Abstract

Despite the wealth of research into provably efficient reinforcement learning algorithms, most works focus on tabular representation and thus struggle to handle exponentially or infinitely large state-action spaces. In this paper, we consider episodic reinforcement learning with a continuous state-action space which is assumed to be equipped with a natural metric that characterizes the proximity between different states and actions. We propose ZOOMRL, an online algorithm that leverages ideas from continuous bandits to learn an adaptive discretization of the joint space by zooming in more promising and frequently visited regions while carefully balancing the exploitation-exploration trade-off. We show that ZOOMRL achieves a worst-case regret $\tilde{O}(H^{2/3}K^{1/6})$ where $H$ is the planning horizon, $K$ is the number of episodes and $d$ is the covering dimension of the space with respect to the metric. Moreover, our algorithm enjoys improved metric-dependent guarantees that reflect the geometry of the underlying space. Finally, we show that our algorithm is robust to small misspecification errors.

1. Introduction

Reinforcement learning (Sutton and Barto, 1998) (RL) is a framework for solving sequential decision-making problems. Through trial and error an agent must learn to act optimally in an unknown environment in order to maximize its expected utility. Efficient learning requires balancing exploration (acting to gain more knowledge) and exploitation (acting optimally according to the available knowledge).

Optimism in the face of uncertainty (OFU) is one of the traditional guiding principles that offers provably efficient learning algorithms. We can distinguish two classes of approaches: confidence-intervals based methods (Kearns and Singh, 2002; Strehl and Littman, 2005; Jaksch et al., 2010) and exploration-bonus based methods (Azar et al., 2017; Jin et al., 2018; Jian et al., 2019). In the former, the agent builds a set of statistically plausible Markov Decision Processes (MDPs) that contains the true MDP with high probability. Then, the agent selects the most optimistic version of its model and acts optimally with respect to it. In the latter, discoveries of poorly understood states and actions are rewarded by an exploration bonus. Such bonus is designed to bound estimation errors on the value function.

In the regime of MDPs with a finite state-action space, the OFU principle has been successfully implemented and efficient algorithms typically achieve regret that scales sublinearly with the number of discrete states and the number of discrete actions. This precludes applying them to arbitrarily large state-action spaces. On the other hand, MDPs with continuous state-actions spaces have been an active area of investigation (Ortner and Ryabko, 2012; Lakshmanan et al., 2015; Song and Sun, 2019). A common theme is to assume some structure knowledge, such as the existence of similarity metric between state-action pairs, and then to use uniform discretisation of the space or nearest-neighbor approximators.

In this work, we focus on the finite-horizon MDP formalism with an unknown transition kernel. We suppose that the state-action space is equipped by a metric that characterizes the proximity between different states and actions. Such metrics have been studied in previous work for state aggregation (Ferns et al., 2004; Ortner, 2007). We assume that the optimal action-value function is Lipschitz continuous with respect to this metric, which means that state-action pairs that are close to each other have similar optimal values.

We propose an online model-free RL algorithm, ZOOMRL, that actively explores the state-action space by learning on-the-fly an adaptive partitioning. Algorithms based on uniform partitions, such as the works in Ortner and Ryabko (2012) and Song and Sun (2019), disregard the shape of the optimal value function and thus could waste effort in partitioning irrelevant regions of the space. Moreover, the granularity of the partition should be tuned and it depends on the time horizon and the covering dimension of joint space. In contrast, ZOOMRL is able to take advantage of the structure of the problem’s instance at hand by adjusting the discretisation to frequently visited and high-rewarding
regions to get better estimates. Zooming approaches have been successfully applied in Lipschitz bandits (Kleinberg et al., 2008) and continuous contextual bandits (Slivkins, 2014). However, in the bandit setting, an algorithm’s cumulative regret can be easily decomposed into regret incurred in each sub-partition which is controlled by the size of the sub-partition itself. In contrast, in the reinforcement learning setting, the errors are propagated through iterations and we need to carefully control how they accumulate over iterations and navigate through sub-partitions. We show that ZoomRL achieves a worst-case regret \( O(H^{\frac{d}{2} + \frac{1}{2}}) \) where \( H \) is the planning horizon, \( K \) is the number of episodes and \( d \) is the covering dimension of the space with respect to the metric. Moreover, ZoomRL enjoys an improved metric-dependent guarantee that reflects the geometry of the underlying space and whose scaling in terms of \( K \) is optimal as it matches the lower bound in continuous contextual bandit (Slivkins, 2014) when \( H = 1 \). Finally, we study how our algorithm cope with the misspecified setting (Assumption 5.3). We show that it is robust to small misspecification error as it suffers only from an additional regret term \( O(HK\epsilon) \) if the true optimal action-value function is Lipschitz up to an additive error uniformly bounded in absolute value by \( \epsilon \).

2. Related Work

**Exploration in metric spaces:** There have been several recent works that study exploration in continuous state-action MDPs under different structured assumptions. Kakade et al. (2003) assume a local continuity of the reward function and the transition kernel with respect to a given metric. They propose a generalization of the \( E^3 \) algorithm of Kearns and Singh (2002) to metric spaces. Their sample complexity depends on the covering number of the space under the continuity metric instead of the number of the states. However, their algorithm requires access to an approximate planning oracle. Lattimore et al. (2013) assume that the true transition kernel belongs to a finite or compact hypothesis class. Their algorithm consists in maintaining a set of transitions models and pruning it over time by eliminating the provable implausible models. They establish a sample complexity that depends polynomially on the cardinality or covering number of the model class. Pazis and Parr (2013) consider a continuous state-action MDP, develop a nearest-neighbor based algorithm under the assumption that all Q-functions encountered are Lipschitz continuous, showing a sample complexity that depends on an approximate covering number. Ortner and Ryabko (2012) develop a model-based algorithm that combines state aggregation with the standard UCRL2 algorithm (Jaksch et al., 2010) under the assumption of Lipschitz or Hölder continuity of rewards and transition kernel and they establish a regret bound scaling in \( K^{\frac{d}{2} + \frac{1}{2}} \) where \( d \) in the dimension of the state space and \( K \) is the number of episodes. Lakshmanan et al. (2015) improve the latter work by considering a kernel density estimator instead of a frequency estimator for the transition probabilities. They achieve a regret bound of \( K^{\frac{d}{2} + \frac{1}{2}} \). Yang et al. (2019) consider a deterministic control system under a Lipschitz assumption of the optimal action-value functions and the transition function and they establish a regret of \( K^{\frac{d}{2} + \frac{1}{2}} \) where \( d \) here is the doubling dimension. Recently, Song and Sun (2019) extended the tabular Q-learning with upper-confidence bound exploration strategy, developed in Jin et al. (2018), to continuous state-action MDPs using a uniform discretisation of the joint space leading to the regret bound \( O(H^{\frac{d}{2} + \frac{1}{2}}) \) where \( H \) is the planning horizon and \( d \) is the covering dimension. They only assume that the optimal action-value function is Lipschitz continuous. This assumption is more general than that used in the aforementioned works as it is known that Lipschitz continuity of the reward function and the transition kernel leads to Lipschitz continuity of the optimal action-value function (Asadi et al., 2018). We use the same condition in this present paper.

**Adaptive discretization:** Our method is closely related to methods that learn partition from continuous bandit literature (Kleinberg et al., 2008; Bubeck et al., 2009; Slivkins, 2014; Azar et al., 2014; Munos et al., 2014). In particular, our method is inspired by the contextual Zooming algorithm introduced in Slivkins (2014) for contextual bandits, that we extend in non-trivial way to episodic RL setting. Our method is similar to two recently proposed algorithms. Zhu and Dunson (2019) propose and analyze an adaptive partitioning algorithm approach in the specific case where the metric space is a subset of \( \mathbb{R}^d \) equipped with \( l_\infty \) distance as similarity metric. Concurrently to our work, Sinclair et al. (2019) extend the latter result to any generic metric space. However, their algorithm ADAPTIVE Q-LEARNING requires, at each re-partition step, a packing oracle that is able to take a region and value \( r \) and outputs an \( r \)-packing of that region. Whereas, our algorithm is oracle-free and creates at most a single sub-region when needed. More comparison with this work requires the introduction of some notations and is therefore deferred to Section 4.

3. Problem Statement

3.1. Episodic Reinforcement Learning and Regret

We consider a finite horizon MDP \((S,A,P,r,H)\) where \( S \) and \( A \) are the state and action space, \( H \) is the planning horizon i.e number of steps in each episode, \( P \) is the transition kernel such that \( P_a(\cdot|s,a) \) gives the distribution over next states if action \( a \) is taken at state \( s \) at step \( h \in [H] \), \( r \) is the reward function such that \( r_h(s,a) \in [0,1] \) is the reward of taking action \( a \) at state \( s \) at time step \( h \). For any step \( h \in [H] \) and \((s,a) \in \)
The state-action value function of a non-stationary policy \( \pi = \{\pi_1, \ldots, \pi_H\} \) is defined as
\[
Q_h^\pi(s, a) = r_h(s, a) + \mathbb{E} \left[ \sum_{i=1}^{H-1} r_i(s_i, \pi_i(s_i)) \mid s_h = s, a_h = a \right],
\]
and the value function is \( V_h^\pi(s) = Q_h^\pi(s, \pi_h(s)) \). As the horizon is finite, under some regularity conditions (Shreve and Bertsekas, 1978), there always exists an optimal policy \( \pi^* \) whose value and action-value functions are defined as \( V_h^{\pi^*}(s, a) = \max_{\pi \in \Pi} V_h^\pi(s, a) \) and \( Q_h^{\pi^*}(s, a) = \max_{\pi \in \Pi} Q_h^\pi(s, a) \). If we denote \( \{V_h^1, \ldots, V_h^H\} \) by \( M(\epsilon) \) and have the same scaling as we have \( M(2\epsilon) \leq N(\epsilon) \leq M(\epsilon) \).

4. The ZOOMRL algorithm

The ZOOMRL algorithm, shown in Algorithm 1, incrementally builds an optimistic estimate of the optimal action-value function over \( \mathcal{X} \). The main idea is to estimate \( Q \)-values precisely in near-optimal regions, while estimating it loosely in sub-optimal regions. To implement this idea, we learn a partition of the space by zooming in more promising and frequently visited regions.

