Fitting higher order moments of empirical financial series with GARCH models

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Abstract

Here we have analysed a GARCH(1,1) model with the aim to fit higher order moments for different companies' stock prices. When we assume a gaussian conditional distribution, we fail to capture any empirical data. We show instead that a double gaussian conditional probability better captures the higher order moments of the data. To demonstrate this point, we construct regions (phase diagrams) in higher order moment space, where a GARCH(1,1) model can be used to fit the higher order moments and compare this with empirical data from different sectors of the economy. We found that, the ability of the GARCH model to fit higher order moments is dictated by the time window our data spans. Primarily, if we have a time series, using a GARCH(1,1) model with a double gaussian conditional probability (a GARCH-double-normal model) we cannot necessarily fit the statistical moments of the time series. Highlighting, that the GARCH-double-normal model only allows fitting of specific lengths of time series. This is indicated by the migration of the companies’ data out of the region of the phase diagram where GARCH is able to fit these higher order moments. In order to overcome the non-stationarity of our modelling, we assume that one of the parameters of the GARCH model, $\alpha_0$, has a time dependence.

Keywords: GARCH, Phase Diagrams, Double Gaussian, Empirical Data

JEL Classification: C10, C40, C50, C80

1. Introduction

Modelling of financial time series is a very extensive area of research. In general, there are large over simplifications of a financial time series, so there has been research into the modelling of time series with a plethora of different mathematical tools. The main issues are; the large tails present in most financial time series, the heteroskedasticity of volatility and the conditional second order moments of price change, volatility; demonstrating the presence of a further stochastic dynamic in addition to those of price change. These stochastic processes evolve in the same system but have different time scales; a fast changing stochastic process (the price) and a slow changing stochastic process (the volatility). This motivates the creation of the Autoregressive Conditional Heteroskedasticity models (ARCH) by Engle [1] and later generalised (GARCH) by Bollersev [2].
The autoregressive processes allow a stochastic model to predict price change probability of a given time series. The level of return at a certain instance is described by a probability distribution (usually gaussian). Where the standard deviation of the process varies with time and is defined by both the standard deviation and level of return at the previous time instance(s). However, GARCH is not limited to simply financial systems but to any system where this two scale stochastic process is seen, for instance the study by Kumar et al. on atmospheric cycles [3] or a study on pathogen growth by Ali in [4].

GARCH has been thoroughly researched. Whilst it captures the heteroskedasticity of volatility (something standard time series models do not) and the two time scales of price dynamics. The original model proposed by Bollerslev, [2], does not capture the leveraging effect of volatility and it does not accommodate for the jumps seen in volatility. When we derive expressions for higher order moments, it is not possible to solve past the sixth order moment analytically for the smallest GARCH model. For longer-time history models, for example, GARCH(2,2), generating higher order moment expressions is a difficult task. Extensive research has been undertaken to adapt the original Bollerslev GARCH model to fit the different empirical observations of time series. For example, there is the A-GARCH (asymmetric GARCH) model to account for the asymmetrical effect of volatility to price change, [5] and also the GJR-GARCH model to account for this asymmetry, [6, 7, 8]. One particularly notable improvement to account for the asymmetry is the E-GARCH model of Nelson, [9]. We can also hypothesise a GARCH model that has more than one GARCH term and so looks at the development of a higher-dimensional system, such models are the BEKK-GARCH and DCC-GARCH models, [10, 11]; they have shown promise alongside the use of a CAPM (capital asset price model). Moreover, the increase in computational power in the early 1990s allowed for the collection of trade (tick) data. With this increased data collection, one needs to consider how to model the type of volatility seen on short time scales, and so realised volatility measures were created and incorporated in to GARCH frameworks, [12, 13]. In the majority of modelling, one assumes a stationarity of time series, however, this is not necessarily the case and as such the fractionally integrated GARCH (FIGARCH) models were proposed, [14, 15, 16]. One final development that motivates the work in this paper is the conversion from discrete time models to continuous time models leading to the creation of the continuous-GARCH (coGARCH) models, [17]. With these developments of the GARCH model, there is an increase in complexity of analytical solutions to the moment equations. Therefore, we wish to seek how effective the original GARCH model is at fitting higher order moments of empirical financial data series for different sectors of the economy.

GARCH is a stochastic model, therefore it enables one to fit time series for perceived stochastic phenomena, for example financial time series for stock prices. Three independent stochastic quantities of a particular time series can, in principle, be fitted by three GARCH parameters. If one cannot fit a GARCH model it is not enough to simply say that the time series we have in question is not stochastic. There are a multitude of factors that could affect the accuracy of the GARCH model that can be fitted to a given time series, as presented in this paper. In particular, we show that there is a limit to the time window that can be modelled by a GARCH process.

The question is, therefore, how to fit prices for a time series. This is a difficult process as selecting values of the GARCH parameters depends on what we wish to achieve. One can set a task to fit a certain set of statistical data moments, for example \( \langle x^2 \rangle \), \( \langle x^4 \rangle \) and \( \langle x^6 \rangle \) by the three parameters of the GARCH(1,1) model. Alternatively, we can choose another set \( \langle x^2 \rangle \), \( \langle x^4 \rangle \) and \( \langle x^8 \rangle \). Obviously, we will have different parameter values for the two cases. Moreover, we can ask if we can or cannot fit three empirically estimated moments of a chosen stock price series by three
GARCH(1,1) parameters. Here we will show, that it is often not possible. And by the use of phase diagrams in \((x^n), (x^m)\) space, we shall see where GARCH(1,1) can fit the corresponding moments. The region where the parameters of the GARCH model can fit empirical moments shall be referred to throughout as the GARCH existence region or the ‘GARCHable’ region.

