Abstract

Quantum-mechanical wave equation for a particle with spin 1 is investigated in presence of external magnetic field in spaces with non-Euclidean geometry with constant positive curvature. Separation of the variable is performed; differential equations in the variable $r$ are solved in hypergeometric functions. The study of $z$-dependence of the wave function has been reduced to a system of three linked ordinary differential 2-nd order equations; till now the system in $z$ variable is not solved.

1. Introduction, setting the problem

In the present paper, we consider a quantum-mechanical problem a particle with spin 1 described by the Duffin–Kemmer in 3-dimensional Riemann space model in presence of the external magnetic field – relevant publications see in [1–30].

Initial matrix wave equation of Duffin–Kemmer for a spin 1 particle has the for (we adhere notation [31])

$$\left\{ \beta^\alpha \left[ i\hbar \left( e_\alpha^\beta \partial_\beta + \frac{1}{2} f^{ab\gamma}_{\alpha} \gamma_{abc} \right) - \frac{e}{c} A_\alpha \right] - mc \right\} \Psi = 0 , \quad (1.1)$$
where $\gamma_{abc}$ stand for Ricci rotation coefficients

$$
\gamma_{bac} = -\gamma_{abc} = -e^{\beta}_{(b)} e^{\alpha}_{(a)} e^{\alpha}_{(c)}.
$$

$A_a = e^\beta_{(a)} A_\beta$ are tetrad components of an electromagnetic 4-vector $A_\beta$; $J_{ab} = (\beta^a \beta^b - \beta^b \beta^a)$ stand for generators of 10-dimensional representation of the Lorentz group. Below we will use shortened notation $e/c \Rightarrow e$, $mc/h \Rightarrow M$.

In Olevsky paper [32] under the number XI the following coordinates are were specified

$$
dS^2 = c^2 dt^2 - \rho^2 \left[ \cos^2 z (dr^2 + \sin^2 r d\phi^2) + dz^2 \right],
$$

$z \in [-\pi/2, +\pi/2]$, $r \in [0, +\pi]$, $\phi \in [0, 2\pi]$.  

(1.2)

Generalization of the concept of an uniform magnetic field for the curved model $S_3$ is given by the following potential

$$
A_\phi = -2B \sin^2 \frac{r}{2} = B (\cos r - 1).  
$$

(1.3)

To this potential there correspond a single non-vanishing component of the electromagnetic tensor $F_{\phi r} = \partial_\phi A_r - \partial_r A_\phi = B \sin r$; this tensor satisfies Maxwell equations in $S_3$.

Let us consider eq. (1.3) in the space $S_3$. To cylindric coordinates $x^\alpha = (t, r, \phi, z)$ there corresponds the tetrad

$$
e^\beta_{(a)}(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos^{-1} z & 0 & 0 \\
0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(1.4)

Relevant Christoffel symbols and Ricci rotation coefficients are

$$
\Gamma^r_{jk} = \begin{pmatrix}
0 & 0 & -\tan z \\
0 & -\sin r \cos r & 0 \\
-\tan z & 0 & 0
\end{pmatrix},
$$

$$
\Gamma^\phi_{jk} = \begin{pmatrix}
0 & \cot r & 0 \\
\cot r & 0 & -\tan z \\
0 & -\tan z & 0
\end{pmatrix},
$$

$$
\Gamma^z_{jk} = \begin{pmatrix}
\sin z \cos z & 0 & 0 \\
0 & \sin z \cos z \sin^2 r & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

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\(\gamma_{122} = \frac{1}{\cos z \tan r}, \quad \gamma_{311} = -\tan z, \quad \gamma_{322} = -\tan z.\) \hspace{1cm} (1.5)

So, general covariant Duffin–Kemmer equation (1.1) takes the form

\[
\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left( i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial \phi}{\sin r} - eB(\cos r - 1) + iJ^{12} \cos r \right) \right.
\]

\[
+ i\beta^3 \frac{\partial}{\partial z} + \frac{\sin z}{\cos z} \left( \beta^1 J^{13} + \beta^2 J^{23} \right) - M \right\} \Psi = 0,
\]

\hspace{1cm} (1.6)

In the limit of flat Minkowaki space, eq. (1.6) becomes simpler

\[
\left\{ i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial \phi + eBr^2/2 + iJ^{12}}{r} + i\beta^3 \frac{\partial}{\partial z} - M \right\} \Psi = 0.
\]

\hspace{1cm} (1.7)

To separate the variable we will need an explicit representation for Duffin–Kemmer matrices \(\beta^a;\) most convenient for us is the cyclic representation; in particular, then \(J^{12}\) is diagonal (we will use blocks structure in accordance with the structure \(1 - 3 - 3 - 3\)):

\[
\beta^0 = \begin{vmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix}, \quad \beta^i = \begin{vmatrix}
0 & 0 & e_i & 0 \\
0 & 0 & 0 & \tau_i \\
-e_i & 0 & 0 & 0 \\
0 & -\tau_i & 0 & 0
\end{vmatrix}, \hspace{1cm} (1.8)
\]

where \(e_i, e_i^t, \tau_i\) designate

\[
e_1 = \frac{1}{\sqrt{2}} (-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}} (1, 0, 1), \quad e_3 = (0, i, 0),
\]

\[
\tau_1 = \frac{1}{\sqrt{2}} \begin{vmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{vmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{vmatrix}
0 & -i & 0 \\
i & 0 & -i \\
0 & i & 0
\end{vmatrix}, \quad \tau_3 = \begin{vmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{vmatrix} = s_3.
\]

\hspace{1cm} (1.9)

Entering eq. (1.6), the matrix \(J^{12}\) is

\[
J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 =
\]

3
\[
\begin{vmatrix}
-e_1 e_2^+ + e_2 e_1^+ & 0 & 0 & 0 \\
0 & -\tau_1 \tau_2 + \tau_2 \tau_1 & 0 & 0 \\
0 & 0 & -e_1^+ e_2 + e_2^+ e_1 & 0 \\
0 & 0 & 0 & -\tau_1 \tau_2 + \tau_2 \tau_1
\end{vmatrix} = \\
-\tau_3 \begin{vmatrix}
0 & 0 & 0 \\
0 & \tau_3 & 0 \\
0 & 0 & \tau_3
\end{vmatrix} = -\tau S_3. \tag{1.8}
\]

2. Separation of the variables

Let us rewrite eq. (1.6) in the form

\[
\left[ i \beta^0 \cos z \frac{\partial}{\partial t} + i \beta^1 \frac{\partial}{\partial r} + \beta^2 i \partial_\phi - e B (\cos r - 1) + i J^{12} \cos r \sin r \\
+ i \beta^3 \cos z \frac{\partial}{\partial z} + i \sin z (\beta^1 J^{13} + \beta^2 J^{23}) - \cos z M \right] \Psi = 0.
\tag{2.1}
\]

To separate the variables, we will use the following substitution for the wave function

\[
\Psi = e^{-i e t} e^{i m \phi} \begin{vmatrix}
\Phi_0 (r, z) \\
\Phi (r, z) \\
\vec{E} (r, z) \\
\vec{H} (r, z)
\end{vmatrix} . \tag{2.2}
\]