ZOOMRL maintains a partition of the space \( \mathcal{X} \) that consists of a growing set of balls, of various sizes. Initially the set contains a single ball which includes the entire state-action space. Over time the set is expanded to include additional balls. The algorithm assigns two quantities to each ball: the number of times the ball is selected and an optimistic estimate of the \( Q \)-value of its center. By interpolating between these estimates using the Lipschitz structure, the algorithm assigns a tighter upper bound (called index) of the \( Q \)-value of each ball’s center. These indices are then used to select the next ball and the next action to execute (line 8-10 of Algorithm 1). Based on the received reward and the observed next state, the algorithm updates the selected ball’s statistics (cf line 14-17 of Algorithm 1). Then, one ball may be created inside the selected ball according to an activation rule that reflects a bias-variance tradeoff (line 19-22 of Algorithm 1).

We denote by \( B_h \) the set of balls at step \( h \in [H] \) that may change from episode to episode. Each ball \( B = \{x \in \mathcal{X}, \text{dist}(x_B, x) \leq \text{rad}(B)\} \) has a radius \( \text{rad}(B) \), center \( x_B = (s_B, a_B) \) and a domain. The domain of ball \( B \in \mathcal{B}_h \), denoted by \( \text{dom}_h(B) \), is defined as the subset of \( B \) that excludes all active balls in \( B_h \) that have radius strictly smaller than \( \text{rad}(B) \), i.e.
\[
\text{dom}_h(B) \triangleq B \setminus \left( \cup_{B' \in \mathcal{B}_h \setminus B} B' \right).
\]
B is called relevant to a state s at step h if (s, a) ∈ domh(B) for some action a ∈ A. We denote the set of relevant balls to a given state s at step h by relh(s) ≡ {B ∈ Bh : ∃a ∈ A, (s, a) ∈ domh(B)}. For each ball B ∈ Bh for some h, we keep track of the number of times B is selected at step h (denoted by nh(B)), as well as a high probability upper bound (denoted by Q̂h(B)) for the optimal Q-value of the center of B (i.e. Q∗(sB, aB)).

Using the Lipschitz continuity assumption, we have that L · rad(B) + Q̂h(B′) + L · dist(xB, xB′) is a valid high probability upper bound on Q∗(sB, aB) for any B′ ∈ Bh. Consequently, we get a tighter (less overoptimistic) upper bound, denoted by indext(B), by taking the minimum of these bounds

\[
\text{indext}(B) \triangleq L \cdot \text{rad}(B) + \min_{B' \in Bh} \{\hat{Q}_h(B') + L \cdot \text{dist}(B, B')\},
\]

where, by abuse of notation, we write dist(B, B′) = dist(xB, xB′).

To facilitate the algorithm’s description, we introduce episode-indexed versions of the quantities, as shown in algorithm 1. We will use ŝk,h, ak,h, and B̂k,h to represent the state, the action and the ball generated at time step h of the k-th episode. Moreover, Q̂k,h(B) and r̂k,h(B) are the statistics associated with each ball B at time step h at the beginning of the k-th episode.

The algorithm proceeds as follows. Initially, ZoomRL creates a ball centered at arbitrary state-action pair with radius 1, hence covering the whole space. At step h of the k-th episode, a state ŝk,h is observed, the algorithm finds the set of relevant balls to ŝk,h (i.e. relh(ŝk,h)) and picks the ball B̂k,h with the largest index (i.e. indext) among the relevant balls. Once the ball is selected, an action ak,h is chosen randomly among actions a satisfying (ŝk,h, a) ∈ domh(B̂k,h). Action ak,h is then executed in the environment, a reward r̂k,h is obtained and next state ŝk+1,h is observed.

Based on the received reward and next state, the algorithm updates the statistics of the selected ball. The number of visits nhk(Bh) is incremented by 1: nhk+1(Bh) = nhk(Bh) + 1. Let t = nhk+1(Bh), the Q-value estimate is updated as follows

\[
\hat{Q}_{h+1}(B^k_h) \leftarrow (1 - \alpha_t) \hat{Q}_{h}(B^k_h) + \alpha_t (r^k_{h} + \bar{V}_{h+1}(s^k_{h+1}) + u_t + 2L \cdot \text{rad}(B^k_h)).
\]

αt ≡ \frac{H + 1}{H + t} is a learning rate and \(\bar{V}_{h+1}(s^k_{h+1}) = \min_{B \in \text{rel}_{h+1}(s^k_{h+1})} \text{indext}(B)\) is the estimate of the next state’s value. The term ut + 2L · rad(Bk,h) corresponds to an exploration bonus used to bound estimation errors on the value function with high probability.

Comparison with Sinclair et al. (2019): Concurrently to our work, Sinclair et al. (2019) use a similar approach to learn an adaptive discretization. We highlight here differences between ZoomRL and their algorithm ADAPTIVE Q-LEARNING:

1. ADAPTIVE Q-LEARNING requires, at each re-partition step, a packing oracle that is able to take a ball B and value r to output an r-packing of B. Whereas, our algorithm is oracle-free and creates at most a single
Algorithm 1 ZOOMRL

1: **Data:** For $h \in [H]$, we have a collection $\mathcal{B}_h$ of balls.
2: **Init:** create ball $B$, with $rad(B) = 1$ and arbitrary center. $\mathcal{B}_h^k \leftarrow \{B\}$ for all $h \in [H]$
3: $\tilde{Q}_h^1(B) = H$ and $n_h^k(B) = 0$, $\forall h \in [H]$
4: for episode $k = 1, \ldots, K$ do
5: \hspace{1em} Observe $x_t^h$
6: \hspace{1em} for step $h = 1, \ldots, H$ do
7: \hspace{2em} // Select action
8: $B_h^k \leftarrow \arg\max_{B \in \mathcal{B}_h^k(s_h^k)} \text{index}^k_h(B)$
9: $a_h^k \leftarrow$ any arm $a$ such that $(s_h^k, a) \in \text{dom}(B_h^k)$
10: Execute action $a_h^k$, observe reward $r_h^k$ and next state $s_{h+1}^k$
11: // Query the next value function
12: $\tilde{V}_{h+1}^k(s_{h+1}^k) \leftarrow \min\{H, \max_{B \in \mathcal{B}_h^k(s_{h+1}^k)} \text{index}_{h+1}^k(B)\}$
13: // Update the selected ball’s statistics
14: $t = n_h^{k+1}(B_h^k) \leftarrow n_h^k(B_h^k) + 1$
15: $u_t \leftarrow 4\sqrt{\frac{H^31}{t}}$
16: $\tilde{Q}_{h+1}^k(B_h^k) \leftarrow (1 - \alpha_t) \tilde{Q}_{h}^k(B_h^k) + \alpha_t (r_{h+1}^k + \tilde{V}_{h+1}^k(s_{h+1}^k) + u_t + 2L \cdot rad(B_h^k))$
17: // New ball’s activation step
18: if $n_h^k(B) \geq \frac{1}{rad(B_h^k)}$ then
19: Create a new ball $B'$ centered in $(s_h^k, a_h^k)$ and radius $rad(B') = rad(B_h^k)/2$
20: $\mathcal{B}_h^{k+1} = \mathcal{B}_h^k \cup B'$
21: $\tilde{Q}_{h+1}^k(B') = H$ and $n_h^{k+1}(B') = 0$, $\forall h \in [H]$
22: end if
23: end for
24: end for

We leverage this new Lipschitz structure to define the ball’s index, which is not used in their algorithm.

3: Sinclair et al. (2019) use an exploration bonus $2\sqrt{H^3 \log(4H^2/K)} + \frac{4}{\sqrt{t}}$ where $t = n_h^k(B_h^k)$. The first term looks similar to our term $u_t$ with $K^2$ instead of $K$ in the log factor. We think it is due to small issue in their proof because there is a missing union bound over $K$ possible values of the random stopping time $t = n_h^k(B_h^k)$ (cf Proof C.3.1 in appendix). The second term, $\frac{4}{\sqrt{t}}$, is different than ours, $L \cdot rad(B_h^k)$.

4: Finally, in Sinclair et al., 2019 each child ball inherits statistics from their parent while in our algorithms the statistics are initialized by zero for $n_h$ and $H$ for $\tilde{Q}_h$.

5. Main results

In this section, we present our main theoretical result which is an upper bound on the total regret of ZOOMRL (see Algorithm 1). We start by showing a pessimistic version of the regret bound.

**Theorem 5.1** (Worst case guarantee). For any $p \in (0, 1)$, with probability $1 - p$, the total regret of ZOOMRL (see Algorithm 1) is at most $O(\sqrt{H^5LK\frac{1}{1-p}})$ where $\epsilon = \log(4HK^2/p)$ and $d$ is the covering dimension of the state-action space.

The bound in Theorem 5.1 matches the regret bound achieved by Net-based Q-learning (NBQL) studied in Song and Sun (2019) which assumes access to an $\epsilon$-net of the whole space as input to the algorithm. Moreover, the $\epsilon$-net should be optimal in the sense that the granularity $\epsilon$ of the covering must be chosen in advance ($\epsilon = K^{-\frac{d}{2}}$). Meanwhile, ZOOMRL builds the partition on the fly and in data-dependent fashion by allocating more effort in promising regions, which would considerably save the memory requirement in favorable problems while preserving the worst-case guarantee (as shown in Theorem 5.1).

Now, we present a refined regret bound that reflects better the geometry of the underlying space.

**Theorem 5.2** (Refined regret bound). For any $p \in (0, 1)$, with probability $1 - p$, the total regret of ZOOMRL (see Algorithm 1) is at most

$$O((L + \sqrt{H\epsilon_1}) \min_{r_0 \in (0, 1)} \left\{ K r_0 + \sum_{r = 2^{-i}}^{\epsilon_1} M(r) \right\} + H^2 + \sqrt{H^5K \epsilon}),$$

where $\epsilon = \log(4HK^2/p)$, $M(r)$ is the $r$-packing number of the state-action space.

Since $M(r)$ is non-increasing in $r$, the leading term (the first term) of the bound of Theorem 5.2 is upper bounded by $\min_{r_0 \in (0, 1)} \left\{ K r_0 + \frac{M(r_0)}{r_0} \log(r_0) \right\}$. By setting $r_0 = K^{-\frac{d}{2}}$, we recover the worst-case bound in Theorem 5.1. Note that the work of Sinclair et al. (2019) achieves the same regret bound.

The $r$-packing number $M(r)$ is here to uniformly upper bound the number of balls of radius $r$ generated by the algorithm, as we will see in the analysis deferred to the next section. Intuitively, balls with small radius would not cover the whole state-action space but rather would be concentrated around near-optimal regions. We expect that their number would be much smaller that $M(r)$ in practice.

