Intertwining operators for $\ell$-conformal Galilei algebras and hierarchy of invariant equations

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Abstract

The $\ell$-conformal Galilei algebra, denoted by $g_\ell(d)$, is a non-semisimple Lie algebra specified by a pair of parameters $(d, \ell)$. The algebra is regarded as a nonrelativistic analogue of the conformal algebra. We derive hierarchies of partial differential equations which have invariance of the group generated by $g_\ell(d)$ with a central extension as kinematical symmetry. This is done by developing a representation theory such as Verma modules, singular vectors of $g_\ell(d)$ and vector field representations for $d = 1, 2$.

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1. Introduction

When a differential equation is given, searching for its symmetry is one of the standard approaches to find solutions to the equation or to study its properties. Sometimes it turns out that the given equation has larger symmetry than expected from physical intuition. Maximal kinematical symmetry of the Schrödinger equation for a free particle is one of such cases. It is obvious that the Galilei group is a symmetry of the free Schrödinger equation. However, Niederer showed that the equation is also invariant under the scale and the special conformal transformations [1] (see also [2]). The Lie group of maximal kinematical symmetry of the free Schrödinger equation is nowadays called the Schrödinger group. Although the Schrödinger group has been known since the period of Lie, it is not so long since the group was recognized as one of the fundamental algebraic objects in physics. The group and its Lie algebra play certain roles in a wide range of physical problems from fluid dynamics to nonrelativistic holography. One may find a very nice review with an appropriate list of references on physical implications of the Schrödinger group and its Lie algebra in [3] (see the foreword by Henkel).

Returning to the problem of the symmetry of a given differential equation, one may reverse the problem. Namely, for a given Lie group we look for differential equations invariant under
the group. When invariance means the usual Lie symmetry one may use an algorithm based on the classical Lie method (see e.g. [4, 5]). In particular, a wide class of equations invariant under the Schrödinger group has been obtained by the algorithm [6–8].

In the present work, we consider the \( \ell \)-conformal Galilei group as a kinematical symmetry for a hierarchy of differential equations. It is defined as the symmetry under the transformation of the independent variables \( x_i \to \tilde{x}_i \) together with the transformation of a solution \( \psi(x_i) \) to another solution of the form \( f(\tilde{x}_i)\psi(\tilde{x}_i) \) where \( f(x_i) \) is a weight function independent of \( \psi \) [1]. As we shall see, such equations are obtained as a byproduct of the representation theory of the Lie algebra of the group. In fact, this is an established fact if the given Lie group is real connected and semisimple. [9, 10]. Although the Lie algebra of the Schrödinger group is not semisimple, it is known that a similar technique to the semisimple case is applicable to develop its representation theory [11, 12]. This enables us to apply a method similar to [9, 10] to find differential equations having the Schrödinger group as kinematical symmetry [11–14]. Here we extend this analysis to a larger group, the \( \ell \)-conformal Galilei group with a central extension. We develop a representation theory of the Lie algebra of the \( \ell \)-conformal Galilei group and, as a consequence, derive differential equations having the group as kinematical symmetry.

By definition, the Schrödinger group is an enlargement of the Galilei group. It has been shown that one can further enlarge the Schrödinger group and have a sequence of Lie groups in which each Lie group is specified by a parameter \( \ell \in \frac{1}{2} \mathbb{N} \), where \( \mathbb{N} \) denotes the set of positive integers [15, 16]. We refer to each group of the sequence as an \( \ell \)-conformal Galilei group. Here we assumed that the dimension of spacetime is fixed. Taking into account the difference of spacetime dimension one may say that each \( \ell \)-conformal Galilei group is specified by a pair of parameters \( (d, \ell) \) where the dimension of spacetime is \( (d + 1) \). The first member with \( \ell = 1/2 \) of the sequence corresponds to the Schrödinger group. As the Schrödinger group, the \( \ell = 1 \) conformal Galilei group and its Lie algebra also appear in a wide range of physical problems; for instance, nonrelativistic electrodynamics [17], classical mechanics with higher order time derivatives [18–20], nonrelativistic analogue of AdS/CFT correspondence [21–25], nonrelativistic spacetime and gravity [26–30], quantum mechanical particle systems [31], conformal mechanics [32, 41], nonrelativistic twistors [33] and so on. Furthermore, one may find applications of the algebraic structure to mathematical studies of topics such as systems of partial differential equations [8, 34, 35]. It is, however, quite recent that conformal Galilei groups and algebras with higher \( \ell \) are studied from both physical and mathematical points of view [36–43]. It has been observed that physical systems having a connection with the \( \ell \)-conformal Galilei (\( \ell > 1/2 \)) group are described by Lagrangians or Hamiltonians with higher order derivatives.

This paper is organized as follows. In the next section we give a short review of generators of the \( \ell \)-conformal Galilei group and its central extensions. Then we consider the case of \( d = 1 \) and half-integer \( \ell \). A representation theory of the conformal Galilei algebra for this case has been developed in [40] and the necessary items to find differential equations with desired symmetry, Verma module and singular vectors in it, have also been obtained. With this information we derive a hierarchy of differential equations having the conformal Galilei group for \( (d, \ell) = (1, \mathbb{N} + \frac{1}{2}) \) as kinematical symmetry. We also present a vector field representation of the conformal Galilei algebra on a coset space. In section 4 and section 5, the same analysis is repeated for the case of \( d = 2 \) and arbitrary values of \( \ell \). In this case, however, the representation theory of the conformal Galilei algebra has not yet been investigated. So we shall start by constructing Verma modules and singular vectors. We then derive differential equations with desired symmetry for the values of \( (2, \ell) \). A vector field representation of the conformal Galilei algebra is also presented. Motivated by the recent active studies of holographic dual of...
3D Minkowski spacetime [28, 29] we repeat the same analysis for \((d, \ell) = (1, 1)\), the case without a central extension, in the appendix.

