A FINITE DIMENSIONAL $A_\infty$ ALGEBRA EXAMPLE

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Dedicated to Tornike Kadeishvili on the occasion of his 60th birthday

Abstract. We construct an example of an $A_\infty$ algebra structure defined over a finite dimensional graded vector space.

Introduction

$A_\infty$ algebras (or sha algebras) and $L_\infty$ (or sh Lie algebras) have been topics of current research. Construction of small examples of these algebras can play a role in gaining insight into deeper properties of these structures. These examples may prove useful in developing a deformation theory as well as a representation theory for these algebras.

In [2], an $L_\infty$ algebra structure on the graded vector space $V = V_0 \oplus V_1$ where $V_0$ is a 2 dimensional vector space, and $V_1$ is a 1 dimensional space, is discussed. This surprisingly rich structure on this small graded vector space was shown by Kadeishvili and Lada, [3], to be an example of an open-closed homotopy algebra (OCHA) defined by Kajiura and Stasheff [4]. In an unpublished note [1] M. Daily constructs a variety of other $L_\infty$ algebra structures on this same vector space.

In this article we add to this collection of structures on the vector space $V$ by providing a detailed construction of non-trivial $A_\infty$ algebra data for $V$.

1. $A_\infty$ Algebras

We first recall the definition of an $A_\infty$ algebra (Stasheff [6]).

Definition 1.1. Let $V$ be a graded vector space. An $A_\infty$ structure on $V$ is a collection of linear maps $m_k : V^\otimes k \to V$ of degree $2 - k$ that satisfy the identity

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where $\alpha = (-1)^{k+\lambda+k+\lambda+k+n+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

This utilizes the cochain complex convention. One may alternatively utilize the chain complex convention by requiring each map $m_k$ to have degree $k - 2$. 

We will define the desuspension of $V$ (denoted $\downarrow V$) as the graded vector space with indices given by $(\downarrow V)_n = V_{n+1}$, and the desuspension operator, $\downarrow : V \to (\downarrow V)$ (resp. suspension operator $\uparrow : (\downarrow V) \to V$) in the natural sense. We will also employ the usual Koszul sign convention in this setting: whenever two symbols (objects or maps) of degree $p$ and $q$ are commuted, a factor of $(-1)^{pq}$ is introduced. Subsequently, $\uparrow \otimes \downarrow \circ \uparrow \otimes \downarrow = (-1)^{\frac{n(n-1)}{2}} id$ and $\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n = (-1)^{\sum_{i=1}^{n}(n-i)|x_i|} \downarrow \otimes (x_1 \otimes x_2 \otimes \cdots \otimes x_n)$.

Stasheff also showed that an $A_{\infty}$ structure on $V$ is equivalent to the existence of a degree 1 coderivation $D : T^* (\downarrow V) \to T^* (\downarrow V)$ with the property $D^2 = 0$. Here, $T^* (\downarrow V)$ is the tensor coalgebra on the graded vector space $\downarrow V$. Such a coderivation is constructed by defining $m'_k : (\downarrow V^\otimes k) \to \downarrow V$ by $m'_k = (-1)^{\frac{k(k-1)}{2}} \downarrow \circ \uparrow \circ \uparrow \otimes \downarrow$ and then extending each $m'_k$ to a coderivation on $T^* (\downarrow V)$. By “abuse of notation”, $m'_k$ can be described by

$$m'_k(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i=0}^{n-1} (1^\otimes i \circ m'_k \circ 1^\otimes n-i-1)(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

$$= \sum_{i=0}^{n-1} (-1)^{k(2|x_i| + \cdots + |x_i|-1)}(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i \circ m'_k(\downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{i+k}) \otimes \downarrow x_{i+k+1} \otimes \cdots \otimes \downarrow x_n)$$

We then define $D := \sum_{k=1}^{\infty} m'_k$.