**Comparison with contextual bandit setting:** We would like to highlight a negative result of the RL setting compar-
We study now how

A straightforward consequence of Assumption 3.1 is:

\[ \text{approximately Lipschitz} \]

This follows from the fact that in contextual bandit

\[ \epsilon \]

where for all

\[ L \]

sume that for any

\[ Q \]

ror. First, we present a formal definition for an approximate

5.1. Result For The Misspecified Case

We study now how ZOOMRL deals with misspecification er-

\[ Z \]

RL deals with misspecification er-

\[ Z \]

\[ 0 \]

\[ \Omega \]

\[ \epsilon \]

\[ \epsilon \]

\[ Ω \]

\[ H \]

\[ K \]

\[ 1 \]

\[ H \]

\[ K \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]

\[ H \]
Lemma 6.3. For any \( p \in (0, 1) \), we have \( \beta_t = 2 \sum_{i=1}^t \alpha_i^t u_i \leq 16 \sqrt{\frac{H_t}{\mu_t}} \) and, with probability at least \( 1 - p/2 \), we have that for all \( (s, a, h, k) \in S \times A \times [H] \times [K] \) and any ball \( B \in B_h^k \) such that \( (s, a) \in dom^k_h(B) \):

\[
\begin{align*}
\text{(a) } Q_h^k(B) & \geq Q_h^k(s, a), \\
\text{(b) } Q_h^k(B) - Q_h^k(x, a) & \leq \alpha_0 \cdot H + \beta_t + 4L \cdot rad(B) + \sum_{i=1}^t \alpha_i^t \cdot (\bar{V}_{h+1}^k - V_{h+1}^k)(s_{h+1}^k).
\end{align*}
\]

where \( t = n_h^k(B) \) and \( k_1, \ldots, k_t < k \) are the episodes where \( B \) was selected at step \( h \).

The next lemma translates the optimism in terms of Q-value estimates to optimism in terms of value function estimates.

Lemma 6.4 (Optimism). Following the same setting as in Lemma 6.3, for any \((h, k, a)\), with probability at least \( 1 - p/2 \), we have for any \( s \in S \), \( V_h^k(s) \geq V_h^*(s) \).

Proof. Let \( s \in S \). We have \( V_h^k(s) = Q_h^k(s, \pi(s)) \). As the set of domains of active balls covers the entire space, there exists \( B^* \in B_h^k \) such that \( (s, \pi(s)) \in dom^k_h(B^*) \). By the definition of index, we have \( \pi(s) = \arg\max V_h^k(s, a) \).

\[
\begin{align*}
\hat{V}_h^k(s) &= \min\{H, \max_{B \in \mathbb{R}_h^k(s)} \pi(s)\} - Q_h^k(s, \pi(s)) \\
&\geq \max_{B \in \mathbb{R}_h^k(s)} \pi(s) - Q_h^k(s, \pi(s)) \\
&= L \cdot rad(B^*) + Q_h^k(B^*) + L \cdot dist(B^*, B^*) \\
&\geq L \cdot rad(B^*) + Q_h^k(s, \pi(s)) - Q_h^k(s, \pi(s)) \\
&\geq L \cdot rad(B^*) + Q_h^k(s, \pi(s)) - Q_h^k(s, \pi(s)) \\
&\geq L \cdot rad(B^*) + Q_h^k(s, \pi(s)) - Q_h^k(s, \pi(s)) \\
&\geq L \cdot rad(B^*) + Q_h^k(s, \pi(s)) - Q_h^k(s, \pi(s)) \\
&\geq 0
\end{align*}
\]

Where \((s_{B^*}, a_{B^*})\) and \((s_{B'}, a_{B'})\) denote respectively the centers of balls \( B^* \) and \( B' \). The first inequality follows from \( Q_h^k(s, a) \leq H \) for any state-action pair \((s, a)\). The third inequality follows from lemma 6.3. The fourth and the last inequalities follow from Lipschitz assumption 3.1

6.1. Regret Analysis

\( \pi_k \) is the policy executed by the algorithm in step \( h \) for \( H \) steps to reach the end of the episode. By the optimism of our estimates with respect to the true value function (see lemma 6.4), we have with probability at least \( 1 - p/2 \)

\[
\text{REGRET}(K) = \sum_{k=1}^K (V_1^* - V_{1}^k)(x_k^1) \leq \sum_{k=1}^K (\bar{V}_1^k - V_{1}^k)(x_1^k)
\]

Denote by \( \delta_h^k \equiv (\bar{V}_h^k - V_{1}^k)(s_h^k) \) and \( \phi_h^k \equiv (\bar{V}_h^k - V_{1}^k)(s_h^k) \). As \( V_h^* \geq V_{1}^k \), we have \( \phi_h^k \leq \delta_h^k \). In the sequel, we aim to upper bound \( \delta_h^k \) as we have \( \text{REGRET}(K) \leq \sum_{k=1}^K \delta_h^k \).

Let \( B_h^k \) be the ball selected at step \( h \) of episode \( k \) and \( B_{init} \) be the initial ball of radius one that covers the whole space. We denote \( B_h^{k, \pi} \) the parent of \( B_h^k \). When \( B_h^k \) is the initial ball, we consider that it is parent of itself.

Lemma 6.5 (Bound on estimation). If we denote \( \xi_h^k = (\mathbb{P}_h - \widehat{\mathbb{P}}_h)(V_{1}^k - V_{1}^k)(s_h^k, a_h^k) \), we have

\[
\begin{align*}
\delta_h^k &\leq H\alpha_0 \cdot n_h^k(B_h^k) \cdot \sum_{s \in s, \pi(s)} L \cdot rad(B^*) + \phi_h^k(B_h^{k, \pi}) - \phi_h^k + \delta_h^{k+1} + \xi_h^{k+1},
\end{align*}
\]

where \( k_i(B_h^{k, \pi}) \) is the \( i \)-th episode where \( B_h^{k, \pi} \) is selected at step \( h \).

Taking the sum over \( k \in [K] \) of the estimation bound in lemma 6.5,

\[
\sum_{k=1}^K \delta_h^k \leq \sum_{k=1}^K (11L + 32\sqrt{H}) \sum_{k=1}^K \text{rad}(B_h^k) \]

where \( \square = H \sum_{k=1}^K \alpha_0 \cdot n_h^k(B_h^k) \cdot \sum_{s \in s, \pi(s)} \sum_{i=1}^K n_h^k(B_h^{k, \pi}) \cdot k_i(B_h^{k, \pi}) \cdot \alpha_0 \cdot n_h^k(B_h^{k, \pi}) \cdot \phi_h^k \). For the first term, we have \( \square = H \sum_{k=1}^K n_h^k(B_h^{k, \pi}) \cdot \sum_{s \in s, \pi(s)} k_i(B_h^{k, \pi}) \cdot n_h^k(B_h^{k, \pi}) = H \). For the second term \( \triangle \), we regroup the summation in a different way. For every \( k' \in [K] \), the term \( \phi_h^{k+1} \) appears in the summation with \( k > k' \) when \( B_h^k \) and \( B_h^{k'} \) share the same parent. The first time it appears we have \( n_h^k(B_h^{k, \pi}) = n_h^k(B_h^{k', \pi}) + 1 \), the second time it appears we
have $n(h^k_{B_{h}^{k,p,a}}) = n(h^k_{B_{h}^{k,p,a}}) + 2$ and so on. Therefore:

$$\Delta \leq \sum_{k=1}^{K} \phi_{h+1}^{k} \sum_{t=n(h^k_{B_{h}^{k,p,a}})}^{\infty} \alpha_{t}^{k}(B_{h}^{k,p,a})$$

$$\leq \left(1 + \frac{1}{H}\right) \sum_{k=1}^{K} \phi_{h+1}^{k}.$$  

We use in the last inequality $\sum_{t=0}^{\infty} \alpha_{t} = 1 + \frac{1}{H}$ (Lemma F.2 in appendix). Therefore, using that $\phi_{h+1}^{k} \leq \delta_{h+1}^{k}$ we have

$$\sum_{k=1}^{K} \delta_{h}^{k} \leq H + (11L + 32\sqrt{H^3t}) \sum_{k=1}^{K} \text{rad}(B_{h}^{k})$$

$$+ \left(1 + \frac{1}{H}\right) \sum_{k=1}^{K} \phi_{h+1}^{k} \sum_{k=1}^{K} (-\phi_{h+1}^{k} + \delta_{h+1}^{k} + \xi_{h+1}^{k})$$

$$\leq H + (11L + 32\sqrt{H^3t}) \sum_{k=1}^{K} \text{rad}(B_{h}^{k})$$

$$+ \left(1 + \frac{1}{H}\right) \sum_{k=1}^{K} \delta_{h}^{k} + \sum_{k=1}^{K} \xi_{h+1}^{k}.$$  

By unrolling the last inequality for $h \in \{H\}$ and using the fact $\delta_{H+1}^{k} = 0 \quad \forall k \in \{K\}$, we obtain

$$\sum_{k=1}^{K} \delta_{h}^{k} \leq \sum_{h=1}^{H} \left(1 + \frac{1}{H}\right)^{h-1} \left(H + (11L + 32\sqrt{H^3t}) \sum_{k=1}^{K} \text{rad}(B_{h}^{k}) + \sum_{k=1}^{K} \xi_{h}^{k}\right)$$

$$\leq 3H^2 + 3(11L + 32\sqrt{H^3t}) \sum_{h=1}^{H} \sum_{k=1}^{K} \text{rad}(B_{h}^{k})$$

$$+ 3 \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^{k}.$$  

(3)

The last inequality follows from the fact $\forall h \in \{H\}, \left(1 + \frac{1}{H}\right)^{h-1} \leq (1 + \frac{1}{H})^{H} \leq \exp(1) \leq 3$.

