CONSTRUCTION AND ASSIGNMENT OF ORTHOGONAL SEQUENCES AND ZERO CORRELATION ZONE SEQUENCES FOR APPLICATIONS IN CDMA SYSTEMS

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ABSTRACT. Orthogonal sequences can be assigned to a regular tessellation of hexagonal cells, typical for synchronised code-division multiple-access (S-CDMA) systems. In this paper, we first construct a new class of orthogonal sequences with increasing the number of users per cell to be $2^m - 2$ for even number $m \geq 4$ (where $2^m$ is the length of the sequences). In addition, based on the above construction we construct a family of orthogonal sequences with zero correlation zone property which can be applied to the quasi-synchronous CDMA (QS-CMDA) spread spectrum systems.

1. INTRODUCTION

Orthogonal sequences play an important role in synchronised CDMA (S-CDMA) spread spectrum systems. To prevent the interference from the neighboring cells in the regular tessellation of hexagonal cells, a basic requirement is that the sequences within any cell are orthogonal to the sequences in the neighboring cells. In addition, the cross-correlation of the sequences in a given cell between non-neighboring cells should be sufficiently small and lie in the range $[2^m/2, 2^{(m+2)/2}]$. A usual way of constructing spreading codes in these systems is to use correlation-constrained sets of Hadamard matrices [7, 8, 10, 12, 13].

To reduce the interference among users in a multipath environment or in an quasi-synchronised CDMA (QS-CDMA) environment [1, 3], the concept of generalized orthogonality was defined and the zero correlation zone (ZCZ) sequences was presented in [4, 9, 10, 11, 14]. In these systems it is of importance that two sequences have zero cross-correlation within certain range time-shifts.

In order to reflect the ability of using the same sequences in non-adjacent cells, the re-use distance $D$ is proposed. The so-called re-use distance $D$ reflects the ability to use the same codewords in non-adjacent cells that are at distance $D$ from

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the cell where these codewords have originally been placed. The re-use distance is discussed in detail in [8, 13].

In this paper, we first construct a new class of sequences with orthogonality in adjacent cells while increasing the number of users per cell to be $2^{m-2}$ and the re-use distance $D = 4$. Then through a vectorial bent direct sum these sequences can be extended to any desired length $2^{m+u}$, where $u \geq 4$ is an even. Meanwhile, the number of orthogonal semi-bent sequences in each cell has a $2^m$-fold increase. Based on the results above, we construct a family of sequences satisfying orthogonality and ZCZ property in each cell and adjacent cells. The assignment of sequences has the same re-use distance $D = 4$.

This paper is organized as follows. In Section 2, the useful notations are given. In Section 3, a new class of orthogonal sequences is constructed, and a family of sequences with ZCZ is presented in Section 4. Finally, the conclusion is presented in Section 5.

2. Preliminaries

Let $\mathbb{F}_{2^m}$ denote the finite field $GF(2^m)$ and $\mathbb{F}_2^m$ denote the corresponding $m$-dimension vector space. An $m$-variable Boolean function $f$ is a function from $\mathbb{F}_2^m$ to $\mathbb{F}_2$. Let $\mathcal{B}_m$ denote the set of all Boolean functions in $m$-variables. The sequence of $f$ is a $(1, -1)$-sequence of length $N = 2^m$ defined by

$$ \mathcal{F} = \{(−1)^{f(0\cdots0)}, (−1)^{f(0\cdots01)}, \ldots, (−1)^{f(1\cdots11)}\}. $$

