Log-Sobolev inequality and proof of Hypothesis of the Gaussian Maximizers for the capacity of quantum noisy homodyning

A. S. Holevo
Steklov Mathematical Institute, RAS, Moscow, Russia

Abstract

In the present paper we give proof that the information-transmission capacity of the approximate position measurement with the oscillator energy constraint, which underlies noisy Gaussian homodyning in quantum optics, is attained on Gaussian encoding. The proof is based on general principles of convex programming. Rather remarkably, for this particular model the method reduces the solution of the optimization problem to a generalization of the celebrated log-Sobolev inequality. We hope that this method should work also for other models lying out of the scope of the “threshold condition” ensuring that the upper bound for the capacity as a difference between the maximum and the minimum output entropies is attainable.

1 Introduction

Quantum Shannon theory was rapidly developing during the past decades. As distinct from the classical case, quantum channel is characterized by a whole variety of different capacities, depending on the type of transmitted information (classical or quantum) and on additional resources used during transmission. Among these capacities, the classical capacity, i.e. the ultimate rate of reliable transmission of classical data via a quantum channel, plays a distinguished role, both historically and because of its central importance for quantum communications. A long-standing problem is the classical capacity
of bosonic Gaussian channels of various kinds. Hypothesis of Gaussian Max-
imizers (HGM) states that the full capacity of such channels is attained on
Gaussian encodings. The conjecture turned out remarkably difficult. After a
series of intermediate steps, a breakthrough was made in the papers [2], [3],
[24], where HGM was proved for important class of multimode gauge co-
or contra-variant channels (called phase-insensitive in quantum optics). In [12],
[20] the solution was extended to a much broader class of channels satisfying
certain “threshold condition”, essentially ensuring that the upper bound for
the capacity as a difference between the maximum and the minimum output
entropies is attainable. In [12], [19] these findings were extended to Gaussian
measurement (quantum-classical) channels. At the same time, HGM remains
open for rather large variety of quantum and quantum-classical channels ly-
ing beyond the scope of the threshold condition [17]. At this point we want
to mention that many authors (see e.g. [5], [25], [29], [22]) explored the
maximum information restricting to the class of Gaussian encodings. In the
absence of proof of HGM such results give only a lower bound for the full
classical capacity of the Gaussian channel.

Among these models there are rather elementary Gaussian channels, such
as approximate position measurement with the oscillator energy constraint,
which essentially underlies noisy Gaussian homodyning in quantum optics.
While this and related models were studied in earlier quantum communica-
tion papers (see e.g. [1], [6]), we could not find a proof of HGM for them in
the literature. In the present paper we give such a proof which is based on
general principles of convex programming. Rather remarkably, for this par-
ticular model the method reduces the solution of the optimization problem
to a generalization of log-Sobolev inequality [4], [23] – one of the highlights
of Analysis in the past century. We think that this method should work also
for other models out of the scope of the threshold condition, and hope to
address it in further publications.

2 The classical capacity of quantum measure-
ment

Let $\mathcal{H}$ be a Hilbert space of quantum system. Statistics of a quantum mea-
surement with the outcomes $y \in \mathcal{Y}$ is described by a probability operator-
valued measure $M = \{M(dy)\}$ (POVM) on $\mathcal{H}$, that is $M(dy) \geq 0$ and
\[ \int M(dy) = I \] (the unit operator on \( \mathcal{H} \)). As shown e.g. in [15], under mild separability assumptions there exists a countably finite measure \( \mu(dy) \) such that for any density operator \( \rho \) the distribution of the measurement outcomes \( \text{Tr}\rho M(dy) \) is absolutely continuous w.r.t. \( \mu(dy) \), thus having the probability density \( p_{\rho}(y) \). The affine map \( M: \rho \to p_{\rho}(\cdot) \) will be called the measurement channel.

An ensemble (encoding) \( \mathcal{E} = \{\pi(dx), \rho(x)\} \) consists of probability measure \( \pi(dx) \) on a “alphabet” \( \mathcal{X} \) and a measurable family of density operators (quantum states) \( x \to \rho(x) \) on the Hilbert space \( \mathcal{H} \) of the quantum system. The average state of the ensemble is the barycenter of this measure

\[
\bar{\rho}_\mathcal{E} = \int_{\mathcal{X}} \rho(x) \pi(dx).
\]

The classical Shannon information between the input \( x \) and the measurement outcome \( y \) is equal to

\[
I(\mathcal{E}, M) = \int \int \pi(dx) \mu(dy) p_{\rho(x)}(y) \ln \frac{p_{\rho(x)}(y)}{p_{\bar{\rho}_\mathcal{E}}(y)}
\]

In what follows we will consider POVMs having (uniformly) bounded operator density, \( M(dy) = m(y)\mu(dy) \), with \( \|m(y)\| \leq b \), so that the probability densities \( p_{\rho}(y) = \text{Tr}\rho m(y) \) are uniformly bounded, \( 0 \leq p_{\rho}(y) \leq b \). Moreover, by including \( b \) into \( \mu(dy) \), we can assume without loss of generality that \( b = 1 \). Then the output differential entropy

\[
h_M(\rho) = - \int p_{\rho}(y) \ln p_{\rho}(y) \mu(dy)
\]

is well defined with values in \([0, +\infty]\) (see [15] for the detail).

