Poisson Quasi-Nijenhuis Structures with Background

PAULO ANTUNES
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal. e-mail: pantunes@mat.uc.pt

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Abstract. We define Poisson quasi-Nijenhuis structures with background on Lie algebroids and we prove that any generalized complex structure on a Courant algebroid which is the double of a Lie algebroid has an associated Poisson quasi-Nijenhuis structure with background. We prove that any Lie algebroid with a Poisson quasi-Nijenhuis structure with background constitutes, with its dual, a quasi-Lie bialgebroid. We also prove that any pair \((\pi, \omega)\) of a Poisson bivector and a 2-form induces a Poisson quasi-Nijenhuis structure with background and we observe that particular cases correspond to already known compatibilities between \(\pi\) and \(\omega\).

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0. Introduction

The aim of this work is to define, on a Lie algebroid, the notion of Poisson quasi-Nijenhuis structure with a (closed) 3-form background. The Poisson quasi-Nijenhuis structures (without background) were introduced by Stiénon and Xu in [16] on the tangent Lie algebroid and then on any Lie algebroid by Caseiro et al. in [2]. In Physics, the Poisson quasi-Nijenhuis geometry was studied by Zucchini [21] as the target space geometry implied by the BV master equations of a Poisson sigma model. In his paper, Zucchini also treated the case with background but we remarked that a condition is missing in the definition proposed there. This extra condition was already considered in [16] and appears naturally in our work when we require some structures to be integrable (or some brackets to satisfy the Jacobi identity).

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In this paper we will use a supermanifold approach \cite{14,18} to describe Lie algebroid structures. Let us consider a vector bundle $A \to M$ and change the parity of the fibre coordinates (considering them odd), then we obtain a supermanifold denoted by $\Pi A$. The algebra of functions on $\Pi A$, which are polynomial in the fibre coordinates, is denoted by $C^\infty(\Pi A)$ and coincides with $\Omega(A) := \Gamma(\bigwedge^\bullet A^*)$, the exterior algebra of $A$-forms. Let us consider a Lie algebroid structure $(\rho, [\cdot,\cdot])$ on $A$. As it is known (see \cite{18}), the Lie algebroid structure on $A$ can equivalently be given by $d$, a degree 1 derivation of $\Omega^1(A)$ such that $d^2 = 0$. In the supermanifold setting, $d$ is a vector field on $\Pi A$ and can be seen as the derivation defined by a hamiltonian on $\Pi A$, i.e., an element $\mu \in C^\infty(T^*\Pi A)$. Then $d = \{\mu,\cdot\}$ where the so-called big bracket \cite{7}, $\{\cdot,\cdot\}$, is the canonical Poisson bracket on the symplectic supermanifold $T^*\Pi A$. The condition $d^2 = 0$ is equivalent to $\{\mu,\mu\} = 0$.

To each $f \in C^\infty(T^*\Pi A)$ is associated a bidegree $(\epsilon, \delta)$. In fact, since using Legendre transform (see \cite{12}) $T^*(\Pi A) \cong T^*(\Pi A^*)$, we can define $\epsilon$ (resp. $\delta$) as the polynomial degree of $f$ in the fibre coordinates of the vector bundle $T^*(\Pi A) \to \Pi A$ (resp. $T^*(\Pi A) \to \Pi A^*$). We define the shifted bidegree of $f$ as the pair $(\epsilon - 1, \delta - 1)$ and the total shifted bidegree as the sum $(\epsilon - 1) + (\delta - 1) = \epsilon + \delta - 2$. Then, a Lie algebroid structure in $A$ is a hamiltonian $\mu \in C^\infty(T^*\Pi A)$ of shifted bidegree $(0, 1)$ such that $\{\mu, \mu\} = 0$.

Instead of the expression “with background” used here, some authors use “twisted”, or in Physics, “H-flux”. In this work, our choice was motivated by the result of the Proposition 4.2. In fact, we prove there that a particular class of Poisson quasi-Nijenhuis structure with background is obtained by twisting, in a way explained in \cite{9,13,17}, a Lie algebroid structure by a Poisson bivector and then by a 2-form. Therefore, to avoid confusion, we will use the word “twist” only when we are dealing with twisting by a 2-form or a bivector as in \cite{9,13,17}.