### 2. A Finite Dimensional Example

Let $V$ denote the graded vector space given by $V = \bigoplus V_n$ where $V_0$ has basis $< v_1, v_2 >$, $V_1$ has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on $V$ by the following linear maps $m_n : V^\otimes n \to V$:

$$m_1(v_1) = m_1(v_2) = w$$

For $n \geq 2$:

$$m_n(v_1 \otimes w^\otimes k \otimes v_1 \otimes w^\otimes (n-2-k)) = (-1)^k s_n v_1, \ 0 \leq k \leq n - 2$$

$$m_n(v_1 \otimes w^\otimes (n-2) \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^\otimes (n-1)) = s_{n+1} w$$

where $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$, and $m_n = 0$ when evaluated on any element of $V^\otimes n$ that is not listed above. It is worth noting that this assumes the cochain convention regarding $A_{\infty}$ algebra structures. Hence $|v_1| = |v_2| = 0$ and $|w| = 1$.

**Theorem 2.1.** The maps defined above give the graded vector space $V$ an $A_{\infty}$ algebra structure.

The proof of this theorem relies on two lemmas:
Lemma 2.2. Let $m'_n := (-1)^{(n-1)/2} \downarrow \circ m_n \circ \uparrow^\otimes : (\downarrow V)^\otimes n \to \downarrow V$. Under the preceding definitions for $m_n$ and $V$, we obtain the following formulas for $m'_n$:

$$m'_1 = \downarrow m_1$$

For $n \geq 2$:

$$m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes k \otimes \downarrow v_1 \otimes (\downarrow w)^\otimes (n-2)-k) = \downarrow v_1, \ 0 \leq k \leq n - 2$$

$$m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes (n-2) \otimes \downarrow v_2) = \downarrow v_1$$

$$m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes (n-1)) = \downarrow w$$

Remark 2.3. Each $m'_n$ is of degree 1.

Proof of Lemma 2.2 $m'_1(\downarrow x) = (-1)^0 \downarrow \circ m_1 \circ \uparrow (\downarrow x) = \downarrow m_1(x)$ for any $x$.

Now let $n \geq 2$. The majority of the work here is centered around computing the signs associated with the graded setting. The elements $x_i$ and the maps $\uparrow, \downarrow$, and $m_n$ all contribute to an overall sign via their degrees. Observing these signs, we find

$$m'_n (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^\otimes (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

$$= \begin{cases} (-1)^{\frac{n}{2} \lfloor \frac{n}{2} \rfloor} \downarrow m_n (x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if } n \text{ is even.} \\ (-1)^{\frac{n}{2}(n-1)/2} \downarrow m_n (x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if } n \text{ is odd.} \end{cases}$$

First consider $m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes k \otimes \downarrow v_1 \otimes (\downarrow w)^\otimes (n-2)-k), \ 0 \leq k \leq n - 2$. This computation may be divided into 4 cases based on the parity of $n$ and $k$. If $n$ and $k$ are both even, then:

$$m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes k \otimes \downarrow v_1 \otimes (\downarrow w)^\otimes (n-2)-k) = (-1)^{v_1 + \lfloor \frac{k}{2} \rfloor + |w|} \downarrow m_n (v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k})$$

$$= (-1)^{0^{+}k-1} (-1)^{n-1} s_n \downarrow v_1$$

$$= (-1)^{\frac{n}{2}k-1} (-1)^{\frac{n+1}{2}(n+2)} \downarrow v_1$$

$$= (-1)^{\frac{n}{2}k-1} (-1)^{\frac{n+1}{2}(\frac{n}{2}+1)} \downarrow v_1 \ (*)$$

If $\frac{n}{2}$ is even, then $(*) = (-1)^{\text{odd}} (-1)^{\text{odd}^*\text{odd}} \downarrow v_1 = \downarrow v_1$ where ‘odd’ denotes an odd number.

If $\frac{n}{2}$ is odd, then $(*) = (-1)^{\text{even}} (-1)^{\text{odd}^*\text{even}} \downarrow v_1 = \downarrow v_1$ where ‘even’ denotes an even number.

