Strong convergence rates of an exponential integrator and finite elements method for time-fractional SPDEs driven by Gaussian and non-Gaussian noises

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Received: date / Accepted: date

Abstract In this work, we provide the first strong convergence result of numerical approximation of a general second order semilinear stochastic fractional order evolution equation involving a Caputo derivative in time of order \( \alpha \in (\frac{3}{4}, 1) \) and driven by Gaussian and non-Gaussian noises simultaneously more useful in concrete applications. The Gaussian noise considered here is a Hilbert space valued Q-Wiener process and the non-Gaussian noise is defined through compensated Poisson random measure associated to a Lévy process. The linear operator is not necessary self-adjoint. The fractional stochastic partial differential equation is discretized in space by the finite element method and in time by a variant of the exponential integrator scheme. We investigate the mean square error estimate of our fully discrete scheme and the result shows how the convergence orders depend on the regularity of the initial data and the power of the fractional derivative.

Keywords Time fractional derivative · Second order semilinear stochastic evolution equation · Mittag-Leffler function · Finite element method · Exponential integrator scheme · Error estimates.

Mathematics Subject Classification (2010) MSC 65C30 · MSC 74S05 · MSC 74S60

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1 Introduction

We consider the following SPDE with initial value

\[
\begin{aligned}
\partial_\alpha^\alpha X(t) &= AX(t) + F(X(t)) + B(X(t)) \frac{dW(t)}{dt} + \int_{X} G(z,X(t)) \tilde{N}(dz,dt), \\
X(0) &= X_0, \quad t \in [0,T],
\end{aligned}
\]  

(1)

on the Hilbert space \( H = (L^2(\Lambda), \langle \cdot, \cdot \rangle_H, \| \cdot \|_H) \), \( \Lambda \subset \mathbb{R}^d, d = 1, 2, 3 \), where \( T > 0 \) is the final time, \( A \) is a linear operator which is unbounded, not necessarily self-adjoint and is assumed to generate an analytic semigroup \( S(t) := e^{tA} \). Note that \( \partial_\alpha^\alpha \) denotes the Caputo fractional derivative with \( \alpha \in \left( \frac{3}{4}, 1 \right) \), \( W(t) = W(x,t) \) is a \( H \)-valued \( Q \)-Wiener process defined in a filtered probability space \( (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0}) \), where the covariance operator \( Q : H \to H \) is a positive and linear self-adjoint operator. The filtration is assumed to fulfil the usual assumptions (see [28, Def 2.1.11]). The \( Q \)-Wiener process \( W(t) \) can be represented as follows [28]

\[
W(x,t) = \sum_{i=0}^{\infty} \beta_i(t)Q^{1/2}e_i(x) = \sum_{i=0}^{\infty} \sqrt{q_i} \beta_i(t)e_i(x).
\]  

(2)

where \( q_i, e_i, i \in \mathbb{N} \) are respectively the eigenvalues and eigenfunctions of the covariance operator \( Q \), and \( \beta_i \) are mutually independent and identically distributed standard normal distributions. The mark set \( X \) is defined by \( X := H - \{0\} \). For a given set \( \Gamma \), we denote by \( \mathcal{B}(\Gamma) \) the smallest \( \sigma \)-algebra containing all open sets of \( \Gamma \). Let \( (X, \mathcal{B}(X), \nu) \) be a \( \sigma \)-finite measurable space and \( \nu \) (with \( \nu \neq 0 \)) a Levy measurable on \( \mathcal{B}(X) \) such that

\[
\nu(\{0\}) = 0, \quad \int_X \min(\|z\|^2, 1)\nu(dz) < \infty.
\]  

(3)

Let \( N(dz,dt) \) be the \( H \)-valued Poisson distributed \( \sigma \)-finite measure on the product \( \sigma \)-algebra \( \mathcal{B}(X) \) and \( \mathcal{B}(\mathbb{R}_+) \) with intensity \( \nu(dz)dt \), where \( dt \) is the Lebesgue measure on \( \mathcal{B}(\mathbb{R}_+) \). In our model problem (1), \( \tilde{N}(dz,dt) \) stands for the compensated Poisson random measure defined by

\[
\tilde{N}(dz,dt) := N(dz,dt) - \nu(dz)dt.
\]  

(4)

Note that \( \tilde{N}(dz,dt) \) is a noncontinuous martingale with mean 0 (see e.g [19]). The Wiener process \( W \) and the compensated Poisson measure \( \tilde{N} \) are supposed to be independent. Precise assumptions on the nonlinear functions \( F, B \) and \( G \) to ensure the existence of the mild solution of (1) will be given in the following section.

In the last few decades, fractional calculus has become of increasing interest to researchers in various fields of science and technology. The theory of fractional partial differential equations has gained considerable interest over time, and since most of these equations have no analytical solutions, numerical schemes are the only tools to provide good approximations. For deterministic
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(\(G = B = 0\)) and self adjoint operator \(A\), Lin and Xu \([15]\) have considered the numerical approximation of time-fractional diffusion equation and proposed an algorithm based on the finite difference scheme in time and Legendre spectral method in space. In the same context, high order finite element method and mixed finite element scheme have been studied in \([12,16]\). The numerical methods to solve the fractional heat equation with Dirichlet condition, involving a Riemann Liouville fractional derivative in time has presented in \([4,29]\). Gao et al. \([6]\) presented a novel fractional numerical method (called \(L^1\)–\(L^2\) formula) to approximate the Caputo fractional derivative order \(\alpha\) (\(0 < \alpha < 1\)) with a modification of the classical \(L^1\) formula and prove that the computational efficiency and the numerical accuracy of the new formula are superior than the standard \(L^1\) formula. The Galerkin finite element approximation for time-fractional Navier-Stokes and of the semilinear time-fractional subdiffusion problem is studied in \([1,14]\). In \([23]\), the authors developed an alternative numerical method based upon the Keller Box method for the subdiffusion equation and in \([3]\) Elzaki et al. used the decomposition method coupled with Elzaki transform to construct appropriate solutions to multi-dimensional wave, Burger and Klein-Gordon equations of fractional order. The authors in \([31]\) presented a new type of discrete fractional Gronwall inequality and they used it to analyse the stability and the convergence of the Galerkin spectral method for a linear time-fractional subdiffusion equation. Note that the time stepping methods used in all the works mentioned until now are based on finite difference methods. However theses schemes are explicit, but unstable, unless the time stepsize is very small. To solve that drawback, numerical method based on exponential integrators of Adams type have been proposed in \([7]\). The price to pay is the computation of Mittag-Leffler (ML) matrix functions. As ML matrix function is the generalized form of the exponential of matrix function, works in \([8,20,26]\) have extended some exponential computational techniques to ML. Note that up to now all the numerical algorithms presented are for time fractional deterministic PDEs with self adjoint linear operators.

However in order to represent real-world physical phenomena more accurately, it is necessary to take into account stochastic disturbances from uncertain input data. The uncertain is usually modelled by including the standard Brownian motion (Gaussian noise) and the corresponding model equation is given by \([11]\) with \(G = 0\). Few works have been done for numerical methods for Gaussian noise and time fractional stochastic partial differential equation \([11]\) with \(G = 0\), even when the linear operator \(A\) is self adjoint. To the best of our knowledge, \([34]\) is the first of basic theory and numerical method for a class of these fractional SPDEs. In \([35]\), the authors developed the fully discrete Galerkin finite element method for solving the time-fractional stochastic diffusion equations based on the approximations of the Mittag-Leffler function. Indeed the temporal integration is similar to the deterministic exponential scheme in \([7]\). In \([9]\), the authors provided rigorous convergence of numerical

\(^1\) So the corresponding linear operator is self adjoint.
A method for solving the stochastic time-fractional partial differential equation where the temporal discretization is done by the backward-Euler convolution quadrature. Note that all the above works have been done for self-adjoint linear operator $A$, so numerical study for (1) with $G = 0$ and non self-adjoint operator $A$ is still an open problem in the field, to the best of our knowledge.

Furthermore in finance for example, the unpredictable nature of many events such as market crashes, announcements made by the central banks, changing credit risk, insurance in a changing risk, changing face of operational risk [2,25] might have sudden and significant impacts on the stock price. In such situation, the more realistic model is built by incorporating a non-Gaussian noise such as Lévy process or Poisson random measure to model such events. The corresponding equation is our model equation given in (1). As we have mentioned, numerical schemes for such SPDE of type (1) driven by Gaussian and non Gaussian noises have been lacked in the scientific literature, our goal will be to fill that gap by extending the exponential scheme [17, 22] to time-fractional SPDE of type (1). The extension is extremely complicated since the ML function is more challenging than the exponential function. Using novel technical results that we have developed here, we have provided the strong convergence of our full discrete scheme for (1). Our strong convergence results examine how the convergence orders depend on the regularity of the initial data and the power of fractional derivative.

The rest of the paper is structured as follows. In Section 2, Mathematical settings for cylindrical Brownian motion, random Poisson measure, Caputo-type fractional derivative, Laplace transform, Mainardi’s Wright-type function are presented, along with the well posedness and regularity results of the mild solution of SPDE (1). In Section 3, numerical schemes based on stochastic exponential integrator scheme for SPDE (1) are presented. We give some regularity estimates of the semi-discrete problem and analyse the spatial error in Section 4. We end the paper in Section 5 by presenting the strong convergence proof of the full scheme for (1) based on finite element for spatial discretization and exponential integrator for temporal discretization.

2 Mathematical setting, main assumptions and well posedness problem

In this section, some notations and preliminary results needed throughout this work are provided. Let $(K, \langle \cdot, \cdot \rangle_K, \|\cdot\|)$ be a separable Hilbert space. For $p \geq 2$ and for a Banach space $U$, we denote by $L^p(\Omega, U)$ the Banach space of $p$-integrable $U$-valued random variables. We denote by $L(U, K)$ the space of bounded linear mapping from $U$ to $K$ endowed with the usual operator norm $\|\cdot\|_{L(U, K)}$ and $L_2(U, K) = HS(U, K)$ the space of Hilbert-Schmidt operators from $U$ to $K$ equipped with the following norm

$$
\|l\|_{L_2(U, K)} := \left( \sum_{i \in \mathbb{N}^d} \|l\psi_i\|^2 \right)^{1/2}, \quad l \in L_2(U, K),
$$

(5)
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where \((\psi_i)_{i\in\mathbb{N}^d}\) is an orthonormal basis on \(U\). The sum in Eq. (4) is independent of the choice of the orthonormal basis of \(U\). We use the notation \(L(U,U) =: L_2(U, U)\) and \(L_1(U) \in L_2(U, U)\). It is well known that for all \(l \in L(U, K)\) and \(l_1 \in L_2(U, K)\) and

\[
\|l_1\|_{L_2(U, K)} \leq \|l\|_{L(U, K)} \|l_1\|_{L_2(U)}.
\]

We denote by \(L^0_2 := HS(Q^\frac{1}{2}(H), H)\) the space of Hilbert-Schmidt operators from \(Q^\frac{1}{2}(H)\) to \(H\) with corresponding norm \(\|\cdot\|_{L^0_2}\) defined by

\[
\|l\|_{L^0_2} := \|lQ^\frac{1}{2}\|_{HS} = \left( \sum_{i \in \mathbb{N}^d} \|lQ^\frac{1}{2}e_i\|_2^2 \right)^{\frac{1}{2}}, \quad l \in L^0_2, \quad (6)
\]

where \((e_i)_{i \in \mathbb{N}^d}\) is an orthonormal basis of \(H\). The sum in Eq. (6) is also independent of the choice of the orthonormal basis of \(H\). Let \(L^2_\nu(\chi \times [0, T]; H)\) be the space of all mappings \(\theta : \chi \times [0, T] \times \Omega \to H\) such that \(\theta\) is jointly measurable and \(\mathcal{F}_t\)-adapted for all \(z \in \chi\), \(0 \leq s \leq T\) satisfying

\[
\int_0^T \int_\chi \|\theta(z, s)\|^2 \nu(dz)ds < \infty.
\]

The following lemma is a result that will be used throughout this paper.

