The generic soliton of the $A_n$ affine Toda field theories

Edwin J. Beggs* and Peter R. Johnson†

*Department of Mathematics,
University of Wales at Swansea,
Singleton Park,
Swansea,
SA2 8PP, UK.

†Department of Physics,
University of Wales at Swansea,
Singleton Park,
Swansea,
SA2 8PP, UK.

Abstract

In this note we show that the single soliton solutions known previously in the 1 + 1 dimensional affine Toda field theories from a variety of different methods [1, 2, 3, 4], are in fact not the most general single soliton solutions. We exhibit single soliton solutions with additional small parameters which reduce to the previously known solutions when these extra parameters are set to zero. The new solution has the same mass and topological charges as the standard solution when these parameters are set to zero. However we cannot yet completely rule out the possibility that other solutions with larger values of these extra parameters are non-singular, in the cases where the number of extra parameters is greater than one, and if so their topological charges would most likely be different.
1 Introduction

The affine Toda field theories have equations of motion based on the root system of an affine algebra $\hat{g}$. Let $\alpha_i, i = 1, \ldots, r$ be the simple roots of the Lie algebra $g$, and $\alpha_0$ be minus the highest root $-\psi$, then the equations of motion of the affine Toda field theories, for $\phi(x,t)$ an $r$-component scalar field, are

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{4\mu^2}{\beta} \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \cdot \phi} = 0. \quad (1.1)$$

Here $m_i$ are certain integers such that

$$\psi \psi^2 = \sum_{i=1}^{r} m_i \alpha_i^2,$$

and $m_0 = 1$, so that $\sum_{i=0}^{r} m_i \alpha_i = 0$. If $\beta$ is purely imaginary then we see that a constant $\phi$ such that $\alpha_i \cdot \phi \in \frac{2\pi}{|\beta|} \mathbb{Z}$ is a solution. These constant values make up the weight lattice of the coroot algebra $\Lambda_W(g^\vee)$, $\phi \in \frac{2\pi}{|\beta|} \Lambda_W(g^\vee)$, and these values lie at the degenerate minima of the potential in the Lagrangian to the affine Toda model. Hence we expect soliton solutions to exist interpolating different values of these minima at $x \to \infty$ and $x \to -\infty$. Some solutions were first found by Hollowood [1] for the $A_n$ theories using the Hirota method, and some other solutions by MacKay and McGhee [2] for other theories also using the Hirota method. This method however is somewhat unsatisfactory and a more powerful method exploiting the representation theory of affine Kac-Moody algebras, and the vertex operators, was developed for all simply-laced algebras in [3, 4]. These methods agreed on what the solutions were for the $A_n$ theories, it was thought that the single soliton solutions were, for the species $i = 1, \ldots, r$ of soliton, associated with a node on the Dynkin diagram, and $\lambda_j$ a fundamental weight of $g$,

$$e^{-\beta \lambda_j \cdot \phi} = \frac{1 + Q_1 \omega^j W_i}{1 + Q_1 W_i}, \quad (1.2)$$

where

$$\omega = e^{\frac{2\pi i}{h}}, \quad W_i(\theta) = e^{m_i (e^{\theta} x_i - e^{-\theta} x_i^-)}, \quad x_{\pm} = t \pm x \quad (1.3)$$

$$h = n + 1, \quad m_i = 2\mu \sin(\frac{\pi i}{h}), \quad Q_1 \in \mathbb{C},$$

and $\theta$ is the rapidity of the soliton. The position of the soliton is proportional to $\log |Q_1|$. The solution (1.2) is also singular at particular values of the phase of $Q_1$ given by the zeroes of either the numerator or the denominator of (1.2). In this case the singular values of $Q_1$ are simply given by straight lines joining the origin and extending to infinity. For simplicity, we restrict our attention to $h = 4$, and $i = 1$, then the singularities in the $Q_1$ plane are with phases $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. 1
The topological charge of the soliton is defined as