If we evaluate the time series and conclude that GARCH(1,1) parameters cannot fit empirical moments, then we can judge that the time series might no longer be purely stationary in nature or a significant modification of the GARCH(1,1) model is needed. Most time series are not purely stationary. As in the case of stocks and shares, the global economic climate/factors are a major stressor when determining the price of a given stock. Such a complex dependence of factors leads to a very fluid economic environment. With the changing of the economic climate, the time series of certain stock prices also changes in response to this. We can draw an analogy to a phase change. When we have a very large economic change, the ‘state’ of the time series changes in response to this. In the case of a small economic change, we can deem the time series of this company to gradually evolve its ‘state’. So in both cases, the GARCH model fitting the empirical moments would also change. Moreover, we may not be able to find a GARCH model fitting long time series, due to this change of state. So it may be plausible that we have a limit to the time horizon GARCH can be used for. However, with this changing state in the stock price, one would expect GARCH models with different sets of parameters to fit the empirical moments for each time window of an empirical series. This presents the GARCH model with parameters evolving in time. Within this time dependence we can see the response of these parameters to the economic environment. The information about the changing of states of time series can identify changing economic factors and trends, including crisis periods [18]. Our final motivation for our work is to allow an individual to evaluate how best to model a series they are interested in to gain the most accurate parameters, if possible, of a GARCH(1,1) model for their particular needs.

The estimation data in this paper is for the period of 6th October 2005 to 6th October 2011 for most data sets presented, unless otherwise stated. This can be divided in to a pre-crisis, post-crisis and crisis period. This division is extremely valuable in deducing the statistics that are evident within an economic crisis. This will be reflected in the results we gain from evaluating certain order moments in the years from 2005 to 2011.

The paper is organised as follows, in section 2, we initially analyse the sixth order moment for several companies and discuss the economic environments. In section 3, we discuss the methods we will be using and how we have created the phase diagrams. Section 4 presents our findings on GARCH models with a gaussian conditional probability (we will proceed calling these GARCH-normal models) for time series fitting, whilst also showing their failure to describe higher order moments of financial time series. In section 5 we discuss GARCH models with a double gaussian conditional probability (we shall proceed calling these the GARCH-double-normal models) to account for this shortfall. We also show how with the assumption of time dependent parameters the data we analyse is non-stationary. Section 6 concludes our paper.

2. Raw Data Analysis

In order to determine the behaviour of the moments of financial time series, we first highlight the time dependence of the sixth order moment for a group of companies and a government bond (gilt) through the previous financial crisis of 2008. To do this we use the daily closing price of each trading day over yearly periods for 8 years, 2002-2010. We then use the below equation to calculate the nth order moment:
\langle x^n(t) \rangle = \frac{1}{N} \sum_{t-N/\delta t < \tau < t+N/\delta t} x^n(\tau) \quad (1)

where we take \( N \) to be a year, each time step \( \delta t \) a trading day and the period we average over is 252 days (due to trading exclusion dates). Here we define \( x \) as the logarithm of price change:

\[ x(\tau) = \ln \left( \frac{y(\tau + \delta t)}{y(\tau)} \right) \quad (2) \]

\( y(\tau) \) is the closing price at day \( \tau \). We also consider the average over all 18 years taking \( M = 4536 \):

\[ \langle x^n \rangle = \frac{1}{M} \sum_{i=1}^{M} x^n_i \quad (3) \]

We also evaluate empirical standardised moments for different ratios of higher order moments:

\[ \Gamma_{2m} = \frac{\langle x^{2m} \rangle}{\langle \sigma^2 \rangle^m} \quad (4) \]

From figure 1 we see that before the crisis (2002-2007) the sixth order statistical moments in general has a very flat dependence for the British companies. This is not mirrored in the American data as there is, in general, a negative rate of change for the sixth order moment.

Figure 1: Sixth order moment time evolution for several global companies, to highlight the behaviour of the sixth order moment in the financial crash of 2008. The top figure shows the evolution for primarily British companies and a gilt, whilst the bottom shows the same information for primarily American companies. The time period analysed for all companies spans 2002-2010.
What is evident from the yearly sixth moment values from 2002 to 2010, figure [1], is that within a financial crisis and subsequent economic collapse, the rate of change of the moments with respect to time increases in periods of financial worry. The nature of the rate of change, if it is positive or negative, can be attributed to the conditions the companies encounter during the financial crisis. If we focus upon some specific companies we can try to highlight the circumstances that cause the nature of the rates of change. Firstly, during the 2007-2009 economic recession, Lloyds Bank was taken over by the UK Government. In figure [1] we can see that during the economic crisis, December 2007-June 2009, there is a negative rate of change of the sixth moment and as such the magnitude of the sixth moment is much lower after the economic recession, in 2009. This negative rate of change is seen in the UK 10 year Government bonds (gilts). Within the economic crisis, the Bank of England interest rates were decreased quite drastically and as such this increased the price of the bonds. This reduction in interest rates was a direct interference, as is the bailing out of a company, [19]. We can infer that this nature of interference, to increase the attractiveness of the security to increase the quantity of security being bought, leads to a decrease in the sixth order statistical moment. In essence, a more deterministic interference results in a lower $\langle x^6 \rangle$ value. Increasing the value of the sixth order moment gains a much broader distribution for the price of the security analysed, highlighting increased uncertainty within the process. Comparing this to the qualitative environment of the company at the time, reflects the level of external interference in the process.

Conversely, if we look at other non-British companies, their sixth order statistical moments increase with time. The largest in magnitude comes from Credit Suisse. Morgan Stanley, UBS, BNY Mellon, Goldman Sachs, HSBC and Credit Suisse also have increasing rates of change of sixth order moment. These companies were not given such direct assistance and therefore the nature of these sixth order moments within this time period is indicative of an organic company within an economic crisis.

