Small data global existence and decay for two dimensional wave maps

Willie Wong∗

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Abstract We prove the global existence and almost-sharp decay for the wave-map equation on \( \mathbb{R}^{1,2} \), when the initial data has compact spatial support, using a variant of the vector field method. The main innovation lies in controlling the dispersive decay of the solution \( \phi \) itself (as opposed to its derivatives), which has not been previously established using purely physical space methods. Our energy-based method carries a log-loss in the time-decay for the linear evolution on \( \mathbb{R}^{1,2} \), when compared with the stationary-phase results; this loss is specific to two spatial dimensions. As the linear theory is largely similar between dimensions, we include a discussion of the method for \( \mathbb{R}^{1,d} \) in general, also showing how to obtain sharp rates of decay when \( d \geq 3 \).

1. Introduction

We consider the equation

\[
\Box_\eta \phi = \phi \cdot \eta(\partial_t \phi, \partial_i \phi)
\]

which models the wave-map equation, in the small data regime. Here \( \eta \) is the Minkowski metric on \( \mathbb{R}^{1,d} \), represented by the matrix \( \text{diag}(-1, 1, \ldots, 1) \) in rectangular coordinates, and so in standard coordinates we can rewrite (1.1) as

\[-\partial_{tt}^2 \phi + \sum_{i=1}^d \partial_{ii}^2 \phi = \phi \left[ - (\partial_t \phi)^2 + \sum_{i=1}^d (\partial_i \phi)^2 \right].
\]

The general wave-map equation, in local coordinates, is the quasidiagonal system of semilinear wave equations for \( \phi = (\phi^1, \ldots, \phi^N): \mathbb{R}^{1,d} \to \mathbb{R}^N \) given by the component-wise equations

\[
\Box_\eta \phi^C = \sum_{A,B=1}^N \Gamma_{AB}^C(\phi) \eta(\partial_t \phi^A, \partial_t \phi^B)
\]

∗Michigan State University, East Lansing, USA; wongwy@member.ams.org
where the Christoffel symbols $\Gamma$ captures the geometry of the target manifold. For solutions which are perturbations of a constant solution, we can perform a Taylor expansion of the functions $\Gamma$ and arrive at (1.1), which captures the leading order contributions. In the small-data regime where decay can be proven, one can rather straightforwardly upgrade results concerning (1.1) to the full wave-map system. For more about the wave-map equation in general, see [12].

Using the dispersive decay of the linear wave equation, one can easily show, using the vector field method of Klainerman, that when the domain has spatial dimension $d \geq 3$ that solutions to (1.1) exist for all time and scatters provided that the initial data is sufficiently small in certain weighted energy norm. Indeed, multiplying (1.1) by $\partial_t \phi$ and integrating by parts on the domain $[0, T] \times \mathbb{R}^d$, we see the energy inequality

\begin{equation}
(1.2) \quad \frac{1}{2} \int_{[T] \times \mathbb{R}^d} (\partial_t \phi)^2 + |\nabla \phi|^2 \, dx \leq \frac{1}{2} \int_{[0] \times \mathbb{R}^d} (\partial_t \phi)^2 + |\nabla \phi|^2 \, dx + \int_0^T \int_{\mathbb{R}^d} |\phi(\partial \phi)|^3 \, dx \, dt.
\end{equation}

By Gronwall, we see then

\begin{equation}
(1.3) \quad \int_{[T] \times \mathbb{R}^d} (\partial_t \phi)^2 + |\nabla \phi|^2 \, dx \leq \left[ \int_{[0] \times \mathbb{R}^d} (\partial_t \phi)^2 + |\nabla \phi|^2 \, dx \right] \cdot \exp \left[ C \int_0^T \sup_{\mathbb{R}^d} |\phi \partial \phi| \, dt \right].
\end{equation}

Using that the expected $L^\infty$ decay for the linear wave equation is at the rate $(1 + t)^{(1-d)/2}$, we see that when $d \geq 3$ the integral within the exponential is expected to converge, giving global uniform bounds on the energy. This argument can be made precise via the Klainerman-Sobolev inequalities [10, 7] which allows us to relate boundedness of weighted energies with $L^\infty$ decay of the solution.

In dimension $d = 2$ we see, however, that the expected $L^\infty$ decay for the linear wave equation is only at the rate $(1 + t)^{-1/2}$, and so $\sup_{\mathbb{R}^2} |\phi \partial \phi|$ is not integrable in time. This difficulty can in principle be overcome by the fact that the nonlinearity $\eta(d\phi, d\phi)$ satisfies the null condition [8, 2, 1, 2], which would imply that for all intents and purposes the term

$$\sup_{\mathbb{R}^2} |\phi \partial \phi| \leq (1 + t)^{-3/2}.$$ 

The extra $t^{-1/2}$ decay upgrades the nonlinearity to be integrable in time.

There is however, one other difficulty in dimension $d = 2$, and this relates to obtaining the decay estimate for $\phi$ itself. Classical Klainerman-Sobolev inequalities, when applied to the standard energy estimates, only control $|\partial \phi|$ in $L^\infty$. One
can na"ively estimate $|\phi(t)| \leq |\phi(0)| + \int_0^t |\partial_t \phi(s)| \, ds$, but at a loss of one factor of $t$ decay. A better method for dimensions $d \geq 3$ instead modifies the vector field method by using the Morawetz energy (where instead of multiplying by $\partial_t \phi$ and integrating by parts, one multiplies by $(t^2 + r^2) \partial_t \phi + 2tr \partial_r \phi + (d - 1)t \phi$). When $d \geq 3$, the resulting energy is coercive on $\|\phi\|_{L^2(\mathbb{R}^d)}$, and direct applications of the classical vector field method yields also pointwise decay for the solution $\phi$ itself. This argument is worked out in detail in [5]. When $d = 2$, however, the coercivity is lost, and while we have good control on $\eta(d\phi, d\phi)$ by virtue of the Klainerman-Sobolev inequalities and the null condition, we have no control on the $\phi$ factor in the nonlinearity in (1.1).

The goal of the present manuscript is to present a modified vector field method that allows us to recover, when $d = 2$, decay estimates on $\phi$ itself up to a logarithmic loss. Through this control we are able to prove:

1.4 Theorem
Let $\phi_0, \phi_1 \in C_0^\infty(B(0,1))$, and fix $k \geq 3$. There exists an $\epsilon > 0$ such that whenever $\|\phi_0\|_{H^{k+1}} + \|\phi_1\|_{H^k} < \epsilon$, there exists a global solution to (1.1) with $\phi(0,x) = \phi_0(x)$ and $\partial_t \phi(0,x) = \phi_1(x)$. Furthermore the solution decays to zero as $t \to +\infty$. ■

We will state and prove a more precise version of this theorem, including the boundedness of weighted energies as well as the pointwise peeling estimates, in Section 6. An interesting feature of our physical space argument is that we essentially make use of the assumption that the initial data has compact support; we hope to overcome this in future work.

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2. Hyperboloidal Global Sobolev Inequalities

We begin by proving a family of weighted global Sobolev inequalities adapted to hyperboloids in Minkowski space. Inequalities of this type was first introduced by Klainerman [9, 6] for the study of the decay of the Klein-Gordon equation, and has been adapted by LeFloch and Ma [11] for treatment of coupled Klein-Gordon–Wave systems. The main novelty in this section of our work is the geometric formulation of our main inequalities as weighted Sobolev inequalities adapted to the Lorentz boosts, in the sense that standard Sobolev inequalities on Euclidean domains are adapted to coordinate partial derivatives. In applications
this means that the only commutator vector fields we will use are Lorentz boosts (in particular, no purely spatial rotations).

We restrict our attention to the interior of the future light-cone in Minkowski space, that is, the set \( \{ t > |x| \} \subset \mathbb{R}^{1,d} \). (We use \( x^1, \ldots, x^d \) for the standard coordinates on \( \mathbb{R}^d \), and use \( x^0 \) or \( t \) for the time coordinate.) Let

\[
\tau \overset{\text{def}}{=} \sqrt{t^2 - |x|^2}
\]

and denote by \( \Sigma_\tau \) its level sets. The \( \Sigma_\tau \) are hyperboloids that asymptote to the forward light-cone centered at the origin.