The paper is organized as follows. In the first section we recall some basic definitions such as Nijenhuis tensors, Poisson bivectors and Poisson Nijenhuis structures on a Lie algebroid and give the corresponding expression in the supermanifold approach. Then, in the second section, we introduce the notion of Poisson quasi-Nijenhuis structure with a 3-form background $H$, on a Lie algebroid $(A, \mu)$. We prove that any complex structure (or more generally any c.p.s. structure, see Definition 2.2) on $(A \oplus A^*, \mu + H)$ induces such a structure. In the third section we generalize a result from \cite{2,16} and prove that any Poisson quasi-Nijenhuis structure with background on $A$ induces a Lie quasi-bialgebroid on $(A^*, A)$. Finally, in the last section we study Poisson quasi-Nijenhuis structures with background defined by a pair $(\pi, \omega)$ of a Poisson bivector and a 2-form. We observe that already known compatible pairs such that complementary 2-forms for Poisson bivectors \cite{19}, Hitchin pairs \cite{3} and $P\Omega$-structures or $\Omega N$-structures \cite{11} are all particular examples of Poisson quasi-Nijenhuis structures with background.
1. Basic Definitions

In this section we will recall some well known structures such as Poisson Nijenhuis structures on a Lie algebroid $(A, \rho, [, .])$ and give their expression in terms of the big bracket and polynomial functions on $T^*\Pi A$.

Let $A$ be a vector bundle over a smooth manifold $M$. A Lie algebroid structure on $A$ is a pair $(\rho, [, .])$ where $\rho : A \to TM$ is a vector bundle morphism and $[.,.]$ is a Lie bracket on the space of smooth sections $\Gamma(A)$, such that the Leibniz rule is satisfied

$$[X, fY] = f[X, Y] + (\rho(X) \cdot f)Y, \quad \forall X, Y \in \Gamma(A), \quad \forall f \in C^\infty(M).$$

The Lie algebroid structure, $(\rho, [, .])$, can be seen [12] as a function $\mu \in C^\infty(T^*\Pi A)$, of shifted bidegree $(0, 1)$, such that $\{\mu, \mu\} = 0$. The pair $(\rho, [, .])$ can be recovered from $\mu$ by using the formulae

- $\rho(X) \cdot f = \{[X, \mu], f\}$;
- $[X, Y] = \{[X, \mu], Y\},$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

Consider a $(1, 1)$-tensor $N \in \Gamma(A \otimes A^*)$. The Nijenhuis torsion of $N$ is defined by

$$TN(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]).$$

In terms of the big bracket and elements of $C^\infty(T^*\Pi A)$, the Nijenhuis torsion is given by

$$TN = \frac{1}{2} \left( \{N, [N, \mu]\} - \left\{N^2, \mu\right\} \right).$$

(1)

If $TN = 0$, $N$ is said to be a Nijenhuis tensor and in this case we define a new Lie algebroid structure on $A$ as follows

$$\begin{align*}
[X, Y]_N &= [NX, NY] + [X, NY] - N[X, Y], \quad X, Y \in \Gamma(A), \\
\rho_N &= \rho \circ \pi^\sharp.
\end{align*}$$

(2)

In the supermanifold setting, the structure $([., .], \rho_N)$ is given by $\{N, \mu\} \in C^\infty(T^*\Pi A)$. We denote by $d_N$ the degree 1 derivation of $\Omega(A)$ induced by this Lie algebroid structure. Then

$$d_N = \{\{N, \mu\} \ldots\}.$$

A bivector $\pi \in \Gamma(\wedge^2 A)$ is said to be Poisson if $[\pi, \pi]_{SN} = 0$, where $[..]_{SN}$ is the Schouten-Nijenhuis bracket naturally defined on $\Gamma(\wedge^* A)$. If $\pi$ is a Poisson bivector we define a Lie algebroid structure on $A^*$ by setting

$$\begin{align*}
[\alpha, \beta]_\pi &= \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)), \quad \alpha, \beta \in \Gamma(A^*), \\
\rho_\pi &= \rho \circ \pi^\sharp.
\end{align*}$$

(3)

In the supermanifold setting, the structure $([., .], \rho_\pi)$ is given by $\{\pi, \mu\} \in C^\infty(T^*\Pi A)$. 
DEFINITION 1.1. A Poisson bivector $\pi$ and a Nijenhuis tensor $N$ are said to be compatible if
\[
\begin{align*}
N \circ \pi^\sharp &= \pi^\sharp \circ \iota_N, \\
C_{\pi,N} &= 0,
\end{align*}
\]
with
\[
C_{\pi,N} = ([.,.]_N)_{\pi} - ([.,.]_{\pi})_N,
\]
a $C^\infty(M)$-bilinear bracket on $\Gamma(A^*)$. When $\pi$ and $N$ are compatible, the triple $(A, \pi, N)$ is called a Poisson Nijenhuis Lie algebroid.

