Abstract. We establish the existence and the pointwise bound of the fundamental solution for the stationary Stokes system with measurable coefficients in the whole space $\mathbb{R}^d$, $d \geq 3$, under the assumption that weak solutions of the system are locally H"older continuous. We also discuss the existence and the pointwise bound of the Green function for the Stokes system with measurable coefficients on $\Omega$, where $\Omega$ is an unbounded domain such that the divergence equation is solvable. Such a domain includes, for example, half space and an exterior domain.

1. Introduction

In this paper, we study the stationary Stokes system
\begin{align}
\mathcal{L}u + \nabla p &= f \\
\text{div} u &= g
\end{align}
(1.1)
in $\mathbb{R}^d$, $d \geq 3$ and half space where $\mathcal{L}$ is an elliptic operator
\[
\mathcal{L}u = -D_\alpha(A^{\alpha\beta}D_\beta u)
\]
acting on vector fields $u = (u^1, \ldots, u^d)^{tr}$. Throughout the paper we use Einstein’s summation convention over repeated indices. The coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ are $d \times d$ matrix valued functions whose entries $A^{\alpha\beta}_{ij}(x)$ are bounded and satisfy the strong ellipticity condition, i.e., there exists a constant $\lambda \in (0, 1)$ such that for any $x \in \mathbb{R}^d$ and $\xi = (\xi^i_\alpha)$, $\eta = (\eta^i_\alpha) \in \mathbb{R}^{d \times d}$, we have
\begin{align}
\sum_{\alpha,\beta,i,j=1}^d A^{\alpha\beta}_{ij}(x)\xi^i_\beta \xi^j_\alpha &\geq \lambda |\xi|^2, \\
\sum_{\alpha,\beta,i,j=1}^d |A^{\alpha\beta}_{ij}(x)\xi^i_\beta \eta^j_\alpha| &\leq \lambda^{-1}||\xi|||\eta|.
\end{align}
(1.2)

Let $\mathcal{M} : \Omega \to \tilde{\Omega}$ be a smooth diffeomorphism whose Jacobian equals one to preserve the incompressibility of the flow. If we set $v(y) = v(\mathcal{M}(x)) = u(x)$ and $q(y) = q(\mathcal{M}(x)) = p(x)$ for all $y = \mathcal{M}(x) \in \tilde{\Omega}$, then we have for $i = 1, 2, \ldots, d$
\[
\frac{\partial v}{\partial y^i} = \frac{\partial u}{\partial x^j} \frac{\partial (\mathcal{M}^{-1})^j}{\partial y^i}, \quad \frac{\partial q}{\partial y^i} = \frac{\partial p}{\partial x^j} \frac{\partial (\mathcal{M}^{-1})^j}{\partial y^i}.
\]
We may regard the directional derivatives as a gradient operator $\nabla_y = \partial \mathcal{M}^{-1} \nabla_x$. Using this operator we can write $\text{div}_y v = (\partial \mathcal{M}^{-1} \nabla_x) \cdot u$ and so $\text{div} v = g$ is equivalent to
\[
\text{div} u = (\partial \mathcal{M}) g.
\]

1

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Similarly, we can rewrite $-\Delta_y v + \nabla_y q = f$ as
\[
\nabla_x \cdot (\partial M^{-1} \nabla_x u) + \nabla_x p = (\partial M) f.
\]
This situation often occurs when one consider the limiting case of the Stokes system in time varying domains. These variable coefficient systems are used also for describing inhomogeneous fluids with density dependent viscosity (see, for instance, [1, 18]). Giaquinta–Modica [13] gave various regularity results for nonlinear systems of the type of the stationary Navier–Stokes system. $L^p$-estimates of these operators were established recently in [8, 9, 10]. This motivates our study of the Stokes system with variable coefficients.

For the classical Stokes system
\[
-\Delta u + \nabla p = f, \quad \text{div} \, u = g
\]
there are a huge number of literatures regarding the Green function, which plays a significant role in the study of mathematical fluid dynamics. One of the most popular references is a monograph [11] written by Galdi. We refer the reader for additional discussions of the fundamental solution to [5, 26] and references therein.

For the study of the Green function subject to Dirichlet boundary conditions on bounded domains in $\mathbb{R}^2$ or $\mathbb{R}^3$, we refer to [21, 22, 4, 17, 23] and references therein.

For mixed boundary value problems in $\mathbb{R}^3$, Maz’ya–Rossmann [20] obtained the pointwise estimate of Green functions. For the two dimensional case, Ott–Kim–Brown [24] obtained corresponding results.

Our aim is to construct the fundamental solution $(V(x, y), \Pi(x, y))$ and to establish the pointwise bound of $V(x, y)$
\[
|V(x, y)| \leq C_0 |x - y|^{2-d}, \quad \forall x, y \in \mathbb{R}^d, \quad 0 < |x - y| \leq R_0
\]
under the assumption that weak solutions $(u, p)$ of either
\[
\mathcal{L} u + \nabla p = 0, \quad \text{div} \, u = 0
\]
or
\[
\mathcal{L}^\ast u + \nabla p = 0, \quad \text{div} \, u = 0
\]
are locally Hölder continuous, where $\mathcal{L}^\ast$ denotes the adjoint operator
\[
\mathcal{L}^\ast u = -D_\alpha (A^{\alpha\beta}(x)^{tr} D_\beta u).
\]
We shall show that the local Hölder continuity assumption is satisfied even in the following general cases.

i) The coefficients $A^{\alpha\beta}$ are merely measurable functions of only one fixed direction.

ii) The coefficients $A^{\alpha\beta}$ are partially BMO (measurable in one direction and having small BMO semi norms in the other variables).

The first case is actually a special case of the second one. However, the pointwise estimate (1.3) holds for all $R_0 \in (0, \infty)$ for the case i), whereas (1.3) holds for some $R_0$ for the case ii); see Section 2 for more explicit statements. We are also interested in the existence and the global pointwise bound of the Green function for the Stokes system (1.1) in an unbounded domain $\Omega \subset \mathbb{R}^d, d \geq 3$. We prove that if the problem
\[
\begin{cases}
\text{div} \, u = g & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
\[
\|Du\|_{L^q(\Omega)} \lesssim \|g\|_{L^q(\Omega)}
\]
is solvable and if weak solutions of the system (1.3) or (1.5) are locally Hölder continuous, then the Green function exists and satisfies a natural growth estimate near the pole; see Theorems 2.7 and 10.4. Moreover, we obtain the global pointwise bound for the Green function under an additional assumption that weak solutions of Dirichlet problem are locally bounded up to the boundary; see Theorems 2.14 and 10.5.

Unlike the classical Stokes system with the Laplace operator, we are not able to find any literature explicitly dealing with the existence and the pointwise estimate of the fundamental solution for the Stokes system with nonsmooth coefficients. In a recent article [8], the existence of the Green function for the general Stokes system with VMO (vanishing mean oscillation) coefficients in a bounded Lipschitz domain has been studied. We note that in this paper, interior and boundary estimates for the pressure \( \Pi(x, y) \) of the Green function are established with precise information on the dependence of the estimates, whereas in [8] \( L^q \)-integrability on a domain for the pressure of the Green function is considered.

Green functions for the linear systems have been studied by many authors. In particular, Hofmann–Kim [15] proved the existence and various estimates of the Green function for the elliptic system with irregular coefficients on any open domain. Kang–Kim [17] established the global pointwise estimate of the Green function for the system. We also refer the reader to [6, 7] for the study of Green functions for elliptic systems with irregular coefficients subject to Neumann or Robin boundary condition. In this paper, we mainly follow the arguments by Hofmann–Kim [15] and Kang–Kim [17], but the technical details are different from those papers because the presence of the pressure term makes the argument more involved. In order to estimate \( V(x, y) \) and \( \Pi(x, y) \), we utilize the solvability of the divergence equation in the domain.

The organization of this paper is as follows. In Section 2, we set up our notations and state our main results. In Section 3, we gather some auxiliary lemmas. From Section 4 to Section 9, we give each proof of our main theorems, Theorem 2.6, Theorem 2.7, Theorem 2.12, Theorem 2.14, Theorem 2.15, and Theorem 2.18. Section 10 is devoted to the study of the Green function on an unbounded domain such as an exterior domain.

Throughout the paper we shall use the following notation.

**Notation 1.** We denote \( A \lesssim B \) if there exists a generic positive constant \( C \) such that \( |A| \leq C|B| \). We add subscript letters like \( A \lesssim_{a,b} B \) to indicate the dependence of the implied constant \( C \) on the parameters \( a \) and \( b \).

### 2. Main results

Before stating our main results, we set up some notations and definitions. We use \( x = (x_1, x') = (x_1, \ldots, x_d) \) to denote a point in \( \mathbb{R}^d \). We fix half space to be

\[
\mathbb{R}^d_+ = \{ x = (x_1, x') \in \mathbb{R}^d : x_1 > 0, x' \in \mathbb{R}^{d-1} \}.
\]

We denote by \( B_r(x) \) usual Euclidean balls of radius \( r > 0 \) centered at \( x \in \mathbb{R}^d \) and by \( B_r^+(x) \) half balls

\[
B_r^+(x) = \{ y \in B_r(x) : y_1 > x_1 \}.
\]

Balls in \( \mathbb{R}^{d-1} \) are denoted by \( B_r(x') = \{ y' \in \mathbb{R}^{d-1} : |x' - y'| < r \} \). We use the following abbreviations \( B_r = B_r(0) \) and \( B_r^+ = B_r^+(0) \), where \( 0 \in \mathbb{R}^d \), and
Definition 2.1 (\(Y^1_q(\Omega)\) spaces). Let \(d \geq 3\) and \(\Omega\) be an open set in \(\mathbb{R}^d\). The space \(Y^1_q(\Omega)\) is defined for \(q \in [1, d)\) to be the family of all weakly differential functions \(u \in L_{dq/(d-q)}(\Omega)\) whose weak derivatives are functions in \(L_q(\Omega)\). The space \(Y^1_q(\Omega)\) is endowed with the norm

\[ \|u\|_{Y^1_q(\Omega)} = \|u\|_{L_{dq/(d-q)}(\Omega)} + \|Du\|_{L_q(\Omega)}. \]

We let \(\hat{W}^1_q(\Omega)\) and \(\hat{Y}^1_q(\Omega)\) be the closure of \(C_c^\infty(\Omega)\) in \(W^1_q(\Omega)\) and \(Y^1_q(\Omega)\), respectively. Here \(W^1_q(\Omega)\) denotes the usual Sobolev space.

Remark 2.2. We note that \(Y^1_q(\mathbb{R}^d) = \hat{Y}^1_q(\mathbb{R}^d)\) (see \([19\), p. 46\]). The Sobolev inequality implies that for all \(u \in \hat{Y}^1_q(\Omega)\)

\[ \|u\|_{L_{dq/(d-q)}(\Omega)} \lesssim_{d, q} \|Du\|_{L_q(\Omega)}. \]

Therefore, \(\hat{Y}^1_q(\Omega)\) can be understood as a Hilbert space with the inner product

\[ \langle u, v \rangle = \int_\Omega D_\alpha u \cdot D_\alpha v \, dx. \]

Notation 2. We denote an average of a function \(u\) on \(\Omega\) by

\[ \langle u \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx. \]

Definition 2.3 (Weak solutions). Let

\[ f \in L_{2d/(d+2)}(\Omega)^d, \quad f_\alpha \in L_2(\Omega)^d, \quad g \in L_2(\Omega). \]

We say that \((u, p) \in \hat{Y}^1(\Omega)^d \times L_2(\Omega)\) is a weak solution to

\[ \begin{align*}
L u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega,
\end{align*} \]

in an unbounded domain \(\Omega = \mathbb{R}^d\) or \(\Omega = \mathbb{R}^d_+\) if \((u, p)\) satisfies the system in the sense of distributions in \(\Omega\). In particular, for any \(\phi \in \hat{Y}^1(\Omega)^d\)

\[ \int_\Omega A^{\alpha \beta} D_\beta u \cdot D_\alpha \phi \, dx - \int_\Omega p \, div \phi \, dx = \int_\Omega f \cdot \phi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \phi \, dx. \]

Similarly, we say that \((u, p) \in \hat{Y}^1(\Omega)^d \times L_2(\Omega)\) is a weak solution to

\[ \begin{align*}
\mathcal{L}^* u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega,
\end{align*} \]

in an unbounded domain \(\Omega = \mathbb{R}^d\) or \(\Omega = \mathbb{R}^d_+\) if \((u, p)\) satisfies the system in the sense of distributions in \(\Omega\). In particular, for any \(\phi \in \hat{Y}^1(\Omega)^d\)

\[ \int_\Omega A^{\alpha \beta} D_\beta \phi \cdot D_\alpha u \, dx - \int_\Omega p \, div \phi \, dx = \int_\Omega f \cdot \phi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \phi \, dx. \]

Definition 2.4 (Green functions on unbounded domains \(\Omega\)). Let \(V(x, y)\) be a \(d \times d\) matrix valued function and \(\Pi(x, y)\) be a \(d \times 1\) vector valued function on \(\Omega \times \Omega\). We say that a pair \((V(x, y), \Pi(x, y))\) is the Green function for the Stokes system if it satisfies the following properties.
(a) For any \( y \in \Omega \), \( V(\cdot, y) \in W^1_{1,\text{loc}}(\Omega)^{d \times d} \) and \( \Pi(\cdot, y) \in L_{1,\text{loc}}(\Omega)^d \). Moreover, 
\((1 - \eta)V(\cdot, y) \in Y^1_0(\Omega)^{d \times d}\) for all \( \eta \in C^\infty_0(\Omega) \) satisfying \( \eta \equiv 1 \) on \( B_r(y) \), where \( 0 < r < d_y \).
(b) For any \( y \in \Omega \), \((V(\cdot, y), \Pi(\cdot, y))\) satisfies
\[
\text{div} \, V(\cdot, y) = 0 \quad \text{in} \quad \Omega \tag{2.1}
\]
and
\[
\mathcal{L} V(\cdot, y) + \nabla \Pi(\cdot, y) = \delta_y \hat{I} \quad \text{in} \quad \Omega
\]
in the sense that for any \( \phi \in C^\infty_0(\Omega)^d \), we have
\[
\int_{\Omega} \alpha_{ij} D_j V(x) \cdot D_i \phi \, dx - \int_{\Omega} \Pi(x, y) \phi \, dx = \phi^k(y),
\]
where \( V^k(x, y) \) is the \( k \)-th \((k \in \{1, \ldots, d\})\) column of \( V(x, y) \).
(c) Suppose that \( f \in C^\infty_c(\Omega)^d \) and \( g \in C^\infty_0(\Omega) \). If \((u, p) \in \hat{Y}^2_0(\Omega)^d \times L_2(\Omega)\) is a weak solution to
\[
\left\{ \begin{array}{l}
\mathcal{L}^* u + \nabla p = f \quad \text{in} \quad \Omega, \\
\text{div} u = g \quad \text{in} \quad \Omega,
\end{array} \right.
\]
then
\[
u(y) = \int_{\Omega} V(\cdot, y)^{1^2} f \, dx - \int_{\Omega} \Pi(\cdot, y) g \, dx.
\]

The Green function for the adjoint Stokes system is defined similarly, and the Green function in \( \Omega = \mathbb{R}^d \) is called the fundamental solution. We point out that the condition (c) in the above definition gives the uniqueness of a Green function.

