Adiabatic Limit and Deformations of Complex Structures

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Abstract. Based on our recent adaptation of the adiabatic limit construction to the case of complex structures, we give a new proof of the fact, that we first proved in 2009 and 2010, that the deformation limiting manifold of any holomorphic family of Moishezon manifolds is Moishezon. Two new ingredients, hopefully of independent interest, are introduced. The first one canonically associates with every compact complex manifold \( X \), in every degree \( k \), a holomorphic vector bundle over \( \mathbb{C} \) of rank equal to the \( k \)-th Betti number of \( X \). This vector bundle shows that the degenerating page of the Fröhlicher spectral sequence of \( X \) is the holomorphic limit, as \( h \in \mathbb{C}^* \) tends to 0, of the \( d_h \)-cohomology of \( X \), where \( d_h = h\partial + \bar{\partial} \). A relative version of this vector bundle is canonically associated with every holomorphic family of compact complex manifolds. The second new ingredient is a relaxation of the notion of strongly Gauduchon (sG) metric that we introduced in 2009. For a given positive integer \( r \), a Gauduchon metric \( \gamma \) on an \( n \)-dimensional compact complex manifold \( X \) is said to be \( E_r \)-sG if \( \partial \gamma^{n-1} \) represents the zero cohomology class on the \( r \)-th page of the Fröhlicher spectral sequence of \( X \). Strongly Gauduchon metrics coincide with \( E_1 \)-sG metrics.

1 Introduction

The main result of this paper is the following statement that was first proved in [Pop09] and [Pop10] in a different, ad hoc way, although the general strategy and some details were the same as in the present, more conceptual, approach.

Theorem 1.1. Let \( \pi : \mathcal{X} \to B \) be a complex analytic family of compact complex manifolds over an open ball \( B \subset \mathbb{C}^N \) about the origin such that the fibre \( X_t := \pi^{-1}(t) \) is a Moishezon manifold for every \( t \in B \setminus \{0\} \). Then \( X_0 := \pi^{-1}(0) \) is again a Moishezon manifold.

As usual, by a complex analytic (or holomorphic) family of compact complex manifolds we mean a proper holomorphic submersion \( \pi : \mathcal{X} \to B \) between two complex manifolds \( \mathcal{X} \) and \( B \) (cf. e.g. [Kod86]). In particular, the fibres \( X_t := \pi^{-1}(t) \) are compact complex manifolds of the same dimension. By a classical theorem of Ehresmann [Ehr47], any such family is locally (hence also globally if the base \( B \) is contractible) \( C^\infty \) trivial. Thus, all the fibres \( X_t \) have the same underlying \( C^\infty \) manifold \( X \) (hence also the same De Rham cohomology groups \( H^k_{DR}(X, \mathbb{C}) \) for all \( k = 0, \ldots, 2n \)), but the complex structure \( J_t \) of \( X_t \) depends, in general, on \( t \in B \).

On the other hand, as usual, by a Moishezon manifold we mean a compact complex manifold \( Y \) for which there exists a projective manifold \( \tilde{Y} \) and a holomorphic bimeromorphic map \( \mu : \tilde{Y} \to Y \) (cf. [Moi67]). By another classical result of [Moi67], we know that a Moishezon manifold is not Kähler unless it is projective.

Our Theorem 1.1 above is a closedness result under deformations of complex structures: any deformation limit of a family of Moishezon manifolds is Moishezon. Indeed, the fibre \( X_0 \) can be regarded as the limit of the fibres \( X_t \) when \( t \in B \) tends to 0 \( \in B \). We can, of course, suppose that \( B \) is an open disc about the origin in \( \mathbb{C} \).
1.1 Brief reminder of the main construction in [Pop17]

The method introduced in this paper originates in our recent adaptation to the case of complex structures (cf. [Pop17]) of the adiabatic limit construction associated with Riemannian foliations (cf., e.g., [Wi85] and [MM90]). Given a compact complex n-dimensional manifold \( X \), for every constant \( h \in \mathbb{C} \), we associate with the splitting \( d = \partial + \bar{\partial} \) defining the complex structure of \( X \) the following 1st-order differential operator:

\[
d_h := h\partial + \bar{\partial} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_{k+1}(X, \mathbb{C}), \quad k = 0, \ldots, 2n,
\]

acting on the space \( C^\infty_k(X, \mathbb{C}) \) of smooth \( k \)-forms on \( X \), for every degree \( k \). Only positive real constants \( h \) were considered in [Pop17], but we now allow \( h \) to be any complex constant. In particular, \( d_h \) depends on the complex structure of \( X \), except when \( h = 1 \), in which case \( d_1 = d \). On the other hand, \( d_0 = \bar{\partial} \).

Meanwhile, for every non-zero \( h \), the linear map defined pointwise on \( k \)-forms by

\[
\theta_h : \Lambda^k T^* X \rightarrow \Lambda^k T^* X, \quad u = \sum_{p+q=k} u^{p,q} \mapsto \theta_h u := \sum_{p+q=k} h^p u^{p,q},
\]

induces an automorphism of the vector bundle \( \Lambda T^* X = \bigoplus_{k=0}^{2n} \Lambda^k T^* X \) and the operators \( d_h \) and \( d \) are related by the identity

\[
d_h = \theta_h d \theta_h^{-1}.
\]

This implies that \( d_h^2 = 0 \), so we can define the \( d_h \)-cohomology of \( X \) (cf. [Pop17]) in every degree \( k \) as

\[
H^k_{d_h}(X, \mathbb{C}) := \ker(d_h : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_{k+1}(X, \mathbb{C}))/\text{Im} (d_h : C^\infty_{k-1}(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})).
\]

Moreover, \( \theta_h \) maps \( d \)-closed forms to \( d_h \)-closed forms and \( d \)-exact forms to \( d_h \)-exact forms, so it induces an isomorphism between the De Rham cohomology and the \( d_h \)-cohomology for every \( h \in \mathbb{C} \setminus \{0\} \):

\[
\theta_h : H^k_{DR}(X, \mathbb{C}) \overset{\simeq}{\longrightarrow} H^k_{d_h}(X, \mathbb{C}), \quad k = 0, \ldots, 2n.
\]

Now, if \( X \) is given a Hermitian metric \( \omega \), we let \( d_h^* \) be the formal adjoint of \( d_h \) w.r.t. the \( L^2 \)-inner product on differential forms induced by \( \omega \). The \( d_h \)-Laplacian w.r.t. \( \omega \) is defined in every degree \( k \) in the expected way:

\[
\Delta_h : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}), \quad \Delta_h := d_h d_h^* + d_h^* d_h.
\]

It turns out that the (non-negative, self-adjoint) 2nd-order differential operator \( \Delta_h \) is elliptic (cf. [Pop17, Lemma 2.7]). Together with the integrability of \( d_h \) (i.e. \( \lambda_2^2 = 0 \)) and the compactness of \( X \), this implies the Hodge isomorphism

\[
\ker(\Delta_h : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})) \simeq H^k_{d_h}(X, \mathbb{C}), \quad k = 0, \ldots, 2n,
\]

for the \( d_h \)-cohomology. By elliptic theory, \( \Delta_h \) has a discrete spectrum \( 0 \leq \lambda_1^{(k)}(h) \leq \lambda_2^{(k)}(h) \leq \cdots \leq \lambda_j^{(k)}(h) \leq \cdots \) whose only accumulation point is \( +\infty \). Moreover, for every \( h \neq 0 \), the isomorphism between the \( d_h \)- and the De Rham cohomologies implies that the multiplicity of 0 as an eigenvalue of \( \Delta_h \) acting on \( k \)-forms is the \( k \)-th Betti number \( b_k \) of \( X \).
1.2 Constructions introduced in this paper

There are two new main ideas that we hope are of independent interest and that we now outline. The details will occupy sections 2, respectively 3.

(I) The first construction (cf. §2.3.1 and §2.3.2) builds on the adiabatic limit for complex structures introduced in [Pop17] and outlined above to prove that the degenerating page of the Frölicher spectral sequence is the holomorphic limit, as $h \in \mathbb{C}$ tends to 0, of the $d_h$-cohomology in every degree $k$. Specifically, with every compact complex $n$-dimensional manifold $X$ and every degree $k \in \{0, \ldots, 2n\}$, we canonically associate a holomorphic vector bundle $A^k$ of rank $b_k$ (= the $k$-th Betti number of $X$) over $\mathbb{C}$ whose fibres are defined as

$$A^k_h := H^k_{d_h}(X, \mathbb{C}) \quad \text{if } h \in \mathbb{C} \setminus \{0\}, \quad \text{and} \quad A^0_0 := \bigoplus_{p+q=k} E^{p,q}_r(X) \quad \text{if } h = 0,$$

where $r \geq 1$ is the smallest positive integer such that the Frölicher spectral sequence of $X$ degenerates at $E_r$. The vector bundle structure over $\mathbb{C} \setminus \{0\}$ is defined to be the one induced by the isomorphisms $\theta_h : H^k_{dR}(X, \mathbb{C}) \to H^k_{d_h}(X, \mathbb{C})$, with $h \neq 0$, from the local system $H^k \to \mathbb{C} \setminus \{0\}$ of fibre $H^k_{dR}(X, \mathbb{C})$.

That the resulting holomorphic vector bundle $A^k \to \mathbb{C} \setminus \{0\}$ extends to a holomorphic vector bundle over $\mathbb{C}$ whose fibre at $h = 0$ is the vector space $A^k_0$ defined above, is asserted by Corollary and Definition 2.8. It can be loosely reworded as

**Theorem 1.2.** For every $k \in \{0, \ldots, 2n\}$, $A^k \to \mathbb{C}$ is a holomorphic vector bundle of rank $b_k$.

We call $A^k$ the Frölicher approximating vector bundle of $X$ in degree $k$. Once we have proved that $A^k \to \mathbb{C}$ is a $C^\infty$ vector bundle, it follows at once that it is actually holomorphic on $\mathbb{C}$ since, thanks to the maps $\theta_h : H^k_{dR}(X, \mathbb{C}) \to H^k_{d_h}(X, \mathbb{C})$ varying in a holomorphic way with $h \in \mathbb{C}^*$, $A^k$ is holomorphic on $\mathbb{C}^*$, hence also on $\mathbb{C}$ where it is already $C^\infty$.

However, the proof of the fact that $A^k$ is indeed a $C^\infty$ vector bundle on $\mathbb{C}$ is technically involved. To this end, we fix an arbitrary Hermitian metric $\omega$ on $X$ and construct a $C^\infty$ family $(\Delta^{(r)}_h)_{h \in \mathbb{C}}$ of elliptic pseudo-differential operators whose kernels are isomorphic to the $d_h$-cohomology group $H^k_{d_h}(X, \mathbb{C})$ for every $h \in \mathbb{C}^*$ and to $\bigoplus_{p+q=k} E^{p,q}_r(X)$ when $h = 0$. In other words, the kernels are isomorphic to the fibres of $A^k$ for all $h \in \mathbb{C}$.

When the Frölicher spectral sequence of $X$ degenerates at $E_1$ (i.e. when $r = 1$), there is nothing new about this construction: $\Delta^{(1)}_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ is even a differential operator for every $h \in \mathbb{C}$ and $\Delta^{(1)}_0$ is the classical $\partial\bar{\partial}$-Laplacian $\Delta'' = \partial\bar{\partial}^* + \bar{\partial}^*\partial$, while for $h \neq 0$, $\Delta^{(1)}_h$ is the $d_h$-Laplacian $\Delta_h$ introduced in [Pop17] and recalled above in §3.1.1. This case occurs if, for example, $X$ is Kähler or merely a $\partial\bar{\partial}$-manifold (in the sense that the $\partial\bar{\partial}$-lemma holds on $X$, see definition reminder below).

When the Frölicher spectral sequence of $X$ first degenerates at $E_2$ (i.e. when $r = 2$), the pseudo-differential operator $\Delta^{(2)}_0 : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ is the one introduced in [Pop16] as

$$\tilde{\Delta} = \partial p'' \partial^* + \partial^* \partial p'' \partial + \Delta''$$

where $p'' : C^\infty_k(X, \mathbb{C}) \to \ker(\Delta'' : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}))$ is the orthogonal projection (w.r.t. the $L^2$ inner product induced by $\omega$) onto $\Delta''$-harmonic forms. We know from [Pop16] that $\ker \tilde{\Delta}$ is isomorphic to $\bigoplus_{p+q=k} E^{p,q}_r(X)$. For $h \in \mathbb{C}^*$, we construct in §2.1 the pseudo-differential operators $\tilde{\Delta}^{(2)}_h = \tilde{\Delta}_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ as $C^\infty$ deformations of $\tilde{\Delta}$ by adding to each factor of $\tilde{\Delta}$ an $h$-dependence $h\tilde{\Delta}^{(2)}_h$.
multiple of its conjugate. We then prove in Lemma 2.2 that \( \ker \tilde{\Delta}_h^{(2)} = \ker \Delta_h \) for every \( h \neq 0 \), so in particular \( \ker \tilde{\Delta}_h^{(2)} \) is isomorphic to the \( d_n \)-cohomology group \( H^k_{d_n}(X, \mathbb{C}) \).

When the Frölicher spectral sequence of \( X \) first degenerates at \( E_r \) for some \( r \geq 3 \), we borrow from our ongoing joint work [PU18] with L. Ugarte the construction of the pseudo-differential operator \( \tilde{\Delta}_h^{(r)} = \tilde{\Delta}^{(r)} : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}) \) whose kernel is isomorphic to \( \oplus_{p+q=k} E^{p,q}_r(X) \). This is a Hodge isomorphism for an arbitrary page \( E_r \), with \( r \geq 3 \), of the Frölicher spectral sequence and the construction is explained in the former part of §2.2. In the latter part of §2.2, we construct the pseudo-differential operators \( \tilde{\Delta}_h^{(r)} : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}) \) as \( C^\infty \) deformations of \( \tilde{\Delta}^{(r)} \) by adding to each factor of \( \tilde{\Delta}^{(r)} \) an \( h \)-multiple of its conjugate (cf. Definition 2.4). Finally, we prove in Lemma 2.5 that \( \ker \tilde{\Delta}_h^{(r)} = \ker \Delta_h \) (hence \( \ker \tilde{\Delta}_h^{(r)} \simeq H^k_{d_n}(X, \mathbb{C}) \)) for every \( h \neq 0 \).

This absolute version of the Frölicher approximating vector bundle has a relative counterpart. Indeed, in §2.3.2, with every holomorphic family \( \pi : \mathcal{X} \to B \) of compact complex \( n \)-dimensional manifolds \( X_t := \pi^{-1}(t) \) over an open ball \( B \subset \mathbb{C}^N \) about the origin such that, for some \( r \in \mathbb{N}^* \), the Frölicher spectral sequence of \( X_t \) degenerates at least at \( E_r \) for all \( t \in B \), we associate a Frölicher approximating vector bundle \( \mathcal{A}^k \to \mathbb{C} \times B \) in every degree \( k \in \{0, \ldots, 2n\} \) as follows.

As usual, we let \( X \) stand for the \( C^\infty \) manifold that underlies the fibres \( X_t \). The operator \( d_{h,t} := h \partial_t + \partial_t : C^\infty_k(X, \mathbb{C}) \to C^\infty_{k+1}(X, \mathbb{C}) \) depends now on both \( h \in \mathbb{C} \) and \( t \in B \) (because it depends on the complex structure \( J_t \) of \( X_t \)) and so does \( \theta_{h,t} : \Lambda^k T^* X \to \Lambda^k T^* X \) acting as \( \theta_{h,t}(\sum_{p+q=k} u^{p,q}_t) := \sum_{p+q=k} h^p u^{p,q}_t \), where the \( u^{p,q}_t \) are the \( (p, q) \)-type components of a given \( k \)-form \( u \) a \( \Lambda^k \) extends to a holomorphic vector bundle \( \tilde{\mathcal{A}}^k \to \mathbb{C} \times B \) for every \( k \) as follows.

We define the fibres of the Frölicher approximating vector bundle over \( \mathbb{C} \times B \) of the family \( (X_t)_{t \in B} \) in degree \( k \) as

\[ \mathcal{A}^k_{h,t} := H^k_{d_n}(X_t, \mathbb{C}) \quad \text{if} \quad (h, t) \in \mathbb{C}^* \times B, \quad \text{and} \quad \mathcal{A}^k_{h,t} := \bigoplus_{p+q=k} E^{p,q}_r(X_t) \quad \text{for} \quad (0, t) \in \{0\} \times B. \]

The vector bundle structure over \( \mathbb{C}^* \times B \) is defined to be the one induced by the cohomology isomorphisms \( \theta_{h,t} : H^k_{d_n}(X_t, \mathbb{C}) \to H^k_{d_n}(X_t, \mathbb{C}) \), with \( (h, t) \in \mathbb{C}^* \times B \), from the local system \( \mathcal{H}^k \to \mathbb{C}^* \times B \) of fibre \( H^k_{d_n}(X, \mathbb{C}) \) (identified with \( H^k_{d_n}(X_t, \mathbb{C}) \) for every \( t \in B \)).

That the resulting holomorphic vector bundle \( \tilde{\mathcal{A}}^k \to \mathbb{C}^* \times B \) extends to a holomorphic vector bundle over \( \mathbb{C} \times B \) whose fibre at every point \( (0, t) \in \{0\} \times B \) is the vector space \( \tilde{\mathcal{A}}^k_{0,t} \) defined above, is asserted by Corollary and Definition 2.14. It can be loosely reworded as

**Theorem 1.3.** For every \( k \in \{0, \ldots, 2n\} \), \( \mathcal{A}^k \to \mathbb{C} \times B \) is a holomorphic vector bundle of rank \( b_k \).

By \( b_k \) we mean the \( k \)-th Betti number of the fibres \( X_t \), or equivalently, of the \( C^\infty \) manifold \( X \) underlying them. The proof of Theorem 1.3 uses the absolute case proved in Theorem 1.2.

(II) The second main idea introduced in this paper is a relaxation (cf. Definition 3.2) of the notion of strongly Gauduchon (sG) metric introduced in [Pop09] and [Pop13]. Starting from the observation that, for every Gauduchon metric \( \gamma \) on a given compact complex \( n \)-dimensional manifold \( X \), the \( (n, n-1) \)-form \( \partial \gamma^{n-1} \) is \( E_r \)-closed (i.e. represents an \( E_r \)-cohomology class on the \( r \)-th page of the Frölicher spectral sequence of \( X \)) for every \( r \in \mathbb{N}^* \), we call \( \gamma \) an \( E_r \)-sG metric if \( \partial \gamma^{n-1} \) is \( E_r \)-exact (i.e. represents the zero \( E_r \)-cohomology class on the \( r \)-th page of the Frölicher spectral sequence of \( X \)). Any \( X \) that carries an \( E_r \)-sG metric is called an \( E_r \)-sG manifold.
For the reader’s convenience, we recall in Proposition 3.1 how the $E_r$-closedness and $E_r$-exactness conditions translate into explicit terms. For every $r \in \mathbb{N}^*$, the $E_r$-sG condition implies the $E_{r+1}$-sG condition, while the strongest of them, the $E_1$-sG condition, is equivalent to the sG condition.