**Lemma 1** (Itô Isometry: [27, (4.30)], [19, (3.56)])

(i) Let \(\phi \in L^2([0, T]; L^0_2)\), then the following holds

\[
E \left[ \left\| \int_0^T \phi(s)dW(s) \right\|^2 \right] = E \left[ \int_0^T \|\phi(s)\|_{L^2_0}^2 ds \right]. \quad (7)
\]

(ii) Let \(\theta \in L^2_\nu(\chi \times [0, T]; H)\), then the following holds

\[
E \left[ \left\| \int_0^T \int_\chi \theta(z, s)\tilde{N}(dz, ds) \right\|^2 \right] = E \left[ \int_0^T \int_\chi \|\theta(z, s)\|^2 \nu(dz)ds \right]. \quad (8)
\]

**Definition 1** The Caputo-type derivative of order \(\alpha\) with respect to \(t\) is defined by

\[
\partial_t^\alpha X(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial X(s)}{\partial s} \frac{ds}{(t-s)^\alpha}, & 0 < \alpha < 1, \\ \alpha = 1, \end{cases}
\]

where \(\Gamma(\cdot)\) is the gamma function.
Let introduce the generalized Mittag-Leffler function \( E_{\alpha, \beta}(t) \) defined as follows:
\[
E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},
\]
and his Laplace transform given by (see [10])
\[
\mathcal{L}(t^{\beta-1}E_{\alpha, \beta}(\lambda t^\alpha)) = \int_0^\infty \exp^{-\varsigma t} t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha) dt = \frac{\varsigma^{\alpha-\beta}}{\varsigma^\alpha - \lambda}.
\]
Remember that the Laplace transform is defined by
\[
\tilde{f}(\varsigma) = \mathcal{L}(f(t)) = \int_0^\infty e^{-\varsigma t} f(t) dt.
\]
Now we give the definition of mild solution to (1).

**Definition 2** ([34, Definition 2.2]) For any \( 0 < \alpha < 1 \), a stochastic process \( \{X(t), t \in [0, T]\} \) is called mild solution of (1) if
1. \( X(t) \) is \( \mathcal{F}_t \)-adapted on the filtration \( (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \),
2. \( \{X(t), t \in [0, T]\} \) is measurable and \( \mathbb{E} \left[ \int_0^T \|X(t)\|^2 dt \right] < \infty \),
3. For all \( t \in [0, T] \),
\[
X(t) = S_1(t)X_0 + \int_0^t (t-s)^{\alpha-1} S_2(t-s)F(X(s))ds \\
+ \int_0^t (t-s)^{\alpha-1} S_2(t-s)B(X(s))dW(s) \\
+ \int_0^t \int_X (t-s)^{\alpha-1} S_2(t-s)G(z, X(s))dN(dz, ds).
\]
hold a.s. Where \( S_1(t) = E_{\alpha, 1}(At^\alpha) \) and \( S_2(t) = E_{\alpha, \alpha}(At^\alpha) \).

**Remark 1** Considering the Mainardi’s Wright-type function (see [18])
\[
M_\alpha(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1 - \alpha(1 + \theta))}, \quad 0 < \alpha < 1, \quad \theta > 0,
\]
then the following results holds
\[
M_\alpha(\theta) \geq 0, \quad \int_0^\infty \theta^\mu M_\alpha(\theta)d\theta = \frac{\Gamma(1 + \mu)}{\Gamma(1 + \alpha \mu)}, \quad -1 < \mu < \infty, \quad \theta > 0,
\]
and
\[
E_{\alpha, 1}(t) = \int_0^\infty M_\alpha(\theta)e^{\theta t} d\theta, \quad E_{\alpha, \alpha}(t) = \int_0^\infty \theta M_\alpha(\theta)e^{\theta t} d\theta.
\]
Using (15), we rewrite operator $S_1(t)$ and $S_2(t)$ as following

$$S_1(t) = E_{\alpha,1}(At^\alpha) = \int_0^\infty M_\alpha(\theta)e^{A\theta t^\alpha}d\theta = \int_0^\infty M_\alpha(\theta)S(\theta t^\alpha)d\theta,$$  \hspace{0.5cm} (16)

and

$$S_2(t) = E_{\alpha,\alpha}(At^\alpha) = \int_0^\infty \alpha \theta M_\alpha(\theta)e^{A\theta t^\alpha}d\theta = \int_0^\infty \alpha \theta M_\alpha(\theta)S(\theta t^\alpha)d\theta. \hspace{0.5cm} (17)$$

Combining (13), (14) and (15), we obtain the result presented in [33, Definition 2.5] and we have the following lemma.

**Lemma 2** ([33, Lemma 2.8]) The operators\{\(S_1(t)\)\}_{t \geq 0} and \{\(S_2(t)\)\}_{t \geq 0} depending of \(\alpha \in (\frac{3}{4},1)\) are bounded linear operators and the following estimates hold

$$\|S_1(t)v\| \leq C_1 e^{-\gamma t}\|v\|, \quad \|S_2(t)v\| \leq \frac{C_1\alpha}{\Gamma(1+\alpha)} e^{-\gamma t}\|v\|, \quad v \in H,$$  \hspace{0.5cm} (18)

and for some constants \(C_1, \gamma > 0\).

In order to ensure the existence and the uniqueness of mild solution for SPDE and for the purpose of convergence analysis we make the following assumptions.

**Assumption 1 (Initial Value)** We assume that the initial data \(X_0 : \Omega \to H\) to be \(\mathcal{F}_0\)-measurable mapping and \(X_0 \in L^2(\Omega, D((\cdot A)^{\beta/2}))\) with \(0 \leq \beta < 2\).

**Assumption 2 (Non linearity term F)** We assume the nonlinear mapping \(F : H \to H\), to be linear growth and Lipschitz continuous ie there exist constant \(L > 0\) such that

$$\|F(u) - F(v)\|^2 \leq L\|u - v\|^2, \quad \|F(v)\|^2 \leq L(1 + \|v\|^2), \quad u, v \in H.$$  \hspace{0.5cm} (19)

**Assumption 3 (Lipschitz condition)** We assume that the diffusion and jump coefficients \(B : H \to L^2_\mathbb{F}\) and \(G : X \times H \to H\) satisfy the global Lipschitz condition ie, there exists a positive constant \(L > 0\) such that:

$$\|B(u) - B(v)\|^2 \leq L\|u - v\|^2,$$

$$\int_X \|G(z, u) - G(z, v)\|^2v(dz) \leq L\|u - v\|^2, \quad u, v \in H.$$  \hspace{0.5cm} (20)

**Assumption 4 (Linear growth)** For \(\tau \in [0,1)\) and some constant \(L > 0\) the following bound holds

$$\|(-A)^\tau B(u)\|^2_{L^2_\mathbb{F}} \leq L(1 + \|(-A)^\tau u\|^2),$$

$$\int_X \|(-A)^\tau G(z, u)\|^2v(dz) \leq L(1 + \|(-A)^\tau u\|^2), \quad u \in H.$$  \hspace{0.5cm} (21)
Theorem 1 (Corollary 3.2) Under the Assumptions 1-4, if
\[ \frac{9}{2} \left( \frac{C_1}{(1+\alpha)} \right)^2 \frac{L}{\Gamma (1+\alpha)} \left( \frac{1}{\alpha - 1} \right)^\frac{1}{2} < 1 \] the SPDEs (1) admits a unique mild solution 
\( X(t) \in (D[0,T], H) \) asymptotic stable in mean square, that is
\[ E \left[ \sup_{0 \leq t \leq T} \| X(t) \|_2 \right] < \infty, \] (22)
where by \( (D[0,T], H) \) we denote the space of all adapted càdlàg processes defined on \([0,T]\) with values in \( H \). Note that \( C_1 \) and \( \gamma \) are given in (18), and \( L \) is the Lipschitz condition from Assumptions 2-4.

In all that follows, \( C \) denotes a positive constant that may change from line to line. In the Banach space \( D((-A)^{\alpha/2}), \alpha \in \mathbb{R}, \) we use the notation \( \|(-A)^{\alpha/2} \cdot\| = \|\cdot\|_\alpha \) and we now present the following regularity results.

3 Regularity of the mild solution

We discuss the space and regularity of the mild solution \( X(t) \) of (11) given by (13) in this section. In the rest of this paper to simplify the presentation, we assume the SPDE (11) to be second order of the following type.
\[ \partial_t^\alpha X(t) = \left[ \nabla \cdot (D\nabla X(t,x)) - q \cdot \nabla X(t,x) + f(x, X(t,x)) \right] + b(x, X(t,x)) \frac{dW(t,x)}{d t} + \int_X g(z,x,X(t,x)) \tilde{N}(dz,dt), \]
where \( f : \Lambda \times \mathbb{R} \rightarrow \mathbb{R} \) is globally continuous, \( b : \Lambda \times \mathbb{R} \rightarrow \mathbb{R} \) is continuously differentiable with globally bounded derivatives and \( g : \mathcal{X} \times \Lambda \times \mathbb{R} \rightarrow \mathbb{R} \) is globally Lipschitz continuous. In the abstract framework (11), the linear operator \( A \) is the \( L^2(\mathbb{R}) \) realization (see [5, p. 812]) of the following differential operator
\[ Au = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( D_{i,j}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i=1}^d q_i(x) \frac{\partial u}{\partial x_i} , \]
\[ D = (D_{i,j})_{1 \leq i,j \leq d}, \quad q = (q_i)_{1 \leq i \leq d}, \]
where \( D_{i,j} \in L^\infty(\Lambda), \) \( q_i \in L^\infty(\Lambda) \). We assume that there exists a positive constant \( c_1 > 0 \) such that
\[ \sum_{i,j=1}^d D_{i,j}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad \xi \in \mathbb{R}^d, \quad x \in \tilde{\Lambda}. \]

