FERMION-CURRENT BASIS AND CORRELATION FUNCTIONS FOR THE INTEGRABLE SPIN 1 CHAIN

C. BABENKO AND F. SMIRNOV

ABSTRACT. We use the fermion-current basis in the space of local operators for the computation of the expectation values for the integrable spin chain of spins 1. Our main tool consists in expressing a given local operators in the fermion-current basis. For this we use the same method as in the spin 1/2 case which is based on the arbitrariness of the Matsubara data.

1. Introduction

In the papers [1, 2] a method was described which allows to compute expectation values of local operators (up to 11 sites long) for the spin 1/2 isotropic spin chain. This method is based on the results of the paper [3] in which for the six-vertex model (possibly inhomogeneous one) the expectation values of local operators in the fermionic basis are computed in terms of a function $\omega$ defined by the Matsubara data. Let us describe briefly the method of [1, 2].

Every local operator allows a decomposition on the fermionic basis with the coefficients depending only on the operator in question. For sufficiently simple Matsubara data the expectation value of the operator can be computed in two ways: directly with the help of the algebraic Bethe ansatz and using the decomposition on the fermionic basis and the function $\omega$. This provides equations for the coefficients of the decomposition for any given Matsubara data. Repeating this procedure for sufficiently large number of Matsubara data one obtains a system of equations for the coefficients which allow to find them.

In the present paper we apply a similar method to the much more complicated case of the integrable isotropic spin-1 chain described by the Hamiltonian

\begin{equation}
H = \sum_{j=-\infty}^{\infty} \left( S_j^a S_{j+1}^a - \left( S_j^a S_{j+1}^a \right)^2 \right),
\end{equation}

where summation over $a$ is implied, $S^a$ are generators of the spin-1 representation of $\mathfrak{sl}_2$, whose expression will be given below. The infinite chain (called Space below) is understood as the limit of finite chains with periodic boundary conditions.

The correlation functions for the model (1.1) were studied in [3]. Later in [6] the problem was considered in the spirit of the fermionic basis construction [5]. The authors of [6] were very much influenced by [3]. In the present paper we use [3] in two ways: indirectly through [6], and directly, comparing exact results on 2 and 3 sites.

Below we formulate our problem. The exposition is close to that of the paper [1] where some more details can be found.

The integrable models are closely related with Quantum Groups, in particular the isotropic model under consideration is related to the $\mathfrak{sl}_2$ Yangian. We denote by $\pi_S$ the representation obtained as the tensor product of the spin-1 representations along the Space. In addition we introduce a finite, possibly inhomogeneous and carrying different
spins, Matsubara chain and corresponding representation $\pi_M$ of the Yangian. We visualise
the lattice on an infinite cylinder with the compact direction been the Matsubara space.
The fundamental object is the evaluation of the universal R-matrix $R$:
$$T_{S,M} = (\pi_S \otimes \pi_M) R.$$ 

The relation with the integrable spin chain is due to the commutativity
$$[H, \text{Tr}_M(T_{S,M})] = 0,$$
which reflects the fact that $H$ is just one element of a huge commutative algebra generated
by the transfer-matrices $\text{Tr}_M(T_{S,M})$ computed for all possible Matsubara chains.

Denote by $M_d$ the data for a given Matsubara chain (length, spins, inhomogeneities).
For a local operators $\mathcal{O}$ localised (acting non-trivially) on a finite subchain of the Space
chain, define the expectation value
$$\langle \mathcal{O} \rangle_{M_d} = \frac{\text{Tr}_S \text{Tr}_M(T_{S,M} \cdot \mathcal{O})}{\text{Tr}_S \text{Tr}_M(T_{S,M})}. \quad (1.2)$$

Using the results of the paper [6] it can be shown that there is a basis of the local
operators for the spin-1 chain created by the action on the unit operators by two fermions
and one Kac-Moody current (details will be given in the main text). We shall call this
the fermion-current basis. Denote the elements of the fermion-current basis by $v_\alpha$. For
any $\mathcal{O}$ we have
$$\mathcal{O} = \sum_\alpha X_\alpha v_\alpha,$$
where $X_\alpha$ are the wanted coefficients of the decomposition depending on the inhomogeneities of the Space. This implies
$$\langle \mathcal{O} \rangle_{M_d} = \sum_\alpha X_\alpha \langle v_\alpha \rangle_{M_d}.$$

For reasonable simple Matsubara data there are independent ways to compute
$\langle \mathcal{O} \rangle_{M_d}$ and $\langle v_\alpha \rangle_{M_d}$. This is how we get equations for $X_\alpha$.

2. Basis

2.1. Homogeneous case. We begin this section by making our notations more detailed.
We have the $\mathfrak{sl}_2$ Yangian $\mathcal{Y}$. Denote by $\pi^{2s}_\lambda$ the $(2s+1)$-dimensional evaluation representation
with the evaluation parameter $\lambda$. In order to handle $T_{S,M}$ in the definitions above,
we use the following expression for the $R$ matrix of the spin 1 chain $R(\lambda, \mu) = (\pi_\lambda^2 \otimes \pi_\mu^2) R$
which depends only on the difference of arguments $\zeta = \lambda - \mu$:
$$R(\zeta) = \begin{pmatrix}
(\zeta + 1)(\zeta + 2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta(\zeta + 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\zeta - 1)\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2(\zeta + 1) & 0 & \zeta(\zeta + 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\zeta + 1)(\zeta + 2)
\end{pmatrix}.$$

In the homogeneous case
$$\pi_S = \cdots \pi_0^2 \otimes \pi_0^2 \otimes \pi_0^2 \otimes \pi_0^2 \otimes \cdots .$$

As has been said we are supposed to begin with a finite periodic Space chain of length $2N$ and then consider the limit $N \to \infty$. However, it is well-known that in the cylindric
geometry adopted in this paper the limiting procedure is trivial, so, we shall consider the
Space chain as an infinite one. There is a well-known infinite family of commuting local integrals of motion which includes the Hamiltonian. The adjoint action of these operators is well-defined on the space of local operators. We denote by $\mathcal{V}$ the corresponding quotient space. For the problem considered in this paper this is the space of interest.

