Online Ranking with Concept Drifts in Streaming Data

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Abstract

Two important problems in preference elicitation are rank aggregation and label ranking. Rank aggregation consists of finding a ranking that best summarizes a collection of preferences of some agents. The latter, label ranking, aims at learning a mapping between data instances and rankings defined over a finite set of categories or labels. This problem can effectively model many real application scenarios such as recommender systems. However, even when the preferences of populations usually change over time, related literature has so far addressed both problems over non-evolving preferences.

This work deals with the problems of rank aggregation and label ranking over non-stationary data streams. In this context, there is a set of \( n \) items and \( m \) agents which provide their votes by casting a ranking of the \( n \) items. The rankings are noisy realizations of an unknown probability distribution that changes over time. Our goal is to learn, in an online manner, the current ground truth distribution of rankings.

We begin by proposing an aggregation function called Forgetful Borda (FBorda) that, using a forgetting mechanism, gives more importance to recently observed preferences. We prove that FBorda is a consistent estimator of the Kemeny ranking and lower bound the number of samples needed to learn the distribution while guaranteeing a certain level of confidence. Then, we develop a \( k \)-nearest neighbor classifier based on the proposed FBorda aggregation algorithm for the label ranking problem and demonstrate its accuracy in several scenarios of label ranking problem over evolving preferences.

1 Introduction

During the last years, stream learning has gained the attention of the machine learning community [10], [16]. Stream learning refers to learning problems for which the data are continuously generated over time. This feature of stream learning imposes severe constraints on the proposed learning algorithms. For instance, in contrast to batch learning, in stream learning scenarios the data no longer can be completely stored and the learning algorithms must be very efficient (typically linear or even sub-linear), to the extreme of
operating close to real-time. These computational complexity constraints usually lead to models that are incrementally updated with the arrival of new data.

In addition, another major challenge when dealing with stream learning arises when the source producing stream data instances evolve over time, which yields to non-stationary data distributions. This phenomenon is known as concept drift (CD) [11]. In the stream learning scenarios with CD, the predictive performance of a learned model when a change (drift) in the data distribution (concept) occurs tends to decrease because it does not adapt suitably to the new data distribution.

In this paper, we focus on the concept drift scenario in which data are given as permutations or rankings. The ranking data are distributed according to the Mallows model [8], which is a usual and natural assumption when dealing with both methodological and applied studies [17]. Concept drift can be modeled by changes in the parameters of the distribution defining the population over time. We study under this context the popular problems of rank aggregation [7] and label ranking [12], which have been extensively studied in an offline setting.

The rank aggregation problem consists of finding a ranking that best summarizes a collection of rankings. One of the most prominent algorithms in the offline setting is Borda [3]. It is computationally cheap and guaranteed to be an accurate approximation of the permutation that minimizes the distances to the sample. The rank aggregation problem has been also studied in the online learning perspective. In particular, [21] first considered this problem and proposed an algorithm based on the pairwise comparisons of the items and a permutation reconstruction algorithm and whose relative loss bound is close to optimal. In [1] the authors present two simple algorithms based on sorting procedures that predict the aggregated ranking efficiently, bounding its maximal expected regret. In [4] the learner, which addresses several rank elicitation problems based on sorting procedures that predict the aggregated ranking efficiently, bounding its maximal expected regret. However, to the best of the authors’ knowledge, there has been no attempt to approach these problems under the assumption that the underlying distribution for rankings changes or evolves over time, despite being a reasonable scenario which has been considered in stream learning.

In this work, we propose a weighted version of the Borda algorithm called Forgetful Borda (FBorda) that is able to work online and that can adapt to the changes in the distribution of the population of rankings. FBorda has a memory parameter that grants more importance to recent ranks and can deal with the rank aggregation problem when the underlying probability distribution evolves over time. We theoretically show that, when the sequence of models that generate a sample of rankings are Mallows models, the FBorda ranking is a consistent estimator of the last central ranking used in the last Mallows models. Moreover, we give bounds for the number of samples in the last stationary period to ensure confidence levels in the quality of the estimator.

A second contribution of this work deals with Label ranking (LR). LR consists of learning a mapping from unlabeled instances (features) to rankings defined over a finite and fixed
set of elements [12]. This problem can be regarded as an alternative formulation of supervised classification problem where a classifier predicts rankings of labels instead of a single class label. LR is of practical importance because it is used to approach real-world problems such as document recommendation, personalized web-based retail, and automated e-mail prioritization, among many others [20]. In order to deal with LR over non-stationary data streams, we propose a classifier that hybridizes a $k$-nearest neighbor classifier with the aforementioned FBorda rank aggregation function, called $k$-nearest Forgetful Classifier (KFC). Given an instance, KFC selects a set of neighbor instances and aggregates them by means of the proposed FBorda, harnessing its forgetting capability for enhanced adaptability to evolving concepts along the stream. The provided experimental results show the efficiency of KFC for the LR problem in the context of evolving preferences.

This paper is organized as follows: Section 2 includes the preliminaries of this paper. Section 3 introduces the rank aggregation problem and the context of evolving preferences. Moreover, the main contribution of this paper, FBorda, is detailed along with its theoretical guarantees. Section 4 introduces the label ranking problem with evolving preferences and Section 5 shows the empirical evaluation of the algorithms. Finally, Section 6 concludes the paper.

2 Preliminaries and notation

Permutations or rankings are bijection of the set $[n] = \{1, 2, \ldots, n\}$ onto itself and are represented as an ordered vector of $[n]$. Along the paper, they will be denoted with the Greek letters $\sigma$ or $\pi$. Following the convention in the literature, we denote by $\tau_{ij}$, or simply $\tau$ when it is clear by the context, a permutation with an inversion in positions $i,j$, i.e., $\tau(i) = j$, $\tau(j) = i$ and $\tau(k) = k$ for $k \neq i,j$. The set of permutations of $[n]$ is denoted $S_n$. Permutations of $n$ items are used to represent preferences by denoting the ranking of item $i$ as $\sigma(i)$. In this sense we say that an item item $i$ is preferred to item $j$ when it has a lower ranking, $\sigma(i) < \sigma(j)$.