Now, we proceed to upper bound the two terms $\sum_{h=1}^{H} \sum_{k=1}^{K} \text{rad}(B_{h}^{k})$ and $\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^{k}$. Using concentration argument, we show with probability at least $1 - p/2$, we have (see Lemma C.1 in appendix)

$$\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^{k} \leq 4\sqrt{2H^3Kt}.$$  

(4)

Bounding $\sum_{h=1}^{H} \sum_{k=1}^{K} \text{rad}(B_{h}^{k})$: Let’s consider all balls of radius $r$ that have been activated at step $h$ throughout the execution of the algorithm. The maximum number of times a ball $B$ of radius $r$ can be selected before it becomes a parent is upper bounded by $\frac{1}{r^2}$. After ball $B$ becomes a parent, a new ball of radius $r/2$ is created every time $B$ is selected. Therefore, we can write the sum over all ball $B \in B_{h}^{K}$ of radius $r$ as the sum over set of rounds which consists of the round when $B$ was created and all rounds when $B$ was selected before being a parent. Let $r_0 \in (0, 1)$, we have

$$\sum_{k=1}^{K} \text{rad}(B_{h}^{k}) = \sum_{r=2^{-i}} \sum_{r < r_0 \text{rad}(B) = r} \sum_{r_0 \leq r \leq r_0} \sum_{k \in \{K\}} r$$

$$= \sum_{r=2^{-i}} \sum_{r < r_0 \text{rad}(B) = r} \sum_{r_0 \leq r \leq r_0} \sum_{k \in \{K\}} r$$

$$\leq Kr_0 + \sum_{r=2^{-i}} \sum_{r_0 \leq r \leq r_0} \sum_{k \in \{K\}} r$$

$$\leq Kr_0 + \sum_{r=2^{-i}} \sum_{r_0 \leq r \leq r_0} \frac{M(r)}{r}.$$  

(5)

The last step follows from lemma 6.1: The set of active balls of radius $r$ is a $r$-packing of $S \times A$. Thus, $\{|B \in B_{h}^{K} : \text{rad}(B) = r| \leq M(r) \}$ where $M(r)$ is the $r$-covering number.

Plugging bounds (4) and (5) in (3) and using the union bound, we obtain the desired regret bound in Theorem 5.2

7. Conclusion

In this paper, we present ZoomRL, a provably efficient model-free reinforcement learning algorithm in continuous state-action spaces under the assumption that the true optimal action-value function is Lipschitz with respect to similarity metric between state-action pairs. Our algorithm takes into account the geometry of the action-value function by allocating more attention to relevant regions. We show that our method achieves sublinear regret that depends on the packing number of the state-action space and that it is robust to small misspecification errors.

Our method requires the knowledge of the Lipschitz constant $L$ as well as the metric $\text{dist}$ to achieve its performance. A natural future question is whether an RL algorithm can be proved to be efficient without knowing $L$ or $\text{dist}$ in advance.
References

Asadi, K., Misra, D., and Littman, M. L. (2018). Lipschitz continuity in model-based reinforcement learning. arXiv preprint arXiv:1804.07193.

Azar, M. G., Lazaric, A., and Brunskill, E. (2014). Online stochastic optimization under correlated bandit feedback. In ICML, pages 1557–1565.

Azar, M. G., Osband, I., and Munos, R. (2017). Minimax regret bounds for reinforcement learning. In Proceedings of the 34th International Conference on Machine Learning—Volume 70, pages 263–272. JMLR. org.

Bubeck, S., Stoltz, G., Szepesvári, C., and Munos, R. (2009). Online optimization in x-armed bandits. In Advances in Neural Information Processing Systems, pages 201–208.

Chow, Y. and Teicher, H. (1998). Probability theory: Independence, interchangeability, martingales. Journal of the American Statistical Association, 93.

Ferns, N., Panangaden, P., and Precup, D. (2004). Metrics for finite markov decision processes. In Proceedings of the 20th conference on Uncertainty in artificial intelligence, pages 162–169. AUAI Press.

Jaksch, T., Ortner, R., and Auer, P. (2010). Near-optimal regret bounds for reinforcement learning. Journal of Machine Learning Research.

Jian, Q., Fruit, R., Pirotta, M., and Lazaric, A. (2019). Exploration bonus for regret minimization in discrete and continuous average reward mdps. In Advances in Neural Information Processing Systems, pages 4891–4900.

Jin, C., Allen-Zhu, Z., Bubeck, S., and Jordan, M. I. (2018). Is q-learning provably efficient? In Advances in Neural Information Processing Systems, pages 4863–4873.

Kakade, S., Kearns, M. J., and Langford, J. (2003). Exploration in metric state spaces. In Proceedings of the 20th International Conference on Machine Learning (ICML-03), pages 306–312.

Kearns, M. and Singh, S. (2002). Near-optimal reinforcement learning in polynomial time. Machine learning.

Kleinberg, R., Slivkins, A., and Upfal, E. (2008). Multiarmed bandits in metric spaces. In Proceedings of the fortieth annual ACM symposium on Theory of computing, pages 681–690. ACM.

Lakshmanan, K., Ortner, R., and Ryabko, D. (2015). Improved regret bounds for undiscounted continuous reinforcement learning. In International Conference on Machine Learning, pages 524–532.

Lattimore, T., Hutter, M., Sunehag, P., et al. (2013). The sample-complexity of general reinforcement learning. In Proceedings of the 30th International Conference on Machine Learning. Journal of Machine Learning Research.

Munos, R. et al. (2014). From bandits to monte-carlo tree search: The optimistic principle applied to optimization and planning. Foundations and Trends® in Machine Learning, 7(1):1–129.

Ortner, R. (2007). Pseudometrics for state aggregation in average reward markov decision processes. In International Conference on Algorithmic Learning Theory, pages 373–387. Springer.

Ortner, R. and Ryabko, D. (2012). Online regret bounds for undiscounted continuous reinforcement learning. In Advances in Neural Information Processing Systems, pages 1763–1771.

Pazis, J. and Parr, R. (2013). Pac optimal exploration in continuous space markov decision processes. In Twenty-Seventh AAAI Conference on Artificial Intelligence.

Qian, J., Fruit, R., Pirotta, M., and Lazaric, A. (2018). Exploration bonus for regret minimization in undiscounted discrete and continuous markov decision processes. arXiv preprint arXiv:1812.04363.

Shreve, S. E. and Bertsekas, D. P. (1978). Alternative theoretical frameworks for finite horizon discrete-time stochastic optimal control. SIAM Journal on control and optimization, 16(6):953–978.

Sinclair, S. R., Banerjee, S., and Yu, C. L. (2019). Adaptive discretization for episodic reinforcement learning in metric spaces. arXiv preprint arXiv:1910.08151.

Slivkins, A. (2014). Contextual bandits with similarity information. The Journal of Machine Learning Research, 15(1):2533–2568.

Song, Z. and Sun, W. (2019). Efficient model-free reinforcement learning in metric spaces. arXiv preprint arXiv:1905.00475.

Strehl, A. L. and Littman, M. L. (2005). A theoretical analysis of model-based interval estimation. In Proceedings of the 22nd international conference on Machine learning. ACM.

Sutton, R. S. and Barto, A. G. (1998). Introduction to reinforcement learning, volume 135. MIT Press Cambridge.

Yang, L. F., Ni, C., and Wang, M. (2019). Learning to control in metric space with optimal regret. arXiv preprint arXiv:1905.01576.

Zhu, X. and Dunson, D. (2019). Stochastic lipschitz q-learning.
A. Outline

The appendix of this paper is organized as follows:

1. Appendix B provides a table of notation for easy reference.
2. Appendix C provides omitted proofs for regret analysis in the Lipschitz setting.
3. Appendix D provides the complete regret analysis in the misspecified setting.
4. Appendix F provides some technical lemmas.

B. Notations

We provide this table for easy reference. Notation will also be defined as it is introduced.

| $L$ | Lipschitz constant of optimal action-value function |
| $\text{rad}(B)$ | radius of a ball $B$ |
| $\text{dist}((s,a),(s',a'))$ | distance between two action-state pairs |
| $(s_B,a_B)$ | center of a ball $B$ |
| $\text{dist}(B,B')$ | distance between centers of balls $B$ and $B'$ |
| $B^{-a}$ | parent of a ball $B$ |
| $s^h$ | state encountered in step $h$ of episode $k$ |
| $a^h$ | action taken by the algorithm in step $h$ of episode $k$ |
| $B^h$ | ball selected by the algorithm in step $h$ of episode $k$ |
| $b^h_{-a}$ | parent of the selected ball at step $h$ in episode $k$ |
| $B^h$ | the set of balls activated by the algorithm in step $h$ before episode $k$ |
| $\hat{Q}_h^k(B)$ | $Q$-value estimate at ball $B$ |
| $n^h_k(B)$ | number of times a ball $B$ is selected by the algorithm |
| $\pi^k_h$ | policy executed by the algorithm in episode $k$ |
| $\text{dom}_{B^h}^k$ | $B \setminus \left( \bigcup_{B' \in B^h_k} \{B' \mid \text{rad}(B') < \text{rad}(B)\} \right)$, domain of a ball $B$ |
| $\text{index}_{B^h}^k$ | $L \cdot \text{rad}(B) + \min_{B' \in B^h_k} \{\hat{Q}_h^k(B') + L \cdot \text{dist}(B,B')\}$ |
| $\hat{V}_{h,s}^k$ | $\min \{H, \max_{B \in \text{dom}_{B^h}^k(B)} \text{index}_{B^h}^k(B)\}$, value function estimate |
| $p$ | failing probability |
| $\log(4H^2k^3)$ | log factor |
| $u_t$ | $4\sqrt{H^3s}$, UCB exploration bonus |
| $\text{REGRET}(K)$ | $\sum_{k=1}^K (V^*_t - V^*_{\pi^k})$$\left(x_t^k\right)$ |
| $\delta_h^k$ | $(\hat{V}_{h,s}^k - V^*_{\pi^k})$$\left(s^k\right)$ |
| $\phi_h^k$ | $(\hat{V}_{h,s}^k - V^*_{\pi^k})$$\left(s^k\right)$ |
| $\hat{P}_{h,s}^k$ | $[\hat{P}_{h,s}^k](s^k_h,a^k_h) = V(s^k_{h+1})$ |
| $\xi^k_{h+1}$ | $[(P_h - \hat{P}_{h,s}^k)(V^*_{h+1} - V^*_{\pi^k})](s^k_h,a^k_h)$ |
| $\alpha_0$ | $\frac{H+1}{\tau_0}$ |
| $\alpha_0^0$ | $\prod_{j=1}^t (1 - \alpha_j)$ |
| $\alpha_0^t$ | $\alpha_0 \prod_{j=1}^t (1 - \alpha_j)$ |
| $M(r)$ | $r$-packing number of the state-action space $S \times A$ |
| $f^h(s,a)$ | the lipschitz term in the misspecified setting (Assumption 5.3) |
| $\Delta_h^k(s,a)$ | $Q^k_h(s,a) - f^k_h(s,a)$ |
| $\epsilon$ | misspecification error (uniform bound on $|\Delta_h(s,a)|$) |
\section*{C. Omitted proofs for the Lipschitz setting}

\subsection*{C.1. Proof of Lemma 6.1}

(a) It is obvious that $\cup_{B \in B_h^k} \text{dom}_B^k(B) \subseteq \cup_{B \in B_h} B$. Let $x \in \cup_{B \in B_h^k} B$. Consider a smallest radius ball $B$ in $B_h^k$ that contains $x$. Hence, $x \in \text{dom}_B^k(B)$. This shows that $\cup_{B \in B_h^k} B \subseteq \cup_{B \in B_h} \text{dom}_B^k(B)$ and consequently $\cup_{B \in B_h^k} B = \cup_{B \in B_h^k} \text{dom}_B^k(B)$. Moreover, $\cup_{B \in B_h^k} B = S \times A$ as it contains the initial ball which covers the whole space.