The functions \( F : H \rightarrow H, \) \( B : H \rightarrow L^0_2 \) and \( G : \mathcal{X} \times H \rightarrow H \) are defined by
\[ (F(v))(x) = f(x, v(x)), \quad (B(v)u)(x) = b(x, v(x)) \cdot u(x), \]
\[ G(z,v)(x) = g(z, x, v(x)). \]
for all $x \in A$, $v \in H$, $u \in Q^{1/2}(H)$ and $z \in X$. As in [17], we introduce two spaces $\mathbb{H}$ and $V$ such that $\mathbb{H} \subset V$; the two spaces depend on the boundary conditions of $A$ and the domain of the operator $A$. For Dirichlet (or first-type) boundary conditions, we take

$$V = \mathbb{H} = H^1_0(A) = \{ v \in H^1(A) : v = 0 \text{ on } \partial A \}. $$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take

$$V = H^1(\Omega), \quad H = \left\{ v \in H^2(\Lambda) : \frac{\partial v}{\partial n} + \alpha_0 v = 0, \quad \text{on } \partial \Lambda \right\}, \quad \alpha_0 \in \mathbb{R}. $$

where $\frac{\partial v}{\partial n}$ is the normal derivative of $v$ and $v_A$ is the exterior pointing normal $n = (n_i)$ to the boundary of $A$ given by

$$\frac{\partial v}{\partial n} = \sum_{i,j=1}^d n_i(x) D_{i,j}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial \Lambda.$$

Using Gårding’s inequality (see e.g. [30]), it holds that there exist two constants $c_0$ and $\lambda_0 > 0$ such that the bilinear form $a(\cdot, \cdot)$ associated to $-A$ satisfies

$$a(v, v) \geq \lambda_0 \| v \|^2_{H^1(\Lambda)} - c_0 \| v \|, \quad v \in V. \quad (23)$$

By adding and subtracting $c_0 X dt$ in both sides of (23), we have a new linear operator still denoted by $A$, and the corresponding bilinear form is also still denoted by $a$. Therefore, the following coercivity property holds

$$a(v, v) \geq \lambda_0 \| v \|^2_1, \quad v \in V. \quad (24)$$

Note that the expression of the nonlinear term $F$ has changed as we included the term $c_0 X$ in the new nonlinear term that we still denote by $F$. The coercivity property (24) implies that $A$ is the infinitesimal generator of a contraction semigroup $S(t) = e^{tA}$ on $L^2(\Lambda)$. Note also that the coercivity property (24) also implies that the real part of the eigenvalues of $-A$ are positive, therefore its fractional powers are well defined for any $\alpha > 0$, by

$$\begin{cases} 
(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tA} dt, \\
(-A)^\alpha = (-A^{-\alpha})^{-1},
\end{cases} \quad (25)$$

where $\Gamma(\alpha)$ is the Gamma function. As the real part of the eigenvalues of $A$ is negative, the generalized Mittag-Leffler operator $E_{\alpha, \beta}(tA)$ is therefore well defined by

$$\| (\lambda I + A)^{-1} \|_{L(L^2(\Lambda))} \leq \frac{C_1}{|\lambda|}, \quad \lambda \in S_\theta,$$

where $S_\theta := \{ \lambda \in \mathbb{C} : \lambda = \rho e^{i\theta}, \rho > 0, \, 0 \leq |\phi| \leq \theta \}$ (see e.g. [11]).

We recall the following properties of the semigroup $S(t)$ generated by $-A$, that will be useful throughout this paper.
Proposition 1 (Smoothing properties of the semigroup) \[ \|(-A)^\delta S(t)\|_{L(H)} \leq Ct^{-\delta}, \quad \|(-A)^{-\gamma}(I - S(t))\|_{L(H)} \leq Ct^\gamma, \quad t > 0, \] (26)

\((-A)^\delta S(t) = S(t)(-A)^\delta \) on \( D((-A)^\delta) \), If \( \delta > \gamma \) then \( D((-A)^\delta) \supset D((-A)^\gamma) \). (27)

Firstly, we have the following useful lemma

Lemma 3 Let \( 0 \leq t_1 < t_2 \leq T \) and \( 0 < a < 1 \), we have

\[ t_2^a - t_1^a \leq (t_2 - t_1)^a. \] (28)

Proof: Using the integral form and the variable change \( t = t_2u + t_1(1 - u) \) yields

\[
t_2^a - t_1^a = \int_{t_1}^{t_2} at^{a-1}dt \\
= a(t_2 - t_1) \int_0^1 [t_2 - (t_2 - t_1)u]^{a-1}du \\
\leq a(t_2 - t_1) \int_0^1 (t_2 - t_1)^{a-1}u^{a-1}du \\
\leq (t_2 - t_1)^a \int_0^1 au^{a-1}du \\
\leq (t_2 - t_1)^a.
\]

The following lemma is an extension of the smoothing properties of the semigroup \( S(t) \), Proposition 1 to the operators \( \{S_1(t)\}_{t \geq 0} \) and \( \{S_2(t)\}_{t \geq 0} \).

Lemma 4 Let \( t \in (0, T) \), \( 0 < t_1 < t_2 \leq T \), \( T < \infty \), \( \frac{1}{2} < \alpha < 1 \), \( \rho \geq 0 \), \( 0 \leq \eta < 1 \) and \( \delta \geq 0 \), there exists a constant \( C > 0 \) such that for all \( i = 1, 2 \)

\[ \|(-A)^\rho S_i(t)\|_{L(H)} \leq Ct^{-\alpha \rho}, \quad \|(-A)^{-\eta}(S_i(t_2) - S_i(t_1))\|_{L(H)} \leq Ct^{\alpha \eta} \] (29)

\[ \|t_1^{\alpha-1}S_1(t_1^\rho) - t_2^{\alpha-1}S_2(t_2^\rho)\|_{L(H)} \leq C(t_2 - t_1)^{1-\alpha}t_1^{\alpha-1}t_2^{\alpha-1}, \] (30)

and

\[ (-A)^\delta S_i(t) = S_i(t)(-A)^\delta \text{ on } D((-A)^\delta). \] (31)

Proof See \[34\] Lemma 3.3 for the proof of the first result of (29) and (31) is just a consequence of (27) using (14) and (15). Concerning the second, using
triangle inequality, Proposition 4, 11 and Lemma 3 we have
\[
\|(-A)^{-\eta}(S_1(t_2) - S_1(t_1))\|_{L(H)} = \left\| \int_0^\infty (-A)^{-\eta}M_\alpha(\theta) (S(\theta t_2^\alpha) - S(\theta t_1^\alpha)) d\theta \right\|_{L(H)} \\
\leq \int_0^\infty M_\alpha(\theta) \|S(\theta t_2^\alpha)\|_{L(H)} \left\| (-A)^{-\eta} \left( e^{A\theta(t_2^\alpha - t_1^\alpha)} - 1 \right) \right\|_{L(H)} d\theta \\
\leq C \int_0^\infty \left[ \theta (t_2^\alpha - t_1^\alpha) \right]^\eta M_\alpha(\theta) d\theta \\
\leq C (t_2^\alpha - t_1^\alpha)^\eta \int_0^\infty \theta^\eta M_\alpha(\theta) d\theta \\
\leq C \frac{\Gamma(1 + \eta)}{\Gamma(1 + \alpha \eta)} (t_2 - t_1)^{\alpha \eta} \leq C (t_2 - t_1)^{\alpha \eta}.
\]

For the proof of (29), we use triangle inequality, Proposition 4 and 11 to obtain
\[
\|t_2^{\alpha - 1}S_2(t_2) - t_1^{\alpha - 1}S_2(t_1)\|_{L(H)} \\
= \left\| \int_0^\infty \alpha \theta M_\alpha(\theta) \left[ t_2^{\alpha - 1}e^{A\theta t_2^2} - t_1^{\alpha - 1}e^{A\theta t_1^2} \right] dt \right\|_{L(H)} \\
= \left\| \int_0^\infty \alpha \theta M_\alpha(\theta) \left( \int_{t_1}^{t_2} \frac{d(t_2^{\alpha - 1}e^{A\theta t_2^2})}{dt} dt \right) \right\|_{L(H)} \\
= \left\| \int_0^\infty \alpha \theta M_\alpha(\theta) \left( \int_{t_1}^{t_2} (\alpha - 1) t_2^{\alpha - 2} e^{A\theta t_2^2} + \alpha \theta t_2^{\alpha - 2} e^{A\theta t_2^2} dt \right) \right\|_{L(H)} \\
\leq \int_0^\infty \alpha \theta M_\alpha(\theta) \left( \int_{t_1}^{t_2} (\alpha - 1) t_2^{\alpha - 2} e^{A\theta t_2^2} \right) d\theta \\
\leq C \int_0^\infty \alpha \theta M_\alpha(\theta) \left( \int_{t_1}^{t_2} (\alpha - 1) t_2^{\alpha - 2} + \alpha \theta t_2^{\alpha - 2} (t_1^{\alpha - 1} - t_2^{\alpha - 1}) dt \right) d\theta \\
\leq C \int_0^\infty \alpha \theta M_\alpha(\theta) \left( \int_{t_1}^{t_2} t_2^{\alpha - 2} dt \right) d\theta \\
\leq C \frac{\alpha}{1 - \alpha} \int_0^\infty \theta M_\alpha(\theta) (t_1^{\alpha - 1} - t_2^{\alpha - 1}) d\theta \\
\leq C \frac{\alpha \Gamma(1 + \alpha)}{1 - \alpha} \frac{t_2^{1 - \alpha} - t_1^{1 - \alpha}}{t_1^{1 - \alpha} t_2^{1 - \alpha}},
\]
applying Lemma 3 with \( a = 1 - \alpha \) yields
\[
\|t_2^{\alpha - 1}S_2(t_2) - t_1^{\alpha - 1}S_2(t_1)\|_{L(H)} \leq C \frac{\alpha \Gamma(1 + \alpha)}{1 - \alpha} \frac{t_2^{1 - \alpha} - t_1^{1 - \alpha}}{t_1^{1 - \alpha} t_2^{1 - \alpha}} \\
\leq C (t_2 - t_1)^{1 - \alpha} t_1^{\alpha - 1} t_2^{\alpha - 1}.
\]
Moreover (23), 31 hold if \( A, S_1 \) and \( S_2 \) are replaced by their discrete versions \( A_h, S_{1h} \) and \( S_{2h} \) respectively defined in Section 4.
Now, we give a spatial regularity result for the solution $X(t)$ in the following lemma.

**Lemma 5** Let Assumptions [7] and [4] be fulfilled. Then the following space regularity holds

$$\|(−A)^{β/2}X(t)\|_{L^2(Ω,H)} ≤ C(1 + \|(−A)^{β/2}X_0\|_{L^2(Ω,H)}), \quad 0 ≤ t ≤ T, \quad (32)$$

Moreover, (32) holds if $A$ and $X$ are replaced by their discrete versions $A_h$ and $X^h$ defined in Section 1.

