Spectral estimates for Schrödinger operator with periodic matrix potentials on the real line

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Abstract

We consider the Schrödinger operator on the real line with a $N \times N$ matrix valued periodic potential, $N > 1$. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define the Lyapunov function, which is analytic on an associated $N$-sheeted Riemann surface. On each sheet the Lyapunov function has the standard properties of the Lyapunov function for the scalar case. The Lyapunov function has (real or complex) branch points, which we call resonances. We determine the asymptotics of the periodic, anti-periodic spectrum and of the resonances at high energy (in terms of the Fourier coefficients of the potential). We show that there exist two types of gaps: i) stable gaps, i.e., the endpoints are periodic and anti-periodic eigenvalues, ii) unstable (resonance) gaps, i.e., the endpoints are resonances (real branch points). Moreover, the following results are obtained: 1) we define the quasimomentum as an analytic function on the Riemann surface of the Lyapunov function; various properties and estimates of the quasimomentum are obtained, 2) we construct the conformal mapping with real part given by the integrated density of states and imaginary part given by the Lyapunov exponent. We obtain various properties of this conformal mapping, which are similar to the case $N=1$, 3) we determine various new trace formulae for potentials, the integrated density of states and the Lyapunov exponent, 4) a priori estimates of gap lengths in terms of potentials are obtained.

1 Introduction and main results

We consider the self-adjoint operator $\mathcal{L} y = -y'' + V(t)y$, acting in $L^2(\mathbb{R})^N$, $N \geq 2$ where the symmetric 1-periodic $N \times N$ matrix potential $V$ belongs to the real Hilbert space $\mathcal{H}$ given by

$$\mathcal{H} = \left\{ V = V^\ast = \{V_{jk}(t)\}_{j,k=1}^N, \ t \in \mathbb{R}/\mathbb{Z}, \ \|V\|^2 = \int_0^1 \text{Tr} V^2(t)dt < \infty \right\}.$$
It is well known (see [DS]) that the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is absolutely continuous and consists of non-degenerated intervals $[\lambda_n^- , \lambda_n^+]$, $n = 1, \ldots, N_G \leq \infty$. These intervals are separated by the gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$ with the length $> 0$. **Without loss of generality we assume**

$$\lambda_0^- = 0, \quad \text{and} \quad V^0 = \int_0^1 V(t)dt = \text{diag}\{V_1^0, V_2^0, \ldots, V_N^0\}, \quad V_1^0 \leq V_2^0 \leq \ldots \leq V_N^0. \quad (1.1)$$

A great number of papers is devoted to the inverse spectral theory for the Hill operator. We mention all papers where the inverse problem including characterization was solved: Marchenko and Ostrovski [MO1], Garnett and Trubowitz [GT1-2], Kappeler [Kap], Kargaev and Korotyaev [KK1], and Korotyaev [K1-3] and for 2 $\times$ 2 Dirac operator Misura [Mi1-2] and Korotyaev [K4-5]. Recently, the author [K6] extended the results of [MO1], [GT1], [K1-2] for the case $-y'' + uy$ to the case of distributions, i.e. $-y'' + u'y$ on $L^2(\mathbb{R})$, where periodic $u \in L^2_{\text{loc}}(\mathbb{R})$. It is important that in these papers new results from analytic function theory (in particular, conformal mapping theory) were obtained. As an example, we mention the proofs by the direct method (see [GT2], [KK1], [K1-3]). These are short, but this approach needs a priori estimates of potential in terms of spectral data. A priori estimates for various parameters of the Hill operator and for the Dirac operator (the norm of a periodic potential, effective masses, gap lengths, height of slits, and so on) were obtained in [GT1], [MO1-2], [KK1], [KK2], [K2-11]. In order to get the required estimates the authors of [GT1], [MO1-2], [Mi1-2], [K2-11]... used the "global quasi-momentum" (the conformal mapping), which was introduced into the spectral theory of the Hill operator by Marchenko-Ostrovski [MO1].

There exist many papers about the periodic systems $N \geq 2$ (see [Ca1-3], [YS]). The basic results for direct spectral theory for the matrix case were obtained by Lyapunov [Ly] (see also interesting papers of Krein [Kr], Gel’fand and Lidskii [GL]). In [BBK] for the case $N = 2$ the following results are obtained: the Lyapunov function is constructed on the 2-sheeted Riemann surface and the existence of real and complex branch points is proved. In [BK] the operator $y''' + qy$, where $q$ is a periodic real potential, was studied. In this case the Lyapunov function is constructed on a 2-sheeted Riemann surface and the existence of real and complex branch points is proved. The asymptotics of gaps and resonances in terms of the Fourier coefficients are obtained.

The main goal of our paper is to reformulate some spectral problem for the differential operator with periodic matrix coefficients as problems of conformal mapping theory. We construct the conformal mapping (averaged quasimomentum) $w$, with real part given by the integrated density of states and imaginary part given by the Lyapunov exponent. We obtain various properties of this conformal mapping, which are similar to the case $N = 1$. For solving these "new" problems we use some techniques from [KK2], [K2], [K6-8]. In particular, we use the Poisson integral for the domain $\mathbb{C}_+ \cup (-1, 1) \cup \mathbb{C}_-$ and the Dirichlet integral for the function $w(z) - z$. Note that the Dirichlet integral was used in [K6-8] for the scalar case to obtain a priori two-sided estimates of the potential in terms of spectral data.

Introduce the fundamental $N \times N$-matrix solutions $\varphi(t, z), \vartheta(t, z)$ of the equation

$$-f'' + V(t)f = z^2f, \quad z \in \mathbb{C}, \quad (1.2)$$

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with the conditions \( \varphi(0, z) = \vartheta'(0, z) = 0, \varphi'(0, z) = \vartheta(0, z) = I_N, \) where \( I_N, N \geq 1 \) is the identity \( N \times N \) matrix. Here and below we use the notation ('') = \( \partial / \partial t \). We define the monodromy \( 2N \times 2N \)-matrix \( M \) and the trace \( T_m, m \geq 1 \) by

\[
M(z) = \mathcal{M}(1, z), \quad \mathcal{M}(t, z) = \begin{pmatrix} \vartheta(t, z) & \varphi(t, z) \\ \vartheta'(t, z) & \varphi'(t, z) \end{pmatrix}, \quad T_m(z) = \frac{\text{Tr} M^m(z)}{2N}.
\] (1.3)

The functions \( M(z) \) and \( T_m, m \geq 1 \) are entire, real for \( z^2 \in \mathbb{R} \) and \( \det M = 1 \). Let \( \tau_m, m = 1, \ldots, 2N \) be the eigenvalues of \( M \). An eigenvalue of \( M(z) \) is called a multiplier. It is a root of the algebraic equation \( D(\tau, z) \equiv \det(M(z) - \tau I_{2N}) = 0, \tau, z \in \mathbb{C} \). The zeros of \( D(1, \sqrt{\lambda}) \) (and \( D(-1, \sqrt{\lambda}) \) (counted with multiplicity) are the periodic (anti-periodic) eigenvalues for the equation \( -y'' + V y = \lambda y \) with periodic (anti-periodic) boundary conditions.

Below we need the following well-known results of Lyapunov (see [YS]).

**Theorem (Lyapunov)** Let \( V \in \mathcal{H} \). Then the following identities are fulfilled:

\[
M^{-1} = -JM^T J = \begin{pmatrix} \varphi(1, \cdot)^T & -\varphi(1, \cdot)^T \\ -\vartheta(1, \cdot)^T & \vartheta(1, \cdot)^T \end{pmatrix}, 
J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \quad (1.4)
\]

\[
D(\tau, \cdot) = \tau^{2N} D(\tau^{-1}, \cdot), \quad \tau \neq 0. \quad (1.5)
\]

If for some \( z \in \mathbb{C} \) (or \( z^2 \in \mathbb{R} \)) \( \tau(z) \) is a multiplier of multiplicity \( d \geq 1 \), then \( \tau^{-1}(z) \) (or \( \tau(z)^{-1} \)) is a multiplier of multiplicity \( d \). Moreover, each \( M(z), z \in \mathbb{C} \), has exactly \( 2N \) multipliers \( \tau^\pm(z), m = 1, \ldots, N \). Furthermore, \( z^2 \in \sigma(V) \) iff \( |\tau_m(z)| = 1 \) for some \( m = 1, \ldots, N \). If \( \tau(z) \) is a simple multiplier and \( |\tau(z)| = 1 \), then \( \tau'(z) \neq 0 \).

It is well known that \( D(\tau, z) = \sum_{m=0}^{2N} \xi_m(z) \tau^{2N-m} \), where the functions \( \xi_m \) are given by

\[
\xi_0 = 1, \quad \xi_1 = -2NT_1, \quad \xi_2 = -\frac{2N}{2} (T_2 + T_1 \xi_1), \quad \ldots, \quad \xi_m = -\frac{2N}{m} \sum_{j=0}^{m-1} T_{m-j} \xi_j, \ldots \quad (1.6)
\]

see [RS]. Using the identity (1.5) we obtain

\[
D(\tau, \cdot) = (\tau^{2N} + 1) + \xi_1(\tau^{2N-1} + \tau) + \ldots + \xi_{N-1}(\tau^{N+1} + \tau^{N-1}) + \xi_N \tau^N. \quad (1.7)
\]

The eigenvalues of \( M(z) \) are the zeros of Eq. \( D(\tau, z) = 0 \). This is an algebraic equation in \( \tau \) of degree \( 2N \). The coefficients \( \xi_m(z) \) are entire in \( z \in \mathbb{C} \). It is well known (see e.g. [Fo],[Sp]) that the roots \( \tau_m(z), m = 1, \ldots, 2N \) constitute one or several branches of one or several analytic functions that have only algebraic singularities in \( \mathbb{C} \). Thus the number of eigenvalues of \( M(z) \) is a constant \( N_e \) with the exception of some special values of \( z \) (see below the definition of a resonance). In general, there is an infinite number of such points on the plane. If the functions \( \tau_m(z), m = 1, \ldots, N \) are all distinct, then \( N_e = 2N \). If some of them are identical, then we get \( N_e < 2N \) and \( M(z) \) is permanently degenerate.

The Riemann surface for the multipliers \( \tau_m(z), m = 1, \ldots, N \) has \( 2N \) sheets, since degree of \( D(\tau, \cdot) = 2N \) see (1.7). If \( N = 1 \), then it has 2 sheets, but the Lyapunov function is entire. Similarly, in the case \( N \geq 2 \) it is more convenient for us to construct the Riemann surface.
for the Lyapunov function, which has $N$ sheets (see Eq. (1.9)). In order to formulate our first result we transform $D(\tau, z)$ to the polynomial $\Phi(\nu, z)$ by

$$
\frac{D(\tau, z)}{(2\tau)^N} = \Phi(\nu, z) = \nu^N + \phi_1(z)\nu^{N-1} + ... + \phi_N(z), \quad \nu = \frac{\tau + \tau^{-1}}{2},
$$

(1.8)

where $\phi_1, \ldots, \phi_N$ are some linear combinations of $\xi_0, \ldots, \xi_N$, see (2.12)-(2.15). In particular, all coefficients $\phi_1(z), \ldots, \phi_N(z)$ are entire functions. Each zero of $\Phi(\nu, z)$ is a Lyapunov function

$$
\Delta_m(z) = \frac{1}{2}(\tau_m(z) + \tau_m^{-1}(z)), \ m = 1, \ldots, N.
$$

Remark. We note that this reduction from the polynomial $D$ with $\deg D = 2N$ to the polynomial $\Phi$ with $\deg \Phi = N$ is crucial for our analysis. It is based on (1.5), which is a consequence of $M$ being a symplectic matrix and on the identity (2.11) for the Chebyshev polynomials.

We need the following preliminary results

**Theorem 1.1.** Let $V \in \mathcal{H}$. Then there exist analytic functions $\tilde{\Delta}_s, s = 1, \ldots, N_0 \leq N$ on the $N_s$-sheeted Riemann surface $\mathcal{R}_s, N_s \geq 1$ having the following properties:

i) There exist disjoint subsets $\omega_s \subset \{1, \ldots, N\}, s = 1, \ldots, N_0, \bigcup \omega_s = \{1, \ldots, N\}$ such that all branches of $\tilde{\Delta}_s, s = 1, 2, \ldots, N_0$ have the form $\Delta_j(z) = \frac{1}{2}(\tau_j(z) + \tau_j^{-1}(z)), \ j \in \omega_s$. Moreover, for any $z, \tau \in \mathbb{C}$ the following identities are fulfilled:

$$
\frac{D(\tau, z)}{(2\tau)^N} = \prod_{1}^{N_0} \Phi_s(\nu, z), \quad \Phi_s(\nu, z) = \prod_{j \in \omega_s} (\nu - \Delta_j(z)), \quad \nu = \frac{\tau + \tau^{-1}}{2}, \quad \tau \neq 0,
$$

(1.9)

where the functions $\Phi_s(\nu, z)$ are entire with respect to $\nu, z \in \mathbb{C}$ and $\Phi_s(\nu, z) \in \mathbb{R}$ for all $\nu, z \in \mathbb{R}$. Moreover, if $\Delta_i = \Delta_j$ for some $i \in \omega_k, j \in \omega_s$, then $\Phi_k = \Phi_s$ and $\tilde{\Delta}_k = \tilde{\Delta}_s$.

ii) (The monotonicity property). Let some branch $\Delta_m, m = 1, \ldots, N$ be real analytic on some interval $Y = (\alpha, \beta) \subset \mathbb{R}$ and $-1 < \Delta_m(z) < 1$ for any $z \in Y$. Then $\Delta_m(z) \neq 0$ for each $z \in Y$.

iii) Each function $\rho_s, s = 1, \ldots, N_0$ given by (1.10) is entire and real on the real line,

$$
\rho = \prod_{1}^{N_0} \rho_s, \quad \rho_s(\cdot) = \prod_{i<j, i,j \in \omega_s} (\Delta_i(\cdot) - \Delta_j(\cdot))^2.
$$

(1.10)

iv) Each gap $\gamma_n = (\lambda_n^-, \lambda_n^+), n \geq 1$ is a bounded interval and $\lambda_n^+ \lambda_n^-$ are either periodic (anti-periodic) eigenvalues or real branch points of $\Delta_m$ (for some $m = 1, \ldots, N$) which are zeros of $\rho$ (below we call such points resonances).

Remark. 1) If $N_0 = N$, then $\mathcal{R}_m = \mathbb{C}$ and each function $\Delta_m, m = 1, \ldots, N$ is entire.

2) We have the following asymptotics (see Sect. 3)

$$
\Delta_m(z) = \cos z + \frac{\sin z}{2z} V_m^0 + O\left(\frac{e^{\sqrt{N}z}}{z^2}\right), \quad m = 1, \ldots, N, \quad |z| \to \infty.
$$

(1.11)
Then firstly, ρ is not a polynomial since ρ is bounded on \(\mathbb{R}\). Secondly, if \(V^0_i \neq V^0_j, i \neq j\), then (1.11) implies \(\Delta_i \neq \Delta_j\).

3) Let the surface \(\mathcal{R} = \cup_1^{N_0} \mathcal{R}_s\) be a union of the disjoint Riemann surfaces \(\mathcal{R}_s\) and let \(\Delta = \{\Delta_s, s = 1, \ldots, N_0\}\) be the corresponding analytic function on \(\mathcal{R}\). Let \(\phi : \mathcal{R} \to \mathbb{C}\) be the projection from the surface \(\mathcal{R}\) into the complex plane. We set \(\zeta \in \mathcal{R}\) and \(z \in \mathbb{C}\).