Eq. (2.1) leads us to (let \( m + B (1 - \cos r) = \nu (r) \))

\[
\left\{ i \beta^0 \cos z \beta^0 + i \beta^1 \frac{\partial}{\partial r} - \beta^2 \frac{\nu (r) - \cos r S_3}{\sin r} \\
+ i \beta^3 \cos z \frac{\partial}{\partial z} + i (\beta^1 J^{13} + \beta^2 J^{23}) \sin z - \cos z M \right\} \begin{vmatrix}
\Phi_0 (r, z) \\
\Phi (r, z) \\
\vec{E} (r, z) \\
\vec{H} (r, z)
\end{vmatrix} = 0 , \tag{2.3}
\]

With the help of auxiliary relations

\[
J^{13} = \beta^1 \beta^3 - \beta^3 \beta^1 =
\]
\[ \begin{bmatrix} -e_1 e_3 + e_3 e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 \tau_3 + \tau_3 \tau_1 & 0 & 0 \\ 0 & 0 & -e_1^+ \cdot e_3 + e_3^+ \cdot e_1 & 0 \\ 0 & 0 & 0 & -\tau_1 \tau_3 + \tau_3 \tau_1 \end{bmatrix} =
\]

\[ = i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{bmatrix} = i S_2 , \]

\[ J^{23} = \beta^2 \beta^3 - \beta^3 \beta^2 =
\]

\[ \begin{bmatrix} -e_2 e_3 + e_3 e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 \tau_3 + \tau_3 \tau_2 & 0 & 0 \\ 0 & 0 & -e_2^+ \cdot e_3 + e_3^+ \cdot e_2 & 0 \\ 0 & 0 & 0 & -\tau_2 \tau_3 + \tau_3 \tau_2 \end{bmatrix} =
\]

\[ = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{bmatrix} = -i S_1 , \]

we get

\[ (\beta^1 J^{13} + \beta^2 J^{23}) =
\]

\[ = i \begin{bmatrix} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{bmatrix} =
\]

\[ = -i \begin{bmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{bmatrix} =
\]

\[ = i \begin{bmatrix} 0 & 0 & e_1 \tau_2 - e_2 \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \tau_2 - \tau_2 \tau_1 \\ 0 & 0 & 0 & 0 \\ 0 & -\tau_1 \tau_2 + \tau_2 \tau_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -2 e_3 \\ 0 & 0 & 0 & -\tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =
\]

\[ \frac{\partial}{\partial r} \]

eq. (2.3) can be presented as

\[
\begin{bmatrix} \epsilon \cos z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + i
\begin{bmatrix} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{bmatrix}
\]
\[
\begin{pmatrix}
-\frac{1}{\sin r} & 0 & 0 & e_2 & 0 \\
0 & 0 & 0 & \tau_2 & 0 \\
-\tau_2 & 0 & 0 & 0 & 0
\end{pmatrix}
(\nu - \cos r \ S_3)
\]
\[+
\begin{pmatrix}
i \cos z & 0 & 0 & e_3 & 0 \\
0 & 0 & 0 & \tau_3 & \frac{\partial}{\partial z} \\
-e_3^+ & 0 & 0 & 0 & 0
\end{pmatrix}
\]
\[+
i \sin z
\begin{pmatrix}
0 & 0 & -2e_3 & 0 \\
0 & 0 & 0 & -\tau_3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Phi_0 \\
\Phi \\
E \\
H
\end{pmatrix}
= 0.
\]

(2.4)

In block form it is written
\[
ie_1 \partial_r \vec{E} - \frac{1}{\sin r} e_2 (\nu - \cos r \ s_3) \vec{E} + i (\cos z \ \partial_z - 2 \sin z) e_3 \vec{E} = M \cos z \ \Phi_0,
\]
\[
ie \cos z \ \vec{E} + i \gamma \partial_r \vec{H} - \frac{\tau_2}{\sin r} (\nu - \cos r \ s_3) \vec{H} + i (\cos z \ \partial_z - \sin z) \tau_3 \vec{H} = M \cos z \ \vec{\Phi},
\]
\[
-\gamma \cos z \ \vec{\Phi} - ie_1^+ \partial_r \Phi_0 + \frac{\nu}{\sin r} e_3^+ \Phi_0 - i \cos z \ e_3^+ \partial_z \Phi_0 = M \cos z \ \vec{E},
\]
\[
-\gamma \tau_1 \partial_r \vec{\Phi} + \frac{(\nu - \cos r \ s_3)}{\sin r} \tau_2 \vec{\Phi} - i (\cos z \ \partial_z - i \sin z) \tau_3 \vec{\Phi} = M \cos z \ \vec{H}.
\]

(2.5)

After simple calculation, we arrive at a system of 10 equations (let \(\gamma = 1/\sqrt{2}\))
\[
\gamma \left( \frac{\partial E_1}{\partial r} - \frac{\partial E_3}{\partial r} \right) - \frac{\gamma}{\sin r} [(\nu - \cos r) E_1 + (\nu + \cos r) E_3] -
\]
\[
-(\cos z \ \partial_z - 2 \sin z) E_2 = M \cos z \ \Phi_0,
\]
\[
ie \cos z \ E_1 + i \gamma \frac{\partial H_2}{\partial r} + i \gamma \frac{\nu}{\sin r} H_2 + i (\cos z \ \partial_z - \sin z) H_1 = M \cos z \ \Phi_1,
\]
\[
ie \cos z \ E_2 + i \gamma \left( \frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r} \right) - i \gamma [(\nu - \cos r) H_1 -
\]
\[
-(\nu + \cos r) H_3] = M \cos z \ \Phi_2,
\]
\[ +i\epsilon\cos zE_3 + i\gamma \frac{\partial H_2}{\partial r} - i\gamma \nu \sin r H_2 - i(\cos z \frac{\partial}{\partial z} - \sin z)H_3 = M \cos z\Phi_3 \]

\[ -i\epsilon \cos z\Phi_1 + \frac{\partial \Phi_0}{\partial r} + \gamma \frac{\nu}{\sin r} \Phi_0 = M \cos zE_1 , \]
\[ -i\epsilon \cos z\Phi_2 - \cos z \frac{\partial \Phi_0}{\partial z} = M \cos zE_2 , \]
\[ -i\epsilon \cos z\Phi_3 - \frac{\partial \Phi_0}{\partial r} + \gamma \frac{\nu}{\sin r} \Phi_0 = M \cos zE_3 , \]

\[ -i\gamma \frac{\partial \Phi_1}{\partial r} - i\gamma \nu \frac{\partial \Phi_2}{\partial r} - i(\cos z \frac{\partial}{\partial z} - \sin z)\Phi_1 = M \cos zH_1 , \]
\[ -i\gamma \frac{\partial \Phi_2}{\partial r} + i\gamma \nu \frac{\partial \Phi_1}{\partial r} + i(\cos z \frac{\partial}{\partial z} - \sin z)\Phi_1 = M \cos zH_2 , \]
\[ -i\gamma \frac{\partial \Phi_3}{\partial r} + i\gamma \frac{\nu}{\sin r} \Phi_2 + i(\cos z \frac{\partial}{\partial z} - \sin z)\Phi_3 = M \cos zH_3 . \]