**Proof** By the definition of mild solution (13),

$$X(t) = S_1(t)X_0 + \int_0^t (t-s)^{α−1}S_2(t-s)F(X(s))ds + \int_0^t (t-s)^{α−1}S_2(t-s)B(X(s))dW(s) + \int_0^t \int_X (t-s)^{α−1}S_2(t-s)G(z,X(s)):\tilde{N}(dz,ds).$$

Then taking the $L^2$-norm, using triangle inequality, the classical estimate $(\sum_{i=1}^n a_i)^2 ≤ n \sum_{i=1}^n a_i^2$, the Itô isometry (7) and (8) to the last two terms yield

$$\|(−A)^{β/2}X(t)\|_{L^2(Ω,H)}^2 ≤ 4\|(−A)^{β/2}S_1(t)X_0\|_{L^2(Ω,H)}^2 + 4\left(\int_0^t (t-s)^{α−1}\|(−A)^{β/2}S_2(t-s)F(X(s))\|_{L^2(Ω,H)}^2 ds\right)^{1/2}$$

$$+ 4\int_0^t (t-s)^{2α−2}\|(-A)^{β/2}S_2(t-s)B(X(s))\|_{L^2(Ω,H)}^2 ds$$

$$+ 4\int_0^t \int_X (t-s)^{2α−2}\|(-A)^{β/2}S_2(t-s)G(z,X(s))\|_{L^2(Ω,H)}^2 v(dz)ds$$

$$= 4 \sum_{i=1}^4 I_i^2. \quad (33)$$

We will bound $I_i^2, i = 1, 2, 3, 4$ one by one. First, by the boundedness of $S_1(t)$ in Lemma 2 and (31), we have

$$I_1^2 := \|(−A)^{β/2}S_1(t)X_0\|_{L^2(Ω,H)}^2 ≤ C\|(−A)^{β/2}X_0\|_{L^2(Ω,H)}^2. \quad (34)$$
For $I_2$, by semigroup property \( (24) \) with $\rho = \frac{\beta}{2}$, Assumption \( (3) \) and \( (22) \), we get
\[
I_2^2 := \left( \int_0^t (t-s)^{\alpha-1} \left\| (-A)^{\beta/2} S_2(t-s) F(X(s)) \right\|_{L^2(\Omega, H)}^2 \right)^{1/2} ds \\
\leq C \left( \int_0^t (t-s)^{\alpha-1} \left( 1 + \|X(s)\|_{L^2(\Omega, H)} \right) \right)^{1/2} ds \\
\leq C \left( \int_0^t (t-s)^{\frac{\alpha-2}{2}} ds \right)^{1/2} \left( 1 + E \left[ \sup_{0 \leq s \leq t} \|X(s)\|^2 \right] \right) \\
\leq C t^{\alpha(2-\beta)} \left( 1 + E \left[ \sup_{0 \leq t \leq T} \|X(t)\|^2 \right] \right) \leq C. \tag{35}
\]

Applying \( (27) \), the boundedness of $S_2(t)$ in Lemma \( (2) \), Assumption \( (4) \) with $\tau = \frac{\beta}{2}$, we deduce
\[
I_3^2 := \int_0^t (t-s)^{2\alpha-2} E \left[ \left\| (-A)^{\beta/2} S_2(t-s) B(X(s)) \right\|_{L^2_0}^2 \right] ds \\
\leq C \int_0^t (t-s)^{2\alpha-2} \|S_2(t-s)\|_{L_2(\Omega, H)}^2 \left( 1 + E \left[ \|(-A)^{\beta/2} X(s)\|^2 \right] \right) ds \\
\leq C \int_0^t (t-s)^{2\alpha-2} \left( 1 + \|(-A)^{\beta/2} X(s)\|^2 \right)_{L^2(\Omega, H)}^2 ds \\
\leq C + C \int_0^t (t-s)^{2\alpha-2} \|(-A)^{\beta/2} X(s)\|_{L^2(\Omega, H)}^2 ds. \tag{36}
\]

For $I_4$, analogous to $I_3$, we use also \( (27) \), the boundedness of $S_2(t)$ in Lemma \( (2) \) Assumption \( (4) \) with $\tau = \frac{\beta}{2}$, to obtain
\[
I_4 := E \int_0^t \int_X (t-s)^{2\alpha-2} \left\| (-A)^{\beta/2} S_2(t-s) G(z, X(s)) \right\|^2 v(dz) ds \\
= E \int_0^t \int_X (t-s)^{2\alpha-2} \|S_2(t-s)\|_{L_2(\Omega, H)}^2 \left\| (-A)^{\beta/2} G(z, X(s)) \right\|^2 v(dz) ds \\
\leq C E \int_0^t (t-s)^{2\alpha-2} \int_X \left\| (-A)^{\beta/2} G(z, X(s)) \right\|^2 v(dz) ds \\
\leq C \int_0^t (t-s)^{2\alpha-2} \left( 1 + \|(-A)^{\beta/2} X(s)\|^2 \right)_{L^2(\Omega, H)}^2 ds \\
\leq C + C \int_0^t (t-s)^{2\alpha-2} \|(-A)^{\beta/2} X(s)\|_{L^2(\Omega, H)}^2 ds. \tag{37}
\]

Putting \( (34) - (37) \) in \( (33) \) hence yields
\[
\|(-A)^{\beta/2} X(t)\|_{L^2(\Omega, H)}^2 \leq C(1 + \|(-A)^{\beta/2} X_0\|^2_{L^2(\Omega, H)}) \\
+ C \int_0^t (t-s)^{2\alpha-2} \|(-A)^{\beta/2} X(s)\|_{L^2(\Omega, H)}^2 ds,
\]
applying fractional Gronwall’s lemma \( (13, 32) \) Lemma A.2 proves \( (32) \).
Now by the following theorem, we provide a temporal regularity of the solution process of (1).

**Theorem 2** Suppose that Assumptions [1] - [2] are fulfilled. Then the following estimate holds

\[ \|X(t_2) - X(t_1)\|_{L^2(\Omega; H)} \leq C(t_2 - t_1)^{\frac{\min\{\alpha, 2-2\eta\}}{2}}, \quad 0 \leq t_1 < t_2 \leq T. \] (38)

Moreover, (38) holds when \( A \) and \( X \) are replaced by their semidiscrete versions \( A_h \) and \( X^h \) respectively, defined in Section [4].

**Proof** For \( 0 \leq t_1 < t_2 \leq T \), we rewrite the mild solution [13] at times \( t = t_2 \) and \( t = t_1 \) and we subtract \( X(t_2) \) by \( X(t_1) \) as

\[
\begin{align*}
X(t_2) - X(t_1) &= (S_1(t_2) - S_1(t_1)) X_0 \\
+ &\int_0^{t_1} \left[ (t_2-s)^{-1} S_2(t_2-s) - (t_1-s)^{-1} S_2(t_1-s) \right] F(X(s)) ds \\
+ &\int_0^{t_1} \left[ (t_2-s)^{-1} S_2(t_2-s) - (t_1-s)^{-1} S_2(t_1-s) \right] B(X(s)) dW(s) \\
+ &\int_0^{t_1} \int_X \left[ (t_2-s)^{-1} S_2(t_2-s) - (t_1-s)^{-1} S_2(t_1-s) \right] G(z, X(s)) \tilde{N}(dz, ds) \\
+ &\int_0^{t_2} (t_2-s)^{-1} S_2(t_2-s) F(X(s)) ds \\
+ &\int_0^{t_2} (t_2-s)^{-1} S_2(t_2-s) B(X(s)) dW(s) \\
+ &\int_0^{t_2} \int_X (t_2-s)^{-1} S_2(t_2-s) G(z, X(s)) \tilde{N}(dz, ds).
\end{align*}
\] (39)

Taking the \( L^2 \) norm in both sides and using triangle inequality yields

\[ \|X(t_2) - X(t_1)\|_{L^2(\Omega; H)} \leq \sum_{i=1}^7 J_i. \] (40)

Inserting an appropriate power of \((-A)\), using Lemma [4] with \( \eta = \frac{\beta}{2} \) and Assumption [1] implies

\[
\begin{align*}
J_1 := &\| (S_1(t_2) - S_1(t_1)) X_0 \|_{L^2(\Omega; H)} \\
= &\|(-A)^{-\frac{\beta}{2}} (S_1(t_2) - S_1(t_1)) (-A)^{\frac{\beta}{2}} X_0 \|_{L^2(\Omega; H)} \\
\leq &\ C(t_2 - t_1)^{\frac{\beta}{2}} \|(-A)^{\frac{\beta}{2}} X_0 \|_{L^2(\Omega; H)} \\
\leq &\ C(t_2 - t_1)^{\frac{\beta}{2}}.
\end{align*}
\] (41)
By triangle inequality, using (30), Assumption 2, (22) and Cauchy-Schwartz inequality, we get

\[
J_2 := \left\| \int_0^{t_1} \left[ (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right] F(X(s)) ds \right\|_{L^2(\Omega; H)} \\
\leq \int_0^{t_1} \left\| (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right\|_{L^2(\Omega; H)} F(X(s)) ds \\
\leq C \int_0^{t_1} \left\| (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right\|_{L(H)} F(X(s)) ds \\
\leq C \left( \int_0^{t_1} (t_2 - t_1)^{1-\alpha} (t_1 - s)^{\alpha - 1} ds \right) \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \| X(s) \| \right] \right)^{1/2} \\
\leq C(t_2 - t_1)^{1-\alpha} \left( \int_0^{t_1} (t_2 - t_1)^{2\alpha - 2} ds \right)^{1/2} \left( \int_0^{t_1} (t_1 - s)^{2\alpha - 2} ds \right)^{1/2} \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \| X(s) \| \right] \right) \\
\leq C(t_2 - t_1)^{1-\alpha} t_1^{2\alpha - 2} \leq C(t_2 - t_1)^{1-\alpha}. \tag{42}
\]

Using the Itô isometry \(\mathbb{E} \left[ \int_Y \left( \int_X \left[ (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right] G(z, X(s)) \tilde{N}(dz, ds) \right]^2 \right] \leq 0\), Assumption with \(r = 0\) and Cauchy-Schwartz inequality, we obtain

\[
J_2^2 = \left\| \int_0^{t_1} \int_X \left[ (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right] G(z, X(s)) \tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)}^2 \\
= \mathbb{E} \left[ \int_0^{t_1} \int_X \left[ (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right]^2 G(z, X(s)) \nu(dz) ds \right] \\
\leq \int_0^{t_1} \left\| (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right\|_{L(H)}^2 \mathbb{E} \left[ \int_X \| G(z, X(s)) \|^2 \nu(dz) \right] ds \\
\leq C \left( \int_0^{t_1} (t_2 - t_1)^{2-2\alpha} (t_1 - s)^{2\alpha - 2} ds \right) \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \| X(s) \| \right] \right) \\
\leq C(t_2 - t_1)^{2-2\alpha} \left( \int_0^{t_1} (t_1 - s)^{2\alpha - 2} ds \right)^{1/2} \left( \int_0^{t_1} (t_1 - s)^{4\alpha - 4} ds \right)^{1/2} \\
\leq C(t_2 - t_1)^{2-2\alpha} \left( \int_0^{t_1} (t_1 - s)^{4\alpha - 4} ds \right)^{1/2} \left( \int_0^{t_1} (t_1 - s)^{4\alpha - 4} ds \right)^{1/2} \\
\leq C(t_2 - t_1)^{2-2\alpha} t_1^{4\alpha - 3} \leq C(t_2 - t_1)^{2-2\alpha}. \tag{43}
\]
By using a similar procedure as bounding $J_4$, using the Itô isometry [7], Assumption [4] with $\tau = 0$, (22) and Cauchy-Schwartz inequality, we have

\[
J_3^2 \leq \left\| \int_0^{t_1} \left[ (t_2 - s)^{\alpha - 1} S_2(t_2 - s) - (t_1 - s)^{\alpha - 1} S_2(t_1 - s) \right] B(X(s))dW(s) \right\|_{L^2(\Omega; H)}^2 \\
\leq C(t_2 - t_1)^{2 - 2\alpha}.
\]  