**Definition.** The number \(z_0\) is a resonance of \(\mathcal{L}\), if \(z_0\) is a zero of \(\rho\) given by (1.10).

Define the real matrices \(V^{sn} = \{V_{jk}^{sn}\}, V^{cn} = \{V_{jk}^{cn}\}\) by

\[
\hat{V}^{(n)} = \{\hat{V}_{jk}^{(n)}\} = \hat{V}^{cn} + i\hat{V}^{sn} = \int_0^1 V(t)e^{i2\pi nt} dt. \tag{1.12}
\]

Denote by \(\lambda_n^{m,\pm}, n \geq 0, m \in \{1, 2, \ldots, N\}\) the eigenvalues of the periodic and anti-periodic problem for the equation \(-f'' + Vf = z^2 f\). The periodic eigenvalues \((n \text{ is even})\) satisfy

\[
0 \leq \lambda^{0,+}_1 \leq \lambda^{0,+}_2 \leq \ldots \leq \lambda^{0,+}_N \leq \lambda^{2,-}_1 \leq \lambda^{2,-}_2 \leq \ldots \leq \lambda^{2,-}_N \leq \lambda^{2,+}_1 \leq \lambda^{2,+}_2 \leq \ldots \leq \lambda^{2,+}_N \leq \lambda^{4,-}_1 \leq \lambda^{4,+}_1 \leq \ldots \tag{1.13}
\]

Recall \(\lambda_0^+ = 0\). The anti-periodic eigenvalues \((n \text{ is odd})\) satisfy

\[
0 \leq \lambda^{-1,-}_1 \leq \lambda^{-1,+}_2 \leq \ldots \leq \lambda^{-1,-}_N \leq \lambda^{-1,+}_1 \leq \lambda^{-3,-}_2 \leq \ldots \leq \lambda^{-3,+}_1 \leq \lambda^{-3,-}_2 \leq \ldots \leq \lambda^{-5,-}_1 \leq \ldots \tag{1.14}
\]

If \(V = 0\), then \(\lambda_n^{m,\pm} = (\pi n)^2, m = 1, \ldots, N\). Let \(z^{n,\pm}_m = \sqrt{\lambda_n^{m,\pm}} > 0\) and \(z^{-n,\pm}_m = -z^{n,\pm}_m, n \geq 0, m \in \{1, 2, \ldots, N\}\). The zeros of \(D(1, z)\) and \(D(-1, z)\) (counted with multiplicity) have the forms \(z^{2n,\pm}_m\) and \(z^{2n+1,\pm}_m\) \(n \in \mathbb{Z}\). Let \(|A|\) denote the operator norm of the matrix \(A\).

**Theorem 1.2.** Let \(V \in \mathcal{H}\). Then the periodic and anti-periodic eigenvalues have the following asymptotics:

\[
\lambda_n^{m,\pm} = (\pi n)^2 + \zeta_n^{m,\pm} + O(n^{-1}), \quad m = 1, \ldots, N, \quad n \to \infty, \tag{1.15}
\]

where \(\zeta_n^{m,\pm}, m = 1, 2, \ldots, N\) are the eigenvalues of the matrix \(\begin{pmatrix} V^0 & \hat{V}^{cn} \\ \hat{V}^{sn} & -V^0 \end{pmatrix}\).

Assume that \(V^0_j \neq V^0_j\) for all \(j \neq j' \in \omega_s\) for some \(s = 1, \ldots, N_0\). Then the function \(\rho_s\) has the zeros \(z^{n,\pm}_\alpha, \alpha = (j, j'), j < j', j', j' \in \omega_s, n \in \mathbb{Z} \setminus \{0\}\), which are real at large \(n\) and satisfy

\[
z^{n,\pm}_\alpha = \pi n + \frac{V^0_i + V^0_j}{4\pi n} + O\left(\left|\frac{\hat{V}^{(n)}}{n} + \frac{1}{n^2}\right|\right), \quad \alpha = (j, j') \quad \text{as} \quad n \to \infty. \tag{1.16}
\]

Let in addition \(V^0 < \ldots < V^0_N\). Then for each \(s = 1, \ldots, N_0\) and for large \(n \to \infty\) there exists a system of real intervals (gaps) \(\zeta_n^{\alpha} = (\lambda_n^{\alpha-}, \lambda_n^{\alpha+}), g_n^{\alpha} = (z_n^{\alpha-}, z_n^{\alpha+}), \) such that

\[
z^{n,\pm}_{jj'} = z^{n,\pm}_{jj'} > 0, \quad \lambda_n^{\pm} = \lambda_n^{\pm}, \quad (z_n^{\pm})^2 = \lambda_n^{\pm} > 0, \quad \alpha = (j, j'), j, j' \in \omega_s,
\]

\[
\lambda_n^{\alpha-} \leq \lambda_n^{\alpha+} < \lambda_n^{\alpha-} < \lambda_n^{\alpha+} < \ldots < \lambda_{j,j\alpha}^{\alpha-} < \lambda_{j,j\alpha}^{\alpha+}, N_s = |\omega_s|.
\]
\[ (-1)^n \Delta_j(z) > 1, \quad z \in g_{j,j}^n, \quad \text{and} \quad \overline{\Delta}_j'(z) = \Delta_j(z), \quad z \in g_{j,j'}, j \neq j' \quad (1.17) \]

i) The branch \( \Delta_j \) is real and is analytic on the set \( (\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}) \setminus \cup_{p \neq j} g_{j,j}^n \) and is not real on \( \cup_{p \neq j} g_{j,j}^n \).

ii) If \( z_{\alpha}^{n,-} \neq z_{\alpha}^{n,+} \) for some \( \alpha = j, j', j \neq j' \), then \( z_{\alpha}^{n,\pm} \) is a simple branch point, i.e., of square root type (resonance) for the functions \( \Delta_j, \overline{\Delta}_j' \). If \( z_{\alpha}^{n,-} = z_{\alpha}^{n,+} \), then \( \Delta_j, \overline{\Delta}_j' \) are analytic at \( z_{\alpha}^{n,\pm} \).

iii) The following asymptotics are fulfilled:

\[ \lambda_{\alpha}^{n,\pm} = (\pi n)^2 + \frac{V_0^2 + V_j^2}{2} \pm |\hat{\nu}^{(n)}| + O\left(|\hat{\nu}^{(n)}| + \frac{1}{n}\right), \quad \alpha = (j, j'). \quad (1.18) \]

Remark. 1) If \( N = 1 \), then the asymptotics (1.18) are well known [Ti]. 2) We describe the surface \( \mathcal{R} \) in the case \( V_1^0 < \ldots < V_N^0 \) and \( N_0 = 1 \) for large \( z \). To "build" the surface \( \mathcal{R} \) for large \( z \), we take \( N \) replicas of the \( z \)-plane and call them sheets \( \mathcal{R}_1, \ldots, \mathcal{R}_N \). Each \( \mathcal{R}_j \) is cut along the real interval \( g_{\alpha}^n, \alpha = (j, j'), j, j' = 1, \ldots, N, j \neq j' \). The cut on each sheet two edges; we label each edge with \( \sharp \) or \( \flat \). Then attach the \( \sharp \) edge of the cut \( g_{\alpha}^n \) on \( \mathcal{R}_j \) to the \( \flat \) edge of the same cut on \( \mathcal{R}_{j'}, j \neq j' \), and attach the \( \flat \) edge of the cut on \( \mathcal{R}_j \) to the \( \sharp \) edge of the same cut on \( \mathcal{R}_{j'} \). Thus, whenever we cross the cut, we pass from one sheet to the other.

3) If \( V_1^0 < \ldots < V_N^0 \), then all resonances are real at high energy. The existence of low energy complex resonances for specific potentials was established in [BKK].

Recall that \( N_G \) is the total number of gaps in the spectrum of \( \mathcal{L} \).

Corollary 1.3. Let \( V \in \mathcal{H} \) and \( V_1^0 < \ldots < V_N^0 \).

(i) If the identity \( V_1^0 + V_N^0 = V_2^0 + V_{N-1}^0 = \ldots = V_N^0 + V_1^0 \) is not fulfilled, then \( N_G < \infty \).

(ii) If \( V_1^0 + V_N^0 = \ldots = V_N^0 + V_1^0 \) holds true and there exists a sequence of indices \( n_k \to \infty \) such that \( |\hat{\nu}^{(n_k)}|^2 + n_k^{-1} = o\left(|\hat{\nu}^{(n_k)}|\right) \), \( k \to \infty \), for each \( m = 1, \ldots, N \), then \( N_G = \infty \).

Remark. 1) Maksudov and Veliev [MV] proved i), for \( N \geq 3 \) in a more general case.

2) If \( V = \text{diag}\{V_{11}, V_{22}, \ldots, V_{NN}\} \) and \( V_1^0 < V_2^0 \), then the number of gaps is \( N_G < +\infty \).

3) Note that the condition \( |\hat{\nu}^{(n)}|^2 + n^{-1} = o\left(|\hat{\nu}^{(n)}|\right) \), \( m = 1, \ldots, N, n \to \infty \), holds true for "generic" potentials from the space \( \mathcal{H} \). This yields the existence of real resonance gaps \( (\lambda_{\alpha}^{n,-}, \lambda_{\alpha}^{n,+}) \) at high energy. The coefficients \( V_{m,N+1-m}, m = 1, \ldots, N \) (the "second diagonal" of the matrix \( V \)) "create" the gaps.

We consider the conformal mapping associated with the operator \( \mathcal{L} \). We need functions from the subharmonic counterpart of the Cartwright class of the entire functions given by

\[ \mathcal{SC} = \left\{ v : \mathbb{C} \to \mathbb{R}, \quad v \text{ is subharmonic in } \mathbb{C} \text{ and harmonic outside } \mathbb{R}, \quad v(z) \equiv v(z), \quad \int_{\mathbb{R}} \frac{v(t)dt}{1+t^2} < \infty, \quad \limsup_{z \to \infty} \frac{v(z)}{|z|} < \infty \right\}. \quad (1.19) \]

We recall the class of functions from [KKK] given by

\[ \mathcal{SK}^+_m = \left\{ v \in \mathcal{SC} : v \geq 0, \quad \lim_{y \to \infty} \frac{v(iy)}{y} = 1, \quad \int_{\mathbb{R}} (1 + t^{2m})v(t)dt < \infty \right\}, \quad m \geq 0. \]
Figure 1: $N = 3$, $V_{1}^{0} < V_{2}^{0} < V_{3}^{0}$, $\frac{1}{2} (V_{1}^{0} + V_{3}^{0}) < V_{2}^{0}$.
We note that $SK_{m+1}^+ \subset SK_m^+$, $m \geq 0$.

Introduce the simple conformal mapping $\eta : \mathbb{C} \setminus [-1,1] \to \{ \zeta \in \mathbb{C} : |\zeta| > 1 \}$ by

$$\eta(z) = z + \sqrt{z^2 - 1}, \quad z \in \mathbb{C} \setminus [-1,1], \quad \text{and} \quad \eta(z) = 2z + o(1), \quad |z| \to \infty. \quad (1.20)$$

Note that $\eta(z) = \overline{\eta(\overline{z})}, z \in \mathbb{C} \setminus [-1,1]$ since $\eta(z) > 1$ for any $z > 1$. Due to the properties of the Lyapunov functions we have $|\eta(\Delta_s(\zeta))| > 1, \zeta \in \mathcal{R}_s^+ = \{ \zeta \in \mathcal{R}_s : \text{Im} \zeta > 0 \}$. Thus we can introduce the quasimomentum $k_m$ (we fix some branch of arccos and $\Delta_m(z)$) and the function $q_m$ by

$$k_m(z) = \text{arccos} \Delta_m(z) = i \log |\Delta_m(z)|, \quad q_m(z) = \text{Im} k_m(z) = \log |\Delta_m(z)|, \quad (1.21)$$

$m = 1, 2, ..., N$ and $z \in \mathcal{R}_0^+ = \mathbb{C}_+ \setminus \beta_+^s, \beta_+^s = \bigcup_{\beta \in B_\Delta \cap \mathbb{C}_+} [\beta, \beta + i\infty]$ where $B_\Delta$ is the set of all branch points of the function $\Delta$. The branch points of $k_m$ belong to $B_\Delta$. Define the averaged quasimomentum $w$, the density $u$ and the Lyapunov exponent $v$ by

$$w(z) = u(z) + iv(z) = \frac{1}{N} \sum_{k=1}^N k_m(z), \quad v(z) = \text{Im} w(z), \quad z \in \mathcal{R}_0^+. \quad (1.22)$$

Define the sets $\sigma(N) = \{ z \in \mathbb{R} : \Delta_1(z), ..., \Delta_N(z) \in [-1,1] \}$ and

$$\sigma(1) = \{ z \in \mathbb{R} : \Delta_m(z) \in (-1,1), \Delta_p(z) \notin [-1,1] \text{ some } m, p = 1, ..., N \}.$$ 

For the function $w(z) = u(z) + iv(z), z = x + iy \in \mathbb{T}_+$ we formally introduce the integrals

$$Q_n = \frac{1}{\pi} \int_{\mathbb{R}} x^n v(x) dx, \quad P_n = \frac{1}{\pi} \int_{\mathbb{R}} x^n u(x) du(x), \quad R_n^D = \frac{1}{\pi} \int_{\mathbb{T}_+} |w_n(z)|^2 dx dy, \quad (1.23)$$

$n \geq 0$, where here and below $w_m(z), z \in \mathbb{C}_+$ is given by

$$w_m(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{t^m v(t)}{t-z} dt = z^m \left( w(z) - z + \sum_{n=0}^{m-1} Q_n z^{-n-1} \right), \quad z \in \mathbb{C}_+.$$ 

Let $C_{us}$ denote the class of all real upper semi-continuous functions $h : \mathbb{R} \to \mathbb{R}$. With any $h \in C_{us}$ we associate the ”upper” domain $W(h) = \{ w = u + iv \in \mathbb{C} : v > h(u), u \in \mathbb{R} \}$. Let $g = \bigcup_{n \in \mathbb{Z}} g_n$ where $g_n = (z_n^-, z_n^+), z_n^\pm = \sqrt{\lambda_n^+} > 0$ and $g_{-n} = g_n, n \geq 1$. We formulate our main result.

**Theorem 1.4.** Let $V \in \mathcal{M}$. Then the averaged quasimomentum $w = \frac{1}{N} \sum_{k=1}^N k_m$ is analytic in $\mathbb{C}_+$ and $w : \mathbb{C}_+ \to w(\mathbb{C}_+) = W(h)$ is a conformal mapping onto $W(h)$ for some $h \in C_{us}$. Moreover, $v = \text{Im } w$ has an harmonic extension from $\mathbb{C}_+$ into $\Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup g$ given by $v(z) = v(\overline{z}), z \in \mathbb{C}_-$ and $v(z) > 0$ for any $z \in \Omega$. Furthermore $v \in SK_{m+1}^+ \cap C(\mathbb{C})$ and there exist branches $k_m, m = 1, ..., N$ such that the following asymptotics, identities and estimates are fulfilled:

$$w(z) = -w(\overline{z}), \quad z \in \mathbb{C}_+. \quad (1.24)$$
\[ w(z) - z = -\frac{Q_0}{z} - \frac{Q_2 + o(1)}{z^3}, \quad \text{as } y > r|x|, \quad y \to \infty, \quad \text{for any } r > 0, \quad (1.25) \]

\[ Q_0 = I_0^D + P_0 = \int_0^1 \frac{\text{Tr} V(t)dt}{2N}, \quad (1.26) \]

\[ Q_2 = I_1^D + P_2 = \int_0^1 \frac{\text{Tr} V^2(t)dt}{2^3N}, \quad (1.27) \]

\[ v|_{\sigma(N)} = 0, \quad 0 < v|_{\sigma(1) \cup g} \leq \sqrt{2Q_0}, \quad (1.28) \]

**Remark.** 1) Craig and Simon [CS] proved that the Lyapunov exponent is subharmonic in \( \mathbb{C} \) for the Schrödinger operator \(-\frac{d^2}{dx^2} + V\) for a large class of scalar potentials.

