QUOTIENTS OF EVEN RINGS

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ABSTRACT. We prove that if \( R \) is an \( \mathbb{E}_2 \)-ring with homotopy concentrated in even degrees, and \( \{x_j\} \) is any sequence of elements in \( \pi_{2k}(R) \), then \( R/(x_1, x_2, \cdots) \) admits the structure of an \( \mathbb{E}_1 \)-\( R \)-algebra. This removes an assumption, common in the literature, that \( \{x_j\} \) be a regular sequence.

1. Introduction

Let \( R \) be a homotopy commutative ring and \( x \in \pi_* R \) an element. Then one can ask whether the \( R \)-module cofiber \( R/x \) admits a product compatible with the action of \( R \), up to homotopy. Of particular interest has been the case when \( R \) has homotopy groups concentrated in even degrees. Variations and special cases of this question have been considered throughout the years:

- Bass and Sullivan [Baa73] realized various quotients of the complex bordism ring \( MU_* \) as coming from homology theories via a geometric construction. Placing products on the resulting homology theories was a delicate process, studied early in various references, such as [SY77, Mor79, Mir75].
- Elmendorf-Kriz-Mandell-May [EKMM97], as an application of their new, highly structured models for ring spectra, showed that various quotients of \( MU \) could be realized as having multiplications, often associative or commutative, inside the homotopy category \( h\text{Mod}_{MU} \).
- Strickland [Str99] generalized the results of [EKMM97] by proving, among other things, that if \( R \) is an even, \( \mathbb{E}_x \)-ring spectrum, and \( x \in \pi_* R \) is not a zero divisor, then \( R/x \) admits the structure of an associative ring in \( h\text{Mod}_R \).
- Angeltveit [Ang08] showed that if \( R \) is an \( \mathbb{E}_x \)-ring and \( x \in \pi_* R \) is not a zero-divisor, then \( R/x \) further admits the structure of an \( \mathbb{E}_1 \)-\( R \)-algebra. He concludes that, if \( (x_1, x_2, \ldots) \) is a regular sequence in \( \pi_* R \), then \( R/(x_1, x_2, \ldots) \) admits the structure of an \( \mathbb{E}_1 \)-\( R \)-algebra. We note that his proof works just as well in the case when \( R \) is an \( \mathbb{E}_2 \)-ring.

If \( x \) is a zero-divisor, then \( \pi_*(R/x) \) is not \( \pi_*(R)/x \), because the relevant long-exact sequence is not short-exact. In particular, \( \pi_*(R/x) \) will not be concentrated in even degrees, and this significantly interferes with any direct attempt to generalize the arguments of Strickland and Angeltveit. Nonetheless, the theory of Thom spectra has been used to remove the assumption in certain special cases. The strongest result of this form is recent work of Basu-Sagave-Schlichtkrull [BSS16], who give a direct construction of \( R \)-algebra quotients \( R/(x_1, x_2, \ldots, x_n) \) in the case when \( R \) is an even, \( \mathbb{E}_x \)-ring and \( |x_i| = 2i \), with no assumption that the sequence be regular. We remark that their proof works also in the case when \( R \) is an \( \mathbb{E}_2 \)-algebra, using, for example, the machinery in [ABL14].

We offer the following addition to the above list, which is a simultaneous strengthening of Angeltveit’s result and the result of Basu-Sagave-Schlichtkrull.

Theorem A. Let \( R \) be an \( \mathbb{E}_2 \)-ring with \( \pi_{2k-1} R = 0 \) for all \( k \in \mathbb{Z} \). Let \( J = \{x_j\} \) be any sequence of elements in \( \pi_{2k} R \). Define

\[
R/J := \bigotimes_{x_j} R/x_j.
\]

Then \( R/J \) admits the structure of an \( \mathbb{E}_1 \)-algebra in left \( R \)-modules.

We will reduce our theorem to Angeltveit’s by way of the following:

Lemma B. For fixed \( k \in \mathbb{Z} \), let

\[
A = S^0[S^{2k}]
\]
denote the free \( \mathbb{E}_1 \)-ring spectrum generated by \( S^{2k} \). Then \( A \) admits the structure of a nonnegatively graded \( \mathbb{E}_2 \)-ring, with degree 0 component \( A_0 = S^0 \). In particular, there is an \( \mathbb{E}_2 \)-algebra augmentation

\[
A \longrightarrow S^0.
\]
Remark. We imagine that Lemma [B] is well-known, but could not find the statement in the literature so we provide a proof below.

Remark. In the case that the sequence of elements \( \{x_j\} \) appear in nonnegative degree, it is possible to use Thom spectra, elaborating on the argument given in [BSS16], to give a proof of Theorem A. However, because \( \text{gl}_1(R) \) depends only on the connective cover of \( R \), it seems impossible for such techniques to handle the case that the \( x_j \) appear in negative degrees, and this case is essential to forthcoming applications of the authors.

2. Reduction

We defer the proof of Lemma [B] to the next section and explain how to deduce Theorem A.

With notation as in the statement of the main theorem, define

\[
S^0[J] := \bigwedge_j S^0[S^{[x_j]}].
\]

Denote by \( t_j \) the class \( S^{[x_j]} \to S^0[J] \).

By Lemma [B] this is an augmented \( \mathbb{E}_2 \)-ring. Thus

\[
R[J] := R \wedge S^0[J]
\]

is an \( \mathbb{E}_2 \)-ring equipped with an \( \mathbb{E}_2 \)-map to \( R \). (Beware, however, that we cannot even make sense of \( R[J] \) being an \( \mathbb{E}_2 \)-\( R \)-algebra in general, since \( \text{LMod}_R \) is only guaranteed to be \( \mathbb{E}_1 \)-monoidal.)