(44)

By triangle inequality, the boundedness of operator $S_2(t)$ [18], Assumption [2] and [22], we get

\[
J_5 := \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} S_2(t_2 - s)F(X(s))ds \right\|_{L^2(\Omega; H)} \\
\leq \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \|S_2(t_2 - s)F(X(s))\|_{L^2(\Omega; H)}ds \\
\leq C \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \|F(X(s))\|_{L^2(\Omega; H)}ds \\
\leq C \left( \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} ds \right) \left( 1 + E \left[ \sup_{0 \leq s \leq T} \|X(s)\|^2 \right] \right)^{\frac{1}{2}} \\
\leq C(t_2 - t_1)^\alpha.
\]  

(45)

Now to bound the sixth term, the Itô isometry property [7] leads

\[
J_6^2 := \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} S_2(t_2 - s)B(X(s))dW(s) \right\|_{L^2(\Omega; H)}^2 \\
= E \left[ \int_{t_1}^{t_2} \| (t_2 - s)^{\alpha - 1} S_2(t_2 - s)B(X(s)) \|_{L^2_0}^2 ds \right].
\]

By the boundedness of operator $S_2(t)$ [18], Assumption [4] with $\tau = 0$ and (22), we have

\[
J_6^2 \leq \int_{t_1}^{t_2} (t_2 - s)^{2\alpha - 2} \|S_2(t_2 - s)\|_{L^2(\Omega; H)}^2 \|B(X(s))\|_{L^2(\Omega; H)}^2 ds \\
\leq C \left( \int_{t_1}^{t_2} (t_2 - s)^{2\alpha - 2} ds \right) \left( 1 + E \left[ \sup_{0 \leq s \leq T} \|X(s)\|^2 \right] \right)^{\frac{1}{2}} \\
\leq C(t_2 - t_1)^{2\alpha - 1}.
\]  

(46)

By using a similar procedure as bounding $J_6$, using the Itô isometry [8], the boundedness of operator $S_2(t)$ [18], Assumption [4] with $r = 0$ and [22], we easily have

\[
J_7^2 = \left\| \int_{t_1}^{t_2} \int_X (t_2 - s)^{\alpha - 1} S_2(t_2 - s)G(z, X(s))\tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)}^2 \\
\leq C(t_2 - t_1)^{2\alpha - 1}.
\]  

(47)
Substituting (14) - (17) in (10) and note that for $\alpha \in \left(\frac{1}{4}, 1\right)$, $2\alpha - 1 > 2 - 2\alpha$ then

$$\|X(t_2) - X(t_1)\|_{L^2(\Omega; H)} \leq C(t_2 - t_1)^{\min(\alpha, 0.2 - 2\alpha)} , \quad 0 < t_1 < t_2 \leq T.$$

(48)

The proof of Theorem 2 is thus completed.

4 Space approximation and error estimates

We consider the discretization of the spatial domain by a finite element triangulation with maximal length $h$ satisfying the usual regularity assumptions. Let $V_h \subset V$ denotes the space of continuous functions that are piecewise linear over triangulation $J_h$. To discretise in space, we introduce $P_h$ from $L^2(\Omega)$ to $V_h$ define for $u \in L^2(\Omega)$ by

$$(P_h u, \xi) = (u, \xi), \quad \forall \xi \in V_h. \quad (49)$$

The discrete operator $A_h : V_h \to V_h$ is defined by

$$(A_h \rho, \xi) = -a(\rho, \xi), \quad \forall \rho, \xi \in V_h, \quad (50)$$

where $a$ is the corresponding bilinear form of $A$. Like the operator $A$, the discrete operator $A_h$ is also the generator of an analytic semigroup $S_h(t) := e^{tA_h}$. The semidiscrete space version of problem (11) is to find $X^h(t) = X^h(\cdot, t)$ such that for $t \in [0, T]$

$$\begin{cases}
\partial_t^\alpha X^h(t) = A_h X^h(t) + P_h F(X^h(t)) + P_h B(X^h(t)) \frac{\text{d}W(t)}{\text{d}t} + \int_0^t P_h G(z; X^h(t)) \tilde{N}(dz, dt) \\
X^h(0) = P_h X_0, \quad t \in [0, T].
\end{cases} \quad (31)$$

Note that $A_h$, $P_h F$, $P_h B$ and $P_h G$ satisfy the same assumptions as $A$, $F$, $B$ and $G$ respectively. The mild solution of (31) can be represented as follows

$$X^h(t) = S_{1h}(t)X_0^h + \int_0^t (t-s)^{\alpha - 1} S_{2h}(t-s) P_h F(X^h(s)) ds$$
$$+ \int_0^t (t-s)^{\alpha - 1} S_{2h}(t-s) P_h B(X^h(s)) dW(s)$$
$$+ \int_0^t \int_\Omega (t-s)^{\alpha - 1} S_{2h}(t-s) P_h G(z, X^h(s)) \tilde{N}(dz, ds), \quad (52)$$

where $S_{1h}$ and $S_{2h}$ are the semi discrete version of $S_1$ and $S_2$ respectively defined by (16) and (17). Let us define the error operators

$$T_h(t) := S(t) - S_h(t)P_h, \quad T_{1h}(t) := S(t) - S_{1h}(t)P_h, \quad T_{2h}(t) := S(t) - S_{2h}(t)P_h.$$  

Then we have the following lemma.
Lemma 6 (i) Let $r \in [0, 2], \rho \leq r, t \in (0, T], v \in D((-A)^\rho)$. Then there exists a positive constant $C$ such that
\[
\|T_h(t)v\| \leq Ch^\gamma t^{-(r-\rho)/2}\|v\|_\rho, \tag{53}
\]
and
\[
\|T_{1h}(t)v\| \leq Ch^{\gamma} t^{-\alpha(r-\rho)/2}\|v\|_\rho, \quad \|T_{2h}(t)v\| \leq Ch^{\gamma} t^{-\alpha(r-\rho)/2}\|v\|_\rho. \tag{54}
\]

(ii) Let $0 \leq \gamma \leq 1$, then there exists a constant $C$ such that
\[
\|\int_0^t s^{\gamma-1}T_{2h}(s)vds\| \leq C h^{2-\gamma}\|v\|_{-\gamma}, \quad v \in D((-A)^{-\gamma}), \quad t > 0. \tag{55}
\]

Proof (i) See [21, Lemma 3.1], for the proof of (53). For the proof of (54), using [10, 17], their semi discrete forms, [53, 134] and [15], we get
\[
\|T_{1h}(t)v\| = \|(S_1(t) - S_{1h}(t)P_h)v\|
\]
\[
= \left\| \int_0^\infty M_\alpha(\theta)(S(\theta t^\alpha) - S_h(\theta t^\alpha)P_h)v d\theta \right\|
\]
\[
\leq \int_0^\infty M_\alpha(\theta)\|T_h(\theta t^\alpha)v\|d\theta
\]
\[
\leq C \int_0^\infty M_\alpha(\theta)h^\gamma(\theta t^\alpha)^{-(r-\rho)/2}\|v\|_\rho d\theta
\]
\[
\leq Ch^\gamma t^{-\alpha(r-\rho)/2} \int_0^\infty M_\alpha(\theta)\theta^{-(r-\rho)/2}\|v\|_\rho d\theta
\]
\[
\leq C \frac{\Gamma(1 - \frac{r-\rho}{2})}{\Gamma(1 - \frac{\alpha(r-\rho)}{2})} h^\gamma t^{-\alpha(r-\rho)/2}\|v\|_\rho
\]
\[
\leq Ch^\gamma t^{-\alpha(r-\rho)/2}\|v\|_\rho,
\]
and
\[
\|T_{2h}(t)v\| = \|(S_2(t) - S_{2h}(t)P_h)v\|
\]
\[
= \left\| \int_0^\infty \alpha t \theta M_\alpha(\theta)(S(\theta t^\alpha) - S_h(\theta t^\alpha)P_h)v d\theta \right\|
\]
\[
\leq \int_0^\infty \alpha t \theta M_\alpha(\theta)\|T_h(\theta t^\alpha)v\|d\theta
\]
\[
\leq C \int_0^\infty \alpha t \theta M_\alpha(\theta)h^\gamma(\theta t^\alpha)^{-(r-\rho)/2}\|v\|_\rho d\theta
\]
\[
\leq C \alpha h^\gamma t^{-\alpha(r-\rho)/2} \int_0^\infty M_\alpha(\theta)\theta^{-(r-\rho)/2}\|v\|_\rho d\theta
\]
\[
\leq C \frac{\Gamma(2 - \frac{r-\rho}{2})}{\Gamma(1 + \alpha (1 - \frac{\alpha(r-\rho)}{2}))} h^\gamma t^{-\alpha(r-\rho)/2}\|v\|_\rho
\]
\[
\leq Ch^\gamma t^{-\alpha(r-\rho)/2}\|v\|_\rho.
\]
(ii) Using (17) and its semidiscrete form, (14), we obtain
\[
\int_0^t \beta_s^{-1}T_{2h}(s)vd\theta = \int_0^t \beta_s^{-1}S_2(s)vd\theta - \int_0^t \beta_s^{-1}S_{2h}(s)P_hvd\theta
\]
\[
\beta = \frac{1}{\alpha}M_0(\beta)S(\beta^\alpha)vd\theta ds - \int_0^t \beta \alpha M_0(\beta)S_h(\beta^\alpha)P_hvd\theta ds
\]
\[
\frac{1}{\alpha}A^{-1}M_0(\beta)\int_0^t A\beta\theta \beta^{-1}S(\beta^\alpha)vd\theta ds - \int_0^t A^{-1}M_0(\beta)\int_0^t A\beta\alpha\theta \beta^{-1}S_h(\beta^\alpha)P_hvd\theta ds
\]
\[
\frac{1}{\alpha}A^{-1}M_0(\beta)\int_0^t A\beta\theta \beta^{-1}S(\beta^\alpha)vd\theta ds - \int_0^t A^{-1}M_0(\beta)\int_0^t A\beta\alpha\theta \beta^{-1}S_h(\beta^\alpha)P_hvd\theta ds
\]
\[
\frac{1}{\alpha}A^{-1}M_h(\beta)(S(\beta^\alpha) - I)vd\theta ds - \int_0^t A^{-1}M_h(\beta)(S_h(\beta^\alpha) - I)P_hvd\theta ds
\]
\[
\frac{1}{\alpha}A^{-1}M_h(\beta)(A^{-1}P_h - A^{-1})vd\theta + \int_0^t M_h(\beta)(A^{-1}S(\beta^\alpha) - A^{-1}S_h(\beta^\alpha)P_h)vd\theta
\]
\[
(A^{-1}P_h - A^{-1})v + \int_0^t M_h(\beta)(A^{-1}S(\beta^\alpha) - A^{-1}S_h(\beta^\alpha)P_h)vd\theta,
\]
and \ref{21} (65), (69) allow to have
\[
\left\| \int_0^t \beta_s^{-1}T_{2h}(s)vd\theta \right\|
\]
\[
\leq \left\| (A^{-1}P_h - A^{-1})v \right\| + \int_0^t M_h(\beta)(A^{-1}S(\beta^\alpha) - A^{-1}S_h(\beta^\alpha)P_h)vd\theta
\]
\[
\leq \left\| (A^{-1}P_h - A^{-1})v \right\| + \int_0^t M_h(\beta)\left\| (A^{-1}S(\beta^\alpha) - A^{-1}S_h(\beta^\alpha)P_h)vd\theta
\]
\[
\leq Ch^{2-\gamma} \left\| v \right\|_{-\gamma} + C \int_0^t M_h(\beta)h^{2-\gamma} \left\| v \right\|_{-\gamma} d\theta
\]
\[
\leq Ch^{2-\gamma} \left\| v \right\|_{-\gamma} + Ch^{2-\gamma} \left\| v \right\|_{-\gamma} \int_0^t M_h(\beta)d\theta
\]
\[
\leq Ch^{2-\gamma} \left\| v \right\|_{-\gamma}.
\]
This completes the proof of Lemma \ref{6}.

