CANONICAL REDUCED EXPRESSION IN AFFINE COXETER GROUPS
PART I - TYPE $\tilde{A}_n$

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ABSTRACT. We classify the elements of $W(\tilde{A}_n)$ by giving a canonical reduced expression for each, using basic tools among which affine length. We give some direct consequences for such a canonical form: a description of left multiplication by a simple reflection, a study of the right descent set, and a proof that the affine length is preserved along the tower of affine Coxeter groups of type $\tilde{A}$, which implies in particular that the corresponding tower of affine Hecke algebras is a faithful tower.

1. INTRODUCTION

This paper is the first of a series in which for the elements of an affine Coxeter group $\tilde{W}$, we produce a canonical reduced expression, together with the set of all distinguished representatives of $\tilde{W}/W$ where $W$ is a maximal parabolic subgroup of $\tilde{W}$. This very paper is meant to detail type $\tilde{A}_n$, we do the same in the second paper for types $\tilde{C}_n$ and $\tilde{B}_n$, while type $\tilde{D}_n$ and the five other types are to be treated in the last two.

1.1. Coxeter systems and related topics (such as Hecke algebras and their quotients, K-L polynomials and the new born: Light leaves) take a place in the heart of representation theory. Reduced expressions are the salt of such systems: Almost every related object is defined starting from a reduced expression or reduced to a reduced expression explanation, especially and not surprisingly objects which are ”independent” from reduced expressions! Such as: Hecke algebras bases and Bruhat order. One may bet that no work concerning/using Coxeter group theory is reduced-expression free. A canonical reduced expression for elements in the infinite families of finite Coxeter groups has been known while ago, we refer to [13] to see an easy explication of such canonical expressions. $W(A_n)$ is a famous extension of the symmetric group $W(A_n)$, known to be the first ”group”.

Let $W(A_n)$ the $A$-type Coxeter group with $n \geq 1$ generators $\{\sigma_1, \sigma_2, \ldots \sigma_n\}$ (AKA Sym$_n+1$). Let $[i, j] = \sigma_i \sigma_{i+1} \ldots \sigma_j$ for $1 \leq i \leq j \leq n$. One of the very basic results is:

\begin{itemize}
  \item [Date: October 25, 2022.]
  \item 2010 Mathematics Subject Classification. Primary 20F55, Secondary 05E16, 20C08.
\end{itemize}
Theorem. \( W(A_n) \) is the set of elements of the following canonical reduced form:
\[ [i_1, j_1] [i_2, j_2] \cdots [i_s, j_s] \]
with \( n \geq j_1 > \cdots > j_s \geq 1 \) and \( j_t \geq i_t \geq 1 \) for \( s \geq t \geq 1 \). Identity is to be considered the case where \( s = 0 \).

This is equivalent to saying that the distinguished representatives of the cosets in \( W(A_n)/W(A_{n-1}) \) are the elements 1 and \( [r, n] \) for \( 1 \leq r \leq n \).

In this work we give an analogue of this assertion for the infinite affine Coxeter group \( W(\tilde{A}_n) \). More precisely: we give a canonical reduced expression for the elements of this group, with a full set of the distinguished coset representatives of \( W(\tilde{A}_n)/W(A_n) \). Then we give some examples of direct consequences of this classification by canonical forms.

1.2. The key word (and almost everywhere used creature in this work) is affine length (Definition 2.4): We let \( w \) represent the following inequalities \( m \) where the very expression of Coxeter generators of \( W \) is Theorem 2.13, of which we give a shortened version as follows:

\[ \text{Theorem 1.1. Any distinguished representative } w \text{ of } W(\tilde{A}_n)/W(A_n) \text{ has a unique canonical reduced expression:} \]
\[ w_a = B(j_1, i_1) B(j_2, i_2) \cdots B(j_m, i_m) \]
where \( m \) is the affine length of \( w \) and \( (j_s, i_s)_{1 \leq s \leq m} \) is a family of integers satisfying the following pairwise inequalities:

- \( 1 \leq j_1 \leq n + 1 \) and \( 0 \leq i_1 \leq n - 1 \); for \( 2 \leq s \leq m \), either \( i_s = 0 \) and \( j_s = 1 \), or \( 1 \leq i_s \leq n - 1 \) and \( 1 \leq j_s \leq n \);
- the sequence \( (j_k) \) (resp. \( i_k) \) is non-increasing (resp. non-decreasing);
- for \( 2 \leq s \leq m \), if \( j_{s-1} > i_{s-1} + 1 \), then \( j_s < j_{s-1} \); if \( j_s > i_{s} + 1 \) then \( i_s > i_{s-1} \).

Vice versa, any such family \( (j_s, i_s)_{1 \leq s \leq m} \) determines by (1.2) a distinguished representative \( w \) of \( W(\tilde{A}_n)/W(A_n) \), in reduced form, of affine length \( m \). We call the very expression \( w_a := B(j_1, i_1) B(j_2, i_2) \cdots B(j_m, i_m) \) the affine block of any element in \( wW(A_n) \).
The proof establishes in an explicit, algorithmic and independent way the existence of such representatives of minimal length, given in canonical form. Appending on the right of an affine block a canonical reduced expression for an element of $W(\tilde{A}_n)$ provides a canonical reduced expression for any element in $W(\tilde{A}_n)$. We note that the lengths of the successive affine bricks in a given affine block form a non-decreasing sequence with first terms increasing strictly up to $n$, and that two of those bricks have the same length if and only if they are identical.

1.3. We pause here to thank the referee of the previous version of this paper who pointed out similarities with section 3.4 in the book [5] by Björner and Brenti on the one hand, and with the paper [14] by Yilmaz, Özel, and Ustaoğlu on the other hand. Therefore we studied those references.

After getting into the context and language of Gröbner-Shirshov bases in [14], it turns out that the canonical form in Theorem 2.13 below is indeed the one given in loc. cit. up to taking inverses. Yet, in our work, the single set of parameters is simpler (to read and to use) than the artificially separated parameters $u$, $v$ and $uv$ in loc. cit.; the proofs give more insight into the Coxeter group structure of $W(\tilde{A}_n)$ (loc. cit. relies on a counting argument); some intermediate calculations are also efficient when working on consequences. In addition, the present paper is the first in a series in which we will deal with other types of affine Weyl groups, following the same plan as here.

We turn to the normal form whose existence and uniqueness are established in [5, §3.4] for any Coxeter group: it is the lexicographically first reduced word, in short the left lex-min form, for a given order on the set $S$ of generators, hence written $S = \{s_1, \ldots, s_{n+1}\}$ (implicitly and conventionally the lexicographic comparison starts on the left of the word and proceeds from left to right). As observed by Stembridge in [13, p.1288] (citing Edelman), the normal form (1.1) for elements of $W(\tilde{A}_n)$ is the reverse, i.e. from right to left, lexicographically first reduced word, in short the right lex-min form. It is easy to check that our canonical form is the right lex-min form for any numbering $\{s_1, \ldots, s_{n+1}\}$ of $\{\sigma_1, \ldots, \sigma_n, a_{n+1}\}$ such that $s_{n+1} = a_{n+1}$, $s_n = \sigma_n$ and $s_{n-1} = \sigma_1$.

Our form depends on the choice of the ”affinizing” generator, denoted by $a_{n+1}$; this choice is essential to define the affine length and is notably relevant when dealing with the full tower of groups $(W(\tilde{A}_n))_{n\geq 1}$ as in section 4.3. We obtain our canonical form by forcing occurrences of $a_{n+1}$ to be minimal and leftmost. By the previous statement, this implies right-lexicographic minimality (we also order the two neighbours of $a_{n+1}$ in the Dynkin diagram – the effect of this choice is mild, changing it amounts to applying rules (2.7)).

Now we make an important remark. In [5] existence and uniqueness are a direct consequence of the existence and uniqueness of a minimal element for the lexicographic order. In the present paper, the existence of a form (1.2) for a distinguished representative of $W(\tilde{A}_n)/W(A_n)$ is easy, but more work has to be done to find necessary and sufficient conditions for such a form to be minimal – that is,
the pairwise inequalities. Getting the general form (1.2), a product of affine bricks, from [5] would be easy, but the pairwise inequalities cannot be deduced from there.

To end this interlude, we thank Bill Casselman for drawing a path for us in the story of normal forms, which developed in the nineties with works of Fokko du Cloux and Bill Casselman in particular [9] [8] [10], and for pointing out the importance of the result of Brink and Howlett that Coxeter groups are automatic [7].

1.4. We give three direct consequences of the canonical form. As a **first consequence**, we show that through left multiplication by a simple reflection in $S_n$, the canonical form behaves exactly as wished! In other terms: the change made by left multiplication by a simple reflection is very localized, it happens in at most one affine brick of the affine block in such a way that we get a canonical form directly, without passing by the algorithm. This is Theorem 4.2, to which we refer for more detailed statements:

**Theorem (Theorem 4.2).** Let $w_n = B(j_1, i_1)B(j_2, i_2)\ldots B(j_m, i_m)$ be an affine block of affine length $m \geq 1$, let $w_a$ be the corresponding element of $W(A_n)$ and let $s$ be in $S_n$. Then:

1. either $sw_a$ cannot be expressed by an affine block, and we have actually $l(sw_a) = l(w_a) + 1$ and $sw_a = w_a\sigma_v$ for some $v$, $1 \leq v \leq n$;
2. or $sw_a$ has a reduced expression that is an affine block $w'_a$ and, other than the obvious two cases when $s = s_{n+1}$ with $h(j_1, i_1)$ trivial or extremal, the two affine blocks $w'_a$ and $w_a$ differ in one and only one $h(j_s, i_s)$ and one and only one entry there, say $j'_s \neq j_s$ or $i'_s \neq i_s$. If $l(sw_a) = l(w_a) + 1$ we have $j'_s = j_s - 1$ or $i'_s = i_s + 1$, while if $l(sw_a) = l(w_a) - 1$ we have $j'_s = j_s + 1$ or $i'_s = i_s - 1$.

This theorem is telling that the canonical form is somehow "stable" by left multiplication by an $s \in S_n$ up to a change in at most one $i_s$ or one $j_s$, but words are but finite sequence of generators! So the canonicity is not bothered by the left multiplications! Actually, after getting acquainted with Fokko du Cloux’s work as explained above, we saw the similarity of this statement with Theorem 2.6 in [10], changing left to right. We chose to leave our statement unchanged with its direct proof, instead of deducing it, however easily, from loc.cit, because our proof includes in fact an automaton to deal with left multiplication of an affine brick, see Lemma 4.4. Even more important, our proof controls the path, i.e. the sequence of braid relations, leading from $sw_a$ to $w'_a$, which is essential in an application to light leaves under way.

