Relaxing Numerical Difficulties Arising from Discretization in the Cylindrical Coordinates

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Abstract. The cylindrical coordinates, \((r, \phi, z)\), are often used in our numerical simulations, in particular, when gas accretes onto a central object. Although the cylindrical coordinates have several advantages in describing rotation, they have apparent singularity at \(r = 0\). The singularity induces some difficulties in numerical simulations. First, it is difficult to solve the flow across the axis numerically. Second, the time interval between two successive time steps is restricted by the innermost cell size, resulting in large numbers of time steps to follow dynamics in the outer regions. Here, we present a new discretization scheme to overcome these difficulties. First, we evaluate the centrifugal force from the dynamical pressure working on the azimuthal cell surface. Our method of the evaluation is based on the vector analysis. Second, we reduce the angular resolution appreciably. Our examples demonstrate that the innermost circular region around the axis can be resolved by only six numerical cells. We increase the angular resolution with the increase in the radius so that each numerical cell has an aspect ratio close to unity. Then all the numerical cells have nearly the same size, and the CFL condition is relaxed. We show our application to an accreting protostar after showing a Sod shock tube problem and a uniform flow. The test problems demonstrate that the apparent singularity around the axis have no critical effects on the results.

1. Introduction
The cylindrical coordinates, \((r, \phi, z)\), are appropriate for describing an accretion flow onto a compact object. They can describe centrifugal balance, angular momentum transfer, and vertical structure along the rotation axis in a simple form. Thus, they have been adopted in the numerical simulations of accreting protostars, neutron stars, and black holes. Although the simulations thus far have given us useful insights on our understanding the astronomical objects, they are often associated with technical problems. First, they often show unphysical disturbances around the \(z\)-axis, which hinder wave propagation and non-axisymmetric structures of physical origin. Second, the time step, \(\Delta t\), is restricted to be very small because of the tight CFL condition around the \(z\)-axis, since the azimuthal cell width, \(r \Delta \phi\), is proportional to the radius, \(r\), for a fixed angular resolution. The small time step wastes our computational resources and inhibits pursuing a long-term evolution. These disadvantages come from the apparent singularity of the cylindrical coordinates on the \(z\)-axis. In this paper, we propose a new discretization method to
overcome the problems arising from the apparent singularity of the cylindrical coordinates. As shown later, this method is easy to be installed and robust. It is distinct from those proposed thus far for alleviating the problems.

Our method originates from the recognition that we should take account of the change in the unit vectors parallel to the coordinates within a cell when applying the finite volume method for a conservation law. The hydrodynamical equations consist of the conservation of mass, momentum, and energy. We express the conservation of momentum as

$$\frac{\partial}{\partial t} (\rho v) + \nabla \cdot T = \rho g,$$

where the symbols $\rho$, $v$, $g$, and $T$ denote the density, velocity, gravity and momentum flux tensor, respectively. The momentum flux includes thermal and dynamical pressure. By integrating Equation (1) over a volume, $V$, we obtain

$$\frac{\partial}{\partial t} \int_V \rho v dV + \int_{\partial V} T \cdot dS = \int_V \rho g dV,$$

where the second term in the left-hand side denotes the surface integral of the momentum flux tensor. The finite volume method is based on this integral form, Equation (2). When we use the cylindrical coordinates, we must realize that the radial direction depends on the azimuthal angle, $\varphi$. The azimuthal direction on the azimuthal cell boundary is not parallel to that at the cell center. Thus, the force normal to the azimuthal cell boundary has a radial component if evaluated at the cell center. The force includes the dynamical pressure, $\rho v^2$, where $v_\varphi$ denotes the azimuthal component of the velocity. We will show in the next section that the radial component of the dynamical pressure coincides with the centrifugal force. Thus, the centrifugal force should be evaluated on the cell boundary while it is often evaluated at the cell center thus far. By evaluating the centrifugal force on the cell surface, we have succeeded in solving a Sod shock tube problem and a uniform flow across the $z$-axis with a low angular resolution, $\Delta \varphi = \pi/3$ for the innermost cells. This low angular resolution resolves the small time step problem.

