Conformal transformations and doubling of the particle states

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Abstract

The 6D and 5D representations of the four-dimensional (4D) interacting fields in the Heisenberg picture and the corresponding equations of motion are studied using equivalence of the conformal transformations of the four-momentum $q_{\mu}$ ($q'_{\mu} = q_{\mu} + h_{\mu}$, $q'_{\mu} = \Lambda_{\mu}^{\nu}q_{\nu}$, $q'_{\mu} = \lambda q_{\mu}$ and $q'_{\mu} = -M^2q_{\mu}/q^2$) and the corresponding rotations on the 6D cone $\kappa_A\kappa^A = 0$ ($A = \mu; 5, 6 \equiv 0, 1, 2, 3, 5, 6$), where $q_{\mu} = M\kappa_{\mu}/(\kappa_5 + \kappa_6)$ and $M$ is the scale parameter. The 4D reduction of the 6D fields on the cone $\kappa_A\kappa^A = 0$ is unambiguously fulfilled by the intermediate 5D projection into two 5D hyperboloids $q_{\mu}q^{\mu} + q_5^2 = M^2$ and $q_{\mu}q^{\mu} - q_5^2 = -M^2$ in order to cover the whole domains $-\infty < q_{\mu}q^{\mu} < \infty$ and $q_5^2 \geq 0$. The resulting 5D and 4D fields in the coordinate space consist of two parts $\varphi_1(x, x_5)$, $\varphi_2(x, x_5)$ and $\Phi_1(x) = \varphi_1(x, x_5 = 0)$, $\Phi_2(x) = \varphi_2(x, x_5 = 0)$, where the Fourier conjugate of $\varphi_1(x, x_5)$ and $\varphi_2(x, x_5)$ are defined on the hyperboloids $q_{\mu}q^{\mu} + q_5^2 = M^2$ and $q_{\mu}q^{\mu} - q_5^2 = -M^2$ respectively. Consequently, the 4D reduction of the 6D fields generate two kinds of the 5D and 4D fields $\varphi_{\pm} = \varphi_1 \pm \varphi_2$ and $\varphi_{\pm}(x, x_5 = 0) = \Phi_{\pm}(x) = \Phi_1(x) \pm \Phi_2(x)$ with the same quantum numbers but with the different masses and the sources. This doubling of the 4D fields $\Phi_{\pm} = \Phi_1 \pm \Phi_2$ can be applied for unified description of the interacting electron and muon fields, $\pi$ and $\pi(1300)$-mesons, $N$ and $N(1440)$-nucleons and other particles with the same quantum numbers but different masses and interactions.
Introduction

The 5D extension of the 4D relativistic theories is the fruitful method that has a long history. The Kaluza-Klein theory and their generalizations for the gauge transformations [1, 2, 3, 4] allow to unify the electromagnetic and gravitation theories. In the traditional Kaluza-Klein theory all partial derivatives with respect to fifth coordinates have been equated to zero and the extra spatial dimension was compacted to a small size circle. The rigorous mathematical approach for the $N + 1$ and $N$ dimensional manifolds (see ch. 2.2 in [3]) allow to embed the 4D equation of motion with the sources into the 5D equation without sources. In the recent 5D field theoretical formulations [5, 6] the extra fifth dimension is required to solve the problems of the renormalizable $SO(10)$ grand unification theories with the breakdown of the gauge coupling. In this approach the fifth dimension enable to reproduce the fermion generations, quark mass hierarchy, flavor mixing and Cabbibbo-Kabayashi-Maskawa matrices and it is argued, that in virtue of the no go theorems it is not possible to achieve these results in the 4D space. Other kind of the 5D relativistic field theories were performed within the invariant time method [7, 8, 9], where the fifth coordinate is the proper time $x_5 = x_o - x$. In these theories $x_5$ is an auxiliary variable and the sought 4D wave functions and fields are reproduced through the 5D wave functions and fields via the boundary conditions for $x_5 = 0$ or $x_5 = \sqrt{t^2 - x^2}$ and the evolution over the fifth coordinates were often described through the equation for the first derivatives of the scalar and fermion fields $i\partial \phi / \partial x_5$ and $i\partial \psi / \partial x_5$.

The general scheme for the 5D extensions of the 4D relativistic theories and the 4D reductions of the 5D relativistic formulations presents the conformal group of the 4D transformations that can be unambiguously represent through the rotations on the 6D cone. In particular, the conformal transformations of the four coordinate $x_\mu$ consists of the following independent motions $x'_\mu = x_\mu + a_\mu$, $x'_\mu = \Lambda^\nu_\mu x_\nu$, $x'_\mu = \lambda x_\mu$ and $x'_\mu = -\ell^2 x_\mu / x^2$ which can be performed through the rotations on the 6D cone $\xi_A \xi^A \equiv \xi_\mu \xi^\mu + \xi_5 \xi^5 - \xi_6 \xi^6 = 0$ [10], where $A = 0, 1, 2, 3; 5, 6$, $x_\mu = \xi_\mu / \xi^A_+$, $\xi_{\pm} = (\xi_5 \pm \xi_6) / \ell$ and $\ell$ is the dimension parameter. The one-to-one relationship between the 4D conformal transformations and 6D rotations allow to construct the one-to-one relationship between an interacting 4D Heisenberg field $\Phi(x)$ and the corresponding 6D field $\varsigma(\xi) \equiv \varsigma(x_0, \xi_1, \xi_2, \xi_3; \xi_5, \xi_6)$ as $\Phi(x) = \left[ \varsigma(x, \xi^+, \xi_-) \right]_{\xi_k^A = 0}$ with the fixed scale parameter $\xi_+$ and $\xi_- / \xi_+ = x^2 / \ell^2$ [10]-[13]. The other 4D reduction of $\varsigma(\xi)$ was used in the manifestly conformal invariant formulation [14]-[24], where the homogeneity of the conformal invariant 6D fields $\left[ \varsigma(\xi) \right]_{\xi_k^A = 0} = \left[ \varsigma(x, \xi^+, \xi_A^A = 0) \right]$ over the scale variable $\xi_+$ is required, i.e. $\varsigma(x, \xi^+, \xi^A_A = 0) = \xi_+ d \phi(x, \xi_A^A = 0)$ and the 4D conformal invariant field is $\Phi(x) \equiv \phi(x, \xi_A^A = 0)$. Location of $\varsigma(\xi)$ on the 6D cone $\xi_A^A = 0$ impose additional condition by conformal transformations. For instance, the Fourier transformation of an arbitrary field $\varsigma(\xi)$ on the
6D cone produces the following condition

\[
\left( \frac{\partial^2}{\partial \kappa^\mu \partial \kappa_\mu} + \frac{\partial^2}{\partial \kappa^5 \partial \kappa_5} - \frac{\partial^2}{\partial \kappa^6 \partial \kappa_6} \right) \int d^6 \xi e^{i \kappa^A \xi^A} \delta \left( \xi_5^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_5^2 - \xi_6^2 \right) \varsigma(\xi) = 0. \quad (I.1a)
\]

where \( \kappa_A \) are Fourier conjugate to \( \xi_A \) and for derivation of (I.1a) the condition \( (\xi_A \xi^A) \delta (\xi_A \xi^A) = 0 \) was used. The intermediate 5D projection of the 6D fields and the condition (I.1a) determine the corresponding 5D fields and the 5D condition.

In order to obtain the 5D and 6D extensions of the 4D equations of motion it is convenient to consider the conformal transformations of the four momentum \( q_\mu \) \( (q'_\mu = q_\mu + h_\mu, q'_\mu = \Lambda_\nu q_\nu, q'_\mu = \lambda q_\mu \) and \( q'_\mu = -M^2 q_\mu / q^2 \). The 6D representation of the conformal transformations for the independent four components of the momentum \( q_\mu \) are similar with the conformal transformations in the coordinate space. The principal difference between the conformal transformations in the coordinate and momentum space is in the translation. In the next section it is shown that the translation of the four momentum \( q'_\mu = q_\mu + h_\mu \) for the Fourier conjugate of \( \Phi(x) \) produces the gauge transformation \( \Phi'(x) = e^{ih_\mu x^\mu} \Phi(x) \). According to the Dirac geometrical model [10]-[18], each of the conformal transformations in the momentum space is unambiguously determined via the appropriate 6D rotation with the invariant 6D form

\[
\kappa_A \kappa^A \equiv \kappa_\mu \kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0, \quad (I.2a)
\]

where the four momentum \( q_\mu \) \( (\mu = 0, 1, 2, 3) \) is defined as \( q_\mu = \kappa_\mu / \kappa_+ \) and \( M \) is a scale parameter. The 6D cone (I.2a) and the corresponding surface

\[
q_\mu q^\mu + M^2 \frac{\kappa_-}{\kappa_+} = 0, \quad \text{with} \quad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M} \quad (I.2b)
\]

are invariant under any combination of the conformal transformations of \( q_\mu \).

In analogy with (I.1a) the conformal transformations of a 4D field \( \Phi(x) \) in the momentum space can be performed via the 6D rotations of the corresponding 6D field operator \( \varsigma(\kappa) \) which is embedded into the cone (I.2a). Therefore, location of \( \varsigma(\kappa) \) on the same 6D cone (I.2a) before and after the conformal transformations in the momentum space impose the condition

\[
\left( \frac{\partial^2}{\partial \xi^\mu \partial \xi_\mu} + \frac{\partial^2}{\partial \xi^5 \partial \xi_5} - \frac{\partial^2}{\partial \xi^6 \partial \xi_6} \right) \int \frac{d^6 \kappa}{(2\pi)^4} e^{i \kappa^A \xi^A} \delta \left( \kappa_\mu \kappa^\mu + \kappa_5^2 - \kappa_6^2 \right) \varsigma(\kappa) = 0. \quad (I.1b)
\]

This paper deals with consistency of the usual 4D equations of motion for 4D interacting Heisenberg field \( \Phi(x) \) and boundary conditions and constraints for the 5D and 6D representations of \( \Phi(x) \) which follows from the conformal group of the transformations in the momentum space. Two particular features generate the special interest to the conformal transformations in the momentum space[20]. First, the observables of the particle interactions, like the cross sections and polarizations are determined in the momentum space. Secondly, the accuracy of the measurement of the particle coordinates is
in principle restricted by the Compton length of this particle. Moreover, determination of the coordinate of the conformal invariant massless particles produces additional essential troubles (see [19] ch. 20 and [26]). The conformal transformations of the fields and the corresponding equations of motion in the momentum space were considered in ref. [11, 13, 20, 25], where the conformal transformations were performed in the configuration space and followed relations in the momentum space were obtained using the Fourier transformation.

