Analytic approach to co-evolving dynamics in complex networks: dissatisfied adaptive snowdrift game

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Abstract. We investigate the formulation of mean-field (MF) approaches for co-evolving dynamic model systems, focusing on the accuracy and validity of different schemes in closing MF equations. Within the context of a recently introduced co-evolutionary snowdrift game in which rational adaptive actions are driven by dissatisfaction in the payoff, we introduce a method to test the validity of closure schemes and analyse the shortcomings of previous schemes. A previous scheme suitable for adaptive epidemic models is shown to be invalid for the model studied here. A binomial-style closure scheme that significantly improves upon the previous schemes is introduced. Fixed-point analysis of the MF equations not only explains the numerical observed transition between a connected state with suppressed cooperation and a highly cooperative disconnected state, but also reveals a previously undetected connected state that exhibits the unusual behaviour of decreasing cooperation as the temptation for uncooperative action drops. We proposed a procedure for selecting proper initial conditions to realize the unusual state in numerical simulations. The effects of the mean number of connections that an agent carries are also studied.
1. Introduction

The physics and applications of complex networks have been a centre of focus for research in recent years. Extensive studies on the properties of real-world networks [1–3], the formation and evolution of networks with desired properties [4, 5] and the evolution of dynamic models on networks [6] have been carried out. Dynamic models on networks typically consist of a group of agents that act according to a set of rules defined in some model paradigm and an underlying network. In general, the dynamic status of every agent can be based on random events, on its current situation and on its environment. The environment of an agent is defined through the network: an agent itself occupies a node on the network, and the agents to which it is linked constitute its environment. The underlying network can significantly alter the evolution of a dynamic model. For example, a critical infection threshold exists for the spread of a susceptible-infected-susceptible (SIS)-type disease on a random graph as well as on a small-world [7] network, but no threshold is found when the disease spreads on a scale-free [8] network [9, 10]. In opinion formation dynamics, the consensus time for a majority-rule model was shown to scale as \( \ln N \) on a complete graph, but as a power law of \( N \) when the underlying network is a lattice [11, 12], where \( N \) is the number of agents.

More recently, the focus of research shifted to the interplay between the dynamic model and the network. Such an interplay typically involves a dynamic model whose evolution depends on both the topology of a complex network and an adaptive reaction that would in turn alter the network structure. Examples include the spreading of disease in a system with agents trying to isolate themselves from infected neighbours [13–15], opinion formation through the agents associating themselves with those who share their opinion [16, 17] and competitive situations following the snowdrift game rules with the agents seeking to associate themselves with cooperative neighbours [18]. Different mechanisms of adaptive reactions have been introduced: for example, through the action of a single individual who may be disappointed with its neighbour’s actions or simply acting out of self-interest to protect its own health [15], creating links at a rate that depends on the state of the dynamic model [19] and cutting links as a result of an individual’s action but creating new links through a system-wide aging process [20]. Such models of co-evolving dynamics are better representations of, say, human interactions in that the neighbourhood of an agent is more likely to be one that is dynamic and preferably selected by
the agent, rather than pre-determined as assumed in many models in which a static underlying network is involved.

Computer simulations form an important part of the study of co-evolving processes. Detailed numerical studies could reveal intricate features, e.g. the existence of oscillatory behaviour and multiple stable regimes, in seemingly simple models. However, analytic treatments are essential for gaining an in-depth understanding. Such approaches are typically mean field (MF) in nature and invoke different approximations. In a strict MF sense, one ignores fluctuations, and very often the resulting theory captures the essential features only qualitatively. For a quantitative comparison, it is necessary to include the effects of fluctuations. A recurrent problem is finding the proper assumptions regarding the nature of these fluctuations and the appropriate criteria to judge their validity.

In the present work, we investigate this problem within the context of a snowdrift game co-evolving with an adaptive network recently proposed by Gräser et al [18]. The model is introduced in section 2. Different ways of treating the effects of fluctuations in a mean field theory (MFT) are discussed in section 3. In section 4, we test the validity of different schemes of closing a set of MF equations. We show that a previous approach [21] suitable for a co-evolving epidemic model [13] becomes inapplicable. It is shown that an improved MFT is able to reproduce the key features of the co-evolving model accurately, including the disconnected–connected transition in the network structure accompanying a transition between a segregated and a mixed-phase population, and the non-monotonic dependence of the extent of cooperation on a temptation parameter \( r \). In section 5, we analyse the improved theory and discover a new set of fixed points previously not found in numerical simulations in the regime of small \( r \). An algorithm is proposed to verify the existence of the predicted fixed points. The effects of the average mean number of neighbours per node on the emergence of cooperation are discussed in section 6, followed by a summary in section 7. A discussion of a Poisson-type scheme for closing the MF equations is given in the appendix. The present study of the proper way of closing a set of MF equations would not only lead to a better understanding of a particular model, but also provide insights into the problem of formulating better theories for a large class of co-evolving models.

2. Dissatisfied-adaptive snowdrift game

Figures 1 and 2 show our model schematically [18]. We consider a total of \( N \) agents occupying the nodes of a network. There are a total of \( L_{\text{tot}} = Nk/2 \) bidirectional links and thus an average of \( k \) links per node. Agents that are connected are referred to as neighbours and together they may encounter competitive situations. In the present work, the snowdrift game is used as the competitive paradigm. These links are not static in an adaptive model; instead they evolve as the agents react to their competitive environment.