$$T = \frac{|\beta|}{2\pi} \left( \phi(\infty, t) - \phi(-\infty, t) \right) \in \Lambda_W(g')$$

and is independent of time $t$. The topological charge is a continuous function of $Q_1$ and takes discrete values, so it will be constant as we vary $Q_1$ within each of the four regions, but will not be defined if $Q_1$ touches the singular lines. In fact we expect the charge to jump and take a different value as $Q_1$ moves between the regions separated by the singularities. This is corroborated in detail by work done by McGhee [5] for the $A_n$ theories, where the charges are computed. For the remaining Toda theories, and the previously known single soliton solutions, the singular regions in the $Q$ plane are also given by straight lines joining the origin and extending to infinity, but the analysis is slightly more complicated because more than one power in $QW$ is present in the numerator and denominator of $e^{-\beta\lambda_j \cdot \phi}$, compare with the formula (1.2) for $A_n$. We shall see that this will not be the case in general for even the simpler $A_n$ theories.

These methods [1, 3, 4] also agreed on a form for the two-soliton solution. Here we follow [3, 4], where the two-soliton solution is understood in terms of a special function $X_{jk}(\theta_j - \theta_k)$, which is a function obtained when we normal order the two vertex operators $F_j(\theta_j)$ and $F_k(\theta_k)$ associated with the solitons of species $j$ and $k$ in the two-soliton solution. Here $\theta_j$ and $\theta_k$ are the rapidities of the two solitons which must be real in order to make physical sense.

$$F_j(\theta_j)F_k(\theta_k) = X_{jk}(\theta_j - \theta_k) : F_j(\theta_j)F_k(\theta_k) :$$

and $X_{jk}(\theta)$ can be given explicitly by

$$X_{jk}(\theta) = \prod_{p=1}^h \left( 1 - e^{\theta} e^{\sum_{p=1}^h (2p + c(j)-c(k))} \gamma_j \cdot \sigma^p \gamma_k. \right) \tag{1.4}$$

Here $c(j) = \pm 1$ is a particular ‘colour’ depending on a bi-colouration of the Dynkin diagram of $g$, where the soliton of species $j$ is associated with a node of the Dynkin diagram. Also $\gamma_j = c(j)\alpha_j$, and $\sigma$ is a special element of the Weyl group known as the Coxeter element [3, 4]. For the $A_n$ theories the two-soliton solution (species $j$ and species $k$) is

$$e^{-\beta\lambda_j \cdot \phi} = \frac{1 + Q_1 \omega^{jk} W_k + Q_2 \omega^{ij} W_j + X^{kj}(\theta_k - \theta_j) Q_1 Q_2 \omega^{(k+j)} W_k W_j}{1 + Q_1 W_k + Q_2 W_j + X^{kj}(\theta_k - \theta_j) Q_1 Q_2 W_k W_j} \tag{1.5}$$

The coefficient $X^{kj}(\theta)$ tells us a surprising amount about the interaction of two solitons, albeit where the single solitons are the ones given by (1.2) and not the more general ones which we are about to discuss. The time delay experienced by the soliton $k$ as it interacts with soliton $j$ is proportional to $\log X^{kj}(\theta)$ [3]. $X^{kj}(\theta)$ can also be extrapolated to the exact S-matrix of the solitons [8].
In this note we are not directly concerned with these properties of \( X^{kj}(\theta) \) related to the interaction of two solitons, because we shall take the case where \( X^{kj}(\theta) \) vanishes. This has not been treated in the literature before because it was previously thought that (1.5) did not have real total energy and momentum when \( \theta_k - \theta_j \) is at the zeroes of \( X^{kj}(\theta) \). It was also thought in [12] that the restricted solution was singular. Some of these solutions were mentioned by Caldi and Zhu [15], but not properly identified as true single solitons and also not fully discussed. We shall see how the restriction works in the next section. However before we do this we shall briefly discuss an alternative scheme developed by us [7, 8], different from [1, 3, 4], for finding soliton solutions, which for the moment is restricted to the \( A_n \) theories. This method is based on the inverse scattering method [9, 10].