Before stating our main theorems, we introduce the following assumption. It is known that if the coefficients are VMO (vanishing mean oscillations), then Assumption 2.5 holds; see [8]. For more examples of the coefficients satisfying Assumption 2.5, see Theorem 2.6.

**Assumption 2.5.** There exist positive real numbers \( R_0, C_0, \) and \( \alpha_0 < 1 \) such that if \((u, p) \in W^2_{2}(B_R(x^0))^d \times L_2(\hat{B}_R(x^0))\) satisfies, in the sense of distributions,
\[
\mathcal{L} u + \nabla p = 0, \quad \text{div} \, u = 0 \quad \text{in} \quad \hat{B}_R(x^0),
\]
for some \( x^0 \in \Omega \) and \( 0 < R \leq \min\{R_0, \text{dist}(x^0, \partial \Omega)\} \), then
\[
[u]_{C^{\alpha_0}(\hat{B}_{R/2}(x^0))} \leq C_0 R^{-\alpha_0} \left( \int_{B_R(x^0)} |u|^2 \, dx \right)^{1/2},
\]
where \([u]_{C^{\alpha_0}}\) denotes the usual Hölder seminorm. The same estimate holds true when \( \mathcal{L} \) is replaced by \( \mathcal{L}^* \).

**Theorem 2.6.** Let \( \Omega = \mathbb{R}^d \), \( d \geq 3 \). If Assumption 2.5 holds true, then there exists a unique fundamental solution \((V(x, y), \Pi(x, y))\) for the Stokes problem in \( \Omega \). Moreover, for any \( x, y \in \Omega \) satisfying \( 0 < |x - y| \leq R_0 \),
\[
|V(x, y)| \lesssim_{d, \lambda, C_0, \alpha_0} |x - y|^{2-d}. \tag{2.4}
\]
Furthermore, if for some \( q_0 > d \)
\[
f \in L^{2d/(d+2)}(\Omega)^d \cap L^{q_0/2, \text{loc}}(\Omega)^d,
\]
\[
f_0 \in L^2(\Omega)^d \cap L^{q_0, \text{loc}}(\Omega)^d,
\]
\[
g \in L^2(\Omega) \cap L^{q_0, \text{loc}}(\Omega),
\]

(2.5)
and \((u, p) \in \hat{Y}^1_2(\Omega)^d \times L_2(\Omega)\) is a weak solution to

\[
\begin{aligned}
\mathcal{L}^* u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega,
\end{aligned}
\]

then

\[
u(y) = \int_\Omega V(\cdot, y)^{tr} f \, dx - \int_\Omega D_\alpha V(\cdot, y)^{tr} f_\alpha \, dx - \int_\Omega \Pi(\cdot, y)g \, dx.
\]

Our next result is about the existence of the Green function for the Stokes system on \(\mathbb{R}^d_+\). We denote \(d_x = \text{dist}(x, \partial \mathbb{R}^d_+)\) for \(x \in \mathbb{R}^d_+\).

**Theorem 2.7.** Let \(\Omega = \mathbb{R}^d_+, d \geq 3\). If Assumption 2.5 holds, then there exists a unique Green function \((V(x, y), \Pi(x, y))\) for the Stokes operator in \(\Omega\). Moreover, for any \(x, y \in \mathbb{R}^d_+\) satisfying \(|x - y| \leq \min\{d_x, d_y, R_0\}\), we have

\[
|V(x, y)| \lesssim_{d, \lambda, C_0, \alpha_0} |x - y|^{2 - d}.
\]

Furthermore, the representation formula (2.7) is valid.

Actually, we will obtain the following corollary in the middle of the proofs of the previous theorems. But, we record it here to place useful information together.

**Corollary 2.8.** Let \(\Omega = \mathbb{R}^d\) or \(\Omega = \mathbb{R}^d_+\). The Green functions constructed in Theorem 2.6 and Theorem 2.7 satisfy the following estimates: for any \(y \in \Omega\) and \(0 < R \leq \min\{R_0, d_y\}\)

1. \(\|V(\cdot, y)\|_{Y^1_2(\Omega \setminus B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1 - d/2},\)
2. \(\|V(\cdot, y)\|_{L_q(B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{2 - d/q}, \quad q \in [1, d/(d - 2)],\)
3. \(\|D V(\cdot, y)\|_{L_q(B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{1 - d/q}, \quad q \in [1, d/(d - 1)],\)
4. \(\|\Pi(\cdot, y)\|_{L_2(\Omega \setminus B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1 - d/2},\)
5. \(\|\Pi(\cdot, y)\|_{L_q(B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{1 - d/q}, \quad q \in [1, d/(d - 1)].\)

**Remark 2.9.** Theorem 2.6, Theorem 2.7, and Corollary 2.8 continue to hold for the adjoint system under Assumption 2.5.

**Corollary 2.10.** Let \(\Omega = \mathbb{R}^d\) or \(\Omega = \mathbb{R}^d_+\). Let \((^*V(x, y), ^*\Pi(x, y))\) be the Green function for the adjoint problem. Then for \(x \neq y\)

\[
V(x, y) = ^*V(y, x)^{tr}.
\]

Moreover, if \((u, p) \in \hat{Y}^1_2(\Omega)^d \times L_2(\Omega)\) satisfies

\[
\begin{aligned}
\mathcal{L} u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

with (2.5), then

\[
u(y) = \int_\Omega V(y, \cdot)^{tr} f \, dx - \int_\Omega D_\alpha V(y, \cdot) f_\alpha \, dx.
\]

**Remark 2.11.** When \(\mathcal{L} = \mathcal{L}^*\), i.e., \(A_{ij}^{\alpha \beta} = (A_{ji})^{\alpha \beta}\), we have \(V(x, y) = V(y, x)^{tr}\) from (2.8).

The following theorem shows some examples satisfying Assumption 2.5.
Theorem 2.12. (a) If the coefficients $A^{\alpha\beta}$ of $L$ are merely measurable functions of only one fixed direction, i.e.,

$$A^{\alpha\beta} = A^{\alpha\beta}(x_k)$$

for some $k \in \{1, \ldots, d\}$,

then for any $\alpha_0 \in (0, 1)$ and $R_0 \in (0, \infty)$, Assumption 2.13 holds with $C_0 = C_0(d, \lambda, \alpha_0)$.

(b) Let $\alpha_0 \in (0, 1)$. There exists a constant $\gamma \in (0, 1)$, depending on $d$, $\lambda$, and $\alpha_0$, such that if

$$\sup_{x \in \mathbb{R}^d} \sup_{r \leq R_0} \int_{B_r(x)} |A^{\alpha\beta}(y_1, y') - \int_{B_r'(x')} A^{\alpha\beta}(y_1, z') \, dz'| \, dy \leq \gamma,$$

for some $R_0 \in (0, \infty)$, then Assumption 2.13 holds with $C_0 = C_0(d, \lambda, \alpha_0)$. The statement remains true, provided that $y_1$ and $y'$ are replaced by $y_k$ and $(y_1, \ldots, y_k-1, y_{k+1}, \ldots, y_d)$, respectively.

Next we consider the pointwise bound for the Green function on half space under the additional assumption.

Assumption 2.13. There exist positive numbers $R_1$ and $C_1$ such that if $(u, p) \in W^1_2(\mathbb{R}^d_+ \cap B_R(x^0)) \times L^2(\mathbb{R}^d_+ \cap B_R(x^0))$ satisfies

$$\begin{cases}
Lu + \nabla p = 0, &\text{div } u = 0 \text{ in } \mathbb{R}^d_+ \cap B_R(x^0), \\
u = 0 &\text{on } \partial \mathbb{R}^d_+ \cap B_R(x^0),
\end{cases}$$

(2.10)

for some $x^0 \in \partial \mathbb{R}^d_+$ and $0 < R \leq R_1$, then

$$\|u\|_{L^\infty(\mathbb{R}^d_+ \cap B_R(x^0))} \leq C_1 \left( \frac{1}{R^d} \int_{\mathbb{R}^d_+ \cap B_R(x^0)} |u|^2 \, dx \right)^{1/2}. \quad (2.11)$$

The same estimate holds true if $L$ is replaced by $L^*.$

Theorem 2.14. Suppose that Assumptions 2.12 and 2.13 hold. Let $(V(x, y), \Pi(x, y))$ be the Green function constructed in Theorem 2.7. Then for any $x, y \in \mathbb{R}^d_+$ satisfying $0 < |x - y| \leq \min \{R_0, R_1\}$,

$$|V(x, y)| \lesssim_{d, \lambda, C_0, \alpha_0, C_1} |x - y|^{2-d}. \quad (2.12)$$

Moreover, for any $y \in \mathbb{R}^d_+$ and $0 < R \leq \min \{R_0, R_1\}$,

\begin{enumerate}
  \item $\|V(\cdot, y)\|_{L^2_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, C_1} R^{1-d/2},$
  \item $\|V(\cdot, y)\|_{L^q(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, C_1, q} R^{2-d+q/d}, \quad q \in [1, d/(d - 2)],$
  \item $\|DV(\cdot, y)\|_{L^q(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, C_1, q} R^{1-d+q/d}, \quad q \in [1, d/(d - 1)],$
  \item $\|\Pi(\cdot, y)\|_{L^2_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, C_1} R^{1-d/2},$
  \item $\|\Pi(\cdot, y)\|_{L^q(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, C_1, q} R^{1-d+q/d}, \quad q \in [1, d/(d - 1)].$
\end{enumerate}

The following theorem shows some examples satisfying Assumption 2.13.

Theorem 2.15. (a) If the coefficients $A^{\alpha\beta}$ of $L$ are merely measurable functions of only $x_1$-direction, i.e.,

$$A^{\alpha\beta} = A^{\alpha\beta}(x_1),$$

then for any $R_1 \in (0, \infty)$ Assumption 2.13 holds for some $C_1 = C_1(d, \lambda)$. 


(b) There exists a number \( \gamma \in (0, 1) \), depending on \( d \) and \( \lambda \), such that if
\[
\sup_{x \in \mathbb{R}^d} \sup_{r \leq R} \int_{B_r(x)} \left| A^{\alpha\beta}(y, y') - \int_{B_r(x')} A^{\alpha\beta}(y_1, z') \, dz' \right| \, dy \leq \gamma,
\]
for some \( R_1 \in (0, \infty) \), then Assumption 2.16 holds for some \( C_1 = C_1(d, \lambda) \).

The following assumption is used to obtain a better estimate for the Green function near the boundary.

**Assumption 2.16.** There exist positive real numbers \( R_2, C_2 \), and \( \alpha_2 < 1 \) such that if \((u, p) \in W^1_2(\mathbb{R}^d_+ \cap B_R(x^0)) \times L^2_2(\mathbb{R}^d_+ \cap B_R(x^0))\) satisfies, in the sense of distributions,
\[
\begin{cases}
\mathcal{L}u + \nabla p = 0, & \text{div} u = 0 \quad \text{in } \mathbb{R}^d_+ \cap B_R(x^0), \\
u = 0 & \text{on } \partial\mathbb{R}^d_+ \cap B_R(x^0),
\end{cases}
\]
(2.13)
for some \( x^0 \in \mathbb{R}^d_+ \) and \( 0 < R \leq R_2 \), then
\[
[u \chi_{\mathbb{R}^d_+ \cap B_R(x^0)}] C^{\alpha_2}(B_{R/2}(x^0)) \leq C_2 R^{-\alpha_2} \left( \int_{\mathbb{R}^d_+ \cap B_R(x^0)} |u|^2 \, dx \right)^{1/2}.
\]
The same estimate holds true when \( \mathcal{L} \) is replaced by \( \mathcal{L}^* \).

**Remark 2.17.** It will be clear from the proof of Theorem 2.15 that Assumption 2.16 holds under the hypothesis in (a) or (b) of Theorem 2.15.

We observe that Assumption 2.16 implies Assumptions 2.13 and 2.13. By Theorem 2.14 under Assumption 2.13, there exists the Green function \((V(x, y), \Pi(x, y))\) for the Stokes problem satisfying the pointwise estimate (2.12) in Theorem 2.14. The following theorem shows that a better estimate for \(V(x, y)\) is available near the boundary \( \partial\mathbb{R}^d_+ \). We denote \( d_x = \text{dist}(x, \partial\mathbb{R}^d_+) \) for \( x \in \mathbb{R}^d_+ \).

**Theorem 2.18.** Suppose that Assumption 2.16 holds. Let \((V(x, y), \Pi(x, y))\) be the Green function constructed in Theorem 2.17. Then for any \( x, y \in \mathbb{R}^d_+ \) with \( x \neq y \),
\[
|V(x, y)| \leq C \min\{d_x, |x - y|, R_2\}^{\alpha_2} \min\{d_y, |x - y|, R_2\}^{\alpha_2} \min\{|x - y|, R_2\}^{2-d-2\alpha_2},
\]
(2.14)
where \( C = C(d, \lambda, C_2, \alpha_2) \).

In a bounded Lipschitz domain, the estimate (2.14) of the Green function for the classical Stokes system with the Laplace operator was proved by Chang-Choe [4] and Kang-Kim [17]. In particular, [17] dealt with the estimate (2.14) of the Green functions for elliptic systems with measurable coefficients in the whole space and half space.

### 3. Auxiliary lemmas

In this section, we review the existence of solutions to the divergence equation. We also gather some auxiliary lemmas about unique solvability results, pressure estimates, and gradient estimates for the Stokes system with measurable coefficients in the whole space and half space.