2) Similar arguments give

\[ Q_4 = I_2^D + P_4 = \int_0^1 \frac{\text{Tr}(V'(t)^2 + 2V^3(t))dt}{2^5N}, \quad \text{if } V' \in \mathcal{H}. \quad (1.29) \]

3) The integral \( P_0 \geq 0 \) is the area between the boundary of \( W(h) \) and the real line. The mapping \( w : \mathbb{C}_+ \to W(h) \) is illustrated in Figure 2. In Figure 1 the upper picture is a domain \( W(h) \) and \( \tilde{A} = w(A), \tilde{B} = w(B), \ldots \). The spectral interval \((A, B)\) (with multiplicity 2) of the \( z \)-domain is mapped on the curve \((\tilde{A}, \tilde{B})\) of the \( w \)-domain, the interval (a gap) \((B, C)\) of the \( z \)-domain is mapped on a vertical slit, which lies on the line \( \text{Re } w = \pi \). The spectral interval \((C, D)\) (with multiplicity 2) of the \( z \)-domain is mapped on the curve \((\tilde{C}, \tilde{D})\) of the \( w \)-domain. The spectral interval \((D, E)\) (with multiplicity 4) of the \( z \)-domain is mapped on the interval \((\tilde{D}, \tilde{C})\) of the \( w \)-domain. The case of the interval \((E, J)\) is similar. The resonance gap \((K, L)\) of the \( z \)-domain is mapped on the vertical slit on the line \( \text{Re } w = 3\pi \). In fact the boundary of \( W(h) \) is given by the graph of the function \( h(u), u \in \mathbb{R} \).

Figure 2: \( N = 2 \). The domain \( W(h) = w(\mathbb{C}_+) \) and gaps in the spectrum.
Let $\sigma(m, V)$ denote the spectrum of $\mathcal{L}$ of multiplicity $2m, m \geq 0$. We have the following simple corollary from Theorem 1.4.

**Corollary 1.5.** Let $\sigma(\mathcal{L}) = \sigma(N, V) = \mathbb{R}_+$ for some $V \in \mathcal{H}$. Then $V = 0$.

**Proof.** Due to $\sigma(N) = \mathbb{R}_+$, we obtain $Q_0 = Q_2 = 0$. Then identity (1.26) yields $\|V\| = 0$.

Recall that in the scalar case, the so-called Borg Theorem follows immediately from the existence of the conformal mapping and the asymptotics of the Lyapunov function at high energy [M]. In general, in order to prove the uniqueness result (the simplest part in the inverse spectral theory) it is necessary to use some results from "function theory". In our case we use conformal mapping theory. Recall that the so-called Borg Theorem for periodic systems was proved in [CHGL],[GKM] for general cases.

We describe the properties of the conformal mapping $w$.

**Theorem 1.6.** Let $V \in \mathcal{H}$. Then the following relations are fulfilled:

\[
u''(z) < 0 < \nu(z), \quad \nu(z) = \text{const} \in \frac{\pi}{N} \mathbb{Z}, \quad \text{for all } z \in g_n = (z_n^-, z_n^+),
\]

\[
v(x) = v_n^0(x) \left(1 + \frac{1}{\pi} \int_{\mathbb{R} \setminus g_n} \frac{\nu(t) dt}{v_n^0(t)|t - x|}\right), \quad x \in g_n, \quad v_n^0(z) = |(z - z_n^-)(z_n^+ - z)|^{1/4},
\]

\[
\sum_n |g_n|^2 \leq 8Q_0, \quad G^2 \equiv \sum |\gamma_n|^2 \leq \frac{8}{N} \|V\|^2,
\]

\[
\|V\| \leq C_0G(1 + G^2), \quad \text{if } \sigma(N, V) = \sigma(\mathcal{L}),
\]

for some absolute constant $C_0$.

**Remark.** Using this theorem we deduce that the function $h(u) = \nu(x(u)), u \in \mathbb{R}$ is continuous on $\mathbb{R} \setminus \{u_n, n \in \mathbb{Z}\}$, where $u_n = u(x), x \in g_n$. In this case we have

\[
h(u_n \pm 0) \leq h(u_n), \quad n \in \mathbb{Z}.
\]

The plan of our paper is as follows. In Sect. 2 we obtain the basic properties of the fundamental solution and prove Theorem 1.1. In Sect. 3 we determine the asymptotics of $M(z)$ and of the Lyapunov function and the multipliers at high energy and prove Theorem 1.2. Sect. 4 is the central part of the paper where we obtain the main properties of the quasimomentum $k_m, m = 1, \ldots, N$. We prove the basic Theorems 1.1 and 1.2. In Sect. 5 using Theorems 1.1, 1.2 and [KK3], devoted to the conformal mapping theory, we prove Theorem 1.4 and 1.6.
2 Fundamental solutions

In this section we study \( \vartheta, \varphi \). We begin with some notational convention. A vector \( h = \{h_n\}_1^N \in \mathbb{C}^N \) has the Euclidean norm \( |h|^2 = \sum_1^N |h_n|^2 \), while a \( N \times N \) matrix \( A \) has the operator norm given by \( |A| = \sup_{|h|=1} |Ah| \).

The fundamental solution \( \varphi \) satisfy the integral equations

\[
\varphi(t, z) = \varphi_0(t, z) + \int_0^t \frac{\sin z(t - s)}{s} V(s) \varphi(s, z) ds, \quad \varphi_0(t, z) = \frac{\sin z t}{z} I_N, \tag{2.1}
\]

where \( (t, z) \in \mathbb{R} \times \mathbb{C} \). The standard iterations in (2.1) yields

\[
\varphi(t, z) = \sum_{n \geq 0} \varphi_n(t, z), \quad \varphi_{n+1}(t, z) = \int_0^t \frac{\sin z(t - s)}{z} V(s) \varphi_n(s, z) ds. \tag{2.2}
\]

The similar expansion \( \vartheta = \sum_{n \geq 0} \vartheta_n \) with \( \vartheta_0(t, z) = I_N \cos z t \) holds. In order to determine the asymptotics of the fundamental solutions we introduce the functions

\[
I_m^0(z) = \int_0^m ds_1 \int_0^{s_1} \cos z(m - 2s_1 + 2s_2) F_2(s) ds_2, \quad m = 1, 2, \tag{2.3}
\]

where \( F_n(s) = \text{Tr} V(s_1) \cdot \ldots \cdot V(s_n), s = (s_1, \ldots, s_n) \in \mathbb{R}^n, n \geq 1 \). They satisfy the simple identity \( I_2^0(z) = 4I_1^0(z) \cos z, \quad z \in \mathbb{C} \), see [BKK]. Define

\[
V^0 = \int_0^1 V(x) dx, \quad B_n = \frac{(V^0)^n}{n!}, \quad n \geq 1, \quad |z|_1 = \max\{1, |z|\}. \tag{2.4}
\]

We prove

**Lemma 2.1.** Let \( V \in \mathcal{H} \). Then each functions \( \varphi(t, z), \vartheta(t, z), t \geq 0 \), are entire and for any integers \( m \geq t \geq 0, n_0 \geq -1 \) the following estimates are fulfilled:

\[
\max \left\{ \left| \vartheta(t, z) - \sum_0^{n_0} \vartheta_n(t, z) \right|, \left| |z|_1 \left( \varphi(t, z) - \sum_0^{n_0} \varphi_n(t, z) \right) \right|, \left| \frac{1}{|z|_1} \left( \vartheta'(t, z) - \sum_0^{n_0} \vartheta_n'(t, z) \right) \right| \right\} \leq \frac{(m \kappa)^{n_0+1}}{(n_0 + 1)!} e^{\kappa |z|_1 + m \kappa}. \tag{2.5}
\]

Moreover, each \( T_m(\cdot), m \geq 1 \) is entire and satisfies

\[
|T_m(z) - \sum_0^{n_0} T_m^{(n)}(z)| \leq \frac{(m \kappa)^{n_0+1}}{(n_0 + 1)!} e^{m |z|_1 + m \kappa}, \tag{2.6}
\]

\[
T_m^{(0)}(z) = \cos mz, \quad T_m^{(1)}(z) = \frac{mB_1}{2Nz} \sin mz, \quad T_m^{(2)}(z) = \frac{1}{4Nz^2} (I_2^{(m)}(z) - m^2 B_2 \cos mz), \ldots
\]
Proof. We prove the estimates of \( \varphi \), the proof for \( \varphi', \vartheta, \vartheta' \) is similar. (2.2) gives

\[
\varphi_n(t, z) = \int_{D_n} f_n(t, s)V(s_1) \cdot \ldots \cdot V(s_n) ds, \quad f_n(t, s) = \varphi_0(s_n, z) \prod_{1}^{n} \frac{\sin(z(s_{k-1} - s_k))}{z}, \quad (2.7)
\]

where \( s = (s_1, \ldots, s_n) \in \mathbb{R}^n, s_0 = t \) and \( D_n = \{0 < s_1 < \ldots < s_2 < s_1 < t\} \). Substituting the estimate \( |\varphi_0(t, z)| = |z^{-1} \sin zt| \leq |z|^{-1} e^{i|z|t} \) into (2.7), we obtain

\[
|\varphi_n(t, z)| \leq \frac{e^{\text{Im} zt}}{|z|^{n+1}} \int_{D_n} |V(s_1)| \cdot \ldots \cdot |V(s_n)| ds \leq \frac{e^{\text{Im} zt}}{|z|^{n+1}} \cdot \frac{1}{n!} \left( \int_0^t |V(x)| dx \right)^n.
\]

This shows that for each \( t \geq 0 \) the series (2.2) converges uniformly on bounded subset of \( \mathbb{C} \). Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.5).

The function \( T_m, m \geq 1 \) is entire, since \( \varphi, \vartheta \) are entire. We have \( (2N)T_m = \text{Tr}M^m(z) = \text{Tr}M_0(m, z), m \geq 1 \).

\[
\text{Tr}M_0(m, z) = 2N \cos mz, \quad \text{Tr}M_n(m, z) = \text{Tr} \varphi_n(m, z) + \text{Tr} \varphi_n'(m, z), \quad n \geq 1. \quad (2.8)
\]

The estimates \( |\text{Tr} \varphi_n'(m, z)| \leq \frac{(mx)^n}{n!} e^{\text{Im} zm} \) and \( |\text{Tr} \varphi_n(m, z)| \leq \frac{(mx)^n}{n!} e^{\text{Im} zm} \) yield

\[
|\text{Tr}M_n(m, z)| \leq (2N) \frac{(mx)^n}{n!} e^{m|\text{Im} z|}, \quad n \geq 0. \quad (2.9)
\]

Using (2.8) we obtain

\[
\text{Tr}M_1(m, z) = \frac{1}{z} \int_0^m (\sin z(m-t) \cos zt + \cos z(m-t) \sin zt) \text{Tr}V(t) dt = \frac{\sin mz}{z} mB_1. \quad (2.10)
\]

and

\[
\text{Tr}M_2(m, z) = \frac{1}{z^2} \int_0^m \int_0^t \sin z(t-s)z(m-t+s)F_2(t, s) dt ds
\]

\[
= \frac{1}{2z^2} \int_0^m \int_0^t (\cos z(m-t+s) - \cos zm)F_2(t, s) dt ds = \frac{1}{2z^2} (I_m^0(z) - m^2 B_2 \cos mz)
\]

since \( \int_0^m \int_0^t F_2(t, s) dt ds = \frac{1}{2} \text{Tr} \left( \int_0^m V(t) dt \right)^2 = m^2 B_2 \).

Note that the fundamental solutions \( \varphi(t, z), \vartheta(x, z) \) and \( M(z), T_m(z), m \geq 1 \) is real for \( z^2 \in \mathbb{R} \). Moreover, all functions \( \varphi(t, z), \varphi'(t, z), \vartheta(t, z), \vartheta'(t, z), M(z), T_m, m \geq 1 \) are even with respect to \( z \in \mathbb{C} \) and then they are entire with respect to \( \lambda = z^2 \).

Using the identity (15) and \( D(\tau, z) = \sum_{0}^{2N} \xi_m(z) \tau^{2N-m} \) we obtain

\[
\frac{\tau^n + \tau^{-n}}{2} = \mathcal{T}_n(\nu) = 2^{N-1} \sum_{0}^{\lfloor \frac{n}{2} \rfloor} c_{n,m} \nu^{n-2m}, \quad c_{n,m} = (-1)^{m} \frac{(n-m-1)!}{(n-2m)!m!} \nu^{2n-2m-N}, \quad (2.11)
\]
\[
\nu = \frac{r + r^{-1}}{2}, \text{ see [AS], we get } \Phi(\nu, z) \text{ given by}
\]
\[
\Phi(\nu, z) = \frac{D(\tau, z)}{(2\tau)^N} = \sum_{m=0}^{N} \phi_m(z)\nu^{N-m}, \quad (2.12)
\]
\[
\phi_0 = 1, \quad \phi_1 = c_{N-1,0}\xi_1 = \frac{\xi_1}{2}, \quad \phi_2 = c_{N,1} + c_{N-2,0}\xi_2, \quad \phi_3 = c_{N-1,1}\xi_1 + c_{N-3,0}\xi_3, \ldots, \quad (2.13)
\]
\[
\phi_{2n} = c_{N,n} + c_{N-2,n-1}\xi_2 + c_{N-4,n-2}\xi_4 + \ldots + c_{N-2n,0}\xi_{2n}, \quad (2.14)
\]
\[
\phi_{2n+1} = c_{N-1,n}\xi_1 + c_{N-3,n-1}\xi_3 + c_{N-5,n-2}\xi_5 + \ldots + c_{N-2n-1,0}\xi_{2n+1}. \quad (2.15)
\]

Let \( D^0, \Phi^0 \) be the determinant \( D \) and the polynomial \( \Phi \) at \( V = 0 \). In this case we have
\[
D^0(\tau, z) = (\tau^2 + 1 - 2\tau \cos z)^N = (2\tau)^N(\nu - \cos z)^N = (2\tau)^N \sum_{m=0}^{N} C_m^N(-\cos z)^m \nu^{N-m}
\]
where \( C_m^N = \frac{N!}{(N-m)!m!} \). Thus \( \Phi^0 = (\nu - \cos z)^N \) and \( \phi_m^0(z) = (-1)^m C_m^N \cos^m z \) at \( V = 0 \).

Substituting the identity for the Chebyshev polynomials (2.11) into the trace formula
\[
\frac{1}{2} \text{Tr } M^n(z) = \sum_{m=1}^{N} \frac{\tau_m^n + \tau_{m-n}^n}{2}, \text{ we obtain}
\]
\[
\sum_{m=1}^{N} \Delta_m(z) = \frac{1}{2} \text{Tr } M^n(z), \quad z \in \mathbb{C}. \quad (2.16)
\]

**Proof of Theorem 1.1** \( \Delta_1(z), \ldots, \Delta_N(z) \) are the roots of \( \Phi(\nu, z) = 0 \) for fixed \( z \in \mathbb{C} \). Recall that \( B_\Delta \) are all branch points of \( \Delta_1(z), \ldots, \Delta_N(z) \).

First simple case, let \( B_\Delta = \emptyset \). Then all functions \( \Delta_1(z), \ldots, \Delta_N(z) \) are entire, the function \( \Phi_j = \nu - \Delta_j, j = 1, \ldots, N \) and \( N_0 = N \).

