Concavity of the meson Regge trajectories

Jiao-Kai Chen

School of Physics and Information Science, Shanxi Normal University, Linfen 041004, China

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It is illustrated by the fitted Regge trajectories for a large majority of mesons that the radial and orbital Regge trajectories for mesons prefer being concave in the \((n_r, M^2)\) and \((l, M^2)\) planes, respectively. The concavity of the meson Regge trajectories is model-independent. The concavity is expected to act as a criterion to choose a newly observed meson or to assign a particle to the unwell-established state. The concavity of the meson Regge trajectories can assist in model construction and in the choice of the appropriate model to describe mesons. The appropriate models should yield the spectra which can produce the concave Regge trajectories according to the concavity of the meson Regge trajectories. If the large majority of the meson Regge trajectories are concave while a few meson Regge trajectories are convex which neither have been confirmed nor have been completely excluded at present, many existing models should be corrected or even be reconstructed, which will lead to the further understanding of the meson dynamics.

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I. INTRODUCTION

It is well-known that the Regge trajectory is one of the effective approaches for studying hadron spectra [1–9]. The famous Chew-Frautschi plots of the Regge trajectories provide a useful way of hadron classification. Linearity is a convenient guide in constructing the Chew-Frautschi plots and the linear Regge trajectories have played an important role in studying hadron spectra. On the other hand, various studies have shown that the Regge trajectories can be nonlinear [10, 11]. The Regge trajectories can be divided into two groups depending on whether they are linear or not. One group is the linear Regge trajectories [8, 9, 12–28]. Another group of the Regge trajectories are nonlinear.

For heavy quarkonia especially for bottomonia, the Regge trajectories deviate significantly from the linearity [29–32]. The nonlinearity of the Regge trajectories for light mesons, for heavy-light mesons and for baryons are also investigated [7, 11, 32–34]. The nonlinear Regge trajectories take different forms. In Refs. [35–41], the authors show that the effective trajectory for large momentum transfer \((-t)\) goes like \(\sqrt{-t}\). In Ref. [42], the authors present the nucleon trajectory obtained from the backward scattering of pions by nucleons, \(\alpha_N(t) = -0.4 + 0.9t + 0.25t^2/2\). In Ref. [43], UA8 propose the effective Pomeron Regge trajectory by analyzing the inclusive differential cross sections for the single-diffractive reactions \(p_i + \bar{p}_i \rightarrow p_f + X\) and \(p + \bar{p}_i \rightarrow X + \bar{p}_f\) at \(\sqrt{s} = 630\) GeV, \(\alpha_R(t) = 1.10 + 0.25t + (0.079\pm0.012)t^2\), where \(t\) is the squared-momentum-transfer. See Refs. [4, 7] for more Regge trajectories.

In Ref. [7], the authors scrutinize the hadron Regge trajectories in the framework of the string and potential models, and they observe the appreciable curvature of the orbital Regge trajectories for mesons and baryons. As our best knowledge, the fitted Regge trajectories which have the inflection points are not observed in the references. Therefore, the nonlinear Regge trajectories are expected to be concave or convex. In Refs. [44, 45], the flattening of the resonance spectrum of hadrons is observed, i.e., the hadron Regge trajectories are concave in the \((J, M^2)\) plane. In Ref. [46], the authors notice the curvature of the Regge trajectories, \(\alpha^0(t) > 0\). The Regge trajectories are concave for the heavy mesons [29, 30] and for many light mesons and baryons [32, 47]. In Refs. [30, 32, 34, 48–52], the fitted curves show evidently the concavity although the authors do not point out explicitly the curvature of the meson Regge trajectories. In Refs. [7, 10, 31], the authors notice the curvature of the Regge trajectories for mesons but only a few Regge trajectories are fitted. We conclude that the generality of the concavity of the meson Regge trajectories has not been pointed out clearly and has not been demonstrated either. In the present work, we illustrate by the fitted Regge trajectories for a large majority of mesons that the meson Regge trajectories, both the radial Regge trajectories and the orbital Regge trajectories, prefer being concave and this property is model-independent. The effects of the concavity of the Regge trajectories on hadron classification and model construction are also discussed.

This paper is organized as follows. In Sec. II, the concavity of the meson Regge trajectories is illustrated. In Sec. III, the general properties of the Regge trajectories and the effects of the concavity of the meson Regge trajectories

*Electronic address: chenjk@sxnu.edu.cn, chenjkphy@outlook.com
on model construction and hadron classification are discussed. We conclude in Sec. IV.

II. CONCAVE MESON REGGE TRAJECTORIES

In this section, we show that the radial and orbital Regge trajectories for the large majority of mesons can be well described by the concave formulas. Some Regge trajectories for the light unflavored mesons are not definite due to scarce data or the unwell-established states. We propose that the radial and orbital meson Regge trajectories prefer being concave in the \((n_r, M^2)\) and \((l, M^2)\) planes, respectively.