While for the **second consequence**: in section 4.2 devoted to right multiplication, we compare the descent set $\mathcal{R}(w)$ of $w$ with the descent set $\mathcal{R}(x)$ of $x$, where $w = w_n x$, $x$ in $W(A_n)$, and $w_n$ is our distinguished representative of $wW(A_n)$, having the affine block $w_a$ of $w$ as a reduced expression. We see in Theorem 2.13 that $\mathcal{R}(w_n) = \{a_{n+1}\}$. We actually have either $\mathcal{R}(w) = \mathcal{R}(x)$ or $\mathcal{R}(w) = \mathcal{R}(x) \cup \{a_{n+1}\}$. We give sufficient conditions on $w$ for $a_{n+1}$ to belong
to $\mathcal{R}(w)$, together with the hat partner (see [3,1]) of $a_{n+1}$ multiplied from the right when the multiplication decreases the length. The cases where $m = 1$ and $m = 2$ are fully described.

A third consequence is to show that the affine length is preserved in the tower of affine groups defined in [1], that is: When seeing $W(\tilde{A}_{n-1})$ as a reflection subgroup of $W(\tilde{A}_n)$ via the map defined in section 4.3:

$$R_n : W(\tilde{A}_{n-1}) \rightarrow W(\tilde{A}_n),$$

then a canonical reduced expression of $(n-1)$-rank is sent to an explicit canonical reduced expression of $(n)$-rank, preserving the affine length, this is Theorem 4.6.

**Theorem (Theorem 4.6).** Let

$$w = h_{n-1}(j_1, i_1)a_nh_{n-1}(j_2, i_2)a_n \cdots h_{n-1}(j_m, i_m)a_nx,$$

with $x \in W(A_{n-1})$, be the canonical reduced form of an element $w$ in $W(\tilde{A}_{n-1})$. Then:

$$R_n(w) = h_{n}(j_1, i_1)a_{n+1}h_{n}(j_2, i'_2)a_{n+1} \cdots h_{n}(j_m, i'_m)a_{n+1}[t, n]x,$$

where, letting $s = \max\{k / 1 \leq k \leq m$ and $n - k - i_k > 0\}$, we have:

$$i'_k = i_k \text{ for } k \leq s, \quad i'_k = i_k + 1 \text{ for } k > s, \quad t = n - s + 1.$$

This implies $L(R_n(w)) = L(w)$ and $l(R_n(w)) = l(w) + 2L(w)$, hence replacing $a_n$ by $\sigma_n a_{n+1} \sigma_n$ in a reduced expression for $w$ produces a reduced expression for $R_n(w)$ if and only if the expression for $w$ is affine length reduced.

The latter theorem gives a necessary and sufficient condition for an element in $W(\tilde{A}_n)$ to belong to the image of $W(\tilde{A}_{n-1})$, that is Corollary 4.7. A worthwhile consequence is that the corresponding Hecke algebras embed one in the other regardless of the ground ring, that is Corollary 4.9.

1.5. We mention briefly farther goals in what follows:

The rigidity of the blocks is a natural field for "cancelling", otherwise called "applying the star operation", to comment this point we need a more advanced calculus, to be done in a forthcoming work centering around the famous Kazhdan-Lusztig cells, and around $W(A_n)$-double cosets since some additional work on the material obtained above (having very strong relations with the second direct consequence) leads to a complete (long) list of canonical reduced expressions of representatives of $W(A_n)$-double classes.

Moreover, in general the canonical form gives us precious data on the space of traces, in particular the embedding of the canonical forms would help a great deal in classifying traces of type Jones on the tower of affine Hecke algebras. Indeed the canonical form given here is easily seen to coincide (up to a notation), on fully commutative elements, with the normal form (actually, a canonical form) established in [3], which is a crucial ingredient in classifying Markov traces on the tower of affine Temperley-Lieb algebras of type $\tilde{A}$ in [2].
In yet another direction, namely an algorithmic way to go towards and come back from the Bernstein presentation, the canonical form indeed gives long ones easily, definitely the third consequence is a tricky way to shorten the two algorithms. It gives as well a way to enumerate elements by affine length for example.

Experts of the theory of light leaves (born in [11]) would be interested in such a canonical form, since their computation starts usually with a reduced expression, thus it is even better to have it canonical. For instance, in an ongoing work starting from the canonical form, David Plaza and the author are providing an explicit and simple way to produce "canonical" light leaves bases for the group $W(\tilde{A}_n)$, where usually the construction depends on many non-canonical choices. It is worth to mention that the algorithm to arrive to our canonical form can start from any reduced expression and not only from affine length reduced ones.

The work is self contained and accessible for any who is familiar with Coxeter systems or otherwise want-to-be, we count only on the simplicity of the canonical form, which shows that $W(\tilde{A}_n)$ is way more "tamed" than Coxeter theory amateurs tend to think, or at least than the author used to think.

2. A CANONICAL REDUCED EXPRESSION

Let $(W(\Gamma), S)$ be a Coxeter system with associated Coxeter diagram $\Gamma$. Let $w \in W(\Gamma)$ or simply $W$. We denote by $l(w)$ the length of $w$ (with respect to $S$). We define $L(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other terms $s$ appears at the left edge of some reduced expression of $w$. We define $R(w)$ similarly, on the right. If $S'$ is a subset of $S$ and $W'$ is the (parabolic) subgroup of $W$ generated by $S'$, each right coset $wW'$ has a unique element of minimal length, say $a$, and for any $x \in W'$ we have $l(ax) = l(a) + l(x)$ (see for instance [12, Lemma 9.7]). We call $a$ the distinguished representative of its coset $aW'$.

2.1. Canonical form in $W(A_n)$. Let $n \geq 2$. Consider the $A$-type Coxeter group with $n$ generators $W(A_n)$, with the following Coxeter diagram:

\[
\begin{array}{cccc}
\sigma_1 & \sigma_2 & \ldots & \sigma_{n-1} & \sigma_n \\
\end{array}
\]

Now let $W(\tilde{A}_n)$ be the affine Coxeter group of $\tilde{A}$-type with set of $n + 1$ generators $S_n = \{\sigma_1, \sigma_2, \ldots, \sigma_n, a_{n+1}\}$, perfectly determined by the following Coxeter diagram:

\[
\begin{array}{cccc}
\sigma_1 & \sigma_2 & \sigma_{n-1} & \sigma_n \\
& & \sigma_{n+1} \\
\end{array}
\]
Since \( W(A_n) \) is a parabolic subgroup of \( W(\tilde{A}_n) \), we have for any \( v \in W(\tilde{A}_n) \), \( v \neq 1 \):

\[
\mathcal{R}(v) = \{a_{n+1}\} \iff \forall x \in W(A_n) \quad l(vx) = l(v) + l(x).
\]

In the group \( W(A_n) \) we let:

\[
[i, j] = \sigma_i \sigma_{i+1} \cdots \sigma_j \quad \text{for} \quad n \geq j \geq i \geq 1 \quad \text{and} \quad [n + 1, n] = 1,
\]

\[
[i, j] = \sigma_i \sigma_{i-1} \cdots \sigma_j \quad \text{for} \quad 1 \leq j \leq i \leq n \quad \text{and} \quad [0, 1] = 1,
\]

\[
h(r, i) = [r, n][i, 1] \quad \text{for} \quad 0 \leq i \leq n - 1, 1 \leq r \leq n + 1.
\]

One can prove by induction on \( n \) (considering right cosets of \( W(A_{n-1}) \) in \( W(A_n) \)) the following well-known theorem.

**Theorem 2.1.** \( W(A_n) \) is the set of elements of the following canonical reduced form:

\[
(i_1, j_1)[i_2, j_2] \cdots [i_s, j_s]
\]

with \( n \geq j_1 > \cdots > j_s \geq 1 \) and \( j_t \geq i_t \geq 1 \) for \( s \geq t \geq 1 \). Identity is to be considered the case where \( s = 0 \).

Notice that if \( \sigma_n \) appears in form \((2.2)\), then \( \sigma_n \) will certainly appear only once, and it is to be equal to \( \sigma_{j_1} \).

**Definition 2.2.** An element \( u \) in \( W(A_n) \) is called extremal if both \( \sigma_n \) and \( \sigma_1 \) belong to \( \text{Supp}(u) \).

**Lemma 2.3.** Let \( P \) be the parabolic subgroup of \( W(A_n) \) generated by \( \sigma_2, \ldots, \sigma_{n-1} \). An element in \( W(A_n) \) can uniquely be written in the following reduced form:

\[
h(r, i) x, \quad 0 \leq i \leq n - 1, 1 \leq r \leq n + 1, \quad x \in P.
\]

The element is extremal if and only if either \( r = 1 \) and \( i = 0 \), or \( i \geq 1 \) and \( r \leq n \).

**Proof:** The elements \([j, n]\) for \( 1 \leq j \leq n + 1 \) constitute the set of distinguished representatives for \( W(A_n)/W(A_{n-1}) \), as is well-known, actually the first step for proving Theorem \( 2.1 \). An easy transformation gives the set of elements \([i, 1]\) for \( 0 \leq i \leq n - 1 \) as the set of distinguished representatives for \( W(A_{n-1})/P \), hence the statement. \( \square \)

As a consequence, we can define what we call the **extremal canonical form** of any \( w \in W(A_n) \):

\[
h(r, i)[i_1, j_1][i_2, j_2] \cdots [i_s, j_s]
\]

with \( 1 \leq r \leq n + 1, 0 \leq i \leq n - 1, n - 1 \geq j_1 > \cdots > j_s \geq 2 \) and \( j_t \geq i_t \geq 2 \) for \( s \geq t \geq 1 \). This form could be used everywhere below instead of the usual canonical form \((2.2)\).
2.2. Affine length.

**Definition 2.4.** We call affine length reduced expression of a given \( u \) in \( W(\tilde{A}_n) \) any reduced expression with minimal occurrence of \( a_{n+1} \), and we call affine length of \( u \) this minimum, we denote it by \( L(u) \).

**Remark 2.5.** We gave the definition of affine length for fully commutative elements in [3]: for such elements the number of occurrences of \( a_{n+1} \) in a reduced expression does not depend on the reduced expression.

**Remark 2.6.** The affine length is constant on the double classes of \( W(A_n) \) in \( W(\tilde{A}_n) \). It satisfies, for any \( v, w \in W(\tilde{A}_n) \):

\[ |L(v) - L(w)| \leq L(vw) \leq L(v) + L(w). \]

**Lemma 2.7.** Let \( w \) be in \( W(\tilde{A}_n) \) with \( L(w) = m \geq 2 \). Fix an affine length reduced expression of \( w \) as follows:

\[ w = u_1a_{n+1}u_2a_{n+1} \cdots u_m a_{n+1} u_{m+1} \text{ with } u_i \in W(A_n) \text{ for } 1 \leq i \leq m+1. \]

Then \( u_2, \ldots, u_m \) are extremal and there is a reduced writing of \( w \) of the form:

\[ (2.4) \quad w = h(j_1,i_1) a_{n+1} h(j_2,i_2) a_{n+1} \cdots h(j_m,i_m) a_{n+1} v_{m+1}, \]

where \( v_{m+1} \) is an element in \( W(A_n) \), \( 1 \leq j_1 \leq n + 1, 0 \leq i_1 \leq n - 1 \), and for \( 2 \leq s \leq m \), either \( i_s = 0 \) and \( j_s = 1 \), or \( 1 \leq i_s \leq n - 1 \) and \( 1 \leq j_s \leq n \).