This paper explains our methods in the following order. First, we rewrite the hydrodynamical equations in the forms appropriate for numerical integration in the cylindrical coordinates in \S2.1. Second, we derive the discretized form of the hydrodynamical equations in \S2.2. The discretized form takes account of the change in the radial direction on the cell surface from that at the cell center. Third, we show the procedures to solve the discretized equations in \S2.3. They include the method of evaluating the physical variables on the cell boundary for computing the numerical flux across it. In \S3, we prove the validity of our numerical method by showing the numerical tests including a Sod shock tube problem, a uniform flow, and an anisotropic accretion onto a protostar. We discuss the application of the same idea to magnetohydrodynamical equations (MHD) and spherical coordinates in \S4.

2. Numerical Methods

2.1. Basic Equations

Here, we consider the flow of an ideal gas under the attraction of gravity. We ignore the magnetic field, radiation, cooling, and heating for simplicity. Then, the following hydrodynamical
describe the gas flow, where \( \rho, P, v, g \) and \( H \) denote the density, pressure, velocity, gravity and the energy of the gas element per unit mass, respectively. The specific heat ratio is assumed to be \( \gamma = 5/3 \). The symbol \( \mathbf{I} \) denotes the identity tensor and \( vv \) does the direct product.

We rewrite Equations (3) through (6) into the conservation form in the cylindrical coordinates, \( (r, \varphi, z) \),

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ \begin{array}{c}
\rho \\
\rho v_r \\
\rho v_\varphi \\
\rho v_z \\
\rho E
\end{array} \right] + \frac{\partial}{\partial r} \left[ \begin{array}{c}
\rho v_r \\
r^2 \rho v_\varphi \\
r \rho v_z \\
r \rho v_r H \\
r \rho v_\varphi H
\end{array} \right] + \frac{\partial}{\partial \varphi} \left[ \begin{array}{c}
\rho v_\varphi \\
\rho v_r v_\varphi \\
r \rho v_z \\
r \rho v_\varphi v_z \\
r \rho v_\varphi H
\end{array} \right] + \frac{\partial}{\partial z} \left[ \begin{array}{c}
\rho v_z \\
r^2 \rho v_\varphi \\
r \rho v_r v_z \\
r \rho v_\varphi v_z \\
r \rho v_z H
\end{array} \right] = \left[ \begin{array}{c}
0 \\
r \rho g_r + \rho v_\varphi^2 + P \\
r^2 \rho g_\varphi \\
r \rho g_z \\
r \rho g \cdot v
\end{array} \right],
\end{align*}
\]

in order to apply the finite volume method, since this form is widely used in the simulations of astrophysical flows. Note that each component except the third one is multiplied by factor, \( r \), while the third by \( r^2 \). The factor, \( r \), comes from the fact that the volume integral is denoted by \( r dr d\varphi dz \). The extra factor, \( r \), for the third component is introduced to denote the conservation of total angular momentum. Equation (7) is the standard form adopted in the numerical simulations of astrophysical flows in the cylindrical coordinates.

The right hand side of Equation (7) denotes the source terms due to gravity and geometry. We should examine the source term due to the geometry, \( P + \rho v_\varphi^2 \), in the second component since it comes from the curvature of the coordinates and is likely to be the origin of unphysical disturbances. The term, \( \rho v_\varphi^2 \), originates from the centrifugal force, \( \nu_\varphi^2 / r \), since we derive the second component of Equation (7) by multiplying the equation of motion by \( r \rho \). The pressure, \( P \), in the source term cancels a surplus term originating from the derivative with respect to \( r \). These geometrical source terms have apparently different origin, but the sum coincides with the total pressure working in the \( \varphi \) direction. This coincidence implies the possibility of evaluating the source term on the cell boundary.