Next we define the quantity (11):

\[
e_M(\rho) = \inf_{\mathcal{E}:\bar{\rho}_\mathcal{E}=\rho} \int h_M(\rho(x))\pi(dx),
\]

which is an analog of the convex closure of the output differential entropy for a quantum channel [28].

Let \( H \) be a Hamiltonian in the Hilbert space \( \mathcal{H} \) of the quantum system, \( E \) a positive number. Then the energy-constrained classical capacity of the measurement channel \( M \) is

\[
C(M, H, E) = \sup_{\mathcal{E}:\text{Tr}\rho_\mathcal{E} H \leq E} I(\mathcal{E}, M),
\]
where maximization is over the input ensembles of states \( \mathcal{E} \) satisfying the energy constraint \( \text{Tr}\bar{\rho}_E H \leq E \). As shown in [20], proposition 1, it is indeed the capacity in the sense of information theory, i.e. the ultimate rate of the classical asymptotically reliable data transmission via the measurement channel. Note that the measurement channel is entanglement-breaking [14] hence its classical capacity is additive and is given by the one-shot expression [3].

If \( h_M(\bar{\rho}_E) < +\infty \), then

\[
I(\mathcal{E}, M) = h_M(\bar{\rho}_E) - \int h_M(\rho(x))\pi(dx).
\]

(4)

By using (4), (2), we obtain

\[
C(M, H, E) = \sup_{\rho : \text{Tr}\rho H \leq E} \left[ h_M(\rho) - e_M(\rho) \right]
\]

(5)

\[
\leq \sup_{\rho : \text{Tr}\rho H \leq E} h_M(\rho) - \inf_{\rho} h_M(\rho).
\]

There are important cases where the last inequality turns into equality thus giving the value of the capacity. This happens when the maximizer of the first term can be represented as a mixture of (pure) states minimizing \( h_M(\rho) \) [10], [20]. In particular, all the instances where the Hypothesis of Gaussian Maximizer was proved for Gaussian channels so far refer to that case. In the present paper we propose a method allowing to prove this hypothesis for the first case where this condition is violated and the inequality in (5) is strict and hence becomes useless. In sec. 5 we illustrate it on the example of approximate position measurement (corresponding to noisy homodyning in quantum optics) where this violation happens in the most extreme form.

### 3 Reduction to a convex programming problem

In this section we propose a method of computation of the quantity \( e_M(\rho) \) based on similarity of the optimization problem in the right side of (2) and general quantum Bayes estimation problem [8], [9], [11]. Consider a measurement channel given by the map \( M : \rho \rightarrow p_\rho(y) = \text{Tr}\rho m(y) \), where \( m(y) \) is a uniformly bounded positive-operator-valued function of \( y \in \mathcal{Y} \), such that \( \int m(y)\mu(dy) = I \). Any ensemble \( \mathcal{E} = \{\pi(dx), \rho(x)\} \) where \( x \in \mathcal{X} \), can be
equivalently considered as a probability distribution \( \pi(d\rho) \) on the whole set of quantum states \( \mathcal{S} = \mathcal{S}(\mathcal{H}) \) (with the carrier concentrated of the states \( \rho(x), x \in \mathcal{X} \)). Another equivalent description of \( \mathcal{E} \) is given by the positive (but not probability!) operator-valued measure \( \Pi(d\rho) = \rho \pi(d\rho) \) with values in \( \mathcal{S} \). The average state is then

\[
\bar{\rho}_E = \int_{\mathcal{S}} \rho \pi(d\rho) = \Pi(\mathcal{S}),
\]

and the minimized functional

\[
F(\mathcal{E}) = \int_{\mathcal{S}} h(p_\rho) \pi(d\rho) = \int_{\mathcal{S}} \text{Tr} K(\rho) \Pi(d\rho),
\]

where

\[
K(\rho) = -\int m(y) \ln p_\rho(y) \mu(dy). \tag{6}
\]

By fixing an average state \( \bar{\rho} \), we arrive at the optimization problem

\[
\int_{\mathcal{S}} \text{Tr} K(\rho) \Pi(d\rho) \longrightarrow \min
\]

\[
\Pi(d\rho) \geq 0 \]

\[
\Pi(\mathcal{S}) = \bar{\rho},
\]

which is formally similar to one arising in the general quantum Bayes problem [9], [11]. The minimized functional is affine in \( \mathcal{E} = \{\Pi(d\rho)\} \) and the constraints are convex, so it is a convex programming problem. Under certain regularity condition the problem was investigated in [11], where the following necessary and sufficient conditions for optimality of an ensemble \( \mathcal{E}_0 = \{\Pi_0(d\rho)\} \) were given, which we reproduce here formally:

There exists Hermitian operator \( \Lambda_0 \) such that

(i) \( \Lambda_0 \leq K(\rho) \) for all \( \rho \in \mathcal{S} \);

(ii) \( [K(\rho) - \Lambda_0] \Pi_0(d\rho) = 0 \).