A similar argument holds in the remaining 3 cases. Hence

$$m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes k \otimes \downarrow v_1 \otimes (\downarrow w)^\otimes (n-2)-k) = \downarrow v_1, \ 0 \leq k \leq n - 2$$

Now consider $m'_n (\downarrow v_1 \otimes (\downarrow w)^\otimes (n-2) \otimes \downarrow v_2)$. This computation may be divided into 2 cases based on the parity of $n$. If $n$ is even, then:
Lemma 2.4. Let $D = \sum_{k=1}^{\infty} m'_{k}$ where $m'_{k}$ is defined in Lemma 2.2. Let $n \geq 2$ be a positive integer. Suppose $D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_m) = 0 \forall x_i \in V$, $1 \leq m \leq n - 1$.

Then $D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j=n+1} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$

Proof. We first note that 

$$D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j \leq n+1} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

since $m'_{k}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_i) = 0$ for $k > l$. So

$$D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j \leq n} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) + \sum_{i+j = n+1} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

Hence it suffices to show that $\sum_{i+j \leq n} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0$

Consider $\sum_{i+j \leq n} m'_{i}m'_{j}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$: Since $i+j \leq n$, we can break this sum up into 4 different types of of elements in $(\downarrow V)^{\otimes k}$ based on whether the first and last terms in the tensor product contain $m'_{i}$ or $m'_{j}$:
• Type 1: Elements with first term $\downarrow x_1$ and last term $\downarrow x_n$
  (example: $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_1(\downarrow x_3) \otimes m'_2(\downarrow x_4 \otimes \downarrow x_5) \otimes \downarrow x_6$)
• Type 2: Elements with first term $\downarrow x_1$ and last term containing $m'_k$ for some $k$
  (example: $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_3(\downarrow x_3 \otimes \downarrow x_4 \otimes \downarrow x_5 \otimes \downarrow x_6)$)
• Type 3: Elements with first term containing $m'_k$ for some $k$ and last term $\downarrow x_n$
  (example: $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes m'_1(\downarrow x_3 \otimes \downarrow x_4 \otimes \downarrow x_5 \otimes \downarrow x_6)$)
• Type 4: Elements with first term containing $m'_k$ and last term containing $m'_l$ for some $k, l$
  (example: $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes \downarrow x_3 \otimes \downarrow x_4 \otimes m'_2(\downarrow x_5 \otimes \downarrow x_6)$)

Now each term of type 1 must be produced by $m'_i m'_j$ with $i + j \leq n - 1$. Hence, by factorization of tensor products, all possible terms of type 1 are given by:

\[
(\frac{-1}{2})^{2|x_1| - 2} \left( \downarrow x_1 \otimes \left( \sum_{i+j \leq n-1} m'_i m'_j(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1}) \right) \right) \otimes \downarrow x_n
= \left( \downarrow x_1 \otimes (D^2(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1})) \right) \otimes \downarrow x_n
= (\downarrow x_1 \otimes 0) \otimes \downarrow x_n
= 0
\]

since $D^2 = 0$ when evaluated on $n - 2$ terms. A similar argument holds for the type 2 and type 3 summands.

We now consider type 4 terms. Consider an arbitrary element of type 4:

\[
m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n)
\]

Consider how this arbitrary element is generated: We begin with

\[
m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)
\]

We then apply $m'_j$ to the last $j$ terms, which yields:

\[
(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)} m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n))
\]

Finally we apply $m'_i$ to the first $i$ terms:

\[
(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)} m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \downarrow x_n) \quad (*)
\]

Each of these arbitrary type 4 elements can be paired up with an element generated by $m'_j m'_i$ as follows: Begin with

\[
m'_j m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)
\]

Then apply $m'_i$ to the first $i$ terms:

\[
m'_i(m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_n)
\]
Finally, apply $m'_j$ to the last $j$ terms:

$$(-1)^{\sum_{i=1}^{n-j} x_i} m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_i \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n))$$

Since these type 4 elements were arbitrary, and $(\ast) + (\ast\ast) = 0$, all type 4 terms added together equal 0. Hence, all type 1, 2, 3, and 4 terms yield 0, and so

$$\sum_{i+j \leq n} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0$$

□

Proof of Theorem 2.1. It is clear that each map $m_n$ is of degree $2 - n$. To prove that these maps yield an $A_\infty$ structure, one may verify that they satisfy the identity given in definition 1.1. However, this is a rather daunting task, due to the varying signs, $s_n$, accompanying the $m_n$ maps. To utilize an alternative method of proof, we construct a degree 1 coderivation, $D$, as described in section 1.

