Optimal Relay Functionality for SNR Maximization in Memoryless Relay Networks

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Abstract

We explore the SNR-optimal relay functionality in a memoryless relay network, i.e. a network where, during each channel use, the signal transmitted by a relay depends only on the last received symbol at that relay. We develop a generalized notion of SNR for the class of memoryless relay functions. The solution to the generalized SNR optimization problem leads to the novel concept of minimum mean square uncorrelated error estimation (MMSUEE). For the elemental case of a single relay, we show that MMSUEE is the SNR-optimal memoryless relay function regardless of the source and relay transmit power, and the modulation scheme. This scheme, that we call estimate and forward (EF), is also shown to be SNR-optimal with PSK modulation in a parallel relay network. We demonstrate that EF performs better than the best of amplify and forward (AF) and demodulate and forward (DF), in both parallel and serial relay networks. We also determine that AF is near-optimal at low transmit power in a parallel network, while DF is near-optimal at high transmit power in a serial network. For hybrid networks that contain both serial and parallel elements, and when robust performance is desired, the advantage of EF over the best of AF and DF is found to be significant. Error probabilities are provided to substantiate the performance gain obtained through SNR optimality. We also show that, for Gaussian inputs, AF, DF and EF become identical.

Index Terms

Estimate and forward, memoryless relay networks, relay function, MMSUE, parallel relay networks, serial relay networks, hybrid relay networks
I. INTRODUCTION

The traditional wireless communication problem is to design effective coding and decoding techniques to enable reliable communication at data rates approaching the capacity of a channel. The channel is defined by a set of \textit{given} assumptions regarding the physical signal propagation environment between the transmitter and the receiver. However, recent focus on cooperative communications presents a remarkable change of paradigm where in addition to the physical environment, \textit{the network is the channel} \cite{1}. In other words, with cooperative communications the effective channel between the original source and the final destination of a message depends not only on the given physical signal propagation conditions but also the signal processing at the cooperating nodes. The change of paradigm is quite significant. With cooperative communications, not only is there a need to optimally design the encoder and decoder at the source and destination, but also to design the channel itself by optimally choosing the functionality of the intermediate relay nodes. The choice of relay function is especially important as it directly affects the potential capacity benefits of cooperation which have been shown to be quite significant \cite{2}–\cite{6}.

A number of relay strategies have been studied in literature. These strategies include amplify-and-forward \cite{7} \cite{8}, where the relay sends a scaled version of its received signal to the destination, demodulate-and-forward \cite{8} in which the relay demodulates individual symbols and retransmits, decode-and-forward \cite{9} in which the relay decodes the entire message, re-encodes it and re-transmits it to the destination, and compress-and-forward \cite{10} \cite{4} where the relay sends a quantized version of its received signal. In \cite{11}, gains are determined for AF relays to minimize the MMSE of the source signal at the destination. It is shown that significant savings in power is achieved if there is no power constraint on the relays. Similarly in \cite{12} gains for AF relays in a multiuser parallel network are determined that realizes a joint minimization of the MMSE of all the source signals at the destination.

From a practical standpoint, the benefits of cooperation are offset by the cost of cooperation in terms of the required processing complexity and transmit power at the relay nodes. The complexity of the signal processing at the relay could range from highly sophisticated decode-and-forward or compress-and-forward techniques \cite{13} that require joint processing of a long sequence of received symbols to memoryless schemes such as amplify-and-forward or demodulate-and-forward that process only one symbol at a time. Clearly, the most desirable schemes are those that approach the limits of cooperative capacity with minimal processing complexity at the relays. Memoryless relay functions are highly relevant for this objective. In addition to their simplicity, memoryless relays are quite powerful in their capacity benefits. For example, the memoryless scheme of amplify-and-forward is known to be the
capacity-optimal relay scheme for many interesting cases [1], [14]–[17]. The effect of finite block-length processing at the relay on the capacity of serial networks is analyzed in [18], [19]. In [20], the memoryless MMSE estimate and forward scheme has been shown to be capacity optimal for a single relay system. For a single relay and with BPSK modulation, the BER-optimal memoryless scheme is found by Faycal and Medard [21]. The BER-optimal relay function turns out to be a Lambert W function normalized by the signal and noise power.

In this paper we explore the SNR-optimal signal processing function for memoryless networks with possibly multiple relays. While SNR optimality does not always guarantee capacity or BER optimality, it is a practically useful performance metric. SNR-optimization is especially interesting for its greater tractability that allows analytical results where capacity and BER optimizations may be intractable, e.g. with multiple relays.

A. Notations

Throughout the paper, $\mathbb{E}[.]$ denotes the standard expectation operator, $^*$ represents the conjugation operation, $|.|$ and $\mathrm{Re}(.)$ denotes the absolute and real part of the argument respectively.

II. SHAPING THE RELAY CHANNEL: AMPLIFICATION, DEMODULATION AND ESTIMATION

In this section, we discuss the relay functions of common memoryless forwarding strategies and provide new perspectives that lead us to a novel and superior memoryless forwarding technique.

A. Soft and Hard Information: Amplify and Demodulate

Within the class of memoryless relay strategies, amplification and demodulation are the most basic forwarding techniques [8]. An AF relay simply forwards the received signal $r$ after scaling it down to satisfy its power constraint. The relay function for AF can be written as

$$f_{AF}(r) = \sqrt{\frac{P_R}{P+1}} r.$$  \hspace{1cm} (1)

Evidently with AF, the relay tries to provide soft information to the destination. A disadvantage with this technique is that significant power is expended at the relay when $|r|$ is high. In DF schemes, demodulation of the received symbol at the relay is followed by modulation with its own power constraint $P_R$. For BPSK modulation, the relay function for DF can be expressed as

$$f_{DF}(r) = \sqrt{P_R} \mathrm{sign}(r),$$  \hspace{1cm} (2)
where \( \text{sign}(r) \) outputs the sign of \( r \). Due to demodulation, the relay transmitted signal carries no information about the degree of uncertainty in the relay’s choice of the optimal demodulated symbol. Demodulation at the relays can lead to severe performance degradation in some scenarios. For example, in a parallel relay network, reliability information can be utilized to achieve better performance over DF.

From the relay functions of AF and DF, one can argue that an optimal relay function should provide soft information when there is an uncertainty in the received symbol, and at the same time should not expend a lot of power when the cost of power out-weighs the value of soft information.