### 2.2. Discretized Equations

In order to examine the possibility mentioned in the previous subsection, we rewrite Equations (4) in the integral form,

\[
\frac{\partial}{\partial t} \iint_V \rho v dV + \iint_{\partial V} \left( \rho vv + P \mathbf{I} \right) \cdot dS = \iint_V \rho g dV.
\]
We apply Equation (8) to a numerical cell bounded by, \(|r - r_i| \leq \Delta r_i/2, |\varphi - \varphi_j| \leq \Delta \varphi_j/2,\) and \(|z - z_k| \leq \Delta z_k/2,\) where the subscripts, \(i, j,\) and \(k,\) specify the cell under consideration. The surface integral has contributions from the cell boundaries on either of \(r = r_i \pm \Delta r_i/2,\) \(\varphi = \varphi_j \pm \Delta \varphi_j/2,\) and \(z = z_k \pm \Delta z_k/2.\) The cell surface element on \(\varphi = \varphi_j \pm \Delta \varphi_j/2\) is expressed as

\[
dS_{i,j,k} = r_i \Delta r_i \Delta z_k \frac{\partial}{\partial t} (\rho v)_{i,j,k} + (\rho v + P)_{i+1/2,j,k} \cdot \left( r_i + \frac{\Delta r_i}{2} \right) \Delta \varphi_j \Delta z_k e_r (\varphi_j) \\
- (\rho v + P)_{i-1/2,j,k} \cdot \left( r_i - \frac{\Delta r_i}{2} \right) \Delta \varphi_j \Delta z_k e_r (\varphi_j) \\
+ (\rho v + P)_{i,j+1/2,k} \cdot \Delta r_i \Delta z_k \cos \left( \frac{\Delta \varphi_j}{2} \right) \left[ e_r (\varphi_j) - \frac{\Delta \varphi_j}{2} e_r (\varphi_j) \right] \\
- (\rho v + P)_{i,j-1/2,k} \cdot \Delta r_i \Delta z_k \cos \left( \frac{\Delta \varphi_j}{2} \right) \left[ e_r (\varphi_j) + \frac{\Delta \varphi_j}{2} e_r (\varphi_j) \right] \\
+ (\rho v + P)_{i,j,k+1/2} \cdot r_i \Delta r_i \Delta \varphi_j e_z \\
- (\rho v + P)_{i,j,k-1/2} \cdot r_i \Delta r_i \Delta \varphi_j e_z = \rho g r_i \Delta r_i \Delta \varphi_j \Delta z_k, \tag{11}
\]

where the subscripts, \(i \pm 1/2, j \pm 1/2,\) and \(k \pm 1/2,\) specify the cell boundaries. Here, the numerical cells are arranged so that the subscripts, \(i, j,\) and \(k,\) increase successively in the \(r,\) \(\varphi-\) and \(z\)-directions, respectively, for simplicity. In other words, the angular resolution, \(\Delta \varphi,\) is assumed to be independent of the radius, \(r,\) in this subsection. We relax this restriction in the next subsection. We use the symbol, \(r_{i+1/2} = r_i \pm \Delta r_i/2,\) to denote the radius of the boundary in the \(r\)-direction in the following to save space.

We substitute the expressions,

\[
v = v_r e_r + v_{\varphi} e_{\varphi} + v_z e_z, \tag{12}
\]
\[
g = g_r e_r + g_{\varphi} e_{\varphi} + g_z e_z, \tag{13}
\]