We start with a description of the snowdrift game in which every agent reacts to a competitive scenario in one of two ways. One can choose to either cooperate with the opponent in working on a laborious task that requires a total cost \( c \) or be uncooperative (or ‘defect’) and hope that the task will be completed by the opponent alone. If the task is completed, i.e. when at least one agent completes the task, both agents will receive a benefit \( b \), where \( b > c \), so that there is still a positive payoff to the one that performs the task alone. In the symmetric formulation of the snowdrift game used here, two cooperators share the laborious task of total cost \( c \) equally.
Figure 1. Schematic representation of the model paradigm. (a) An instantaneous network of agents; the character of an agent is not known to its neighbours until they compete. (b) Possible pairs of players and opponents. (c) An agent who played against a $D$-opponent is dissatisfied (with dissatisfaction $S$) and will respond accordingly (see figure 2).

\[ S = r \]
\[ S = 1 + r \]

Figure 2. A dissatisfied player (here a $C$-node) chooses one of two possible reactions: change its character (left) or cut the dissatisfied link and rewire (right). The probabilities for choosing either reaction are determined by the temptation parameter $r$ as defined in the text. Changes in the macroscopic variables $L_{CC}$, $L_{CD}$ and $N_C$ in the scenario shown are tabulated below the sketch.

\[ \Delta L_{CD} = -1 \quad \Delta L_{CD} = -2 \]
\[ \Delta L_{CC} = 1 \quad \Delta L_{CC} = -1 \]
\[ \Delta N_c = 0 \quad \Delta N_c = -1 \]

---

CC− link (satisfied)
CD− link (semi-dissatisfied)
DD− link (dissatisfied)
The game can thus be represented by the following payoff matrix:

\[
A = \begin{pmatrix} R & S \\ T & P \end{pmatrix} = \begin{pmatrix} b - \frac{c}{2} & b - c \\ b & 0 \end{pmatrix}.
\]

(1)

Specifically, one can associate a vector \(e_C = (1, 0)^T\) with a cooperative agent (or C-node) and a vector \(e_D = (0, 1)^T\) with an uncooperative agent (or D-node). The payoff to an agent of character \(\alpha\) playing against an opponent of character \(\beta\) is then given by

\[
A(\alpha, \beta) = e_\alpha^T A e_\beta,
\]

(2)

where \(\alpha, \beta = C\) or \(D\). For the snowdrift game, the payoffs follow the ranking \(T > R > S > \bar{P}\). A slight change in the ranking to \(T > R > \bar{P} > S\) gives the prisoner’s dilemma. Due to the difficulty in measuring the payoffs accurately, the snowdrift game is regarded as an important alternative to the prisoner’s dilemma in studying the emergence of cooperation. In different contexts such as politics, the snowdrift game is also used as a model paradigm under the name ‘Game of Chicken’ [22]. Dividing each matrix element by \(b - c/2\) and defining \(r = c/(2b - c)\), the payoff matrix can be rewritten in terms of a single parameter \(r\) that represents the temptation to defect:

\[
A = \begin{pmatrix} 1 & 1 - r \\ 1 + r & 0 \end{pmatrix}.
\]

(3)

For small \(r\), cooperation is preferred since exploiting other agents leads to only a little extra benefit. Instead, an uncooperative agent might lose the entire benefit if the opponent is also uncooperative. In contrast, for large \(r\), defecting becomes more promising since the possible extra benefit outweighs the loss when playing against another defector.

Each agent adopts a current character (C or D), which is not a priori known to its neighbours (see figure 1). In a time step \(\delta t\), an agent—the player—is chosen at random, and plays the snowdrift game against one of its neighbours—the opponent—also chosen at random (see figure 1). We assume that the frequency with which an agent enters such a competing situation is a constant, and hence \(\delta t \propto 1/N\). From equation (3), a player will hope for a cooperative opponent so as to receive the higher payoff regardless of its own action. Thus, he will be dissatisfied if the outcome of the game shows that his opponent is of D-character. Hence, a player–opponent link of the DD and CD type will cause dissatisfaction on the part of the player.

A dissatisfied player will react in two different ways: either by adapting itself via a switch in character from cooperative to uncooperative or vice versa, or by changing its competing environment via a rewiring process in which the player cuts the connection to the opponent and rewires to another randomly chosen agent (see figure 2). As an adaptive reaction occurs only when the player is dissatisfied, we refer to the model as the dissatisfied-adaptive snowdrift game (DASG). A crucial step that determines the relevance of the model’s results is that of defining the probabilities of switching and rewiring. The guiding principle is that the agents act in a rational manner [18]:

(i) Every node will react if and only if it is dissatisfied with the outcome, i.e. if its opponent was a D-node;

(ii) The likelihood of switching or rewiring is based on the player’s degree of dissatisfaction, i.e. the difference in payoffs between what he expected to receive and what he actually received, and should reflect the difference in benefit that the change is expected to bring about.
To quantify the idea, we define the disappointment $S_a$ of a player of character $\alpha$ playing against a $D$-opponent as $S_a = A(\alpha, C) - A(\alpha, D)$, where $A = C$ or $D$ (see figure 1(c)). For a $D$-player, the rewiring probability is taken to be $P_D^{(R)} = \frac{1}{2}S_D = \frac{1}{2}(1 + r)$ and the switching probability to be $P_D^{(S)} = 1 - P_D^{(R)} = \frac{1}{2}(1 - r)$. Similarly, the switching probability of a dissatisfied $C$-player is $P_C^{(S)} = \frac{1}{2}S_C = \frac{1}{2}r$ and the corresponding rewiring probability is $P_C^{(R)} = 1 - P_C^{(S)} = 1 - \frac{1}{2}r$.