The simplest operator $I$ acts as a unit operator in every tensor component. In [6] several operators were introduced acting on the space of local operators, let us describe them. We start with the operators $j^-(\lambda), j^0(\lambda), j^+(\lambda), b^*(\lambda), c^*(\lambda)$, for which we shall often use the universal notation $x^{(1,2)} = b^*, x^{(2,1)} = c^*, x^{(1,3)} = j^+, x^{(2,2)} = j^0, x^{(3,1)} = j^-$. The indices $\{1, 2\}$ etc. are natural in the framework of [6]. All these operators are understood as generating functions

$$x^\varepsilon(\lambda) = \sum_{p=-\infty}^{\infty} \lambda^{p-1}x^\varepsilon_p.$$ 

It is almost correct that the space $\mathcal{V}$ is created by action of $x^\varepsilon_p$ with $p > 0$, but some refinements are needed. The first of them concerns the normal ordering. The operators $j^-(\lambda), j^0(\lambda), j^+(\lambda)$ form an $\hat{sl}_2$ Kac-Moody algebra at level 1. The fermions $b^*(\lambda), c^*(\lambda)$ form an $\hat{sl}_2$ doublet. That leads to the natural commutation relations and, most importantly for our goals, to the rules of the normal ordering:

$$\begin{align*}
: j^0(\lambda) j^0(\mu) := j^0(\lambda) j^0(\mu) - \frac{2}{(\lambda - \mu)^2}, & \quad : j^+(\lambda) j^-(\mu) := j^+(\lambda) j^-(\mu) + \frac{2 j^0(\mu)}{\lambda - \mu} + \frac{1}{(\lambda - \mu)^2}, \\
: j^+(\lambda) j^0(\mu) := j^+(\lambda) j^0(\mu) + \frac{2 j^+(\mu)}{\lambda - \mu}, & \quad : j^0(\lambda) j^-(\mu) := j^0(\lambda) j^-(\mu) + \frac{2 j^-(\mu)}{\lambda - \mu}, \\
b^*(\lambda) j^-(\mu) := b^*(\lambda) j^-(\mu) - \frac{c^*(\mu)}{\lambda - \mu}, & \quad : c^*(\lambda) j^+(\mu) := c^*(\lambda) j^+(\mu) + b^*(\lambda) j^-(\mu) + \frac{c^*(\mu)}{\lambda - \mu}, \\
b^*(\lambda) j^0(\mu) := b^*(\lambda) j^0(\mu) + \frac{b^*(\mu)}{\lambda - \mu}, & \quad : c^*(\lambda) j^0(\mu) := c^*(\lambda) j^0(\mu) - \frac{c^*(\mu)}{\lambda - \mu}.
\end{align*}$$

So, the local operators are created by acting on the unit operator by normal ordered products

$$: x^{p_1}_{i_1}\cdots x^{p_l}_{i_l} : I, \quad p_j > 0.$$ 

Introduce the ordering $\{1, 2\} \prec \{2, 1\} \prec \{1, 3\} \prec \{2, 2\} \prec \{3, 1\}$. For the sake of definiteness we shall require $\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_l$. The second problem is that of completeness. Contrary to the case of the spin 1/2 chain [7] we do not have a formal proof of the completeness in the present situation. On the other hand the Russian doll construction discussed below makes the completeness quite plausible.

Let us discuss now the most complicated issue. An important question is that of how the operators located exactly on the interval $[1, n]$ look like in our fermion-current basis. In the spin 1/2 case we had only fermionic operators $b^*_p, c^*_p$. For the operators

$$(2.1) \quad b^*_p \cdots b^*_p c^*_q \cdots c^*_q I,$$

to be located on $[1, n]$ one imposes first of all two necessary conditions:

$$(2.2) \quad \begin{align*}
1) & \quad k + l \leq n, \\
2) & \quad p_j \leq n, q_j \leq n \quad \forall j.
\end{align*}$$

Then there are more subtle necessary conditions explained in details in [11, 12]. Taking into account all the necessary conditions we come to the subspace of the fermionic space,
whose elements may be located on $[1,n]$, of rather reasonable size. Notice also that in [1] [2] as well as in the present paper we are interested in operators invariant under the action of global $\mathfrak{s}l_2$. This requires $k = l$ in [2,1].

For the spin 1 case, let us write the element of the fermion-current basis in complete notations

$$b^*_p \cdots b^*_{p_k} c^*_q \cdots c^*_{q_k} j^+_r \cdots j^+_r \ j^+_s \cdots j^+_s \ j^-_t \cdots j^-_t : I.$$  

(2.4)

There is one necessary condition which remains unchanged:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq n.$$  

(2.5)

The requirement of $\mathfrak{s}l_2$-invariance of the operators is equivalent to

$$\lambda_1 - \lambda_2 + 2\lambda_3 - 2\lambda_5 = 0.$$  

For fermions the condition [2,3] and additional conditions from [1,2] (null-vectors) still hold. However, we were not able to formulate reasonable conditions for the currents. That is why in what follows, we are forced to take much more complicated and less efficient ways to calculate the correlations functions of the fermion-current basis, than in [1,2].

2.2. Introducing Matsubara. The Matsubara chain is inhomogeneous

$$\pi_M = \pi_{r_1}^{2s_1} \otimes \pi_{r_2}^{2s_2} \otimes \cdots \otimes \pi_{r_L}^{2s_L}.$$  

Let us introduce the transfer-matrix

$$T_M(\lambda) = (\text{Tr} \otimes \text{id}) (\pi^{(2)}_L \otimes \pi_M)(\mathcal{R}).$$  

This is a commutative family, for generic Matsubara data there is a unique eigenvector with the maximal in absolute value eigenvalue of $T_M(0)$. We shall denote this eigenvector by $|\Psi\rangle$. The corresponding eigenvalue of the transfer-matrix will be denoted by $T(\lambda)$.