We are now in position to prove one of our main results, which provides an estimate in mean square sense of the error between the solution of SPDE \ref{1} and the spatially semidiscrete approximation \ref{54}.

**Theorem 3 (Space error)** Let \(X\) and \(X^h\) be the mild solution of \ref{1} and \ref{57}, respectively. Suppose that Assumptions \ref{4} - \ref{7} hold. We have the following estimates depending on the regularity parameter \(\beta\) of the initial solution \(X_0\),
\[
\left\| X(t) - X^h(t) \right\|_{L^2(\Omega; H)} \leq Ch^{\beta}, \quad 0 \leq t \leq T.
\]
Proof Define $e(t) := X(t) - X^h(t)$. By $[13]$ and $[12]$, we deduce

$$
e(t) = S_1(t) X_0 - S_{1h}(t) P_h X_0$$
$$+ \int_0^t (t-s)^{\alpha-1} S_2(t-s) F(X(s)) ds - \int_0^t (t-s)^{\alpha-1} S_{2h}(t-s) P_h F(X^h(s)) ds$$
$$+ \int_0^t (t-s)^{\alpha-1} S_2(t-s) B(X(s)) dW(s) - \int_0^t (t-s)^{\alpha-1} S_{2h}(t-s) P_h B(X^h(s)) dW(s)$$
$$+ \int_0^t \int_{\mathcal{L}} (t-s)^{\alpha-1} S_2(t-s) G(z, X(s)) \tilde{N}(dz, ds)$$
$$- \int_0^t \int_{\mathcal{L}} (t-s)^{\alpha-1} S_{2h}(t-s) P_h G(z, X(s)) \tilde{N}(dz, ds)$$
$$=: I + II + III + IV.$$}

Thus taking the $L^2$ norm and using triangle inequality we have

$$\|e(t)\|_{L^2(\Omega; H)} \leq \|I\|_{L^2(\Omega; H)} + \|II\|_{L^2(\Omega; H)} + \|III\|_{L^2(\Omega; H)}$$
$$+ \|IV\|_{L^2(\Omega; H)}.$$

(58)

We will bound the above terms one by one. For the first term $\|I\|_{L^2(\Omega; H)}$, using $[53]$ with $r = \rho = \beta$ and Assumption 1 yields

$$\|I\|_{L^2(\Omega; H)} := \|S_1(t) X_0 - S_{1h}(t) P_h X_0\|_{L^2(\Omega; H)}$$
$$= \|T_{1h}(t) X_0\|_{L^2(\Omega; H)}$$
$$\leq C h^2 \|(-A)^{3/2} X_0\|_{L^2(\Omega; H)} \leq C h^2.$$

(59)

For the second term $\|II\|_{L^2(\Omega; H)}$, adding and substracting a term, using triangle inequality gives

$$\|II\|_{L^2(\Omega; H)} = \left\| \int_0^t (t-s)^{\alpha-1} S_2(t-s) F(X(s)) ds \right\|_{L^2(\Omega; H)}$$
$$- \left\| \int_0^t (t-s)^{\alpha-1} S_{2h}(t-s) P_h F(X^h(s)) ds \right\|_{L^2(\Omega; H)}$$
$$\leq \left\| \int_0^t (t-s)^{\alpha-1}(S_2(t-s) - S_{2h}(t-s)) P_h F(X(s)) ds \right\|_{L^2(\Omega; H)}$$
$$+ \left\| \int_0^t (t-s)^{\alpha-1} S_{2h}(t-s) P_h (F(X(s)) - F(X^h(s))) ds \right\|_{L^2(\Omega; H)}$$
$$=: \|II_1\|_{L^2(\Omega; H)} + \|II_2\|_{L^2(\Omega; H)}.$$

(60)
To estimate the term $\|II_1\|_{L^2(\Omega,H)}$, we also add and subtract a term. Using the triangle inequality yields

$$\|II_1\|_{L^2(\Omega,H)} := \left\| \int_0^t (t-s)^{\alpha-1} (S_2(t-s) - S_{2h}(t-s)P_h)F(X(s))ds \right\|_{L^2(\Omega,H)}$$

$$\leq \left\| \int_0^t (t-s)^{\alpha-1} (S_2(t-s) - S_{2h}(t-s)P_h)(F(X(s)) - F(X(t)))ds \right\|_{L^2(\Omega,H)}$$

$$+ \left\| \int_0^t (t-s)^{\alpha-1} (S_2(t-s) - S_{2h}(t-s)P_h)F(X(t))ds \right\|_{L^2(\Omega,H)}$$

$$=: \|II_{11}\|_{L^2(\Omega,H)} + \|II_{12}\|_{L^2(\Omega,H)}. \quad (61)$$

We estimate these two terms separately. Using Cauchy-Schwartz inequality, with $r = \beta$, $\rho = 0$, Assumption 2 and Theorem 2, leads to

$$\|II_{11}\|_{L^2(\Omega,H)} := \left\| \int_0^t (t-s)^{\alpha-1} T_{2h}(t-s)(F(X(s)) - F(X(t)))ds \right\|_{L^2(\Omega,H)}$$

$$\leq \int_0^t (t-s)^{\alpha-1} \|T_{2h}(t-s)(F(X(s)) - F(X(t)))\|_{L^2(\Omega,H)} ds$$

$$\leq C h^\beta \int_0^t (t-s)^{\alpha-1} (t-s)^{-\frac{\alpha\beta}{2}} (t-s)^{\frac{\alpha(4-\beta)-3}{2}} ds$$

$$\leq C h^\beta \int_0^t (t-s)^{\min(\alpha-1,\frac{\alpha(4-\beta)-3}{2})} ds$$

$$\leq C h^\beta \int \min(\alpha-1,\frac{\alpha(4-\beta)-3}{2}) \leq C h^\beta. \quad (62)$$

As with the term $\|II_{11}\|_{L^2(\Omega,H)}$, by applying (55) with $\gamma = 0$, Assumption 2 and Theorem 1, we get

$$\|II_{12}\|_{L^2(\Omega,H)} = \left\| \int_0^t (t-s)^{\alpha-1} T_{2h}(t-s)F(X(t))ds \right\|_{L^2(\Omega,H)}$$

$$\leq C h^2 \|F(X(t))\|_{L^2(\Omega,H)}$$

$$\leq C h^2 \left( 1 + \mathbb{E} \sup_{0 \leq t \leq T} \|X(t)\|^2 \right)^{\frac{1}{2}} \leq C h^2. \quad (63)$$

For the term $\|II_2\|_{L^2(\Omega,H)}$, using Cauchy-Schwartz inequality and the fact that $S_{2h}(t-s)$ and $P_h$ are bounded, Assumption 2, we have

$$\|II_2\|^2_{L^2(\Omega,H)} := \left\| \int_0^t (t-s)^{\alpha-1} S_{2h}(t-s)P_h(F(X(s)) - F(X^h(s)))ds \right\|_{L^2(\Omega,H)}^2$$

$$\leq C \int_0^t (t-s)^{2\alpha-2} \|S_{2h}(t-s)P_h\|^2_{L^2(\Omega,H)} \|F(X(s)) - F(X^h(s))\|^2_{L^2(\Omega,H)} ds$$

$$\leq C \int_0^t (t-s)^{2\alpha-2} \|e(s)\|^2_{L^2(\Omega,H)} ds. \quad (64)$$
Substituting (62), (63) in (61), hence putting (61) and (64) in (60) gives

\[
\|II\|_{L^2(\Omega; H)}^2 \leq C h^{2\beta} + C h^4 + C \int_0^t (t-s)^{2\alpha - 2} \|e(s)\|_{L^2(\Omega; H)}^2 \, ds \\
\leq C h^{2\beta} + C \int_0^t (t-s)^{2\alpha - 2} \|e(s)\|_{L^2(\Omega; H)}^2 \, ds. \tag{65}
\]

For the fourth term \(\|IV\|_{L^2(\Omega; H)}\), by adding and subtracting a term, the use of the triangle inequality yields

\[
\|IV\|_{L^2(\Omega; H)} = \left\| \int_0^t \int_{\mathcal{X}} (t-s)^{\alpha - 1} S_2(t-s)G(z, X(s))\tilde{N}(dz, ds) \right. \\
- \int_0^t \int_{\mathcal{X}} (t-s)^{\alpha - 1} S_{2h}(t-s)P_h G(z, X(s))\tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)} \\
\leq \left\| \int_0^t \int_{\mathcal{X}} (t-s)^{\alpha - 1} (S_2(t-s) - S_{2h}(t-s)P_h) G(z, X(s))\tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)} \\
+ \left\| \int_0^t \int_{\mathcal{X}} (t-s)^{\alpha - 1} S_{2h}(t-s)P_h (G(z, X(s)) - G(z, X^h(s)))\tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)} \\
=: \|IV_1\|_{L^2(\Omega; H)} + \|IV_2\|_{L^2(\Omega; H)}.
\]

In a similar way as for \(\|II\|_{L^2(\Omega; H)}^2\), using Itô isometry [3], the boundedness of the operators \(S_{2h}(t-s)\) and \(P_h\), Assumption [3] we deduce

\[
\|IV_2\|_{L^2(\Omega; H)}^2 = \left\| \int_0^t \int_{\mathcal{X}} (t-s)^{\alpha - 1} S_{2h}(t-s)P_h (G(z, X(s)) - G(z, X^h(s)))\tilde{N}(dz, ds) \right\|_{L^2(\Omega; H)}^2 \\
= \mathbb{E} \left[ \int_0^t \int_{\mathcal{X}} (t-s)^{2\alpha - 2} \|S_{2h}(t-s)P_h\|_{L^2(H)}^2 \|G(z, X(s)) - G(z, X^h(s))\|^2 \, v(dz) ds \right] \\
\leq C \mathbb{E} \left[ \int_0^t (t-s)^{2\alpha - 2} \left( \int_{\mathcal{X}} \|G(z, X(s)) - G(z, X^h(s))\|^2 \, v(dz) \right) \, ds \right] \\
\leq C \mathbb{E} \left[ \int_0^t (t-s)^{2\alpha - 2} \|X(s) - X^h(s)\|^2 \, ds \right] \\
\leq C \int_0^t (t-s)^{2\alpha - 2} \|e(s)\|_{L^2(\Omega; H)}^2 \, ds. \tag{66}
\]
For the estimate \( \|IV_1\|_{L^2(Ω, H)}^2 \), using Itô isometry \( [38] \), we rewrite this numerical approximation.