The 4D reduction of the 6D operators $\zeta(\kappa) \equiv \zeta(q, \kappa_+, \kappa_-)$ generates the 5D operators as the intermediate 5D projections. There exists only two 5D De Sitter spaces with the constant curvature which have the invariant forms $q_\mu q^\mu \pm q_5^2 \mp M^2 = 0$ ($q_5^2 \geq 0$) of the $O(2, 3)$ and $O(1, 4)$ rotational groups [18, 24, 28]. The single 5D hyperboloid is not enough for reproduction of the whole values of $-\infty < q^2 < \infty$. Therefore, we use the domains from the both 5D hyperboloids which are connected by inversion $q_\mu = -M^2 q_\mu/q^2$. Thus for the intermediate 5D projections of the 6D cone (I.2a) we shall use the invariant forms

$$q_\mu q^\mu + q_5^2 = M^2 \quad \text{with} \quad \frac{q_5^2}{M^2} = \frac{\kappa_-}{\kappa_+} + 1, \quad \text{and} \quad q_5^2 \geq 0 \quad \text{(I.3a)}$$

$$q_\mu q^\mu - q_5^2 = -M^2 \quad \text{with} \quad \frac{q_5^2}{M^2} = -\frac{\kappa_-}{\kappa_+} + 1 \quad \text{and} \quad q_5^2 \geq 0. \quad \text{(I.3b)}$$

In (I.3a,b) the fifth variable $q_5^2$ is positive $0 \leq q_5^2 \leq \infty$. Consequently, for the positive $q^2 \equiv q_\mu q^\mu \geq 0$ the corresponding four-momenta $q_\mu$ are distributed between the domains $0 \leq q^2 \leq M^2$ and $q^2 > M^2$ on the hyperboloids (I.3a) and (I.3b) respectively. These domains are connected by inversion $q'_\mu = -M^2 q_\mu/q^2$. The values of the $q^2$ on the hyperboloids (I.3a) and (I.3b) are also connected through the reflection $q^2 \leftrightarrow -q^2$. Therefore, for the negative $q^2 < 0$ one has $-M^2 < q^2 < 0$ on the hyperboloid (I.3b) and $-\infty < q^2 < -M^2$ is on the hyperboloid (I.3a). More detailed the distributions of $q^2$ on the hyperboloids (I.3a) and (I.3b) are listed in Table 1 of Section 2.

In order to determine the 5D and 4D projections of the 6D fields $\delta(\kappa A \kappa^A) \zeta(\kappa)$ (I.2a) we shall introduce the following 5D fields $\varphi_1(x, x_5)$ and $\varphi_2(x, x_5)$

$$\varphi_1(x, x_5) = \int \frac{d^4q}{(2\pi)^4} \frac{d^2q_5 e^{-iqx-iq^2x_5}}{2\pi} \delta(q^2 + q_5^2 - M^2) [\theta(q^2)\theta(M^2 - q^2) + \theta(-q^2)\theta(-M^2 + q^2)] \phi(q, q_5^2), \quad \text{(I.4a)}$$

$$\varphi_2(x, x_5) = \int \frac{d^4q}{(2\pi)^4} \frac{d^2q_5 e^{-iqx-iq^2x_5}}{2\pi} \delta(q^2 - q_5^2 + M^2) [\theta(q^2)\theta(-M^2 + q^2) + \theta(-q^2)\theta(M^2 + q^2)] \phi(q, q_5^2). \quad \text{(I.4b)}$$

where

$$\phi(q, q_5^2) = \frac{M^2}{2} \int \kappa_+^3 d\kappa_+ \theta(\kappa_+) \zeta(q, q_5^2, \kappa_+). \quad \text{(I.5)}$$
For the sake of simplicity the scale variable \( \kappa_+ \) \((2b)\) is taken in positive i.e. \( \phi(\kappa) = \theta(\kappa_+)^\kappa(\kappa), \) where \( \theta(\kappa_+) = 1 \) for \( \kappa_+ > 0 \) and \( \theta(\kappa_+) = 0 \) for \( \kappa_+ < 0. \)

The Fourier conjugate of \( \varphi_1(x, x_5) \) and \( \varphi_2(x, x_5) \) are located into hyperboloids \((3a)\) and \((3b)\) respectively. Therefore they satisfy the conditions

\[
\begin{align*}
\frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x^{\alpha} \partial x_\alpha} + M^2 \varphi_1(x, x_5) &= 0, \\
\frac{\partial^2}{\partial x^\mu \partial x_\mu} - \frac{\partial^2}{\partial x^{\alpha} \partial x_\alpha} - M^2 \varphi_2(x, x_5) &= 0.
\end{align*}
\]  

\((I.6)\)

The fields \( \varphi_1 \) \((4a)\) and \( \varphi_2 \) \((4b)\) produce two independent 5D fields

\[
\varphi_+(x, x_5) = \varphi_1(x, x_5) + \varphi_2(x, x_5); \quad \varphi_-(x, x_5) = \varphi_1(x, x_5) - \varphi_2(x, x_5),
\]

\((I.7)\)

which Fourier conjugate are defined in the whole domains \((-\infty < q_\mu < +\infty)\) and \((-\infty < q^2 < +\infty)\).

The usual boundary condition for the 5D fields \( \varphi_\pm(x, x_5) \) at \( x_5 = 0 \) allows to get the 4D fields \( \Phi_\pm(x) \)

\[
\Phi_\pm(x) = \varphi_\pm(x, x_5) = 0.
\]

\((I.8)\)

According to \((4a,b), (7)\) and \((8)\) the Fourier conjugate of \( \Phi_\pm(x) \)

\[
\Phi_\pm(q) = \int d^4xe^{iqx} \Phi_\pm(x)
\]

\((I.9)\)

have the following structure

\[
\Phi_\pm(q) = \sum_{N=I,III} \Phi_N(q) \pm \sum_{N=II,IV} \Phi_N(q);
\]

\((I.10)\)

where

\[
\begin{align*}
\Phi_I(q) &= \theta(q^2)\theta(M^2-q^2)\phi(q, q_5^2 = M^2-q^2); \quad \Phi_{II}(q) = \theta(q^2)\theta(-M^2+q^2)\phi(q, q_5^2 = M^2+q^2); \\
\Phi_{III}(q) &= \theta(-q^2)\theta(-M^2-q^2)\phi(q, q_5^2 = M^2-q^2); \quad \Phi_{IV}(q) = \theta(-q^2)\theta(M^2+q^2)\phi(q, q_5^2 = M^2+q^2),
\end{align*}
\]

\((I.11)\)

where the lower index \( I, II, III \) and \( IV \) of \( \Phi(q) \) corresponds to the domains of \( q^2 \) which are listed in Table 1 of Section 2. The details of the relationship between the 4D and 5D scalar field are given in in Section 3.

The equations \((4a,b)-(10)\) presents the relationship between the 4D, 5D and 6D fields \( \delta(\kappa_A \kappa^A)\delta(\kappa) \). The 4D interacting Heisenberg fields \( \Phi_\pm(x) \) and their 6D representations \( \delta(\kappa_A \kappa^A)\delta(\kappa) \) are not invariant under the conformal transformations. Nevertheless, location of \( \delta(\kappa_A \kappa^A)\delta(\kappa) \) on the 6D cone \((2a)\) and the corresponding location of the Fourier conjugate of \( \varphi_\pm(x, x_5) \) on the hyperboloids \((3a,b)\) produce the conditions \((1b)\) and \((6)\) for any 6D field and its 5D projections. The consistency of these conditions with the equation of motion for \( \varphi_\pm(x, x_5) \) and \( \Phi_\pm(x) \) and the boundary conditions \((8)\) are considered in Sect. 4 and 5.
The intermediate projection of the 6D field \( \delta(\kappa_A \kappa^A) \zeta(\kappa) \) on the invariant forms (I.3a,b) of the \( O(2, 3) \) and \( O(1, 4) \) subgroups of the conformal group \( O(2, 4) \) need to introduce the two independent 5D fields \( \varphi_+ \) and \( \varphi_- \) (I.7) which are constructed from the same parts \( \varphi_1 \) and \( \varphi_2 \) (I.4a,b). The invariant forms (I.3a,b) of the \( O(2, 3) \) and \( O(1, 4) \) form the definition area of \( \varphi_1 \) and \( \varphi_2 \) and correspondingly of the field \( \Phi_N(q) \) in (I.10) and (I.11). The 4D fields \( \Phi_+ \) and \( \Phi_- \) (I.10) have the same quantum numbers. But \( \Phi_+ \) and \( \Phi_- \) can have the different masses and the different sources. Other details of the inversion and related constructions of the 4D fields are given in the next Section.

The reduction formulas (I.5) of the 6D field \( \zeta(\kappa) \) on the cone (I.2a) \( \zeta(\kappa) \equiv \zeta(q_\mu, \kappa_+, \kappa_A \kappa^A \neq 0) \equiv \zeta(q, q_5^2, \kappa_+) \) differ from the reduction formula in the manifestly covariant formulation [14]-[24], where the homogeneity of \( \zeta(\kappa) = \phi(q_\mu, \kappa_+, \kappa_A \kappa^A = 0) \) over the scale variable \( \kappa_+ \) is required, i.e. \( \phi(q_\mu, \kappa_+, \kappa_A \kappa^A = 0) = \kappa_+^d \phi(q_\mu) \), where \( d \) is the scale dimension of \( \phi(\kappa) \). In order to reproduce this property in the present approach one can use an additional condition in (I.5)

\[
\zeta(q, q_5^2, \kappa_+) = \delta(\kappa_+ - M) \phi(q, q_5^2, M); \quad \text{or} \quad \zeta(q, q_5^2, \kappa_+) = \theta(\kappa_+ - M) \phi(q, q_5^2, M) \quad (I.12)
\]

where \( M \) is a fixed scale parameter.