**Lemma 3.1.** Let \( 1 < q < \infty \).
(a) Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Then for any $g \in L_q(\Omega)$ satisfying $(g)_{\Omega} = 0$, there exists $u \in W^1_q(\Omega)^d$ such that
\[
\text{div } u = g \text{ in } \Omega, \quad \|Du\|_{L_q(\Omega)} \lesssim_{d,q,\text{Lip}(\Omega)} \|g\|_{L_q(\Omega)}
\]
where Lip$(\Omega)$ denotes the Lipschitz constant of $\Omega$.

(b) Let $\Omega = B_R$. Then for any $g \in L_q(\Omega)$ satisfying $(g)_{\Omega} = 0$, there exists $u \in W^1_q(\Omega)^d$ such that
\[
\text{div } u = g \text{ in } \Omega, \quad \|Du\|_{L_q(\Omega)} \lesssim_{d,q} \|g\|_{L_q(\Omega)}.
\]

This remains true when $B_R$ is replaced by $B^+_R$, $B_R \setminus \overline{B_R/2}$, or $B^+_R \setminus \overline{B_R/2}$.

Proof. For the proof of (a) we refer to [2]. Using (a) and scaling, one can show (b). \hfill \square

The problem of the existence of solutions to the divergence equation in various domains $\Omega$ has been studied by many authors upon the regularity assumptions made on $\Omega$ and the construction methods of solutions $u$. We note that the existence of solutions to the divergence equation in the whole space and half space can be deduced from Lemma 3.1 with scaling; see also [11, p. 261, Corollary IV.3.1]. For the half space case, there is a method based on some explicit representation formula, which was studied in detail by Cattabriga [3] and Solonnikov [25].

**Lemma 3.2.** Let $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d_+$. If $1 < q < d$ and $g \in L_q(\Omega)$, then there exists $u \in \dot{Y}^1_q(\Omega)^d$ such that
\[
\text{div } u = g \text{ in } \Omega, \quad \|Du\|_{L_q(\Omega)} \lesssim_{d,q} \|g\|_{L_q(\Omega)}.
\]

**Lemma 3.3.** Let $\Omega = \mathbb{R}^d$ or $\mathbb{R}^d_+$. Then for $f \in L_{2d/(d+2)}(\Omega)^d$, $f_\alpha \in L_2(\Omega)^d$, and $g \in L_2(\Omega)$, there exists a unique weak solution $(u,p) \in \dot{Y}^1_q(\Omega)^d \times L_2(\Omega)$ to the problem
\[
\begin{aligned}
L u + \nabla p &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega.
\end{aligned}
\]
Moreover,
\[
\|Du\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)} \lesssim_{d,\lambda} \|f\|_{L_{2d/(d+2)}(\Omega)^d} + \|f_\alpha\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)}. \quad (3.1)
\]

Proof. The proof is based on Lemma 3.2 and the Lax-Milgram theorem. We omit the proof because it is almost the same as that of [8, Lemma 3.1]. \hfill \square

**Lemma 3.4.** Let $R > 0$. If $(u,p) \in W^1_2(B_R)^d \times L_2(B_R)$ satisfies
\[
Lu + \nabla p = 0 \quad \text{in } B_R,
\]
then
\[
\int_{B_R} |p - (p)_{B_R}|^2 dx \lesssim_{d,\lambda} \int_{B_R} |Du|^2 dx.
\]
The same estimate holds true if $B_R$ is replaced by $B^+_R$, $B_R \setminus \overline{B_R/2}$, or $B^+_R \setminus \overline{B_R/2}$.

Proof. The proof is almost the same as the classical case. For reader’s convenience we sketch the proof. From the solvability of the divergence equation, there exists $\phi \in \dot{W}^1_2(B_R)^d$ such that
\[
\text{div } \phi = p - (p)_{B_R} \quad \text{in } B_R, \quad \|D\phi\|_{L_2(B_R)} \lesssim_{d} \|p - (p)_{B_R}\|_{L_2(B_R)}.
\]
Using $\phi$ as a test function we obtain
\[
\|p - (p)_{B_R}\|_{L^2(B_R)}^2 = \int (p - (p)_{B_R}) \div \phi = \int \mathcal{L}u \cdot \phi.
\]
The result follows from the strong ellipticity condition with the Cauchy inequality.

\[\square\]

**Lemma 3.5.** Let $R > 0$.

(a) If $(u, p) \in W^1_2(B_R)^d \times L^2(B_R)$ satisfies the system
\[
\begin{cases}
\mathcal{L}u + \nabla p = 0 & \text{in } B_R, \\
\div u = 0 & \text{in } B_R,
\end{cases}
\]
then we have
\[
\int_{B_{R/2}} |Du|^2 \, dx \lesssim_{d, \lambda} R^{-2} \int_{B_R} |u|^2 \, dx.
\]
The statement remains true, provided that $B_R$ and $B_{R/2}$ are replaced by $B_{5R/4} \setminus \overline{B_{R/4}}$ and $B_{R} \setminus \overline{B_{R/2}}$, respectively.

(b) If $(u, p) \in W^1_2(B_R^+)^d \times L^2(B_R^+)$ satisfies the system
\[
\begin{cases}
\mathcal{L}u + \nabla p = 0 & \text{in } B_R^+, \\
\div u = 0 & \text{in } B_R^+, \\
u = 0 & \text{on } B_R \cap \partial \mathbb{R}^d_+,
\end{cases}
\]
then we have
\[
\int_{B_{R/2}^+} |Du|^2 \, dx \lesssim_{d, \lambda} R^{-2} \int_{B_R^+} |u|^2 \, dx.
\]
The statement remains true, provided that $B_R^+$, $B_{R/2}^+$, and $B_R$ are replaced by $B_{5R/4}^+ \setminus \overline{B_{R/4}}$, $B_{R}^+ \setminus \overline{B_{R/2}}$, and $B_{5R/4}^+ \setminus \overline{B_{R/4}}$, respectively.

**Proof.** For a proof, one can just refer to the proofs of [16, Lemma 3.2] and [9, Lemma 3.6] with obvious modifications. For reader’s convenience we sketch the proof for the case when $(u, p) \in W^1_2(B_{5R/4}^+ \setminus \overline{B_{R/4}})^d \times L^2(B_{5R/4}^+ \setminus \overline{B_{R/4}})$ in (b).

We denote for $r > 0$
\[
C_r = B_{R+r} \setminus B_{R/2-r} \quad \text{and} \quad C_r^+ = B_{R+r}^+ \setminus B_{R/2-r}^+.
\]
Let $0 < \rho < r \leq R/4$ and $\eta$ be a smooth function on $\mathbb{R}^d$ satisfying
\[
0 \leq \eta \leq 1, \quad \eta = 1 \quad \text{on } C_\rho, \quad \text{supp } \eta \subset C_r, \quad |D\eta| \lesssim (r - \rho)^{-1}.
\]
Using $\eta^2 u$ as a test function to
\[
\mathcal{L}u + \nabla p = 0 \quad \text{in } C_r^+,
\]
we obtain the Caccioppoli type inequality; for all $\varepsilon > 0$
\[
\int_{C_r^+} |Du|^2 \, dx \leq \varepsilon \int_{C_r^+} |p - (p)_{C_r^+}|^2 \, dx + \frac{C(d, \lambda, \varepsilon)}{(r - \rho)^2} \int_{C_r^+} |u|^2 \, dx.
\]
Using the pressure estimate, Lemma [5.4] we have for all $0 < \rho < r \leq R/4$
\[
\int_{C_r^+} |Du|^2 \, dx \leq \varepsilon \int_{C_r^+} |Du|^2 \, dx + \frac{C}{(r - \rho)^2} \int_{C_r^+} |u|^2 \, dx.
\] (3.2)
For $k = 0, 1, 2, \ldots$ we set

$$
\varepsilon = \frac{1}{8}, \quad r_k = \frac{R}{4} \left(1 - \frac{1}{2^k}\right)
$$

so that (3.2) becomes

$$
|Du|_k^2 \leq \varepsilon |Du|_{k+1}^2 + \frac{C4^k}{R^2} \int_{r_k} |u|^2 dx.
$$

Multiplying $\varepsilon^k$ and summing the estimates we obtain the required result.

**Lemma 3.6.** (a) Let Assumption 2.9 hold. If $(u, p) \in W^1_2(B_R(x^0)) \times L_2(B_R(x^0))$ satisfies (2.3) with $x^0 \in \mathbb{R}^d$ and $0 < R \leq R_0$, then

$$
\|u\|_{L^\infty(B_{r/2}(x^0))} \lesssim d, C_0, \alpha_0 R^{-d} \|u\|_{L_1(B_R(x^0))}.
$$

(b) Let Assumptions 2.9 and 2.10 hold. If $(u, p) \in W^1_2(B_R^+(x^0)) \times L_2(B_R^+(x^0))$ satisfies (2.10) with $x^0 \in \partial \mathbb{B}^+_4$ and $0 < R \leq \min\{R_0, R_1\}$, then

$$
\|u\|_{L^\infty(B_{r/2}^+(x^0))} \lesssim d, C_0, \alpha_0, C_1 R^{-d} \|u\|_{L_1(B^+_R(x^0))}.
$$

**Proof.** We only prove the second assertion of the lemma because the first one is the same with obvious modifications. Let $0 < r < R$ and set $\rho = \frac{R-r}{R}$. We can choose $y^0 \in B^+_r(x^0)$ satisfying

$$
\frac{1}{2} \sup_{B^+_r(x^0)} |u|^2 \leq \sup_{B^+_r(y^0) \cap \mathbb{R}^d} |u|^2.
$$

If $2\rho \leq \text{dist}(y^0, \partial \mathbb{B}^+_4)$, then by Assumption 2.5

$$
\sup_{B^+_r(y^0)} |u|^2 \lesssim \int_{B^+_2(y^0)} |u|^2 dx \lesssim (R-r)^{-d} \int_{B^+_R(x^0)} |u|^2 dx.
$$

On the other hand, if $2\rho > \text{dist}(y^0, \partial \mathbb{B}^+_4)$, then by Assumption 2.13

$$
\sup_{B^+_r(y^0) \cap \mathbb{B}^+_4} |u|^2 \lesssim \sup_{B^+_r(y^0)} |u|^2 \lesssim \int_{B^+_r(y^0)} |u|^2 dx \lesssim (R-r)^{-d} \int_{B^+_R(x^0)} |u|^2 dx,
$$

where $z^0 = (0, y^0_2, \ldots, y^0_d)$. Hence Young’s inequality yields that for $0 < r < R$ and $\varepsilon > 0$

$$
\sup_{B^+_r(x^0)} |u| \lesssim d, C_0, \alpha_0, C_1 (R-r)^{-d/2} \|u\|_{L_2(B^+_R(x^0))} \leq \varepsilon \sup_{B^+_R(x^0)} |u| + C_0 (R-r)^{-d} \|u\|_{L_1(B^+_R(x^0))}.
$$

Now, the result follows from a standard iteration argument in [12, pp. 80-82].

**4. Proof of Theorem 2.6**

The proof is a modification of the argument for elliptic systems found in Hofmann–Kim [13, Theorem 3.1]. Throughout this section, $R_0$, $C_0$, and $\alpha_0$ are constants in Assumption 2.5 and we divide the proof into several steps.
**Step 1** First we define an averaged fundamental solution on \( \mathbb{R}^d \) as follows. For each \( \varepsilon > 0, y \in \mathbb{R}^d \), and \( k \in \{1, \ldots, d\} \) we denote

\[
f_{\varepsilon;y,k} = \frac{\chi_{B_\varepsilon(y)}}{|B_\varepsilon(y)|} e_k
\]

where \( \chi_{B_\varepsilon(y)} \) is the characteristic function and \( e_k \) is the \( k \)-th unit vector in \( \mathbb{R}^d \). By Lemma 3.3 there is a unique weak solution \( (v_{\varepsilon;y,k}, \pi_{\varepsilon;y,k}) \in Y^1_2(\mathbb{R}^d)^d \times L_2(\mathbb{R}^d) \) to

\[
\begin{cases}
L v + \nabla \pi = f_{\varepsilon;y,k} & \text{in } \mathbb{R}^d, \\
\text{div } v = 0 & \text{in } \mathbb{R}^d.
\end{cases}
\]

We define *the averaged fundamental solution* \( (V_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y)) \) by

\[
V_\varepsilon^k(\cdot, y) = v_{\varepsilon;y,k}^k \quad \text{and} \quad \Pi_\varepsilon^k(\cdot, y) = \pi_{\varepsilon;y,k}.
\]

Hereafter, we denote by \( V_\varepsilon^k(x, y) \) the \( k \)-th column of \( V_\varepsilon(x, y) \). Then

\[
\int_{\mathbb{R}^d} A^{\alpha \beta} D_\alpha V_\varepsilon^k(\cdot, y) \cdot D_\beta \phi \, dx - \int_{\mathbb{R}^d} \Pi_\varepsilon^k(\cdot, y) \text{div } \phi \, dx = \int_{B_\varepsilon(y)} \phi^k \, dx
\]  

(4.1)

for all \( \phi \in C_0^\infty(\mathbb{R}^d)^d \). Moreover, from (3.1),

\[
\|DV_\varepsilon(\cdot, y)\|_{L_2(\mathbb{R}^d)} + \|\Pi_\varepsilon(\cdot, y)\|_{L_2(\mathbb{R}^d)} \lesssim \varepsilon^{1-d/2}.
\]  

(4.2)

**Step 2** We prove the local pointwise estimate for \( V_\varepsilon(x, y) \).

**Lemma 4.1.** If Assumption 2.3 holds, then

\[
|V_\varepsilon(x, y)| \lesssim_{d, \lambda, C_{0, \alpha_0}} |x - y|^{2-d}
\]

for all \( x, y \in \mathbb{R}^d \) and \( \varepsilon > 0 \) satisfying \( 0 < \varepsilon \leq |x - y|/3 \leq R_0/2 \).