Clearly for any local operator located on the interval $[1,n]$ we have

$$\lim_{N \to \infty} \frac{\text{Tr}_S \text{Tr}_M (T_{S,M} : 0)}{\text{Tr}_S \text{Tr}_M (T_{S,M})} = \langle \Psi | \text{Tr}_{[1,n]} (T_{[1,n],M} \mathcal{O}) | \Psi \rangle \bigg/ T(0)^n \langle \Psi | \Psi \rangle,$$

(2.6)

where $T_{[1,n],M}$ is the restriction of $T_{S,M}$ for the Space taken to be the finite interval $[1,n]$, its explicit expression is given below for the inhomogeneous case. Our way of computing the right hand side does not depend on the fact that the eigenvalue is maximal being applicable to any eigenvector of the transfer-matrix.

2.3. Inhomogeneous case, Russian doll. The Russian doll construction is present indirectly already in the paper [4], however, in [6] it becomes really indispensable. The construction requires some definitions which we are going to give.

We shall need an inhomogeneous space chain:

$$\pi_S = \pi_0^2 \otimes \pi_0^2 \otimes \pi_1^2 \otimes \cdots \otimes \pi_{n_1}^2 \otimes \pi_0^2 \otimes \pi_0^2 \otimes \cdots.$$  

The inhomogeneity is located on a finite subchain $[1,n]$. Consider the space of all the operators located on this interval. Consider the expectation value [1,2] for the inhomogeneous case assuming that the local operator $\mathcal{O}$ is located on the interval $[1,n]$. Denote the corresponding spaces, isomorphic to $\mathbb{C}^3$, by $V_1, \cdots, V_n$.

In order to describe a suitable for our goals basis in $V_1 \otimes \cdots \otimes V_n$ we introduce nine operators $g^\epsilon(\lambda_k)$ ($\epsilon = \{i,j\}$, $i,j = 1,2,3$) and act by these operators on $I$ consequently:

$$g^{\epsilon_1}(\lambda_n) g^{\epsilon_{n-1}}(\lambda_{n-1}) \cdots g^{\epsilon_2}(\lambda_1) I.$$  

For generic $\lambda_1, \cdots, \lambda_n$ this gives a basis of the space of operators localised on the interval $[1,n]$. We have the equality $g^{\{1,1\}}(\lambda) = \text{id}$. The expectation values considered in the
These operators are such that in the weak sense (holding when considered in correlation functions)

\[ g^{3,3}\{(\lambda)\} = g^{1,1}\{(\lambda)\}, \quad g^{2,3}\{(\lambda)\} = g^{1,2}\{(\lambda)\}, \quad g^{3,2}\{(\lambda)\} = g^{2,1}\{(\lambda)\}. \]

So, effectively we are left with the same set of indices counting the operators \( g \) as we had before for \( x \).

As usual the monodromy matrix \((\pi^2_{\lambda j} \otimes \pi_{\mu} v_j) (R)\) with the first tensor component identified with \( V_j \) will be devoted by \( T_{j,M}(\lambda_j) \). The formula (2.6) remains valid for \( 0 \) being located on the interval \([1, n]\), and, certainly,

\[ T_{[1,n]} = T_{1,M}(\lambda_1) \cdots T_{n,M}(\lambda_n). \]

These operators \( g^w \) are in one-to-one correspondence with \( x \)'s. Wanting to pass to the homogeneous case one has to apply the normal ordering, the rules are the same as above. The Russian doll construction is based on the identity

\[ \lim_{N \to \infty} \frac{\text{Tr}_S \text{Tr}_M (T_{S,M} : x^{e_n}(\lambda_n) \cdots x^{e_1}(\lambda_1) : I)}{\text{Tr}_S \text{Tr}_M (T_{S,M})} = \frac{\langle \Psi | \text{Tr}_{[1,n]} (T_{1,M}(\lambda_1) \cdots T_{n,M}(\lambda_n) : g^{e_n}(\lambda_n) \cdots g^{e_1}(\lambda_1) : I) | \Psi \rangle}{\prod_{j=1}^n T(\lambda_j) \langle \Psi | \Psi \rangle}. \]

This formula establishes an identity between the expectation values of a family of local operators of different lengths for the homogeneous case with the expectation values for the operators of length \( n \) in the inhomogeneous case. For our goals, rather complicated reasonings concerning this formula which are given in [6] can be avoided just by saying that the explicit computation of the right hand side (which will be given soon for any Matsubara data), defines the operators \( x \) in the left hand side.

Still there is another way to apply this formula. Suppose one computes the right hand side and then sets all \( \lambda_j \) to zero. In that case the right hand side gives the expectation value of a local operator located on \([1, n]\) for the homogeneous chain, this allows to identify the local operators of length \( n \) in the left hand side. We shall explain how to apply this idea in practice later.

2.4. Fusion. Consider the tensor product of \( 2n \) two-dimensional spaces \( v_j \). Introduce the projector \( P_j : v_{2j-1} \otimes v_{2j} \to V_j \) onto the symmetric component. Consider the product \( \mathcal{P} = P_1 \otimes \cdots \otimes P_n \). Denote by \( T_{j,M}(\lambda) \) the monodromy matrix whose first tensor component acts in \( v_j \). We have the fusion

\[ T_{1,M}(\lambda_1 - 1/2) T_{2,M}(\lambda_1 + 1/2) \cdots T_{2n-1,M}(\lambda_n - 1/2) T_{2n,M}(\lambda_n + 1/2) \mathcal{P} = \mathcal{P} T_{1,M}(\lambda_1) \cdots T_{n,M}(\lambda_n). \]

