Continuation and stability of rotating waves in the magnetized spherical Couette system: Secondary transitions and multistability

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Abstract

Rotating waves (RW) bifurcating from the axisymmetric basic magnetized spherical Couette (MSC) flow are computed by means of Newton-Krylov continuation techniques for periodic orbits. In addition, their stability is analysed in the framework of Floquet theory. The inner sphere is rotating whilst the outer is kept at rest and the fluid is exposed to an axial magnetic field. For a moderate Reynolds number $\text{Re} = 10^3$ (measuring inner rotation) the effect of increasing the strength of the magnetic field (measured by the Hartmann number $\text{Ha}$) is addressed in the range $\text{Ha} \in (0, 80)$ corresponding to the working conditions of the HEDGEHOG experiment at Helmholtz-Zentrum Dresden-Rossendorf. The study reveals several regions of multistability of waves with azimuthal wave number $m = 2, 3, 4$, and several transitions to quasiperiodic flows, i.e modulated rotating waves (MRW). These nonlinear flows can be classified as the three different instabilities of the radial jet, the return flow and the shear-layer, as found in previous studies. The two latter are continuously linked when increasing the magnetic forcing. In the middle of both instabilities, at a certain critical $\text{Ha}$, the nonaxisymmetric component of the flow is maximum.

1 Introduction

The origin of the magnetic fields of planets, stars and galaxies constitutes one of the most challenging problems of modern physics. Larmor was the first to suggest that the magnetic fields of the Sun, the Earth and cosmic bodies are supported by electrically conducting fluid motions in their interiors \cite{1}. Many years later, the first successful dynamo experiments with liquid sodium \cite{2} were supportive for the the acceptance of this idea. The dynamo problem involves a large range of scales, and hence provides tremendous experimental, analytical and numerical challenges in the parameter regime relevant to geophysical and astrophysical applications. We refer to the review article \cite{3} and a book \cite{4} for detailed references and history of the field.

One of the paradigms of magnetohydrodynamic flows in spherical bodies such as planets and stars is the magnetized spherical Couette (MSC) flow. An electrically conducting liquid is confined between two differentially rotating spheres and is subjected to a magnetic field. Despite its simplicity, this model gives rise to a rich variety of instabilities, and it is also important from an astrophysical point of view.
For instance, simulations of the spherical Couette (SC) flow were used to compute the gravitational wave signal generated by global nonaxisymmetric shear flows in a neutron star [5]. In addition, instabilities observed in a liquid sodium flow between differentially rotating spheres in the presence of a magnetic field were attributed in [6] to the magnetorotational instability (MRI), but see [7, 8, 9] for alternative interpretations. Since the pioneering work of [10], MRI has been considered the best candidate to explain the mechanism of transporting angular momentum in accretion disks around black holes and stars, and also in protoplanetary disks [11], allowing matter to fall into the center. Various types of the MRI have also been studied experimentally at Helmholtz-Zentrum Dresden-Rossendorf (HZDR) [12, 13, 14].

Although the MSC represents one of the simplest paradigms of astrophysical magnetohydrodynamics, it possesses several peculiarities that make the problem difficult. From the analytical point of view, the nonlinear nature of the Navier-Stokes equations and the spherical geometry of the domain leads to mathematical complications that are indeed hard to treat. For this reason, the development and improvement of appropriate numerical techniques is of key importance for a deep understanding of nonlinear flows, even in the weakly supercritical regime. In addition, thin Ekman or Ekman-Hartmann boundary layers, depending on the strength of the magnetic field, appear when the no-slip condition, used to model planetary dynamos and for comparison with laboratory experiments, is imposed at one boundary. Even in the absence of a magnetic field, [15] showed analytically the existence of a thin shear layer (Stewartson layer) at the tangent cylinder (containing the inner sphere and parallel to the rotation axis) which separates regions of different flow behavior. These thin shear layers make the numerical treatment extremely challenging because of the higher spatial resolutions which are computationally most demanding.

In the absence of magnetic fields, the solutions of the SC problem including the basic flow, the first instabilities, and even turbulent states have been widely studied experimentally [16, 17], analytically [15, 18] and numerically using direct numerical simulations (DNS) [19, 20] or continuation methods [21, 22]. Much less numerical studies exist in which the magnetic field is taken into account. Most studies deal with the linear stability analysis of the basic flow, or are build on a basis of very few nonlinear solutions, and mainly rely on considering different types of boundary conditions for the magnetic field [23, 24] (insulating or conducting inner sphere allowing magnetic lines to pass), different topologies of the applied magnetic field (dipolar [24, 25], axial [6, 7, 25], or a combination of both [26]). Sophisticated tools of hydrodynamic stability theory, such as continuation techniques, are in many aspects superior to simple DNS. For instance, time integration methods are unable to obtain unstable oscillatory solutions when all the symmetries of the flow are broken. These solutions might be relevant in organizing the global dynamics [27]. Bifurcation and continuation methods have been successfully applied during the last years to a great variety of problems in the fluid dynamics context [27, 28]. Computations based on continuation of periodic orbits of nontrivial time dependence [29, 30] and even tori [31] or other invariant objects [32] have provided useful information to clarify the dynamics.

The solution (basic state) of the SC equations is unconditionally stable up to a certain critical value of the forcing parameter (the Reynolds number, Re, measuring the strength of differential rotation). Beyond this threshold an instability develops and a branch of stable or unstable solutions bifurcates
and extends into a certain region of the parameter space. The appearance of multiple states in experimental flows strongly depends on the initial state [16], and in DNS the type of perturbation applied determines the type of instability or the mode that will be selected among the bifurcated solutions [20]. To get a complete picture of the skeleton of the phase space, and thus to provide a better characterization of the instabilities and the physically realizable flows, a continuation method is necessary [33, 22, 34, 30] (see [35] for a nice tutorial).

Due to the spherical geometry and rotation the SC system has a $SO(2)$ symmetry. Therefore, the instability of the basic flow usually gives rise to waves traveling in the azimuthal direction, i.e. rotating waves (RW), which break the axisymmetry of the basic state [36]. A secondary Hopf bifurcation results in an amplitude or shape modulation of the flow pattern, i.e. the occurrence of modulated rotating waves (MRW), which may have different types of spatio-temporal symmetries [37, 38]. In the somewhat different case of the Taylor-Couette system, several types of RW and MRW have been identified and characterized depending on their symmetry [39]. In spherical geometry [40] had shown that the destabilization of the Stewartson layer, which is characteristic for the basic state, gives rise to a Rossby wave of fixed azimuthal wave number, travelling in the azimuthal direction due to the curvature of the boundaries. In addition, by means of fully three-dimensional simulations, [20] reported further transitions in the supercritical regime in which the original azimuthal symmetry is replaced by a so-called shift-and-reflect symmetry [11, 36]. In case of the MSC problem the existence of RW and MRW has been confirmed by experimental studies [42], and by DNS [23, 7, 8].

