2 \). Then we see
that
\[
\langle \Psi_{\infty}^{(2)}(V_{k+1}^{(2)}, t) \rangle
= \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx) \frac{Q_{k+1}^{(2)}(x) dx}{\pi \sqrt{4-x^2}}
\]
\[
= \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx) \left\{ xQ_{k}^{(2)}(x) - Q_{k-1}^{(2)}(x) \right\} \frac{dx}{\pi \sqrt{4-x^2}}
\]
\[
= \frac{1}{i \frac{d}{dt}} \left( \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx) \frac{Q_{k}^{(2)}(x) dx}{\pi \sqrt{4-x^2}} \right) - \frac{1}{\sqrt{2}} \int_{-2}^{2} \exp(itx) \frac{Q_{k-1}^{(2)}(x) dx}{\pi \sqrt{4-x^2}}
\]
\[
= \frac{1}{i \frac{d}{dt}} (\sqrt{2} i^k J_k(2t)) - \sqrt{2} i^{k-1} J_{k-1}(2t)
\]
\[
= \sqrt{2} i^{k+1} J_{k+1}(2t).
\]

The second equality follows from the definition of $Q_{k}^{(2)}(x)$. By induction, we have the fourth equality. For the last equality, we use a recurrence formula for the Bessel coefficients: $2J_{k}^{(2)}(2t) = J_{k-1}(2t) - J_{k+1}(2t)$ (see (2) in page 17 of [23]).

As a consequence, we have

**Corollary 1.**

\[ P_{\infty}^{(2)}(V_0^{(2)}, t) = J_0^2(2t), \quad P_{\infty}^{(2)}(V_k^{(2)}, t) = 2J_k^2(2t), \quad (k = 1, 2, \ldots). \]

We confirm that

\[
\sum_{k=0}^{\infty} P_{\infty}^{(2)}(V_k^{(2)}, t) = 1,
\]

since it follows from $J_0^2(2t) + 2 \sum_{k=1}^{\infty} J_k^2(2t) = 1$ (see (3) in page 31 in [23]). Noting that $V_k^{(2)} = \{-k, k\}$ for any $k \geq 0$, we have the same result given by [10]:

\[ P_{\infty}^{(2)}(n, t) = J_n^2(2t), \]

for any $n \in \mathbb{Z}$ and $t \geq 0$.

**III. QUANTUM CENTRAL LIMIT THEOREM**

To state a quantum central limit theorem in our case, it is convenient to rewrite as

\[
\langle \Phi_{k}^{(p)} \mid \exp(itH_{\infty}^{(p)}) \mid \Phi_{0}^{(p)} \rangle = |\Psi_{\infty}^{(p)}(V_k^{(p)}, t)\rangle,
\]

\[\]
where

\[ \Phi_k(p) = \frac{1}{\sqrt{|V_k(p)|}} \sum_{n \in V_k(p)} I_n, \]

and \( I_n \) denotes the indicator function of the singleton \( \{n\} \). It is easily obtained that

\[ \lim_{p \to \infty} \langle \Phi_k(p) | \exp \left( it \frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) | \Phi_0^{(p)} \rangle = 0, \]

for any \( k \geq 0 \). Then we have the following quantum central limit theorem:

**Theorem 1.**

\[ \lim_{p \to \infty} \langle \Phi_k(p) | \exp \left( it \frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) | \Phi_0^{(p)} \rangle = (k + 1) i^k \frac{J_{k+1}(2t)}{t}, \]

for \( k = 0, 1, 2, \ldots \).

**Proof.** We induct on \( k \). First we consider \( k = 0 \) case. We see that

\[ \lim_{p \to \infty} \langle \Phi_0^{(p)} | \exp \left( it \frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) | \Phi_0^{(p)} \rangle = \frac{J_1(2t)}{t}. \]