We began to consider the tensor product of \( 2n \) spaces \( v_j \) isomorphic to \( \mathbb{C}^2 \). In the framework of the present paper the interest of this consideration is due to the fact that \( \pi^2_{\lambda_1} \otimes \cdots \otimes \pi^2_{\lambda_n} \) is a submodule of \( \pi_{\lambda_1 - 1/2} \otimes \pi_{\lambda_1 + 1/2} \otimes \cdots \otimes \pi_{\lambda_n - 1/2} \otimes \pi_{\lambda_n + 1/2} \). In what follows it will be useful to consider a more general module \( \pi_{\mu_1} \otimes \cdots \otimes \pi_{\mu_{2n}} \) with generic \( \mu_1, \cdots \mu_{2n} \) specialising to \( \mu_j = \lambda_{[j+1]} + \frac{(-1)^{j}}{2} \) when needed. We have operators \( g^\sigma(\mu_j) \) (\( \sigma = \{1, 2\}, \{2, 1\} \)) acting on the latter space. The Matsubara expectation values are
computed via a particular case of the main fermionic basis formula \[5\]:

\[
\langle \Psi | \text{Tr}_{[1,2n]} (T_{1,M}(\mu_1) \cdots T_{2n,M}(\mu_{2n}) g^{\sigma_2 n}(\mu_{2n}) \cdots g^{\sigma_1}(\mu_1) I) | \Psi \rangle = (-1)^{\text{sgn}(\pi)} \det |\omega(\mu_i, \mu_j)|_{i;\sigma_i=\{2,1\}, j;\sigma_j=\{1,2\}},
\]

where \(\pi\) is the permutation putting all \(i\) such that \(\sigma_i = \{2,1\}\) to the left. The functions \(\omega(\lambda, \mu)\) depends on the Matsubara data as on parameters. We shall not repeat the definition which can be found in [1, 2].

Using the formula above one computes the right hand side of (2.8) using the following formulae

\[
\begin{align*}
g^{(1,2)}(\lambda) &= g^{(1,2)}(\lambda + 1/2) + g^{(1,2)}(\lambda - 1/2), \\
g^{(2,1)}(\lambda) &= g^{(2,1)}(\lambda + 1/2) + g^{(2,1)}(\lambda - 1/2), \\
g^{(1,3)}(\lambda) &= g^{(1,2)}(\lambda + 1/2)g^{(1,2)}(\lambda - 1/2), \\
g^{(3,1)}(\lambda) &= g^{(2,1)}(\lambda + 1/2)g^{(2,1)}(\lambda - 1/2), \\
g^{(2,2)}(\lambda) &= g^{(2,1)}(\lambda + 1/2)g^{(1,2)}(\lambda - 1/2) + g^{(1,2)}(\lambda + 1/2)g^{(2,1)}(\lambda - 1/2).
\end{align*}
\]

It is important to notice that \(g^{\epsilon_n}(\lambda_n) \cdots g^{\epsilon_1}(\lambda_1) I\) in which \(g\) are defined by (2.10) satisfies the identity

\[
g^{\epsilon_n}(\lambda_n) \cdots g^{\epsilon_1}(\lambda_1) I = \mathcal{P} g^{\epsilon_n}(\lambda_n) \cdots g^{\epsilon_1}(\lambda_1) I,
\]

which provides the self-consistence of the fusion.

This procedure expresses the right hand side of (2.8) in terms of determinants of matrices with the matrix elements being expressed in terms of the function \(\omega(\lambda, \mu)\) and the normalisation

\[
N(\lambda) = \frac{T(\lambda)}{T(\lambda + \frac{1}{2})T(\lambda - \frac{1}{2})},
\]

as follows

\[
\frac{\langle \Psi | \text{Tr}_{[1,n]} (T_{1,M}(\lambda_1) \cdots T_{n,M}(\lambda_n) g^{\epsilon_n}(\lambda_n) \cdots g^{\epsilon_1}(\lambda_1) I) | \Psi \rangle}{\prod_{j=1}^n T(\lambda_j) \langle \Psi | \Psi \rangle} = \prod_{j=1}^n \frac{1}{N(\lambda_j)}
\]

\[
\times \mathcal{F}^{\epsilon_1, \cdots, \epsilon_n}_{\sigma_1, \cdots, \sigma_{2n}} \langle \Psi | \text{Tr}_{[1,2n]} (T_{1,M}(\mu_1) \cdots T_{2n,M}(\mu_{2n}) g^{\sigma_2 n}(\mu_{2n}) \cdots g^{\sigma_1}(\mu_1) I) | \Psi \rangle \prod_{j=1}^n T(\mu_j) \langle \Psi | \Psi \rangle,
\]

where

\[
\{\mu_1, \mu_2, \cdots, \mu_{2n-1}, \mu_{2n}\} = \{\lambda_1 - \frac{1}{2}, \lambda_1 + \frac{1}{2}, \cdots, \lambda_n - \frac{1}{2}, \lambda_n + \frac{1}{2}\},
\]

(2.12)

\(\mathcal{F}^{\epsilon_1, \cdots, \epsilon_n}_{\sigma_1, \cdots, \sigma_{2n}}\) is a tensor easily read from (2.10).

3. Computational procedure and results

3.1. General procedure. In the homogeneous case consider an operator localised on the interval \([1, n]\). As usual we simplify the notations in (2.4) introducing multi-indices:

\[
: b_p^* c_q^* j^+_R j^-_S j^0_I :.
\]
Consider an operator $\mathcal{O}$ localised on the interval $[1, n]$. Our goal is to find the decomposition

$$
\mathcal{O} \equiv \sum_{P,Q,R,S,T} X_{P,Q,R,S,T} : b_P^* c_Q^* j_R^+ j_S^- I : ,
$$

where $\equiv$ means equality in the quotient by the action of the local integrals of motion space. We would like to proceed as in $[1,2]$, namely, to use sufficiently simple Matsubara data in order to obtain equations for the coefficients $X$ by computing independently the expectation values of operators on the right hand side and on the left hand side. However, in the present case there are some complications. The first is the normal ordering. The second is the multiplier containing $N$ in $[2,11]$, it looks quite innocent, but actually it is not. Also, as has been discussed, we did not find an efficient way (similar to $[1,2]$) to restrict the number of terms in the right hand side. With all that in mind we decided to take a simpler way based on the inhomogeneous chain.

