Wilson loop for large N Yang-Mills theory on a two-dimensional sphere

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Abstract

We calculate various Wilson loop averages in a pure SU(N)-gauge theory on a two-dimensional sphere, in the large N limit. The results can be expressed through the density of rows in the most probable Young tableau found in [1]. They are valid in both phases (small and large areas of the sphere). All averages for self-intersecting loops can be reproduced from the average for a simple (non self-intersecting) loop by means of loop equations.

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1 Simple non self-intersecting loop

Two-dimensional Yang-Mills theory is quite a trivial example of a quantum field theory, since it has no propagating degrees of freedom. Nevertheless, the study of different gauge invariant quantities shows quite a rich structure and can be viewed as a useful tool for the verification of mathematical hypotheses about the corresponding theory in physically relevant dimension; say, for the search of a string theory describing multicolour QCD.

As was demonstrated in [6, 4] on the example of Wilson loops, and in [3] on the example of partition functions on the compact two-dimensional manifolds, the results can be represented in terms of a sum over minimal coverings (“soap films”) spanned on a contour, or wrapping over the closed manifold. In this respect, the theory can have a beautiful mathematical application, as was the case with the application of three-dimensional Chern-Simons gauge theory.

First we shall consider a simple loop on a sphere, separating areas $A_1$ and $A_2$, with which a holonomy $U$ is associated. The average over all possible gauge fields inside and outside the loop gives the partition function of corresponding discs with fixed holonomies $U$ and $U^+$ around the boundary, each given by the heat kernel (on the group)

$$
\langle 1| e^{-\frac{A}{N}\Delta} | U \rangle = \sum_R d_R e^{-\frac{A}{2N}c_R^{(2)}}
$$

and

$$
W(A_1, A_2) = \langle \frac{1}{N} \text{Tr} U \rangle = \frac{1}{Z} \int dU \frac{1}{N} \text{Tr} U\langle 1| e^{-\frac{A}{2N}\Delta} | U \rangle \langle U| e^{-\frac{A}{2N}\Delta} | 1 \rangle
$$

where $\Delta$ denotes the laplacian on $SU(N)$, with characters as eigenfunctions:

$$(\Delta \chi_R)(U) = c_R^{(2)} \chi_R(U),
$$

$c_R^{(2)}$ being the quadratic Casimir of the irreducible representation $R$.

The character expansion of $W(A_1, A_2)$ reads:

$$
W(A_1, A_2) = \frac{1}{Z} \sum_{R_1, R_2} d_1 d_2 \frac{1}{N} \int dU \text{Tr} U \chi_1(U) \chi_2(U) e^{-\frac{A}{2N}c_1 + \frac{A}{2N}c_2}
$$

where $\int dU \text{Tr} U \chi_1(U) \chi_2(U)$ is the number of occurrences of $R_2$ in fundamental $\otimes R_1$, that is: 1 when the Young diagram of $R_2$ is obtained by adding one box to the diagram of $R_1$ or 0 otherwise.

Now, if the rows in the first diagram have lengths $(n_i)_{i=1...N}$, the corresponding dimension is

$$
d = \prod_{1\leq i<j\leq N} \frac{j - n_j - i + n_i}{j - i}
$$

and the Casimir

$$
c = \sum_{i=1}^{N} n_i(n_i + N + 1 - 2i) - N \langle n \rangle^2
$$
with \( \langle n \rangle = \frac{1}{N} \sum_j n_j \). If the \( i^{th} \) row can be added a box, the dimension is multiplied by

\[
\frac{d_2}{d_1} = \prod_{j,j\neq i} \frac{j - n_j - i + n_i + 1}{j - n_j - i + n_i}
\]  

(6)

and the Casimir changes according to

\[
c_2/N = c_1/N + 2\left( \frac{n_i - i}{N} + \frac{1}{2} - \frac{\langle n \rangle}{N} \right)
\]  

(7)

We use the notation \( \Phi_i = i - \frac{N}{2} - n_i + \langle n \rangle \) and \( \phi(x) \) its large N limit: \( \phi(i/N) = \Phi_i/N \); we have to compute

\[
W(A_1, A_2) = \frac{1}{Z} \sum_{R_1} \frac{1}{N} \sum_i d_1^2 \prod_{j,j\neq i} \left( 1 + \frac{1}{\Phi_j - \Phi_i} \right) e^{-\frac{A_1 + A_2}{2N} c_1} e^{A_2 \phi(i/N)}
\]  