These switching and rewiring probabilities reflect rational reactions. In the small $r$ regime, there is little benefit for a $C$-player to switch character: it would obtain only a marginally higher benefit ($1 + r$ versus $1$ with $r \ll 1$) when playing against its $C$-neighbours, but lose all the remaining benefit when playing against the same or other $D$-neighbours again ($0$ versus $1 - r$). In this case, rewiring should be more likely, as the form of $P_C^{(R)}$ indicates. In the high $r$ regime, the risk of losing the very small remaining benefit when switching from $C$ and $D$ balances the expectation of profiting from playing with $C$-neighbours after switching. Hence, rewiring and switching are equally likely. Similarly, a dissatisfied $D$-player would gain nearly nothing from switching to $C$ in the high $r$ regime (as $1 - r \approx 0$ for $r \sim 1$) when competing with the same or other $D$-neighbours again, but would lose its benefit ($1 + r \approx 2$ for $r \sim 1$) when playing against $C$-neighbours after switching. Hence, dissatisfied $D$-players will prefer to rewire than to switch character for $r \approx 1$, as given by $P_D^{(R)}$ and $P_C^{(S)}$.

The model is an example of node-driven dynamics, as the number of links that a node possesses does not determine its importance to the network’s evolution. We remark that the dynamics of a model can also be link-driven, e.g. every link in epidemic dynamics can spread the disease and nodes with more neighbours can play a pivotal role in spreading the disease. In DASG, the dissatisfaction depends only on the opponent’s character and a player takes action whenever a $D$-player is chosen as an opponent. This has the advantages of making use of the minimal local information that a player can collect in a competing situation and having a particular link for cutting and rewiring. The resulting dynamics of DASG are presented in [18]. The model shows several coupled transitions in the long-time dynamics as a function of the payoff parameter $r$. For small $r$, the system reaches a disconnected network structure in which all the $C$-agents form a highly connected cluster and isolated $D$-agents are expelled from the cluster. It is a frozen state with no further evolution and has a high level of cooperation. For large $r$, the system reaches a connected network structure consisting of nodes of both characters. It keeps on evolving, but the distributions of properties are stationary, and it has a low level of cooperation. In an intermediate regime of $0.15 \leq r \leq 0.4$, the final state is dependent on the initial fraction of $C$-agents: the more cooperators a network initially has, the more likely it is to expel all defectors and approach a disconnected state. The results of [18] also revealed that in the vicinity of the transition, the degree of cooperation shows non-monotonic behaviour as a function of $r$.

3. Mean-field theories

In [18], we outlined an MFT for DASG that was able to capture the key features qualitatively. A key ingredient of the theory is to treat the $C$-nodes and $D$-nodes separately, but assuming an identical neighbourhood for every node of the same type. Here, we focus on how to formulate MFTs to achieve better accuracy. The discussion will be kept as general as possible, since several features of our model—node-driven dynamics, binary character of the nodes, two possible reactions: switching or rewiring—are shared by many co-evolving adaptive models.
For a given neighbourhood, the change in the macroscopic quantities \( \Delta f_c \) \( \Delta L_{CD} \) \( \Delta L_{CC} \) \( \Delta N_c \) is characterized by its degree \( k \) \( f_c \) \( \lambda_{CD} \) \( \lambda_{CD} \). Locally, the neighbourhood of a specific node \( l_{CC} = L_{CC} / N \) and the number of \( CD \)-links per node \( l_{CD} = L_{CD} / N \) as the macroscopic variables, where \( N_c \) is the number of cooperative agents, \( L_{CC} \) is the number of \( CC \)-links and \( L_{CD} \) is the number of \( CD \)-links. These variables change with time as the co-evolving dynamics proceed. The fraction of \( D \)-nodes and the number of \( DD \)-links per node can be calculated from these variables. Other variables used in the literature such as the ‘magnetisation’ \( m \) \( m = \frac{1}{2} \) that gives the difference between the fractions of \( C \) - and \( D \)-nodes, the density of \( CD \)-links \( L_{CD} / L_{tot} = 2l_{CD} / k \) and the ‘link magnetisation’ \( m_f = (L_{CC} - L_{DD}) / L_{tot} = 2(l_{CC} + l_{CD} - 1) / k \) can also be expressed in terms of the three macroscopic variables.

In the phase space spanned by \( f_c \), \( l_{CC} \) and \( l_{CD} \), the system at any time can be represented by a point. The accessible region of the phase space is enclosed by the boundaries \( f_c \in [0, 1] \), \( l_{CD} \in [0, k/2] \) and \( l_{CC} \in [0, k/2 - l_{CD}] \). To formulate MFTs, we start with the equations on the change in \( N_c \), \( L_{CC} \), and \( L_{CD} \), when a switching or rewiring process occurs. These processes take place at a local node when it is chosen as the player. Locally, the neighbourhood of a specific node is characterized by its degree \( \kappa \) and the number of \( CD \)-links \( \lambda_{CD} \) among its \( \kappa \) local links. For a given neighbourhood, the change in the macroscopic quantities \( \Delta X(\kappa, \lambda_{CD}) \) \( X = N_c, L_{CC}, L_{CD} \) can be readily calculated for each of the four possible events: \( C \rightarrow D \) switching \( CS \), rewiring of a \( C \)-node \( CR \), \( D \rightarrow C \) switching of a \( D \)-node \( DS \) and rewiring of a \( D \)-node \( DR \), as listed in Table 1.

Taking into account the probabilities for the various events, we can express the expected change of each macroscopic variable formally as

\[
\Delta X_{tot} = \sum_{T = C, D} P_T^{(T)} \sum_{\kappa} P_T^{(\kappa)} \sum_{\lambda_{CD} = 0}^{\kappa} P_T^{(2\lambda_{CD})} P_{\kappa, \lambda_{CD}, T}^{(D)} \sum_{\kappa = \kappa, \kappa} P_T^{(E)} \Delta X(\kappa, \lambda_{CD}).
\]  

Table 1. Changes in the macroscopic variables \( L_{CD}, L_{CC} \) and \( N_c \) for each possible event that can occur at a node with \( \lambda_{CD} \) \( CD \)-links among its \( \kappa \) links.