Denote by \( L^i, i \in \{1, \ldots, d\} \) the Lorentz boost vector fields

\[
L^i \overset{\text{def}}{=} t \partial_{x^i} + x^i \partial_t.
\]

By definition \( L^i \) are tangent to the hypersurfaces \( \Sigma_\tau \). The set \( \{L^i\} \) is linearly independent and span the tangent space of \( \Sigma_\tau \). We will also mention the rotational vector fields

\[
\Omega_{ij} \overset{\text{def}}{=} x^i \partial_{x^j} - x^j \partial_{x^i}.
\]

The \( \Omega_{ij} \) are also tangent to \( \Sigma_\tau \), and they admit the decomposition

\[
\Omega_{ij} \overset{\text{def}}{=} \frac{x^i}{t} L^j - \frac{x^j}{t} L^i.
\]

The boosts and rotations form an algebra under commutation:

\[
\begin{align*}
[L^i, L^j] &= \Omega_{ij} \\
[\Omega_{ij}, \Omega_{jk}] &= \Omega_{ik} \\
[L^i, \Omega_{ij}] &= L^j
\end{align*}
\]

We can define a system of radial coordinates on \( \{ t > |x| \} \setminus \{ x = 0 \} \). Let

\[
(\tau, \rho, \theta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S^{d-1}
\]

(where we identify \( S^{d-1} \) canonically as the unit sphere in \( \mathbb{R}^d \)) be the coordinate of the point

\[
\begin{align*}
t &= \tau \cosh(\rho), \\
x &= \tau \sinh(\rho) \cdot \theta.
\end{align*}
\]
The Minkowski metric takes the warped product form

\begin{equation}
\eta = -dt^2 + \sum_{i=1}^{d} (dx^i)^2 = -d\tau^2 + \tau^2 \, d\rho^2 + \tau^2 \sinh(\rho)^2 \, d\theta^2
\end{equation}

were by \(d\theta^2\) we refer to the standard metric on \(S^{d-1}\). Then clearly the induced Riemannian metrics on \(\Sigma_\tau\), which we denote by \(h_\tau\), have the coordinate expressions relative to the \((\rho, \theta)\) coordinates

\begin{equation}
h_\tau = \tau^2 (d\rho^2 + \sinh(\rho)^2 \, d\theta^2)
\end{equation}

\begin{equation}
(h_\tau)^{-1} = \frac{1}{\tau^2} \left( \partial_\rho \otimes \partial_\rho + \frac{1}{\sinh(\rho)^2} \partial_\theta \otimes \partial_\theta \right)
\end{equation}

where \(\partial_\theta \otimes \partial_\theta\) is the inverse standard metric on \(S^{d-1}\).

One can check the identity

\begin{equation}
\sum_{i=1}^{d} L^i \otimes L^i = \partial_\rho \otimes \partial_\rho + \frac{\cosh(\rho)^2}{\sinh(\rho)^2} \partial_\theta \otimes \partial_\theta
\end{equation}

which implies that

\begin{equation}
(\tau^{-2} h_\tau)^{-1} + \sum_{i<j} \Omega_{ij} \otimes \Omega_{ij} = \sum_{i=1}^{d} L^i \otimes L^i
\end{equation}

as tensor fields along \(\Sigma_\tau\). Note that \((\Sigma_\tau, \tau^{-2} h_\tau)\) is isometric to the standard hyperbolic space \((\mathbb{H}^d, h)\). A particular consequence is the following coercivity property:

2.12 **Lemma**

Let \(f\) be a smooth function on \(\Sigma_\tau\). Then

\[ (h_\tau^{-1})(df, df) \leq \tau^{-2} \sum_{i=1}^{d} \left| L^i f \right|^2. \]

2.13 **Convention**

Throughout the notation \(\partial_t\) denotes the vector field associated to the \(\partial_t\) partial differential in the standard coordinates of \(\mathbb{R}^{1,d}\); similarly \(\partial_t\) and \(\partial_\rho\) the vector fields associated to the corresponding partial derivatives in the \((\tau, \rho, \theta)\) coordinate system.
2.14 Remark
Observe that $\partial_\tau$ and $\partial_\rho$ have the decompositions

$$
\partial_\tau = \frac{1}{\tau} (t \partial_t + \sum_{i=1}^d x_i \partial x_i);
$$

$$
\partial_\rho = t \partial_\tau + r \partial_1 = \sum_{i=1}^d \frac{x^i}{r} L^i.
$$

The global Sobolev inequalities that we use will be derived from the following standard Sobolev inequalities which we state without proof. For convenience we introduce the notation

$$
\mathfrak{s}(d) \overset{\text{def}}{=} \left\lceil \frac{d}{2} \right\rceil + 1
$$

for the minimum integer satisfying $H^{\mathfrak{s}(d)} \hookrightarrow L^\infty$.

2.16 Proposition (Standard Sobolev inequalities)
Below the symbol $\nabla$ stands for the Levi-Civita connection on the corresponding Riemannian manifolds.

1. Let $f$ be a function on the standard hyperbolic space $(\mathbb{H}^d, h)$, which we represent in polar coordinates $(\rho, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ as above. Then

$$
\sup_{\rho < \frac{4}{3}} |f(\rho, \theta)|^2 \lesssim \sum_{k \leq \mathfrak{s}(d)} \int_0^2 \int_{\mathbb{S}^{d-1}} |\nabla^k f|^2 h^{\rho} \sinh(\rho) \, d\theta \, d\rho.
$$

2. Let $f$ be a function on the cylinder $\mathbb{R}_+ \times \mathbb{S}^{d-1}$ with the product metric. Then

$$
\sup_{\rho > \frac{4}{3}} |f(\rho, \theta)|^2 \lesssim \sum_{k \leq \mathfrak{s}(d)} \int_1^\infty \int_{\mathbb{S}^{d-1}} |\nabla^k f|^2 \, d\theta \, d\rho.
$$

For convenience\footnote{We cannot use the standard multi-index notation, whose definition uses the fact that partial derivatives commute. Similarly, we cannot use the natural generalization of the multi-index notation as usually used in discussion of the vector field method, since there the set of first order operators used generate a Lie algebra (are closed under commutations). As we saw in \cite{25}, the Lorentz boost vector fields themselves are not closed under commutation.}, define the index set $I^m = \{1, \ldots, d\}^m$, and set

$$
I^{\leq m} \overset{\text{def}}{=} \bigcup_{0 \leq m' \leq m} I^{m'}; \quad I \overset{\text{def}}{=} \bigcup_{0 \leq m'} I^{m'}.
$$
For $\alpha \in I$, its order (denoted $|\alpha|$), is the (unique) non-negative integer $m$ such that $\alpha \in I^m$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in I^m$, we denote by $L^\alpha$ the $m$th order differential operator

$$(2.17) \quad L^\alpha \phi \overset{\text{def}}{=} L^{\alpha_m} \cdots L^{\alpha_1} \phi.$$ 

By convention, for the unique element $\alpha \in I^0$ we define the corresponding $L^\alpha \phi \overset{\text{def}}{=} \phi$.

Putting everything together we have the following global Sobolev inequality adapted to Lorentz boosts.

**Theorem (Global Sobolev Inequality)**

Let $\ell \in \mathbb{R}$ be fixed. We have the following uniform estimate (the implicit constant depending on $d$ and $\ell$) for functions $f$ defined on the set $\{t > |x|\} \subset \mathbb{R}^{1,d}$:

$$|f(\tau, \rho, \theta)|^2 \lesssim \tau^d \cosh(\rho)^{1-d-\ell} \sum_{\alpha \in I^{\leq s(d)}} \int_{\Sigma_{\tau} \cap \{\rho < 2\}} \cosh(\rho)^{\ell} |L^\alpha f|^2 \frac{\tau^d \sinh(\rho)^{d-1} \, d\tau \, d\rho}{\text{dvol}_{\Sigma_{\tau}}}.$$

**Proof** First we prove the estimate for $\rho < \frac{\ell}{2}$: using that $(\Sigma_{\tau}, h_{\tau})$ is conformal to $(\mathbb{H}^d, h)$ with $h_{\tau} = \tau^2 h$, we see that they have the same Levi-Civita connection. So the first part of Proposition 2.16 implies

$$\sup_{\rho < \frac{\ell}{2}} |f|^2 \lesssim \sum_{k \leq s(d)} \int_{\Sigma_{\tau} \cap \{\rho < 2\}} |\nabla^k f|_{\tau^{-2} h_{\tau}}^2 \text{dvol}_{\tau^{-2} h_{\tau}}.$$

Next, observe that for a fixed $(\rho, \theta)$, the higher derivative $|\nabla^k L^i|_{\tau^{-2} h_{\tau}}$ is independent of $\tau$, and hence has universal bounds when $\rho < 2$. Observing that

$$L^i \nabla_a f = \nabla_a L^i(f) - \nabla_{\nabla_a L^i} f,$$

we see after applying Lemma 2.12 repeatedly by induction that

$$\sum_{k \leq s(d)} |\nabla^k f|_{\tau^{-2} h_{\tau}}^2 \lesssim \sum_{\alpha \in I^{\leq s(d)}} |L^\alpha f|^2$$

on the domain $\Sigma_{\tau} \cap \{\rho < 2\}$, for some universal constant depending only on the dimension $d$. Using that $\cosh(\rho)$ is bounded both above and below on the domain $\{\rho < 2\}$, we see our claim follows, with the $\tau^{-d}$ decay originating from $\tau^{-d} \text{dvol}_{\Sigma_{\tau}} = \text{dvol}_{\tau^{-2} h_{\tau}}$.