Given two real-valued sequences $a = (a_0, \ldots, a_{N-1})$ and $b = (b_0, \ldots, b_{N-1})$, their componentwise product is defined by $a \ast b = (a_0b_0, \ldots, a_{N-1}b_{N-1})$. For any $f_1, f_2 \in \mathcal{B}_m$, we always have $\mathcal{F}_1 \ast \mathcal{F}_2 = \mathcal{F}_1 + \mathcal{F}_2$. The walsh transform of $f \in \mathcal{B}_m$ at point $\omega$ is denoted by $W_f(\omega)$ and it is computed as

$$ W_f(\omega) = \sum_{x \in \mathbb{F}_2^m} (−1)^{f(x)+\omega \cdot x}. $$

**Definition 2.1.** A function $f \in \mathcal{B}_m$ satisfying that $W_f(\omega) \in \{±2^m/2\}$, for all $\omega \in \mathbb{F}_2^m$, is called bent [6]. A function $f \in \mathcal{B}_m$ satisfying that $W_f(\omega) \in \{0, ±2^{(m+2)/2}\}$, for all $\omega \in \mathbb{F}_2^m$, is called semi-bent [2]. $\mathcal{F}$ is called a semi-bent sequence if $f$ is a semi-bent function.

Let $f_1, f_2 \in \mathcal{B}_m$. $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal, denoted by $\mathcal{F}_1 \perp \mathcal{F}_2$, if

$$ \mathcal{F}_1 \cdot \mathcal{F}_2 = \sum_{x \in \mathbb{F}_2^m} (−1)^{f_1(x)+f_2(x)} = 0. $$

**Definition 2.2.** An $m$-variable $t$-dimensional vectorial function is a mapping $F$ from $\mathbb{F}_2^m$ to $\mathbb{F}_2^t$, which can also be viewed as a collection of $t$ Boolean functions so that $F(x) = (f_1, \ldots, f_t)$, where $f_1, \ldots, f_t \in \mathcal{B}_m$. $F$ is said to be a vectorial semi-bent function if any nonzero linear combination of the component functions $f_1, \ldots, f_t$ is a semi-bent Boolean function with Walsh spectra taken values in $\{0, ±2^{(m+2)/2}\}$. $F$ is said to be a vectorial bent function if any nonzero linear combination of $f_1, \ldots, f_t$ is a bent function with Walsh spectra taken values in $\{±2^m/2\}$.

**Definition 2.3.** Let $a = (a_0, a_1, \ldots, a_{N-1})$ and $b = (b_0, b_1, \ldots, b_{N-1})$ be two sequences, the aperiodic correlation function of $a$ and $b$ at shift $\tau$ is given as follows:

$$ R_{a,b}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} a_i b_i + \tau, & 0 \leq \tau \leq N - 1; \\ \sum_{i=0}^{N-1+\tau} a_{i-\tau} b_i, & -N + 1 \leq \tau \leq -1; \\ 0, & |\tau| \geq N. \end{cases} $$
\( R_{a,b}(\tau) \) is called aperiodic cross-correlation function if \( a \neq b \); otherwise, it is said the auto-correlation function, denoted as \( R_{a}(\tau) \).

**Definition 2.4.** A pair of sequences \((C_1, D_1)\) of period \( N \) is said to be a complementary pair if
\[
R_{C_1}(\tau) + R_{D_1}(\tau) = 0, 0 < |\tau| < N
\]
Another complementary pair \((C_2, D_2)\) of the same period is called a mate of \((C_1, D_1)\) if
\[
R_{C_1, C_2}(\tau) + R_{D_1, D_2}(\tau) = 0, 0 \leq |\tau| < N
\]
A complementary pair and its mate are called two uncorrelated complementary pairs.

**Definition 2.5.** Let \( A = \{ A_1, A_2, \cdots, A_K \} \) be a set of \( K \) sequences, each of length \( L \), i.e.,
\[
A_i = (a_{i0}, a_{i1}, \cdots, a_{iL-1}), 1 \leq i \leq K
\]
A is said to be a ZCZ sequence set if and only if satisfies the following two conditions:
\[ i) \quad R_{A_i}(\tau) = 0 \text{ holds for any } 1 \leq i \leq K \text{ and } 1 \leq |\tau| \leq Z; \]
\[ ii) \quad R_{A_i,A_j}(\tau) = 0 \text{ holds for any } i \neq j \text{ and } 0 \leq |\tau| \leq Z. \]
The zero correlation spread (ZCS) of \( A \) is \( Z \).