| \( CS \) | \( CR \) | \( DS \) | \( DR \) |
|---|---|---|---|
| \( \Delta L_{CD} \) | \( \kappa - 2\lambda_{CD} \) | \( -f_c \) | \( \kappa - 2\lambda_{CD} \) | \( f_c \) |
| \( \Delta L_{CC} \) | \( \lambda_{CD} \) | \( -f_c \) | \( \lambda_{CD} \) | \( 0 \) |
| \( \Delta N_c \) | \( -1 \) | \( 0 \) | \( 1 \) | \( 0 \) |

Finally, \( P_T^{(E)} \) is the probability for a node of type \( T \) to take the action \( E \) \( (E = S \) (switching) or \( R \) (rewiring)) as defined in section 2. Without making any specific assumptions, equation (4)
can be expressed in terms of expectation values of the functions of $\lambda_{CD}$ and $\kappa$:

\[
\Delta N_C = (1 - f_c) \left( 1 - \frac{\lambda_{CD}}{\kappa} \right)_D P^{(S)}_D - f_c \left( \frac{\lambda_{CD}}{\kappa} \right)_C P^{(S)}_C
\]

(6)

\[
\Delta L_{CD} = f_c P^{(S)}_C \left( \langle \lambda_{CD} \rangle_C - 2 \frac{\lambda_{CD}^2}{\kappa} \right)_C - f_c^2 P^{(R)}_C \frac{\lambda_{CD}}{\kappa} \left( \frac{\lambda_{CD}}{\kappa} \right)_C + (1 - f_c) P^{(S)}_D \left( \langle \kappa \rangle_D - 3 \langle \lambda_{CD} \rangle_D \right.

+ \left. 2 \frac{\lambda_{CD}^2}{\kappa} \right)_D
\]

(7)

\[
\Delta L_{CC} = f_c P^{(S)}_C \left( \frac{\lambda_{CD}^2}{\kappa} \right)_C - \langle \lambda_{CD} \rangle_C \left( 1 - \frac{\lambda_{CD}^2}{\kappa} \right)_C + (1 - f_c) P^{(S)}_D \left( \langle \lambda_{CD} \rangle_D - \frac{\lambda_{CD}^2}{\kappa} \right)_D.
\]

(8)

where $\langle \cdots \rangle_C$ and $\langle \cdots \rangle_D$ denote the expectation values or averages of the quantity taken with respect to $C$- and $D$-nodes, respectively. Note that the dependence on $r$ is included in the switching and rewiring probabilities.

Different ways of treating the expectation values will result in different sets of MF equations. To proceed, first we assume that the average of a fraction can be decoupled into a fraction of averages, i.e.,

\[
\frac{\langle \lambda_{CD} \rangle_C}{\kappa} = \frac{\langle \lambda_{CD} \rangle_{C/D}}{\langle \kappa \rangle_{C/D}},
\]

(9)

where the subscript $C/D$ means that there are actually two equations, one for averages over $C$-nodes and another for averages over $D$-nodes. Although this approximation is not strictly satisfied, it is however less restrictive than the commonly chosen approximation that all nodes, or at least the nodes of one type, have an identical degree \[16\]. The first moments are identical to the averages over the macroscopic variables, i.e.

\[
\langle \lambda_{CD} \rangle_C = \frac{l_{CD}}{f_c},
\]

\[
\langle \lambda_{CD} \rangle_D = \frac{l_{CD}}{1 - f_c},
\]

\[
\langle \kappa \rangle_C = \frac{2l_{CC} + l_{CD}}{f_c},
\]

\[
\langle \kappa \rangle_D = \frac{k - 2l_{CC} - l_{CD}}{1 - f_c}.
\]

(10)

Additional assumptions are needed for the averages $\langle \lambda_{CD}^2 \rangle_C$ and $\langle \lambda_{CD}^2 \rangle_D$, which deal with the second moments. The simplest approximation amounts to assuming all nodes of the same type have an identical neighbourhood, yet allowing for different neighbourhoods for $C$- and $D$-nodes. Within this approximation, the second moments equal the first moments squared:

\[
\langle \lambda_{CD}^2 \rangle_C^{(SSC)} = \langle \lambda_{CD} \rangle_C^2 = \frac{l_{CD}^2}{f_c^2},
\]

\[
\langle \lambda_{CD}^2 \rangle_D^{(SSC)} = \langle \lambda_{CD} \rangle_D^2 = \frac{l_{CD}^2}{(1 - f_c)^2}.
\]

(11)
Applying this simple squared closure (SSC) to equations (6)–(8) gives a closed set of equations for the simplest MFT, equivalent to that used in [18]. Results of this simple closure explain the disconnected–connected transition qualitatively, but there are quantitative discrepancies.

To construct an improved MFT, we note that it is unlikely that all $C$ (or $D$) agents have an identical competing environment. Since the second moment, in general, is related to the variance through $\sigma^2 = \langle X^2 \rangle - \langle X \rangle^2$, the SSC assumes a vanishing variance and thus constitutes a lower bound for the second moments. Any variations in the nodes’ environment will lead to an increase in the second moment with respect to SSC. To go beyond SSC, it is necessary to include the effects of fluctuations. One approach was proposed by Keeling and Eames [21] and applied to adaptive epidemic (SIS) model by Gross et al [13]. This approach will be referred to as the Poisson-type closure and it will be discussed in the appendix. Another approach is to account for the different possible numbers of neighbours of a certain kind. For example, given a $C$-player with $\kappa$ neighbours, the number of $D$-neighbours could take on values from zero to $\kappa$. In addition, there are $\binom{\kappa}{1}$ possible configurations for the case of only one $D$-neighbour, $\binom{\kappa}{2}$ possible configurations for exactly two $D$-neighbours and so on. Applying this consideration of a binomial distribution to evaluating the averages, the second moments become

$$
\langle \lambda_{CD}^2 \rangle^{\text{BINO}}_c = \frac{l_{CD}^2}{f_c^2} + \frac{l_{CD}}{f_c} - \frac{l_{CD}^2}{f_c(2l_{CC} + l_{CD})},
$$

$$
\langle \lambda_{CD}^2 \rangle^{\text{BINO}}_d = \frac{l_{CD}^2}{(1 - f_c)^2} + \frac{l_{CD}}{1 - f_c} - \frac{l_{CD}^2}{(1 - f_c)(k - 2l_{CC} - l_{CD})}.
$$

This binomial closure also leads to a set of closed equations from which $N_C$, $L_{CD}$ and $L_{CC}$ can be calculated as a function of time and in the long-time limit.