The integrability of the affine Toda systems follows from the zero-curvature condition

\[
[\partial_+ + A^+, \partial_- + A^-] = 0,
\]

(1.6)

where \( A^\pm \) is given by

\[
A^\pm = \pm \frac{1}{2} \beta \partial_\pm (\phi, H) \pm \lambda^\pm \mu e^{\pm \frac{1}{2} \beta \phi, H} E_{\pm 1} e^{\mp \frac{1}{2} \beta \phi, H},
\]

(1.7)

and where \( H \) is the Cartan-subalgebra of \( g \), and \( E_{+1} = E_{-1}^\dagger = \sum_{i=0}^{r} \sqrt{m_i} E_{\alpha_i} \in g \), for \( E_{\alpha} \) the step operator in \( g \) corresponding to the root \( \alpha \). It is easy to see that \([E_{+1}, E_{-1}] = 0\). The compatibility condition \( \partial_+ \partial_- \Phi = \partial_- \partial_+ \Phi \) for \( \Phi \) the solution to the linear system

\[
\partial_\pm \Phi = A^\pm \Phi,
\]

(1.8)

implies the zero-curvature condition (1.6), and here \( \Phi(x, t, \lambda) \) is valued in the loop group \( \hat{G} \) of \( \hat{g} \). Therefore any solution \( \Phi \) to (1.8) in \( \hat{G} \) will generate a solution to the affine Toda field equations of motion (1.1). It turns out however that solutions \( \Phi(\lambda) \) which are analytic in \( \lambda \) (except for some essential singularities at \( \lambda = 0, \infty \), which can always be subtracted off) generate the trivial solution \( \phi = 0 \) to (1.1). It is the solutions with poles which precisely generate the soliton solutions. The number of poles present (modulo a discrete rotational symmetry in \( \lambda \)) gives the number of solitons generated, and the residues of the poles determine the species and positions, and the other degrees of freedom.

We can exhibit the solutions \( \Phi(\lambda, x, t) \) which generate a single soliton explicitly, these have a pole at \( \lambda = \omega \alpha \), for some \( \alpha \in \mathbb{C} \). However there is also a discrete rotational symmetry \( U \Phi(\lambda) U^\dagger = f(\lambda) \Phi(\omega \lambda) \), where \( f(\lambda) \) is a scalar function, for some matrix \( U \) which commutes with \( H \), and \( U^h = 1 \), this follows from the linear system (1.8) and the property \( U E_{\pm 1} U^\dagger = \omega^\pm E_{\pm 1} \). Therefore there must also be poles in \( \Phi(\lambda, x, t) \) at \( \lambda = \omega \alpha, \omega^2 \alpha, \ldots, \omega^{h-1} \alpha, \alpha \). The general solution for \( \Phi(\lambda) \) incorporating this symmetry is
\[ \Phi(\lambda) = \left( P + UPU^\dagger \frac{\lambda - \alpha}{\lambda - \omega \alpha} \right) + U^2PU^\dagger 2 \left( \frac{\lambda - \alpha}{\lambda - \omega^2 \alpha} \right) + \cdots + U^{h-1}PU^{h-1} \left( \frac{\lambda - \alpha}{\lambda - \omega^{h-1} \alpha} \right) e^{-\beta \phi H/2}, \quad (1.9) \]

where \( P \) is the unique matrix valued projection such that

\[ P + UPU^\dagger + \cdots + U^{h-1}PU^{h-1} = 1, \quad PUP = PU^2P = \cdots = PU^{h-1}P = 0, \quad (1.10) \]

as explained in \cite{7, 8}. Multi-soliton solutions are generated by multiplying these (1.9) together in the loop group \( \hat{G} \).