Moreover, \( \Lambda_0 \) is the solution of the dual problem

\[
\max \{ \text{Tr} \bar{\rho} \Lambda : \Lambda^* = \Lambda, \ \Lambda \leq K(\rho) \text{ for all } \rho \in \mathcal{S} \}. \tag{7}
\]

By integrating (ii), we obtain the equation for determination of \( \Lambda_0 \)

\[
\int_{\mathcal{S}} K(\rho) \Pi_0(d\rho) = \int_{\mathcal{S}} K(\rho) \rho \pi_0(d\rho) = \Lambda_0 \bar{\rho}. \tag{8}
\]
The sufficiency of the conditions (i), (ii) is easy to demonstrate formally (cf. [8]): for any $E = \{\Pi(d\rho)\}$

$$F(E) = \int_E \text{Tr} K(\rho)\Pi(d\rho) \geq \int_E \text{Tr} \Lambda_0\Pi(d\rho) = \text{Tr} \Lambda_0\mathcal{P} = \int_E \text{Tr} K(\rho)\Pi_0(d\rho) = F(E_0).$$

Coming back to the parametric representation $E = \{\pi(dx), \rho(x)\}$, we can write the condition (ii) as

$$[K(\rho(x)) - \Lambda_0] \rho(x) = 0, \quad \text{a.e. } x \in \mathcal{X},$$

which means that the equality holds a.e. with respect to the measure $\pi_0(dx)$. The equation (8) becomes

$$\int K(\rho(x)) \rho(x) \pi_0(dx) = \Lambda_0\mathcal{P}. \quad (9)$$

In the case of the measurement channel $M : \rho \rightarrow p_\rho(y) = \text{Tr} \rho m(y)$, this equation reduces to

$$- \int \int m(y) \ln p_\rho(x)(y) \rho(x) \mu(dy) \pi_0(dx) = \Lambda_0\mathcal{P}.$$

In specific applications, like HGM for bosonic Gaussian channel, the candidate for an optimal encoding usually can be found by optimizing in the class of Gaussian encodings. Then the condition (ii) for this candidate can be verified and the operator $\Lambda_0$ found, while a major difficulty may be the check of the operator inequality (i).

### 4 Gaussian systems

We will systematically use some notations and results from the books [13], [14]. Consider the finite-dimensional symplectic vector space $(Z, \Delta)$ with $Z = \mathbb{R}^{2s}$ and the canonical symplectic matrix

$$\Delta = \text{diag} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{j=1,\ldots,s}. \quad (10)$$
In what follows \( \mathcal{H} \) will be the space of an irreducible representation \( z \to W(z); z \in Z \), of the canonical commutation relations (CCR)

\[
W(z)W(z') = \exp[-\frac{i}{2}z't \Delta z'] W(z + z').
\] (11)

Here \( W(z) = \exp i Rz \) are the unitary Weyl operators with the generators

\[
Rz = \sum_{j=1}^{s} (x_j q_j + y_j p_j), \quad z = [x_j y_j]_{j=1,...,s}^{t}
\] (12)

and \( R = [q_1, p_1, \ldots, q_s, p_s] \) is the row vector of the bosonic position-momentum observables, satisfying the canonical commutation relation

\[
q_j p_k - p_k q_j \subseteq i \delta_{jk} I, \quad j, k = 1, \ldots, s.
\]

In quantum communication theory, \( q_j, p_j \) describe the relevant modes of the field on receiver’s aperture (see, e.g. [26]). A number of analytical complications related to unboundedness of operators unavoidably arises in connection with Bosonic systems and CCR. In our treatment of CCR we focus on the algebraic aspects essential for applications while a presentation of the related analytical detail such as domains of definition, selfadjointness etc. can be found in the literature.

The displacement operators \( D(z) = W(-\Delta^{-1}z) \) satisfy the equation that follows from the canonical commutation relations (11)

\[
D(z)^* W(w) D(z) = \exp (i w^t z) W(w).
\] (13)

The quantum Fourier transform of a trace class operator \( \rho \) is defined as

\[
\text{Tr} \rho D(w)
\]

When \( \rho \) is a density operator, this is called the quantum characteristic function of the state \( \rho \). The quantum Parceval formula holds [13]:

\[
\text{Tr} \rho \sigma^* = \int \text{Tr} \rho D(w) \overline{\text{Tr} \sigma D(w)} \frac{d^{2s} w}{(2\pi)^s},
\] (14)

---

1See e.g. [13], [14] for the detail of mathematical treatment of expressions with the unbounded operators related to \( R \).
From now on we will consider states $\rho$ with finite second moments:

$$\sum_{j=1}^s \left( \text{Tr}\rho q_j^2 + \text{Tr}\rho p_j^2 \right) \equiv \text{Tr}\rho RR^t < \infty.$$  

The set of these states will be denoted $\mathfrak{S}_2$. For such states the matrix of second moments is defined as

$$\alpha^{(2)} = \text{Re Tr} R^t \rho R, \tag{15}$$

and the covariance matrix as

$$\alpha = \text{Re Tr} (R - m)^t \rho (R - m) = \alpha^{(2)} - m^t m \leq \alpha^{(2)},$$

where $m = \text{Tr}\rho R$ is the row-vector of the first moments (the mean vector).

It is a real symmetric $2s \times 2s$-matrix satisfying

$$\alpha \geq \pm \frac{i}{2} \Delta, \tag{16}$$

The state is centered if $m = 0$. For centered states the covariance matrix and the matrix of second moments coincide and are equal to $\alpha^{(2)}$.