A. Concavity of the Regge trajectories for heavy quarkonia

In Ref. \[29\], we propose one new form of the Regge trajectories from the quadratic form of the spinless Salpeter equation \[29, 53–59\] by employing the Bohr-Sommerfeld quantization approach \[60, 61\],

\[
M^2 = \beta_l (l + b_l)^{2/3} + c_l,
M^2 = \beta_{n_r} (n_r + b_{n_r})^{2/3} + c_{n_r},
\]

where \(M\) is the meson mass, \(l\) is the orbital angular momentum, \(n_r\) is the radial quantum number. \(\beta_l, \beta_{n_r}, b_l, b_{n_r}, c_l\) and \(c_{n_r}\) are parameters. For heavy quarkonia, \(\beta_l\) and \(\beta_{n_r}\) are universal parameters.

Applying the concave formulas (1) to fit the spectra of bottomonia and charmonia, the fitted Regge trajectories are in good agreement with the experimental data and the theoretical predictions \[29\]. The Regge trajectories will be concave if the spectra can be described by the concave formulas, therefore, the Regge trajectories for heavy quarkonia are concave. The concavity of these Regge trajectories is independent of the models from which the employed formulas come.

B. Concavity of the Regge trajectories for the mesons consisting of different quarks

In Ref. \[33\], we show that the Regge trajectories (1) are also appropriate for the mesons composed of unequally massive quarks. We employ Eq. (1) to fit the spectra of the strange mesons, the heavy-light mesons (the \(D, D_s, B\) and \(B_s\) mesons) and the bottom-charmed mesons. The fitted Regge trajectories are consistent with the experimental data and the theoretical values. Thus, the Regge trajectories for these mesons are concave because the data can be described by the concave formulas. The concavity of the Regge trajectories is model-independent.

C. Concavity of the Regge trajectories for the light unflavored mesons

For the light unflavored mesons, there are eight radial Regge trajectories and ten orbital Regge trajectories can be described by Eq. (1) which are concave. There are eight radial Regge trajectories and eight orbital Regge trajectories seem convex. No convex Regge trajectories for the light unflavored mesons have been confirmed until now. More theoretical analyses and more experimental data are needed to confirm or exclude the possible convexity of the Regge trajectories. See Appendix A for more discussions.

The parameters \(\beta_l\) and \(\beta_{n_r}\) are universal parameters for heavy quarkonia and the mesons constituting of unequally massive quarks. But for the light unflavored mesons, the universal description of these two parameters does not hold again, see Tables II and III.

D. Convex Regge trajectories

Different from the concave Regge trajectories mentioned above, the Regge trajectories for some light unflavored mesons look like convex. In this subsection, we show by example that the convex Regge trajectories are problematic.

Takes as an example, the radial Regge trajectory for \(a_1(1260), a_1(1640)\) and \(a_1(2095)\) is discussed. \(a_1(1260), a_1(1640)\) are the well-established states, assigned to be the \(1^3P_1\) and \(2^3P_1\) states, respectively. In Refs. \[28, 34, 62\], \(a_1(2095)\) is assigned to be the \(3^3P_1\) state. The Regge trajectory for these three states are convex and cannot be described by the concave Regge trajectories (1). Using the interpolated formula in Eq. (B7), the fitted results are
consistent with the experimental data, see Fig. 1(a). However, the fitted parameter $\gamma_2$ is positive which should be negative. It is not acceptable physically because the first term in Eq. (B7) comes from the color Coulomb potential and should be negative. The same problem exists for the Regge trajectory formula [48, 64, 65]

$$M^2 = \tau_1 (J + 2n_r) + \frac{\tau_2}{(n_r + J + 1)^2} + \tau_0,$$

(2)

where $\tau_1$ should be positive while $\tau_2$ should be negative. The fitted Regge trajectory for the $a_1$ mesons by using Eq. (2) is also unacceptable because $\tau_2 > 0$, see Fig. 1(a).

If the Regge trajectory for the $a_1$ mesons is really convex, Eqs. (B7) and (2) can be the temporary formulas to fit the data. If the correct Regge trajectory for the $a_1$ mesons is concave, the improper Regge trajectory maybe results from the insufficient experimental data or the inappropriate assignment of states. As an example of the insufficient data case, the experimental data can be described by the concave formula as the candidate state $a_1(2270)$ is considered, see Fig. 1(b). In the latter case, the Regge trajectory becomes concave if not $a_1(2095)$ but $a_1(1930)$ is assigned as $3P_1$ state [66], see Fig. 1(c). Some other Regge trajectories for the light unflavored mesons are in this case. Up to present, the convex Regge trajectories for mesons have not been definitely confirmed and have not been completely excluded either.

