Spectral features of solar wind turbulent plasma

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ABSTRACT
Spectral properties of a fully compressible solar wind Hall Magnetohydrodynamic plasma are investigated by means of time dependent three dimensional Hall MHD simulations. Our simulations, in agreement with spacecraft data, identify a spectral break in turbulence spectra at characteristic length-scales associated with electromagnetic fluctuations that are smaller than the ion gyroradius. In this regime, our 3D simulations show that turbulent spectral cascades in the presence of a mean magnetic field follow an omnidirectional anisotropic inertial range spectrum close to \( k^{-7/3} \). The onset of the spectral break in our simulations can be ascribed to the presence of nonlinear Hall interactions that modify the spectral cascades. Our simulations further show that the underlying characteristic turbulent fluctuations are spectrally anisotropic, the extent of which depends critically on the local wavenumber. The fluctuations associated with length scales smaller than the ion gyroradius are highly compressible and tend to exhibit a near equipartition in the velocity and magnetic fields. Finally, we find that the orientation of velocity and magnetic field fluctuations critically determine the character of nonlinear interactions that predominantly govern a Hall MHD plasma, like the solar wind.

Key words: (magnetohydrodynamics) MHD, (Sun:) solar wind, Sun: magnetic fields, ISM: magnetic fields

1 INTRODUCTION
The solar wind plasma is predominantly in a turbulent state. Nonlinear turbulent processes in the magnetized solar wind plasma fluid yield a multitude of spatial and temporal length-scales associated with an admixture of waves, fluctuations, structures and nonlinear turbulent interactions. In-situ spacecraft measurements (Matthaeus & Brown 1988, Goldstein et al 1995, Ghosh et al 1996) reveal that the solar wind fluctuations, extending over several orders of magnitude in frequency and wavenumber, when described by power spectral density (PSD) spectrum, can be divided into three distinct regions (Goldstein et al 1995, Leamon et al 1999) depending on the frequency and wavenumber. The first region corresponds to a flatter spectrum, associated with lower frequencies, and it is consistent with \( k^{-1} \) (where \( k \) is wavenumber). Second region follows and extends to the ion/proton gyrofrequency, and the spectral slope has an index ranging from \(-3/2\) to \(-5/3\). This region is typically characterized as corresponding to fully developed turbulence, and can be described by the usual incompressible magnetohydrodynamic (MHD) description. The turbulent interactions in this regime are governed entirely by Alfvénic cascades. Spacecraft observations (Leamon et al 1999, Bale et al. 2005, Alexandrova et al 2007, Sahraoui et al 2009) further reveal that at length scales beyond the MHD regime, i.e. length scales less than ion gyroradius \( (k\rho_i \gg 1) \) and temporal scales greater than the ion cyclotron frequency \( \omega > \omega_{ci} = eB_0/m_e c \) (where \( k, \rho_i, \omega_{ci}, e, B_0, m_e, c \) are respectively characteristic mode, ion gyroradius, ion cyclotron frequency, electronic charge, mean magnetic field, mass of electron, and speed of light), the spectrum exhibits a spectral break, and the spectral index of the solar wind turbulent fluctuations varies between \(-2\) and \(-5\) (Smith et al 1990, Goldstein et al 1994, Leamon et al 1999, Bale et al 2005, Shaikh & Shukla 2009, Sahraoui et al 2009). Higher time resolution observations find that at the spectral break, Alfvénic MHD cascades (Smith et al 1990, Goldstein et al 1994, Leamon et al 1999, Bale et al 2005, Shaikh & Shukla 2009, Sahraoui et al 2009) close. The characteristic modes in this region appear to evolve typically on timescales associated with dispersive kinetic Alfvénic fluctuations.

The onset of the second or the kinetic Alfvén inertial range is not understood. Some suggestions have however been made. The spectral break may result from energy transfer processes associated with possibly kinetic Alfvén waves (KAWs) (Hasegawa 1976), electromagnetic ion-cyclotron-Alfvén (EMICA) waves \( \text{Gary et al 2003} \), \( \text{Wu & Yoon 2007} \), or by fluctuations described by a Hall
MHD (HMHD) plasma model (Alexandrova et al 2007, 2008; Shaikh & Shukla 2008, 2008a). Stawicki et al (2001) argue that Alfvén fluctuations are suppressed by proton cyclotron damping at intermediate wavenumbers so the observed power spectra are likely to comprise weakly damped dispersive magnetosonic and/or whistler waves (unlike Alfvén waves). Beinroth & Neubauer (1981) and Den skat & Neubauer (1982) have reported the presence of whistler waves based on Helios 1 & 2 observations in this high frequency regime. A comprehensive data analysis by Goldstein et al. (1994), based on correlations of sign of magnetic helicity with direction of magnetic field, indicates the possibility of the existence of multiscale waves (Alfvénic, whistlers and cyclotron waves) with a single polarization in the dissipation regime. Counter intuitively, in the $\omega < \omega_{ci}$ regime, or Alfvénic regime, Howes et al. (2008) noted the possibility that highly obliquely propagating KAWs are present (with $\omega \ll \omega_{ci}$) making the possibility that damping of ion cyclotron waves is responsible for the spectral break-point questionable.

Fluid (Shaikh & Shukla 2009) and kinetic (Howes et al. 2008) simulations, in qualitative agreement with spacecraft data as described above, have been able to obtain the spectral break point near the characteristic turbulent length scales that are comparable with ion inertial length scale ($d_i$). These simulations demonstrated a Kolmogorov-like $k^{-5/3}$ spectra for the length scales larger than the ion inertial length scales where MHD is typically a valid description. By contrast, smaller (than $d_i$) scales are shown to follow a steeper spectrum that is close to $k^{-7/3}$ (Howes et al. 2008, Shaikh & Shukla 2009). Spacecraft data and simulations thus reveal that migration of turbulent energy proceeds essentially through different regions in the $k$-space, i.e. $k^{-1}, k^{-5/3}$ and $k^{-7/3}$. Needless to mention that turbulent cascade does not entirely terminate immediately beyond the $k^{-7/3}$ spectrum. Fluid and kinetic simulations (Biskamp 1996, Galtier 2006, Galtier & Buchlin 2007, Cho and Lazar ian 2004, Shaikh & Zank 2005, Shaikh 2009, Gary et al. 2008, Saito et al. 2008, Howes et al. 2008) have further shown that spectral transfer of energy extends even beyond the $k^{-7/3}$ spectrum and it is governed predominantly by small scale, high frequency, whistler turbulence. The latter also exhibits a definite power law.

The physical processes describing MHD, KAW or Hall MHD and whistler spectra are rich and complex. They differ significantly from others and continue to pose serious challenges in our understanding of multiscale solar wind turbulence. One of the major goals of this paper is to describe the connection between different scales associated with the MHD, KAW or Hall MHD and whistler spectra. In the following, we describe extended part of the spectra that are predicted by theory and simulations.