In the inhomogeneous case the analogue of (3.1) looks like

$$
\mathcal{O} \equiv \sum_{\epsilon_1, \cdots, \epsilon_n} X_{\epsilon_1, \cdots, \epsilon_n}(\lambda_1, \cdots, \lambda_n) : g^{\epsilon_n}(\lambda_n) \cdots g^{\epsilon_1}(\lambda_1) : I ,
$$

having in mind $[2,7]$ we reduce the indices to $\epsilon_\lambda = \{1, 1\}, \{1, 2\}, \{2, 1\}, \{1, 3\}, \{3, 1\}, \{2, 2\}$ remembering that $g^{(1,1)}(\lambda) = \text{id}$, and $\equiv$ stands for equality of the expectation values for all Matsubara data in the geometry accepted in the present paper, in other words for the case when the left and the right Matsubara states are equal (we denote them by $|\Psi\rangle$). This is the inhomogeneous version of the quotient by the action of the local integrals.

The computation of the expectation value of (3.2) follows closely that explained in $[1,2]$. In the left hand side we have

$$
\frac{\langle \Psi | T_{i_1,j_1}(\lambda_1) \cdots T_{i_n,j_n}(\lambda_n) \mathcal{O} | \Psi \rangle}{\prod_{j=1}^n T(\lambda_j) \langle \Psi | \Psi \rangle}
$$

The choice of Matsubara data is explained in $[1]$. The numerator of this expression is a linear combination of terms of the kind

$$
\langle \Psi | T_{i_1,j_1}(\lambda_1) \cdots T_{i_n,j_n}(\lambda_n) | \Psi \rangle.
$$

where $T_{i_k,j_k}(\lambda_k) \in \text{End}(M)$ stands for the coefficient at position $i_k,j_k$ of $T_{k,M}(\lambda_k) = (T_{i_k,j_k}(\lambda_k))_{1 \leq i_k,j_k \leq 3}$.

Using fusion, the computations are reduced to the ones explained in details in $[2]$. The norm $\langle \Psi | \Psi \rangle$ is computed by Gaudin formula, the eigenvalue

$$
T(\lambda) = T(\lambda - \frac{1}{2})T(\lambda + \frac{1}{2}) - \Delta(\lambda),
$$

$\Delta(\lambda)$ being the quantum determinant.

The right hand side of (3.2) is computed applying consequently the rules of the normal ordering, that is the formulae $(2.10)$, then we express the result in terms of the functions $\omega(\lambda, \mu)$ and $N(\lambda)$.

Notice that $\omega$ appear only in expressions of the form $\omega(\lambda + \frac{1}{2}, \mu + \frac{1}{2})$ and as well as $N$ is computed from formulae given in $[1,2]$. However $\omega$ need to be made compatible with the definition of the normal order. To this end, we introduce an auxiliary function $\varphi$:

$$
\varphi(z) = \frac{1}{4} \left( - \frac{3}{z + 1} - \frac{1}{z - 1} + \frac{3}{z} + \frac{1}{z + 2} \right)
$$

and consider the two redefinitions :

$$
\tilde{\omega}(\lambda + \frac{1}{2}, \mu - \frac{1}{2}) = \omega(\lambda + \frac{1}{2}, \mu - \frac{1}{2}) + \varphi(\lambda - \mu), \quad \tilde{\omega}(\lambda - \frac{1}{2}, \mu + \frac{1}{2}) = \omega(\lambda - \frac{1}{2}, \mu + \frac{1}{2}) + \varphi(\lambda - \mu - 1),
$$
where \( \omega \) is taken as such from \[ \text{[1, 2]} \].

Below we give some examples of the expressions of the simplest elements of the fermion-current basis in terms of \( \omega \), and of how the normal ordering works in practice:

\[
\langle b^* (\lambda) e^* (\mu) \rangle = N(\lambda)N(\mu) \left( \tilde{\omega}(\lambda + \frac{1}{2}, \mu + \frac{1}{2}) + \tilde{\omega}(\lambda + \frac{1}{2}, \mu - \frac{1}{2}) + \tilde{\omega}(\lambda - \frac{1}{2}, \mu + \frac{1}{2}) + \tilde{\omega}(\lambda - \frac{1}{2}, \mu - \frac{1}{2}) \right),
\]

\[
\langle j^+(z) j^-(w) \rangle = -N(\lambda)N(\mu) \left( \tilde{\omega}(\lambda + \frac{1}{2}, \mu + \frac{1}{2}) \tilde{\omega}(\lambda + \frac{1}{2}, \mu - \frac{1}{2}) + \tilde{\omega}(\lambda - \frac{1}{2}, \mu + \frac{1}{2}) \tilde{\omega}(\lambda - \frac{1}{2}, \mu - \frac{1}{2}) \right) + \frac{1}{(\lambda - \mu)^2}.
\]

This type of formulae are easy to compute for given small Matsubara data and numerical \( \lambda_j \). Doing that we find experimentally how many different Matsubara data we need to get the expansion \([3.2]\). Denote by \( L \) the length of the Matsubara chain and for \( B \) the number of Bethe roots. For example, for the most complicated case considered in the present paper, \( n = 5 \), the following stock of Matsubara data is sufficient: 22 with \( L = 1, B = 0, 149 \) with \( L = 2, B = 0, 25 \) with \( L = 3, B = 0, 8 \) with \( L = 2, B = 1, 35 \) with \( L = 3, B = 1, 1 \) with \( L = 4, B = 2 \).

Up to \( n = 3 \) the computation is simple. The structure of the coefficients is as follows

\[
(3.5) \quad X(\lambda_1, \cdots, \lambda_n) = \prod_{i < j} \frac{1}{(\lambda_i - \lambda_j)^{d_{i,j}}} \frac{P(\lambda_1, \cdots, \lambda_n)}{R(\lambda_1, \cdots, \lambda_n)},
\]

where \( d_{i,j}, P, R \) depend on \( \epsilon_1, \cdots, \epsilon_n, R(0, \cdots, 0) \neq 0 \). The degrees \( d_{i,j} \) are easy to find: we take all \( \lambda \)'s sufficiently distant except \( \lambda_i \) and \( \lambda_j \) for which we consider two separations, say, \( 10^{-12}, 1 + 10^{-13} \). Obviously, this allows to define \( d_{i,j} \).