**Proof.** Let \( y \in W(A_n) \) such that \( a_{n+1} y a_{n+1} \) is an affine length reduced expression. We use Lemma 2.3 to write \( y = h(r,i) x \) with \( x \in P \). Since \( x \) and \( a_{n+1} \) commute, the element \( a_{n+1} h(r,i) a_{n+1} \) must be affine length reduced. Since the braids \( a_{n+1} \sigma_1 a_{n+1} \) and \( a_{n+1} \sigma_n a_{n+1} \) are to be excluded, both \( \sigma_1 \) and \( \sigma_n \) must appear in \( h(r,i) \) so \( y \) is extremal.

Now we proceed from left to right, using Lemma 2.3 at each step. We write \( u_1 = h(j_1,i_1) x_1 \) with \( x_1 \in P \), so that \( u_1 a_{n+1} u_2 = h(j_1,i_1) a_{n+1} x_1 u_2 \). We repeat with \( x_1 u_2 a_{n+1} = h(j_2,i_2) a_{n+1} x_2 \) with \( x_2 \in P \) and so on, getting \( (2.4) \). We started with a reduced expression of \( w \) so we obtain a reduced expression. \( \square \)

Yet, an expression as \( (2.4) \) may be reduced without being affine length reduced, as in the following example:

\[ a_{n+1} \sigma_n \cdots \sigma_1 a_{n+1} \sigma_1 \cdots \sigma_n a_{n+1} = \sigma_n a_{n+1} \sigma_n \cdots \sigma_1 \cdots \sigma_n a_{n+1} \sigma_n. \]

**Lemma 2.8.** An element of affine length 1 can be written in a unique way as

\[ h(r,i) a_{n+1} x, \quad 0 \leq i \leq n - 1, \quad 1 \leq r \leq n + 1, \quad x \in W(A_n), \]

and such an expression is always reduced. The commutant of \( a_{n+1} \) in \( W(A_n) \) is \( P \).

**Proof.** The existence of such an expression comes from Lemma 2.3. Showing that the expression is reduced amounts, by (2.1), to showing that \( \mathcal{R}(h(r,i) a_{n+1}) = \{a_{n+1}\} \). Indeed, if \( 2 \leq k \leq n - 1 \), then \( w \sigma_k = h(r,i) \sigma_k a_{n+1} \) has length \( l(w) + 1 \). Now assume \( k = 1 \) or \( k = n \), and \( l(w \sigma_k) < l(w) \). By the exchange condition there is a \( \sigma_u \) appearing in \( h(r,i) \) such that \( h(r,i) a_{n+1} \sigma_k = \hat{h}(r,i) a_{n+1} \) where \( \hat{h}(r,i) \) is
what becomes $h(r, i)$ after omitting $\sigma_n$. We multiply by $a_{n+1}$ on the right and get $h(r, i)\sigma_k a_{n+1} = h(r, i)$, impossible considering supports.

Uniqueness amounts to proving that $h(j, i)a_{n+1} = h(j', i')a_{n+1}x$ (with obvious notation) implies $x = 1$, immediate from $R(h(j, i)a_{n+1}) = \{a_{n+1}\}$ and (2.1). The last assertion is a consequence of uniqueness.

**Definition 2.9.** We call affine brick and denote by $B(r, i)$, or $B_n(r, i)$ when we need to emphasize the dependency in $n$, the expression

$$B(r, i) = h(r, i)a_{n+1}, \quad 0 \leq i \leq n - 1, \ 1 \leq r \leq n + 1.$$  

The length of an affine brick $B(r, i)$ is $n + 1 + i + 1 - r$. We call an affine brick short if its length is at most $n$, i.e. $r \geq i + 1$. Otherwise we call it long.

We will keep in mind that the two segments of a short affine brick commute:

$$B(r, i) = [r, n][i, 1]a_{n+1} = [i, 1][r, n]a_{n+1} \ \text{for} \ r \geq i + 1.$$  

Other cases are listed in (2.6) below.

### 2.3. Affine length reduced expressions

The property $R(h(r, i)a_{n+1}) = \{a_{n+1}\}$ does not extend to elements in form (2.4) with $v_{m+1} = 1$. For instance, the relations:

$$\begin{align*}
\sigma_n a_{n+1}\sigma_n\sigma_1 a_{n+1} &= a_{n+1}\sigma_n\sigma_1 a_{n+1}\sigma_1 \\
\sigma_1 a_{n+1}\sigma_n\sigma_1 a_{n+1} &= a_{n+1}\sigma_n\sigma_1 a_{n+1}\sigma_n
\end{align*}$$  

imply: $\sigma_1 \in R(\sigma_n a_{n+1}\sigma_n\sigma_1 a_{n+1})$ and $\sigma_n \in R(\sigma_1 a_{n+1}\sigma_n\sigma_1 a_{n+1})$. So the general form (2.4) need not be reduced, we must impose more conditions. As in Lemma 2.7, we want to push to the right the simple reflections $\sigma_k, 1 \leq k \leq n$, whenever possible. To do this we bring out the following formulas:

**Proposition 2.10.** Let $1 \leq r \leq n + 1$, $0 \leq u \leq n - 1, 1 \leq s \leq n$ and $1 \leq v \leq n - 1$. We have the following rules.

1. If $r > u + 1$ and $s \geq r$: $B(r, u)B(s, v) = B(s + 1, u)B(r, v)\sigma_1$.

2. If $s > u + 1 \geq v + 1$: $B(r, u)B(s, v) = B(r, v - 1)B(s, u)\sigma_n$.

3. If $v + 1 < s \leq u + 1$: $B(r, u)B(s, v) = B(r, v - 1)B(s - 1, u - 1)\sigma_n$.

4. If $s \leq v + 1$ and $v < u$: $B(r, u)B(s, v) = B(r, v)B(s, u - 1)\sigma_n$.

5. If $r \leq u + 1 < s$: $B(r, u)B(s, v) = B(s + 1, u + 1)B(r + 1, v)\sigma_1$.

6. If $r < s \leq u + 1$: $B(r, u)B(s, v) = B(s, u)B(r + 1, v)\sigma_1$.

**Proof:** These are straightforward computations based on (2.6), relying on the rules: $[r, s] \sigma_k = \sigma_{k+1}[r, s]$ if $r \leq k < s$; $[r, s] \sigma_k = \sigma_{k-1}[r, s]$ if $r \geq k > s$. 

Definition 2.12. Let the pairwise inequalities (2.6)

\[
\begin{align*}
[a, 1] [b, n] &= [b - 1, n] [a - 1, 1] & \text{if } 1 < b \leq a + 1 \leq n + 1; \\
[a, 1] [b, n] &= [b, n] [a, 1] & \text{if } n + 1 \geq b > a + 1; \\
[a, 1] [1, n] &= [a + 1, n] & \text{if } 0 \leq a \leq n; \\
[a, n] [b, n] &= [b, n] [a - 1, n - 1] & \text{if } n + 1 \geq a > b \geq 1; \\
[a, n] [b, n] &= [b + 1, n] [a, n - 1] & \text{if } 1 \leq a \leq b \leq n; \\
[a, 1] [b, 1] &= [b, 1] [a + 1, 2] & \text{if } 1 \leq a < b; \\
[a, 1] [b, 1] &= [b - 1, 1] [a, 2] & \text{if } a \geq b.
\end{align*}
\]

(2.6)

We remark that equalities (1) to (6) involve expressions of the same length. They are actually all reduced (Lemma 3.7).

\[\Box\]

Corollary 2.11. Let \(w\) be in \(W(\tilde{A}_n)\) with \(L(w) = m \geq 1\). Among the affine length reduced expressions of \(w\):

\[w = u_1 a_{n+1} u_2 a_{n+1} \ldots u_m a_{n+1} u_{m+1}\]

with \(u_i \in W(A_n)\) for \(1 \leq i \leq m + 1\) we fix one with leftmost occurrences of \(a_{n+1}\). We have the following, where in (2) and (3) we assume \(2 \leq s \leq m\).

1. For \(1 \leq s \leq m\), there exist integers \(j_s, i_s\) such that \(u_s = h(j_s, i_s)\). They satisfy \(1 \leq j_1 \leq n + 1, 0 \leq i_1 \leq n - 1\), and, for \(2 \leq s \leq m\):
   1. either \(i_s = 0\) and \(j_s = 1\), or \(1 \leq i_s \leq n - 1\) and \(1 \leq j_s \leq n\).
   2. If \(j_{s-1} > i_{s-1} + 1\), then \(j_s < j_{s-1}\) and \(i_s \geq i_{s-1}\); if \(j_s > i_s + 1\) then \(i_s > i_{s-1}\).
   3. If \(j_{s-1} \leq i_{s-1} + 1\), then \(j_s \leq j_{s-1}\) and \(i_s \geq i_{s-1}\) (so \(j_s \leq i_s + 1\) also).

Proof: All numbered references below refer to Proposition 2.10 used to produce contradictions to the assumption that occurrences of \(a_{n+1}\) are leftmost.

1. follows directly from Lemma 2.3 and Lemma 2.7
2. We assume \(j_{s-1} > i_{s-1} + 1\). If \(j_{s-1} = n + 1\) (so \(s = 1\) then \(j_s < j_{s-1}\).
   If \(j_{s-1} \leq n\) and \(j_s \geq j_{s-1}\), then (1) gives a contradiction since the two \(a_{n+1}\) have moved left. Hence \(j_s < j_{s-1}\).
   If also \(j_s > i_s + 1\), then \(i_s\) cannot be 0 (since \(h(j_s, i_s)\) is extremal), so if \(i_{s-1} = 0\) we have indeed \(i_s > i_{s-1}\). Now if \(i_{s-1} > 0\) and \(i_s \leq i_{s-1}\), (2) gives a contradiction, whatever the value of \(j_{s-1}\).
   We turn to \(j_s \leq i_s + 1\). If \(i_{s-1} = 0\) we do have \(i_s \geq i_{s-1}\). If \(i_{s-1} > 0\) and \(i_s < i_{s-1}\), (4) gives a contradiction, hence \(i_s \geq i_{s-1}\).
3. We now assume \(j_{s-1} \leq i_{s-1} + 1\). If \(j_s > j_{s-1}\), (5) or (6) give a contradiction. We conclude that \(j_s \leq j_{s-1}\). Now if \(i_s < i_{s-1}\) we are either in case (3) or in case (4), and both give a contradiction, so \(i_s \geq i_{s-1}\).

\[\Box\]

Definition 2.12. Let \(m \geq 1\). A family of integers \((j_s, i_s)_{1 \leq s \leq m}\) is said to satisfy the pairwise inequalities if the following conditions hold:

1. \(1 \leq j_1 \leq n + 1\) and \(0 \leq i_1 \leq n - 1\);
We call the expression $B_{(\tilde{w})}$ the canonical reduced expression for the minimal length representatives of the right cosets of $W(A_n)$ in $W(\tilde{A}_n)$.