2 EXTENDED COMPOSITE SPECTRA

Theory and simulations indicate that turbulent fluctuations in the high frequency and $k \rho_i \gg 1$ (where $\rho_i$ is ion gyro radius) regime correspond to a regime in which electron motions are decoupled from the ion motions (Kings et al 1990, Biskamp et al 1996, Dastgeer et al 2000a, Dastgeer et al 2000b, Shaikh & Zank 2003, Cho & Lazarian 2004, Saito et al 2008, Gary et al 2008). Correspondingly, ions are essentially unmagnetized and can be treated as an immobile neutralizing background fluid. This regime corresponds to the whistler wave band of the spectrum and comprises characteristic scales that are smaller than those that describe MHD, KAW or Hall MHD processes. An extended composite schematic describing the whistler modes spectra, in addition to the observed $k^{-1}, k^{-5/3}$ and $k^{-7/3}$ spectra and are consistent with the observations (Leamon et al 1999, Bale et al. 2005, Alexandrova et al 2007, Sahraoui et al 2009). The boundary of regions III and IV represents a wavenumber band in spectral space that corresponds to the decoupling of electron and ion motions. The wavenumbers above this boundary characterize the onset of whistler turbulence. The spectral cascades associated with whistler turbulence are described extensively by Biskamp et al (1996), Dastgeer et al (2000a), Dastgeer et al (2000b), Shaikh & Zank (2003), Shaikh & Zank (2005), Shaikh (2009a), Shaikh (20009b). Cho & Lazarian 2004 describe scale dependent anisotropy that is mediated by whistler waves in the context of electron MHD plasma. Gary et al. (2008) and Saito et al. (2008) have reported two-dimensional electromagnetic particle-in-cell simulations of electron MHD model to demonstrate the forward cascade of whistler turbulence. Their work show that magnetic spectra of the cascading fluctuations become more anisotropic with increasing fluc-
tuation energy. Interestingly, whistler turbulence associated with longer wavelengths in region IV exhibits a power spectrum $k^{-7/3}$ that is similar to the short wavelength spectrum of kinetic Alfvén waves (KAW), as shown in region III of Fig. 1. The underlying physical processes, responsible for the spectrum differ significantly for KAW and whistler waves. These differences are discussed in Section 7.

It is to be noted that the Hall MHD description of magnetized plasma is valid up to region III where characteristic turbulent scales are smaller than ion inertial length scales ($kd_i > 1$). Beyond this point, the high frequency motion of plasma is governed predominantly by the electron motions only. By contrast, ions form static neutralizing background. Consequently, the ion motions decouple significantly from the electrons. These aspects of the spectra, depicted necessarily by regions IV & V in Fig (1), can be described adequately by whistler wave model. The Hall MHD models are therefore not applicable in regions IV, V and beyond. Neither they can describe kinetic physics associated with the dissipative regime. Since the high frequency regime (i.e. regions IV & V) is dominated by the electron motions, there exists intrinsic length scale corresponding to electron inertial length scale $d_e = c/\omega_{pe}$ (where $c$ is the speed of light and $\omega_{pe}$ is the electron plasma frequency). The characteristic turbulent length scales in regions IV & V are essentially comparable with $d_e$ and therefore they can describe scales larger (i.e. $kd_e < 1$ in region IV) and smaller (i.e. $kd_e > 1$ in region V) than the electron inertial scale. While whistler wave model can describe nonlinear processes associated with length scales as small as the electron inertial length scale, they fail to describe finite electron Larmor radii effects for which a fully kinetic description of plasma must be sought.

The schematic of solar wind turbulence depicted in Fig. 1 raises numerous unresolved questions. Beside those described above, we do not understand what leads to the decoupling of ion and electron motions near the boundary of region III and IV for example. Although the turbulent spectra are described by similar spectral indices, the nonlinear processes are fundamentally different in region III and IV. With this paper, we attempt to identify and understand the processes that lead to the spectra in III and IV. It appears that the character of the nonlinear interaction is determined primarily by the orientation of turbulent velocity ($\mathbf{V}$) and magnetic ($\mathbf{B}$) field fluctuations with respect to each other, relative to the mean magnetic field. The spatially varying angular distribution of perpendicular velocity and magnetic field fluctuations relative to the mean magnetic field was predicted theoretically by Boldyrev (2006) and is a conjecture by Podesta et al (2008). It nevertheless remains to be seen how the orientation of turbulent $\mathbf{V}$ and $\mathbf{B}$ fluctuations govern the nonlinear spectral transfer of energy in solar wind turbulent plasma. In the context of MHD turbulence, Servidio et al (2008) show that the orientation between the velocity and magnetic field fluctuations plays a critical role in depleting the nonlinear interactions that lead to the relaxation of MHD turbulence.

Spectral anisotropy is another issue that is not yet properly understood for characteristic modes with scales greater than $kd_e \sim 1$ in Hall MHD plasma. In the case of MHD turbulence, the presence of a mean magnetic field leads to the asymmetric transfer of spectral energy along and across the mean magnetic field (Shebalin et al. 1983). An excellent analysis of anisotropic cascades in the framework of Hall MHD is presented by Ghosh & Goldstein (1997). They describe how the degree of turbulent anisotropy in a Hall MHD plasma varies as the plasma beta (the ratio of magnetic to pressure energy) changes from greater than to less than 1. A more general characterization of spectral anisotropy in the vicinity of modes $kd_e \sim 1$ is still unclear. Specifically, the scale dependence of the turbulent anisotropy in the KAW regime and its connection with that in the whistler regime is not yet established. Spectral anisotropy in the whistler regime was independently investigated by Shaikh & Zank (2003), Dastgeer et al (2000a, b). In this paper, we relate these results to those associated with anisotropic KAW modes.

To address the issues described above, we use time dependent, fully compressible three dimensional simulations of Hall MHD plasma in a triply periodic domain. This represents a local or regional volume of the solar wind plasma. Note that the dynamics of length-scales associated with region III, i.e. corresponding to the KAW modes, cannot be described by the usual MHD models since we are interested in characteristic frequencies smaller than an ion gyro frequency. At 1 AU, ion inertial length scales are smaller than ion gyro radii in the solar wind (Goldstein et al 1993). Plasma effects due to finite Larmor radii can readily be incorporated in MHD models by introducing Hall terms to accommodate ion gyro scales up to scales as small as ion inertial length scales. In section 3, we describe the underlying Hall MHD model along with the intrinsic assumptions and discuss the Hall MHD linear dispersion relation. Section 4 addresses our 3D fluid simulations of the nonlinear Hall MHD equations. Our results suggest that the secondary inertial range spectrum has a form that is close to $k^{-7/3}$ above the spectral break in the solar wind plasma, and mediated by the Hall terms in the short wavelength (in comparison with the ion skin-depth) KAW regime. Anisotropy in spectral cascade behavior is explored in section 5. We find that long length scale fluctuations in the $kd_i > 1$ KAW regime exhibit a more anisotropic energy cascade compared to smaller scales. The dynamical alignment and angular distribution of turbulent velocity and magnetic field fluctuations is described in section 6. We find that characteristic turbulent fluctuations in the $kd_i > 1$ regime relax towards a state of orthogonality such that the majority of turbulent scales contain fluctuations in which the velocity and magnetic fields are nearly orthogonal, i.e. $\mathbf{V} \perp \mathbf{B}$. Section 7 compares the KAW and whistler spectra. Finally, in section 8 we provide a summary and conclusions.