**Proof.** Let

\[
0 < \varepsilon \leq R := \frac{|x - y|}{3} \leq \frac{R_0}{2}.
\]

Since \( (V_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y)) \) satisfies

\[
\begin{cases}
L V_\varepsilon^k(\cdot, y) + \nabla \Pi_\varepsilon^k(\cdot, y) = 0 & \text{in } B_R(x), \\
\text{div } V_\varepsilon^k(\cdot, y) = 0 & \text{in } B_R(x),
\end{cases}
\]

By Lemma 3.6

\[
|V_\varepsilon^k(x, y)| \lesssim_{d, C_{0, \alpha_0}} R^{-d} \|V_\varepsilon^k(\cdot, y)\|_{L_1(B_R(x))}.
\]

Thus, it suffices to show that

\[
\|V_\varepsilon^k(\cdot, y)\|_{L_1(B_R(x))} \lesssim R^2.
\]  

(4.3)

Let \( f \in L_\infty(\mathbb{R}^d)^d \) with \( \text{supp } f \subset B_R(x) \) and \( (u, p) \in Y^1_2(\mathbb{R}^d)^d \times L_2(\mathbb{R}^d) \) be the weak solution to

\[
\begin{cases}
L^* u + \nabla p = f & \text{in } \mathbb{R}^d, \\
\text{div } u = 0 & \text{in } \mathbb{R}^d.
\end{cases}
\]

By testing with \( V_\varepsilon^k(x, y) \) in the above system,

\[
\int_{\mathbb{R}^d} A^{\alpha \beta} D_\alpha V_\varepsilon^k(\cdot, y) \cdot D_\beta u \, dx = \int_{B_R(x)} V_\varepsilon^k(\cdot, y) \cdot f \, dx.
\]
Also, by testing with $\phi = u$ in (4.1),
\[
\int_{\mathbb{R}^d} A^{\alpha \beta} \partial_{\beta} v^{\varepsilon}(\cdot, y) \cdot D_{\alpha} u \, dx = \int_{B_r(y)} u^k \, dx.
\]

Hence
\[
\int_{B_R(x)} v^{\varepsilon}(\cdot, y) \cdot f \, dx = \int_{B_r(y)} u^k \, dx. \quad (4.4)
\]

Since $(u, p)$ satisfies
\[
\begin{cases}
\mathcal{L} u + \nabla p = 0 & \text{in } B_{2R}(y), \\
\text{div } u = 0 & \text{in } B_{2R}(y),
\end{cases}
\]
we use Lemma 3.6, Hölder’s inequality, and the Sobolev inequality to obtain
\[
\|u\|_{L_\infty(B_R(y))} \lesssim R^{1-d/2} \|u\|_{L_{2d/(d-2)}(\mathbb{R}^d)} \lesssim R^{1-d/2} \|Du\|_{L_2(\mathbb{R}^d)}.
\]

Thus, from the estimate (3.1) we conclude that
\[
\|u\|_{L_\infty(B_R(y))} \lesssim d, \lambda, c_0, a_0 R^2 \|f\|_{L_\infty(B_R(x))}.
\]

Using this together with (4.4) and the duality argument, we get (4.3). □

**Step 3)** We prove the uniform estimates for $V^{\varepsilon}(\cdot, y)$.

**Lemma 4.2.** If Assumption 2.5 holds, then for any $y \in \mathbb{R}^d$, $0 < R \leq R_0$, and $\varepsilon > 0$
\[
\|V^{\varepsilon}(\cdot, y)\|_{Y^1_2(\mathbb{R}^d \setminus B_R(y))} \lesssim d, \lambda, c_0, a_0 R^{1-d/2}. \quad (4.5)
\]

**Proof.** When $\varepsilon \geq R/12$, we have, from (4.2) and the Sobolev inequality,
\[
\|V^{\varepsilon}(\cdot, y)\|_{Y^1_2(\mathbb{R}^d \setminus B_R(y))} \leq \|V^{\varepsilon}(\cdot, y)\|_{Y^1_2(\mathbb{R}^d)} \lesssim R^{1-d/2}.
\]

So, we assume $\varepsilon \in (0, R/12)$. Denote $\mathcal{D} = B_R(y) \setminus \overline{B_{R/2}(y)}$ and let $\eta$ be a smooth function on $\mathbb{R}^d$ satisfying
\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{R/2}(y), \quad \text{supp } \eta \subset B_R(y), \quad |D\eta| \lesssim R^{-1}.
\]

Then
\[
\|V^{\varepsilon}_k(\cdot, y)\|_{L_{2d/(d-2)}(\mathbb{R}^d \setminus B_R(y))} \lesssim \|1 - \eta^2\|_{L_{2d/(d-2)}(\mathbb{R}^d)} \lesssim \|D((1 - \eta^2)V^{\varepsilon}_k(\cdot, y))\|_{L_2(\mathbb{R}^d)} \lesssim \|D((1 - \eta^2)DV^{\varepsilon}_k(\cdot, y))\|_{L_2(\mathcal{D})} + R^{-1}\|V^{\varepsilon}_k(\cdot, y)\|_{L_2(\mathcal{D})}. \quad (4.6)
\]

We shall show that
\[
\|D((1 - \eta^2)DV^{\varepsilon}_k(\cdot, y))\|_{L_2(\mathbb{R}^d)} \lesssim R^{-1}\|V^{\varepsilon}_k(\cdot, y)\|_{L_2(\mathcal{D}_0)} \quad (4.7)
\]
where $\mathcal{D}_0 = B_{5R/4}(y) \setminus \overline{B_{R/4}(y)}$. To show this, we observe first that
\[
\int_{\mathbb{R}^d} \text{div } ((1 - \eta^2)V^{\varepsilon}_k(\cdot, y)) \, dx = 0,
\]
so we can subtract an average to get
\[ \left| \int_{\mathbb{R}^d} \Pi^k_\varepsilon (\cdot, y) \operatorname{div} \left( (1 - \eta^2) V^{k, \varepsilon} (\cdot, y) \right) \, dx \right| \]
\[ = \left| \int_{\mathbb{R}^d} \left( \Pi^k_\varepsilon (\cdot, y) - (\Pi^k_\varepsilon (\cdot, y))_D \right) 2\eta D\eta \cdot V^{k, \varepsilon} (\cdot, y) \, dx \right| \]
\[ \lesssim \int_D |\Pi^k_\varepsilon (\cdot, y) - (\Pi^k_\varepsilon (\cdot, y))_D|^2 \, dx + R^{-2} \int_D |V^{k, \varepsilon} (\cdot, y)|^2 \, dx. \] (4.8)

Using the test function \( \phi = (1 - \eta^2) V^{k, \varepsilon} (\cdot, y) \) in (4.1) and using (4.8), we get
\[ \int_{\mathbb{R}^d} (1 - \eta^2) |DV^{k, \varepsilon} (\cdot, y)|^2 \, dx \]
\[ \lesssim \int_D |\Pi^k_\varepsilon (\cdot, y) - (\Pi^k_\varepsilon (\cdot, y))_D|^2 \, dx + R^{-2} \int_D |V^{k, \varepsilon} (\cdot, y)|^2 \, dx \]
\[ + \int_D |DV^{k, \varepsilon} (\cdot, y)|^2 \, dx. \]

Thus, using Lemma 3.4 and Lemma 3.5 (a) we get (4.4).

Finally, using Lemma 4.1 and the fact
\[ 0 < \varepsilon \frac{R}{12} < \frac{|x - y|}{3} < \frac{5R}{12} < \frac{R_0}{2}, \quad \forall x \in D_0, \]
we have
\[ R^{-2} \int_{D_0} |V^{k, \varepsilon} (\cdot, y)|^2 \, dx \lesssim R^{2 - d}. \]

Combining this with (4.6) and (4.7) yields the estimate (4.9).

**Step 4** We prove uniform \( L^q \)-estimates for \( V^{\varepsilon} (\cdot, y) \) and \( DV^{\varepsilon} (\cdot, y) \).

**Lemma 4.3.** If Assumption 2.2 holds, then for any \( y \in \mathbb{R}^d \), \( 0 < R \leq R_0 \), and \( \varepsilon > 0 \)
\[ \| V^{\varepsilon} (\cdot, y) \|_{L_q(B_R(y))} \lesssim d, \lambda, \alpha_0, q \quad R^{2 - d + d/q}, \quad q \in [1, d/(d - 2)], \] (4.9)
\[ \| DV^{\varepsilon} (\cdot, y) \|_{L_q(B_R(y))} \lesssim d, \lambda, \alpha_0, q \quad R^{1 - d + d/q}, \quad q \in [1, d/(d - 1)]. \] (4.10)

**Proof.** From the previous lemma we have for all \( 0 < \rho \leq R_0 \)
\[ \int_{\mathbb{R}^d \setminus B_\rho(y)} |V^{\varepsilon} (\cdot, y)|^{2d/(d-2)} \, dx \lesssim \rho^{-d}. \]

Let \( 0 < t < \infty \) and denote
\[ A(t) = \{ x \in \mathbb{R}^d : |V^{\varepsilon}(x, y)| > t \}. \]

Then for all \( 0 < \rho \leq R_0 \)
\[ |A(t)| = |A(t) \cap B_\rho(y)| + |A(t) \setminus B_\rho(y)| \]
\[ \lesssim \rho^d + t^{-2d/(d-2)} \int_{A(t) \setminus B_\rho(y)} |V^{\varepsilon} (\cdot, y)|^{2d/(d-2)} \, dx \]
\[ \lesssim \rho^d + t^{-2d/(d-2)} \rho^{-d}. \]

If \( t \geq R_0^{-d} \), then we can take \( \rho = t^{-1/(d-2)} \) so that
\[ |A(t)| \lesssim t^{-d/(d-2)}. \]
Hence, for all $R_0^2 - d < T < \infty$

$$
\int_{B_R(y)} |V_\varepsilon(\cdot, y)|^q \, dx \lesssim \int_0^T t^{q-1} |B_R(y) \cap A(t)| \, dt \\
\lesssim \int_0^T t^{q-1} R^d \, dt + \int_T^\infty t^{q-1} t^{-d/(d-2)} \, dt \\
\lesssim T^q R^d + T^{q-d/(d-2)}.
$$

In the last estimate, we have used the condition $q < d/(d-2)$. If $0 < R \leq R_0$, then we can take $T = R^{2-d}$ so that

$$
\int_{B_R(y)} |V_\varepsilon(\cdot, y)|^q \, dx \lesssim R^{(2-d)q + d}.
$$

This proves the estimate (4.9)

The proof of (4.10) is similar. From the previous lemma we have for all $0 < \rho \leq R_0$

$$
\int_{R^d \setminus B_\rho(y)} |DV_\varepsilon(\cdot, y)|^2 \, dx \lesssim \rho^{2-d}.
$$

Let $0 < t < \infty$ and denote

$$
B(t) = \{x \in R^d : |DV_\varepsilon(x, y)| > t\}.
$$

By performing the same procedure, we can obtain (4.10). \qed

**Step 5)** Similar to the previous lemmas, we prove uniform estimates for $\Pi_\varepsilon(\cdot, y)$.

**Lemma 4.4.** If Assumption 2.5 holds, then for any $y \in R^d$, $0 < R \leq R_0$, and $\varepsilon > 0$

$$
\|\Pi_\varepsilon(\cdot, y)\|_{L^2(R^d \setminus B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1-d/2}. \quad (4.11)
$$

Moreover, for any $y \in R^d$, $0 < R \leq R_0$, and $\varepsilon > 0$

$$
\|\Pi_\varepsilon(\cdot, y)\|_{L^q(B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{1-d+d/q}, \quad q \in [1, d/(d-1)]. \quad (4.12)
$$

**Proof.** If $\varepsilon \geq R/2$, then one can easily check (4.11) from (4.12). So, we assume $\varepsilon \in (0, R/2)$. Let $D$ and $\eta$ be as in the proof of Lemma 4.2 Let $\varphi \in Y^1_2(R^d)$ be a solution to the divergence equation

$$
\text{div} \varphi = \Pi^{\varepsilon}_\varphi(\cdot, y) \chi_{R^d \setminus \overline{B_R(y)}} \text{ in } R^d
$$

satisfying

$$
\|\varphi\|_{Y^1_2(R^d)} \lesssim \|\Pi^{\varepsilon}_\varphi(\cdot, y)\|_{L^2(R^d \setminus \overline{B_R(y)})}. \quad (4.13)
$$

From the definition of the averaged fundamental solution $(V_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y))$ with a test function $(1 - \eta)\varphi$, we obtain

$$
\int_{R^d} A^{\alpha\beta} D_\beta V_\varepsilon^{k}(\cdot, y) \cdot D_\alpha ((1 - \eta)\varphi) \, dx - \int_{R^d} \Pi^{\varepsilon}_\varphi(\cdot, y) \text{div}((1 - \eta)\varphi) \, dx \\
= \int_{B_{\varepsilon}(y)} (1 - \eta)\varphi^{k} \, dx = 0, \quad (4.14)
$$
where the last equality follows from the fact that the integrand vanishes in the domain of integration. We notice that

\[
\int_{\mathbb{R}^d} \Pi_k^{\varepsilon}(:,y) \, \text{div}(\varepsilon \eta \varphi) \, dx
\]

\[
= \int_{\mathbb{R}^d} \Pi_k^{\varepsilon}(:,y) \, \varphi \, dx - \int_{\mathbb{R}^d} \Pi_k^{\varepsilon}(:,y) \, \text{div}(\eta \varphi) \, dx
\]

\[
= \int_{\mathbb{R}^d \setminus B_R(y)} |\Pi_k^{\varepsilon}(:,y)|^2 \, dx - \int_{B_R(y)} \left( \Pi_k^{\varepsilon}(:,y) - (\Pi_k^{\varepsilon}(:,y))_D \right) D\eta \cdot \varphi \, dx
\]

due to \( \text{div} \varphi = 0 \) in \( B_R(y) \). Since

\[
\mathcal{L}V^{\varepsilon,k}(\cdot,y) + \nabla \Pi_k^{\varepsilon}(\cdot,y) = 0 \quad \text{in} \quad D,
\]

it follows from Lemma 3.4 that

\[
\int_D |\Pi_k^{\varepsilon}(\cdot,y) - (\Pi_k^{\varepsilon}(\cdot,y))_D|^2 \, dx \lesssim_{d,\lambda,C_0,\alpha_0} \int_D |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx.
\]

Using Young’s inequality, Hölder’s inequality, (4.13), and (4.16) we obtain that

\[
\int_{B_R(y)} \left( \Pi_k^{\varepsilon}(\cdot,y) - (\Pi_k^{\varepsilon}(\cdot,y))_D \right) D\eta \cdot \varphi \, dx \lesssim_{d,\lambda,C_0,\alpha_0} \int_D |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx.
\]

Similarly, using Young’s inequality, Hölder’s inequality, and (4.13), we obtain that for all positive number \( \varepsilon \)

\[
\int_{\mathbb{R}^d} \mathcal{A}^{\alpha\beta} \mathcal{D}_\beta V_k^{\varepsilon}(\cdot,y) \cdot \mathcal{D}_\alpha ((1 - \eta) \varphi) \, dx
\]

\[
\lesssim \varepsilon \int_{\mathbb{R}^d} |D((1 - \eta) \varphi)|^2 \, dx + C_\varepsilon \int_{\mathbb{R}^d \setminus B_R/2(y)} |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx
\]

\[
\lesssim \varepsilon \int_{\mathbb{R}^d \setminus B_R(y)} |\Pi_k^{\varepsilon}(\cdot,y)|^2 \, dx + C_\varepsilon \int_{\mathbb{R}^d \setminus B_R/2(y)} |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx.
\]

By choosing a small \( \varepsilon \) and combining (4.13), (4.15), (4.17), and (4.18), we get

\[
\int_{\mathbb{R}^d \setminus B_R(y)} |\Pi_k^{\varepsilon}(\cdot,y)|^2 \, dx \lesssim_{d,\lambda} \int_{\mathbb{R}^d \setminus B_R/2(y)} |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx.
\]

Finally, we have from (4.16)

\[
\int_D |DV^{\varepsilon,k}(\cdot,y)|^2 \, dx \lesssim_{d,\lambda,C_0,\alpha_0} R^{2-d},
\]

so we get desired estimate (4.11).