Addressing the influence of the applied magnetic field, [23] have shown that the axisymmetric basic state of the axially MSC problem is equatorially symmetric and remains stable for all Hartmann numbers $Ha$ (measuring the strength of the applied magnetic field) if the Reynolds number is sufficiently small. It can be described as a strong azimuthal flow associated with a meridional recirculation. As the Reynolds number is increased, the basic state becomes unstable to non-axisymmetric perturbations. At low Hartmann number these perturbations are equatorially antisymmetric giving rise to an instability which is essentially hydrodynamic and related to a Kelvin-Helmholtz instability (KHI) of the radial jet at the equatorial plane. At sufficiently large Hartmann numbers the perturbations become equatorially symmetric. For small rotation rates this instability is related to a shear layer at the tangent cylinder [23, 7] while at higher rotations the instability is located at the base of the meridional return flow [7]. In this case, increasing further the Reynolds number stabilizes the basic flow [7, 43]. When holding the Reynolds number fixed, which is the approach of the present study and of the preliminary experiments performed in [9], the two types of instability are separated by a stable regime which occurs for intermediate Hartmann numbers.

Our analysis provides a further step to fill the gap between the very high Reynolds number turbulent regime reached in some of the experiments [6] and some numerical studies [24] and the low Reynolds number laminar regime in which the linear stability of the basic state has been deeply analyzed but the nonlinear saturation of the instabilities has only been studied using a few nonlinear simulations [23, 7, 43]. The innovative use of continuation techniques, as sketched above, allows us to obtain precise bifurcation diagrams and to determine the stability regions of RW with azimuthal wave number $m = 2, 3, 4$ in the range of $Ha \in (0, 80)$. The paper is organized as follows: In §2 we introduce
the formulation of the problem, and the numerical method used to integrate the model equations. Next, the continuation method and the basic ingredients for the stability analysis are briefly described in §3. In §4 the bifurcation diagrams as a function of Ha, the stability of RW and the patterns of convection are analyzed. Finally, in §5 the paper closes with a brief summary on the results obtained.

2 The Model

Let us consider a spherical shell of inner and outer radii \( r_i \) and \( r_o \). The outer sphere is at rest while the inner is rotating at a constant angular velocity \( \Omega \) around the \( \hat{e}_z \) axis. The shell is filled with a homogeneous and conducting fluid of constant density \( \rho \), dynamic viscosity \( \mu \), magnetic diffusivity \( \lambda \) and electrical conductivity \( \sigma = 1/(\lambda \mu_0) \), with \( \mu_0 \) being the free-space value for the magnetic permeability.

We are interested in a comparison with laboratory experiments [9] which expose the flow to a uniform axial magnetic field \( B_0 = B_0 \cos(\theta) \hat{e}_r - B_0 \sin(\theta) \hat{e}_\theta, \theta \) being the colatitude and \( B_0 \) the magnetic field strength. With the use of the eutectic alloy GaInSn as the working fluid of the HEDGEHOG experiment, the inductionless approximation can be adopted. The latter is valid in the limit of low magnetic Reynolds number \( Rm = \frac{PmRe}{\mu_0} \ll 1 \), which applies in the case of the previous experiment because of its very low magnetic Prandtl number fluid (GaInSn) with \( Pm \sim O(10^{-6}) \) and the moderate Reynolds numbers (inner sphere’s rotation rates) considered \( Re \sim 10^3 \).

By scaling length, time, velocity and magnetic field by \( d, d^2/\nu, r_i \Omega \) and \( B_0 \), respectively, expanding magnetic field \( B = \hat{e}_z + Rm \mathbf{b} \) and neglecting terms \( O(Rm) \), the Navier-Stokes and induction equations become

\[
\begin{align*}
\partial_t \mathbf{v} + \text{Re} (\mathbf{v} \cdot \nabla) \mathbf{v} & = -\nabla p + \nabla^2 \mathbf{v} + \text{Ha}^2(\nabla \times \mathbf{b}) \times \hat{e}_z, \\
0 & = \nabla \times (\mathbf{v} \times \hat{e}_z) + \nabla^2 \mathbf{b}, \\
\nabla \cdot \mathbf{v} & = 0, \quad \nabla \cdot \mathbf{b} = 0.
\end{align*}
\]

In this inductionless approximation the system is governed by only three non-dimensional numbers, namely the Reynolds number, the Hartmann number and the aspect ratio:

\[
\text{Re} = \frac{\Omega r_i d}{\nu}, \quad \text{Ha} = \frac{B_0 d}{\sqrt{\mu_0 \rho \nu \lambda}} = B_0 d \sqrt{\frac{\sigma}{\rho \nu}}, \quad \eta = \frac{r_i}{r_o}.
\]

No-slip \( (v_r = v_\theta = v_\phi = 0) \) at \( r = r_o \) and constant rotation \( (v_r = v_\theta = 0, \ v_\phi = \sin \theta \ \hat{e}_\phi) \) at \( r = r_i \) are the boundary conditions imposed to the velocity field. For the magnetic field, insulating exterior regions are considered in accordance with the experimental setting, see [23] for more details.

The equations are discretized and integrated with the same method as described in [44] and references therein. The velocity and magnetic fields are expressed in terms of toroidal and poloidal potentials and are expanded in spherical harmonics in the angular coordinates, and in the radial direction a collocation method on a Gauss–Lobatto mesh is used. The code is parallelized in the spectral and in the physical space by using OpenMP directives. We use optimized libraries (FFTW3 [45]) for
the FFTs in $\varphi$ and matrix-matrix products (dgemm GOTO [46]) for the Legendre transforms in $\theta$ when computing the nonlinear terms.

For the time integration high order implicit-explicit backward differentiation formulas (IMEX–BDF) [44] are used. In the IMEX method we treat the nonlinear terms explicitly in order to avoid solving nonlinear equations at each time step. The Lorenz term is treated also explicitly, which may reduce the time step in comparison with an implicit treatment. However, this is not a serious issue when moderate Ha are considered, as is the case of the present study. The use of matrix-free Krylov methods (GMRES [47] in our case) for the linear systems facilitates the implementation of a suitable order and time stepsize control.

3 Computation and stability of RW

The system of Eqs. (1-3) is $SO(2) \times Z_2$-equivariant, $SO(2)$ generated by azimuthal rotations, and $Z_2$ by reflections with respect to the equatorial plane. According to bifurcation theory [48, 38], the first bifurcation, which breaks the axisymmetry of the basic state, is a Hopf bifurcation giving rise to a rotating wave (RW). The linear stability analysis of the basic state [43] provides the critical values $Ha_c$ and the drifting frequencies $\omega_c$ of these nonaxisymmetric instabilities as a function of $Re$ and $\eta$.