We remark that in an affine block, the affine brick on the left (resp. on the right) of a short affine brick of length $l$ has length at most $l - 2$ (resp. at least $l + 1$), while the lengths of long affine bricks form a non-decreasing sequence from left to right, in which two affine bricks with the same length are equal.

3. Proof of Theorem 2.13

3.1. Skeleton of the proof. Let $j_s, i_s, 1 \leq s \leq m$, be any family of integers satisfying the pairwise inequalities in Definition 2.12. It suffices to prove what we call for short the key statement:

The expression $w = h(j_1, i_1)a_{n+1}h(j_2, i_2)a_{n+1} \cdots h(j_m, i_m)a_{n+1}$ is reduced and affine length reduced, and satisfies $R(w) = \{a_{n+1}\}$. Furthermore it is the
unique such expression of \( w \) satisfying the conditions in Theorem 2.13

By Lemma 3.1, our key statement is equivalent to the following set of six statements, letting

\[ w_m = h(j_1, i_1) a_{n+1} h(j_2, i_2) a_{n+1} \dots h(j_m, i_m) : \]

1. The expression \( w_m a_{n+1} \) is reduced.
2. The expression \( w_m a_{n+1} \sigma_k \) is reduced for \( 2 \leq k \leq n - 1 \).
3. The expression \( w_m a_{n+1} \sigma_1 \) is reduced.
4. The expression \( w_m a_{n+1} \sigma_n \) is reduced.
5. The element expressed by \( w_m a_{n+1} \) has affine length \( m \).
6. The expression \( w_m a_{n+1} \) is unique with the given conditions.

Our main tool is the criterion given in Bourbaki [6, Ch. IV, §1.4]. Given a Coxeter system \( (W, S) \), we attach to any finite sequence \( s = (s_1, \cdots, s_r) \) of elements in \( S \), the sequence \( t_s = (t_{s_1}, \cdots, t_{s_r}) \) of elements in \( W \) defined by:

\[ t_s(s_j) = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1} \quad \text{for } 1 \leq j \leq r. \]

We call \( t_s(s_j) \) the reflection attached to \( s_j \) (in the expression \( s \)). We shorten the notation sometimes by writing the expression on the left into brackets and writing \([\ldots]^{-1}\) for its inverse, namely we write:

\[ t_s(s_j) = [s_1 \cdots s_{j-1}] s_j [\ldots]^{-1}. \]

We know from [6, Ch. IV, §1, Lemma 2] that the product \( s_1 \cdots s_r \) is a reduced expression (of the element \( s_1 \cdots s_r \) in \( W \)) if and only if all terms in the sequence \( t_s \) are distinct. We will use this in the following form:

**Lemma 3.1.** Let \( s = (s_1, \cdots, s_r) \) be a sequence of elements in \( S \). Assume that \( s_1 \cdots s_{r-1} \) is a reduced expression. The expression \( s_1 \cdots s_r \) is not reduced if and only if there exists an integer \( j, 1 \leq j \leq r - 1 \), such that \( t_s(s_j) = t_s(s_{r-1}) \). Such an integer \( j \), if it exists, is unique.

We remark from the proof in [6] that having \( t_s(s_j) = t_s(s_{r-1}) \) for some \( j \leq r - 1 \) is equivalent to the equality \( s_1 \cdots s_j \cdots s_r = s_1 \cdots \hat{s}_j \cdots s_r \) in \( W \), where the hat \( \hat{s}_j \) over \( s_j \) means that \( s_j \) is removed from the expression. We call for short the \( j \)-th element \( s_j \) of the sequence the hat partner of \( s_r \).

We illustrate the use of this Lemma with the following statement:

**Lemma 3.2.** Let \( w \in W(\tilde{A}_n) \) and \( p \in P \) such that \( wp \) is reduced. Then \( wp a_{n+1} \) is reduced if and only if \( wa_{n+1} \) is reduced.

**Proof:** The proof by induction on the length of \( p \) is immediate once the length 1 case is established. Assume \( w \sigma_k \) is reduced for some \( k, 2 \leq k \leq n - 1 \) and pick a reduced expression \( w \) for \( w \). From Lemma 3.1 we see that \( w \sigma_k a_{n+1} \) is not reduced if there is a simple reflection \( s \) in \( w \sigma_k \), actually in \( w \), such that \( t_{w \sigma_k a_{n+1}}(a_{n+1}) = t_{w \sigma_k a_{n+1}}(s) \). Since \( \sigma_k \) commutes with \( a_{n+1} \) this equality reads exactly \( t_{wa_{n+1}}(a_{n+1}) = t_{wa_{n+1}}(s) \) for some \( s \) in \( w \), which is equivalent to \( wa_{n+1} \) being not reduced. \( \square \)
The proof of Theorem 2.13 translated into the set of statements (1) to (6) above, proceeds by induction on \( m \). The key statement holds for \( m = 1 \): it is given by Lemma 2.8. Uniqueness follows from Lemma 2.3. In subsections 3.4 to 3.8 we let \( m \geq 2 \) and, assuming that properties (1) to (6) hold for \( w_k \) for any \( k \leq m - 1 \), we prove successively properties (1) to (6) for \( w_m \). To do this we rely on Lemma 3.1.

We start with a sequence \( d = (s_1, \ldots, s_r) \) and a simple reflection \( s \) such that the expression \( s_1 \cdots s_r \) is reduced and we want to show that \( s_1 \cdots s_r s \) is also reduced. We transform the reflection \( t_d(s) \) attached to \( s \) in the expression \( s_1 \cdots s_r s \) into the reflection attached to some simple reflection \( s' \) in another expression \( s'_1 \cdots s'_k s' \) which is known to be reduced by induction hypothesis.

We recall (2.5) and Corollary 2.11 we need the pairwise inequalities. In other words: there will be computation, mostly contained in preliminary lemmas.

3.2. **Rigidity Lemma.** We start with an important Lemma.

**Lemma 3.3 (Rigidity Lemma).** Let \( w = u \sigma_1 \cdots \sigma_n \) be reduced: \( l(w) = l(u) + n \), with \( u \in W(\tilde{A}_n) \). Then \( a_{n+1} \) does not belong to \( A(w) \), in other words \( u \sigma_1 \cdots \sigma_n a_{n+1} \) is reduced.

**Proof:** We proceed by induction on \( r = l(u) \), the case \( r = 0 \) being trivial and the case \( r = 1 \) contained in Lemmas 2.3 and 2.8. We assume that \( r \geq 2 \) and that the assertion holds for any \( u \) such that \( l(u) \leq r - 1 \). We take \( u \) with \( l(u) = r \) and pick a reduced expression \( u = s_1 \cdots s_r, s_i \in S_n \). Assume for a contradiction that \( w a_{n+1} \) is not reduced. By Lemma 3.1 and the induction hypothesis the hat partner of \( a_{n+1} \) on the right is the leftmost term \( s_1 \), i.e. we have the following equality:

\[
(1.1) \quad s_1 \cdots s_r \sigma_1 \cdots \sigma_n = s_2 \cdots s_r \sigma_1 \cdots \sigma_n a_{n+1} \quad \text{(both sides reduced)}.
\]

We discuss according to \( s_r \), which is not equal to \( \sigma_1 \).

(a) If \( s_r = \sigma_k \) for \( 2 < k \leq n \), equality (1.1) becomes

\[
\left(s_1 \cdots s_{r-1} \sigma_1 \cdots \sigma_n \sigma_{k-1} = s_2 \cdots s_{r-1} \sigma_1 \cdots \sigma_n \sigma_{k+1} \right)
\]

which, after canceling \( \sigma_{k-1} \), contradicts the induction hypothesis for \( r - 1 \).

(b) If \( s_r = a_{n+1} \) and \( s_{r-1} = \sigma_1 \) we transform (1.1) with the braid on \( a_{n+1} \) and \( \sigma_1 \) to get

\[
\left(s_1 \cdots s_{r-2} a_{n+1} \sigma_1 \cdots \sigma_n a_{n+1} \sigma_n = s_2 \cdots s_{r-2} a_{n+1} \sigma_1 \cdots \sigma_n a_{n+1} \sigma_n a_{n+1} \right)
\]

where both sides are reduced, and after using the braid on \( a_{n+1} \) and \( \sigma_n \) on the right and canceling the rightmost terms we get:

\[
\left(s_1 \cdots s_{r-2} a_{n+1} \sigma_1 \cdots \sigma_n a_{n+1} \sigma_n = s_2 \cdots s_{r-2} a_{n+1} \sigma_1 \cdots \sigma_n a_{n+1} \sigma_n \right)
\]

where both sides are reduced. Using the Dynkin automorphism \( a_{n+1} \rightarrow \sigma_1 \rightarrow \sigma_2 \cdots \), denoted by \( s \mapsto s' \), we obtain

\[
\left(s'_1 \cdots s'_{r-2} \sigma_1 \cdots \sigma_n = s'_2 \cdots s'_{r-2} \sigma_1 \cdots \sigma_n a_{n+1} \right) \quad \text{(both sides reduced)}
\]

which contradicts our induction hypothesis for \( r - 2 \).
(c) If \( s_r = \sigma_2 \) equality (3.1) reads
\[
s_1 \cdots s_{r-1} \sigma_2 \sigma_1 \cdots \sigma_n = s_2 \cdots s_{r-1} \sigma_2 \sigma_1 \cdots \sigma_n a_{n+1}
\]
that transforms under the inverse of the Dynkin automorphism above, denoted by \( s \mapsto s'' \), into:
\[
s_1'' \cdots s_{r-1}'' \sigma_1 a_{n+1} \sigma_1 \cdots \sigma_{n-1} = s_2'' \cdots s_{r-1}'' \sigma_1 a_{n+1} \sigma_1 \cdots \sigma_n
\]
where both sides are reduced. Now right multiplication by \( a_{n+1} \) clearly reduces the right-hand-side expression, whereas it cannot reduce the left-hand-side expression by (b): observe in the proof of (b) that the induction hypothesis for \( r - 1 \) actually gives us case (b) for length \( r + 1 \), which is what we need here.