(8)

where the sum over \( i \) corresponds to all possible ways of adding one box to the diagram. This is the expectation value

\[
\langle \frac{1}{N} \sum_i \prod_{j,j\neq i} \left( 1 + \frac{1}{\Phi_j - \Phi_i} \right) e^{A_2 \phi(i/N)} \rangle
\]  

(9)

computed through averaging over all possible Young diagrams with the same weights as those which appear in the partition function of a sphere with total area \( A_1 + A_2 \); but in [1] has been found the shape of the most probable diagrams: if \( x \) corresponds to a line to which a box can be added, we have \( (A_1 + A_2)\phi(x) = 2 \int_0^1 dy \frac{1}{\phi(x) - \phi(y)} \) and we can solve for \( \rho(\phi) = \frac{d\phi}{dx} \), obtaining a semi-circle law if \( A = A_1 + A_2 < \pi^2 \) or an expression involving the complete elliptic integral of the third kind if \( A = A_1 + A_2 > \pi^2 \) (see [1]). Indeed, as the same weight is given to any representation and to its conjugate (represented by \( n_i' = 2\langle n \rangle - n_{N+1-i} \)), the unique solution for \( \phi \) is symmetric: \( \phi(1-x) = -\phi(x) \), and corresponds to (pseudo-)real representations.

In a similar way, we can average over \( R_2 \) and sum over all possible ways of suppressing one box, so that \( W(A_1, A_2) \) is also given by the average

\[
\langle \frac{1}{N} \sum_i \prod_{j,j\neq i} \left( 1 + \frac{1}{\Phi_j - \Phi_i} \right)^{-1} e^{-A_1 \phi(i/N)} \rangle
\]  

(10)

We turn to the computation of

\[
\exp \left( \sum_{j,j\neq i} \ln \left( 1 - \frac{1}{\Phi_i - \Phi_j} \right) \right)
\]  

(11)

where the \( \Phi_j \)'s are distributed according to the above-mentioned law, with fluctuations of order 1. When \( j \) is far apart from \( i \) (\(|j-i|\) of order \( N \)), the logarithm is of order \( 1/N \),

\[
^2\rho(\phi) = \frac{4}{2\pi} \sqrt{\frac{1}{4} - \phi^2}
\]

The even density is \( \rho(\phi) = \frac{2}{\pi a\phi} \sqrt{(a^2 - \phi^2)(\phi^2 - b^2)} \Pi_1 \left( -\frac{b^2}{\phi^2}, k = \frac{b}{a} \right) \) when \( b < \phi < a \) or \( \rho = 1 \) when \( -b < \phi < b \), with \( aA = 4K \), \( a(2E - k^2 K) = 1 \).
and there are $N$ such terms: these give a finite contribution to the sum. But the same is true for those $j$’s that are close to $i$: there are a few such terms, each of order one, and one has to take them into account, adding their contribution to the (principal value) integral which accounts for the terms of the first kind: $- \int \frac{dy}{\phi(x) - \phi(y)}$. Note that the latter terms were discarded in the computation of the shape of the most probable diagrams: looking for a maximum of $d^2 \exp \left( - \frac{Ac}{2\pi} \right)$ with its continuous formulation really means looking for a diagram with maximal weight with respect to long-wavelength perturbations $(\delta n_i - \delta n_{i+1} = 0$ but for few $i$’s, far apart from one another), in which case the contribution we are going to consider vanishes; while here it does not, because we consider a short-wavelength perturbation $\delta n_i = \delta_{i,i_0}$. So, the solution of [1] has to be understood in this way, and we have to consider the contributions of all the “most probable” diagrams, which differ from one another through short-wavelength fluctuations.