(b) Let $(B, B') \in B_h^k$ two balls of radius $r > 0$. Without loss of generality, we suppose that $B$ is created in episode $\tau \leq k$ with parent ball $B_{\text{pa}}$ and $B'$ is created before $\tau$. According to the activation step in ZOOMRL algorithm, $(s_h^*, a_h^*)$ is the center of $B$ and $(s_h^*, a_h^*) \notin B'$, which proves that $\text{dist}(B, B') > r$.

\subsection*{C.2. Proof of Lemma 6.2}

\begin{proof}

We fix $B \in B_h^k$. For notation simplicity, denote $t = n_k(B)$. We have:

\[ Q_h^k(B) = (1 - \alpha_t) \cdot Q_h^k(B) + \alpha_t \cdot \left( r_h(x_h^{k_i}, a_h^{k_i}) + \hat{V}_{h+1}^{k_i}(x_{h+1}) + u_t + 2L \cdot \text{rad}(B) \right) \]

\[ = (1 - \alpha_t) \cdot \left( (1 - \alpha_{t-1}) \cdot Q_h^{k_{i-1}}(B) + \alpha_{t-1} \cdot \left( r_h(x_h^{k_{i-1}}, a_h^{k_{i-1}}) + \hat{V}_{h+1}^{k_{i-1}}(x_{h+1}) + u_{t-1} + 2L \cdot \text{rad}(B) \right) \right) + \alpha_t \cdot \left( r_h(x_h^{k_i}, a_h^{k_i}) + \hat{V}_{h+1}^{k_i}(x_{h+1}) + u_t + 2L \cdot \text{rad}(B) \right) \]

\[ = \ldots \]

\[ = \prod_{i=1}^{t} (1 - \alpha_i) H + \sum_{i=1}^{t} \alpha_i \prod_{j=i+1}^{t} (1 - \alpha_j) \left( r_h(x_h^{k_i}, a_h^{k_i}) + \hat{V}_{h+1}^{k_i}(x_{h+1}) + u_i + 2L \cdot \text{rad}(B) \right) \]

where the first step follows from the update rule of $Q_h^k$, the second step follows from the update rule for $Q_h^{k_{i-1}}$, the third step follows from recursively representing $Q_h^{k_i}$ using $Q_h^{k_{i-1}}$ until $i = 1$, and the last step follows from the definition of $\alpha_i$ and $\alpha_t$.
\end{proof}

\subsection*{C.3. Proof of Lemma 6.3}

\begin{proof}

Let $B \in B_h^k$ and $(s, a) \in \text{dom}_B^k(B)$.

Since $\sum_{i=0}^{t} \alpha_i = 1$, we have that $Q_h^*(s, a) = \alpha_0 Q_h^*(s, a) + \sum_{i=1}^{t} \alpha_i Q_h^*(s, a)$.

By the Lipschitz assumption 3.1 and the fact $\forall i \in [t], (s_h^{k_i}, a_h^{k_i}) \in B$ and $(s, a) \in B$, we have:

\[ |Q_h^*(s_h^{k_i}, a_h^{k_i}) - Q_h^*(s, a)| \leq L \cdot \text{dist}(x_h^{k_i}, a_h^{k_i}, (s, a)) \leq 2L \cdot \text{rad}(B). \]

Then we have

\[ Q_h^*(s, a) \geq \alpha_0 Q_h^*(s, a) + \sum_{i=1}^{t} \alpha_i \left( Q_h^*(s_h^{k_i}, a_h^{k_i}) - 2L \cdot \text{rad}(B) \right) \]

\[ Q_h^*(s, a) \leq \alpha_0 Q_h^*(s, a) + \sum_{i=1}^{t} \alpha_i \left( Q_h^*(s_h^{k_i}, a_h^{k_i}) + 2L \cdot \text{rad}(B) \right). \]

By Bellman equation, we have $Q_h^*(s_h^{k_i}, a_h^{k_i}) = r_h(s_h^{k_i}, a_h^{k_i}) + [P_h V_{h+1}^*](s_h^{k_i}, a_h^{k_i})$. Recall $[\hat{P}_h V_{h+1}^*](s_h^{k_i}, a_h^{k_i}) = V_{h+1}(s_{h+1}^{k_i})$, we have:

\[ Q_h^*(s_h^{k_i}, a_h^{k_i}) = r_h(s_h^{k_i}, a_h^{k_i}) + ([P_h - \hat{P}_h] V_{h+1}^*)(s_h^{k_i}, a_h^{k_i}) + V_{h+1}^*(s_{h+1}^{k_i}). \]
Substitute the above equality into Eq. (6) and (7), we have:

\[
Q_h^*(s, a) \geq \alpha_i^0 \cdot Q_h^*(x, a) + \sum_{i=1}^{t} \alpha_i^i \left( r_h(x_{k_i}^i, a_{k_i}^i) + \left[ (P_h - \hat{P}_h^{k_i}) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) + V_{h+1}^*(x_{k_i}^i, a_{k_i}^i) - 2L \cdot \varepsilon \text{ad}(B) \right)
\]

\[
Q_h^*(s, a) \leq \alpha_i^0 \cdot Q_h^*(x, a) + \sum_{i=1}^{t} \alpha_i^i \left( r_h(x_{k_i}^i, a_{k_i}^i) + \left[ (P_h - \hat{P}_h^{k_i}) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) + V_{h+1}^*(x_{k_i}^i, a_{k_i}^i) + 2L \cdot \varepsilon \text{ad}(B) \right)
\]

Subtracting the formula in Lemma 6.2 from the two above inequalities, we have:

\[
\tilde{Q}_h^k(B) - Q_h^*(s, a) \geq \sum_{i=1}^{t} \alpha_i^i \left( \tilde{V}_{h+1}^i(x_{k_i}^i, a_{k_i}^i) + \left[ (\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) + u_i \right)
\]

(8)

\[
\hat{Q}_h^k(B) - Q_h^*(s, a) \leq \alpha_i^0 H + \sum_{i=1}^{t} \alpha_i^i \left( \tilde{V}_{h+1}^i(x_{k_i}^i, a_{k_i}^i) + \left[ (\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) + u_i + 4L \cdot \varepsilon \text{ad}(B) \right)
\]

(9)

C.3.1. HIGH PROBABILITY BOUNDS ON THE SAMPLING NOISE

To ensure that our estimates converge around the true optimal $Q$-values, we need to ensure that the noise terms \([\hat{P}_h^k - P_h] V_{h+1}^* \) such that due to the next states sampling, are not large.

For each ball $B \in B_h^k$, $k_i$ is the episode of which $B$ was selected as step $h$ for the $i$-th time. Let $F_i$ be the $\sigma$-field generated by all the random variables until episode $i$, step $h$. As $\{k_i = i\} \subset F_i$, the random variable $k_i$ is a stopping time. By definition for any $i \geq 0$, $k_i \leq k_{i+1}$ so the $\sigma$-algebra $F_i$, at time $k_i$ satisfies $F_i \subset F_{i+1}$ (see Lemma F.5). Let’s denote $G_i = F_{i+1}$. Then, $(G_i)_{i \geq 0}$ is a filtration. Moreover, via optional stopping (Chow and Teicher, 1998), $(\hat{P}_h^k - P_h) [\hat{P}_h^k - P_h] V_{h+1}^* \) w.r.t the filtration $(G_i)_{i \geq 0}$. By Azuma-Hoeffding F.1, we have $\forall t > 0, \tau \leq [K]$

\[
\text{Pr} \left[ \sum_{i=1}^{\tau} \alpha_i^* \cdot \mathbb{I}(k_i \leq K) \cdot [(\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) \geq t \right] \leq 2 \text{exp} \left( \frac{-t^2}{8H^2 \sum_{i=1}^{\tau} (\alpha_i^*)^2} \right)
\]

Let $p \in (0, 1)$, by setting $2 \text{exp} \left( \frac{-t^2}{8H^2 \sum_{i=1}^{\tau} (\alpha_i^*)^2} \right) = \frac{p}{2H^2\tau}$, we have that for all $\tau \in [K]$ with probability at least $1 - \frac{p}{2H^2\tau}$:

\[
\sum_{i=1}^{\tau} \alpha_i^* \cdot \mathbb{I}(k_i \leq K) \cdot [(\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) \leq 2\sqrt{2H} \cdot \sum_{i=1}^{\tau} (\alpha_i^*)^2 \cdot \ln(4HK^2/p) \leq 4 \sqrt{H^3 \ln(AHK^2/p)} \leq 4 \sqrt{H^3 \tau}
\]

where the second inequality follows from $\sum_{i=1}^{\tau} (\alpha_i^*)^2 \leq \frac{2H}{\tau}$ for any $\tau > 0$ (see lemma F.2). Then by union bound over $\tau \in [K]$, we have with probability at least $1 - \frac{p}{2H^2K}$

\[
\forall \tau \in [K], \sum_{i=1}^{\tau} \alpha_i^* \cdot \mathbb{I}(k_i \leq K) \cdot [(\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) \leq 4 \sqrt{H^3 \tau}
\]

Since the above inequality holds for all $\tau \in [K]$, it also holds for $\tau = t = n_h^b(B) \leq K$. We also have that $\mathbb{I}(k_i \leq K) = 1$ for any $i \leq n_h^b(B)$. As $\hat{P}_h^k \leq K$ for all $(h, k) \in [H] \times [K]$, using union bound for all balls and for all steps, we have with probability at least $1 - p/2$: $\forall (h, k) \in [H] \times [K]$ and for all ball $B \in B_h^k$,

\[
\sum_{i=1}^{t} \alpha_i^i \cdot [(\hat{P}_h^k - P_h) V_{h+1}^* \right](x_{k_i}^i, a_{k_i}^i) \leq 4 \sqrt{H^3 t}, \text{ where } t = n_h^b(B)
\]