Next we prove the estimate for $\rho > \frac{\ell}{2}$: for this we apply the second part of Proposition 2.16 to the function $f \cosh(\rho)^{\ell/2} \sinh(\rho)^{(d-1)/2}$. We note that when...
\( \rho > 1 \), both \( \cosh(\rho) \) and \( \sinh(\rho) \) are uniformly comparable to \( e^\rho \), and so we have \( (\partial_\rho)^{d/2} \sinh(\rho)^{(d-1)/2} \approx \cosh(\rho)^{d/2} \sinh(\rho)^{(d-1)/2} \). Hence immediately we get

\[
(2.19) \quad \sup_{\rho > \frac{\pi}{2}} [f]^2 \cosh(\rho)^{\ell} \sinh(\rho)^{d-1} \lesssim \sum_{k \leq s(d)} \int_{1}^{\infty} \cosh(\rho)^{\ell} |\nabla^k f|^2 \sinh(\rho)^{d-1} \ d\theta \ d\rho;
\]

we remark that the Levi-Civita connection and the norm are both taken with respect to the product metric on \( \mathbb{R}_+ \times S^{d-1} \). Note that the inverse product metric has the decomposition

\[
\partial_\rho \otimes \partial_\rho + \sum_{i < j} \Omega_{ij} \otimes \Omega_{ij}
\]

and by (2.4) we have \( \Omega \approx \frac{\pi \ell}{\ell} L \), and we know that \( \partial_\rho \approx \frac{\pi \ell}{\ell} L \). Since the functions \( x/r = \theta \) and \( x/t = \theta \tanh(\rho) \) have bounded derivatives along \( \Sigma_\tau \) (with respect to the product metric) to all orders away from \( \rho = 0 \), arguing as above we conclude that the quantity with respect to the product metric

\[
\sum_{k \leq s(d)} |\nabla^k f|^2 \lesssim \sum_{\alpha \in I} \sum_{1}^{s(d)} |L^i f|^2.
\]

And thus our claim follows. \( \square \)

3. Decay of linear waves: \( \partial_\ell \)-energy

In this section we apply Theorem 2.18 to get point-wise decay for the derivatives \( \partial_\ell \phi \) of solutions to the linear wave equation \( \Box \phi = 0 \) on \( \mathbb{R}^{1,d} \). A notable feature of our result is that we see improved decay in “good directions” already at the level of first derivatives.

Let \( \phi : \mathbb{R}^{1,d} \to \mathbb{R} \), define the standard energy quantity

\[
(3.1) \quad (E_0[\phi])^2 \overset{\text{def}}{=} \int_{[0] \times \mathbb{R}^d} |\partial_\ell \phi|^2 + |\nabla \phi|^2 \ dx.
\]

3.2 Proposition

Let \( \phi \) be a solution to \( \Box \phi = 0 \). Then we have the estimate in the region \( \{ t > |x| \} \)

\[
|L^i \phi| \lesssim \frac{\tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}}}{t^{1-\frac{d}{2}}} \sum_{\alpha \in I} E_0[L^\alpha \phi],
\]

\[
|\partial_\ell \phi| \lesssim \frac{\tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}}}{t^{1-\frac{d}{2}}(t+r)^{-1/2}(t-r)^{-1/2}} \sum_{\alpha \in I} E_0[L^\alpha \phi]. \quad \square
\]
3.3 Remark
Note that the coefficients of the vector field $L^i$ in rectangular coordinates have size $\approx t$, so the first inequality really states that certain "good derivatives" (the ones in the span of $L^i$ with size $\approx 1/t$ coefficients) decay like $t^{-d/2}$, which captures exactly the expected improve decay over the naive $t^{-(d-1)/2}$. We also see from the second inequality that in the "generic direction" $\partial_t \phi$ decays only uniformly like $t^{-(d-1)/2}$ as expected; however, we also see improved interior decay by $(t-r)^{-1/2}$.

These peeling properties do not obviously hold using the classical vector field method, using the $\partial_t$-energy, for the lowest order derivatives of the solution. ■

3.4 Remark
We do not need compact support for the estimates derived in this section. See also Footnote 2 below.

An interesting feature of our decay estimates, when compared to the classical argument of Klainerman, is that we do not need to commute with the full set of generators of Poincaré group, nor do we need to commute with the scaling vector field $S = t \partial_t + \sum x^i \partial_{x^i}$. Here we only commute with the Lorentz boosts and not spatial rotations. ■

Proof Let $Q_{ab}$ as usual denote the stress energy tensor $\partial_a \phi \partial_b \phi - \frac{1}{2} \eta_{ab} (d\phi, d\phi)$. Consider the energy current

$$J_a = Q_{ab} (\partial_b \phi).$$

It is standard that the space-time divergence $\nabla^a J_a = \Box \phi \partial_t \phi$, which in our case vanishes.

Integrating in the region between $\Sigma_\tau$ and $\{0\} \times \mathbb{R}^d$, by the divergence theorem we have

$$\int_{\Sigma_\tau} Q(\partial_\tau, \partial_t) \, d\text{vol}_{\Sigma_\tau} \leq \frac{1}{2} E_0 \|\phi\|^2.$$

We can compute that

$$\partial_t = \cosh(\rho) \partial_\tau - \tau^{-1} \sinh(\rho) \partial_\rho$$

which implies

$$Q(\partial_t, \partial_\tau) = \cosh(\rho) Q(\partial_\tau, \partial_\tau) - \tau^{-1} \sinh(\rho) Q(\partial_\rho, \partial_\tau).$$

By orthogonality we have

$$Q(\partial_\rho, \partial_\tau) = \partial_\rho \phi \partial_\tau \phi;$$

Note that we don't necessarily have a conservation law; there is formally another boundary integral along null infinity that captures the energy radiated away. By the construction of the energy current, this quantity is signed and its omission in the formula below is what gives the $\leq$ sign instead of the $=$ sign.
and a standard computation gives

\[ Q(\partial_\tau, \partial_\tau) = \frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2\tau^2}(\partial_\rho \phi)^2 + \frac{1}{2\tau^2 \sinh(\rho)^2} |\partial_\theta \phi|^2. \]

Combining the two and completing the square we get the following identity

\[ 2Q(\partial_\tau, \partial_\tau) = \frac{\cosh(\rho)}{\tau^2 \sinh(\rho)^2} |\partial_\theta \phi|^2 + \frac{1}{\tau^2 \cosh(\rho)} (\partial_\rho \phi)^2 + \frac{1}{\cosh(\rho)} (\partial_\tau \phi)^2. \]

By (2.10) we finally have

\[ (3.5) \quad Q(\partial_\tau, \partial_\tau) = \frac{1}{2\tau^2 \cosh(\rho)} \sum_{i=1}^d (L^i \phi)^2 + \frac{1}{2 \cosh(\rho)} (\partial_\tau \phi)^2. \]

Now, as \( L^i \) commutes with the wave operator we have that if \( \phi \) solves \( \Box \phi = 0 \), so does \( L^\alpha \phi \). This implies

\[ (3.6) \quad \sum_{\alpha \in \Lambda^{(d)}} \int \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^d (L^i L^\alpha \phi)^2 \]

\[ + \frac{1}{\cosh(\rho)} (\partial_\tau L^\alpha \phi)^2 \text{ dvol}_{\Sigma_\tau} \leq \sum_{\alpha \in \Lambda^{(d)}} E_0 [L^\alpha \phi]^2. \]

The pointwise estimates for \( L^i \phi \) now follows immediately from Theorem 2.18.