B. Estimate and Forward: A Novel Memoryless Forwarding Strategy

The forwarding schemes can also be related to the fundamental signal processing operations: detection and estimation. In DF, the relay demodulates the received symbol employing MAP detection rule, which is the optimal detection technique. So a DF function can be viewed as a MAP detector followed by a modulator. In a similar vein, AF can be viewed as a linear MMSE\(^1\) estimation scheme followed by normalization to satisfy the relay power constraint.

\[
f_{AF}(r) = \beta' \hat{X}_{\text{linear}}(r)
\]

where the linear estimate \( \hat{X}_{\text{linear}}(r) \) obtained at the relay is given by

\[
\hat{X}_{\text{linear}}(r) = \frac{P}{P + 1} r,
\]

and

\[
\beta' = \sqrt{\frac{P_R (P + 1)}{P^2}}.
\]

Viewing AF as linear MMSE leads naturally to the forwarding scheme of EF where the unconstrained MMSE estimate is forwarded. The unconstrained MMSE estimator which minimizes the distortion is given by

\[
\hat{X}(r) = E(x|r).
\]

\(^1\)Because of the normalization associated with the relay power constraint all linear estimates are equivalent
1) Relay Function for EF with BPSK modulation: When the source employs BPSK modulation, the MMSE estimate at the relay is given by

\[ \hat{X}(r) = \sqrt{P} \tanh(\sqrt{P}r), \]

where \( \tanh(z) \) returns the hyperbolic tangent of \( z \). The relay function is therefore,

\[ f_{EF}(r) = \sqrt{\frac{P_R}{\mathbb{E}[\tanh^2(\sqrt{P}r)]}} \tanh(\sqrt{P}r) \]

Note that all the memoryless schemes operate at sampled output of the matched filter. In this regard all the schemes have similar processing complexity. It is worth noting that while EF and AF require amplitude digitization, DF does not. Figure II shows the relay functions for AF, EF and DF for \( P = 1 \). It can be seen that the relay function \( f_{EF} \) is linear for small values of \(|r|\). Its slope reduces gradually and ultimately becomes flat similar to \( f_{DF} \). The function \( f_{EF} = \sqrt{P} \tanh(\sqrt{P}r) \) is intuitively appealing for the following characteristics.

- Soft information in region of uncertainty.
- Limited power in region of high power cost.

The insights obtained in this section will be useful in understanding optimum relay functionalities in a multiple relay network. In the next section, we determine the optimal memoryless strategy in a single relay network.
III. SINGLE RELAY CHANNEL

A. Problem Statement

Consider an elemental relay channel model as shown in the figure below, in which a single relay R assists the communication between the source S and the destination D. Both S-R and R-D links are assumed to be non-fading.

![Elementary Relay Channel](image)

There is no direct link between the source and the destination, which may be due to the half duplex constraint of the nodes, where in the first slot D serves a different set of nodes. The transmit power at the source and the relay is $P$ and $P_R$ respectively. At both the relay and the destination, the received symbol is corrupted by additive white Gaussian noise of unit power. Relay R observes $r$, a noisy version of the transmitted symbol $x$. Based on the observation $r$, the relay transmits a symbol $f(r)$ which is received at the destination along with its noise $n_2$.

\[
\begin{align*}
    r &= x + n_1 \\
    y &= f(r) + n_2
\end{align*}
\]

The relay function $f$ satisfies the average power constraint, i.e. $\mathcal{E}_r \left[ f(r)^2 \right] = P_R$. Without loss of generality, channel gains for the source-relay and the relay-destination link can be incorporated into the model by modifying $P$ and $P_R$ appropriately. We seek to determine the memoryless relay function $f(\cdot)$ that maximizes SNR at the destination.

B. What is the definition of SNR?

Given an observation $y$, that contains a desired signal $x$ as well as some distortion (noise), SNR is traditionally defined as the power of the signal $P_x$ divided by the power in the noise component $P_n$. For observations of the form $y = x + n$ where the observed power $P_y = P_x + P_n$ (i.e. signal and noise are uncorrelated) it is easy to separately identify the contribution of the signal power and the noise power to the observed power. However, what is the definition of SNR if the observation $y$ is not already explicitly presented in the standard form $y = x + n$ with
uncorrelated signal and noise components? In general, the observation $y$ may have an arbitrary and possibly non-linear dependence on the desired signal $x$. For example, consider the signal at the destination: $y = f(x + n_1) + n_2$ with an arbitrary function $f()$ describing the memoryless relay functionality. In order to define SNR one needs to separately identify the power contributions of the signal and noise components to the observed signal $y$. If we can identify $P_y = P_x + P_n$ then the definition of SNR readily follows as $\frac{P_x}{P_n}$. In other words, the definition of SNR follows from a representation of the observation $y$ in the form $y = x + n$, with uncorrelated signal and noise components. To this end, we view the signal $y$ as a scaled version of the sum of the signal $x$ and an error component $e_u$ uncorrelated with $x$.

$$y = f(x + n) = \frac{E[x^*y]}{E[|x|^2]}(x + e_u).$$ \hspace{1cm} (5)

Notice that any signal $y$ can be expressed as above regardless of whether $y$ is a linear or non-linear function of $x + n$. Rearranging (5),

$$e_u = \frac{E[|x|^2]}{E[x^*y]}y - x.$$ \hspace{1cm} (6)

It is easy to verify that $e_u$ is uncorrelated to $x$.

$$E[x^*e_u] = E[x^*(\frac{E[|x|^2]}{E[x^*y]}y - x)]$$ \hspace{1cm} (7)

$$= \frac{E[|x|^2]}{E[x^*y]}E[x^*y] - E[x^*x] = 0$$ \hspace{1cm} (8)

To calculate the SNR of the received signal $y$, we need to identify the error term in the received signal $y$, that is uncorrelated to the signal $x$. The scaling factor $\frac{E[x^*y]}{E[|x|^2]}$ in (5) is common to both the signal and error terms. Therefore, the generalized SNR is defined as follows:

$$\text{GSNR} = \frac{E[|x|^2]}{E[|e_u|^2]} = \frac{E[|x|^2]}{E[|\alpha y - x|^2]},$$ \hspace{1cm} (9)

where $\alpha = \frac{E[|x|^2]}{E[x^*y]}$. The advantage of the generalized definition lies is its applicability to both linear and nonlinear relay functions. Note that the conventional definition of SNR for point to point links is a special case of the generalized SNR. For example, consider a received signal $y = hx + n$. The conventional SNR is $|h|^2P$. To obtain the generalized SNR, we need to express $y$ as a sum of the signal $x$ and uncorrelated error $e_u$ in the following
form.

\[ y = \frac{\mathcal{E}[x^*y]}{\mathcal{E}[\|x\|^2]}(x + e_u), \]

where \( \frac{\mathcal{E}[x^*y]}{\mathcal{E}[\|x\|^2]} = h \), in this case. Therefore \( e_u = \frac{n}{h} \). The generalized SNR from (9) is,

\[ \text{GSNR} = \frac{P}{|h|^2} = |h|^2 P, \]

which is also the conventional definition of SNR.