(10)
According to Lemma F.2, we have $1/\sqrt{t} \leq \sum_{i=1}^{t} \frac{a_i}{t} \leq 2/\sqrt{t}$. This implies
\[
4\sqrt{\frac{H^3 t}{t}} \leq 4\sqrt{H^3 \cdot \sum_{i=1}^{t} \frac{a_i^2}{t}} = \sum_{i=1}^{t} \alpha_i u_i = \beta_t/2 \leq 8\sqrt{\frac{H^3 t}{t}}.
\]

Then Eq (10) gives that, with probability at least $1 - p/2$: $\forall (h, k) \in [H] \times [K]$ and for all ball $B \in B^k_h$,
\[
\left| \sum_{i=1}^{t} \alpha_i \cdot ((\hat{P}_h^{k_i} - P_h)V_{h+1}^*)(x_h^{k_i}, a_h^{k_i}) \right| \leq \beta_t/2, \text{where } t = n^k_h(B).
\]

C.3.2 Optimism of $Q$-values: Lemma 6.3 (a)

We proceed by induction. By definition, we have $\hat{Q}^k_{h+1} = Q^k_{h+1} = 0$ which implies $Q^k_{h+1}(B) - Q^k_{h+1}(s,a) = 0$. Assume that $Q^k_{h+1}(B) - Q^k_{h+1}(s,a) \geq 0$.

Let $i \in [1,t]$. We have $V^*_h(s_h^{k_i}) = Q^k_{h+1}(s_h^{k_i}, \pi^*_h(s_h^{k_i})).$ As the set of domains of active balls covers the entire space, there exists $B^* \in B^k_{h+1}$ such that $(s_h^{k_i}, \pi^*_h(s_h^{k_i})) \in \text{dom}_h^k(B^*)$. By the definition of index, we have $\text{index}_{h+1}^k(B^*) = L \cdot \text{rad}(B^*) + \hat{Q}_{h+1}^k(B^*) + L \cdot \text{dist}(B^*, B^*)$ for some active ball $B^*$.

We have
\[
\hat{V}_{h+1}^k(s_h^{k_i}) - V^*_h(s_h^{k_i}) = \min \{ H, \max_{B \in \text{dom}_h^k(s_h^{k_i})} \text{index}_{h+1}^k(B) \} - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
\geq \max_{B \in \text{dom}_h^k(s_h^{k_i})} \text{index}_{h+1}^k(B) - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
\geq \text{index}_{h+1}^k(B^*) - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
= L \cdot \text{rad}(B^*) + \hat{Q}_{h+1}^k(B^*) + L \cdot \text{dist}(B^*, B^*) - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
\geq L \cdot \text{rad}(B^*) + Q^k_{h+1}(s_{B^*}, a_{B^*}) + L \cdot \text{dist}(B^*, B^*) - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
\geq L \cdot \text{rad}(B^*) + Q^k_{h+1}(s_{B^*}, a_{B^*}) - Q^k_{h+1}(s_h^{k_i}, \pi^*(s_h^{k_i}))
\]
\[
\geq 0,
\]

where $(s_{B^*}, a_{B^*})$ and $(s_{\hat{B}^*}, a_{\hat{B}^*})$ denote respectively the centers of balls $B^*$ and $\hat{B}^*$. The first inequality follows from $Q^k_{h+1}(s,a) \leq H$ for any state-action pair $(s,a)$. The third inequality follows from the induction hypothesis. The fourth and the last inequalities follow from Lipschitz assumption 3.1

Therefore, we have
\[
Q^k_h(B) - Q^k_h(s,a) \geq \sum_{i=1}^{t} \alpha_i \cdot \left( (\hat{V}_{h+1}^k - V^*_h)(s_h^{k_i}) + [(\hat{P}_h^{k_i} - P_h)V_{h+1}^k](s_h^{k_i}, a_h^{k_i}) + u_i \right)
\]
\[
\geq -\beta_t/2 + \beta_t/2 = 0.
\]

C.3.3 Upper bound: Lemma 6.3 (b)

We have:
\[
\hat{Q}^k_{h}(B) - Q^k_{h}(s,a) \leq \alpha^0_t \cdot H + \sum_{i=1}^{t} \alpha_i \cdot \left( (\hat{V}_{h+1}^k - V^*_h)(x_h^{k_i}) + [(\hat{P}_h^{k_i} - P_h)V_{h+1}^*](x_h^{k_i}, a_h^{k_i}) + u_i + 4r(B) \right)
\]
\[
\leq \alpha^0_t H + \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}_{h+1}^k - V^*_h)(x_h^{k_i}) + \beta_t/2 + \sum_{i=1}^{t} \alpha_i u_i + 4L \sum_{i=1}^{t} \alpha_i \text{ rad}(B)
\]
\[
\leq \alpha^0_t H + \beta_t + 4L \cdot \text{rad}(B) + \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}_{h+1}^k - V^*_h)(x_h^{k_i}),
\]
where the second inequality follows from the inequality 11. The third inequality follows from $\sum_{i=1}^{t} \alpha_i \leq 1$

C.4. Proof of lemma 6.5

Let $B_h^k$ the ball selected at step $h$ of episode $k$ and $B_{\text{init}}$ be the initial ball of radius one that covers the whole space. We need to distinguish between cases where $B_h^k = B_{\text{init}}$ or not. By the selection step in ZOOMRL algorithm, we have $\max_{B \in \text{rel}_h^k(s_h^k)} \text{index}_h^k(B) = \text{index}_h^k(B_h^k)$ and $\pi_k(s_h^k) = a_h^k$.

1. Case of $B_h^k \neq B_{\text{init}}$: We denote $B_h^{k,pa}$ the parent of $B_h^k$.

$$\delta_h^k = \hat{V}_h^k - V_h^{\pi_k}(s_h^k) \leq \max_{B \in \text{rel}_h^k(s_h^k)} \text{index}_h^k(B) - V_h^{\pi_k}(s_h^k)$$

$$= \max_{B \in \text{rel}_h^k(s_h^k)} \text{index}_h^k(B_h^k) - Q_h^{\pi_k}(s_h^k, a_h^k)$$

$$\leq L \cdot \text{rad}(B_h^k) + \hat{Q}_h(B_h^{k,pa}) + L \cdot \text{dist}(B_h^{k,pa}, B_h^k) - Q_h^{\pi_k}(s_h^k, a_h^k)$$

$$\leq L \cdot \text{rad}(B_h^k) + \hat{Q}_h(B_h^{k,pa}) + L \cdot \text{rad}(B_h^{k,pa}) - Q_h^{\pi_k}(s_h^k, a_h^k)$$

$$= 3L \cdot \text{rad}(B_h^k) + \hat{Q}_h(B_h^{k,par}) - Q_h^*(s_h^k, a_h^k) + (Q_h^* - Q_h^{\pi_k})(s_h^k, a_h^k)$$

The third inequality follows from the fact that the center of $B_h^k$ is in $B_h^{k,pa}$ and the last equality follows from $\text{rad}(B_h^{k,pa}) = \text{rad}(B_h^k)$. Since $(x_h^k, a_h^k) \in \text{dom}(B_h^k) \subset B_h^{k,pa}$, we have by Lemma 6.3

$$q_1 \leq a_0^0\text{n}_h^{(B_h^{k,pa})} H + \beta_{n_h^{(B_h^{k,pa})}} + 4L \cdot \text{rad}(B_h^{k,pa}) + \sum_{i=1}^{n_h^{(B_h^{k,pa})}} \alpha_i^{(B_h^{k,pa})} (V_{h+1}(B_h^{k,pa}^i) - V_h(s_h^k, a_h^k))$$

where we denote by $k_i(B) \in [1, n_h^{(B_h^{k,pa})}]$ the $i$-th episode where $B$ was selected by the algorithm at step $h$. As $B_h^{k,pa}$ is a parent, we have $n_h^{(B_h^{k,pa})} > 0$ implying that $\alpha_0^{(B_h^{k,pa})} = 1\{n_h^{(B_h^{k,pa})} = 0\} = 0$. Moreover, by the activation rule, we have $\frac{1}{\sqrt{n_h^{(B_h^{k,pa})}}} \leq \text{rad}(B_h^{k,pa})$, implying that $\beta_{n_h^{(B_h^{k,pa})}} \leq 16\sqrt{\frac{H_3}{n_h^{(B_h^{k,pa})}}} \leq 16\sqrt{H_3} \cdot \text{rad}(B_h^{k,pa}) = 32\sqrt{H_3} \cdot \text{rad}(B_h^k)$. Consequently,

$$q_1 \leq (8L + 32\sqrt{H_3}) \cdot \text{rad}(B_h^k) + \sum_{i=1}^{n_h^{(B_h^{k,pa})}} \alpha_i^{(B_h^{k,pa})} k_i(B_h^{k,pa})$$

and therefore,

$$\delta_h^k \leq (11L + 32\sqrt{H_3}) \cdot \text{rad}(B_h^k) + \sum_{i=1}^{n_h^{(B_h^{k,pa})}} \alpha_i^{(B_h^{k,pa})} k_i(B_h^{k,pa}) + (Q_h^* - Q_h^{\pi_k})(s_h^k, a_h^k) \tag{12}$$

2. Case of $B_h^k = B_{\text{init}}$: 

---

**ZoomRL**
The third inequality follows from lemma 6.3 and the last inequality follows from the fact that $\text{rad}(B_h^k) = \text{rad}(B_{\text{init}}) = 1$

