THE "ACTION" VARIABLE IS NOT AN INVARIANT FOR THE UNIQUENESS IN THE INVERSE SCATTERING PROBLEM

A. KHEIFETS AND P. YUDITSKII

Abstract. We give a simple example of non-uniqueness in the inverse scattering for Jacobi matrices: roughly speaking, $S$-matrix is analytic. Then, multiplying a reflection coefficient by an inner function, we repair this matrix in such a way that it does uniquely determine a Jacobi matrix of Szegő class; on the other hand, the transmission coefficient remains the same. This implies the statement given in the title.

1. Jacobi matrices of Szegő class. Direct scattering — Bernstein–Szegő type theorem

As it is well known in the theory of completely integrable systems the absolute values of a reflection coefficient have played the role of the "action" variables and the arguments of this function have meaning of the "angle" variables (in this case we think on the Toda lattice as on an integrable system) [4]. Combining results of our previous works [2] (see also [3]) and [5], we give a wide set of examples where two reflection coefficients, having the same absolute values, possess completely different properties: the first one uniquely determines a Jacobi matrix of Szegő class and the second one does not. Note that the proof of the main theorem in [2] (and, therefore, our result) essentially uses the analysis of [1] on regularization of so-called Arov–singular matrix functions.

Let $J$ be a Jacobi matrix defining a bounded self–adjoint operator on $l^2(\mathbb{Z})$:

$$Je_n = p_n e_{n-1} + q_n e_n + p_{n+1} e_{n+1}, \quad n \in \mathbb{Z},$$

where $\{e_n\}$ is the standard basis in $l^2(\mathbb{Z})$, $p_n > 0$. The resolvent matrix–function is defined by the relation

$$R(z) = R(z, J) = E^*(J - z)^{-1} E,$$

where $E : \mathbb{C}^2 \to l^2(\mathbb{Z})$ is such that

$$E \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = e_{-1} c_{-1} + e_0 c_0.$$

This matrix–function possesses an integral representation

$$R(z) = \int \frac{d\sigma}{x - z}$$

with a $2 \times 2$ matrix–measure having a compact support on $\mathbb{R}$. $J$ is unitary equivalent to the multiplication operator by an independent variable on

$$L^2_{ds} = \left\{ f = \begin{bmatrix} f_{-1}(x) \\ f_0(x) \end{bmatrix} : \int f^* \, d\sigma \, f < \infty \right\}.$$
The spectrum of $J$ is called absolutely continuous if the measure $d\sigma$ is absolutely continuous with respect to the Lebesgue measure on the real axis,

$$d\sigma(x) = \rho(x) \, dx.$$  

(4)

It is natural to ask how properties of coefficients of $J$ are reflected on its spectral properties. One is especially interested in $J$’s “close” to the “free” matrix $J_0$ with constant coefficients, $p_n = 1$, $q_n = 0$ (so called Chebyshev matrix).

Let us mention that $J_0$ has the following functional representation, besides the general one mentioned above. The resolvent set of $J_0$ is the domain $\mathbb{C} \setminus [-2, 2]$. Let $z(\zeta) : \mathbb{D} \to \mathbb{C} \setminus [-2, 2]$ be a uniformization of this domain, $z(\zeta) = 1/\zeta + \zeta$. With respect to the standard basis $\{t^n\}_{n \in \mathbb{Z}}$ in $L^2 = \{f(t) : \int_T |f|^2 \, dm\}$, the matrix of the operator of multiplication by $z(t)$, $t \in T$, is the Jacobi matrix $J_0$, since $z(t)t^n = t^{n-1} + t^{n+1}$.