In the supermanifold setting, we have
\[
C_{\pi,N} = \{\pi, [N, \mu]\} + [N, \{\pi, \mu\}].
\]

THEOREM 1.2 [5]. If $(A, \pi, N)$ is a Poisson Nijenhuis Lie algebroid, then $(A_N, A^*_\pi)$ is a Lie bialgebroid, where $A_N$ and $A^*_\pi$ are the Lie algebroids defined, respectively, by (2) and (3).

Remark 1.3. When $A = TM$ and $\mu$ is the standard Lie algebroid structure, the implication of the previous theorem becomes an equivalence (see [8]).

The Lie bialgebroid $(A_N, A^*_\pi)$ induces a Courant algebroid structure on $A \oplus A^*$, which is given in the supermanifold setting by
\[
S = \{\pi, \mu\} + [N, \mu] = \{\pi + N, \mu\}.
\]

In the next sections we will weaken the Poisson Nijenhuis Lie algebroid $(A, \pi, N)$ and study the structures that we get on $A \oplus A^*$.

2. Poisson Quasi-Nijenhuis with Background and Generalized Geometry

Let $S$ be a Courant algebroid structure on $A \oplus A^*$, i.e., $S \in C^\infty(T^*\Pi A)$ is of total shifted degree 1 and $[S, S] = 0$. Consider also a $(1, 1)$-tensor $J$ on $A \oplus A^*$, seen as a map $J : A \oplus A^* \rightarrow A \oplus A^*$. We call $J$ orthogonal if
\[
\langle J(\mathcal{X}), \mathcal{Y} \rangle + \langle \mathcal{X}, J(\mathcal{Y}) \rangle = 0,
\]
for all $\mathcal{X}, \mathcal{Y} \in \Gamma(A \oplus A^*)$, with $\langle ., . \rangle$ defined by $\langle X + \alpha, Y + \beta \rangle = \beta(X) + \alpha(Y)$ for all $X, Y \in \Gamma(A)$, $\alpha, \beta \in \Gamma(A^*)$.

As in the Lie algebroid case, we can define a new bracket $[,]_J$ deforming by $J$ the Courant structure on $A \oplus A^*$ by setting
\[
[\mathcal{X}, \mathcal{Y}]_J = [J\mathcal{X}, \mathcal{Y}] + [\mathcal{X}, J\mathcal{Y}] - J[\mathcal{X}, \mathcal{Y}],
\]

for all \( \mathcal{X}, \mathcal{Y} \in \Gamma(A \oplus A^*) \), where \([.,.]\) is the Dorfman bracket on \( A \oplus A^* \). When \( J \) is an orthogonal \((1, 1)\)-tensor on \( A \oplus A^* \), this deformed bracket is given by the hamiltonian

\[
S_J := \{J, S\} \in C^\infty(T^* \Pi A).
\]

We also define the Nijenhuis torsion of \( J \),

\[
T_J(\mathcal{X}, \mathcal{Y}) = [J \mathcal{X}, J \mathcal{Y}] - J([\mathcal{X}, \mathcal{Y}]),
\]

for all \( \mathcal{X}, \mathcal{Y} \in \Gamma(A \oplus A^*) \).

**Proposition 2.1** [1]. 1. The hamiltonian \( S_J \) defines a Courant structure on \( A \oplus A^* \) if and only if \( \{S, T_J\} = 0 \).

2. \( J \) is a Courant morphism from \((A \oplus A^*, S_J)\) to \((A \oplus A^*, S)\) if and only if \( T_J = 0 \).

**Definition 2.2.** An orthogonal \((1, 1)\)-tensor \( J \), on \( A \oplus A^* \), is an almost c.p.s. structure if \( J^2 = \lambda id_{A \oplus A^*} \), with \( \lambda \in \{-1, 0, 1\} \). The almost c.p.s. structure \( J \) is integrable when \( T_J = 0 \).