For \( \partial_\tau \phi \), we need to control the integral of \( (L^\alpha \partial_\tau \phi)^2 \); the energy estimate only controls \( (\partial_\tau L^\alpha \phi)^2 \). We do so by computing the commutator:

\[ [L^i, \partial_\tau] = -\partial_{x_i}, \quad [L^i, \partial_{x_j}] = -\delta_{ij} \partial_\tau. \]

By induction this implies

\[ |L^\alpha \partial_\tau \phi| \leq \sum_{|\beta| \leq |\alpha|} |\partial_\tau L^\beta \phi| + \sum_{|\beta| \leq |\alpha| - 1} \sum_{i=1}^d |\partial_{x_i} L^\beta \phi|. \]

Noting that

\[ \partial_{x_i} = \frac{1}{t} (L^i - x_i \partial_\tau) \]

we can bound

\[ \sum_{|\beta| \leq |\alpha| - 1} \sum_{i=1}^d |\partial_{x_i} L^\beta \phi| \leq \sum_{|\beta| \leq |\alpha| - 1} |\partial_\tau L^\beta \phi| + \frac{1}{\tau \cosh \rho} \sum_{|\beta| \leq |\alpha|} |L^\beta \phi|. \]

10
And therefore we have the estimate
\[
\sum_{\alpha \in I} \int_{\Sigma_{\tau}} \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^{d} (L^i L^a \phi)^2 + \frac{1}{\cosh(\rho)} (\partial_t L^a \phi)^2 \ d\text{vol}_{\Sigma_{\tau}} \\
\leq \sum_{\alpha \in I} \int_{\Sigma_{\tau}} \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^{d} (L^i L^a \phi)^2 + \frac{1}{\cosh(\rho)} (\partial_t L^a \phi)^2 \ d\text{vol}_{\Sigma_{\tau}}.
\]

Applying Theorem 2.18 we also get the decay estimate for $\partial_t \phi$. □

4. Decay of linear waves: K-energy

The Morawetz energy can also be used with the hyperboloidal foliation; this gives improved decay properties. For convenience, we shall assume that the data has compact support in this section.

4.1 Proposition
Let $\phi$ be a solution to $\Box \phi = 0$. Suppose $\phi(2, x)$ and $\partial_t \phi(2, x)$ are both supported on the ball of radius 1 centered at the origin. Then in the region $t > \max(2, |x|)$ we have
\[
\left| L^i \phi \right| \lesssim \tau^{-\frac{d}{2}} \cosh(\rho)^{1-\frac{d}{2}} \left( \left\| \partial_t \phi(2, -) \right\|_{H^{s}(d)} + \left\| \phi(2, -) \right\|_{H^{s}(d+1)} \right).
\]

4.2 Remark
We see that this gives a gain of an additional factor of $\tau^{-1}$ decay for derivatives in the “good directions”. In terms of uniform-in-$x$ decay, this means a gain of $t^{-1/2}$ (as well as a gain of $(t-r)^{-1/2}$ which is not uniform on constant $t$ hyperplanes).

Proof Define the modified current
\[
K_a = Q_{ab} K^b + \frac{d-1}{2} t \partial_a \phi^2 - \frac{d-1}{2} \phi^2 \partial_a t
\]
where $K$ is the vector field
\[
K \overset{\text{def}}{=} (t^2 + |x|^2) \partial_t + 2tr \partial_r = \tau^2 \cosh(\rho) \partial_t + \tau \sinh(\rho) \partial_\rho.
\]
It is a simple exercise to check that the space-time divergence
\[
\nabla^a K_a = \Box \phi [K \phi + (d-1) t \phi].
\]
By virtue of finite speed of propagation and the divergence theorem, we have that
\[
\int_{\Sigma_t} K_a(\partial_t)^a \ d\text{vol}_{\Sigma_{t}} = \int_{[2] \times \mathbb{R}^d} K_a(\partial_t)^a \ dx.
\]
The integral on the right is entirely determined by the initial data, so we focus on evaluating the integral over \( \Sigma_\tau \). The integrand is
\[
\mathcal{K}_a(\partial_\tau)^a = Q(K, \partial_\tau) + \frac{d-1}{2} \tau \cosh(\rho) \partial_\tau (\phi^2) - \frac{d-1}{2} \phi^2 \cosh(\rho).
\]
The middle term, observe, can be re-written in terms of \( K \) and \( \partial_\rho \).
\[
\mathcal{K}_a(\partial_\tau)^a = Q(K, \partial_\tau) + \frac{d-1}{2} \tau K(\phi^2) - \frac{d-1}{2} \phi^2 \cosh(\rho) \partial_\rho (\sinh(\rho) \phi^2).
\]
We wish to integrate
\[
\int_{\Sigma_\tau} \mathcal{K}_a(\partial_\tau)^a \, d\text{vol}_{\Sigma_\tau} =
\int_0^\infty \int_{S^{d-1}} \left\{ Q(K, \partial_\tau) + \frac{d-1}{2} \tau K(\phi^2) - \frac{d-1}{2} \phi^2 \cosh(\rho) \partial_\rho (\sinh(\rho) \phi^2) \right\} \tau^d \sinh(\rho)^{d-1} \, d\theta \, d\rho.
\]
Using the finite-speed of propagation property again, which asserts that \( \phi \) has compact support on each \( \Sigma_\tau \), we can integrate the final term in the braces by parts to get
\[
\int_{\Sigma_\tau} \mathcal{K}_a(\partial_\tau)^a \, d\text{vol}_{\Sigma_\tau} =
\int_0^\infty \int_0^{\infty} \left\{ Q(K, \partial_\tau) + \frac{d-1}{2} \tau K(\phi^2) + \frac{(d-1)^2}{2} \cosh(\rho) \phi^2 \right\} \tau^d \sinh(\rho)^{d-1} \, d\theta \, d\rho.
\]
Next, we can write
\[
Q(K, \partial_\tau) = \tau^2 \cosh(\rho) Q(\partial_\tau, \partial_\tau) + \tau \sinh(\rho) Q(\partial_\tau, \partial_\rho)
\]
\[
\quad = \frac{1}{2} \cosh(\rho) \left[ \tau^2 (\partial_\tau \phi)^2 + (\partial_\rho \phi)^2 + \frac{1}{\sinh(\rho)^2} |\partial_\theta \phi|^2 \right] + \tau \sinh(\rho) \partial_\tau \phi \partial_\rho \phi
\]
\[
\quad = \frac{1}{2} \cosh(\rho) \left[ \left( \tau \cosh(\rho) \partial_\tau \phi + \sinh(\rho) \partial_\rho \phi \right)^2 + (\partial_\rho \phi)^2 + \frac{\cosh(\rho)^2}{\sinh(\rho)^2} |\partial_\theta \phi|^2 \right]
\]
\[
\quad = \frac{1}{2} \cosh(\rho) \left[ \frac{1}{\tau^2} (K \phi)^2 + \sum_{i=1}^{d} (L^i \phi)^2 \right].
\]
So finally, completing the square one more time we get

\[ (4.6) \quad \int_{\Sigma_t} K_x (\partial_\tau)^a \text{dvol}_{\Sigma_t} = \int_{\Sigma_t} \frac{1}{2 \cosh(\rho)} \sum_{i=1}^d (L^i \phi)^2 + \frac{1}{2 \tau^2 \cosh(\rho)} \left[ K \phi + (d - 1) t \phi \right]^2 \text{dvol}_{\Sigma_t}. \]

Applying this estimate to \( L^a \phi \) for \( \alpha \in I^{\leq d} \), we see that

\[ \sum_{\alpha \in I^{\leq d}} \int_{\Sigma_t} \frac{1}{\cosh(\rho)} \sum_{i=1}^d |L^i L^a \phi|^2 \text{dvol}_{\Sigma_t} \]

is uniformly bounded by the initial data, which in turn is bounded by

\[ \left\| \partial_t \phi(2, -) \right\|_{H^1}^2 + \left\| \phi(2, -) \right\|_{H^2}^2 \]

(we make use of the compact support again to control the various weights). One application of Theorem 2.18 gives the desired decay.

\[ \square \]

4.7 Remark

Comparing the energy quantities (4.6) and (3.5), we see a clearly hierarchy. This is related to the \( r^p \)-weighted energy estimates of Dafermos and Rodnianski [4].

5. Decay of \( \phi \) itself

In this section we prove that the energy integrals considered in the previous two sections given in (3.5) and (4.6) have nice coercivity properties on the \( L^2 \)-integral for \( \phi \) itself. The argument we present contains two novel aspects:

1. First, it shows that the Morawetz energy is not necessary for controlling the decay of the solution \( \phi \) itself. This has applications to situations where the Morawetz energy is not available, due to the lack of conformal symmetry of the underlying equation. A particular such application is to the study of wave equations on Kaluza-Klein backgrounds. See the discussion in [Remark 5.7].

2. Second, it shows that one can in fact obtain estimates on the solution \( \phi \) itself using purely physical space techniques, in dimension \( d = 2 \). Such estimates were previously unavailable. The downside to our argument is that compact support of initial data seems essential in this case.
The basic technical tool for obtaining the coercivity are the following Hardy inequalities.