E. Discussions

The dynamics of the light mesons is more complicated than that of heavy quarkonium because the light mesons are rather sensitive to detailed properties of the confinement mechanism. Therefore, the fitted Regge trajectories for heavy mesons will be better than the Regge trajectories for the light mesons. All fitted Regge trajectories for heavy quarkonia are concave [29]. All the bottom-charmed mesons, the heavy-light mesons (the $D/D_s/B/B_s$ mesons) and the strange mesons can be described by the concave Regge trajectories [Eq. (1)] [33]. Most of the Regge trajectories for the light unflavored mesons are concave. Therefore, the large majority of the Regge trajectories for mesons are concave [7, 10, 32–34, 51, 52]. On the other hand, no Regge trajectories having the inflection points are observed and only a few convex Regge trajectories are given in references [67–70]. Up to date, no convex Regge trajectories for mesons are definitely confirmed. Combining the discussions in this section in Appendix A, we can conclude that the radial and orbital meson Regge trajectories prefer being concave in the $(n_r, M^2)$ and $(l, M^2)$ planes, respectively. The concavity of the meson Regge trajectories is model-independent although the used formulas are obtained from different models.

The curvature of the meson Regge trajectories can be used to classify hadrons. For mesons, the concavity can act as a criterion to choose a newly observed meson or to assign a particle to the unwell-established state. If the great majority of the meson Regge trajectories are concave while a few meson Regge trajectories are convex which have not been confirmed and have not been completely excluded either at present, mesons can be classified according to different curvature.

III. EFFECTS OF THE CONCAVITY OF THE MESON REGGE TRAJECTORIES ON MODELS

In this section, the general properties of the Regge trajectories are presented. The effects of the concavity of the mesons Regge trajectories on model construction are discussed.
A. General properties of the Regge trajectories

Based on the meson spectra obtained experimentally [63] and predicted theoretically [48, 62, 71–77], it is evident that the orbital Regge trajectories $M(l)$ and the radial Regge trajectories $M(n_r)$ should be the strictly increasing functions,

$$\frac{dM^2(l)}{dl} > 0, \quad \frac{dM^2(n_r)}{dn_r} > 0,$$

where $M$ is the meson mass, $l$ is the orbital angular momentum and $n_r$ is the radial quantum number. Let $\kappa$ be the curvature of the Regge trajectories. If the Regge trajectories are linear, $\kappa = 0$,

$$\frac{dM^2(l)}{dl} = \text{const.}, \quad \frac{dM^2(n_r)}{dn_r} = \text{const.}$$

(4)

If the Regge trajectories are convex upwards in the $(l, M^2)$ and $(n_r, M^2)$ planes, $\kappa > 0$,

$$\frac{d^2M^2(l)}{dl^2} > 0, \quad \frac{d^2M^2(n_r)}{dn_r^2} > 0.$$  

(5)

If the Regge trajectories are concave functions, $\kappa < 0$,

$$\frac{d^2M^2(l)}{dl^2} < 0, \quad \frac{d^2M^2(n_r)}{dn_r^2} < 0.$$  

(6)

Considering Eqs. (3) and (6), we can obtain

$$\lim_{l \to \infty} \frac{d^2M^2(l)}{dl^2} = 0, \quad \lim_{n_r \to \infty} \frac{d^2M^2(n_r)}{dn_r^2} = 0$$

(7)

if the variables $l$ and $n_r$ can be extended to infinity. If the Regge trajectories are concave, they should satisfy Eqs. (3), (6) and/or (7). If the Regge trajectories are convex, they should meet Eqs. (3) and (5).

Eq. (7) does not hold if $l$ and $n_r$ have the upper bounds, for example, the Regge trajectory $t = -0.139l^2 + 2.55l + 115.98$ listed in [10], in which $l$ should be smaller than or equal to the maximum value $l_{\text{max}}$. If the Regge trajectories have the inflection points on the domain, they will change from being concave to being convex or vice versa. This kind of Regge trajectories are not favored because the changing from being concave to being convex suggests the possible changing of dynamics. In fact, no meson Regge trajectories having the inflection points have been observed. The Regge trajectories $M^2 = \tau_1(J+2n_r)+\tau_2(n_r+J+1)^{-2}+\tau_0$ [Eq. (2)] [48, 64, 65] and $M^2 = \gamma_1(l+n_r+1)^{-2}+\beta_1(l+b_1)^{2/3}+c_1$ [Eq. (B7)] can be concave, convex or have the inflections if the physical constraints on the parameters are not considered. As the physical constraints on the parameters are considered, they should be concave.

B. Dynamic equations

The potential models are the basic tools of the phenomenological approach to model the features of QCD relevant to hadron with the aim to produce concrete results [48, 62, 71–73, 78–85]. Different dynamic equations and different potentials will lead to different Regge trajectories [29, 33, 53, 54, 64, 65, 85–93]. The concavity of the meson Regge trajectories is of significance for the potential models, because it can assist in the choice of the appropriate dynamic equation and the corresponding potential.