The space-time dependence of the projection can be easily found \cite{7, 8}, it is the unique projection which projects onto the space

\[ V = e^{-\mu(\omega \alpha E_{+1} x_{+} - \omega^{-1} \alpha^{-1} E_{-1} x_{-})} V_0, \quad (1.11) \]

where \( V_0 \) is some initial arbitrary one-dimensional space, and which satisfies the conditions (1.10). In a basis where \( H \) is diagonal,

\[ E_{+1} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad E_{-1} = E_{+1}^\dagger = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \ddots & 1 & 0 \end{pmatrix}, \quad (1.12) \]

the eigenvectors of \( E_{\pm 1} \) are

\[ v_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{h-1} \end{pmatrix}, \cdots, \quad v_r = \begin{pmatrix} 1 \\ \omega^r \\ \omega^{2r} \\ \vdots \\ \omega^{r(h-1)} \end{pmatrix}, \cdots, \quad v_{h-1} = \begin{pmatrix} 1 \\ \omega^{-1} \\ \omega^{-2} \\ \vdots \\ \omega^{-(h-1)} \end{pmatrix}, \quad (1.13) \]

with eigenvalues

\[ E_{\pm 1} v_r = \omega^{\pm r} v_r, \quad r = 0, \ldots, h - 1. \]

If we write

\[ V = \langle \begin{pmatrix} 1 \\ A_1 \\ \vdots \\ A_n \end{pmatrix} \rangle, \quad (1.13) \]
then the soliton solutions are given by
\[ e^{-\beta \lambda_i \cdot \phi} = A_i, \]
(1.14)
as explained in [8]. So the choice of initial subspace \( V_0 \) gives us different single soliton solutions. We see that the choice
\[ V_0 = < v_0 + Q_1 v_j > \]
gives us the one-soliton solution (1.2) for a soliton of species \( j \), this follows because after normalizing the first component of \( V \) to agree with (1.13), and solving for the space-time dependence using (1.11), we find that
\[ A_i = \frac{1 + Q_1 \omega^j W^j_j}{1 + Q_1 W_j^j}. \]
However we also see that the choice
\[ V_0 = < v_0 + Q_1 v_k + Q_2 v_j > \]
is just as good as the first choice (1.15), it is just a more general one-dimensional subspace than (1.15). We shall see that this solution is actually a restriction of the two-soliton solution (of species \( j \) and species \( k \)) such that \( X^{kj}(\theta) = 0 \), (compare with equation (1.5)), but more importantly its energy is real and it has the same mass as the soliton of species \( j \). Furthermore, provided \( Q_1 \) is sufficiently small when compared with \( Q_2 \), it is non-singular for all \( x \) and \( t \) and is therefore a bona-fide solution.

2 The more general one-soliton solution

2.1 The two parameter case

The solution given by the initial subspace (1.16) is from (1.11), (1.13) and (1.14)
\[ e^{-\beta \lambda_i \cdot \phi} = \frac{1 + Q_1 \omega^{jk} U + Q_2 \omega^{ij} W^j_j(\theta_j)}{1 + Q_1 U + Q_2 W^j_j(\theta_j)}, \]
(2.1)
where \( W^j_j(\theta) \) is given by equation (1.3), and we have chosen the phase of \( \omega_\alpha \) so that \( \omega_\alpha = i \omega^{-j/2} e^{-\theta} \), for \( \theta \) a real rapidity. Then we have
\[ U = e^{\mu 2 \sin \left( \frac{\theta}{h} \right)} (e^{\frac{\pi i (j-k)}{h}} e^{-\theta x_+} - e^{\frac{\pi i (j-k)}{h}} e^{\theta x_-}) \]
\[ = W_k \left( \theta - \frac{\pi i (j-k)}{h} \right). \]
(2.2)
For simplicity we now put the soliton at rest by setting \( \theta = 0 \), and then
\[ W^j_j = e^{\rho j(2x)} \]
\[ U = e^{m_k \left( 2 \cos \left( \frac{(j-k)\pi}{h} \right)x + i 2 \sin \left( \frac{(j-k)\pi}{h} \right)t \right)} \]  

(2.3)

We can then perform a Lorentz transformation on this to get back to a moving soliton with \( \theta \neq 0 \).