In the context of Theorem 2.1, we may use the definition for $m'_k$ given by Lemma 2.2 to construct $D$. It then suffices to show that $D^2 = 0$.

We aim to prove $D^2 = 0$ by induction on the number of inputs for $D$. It is worth first noting that $D = \sum_{k=1}^\infty m'_k$, however $D (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = \sum_{k=1}^n m'_k (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ since $m'_k (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0$ for $k \geq n$.

For $n = 1$, we have $D^2(\downarrow x) = m'_1 m'_1 (\downarrow x) = m'_1^2 (x) = 0 \forall x \in V$.

Now assume $D^2(\downarrow x_1 \otimes \cdots \otimes \downarrow x_{n-1}) = 0$. We aim to show that $D^2(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0$:

**Remark 2.5.** Since $m'_i$ and $m'_j$ are linear, it is sufficient to show that $D^2 = 0$ on only basis elements.

By Lemma 2.4, $D^2(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j \geq n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$, hence it suffices to show that

$$\sum_{i+j = n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0, \forall x_1 \cdots x_n \in V.$$

It is advantageous to approach this problem from the bottom up, since $x_1 \cdots x_n \in V$ implies calculating $3^n$ different combinations of elements. That is, we consider only nontrivial (nonzero) elements in the sum $\sum_{i+j = n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$. Now since $i + j = n + 1$, we observe that $m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) \in (\downarrow V)^{\otimes 1}$. Since, by definition, $m'_i$ cannot produce
the element $\downarrow v_2$, the seemingly large task of considering nontrivial $m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ yields only two possibilities:

$$m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = c \downarrow v_1$$

or

$$m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = c \downarrow w$$

for some constant, $c$. Therefore if $m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) \neq 0$ for some $i + j = n + 1$, then $\sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ is a sum of $\downarrow v_1$’s or $\downarrow w$’s.

We first consider the manner in which $m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ yields a $\downarrow w$:

By definition of $m'_i$, $\downarrow w$ must be produced by $m'_i (\downarrow v_1 \otimes (\downarrow w)^{\otimes (i-1)}) \ast$. To accomplish this, the original arrangement $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$ must satisfy $x_1 = v_1$ and must contain exactly one more ‘v’ ($v = v_1$ or $v_2$).

1. **Case 1:** $v = v_1$. Let us consider $m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k})$, $0 \leq k \leq n - 2$. Now, to produce $\ast$, $m'_j$ must ‘catch’ (1) both $\downarrow v_1$’s, or (2) only the second $\downarrow v_1$.

   (1) We have $m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) = \downarrow v_1$, $k + 2 \leq j \leq n$.

   This yields $m'_i (\downarrow v_1 \otimes (\downarrow w)^{\otimes (n-j)} ) = \downarrow w$. Now since $k + 2 \leq j \leq n$, there are $n - (k + 2) + 1 = n - k - 1$ such terms in $\sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k})$.

   (2) We have $(-1)^{|v_1|+|k|+|w|-k|+1}) m'_i (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes [m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes (j-1)})] \otimes (\downarrow w)^{\otimes (n-2)-k-(j-1)}) = -\downarrow w$, $1 \leq j \leq n - k - 1$. Similarly, there are $(n - k - 1) - 1 + 1 = n - k - 1$ such terms in $\sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k})$.

   \[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) = (n - k - 1) \downarrow w - (n - k - 1) \downarrow w = 0. \]

2. **Case 2:** $v = v_2$. Let us consider $m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes (n-2)-k})$, $0 \leq k \leq n - 2$. Similarly, to produce $\ast$, $m'_j$ must ‘catch’ (1) both $\downarrow v_1$ and $\downarrow v_2$, or (2) only $\downarrow v_2$.