### 3 HALL MHD SIMULATION MODEL

Our 3D simulations are based on a two fluid nonlinear Hall MHD plasma model. The model assumes that the electrons are inertial-less, while the ions are inertial (Krishan & Mahajan 2004). Hence, the electrons and ions have a differential drift, unlike the one fluid MHD model for which the electron and ion flow velocities are identical. Hall MHD description of magnetized plasma has previously been employed by a number of workers to investigate wave and turbulence processes in the context of solar wind plasma. In an excellent work, Sahraoui et al. (2007) extended the
ordinary MHD system to include spatial scales down to the ion skin depth or frequencies comparable to the ion gyrofrequency in an incompressible limit. They further analyzed the differences in the incompressible Hall MHD and MHD models within the framework of linear modes, their dispersion and polarizations. Gahtier (2006) developed a wave turbulence theory in the context of an incompressible Hall MHD system to examine the steepening of the magnetic fluctuation power law spectra in the solar wind plasma. Furthermore, Gahtier and Buchlin (2007) have developed 3D dispersive Hall magnetohydrodynamics simulations within the paradigm of a highly turbulent shell model and demonstrated that the large-scale magnetic fluctuations are characterized by a $k^{-3/3}$-type spectrum that steepens at scales smaller than the ion inertial length $d_i$ to $k^{-7/3}$.

Here we start from the electron momentum equation,

$$m_e n_e \left( \frac{\partial}{\partial t} + V_e \cdot \nabla \right) V_e = -\nabla P_e - e n_e \left( E + \frac{1}{c} V_e \times B \right)$$

(1)

where $m_e$, $n_e$, $V_e$, $P_e$ are respectively mass, density, velocity and pressure of electrons, and $E$ and $B$ are the electric and magnetic fields. In the presence of low-frequency (compared with the electron gyrofrequency) electromagnetic fields, the electric force acting on the inertial-less electrons is balanced by the electron Lorentz force and the pressure gradient. This yields a more general form of Ohm’s law than is typically used in the MHD description and is described as Hall MHD. We assume a quasineutral solar wind plasma where the density of electrons ($n_e$) and ions ($n_i$) is nearly equal such that $n_e \approx n_i = n$. Thus in the inertial-less (Ohm’s law) electron limit, the electron momentum equation yields the electric field as

$$E = -\frac{1}{n_e} \nabla P_e - \frac{1}{c} V_e \times B$$

(2)

The electric field arising from separation introduced by the inertial-less electron momentum equation can be substituted into the ion momentum equation

$$m_i n_i \left( \frac{\partial}{\partial t} + V_i \cdot \nabla \right) V_i = -\nabla P_i - e n_i \left( E + \frac{1}{c} V_i \times B \right),$$

(3)

where $m_e$, $n_e$, $V_e$, $P_e$ are mass, density, velocity and ion pressure respectively, which then yields

$$\rho \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) V = -\nabla P + \frac{1}{c} J \times B.$$ (4)

Here we used $\rho = m_e n$ (solar wind plasma mass density), $J = n_e (V_e - V_i)$ (plasma current) and $P = P_e + P_i$ (total plasma pressure). We next substitute the electric field into Faraday’s law which yields

$$\nabla \times E = \nabla \times (V \times B).$$

(5)

On eliminating the electron fluid velocity from the current equation i.e. $V_e = V_i - J / n_e$, we obtain

$$\frac{\partial B}{\partial t} = \nabla \times \left( V_i \times B - \frac{J \times B}{ne} \right).$$

(6)

In the inertial-less electron ($m_e \to 0$) limit the electron fluid does not influence the momentum of solar wind plasma directly except through the current. Since the electron fluid contributes to the electric field, plasma currents and the magnetic field are affected by electron oscillations. The combination of electron dynamics and ion motions distinguishes the Hall MHD model from its single fluid MHD counterpart. Thanks to the inclusion of electron dynamics, Hall MHD can describe solar wind plasma fluctuations that are associated with a finite ion Larmor radius and thus a characteristic plasma frequency is $\omega > \omega_{ci}$. Because Hall MHD contains both ion and electron effects, there is a regime at which the dominance of one set of plasma fluctuations crosses over to be dominated by the other. This therefore introduces naturally an intrinsic scale length/time scale (frequency) that separates ion dominated behavior in the plasma from electron dominated. It is the Hall term corresponding to the $J \times B$ term in Faraday’s law that is primarily responsible for decoupling electron and ion motion on ion inertial length and ion cyclotron time scales (and introducing an intrinsic length scale). It is this feature that makes Hall MHD useful in describing dissipative solar wind processes where single fluid MHD is not applicable (the MHD model breaks down at $\omega \sim \omega_{ci}$). Hall MHD allows us to study inertial range cascades beyond $\omega \sim \omega_{ci}$, and can be extended to study dissipative heating processes where ion cyclotron waves are damped. The extreme limit of a fluid modeling applied to solar wind processes (even beyond the limit of the Hall MHD regime) is the use of an electron MHD model in which high frequency electron dynamics is treated by assuming stationary ions that act to neutralize the plasma background. We discuss this regime at the end of this paper.

The quasi-neutral plasma density ($\rho$), the velocity ($V$), the magnetic field ($B$) and the plasma pressure ($P = P_e + P_i$) can be described by the nonlinear Hall MHD equations. For the purpose of numerical simulations, it is convenient to express Hall MHD in the conservative form,

$$\frac{\partial F}{\partial t} + \nabla \cdot Q = 0,$$ (7)

where,

$$F = \begin{bmatrix} \rho & \rho V & \rho V^2 \\ \rho V & \rho V^2 + P - \frac{\mu}{\gamma - 1} \frac{V B}{\gamma} & \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} \\ V B & \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} & \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} \end{bmatrix}, Q = \begin{bmatrix} \rho V V + \frac{\mu}{\gamma - 1} \frac{V B}{\gamma} + \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} - B \cdot J B \\ \int (V B) \cdot (V - \bar{V}) - \frac{\eta}{\gamma - 1} \int \nabla \cdot (V \cdot B) \\ \frac{\eta}{\gamma - 1} \int \nabla \cdot (V \times B) \\ \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} \end{bmatrix};$$

$$\bar{V} = \frac{1}{\rho} \int \frac{\gamma P}{\gamma - 1} + \frac{\mu}{\gamma - 1} \frac{V B}{\gamma} + \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma} + \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma}.$$ (8)

$e = \frac{1}{2} \rho V^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8 \pi}$

and

$$\bar{V} = \frac{1}{\rho} \int \frac{\gamma P V}{\gamma - 1} + \frac{\mu}{\gamma - 1} \frac{V B}{\gamma} + \frac{\mu}{\gamma - 1} \frac{V B^2}{\gamma}.$$ (9)

The suffix $i$ is dropped from the ion fluid velocity $V_i$ in the conservative form.