into Equation (11) and take the inner product with \(e_r (\varphi_j)\) to obtain

\[
r_i \Delta r_i \Delta \varphi_j \Delta z_k \frac{\partial}{\partial t} (\rho v_r) + r_{i+1/2} \Delta \varphi_j \Delta z_k \left( \rho v_r^2 + P \right)_{i+1/2,j,k} \\
- r_{i-1/2} \Delta \varphi_j \Delta z_k \left( \rho v_r^2 + P \right)_{i-1/2,j,k} \\
+ \Delta r_i \Delta z_k \cos \left( \frac{\Delta \varphi_j}{2} \right) \left[ (\rho v_{\varphi} v_r)_{i,j+1/2,k} - (\rho v_{\varphi} v_r)_{i,j-1/2,k} \right] \\
+ r_i \Delta r_i \Delta \varphi_j \left[ (\rho v_{\varphi} v_r)_{i,j,k+1/2} - (\rho v_{\varphi} v_r)_{i,j,k-1/2} \right] \\
- \Delta r_i \Delta z_k \sin \left( \frac{\Delta \varphi_j}{2} \right) \left[ (\rho v_{\varphi}^2 + P)_{i,j+1/2,k} + (\rho v_{\varphi}^2 + P)_{i,j-1/2,k} \right] \\
= r_i \Delta r_i \Delta \varphi_j \Delta z_k (\rho g r_i)_{i,j,k}. \tag{14}
\]
Equation (14) indicates that the source term due to the geometry should indeed be evaluated on the cell surface. However, Equation (14) gives a false pressure force when the pressure is uniform. In order to fix this problem, we modify Equation (14) into

\[ r_i \Delta r_i \Delta \phi j \Delta z_k \frac{\partial}{\partial t} (r\rho v_r) + r_{i+1/2} \Delta \phi j \Delta z_k \left( \rho v_r^2 + P \right)_{i+1/2,j,k} - r_{i-1/2} \Delta \phi j \Delta z_k \left( \rho v_r^2 + P \right)_{i-1/2,j,k} + \Delta r_i \Delta z_k \left[ (\rho v_r v_r)_{i,j+1/2,k} - (\rho v_r v_r)_{i,j-1/2,k} \right] \cos \left( \frac{\Delta \phi}{2} \right) + r_i \Delta r_i \Delta \phi j \left( \rho v_z v_r \right)_{i,j,k+1/2} - (\rho v_z v_r)_{i,j,k-1/2} - \frac{\Delta r_i \Delta \phi j \Delta z_k}{2} \left[ \left( \rho v_r^2 + P \right)_{i,j+1/2,k} + \left( \rho v_r^2 + P \right)_{i,j-1/2,k} \right] = r_i \Delta r_i \Delta \phi j \Delta z_k \left( \rho g_r \right)_{i,j,k}. \]  

(15)

The other components of Equation (7) have the discretized form,

\[ r_i \Delta r_i \Delta \phi j \Delta z_k \frac{\partial}{\partial t} \rho \left( v_{i,j,k} \right) + \Delta \phi j \Delta z_k \left[ r_{i+1/2} \left( \rho v_r \right)_{i+1/2,j,k} - r_{i-1/2} \left( \rho v_r \right)_{i-1/2,j,k} \right] + \Delta r_i \Delta z_k \left[ \left( \rho v_r \right)_{i,j+1/2,k} - \left( \rho v_r \right)_{i,j-1/2,k} \right] + r_i \Delta r_i \Delta \phi j \left( \rho v_z \right)_{i,j,k+1/2} - \left( \rho v_z \right)_{i,j,k-1/2} = 0, \]  

(16)

\[ r_i \Delta r_i \Delta \phi j \Delta z_k \frac{\partial}{\partial t} \left( \rho v_z \right)_{i,j,k} + \Delta \phi j \Delta z_k \left[ r_{i+1/2} \left( \rho v_z \right)_{i+1/2,j,k} - r_{i-1/2} \left( \rho v_z \right)_{i-1/2,j,k} \right] + r_i \Delta r_i \Delta \phi j \left( \rho v_z \right)_{i,j,k+1/2} - \left( \rho v_z \right)_{i,j,k-1/2} = r_i^2 \Delta r_i \Delta \phi j \Delta z_k \left( \rho g_z \right)_{i,j,k}, \]  