(d) We are reduced to the case where no reduced expression of \( u \) falls into cases (a), (b) or (c). Then any reduced expression of \( u \) ends (on the right) with \( \sigma_n a_{n+1} \) and our claim follows from another Lemma:

**Lemma 3.4.** Let \( u \) be an element of \( W(\tilde{A}_n) \) of length \( r \geq 2 \) such that all reduced expressions of \( u \) end with \( \sigma_n a_{n+1} \) (on the right). Then \( u \) is rigid (has a unique reduced expression) and is a left truncation of
\[
(\sigma_1 \cdots \sigma_n a_{n+1})^k \quad (k \geq 1),
\]
which is a rigid hence reduced expression.

**Proof.** The rigidity of (3.2) is clear and well-known. We show by induction that for \( 2 \leq t \leq r \), all reduced expressions of \( u \) end on the right with the rightmost \( t \) terms in \( (\sigma_1 \cdots \sigma_n a_{n+1})^k \). This holds for \( t = 2 \), we assume it holds up to \( t - 1 \) and prove it for \( t \leq 3 \). Write a reduced expression for \( u \) as \( s_r \cdots s_1 \). By induction \( s_1 \) to \( s_{t-1} \) are uniquely determined. There is actually no choice for \( s_t \) (it cannot commute with \( s_{t-1} \) (otherwise we would get another expression with different rightmost \( t - 1 \) terms), which itself does not commute with \( s_{t-2} \), and \( s_t s_{t-1} s_{t-2} \) cannot be a braid for the same reason. Hence \( s_t \) is the unique neighbour of \( s_{t-1} \) in the Dynkin diagram of \( \tilde{A}_n \) different from \( s_{t-2} \). □

So in case (d), \( w a_{n+1} \) is a left-truncation of some element of form (3.2), hence reduced.

□

**Remark 3.5.** The two lemmas above clearly hold when replacing \( \sigma_1 \cdots \sigma_n \) by \( \sigma_n \cdots \sigma_1 \), using the Dynkin automorphism of \( A_n \).

### 3.3. A few more lemmas.

We proceed with more lemmas needed in the proof.

**Lemma 3.6.** The expression \( D = a_{n+1} \sigma_1 \cdots \sigma_n \sigma_1 a_{n+1} \) is reduced and affine length reduced.

**Proof.** The expressions \( a_{n+1} \sigma_1 \cdots \sigma_n \sigma_1 \) and \( \sigma_1 \cdots \sigma_n \sigma_1 a_{n+1} \) are reduced. If the given expression was not reduced, the hat partner of the \( a_{n+1} \) on the right
could only be the $a_{n+1}$ on the left, contradicting uniqueness in Lemma 2.8. Assuming the affine length is 1, we get $a_{n+1}\sigma_1 \cdots \sigma_n = h(r, i)a_{n+1}x$ with the notation in Lemma 2.8. Since $x \in P$ is clearly impossible ($a_{n+1}$ cancels out), either $\sigma_1$ or $\sigma_n$ belongs to $\mathcal{R}(x)$ hence to $\mathcal{R}(D)$. Since $D$ also equals $a_{n+1}\sigma_n \cdots \sigma_1 \cdots \sigma_n a_{n+1}$, it is enough to deal with $\sigma_1$, so we assume $D\sigma_1$ is not reduced. Then the $\sigma_1$ on the right has a hat partner $s$ in $D$. This $s$ must be the $a_{n+1}$ on the left (for otherwise the expression $\sigma_1 \cdots \sigma_n a_{n+1}$ would not be reduced, contrary to Lemma 2.8). Transforming the resulting equality with a braid and cancellations we obtain $a_{n+1}\sigma_1 \cdots \sigma_n = \sigma_1 \cdots \sigma_n a_{n+1}$, contradicting Lemma 2.8 again. □

**Lemma 3.7.** We consider an expression of the following form:

$$h(j_1, i_1)a_{n+1}h(j, i)a_{n+1}, \quad 0 \leq i_1, i \leq n - 1, \quad 1 \leq j_1, j \leq n + 1,$$

with $h(j, i) \neq 1$. This expression is reduced except in the four “deficient” cases listed below together with the hat partner of the rightmost $a_{n+1}$:

1. $h(j, i) = [i, 1]$ and $i_1 \geq i \geq 1$,
   the hat partner is the $\sigma_1$ in $h(j_1, i_1) = [j_1, n][\sigma_{i_1} \cdots \sigma_i \cdots \sigma_1$;

2. $h(j, i) = [j, n]$ and $1 < j \leq n$, $j_1 \leq j$, $i_1 < j - 1$,
   the hat partner is the $\sigma_j$ in $h(j_1, i_1) = \sigma_j \cdots \sigma_{j_1} \cdots \sigma_n[i_1, 1]$;

3. $h(j, i) = [j, n]$ and $2 < j \leq n$, $j_1 < j$, $i_1 \geq j - 1$,
   the hat partner is the $\sigma_{j-1}$ in $h(j_1, i_1) = \sigma_{j-1} \cdots \sigma_{j_1} \cdots \sigma_n[i_1, 1]$;

4. $h(j, i) = [2, n]$ and $j_1 = 1$, $i_1 = 1$,
   the hat partner is the leftmost $\sigma_1$ in $h(j_1, i_1) = \sigma_1 \cdots \sigma_n \sigma_1$.

In particular, if $h(j, i)$ is extremal, the expression is reduced.

**Proof.** From Lemma 2.8 we know that $h(j_1, i_1)a_{n+1}h(j, i)$ is reduced. Assume that $h(j_1, i_1)a_{n+1}h(j, i)a_{n+1}$ is not. The hat partner of the rightmost $a_{n+1}$ cannot be the leftmost $a_{n+1}$ because the commutant of $a_{n+1}$ in $W(A_n)$ is $P$. So $h(j_1, i_1)$ is not equal to 1 and the hat partner is a reflection $s$ in $h(j_1, i_1)$. Truncating the elements on the left of $s$ we obtain an equality $h(j_1', i_1')a_{n+1}h(j, i)a_{n+1} = h(j_1', i_1')a_{n+1}h(j, i)$ where $h(j_1', i_1')$ is obtained from $h(j_1', i_1')$ by removing the leftmost reflection. We rewrite this as:

$$a_{n+1}h(j_1', i_1')^{-1}h(j_1', i_1')a_{n+1} = h(j, i)a_{n+1}h(j, i)^{-1}.$$

Let $V(j_1', i_1')$ be the expression on the left hand side. We compute:

$$V(j_1', i_1') = \begin{cases} [i_1', 1]a_{n+1}[1, i_1'] & \text{if } j_1' = n + 1; \\
[j_1', n]a_{n+1}[n, j_1'] & \text{if } 1 < j_1' \leq n \text{ and } i_1' < j_1' - 1; \\
D & \text{if } 1 < j_1' \leq n \text{ and } i_1' = j_1' - 1; \\
[j_1' + 1, n]a_{n+1}[n, j_1' + 1] & \text{if } 1 < j_1' \leq n \text{ and } i_1' \geq j_1'; \\
D & \text{if } j_1' = 1 \text{ and } i_1' \neq 1; \\
[2, n]a_{n+1}[n, 2] & \text{if } j_1' = 1 \text{ and } i_1' = 1. \end{cases} (3.3)$$
Our equality implies that $V(j_1', i_1')$ has affine length 1, which excludes the cases where it is equal to $D$, by Lemma 3.6. The uniqueness in Lemma 2.8 now implies that $h(j, i)$ is equal to one of the following: $[i_1', 1], [j_1', n], [j_1' + 1, n]$ or $[2, n]$, it remains to plug in the conditions in (3.3).

□

Lemma 3.8. Let $m \geq 2$, assume the pairwise inequalities hold and $j_m > 1$. The element $h(j_{m-1}, i_{m-1})j_m, n]$ is reduced and equal to one of the following reduced elements:

\[ h(j_{m-1}, i_{m-1})[j_{m-1} - 1, n - 1] \quad \text{if} \ j_{m-1} > j_m > i_{m-1} + 1 \]

\[ h(j_{m-1}, i_{m-1} - 1)[j_{m-1} - 1, n - 1] \quad \text{if} \ j_{m-1} > i_{m-1} + 1 \geq j_m > 1 \]

\[ h(j_{m-1}, i_{m-1})[j_{m-1}, n - 1] \quad \text{if} \ i_{m-1} + 1 \geq j_{m-1} \geq j_m > 1 \]

Writing this as $h(j_{m-1}, i_{m-1})j_m, n] = h(j_{m-1}, i_{m-1})u_m, n - 1]$ with $u_m \geq 2$, the sequence $\{(j_1, i_1), \cdots, (j_{m-2}, i_{m-2}), (j_{m-1}', i_{m-1}')\}$ satisfies the pairwise inequalities.

Proof. We note the following formulas, for $0 \leq a \leq n - 1, 1 \leq b \leq n + 1, 1 \leq c \leq n$:

\[ [b, n][a, 1][c, n] = [c, n][a, 1][b - 1, n - 1] \quad \text{if} \ c > a + 1, b > c; \]

\[ = [b + 1, n][a, 1][b, n - 1] \quad \text{if} \ c > a + 1, b = c; \]

\[ = [c - 1, n][a - 1, 1][b - 1, n - 1] \quad \text{if} \ 1 < c \leq a + 1 < b; \]

\[ = [c - 1, n][a, 1][b, n - 1] \quad \text{if} \ 1 < c \leq b \leq a + 1. \]

They imply the equalities in the Lemma, with $a = i_{m-1} \geq 0, c = j_m > 1, b = j_{m-1} \geq c > 1$. The pairwise inequalities are easy to check. The expressions obtained are reduced by Lemma 2.8 and have the same length that the initial expression.

□

3.4. The expression $w_ma_{n+1}$ is reduced. The case $m = 2$ has been dealt with in Lemma 3.7 so we let $m \geq 3$. Furthermore the Rigidity Lemma 3.8 gives the result if $i_m = 0$, or if $i_m = n - 1$, or if $j_m = 1$, hence we assume $j_m > 1$ and $1 \leq i_m < n - 1$.