So the result holds for \( k = 0 \). Similarly we obtain

\[ \lim_{p \to \infty} \langle \Phi_1^{(p)} | \exp \left( it \frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) | \Phi_0^{(p)} \rangle = \frac{2i J_2(2t)}{t}, \]

\[ \lim_{p \to \infty} \langle \Phi_2^{(p)} | \exp \left( it \frac{H_{\infty}^{(p)}}{\sqrt{p}} \right) | \Phi_0^{(p)} \rangle = -\frac{3 J_3(2t)}{t}. \]
Next we suppose that the result holds for all values up to \( k \), where \( k \geq 2 \). Then we have

\[
\lim_{p \to \infty} \left< \Phi_{k+1}^{(p)} \left| \exp \left( it \frac{H^{(p)}_{\infty}}{\sqrt{p}} \right) \right| \Phi_0^{(p)} \right>
= \lim_{p \to \infty} \frac{1}{\sqrt{|V_{k+1}^{(p)}|}} \int_{\mathbb{R}} \exp \left( it \frac{x}{\sqrt{p}} \right) Q_{k+1}^{(p)}(x) \mu_{\infty}^{(p)}(x) \, dx
= \lim_{p \to \infty} \frac{1}{\sqrt{p(p-1)^k}} \int_{-2}^{2} \exp (itx) \frac{\sqrt{2^2 - x^2}}{2\pi} \, dx \frac{\overline{\sqrt{(2(p-1)/p)^2 - x^2}}}{2\pi} \, dx
= \int_{-2}^{2} \exp (itx) Q_{k+1}^{(\infty)}(x) \frac{\sqrt{2^2 - x^2}}{2\pi} \, dx
= \frac{1}{i} \left( \int_{-1}^{1} \exp (2itx) Q_{k}^{(\infty)}(2x) \frac{2\sqrt{1-x^2}}{\pi} \, dx \right)
- \int_{-1}^{1} \exp (2itx) Q_{k-1}^{(\infty)}(2x) \frac{2\sqrt{1-x^2}}{\pi} \, dx
= i^{k-1} \left\{ (k+1) \frac{d}{dt} \left( \frac{J_{k+1}(2t)}{t} \right) - k \frac{J_k(2t)}{t} \right\}
\]

where the last equality is given by the induction and \( Q_k^{(\infty)}(x) = \lim_{p \to \infty} Q_k^{(p)}(\sqrt{p}x)/\sqrt{p(p-1)^{k-1}} \), if the right hand side exists. We confirm that the limit exists for any \( k \geq 1 \). For example, we compute \( Q_1^{(\infty)}(x) = x \), \( Q_2^{(\infty)}(x) = x^2 - 1 \), \( Q_3^{(\infty)}(x) = x^3 - 2x \), \( Q_4^{(\infty)}(x) = x^4 - 3x^2 + 1 \ldots \) In order to prove the result, it suffices to check the following relation:

\[
(k+1) \frac{d}{dt} \left( \frac{J_{k+1}(2t)}{t} \right) - k \frac{J_k(2t)}{t} = -(k+2) \frac{J_{k+2}(2t)}{t}.
\]

The left hand side of this equation becomes

\[
(k+1) \frac{2J_{k+1}(2t)}{t^2} - (k+1) \frac{J_{k+1}(2t)}{t^2} - k \frac{J_k(2t)}{t}
= \frac{J_k(2t)}{t} - (k+1) \frac{J_{k+1}(2t)}{t^2} - (k+1) \frac{J_k(2t)}{t}
= -(k+2) \frac{J_{k+2}(2t)}{t},
\]

since the first and second equalities are obtained from recurrence formulas for the Bessel coefficients: \( 2J_{k+1}'(2t) = J_k(2t) - J_{k+2}(2t) \) and \( J_k(2t) + J_{k+2}(2t) = (k+1)J_{k+1}(2t)/t \) (see (1) in page 17 of [23]), respectively. This finishes the proof of the theorem. \[\square\]
IV. A NEW TYPE OF LIMIT THEOREMS

We can now state the main result of this paper. To do so, let define
\[ |\tilde{\Psi}_\infty(V_{k_0}(\infty), t)| = \lim_{p \to \infty} \left| \Phi_p(k) \right| \exp \left( \frac{it}{\sqrt{p}} H^{(p)}_\infty \right) \Phi_0(p), \]
and
\[ \tilde{P}_\infty(V_{k_0}(\infty), t) = \langle \tilde{\Psi}_\infty(V_{k_0}(\infty), t) | \tilde{\Psi}_\infty(V_{k_0}(\infty), t) \rangle. \]