3. Stochastic Model

In this section we focus upon a GARCH-normal(1,1) model. One can see from Bollersev’s work [2] that for such a model; $x_t$ is a random variable with zero mean and possesses the variance, $\sigma_t^2$. We define $x_t \equiv \zeta_t \sigma_t$. Here $\zeta_t$ is a random process with standard deviation equal to one. Depending on the system we wish to model, the variable $\zeta_t$ can be described by different probability distributions, see for example, [8, 17, 20, 21, 22]. However, as mentioned by Mantegna et al. in [23], it is often chosen to be gaussian. We shall also simulate a gaussian random variable, hence we denote this model the GARCH-normal model. The GARCH(1,1) class of stochastic processes are defined via the relation:

$$\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

If we know the exact probability density, $p(x)$, of a process, we can write the definition of moments by, $E[x^n] = \int_{-\infty}^{\infty} P(x)x^ndx$. However, we do not know the analytical expression for the probability distribution of the GARCH process. To resolve this problem Bollerslev, [2], proposed recurrence relations for moments of the GARCH-normal(1,1) model:

$$E(x_t^{2m}) = \frac{\sum_{n=0}^{m-1} a_n^{-1}(E(x_t^{2n})\alpha_0^{m-n} m \mu_\alpha(\alpha_1, \beta_1, n))}{1 - \mu_\alpha(\alpha_1, \beta_1, m)}$$

5
where,
\[
\mu(\alpha_1, \beta_1, m) = \sum_{j=0}^{m} \binom{m}{j} a_i \alpha_1^j \beta_1^{m-j}, \quad a_j = \prod_{i=1}^{j} (2i - 1)
\] (7)

Therefore, we can derive equations for the variance, fourth order and sixth order standardised moments:
\[
\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}
\] (8)
\[
\Gamma_4 = \frac{E(x_t^4)}{E(x_t^2)^2} = 3 + \frac{6\alpha_1^2}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2}
\] (9)
\[
\Gamma_6 = \frac{E(x_t^{2(3)})}{(E(x_t^{2(1)})^3} = \frac{15(1 - \alpha_1 - \beta_1)^3(1 + \frac{3(\alpha_1 + \beta_1)}{1 - \alpha_1 - \beta_1} + \frac{3(1 + \frac{3(\alpha_1 + \beta_1)}{1 - \alpha_1 - \beta_1})(\beta_1^2 + 2\alpha_1\beta_1 + 3\alpha_1^2)}{1 - 15\alpha_1^2 - 9\alpha_1\beta_1 - 3\alpha_1\beta_1^2 - \beta_1^4})}{1 - 15\alpha_1^2 - 9\alpha_1\beta_1 - 3\alpha_1\beta_1^2 - \beta_1^4}
\] (10)

The relation (8) and (7) are defined for all moments if we fix the three GARCH parameters, \(\alpha_0\), \(\alpha_1\) and \(\beta_1\). For a moment to exist it is clear that \(\mu(\alpha_1, \beta_1) < 1\). Therefore, we can solve \(\mu(\alpha_1, \beta_1)\) for \(\beta_1 = \beta_1(\alpha_1)\). In doing so, we can create figure 2. In this figure, we see the different curves of \(\beta_1 = \beta_1^{(n)}(\alpha_1)\) where \(n\) takes the value; 2, 4, 6, 8, 10, and 12. When \(\beta_1 < \beta_1^{(n)}(\alpha_1)\) the corresponding moments have finite value, for \(\beta_1 > \beta_1^{(n)}(\alpha_1)\), these moments diverge. Since the particular line \(\beta_1 = \beta_1^{(n)}(\alpha_1)\) separates the region of parameters where the sixth moment exists and where it does not, we can interpret this as a phase diagram in parameter space, 2. Below, we extend this idea. However, the inverse problem to estimate the three GARCH parameters, if three moments are known, is much more complicated and reduces to a set of transcendental equations which can be hard to solve. In figure 2 we present a filled area that shows the region of existence of the sixth order moment. The red circle in this figure represents parameter values that allow for the existence of the second, fourth and sixth moment but not the eighth. While for the black square in figure 2 only the second and fourth moments are finite. We can daw the conclusion, that GARCH-normal(1,1) cannot model time series if the empirical fourth order standardised moment value, \(\Gamma_4 > 6\), as seen by the work undertaken in Appendix B. We therefore, need to ask how is it possible to access higher values of \(\Gamma_4\) for empirical data where a GARCH-normal model is no longer sufficient.

4. GARCH-Normal Models

If we want to fit the second, fourth and sixth moments, the values of the parameters must be below the divergence curve; \(\beta_1 < \beta_1^{(6)}(\alpha_1)\) which does not cover all parameter space for the existence of the fourth \((\beta_1 < \beta_1^{(4)}(\alpha_1)\) and the second \((\beta_1 < \beta_1^{(2)}(\alpha_1)\) order moments. This can result in some values of the fourth and second moment, or fourth order standardised moment and the second order moment being unreachable for GARCH modelling.