(17)

\[ r_i \Delta r_i \Delta \phi j \Delta z_k \frac{\partial}{\partial t} \left( \rho E \right)_{i,j,k} + \Delta \phi j \Delta z_k \left\{ r_{i+1/2} \left[ \left( \rho E + P \right) v_r \right]_{i+1/2,j,k} - r_{i-1/2} \left[ \left( \rho E + P \right) v_r \right]_{i-1/2,j,k} \right\} + \Delta r_i \Delta z_k \left\{ \left[ \left( \rho E + P \right) v_z \right]_{i,j+1/2,k} - \left[ \left( \rho E + P \right) v_z \right]_{i,j-1/2,k} \right\} + r_i \Delta r_i \Delta \phi j \left\{ \left[ \left( \rho E + P \right) v_z \right]_{i,j,k+1/2} - \left[ \left( \rho E + P \right) v_z \right]_{i,j,k-1/2} \right\} = \Delta r_i \Delta \phi j \Delta z_k \left( \rho v \cdot g \right)_{i,j,k}. \]  

(19)

We integrate Equations (15) through (19) with the two-step Runge-Kutta, although we can employ other methods to integrate them.

For simplicity, we have described the gravity in Equations (15) through (19) as if it were defined at the cell center. However, we defined the gravity on the cell boundaries in order to ensure the conservation of energy in the described form. See Hanawa (2019) for more details.
2.3. Numerical Grid and Flux

We use the numerical grid shown in Figure 1 in order to relax the tight CFL condition for a fixed angular resolution. The black curves and lines denote the cell boundaries on the $r - \varphi$ plane. The central circle ($i = 1$) is divided into 6 numerical cells while the adjacent ring ($i = 2$) is divided into 12 numerical cells. The angular resolution is set to be a function of $r$ so that

$$\frac{\Delta r_i}{\sqrt{2} \varphi_i} < \Delta \varphi_i = \frac{\pi}{3 \times 2^n} < \frac{\sqrt{2} \Delta r_i}{r_i},$$

where $n$ denotes an integer in the range $0 \leq n \leq n_{\text{max}}$ and a function of $i$. The cell boundary is set on $\varphi = 0$ throughout the computation region to synchronize the azimuthal boundaries. The radial cell width is set to be constant up to the $2^n_{\text{max}}$-th cell, beyond which it increases by a factor $1 + 2^{-n_{\text{max}}}$ each with increase in the cell number, $i$, in the $r$-direction to keep the aspect ratio, $r_i \Delta \varphi_i/\Delta r_i$, close to unity. The cell width in the $z$-direction, $\Delta z_k$, is set to be constant in the region around the mid plane, $z = 0$, while it increases as $|z|$ increases.

![Figure 1](image_url)

Figure 1. These two panels show the numerical cells used in our simulations. The solid black curves denote the cell boundaries in the $r$-direction while the solid black lines do those in the $\varphi$-direction. The left and right panels illustrate the data reconstruction on the $\varphi$- and $r$-directions, respectively for computing the numerical flux across the orange cell boundaries. We reconstruct the data on the left and right sides of the boundary along the broken blue curves and lines.

We employ MUSCL (Monotonic Upwind Scheme for Conservation Law) approach to achieve a second-order accuracy in space.

The left panel of Figure 1 illustrates the data reconstruction for computing the flux in the $\varphi$-direction. We use the four data points marked orange in the panel to evaluate the values on the left- and right-hand sides of the cell boundary marked orange. We obtain the values of $\rho$, $P$, and $v_z$ on the left and right hand sides of the cell boundary from those on the blue dashed curve by interpolation with the minmod limiter. When evaluating $v_r$ and $v_\varphi$ on the cell boundary, we take account of the rotation of the unit vectors, $e_r$ and $e_\varphi$. Before the interpolation, we transform $v_r$ and $v_\varphi$ into the tangential and normal components referred to the cell boundary on $\varphi = \varphi_{j+1/2}$ by

$$v_t(\varphi) = \cos (\varphi - \varphi_{j+1/2}) v_r(\varphi) - \sin (\varphi - \varphi_{j+1/2}) v_\varphi(\varphi),$$

$$v_n(\varphi) = \sin (\varphi - \varphi_{j+1/2}) v_r(\varphi) + \cos (\varphi - \varphi_{j+1/2}) v_\varphi(\varphi).$$
We evaluate $v_r$ and $v_\phi$ on the cell boundary by interpolating $v_t$ and $v_n$, respectively, with the minmod limiter.