The GSNR concept can be viewed as a decomposition of an observation into a component along the desired signal space and its orthogonal signal (uncorrelated noise) space. The orthogonal projections are evident in the second moment constraint \( P_y = P_x + P_n \) (Pythagoras Theorem). GSNR is therefore as natural and meaningful a metric as the orthogonal projections themselves. While GSNR optimization does not guarantee capacity or BER optimality it is interesting to note that all three metrics (BER, capacity, GSNR) lead to very similar optimal relay functions for BPSK. The BER optimal Lambert-W function is very similar to the GSNR optimal tan-hyperbolic function (EF). Moreover, in a separate work we have shown that, numerically, the GSNR optimal EF function is also capacity optimal for BPSK [20]. To summarize, GSNR optimality is related to capacity and BER optimality and offers a tractable performance optimization metric.

C. Optimal Relay Function

We first derive the optimal estimation method that maximizes GSNR. Based on this result, we determine the optimal relay function.

**Theorem 1:** Given an observation \( r \) that contains both the signal \( x \) and noise \( n \), the MMSUE (SNR maximizing estimate) of \( x \) is

\[ \hat{X}(r) = \frac{\mathcal{E}[\|x\|^2]}{\mathcal{E}[x^*E(x|r)\]E[x|r]}, \]

regardless of the input and the noise distributions.

**Proof:** Without loss of generality, any estimator \( \hat{X}(r) \) can be expressed as

\[ \hat{X}(r) = x + e_u, \]
where $e_u$ is uncorrelated with $x$. It is clear from (9) that minimizing the mean square uncorrelated estimation error (MMSUEE), $\mathcal{E}[|e_u|^2]$ amounts to maximizing SNR. The optimization problem is therefore to minimize $\mathcal{E}[|e_u|^2]$ with respect to $\hat{X}(r)$ subject to the constraint that $e_u$ is uncorrelated with $x$. The constraint is equivalent to $\mathcal{E}[xe_u^*] = 0$ as $\mathcal{E}[x] = 0$ for all signal constellations. Employing Lagrange multipliers, we write the constrained minimization as the minimization of

$$E = \mathcal{E}[|e_u|^2] - \lambda \mathcal{E}[x^* e_u] - \lambda^* \mathcal{E}[xe_u^*]$$

$$= \mathcal{E}[|\hat{X}(r) - x|^2] - \lambda \mathcal{E}[x^*(\hat{X}(r) - x)] - \lambda^* \mathcal{E}[x(\hat{X}^*(r) - x^*)]$$

$$= \mathcal{E}[|\hat{X}(r)|^2] + (-\lambda + 1) \mathcal{E}[x^* \hat{X}(r)] + (-\lambda^* + 1) \mathcal{E}[x \hat{X}^*(r)] - P(-\lambda - \lambda^* - 1)$$

$$= \mathcal{E}_r \left[ |\hat{X}(r)|^2 - (\lambda - 1) \hat{X}(r) \mathcal{E}[x^* |r] - (\lambda^* - 1) \hat{X}^*(r) \mathcal{E}[x |r] \right] + P(\lambda + \lambda^* + 1)$$

$$= \mathcal{E}_r \left[ |\hat{X}(r) - (\lambda - 1) \mathcal{E}[x |r]|^2 - |(\lambda - 1) \mathcal{E}[x |r]|^2 \right] + P(2 \Re(\lambda) + 1)$$

From the above equation, it is clear that $E$ is minimized when

$$\hat{X}(r) = (\lambda - 1) \mathcal{E}[x |r],$$

and the minimum mean squared uncorrelated estimation error is

$$E^* = P(2 \Re(\lambda) + 1) - |\lambda - 1|^2 \mathcal{E}_r \left[ |\mathcal{E}[x |r]|^2 \right] \quad (10)$$

From the constraint that

$$\mathcal{E}[x^* e_u] = 0 \Rightarrow \mathcal{E}[x^* \hat{X}(r)] = P,$$

we have

$$\lambda - 1 = \frac{P}{\mathcal{E}[x^* \mathcal{E}(x |r)]}.$$

Therefore $\hat{X}(r) = \frac{P}{\mathcal{E}[x^* \mathcal{E}(x |r)]} \mathcal{E}[x |r]$ is the SNR maximizing estimate. It should be noted that the above result is completely general and is valid for all input and noise distributions.

This result implies that any scaled version of MMSE estimator is GSNR optimal. Thus regardless of the power constraint at the relay, EF maximizes GSNR at the output of the relay.

As the constraint $\mathcal{E}[xe_u^*]$ is a complex quantity, the Lagrange multipliers are $\lambda$ and $\lambda^*$ corresponding to $\mathcal{E}[xe_u^*]$ and $\mathcal{E}[x^* e_u]$ respectively.
Theorem 2: In a single relay network, maximizing GSNR at the output of the relay amounts to maximizing GSNR at the destination.

Proof: Consider any estimate \( \hat{X}(r) = x + e_u \), with \( E[|e_u|^2] = E \), the uncorrelated estimation error power.

Let \( f(r) = \alpha \hat{X}(r) \)

\[ \text{where } \alpha^2 = \frac{P_R}{P + E} \]

satisfies the relay power constraint. The received symbol at the destination is

\[ y = \alpha \hat{X}(r) + n = \alpha(x + e_u) + n \]

The GSNR at the destination is given by

\[ \text{GSNR}_D = \frac{\alpha^2 P}{\alpha^2 E + 1} = \frac{P}{E + \frac{P + E}{\alpha^2}}. \]  

Clearly minimizing \( E \) amounts to maximizing GSNR at the destination.