One of the most popular noisy models for permutations is the Mallows model (MM). The MM is an exponential model based on an spread parameter $\theta$, a location parameter $\pi$ and the definition of a distance for permutations. When preferences are involved, this distance will be the Kendall’s-$\tau$ distance. In general, the probability of any $\sigma \in S_n$ under the MM can be written as:

$$p(\sigma) = \frac{\exp(-\theta \cdot d(\sigma, \pi))}{\psi(\theta)}. \quad (1)$$

The location parameter $\pi$ is referred to as consensus ranking for being the mode of the distribution. The spread parameter $\theta$ controls how sharp the distribution is around the mode, that is, as $\theta$ increases the distribution becomes sharper. We will usually use $\pi$ to refer to central rankings of MM and $\sigma$ to refer to random MM permutations, so $\sigma \sim MM(\pi, \theta)$ denotes that $\sigma$ is a random permutation sampled from a MM centered at
\[ \pi \text{ and with dispersion parameter } \theta. \text{ We consider that } \theta > 0 \text{ in general.} \]

Despite its appearance of computational infeasibility, its main operations can be performed efficiently [8]. For example, the normalization constant \( \psi(\theta) \) can be computed in linear time. Moreover, the expected distance of a random Mallows permutation from the mode is:

\[ E_\theta[D] = \frac{n \exp(-\theta)}{1 - \exp(-\theta)} - \sum_{j=0}^{k-1} \frac{j \exp(-\theta j)}{1 - \exp(-\theta j)}. \tag{2} \]

The maximum likelihood estimation of both parameters of a MM sample \( S \) is done sequentially. First, the MLE \( \bar{\pi} \) for the consensus permutation must be obtained and it happens that the MLE for \( \bar{\pi} \) is the Kemeny ranking. Since computing this ranking is computationally hard, usually the Borda ranking is used because (i) it is a consistent estimator of \( \bar{\pi} \) as we will later see and (ii) it can be computed in quasi-linear time, the complexity comes from sorting an array. Then, the MLE dispersion parameter can be obtained by solving the next equality numerically (e.g. Newton-Raphson procedure):

\[ \frac{n - 1}{\exp(\theta) - 1} - \sum_{k=2}^{n} \frac{k \exp(-\theta k)}{1 - \exp(-\theta k)} = \bar{d}, \tag{3} \]

where \( \bar{d} = \sum_{\sigma \in S} d(\sigma, \bar{\pi})/|S| \).

### 2.1 Evolving Mallows Models

As far as we know, the question of preferences that change over time has not been explicitly considered in the literature so far. Our proposal to model this scenario is to assume that we have a sequence of MM, where the \( t \)-th model \( MM_t(\pi_t, \theta_t) \) has parameters \( \pi_t \) and \( \theta_t \).

When two consecutive models, \( MM_t(\pi_t, \theta_t) \) and \( MM_{t+1}(\pi_{t+1}, \theta_{t+1}) \) differ on the location parameter, \( \pi_t \neq \pi_{t+1} \), there has been a drift in the consensus of the preferences of the population at time \( t \), i.e., both models do not longer agree in the ordering of the items. If, on the other hand, is the spread parameter what changes over time, \( \theta_t \neq \theta_{t+1} \), the drift represents the idea that the concentration of the underlying distribution changes. For the rest of the paper we focus on the scenario in which the mode of the distribution changes over time since it is a more interesting scenario.

It is possible to have two different perspectives when we refer to the type of drift. Frequently, drifts can be classified as gradual and abrupt in terms of speed, being abrupt when a change happens suddenly between two concepts, and gradual when there is a smooth transition between both concepts. In this work, we have considered abrupt drifts in what refers to the speed of change (instances of the old concept disappear suddenly and the new ones appear), and gradual drift in what refers to the changes in the order preference of the labels (the order changes smoothly between drifts).
3 Rank Aggregation and evolving preferences

The rank aggregation has been studied for years and recently, it has become a problem of interest in the machine learning community dealing with preferences. Rank aggregation consists of finding a ranking that best summarizes a set of rankings which are representing a set of votes over a set of alternatives. In other words, given a collection of rankings of a population, find a mean ranking that is a consensus for the whole population.

In a static environment, for non-evolving preferences, a very popular choice for aggregating a collection of rankings is the Kemeny ranking rule [19]. This rule computes the so-called Kemeny ranking, the permutation that minimizes the number of pairwise disagreements between the final ranking and the existing rankings, or equivalently, that minimizes the sum of the Kendall’s-τ distances to the permutations in the sample, being the Kendall’s-τ distance defined as:

\[ d(\sigma, \sigma') = |(i, j) : \sigma(i) > \sigma(j) \land \sigma'(i) < \sigma'(j)| \] (4)

Interestingly, the Kemeny ranking is the MLE for the central permutation of a MM. Unfortunately, the problem of obtaining the Kemeny ranking is known to be NP-hard [7]. Nevertheless, the Borda ranking has been shown, both empirically and theoretically to be a good estimator for the Kemeny ranking [2], [9], particularly if the sample follows a strongly unimodal model such as the Mallows distribution. The Borda ranking of a given set of rankings \( S \) is given by:

\[ B(S) = r(\sum_{\sigma \in S} \sigma), \] (5)

where given a vector \( v \) of size \( n \), \( r(\cdot) \) denotes a function that maps \( v \in \mathbb{R}^n \) into \( \pi \in S_n \) where \( \pi(i) \) is the ranking associated to the (increasing) ordering of \( v \). For instance, giving \( v = (0.3, 2.2, 1.7, 0.1) \), \( r(v) = (2, 4, 3, 1) \). The complexity of Borda is quasi-linear, i.e., \( O(nm) \) for adding the terms up and \( O(n \log n) \) for the reordering of the score vector in function \( r(\cdot) \).

3.1 FBorda

We call forgetful Borda (FBorda) to the proposed estimate of the Kemeny ranking for sequences of rankings with concept drifts. FBorda is inspired in the Borda estimate and it gives more importance to recent rankings of the sequence by means of weights with an exponential decay. Formally, FBorda is defined as:

\[ B_\rho(\sigma_t)_{t \geq 0} = r(\sum_{t \geq 0} \rho^t \sigma_t) \]
\[ = r(\rho \cdot w + \sigma_0) \] (6)

where \( w = \sum_{t > 0} \rho^t \sigma_t \).
FBorda gives more importance to recent rankings by giving a weight of $\rho^t$ to ranking $\sigma_t$ for $t \geq 0$ and $\rho \in [0, 1]$. FBorda, the same as Borda, can deal with a collection of rankings in quasi-linear time: $O(n \times m)$ for adding the permutations up and $O(n \log n)$ to order the sums, what we called function $r(\cdot)$. The same as Borda, FBorda can be easily adapted to work in an online manner. Moreover, with the trick of multiplying the permutation and the weight, $\rho^t \sigma_t$, FBorda is able of incorporating a forgetting mechanism while keeping the quasi-linear time complexity at each step of the online process and the linear space complexity.