By Theorem 1 and the definition of \( |\tilde{\Psi}_\infty(V_{k_0}(\infty), t)| \), we see that
\[ \sum_{k=0}^{\infty} \tilde{P}_\infty(V_{k_0}(\infty), t) = \sum_{k=1}^{\infty} k^2 J_k^2(2t) = 1. \] (3)

The second equality comes from an expansion of \( z^2 \) as a series of squares of Bessel coefficients (see page 37 in [23]):
\[ z^2 = 4 \sum_{k=1}^{\infty} k^2 J_k^2(z). \]

Noting the result (3), here we define another continuous-time quantum walk \( Y(t) \) starting from the root defined by
\[ P(Y(t) = k) = \tilde{P}_\infty(V_{k_0}(\infty), t) = (k + 1)^2 J_{k+1}^2(2t) \frac{t^2}{k^2}. \]

Therefore we obtain

**Theorem 2.**
\[ \frac{Y(t)}{t} \Rightarrow Z, \]
as \( t \to \infty \), where \( \Rightarrow \) means the weak convergence and \( Z \) has the following density function:
\[ I_{(0,2)}(x) = \frac{x^2}{\pi \sqrt{4 - x^2}}. \]

**Proof.** From Theorem 1, we begin with computing
\[ E\left( \exp \left( \frac{-i\xi Y(t)}{t} \right) \right) = \exp \left( \frac{-i\xi}{t} \sum_{k=1}^{\infty} \exp \left( \frac{i\xi}{t} k \right) k^2 J_{k+1}^2(2t) \right), \]
for $\xi \in \mathbb{R}$. By Neumann’s addition theorem (see p.358 in [23]), we have

$$J_0(\sqrt{a^2 + b^2 - 2ab \cos(\xi)}) = \sum_{k=-\infty}^{\infty} J_k(a)J_k(b) \exp(ik\xi).$$

Taking $t = a = b$ in this equation gives

$$J_0(4t \sqrt{\sin(\xi/2)}) = \sum_{k=-\infty}^{\infty} J_k^2(t) \exp(ik\xi).$$

By differentiating both sides of the equation with respect to $t$ twice, we see

$$\sum_{k=1}^{\infty} k^2 J_k^2(t) \exp(ik\xi) = \frac{1}{2} \sum_{k=-\infty}^{\infty} k^2 J_k^2(t) \exp(ik\xi)$$

$$= \frac{t}{4} \sin\left(\frac{\xi}{2}\right) J_0'(2t \sin\left(\frac{\xi}{2}\right)) - \frac{t^2}{4} \cos^2\left(\frac{\xi}{2}\right) J_0''(2t \sin\left(\frac{\xi}{2}\right)).$$

Therefore we obtain

$$E\left(\exp\left(i\xi \frac{Y(t)}{t}\right)\right) = \exp\left(-\frac{i\xi}{t}\right) \left\{ \frac{1}{2t} \sin\left(\frac{\xi}{2}\right) J_0'(4t \sin\left(\frac{\xi}{2}\right)) \right.\right.$$

$$\left. - \frac{t}{2} \cos^2\left(\frac{\xi}{2}\right) J_0''(4t \sin\left(\frac{\xi}{2}\right)) \right\}.$$

Then a similar argument in [10] yields

$$\lim_{t \to \infty} E\left(\exp\left(i\xi \frac{Y(t)}{t}\right)\right) = -2J_0''(2\xi).$$

On the other hand, (2) with $\nu = 0$ gives

$$J_0''(2\xi) = -\int_{-1}^{1} \exp(2i\xi x) \frac{x^2}{\pi \sqrt{1 - x^2}} dx.$$

From the last two equations, we conclude that

$$\lim_{t \to \infty} E\left(\exp\left(i\xi \frac{Y(t)}{t}\right)\right) = \int_{0}^{2} \exp(i\xi x) \frac{x^2}{\pi \sqrt{4 - x^2}} dx.$$

It is interesting to remark that when $p = 2$ case, i.e., $\mathbb{Z}^1$, a similar type of density function was derived from a limit theorem for $X(t)$ (see [10]):

$$\lim_{t \to \infty} E\left(\exp\left(i\xi \frac{X(t)}{t}\right)\right) = \int_{0}^{1} \exp(i\xi x) \frac{2}{\pi \sqrt{1 - x^2}} dx.$$
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