4. Validity and accuracy of closure schemes

To test the validity of the different schemes of closing equations (6)–(8), we propose to study the quantity

$$
\Lambda_{C/D} = \langle \lambda_{CD}^2 \rangle_{C/D} - \langle \lambda_{CD} \rangle_{C/D},
$$

as a function of

$$
\mathcal{P}^{(\text{opp})}_{C/D} = \langle \lambda_{CD} \rangle_{C/D}.
$$

Here, $\mathcal{P}^{(\text{opp})}_{C/D}$ is meant to be the probability of picking a neighbour of the opposite type. These quantities and their relationships can be calculated exactly by numerical simulations, and the results can be tested against those obtained by the three closure schemes together with the decoupling approximation (equation (9)). The quantities $\Lambda_{C/D}$ must satisfy the conditions that:

(i) $\Lambda_{C/D} \leq 0$ within the interval $0 \leq \mathcal{P}^{(\text{opp})}_{C/D} \leq 1$, as $\langle \lambda_{CD}^2 \rangle_{C/D} \leq \langle \lambda_{CD} \rangle_{C/D} \langle \lambda_{CD} \rangle_{C/D} \leq \langle \lambda_{CD} \rangle_{C/D}$ as no node could have more $CD$-links than its number of neighbours. (ii) The quantities $\Lambda_{C/D}$ should vanish at the boundaries of the range of $\mathcal{P}^{(\text{opp})}_{C/D}$, i.e. $\Lambda_{C/D} \to 0$ as $\mathcal{P}^{(\text{opp})}_{C/D} \to 0$ and $\mathcal{P}^{(\text{opp})}_{C/D} \to 1$. In addition, $\Lambda$ is also related to the terms in the MF equations. For example, at $C$-nodes ($D$-nodes), $\Lambda$ ($-\Lambda$) is the expectation value of $\Delta L_{CC}$ for the event of a $C \to D$
Figure 3. The quantity $\Lambda$ as a function of $P^{(\text{opp})}$ at $D$-nodes (a, b) and $C$-nodes (c, d), for simulations with $r = 0.1$ (a, c) and $r = 0.5$ (b, d). All figures show plots of $\Lambda$ against $\langle \lambda_{CD}/\kappa \rangle$ as calculated directly from simulations (blue symbols), and using the MF equations (equations (6)–(8)) obtained by the SSC (blue dashed line), the binomial closure (orange solid line) and the Poisson-type closure (green dash-dotted line) as discussed in the appendix.

$(D \rightarrow C)$ switch, as given in equation (8). Therefore, possible errors in obtaining $\Lambda_{C/D}$ and, in particular, violations of the necessary conditions could provide insights into the accuracy of the closure schemes.

It turns out that not all the closure schemes satisfy the conditions. The symbols in figure 3 show $\Lambda$ versus $P^{(\text{opp})}$ for $D$-nodes (figures 3(a) and (b)) and $C$-nodes (figures 3(c) and (d)), as calculated by two simulation runs with $r = 0.1$ and $r = 0.5$, respectively. The initial condition is a random network with $f_{c}^{i} = 0.01$. Note that the abscissas in figures 3(a) and (b) refer to $P_{D}^{(\text{opp})}$ and in figures 3(c) and (d) refer to $P_{C}^{(\text{opp})}$. The simulation results are obtained by calculating both $\Lambda$ and $P^{(\text{opp})}$ directly as the system evolves, with the isolated nodes treated as $\lambda_{CD}/\kappa = \lambda_{CD}/\kappa \equiv 0$ in the averaging process, in accordance with the meaning of $P^{(\text{opp})}$. The data points give $\Lambda_{C/D}$ and $P^{(\text{opp})}$ obtained by averaging over nodes in the system taken at regular intervals of 1000 time steps in the simulation. The variation in the value of $P^{(\text{opp})}$ as shown in the abscissa thus also reflects the time evolution of the network. In each panel, there are also three lines showing the relationship between $\Lambda$ and $P^{(\text{opp})}$ as calculated by the three closure schemes. For each line, the macroscopic variables $l_{CC}$, $l_{CD}$ and $f_{c}$ were taken directly from simulation at regular intervals. These values were then used to evaluate $\Lambda$ and $P^{(\text{opp})}$ using the different closure schemes (see equations (11) and (12)), together with the decoupling approximation (equation (9)). The simple squared closure (SSC) fulfills the conditions on $\Lambda$. However, SCC underestimates $\Lambda_{C/D}$ significantly as it underestimates the variance. The binomial closure also fulfills the necessary conditions and it gives results that agree very well with simulation results. Deviations only occur in the long-time limit for $D$-nodes in the $r = 0.5$ case (see figure 3(b)). In this case, although the system ends in a mixed-character phase with $C$-nodes and $D$-nodes in a connected network, not all the $D$-nodes are always in the connected network and some may
temporarily be isolated. These isolated nodes make the decoupling approximation (equation (9)) inaccurate. The deviation will therefore become smaller when the mean degree $k$ in the network increases, as we will discuss in section 6. The results for the Poisson-type closure are also included in figure 3 for comparison. This closure scheme, however, does not satisfy the necessary conditions and thus is invalid for DASG. A discussion on the Poisson-type closure scheme is given in the appendix.