Using this as an Ansatz in the general case is difficult mostly because of the denominator \( R(\lambda_1, \cdots, \lambda_n) \). On the other hand we are not really interested in all the details of this denominator having in mind further application to the homogeneous case. Let us explain that.

Consider the right hand side of \([3.2]\). The normally ordered expression: \( g^m(\lambda_n) \cdots g^i(\lambda_1) : I \) is regular at the point \( \lambda_1 = 0, \cdots, \lambda_n = 0 \). The left hand side of \([3.2]\) does not depend on \( \lambda \)'s. So, setting \( \lambda_j = \epsilon \lambda_j' \) and sending \( \epsilon \) to 0 one concludes that in the function

\[
F(\lambda_1, \cdots, \lambda_n) = \frac{P(\lambda_1, \cdots, \lambda_n)}{R(\lambda_1, \cdots, \lambda_n)}
\]

among the terms with \( \epsilon^D \), only those with \( D = \sum d_{i,j} \) may contribute. The terms with \( D > \sum d_{i,j} \) vanish in the limit. The singular terms with \( D < \sum d_{i,j} \) must vanish, this gives rise to null-operators whose expectation values vanish regardless of the choice of the Matsubara data.

Experiments show that \( F(\lambda_1, \cdots, \lambda_n) \) is invariant under simultaneous shift of arguments. So we need the expansion

\[
F(\lambda_1, \cdots, \lambda_n) = \sum_{\Sigma m_j \leq \sum d_{i,j}} \prod_{j=2}^n (\lambda_j - \lambda_1)^{m_j} F_{m_2, \cdots, m_n}.
\]

Practical computations are easier in this form: we do not need to know the denominator \( R \). The computation of the Taylor series are performed taking sufficiently small \( \lambda \)'s and determining the Taylor coefficients \( F \) step by step. The coefficients of the Taylor series grow rapidly with the length of the interval \( n \), hence the inconvenience of the present procedure: for \( n = 5 \) we are forced to take \( \lambda \)'s of the order of \( 10^{-30} \). This makes computations rather slow.
Having the coefficients $F$, we arrive after a simple computation at the final formula (3.1).

3.2. Examples. The simplest $\mathfrak{sl}_2$-invariant operator of length $n$ is $\sum_{a=1}^{3} S_a^a$. It is defined by:

$$
\sum_{a=1}^{3} S_a^a = \frac{1}{2} h \otimes I_{n-2} \otimes h + e \otimes I_{n-2} \otimes f + f \otimes I_{n-2} \otimes e
$$

where we have the usual $\mathfrak{sl}_2$ spin 1 operators:

$$
h = \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{pmatrix},
$$

$$
e = \begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
$$

$$
f = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 2 & 0
\end{pmatrix}.
$$

For $n = 2, 3$ we compute

$$
\sum_{a=1}^{3} S_a^a = -\frac{34}{3} - 4b_1^* c_1^* - \frac{8}{3} j_1^+ j_1^-,
$$

$$
\sum_{a=1}^{3} S_a^a = -478 + \frac{384}{5} b_1^* c_1^* + \frac{176}{3} (b_2^* c_2^* - b_3^* c_3^*) - \frac{13216}{15} j_1^+ j_1^- + \frac{1024}{15} (j_2^+ j_4^+ - j_3^+ j_2^- - j_3^+ j_3^- - j_3^+ j_3^-)
$$

$$
+ 224 (j_3^+ j_1^- - j_3^+ j_2^-) + 240 b_1^* b_2^* j_1^- + \frac{832}{15} (b_1^* b_2^* j_2^- - b_2^* b_1^* j_1^- - b_1^* b_2^* j_3^-).
$$

The first results are derived from the inhomogeneous formula $(n = 2, 3)$ which are presented in the Appendix. In the case of the infinite volume and zero temperature the function $\omega(\lambda, \mu)$ simplifies a lot. First, in this case it depends only on the difference of the arguments: $\omega(\lambda, \mu) = \omega(\lambda - \mu)$. Second, we have the functional equation

$$
\omega(\lambda + 1) + \omega(\lambda) = \frac{\pi}{2 \sin(\pi \lambda)} - \varphi(\lambda),
$$

where $\varphi$ is defined in (3.4). The equation (3.7) is easy to solve, but actually the explicit solution is never needed in our computations: the final results are expressed only through the shifted sum of two $\omega$'s in the left hand side of (3.7). This explains why the final results are given by sums of even powers of $\pi$ with rational coefficients. For two and three sites we have

$$
\langle \sum_{a=1}^{3} S_a^a \rangle = \frac{8 \pi^2}{9} - \frac{34}{3} = \frac{-2.560351643}{15},
$$

$$
\langle \sum_{a=1}^{3} S_a^a \rangle = -478 + \frac{13216 \pi^2}{45} - \frac{224 \pi^4}{5} + \frac{4096 \pi^6}{2025} = \frac{1.283223553}{15}.
$$

in full agreement with [3].

We found expressions similar to (3.6) for $n = 4, 5$ which are unfortunately too long to be presented here. They are available upon a request. But the results for the infinite
volume and zero temperature are of reasonable size:

\[
\left\langle \sum_{a=1}^{3} S_1^a S_4^a \right\rangle = \frac{74317166}{75} - \frac{54372392\pi^2}{27} + \frac{14677235264\pi^4}{10125} - \frac{6743857664\pi^6}{14175} \\
+ \frac{238274860288\pi^8}{3189375} - \frac{1509154816\pi^{10}}{273375} + \frac{17291214848\pi^{12}}{111628125} = -1.083843468,
\]

\[
\left\langle \sum_{a=1}^{3} S_1^a S_5^a \right\rangle = \frac{30764875058782}{175} - \frac{5889239056193536\pi^2}{6615} + \frac{129766077160539584\pi^4}{70875} \\
- \frac{179533248577890184\pi^6}{893025} + \frac{609942688710268901888\pi^8}{468838125} \\
- \frac{6922910606153603072\pi^{10}}{13395375} + \frac{2684747793382087192576\pi^{12}}{21097715625} \\
- \frac{339956010411039064064\pi^{14}}{17722081125} + \frac{7217056126203854848\pi^{16}}{4219543125} \\
- \frac{2439025898062610432\pi^{18}}{29536801875} + \frac{572648486718144512\pi^{20}}{344506021875} = 0.8330261734. 
\]

From the expressions above one conjectures that \( \left\langle \sum_{a=1}^{3} S_1^a S_n^a \right\rangle \) is a polynomial in \( \pi^2 \) of degree \( n(n - 1)/2 \) with rational coefficients.