The two main constructions of this paper are brought together in the following result (see Theorem 3.4 for a more precise statement).

**Theorem 1.4.** If in a holomorphic family $(X_t)_{t \in B}$ of compact complex manifolds all the fibres $X_t$ with $t \in B \setminus \{0\}$ are $\partial \bar{\partial}$-manifolds, then the limiting fibre $X_0$ is an $E_r$-sG manifold, where $E_r$ is the first page at which the Frölicher spectral sequence of $X_0$ degenerates.

Recall that a $\partial \bar{\partial}$-manifold is, by definition, a compact complex manifold $X$ that satisfies the $\partial \bar{\partial}$-lemma in the following sense:

for every $C^\infty$ $d$-closed pure-type form $u$ on $X$, the following exactness conditions are equivalent:

$$
 u \in \text{Im} \ d \iff u \in \text{Im} \partial \iff u \in \text{Im} \bar{\partial} \iff u \in \text{Im} \partial \bar{\partial}.
$$

The $\partial \bar{\partial}$-property is equivalent to all the canonical linear maps $H^p_{BC}(X, \mathbb{C}) \to H_A^p(X, \mathbb{C})$, from the Bott-Chern to the Aeppli cohomology, being isomorphisms. Since both of these cohomologies can be computed using either smooth forms or currents, the $\partial \bar{\partial}$-property is also equivalent to the equivalences (1) holding for every $d$-closed pure-type current on $X$.

A standard result in Hodge theory asserts that every compact Kähler manifold is a $\partial \bar{\partial}$-manifold. Moreover, every class $\mathcal{C}$ manifold (by definition, these are the compact complex manifolds that are bimeromorphically equivalent to compact Kähler manifolds), hence also every Moishezon manifold, is a $\partial \bar{\partial}$-manifold, but the class of $\partial \bar{\partial}$-manifolds strictly contains the class $\mathcal{C}$. (See, e.g., [Pop14] for further details.)

A stronger result than Theorem 1.4 was proved in Proposition 4.1 of [Pop09]: any deformation limit of $\partial \bar{\partial}$-manifolds is a strongly Gauduchon (sG) manifold. In the present paper, we use our Frölicher approximating vector bundle of Corollary and Definition 2.14 to obtain the possibly weaker $E_r$-sG conclusion on the limiting fibre under the same assumption on the other fibres. However, we will see that this weaker conclusion on the deformation limits of $\partial \bar{\partial}$-manifolds will yield the same optimal conclusion, captured in Theorem 1.1, on the deformation limits of projective and Moishezon manifolds as the one obtained in [Pop09] and [Pop10]. Moreover, the new method introduced in the present paper has the advantage of being more conceptual than the ad hoc arguments of [Pop09]. It effectively puts those arguments on a more theoretical footing via the machinery of the Frölicher spectral sequence and our new Frölicher approximating vector bundle.

Besides Theorem 1.4, the other main building block (cf. Theorem 3.7) of the proof of Theorem 1.1 is the use of a $C^\infty$ family $(\gamma_t)_{t \in B}$ of $E_r$-sG metrics on the fibres $(X_t)_{t \in B}$, whose existence is mainly guaranteed by Theorem 1.4, to uniformly control the volumes of the relative (i.e. contained in the fibres) divisors that form an irreducible component of the relative Barlet space of divisors (cf. [Bar75]) associated with the family $(X_t)_{t \in B}$. Finitely many integrations by parts are used.

## 2 h-theory for the Frölicher spectral sequence

Recall that $(\Delta_h)_{h \in \mathbb{C}}$ is a $C^\infty$ family of elliptic differential operators such that $\Delta_0 = \Delta''$. So, the $\Delta_h$’s can be regarded as an approximation (allowing for more flexibility) of the standard $\bar{\partial}$-Laplacian $\Delta''$. 

2.1 Second page: the pseudo-differential Laplacians $\tilde{\Delta}_h$

We will now introduce and study a similar approximation of the pseudo-differential Laplacian

$$\tilde{\Delta} = \partial\partial^{\ast} + \partial^{\ast}\partial + \Delta'' : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C}), \quad p, q = 0, \ldots, n,$$

introduced in [Pop16] and proved there to define a Hodge theory for the second page of the Frölicher spectral sequence, namely a Hodge isomorphism

$$\mathcal{H}^p_q(X, \mathbb{C}) := \ker(\tilde{\Delta} : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})) \simeq E^p_{2,q}(X)$$

in every bidegree $(p, q)$. Note that $\tilde{\Delta} = (\partial\partial')(\partial\partial')^{\ast} + (\partial\partial')^{\ast}(\partial\partial') + \Delta''$, so we will approximate $\partial\partial''$ and $\partial\partial'$ by adding to each a small $h$-multiple of its conjugate, while still approximating $\Delta''$ by $\Delta_h$.

**Definition 2.1.** Let $(X, \omega)$ be a compact complex Hermitian manifold with dim$_\mathbb{C}X = n$. For every $h \in \mathbb{C}$ and every $k = 0, \ldots, 2n$, we let

$$\tilde{\Delta}_h = (\partial\partial'' + h\partial\partial')(\partial\partial'' + h\partial\partial')^{\ast} + (\partial\partial'' + h\partial\partial')^{\ast}(\partial\partial'' + h\partial\partial') + \Delta_h : C_k^\infty(X, \mathbb{C}) \to C_k^\infty(X, \mathbb{C}),$$

where $p' = p'_\omega : C_{p,q}^\infty(X, \mathbb{C}) \to \ker(\tilde{\Delta} : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})) = \mathcal{H}^p_{\Delta''}(X, \mathbb{C})$ and $p'' = p''_\omega : C_{p,q}^\infty(X, \mathbb{C}) \to \ker(\Delta'' : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})) = \mathcal{H}_{\Delta''}^p(X, \mathbb{C})$ are the orthogonal projections onto the $\Delta'$-, resp. $\Delta'$-harmonic spaces of any fixed bidegree $(p, q)$. These projections are then extended by linearity to

$$p' = p'_\omega : C_k^\infty(X, \mathbb{C}) \to \mathcal{H}^k_{\Delta'}(X, \mathbb{C}), \quad p'' = p''_\omega : C_k^\infty(X, \mathbb{C}) \to \mathcal{H}^k_{\Delta''}(X, \mathbb{C}),$$

where $\mathcal{H}^k_{\Delta'}(X, \mathbb{C}) := \oplus_{p+q=k} \mathcal{H}^p_{\Delta''}(X, \mathbb{C})$ and $\mathcal{H}^k_{\Delta''}(X, \mathbb{C}) := \oplus_{p+q=k} \mathcal{H}^p_{\Delta''}(X, \mathbb{C})$.

For every $h \in \mathbb{C}$, $\tilde{\Delta}_h$ is a non-negative, self-adjoint pseudo-differential operator and $\tilde{\Delta}_0 = \tilde{\Delta}$. Further properties include the following.

**Lemma 2.2.** For every $h \in \mathbb{C} \setminus \{0\}$, $\tilde{\Delta}_h$ is an elliptic pseudo-differential operator whose kernel is

$$\ker \tilde{\Delta}_h = \ker(\partial\partial'' + h\partial\partial')^{\ast} \cap \ker((\partial\partial' + h\partial\partial')^{\ast}(\partial\partial' + h\partial\partial')) \cap \ker d_h \cap \ker d_h^\ast = \ker \Delta_h, \quad k = 0, \ldots, 2n. \quad (2)$$

Hence, the 3-space orthogonal decompositions induced by $\tilde{\Delta}_h$ and $\Delta_h$ coincide when $h \in \mathbb{C} \setminus \{0\}$:

$$C_k^\infty(X, \mathbb{C}) = \ker \tilde{\Delta}_h \oplus \text{Im} d_h \oplus \text{Im} d_h^\ast, \quad k = 0, \ldots, 2n, \quad (3)$$

where $\ker d_h = \ker \tilde{\Delta}_h \oplus \text{Im} d_h$, $\ker d_h^\ast = \ker \tilde{\Delta}_h \oplus \text{Im} d_h^\ast$, and $\text{Im} \tilde{\Delta}_h = \text{Im} d_h \oplus \text{Im} d_h^\ast$.

Consequently, we have the Hodge isomorphism:

$$\mathcal{H}^k_{\Delta_h}(X, \mathbb{C}) = \mathcal{H}^k_{\tilde{\Delta}_h}(X, \mathbb{C}) \simeq H^k_{d_h}(X, \mathbb{C}), \quad k = 0, \ldots, 2n, \quad h \in \mathbb{C} \setminus \{0\}. \quad (4)$$

Moreover, the decomposition (3) is stable under $\tilde{\Delta}_h$, namely

$$\tilde{\Delta}_h(\text{Im} d_h) \subset \text{Im} d_h \quad \text{and} \quad \tilde{\Delta}_h(\text{Im} d_h^\ast) \subset \text{Im} d_h^\ast. \quad (5)$$
Proof. The first identity in (2) follows immediately from the fact that $\tilde{\Delta}_h$ is a sum of non-negative operators of the shape $A^\ast A$ and $\ker(A^\ast A) = \ker A$ for every $A$, since $\langle \langle A^\ast Au, u \rangle \rangle = ||Au||^2$.

To prove the second identity in (2), we will prove the inclusions $\ker d_h \subset \ker(p''\partial + h \partial p')$ and $\ker d_h^* \subset \ker(\partial p'' + h \partial p')^\ast$.

Let $u = \sum_{r+s=k} u^{r,s}$ be a smooth $k$-form such that $d_h u = 0$. This amounts to $h\partial u^{r,s} + \bar{\partial} u^{r+1,s-1} = 0$ whenever $r + s = k$. Applying $p'$ and respectively $p''$, we get

$$p'\bar{\partial} u^{r+1,s-1} = 0 \quad \text{and} \quad p''\partial u^{r,s} = 0,$$

since $h \neq 0$, while $p'\partial = 0$ and $p''\bar{\partial} = 0$. Hence,

$$(p''\partial + h p'\bar{\partial}) u = \sum_{r+s=k} (p''\partial u^{r,s} + h p'\bar{\partial} u^{r+1,s-1}) = 0.$$

This proves the inclusion $\ker d_h \subset \ker(p''\partial + h p'\bar{\partial})$.

The ellipticity of the (pseudo)-differential operators $\Delta_h$ and $\tilde{\Delta}_h$, combined with the compactness of the manifold $X$, implies that the images of $d_h$ and $\partial p'' + h \partial p'$ are closed in $C_k^\infty(X, \mathbb{C})$. Hence, these images coincide with the orthogonal complements of the kernels of the adjoint operators $d_h^\ast$ and $(\partial p'' + h \partial p')^\ast$. Therefore, proving the inclusion $\ker d_h^* \subset \ker(\partial p'' + h \partial p')^\ast$ is equivalent to proving the inclusion $\text{Im} (\partial p'' + h \partial p') \subset \text{Im} d_h$. (Actually, the closedness of these images is not needed here, we would have taken closures otherwise.)

Let $u = \partial p'' v + h \partial p' v$ be a smooth $k$-form lying in the image of $\partial p'' + h \partial p'$. Since $\partial p' = 0$ and $\bar{\partial} p'' = 0$, while $h \neq 0$, we get

$$u = (h\partial) (\frac{1}{h} p'' v + h p' v) + \bar{\partial} (\frac{1}{h} p'' v + h p' v) = d_h (\frac{1}{h} p'' v + h p' v) \in \text{Im} d_h.$$

This completes the proof of (2).

Since $\Delta_h$ commutes with both $d_h$ and $d_h^\ast$, to prove (5) it suffices to prove the stability of $\text{Im} d_h$ and $\text{Im} d_h^\ast$ under $\tilde{\Delta}_h - \Delta_h$. Now, since $(p''\partial + h p'\bar{\partial}) d_h = 0$ (immediate verification), we get

$$(\tilde{\Delta}_h - \Delta_h) d_h = (\partial p'' + h \partial p') (p''\partial + h \partial p')^\ast (h\partial + \bar{\partial}).$$

Since $\text{Im} (\partial p'' + h \partial p') \subset \text{Im} d_h$ (as seen above), we get $(\tilde{\Delta}_h - \Delta_h)(\text{Im} d_h) \subset \text{Im} d_h$. Similarly, an immediate verification shows that $(p''\partial + h p'\bar{\partial})^\ast d_h^* = 0$. Consequently,

$$(\tilde{\Delta}_h - \Delta_h) d_h^* = (p''\partial + h p'\bar{\partial})^\ast (p''\partial + h p'\bar{\partial}) d_h^*.$$

Meanwhile, $\text{Im} (p''\partial + h p'\bar{\partial})^\ast \subset \text{Im} d_h^\ast$ (since this is equivalent to the inclusion $\ker d_h \subset \ker(p''\partial + h \partial p')$ that was proved above). Therefore, $(\tilde{\Delta}_h - \Delta_h)(\text{Im} d_h^\ast) \subset \text{Im} d_h^\ast$. The proof of (5) is complete.

The remaining statements follow from the standard elliptic theory as in [Pop17].

Conclusion 2.3. Let $(X, \omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every degree $k \in \{0, \ldots, 2n\}$, we have $C_k^\infty$ families of elliptic differential operators $(\Delta_h)_{h \in \mathbb{C}}$ and, respectively, elliptic pseudo-differential operators $(\tilde{\Delta}_h)_{h \in \mathbb{C}}$ from $C_k^\infty(X, \mathbb{C})$ to $C_k^\infty(X, \mathbb{C})$ such that

(i) $\Delta_0 = \Delta''$ and $\tilde{\Delta}_0 = \tilde{\Delta}$;

(ii) $H^k_{\Delta_h}(X, \mathbb{C}) = H^k_{\Delta_h}(X, \mathbb{C}) \simeq H^k_{d_h}(X, \mathbb{C})$ for all $h \in \mathbb{C} \setminus \{0\}$. 

7
(iii) $\mathcal{H}^k_{\Delta_0}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$ and $\mathcal{H}^k_{\Delta_{\mathbb{R}}}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_2^{p,q}(X)$. 

Proof. Only the latter part of (iii) still needs a proof. Since $\widetilde{\Delta}$ preserves the pure type of forms and since the kernel of $\widetilde{\Delta}: C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})$ is isomorphic to $E_2^{p,q}(X, \mathbb{C})$ for every bidegree $(p, q)$ (cf. [Pop16, Theorem 1.1]), the isomorphism follows. □

2.2 Page $r \geq 3$: the pseudo-differential Laplacians $\widetilde{\Delta}_h^{(r)}$

We start by recalling the bare bones of a construction from [Pop17, §3.2] and [PU18, §2.1]. Given an arbitrary compact Hermitian manifold $(X, \omega)$ with $\text{dim}_\mathbb{C}X = n$, in every bidegree $(p, q)$ one defines a sequence of $\omega$-harmonic spaces:

$$C^\infty_{p,q}(X, \mathbb{C}) \supset \mathcal{H}^{p,q}_1 \supset \cdots \supset \mathcal{H}^{p,q}_r \supset \mathcal{H}^{p,q}_{r+1} \supset \cdots$$

such that, for every $r \in \mathbb{N}^*$, the space $\mathcal{H}^{p,q}_r$ (depending on $\omega$) is isomorphic to the $E_r$-cohomology space $E_r^{p,q}(X)$ on the $r$-th page of the Frölicher spectral sequence. Specifically,

- every space $C^\infty_{p,q}(X, \mathbb{C})$ splits successively into mutually $L^2_\omega$-orthogonal subspaces in the following way (cf. Proposition 2.3 in [PU18]):

$$C^\infty_{p,q}(X, \mathbb{C}) = \text{Im} d_0 \oplus \mathcal{H}^{p,q}_1 \oplus \text{Im} d_0^* \oplus \text{Im} d_0^{(\omega)} \oplus \mathcal{H}^{p,q}_2 \oplus \text{Im} (d_1^{(\omega)})^* \oplus \cdots$$

where, on the top row, $d_0 := \partial$ and $\mathcal{H}^{p,q}_1 := \mathcal{H}_{\Delta}'(X, \mathbb{C})$ is the kernel of the $\bar{\partial}$-Laplacian $\Delta'' = \bar{\partial}\partial + \bar{\partial}^*\partial: C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})$. Of course, $\mathcal{H}^{p,q}_1 \simeq E_1^{p,q}(X)$.