Second case, let \( B_\Delta \neq \emptyset \). Consider a simply-connected domain \( \Omega_1 \subset \mathbb{C} \) containing only one branch point \( z_1 \in B_\Delta \). We consider the behavior of the roots \( \Delta_m, m = 1, \ldots, N \) in the neighborhood of the branch points \( z_1 \). Let \( B'(z_1, r) = B(z_1, r) \setminus \{z_1\} \) be the small disk near \( z_1 \) with radius \( r > 0 \) but excluding \( z_1 \). The functions \( \Delta_m, m = 1, \ldots, N \) are branches of analytic functions (defined in \( \Omega_2 \setminus \{z_1, z_2\} \)) with a branch point (if \( p > 1 \)) at \( z_1 \) and \( z_2 \).

are analytic of \( z \in B'(z_1, r) \). If \( B'(z_1, r) \) is moved continuously around \( z_1 \), then \( N \) functions can be continued analytically. When \( B'(z_1, r) \) has been brought to its initial position after one revolution around \( z_1 \), the functions \( \Delta_m, m = 1, \ldots, N \) will have undergone a permutation among themselves. These functions may therefore be grouped in the manner
\[
\{\Delta_1(z), \ldots, \Delta_{p_i}(z)\}, \{\Delta_{p_i+1}(z), \ldots, \Delta_{p_i+q_i}(z)\}, \ldots \quad (2.17)
\]
in such a way that each group undergoes a cyclic permutation by a revolution of \( B'(z_1, r_1) \) of the kind described. For brevity each group will be called a cycle at the branch point \( z_1 \), and the number of elements of a cycle will be called its period.

It is obvious that the elements of a cycle of period \( p_1 \) constitute a branch of an analytic function (defined near \( z_1 \)) with a branch point (if \( p_1 > 1 \)) at \( z_1 \). We have Puiseux series as
\[
\Delta_j(z) = \Delta_1(\zeta) + a_1 t + a_2 t^2 + \ldots, \quad t = e^{j\frac{2\pi}{p_1}}(z - z_1)^\frac{1}{p_1}, \quad j = 1, \ldots, p_1.
\]
Consider a simply-connected domain $\Omega_2 \subset \mathbb{C}$, containing only two distinct branch point $z_1, z_2 \in \mathcal{B}_\Delta$. The similar argument as above give that functions $\Delta_1, \ldots, \Delta_N$ may therefore be grouped in the manner

$$\{\Delta_1(z), \ldots, \Delta_{p_1}(z)\}, \{\Delta_{p_1+1}(z), \ldots, \Delta_{p_2}(z)\}, \ldots, \text{where } p_1 \leq p_2, \ldots \quad (2.18)$$

and $\Delta_i \neq \Delta_j, 1 \leq i < j \leq p_1, \ldots$. Here the elements of a cycle of period $p_2$ constitute a branch of an analytic function (defined in $\Omega_2 \setminus \{\{z_1, z_2\}\}$) with a branch point (if $p_2 > 1$) at $z_1$ and $z_2$.

If we take a sequence of domain $\Omega_n \subset \Omega_{n+1}, n \geq 1$, containing only $n$ distinct branch points $z_1, \ldots, z_n \in \mathcal{B}_\Delta \cap \Omega_n$ and let $\lim \Omega_n = \mathbb{C}$, then the similar argument as above give that functions $\Delta_1, \ldots, \Delta_N$ may therefore be grouped in the manner

$$\{\Delta_j(z), j \in \omega_1\}, \{\Delta_j(z), j \in \omega_2\}, \ldots, \{\Delta_j(z), j \in \omega_N\}, \text{ where } \Delta_i \neq \Delta_j, i, j \in \omega_s, \quad (2.19)$$

$i \neq j, s = 1, \ldots, N_0$. Here the elements of $\{\Delta_j(z), j \in \omega_1\}$ constitute a branch of an analytic function $\Lambda_1$ (defined in $\mathbb{C} \setminus \mathcal{B}_\Delta$) with a branch point (if the of numbers of elements of $\omega_1$ is $> 1$) at some points $z_n \in \mathcal{B}_\Delta, z \geq 1$.

There exists an interval $Y \subset \mathbb{R}, Y \neq \emptyset$ such that the spectral interval $Y \subset \sigma(L)$ has multiplicity $2N$ (see [MV]). Thus all functions $\Delta_m, m = 1, \ldots, N$ are real on $Y$. Hence each entire function $\Phi_{s}(\nu, z), s = 1, \ldots, N_0$ is real for all $\nu, z \in \mathbb{R}$.

ii) We have $\Delta'_m(z) = -\frac{1}{2}(1 - \tau^{-2}(z))\tau'(z) \neq 0, z \in Y,$ since by the Lyapunov Theorem, $\tau'(z) \neq 0$ for all $z \in Y$.

iii) Recall that the resultant for the polynomials $f = \tau^n + a_1\tau^{n-1} + \ldots + a_n, g = b_0\tau^s + b_1\tau^{s-1} + \ldots + b_s$ is given by

$$R(f, g) = \begin{vmatrix} 1 & a_1 & \ldots & a_n & 0 & 0 & \ldots & 0 \\ 0 & 1 & a_1 & \ldots & a_n & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_s & 0 & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & b_s & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_s \end{vmatrix} \quad (2.20)$$

The discriminant of the polynomial $f$ with zeros $\tau_1, \ldots, \tau_n$ is given by

$$\text{Dis} f = \prod_{i<j}(\tau_i - \tau_j)^2 = (-1)^{\frac{n(n-1)}{2}}R(f, f')$$

Thus we have $\text{Dis} \Phi_{s}(\tau, z) = \prod_{i<s,i,s<\omega}(\Delta_i(z) - \Delta_s(z))^2 = (-1)^{\frac{N_j(N_j-1)}{2}}R(\Phi_{s}, \Phi'_{s})$ is entire, since $\Phi_{s}(\tau, z)$ is the entire function. Then the function $\rho = \prod_{i=1}^{N_0} \text{Dis} \Phi_{j}$ is entire.

iv) We consider the gap $g_n = (z_n^+, z_n^-)$ in the variable $z$, where a gap $\gamma_n = (\lambda_n^-, \lambda_n^+), \lambda_n^\pm = (z_n^\pm)^2$. Each gap $g_n = (z_n^+, z_n^-) = \bigcup_{g_{n,p}, p = 1, \ldots, p_n}$, where $g_{n,p} = (z_{n,p}^-, z_{n,p}^+)$ is finite interval such that $\Delta_m(z) \notin [-1, 1]$ for all $z \in g_{n,p}$ for some $m = m(p)$. Note that $\Delta_m(z_{n,p}^\pm) = \pm 1$ or $z_{n,p}^-$ is the branch point $\in \mathcal{B}_\Delta$, otherwise we have a contradiction. ■
3 Spectral asymptotics

Below we need the identities for $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, $J_1 = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$:

$$J^2 = -I, \quad J_1 J_2 = J, \quad J_1 J_1 = -J_2, \quad J_2 J_2 = J_1,$$

(3.1)

$$e^{zJ} = I_{2N} \cos z + J \sin z, \quad z \in \mathbb{C}. \quad (3.2)$$

We have also

$$D^0(z, \tau) = \det(e^{zJ} - \tau I_{2N}) = (\tau - e^{iz})^N(\tau - e^{-iz})^N, \quad D^0(z, \pm 1) = 2^N(1 \mp \cos z)^N, \quad (3.3)$$

$$(e^{zJ} - \tau I_{2N})^{-1} = (\tau - e^{iz})^{-1}(\tau - e^{-iz})^{-1}(e^{-izJ} - \tau I_{2N}). \quad (3.4)$$

We shall obtain the simple properties of the monodromy matrix. We introduce the modified monodromy matrix

$$\tilde{M}(z) = \begin{pmatrix} I_N & 0 \\ 0 & zI_N \end{pmatrix}^{-1} M(z) \begin{pmatrix} I_N & 0 \\ 0 & zI_N \end{pmatrix} = \begin{pmatrix} \vartheta(1, z) & z\varphi(1, z) \\ z^{-1}\varphi'(1, z) & \varphi'(1, z) \end{pmatrix}, \quad z \in \mathbb{C}, \quad (3.5)$$

with the same eigenvalues and the same traces. We will show following asymptotics

$$\tilde{M}(z) = e^{zJ}(I + \frac{-V_0 J + \hat{V}(z) J_2}{2z}) + O(z^{-2}e^{1|\text{Im} z|}), \quad \hat{V}(z) = \int_0^1 V(t)e^{-2tzJ}dt, \quad |z| \to \infty. \quad (3.6)$$

Indeed, using (2.5), (3.5) we get $\tilde{M}(z) = e^{zJ} + \tilde{M}_1(z) + O(z^{-2}e^{1|\text{Im} z|})$, where $\tilde{M}_1$ is given by

$$\tilde{M}_1(z) = \frac{1}{2z} \int_0^1 V(t)f(t, z)dt, \quad f = 2 \begin{pmatrix} \sin z(1 - t) \cos tzI_N & \sin z(1 - t) \sin tzI_N \\ \cos z(1 - t) \cos tzI_N & \cos z(1 - t) \sin tzI_N \end{pmatrix}. \quad (3.7)$$

Let $c = \cos z, s = \sin z$ and $a = z(1 - 2t)$. Then substituting the identity

$$f = \begin{pmatrix} (s + \sin a)I_N & (-c + \cos a)I_N \\ (c + \cos a)I_N & (s - \sin a)I_N \end{pmatrix} = (s - cJ) + (\sin aJ_1 - \cos aJ_2) = -Je^{zJ} + e^{aJ}J_2$$

into (3.7) we obtain (3.6).

Define the matrix $L = \frac{1}{2}(\tilde{M} + \tilde{M}^{-1})$ with eigenvalues $\Delta_m = \frac{1}{2}(\tau_m + \tau_m^{-1}), m = 1, 2, ..., N$, of multiplicity two. The identity (1.4) yields

$$L = \frac{\tilde{M} + \tilde{M}^{-1}}{2} = \frac{1}{2} \begin{pmatrix} \vartheta(1, \cdot) + \varphi'(1, \cdot)^\top & z(\varphi(1, \cdot) - \varphi(1, \cdot)^\top) \\ z^{-1}(\varphi'(1, \cdot) - \varphi'(1, \cdot)^\top) & \vartheta(1, \cdot)^\top + \varphi'(1, \cdot) \end{pmatrix}. \quad (3.8)$$

Using (2.5) we obtain

$$L(z) = L_1(z) + \frac{L_2(z)}{2z^2} + \frac{L_3(z)}{2z^3} + O(e^{1|\text{Im} z|})/z^4, \quad L_1(z) = \cos z + \frac{\sin z}{2z} V^0, \quad |z| \to \infty. \quad (3.9)$$
where

\[ L_2(z) = \int_0^1 dt \int_0^t \sin z(t-s) \begin{pmatrix} a_{11}(t, s, z) & a_{12}(t, s, z) \\ a_{21}(t, s, z) & a_{22}(t, s, z) \end{pmatrix} ds, \quad z \in \mathbb{C}, \quad (3.10) \]

\[ a_{11}(t, s, z) = \sin z(1-t) \cos zsV(t)V(s) + \cos z(1-t) \sin zsV(t)V(s), \quad a_{22} = a_{11}^T, \quad (3.11) \]

\[ a_{12}(t, s, z) = \sin z(1-t) \sin zsV(t)V(s) - V(s)V(t), \quad (3.12) \]

\[ a_{21}(t, s, z) = \cos z(1-t) \cos zsV(t)V(s) - V(s)V(t), \quad (3.13) \]

and

\[ L_3(z) = \int_0^1 dt \int_0^t ds \int_0^s \sin z(t-s) \sin z(s-u) \begin{pmatrix} b_{11}(t, s, u, z) & b_{12}(t, s, u, z) \\ b_{21}(t, s, u, z) & b_{22}(t, s, u, z) \end{pmatrix} du, \quad (3.14) \]

\[ b_{11} = \sin z(1-t) \cos z uV(t)V(s)V(u) + \cos z(1-t) \sin z uV(t)V(s)V(u), \quad b_{22} = b_{11}^T, \quad (3.15) \]

\[ b_{12} = \sin z(1-t) \sin z uV(t)V(s)V(u) - V(u)V(s)V(t), \quad (3.16) \]

\[ b_{21} = \cos z(1-t) \cos z uV(t)V(s)V(u) - V(u)V(s)V(t). \quad (3.17) \]

Recall that \( V^0 = \int_0^1 V(x)dx, \quad B_n = \text{Tr} \frac{(V^0)^{2n}}{n!} \).

**Lemma 3.1.** For each \((r, V) \in \mathbb{R}_+ \times \mathcal{H}\) asymptotics \(\mathbf{(1.11)}\) and the following ones

\[ 2^{2N} \text{det} L(z) = \exp \left( -2Niz + i \frac{\text{Tr} V^0}{z} + i \frac{||V||^2 + o(1)}{4z^3} \right), \quad (3.18) \]

\[ \text{Tr} L_2(z) = -B_2 + i \frac{||V||^2 + o(1)}{2z} \cos z, \quad (3.19) \]

\[ \text{Tr} V_0 L_2(z) = -3B_3 \cos z + O(e^{\text{Im} z}/z), \quad (3.20) \]

\[ \text{Tr} L_3(z) = -iB_3 \frac{\cos z}{z} + o(e^{\text{Im} z}), \quad (3.21) \]

hold as \( y \geq r|x|, y \to \infty \). Moreover, the following identity and the asymptotics are fulfilled

\[ L_2(\pi n) = \frac{(-1)^n}{4} \left( X^{(n)} + [\hat{V}^{cn}, V^0] [\hat{V}^{sn}, V^0 - \hat{V}^{cn}] [\hat{V}^{sn}, V^0 + \hat{V}^{cn}] X^{(n)} - [\hat{V}^{cn}, V^0] \right), \quad (3.22) \]

\[ \Delta_m(z) = \cos \left( z - \frac{V_{m0}}{2z} \right) + O(\frac{||V^{(n)}|| + n^{-1}}{n^2}) \quad \text{as} \quad z = \pi n + O(1/n), \quad m = 1, \ldots, N, \quad (3.23) \]

where \( X^{(n)} = -(V^0)^2 + (\hat{V}^{cn})^2 + (\hat{V}^{sn})^2 \) and \([A, B] = AB - BA\) for matrix \( A, B \).
Proof. Recall the simple fact: Let \( A, B \) be matrices and and \( \sigma(B) \) be spectra of \( B \). If \( A \) be normal, then dist \( \{ \sigma(A), \sigma(A + B) \} \leq |B| \) (see [Ka,p.291]).

The diagonal operator \( L_1(z) \) has the eigenvalues \( \Delta_m^0(z) = \cos z - V_0^0 \frac{\sin z}{z} \), \( m = 1, \ldots, N \) with the multiplicity 2. Using the result from [Ka] and asymptotics (3.9) we deduce that the eigenvalues \( \Delta_m(z) \) of matrix \( L(z) \) satisfy the asymptotics (1.11).

Using (3.10) we obtain

\[
\operatorname{Tr} L_2(z) = 2 \int_0^1 \int_0^t \sin z(t-s) \sin z(1-t+s) F_2(t,s) dt ds
\]

\[
= \int_0^1 \int_0^t (\cos(1-t+s) - \cos zm) F_2(t,s) dt ds = I_1(z) - B_2 \cos mz,
\]

since

\[
\int_0^1 \int_0^t F_2(t,s) dt ds = \frac{1}{2} \operatorname{Tr} \left( \int_0^1 V(t) dt \right)^2 = B_2.
\]

Due to (6.1) we have \( I_1(z) = i \|V\|^2 - o(1) \cos z \), which yields (3.19).