As scaling does not alter the GSNR of the estimate, forwarding MMSUE and MMSE results in the same relay function. The fundamental relationship between these estimation methods is discussed in Appendix A. From the results of theorem 1 and theorem 2 we have the following theorem for the optimal relay function.

Theorem 3: For a network with a single relay that has a power constraint \( P_R \), the relay function that maximizes GSNR at the destination is

\[ f(r) = \sqrt{\frac{P_R}{\mathbb{E}_r[|\mathbb{E}(x|r)|^2]}} \mathbb{E}[x|r], \]

regardless of the input and noise distributions.

Thus the new memoryless forwarding strategy, estimate and forward is GSNR optimal in a single relay network. In the next section, we compare the performance.

IV. COMPARATIVE ANALYSIS

From the mean square uncorrelated estimation error at the relay, the SNR at the destination for any forwarding scheme can be obtained directly from (12). Therefore calculation of uncorrelated error power of DF and AF allows a direct comparison of these schemes with the SNR optimal EF. To determine the estimate whose error is uncorrelated to the signal \( x \) from the actual relay function, we only need to obtain the scaling factor that allows the relay function to be expressed as in (5). The relay function for DF depends on the modulation scheme as discussed earlier. In this
section, we compare the schemes for BPSK modulation and illustrate the concept of generalized SNR.

A. Demodulate and Forward

We express the relay function of a demodulating relay as

$$f_{DF}(x + n) = \sqrt{P_R} \text{sign}(x + n) = \sqrt{\frac{P_R}{P}} (x + d)$$

where $d$ is the Euclidean distance between the input symbol $x$ and the demodulated symbol. The distribution of $d$ conditioned on $x$ is given by

$$d = \begin{cases} 
0 & 1 - \epsilon \\
-2x & \epsilon 
\end{cases}$$

(13)

where $\epsilon = Q\left(\sqrt{P}\right)$, the probability of symbol error. As seen from the error distribution, the demodulation error $d$ is correlated with $x$. The correlation between the input and the error is given by

$$\mathcal{E}(xd) = -2P\epsilon.$$  

(14)

The uncorrelated error can be calculated from (5) according to which

$$e_u = \frac{P}{\mathcal{E}(xf_{DF}(x + n))} f_{DF}(x + n) - x = \frac{P}{P - 2P\epsilon} (x + d)$$

(15)

From (14), the power of the uncorrelated error in (15) can be calculated and is given by

$$\text{MSUEE}_{DF} = \frac{4P\epsilon(1 - \epsilon)}{(1 - 2\epsilon)^2}$$

(16)

To characterize the mean squared uncorrelated error at the output of the DF relay, we first consider $\epsilon$, the probability of decision error at the relay.

$$\epsilon = Q\left(\sqrt{P}\right) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{P}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{\sqrt{P}} \exp\left(-\frac{x^2}{2}\right) dx$$

For small values of $P$,

$$\epsilon = \frac{1}{2} - \sqrt{\frac{P}{2\pi}}$$
Therefore at low source transmit power, the mean squared uncorrelated error can be expressed as a function of $P$.

$$MSUE_{DF}(P) = 4P \left( \frac{1}{2} - \sqrt{\frac{P}{2\pi}} \right) \left( \frac{1}{2} + \sqrt{\frac{P}{2\pi}} \right) \frac{2}{\left( 1 - \left( \frac{2P}{\pi} \right)^{1/2} \right)^2} = 2\pi \left[ \frac{1}{4} - \frac{P}{2\pi} \right]$$  \hspace{1cm} (17)

As $P \to 0$, the uncorrelated error power shoots up to $\frac{\pi}{2}$. It should be noted that the noise variance at the relay is 1. This suggests that DF is not preferable at low $P$.

**B. Amplify and Forward**

As the relay function of an AF relay is a scaled version of the received signal $r$, it is simple to determine the mean squared uncorrelated error. From (5), we have

$$e_u = \frac{P}{\mathbb{E}[x^*f_{AF}(x+n)]} f_{AF}(x+n) - x = n$$

The uncorrelated error power is therefore the same as the noise variance, $MSUE_{AF} = 1$, interestingly independent of the source transmit power.

Fig. 2 plots the uncorrelated estimation error as a function of transmit power for all the three schemes. Several interesting observations can be made. It can be seen that AF is close to optimum (EF) at low $P$ while DF is near optimal at high $P$. In the intermediate range, both AF and DF are far from optimal. It is well known that AF suffers from noise amplification at low SNR [22], which is in contrast to the results here. When we view the relay operation as an estimation, it is only natural that the estimation error is high at low $P$, which results in noise.
amplification. In fact AF is very close to optimum among all memoryless function at low $P$. Rather it is DF that suffers the most from noise/error$^3$ amplification. However AF is inefficient at high $P$ as $\text{MSU}E_{AF} = 1$ does not decrease with $P$, while uncorrelated error in DF and EF vanishes at high $P$. The mean squared uncorrelated error power of the schemes for extreme values of $P$ is listed in Table I.

| Relay Function | $P \rightarrow 0$ | $P \rightarrow \infty$ |
|---------------|------------------|-----------------|
| Amplify       | 1                | 1               |
| Demodulate    | $\frac{\pi}{4}$ | 0               |
| Estimate      | 1                | 0               |

**Table I**  
**Uncorrelated error power at output of relay for BPSK modulation.**

C. **Higher Order Constellations**

We know from theorem 3 that EF is GSNR optimal for all modulation schemes. For fixed input power $P$, increasing the number of constellation points $M$ will result in an increased mean squared uncorrelated power for EF. This is rather intuitive from the fact that increasing the number of constellation points for fixed power increases the estimation error. Fig. 3 shows the relay functions for 4-PAM constellation set. Interestingly, the relay functions of the schemes become more and more similar with increase in constellation points.