### 3.1.1 FBorda Guaranties

In this section we analyze theoretically the behaviour of FBorda in online rank aggregation under evolving preferences. In particular, we will prove that the FBorda is an consistent estimator of the last consensus permutation on an evolving MM. Moreover, we conjecture that it is a valid approach for several models, particularly for strongly unimodal models with drifts.

First, we provide some known and intermediate result regarding the expected value of rankings obtained from a MM. In fact, the first two results have been adapted from [9]. We denote by $E_\pi[\sigma(i)]$ the expected rank of item $i$ for $\sigma \sim MM(\pi, \theta)$ (or $E[\sigma(i)]$ if it is clear from the context) and by $c_{ij}\theta$ (or simply $c_{ij}$) the expected difference between $\sigma(j)$ and $\sigma(i)$. Unfortunately, not $E_\pi[\sigma(i)]$ neither $c_{ij}$ have a close expression, but the latter can be expressed conveniently as follows.

$$c_{ij}\theta = E[\sigma(j)] - E[\sigma(i)] = E[\sigma(j) - \sigma(i)]$$

$$= \sum_{\sigma(i)<\sigma(j)} (\sigma(j) - \sigma(i))(p(\sigma) - p(\sigma_{ij})) \quad (7)$$

Another interesting conclusion that can be drawn with this representation is that $c_{ij}\theta > 0$ for every MM which $\theta > 0$. In order to see this, note that the sum ranges for every permutation such that $\sigma(i) < \sigma(j)$ so the first term of the product, $(\sigma(j) - \sigma(i))$, is always positive. Moreover, we know that MM has complete consensus when $\theta > 0$ [6], which means that for a consensus ranking $\pi$ with $\pi(i) < \pi(j)$ and any permutation $\sigma$ such that $\sigma(i) < \sigma(j)$ then $p(\pi) \geq p(\sigma_{ij})$. Consequently, the second part of the product, $(p(\sigma) - p(\sigma_{ij}))$, is also positive and $c_{ij}\theta > 0$.

**Lemma 1.** The Borda ranking is an consistent estimator of $\pi$.

The proof can be found in the supplementary material.

In the following Lemma we consider the evolving preferences scenario: first, the central permutation is $\pi$ and later the central permutation is $\pi\tau$.  


Lemma 2. Let $\tau$ be an inversion of $i$ and $j$ so that $d(\pi\tau, \pi) = 1$. Let $E_{\pi}[\sigma(i)]$ be the expected value of $\sigma(i)$ for $\sigma \sim MM(\pi)$. It holds that $E_{\pi}[\sigma]$, is related to $E_{\pi\tau}[\sigma]$ as follows:

$$E_{\pi\tau}[\sigma(i)] = E_{\pi}[\sigma(j)] = E_{\pi}[\sigma(i)] + c_{ij}$$

Next, we provide the main theoretical result regarding the FBorda ranking, $\bar{\pi}$ for the case in which the preferences of the population evolve over time. We give conditions regarding the value of the parameter $\rho$ and number of rankings generated since the last drift for which the FBorda ranking satisfies that $\bar{\pi}(i) < \bar{\pi}(j)$ if and only if $\pi(i) < \pi(j)$, where $\pi$ is the last consensus ranking of the evolving MM.

Theorem 3. Let item $i$ be preferred to $j$ by ranking $\pi$, $\pi(i) < \pi(j)$ and $\tau$ be an inversion of $i$ and $j$ so that $d(\pi\tau, \pi) = 1$ (therefore, $\pi(\tau(i)) > \pi(\tau(j))$). Let an evolving MM generate a possibly infinite sequence of permutations $\sigma_t$ such that $\sigma_t \sim MM(\pi)$ for $t \geq m$ and $\sigma_t \sim MM(\pi\tau)$ for $t < m$. There exist $m$ and $\rho$ for which FBorda is an consistent estimator of $\pi\tau$, which are as follows:

$$m < \frac{\log 0.5}{\log \rho}.$$

Proof. The same as Borda, FBorda ranking ranks the items in $[n]$ w.r.t. the sum of the scores which are computed as $\sum_t \rho_t \sigma_t(i)$ of the sample $S$, so it is consistent if there exists $m$ and $\rho$ that $\pi\tau(i) < \pi\tau(i) \Rightarrow E[\sum_t \rho_t \sigma_t(i)] < E[\sum_t \rho_t \sigma_t(j)]$.

$$E[\sum_t \rho_t \sigma(i)] = \sum_{t>m} \rho_t E_{\pi}[\sigma(i)] + \sum_{t\leq m} \rho_t E_{\pi}[\sigma(i)]$$

$$= \sum_{t>m} \rho_t E_{\pi}[\sigma(i)] + \sum_{t\leq m} \rho_t E_{\pi}[\sigma(i)] + \sum_{t\leq m} \rho_t c_{ij}$$

$$E[\sum_t \rho_t \sigma(j)] = \sum_{t>m} \rho_t E_{\pi}[\sigma(j)] + \sum_{t\leq m} \rho_t E_{\pi}[\sigma(j)]$$

$$= \sum_{t>m} \rho_t E_{\pi}[\sigma(i)] + \sum_{t>m} \rho_t c_{ij} + \sum_{t\leq m} \rho_t E_{\pi}[\sigma(i)]$$

Therefore, FBorda is an consistent estimator if and only it holds that:

$$E[\sum_t \rho_t \sigma(i)] < E[\sum_t \rho_t \sigma(j)] \iff \sum_{t=0}^{m-1} \rho_t < \sum_{t=m}^{\infty} \rho_t.$$  

(10)
Thus, given $m > 0$ we can select $\rho$ for ensuring the inequality of Equation (10) as follows:

$$m < \frac{\log 0.5}{\log \rho}. \tag{11}$$

Note that the right hand side expression increases with $\rho \in (0, 1)$ and in the limit $\lim_{\rho \to 0} \frac{\log 0.5}{\log \rho} = 0$. In other words, for any value of $m$ (number of samples from the last drift) it is possible to select a $\rho$ for which FBorda is an consistent estimator of $\pi_t$. The intuitive conclusion is that as $\rho$ decreases FBorda is more reactive, that is, on average it needs less samples to accommodate to the drift.