The 5D quantum field theory with the invariant forms \( q_\mu q^\mu + q_5^2 = M^2 \) or \( q_\mu q^\mu - q_5^2 = -M^2 \) separately was firstly studied in refs. [27, 28, 29], where \( M \) was interpreted as the fundamental (maximal) mass and its inverse \( 1/M \) as the fundamental (minimal) length [30, 31]. The conformal transformations in the momentum space for the complete fields \( \varphi_+ = \varphi_1 + \varphi_2 \) was suggested in [32], where \( M \) is determined via \( m_\pi \) and \( m_{Higgs} \) according to the chiral symmetry breaking mechanism within the 5D chiral models. In the present paper is studied coupling between the 5D and 4D equations of motion for the fields \( \varphi_+, \Phi_+ \) and \( \varphi_-, \Phi_- \). Besides the present paper contains more general and self-consistent formulation of translations and inversions in the 4D momentum space for the 4D charged and neutral fields.

In Section 1 the conformal transformations in the 4D momentum space and the corresponding transformations of the interacting fields \( \Phi(x) \) are considered. The domains of the variables \( q^2 = q_\mu q^\mu \) and \( q_5^2 \) (I.3a,b) are determined in Sect. 2. The 4D reduction of the 5D fields and the related projections and convolution formulas are given in Sect. 3. In Sections 4 and 5 the 4D and 5D equations of motion for the scalar fields and the corresponding constrains for \( x_5 \) are considered. Sections 6 is devoted to the 5D and 4D Lagrangians. The 4D and 5D equations of motion for the fermion fields with the electromagnetic interaction and the constrains for the fifth coordinates \( x_5 \) are considered in Sect. 7 and 8. In Section 9 the 5D extension of the standard \( SU(2) \times U(1) \) model for the electron and muon fields is given as an example of the suggested doubling for the fermion states. Besides in this Section is shortly discussed consistency of the present scheme and the purely 5D models [5, 6] of the grand unification theory. The generalized translations in the momentum space as the gauge transformations are considered in Sect 10. The summary is given in Sect. 11.
1. Conformal transformations in the 4D momentum space.

Conformal transformations of the four-momentum $q_\mu (\mu = 0, 1, 2, 3)$ consists of

- **translations**
  $$ q_\mu \rightarrow q'_\mu = q_\mu + h_\mu, \quad (1.1a) $$

- **rotations**
  $$ q_\mu \rightarrow q'_\mu = \Lambda^\mu_\nu q_\nu, \quad (1.1b) $$

- **dilatation**
  $$ q_\mu \rightarrow q'_\mu = e^{\kappa} q_\mu, \quad (1.1c) $$

- **and inversion**
  $$ q_\mu \rightarrow q'_\mu = -M^2 q_\mu / q^2, \quad (1.1d) $$

where $M$ is a mass parameter that insures the correct dimension of $q_\mu$. Translations and inversions form

**special conformal transformation**

$$ q_\mu \rightarrow q'_\mu = \frac{q_\mu - h_\mu q^2 / M^2}{1 - 2 h_\mu q_\mu / M^2 + h_\mu q^2 / M^4} \quad (1.1e) $$

$q_\mu$ in (1.1a)-(1.1e) is off mass shell, i.e. $q_\mu$ is an independent variable and $q_\mu \neq \sqrt{q^2 + m^2}$.

According to the Dirac geometrical model [10], transformations (1.1a)-(1.1e) are equivalent to the rotations on the 6D cone $\kappa^2 \equiv \kappa_A \kappa^A = 0$ (I.2a) with the metric tensor $g_{AB} = \text{diag}(+1, -1, -1, -1, +1, -1)$. In particular, translation (1.1a) and the special conformal transformation are generated by the combinations of the rotations in the planes $(\mu, 5)$ and $(\mu, 6)$, dilatation is obtained via the rotation in the plane $(5, 6)$ and inversion follows from transposition of the of the variables $\kappa'_5 = \kappa_5$ and $\kappa'_6 = -\kappa_6$.

**translation**: $\kappa'_\mu = \kappa_\mu + h_\mu \kappa_+; \quad \kappa'_+ = \kappa_+; \quad \kappa'_- = -\frac{2 h_\mu \kappa^\mu}{\kappa_+} - \frac{\kappa_\mu \kappa^\mu}{\kappa_+} \quad (1.2a)$

**rotation**: $\kappa'_\mu = \Lambda^\mu_\nu \kappa_\nu; \quad \kappa'_+ = \kappa_+; \quad \kappa'_- = \kappa_- \quad (1.2b)$

**dilatation**: $\kappa'_\mu = \kappa_\mu; \quad \kappa'_+ = e^{-\lambda} \kappa_+; \quad \kappa'_- = e^{\lambda} \kappa_- \quad (1.2c)$

**inversion**: $\kappa'_\mu = \kappa_\mu; \quad \kappa'_+ = \kappa_-; \quad \kappa'_- = \kappa_+ \quad (1.2d)$

where

$$ q_\mu = \frac{\kappa_\mu}{\kappa_+}; \quad \kappa_\pm = \kappa_5 \pm \kappa_6; \quad M = \mu = 0, 1, 2, 3. \quad (1.3) $$

The equivalence of the 4D and 6D conformal transformations implies that translation, rotation, dilatation and inversion of the 4D field $\Phi(q)$ (I.9) are unambiguously (isomorphic) determined through the corresponding 6D rotations of the 6D field $\varsigma(\kappa) \equiv \varsigma(\kappa_\mu; \kappa_+, \kappa_-)$

$$ \Phi(q'_\mu = q_\mu + h_\mu) \iff \varsigma(\kappa_\mu = \kappa_\mu + h_\mu \kappa_+, \kappa_+, \kappa_- = \kappa_- = -\frac{2 h_\mu \kappa^\mu}{\kappa_+} - \frac{\kappa_\mu \kappa^\mu}{\kappa_+}) \quad (1.4a) $$

$$ \Phi(q'_\mu = \Lambda^\mu_\nu q_\nu) \iff \varsigma(\kappa'_\mu = \Lambda^\mu_\nu \kappa_\nu, \kappa_+, \kappa_-) \quad (1.4b) $$

$$ \Phi(q'_\mu = \lambda q_\nu) \iff \varsigma(\kappa_\mu = \lambda q_\nu, \kappa_+, \kappa_-) \quad (1.4c) $$

$$ \Phi(q'_\mu = -q_\mu / q^2) \iff \varsigma(\kappa_\mu, \kappa_-, \kappa_+) \quad (1.4d) $$
These relationships between the 4D and 6D operators $\Phi(q)$ and $\varsigma(\kappa)$ is achieved in (1.1b) through $\delta(\kappa, \kappa^4)$.

An interacting scalar field $\Phi(x)$ is usually decomposed in the positive and in the negative frequency parts in the 3D Fock space

$$\Phi(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_p} [a_p(x_0)e^{-ipx} + b_p^+(x_0)e^{ipx}]; \quad p_o \equiv \omega_p = \sqrt{p^2 + m^2}, \quad (1.5)$$

where in the asymptotic regions $a_p(x_0)$ and $b_p^+(x_0)$ transforms into particle (antiparticle) annihilation (creation) operators. On the other hand, $\Phi(x)$ can be decomposed in the 4D momentum space as

$$\Phi(x) = \int \frac{d^4 q}{(2\pi)^4} [\Phi^+(q)e^{-iqx} + \Phi^-(q)e^{iqx}]; \quad \text{or} \quad \Phi(x) = \int \frac{d^4 q}{(2\pi)^4} \Phi(q)e^{-iqx}. \quad (1.6a)$$

where

$$\Phi(q) = \Phi^+(q) + \Phi^-(q) \quad (1.6b)$$

Comparison of (1.5) and (1.6a) gives

$$\frac{e^{-i\omega_p x_0}}{2\omega_p}a_p(x_0) = \int \frac{dq_o}{2\pi} \Phi^+(q_o, p)e^{-iq_o x_0} \quad (1.7a)$$

and

$$\frac{e^{i\omega_p x_0}}{2\omega_p}b_p^+(x_0) = \int \frac{dq_o}{2\pi} \Phi^-(q_o, p)e^{iq_o x_0}. \quad (1.7b)$$

The field operators $a_p(x_0)$ and $b_p^+(x_0)$ are simply determined via the corresponding source operator $\partial a_p(x_0)/\partial x_0 = i \int d^3 x e^{ipx} j(x)$, where $\left(\partial^2/\partial x_\mu \partial x^\mu + m^2\right)\Phi(x) = j(x)$.

Moreover, these operators determine the transition $S$-matrix

$$S_{mn} \equiv \langle \text{out: } p_1', ..., p_m' | p_1, ..., p_n; \text{in} \rangle = \prod_{i=1}^m \left[ \int d^3 x_{i'} \frac{d}{dx_{i'}^0} \right] \prod_{j=1}^n \int d^3 x_j \frac{d}{dx_j^0} < 0 | \mathcal{T} \left( a_{p_{1}'}(x_{1}'), ..., a_{p_{m}'}(x_{m}'), a_{p_{1}}^+(x_{1}^0), ..., a_{p_{n}}^+(x_{n}^0) \right) | 0 >. \quad (1.8)$$

The translation (1.1a) for the 4D field in the momentum space $\Phi'(q) = \Phi(q + h)$ generates the corresponding gauge transformation for $\Phi(x)$. In particular, for the complex charged field the four-momentum translation (1.1a) produces the simplest gauge transformation

$$\Phi'(x) = e^{ih_\mu x^\mu} \Phi(x); \quad i \frac{\partial}{\partial x'_\mu} = i \frac{\partial}{\partial x_\mu} + h_\mu, \quad (1.9)$$

where generally $h_\mu$ is a complex constant.
For the changeless real fields \( \Phi(x) \) the gauge transformation (1.9) is consistently defined for the pure imaginary \( h_\mu = ir_\mu \) with the real \( r_\mu \). In particular, for the real scalar fields we get

\[
\Phi'(x) = e^{-r_\mu x^\mu} \Phi(x); \quad \frac{\partial}{\partial x'_\mu} = \frac{\partial}{\partial x_\mu} + r_\mu, \tag{1.10}
\]

The generalization of the gauge transformations (1.10) for the real scalar fields within the nonlinear \( \sigma \) model was performed in [36, 16]. These formulation we shall consider at end of Sect. 10.