A Gaussian state $\rho_{m,\alpha}$ is determined by its quantum characteristic function

$$\text{Tr} \rho_{m,\alpha} W(z) = \exp \left( im^t z - \frac{1}{2} z^t \alpha z \right). \tag{17}$$

Here $\alpha$ is the covariance matrix and $m$ is the mean vector. For a centered state we denote $\rho_\alpha = \rho_{0,\alpha}$.

For $\rho \in \mathfrak{S}_2$ we have $h_M(\rho) \leq h_M(\rho_\alpha) < +\infty$, where $\alpha$ is the matrix of the second moments of the state $\rho$ by the maximum entropy principle. With the quadratic Hamiltonian

$$H = R\epsilon R^t, \tag{18}$$

where $\epsilon$ is real positive definite $2s \times 2s$-matrix, the energy constraint reduces to

$$\text{Sp} \alpha \epsilon \leq E. \tag{19}$$

We denote the set of all states $\rho$ with the fixed matrix of second moments $\alpha$ by $\mathfrak{S}(\alpha)$ and we will study the following $\alpha$-constrained capacity

$$C(M;\alpha) = \sup_{\mathcal{E}:\rho\in\mathfrak{S}(\alpha)} I(\mathcal{E}, M) = \sup_{\rho\in\mathfrak{S}(\alpha)} \left[ h_M(\rho) - e_M(\rho) \right]. \tag{20}$$

\textsuperscript{2}We denote Sp trace of $s \times s$-matrices as distinct from trace of operators on $\mathcal{H}$. 

The energy-constrained classical capacity \( C(M; H, E) \) of the measurement channel \( M \) is
\[
C(M; H, E) = \sup_{\alpha: S\alpha \leq E} C(M; \alpha).
\]

A Gaussian measurement channel \( M \) in the sense of [14], [18] is defined via the operator-valued characteristic function of the form
\[
\int e^{iz^t w} M(dz) = \exp \left( i R K w - \frac{1}{2} w^t \beta w \right), \tag{21}
\]
where \( K \) is a scaling matrix, \( \beta \) is the measurement noise covariance matrix, \( \beta \geq \pm \frac{1}{2} K^t \Delta K \). The case \( K = I_{2s} \) (as well as of a general nondegenerate \( K \)) corresponds to the type 1 Gaussian measurement channel (with the multimode noisy heterodyning, see e.g [1], [26] as the prototype). However (21) includes also type 2 and 3 measurement channels (noisy and noiseless multimode homodyning) in which case \( K \) is a projection onto an isotropic subspace of \( Z \) (i.e. one on which the symplectic form \( \Delta \) vanish). The following theorem was proved in [17]:

**Theorem 1.** Let \( M \) be a general Gaussian measurement channel. The optimizing density operator \( \rho \) in (20) is a (centered) Gaussian density operator \( \rho_\alpha \):
\[
C(M; \alpha) = h_M(\rho_\alpha) - e_M(\rho_\alpha), \tag{22}
\]
and hence for a quadratic Hamiltonian [18]
\[
C(M, H, E) = \max_{\alpha: S\alpha \leq E} C(M; \alpha) = \max_{\alpha: S\alpha \leq E} [h_M(\rho_\alpha) - e_M(\rho_\alpha)]. \tag{23}
\]

The theorem asserts that the average state of an optimal encoding for a Gaussian measurement is Gaussian but says nothing about the detailed structure of the ensemble. It is well known that a non-Gaussian ensemble can have Gaussian average state (a canonical example is ensemble of the Fock states with the geometric distribution).

**Hypothesis of Gaussian Maximizers (HGM):** Let \( M \) be an arbitrary Gaussian measurement channel. Then there exists an optimal ensemble for (2) and hence for (3) which is Gaussian, more precisely it consists of (properly squeezed) coherent states with the displacement parameter having Gaussian probability distribution.

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For Gaussian measurement channels of the type 1 the minimum output differential entropy
\[
\inf_\rho h_M(\rho)
\]
is attained on a pure Gaussian (i.e. squeezed) state (this follows from the result of [3] applied to the entropy function and the complex structure associated with \(M\)). If in addition the Gaussian state \(\rho_\alpha\) satisfies the “threshold condition” which means that the covariance matrix \(\alpha\) dominates the covariance matrix of the entropy-minimizing squeezed state, then \(\rho_\alpha\) can be represented as a Gaussian mixture of these squeezed states, thus giving the optimal ensemble. This implies the validity of the HGM and an efficient computation of the \(\alpha\)-constrained capacity as
\[
C(M, H, E) = h_M(\rho_\alpha) - \min_\rho h_M(\rho)
\]
see [20]. On the other hand, this does not work when the “threshold condition” is violated, and notably, for all Gaussian measurement channels of the type 2 (noisy homodyning), with the generic example of the energy-constrained approximate measurement of the position \(q = [q_1, \ldots, q_s]\) subject to Gaussian noise. In the section 6 we will apply the method from section 3 to prove the HGM in this case for a single mode system. Our strategy will be the computation of \(e_M(\rho_\alpha)\) with the optimality conditions of that section and relying on the formula (23).