Having developed the fermion-current basis it is easy to compute the correlators \( \left\langle \sum_{a=1}^{3} S_1^a S_n^a \right\rangle \) \( (n = 2, 3, 4, 5) \) for finite temperature (like in \[2\]), or for the generalised Gibbs ensemble.

Another interesting application consists in the computation of the density matrix \( D(n) \) for the interval of length \( n \) in the infinite antiferromagnetic chain and of the entanglement entropy. Our methods of computation are far from perfection, so, we are doing much worse than in the paper \[2\], namely, only up to \( n = 4 \). This is not enough to compare the entanglement entropy \( s(n) = -\text{Tr}(D(n) \log D(n)) \) with the CFT prediction \[8\]

\[
s(n) \approx \frac{c}{3} \log n + a = \frac{1}{2} \log n + a, 
\]

where \( a \) is a non-universal constant. We remind that the scaling limit of the model is described by a CFT with \( c = 3/2 \). Still some resemblance with the scaling behaviour is already observed in the table which present the results of our computations.

| \( n \) | \( s(n) \) | \( s(n) - \frac{1}{2} \log n \)
|---|---|---|
| 2 | 1.5005420731509647 | 1.153968482870992 |
| 3 | 1.7187172552051159 | 1.169411110871061 |
| 4 | 1.8681251161018912 | 1.174977935541946 |

4. Conclusion

We have shown that the fermion-current basis works for small subchains of an infinite spin 1 integrable chain. In particular, the completeness holds at least up to intervals of length 5. We produced exact results for lengths \( n = 4, 5 \) which were not available previously. However, we are far from the length 11 achieved in \[1\]. There are two reasons for that. First, there is an objective reason: the model is far more complicated and the fermion-current basis contains much more elements than the fermionic basis for the spin 1/2 case. Second, there is a subjective reason: our method of computation is not perfect, we did not find how to work with the homogeneous case directly, so, we are forced to mix it with the inhomogeneous one, in a rather involved way which requires a lot of computer memory.
5. Appendix

Here we give formulae for the inhomogeneous case in a weak sense having in mind (2.7). The inhomogeneities are \(\lambda_1, \cdots, \lambda_n\). The coefficients do not depend on a simultaneous shift of inhomogeneities for that reason we shall use

\[
\mu_j = \lambda_{j+1} - \lambda_1.
\]

For \(n = 2\) we have

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = \frac{2(17 - 6\mu_1^2 + \mu_1^4)}{3(\mu_1^2 - 1)} + (\mu_1^2 - 4)\mathbf{g}^1,2(\lambda_1)\mathbf{g}^2,1(\lambda_2) - \frac{2}{3}(\mu_1^2 - 4)(\mu_1^2 - 1)\mathbf{g}^1,3(\lambda_1)\mathbf{g}^3,1(\lambda_2)
\]

For \(n = 3\) we have

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{45(\mu_1^2 - 1)((\mu_1 - \mu_2)^2 - 1)(\mu_2^2 - 1)} \times (-10755 + 4406\mu_1^2 - 943\mu_1^4 + 20\mu_1^6 - 4406\mu_1 \mu_2 + 1886\mu_1^2 \mu_2
\]

\[
- 60\mu_1^2 \mu_2 + 2424\mu_2^2 - 2349\mu_1^2 \mu_2^2 + 342\mu_1 \mu_2^2 + 4\mu_1^2 \mu_2^2 + 1556\mu_1 \mu_2^2 - 584\mu_1^2 \mu_2^2 - 12\mu_1^3 \mu_2^2 - 793\mu_1^4 \mu_2^2 + 372\mu_1^3 \mu_2^3 + \mu_1^4 \mu_2^3
\]

\[
- 90\mu_1 \mu_2^3 + 18\mu_1^2 \mu_2^3 + 35\mu_2^3 - 11\mu_1^2 \mu_2^3)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{15(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 1)((\mu_1 - \mu_2)^2 - 1)(\mu_2^2 - 1)} \times (2720 - 1176\mu_1 - 428\mu_1^2 + 48\mu_1^3 - \mu_1^4 + 1176\mu_1 \mu_2 - 856\mu_1^2 \mu_2
\]

\[
- 144\mu_1^3 \mu_2 - 36\mu_1^4 \mu_2 - 364\mu_1 \mu_2^3 - 284\mu_1^2 \mu_2^3 + 284\mu_1 \mu_2^2 + 364\mu_1^2 \mu_2^2 - 72\mu_1^3 \mu_2^2 + 364\mu_1^2 \mu_2^2 - 25\mu_1 \mu_2^3 + 364\mu_1^2 \mu_2^3 - 55\mu_1^2 \mu_2^3
\]

\[
+ 55\mu_1^3 \mu_2 + 15\mu_1^2 \mu_2^2 - 15\mu_1 \mu_2^3)\mathbf{g}^1,2(\lambda_1)\mathbf{g}^2,1(\lambda_1)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{15(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 1)((\mu_1 - \mu_2)^2 - 1)(\mu_2^2 - 1)} \times (-96 + 32\mu_1 - 2\mu_1^2 + 67\mu_1 \mu_2 - 18\mu_1^2 \mu_2 - 67\mu_2 \mu_2^2 + 7\mu_2^2 - 5\mu_1^2 \mu_2^2
\]