With the help of substitutions
\[ H_1 = \frac{h_1}{\cos z} , \quad (\cos z \frac{\partial}{\partial z} - \sin z)H_1 = \frac{\partial h_1}{\partial z} , \]
\[ H_3 = \frac{h_3}{\cos z} , \quad (\cos z \frac{\partial}{\partial z} - \sin z)H_3 = \frac{\partial h_3}{\partial z} , \]
\[ \Phi_1 = \frac{\varphi_1}{\cos z} , \quad (\cos z \frac{\partial}{\partial z} - \sin z)\Phi_1 = \frac{\partial \varphi_1}{\partial z} , \]
\[ \Phi_3 = \frac{\varphi_3}{\cos z} , \quad (\cos z \frac{\partial}{\partial z} - \sin z)\Phi_3 = \frac{\partial \varphi_3}{\partial z} , \]
\[ E_2 = \frac{e_2}{\cos^2 z} , \quad (\cos z \frac{\partial}{\partial z} - 2 \sin z)E_2 = \frac{1}{\cos z} \frac{\partial e_2}{\partial z} , \]
\[ E_1 = \frac{e_1}{\cos z} , \quad E_3 = \frac{e_3}{\cos z} , \]
\[ \Phi_0 = \frac{\varphi_0}{\cos^2 z} , \quad \Phi_2 = \frac{\varphi_2}{\cos^2 z} , \quad H_2 = \frac{h_2}{\cos^2 z} , \]

we get a more simple system

(2.9)
\[\gamma \left( \frac{\partial e_1}{\partial r} - \frac{\partial e_3}{\partial r} \right) - \frac{\gamma}{\sin r} \left[ (\nu - \cos r)e_1 + (\nu + \cos r)e_3 \right] - \frac{\partial e_2}{\partial z} = M\varphi_0, \]

\[+ i\epsilon e_1 + \frac{i \gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) h_2 + i \frac{\partial h_1}{\partial z} = M\varphi_1, \]

\[+ i\epsilon e_2 + i\gamma \frac{\partial h_1}{\partial r} + \frac{\partial h_3}{\partial r} - \frac{i \gamma}{\sin r} \left[ (\nu - \cos r)h_1 - (\nu + \cos r)h_3 \right] = M\varphi_2, \]

\[+ i\epsilon e_3 + \frac{i \gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) h_2 - i \frac{\partial h_3}{\partial z} = M\varphi_3. \]

(2.10)

\[-i\epsilon \varphi_1 + \frac{\gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) \varphi_0 = Me_1, \]

\[-i\epsilon \varphi_2 - \left( \frac{\partial}{\partial z} + 2 \frac{\sin z}{\cos z} \right) \varphi_0 = Me_2, \]

\[-i\epsilon \varphi_3 - \frac{\gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) \varphi_0 = Me_3, \]

(2.11)

\[-i \frac{\gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) \varphi_2 - i \frac{\partial \varphi_1}{\partial z} = Mh_1, \]

\[-i \gamma \left( \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_3}{\partial r} \right) + \frac{i \gamma}{\sin r} \left[ (\nu - \cos r)\varphi_1 - (\nu + \cos r)\varphi_3 \right] = Mh_2, \]

\[-i \frac{\gamma}{\cos^2 z} \left( \frac{\partial \varphi_2}{\partial r} - \frac{\nu}{\sin r} \right) \varphi_2 + i \frac{\partial \varphi_3}{\partial z} = Mh_3. \]

(2.12)

These equations can be transformed to the form

\[\gamma \left( \frac{\partial}{\partial r} - \frac{\nu - \cos r}{\sin r} \right) e_1 - \gamma \left( \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) e_3 - \frac{\partial e_2}{\partial z} = M\varphi_0, \]

\[i\gamma \left( \frac{\partial}{\partial r} - \frac{\nu - \cos r}{\sin r} \right) h_1 + i\gamma \left( \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) h_3 + i\epsilon e_2 = M\varphi_2, \]

\[i\gamma \left( \frac{\partial}{\partial r} - \frac{\nu - \cos r}{\sin r} \right) h_2 + i\epsilon e_1 + i \frac{\partial h_1}{\partial z} = M\varphi_1, \]

\[i\gamma \left( \frac{\partial}{\partial r} - \frac{\nu - \cos r}{\sin r} \right) h_2 + i\epsilon e_3 - i \frac{\partial h_3}{\partial z} = M\varphi_3. \]
\[
\gamma \cos^2 z \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) \varphi_0 - i \epsilon \varphi_1 = M e_1 ,
\]
\[
- \frac{i \gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) \varphi_2 - i \frac{\partial \varphi_1}{\partial z} = M h_1 ,
\]
\[
- \frac{\gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) \varphi_0 - i \epsilon \varphi_3 = M e_3 ,
\]
\[
- \frac{i \gamma}{\cos^2 z} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) \varphi_2 + i \frac{\partial \varphi_3}{\partial z} = M h_3 ,
\]
\[
-i \epsilon \varphi_2 - \left( \frac{\partial}{\partial z} + 2 \frac{\sin z}{\cos z} \right) \varphi_0 = M e_2 ,
\]
\[
- i \gamma \left( \frac{\partial}{\partial r} - \frac{\nu - \cos r}{\sin r} \right) \varphi_1 - i \gamma \left( \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) \varphi_3 = M h_2 .
\]

Let us introduce a shortened notation
\[
\gamma \left( \frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r} \right) = \hat{a}_-, \gamma \left( \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) = \hat{a}_+, \gamma \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) = \hat{a},
\]
\[
\gamma \left( - \frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r} \right) = \hat{b}_-, \gamma \left( - \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) = \hat{b}_+, \gamma \left( - \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) = \hat{b} ,
\]
then the above equations read
\[
- \hat{b}_- e_1 - \hat{a}_+ e_3 - \frac{\partial e_2}{\partial z} = M \varphi_0 ,
\]
\[
- i \hat{b}_- h_1 + i \hat{a}_+ h_3 + i \epsilon e_2 = M \varphi_2 ,
\]
\[
\frac{i}{\cos^2 z} \hat{a} h_2 + i \epsilon e_1 + i \frac{\partial h_1}{\partial z} = M \varphi_1 ,
\]
\[
- \frac{i}{\cos^2 z} \hat{b} h_2 + i \epsilon e_3 - i \frac{\partial h_3}{\partial z} = M \varphi_3 ,
\]
\[
\frac{1}{\cos^2 z} \hat{a} \varphi_0 - i \epsilon \varphi_1 = M e_1 ,
\]
\[
- \frac{i}{\cos^2 z} \hat{a} \varphi_2 - i \frac{\partial \varphi_1}{\partial z} = M h_1 ,
\]
\[
\begin{align*}
\frac{1}{\cos^2 z} \dot{b} \varphi_0 - i \epsilon \varphi_3 &= M e_3 , \\
\frac{i}{\cos^2 z} \dot{b} \varphi_2 + i \frac{\partial \varphi_3}{\partial z} &= M h_3 , \\
-ic \varphi_2 - \frac{\partial}{\partial z} + 2 \frac{\sin z}{\cos z} \varphi_0 &= M e_2 , \\
i \dot{b}_- \varphi_1 - i \dot{a}_+ \varphi_3 &= M h_2 ,
\end{align*}
\]