The transformation of \( \Phi(x) \) under the rotation and dilatation of \( q_\mu (1.1b,c) \) can be reproduced through the Fourier transformations in (1.6a) using the inverse rotation and dilatation of \( x_\mu \) in \( \exp(-iqx) \).

The doubling of the 5D fields \( \varphi_\pm(x, x_5) \) in (I.4a,b) and (I.7) is generated by the intermediate projections onto domains of the 5D hyperboloids (I.3a,b). These domains cover unambiguously the whole area of \( q_\mu \) and \( q^2 \) and cover the whole domain of the variables of the fields \( \Phi_\pm(q) \) (I.10). Inversion \( q_\mu' = -M^2q_\mu/q^2 \) transform the domains of the variables of \( \varphi_1(q, q_5^2) \) into the domain of the variables of \( \varphi_1(q, q_5^2) \) and vice versa. But inversion transforms also these fields, i.e. inversion replaces \( \varphi_1(x, x_5) \) (I.4a) and \( \varphi_2(x, x_5) \) (I.4b) with the \( \varphi_2^{(I)} \) and \( \varphi_1^{(I)} \)

\[
\varphi_1 \leftrightarrow \varphi_2^{(I)} \quad \text{i.e.} \quad \varphi_+ \leftrightarrow \varphi_+^{(I)}; \quad \varphi_- \leftrightarrow -\varphi_-^{(I)}, \tag{1.12}
\]

where the upper index \( ^{(I)} \) denotes the inversion of the corresponding operator. One needs to introduce this index because the 4D equation of motion for the massive particles are not invariant under the inversion. For instance, the equation of motion \( (q^2 - m^2)\Phi(q) = j(q) \) after inversion transforms into new type equation \( (M^2 - q^2 m^2/M^2)\phi^{(I)}(q) = q^2 j^{(I)}(q)/M^2 \).

### 2. Domains of \( q^2 \) and \( q_5^2 \)

The invariant form \( \kappa_A \kappa^A = 0 \) (I.2a) of the \( O(2, 4) \) group can be represented for \( q^2 \) as

\[
q^2 + M^2 \frac{\kappa_-}{\kappa_+} = 0, \tag{2.1a}
\]

where

\[
q_\mu = \frac{\kappa_\mu}{\kappa_+}; \quad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M}. \tag{2.1b}
\]

It is convenient to use the auxiliary fifth momentum \( q_5^2 \) instead of \( \kappa_5/\kappa_6 \) in (2.1a). This procedure implies a projection of the 6D rotational invariant form \( \kappa_A \kappa^A = 0 \) into corresponding 5D forms. In order to cover unambiguously the whole domain \(-\infty \leq q^2 \equiv q_\mu q^\mu \leq \infty \) we have distributed \( q^2 \) and corresponding \( q_5^2 \) between the domains of the two 5D hyperboloids

\[
q^2 + q_5^2 = M^2 \quad \text{with} \quad q_5^2 = M^2 \frac{2\kappa_5}{\kappa_5 + \kappa_6}, \tag{2.2a}
\]

and
The hyperboloids (2.2a,b) presents the simple intermediate 5D projection of the 6D cone (I.2a) into the 4D momentum space with only one auxiliary variable $q_5^2$.

$\kappa_+$ and $q_5^2$ in hyperboloids (2.2a,b) are defined in positive. Consequently, $\kappa_5$ on the hyperboloid (2.2a) and $\kappa_6$ in (2.2b) are also positive. The domains of the variables $q^2$, $q_5^2$, $\kappa_5$ and $\kappa_6$ defined on the 6D cone (2.1a) and on the corresponding 5D hyperboloids $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b) are listed in Table 1. The border points $q^2 = 0$, $q^2 = M^2$ and $q^2 = -M^2$ are included in the domain I and IV.

It must be emphasized, that the domain $M^2 < q_5^2 < 2M^2$ of the variables of $\phi(q, q_5^2)$ is excluded by construction of $\psi_{1,2}(x, x_5)$ (I.4a,b). Correspondingly, the variables $\kappa_5$ and $\kappa_6$ cover only the part of the 6D cone $\kappa_5 \kappa_6 = 0$ (I.2a). The principal restriction for the auxiliary variables $\kappa_5$, $\kappa_6$ and $q_5^2$ is that the corresponding $q^2$ must cover the whole domain ($-\infty, +\infty$). This property allows to construct unambiguously the 4D field $\Phi_\pm(x) = \varphi \pm(x, x_5 = 0)$.

Table 1  Domains of $q^2$, $q_5^2$, $\kappa_5$ and $\kappa_6$ placed on the hyperboloids $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b) and on the surface (2.1a).

|    | I         | II        | III       | IV         |
|----|-----------|-----------|-----------|-----------|
| $q^2$ | $q^2 + q_5^2 = M^2$ | $q^2 - q_5^2 = -M^2$ | $q^2 + q_5^2 = M^2$ | $q^2 - q_5^2 = -M^2$ |
| $q_5^2$ | $0 \leq q_5^2 \leq M^2$ | $M^2 < q_5^2 < \infty$ | $-\infty < q^2 < -M^2$ | $-M^2 \leq q^2 < 0$ |
| $\kappa_5$ & $\kappa_6$ | $0 \leq \kappa_5 \leq \kappa_6$ | $\kappa_5 > 0$; $\kappa_5 < 0$; $\kappa_5 + \kappa_6 > 0$ | $\kappa_5 > 0$; $\kappa_6 < 0$; $\kappa_5 + \kappa_6 > 0$ |
|    | $0 \leq \kappa_6 < \kappa_5$ | $0 \leq \kappa_6 < \kappa_5$ | $0 \leq \kappa_6 < \kappa_5$ |

The hyperboloids $q^2 + q_5^2 = M^2$ and $q^2 - q_5^2 = -M^2$ and the corresponding domains I, III and II, IV transforms also into each other by reflection $q^2 \leftrightarrow -q^2$ which is generated by transposition of the variables $\kappa_5$ and $\kappa_6$

$$q^2 \leftrightarrow -q^2 \quad \kappa_5 = \kappa_6, \quad \kappa_5 = \kappa_5; \quad \kappa_+ = \kappa_+, \quad \kappa_- = -\kappa_-.$$ \hspace{1cm} (2.3)

The similar transpositions of the hyperboloids $q^2 + q_5^2 = M^2$ and $q^2 - q_5^2 = -M^2$ produces inversion $q_\mu^I = -M^2 q_\mu / q^2$ (1.1d) which is generated by transposition of the variables $\kappa_+$ and $\kappa_-$ according to (1.2d).

The choice of the 5D hyperboloids is not unique. For instance, instead of the two 5D hyperboloids (2.2a,b) one can take other hyperboloids $q^2 \pm q_5^2 = M^2$ with $q_5^2 = \pm M^2 \kappa_5 / \kappa_5 + \kappa_6$, where $\kappa_+$ is fixed $\kappa_+ = \mathcal{M}/M$ and $\mathcal{M}$ is a mass parameter. But
these domains of \( q^2 \) are not symmetric under the reflection and inversion of \( q^2 \) unlike the domains in Table 1. Therefore we do not consider them.

An other choice of the intermediate 5D projections presents the stereographic projection, where \( \kappa_0 = -1 \), and the auxiliary momenta \( Q_\mu = q_\mu/(1 - q^2) \) and \( Q_4 = (1 + q^2)/(1 - q^2) \), i.e. \( q_\mu = Q_\mu/(1 + Q_4) \). \( q^2 = (Q_4 - 1)/(1 + Q_4) \) are introduced (see, for example, eq. (13.43) in [7]). This choice of the variables require only one 5D hyperboloid \( Q^2 - Q^2_4 = -1 \) with the 5 auxiliary variables \( Q_\mu \) and \( Q_4 \) for the intermediate 5D projections. Certainly, one can represent this projection via the considered projections on the hyperboloids \( q^2 \pm q^2_5 = \pm M^2 \) (2.2a,b) where only one auxiliary variable \( q^2_5 \) is used.

3. 4D reductions of the 5D fields.

The 5D fields \( \varphi_\pm = \varphi_1 \pm \varphi_2 \) (I.7) consist of the two parts \( \varphi_1 \) (I.4a) and \( \varphi_1 \) (I.4b) which satisfy the conditions (I.6), because their Fourier conjugates are defined on the 5D hyperboloids \( q^2 \pm q^2_5 = \pm M^2 \) (I.3a,b). For \( \varphi_\pm \) these conditions can be represented as

\[
\frac{\partial^2 \varphi_\pm(x, x_5)}{\partial x^\mu \partial x_\mu} + \left( \frac{\partial^2}{\partial x^5 \partial x_5} + M^2 \right) \varphi_\pm(x, x_5) = 0. \tag{3.1}
\]

Integration over \( q^2_5 \) in (I.4a,b) yields

\[
\varphi_\pm(x, x_5) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \left[ \phi(q, Q_2^2) \Lambda_1(q^2) e^{-iQ_1 x_5} \pm \phi(q, Q_2^2) \Lambda_2(q^2) e^{-iQ_2 x_5} \right]. \tag{3.2}
\]

The expression (3.2) can be represented via the 5D convolution formula

\[
\varphi_\pm(x, x_5) = \int d^5 y \phi(x - y, x_5 - y_5) \mathcal{P}_\pm(y, y_5), \tag{3.3}
\]

where \( \phi(x, x_5) \) is the Fourier conjugate of the full 5D function \( \phi(q, q^2_5) \) in (I.4a,b) and

\[
\Lambda_a(q^2) = \begin{cases} 
\theta(q^2)\theta(M^2 - q^2) + \theta(-q^2)\theta(-M^2 - q^2) & \text{if } a = 1; \\
\theta(q^2)\theta(-M^2 + q^2) + \theta(-q^2)\theta(M^2 + q^2) & \text{if } a = 2,
\end{cases} \tag{3.4}
\]

\[
Q^2_\alpha = \begin{cases} 
M^2 - q^2 & \text{if } a = 1; \\
M^2 + q^2 & \text{if } a = 2,
\end{cases} \quad \text{and} \quad Q_a = \sqrt{Q^2_\alpha}. \tag{3.5}
\]