The concavity of the Regge trajectories gives the constraints on the confinement potential if the dynamic equation is ascertained. For the Schrödinger equation, the confinement potential should be $r^a$ with $0 < a < 2/3$ [60, 88]. For the Dirac equation [89], the spinless Salpeter equation [60, 85, 86, 93] and the Klein-Gordon equation [64, 65, 86, 90, 91], the confinement potential should be $r^a$ with $0 < a < 1$. For the quadratic form of the spinless Salpeter-type equation [29] or the eigenvalue equation of the square mass operator [29, 53], the confinement potential should take the form $r^a$ with $0 < a < 2$.

If the confinement potential is $r^a$ with $0 < a < 2/3$, all the dynamic equations mentioned above can produce the concave Regge trajectories. In this case, we need more information to choose the appropriate dynamic equation. For example, the Martin potential $-8.064 + 6.898r^{0.1}$ [94, 95], the Song-Lin potential $-Ar^{-1/2} + kr^{1/2} + V_0$ [96] and the Motyka-Zalewiski potential $-Ar^{-1} + kr^{1/2} + V_0$ [97, 98] are in this case. $A$ is the strong interaction constant, $k$ is the confinement constant.
C. Linear confinement potential

The popular confinement is a linearly rising potential which has been validated by lattice QCD calculations [80, 84, 99–102]. The linear confinement potential represents the nonperturbative behavior of QCD forces between the static heavy quarks at long distances and it corresponds to the square-law limit for the Wilson loop. The linear confinement potential is applied not only to the heavy mesons but also to the light mesons [24, 27, 62, 71, 103–106].

The linear confinement potential is used with different dynamic equations [71, 78, 106–109]. As discussed in III B, the Schrödinger equation with the linear potential gives the convex Regge trajectories while the spinless Salpeter equation, the Dirac equation and the Klein-Gordon equation produce the linear Regge trajectories. The quadratic form of the spinless Salpeter-type equation [29, 33] and the eigenvalue equation for the square mass operator [54–57, 76] lead to the concave Regge trajectories which are in good agreement with the experimental data (see Sec. II). Therefore, if the confinement potential is linear, the quadratic form of the spinless Salpeter-type equation and the eigenvalue equation for the square mass operator are preferred according to the concavity of the meson Regge trajectories.

D. Nonlinear Regge trajectories from different approaches

The curvature of the meson Regge trajectories leads to different inferences for different models. For example, in a string model, the massless string leads to the linear Regge trajectories while the relativistic string with massive ends can generate the nonlinear Regge trajectories [32, 50, 110].

In Refs. [48, 64, 65], the authors propose an interpolated formula (2). Without the physical constraints on the parameters, formula (2) can be concave, convex or has the inflection point. As the physical constraints on the parameters are considered, the formulas will be only concave, which cannot describe the convex Regge trajectories. The interpolated formulas in Eq. (B7) are also in this case.

In Ref. [111], the author obtain the concave Regge trajectories from the $\kappa$-deformed Poincaré phenomenology,

$$M = \frac{2}{\epsilon} \sinh^{-1} \left[ \left( \frac{\epsilon}{2} \right)^2 \left( \frac{1}{\alpha'} + \frac{n_\rho}{\beta'} + \frac{S}{\gamma'} + \frac{J}{\delta'} \right) + \sinh^2 \left( \frac{m\epsilon}{2} \right) \right]^{1/2},$$

where $\epsilon$ is the inverse of the deformation parameter $\kappa$, $m$ is the quark mass. $M$ is the bound state mass, $J$ is the angular momentum, and $n$ is the radial quantum number. The spin-spin and spin-orbit and tensor effects can be included through the terms involving $J$ and $S$. $\alpha'$, $\beta'$, $\gamma'$ and $\delta'$ are the parameters. Eq. (8) is a concave function, which cannot be convex or have the inflection point.

In Ref. [10], the author discuss the generalized string with the massless ends. They obtain the square-root trajectory

$$M^2 = \frac{\sigma}{\mu}^2 - \left( \frac{\sigma}{\mu} - \pi\mu J \right)^2 \tag{9}$$

from the generalized massless string with the potential $V(\rho) = \sigma/(\pi\mu) \arctan(\pi\mu \rho)$ and the logarithmic trajectory

$$M^2 = \frac{\sigma}{\mu}^2 \left( 1 - e^{-2\pi\mu^2 J/\sigma} \right) \tag{10}$$

from the generalized massless string with the potential $V(\rho) = \sigma/(2\pi\mu)(2 \arctan(2\pi\mu \rho) - \log[1 + (2\pi\mu \rho)^2]/(2\pi\mu \rho))$, respectively. In the above two equations, $\sigma$ is the string tension and $\mu$ is a parameter. Eqs. (9) and (10) are concave functions.