The dynamical variables are functions of three space coordinates and time, i.e. $(x, y, z, t)$ and are normalized by typical length $\ell_0$ and time $t_0 = \ell_0 / V_A$ in our simulations, $V_A = B_0 / (4 \pi \rho_0)^{1/2}$ the Alfvén speed, such that $\nabla = \ell_0 \nabla, \partial / \partial t = t_0 \partial / \partial t, V = V / V_A, B = B / V_A (4 \pi \rho_0)^{1/2}, \bar{E} = \bar{E} / V_A (4 \pi \rho_0)^{1/2}, \bar{P} = P / \rho_0 V_A^2, \bar{\rho} = \rho / \rho_0, \bar{e} = e / V_A$. $\bar{I}$ is the unit tensor. The parameters $\mu$ and $\eta$ represent ion-electron viscous drag and magnetic field diffusivity, respectively. While the viscous drag modifies the dissipation in
the plasma momentum in a nonlinear fashion, the magnetic diffusivity damps small scale magnetic field fluctuations linearly. The dimensionless parameter $d_i (= d_i/\ell_0)$, where $d_i = c/\omega_{pi}$ is the ion skin depth and $\omega_{pi}$ is the ion plasma frequency) in Faraday’s law identifies the Hall effect. The ion skin depth is a natural or an intrinsic length scale representative of the Hall MHD plasma model, and the Hall term plays an important role for high-frequency fluctuations with $kd_i \geq 1$. It turns out that the Hall physics dominates the magnetoplasma dynamics when $(1/\rho)\mathbf{J} \times \mathbf{B} > \mathbf{V} \times \mathbf{B}$ in the magnetic field induction equation. Finally, our nonlinear Hall MHD model also includes the full energy equation, rather than an adiabatic equation of state connecting the plasma pressure and the density. The use of the energy equation enables us to follow self-consistently the evolution of turbulent plasma heating resulting from the nonlinear cascading of energy.

The rhs in the momentum equation corresponds to a forcing function ($f_{\mathbf{M}}(\mathbf{r}, t)$) that drives the plasma momentum at large length scales in our simulation. With the help of this function, we introduce energy in large scale eddies to sustain the magnetized turbulent interactions. In the absence of forcing, the turbulence decays freely. While the driving term modifies the plasma momentum, we conserve density (since we neglect photoionization and recombination). The large-scale random driving of turbulence can correspond to external forces or instabilities, for example fast and slow streams, merged interaction regions etc, in the solar wind, (or even supernova explosions and stellar winds in the ISM, etc).

The linear Hall MHD equations admit the dispersion relation (Brodin & Stenflo 1990),

$$(\omega^2 - k^2V_A^2)D_m(\omega, k) = \frac{\omega}{\omega_{ci}}(\omega^2 - k^2V_s^2)k_i^2k_s^2V_A^4,$$

which has been written in a form to clearly identify the relationship of MHD to Hall MHD. The expression $D_m(\omega, k) = \omega^4 - \omega^2k^2(V_s^2 + V_A^2) + k_i^2k_s^2V_A^2V_s^2$ is the MHD dispersion relation for fast and slow magnetosonic modes. Here $\omega$ is the wave frequency, $k(= k_i + k_s\mathbf{\hat{z}})$ the wave vector, the subscripts $\perp$ and $\parallel$ represent components across and along the external magnetic fields $B_0\mathbf{\hat{z}}$, and $V_A$ and $V_s$ are the Alfvén and sound speeds respectively. Besides the fast and slow magnetosonic modes, KAWs, EMICA waves, and electron whistlers are identified by the remaining three roots of the above dispersion relation. Warm Hall MHD plasma thus supports a great variety of waves of different wavelengths (comparable to the ion skin depth associated with the Hall drift, the ion sound gyroradius associated with the electron pressure and the perpendicular ion inertia/ion polarization drift).

4 NONLINEAR SIMULATION RESULTS

To study the evolution of turbulent cascades in the solar wind turbulent plasma, we have developed a fully 3D compressible Hall MHD code. The three dimensional computations are numerically expensive. But, with the advent of high speed vector and parallel distributed memory clusters, and efficient numerical algorithms such as those designed for Message Passing Interface (MPI) libraries, it is now possible to perform magnetofluid turbulence studies at substantially higher resolutions. Based on MPI libraries, we have developed a three dimensional, time dependent, compressible, non-adiabatic, driven and fully parallelized Hall magnetohydrodynamic (MHD) nonlinear code that runs efficiently on both distributed memory clusters like distributed-memory supercomputers or shared memory parallel computers. This allows us to achieve a very high resolution in Fourier spectral space. The code is scalable and transportable to different cluster machines. The spatial descretization employs a conservative and parallelized descretization of Fourier harmonics, and the temporal integration uses a Runge Kutta (RK) 4th order method. The boundary conditions are periodic along the $x, y$ and $z$ directions in the local rectangular region of the solar wind plasma. The MHD counterpart of this code was used in several other studies by us, see e.g. (Shaikh & Zank 2006; Shaikh & Zank 2007; Shaikh 2009d).

Our code treats the solar wind plasma fluctuations as statistically isotropic, locally anisotropic, homogeneous and random. Such a representation is further consistent with ACE spacecraft measurements (Smith et al 2006). The numerical algorithm accurately describes the physical variables (density, temperature, magnetic field, velocity field, pressure etc) in our code and they are also less dissipative. Because of the latter, nonlinear mode coupling interactions preserve ideal ruged invariants of fluid flows, unlike finite difference or finite volume methods. The conservation of ideal invariants (energy, enstrophy, magnetic potential, helicity) in inertial range turbulence is an extremely important feature because these quantities describe the cascade of energy in the inertial regime, where turbulence is, in principle, free from large-scale forcing as well as small scale dissipation. Damping of plasma fluctuations may nonetheless occur as a

Figure 2. Scaling of our three dimensional Hall MHD code with respect to number of processors. The spectral resolution is fixed at 60^3 in a cubic box of volume $\pi^3$. By increasing the number of shared memory processors, the total computational time (measured in CPU unit time; real time) to achieve steady state turbulent interactions can be reduced significantly. Our code clearly scales efficiently with an increasing number of processors.
result of intrinsic non-ideal effects such those introduced by the finite Larmor radius.

To test the scalability of our code, we have performed simulations by increasing the number of processors for a fixed number of modes. Our scaling results are depicted in Fig. 2. Clearly, by increasing the number of shared memory processors, the computational time to achieve steady state turbulent interactions can be reduced significantly.