For DASG, the simple square closure and binomial closure are valid. Figure 4 compares the results obtained by equations (6)–(8) using the simple square closure and binomial closure with simulation results. For each closure scheme, we iterate the resulting set of equations to obtain the long-time limit for a given initial fraction of cooperators. The binomial closure (solid coloured lines) gives results in very good agreement with simulation results, particularly in the small $r$ regime where the system ends in a disconnected state. In this regime, SSC (dashed lines) only gives results in qualitative agreement, as pointed out in [18]. For large $r$, a small improvement can be seen, but deviations remain visible. The deviations are mainly due to the decoupling assumption in equation (9).

5. Fixed-point analysis

Equations (6)–(8) with the binomial closure thus provide an improved MFT. Explicitly, the binomial closure gives the following set of equations:

$$\Delta N_C = \left(\frac{1-r}{2}\right)(1-f_c)\left(1-f_c\right) - \left(\frac{r}{2}\right)f_c\left(\frac{l_{CD}}{2l_{CC}+l_{CD}}\right),$$

Figure 4. Long-time limits $f_c(\infty)$ obtained by simulations (symbols) and equations (6)–(8) using the binomial closure (lines). Results are shown for different initial fractions of cooperative nodes $f_c' = 0.9$ (red, circles), $f_c' = 0.5$ (blue, triangles) and $f_c' = 0.1$ (green, squares), all starting from a random network configuration. Dashed (black) lines: previous results using the SSC.
\[ \Delta L_{CD} = \left( \frac{r}{2} \right) f_c \left( \frac{l_{CD}}{f_c} - \frac{2}{2l_{CC} + l_{CD}} \left( \frac{l_{CD}^2}{f_c} + l_{CD} - \frac{l_{CD}^2}{2l_{CC} + l_{CD}} \right) \right) - \left( \frac{2 - r}{2} \right) f_c^2 \left( \frac{l_{CD}}{2l_{CC} + l_{CD}} \right) + \left( \frac{1 - r}{2} \right) \left( k - 2l_{CC} - 4l_{CD} + \frac{2(1 - f_c)}{k - 2l_{CC} - l_{CD}} \left( \frac{l_{CD}^2}{f_c} + l_{CD} - \frac{l_{CD}^2}{k - 2l_{CC} - l_{CD}} \right) \right) + \left( \frac{1 + r}{2} \right) f_c(1 - f_c) \left( 1 - \frac{l_{CD}}{k - 2l_{CC} - l_{CD}} \right), \]  
\[ \Delta L_{CC} = \left( \frac{r}{2} \right) f_c \left( \frac{1}{2l_{CC} + l_{CD}} \left( \frac{l_{CD}^2}{f_c} + l_{CD} - \frac{l_{CD}^2}{2l_{CC} + l_{CD}} \right) - \frac{l_{CD}}{f_c} \right) + \left( \frac{2 - r}{2} \right) f_c^2 \left( \frac{l_{CD}}{2l_{CC} + l_{CD}} \right) + \left( \frac{1 - r}{2} \right) \left( l_{CC} - \frac{1 - f_c}{k - 2l_{CC} - l_{CD}} \left( \frac{l_{CD}^2}{f_c} + l_{CD} - \frac{l_{CD}^2}{k - 2l_{CC} - l_{CD}} \right) \right). \]  

As \( f_c = N_C/N, \ l_{CC} = L_{CC}/N, \ l_{CD} = L_{CD}/N \), this is a closed set of equations for the three global variables. The set of equations can be used to explore the transitions from a disconnected frozen state with segregation in character to a mixed-character state in a deeper way by carrying out a fixed-point analysis. A system reaches a frozen state when only CC-links are left. Recall that in the phase space spanned by \( f_c, l_{CC} \) and \( l_{CD} \), the accessible region is enclosed by the boundaries \( f_c \in [0, 1], l_{CD} \in [0, k/2] \) and \( l_{CC} \in [0, k/2 - l_{CD}] \). In the disconnected frozen state, \( l_{CD} = 0 \) and the fixed points lie on the line defined by \( l_{CC} = k/2 \). These fixed points are sticky in that once reached, random fluctuations cannot change the system’s configuration anymore. The results in [18] show that for sufficiently small \( r \), a system that started from a random configuration is attracted towards these sticky fixed points. For larger values of \( 0.15 < r < 0.4 \), the system ends up in a sticky fixed point if its initial fraction of cooperators is high enough.

Besides the line of sticky fixed points, however, there exists another fixed point of physical interest that can be obtained by setting each of the improved MF equations to zero. There is one such fixed point for every value of \( 0 < r < 1 \). The \( f_c \)-coordinate of this fixed point is shown as a function of \( r \) in figure 5(a). For large \( r \), the fixed point can be identified with the connected mixed-character phase, which also emerges from iterating the MFT to convergence. For small values of \( r \), the MFT gives a fixed point with \( l_{CD} > 0 \) and \( l_{DD} > 0 \) and thus also corresponds to a mixed-character phase, while iterating the MFT equations to convergence gives the disconnected state as shown by the dashed line in figure 5(a) for an initial fraction \( f_c = 0.5 \). The disconnected state is also obtained in numerical simulations of the system.

The eigenvalue of the Jacobian matrix at the fixed point provides useful information on the stability of the fixed point. If all eigenvalues carry a negative real part, then the fixed point is stable against small fluctuations. Figure 5(b) shows the negated real parts of the three eigenvalues as a function of \( r \), displayed on a log scale for clarity. Strictly speaking, all three eigenvalues have negative real parts, with two of them taking on identical values and approaching zero as \( r \to 0 \) (dashed line in figure 5(b)) and one becoming increasingly negative as \( r \to 0 \). Thus, unlike other adaptive models [15, 23], our system does not undergo any bifurcations. Instead, the stable fixed point has a shrinking basin of attraction as \( r \) decreases.