Rotating waves, $u(r, \theta, \varphi - \omega t) = \tilde{u}(r, \theta, \tilde{\varphi})$, with $\tilde{\varphi} = \varphi - \omega t$, can be obtained efficiently by Newton-Krylov continuation methods as steady solutions of the equations written in a reference frame which is rotating with the wave [34]. However, as mentioned in [30], they can also be found as periodic orbits which is the approach followed in the present study. Although this method is not so efficient in large scale $SO(2) \times Z_2$-equivariant systems, it can be significantly more easy to implement because it relies on time integrations and does not require the use of preconditioning techniques, to accelerate the convergence of the linear solver, specifically designed to the system as required in the method proposed in [34].

Some background necessary to follow easily this section is now provided. The discretization of Eqs. (13) leads to a system of ordinary differential equations (ODE) of dimension $n = (2L_{\text{max}}^2 + 4L_{\text{max}})(N_r - 1)$, $L_{\text{max}}$ and $N_r$ being the spherical harmonics truncation parameter and the number of radial collocation points, respectively. The ODE system takes the form

$$L_0 \partial_t u = Lu + B(u, u),$$

where $u$ is the vector containing the spherical harmonic coefficients of the velocity potentials at the inner radial collocation points, and $L_0$ and $L$ are linear operators which include the boundary conditions, $L_0$ being invertible. The operator $L$ includes all the linear terms and depends on the Hartmann number $Ha$, which will be the control parameter of this study. The rest of parameters are kept fixed to $\eta = 0.5$ and $Re = 10^3$. Therefore, $p = Ha$ and $L = L(p)$. The bilinear operator $B$ only contains the non-linear (quadratic) terms.
3.1 Continuation of RW

To study the dependence of RW, rotating at a frequency $\omega$ and with $m$-fold azimuthal wave number, on the parameter $p = Ha$, pseudo-arclength continuation methods for periodic orbits are used [49, 35]. They allow to obtain the curve of periodic solutions $x(s) = (u(s), \tau(s), p(s)) \in \mathbb{R}^{n+2}$, $s$ being the arclength parameter and $\tau = 2\pi/(m\omega)$ the rotation period, by adding the pseudo-arclength condition

$$m(u, \tau, p) \equiv \langle w, x - x^0 \rangle = 0,$$

$x^0 = (u^0, \tau^0, p^0)$ and $w = (w_u, w_\tau, w_p)$ being the predicted point and the tangent to the curve of solutions, respectively, obtained by extrapolation of the previous points along the curve.

The system which determines a single solution, $x = (u, \tau, p)$ is

$$H(u, \tau, p) = \begin{pmatrix} u - \phi(\tau, u, p) \\ g(u) \\ m(u, \tau, p) \end{pmatrix} = 0,$$ (5)

where $\phi(\tau, u, p)$ is a solution of Eq. (4) at time $\tau = 2\pi/(m\omega)$ and initial condition $u$ for fixed $p$. The condition $g(u) = 0$ is selected to fix the undetermined phase of the RW. We use $g(u) = \langle u, \partial_u u_c \rangle$ where $u_c$ is a reference solution (a previously computed solution, or an eigenfunction provided by the linear stability analysis, for instance). It is a necessary condition for $\|u - u_c\|^2$ to be minimal with respect to the phase (see [34]). For the computation of the inner products $\langle \cdot, \cdot \rangle$ between two functions expanded in spherical harmonics we use the definitions of [34].

To solve the large non-linear system defined by Eq. (5) we use Newton-Krylov methods. They are matrix-free methods that do not require the explicit computation of the Jacobian $D_{(u,\tau,p)}H(u,\tau,p)$, but only its action on a given vector. For the linear systems we use GMRES [47]. Due to the particular form of the spectrum of $D_{(u,\tau,p)}H(u,\tau,p)$ for dissipative systems, GMRES does not need preconditioning (see [49] for details).

The action of the Jacobian $D_{(u,\tau,p)}H(u,\tau,p)$ on $\delta x = (\delta u, \delta \tau, \delta p) \in \mathbb{R}^{n+2}$ is

$$\begin{pmatrix} \delta u - v(\tau) - z(\tau)\delta \tau \\ D_u g(u)\delta u \\ D_x m(x)\delta x \end{pmatrix} \in \mathbb{R}^{n+2}.$$ 

Here $z(\tau), v(\tau) \in \mathbb{R}^n$ are the solutions, at time $t = \tau$, of the system

$$\begin{align*}
\partial_t z &= L_0^{-1}(L(p)z + B(z, z)), \\
\partial_t v &= L_0^{-1}(L(p)v + B(z, v) + B(v, z)) + 2p\delta pL_0^{-1}L^{(2)}z,
\end{align*}$$

with initial conditions $z(0) = u$ and $v(0) = \delta u$, with fixed $p$. The dependence of $L$ on $p$ has been assumed to be of the form $L(p) = L^{(1)} + p^2 L^{(2)}$. Each GMRES iteration will require one evaluation of the Jacobian, therefore most of the computational cost is consumed in the integration over one tentative rotation period $\tau$ of a large ODE system of dimension $2n$. An efficient time-stepping code is hence a key ingredient for a successful application of the method.
3.2 Stability of RW

Suppose a RW \((u, \tau, p) \in \mathbb{R}^{n+2}\) has been found (we recall \(\tau = 2\pi/(m\omega)\)). To study the stability of this periodic solution, Floquet theory is applied. Handling the full Jacobian matrix \(D_u \phi(\tau, u, p)\), \(\phi(\tau, u, p)\) being the solution of Eq. (4) at time \(t = \tau\) with initial condition \(u\) and for fixed \(p\), would require a prohibitive amount of memory due the high resolutions employed in the present study. Fortunately, it is enough to compute the leading eigenvalues and eigenvectors of the map \(\delta u \rightarrow D_u \phi(\tau, u, p) \delta u = v(\tau)\), with \(v(\tau)\) being the solution of the first variational equation, obtained by integrating the system

\[
\begin{align*}
\partial_t z &= L_\omega^{-1}(L(p)z + B(z, z)), \\
\partial_t v &= L_\omega^{-1}(L(p)v + B(z, v) + B(v, z)),
\end{align*}
\]

of dimension \(2n\), with initial conditions \(z(0) = u\) and \(v(0) = \delta u\), over a rotation period \(\tau\), with fixed \(p\).