We show (3.20). Let \( G(t,s) = \operatorname{Tr} V_0(V(t)V(s) + V(s)V(t)) \). Using (3.10), (6.1) we have

\[
\operatorname{Tr} V_0 L_2(z) = \operatorname{Tr} V_0 \int_0^1 \int_0^t \sin z(t-s) z(1-t+s) (V(t)V(s) + V(s)V(t)) ds ds
\]

\[
= \frac{1}{2} \int_0^1 dt \int_0^t (- \cos z + \cos z(1-t+s)) G(t,s) ds = -3B_3 \cos z + O(e^{1m z}/z),
\]

since

\[
\int_0^1 dt \int_0^t \operatorname{Tr} V_0(V(t)V(s) + V(s)V(t)) ds = \operatorname{Tr} V_0 \left( \int_0^1 V(t) dt \right)^2 = 6B_3.
\]

Consider \( \operatorname{Tr} L_3 \). The identity (3.14) gives

\[
\operatorname{Tr} L_3(z) = \int_0^1 dt \int_0^t ds \int_0^s \sin z(t-s) z(s-u) \operatorname{Tr}(b_{11} + b_{22}) du
\]

\[
= \int_0^1 dt \int_0^t ds \int_0^s \sin z(t-s) \sin z(1-t+u) R du,
\]

where \( R = \operatorname{Tr}(V(t)V(s)V(u) + V(u)V(s)V(t)) \). Using the identity

\[
4 \sin z(t-s) \sin z(s-u) \sin z(1-t+u) = - \sin z + P,
\]

\[
P = \sin z(1 - 2s + 2u) - \sin z(1 - 2t + 2s) - \sin z(1 - 2t + 2u),
\]

we get

\[
L_3 = -L_3^0 \frac{\sin z}{2} + L_3^1, \quad L_3^0 = \frac{1}{2} \int_0^1 dt \int_0^t ds \int_0^s R du, \quad L_3^1 = \frac{1}{4} \int_0^1 dt \int_0^t ds \int_0^s P R du,
\]

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where
\[
L_0^3 = \frac{\text{Tr}}{2} \int_0^1 V(t)dt \int_0^t ds \int_0^u (V(s)V(u) + V(u)V(s))du = \frac{\text{Tr}}{2} \int_0^1 V(t) \left( \int_0^t V(s)ds \right)^2 = B_3.
\]

Due to (6.3) we obtain \( L_1^3 = o(e^{\|m\|}) \), which yields (3.21).

We will show (3.18). Asymptotics (3.9) yields
\[
\frac{L}{\cos z} = I_{2N} + S, \quad S = i\frac{V^0 I_{2N}}{2z} + \frac{L_2}{2z^2 \cos z} + \frac{L_3}{2z^3 \cos z} + O(z^{-4}),
\]
(3.24)
as \( |z| \to \infty, y \geq r|x| \), since \( \tan z = i + O(e^{-y}) \). In order to use the identity
\[
\det(I + S) = \exp \left( \text{Tr} \, S - \text{Tr} \, \frac{S^2}{2} + \text{Tr} \, \frac{S^3}{3} + \ldots \right), \quad |S| \to 0,
\]
we need the traces of \( S^m, m = 1, 2, 3 \). Due to (3.19)–(3.22), we get
\[
\frac{\text{Tr} \, S^3}{3} = -i \text{Tr} \, \left( \frac{V^0)^2}{3(2z)^3} I_{2N} + O(z^{-4}) \right) = -i \frac{B_3}{2z^3} + O(z^{-4}),
\]
(3.25)
\[
- \text{Tr} \, \frac{S^2}{2} = \frac{\text{Tr}}{2} \left( \frac{(V^0)^2}{(2z)^2} I_{2N} - 2i \frac{V^0 L_2}{4z^3 \cos z} + O(z^{-4}) \right) = \frac{B_2}{2z^2} + i \frac{3B_3}{4z^3} + O(z^{-4}),
\]
(3.26)
and
\[
\text{Tr} \, S = \text{Tr} \left( i \frac{V^0}{2z} + \frac{L_2}{2z^2 \cos z} + \frac{L_3 + O(z^{-1})}{2z^3 \cos z} \right) = i \frac{B_1}{z} + \left( - \frac{B_2}{2z^2} + i \frac{\|V\|^2}{4z^3} \right) - i \frac{B_3 + O(1)}{4z^3},
\]
(3.27)
and summing (3.25)–(3.27) we get (3.18).

We will show (3.22). Let \( z = \pi n, c_t = \cos \pi nt, s_t = \sin \pi nt \), and \( K_{ts}^+ = V(t)V(s) \pm V(s)V(t) \). Using (3.11), we have
\[
4(-1)^n \sin \pi n(t-s) a_{11}(t, s, \pi n) = 4(s_tC_s - s_s c_t)(-s_tc_s V(t)V(s) + s_sc_t V(s)V(t))
\]
\[
= ((c_{2t} - 1)(1 + c_{2s}) + s_{2s} s_{2t}) V(t)V(s) + ((c_{2t} + 1)(1 - c_{2s}) + s_{2s} s_{2t}) V(s)V(t)
\]
\[
= (-1 + c_{2t} c_{2s} + s_{2t} s_{2s}) K_{ts}^+ + ((c_{2t} - c_{2s}) K_{ts}^-)
\]
and the integration yields
\[
\int_0^1 dt \int_0^t \sin z(t-s) a_{11}(t, s, \pi n) ds = \frac{1}{2} \int_0^1 \int_0^t \sin z(t-s) a_{11}(t, s, \pi n) dt ds = \frac{(-1)^n}{4} \left( X^{(n)} + [V^{cn}, V^0] \right),
\]
(3.28)
where \( X^{(n)} = -(V^0)^2 + (V^{cn})^2 + (V^{sn})^2 \). Using (3.22), we obtain
\[
(-1)^n \sin \pi n(t-s) a_{12}(t, s, \pi n) = (s_tC_s - s_s c_t)(-s_t s_s) K_{ts}^- = (s_{2t} - s_{2s} + s_{2s} c_{2t} - c_{2s} s_{2t}) \frac{K_{ts}^-}{4}
\]
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and the integration yields
\[ \int_0^1 \int_0^t \sin z(t-s) a_{12}(t,s,z) ds = \frac{(-1)^n}{4} \left( \hat{V}^{sn} V^0 - V^0 \hat{V}^{sn} - \hat{V}^{sn} \hat{V}^{cn} + \hat{V}^{cn} \hat{V}^{sn} \right). \] (3.29)

The similar arguments give
\[ \int_0^1 \int_0^t \sin z(t-s) a_{21}(t,s,z) ds = \frac{(-1)^n}{4} \left( V^0 \hat{V}^{sn} - \hat{V}^{sn} V^0 - \hat{V}^{sn} \hat{V}^{cn} + \hat{V}^{cn} \hat{V}^{sn} \right). \] (3.30)

Combining the identities (3.28) - (3.30) and using \( a_{22} = a_{11}^T \) we obtain (3.22).

The diagonal operator \( L_1(z) \) has the eigenvalues \( \Delta_m^0(z) = \cos z - V^0 \frac{\sin z}{m} \), \( m = 1, \ldots, N \) with the multiplicity 2. Using the result from [Ka] and asymptotics (3.22) we deduce that the eigenvalues \( \Delta_m(z) \) of matrix \( L(z) \) satisfy the asymptotics (3.23) as \( z = \pi n + O(1/n) \).

Recall that \( D^0(\pm 1, z) = (e^{iz} \mp 1)^N (e^{-iz} \mp 1)^N = 2^N (1 \mp \cos z)^N \).

**Lemma 3.2.** Let \( V \in \mathcal{H} \) and let \( A = e^{i m \pi_i \pm \kappa} \), \( \kappa = \frac{||V||_{\mathcal{H}}}{|z|}, |z| = \max\{|1, |z||} \). Then

i) The following estimates are fulfilled:
\[ |\xi_m(z)| \leq (2NA)^m, \quad |\xi_m(z) - \xi^0_m(z)| \leq 2\kappa(2NA)^m, \quad m = 1, \ldots, N. \] (3.31)
\[ |D(\pm 1, z) - D^0(\pm 1, z)| < C_N \kappa A^N, \quad C_N = 4(2N)^N. \] (3.32)

ii) For each integer \( n_0 > 4^N C_N \|V\| \) the function \( D(1, z) \) has exactly \( N(2n_0 + 1) \) roots, counted with multiplicity, in the disc \( \{|z| < \pi(2n_0 + 1)\} \) and for each \(|n| > n_0 \), exactly \( 2N \) roots, counted with multiplicity, in the domain \( \{|z - 2\pi n| < \frac{\pi}{2}\} \). There are no other roots.

iii) For each integer \( n_0 > 4^N C_N \|V\| \) the function \( D(-1, \lambda) \) has exactly \( 2Nn_0 \) roots, counted with multiplicity, in the disc \( \{|z| < 2\pi n_0\} \) and for each \(|n| > n_0 \), exactly \( 2N \) roots, counted with multiplicity, in the domain \( \{|z - \pi(2n_0 + 1)| < \frac{\pi}{2}\} \). There are no other roots.

iv) Assume that \( V_i \neq V_j \) for all \( i \neq j \in \omega_s \) for some \( s = 1, \ldots, N_0 \). Then there exists integer \( n_0 \geq 1 \) such that the function \( \rho_s \) has exactly \( 2N_n(N_n - 1)n_0 \) roots, counted with multiplicity, in the disc \( \{|z| < \pi(n_0 + \frac{1}{2})\} \) and for each \(|n| > n_0 \), exactly \( N_s(N_s - 1) \) roots, counted with multiplicity, in the domain \( \{|z - \pi n| < \frac{\pi}{2}\} \). There are no other roots.

**Proof.** We prove (3.31) by induction. Let the first estimate in (3.31) hold for \( \xi_m \). Using (2.6) we obtain \( |\xi_m(z)| \leq \mu^m, \mu = 2NA \) for \( m = 0, 1, 2 \). If \( m \geq 2 \), then substituting the estimate \( |\xi_j(z)| \leq \mu^j \) and \( |T_j(z)| \leq A^j \) (see (2.6)) into (1.6) we have
\[ |\xi_{m+1}(z)| \leq \frac{2N}{m+1} \sum_{j=0}^{m} A^{m+1-j} \mu^i \leq \frac{2NA^{m+1}((2N)^{m+1} - 1)}{(m+1)(2N-1)} \leq \mu^{m+1}. \]

We shall show the second estimate in (3.31). Let \( p_n = |\xi_n(z) - \xi_n^0(z)| \). The recurrent identities (1.6) and \( |T_j(z) - T_j^0(z)| \leq \kappa_j A^j, |\xi_j(z)| \leq (2NA)^j \) and (2.6) give
\[ p_n \leq \frac{2N}{n} \sum_{k=0}^{n-1} |T_{n-k}(z)\xi_k(z) - T_{n-k}^0(z)\xi_k^0(z)| \leq \frac{2N}{n} \sum_{k=0}^{n-1} \left( |\xi_k(z)||T_{n-k}(z) - T_{n-k}^0(z)| + |T_{n-k}(z)|p_k \right) \]

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\[
\leq 2\frac{N}{n} \sum_{k=1}^{n-1} \mu^k r(n-k)A^{n-k} + A^{n-k} r\mu^k = 2NA^n r \sum_{k=1}^{n-1} (2N)^k \leq 2r\mu^n
\]
where we used \(p_1 \leq rA\), see (3.26). Thus we get (3.31).

ii) We have \(D(1, \cdot) = \xi_N + 2 \sum_{k=1}^{N-1} \xi_k\), where \(\xi_k\) are given by (1.6). Recall that \(\xi_0^0 = \xi_0^*\) at \(V = 0\). Hence using (3.31) and \(\mu = 2NA\) we have

\[
|D(1, z) - D^0(1, z)| \leq |\xi_N(z) - \xi_0^0(z)| + 2 \sum_{k=1}^{N-1} |\xi_k(z) - \xi_0^0(z)| \leq 2r\mu^N + 2 \sum_{k=1}^{N-1} 2r\mu^n. \tag{3.33}
\]

Thus \(|D(1, z) - D^0(1, z)| \leq 2\mu^N (1 + \frac{2}{\mu-1}) < 4r\mu^N\) since \(\mu \geq 4\) which yields (3.33).

Let \(n_1 > n_0\) be another integer. Introduce the contour \(C_n(r) = \{z : |z - \pi n| = \pi r\}\). Consider the contours \(C_0(2n_0 + 1), C_0(2n_1 + 1), C_2n(\frac{1}{4}), |n| > n_0\). Note that \(r \leq \frac{1}{4}\) on all contours. Then (3.27) and the estimate \(e^\frac{1}{2}|z| < 4|\sin \frac{z}{2}|\) on all contours yield

\[
|D(1, \lambda) - \left(2\sin \frac{z}{2}\right)^2 | \leq C nA^N = Ce^{N|z|} \leq (C^4N^2)|2\sin \frac{z}{2}| < \frac{1}{2}|2\sin \frac{z}{2}|^2 \tag{3.34}
\]

where \(C = |z|^{-1}C_n||V||e^{N}\). Hence, by Rouché’s theorem, \(D(1, z)\) has as many roots, counted with multiplicities, as \(\sin^{2N} \frac{z}{2}\) in each of the bounded domains and the remaining unbounded domain. Since \(\sin^{2N} \frac{z}{2}\) has exactly one root of the multiplicity 2 \(2N\) at \(2\pi n\), and since \(n_1 > n_0\) can be chosen arbitrarily large, the point ii) follows.

iii) The proof for \(D(-1, \lambda)\) is similar.

iv) Consider the case \(N_0 = 1\), the proof for \(N_0 \geq 2\) is similar. Asymptotics (1.11) yields

\[
\Delta_j(z) - \Delta_m(z) = (V_j^0 - V_m^0) \sin z + O(e^{-|z|}), \quad |z| \to \infty.
\]

Then

\[
\rho(z) = \prod_{j < m} (\Delta_j(z) - \Delta_m(z))^2 = \left(\sin \frac{z}{2}\right)^2 + O(e^{-|z|}) \right)^{N(N-1)}, \quad c_0 = \prod_{j < m} (V_j^0 - V_m^0)^2. \tag{3.35}
\]

Let \(n_1 > n_0\) be another integer. Introduce the contour \(C_n(r) = \{z : |z - \pi n| = \pi r\}\). Consider the contours \(C_0(2n_0 + 1), C_0(2n_1 + 1), C_2n(\frac{1}{4}), |n| > n_0\). Note that \(r \leq \frac{1}{4}\) on all contours. Then (3.35) and the estimate \(e^{-|z|} < 4|\sin z|\) on all contours (for large \(n_0\)) yield

\[
\rho(z) = \rho^0(z)(1 + O(z^{-1})), \quad \rho^0(z) = c_0 \left(\frac{\sin z}{2}\right)^{N(N-1)}.
\]

Hence, by Rouché’s theorem, \(\rho\) has as many roots, counted with multiplicities, as \(\rho^0\) in each of the bounded domains and the remaining unbounded domain. Since \(\rho^0\) has exactly one root of the multiplicity \(N(N-1)\) at \(\pi n \neq 0\), and since \(n_1 > n_0\) can be chosen arbitrarily large, the point iv) follows.