We choose \( j \) and \( k \) so that

\[ m_j > m_k \cos \left( \frac{(j-k)\pi}{h} \right), \]

hence for large \( x \), \( W_j \) will dominate \( U \) and \( e^{-\beta \lambda_i \phi} \) will have the same limit as when \( Q_1 = 0 \). Since the mass of the soliton is only a function of the limit of the solution \( \phi \) when \( x \to \pm \infty \) (the energy density can be written as a total derivative \[3\]), and the limit as \( x \to -\infty \) of \( e^{-\beta \lambda_i \phi} \) is 1 for all \( Q_1 \in \mathbb{C} \), we conclude that the mass of the solution (2.1) is real and the same as for the case \( Q_1 = 0 \). This mass is \( m_j = 2\mu \sin \left( \frac{j\pi}{h} \right) \), see [3]. If we re-insert the \( \theta \) dependence into \( W_j \) and \( U \), then the total energy and momentum \( P^\pm \) in light-cone coordinates is

\[ P^\pm = m_j e^{\mp \theta_j} \]

This shows that the phase of \( \omega \phi \) has been chosen correctly so that for \( \theta \) real, we get real total energy and momentum, and that \( \theta \) agrees with the standard definition of rapidity.

If we now take the two-soliton solution (1.5)

\[ e^{-\beta \lambda_i \phi} = \frac{1 + Q_1 \omega^{jk} W_k(\theta_k) + Q_2 \omega^{ij} W_j(\theta_j) + X^{kj}(\theta_k - \theta_j)Q_1 Q_2 \omega^{ij(k+j)} W_k(\theta_k) W_j(\theta_j)}{1 + Q_1 W_k(\theta_k) + Q_2 W_j(\theta_j) + X^{kj}(\theta_k - \theta_j)Q_1 Q_2 W_k(\theta_k) W_j(\theta_j)}. \]

(2.4)

This equation makes physical sense for \( \theta_k \) and \( \theta_j \) real, but will still satisfy the equations of motion (1.1) after analytically continuing \( \theta_k \) and \( \theta_j \). In particular we can set \( \theta_k - \theta_j \) to be at a zero of \( X^{kj}(\theta_k - \theta_j) \). For the \( A_n \) theories it is known [8] from the formula (1.4) that this zero is at

\[ \theta_k - \theta_j = i\frac{\pi(k - j)}{h}. \]

(2.5)

Hence with this restriction, and with \( \theta = \theta_j \), we recover the formula (2.1) from (2.2) and (2.4). The energy and momentum \( P^\pm \) of the generic two-soliton solution (2.4) is from [3]

\[ P^\pm = m_j e^{\mp \theta_j} + m_k e^{\mp \theta_k}. \]

(2.6)

When evaluated at the analytically continued values (2.5), this does not agree with our energy-momentum formula \( P^\pm = m_j e^{\mp \theta_j} \) derived for the restricted solution (2.1). This is somewhat surprising, but a closer examination of the proof of the two-soliton result (2.6) in [4] requires \( X^{kj}(\theta) \neq 0 \), so that the dominant term for large \( x \) in both the numerator and denominator of (2.4) is \( W_k W_j \).