5.1 Lemma (Hardy, $d \geq 3$)

Let $d \geq 3$, then

$$
\int_{\Sigma_{\tau}} \frac{1}{\cosh(\rho)} \phi^2 \text{dvol}_{\Sigma_{\tau}} \leq \frac{4}{(d-2)^2} \int_{\Sigma_{\tau}} \frac{1}{\cosh(\rho)} \sum_{i=1}^{d} (L_i \phi)^2 \text{dvol}_{\Sigma_{\tau}}.
$$

5.2 Lemma (Hardy, $d = 2$)

When $d = 2$, then

$$
\int_{\Sigma_{\tau}} \frac{1}{\cosh(\rho)} \phi^2 \text{dvol}_{\Sigma_{\tau}} \lesssim \int_{\Sigma_{\tau}} \frac{(1+\rho^2)}{\cosh(\rho)} \sum_{i=1}^{d} (L_i \phi)^2 \text{dvol}_{\Sigma_{\tau}}.
$$

5.3 Remark

When $d \geq 3$, noting that sinh($\rho$)$^{d-1}/\cosh(\rho)$ grows asymptotically like sinh($\rho$)$^{d-2}$, we see that our Hardy inequality can be viewed as the statement that the hyperbolic space $\mathbb{H}^{d-1}$ of one lower dimension has a spectral gap (and hence $\dot{H}^1$ controls $L^2$). When $d = 2$ this obviously degenerates, and the best we can have is essentially the Hardy inequality on the half-line. Note that compared to the exponential weight in $\rho$, the polynomial $\rho^2$ is effectively a logarithm.

Proof (Lemma 5.1) For any $f : \mathbb{R}_+ \to \mathbb{R}$ that decays rapidly as $\rho \to \infty$, observe that

$$
\int_{0}^{\infty} \frac{1}{\cosh(\rho)} \phi^2 \text{dvol}_{\Sigma_{\tau}} \leq \frac{4}{(d-2)^2} \int_{\Sigma_{\tau}} \frac{1}{\cosh(\rho)} \sum_{i=1}^{d} (L_i \phi)^2 \text{dvol}_{\Sigma_{\tau}}.
$$

Now set $\alpha = d - 2$, so $\alpha + 1 = d - 1$. We can integrating in the spherical directions and use (2.10) to control $(\partial_\rho f)^2$. On the left we apply the simple observation that tanh($\rho$) $\leq 1$. This leads to exactly the claimed inequality.

14
Proof (Lemma 5.2) For any \( f: \mathbb{R}_+ \to \mathbb{R} \) that decays rapidly as \( \rho \to \infty \), we observe that
\[
\int_0^\infty \partial_\rho [\rho f(\rho)^2] \ d\rho = 0
\]
which implies
\[
\int_0^\infty f(\rho)^2 \ d\rho \leq 2 \int_0^\infty f(\rho)f'(\rho) \rho \ d(\rho)
\]
\[
\leq 2 \left[ \int_0^\infty \frac{\rho}{\sqrt{1 + \rho^2}} f(\rho)^2 \ d\rho \right]^{\frac{1}{2}} \left[ \int_0^\infty \rho \sqrt{1 + \rho^2} [f'(\rho)]^2 \ d\rho \right]^{\frac{1}{2}}
\]
by Cauchy-Schwarz. This implies, since \( \rho/\sqrt{1 + \rho^2} < 1 \),
\[
\int_0^\infty f(\rho)^2 \ d\rho \leq 4 \int_0^\infty [f'(\rho)]^2 \rho \sqrt{1 + \rho^2} \ d\rho.
\]
Now since \( \lim_{\rho \to 0} \tan(\rho)/\rho = 1 \), there exists some constant \( C \) such that \( \rho \leq C \tan(\rho) \sqrt{1 + \rho} \) for all \( \rho > 0 \). This implies
\[
\int_0^\infty f(\rho)^2 \tanh(\rho) \ d\rho \leq C \int_0^\infty (1 + \rho^2) \tanh(\rho) [f'(\rho)]^2 \ d\rho
\]
which, after integrating in the spherical directions and using (2.10) is exactly the claimed inequality. \( \square \)

Applying the Hardy inequalities to the energy estimates (3.6) and (4.6) we get the following decay estimates for \( \phi \) itself when dimension \( d \geq 3 \). We omit the obvious proofs.

5.4 Proposition (\( d \geq 3 \), \( \partial_I \)-energy)
Let \( \phi \) solve the linear wave equation on \( \mathbb{R}^{1,d} \) with finite energy initial data, in the sense that \( \sum_{a \in \mathbb{N}^{1,d}} \mathcal{E}_0[L^a \phi] < \infty \). Then
\[
|\phi| \lesssim t^{1-d/2}.
\]
5.5 Proposition \((d \geq 3, \text{K-energy})\)

Let \(\phi\) solve the linear wave equation on \(\mathbb{R}^{1,d}\) with compactly supported initial data (e.g. satisfying the hypotheses of Proposition 4.1), then

\[ |\phi| \lesssim \frac{1}{t^{\frac{d}{2}} - \frac{1}{\sqrt{(t+r)(t-r)}}}. \]

\[ \blacksquare \]

5.6 Remark

We note that the estimates are sharp. The finite \(\partial_t\)-energy condition is compatible with initial data \(\phi(0,x) = 0\) and \(\partial_t \phi(0,x) = (1 + r^2)^{-d/4 - \varepsilon}\). One can check using the fundamental solution that the solution \(\phi(t,0)\) for this data decays like \(t^{1-d/2-2\varepsilon}\).

For the case with the \(K\)-energy, we note that the rate \(t^{-(d-1)/2}\) is exactly the standard \(L^1 - L^\infty\) rate predicted by the fundamental solution (or alternatively stationary phase arguments). This is correlated with the extra spatial weights of the \(K\) energy:

\[ Q(K, \partial_t)|_{t=0} = r^2 Q(\partial_t, \partial_t). \]

\[ \blacksquare \]

5.7 Remark (Kaluza-Klein backgrounds)

That estimates for \(\phi\) itself is available using only the \(\partial_t\) energy has important applications. Previously the only control for \(\phi\) itself in the vector field method is via the \(K\) energy as described in [3]. The availability of the \(K\) energy however depends on the conformal symmetries of the wave equation: indeed, the vector field \(K\) is also known as the “conformally inverted time translation” and can be obtained by pushing forward the vector field \(\partial_t\) under the Lorentzian Kelvin transform. Equations not exhibiting (an approximate version of) this conformal symmetry cannot be expected to have the same inequalities hold. One place where the \(K\) energy is not available is the case of Klein-Gordon equations. However, as was shown originally by Klainerman [10], the \(t^{-d/2}\) decay of \(\phi\) itself is available in using the \(\partial_t\) energy. In our formulation, this observation is due to the fact that, for the Klein-Gordon equation

\[ \Box \phi - m^2 \phi = 0 \]

the analogue of the estimate (3.6) takes the form

\[ \int_{\Sigma_t} \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^{d} (L_i \phi)^2 + \frac{1}{\cosh(\rho)} (\partial_t \phi)^2 + \cosh(\rho) m^2 \phi^2 \, d\text{vol}_{\Sigma_t} \]

\[ \leq \int_{\{0\} \times \mathbb{R}^d} (\partial_t \phi)^2 + |\nabla \phi|^2 + m^2 \phi^2 \, dx. \]
Theorem 2.18 implies that, after commuting with $L^\alpha$, 
\[
|\phi| \lesssim t^{-d/2}
\]
directly. These type of arguments were also used in [11] for handling coupled systems of wave and Klein-Gordon equations.

We note here that similar arguments can also be made for linear waves on Kaluza-Klein backgrounds. More precisely, consider the space-time $\mathbb{R}^{1,d} \times S$ where $S$ is compact and equipped with some Riemannian metric $g_S$. Take the spacetime metric to be the product metric $g = \eta + g_S$. Consider a solution $\phi$ to the linear wave equation $\Box_g \phi = 0$ on this background. Due to the lack of conformal symmetry only the $\partial_t$-energy is available. For convenience denote by $\nabla$ the derivatives tangent to $S$, then one can check that the analogue of (3.6) in this case would be

\[
(5.8) \quad \int_{\Sigma_t \times S} \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^{d} (L^i \phi)^2 + \frac{1}{\cosh(\rho)} (\partial_t \phi)^2 + \cosh(\rho)|\nabla \phi|^2 \, \text{dvol}_{\Sigma_t \times S} \\
\leq \int_{[0] \times \mathbb{R}^d \times S} (\partial_t \phi)^2 + |\nabla \phi|^2 + |\nabla \phi|^2 \, \text{d}x \, \text{d}\sigma.
\]

And the Global Sobolev inequalities then implies the following decay rates in the forward lightcone $\{ t > r \}$:

\[
(5.9a) \quad |\phi| \lesssim t^{1-d/2} \quad \text{(only when } d \geq 3) \\
(5.9b) \quad |L^i \phi| \lesssim t^{1-d/2} \\
(5.9c) \quad |\partial_t \phi| \lesssim t^{1-d/2}(t + r)^{-1/2}(t - r)^{-1/2} \\
(5.9d) \quad |\nabla \phi| \lesssim t^{-d/2}
\]

which are exactly what one would expect from taking spectral projections on the $S$ component and treating $\phi$ as a sum of solutions to the wave equation and an infinite family of Klein-Gordon equations.