We say that $J$ with absolutely continuous spectrum $[-2, 2]$ is of Szegő class if its spectral density (4) satisfies

$$\log \det \rho(z(t)) \in L^1.$$  

(5)

Theorem 1.1. Let $J$ be of Szegő class. Then

$$p_n \to 1, \quad q_n \to 0, \quad n \to \pm\infty.$$  

(6)

Moreover, there exist generalized eigenvectors

$$p_n e^+(n - 1, t) + q_n e^+(n, t) + p_{n+1} e^+(n + 1, t) = z(t) e^+(n, t)$$  

$$p_n e^-(n, t) + q_n e^-(n - 1, t) + p_{n+1} e^-(n - 2, t) = z(t) e^-(n - 1, t)$$  

(7)

such that the following asymptotics hold true

$$s(t)e^+(n, t) = s(t)t^n + o(1), \quad n \to +\infty$$  

$$s(t)e^+(n, t) = t^n + s_+(t)t^{-n-1} + o(1), \quad n \to -\infty$$  

(8)

in $L^2$.

Thus the eigenvectors of $J$ behave asymptotically as the eigenvectors of $J_0$ (later we make more precise statement).

The matrix formed by the coefficients of (8)

$$S(t) = \begin{bmatrix} s_- & s \\ s & s_+ \end{bmatrix}(t), \quad t \in \mathbb{T},$$  

(9)

is called the scattering matrix of $J$. It is unitary–valued, possesses the symmetry property $S^*(t) = S(t)$ and the following analytic property: $s(t)$ is an outer function.

In what follows every matrix–function of the form (9) with the above listed properties is called a scattering matrix. Of course we have a good reason for this, since with every matrix $S(t)$ of this kind one can associate a Jacobi matrix $J$ whose scattering matrix (associated to $J$ according to Theorem 1.1) is the initial matrix–function $S(t)$. However, $S(t)$, generally speaking, does not determine $J$ uniquely.
To clarify all above statements we need some notation and definitions. First of all for a given functions \( s_\pm \) we define the metric
\[
||f||^2_{s_\pm} = \frac{1}{2} \left( \left[ \begin{array}{c} 1 \\ s_\pm(t) \\ 1 \end{array} \right] \left[ \begin{array}{c} f(t) \\ \bar{f}(t) \\ \bar{f}(t) \end{array} \right] \right)
\]
and we denote by \( L^2_{dm,s_\pm} \) or \( L^2_{s_\pm} \) (for shortness) the closure of \( L^2 \) with respect to this new metric.

The following relations set a unitary map from \( L^2_{s_+} \) to \( L^2_{s_-} \):
\[
\left[ \begin{array}{c} s f^+ \\ s f^- \end{array} \right](t) = \left[ \begin{array}{c} s \\ 0 \\ s_+ \\ 1 \\ s_- \\ 0 \end{array} \right](t) \left[ \begin{array}{c} f^+(t) \\ \bar{f}^+(t) \\ \bar{f}^-(t) \\ f^-(t) \end{array} \right].
\]
Moreover, in this case,
\[
||f^+||^2_{s_+} = ||f^-||^2_{s_-} = \frac{1}{2} (||sf^+||^2 + ||sf^-||^2).
\]

It is worth to give a scalar variant of relations between \( f^+ \in L^2_{s_+} \) and \( f^- \in L^2_{s_-} \):
\[
s(t)f^\mp(t) = \bar{t}f^\pm(\bar{t}) + s_\pm(t)f^\pm(t).
\]

**Theorem 1.2.** \( J \) is a Jacobi matrix of Szegő class with the spectrum \( E = [-2,2] \) if and only if \( J \) possesses the scattering representation, i.e.: there exists a unique matrix–function \( S(t) \) of the form \( \text{(9)} \) (with the listed properties) and a unique pair of Fourier transforms
\[
\mathcal{F}^\pm : l^2(\mathbb{Z}) \to L^2_{s_\pm}, \quad (\mathcal{F}^\pm J f)(t) = z(t)(\mathcal{F}^\pm f)(t),
\]
determining each other by the relations
\[
s(t)(\mathcal{F}^\pm f)(t) = \bar{t}(\mathcal{F}^\mp f)(\bar{t}) + s_\mp(t)(\mathcal{F}^\mp f)(t),
\]
and having the following analytic properties
\[
s\mathcal{F}^\pm(l^2(\mathbb{Z}_\pm)) \subset H^2;
\]
and asymptotic properties
\[
e^\pm(n,t) = t^n + o(1) \quad \text{in} \ L^2_{s_\pm}, \quad n \to +\infty,
\]
where
\[
e^+ n, t) = (\mathcal{F}^+ e_n)(t), \quad e^-(n, t) = (\mathcal{F}^- e_{-n-1})(t),
\]
with \( \{e_n\} \) being the standard basis in \( l^2(\mathbb{Z}) \).