The abbreviation “c.p.s.” is due to Vaisman [20] and corresponds to the three different structures we are considering: if \( \lambda = -1 \), \( J \) is an almost complex structure [6]; if \( \lambda = 1 \), \( J \) is an almost product structure; and if \( \lambda = 0 \), \( J \) is an almost subtangent structure.

As it was noticed in [3,20], \( J \) is an almost c.p.s. structure if and only if \( J \) can be represented in a matrix form by

\[
J \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} N & \pi^z \\ \sigma^b & -t N \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}
\]

for all \( X \in \Gamma(A) \) and \( \alpha \in \Gamma(A^*) \), where \( \pi \in \Gamma(\bigwedge^2 A) \), \( \sigma \in \Gamma(\bigwedge^2 A^*) \) and \( N \in \Gamma(A \otimes A^*) \) satisfy

\[
\begin{cases}
N \circ \pi^z = \pi^z \circ t N, \\
\sigma^b \circ N = t N \circ \sigma^b, \\
N^2 + \pi^z \circ \sigma^b = \lambda id_A.
\end{cases}
\]

In the supermanifold setting,

\[
J = \pi + N + \sigma
\]

\[^1\text{In [4], the notion of irreducible Courant algebroid is introduced, as a Courant algebroid where each orthogonal Nijenhuis tensor is proportional to a c.p.s. structure. It is proved there that, for example, the classical Courant algebroid structure on } TM \oplus T^* M \text{ is irreducible.}\]
in the sense that \( J(.) = \{., \pi + N + \sigma \} \). Moreover, the integrability condition of an almost c.p.s. structure, \( T_J = 0 \), is expressed by (see [4])

\[
\{[J, S], J\} + \lambda S = 0. \tag{5}
\]

Let us now consider the case \( S = \mu + H \), where \( \mu \in C^\infty(T^*\Pi A) \) defines a Lie algebroid structure on \( A \), and \( H \in \Gamma(\wedge^3 A^*) \) is a closed 3-form. Then \( \{S, S\} = 0 \) and \( S \) defines a Courant algebroid structure on \( A \oplus A^* \).

The goal of this section is to relate c.p.s. structures on \( (A \oplus A^*, \mu + H) \) with the Poisson quasi-Nijenhuis structures with background which we now define.

**DEFINITION 2.3.** A *Poisson quasi-Nijenhuis structure with background* on a Lie algebroid \( A \) is a quadruple \( (\pi, N, \psi, H) \) where \( \pi \in \Gamma(\wedge^2 A) \), \( N \in \Gamma(A \otimes A^*) \), \( \psi \in \Gamma(\wedge^3 A^*) \) and \( H \in \Gamma(\wedge^3 A^*) \) are such that \( N \circ \pi^\sharp = \pi^\sharp \circ t N \), \( d\psi = 0 \), \( dH = 0 \) and the following conditions hold:

\[
\begin{align*}
\pi & \text{ is a Poisson bivector,} \\
C_{\pi, N}(\alpha, \beta) &= 2i\pi^\sharp \alpha \wedge \pi^\sharp \beta H, \\
T_N(X, Y) &= \pi^\sharp (i_N X \wedge Y H - i_Y X H + i_X Y \psi), \\
d_N \psi &= dH,
\end{align*}
\tag{6}
\]

for all \( X, Y \in \Gamma(A) \), \( \alpha, \beta \in \Gamma(A^*) \) and where \( H \) is the 3-form defined by

\[
H(X, Y, Z) = \circ_{X,Y,Z} H(NX, NY, Z),
\tag{7}
\]

for all \( X, Y, Z \in \Gamma(A) \).

**Remark 2.4.**

1. In terms of the big bracket and elements of \( C^\infty(T^*\Pi A) \), the conditions (6) correspond to

\[
\begin{align*}
\{\{\pi, \mu\}, \pi\} &= 0, \\
\{\{\pi, \mu\}, N\} + \{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} &= 0, \\
\{\{N, \mu\}, N\} + \{N^2, \mu\} - 2\{\pi, \psi\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\} &= 0, \\
2\{\{N, \mu\}, \psi\} &= \{\mu, \{N, [N, H]\} - \{N^2, H\}\}.
\end{align*}
\tag{8}
\]

2. If \( H = 0 \) we recover the Poisson quasi-Nijenhuis structures defined in [2,16].
3. The last condition of (6) is missing in the definition proposed by Zucchini [21]. In our study this condition appears naturally and is necessary in order to include the case without background, described in [2,16].