\(^3\) Strictly speaking, the MFT is not well defined when there exist isolated nodes, for which a ratio of the form \( 0/0 \) appears. According to the model definition, we can amend the MFT by defining \( \Delta f_c(l_{CC} = k/2) = \Delta l_{CC}(l_{CC} = k/2) = \Delta l_{CD}(l_{CC} = k/2) = 0 \).
and the number of initial configurations drawn towards the fixed point drops. Systems starting from a state outside the basin of attraction would then be driven towards the sticky fixed point corresponding to the disconnected state.

The question remains whether the isolated fixed point corresponding to a mixed-character phase represents a viable final state in the regime of small $r$. To test this, we performed numerical simulations by gradually lowering the value of $r$ in steps of $\Delta r = 0.005$, i.e. from $r$ to $r - \Delta r$, using a configuration in the steady state corresponding to the value $r$ as the initial state for the simulations at $r - \Delta r$. For each value of $r$, $10^4$ Monte Carlo steps are sufficient for the system to become steady and the mean, standard deviation and the maximal and minimal values of the macroscopic variables are recorded. Results are shown in figure 6. Figure 6(a) shows the mean of $f_c$ as a function of $r$. The range of values observed during the run is shown as an error bar. Interestingly, $f_c$ follows the trend as predicted by the fixed points of the MF equations. This confirms the existence of the stable fixed point and thus the corresponding connected state down to $r = 0.05$. It should be stressed that simulations using an arbitrary initial state result in a disconnected state for values of $r$ well above this value. For $r < 0.05$, the CD-link density drops to zero faster (triangles in figure 6(b)) than the fixed point predicts and the system is taken to the axis of sticky fixed points. Here, fluctuations also become increasingly important as $r$ decreases, as shown by the behaviour of the standard deviations in figure 6(c).

The decreasing stability of the fixed point and the enhanced fluctuations as $r$ decreases are the reasons why the fixed points corresponding to connected states are usually not observed if one starts from a random configuration. Here, we confirmed the existence of these states even at small $r$. These states are unusual in that $f_c$ drops as $r$ decreases (see figures 6(a) and 4),

![Figure 5](image-url)
Figure 6. Results obtained by tracing $r \to 0$ in steps of small intervals $\Delta r$.
(a) The average of $f_c$ over a run as $r$ is gradually lowered, with error bars indicating the maximum and minimum values observed in the run. (b) $l_{CC}$ (circles) and $l_{CD}$ (triangles). (c) Standard deviation of $l_{CC}$ (circles) and $l_{CD}$ (triangles). The standard deviations increase in the small $r$ regime, in accordance with the fixed point becoming less stable as discussed in figure 5. All panels: symbols are simulation results and lines are results from MF equations.

while a decreasing temptation to be uncooperative should usually promote rather than suppress cooperation. It should be stressed that the actions of the agents are rational, yet the overall response of the system could become irrational in this regime. Our results also illustrate that the non-monotonic behaviour of $f_c$ (see figure 4) at values of $r$ slightly above the transition has its root in the $r$-dependence of the fixed point. It is, however, difficult to obtain analytically where the peak occurs as the MF equations give an equation for $f_c(r)$ that is of very high degree. As $r$ becomes smaller, the fixed point becomes less stable and fluctuations take the system into the disconnected state observed in numerical simulations.

6. Dependence on the mean degree

So far, we presented results for a system of $N = 10\,000$ agents with an average of $k = 4$ neighbours per agent. The mean degree is expected to influence the local competing environment based on which agents take action. Figure 7 shows the long-time limit $f_c(\infty)$ for systems with mean degrees of $k = 2$, $k = 10$ and $k = 100$. Note that for $k = 2$, a connected network results only when the agents form a ring. The solid symbols are the results of simulations using an initial random network with $f_i^2 = 0.5$, whereas the open symbols are the results when $r$ is gradually lowered with the initial configuration at $r - \Delta r$ being that of the steady state at $r$. Qualitatively, the systems show similar behaviour. For a higher mean degree, it will be less unlikely to isolate the $D$-nodes and thus the character-segregated disconnected
Figure 7. Long-time limits for $f_c(\infty)$ as a function of $r$ for systems with a mean degree of $k = 100$ (red triangles), $k = 10$ (purple circles) and $k = 2$ (blue squares). For systems that started from a random network with $f_i^c = 0.5$, simulations (solid symbols) and MF equations (solid lines) show that the values of $r$ at which the transition between the disconnected and connected states occurs depend on $k$. Gradually tracing $r$ results in a connected network even at small $r$, as obtained by simulations (open symbols) and by the fixed points of the MF equations (dashed lines).

state is expected to be less likely to emerge. This is indeed the case. For systems starting from a random network configuration, the transition value of $r$ shifts to a lower value as $k$ increases. The fraction of cooperation in the disconnected state also increases with the mean degree. Tracing $r$ downwards also brings the system to a connected state at small values of $r$, for which the system would otherwise be in the disconnected state.

Results of iterating the MF equations are shown by the solid lines and results of fixed-point analysis are shown by the dashed lines in figure 7 for comparison. While deviations are more notable in the connected states for systems with a low mean degree, the deviations are barely noticeable for networks with $k = 100$. It is reasonable in that the effects of fluctuations diminish as $k$ increases, analogous to the better performance of MFTs in higher spatial dimensions. In addition, the decoupling approximation (equation (9)) also works better at large values of $k$, where the problem related to the inclusion of isolated $D$-nodes into the system averages occurs less frequently.

