Finding, by means of a scattered sound, the geometric parameters of a finite elastic cylinder located near the half-space border

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Abstract.

The solution of the inverse problem of sound diffraction on a finite circular elastic cylinder is considered. The cylinder \( T \) is located near the surface \( \Pi \) of acoustic half-space. The following cylinder parameters are finding according to the scattered sound field: radius, height, distance from the half-space surface, and axis orientation. The solution of the diffraction problem is conducted by expanding the area of the problem to the full space. With that an additional obstacle \( T' \), which is a copy of \( T \), is introduced. The additional obstacle is a mirror reflection of the original one with respect to the surface \( \Pi \). In addition, a second incident wave, which corresponds to the type of the half-space boundary, is introduced. The solving is based on the linear elastic theory and the model of the small vibrations propagation in ideal fluid. In the external part of the fluid, the solution is sought analytically in the form of expansion in spherical harmonics and Bessel functions. In the ball area, which includes two balls cylinders and an adjacent layer of the fluid, the finite element method (FEM) is used. The cylinder required parameters are detected on the basis of the minimization of the norm of the vector difference \( (p_{jm}^*) \) and \( (p_{jm}) \). Where \( (p_{jm}^*) \) is the vector of measured pressure values in the scattered sound field, and \( (p_{jm}) \) is the pressure vector obtained from the theoretical solution of the diffraction problem.

1. Introduction

Engineering applications in the ultrasound diagnostics, sonar measures, musical and architectural acoustics are important areas of use of sound diffraction problems solutions [1-4]. In the biggest part of applications, the solution of the so-called inverse problems is required. The parameters of the medium and the scatterers characteristics are determined in inverse problems according to the known parameters of the radiation and the scattered wave [5, 6].

This paper presents the solution to the problem of determining the coordinates of the center, radius, height and orientation of the finite elastic cylinder located in the acoustic half-space near its ideal surface (absolutely rigid or soft). The cylinder parameters are identified according to the known scattered field of the plane harmonic sound wave.

The biggest part of the solutions to the problems of diffraction of sound on cylindrical objects is obtained for cylinders of infinite length in the two-dimensional case. Some approaches of solving the problem of scattering of sound waves by the elastic finite cylinder are proposed [7, 8].

The effect on the sound scattering by the elastic object of the bounding surface located near it is interesting. In the problem under study, the variant of the simplest type of the surface, the
ideal one, is considered.

2. Problem formulation
Assume that the finite elastic cylinder with a circular cross-section is located inside the acoustic half-space near its boundary. The plane sound wave is incident on the cylinder. From the known pressure in the scattered acoustic wave it's required to evaluate the geometric characteristics of the cylinder.

In figure 1 a) the geometric set of the problem is presented.

By $T$ denote the elastic circular cylinder. The figure shows its center $O_1$ and the axis of rotation $L$. The cylinder is in the half-space $\Omega_0$ filled with ideal liquid. The density $\rho_0$ and the speed of sound $c_0$ are known for the fluid. The density $\rho$ and the elasticity modulus $\lambda$, $\mu$ of the cylinder material are also known. By $\Pi$ denote the surface of the half-space. The plane $\Pi$ is showed by the rectangle represented by a dotted line. From the inner part of the acoustic half-space, a plane harmonic sound wave is incident on the object $T$ and the plane $\Pi$. It’s assumed that the displacement potential $\Psi_0$ of the fluid particles in it is given by

$$ \Psi_0 = \exp[i(k_0 \cdot r - \omega t)], $$

where $k_0$ is the wave vector of the incident wave ($|k_0| = \frac{\omega}{c_0}$); $\omega$ is the circular frequency; $r$ is the radius vector of a point in space; $t$ is time. $|\Psi_0| = 1$ is assumed to be without loss of generality.

Assume that the parameters of the incident wave and the physical characteristics of the containing medium and the elastic material of the cylinder are given. The geometric characteristics of the cylinder are unknown: radius $R$, height $2H$ and its position with respect to the plane $\Pi$ and to the direction of the wave propagation $k_0$. More precisely, assume that the change interval of these characteristics are given. It’s required to find their values by the known scattered field $\Psi_s$.