**THEOREM 2.5.** If an endomorphism \( J \), defined by (4), is a c.p.s. structure on \( (A \oplus A^*, \mu + H) \) then \( (\pi, N, -d\sigma, H) \) is a Poisson quasi-Nijenhuis structure with background on \( A \).
Proof. The result follows directly by writing the integrability condition (5) with $J = \pi + N + \sigma$ and $S = \mu + H$. Using the bilinearity of $\{.,.\}$ and taking into account the bidegree of each term we obtain the following system of equations

$$\begin{align*}
\{\{\pi, \mu\}, \pi\} &= 0, \\
\{\{\pi, \mu\}, N\} + \{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} &= 0, \\
\{\{N, \mu\}, N\} + 2\{\{\pi, \{\mu, \sigma\}\}, \mu\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\} + \lambda \mu &= 0, \\
\{\{N, \mu\}, \sigma\} + \{\{\sigma, \mu\}, N\} + \{\{\pi, \sigma\}, H\} + \{\{N, H\}, N\} + \{\{\pi, \sigma\}, H\} + \lambda H &= 0.
\end{align*}$$

In the last two equations of the system we now use the algebraic conditions for $J$ to be a c.p.s. structure and more precisely the condition $N^2 + \pi^\sharp \circ \sigma^\flat = \lambda \text{id}_A$ which is written in terms of the big bracket and elements of $C^\infty(T^*\Pi A)$ as

$$\{\pi, \sigma\} = N^2 - \lambda \text{id}_A.$$

We obtain

$$\begin{align*}
\{\{\pi, \mu\}, \pi\} &= 0, \\
\{\{\pi, \mu\}, N\} + \{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} &= 0, \\
\{\{N, \mu\}, N\} + \{\{N^2, \mu\}, \pi\} + 2\{\{\pi, \{\mu, \sigma\}\}, \mu\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\} &= 0, \\
\{\{N, \mu\}, \sigma\} + \{\{\sigma, \mu\}, N\} + \{\{N, H\}, N\} + \{\{N^2, H\}, \pi\} + 2\lambda H &= 0.
\end{align*}$$

The proof is achieved after interpreting the previous system of equations as

$$\begin{align*}
\pi &\text{ is a Poisson bivector,} \\
C_{\pi, N}(\alpha, \beta) &= 2i_\pi^\sharp \alpha \wedge i_\pi^\sharp \beta H, \\
T_N(X, Y) &= \pi^\sharp (i_{NX\wedge Y} H - i_{NY\wedge X} H - i_{X\wedge Y} d\sigma), \\
2i_N d\sigma - d(i_N\sigma) &= 2(H + \lambda H),
\end{align*}$$

(9)

for all $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$ and $H$ defined by Equation (7).

Remark 2.6. In [20], Vaisman studied the integrability of almost c.p.s. structures on $TM \oplus T^*M$ considering both the usual Courant bracket, and also the case with a 3-form background. The conditions obtained in Remark 1.5 of [20] coincide with the system of conditions (9).

Note that, in the previous proof, the last equation of (9) is only a sufficient condition for the last equation of (6). We can get an equivalence if we impose additional conditions on the quadruple ($\pi, N, \sigma, H$).

THEOREM 2.7. An endomorphism $J$, defined by (4), is a c.p.s. structure on $(A \oplus A^*, \mu + H)$ if and only if $(\pi, N, -d\sigma, H)$ is a Poisson quasi-Nijenhuis structure on $A$ with background $H$ such that

$$\begin{align*}
N^2 + \pi^\sharp \circ \sigma^\flat &= \lambda \text{id}_A, \\
\sigma^\flat \circ N &= i N \circ \sigma^\flat, \\
2(i_N d\sigma - H) &= d(i_N\sigma) + 2\lambda H.
\end{align*}$$
3. Poisson Quasi-Nijenhuis with Background and Lie Quasi-Bialgebroids

In this section we will generalize a result proved for structures without background in [2,16]. Let \((A, \mu)\) be a Lie algebroid over a smooth manifold \(M\).