Suppose for a contradiction that $w_m a_{n+1}$ is not reduced and let $s$ be the hat partner of the $a_{n+1}$ on the right (Lemma 3.1). By induction hypothesis the expression $h(j_2, i_2) a_{n+1} \cdots h(j_m, i_m) a_{n+1}$ is reduced so $s$ is to be removed from the leftmost part $h(j_1, i_1) a_{n+1}$. From Lemma 3.1 we have $t_{w_m a_{n+1}}(a_{n+1}) = t_{w_m a_{n+1}}(s)$, with $t_{w_m a_{n+1}}(s) = t_{h(j_1, i_1) a_{n+1}}(s)$, so:

\[ [h(j_1, i_1) a_{n+1} \cdots h(j_m, i_m)] a_{n+1} [\cdots]^{-1} = t_{h(j_1, i_1) a_{n+1}}(s). \]

Recalling our assumptions $j_m > 1$ and $1 \leq i_m < n - 1$, we compute

\[ X = [h(j_{m-1}, i_{m-1}) a_{n+1} h(j_m, i_m)] a_{n+1} [\cdots]^{-1} \]

\[ = [h(j_{m-1}, i_{m-1}) a_{n+1} h(j_m, i_m)[i_m, 2]] \sigma_1 a_{n+1} \sigma_1 [\cdots]^{-1} \]

\[ = [h(j_{m-1}, i_{m-1}) a_{n+1} h(j_m, i_m)[i_m, 2]] \sigma_1 [\cdots]^{-1} \]

\[ = [h(j_{m-1}, i_{m-1}) a_{n+1} h(j_m, i_m)[i_m, 2]] \sigma_1 [\cdots]^{-1} \]
We let \( h(j_{m-1}, i_{m-1})[j_m, n] = h(j'_{m-1}, i'_{m-1})x, x \in P, \) and
\[
v = h(j_1, i_1)a_{n+1} \ldots h(j_{m-2}, i_{m-2})a_{n+1}h(j'_{m-1}, i'_{m-1})a_{n+1}
\]
With Lemma \[3.3\] we know that the expression \( v \) satisfies the conditions in the key statement for \( m - 1 \), so it is reduced and for any reduced expression \( y \) of an element in \( W(A_n) \), \( vy \) is reduced. Let \( y \) be a reduced form of \( x\sigma_n[i_m, 2] \) (\( \sigma_1 \) is not in the support). The expression \( vy\sigma_1 \) is reduced with leftmost terms \( h(j_1, i_1)a_{n+1} \) \( (m \geq 3) \), so with Lemma \[3.1\] \( vy\sigma_1y^{-1}v^{-1} \) cannot be equal to \( t_{vy\sigma_1}(s) = t_{h(j_1, i_1)a_{n+1}(s)} \), a contradiction with \( w_m a_{n+1}w_m = vy\sigma_1y^{-1}v^{-1} \).

3.5. The expression \( w_m a_{n+1}\sigma_k \) is reduced for \( 2 \leq k \leq n - 1 \). We just proved that \( w_m a_{n+1} \) is reduced, so this follows from Lemmas \[2.3\] and \[3.2\].

3.6. The expression \( w_m a_{n+1}\sigma_1 \) is reduced. Let \( m \geq 2 \). We have shown that \( w_m a_{n+1} \) is a reduced expression. Suppose for a contradiction that \( w_m a_{n+1}\sigma_1 \) is not and let \( s \) be the hat partner of \( \sigma_1 \) (Lemma \[3.1\]). By induction hypothesis \( s \) belongs to the leftmost part of the expression: \( h(j_1, i_1)a_{n+1} \). We have
\[
t_{w_m a_{n+1}\sigma_1}(\sigma_1) = w_m a_{n+1}\sigma_1 a_{n+1}w_m^{-1} = w_m \sigma_1 a_{n+1}w_m^{-1} = t_{w_m a_{n+1}}(a_{n+1})
\]
while \( t_{w_m a_{n+1}\sigma_1}(s) = t_{w_m a_{n+1}}(s) \) since the two expressions have the same leftmost part \( h(j_1, i_1)a_{n+1} \).

If \( i_m = 0 \) the expression \( w_m \sigma_1 \) is obtained from \( w_m \) by replacing \( h(j_m, 0) \) with \( h(j_m, 1) \). It satisfies the conditions in the key statement, so \( w_m a_{n+1} \) is reduced and \( t_{w_m a_{n+1}}(a_{n+1}) \) cannot be equal to \( t_{w_m a_{n+1}}(s) \).

If \( i_m \geq 1 \), we have the following reduced expression for \( w_m \sigma_1 \):
\[
y = h(j_1, i_1)a_{n+1} \ldots h(j_{m-1}, i_{m-1})a_{n+1}[j_m, n][i_m, 2].
\]
A contradiction will follow if we prove that \( ya_{n+1} \) is reduced or, equivalently by Lemma \[3.2\] that
\[
z = h(j_1, i_1)a_{n+1} \ldots h(j_{m-1}, i_{m-1})a_{n+1}[j_m, n]a_{n+1}
\]
is reduced. Lemma \[3.3\] does the work if \( j_m = 1 \). If \( j_m > 1 \), we observe that
\[
[h(j_{m-1}, i_{m-1})a_{n+1}[j_m, n]a_{n+1} \ldots]^{-1} = [h(j_{m-1}, i_{m-1})[j_m, n]a_{n+1} \ldots]^{-1}.
\]
By Lemma \[3.8\] the expression \( h(j_{m-1}, i_{m-1})[j_m, n] \) is reduced hence, by induction, so is \( x = h(j_1, i_1)a_{n+1} \ldots h(j_{m-1}, i_{m-1})[j_m, n] \). If \( m > 2 \), we obtain \( t_x(\sigma_n) = t_x(s) \), a contradiction. If \( m = 2 \), we see directly that \( z = h(j_1, i_1)a_{n+1}[j_2, n]a_{n+1} \) is reduced using a braid, Lemma \[3.8\] and Lemma \[2.8\].

3.7. The expression \( w_m a_{n+1}\sigma_n \) is reduced. We follow the same track as for \( \sigma_1 \) and examine the expression \( w_m \sigma_n a_{n+1} \).

If \( i_m = n - 1 \) and \( j_m = n \), the expression \( w_m \sigma_n \) is obtained from \( w_m \) by replacing \( h(n, n - 1) \) with \( h(n - 1, n - 1) \) at the \( m \)-th rank. It satisfies the pairwise inequalities, so \( w_m \sigma_n a_{n+1} \) is reduced.

If \( i_m = n - 1 \) and \( j_m \leq n - 1 \), we have
\[
[j_m, n][n - 1, 1]a_{n+1} = [j_m, n - 2][n, 1] = [n, 1][j_m + 1, n - 1]
\]
and the expression $h(j_1, i_1)a_{n+1} \ldots h(j_{m-1}, i_{m-1})a_{n+1}[n, 1]j_m + 1, n-1]a_{n+1}$ is reduced by Lemmas 3.2 and 3.3.

If $i_m < n - 1$, we have $u := [j_m, n][i_m, 1]a_n = [j_m, n-1][i_m, 1]$. If $j_m > i_m + 1$, then $u = [i_m, 1][j_m, n-1]$; if $j_m \leq i_m + 1$, then $u = [i_m, 1][j_m + 1, n-1]$ (2.6). The piece $[\ldots, n-1]$ belongs to $P$ and can be left out (Lemma 3.2). We get the expression $h(j_1, i_1)a_{n+1} \ldots h(j_{m-1}, i_{m-1})a_{n+1}[i', 1]a_{n+1}$ with $i' = i_m$ or $i_m + 1$ and $i_m-1 < i'$. For $m = 2$ Lemma 3.7 ensures that this expression is reduced (since $i_1 < i'$) and we are done. For $m > 2$ we let

$$v = h(j_{m-1}, i_{m-1})a_{n+1}[i', 1]a_{n+1} = h(j_{m-1}, i_{m-1})[i', 1]a_{n+1}a_{n+1}\sigma_1.$$

If $i_{m-1} = 0$, or if $i_{m-1} \geq 1$ with (2.6), we have:

$$v = h(j_{m-1}, i'[i_{m-1} + 1, 2]a_{n+1}a_{n+1}\sigma_1.$$

Since $h(j_{m-1}, i')$ satisfies the pairwise inequalities, we get a reduced expression hence the contradiction needed.

3.8. **Affine length and uniqueness.** We already know that an element of affine length $k$ can be written as

$$h(j_1', i_1')a_{n+1}h(j_2', i_2')a_{n+1} \ldots h(j_k', i_k')a_{n+1}x$$

where $x \in P$ and the family of integers $j_s', i_s', 1 \leq s \leq k$, satisfies the pairwise inequalities, and we just proved that for $k \leq m$ this expression is reduced. Assume for a contradiction that either $w_m a_{n+1}$ has affine length less than $m$, or there is another expression of this element satisfying the required conditions. Either way, we have an integer $k \leq m$ and a family of integers $j_s', i_s', 1 \leq s \leq k$, satisfying the pairwise inequalities, such that

$$w = h(j_1, i_1)a_{n+1}h(j_2, i_2)a_{n+1} \ldots h(j_m, i_m)a_{n+1}$$

$$= h(j_1', i_1')a_{n+1}h(j_2', i_2')a_{n+1} \ldots h(j_k', i_k')a_{n+1}x$$

with $x \in P$ and both expressions reduced. We already proved that $R(w) = \{a_{n+1}\}$, hence $x = 1$ and we can cancel out the term $a_{n+1}$ on the right. By induction the element expressed by $w_m = h(j_1, i_1)a_{n+1}h(j_2, i_2)a_{n+1} \ldots h(j_m, i_m)$ has affine length $m - 1$ and can be uniquely written in this form, so $k = m$ and $(j_s', i_s') = (j_s, i_s)$ for any $s, 1 \leq s \leq m$.

4. **First consequences**

4.1. **Left multiplication.** We need some insight into left multiplication of affine blocks by a simple reflection. We recall first a well-known property of Coxeter groups.

**Proposition 4.1.** Let $(W, S)$ be a Coxeter group and let $I$ be a strict subset of $S$, generating the parabolic subgroup $W_I$. Let $W^I$ be the set of minimal coset representatives of $W/W_I$. For $s \in I$ and $w \in W^I$, either $sw \in W^I$, or there is $r \in I$ such that $sw = wr$ (in particular $l(sw) = l(w) + 1$).
Theorem 4.2. Let \( w_a = B(j_1, i_1)B(j_2, i_2)\ldots B(j_m, i_m) \) be an affine block of affine length \( m \geq 1 \), let \( w_a \) be the corresponding element in \( W(\tilde{A}_n) \), and let \( s \in S_n \). Then:

1. either \( sw_a \) is not of minimal length in its right \( W(\tilde{A}_n) \)-coset, and we have actually \( l(sw_a) = l(w_a) + 1 \) and \( sw_a = w_a\sigma_v \) for some \( v, 1 \leq v \leq n \);
2. or \( sw_a \) has minimal length in its right \( W(\tilde{A}_n) \)-coset and one of the following holds:
   a. \( s = a_{n+1} \) and \( h(j_1, i_1) = 1 \), so \( a_{n+1}w_a \) reduces to the affine block
      \[ B(j_2, i_2)\ldots B(j_m, i_m) \] (1 if \( m = 1 \)).
   b. \( s = a_{n+1} \) and \( h(j_1, i_1) \) is extremal, so \( a_{n+1}w_a \) is the affine block
      \[ a_{n+1}B(j_1, i_1)B(j_2, i_2)\ldots B(j_m, i_m) \].
   c. Otherwise, \( sw_a \) is expressed as an affine block of the following form:
      \[ B(j'_1, i'_1)B(j'_2, i'_2)\ldots B(j'_m, i'_m) \]
      where the \( 2m \)-tuples \( (j_1, i_1, \ldots, j_m, i_m) \) and \( (j'_1, i'_1, \ldots, j'_m, i'_m) \) differ in one and only one entry, say \( j'_r \neq j_r \) or \( i'_r \neq i_r \). If \( l(sw_a) = l(w_a) + 1 \) we have \( j'_r = j_r - 1 \) or \( i'_r = i_r + 1 \), while if \( l(sw_a) = l(w_a) - 1 \) we have \( j'_r = j_r + 1 \) or \( i'_r = i_r - 1 \).