We emphasize that classical vector field method, integrating along the hyperplanes $\{t\} \times \mathbb{R}^d \times S$, can only recover
\[
|\partial \phi|, |L^i \partial \phi|, |\nabla \partial \phi| \lesssim t^{(1-d)/2}(t - r)^{-1/2}
\]
and in particular cannot see any of the improved decay for $\nabla \phi$ and its derivatives.
We conclude this section with the decay estimates for $\phi$ when the dimension $d = 2$. For technical reasons our proof only works when $\phi$ has compactly supported initial data, and hence we only state the version available from the $K$ energy.

5.10 Proposition ($d = 2$, $K$-energy)
Let $\phi$ solve the linear wave equation on $\mathbb{R}^{1,2}$ with compactly supported initial data (e.g. satisfying the hypotheses of Proposition 4.1). Then

$$|\phi| \lesssim \ln[(t+r)(t-r)] \left\| \right\| \frac{\sqrt{(t+r)(t-r)}}{(t+r)(t-r)}.$$

Proof Without loss of generality we shall assume that $\phi$ satisfies the conditions of Proposition 4.1, this implies the quantity in (4.6) is bounded by the initial data.

Observe as the data, prescribed at time $t = 2$, is supported in the unit ball, this means by finite speed of propagation, on the support of $\phi$, when $t \geq 2$ it must also satisfy $t \geq r+1$. Within the forward lightcone from the origin this inequality can be rewritten as

$$\tau (\cosh \rho - \sinh \rho) \geq 1 \iff \tau e^{-\rho} \geq 1 \iff \ln \tau \geq \rho.$$

Returning to (5.2) we see that, when $\tau \geq 2$,

$$\int_{\Sigma_\tau} \frac{1}{\cosh(\rho)} \phi^2 \, d\text{vol}_{\Sigma_\tau} \leq \int_{\Sigma_\tau} \frac{(1 + \rho^2)}{\cosh(\rho)} \sum (L^i \phi)^2 \, d\text{vol}_{\Sigma_\tau},$$

$$\leq (1 + \ln \tau) \int_{\Sigma_\tau} \frac{1}{\cosh(\rho)} \sum (L^i \phi)^2 \, d\text{vol}_{\Sigma_\tau},$$

$$\leq (1 + \ln \tau) \int_{\Sigma_\tau} K_a (\partial_\tau)^a \, d\text{vol}_{\Sigma_\tau}.$$}

Here, in the second inequality, we can use (5.11) thanks to the compact-data assumption.

Applying this estimate to $\phi$ and $L^a \phi$ as before, we see by Theorem 2.18 that $|\phi| \lesssim \tau^{-1} \ln(\tau)$ as claimed.

6. Two-dimensional wave-maps

Let us now apply the method developed in the previous sections to study the global existence and decay of solutions to (1.1) under the assumption that its initial data has compact support. First we state the precise version of our theorem.
6.1 Theorem
Consider the future-evolution governed by (1.1) on $\mathbb{R}^{1,2}$, with initial data prescribed at time $t = 2$ such that $\phi(2, x)$ and $\partial_t \phi(2, x)$ are supported in the ball of radius 1. Let $k \geq 4$ be a positive integer. Then there exists some $\epsilon > 0$ such that if

$$\|\phi(2, \cdot)\|_{H^{k+1}} + \|\phi(2, \cdot)\|_{H^k} < \epsilon,$$

then there exists a future-global solution $\phi : [2, \infty) \times \mathbb{R}^d \to \mathbb{R}$. Furthermore, this solution obeys the following pointwise decay estimates:

$$|\partial_t L^\alpha \phi| \leq \frac{1}{\tau}, \quad |\alpha| \leq k - 2;$$

$$|L^\alpha \phi| \leq \frac{1}{\tau}, \quad 1 \leq |\alpha| \leq k - 2;$$

$$|L^\alpha \phi| \leq \frac{\ln \tau}{\tau}, \quad |\alpha| = 0 \text{ or } |\alpha| = k - 1.$$

Here $\tau = \sqrt{t^2 - |x|^2}$ as defined previously. (Note that by finite speed of propagation, within the support of $\phi$ we have the lower bound $\tau \geq \sqrt{2}$.)

As the equation is semilinear, local existence theory using energy method is standard. In particular, for sufficiently small initial data it is clear that the solution exists at least up to, and including $\{\tau = 2\}$ (here we also use finite speed of propagation and boundedness of initial support). We concentrate on obtaining a priori energy bounds when $\tau \geq 2$. Define

$$E_\tau[\phi] \overset{\text{def}}{=} \int_{\Sigma_\tau} \frac{1}{\tau^2 \cosh(\rho)} \sum_{i=1}^2 (L_i^\phi)^2 + \frac{1}{\cosh(\rho)} (\partial_t \phi)^2 \, d\text{vol}_{\Sigma_\tau};$$

(6.2)

$$F_\tau[\phi] \overset{\text{def}}{=} \int_{\Sigma_\tau} \frac{1}{\cosh(\rho)} \sum_{i=1}^2 (L_i^\phi)^2 + \frac{1}{\tau^2 \cosh(\rho)} (K \phi + t \phi)^2 \, d\text{vol}_{\Sigma_\tau};$$

(6.3)

We have the following energy identities for $\tau_1 > \tau_0 > 2$

$$E_{\tau_1}[\phi] \leq E_{\tau_0}[\phi] + 2 \int_{\tau_0}^{\tau_1} |\Box \phi \cdot (\partial_t \phi)| \, d\text{vol}_{\Sigma_\tau} \, d\tau$$

(6.4)

$$F_{\tau_1}[\phi] \leq F_{\tau_0}[\phi] + 2 \int_{\tau_0}^{\tau_1} |\Box \phi \cdot (K \phi + t \phi)| \, d\text{vol}_{\Sigma_\tau} \, d\tau$$

(6.5)
where we used that the space-time volume-element is exactly $d\text{vol}_{\Sigma_t}$ $d\tau$ as seen in the metric decomposition (2.8).

The basic strategy is standard: we wish to estimate $E[L^\alpha \phi]$ and $F[L^\alpha \phi]$ for all $|\alpha|$ less than some fixed constant. This will require estimating integrals of the forms

\begin{equation}
\int_{\Sigma_t} L^{\alpha_1} \phi \cdot \eta(dL^{\alpha_2} \phi, dL^{\alpha_3} \phi) \cdot \partial_t L^\alpha \phi \ d\text{vol}_{\Sigma_t}
\end{equation}

and

\begin{equation}
\int_{\Sigma_t} L^{\alpha_1} \phi \cdot \eta(dL^{\alpha_2} \phi, dL^{\alpha_3} \phi) \cdot (K + t) L^\alpha \phi \ d\text{vol}_{\Sigma_t}
\end{equation}

for $|\alpha_1| + |\alpha_2| + |\alpha_3| = |\alpha|$. In writing down the integrals above we implicitly used that vector fields acts on scalars by Lie differentiation, and $\eta$ is invariant under Lorentz boosts $L^i$, and that exterior differentiation commutes with Lie differentiation, so that if $\phi$ solves (1.1), then $L^\alpha \phi$ solves an equation of the form

\begin{equation}
\Box L^\alpha \phi = \sum c_{\alpha_1, \alpha_2, \alpha_3, \alpha} L^{\alpha_1} \phi \eta(dL^{\alpha_2} \phi, dL^{\alpha_3} \phi)
\end{equation}

where $|\alpha_1| + |\alpha_2| + |\alpha_3| = |\alpha|$, and $c_{\alpha_1, \alpha_2, \alpha_3, \alpha}$ are combinatorial constants.