**DEFINITION 3.1.** A Lie quasi-bialgebroid is a triple \((A, \delta, \varphi)\) where \(A\) is a Lie algebroid, \(\delta\) is a degree one derivation of the Gerstenhaber algebra \(\Gamma(\bigwedge^\bullet A, \wedge, [ , ] )\) and \(\varphi \in \Gamma(\bigwedge^3 A)\) is such that \(\delta^2 = [\varphi, .]\) and \(\delta \varphi = 0\).

The main result of the section is the following

**THEOREM 3.2.** If \((\pi, N, \psi, H)\) is a Poisson quasi-Nijenhuis structure with background on \(A\) then \((A^*, d_H^N, \gamma + i_N H)\) is a Lie quasi-bialgebroid, where \(d_H^N(\alpha) = d_N(\alpha) - i_{\pi}(\alpha) H\), for all \(\alpha \in \Gamma(A^*)\).

**Proof.** The hamiltonian on \(C^\infty(T^*\Pi A)\) which induces the structure \((A^*, d_H^N, \gamma + i_N H)\) is \(\tilde{S} = \{\pi + N, \mu + H\} + \psi\). Considering the bidegree of each term, the following system of equations holds:

\[
\begin{align*}
\{\{\pi, \mu\}, \{\pi, \mu\}\} &= 0, \\
\{\{\pi, \mu\}, \{\pi, H\}\} + \{\{\pi, \mu\}, \{N, \mu\}\} &= 0, \\
\{\{\pi, H\}, \{\pi, H\}\} &+ \{\{N, \mu\}, \{N, \mu\}\} + 2 \{\{\pi, \mu\}, \{N, H\}\} + 2 \{\{\pi, H\}, \{N, \mu\}\} + 2 \{\{\pi, \mu\}, \psi\} = 0, \\
\{\{\pi, H\}, \{N, H\}\} &+ \{\{N, \mu\}, \{N, H\}\} + \{\{\pi, H\}, \psi\} + \{\{N, \mu\}, \psi\} = 0.
\end{align*}
\]  

(10)

It is now straightforward to observe that the system of equations (8) implies the system (10).

**COROLLARY 3.3.** If \((\pi, N, -d\sigma, H)\) is a Poisson quasi-Nijenhuis structure with background on \(A\), then \(\{\pi + N + \sigma, \mu + H\}\) is a structure of Lie quasi-bialgebroid on \((A^*, A)\) or equivalently a Courant algebroid structure on \(A \oplus A^*\).

**Proof.** If we consider \(\psi = -d\sigma\) in the previous proof, we obtain \(\tilde{S} = \{\pi + N + \sigma, \mu + H\}\). Then, as we have already seen, \(\{\tilde{S}, \tilde{S}\} = 0\).

**Remark 3.4.** In the corollary above, \(J = \pi + N + \sigma\) is not necessarily integrable, i.e., the Nijenhuis torsion \(T_J\) may not vanish (see necessary conditions in Theorem 2.7). But the previous corollary proves that the deformed structure \(S_J (= \tilde{S})\) defines a Courant algebroid structure in \(A \oplus A^*\), i.e., that \(\{S, T_J\} = 0\) (see Proposition 2.1).
4. Poisson Quasi-Nijenhuis with Background and Compatible Second-Order Tensors

In this section we shall consider \( \pi \in \Gamma(\bigwedge^2 A) \) a Poisson bivector and a 2-form \( \omega \in \Gamma(\bigwedge^2 A^\ast) \). Let us denote
\[
\pi^{\sharp}(\alpha) = \pi(\alpha, .), \quad \forall \alpha \in \Gamma(A^\ast), \quad \omega^{\flat}(X) = \omega(X, .), \quad \forall X \in \Gamma(A),
\]
\[
N = \pi^{\sharp} \circ \omega^{\flat}, \quad \omega_N = \omega(N, .).
\]

Then, the main result of this section is the following:

**THEOREM 4.1.** The quadruple \((\pi, N, d\omega_N, -d\omega)\) is a Poisson quasi-Nijenhuis structure with background on \( A \).