Let us introduce the global orthogonal Cartesian coordinate system $x$, $y$, $z$ (the reference system). Assume that the plane $\Pi$ coincides with the coordinate surface $z = 0$ and the $z$-axis is directed into the acoustic half-space. An illustration of the introduction of the coordinate system is presented in figure 1 b). The direction of the $x$-axis is chosen so that the displacements in the
incident wave occur only in the \(xOz\) plane. The direction of the wave vector in (1) is given by the angle \(\theta_0\) between the \(z\)-axis and \(k_0\). Then in the \(x,y,z\) system the wave vector of the incident wave can be written as \(k_0 = (k_0 \sin \theta_0, 0, k_0 \cos \theta_0)\).

Let us denote the unknown coordinates of the cylinder center \(O_1\) by \((a, b, c)\). We also introduce the local coordinate system \(x_1, y_1, z_1\). In this coordinate system the origin is at the point \(O_1\) and the \(z_1\)-axis is directed along \(L\). Then the equation of the surface of the cylinder \(T\) is represented by the set
\[
\Gamma_1 = \{(x_1, y_1) : x_1^2 + y_1^2 = R^2 \quad (|z_1| < H) \quad x_1^2 + y_1^2 \leq R^2 \quad (|z_1| = H)\}.
\]
Accordingly, the whole area \(\Omega_1\) occupied by the cylinder is determined by the formula
\[
\Omega_1 = \{(x_1, y_1) : x_1^2 + y_1^2 \leq R^2 \quad (|z_1| \leq H)\}.
\]

Define the direction of the \(z_1\)-axis by the angles \(\alpha, \beta\). Where \(\alpha\) is the angle between \(O_1z_1\) and \(Oz\), and \(\beta\) is the angle between the projection of the \(O_1z_1\)-axis on \(\Pi\) and the \(Ox\)-axis. Then the relation between the local and global coordinate systems can be defined as
\[
x = z_1 \cos \beta \sin \alpha + x_1 \cos \beta \cos \alpha - y_1 \sin \beta + a,
y = z_1 \sin \beta \sin \alpha + x_1 \sin \beta \cos \alpha + y_1 \cos \beta + b,
z = z_1 \cos \alpha - x_1 \sin \alpha + c.
\]

Thus, the cylinder size, its position and orientation in the frame of reference reference coordinate system is fully specified by the vector of seven parameters \(\xi = (R, H, a, b, c, \alpha, \beta)\). In the inverse problem formulated above the parameters \(\xi\) must be found through the scattered field of the sound wave (by \(\Psi_s\) or \(\Psi_0 + \Psi_s\)). It is assumed that to determine the \(\xi\) it might be necessary to carry out the analysis of the scattered field for several values of \(k_0\) and \(\theta_0\) in the incident wave (1).

Let us suppose that for each parameter of the \(\xi\) an interval of its change is known, for example, for radius \(R_1 \leq R \leq R_2\). Then the position of the cylinder \(T\) can be localized by a certain spherical surface \(r = R_0\) centered at \(O\) such that
\[
\forall \mathbf{r} \in \Omega_1 : |\mathbf{r}| < R_0,
\]
where \(\mathbf{r}\) is the radius vector of a point in space in the global coordinate system.

We assume that to determine \(\xi\), the scattered sound field can be measured as the acoustic pressure \(p = i\omega \rho_0 (\Psi_0 + \Psi_s)\) [9] in a limited finite number of points \(V_1, V_2, \ldots, V_J\). Then \(V_j\) can be placed on the surface \(|\mathbf{r}| = R_0\) or outside it. In figure 2 variants for the arrangement of measurement points of the scattered field are presented. Figure 2 a) illustrates the layout location of \(V_j\) on the surface \(|\mathbf{r}| = R_0\). In the figure 2 b) a possible option of placing of measurement points in a square area on the plane \(z = R_0\) parallel to \(\Pi\) is presented.