Combining (59), (65), (68), (69) and applying the fractional Gronwall’s lemma (see \( [13, 32] \)) completes the proof of Theorem 3.

Combining (66) and (67) it results that

\[
\|IV_1\|_{L^2(Ω, H)}^2 := \left\| \int_0^t \int_X (t-s)^{α-1}T_{2h}(t-s)G(z, X(s))\tilde{N}(dz, ds) \right\|_{L^2(Ω, H)}^2 \\
= \mathbb{E} \left[ \int_0^t \int_X (t-s)^{2α-2} \|T_{2h}(t-s)G(z, X(s))\|^2 v(dz)ds \right] \\
\leq Ch^{2β}\mathbb{E} \left[ \int_0^t (t-s)^{2α-2} \left( \int_X \|(-A)^{β}G(z, X(s))\|^2 v(dz) \right) ds \right] \\
\leq Ch^{2β}\int_0^t (t-s)^{2α-2} \left( 1 + \|(-A)^{β}X(t-s)\|_{L^2(Ω, H)}^2 \right) ds \\
\leq Ch^{2β}t^{2α-1} \left( 1 + \|(-A)^{β}X_0\|_{L^2(Ω, H)}^2 \right) \leq Ch^{2β}.
\]

Combining (66) and (67) it results that

\[
\|IV\|_{L^2(Ω, H)}^2 \leq Ch^{2β} + C \int_0^t (t-s)^{2α-2} \|v(s)\|_{L^2(Ω, H)}^2 ds.
\]

Using the similar procedure as in \( \|IV_1\|_{L^2(Ω, H)}^2 \), we have the following estimate

\[
\|III\|_{L^2(Ω, H)}^2 \leq Ch^{2β} + C \int_0^t (t-s)^{2α-2} \|v(s)\|_{L^2(Ω, H)}^2 ds.
\]

Combining \( [59], [65], [68], [69] \) and applying the fractional Gronwall’s lemma (see \( [13, 32] \)) completes the proof of Theorem 3.

5 Fully discrete Euler scheme and its error estimates

In this section, we consider a fully discrete approximation of SPDE \( [51] \). Before defining our numerical approximation of the mild solution of the semidiscrete problem \( [51] \), we present a useful to well rewrite this numerical approximation.

We recall that the mild solution at \( t_m = mΔt, Δt > 0 \) of the semi discrete problem \( [51] \) is given by

\[
X^h(t_m) = S_{1h}(t_m)X_0^h + \int_0^{t_m} (t_m-s)^{α-1}S_{2h}(t_m-s)P_hF(X^h(s))ds \\
+ \int_0^{t_m} (t_m-s)^{α-1}S_{2h}(t_m-s)P_hB(X^h(s))dW(s) \\
+ \int_0^{t_m} \int_X (t_m-s)^{α-1}S_{2h}(t_m-s)P_hG(z, X^h(s))N(dz, ds).
\]

By decomposing the integrals of the right-hand side of the previous equality using the Chasles relation, we obtain
We can define our approximation the following theorem. The strong convergence result of the fully discrete scheme are formulated in (70).

To build our numerical scheme we use the following approximations for all $s \in [t_j, t_{j+1})$ with $j \in \{0, 1, \ldots, m-1\}$

$$(t_m-s)^{\alpha-1} S_{2h}(t_{m+1}-s) P_h F(X^h(s)) \approx (t_m-t_j)^{\alpha-1} S_{2h}(t_m-t_j) P_h F(X^h(t_j)),$$

$$(t_{m+1}-s)^{\alpha-1} S_{2h}(t_{m+1}-s) P_h B(X^h(s)) \approx (t_{m+1}-t_j)^{\alpha-1} S_{2h}(t_{m+1}-t_j) P_h B(X^h(t_j)),$$

$$(t_{m+1}-s)^{\alpha-1} S_{2h}(t_{m+1}-s) P_h G(z, X^h(s)) \approx (t_{m+1}-t_j)^{\alpha-1} S_{2h}(t_{m+1}-t_j) P_h G(z, X^h(t_j)).$$

We can define our approximation $X^h_m$ of $X^h(m \Delta t)$ by

$$X^h_m = S_{1h}(t_m) X^h_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-s)^{\alpha-1} S_{2h}(t_m-s) P_h F(X^h(s)) ds$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-t_j)^{\alpha-1} S_{2h}(t_m-t_j) P_h B(X^h(t_j)) dW(s)$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_X (t_m-t_j)^{\alpha-1} S_{2h}(t_m-t_j) P_h G(z, X^h(t_j)) \tilde{N}(dz, ds). \quad (71)$$

Hence using (16) and (17), it holds that

$$X^h_m = E_{\alpha,1}(A^\alpha h t_m^\alpha) X^h_m + \Delta t \sum_{j=0}^{m-1} (t_m-t_j)^{\alpha-1} E_{\alpha,\alpha}(A_h(t_m-t_j)^\alpha) S_{2h} P_h F(X^h_j)$$

$$+ \sum_{j=0}^{m-1} (t_m-t_j)^{\alpha-1} E_{\alpha,\alpha}(A_h(t_m-t_j)^\alpha) P_h B(X^h_j)(W_{t_{j+1}} - W_{t_j})$$

$$+ \sum_{j=0}^{m-1} (t_m-t_j)^{\alpha-1} E_{\alpha,\alpha}(A_h(t_m-t_j)^\alpha) \int_{t_j}^{t_{j+1}} \int_X P_h G(z, X^h_j) \tilde{N}(dz, ds). \quad (72)$$

The strong convergence result of the fully discrete scheme are formulated in the following theorem.
Theorem 4 (Main result). Let Assumptions 1 - 4 are fulfilled. Let $X^m_h$ be the numerical approximation defined in (71). We have the following estimates depending on the regularity parameter $\beta$ of the initial solution $X_0$ and the power of the time fractional derivative $\alpha$

$$\|X(t_m) - X^m_h\|_{L^2(\Omega; H)} \leq C \left( h^\beta + \Delta t^{\min(\alpha, \beta, 2-2\alpha)} \right).$$ (73)

Proof Using triangle inequality yields

$$\|X(t_m) - X^m_h\|_{L^2(\Omega; H)} \leq \|X(t_m) - X^m(t_m)\|_{L^2(\Omega; H)} + \|X^m(t_m) - X^m_h\|_{L^2(\Omega; H)}.$$ (74)

The space error is estimated in Theorem 3. It remains to estimate the time error. Recall the mild solution given by (70)

$$X^h(t_m) = S_{1h}(t_m)X^h_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - s)^{\alpha-1} S_{2h}(t_m - s)P_h F(X^h(s))ds$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - s)^{\alpha-1} S_{2h}(t_m - s)P_h B(X^h(s))dW(s)$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{X} (t_m - s)^{\alpha-1} S_{2h}(t_m - s)P_h G(z, X^h(s))\tilde{N}(dz, ds),$$

and the numerical solution $X^m_h$ given by (71)

$$X^m_h = S_{1h}(t_m)X^m_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j)P_h F(X^h_j)ds$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j)P_h B(X^h_j)dW(s)$$

$$+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{X} (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j)P_h G(z, X^h_j)\tilde{N}(dz, ds).$$
Subtracting these two previous equalities yields

\[
X^h(t_m) - X^h_m = \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( (t_m - s)^{\alpha - 1} S_{2h}(t_m - s) - (t_m - t_j)^{\alpha - 1} S_{2h}(t_m - t_j) \right) P_h F(X^h(s)) \, ds
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( (t_m - s)^{\alpha - 1} S_{2h}(t_m - s) P_h B(X^h(s)) \right) \, dW(s)
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( (t_m - s)^{\alpha - 1} S_{2h}(t_m - s) - (t_m - t_j)^{\alpha - 1} S_{2h}(t_m - t_j) \right) \, dW(s)
\]

\[
= K_1 + K_2 + K_3.
\]

Using the triangle inequality

\[
\|X^h(t_m) - X^h_m\|_{L^2(\Omega; H)} \leq \|K_1\|_{L^2(\Omega; H)} + \|K_2\|_{L^2(\Omega; H)} + \|K_3\|_{L^2(\Omega; H)}.
\]

By adding and subtracting a term, we recast \(K_1\) as follows

\[
K_1 = \sum_{j=0}^{m-1} \int_{t_k}^{t_{k+1}} \left( (t_m - s)^{\alpha - 1} S_{2h}(t_m - s) - (t_m - t_j)^{\alpha - 1} S_{2h}(t_m - t_j) \right) P_h F(X^h(s)) \, ds
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - t_j)^{\alpha - 1} S_{2h}(t_m - t_j) P_h \left[ F(X^h(s)) - F(X^h(t_k)) \right] \, ds
\]

\[
+ \sum_{j=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - t_j)^{\alpha - 1} S_{2h}(t_m - t_j) P_h \left[ F(X^h(t_k)) - F(X^h_m) \right] \, ds
\]

\[
= K_{11} + K_{12} + K_{13}.
\]

Using triangle inequality, the discrete version of (30), Assumption 2 with, boundedness of \(P_h\) and the discrete version of (22), Cauchy-Schwartz inequality and the variable change \(k = m - j\) leads
\[ \|K_{11}\|_{L^2(\Omega;H)} \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|(t_m - s)^{\alpha-1} S_{2h}(t_m - s) - (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j)\|_{L^2(H)} ds \]

\[ \leq C \left( \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (s - t_j)^{1-\alpha} (t_m - t_j)^{\alpha-1} (t_m - s)^{\alpha-1} ds \right) \left( 1 + E \left[ \sup_{0 \leq s \leq T} \|X^h(s)\|^2 \right] \right)^{\frac{1}{2}} \]

\[ \leq C \Delta t^{1-\alpha} \left( \sum_{j=0}^{m-1} (t_m - t_j)^{\alpha-1} \int_{t_j}^{t_{j+1}} (t_m - s)^{\alpha-1} ds \right) \]

\[ \leq C \Delta t^{1-\alpha} \left( \sum_{j=0}^{m-1} (t_m - t_j)^{\alpha-1} \right)^{\frac{1}{2}} \left( \sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} (t_m - s)^{2\alpha-2} ds \right)^2 \right)^{\frac{1}{2}} \]

\[ \leq C \Delta t^{1-\alpha} \left( \sum_{k=1}^{m} t_k^{-1+2\alpha-1} \int_{t_{k-1}}^{t_k} (t_m - s)^{2\alpha-2} ds \right)^{\frac{1}{2}} \]

\[ \leq C \Delta t^{1-\alpha} \left( \sum_{k=1}^{m} t_k^{-1+2\alpha-1} \Delta t \right)^{\frac{1}{2}} t_m^{-\frac{\alpha}{2}}. \quad (77) \]