The turbulent fluctuations are initialized by using a uniform isotropic random spectral distribution of Fourier modes concentrated in a smaller band of lower wavenumbers ($k < 0.1 k_{max}$). While spectral amplitudes of the fluctuations are random for each Fourier coefficient, it follows a certain initial spectral distribution proportional to $k^{-\alpha}$, where $\alpha$ is the initial spectral index. The spectral distribution set up in this manner initializes random scale turbulent fluctuations. We note that a constant magnetic field is included along the $z$ direction (i.e. $B_0 = B_0 \hat{z}$) to accommodate the large scale (or the background solar wind) magnetic field. The plasma beta, ratio of plasma pressure to magnetic field energy, is close to unity in our simulation $\beta \approx 1$, as typically observed in solar wind (Goldstein et al 1995; Smith et al 2006). Turbulent fluctuations in our 3D Hall MHD simulations are driven either at the lowest Fourier modes or evolve freely under the influence of the self-consistent dynamics described by the set of Equation 7. The inertial range turbulent spectra in either cases lead to nearly identical inertial-range turbulent spectra. We have further carried out 3D simulations for a range of various parameters and spectral distributions to ensure the validity, as well as the consistency, of our codes and the physical results. The simulation parameters are: spectral resolution is $128^3$, $\eta = \nu = 10^{-3}$, $\beta = 1.0$, $kd_i \sim 0.1 - 10$, $L_x = L_y = L_z = 2\pi$. The nonlinear coupling of velocity and magnetic field fluctuations, amidst density perturbations, excites high-frequency and short wavelength (by the $\omega/\omega_{ci}$ effect) compressional dispersive KAWs. While the nonlinear mode coupling interactions must influence local turbulent fluctuations in the inertial range, their role in the spectral energy cascade is not yet fully understood. We come back to this issue below.

The nonlinear spectral cascade in the modified KAW regime leads to a secondary inertial range in the vicinity of $kd_i \approx 1$, where the turbulent magnetic and velocity fluctuations form spectra close to $k^{-7/3}$. This is displayed in Fig. 3, which also shows that for length scales larger than the ion thermal gyroradius, an MHD inertial range spectrum close to $k^{-5/3}$ is formed. Figure 3 results from averaging the spectra over ten simulation runs that are initialized with different random numbers. The spectra obtained in each simulation further consist of 20 different spectra that are averaged over the simulation period ($15 - 20$) $l_0/V_A$. The resultant spectrum is further processed using a higher order polynomial fit leading thereby to a well-behaved omnidirectional power law. The characteristic turbulent spectrum in the KAW regime is steeper than that of the MHD inertial range. Identifying the onset of the secondary inertial range has been the subject of some debate because of the presence of multiple processes in the KAW regime that can mediate the spectral transfer of energy. These processes include, for instance, the dispersion and damping of EMICA waves, turbulent dissipation, etc. In the context of our 3D Hall MHD simulations, the observed $k^{-7/3}$ scaling in the turbulent plasma can be understood from effect of the Hall term on the energy cascade process. The time-scale associated with Hall MHD is $\tau_H$, which is smaller than the MHD characteristic time scale $\tau_{mHD}$. Nonlinear cascades in MHD turbulence are typically governed by

$$\tau_{mHD} \sim \frac{1}{ke k},$$

where $v_k = |V(k,t)|$ is the velocity field in $k$-space. By contrast, spectral transfer of turbulent energy in the Hall MHD plasma has a typical timescale of

$$\tau_H \sim \frac{1}{k^2 B_k^2}.$$

The energy transfer rate in the KAW regime is, therefore,

$$\varepsilon \sim \frac{v_k^2}{\tau_H}.$$

On assuming a turbulent equipartition relation (Alexandrova et al 2007, Alexandrova et al 2008) between the velocity and magnetic fields $B_k^2 \sim v_k^2$, and using $\tau_H$, the energy transfer rate, Eq. (8) reduces to

$$\varepsilon \sim k^2 B_k^3.$$

The inertial range modes in our 3D Hall MHD simulations exhibit near equipartition, a result that is consistent with other work (Alexandrova et al 2007, Alexandrova et al 2008). This is shown in Fig. 4. It is noteworthy that equipartition was assumed by Alexandrova et al (2007) and Alexandrova et al (2008) and is not directly observed in the solar wind plasma in the $kd_i > 1$ regime. Our simulation results shown in Fig. 4 thus provide theoretical support to arguments that invoke turbulent equipartition in the MHD/KAW range (Alexandrova et al 2007, Alexandrova et al 2008). Fig. 4 shows turbulent equipar-
tition where spectrally averaged (i.e. averaged over entire k-spectrum) fluctuations exhibit turbulent equipartition during the nonlinear evolution. In Fig. 3, the ratio of magnetic and velocity field energy remains close to unity i.e. $|\delta B|^2 / |\delta v|^2 \sim 1$. Fig. 4 describes spectral distribution of $|\delta v|^2 / |\delta B|^2$ as a function of $k$. This figure also shows that the ratio $|\delta v|^2 / |\delta B|^2$ stays more or less close to the unity for the inertial range turbulent scales ($kd_i > 1$) in our simulations.

In the $kd_i < 1$ Alfvénic regime, turbulent equipartition in the solar wind plasma remains a subject of debate (Podesta et al. 2008; Tu & Marsch 1993; Yokoi 2006; Yokoi & Hamba 2007). Turbulent equipartition observed in the solar wind plasma in the $kd_i < 1$ Alfvénic regime is not exact and $v_{12}^2 / B_{12}^2 \leq 1$. A precise mechanism leading to the non-exact turbulent equipartition is unclear in the solar wind plasma (in the $kd_i < 1$ Alfvénic regime). The deviation from equipartition may result from inhomogeneity of the magnetic field (Yokoi 2006), possible differences in the dissipation of kinetic and magnetic energies (Yokoi & Hamba 2007), or compressibility of the density fluctuations (Podesta et al. 2008). Our objective however is not to address the mechanism that lead to unequal turbulent equipartition in the solar wind plasma in the $kd_i < 1$ Alfvénic regime. The readers can refer to the work of (Podesta et al. 2008; Tu & Marsch 1993; Yokoi 2006; Yokoi & Hamba 2007) for a more detailed analysis of turbulent equipartition in the $kd_i < 1$ Alfvénic regime in the solar wind plasma. We simply use the equipartition relationship here to derive Kolmogorov-like energy spectrum in Hall MHD plasma.

To derive the expression for the energy spectrum describing the energy cascade in Hall MHD plasma, we apply Kolmogorov’s phenomenology (Kolmogorov 1941; Kraichnan 1965; Iroshnikov 1963) in which the energy cascade in the inertial range is local and depends on Fourier modes and the energy dissipation rates (Kolmogorov 1941; Kraichnan 1965; Iroshnikov 1963). We obtain $k^{-1} B_{12}^2 \sim (B_{12}^3 k^2)^{\alpha} k^3$. On equating indices, we obtain $\alpha = 2/3$ and $\beta = -7/3$. This results in an energy spectrum of the form $E_k \propto k^{-7/3}$, which is consistent with our 3D Hall MHD simulations [see Fig. 3] and solar wind observations (Goldstein et al. 1995; Leamon et al. 1999, Bale et al. 2005, Sahraoui et al. 2009, Alexandrova et al. 2007, 2008). On the other hand, the use of $\tau_{\text{MHD}}$ in estimating the energy dissipation rates recovers the $k^{-5/3}$ inertial range MHD spectrum. This suggests that the Hall effect may be responsible for the spectral steepening in the solar wind plasma fluctuations in the $kd_i > 1$ regime.