Remark 4.3. In the case when \( l(sw_a) = l(w_a) - 1 \), Theorem 4.2 says that the “hat partner” of \( s \) is a \( \sigma_{j_r} \) or a \( \sigma_{i_r} \) and that the resulting expression is in canonical form, i.e. an affine block.

Proof. We establish first our statement in the case when \( s = \sigma_u \) with \( 1 \leq u \leq n \). The case of affine length 1 is detailed in the following Lemma, easily checked, in fact an automaton describing left multiplication of an affine brick \( B(j, i) \) by \( \sigma_v \). The result is either \( B(j, i)\sigma_v \) for some \( v \), or an affine brick of length \( l(B(j, i)) \pm 1 \).
Lemma 4.4. For $1 \leq j \leq n + 1$ and $n - 1 \geq i \geq 0$, we have if $1 \leq u \leq n$:

$$
\sigma_u \mathcal{B}(j, i) = \begin{cases} 
\mathcal{B}(j - 1, i) & \text{if } u = j - 1, \\
\mathcal{B}(j, i + 1) & \text{if } u = j, \\
\mathcal{B}(j, i - 1) & \text{if } u = j + 1.
\end{cases}
$$

In particular, $\mathcal{L}(\mathcal{B}(j, i))$ is the set $\{\sigma_j, \sigma_i\}$ if $\mathcal{B}(j, i)$ is short, the set $\{\sigma_j, \sigma_{i+1}\}$ if $\mathcal{B}(j, i)$ is long.

We prove the general case by induction on $a = (j, i)$. If some $\sigma$ appears on the right we are in case (1). Assume now $\mathcal{B}(j, i) = \mathcal{B}(j_m, i_m)$. If $j'_m = j_m - 1$ or $i'_m = i_m + 1$, we are in case (2c) since we get an affine block. If $j'_m = j_m + 1$ or $i'_m = i_m - 1$, it seems at first that the resulting expression might not be canonical, depending on the value of $j_{m-1}$ or $i_{m-1}$. But actually the expression has no other choice than being canonical. Indeed we are in a case where $l(\sigma_u w_a) = l(w_a) - 1$, hence $\sigma_u w_a$ has minimal length in its right coset and by Proposition 2.10 the required inequalities are satisfied.

- If $\sigma_u w'_a$ is of minimal length in its coset, we write $\sigma_u w'_a = w'_a \sigma_v$ for some $v$, $1 \leq v \leq m$, so that

$$
\sigma_u w_a = w'_a \sigma_v \mathcal{B}(j_m, i_m).
$$

We deal with $\sigma_u \mathcal{B}(j_m, i_m)$ using the previous Lemma. If some $\sigma_z$ appears on the right we are in case (1). Assume now $\sigma_u \mathcal{B}(j_m, i_m) = \mathcal{B}(j'_m, i'_m)$. If $j'_m = j_m - 1$ or $i'_m = i_m + 1$, we are in case (2c) since we get an affine block. If $j'_m = j_m + 1$ or $i'_m = i_m - 1$, it seems at first that the resulting expression might not be canonical, depending on the value of $j_{m-1}$ or $i_{m-1}$. But actually the expression has no other choice than being canonical. Indeed we are in a case where $l(\sigma_u w_a) = l(w_a) - 1$, hence $\sigma_u w_a$ has minimal length in its right coset and by Proposition 2.10 the required inequalities are satisfied.
In cases (1), (2), (3) above, the element \( w \) does not satisfy the required inequalities. In such a case we apply Proposition 2.10 and get

\[
\sigma_u w_a = B(j_1, i_1)B(j_2, i_2) \cdots B(j_{m-1}, i_{m-1})B(j_m, i_m)\sigma_t
\]

with \( t = 1 \) or \( n \). Proposition 4.1 leaves only one choice, namely \( \sigma_u w_a = w_a \sigma_t \). This finishes the proof in the case \( s = \sigma_u \).

We take next \( s = a_{n+1} \). The cases when \( h(j_1, i_1) \) is extremal or equal to 1 are obvious. Otherwise we have \( h(j_1, i_1) = [j_1, n] \) with \( 1 < j_1 \leq n \) or \( h(j_1, i_1) = [i_1, 1] \) with \( i_1 \geq 1 \). Using a braid we reduce the claim to the one we have already proved for \( s = \sigma_n \) or \( s = \sigma_1 \), left-multiplying the affine block starting at \( h(j_2, i_2) \). Checking that the resulting expression satisfies the pairwise inequalities is straightforward and left to the reader.

4.2. **Right descent set.** In this subsection we study the right descent set \( \mathcal{R}(w) \) of an element \( w \) in \( W(A_n) \) with \( L(w) = m > 0 \), given canonically as

\[
w = B(j_1, i_1)B(j_2, i_2) \cdots B(j_m, i_m)x, \quad x \in W(A_n),
\]

(hence the family \( (j_s, i_s)_{1 \leq s \leq m} \) satisfies the pairwise inequalities).

The first observation is the following: \( \mathcal{R}(x) \subseteq \mathcal{R}(w) \subseteq \mathcal{R}(x) \cup \{a_{n+1}\} \). Indeed if a simple reflection \( s \) other than \( a_{n+1} \) does not belong to \( \mathcal{R}(x) \), then \( ws \) is reduced by Theorem 2.13.

The determination of \( \mathcal{R}(w) \) then amounts to giving the conditions for \( a_{n+1} \) to belong to this set. Writing \( x = h(j, i)p, p \in P \), Lemma 3.2 shows that these conditions depend only on the \( h(j, i) \) part of \( x \), not on \( p \). Of course Theorem 2.13 ensures that if \( (j_m, i_m), (j, i) \) satisfy the pairwise inequalities, then \( a_{n+1} \) does not belong to \( \mathcal{R}(w) \). It is tempting to believe that if \( x \) is extremal, then \( wa_{n+1} \) is reduced. This holds for \( m = 1 \) (Lemma 3.7) but it is not true in general, as we can see in the following Lemma that gives a full account of the case \( m = 2 \).

**Lemma 4.5.** We consider an expression of the following form:

\[
B(j_1, i_1)B(j_2, i_2)x a_{n+1}
\]

where \( x \in W(A_n) \) and \((j_1, i_1), (j_2, i_2)\) satisfy the pairwise inequalities, and we write \( x = h(j, i)p, p \in P \). If \( h(j, i) \neq 1 \) this expression is reduced except:

- in the four “deficient” cases listed in Lemma 3.7 with \( j_1, i_1 \) replaced by \( j_2, i_2 \),
- in the cases listed below together with the hat partner of the rightmost \( a_{n+1} \):
  1. \( h(j, i) = \sigma_n \sigma_1 \) and \( j_2 > 1 \) and \( 1 \leq i_2 < n - 1 \), the hat partner is the leftmost \( a_{n+1} \):
  2. \( h(j, i) = h(n, i) \) and \( 1 \leq i \leq i_2 < n - 1 \), \( i < j_2 \), and \( i_1 \geq i - 1 \), the hat partner is the \( \sigma_{i-1} \) in \( h(j_1, i_1) = [j_1, n] \sigma_{i} \cdots \sigma_{i-1} \cdots \sigma_{1} \);
  3. \( h(j, i) = h(n, i) \) and \( 1 \leq i \leq i_2 < n - 1 \), \( i \geq j_2 \), and \( i_1 \geq i \), the hat partner is the \( \sigma_{i} \) in \( h(j_1, i_1) = [j_1, n] \sigma_{i} \cdots \sigma_{i} \cdots \sigma_{1} \).

We note that in cases (1), (2), (3) above, the element \( x \) is extremal.
We skip the (technical) proof of this Lemma. Further computation shows that for $m = 3$ the list of non reduced cases grows bigger, therefore we do not pursue this matter for now.

Observing that actually, for $m \geq 2$: 
\[ \mathcal{R}(x) \subseteq \mathcal{R}(B(j_m, i_m)x) \subseteq \mathcal{R}(B(j_{m-1}, i_{m-1})B(j_m, i_m)x) \subseteq \mathcal{R}(w) \subseteq \mathcal{R}(x) \cup \{a_{n+1}\} \]
we draw from Lemmas [3,7] and [4,5] a list of cases in which $a_{n+1}$ does belong to $\mathcal{R}(w)$, together with its hat partner:

(1) (a) $h(j, i) = [i, 1]$ and $i_m \geq i \geq 1$,
the hat partner is the $\sigma_i$ in $h(j_m, i_m) = [j_m, n]\sigma_m \cdots \sigma_i \cdots \sigma_1$;
(b) $h(j, i) = [j, n]$ and $1 < j \leq n$, $j_m \leq j$, $i_m < j - 1$,
the hat partner is the $\sigma_j$ in $h(j_m, i_m) = \sigma_j \cdots \sigma_j \cdots \sigma_n[i_m, 1]$;
(c) $h(j, i) = [j, n]$ and $2 < j \leq n$, $j_m < j$, $i_m \geq j - 1$,
the hat partner is the $\sigma_{j-1}$ in $h(j_m, i_m) = \sigma_j \cdots \sigma_{j-1} \cdots \sigma_n[i_m, 1]$;
(d) $h(j, i) = [2, n]$ and $j_m = 1$, $i_m = 1$,
the hat partner is the leftmost $\sigma_1$ in $h(j_m, i_m) = \sigma_1 \cdots \sigma_n \sigma_1$.

(2) (a) $h(j, i) = \sigma_n \sigma_1$ and $j_m > 1$ and $1 \leq i_m < n - 1$,
the hat partner is the $\sigma_n$ in the $a_{n+1}$ on the left of $h(j_m, i_m)$;
(b) $h(j, i) = h(n, i)$ and $1 \leq i \leq i_m < n - 1$, $i < j_m$, and $i_{m-1} = i - 1$,
the hat partner is the $\sigma_{i-1}$ in $h(j_{m-1}, i_{m-1}) = [j_{m-1}, n]\sigma_{m-1} \cdots \sigma_{i-1} \cdots \sigma_1$;
(c) $h(j, i) = h(n, i)$ and $1 \leq i \leq i_m < n - 1$, $i \geq j_m$, and $i_{m-1} \geq i$,
the hat partner is the $\sigma_i$ in $h(j_{m-1}, i_{m-1}) = [j_{m-1}, n]\sigma_{m-1} \cdots \sigma_i \cdots \sigma_1$.