Let us denote the actual (unknown) values of the parameters of the cylinder position and orientation by \(\xi^* = (R^*, H^*, a^*, b^*, c^*, \alpha^*, \beta^*)\). Then the measured pressure values \(p_{jm}^*\) characterize the parameters \(\xi^*\). Here the \(j = 1, J\) defines the measurement point and the \(m = 1, M\) defines the combination of the parameters \(k_0\) and \(\theta_0\) in the incident wave \(\Psi_0\).

For a fixed set of parameters \(k_0\) and \(\theta_0\) at points \(V_1, V_2, \ldots, V_J\), we perform pressure calculations \(p_{jm}\) for some combinations of parameters \(\xi = (R, H, a, b, c, \alpha, \beta)\). Where \(R, H, a, b, c, \alpha, \beta\) are taken from given intervals. Deviations \(p_{jm}\) from \(p_{jm}^*\) shows the degree of mismatch between \(\xi\) and \(\xi^*\). Therefore, to estimate the degree of closeness of the selected for the calculation parameters \(\xi\) to their actual values \(\xi^*\), we introduce the function
\[
\delta(\xi) = \sum_{j=1}^{J} \sum_{m=1}^{M} (p_{jm} - p_{jm}^*)^2.
\]
Figure 2. The placement variants of acoustic pressure measurement points.

Obviously, with an adequate model for calculating pressure $p_{jm}$ when $\xi=\xi^*$ the sum (2) should be close to zero. Therefore, the search for actual parameters $\xi^*$ based on the deviations of measured and calculated pressure values can be represented as the minimization problem

$$\delta(\xi) \rightarrow \min_{\xi \in D},$$

where $D$ is the set of allowable populations set of parameters $(R, H, a, b, c, \alpha, \beta)$ determined by the intervals of possible values of individual parameters.

The vector of parameters

$$\xi_0 = \arg \min_{\xi \in D} \delta(\xi)$$

is a generalized solution to the problem of searching for actual parameters $\xi^*$, since in the general case the solution to the inverse problem is not unique [10]. In addition, deviation of $\xi_0$ from $\xi^*$ may be due to uncertainty of measurements of $p_{jm}^*$ and errors in the calculations of $p_{jm}$.

A numerical-analytical solution of the diffraction problem similar to one used in paper [11] is used to obtaining the calculated values of pressure. At the first stage of the solution decision process based on the well-known approach the surface $\Pi$ of the half-space is excluded from the model. With that, the area filled with acoustic fluid expands till whole space. For preserve the ideality condition of the surface $z = 0$ a second elastic cylinder $T'$ is introduced. This is a mirror copy of the initial original cylinder $T$ with respect to $\Pi$. In addition, the second incident wave $\Psi_1$ is introduced, which, like (1), is written as

$$\Psi_1 = B \exp[i(k_1 \cdot r - \omega t)],$$

where the wave vector is $k_1 = (k_0 \sin \theta_0, 0, -k_0 \cos \theta_0)$; in the case of a rigid surface $\Pi$ the coefficient $B = 1$, and by soft surface $\Pi$ the coefficient $B = -1$.

These changes in the setting of the problem are illustrated in figure 3.

Further, the approach based on the finite element method (FEM) proposed in [12, 13] is used in solving the problem of diffraction of two waves on two elastic finite cylinders.

In accordance with this approach, in the area of a complex obstacle a layer of the exterior fluid with the external spherical surface is extracted. In figure 4, this layer is denoted by $\Omega'_0$. Its
outer surface is the sphere of radius $R_0$ with the center at the point $O$. In the figure the area occupied by the cylinder $T'$ is denoted by $\Omega_2$ and its surface by $\Gamma_1$.

When solving the problem of sound diffraction, the following unknowns are introduced: $u_1$, $u_2$ are particle displacements of the elastic medium of cylinders in the areas $\Omega_1$ and $\Omega_2$ respectively; $\Psi'$ is the fluid particles displacement potential in the layer $\Omega_0'$; $\Psi_s$ is the displacement potential in the reflected sound wave beyond the $\Gamma_0$.