The leading eigenvalues of the map, which correspond to the leading Floquet multipliers, are computed by using the ARPACK package. RW with leading Floquet multipliers with modulus larger (smaller) than +1 are unstable (stable). Note that in this problem, for any value of \(p\), there is a marginal (+1) Floquet multiplier due to the invariance under azimuthal rotations, with associated eigenfunction \(v_1 = \partial_\tau u\). To avoid unnecessary computations it can be deflated by computing the eigenvalues of the map \(\delta u \rightarrow v(\tau) - \langle v(\tau), v_1 \rangle v_1\). This method to determine the stability of the solutions is very robust but computationally expensive because it requires the time integration of an ODE system of dimension \(2n\) over one rotation period. Because the periodic orbit is a RW there is a more efficient alternative to this procedure \[34\] which consist in studying the stability as a fixed point of a vector field. However, matrix transformations must be used (real shift invert, Cayley, etc.) to extract the right-most eigenvalues of the associated eigenvalue problem. Again, preconditioning techniques, depending on the particular spatial discretization, must be used.

4 Results

This study constitutes a further step towards the modelling of the HEDGEHOG experiment \[25, 9\]. This study is restricted to \(\eta = 0.5\) and \(Re = 10^3\) and \(Ha \in (0, 80)\) for which some experimental data is available \[9\]. For this range of parameters the linear stability analysis of the basic axisymmetric state has already been performed in \[43\]. At \(Re = 10^3\) and without magnetic field (\(Ha = 0\)) the basic state is unstable to equatorially antisymmetric nonaxisymmetric perturbations developing the so-called radial jet instability \[20\]. This instability, with azimuthal wave number \(m = 3\), is maintained by increasing magnetic strength, but the basic state restabilises again at a critical Hartmann number \(Ha_c = 12.2\) (see Table 3, \[43\]). By increasing \(Ha\) further beyond \(Ha_c = 25.8\) another Hopf bifurcation gives rise to a RW, with \(m = 4\) and equatorially symmetric, which corresponds to the the return flow instability \[7\]. The latter is characterised by a meridional circulation from the equatorial plane in the middle of the shell. At sufficiently large \(Ha\) the flow instabilities become magnetically confined.
Figure 1: Bifurcation diagrams varying $Ha$ at fixed Reynolds number $Re = 10^3$ and aspect ratio 0.5. (a) Time and volume averaged nonaxisymmetric kinetic energies $K_{na}$. (b) Time and volume averaged axisymmetric kinetic energies $K_a$. (c) Ratios of the time and volume averaged global kinetic energies and volume averaged axisymmetric energies $K/K_a$. (d) Rotation frequency. Solid/dashed lines mean stable/unstable waves. In each panel the branches on the left (low $Ha$) are equatorially asymmetric while those on the right (high $Ha$) are symmetric.

within the tangential cylinder and strong shear layers develop [23]. This is the so-called shear-layer instability.

The bifurcation diagrams, i.e branches of RW, for the three types of instabilities previously described are presented in the following. Azimuthal wave numbers $m = 2, 3, 4$ are selected because they correspond to those which are preferred at the onset of the instabilities. They are indeed in concordance with those that can be measured with ultrasonic Doppler velocimetry (UDV) probes mounted
on the HEDGEHOG experiment [9]. We also describe the flow topology of the instabilities and report the regions of multistability of the waves and the critical Hartmann numbers for the secondary bifurcations to quasiperiodic flows. The symmetry of the latter is identified and some examples, for both equatorially symmetric and antisymmetric instabilities, are provided. Finally, some numerical tests to validate the results are briefly commented.

4.1 Bifurcation Diagrams

Figure 1 contains the branches of RW with azimuthal wave numbers $m = 2, 3, 4$ versus the Hartmann number $Ha$, each panel displaying a different quantity. Volume-averaged nonaxisymmetric $K_{na}$ and axisymmetric $K_a$ kinetic energies, and the ratio of total over axisymmetric volume-averaged kinetic energy $K/K_a$, are represented in Fig. 1(a), (b), and (c), respectively. Finally, the rotation frequency $\omega$ is shown in Fig. 1(d). In all the panels, the branches of RW on the right correspond to the return flow and shear-layer instabilities, both equatorially symmetric, and those on the left correspond to the radial jet instability which is equatorially asymmetric. Unstable/stable RW are denoted with a dashed/solid line. The general picture of the situation is best displayed in Fig. 1(a) (also (c)). Each branch bifurcates from the axisymmetric basic state (horizontal axis, $K_{na} = 0$) at $Ha_{1c}(m)$, $Ha_{2c}(m)$ and $Ha_{3c}(m)$ with $Ha_{1c}(m) < Ha_{2c}(m) < Ha_{3c}(m)$. The azimuthal wave numbers giving rise to the critical values are $m = 3$, $m = 4$ and $m = 2$, respectively, representative of the radial jet, return flow and shear-layer instabilities. As commented before $Ha_{1c}(m = 3) = 12.2$, $Ha_{2c}(m = 4) = 25.8$ (see [43]) and we have found $Ha_{3c}(m = 2) = 79.4$. According to bifurcation theory, RW bifurcating at these critical values are stable, otherwise are unstable, i.e if they bifurcate from nondoninant eigenfunctions.

At the weakly magnetised regime $Ha < 12.2$, corresponding to the radial jet instability, only the $m = 3$ RW is found stable. All waves are characterised by a pronounced increase of nonaxisymmetry, although solutions are still nearly axisymmetric when $Ha = 0$ (see Fig. 1(c)). The difference between kinetic energies or frequencies of the branches is not so large, especially at $Ha \in (5, 12.2)$ and $K_a$ and $\omega$ vary smoothly with $Ha$. The characteristic flow topology of these radial jet instabilities, exhaustively described in [20], can be seen in Fig. 2. The latter displays a spherical and meridional section of the radial velocity $v_r$ (1st and 2nd plots, from left to right) and a meridional section of the azimuthal velocity $v_\phi$ at $Ha = 4.3230263$. The meridional sections cut a relative maximum. Right three plots: Same sections for $Ha = 9.2907964$.

Figure 2: Contour plots of equatorially asymmetric rotating waves, corresponding to the radial jet instability, on the $m = 2$ branch. The left three plots are the equatorial and meridional sections of radial velocity $v_r$, and the meridional section of the azimuthal velocity $v_\phi$ at $Ha = 4.3230263$. The meridional sections cut a relative maximum. Right three plots: Same sections for $Ha = 9.2907964$.
velocity $v_\varphi$ (3rd plot) for a RW with $m = 2$ azimuthal wave number and $Ha = 4.3230263$. Same sections are on the right group of three plots but for the $m = 2$ RW at $Ha = 9.2907964$. The flow results in an equatorial radial jet which emerges from the inner boundary. The flow’s meridional circulation is enhanced, see meridional sections of $v_\varphi$ in Fig. 2 when the radial jet reaches the outer boundary. Because of the drifting nature of the nonaxisymmetric flow the time evolution of the pattern is seen as an oscillation of the jet around the equatorial plane. The larger the nonaxisymmetric component (i.e. the smaller $Ha$), the higher latitudes the jet can reach (compare meridional sections of $v_r$ in Fig. 2). The study of the location of convection is important from an experimental point of view since it helps to decide on the optimal position of measuring probes.