- Setting $p_1 := p'' : C^\infty_{p,q}(X, \mathbb{C}) \rightarrow \mathcal{H}^{p,q}_1$ the orthogonal projection (w.r.t. the $L^2$ inner product induced by $\omega$) onto $\mathcal{H}^{p,q}_1$, we then define the metric realisation $d_1^{(\omega)} := p_1(\partial d_1 : \mathcal{H}^{p,q}_1 \rightarrow \mathcal{H}^{p+1,q}_1)$ of the Frölicher differential $d_1 : E_1^{p,q}(X) \rightarrow E_1^{p+1,q}(X)$ so that the following diagram is commutative:
We then consider the adjoint operator $(d_1^{(\omega)})^* = p_1 \partial^* p_1 : \mathcal{H}_1^{p+1,q} \to \mathcal{H}_1^{p,q}$ and its associated "Laplacian" $\widetilde{\Delta}^{(\omega)}_{(2)} : \mathcal{H}_1^{p,q} \to \mathcal{H}_1^{p,q}$ defined in the usual way as

$$\widetilde{\Delta}^{(\omega)}_{(2)} = d_1^{(\omega)} (d_1^{(\omega)})^* + (d_1^{(\omega)})^* d_1^{(\omega)} = p_1 (\partial p_1 \partial^* + \partial^* p_1 \partial) p_1 = p_1 (\partial p_1 \partial^* + \partial^* p_1 \partial + \Delta^u) p_1 = p_1 \widetilde{\Delta} p_1^{(2)} = p_1 \left( (\partial p_1)(\partial p_1)^* + (\partial p_1)^*(\partial p_1) + \Delta^u \right) p_1,$$

where $\widetilde{\Delta} : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C})$ is the pseudo-differential Laplacian of [Pop16] whose kernel is isomorphic to $E_2^{p,q}(X)$ (also considered in the previous subsection). For reasons that will become apparent in the inductive construction below, we also denote $\Delta$ by $\Delta^{(2)}$. (Note that $\Delta^u p_1 = 0$, by construction.) We let $\mathcal{H}_2^{p,q}$ denote the kernel of $\widetilde{\Delta}^{(\omega)}_{(2)}$ and we get

$$\mathcal{H}_2^{p,q} = \ker \widetilde{\Delta}^{(\omega)}_{(2)} = \ker d_1^{(\omega)} \cap \ker (d_1^{(\omega)})^* = \ker \Delta \subset \mathcal{H}_1^{p,q} \subset C^\infty_{p,q}(X, \mathbb{C}).$$

- We then continue by induction on $r \geq 1$. Once $\mathcal{H}_r^{p,q}$ has been constructed (as a subspace of $\mathcal{H}_{r-1}^{p,q}$), we let $p_r : C^\infty_{p,q}(X, \mathbb{C}) \to \mathcal{H}_r^{p,q}$ be the orthogonal projection (w.r.t. the $L^2$ inner product induced by $\omega$) onto $\mathcal{H}_r^{p,q}$. Then, we define the metric realisation

$$d_r^{(\omega)} := p_r [\partial (\Delta^u - 1 \partial^* \partial)^{r-1}] p_r : \mathcal{H}_r^{p,q} \to \mathcal{H}_{r}^{p+r,q-r+1}$$

of the Frölicher differential $d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X)$ so that the following diagram is commutative:

$$\begin{array}{ccc}
E_r^{p,q}(X) & \xrightarrow{d_r} & E_r^{p+r,q-r+1}(X) \\
\approx & \downarrow & \approx \\
\mathcal{H}_r^{p,q} & \xrightarrow{d_r^{(\omega)} = (p_r \partial) (\Delta^u - 1 \partial^* \partial)^{r-1} p_r} & \mathcal{H}_{r+1}^{p+r,q-r+1}.
\end{array}$$

A word of explanation is in order to account for the above definition of $d_r^{(\omega)}$, first given in [PU18, Proposition 2.3]. For any representative $\alpha$ of an $E_r$-cohomology class $\{\alpha\}_{E_r} \in E_r^{p,q}(X)$, there exist (cf., e.g., Proposition 3.1 below or an easy application of the definitions) non-unique forms $u_1, \ldots, u_{r-1}$ such that

$$\bar{\partial} \alpha = 0, \quad \partial \alpha = \bar{\partial} u_1, \quad \partial u_1 = \bar{\partial} u_2, \ldots, \partial u_{r-2} = \bar{\partial} u_{r-1}.$$

Moreover, the Frölicher differential $d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X)$ acts as $d_r(\{\alpha\}_{E_r}) = \{\partial u_{r-1}\}_{E_r}$ and this expression is independent of the choices of "potentials" $u_1, \ldots, u_{r-1}$ with the above properties. Now, if a Hermitian metric $\omega$ has been fixed on $X$, we can make the (unique) choices of
where $\Delta''$ is the Green operator of $\Delta'$. Therefore, $d_r(\{\alpha\})_E = \{\partial u_{r-1}\}_E = \{\partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} \alpha\}_E$. Thus, if we represent the class $\{\alpha\}_E \in E^p,q_r(X)$ by the unique form $\alpha$ that lies in $\mathcal{H}^{p,q}_r$, we have $\alpha = p_r \alpha$. Meanwhile, the representative $\partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} \alpha = \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \alpha$ of $d_r(\{\alpha\})_E$ need not lie in $\mathcal{H}^{p,q}_r$, so we project it to $p_r \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \alpha \in \mathcal{H}^{p,q}_r$. This projection does not change the $E_r$-cohomology class. Consequently, to make the above diagram commutative, we must set $d_r^{(\omega)} \alpha := p_r \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \alpha$ for every $\alpha \in \mathcal{H}^{p,q}_r$, which is the definition given above for $d_r^{(\omega)}$.

The associated “Laplacian” $\tilde{\Delta}^{(\omega)}_{(r+1)} : \mathcal{H}^{p,q}_r \rightarrow \mathcal{H}^{p,q}_r$ is then defined in the usual way as

$$
\tilde{\Delta}^{(\omega)}_{(r+1)} = d_r^{(\omega)} (d_r^{(\omega)})^* + (d_r^{(\omega)})^* d_r^{(\omega)} = p_r \left[ \partial (\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \left( \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} \right) \right] p_r
$$

$$
= p_r \tilde{\Delta}^{(r+1)} p_r,
$$

where $\tilde{\Delta}^{(r+1)} : C^\infty_{p,q} (X, \mathbb{C}) \rightarrow C^\infty_{p,q} (X, \mathbb{C})$ is defined as

$$
\tilde{\Delta}^{(r+1)} = \left( \partial (\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \right) \left( \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} p_r \right)^* + \left( p_r \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} \right)^* \left( p_r \partial(\Delta''-\bar{\partial}\bar{\partial})^{-1} \right)
$$

$$
+ \tilde{\Delta}^{(r)}
$$

(6)

and $\tilde{\Delta}^{(r)} : C^\infty_{p,q} (X, \mathbb{C}) \rightarrow C^\infty_{p,q} (X, \mathbb{C})$ was defined at the previous induction step such that $\ker \tilde{\Delta}^{(r)} = \mathcal{H}^{p,q}_r$. (Note that $\tilde{\Delta}^{(r)} p_r = 0$, by construction.) We let $\mathcal{H}^{p,q}_{r+1}$ denote the kernel of $\tilde{\Delta}^{(\omega)}_{(r+1)}$ and we get

$$
E^{p,q}_{r+1} (X) \simeq \mathcal{H}^{p,q}_{r+1} = \ker \tilde{\Delta}^{(\omega)}_{(r+1)} = \ker d_r^{(\omega)} \cap \ker (d_r^{(\omega)})^* = \ker \tilde{\Delta}^{(r+1)} \subset \mathcal{H}^{p,q}_r \subset \cdots \subset \mathcal{H}^{p,q}_1 \subset C^\infty_{p,q} (X, \mathbb{C}).
$$

We also extend the operators $\tilde{\Delta}^{(r)} : C^\infty_{p,q} (X, \mathbb{C}) \rightarrow C^\infty_{p,q} (X, \mathbb{C})$ by linearity to $\tilde{\Delta}^{(r)} : C^\infty_k (X, \mathbb{C}) \rightarrow C^\infty_k (X, \mathbb{C})$ and denote the corresponding kernels by $\mathcal{H}^{k}_{\tilde{\Delta}^{(r)}} (X, \mathbb{C}) = \oplus_{p+q=k} \mathcal{H}^{p,q}_r \simeq \oplus_{p+q=k} E^{p,q}_r (X)$.

With this summary of the construction from [Pop17, §3.2] and [PU18, §2.1] in place, we will now introduce, for every $r \in \mathbb{N}^*$, a smooth family $\tilde{\Delta}^{(r+1)}_{h \in C}$ of pseudo-differential operators whose member for $h = 0$ is the pseudo-differential Laplacian $\tilde{\Delta}^{(r+1)}$ constructed above. When $r = 1$, this will be the smooth family $\tilde{\Delta}_{h \in C}$ constructed in the previous subsection as an approximation of the pseudo-differential Laplacian $\tilde{\Delta}^{(2)} = \tilde{\Delta}$. Following the model of Definition 2.1, we will approximate each factor in the above definition of $\tilde{\Delta}^{(r+1)}$ by adding to it a small $h$-multiple of its conjugate.
Lemma 2.5. Let $(X, \omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. For every $h \in \mathbb{C}$ and every $k = 0, \ldots, 2n$, we define the pseudo-differential operator $\tilde{\Delta}^{(r+1)}_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ by induction on $r \geq 2$ as follows:

$$
\tilde{\Delta}^{(r+1)}_h = \left( \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} p_r + \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \bar{p}_r \right) \left( \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} p_r + \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \bar{p}_r \right)^* 
$$

$$
+ \left( p_r \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} + \bar{p}_r \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \right) \left( p_r \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} + \bar{p}_r \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \right)^* + \tilde{\Delta}^{(r)}_h,
$$

where $\tilde{\Delta}^{(r)}_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ has been defined at the previous induction step and $\tilde{\Delta}^{(2)}_h := \tilde{\Delta}_h$ was defined in Definition 2.1. For every bidegree $(p, q)$, by $\bar{p}_r : C^\infty_{p,q}(X, \mathbb{C}) \to \ker(\Delta^{(r)} : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p,q}(X, \mathbb{C}))$ we mean the orthogonal projection onto the kernel of the conjugate of $\Delta^{(r)}$ acting in bidegree $(p, q)$. Both the projections $p_r$ and $\bar{p}_r$ are then extended by linearity to the whole space $C^\infty_k(X, \mathbb{C})$.

As in the case of $\tilde{\Delta}_h = \tilde{\Delta}^{(2)}_h$ (cf. Lemma 2.2), we need to prove that $\tilde{\Delta}^{(r+1)}_h$ has the same kernel as $\Delta_h$ for every $r \geq 2$. A priori, the kernel of $\tilde{\Delta}^{(r+1)}_h$ might be smaller than that of $\Delta_h$.

Lemma 2.5. For every $h \in \mathbb{C} \setminus \{0\}$, the following identities of kernels hold:

$$
\ker \Delta_h = \ker \tilde{\Delta}^{(2)}_h = \cdots = \ker \tilde{\Delta}^{(r)}_h = \ker \tilde{\Delta}^{(r+1)}_h = \cdots
$$

in every degree $k = 0, \ldots, 2n$.

Proof. Fix any $k$. We will prove by induction on $r \geq 1$ that $\ker \tilde{\Delta}^{(r+1)}_h = \ker \Delta_h$ in degree $k$. The case $r = 1$ was proved in Lemma 2.2. Since each operator $\tilde{\Delta}^{(r+1)}_h$ is a sum of non-negative self-adjoint operators of the shape $\Lambda \Lambda^*$ and since $\ker(\Lambda \Lambda^*) = \ker \Lambda^*$, we have:

$$
\ker \tilde{\Delta}^{(r+1)}_h = \ker \left( \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} p_r + \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \bar{p}_r \right)^* 
$$

$$
\cap \ker \left( p_r \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} + \bar{p}_r \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \right) \cap \ker \tilde{\Delta}^{(r)}_h.
$$

In particular, $\ker \tilde{\Delta}^{(r+1)}_h \subset \ker \tilde{\Delta}^{(r)}_h \subset \cdots \subset \ker \tilde{\Delta}^{(2)}_h \subset \ker \Delta_h$ for every $r$ and $\ker \tilde{\Delta}^{(2)}_h = \ker \Delta_h$ thanks to Lemma 2.2.

Suppose, as the induction hypothesis, that $\ker \tilde{\Delta}^{(r)}_h = \ker \Delta_h$ for some $r \geq 2$. Since $\ker \Delta_h = \ker d_h \cap \ker d_h^*$, to prove that $\ker \tilde{\Delta}^{(r+1)}_h = \ker \Delta_h$, it suffices to prove the inclusions

$$
\ker (h \partial + \bar{\partial}) \subset \ker \left( p_r \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} + \bar{p}_r \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \right)^* \quad (7)
$$

$$
\ker (h \partial^* + \bar{\partial}^*) \subset \ker \left( \partial (\Delta^{''-1} \bar{\partial}^r \partial)^{1-r} p_r + \bar{\partial} (\Delta^{'-1} \partial^r \partial)^{1-r} \bar{p}_r \right)^* \quad (8)
$$

in every degree $k = 0, \ldots, 2n$. The proof of these inclusions will be given in the next section.
To prove inclusion (7), let $u = \sum_{l+s=k} u^{l,s} \in \ker(h \partial + \bar{\partial})$. This amounts to $h \partial u^{l,s} + \bar{\partial} u^{l+1,s-1} = 0$ for all $l, s$ such that $l + s = k$. Applying $p_r$ and $\bar{p}_r$ to this identity and using the fact that $h \neq 0$, we get

$$p_r \partial u^{l,s} = 0 \quad \text{and} \quad \bar{p}_r \partial u^{l+1,s-1} = 0 \quad \text{for all } l, s \text{ such that } l + s = k,$$

(9)

since $p_r \bar{\partial} = 0$ and $\bar{p}_r \partial = 0$. The last two identities follow from the fact that $\text{Im} \bar{\partial}$ (resp. $\text{Im} \partial$) is orthogonal to $\ker \Delta''$ (resp. $\ker \Delta'$), hence also to its subspace $\mathcal{H}_r^{p,q}$ (resp. $\mathcal{H}_r^{p,-q}$) onto which $p_r$ (resp. $\bar{p}_r$) projects orthogonally.

Meanwhile, for such a $u$, we have:

$$\left( p_r \partial (\Delta'' \bar{\partial}^*)^{r-1} + h \bar{p}_r \bar{\partial} (\Delta' \partial^*)^{r-1} \right) u$$

$$= \sum_{l+s=k} \left( p_r \partial (\Delta'' \bar{\partial}^*)^{r-2} \Delta'' \bar{\partial}^* (\partial u^{l,s}) + h \bar{p}_r \bar{\partial} (\Delta' \partial^*)^{r-2} \Delta' \partial^* (\bar{\partial} u^{l,s}) \right)$$

$$= \sum_{l+s=k} \left( -\frac{1}{h} p_r \partial (\Delta'' \bar{\partial}^*)^{r-2} \Delta'' \bar{\partial}^* (\bar{\partial} u^{l+1,s-1}) - h^2 \bar{p}_r \bar{\partial} (\Delta' \partial^*)^{r-2} \Delta' \partial^* (\bar{\partial} u^{l-1,s+1}) \right),$$

where the last line followed from the properties of the forms $u^{l,s}$: $\partial u^{l,s} = -\frac{1}{h} \bar{\partial} u^{l+1,s-1}$ and $\bar{\partial} u^{l,s} = -h \partial u^{l-1,s+1}$.

Now, the orthogonal decomposition $C_{-1}^{\infty} (X, \mathbb{C}) = \text{Im} \bar{\partial} \oplus \ker \bar{\partial}^*$ induces a splitting $u^{l+1,s-1} = \bar{\partial} \xi^{l+1,s-2} + \eta^{l+1,s-1}$ with $\eta^{l+1,s-1} \in \ker \bar{\partial}^*$. Similarly, the orthogonal decomposition $C_{-1}^{\infty} (X, \mathbb{C}) = \text{Im} \partial \oplus \ker \partial^*$ induces a splitting $u^{l-1,s+1} = \partial \xi^{l-2,s+1} + \rho^{l-1,s+1}$ with $\rho^{l-1,s+1} \in \ker \partial^*$. Therefore, in the last sum over $l + s = k$, we can re-write the following quantities as follows:

$$\Delta'' \bar{\partial}^* (\partial u^{l+1,s-1}) = \Delta'' \bar{\partial}^* \partial \eta^{l+1,s-1} = \Delta'' \bar{\partial}^* \eta^{l+1,s-1} = \eta^{l+1,s-1}$$

and

$$\Delta' \partial^* (\bar{\partial} u^{l-1,s+1}) = \Delta' \partial^* \bar{\partial} \rho^{l-1,s+1} = \Delta' \partial^* \rho^{l-1,s+1} = \rho^{l-1,s+1}.$$
Thus, inclusion (7) is proved in the case when \( r = 2 \). To prove it for \( r \geq 3 \), we iterate the above arguments by constructing two sequences of forms whose first terms are \( \eta^{l+1, s-1} \) and respectively \( \rho^{l-1, s+1} \) in the following way. The first sequence \( (\eta^{l(j)+1, s-j-1})_{j \geq 0} \) consists of forms of the shown bidegrees (with the understanding that if one of the two degrees exceeds \( n \) or is negative, the form is zero) such that \( \eta^{l+1, s-1} = \eta^{l+1, s-1} \) and for every \( j \geq 1 \) we have:

\[
\begin{align*}
\eta^{l(j)+1, s-j-1} &= (-1)^{j} \frac{1}{h^{j}} u^{l(j)+1, s-j-1} + \partial \xi^{l(j)+1, s-j-1} - \bar{\partial} \epsilon^{l(j)+1, s-j-2} \in \ker \bar{\partial}^* \\
\bar{\partial} \eta^{l(j)+1, s-j-1} &= \partial \eta^{l(j)+1, s-j} \\
\partial \eta^{l(j)+1, s-j-1} &= \bar{\partial} \left((-1)^{j+1} \frac{1}{h^{j+1}} u^{l(j)+2, s-j-2} + \partial \epsilon^{l(j)+1, s-j-2}\right), \\
\end{align*}
\]

where \( (\xi^{l(j)+1, s-j-2})_{j \geq 0} \) is a sequence of forms of the shown bidegrees whose term for \( j = 0 \) is \( \xi^{l+1, s-2} \).

The construction runs by induction on \( j \geq 0 \) as follows. From the property \( \partial u^{l+1, s-1} = -\frac{1}{h} \bar{\partial} u^{l+2, s-2} \) and the definition of \( \eta^{l+1, s-1} \), we get

\[
\partial \eta^{l+1, s-1} = \bar{\partial} \left(-\frac{1}{h} u^{l+2, s-2} + \partial \xi^{l+1, s-2}\right).
\]

Now, the orthogonal decomposition \( C^\infty_{l+2, s-2}(X, \mathbb{C}) = \text{Im} \bar{\partial} \oplus \ker \bar{\partial}^* \) induces a splitting

\[
-\frac{1}{h} u^{l+2, s-2} + \partial \epsilon^{l+1, s-2} = \bar{\partial} \xi^{l+2, s-3} + \eta^{l+2, s-2}
\]

with forms \( \xi^{l+2, s-3} \) and \( \eta^{l+2, s-2} \in \ker \bar{\partial}^* \) of the shown types. An immediate verification shows that they satisfy the three properties under (10) for \( j = 1 \).

Suppose we have run the construction of forms with the properties (10) up to some index \( j \). The orthogonal decomposition \( C^\infty_{l+j+2, s-j-2}(X, \mathbb{C}) = \text{Im} \bar{\partial} \oplus \ker \bar{\partial}^* \) induces a splitting

\[
(-1)^{j+1} \frac{1}{h^{j+1}} u^{l+j+2, s-j-2} + \partial \epsilon^{l+j+1, s-j-2} = \bar{\partial} \xi^{l+j+2, s-j-3} + \eta^{l+j+2, s-j-2},
\]

with forms \( \xi^{l+j+2, s-j-3} \) and \( \eta^{l+j+2, s-j-2} \in \ker \bar{\partial}^* \) of the shown types. An immediate verification shows that they satisfy the three properties under (10) when \( j \) is replaced by \( j + 1 \).