Also note that, as remarked upon in [6], the case \( j = k \) is empty, because \( X^{jj}(0) = 0 \), and we recover the standard one-soliton solution in the form

\[ e^{-\beta \lambda_i \phi} = \frac{1 + (Q_1 + Q_2) \omega^{ij} W_j}{1 + (Q_1 + Q_2) W_j}. \]
We now study the singularities of the solution (2.1), in the same way that we saw the singularities of (1.2). For illustrative purposes we restrict our attention to $h = 4, j = 1, k = 3$. With these values $m_j > m_k \cos \left( \frac{(j-k)\pi}{h} \right)$ (actually $j$ and $k$ are anti-solitons of each other, and $m_j = m_k$). Suppose that $Q_2$ is given and that $\text{Re } Q_2 > 0$, and $\text{Im } Q_2 < 0$, say, for definiteness. We then sketch in the $Q_1$ plane the values where the numerator and denominator of (2.1) vanish, in the rest frame of the soliton ($\theta = 0$), as before. We also implicitly absorb the time dependence $e^{im_k \sin \frac{(j-k)\pi}{h} t}$ from (2.3) into $Q_1$.

![Figure 1: Singularities of (2.1) in the $Q_1$ plane](image)

The label 0 refers to the denominator of (2.1), and the labels $j = 1, 2, 3$ refer to the numerator of $e^{-\beta \lambda_j \cdot \phi}$.

We see that provided $|Q_1|$ is sufficiently small, a circle of radius $|Q_1|$ centered at the origin will not intersect any of the singular curves, and hence the solution (2.1) is free of singularities for all time $t$. As $t$ increases (in the rest frame) we move round the circle as shown. This time dependence is interesting because it shows that in the rest frame of the soliton, it is not completely motionless, in constrast to the case with $Q_1 = 0$, and that there is an incipient small beating motion, similar to the way that a breather breathes. This is shown in Figure 5. Observe the difference between the simple step shown in Figure 4, where $Q_1 = 0$, and the
Figure 5. The breather is an analytic continuation of a soliton–anti-soliton solution \[4, 12\] which gives a real total energy and momentum, but our solutions are certainly different from these breathers. Indeed in \[12\] it was thought that the analytic continuation of a two soliton solution so that \(X^{jk}(\theta) = 0\) always gave a singular solution, but the analysis in Figure 1 shows that this is not necessarily the case.

Now Figure 1 was drawn given the assumption that \(Q_2\) was known, and that we had not accidently chosen a value which always gave a singular solution. The phase of \(Q_2\) is shown in the figure, and it is clear by inspection that if we smoothly adjust the phase of \(Q_2\) to either \(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\), one of the four curves each in turn will become a straight line passing through the origin, and the solution must then be singular however small we choose \(|Q_1|\). As the phase of \(Q_2\) becomes close to these four values we must choose successively smaller values of \(|Q_1|\) for the solution to be non-singular. Hence in the \(Q_2\) plane the singularities are the same as the case with \(Q_1 = 0\), but with the understanding that \(|Q_1|\) must be sufficiently small for a non-singular solution. Also note that as \(|Q_2|\) becomes smaller, but with the phase of \(Q_2\) fixed, the turning points of the curves in Figure 1 move towards the origin, and tend to it in the limit when \(|Q_2| \to 0\). Hence we must also adjust \(|Q_1|\) depending on \(|Q_2|\).

We can also observe from Figure 1, that there are no new solutions for any large values of \(|Q_1|\), since the curves all go off to infinity. A circle of large radius must intersect the curves.

Now a non-singular solution with appropriately small \(|Q_1|\) is continuously connected to the standard solution with \(Q_1 = 0\), therefore we conclude from the continuity of the topological charge that the charge is the same as the solution with \(Q_1 = 0\). These are already well understood, at least for the \(A_n\) theories, and have been calculated \[5\].

### 2.2 More than two parameters

We pick a soliton species \(j\), and consider all soliton species \(k\) such that

\[
m_j > m_k \cos \left( \frac{(j - k)\pi}{h} \right). \tag{2.7}\]

Define the set \(B_j\) to consist of all the integers \(k\) such that (2.7) holds. The number of possibilities for \(k\) will depend on \(j\), but as we have seen in the first case, we can always take the anti-soliton \(k\) to the soliton \(j\) provided \(j \neq k\). If the anti-soliton species is the same as the soliton species, then \(n\) must be odd, and \(k = \frac{n}{2} + 1\). The property (2.7) must then be true for all species \(k\), since the soliton \(j\) is the heaviest. Also note that for \(j = 1\) the only possibility for \(k \in B_j\) is \(k = n\).