Potentials $\Psi'$, $\Psi_s$ must satisfy the Helmholtz equation

$$\Delta \Psi' + k_0^2 \Psi' = 0, \quad \Delta \Psi_s + k_0^2 \Psi_s = 0.$$ \(4\)

With that, for the function $\Psi_s$ the radiation conditions at infinity must be fulfilled

$$\left(\frac{\partial \Psi_s}{\partial r} - ik_0 \Psi_s\right)_{r \to \infty} = O\left(\frac{1}{r^2}\right), \quad \Psi_s|_{r \to \infty} = O\left(\frac{1}{r}\right),$$ \(5\)

where $r = |\mathbf{r}|$.

Particle displacements of the elastic medium must satisfy the motion equation [14]

$$\text{Div} \sigma_1 = -\rho \omega^2 u_1, \quad \text{Div} \sigma_2 = -\rho \omega^2 u_2,$$ \(6\)

where $\rho$ is the density of the elastic medium of the cylinder; $\sigma_q$ is the stress tensor in the area $\Omega_q (q = 1, 2)$; $\text{Div} \sigma_q$ is the first invariant of the covariant derivative of the stress tensor $\sigma_q$.

The stress tensor is expressed by the displacement vector components by means of Hooke’s law [14]. Thus, equations (6) can be considered as systems of second-order differential equations for the displacement vectors $u_1$ and $u_2$ components.

On the outer surface $\Gamma_q$ of the cylinders, where the fluid and the elastic material are in contact, the normal component of the displacement vector and the stress vector must be continuous

$$u_{qn}|_{\Gamma_q} = u'_n, \quad \sigma_{qnn}|_{\Gamma_q} = p', \quad \sigma_{qnr}|_{\Gamma_1} = 0 \quad (\tau = 1, 2);$$ \(7\)
here the index \( n \) corresponds to the projection on the normal (the index \( \tau \) on the tangents) to the surface \( \Gamma_q \); values \( u'_n, p' \) are expressed by the potential \( \Psi' \).

Finally, on the surface \( \Gamma_0 \) it is necessary to introduce the matching conditions for the parameters of fluid motion in \( \Omega'_0 \) and in the exterior medium \( \Omega_0 \)

\[
\left. \frac{\partial \Psi'}{\partial r} \right|_{r=R} = \left. \frac{\partial (\Psi_0 + \Psi_1 + \Psi_s)}{\partial r} \right|_{r=R}, \quad \Psi'\big|_{r=R} = \Psi_0 + \Psi_1 + \Psi_s. \tag{8}
\]

Here the first condition expresses the requirement of equality of normal displacements in particles located on both sides of the \( \Gamma_0 \). The second condition expresses the requirement of equality of pressures.

Thus, it is required to solve equations (4), (6) with regard to the boundary conditions (7), (8) and the radiation conditions (5).

3. The solution of the diffraction problem

In the beginning we consider the solution of the diffraction problem of a single incident wave (1).

Let us break up all subdomains of the ball \( \Omega = \Omega'_0 \cup \Omega_1 \cup \Omega_2 \) into finite elements in the shape of tetrahedral tetrahedron. A two-dimensional illustration of this procedure is presented in figure 5.

Represent all unknown functions of the \( \Omega \) in the form of linear combinations of nodal coordinate functions [12]. In particular, the potential \( \Psi \) we write in the form

\[
\Psi'(r) = \sum_{k=1}^{K} \psi_k f_k(r), \tag{9}
\]

where \( \psi_k \) is the nodal value of the potential in the \( \Omega \) area; \( f_k(r) \) is coordinate functions of the finite element model; \( K \) is the number of nodes. We assume that the set of values \( k = 1...K \) covers the pivots nodes of the entire finite element mesh of the \( \Omega \) area. With that, in the nodes that are not related to \( \Omega'_0 \) let us assume that \( \psi_k \equiv 0 \).

In the exterior area of the containing fluid we search for the displacement potential \( \Psi_s \) in the scattered wave in the form of spherical harmonics expansions taking into account the radiation
conditions
\[ \Psi_s = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} h_n(k_0r) P_n^m(\cos \theta) \exp(i m \varphi), \] (10)
where \( h_n(x) \) is the spherical Hankel function of the first kind of order \( n \); \( P_n^m(x) \) is the associated Legendre polynomial of degree \( n \) of order \( m \); \( r, \theta, \varphi \) are coordinates of the spherical coordinate system correlated with the \( x, y, z \) system; \( A_{nm} \) are unknown coefficients that are to be obtained from the boundary conditions.