### 3. A NEW METHOD TO CONSTRUCT ORTHOGONAL SEQUENCE SETS FOR EVEN INTEGER CASES

In this section, we construct a new class of orthogonal sequences which increases the number of users locally to be \( 2^{m-2} \) with re-use distance \( D = 4 \) for \( m = 4 \). Then through a suitable vectorial bent direct sum these sequences can be extended to any desired length \( 2^{m+u} \), where \( u \geq 4 \) is an even. At the same time, the number of orthogonal semi-bent sequences in each \( S_{c,a} \) is increased by a factor of \( 2^u \).

**Construction 1.** Let \( m = 4 \) and \( \gamma \in \mathbb{F}_{2^3} \) be a root of the primitive polynomial \( z^3 + z + 1 \). For \( i = 1, 2 \), let \( \phi_i : \mathbb{F}_2 \rightarrow \mathbb{F}_2^3 \) be an injective mapping with
\[
\phi_1(0) = 010, \phi_1(1) = 001; \\
\phi_2(0) = 001, \phi_2(1) = 110.
\]
For \( y \in \mathbb{F}_2, x = (x_1, x_2, x_3) \in \mathbb{F}_2^3 \), a vectorial semi-bent function \( F : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^2 \) is constructed as
\[
F(y, x) = (f_1, f_2),
\]
where
\[
f_1(y, x) = \phi_1(y) \cdot x = (y + 1)x_2 + yx_3, \\
f_2(y, x) = \phi_2(y) \cdot x = (y + 1)x_3 + y(x_1 + x_2).
\]
For \( c \in \mathbb{F}_2^3 \), let \( f_c(y, x) = c \cdot F(y, x) \). Specifying the sequences by only using the signs instead of \(+1, -1\), we have
\[
\overline{f_{00}} = (+ + + + + + + + + + + + + + + + + +) \\
\overline{f_{01}} = (+ - + + + + + + + + + + + + + + + + +) \\
\overline{f_{10}} = (+ + - - + + + + + + + + + + + + + + +) \\
\overline{f_{11}} = (+ - + + - - - + + + + + + + + + + +)
For \( \alpha \in \mathbb{F}_2^3 \), let \( \mathcal{L}_\alpha = \{ \mathcal{I} \mid l \in L_\alpha \} \), where
\[
L_\alpha = \{ (\beta, \alpha) \cdot (y, x) \mid \beta \in \mathbb{F}_2 \}.
\]
Let \( \mathcal{H}_{00} = \mathcal{L}_{000} \cup \mathcal{L}_{100}, \mathcal{H}_{01} = \mathcal{L}_{001} \cup \mathcal{L}_{101}, \mathcal{H}_{10} = \mathcal{L}_{010} \cup \mathcal{L}_{110}, \) and \( \mathcal{H}_{11} = \mathcal{L}_{011} \cup \mathcal{L}_{111} \). Then we have
\[
\mathcal{H}_{00} : \begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +&
Table 1. Orthogonality between $f_c$ and $H_a$

|     | $H_{00}$ | $H_{10}$ | $H_{01}$ | $H_{11}$ |
|-----|---------|---------|---------|---------|
| $f_{00}$ | ⊥       | ⊥       | ⊥       | ⊥       |
| $f_{10}$ | ⊥       | ⊥       | ⊥       | ⊥       |
| $f_{01}$ | ⊥       | ⊥       | ⊥       | ⊥       |
| $f_{11}$ | ⊥       | ⊥       | ⊥       | ⊥       |

Figure 1. Assignment of orthogonal sets to a lattice of regular hexagonal cells

All the $2^{m+u}$-length sequences in $S_{c,a}$ are semi-bent sequences. We can easily use all the 16 sets of orthogonal sequences to get an assignment with the re-use distance $D = 4$ as depicted in Figure 1.