An equatorially symmetric (return flow instability) $m = 4$ RW emerges at $Ha = 25.8$ (see Fig. 1a)). This is quite close to $Ha = 26.3$, corresponding to the bifurcation of the $m = 3$ RW. Then, similarly as in [34] for a thermal convection in spherical geometry problem, a double-Hopf bifurcation could be found by moving a second parameter (Re or $\eta$). At $Ha = 28.5$ the remaining $m = 2$ RW branch starts. Because the unstable $m = 2$ and $m = 3$ RW become stable very soon, especially in the case of $m = 3$ RW, several regions of multistability arise. They correspond to the return flow instability and will be accurately computed in § 4.2. From the begining of the return flow instability $K_{na}$ and $\omega$ increase sharply whilst $K_a$ decreases for all the branches. In contrast to what occured in the radial jet instability, the differences between $K_{na}$ or $K_a$ of the $m = 2, 3, 4$ branches start to rise significantly for $Ha > 30$. The same applies to the difference of $\omega$ for $Ha > 40$. Detecting these differences is of experimental relevance because it helps to select the proper azimuthal wave number (that with a maximum signal) and thus guide in the measurement design, by positioning accordingly the UDV.

The return flow instability of the basic state turns gradually, by increasing $Ha$, into the shear-layer instability [7]. This also occurs when nonlinear effects are included giving rise to equatorially

Figure 3: Contour plots of equatorially symmetric $m = 2$ RW showing the continuous transformation from the return flow to the shear-layer instabilities with increasing $Ha$. From left to right $Ha = 29$, $Ha = 32.892543$, $Ha = 44.126389$, $Ha = 60.233282$, $Ha = 72.000970$, and $Ha = 79.4$. Spherical (first row) and equatorial (second row) sections of radial velocity $v_r$. The spherical sections cut a relative maximum at radius $r = r_i + \alpha d$. The values of $\alpha$ are (from left to right section) 0.19, 0.2, 0.19, 0.15, 0.12 and 0.08.
symmetric RW. Because branches of different \( m \) start and end from and at the basic state \( (K_{na} = 0) \) there should be some \( Ha^* \) which maximises \( K_{na} \) as shown in Fig. 1(a). We have found that close to this \( Ha^* \) the instability can no longer be strictly classified as return flow or shear-layer because it contains features of both. Then, it seems reasonable to consider this \( Ha^* \) as the critical Hartmann number defining the boundary between return flow and shear-layer instabilities. Notice in Fig. 1(a) that \( Ha^* \) can be defined for each branch, i.e, \( Ha^* \equiv Ha^*(m) \). According to our results \( Ha^*(m_1) > Ha^*(m_2) \) and \( K_{na}^{max}(m_1) > K_{na}^{max}(m_2) \) provided that \( m_1 < m_2 \) (the maximum value of the nonaxisymmetric kinetic energy in each branch \( m \) is \( K_{na}^{max}(m) \)). Then, the smaller \( m \) the larger \( K_{na}^{max}(m) \) which again is interesting for experimental purposes: Nonaxisymmetric signals \( (m > 0) \) may be maximised by choosing properly \( Ha \) close to \( Ha^*(m) \). In addition, mounting sensors to measure low azimuthal wave
number properties may also give rise to larger signal. The existence of this critical $H_a^*$ separating two different flow behaviours can also be inferred from the frequency dependence seen in the branches of Fig. 4(d). Approximately at $H_a^*$ there is a change of slope (best seen for $m = 2$), indicating that frequencies change smoothly when shear-layer instabilities are selected.

Figure 1(a) shows that the smaller the $m$ the larger is the corresponding equatorially symmetric branch. This is in correspondence with the shape of the marginal stability curves corresponding to the basic state shown in Fig. 4 of [43]. Looking at this figure and comparing with Fig. 1(a) we could imagine the $m = 5$ and $m = 6$ branches of RW to lie just below the $m = 4$ branch having both very small $K_{na}$. On the other hand, because all branches terminate at the basic state, Fig. 1(b) provides some information about the basic axisymmetric flow. This figure shows how the amplitude of the axisymmetric flow is increased from radial jet to return flow instability but is strongly decreased when shear-layer instabilities are selected at the higher $H_a$.

The features of the equatorially symmetric RW exposed above can be better visualised with the help of contour plots. The first row of Fig. 3 displays the contour plots of the radial velocity $v_r$ on the equatorial plane for a sequence of $m = 2$ RW with increasing Hartmann number (from left to right $H_a = 29$, $H_a = 32.892543$, $H_a = 44.126389$, $H_a = 60.233282$, $H_a = 72.000970$, and $H_a = 79.4$). The left and right equatorial sections correspond to the return flow and shear-layer instabilities, respectively. The equatorial sections in between show the continuous transformation between both instabilities and the corresponding increase and decrease of the amplitude of nonaxisymmetric flow. Spherical sections of $v_r$ are given in the 2nd row. Their radial positions correspond to those of the maximum, $r = r_i + \alpha d$ with $\alpha = 0.19, 0.2, 0.19, 0.15, 0.12$ (from left to right). For the return flow instability the position of maximum $v_r$ becomes fairly constant (three left spherical sections, $H_a < 45$) and increasing $H_a$ results in a wider radial jet which spreads to higher latitudes. Beyond $H_a^* \approx 45$ the RW should be classified as shear-layer instability. The radial jet is progressively stretched to the inner boundary reaching higher latitudes. At $H_a = 72.000970$ convection is almost absent in the bulk of the shell. A view of the meridional sections of radial $v_r$, colatitudinal $v_\theta$, and azimuthal $v_\phi$ velocity of Fig. 4 (1st, 2nd, and 3rd row, respectively), completes the picture about the transformation of return flow into the shear-layer instability. Again the sections cut a maximum of each component of the velocity. The stretching of the return flow instability is clearly seen on the sections of $v_r$ (first row). The latter also depicts the development of strong shear layers, and the progressive alignment of the flow with the rotation axis, when the corresponding instability is selected (right sections, $H_a > 45$). The sections of $v_\theta$ (2nd row), or those for $v_\phi$ (3rd row), confirm this tendency. At the shear-layer instability the strong magnetic field inhibits convection outside the tangent cylinder.