For Gaussian inputs, the unconstrained MMSE estimate and the linear MMSE estimate are equivalent.

$$\mathcal{E}[x|r] = \frac{P}{P + 1}$$

Thus AF and EF strategies are the same for a Gaussian source. In this context, it can also be shown that DF and AF are equivalent for Gaussian inputs. The notion of demodulation of symbols from a Gaussian source is explained through the following. A Gaussian distribution is quantized into a number of states with the probability of the $i^{th}$ state given by,

$$\Pr(x_i) = \frac{1}{\sqrt{2\pi P}} \int_{(i-1)\Delta x}^{i\Delta x} \exp\left(-\frac{x^2}{2P}\right) dx.$$  

Suppose the source transmits symbols $x_i$ according to the probability distribution above, then the MAP detection

$^3$the term ‘error’ is more appropriate as noise process is usually independent of the input.
Fig. 3. Relay Functions for 4-PAM modulation

rule at the relay is given by

\[ \hat{X}(r) = \arg \max_{x_i} \Pr(x|r) \]

In the limit \( \Delta x \to 0 \), \( x \) and \( r \) become jointly Gaussian. It is well known that the conditional mean \( \mathcal{E}(x|r) \) maximizes the joint probability. Therefore \( \mathcal{E}(x|r) \) which is also the MMSE estimate is the output of the ML detector. Thus for Gaussian inputs \( AF, EF \) and \( DF \) are identical.

V. PARALLEL RELAY NETWORK

A Gaussian parallel relay channel [14] is shown in Fig. 4. It consists of a single source destination pair with \( L \) relays that assist in the communication. All the links are assumed to be non-fading with unequal channel gains and

![Graphical representation of a parallel relay network](image)

Fig. 4. Gaussian Parallel Relay Channel

information is transferred in two time slots. The relays observe \( \{r_i\}_{i=1}^L \), the noisy version of the transmitted signal \( x \).

\[ r_i = g_i x + n_i \quad (18) \]
where $g_i$ is the gain of the link between the source and the $i$th relay. $n_i$ denotes an additive Gaussian noise with $\sigma^2=1$. Since the relays are assumed to be memoryless, each relay $R_i$ transmits a signal that is a function of its observation $r_i$. We assume that the relay function in a parallel relay network is the same for all the relays, although the channel gains of the relays may be different. Strictly speaking, an optimal power allocation based on the channel gains is necessary. However, it is beyond the scope of the paper. For ease of notation, we denote the transmit power at the source as $P$ and the relay transmit power as $\{P_i\}_{i=1}^L$. Without loss of generality, the channel gain for the relay destination links can be introduced through the relay transmit power. The destination receives the sum of all the relay observations along with its own noise.

$$y = \sum_{i=1}^L f(r_i) + n$$

By viewing relay operation as an estimation we have,

$$f(r_i) = f(g_i x + n_i) = \alpha_i (x + e_i)$$

where $e_i$ is the uncorrelated estimation error at the $i$th relay, $\alpha_i = \sqrt{\frac{P}{P + E_i}}$ and $E_i = \mathcal{E}[|e_i|^2]$, the mean square uncorrelated error power associated with the relay function. The received signal at the destination can be expressed as

$$y = \sum_{i=1}^L \alpha_i (x + e_i) + n \quad (19)$$

For any forwarding scheme, the SNR at the destination is given by

$$\text{GSNR} = \frac{(\sum_{i=1}^L \alpha_i)^2 P}{\sum_{i=1}^L \alpha_i^2 E_i + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \alpha_i \alpha_j C_{ij} + 1} \quad (20)$$

where $C_{ij} = \mathcal{E}(e_i^* e_j)$ is the correlation between errors $e_i$ and $e_j$ at relays $i$ and $j$, $i \neq j$. For the zero correlation case ($C_{ij} = 0, \ \forall i,j$), it is clear from (20) that the SNR is maximized when the uncorrelated estimation error at the relays ($E_i$) are minimized. This can also be inferred from the generalized definition of SNR in (5). Error in the received symbol at the destination in (19) is a linear combination of errors at the output of relays and the destination noise. When the errors are uncorrelated, minimizing the error at the output of each of the relays clearly amounts to maximizing SNR at the destination.

For AF, the correlation $C$ is always zero as the error terms represent independent AWGN noise. For both DF
and EF, the correlation depends on the modulation scheme and is not always zero. Although each $E_i$ is minimized by EF, due to the possibility of error correlation, maximum SNR is not always guaranteed. However for most constellation sets, the error correlation can be shown to be either very close or exactly equal to zero. Theorem 5 in Appendix B characterizes the rotational property of estimate and forward for MPSK constellation sets and will be useful to prove zero correlation property of EF. It proves that, due to the symmetry in the constellation, MMSE estimate of a signal rotated by an angle that belongs to a constellation point is the same as the rotated version of the MMSE estimate of the signal.

$$\hat{X}(re^{j\theta_m}) = e^{j\theta_m}\hat{X}(r)$$

where $\theta_m = \frac{2\pi m}{M}$, $m = 0, 1, \ldots, M - 1$, the signal phases of MPSK.

**Theorem 4:** Error at the relays that estimate and forward are uncorrelated with each other if MPSK modulation is employed at the source.

**Proof:** In Appendix C. Thus $\mathbb{E}[e_1e_2^*]=0$ for all MPSK constellation set inputs. This directly suggests that SNR achieved at the destination is always the highest with estimate and forward.

## A. Effect of Error Correlation on EF

The correlation between errors at the output of EF relays, in general, is not zero for all constellations. However it is negligible for many constellation sets like M-QAM and it does not result in any tangible SNR loss. In fact, the correlation can be expected to decrease for large QAM constellations where the ‘edge effects’ become insignificant. However due to the combination of $L(L - 1)$ terms for the correlation expression in (20), we can expect the performance of the system to degrade for very large values of $L$. In a symmetrical relay network where the channel gain of all the links are equal, the SNR for any relay function, as a function of correlation between errors is obtained from (20) as

$$\text{GSNR} = \frac{L^2P}{LE + L(L - 1)C + 1 + \frac{E}{P}}. \quad (21)$$

Suppose the source employs a modulation scheme that results in nonzero correlation between errors with EF in a parallel network with unit channel gains, $\text{GSNR}_{AF} > \text{GSNR}_{EF}$ only if

$$C > \frac{1 - E}{L(L - 1)} \left( L + \frac{1}{P} \right) \quad (22)$$

As the scaling associated with the correlation is $L(L - 1)$, its effect is prominent for large $L$. For a given error
correlation $C$, $\text{GSNR}_{AF} > \text{GSNR}_{EF}$ if

$$L \gtrapprox \frac{1 - E}{C} + 1$$

(23)

The above relation along with Fig. 5 suggests that, even if the correlation with EF is nonzero, the number of relays has to be very large for AF to outperform EF at high SNR, with modulation schemes like M-QAM.