\[
+ 50\mu_1^3 \mu_2 - 25\mu_1^2 \mu_2^2 + 30\mu_1 \mu_2^3 - 10\mu_2^3)\mathbf{g}^1,3(\lambda_1)\mathbf{g}^3,1(\lambda_2)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{45(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 1)((\mu_1 - \mu_2)^2 - 1)(\mu_2^2 - 1)} \times (-384 + 992\mu_1 - 872\mu_1^2 + 306\mu_1^3 + 44\mu_1^4 + 2\mu_1^5 + 268\mu_1 \mu_2 + 987\mu_2 \mu_2
\]

\[
- 602\mu_1 \mu_2 + 147\mu_1^2 \mu_2^2 - 8\mu_1^2 \mu_2^2 - 748\mu_1 + 183\mu_2 + 188\mu_1^2 \mu_2 + 187\mu_1 \mu_2^2 - 109\mu_1 \mu_2^2 - 95\mu_1 \mu_2^3 + 274\mu_1 \mu_2^3 + 52\mu_1 \mu_2^3
\]

\[
- 21\mu_1^2 \mu_2^2 + 708\mu_1 \mu_2^2 + 194\mu_1^2 \mu_2^2 + 92\mu_1^2 \mu_2^2 + 54\mu_1 \mu_2^3 + 140\mu_1 \mu_2^3 - 74\mu_2 + 104\mu_2 - 20\mu_2^2
\]

\[
+ 52\mu_2^3 + 50\mu_2 + 3\mu_2^3)\mathbf{g}^1,3(\lambda_1)\mathbf{g}^3,1(\lambda_1)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{4}{45(\mu_1^2 - 1)(\mu_1 - \mu_2)(\mu_2^2 - 1)} \times (-192 + 256\mu_1 - 68\mu_1^2 + 4\mu_1^3 + 40\mu_1 \mu_2 + 40\mu_1 \mu_2^2 - 374\mu_1^2 + 540\mu_2 \mu_2
\]

\[
- 142\mu_1 \mu_2 + 866\mu_1 \mu_2 - 561\mu_1 \mu_2 - 111\mu_1 \mu_2^2 + 26\mu_1 \mu_2^2 + 227\mu_1 \mu_2^2 - 128\mu_1 \mu_2^2 - 3\mu_1 \mu_2^2
\]

\[
- 5\mu_1 \mu_2^2 + 56\mu_1 \mu_2 + 3\mu_1 \mu_2^2 - 5\mu_1 \mu_2 - 3\mu_1 \mu_2^2)\mathbf{g}^1,3(\lambda_1)\mathbf{g}^3,1(\lambda_2)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{15(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 1)(\mu_1 - \mu_2)^2 - 1)} \times (-26 + 7\mu_1^2 + 12\mu_1 \mu_2 - 7\mu_2 - 10\mu_1 \mu_2^2 + 5\mu_1 \mu_2^3
\]

\[
+ 15\mu_1 (\mu_1 - \mu_2)^2 - 1)(\mu_1 - \mu_2)^2 - 1))\mathbf{g}^1,3(\lambda_2)\mathbf{g}^2,1(\lambda_2)\mathbf{g}^3,1(\lambda_3)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{2}{15(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 4)(\mu_2^2 - 1)} \times (-26 + 7\mu_1^2 + 12\mu_1 \mu_2 - 7\mu_1 \mu_2^2 + 5\mu_1 \mu_2^3
\]

\[
+ 15\mu_1 (\mu_1 - \mu_2)^2 - 1)(\mu_1 - \mu_2)^2 - 1))\mathbf{g}^1,3(\lambda_2)\mathbf{g}^2,1(\lambda_2)\mathbf{g}^3,1(\lambda_3)
\]

\[
\sum_{a=1}^{3} S^a_1 S^a_2 = -\frac{4}{45(\mu_1^2 - 1)(\mu_1 - \mu_2)^2 - 4)(\mu_2^2 - 1)} \times (-12 + \mu_1^2 - \mu_1 \mu_2 + \mu_2^2))\mathbf{g}^1,3(\lambda_1)\mathbf{g}^3,1(\lambda_2)\mathbf{g}^3,2(\lambda_3)
\]
References

[1] Ph. DiFrancesco, F. Smirnov. OPE for XXX, *Reviews in Mathematical Physics* **30** No. 06, 1840006 (2018)
[2] T. Miwa, F. Smirnov. New exact results on density matrix for XXX spin chain. *Lett Math Phys* (2018). https://doi.org/10.1007/s11005-018-01143-x
[3] A. Klümper, D. Nawrath and J. Suzuki. Correlation functions of the integrable isotropic spin-1 chain: algebraic expressions for arbitrary temperature *J. Stat. Mech.* **2013** (2013) P08009
[4] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, Y. Takeyama, *Hidden Grassmann Structure in the XXZ Model II: Creation Operators*. Commun.Math.Phys.286 (2009) 875
[5] M. Jimbo, T. Miwa, and F. Smirnov. Hidden Grassmann structure in the XXZ model III: Introducing Matsubara direction. *J. Phys. A:Math.Theor.*, **42** (2009) 304018
[6] M. Jimbo, T. Miwa, F. Smirnov, Creation operators for the Fateev-Zamolodchikov spin chain, *Theoretical and Mathematical Physics* **181** (2014) 1169-1193
[7] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, *Completeness of a fermionic basis in the homogeneous XXZ model*. J. Math. Phys. 50, (2009) P095206
[8] C. Holzhey C, F. Larsen, F. Wilczek. Geometric and renormalized entropy in conformal field theory *Nucl. Phys. B* **424** (1994) 443-467

CB, FS\textsuperscript{1} SORBONNE UNIVERSITE, UPMC UNIV PARIS 06, CNRS, UMR 7589, LPTHE, F-75005, PARIS, FRANCE

E-mail address: cbabenko@lpthe.jussieu.fr, smirnov@lpthe.jussieu.fr

\textsuperscript{1}Membre du CNRS