In Ref. [112], one concave formula is given

$$M^2 = \left( \sqrt{\pi\sigma l} + m_Q \right)^2 \tag{11}$$

where $m_Q$ is the heavy quark mass. In Ref. [110], the authors give the derivation of Eq. (11) by following the Nambu’s picture for hadrons in which quarks are connected by a gluon flux tube. Eq. (11) is concave and cannot be convex or have the inflection point. The formula [50]

$$M^2 = \left( \sqrt{2\pi\sigma (n + b)} + 2m \right)^2 \tag{12}$$
is also in this case, where \( n \) is the radial quantum number. Other formulas which can give the concave Regge trajectories are not discussed due to their many parameters [7, 32].

The curvature of the meson Regge trajectories can assist in the choice of the appropriate models to describe mesons. Thus it can be taken as a guide in constructing models, because the curvature of the meson Regge trajectories is model-independent. The appropriate models should give the spectra which can produce the concave Regge trajectories according to the concavity of the meson Regge trajectories. If the majority of the meson Regge trajectories are concave while a few meson Regge trajectories are convex which neither have been confirmed nor excluded completely at present, mesons should be classified according to different curvature.

The curvature of the meson Regge trajectories can assist in the choice of the appropriate models to describe mesons. Therefore, the curvature of the meson Regge trajectories can be used to classify hadrons. If the convex Regge trajectories do not exist, the concavity can act as a criterion to choose a newly observed meson or to assign a particle to the unwell-established state. If the vast majority of the meson Regge trajectories are concave while a few meson Regge trajectories are convex which neither have been confirmed nor have been completely excluded at present, mesons should be classified according to different curvature.

The curvature of the meson Regge trajectories can assist in the choice of the appropriate models to describe mesons. If the large majority of the meson Regge trajectories are concave, whereas a few of the meson Regge trajectories are convex, many existing models should be corrected or even be reconstructed, which will lead to the further understanding of the meson dynamics.

IV. CONCLUSIONS

We have shown by fitting the spectra of a large majority of mesons that the radial and orbital Regge trajectories for mesons prefer being concave in the \((n_r, M^2)\) and \((l, M^2)\) planes, respectively. The concavity of the meson Regge trajectories is model-independent. No meson Regge trajectories having the inflection points are observed. The convex meson Regge trajectories are neither definitely confirmed nor excluded completely until now. More theoretical analyses and more experimental data are needed to construct the Regge trajectories which seem convex at present.

The curvature of the meson Regge trajectories is a fundamental property of the meson Regge trajectories. The curvature of the meson Regge trajectories can be used to classify hadrons. If the convex Regge trajectories do not exist, the concavity can act as a criterion to choose a newly observed meson or to assign a particle to the unwell-established state. If the vast majority of the meson Regge trajectories are concave while a few meson Regge trajectories are convex which neither have been confirmed nor have been completely excluded at present, mesons should be classified according to different curvature.

The curvature of the meson Regge trajectories can assist in the choice of the appropriate models to describe mesons. Therefore, the curvature of the meson Regge trajectories can be used as a guide in constructing models. The appropriate models should yield the spectra which can produce the concave Regge trajectories according to the concavity of the meson Regge trajectories. If the large majority of the meson Regge trajectories are concave, whereas a few of the meson Regge trajectories are convex, many existing models should be corrected or even be reconstructed, which will lead to the further understanding of the meson dynamics.

Appendix A: Regge trajectories for the light unflavored mesons

1. Regge trajectories for the light unflavored mesons

The spectra of the light unflavored mesons are listed in Table I. The states are assigned according to Refs. [24, 28, 34, 63, 113, 114]. We employ Eq. (1) to fit the spectra of the light unflavored mesons in Table I. As shown in Tables II and III, there are eight radial Regge trajectories and ten orbital Regge trajectories can be described by Eq. (1) which are concave. There are eight radial Regge trajectories (for \(a_0(980)\), \(\rho(1700)\), \(\pi_2(1670)\), \(\omega(782)\), \(f_2(1270)\), \(\omega_3(1670)\), \(\phi(1020)\) and \(\eta(958)\)) and eight orbital Regge trajectories (for \(\pi(1800)\), \(\rho(1450)\), \(a_1(1260)\), \(a_1(1640)\), \(\eta\), \(\eta(1295)\), \(\eta(1760)\) and \(\eta(958)\)) look like convex. We notice that these sixteen convex Regge trajectories are only possibly convex. More theoretical studies and more experimental data are needed to confirm the possible convexity of these Regge trajectories.