**Corollary 4.** The spread parameter of the last of the drifted MM is the one that satisfies the expression:

$$\frac{n - 1}{\exp(\theta) - 1} - \sum_{k=2}^{n} \frac{k \exp(-\theta k)}{1 - \exp(-\theta k)} = \frac{\sum_{k} \rho^t d(\sigma_t, \sigma^*)}{\sum_{k} \rho^k} \tag{12}$$

Note that, given $\rho$, the expression is a convex function on $\theta$ so the solution can be found efficiently with any numerical method.

### 3.1.2 Concentration bounds

Previously, we have shown that FBorda is a consistent estimator of the central ranking under evolving preferences. Intuitively, it states that when $\rho$ decreases FBorda quickly forgets the old preferences and focuses on the recently generated ones so it can adapt to recent drifts. One might feel tempted of lowering $\rho$ to guarantee, on average, that FBorda recovers the last central ranking. However, as we decrease $\rho$ the variance on the estimated central ranking increases. In this way, the next results (i) show how the confidence of our estimated central ranking decreases as $\rho$ decreases and (ii) give a lower bound on the number of samples required after a drift for ensuring, with a high probability, that FBorda recovers the central ranking of the model. In particular, the next result shows the deviation between the weighted sum of $\sigma(i)$ and its expectation in the absence of drift. Intuitively, it states that with probability $1 - \delta$ the confidence interval decreases with $\rho$ for given values of $m$ and $\delta$.

**Lemma 5.** Let $\sigma_t$ for $t < m$ be rankings i.i.d. distributed according to $MM(\pi)$ (we do not consider drifts for this result). In the absence of drifts, the difference between the weighted sum of $\sigma(i)$, $\sum_t \rho^t \sigma_t(i)$, and its expectation $E[\sum_t \rho^t \sigma_t(i)]$ is smaller than $c = \sqrt{n^2 \sum_{t=0}^{m-1} \rho^2 t^2 \log(\frac{1}{\delta})}$ with probability $1 - \delta$. Moreover, for fixed $m > 0$ and $\delta \in [0, 1]$, $c$ decreases with $\rho$. 

8
The proof can be found in the supplementary material. An empirical evaluation for illustration purposes of this evolution for $n = 5$ is shown in Figure 1. The intuitive conclusion is that the larger is $\rho$ the higher will be our confidence on $\sum_t \rho^t \sigma_t(i)$ having converged to $E[\sum_t \rho^t \sigma_t(i)]$.

The following theorem gives a lower bound on the number of samples that are needed to adapt to a drift as a function of the dispersion of the distribution and $\rho$.

**Theorem 6.** Let item $i$ be preferred to $j$ by ranking $\pi$, $\pi(i) < \pi(j)$ and $\tau$ be an inversion of $i$ and $j$ so that $d(\pi \tau, \pi) = 1$ (therefore, $\pi \tau(i) > \pi \tau(j)$). Let an evolving MM generate a possibly infinite sequence of permutations $\sigma_t$ such that $\sigma_t \sim \text{MM}(\pi)$ for $t \geq m$ and $\sigma_t \sim \text{MM}(\pi \tau)$ for $t < m$. The number of samples that FBorda needs to recover $\pi \tau$ (i.e., adapt to the drift) with probability $1 - \delta$ is at least

$$m > \mathcal{O}\left(\log_\rho \frac{1}{c_{ij}} \sqrt{\frac{\log \delta}{\rho^2 - 1}}\right)$$

**Proof.** Following the idea in the previous results, Theorem 3 and Lemma 5, the estimation of FBorda is consistent when the weighted sum of $\sigma(i)$ is smaller than the weighted sum of $\sigma(j)$.
\[
\sum_{t=m}^{\infty} \rho^t E_\pi[\sigma(j)] + \sum_{t=0}^{m-1} \rho^t E_\pi[\sigma(j)] < \\
\sum_{t=m}^{\infty} \rho^t E_\pi[\sigma(i)] + \sum_{t=0}^{m-1} \rho^t E_\pi[\sigma(i)]
\]

\[
\sum_{t=m}^{\infty} \rho^t E_\pi[\sigma(j)] - \rho^t E_\pi[\sigma(i)] < \\
\sum_{t=0}^{m-1} \rho^t E_\pi[\sigma(i)] - \rho^t E_\pi[\sigma(j)]
\]

(13)

\[
c_{ij\theta} \sum_{t=m}^{\infty} \rho^t < c_{ij\theta} \sum_{t=0}^{m-1} \rho^t
\]

The last line is based on Equation (7). Equation (13) would be an accurate measure for the sample complexity if the expected value did not deviate at all from the sum. However, Lemma 5 bounds the error that with probability \(1 - \delta\) will happen. We will include this errors in our sample complexity analysis after defining the errors happened before the drift \(e_{old}\) and after the drift \(e_{new}\).

\[
e_{old} = \sqrt{\frac{\sum_{t=m}^{\infty} \rho^t n^2 \log \frac{1}{\delta}}{(\sum_{t=0}^{m-1} \rho^t)^2}}, \quad e_{new} = \sqrt{\frac{\sum_{t=0}^{m-1} \rho^t n^2 \log \frac{1}{\delta}}{(\sum_{t=0}^{m-1} \rho^t)^2}}
\]

(14)

Plugging these errors a the worst case that can happen in Equation (13) we can give a tighter analysis. In this way, we can say that with probability \(1 - \delta\) the FBorda ranking will be an consistent estimator of \(\pi_\tau\) when the current expression holds:

\[
c_{ij\theta} \sum_{t=0}^{m-1} \rho^t - e_{new} > c_{ij\theta} \sum_{t=m}^{\infty} \rho^t + e_{old}
\]

(15)

This is the worst case scenario, in which the recent samples suffer an error by defect and the old ones by excess. After some algebra, we find a lower bound for \(m\), which concludes
the proof.