**Remark 1.** According to the Definition 2.1, if the boolean function $g_c(z, y, x) = h_c(z) + f_c(y, x)$ is a semi-bent function, the corresponding sequence $f_c$ is a semi-bent sequence. In Construction 1, note that the set $S_{c,a} = \{f_c * l \mid l \in \mathcal{L}_a\}$, for $c \in \mathbb{F}_2^2$, $a \in \mathbb{F}_2^2$. We prove the function $g_c(z, y, x)$ is semi-bent function as follows.

For any $\omega = (\gamma, \beta, \alpha) \in \mathbb{F}_2^3 \times \mathbb{F}_2 \times \mathbb{F}_2^3$, we have

$W_{g_c}(\omega) = \sum_{(z, y, x) \in \mathbb{F}_2^{m+u}} (-1)^{g_c(z, y, x) + \omega \cdot (z, y, x)}$

$= \sum_{z \in \mathbb{F}_2^u} (-1)^{h_c(z) + \gamma \cdot z} \cdot \sum_{y, x \in \mathbb{F}_2 \times \mathbb{F}_2^3} (-1)^{f_c(y, x) + (\beta, \alpha) \cdot (y, x)}$

$= \pm 2^{n/2} \cdot \sum_{y, x \in \mathbb{F}_2 \times \mathbb{F}_2^3} (-1)^{\phi_c(y) \cdot x + \beta \cdot y + \alpha \cdot x}$

$= \pm 2^{n/2} \cdot \sum_{y \in \mathbb{F}_2} (-1)^{\beta \cdot y} \cdot \sum_{x \in \mathbb{F}_2^3} (-1)^{\phi_c(y) \cdot x + \alpha \cdot x}$

$= \begin{cases} 
\pm 2^{n/2+3}, & \text{if } \phi_c(y) = \alpha, \\
0, & \text{otherwise.}
\end{cases}$
Since the boolean function $h_c(z)$ is a $u$-variable bent function, we have $W_{h_c}(\gamma) = \sum_{z \in \mathbb{F}_2} (-1)^{h_c(z)\cdot\gamma} = \pm 2^{u/2}$. For any $\omega = (\gamma, \beta, \alpha) \in \mathbb{F}_2^u \times \mathbb{F}_2 \times \mathbb{F}_2^3$, there exists an $\alpha$ such that $\phi_1(y) = \alpha$. We can obtain $g_c(z, y, x)$ is a semi-bent function. For example, for $c = 1$, we have $\phi_1(0) = 010, \phi_1(1) = 100$. When $\omega_1 = (\gamma, \beta, 010)$ and $\omega_2 = (\gamma, \beta, 100)$, where $(\gamma, \beta) \in \mathbb{F}_2^u \times \mathbb{F}_2$, then $W_{g_1}(\omega_1) = \pm 2^{u/2+3}$, and $W_{g_1}(\omega_2) = \pm 2^{u/2+3}$. We have $\overline{W_1}$ is a semi-bent sequence.

Remark 2. Re-use distance is defined as the distance between the centers of hexagonal cells that use the same channels [5]. Since path loss is a function of propagation distance, the re-use distance $D$ between cells using the same channel is an important parameter in determining average intercell interference power. Assume the tessellation of hexagonal cells is regular and let the distance between the centers of adjacent cells be 1. According to the Figure 1, the minimum distance between the same hexagonal cell is 4. For example, for the hexagonal cell $S_{00,10}$, the distance between the same cell $S_{00,10}$ is 4, and so is any other cell $S_{c,\alpha}$, for $c \in \mathbb{F}_2^u$, and $\alpha \in \mathbb{F}_2^3$. Therefore, the re-use distance of the hexagonal cells is 4 in Figure 1.

In Construction 1, we can see that orthogonal sequences can be applied to the S-CDMA systems successfully. But orthogonal sequences cannot be applied to the QS-CDMA systems directly. This problem is solved in the next section by constructing large sets of orthogonal sequences with ZCZ property.

4. A FAMILY OF ORTHOGONAL SEQUENCES WITH ZCZ

Based on the orthogonal sequences obtained in Section 3, we next construct a family of sequences satisfying ZCZ properties, which implies that the sequences can be applied to QS-CDMA systems. Furthermore, the re-use distance $D$ is also 4.