We can note that turning back to \( \Phi_0 \), we get a simple system as well

\[
\begin{align*}
-\dot{b}_- e_1 - \dot{a}_+ e_3 - \frac{\partial e_2}{\partial z} &= M \cos^2 z \Phi_0 , \\
-ib_+ h_1 + i \dot{a}_+ h_3 + i \epsilon e_2 &= M \varphi_2 , \\
\frac{i}{\cos^2 z} \dot{a} h_2 + i \epsilon e_1 + i \frac{\partial h_1}{\partial z} &= M \varphi_1 , \\
-\frac{i}{\cos^2 z} \dot{b} h_2 + i \epsilon e_3 - i \frac{\partial h_3}{\partial z} &= M \varphi_3 ,
\end{align*}
\]

\[
\begin{align*}
\dot{a} \Phi_0 - i \epsilon \varphi_1 &= M e_1 , \\
-\frac{i}{\cos^2 z} \dot{a} \varphi_2 - i \frac{\partial \varphi_1}{\partial z} &= M h_1 , \\
\dot{b} \Phi_0 - i \epsilon \varphi_3 &= M e_3 , \\
\frac{i}{\cos^2 z} \dot{b} \varphi_2 + i \frac{\partial \varphi_3}{\partial z} &= M h_3 , \\
-ic \varphi_2 - \cos^2 z \frac{\partial \Phi_0}{\partial z} &= M e_2 , \\
i \dot{b}_- \varphi_1 - i \dot{a}_+ \varphi_3 &= M h_2 .
\end{align*}
\]

Below we will work with equations (2.17) – (2.18).
3. Transition to a non-relativistic approximation

Excluding from (2.17)–(2.18) non-dynamical variables \(\Phi_0, h_1, h_2, h_3\):

\[
\frac{1}{\cos^2 z} (-\hat{b} - e_1 - \hat{a} + e_3 - \frac{\partial e_2}{\partial z}) = M \Phi_0 ,
\]

\[
-\frac{i}{\cos^2 z} \hat{a} \varphi_2 - i \frac{\partial \varphi_1}{\partial z} = M h_1 ,
\]

\[
\hat{b} - \varphi_1 - i \hat{a} + \varphi_3 = M h_2 ,
\]

\[
\frac{i}{\cos^2 z} \hat{b} \varphi_2 + i \frac{\partial \varphi_3}{\partial z} = M h_3 .
\]

we obtain 6 equations (grouping them in pair)

\[
\frac{i}{\cos^2 z} \hat{a} (\hat{b} - \varphi_1 - i \hat{a} + \varphi_3) + i \epsilon M e_1 + i \frac{\partial}{\partial z} (-\frac{i}{\cos^2 z} \hat{a} \varphi_2 - i \frac{\partial \varphi_1}{\partial z}) = M^2 \varphi_1 ,
\]

\[
\hat{a} \frac{1}{\cos^2 z} (-\hat{b} - e_1 - \hat{a} + e_3 - \frac{\partial e_2}{\partial z}) - i \epsilon M \varphi_1 = M^2 e_1 ,
\]

(3.2a)

\[
-\hat{b} - \left( -\frac{i}{\cos^2 z} \hat{a} \varphi_2 - i \frac{\partial \varphi_1}{\partial z} \right) + i \hat{a} + \left( \frac{i}{\cos^2 z} \hat{b} \varphi_2 + i \frac{\partial \varphi_3}{\partial z} \right) + i \epsilon M e_2 = M^2 \varphi_2 ,
\]

\[
-\epsilon M \varphi_2 - \cos^2 z \frac{\partial}{\partial z} \frac{1}{\cos^2 z} (-\hat{b} - e_1 - \hat{a} + e_3 - \frac{\partial e_2}{\partial z}) = M^2 e_2 ,
\]

(3.2b)

\[
-\frac{i}{\cos^2 z} \hat{b} (\hat{b} - \varphi_1 - i \hat{a} + \varphi_3) + i \epsilon M e_3 - i \frac{\partial}{\partial z} \left( \frac{i}{\cos^2 z} \hat{b} \varphi_2 + i \frac{\partial \varphi_3}{\partial z} \right) = M^2 \varphi_3 ,
\]

\[
\hat{b} \frac{1}{\cos^2 z} (-\hat{b} - e_1 - \hat{a} + e_3 - \frac{\partial e_2}{\partial z}) - i \epsilon M \varphi_3 = M^2 e_3 .
\]

(3.2c)

Now we should introduce big \(\Psi_1\) and small \(\psi_i\) components

\[
\varphi_1 = \Psi_1 + \psi_1 \quad \text{ie}_1 = \Psi_1 - \psi_1 ,
\]

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\( \varphi_2 = \Psi_2 + \psi_2 , \quad i\epsilon_2 = \Psi_2 - \psi_2 , \)
\( \varphi_3 = \Psi_3 + \psi_3 , \quad i\epsilon_3 = \Psi_3 - \psi_3 , \)

and in the same time separate the rest energy by formal change \( \epsilon \mapsto (\epsilon + M) \)

so we arrive at

\[
- \frac{\hat{\epsilon}_b}{\cos^2 z} (\Psi_1 + \psi_1) + \frac{\hat{\epsilon}_a}{\cos^2 z} (\Psi_3 + \psi_3) + \frac{\hat{\epsilon}_a}{\cos^2 z} (\frac{\partial}{\partial z} + \frac{2\sin z}{\cos z})(\Psi_2 + \psi_2) \\
+ \frac{\partial^2}{\partial z^2} (\Psi_1 + \psi_1) + (\epsilon + M)M(\Psi_1 - \psi_1) = M^2(\Psi_1 + \psi_1) ,
\]

\[
- \frac{\hat{\epsilon}_b}{\cos^2 z} (\Psi_1 - \psi_1) - \frac{\hat{\epsilon}_a}{\cos^2 z} (\Psi_3 - \psi_3) - \frac{\hat{\epsilon}_a}{\cos^2 z} (\frac{\partial}{\partial z})(\Psi_2 - \psi_2) \\
+ (\epsilon + M)M(\Psi_1 + \psi_1) = M^2(\Psi_1 - \psi_1) ;
\] (3.3a)

\[
- \frac{\hat{\epsilon}_b}{\cos^2 z} (\Psi_2 + \psi_2) - \frac{\hat{\epsilon}_a}{\cos^2 z} (\Psi_1 + \psi_1) - \frac{\hat{\epsilon}_a}{\cos^2 z} (\Psi_3 + \psi_3) \\
+ (\epsilon + M)M(\Psi_2 - \psi_2) = M^2(\Psi_2 + \psi_2) ,
\]

\[
\hat{\epsilon}_b \cos^2 z \frac{\partial}{\partial z} \frac{1}{\cos^2 z} (\Psi_1 - \psi_1) + \frac{\hat{\epsilon}_a}{\cos^2 z} \frac{\partial}{\partial z} \frac{1}{\cos^2 z} (\Psi_3 - \psi_3) \\
+ \cos^2 z \frac{\partial}{\partial z} \frac{1}{\cos^2 z} \frac{\partial}{\partial z} (\Psi_2 - \psi_2) + (\epsilon + M)M(\Psi_2 + \psi_2) = M^2(\Psi_2 - \psi_2) ;
\] (3.3b)