4.2 Stability Analysis: Transition to Quasiperiodic Flows

According to bifurcation (Floquet) theory a periodic orbit is stable for a given $H_a$ as far as all its Floquet multipliers $\lambda_k \in \mathbb{C}$ lie within the unit circle $|\lambda_k| < 1$, $k = 1, ..., n$. When, by varying $H_a$, one of these multipliers crosses the unit circle the periodic orbit becomes unstable. That multiplier gives information about the type of bifurcation and the type of solutions of the branch emerging at
Table 1: Critical parameters of the asymmetric RWs at the bifurcations where they change the stability ($|\lambda| = 1$). They are obtained by inverse interpolation with a polynomial of degree 5. Transition from stable to unstable is denoted by Ch. st. 0/1. If Ch. st. is equal 0 the transition is for the 2nd leading Floquet multiplier (the 1st is unstable).

| $m$ | Ch. st. | $\text{Ha}_c$ | $\omega$ | $\text{Arg}(\lambda)$ | $m$ | Eig. |
|-----|---------|----------------|---------|----------------------|-----|------|
| 2   | 0       | 9.43           | 133.82  | 0.10                 | 2   |      |
| 3   | 0/1     | 3.95           | 139.07  | 0.77                 | 1   |      |
| 4   | 0       | 9.98           | 136.08  | 3.13                 | 2   |      |
| 4   | 0       | 9.97           | 136.08  | 1.49                 | 1   |      |

the critical point. For instance if $\lambda = \lambda_r + i\lambda_i$ is real ($\lambda_i = 0$) then a pitchfork or period doubling bifurcation, giving rise to periodic orbits, occurs depending on whether $\lambda_r = 1$ or $\lambda_r = -1$, respectively. If by contrast $\lambda$ is a complex eigenvalue with $\lambda_r \neq 0$ the bifurcation is of Hopf type and the new solution branch is of an invariant torus, i.e, a quasiperiodic orbit with two frequencies. The latter scenario is found in all branches of RW, either equatorially asymmetric or symmetric.

In the absence of magnetic field, $\text{Ha} = 0$, the critical Reynolds number is around $\text{Re}_c \approx 550$ (see [43]). Because we have chosen a significantly supercritical Reynolds of $\text{Re} = 10^3$, rotating waves with $\text{Ha} \sim 0$ are expected to be unstable. This is what occurs in the $m = 2, 3, 4$ branches of radial jet instability having several multipliers outside the unit circle. Because of the stabilising effect of the magnetic field the unstable multipliers eventually cross the unit circle giving rise to Hopf bifurcations. In the case of $m = 2$ or $m = 4$ RW there is always one unstable multiplier whereas the $m = 3$ RW stabilises at $\text{Ha}_c = 3.95$ until the critical Hartmann for the basic state is reached ($\text{Ha} = 12.2$). At the critical parameter $\text{Ha}_c = 3.95$ the azimuthal wave number of the eigenfunction is $m = 1$ (see Table 1) and then a branch of azimuthally asymmetric ($m = 1$) modulated rotating waves MRW is born. Whether MRW are stable or not depends on whether the bifurcation is subcritical or not. We have found that the bifurcation is subcritical and then MRW are stable with $\text{Ha} \lesssim \text{Ha}_c$. An example of MRW obtained with DNS will be shown later on. In addition, for $\text{Ha} \lesssim 3.33$ the $m = 3$ RW’s have a 2nd unstable eigenfunction with azimuthal wave number $m = 3$. Then, at $\text{Ha}_c = 3.33$ an unstable branch of MRW with azimuthal wave number $m = 3$ is born. Detecting the existence of this branches is important for a deeper understanding of temporal chaotic flows in the range $\text{Ha} \in [0, 5]$.

For all $\text{Ha} < 12.2$ explored the $m = 2$ RW’s are unstable with leading eigenfunction with $m = 1$. At $\text{Ha} \gtrsim 9.43$ the $m = 2$ RW’s have a 2nd unstable eigenfunction with $m = 2$ and thus a branch of unstable $m = 2$ MRW could be found near this range. Since these MRW retain the $m = 2$ symmetry they can be found with $m = 2$ azimuthally constrained DNS. A similar behaviour is found for the $m = 4$ RW. For $\text{Ha} \lesssim 9.98$ they are unstable with an eigenfunction with $m = 2$. In addition, for $\text{Ha} \gtrsim 9.97$ the $m = 4$ RW’s are unstable with an eigenfunction with $m = 1$. Then, for $\text{Ha} \in (9.97, 9.98)$ the $m = 4$
Figure 5: Contour plots of an \( m = 2 \) unstable equatorially asymmetric rotating wave at \( \text{Ha} = 9.2907964 \). First row: The left three plots are the spherical, equatorial and meridional sections of the radial velocity \( v_r \). Spherical and meridional sections are taken where \( v_r \) has a relative maximum. Right three plots: Same sections for \( v_\varphi \). The spherical section is at \( r = r_i + 0.1d \) and the meridional section cuts a relative maximum. Second/third row: As first row but for the first/second leading eigenfunctions, respectively. In this case the spherical and meridional sections are taken at a relative maximum.

RW have two unstable eigenfunctions, while for \( \text{Ha} \) out (but close) to this interval RW’s have only one unstable eigenfunction. Because this interval is too narrow and we may consider errors around 1% coming from spatial discretisation we could consider \( \text{Ha} \approx 9.98 \) as a double-Hopf bifurcation.

Figures 5, 6, and 7 display the patterns (radial and azimuthal velocity) of RW and their leading eigenfunctions lying on the \( m = 2 \), \( m = 3 \) and \( m = 4 \) branches, respectively. The values of \( \text{Ha} \) are selected close to bifurcation points. While the patterns of the different \( m \) RW remain quite similar, those of the eigenfunctions appear to be noticeably different. For small magnetic forcings \( \text{Ha} < 5 \) the instability giving rise to MRW is mainly concentrated within the tangent cylinder, i.e, in the polar regions, see Fig. 6 corresponding to a \( m = 3 \) RW at \( \text{Ha} = 3.1316477 \). Because these \( m = 3 \) RW are equatorially asymmetric and their leading eigenfunction has no azimuthal symmetry (\( m = 1 \)), the bifurcated MRW will have all the spatial symmetries broken. In the context of convection in spherical shells these solutions are quite rare and have never been reported before. In contrast, for larger values
Figure 6: As Fig. 5 but for an $m = 3$ RW at $Ha = 3.1316477$.

