The $L^2$-torsion is a variant defined for compact $L^2$-acyclic manifolds of determinant class, for example odd dimensional hyperbolic manifolds. It was introduced by John Lott [Lot92] and Varghese Mathai [Mat92] and computed for hyperbolic manifolds in low dimensions. Our definition of the $L^2$-torsion coincides with that of John Lott, which is twice the logarithm of that of Varghese Mathai.

In this paper you will find a proof of the fact that the $L^2$-torsion of hyperbolic manifolds of arbitrary odd dimension does not vanish. This was conjectured by John Lott in [Lot92, p.484, Proposition 16 infra]. Some concrete values are computed and an estimate of their growth with the dimension is given. The values we compute for dimensions 5 and 7 differ from those published in [Lot92, Proposition 16]. The result has been independently achieved by both authors and will be part of the dissertation of Eckehard Hess at the university of Mainz. For an introduction into $L^2$-theory see [Lü97].

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1. Definition. Following [Lot92, p.482] we define the analytic $L^2$-torsion of an $L^2$-acyclic Riemannian $(d = 2n + 1)$-dimensional manifold of determinant class by

$$\text{Tor}_{(2)}(M) = 2 \sum_{j=0}^{n} (-1)^{j+1} \log \det_G(\Delta_j)$$

Here $G$ is the fundamental group of $M$, $\Delta_j$ is the Laplacian restricted to coclosed forms on the universal covering $\tilde{M}$ and the logarithm of the $G$-determinant is computed from the local trace of the heat kernel as follows

$$\log \det_G(\Delta_j) = \int_{F} \left\{ \frac{d}{ds} \left|_{s=0} \left[ \frac{1}{\Gamma(s)} \int_{0}^{1} t^{s-1} \text{tr}_{C} e^{-t\Delta_j}(x,x) dt \right] \right. \right.$$  

$$\left. + \int_{1}^{\infty} t^{-1} \text{tr}_{C} e^{-t\Delta_j}(x,x) dt \right\} dx$$
Here $F$ is a fundamental domain of $M$ in $\tilde{M}$, the first integral exists for $s$ sufficiently large and one has to take the meromorphic extension at 0. $M$ being of determinant class ensures the second integral to converge.

2. Theorem. There is a constant $\alpha_d > 0$, such that for every $(d = 2n + 1)$-dimensional closed hyperbolic manifold

$$\text{Tor}(M) = (-1)^n \alpha_d \text{Vol}(M)$$

The values for $\alpha_d$ have been computed as follows. Although exact values were computed for $d \leq 251$, we will give only exact numbers for $d \leq 11$ and approximate numbers for $d \leq 39$:

| $d$ | $\alpha_d$ | $\approx \alpha_d$ | $d$ | $\approx \alpha_d$ |
|-----|-------------|---------------------|-----|---------------------|
| 3   | $\frac{1}{3\pi}$ | 0.106103            | 13  | 1.61885             |
| 5   | $\frac{62}{45\pi^2}$ | 0.139598            | 15  | 4.22925             |
| 7   | $\frac{221}{35\pi^2}$ | 0.203645            | 17  | 12.3578             |
| 9   | $\frac{32204}{2193\pi^4}$ | 0.349847            | 19  | 39.9606             |
| 11  | $\frac{1339661}{6237\pi^6}$ | 0.701891            | 21  | 141.729             |
|     |             |                     | 23  | 547.188             |

3. Lemma. Let $\Delta_j = \delta_j d_j|_{\ker(\delta_j+1)}$ be the Laplacian, restricted to coclosed $L^2$-forms on the $(d = 2n + 1)$-dimensional hyperbolic space $H^d$. Then for every closed $d$-dimensional hyperbolic manifold $M$ with fundamental group $G$ and $j \neq n$

$$\log \det_G(\Delta_j) = \text{Vol}(M)C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K^n_{k,j} (-1)^{k+1} \frac{2\pi}{2k+1} (n - j)^{2k+1}$$

with $C = \frac{(4\pi)^{-(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}$ and $a = n - j$