\[
\hat{\epsilon}_b \cos^2 z (\Psi_1 + \psi_1) - \hat{\epsilon}_a (\Psi_3 + \psi_3) + \frac{\hat{\epsilon}_a}{\cos^2 z} (\frac{\partial}{\partial z} + \frac{2\sin z}{\cos z})(\Psi_2 + \psi_2) + \\
\frac{\partial^2}{\partial z^2} (\Psi_3 + \psi_3) + (\epsilon + M)M(\Psi_3 - \psi_3) = M^2(\Psi_3 + \psi_3) ,
\]

\[
- \frac{\hat{\epsilon}_b}{\cos^2 z} (\Psi_1 - \psi_1) - \frac{\hat{\epsilon}_a}{\cos^2 z} (\Psi_3 - \psi_3) - \frac{\hat{\epsilon}_b}{\cos^2 z} (\frac{\partial}{\partial z})(\Psi_2 - \psi_2) \\
+ (\epsilon + M)M(\Psi_3 + \psi_3) = M^2(\Psi_3 - \psi_3) .
\] (3.3c)

Summing equation for each pair and neglecting small components \( \psi_k \) in comparison with big ones \( \Psi_k \), we get
\[
\begin{align*}
&\left( -\frac{2}{\cos^2 z} \hat{a}\hat{b}_- + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_1 + 2 \frac{\sin z}{\cos^2 z} \hat{a} \Psi_2 = 0, \\
&\left( -\frac{2}{\cos^2 z} \hat{b}\hat{a}_+ + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_3 + 2 \frac{\sin z}{\cos^2 z} \hat{b} \Psi_2 = 0, \\
&\left( -\frac{1}{\cos^2 z} (\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) + 2\epsilon M + \frac{\partial^2}{\partial z^2} + 2 \frac{\sin z}{\cos z} \frac{\partial}{\partial z} \right) \Psi_2 \\
&\quad + 2 \frac{\sin z}{\cos z} (\hat{b}_-\Psi_1 + \hat{a}_+\Psi_3) = 0. \tag{3.4a}
\end{align*}
\]

It is a needed system in Pauli approximation. In particular, for the case of flat space model we get much more simple system of three separated equations

\[
\begin{align*}
&\left( -2\hat{a}\hat{b}_- + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_1 = 0, \\
&\left( -2\hat{b}\hat{a}_+ + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_3 = 0, \\
&\left( -(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) + 2\epsilon M + \frac{\partial^2}{\partial z^2} \right) \Psi_2 = 0,
\end{align*}
\]

where in definitions for \( \hat{a}, \hat{b}, \hat{a}_-, \hat{b}_-, \hat{a}_+, \hat{b}_+ \) some simplifications are to be performed – see (2.14).

Equations (3.4a) can be transformed to a more symmetrical form if one make a substitution

\[
\Psi_2 = \cos z \bar{\Psi}_2,
\]

\[
\left( \frac{\partial^2}{\partial z^2} + 2 \frac{\sin z}{\cos z} \frac{\partial}{\partial z} \right) \cos z \bar{\Psi}_2 = \cos z \left( \frac{\partial^2}{\partial z^2} - \frac{2}{\cos^2 z} \right) + 1) \bar{\Psi}_2, \tag{3.4b}
\]

Then, eqs. (3.4a) read

\[
\begin{align*}
&\left( -\frac{2}{\cos^2 z} \hat{b}_- + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_1 + 2 \frac{\sin z}{\cos^2 z} \hat{a} \Psi_2 = 0, \\
&\left( -\frac{2}{\cos^2 z} \hat{a}_+ + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) \Psi_3 + 2 \frac{\sin z}{\cos^2 z} \hat{b} \Psi_2 = 0, \\
&\left( -(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) + 2\epsilon M + \frac{\partial^2}{\partial z^2} \right) \bar{\Psi}_2 \\
&\quad + \frac{2}{\cos^2 z} (\hat{b}_-\Psi_1 + \hat{a}_+\Psi_3) = 0.
\end{align*}
\]
Let us introduce new functions

\[ \hat{b}_- \Psi_1 = G_1, \quad \Psi_2 = G_2, \quad \hat{a}_+ \Psi_3 = G_3, \]  

(3.5a)

eqs. (3.4a) will give

\[ \left( -\frac{2}{\cos^2 z} \hat{b}_- \hat{a} + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) G_1 + 2\frac{\sin z}{\cos^2 z} \hat{b}_- \hat{a} G_2 = 0, \]

\[ \left( -\frac{2}{\cos^2 z} \hat{a}_+ \hat{b} + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) G_3 + 2\frac{\sin z}{\cos^2 z} \hat{a}_+ \hat{b} G_2 = 0, \]

\[ \left( \frac{1}{\cos^2 z} (\hat{b}_- \hat{a} + \hat{a}_+ \hat{b} + 2) + \frac{\partial^2}{\partial z^2} + 2\epsilon M + 1 \right) G_2 \]

\[ + 2\frac{\sin z}{\cos^2 z} (G_1 + G_3) = 0. \]  

(3.5b)

Now we should define a factorized form for three functions

\[ G_1 = Z_1(z) R_1(r), \quad G_2 = Z_2(z) R_2(r), \quad G_3 = Z_3(z) R_3(r); \]  

(3.6a)

then eqs. (3.5b) read

\[ \left( -\frac{2}{\cos^2 z} \hat{b}_- \hat{a} + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) Z_1 R_1 + 2\frac{\sin z}{\cos^2 z} \hat{b}_- \hat{a} Z_2 R_2 = 0, \]

\[ \left( -\frac{2}{\cos^2 z} \hat{a}_+ \hat{b} + \frac{\partial^2}{\partial z^2} + 2\epsilon M \right) Z_3 R_3 + 2\frac{\sin z}{\cos^2 z} \hat{a}_+ \hat{b} Z_2 R_2 = 0, \]

\[ \left( \frac{1}{\cos^2 z} (\hat{b}_- \hat{a} + \hat{a}_+ \hat{b} + 2) + \frac{\partial^2}{\partial z^2} + 2\epsilon M + 1 \right) Z_2 R_2 \]

\[ + 2\frac{\sin z}{\cos^2 z} (Z_1 R_1 + Z_3 R_3) = 0. \]  

(3.6b)

Note that the first equation in (3.6b) does not change if one acts from the left by the operator \( \hat{b}_- \hat{a} \); similarly the second equation preserves its form if one acts from the left by the operator \( \hat{a}_+ \hat{b} \). Therefore, one can assume existence of the following radial relationships

\[ \hat{b}_- \hat{a} R_1 = \lambda R_1, \quad \hat{b}_- \hat{a} R_2 = \lambda R_2, \quad R_1 = R_2 = R; \]  

(3.7a)
and
\[
\hat{a}_+ \hat{b} R_3 = \lambda' R_3, \quad \hat{a}_+ \hat{b} R_2 = \lambda' R_2, \quad R_2 = R_3 = R. \quad (3.7b)
\]