Table 2: Critical parameters of the symmetric RWs at the bifurcations where they change the stability ($|\lambda| = 1$). They are obtained by inverse interpolation with a polynomial of degree 5. Transition from stable to unstable (resp. unstable to stable) is denoted by Ch. st. 0/1 (resp. 1/0).

| $m$ | Ch. st. | $Ha_c$ | $\omega$ | Arg($\lambda$) | $m$ Eig. |
|-----|---------|--------|----------|----------------|----------|
| 2   | 0/1     | 30.04  | 56.83    | 0.31           | 1        |
| 3   | 0/1     | 26.68  | 54.98    | 2.52           | 1        |
| 3   | 1/0     | 35.24  | 110.36   | 2.83           | 1        |
| 4   | 1/0     | 31.95  | 101.62   | 2.077          | 1        |

Table 2: Critical parameters of the symmetric RWs at the bifurcations where they change the stability ($|\lambda| = 1$). They are obtained by inverse interpolation with a polynomial of degree 5. Transition from stable to unstable (resp. unstable to stable) is denoted by Ch. st. 0/1 (resp. 1/0).

of $Ha$ the instabilities lie more close to the outer surface in the equatorial plane. Nevertheless, the instability also affects higher latitudes as it reflects the spherical sections of $v_\phi$ shown in Figs. 5 and 7, with the $m = 2$ instabilities being the most altered in this region.

The stability analysis of the return flow and shear-layer type equatorially symmetric RW reveals several regions of multistability. They can be constructed from table 2:

- For $Ha \in [25.8, 26.68]$ only the $m = 4$ branch is stable.
For \( \text{Ha} \in [26.68, 30.04] \) the branches of \( m = 4 \) and \( m = 3 \) are stable.

- For \( \text{Ha} \in [30.04, 31.95] \) all branches are stable.
- For \( \text{Ha} \in [31.95, 35.24] \) the branches \( m = 3 \) and \( m = 2 \) are stable.
- For \( \text{Ha} \in [35.24, 79.42] \) only the branch \( m = 2 \) is stable.

By means of DNS explorations we have found that the Hopf bifurcation where \( m = 4 \) RW loose the stability is supercritical. Then, there exist some \( \epsilon > 0 \) in which for all \( \text{Ha} \in [31.95, 31.95 + \epsilon] \) a MRW is stable giving rise to a tri-stability region. The eigenfunctions at the bifurcation point have \( m = 1 \) and so do the bifurcated MRW. This also occurs at the bifurcations on the \( m = 2 \) and \( m = 3 \) branches.

The RW and their leading eigenfunctions convective patterns at the bifurcations points of Table 2 from top to bottom, are displayed in Figs. 8, 9, 10, and 11 respectively. They are all representative of return flow instability because the bifurcations occur at relatively moderate magnetic forcings \( \text{Ha} < 45 \). In all the cases the eigenfunctions are equatorially symmetric and the instability is concentrated in the bulk of the shell, specifically located on the edge of the radial jet. Notice that although the azimuthal wave number is \( m = 1 \), in the case of \( m = 2 \) and \( m = 4 \) the eigenfunctions satisfy some additional symmetry, namely, they are invariant under azimuthal rotations of \( \pi \) degrees and change of sign. This
Figure 8: Contour plots of $m = 2$ equatorially symmetric rotating wave at $Ha = 30.074289$ (close to critical). First row: The left three plots are the spherical, equatorial and meridional sections of the radial velocity $v_r$. Spherical and meridional sections are taken where $v_r$ has a relative maximum. Right three plots: Same sections for $v_\phi$. The spherical section is at $r = r_i + 0.1d$ and the meridional section cuts a relative maximum. Second: As first row but for the 1st leading eigenfunction. In this case all the spherical and meridional sections are taken at a relative maximum.

Figure 9: As in Fig. 8 but for an $m = 3$ rotating wave at $Ha = 26.45$.

does not occur on the $m = 3$ branch where the azimuthal symmetry is strictly $m = 1$, i.e only invariant under $2\pi$ azimuthal rotations.

As previously mentioned we have found, by means of time-stepping the model equations close to a bifurcation points, two branches of stable MRW. Specifically, one of those branches bifurcates
Figure 10: As in Fig. 8 but for an $m = 3$ RW at $Ha = 35.3$.

Figure 11: As in Fig. 8 but for an $m = 4$ RW at $Ha = 30.15$.

subcritically from the equatorially asymmetric $m = 3$ RW at $Ha = 3.95$, characteristic of the radial jet instability, and the other bifurcates supercritically from the equatorially symmetric $m = 4$ RW at $Ha = 31.95$, characteristic of the return flow instability. Our explorations with DNS suggest that the interval of stability of these MRW is quite small and because of this very long initial transients are required to saturate the instability, even with close initial conditions. This makes quite challenging their finding by means of DNS and this is why very few MRW, i.e purely time dependent solutions, have been reported previously in the literature for this problem (see [7] for instance). The computation of MRW branches is important because they give rise to 3 frequency solutions [30], which are the last stage in the route to turbulence [50].
Figure 12: Poincaré sections of the radial jet and return flow instabilities (a) The Poincaré section is defined by $v_r(r_s, \theta_s, \varphi_s) = 0$, with $(r_s, \theta_s, \varphi_s) = (1.5, 3\pi/4, 0)$. Invariant tori at $Ha = 3.7986668$ (inner closed curve) and at $Ha = 3.5410819$ (outer closed curve) bifurcated from the $m = 3$ asymmetric branch. The inner point corresponds to a stable $m = 3$ periodic orbit at $Ha = 4.1874771$. (b) The Poincaré section is defined by $v_r(r_s, \theta_s, \varphi_s) = 0$, with $(r_s, \theta_s, \varphi_s) = (1.854, 5\pi/8, 0)$. Invariant tori at $Ha = 32$ (inner closed curve) and at $Ha = 32.02$ bifurcated from the $m = 4$ symmetric branch. The inner dot corresponds to a stable $m = 4$ periodic orbit at $Ha = 31.9$. The axis are defined by $v_r^i = v_r(r_i, \theta_i, \varphi_i)$, $i = 1, 2$ with $(r_1, \theta_1, \varphi_1) = (1.5, 5\pi/8, 0)$ and $(r_2, \theta_2, \varphi_2) = (1.146, 5\pi/8, 0)$.