Likewise let us expand the displacement potential in the incident plane wave into the spherical harmonics
\[ \Psi_0 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \gamma_{nm} j_n(k_0r) P_n^m(\cos \theta) \exp(i m \varphi), \] (11)
where \( \gamma_{nm} = \frac{i^n(2n+1)(n-m)!}{(n+m)!} P_n^m(\cos \theta_0) \exp(-i m \varphi_0) \); \( j_n(x) \) is the spherical Bessel function of the first kind of order \( n \).

Substituting (9), (10), (11) into the second boundary condition (8) and using the orthogonality of spherical harmonics, we obtain the \( A_{nm} \) expressions through nodal values of the \( \Psi' \) on the surface \( r = R_0 \)
\[ A_{nm} = -\gamma_{nm} \frac{j_n(k_0R_0)}{h_n(k_0R_0)} + \frac{1}{h_n(k_0r) N_{nm}} \sum_{k=1}^{K_0} \psi_k(f_j, Y_{nm}), \] (12)
where \( N_{nm} = \frac{4\pi(n+m)!}{(2n+1)(n-m)!} \) is the spherical harmonic norm of the \( Y_{nm}(\theta, \varphi) = P_n^m(\cos \theta) \exp(i m \varphi); (f_j, Y_{nm}) = \int \int f_j(R_0, \theta, \varphi) Y_{nm}(\theta, \varphi) \sin \theta d\theta d\varphi \) is the scalar product of the coordinate function \( f_k(R, \theta, \varphi) \) and spherical harmonics \( Y_{nm}(\theta, \varphi) \) on the surface \( \Gamma_0 \); here the set of values \( k = 1, \ldots, K_0 \) corresponds to the set of nodes located on the surface \( \Gamma_0 \).

Further, substitute the expression (12) for the \( A_{nm} \) in the first boundary condition (8).

In the form similar to (9) we search for find the displacement in the elastic part of the obstacle (in the areas \( \Omega_1, \Omega_2 \))
\[ u(r) = \sum_{k=1}^{K} U_k f_k(r). \]

Here both displacements \( u_1, u_2 \) introduced above is denoted by \( u \).

As a result, the boundary conditions (7) and the first (8) contain only nodal values of the \( \Psi', u_1, u_2 \) functions from the bounded area \( \Omega \) as unknowns. Further, it is possible to solve the boundary value problem for equations (6) with the indicated boundary conditions by the standard FEM technology [12]. As a result of the solution, we find all the node values \( \psi_k, U_k( k = 1, 2, \ldots, K) \) of the unknown functions. We substitute the found values of the \( \psi_k( k = 1, 2, \ldots, K_0) \) in (12). Thus, we find the coefficients in the series (10) of the displacement potential in the scattered field.

Analogously, let us solve the scattering problem for the second incident wave \( \Psi_1 \). We find the \( A'_{nm} \) coefficients in the splitting series of the displacement potential in the scattered field similar to (10). Then the displacement potential in the scattered field obtained from the action of two waves can be represented as
\[ \Psi_s(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{nm} h_n(k_0r) Y_{nm}(\theta, \varphi), \]
where \( B_{nm} = A_{nm} + A'_{nm} \).
4. Numerical results

The obtained solution of the diffraction problem is used for the computation of the \( p_{jm} \) and for the simulation of the \( p^*_{jm} \). In the model under consideration no measurement errors are assumed.

Computational investigations are conducted for the case when the density and elasticity modulus of the cylinder material are given as follows: \( \rho = 2700 \text{ kg/m}^3 \), \( \lambda = 5.3 \cdot 10^{10} \text{ N/m}^2 \), \( \mu = 2.6 \cdot 10^{10} \text{ N/m}^2 \). The fluid with a density of \( \rho_0 = 1000 \text{ kg/m}^3 \) and the sound speed \( c_0 = 1485 \text{ m/s} \) is used as the ideal fluid filling the half-space. The border of the \( \Pi \) is assumed absolutely rigid. The case of a plane wave with the single frequency and in the fixed direction \( (\theta_0 = 120^\circ) \) is considered.