\[
\begin{align*}
\sum_{t=0}^{m-1} \rho^t c_{ij} \theta & - \sum_{t=m}^{\infty} \rho^t c_{ij} \theta > e_{\text{old}} + e_{\text{new}} > e_{\text{total}} \\
\frac{\rho^m - 1}{\rho - 1} - \frac{\rho^m}{1 - \rho} > \frac{e_{\text{total}}}{c_{ij}} \\
m > O\left(\log_{\rho} \frac{1}{c_{ij} \theta} \sqrt{\frac{\log \delta}{\rho^2 - 1}}\right)
\end{align*}
\]

\[
\square
\]

4 Label Ranking with evolving preferences

LR (with non-evolving preferences) can be thought as being a generalization of supervised classification. While in supervised classification the goal is to assign the label to each unlabeled instance \(x\) from a finite set of labels, in LR the goal is to assign a ranking of labels where the lowest the ranking of a label the highest its preference. In this work we focus on the problem of assigning a complete ranking of labels rather than an incomplete one. Let \(X\) be a random variable called feature with support \(\mathcal{X}\), and let \(\Sigma\) a be random variable called LR with support over the space of permutations of size \(n, S_n\).

In LR pair \((X, \Sigma)\) is distributed according to some \(p\), and a particular instance of \((X, \Sigma)\) is denoted by \((x, \sigma)\). In order to deal with label ranking problem we construct a classifier, that is, a function \(f\) that maps \(X\) into \(\Sigma\) and its quality can be measured in terms of the (label ranking) error function:

\[
\epsilon(f) = E_p[d(f(x), \sigma)],
\]

where \(d\) is the Kendall’s-\(\tau\) distance (see Equation (4)). In words, the error corresponds to the expected Kendall’s-\(\tau\) distance between the predicted and the true rankings. We call the classifier that minimizes the error, the Bayes classifier and it is defined as:

\[
f_B(x) = \arg \max_{\sigma \in S_n} p(x, \sigma) \equiv \arg \max_{\sigma \in S_n} p(\sigma|x),
\]

Unfortunately, \(p\) is unknown and we can only have access to it through a set of instances \(D\) distributed according to \(p\).

While in LR \(p\) is stationary, in the LR with concept drift \(p\) evolves over the time. Instead of having a set of instances \(D\), in the LR with CD problem we have a sequence of instances \((x_t, \sigma_t)\), for \(t \geq 0\) where the instances are indexed by their antiquity, \(t\). In LR with CD we assume that each instance \((x_t, \sigma_t)\) is sampled from a distribution \(p_t\), for \(t \geq 0\). Again, we say that there is a drift at time \(t\) when \(p_{t+1} \neq p_t\). For the rest of the paper
Algorithm 1: K-nearest Forgetful Classifier

**Input:** A sequence \((x^t, \sigma^t)\) for \(t \geq 1\), an unlabeled instance \(x\), a weight \(0 < \rho \leq 1\), and a positive integer \(k\).

**Output:** A (label) ranking \(\sigma^*\) for \(x\).

- Select a subset of rankings \(D_k \subseteq \{\sigma^t : t \geq 1\}\) of maximum size with the associated features closest to the unlabeled instance \(x\), that satisfy \(\sum_{\sigma^t \in D_k} \rho^t \leq k\).
- **Return** The FBorda ranking associated to the set of rankings \(\sigma^t \in D_k\) with weights \(\rho^t\), respectively.

we focus on the case where the concept drift happens in the distribution for rankings, and denote the problem as *label ranking with evolving preferences* (LREP).

The LREP problem consists of predicting \(\sigma_0\) given \(x_0\) and the subsequence \((x_t, \sigma_t)\) for \(t \geq 1\). Thus, we measure the quality of a classifier in LREP using the next (empirical) error function:

\[
\hat{\epsilon}(f) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} d(f(x_t), \sigma_t),
\]

where \(d\) is the Kendall’s-\(\tau\) distance. Clearly, in order to minimize the error, a classifier \(f\) has to be able to adapt to the CDs.

### 4.1 k-nearest Forgetful Classifier

In this section we present the *k*-nearest Forgetful Classifier (KFC), a classifier for the LREP problem. This classifier combines the strengths of FBorda for dealing with the drifting rank aggregation problem and the *k*-nearest neighbor classifiers for dealing with standard supervised classification problem. The pseudocode of KFC is shown in Algorithm 1. Given an unlabeled instance \(x\), KFC selects the *k* closest instances to \(x\). Then, KFC estimates as the associated ranking of \(x\) the result of aggregating the rankings of the selected instances using FBorda. In practice, when the weights of a ranking \(\sigma_t, \rho^t\), is below an small threshold the ranking is no longer taken into account.

This classifier inherits the theoretical properties of FBorda. For instance, under mild conditions regarding to the strong unimodality of the rankings of the instances in the neighborhood of an unlabeled instance \(x\), it can be proved that on average KFC can recover the most probable ranking.

### 5 Experiments

In this section, we provide empirical evidences of the strengths of KFC for dealing with LR problem over evolving data streams. The experiments are divided into two parts. The first
set of experiments illustrate that FBorda procedure can be used to find the most probable ranking of an evolving strong unimodal probability distribution over rankings. The second set of experiments is implemented for evaluating KFC in a LR over evolving data streams.

5.1 Rank Aggregation for evolving preferences

This first set of experiments analyses the performance of FBorda as a rank aggregation algorithm in the situation in which the consensus of the population changes over time. This is evaluated with a synthetic dataset.

We show in the next lines how to generate the sample from an evolving MM. First, generate an ordered sequence of permutations $\Pi = (\pi_0, \ldots, \pi_N)$ for $N = n(n - 1)/2$ such that $d(\pi_i, \pi_j) = j - i$ holds for every $i, j$. A consensus $\pi_t$ iterating in $\Pi$ represents that the preferences change slightly from $\pi_t$ to $\pi_{t+1}$ but the rankings $\pi_0$ and $\pi_N$ are the reverse, e.g., $\pi_0 = 12345$ and $\pi_{10} = 54321$.

The final sample consists of sampling an evolving MM whose sequence of consensus is $\Pi$. In other words, for each $\pi_t \in \Pi$ we define the distribution $MM(\pi_t, \theta)$, and generate the sample $\{\sigma_{t1}, \ldots, \sigma_{tm}\}$ by sampling that model $\sigma_{ti} \sim MM(\pi_t, \theta)$ [15]. The spread parameter $\theta$ has been chosen so that the expected distance of a random Mallows ranking ($d(\sigma_{ti}, \pi^i)$ defined in Equation (2)) is $1/3$ of the expected distance at uniformity and $m = 100$.