### B. Error Correlation in DF

Similar to EF, the error correlation in DF depends on the modulation scheme at the source. For BPSK modulation, the error term in uncorrelated estimate of DF from (15) is

$$e_i = \frac{d_i + 2\epsilon_i x}{1 - 2\epsilon_i},$$

(24)

where $\epsilon_i$ depends on the transmit power and the source-relay channel. From the distribution of $d_i$ in (13), we can calculate the correlation between errors $e_i$ and $e_j$ at relay $i$ and $j$.

$$C = \mathcal{E}(e_i e_j) = \frac{\mathcal{E}[d_i d_j] + 2\epsilon_i \mathcal{E}[xd_i] + 2\epsilon_j \mathcal{E}[xd_j] + 4\epsilon_i \epsilon_j \mathcal{E}[|x|^2]}{(1 - 2\epsilon_i)(1 - 2\epsilon_j)}$$

$$= \frac{4\epsilon_i \epsilon_j P - 8\epsilon_i \epsilon_j P + 4\epsilon_i \epsilon_j P}{(1 - 2\epsilon_i)(1 - 2\epsilon_j)} = 0$$

(25)

For BPSK modulation and with unit channel gain for all the links, the effective SNR at the destination is

$$\text{GSNR}_{DF} = \frac{PL^2(1 - 2\epsilon)^2}{4PL\epsilon(1 - \epsilon) + 1}$$

(26)

For large M-QAM constellation ignoring edge effects, the demodulation error can be assumed to be independent
of the transmitted symbol, with the distribution given by

\[ d = \begin{cases} 
0 & 1 - \varepsilon \\
d_{\min} & \varepsilon \\
-d_{\min} & \varepsilon \\
jd_{\min} & \varepsilon \\
-jd_{\min} & \varepsilon 
\end{cases} \]  \tag{27}

where, from [23] we have

\[ d_{\min} = \sqrt{\frac{6P}{M - 1}} \]
\[ \varepsilon \leq 4 \left( 1 - \frac{1}{\sqrt{M}} \right) Q \left( \sqrt{\frac{3P}{M - 1}} \right) \]  \tag{28}

Here we assume that decisions error occur only among the nearest neighbors. Although this is an optimistic assumption, it closely predicts the performance of the system at medium and high SNR where the assumption is justified. The effective SNR at the destination is,

\[ \text{GSNR}_{DF} \approx \frac{P L^2}{L d_{\min}^2 \varepsilon + 1}. \]  \tag{29}

C. Asymptotic GSNR Comparison

While EF is superior to AF and DF at all SNR regardless of the number of relays, it will be interesting to characterize the asymptotic gain of EF as a function of L and P. For ease of analysis, we restrict the channel gains to be equal. From the SNR expressions of EF and AF, we have the ratio,

\[ \frac{\text{GSNR}_{EF}}{\text{GSNR}_{AF}} = \frac{LP + P + 1}{LPE(P) + P + E(P)}, \]

where \( E(P) \) is the mean squared uncorrelated error power of EF that is a function of \( P \). Note that the above expression does not include the correlation term. Therefore, it is valid only when there is zero correlation between the error terms. For any input distribution \( p_X(x) \), i.e. for all modulation schemes,

\[ E(P) \leq 1, \quad \forall P. \]
1) **Fixed $P$**: With large number of relays,

\[
L \to \infty, \quad \frac{\text{GSNR}_{EF}(P)}{\text{GSNR}_{AF}(P)} = \frac{\text{MSUEE}_{AF}}{\text{MSUEE}_{EF}} = \frac{1}{E(P)}
\]

\[
\frac{\text{GSNR}_{EF}(P)}{\text{GSNR}_{DF}(P)} = \frac{\text{MSUEE}_{DF}}{\text{MSUEE}_{EF}} = \frac{4P \epsilon(1 - \epsilon)}{(1 - 2\epsilon)^2 E(P)}
\]

Notice that MSUEE of the schemes determine the gain. We know from Section. [IV](#) that $E(P)$ decreases with $P$ and ultimately becomes zero as $P \to \infty$. This implies that in a large relay network, maximum gain over AF is obtained for high source transmit power $P$. Similarly maximum gain over DF is obtained at low $P$. This is due to the fact that DF is inefficient at low $P$ as indicated by its mean squared uncorrelated error power.

2) **Fixed $L$**: For a fixed number of relays, the GSNR gain of EF over AF at high $P$ is approximately $L + 1$. Similarly the GSNR gain of EF over DF at low $P$ is very high as indicated in the following expressions.

\[
P \to \infty, \quad \frac{\text{GSNR}_{EF}(L)}{\text{GSNR}_{AF}(L)} = L + 1 \quad \frac{\text{GSNR}_{EF}(L)}{\text{GSNR}_{DF}(L)} = 1
\]

\[
P \to 0, \quad \frac{\text{GSNR}_{EF}(L)}{\text{GSNR}_{AF}(L)} = 1 \quad \frac{\text{GSNR}_{EF}(L)}{\text{GSNR}_{DF}(L)} = \frac{\pi}{2}
\]

Above expressions clearly demonstrate the inefficiency of DF and AF at low and high SNR respectively.

**D. Numerical Results**

Fig. 6 provides the GSNR performance of the schemes in a parallel relay network with equal channel gains. The corresponding error probabilities are provided in Fig. 7. The error probabilities closely follow the trend exhibited in GSNR. It can be seen that EF achieves substantial error probability gains over AF and DF for all values of $P$. As seen in GSNR plot, AF is superior to DF at low $P$ while DF performs better than AF at high $P$. This is also indicated by (30) and (31). For our system model, the AF relaying scheme is equivalent to the one proposed in [12]. It will be interesting to compare the performance of the schemes with optimal power allocation similar to [12].

**VI. SERIAL RELAY NETWORK**

A serial relay network with Gaussian noise at all receivers in shown in Fig. 8. All the relays are memoryless and employ a relay function to transmit a symbol based on its received symbol. It should be noted that the relay functions, in general, need not be the same for all the relays unlike in a parallel network. This is due to the fact that
the noise distribution gets altered at every hop depending on the relay function of the preceding relay for multiple relay networks. We assume unit channel gains for the links and equal transmit power at all the nodes for simplicity of exposition.