We point out that asymptotic relations \( \text{(8)} \) and \( \text{(13)} \) are equivalent, moreover \( \text{(14)} \) directly shows that the eigenvectors of \( J \) asymptotically are the eigenvectors of \( J \).
2. Uniqueness and Completeness

Before we proceed with the uniqueness theorem, we show how to construct at least one \( J \) with the given scattering matrix \( S(t) \).

Consider the space

\[ H^2_{s_+} = \text{clos}_{L^2} H^2, \]

and introduce the Hankel operator \( H_{s_+} : H^2 \rightarrow H^2 \),

\[ H_{s_+} f = P_+ \bar{t}(s+ f)(\bar{t}), \quad f \in H^2, \]

where \( P_+ \) is the Riesz projection from \( L^2 \) onto \( H^2 \). This operator determines the metric in \( H^2_{s_+} \):

\[ ||f||^2_{s_+} = \langle f(t) + \bar{t}(s+ f)(\bar{t}), f(t) \rangle = \langle (I + H_{s_+}) f, f \rangle, \quad \forall f \in H^2. \]

**Theorem 2.1.** Let \( S(t) \) be a scattering matrix, i.e., the matrix of the form \( [I] \) with listed properties. Then the space \( H^2_{s_+} \) is a space of holomorphic functions with a reproducing kernel. Moreover, the reproducing vector \( k_{s_+} \):

\[ \langle f, k_{s_+} \rangle = f(0), \quad \forall f \in H^2_{s_+}, \]

is of the form

\[ k_{s_+} = (I + H_{s_+})^{-1} \mathbf{1} = \lim_{\epsilon \to 0^+} \epsilon (I + H_{s_+})^{-1} \mathbf{1} \quad \text{in} \quad L^2_{s_+}. \]

Put \( K_{s_+}(t) = k_{s_+}(t)/\sqrt{k_{s_+}(0)} \). Then the system of functions \( \{ t^n K_{s_+} z_n(t) \}_{n \in \mathbb{Z}} \) forms an orthonormal basis in \( L^2_{s_+} \). With respect to this basis, the multiplication operator by \( z(t) \) is a Jacobi matrix \( J = J[s_+] \) of Szegő class. Moreover, the initial \( S(t) \) serves as the scattering matrix–function, associated with given \( J \) by Theorem 2.1 and the relations

\[ \mathcal{F}^+(e_n) = t^n K_{s_+} z_n(t) \]

determine uniquely corresponding Fourier transforms.

Let us fix the notation \( J[s_+] \) for the Jacobi matrix associated with \( S(t) \) by Theorem 2.1. On the other hand, the system of functions \( \{ t^n K_{s_-} z_n(t) \}_{n \in \mathbb{Z}} \) forms an orthonormal basis in \( L^2_{s_-} \), and we can define a Jacobi matrix \( \bar{J} = J[s_-] \) by the relation

\[ z(t) \bar{e}^+(n, t) = \hat{p}_n \hat{e}^+(n-1, t) + \hat{q}_n \hat{e}^+(n, t) + \hat{p}_{n+1} \hat{e}^+(n + 1, t), \]

where \( \{ \hat{e}^+(n, t) \} \) is the dual system to the system \( \{ t^n K_{s_-} z_n(t) \} \) (see \( \mathcal{F}_Z \)), i.e.:

\[ s(t) \hat{e}^+(n-1, t) = t^{n+1} K_{s_-} z_n(t) \hat{f}(0) + s(t) t^n K_{s_-} z_n(t). \]

None guarantees that operators \( J[s_+] \) and \( J[s_-] \) are the same (see beginning of the next section). However, if \( J[s_+] = J[s_-] \), then the uniqueness theorem takes place.