The evaluation process is done following the test-then-train strategy, a common approach in stream learning in which an instance $\sigma_{it}$ is generated, then used for evaluation and then fed to the training model. The error of FBorda is measured as Kendall’s-$\tau$ distance $d(\bar{\pi}_i, \pi_t)$, where $\bar{\pi}_i$ is the FBorda ranking after sampling $\sigma_{ti}$ and $\pi_t$ the current consensus ranking. Once the evaluation is done, the new instance $\sigma_{ti}$ is appended to the sample, so it can be considered by FBorda in the next iteration of the process.

Note that we can handle a possibly infinite stream of rankings. Interestingly, this infinite stream of rankings is stored with linear space complexity.

The results of $n = 7$ are shown in Figure 2, where the X-axis orders the sequence of rankings chronologically and the Y-axis shows the error $d(\bar{\pi}_i, \pi_t)$. We use different fading factors $\rho \in \{0.8, 0.9, 1\}$, each corresponding to a different line in the plot. The three boxes correspond to the results after the 3rd, 10th and 20th drifts. For each parameter configuration the results are run 10 times and the average results are shown.

The first evaluations after a drift occurs, the error increases. As expected, as the number of rankings of the same distribution increases, the error tends to decrease. We can see that the the model in which no fading factor is considered, with $\rho = 1$, performs the worse. This is because it is assuming that the last and the first permutations seen are equally important, and the consensus of the population does not change in time. This is equivalent to a standard online learning algorithm.

Choosing a fading factor too small makes FBorda forget quicker the previous permutations and this can lead to a situation in which just a few of the last permutations are considered to estimate the consensus. The more chaotic behavior of the smallest value of
\[ \rho = 0.8 \] is related to this phenomenon in which few permutations are contributing in the estimation of the consensus, i.e., FBorda is aggregating a small number of rankings.

Finally, when \( \rho = 0.9 \) FBorda has the most accurate results. When there is a drift, at the left hand size of each of the plots, the error increases. However, FBorda needs a small number of permutations under the same distribution to accurately recover the new mode.

### 5.2 Label ranking with evolving preferences

To the best of the authors’ knowledge there is not any real-world dataset that considers a LR problem in which the user preferences evolve over time. We argue that LREP is a useful scenario since the preferences are usually dynamic relations that evolve over time. We propose a framework to simulate data in which we can define populations with similar features \( X \) that have similar preferences in \( Y \). We can easily generate data with several configurations. For example, we can have different number of sub-populations, the drifts are abrupt or the sequence of preferences in a drift of \( Y \) are far apart or close. Due to the lack of space this framework is introduced in the supplementary material. Moreover, we include a set of experiments that validate KFC for \( n \in \{7, 10, 15\} \).

For further testing our approach we consider a real dataset from the label ranking community. In particular, we use 5 different databases from the bioinformatics field which are related to genetics [12].

These data can be used without any further modification in the context of online learning by feeding the learning algorithm one sample at a time, taking the order given in the input file. However, there is no concept drift in it and therefore the drift has to be simulated. To simulate \( D \) drifts in the preferences, we (i) generate a sequence of permutations \( \Pi = \pi_0, \pi_2, \ldots, \pi_D \) and where \( d(\pi_i, \pi_j) = j - i \) holds for every \( i, j \) and then (ii) partition the sample in \( D + 1 \) nearly equally sized blocks and compose the first block...
with $\pi_0$, the second block with $\pi_1$ and so on.

The results are summarized in Figure 3. The X-axis displays $D$, i.e., the first row in the figure shows the results of the original dataset in which no drift has been applied to the dataset, while the bottom row, on the other hand, there have been $n(n-1)/2$ drifts.

The columns show the forgetting parameter considered in the learning process. Recall that $\rho = 1$ is the case in which no forgetting is considered, while as we decrease $\rho$ we put higher emphasis on the recently generated permutations.

The error is measured as the Kendall’s-$\tau$ correlation between the predicted ranking and the real one. In the case where no drift has been made, first row of the heat-map in Figure 3, not forgetting old data (setting $\rho = 1$) is the best alternative. On the other hand, when there are many drifts in the rankings better results are obtained as we decrease $\rho$. It should be noted how the best $\rho$ decreases as the number of drifts increases.

As a conclusion, we can state that in an evolving preference scenario our proposed KFC algorithm, based on FBorda, is able adapt to drifts in the distribution.

6 Conclusions and Further Work

In this paper, we have considered a novel scenario for rank elicitation which assumes online learning scenarios in which the distribution modeling the preferences changes as time goes by. Under this realistic prism, we have studied two well-known ranking problems: rank aggregation and label ranking.

Our main contribution, and core of the algorithms proposed for both problems is the Forgetful Borda algorithm, FBorda. It is an online version of the well-known Borda aggregation function which gives more importance to recent rankings by using weights with an exponential decay. Interestingly, it has quasi-linear time complexity and linear space complexity. We have theoretically analyzed FBorda, showing its consistency and giving
bounds for the sample complexity while ensuring quality guarantees. Moreover, we have shown its efficiency in several empirical scenarios.

In this paper, we raise the question of evolving preferences, and this idea opens several interesting research lines. For example, we plan on considering the design of new aggregation algorithms for evolving Mallows models under different distances. Similar questions have already been considered in an offline setting for the Cayley [13] and Hamming [14] distances. Moreover, other ranking models such as Plackett-Luce [18] or Babington Smith [6] distributions are worth considered under this point of view.