Definition 4.1. Let $b = (b_0, b_1, \ldots, b_{n-1})$ be a sequence of period of $n$, and $A = (A_0, A_1, \ldots, A_{n-1})$ be a row vector consisting of $n$ sequences $A_i$, $0 \leq i < n$. We define

$$b \odot A = (b_0 \cdot A_0, b_1 \cdot A_1, \ldots, b_{n-1} \cdot A_{n-1})$$

where $b_i \cdot A_i$ is the scalar multiplication of $b_i$ by the scalar $A_i$ for $0 \leq i < n$.

Construction 2. Let $(C_1, D_1)$ and $(C_2, D_2)$ of length $\mathcal{N}$ be two uncorrelated complementary pairs. Let

$$C = (C_1, C_2, C_1, C_2, \ldots, C_1, C_2)$$

$$D = (D_1, D_2, D_1, D_2, \ldots, D_1, D_2)$$

$$\tilde{C} = (C_2, C_1, C_2, C_1, \ldots, C_2, C_1)$$

$$\tilde{D} = (D_2, D_1, D_2, D_1, \ldots, D_2, D_1)$$

be the $2^{n+u}$-tuples. Let $S_{c,\alpha}$ be the set of sequences generated by the Construction 1 in the Section 3. We construct 16 disjoint sets of $(2^{n+u+1} \mathcal{N} + N - 1)$-length sequences with ZCZ as follows:

$$T_{c,\alpha} = \{ (S_{c,\alpha}^i \odot C, O_{N-1}, S_{c,\alpha}^i \odot D) \mid S_{c,\alpha}^i \in S_{c,\alpha} \}$$

(13)

$$\bigcup \{ (S_{c,\alpha}^i \odot \tilde{C}, O_{N-1}, S_{c,\alpha}^i \odot \tilde{D}) \mid S_{c,\alpha}^i \in S_{c,\alpha} \}.$$

We can easily use all the 16 sets of disjoint sequences to get an assignment with the re-use distance $D = 4$ as depicted in Figure 1.
Theorem 4.2. For \( c \in \mathbb{F}_2^2, \alpha \in \mathbb{F}_2^2 \), let the set of sequences \( T_{c, \alpha} \) be defined by (13) as in Construction 2. Then \( T_{c, \alpha} \) forms a \((2^{m+u+1}N + N - 1)\)-length ZCZ sequence set with \( Z_{c, \alpha} = N \). Let \( T_{c, \alpha}^i, T_{c', \alpha'}^j \) be two sequences taken from the sequence sets \( T_{c, \alpha} \) and \( T_{c', \alpha'} \), respectively. We have

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(\tau) = 0, \quad \text{if } 1 \leq |\tau| \leq N - 1,
\]

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0) = 0, \quad \text{if } c = c', \alpha = \alpha' \text{ and } i \neq j.
\]

When \( T_{c, \alpha} \) and \( T_{c', \alpha'} \) are two adjacent cells, we have

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0) = 0.
\]

When \( T_{c, \alpha} \) and \( T_{c', \alpha'} \) are two non-adjacent cells, we have

\[
|R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0)| \leq 2^{(m+u+1)/2}N.
\]

Proof. Let \( T_{c, \alpha}^i = (S_{c, \alpha} \circ C, 0_{N-1}, S_{c, \alpha}^j \circ D) \) and \( T_{c', \alpha'}^j = (S_{c', \alpha'}^j \circ C, 0_{N-1}, S_{c', \alpha'}^j \circ D) \) be the sequences defined as Construction 2, where \( S_{c, \alpha}^i = (s_0, s_1, \cdots, s_{2^m+u-1}) \), and \( S_{c', \alpha'}^j = (s_0, s_1, \cdots, s_{2^m+u-1-1}) \). We have