It is worth mentioning that Fig. 3 represents spectral average of the entire mode spectrum. On the other hand, spectra of $|\delta B(k)|^2 / |\delta V(k)|^2$ and $|B(k)|^2 / |V(k)|^2$ in steady state show that turbulent equipartition in Hall MHD deviates for higher $k$’s (Shaikh & Shukla 2009). Interestingly, nonlinear fluid simulations in whistler wave regime (Dastgeer et al. 2000a, Shaikh 2009) corresponding to region IV in Fig (1) show that magnetic and velocity fields in whistler modes tend to establish turbulent equipartition. This is essentially a regime where the characteristic length scales are bigger than electron inertial length, i.e. $kd_e < 1$. In this regime, dispersive wave effects dominate the cascade processes by establishing turbulent equipartition amongst modes. In the $kd_e > 1$ regime (region V in Fig (1)), Dastgeer et al. (2000b) find that turbulent equipartition does not hold for short scales in EMHD turbulence. The latter behaves like hydrodynamic regime of fluid turbulence where total energy is dominated entirely by kinetic energy, whereas magnetic energy becomes sub-dominant.

5 ANISOTROPIC CASCADES

Here we discuss the degree of anisotropy in the turbulence, introduced by the presence of a large scale magnetic field in
3D Hall MHD. A mean magnetic field in the z-direction is included in our simulation, this meant to mimic the large scale solar wind background magnetic field. The small scale fluctuations are influenced by the presence of the large scale mean magnetic field and tend to evolve in an anisotropic manner in the sense that the turbulent cascade along and across the mean magnetic field behave differently. In 3D turbulence, fluctuations in a plane orthogonal to the mean magnetic field remains isotropic, whereas those aligned in the direction of the external magnetic field are affected. The latter is established for $\beta < 1$ and $\beta \approx 1$ by Zank & Matthaeus (1990, 1993) in the context of a nearly incompressible MHD theory. Thus, turbulent anisotropy in the inertial range spectrum corresponds to the preferential transfer of spectral energy that feeds perpendicular modes $k_\perp$ while the parallel ($k_\parallel$) cascades are suppressed. The anisotropy in the initially isotropic turbulent spectrum is triggered essentially by background anisotropic gradients that nonlinearly migrate spectral energy in a particular direction. To measure the degree of anisotropic cascades, we employ the following diagnostics to monitor the evolution of the $k_\perp$ mode in time. The averaged $k_\perp$ mode is determined by averaging over the entire turbulent spectrum weighted by $k_\perp$, thus

$$\langle k_\perp(t) \rangle = \sqrt{\frac{\sum_k |k_\perp Q(k, t)|^2}{\sum_k |Q(k, t)|^2}}.$$  

Here $\langle \cdots \rangle$ represents an average over the entire Fourier spectrum, $k_\perp = \sqrt{k_x^2 + k_y^2}$ and $Q$ represents any of $B, V, \rho, \nabla \times B$ and $\nabla \times V$. Similarly, the evolution of the $k_\parallel$ mode is determined by the following relation,

$$\langle k_\parallel(t) \rangle = \sqrt{\frac{\sum_k |k_\parallel Q(k, t)|^2}{\sum_k |Q(k, t)|^2}}.$$  

It is clear from these expressions that the $\langle k_\perp(t) \rangle$ and $\langle k_\parallel(t) \rangle$ modes exhibit isotropy when $\langle k_\perp(t) \rangle \approx \langle k_\parallel(t) \rangle$. Any deviation from this equality corresponds to spectral anisotropy. We follow the evolution of the $\langle k_\perp(t) \rangle$ and $\langle k_\parallel(t) \rangle$ modes in our simulations. Our simulation results describing the evolution of $\langle k_\perp(t) \rangle$ and $\langle k_\parallel(t) \rangle$ modes are shown in Fig. 6. It is evident from Fig. 6 that the initially isotropic modes $\langle k_\perp(t) \rangle \approx \langle k_\parallel(t) \rangle$ gradually evolve towards a highly anisotropic state in that spectral transfer preferentially occurs in the $\langle k_\perp(t) \rangle$ mode, and is suppressed in $\langle k_\parallel(t) \rangle$ mode. Consequently, spectral transfer in the $\langle k_\perp(t) \rangle$ mode dominates the nonlinear evolution of fluctuations in Hall MHD, and mode structures become elongated along the mean magnetic field or z-direction. Hence nonlinear interactions led by the nonlinear terms in the presence of background gradients lead to anisotropic turbulent cascades in the inertial range turbulent spectra.

While the spectral transfer of energy differs in the $k_\perp$ and $k_\parallel$ modes, the 3D volume averaged turbulent spectrum follows a $k^{-7/3}$ power law, as shown in Fig. 7. The steepness of the observed spectrum can be ascribed to the co-existence of partially anisotropic flows and turbulent fluctuations in steady state Hall MHD turbulence. Fig. 7 illustrates anisotropy corresponding to an averaged $k$ mode but the anisotropy exhibited by the small and large scale $B$ and $v$ fluctuations is not distinctively clear, nor is the degree of anisotropy in $B$ and $v$ fields clear from Fig. 7.

The scale dependence of turbulent anisotropy is described in Fig. 7. Fig. 7 shows discrepancy in $k_\perp$ and $k_\parallel$ is prominent at the smaller $k$'s. This essentially means that the large scale turbulent fluctuations are more anisotropic than the smaller ones in a regime where characteristic length scales are smaller than $d_i$, i.e. $kd_i > 1$. It further appears from Fig. 7 that the smaller scales in the $kd_i > 1$ are virtually unaffected by anisotropic kinetic Alfvén waves that propagate along the externally imposed mean magnetic field $B_0$. Turbulent fluctuations with small characteristic scales in the

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**Figure 6.** Evolution of $\langle k_\perp(t) \rangle$ and $\langle k_\parallel(t) \rangle$ as a measure of anisotropy in Hall MHD. Initially, $\langle k_\perp(t = 0) \rangle = k_\parallel(t = 0)$. As time evolves, anisotropy in spectral transfer progressively develops such that $\langle k_\perp(t) \rangle > \langle k_\parallel(t) \rangle$. Hence the presence of a large scale magnetic field suppresses the turbulent cascade along the direction of the field, while the perpendicular cascade remains nearly unaffected.

**Figure 7.** Spectrum of $k_\perp$ and $k_\parallel$ as a function of $k$. The large scale inertial range turbulent fluctuations are more anisotropic as compared to the smaller ones.
Figure 8. Evolution of the angular anisotropic distribution in Hall MHD turbulent plasma fluctuations. Shown in the figure is the angular evolution of anisotropy associated with the rotational velocity and magnetic field fluctuations. The initial angular distribution is identical. As the simulation progresses, small scale velocity field fluctuations tend to become more anisotropic than their magnetic field counterpart. The implication is that the smaller scale magnetic field fluctuations are less anisotropic than the velocity field fluctuations.

For isotropy, $k_\perp \approx k_\parallel$, hence the angle of anisotropy is close to $\theta \simeq 45^\circ$. Any deviation from the isotropic spectrum can be seen if $\theta \neq 45^\circ$. It is also clear from Eqs. (9) & (10) that $\theta > 45^\circ$ corresponds to the inertial range anisotropic turbulent spectrum that is dominated by the cascades in the perpendicular direction such that $k_\perp > k_\parallel$ whereas $\theta < 45^\circ$ describes $k_\perp < k_\parallel$ cascades. The evolution of the angular anisotropy associated with small scale inertial range turbulent fluctuations described by Eqs. (9) & (10) is depicted in Fig. 8. Several important points emerge from Fig. 8.