As the integers $\Phi_j$ have independent fluctuations of order 1, it seems reasonable to replace $\Phi_i - \Phi_j$ (for $j - i$ of order 1) by its continuous approximation $N(\phi(i/N) - \phi(j/N))$, which is: $(i - j)\phi'(i/N)$ in the large $N$ limit. Thus, the contribution to the sum of logarithms is

$$- \sum_{j,j \neq i} \sum_{k \geq 1} \frac{1}{k} (i - j)^{-k} \rho^k(\phi)$$  \hspace{1cm} (12)

If we introduce $f(x) = \sum_{k \geq 1} \frac{1}{2k} x^{2k} (B_{2k}/(2k)!)$ ($B_p$ = $p^{th}$ Bernoulli number), and use the characterization of the most probable distribution, we obtain:

$$W(A_1, A_2) = \frac{1}{N} \sum_i \exp \left[ \frac{A_2 - A_1}{2} \frac{i}{N} \phi(N) + f(2i\pi \rho(\phi)) \right]$$  \hspace{1cm} (13)

Noting that $xf'(x) = x/2 - 1 + x/(e^x - 1)$, we have $f(x) = \ln \left( \frac{e^x - 1}{x} \right) - \frac{x}{2}$ and

$$W(A_1, A_2) = \frac{1}{N} \sum_i e^{\phi} \frac{\delta A/2 \sin(\pi \rho)}{\pi \rho}$$  \hspace{1cm} (14)

with $\delta A = A_2 - A_1$: starting from an average over $R_2$, and considering all possible ways of suppressing one box, we would obtain the same expression with $A_2$ and $A_1$ interchanged: indeed, the sign of $\delta A$ does not matter as the even distribution $\rho$ has only even moments.

What about the sum over all acceptable lines? Defining $n(x)$ to be $n_i/N$ ($x = i/N$), we see that if $|\frac{d n}{d x}| > 1$, generically we have $n_i > n_{i+1}$ and all lines are likely to be extended. However, if $|\frac{d n}{d x}| < 1$, there is a finite fraction of lines for which $n_i = n_{i+1}$, and the non-increasing condition on the $n_i$’s prevents a finite fraction of lines from being extended. In this case, the contribution of $\frac{1}{N} \sum_i$ in the range $dx$ is $dx |\frac{dn}{dx}| = dx (\frac{1}{\rho} - 1)$.

Finally, we would expect:

$$W(A_1, A_2) = \int d\phi \text{Min}(\frac{1}{\rho} - 1, 1) e^{\delta A/2} \frac{\phi \sin(\pi \rho)}{\pi}$$  \hspace{1cm} (15)

However, as Boulatov first noticed [2], one has to be careful in expressing the product $\prod_{j,j \neq i} (1 + \frac{1}{\Phi_j - \Phi_i})$ in terms of the continuous function $\phi$, when $|\frac{dn}{dx}| < 1$. In this case, he
groups factors in packets and obtains a continuous approximation to \( \frac{d_2}{d_1} \) which is the same as the preceding one, but for a factor \( \frac{1}{1-\rho} \) that compensates for the \( \text{Min}(...) \). Indeed, when \( |\frac{dn}{dx}| < 1 \), fluctuations of lengths of the rows are prevented by the non-increasing condition on the \( n_i \)'s: this is the very reason why a direct continuous approximation is not valid. We shall rather consider the heights of the columns, which are free to fluctuate, so that the argument given above for lines gets transposed.

When \( |\frac{dn}{dx}| < 1 \), generically we have: \( n_i - n_{i+1} = 0 \) or 1, so columns are labelled by \( n \); summing over all possible additions of one box means summing over all \( n \)'s. Just as in the preceding case, the change in the Casimir is \( c_2 - c_1 = -2\phi \); as for dimensions, the contribution of \( j \)'s far apart from \( i \) reduces to the exponential of the same principal value integral, recognized as \( \exp\left(-\frac{A_1 + A_2}{2}\phi\right) \). To study the contribution of \( j \)'s close to \( i \) we consider a truncated diagram, formed with a finite number of lines extending above and under \( i \); let \( h_n \) denote the height of the \( n^{th} \) truncated column \( (\nu_1 \leq n \leq \nu_2) \) and set \( \tilde{\Phi}_n = n - h_n \), \( \tilde{\phi} = \tilde{\Phi}/N \). How does the dimension change when we add a box at the end of line \( i \), that is at the bottom of column \( n = n_i + 1 \)?

Before the extension, \( \prod_{j<i}(\Phi_i - \Phi_j) \) contains all integers ranging from 1 to \( h_n + \nu_2 - n + 1 \) but the \( h_n - \nu_2 + \nu - n + 1 \), \( \nu = n + 1, \ldots, \nu_2 \). Similarly, the product \( \prod_{j>i}(\Phi_j - \Phi_i) \) evaluates to \( (\tilde{\Phi}_n - \tilde{\Phi}_{\nu_1} - 2)! \prod_{\nu_1 < \nu < n}(\tilde{\Phi}_n - \tilde{\Phi}_\nu - 1)! \).