2. The \(\ell\)-conformal Galilei algebra

We present the \(\ell\)-conformal Galilei algebra, denoted by \(g_{\ell}(d)\), as a set of generators of transformation of coordinates in \((d + 1)\) dimensional spacetime [16]. Consider the transformations generated by

\[
H = \frac{\partial}{\partial t}, \quad D = -2i\frac{\partial}{\partial t} - 2\ell x_i \frac{\partial}{\partial x_i}, \quad C = r^2 \frac{\partial}{\partial t} + 2\ell x_i \frac{\partial}{\partial x_i},
\]

\[
M_{ij} = -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \quad P^{(n)}_i = (-t)^n \frac{\partial}{\partial x_i},
\]

where \(i = 1, \ldots, d\) specify the space coordinates and \(n = 0, \ldots, 2\ell\) with \(\ell \in \frac{1}{2}\mathbb{N}\). We thus have \(d(d - 1)/2 + (2\ell + 1)d + 3\) generators. The time translation \(H\), space translations \(P^{(0)}_i\), spatial rotations \(M_{ij}\), and Galilei transformations \(P^{(1)}_i\) form the Galilei group of nonrelativistic kinematics. In addition we have numbers of transformation to a coordinate system with acceleration \(P^{(n)}_i (n \geq 2)\) according to the values of the parameter \(\ell\). The scale transformation \(D\) and special conformal transformation \(C\) also depend on \(\ell\). One may see that space and time are equally scaled only for \(\ell = 1\).

One can immediately see that the generators (1) have closed commutators and form a Lie algebra. Nonvanishing commutators of \(g_{\ell}(d)\) are given by

\[
[D, H] = 2H, \quad [D, C] = -2C, \quad [C, H] = D,
\]

\[
[M_{ij}, M_{kl}] = -\delta_{ik}M_{jl} - \delta_{jk}M_{il} + \delta_{il}M_{jk} + \delta_{jl}M_{ik},
\]

\[
[H, P^{(n)}_i] = -n P^{(n-1)}_i, \quad [D, P^{(n)}_i] = 2(\ell - n)P^{(n)}_i,
\]

\[
[C, P^{(n)}_i] = (2\ell - n)P^{(n+1)}_i, \quad [M_{ij}, P^{(n)}_i] = -\delta_{ik}P^{(n)}_j + \delta_{jk}P^{(n)}_i.
\]

The algebra \(g_{\ell}(d)\) has a subalgebra consisting of a direct sum of \(sl(2, \mathbb{R}) \simeq so(2, 1)\) generated by \(H, D, C\) and \(so(d)\) by \(M_{ij}\). While \(P^{(n)}_i\) forms an Abelian ideal of \(g_{\ell}(d)\) so that \(g_{\ell}(d)\) is not a semisimple Lie algebra.

It is known that \(g_{\ell}(d)\) has two distinct types of central extension according to the values of \(d\) and \(\ell\).

(i) A mass extension existing for any \(d\) and \(\ell \in \mathbb{N} + \frac{1}{2}\)

\[
\tilde{P}^{(m)}_i, \tilde{P}^{(m)}_j = \delta_{ij} \delta_{m+2\ell} L_m M, \quad I_m = (-1)^{m+\ell+\frac{1}{2}} (2\ell - m)! m!.
\]

(ii) An exotic extension existing only for \(d = 2\) and \(\ell \in \mathbb{N}\)

\[
\tilde{P}^{(m)}_i, \tilde{P}^{(m)}_j = \epsilon_{ij} \delta_{m+2\ell} L_m \Theta, \quad I_m = (-1)^m (2\ell - m)! m!.
\]

where \(\epsilon_{ij}\) is the antisymmetric tensor with \(\epsilon_{12} = 1\). We note that the structure constants are chosen to agree, up to an overall factor, with the ones used in [36] and [38]. A simple explanation for the existence of two distinct central extensions is found in [21]. The Schrödinger algebra considered by Niederer corresponds to \(g_{1/2}(2)\) with the mass central extension. The exotic extension was first found for \(\ell = 1\) in the study of classical mechanics having higher order time derivatives [18–20].

In the following sections we discuss the lowest weight representations (especially Verma modules) of \(g_{\ell}(d)\) with a central extension. Since we are interested in an algebra with central extensions, we denote \(g_{\ell}(d)\) with the mass and exotic central extensions by \(\tilde{g}_{\ell}(d)\) and \(\tilde{g}_{\ell}\), respectively. By using the representations we derive differential equations having a group generated by \(\tilde{g}_{\ell}(d)\) or \(\tilde{g}_{\ell}\) as kinematical symmetry.
3. The \( d = 1 \) conformal Galilei group with a mass central extension

3.1. Equations and symmetry

The simplest example of an \( \ell \)-conformal Galilei algebra with a central extension is \( \hat{g}_\ell(1) \), namely a \( d = 1 \) algebra with a mass central extension (\( \ell \in \mathbb{N} + \frac{1}{2} \)). Thus we start our construction of invariant equations with this simplest algebra \( \hat{g}_\ell(1) \). We also outline the method of [10] throughout the construction. The method consists of two steps. The first step is an abstract representation theory, i.e., a Verma module over \( \hat{g}_\ell(1) \) and singular vectors in it. The second step is a realization of the Verma module on a space of \( C^\infty \) functions with a special property. The space of \( C^\infty \) functions is also a representation space of the group generated by \( \hat{g}_\ell(1) \).

Verma modules over \( \hat{g}_\ell(1) \) and their singular vectors have already been studied in detail [40]. The fact that one can appropriately define Verma modules over \( \hat{g}_\ell(1) \) is, in fact, the reason why one may apply the method of [10], which is established for semisimple Lie groups, to non-semisimple \( \hat{g}_\ell(1) \). To complete the first step we summarize the results of [40] below (highest weight Verma modules are considered in [40], here we convert it to lowest weight modules for later convenience).