Firstly, spectral anisotropy associated with small scale inertial range turbulent fluctuations in the velocity field is more pronounced than that in the magnetic field fluctuations at large times. The angular evolution associated with the anisotropic velocity field deviates markedly from the isotropic angle $\theta = 45^\circ$. On the other hand, magnetic field fluctuations show a smaller degree of anisotropy compared to the velocity field fluctuations. It then appears, in view of Fig. 7, that the velocity field spectrum is dominated by relatively large scale turbulent fluctuations as compared to the magnetic field.

6 DYNAMICAL ALIGNMENT OF TURBULENT FLUCTUATIONS

The nature and strength of nonlinear interactions resides critically with their orientation with respect to each other in the presence of an externally imposed or self-consistently generated large scale magnetic field. For instance, the dominant nonlinear interactions in a Hall MHD fluid are governed by the Lorentz force that contains a nonlinear term corresponding to the $\mathbf{V} \times \mathbf{B}$ nonlinearity. Hence the orientation of the velocity field relative to the magnetic field fluctuations plays a critical role in determining the strength of nonlinear interactions. This clearly means that nonlinear interactions are dominated by those fluctuations that possess the velocity field aligned perfectly orthogonal (at a 90° angle) to the magnetic field, i.e. $\mathbf{V} \perp \mathbf{B}$. The obliqueness, introduced primarily by deviation from the orthogonality, tends to weaken the strength of nonlinear interactions mediated by the $\mathbf{V} \times \mathbf{B}$ nonlinearity. The $\mathbf{V} \times \mathbf{B}$ nonlinearity disap-
pears (i.e. $V \parallel B = 0$) for fluctuations that carry parallel or anti-parallel velocity and magnetic field fluctuations.

To understand the strength of the nonlinear interactions in Hall MHD solar wind plasma, we determine the degree of alignment of the velocity and magnetic field fluctuations by defining the following alignment parameter (Podesta et al. 2008) that spans the entire $k$-spectrum in both the $kd_i > 1$ (Hall MHD) and $kd_i < 1$ (usual MHD) regimes.

$$\Theta(t) = \cos^{-1} \left( \frac{\sum_k V_k(t) \cdot B_k(t)}{\sum_k |V_k(t)||B_k(t)|} \right).$$

The summation is determined from the modes by summing over the entire spectrum. In this sense, the alignment parameter depicts an average evolution of the alignment of velocity relative to the magnetic field fluctuations. Note carefully that this alignment can vary locally from smaller to larger scales, but the averaging (i.e. summing over the entire spectrum) rules out any such possibility in our simulations. Nonetheless, $\Theta$ defined as above enables us to quantitatively measure the average alignment of the magnetic and velocity field fluctuations while the nonlinear interactions evolve in a turbulent solar wind plasma.

Unlike Podesta et al. (2008), where a probability distribution of angular orientation corresponding to each mode is determined, we follow the total (i.e. spectrally volume averaged) evolution of the angle of alignment for the velocity and magnetic field fluctuations in both the Alfvénic cascade (i.e. MHD) and kinetic Alfvén wave (i.e. Hall MHD) regimes. Our results are plotted respectively in Figs (9) & (10). The Alfvénic cascade regime of MHD turbulence $kd_i < 1$ in the solar wind plasma possesses relatively large scales ($kd_i < 1$) in which the velocity and magnetic field fluctuations are observed to be somewhat obliquely aligned. Hence our simulations show that the angle of alignment evolves towards $\Theta < 90^\circ$, as depicted in Fig. (9). Hence the strength of the nonlinear interactions corresponding to the $V \times B$ nonlinearity is relatively weak. This result is to be contrasted with characteristic turbulent length scales in the $kd_i > 1$ regime. The angle of alignment for the smaller scales corresponding to the $kd_i > 1$ regime is shown in Fig. (10). Significant differences are apparent in the angle of alignments associated with the large and small scales Figs (9) & (10). It appears from our simulations that the small scale fluctuations ($kd_i > 1$) are nearly orthogonal as seen in Fig (10). By contrast, the large scale fluctuations ($kd_i < 1$) in Fig (9) show a significant departure from the orthogonality.

It is interesting to note that characteristic length scales that are large compared to the ion inertial skin depth ($kd_i < 1$) tend to deviate from orthogonality (a tendency towards dynamic alignment). Furthermore, these are the scales at which anisotropic cascades are dominated by the higher rate of spectral transfer of energy in the $k_\perp$ modes. This suggests the possibility that anisotropic cascades are related to the orientation of the velocity field relative to the magnetic field fluctuations. Our simulation results described in Figs (7), (8), (9) & (10) hint that the predominance of spectral anisotropy at large scales and the obliqueness are related. A heuristic and theoretical analysis of this relationship is beyond the scope of this paper.

7 COMPARISON BETWEEN KAW AND WHISTLER SPECTRA

It is interesting to note the similarities and differences between the characteristic Hall MHD and whistler wave turbulence spectra. Here we discuss properties of whistler turbulence from the previous works by Shaikh & Zank 2003; Shaikh & Zank 2005; Shaikh 2009a; Shaikh 2009b; Biskamp et al 1996 and compare them with our present simulation results.

While the two spectra correspond to essentially different frequencies as depicted schematically in Fig. (11), they are described by nearly identical spectral power laws in the vicinity of high frequency KAW modes (see the interface of boundary between III and IV in Fig. (11)). Thus, the small scale KAW in the $kd_i > 1$ regime and relatively large scale whistler fluctuations exhibit a $k^{-7/3}$ spectrum. Although the slope of the inertial range spectrum for these two modes is identical, the nonlinear physical processes yielding the spectra are distinctively different. Whistler modes are essentially governed by the motion of electrons in a static neutralizing ion background. The characteristic frequencies therefore reside between the ion and electron gyro frequencies ($\omega_{ci} \ll \omega \ll \omega_{ce}$) and characteristic length scales lie between the ion ($d_i$) and electron ($d_e$) inertial lengths i.e., $d_i = c/\omega_{pi} < \ell < d_e = c/\omega_{pe}$, where $\omega_{pi}$ and $\omega_{pe}$ are respectively the ion and electron plasma frequency. By virtue of $d_e$, there exist two inertial ranges that correspond to smaller $kd_i > 1$ and larger $kd_i < 1$ lengthscales. Correspondingly, forward cascade turbulent spectra in these regimes exhibit $k^{-7/3}$ and $k^{-7/3}$ respectively as shown in schematic Fig. (11). The whistler spectrum $k^{-7/3}$ is produced essentially by fluctuations in electron fluid while ions are at rest. This is demonstrated in 2D (Biskamp et al 1996; Shaikh & Zank 2003) as well as in 3D (Shaikh 2009b) simulations. As described in (Shaikh 2009b), the whistler waves cascade the inertial range turbulent energy typically through the convective electron fluid velocity $v_e \sim \nabla \times B$. Thus the typical velocity of the magnetic field
The eddy turnover time is then given by
\[ \tau_{\text{whis}} \sim \frac{\ell}{v_e} \sim \frac{\ell^2}{B_i}. \] (11)