**Proof of Theorem 1.2** i) We determine asymptotics (1.15) for \(z_{m,m}^{\pm} = \sqrt{\lambda_{m,m}^{\pm}}\) as \(n \to \infty\). Lemma 3.2 yields \(|z_{m,m}^{\pm} - \pi n| < \frac{\pi}{2}\) as \(n \to \infty\), \(m = 1, 2, \ldots, N\). Lemma 3.4 gives \(\Delta_{m}(z) = \cos(z - \frac{V_0}{2z}) + O(1/z^2)\) as \(z = \pi n + O(1)\). Using the identity \(\Delta_{j}(z_{m,m}^{\pm}) = (-1)^{n}\), we have \(z_{m,m}^{\pm} = \pi n + O(1/n)\). We shall determine sharper asymptotics.
Recall that the modified monodromy matrix $\tilde{M}(z)$ is given by (3.36) and it has the same eigenvalues as $M(z)$. Define the local parameter $\mu$ by $z = \pi n + \varepsilon \mu, \varepsilon = \frac{1}{2\pi n}$. Note that $\lambda = z^2 = (\pi n)^2 + \mu + (\varepsilon \mu)^2$. Asymptotics (3.36) gives
\[
(-1)^n \tilde{M}(z) = e^{\varepsilon \mu J} \left( I_{2N} - \varepsilon J(V_0 + J\hat{V}(\pi n), J_2) + O(\varepsilon) \right) = I_{2N} - \varepsilon J(V_0 + \hat{V} + O(\varepsilon) - \mu),
\]
where
\[
\hat{V}^n = J\hat{V}(\pi n) J_2 = J \int_0^1 V(t) e^{-2tz^J} dt J_2 = J(\hat{V}^{nc} - J\hat{V}^{ns}) J_2 = J_1 \hat{V}^{nc} + J_2 \hat{V}^{ns}.
\]
Hence we have the asymptotics (1.15), since
\[
0 = \det\left((-1)^n \tilde{M}(z) - I_{2N}\right) = \det(-\varepsilon J) \det\left(V_0 + \hat{V} + O(\varepsilon) - \mu\right).
\]
Consider the case $V_0 < ... < V_0$. We shall determine asymptotics (1.18) for the case $\alpha = (m, m), m = 1, ..., N$. Let $z_{m,m}^\pm = \pi n + \varepsilon \mu$ and $\mu - V_0 = \xi \to 0$. Using the simple transforamation (unitary), i.e., changing the lines and columns, we obtain
\[
\det(V^0 + \hat{V}^{(n)} + O(\varepsilon) - \mu) = \det\left(\begin{array}{cc} A_1 - \xi & A_2 \\ A_3 & A_4 - \xi \end{array}\right) = \det(A_4 - \xi) \det K(\xi),
\]
\[
A_1 = \left(\begin{array}{cc} V_{cn}^{\ri \pi n} & V_{sn}^{\ri \pi n} \\ V_{sn}^{\ri \pi n} & V_{cn}^{\ri \pi n} \end{array}\right) + O(\varepsilon), \quad A_4 = \text{diag}\{V_2^0 - V_1^0, ..., V_N^0 - V_1^0\} + A_5,
\]
\[
A_2, A_3, A_5 = O(\delta_n), \quad \delta_n = |\hat{V}^n| + \varepsilon, \quad K(\xi) = A_1 - \xi - A_2(A_4 - \xi)^{-1} A_3 + O(\varepsilon)
\]
We have $K(\xi) = A_1 - \xi + O(\phi), \phi = \varepsilon + |\hat{V}^n|^2$, which yields
\[
0 = \det K(\xi) = \xi^2 - |\hat{V}^n|_1^2 - \xi b_1 + V_{11}^c b_2 + V_{11}^s b_3 + O(\phi^2), \quad b_1, b_2, b_3 = O(\phi)
\]
where $b_1, b_2, b_3$ are analytic functions of $\xi$. Rewriting the last equation in the form $(\xi + \alpha)^2 = (|\hat{V}^n|_1 + \beta)^2 + O(\phi^2), \alpha, \beta = O(\phi)$ and using the estimate $\sqrt{x^2 + y^2} - x \leq y$ for $x, y \geq 0$ we get $\xi = \pm |\hat{V}^n|_1 + O(\phi)$, which yields (1.18) for the case $\alpha = (m, m)$.

Consider the resonances. Assume that $V_i^0 \neq V_j^0$ for all $i \neq j \in \omega_s$ for some $s = 1, ..., N$. By Lemma 3.2, the zeros of $\rho_s$ have the form $z_{\alpha}^\pm, \alpha = (j, j'), j, j' \in \omega_s, j < j', n \in \mathbb{Z} \setminus \{0\}$ and satisfy $|z_{\alpha}^\pm - \pi n| < \pi/2$.

Asymptotics (1.11) yields $\Delta_j(z) - \Delta_{j'}(z) = (V_j^0 - V_j^0 \sin \frac{\pi z}{2\pi n} + O(z^{-2}e^{1\text{Im}z})), \quad |z| \to \infty$. Then $|z_{\alpha}^\pm - \pi n| < \pi/2$ yields $z_{\alpha}^\pm = \pi n + O(1/n)$ as $n \to \infty$.

We have the identity $\Delta_j(z) - \Delta_{j'}(z) = 0$ at $z = z_{\alpha}^\pm$. Then using (3.23) we have
\[
\cos\left(z_{\alpha}^\pm - \frac{V_j^0}{2\pi n}\right) - \cos\left(z_{\alpha}^\pm - \frac{V_j^0}{2\pi n}\right) = 2(-1)^n \sin \frac{V_j^0 - V_j^0}{4\pi n} \sin \left(z_{\alpha}^\pm - \pi n - \frac{V_j^0 + V_j^0}{4\pi n}\right) = O(\delta_n)^2
\]
which yields (1.16), i.e.,
\[
z_{\alpha}^\pm = \pi n + \varepsilon(a_+ + z_{\alpha}^\pm), \quad a_+ = \frac{V_j^0 \pm V_j^0}{2}, \quad \varepsilon = \frac{1}{2\pi n}, \quad wt z_{\alpha}^\pm = \delta_n. \quad (3.37)
\]
We shall show that for large $n$ in the neighborhood of each $\pi n + \varepsilon a_+$ the function $(\Delta_0(z) - \Delta_j(z))^2$ has two real zeros resonances (counted with multiplicity). Introduce the functions

$$f_m(\mu) = 2(2\pi n)^2(1 - (-1)^n \Delta_m(\pi n + \varepsilon \mu)) = (\mu - V_m^0)^2 + O(\delta_n), \quad \delta_n = |\hat{V}(n)| + \varepsilon.$$

For the case $\mu \to a_+$ we get

$$f_m(\mu) = (a_+ - V_m^0)^2 + o(1), \quad m = 1, \ldots, N, \text{ and } f_m(\mu) = a_+^2 + o(1), \quad m = j, j'.$$

Hence the function $f_j - f_j'$ (maybe) has the zeros, but the functions $f_j - f_m, m \neq j, j'$ have not zeros in the neighborhood of the point $a_+$.

Note that these functions are real outside the small neighborhood of $a_+$, otherwise for any complex branches there exists a complex conjugate branch, but the asymptotics (3.38) show that such branches are absent.

We have two cases: (1) let $f_m(\mu), m = j, j'$ be real in some small neighborhood of $a_+$.

Then the function $f_j - f_j'$ has at least one real zero, since by Theorem 1.1, the functions $f_j, f_j'$ are strongly monotone. Thus $(f_j - f_j')^2$ has at least 2 real zeros.

(2) Let $f_m(\mu), m = j, j'$ be complex in some small neighborhood of $a_+$. Then they have at least two real branch points. Thus $(f_j - f_j')^2$ has at least 2 real zeros.

Hence $(f_j - f_j')^2$ has exactly two real zeros, since the number of resonances (in the neighborhood of the point $\pi n$) is equal to $N_s(N_s - 1)$.

We determine the sharp asymptotics of resonances. Define the unitary matrix $P = \frac{1}{\sqrt{2}}(J_1 + iJ_0) = P^*, P^2 = I_{2N}$. Using the identities

$$PJP = -iJ_1, \quad PJ_0P = iJ_0, \quad PJ_2P = -J_2,$$

$$\mathcal{V}_n = -iJ_1P\hat{V}^nP = -iJ_1(iJ\hat{V}^{cn} - J_2\hat{V}^{sn}) = J_2\hat{V}^{cn} + i\hat{V}^{sn} = \begin{pmatrix} 0 & \hat{V}(n) \\ \hat{V}(n)^* & 0 \end{pmatrix},$$

we have

$$A = P\left(\frac{(-1)^n\tilde{M}(z) - I}{i\varepsilon}\right)P = J_1(V_0 + PV^nP + O(\varepsilon) - \mu) = J_1(V_0 - \mu) + i\mathcal{V}_n + O(\varepsilon).$$

The operator $A - a_-$ has the eigenvalue $\xi_0 = \frac{(-1)^n\tau_{n,s,-1} - a_-}{i\varepsilon}$ of multiplicity two, since $\tau_{n,s} = \tau(z_n^{s+}) = (-1)^n e^{i\varepsilon(a_- + o(1))}$. The operator $J_1(V_0 - a_+) - a_- = \begin{pmatrix} V_0 - V_j^0 & 0 \\ 0 & -V_0 + V_j^0 \end{pmatrix}$ has two eigenvalue $(= 0)$ and other eigenvalues are not zeros. Using the simple transformation (unitary), i.e., changing the lines and columns, $\mu = a_+ + r \in \mathbb{R}, r = \frac{z^{s+}_n}{a} \to 0, v = \hat{V}_{j,j'}^{r(n)}$, we obtain

$$F(\xi) = \det(A - a_- - \xi) = \det\begin{pmatrix} A_1 - \xi & A_2 \\ A_4 & A_3 - \xi \end{pmatrix} = \det(A_3 - \xi) \det(K(\xi) - \xi I_2),$$

$$A_4 = \begin{pmatrix} -r & rv \\ iv & r \end{pmatrix} + O(\varepsilon), \quad K(\xi) = A_1 - A_2(A_3 - \xi)^{-1}A_4 = \begin{pmatrix} -r + a_1 & iv + a_4 \\ iv + a_3 & r + a_2 \end{pmatrix},$$

where $a_1 = \frac{a}{\pi n}, a_2 = \frac{a}{i\pi n}, a_3 = \frac{a}{\pi n}, a_4 = \frac{a}{i\pi n}$. Where $a$ is the coefficient of the spectral function $\mu = a_+ + r, r \in \mathbb{R}, r = \frac{z^{s+}_n}{a} \to 0, v = \hat{V}_{j,j'}^{r(n)}$, we obtain

$$A_1 = \begin{pmatrix} -r & rv \\ iv & r \end{pmatrix} + O(\varepsilon), \quad K(\xi) = A_1 - A_2(A_3 - \xi)^{-1}A_4 = \begin{pmatrix} -r + a_1 & iv + a_4 \\ iv + a_3 & r + a_2 \end{pmatrix},$$

where $a_1 = \frac{a}{\pi n}, a_2 = \frac{a}{i\pi n}, a_3 = \frac{a}{\pi n}, a_4 = \frac{a}{i\pi n}$. Where $a$ is the coefficient of the spectral function $\mu = a_+ + r, r \in \mathbb{R}, r = \frac{z^{s+}_n}{a} \to 0, v = \hat{V}_{j,j'}^{r(n)}$, we obtain

$$A_1 = \begin{pmatrix} -r & rv \\ iv & r \end{pmatrix} + O(\varepsilon), \quad K(\xi) = A_1 - A_2(A_3 - \xi)^{-1}A_4 = \begin{pmatrix} -r + a_1 & iv + a_4 \\ iv + a_3 & r + a_2 \end{pmatrix},$$
$A_3 = \text{diag}\{V_m^0 - V_j^0 - r, m \neq j\} \oplus \text{diag}\{V_m^0 - V_j^0 + r, m \neq j\} + A_4, \quad A_2, A - 3, A_4 = O(\delta_n),$ 
the function $a_1, a_2, a_3, a_4 = O(\phi), \phi = \varepsilon + |\dot{V}(n)|^2$ and they analytic with respect to $\xi$ in some small disk. The function $F = \det(K(\xi) - \xi I_2)$ has the form

$$F(\xi) = \xi^2 - r^2 + |v|^2 + a_1(r - \xi) + a_2(-r - \xi) - iv\bar{a}_3 - iv\bar{a}_4a_3 = (\xi - \xi_0)^2(1 + O(\xi - \xi_0)).$$

for $\xi \to 0$ where $\xi_0 = ((-1)^n \approx_n - 1) - a_-$ is the zero of $F$ of multiplicity two. Then $\xi_0 = O(\phi)$ and we have $(r - \alpha)^2 = (|v| - \beta)^2 + O(\phi^2)$ where $\alpha, \beta = O(\phi).$ Then using the estimate $\sqrt{x^2 + y^2} - x \leq y$ for $x, y \geq 0$ we get $r = |v| + O(\phi).$

\textbf{Proof of Corollary 4.3} (i) Let $N_G = \infty.$ Then, due to the Lyapunov Theorem, Theorem 4.2 there exists a real sequence $\lambda_k \to +\infty$ as $k \to \infty,$ such that $\lambda_k \in \gamma_{j(m)}^{(j(m)),m}$ for each $m = 1, \ldots, N.$ Hence, $\cap_{m=1}^N \gamma_{j(m),m}^{(j(m)),m} \neq \emptyset.$ Using asymptotics [1.18] and $k \to \infty,$ we obtain $V_1^0 + V_1^0 = \ldots = V_N^0 + V_j^0.$ Moreover, the estimates $V_1^0 < \ldots < V_N^0$ yield $V_j^0 = \ldots = V_j^0,$ i.e. $j(1) = N, j(2) = N - 1, \ldots$ Then, $V_1^0 + V_N^0 = V_2^0 + V_N^0 = \ldots,$ which give a contradiction.

(ii) Let $2a = V_1^0 + V_N^0 = V_2^0 + V_N^0 = \ldots.$ Due to [1.18], $(\pi n_k)^2 + a \in \cap_{m=1}^N \gamma_{n_k}^{N+1-m,m}$ as $k \to \infty.$ Then the Lyapunov Theorem yields $(\pi n_k)^2 + a \notin \sigma(\mathcal{L}),$ $k \to \infty,$ i.e. $N_G = \infty.$

\section{Harmonic functions}

In this Sect. we will prove Theorems 4.1 and 4.2 about the properties of the quasimomentum. Recall that the Lyapunov function $\Delta_s(\zeta)$ is analytic on some $N_s$-sheeted Riemann surface $\mathcal{R}_s$ and $\mathcal{R} = \cup_{1}^{N_0} \mathcal{R}_s.$ Let $z = x + iy \in \mathbb{C}$ be the natural projection of $\zeta \in \mathcal{R},$ $\mathcal{B}_s$ be the set of all branch points of the Lyapunov function and $\mathcal{B}^\pm = \{ \zeta \in \mathcal{R} : \pm \text{Im} \zeta > 0 \}.$ We define the simply connected domains $\mathcal{R}_0^\pm \subset \mathbb{C}_\pm$ and a domain $\mathcal{R}_0$ by

$$\mathcal{R}_0^\pm = \mathbb{C}_\pm \setminus \beta_\pm, \quad \beta_\pm = \bigcup_{\beta \in \mathcal{B}_s \cap \mathbb{C}_\pm} [\beta, \beta \pm i\infty].$$

$$\mathcal{R}_0 = \mathbb{C} \setminus (\beta_+ \cup \beta_- \cup \beta_0), \quad \beta_0 = \{ z \in \mathbb{R} : \Delta_m(z) \notin \mathbb{R} \text{ for some } m \in \{1, \ldots, N\}\}$$

Due to the Lyapunov Theorem, $\Delta(\zeta) \notin [-1, 1], \zeta \in \mathcal{R}^\pm.$ Recall that $q(\zeta) = |\log \eta(\Delta(\zeta))|$ is the single-valued on $\mathcal{R}^\pm$ imaginary part of the (in general, many-valued on $\mathcal{R}^\pm$) quasimomentum $k(\zeta) = p(\zeta) + iq(\zeta) = \text{arccos} \Delta(\zeta) = i\log \eta(\Delta(\zeta)),$ where

$$\eta(z) \equiv z + \sqrt{z^2 - 1}, \quad \eta : \mathbb{C} \setminus [-1, 1] \to \{ z \in \mathbb{C} : |z| > 1 \}.$$  

We denote by $q_m(z), z \in \mathbb{C}_+, m = 1, \ldots, N,$ the branches of $q(\zeta)$ and by $p_m(z), k_m(z), z \in \mathcal{R}_0^+,$ the single-valued branches of $p(\zeta), k(\zeta),$ respectively.