Taking into account these restrictions from (3.6b) we obtain the system in \( z \) variable
\[
\left( -\frac{2\lambda}{\cos^2 z} + \frac{d^2}{dz^2} + 2\epsilon M \right) Z_1 + 2\lambda \frac{\sin z}{\cos^2 z} Z_2 = 0, \\
\left( -\frac{2\lambda'}{\cos^2 z} + \frac{d^2}{dz^2} + 2\epsilon M \right) Z_3 + 2\lambda' \frac{\sin z}{\cos^2 z} Z_2 = 0, \\
\left( -\frac{1}{\cos^2 z} (\lambda + \lambda' + 2) + \frac{d^2}{dz^2} + 2\epsilon M + 1 \right) Z_2 + 2\frac{\sin z}{\cos^2 z} (Z_1 + Z_3) = 0.
\]

(3.8)

With the use of explicit expressions for operators \( \hat{a}, \hat{a}_+, \hat{b}, \hat{b}_- \), we derive
\[
\hat{b}_- \hat{a} = \frac{1}{2} \left( -\frac{d^2}{dr^2} - \frac{\cos r}{\sin r} \frac{d}{dr} - B + \frac{\nu^2(r)}{\sin^2 r} \right), \\
\hat{a}_+ \hat{b} = \frac{1}{2} \left( -\frac{d^2}{dr^2} - \frac{\cos r}{\sin r} \frac{d}{dr} + B + \frac{\nu^2(r)}{\sin^2 r} \right), \\
\hat{a}_+ \hat{b} = \hat{b}_- \hat{a} - B,
\]
so the first radial equation for \( R_2 \) takes the form
\[
\hat{b}_- \hat{a} R_2 = \lambda R_2 \quad \Rightarrow \quad \left( \frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} + B - \frac{\nu^2(r)}{\sin^2 r} + 2\lambda \right) R_2 = 0; \quad (3.9a)
\]
the second equation for \( R_1 \) gives the same only if two parameters \( \lambda \) and \( \lambda' \) obey a special additional constraint
\[
\hat{a}_+ \hat{b} R_2 = \lambda' R_2 \quad \Rightarrow \quad \hat{b}_- \hat{a} R_2 = (\lambda' + B) R_2,
\]
that is
\[
\lambda' = \lambda - B. \quad (3.9b)
\]

Let us consider eq. (3.9a) in more detail
\[
\frac{d^2}{dr^2} R + \frac{1}{\tan r} \frac{dR}{dr} - \frac{1}{\sin^2 r} \left[ m + B (1 - \cos r) \right]^2 R + (B + 2\lambda) R = 0.
\]

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In a new variable

\[ 1 - \cos r = 2y , \quad y = \sin^2 \frac{r}{2} \in [0, 1], \]

\[
\left[ y(1 - y) \frac{d^2}{dy^2} + (1 - 2y) \frac{d}{dy} - \frac{1}{4} \left( \frac{m^2}{y} - 4B^2 + \frac{(m + 2B)^2}{1 - y} \right) \right] R = 0 . \tag{3.10}
\]

With the substitution \( R = y^a(1 - y)^b \ F, \) eq. (3.10) gives

\[
y(1 - y) F'' + \left[ a(1 - y) - by + a(1 - y) - by + (1 - 2y) \right] F' \\
+ \frac{1}{y} \left[ a(a - 1) + a - \frac{m^2}{4} \right] F + \frac{1}{1 - y} \left[ b(b - 1) + b - \frac{(m + 2B)^2}{4} \right] F \\
- [a(a + 1) + 2ab + b(b + 1) - B^2 - (B + 2\lambda)] F = 0 .
\]

If parameters obey restriction below

\[ a = \pm \frac{|m|}{2}, \quad b = \pm \frac{m + 2B}{2} ; \tag{3.11a} \]

we arrive at a more simple equation

\[
y(1 - y) F'' + \left[ (2a + 1) - 2(a + b + 1)y \right] F' \\
- [a(a + 1) + 2ab + b(b + 1) - B^2 - (B + 2\lambda)] F = 0 , \tag{3.11b}
\]

which is recognized as a hypergeometric one

\[
y(1 - y) F + \left[ \gamma - (\alpha + \beta + 1)y \right] F' - \alpha \beta F = 0 . \tag{3.11c}
\]

So we have (to obtain solutions for bound states we must assume positive \( a \) and \( b \))

\[ y = \sin^2 \frac{r}{2} , \quad y \in [0, +1], \quad r \in [0, +\pi], \]

\[
R = \left( \sin \frac{r}{2} \right)^{|m|} \left( \cos \frac{r}{2} \right)^{|m+2B|} F(\alpha, \beta, \gamma; -\sin^2 \frac{r}{2}); \tag{3.11d}
\]

parameters \((\alpha, \beta, \gamma)\) are determined by

\[ \gamma = + |m| + 1 , \quad a = + \frac{|m|}{2} , \quad b = + \frac{m + 2B}{2} . \]

\[
\begin{align*}
\alpha + \beta &= 2a + 2b + 1, \\
\alpha \beta &= (a + b)(a + b + 1) - B^2 - (B + 2\lambda); \tag{3.12a}
\end{align*}
\]

that is

\[
\gamma = + |m| + 1, \quad a = + \frac{\left| m \right|}{2}, \quad b = + \frac{\left| m + 2B \right|}{2},
\]

\[
\alpha = a + b + \frac{1}{2} - \sqrt{\left( B + \frac{1}{2} \right)^2 + 2\lambda},
\]

\[
\beta = a + b + \frac{1}{2} + \sqrt{\left( B + \frac{1}{2} \right)^2 + 2\lambda}. \tag{3.12b}
\]

To obtain solutions in polynomials, we must assume positivity of the expression under the sign of square root and must impose restriction on the \( \alpha \)

\[
\alpha = a + b + \frac{1}{2} - \sqrt{\left( B + \frac{1}{2} \right)^2 + 2\lambda} = -n = 0, -1, -2, \ldots,
\]

from whence it follows the quantization rule

\[
2\lambda + \left( B + \frac{1}{2} \right)^2 = (a + b + \frac{1}{2} + n)^2 > 0, \tag{3.13b}
\]

solutions corresponding to bound states are given by

\[
R = (\sin \frac{r}{2})^{+\left| m \right|} (\cos \frac{r}{2})^{+\left| m + 2B \right|} \\
\times F(-n, \left| m \right| + 1 + \left| m + 2B \right| + 1 + n, \left| m \right| + 1; -\sin^2 \frac{r}{2}). \tag{3.13c}
\]

Below, we will use notation

\[
\Lambda = \Lambda - \frac{B}{2}, \tag{3.14a}
\]

then the formula for spectrum (3.13b) will read

\[
2\Lambda + B^2 = N(N + 1), \quad N = a + b + n. \tag{3.13b}
\]
4. Behavior of solutions in \( z \) variable near singular points

Let us turn to the system (3.8)

\[
\begin{align*}
\left( \frac{d^2}{dz^2} - \frac{2\lambda}{\cos^2 z} + 2\epsilon M \right) Z_1 + 2\lambda \frac{\sin z}{\cos^2 z} Z_2 = 0 , \\
\left( \frac{d^2}{dz^2} - \frac{2\lambda'}{\cos^2 z} + 2\epsilon M \right) Z_3 + 2\lambda' \frac{\sin z}{\cos^2 z} Z_2 = 0 , \\
\left( \frac{d^2}{dz^2} - \frac{\lambda + \lambda' + 2}{\cos^2 z} + 2\epsilon M + 1 \right) \bar{Z}_2 + 2\frac{\sin z}{\cos^2 z} (Z_1 + Z_3) = 0 .
\end{align*}
\]