The parameters of the cylinder size are assumed to be fixed such that \( H/R = 1.25 \). The frequency of the incident wave is chosen such that \( k_0 R' = 3 \). Where \( R' = \sqrt{R^2 + H^2} \) is the typical size of the cylinder.

In the beginning, a series of calculations of the acoustic pressure amplitude in the near field at changing of individual parameters from the \( \xi \) is carried out. They show that the change in these parameters is noticeably become apparent in the field of the scattered sound wave even at such a low frequency of the wave.

In figure 6 the polar graphs diagrams of the pressure distribution at the points of the section \( y = 0 \) of the upper hemisphere \( r = R_0 \) are shown. In the central part of the figures, the dotted line shows the axis of the cylinder and the section of its surface. With such cylinder position the pressure distribution is shown with the solid line. In both graphs the dashed line represents the dependence of the pressure on the polar angle \( \theta \) for the case of fixed parameters: \( a = 0, b = 0, c = 2R, \alpha = 0, \beta = 0 \).

![Figure 6](image.png)

**Figure 6.** An effect of the \( \xi \) parameters on the pressure in the near field.

In figure 6 a) the solid line represents the pressure dependence for the case when the cylinder is displaced along the \( x \)-axis by \( R (a = R) \). In the figure 6 b) the solid line represents the pressure for the cylinder rotated by the angle \( \alpha (\alpha = \pi/6) \).

Comparison of the graphs shows that the change in these parameters leads to the change in the amplitude of pressure in the sound field by up to 20% for individual values of the \( \theta \).

To illustrate the solution of the problem (3), only the \( a, b \) coordinates determining the position of the cylinder in a plane parallel to \( \Pi \) are considered as the required parameters. It is assumed that the intervals of parameter variations are given by inequalities

\[-2R \leq a \leq 2R, \quad -2R \leq b \leq 2R,\]

that define the area \( D \) for \( (a, b) \) as \([-2R, 2R] \times [-2R, 2R] \). Other parameters are fixed by the values \( c = 2R, \alpha = 0, \beta = 0 \).
In figure 7 the correspondence of the function $\delta(\xi)$ in the form of the surface built on the area $D$ for $a^* = R/2$, $b^* = -R$ is shown. When calculating the function (2), the quantity $M = 1$, since only one set of $k_0, \theta_0$ is considered.

![Figure 7](image1)

**Figure 7.** The surface $\delta(\xi)$ at $a^* = R/2$, $b^* = -R$.

In figure 7 a) the result of calculations for the case of placing measurement points on the hemisphere (see figure 2 a)) of radius $R_0$ is shown. The number of measurement points $J = 38$. The figure 7 b) shows the results of the calculation of the $\delta$ for the case of placing the measurement points on the hemisphere (see figure 2 a)) at $a^* = -R$, $b^* = -R$.

![Figure 8](image2)

**Figure 8.** The surface $\delta(\xi)$ at $a^* = -R$, $b^* = -R$.
measurement points in a square area (see figure 2 b)). With that the length of the side of the square is equal to $1.6R_0$, and the number of measurement points $J = 36$.

The selected point on the surface $\delta = 0$ corresponds to the wanted parameter $\xi^*$. Mesh surfaces of calculated values of the $\delta(\xi)$ have an absolute minimum at a point $\xi = \xi^*$. Thus, the problem solution (3) leads to $\xi_0 = \xi^*$.

In figure 8 the similar correspondences for another parameters set are shown ($\xi^* : a^* = -R$, $b^* = -R$). And in this case the function $\delta(\xi)$ has absolute minimum at the point $\xi = \xi^*$. This shows that for the considered examined parameters of the cylinder position, the procedure of finding their values by the scattered field in accordance with (3) is quite effective.

5. Conclusion
Numerical studies show that the proposed model is productive. Geometrical parameters of elastic objects located near the ideal surface of an acoustic half-space can be obtained from the sound wave scattered field.

Acknowledgments
This work was supported by the Russian Science Foundation (project no. 18-11-00199).

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This work was supported by the Russian Science Foundation (project no. 18-11-00199).

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