It is this time scale that characterizes the nonlinear spectral transfer of energy in fully developed whistler wave turbulence. Kolmogorov-like dimensional arguments further yield a \( k^{-7/3} \) spectrum in the \( kd_i < 1 \) characteristic turbulent regime, as described in Fig. (1). While the inertial range nonlinear cascades are determined essentially by the eddy turn over or spectral transfer time scale, the characteristic length scales longer and less than \( d_e \) are significantly influenced by the whistler interaction time scales in a disparate manner (Shaikh 2009d). By contrast, as described above, the KAW spectrum mediated by Hall MHD processes in the \( kd_i > 1 \) regime is produced by the combined motion of electron and ion fluids amidst density fluctuations that evolve on dispersive KAW time and length scales. The KAW fluctuations relax towards a \( k^{-7/3} \) spectrum in the \( kd_i > 1 \) regime essentially by means of nonlinear mode coupling interactions in which both the electron and ion fluid perturbations participate on an equal footing. The most notable difference in the energy cascade processes associated with the KAW and whistler modes thus emerges from Equations (3) & (11), where the energy transfer rates are determined typically by different nonlinear fluid velocities.

8 SUMMARY
In summary, we have investigated the nonlinear and turbulent behavior of a two fluid, compressible, three dimensional Hall MHD model. Hall MHD model is relevant to describe solar wind plasma and magnetospheric and laboratory plasma environments. As exhibited by the solar wind plasma, multiple scale characteristic fluctuations lead to a complex power spectral density (or power spectrum) spanning very low frequencies, corresponding to a flatter \( (k^{-1}) \) spectrum, followed by the usual “Alfvénic” spectrum corresponding to a \( k^{-5/3} \) spectrum. How the Alfvénic cascade is terminated in the solar wind plasma, thus introducing a spectral discontinuity near the ion gyro frequency, continues to challenge our understanding of nonlinear solar wind turbulent processes near the \( kd_i \sim 1 \) band of wavenumbers. The slope of inertial range beyond the \( kd_i \sim 1 \) wavenumbers varies between -2 and -5 (Smith et al 1990, Goldstein et al 1994, Leamon et al 1999), depending upon what determines the underlying nonlinear interactions. Several mechanisms have been proposed to describe the spectral discontinuity of turbulent energy transfer in the solar wind plasma; these include kinetic Alfvén waves (KAWs) (Hasegawa 1976), electromagnetic ion-cyclotron-Alfvén (EMICA) waves (Gary et al 2008, Wu & Yoon 2007), nonlinear Hall MHD interactions (Alexandrova et al 2007, 2008; Shaikh & Shukla 2008, 2008a), and suppression of Alfvén fluctuations by proton cyclotron damping at intermediate wavenumbers (Stawicki et al 2001), so exciting whistler waves which are dispersive unlike Alfvén waves. More complex interactions may be expected due to the presence of highly obliquely propagating un-damped kinetic Alfvén wave (with \( \omega \ll \omega_i \)) as proposed by (Howes et al. 2008).

In this paper, our simulation results make interesting non trivial connections between regions that correspond to distinct wavenumbers in the composite spectra sketched in Fig. (1). We find that the spectra exhibit a break, consistent with the spacecraft observations (Smith et al 1990, Goldstein et al 1994, Leamon et al 1999), in the vicinity of the \( kd_i \sim 1 \) wavenumbers. The characteristic turbulent modes below these wavenumbers exhibit a \( k^{-5/3} \) like MHD spectrum. By contrast, turbulent fluctuations scale as \( k^{-7/3} \) in the inertial range characterized by \( kd_i > 1 \). The spectral index \( -7/3 \) is mediated essentially by equipartition processes between the turbulent velocity and magnetic field fluctuations. Note however that turbulent equipartition, for instance \( |B_i|^2 \sim |V_i|^2 \), is not directly observed in the spacecraft data. Small scale inertial range fluctuations in the \( kd_i > 1 \) regime tend to establish turbulent equipartition amongst the characteristic modes in our simulations. The inertial range \( k^{-7/3} \) spectrum is a unique feature of nonlinearly interacting multi-scale electromagnetic fluctuations that follows from the turbulent equipartition of high frequency kinetic Alfvén modes. We further find a considerable departure from turbulent equipartition for the higher \( k \) modes. The departure from equipartition in our simulations can be ascribed essentially to compressibility, which we find to be enhanced for the higher \( k \) modes in the inertial range KAW spectra.

In the presence of a large scale mean background magnetic field, small scale turbulent fluctuations exhibit anisotropic cascades. We find that the long length scales in the \( kd_i > 1 \) KAW regime are more anisotropic compared to the shorter scales. Dynamical alignment and angular distribution of turbulence velocity and magnetic field fluctuations is found to play a critical role in determining the degree of nonlinear interactions. We find that characteristic turbulent fluctuations in the \( kd_i > 1 \) regime relax towards orthogonality, so that most of turbulent scale fluctuations have velocity and magnetic fields that are nearly orthogonal, i.e. \( \mathbf{V} \perp \mathbf{B} \). For large scale fluctuations corresponding to the MHD regime, magnetic and velocity fields are not perfectly orthogonal, being instead on average nearly 70° to each other. By contrast, small scale fluctuations in the \( kd_i > 1 \) KAW regime exhibit nearly perfect orthogonality in that the average magnetic and velocity fields make an angle of nearly 90° with respect to each other. We finally noted the similarities and differences between the KAW and whistler spectra. Interestingly, the two modes exhibit a similar \( k^{-7/3} \) spectrum in region III and IV of Fig. (1) that corresponds respectively to the \( kd_i > 1 \) and \( kd_i < 1 \) regimes in wavenumber space. While the KAW spectrum is dominated by the combined motion of ions and electrons, the whistler wave spectrum is governed predominantly by electron motion only in the presence of a static neutralizing ion background.

The spectral properties of nonlinear Hall MHD are particularly relevant for understanding the observed solar wind and heliospheric turbulence. Hall MHD may also be useful for understanding multi-scale electromagnetic fluctuations and magnetic field reconnection in the Earth’s magnetosphere (Matthaeus et al 2003, Phan et al 2006) and in laboratory plasmas (Egedal et al 2007, Ono et al 1991, Hsu et al 2006, Carter et al 2006).
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$60^3$ modes
@ $5.0 \times 10^{-2} \omega_c$
\[ \frac{|V_k|^2}{|B_k|^2} \sim 1 \]
\[ |\delta V_k|^2 / |\delta B_k|^2 \sim 1 \]

\[ kd_i > 1 \]
The diagram shows the relationship between the Wavenumber and time. The axes are labeled:

- Y-axis: Wavenumber
- X-axis: Time (\( t_0/V_a \))

The diagram includes two curves labeled \( k_\parallel \) and \( k_\perp \). The curves indicate an increase in Wavenumber with time, with \( k_\parallel \) rising more steeply than \( k_\perp \).
kd_i < 1