**Theorem 2.2.** A scattering matrix \( S(t) \) determines a Jacobi matrix \( J \) of Szegő class in a unique way if and only if the following relations take place

\[ s(0) K_{s_+}(0) K_{s_-}^{-1}(0) = 1. \]
Of course it is hard to check identities, especially using computer simulation, but, in fact, (16) has a specific approximating meaning.

Let us return to the matrix $J[s_-]$. According to (13) the space $F^+(l^2(\mathbb{Z}_+))$ is a subspace of $L^2_{s^+}$ consisting of holomorphic in $\mathbb{D}$ functions. In the given case we can even specify this space in the form
\begin{equation}
F^+(l^2(\mathbb{Z}_+)) = \mathcal{H}^2_{s^+} := \{ f \in L^2_{s^+} : sf \in H^2 \}.
\end{equation}
Every function $f$ from $H^2_{s^+}$ possesses the property $sf \in H^2$, but this means only inclusion:
\begin{equation}
\mathcal{H}^2_{s^+} \supset H^2_{s^+}.
\end{equation}
The meaning of (16) is that every function from $\mathcal{H}^2_{s^+}$ can be approximated by functions from $H^2$ in $L^2_{s^+}$-norm ($H^2$ is dense in $\mathcal{H}^2_{s^+}$). In fact, it is enough to prove that we can approximate just two functions $\hat{e}^+(0,t)$ and $\hat{e}^+(1,t)$ from $\mathcal{H}^2_{s^+}$. This would guarantee (16), completeness and the uniqueness theorem.

3. The result

Our first remark is almost evident. Let us pick an analytic scattering matrix (all entries are in $H^\infty$). For example,
\begin{equation}
S(t) = \begin{bmatrix}
\frac{1+\Delta}{1-2} & \frac{1-\Delta}{1-2} \\
\frac{1}{2} & \frac{1}{2} \\
\end{bmatrix} (t),
\end{equation}
where $\Delta(t) = \overline{\Delta(t)}$ is a symmetric inner function. Then the analyticity of $s_+$ implies that $\mathcal{H}_{s^+} = 0$, $H^2_{s^+} = H^2$ and $t^n K_{s^+}X_n(t) = t^n$, $n \geq 0$. Thus,
\begin{equation}
J[s_+] | l^2(\mathbb{Z}_+ (1)) = J_0 | l^2(\mathbb{Z}_+ (1)),
\end{equation}
where $\mathbb{Z}_+(m) = \{ n \in \mathbb{Z} : n \geq m \}$. The analyticity of $s_-$, in its turn, implies that
\begin{equation}
J[s_-] | l^2(\mathbb{Z}_- (-1)) = J_0 | l^2(\mathbb{Z}_- (-1)),
\end{equation}
$\mathbb{Z}_- (m) = \{ n \in \mathbb{Z} : n \leq m - 1 \}$.

Thus, the uniqueness would imply that $J$ associated to given $S(t)$ coincides with $J_0$ up to three coefficients: $p_0, q_0, q_{-1}$. For fun we give an exact formula for $S(t)$ related to this case:
\begin{equation}
S(t) = \frac{1}{\phi(t)} \begin{bmatrix}
\psi_+(t) & p_0 (1 - t^2) \\
p_0 (1 - t^2) & \psi(t) \\
\end{bmatrix},
\end{equation}
where $\phi(t) = (1 - q_0 t)(1 - q_{-1} t) - p_0^2 t^2$, $\psi(t) = (1 - q_0 t)(q_{-1} - t) + p_0^2 t^2$ and $\psi_+(t) = t^2 \overline{\psi(t)}$. Note that this matrix has no bound states ($s(t)$ is holomorphic in $\mathbb{D}$) if
\begin{equation}
|| \begin{bmatrix}
q_{-1} & p_0 \\
p_0 & q_0 \\
\end{bmatrix} || \leq 1.
\end{equation}
Therefore, once the scattering matrix is not a rational function of degree two, then, for sure, the holomorphic $S(t)$ does not determine $J$ uniquely.