**Remark 1.** In the case of the oscillator-type Hamiltonian
\[
H = \sum_{j,k=1}^s \left( q_j \epsilon_j q_k^\dagger + p_j \epsilon_j p_k^\dagger \right),
\]
where \(q = [q_1, \ldots, q_s], p = [p_1, \ldots, p_s]\), the energy constraint is
\[
\text{Sp} \epsilon_q \alpha_q + \text{Sp} \epsilon_p \alpha_p \leq E.
\]
Then the maximization in (23) can be taken over only block-diagonal covariance matrices \(\alpha = \begin{bmatrix} \alpha_q & 0 \\ 0 & \alpha_p \end{bmatrix}\). The argument relying upon concavity of the capacity as the function of the average state of the ensemble [27], is similar to one given in sec. 4 of [21] for the case of entanglement-assisted capacity.
5 The classical capacity of approximate position measurement

The approximate (unsharp) measurement of position $q$ in one mode $q,p$ (a mathematical model for noisy homodyning) is given by POVM $M(dy) = m(y)dy$, where

$$m(y) = \frac{1}{\sqrt{2\pi\beta}} \exp \left[ -\frac{(q - y)^2}{2\beta} \right] = \frac{1}{\sqrt{2\pi\beta}} D(y) \exp \left[ -\frac{q^2}{2\beta} \right] D(y)^*, \quad (26)$$

where $\beta > 0$ is the power of the Gaussian noise, $D(y) = \exp (-iyp)$. The Gaussian measurement channel given by this POVM acts on a centered Gaussian state $\rho_\alpha$ with the covariance matrix $\alpha = \begin{bmatrix} \alpha_q & 0 \\ 0 & \alpha_p \end{bmatrix}$ by the formula

$$M : \rho_\alpha \rightarrow \exp \left[ -\frac{y^2}{2(\alpha_q + \beta)} \right] \frac{dy}{\sqrt{2\pi(\alpha_q + \beta)}}. \quad (27)$$

Take the oscillator Hamiltonian $H = \frac{1}{2} (q^2 + p^2)$. The problem is to compute the classical capacity

$$C(M,H,E) = \max_{E : \text{Tr} \rho E H \leq E} I(E, M), \quad (28)$$

and the maximization is over the input ensembles of states $E$ (encodings) satisfying the energy constraint $\text{Tr} \rho E H \leq E$. Remark 1 above shows that we can restrict to ensembles with average state having the diagonal matrix $\alpha$ as above.

In other words, one makes the “classical” measurement of the observable

$$Y = q + \xi, \quad \xi \sim \mathcal{N}(0,\beta),$$

with the quantum energy constraint $\text{Tr} \rho (q^2 + p^2) \leq 2E$, aiming to transmit the maximum information. The difficulty is that one measures $q$, while imposing the constraint on the energy $H = \frac{1}{2} (q^2 + p^2)$, involving an implicit constraint on $p$ which does not commute with $q$.

As we have mentioned, there is no general “Gaussian maximizer” result for $C$ in such cases, therefore we will first find the maximum over (special) Gaussian ensembles. The final goal will be to prove the HGM showing thus
that the found Gaussian ensemble is in fact a solution of the capacity problem (28) by checking the optimality conditions from sec. 3.

**HGM for approximate position measurement** [16]: the maximum is attained on the Gaussian ensemble $\mathcal{E}_{\text{gauss}} = \{\pi_0(dx), \rho_0(x)\}$ where $\rho_0(x) = |x\rangle_\delta \langle x|$ is the pure Gaussian (squeezed) state with the vector $|x\rangle_\delta = D(x) |0\rangle_\delta$, and the squeezed vacuum has zero mean and the following second moments:

$$\delta \langle 0| q^2 |0\rangle_\delta = \delta, \quad \text{Re} \delta \langle 0| qp |0\rangle_\delta = 0, \quad \delta \langle 0| p^2 |0\rangle_\delta = \frac{1}{4\delta}.$$  

The distribution $\pi_0(dx) = \frac{1}{\sqrt{2\pi\gamma}} \exp\left[-\frac{x^2}{2\gamma}\right] dx$. For the fixed centered Gaussian state $\rho_\alpha$ with the covariance matrix $\alpha = \begin{bmatrix} \alpha_q & 0 \\ 0 & \alpha_p \end{bmatrix}$, in order that the average state of the ensemble to be $\int \rho(x) \pi_0(dx) = \rho_\alpha$ the parameters should satisfy $\frac{1}{4\beta} = \alpha_p$, $\delta + \gamma = \alpha_q$, whence

$$\delta = \frac{1}{4\alpha_p}, \quad \gamma = \alpha_q - \frac{1}{4\alpha_p}. \quad (29)$$

This ensemble encodes the information solely into the displacement $x$ of the position leaving the momentum intact.