The right panel of Figure 1 illustrates the process for evaluating the numerical flux in the $r$-direction. The angular resolution is higher at a larger distance from the origin. We employ the highest angular resolution from the four adjacent rings under consideration for evaluating the radial numerical flux. We used the cell center values marked blue in the panel without interpolation in the $\phi$-direction. At the time of ASTRONUM 2019, we did not take account of the rotation of the unit vectors. However, we have found later that the solution is improved substantially by taking account of the rotation.

We derive the numerical flux by using an approximate Riemann solution to the reconstructed density, velocity, and pressure. We have used the approximate solution of HLLD for vanishing magnetic field. The time step is set so that the CFL number be 0.3, i.e., $\Delta t = 0.3 \Delta r_1 / \max(c_s + \sqrt{v_r^2 + v_\phi^2 + v_z^2})$, where $c_s$ denotes the sound speed.

3. Numerical Examples

3.1. Sod Shock Tube Problem

We have solved the Sod shock tube problem since it is a standard test problem. The initial condition is expressed as

$$
(\rho, P, v) = \begin{cases} 
(1, 1, 0) & (\sin \phi \leq 0) \\
(0.1, 0.125, 0) & (\sin \phi > 0)
\end{cases},
$$

so that the density and pressure have higher values in the lower half on the $r-\phi$ plane than in the upper half. While this problem is one dimensional in the Cartesian coordinates, it is two dimensional in the cylindrical coordinates. The radial cell width is taken to be $\Delta r = 0.05$ and the highest angular resolution is set by $n_{\text{max}} = 6$. We set the fixed boundary on $r = 3.2$ for simplicity.

Figure 2 compares the density distribution obtained with our new scheme (left) and that with a conventional scheme (right). The latter evaluates the geometrical source term from $\rho$, $P$, and $v_\phi$ at the cell center. Both the panels denote $\rho$ on the $x-y$ plane by color at the 45th time step later from the initial state. The wave fronts are apparently angular because of the cylindrical grids. However, both the schemes seem to provide proper density distribution at this early stage.

![Figure 2](image-url)
Figure 3 is the same as Figure 2 but for $v_y = v_r \sin \varphi + v_\varphi \cos \varphi$, the velocity normal to the wave front. Although the normal velocity should be constant between the shock front and inner edge of the rarefaction wave, we can see a clear deviation in the right panel. The normal velocity is lower around the axis, i.e., along the flow across the origin. This problem is improved greatly in our new scheme.

![Sod Shock Tube [new] t = 0.147](image1)

![Sod Shock Tube [old] t = 0.151](image2)

Figure 3. The same as Figure 2 but for $v_y$, the velocity normal to the wave front.

The tangential velocity has the maximum value, $|v_x|_{\text{max}} = 0.035$ at this stage in the solution obtained with our new scheme while $|v_x|_{\text{max}} = 0.115$ in that with the conventional one. It should vanish ($v_x = 0$) in the ideal solution. Thus, this lower tangential velocity is a proof of the superiority of our scheme.

The deviation from the ideal solution is larger at a later stage in the solution obtained with the conventional scheme. Figure 4 compares the density distribution at the 200th time step. The conventional scheme produces dips on the contact discontinuity ($y \approx 0.3$) and on the shock front ($y \approx 0.5$). Our new scheme makes these dips much shallower.

![Sod Shock Tube [new] t = 0.563](image3)

![Sod Shock Tube [old] t = 0.568](image4)

Figure 4. The same as Figure 2 but for a later stage and a larger area.