Here $K^n_{k,j}$ is the coefficient of $\nu^{2k}$ in the polynomial

$$P(\nu) := \prod_{i=0}^{n} \frac{(\nu^2 + i^2)}{\nu^2 + (n - j)^2} \quad (*)$$

Note that $P(\nu)$ indeed is a polynomial rather than a rational function, as $|a| \in \{1...n\}$. In addition

$$\log \det_G(\Delta_n) = 0$$

Proof. Following [Lot92] prop. 15 the local trace of the heat kernel of $\Delta_j$ is

$$\text{tr} e^{-t\Delta_j}(x,x) = C \left( \frac{2n}{j} \right) \int_{-\infty}^{\infty} e^{-t(\nu^2 + a^2)} \prod_{k=0}^{n} \frac{(\nu^2 + k^2)}{\nu^2 + a^2} d\nu$$

According to the above remark let

$$\prod_{k=0}^{n} \frac{(\nu^2 + k^2)}{\nu^2 + a^2} = \sum_{k=0}^{n} K^n_{k,j} \nu^{2k}$$
Evaluation of the above integral yields
\[ \text{tr}_C e^{-t\triangle_j}(x,x) = C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n e^{-\frac{1}{2}a^2 t^{k + \frac{1}{2}}} \Gamma \left( k + \frac{1}{2} \right) \]

Now we have to compute
\[ L_j := \frac{\log\det(\triangle_j)}{\text{Vol}(M)} \]
\[ = \frac{d}{ds} \bigg|_{s=0} \left[ \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \text{tr}_C e^{-t\triangle_j}(x,x) dt \right] + \int_1^\infty t^{-1} \text{tr}_C e^{-t\triangle_j}(x,x) dt \]

John Lott showed in [Lot92, Lemma 13, p.481] that
\[ L_n = 0 \]

For \( j \neq n \), that is \( a = n - j \neq 0 \)
\[ L_j = C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n \Gamma \left( k + \frac{1}{2} \right) \frac{d}{ds} \bigg|_{s=0} \left( \frac{1}{\Gamma(s)} \int_0^\infty e^{-\frac{1}{2}a^2 t^{s-k-\frac{3}{2}}} dt \right) \]

\( J \) exists for \( s \) sufficiently large. Its meromorphic extension leads to
\[ L_j = C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n \Gamma \left( k + \frac{1}{2} \right) \Gamma \left( -k - \frac{1}{2} \right) a^{2k+1} \]
\[ = C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n (-1)^{k+1} \frac{2\pi}{2k+1} a^{2k+1} \]

4. **Corollary.** For any closed hyperbolic manifold of dimension \( d = 2n+1 \) we have by Definition 3
\[ \frac{\text{Tor}(2)(M)}{\text{Vol}(M)} = 2 \sum_{j=0}^{n-1} (-1)^{j+1} C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n (-1)^{k+1} \frac{2\pi}{2k+1} (n-j)^{2k+1} \]

The numerical values were computed using this formula and Mathematica.

5. **Lemma.** Let \( M \) be a closed \((d = 2n+1)\)-dimensional manifold. Then
\[ (-1)^{j+1} \log\det_G(\triangle_j) = (-1)^n \log\det_G(\triangle_j) \]

with
\[ |\log\det_G(\triangle_j)| > 0 \text{ for } j \neq n \]

In particular
\[ (-1)^n \text{Tor}(2)(M) > 0 \]
Proof. Let \( j \neq n \). Then we have

\[
L_j = -2\pi C \left( \frac{2n}{j} \right) \sum_{k=0}^{n} K_{k,j}^n (-1)^{k} \frac{1}{2k+1} a^{2k+1}
\]

\[
= -2\pi C \left( \frac{2n}{j} \right) \int_0^a \sum_{k=0}^{n} K_{k,j}^n (ix)^{2k} dx
\]

Using the definition (*) of the coefficients \( K_{k,j}^n \) in Lemma 3 one gets

\[
L_j = -2\pi C \left( \frac{2n}{j} \right) \int_0^a \prod_{k=0}^{n} \frac{a^{2k+1}}{a^{2} - x^2} dx
\]

One has

\[
\int_0^a \prod_{k=0}^{n} \frac{(k^2 - x^2)}{a^2 - x^2} dx
\]

\[
= (-1)^{n+1} \int_0^a \frac{x}{(a+x)(a-x)} \prod_{k=-n}^{n} (x+k) dx
\]

\[
= (-1)^{n+1} \sum_{r=0}^{a-1} \int_0^1 f_r(t) dt
\]

where for \( t \in ]0,1[ \), \( r \in \{0, \ldots, a-1\} \) we define

\[
f_r(t) = \frac{(a+t+r)(a-t-r)}{(a-t-r)_{>0}} \prod_{k=-n+r}^{n+r} (t+k) \]

\[
< 0 \text{ for } k < 0
\]

\[
> 0 \text{ otherwise}
\]

\[
= (-1)^{n-r} |f_r(t)|
\]