Similarly to (27)

$$p_{\rho_0(x)}(y) = \frac{1}{\sqrt{2\pi(\beta + \delta)}} \exp\left[-\frac{(y-x)^2}{2(\beta + \delta)}\right]. \quad (30)$$

Using this and (27) we get the “Gaussian” values

$$h_M(\rho_\alpha) = \frac{1}{2} \ln (\alpha_q + \beta) + \frac{1}{2} \ln(2\pi e), \quad (31)$$

$$e_M(\rho_\alpha) = \frac{1}{2} \ln \left(\frac{1}{4\alpha_p} + \beta\right) + \frac{1}{2} \ln(2\pi e), \quad (32)$$

hence taking into account (23),

$$C_{\text{gauss}}(M; \alpha) = h_M(\rho_\alpha) - e_M(\rho_\alpha) = \frac{1}{2} \ln \frac{\alpha_q + \beta}{\frac{1}{4\alpha_p} + \beta}. \quad (33)$$

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The Gaussian constrained capacity is thus

\[ C_{\text{gauss}}(M, H, E) = \max \alpha \left\{ \frac{1}{2} \left[ \ln (\alpha_q + \beta) - \ln \left( \frac{1}{4\alpha_p} + \beta \right) \right] \right\} \]

(34)

where in the second line we took the maximal value \( \alpha_q = 2E - \alpha_p \). Differentiating, we obtain the equation for the optimal value \( \alpha_p \):

\[ 4\beta \alpha_p^2 + 2\alpha_p - (2E + \beta) = 0, \]

the positive solution of which is

\[ \alpha_p = \frac{1}{4\beta} \left( \sqrt{1 + 8E\beta + 4\beta^2} - 1 \right), \]

(35)

whence

\[ C_{\text{gauss}}(M, H, E) = \ln \left( \frac{\sqrt{1 + 8E\beta + 4\beta^2} - 1}{2\beta} \right). \]

(36)

The parameters of the optimal Gaussian ensemble are obtained by substituting the value (35) into (29) with \( \alpha_q = 2E - \alpha_p \).

The case of sharp position measurement (\( \beta = 0 \)) formally corresponding to \( M(dy) = \delta(q-y)dy \), is not included in the discussion above. Yet for \( \beta = 0 \) the formula (36) gives \( \delta = 1/4E \) and

\[ C(M, H, E) = C_{\text{gauss}}(M, H, E) = \ln 2E. \]

The last formula was obtained in the paper [6] where also a general upper bound

\[ \ln \left( \frac{1 + \frac{E - 1/2}{\beta + 1/2}}{\frac{1}{2}} \right) = \ln \left( \frac{2(E + \beta)}{1 + 2\beta} \right) \]

(37)

for (28) was given (Eq. (28) in [6], see also Eq. (5.39) in [1]) .

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\(^3\)Notably, the expression (36) is of the same type as the one obtained in [5] by optimizing the information from applying sharp position measurement to noisy optimally squeezed states (the author is indebted to M. J. W. Hall for this observation).
6 Checking the optimality condition

Starting from this section it will be convenient to use natural logarithms; then one can return to the binary logarithms if necessary.

**Theorem 2.** The Gaussian encoding $E_{gauss}$ is optimal for the approximate Gaussian position measurement $M$ and the oscillator energy constraint as described in previous section. Its constrained capacity $C(M, H, E)$ is equal to (36).

**Proof.** We use the method of sec. 3 and check the optimality conditions (i), (ii) for the Gaussian encoding $E_{gauss} = \{\pi_0(dx), \rho_0(x)\}$. We start with computation of $\Lambda_0$ for $E_{gauss}$.

By (6), (26), (30),

$$K(\rho_0(x)) = \int \frac{1}{\sqrt{2\pi\beta}} \exp\left[-\frac{(q-y)^2}{2\beta}\right] \left[\ln\sqrt{2\pi(\beta+\delta)} + \frac{(y-x)^2}{2(\beta+\delta)}\right] dy$$  

$$= c + \frac{(q-x)^2 + \beta}{2(\beta+\delta)}, \quad c = \ln\sqrt{2\pi(\beta+\delta)}.$$
Hence

\[ K(\rho_0(x))\rho_0(x) = \left[ c + \frac{(q - x)^2 + \beta}{2(\beta + \delta)} \right] D(x) \langle 0 \rangle_\delta \langle x \rangle \]

\[ = D(x) \left[ c + \frac{q^2 + \beta}{2(\beta + \delta)} \right] \langle 0 \rangle_\delta \langle x \rangle \]

\[ = D(x) \left[ c + \frac{\beta}{2(\beta + \delta)} + \frac{\delta}{(\beta + \delta)} \left( \frac{q^2}{2\delta} + 2\delta p^2 \right) - \frac{2\delta^2 p^2}{(\beta + \delta)} \right] \langle 0 \rangle_\delta \langle x \rangle. \]

Taking into account that the squeezed vacuum \( |0\rangle_\delta \) is the ground state for the corresponding oscillator Hamiltonian

\[ \left( \frac{q^2}{2\delta} + 2\delta p^2 \right) \langle 0 \rangle_\delta = \langle 0 \rangle_\delta, \]

and that \( D(x) \) commute with \( p^2 \), we have

\[ K(\rho_0(x))\rho_0(x) = \left[ c + \frac{\beta + 2\delta}{2(\beta + \delta)} - \frac{2\delta^2 p^2}{(\beta + \delta)} \right] \langle x \rangle_\delta \langle x \rangle. \]

Integrating over \( \pi_0(dx) \), and taking into account that \( \int \langle x \rangle_\delta \langle x \rangle \pi_0(dx) = \rho_\alpha \), we obtain

\[ \int K(\rho_0(x))\rho_0(x)\pi_0(dx) = \left[ c + \frac{\beta + 2\delta}{2(\beta + \delta)} - \frac{2\delta^2 p^2}{(\beta + \delta)} \right] \rho_\alpha. \]

Comparing with (38), we obtain

\[ \Lambda_0 = c + \frac{\beta + 2\delta}{2(\beta + \delta)} - \frac{2\delta^2 p^2}{(\beta + \delta)}. \]  

(38)

By construction, this is Hermitian operator satisfying \( [K(\rho_0(x)) - \Lambda_0] \rho_0(x) = 0 \), i.e. the condition (ii) of section 3.