We make the following vector space decomposition of \( g = \hat{g}_\ell(1) \)

\[
\begin{align*}
g^+ &= \{ H, \hat{p}^{(0)}, \hat{p}^{(1)}, \ldots, \hat{p}^{(\ell - \frac{1}{2})} \}, \\
g^0 &= \{ D, M \}, \\
g^- &= \{ C, \hat{p}^{(\ell + \frac{1}{2})}, \hat{p}^{(\ell + \frac{3}{2})}, \ldots, \hat{p}^{(2\ell)} \}.
\end{align*}
\]

Here we omit the index for the space coordinate. Then one may see that \([g^0, g^\pm] \subset g^\pm\), that is, this is an analogue of the triangular decomposition of semisimple Lie algebras. Suppose that there exists a lowest weight vector defined by

\[
D [\delta, \mu] = -\delta [\delta, \mu], \quad M [\delta, \mu] = -\mu [\delta, \mu], \\
X [\delta, \mu] = 0, \quad \forall X \in g^-.
\]

We define a Verma module \( V^{h,\mu} \) over \( g \) by

\[
V^{h,\mu} = \left\{ \left. H^{\ell - \frac{1}{2}} \prod_{j=0}^{\ell - \frac{1}{2}} (\hat{p}^{(\ell - j)})^{k_j} [\delta, \mu] \right| h, k_0, k_1, \ldots, k_{\ell - \frac{1}{2}} \in \mathbb{Z}_{\geq 0} \right\}.
\]

The vector space \( V^{h,\mu} \) carries a representation of \( g \) specified by \( \delta \) and \( \mu \). In general, the representation is not irreducible (ducible). The reducibility of \( V^{h,\mu} \) is detected by singular vectors. A singular vector is a vector having the same property as \( [\delta, \mu] \) but different eigenvalues. If the Verma module has a singular vector, then the representation is reducible. For the Verma module given by (7) one may prove the following. If \( 2\delta - 2(q - 1) + (\ell + \frac{1}{2})^2 = 0 \) for \( q \in \mathbb{Z}_{\geq 0} \) then \( V^{h,\mu} \) has precisely one singular vector given by

\[
[v_q] = S^q[\delta, \mu], \quad S = a_{\ell}\mu H + (\hat{p}^{(\ell - \frac{1}{2})})^2, \quad a_{\ell} = 2 \left((\ell - \frac{1}{2})!\right)^2.
\]

Namely, \([v_q]\) satisfies the relations:

\[
D [v_q] = (2q - \delta)[v_q], \quad M [v_q] = -\mu[v_q], \\
X [v_q] = 0, \quad \forall X \in g^-.
\]

One may build another Verma module \( V^{2q-h,\mu} \) on the singular vector \([v_q]\) by replacing \([\delta, \mu]\) with \([v_q]\) in (7).
We now proceed to the second step. We introduced a vector space decomposition (5) of \( g \). It follows that the Lie group \( G \) generated by \( g \) also has the corresponding decomposition, \( G = G^+ G^0 G^- \) where \( G^\pm = \exp(g^\pm) \) and \( G^0 = \exp(g^0) \). Consider a \( C^\infty \) function on \( G \) having the property of right covariance:

\[
f(\hat{g} g^0 g^-) = e^{\Lambda(X)} f(g), \quad \forall g \in G, \quad \forall g^0 \in G^0, \quad \forall g^- \in G^- \tag{10}
\]

where \( \Lambda(X) \) is an eigenvalue of \( X \in g^0 \). Thus the function \( f(g) \) is actually a function on the coset \( G/G^0 G^- \). Now consider the space \( C^\Lambda \) of right covariant functions on \( G \). We introduce the right action of \( g \) on \( C^\Lambda \) by the standard formula:

\[
\pi_R(X) f(g) = \frac{d}{d\tau} f(g e^{\tau X}) \bigg|_{\tau=0}, \quad X \in g, \quad g \in G. \tag{11}
\]

When can then verify by the right covariance (10) that \( f(g) \) has the properties of the lowest weight vector:

\[
\pi_R(D) f(g) = \Lambda(D) f(g), \quad \pi_R(M) f(g) = \Lambda(M) f(g),
\]

\[
\pi_R(X) f(g) = 0, \quad X \in g^-. \tag{12}
\]

Parameterizing an element of \( G^+ \) as \( g^+ = \exp(tH + \sum_{j=1}^{\ell-1} x_j \hat{P}^{(j)}) \) the right action of \( H \) and \( \hat{P}^{(k)} \) become differential operators on \( C^\Lambda \):

\[
\pi_R(H) = \frac{\partial}{\partial t} + \sum_{j=1}^{\ell-1} jx_j \frac{\partial}{\partial x_{j-1}}, \quad \pi_R(\hat{P}^{(k)}) = \frac{\partial}{\partial x_k}. \tag{13}
\]

Thus one may construct the Verma module \( V^{h,\mu} \) by regarding \( f(g) \) as the lowest weight vector \( |\delta, \mu \rangle \) with an identification \( \Lambda(D) = -\delta, \quad \Lambda(M) = -\mu \). The singular vector \( |v_q \rangle \) corresponds to \( f_q(g) \equiv \pi_R(S^q) f(g) \).

Now we are ready to write down differential equations having the group \( G = \exp(\hat{g}_x(1)) \) as kinematical symmetry. Suppose that the operator \( \pi_R(S^q) \) has a nontrivial kernel:

\[
\pi_R(S^q) \psi(t, x_i) = \left( a_\ell \mu \left( \frac{\partial}{\partial t} + \sum_{j=1}^{\ell-1} jx_j \frac{\partial}{\partial x_{j-1}} \right) + \frac{\partial^2}{\partial^2 x_{\ell-1}} \right)^q \psi(t, x_i) = 0 \tag{14}
\]

for some function \( \psi \neq 0 \). Then the differential equations (14) have the desired symmetry. This is verified in the following way. Consider the left regular representation of \( G \) on \( C^\Lambda \) defined by

\[
(T^{h,\mu}(g) f)(g') = f(g^{-1} g'). \tag{15}
\]

If we take the singular vector \( f_q(g') \) instead of \( f(g') \) in (15), then we have another representation \( T^{2q-h,\mu} \). Furthermore, the operator \( \pi_R(S^q) \) is an intertwining operator of the representations

\[
\pi_R(S^q) T^{h,\mu} = T^{2q-h,\mu} \pi_R(S^q). \tag{16}
\]

If follows that if \( \psi(t, x_i) \) is a solution to the equation (14) then a transformed function \( T^{h,\mu}(g) \psi \) is also a solution to (14):

\[
\pi_R(S^q) (T^{h,\mu}(g) \psi) = T^{2q-h,\mu}(g) \pi_R(S^q) \psi = 0.
\]