Therefore we are led to consider integrals of the form

\[\int_{\Sigma_t} \zeta \eta(d\psi, d\phi) \partial_t \xi \ d\text{vol}_{\Sigma_t} \quad \text{and} \quad \int_{\Sigma_t} \zeta \eta(d\psi, d\phi)(K \xi + t \xi) \ d\text{vol}_{\Sigma_t}\]

where the functions $\zeta, \psi, \phi, \xi$ stand in place of $L^\alpha$ derivatives of the original unknown. To get good control, we begin by decomposing the form

\[\eta(d\psi, d\phi) = (h_\tau)^{-1}(d\psi, d\phi) - (\partial_\tau \psi)(\partial_\tau \phi).\]

Using that

\[\partial_\tau = \frac{1}{t \tau} K - \frac{r}{t \tau} \partial_\rho\]

\[= \frac{r}{t \tau} \partial_t + \frac{r}{t \tau} \partial_\rho\]

we have

\[\eta(d\psi, d\phi) = (h_\tau)^{-1}(d\psi, d\phi) - \left[\frac{1}{t \tau} K \psi - \frac{r}{t \tau} \partial_\rho \psi\right] \left[\frac{r}{t} \partial_\tau \phi + \frac{r}{t \tau} \partial_\rho \phi\right].\]
Introducing the short hand $|L\phi| \overset{\text{def}}{=} \left[ \sum_{i=1}^{2} (L^i \phi)^2 \right]^{\frac{1}{2}}$ we can bound

$$
|\eta(d\psi,d\phi)| \lesssim \frac{1}{\tau^2} |L\psi||L\phi| + \frac{1}{t^2} |K\psi\partial_t \phi| + \frac{r}{t^2} |\partial_t \psi \partial_t \phi| + \frac{r^2}{t^4 \tau^2} (\partial_t \phi)(\partial_t \psi).
$$

This we can bound by

$$(6.8) \quad |\eta(d\psi,d\phi)| \lesssim \frac{1}{\tau^2} |L\psi||L\phi| + \frac{1}{t^2} |K\psi\partial_t \phi| + \frac{1}{t} |\partial_t \phi||L\psi|
$$

where we simplified using $r/t \leq 1$. This decomposition is good for taking care of terms where the quantity $\alpha_1$ in (6.6a) and (6.6b) has order $|\alpha_1| \leq |\alpha| - 1$, through the following integral estimate.

**6.9 Lemma (Non-borderline estimates)**

We have the estimates (the $L^\infty$ norms are taken along $\Sigma_{\tau}$)

$$
\int_{\Sigma_{\tau}} \zeta \eta(d\psi,d\phi) \partial_t \xi \, d\text{vol}_{\Sigma_{\tau}} \lesssim \frac{\ln \tau}{\tau} \left[ \|L\psi\|_{L^\infty} + \|\partial_t \phi\|_{L^\infty} \right] F_{\tau}[\psi] E_{\tau}[\xi],
$$

$$
\int_{\Sigma_{\tau}} \zeta \eta(d\psi,d\phi)(K\xi + t\xi) \, d\text{vol}_{\Sigma_{\tau}} \lesssim \ln \tau \left[ \|L\psi\|_{L^\infty} + \|\partial_t \phi\|_{L^\infty} \right] F_{\tau}[\psi] F_{\tau}[\xi].
$$

**Proof** The proof of the two cases are similar, we focus on the harder case which is the second inequality. The basic idea is to put all the terms involving $\xi$ and $\psi$ in weighted $L^2$ (controlled by $E[\xi], F[\xi], F[\psi]$), and the remainder ($\zeta$ and $\phi$) in $L^\infty$. At parts of the argument we will also use the fact that by our finite speed of propagation property, $\cosh(\rho) \leq e^\rho \leq \tau$ and hence $\frac{1}{\tau} \leq 1$.

Observe that the $F[\xi]^2$ controls the square integral of $(K\xi + t\xi)/\sqrt{\tau}$. So it suffices to show that $\sqrt{\tau \eta}(d\psi,d\phi)$ is square integrable on $\Sigma_{\tau}$. Observe now that $F[\psi]^2$ controls the square integral of $|L\psi|/\sqrt{\cosh(\rho)}$, $(K\psi + t\psi)/\sqrt{\tau}$ and $|\psi|/|\ln(\tau)\sqrt{\cosh(\rho)}|$ (the last through Lemma 5.2 and finite speed of propagation). So we can check
each term that appears on the right of (6.8):

\[
\frac{\sqrt{t\tau}}{\tau^2} |L\psi||L\phi| = \frac{\cosh \rho}{\tau} |L\phi| \cdot \frac{1}{\sqrt{\cosh \rho}} |L\psi|
\]

\[
\frac{\sqrt{t\tau}}{t} |L\psi||\partial_t\phi| = |\partial_t\phi| \cdot \frac{1}{\sqrt{\cosh \rho}} |L\psi|
\]

\[
\frac{\sqrt{t\tau}}{t^2} |(K + t)\psi||\partial_t\phi| = \frac{\tau}{t} |\partial_t\phi| \cdot \frac{1}{\sqrt{t\tau}} |(K + t)\psi|
\]

\[
\frac{\sqrt{t\tau}}{t^2} |(K + t)\psi||L\phi| = \frac{1}{\tau} |L\phi| \cdot \frac{1}{\sqrt{t\tau}} |(K + t)\psi|
\]

\[
\frac{\sqrt{t\tau}}{t^2} |\psi||L\phi| = \frac{t \ln \tau}{t^2} |L\phi| \cdot \frac{1}{\ln(\tau) \sqrt{\cosh \rho}} |\psi|
\]

Using that \(t/\tau^2 \leq 1\) by finite speed of propagation we see our claim is proved. □

For taking care of the borderline terms where \(|\alpha_1| = |\alpha|\), we will also use the following alternative decomposition

\[
\eta(d\psi, d\phi) = (h_\tau)^{-1}(d\psi, d\phi) - \left[\frac{\tau}{t} \partial_t \phi + \frac{r}{t\tau} \partial_{\rho} \phi\right]^2
\]

which leads to the estimate

\[
(6.10) \quad |\eta(d\psi, d\phi)| \leq \frac{1}{\tau^2} |L\psi||L\phi| + \frac{\tau^2}{t^2} |\partial_t \phi||\partial_t \psi|.
\]

The corresponding integral estimates are

**6.11 Lemma (Borderline estimates)**

We have the estimates (the \(L^\infty\) norms are taken along \(\Sigma_\tau\))

\[
\int_{\Sigma_\tau} (L\zeta)\eta(d\psi, d\phi) \partial_\xi \psi d\text{vol}_{\Sigma_\tau} \leq \frac{1}{\tau} \|L\psi\|_{L^\infty} \|L\phi\|_{L^\infty} + \|\partial_t \psi\|_{L^\infty} \|\partial_t \phi\|_{L^\infty} \mathcal{F}[\zeta] \mathcal{E}[\xi],
\]

\[
\int_{\Sigma_\tau} (L\zeta)\eta(d\psi, d\phi)(K\xi + t\xi) d\text{vol}_{\Sigma_\tau} \leq \left[\|L\psi\|_{L^\infty} \|L\phi\|_{L^\infty} + \tau \|\partial_t \psi\|_{L^\infty} \|\partial_t \phi\|_{L^\infty}\right] \mathcal{F}[\zeta] \mathcal{F}[\xi].
\]
Proof. Again we focus on the second inequality, the first is similar. In this situation we must put $L \zeta$ in $L^2$, and both $\phi$ and $\psi$ in $L^\infty$. The square integral of $(L \zeta) / \sqrt{\cosh \rho}$ is controlled by $F_\tau[\zeta]^2$, so it suffices to compute the $L^\infty$ estimates of $t \eta(d\psi, d\phi)$. The results follow straightforwardly from (6.10), after using again that $\cosh(\rho) \leq \tau$ from finite speed of propagation. \qed

Next let us introduce the following shorthand notations:

$$E_{\tau_0, \tau_1, k} = \sup_{\tau \in [\tau_0, \tau_1]} \sum_{\alpha \in I^k} E_{\tau}[L^\alpha \phi]$$

$$F_{\tau_0, \tau_1, k} = \sup_{\tau \in [\tau_0, \tau_1]} \sum_{\alpha \in I^k} F_{\tau}[L^\alpha \phi]$$

$$E_{\tau_0, \tau_1, \leq k} = \sum_{j=0}^k E_{\tau_0, \tau_1, j}$$

$$F_{\tau_0, \tau_1, \leq k} = \sum_{j=0}^k F_{\tau_0, \tau_1, j}$$

The global Sobolev inequality Theorem 2.18 states, when $d = 2$, that, for $\tau \in [\tau_0, \tau_1]$

$$|\partial_t \psi| \leq \tau^{-1} E_{\tau_0, \tau_1, \leq 2} \quad (6.12a)$$

$$|L \psi| \leq E_{\tau_0, \tau_1, \leq 2} \quad (6.12b)$$

$$|L^2 \psi| \leq \tau^{-1} F_{\tau_0, \tau_1, \leq 2} \quad (6.12c)$$

$$|\psi| \leq \ln(\tau) \tau^{-1} F_{\tau_0, \tau_1, \leq 2} \quad (6.12d)$$

where in the last estimate we used the compact support assumption.