For \( t \in ]0,1[ \) and \( 0 \leq r < r + 1 \leq a - 1 \) one computes

\[
\left| \frac{f_{r+1}(t)}{f_r(t)} \right| > 1
\]

Hence

\[
\int_0^1 |f_{r+1}(t)| dt \geq \int_0^1 |f_r(t)| dt
\]

Now the sum

\[
L_j = -2\pi C \left( \frac{2n}{j} \right) \sum_{r=0}^{a-1} (-1)^{r+1} \int_0^1 |f_r(t)| dt
\]

is an alternating sum and the absolute values of the summands are strictly increasing. So it is not 0 and the sign is that of the last summand. One concludes

\[
\log \det G(\triangle_j) = (-1)^{n-j-1} |\log \det G(\triangle_j)|
\]

This also finishes the proof of Theorem 4. \( \square \)
6. Proposition. The constants $\alpha_d$ of Theorem 4 strictly increase and

$$\alpha_{2n+1} \geq \frac{n}{2\pi} \alpha_{2n-1}$$

In particular

$$\alpha_{2n+1} \geq \frac{2}{3} \frac{n!}{(2\pi)^n}$$

Proof. An elementary computation shows

$$\left| \frac{f_{a-1}(t)}{f_{a-2}(t)} \right| \geq 2$$

Now one has

$$\int_0^a \frac{\prod_{k=0}^n(k^2 - x^2)}{a^2 - x^2} \, dx \geq \frac{1}{2} \int_{a-1}^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-j)^2 - x^2} \right| \geq \int_0^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-j)^2 - x^2} \right| \, dx$$

and

$$\alpha_{2n+1} \geq 2\pi \sum_{j=0}^{n-1} \frac{(4\pi)^{(n+\frac{1}{2})}}{\Gamma(n + \frac{1}{2})} \left( \begin{array}{c} 2n \\ j \end{array} \right) \frac{\prod_{k=0}^{n-j}(k^2 - x^2)}{(n-j)^2 - x^2} \int_{a-1}^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-j)^2 - x^2} \right| \, dx$$

$$\geq 2\pi \sum_{j=1}^{n-1} \frac{(4\pi)^{(n+\frac{1}{2})}}{\Gamma(n + \frac{1}{2})} 2n(2n-1) \left( \begin{array}{c} 2n-2 \\ j-1 \end{array} \right) \frac{\prod_{k=0}^{n-j}(k^2 - x^2)}{(n-j)^2 - x^2} \int_{a-1}^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-j)^2 - x^2} \right| \, dx$$

$$= 4\pi \sum_{j=1}^{n-1} \frac{2n}{4\pi} \frac{(4\pi)^{(n+\frac{1}{2})}}{\Gamma(n - 1 + \frac{1}{2})} \left( \begin{array}{c} 2n-2 \\ j-1 \end{array} \right) \frac{\prod_{k=0}^{n-j}(k^2 - x^2)}{(n-j)^2 - x^2} \int_{a-1}^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-j)^2 - x^2} \right| \, dx$$

$$\geq \frac{n}{2\pi} \sum_{l=0}^{n-2} \frac{(4\pi)^{(n+\frac{1}{2})}}{\Gamma(n - 1 + \frac{1}{2})} \left( \begin{array}{c} 2(n-1) \\ l \end{array} \right) \frac{\prod_{k=0}^{n-1-l}(k^2 - x^2)}{(n-1-l)^2 - x^2} \int_0^a \left| \frac{\prod_{k=0}^{n-1}(k^2 - x^2)}{(n-1-l)^2 - x^2} \right| \, dx$$

$$\geq \frac{n}{2\pi} \sum_{l=0}^{n-2} \frac{(4\pi)^{(n+\frac{1}{2})}}{\Gamma(n - 1 + \frac{1}{2})} \left( \begin{array}{c} 2(n-1) \\ l \end{array} \right) \int_0^a \left| \frac{\prod_{k=0}^{n-1-l}(k^2 - x^2)}{(n-1-l)^2 - x^2} \right| \, dx$$

$$= \frac{n}{2\pi} \alpha_{2n-1}$$

For $n \leq 7$ the growth follows from the table. \qed

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