Thus the group \( G \) is the kinematical symmetry of equation (14) (see [10] for more detail). We remark that the symmetry transformations are determined by the left action of \( G \) on \( C^\Lambda \), i.e., equation (15). On the other hand the differential equations (14) are obtained by using the right action of \( g \) on \( C^\Lambda \), i.e., equation (13). Thus (13) is not the generator of symmetry.
We have obtained a hierarchy of differential equations (14). For $\ell = 1/2$ (14) recovers a hierarchy of heat/Schrödinger equations in one space dimension obtained in [11, 13, 14]:

$$\left( 2\mu \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right)^q \psi(t, x) = 0.$$  

The central charge $\mu$ is interpreted as a (imaginary) mass. For higher values of $\ell$ we observe an interesting deviation from the heat/Schrödinger equation and the obtained equations are highly nontrivial. As an illustration we show the hierarchy of equations for $\ell = 3/2$ and $5/2$.

$$(2\mu \left( \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_0} \right) + \frac{\partial^2}{\partial x_1^2})^q \psi(t, x_0, x_1) = 0, \quad \ell = 3/2$$

$$(8\mu \left( \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_0} + 2x_2 \frac{\partial}{\partial x_1} \right) + \frac{\partial^2}{\partial x_2^2})^q \psi(t, x_0, x_1, x_2) = 0, \quad \ell = 5/2.$$ 

The first member of the hierarchy ($q = 1$) is always a second order differential equation for any values of $\ell$. This is a sharp contrast to the results in [38, 39, 42, 43] where the $\ell$-conformal Galilei algebra relates to systems with higher order derivatives. Other examples of second order differential equations relating to the algebraic structure are found in [21, 35].

### 3.2. The vector field representation of $\hat{g}_\ell$(1)

We used the right action of $g$ on the space of right covariant functions to derive a hierarchy of differential equations. One may, of course, consider the left action of $g$ on the same space defined by

$$\pi_L(X) f(g) = \left. \frac{d}{d\tau} f(e^{-\tau X} g) \right|_{\tau = 0}, \quad X \in g.$$  

(17)

The left action gives a vector field representation of $g$, the algebra with the mass central extension. It is highly nontrivial so we give its explicit formula. The representation of generators in $g^0$ is given by

$$\pi_L(D) = \delta - 2\ell \frac{\partial}{\partial t} - \sum_{j=0}^{\ell - 1} 2(\ell - j) x_j \frac{\partial}{\partial x_j}, \quad \pi_L(M) = \mu.$$ 

Generators in $g^+$ are represented as

$$\pi_L(H) = -\frac{\partial}{\partial t}, \quad \pi_L(\hat{P}^{(k)}) = -\sum_{j=0}^{k} \binom{k}{j} t^{k-j} \frac{\partial}{\partial x_j},$$

and those in $g^-$ as

$$\pi_L(C) = t\pi_L(D) + t^2 \frac{\partial}{\partial t} + \frac{\mu}{2} \left( (\ell + 1) \right)^2 x_{\ell - 1} - \sum_{j=0}^{\ell - 1} (2\ell - j) x_j \frac{\partial}{\partial x_{j+1}},$$

$$\pi_L(\hat{P}^{(k)}) = \mu \sum_{j=2\ell-k}^{\ell - 1} \binom{k}{2\ell - j} t^{k-2\ell+j} x_j - \sum_{j=0}^{\ell - 1} \binom{k}{j} t^{k-j} \frac{\partial}{\partial x_j},$$

where $\binom{k}{j}$ is the binomial coefficient. It can be easily verified that these representations satisfy the appropriate commutation relations.
4. The $d = 2$ conformal Galilei group with a mass central extension

4.1. The Verma module and singular vector

Employing the same procedure as in the case of $\hat{g}_\ell(1)$, we investigate differential equations having the Lie group generated by $\hat{g}_\ell(2)$ as kinematical symmetry. We remark that $\ell$ is restricted to half-integer values and we have a spatial rotation $so(2)$ in this case. We redefine the generators in order to introduce the decomposition analogous to triangular one:

$$\hat{P}^{(n)} = \hat{p}_1^{(n)} \pm i\hat{p}_2^{(n)}, \quad J = -iM_{12}, \quad M \rightarrow 2M.$$  \hfill (18)

Then the nonvanishing commutators of $\hat{g}_\ell(2)$ are given by

$$[D, H] = 2H, \quad [D, C] = -2C, \quad [C, H] = D,$$

$$[H, \hat{P}_\pm^{(n)}] = -n\hat{P}_\pm^{(n-1)}, \quad [D, \hat{P}_\pm^{(n)}] = 2(\ell - n)\hat{P}_\pm^{(n)},$$

$$[C, \hat{P}_\pm^{(n)}] = (2\ell - n)\hat{P}_\pm^{(n+1)}, \quad [J, \hat{P}_\pm^{(n)}] = \pm\hat{P}_\pm^{(n)},$$

$$[\hat{P}_\pm^{(n)}, \hat{P}_\mp^{(m)}] = \delta_{m+n,2}I_0M.$$  \hfill (19)

Here $I_0$ is given by (3). We introduce a vector space decomposition of $\mathfrak{g} = \hat{g}_\ell(2)$:

$$\mathfrak{g}^+ = \{H, \hat{P}_\pm^{(n)}\}, \quad n = 0, 1, \ldots, \ell - \frac{1}{2},$$

$$\mathfrak{g}^0 = \{D, J, M\},$$

$$\mathfrak{g}^- = \{C, \hat{P}_\pm^{(n)}\}, \quad n = \ell + \frac{1}{2}, \ldots, 2\ell,$$  \hfill (20)

which satisfies the relation $[\mathfrak{g}^0, \mathfrak{g}^\pm] \subset \mathfrak{g}^\pm$ analogous to the triangular decomposition.