For the fourth and sixth order moment we can obtain the divergence line explicitly, and so derive:
\[
\beta_1^{(4)} = \sqrt{1 - 2\alpha_1^2 - \alpha_1}
\] (11)
Figure 2: The ‘Bollerslev Phase Diagram’, [2], showing the divergence of moments in a GARCH-normal(1,1) model. The highlighted area shows the existence region for the sixth order moment. The red circle presents an example of $\alpha_1, \beta_1$ values that allow for the sixth, fourth and second order moments to exist, whilst the black square shows an example of $\alpha_1, \beta_1$ values that allow for only the fourth and second order moment to exist.
\[ \beta_1^{(6)} = \frac{(-8\alpha_1^3 + \sqrt{96\alpha_1^5} - 16\alpha_1^4 + 1 + 1)^{\frac{1}{3}}}{2^{\frac{1}{3}}} - \alpha_1 - \frac{2(2)^{\frac{1}{3}} \alpha_1^2}{(-8\alpha_1^3 + \sqrt{96\alpha_1^5} - 16\alpha_1^4 + 1 + 1)^{\frac{1}{3}}} \]  

(12)

For higher order moments, the divergence lines are defined by high order algebraic equations, which cannot be solved analytically.

4.1. Company Trajectories

Here we will consider the situation of when we need to fit only the second and fourth moments, or equivalently, fitting the variance \( \langle x^2 \rangle \) and fourth order standardised moment \( \Gamma_4 \). Since the GARCH-normal(1,1) model has three parameters we can conclude that we can express two GARCH parameters, for instance, \( \alpha_1 \) and \( \beta_1 \), as a function of the third parameters \( \alpha_0 \). In doing so, we derive:

\[ \beta_1 = 1 - \frac{\alpha_0}{\sigma_{emp}^2} - \sqrt{\frac{2\alpha_0}{\sigma_{emp}^2} - \frac{\alpha_0^2}{(\sigma_{emp}^2)^2}} \]

(13)

\[ \alpha_1 = \sqrt{\frac{2\alpha_0}{\sigma_{emp}^2} - \frac{\alpha_0^2}{(\sigma_{emp}^2)^2}} \]

(14)

It is clear from these equations that for any value of \( \Gamma_{4,emp} > 3 \) and \( \sigma_{emp}^2 > 0 \), we can find a family of one-parametric GARCH models. So, we obtain the parametric curves; \((\alpha_1(\alpha_0), \beta_1(\alpha_0))\), in \((\alpha_1, \beta_1)\) space. Such curves represent the ‘company trajectories’ with already fixed (empirical) variance, \( \sigma_{emp}^2 \), and empirical fourth order standardised moments, \( \Gamma_{4,emp} \).

In figure 3, we see an extension of figure 2 for a banking stock, a commodity, a pharmaceutical and a mining company, respectively. The dotted lines represent the parameters of the GARCH-normal model for the given company. They allow one to see the stability of the time series, in essence, what statistical moment can exist for the GARCH description of the empirical data. It is evident, for the longest time period (18 years) that apart from the gold ETF, trajectories of all other companies lie above the divergence line of the sixth order moment. Implying, the values of the fourth and second order empirical statistical moments do not allow for any higher moments to be fitted via a GARCH-normal model.

If one decreases the time window of data collection, for example a year, 6th October 2017 to 6th October 2018, or even six months, 6th April 2018 to 6th October 2018, then we can see the migration of the company trajectory to deeper inside the stability region in the \((\alpha_1, \beta_1)\) plane, where higher moments are finite (see figures 3a, 3b, 3c, 3d). We have also examined the time lengths of nine months, fifteen months and three years. In these figures (figures 3a, 3b, 3c, 3d), it is clear that the Lloyds Bank and Rio Tinto 6 month time series allows the largest number of higher order moments to exist. Apart from Lloyds Bank and Rio Tinto all other time series present only with the sixth moment as the highest finite moment. However, in general, it is clear to see that the shorter a time series we take, the more moments exist for a GARCH-normal(1,1) model.

As we traverse a company’s trajectories in \((\alpha_1, \beta_1)\) space we can work out the value of the sixth order standardised moment generated from the GARCH-normal(1,1) model for the specific \( \alpha_1 \) and \( \beta_1 \) values. In table 1 we see the minimum and maximum of \( \Gamma_6 \) generated, when the sixth order moment exists. We can see in general \( \Gamma_6 \) does not vary significantly along the company trajectory.
Figure 3: The stability phase diagram for the GARCH-normal(1,1) moments with an overlap of company trajectories. Panel (a) shows the trajectory for Lloyds Bank, where the shortest time window allows up to the eighth order moment to exist. Panel (b) shows the trajectory of Gold ETFs, where the shortest time window allows to the sixth moment to exist. Panel (c) shows the same for GSK, which also allows up to the sixth moment to exist and lastly, panel (d) shows the same for Rio Tinto, which allows up to the eighth order moment to exist.
| Company       | Minimum   | Maximum   |
|--------------|-----------|-----------|
| Lloyds- 6 Months | 33.3700   | 37.0000   |
| GSK- 6 Months  | 60.3386   | 76.5162   |
| Gold- 3 Year   | 90.3439   | 253.4463  |
| Gold- 1 Year   | 79.3684   | 188.9430  |
| RioTinto- 3 Year| 36.1730   | 42.5383   |
| RioTinto- 1 Year| 34.1419   | 39.9972   |
| RioTinto- 6 Months | 30.9564   | 35.3291   |

Table 1: The minimum and maximum values of $\Gamma_6$ along the company’s trajectories in $(\alpha_1, \beta_1)$ space.