7 Proof of Lemma 1

Proof. Let $\sigma$ be a random Mallows ranking sampled from a MM centered at $\pi$ where $\pi(i) < \pi(j)$, $\sigma \sim MM(\pi, \theta)$. We can express the expected value of $\sigma(i)$ as follows.

$$E_\pi[\sigma(i)] = \sum_{\sigma(i)<\sigma(j)} \sigma(i)p(\sigma) + \sigma(j)p(\sigma) \quad (18)$$

$$E_\pi[\sigma(j)] = \sum_{\sigma(i)<\sigma(j)} \sigma(j)p(\sigma) + \sigma(i)p(\sigma)$$

By defining $a, b < n$ as $\sigma(j) = \sigma(i) + a$ and $p(\sigma) = p(\sigma) + b$, the previous expressions can be rewritten as follows.

$$E_\pi[\sigma(i)] = \sum_{\sigma(i)<\sigma(j)} \sigma(i)(p(\sigma) + b) + (\sigma(i) + a)p(\sigma)$$

$$= \sum_{\sigma(i)<\sigma(j)} \sigma(i)p(\sigma) + \sigma(i)b + \sigma(i)p(\sigma) + ap(\sigma) \quad (19)$$

$$E_\pi[\sigma(j)] = \sum_{\sigma(i)<\sigma(j)} (\sigma(i) + a)(p(\sigma) + b) + \sigma(i)p(\sigma)$$

$$= \sum_{\sigma(i)<\sigma(j)} \sigma(i)p(\sigma) + \sigma(i)b + ap(\sigma) + ab + \sigma(i)p(\sigma)$$

Note that we can now use the expression in Equation (7) and note that is always positive if $\pi(i) < \pi(j)$ and $\sigma$ is drawn from $MM(\pi)$ with $\theta > 0$. Then,

$$E_\pi[\sigma(j)] = E_\pi[\sigma(i)] + ab = E_\pi[\sigma(i)] + c_{ij}$$

$$= E_\pi[\sigma(i)] + \sum_{\sigma(i)<\sigma(j)} (\sigma(j) - \sigma(i))(p(\sigma) - p(\sigma)) \quad (20)$$

$$> E_\pi[\sigma(i)]$$
Equation (20) implies that for the central permutation of a MM \( \pi \) with \( \pi(i) < \pi(j) \) then \( E[\sigma(i)] < E[\sigma(j)] \). For this case, Borda, in expectation will rank \( i \) before \( j \), since Borda ranks the items in a sample regarding the sum of their rankings. Given that preferences are transitive relations, we can state that this holds for every pair \( i, j \) and therefore, the Borda ranking is an consistent estimator of \( \pi \). \( \square \)

8 Proof of Lemma 5

Proof. First, we give an expression of the confidence interval for this difference, based on the Hoeffding inequality.

\[
P \left( \frac{1}{\rho} \sum_{t=0}^{m-1} \rho^t (\sigma_t(i) - E[\sigma_t(i)]) \geq c \right)
= P \left( \sum_{t=0}^{m-1} \rho^t (\sigma_t(i) - E[\sigma_t(i)]) \geq c \sum_{t=0}^{m-1} \rho^t \right)
= P \left( \exp(s \left( \sum_{t=0}^{m-1} \rho^t (\sigma_t(i) - E[\sigma_t(i)]) \right)) \geq \exp(s \cdot c \sum_{t=0}^{m-1} \rho^t) \right)
\]

by Markov’s inequality it holds that

\[
\leq \exp(-sc \sum_{t=0}^{m-1} \rho^t) E[\exp(s \left( \sum_{t=0}^{m-1} \rho^t (\sigma_t(i) - E[\sigma_t(i)]) \right))]
= \exp(-sc \sum_{t=0}^{m-1} \rho^t) \prod_{t=1}^{m} E[\exp(\rho^t s(\sigma_t(i) - E[\sigma_t(i)]))]
\]

and by adapting the Hoeffding’s Lemma for the weighted case we get

\[
\leq \exp(-sc \sum_{t=0}^{m-1} \rho^t) \prod_{t=1}^{m} \exp \left( \frac{1}{8} s^2 \rho^{2t} (b - a)^2 \right)
= \exp(-sc \sum_{t=0}^{m-1} \rho^t) \exp \left( \frac{1}{8} s^2 \sum_{t=0}^{m-1} \rho^{2t} (b - a)^2 \right)
= \exp(-sc \sum_{t=0}^{m-1} \rho^t) \exp \left( \frac{1}{8} s^2 n^2 \sum_{t=0}^{m-1} \rho^{2t} \right)
= \exp \left( \frac{1}{8} s^2 n^2 \sum_{t=0}^{m-1} \rho^{2t} - sc \sum_{t=0}^{m-1} \rho^t \right)
\]
We define \( g(s) = \frac{1}{8} s^2 n^2 \sum_{t=0}^{m-1} \rho^{2t} - sc \sum_{t=0}^{m-1} \rho^t \). That is a quadratic function whose minimum is at

\[
\min(g(s)) = \frac{-b}{2a} = \frac{-2c^2 (\sum_{t=0}^{m-1} \rho^t)^2}{n^2 \sum_{t=0}^{m-1} \rho^{2t}}
\]

Plugging this expression into the previous one we get

\[
P(\frac{1}{\sum_{t=0}^{m-1} \rho^t} \sum_{t=0}^{m-1} \rho^t (\sigma_t(i) - E[\sigma_t(i)]) \geq c) < \exp\left(\frac{-2c^2 (\sum_{t=0}^{m-1} \rho^t)^2}{n^2 \sum_{t=0}^{m-1} \rho^{2t}}\right)
\]

Therefore, taking the following value of the confidence interval we can state that with probability \( 1 - \delta \) the difference of the weighted sum of \( \sigma(i) \) and its expected value will be at most \( c \).

\[
c = \sqrt{\frac{\sum_{t=0}^{m-1} \rho^{2t} n^2}{\sum_{t=0}^{m-1} \rho^t} \frac{1}{2 \log \frac{1}{\delta}}}
\]

To conclude the proof we have to see that \( c \) grows as

\[
\frac{\sqrt{\sum_{t=0}^{m-1} \rho^{2t}}}{\sum_{t=0}^{m-1} \rho^t} = \frac{(\rho^{2m} - 1)(\rho - 1)}{(\rho^m - 1)^2}
\]

which is decreasing with \( \rho \in (0, 1) \) for fixed \( m > 0 \) and \( \delta \in [0, 1] \), which concludes the proof. An illustration of this evolution for \( n = 5 \) is shown in Figure 1 in the main paper.

\[\square\]