Suppose the existence of the lowest weight vector $|\delta, r, \mu\rangle$ defined by

$$D|\delta, r, \mu\rangle = -\delta|\delta, r, \mu\rangle, \quad J|\delta, r, \mu\rangle = -r|\delta, r, \mu\rangle,$$

$$M|\delta, r, \mu\rangle = -\mu|\delta, r, \mu\rangle, \quad X|\delta, r, \mu\rangle = 0, \quad ^\forall X \in \mathfrak{g}^-.$$  \hfill (21)

One may build a Verma module $V^{k, r, \mu}$ on $|\delta, r, \mu\rangle$ whose basis is given by

$$|k, \underline{a}, \underline{b}\rangle = H^k \prod_{n=0}^{\ell - \frac{1}{2}} (\hat{P}_+^{(n)})^{a_n} (\hat{P}_-^{(n)})^{b_n} |\delta, r, \mu\rangle,$$  \hfill (22)

where $\underline{a}$ and $\underline{b}$ are $\ell + \frac{1}{2}$ components vectors whose entries are non-negative integers:

$$\underline{a} = (a_0, a_1, \ldots, a_{\ell - \frac{1}{2}}), \quad \underline{b} = (b_0, b_1, \ldots, b_{\ell - \frac{1}{2}}).$$

The action of $\mathfrak{g}$ on $V^{k, r, \mu}$ is obtained by a straightforward computation. Elements of $\mathfrak{g}^0$ act on (22) diagonally:

$$M|k, \underline{a}, \underline{b}\rangle = -\mu|k, \underline{a}, \underline{b}\rangle,$$

$$D|k, \underline{a}, \underline{b}\rangle = -\delta + 2k + \sum_{n=0}^{\ell - \frac{1}{2}} 2(\ell - n)(a_n + b_n) |k, \underline{a}, \underline{b}\rangle,$$

$$J|k, \underline{a}, \underline{b}\rangle = -r + \sum_{n=0}^{\ell - \frac{1}{2}} (a_n - b_n) |k, \underline{a}, \underline{b}\rangle.$$  \hfill (23)

To have a simple expression of the formulas for $\mathfrak{g}^\pm$ we introduce an $\ell + 1/2$ components vector $\underline{\delta}_n$ having 1 on $n$th entry and 0 for all other entries:

$$\underline{\delta}_n = (0, \ldots, 1, \ldots, 0).$$
With this the $\delta$ action of $g^+$ is given by
\[ H(k, a, b) = |k + 1, a, b\rangle, \]
\[ \hat{P}^+(n) |k, a, b\rangle = \sum_{i=0}^{n} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) |k - i, a + \delta_{n-i}, b\rangle, \]
\[ \hat{P}^-(n) |k, a, b\rangle = \sum_{i=0}^{k} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) |k - i, a, b + \delta_{n-i}\rangle, \] (24)
and the action of $g^-$ is given by
\[ C(k, a, b) = k \left( -\delta + k - 1 + \sum_{n=0}^{\ell-1/2} 2(\ell - n)(a_n + b_n) \right) |k - 1, a, b\rangle \]
\[ -\mu \left( \ell + \frac{1}{2} \right) a_{\ell-1/2} b_{\ell-1/2} |k, a - \delta_{\ell-1/2}, b - \delta_{\ell-1/2}\rangle \]
\[ + \sum_{n=0}^{\ell-3/2} (2\ell - n)(a_n |k, a - \delta_n + \delta_{n+1}, b\rangle + b_n |k, a, b - \delta_n + \delta_{n+1}\rangle). \]
\[ \hat{P}^+(n) |k, a, b\rangle = -\mu \sum_{i=0}^{n-\ell+1/2} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) b_{2\ell-n+i} |k - i, a, b + \delta_{2\ell-n+i}\rangle, \]
\[ + \sum_{i=n-\ell+1/2}^{k} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) |k - i, a + \delta_{n-i}, b\rangle, \]
\[ \hat{P}^-(n) |k, a, b\rangle = -\mu \sum_{i=0}^{n-\ell+1/2} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) a_{2\ell-n+i} |k - i, a - \delta_{2\ell-n+i}, b\rangle, \]
\[ + \sum_{i=n-\ell+1/2}^{k} i! \left( \frac{k}{i} \right) \left( \frac{n}{i} \right) |k - i, a, b + \delta_{n-i}\rangle. \] (25)
In the above formulas we understand $|k, a, b\rangle = 0$ for $k < 0$. It is also verified by straightforward but lengthy computations that the above formulas are consistent with the defining commutation relations of $g$.

The Verma module $V^{k, r, \mu}$ has a singular vector if $\delta - q + (\ell + \frac{1}{2})^2 + 1 = 0$ for a non-negative integer $q$. This is confirmed by checking that the vector
\[ |v_q\rangle = (a_\ell \mu H + \hat{P}^{(-\ell-1)} \hat{P}^{(-\ell)} \delta |\delta, r, \mu\rangle, \quad \alpha_\ell = \left( \left( \ell - \frac{1}{2} \right) \right)^2 \] (26)
satisfies the relations
\[ D|v_q\rangle = (2q - \delta)|v_q\rangle, \quad J|v_q\rangle = -r|v_q\rangle, \quad M|v_q\rangle = -\mu|v_q\rangle, \quad X|v_q\rangle = 0, \quad \forall X \in g^- \] (27)

4.2. Differential equations with kinematical symmetry

Consider the space of right covariant functions (10) on the group $G$ generated by $g = \hat{g}_\ell(2)$. As in the case of $\hat{g}_\ell(1)$ a function $f(g)$ in the space exhibits the property of the lowest weight vector of the Verma module $V^{k, r, \mu}$ under the right action defined by (11). Parameterizing an element $g \in \exp(g^+)$ as
By the parameterization (28) and the left action (17) one may write down a vector field representation of the vector field. The generators in $\pi$ are represented as

$$g = \exp(tH) \exp \left( \sum_{n=0}^{\ell-\frac{1}{2}} (x_n \dot{P}_n^+ + y_n \dot{P}_n^-) \right),$$

(28)

then the right action of elements in $\mathfrak{g}^+$ on $f(g)$ becomes differential operators:

$$\pi_R(H) = \frac{\partial}{\partial t} + \sum_{n=1}^{\ell-\frac{1}{2}} n \left( x_n \frac{\partial}{\partial x_{n-1}} + y_n \frac{\partial}{\partial y_{n-1}} \right),$$

$$\pi_R(\dot{P}_n^+) = \frac{\partial}{\partial x_n}, \quad \pi_R(\dot{P}_n^-) = \frac{\partial}{\partial y_n}.$$

(29)

The same argument as $\hat{g}_L(1)$ concludes that the following equation has the group $G$ generated by $\hat{g}_L(2)$ as kinematical symmetry:

$$\left[ a_{\mu} \left( \frac{\partial}{\partial t} + \sum_{n=0}^{\ell-\frac{1}{2}} n \left( x_n \frac{\partial}{\partial x_{n-1}} + y_n \frac{\partial}{\partial y_{n-1}} \right) \right) + \frac{\partial^2}{\partial x_{\ell-\frac{1}{2}} \partial y_{\ell-\frac{1}{2}}} \right]^q \psi(t, x, y) = 0,$$

(30)

with $q \in \mathbb{N}$. For $\ell = 1/2$ (30) yields the following form:

$$\left( \mu \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x \partial y} \right)^q \psi(t, x, y) = 0.$$

By an appropriate change of variables, this recovers a hierarchy of heat/Schrödinger equations in two space dimension obtained in [12–14].