The operators

\[
\mathcal{P}_\pm(x, x_5) = \mathcal{P}_1(x, x_5) \pm \mathcal{P}_2(x, x_5) \tag{3.6a}
\]

consist of the two parts that are placed onto hyperboloids \( q^2 + q^2_5 = M^2 \) and \( q^2 - q^2_5 = -M^2 \)

\[
\mathcal{P}_a(x, x_5) = \begin{cases} 
\int dq^2 e^{-iq x_5} d^4 q/(2\pi)^4 e^{-iqx} \Lambda_1(q^2) \delta(q^2 + q^2_5 - M^2) & \text{if } a = 1; \\
\int dq^2 e^{-iq x_5} d^4 q/(2\pi)^4 e^{-iqx} \Lambda_2(q^2) \delta(q^2 - q^2_5 + M^2) & \text{if } a = 2,
\end{cases} \tag{3.6b}
\]
that satisfy the orthogonality and completeness conditions at \( x_5 = 0 \)

\[
\int d^4y \mathcal{P}_a(x - y, 0) \mathcal{P}_b(y, 0) = \delta_{ab} \mathcal{P}_a(x, 0); \tag{3.7}
\]

\[
\mathcal{P}_+(x, 0) = \delta^{(4)}(x) \tag{3.8}
\]

The relation (3.3) presents the projections of the complete 5D field \( \phi(x, x_5) \) onto the two independent 5D fields \( \varphi_{\pm}(x, x_5) \) which for \( x_5 = 0 \) produce the 4D fields \( \Phi_{\pm}(x) \) (I.8). The Fourier conjugate \( \Phi_{\pm}(q) \) (I.9) and the Fourier conjugate of the 5D fields \( \varphi_{\pm} \) (3.3) consist of the same four parts that are given in (I.10) and (I.11). Therefore from (3.3) we get

\[
\Phi_{\pm}(x) = \varphi_{\pm}(x, x_5 = 0) = \int \frac{d^4q}{(2\pi)^4} e^{-iqx} \left[ \phi(q, q_5^2 = M^2 - q^2) \Lambda_1(q^2) \pm \phi(q, q_5^2 = M^2 + q^2) \Lambda_2(q^2) \right], \tag{3.9}
\]

The equations (I.5) and (3.9) presents the 4D reduction of the 6D field \( \varsigma(\kappa, \kappa_-, \kappa_+) \) and 5D field \( \phi(q, q_5^2) \) onto the 4D field \( \Phi_{\pm}(q) \) (I.10).

It must be noted that \( \phi(q, M^2 < q_5^2 < 2M^2) \) does not contribute in the 4D fields \( \Phi_{\pm}(x) \) because the region \( M^2 < q_5^2 < 2M^2 \) is excluded from the domains in Table 1.

(3.3) and (3.8) allow to represent (3.9) m as the 4D convolution formula

\[
\varphi_{\pm}(x, x_5) = \int d^4y \Phi_{\pm}(x - y) \mathcal{P}_+(y, x_5). \tag{3.10}
\]

that determines the two 5D fields \( \varphi_{\pm} \) via the corresponding 4D fields \( \Phi_{\pm} \).

4. 4D and 5D equations of motion.

According to the boundary condition (3.9) and the convolution formula (3.10) the 5D and 4D scalar fields \( \varphi_{\pm} \) and \( \Phi_{\pm} \) satisfy the similar equation of motion

\[
\left( \frac{\partial^2}{\partial x^\mu \partial x_\mu} + m_{\pm}^2 \right) \varphi_{\pm}(x, x_5) = j_{\pm}(x, x_5), \tag{4.1a}
\]

\[
\left( \frac{\partial^2}{\partial x^\mu \partial x_\mu} + m_{\pm}^2 \right) \Phi_{\pm}(x) = J_{\pm}(x) \tag{4.1b}
\]

where

\[
J_{\pm}(x) = j_{\pm}(x, x_5 = 0). \tag{4.2}
\]

The 5D source operators \( j_{\pm}(x, x_5) \) in (4.1a) consist of the products of the 5D fields \( \varphi_{\pm} \) and their derivatives. But the Fourier conjugate of the multiplication of \( \varphi_{\pm}(x, x_5) \) are not placed on the hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \). The source operators \( j_{\pm}(x, x_5) \) in (4.1a) must be placed on the hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) as well as \( \varphi_{\pm}(x, x_5) \) in (3.1). Consequently, \( j_{\pm}(x, x_5) \) must satisfy the additional 5D constrains in analogy with (3.1)
for $\varphi_\pm(x, x_5)$. In order to obtain these conditions for $j_\pm$ we combine the equations (4.1a,b) and the conditions (3.1) as

$$\frac{\partial^2 j_\pm(x, x_5)}{\partial x^\mu \partial x_\mu} + \left(\frac{\partial^2}{\partial x^5 \partial x_5} + M^2\right) j_\pm(x, x_5) =$$

$$\left(\frac{\partial^2}{\partial x_\mu \partial x^\mu} \frac{\partial^2}{\partial x_5 \partial x_5} - \frac{\partial^2}{\partial x_5 \partial x^5} \frac{\partial^2}{\partial x_\mu \partial x_\mu}\right) \varphi_\pm(x, x_5) + \left(m^2_\pm - m^2_\mp\right) \frac{\partial^2 \varphi_\pm(x, x_5)}{\partial x_\mu \partial x^\mu}.$$  \hspace{1cm} (4.3)

For the independent variables $x_\mu$ and $x_5$ the operator $\partial^2 / \partial x_\mu \partial x^\mu$ and $\partial^2 / \partial x_5 \partial x^5$ commute. Therefore we get

$$\frac{\partial^2 \tilde{j}_\pm(x, x_5)}{\partial x^\mu \partial x_\mu} + \left(\frac{\partial^2}{\partial x^5 \partial x_5} + M^2\right) \tilde{j}_\pm(x, x_5) = 0,$$  \hspace{1cm} (4.4a)

where

$$\tilde{j}_\pm(x, x_5) = j_\pm(x, x_5) - m^2_\pm \varphi_\pm(x, x_5) \hspace{1cm} (4.4b)$$

Thus $\tilde{j}_\pm$ satisfy also the same sourceless equation (3.1). Using (4.1a) one can rewrite (4.4a) as

$$\frac{\partial^2}{\partial x^\nu \partial x_\nu} \frac{\partial^2 \varphi_\pm(x, x_5)}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x_\nu \partial x_\mu} \left(\frac{\partial^2}{\partial x^5 \partial x_5} + M^2\right) \varphi_\pm(x, x_5) = 0,$$  \hspace{1cm} (4.4c)

which indicates that the Fourier conjugate of $\partial^2 \varphi_\pm(x, x_5) / \partial x_\mu \partial x_\mu$ and $\tilde{j}_\pm$ in (4.1a) are placed on the hyperboloids $q^2 \pm q_5^2 = \pm M^2$. Therefore, in analogy with (I.4a,b) we get

$$j_1(x, x_5) = \int \frac{d^4q}{(2\pi)^3} d^2 q_5 e^{-iqx - iq^5 x_5} \delta(q^2 + q_5^2 - M^2) \left[\theta(q^2)\theta(M^2 - q^2)\theta(-q^2)\theta(-M^2 - q^2)\right] J(q, q_5^2),$$  \hspace{1cm} (4.5a)

$$j_2(x, x_5) = \int \frac{d^4q}{(2\pi)^3} d^2 q_5 e^{-iqx - iq^5 x_5} \delta(q^2 - q_5^2 + M^2) \left[\theta(q^2)\theta(-M^2 + q^2)\theta(-q^2)\theta(M^2 + q^2)\right] J(q, q_5^2).$$  \hspace{1cm} (4.5b)

where $J$ denote the full 5D source operator

$$J(x, x_5) = \int \frac{d^5q}{(2\pi)^4} e^{-iqx - iq_5 x_5} J(q, q_5^2)$$  \hspace{1cm} (4.5c)

Afterwards the 5D sources $j_\pm$ can be represented as

$$j_\pm(x, x_5) = \int \frac{d^4q e^{-iqx}}{(2\pi)^4} \left[J(q, Q_1^2)\Lambda_1(q^2)e^{-iq_1 x_5} \pm J(q, Q_2^2)\Lambda_2(q^2)e^{-iq_2 x_5}\right],$$  \hspace{1cm} (4.6a)

that in analogy with (3.3) and (3.10) allow to construct the 5D sources $j_\pm(x, x_5)$ through the 4D sources $J_\pm(x)$ or the full 5D sources $J$ using the following convolution formulas.
\[ j_\pm(x, x_5) = \int d^4y J_\pm(x - y) P_\pm(y, x_5), \] (4.6b)

\[ j_\pm(x, x_5) = \int d^5y J(x - y, x_5 - y_5) P_\pm(y, y_5). \] (4.6c)

According to (4.6a) the sources \( J_\pm(x) \) and their Fourier conjugate \( J_\pm(q) \) consist of the four parts as well as the 4D field \( \Phi_\pm(q) \) in (1.10)

\[ J_\pm(q) = \sum_{N=I,III} J_N(q) \pm \sum_{N=II,IV} J_N(q), \] (4.7)

where

\[ J_I(q) = \theta(q^2)\theta(M^2 - q^2)J(q, q_5^2 = M^2 - q^2); \quad J_{III}(q) = \theta(-q^2)\theta(-M^2 - q^2)J(q, q_5^2 = M^2 - q^2); \]

\[ J_{II}(q) = \theta(q^2)\theta(-M^2 + q^2)J(q, q_5^2 = M^2 + q^2); \quad J_{IV}(q) = \theta(-q^2)\theta(M^2 + q^2)J(q, q_5^2 = M^2 + q^2), \] (4.8a)

where the lower index \( N = I, II, III, IV \) in \( J_N \) indicates the domains of \( q^2 \) in Table 1.

In (4.6a) \( j_\pm(x, x_5) \) is constructed through the 4D sources \( J_\pm(q) \) (4.8a,b) that are defined via the 4D fields \( \Phi_\pm \), i.e. (4.6a) allow to determine the 5D source \( j_\pm \) through the 4D fields \( \Phi_\pm \). In (4.6b) \( j_{5\pm}(x, x_5) \) is constructed is constructed via the 5D sources \( J(x, x_5) \) (4.5c) which consists of the products of the complete 5D fields \( \phi(x, x_5) \). The 5D source \( J \) (4.5c) can be determined via the fields \( \phi \) according to the 5D extension of the Klein-Gordon equation

\[ \left( \frac{\partial^2}{\partial x^\mu \partial x_\mu} + \tilde{m}^2 \right) \phi(x, x_5) = J(x, x_5). \] (4.9)

In the present formulation the terms with \( \partial \phi_\pm/\partial x_5 \) are determined through the constrains which ensure the consistency of the condition (3.1) and the equation of motion (4.1a). These constrains are considered in the next section. In (4.9) the projections of \( \partial \phi_\pm/\partial x_5 \) can be included in \( J \). But these terms must be consistent with the constrains for \( \partial \phi/\partial x_5 \).