Now, we can unify the bound (12) obtained in the first case where the algorithm selects a ball other than the initial ball and the bound (13) in second case where the initial ball is selected. To do that, we consider, by abuse of notation, that the initial ball is parent of itself i.e when $B_h^k = B_{\text{init}}$ we have $B_h^{k,\text{parent}} = B_{\text{init}}$ and we take the maximum over the two bounds

\[
\delta_h^k \leq \alpha_{n_h^k(B_h^k)}(B_{\text{init}}) \cdot H_{\text{max}}(B_h^k = B_{\text{init}}) + (11L + 32\sqrt{H^3}t) \text{rad}(B_h^k) + \sum_{i=1}^{n_h^k(B_h^k,\text{parent})} \alpha_{n_h^k(B_h^k,\text{parent})}^i(B_h^k,\text{parent}) \cdot (Q_h^k - Q_h^{\text{parent}})(s_h^k, a_h^k)
\]

we obtain the desired result but noting that

\[
(Q_h^k - Q_h^{\text{parent}})(s_h^k, a_h^k) = [(\mathbb{P}_h - \mathbb{P}_h^*)(V_h^* - V_h^{\text{parent}})](s_h^k, a_h^k) = [(\mathbb{P}_h - \mathbb{P}_h)(V_h^{k+1} - V_h^*)](s_h^k, a_h^k) + (V_h^* - V_h^{\text{parent}})(s_h^k, a_h^k)
\]

\[
= [(\mathbb{P}_h - \mathbb{P}_h)(V_h^{k+1} - V_h^*)](s_h^k, a_h^k) + (V^* - \tilde{V}_h^{\text{parent}})(s_h^k, a_h^k) + (\tilde{V}_h^k - V_h^{\text{parent}})(s_h^k, a_h^k)
\]

C.5. Bounding $\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k$

Lemma C.1. With probability at least $1 - p/2$, we have

\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k \leq 4\sqrt{2H^3K^4}
\]

Let $\mathcal{F}_{k,h}$ be the $\sigma$-field generated by all the random variables until episode $k$, step $h$. Then, $\xi_{h+1}^k = [(\mathbb{P}_h - \mathbb{P}_h)(V_h^{k+1} - V_h^{\text{parent}})](s_h^k, a_h^k)$ is a martingale difference sequence w.r.t the filtration $\{\mathcal{F}_{k,h}\}_{k,h \geq 0}$ bounded by $4H$. By Azuma-Hoeffding (lemma F.1), we have $\forall t > 0$, $\text{Pr} \left[ \left| \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k \right| \geq t \right] \leq 2 \exp \left( \frac{-t^2}{32H^3K^4} \right)$. Therefore,

\[
\text{Pr} \left[ \left| \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k \right| \geq 4\sqrt{2H^3K^4} \right] \leq 2 \exp \left( \frac{-32H^3K^4}{32H^3K^4} \right) = 2 \frac{p}{4H^2K^2} \leq p/2
\]

Hence, with probability at least $1 - p/2$, we have

\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k \leq 4\sqrt{2H^3K^4}
\]
D. Misspecified Setting: Approximately Lipschtiz Case

The proof structure is similar to the structure in Appendix B. We will particularly focus on the parts that require different treatments in the misspecified setting.

D.1. Recursive Formula of \( \hat{Q}_h^k(B) - Q_h^*(s,a) \)

Let \( B \in \mathcal{B}_h^k \) and \((s,a) \in \text{dom}_h^k(B) \).

Since \( \sum_{i=0}^t \alpha_i = 1 \), we have that \( Q_h^*(s,a) = \alpha_0^0 Q_h^*_0(s,a) + \sum_{i=1}^t \alpha_i^i Q_h^*_i(s,a) \).

By the \( \epsilon \)-approximately Lipschtiz assumption 5.3 and the fact \( \forall i \in [t] \), \((s_h^k, a_h^k) \in B \) and \((s,a) \in B \), we have:

\[
|Q_h^*(s_h^k, a_h^k) - Q_h^*(s,a)| \leq L \cdot \text{dist}((x_h^k, a_h^k), (x,a)) + 2\epsilon \leq 2L \cdot \text{rad}(B) + 2\epsilon.
\]

Then we have

\[
Q_h^*(s,a) \geq \alpha_0^0 Q_h^*_0(s,a) + \sum_{i=1}^t \alpha_i^i \left( Q_h^*(s_h^k, a_h^k) - 2L \cdot \text{rad}(B) - 2\epsilon \right) \tag{15}
\]

\[
Q_h^*(s,a) \leq \alpha_0^0 Q_h^*_0(s,a) + \sum_{i=1}^t \alpha_i^i \left( Q_h^*(s_h^k, a_h^k) + 2L \cdot \text{rad}(B) + 2\epsilon \right). \tag{16}
\]

By Bellman equation, we have \( Q_h^*(s_h^k, a_h^k) = r_h(s_h^k, a_h^k) + [P_h V_h^*](s_h^k, a_h^k) \). Recall \([\hat{P}_h V_h^*](s_h^k, a_h^k) = V_{h+1}(s_{h+1}^k)\), we have:

\[
Q_h^*(s_h^k, a_h^k) = r_h(s_h^k, a_h^k) + [(P_h - \hat{P}_h) V_{h+1}^*](s_h^k, a_h^k) + V_{h+1}^*(s_{h+1}^k).
\]

Substitute the above equality into Eq. (15) and (16), we have:

\[
Q_h^*(s,a) \geq \alpha_0^0 Q_h^*_0(s,a) + \sum_{i=1}^t \alpha_i^i \left( r_h(s_h^k, a_h^k) + [(P_h - \hat{P}_h) V_{h+1}^*](s_h^k, a_h^k) + V_{h+1}^*(s_{h+1}^k) - 2L \cdot \text{rad}(B) - 2\epsilon \right)
\]

\[
Q_h^*(s,a) \leq \alpha_0^0 Q_h^*_0(s,a) + \sum_{i=1}^t \alpha_i^i \left( r_h(s_h^k, a_h^k) + [(P_h - \hat{P}_h) V_{h+1}^*](s_h^k, a_h^k) + V_{h+1}^*(s_{h+1}^k) + 2L \cdot \text{rad}(B) + 2\epsilon \right)
\]

Subtracting the formula in Lemma 6.2 from the two above inequalities, we have:

\[
\hat{Q}_h^k(B) - Q_h^*(s,a) \geq \sum_{i=1}^t \alpha_i^i \left( \left( V_{h+1}^* - \hat{V}_{h+1}^* \right)(s_{h+1}^k) + [(P_h - \hat{P}_h) V_{h+1}^*](s_h^k, a_h^k) + u_i - 2\epsilon \right) \tag{17}
\]

\[
\check{Q}_h^k(B) - Q_h^*(s,a) \leq \alpha_0^0 H + \sum_{i=1}^t \alpha_i^i \left( \left( V_{h+1}^* - \hat{V}_{h+1}^* \right)(s_{h+1}^k) + [(P_h - \hat{P}_h) V_{h+1}^*](s_h^k, a_h^k) + u_i + 4L \cdot \text{rad}(B) + 2\epsilon \right) \tag{18}
\]

E. Bounding of \( Q_h^k(B) - Q_h^*(s,a) \)

Lemma E.1. Suppose Assumption 5.3 holds. For any \( p \in (0, 1) \), we have \( \beta_t = 2 \sum_{i=1}^t \alpha_i^i u_i \leq 16 \sqrt{\frac{H^2 \epsilon}{p}} \) and, with probability at least \( 1 - p/2 \), we have that for all \((s,a,h,k) \in \mathcal{S} \times \mathcal{A} \times [H] \times [K] \) and any ball \( B \) such that \((s,a) \in \text{dom}_h^k(B) \):

(a) \( \hat{Q}_h^k(B) - Q_h^*(s,a) \geq -4(H-h+1)\epsilon \)

(b) \( \check{Q}_h^k(B) - Q_h^*(s,a) \leq \alpha_0^0 H + \beta_t + 4L \cdot \text{rad}(B) + 2\epsilon + \sum_{i=1}^t \alpha_i^i \left( \left( V_{h+1}^* - \hat{V}_{h+1}^* \right)(s_{h+1}^k) \right) \)

where \( t = n_h^k(B) \) and \( k_1, \ldots, k_t < k \) are the episodes where \( B \) was selected at step \( h \).
E.1. High Probability Bound On The Sampling Noise

The same reasoning as in the subsection C.3.1 in the exact lipschitz case gives: with probability at least 1 – p/2: ∀(h,k) ∈ [H] × [K] and for all ball \( B \in \mathcal{B}_k^h \),

\[
\sum_{i=1}^{t} \alpha_i \cdot |(\hat{P}_h^k - \mathbb{P}_h)V^*_h(x^k_i, a^*_h)| \leq \beta_t/2, \text{where } t = n^k_h(B) \tag{19}
\]

E.2. Approximate Optimism Of \( Q \)-values

We proceed by induction. By definition, we have \( \hat{Q}^k_{H+1} = Q^*_h = 0 \) which implies \( Q^k_{H+1}(B) - Q^*_h(s,a) = -4(H - (H + 1) + 1)\epsilon \). Assume that \( Q^k_{H+1}(B) - Q^*_h(s,a) \geq -4(H - (H + 1) + 1)\epsilon = -4(H - h)\epsilon \).

Let \( i \in [1, t] \). We have \( V^*_h(s^k_{h+1}(s^k_{h+1})) = Q^*_h(s^k_{h+1}(s^k_{h+1})) \). As the set of domains of active balls covers the entire space, there exists \( B^* \in \mathcal{B}_k^h \) such that \( (s^k_{h+1}, \pi^*_h(s^k_{h+1})) \in \text{dom}_{h+1}^k(B^*) \). By the definition of index, we have \( \text{index}^k_{h+1}(B^*) = L \cdot \text{rad}(B^*) + \hat{Q}^k_{h+1}(B^*) + L \cdot \text{dist}(\hat{B}^*, B^*) \) for some ball \( \hat{B}^* \).

We have

\[ \hat{V}^k_{h+1}(s^k_{h+1}) - V^*_h(s^k_{h+1}) = \min\{H, \max_{B \in \mathcal{B}_k^h} \text{index}^k_{h+1}(B) - Q^*_h(s^k_{h+1}, \pi^*_h(s^k_{h+1})) \geq \text{index}^k_{h+1}(B^*) - Q^*_h(s^k_{h+1}, \pi^*_h(s^k_{h+1})) \geq L \cdot \text{rad}(B^*) + \hat{Q}^k_{h+1}(B^*) + L \cdot \text{dist}(\hat{B}^*, B^*) - Q^*_h(s^k_{h+1}, \pi^*_h(s^k_{h+1})) \geq L \cdot \text{rad}(B^*) - 4(h - h)\epsilon + Q^*_h(s^k_{h+1}, \pi^*_h(s^k_{h+1})) \geq -4(H - h)\epsilon \]

Where \( s^{k'}_{h'}, a_{B_{B'}} \) and \( s^{k'}_{\hat{B}}, a_{\hat{B}_{B'}} \) denote respectively the centers of balls \( B^* \) and \( \hat{B}^* \). The first inequality follows from \( Q^*_h(s,a) \leq H \) for any state-action pair \( (s,a) \). The third inequality follows from the induction hypothesis. The fourth and the last inequalities follow from the assumption 5.3.