4.3. The vector field representation of $\hat{g}_L(2)$

By the parameterization (28) and the left action (17) one may write down a vector field representation of $\hat{g}_L(2)$ on the manifold $G/G^2G$. It is not difficult to verify the following formula of the vector field representation. The generators in $\mathfrak{g}^+$ are represented as

$$\pi_L(D) = \delta - 2\ell \frac{\partial}{\partial t} - \sum_{n=0}^{\ell-\frac{1}{2}} \frac{2(\ell - n)}{2} \left( x_n \frac{\partial}{\partial x_{n-1}} + y_n \frac{\partial}{\partial y_{n-1}} \right),$$

$$\pi_L(J) = r - \sum_{n=0}^{\ell-\frac{1}{2}} \frac{n}{2} \left( x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n} \right),$$

$$\pi_L(M) = \mu.$$

Those in $\mathfrak{g}^+$ are represented as

$$\pi_L(H) = -\frac{\partial}{\partial t},$$

$$\pi_L(\dot{P}_n^+) = -\sum_{k=0}^{n} \left( n \right) \left( k \right) \mu \frac{\partial}{\partial x_{n-k}}, \quad \pi_L(\dot{P}_n^-) = -\sum_{k=0}^{n} \left( n \right) \left( k \right) \mu \frac{\partial}{\partial y_{n-k}},$$

and those in $\mathfrak{g}^-$ are given by

$$\pi_L(C) = t\pi_L(D) + t^2 \frac{\partial}{\partial t} + \left( \ell + \frac{1}{2} \right) \left( x_{\ell+\frac{1}{2}} \frac{\partial}{\partial x_{\ell+\frac{1}{2}}} + y_{\ell+\frac{1}{2}} \frac{\partial}{\partial y_{\ell+\frac{1}{2}}} \right) - \sum_{n=0}^{\ell-\frac{3}{2}} \left( 2\ell - n \right) \left( x_n \frac{\partial}{\partial x_{n+1}} + y_n \frac{\partial}{\partial y_{n+1}} \right),$$

$$\pi_L(\dot{P}_n^+) = \mu \sum_{n=2\ell-n}^{\ell-\frac{1}{2}} \left( n \right) \left( k \right) I_{2\ell-k} x_{\ell+\frac{1}{2}} - \sum_{k=0}^{\ell-\frac{1}{2}} \left( n \right) \left( k \right) \mu \frac{\partial}{\partial x_{\ell-k}},$$

$$\pi_L(\dot{P}_n^-) = \mu \sum_{n=2\ell-n}^{\ell-\frac{1}{2}} \left( n \right) \left( k \right) I_{2\ell-k} x_{\ell+\frac{1}{2}} - \sum_{k=0}^{\ell-\frac{1}{2}} \left( n \right) \left( k \right) \mu \frac{\partial}{\partial y_{\ell-k}}.$$
5. The $d = 2$ conformal Galilei group with an exotic central extension

5.1. The Verma module and singular vector

In this section we study the case of $\hat{g}_{\ell}$. In contrast to the analysis in section 3 and section 4, $\ell$ takes positive integral values for $\hat{g}_{\ell}$. We redefine the generators to introduce a decomposition analogous to the triangular one:

$$\hat{P}^{(n)}_{\pm} = \hat{P}^{(n)}_{12} \pm i \hat{P}^{(n)}_{2} \quad \text{and} \quad J = -i M_{12}, \quad \Theta \to -2i \Theta. \quad (31)$$

Then the nonvanishing commutators of $\hat{g}_{\ell}$ are identical to (19) except for the central extension. The central extension for $\hat{g}_{\ell}$ in terms of new generators is given by

$$[\hat{P}^{(n)}_{\pm}, \hat{P}^{(n)}_{\mp}] = \pm \delta_{m+n, 2\ell} I_{m} \Theta \quad (32)$$

with $I_{m}$ defined in (4). We make a triangular like decomposition of $g_{\ell}$:

$$g^{+} = \{H, \hat{P}^{(n)}_{+}, \hat{P}^{(n)}_{\mp}\}, \quad n = 0, 1, \ldots, \ell - 1$$

$$g^{0} = \{D, J, \Theta\},$$

$$g^{-} = \{C, \hat{P}^{(n)}_{-}, \hat{P}^{(n)}_{\mp}\}, \quad n = \ell + 1, \ell + 2, \ldots, 2\ell. \quad (33)$$

It is an easy task to see that $[g^{0}, g^{\pm}] \subset g^{\pm}$.

The lowest weight vector $|\delta, r, \theta\rangle$ is defined as usual:

$$D|\delta, r, \theta\rangle = -\delta|\delta, r, \theta\rangle, \quad J|\delta, r, \theta\rangle = -r|\delta, r, \theta\rangle,$$

$$\Theta|\delta, r, \theta\rangle = \theta|\delta, r, \theta\rangle, \quad X|\delta, r, \theta\rangle = 0 \text{ for } X \in g^{-}. \quad (34)$$

We construct a Verma module $V^{k, r, \theta}$ by repeated applications of an element of $g^{+}$ on $|\delta, r, \theta\rangle$.