The main result of this note is as follows.

**Theorem 3.1.** Given $S(t)$ from $H^\infty$, one can find an inner function $\Phi(t) = \overline{\Phi(t)}$ such that
\begin{equation}
S_{\Phi}(t) = \begin{bmatrix}
s_- \Phi & s \\
\Phi & s_+ \Phi \\
\end{bmatrix} (t), \quad t \in \mathbb{T},
\end{equation}
determines a Jacobi matrix of Szegö class uniquely.

Let us point out that we did not change the transmission coefficient $s$. Moreover
the new reflection coefficient $s_\Phi $ is still analytic one, so, the corresponding Jacobi
matrix $J$ coincides with $J_0$, when both are restricted on $l^2(\mathbb{Z}_\omega (-1))$.

Theorem 3.1 is the direct consequence of the main result of [2] and the following
two lemmas.

**Lemma 3.2.** Let the reflection coefficient $s_- \in H^\infty $, and $s_-(\zeta ) = a \zeta + \ldots $, i.e.: $s_-(0) = 0$. Then (20) holds if

\[
\begin{bmatrix}
1 & s_- \\
0 & s
\end{bmatrix}
\begin{bmatrix}
\bar{t} & 1 \\
1 & \bar{t}
\end{bmatrix}
= \lim_{n \to \infty }
\begin{bmatrix}
s & 0 \\
\bar{s}_+ & 1
\end{bmatrix}
\begin{bmatrix}
f_n & \bar{t}_n
\end{bmatrix}
\begin{bmatrix}
\bar{g}_n & u_n
\end{bmatrix}
\]

with a suitable choice of sequences of functions $f_n, g_n$ from $H^2$, that is, $\frac{1+s_-/t}{s} \in H^2_{s_+}$ and $\frac{1+s_-}{s} \in H^2_{s_+ t^{-2}}$.

**Proof.** Since $s_-$ is holomorphic, we have $\mathcal{H}_{s_-} = 0$. Moreover, even $\mathcal{H}_{s_- t^{-2}} = a\mathbf{1}$, since $s_-(0) = 0$. Therefore,

\[
K_{s_-} = 1 \quad \text{and} \quad K_{s_- t^{-2}} = \frac{1}{\sqrt{1+a}}.
\]

We work with the second vector. Because of

\[
\begin{bmatrix}
s & 0 \\
\bar{s}_+ & 1
\end{bmatrix}
\begin{bmatrix}
f^+(t) \\
\bar{f}^+ (\bar{t})
\end{bmatrix}
= \lim_{n \to \infty }
\begin{bmatrix}
s & 0 \\
\bar{s}_+ & 1
\end{bmatrix}
\begin{bmatrix}
f_n & \bar{t}_n
\end{bmatrix}
\begin{bmatrix}
\bar{g}_n & u_n
\end{bmatrix}
= 2(s f^+)(0), \quad f^+ \in H^2,
\]

and (20), the vector $\frac{1+s_-/t}{s\sqrt{1+a}}$ belongs to $H^2_{s_+}$ and it is collinear with the reproducing kernel of this space. Since its norm is one, we get

\[
K_{s_+} (\zeta ) = \frac{1+s_- (\zeta )/\zeta }{s(\zeta )\sqrt{1+a}}.
\]

Putting here $\zeta = 0$, we get $s(0) K_{s_+}(0) K_{s_- t^{-2}}(0) = 1$. The proof of the second identity is very similar.