\textbf{Theorem 4.1.} Assume that $V \in \mathcal{K}$ and fix some $j = 1, \ldots, N_0.$ Then the function $\tilde{q}_s(\zeta) = \log |\eta(\tilde{\Delta}_s(\zeta))|$ is subharmonic on the Riemann surface $\mathcal{R}_s$ and the following asymptotics are fulfilled:

$$\tilde{q}_s(\zeta) = y + O(1/|z|), \quad y > r|x|, \quad \text{any } r > 0, \quad (4.1)$$
\[ \tilde{q}_s(\zeta) = y + O(|z|^{-\frac{1}{2}}), \tag{4.2} \]

as \(|\zeta| \to \infty, \zeta \in \mathbb{R}_s\). Moreover, let \(\Delta_j\) be analytic on some bounded interval \(Y = (\alpha, \beta) \subset \mathbb{R}\) for some \(j \in \omega_s\). Then

i) If \(\Delta_j(z) \in \mathbb{R} \setminus [-1,1]\) for all \(z \in Y\), then \(k_j(\cdot)\) has an analytic extension from \(\mathbb{R}_0^+\) into \(\mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y\) such that

\[
\Re k_j(z) = \text{const} \in \pi \mathbb{Z}, \quad z \in Y, \tag{4.3}
\]

\[
q_j(z) = q_j(\overline{z}) > 0, \quad z \in \mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y. \tag{4.4}
\]

ii) If we assume \(\Delta_j(z) \notin \mathbb{R}\) for any \(z \in Y\), then there exists a branch \(\Delta_i, i \in \omega_s\) such that \(\Delta_i(z) = \Delta_j(z)\) for any \(z \in Y\). The functions \(\Delta_j(z)\) and \(k_i + k_j\) have analytic extensions from \(\mathbb{R}_0^+\) into \(\mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y\) such that

\[
\Delta_j(z) = \begin{cases} \Delta_j(z) & \text{if } z \in \mathbb{R}_0^+, \\ \Delta_i(z) & \text{if } z \in \mathbb{R}_0^- . \end{cases} \tag{4.5}
\]

\[
p_j(z) + p_i(z) = \text{const} \in 2\pi \mathbb{Z}, \quad z \in Y, \quad q_j(z) = q_i(z), \quad z \in Y, \tag{4.6}
\]

\[
q_j(z) + q_i(z) = q_j(\overline{z}) + q_i(\overline{z}) > 0, \quad z \in \mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y. \tag{4.7}
\]

**Proof.** By Theorem 1.1, the function \(\tilde{s}_s\) is analytic on \(\mathbb{R}_s\) and the function \(\eta(\cdot)\) is subharmonic on \(\mathbb{C}\). Then \(\tilde{q}_s(\zeta) = \log |\eta(\tilde{s}_s(\zeta))|\) is subharmonic on \(\mathbb{R}_s\). Using the asymptotics \(1.20\), \(1.11\) we obtain

\[
\tilde{q}_s(\zeta) = \log |2 \cos z + O(y^{-1})| = y + O(y^{-1}), \quad |z| \to \infty, \quad y > r|x|,
\]

which yields \(1.1\). Due to \(1.11\) and Lemma 6.2 we have

\[
\tilde{q}_s(\zeta) = \log |\eta(\cos z + O(e^{\Im z}/z))| = \log |\eta(\cos z)| + O(1/\sqrt{z}) = y + O(1/\sqrt{z})
\]

which yields \(4.2\).

i) Due to \(\Delta_j(z) = \cos k_j(z)\) we obtain \(4.3\). The real part of \(k_j\) is a constant on \(Y\), then \(k_j\) has an analytic extension from \(\mathbb{R}_0^+\) into \(\mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y\). Moreover, \(q_j\) has an harmonic extension from \(\mathbb{R}_0^+\) into \(\mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y\) by \(q_j(z) = q_j(\overline{z}), \ z \in \mathbb{R}_0^+\).

ii) By Theorem 1.1 each polynomial \(\Phi_s(\nu, \nu) = \prod_{n \in \omega_s}(\nu - \Delta_n(z)), \ z \in \mathbb{R}\) is real for \(\nu \in \mathbb{R}\). Then for \(\Delta_j\) there exists a \(\Delta_i\) such that \(\Delta_j(x) = \Delta_i(x), \ x \in Y\). Then by the Morer Theorem, the function \(\Delta_j\) has an analytic extension given by \(4.5\) from \(\mathbb{R}_0^+\) into \(\mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y\). Using \(\Delta_m(z) = \cos k_m(z), m = j, i\) we obtain \(0 = \Delta_j(x) - \Delta_i(x) = -2 \sin \frac{k_j(x) + \overline{k_i(x)}}{2} \sin \frac{k_j(x) - \overline{k_i(x)}}{2}\). Thus we get \(4.6, 4.7\) since \(q_j(x) > 0, q_i(x) > 0\) on \(Y\). ■

Recall the needed properties of the functions \(v \in SC\) defined in Sect. 1 and \(w = u + iv\). It is well known, that \(u \in C(\mathbb{C}_+)\) and \(\frac{1}{2\pi} \Delta v = \mu_v\) (in a sense of distribution) is a so-called Riesz measure of the function \(v\). Moreover, the following identities are fulfilled:

\[
\pi \mu_v((x_1, x_2)) = u(x_2) - u(x_1), \quad \text{for any } x_1 < x_2, \ x_1, x_2 \in \mathbb{R}, \tag{4.8}
\]

\[
\frac{\partial v(z)}{\partial y} = y \int_\mathbb{R} \frac{d \mu_v(t)}{(t - x)^2 + y^2}, \quad z = x + iy \in \mathbb{C}_+, \tag{4.9}
\]
which yields \( \frac{\partial v(z)}{\partial y} \geq 0, z \in \mathbb{C}_+ \). Moreover, \( v(x) = v(x \pm i0), x \in \mathbb{R} \). It is well known that if \( v \in \mathcal{SC} \), then

\[
\int_{\mathbb{R}} \frac{d\mu_v(t)}{1 + t^2} < +\infty, \quad \lim_{z \to \infty} v(z) = \lim_{y \to +\infty} \frac{w(iy)}{iy} = \lim_{x \to +\infty} \frac{v(iy)}{y} = \lim_{x \to +\infty} \frac{u(x)}{x} > 0. \tag{4.10}
\]

Now we recall the well known fact (see [Ah]).

**Theorem (Nevanlinna).** i) Let \( \mu \) be a Borel measure on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} (1 + x^{2p})d\mu(x) < +\infty \) for some \( p \in \mathbb{Z}_+ \). Then for each \( r > 0 \) the following asymptotics is valid

\[
\int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = -\sum_{k=0}^{2p} \frac{Q_k}{z^{k+1}} + \frac{O(1)}{z^{2p+1}}, \quad |z| \to \infty, \quad y > r|x|, \quad Q_n = \int_{\mathbb{R}} x^{n}d\mu(x), \quad 0 \leq n \leq 2p.
\]

ii) Let \( F \) be an analytic function in \( \mathbb{C}_+ \) such that \( \text{Im} F(z) \geq 0, \ z \in \mathbb{C}_+ \) and

\[
\text{Im} F(iy) = c_0 y^{-1} + \ldots + c_{2p-1} y^{-2p} + O(y^{-2p-1}) \quad \text{as} \quad y \to \infty \tag{4.11}
\]

for some \( c_0, \ldots, c_{2p-1} \) and \( p \geq 0 \). Then \( F(z) = C + \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}, \ z \in \mathbb{C}_+ \), for some Borel measure \( \mu \) on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} (1 + x^{2p})d\mu(x) < +\infty \) and \( C \in \mathbb{R} \).

**Theorem 4.2.** Assume that \( V \in \mathcal{H} \). Then the function \( w = \frac{1}{N} \sum_{m=1}^{N} k_m \) is analytic in \( \mathbb{C}_+ \); the function \( v = \text{Im} w = \frac{1}{N} \sum_{m=1}^{N} q_m \) belongs to \( \mathcal{SK}^+ \cap C(\mathbb{C}) \) and it is positive harmonic in \( \Omega = \mathbb{C}_+ \cup \mathbb{C}_- \cup g \). Moreover, the following asymptotics and identities hold

\[
u(z) = \text{const} \in \left[ \frac{\pi}{N} \mathbb{Z} \right], \quad v(z) > 0, \quad z \in g_n, \tag{4.12} \\
v(z) = y + O(1/y), \quad |z| \to \infty, \quad y > r|x|, \quad \text{any} \quad r > 0, \tag{4.13} \\
v(z) = y + O(|z|^{-\frac{1}{2}}) \quad |z| \to \infty. \tag{4.14}
\]

**Proof.** Recall that \( k_m(z) = i \log \eta(\Delta_m(z)), \ z \in \mathcal{R}_0^+, \) where \( \Delta_m(z), \ m = 1, \ldots, N, \) are the branches of the function \( \Delta \) analytic on \( \mathcal{R} \). Since \( \Delta(\zeta) \notin [-1, 1], \ \zeta \in \mathcal{R}^+ \), the function \( \eta(\Delta(\zeta)) \) is analytic on \( \mathcal{R}^+ \) and \( |\eta(\Delta(\zeta))| > 1, \ \zeta \in \mathcal{R}^+ \). Recall that \( B_{\Delta} \) is the set of all branch points of the function \( \Delta \). Define the function \( F(z) := \prod_{m=1}^{N} \eta(\Delta_m(z)), \ z \in \mathbb{C}_+ \). Due to the symmetry of \( F \) with respect to the permutations of \( \Delta_1, \ldots, \Delta_n \), we deduce that \( F \) is analytic in the domain \( B(z_0, r) \setminus \{z_0\} \) for any \( z_0 \in \mathbb{C}_+ \) and some \( r = r(z_0) \), including the case \( z_0 \in \mathbb{C}_+ \cap B_{\Delta} \). Note that \( F \) is bounded in \( B(z_0, r) \). This implies \( F \) is analytic in \( B(z_0, r) \) for any \( z_0 \in \mathbb{C}_+ \), i.e. \( F \) is analytic in \( \mathbb{C}_+ \) and \( |F(z)| > 1, \ z \in \mathbb{C}_+ \). Therefore, the averaged quasimomentum

\[
w(z) = \frac{i}{N} \log F(z) = \frac{1}{N} \sum_{m=1}^{N} i \log \eta(\Delta_m(z)) = \frac{1}{N} \sum_{m=1}^{N} k_m(z) \tag{4.15}
\]

is analytic in \( \mathbb{C}_+ \) and \( v(z) = \text{Im} w(z) > 0, \ z \in \mathbb{C}_+ \). Moreover, \( w \) is continuous in \( \mathbb{C}_+ \). Using this fact and Theorem 4.1 we deduce that \( v \) has an harmonic extension from \( \mathbb{C}_+ \) into \( \Omega \) by
We deduce that since \( v \) is continuous in \( \mathbb{C} \) and \( v \) is harmonic in \( \mathbb{C}_+ \cup \mathbb{C}_- \cup g \) we have to show that \( v \) is subharmonic at \( z_0 \in \mathbb{R} \setminus g \). Note that \( v \) is continuous in \( \mathbb{C} \).

Introduce the sets
\[
\alpha_s(z) = \{ m : \Delta_m(z) \in [-1, 1] \}, \quad \alpha_g(z) = \{ m : \Delta_m(z) \in \mathbb{R} \setminus [-1, 1] \},
\]
and \( \alpha_c(z) = \{ m : \Delta_m(z) \notin \mathbb{R} \} \). We have the decomposition
\[
v = v_s + v_c + v_g, \quad v_j = \sum_{m \in \alpha_j(z_0)} q_m \frac{m}{N}, \quad j = s, c, g.
\]

We have 2 cases: the first case \( z_0 \notin \mathcal{B}_\Delta \).

Consider \( v_g \). We take the interval \( Y = (z_0 - \varepsilon, z_0 + \varepsilon) \) for some \( \varepsilon > 0 \) such that \( Y \cap \mathcal{B}_\Delta = \emptyset \). Due to Theorem 4.1 each \( q_m(z), m \in \alpha_g(z_0) \) has an harmonic extension from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \cup \mathbb{R}^- \cup \mathbb{Y} \) by \( q_m(z) = q_m(\mathbb{1}) \), \( z \in \mathbb{R}^+ \) for sufficiently small \( \varepsilon \). Thus \( v_g \) has a harmonic extension from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \cup \mathbb{R}^- \cup \mathbb{Y} \) by \( v_g(z) = v_g(\mathbb{1}) \), \( z \in \mathbb{R}^+ \).

Consider \( v_c \). Due to Theorem 4.1 we deduce that
\[
v_c = \sum_{m \in \alpha_c(z_0)} q_m(z) = \frac{1}{2} \sum_{m \in \alpha_c(z_0)} q_m(z) + q_{\bar{m}}(z),
\]
where \( k_{\bar{m}}(z), \bar{m} \in \alpha_c, \) is the quasimomentum such that \( \Delta_{\bar{m}}(z) = \Delta_m(z), z \in \mathbb{1} \). Due to Theorem 4.1 each \( v_m = q_m(z) + q_{\bar{m}}, m \in \alpha_g(z_0) \) has an harmonic extension from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \cup \mathbb{R}^- \cup \mathbb{Y} \) by \( v_m(z) = v_m(\mathbb{1}) \), \( z \in \mathbb{R}^+ \) for sufficiently small \( \varepsilon \). Thus \( v_c \) has an harmonic extension from \( \mathbb{R}_0^+ \) into \( \mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup \mathbb{Y} \) by \( v_c(z) = v_c(\mathbb{1}) \), \( z \in \mathbb{R}^+ \).

Hence due to \( v_s(z_0) = 0 \) we obtain \( v(z_0) \leq \frac{1}{2 \pi} \int_0^{2 \pi} v(z_0 + re^{i\phi})d\phi \), for any small \( r > 0 \), and thus \( v \) is subharmonic in some neighborhood of \( z_0 \).

The second case \( z_0 \in \mathcal{B}_\Delta \). Define two intervals \( Y_- = (z_0 - 2\varepsilon, z_0), Y_+ = (z_0 + 2\varepsilon, z_0 + 2\varepsilon) \), for some \( \varepsilon > 0 \) such that \( Y_\pm \cap \mathcal{B}_\Delta = \emptyset \) and each \( q_m \) is harmonic in \( \{|z - z_0| < \varepsilon\} \cap \mathbb{C}_+ \). Define the functions
\[
v = v_1 + v_0, \quad v_0(z) = \sum_{m \in \alpha_0} q_m(z), \quad \alpha_0 = \alpha_s(z_0 + \varepsilon) \cup \alpha_s(z_0 + \varepsilon).
\]
Note that \( v_0(z_0) = 0 \). The above arguments show that the function \( v_1 \) has a harmonic extension from \( \mathbb{R}_0^+ \) into \( \mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup Y_- \cup Y_+ \) by \( v_1(z) = v_1(\mathbb{1}) \) for sufficiently small \( \varepsilon > 0 \). Then the function \( v_1 \) has a harmonic extension from \( \mathbb{R}_0^+ \) into \( \mathbb{R}_0^+ \cup \mathbb{R}_0^- \cup (z_0 - 2\varepsilon, z_0 + 2\varepsilon) \) by \( v_1(z) = v_1(\mathbb{1}) \) for sufficiently small \( \varepsilon \), since the function \( v_1 \) is bounded in the disk \( B(z_0, t) = \{z : |z - z_0| < \varepsilon\} \) for some small \( \varepsilon > 0 \). Thus we obtain \( v(z_0) \leq \frac{1}{2 \pi} \int_0^{2 \pi} v(z_0 + re^{i\phi})d\phi \) for any small \( r > 0 \), since \( v_0(z_0) = 0 \). Hence the function \( v \) is subharmonic in \( \mathbb{C} \).

Asymptotics (1.1), (1.2) give (1.12), (1.13). Moreover, (1.13) yields \( \int_{\mathbb{R}} \frac{w(t)}{1 + t} dt < \infty \) and \( \limsup_{t \to \infty} \frac{w(t)}{1 + t} = 1 \), i.e., \( v \in \mathcal{S}\mathcal{C} \).

The function \( v(z) - y \) is harmonic in \( \mathbb{C}_+ \), it is positive on the real line and \( v(z) - y = o(1) \) as \( |z| \to \infty, z \in \mathbb{C}_+ \). Then \( v(z) - y \) is positive on \( \mathbb{C}_+ \). Using (1.12) and the Nevanlinna Theorem we deduce that \( \int_{\mathbb{R}} v(t) dt < \infty \). Then \( v \in \mathcal{S}\mathcal{K}_0^+ \).
5 The conformal mappings

We need the result from [KK1].