The projections of the 5D field \( \phi \) with mass \( \tilde{m} \) and source \( J \) into two 5D fields \( \varphi_\pm \) with the two different masses \( m_\pm \) and the two sources \( j_\pm \) are performed in (3.3) and (4.6c) using the projection operators \( P_\pm \) (3.6a). In these projections only the parts of the complete 5D field and its source are used for construction of the 5D fields \( \varphi_\pm \) and \( j_\pm \) because the parts of these fields in the domains \( N = I, II, III, IV \) in Table 1 cover only the part of the full 5D space.

The input 5D fields \( \phi, J \) and their projections \( \varphi_\pm, j_\pm \) can be constructed within the various relativistic invariant time models [8, 9].

It must be noted, that \( \partial \varphi_\pm(x, x_5)/\partial x_5, j_\pm(x, x_5) \) and \( m_\pm^2 \) satisfy an additional conditions that can be obtained combining (4.1a) and (3.1)

\[ M^2 \left[ 1 + \left( \frac{1}{M} \frac{\partial}{\partial x_5} \right)^2 \right] \varphi_\pm = m_\pm^2 \varphi_\pm - j_\pm. \] (4.10)
The conditions (4.10) presents the relationship between $j_\pm, m_\pm$ and the operators $\left[1 + \left(\frac{1}{M} \partial/\partial x_5\right)^2\right] \varphi_\pm$. The factorization of $\left[1 + \left(\frac{1}{M} \partial/\partial x_5\right)^2\right]$ allows to determine $\partial \varphi_\pm/\partial x_5$ through $j_\pm(x, x_5), m_\pm^2$ and vice versa. This problem will be considered in the next section.

5. Constrains for $\partial \varphi_\pm/\partial x^5$.

In order to obtain the constrains for $\partial \varphi_\pm/\partial x^5$ we shall factorize (4.10). The general form of the sought linear conditions for $\partial \varphi_\pm/\partial x^5$ of the scalar charged fields are

$$\frac{i}{M} \frac{\partial \varphi_\pm}{\partial x_5} = \alpha_\pm \varphi_+ + \beta_\pm \varphi_- + C_\pm,$$

(5.1)

where $C_\pm$ consist of the products of $\varphi_\pm$. The Fourier conjugate of these products are not located on the hyperboloids $q^2 \pm q_5^2 = \pm M^2$. Therefore, $C_\pm$ are defined as

$$C_\pm(x, x_5) = \int \tilde{C}_\pm(x - y, x_5 - y_5) d^5 y \mathcal{P}_+(y, y_5).$$

(5.2)

Substituting (5.1) in (4.10) one obtains

$$m_\pm^2 = -2M^2 \alpha_+ \left(1 - \alpha_+^2 / \beta_+\right), \quad j_\pm = M^2 \left[\frac{1 - \alpha_+^2}{\beta_+} C_+ + \alpha_+ C_- + \frac{i}{M} \partial C_- / \partial x_5\right];$$

(5.3a)

$$m_\pm^2 = -2M^2 \alpha_+ \beta_+; \quad j_\pm = M^2 \left[\alpha_+ C_+ + \beta_+ C_- + \frac{i}{M} \partial C_+ / \partial x_5\right].$$

(5.3b)

where are using the conditions

$$\alpha_+^2 + \alpha_- \beta_+ = 1; \quad \beta_+^2 + \alpha_- \beta_+ = -1$$

(5.4)

that insures reproduction of the mass terms in (4.10).

From (5.4) follows that $\alpha_+ = \beta_-$ and the condition $m_\pm^2 > 0$ requires that $-1 < \alpha_+ < 1$. If $\alpha_+ = -\beta_-$, then $m_+ = m_- = 0$.

The relations (5.3a,b) determine $m_\pm^2$ through the three parameters $M^2, \alpha_+$ and $\beta_+$. The same parameters, $C_\pm$ and $i/M \partial C_\pm / \partial x_5$ determine the sources $j_\pm$.

For the neutral particles the analogue of (5.1) is

$$\frac{1}{M} \frac{\partial \varphi_\pm}{\partial x_5} = \alpha_\pm \varphi_+ + \beta_\pm \varphi_- + C_\pm,$$

(5.5)

Substituting this condition in (4.10) we get

$$m_\pm^2 = -2M^2 \alpha_+ \left(1 + \alpha_+^2 / \beta_+\right), \quad j_\pm = M^2 \left[\frac{1 + \alpha_+^2}{\beta_+} C_+ + \alpha_+ C_- + \frac{1}{M} \partial C_- / \partial x_5\right];$$

(5.6a)

$$m_\pm^2 = 2M^2 \alpha_+ \beta_+; \quad j_\pm = M^2 \left[\alpha_+ C_+ + \beta_+ C_- + \frac{1}{M} \partial C_+ / \partial x_5\right],$$

(5.6b)
where the following conditions for the parameters are used

\[ \alpha_+^2 + \alpha_- \beta_+ = -1; \quad \beta_+^2 + \alpha_- \beta_+ = -1. \] (5.6c)

The ratio \( m_+^2/m^-^2 = -(\alpha_+^2 + 1)/\beta_+^2 \) in this case is negative. Therefore, the constrains (5.5) can not generate the positive \( m_+^2 \) and \( m^-^2 \) simultaneously for the neutral fields.

The positive masses \( m_\pm^2 \geq 0 \) for the neutral particles in (4.10) can be reproduced using the other kind of the constrains

\[ \varphi_- = \alpha \varphi_+ + G(\varphi_+, \varphi_-), \] (5.7a)

where \( G \) does not contain the linear over \( \varphi_\pm \) terms. For the sake of simplicity the dependence over \( \partial \varphi_\pm / \partial x_\mu \) and \( \partial \varphi_\pm / \partial x_5 \) in \( G \) (5.7a) are omitted.

Acting with \( \left[ 1 + \left( \frac{1}{M} \partial / \partial x_5 \right)^2 \right] \) in (5.7a) and using the conditions

\[ m_+^2 = \alpha^2 m_-^2; \quad F = M^{-2} \left[ 1 + \alpha \frac{m_-^2}{M^2} + \left( \frac{1}{M} \frac{\partial}{\partial x_5} \right)^2 \right]^{-1} G \] (5.7b)

one determine \( j_- \) via \( j_+ \) as

\[ \alpha j_+ - j_- = F. \] (5.7c)

In particular, for the \( \Phi_\pm^4 \) model with the 4D source \( J_+ = a\Phi_+^2 + b\Phi_3^3 \) and mass \( m_+ \) the relations (5.7a,b,c) allow to reproduce the 5D equations (4.1a) with the 5D source operator \( j_\pm \) and the masses \( m_\pm^2 \). These 5D equation generates the 4D equation of motion with \( J_- = (J_+ + F) / \alpha \) and the real mass \( m_- = m_+ / \alpha \).

For the charged fields it is convenient to use the following constrains

\[ \left[ 1 - \frac{i}{M} \frac{\partial}{\partial x_5} \right] \varphi_\pm = C_\pm. \] (5.8a)

where \( C_\pm \) depends generally on the products of \( \varphi_\pm^4 \) and \( \varphi_\pm \).

\[ M^2 \left[ 1 + \frac{i}{M} \frac{\partial}{\partial x_5} \right] C_\pm = m_\pm^2 \varphi_\pm - j_\pm. \] (5.8b)

Present formulation has in common with the other 5D field-theoretical approaches within the relativistic invariant time method [7, 8, 9] with \( x_5^2 = x_0^2 - x^2 \equiv x_\mu x^\mu \). Unlike these 5D formulations our approach based on the invariance of the 6D and 5D forms (I.2a,b) and (I.3a,b) under the conformal transformations in the momentum space. The main difference between the our conditions for \( \partial \varphi_\pm / \partial x_5 \) (4.10), (5.1) and (5.7c) and the evaluation-type equations over the \( x_5 \) other authors is that in the present formulation \( \partial \varphi_+/ \partial x_5 \) is determined through the source and mass of the coupling field \( j_- \) and \( m_- \). Nevertheless, one can use the 5D models in [7, 8, 9] for input 5D field \( \phi(q,q_5^2) \) in (1.4a,b) and input 5D and 4D source operators \( J_\pm(x) = j_\pm(x,x_5 = 0) \) in (4.1,b).

6. The 5D and 4D Lagrangians.
In this section we shall consider the 5D Lagrangians $L_{\pm}(x, x_5)$ for the two interacted scalar 5D fields $\varphi_{\pm}$ with the same quantum numbers, but with the different masses $m_{\pm}$ and sources $j_{\pm}(x, x_5)$. These Lagrangians must reproduce the 4D Lagrangians $L_{\pm}(x)$ at $x_5 = 0$ and the 5D equations of motion (4.1a). The sought Lagrangian can be represented as

$$L_{\pm}(x, x_5) = (L_{\pm})_o(x, x_5) + (L_{\pm})_{\text{INT}}(x, x_5) + (L_{\pm})^{C}(x, x_5)$$

(6.1)

where $(L_{\pm})_o$ and $(L_{\pm})_{\text{INT}}$ denote the non-interacted and interacted parts of these Lagrangians. The third term $(L_{\pm})^{C}$ reproduce the constrains for the $\partial \varphi_{\pm}/\partial x_5$ in (5.1), (5.5), (5.7a) and (5.8a,b). Any one of these constrains together with the 5D equation of motion (4.1a) generate the conditions (3.1). In particular, $(L_{\pm})^{C}$ for the constrains (5.8a,b) are

$$L_{\pm}^{C} = M^2\left| \frac{i}{M} \frac{\partial \varphi_{\pm}}{\partial x_5} - \varphi_{\pm} + C_{\pm} \right|^2 + M^2\left| i \frac{\partial C_{\pm}}{M \partial x_5} + C_{\pm} - \frac{m_{\pm}^2 \varphi_{\pm} - j_{\pm}}{M^2} \right|^2,$$

(6.2)

where $\delta L_{\pm}^{C}/\delta C_{\pm}$ and $\delta L_{\pm}^{C}/\delta (\partial C_{\pm}/\partial x_5)$ reproduce (5.8a) and (5.8b) correspondingly.