A. Amplify and Forward

With AF relays in series, the received symbol at the destination can be expressed as

\[ y_{L+1} = \beta^L x + \sum_{i=1}^{L} \beta^i n_i + n. \]  

(32)

\[ \text{GSNR}_{AF} = \frac{\beta^{2L} P}{1 + \sum_{i=1}^{L} \beta^{2i}} \]  

(33)

B. Demodulate and Forward

As in Section V-B we express the received signal at the destination as

\[ y_{L+1} = x + \sum_{i=1}^{L} d_i + n \]  

(34)
From (13) and (27), the effective SNR at the destination with BPSK modulation and for a large QAM modulation is obtained.

\[
\text{GSNR}_{\text{DF}}^{\text{BPSK}} = \frac{P(1 - 2\epsilon)^2}{4PL\epsilon(1 - \epsilon) + 1} \quad \text{GSNR}_{\text{DF}}^{\text{QAM}} \approx \frac{P}{Ld_{\text{min}}^2\epsilon + 1}
\]

C. Estimate and Forward

With estimate and forward at all the relays, the corresponding relay functions varies with each relay as the noise distribution gets altered at each link due to nonlinear operations performed at the preceding relay. The relay function for the \(i^{th}\) is given by

\[f_i(r_i) = \frac{\alpha_i \text{E}[x|r_i = f_{i-1}(r_{i-1}) + n_i]}{1 - \epsilon^{2i}}\]

**Proposition 1**: In a serial relay network, the last relay should perform estimate and forward for maximizing SNR at the destination, regardless of the relay function in the preceding relays.

**Proof**: Regardless of the relay functions at the preceding relays, the received signal at the last relay can be expressed in the same form as (5). From theorem 1 which is valid for all input and noise distributions, it is straightforward that EF at the last relay maximizes the SNR at the destination.

D. Performance Comparison

Fig. 9 compares the destination SNR of the schemes for two serial relays. Here the relay functions for DF and AF remain the same for both the relays. For EF, \(f_1(r_1) = \alpha_1 \tanh(\sqrt{P}r_1)\) and \(f_2(r_2) = \alpha_2 \text{E}[x|r_2 = \alpha_1 \tanh(\sqrt{P}r_1) + n_2]\). As expected, EF is the best performing scheme and DF closely follows it.

It can be easily noticed that in a serial network, the effective SNR decreases with each stage. AF, being power inefficient at high SNR, suffers the most due to multi-hop communication.

\[\text{GSNR}_{\text{AF}} = \frac{P}{1 + \sum_{k=1}^{L} \frac{1}{\beta^k}} < \frac{P}{L + 1}\]  \hspace{1cm} (35)

For large M-QAM modulation, effective SNR at the destination with DF scheme can be approximated as

\[\text{GSNR}_{\text{DF}} = \frac{P}{Ld_{\text{min}}^2\epsilon + 1}\]  \hspace{1cm} (36)

Clearly when \(d_{\text{min}}^2\epsilon < 1\) (at high SNR regime),

\[\text{GSNR}_{\text{DF}} \geq \frac{P}{L + 1}\]  \hspace{1cm} (37)
Fig. 9. Comparison of SNR at the destination as a function of transmit power $P = P_R$ for a serial network ($L = 2$)

Fig. 10. BER of schemes in a parallel network ($L = 2$) for BPSK modulation

indicates that DF is superior to AF at high SNR. We can also observe the case where $\text{GSNR}_{AF} > \text{GSNR}_{DF}$ at low SNR (when $d^2_{min} \epsilon > 1$). Note that the variance of the error components associated with DF ($d^2_{min} \epsilon$) decreases exponentially with $P$, while those in AF ($\frac{1}{\gamma_{DF}}$ for the $k^{th}$ relay) decreases linearly with $P$. These observations can be clearly seen in Fig. 9 where AF performs slightly better than DF at very low SNR. Gradually with increase in $P$, DF outperforms AF and the performance gap widens with further increase in $P$.

E. Hybrid Relay Networks

From the previous sections, we determine that EF is well suited to both parallel and serial relay network regardless of $P$, and substantial performance gain can be obtained over AF and DF in many scenarios. We also observe that DF is close to optimal in a serial relay network at high SNR while as AF is near-optimal in parallel relay networks at low SNR. In these regimes, the performance gain of EF is limited. Thus, it is interesting to determine the performance gain of EF in general memoryless relay networks. Consider a network consisting of both parallel and serial subnetworks as shown in Fig. 11. Due to the presence of parallel and serial elements together in the network, we find a significant performance degradation in both AF and DF at all $P$. Precisely, this is a scenario where EF obtains a large gain over the best of DF and AF. Fig. 12 compares the performance of schemes for the hybrid network in Fig. 11. It can be noticed that EF performs significantly better than the best of DF and AF. Fig. 13 displays the error probability of the schemes for the hybrid network. It can be seen that substantial gain is obtained over the best of DF and AF. The performance gain will increase for a large network with both parallel and serial
VII. CONCLUSION

In this work, we address a fundamental problem in relay networks that involves determining the set of relay functions in a memoryless relay network that maximizes performance. From an estimation point of view, we develop a general framework for determining SNR at the destination for all memoryless relay processing. For the single relay case, the generalized SNR was shown to be optimized by minimum mean square uncorrelated error (MMSUE) estimate which is related to the traditional MMSE estimate by a constant scaling factor. For both parallel and serial relay networks, we establish the superiority of EF over DF and AF. We show that, with MPSK modulation at the source, maximum SNR is obtained at the destination when the relays estimate and forward, regardless of the number of relays. Further, we demonstrate that the last stage of a serial relay network must employ EF for maximizing SNR at the destination. For hybrid networks that contain both serial and parallel elements, the advantage of EF

![Fig. 11. Hybrid Serial Parallel Network](image)

![Fig. 12. Comparison of SNR at destination as a function of transmit power $P = P_R$ for a hybrid network with both serial and parallel elements, for BPSK modulation at source](image)

![Fig. 13. BER of the forwarding schemes for BPSK modulation for the hybrid network in Fig. 11](image)
over the best of AF and DF is found to be significant.

APPENDIX

A. Relation between MMSUEE and MMSEE

Although the relay functions arising out of MMSUE and MMSE estimates are identical, they are fundamentally
distinct as the objectives optimized by them are different. MMSUE is the minimum achievable uncorrelated error
power while MMSE is the minimum achievable distortion. By proving that the correlation of the MMSE error $e$
and the input $x$, we obtain the relationship between MMSE and MMSUE.