**Proof.** Let us set \( \psi = d\omega_N \) and \( H = -d\omega \). In terms of elements of \( C^\infty(T^\ast \Pi A) \), we have the following correspondences
\[
\begin{cases}
N = \{\omega, \pi\}, \\
\psi = \frac{1}{2} \{\mu, \{N, \omega\}\}, \\
H = \{\omega, \mu\}.
\end{cases}
\]

We easily check that \( \psi \) and \( H \) are closed and that \( N \circ \pi^{\sharp} = \pi^{\sharp} \circ t^\top N \). To prove that \((\pi, N, \psi, H)\) is a Poisson quasi-Nijenhuis structure with background we need to verify the set of conditions (6) [or equivalently the conditions (8)].

1. \( \pi \) is a Poisson bivector by assumption.
2. Considering the fact that \( \pi \) is a Poisson bivector, i.e., that
\[
\{(\pi, \mu), \pi\} = 0
\]
and applying \{\omega, .\} to both sides, we get
\[
\{\omega, \{(\pi, \mu), \pi\}\} = 0.
\]

Then, we use the Jacobi identity to obtain
\[
\{\omega, \{(\pi, \mu), \pi\}\} + \{(\pi, \mu), \{\omega, \pi\}\} = 0.
\]

Using once more the Jacobi identity in the first term of the l.h.s. we have
\[
\{\{(\omega, \pi), \mu\}, \pi\} + \{(\pi, [\omega, \mu]), \pi\} + \{(\pi, \mu), \{\omega, \pi\}\} = 0,
\]
which is the second condition of (8)
\[
\{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} + \{\{\pi, \mu\}, N\} = 0.
\]

3. As above, we start from the previous condition
\[
\{\{N, \mu\}, \pi\} + \{\{\pi, H\}, \pi\} + \{\{\pi, \mu\}, N\} = 0,
\]
and apply \{\omega, .\} to both sides. We obtain
\[
\{\omega, \{\{N, \mu\}, \pi\}\} + \{\omega, \{\{\pi, H\}, \pi\}\} + \{\omega, \{\{\pi, \mu\}, N\}\} = 0,
\]
and using the Jacobi identity twice we get the required equation
\[
\{\{N, \mu\}, N\} + \left\{N^2, \mu\right\} - 2\{\pi, \psi\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\} = 0.
\]

4. The way of proving this condition is the same as above. We start from the previous condition and apply \{\omega, .\} to both sides. Then, using the Jacobi identity, we get
\[
\{\{N, H\}, N\} + \left\{N^2, H\right\} - 2\{N, \psi\} - \left\{\left\{N^2, \omega\right\}, \mu\right\} = 0.
\]
Finally, applying \{\mu, .\} we obtain
\[
\left\{\mu, \{\{N, H\}, N\} + \left\{N^2, H\right\}\right\} - 2\{\mu, \{N, \psi\}\} = 0.
\]
Using again the Jacobi identity and the fact that \psi is closed we get
\[
2\{\{N, \mu\}, \psi\} = \left\{\mu, \{N, \{N, H\}\} - \left\{N^2, H\right\}\right\}.
\]

The proof of the previous theorem suggests that starting from a Poisson bivector and composing iteratively, in a certain way, with a 2-form we get all the conditions of the definition of a Poisson quasi-Nijenhuis structure with background. The precise way to describe this fact is using the twist of a structure by a bivector or a 2-form as in [9,13,17].

**Proposition 4.2.** If we denote by \(S\) the Lie quasi-bialgebroid structure induced by the Poisson quasi-Nijenhuis structure with background \((\pi, N, d\omega_N, -d\omega)\), then \(S = e^{-\omega} \circ (e^{-\pi} \mu - \mu)\), or equivalently
\[
S = e^{-\omega}(\mu_{\pi}),
\]
where \(\mu_{\pi}\) is the Lie algebroid structure defined by (3).

In the next proposition we will see that the Poisson quasi-Nijenhuis structure with background \((\pi, N, d\omega_N, -d\omega)\) is induced (as shown in Theorem 2.5) by a subtangent structure.

**Proposition 4.3.** The \((1, 1)\)-tensor \(J = \left( \begin{array}{cc} N & \pi^\sharp_N \\ -\omega_N \delta & -\delta^! N \end{array} \right) \) is a subtangent structure (i.e., a c.p.s. structure with \(\lambda = 0\)) on \(A \oplus \Lambda^0, \mu - d\omega\).
Proof. Using the Theorems 4.1 and 2.7 we only need to prove
\[
\begin{aligned}
N^2 - \pi^r \circ \omega_N^b &= 0, \\
\omega_N^b \circ N &= N \circ \omega_N^b, \\
2(i_N d\omega_N + \mathcal{H}) &= d(i_N \omega_N).
\end{aligned}
\]
But the verification of the two first conditions is straightforward and, using the fact that \(i_N \omega_N = i_{N^2} \omega\), the last condition is equivalent to (11).