With this construction in place, we are able to show the vanishing of the first of the two terms whose vanishing we need. Indeed, using repeatedly the second property in (10) and the \( \bar{\partial}^* \)-closedness of the forms \( \eta^{l(j)+1, s-j-1} \), we get

\[
p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-1} u^{l, s} = -\frac{1}{h} p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-2} \eta^{l+1, s-1} = -\frac{1}{h} p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-3} \Delta^{r-1} \bar{\partial}^* \epsilon^{l+2, s-2} \\
= -\frac{1}{h} p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-3} \xi^{l+2, s-2} = -\frac{1}{h} p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-4} \Delta^{r-1} \bar{\partial}^* \eta^{l+3, s-3} \\
= -\frac{1}{h} p_r \partial (\Delta^{r-1} \bar{\partial}^* \partial)^{r-4} \xi^{l+3, s-3} \\
\vdots \\
= -\frac{1}{h} p_r \partial \eta^{l+r-1, s-r+1} = -\frac{1}{h} p_r \partial \eta^{l+r, s-r} = 0
\]
for all \( l, s \) such that \( l + s = k \), the last identity being the result of the general identity of operators \( p_r \bar{\partial} = 0 \) that has already been explained and used.

The vanishing of the other term \( \bar{p}_r \partial_r (\Delta'^{-1} \partial^* \bar{\partial}')^{r-1} u^{l,s} \) can be proved by an analogous (but “conjugated”) construction that uses the orthogonal decomposition \( C_{l,j,s+1}^\infty (X, \mathbb{C}) = \text{Im} \partial \oplus \ker \partial^* \) starting from the forms \( \rho'^{l-1,s+1} \) and \( \zeta'^{l-2,s+1} \) constructed in the case \( r = 2 \). The details will not be repeated.

Thus, inclusion (7) is proved for every \( r \geq 2 \).

- Proving inclusion (8) amounts to proving the inclusion:

\[
\ker (h \partial^* + \bar{\partial}^*) \subset \ker \left( p_r (\partial^* \bar{\partial} \Delta'^{-1})^{r-1} \partial^* + h \bar{p}_r (\bar{\partial}^* \partial \Delta'^{-1})^{r-1} \bar{\partial}^* \right).
\]

Let \( u = \sum_{l+s=k} u^{l,s} \in \ker (h \partial^* + \bar{\partial}^*) \). This amounts to \( h \partial^* u^{l,s} + \bar{\partial}^* u^{l-1,s+1} = 0 \) for all \( l, s \) such that \( l + s = k \). Applying \( p_r \) and \( \bar{p}_r \) to this identity and using the fact that \( h \neq 0 \), we get

\[
p_r \partial^* u^{l,s} = 0 \quad \text{and} \quad \bar{p}_r \bar{\partial}^* u^{l-1,s+1} = 0 \quad \text{for all} \ l, s \ \text{such that} \ l + s = k, \tag{11}
\]

since \( p_r \partial^* = 0 \) and \( \bar{p}_r \bar{\partial}^* = 0 \). The last two identities follow from the fact that \( \text{Im} \bar{\partial}^* \) (resp. \( \text{Im} \partial^* \)) is orthogonal to \( \ker \Delta'' \) (resp. \( \ker \Delta' \)), hence also to its subspace \( \mathcal{H}_r^{p,q} \) (resp. \( \mathcal{H}_r^{p,q} \)) onto which \( p_r \) (resp. \( \bar{p}_r \)) projects orthogonally.

Meanwhile, for such a \( u \), we have:

\[
\left( p_r (\partial^* \bar{\partial} \Delta'^{-1})^{r-1} \partial^* + h \bar{p}_r (\bar{\partial}^* \partial \Delta'^{-1})^{r-1} \bar{\partial}^* \right) u
\]

\[
\begin{aligned}
&= \sum_{l+s=k} \left( p_r (\partial^* \bar{\partial} \Delta'^{-1})^{r-2} \partial^* \bar{\partial} \Delta'^{-1} (\partial^* u^{l,s}) + h \bar{p}_r (\bar{\partial}^* \partial \Delta'^{-1})^{r-2} \bar{\partial}^* \partial \Delta'^{-1} (\partial^* u^{l,s}) \right) \\
&= \sum_{l+s=k} \left( -\frac{1}{h} p_r (\partial^* \bar{\partial} \Delta'^{-1})^{r-2} \partial^* (\Delta'^{-1} \bar{\partial} \partial^* u^{l-1,s+1}) - h^2 \bar{p}_r (\bar{\partial}^* \partial \Delta'^{-1})^{r-2} \bar{\partial}^* (\Delta'^{-1} \partial^* u^{l+1,s-1}) \right),
\end{aligned}
\]

where the last line followed from the properties of the forms \( u^{l,s} \) and from the commutation properties \( \bar{\partial} \Delta'^{-1} = \Delta''^{-1} \bar{\partial} \) and \( \partial \Delta'^{-1} = \Delta'^{-1} \partial \).

Now, the orthogonal decomposition \( C_{l-1,s+1}^\infty (X, \mathbb{C}) = \ker \bar{\partial} \oplus \text{Im} \bar{\partial}^* \) induces a splitting \( u^{l-1,s+1} = \bar{\partial}^* u^{l-1,s+2} + b^{l-1,s+1} \) with \( b^{l-1,s+1} \in \ker \bar{\partial} \). Similarly, the orthogonal decomposition \( C_{l+1,s-1}^\infty (X, \mathbb{C}) = \ker \partial \oplus \text{Im} \partial^* \) induces a splitting \( u^{l+1,s-1} = \partial^* c^{l+2,s-1} + d^{l+1,s-1} \) with \( d^{l+1,s-1} \in \ker \partial \). Therefore, in the last sum over \( l + s = k \), we can re-write the following quantities as follows:

\[
\begin{aligned}
\Delta''^{-1} \bar{\partial} \partial^* u^{l-1,s+1} &= \Delta''^{-1} \bar{\partial} \partial^* b^{l-1,s+1} = \Delta''^{-1} \Delta'' b^{l-1,s+1} = b^{l-1,s+1} \\
\Delta'^{-1} \partial \partial^* u^{l+1,s-1} &= \Delta'^{-1} \partial \partial^* d^{l+1,s-1} = \Delta'^{-1} \Delta' d^{l+1,s-1} = d^{l+1,s-1}.
\end{aligned}
\]

Suppose that \( r = 2 \). We get
\[
\left( p_r (\partial^* \bar{\nabla}^{r-1}) \partial^* + h \bar{p}_r (\partial^* \bar{\nabla}^{r-1} \partial^*) \right) u
\]
\[= \sum_{l+s=k} \left( - \frac{1}{h} p_r \partial^* b^{l-1, s+1} - h^2 \bar{p}_r \partial^* d^{l+1, s-1} \right) \]
\[= \sum_{l+s=k} \left( - \frac{1}{h} p_r \partial^* (\bar{\nabla}^{l-1, s+2} + b^{l-1, s+1}) - h^2 \bar{p}_r \partial^* (\bar{\nabla}^{l+2, s+1} + d^{l+1, s-1}) \right) \]
\[= \sum_{l+s=k} \left( - \frac{1}{h} p_r \partial^* u^{l-1, s+1} - h^2 \bar{p}_r \partial^* u^{l+1, s-1} \right) = 0,
\]
where the identity on the third row follows from \( p_r \partial^* \bar{\nabla}^{l-1, s+2} = -(p_r \partial^*) \bar{\nabla}^{l-1, s+2} = 0 \) (since \( p_r \partial^* = 0 \) as already explained) and \( \bar{p}_r \partial^* e^{l+2, s-1} = -(\bar{p}_r \partial^*) \bar{\nabla}^{l+2, s-1} = 0 \) (since \( \bar{p}_r \partial^* = 0 \) as already explained), while the last identity follows from (11).

Thus, inclusion (8) is proved in the case when \( r = 2 \).

To prove inclusion (8) for \( r \geq 3 \), one can run constructions of families of forms satisfying properties similar (but “dual” and, for one of the two families, “conjugated”) to those in (10). Indeed, as in the case \( r = 2 \) that serves as the first stage of the inductive construction, the relevant orthogonal decompositions are \( C_{l-j, s+j} \infty (X, \mathbb{C}) = \ker \bar{\nabla} \oplus \im \partial^* \) and \( C_{l+j, s-j} \infty (X, \mathbb{C}) = \ker \partial \oplus \im \partial^* \). The details will not be repeated.

Summing up, as in the case of \( \Delta_h = \Delta_h^{(2)} \) described in Conclusion 2.3, we get an analogous family of pseudo-differential operators \( (\Delta_h^{(r)})_{h \in \mathbb{C}} \) for every integer \( r \geq 2 \) (and the already discussed family of differential operators \( (\Delta_h)_{h \in \mathbb{C}} \) for \( r = 1 \)). The kernel of \( \Delta_h^{(r)} : C_\infty^\infty (X, \mathbb{C}) \rightarrow C_\infty^\infty (X, \mathbb{C}) \) will be denoted by \( H_{\Delta_h^{(r)}} (X, \mathbb{C}) \) and the analogous notation is used for \( \Delta_h \).

**Conclusion 2.6.** Let \((X, \omega)\) be a compact complex Hermitian manifold with \( \dim \mathbb{C} X = n \). For every integer \( r \geq 2 \) and every degree \( k \in \{0, \ldots, 2n\} \), we have \( C_\infty^\infty \) families of elliptic differential operators \((\Delta_h^{(r)})_{h \in \mathbb{C}}\) (independent of \( r \)) and, respectively, elliptic pseudo-differential operators \((\Delta_h^{(r)})_{h \in \mathbb{C}}\) from \( C_\infty^k (X, \mathbb{C}) \) to \( C_\infty^k (X, \mathbb{C}) \) such that

(i) \( \Delta_0 = \Delta'' \) and \( \Delta_h^{(0)} = \Delta^{(r)} \), where \( \Delta^{(r)} \) was defined in (6);

(ii) \( H_{\Delta_h^{(r)}}^k (X, \mathbb{C}) = H_{\Delta_h^{(r)}}^k (X, \mathbb{C}) \simeq H_0^k (X, \mathbb{C}) \) for all \( h \in \mathbb{C} \setminus \{0\} \);

(iii) \( H_{\Delta_0}^k (X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p, q} (X, \mathbb{C}) \) and \( H_{\Delta_h^{(r)}}^k (X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_{\partial}^{p, q} (X) \).

**2.3 The Frölicher approximating vector bundle**

We start with a preliminary observation. When \( h = 0, d_h \) becomes \( \bar{\nabla} \), so \( \Delta_0 \) becomes \( \Delta' : C_\infty^\infty (X, \mathbb{C}) \rightarrow C_\infty^\infty (X, \mathbb{C}) \) and \( H_0^k (X, \mathbb{C}) = \bigoplus_{p+q=k} E^{p, q} (X) \). The linear map \( \theta_0 : H^k_{\partial R} (X, \mathbb{C}) \rightarrow H_0^k (X, \mathbb{C}) \) reduces to

\[
\theta_0 : H^k_{\partial R} (X, \mathbb{C}) \rightarrow H^0_{\bar{\partial}}^k (X, \mathbb{C}) \subset H_0^k (X, \mathbb{C}), \quad \{u\}_{\partial R} \mapsto [u^{0, k}]_{\bar{\partial}},
\]
where \( u^{0, k} \) is the component of type \((0, k)\) of any given \( k \)-form \( u \). It is not bijective and may not even be surjective in general.
However, we shall now see how the space $H^0_{\bar{\partial}}(X, \mathbb{C}) = E^0_{1}(X)$ can be adjusted to make $\theta_0$ surjective in cohomology. The following statement also shows that no adjustment is necessary in the special case when $E_1(X) = E_{\infty}(X)$.

**Lemma 2.7.** Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $k \in \{0, \ldots, 2n\}$, the $\mathbb{C}$-linear map $\theta_0 : H^k_{DR}(X, \mathbb{C}) \to H^k_{0,\bar{\partial}}(X, \mathbb{C})$ reduces to the surjective $\mathbb{C}$-linear map:

\[
\theta_0 : H^k_{DR}(X, \mathbb{C}) \to E^0_{k+2}(X), \quad \{u\}_{DR} \mapsto \{u^0,k\}_{E_{k+2}},
\]

where $\{ \}$ is stands for the $E_{k+2}$-cohomology class.

Also note that if $r$ is the smallest positive integer such that the Frölicher spectral sequence of $X$ degenerates at $E_r$, then $E^0_{k+2}(X) = E^r_{k}(X)$ for every $k \in \{0, \ldots, 2n\}$.

**Proof.** If $u = \sum_{r+s=k} u^{r,s}$ is a $k$-form, $\theta_h u = \sum_{r+s=k} h^r u^{r,s}$ for every $h \in \mathbb{C}$. So, $\theta_0 u = u^0,k$. Thus, at the level of differential forms, the linear map $\theta_0 : C^\infty_k(X, \mathbb{C}) \to C^\infty_{0,k}(X, \mathbb{C})$ is always surjective.

Now, a smooth $k$-form $u = \sum_{r+s=k} u^{r,s}$ is $d$-closed if and only if

\[
\partial u^{0,k} = 0, \quad \ldots \quad \partial u^{l,k-l} = -\bar{\partial} u^{l+1,k-l-1}, \quad \ldots \quad \partial u^{k,0} = 0,
\]

for all $l \in \{0, \ldots, k-1\}$, while $u$ is $d$-exact if and only if there exists a smooth $(k-1)$-form $v = \sum_{s=0}^{k-1} v^{s,k-s} u^{r,s}$ such that

\[
u^{0,k} = \partial v^{0,k-1}, \quad \ldots, \quad u^{l,k-l} = \partial v^{l-1,k-l} + \bar{\partial} v^{l,k-l-1}, \quad \ldots, \quad u^{k,0} = \partial v^{k-1,0},
\]

for all $l \in \{1, \ldots, k-1\}$.

Condition (12) is precisely the necessary and sufficient condition for an arbitrary $(0,k)$-form $u^{0,k}$ to represent an $E_{k+2}$-cohomology class. (See, e.g. (i) of Proposition 3.1. Note also that, for bidegree reasons, the last condition $\partial u^{k,0} = 0$ in (12) is equivalent to $\partial u^{k,0} \in \text{Im} \bar{\partial}$.) Thus, the class $\{u^{0,k}\}_{E_{k+2}}$ is meaningful for every $d$-closed $k$-form $u$ and, conversely, every $(0,k)$-form $u^{0,k}$ representing an $E_{k+2}$-cohomology class can be completed to a $d$-closed $k$-form $u$ by the addition of forms $u^{l,k-l}$ solving the equations in (12) corresponding to $l \in \{0, \ldots, k-1\}$. The latter fact will prove that the map $H^k_{DR}(X, \mathbb{C}) \ni \{u\}_{DR} \mapsto \{u^{0,k}\}_{E_{k+2}} \in E^0_{k+2}(X)$ is surjective once we have finished proving its well-definedness.

Meanwhile, the first property $u^{0,k} \in \text{Im} \bar{\partial}$ in (13) suffices to guarantee that $\{u^{0,k}\}_{E_{k+2}} = 0$ whenever $u$ is $d$-exact. Therefore, the class $\{u^{0,k}\}_{E_{k+2}}$ is independent of the choice of representative $u$ of the De Rham class $\{u\}_{DR} \in H^k_{DR}(X, \mathbb{C})$. It follows that the map $H^k_{DR}(X, \mathbb{C}) \ni \{u\}_{DR} \mapsto \{u^{0,k}\}_{E_{k+2}} \in E^0_{k+2}(X)$ is well defined.

To prove the last conclusion of Lemma 2.7, recall that $E_{r}^{a,b}(X) = E_{s}^{a,b}(X)$ for all $s \geq r$ and all $a,b$. In particular, $E^{k+2}_{k+2}(X) = E^{r}_{r}(X)$ if $k+2 \geq r$. If $k+2 < r$, all the maps $d^0_{l,k} : E^{0,k}_{l}(X) \to E^{k,k-l+1}_{l}(X)$ vanish identically when $l \geq k+2$ because $k-l+1 \leq -1 < 0$, so $E^{0,k}_{l}(X) = 0$. Since the map $d^0_{k+2,2k+1} : E^{k+2}_{k+2,2k+1}(X) \to E^{0,k}_{k+2}(X)$ and its counterparts $d^0_{l,k+1}$ for all $l \geq k+2$ vanish identically for bidegree reasons, we get $E^{0,k}_{k+2}(X) = E^{0,k}_{l}(X) = E^{0,k}_{r}(X)$ for all $l \in \{k+2, \ldots, r\}$. \qed
2.3.1 The absolute case

As a first application of the pseudo-differential operators $\tilde{\Delta}_h$, we obtain a holomorphic vector bundle over $\mathbb{C}$ whose fibre above 0 is defined by the page in the Frölicher spectral sequence of $X$ on which degeneration occurs.

**Corollary and Definition 2.8.** Let $X$ be a compact complex manifold with $\dim_\mathbb{C} X = n$. Let $r \in \mathbb{N}^*$ be the smallest positive integer such that the Frölicher spectral sequence of $X$ degenerates at $E_r$.

For every $k \in \{0, \ldots, 2n\}$, there exists a holomorphic vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C}$, of rank equal to the $k$-th Betti number $b_k$ of $X$, whose fibres are

$$\mathcal{A}^k_h = H^k_{dh}(X, \mathbb{C}) \quad \text{if } h \in \mathbb{C} \setminus \{0\}, \quad \text{and} \quad \mathcal{A}^k_0 = \bigoplus_{p+q=k} E^{p,q}_0(X) \quad \text{if } h = 0,$$

and whose restriction to $\mathbb{C} \setminus \{0\}$ is isomorphic to the constant vector bundle $\mathcal{H}^k_{\mathbb{C}} \longrightarrow \mathbb{C} \setminus \{0\}$ of fibre $H^k_{DR}(X, \mathbb{C})$ under the holomorphic vector bundle isomorphism $\theta = (\theta_h)_{h \in \mathbb{C}} : \mathcal{H}^k_{\mathbb{C}} \longrightarrow \mathcal{A}^k_{\mathbb{C}}$.

The vector bundle $\mathcal{A}^k \longrightarrow \mathbb{C}$ will be called the Frölicher approximating vector bundle of $X$ in degree $k$.

**Proof.** Recall that $\dim_\mathbb{C} H^k_{dh}(X, \mathbb{C}) = b_k$ for every $h \neq 0$. Fix any Hermitian metric $\omega$ on $X$.