In order to specify a basis of $V^{k, r, \theta}$ we introduce the following notations. Let $a$ and $b$ be $(\ell + 1)$ and $\ell$ components vectors, respectively. Their entries are non-negative integers:

$$a = (a_0, a_1, \ldots, a_\ell), \quad b = (b_0, b_1, \ldots, b_{\ell-1}).$$

We also introduce the $(\ell + 1)$ components vector $\xi$ with 1 in the $i$th entry and 0 elsewhere. Similarly, the $\ell$ components vector $\delta$ has 1 in the $i$th entry and 0 elsewhere, i.e.,

$$\xi = (0, \ldots, 1, \ldots, 0), \quad \delta = (0, \ldots, 1, \ldots, 0).$$

With these notations the following vectors form a basis of $V^{k, r, \theta}$

$$|h, a, b\rangle = H^{b} (\hat{P}^{(n)}_{+})^{a_0} \prod_{n=0}^{\ell-1} (\hat{P}^{(n)}_{-})^{a_n} (\hat{P}^{(n)}_{\mp})^{b}|\delta, r, \mu\rangle. \quad (35)$$

It is not difficult, but requires a lengthy computation, to write down the action of $g$ on $V^{k, r, \theta}$ and to verify that $V^{k, r, \theta}$ indeed carries a representation of $g$. The action of $g^{0}$ on $|h, a, b\rangle$ is diagonal:

$$\Theta|h, a, b\rangle = \theta|h, a, b\rangle,$$

$$D|h, a, b\rangle = \left( -\delta + 2h + \sum_{n=0}^{\ell-1} (\ell - n)(a_n + b_n) \right)|h, a, b\rangle,$$

$$J|h, a, b\rangle = \left( -r + \sum_{n=0}^{\ell-1} (a_n - b_n) + a_\ell \right)|h, a, b\rangle. \quad (36)$$
While the action of $g^+$ is given by
\[
H|h, a, b\rangle = |h + 1, a, b\rangle, \\
\tilde{P}_+^{(n)}|h, a, b\rangle = \sum_{i=0}^{n} i! \left(\begin{array}{c} h \\ i \end{array} \right) |h - i, a + \epsilon, b + \delta_i\rangle, \\
\tilde{P}_-^{(n)}|h, a, b\rangle = \sum_{i=0}^{n} i! \left(\begin{array}{c} h \\ i \end{array} \right) |h - i, a, b + \delta_i\rangle, \\
\]
and the action of $g^-$ by
\[
C|h, a, b\rangle = h \left( -\delta + h - 1 + \sum_{n=0}^{h-1} 2(\ell - n)(a_n + b_n) \right) |h - 1, a, b\rangle \\
+ \sum_{n=0}^{\ell-1} (2\ell - n)a_n|h, a - \epsilon, b - \delta_{n-1}\rangle \\
+ \sum_{n=0}^{\ell-1} (2\ell - n)b_n|h, a, b - \delta_n\rangle, \\
\tilde{P}_+^{(n)}|h, a, b\rangle = \sum_{i=0}^{n-\ell-1} i! \left(\begin{array}{c} h \\ i \end{array} \right) b_{2\ell-n+1}\theta|h - i, a, b - \delta_{n-1}\rangle \\
+ \sum_{i=0}^{h} i! \left(\begin{array}{c} h \\ i \end{array} \right) |h - i, a + \epsilon, b\rangle, \\
\tilde{P}_-^{(n)}|h, a, b\rangle = -\sum_{i=0}^{n-\ell} i! \left(\begin{array}{c} h \\ i \end{array} \right) a_{2\ell-n+1}\theta|h - i, a - \epsilon, b\rangle \\
+ \sum_{i=0}^{h} i! \left(\begin{array}{c} h \\ i \end{array} \right) |h - i, a, b + \delta_i\rangle. \\
\]
In these formulas we understand $|h, a, b\rangle = 0$ for $h < 0$.

The Verma module $V^{\delta,\epsilon,\theta}$ has a singular vector if $\delta - q + \ell(\ell + 1) + 1 = 0$ for a positive integer $q$ given by
\[
|v_q\rangle = (\epsilon!\theta H + (-1)^\ell \tilde{P}_+^{(\ell)}\theta)|\delta, r, \theta\rangle, \quad \alpha_\ell = \ell! (\ell - 1)!. \\
\]
It is straightforward to verify that the vector $|v_q\rangle$ satisfies
\[
D|v_q\rangle = (2q - \delta)|v_q\rangle, \quad J|v_q\rangle = -r|v_q\rangle, \quad \Theta|v_q\rangle = \theta|v_q\rangle, \\
X|v_q\rangle = 0, \quad \forall X \in g^-. \\
\]
This generalizes the result for $\ell = 1$ investigated in [44].

5.2. Differential equations with kinematical symmetry

We parameterize an element of $g \in \exp(g^+)$ as
\[
g = e^{xH} \exp \left( \sum_{n=0}^{\ell-1} (x_n\tilde{P}_+^{(n)} + y_n\tilde{P}_-^{(n)}) + x_1\tilde{P}_+^{(\ell)} \right). \\
\]
A right covariant function on \( G = \exp(\mathfrak{g}_\ell) \) exhibits the properties of the lowest weight vector (34). It is also easily seen that the right action (11) of \( g^+ \) is given by

\[
\pi_R(H) = \frac{\partial}{\partial t} + \sum_{n=0}^{\ell-1} (n+1)x_n + \sum_{n=0}^{\ell-2} y_{n+1},
\]

\[
\pi_R(\tilde{P}_+^{(n)}) = \frac{\partial}{\partial x_n}, \quad \pi_R(\tilde{P}_-^{(n)}) = -\frac{\partial}{\partial y_n}.
\]  

(42)

It follows, together with (39), that following equations have the group generated by \( \tilde{\mathfrak{g}}_\ell \) as kinematical symmetry:

\[
\left[ \alpha(t) \left( \frac{\partial}{\partial t} + \sum_{n=1}^{\ell} n x_n \frac{\partial}{\partial x_{n-1}} + \sum_{n=1}^{\ell-1} n y_n \frac{\partial}{\partial y_{n-1}} \right) + (-1)^{\ell} \frac{\partial^2}{\partial y_{\ell-1} \partial x_\ell} \right]^q \psi = 0.
\] 

One may see the symmetry by the same argument as section 3.1.