For (1), the only nontrivial way to do this yields:

$$m'_{n-k-1}(m'_{k+2} (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2) \otimes (\downarrow w)^{\otimes (n-2)-k}) = \downarrow w$$
and for (2), the only nontrivial way to do this yields:

\[
(-1)^{|n_1|+|w|-(k+1)} m_n'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes m_1'(\downarrow v_2) \otimes (\downarrow w)^{(n-2)-k}) = -\downarrow w
\]

\[
\Rightarrow \sum_{i+j=n+1} m_i'm_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(n-2)-k}) = \downarrow w - \downarrow w = 0.
\]

In either case, if \(m_i'm_j'(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)\) produces \(\downarrow w\)'s, then

\[
\sum_{i+j=n+1} m_i'm_j'(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0.
\]

We now consider the manner in which \(m_i'm_j'(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)\) yields a \(\downarrow v_1\):

By definition of \(m_n'\), \(\downarrow v_1\) must be produced by either \(m_i'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(i-2)-k})\) or \(m_i'(\downarrow v_1 \otimes (\downarrow w)^{(i-2)} \otimes \downarrow v_2)\).

- **Case 1:** \(\downarrow v_1\) is produced by \(m_i'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(i-2)-k})\).

We examine the 4 different possibilities for which \(m_i'\) can yield this arrangement:

1. \(m_i'\) produces the first \(\downarrow v_1\).  
2. \(m_i'\) produces a \(\downarrow w\) in \((\downarrow w)^{\otimes k}\).  
3. \(m_i'\) produces the second \(\downarrow v_1\).  
4. \(m_i'\) produces a \(\downarrow w\) in \((\downarrow w)^{(i-2)-k}\).

A key observation to make here is that (i), (ii), (iii), and (iv) imply that the original arrangement \(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n\) must contain exactly 3 \(v\)'s, once again with \(x_1 = v_1\). This yields 4 subcases:

- **Subcase 1:** We have \(m_i'm_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(n-k-l-3)}\):
- **Subcase 2:** We have \(m_i'm_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(n-k-l-3)}\):
- **Subcase 3:** We have \(m_i'm_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(n-k-l-3)}\):
- **Subcase 4:** We have \(m_i'm_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{(n-k-l-3)}\):

Let us consider subcase 1:

1. \(m_i'\) must take the first two \(\downarrow v_1\)'s. We have:

\[
m_i'(\left[ m_j'(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{(j-2)-k}) \otimes (\downarrow w)^{(l-(j-k-2)} \right] \otimes \downarrow v_1 \otimes (\downarrow w)^{(n-k-l-3)}) = \downarrow v_1
\]

Now \(k + 2 \leq j \leq l + k + 2\), so there are \((l + k + 2) - (k + 2) + 1 = l + 1\) such terms.
(ii) \( m'_j \) must take only the second \( \downarrow v_1 \). We have:
\[
(1)^{|k| + j|w| - (k+1)}m'_j\left( \downarrow v_1 \otimes \downarrow w \right)^{\otimes k} \left[ m'_j\left( \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{(j-1)} \right)^{\otimes (j-1)} \right) \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{n-k-l-3} \right) \right] = - \downarrow v_1
\]
Now \( 1 \leq j \leq l+1 \), so there are \( (l+1) - 1 + 1 = l + 1 \) such terms.

(iii) \( m'_j \) must take the second and third \( \downarrow v_1 \)'s. We have:
\[
(1)^{|k| + j|w| - (k+1)}m'_i\left( \downarrow v_1 \otimes \downarrow w \right)^{\otimes k} \left[ m'_i\left( \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{(j-2)} \right)^{\otimes (j-2)} \right) \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{n-k-l-1} \right) \right] = - \downarrow v_1
\]
Now \( l + 2 \leq j \leq n - k - 1 \), so there are \( (n - k - 1) - (l + 2) + 1 = n - k - l - 2 \) such terms.