**Theorem 5.1.** Let \( v \in SK_0^+ \) and \( v \neq \text{const} \). Then \( w : \mathbb{C}_+ \to w(\mathbb{C}_+) = W(h) \) is a conformal mapping for some \( h \in C_{us}, h \geq 0 \). Moreover, the following asymptotics, estimates and identities are fulfilled:

\[
w(z) = z - \frac{Q_0 + o(1)}{z}, \quad |z| \to \infty, \quad y \geq r|x|, \quad \text{for any } r > 0, \tag{5.1}
\]

\[
I_0^D + \mathcal{P}_0 = Q_0, \quad \sup_{x \in \mathbb{R}} v^2(x) \leq 2Q_0. \tag{5.2}
\]

If in addition, \( Q_2 < \infty \), then \( Q_2 = I_1^D + \mathcal{P}_2 \), where \( \mathcal{P}_n, I_n^D, Q_n, n \geq 0 \) are given by (123).

**Proof of Theorem 1.4.** By Theorem 1.2 the function \( v = \frac{1}{N} \sum_{m=1}^{N} q_m \in SK_0^+ \). Then Theorem 5.1 gives that \( w : \mathbb{C}_+ \to w(\mathbb{C}_+) = W(h) \) is a conformal mapping for some \( h \in C_{us} \).

The function \( M(iy), y > 0 \) is real, then \( M(z) = M(-z), z \in \mathbb{C}_+ \). This yields that the set \( \{\tau_m(z)\}_{1}^{N} = \{\tau_m(-z)\}_{1}^{N}, z \in \overline{\mathbb{C}}_+ \), which gives \( v(z) = v(-z), z \in \mathbb{C}_+ \). Thus \( v(x) = v(-x) \) for all \( x \in \mathbb{R} \) and the identities

\[
w(-z) = -\overline{z} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(t)dt}{t + \overline{z}} = -\overline{z} - \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(s)dt}{s - \overline{z}} = -\overline{w(z)}, z \in \mathbb{C}_+
\]

give \( -w_0(-z) = \overline{w}_0(z), z \in \mathbb{C}_+ \).

Using (1.11) and \( q_m \geq 0 \) we obtain \( q_m(iy) = y + o(1) \) and

\[
\det L(z) = \prod_{m=1}^{N} \Delta_{m}^2(z) = \frac{(1 + e^{-4NyO(1)})}{4^N} e^{-2i\sum_{m=1}^{N} k_m(z)} = \frac{(1 + e^{-4NyO(1)})}{4^N} e^{-2iW(z)} \tag{5.3}
\]

as \( z = iy, y \to \infty \) and (3.18) yields

\[
\det L(z) = (\cos^2N z) \exp\left(\frac{i \text{Tr} V^0}{z} + \frac{i\|V\|^2 + o(1)}{4z^3}\right). \tag{5.4}
\]

Thus due to \( \text{Re } w(iy) = 0 \) we get

\[
w(z) = z - \frac{\text{Tr} V^0}{2Nz} - \frac{\|V\|^2 + o(1)}{8Nz^3}, \quad z = iy, \quad y \to \infty. \tag{5.5}
\]

Then by the Nevanlinna Theorem and Theorem 5.1 we have asymptotics (1.25) and identities (1.26), (1.27).

Estimate (5.2) gives \( v |_{\mathbb{R}} \leq \sqrt{2Q_0} \). If \( z \in \sigma_{(N)}, \) then \( v(z) = 0 \), since \( \Delta_m(z) \in [-1,1] \) and \( q_m(z) = 0 \) for all \( m = 1, \ldots, N \). If \( z \in \sigma_{(1)} \cup g, \) then \( \Delta_m(z) \notin [-1,1] \) for some \( m = 1, \ldots, N \). Thus \( q_m(z) > 0 \) and \( v(z) > 0 \), which gives (1.28).

**Proof of Theorem 1.6.** i) We show (1.30). Using \( w(z) = z + \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(t)dt}{(t-x)^2}, z \in \mathbb{C}_+ \) and (1.28) we obtain \( w'(z) = 1 + \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(t)dt}{(t-x)^2} > 1, \quad z \in \sigma_{(N)} \), which yields \( u'_z(z) > 1, z \in \sigma_{(N)} \). We used the fact that due to (1.17) \( u'_z(z) = 1 \) for some \( z \in \sigma_{(N)} \) iff \( V = 0 \).
Let $x \in \sigma_{(1)}$. Then some branch $\Delta_m(x) \in (-1, 1)$ for all $x \in Y \subset \sigma_{(1)}$ for some small interval $Y = (\alpha, \beta)$. We have $\Delta_m(x) = \cos k_m(x)$. Thus we get $k'_m(x) = -\frac{\Delta_m(x)}{\sin k_m(x)} \neq 0$. Hence we get $u'''_m(x) = k'_m(x) > 0$, since $q_m(x) = 0$ and (1.9) yields $u(x) \geq 0$ for $x \in Y$, which gives (1.30).

We show (1.31). Note that $v(z) > 0, z \in g$, otherwise we have not a gap. Using (1.9) we obtain $-\frac{\partial^2 v(z)}{\partial x^2} = \frac{\partial^2 v(z)}{\partial y^2} = \int_\mathbb{R} \frac{dv(t)}{(t-x)^2} > 0, \quad z = x \in g_n, \text{ which yields } v''(z) < 0, z \in g_n$. Consider the function $u(z), z \in g_n$. Theorem 4.1 yields $\Re w(z) = \sum_1^N \Re k_m = \sum_1^N \pi n_m = \pi N_n$, which gives (1.31).

We recall the result from [KK2]. Let a function $f$ be harmonic and positive in the domain $\mathbb{C} \setminus [-a, a], a > 0$ and $f(iy) = y(1 + o(1))$ as $y \to \infty$. Assume $f(z) = f(\overline{z}), z \in \mathbb{C} \setminus [-a, a]$ and $f \in C(\mathbb{C}_x)$. Then

$$f(x) = \sqrt{a^2 - x^2} \left(1 + \frac{1}{\pi} \int_{\mathbb{R}\setminus \{0\}} \frac{f(t)dt}{|t-x||t^2-a^2|^{1/2}}\right), \quad x \in [-a, a].$$

Hence the last identity and properties of $v$ yield (1.32) and the estimate $v(z) \geq v_n^0(z) = |(z-z_n^+)(z_n^- - z)|^{1/2}, \quad z \in g_n = (z_n^+, z_n^-)$. Consider the case $g_n \subset \mathbb{R}_+$ (the proof for the cases $g_n \subset \mathbb{R}_-$ is similar). Define $z_n^0 = \frac{z_n^+ + z_n^-}{2}, r = \frac{|g_n|}{2}$ and using $(z_n^+ + x)^{2p} + (z_n^- - x)^{2p} \geq 2z_n^{2p}$ for all $p \geq 0$ we have

$$\frac{1}{\pi} \int_{g_n} t^{2p} v(t) dt \geq \int_{g_n} \frac{t^{2p} v_n^0(t) dt}{\overline{\pi}} \geq \frac{1}{\pi} \int_0^r \sqrt{r^2 - x^2} ((z_n^0 + x)^{2p} + (z_n^- - x)^{2p}) dx \geq \frac{r^2}{2}(z_n^0)^{2p}.$$ 

If $p = 0$, then we get $Q_0 = \frac{1}{\pi} \int_\mathbb{R} v(t) dt \geq \frac{1}{8} \sum |g_n|^2$, which yields the first estimate in (1.33).

If $p = 1$, then using $\gamma_n = \lambda_n^+ - \lambda_n^- \geq 0, \lambda_n^\pm = (z_n^\pm)^2$ we get

$$Q_2 = \frac{1}{\pi} \int_\mathbb{R} t^2 v(t) dt \geq \frac{1}{8} \sum |g_n|^2 \left(\frac{z_n^+ + z_n^-}{4}\right)^2 = \frac{1}{16} \sum |\gamma_n|,$$

which yields the last inequality in (1.33).

Recall estimates from [K10]. The conformal mapping $w : \mathbb{C}_+ \to W(h)$ for the case $w(z) = z + o(1)$ as $|z| \to \infty$ and $h(u) = 0, u \neq \frac{\pi}{N} \in \mathbb{Z}$ and $\{h(\frac{\pi}{N})\}_{N \to \infty} \in L^1_1$ was studied in [K7].

For some absolute constant $C_0$ the following estimate was obtained: $Q_2 \leq C_0 G^2(1 + G^4)$, which together with $Q_2 = \frac{|V|^2}{8N}$ gives the estimate (1.33).

6 Appendix

**Lemma 6.1.** Let function $F(s, t), R(t, s, u), t, s, u \in (0, 1)$ satisfy:

i) $F(\cdot, \cdot) \in L^2((0, 1)^2)$ and $F(t, t) \in L^1((0, 1))$,

ii) $R(\cdot, \cdot, \cdot) \in L^2((0, 1)^3)$ and $R(t, t, s), R(t, s, t), R(s, t, t) \in L^1((0, 1)^2)$.

Then

$$f(z) = \int_0^1 dt \int_0^t \cos z(1 - 2t + 2s)F(t, s)ds = \frac{i \cos z}{2z} \left(\int_0^1 F(t, t)dt + o(1)\right) \quad (6.1)$$
Firstly, let $z$, $z$ be a smooth function such that $|z|^2$ is subharmonic in $C$. Moreover, for some absolute constant $C$ the following estimate is fulfilled:

$$|f(z) - f_0(z)| = O\left(\frac{e^{2y}}{y^2}\right).$$

Proof. The function $f(z) = \log |\xi(z)|$, $z \in \mathbb{C} \setminus [-1, 1]$ is subharmonic and continuous in $C$. Moreover, for some absolute constant $C$ the following estimate is fulfilled:

$$|f(z) - f(z_0)| = C\varepsilon^{3/2}, \quad \text{if} \quad |z - z_0| = \varepsilon \max\{2, |z_0|\}, \quad 0 \leq \varepsilon \leq \frac{1}{8}, \quad z, z_0 \in \mathbb{C}. \quad (6.7)$$

Proof. The function $f(z) = \log |\xi(z)|0 >, z \in \mathbb{C} \setminus [-1, 1]$ is harmonic and $f(z) = 0, z \in [-1, 1]$. Then $f$ is subharmonic in $C$.

1. Let $|z_0| \leq 2$. Consider the case $|z_0 - 1| \leq 1/2$. The proof of other cases is similar. Firstly, let $z, z_0 \in \mathbb{C}$. Then

$$f(z) - f(z_0) = \Re \left[ \int_{z_0}^{z} \frac{dt}{\sqrt{t^2 - 1}} \right] = \Re \left[ \int_{z_0-1}^{z-1} \frac{ds}{\sqrt{s(2 + s)}} \right], \quad \frac{1}{\sqrt{s(2 + s)}} = \frac{1}{\sqrt{s}} + H(s),$$

as $r|x| < y \to \infty$ for any fixed $r > 0$, where $\zeta$ is one of functions: $s - u, t - u$ or $t - s$.

Proof. We have $f(z) = e^{iz} f^+(z) + e^{-iz} f^-(z)$, where

$$f^-(z) = \frac{1}{2} \int_{0}^{1} dt \int_{0}^{t} e^{iz(t-s)} F(t, s) ds = \frac{1}{4} \int_{0}^{1} ds \int_{0}^{t} e^{iz[t-s]} F(t, s) ds$$

where $F(t, s) = F(s, t), t, s \in (0, 1)$. Substituting the identities

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{ipk} \frac{dk}{k^2 - 4z^2} = \frac{i}{4z} e^{iz|p|}, \quad p \in \mathbb{R}, \ z \in \mathbb{C}+, \quad \hat{F}(r, k) = \frac{1}{2\pi} \int_{[0,1]^2} e^{-irt - iks} F(t, s) dt ds,$$

into (6.4) we obtain

$$f^- = \frac{z}{i} \int_{\mathbb{R}} \hat{F}(-k, k) \frac{dk}{k^2 - 4z^2} = \frac{1}{14z} \int_{\mathbb{R}} \left(1 + \frac{k^2}{k^2 - 4z^2}\right) \hat{F}(-k, k) \frac{dk}{k^2 - 4z^2} = \frac{1}{14z} \left(\int_{0}^{1} F(t, t) dt + o(1)\right).$$

Consider $f^+$. Let $\|F\|^2 = \int_{0}^{1} dt \int_{0}^{t} |F(t, s)|^2 ds$. We have

$$|f^+(z)|^2 \leq g(z) \|F\|^2, \quad g(z) = \int_{0}^{1} dt \int_{0}^{t} e^{4yt(t-s)} ds \leq \int_{0}^{1} e^{-yt} \frac{e^{2yt}}{4y} dt \leq \frac{e^{2yt}}{(4y)^2}. \quad (6.5)$$

Let $F_0$ be a smooth function such that $\|F - F_0\| \leq \varepsilon$ for some small $\varepsilon > 0$. Define the function $f_0^+(z) = \int_{0}^{1} dt \int_{0}^{t} e^{-iz(t-s)} F_0(t, s) ds$. Using (6.5) we obtain

$$|f^+(z)| \leq |f_0^+(z)| + |f^+(z) - f_0^+(z)| \leq |f_0^+(z)| + \|F - F_0\| \frac{e^{2yt}}{(4y)^2} \quad (6.6)$$

and the integration by parts yields $f_0^+(z) = O\left(\frac{e^{2yt}}{y^2}\right)$. Thus we obtain (6.2).

The proof for (6.3) is similar. ■

Lemma 6.2. The function $f(z) = \log |\xi(z)|, z \in \mathbb{C} \setminus [-1, 1]$ is subharmonic and continuous in $C$. Moreover, for some absolute constant $C$ the following estimate is fulfilled:

$$|f(z) - f(z_0)| = C\varepsilon^{3/2}, \quad \text{if} \quad |z - z_0| = \varepsilon \max\{2, |z_0|\}, \quad 0 \leq \varepsilon \leq \frac{1}{8}, \quad z, z_0 \in \mathbb{C}. \quad (6.7)$$
where $H(\cdot)$ is analytic and bounded in the disk $B(0,1)$. Then

$$f(z) - f(z_0) = \text{Re} \left( 2\sqrt{s} \right) \frac{z-1}{\overline{z_0} - 1} + O(\varepsilon)$$

which yields (6.7).

Consider the case $z \in \mathbb{C}_-, z_0 \in \mathbb{C}_+$. The identity $f(z) = f(\overline{z})$ gives $f(z) - f(z_0) = f(\overline{z}) - f(\overline{z_0})$, $\overline{z}, \overline{z_0} \in \mathbb{C}_-$. Then the simple estimate $|z - z_0| \leq |z - \overline{z_0}|$ and the case $z, z_0 \in \mathbb{C}_+$ imply (6.7).

Let $|z_0| > 2$. Let $a_0 = z_0/|z_0|$, $a = z/|z_0|$. Note that $|a - a_0| \leq \varepsilon$.

$$f(z) - f(z_0) = \text{Re} \int_{z_0}^{z} \frac{dt}{\sqrt{t^2 - 1}} = \text{Re} \int_{a_0}^{a} F(s) ds, \quad F(s) = \left( s^2 - \frac{1}{|z_0|^2} \right)^{-\frac{1}{2}}, \quad s = t/|z_0|.$$ 

We obtain

$$\left| s^2 - \frac{1}{|z_0|^2} \right| \geq |s|^2 - \frac{1}{4} \geq (1 - \varepsilon) - \frac{1}{4} \geq \frac{1}{2}, \quad \text{for } |s - a_0| \leq \varepsilon.$$ 

Thus the function $F(s) = \left( s^2 - \frac{1}{|z_0|^2} \right)^{-\frac{1}{2}}$ is analytic and bounded in $s$, $|a_0 - s| \leq \varepsilon$. Thus we deduce that $|f(z) - f(z_0)| \leq \varepsilon C$ for some absolute constant $C$. ■

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