4.2. Methods of Parameter Fitting

Let us consider the algorithms we can use to fit empirical values of $\langle x^2 \rangle$, $\langle x^4 \rangle$ and $\langle x^6 \rangle$ which can be reformulated as: variance, $\sigma_{emp}^2$, as well as the fourth and sixth order standardised moments, $\Gamma_{4, emp}$ and $\Gamma_{6, emp}$ respectively. In the first approach, we present $\alpha_1$ and $\beta_1$, as a function of $\alpha_0$, that is $\alpha_1(\alpha_0, \sigma_{emp}^2, \Gamma_{4, emp})$ and $\beta_1(\alpha_0, \sigma_{emp}^2, \Gamma_{4, emp})$, from equations (13) and (14), then numerically solve the equation:

$$\Gamma_6(\alpha_1(\alpha_0, \sigma_{emp}^2, \Gamma_{4, emp}), \beta_1(\alpha_0, \sigma_{emp}^2, \Gamma_{4, emp}), \alpha_0) = \Gamma_{6, emp}$$

(15)

to find the value of $\alpha_0$. This method is inspired by the trajectory analysis we use in the previous section. We search for $\alpha_0$ by traversing the trajectory and trying to fit the empirical sixth order standardised moment. However, if $\Gamma_6$ is lower than the minimum or larger than the maximum of possible $\Gamma_6$ stated in table 1, this equation cannot be solved. Indicating that the GARCH-normal model with such a value of $\Gamma_6$ does not exist.

In the second approach to fit empirical values of $x^2$, $x^4$ and $x^6$, we first fit the empirical fourth and sixth order standardised moments using the fact that $\Gamma_{4}(\alpha_1, \beta_1)$ and $\Gamma_{6}(\alpha_1, \beta_1)$ do not depend on $\alpha_0$, see equations (9) and (10). Therefore, we can reduce the problem to two equations:

$$\Gamma_{4}(\alpha_1, \beta_1) = \Gamma_{4, emp}$$

$$\Gamma_{6}(\alpha_1, \beta_1) = \Gamma_{6, emp}$$

(16)

allowing us to evaluate values of $\alpha_1, \beta_1$ and reserve $\alpha_0$ to the fitting of variance; $\alpha_0 = \sigma_{emp}^2(1 - \alpha_1 - \beta_1)$. The set of equations (16) can be further reduced to one equation by eliminating $\beta_1$ using the first equation of the set namely:

$$\beta_1 = \sqrt{1 - 2\alpha_1^2 - \frac{6\alpha_1^2}{\Gamma_{4, emp} - 3} - \alpha_1}$$

(17)

and substituting it to the second equation of (16). This enables us to write the one-variable equation; $\Gamma_6(\beta_1(\alpha_1, \Gamma_{4, emp}), \alpha_1) = \Gamma_{6, emp}$. Note, that we similarly can exclude $\alpha_1$, resulting in equations for $\beta_1$.

4.3. Phase Diagram

The equations (16) can only be solved for some region in standardised moment space, $(\Gamma_4, \Gamma_6)$, in figure 4 we see two highlighted points. The first point $(1.7, 8)$ is inside the ‘GARCHable’
Figure 4: The phase diagram for $\ln(\Gamma_4), \ln(\Gamma_6)$ space. The filled yellow region shows the ‘GARCHable’ region, whilst the rest of the space is where the certain values of $\Gamma_4$ and $\Gamma_6$ cannot be fitted via a GARCH-normal(1,1) model. The two highlighted points (1.7, 8) and (2.5, 8) show the values of $\Gamma_4$ and $\Gamma_6$ that can be fitted by a GARCH model and those that cannot, respectively. The other data points in the space, represent the empirical data for several companies, truncated to 10-90 day windows, incremented in 10 day periods. It is shown here that no empirical data can be fitted via a GARCH-normal(1,1) model.

region. This is the region of phase space where the respective values of the fourth and sixth order standardised moments can be fitted via a GARCH-normal model. The second point (2.5, 8), is outside of the ‘GARCHable’ region, highlighting that these moment values cannot be fitted via a GARCH-normal model. Therefore, no solution is possible to the equations (16).

To evaluate the appropriateness of a GARCH-normal(1,1) model for the fitting of higher order moments in stock market data, we shall be investigating time series for companies of different sectors of the economy by estimating their empirical values of the 4th and 6th order standardised moments and comparing to the GARCH-normal(1,1) phase region in $(\Gamma_4, \Gamma_6)$ space, figure 4. To see the effect of the length of the time window, the interval of time we take the data over, on the ability of GARCH to fit empirical moments, we divide data in to different economic periods. In figure 5 we show the different regions we wish to evaluate, we start with taking six month periods; 6th April to the 6th October in the years 2005, 2008, 2011, 2014 and 2017. We break these six month time periods in to 10, 20, 30, 40, 50, 60, 70, 80 and 90 day windows, an example of this can be seen for the several stocks in figure 4. We then overlap these data points on top of the ‘GARCHable’ region detailed above.

We never see the empirical data inside of the GARCH-normal(1,1) phase region for the time periods analysed. Therefore, we can say that a GARCH-normal(1,1) model is unable to fit higher order moments of the time series analysed.
5. GARCH-Double-Normal Models

It is clear from this initial analysis of the GARCH-normal model that gaussian conditional distributions do not allow a fitting for $\Gamma_4$ with a finite value of $\Gamma_6$. This problem can potentially be resolved if we replace the conditional gaussian distribution with a distribution allowing to have a better flexibility in higher order GARCH standardised moments. Therefore, we seek a distribution with a larger fourth order standardised moment value in the region where $\Gamma_6$ exists. We introduce the double gaussian distribution \[24\], where $p(x)$:

$$p(x) = ae^{-\frac{x^2}{\sigma_1^2}} + be^{-\frac{x^2}{\sigma_2^2}} \tag{18}$$

In addition to an obvious normalisation condition:

$$a + b = 1 \tag{19}$$

we also have constraints on the 2nd moment:

$$E[x^2] = a\sigma_1^2 + b\sigma_2^2 = 1 \tag{20}$$

due to the requirement that the conditional distribution for a GARCH process should have variance equal to one. We can introduce two more variables (higher order moments; 4th and 6th moments of the conditional distribution) which fully define all parameters in equation \[18\]:

$$E[x^4] = a\sigma_1^4 + b\sigma_2^4 = \mu_4 = \frac{\eta_4}{3} \tag{21}$$

$$E[x^6] = a\sigma_1^6 + b\sigma_2^6 = \mu_6 = \frac{\eta_6}{15} \tag{22}$$

The parameterisation \[21\]-\[22\] of the double Gaussian distribution allows us to generalise Bollerslev’s equation \[9\]. The 2nd order moment is not affected and is still determined by equation \[8\], while the 4th and 6th order standardised moments for GARCH with double-gaussian distribution can be written as:

$$\Gamma_4 = \frac{\eta_4(1 - \alpha_1 - \beta_1)^2(1 + \frac{2(\alpha_1 + \beta_1)}{1 - \alpha_1 - \beta_1})}{1 - \eta_4\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \tag{23}$$

$$\Gamma_6 = \frac{\eta_6(1 - \alpha_1 - \beta_1)^3(1 + \frac{3(\alpha_1 + \beta_1)}{1 - \alpha_1 - \beta_1}) + 3(1 + \frac{2(\alpha_1 + \beta_1)(\eta_4\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2)}{1 - \eta_4\alpha_1 - 2\alpha_1\beta_1 - \beta_1^2})}{1 - \eta_6\alpha_1^4 - 3\eta_4\alpha_1^2\beta_1 - 3\alpha_1\beta_1^2 - \beta_1^4} \tag{24}$$
Using the methods described prior, based on the existence of solutions of the set of equations (16) and equations (23) and (24), we create a family of phase diagrams parameterised by $\eta_4$ and $\eta_6$. To understand which empirical values are achievable using GARCH-double-normal, we need to understand restrictions for the whole family of phase diagrams. We see that these are bounded due to limitations for $\eta_4$ and $\eta_6$ obtained in Appendix A (conditions A.9 and A.10). These limitations require all phase diagrams be started from points above or on the dashed line, figure 6. Only data above the dashed line can be described by a GARCH-double-normal model (which is the case for the empirical data collected for the securities we have considered here).

![Figure 6: The phase diagrams for ($\Gamma_4$, $\Gamma_6$) space for GARCH-double-normal(1,1) models corresponding to different parameters given in table 2. From equations (21) and (22), we gain the condition for the distribution to be double gaussian, seen by the dotted line. We then overlay the empirical data for Bank of America, truncated from 1% to 100% of the length of the time series, incremented in one percent steps. To highlight the ability of the GARCH-double-normal model to fit higher order moments for specific lengths of time windows we present three ‘GARCHable’ regions. Each has a different time window that it can fit, shown by the letter, associated with table 2.](image)

5.1. Time Windows

In figure 6, we see three phase diagrams for three different double gaussian distributions. Parameters for these phase diagrams are given in table 2.

Figure 6 demonstrates how altering the parameters $\eta_4$ and $\eta_6$ of the GARCH-double-normal model enables us to capture different lengths of time windows of the empirical data. The data used in figure 6 is for the Bank of America time series for 6th October 2000 to 6th October 2018. We truncate the time series in to different lengths. We start with 1% of the overall length and increment by 1% up to the whole length of the time series. In other words, the first, most left point corresponds to 40 days of data, (from 06/10/2000 to 01/12/2000), the second point corresponds to moments obtained for 80 days of data (from 06/10/2000 to 25/01/2001) and so on.
Position | \( \eta_4 \) | \( \eta_6 \) | \( t_{\text{min}} \text{(Days)} \) | \( t_{\text{max}} \text{(Days)} \)  
--- | --- | --- | --- | ---  
Leftmost (Red) | 5 | 41.7 | 171 (A) | 600 (B)  
Centre (Blue) | 7 | 81.7 | 943 (C) | 1500 (D)  
Rightmost (Green) | 12 | 240 | 1586 (E) | 1587  

Table 2: Parameters of the conditional double gaussian distributions used to construct ‘GARCHable’ regions in figure [6]. The table summarises the parameters of the distributions used to model the time windows (A) to (B), (C) to (D) and (E). These are the limits of the time horizon in days that the particular instance of the double gaussian distribution can be used to fit the higher order moments of the empirical data of the Bank of America.

For the leftmost phase diagram we use a double gaussian distribution with \( \eta_4 = 5, \eta_6 = 41.7 \). This allows to fit \( \sigma^2, \Gamma_4 \) and \( \Gamma_6 \) for the time window of duration in the interval, \( 171 \leq t \leq 600 \) days. When fitting higher order moments for longer time windows, we need to use double gaussian distributions with parameters summarised in table [2]. It is not possible to gain a GARCH process with a double gaussian distribution to capture all of the empirical data’s higher order standardised moments in this phase space. We denote this behaviour as the local ability to model higher order moments of financial time series by the GARCH-double-normal model. Figure [6] uses the data from the Bank of America, however, this behaviour is seen throughout the empirical data we have studied. In order to capture empirical data one must first decide on the time window they wish to model and then ascertain a suitable distribution that will capture this window.

5.2. Time Dependence of GARCH-double-normal Parameters

Once we have fixed the time window we wish to model, we can study what happens when the window with this fixed duration shifts in time. This can be done by attributing to the higher order moments a time moment, \( t \), corresponding to the middle point (the median) of the time window we wish to model. If we fix the double gaussian distribution (in essence select certain \( \eta_4 \) and \( \eta_6 \)), we can gain the set of GARCH parameters, \( \alpha_0, \alpha_1 \) and \( \beta_1 \) that describes the particular time median. If we change the instance in time we look at by moving the time median of the window, then the GARCH parameters \( \alpha_0, \alpha_1 \) and \( \beta_1 \) also change. Below, we observe that the GARCH parameters \( \alpha_0(t), \alpha_1(t) \) and \( \beta_1(t) \) significantly vary with time, highlighting the non-stationarity of our modelling.