5.3. The vector field representation of \( \tilde{\mathfrak{g}}_\ell \)

The parameterization (41) and the left action (17) give the following vector field representation of \( \tilde{\mathfrak{g}}_\ell \) on the coset \( G/G^0 G^- \). The generators in \( \mathfrak{g}^0 \) are represented as

\[
\pi_L(D) = \delta - 2t \frac{\partial}{\partial t} - \sum_{n=0}^{\ell-1} (2n - \ell + n) \left( x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n} \right),
\]

\[
\pi_L(J) = \gamma - \sum_{n=0}^{\ell} x_n \frac{\partial}{\partial x_n} + \sum_{n=0}^{\ell} y_n \frac{\partial}{\partial y_n}, \quad \pi_L(\Theta) = -\theta,
\]

while the generators in \( \mathfrak{g}^+ \) are represented as

\[
\pi_L(H) = -\frac{\partial}{\partial t}, \quad \pi_L(\tilde{P}_+^{(n)}) = -\sum_{k=0}^{n} \binom{n}{k} t^k \frac{\partial}{\partial x_{n-k}},
\]

\[
\pi_L(\tilde{P}_-^{(n)}) = -\sum_{k=0}^{n} \binom{n}{k} t^k \frac{\partial}{\partial y_{n-k}},
\]

and those in \( \mathfrak{g}^- \) as

\[
\pi_L(C) = \ell \pi_L(D) + t^2 \frac{\partial}{\partial t} - \ell \ell + 1 \theta x_{\ell+1} y_{\ell+1} - \sum_{n=0}^{\ell-1} (2n - \ell + n) x_n \frac{\partial}{\partial x_{n+1}} - \sum_{n=0}^{\ell-2} (2n - \ell + n) y_n \frac{\partial}{\partial y_{n+1}},
\]

\[
\pi_L(\tilde{P}_+^{(n)}) = -\theta \sum_{k=0}^{n-\ell} \binom{n}{k} t^k y_{2\ell-\ell+n+k} - \sum_{k=0}^{n-\ell} \binom{n}{k} t^k \frac{\partial}{\partial x_{n-k}},
\]

\[
\pi_L(\tilde{P}_-^{(n)}) = \theta \sum_{k=0}^{n-\ell} \binom{n}{k} t^k x_{2\ell-\ell+n+k} - \sum_{k=0}^{n-\ell} \binom{n}{k} t^k \frac{\partial}{\partial y_{n-k}}.
\]

6. Concluding remarks

We have investigated the representations of the \( \ell \)-conformal Galilei algebra with central extensions. For \( d = 2 \) we showed that some Verma modules are not irreducible by giving the singular vectors explicitly. By employing the method in [10] we derived partial differential equations having the group generated by \( \tilde{\mathfrak{g}}_\ell(1) \) or \( \tilde{\mathfrak{g}}_\ell(2) \) or \( \tilde{\mathfrak{g}}_\ell \) as kinematical symmetry.
The obtained differential equations have some common properties. They form a hierarchy of linear differential equations. Each hierarchy contains precisely one differential equation of second order. This is a big difference of the present result from the previous works considering physical systems relating to the $\ell$-conformal Galilei group with higher $\ell$.

We restrict ourselves to the algebras of $d = 1, 2$ and having a central extension. One may repeat the same analysis for higher values of $d$ or the algebras without a central extension. The representation theory may be more involved for such cases. However, the case of $d = 3$ is of special interest since spatial rotations become non-Abelian and have a contribution to all sectors in the triangular decomposition. This implies that the differential equation will have different structure from the cases of $d = 1, 2$. Another important, but mathematical, problem to be done is a precise investigation of irreducible representations. Namely, one may try to classify irreducible Verma modules for $d > 2$ and higher $\ell$ as done in [11, 40, 44]. This will be a future work.

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**Appendix**

In this appendix we investigate the centreless algebra $g_1(1)$, i.e., $(d, \ell) = (1, 1)$. This is motivated by recent studies of the field theory potentially dual to 3D Minkowski spacetime [28–30]. The dimension of the algebra $g_1(1)$ is six and the generators are denoted by $D, H, C, P^{(0)}, P^{(1)}, P^{(2)}$. Nonvanishing commutators are given by

$$
[D, H] = 2H, \quad [D, C] = -2C, \quad [C, H] = D,
$$

$$
[H, P^{(\alpha)}] = -nP^{(\alpha - 1)}, \quad [D, P^{(\alpha)}] = 2(1 - n)P^{(\alpha)},
$$

$$
[C, P^{(\alpha)}] = (2 - n)P^{(\alpha + 1)}.
$$

(A.1)

We introduce the triangular-like decomposition:

$$
g_1(1)^- = \{ H, P^{(0)} \}, \quad g_1(1)^0 = \{ D, P^{(1)} \}, \quad g_1(1)^+ = \{ C, P^{(2)} \}.
$$

The lowest weight vector $|\delta, \kappa\rangle$ and the Verma module $V^{\delta, \kappa}$ are defined as usual:

$$
H|\delta, \kappa\rangle = -\delta|\delta, \kappa\rangle, \quad P^{(0)}|\delta, \kappa\rangle = -\kappa|\delta, \kappa\rangle,
$$

$$
V^{\delta, \kappa} = \{ C^k(P^{(2)})^k|\delta, \kappa\rangle \mid h, k \in \mathbb{Z}_{\geq 0} \}.
$$

(A.2)

Writing $|h, k\rangle = C^h(P^{(2)})^k|\delta, \kappa\rangle$ we have

$$
D|h, k\rangle = -(\delta + 2h + 2k)|h, k\rangle.
$$

(A.3)

We set $p = h + k$ and call the non-negative integer $p$ level. Then the following statements on the existence of singular vectors in $V^{\delta, \kappa}$ hold true.

- If $\kappa \neq 0$ then there exists no singular vector in $V^{\delta, \kappa}$.
- If $\kappa = 0$ there exist unique singular vector at each level $p$ and it is given (up to an overall constant) by $(P^{(2)})^p|\delta, 0\rangle$.

Thus $V^{\delta, \kappa}$ is irreducible if $\kappa \neq 0$. One may prove them in a way similar to [11, 44], although we omit the proof. Parameterizing the group element as $g = \exp(iH + xP^{(2)})$ the right action of $P^{(2)}$ is given by

$$
\pi_R(P^{(2)}) = \frac{\delta}{\delta x}.
$$

(A.5)
This leads the equations having a group generated by $g_1(1)$ as kinematical symmetry:

$$\left( \frac{\partial}{\partial x} \right)^\rho \psi(t, x) = 0.$$  \hspace{1cm} (A.6)

The equations are not of physical interest. This situation is similar to the Schrödinger algebra without a central extension [11].

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