The proof of (4.12) is very similar but using (4.11) instead of (4.5).

**Step 6** Let \( y \in \mathbb{R}^d \) and \( q < d/(d - 1) \). By Lemma 4.2, Lemma 4.4 and the weak compactness, there exists functions

\[
V_{\text{ext}} \in Y_2^1(\mathbb{R}^d \setminus \overline{B_{R_0/2}(y)}) \times \mathbb{R}^d, \quad V_{\text{int}} \in W_1^1(B_{R_0}(y)) \times \mathbb{R}^d,
\]

\[
\Pi_{\text{ext}} \in L^2(\mathbb{R}^d \setminus \overline{B_{R_0/2}(y)}), \quad \Pi_{\text{int}} \in L^q(B_{R_0}(y))
\]
and a sequence \( \{\epsilon_\rho\}_{\rho=1}^\infty \) tending to zero such that
\[
V_{\epsilon_\rho}(\cdot, y) \to V_{\text{ext}} \quad \text{weakly in } Y_2^1(\mathbb{R}^d \setminus \overline{B_{R_0/2}(y)}),
\]
\[
V_{\epsilon_\rho}(\cdot, y) \to V_{\text{int}} \quad \text{weakly in } W_1^1(B_{R_0}(y)),
\]
(4.19)
and
\[
\Pi_{\epsilon_\rho}(\cdot, y) \to \Pi_{\text{ext}} \quad \text{in } L_2(\mathbb{R}^d \setminus \overline{B_{R_0/2}(y)}),
\]
\[
\Pi_{\epsilon_\rho}(\cdot, y) \to \Pi_{\text{int}} \quad \text{weakly in } L_q(B_{R_0}(y)).
\]
(4.20)
Observe that
\[
V_{\text{ext}} = V_{\text{int}} \text{ on } B_{R_0}(y) \setminus \overline{B_{R_0/2}(y)},
\]
and we define
\[
V(\cdot, y) :=
\begin{cases}
V_{\text{ext}} & \text{on } \mathbb{R}^d \setminus \overline{B_{R_0}(y)}, \\
V_{\text{int}} & \text{on } B_{R_0}(y) \setminus \overline{B_{R_0/2}(y)}, \\
V_{\text{int}} & \text{on } B_{R_0/2}(y),
\end{cases}
\]
and similarly
\[
\Pi(\cdot, y) :=
\begin{cases}
\Pi_{\text{ext}} & \text{on } \mathbb{R}^d \setminus \overline{B_{R_0}(y)}, \\
\Pi_{\text{int}} & \text{on } B_{R_0}(y) \setminus \overline{B_{R_0/2}(y)}, \\
\Pi_{\text{int}} & \text{on } B_{R_0/2}(y).
\end{cases}
\]
By (4.19), (4.21), and a diagonalization process, there exists a subsequence, still denoted by \( \{\epsilon_\rho\}_{\rho=1}^\infty \), such that
\[
V_{\epsilon_\rho}(\cdot, y) \to V(\cdot, y) \quad \text{weakly in } Y_2^1(\mathbb{R}^d \setminus \overline{B_{r}(y)}), \quad \forall r > 0,
\]
(4.21)
and
\[
\Pi_{\epsilon_\rho}(\cdot, y) \to \Pi(\cdot, y) \quad \text{in } L_2(\mathbb{R}^d \setminus \overline{B_{r}(y)}), \quad \forall r > 0.
\]
(4.22)

**Step 7** We shall show \((V, \Pi)\) satisfies the conditions in Definition 4.4. Obviously, it satisfies the condition (a).

*Verifying (b).* Let \( y \in \mathbb{R}^d \). Since \( \text{div} V_{\epsilon_\rho}(\cdot, y) = 0 \) in \( \mathbb{R}^d \), by using (4.19) and (4.21), one can easily check that (4.21) holds. To show (4.22), we notice from (4.1) that
\[
\phi^k(y) = \lim_{\rho \to \infty} \int_{B_{\rho}(y)} \phi^k(x) \, dx
\]
\[
= \lim_{\rho \to \infty} \left( \int_{\mathbb{R}^d} A^{\alpha\beta} D_\beta V_{\epsilon_\rho}(\cdot, y) \cdot D_\alpha \phi \, dx - \int_{\mathbb{R}^d} \Pi_{\epsilon_\rho}(\cdot, y) \, \text{div} \phi \, dx \right)
\]
(4.23)
for any \( \phi \in C_0^\infty(\mathbb{R}^d) \). Using (4.19) and (4.21), we have
\[
\lim_{\rho \to \infty} \int_{\mathbb{R}^d} A^{\alpha\beta} D_\beta V_{\epsilon_\rho}(\cdot, y) \cdot D_\alpha \phi \, dx
\]
\[
= \lim_{\rho \to \infty} \left( \int_{B_{R_0}(y)} A^{\alpha\beta} D_\beta V_{\epsilon_\rho}(\cdot, y) \cdot D_\alpha \phi \, dx + \int_{\mathbb{R}^d \setminus B_{R_0}(y)} A^{\alpha\beta} D_\beta V_{\epsilon_\rho}(\cdot, y) \cdot D_\alpha \phi \, dx \right)
\]
\[
= \int_{B_{R_0}(y)} A^{\alpha\beta} D_\beta V^k(\cdot, y) \cdot D_\alpha \phi \, dx + \int_{\mathbb{R}^d \setminus B_{R_0}(y)} A^{\alpha\beta} D_\beta V^k(\cdot, y) \cdot D_\alpha \phi \, dx
\]
\[
= \int_{\mathbb{R}^d} A^{\alpha\beta} D_\beta V^k(\cdot, y) \cdot D_\alpha \phi \, dx.
\]
(4.24)
Similarly, we obtain by (4.20) and (4.22) that
\[ \lim_{\rho \to \infty} \int_{\mathbb{R}^d} \Pi^k_{\epsilon,\rho}(\cdot, y) \div \phi \, dx = \int_{\mathbb{R}^d} \Pi^k(\cdot, y) \div \phi \, dx. \]
From this together with (4.23) and (4.24), we get (2.2).

Verifying (c). It suffices to prove that (2.8) holds under the assumptions (2.5) and (2.6). Let \( q_0 > d \). By the uniform estimates (4.9), (4.10) and (4.12), we may assume that
\[ V_{\epsilon,\rho}(\cdot, y) \to V(\cdot, y) \quad \text{weakly in } L_{q_0/(q_0-2)}(B_{R_0}(y)), \]
\[ DV_{\epsilon,\rho}(\cdot, y) \to DV(\cdot, y) \quad \text{weakly in } L_{q_0/(q_0-1)}(B_{R_0}(y)), \]
\[ \Pi_{\epsilon,\rho}(\cdot, y) \to \Pi(\cdot, y) \quad \text{weakly in } L_{q_0/(q_0-1)}(B_{R_0}(y)). \]
Let \((u, p) \in Y^1_0(\mathbb{R}^d) \times L_2(\mathbb{R}^d)\) be the weak solution of (2.6). Then by testing with \( V_{\epsilon,\rho}^k(\cdot, y) \) to (2.6) and setting \( \phi = u \) in (4.1), we have (see e.g., (4.26))
\[ \int_{B_{R_0}(y)} u^k \, dx = \int_{\mathbb{R}^d} V_{\epsilon,\rho}^k(\cdot, y) \cdot f \, dx - \int_{\mathbb{R}^d} D_\alpha V_{\epsilon,\rho}^k(\cdot, y) \cdot f_\alpha \, dx - \int_{\mathbb{R}^d} \Pi_{\epsilon,\rho}^k(\cdot, y) g \, dx. \]
Then similar to the proof of (b), by using (4.21), (4.22), and (4.25), we conclude that
\[ u^k(y) = \int_{\mathbb{R}^d} V^k(\cdot, y) \cdot f \, dx - \int_{\mathbb{R}^d} D_\alpha V^k(\cdot, y) \cdot f_\alpha \, dx - \int_{\mathbb{R}^d} \Pi^k(\cdot, y) g \, dx, \]
which implies the identity (2.7).

**Step 8** Let us fix \( y \in \mathbb{R}^d \) and \( R \in (0, R_0] \). By (4.15) and (4.21), we obtain for \( \phi \in C_0^\infty(\mathbb{R}^d)^d \)
\[ \left| \int_{\mathbb{R}^d \setminus B_R(y)} V^k(\cdot, y) \cdot \phi \, dx \right| = \lim_{\rho \to \infty} \left| \int_{\mathbb{R}^d \setminus B_R(y)} V_{\epsilon,\rho}^k(\cdot, y) \cdot \phi \, dx \right| \lesssim_{d, \lambda, C_0, \alpha_0} R^{1-d/2} \| \phi \|_{L^2_{2d/(d-2)}(\mathbb{R}^d \setminus B_R(y))}, \]
which implies
\[ \| V^k(\cdot, y) \|_{L^2_{2d/(d-2)}(\mathbb{R}^d \setminus B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1-d/2}. \]
Using this argument together with Lemmas 4.2 and 4.3, it is routine to check the estimates i) - v) in Corollary 2.8.

To get the pointwise estimate (2.3), let \( x, y \in \mathbb{R}^d \), and \( 0 < R := |x - y| \leq R_0 \). By the condition (b) in the definition, we find that \((V^k(\cdot, y), \Pi^k(\cdot, y))\)
\[ \left\{ \begin{array}{l}
\div V^k(\cdot, y) = 0 \quad \text{in } B_{R/2}(x), \\
L V^k(\cdot, y) + \nabla \Pi^k(\cdot, y) = 0 \quad \text{in } B_{R/2}(x).
\end{array} \right. \]
Therefore, by Lemma 3.6 and i) in Corollary 2.8 we conclude that
\[ |V^k(x, y)| \lesssim R^{-d} \| V^k(\cdot, y) \|_{L_1(B_{R/2}(x))} \lesssim R^{-d/2} \| V^k(\cdot, y) \|_{L^2_{2d/(d-2)}(\mathbb{R}^d \setminus B_{R/2}(y))} \lesssim R^{2-d}, \]
which implies the pointwise estimate (2.4).
Step 9) Finally, we prove the uniqueness of the fundamental solution \((V, \Pi)\). Let \((\tilde{V}, \tilde{\Pi})\) be another pair satisfying the condition \((c)\) in Definition 2.4. By the unique solvability of Stokes system
\[
\int_{\mathbb{R}^d} V(\cdot, y)^{tr} f \, dx - \int_{\mathbb{R}^d} \Pi(\cdot, y) g \, dx = \int_{\mathbb{R}^d} \tilde{V}(\cdot, y)^{tr} f \, dx - \int_{\mathbb{R}^d} \tilde{\Pi}(\cdot, y) g \, dx
\]
for all \(f \in C_0^\infty(\mathbb{R}^d)^d\) and \(g \in C_0^\infty(\mathbb{R}^d)\). Thus, we should have for almost all \(x, y \in \mathbb{R}^d\)
\[
(V(x, y), \Pi(x, y)) = (\tilde{V}(x, y), \tilde{\Pi}(x, y)).
\]
This completes the proof of Theorem 2.6.

We end this section by giving the proof of Corollary 2.10, which is a slight modification of that of [8, Eq. (2.5)].

Let \(\text{c*(V, \Pi)}\) and \(\text{c*(V_\delta, \Pi_\delta)}\) be the fundamental solution and the averaged fundamental solution for \(L^*\), respectively; i.e., for \(y \in \mathbb{R}^d\) and \(k \in \{1, \ldots, d\}\), the pair \((\text{c*(V_\delta^k(\cdot, y), \Pi_\delta^k(\cdot, y))})\), where \(\text{c*(V_\delta^k(\cdot, y))}\) is the \(k\)-th column of \(\text{c*(V_\delta(\cdot, y))}\), is the weak solution in \(Y_2(\mathbb{R}^d)^d \times L_2(\mathbb{R}^d)\) of
\[
\begin{aligned}
\mathcal{L}^*(\text{c*(V_\delta^k(\cdot, y))}) + \nabla(\text{c*(\Pi_\delta^k(\cdot, y))}) &= \mathbb{I}_{B(y)}(y) e_k \quad \text{in } \mathbb{R}^d, \\
\text{div}(\text{c*(V_\delta^k(\cdot, y))}) &= 0 \quad \text{in } \mathbb{R}^d.
\end{aligned}
\]

Then \(\text{c*(V, \Pi)}\) and \(\text{c*(V_\delta, \Pi_\delta)}\) satisfy counterparts of results in Theorem 2.6.