Combining Lemma 6.9 with (6.4) we arrive at

$$\sup_{\tau \in [\tau_0, \tau_1]} E_{\tau}[L^\alpha \phi] - E_{\tau_0}[L^\alpha \phi] \leq$$

$$\int_{\tau_0}^{\tau_1} \frac{\ln \tau}{\tau} \sum_{\alpha_1, \alpha_2, \alpha_3} \|L^{\alpha_1} \phi\|_{L^\infty} \left[ \|LL^{\alpha_2} \phi\|_{L^\infty} + \|\partial_t L^{\alpha_2} \phi\|_{L^\infty} \right] F_{\tau}[L^{\alpha_3} \phi] \, d\tau.$$

Note that when $|\alpha_1| = 0$ we can use (6.12d) to control $|\phi|$, and when $|\alpha_1| > 0$ by (6.12c) we have that $|L^\alpha \phi| \leq \tau^{-1} F_{\tau_0, \tau_1, \leq |\alpha_1|+1}$. This implies see whenever $k \geq 1$ we
have

\[
\mathcal{E}_{t_0, t_1, \leq k} - \mathcal{E}_{t_0, t_0, \leq k} \leq \int_{\tau_0}^{\tau_1} \frac{(\ln \tau)^2}{\tau^3} \left( \mathcal{F}_{t_0, \tau, \leq k/2+2} + \mathcal{E}_{t_0, \tau, \leq k/2+2} \right) \mathcal{F}_{t_0, \tau, \leq k/2+2} \mathcal{F}_{t_0, \tau, \leq k+1} \ d\tau.
\]

Similarly we get for the K-energy

\[
\mathcal{F}_{t_0, t_1, \leq k} - \mathcal{F}_{t_0, t_0, \leq k} \leq \int_{\tau_0}^{\tau_1} \frac{(\ln \tau)^2}{\tau^2} \left( \mathcal{F}_{t_0, \tau, \leq k/2+2} + \mathcal{E}_{t_0, \tau, \leq k/2+2} \right) \mathcal{F}_{t_0, \tau, \leq k/2+2} \mathcal{F}_{t_0, \tau, \leq k} \ d\tau.
\]

On the other hand, we can also use Lemma 6.11 to handle the case when \(|\alpha_1| = |\alpha|\). This gives instead

\[
\mathcal{E}_{t_0, t_1, \leq k} - \mathcal{E}_{t_0, t_0, \leq k} \leq \int_{\tau_0}^{\tau_1} \frac{1}{\tau^2} \left( \mathcal{F}_{t_0, \tau, \leq k/2+2} + \mathcal{E}_{t_0, \tau, \leq k/2+2} \right) \mathcal{F}_{t_0, \tau, \leq k-1} \ d\tau
\]

and

\[
\mathcal{F}_{t_0, t_1, k} - \mathcal{F}_{t_0, t_0, k} \leq \int_{\tau_0}^{\tau_1} \frac{1}{\tau^2} \left( \mathcal{F}_{t_0, \tau, \leq k/2+2} + \mathcal{E}_{t_0, \tau, \leq k/2+2} \right) \mathcal{F}_{t_0, \tau, \leq k-1} \ d\tau
\]

and

\[
\mathcal{F}_{t_0, t_1, k} - \mathcal{F}_{t_0, t_0, k} \leq \int_{\tau_0}^{\tau_1} \frac{1}{\tau^2} \left( \mathcal{F}_{t_0, \tau, \leq k/2+2} + \mathcal{E}_{t_0, \tau, \leq k/2+2} \right) \mathcal{F}_{t_0, \tau, \leq k-1} \ d\tau
\]

The above estimates imply the following bootstrapping estimate:

**6.17 Proposition (Bootstrap)**

Fix \( k \geq 4 \) to be an integer. There exists a universal constant \( C \) depending on \( k \) such that the following holds: Let \( \phi \) solve \((1.1)\) on the spacetime region sandwiched between \( \Sigma_{t_0} \) and \( \Sigma_{t_1} \) with \( 2 \leq t_0 < t_1 \) be such that \( \phi \) vanishes when \( e^\rho \geq \tau \). Suppose \( \phi \) satisfies the following assumptions: we have the initial data bound

\[
\mathcal{E}_{t_0, t_0, \leq k} + \mathcal{F}_{t_0, t_0, \leq k} \leq \epsilon_0;
\]
and the bootstrap bounds

\begin{align}
(6.19) & \quad \mathcal{E}_{\tau_0, \tau_1, \leq k} + \mathcal{F}_{\tau_0, \tau_1, \leq k-1} \leq \delta_0, \\
(6.20) & \quad \mathcal{F}_{\tau_0, \tau, k} \leq \delta_0 \ln \tau, \quad \forall \tau \in [\tau_0, \tau_1].
\end{align}

Then we have the improved bounds

\begin{align}
(6.21) & \quad \mathcal{E}_{\tau_0, \tau_1, \leq k} + \mathcal{F}_{\tau_0, \tau_1, \leq k-1} \leq \epsilon_0 + C\delta_0^3, \\
(6.22) & \quad \mathcal{F}_{\tau_0, \tau, k} \leq \epsilon_0 + C\delta_0^3 \ln \tau, \quad \forall \tau \in [\tau_0, \tau_1].
\end{align}

\[\square\]

**Proof** We prove the estimates one by one. First note that the quantities \( \mathcal{E}_{\tau_0, \tau, k} \) etc. are increasing in \( \tau \). So the bootstrap assumptions applied to (6.15) implies

\[ \mathcal{E}_{\tau_0, \tau_1, \leq k} \leq k - \epsilon_0 \lesssim \tau_1 \int_{\tau_0}^{\tau_1} \frac{1}{\tau^2} \delta_0^3 \, d\tau + \int_{\tau_0}^{\tau_1} \frac{(\ln \tau)^2}{\tau^3} \delta_0^3 (\ln \tau)^3 \, d\tau \]

where in the second integral, the factor \((\ln \tau)^3\) comes from the possibility that \( k/2 + 2 = k \) when \( k = 4 \), and the top order \( K \)-energy is allowed to grow logarithmically. However, as both \( \tau^{-2} \) and \((\ln \tau)^5 \tau^{-3}\) are integrable on \( \tau \in [2, \infty) \), we pick up our desired estimate.

Similarly, the bootstrap assumptions applied to (6.14) implies

\[ \mathcal{F}_{\tau_0, \tau_1, \leq k-1} - \epsilon_0 \lesssim \int_{\tau_0}^{\tau_1} \frac{(\ln \tau)^2}{\tau^2} \delta_0^3 \ln(\tau) \, d\tau \]

and the desired estimate follows again via integrability of the temporal weights.

Finally, for the top order estimate, we use (6.16); here we note that the top order \( K \)-energy does not appear in the first integral. And therefore we have

\[ \mathcal{F}_{\tau_0, \tau, k} - \epsilon_0 \lesssim \tau^{-1} \delta_0^3 \, d\tau + \int_{\tau_0}^{\tau} \frac{(\ln \tau)^2}{\tau^2} \delta_0^3 \ln(\tau)^3 \, d\tau \]

from which we get the desired estimate. \[\square\]

In particular, if \( \delta_0 \) is such that \( C\delta_0^2 \leq \frac{1}{3} \), and \( \epsilon_0 \leq \frac{1}{3} \ln 2\delta_0 \), then the conclusion of the above proposition implies

\begin{align}
(6.23) & \quad \mathcal{E}_{\tau_0, \tau_1, \leq k} + \mathcal{F}_{\tau_0, \tau_1, \leq k-1} \leq \frac{2}{3} \delta_0, \\
(6.24) & \quad \mathcal{F}_{\tau_0, \tau, k} \leq \frac{2}{3} \delta_0 \ln \tau, \quad \forall \tau \in [\tau_0, \tau_1].
\end{align}
In this case, the (semi-)global existence part of our main Theorem 6.1 follows by a continuity argument, and the decay estimates follow from an application of Theorem 2.18 to the energy bounds derived above.

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