After the extension these factors are respectively \( \prod_{n < \nu \leq \nu_2}(\Phi_\nu - \Phi_n) \) and \( \prod_{\nu_1 < \nu < n}(\tilde{\Phi}_n - \tilde{\Phi}_\nu - 1)! \), so that:

\[
\frac{\tilde{d}_2}{d_1} = \prod_{\nu_1 < \nu \leq \nu_2} \left(1 + \frac{1}{\Phi_\nu - \Phi_n}\right) \times \frac{\tilde{\Phi}_n - \tilde{\Phi}_{\nu_1} - 1}{\nu_2 - \tilde{\Phi}_n + 1} \tag{16}
\]

(we have just expressed the (change in) dimension in terms of heights of columns for “steep” diagrams: \( n_i - n_{i+1} = 0 \) or 1).

The last factor goes to 1 for large truncated diagrams, extending symmetrically around \( i \) (use \( h_p - h_n = \frac{dn}{dx}(p - n) \) for large \( N \), and note that this symmetry corresponds to the cancellation of \( \zeta(2p + 1) \) in (12) and to the identification of the other terms with a principal value integral).

Finally, arguing that the heights of columns are integers with fluctuations of order 1, and can be approximated by a continuous function, we apply the same computation as before to get:

\[
W(A_1, A_2) = \frac{1}{N} \sum_n e^{\delta A/2} \phi \frac{\sin(\pi \tilde{\rho})}{\pi \tilde{\rho}} \tag{17}
\]

where \( \tilde{\rho} = \frac{dn}{d\phi} = -\frac{dn}{d\phi} = 1 - \rho \). This is exactly:

\[
W(A_1, A_2) = \int d\phi e^{\delta A/2} \phi \frac{\sin(\pi \rho)}{\pi} \tag{18}
\]

The same result was independently obtained by Boulatov [2].
2 Other loops

Before showing how the loop equations (5, 6) allow us to compute self-intersecting Wilson loops, we cast (18) into a more convenient form for later purposes.

Define \( G(z) = \int d\phi \rho(\frac{\phi}{z}) \): if \( x \) lies in the support of \( \rho \) and is such that \( \rho(x) < 1 \) (\( x \) corresponds to a region where boxes can be added) we have

\[
G(x \pm i\epsilon) = \frac{A}{2} x \mp i\pi \rho(x)
\]

with \( A \) the total area; if \( \rho(x) = 1 \) then \( \sin(\pi \rho) \) vanishes. Thus, we can write

\[
W(A_1, A_2) = \frac{1}{2i\pi} \oint dz e^{G(z) - A_1 z} = \frac{1}{2i\pi} \oint dz e^{G(z) - A_2 z}
\]

where the contour of integration goes around the cut of \( G \) in the positive direction.

Other loops can be easily obtained by the methods of (6) and expressed as contour integrals. Consider for instance the following case:

![An 8-like contour on the sphere](image)

The associated Wilson average \( W_{1,2} \) satisfies \(- (\partial_1 + \partial_2)W_{1,2} = W_1W_2 \), where we note \( W_i \) for a simple loop enclosing area \( i \) and \( \partial_i \) for derivation with respect to area \( i \) (the total area \( A \) being kept fixed). We now integrate this equation to obtain:

\[
W_{1,2} = \frac{1}{(2i\pi)^2} \oint dx \, dy \frac{e^{G(x) + G(y) - A_1 x - A_2 y}}{x + y}
\]

Indeed, we have just seen that both sides vary by the same quantity when the areas 1 and 2 are changed by the same amount; so, we only have to check that they coincide when one of the areas vanishes. Suppose \( A_2 = 0 \): the left-hand side of (21) reduces to \( W_1 \), and in the right-hand side, we use \( G(y) \sim \frac{1}{y} \) for \( y \to \infty \) to compute the residue at infinity of the \( y \) integral, and find \( W_1 \) again.

So, (21) holds provided the smallest area is associated with the outer contour. But contours can be interchanged: the difference between the integral with \( y \) on the outer contour and the one with \( y \) on the inner contour is given by the (sum of) residue(s) at \(-x \) (for each value of \( x \)), that is

\[
\frac{1}{2i\pi} \oint dx \, e^{G(x) + G(-x) - (A_1 - A_2)x}
\]
that is 0, because $G$ is an odd function. We conclude that contours can be interchanged: their relative position does not matter, provided they do not intersect (more properly: $x + y$ shall not vanish; otherwise, the double integral is still convergent, but it does not give the right answer).