**Lemma 4.5.** Let \(y \in \mathbb{R}^d\). For any compact set \(K \subset \mathbb{R}^d \setminus \{y\}\), there exist sequences \(\{\varepsilon_\rho\}_{\rho=1}^\infty\) and \(\{\delta_\tau\}_{\tau=1}^\infty\) tending to zero such that
\[
V_{\varepsilon_\rho}(\cdot, y) \to V(\cdot, y) \quad \text{uniformly on } K,
\]
\[
\text{c*V_\delta(\cdot, y) \to c*V(\cdot, y) \quad \text{uniformly on } K.}
\]

**Proof.** The proof is the same as that of [8, Lemma 4.4]. \(\square\)

Now we are ready to prove the case \(\Omega = \mathbb{R}^d\) in Corollary 2.10. Let \(x, y \in \mathbb{R}^d\), \(x \neq y\), and \(k, \ell = 1, \ldots, d\). By setting \(\phi = \text{c*V_\delta^k(\cdot, x)}\) in (4.1) and by using \(V_{\varepsilon_\rho}(\cdot, y)\) as a test function to (4.26), we get
\[
\Gamma_{\varepsilon_\rho, \delta}^{k\ell} := \int_{B(\delta)} \text{c*V_\delta^k(\cdot, x)} \, dz = \int_{B(\delta)} \text{c*V_\rho^k(\cdot, y)} \, dz.
\]

Let \(\{\varepsilon_\rho\}\) and \(\{\delta_\tau\}\) be sequences in Lemma 4.5. Then by the continuity of \(V_{\varepsilon_\rho}(\cdot, y)\) and Lemma 4.5 we have
\[
\lim_{\rho \to \infty} \lim_{\tau \to \infty} \text{c*V_\rho(\cdot, y)} = \lim_{\rho \to \infty} \lim_{\tau \to \infty} \int_{B(\delta)} \text{c*V_\rho(\cdot, y)} \, dz = V^{k\ell}(x, y).
\]

Similarly, by the continuity of \(\text{c*V(\cdot, y)}\) and Lemma 4.5 we obtain
\[
\lim_{\rho \to \infty} \lim_{\tau \to \infty} \text{c*V_\rho(\cdot, y)} = \lim_{\rho \to \infty} \lim_{\tau \to \infty} \int_{B(\delta)} \text{c*V_\rho(\cdot, y)} \, dz = \text{c*V_\rho(\cdot, y)}.
\]

We thus have
\[
V^{k\ell}(x, y) = \text{c*V_\rho(\cdot, y)},
\]
Moreover, for all $\varepsilon > 0$ we obtain the pointwise estimate (4.28) can also yield the following uniform estimates.

This justifies why we call it the averaged fundamental solution. Finally, the representation formula (2.9) is an easy consequence of the identity (4.28) and the counterpart of (2.7).

This completes the proof of the case $\Omega = \mathbb{R}^d$ in Corollary 2.10. The case of $\Omega = \mathbb{R}^d_+$ can be treated in a similar way.

5. PROOF OF THEOREM 2.7

The proof is a slight modification of the proof of Theorem 2.6. For each $\varepsilon > 0$, $y \in \mathbb{R}^d_+$, and $k \in \{1, \ldots, d\}$ we denote

$$f_{\varepsilon, y, k} = \frac{\chi_{B_{\epsilon}(y)}^{d^+} \cap B_{\epsilon}(y)}{|B_{\epsilon}(y)|} e_k$$

where $\chi_E$ is the characteristic function and $e_k$ is the $k$-th unit vector in $\mathbb{R}^d$. We define an averaged Green function $(V^{k\varepsilon}(.), \Pi^{k\varepsilon}(., \cdot)) \in \mathcal{Y}_2^1(\mathbb{R}^d_+) \times L_2(\mathbb{R}^d_+)$ as the unique weak solution to the problem

$$\begin{cases}
Lu + \nabla p = f_{\varepsilon, y, k} & \text{in } \mathbb{R}^d,

\text{div } u = 0 & \text{in } \mathbb{R}^d_+.
\end{cases}$$

Using (3.1) we have

$$\|DV\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+)} + \|\Pi\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+)} \lesssim_{d, \lambda} \varepsilon^{1 - d/2}, \quad \forall \varepsilon > 0. \quad (5.1)$$

Moreover, for all $x, y \in \mathbb{R}^d_+$ and $\varepsilon > 0$ satisfying

$$0 < \varepsilon \leq \frac{|x - y|}{3} \leq \frac{1}{2} \min\{d_x, d_y, R_0\},$$

we obtain the pointwise estimate

$$|V\varepsilon(x, y)| \lesssim_{d, \lambda, C_0, \alpha_0} |x - y|^{2 - d} \quad (5.2)$$

by repeating the same argument as in the proof of Lemma 4.1. The pointwise estimate (5.2) can also yield the following uniform estimates.

**Lemma 5.1.** For any $y \in \mathbb{R}^d_+$, $0 < R \leq \min\{d_y, R_0\}$, and $\varepsilon > 0$,

$$\|V\varepsilon(.; y)\|_{L_2^1(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1 - d/2}, \quad (5.3)$$

$$\|\Pi\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0} R^{1 - d/2}. \quad (5.4)$$

Moreover, for any $y \in \mathbb{R}^d_+$, $0 < R \leq \min\{d_y, R_0\}$, and $\varepsilon > 0$,

$$\|V\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{2 - d + dq}, \quad q \in [1, d/(d - 2)), \quad (5.5)$$

$$\|DV\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{1 - d + dq}, \quad q \in [1, d/(d - 1)), \quad (5.6)$$

$$\|\Pi\varepsilon(.; y)\|_{L_2(\mathbb{R}^d_+ \cap B_R(y))} \lesssim_{d, \lambda, C_0, \alpha_0, q} R^{1 - d + dq}, \quad q \in [1, d/(d - 1)). \quad (5.7)$$
The estimates (5.5) – (5.7) are deduced from (5.3) and (5.4) in the same way as
Using this and (5.3), one can easily obtain (5.4) just following the proof of (4.11).

To show (5.4), we notice from Lemma 3.2 that there exists \( \varphi \) such that

\[
\text{div} \varphi = \Pi^k(\cdot, y)I_{R^d}^{d}y \in R^d
\]

satisfying

\[
\| \varphi \|_{Y^2_R} \leq N(d)\| \Pi^k(\cdot, y) \|_{L^2(R^d \setminus B_R(y))}.
\]

Using this and (5.3), one can easily obtain (5.4) just following the proof of (4.11).
The estimates (5.6) – (5.7) are deduced from (5.3) and (5.4) in the same way as
(4.10), and (4.11) are deduced from (4.10) and (4.11). We omit the details. \( \square \)

The proof of Theorem 2.7 is based on Lemma 5.1 and exactly the same argument
in the proof of Theorem 2.6. We can find the Green function \((V, \Pi)\) satisfying the
pointwise estimate in Theorem 2.7 and all the estimates for \( \Omega = R^d \) in Corollary
2.8. We omit the repeated details.

6. PROOF OF THEOREM 2.12

Suppose \( A^\alpha_0 = A^\alpha_0(x_1) \) satisfy (1.2) and denote

\[
\mathcal{L}_0 u = -D_\alpha(A^\alpha_0 D_\beta u).
\]

In the lemma below, we provide interior \( L_\infty \)-estimates for \( Du \) and \( p \), where \((u, p)\)
is a solution of

\[
\mathcal{L}_0 u + \nabla p = 0, \quad \text{div} u = 0.
\]

The results in the following lemma were proved by Dong–Kim [9, Section 4]. Actually,
they proved \( L_\infty \)-estimates of \( D_x u \) and certain linear combinations of \( Du \) and
\( p \). Using this and the argument in [9, Section 6], one can easily show \( L_\infty \)-estimates
for \( Du \) and \( p \). Here, we reproduce it for the reader’s convenience by rearranging
the proof in [9].

Lemma 6.1. If \((u, p) \in W^2_2(B_2)^d \times L_2(B_2)\) satisfies (6.2) in \( B_2 \), then

\[
\| Du \|_{L_\infty(B_1)} \lesssim_{d, \lambda} \| Du \|_{L_2(B_2)}
\]

and

\[
\| p \|_{L_\infty(B_1)} \lesssim_{d, \lambda} \| Du \|_{L_2(B_2)} + \| p \|_{L_2(B_2)}.
\]

Proof. From [9] Lemma 4.3], we have

\[
\| D_x u \|_{L_\infty(B_1)} + \sum_{i=2}^d \| U^i \|_{L_\infty(B_1)} \lesssim \| Du \|_{L_2(B_2)}
\]

and

\[
\| U^1 \|_{L_\infty(B_1)} \lesssim \| Du \|_{L_2(B_2)} + \| p \|_{L_2(B_2)},
\]

where

\[
U^1 = (A^\beta_0)_{ij} D_\beta u^j + p e_1, \quad U_i = (A^\beta_0)_{ij} D_\beta u^j, \quad i = 2, \ldots, d.
\]

Since \( \text{div} u = 0 \), we obtain from (6.5) that

\[
\| D_1 u^1 \|_{L_\infty(B_1)} \lesssim \| Du \|_{L_2(B_2)}.
\]
Thus, by the ellipticity condition (1.2) and Young’s inequality, we have
\[ B(x) \leq x, \quad \text{for almost all } x \in \Omega, \]
we multiply both sides by \( D_1 u^i \) and then sum over \( i = 2, \ldots, d \) to obtain
\[ \sum_{i,j=2}^{d} A_{ij}^{11} D_1 u^i D_1 u^i = \sum_{i=2}^{d} U^i D_1 u^i - \sum_{j=1}^{d} \sum_{i,j=2}^{d} (A_{ij}^{11}) D_1 u^i D_1 u^i - \sum_{i=2}^{d} (A_{ij}^{11}) D_1 u^i D_1 u^i. \]
Thus, by the ellipticity condition (1.2) and Young’s inequality, we have
\[ \sum_{j=2}^{d} |D_1 u^i(x)|^2 \lesssim_{d, \lambda} \sum_{i=2}^{d} |U^i(x)|^2 + |D_1 u(x)|^2 + |D_1 u^i(x)|^2 \]
for almost all \( x \in B_1 \). Taking the norm \( \| \cdot \|_{L_\infty(B_1)} \) to both sides of the above inequality, and then using (6.5) and (6.7), we get (6.3). Finally, since
\[ p e_1 = U^1 - (A_{ij}^{11}) D_1 u^i, \]
we get (6.4) from (6.3) and (6.6). \( \square \)

Corollary 6.2. Let \( 0 < r < R \). If \((u, p) \in W^1(B_R) \times L^2(B_R)\) satisfies \((6.2)\) in \(B_R\), then
\[ \| D u \|_{L_\infty(B_r)} \lesssim_{d, \lambda} (R - r)^{-d/2} \| D u \|_{L^2(B_R)}, \]
and
\[ \| p \|_{L_\infty(B_r)} \lesssim_{d, \lambda} (R - r)^{-d/2} \left( \| D u \|_{L^2(B_R)} + \| p \|_{L^2(B_R)} \right). \]

Proof. Based on Lemma 6.1 with scaling and a well known argument in \([12, p. 80]\), one can easily obtain the desired estimates. We omit the details. \( \square \)

Now we are ready to prove Theorem 2.12. We only prove the case (b) because (a) is its special case.

Step 1) Set
\[ \omega(R) := \sup_{x \in \mathbb{R}^d} \sup_{y \in B_r(x)} \int_{B_r(x)} \left| A^{\alpha \beta}(y_1, y') - \int_{B_r'(x')} A^{\alpha \beta}(y_1, z') dz' \right| dy, \]
where \( A^{\alpha \beta} \) are coefficients of \( \mathcal{L} \). Assume
\[ \omega(R_0) \leq \gamma < 1 \]
where \( \gamma \) is a positive constant to be chosen later. Let \((u, p) \in W^1(B_R(x^0)) \times L^2(B_R(x^0))\) satisfy for \(0 < R \leq R_0\)
\[ \begin{cases} \mathcal{L} u + \nabla p = 0 \quad \text{in } B_R(x^0), \\ \text{div } u = 0 \quad \text{in } B_R(x^0). \end{cases} \]

Step 2) Let \( y = (y_1, y') \) and \( B_{2r}(y) \subseteq B_R(x^0) \). We denote
\[ \mathcal{L}_0 u = -D_\alpha (A^{\alpha \beta}_0 D_\beta u), \]
where
\[ A^{\alpha \beta}_0 = A^{\alpha \beta}_0(x_1) = \int_{B_{2r}'(y')} A^{\alpha \beta}(x_1, z') dz'. \]
Step 3) Since \( \|Du\|_{L^2(B_r(y))} \leq d,\lambda \|\beta A_0 - A_0\beta\|_{L^2(B_r(y))} \), there exists a unique pair \((u_1,p_1) \in W^1_2(B_r(y)) \times L^2(B_r(y))\) satisfying \( \int_{B_r(y)} p_1 \, dx = 0 \) and
\[
\begin{align*}
L_0 u_1 + \nabla p_1 &= -L u + L_0 u \quad \text{in } B_r(y), \\
\text{div } u_1 &= 0 \quad \text{in } B_r(y).
\end{align*}
\]

Moreover, we have the following \( L^2 \)-estimate:
\[
\|Du_1\|_{L^2(B_r(y))} \lesssim d,\lambda \|\beta A_0 - A_0\beta\|_{L^2(B_r(y))}. \tag{6.9}
\]

By the reverse Hölder inequality (see Lemma 8.2), there exists a constant \( q_0 = q_0(d,\lambda) > 2 \)

\[
\left( \int_{B_r(y)} |Du|^{q_0} \, dx \right)^{1/q_0} \lesssim d,\lambda \left( \int_{B_r(y)} |Du|^2 \, dx \right)^{1/2}. \tag{6.10}
\]

Applying Hölder’s inequality and (6.10) to (6.9), we have
\[
\|Du_1\|_{L^2(B_r(y))} \lesssim d,\lambda r^{\frac{2}{2}} \left( \int_{B_r(y)} |A^{\alpha\beta} - A_0^{\alpha\beta}|^{\frac{2q_0}{q_0-2}} \, dx \right)^{\frac{q_0-2}{q_0}} \left( \int_{B_r(y)} |Du|^{q_0} \, dx \right)^{\frac{1}{q_0}}.
\]

\[
\lesssim d,\lambda r^{\frac{2}{2}} \left( \int_{B_r(y)} |Du|^2 \, dx \right)^{\frac{1}{q_0}} \lesssim d,\lambda \gamma \|Du\|_{L^2(B_{2r}(y))}. \tag{6.11}
\]

Step 3) Since \((u_2,p_2) := (u - u_1, p - p_1)\) satisfies
\[
\begin{align*}
L_0 u_2 + \nabla p_2 &= 0 \quad \text{in } B_r(y), \\
\text{div } u_2 &= 0 \quad \text{in } B_r(y),
\end{align*}
\]
Corollary 6.2 implies that for \( 0 < \rho < r \)
\[
\|Du_2\|_{L^2(B_r(y))} \lesssim \left( \frac{\rho}{r} \right)^{d/2} \|Du_2\|_{L^2(B_r(y))}.
\]