If $r = 1$, the dimension of $\oplus_{p+q=k} E^{p,q}_1(X, \mathbb{C})$ equals $b_k$ and the fibre $\mathcal{A}^k_h$ is isomorphic to the kernel of $\Delta'' = \Delta_0 : C^\infty_1(X, \mathbb{C}) \longrightarrow C^\infty_1(X, \mathbb{C})$. Thus, the $C^\infty$ family $(\Delta_h)_{h \in \mathbb{C}}$ of elliptic differential operators has the property that the dimension of the kernel of $\Delta_h : C^\infty_1(X, \mathbb{C}) \longrightarrow C^\infty_1(X, \mathbb{C})$ is independent of $h \in \mathbb{C}$. The classical Theorem 5 of Kodaira-Spencer [KS60] ensures that the harmonic spaces $\mathcal{H}^k_{\Delta_h}(X, \mathbb{C})$ depend in a $C^\infty$ way on $h \in \mathbb{C}$. Therefore, they form a $C^\infty$ vector bundle over $\mathbb{C}$, as do the vector spaces $\mathcal{A}^k_h$ to which they are isomorphic.

If $r = 2$, the dimension of $\oplus_{p+q=k} E^{p,q}_2(X, \mathbb{C})$ equals $b_k$ and the fibre $\mathcal{A}^k_h$ is isomorphic to the kernel of $\tilde{\Delta} = \tilde{\Delta}_0 : C^\infty_2(X, \mathbb{C}) \longrightarrow C^\infty_2(X, \mathbb{C})$ by Theorem 1.1 in [Pop16]. The classical Theorem 5 of Kodaira-Spencer [KS60] still applies to the $C^\infty$ family $(\tilde{\Delta}_h)_{h \in \mathbb{C}}$ of elliptic pseudo-differential operators (cf. argument in [Mas18] for the case $h = 0$), whose kernels have dimension independent of $h \in \mathbb{C}$ (and equal to $b_k$, see Conclusion 2.3), to ensure that the harmonic spaces $\mathcal{H}^k_{\tilde{\Delta}_h}(X, \mathbb{C})$ depend in a $C^\infty$ way on $h \in \mathbb{C}$. As above, we infer that the vector spaces $\mathcal{A}^k_h$, to which the harmonic spaces $\mathcal{H}^k_{\tilde{\Delta}_h}(X, \mathbb{C})$ are isomorphic for all $h \in \mathbb{C}$ (cf. Conclusion 2.3), form a $C^\infty$ vector bundle over $\mathbb{C}$.

If $r \geq 3$, the dimension of $\oplus_{p+q=k} E^{p,q}_r(X, \mathbb{C})$ equals $b_k$ and the fibre $\mathcal{A}^k_h$ is isomorphic to the kernel of $\tilde{\Delta}^{(r)} = \tilde{\Delta}^{(r)}_0 : C^\infty_r(X, \mathbb{C}) \rightarrow C^\infty_r(X, \mathbb{C})$ (cf. Conclusion 2.6). The classical Theorem 5 of Kodaira-Spencer [KS60] still applies to the $C^\infty$ family $(\tilde{\Delta}^{(r)}_h)_{h \in \mathbb{C}}$ of elliptic pseudo-differential operators (cf. argument in [Mas18] for the case of $\Delta$) whose kernels have dimension independent of $h \in \mathbb{C}$ (and equal to $b_k$) to ensure that the harmonic spaces $\mathcal{H}^k_{\tilde{\Delta}^{(r)}_h}(X, \mathbb{C})$ depend in a $C^\infty$ way on $h \in \mathbb{C}$. We infer as above that the vector spaces $\mathcal{A}^k_h$, to which the harmonic spaces $\mathcal{H}^k_{\tilde{\Delta}^{(r)}_h}(X, \mathbb{C})$ are isomorphic for all $h \in \mathbb{C}$ (cf. Conclusion 2.6), form a $C^\infty$ vector bundle over $\mathbb{C}$.

Meanwhile, we know from [Pop17, Lemma 2.5] (see also Introduction) that for every $h \neq 0$, the linear map $\theta_h : H^k_{DR}(X, \mathbb{C}) \longrightarrow H^k_{dh}(X, \mathbb{C})$ defined by $\theta_h(\{u\}_{DR}) = \{\theta_h u\}_{dh}$ is an isomorphism of $\mathbb{C}$-vector spaces. Since $\theta_h$ depends holomorphically on $h$ and the space $H^k_{DR}(X, \mathbb{C})$ is independent
of \( h \), we infer that the \( \mathbb{C} \)-vector spaces \( H^k_{d_h}(X, \mathbb{C}) \) form a holomorphic vector bundle over \( \mathbb{C} \setminus \{0\} \). However, we know from the above argument that this holomorphic vector bundle extends in a \( C^\infty \) way across 0 to the whole of \( \mathbb{C} \). This extension must then be holomorphic. \( \square \)

The discussion that follows in the remainder of this §2.3.1 will not be used in the proof of Theorem 1.1, so the reader only interested in that proof may wish to skip it.

We will now define a natural analogue of a natural connection on every vector bundle \( \mathcal{A}^k \rightarrow \mathbb{C} \). We need the following simple observation.

**Lemma 2.9.** Let \( X \) be any complex manifold. For every \( h \in \mathbb{C} \), the pointwise linear map \( \theta_h : \oplus_k \Lambda^k T^* X \rightarrow \oplus_k \Lambda^k T^* X \) has the following properties:

\[
\theta_h(u \wedge v) = \theta_h u \wedge \theta_h v, \quad u, v \in \oplus_k \Lambda^k T^* X
\]

\[
\theta_{h_1h_2} = \theta_{h_1} \circ \theta_{h_2}, \quad h_1, h_2 \in \mathbb{C}.
\]

Moreover, \( \theta_1 \) is the identity map and \( \theta_h^{-1} = \theta_{h^{-1}} \) for every \( h \in \mathbb{C} \setminus \{0\} \). Meanwhile, \( d_h \) satisfies the Leibniz rule:

\[
d_h(u \wedge v) = d_h u \wedge v + (-1)^{\deg u} u \wedge d_h v, \quad u, v \in \oplus_k \Lambda^k T^* X,
\]

which also holds for \( h = 0 \).

**Proof.** Let \( u = \sum_{p+q=l} u^{p,q} \) and \( v = \sum_{r+s=m} v^{r,s} \) be forms of respective degrees \( l \) and \( m \). Then

\[
\theta_h(u \wedge v) = \sum_{p+q=l, r+s=m} \theta_h(u^{p,q} \wedge v^{r,s}) = \sum_{p+q=l} h^{p+r} u^{p,q} \wedge v^{r,s} = \sum_{p+q=l} (h^p u^{p,q}) \wedge (h^r v^{r,s}) = \theta_h u \wedge \theta_h v.
\]

In particular, \( \theta_1 \) is the identity map.

If \( h_1, h_2 \in \mathbb{C} \), then \( \theta_{h_1h_2} u = \sum_{p+q=l} (h_1h_2)^p u^{p,q} = \sum_{p+q=l} \theta_{h_1}(\theta_{h_2}u^{p,q}) = (\theta_{h_1} \circ \theta_{h_2})(\sum_{p+q=l} u^{p,q}). \) In particular, \( \theta_h \circ \theta_{h^{-1}} = \theta_1 \) is the identity map.

The Leibniz rule for \( d_h \) with \( h \neq 0 \) follows from \( d_h = \theta_h d \theta_h^{-1} \) and from the above properties of \( \theta_h \). The Leibniz rule can also be checked independently of \( \theta_h \) and also holds for \( h = 0 \) since \( d_0 = \bar{\partial} \). \( \square \)

We can define analogues \( D_h \) of the differential operators \( d_h \) for vector-bundle-valued differential forms by requiring \( D_h \) to coincide with \( d_h \) on scalar-valued forms and to satisfy the Leibniz rule. Thus, when \( h \neq 1 \), \( D_h \) differs from a standard connection only by the fact that it does not coincide with \( d \) on scalar-valued forms.

**Definition 2.10.** Let \( E \rightarrow X \) be a \( C^\infty \) complex vector bundle on a complex manifold. Fix an arbitrary constant \( h \in \mathbb{C} \). An \( h \)-connection on \( E \) is a linear differential operator \( D_h : C^\infty_\bullet(X, E) \rightarrow C^\infty_\bullet(X, E) \) of order 1 that satisfies the following conditions for all integers \( k, l \):

\[
(a) \quad D_h : C^\infty_k(X, E) \rightarrow C^\infty_{k+1}(X, E);
\]

\[
(b) \quad D_h(f \wedge s) = d_h f \wedge s + (-1)^k f \wedge D_h s \text{ for every } f \in C^\infty_k(X, \mathbb{C}) \text{ and every } s \in C^\infty_l(X, E).
\]
For example, if \( D = D' + D'' \) is any connection on a \( C^\infty \) complex vector bundle \( E \to X \) over a complex manifold, then for every \( h \in \mathbb{C} \), \( D_h = hD' + D'' \) is an \( h \)-connection on \( E \). Thus, the family \((D_h)_{h \in \mathbb{C}}\) of differential operators defines a smooth homotopy for \( h \in [0, 1] \) between the original connection \( D \) and its \((0, 1)\)-connection \( D'' \).

Going back to the specific case of our Frölicher approximating vector bundle \( \mathcal{A}^k \to \mathbb{C} \), the construction of a connection-like object starts with the following

**Definition 2.11.** In the setup of Corollary and Definition \ref{cor:def}, let \( \bar{\nabla} \) be the trivial connection (extension of \( d \)) on the constant vector bundle \( \mathcal{H}^k \to \mathbb{C} \) of fibre \( H_{\mathcal{D}R}^k(X, \mathbb{C}) \).

For every \( l \in \{0, 1, 2\} \), let \( D = D^{(k)} : C^\infty_c(\mathbb{C} \setminus \{0\}, \mathcal{A}^k) \to C^\infty_c(\mathbb{C} \setminus \{0\}, \mathcal{A}^k) \) be the linear \( 1 \)-st order differential operator defined as \( D_s := (\theta \bar{\nabla} \theta^{-1}) s \) for every \( s \in C^\infty_c(\mathbb{C} \setminus \{0\}, \mathcal{A}^k) \).

However, this definition is very unsatisfactory since it only deals with the restriction of \( \mathcal{A}^k \) to \( \mathbb{C}^* \). We will now define the analogue of a connection (a kind of \( h \)-connection but with a moving \( h \)) on the whole of the holomorphic vector bundle \( \mathcal{A}^k \to \mathbb{C} \).

We start by expressing the above \( D \) in a local trivialisation. Let \( \{e_1, \ldots, e_b\} \) be a \( \mathbb{C} \)-basis of \( H_{\mathcal{D}R}^k(X, \mathbb{C}) \). Then, for every \( h \in \mathbb{C}^* \), \( \{\theta_1 e_1, \ldots, \theta_b e_b\} \) is a \( \mathbb{C} \)-basis of \( H_{\mathcal{D}R}^k(X, \mathbb{C}) \). This defines a holomorphic frame for \( \mathcal{A}^k_{|\mathbb{C}^*} \). (The restriction of \( \mathcal{A}^k \) to \( \mathbb{C}^* \) is thus seen to be the trivial vector bundle of rank \( b_k \).) If \( s \in C^\infty(\mathbb{C}^*, \mathcal{A}^k) \) is a smooth section, then \( s(h) = \sum_{j=1}^{b_k} s_j(h) \otimes \theta_j e_j \) for all \( h \in \mathbb{C}^* \), where the \( s_j \)'s are smooth \( \mathbb{C} \)-valued functions on \( \mathbb{C}^* \). Hence, from Definition \ref{def:2.11} we get

\[
(Ds)(h) = \sum_{j=1}^{b_k} \theta(ds_j)(h) \otimes \theta_j e_j = \sum_{j=1}^{b_k} (d.hs_j)(h) \otimes \theta_j e_j, \quad h \in \mathbb{C}^*,
\]

where we put

\[
(d.hs_j)(h) := (h \partial s_j + \bar{\partial} s_j)(h) = h \frac{\partial s_j}{\partial h}(h) dh + \frac{\partial s_j}{\partial \bar{h}}(h) d\bar{h}.
\]

Note that \( h \) is at once the variable in \( \mathbb{C} \), with respect to which the partial differentiations are performed, and the factor by which one of them is multiplied. So, this operator \( d_h \) on \( \mathbb{C} \) (where \( h \) is moving) is not quite the same as the one used so far (where \( h \) was fixed and served as the coefficient of a \( \partial \) computed w.r.t. variables independent of \( h \)).

In order to extend the definition of \( D \) to \( h = 0 \), the natural thing to do appears to be the replacement of \( d_h \) by \( d_0 = \bar{\partial} \). However, \( \theta_0 \) is not an isomorphism and there is, in general, (unless we make an assumption on \( X \), for example assuming that \( X \) is a \( \partial \bar{\partial} \)-manifold, but we will stick with our general setting) no canonical isomorphism between \( H_{\mathcal{D}R}^k(X, \mathbb{C}) \) and \( \oplus_{p+q=k} E_{p,q}(X) \). However, with every isomorphism \( \tilde{\theta}_0 \) between these two \( \mathbb{C} \)-vector spaces, we will associate connection-like objects on the vector bundle \( \mathcal{A}^k \to \mathbb{C} \) after duly modifying the above formula for \( D \) by changing the \( \theta_j \)'s to \( \tilde{\theta}_j \)'s whose limit when \( h \to 0 \) is \( \theta_0 \). Indeed, every isomorphism \( \tilde{\theta}_0 : H_{\mathcal{D}R}^k(X, \mathbb{C}) \to \oplus_{p+q=k} E_{p,q}(X) \) can be deformed holomorphically (in a non-unique and non-canonical way) to isomorphisms \( \tilde{\theta}_h : H_{\mathcal{D}R}^k(X, \mathbb{C}) \to H_{d_h}^k(X, \mathbb{C}) \) with \( h \) ranging over a small open subset \( U \subset \mathbb{C} \) containing 0. This is done in the obvious way: pick any \( \mathbb{C} \)-basis \( \{e_1, \ldots, e_b\} \) of \( H_{\mathcal{D}R}^k(X, \mathbb{C}) \); consider the induced \( \mathbb{C} \)-basis \( \{\tilde{\theta}_0 e_1, \ldots, \tilde{\theta}_0 e_b\} \) of \( \oplus_{p+q=k} E_{p,q}(X) \) and then arbitrary extensions of the \( \tilde{\theta}_0 \)'s to holomorphic sections \( \{\tilde{e}_1, \ldots, \tilde{e}_b\} \) (which form a \( \mathbb{C} \)-basis of \( H_{d_h}^k(X, \mathbb{C}) \) at every point \( h \in U \) and \( \tilde{e}_j(0) = \tilde{\theta}_0 e_j \)
of the holomorphic vector bundle $\mathcal{A}^k$ over some small neighbourhood $U$ of 0 in $\mathbb{C}$ over which $\mathcal{A}^k$ is trivial; for every $h \in U$, define $\tilde{\theta}_h : H^k_{DR}(X, \mathbb{C}) \rightarrow H^k_d(X, \mathbb{C})$ as the isomorphism taking the basis $\{e_1, \ldots, e_{b_k}\}$ to the basis $\{\tilde{e}_1(h), \ldots, \tilde{e}_{b_k}(h)\}$.

We now propose the following definition of connection-like objects on our Fröhlicher approximating vector bundle $\mathcal{A}^k \rightarrow \mathbb{C}$.

**Definition 2.12.** The setup is that of Corollary and Definition 2.8. With every holomorphic section $(\tilde{\theta}_h)_{h \in U} \in H^0(U, \text{End}(\mathcal{H}^k, \mathcal{A}^k))$ consisting of isomorphisms $\tilde{\theta}_h : H^k_{DR}(X, \mathbb{C}) \rightarrow H^k_d(X, \mathbb{C})$ over an open neighbourhood $U \subset \mathbb{C}$ of 0, and every $C^\infty$ function $\chi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\chi = 1$ on $\frac{1}{2}U$, $\chi = 0$ on $\mathbb{C} \setminus U$, we associate the following linear differential operator $\nabla : C^\infty_{\mathfrak{g}}(\mathbb{C}, \mathcal{A}^k) \rightarrow C^\infty_{\mathfrak{g}+1}(\mathbb{C}, \mathcal{A}^k)$ of order 1 for every $l \in \{0, 1\}$.

For every $\mathcal{A}^k$-valued smooth $l$-form $s \in C^\infty_{\mathfrak{g}}(\mathbb{C}, \mathcal{A}^k)$, we put

$$(\nabla s)(h) := \chi(h) \sum_{j=1}^{b_k} (d_h \tilde{s}_j)(h) \otimes \tilde{\theta}_h e_j + (1 - \chi(h)) \sum_{j=1}^{b_k} (d_h s_j)(h) \otimes \theta_h e_j, \quad h \in \mathbb{C},$$

where $s(h) = \sum_{j=1}^{b_k} \tilde{s}_j(h) \otimes \tilde{\theta}_h e_j$ for $h \in U$ and $s(h) = \sum_{j=1}^{b_k} s_j(h) \otimes \theta_h e_j$ for $h \in \mathbb{C}^\ast$.

To prove that $\nabla$ satisfies the Leibniz rule, we first prove this property for the operator $d_h$ (with a moving $h \in \mathbb{C}$) defined in (14) as acting on $\mathbb{C}$-valued forms on $\mathbb{C}$. Since $d_h(dh) = 0$ and $d_h(d\bar{h}) = 0$, $d_h$ acts non-trivially only on the (function) coefficients of forms on $\mathbb{C}$. Meanwhile, for any form $f$ on $\mathbb{C}$, we will use the standard notation $\partial f/\partial h$, resp. $\partial f/\partial \bar{h}$, for the form obtained by differentiating w.r.t. $h$, resp. $\bar{h}$, the coefficients of $f$. The very simple computations are summed up in

**Lemma 2.13.** (i) For any $\mathbb{C}$-valued differential forms $f, g$ on $\mathbb{C}$, we have

$$d_h(f \wedge g) = d_h f \wedge g + (-1)^{\deg f} f \wedge d_h g.$$

(ii) For any smooth $\mathbb{C}$-valued form $f$ on $\mathbb{C}$ and any smooth $\mathcal{A}^k$-valued form $s$ on $\mathbb{C}$, we have

$$\nabla(f \wedge s) = d_h f \wedge s + (-1)^{\deg f} f \wedge \nabla s.$$

**Proof.** (i) It can be trivially checked that for every $\mathbb{C}$-valued form $f$ on $\mathbb{C}$, we have

$$d_h f = (-1)^{\deg f} \left( h \frac{\partial f}{\partial h} \wedge dh + \frac{\partial f}{\partial \bar{h}} \wedge d\bar{h} \right).$$

From this and from $\partial(f \wedge g)/\partial h = (\partial f/\partial h) \wedge g + f \wedge (\partial g/\partial h)$, we immediately get: $d_h(f \wedge g) =

\begin{align*}
(-1)^{\deg(f \wedge g) + \deg g} & \left( h \frac{\partial f}{\partial h} \wedge dh + \frac{\partial f}{\partial \bar{h}} \wedge d\bar{h} \right) \wedge g + (-1)^{\deg(f \wedge g)} f \wedge \left( h \frac{\partial g}{\partial h} \wedge dh + \frac{\partial g}{\partial \bar{h}} \wedge d\bar{h} \right) \\
&= (-1)^{\deg(f \wedge g) + \deg g} (-1)^{\deg f} d_h f \wedge g + (-1)^{\deg(f \wedge g)} (-1)^{\deg g} f \wedge d_h g,
\end{align*}

which proves the contention.