The 5D Lagrangians (6.1) can be simply constructed starting from the two 4D Lagrangians $L_{\pm}(x)$ for the two interacted scalar fields $\Phi_{\pm}(x)$

$$L_{\pm}(x) = (L_{\pm})_o(x) + (L_{\pm})_{\text{INT}}(x),$$

(6.3)

where the non-interacting part $(L_{\pm})_o$

$$(L_{\pm})_o(x) = \frac{\partial \Phi_{\pm}^+}{\partial x_\mu} \frac{\partial \Phi_{\pm}}{\partial x_\mu} - m_{\pm}^2 \Phi_{\pm}^+ \Phi_{\pm}$$

(6.4a)

determines the non-interacted part of the 5D Lagrangians (6.1)

$$(L_{\pm})_o(x, x_5) = \frac{\partial \varphi_{\pm}^+}{\partial x_\mu} \frac{\partial \varphi_{\pm}}{\partial x_\mu} - m_{\pm}^2 \varphi_{\pm}^+ \varphi_{\pm}.$$  

(6.4b)

The 5D source $j_{\pm}$ in (4.1a) can be constructed from the 4D source $J_{\pm}(4.1b)$ using the convolution formula (4.6b). The sought 5D Lagrangian $(L_{\pm})_{\text{INT}}$ consists of the fields $\varphi_{\pm}$. The replacement of the 4D fields $\Phi_{\pm}$ in $(L_{\pm})_{\text{INT}}$ with the 5D fields $\varphi_{\pm}$ give

$$(L_{\pm})_{\text{INT}}(x, x_5) = \int d^4y (L_{\pm})_{\text{INT}}(x-y, x_5) P_{\pm}(y, x_5), \text{ where } (L_{\pm})_{\text{INT}}(x, x_5) = (L_{\pm})_{\text{INT}}(\Phi_{\pm} \leftrightarrow \varphi_{\pm}).$$

(6.5)

The 5D sources $j_{\pm}$ satisfy the Euler-Lagrange equations

$$j_{\pm}(x, x_5) = \frac{\partial (L_{\pm})_{\text{INT}}}{\partial \varphi_{\pm}^+} - \frac{d}{dx_\mu} \left( \frac{\partial (L_{\pm})_{\text{INT}}}{\partial (\partial \varphi_{\pm}^+/\partial x_\mu)} \right).$$

(6.6)

The other kind of the 5D sources $j_{\pm}$ can be obtained from the general 5D Lagrangian $\ell(\phi, \phi^+)$ for the 5D scalar field $\phi(q, q_5^2)$ that parts are used in (I.4a,b) and (3.2) for reproduction of $\varphi_{\pm}$.
\[
\ell(\phi, \phi^+) = \frac{\partial \phi^+}{\partial x^\mu} \frac{\partial \phi}{\partial x^\mu} + \tilde{m}^2 \phi^+ \phi + \ell_{\text{INT}}(\phi, \phi^+),
\]

where the terms with \(\partial \phi/\partial x^5\) are included in \(\ell_{\text{INT}}\). The Lagrangian (6.7) generate the 5D equation of motion (4.9). In order to reproduce \(j_\pm\) from \(\ell(\phi, \phi^+)\) (6.7) one must determine \(\partial \varphi/\partial x^5\) according to constrains for \(\partial \varphi/\partial x^5\) that are given in (\(L_\pm\))^C (6.1). For this aim we shall consider the projection of the equation of motion (4.9) according to (3.3) and (4.6c)

\[
\int d^5 y \left[ \frac{\partial^2 \phi(x-y, x^5 - y^5)}{\partial(x-y)^\mu} \delta (x-y) \right] \psi_\pm(y, y^5) = 0 \quad (6.8a)
\]

This equation has the form of the equation (4.1a) if

\[
\int d^5 y \int \mathcal{J}(x-y, x^5 - y^5) \psi_\pm(y, y^5) = j_\pm(x, x^5) - \left( m^2_\pm - \tilde{m}^2 \right) \varphi_\pm(x, x^5) \quad (6.8b)
\]

The sources \(j_\pm\) in (6.8a,b) are defined in (4.6c) via the projections of \(\mathcal{J}\). The complete 5D source \(\mathcal{J}\) consists of the products of the fields \(\phi\) and their derivatives. Therefore, \(\mathcal{J}\) can not be reduced to the combination of the fields \(\varphi_\pm\) that are the parts of \(\phi(x, x^5)\). Thus in opposite to \(j_\pm\) constructed from the Lagrangian (6.5), the projections of \(\mathcal{J}\) in (6.8a,b) does not satisfy the equation (6.6) and the Lagrangian \(\ell(\phi, \phi^+)\) (6.7) can not be reduced to the Lagrangians (6.1) \((L_\pm)^C\). The consistency conditions of the projections of \(\mathcal{J}\) in (6.8a,b) in the equation of motion (4.1a) and the condition (3.1) can be obtained using the constrains for \(\partial \varphi/\partial x^5\) considered in the previous section.

The third form of the 5D Lagrangians (6.1) reproduce exactly the constrains (3.1) and (5.8a,b) with the a priory given source \(j_\pm\)

\[
\mathcal{L}(x, x^5) = \frac{\partial \varphi_+^+}{\partial x^\mu} \frac{\partial \varphi_+}{\partial x^\mu} + M^2 \varphi_+^+ \varphi_+ + (\mathcal{L}_+)^C + \frac{\partial \varphi_-^+}{\partial x^\mu} \frac{\partial \varphi_-}{\partial x^\mu} + M^2 \varphi_-^+ \varphi_- + (\mathcal{L}_-)^C \quad (6.11)
\]

where the Lagrangians \((L_\pm)^C\) generate the conditions (5.8a,b) and variation over \(\varphi_\pm^+\) and \(\partial \varphi_\pm^+/\partial x^\mu\) produces (3.1). The combination of (3.1) and (5.8a,b) gives the equation of motion (4.1a).

7. The 4D and 5D equation of motion for the fermion fields with the electromagnetic interactions.

The 4D and 5D equation of motion for the two fermion fields \(\Psi_+(x) = \psi_+(x, x^5 = 0)\) and \(\Psi_-(x) = \psi_-(x, x^5 = 0)\) with the same quantum numbers but with the different masses \(m_\pm\) and source operators can be represented in analogy with the scalar fields. In order to place the Fourier conjugate of the 5D Dirac fields \(\psi_{1,2}(x, x^5) = 1/2 \left( \psi_\pm(x, x^5) \pm \psi_\mp(x, x^5) \right)\)
on the hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) the fields \( \psi_+ (x, x_5) \) and \( \psi_- (x, x_5) \) must satisfy the condition (3.1)

\[
\frac{\partial^2 \psi_\pm (x, x_5)}{\partial x^\mu \partial x_\mu} + \left( \frac{\partial^2}{\partial x^5 \partial x_5} + M^2 \right) \psi_\pm (x, x_5) = 0.
\] (7.1)

Similarly with \( \varphi_\pm \) in (1.10), we divide the Fourier conjugate of \( \psi_\pm (x, x_5) \) into four parts defined in the domains I,II,III and IV in Table 1

\[
\psi_I (q, q_5^2) = \theta(q^2)\theta(M^2-q_5^2)\Upsilon(q, q_5^2); \quad \psi_{II} (q, q_5^2) = \theta(q^2)\theta(-M^2+q_5^2)\Upsilon(q, q_5^2); \\
\psi_{III} (q, q_5^2) = \theta(-q^2)\theta(-M^2-q_5^2)\Upsilon(q, q_5^2); \quad \psi_{IV} (q, q_5^2) = \theta(-q^2)\theta(M^2+q_5^2)\Upsilon(q, q_5^2),
\] (7.2)

Then as in (3.2) for \( \varphi_\pm (x, x_5) \), for \( \psi_\pm (x, x_5) \) we have

\[
\psi_\pm (x, x_5) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \left[ \sum_{N=I,II,III} \psi_N (q, Q_1^2) e^{-iQ_1 x_5} \pm \sum_{N=II,IV} \psi_N (q, Q_2^2) e^{-iQ_2 x_5} \right],
\] (7.3)

and \( \psi_\pm (x, x_5) \) satisfy automatically the condition (7.1).

For \( x_5 = 0 \) \( \psi_\pm (7.3) \) generate the physical 4D fields \( \Psi_\pm (x) \)

\[
\Psi_\pm (x) = \psi_\pm (x, x_5 = 0) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \left[ \sum_{N=I,II,III} \psi_N (q) \pm \sum_{N=II,IV} \psi_N (q) \right],
\] (7.4)

where

\[
\Psi_I (q) = \psi_I (q, q_5^2 = M^2 - q_5^2); \quad \Psi_{II} (q) = \psi_{II} (q, q_5^2 = M^2 + q_5^2); \\
\Psi_{III} (q) = \psi_{III} (q, q_5^2 = M^2 + q_5^2); \quad \Psi_{IV} (q) = \psi_{IV} (q, q_5^2 = M^2 - q_5^2)
\] (7.5a)

and

\[
\Psi_\pm (q) = \sum_{N=I,II,III} \psi_N (q) \pm \sum_{N=II,IV} \psi_N (q).
\] (7.5b)

According to (7.4) and (7.5a,b) the Fourier conjugate of the 5D fields \( \psi_\pm (x, x_5) \) (7.3) contains the same four parts as the Fourier conjugate of the 4D fields \( \Psi_\pm (x) \). Therefore, one can represent the \( \psi_\pm (x, x_5) \) through \( \Psi (x) \) as

\[
\psi_\pm (x, x_5) = \int d^4 y \Psi_\pm (x - y) \mathcal{P}_+(y, x_5).
\] (7.6)

The convolution formula (7.6) is similar to (3.10) for the scalar fields \( \varphi_\pm (x, x_5) \).