We point out again that this list is not exhaustive if $m \geq 3$.

### 4.3. A tower of canonical reduced expressions

We study the affine length in the tower of injections $W(A_{n-1}) \hookrightarrow W(\tilde{A}_n)$ built with the group monomorphism 
\[ R_n : W(\tilde{A}_{n-1}) \to W(\tilde{A}_n) \]
\[ \sigma_i \mapsto \sigma_i \text{ for } 1 \leq i \leq n - 1 \]
\[ a_n \mapsto a_n a_{n+1} \sigma_n \]
from [3, Lemma 4.1]. We produce below the canonical reduced expression of $R_n(w)$ given the canonical reduced expression of $w \in W(\tilde{A}_{n-1})$ from Theorem 2.13. In particular, $R_n(w)$ and $w$ have the same affine length and the Coxeter length of $R_n(w)$ is fully determined by the Coxeter length and affine length of $w$.

In this subsection we need to include the dependency on $n$ in the notation, so we write $h_n(r, i) = [r, n][i, 1]$.

**Theorem 4.6.** Let 
\[ w = h_{n-1}(j_1, i_1)a_n h_{n-1}(j_2, i_2)a_n \cdots h_{n-1}(j_m, i_m)a_n x \]
be the canonical reduced expression of an element $w$ in $W(\tilde{A}_{n-1})$, where $x$ is the canonical reduced expression of an element in $W(A_{n-1})$. Substituting $\sigma_n a_{n+1} \sigma_n$ for $a_n$ in this expression produces a reduced expression which can be transformed into the canonical reduced expression of $R_n(w)$, that has the following shape:
Recalling (4.1)
\[ R_n(w) = R_n(j_1, i_1) a_{n+1} h_n(j_2, i_2') a_{n+1} \ldots h_n(j_m, i'_m) a_{n+1} [t, n] x \]
where, letting
\[ s = \max\{k \mid 1 \leq k \leq m \text{ and } n - k - i_k > 0\}, \]
we have:
\[ i'_k = i_k \text{ for } k \leq s, \quad i'_k = i_k + 1 \text{ for } k > s, \quad t = n - s + 1. \]
This implies
\[ L(R_n(w)) = L(w), \quad l(R_n(w)) = l(w) + 2L(w), \]
hence replacing \( a_n \) by \( \sigma_n a_{n+1} \sigma_n \) in a reduced expression for \( w \) produces a reduced expression for \( R_n(w) \) if and only if the expression for \( w \) is affine length reduced.

Note that we have \( s \leq n - 1 \).

**Proof.** We observe first that the expression (4.1) given for \( R_n(w) \) is canonical: the pairwise inequalities are clearly satisfied, and the fact that \( [t, n] x, x \in W(A_{n-1}) \), is reduced, has been used since the beginning of this paper. The last part of the Proposition states immediate consequences. We only have to produce form (4.1).

Substituting \( \sigma_n a_{n+1} \sigma_n \) for \( a_n \) in the canonical reduced expression of \( w \) gives:
\[ R_n(w) = h_{n-1}(j_1, i_1) \sigma_n a_{n+1} \sigma_n h_{n-1}(j_2, i_2) \sigma_n a_{n+1} \sigma_n \ldots h_{n-1}(j_m, i_m) \sigma_n a_{n+1} \sigma_n x. \]
For the leftmost term, we have \( h_{n-1}(j_1, i_1) \sigma_n = h_n(j_1, i_1) \) since \( i_1 \leq n - 2 \). For the next one we have
\[ \sigma_n h_{n-1}(j_2, i_2) \sigma_n = [j_2, n - 2] \sigma_n [i_2, 1] = [j_2, n] \sigma_{n-1} [i_2, 1]. \]
If \( i_2 = n - 2 \), we obtain \( h_n(j_2, n - 1) \), otherwise \( \sigma_{n-1} \) travels to the right; so if \( m = 1 \) or \( m = 2 \) our claim holds. Assuming the claim holds up to \( m - 1 \geq 2 \), we prove it for \( m \). Let \( s = s_{m-1} = \max\{k \mid 1 \leq k \leq m - 1 \text{ and } n - k - i_k > 0\} \)
and \( t_{m-1} = n - s_{m-1} + 1 \). We have
\[ R_n(w) = h_n(j_1, i_1) a_{n+1} \ldots h_n(j_{m-1}, i'_{m-1}) a_{n+1} [t_{m-1}, n] h_{n-1}(j_m, i_m) \sigma_n a_{n+1} \sigma_n x. \]

We show first: \( t_{m-1} > j_m \). Indeed we have \( t_{m-1} > i_s + 1 \) in particular \( t_{m-1} - 1 > 1 \), to be used soon. If \( j_s \leq i_s + 1 \) we are done, otherwise the sequence \( (j_r) \) decreases strictly for \( r \leq s + 1 \) hence \( j_{s+1} \leq n - (s + 1) + 1 < t_{m-1} \).

We can now compute:
\[ [t_{m-1}, n] h_{n-1}(j_m, i_m) \sigma_n = [j_m, n] [t_{m-1} - 1, n - 1] [i_m, 1] \]
equal to
1. \( [j_m, n] [i_m, 1] [t_{m-1} - 1, n - 1] \) if \( t_{m-1} - 1 > i_m + 1 \);
2. \( [j_m, n] [i_m + 1, 1] [t_{m-1} - 1, n - 1] \) if \( t_{m-1} - 1 \leq i_m + 1 \).
Recalling \( t_{m-1} - 1 > 1 \), we obtain in these two cases, respectively:
1. \( R_n(w) = h_n(j_1, i_1) a_{n+1} \ldots h_n(j_{m-1}, i'_{m-1}) a_{n+1} h_n(j_m, i_m) a_{n+1} [t_{m-1} - 1, n] x; \)
2. \( R_n(w) = h_n(j_1, i_1) a_{n+1} \ldots h_n(j_{m-1}, i'_{m-1}) a_{n+1} h_n(j_m, i_m + 1) a_{n+1} [t_{m-1}, n] x. \)
Both have the expected form, by induction, once we observe the following. If $i_{m-1}' = i_{m-1} + 1$, then also $i_m' = i_m + 1$: certainly $i_{m-1}' = i_{m-1} + 1$ implies $t_{m-1}' = t_{m-2} \leq i_{m-1} + 2$. Hence $t_{m-1} \leq i_m + 2$, so finally $t_{m-1} = t_m$ and $i_m' = i_m + 1$.

**Corollary 4.7.** Let $w \in W(\tilde{A}_n)$ be given in its canonical form:

$$w = h(j_1, i_1)a_{n+1}h(j_2, i_2)a_{n+1} \ldots h(j_m, i_m)a_{n+1}x, \quad x \in W(A_n),$$

then $w \in R_n(W(\tilde{A}_{n-1}))$ if and only if the following conditions hold:

1. $j_1 \leq n$ and $i_1 < n - 1$;
2. letting $s = \max\{k \mid 1 \leq k \leq m \text{ and } n - k - i_k > 0\}$, we have:
   $$n - (s + 1) - i_{s+1} < 0;$$
3. $x = [n - s + 1, n]y$ with $y \in W(A_{n-1})$.

**Proof.** The only thing to check is that, letting $\tilde{i}_t = i_t$ if $t \leq s$ and $\tilde{i}_t = i_t - 1$ if $t > s$, the family $(j_t, \tilde{i}_t)_{1 \leq t \leq m}$ satisfies the pairwise inequalities. This is left to the reader. \qed

The corollary tells that for a $w$ in $W(\tilde{A}_n)$: belonging to the image $R_n(W(\tilde{A}_{n-1}))$ depends only on the $n$ leftmost affine bricks of the affine block $w_n$ of $w$ and the finite part $x \in W(A_n)$! And that for every affine block $w_n$ verifying conditions (1) and (2) there are exactly $n!$ elements $x \in W(A_n)$ such that $w_n.x$ is in $R_n(W(\tilde{A}_{n-1}))$. And finally that every element in $W(\tilde{A}_{n-1})$ can be attained in such a way.

We can deduce from this the faithfulness of the tower of Hecke algebras on any ring, following the tracks of [4, Theorem 3.2], with exactly the same proofs. In what follows, by algebra we mean $K$-algebra, where $K$ is an arbitrary commutative ring with identity. We fix an invertible element $q$ in $K$. There is a unique algebra structure on the free $K$-module with basis $\{g_w \mid w \in W(\tilde{A}_n)\}$ satisfying for $s \in S_n$:

$$g_sg_w = g_sw \quad \text{if } s \notin \mathcal{L}(w),$$
$$g_sg_w = qg_sw + (q - 1)g_w \quad \text{if } s \in \mathcal{L}(w).$$

This algebra is the Hecke algebra of type $\tilde{A}_n$, denoted by $H\tilde{A}_n(q)$. It has a presentation given by generators $\{g_s \mid s \in S_n\}$ and well-known relations. The generators $g_s, s \in S_n$, are invertible.

The morphism $R_n$ defined in the beginning of this subsection has a counterpart in the setting of Hecke algebras, namely the following morphism of algebras (where we write carefully $e_w$ for the basis elements of $H\tilde{A}_{n-1}(q)$, to be reminded of the possible lack of injectivity):

$$HR_n : H\tilde{A}_{n-1}(q) \rightarrow H\tilde{A}_n(q)$$

by

$$e_{\sigma_i} \mapsto g_{\sigma_i}, \quad \text{for } 1 \leq i \leq n - 1$$
$$e_{\sigma_n} \mapsto g_{\sigma_n}g_{\sigma_n+1}g_{\sigma_n}^{-1}.$$
We have shown in [1, Proposition 4.3.3] that this homomorphism is injective for $K = \mathbb{Z}[q, q^{-1}]$ where $q$ is an indeterminate. With a general $K$ as above, we can obtain injectivity using the following technical but crucial result, an immediate consequence of Theorem 4.6 (see [4, Proposition 3.1]):

**Proposition 4.8.** Let $w$ be any element in $W(\tilde{A}_{n-1})$, then there exist $A_w \in q^Z$ and elements $\lambda_x \in K$ such that

$$HR_n(e_w) = A_w g_{R_n(w)} + \sum_{x \in W(\tilde{A}_n), \quad \ell(x) < l(R_n(w))} \lambda_x g_x,$$

$$L(x) \leq L(w)$$

With this, the proof of [4, Theorem 3.2] applies, we obtain:

**Corollary 4.9.** Let $K$ be a ring and $q$ invertible in $K$. The tower of affine Hecke algebras:

$$H\tilde{A}_1(q) \overset{HR_2}{\longrightarrow} H\tilde{A}_2(q) \overset{HR_3}{\longrightarrow} \cdots H\tilde{A}_{n-1}(q) \overset{HR_n}{\longrightarrow} H\tilde{A}_n(q) \longrightarrow \cdots$$

is a tower of faithful arrows.

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