To check the condition (i) it is sufficient to prove

\[ \langle \psi | \Lambda_0 | \psi \rangle \leq \langle \psi | K(\rho) | \psi \rangle \]  

(39)

for arbitrary density operator \( \rho \) and a dense subset of \( \psi \in \mathcal{H} \). We can assume
that $\psi$ is a unit vector. Since

$$\langle \psi | K(\rho) | \psi \rangle = - \int \langle \psi | m(y) | \psi \rangle \ln \text{Tr} m(y)$$

$$= - \int \langle \psi | m(y) | \psi \rangle \ln \langle \psi | m(y) | \psi \rangle$$

$$+ \int \langle \psi | m(y) | \psi \rangle \ln \frac{\langle \psi | m(y) | \psi \rangle}{\text{Tr} m(y)}$$

$$\geq - \int \langle \psi | m(y) | \psi \rangle \ln \langle \psi | m(y) | \psi \rangle,$$

due to nonnegativity of the relative entropy of the two probability densities. Thus (39) will follow if we prove

$$\langle \psi | \Lambda_0 | \psi \rangle \leq - \int \langle \psi | m(y) | \psi \rangle \ln \langle \psi | m(y) | \psi \rangle \, dy$$

(40)

for unit vectors $\psi$. With $\Lambda_0$ given by (38) it amounts to

$$\int \langle \psi | m(y) | \psi \rangle \ln \langle \psi | m(y) | \psi \rangle \, dy + \ln \sqrt{2\pi} (\beta + \delta) + \frac{\beta + 2\delta}{2 (\beta + \delta)}$$

$$\leq \frac{2\delta^2}{(\beta + \delta)} \int |\psi(x)|^2 \, dx.$$  (41)

The proof of this inequality is the subject of the following section.

7 A generalization of log-Sobolev inequality

Let $f(x) = |\psi(x)|^2$ be a smooth probability density on $\mathbb{R}$ and

$$T_t f(y) = \frac{1}{\sqrt{2\pi t}} \int \exp \left( -\frac{(y-x)^2}{2t} \right) f(x) \, dx.$$

Then the inequality we wish to prove, replacing $\beta$ by $t$:

$$\int T_t f(y) \ln T_t f(y) \, dy + \ln \sqrt{2\pi e (t + \delta)} + \frac{\delta}{2 (t + \delta)} \leq \frac{2\delta^2}{(t + \delta)} \int |\psi(x)|^2 \, dx$$

(42)
for $t, \delta \geq 0$. For $t = 0, \delta = 1$ this is the logarithmic Sobolev inequality \cite{4}. For $t = 0, \delta > 0$ it can be obtained by a change of variable (see also (44) below).

Proof of (42). We start from the version of the log-Sobolev inequality in \cite{23} (with dimensionality $n = 1$):

\[
\int |\psi(x)|^2 \ln \frac{|\psi(x)|^2}{\|\psi\|_2^2} \, dx + \ln a + 1 \leq \frac{a^2}{\pi} \int |\psi(x)|^2 \, dx. \tag{43}
\]

Let $\|\psi\|_2 = 1$ then $f(x) = |\psi(x)|^2$ is a probability density, $\int f(x) \, dx = 1$. Also take $a = \sqrt{2\pi\delta}$, then (43) becomes

\[
\int f(x) \ln f(x) \, dx + \ln \sqrt{2\pi\delta} + 1 \leq 2 \delta \int |\psi'(x)|^2 \, dx, \tag{44}
\]

which is the same as (42) for $t = 0$.

Denote

\[
F(t, \delta) = (t + \delta) \int T_t f(x) \ln T_t f(x) \, dx \\
+ (t + \delta) \ln \sqrt{2\pi e (t + \delta)} + \frac{\delta}{2} - 2\delta \int |\psi'(x)|^2 \, dx.
\]

We have to prove

\[
F(t, \delta) \leq 0; \quad t, \delta > 0. \tag{45}
\]

We have just proved that $F(0, \delta) \leq 0$. If we prove that $\frac{\partial}{\partial t} F(t, \delta) \leq 0$, then (45) and hence (42) will follow. We have

\[
\frac{\partial}{\partial t} F(t, \delta) = \int T_t f(x) \ln T_t f(x) \, dx \\
+ (t + \delta) \int [\ln T_t f(x) + 1] \frac{\partial}{\partial t} T_t f(x) \, dx + \ln \sqrt{2\pi e (t + \delta)} + \frac{1}{2}.
\]