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0) = \sum_{k=0}^{2^m+u-1} s_k s'_k R_C(0) + \sum_{k=0}^{2^m+u-1} s_k s'_k R_D(0)
\]

\[
= 2N \sum_{k=0}^{2^m+u-1} s_k s'_k
\]

\[
= 2NR_{S_{c, \alpha}^i, S_{c', \alpha'}^j}(0).
\]

From the above formula, we show that the set \( T_{c, \alpha} \) and \( S_{c, \alpha} \) have one-to-one correspondence. According to the Table 1, when \( c = c', \alpha = \alpha' \), and \( i \neq j \), we have \( R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0) = 2NR_{S_{c, \alpha}^i, S_{c', \alpha'}^j}(0) = 0 \). When \( c \neq c' \), or \( \alpha \neq \alpha' \), in the case of \( S_{c, \alpha} \) and \( S_{c', \alpha'} \) being two adjacent cells, we have \( R_{S_{c, \alpha}^i, S_{c', \alpha'}^j}(0) = 0 \), where \( S_{c, \alpha}^i \in S_{c, \alpha} \), and \( S_{c', \alpha'}^j \in S_{c', \alpha'} \). Therefore, when \( T_{c, \alpha} \) and \( T_{c', \alpha'} \) are two adjacent cells, we have

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0) = 0.
\]

In the case of \( S_{c, \alpha} \) and \( S_{c', \alpha'} \) being two non-adjacent cells, according to the Construction 1 and the Remark 1, we have \( |R_{S_{c, \alpha}^i, S_{c', \alpha'}^j}(0)| \leq 2^{(m+u+1)/2} \). Therefore, when \( T_{c, \alpha} \) and \( T_{c', \alpha'} \) are two non-adjacent cells, we can obtain

\[
|R_{T_{c, \alpha}, T_{c', \alpha'}^j}(0)| = 2N|R_{S_{c, \alpha}^i, S_{c', \alpha'}^j}(0)| \leq 2N2^{m+u+1} = 2^{(m+u+1)/2}N.
\]

For \( \tau \neq 0 \), we consider the following four cases.

Case 1. \( T_{c, \alpha}^i = (S_{c, \alpha} \circ C, 0_{N-1}, S_{c, \alpha}^j \circ D) \) and \( T_{c', \alpha'}^j = (S_{c', \alpha'}^j \circ C, 0_{N-1}, S_{c', \alpha'}^j \circ D) \).

Without loss of generality, we only consider the case of \( 1 \leq |\tau| \leq N - 1 \). For the case of \(-N \leq \tau \leq -1 \), we can obtain it in a similar way. Let \( \tau' = N - \tau \). Note that

\[
R_{T_{c, \alpha}, T_{c', \alpha'}^j}(\tau) = \sum_{i=0}^{2^m+u-1-1} s_{2i}s'_{2i} R_C(\tau) + s_{2i+1}s'_{2i+1} R_C(\tau)
\]
replace the set \( R \) with \( Z \), we always have \( Z \) for any \( R \) as follows:

\[
2^{m+u-1-1} + \sum_{i=0}^{2^{m+u-1-1}} s_{2i+1}^2 s_{2i} R_{C_2, C_1}(\tau') + s_{2i}^2 s_{2i+1} R_{C_1, C_2}(\tau')
\]

\[
+ \sum_{i=0}^{2^{m+u-1-1}} s_{2i} s_{2i} R_{D_1}(\tau) + s_{2i+1} s_{2i+1} R_{D_2}(\tau)
\]

\[
+ \sum_{i=0}^{2^{m+u-1-1}} s_{2i}^2 s_{2i+1} R_{D_2, D_1}(\tau') + s_{2i+1}^2 s_{2i+1} R_{D_1, D_2}(\tau')
\]

\[
= \sum_{i=0}^{2^{m+u-1-1}} s_{2i} s_{2i}(R_{C_1}(\tau) + R_{D_1}(\tau)) + s_{2i+1} s_{2i+1}(R_{C_2}(\tau) + R_{D_2}(\tau))
\]

\[
+ \sum_{i=0}^{2^{m+u-1-1}} s_{2i+1}^2 s_{2i+1}(R_{C_2, C_1}(\tau') + R_{D_2, D_1}(\tau'))
\]

\[
(14) + \sum_{i=0}^{2^{m+u-1-1}} s_{2i}^2 s_{2i+1}(R_{C_1, C_2}(\tau') + R_{D_1, D_2}(\tau'))
\]