Let us recall the following estimate, for \( \epsilon > 0 \) small enough

\[ \sum_{k=1}^{m} t_k^{-1+\epsilon} \Delta t \leq C. \quad (78) \]

Inserting (78) in (77) with \( \epsilon = 2\alpha - 1 \) yields

\[ \|K_{11}\|_{L^2(\Omega;H)} \leq C \Delta t^{1-\alpha}. \quad (79) \]

For the second estimate \( \|K_{12}\|_{L^2(\Omega;H)} \), applying triangle inequality, the boundedness of \( S_{2h}(t) \) and \( P_h \), Assumption 2, (38), the variable change \( k = m - j \) and (78) with \( \epsilon = \alpha \) yields

\[ \|K_{12}\|_{L^2(\Omega;H)} \leq C \Delta t^{1-\alpha}. \]
\[ \|K_{12}\|_{L^2(\Omega, H)} \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \| (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h (F(X^h(t_j)) - F(X^h(t_j))) \|_{L^2(\Omega, H)} \, ds \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} \| S_{2h}(t_m - t_j) P_h \|_{L(H)} \| F(X^h(t_j)) - F(X^h(t_j)) \|_{L^2(\Omega, H)} \, ds \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} (s - t_j)^{\frac{\min(\alpha, \beta - 2\alpha)}{2}} \, ds \]
\[ \leq C \Delta t \frac{\min(\alpha, \beta - 2\alpha)}{2} \left( \sum_{j=0}^{m-1} (t_m - t_j)^{-1 + \alpha} \Delta t \right) \leq C \Delta t \frac{\min(\alpha, \beta - 2\alpha)}{2}, \quad (80) \]

and using also triangle inequality boundedness of \( S_{2h}(t) \) and \( P_h \), Assumption 2 we estimate \( \|K_{13}\|_{L^2(\Omega, H)} \) as follows

\[ \|K_{13}\|_{L^2(\Omega, H)} \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \| (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h (F(X^h(t_j)) - F(X^h(t_j))) \|_{L^2(\Omega, H)} \, ds \]
\[ \leq \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} \| S_{2h}(t_m - t_j) P_h \|_{L(H)} \| F(X^h(t_j)) - F(X^h(t_j)) \|_{L^2(\Omega, H)} \, ds \]
\[ \leq C \Delta t \sum_{j=0}^{m-1} (t_m - t_j)^{\alpha-1} \| X^h(t_j) - X^h_j \|_{L^2(\Omega, H)} \]
\[ \leq C \Delta t^\alpha \sum_{j=0}^{m-1} \| X^h(t_j) - X^h_j \|_{L^2(\Omega, H)} \quad (81) \]

Adding (79) - (81), it holds that

\[ \|K_1\|_{L^2(\Omega, H)} \leq C \Delta t \frac{\min(\alpha, \beta - 2\alpha)}{2} + C \Delta t^\alpha \sum_{k=0}^{m-1} \| X^h(t_k) - X^h_k \|_{L^2(\Omega, H)}. \quad (82) \]

We will not give details of the estimate of \( K_2 \) as it is similar to that of \( K_3 \). Let us now estimate the norm of \( K_3 \). By adding and subtracting the same term,
we rewrite it in three terms as follows

\[
K_3 = \sum_{j=0}^{m-1} \int_0^{t_{j+1}} \int_X \left[ (t_m - s)^{\alpha-1} S_{2h}(t_m - s) P_h G(z, X^h(s)) - (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) \right] P_h G(z, X^h(s)) \tilde{N}(dz, ds)
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_X (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h \left[ G(z, X^h(s)) - G(z, X^h(t_j)) \right] \tilde{N}(dz, ds)
+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_X (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h \left[ G(z, X^h(t_j)) - G(z, X^h_t) \right] \tilde{N}(dz, ds)
=: K_{31} + K_{32} + K_{33}.
\]

Applying again the Itô isometry property (8), the fact that the variation of the compensated Poisson measure are independent, the discrete version of (30) with \( t_2 = t_m - t_1 \) and \( t_1 = t_m - s \), Assumption (4) with \( \tau = 0 \), boundedness of \( P_h \), the discrete version of (22) and Cauchy-Schwartz inequality leads

\[
\begin{align*}
\|K_{33}\|_{L^2(\Omega, H)}^2 & = \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_X \left[ (t_m - s)^{\alpha-1} S_{2h}(t_m - s) - (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) \right] \|P_h G(z, X^h(s))\|_H^2 v(dz) ds \\
& = \int_{t_j}^{t_{j+1}} \left( s - t_j \right)^{2-2\alpha} \left( t_m - t_j \right)^{2\alpha-2} \left( t_m - s \right)^{2\alpha-2} \mathbb{E} \left[ \int_X \|P_h G(z, X^h(s))\|_H^2 v(dz) \right] ds \\
& \leq C \int_{t_j}^{t_{j+1}} \left( s - t_j \right)^{2-2\alpha} \left( t_m - t_j \right)^{2\alpha-2} \left( t_m - s \right)^{2\alpha-2} \left( 1 + \mathbb{E} \left[ \|X^h(s)\|_H^2 \right] \right) ds \\
& \leq C \Delta t^{2-2\alpha} \int_{t_j}^{t_{j+1}} \left( t_m - s \right)^{2\alpha-2} ds \left( 1 + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \|X^h(s)\|_H^2 \right] \right) \\
& \leq C \Delta t^{2-2\alpha} \left( \sum_{j=0}^{m-1} \left( t_m - t_j \right)^{4\alpha-4} \right) \left( \sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} \left( t_m - s \right)^{2\alpha-2} ds \right)^2 \right)^{\frac{1}{2}},
\end{align*}
\]
using additionally the variable change \( k = m - j \) and \( \epsilon = 4\alpha - 3 \) yields

\[
\|K_{31}\|_{L^2(\Omega;H)}^2 \leq C\Delta t^{2-2\alpha} \left( \sum_{k=1}^{m} \left( \sum_{j=0}^{m-1} (t_m - s_j)^{\alpha-\epsilon} \right) \Delta t \right)^\frac{1}{2} \left( \sum_{k=1}^{m} \int_{t_j}^{t_{j+1}} (t_m - s)^{\alpha-\epsilon} ds \right)^{\frac{1}{2}} \\
\leq C\Delta t^{2-2\alpha} \left( \sum_{k=1}^{m} (t_m - s_k)^{\alpha-\epsilon} \Delta t \right)^\frac{1}{2} \left( \frac{1}{2} \left( \sum_{k=1}^{m} (t_m - s_k)^{\alpha-\epsilon} \Delta t \right)^{\frac{1}{2}} \right)^\frac{1}{2} \\
\leq C\Delta t^{2-2\alpha} \left( \sum_{k=1}^{m} (t_m - s_k)^{\alpha-\epsilon} \Delta t \right) \leq C\Delta t^{2-2\alpha}. \tag{85}
\]

To estimate the second term \( \|K_{32}\|_{L^2(\Omega;H)}^2 \), applying Itô isometry \( \mathbb{E} \), using boundedness of \( S_{2h}(t) \), \( P_h \) and Assumption 3, Theorem 2, the variable change \( k = m - j \) and \( \epsilon = 2\alpha - 1 \) yields

\[
\|K_{32}\|_{L^2(\Omega;H)}^2 = \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{\mathcal{X}} (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h \left[ G(z, X^h(s)) - G(z, X^h(t_j)) \right] \right\|_{L^2(\Omega;H)}^2 \\
\leq \sum_{j=0}^{m-1} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \int_{\mathcal{X}} \left( (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h \left[ G(z, X^h(s)) - G(z, X^h(t_j)) \right] \right)^2 v(dz)ds \right] \\
\leq \sum_{j=0}^{m-1} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \left( t_m - t_j \right)^{2\alpha-2} \left\| S_{2h}(t_m - t_j) P_h \left[ G(z, X^h(s)) - G(z, X^h(t_j)) \right] \right\|_{L^2(\Omega;H)}^2 v(dz)ds \right] \\
\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( t_m - t_j \right)^{2\alpha-2} \left\| X^h(s) - X^h(t_j) \right\|_{L^2(\Omega;H)}^2 ds \\
\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left( t_m - t_j \right)^{2\alpha-2} (s - t_j)^{\min(\alpha \beta, 2 - 2\alpha)} ds \\
\leq C \Delta t^{1+\min(\alpha \beta, 2 - 2\alpha)} \left( \sum_{j=0}^{m-1} (t_m - t_j)^{2\alpha-2} \right) \\
\leq C \Delta t^{\min(\alpha \beta, 2 - 2\alpha)} \left( \sum_{k=1}^{m} t_k^{\alpha-2} \Delta t \right) \leq C \Delta t^{\min(\alpha \beta, 2 - 2\alpha)}. \tag{86}
\]

To estimate the third term $\|K_3\|_{L^2(\Omega;H)}^2$, applying Itô isometry, using boundedness of $S_{2h}(t)$, $P_h$ and Assumption yields

\[
\|K_3\|_{L^2(\Omega;H)}^2 = \left\| \sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h [G(z,X^h(t_j)) - G(z,X^h)] \tilde{N}(dz,ds) \right) \right\|_{L^2(\Omega;H)}^2
\]

\[
= \sum_{j=0}^{m-1} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \left\| (t_m - t_j)^{\alpha-1} S_{2h}(t_m - t_j) P_h [G(z,X^h(t_j)) - G(z,X^h)] \right\|^2 v(dz)ds \right]
\]

\[
\leq C \sum_{j=0}^{m-1} (t_m - t_j)^{2\alpha-2} \left\| X^h(t_j) - X^h \right\|_{L^2(\Omega;H)}^2 ds
\]

\[
\leq C \Delta t \sum_{j=0}^{m-1} (t_m - t_j)^{2\alpha-2} \left\| X^h(t_j) - X^h \right\|_{L^2(\Omega;H)}^2
\]

Substituting (85) - (87) in (83) leads

\[
\|K_3\|_{L^2(\Omega;H)}^2 \leq C \Delta t^{\min(\alpha,\beta,2-2\alpha)} + C \Delta t^{2\alpha-1} \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h \right\|_{L^2(\Omega;H)}^2(88)
\]

Using the similar procedure as for $K_3$, we obtain the following estimate of the norm of $K_2$

\[
\|K_2\|_{L^2(\Omega;H)}^2 \leq C \Delta t^{\min(\alpha,\beta,2-2\alpha)} + C \Delta t^{2\alpha-1} \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h \right\|_{L^2(\Omega;H)}^2(89)
\]

Substituting (82), (85) and (89) in (75) yields

\[
\left\| X(t_m) - X_m^h \right\|_{L^2(\Omega;H)}^2 \leq C \Delta t^{\min(\alpha,\beta,2-2\alpha)} + C \Delta t^{2\alpha-1} \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h \right\|_{L^2(\Omega;H)}^2.
\]

Applying discrete Gronwall’s lemma to (90) and taking the square root leads

\[
\left\| X(t_m) - X_m^h \right\|_{L^2(\Omega;H)} \leq C \Delta t^{\min(\alpha,\beta,2-2\alpha)/2}.
\]

Combining (91) and Theorem completes the proof of Theorem 4.
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