In the remaining part of this section, we will see that if we impose some restrictions on the 2-form \(\omega\), in Theorem 4.1, we get already known structures stronger than Poisson quasi-Nijenhuis with background. We also notice that the pairs \((\pi, \omega), (\pi, N)\) and \((\omega, N)\) thus obtained correspond to (or slightly generalize) already known compatible pairs.

COROLLARY 4.4. (Poisson Nijenhuis) If \(\pi \in \Gamma(\bigwedge^2 A)\) is a Poisson bivector and \(\omega \in \Gamma(\bigwedge^2 A^*)\) is a closed 2-form such that \(d\omega_N = 0\), then \((\pi, N)\) is a Poisson Nijenhuis structure on \(A\).

Remark 4.5. 1. A pair \((\pi, \omega)\) in the conditions of the corollary above is exactly what is called a \(P\Omega\)-structure in [11].
2. The condition \(d\omega_N = 0\) is the compatibility condition for \((\omega, N)\) to be a Hitchin pair as it is defined in [3] for \(A = TM\). The pair \((\omega, N)\) above is more general because \(\omega\) is not necessarily symplectic.
3. Using the fact that \(\omega\) is a closed form, we can prove that the compatibility condition \(d\omega_N = 0\) is equivalent to two other known compatibility conditions:
   - \(\omega\) is a complementary 2-form for \(\pi\) as in [19];
   - \((\omega, N)\) is a \(\Omega N\)-structure as in [11].

Let us justify briefly the last remark. In [19], Vaisman defines \(\omega\) as a complementary 2-form for \(\pi\) when
\[
[\omega, \omega]_{\pi} = 0,
\]
where \([., .]_{\pi}\) is the natural extension to \(\Gamma(\bigwedge^\bullet A^*)\) of the bracket \([., .]_{\pi}\) defined in (3). But in terms of the big bracket and elements of \(C^\infty(T^*\Pi A)\), we have
\[
[\omega, \omega]_{\pi} = \{(\omega, \{\pi, \mu\}), \omega\},
\]
and using the Jacobi identity twice we obtain
\[
[\omega, \omega]_{\pi} = 2\{N, \{\mu, \omega\}\} - \{\mu, \{N, \omega\}\},
\]
which corresponds to
\[
[\omega, \omega]_{\pi} = 2i_N d\omega - 2d(\omega_N). \quad (12)
\]
In [11], Magri and Morosi define a pair \((\omega, N)\) to be a \(\Omega N\)-structure if a particular 3-form \(S(\omega, N)\) vanishes. But we can write

\[
S(\omega, N) = -i_N d\omega + d(\omega N). \tag{13}
\]

Therefore, using (12) and (13) the vanishing of \(d\omega_N\) is equivalent, when \(d\omega = 0\), to the vanishing of \([\omega, \omega]_\pi\) or the vanishing of \(S(\omega, N)\).

**COROLLARY 4.6. (Poisson quasi-Nijenhuis)** If \(\pi \in \Gamma(\bigwedge^2 A)\) is a Poisson bivector and \(\omega \in \Gamma(\bigwedge^2 A^*)\) is a closed 2-form then \((\pi, N, d\omega_N)\) is a Poisson quasi-Nijenhuis structure on \(A\) (without background).

We can also define a Poisson Nijenhuis structure with background \((\pi, N, H)\) by considering \(\psi = 0\) in the Definition 2.3. Up to our knowledge, this structure was never studied before. We have the following result.

**COROLLARY 4.7. (Poisson Nijenhuis with background)** If \(\pi \in \Gamma(\bigwedge^2 A)\) is a Poisson bivector and \(\omega \in \Gamma(\bigwedge^2 A^*)\) is a 2-form such that \(d\omega_N = 0\), then \((\pi, N, -d\omega)\) is a Poisson Nijenhuis structure with background on \(A\).

**OBSERVATION 4.8.** In the above results, the bivector \(\pi\) is a true Poisson bivector. So the last structure we obtain is different from a possible compatibility between a Poisson structure with background [15] and a Nijenhuis tensor.

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