3.2. Uniform Flow
In order to quantify the numerical error due to discretization we use a uniform flow in which the density, pressure, and velocity are set $\rho = 1$, $p = 1$, and $(v_r, v_\varphi, v_z) = (\cos \varphi, -\sin \varphi, 0)$, respectively. Any change from this initial state should be numerical error. Figures 5 and 6 show
\[ \rho \text{ and } v_x \text{ at the 300th time step, respectively. The density ranges } 0.946 < \rho < 1.030 \text{ in the solution obtained with our scheme (left), while it ranges } 0.856 < \rho < 1.133 \text{ in that with the conventional scheme (right). The velocity ranges } 0.943 < v_x < 1.010 \text{ and } -0.054 < v_y < 0.054 \text{ in the former, while the error in the velocity is unacceptably large in the latter.} \]

Figure 5. The density distribution at the 300th time step in the uniform flow problem. The left panel shows the solution obtained with our new scheme, while the right does that with the conventional one.

Figure 6. The same as Figure 5 but for \( v_x \).

After the conference, we have found that we can reduce the former numerical error by taking account of the rotation of the unit vector when computing the numerical \( u_x \) in the radial direction. Furthermore, we can eliminate it by modifying the surface area, \( dS_{r} \). We will report these improvements somewhere else.

3.3. Gas Accretion onto a Protostar

We have confirmed that our scheme works well in the presence of gravity. Figure 7 shows a numerical simulation of an anisotropic accretion onto a protostar as an application of the present scheme. The gravity is assumed to be

\[ \mathbf{g} = \begin{cases} -re_r & r < 1 \\ -r^{-2}e_r & r \geq 1 \end{cases} \]  \tag{24} \]
The spatial resolution is $\Delta r_1 = 0.03$ and $\Delta \phi_{\text{min}} = 3.27 \times 10^{-2}$. The numerical grid has 150 and 181 cells in the $r$- and $z$-directions, respectively. The left panel shows a rotating gas disk with accreting gas from $\phi = 0$ at $t = 38.1$. The panel denotes the density by volume rendering (center) and two cross sections (right and back). The right panel is an enlarged view of the left.

Figure 7. Numerical simulation of a protostar with disk onto which gas accretes from the right-hand side. Only dense gas is denoted by color to clarify the flow and disk.

4. Discussions

In this paper, we have proposed two technical improvements, (1) evaluating the centrifugal force using the numerical flux on the cell surface and (2) reducing the angular resolution near the $z$-axis. The latter idea has been employed in some earlier works. However, it works only when the numerical error is small at a low angular resolution. The error is reduced to be an acceptable level by the former. Thus, the former improvement is essential in our new scheme.

We have shown in the previous section that our numerical solution depends strongly on the evaluation of the centrifugal force, i.e., the source term due to the curvature of the coordinates. If we use the velocity at the cell center to evaluate it, the numerical solutions suffer from spurious features around the origin, i.e., around the $z$-axis. Such disturbances disappear if we evaluate the centrifugal force from the dynamical pressure working on the cell surface.

We think that the same idea can be applied to other system equations in which equates the time derivative of a vector field with the divergence of a tensor field. Such system equations include magnetohydrodynamical (MHD) equations and radiation hydrodynamical (RHD) equations. Note that the magnetic force can be expressed as the divergence of the Maxwell tensor,

$$j \times B = \frac{1}{4\pi} \nabla \cdot \left( BB - \frac{B^2}{2} \right), \quad (25)$$

where $B$ and $j$ denote the magnetic field and electric current density, respectively. Similarly, the induction equation can be expressed in the form,

$$\frac{\partial B}{\partial t} + \nabla \cdot (vB - Bv) = 0. \quad (26)$$
In the cylindrical coordinates, equations (25) and (26) have apparent source terms which should be evaluated on the cell surface.

We expect that our method can be applied also to spherical coordinates, \((r, \theta, \varphi)\), and other orthogonal curved coordinates. Such extension will be discussed in an upcoming manuscript.

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