7. Summary and conclusion

In summary, we studied the formalism of MF equations for co-evolving dynamics in networks, within the context of the dissatisfied-adaptive snowdrift game. In particular, we focused on the assumptions, validity and accuracy of different schemes in closing a set of MF equations. An improved closing scheme that goes beyond the assumption of an identical competing environment for each type of node by accounting for fluctuations in the competing environment
via a binomial distribution is shown to give results in good agreement with simulation results. Using the improved set of MF equations, a fixed point that is stable for each value of $r$ is revealed. For large $r$, the fixed point is identified with the connected mixed-phase state that is found in numerical simulations and iterating the equations to the steady state, starting from initial conditions such as those of random networks. For small $r$, the fixed point corresponds to a connected state of suppressed cooperation, while a highly cooperative disconnected state is observed in numerical simulations and iterating the equations to convergence. This previously undetected state can indeed be observed in numerical simulations when the value of $r$ is gradually lowered, using the configuration in the steady state at $r$ as the initial condition for $r - \Delta r$. The connected state in the small-$r$ regime shows counter-intuitive and irrational behaviour—a lower temptation parameter $r$ results in a suppression of cooperation—even though every agent behaves rationally. This state is eventually destroyed by fluctuations for very small $r$. We also studied the effects of the mean degree of nodes and found that the MF equations work better for higher mean degrees.

In the model studied here, the dissatisfaction is evaluated with reference only to one opponent among all the neighbours. It is meant to make use of the minimal local information as well as having a clear target on a particular link for cutting and rewiring. It is also consistent with the idea of rewiring to a randomly chosen player rather than targeting a $C$-player for rewiring, as the latter requires more information. When a player is connected to a majority of friendly $C$-neighbours, it is still possible for the player to switch character when a $D$-opponent is chosen from the minority group, but the probabilities of such character switching are high only when the risk is worth taking. An interesting alternative model would be one in which a player’s dissatisfaction is dependent on the average payoff from all the neighbours, and thus more local information is assumed. Contrasting the two models, the random pick of an opponent in DASG serves to sample the competing environment and a minority $D$-opponent is less likely to be chosen. However, once a $D$-opponent is chosen the player will take action, implying that the model studied here allows for more fluctuations. In the alternative model, once a more friendly competing environment is established, the probabilities of taking an action will be smaller. It amounts to generalizing DASG to allow for multiple games with the neighbours before the dissatisfaction is evaluated. This will probably suppress fluctuations. In the alternative model, a link for cutting and rewiring has to be chosen at some point after the dissatisfaction is evaluated from the average payoff. It will be interesting to apply the ideas presented here on setting up a MFT to the alternative model.

In closing, we remark that the present study provides useful insights into the formulation of MF approaches for other co-evolving models. While numerical simulations are often crucial, analysing the fixed points and their stability of a set of valid equations could point out non-trivial and possibly overlooked dynamical features such as the connected state at small $r$ shown here for DASG. By following the approach in the present paper, we expect that additional insights could be gained in a variety of social, economic or biological adaptive models.

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Appendix. The Poisson-type closure

Besides the simple closure and the binomial closure discussed in the main text, another approach was proposed by Keeling and Eames [21] and applied to the adaptive epidemic (SIS) model by Gross et al [13]. Within the context of our model, the approximation is equivalent to closing equations (6)–(8) by setting

$$\langle \lambda_{CD}^{p_{\text{Pois}}} \rangle_{C/D} = \langle \lambda_{CD} \rangle_{C/D}^2 + \langle \lambda_{CD} \rangle_{C/D}. \quad (A.1)$$

It amounts to assuming that the variance and the mean are equal, and thus we refer to it as the Poisson-type closure scheme. The validity of this scheme is also checked and results are included in figure 3 for comparison with the two other schemes discussed in the main text. For the Poisson-type closure, $\Lambda_{C/D}$ takes on positive values in a range of $\mathcal{P}(\text{opp})$ close to unity and it does not vanish at $\mathcal{P}(\text{opp}) = 1$. Thus, this closure violates the conditions on $\Lambda_{C/D}$. The problem is more prominent for small values of $r$. In this regime, the $D$-nodes might be expelled entirely. If they have remaining connections, the links will almost entirely be $DC$-links. However, the Poisson-type closure would still predict $DD$-links being converted to $CD$-links for $D \rightarrow C$ switching events even when there was no $DD$-link left in the system. In fact, integrating the set of equations resulting from the Poisson-type closure causes the system to pass through unphysical states with some of the global variables taking on negative values. This closure scheme is thus invalid for DASG, although it is successful at treating the adaptive SIS networks [13].

It is illustrative to compare the adaptive SIS model and our DASG by pointing out two differences. Firstly, in the competing environment of a snowdrift game, the network dynamics do not strictly oppose connections between nodes of opposite types; instead they favour connections to nodes of one type—cooperators—over connections to the other. Hence, a $D$-node has predominantly $C$-nodes as neighbours. In contrast, in epidemic dynamics, healthy nodes intend to isolate themselves from infected nodes. As a result, both healthy and infected nodes would have more connections with nodes of their own type than with nodes of the opposite type. In terms of the probabilities $\mathcal{P}(\text{opp})$ of picking a link to a node of the opposite type, the SIS model remains mostly in the $\mathcal{P}(\text{opp}) \approx 0$ regime, where the Poisson-type closure is expected to be a good approximation. In contrast, $D$-nodes in DASG are typically in the $\mathcal{P}(\text{opp}) \approx 1$ regime unless the temptation parameter $r$ is close to unity. In this regime, the Poisson-type closure breaks down. Secondly, it should be noted that typically no information sharing exists in a competitive situation, i.e. a node has no knowledge of its neighbour’s character until it encounters the neighbour as an opponent in a trial. Thus, rewiring in DASG is towards a randomly chosen node, instead of picking a $C$-node preferentially. In adaptive epidemic dynamics, the health of the individuals is known to others and rewiring occurs only towards the healthy nodes, leading to a very high degree for these nodes. Upon contracting the disease, these nodes feature many links between nodes of the opposite types, causing a larger variance in the $SI$-link distribution. Such a large variance can be modelled by Poisson-type closure, since it assumes a variance that exceeds that of a pure random distribution.

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