(ii) It follows immediately from the definition and from (i). \(\square\)
We will now define the Frölicher approximating vector bundles of a holomorphic family \((X_t)_{t \in B}\) of compact complex \(n\)-dimensional manifolds induced by a proper holomorphic submersion \(\pi : X \to B\) whose base \(B \subset \mathbb{C}^N\) is an open ball about the origin in some complex Euclidean vector space.

By the classical Ehresmann Theorem, the differential structure of the fibres \(X_t\) is independent of \(t \in B\), hence so is the Poincaré differential \(d\), which splits differently as \(d = \partial_t + \bar{\partial}_t\) as the complex structure of \(X_t\) varies. In particular, the differential operators \(d_h\) depend on \(t\) (except when \(h = 1\)), so we put

\[
d_{h, t} := h \partial_t + \bar{\partial}_t : C^\infty(X, \mathbb{C}) \to C^\infty_{k+1}(X, \mathbb{C}), \quad h \in \mathbb{C}, \ t \in B, \ k \in \{0, \ldots, 2n\},
\]

where \(X\) is the \(C^\infty\) manifold underlying the fibres \(X_t\). Likewise, the pointwise linear maps \(\theta_h\) (which are isomorphisms when \(h \neq 0\)) depend on \(t\) (because the splitting of \(k\)-forms into pure-type-forms depends on the complex structure of \(X_t\)), so we put

\[
\theta_{h, t} : \Lambda^k T^* X \to \Lambda^k T^* X, \quad u = \sum_{p+q=k} u^{p,q}_t \mapsto \theta_{h, t} u := \sum_{p+q=k} h^p u^{p,q}_t.
\]

When \(h \neq 0\), this induces an isomorphism in cohomology \(\theta_{h, t} : H^k_{DR}(X, \mathbb{C}) \to H^k_{dh}(X_t, \mathbb{C})\) defined by \(\theta_{h, t}(\{u\}_{DR}) = \{\theta_{h, t} u\}_{dh, t}\), since \(\partial_{h, t} d = d_{h, t} \partial_{h, t}\). When \(h = 0\), we saw in Lemma 2.7 that \(\theta_{0, t}\) induces a surjective linear map \(\theta_{0, t} : H^k_{DR}(X, \mathbb{C}) \to E^k_{k+2}(X_t)\) for every \(t \in B\) defined by \(\theta_{0, t}(\{u\}_{DR}) = \{u^0_{0,k}\}_{E_{k+2}}\), where \(u^0_{0,k}\) is the component of type \((0, k)\) of \(u\) w.r.t. the complex structure of \(X_t\).

For every \(k\), let \(\mathcal{H}^k \to B\) be the constant vector bundle of rank \(b_k = b_k(X)\) (the \(k\)th Betti number of \(X\), or equivalently of any \(X_t\)) whose fibre is the \(k\)th De Rham cohomology group \(H^k(X, \mathbb{C})\) of \(X\) (= of any \(X_t\)). Thus, \(\mathcal{H}^k_t = H^k_{DR}(X_t, \mathbb{C}) = H^k_{DR}(X, \mathbb{C})\) for every \(t \in B\). Let \(\nabla\) be the Gauss-Manin connection on \(\mathcal{H}^k\). Recall that this is the trivial connection, given in the local trivialisations of \(\mathcal{H}^k\) by the usual differentiation \(d\) (i.e. \(\nabla(\sum_j f_j \otimes e_j) = \sum_j (df_j) \otimes e_j\) for any local frame \(\{e_j\}\) of \(\mathcal{H}^k\) and any locally defined functions \(f_j\) thanks to the transition matrices of \(\mathcal{H}^k\) having constant entries.

Recall that the degeneration at \(E_1\) of the Frölicher spectral sequence is a deformation open property of compact complex manifolds. Thus, if \(E_1(X_0) = E_\infty(X_0)\), then \(E_1(X_t) = E_\infty(X_t)\) for every \(t \in B\), after possibly shrinking \(B\) about \(0\). (This follows at once from the upper semicontinuity of the Hodge numbers \(h^{p,q}(t)\) and from the invariance of the Betti numbers \(b_k\) of the fibres \(X_t\).

However, when \(r \geq 2\), the degeneration at \(E_r\) of the Frölicher spectral sequence is not deformation open, so we will have to assume it on all the fibres \(X_t\) for the sake of convenience.

**Corollary and Definition 2.14.** Let \(\pi : X \to B\) be a holomorphic family of compact complex \(n\)-dimensional manifolds over an open ball \(B \subset \mathbb{C}^N\) about the origin. Suppose that for an \(r \in \mathbb{N}^*\), the Frölicher spectral sequence of \(X_t\) degenerates (at least) at \(E_r\) for all \(t \in B\) and that \(r\) is the smallest positive integer with this property.

For every \(k \in \{0, \ldots, 2n\}\), there exists a holomorphic vector bundle \(\mathcal{A}^k \to \mathbb{C} \times B\), of rank equal to the \(k\)-th Betti number \(b_k\) of \(X\) (= of any fibre \(X_t\)), whose fibres are

\[
\mathcal{A}^k_{h, t} = H^k_{dh}(X_t, \mathbb{C}) \quad \text{for} \quad (h, t) \in \mathbb{C}^* \times B, \quad \text{and} \quad \mathcal{A}^k_{0, t} = \bigoplus_{p+q=k} E_r^{p,q}(X_t) \quad \text{for} \quad (0, t) \in \{0\} \times B,
\]
and whose restriction to \( \mathbb{C}^* \times B \) is isomorphic to the constant vector bundle \( \mathcal{H}_{\mathbb{C}^* \times B}^k \rightarrow \mathbb{C}^* \times B \) of fibre \( H_{DR}^k(X, \mathbb{C}) \) under the holomorphic vector bundle isomorphism \( \theta = (\theta_{h,t})_{(h,t) \in \mathbb{C}^* \times B} : \mathcal{H}_{\mathbb{C}^* \times B}^k \rightarrow \mathcal{A}_{\mathbb{C}^* \times B}^k \).

The vector bundle \( \mathcal{A}_k \rightarrow \mathbb{C} \times B \) is called the Frölicher approximating vector bundle of the family \( (X_t)_{t \in B} \) in degree \( k \).

**Proof.** We know that \( \dim_{\mathbb{C}} H_{dR}^k(X_t, \mathbb{C}) = b_k \) for all \( h \neq 0 \) and \( t \in B \). Moreover, thanks to the \( E_r \)-degeneration assumption on every fibre \( X_t \), \( \dim_{\mathbb{C}} \oplus_{p+q=k} E_{r}^{p,q}(X_t, \mathbb{C}) = b_k \) for all \( t \in B \). Thus, \( \dim_{\mathbb{C}} \mathcal{A}_{h,t} = b_k \) for all \( (h, t) \in \mathbb{C} \times B \).

Now, fix an arbitrary \( C^\infty \) family \( (\omega_t)_{t \in B} \) of Hermitian metrics on the fibres \( (X_t)_{t \in B} \) and consider the \( C^\infty \) family \( (\Delta_{h,t})_{(h,t) \in \mathbb{C}^* \times B} \) of elliptic differential operators defined in every degree \( k \) by analogy with the absolute case as

\[
\Delta_{h,t} = d_{h,t} d_{h,t}^* + d_{h,t}^* d_{h,t},
\]

where the formal adjoint \( d_{h,t}^* \) is computed w.r.t. the metric \( \omega_t \). The kernels \( \ker \Delta_{h,t} \) are isomorphic to the vector spaces \( \mathcal{A}_{h,t}^k \), hence they have a dimension independent of \( (h, t) \in \mathbb{C}^* \times B \) (and equal to \( b_k \)). This implies, via the Kodaira-Spencer theory [KS60], that \( \mathcal{A}_k \rightarrow \mathbb{C}^* \times B \) is a \( C^\infty \) complex vector bundle of rank \( b_k \). This vector bundle is even holomorphic since, as pointed out in the statement, the \( C^\infty \) vector bundle isomorphism \( \theta = (\theta_{h,t})_{(h,t) \in \mathbb{C}^* \times B} : \mathcal{H}_k \rightarrow \mathcal{A}_k \), viewed as a section of \( \text{End}(\mathcal{H}_k, \mathcal{A}_k) \), depends in a holomorphic way on \( (h, t) \in \mathbb{C}^* \times B \). Note that no assumption on the spectral sequence is necessary to get this conclusion on \( \mathbb{C}^* \times B \).

On the other hand, for every fixed \( t \in B \), we know from the absolute case of Corollary and Definition 2.8 that \( \mathbb{C} \ni h \mapsto \mathcal{A}_{h,t}^k \) is a holomorphic vector bundle (of rank \( b_k \)) over \( \mathbb{C} \).

We conclude that near the points of the hypersurface \( \{0\} \times B \subset \mathbb{C} \times B \), the entries of the transition matrices of the vector bundle \( \mathcal{A}_k \rightarrow \mathbb{C}^* \times B \) are functions \( g(h, t) \) on open subsets \( U \setminus \{0\} \subset \mathbb{C} \times B \) (where \( U \) is an open subset of \( \mathbb{C} \times B \)) with the following two properties:

- the function \( (h, t) \mapsto g(h, t) \) is holomorphic in the complement of the hypersurface \( U \cap \{0\} \times B \) in \( U \);
- for every \( t \in B \), the holomorphic function \( 0 \neq h \mapsto g(h, t) \) extends holomorphically across 0.

Therefore, the resulting functions \( g(h, t) \), defined for all \( (h, t) \in U \subset \mathbb{C} \times B \), must be holomorphic on the whole of \( U \), proving that \( \mathbb{C} \times B \ni (h, t) \mapsto \mathcal{A}_{h,t}^k \) is a holomorphic vector bundle over \( \mathbb{C} \times B \).

\[ \square \]

We now discuss a family counterpart to the \( h \)-connection of §2.3.1. As this discussion plays no role in the proof of Theorem 1.1, some readers may wish to skip to §3.

For every \( h \in \mathbb{C} \setminus \{0\} \), let us consider the holomorphic vector bundles \( \mathcal{H}_h^k := \mathcal{H}_h^k, \mathcal{A}_h^k \rightarrow B \) and \( \mathcal{A}_h^k := \mathcal{A}_h^k, \mathcal{A}_h^k \rightarrow B \), as well as the maps \( \theta_h : C^\infty(B, \mathcal{H}_h^k) \rightarrow C^\infty(B, \mathcal{A}_h^k) \) between their spaces of global \( C^\infty \) sections induced by the isomorphisms \( \theta_{h,t} : H_{DR}^k(X_t, \mathbb{C}) \rightarrow H_{dR}^k(X_t, \mathbb{C}) \) with \( h \neq 0 \).

We will need the following extension of the maps \( \theta_h : C^\infty(B, \mathcal{H}_h^k) \rightarrow C^\infty(B, \mathcal{A}_h^k) \) to \( C^\infty \) forms of arbitrary degree \( l \) with values in these vector bundles.

**Definition 2.15.** For every \( l \in \{0, \ldots, 2N\} \) and every \( h \in \mathbb{C} \setminus \{0\} \), we define the map \( \theta_h : C^\infty_l(B, \mathcal{H}_h^k) \rightarrow C^\infty_l(B, \mathcal{A}_h^k) \) by

\[ 22 \]
\[
\theta_h \left( \sum_{p+q=l, \ 1 \leq j \leq b_k} u_j^{p,q} \otimes e_j \right) := \sum_{p+q=l, \ 1 \leq j \leq b_k} (\theta_h u_j^{p,q}) \otimes (\theta_h e_j) = \sum_{p+q=l, \ 1 \leq j \leq b_k} h^p u_j^{p,q} \otimes (\theta_h e_j),
\]

where \(\{e_j\}\) is any local frame of the vector bundle \(\mathcal{H}_h^k \to B\).

It is immediate to check that the above definition of \(\theta_h\) is independent of the choice of local trivialisation (= of local frame) of \(\mathcal{H}_h^k \to B\). Moreover, this definition and Lemma 2.9 show that

\[
\theta_h(f \wedge s) = \theta_h f \wedge \theta_h s, \quad f \in C_{l_1}^\infty(B, \mathbb{C}), \ s \in C_{l_2}^\infty(B, \mathcal{H}_h^k),
\]

for all \(l_1, l_2 \in \{0, \ldots, 2N\}\).

**Proposition 2.16.** The setup is that of Corollary and Definition 2.14. For every \(h \in \mathbb{C} \setminus \{0\}\), let \(\theta_h := \theta_{h, \bullet}\) be the holomorphic isomorphism between the holomorphic vector bundles \(\mathcal{H}_h^k := \mathcal{H}_{h, \bullet}^k \to B\) and \(\mathcal{A}_h^k := \mathcal{A}_{h, \bullet}^k \to B\). For every \(l = 0, \ldots, 2N\), consider the 1\(^{st}\)-order differential operator

\[
\nabla_h : C_{l_1}^\infty(B, \mathcal{A}_h^k) \to C_{l_1+1}^\infty(B, \mathcal{A}_h^k), \quad s \mapsto \nabla_h s := (\theta_h \tilde{\nabla} \theta_h^{-1}) s,
\]

where \(\tilde{\nabla}\) is the Gauss-Manin connection on the constant vector bundle \(\mathcal{H}_h^k \to B\).

Then, \(\nabla_h\) is an \(h\)-connection on \(\mathcal{A}_h^k\).

**Proof.** To check that \(\nabla_h\) satisfies the Leibniz rule of Definition 2.10, let \(f \in C_{l_1}^\infty(B, \mathbb{C})\) and \(s \in C_{l_2}^\infty(B, \mathcal{A}_h^k)\). We have

\[
\nabla_h(f \wedge s) = \theta_h \tilde{\nabla}(\theta_h^{-1} f \wedge \theta_h^{-1} s) = \theta_h (d\theta_h^{-1} f \wedge \theta_h^{-1} s) + (-1)^{\deg f} \theta_h (\theta_h^{-1} f \wedge \tilde{\nabla} \theta_h^{-1} s) = dh f \wedge s + (-1)^{\deg f} f \wedge \nabla_h s,
\]

where we have used (15), its analogue for \(\theta_h^{-1}\), the Leibniz rule for \(\tilde{\nabla}\) and the formula \(dh = \theta_h d\theta_h^{-1}\) for \(\mathbb{C}\)-valued forms.

3 \quad \(E_r\)-sG manifolds and deformations of complex structures

In this section, we apply the Frölicher approximating vector bundle constructed in §2.3 to the study of limits of \(\partial \bar{\partial}\)-manifolds under holomorphic deformations.

We begin by generalising the notion of strongly Gauduchon (sG) metric introduced in [Pop09] and [Pop13]. Recall that a Gauduchon metric on a compact complex \(n\)-dimensional manifold \(X\) is a positive definite, \(C^\infty(1,1)\)-form \(\gamma\) on \(X\) such that \(\partial \bar{\partial} \gamma^{n-1} = 0\) (or, equivalently, \(\partial \gamma^{n-1}\) is \(\bar{\partial}\)-closed). Thanks to [Gau77], such metrics always exist. If the stronger requirement that \(\partial \gamma^{n-1}\) be \(\bar{\partial}\)-exact (= \(E_1\)-exact w.r.t. the Frölicher spectral sequence) is imposed, \(\gamma\) is said to be strongly Gauduchon (sG) (cf. [Pop09] and [Pop13]). We will relax this definition by requiring \(E_r\)-exactness instead, for a possibly larger \(r \geq 1\).

To fix the notation, recall the following known fact (also spelt out with further details in Proposition 2.1 in [PU18]).
Proposition 3.1. (i) Fix \( r \geq 1 \). A form \( \alpha \in C_{p,q}^\infty(X, \mathbb{C}) \) is \( E_r \)-closed (i.e. \( \alpha \) represents an \( E_r \)-cohomology class) if and only if there exist forms \( u_l \in C_{p+l,q-l}^\infty(X, \mathbb{C}) \) with \( l \in \{1, \ldots, r-1\} \) satisfying the following \( r \) equations:

\[
\begin{align*}
\bar{\partial} \alpha &= 0 \\
\partial \alpha &= \bar{\partial} u_1 \\
\partial u_1 &= \bar{\partial} u_2 \\
&\vdots \\
\partial u_{r-2} &= \bar{\partial} u_{r-1}.
\end{align*}
\]

(When \( r = 1 \), the above equations reduce to \( \bar{\partial} \alpha = 0 \).)

(ii) Fix \( r \geq 1 \). A form \( \alpha \in C_{p,q}^\infty(X, \mathbb{C}) \) is \( E_r \)-exact (i.e. \( \alpha \) represents the zero \( E_r \)-cohomology class) if and only if there exist forms \( \zeta_{r-2} \in C_{p-1,q}^\infty(X, \mathbb{C}) \) and \( \xi_0 \in C_{p,q-1}^\infty(X, \mathbb{C}) \) such that

\[
\alpha = \partial \zeta_{r-2} + \bar{\partial} \xi_0,
\]

with \( \xi_0 \) arbitrary and \( \zeta_{r-2} \) satisfying the following additional condition (which is empty when \( r = 1 \) and reduces to requiring that \( \zeta_{r-2} = \xi_0 \) be \( \bar{\partial} \)-closed when \( r = 2 \)).

There exist \( C^\infty \) forms \( v_0^{(r-2)}, v_1^{(r-2)}, \ldots, v_{r-3}^{(r-2)} \) satisfying the following \( (r-1) \) equations:

\[
\begin{align*}
\bar{\partial} \zeta_{r-2} &= \partial v_{r-3}^{(r-2)} \\
\bar{\partial} v_{r-3}^{(r-2)} &= \partial v_{r-4}^{(r-2)} \\
&\vdots \\
\bar{\partial} v_1^{(r-2)} &= \partial v_0^{(r-2)} \\
\bar{\partial} v_0^{(r-2)} &= 0,
\end{align*}
\]

with the convention that any form \( v_l^{(r-2)} \) with \( l < 0 \) vanishes.