As another example, we consider the Pochhammer contour

![Pochhammer contour diagram]

for which the loop equations give:

\[
\begin{align*}
(\partial_3 - \partial_4 - \partial_1)W_{1,2,3,4} &= W_{1+2}W_{2+4} \\
(-\partial_4 + \partial_2 - \partial_1)W_{1,2,3,4} &= W_{1+3}W_{3+4} \\
(\partial_2 - \partial_4 + \partial_3 - \partial_1)W_{1,2,3,4} &= W_{1+2+3}W_{2+3+4}
\end{align*}
\] (23)

so that $\partial_3W_{1,2,3,4} = W_{1+2+3}W_{2+3+4} - W_{1+3}W_{3+4}$ and

\[
W_{1,2,3,4} = \frac{1}{(2i\pi)^2} \oint dx \, dy \, e^{G(x)+G(y)-A_1x-A_4y-A_3(x+y)} \frac{1-e^{-A_2(x+y)}}{x+y} + F_{1,2,4}
\] (24)

We take $A_3 = 0$ to compute $F_{1,2,4}$: in this case the Pochhammer contour reduces to an 8-like loop ($W_{1,4}$), and

\[
F_{1,2,4} = \frac{1}{(2i\pi)^2} \oint dx \, dy \, e^{G(x)+G(y)-A_1x-A_4y} \frac{e^{-A_2(x+y)}}{x+y}
\] (25)

Finally, we have:

\[
W_{1,2,3,4} = \frac{1}{(2i\pi)^2} \oint dx \, dy \, \frac{e^{G(x)+G(y)-A_1x-A_4y}}{x+y} \left(1 - (1-e^{-A_3(x+y)})(1-e^{-A_2(x+y)})\right)
\] (26)

(here again, the relative position of contours is arbitrary: we only require $x + y$ not to vanish) or $W_{1,2,3,4} = W_{3+1,3+4} + W_{2+1,2+4} - W_{2+3+1,2+3+4}$.

3 The small area phase

When the total area $A = A_1 + A_2$ is less than $\pi^2$, the density $\rho$ obeys a semi-circle law

\[
\rho(\phi) = \frac{A}{2\pi \sqrt{4 - \phi^2}}
\] (27)
so that (18) gives:

\[ W(A_1, A_2) = \frac{2}{\pi} \int_0^{2/\sqrt{A}} d\phi \cosh \left( \frac{\delta A}{2} \phi \right) \sin \left( \frac{A}{2} \sqrt{\frac{4}{A} - \phi^2} \right) = \sqrt{\frac{A}{A_1 A_2}} J_1 \left( 2 \sqrt{\frac{A_1 A_2}{A}} \right) \] (28)

This result had already been obtained in \[1\] by a direct estimation of the heat kernel \[4\], with windings neglected:

\[ W(A_1, A_2) \sim \int d\theta_1...N \prod_{i<j} (\theta_i - \theta_j)^2 e^{-\frac{D(\theta_i + \theta_j)}{N^2}} \sum_i \frac{1}{N} \sum \cos(\theta_i) \] (29)

At large \(N\), the angles \(\theta\) are distributed according to:

\[ \rho(\theta) = \frac{1}{2\pi} \sqrt{\frac{A}{A_1 A_2}} \sqrt{\frac{4A_1 A_2}{A} - \theta^2} \] (30)

and finally:

\[ W(A_1, A_2) = \langle \cos(\theta) \rangle = (28) \] (31)

Apparently, windings (that is, periodicity for \(\theta\)’s) can be neglected consistently provided the support of \(\rho\) doesn’t extend out of \([-\pi, \pi]\), which is the case as long as \(\frac{1}{A_1} + \frac{1}{A_2} > \frac{4}{\pi^2}\); in particular, one of the areas can be arbitrarily large. However, this is not the case with the derivation here given: the total area has to be less than \(\pi^2\) for the semi-circle law to be valid; and, for one large area, Boulatov \[2\] checked the exponential law: indeed, he could reproduce the large-area expansion given by Gross and Taylor’s prescriptions \[3\].

A deeper understanding of the contribution of windings seems needed to directly handle the heat kernel in these computations.

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