In the variable 
\[
\sin z = x , \quad x \in [-1, +1] ,
\]

we get

\[
\begin{align*}
\left( (1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2\lambda}{1 - x^2} + 2\epsilon M \right) Z_1 + \frac{2\lambda x}{1 - x^2} \bar{Z}_2 = 0 , \\
\left( (1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2\lambda'}{1 - x^2} + 2\epsilon M \right) Z_3 + \frac{2\lambda' x}{1 - x^2} \bar{Z}_2 = 0 , \\
\left( (1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2 + \lambda + \lambda'}{1 - x^2} + 2\epsilon M + 1 \right) \bar{Z}_2 + \frac{2x}{1 - x^2} (Z_1 + Z_3) = 0 .
\end{align*}
\]

Near the point \( z = +\pi/2 \) we have

\[
\begin{align*}
\left( 2(1 - x) \frac{d^2}{dx^2} - \frac{d}{dx} - \frac{\lambda}{1 - x} \right) Z_1 + \frac{\lambda}{1 - x} Z_2 = 0 , \\
\left( 2(1 - x) \frac{d^2}{dx^2} - \frac{d}{dx} - \frac{\lambda'}{1 - x} \right) Z_3 + \frac{\lambda'}{1 - x} Z_2 = 0 , \\
\left( 2(1 - x) \frac{d^2}{dx^2} - \frac{d}{dx} - \frac{2 + \lambda + \lambda'}{1 - x} \right) \bar{Z}_2 + \frac{1}{1 - x} (Z_1 + Z_3) = 0 ;
\end{align*}
\]

so the possible solution is

\[
Z_1 = A_1 (1 - x)^\alpha , \quad \bar{Z}_2 = A_2 (1 - x)^\alpha , \quad Z_3 = A_3 (1 - x)^\alpha .
\]
Substituting (4.3b) into (4.3a), we obtain linear system with respect to \(A_1, A_2, A_3\):

\[
\begin{align*}
(2a^2 - a - \lambda)A_1 + \lambda A_2 &= 0, \\
(2a^2 - a - \lambda')A_3 + \lambda' A_2 &= 0, \\
(2a^2 - a - \frac{2 + \lambda + \lambda'}{2})A_2 + A_1 + A_3 &= 0.
\end{align*}
\] (4.3c)

In similar manner consider behavior of solution near the second singular point

\[
\begin{align*}
Z_1 &- \lambda B_1 = 0, \\
Z_2 &- \lambda B_2 = 0, \\
Z_3 &- \lambda B_3 = 0.
\end{align*}
\] (4.4a)

that is

\[
Z_1 = B_1(1 + x)^b, \quad Z_2 = B_2(1 + x)^b, \quad Z_3 = B_3(1 + x)^b. \] (4.4b)

and coefficients \(B_1, B_2, B_3\) obey the linear system as well

\[
\begin{align*}
(2b^2 - b - \lambda)B_1 - \lambda B_2 &= 0, \\
(2b^2 - b - \lambda')B_3 - \lambda' B_2 &= 0, \\
(2b^2 - b - \frac{2 + \lambda + \lambda'}{2})B_2 - B_1 - B_3 &= 0.
\end{align*}
\] (4.4c)

With the notation

\[
\begin{align*}
2a^2 - a &= A, & 2b^2 - b &= B, \\
a &= \frac{1 \pm \sqrt{1 + 8A}}{4}, & b &= \frac{1 \pm \sqrt{1 + 8B}}{4};
\end{align*}
\] (4.5a)

two linear system are written as

\[
\begin{align*}
(A - \lambda)A_1 + \lambda A_2 &= 0, \\
(A - \lambda')A_3 + \lambda' A_2 &= 0.
\end{align*}
\]
\[(A - \frac{2 + \lambda + \lambda'}{2})A_2 + A_1 + A_3 = 0; \quad (4.5b)\]

and
\[(B - \lambda)B_1 - \lambda B_2 = 0,\]
\[(B - \lambda')B_3 - \lambda'B_2 = 0,\]
\[(B - \frac{2 + \lambda + \lambda'}{2})B_2 - B_1 - B_3 = 0; \quad (4.5c)\]

Further we get one the same eigenvalue equation for values \(A\) and \(B\)

\[(A - \lambda)\lambda' + (A - \lambda')\lambda - (A - \lambda)(A - \lambda')(A - \frac{2 + \lambda + \lambda'}{2}) = 0,\]
\[(B - \lambda)\lambda' + (B - \lambda')\lambda - (B - \lambda)(B - \frac{2 + \lambda + \lambda'}{2}) = 0; \quad (4.6)\]

respective solutions are given as

\[A_1 = (A_2) \frac{\lambda}{\lambda' - A}, \quad A_3 = (A_2) \frac{\lambda'}{\lambda' - A}; \quad (4.7a)\]
\[B_1 = (-B_2) \frac{\lambda}{\lambda' - B}, \quad B_3 = (-B_2) \frac{\lambda'}{\lambda' - B}. \quad (4.7b)\]

Now, let us examine a third order equation (4.6) – for definiteness consider the case of \(A\):

\[2(A - \lambda)\lambda' + 2(A - \lambda')\lambda - (A - \lambda)(A - \lambda')(2A - 2 - \lambda - \lambda') = 0; \quad (4.8)\]

the equation arising is symmetric with respect to formal replacement \(\lambda \leftrightarrow \lambda'\).

Explicitly the equation read

\[2A(\lambda + \lambda') - 4\lambda \lambda' + [A^2 - A(\lambda + \lambda') + \lambda \lambda'][-2A + 2 + (\lambda + \lambda')] = 0 \quad \Rightarrow \]

\[2A(\lambda + \lambda') - 4\lambda \lambda' - 2A^3 + 2A^2 + A^2(\lambda + \lambda') + 2A^2(\lambda + \lambda') - 2A(\lambda + \lambda')^2 - 2A\lambda \lambda' + 2\lambda \lambda' + \lambda \lambda'(\lambda + \lambda') = 0 \quad \Rightarrow \]

\[-2A^3 + A^2 [2 + 3(\lambda + \lambda')] - A [2(\lambda + \lambda')^2 + 2\lambda \lambda'] + \lambda \lambda' [2(\lambda + \lambda') - 2] = 0. \quad (4.9a)\]
Remembering on \( \lambda' = \lambda - B \), one can introduce other parameters

\[
\lambda' - \frac{B}{2} = \lambda + \frac{B}{2} \equiv \Lambda ,
\]

\[
\lambda + \lambda' = 2\Lambda , \quad \lambda \lambda' = \Lambda^2 - \frac{B^2}{4} .
\] (4.9b)

Then eq. (4.9a) reads

\[
A^3 - A^2 (3\Lambda + 1) + A \left(3\Lambda^2 - \frac{B^2}{4}\right) - (\Lambda^2 - \frac{B^2}{4}) (\Lambda - 1) = 0 .
\] (4.9c)