(Note that, thanks to (i), equations (16), when read from bottom to top, express precisely the condition that the form \( v_0^{(r-2)} \in C_{p+r+1,q+r-2}^\infty(X, \mathbb{C}) \) be \( E_{r-1} \)-closed. Moreover, the form \( \partial \zeta_{r-2} \) featuring on the r.h.s. of the above expression for \( \alpha \) represents the \( E_{r-1} \)-class \( d_{r-1}(\{v_0^{(r-2)}\}_{E_{r-1}}) \).)

Proof. It is a straightforward consequence of the definition of the Frölicher spectral sequence and can be left to the reader. \( \square \)

Finally, note that for any Gauduchon metric \( \gamma \) on \( X \), the \((n, n-1)\)-form \( \partial \gamma^{n-1} \) is \( E_r \)-closed for every \( r \in \mathbb{N}^* \). Indeed, in (i) of Proposition 3.1 we can choose \( u_1 = \cdots = u_{r-1} = 0 \).

Definition 3.2. Let \( \gamma \) be a Gauduchon metric on a compact complex manifold \( X \) with \( \dim_{\mathbb{C}} X = n \). Fix an arbitrary integer \( r \geq 1 \).

(i) We say that \( \gamma \) is an \( E_r \)-sG metric if \( \partial \gamma^{n-1} \) is \( E_r \)-exact.

(ii) A compact complex manifold \( X \) is said to be an \( E_r \)-sG manifold if an \( E_r \)-sG metric exists on \( X \).

(iii) A compact complex manifold \( X \) is said to be an \( E_r \)-sGG manifold if every Gauduchon metric on \( X \) is an \( E_r \)-sG metric.
The term chosen in the last definition is a nod to the notion of sGG manifold that we introduced jointly with L. Ugarte in [PU14] as any compact complex manifold on which every Gauduchon metric is strongly Gauduchon. It follows from the above definitions that the \( E_1 \)-sG property is equivalent to the sG property and that the following implications hold for any Hermitian metric \( \gamma \) and every \( r \in \mathbb{N}^* \):

\[
\gamma \text{ is } E_1 \text{-sG} \implies \gamma \text{ is } E_2 \text{-sG} \implies \cdots \implies \gamma \text{ is } E_r \text{-sG} \implies \gamma \text{ is } E_{r+1} \text{-sG} \implies \cdots
\]

Actually, for bidegree reasons, if a Hermitian metric \( \gamma \) is \( E_r \)-sG for some integer \( r \geq 1 \), then \( r \leq 3 \). Indeed, if \( (p, q) = (n, n - 1) \), the tower of relations (16) reduces to its first two lines since \( \zeta_{r-2} \) is of bidegree \( (n - 1, n - 1) \), hence \( v^{(r-2)}_{r-3} \) is of bidegree \( (n - 2, n) \), hence \( \bar{\partial}v^{(r-2)}_{r-3} = 0 \) for bidegree reasons, so \( v^{(r-2)}_{r-4}, \ldots, v^{(r-2)}_0 \) can all be chosen to be zero.

We now notice that the \( E_r \)-sG property is open under deformations of the complex structure.

**Lemma 3.3.** Let \( \pi : \mathcal{X} \to B \) be a \( C^\infty \) family of compact complex \( n \)-dimensional manifolds over an open ball \( B \subset \mathbb{C}^N \) about the origin. Fix an integer \( r \geq 1 \).

If \( \gamma_0 \) is an \( E_r \)-sG metric on \( X_0 := \pi^{-1}(0) \), after possibly shrinking \( B \) about 0 there exists a \( C^\infty \) family \( (\gamma_t)_{t \in B} \) of \( E_r \)-sG metrics on the respective fibres \( X_t := \pi^{-1}(t) \) whose element for \( t = 0 \) is the original \( \gamma_0 \).

Moreover, this family can be chosen such that \( \partial_t \gamma_t^{n-1} = \bar{\partial}_t \Gamma_t^{n, n-2} + \partial_t \zeta_{r-2,t} \) for all \( t \), with \( J_t \)-type \( (n, n - 2) \)-forms \( \Gamma_t^{n, n-2} \) and \( J_t \)-type \( (n - 1, n - 1) \)-forms \( \zeta_{r-2,t} \) depending in a \( C^\infty \) way on \( t \).

The forms \( \Gamma_t^{n, n-2}, \zeta_{r-2,t} \) and the induced \( v^{(r-2)}_{k,t} \) (with \( 0 \leq k \leq r - 3 \)) satisfying the tower of relations (16) that are (non-uniquely) associated with an \( E_r \)-sG metric \( \gamma_t \) will be called **potentials** of \( \gamma_t \). So, the above lemma says that not only can any \( E_r \)-sG metric \( \gamma_0 \) on \( X_0 \) be deformed in a smooth way to \( E_r \)-sG metrics \( \gamma_t \) on the nearby fibres \( X_t \), but so can its potentials.

**Proof of Lemma 3.3.** By (ii) of Proposition 3.1, the \( E_r \)-sG assumption on \( \gamma_0 \) implies the existence of a \( J_0 \)-type \( (n, n - 2) \)-form \( \Gamma_0^{n, n-2} \) and of a \( J_0 \)-type \( (n - 1, n - 1) \)-form \( \zeta_{r-2,0} \) such that \( \partial_0 \gamma_0^{n-1} = \bar{\partial}_0 \Gamma_0^{n, n-2} + \partial_0 \zeta_{r-2,0} \) and such that

\[
\bar{\partial}_0 \zeta_{r-2,0} = \partial_0 v^{(r-2)}_{r-3,0}, \quad \text{and} \quad \bar{\partial}_0 v^{(r-2)}_{r-3,0} = 0,
\]

(17)

for some \( J_0 \)-type \( (n - 2, n) \)-form \( v^{(r-2)}_{r-3,0} \). (As already pointed out, for bidegree reasons, the general tower (16) reduces to (17) in this case.)

We get \( \partial_0 (\gamma_0^{n-1} - \zeta_{r-2,0}) = \bar{\partial}_0 (\Gamma_0^{n, n-2} - v^{(r-2)}_{r-3,0}) \), so the \( (2n - 2) \)-form

\[
\Omega := -(\Gamma_0^{n, n-2} - v^{(r-2)}_{r-3,0}) + (\gamma_0^{n-1} - \zeta_{r-2,0} - \bar{\partial}_0 \zeta_{r-2,0}) = (\Gamma_0^{n, n-2} - v^{(r-2)}_{r-3,0})
\]

is real and \( d \)-closed and its \( J_0 \)-pure-type components \( \Omega_0^{n, n-2}, \Omega_0^{n-1, n-1}, \Omega_0^{n-2, n} \) are given by the respective paratheses, with their respective signs, on the right of the above identity defining \( \Omega \).

If \( \Omega_t^{n, n-2}, \Omega_t^{n-1, n-1}, \Omega_t^{n-2, n} \) stand for the \( J_t \)-pure-type components of \( \Omega \) for any \( t \in B \), they all depend in a \( C^\infty \) way on \( t \). On the other hand, deforming identities (17) in a \( C^\infty \) way when the complex structure \( J_0 \) deforms to \( J_t \), we find (non-unique) \( C^\infty \) families of \( J_t \)-type \( (n - 1, n - 1) \)-forms.
\[ (\zeta_{r-2,t})_{t \in B} \text{ and } J_t\text{-type } (n-2, n)\text{-forms } (v_{r-3,t}^{(r-2)})_{t \in B}, \text{ whose elements for } t = 0 \text{ are } \zeta_{r-2,0}, \text{ respectively } v_{r-3,0}^{(r-2)}, \text{ such that } \partial_t \zeta_{r-2,t} = \partial_t v_{r-3,t}^{(r-2)} \text{ and } \partial_t v_{r-3,t}^{(r-2)} = 0 \text{ for } t \in B. \text{ Then, the } J_t\text{-type } (n-1, n-1)\text{-form } \Omega_{t}^{n-1,n-1} + \zeta_{r-2,t} + \overline{\zeta_{r-2,t}} \text{ depends in a } C^\infty \text{ way on } t \in B. \text{ When } t = 0, \text{ it equals } \gamma_0^{n-1}, \text{ so it is positive definite. By continuity, it remains positive definite for all } t \in B \text{ sufficiently close to } 0 \in B, \text{ so it has a unique } (n-1)\text{-st root and the root is positive definite. In other words, there exists a unique } C^\infty \text{ positive definite } J_t\text{-type } (1, 1)\text{-form } \gamma_t \text{ such that }

\[ \gamma_t^{n-1} = \Omega_t^{n-1,n-1} + \zeta_{r-2,t} + \overline{\zeta_{r-2,t}} > 0, \quad t \in B, \]

after possibly shrinking } B \text{ about } 0. \text{ By construction, } \gamma_t \text{ depends in a } C^\infty \text{ way on } t. 

If we set } \Gamma_t^{n,n-2} := -\Omega_t^{n,n-2} + v_{r-3,t}^{(r-2)} \text{ for all } t \in B \text{ close to } 0, \text{ we get } \partial_t \gamma_t^{n-1} = \partial_t \Gamma_t^{n,n-2} + \partial_t \zeta_{r-2,t}. \text{ Since } \partial_t \zeta_{r-2,t} = \partial_t v_{r-3,t}^{(r-2)} \text{ and } \partial_t v_{r-3,t}^{(r-2)} = 0, \text{ we conclude that } \gamma_t \text{ is an } E_r\text{-sG metric for the complex structure } J_t \text{ for all } t \in B \text{ close to } 0. \]

We are now in a position to prove the first main result of this paper on the deformation limits of a specific class of compact complex manifolds (cf. Theorem 1.4 and the comments thereafter.) While it is one of the two building blocks that will yield a proof of Theorem 1.1, we hope that it also holds an independent interest.

**Theorem 3.4.** Let } \pi : X \longrightarrow B \text{ be a holomorphic family of complex compact } n\text{-dimensional manifolds over an open ball } B \subset \mathbb{C}^N \text{ about the origin. Suppose that the fibre } X_t := \pi^{-1}(t) \text{ is a } \partial\bar{\partial}-\text{manifold for all } t \in B \setminus \{0\}. 

Then, the fibre } X_0 := \pi^{-1}(0) \text{ is an } E_r\text{-sG manifold, where } r \text{ is the smallest positive integer such that the Frölicher spectral sequence of } X_0 \text{ degenerates at } E_r. 

Furthermore, } X_0 \text{ is even an } E_r\text{-sGG manifold.} 

**Proof.** Let } \gamma_0 \text{ be an arbitrary Gauduchon metric on } X_0. \text{ It is known that, after possibly shrinking } B \text{ about } 0, \gamma_0 \text{ can be extended to a } C^\infty \text{ family } (\gamma_t)_{t \in B} \text{ of } C^\infty \text{ 2-forms on } X (= \text{ the } C^\infty \text{ manifold underlying the complex manifolds } X_t) \text{ such that } \gamma_t \text{ is a Gauduchon metric on } X_t \text{ for every } t \in B \text{ (see, e.g., [Pop13, section 3]). Let } n \text{ be the complex dimension of the fibres } X_t. 

The Gauduchon property of the } \gamma_t\text{'s implies that } d_{h,t}(\partial_t \gamma_t^{n-1}) = 0 \text{ for all } (h, t) \in \mathbb{C}^* \times B \text{ and that } \partial_t \gamma_t^{n-1} \text{ is } E_r(X_t)\text{-closed for all } t \in B. \text{ Thus, the following object is well defined: }

\[ \sigma(h, t) := \begin{cases} 
\{ \partial_t \gamma_t^{n-1} \}_{d_{h,t} \in H^{2n-1,0}(X_t, \mathbb{C}) = \mathcal{A}^{2n-1}_{h,t},} & \text{if } (h, t) \in \mathbb{C}^* \times B, \\
\{ \partial_t \gamma_t^{n-1} \}_{E_r(X_t) \in \bigoplus_{p+q=2n-1} E_r^{p,q}(X_t) = \mathcal{A}^{2n-1}_{0,t},} & \text{if } (h, t) = (0, t) \in \{0\} \times B, 
\end{cases} \]

where } \mathcal{A}^{2n-1} \longrightarrow \mathbb{C} \times B \text{ is the Frölicher approximating vector bundle of the family } (X_t)_{t \in B} \text{ in degree } 2n-1 \text{ defined in Corollary and Definition 2.14. Note that the } \partial \bar{\partial}\text{-assumption on the fibres } X_t \text{ with } t \neq 0 \text{ implies that the Frölicher spectral sequence of each of these fibres degenerates at } E_1, \text{ hence also at any } E_r \text{ with } r \geq 1. \text{ Thus, the assumption of Corollary and Definition 2.14 is satisfied and that result ensures that } \mathcal{A}^{2n-1} \longrightarrow \mathbb{C} \times B \text{ is a holomorphic vector bundle of rank } b_{2n-1} = b_1 (= \text{ the } (2n-1)\text{-st, respectively the first Betti numbers of } X, \text{ that are equal by Poincaré duality).} 

This last fact, in turn, implies that } \sigma \text{ is a global } C^\infty \text{ section of } \mathcal{A}^{2n-1} \text{ on } \mathbb{C} \times B. \text{ Indeed, } \partial_t \text{ varies holomorphically with } t \in B, \gamma_t^{n-1} \text{ varies in a } C^\infty \text{ way with } t \in B, \text{ while the vector space } \mathcal{A}^{2n-1}_{h,t} \text{ varies holomorphically with } (h, t) \in \mathbb{C} \times B. \]
Meanwhile, the $\partial\bar{\partial}$-assumption on every $X_t$ with $t \in B^*$ implies that the $d$-closed $\partial$-exact $(n, n - 1)$-form $\partial \gamma_t^{n-1}$ is $(\partial_t \bar{\partial})$-exact, hence also $d_{h_t}$-exact for every $h \in \mathbb{C}$. (Indeed, if $\partial_t \gamma_t^{n-1} = \partial_t \bar{\partial} u_t$, then $\partial_t \gamma_t^{n-1} = d_{h_t}(-\partial_t u_t)$.) This translates to $\sigma(h, t) = \{\partial_t \gamma_t^{n-1}\}_{d_{h_t}} = 0 \in \mathcal{A}^{2n-1}_{h, t}$ for all $(h, t) \in \mathbb{C}^* \times B^*$. (We even have $\sigma(h, t) = 0$ for all $(h, t) \in \mathbb{C} \times B^*$.)

Thus, the restriction of $\sigma$ to $\mathbb{C}^* \times B^*$ is identically zero. Then, by continuity, $\sigma$ must be identically zero on $\mathbb{C} \times B$. In particular,

$$\sigma(0, t) = \{\partial_t \gamma_t^{n-1}\}_{E_r(X_t)} = 0 \in \mathcal{A}^{2n-1}_{0, t} \quad \text{for all } t \in B,$$

which means precisely that $\partial_t \gamma_t^{n-1}$ is $E_r(X_t)$-exact for every $t \in B$. In other words, $\gamma_t$ is an $E_r$-sG metric on $X_t$ for every $t \in B$, including $t = 0$. In particular, $X_0$ is an $E_r$-sG manifold and even an $E_r$-sGG manifold since the Gauduchon metric $\gamma_0$ was chosen arbitrarily on $X_0$ in the first place. \square

We need a simple observation before proceeding. If $X$ is a compact complex $n$-dimensional manifold, for every degree $k \in \{0, \ldots, 2n\}$ there exists a canonical, well-known, linear map:

$$T^{(k)} : \bigoplus_{p+q=k} H^{p, q}_{BC}(X, \mathbb{C}) \longrightarrow H^{k}_{DR}(X, \mathbb{C}), \quad ([\alpha^{p, q}]_{BC})_{p+q=k} \mapsto \{ \sum_{p+q=k} \alpha^{p, q} \}_{DR},$$

from the Bott-Chern to the De Rham cohomology of degree $k$. In general, $T^{(k)}$ is neither injective, nor surjective. However, a given De Rham class $\{\alpha\}_{DR}$ of degree $k$ can be represented by a form $\alpha$ whose all pure-type components are $d$-closed if and only if $\{\alpha\}_{DR}$ lies in the image of $T^{(k)}$, so $T^{(k)}$ is surjective if and only if every De Rham class of degree $k$ has such a representative. On the other hand, if $X$ is a $\partial\bar{\partial}$-manifold, the map $T^{(k)}$ is an isomorphism for all $k = 0, \ldots, 2n$. We will need the following simple

**Lemma 3.5.** Let $\pi : X \longrightarrow B$ be a holomorphic family of compact complex $n$-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a $\partial\bar{\partial}$-manifold for all $t \in B \setminus \{0\}$. Then, for every $k \in \{0, \ldots, 2n\}$, the canonical map

$$T^{(k)}_0 : \bigoplus_{p+q=k} H^{p, q}_{BC}(X_0, \mathbb{C}) \longrightarrow H^{k}_{DR}(X, \mathbb{C}), \quad ([\alpha^{p, q}]_{BC})_{p+q=k} \mapsto \{ \sum_{p+q=k} \alpha^{p, q} \}_{DR},$$

is surjective, where $X_0 := \pi^{-1}(0)$ and $X$ is the $C^\infty$ manifold underlying the fibres $X_t$.

**Proof.** Due to the $\partial\bar{\partial}$-assumption on $X_t$ with $t \neq 0$, the canonical map $T^{(k)}_t : \bigoplus_{p+q=k} H^{p, q}_{BC}(X_t, \mathbb{C}) \longrightarrow H^{k}_{DR}(X, \mathbb{C})$ is an isomorphism for every $t \in B \setminus \{0\}$ and every $k \in \{0, \ldots, 2n\}$. In particular, at the level of the dimensions of the vector spaces involved, we have $\sum_{p+q=k} h^{p, q}_{BC}(t) = b_k$ (with obvious notation) for $t \in B \setminus \{0\}$ and $k \in \{0, \ldots, 2n\}$. Since every $h^{p, q}_{BC}(t)$ varies upper semicontinuously with $t \in B$ ([KS60]) while the Betti number $b_k$ is independent of $t$, we get

$$\sum_{p+q=k} h^{p, q}_{BC}(0) \geq b_k \quad k \in \{0, \ldots, 2n\}.$$

This is an obvious necessary condition for the map $T^{(k)}_0$ to be surjective.

Fix any $C^\infty$ family $(\gamma_t)_{t \in B}$ of Hermitian metrics on the fibres $(X_t)_{t \in B}$ and consider the associated $C^\infty$ family $(\Delta^{(t)}_{BC})_{t \in B}$ of Bott-Chern Laplacians acting on the forms of the $X_t$’s. As is well known,
these Laplacians are elliptic differential operators of order 4 (cf. [KS60, §6, where $\Delta_{BC}^{(t)}$ is denoted by $E_t$, also [Sch07, §2.2b]) and the Hodge isomorphisms they induce identify each $\Delta_{BC}^{(t)}$-harmonic space in each bidegree $(p, q)$ to the corresponding Bott-Chern cohomology group $H_{BC}^{p,q}(X_t, \mathbb{C})$.