Given equations (8) and (23), we are able to define trajectories in \((\alpha_1, \beta_1)\) space for a fixed value of \( \sigma_{\text{emp}}^2 \) and \( \Gamma_{4,\text{emp}} \). Unlike the GARCH-normal methods, we now have the trajectories as a function of \( \eta_4 \). These can be seen below:

\[
\alpha_1 = \frac{2\alpha_0 - \left( \frac{\alpha_0}{\sigma_{\text{emp}}^2} \right)^2(\Gamma_{4,\text{emp}} - \eta_4)}{\left( \Gamma_{4,\text{emp}} - \eta_4 \right)(\eta_4 - 1) - (\eta_2 - \eta_4)} 
\]

(25)

\[
\beta_1 = 1 - \frac{\alpha_0}{\sigma_{\text{emp}}^2} - \frac{\left( \frac{2\alpha_0}{\sigma_{\text{emp}}^2} - \left( \frac{\alpha_0}{\sigma_{\text{emp}}^2} \right)^2(\Gamma_{4,\text{emp}} - \eta_4) \right)}{\left( \Gamma_{4,\text{emp}} - \eta_4 \right)(\eta_4 - 1) - (\eta_2 - \eta_4)} 
\]

(26)

Now for each desired data set we can use the trajectories in the same manner as we have done with the GARCH-normal model. We can plot \( \Gamma_6 \) along the trajectories of \((\alpha_1, \beta_1)\) using the running
parameter $\alpha_0$. Plotting $\Gamma_6$ as a function of the running parameter, $\alpha_0$, and overlaying this with the empirical value we gain figure 7.

![The Empirical and Analytical Solution for Parameter Fitting](image)

Figure 7: $\Gamma_6$ as a function of the running parameter $\alpha_0$. Here we show how we calculate the value of $\alpha_0$ for a particular time window. The orange line is the value of $\Gamma_6$ for the empirical time window we are modelling, whilst the blue line shows $\Gamma_6(\alpha_0)$, equation (24), when $\alpha_1(\eta_4, \alpha_0, \sigma^2_{\text{emp}}, \Gamma_{4,\text{emp}})$ (equation (25)) and $\beta_1(\eta_4, \alpha_0, \sigma^2_{\text{emp}}, \Gamma_{4,\text{emp}})$ (equation (26)), for the GARCH-double-normal model. The intercept of the two lines shows the value of $\alpha_0$ which allows us to model data for a certain median time and a certain time window within the GARCH-double-normal(1,1) model.

From the above method we can recover the value of $\alpha_0(t)$ that allows the fitting of $\sigma^2_{\text{emp}}(t)$, $\Gamma_{4,\text{emp}}(t)$ and $\Gamma_{6,\text{emp}}(t)$, where $t$ is the median of the running window, allowing us to create figure 8. This is done for several banks: Lloyds Bank, Barclays Bank, Bank of America, HSBC and Santander, a commodity, Gold ETFs and a pharmaceutical company, GSK. We seek to find a fingerprint of the companies’ GARCH parameters through the financial crisis. It is evident from this figure that the banking sector has a unique behaviour in response to the crisis. We see an initial fluctuating signal, but when the crisis period occurs we see a ‘cliff-edged’ drop in the parameter value. This behaviour is not seen in the commodity or the pharmaceutical and so we infer that this is due to the response to the crisis period of 2008. It is our belief that this specific behaviour exhibited by the banking companies can be used as an indicator for future banking crisis periods.

6. Conclusion

We use the time series of The Bank of America, Barclays Bank, Citi Bank, HSBC, Gold ETFs, GlaxoSmithKlein and Lloyds Bank, among others, to highlight the inability of the gaussian conditional distribution within a GARCH model to fit higher order moments of market time series.
Figure 8: $\alpha_0(t)$ for several companies between 2005 and 2011. We show here the evolution of $\alpha_0$ when we calculate the parameter value in the shifting 6 month time window. In panel (a), we show the time evolution of $\alpha_0$ for Barclays Bank, Gold ETFs, GSK and Lloyds Bank, by the red, blue, yellow and purple lines respectively. Whilst in panel (b), we show the Bank of America’s evolution by the blue line, the evolution of HSBC by the red and Santander’s by the yellow line.

In discovering this, we turn our attention towards different conditional distributions to try to capture the empirical data. We show that with the use of a GARCH-double-normal model we can fit the empirical data’s higher order moment. However through this enquiry, we still cannot capture the long run dynamics of the empirical data. We show that it is only possible to fit a model to empirical data within certain time horizons. To model a different time horizon we have to change the parameters of the double gaussian distribution we use to model.

Fixing the distribution within certain time horizons to enable the fitting of higher order moments, highlights that the obtained GARCH-double-gaussian(1,1) model describes a non-stationary process. Therefore, if we wish to describe a long time series by a GARCH-double-normal model, we have to truncate it to smaller time windows. In doing so, we have to potentially fit different GARCH-double-normal models to each time window. Therefore, we produce a time dependence of the GARCH model’s parameters. We are able to build up a time signature of the $\alpha_0$ parameter through the 2008 financial crash for several companies. We focus our attention on the banking sector to distinguish any shared behaviour in time dependence of the GARCH parameter $\alpha_0$ through this crisis period. The banking companies’ values of $\alpha_0$ have a distinct behaviour from companies belonging to other sectors of the economy through the period, giving hope of a standardised signal of these periods. It is seen through their empirical data that before the financial crash there is an increase in $\alpha_0$ and during the financial period, the value of $\alpha_0$ reduces extremely quickly. A behaviour that is shared with banking companies but not other securities’ time series.