We then choose the initial space

\[
V_0 = < v_0 + \sum_{k \in B_j} Q_k v_k + Q_j v_j >
\]
which generates the single soliton solution

\[ e^{-\beta \lambda_i \phi} = \frac{1 + \sum_{k \in B_j} Q_k \omega^{ik} U_k + Q_j \omega^{ij} W_j(\theta)}{1 + \sum_{k \in B_j} Q_k U_k + Q_j W_j(\theta)}. \]  \tag{2.8}

We have chosen the phase of \( \omega \alpha = i \omega^{-j/2} e^{-\theta} \), for \( \theta \) real. Then

\[ U_k = e^{i n \sin \left( \frac{k \pi}{n} \right) \left( e^{\pi i (j-k) h} e^{-\theta x} - e^{-\pi i (j-k) h} e^{\theta x} \right) \} = W_k \left( \theta - \frac{\pi i (j-k)}{h} \right), \]  \tag{2.9}

and \( U_j = W_j \). The set \( B_j \) has been defined so that \( W_j \) dominates for large \( x \) over the purely exponential parts of \( U_k \). This is enough to guarantee that the mass of the solution \( \text{(2.8)} \) is \( m_j \).

It is clear that we can also derive \( \text{(2.8)} \) from a restriction of a multi-soliton solution. We take the multi-soliton solution with solitons of species in \( B_j \) and also of species \( j \), and with separate rapidities \( \theta_k, k \in B_j \) and \( \theta_j \). The multi-soliton solution is \[ e^{-\beta \lambda_i \phi} = \frac{1 + \sum_{k \in B_j} Q_k \omega^{ik} W_k(\theta_k) + Q_j \omega^{ij} W_j(\theta_j) + \text{higher terms}}{1 + \sum_{k \in B_j} Q_k W_k(\theta_k) + Q_j W_j(\theta_j) + \text{higher terms}}. \]

The higher terms all involve more than one power of \( W \). They are all multiplied by products of \( X \)'s which vanish when we take

\[ \theta_k - \theta_j = \frac{i \pi (k-j)}{h}. \]  \tag{2.10}

For example, a term \( W_{k_1} W_{k_2} \) for \( k_1, k_2 \in B_j \), is multiplied by \( X^{k_1 k_2} (\theta_{k_1} - \theta_{k_2}) \), but from \( \text{(2.10)} \) \[ \theta_{k_1} - \theta_{k_2} = i \frac{\pi (k_1 - k_2)}{h}, \] and \( X^{k_1 k_2} (\theta_{k_1} - \theta_{k_2}) \) has a zero at this point.

We can now discuss the structure of the singularities of \( \text{(2.8)} \), after setting \( \theta = 0 \). We pick an \( r \in B_j \), and assume that the remaining \( Q_k, k \in B_j \setminus \{ r \} \) are given, but are very small. We also assume that \( Q_j \) is given and can be as large as we please. As before, we absorb the time dependent phase of \( U_r \) into \( Q_r \), but we cannot do this for the other phases. We then sketch the singularities of \( \text{(2.8)} \). We first of all sketch the curves for the situation \( h = 4, j = 2, r = 1 \), with all \( Q_k = 0 \), and \( k \in B_j \setminus \{ r \} \). This is done in Figure 2. Observe that the singular phases of \( Q_2 \) are now 0 and \( \pi \). We now make \( Q_k, k \in B_j \setminus \{ r \} \) non-zero, but very small. We fix time \( t \), and sketch the curves for all \( x \). The curve will have the same asymptotic behaviour as the previous case, but will have moved by a small amount. In Figure 3, regions are sketched approximately by varying all \( x \) and \( t \). Hence the solution is singular at some \( x \) and \( t \) in the singular regions, but it may be possible for the circle centered at the origin with the time dependent phase to avoid a singularity even though it may pass through a singular region. In fact for sufficiently large \( Q_k, k \in B_j \setminus \{ r \} \), the singular regions may cover the
origin, and in that case $|Q_1|$ would have to be non-zero for there to be the possibility of a non-singular solution. This situation is too complicated for us to say anything definite, for the time being. On the other hand, we certainly avoid all singularities if $|Q_r|$ is sufficiently small so that $Q_r$ does not enter the singular regions. It is also clear that for the cases where $Q_k, k \in B_j$ are small, the singular lines in the $Q_j$ plane are the same as for the naive solution with $Q_k = 0, k \in B_j$.