Now, the elliptic theory and the compactness of the fibres $X_t$ ensure that each space of forms $C^\infty_{p,q}(X_t, \mathbb{C})$ has a countable orthonormal basis $(e_{t,j}^{p,q}(t))_{j \in \mathbb{N}}$ consisting of eigenvectors of $\Delta_{BC}^{(t)}$, for every $t \in B$. On the other hand, if we choose $\varepsilon > 0$ so small that no eigenvalue of $\Delta_{BC}^{(0)}$ lies in the interval $(0, \varepsilon)$ for any bidegree $(p, q)$, a key result of Kodaira-Spencer [KS60, Lemma 7] ensures the existence of a small open ball $B(0, \delta) \subset \mathbb{C}^N$ such that for every $(p, q)$,

$$B(0, \delta) \ni t \mapsto \bigoplus_{0 \leq \lambda(t) < \varepsilon} E_{\lambda(t)}^{p,q}(\Delta_{BC}^{(t)})$$

defines a $C^\infty$ vector bundle, where $E_{\lambda(t)}^{p,q}(\Delta_{BC}^{(t)})$ stands for the eigenspace, corresponding to the eigenvalue $\lambda(t)$, of $\Delta_{BC}^{(t)}$ acting in bidegree $(p, q)$. The rank of this vector bundle is $h_{BC}^{p,q}(0)$. Let $(e_{t,j}^{p,q}(t))_{1 \leq j \leq h_{BC}^{p,q}(0)}$ be an orthonormal frame of this bundle, consisting of eigenvectors of $\Delta_{BC}^{(t)}$, such that $e_{t,j}^{p,q}(t) \in \ker \Delta_{BC}^{(t)}$ for every $t \in B(0, \delta) \setminus \{0\}$ and every $1 \leq j \leq h_{BC}^{p,q}(t) \leq h_{BC}^{p,q}(0)$. (Shrink $\delta > 0$ if necessary.) Of course, $e_{t,j}^{p,q}(0) \in \ker \Delta_{BC}^{(0)}$ for every $1 \leq j \leq h_{BC}^{p,q}(0)$. Finally, let us fix a class $\{\alpha\}_{DR} \in H^k_{DR}(X, \mathbb{C})$. Since $T_{t}^{(k)}$ is an isomorphism for every $t \neq 0$, there is a unique choice of classes $[\alpha_{t}^{p,q}]_{BC} \in H^p_{BC}(X_t, \mathbb{C})$ (that we identify with the corresponding $\Delta_{BC}^{(t)}$-harmonic forms) such that

$$\{\alpha\}_{DR} = \sum_{p+q=k} [\alpha_{t}^{p,q}]_{BC} = \sum_{p+q=k} \sum_{j=1}^{h_{BC}^{p,q}(t)} e_{t,j}^{p,q}(t) [e_{t,j}^{p,q}(t)]_{BC}, \quad t \in B(0, \delta) \setminus \{0\},$$

with coefficients $e_{t,j}^{p,q}(t) \in \mathbb{C}$ such that $\rho := \sum_{p+q=k} \sum_{j=1}^{h_{BC}^{p,q}(t)} |e_{t,j}^{p,q}(t) |^2$ is independent of $t \in B(0, \delta) \setminus \{0\}$.

By compactness of the sphere of radius $\rho$ in $\mathbb{C}^M$, where $M := \sum_{p+q=k} h_{BC}^{p,q}(t)$ with $t \neq 0$, we get a sequence $B(0, \delta) \setminus \{0\} \ni t_\nu \to 0$ such that, for every $j = 1, \ldots, h_{BC}^{p,q}(t), e_{t,j}^{p,q}(t_\nu)$ converges to some $e_{t,j}^{p,q}(0) \in \mathbb{C}$ when $\nu \to +\infty$. Then, $\{\alpha\}_{DR} \in H^k_{DR}(X, \mathbb{C})$ is the image under $T_0^{(k)}$ of

$$\sum_{p+q=k} \sum_{j=1}^{h_{BC}^{p,q}(0)} e_{t,j}^{p,q}(0) [e_{t,j}^{p,q}(0)]_{BC} \in \bigoplus_{p+q=k} H^p_{BC}(X_0, \mathbb{C}),$$

where, in the second sum above, $h_{BC}^{p,q}(t)$ stands for the Bott-Chern number of bidegree $(p, q)$ of $X_t$ for any $t \neq 0$ close to $0 \in B$. This proves the surjectivity of $T_0^{(k)}$.

\[\Box\]

We will also need the following obvious

**Lemma 3.6.** If $\{\alpha\}_{DR}$ is a real De Rham cohomology class on a complex manifold that can be represented by a form $\xi$ whose all pure-type components are $d$-closed, then $\{\alpha\}_{DR}$ can be represented by a real form $\zeta$ whose all pure-type components are $d$-closed.
Proof. Let $\alpha$ be a real representative of the class $\{\alpha\}_{\text{DR}}$. Then, for some form $u$, $\alpha = \xi + du$. Conjugating, we get $\alpha = \bar{\xi} + d\bar{u}$, hence $\alpha = \frac{\xi + \bar{\xi}}{2} + d\left(\frac{\xi + \bar{\xi}}{2}\right)$. Thus, $\zeta := \frac{\xi + \bar{\xi}}{2}$ is a real representative of the class $\{\alpha\}_{\text{DR}}$ and for every bidegree $(p, q)$, $d\left(\frac{\xi + \bar{\xi}}{2}\right)^{p,q} = \frac{1}{2} d\xi^{p,q} + \frac{1}{2} d\bar{\xi}^{p,q} = 0$. \hfill $\square$

We shall now show that the $E_r$-sG property of the limiting fibre $X_0$ proved in Theorem 3.4 suffices to prove that any deformation limit of Moishezon manifolds is again Moishezon (cf. Theorem 1.1 and the main result in [Pop10]). The result that, together with Theorem 3.4, will prove this fact is the following

**Theorem 3.7.** Let $\pi : \mathcal{X} \rightarrow B$ be a holomorphic family of compact complex $n$-dimensional manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a $\partial\bar{\partial}$-manifold for all $t \in B \setminus \{0\}$. Let $X$ be the $C^\infty$ manifold that underlies the fibres $(X_t)_{t \in B}$ and let $J_t$ be the complex structure of $X_t$.

Suppose there exists a $C^\infty$ family $(\tilde{\omega}_t)_{t \in B}$ of $d$-closed, smooth, real 2-forms on $X$ such that, for every $t \in B$, the $J_t$-pure-type components of $\tilde{\omega}_t$ are $d$-closed. Fix an integer $r \geq 1$ and suppose there exists a $C^\infty$ family $(\gamma_t)_{t \in B}$ of $E_r$-sG metrics on the fibres $(X_t)_{t \in B}$ with potentials depending in a $C^\infty$ way on $t$.

(i) If, for every $t \in B^*$, there exists a Kähler metric $\omega_t$ on $X_t$ that is De Rham-cohomologous to $\tilde{\omega}_t$, then there exists a constant $C > 0$ independent of $t \in B^*$ such that the $\gamma_t$-masses of the metrics $\omega_t$ are uniformly bounded above by $C$:

$$0 \leq M_{\gamma_t}(\omega_t) := \int_X \omega_t \wedge \gamma_t^{n-1} < C < +\infty, \quad t \in B^*.$$ 

In particular, there exists a sequence of points $t_j \in B^*$ converging to $0 \in B$ and a $d$-closed positive $J_0$-$(1, 1)$-current $T$ on $X_0$ such that $\omega_{t_j}$ converges in the weak topology of currents to $T$ as $j \to +\infty$.

(ii) If, for every $t \in B^*$, there exists an effective analytic $(n-1)$-cycle $Z_t = \sum_i n_i(t) Z_i(t)$ on $X_t$ (i.e. a finite linear combination with integer coefficients $n_i(t) \in \mathbb{N}^*$ of irreducible analytic subsets $Z_i(t) \subset X_t$ of codimension 1) that is De Rham-cohomologous to $\tilde{\omega}_t$, then there exists a constant $C > 0$ independent of $t \in B^*$ such that the $\gamma_t$-volumes of the cycles $Z_t$ are uniformly bounded above by $C$:

$$0 \leq v_{\gamma_t}(Z_t) := \int_X [Z_t] \wedge \gamma_t^{n-1} < C < +\infty, \quad t \in B^*.$$ 

Proof. We will prove (ii). The proof of (i) is very similar and we will indicate the minor differences after the proof of (ii). The method is almost the same as the one in [Pop10].

Since the positive $(1, 1)$-current $[Z_t] = \sum_i n_i(t) [Z_i(t)]$ (a linear combination of the currents $[Z_i(t)]$ of integration on the hypersurfaces $Z_i$) on $X_t$ is De Rham cohomologous to $\tilde{\omega}_t$ for every $t \in B^*$, there exists a real current $\beta_t^0$ of degree 1 on $X$ such that

$$\tilde{\omega}_t = [Z_t] + d\beta_t^0, \quad t \in B^.*$$

This implies that

$$\bar{\partial}_t \beta_t^{0,1} = \tilde{\omega}_t^{0,2}, \quad t \in B^.*$$

(18)
In particular, \( \tilde{\omega}_t^{0,2} \) is \( \bar{\partial}\ell \)-exact for every \( t \in B^* \), so it can be regarded as the right-hand side term of equation (19) whose unknown is \( \beta_t^{0,1} \).

For every \( t \in B^* \), let \( \beta_t^{0,1} \) be the minimal \( L^2_{\gamma_t} \) -norm solution of equation (19). Thus, \( \beta_t^{0,1} \) is the \( C^\infty \) \( J_t \)-type (0, 1)-form given by the Neumann formula

\[
\beta_t^{0,1} = \Delta_t''^{-1}\bar{\partial}_t^\ast \omega_t^{0,2}, \quad t \in B^*,
\]

where \( \Delta_t''^{-1} \) is the Green operator of the \( \bar{\partial} \)-Laplacian \( \Delta_t'' := \bar{\partial}_t \bar{\partial}_t^\ast + \bar{\partial}_t \bar{\partial}_t \) induced by the metric \( \gamma_t \) on the forms of \( X_t \). The difficulty we are faced with is that the family of operators \( \Delta_t''^{-1} \) , hence also the family of forms \( \beta_t^{0,1} \), need not extend in a continuous way to \( t = 0 \) if the Hodge number \( h^{0,1}(t) \) of \( X_t \) jumps at \( t = 0 \) (i.e., if \( h^{0,1}(0) > h^{0,1}(t) \) for \( t \in B^* \) close to 0).

As in [Pop10], the way around this goes through the use of special metrics on the fibres \( X_t \). Set

\[
\beta_t^{1,0} := \beta_t^{0,1} \quad \text{and} \quad \beta_t := \beta_t^{1,0} + \beta_t^{0,1}, \quad t \in B^*.
\]

Since \( \tilde{\omega}_t \) is real, this and equation (19) satisfied by \( \beta_t^{0,1} \) imply that \( \tilde{\omega}_t - [Z_t] - d\beta_t \) is a \( J_t \)-type (1, 1)-current. Since this current is \( d \)-exact (it equals \( d(\beta_t^1 - \beta_t) \)) and since every fibre \( X_t \) with \( t \in B^* \) is supposed to be a \( \partial\bar{\partial} \)-manifold, we infer that the current \( \tilde{\omega}_t - [Z_t] - d\beta_t \) is \( \bar{\partial}_t \bar{\partial}_t \)-exact. (See analogue of (1) for currents and the comment in the Introduction on its equivalence to the smooth-form version of the \( \partial\bar{\partial} \)-hypothesis.) Hence, there exists a family of distributions \( (R_t)_{t \in B^*} \) on \( (X_t)_{t \in B^*} \) such that

\[
\tilde{\omega}_t = [Z_t] + d\beta_t + \partial_t \bar{\partial}_t R_t \quad \text{on} \quad X_t \quad \text{for all} \quad t \in B^*.
\]

Consequently, for the \( \gamma_t \)-volume of the divisor \( Z_t \) we get:

\[
u_{\gamma_t}(Z_t) := \int_X [Z_t] \wedge \gamma_t^{n-1} = \int_X \tilde{\omega}_t \wedge \gamma_t^{n-1} - \int_X d\beta_t \wedge \gamma_t^{n-1}, \quad t \in B^*, \quad (22)
\]

since \( \int_X \partial_t \bar{\partial}_t R_t \wedge \gamma_t^{n-1} = 0 \) thanks to the Gauduchon property of \( \gamma_t \) and to integration by parts. Now, the families of forms \( (\tilde{\omega}_t)_{t \in B} \) and \( (\gamma_t^{n-1})_{t \in B} \) depend in a \( C^\infty \) way on \( t \) up to \( t = 0 \), so the quantity \( \int_X \tilde{\omega}_t \wedge \gamma_t^{n-1} \) is bounded as \( t \in B^* \) converges to 0 \( \in B \). Thus, we are left with proving the boundedness of the quantity \( \int_X d\beta_t \wedge \gamma_t^{n-1} = \int_X \partial_t \beta_t^{0,1} \wedge \gamma_t^{n-1} + \int_X \bar{\partial}_t \beta_t^{1,0} \wedge \gamma_t^{n-1} \) whose two terms are conjugated to each other. Consequently, it suffices to prove the boundedness of the quantity

\[
I_t := \int_X \partial_t \beta_t^{0,1} \wedge \gamma_t^{n-1} = \int_X \beta_t^{0,1} \wedge \partial_t \gamma_t^{n-1}, \quad t \in B^*,
\]

as \( t \) approaches \( 0 \in B \).

So far, the proof has been identical to the one in [Pop10]. The assumption made on the \( C^\infty \) family \( (\gamma_t)_{t \in B} \) of \( E_r \)-sG metrics implies the existence of \( C^\infty \) families of \( J_t \)-type \( (n, n-2) \)-forms \( (\Gamma_t^{n,n-2})_{t \in B} \) and of \( J_t \)-type \( (n-1, n-1) \)-forms \( (\zeta_{r-2,t})_{t \in B} \) such that

\[
\partial_t \gamma_t^{n-1} = \bar{\partial}_t \Gamma_t^{n,n-2} + \partial_t \zeta_{r-2,t}, \quad t \in B, \quad (23)
\]

and

\[
\bar{\partial}_t \zeta_{r-2,t} = \partial_t v_{r-3,t}^{(r-2)}, \quad (24)
\]

\[
\bar{\partial}_t v_{r-3,t}^{(r-2)} = 0.
\]

30
Theorem 3.8.

Theorem 3.8.

Theorem 3.8.
exists a $C^\infty$ family $(\gamma_t)_{t \in B}$ of $E_r$-sG metrics on the fibres $(X_t)_{t \in B}$ whose potentials depend in a $C^\infty$ way on $t \in B$.

Let $(Z_t)_{t \in B^*}$ be a $C^\infty$ family of effective analytic divisors such that $Z_t \subset X_t$ for all $t \in B^*$. The De Rham cohomology class $\{[Z_t]\}_{DR} \in H^2(X, \mathbb{R})$ of the current $[Z_t]$ of integration over $Z_t = \sum \gamma_t(t) Z(t)$ (where $\gamma_t(t) \in \mathbb{N}^*$ and the $Z_t(t)$’s are irreducible analytic hypersurfaces of $X_t$) is integral. Therefore, the continuous, integral-class-valued map

$$B^* \ni t \mapsto \{[Z_t]\}_{DR} \in H^2(X, \mathbb{Z})$$

must be constant, equal to an integral De Rham 2-class that we denote by $\{\alpha\}$. By Lemmas 3.5 and 3.6, there exists a $C^\infty$ family $(\tilde{\omega}_t)_{t \in B}$ of $d$-closed, smooth, real 2-forms on $X$ lying in the De Rham class $\{\alpha\}$ such that, for every $t \in B$, the $J_t$-pure-type components of $\tilde{\omega}_t$ are $d$-closed. In particular, for every $t \in B^*$, the current $[Z_t]$ is De Rham-cohomologous to $\tilde{\omega}_t$.

Thus, all the hypotheses of Theorem 3.7 are satisfied. From (ii) of that theorem we get that the $\gamma_t$-volumes $(v_{\gamma_t}(Z_t))_{t \in B^*}$ of the divisors $Z_t$ are uniformly bounded. This implies, thanks to Lieberman’s strengthened form ([Lie78, Theorem 1.1]) of Bishop’s Theorem [Bis64], that a limiting effective divisor $Z_0 \subset X_0$ for the family of relative effective divisors $(Z_t)_{t \in B^*}$ exists. Since this family has been chosen arbitrarily, it follows that $X_0$ has at least as many divisors as the nearby fibres $X_t$ with $t \neq 0$ and $t$ close to 0. Meanwhile, we know (see, e.g., [CP94, Remark 2.22]) that the algebraic dimension of any compact complex manifold $X$ is the maximal number of effective prime divisors meeting transversally at a generic point of $X$. It follows that the algebraic dimension of $X_0$ is $\geq$ the algebraic dimension of the generic fibre $X_t$ with $t \in B^*$ close to 0.

Note that Theorem 3.8 is an upper semicontinuity result for the algebraic dimensions of the fibres of a holomorphic family of compact complex manifolds whose generic fibre is assumed to be a $\partial \bar{\partial}$-manifold. Without the $\partial \bar{\partial}$-assumption on $X_t$ with $t \neq 0$, the statement is known to fail even when the fibres are complex surfaces. An example of a family of compact complex surfaces of class VII (hence non-Kähler and even non-$\partial \bar{\partial}$), whose algebraic dimension drops from 1 on the generic fibre $X_t$ to 0 on the limiting fibre $X_0$, was constructed by Fujiki and Pontecorvo in [FP10].

**Proof of Theorem 1.1.** Let $n = \dim_{\mathbb{C}} X_t$ for all $t \in B$. The Moishezon property is well known to imply the $\partial \bar{\partial}$-property, so the fibre $X_t$ is a $\partial \bar{\partial}$-manifold for every $t \in B \setminus \{0\}$. Therefore, Theorem 3.8 tells us that $a(X_0) \geq a(X_t)$ for all $t \in B \setminus \{0\}$. Meanwhile, $a(X_t) = n$ for every $t \in B \setminus \{0\}$ by the Moishezon assumption on every $X_t$ with $t \in B \setminus \{0\}$. Since $a(X_0) \leq \dim_{\mathbb{C}} X_0 = n$, we must have $a(X_0) = n$. Hence, $X_0$ must be Moishezon.

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