| \(nL_J^{2S+1}I\) | \(J^{PC}\) | \(I = 1(\bar{q}q)\) | PDG\([63]\) | \(I = 0(\bar{q}q)\) | PDG\([63]\) | \(I = 0(s\bar{s})\) | PDG\([63]\) |
|---|---|---|---|---|---|---|---|
| 1 \(^1S_0\) | 0 \(+\) | \(\pi^\pm\) | 134.9770\(\pm0.0005\) | \(\eta\) | 547.862\(\pm0.017\) | \(\eta(958)\) | 957.78 \(\pm0.06\) |
| 2 \(^1S_0\) | 0 \(+\) | \(\pi(1300)\) | 1300\(\pm100\) | \(\eta(1295)\) | 1294\(\pm4\) | \(\eta(1475)\) | 1476\(\pm4\) |
| 3 \(^1S_0\) | 0 \(+\) | \(\pi(1800)\) | 1812\(\pm12\) | \(\eta(1760)\) | 1751\(\pm15\) | X\(1835)\) | 1826.5\(\pm13.0\) |
| 4 \(^1S_0\) | 0 \(+\) | \(\pi(2070)\) | 2070\(\pm35\) | \(\eta(2010)\) | 2010\(\pm35\) | \(\eta(2225)\) | 2221\(\pm13\) |
| 5 \(^1S_0\) | 0 \(+\) | \(\pi(2360)\) | 2360\(\pm25\) | \(\eta(2320)\) | 2320\(\pm15\) | \(\eta(2920)\) | 2920\(\pm20\) |
| 1 \(^1S_0\) | 1 \(\pm\) | \(\rho(770)\) | 775.26\(\pm0.25\) | \(\omega(782)\) | 782.65\(\pm0.12\) | \(\phi(1020)\) | 1019.461\(\pm0.016\) |
| 2 \(^1S_0\) | 1 \(\pm\) | \(\rho(1450)\) | 1465\(\pm25\) | \(\omega(1420)\) | 1400-1450 | \(\phi(1680)\) | 1680\(\pm20\) |
| 3 \(^1S_0\) | 1 \(\pm\) | \(\rho(1900)\) | 1909\(\pm17\) | \(\omega(1650)\) | 1670\(\pm30\) | \(\phi(2170)\) | 2188\(\pm10\) |
| 4 \(^1S_0\) | 1 \(\pm\) | \(\rho(2150)\) | 2150 \(\pm40\) | \(\omega(1960)\) | 1960\(\pm25\) | \(\phi(2290)\) | 2290 \(\pm20\) |
TABLE I: (continued)

| $n_i^{2S+1L_J}$ | $J^{PC}$ | $I = 1$ | PDG[63] | $I = 0$ | PDG[63] | $I = 0$ | PDG[63] |
|-----------------|----------|--------|---------|--------|---------|--------|---------|
| $1^3P_0$       | 0++      | $a_0(980)$ | 980±20  | $f_0(1370)$ | 1200-1500 | $f_0(1710)$ | 1723±21 |
| $1^3P_0$       | 0++      | $a_0(1450)$ | 1474±19 |        |         |        |         |
| $1^3P_0$       | 0++      | $a_0(2020)$ | 2025±30 |        |         |        |         |

| $1^3P_1$       | 1++      | $a_1(1260)$ | 1230±40 | $f_1(1285)$ | 1281.9±0.5 | $f_1(1420)$ | 1426.4±0.9 |
| $1^3P_1$       | 1++      | $a_1(1640)$ | 1654±19 |        |         |        |         |
| $1^3P_1$       | 1++      | $a_1(2095)$ | 2096±1712 |        |         |        |         |
| $4^3P_1$       | 1++      | $a_3(2270)$ | 2270±35 | $f_1(2310)$ | 2310±60 |        |         |

| $1^3P_2$       | 2++      | $a_2(1320)$ | 1318.3±0.6 | $f_2(1270)$ | 1275.5±0.8 | $f_2(1525)$ | 1525±5 |
| $1^3P_2$       | 2++      | $a_2(1700)$ | 1732±9 | $f_2(1750)$ | 1755±10 | $f_2(1950)$ | 1944±12 |
| $1^3P_2$       | 2++      | $a_2(1990)$ | 2050±10±40 | $f_2(2150)$ | 2157±12 |        |         |

| $1^3P_0$       | 1++      | $b_1(1235)$ | 1229.5±3.2 | $h_1(1170)$ | 1170±20 | $h_1(1380)$ | 1407±12 |
| $1^3P_0$       | 1++      | $b_1(1960)$ | 1960±35 | $h_1(1965)$ | 1965±45 |        |         |
| $1^3P_0$       | 1++      | $b_1(2240)$ | 2240±35 | $h_1(2215)$ | 2215±40 |        |         |

| $1^3D_1$       | 1--      | $p(1700)$ | 1720±20 | ω(1650) | 1670±30 |        |         |
| $1^3D_1$       | 1--      | $p(2000)$ | 2000±30 |        |         |        |         |
| $1^3D_1$       | 1--      | $p(2270)$ | 2265±40 |        |         |        |         |

| $1^3D_2$       | 2--      | $p_{1/2}(1940)$ | 1940±40 | ω_{1/2}(1975) | 1975±20 |        |         |
| $1^3D_2$       | 2--      | $p_{3/2}(2225)$ | 2225±35 | ω_{3/2}(2195) | 2195±30 |        |         |