Taking into account that $\frac{\partial}{\partial t} T_t f(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} T_t f(x)$ and integrating by parts in the second integral, we can transform it as

\[
\int [\ln T_t f(x) + 1] \frac{\partial}{\partial t} T_t f(x) \, dx \\
= -\frac{1}{2} \int \frac{\partial}{\partial x} [\ln T_t f(x) + 1] \frac{\partial}{\partial x} T_t f(x) \, dx \\
= -2 \int |\frac{\partial}{\partial x} \sqrt{T_t f(x)}|^2 \, dx.
\]
Denote $g(x) = T_t f(x)$, then it is also a probability density, and denoting $t + \delta = \tilde{\delta}$ we obtain

$$\frac{\partial}{\partial t} F(t, \delta) = \int g(x) \ln g(x) dx + \ln \sqrt{2\pi\tilde{\delta}} + 1 - 2\tilde{\delta} \int \left| \frac{d}{dx} \sqrt{g(x)} \right|^2 dx.$$  

However by (44) this is nonpositive. Thus (45) and hence (42) follows. This also completes the proof of theorem 2.

8 Comment on the proof of the estimate for the convex closure of the output entropy

The sufficient conditions for optimality from section 3 were applied in our proof of theorem 6 to unbounded operators and thus require a corresponding refinement. While this can be done in general, here we wish point out that given the inequality (41), there is another, direct way to rigorous proof of theorem 6. Then the merit of the convex programming approach is in that it allowed to generate the conjectured inequality (41).

First, we note that (41) can be extended to functions $\psi$ from the Sobolev space $H^1(\mathbb{R})$, which are square-integrable along with its first generalized derivative $\psi'$. This space is the natural domain of definition of the momentum operator $p$.

To complete the proof of theorem 6 in view of (22) and (31), we have only to prove that

$$e_M(\rho_\alpha) \equiv \inf_{\mathcal{E}, \rho_\alpha} \int h_M(\rho(\xi)) \pi(d\xi) = \frac{1}{2} \ln \left( \frac{1}{4\alpha_p} + \beta \right) + \frac{1}{2} \ln 2\pi e,$$

(46)

where the infimum is taken over encodings $\mathcal{E} = \{ \pi(d\xi), \rho(\xi) \}$ satisfying $\rho_\alpha = \rho_\alpha$. The concavity of $h_M(\rho)$ implies that we can restrict to ensembles of pure states $\rho(\xi) = |\psi_\xi\rangle \langle \psi_\xi|$, so that

$$\int |\psi_\xi\rangle \langle \psi_\xi| \pi(d\xi) = \rho_\alpha,$$

since we can always perform the convex decomposition for all density operators $\rho(\xi)$ into pure states without changing the barycenter and without
increasing the value of the minimized functional. It follows that
\[
\int \|p\psi_\xi\|^2 \pi(d\xi) = \text{Tr} \rho_\alpha p^2 < \infty,
\]  
(47)
hence \(\psi_\xi \in H^1(\mathbb{R})\) for \(\pi\)-almost all \(\xi\). Applying the inequality (41) to \(\psi_\xi\) and rearranging terms, we get
\[
h_M(|\psi_\xi\rangle \langle \psi_\xi|) \geq \ln \sqrt{2\pi (\beta + \delta)} + \frac{\beta + 2\delta}{2(\beta + \delta)} - \frac{2\delta^2}{\beta + \delta} \|p\psi_\xi\|^2.
\]  
(48)
Integrating with respect to \(\pi(d\xi)\) and taking into account (47) we get
\[
\int h_M(\rho(\xi))\pi(d\xi) \geq \ln \sqrt{2\pi (\beta + \delta)} + \frac{\beta + 2\delta}{2(\beta + \delta)} - \frac{2\delta^2}{\beta + \delta} \alpha_p.
\]
With \(\delta = \frac{1}{4\alpha_p}\) we get the value at the right-hand side of (46), which is attained for the encoding of theorem 6. This proves (46) and hence the theorem.

Similar comment applies to the proof of HGM for approximate joint position-momentum measurement channel (noisy heterodyning) in our subsequent e-print arXiv:2206.02133.

**Appendix**

Let us illustrate the inequality (42) for Gaussians
\[
f(x) = \frac{1}{\sqrt{2\pi a}} \int \exp \left(-\frac{x^2}{2a}\right).
\]  
(49)
Then (42) reduces to
\[
\ln \left(\frac{t + a}{t + \delta}\right) \geq \frac{\delta}{t + \delta} \left(1 - \frac{\delta}{a}\right)
\]
or introducing \(u = a/t, \ v = \delta/t\),
\[
\ln \left(\frac{1 + u}{1 + v}\right) - \frac{v}{1 + v} \left(1 - \frac{v}{u}\right) \geq 0; \ u, v \geq 0.
\]  
(50)
To prove this inequality, notice it becomes equality for \(u = v\). The derivative \(d/du\) is
\[
\frac{1}{1 + u} - \frac{v^2}{(1 + v)u^2} = \frac{(u - v)(u + v + uv)}{(1 + u)(1 + v)u^2}
\]  
(51)
which is \( \geq 0 \ (\leq 0) \) if \( u \geq v \ (u \leq v) \). Hence (50) follows. Also we have obtained that (42) is exact: it turns into equality for Gaussian (49) with \( a = \delta \).

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