*Proposition 2:* Correlation between the MMSE error $e$ and the input $x$ is always non positive.

*Proof:* We express the MMSE estimate as

$$
\hat{X}(r) = x + e = x + \frac{\mu}{P}x + e_u,
$$

where $\mu = \mathcal{E}[x^*e]$ and $e_u$ is uncorrelated to $x$,

$$
e_u = e - \frac{\mu}{P}x 
$$

(38)

Consider another estimate which is a scaled version of the MMSE estimate such that

$$
\hat{X}_{new}(r) = \frac{\hat{X}(r)}{1 + \frac{\mu}{P}} = x + \frac{e_u}{1 + \frac{\mu}{P}}.
$$

As MMSE estimation is optimal distortion minimizing method, we have the relation

$$
\mathcal{E}[|e|^2] \leq \frac{1}{(1 + \frac{\mu}{P})^2}\mathcal{E}[|e_u|^2] 
$$

(39)

$$
\leq \frac{1}{(1 + \frac{\mu}{P})^2} \left( \mathcal{E}[|e|^2] - \frac{\mu^2}{P} \right) 
$$

(40)

$$
\leq \frac{1}{(1 + \frac{\mu}{P})^2}\mathcal{E}[|e|^2] 
$$

(41)

which implies $\mu \leq 0$.

For Gaussian inputs, a unique relationship between MMSE estimate and the correlation exists, which is $\mu = \mathcal{E}(Xe) = -$$\text{MMSEE} = \frac{P}{P+1}$. A direct consequence of the negative correlation of the error with the signal $x$ leads
to the following inequality.

$$\text{SNR} \leq \frac{P}{\text{MMSEE}}.$$ 

**Proposition 3:** The minimum mean squared uncorrelated estimation error cannot be less than MMSEE. The precise relationship between MMSUEE and MMSEE is

$$\text{MMSUEE} = \text{MMSEE} - \frac{\mu^2}{(1 + \frac{\mu}{P})^2}.$$

**Proof:** We have MMSUEE ≥ MMSE, by observing $e_u$ to be the distortion arising out of another estimation method that cannot achieve a mean squared estimation error less than MMSEE. The exact relationship between the mean square error of these methods can be obtained from (38).

**B. Rotational Property of EF**

**Theorem 5:** For all MPSK constellation inputs, the MMSE estimate has the property

$$\hat{X}(re^{j\theta_m}) = e^{j\theta_m} \hat{X}(r),$$

where $\theta_m = \frac{2\pi m}{M}, m = 0, 1, ... M - 1$, the signal phases of MPSK.

**Proof:**

$$\mathcal{E}[x|r = re^{j\theta_m}] = \frac{\sqrt{P}}{M} \sum_{k=0}^{M-1} e^{j\theta_k} \text{Pr}[x = \sqrt{P}e^{j\theta_k}|re^{j\theta_m}]$$

$$= \frac{\sqrt{P}}{M} \sum_{k=0}^{M-1} e^{j\theta_k} \text{Pr}[x = \sqrt{P}e^{j\theta_k}e^{-j\theta_m}|r]$$

$$= \frac{\sqrt{P}}{M} e^{j\theta_m} \sum_{k=0}^{M-1} e^{j\theta_{(k-m)}} \text{Pr}[x = \sqrt{P}e^{j\theta_{(k-m)}}|r]$$

$$= e^{j\theta_m} \mathcal{E}[x|r]$$
C. Proof for Zero Error Correlation of EF

Expressing error as the difference of the estimate and the actual symbol, we have

\[ C = \mathcal{E}[e_1 e_2^*] = \mathcal{E}[(\hat{X}(r_1) - x)(\hat{X}^*(r_2) - x^*)] \]

\[ = \left[ \frac{P^2}{\mathcal{E}[\mathcal{E}[x^*] r_1 = g_1 x + n_1] \mathcal{E}[\mathcal{E}[x^*] r_2 = g_2 x + n_1]} \mathcal{E}[\mathcal{E}[x] r_1] \mathcal{E}[\mathcal{E}[x^*] r_2] \right] - P \]  

\[ \mathcal{E}[\mathcal{E}[x] r_1] \mathcal{E}[\mathcal{E}[x^*] r_2] = \frac{1}{M} \sum_{i=0}^{M-1} \mathcal{E}_{n_i} \mathcal{E}[x] r = g_1 x_i + n_1] \mathcal{E}_{n_i} \mathcal{E}[x^*] r_2 = g_2 x_i + n_2] \]

\[ = \frac{1}{M} \sum_{i=0}^{M-1} \mathcal{E}_{n_i} \mathcal{E}[x] r = g_1 x_i + n] \mathcal{E}_{n}^* \mathcal{E}[x] r = g_2 x_i + n] \]  

\[ = \mathcal{E}_{n} \mathcal{E}[x] r = g_1 x_0 + n] \mathcal{E}_{n}^* \mathcal{E}[x] r = g_2 x_0 + n], \]  

where (47) is obtained by applying theorem 5 in (46).

\[ \mathcal{E}[\mathcal{E}[x] r_1] \mathcal{E}[\mathcal{E}[x^*] r_2] = \frac{1}{M} \sum_{i=0}^{M-1} x_i \mathcal{E}_{n_i} \mathcal{E}[x] r = x_i + n] \]

\[ = \sqrt{P} \frac{1}{M} \sum_{i=0}^{M-1} \mathcal{E}_{n} \mathcal{E}[x] r = x_0 + n] \]

\[ = \sqrt{P} \mathcal{E}_{0} \mathcal{E}[x] r = g_1 x_0 + n], \]  

where (48) is reduced to (49) using theorem 5. Substituting (47) and (49) in (45), we have

\[ C = \mathcal{E}[e_1 e_2^*] = \frac{P^2 \mathcal{E}_{n} \mathcal{E}[x] r = g_1 x_0 + n] \mathcal{E}_{n}^* \mathcal{E}[x] r = g_2 x_0 + n]}{P \mathcal{E}_{n} \mathcal{E}[x] r = g_1 x_0 + n] \mathcal{E}_{n}^* \mathcal{E}[x] r = g_2 x_0 + n]} - P \]

\[ = 0 \]  

(50)

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