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Appendix A. Relations between the parameters of the double gaussian distribution and its higher order moments

The normalisation condition for the double gaussian distribution described by equation (18) is $a + b = 1$. Substituting $a = 1 - b$ into the equation (20), $E[x^2]$, we get:

$$b = \frac{1 - \sigma_1^2}{\sigma_2^2 - \sigma_1^2}$$  \hspace{1cm} (A.1)

and,

$$a = \frac{\sigma_2^2 - 1}{\sigma_2^2 - \sigma_1^2}$$  \hspace{1cm} (A.2)

Assuming $\sigma_1^2 < 1 < \sigma_2^2$, and substituting the equations for $a$ and $b$ into the fourth and sixth moment equations we derive:

$$\mu_4 = \sigma_2^4 + \sigma_1^4 - \sigma_1^2 \sigma_2^2$$  \hspace{1cm} (A.3)

and,

$$\mu_6 = (\sigma_1^2 + \sigma_2^2)^2 - \sigma_1^4 \sigma_2^2 - \sigma_1^2 \sigma_2^4 (\sigma_1^2 + \sigma_2^2)$$  \hspace{1cm} (A.4)
where, $\mu_4 = \frac{44}{11}$ and $\mu_6 = \frac{10}{15}$. Introducing the new variables, $X = \sigma_2^4 + \sigma_1^2$ and $Y = \sigma_1^2\sigma_2^2$, we can simplify the obtained equations:

\[ Y = X - \mu_4 \quad \text{(A.5)} \]
\[ X(\mu_4 - 1) + \mu_4 = \mu_6 \quad \text{(A.6)} \]

Solving the above equations for $X$ and $Y$ we finally obtain:

\[ X = \frac{\mu_6 - \mu_4}{\mu_4 - 1} \quad \text{(A.7)} \]
\[ Y = \frac{\mu_6 - \mu_4^2}{\mu_4 - 1} \quad \text{(A.8)} \]

Since $X$ and $Y$ must be positive, this gives us three conditions: $\mu_4 > 1$, $\mu_6 > \mu_4$ and $\mu_6 > \mu_4^2$. Due to the first one we can disregard the second as $\mu_4^2 > \mu_4$. Using relations between $\mu_6$ and $\mu_4$ and $\eta_4$ and $\eta_6$ we obtain:

\[ \eta_4 > 3 \quad \text{(A.9)} \]
\[ \eta_6 > \frac{15}{9} \eta_4^2 \quad \text{(A.10)} \]

We can then set-up equations for solving $\sigma_1^2$ or $\sigma_2^2$:

\[ \sigma_2^4 - X\sigma_2^2 + Y = 0 \quad \text{(A.11)} \]

and,

\[ \sigma_1^2 = \frac{Y}{\sigma_2^2} \quad \text{(A.12)} \]

Solving for $\sigma_2^2$ we can obtain relations for the parameters of the double gaussian distribution:

\[ \sigma_2^2 = \frac{1}{2}(X + \sqrt{X^2 - 4Y}) \quad \text{(A.13)} \]

and so,

\[ \sigma_1^2 = \frac{2Y}{(X + \sqrt{X^2 - 4Y})}. \quad \text{(A.14)} \]

Since, $\sigma_1^2$ and $\sigma_2^2$ must be both real and positive, this gives us the relation; $X > 4Y$. As such we get the following inequality:

\[ \mu_6^2 - 6\mu_4\mu_6 - 4\mu_6 - \mu_4^2(3 - 4\mu_4) > 0 \quad \text{(A.15)} \]

Solving this inequality for $\mu_6$, we get the condition: $\mu_4 > -1$. Obviously, $\mu_4$ is always larger than $-1$, and so we always satisfy the condition shown in equation (A.15). As such, we have to only obey the conditions shown in equations (A.9) and (A.10).
Appendix B. Conditions for $\Gamma_4$

For a general GARCH conditional probability distribution with variance equal to one, the equation for the sixth order divergence line (the denominator of equation (10)) becomes:

$$1 - \beta_1^3 - 3\alpha_1\beta_1^2 - 3\eta_4\alpha_1^2\beta_1 - \eta_6\alpha_1^3 = 0$$  \hspace{1cm} (B.1)\]

Expanding $\beta_1$ in a series with respect to $\alpha_1$ we derive:

$$\beta_1 = 1 - A\alpha_1 - B\alpha_1^2 - C\alpha_1^3 - \ldots$$  \hspace{1cm} (B.2)\]

Substituting this into our sixth order divergence line we can equate coefficients up to the second order and so $\beta_1$ becomes:

$$\beta_1 = 1 - \alpha_1 - (\eta_4 + 1)\alpha_1^2 + O(\alpha_1^3)$$  \hspace{1cm} (B.3)\]

If we now neglect $\alpha_1$ orders higher than the second we get the equation: $\beta_1 = 1 - \alpha_1 - (\eta_4 + 1)\alpha_1^2$. Substituting this into our equation for the fourth order standardised moment, we obtain:

$$\Gamma_4 = \frac{\eta_4(1 - \alpha_1 - (1 - \alpha_1 - (\eta_4 + 1)\alpha_1^2))}{1 - \eta_4\alpha_1^2 - 2\alpha_1(1 - \alpha_1 - (\eta_4 + 1)\alpha_1^2) - (1 - \alpha_1 - (\eta_4 + 1)\alpha_1^2)^2}$$  \hspace{1cm} (B.4)\]

Considering the limit when $\alpha_1 \rightarrow 1$ we finally obtain:

$$\lim_{\alpha_1 \rightarrow 1} \Gamma_4 = 2\eta_4$$  \hspace{1cm} (B.5)\]