Thus, from (6.11), we get
\[
\|Du\|_{L^2(B_r(y))} \leq \|Du_1\|_{L^2(B_r(y))} + \|Du_2\|_{L^2(B_r(y))}
\lesssim d,\lambda \left( \left( \frac{\rho}{r} \right)^{d/2} + \gamma \right) \|Du\|_{L^2(B_{2r}(y))}. \tag{6.12}
\]

We note that it is trivially hold for \( \rho \in [r,2r] \) and \( B_{2r}(y) \subseteq B_R(x^0) \). Let \( B_r(y) \subseteq B_R(x^0) \) and \( \alpha_0 \in (0,1) \). We can take \( \gamma = \tau^{d/2} \) and choose a sufficiently small \( \tau(d,\lambda,\alpha_0) \in (0,1) \) so that
\[
\|Du\|_{L^2(B_r(y))} \leq \tau^{\frac{d}{2} - 1 + \alpha_0} \|Du\|_{L^2(B_r(y))}.
\]

Hence, by an iteration, we obtain that for \( 0 < \rho < r \)
\[
\|Du\|_{L^2(B_r(y))} \lesssim d,\lambda,\alpha_0 \left( \frac{\rho}{r} \right)^{\frac{d}{2} - 1 + \alpha_0} \|Du\|_{L^2(B_r(y))}. \tag{6.13}
\]
Step 1) From (6.13) we have for \( y \in B_{R/4}(x^0) \) and \( \rho \in (0, R/4) \)
\[
\| Du \|_{L^2(B_{\rho}(y))} \lesssim_{d, \lambda, \alpha_0} \left( \frac{\rho}{R} \right)^{\frac{d}{2} - 1 + \alpha_0} \| Du \|_{L^2(B_{R/4}(y))}
\]
\[
\lesssim_{d, \lambda, \alpha_0} \left( \frac{\rho}{R} \right)^{\frac{d}{2} - 1 + \alpha_0} \| Du \|_{L^2(B_{R/2}(x^0))}.
\]
From (3.3), we get
\[
\| Du \|_{L^2(B_{R}(y))} \lesssim_{d, \lambda, \alpha_0} \frac{\rho^{\frac{d}{2} - 1 + \alpha_0} R^{\frac{d}{2} + \alpha_0}}{R^{d + \alpha_0}} \| u \|_{L^2(B_{R}(x^0))}.
\]
Therefore, the Morrey-Campanato theorem yields
\[
[u]_{C^{\alpha_0}(B_{R/4}(x^0))} \leq CR^{-\frac{d}{2} - \alpha_0} \| u \|_{L^2(B_{R}(x^0))}.
\]
Finally, a standard covering argument yields
\[
[u]_{C^{\alpha_0}(B_{R/2}(x^0))} \lesssim_{d, \lambda, \alpha_0} R^{-\alpha_0} \left( \int_{B_{R}(x^0)} |u|^2 \, dx \right)^{1/2}, \quad 0 < R \leq R_0.
\]
This completes the proof of Theorem 2.12.

7. Proof of Theorem 2.14

The proof of the estimate (2.12) is a modification of the argument for elliptic systems found in Kang–Kim [17, Theorem 3.3]. We divide the proof into several steps.

Step 1) Let \( x, y \in \mathbb{R}^d_+ \) and \( 0 < R := |x - y| \leq \min\{R_0, R_1\} \). We note that \((V^k(\cdot, y), \Pi^k(\cdot, y))\) satisfies
\[
\begin{cases}
L V^k(\cdot, y) + \nabla \Pi^k(\cdot, y) = 0 & \text{in } \mathbb{R}^d_+ \cap B_{R/2}(x), \\
\text{div } V^k(\cdot, y) = 0 & \text{in } \mathbb{R}^d_+ \cap B_{R/2}(x), \\
V^k(\cdot, y) = 0 & \text{on } \partial \mathbb{R}^d_+.
\end{cases}
\]
If \( d_x > R/8 \), then since \( B_{R/8}(x) \subset \mathbb{R}^d_+ \), by Lemma 3.6 (a), we have
\[
|V^k(x, y)| \lesssim_{d, C_0, \alpha_0} R^{-d} \| V^k(\cdot, y) \|_{L^1(B_{R/8}(x))}
\]
\[
\lesssim_{d, C_0, \alpha_0} R^{-d} \| V^k(\cdot, y) \|_{L^1(\mathbb{R}^d_+ \cap B_{R/2}(x))}. \tag{7.1}
\]
If \( d_x \leq R/8 \), then we take \( x^0 \in \partial \mathbb{R}^d_+ \) satisfying \( \text{dist}(x, \partial \mathbb{R}^d_+) = |x - x^0| \) so that \( x \in B_{3R/16}^+(x^0) \subset B_{3R/8}^+(x^0) \subset (\mathbb{R}^d_+ \cap B_{R/2}(x)) \).

By Lemma 3.6 (b)
\[
|V^k(x, y)| \lesssim_{d, C_0, \alpha_0, C_1} R^{-d} \| V^k(\cdot, y) \|_{L^1(B_{R/8}^+(x^0))}
\]
\[
\lesssim_{d, C_0, \alpha_0, C_1} R^{-d} \| V^k(\cdot, y) \|_{L^1(\mathbb{R}^d_+ \cap B_{R/2}(x))}. \tag{7.2}
\]
Combining (7.1) and (7.2), we obtain
\[
|V(x, y)| \lesssim_{d, C_0, \alpha_0, C_1} R^{-d} \| V(\cdot, y) \|_{L^1(\mathbb{R}^d_+ \cap B_{R/2}(x))}. \tag{7.3}
\]
Step 2) We now prove the estimate (2.12). Let \( x, y \in \mathbb{R}_+^d \) and \( 0 < R := |x - y| \leq \min\{R_0, R_1\} \). If \((u, p) \in Y^1_2(\mathbb{R}_+^d) \times L_2(\mathbb{R}_+^d)\) satisfies

\[
\begin{aligned}
\mathcal{L}^* u + \nabla p &= f \quad \text{in} \; \mathbb{R}_+^d, \\
\text{div} u &= 0 \quad \text{in} \; \mathbb{R}_+^d,
\end{aligned}
\]

where \( f \in L_\infty(\mathbb{R}_+^d) \) with \( \text{supp } f \subset (\mathbb{R}_+^d \cap B_{R/2}(x)) \), then by the condition (c) in Definition 2.3, we have

\[
u(y) = \int_{\mathbb{R}_+^d \cap B_{R/2}(x)} V(z, y)^{1/2} f(z) \, dz.
\]

Moreover, since

\[
\begin{aligned}
\mathcal{L}^* u + \nabla p &= 0 \quad \text{in} \; \mathbb{R}_+^d \cap B_{R/2}(y), \\
\text{div} u &= 0 \quad \text{in} \; \mathbb{R}_+^d \cap B_{R/2}(y), \\
u &= 0 \quad \text{on} \; \partial \mathbb{R}_+^d,
\end{aligned}
\]

we obtain that (see (7.3))

\[
\|u\|_{L_\infty(B_{R/16}(y))} \lesssim R^{-d} \|u\|_{L_1(\mathbb{R}_+^d \cap B_{R/2}(y))}.
\]

From this together with (5.1), we get

\[
\|u\|_{L_\infty(B_{R/16}(y))} \lesssim d, \lambda, C_0, \alpha, C_1 R_1^{-d/2} \|u\|_{L_2(d)(\mathbb{R}_+^d \cap B_{R/2}(y))} \lesssim d, \lambda, C_0, \alpha, C_1 R^2 \|f\|_{L_\infty(\mathbb{R}_+^d \cap B_{R/2}(y))}.
\]

Combining this with (7.4), and then using the duality argument, we obtain

\[
\|V(\cdot, y)\|_{L_1(\mathbb{R}_+^d \cap B_{R/2}(z))} \lesssim R^2,
\]

which together with (7.3) implies the desired estimate (2.12).

Step 3) To show estimates i) - v) in Theorem 2.14 due to Corollary 2.8, we may consider only the case \( y \in \mathbb{R}_+^d \) and

\[
16d_y \leq R \leq \min\{R_0, R_1\}.
\]

Take \( y^0 \in \partial \mathbb{R}_+^d \) satisfying dist\((y, \partial \mathbb{R}_+^d) = |y - y^0|\). Then

\[
(\mathbb{R}_+^d \cap B_{R/16}(y)) \subset B_{R/16}(y^0) \subset B_{R/2}(y^0) \subset (\mathbb{R}_+^d \setminus B_R(y)).
\]

Let \( \eta \) be a smooth functions on \( \mathbb{R}_+^d \) satisfying

\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \; \text{on} \; B_{R/4}(y^0), \; \text{supp } \eta \subset B_{R/2}(y^0), \; |D\eta| \lesssim R^{-1}.
\]

Like the estimate (4.8), we have

\[
\left| \int_{\mathbb{R}_+^d} \Pi^k(\cdot, y) \, \text{div} \left( (1 - \eta^2) V^{*k}(\cdot, y) \right) \, dx \right| \lesssim \int_{D^+} |\Pi^k(\cdot, y) - (\Pi^k(\cdot, y))_{D^+}|^2 \, dx + R^{-2} \int_{D^+} |DV^{*k}(\cdot, y)|^2 \, dx,
\]

where \( D^+ = B_{R/2}(y^0) \setminus B_{R/4}(y^0) \). Like the estimate (4.7), we have, by using Lemma 3.3 (b),

\[
\int_{\mathbb{R}_+^d} (1 - \eta^2)|DV^{*k}(\cdot, y)|^2 \, dx \lesssim R^{-2} \int_{D^+} |V^{*k}(\cdot, y)|^2 \, dx,
\]

(7.5)
where $D_0^+ = B_{5R/s}(y^0) \setminus B_R(y^0)$. Since
\[ |x - y| \leq \frac{5R}{8}, \quad \forall x \in D_0^+, \]
we apply (2.12) to (7.5) and then follow the same steps used in the proof of (1.5), we obtain the estimate i). The proof of ii) and iii) are the same as that of Lemma 4.3.

We shall sketch the proof of iv), which is similar to the proof of Lemma 4.3. Let $\varphi \in \dot{Y}_2^1(\mathbb{R}^d)$ be a solution to the divergence equation
\[ \text{div } \varphi = \Pi^k(\cdot, y)I_{\mathbb{R}^d \setminus B_{R/2}(y^0)} \quad \text{in } \mathbb{R}^d_+, \]
satisfying
\[ \|\varphi\|_{\dot{Y}_2^1(\mathbb{R}^d_+)} \lesssim \|\Pi^k(\cdot, y)\|_{L_4(\mathbb{R}^d_+ \setminus B_{R/2}(y^0))}. \]

Using $(1 - \eta)\varphi$ as a test function, we obtain
\[ \int_{\mathbb{R}^d_+ \setminus B_{R/2}(y^0)} |\Pi^k(\cdot, y)|^2 dx \lesssim \int_{D_+} |\Pi^k(\cdot, y) - (\Pi^k(\cdot, y))_{D^+}|^2 dx + \int_{\mathbb{R}^d_+ \setminus B_{R/2}(y^0)} |DV^k(\cdot, y)|^2 dx. \quad (7.6) \]

Since
\[ \mathcal{L}V_{\varepsilon}^k(\cdot, y) + \nabla \Pi^k(\cdot, y) = 0 \quad \text{in } D^+, \]
it follows from Lemma 3.4 that
\[ \int_{D_+} |\Pi^k(\cdot, y) - (\Pi^k(\cdot, y))_{D^+}|^2 dx \lesssim \int_{D_+} |DV_{\varepsilon}^k(\cdot, y)| dx. \quad (7.7) \]

Note that $D^+ \subset (\mathbb{R}^d_+ \setminus B_{R/2}(y^0))$. Combining (7.6) and (7.7) we obtain
\[ \int_{\mathbb{R}^d_+ \setminus B_{R/2}(y^0)} |\Pi^k(\cdot, y)|^2 dx \lesssim \int_{\mathbb{R}^d_+ \setminus B_{R/2}(y^0)} |DV_{\varepsilon}^k(\cdot, y)|^2 dx. \]

Thus, the desired estimate iv) follows from i). We omit the proof of v) because it is very similar.

This completes the proof of Theorem 2.15.

8. PROOF OF THEOREM 2.15

**Lemma 8.1.** Let $\mathcal{L}_0$ be the operator in (6.1) and let $0 < r < R$. If $(u, p) \in W^1_2(B_R^+) \times L_2(B_R^+)$ satisfies
\[
\begin{cases}
\mathcal{L}_0 u + \nabla p = 0 & \text{in } B_R^+,

\text{div } u = 0 & \text{in } B_R^+,

u = 0 & \text{on } B_R \cap \partial \mathbb{R}^d_+,
\end{cases}
\]
then
\[ \|Du\|_{L_\infty(B_R^+)} \lesssim_{d, \lambda} (R - r)^{-d/2} \|Du\|_{L_2(B_R^+)} \quad (8.1) \]
and
\[ \|p\|_{L_\infty(B_R^+)} \lesssim_{d, \lambda} (R - r)^{-d/2} \left( \|Du\|_{L_2(B_R^+)} + \|p\|_{L_2(B_R^+)} \right). \quad (8.2) \]
Proof. Using [9, Lemma 4.4] and repeating the same arguments in the proofs of Lemma 6.1 and Corollary 6.2, one can easily show that the estimates (8.1) and (8.2) hold. We omit the details.

We note that the following lemma is well known (see, for instance, [13]). We present that for the sake of completeness.