It can be presented symbolically as

\[
A^3 + aA^2 + bA + c = 0 ,
\] (4.10a)

where

\[
a = -(3\Lambda + 1) ,
\]

\[
b = (3\Lambda^2 - \frac{B^2}{4}) ,
\]

\[
c = -(\Lambda^2 - \frac{B^2}{4}) (\Lambda - 1) .
\] (4.10b)

Through change in the variable \( A \rightarrow Y \)

\[
A = Y - \frac{a}{3} = Y + \Lambda + \frac{1}{3}
\] (4.11a)

we remove a quadratic term

\[
Y^3 + pY + q = 0 ,
\] (4.11b)

where

\[
p = -\frac{a^2}{3} + b = -(2\Lambda + \frac{B^2}{4} + \frac{1}{3}),
\]

\[
q = \frac{2a^3}{27} - \frac{ab}{3} + c = -\left(\frac{2}{3}\Lambda + \frac{B^2}{3} + \frac{2}{27}\right) .
\] (4.11c)

Note substantial inequalities

\[
p < 0 , \quad q < 0 , \quad |p| > |q| .
\]

Formulas, giving solutions of eq. (4.11b) are well known
\[ Y = \left[ -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{1/3} + \left[ -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{1/3}. \] (4.12a)

Applying (4.12a), one must use correlated roots
\[ \alpha = \left[ -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{1/3} \] (4.12b)
and
\[ \beta = \left[ -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{1/3} \] (4.12c)
so that the following restriction hold
\[ \alpha \beta = -\frac{p}{3}. \] (4.12d)

Besides, the roots can be searched according to the formulas
\[ Y_1 = \alpha_1 + \beta_1, \]
\[ Y_2 = -\frac{1}{2}(\alpha_1 + \beta_1) + i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1) \]
\[ Y_3 = -\frac{1}{2}(\alpha_1 + \beta_1) - i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1) \] (4.13a)
where \( \alpha_1 \) stands for any root in (4.12b), but a root \( \beta_1 \) in (4.12c) must obey
\[ \alpha_1\beta_1 = -\frac{p}{3}. \] (4.13b)

Let us additionally detail expressions (4.13a,b) for three roots. Allowing for
\[ \alpha = \left[ -\frac{q}{2} + i\sqrt{\left(-\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2} \right]^{1/3} \]
\[ = \left[ \left(-\frac{p}{3}\right)^{3/2}(\cos \phi + i \sin \phi) \right]^{1/3} \]
\[ = \sqrt{-\frac{p}{3}} \left\{ e^{i\phi/3}, e^{i(\phi/3+2\pi/3)}, e^{i(\phi/3+4\pi/3)} \right\}, \] (4.14a)
where
\[
\cos \phi = \frac{-q/2}{(-p/3)^{3/2}}, \quad \sin \phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}}. \quad (4.14b)
\]

It is readily to specify the quantity \( \beta \):

\[
\beta = \left[ -\frac{q}{2} - i\sqrt{(-p/3)^3 - (q/2)^2} \right]^{1/3} = \\
= \left[ (-p/3)^{3/2} \left( \cos \phi - i \sin \phi \right) \right]^{1/3} = \\
= \sqrt{-p/3} \left\{ e^{-i\phi/3}, e^{i(-\phi/3+2\pi/3)}, e^{i(-\phi/3+4\pi/3)} \right\}, \quad (4.14a)
\]

where

\[
\cos \phi = \frac{-q/2}{(-p/3)^{3/2}}, \quad \sin \phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}}. \quad (4.14b)
\]

As \( \alpha_1 \) and \( \beta_1 \) we will take

\[
\alpha_1 = \sqrt{-p/3} e^{+i\phi/3}, \quad \beta_1 = \sqrt{-p/3} e^{-i\phi/3};
\]

\[
\cos \phi = \frac{-q/2}{(-p/3)^{3/2}}, \quad \sin \phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}}. \quad (4.15a)
\]

And further we readily find

\[
\alpha_1 + \beta_1 = 2\sqrt{-p/3} \cos \frac{\phi}{3}, \quad \alpha_1 - \beta_1 = 2i \sqrt{-p/3} \sin \frac{\phi}{3}. \quad (4.15b)
\]

Thus, three different (real-valued) roots are determined by the formulas

\[
Y_1 = \sqrt{-p/3} \left( 2 \cos \frac{\phi}{3} \right), \\
Y_2 = \sqrt{-p/3} \left( - \cos \frac{\phi}{3} - \sqrt{3} \sin \frac{\phi}{3} \right), \\
Y_3 = \sqrt{-p/3} \left( - \cos \frac{\phi}{3} + \sqrt{3} \sin \frac{\phi}{3} \right). \quad (4.16)
\]

One can additionally check the results: from the identity

\[
Y^3 + pY + q = (Y - Y_1)(Y - Y_2)(Y - Y_3)
\]
it follows
\[ 0 = Y_1 + Y_2 + Y_3, \]
\[ p = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3, \quad q = -Y_1 Y_2 Y_3. \]  
(4.17)

First we readily verify two identity
\[ 0 = Y_1 + Y_2 + Y_3, \quad p = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3. \]

Turning to the third ine, let us calculate
\[ -Y_1 Y_2 Y_3 = -\frac{2\sqrt{3}}{9} (-p)^{3/2} \left[ 4 \cos^2 \frac{\phi}{3} - 3 \right] \cos \frac{\phi}{3}; \]  
(4.18a)

further with the help of elementary relation
\[ \cos \alpha \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}, \]
we get
\[ \left[ 4 \cos^2 \frac{\phi}{3} - 3 \right] \cos \frac{\phi}{3} = (-1 + 2 \cos \frac{2\phi}{3}) \cos \frac{\phi}{3} = \cos \phi; \]  
(4.18b)

and thus we prove the third identity (remembering on (4.15a))
\[ -Y_1 Y_2 Y_3 = -\frac{2\sqrt{3}}{9} (-p)^{3/2} \cos \phi = \frac{2\sqrt{3}}{9} (-p)^{3/2} \frac{-q/2}{(-p/3)^{3/2}} = -q. \]  
(4.18c)

Unfortunately we have not gained success in solving the main system of 3 equation in \( z \) variable
\[ \left( 1 - x^2 \right) \frac{d^2}{dx^2} - x \frac{dx}{dx} - \frac{2\lambda}{1 - x^2} + 2\epsilon M \right) Z_1 + \frac{2\lambda x}{1 - x^2} \bar{Z}_2 = 0, \]
\[ \left( 1 - x^2 \right) \frac{d^2}{dx^2} - x \frac{dx}{dx} - \frac{2\lambda'}{1 - x^2} + 2\epsilon M \right) Z_3 + \frac{2\lambda' x}{1 - x^2} \bar{Z}_2 = 0, \]
\[ \left( 1 - x^2 \right) \frac{d^2}{dx^2} - x \frac{dx}{dx} - \frac{2 + \lambda + \lambda'}{1 - x^2} + 2\epsilon M + 1 \right) \bar{Z}_2 + \frac{2x}{1 - x^2} (Z_1 + Z_3) = 0. \]

So this analysis can be considered as completed.

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