![Figure 2](image)

Figure 2: Singularities of $h = 4, j = 2, r = 1$, in the $Q_1$ plane

3 Discussion and Conclusions

We have shown that the single soliton solutions in the affine Toda field theories are not as simple as first thought, but that the masses and topological charges of the new solutions with additional small parameters are the same as the previous solutions obtained by setting these extra parameters to zero. The inverse scattering method is seen to be superior to the other methods, from the point of view of being able to generate all single solitonic objects from a single simple pole factor (1.8). It is not so obvious from the restriction of the two-soliton solution, that the solutions that we have derived are single soliton solutions. We also remark that constructions of new solutions will follow in exactly the same way for the remaining affine Toda theories, which we do not discuss.
It is worth considering these new classical solutions in the context of the quantum theory. There are two issues here. The first is that semi-classical techniques to finding quantum corrections to masses of single solitons in [13] and also in [14] only considered the naive single soliton solutions. It is most likely that the fluctuations indicated by the extra parameters introduce extra corrections, and thus may invalidate the work in [13] and [14]. The second point is that the circular degrees of freedom given by the phases of the extra small parameters should lead to an excited set of states all related to the same classical solution, when quantized. Such circular degrees of freedom do not exist in the naive solutions. One of us has already found exact S-matrices for the solitons in simply-laced affine Toda field theories [8], and there it was discovered that associated with a pole due to the fusing of solitons there was a string of further poles consistent with the interpretation that they represent excited soliton states, with increasing masses. Hopefully these new single soliton solutions will shed light on the excited states seen in the S-matrix. It should be possible, for example, to compute their masses and to compare them with the positions of the poles of the string in the S-matrix.

We also remark that we have failed to find new solutions with different topological charges from the single soliton solutions already known, although we have not completely ruled out the possibility of their existence, because of the different rates of rotation of $U_k$, and therefore...
the possibility of non-singular solutions even when the origin is contained within a singular region, or the circle in the $Q_1$ plane passes through a singular region in Figure 3. This was our intention when we first looked at the single soliton solutions in more detail. If such solutions exist they may not be continuously connected to the naive ones, and their charges would most likely be different. It is desirable to find more single soliton solutions which will fill the missing weights in the fundamental representations (the soliton of species $i$ has topological charges which are in general a subset of the weights of the fundamental representation $V_i$ (3)), a problem which requires urgent attention in the context of the quantum theory and the exact S-matrices [8]. It is also extremely intriguing that the weights of the fundamental representations $V_1$ and $V_n$ are filled for the $A_n$ theories, and it is only for these soliton species 1 and $n$ that there are precisely only a maximum of two parameters in the single soliton solutions, the case discussed in §2.1, so that the complication due to the different rates of rotation of the phases of $U_k$ does not occur, and that the singular regions in the $Q$-plane are only curves.

![Figure 4](image-url)

**Figure 4**: Plot of $\text{Re}(\lambda_1 \cdot \phi)$ of equation (2.1) for $Q_1 = 0$

![Figure 5](image-url)

**Figure 5**: Plot of $\text{Re}(\lambda_1 \cdot \phi)$ of equation (2.1) for $Q_1 \neq 0$

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