Lemma 8.2 (Reverse Hölder inequality). Let \( \Omega_R(x^0) = \mathbb{R}^d_+ \cap B_R(x^0) \) with \( x^0 \in \mathbb{R}^d_+ \) and \( R > 0 \). If \((u,p) \in W^1_2(\Omega_R(x^0)) \times L^2(\Omega_R(x^0))\) satisfies

\[
\begin{align*}
\mathcal{L}u + \nabla p &= 0 \quad \text{in } \Omega_R(x^0), \\
\text{div } u &= 0 \quad \text{in } \Omega_R(x^0), \\
u &= 0 \quad \text{on } B_R(x^0) \cap \partial \mathbb{R}^d_+,
\end{align*}
\]

then there exists a constant \( q_0 = q_0(d,\lambda) > 2 \) such that

\[
\left( \int_{\Omega_{R/2}(x^0)} |Du|^q_0 \, dx \right)^{1/q_0} \lesssim_{d,\lambda} \left( \int_{\Omega_R(x^0)} |Du|^2 \, dx \right)^{1/2}. \tag{8.3}
\]

Proof. Throughout the proof, we regard \( u \) as a function in \( W^1_2(B_R) \) by setting \( u = 0 \) in \( B_R \setminus B_R^r \). Set \( q_1 = 2d/(d+2) \) and \( U = |Du|^{q_1} \). We claim that for any \( y \in B_R(x^0), 0 < r \leq \text{dist}(y, \partial B_R(x^0)) \), and \( 0 < \delta < 1 \), we have

\[
\int_{B_r(y)} U^{2/q_1} \, dx \leq \delta \int_{B_r(y)} U^{2/q_1} \, dx + C(d,\lambda,\delta) \left( \int_{B_r(y)} U \, dx \right)^{2/q_1}. \tag{8.4}
\]

Let \( y \in B_R(x^0) \) and \( 0 < r \leq \text{dist}(y, \partial B_R(x^0)) \). We consider two cases when \( r/6 \leq \text{dist}(y, \partial \mathbb{R}^d_+) \) and \( r/6 \geq \text{dist}(y, \partial \mathbb{R}^d_+) \). Assume that \( r/6 \leq \text{dist}(y, \partial \mathbb{R}^d_+) \).

Since it holds that

\[
\begin{align*}
\mathcal{L}(u - (u)_{B_r/6}(y)) + \nabla p &= 0 \quad \text{in } B_r/6(y), \\
\text{div } (u - (u)_{B_r/6}(y)) &= 0 \quad \text{in } B_r/6(y),
\end{align*}
\]

by Lemma 3.5 Hölder’s inequality, and Poincaré’s inequality, we have

\[
\|Du\|_{L^2(B_r/12(y))} \lesssim r^{-1} \left\| u - (u)_{B_r/6}(y) \right\|_{L^2(B_r/6(y))} \lesssim r^{-1} \left\| u - (u)_{B_r/6}(y) \right\|_{L^{2d/(d-2)}(B_r/6(y))} \left\| u - (u)_{B_r/6}(y) \right\|_{L^{q_1}(B_r/6(y))} \lesssim \|Du\|_{L^2(B_r(y))} \|Du\|_{L^{q_1}(B_r(y))}.
\]

Using this together with Young’s inequality, we obtain the estimate (8.4). If \( r/6 \geq \text{dist}(y, \partial \mathbb{R}^d_+) \), then we take \( y^0 \in \partial \mathbb{R}^d_+ \cap B_r/6(y) \) satisfying \( \text{dist}(y^0, \partial \mathbb{R}^d_+) = |y - y^0| \).

Since \( B_{r/6}(y) \subset B_{r/3}(y^0) \subset B_{2r/3}(y^0) \subset B_r(y) \),

Then by Lemma 3.5 Hölder’s inequality, and Poincaré’s inequality (see, for instance, [14, Eq. (7.45), p. 164]), we have

\[
\|Du\|_{L^2(B_r/6(y))} \leq \|Du\|_{L^2(B_{2r/3}(y^0))} \lesssim r^{-1} \left\| u \right\|_{L^2(B_{2r/3}(y^0))} \lesssim r^{-1} \left\| u \right\|_{L^{2d/(d-2)}(B_{2r/3}(y^0))} \left\| u \right\|_{L^{q_1}(B_{2r/3}(y^0))} \lesssim \|Du\|_{L^2(B_r(y))} \|Du\|_{L^{q_1}(B_r(y))}.
\]

Using this together with Young’s inequality, we obtain the estimate (8.4).
We are now ready to prove the lemma. By (8.4) and a standard covering argument, we see that

$$\int_{B_{r/2}(y)} U^{2/q} \, dx \leq \frac{1}{2} \int_{B_r(y)} U^{2/q} \, dx + C(d, \lambda) \left( \int_{B_r(y)} U \, dx \right)^{2/q},$$

for any $B_r(y) \subset B_R(x^0)$. Therefore, applying a version of Gehring’s lemma (see, for instance, [8, Lemma 4.5]) and using the definition of $B$ for any $\rho \in (0, R)$, we see that (8.5) and (8.6), one can easily obtain that (8.5). We only prove the case (b) of Theorem 2.15 because (a) is its special case. We are now ready to prove the lemma. By (8.4) and a standard covering argument, we have

$$\|Du\|_{L^2(B_{r/2}(y))} \lesssim \left( \frac{\rho}{r} \right)^{d/2} + \gamma \|Du\|_{L^2(B_r(y))}$$

for any $0 < \rho \leq 2r$. Similar to (6.12), we also have

$$\|Du\|_{L^2(B_r(y))} \lesssim \left( \frac{\rho}{r} \right)^{d/2} + \gamma \|Du\|_{L^2(B_r(y))}$$

for any $B_r(y) \subset B_R(x^0)$ and $0 < \rho \leq r$.

Now we extend $u$ to $B_R(x^0)$ by setting $u \equiv 0$ on $B_R(x^0) \setminus B_R^+(x^0)$. Then by (8.5) and (8.6), one can easily obtain that

$$\|Du\|_{L^2(B_r(y))} \lesssim \left( \frac{\rho}{r} \right)^{d/2} + \gamma \|Du\|_{L^2(B_r(y))}$$

for any $B_r(y) \subset B_R(x^0)$ and $0 < \rho < r$. Exactly the same steps as in the proof of Theorem 2.12 yield the estimate 2.11. This completes the proof of Theorem 2.15.

9. Proof of Theorem 2.18

We mainly follow the proof in Kang–Kim [17, Theorem 3.13]. For $x \in \mathbb{R}^d_+$ and $R \leq R_2$, we denote $\Omega_r(x) = \mathbb{R}^d_+ \cap B_r(x)$.

**Step 1** Assume that $(u, p) \in W^1_2(\Omega_R(x^0)) \times L^2(\Omega_R(x^0))$ satisfies (2.13), where $x^0 \in \mathbb{R}^d_+$ and $0 < R \leq R_2$ satisfying $d_{x_0} < R/4$. Using Assumption 2.16 and the Poincaré inequality, we have

$$\left[ u \chi_{\Omega_R(x^0)} \right]_{C^{\alpha_2}(B_{R/2}(x^0))} \lesssim R^{-\alpha_2} \left( \int_{\Omega_R(x^0)} |u|^2 \, dx \right)^{1/2} \lesssim R^{1-d/2-\alpha_2} \left( \int_{\Omega_R(x^0)} |Du|^2 \, dx \right)^{1/2}.$$
we have
\[ |u(x)| \lesssim (d_x)^{\alpha_2} R^{1-d/2-\alpha_2} \|Du\|_{L^2(\Omega_R(x^0))} \]  
for \( x^0 \in \mathbb{R}^d_+ \) and \( 0 < R \leq R_2 \) satisfying \( d_{x^0} < R/4 \).

**Step 2)** In this step, we first claim that
\[ |V(x, y)| \lesssim \min\{d_x, |x - y|\} \alpha_2 |x - y|^{2-d-\alpha_2} \]  
for any \( x, y \in \mathbb{R}^d_+ \) satisfying \( 0 < |x - y| < R_2 \). Due to (9.2), it suffices to show that
\[ |V(x, y)| \lesssim (d_x)^{\alpha_2} |x - y|^{2-d-\alpha_2} \]  
if \( 4d_x < R := \frac{|x - y|}{2} \). (9.3)

By (9.1), we have
\[ |V(x, y)| \lesssim (d_x)^{\alpha_2} R^{1-d/2-\alpha_2} \|DV(\cdot, y)\|_{L^2(\Omega_R(x))}. \]
Using this together with the estimate iii) in Theorem 2.14, we have
\[ |V(x, y)| \lesssim (d_x)^{\alpha_2} R^{1-d/2-\alpha_2} \|DV(\cdot, y)\|_{L^2(\mathbb{R}^d_+ \setminus B_R(y))} \lesssim (d_x)^{\alpha_2} R^{2-d-\alpha_2}, \]
which gives the estimate (9.3).

Next, we claim that
\[ |V(x, y)| \lesssim \min\{d_x, |x - y|\} \alpha_2 \min\{d_y, |x - y|\} \alpha_2 |x - y|^{2-d-2\alpha_2} \]  
for any \( x, y \in \mathbb{R}^d_+ \) satisfying \( 0 < |x - y| < R_2/2 \). We may assume that \( 4d_y < R := |x - y|/4 \) to prove (9.4) because otherwise would follow from (9.2). Using Corollary 2.10, and Caccioppoli’s inequality (see, for instance, Lemma 3.5 (b)), we have
\[ |V(x, y)| \lesssim (d_y)^{\alpha_2} R^{1-d/2-\alpha_2} \|D^*V(\cdot, x)\|_{L^2(\Omega_R(y))} \lesssim (d_y)^{\alpha_2} R^{2-d/2-\alpha_2} \|V(\cdot, x)\|_{L^2(\Omega_2R(y))}. \]  
(9.5)

Since it holds that
\[ 2R < |x - z| < 6R \]  
for all \( z \in \Omega_{2R}(y) \), we obtain by (9.2) and (9.3) that
\[ |V(x, y)| \lesssim (d_y)^{\alpha_2} \min\{d_x, |x - y|\} \alpha_2 R^{2-d-2\alpha_2}. \]

**Step 3)** To prove the estimate (2.14), it suffices to show that
\[ |V(x, y)| \lesssim \min\{d_x, R_2\} \alpha_2 \min\{d_y, R_2\} \alpha_2 R_2^{2-d-2\alpha_2} \]  
for any \( x, y \in \mathbb{R}^d_+ \) satisfying \( |x - y| \geq R_2/2 \). Set \( R = R_2/4 \). Note that \( (V(\cdot, y), \Pi(\cdot, y)) \) satisfies
\[ \begin{cases} \mathcal{L}u + \nabla p = 0, & \text{div} u = 0 \quad \text{in} \ \Omega_R(x), \\ u = 0 \quad \text{on} \ \partial\mathbb{R}^d_+. \end{cases} \]
From Lemma 3.6 and i) in Theorem 2.14 it follows that
\[ |V(x, y)| \lesssim R^{-d} \|V(\cdot, y)\|_{L^1(\Omega_R(x))} \lesssim R^{(2-d)/2} \|V(\cdot, y)\|_{L^2(\mathbb{R}^d_+ \setminus B_R(y))} \lesssim R_2^{2-d}. \]

By utilizing the above inequality, and following the same steps used in deriving (9.4), we concluded the estimate (9.6).

This completes the proof of Theorem 2.18.
10. Green functions on unbounded domains

In this section we consider the existence of the Green function for the Stokes system on a domain $\Omega$ with $|\Omega| = \infty$. We impose the following assumption on $\Omega$ in Theorem 10.4 below.

**Assumption 10.1.** There exists a constant $C_3 > 0$ such that the following holds: for any $g \in L_2(\Omega)$, there exists $u \in Y_2^1(\Omega)^d$ satisfying

$$\text{div } u = g \quad \text{in } \Omega, \quad \|Du\|_{L_2(\Omega)} \leq C_3\|g\|_{L_2(\Omega)}.$$

**Remark 10.2.** Below are some examples of cases when Assumption 10.1 holds.

(i) $\Omega$ is the whole space or half space. More generally,

$$\Omega = \{x \in \mathbb{R}^d : x_1 > 0, x_2 > 0, \text{ or } x_d > 0\}.$$

(ii) $\Omega$ is a locally Lipschitz and exterior domain (see [11, Theorem III.3.6]).

**Remark 10.3.** Note that if $\Omega$ is a domain in $\mathbb{R}^d$, $d \geq 3$, with $|\Omega| = \infty$, then under Assumption 10.1, we obtain the $L_2$-solvability of the Stokes systems (with measurable coefficients) and the estimate (2.1).

Under Assumptions 2.5 and 10.1 using Remark 10.3 and repeating the same arguments in the proof of Theorem 2.7, one can prove the existence of the Green function on $\Omega$. We think it is worth to present the precise statement. We denote $d_x = \text{dist}(x, \partial \Omega)$ for $x \in \Omega$.

**Theorem 10.4.** Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 3$, with $|\Omega| = \infty$. If Assumptions 2.5 and 10.1 hold, then there exists a unique Green function $((V(x,y), \Pi(x,y))$ for the Stokes operator on $\Omega$. Moreover, for any $x, y \in \Omega$ satisfying $0 < |x - y| \leq \min\{d_x, d_y, R_0\}$, we have

$$|V(x,y)| \lesssim d, \lambda, C_0, \alpha_0, C_3 |x - y|^{2-d}.$$ 

Furthermore, the Green function satisfies the representation formulas (2.7) and (2.8), and it also satisfies the estimates i) – v) in Corollary 2.8.

By modifying the proof of (2.12), one can prove the following pointwise bound.

**Theorem 10.5.** Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 3$, with $|\Omega| = \infty$. Suppose that Assumptions 2.5 and 10.1 hold. Let $(V(x,y), \Pi(x,y))$ be the Green function constructed in Theorem 10.4. If Assumption 2.7 holds with $\Omega$ in place of $\mathbb{R}^d_+$, respectively, then for any $x, y \in \Omega$ satisfying $0 < |x - y| \leq \min\{R_0, R_1\}$, we have

$$|V(x,y)| \lesssim d, \lambda, C_0, \alpha_0, C_3, C_1 |x - y|^{2-d}.$$ 

We note that Caccioppoli’s inequality holds for the Stokes system on a Lipschitz domain. Then by following the proof of Theorem 2.13, we obtain the following estimate.

**Theorem 10.6.** Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 3$, with $|\Omega| = \infty$. Suppose that $\Omega$ has a Lipschitz boundary with a bounded Lipschitz constant. If Assumption 10.1 holds, and if Assumption 2.7 holds with $\Omega$ in place of $\mathbb{R}^d_+$, then for any $x, y \in \Omega$ with $x \neq y$,

$$|V(x,y)| \leq C \min\{d_x, |x - y|, R_2\}^{\alpha_2} \min\{d_y, |x - y|, R_2\}^{\alpha_2} \min\{|x - y|, R_2\}^{2-d-2\alpha_2},$$

where $C = C(d, \lambda, C_2, \alpha_2, C_3, \Omega)$. 

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