9 Supplementary experiments

9.1 Label ranking with evolving preferences, synthetic data simulation

To the best of the authors’ knowledge there is not any real-world dataset that considers a LR problem in which the user preferences evolve over time. We argue that LR with CD is a useful scenario since the preferences are usually dynamic relations that evolve over time. We propose a framework to simulate data in which we can define populations with similar features \( X \) that have similar preferences in \( Y \). We can easily generate data with several configurations. For example, we can have different number of sub-populations, the drifts are abrupt or the sequence of preferences in a drift of \( Y \) are far apart or close.
As in the general classification scenario, we assume that $\mathcal{X} \in \mathbb{R}^k$. As usually, population $p$ in the $X$ space is defined in this framework as a multivariate Gaussian distribution, $N(\mu^p, \sigma^p)$. Each of the populations in $X$ has an associated population in the $\Sigma$ space. We associate an evolving MM to each of the populations in $X$. Let $\Pi^p$ be the sequence of modes of these MM and $\theta$ the sequence of spread parameters. We denote those distributions as $MM(\Pi^p, \theta)$. Therefore, the distribution over the pairs in $(X, \Sigma)$ is:

$$p(X, \Sigma) = \sum_p N(\mu^p, \sigma^p) \cdot MM(\Pi^p, \theta).$$

The generation of instances for a LREP problem is basically done by iterating over drifts $t$ and for the number of instances per drift (i) randomly choose one of the populations, $p$, (ii) sample the Gaussian $N(\mu^p, \sigma^p)$ (iii) sample a permutation from the the MM with parameters $\pi^p_t, \theta^p_t$. See Algorithm 2 for a complete description of the process.

**Algorithm 2: disjoint decomposition**

**input:** $P$ populations in $X$ ($N(\mu^p, \sigma^p)$), $P$ drifting distributions in $Y$ (the sequence of MM params $(\Pi^p, \theta^p)$ for $p > P$)

**output:** $\mathcal{L}$ a collection of pairs $(x,y)$

for $t \in$ drifts do
  for $m$ times do
    p ← choose int in $[P]$;
    x ← sample from $N(\mu^p, \sigma^p)$;
    y ← sample from $MM(\pi^p_t, \theta^p_t)$;
    $\mathcal{L}$ ← append $(x, y)$;
  end
end

Table 1 shows an example of a collection of data points which were sampled in streaming fashion from positions 22 to 25 from a model in which $X$ is bidimensional and with two subpopulations. The feature space of population 0 is the Gaussian distribution centered at $(0, 0)$, variance 0.1 and a MM centered at $\sigma_0 = 1234$ and the population 1 is the Gaussian distribution centered at $(0, 0)$, variance 0.1 and a MM centered at 4321.

For the experimental evaluation the feature space $X$ is a bidimensional space of real numbers. Considering the LR example in the introduction of this paper, we could think of these variables $x_1, x_2$ as the features income and age of the users in our LR example for a car rental service. We will consider two sub-populations, one of with will be centered at the point $P_0$ where $P_0(x_1, x_2) = (0, 0)$ while the second one is centered at $P_1(x_1, x_2) = (1, 1)$. This points will be the modes of the Gaussian distributions, which will have variance 0.1.

The two distributions in the $\Sigma$ space are two drifting MM, each defined by the sequences of modes $\Pi^1$ and $\Pi^2$ and $\theta$. The evolving consensus $\Pi^1$ and $\Pi^2$ are defined by $(\pi_0^1, \ldots, \pi_N^1)$
for $N = n(n-1)/2$, for $r \in \{1, 2\}$, an ordered sequence of rankings in which $d(\pi^r_i, \pi^r_j) = j-i$ holds for every $i, j$. A consensus $\pi^r_i$ iterating in $\Pi_1$ or $\Pi_2$ represents that the preferences change slightly from $\pi^r_i$ to $\pi^r_{i+1}$ but the rankings $\pi^r_0$ and $\pi^r_N$ are the reverse, e.g., $\pi^r_0 = 12345$ and $\pi^r_{10} = 54321$. Both $\pi^r_0$ are randomly generated. The spread parameter is the same for all the populations. In particular, we choose the $\theta$ that makes the distance be at $1/3$ of the expected distance at the uniform distribution.

The evaluation process is done following the test-then-train strategy, a common approach in stream learning in which and instance is generated, then used for evaluation and then feed to the training model. The evaluation of a particular instance is similar to that in [5]. We look in the feature space $\mathcal{X}$ for the $k = O(|L|)$ closest instances. Their associated rankings are aggregated with FBorda and returned by KFC as ground truth. The weight of each instance will be decrease exponentially for older samples $\rho^t$. The error of the estimator is the Kendall’s-$\tau$ distance between the estimated and the mode of the distribution, $y_0$ of $y_1$. An example of the dataset is illustrated in Table 1.

| drift | order | subpop | $x_1$ | $x_2$ | $\pi$ |
|-------|-------|--------|-------|-------|-------|
| 1     | 22    | 0      | 0.01  | 0.23  | 1243  |
| 1     | 23    | 1      | 0.92  | 1.3   | 3421  |
| 1     | 24    | 1      | 1.2   | 1.1   | 4231  |
| 1     | 25    | 0      | -0.2  | 0.3   | 2134  |

Table 1: Illustrative example of a LR dataset.

9.1.1 Results

The results of the experiments for $n \in \{7, 10, 15\}$ are shown in Figure 4. Here again, the X-axis represents each of the instances generated in chronological order, from left to right (due to the lack of space we show here the last half of the drifts). The drifts are highlighted with a vertical grid line and there are $m = 100$ permutations in each drift. In the Y-axis we can see the error of each of the estimated consensus for the LR problem, measured as the Kendall’s-$\tau$ distance from the original central permutation to the estimated permutation.

In the case where $\rho = 1$ (there is no forgetting mechanism and is equivalent to a regular online learning approach), the error tends to be increase as new central permutations are considered, since, again, every permutation has the same importance, no matter if was generated recently or time ago. For the smaller forgetting parameter, for $\rho = 0.9$, the results are more stable along time and with smaller error but have a chaotic behavior. This is consistent with the results along the paper, where we argued that an small value of $\rho$ makes the model take into account just a very small fraction of instances (i.e., being the weights very concentrated in just a few of the most recent instances generated) what leads to a high variance in the estimator and also to a larger error overall.

The fading factor of $\rho = 0.99$ is the more stable overall: its error tends to increase when
drifts happen, but FBorda recovers the a good estimator for the ground truth ranking quickly after the drift.

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Figure 4: Error in the estimated LR, last half of the drifts.