Therefore, we have

\[ Q^k_{h}(B) - Q^*_h(s,a) \geq \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}^k_{h+1} - V^*_h(s^k_{h+1}))(s^k_{h+1}) + [(\hat{P}_h^k - \mathbb{P}_h)\mathbb{V}^*_h(s^k_{h+1}, a^*_h) + u_t - 2\epsilon] \geq -4(H - h)\epsilon - \beta_t/2 + \beta_t/2 - 2\epsilon = -4(H - h + 1)\epsilon \]

E.3. Upper Bound of \( \hat{Q}^k_h(B) - Q^*_h(s,a) \)

We have:

\[ \hat{Q}^k_h(B) - Q^*_h(s,a) \leq \alpha_i \cdot H + \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}^k_{h+1} - V^*_h(s^k_{h+1}))(x^k_{h+1}) + [(\hat{P}_h^k - \mathbb{P}_h)\mathbb{V}^*_h(s^k_{h+1}, a^*_h) + u_t + 4 \cdot \text{rad}(B) + 2\epsilon] \leq \alpha_i H + \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}^k_{h+1} - V^*_h(s^k_{h+1}))(x^k_{h+1}) + \beta_t/2 + \sum_{i=1}^{t} \alpha_i u_t + \sum_{i=1}^{t} \alpha_i (4L \cdot \text{rad}(B) + 2\epsilon) \leq \alpha_i H + \beta_t + 4L \cdot \text{rad}(B) + 2\epsilon + \sum_{i=1}^{t} \alpha_i \cdot (\hat{V}^k_{h+1} - V^*_h(s^k_{h+1})) \]

where the second inequality follows from the inequality 19. The third inequality follows from \( \sum_{i=1}^{t} \alpha_i \leq 1 \).
Lemma E.2 (Approximate Optimism). Following the same setting as in Lemma E.1, for any \((h, k)\), with probability at least \(1 - \frac{p}{2}\), we have for any \(s \in S\):

\[
\hat{V}_h^k(s) \geq V_h^*(s) - (4(H - h) + 5)\epsilon
\]

Proof. Let \(s \in S\). We have \(V_h^*(s) = Q_h^*(s, \pi_h^*(s))\). As the set of domains of active balls covers the entire space, there exists \(B^* \in B_{h+1}^k\) such that \((s, \pi_h^*(s)) \in \text{dom}_h^k(B^*)\). By the definition of index, we have \(\text{index}_h^k(B^*) = L \cdot \text{rad}(B^*) + \hat{Q}_h^k(B^*) + L \cdot \text{dist}(\hat{B}^*, B^*)\) for some active ball \(\hat{B}^*\).

We have

\[
\hat{V}_h^k(s) - V_h^*(s) = \min\{H, \max_{B \in \text{rel}_h^k(s)} \text{index}_h^k(B)\} - Q_h^*(s, \pi_h^*(s))
\]

\[
\geq \max_{B \in \text{rel}_h^k(s)} \text{index}_h^k(B) - Q_h^*(s, \pi_h^*(s))
\]

\[
\geq \text{index}_h^k(B^*) - Q_h^*(s, \pi_h^*(s))
\]

\[
= L \cdot \text{rad}(B^*) + \hat{Q}_h^k(B^*) + L \cdot \text{dist}(\hat{B}^*, B^*) - Q_h^*(s, \pi_h^*(s))
\]

\[
\geq L \cdot \text{rad}(B^*) + Q_h^*(s_{\hat{B}^*}, a_{\hat{B}^*}) - 4(H - h + 1)\epsilon + L \cdot \text{dist}(\hat{B}^*, B^*) - Q_h^*(s, \pi_h^*(s))
\]

\[
\geq L \cdot \text{rad}(B^*) + Q_h^*(s_{\hat{B}^*}, a_{\hat{B}^*}) - 2\epsilon - 4(H - h + 1)\epsilon - Q_h^*(s, \pi_h^*(s))
\]

\[
\geq -4\epsilon - 4(H - h + 1)\epsilon = -(4H - h) + 5)\epsilon
\]

Where \((s_{\hat{B}^*}, a_{\hat{B}^*})\) and \((s_{\hat{B}^*}, a_{\hat{B}^*})\) denote respectively the centers of balls \(B^*\) and \(\hat{B}^*\). The first inequality follows from \(Q_h^*(s, a) \leq H\) for any state-action pair \((s, a)\). The third inequality follows from lemma E.1. The fourth and the last inequalities follow from assumption 5.3.

E.4. Regret Analysis

By the approximate optimism of our estimates with respect to the true value function (see lemma E.2), we have with probability at least \(1 - \frac{p}{2}\)

\[
\text{REGRET}(K) = \sum_{k=1}^{K} (V_1^1 - V_1^{\pi_k})(x_1^1) \leq \sum_{k=1}^{K} (\hat{V}_1^k - V_1^{\pi_k})(x_1^1) + K(4H + 1)\epsilon = \sum_{k=1}^{K} \delta_h^k + K(4H + 1)\epsilon
\]

Similarly to the Lemma 6.5, we can show that using Lemma E.1 applied on \(B_{h-1}^k\) the parent of the selected ball at step \(h\) of the episode \(k\).

\[
\delta_h^k \leq \alpha_{s_h^k(B_h^k)} H_{h-1} + (11L + 32\sqrt{H^3}) \text{rad}(B_h^k) + \sum_{i=1}^{n_h^k(B_h^k \sigma_h^k)} \alpha_{s_h^k(B_h^k \sigma_h^k)} \phi_h^k(B_h^k) - \phi_{h+1}^k + \xi_{h+1}^k + 2\epsilon
\]

Following the same steps of the Section 6.1 in the exact lipschitz setting, we obtain

\[
\sum_{k=1}^{K} \delta_h^k \leq H + (11L + 32\sqrt{H^3}) \sum_{k=1}^{K} \text{rad}(B_h^k) + \left(1 + \frac{1}{H}\right) \sum_{k=1}^{K} \delta_{h+1}^k + \sum_{k=1}^{K} \xi_{h+1}^k + 2K\epsilon
\]

By unrolling the last inequality for \(h \in [H]\) and using the fact \(\delta_{H+1}^k = 0\) \(\forall k \in [K]\), we obtain

\[
\sum_{k=1}^{K} \delta_1^k \leq \sum_{h=1}^{H} \left(1 + \frac{1}{H}\right)^{h-1} \left(H + (11L + 32\sqrt{H^3}) \sum_{k=1}^{K} \text{rad}(B_h^k) + \sum_{k=1}^{K} \xi_{h+1}^k + 2K\epsilon\right)
\]

\[
\leq 3H^2 + 3(11L + 32\sqrt{H^3}) \sum_{h=1}^{H} \sum_{k=1}^{K} \text{rad}(B_h^k) + 3 \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k + 6HK\epsilon
\]
Plugging bounds (4) and (5) from Section 6.1 in (20) and using union bound, we have with probability $1 - p$

$$\sum_{k=1}^{K} \delta_k \leq O \left( H^2 + \sqrt{H^3 K} + (L + \sqrt{H^3}) \min_{r_0 \in (0, 1)} \left\{ K r_0 + \sum_{\substack{r=2^{r_0} \quad r \geq r_0}}^{\infty} \frac{M(r)}{r} \right\} + HK \epsilon \right)$$

We obtain the regret bound in theorem 5.4 by noting that $\text{REGRET}(K) \leq \sum_{k=1}^{K} \delta_k + O(HK \epsilon)$. 
F. Technical Lemmas

**Lemma F.1** (Azuma-Hoeffding inequality). Suppose \( \{X_k: k = 0, 1, 2, 3, \cdots \} \) is a martingale and \( |X_k - X_{k-1}| < c_k \), almost surely. Then for all positive integers \( N \) and all positive reals \( t \),

\[
\Pr[|X_N - X_0| \geq t] \leq 2 \exp \left( \frac{-t^2}{2 \sum_{k=1}^{N} c_k^2} \right).
\]

**Lemma F.2** (Lemma 4.1 in Jin et al. (2018)). The following properties hold for \( \alpha_i t \):

(a) \( \frac{1}{\sqrt{t}} \leq \sum_{i=1}^{t} \alpha_i \frac{1}{\sqrt{t}} \leq 2 \sqrt{t} \) for every \( t \geq 1 \).

(b) \( \max_{i \leq t} \alpha_i \leq \frac{2H}{\sqrt{t}} \) and \( \sum_{i=1}^{t} (\alpha_i)^2 \leq \frac{2Ht}{2} \) for every \( t \geq 1 \).

(c) \( \sum_{i=1}^{\infty} \alpha_i = 1 + \frac{1}{H} \) for every \( i \geq 1 \).

F.1. Few Reminders on Probability Theory

We consider a probability space \((\Omega, F, P)\). We borrow notation from Qian et al. (2018). We call filtration any increasing (for the inclusion) sequence of sub-\( \sigma \)-algebras of \( F \), i.e., \((F_n)_{n \in \mathbb{N}}\) where \( \forall n \in \mathbb{N}, F_n \subset F_{n+1} \subset F \). We denote by \( F_\infty = \bigcup_{n \in \mathbb{N}} F_n \).

**Definition F.3** (Stopping time). A random variable \( \tau : \Omega \rightarrow \mathbb{N} \cup \{+\infty\} \) is called stopping time with respect to a filtration \((F_n)_{n \in \mathbb{N}}\) if for all \( n \in \mathbb{N}, \{\tau = n\} \in F_n \).

**Definition F.4** (\( \sigma \)-algebra at stopping time). Let \( \tau \) be a stopping time. An event prior to \( \tau \) is any event \( A \in F_\infty \) s.t \( A \cap \tau = n \in F_n \) for all \( n \in \mathbb{N} \). The set of events prior to \( \tau \) is a \( \sigma \)-algebra denoted \( F_\tau \) and called \( \sigma \)-algebra at time \( \tau \):

\[
F_\tau = \{ A \in F_\infty, \forall n \in \mathbb{N}, A \cap \tau = n \in F_n \}
\]

**Lemma F.5.** Let \( \tau_1 \) and \( \tau_2 \) be two stopping times with respect to the same filtration \((F_n)_{n \in \mathbb{N}}\) s.t \( \tau_1 \leq \tau_2 \) almost surely. Then \( F_{\tau_1} \subset F_{\tau_2} \).