**Lemma 3.3.** Let the following approximation hold true

\[
\begin{bmatrix}
\bar{t} s_- \\
s - s(0) t
\end{bmatrix}
= \lim_{n \to \infty }
\begin{bmatrix}
u_n & s u_n \\
P_{s_+ u n} & \bar{s}_+ u n
\end{bmatrix}, \quad u_n \in H^2
\]

(we still assume that $s_-(0) = 0$). Then (20) is also true.

**Proof.** We put

\[
\bar{t} f_n (\bar{t}) = s(0) \bar{t} + u_n (\bar{t}) - P_{s_+ u n}
\]

and $g_n = f_n - (1-t^2) u_n$. In this case

\[
\begin{bmatrix}
f_n & \bar{t}_n \\
\bar{t} f_n (\bar{t}) & g_n (\bar{t})
\end{bmatrix}
= \begin{bmatrix} f_n - u_n & t u_n \\
\bar{t}_n (\bar{t}) - u_n (\bar{t}) & \bar{t} \end{bmatrix}
\begin{bmatrix}
1 & \bar{t} \\
\bar{t} & 1
\end{bmatrix}.
\]

In fact, we are going to show that

\[
\begin{bmatrix}
1 & s_+ t \\
0 & \bar{s}_+ t
\end{bmatrix}
= \lim_{n \to \infty }
\begin{bmatrix}
s & 0 \\
\bar{s}_+ & 1
\end{bmatrix}
\begin{bmatrix}
f_n - u_n & u_n \\
\bar{t}_n (\bar{t}) - u_n (\bar{t}) & \bar{t} \end{bmatrix}.
\]
Let us note that the equality in the second column of (23) is just a direct consequence of (23) and (24). Therefore, we need to check the equality in the first column. In the first entry of this column we have
\[
\lim_{n \to \infty} s(f_n - u_n) = \lim_{n \to \infty} \{s f_n - s_- \ell\}
\]
\[
= \lim_{n \to \infty} \{s s(0) + s u_n - s P_+ \overline{s} s(\ell) u_n(\ell) - s_- \ell\}
\]
\[
= \lim_{n \to \infty} \{s s(0) + s_- \overline{s} u_n(\ell) + s P_- \overline{s} s(\ell) u_n(\ell) - s_- \ell\}.
\]
Using the unitary property of \( S \)-matrix and (23), we get
\[
\lim_{n \to \infty} s(f_n - u_n) = \lim_{n \to \infty} \{s s(0) + s_- s \overline{s} u_n(\ell) + s P_- \overline{s} u_n(\ell) + (1 - |s_-|^2) \overline{s} u_n(\ell)}
\]
\[
= \lim_{n \to \infty} \{s s(0) + s_- \overline{s} (s - s(0)) + s \overline{s} u_n(\ell)\}
\]
\[
= s_+ s(0) + s_+ (s - s(0)) + s \overline{s} = 0.
\]

References

1. V. Katsnelson, Left and right Blaschke-Potapov products and Arov-singular matrix-valued functions, Integral Equations Operator Theory 13 (1990), no. 6, 836–848.
2. A. Kheifets, On regularization of \( \gamma \)-generating pairs, J. Funct. Anal. 130 (1995), no. 2, 310–333.
3. A. Kheifets, Nehari’s interpolation problem and exposed points of the unit ball in the Hardy space \( H^1 \), Proceedings of the Ashkelon Workshop on Complex Function Theory (1996), 145–151, Israel Math. Conf. Proc., 11, Bar-Ilan Univ., Ramat Gan, 1997.
4. G. Teschl, Jacobi operators and completely integrable nonlinear lattices Mathematical Surveys and Monographs, 72. American Mathematical Society, Providence, RI, 2000
5. A. Volberg and P. Yuditski, On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of intervals or a Cantor set of positive length, Comm. Math. Phys. (to appear).

Department of Mathematics, The College Of William and Mary, P. O. Box 8795, Williamsburg, Virginia 23187-8795
E-mail address: sykhei@wm.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824
E-mail address: yuditski@math.msu.edu