| $1^3D_3$       | 3--      | $p_{3/2}(1690)$ | 1688.8±2.1 | ω_{3/2}(1670) | 1667±4 | φ_{3}(1850) | 1854±7 |
| $2^3D_3$       | 3--      | $ω_{3/2}(1945)$ | 1945±20 |        |         |        |         |
| $3^3D_3$       | 3--      | $ω_{3/2}(2255)$ | 2255±15 |        |         |        |         |

| $1^3D_2$       | 2--      | $π_2(1670)$ | 1672±3.0 | η_{2}(1645) | 1617±5 | η_{2}(1870) | 1842±8 |
| $2^3D_2$       | 2--      | $π_{2/2}(2005)$ | 1974±14±83 | η_{2}(2030) | 2030±5±15 |        |         |
| $3^3D_2$       | 2--      | $π_{3/2}(2285)$ | 2285±20±25 | η_{2}(2250) | 2248±20 |        |         |

| $1^5F_2$       | 2++      | $f_2(1810)$ | 1815±12 | $f_2(2150)$ | 2157±12 |        |         |
| $2^5F_2$       | 2++      | $f_2(2140)$ | 2141±12 |        |         |        |         |

| $1^3F_3$       | 3++      | $a_3(1875)$ | 1874±43±96 |        |         |        |         |
| $2^3F_3$       | 3++      | $a_3(2275)$ | 2275±35 |        |         |        |         |
| $1^3F_3$       | 4++      | $a_4(2040)$ | 1993±10 | $f_4(2050)$ | 2018±11 |        |         |
| $2^3F_3$       | 4++      | $a_4(2255)$ | 2237±5 |        |         |        |         |
| $1^3F_3$       | 3++      | $b_3(2030)$ | 2032±12 | $h_3(2025)$ | 2025±20 | $h_3(2275)$ | 2275±25 |
| $2^3F_3$       | 3++      | $b_3(2245)$ | 2245±50 |        |         |        |         |

| $1^5G_3$       | 3--      | $p_{3/2}(1990)$ | 1982±14 |        |         |        |         |
| $2^5G_3$       | 3--      | $p_{3/2}(2250)$ | 2260±20 |        |         |        |         |

| $1^5G_3$       | 4--      | $p_{1/2}(2230)$ | 2230±25 | ω_{1/2}(2250) | 2250±30 |        |         |
| $1^5G_3$       | 5--      | $p_{3/2}(2350)$ | 2330±35 | ω_{5/2}(2250) | 2250±70 |        |         |
| $1^5G_3$       | 6--      | $p_{3/2}(2550)$ | 2530±45 | ω_{3/2}(2330) | 2320±38 |        |         |
| $1^5H_4$       | 4++      | $f_i(2220)$ | 2221.1±3.5 |        |         |        |         |
| $1^5H_5$       | 6++      | $a_6(2450)$ | 2450±130 | $f_6(2510)$ | 2469±29 |        |         |

TABLE II: Parameters of the radial Regge trajectories of the form (1) for the light unflavored mesons.

| $\pi^+$ | $\beta_{\pi^+}$ | $b_{\pi^+}$ | $c_{\pi^+}$ |
|---------|-----------------|--------------|-------------|
| $\rho(770)$ | 2.78 | 0.79 | -2.36 |
| $a_1(1260)$ | 3.23 | 3.58 | -6.04 |
| $a_2(1320)$ | 3.52 | 5.96 | -9.85 |
| $b_1(1295)$ | 3.94 | 10.0 | -16.79 |
| $h_1(1170)$ | 4.03 | 10.0 | -17.37 |
| $\eta(1645)$ | 1.60 | 0.02 | 2.50 |
TABLE III: Parameters of the orbital Regge trajectories of the form (1) for the light unflavored mesons.

|   | $a_i$ | $b_i$ | $c_i$ |
|---|-------|-------|-------|
| $\pi^0$ | 2.85 | 1.50 | -3.69 |
| $\pi(1300)$ | 3.76 | 10.0 | -15.78 |
| $\rho(770)$ | 3.89 | 10.0 | -17.48 |
| $a_0(980)$ | 1.42 | -1.0 | 1.16 |
| $a_0(1450)$ | 1.41 | -1.0 | 2.32 |
| $\omega(782)$ | 4.16 | 13.0 | -22.47 |
| $\omega(1420)$ | 1.22 | 0.09 | 1.79 |
| $\omega(1650)$ | 1.49 | 0.0 | 2.89 |
| $f_0(1370)$ | 2.28 | 4.69 | -5.42 |
| $\phi(1020)$ | 2.36 | 1.37 | -1.88 |

2. Discussions

Twelve Regge trajectories are constructed by fitting three states and four Regge trajectories are obtained by fitting more than three states among sixteen possible convex Regge trajectories. Same as the example discussed in IID, twelve Regge trajectories obtained by fitting three states maybe arise from the insufficient experimental data or the inappropriate assignment of states. The convex radial Regge trajectory for $\omega(782)$ including five states will become concave if $\omega(2290)$ is excluded which is assigned to the unwell-established $5^3S_1$ state. The radial Regge trajectory for the $0^+ s\bar{s}$ state is convex if $\eta(2225)$ is included [28, 113] and becomes concave if $\eta(2225)$ is excluded or $X(2500)$ is assigned as the $5^1S_0$ state [113, 115]. The orbital Regge trajectory for the $0^+ s^3S_1$ state will be convex if $\eta'(958)$ is assigned as the $0^+ s\bar{s}$ state [28, 113] while it comes to be concave if $\eta$ is assumed to be the $1^3S_0$ $s\bar{s}$ state [63].