Now note that \( R[J] \) is an \( \mathbb{E}_2 \)-ring with homotopy concentrated in even degrees and that

\[
(x_1 - t_1, x_2 - t_2, \ldots) \subseteq \pi_* R[J]
\]

is an ideal generated by a regular sequence. Thus, by Angeltveit’s theorem, we may promote

\[
R[J]/(x_1 - t_1, x_2 - t_2, \ldots)
\]

to an \( \mathbb{E}_1 \)-\( R[J] \)-algebra. Since the augmentation \( R[J] \to R \) is an \( \mathbb{E}_2 \)-map, base change is \( \mathbb{E}_1 \)-monoidal and we deduce that

\[
R \otimes_{R[J]} (R[J]/(x_1 - t_1, x_2 - t_2, \ldots))
\]

becomes an \( \mathbb{E}_1 \)-\( R \)-algebra. Rearranging the tensor product yields Theorem A.

3. Augmentation

We are left with proving Lemma [B] which states that \( S^0[S^{2k}] \) admits the structure of a nonnegatively graded \( \mathbb{E}_2 \)-ring with \( S^0 \) in degree zero.

Recall that the homotopy theory of graded spectra is given by \( \text{Fun}(\mathbb{Z}^{\text{ds}}_{\geq 0}, \text{Sp}) \) where \( \mathbb{Z}^{\text{ds}}_{\geq 0} \) denotes the set of nonnegative integers regarded as a discrete \( \infty \)-category. The procedure of taking a colimit gives a functor

\[
\text{Fun}(\mathbb{Z}^{\text{ds}}_{\geq 0}, \text{Sp}) \to \text{Sp}
\]
given informally on objects by the formula \( \{X_n\} \mapsto \bigoplus_{n \geq 0} X_n \). We refer to this as the underlying spectrum of the graded spectrum \( \{X_n\} \).

The homotopy theory of graded spectra can be promoted to a symmetric monoidal \( \infty \)-category via Day convolution, using addition to give \( \mathbb{Z}^{\text{ds}}_{\geq 0} \) a symmetric monoidal structure. With respect to this structure, we have an equivalence

\[
\text{Alg}_{\mathbb{E}_n}(\text{Fun}(\mathbb{Z}^{\text{ds}}_{\geq 0}, \text{Sp})) \cong \text{Fun}^{\text{lax-}\text{Ex}}_{\mathbb{E}_n}(\mathbb{Z}^{\text{ds}}_{\geq 0}, \text{Sp}),
\]

where the latter \( \infty \)-category is the \( \infty \)-category of lax \( \mathbb{E}_n \)-monoidal functors \( \mathbb{Z}^{\text{ds}}_{\geq 0} \to \text{Sp} \). We note that the functor assigning to a graded spectrum its underlying spectrum can be made symmetric monoidal in an essentially unique way.

Now recall that the assignment \( n \mapsto S^{2n} \) may be promoted to an \( \mathbb{E}_2 \)-monoidal functor

\[
\mathbb{Z}^{\text{ds}}_{\geq 0} \to \text{Pic}(S^0).
\]

(See, for example, [Lur15, Proposition 5.1.13]). If \( k \geq 0 \) define \( A_k \in \text{Fun}(\mathbb{Z}^{\text{ds}}_{\geq 0}, \text{Sp}) \) as the composite

\[
\mathbb{Z}^{\text{ds}}_{\geq 0} \xrightarrow{k} \mathbb{Z}_{\geq 0} \xrightarrow{\text{Pic}(S^0)} \text{Sp}
\]

If \( k < 0 \), define \( A_k \) as the composite

\[
\mathbb{Z}^{\text{ds}}_{\geq 0} \xrightarrow{(-k)} \mathbb{Z}_{\geq 0} \xrightarrow{\text{Pic}(S^0)} \text{Sp}
\]
where $D : \text{Pic}(S^0) \to \text{Pic}(S^0)$ denotes Spanier-Whitehead duality. In both cases, we see that $A_k$ is $\mathbb{E}_2$-monoidal, being the composite of $\mathbb{E}_2$-monoidal maps, and hence we may regard $A_k$ as an $\mathbb{E}_2$-algebra object in $\text{Fun}(\mathbb{Z}_{\geq 0}, \mathbb{Sp})$. The underlying spectrum is readily checked to be $S^0[\mathbb{S}^2]$ , so we have proven Lemma B.

We end by justifying the claim that nonnegatively graded $\mathbb{E}_n$-algebras can be canonically augmented over their restriction to degree zero. Denote by $\text{Fun}(\mathbb{Z}_{\geq 0}, \mathbb{Sp})_{=0}$ the full subcategory of graded spectra which are concentrated in degree 0. This is a localization of $\text{Fun}(\mathbb{Z}_{\geq 0}, \mathbb{Sp})$ which is compatible with the symmetric monoidal structure since, if $X \to Y$ is a map of nonnegatively graded spectra such that $X_0 \to Y_0$ is an equivalence, and $Z$ is any other nonnegatively graded spectrum, then

$$(X \otimes Z)_0 \simeq X_0 \otimes Z_0 \to Y_0 \otimes Z_0 \simeq (Y \otimes Z)_0$$

is an equivalence. (Notice that this fails if we take $\mathbb{Z}$-graded spectra, and, indeed, many $\mathbb{Z}$-graded algebra do not admit augmentations). It now follows from [Lur17, 2.2.1.9] that, if $A$ is an $\mathbb{E}_n$-algebra in $\text{Fun}(\mathbb{Z}_{\geq 0}, \mathbb{Sp})$, the counit $A \to A_0$ is a map of graded $\mathbb{E}_n$-algebras. In particular, the underlying $\mathbb{E}_n$-algebra of $A$ admits an $\mathbb{E}_n$-algebra map to $A_0$.

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