By (5), (6), \( R_{C_1}(\tau) + R_{D_1}(\tau) = 0 \) and \( R_{C_2}(\tau) + R_{D_2}(\tau) = 0 \), for \( 1 \leq \tau \leq N - 1 \); \( R_{C_2, C_1}(\tau') + R_{D_2, D_1}(\tau') = 0 \) and \( R_{C_1, C_2}(\tau') + R_{D_1, D_2}(\tau') = 0 \), for \( 0 \leq \tau' \leq N - 1 \). Then we have

\[
R_{T_{c, \alpha}^i, T_{c', \alpha'}^j}(\tau) = 0.
\]

**Case 2.** \( T_{c, \alpha}^i = (S_{c, \alpha}^i \odot C, 0_{N-1}, S_{c, \alpha}^i \odot D) \) and \( T_{c', \alpha'}^j = (S_{c', \alpha'}^j \odot \tilde{C}, 0_{N-1}, S_{c', \alpha'}^j \odot \tilde{D}) \).

**Case 3.** \( T_{c, \alpha}^i = (S_{c, \alpha}^i \odot \tilde{C}, 0_{N-1}, S_{c, \alpha}^i \odot \tilde{D}) \), and \( T_{c', \alpha'}^j = (S_{c', \alpha'}^j \odot C, 0_{N-1}, S_{c', \alpha'}^j \odot D) \).

**Case 4.** \( T_{c, \alpha}^i = (S_{c, \alpha}^i \odot \tilde{C}, 0_{N-1}, S_{c, \alpha}^i \odot \tilde{D}) \), and \( T_{c', \alpha'}^j = (S_{c', \alpha'}^j \odot \tilde{C}, 0_{N-1}, S_{c', \alpha'}^j \odot \tilde{D}) \).

The proofs of Case 2, Case 3 and Case 4 are similar with the proof of Case 1. We always have \( R_{T_{c, \alpha}^i, T_{c', \alpha'}^j}(\tau) = 0 \), for \( 1 \leq |\tau| \leq N - 1 \).

Combining the four cases above, we can determine that \( T_{c, \alpha} \) is a ZCZ sequence set with \( Z_{c} = N \).

**Remark 3.** The 16 sets \( T_{c, \alpha} \), where \( c \in \mathbb{F}_2^3 \), \( \alpha \in \mathbb{F}_2^3 \), can be arranged as in Figure 1, just replacing the set \( S_{c, \alpha} \) with \( T_{c, \alpha} \). At the same time, the orthogonality is updated to ZCZ property accordingly. Note that the number of sequences is \( 2^{m+u-1} \) in each set \( T_{c, \alpha} \). If we define

\[
(15) T_{c, \alpha} = \{(S_{c, \alpha}^i \odot C, 0_{2N-1}, S_{c, \alpha}^i \odot D) \mid S_{c, \alpha}^i \in S_{c, \alpha}\},
\]

then we have that \#\( T_{c, \alpha} = 2^{m+u-2} \) and \( Z_{c} = 2N \).

5. Conclusions

In this paper, we construct a class of sequence sets, and assign them to a regular tessellation of hexagonal cells with the re-use distance \( D = 4 \) and the number of per cell is equal to \( 2^{m-2} \) for even \( m \). Next we extended the orthogonal sequences to ZCZ sequences which can be applied to QS-CDMA system, and analyzed the properties of the new sequence sets. One interesting topic for further research is to find a family of ZCZ sequence sets with larger ZCZ widths. Another interesting
topic for the further research is to find a family of ZCZ sequence sets which has more sequences per cell.

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