The orbital Regge trajectory for the $0^{++} q\bar{q}$ state is obtained by fitting the spectra of five states, which is convex whether $\eta$ or $\eta'(958)$ is assigned as the $1^1S_0$ $q\bar{q}$ state [28, 63, 113]. $h_3(2025)$ and $\eta_4(2330)$ are unconfirmed states among the five states [34]. If $\eta'(958)$ is assigned as the $1^3S_0$ $q\bar{q}$ state [63], both the radial and orbital Regge trajectories for the $1^3S_0$ $q\bar{q}$ state become convex. In summary, the convexity of the orbital Regge trajectory for the $0^{++} q\bar{q}$ state is not confirmed.

We can conclude that no convex Regge trajectories for the light unflavored mesons have been confirmed until now, therefore, the Regge trajectories for the light unflavored mesons prefer being concave. If some of the Regge trajectories for meson are convex illustrated by the future analyses, the Regge trajectory for the $0^{++} q\bar{q}$ state is likely to be the fist one to be confirmed.

Appendix B: Interpolated formulas

The Regge trajectories in Eq. (1) are obtained from the quadratic form of the spinless Salpeter-type equation by employing the Bohr-Sommerfeld quantization approach [60, 61]. In the Regge trajectories in (1), the long-range potential’s contributions are considered while the short-range potential’s contributions are neglected. In this subsection, we consider the contributions from the short-range potential and then present one interpolated form of the Regge trajectories by incorporating both the long-range contributions and the short-range contributions.

The quadratic form of the spinless Salpeter-type equation reads [29, 53–59]

\[ M^2\Psi(r) = M_0^2\Psi(r) + \mathcal{U}\Psi(r), \quad M_0 = \omega_1 + \omega_2, \]  

(B1)

where $\omega_i$ is the square-root operator of the relativistic kinetic energy of constituent

\[ \omega_i = \sqrt{m_i^2 + p^2} = \sqrt{m_i^2 - \Delta}. \]  

(B2)

$m_i$ is the effective mass in the phenomenological model. For simplicity, we assume that $\mathcal{U}$ takes the form

\[ \mathcal{U} = -\frac{A}{r} + Br, \]  

(B3)

which is a variant of the well-known Cornell potential [74].

In the long range, the linear confinement potential is dominant while the color Coulomb potential is subordinate, then the Regge trajectories (1) are obtained. In the shortly distant region, the color Coulomb part will dominate and
the linear confinement potential can be ignored. Using Eqs. (B1) and (B3), we have two auxiliary equations

\[ M_1^2 \Psi(r) = 4(p^2 + m_1^2)\Psi(r) - \frac{A}{r}\Psi(r), \]
\[ M_2^2 \Psi(r) = 4(p^2 + m_2^2)\Psi(r) - \frac{A}{r}\Psi(r). \] (B4)

The obtained eigenvalues read, respectively,

\[ M_1^2 = 4m_1^2 - \frac{A^2}{16} \frac{1}{(n_r + l + 1)^2}, \]
\[ M_2^2 = 4m_2^2 - \frac{A^2}{16} \frac{1}{(n_r + l + 1)^2}. \] (B5)

Because \( 4\omega_1^2 \leq (\omega_1 + \omega_2)^2 \leq 4\omega_2^2 \) if \( m_1 \leq m_2 \), the eigenvalues of Eq. (B1) in the short-range case are expected to be of the form

\[ M^2 = 4\tilde{m}^2 - \frac{\tilde{A}^2}{(n_r + l + 1)^2}, \] (B6)

where \( \tilde{m} \) and \( \tilde{A} \) are parameters. Combining Eqs. (1) and (B6), we propose the parameterized formulas

\[ M^2 = \frac{\gamma_1}{(l + n_r + 1)^2} + \beta_1(l + b_1)^{2/3} + c_1, \]
\[ M^2 = \frac{\gamma_2}{(l + n_r + 1)^2} + \beta_2(n_r + b_2)^{2/3} + c_2. \] (B7)

The first terms are from the color Coulomb potential and the second terms are from the linear confinement potential. Therefore, \( \gamma_1 \) and \( \gamma_2 \) should be negative while \( \beta_1 \) and \( \beta_2 \) should be positive. Different from the Regge trajectories (1) which are concave, the formulas in (B7) can be concave or convex, and can have the inflection points as the physics constraints on the parameters are not considered. The physical constraints on the parameters make the formulas in (B7) be concave.

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