(iv) \( m'_j \) must take only the third \( \downarrow v_1 \). We have:
\[
(1)^{|k| + j|w| - (k+1)}m'_i\left( \downarrow v_1 \otimes \downarrow w \right)^{\otimes k} \left[ m'_i\left( \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{(j-1)} \right)^{\otimes (j-1)} \right) \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{n-k-l-2} \right) \right] = \downarrow v_1
\]
Now \( 1 \leq j \leq n - k - l - 2 \), so there are \( (n - k - l - 2) - 1 + 1 = n - k - l - 2 \) such terms.

\[
\Rightarrow \sum_{i+j=n+1} m'_i m'_j\left( \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes k} \otimes \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes (j-1)} \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{n-k-l-3} \right) \right) = (l+1) \downarrow v_1 - (l+1) \downarrow v_1
\]

A similar argument holds for subcases 2, 3, and 4. Hence, our result holds for case 1.

- **Case 2:** \( \downarrow v_1 \) is produced by \( m'_i\left( \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes (i-2)} \otimes \downarrow v_2 \right) \).

We examine the 2 different possibilities for which \( m'_j \) can yield this arrangement:

(i) \( m'_j \) produces the \( \downarrow v_1 \).

(ii) \( m'_j \) produces a \( \downarrow w \) in \( \left( \downarrow w \right)^{(i-2)} \).

A similar observation to case 1 can be made here regarding the original arrangement \( \downarrow x_1 \otimes \cdots \otimes \downarrow x_n \) containing exactly 3 \( v \)'s, once again with \( x_1 = v_1 \). In this case, \( x_n = v_2 \). This yields 2 subcases:

- **Subcase 1:** We have \( m'_i m'_j\left( \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes k} \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{(n-k-3)} \otimes \downarrow v_2 \right) \)

- **Subcase 2:** We have \( m'_i m'_j\left( \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes k} \otimes \downarrow v_2 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{(n-k-3)} \otimes \downarrow v_2 \right) \)

Let us consider subcase 1:

(i) \( m'_j \) must take both \( \downarrow v_1 \)'s. We have:
\[
m'_i\left[ m'_i\left( \downarrow v_1 \otimes \left( \downarrow w \right)^{\otimes k} \otimes \downarrow v_1 \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{j-k-2} \right) \right) \right] \otimes \left( \downarrow w \otimes \left( \downarrow w \otimes \left( \downarrow w \right)^{n-j-1} \otimes \downarrow v_2 \right) \right) = \downarrow v_1
\]
Now \( k + 2 \leq j \leq n - 1 \), so there are \((n - 1) - (k + 2) + 1 = n - k - 2\) such terms.

(ii) \( m'_j \) must take the second \( \downarrow v_1 \) only. We have:

\[
(-1)^{|v_1|+|w|-(k+1)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes j-1}) \otimes (\downarrow w)^{\otimes n-k-j-2} \right) \otimes \downarrow v_2 = - \downarrow v_1
\]

Now \( 1 \leq j \leq n - k - 2 \), so there are \((n - k - 2) - (1) + 1 = n - k - 2\) such terms. This implies that

\[
\sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-k-3)} \otimes \downarrow v_2) = (n - k - 2) \downarrow v_1 - (n - k - 2) \downarrow v_1 = 0.
\]

A similar argument may be made for subcase 2. Hence, our result holds for case 2.

So \( \sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0, \forall x_1 \cdots x_n \in V. \)

Thus \( D^2(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0 \)

By induction, \( D^2 = 0 \) on any number of inputs.

Hence the preceding maps \( m_n \) defined on the graded vector space \( V \) form an \( A_{\infty} \) algebra structure. \( \square \)

3. Induced \( L_\infty \) Algebra

The \( A_{\infty} \) algebra structure on \( V = V_0 \oplus V_1 \) that was constructed in this note can be skew symmetrized to yield an \( L_\infty \) algebra structure on \( V \); see Theorem 3.1 in [5] for details. This \( L_\infty \) algebra will thus join the collection of previously defined such structures on \( V \). The relationship among these algebras will be a topic for future research.

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