Quasiperiodic MRW are easily identified because their azimuthally-averaged properties are periodic, i.e., one frequency corresponds to the azimuthal drift of the wave and the other to the modulation. However, in some situations the modulation frequency could undergo period doubling bifurcations (see [51] for instance) making difficult the identification of the quasiperiodic character of the solution, even with the help of a frequency spectrum. For a better classification, Poincaré sections provide additional information. By means of the latter, periodic orbits are represented by a single point and quasiperiodic solutions as a closed curve, having loops in case of period doubling bifurcations [51]. The Poincaré sections of the above mentioned MRW, characteristic of the radial jet and return flow instability are plotted in Fig. 12(a) and (b), respectively. The sections are defined by means of the radial velocity $v_r(r_s, \theta_s, \varphi_s) = 0$, $(r_s, \theta_s, \varphi_s) = (1.5, 3\pi/4, 0)$ being a point in the middle of the shell at mid latitudes. On both panels the horizontal axis represents $v_r(1.5, 5\pi/8, 0)$ and the vertical axis $v_r(1.146, 5\pi/8, 0)$. On Fig. 12(a) the point represents a stable RW at $Ha = 4.1874771$. By decreasing $Ha$ and because of the bifurcation is subcritical a branch of MRW with azimuthal wave number $m = 1$ is found. The two closed curves of Fig. 12(a) are representative of solutions belonging to this branch. The inner curve corresponds to $Ha = 3.7986668$ and the outer to $Ha = 3.5410819$. Analogously in Fig. 12(b) the two closed curves are MRW, but in this case the region of stability is smaller. The point corresponds to $Ha = 31.9$, the inner curve to $Ha = 32$ and the outer to $Ha = 32.02$. Examining
Table 3: Spatial discretization study. Frequency $\omega$ and leading Floquet multiplier $\lambda = \lambda_r + i\lambda_i$ as function of the number of radial collocation points $N_r$ and the spherical harmonics truncation parameter $L_{\text{max}}$. The total number of degrees of freedom of the system $n = (2L_{\text{max}}^2 + 4L_{\text{max}})(N_r - 1)$ is also shown.

| $N_r$ | $L_{\text{max}}$ | $n$ | $m$ | Eq. sym | Ha | $\omega$ | $\lambda_r$ | $\lambda_i$ |
|-------|------------------|-----|-----|---------|----|----------|-------------|-------------|
| 40    | 84               | 563472 | 3  | 0       | 3.1316477 | 139.3432 | -0.8340318 | 0.6318775   |
| 60    | 126              | 1903104 | 3  | 0       | 3.1316477 | 138.9097 | -0.8420450 | 0.6276191   |
| 40    | 84               | 563472 | 4  | 1       | 27.327395 | 71.12972 | -0.5921600 | 0.6883127   |
| 60    | 128              | 1963520 | 4  | 1       | 27.327395 | 71.12765 | -0.5921522 | 0.6883861   |

the range of variation shown in the axis on both figures helps to get an idea of the amplitude of modulations. The larger the longitude of the curve, the larger the modulations, i.e., decreasing the modulations makes the curve tend to the point. By comparing the axis ranges of Fig. 12(a) and (b) it is clear that the oscillations of the equatorially symmetric MRW are less pronounced than the corresponding of the equatorially asymmetric MRW. This is not surprising because the former are stable in a very small interval close to the bifurcation point.

Throughout the study several numerical tests, that we summarise in the following, have been performed to validate the results. For the continuation of the waves as well as the computation of its stability the fixed step time integration is checked with an VSVO implicit method and very low tolerances (see [44] for details on the time integration methods). Moreover, changing tolerances, Krylov dimension or number of desired eigenvalues when dealing with Arnoldi methods (Arpack) helps to identify changes of stability. In addition, the spatial resolution is changed from time to time to look for discretization errors, see table 3. On the latter, the values of the drifting frequency and the real and imaginary part of the leading Floquet multiplier are shown for two different solutions belonging to the equatorially asymmetric and symmetric regions, respectively. Errors below 1% are obtained when increasing the resolution from $N_r = 40, L_{\text{max}} = 84$ to $N_r = 60, L_{\text{max}} = 128$. Notice that for the latter resolution the number of degrees of freedom rises to nearly $2 \times 10^6$. For such a large dimensional systems very few studies based on continuation and stability of periodic orbits exist, even in the more general context of fluid dynamics. Finally we mention that several DNS have been performed to further validate the results.

5 Summary

The present analysis constitutes a further step towards a better understanding of weakly nonlinear regime of the axially magnetised spherical Couette problem by extending previous studies [7, 43, 25] based on very few nonlinear solutions and on sketches of bifurcation diagrams. Our study relies on the
novel application of Newton-Krylov continuation and stability analysis of the solutions. For such high dimensional systems there exist very few studies based on continuation methods, even in the more general context of fluid dynamics [28, 35].

Thanks to the use of continuation techniques nearly 600 nonlinear rotating waves have been obtained and the stability of around one third of the solutions have been established in a parameter regime of experimental interest, i.e Reynolds number \( \text{Re} = 10^3 \), aspect ratio \( \eta = 0.5 \) and Hartmann numbers \( \text{Ha} \in (0, 80) \) corresponding to the working conditions of the HEDGEHOG experimental device at the Helmholtz-Zentrum Dresden-Rossendorf [9]. In this regime radial jet, return flow and shear-layer instabilities have been previously described [43, 25].

For the fixed Reynolds number we have accurately determined several regions of multistability and several transitions to quasiperiodic flows (modulated waves) of the first nonaxisymmetric instabilities with azimuthal wave numbers \( m = 2, 3, 4 \) that occur when the Hartmann number is varied. The symmetry of the bifurcated MRW is identified and some examples are provided. The RW and their eigenfunctions at the bifurcation points provide initial conditions for the continuation of MRW that could be performed as in [30].

The patterns, rotation frequencies (i.e time scales) and some physical properties (nonaxisymmetric kinetic energy, for instance) of the radial jet, return flow and shear-layer instabilities have been exhaustively described in the range of \( \text{Ha} \in (0, 80) \). For each azimuthal wave number the return flow instability continuously changes with increasing \( \text{Ha} \) to become the shear-layer instability. At the boundary between both instabilities, defined by a critical \( \text{Ha} \), the nonaxisymmetric part of the flow is maximum. Nonaxisymmetry increases with decreasing wave number \( m \). The critical \( \text{Ha} \) where nonaxisymmetric flow is maximum increases with decreasing \( m \), too.

The determination of stability regions is crucial for comparison with experiments. They allow, for instance, to determine the azimuthal wave number of the most physically realisable solutions and thus help to design appropriate measurement set-ups. The azimuthal wave numbers of the branches chosen for this study matches with those that can be measured in the experiments [9]. The analysis of physical properties, such as the volume-averaged kinetic energies, serve as prior estimate of their experimental values and thus as a guide for tuning measurement techniques. Once the solutions are obtained with continuation methods they can be later on easily processed to obtain other measurable properties not shown in this study, for instance local velocities inside the shell or the torque acting on the outer sphere, for a further comparison with experiments.

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