The equation of motion for the 5D fields \( \psi_\pm \) can be derived using the gauge transformation of the 5D Dirac equations for the two non-interacting fields \( \psi^{(0)}_\pm \) with the different masses \( m_\pm \)
\begin{equation}
(i\gamma^\mu \frac{\partial}{\partial x_\mu} - m_\pm)\psi_\pm^{(o)}(x, x_5) = 0, \quad (7.7)
\end{equation}

where

\begin{equation}
\psi_\pm^{(o)}(x, x_5) = \int d^4 y \exp \left(i\epsilon_\Lambda(x - y, x_5)\right)\psi_\pm(x - y, x_5)\mathcal{P}_+(y, x_5); \quad (7.8a)
\end{equation}

\begin{equation}
ie A^\mu(x, x_5) = \exp \left(-i\epsilon_\Lambda(x, x_5)\right)\frac{\partial}{\partial x_\mu}\exp \left(i\epsilon_\Lambda(x, x_5)\right). \quad (7.8b)
\end{equation}

Substituting (7.8a,b) into (7.7) we obtain

\begin{equation}
\int d^4 y e^{i\epsilon_\Lambda(x - y, x_5)} \left(i\gamma^\mu \frac{\partial}{\partial x_\mu} - e\gamma^\mu A^\mu(x - y, x_5) - m_\pm\right)\psi_\pm(x - y, x_5)\mathcal{P}_+(y, x_5) = 0. \quad (7.9)
\end{equation}

According to (3.8) the projection operator \(\mathcal{P}_+(y, x_5)\) for \(x_5 = 0\) transforms into \(\delta^4(y)\). Therefore, for \(x_5 = 0\) the 5D non-local equations (7.9) generate the usual 4D Dirac equation for the fermion field with the electromagnetic interaction

\begin{equation}
\left(i\gamma^\mu \frac{\partial}{\partial x_\mu} - e\gamma^\mu A^\mu(x) - m_\pm\right)\Psi_\pm(x) = 0. \quad (7.10)
\end{equation}

where \(A^\mu(x) = A^\mu(x, x_5 = 0)\), \(\Psi_\pm^{(o)}(x) = \psi_\pm^{(o)}(x, x_5 = 0)\) are the asymptotic "in" or "out" fields which satisfy the equations

\begin{equation}
\left(i\gamma^\mu \frac{\partial}{\partial x_\mu} - m_\pm\right)\Psi_\pm^{(o)}(x) = 0; \quad (7.11a)
\end{equation}

\begin{equation}
\Psi_\pm^{(o)}(x) = \exp \left(i\epsilon_\Lambda(x, 0)\right)\Psi_\pm(x); \quad ie A^\mu(x) = \exp \left(-i\epsilon_\Lambda(x, 0)\right)\frac{\partial}{\partial x_\mu}\exp \left(i\epsilon_\Lambda(x, 0)\right). \quad (7.11b)
\end{equation}

The Fourier conjugate of the 5D gauge fields \(A^\mu(x, x_5)\) are embedded on the hyperboloids \(q^2 + q_5^2 = \pm M^2\) and they satisfy the condition (3.1) As well as the scalar fields \(\varphi_\pm\). Due to gauge invariance one can use any combination of \(A^\mu_+\) and \(A^\mu_-\) in the gauge transformations (7.8b). Therefore, in (7.8a,b)-(7.11a,b) and in the following formulas the lower indexes \(\pm\) of \(\Lambda, A^\mu\) and \(A^\mu\) are omitted.

The 5D representations of the interacted and non-interacted 5D fields \(\psi_\pm\) and \(\psi_\pm^{(o)}\) are single-valued determined through the their 4D reductions \(\Psi_\pm\) and \(\Psi_\pm^{(o)}\) according to (7.6). Therefore \(\psi_\pm^{(o)}\) satisfies also the condition (7.1)

\begin{equation}
\frac{\partial^2 \psi_\pm^{(o)}(x, x_5)}{\partial x_\mu \partial x_\nu} + \left(\frac{\partial^2}{\partial x^\nu \partial x_5} + M^2\right)\psi_\pm^{(o)}(x, x_5) = 0. \quad (7.12)
\end{equation}

The gauge transformation (7.8a) in (7.12) together with (7.1) yields

\[20\]
\[ \eta_\pm(x, x_5) = -\eta_\pm^{(5)}(x, x_5), \] (7.13)

where

\[ \eta_\pm(x, x_5) = \left( ie \frac{\partial}{\partial x^\mu} A^\mu(x, x_5) + ie A^\mu(x, x_5) \frac{\partial}{\partial x^\mu} - e^2 A^\mu(x, x_5) A_\mu(x, x_5) \right) \psi_\pm(x, x_5), \] (7.14a)

and the auxiliary source operator \( \eta_\pm^{(5)} \) is defined via \( A^5 \) as

\[ \eta_\pm^{(5)}(x, x_5) = \left( ie \frac{\partial}{\partial x^5} A^5(x, x_5) + ie A^5(x, x_5) \frac{\partial}{\partial x^5} - e^2 A^5(x, x_5) A_5(x, x_5) \right) \psi_\pm(x, x_5), \] (7.14b)

\[ ie A^5(x, x_5) = \exp \left( -ie \Lambda(x, x_5) \right) \frac{\partial}{\partial x^5} \exp \left( ie \Lambda(x, x_5) \right). \] (7.14c)

Condition (7.13) allows to define \( A^5 \) through \( \eta_\pm \) and \( \psi_\pm \).

The complete 5D fields \( \Upsilon(x, x_5) \) and their Fourier conjugate \( \Upsilon(q, q_2^5) \) in (7.2) can be defined in the relativistic invariant time models \([8, 9]\). The projections of \( \Upsilon(x, x_5) \) and \( \Upsilon^{(o)}(x, x_5) \) onto hyperboloids \( q^2 \pm q_2^{(5)} = \pm M^2 \) can be determined also via the 5D convolution formulas

\[ \psi_\pm(x, x_5) = \int d^5 y \Upsilon(y, y_5) \mathcal{P}_\pm(y, y_5); \quad \psi^{(o)}_\pm(x, x_5) = \int d^5 y \Upsilon^{(o)}(y, y_5) \mathcal{P}_\pm(y, y_5), \] (7.15)

8. Constrains for \( \partial \psi_\pm / \partial x_5 \).

The consistency condition between the 5D Dirac equations (7.9) and the constrains (7.1) can be represent through the linear equations for \( \partial \psi_\pm / \partial x_5 \). For this aim it is convenient to rewrite the (7.9) in the form of the Klein-Gordon equations. Action of the operator \( i\gamma_\mu \partial / \partial x_\mu + m_\pm \) on (7.7) produces the following Klein-Gordon equations

\[ \left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} + m_\pm^2 \right) \psi^{(o)}_\pm(x, x_5) = 0. \] (8.1)

Then the gauge transformation (7.8a) yields

\[ \int d^4 y e^{ie \Lambda(x-y, x_5)} \left[ \left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} - m_\pm^2 \right) \psi_\pm(x-y, x_5) - \eta_\pm(x-y, x_5) \right] \mathcal{P}_+(y, x_5) = 0, \] (8.2)

where \( \eta_\pm(x, x_5) \) is defined in (7.14a).

One can replace (7.1) with
\[ \int d^4 y e^{i\lambda(x-y, x_5)} \left[ \frac{\partial^2 \psi_+ (x-y, x_5)}{\partial x_\mu \partial x^\mu} + \left( M^2 + \frac{\partial^2}{\partial x_5 \partial x^5} \right) \psi_+ (x-y, x_5) \right] \mathcal{P}_+(y, x_5) = 0. \quad (8.3) \]

Afterwards we obtain the following consistency condition of (8.3) and (8.2) for the 5D Dirac fields

\[ \int d^4 y e^{i\lambda(x-y, x_5)} \left\{ \left[ M^2 + \left( \frac{\partial}{\partial x_5} \right)^2 \right] \psi_\pm (x-y, x_5) - m_\pm^2 \psi_\pm (x-y, x_5) + \eta_\pm (x-y, x_5) \right\} \mathcal{P}_+(y, x_5) = 0. \quad (8.4) \]

According to (8.4) \[ \left[ M^2 + \left( \frac{\partial}{\partial x_5} \right)^2 \right] \psi_+ \] is determined by the source and mass of the \( \psi_- \) fields in (8.2) and vice versa. Unlike (4.10) the condition (8.4) is given in the form of the 4D convolution formula.

One can factorise (8.4) in the same way as (4.10) using the constrain for \( \partial \psi_\pm / \partial x_5 \).

In particular,

\[ i \frac{\partial}{M \partial x_5} \psi_\pm = \alpha_\pm \psi_+ + \beta_\pm \psi_- + C_\pm, \quad (8.5) \]

where \( C_\pm \) are defined onto hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) according to (5.2) and \( C_\pm \) does not contain the linear terms over \( \psi_\pm \).

The constrains (8.5) allow to construct the relationship between the masses \( m_\pm^2 \), sources \( j_\pm \) and the parameters \( M^2, \alpha_+, \beta_+ \) and \( C_\pm \) in the same way as in (5.3a,b). In particular, the choice of the parameters according to (5.4) gives

\[ \frac{m_+^2}{m_-^2} = \frac{1 - \alpha_+^2}{\beta_+^2}; \quad \frac{m_+^2 m_-^2}{4M^4} = \alpha_+^2 (1 - \alpha_-^2); \quad \alpha_+^2 = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{m_+^2 m_-^2}{M^4}} \quad (8.6) \]

The ratio \( m_+^2 / m_-^2 \) and the product \( m_+^2 m_-^2 / 4M^4 \) in (8.6) are positive if

\[ 0 < \alpha_+^2 < 1; \quad \frac{m_+^2 m_-^2}{M^4} \leq 1. \quad (8.7) \]

Therefore, the constrain (8.5) generate the positive masses \( m_+^2 \) and \( m_-^2 \) simultaneously if \( M^2 \geq m_+ m_- \).

9. 5D extension of the standard \( SU(2) \times U(1) \) model for the electron and muon.

As example of unification of the 4D fields with the same quantum numbers and different masses and sources, we shall consider the relationship between the 4D electron and muon fields \( \Psi_+ \equiv \Psi_{el} \) and \( \Psi_- \equiv \Psi_{muon} \), \( \Psi_+ \equiv \Psi_{el} \) and \( \Psi_- \equiv \Psi_{muon} \) according to (7.3)-(7.6). In the standard Weinberg-Salam \( SU(2) \times U(1) \) model \[35\] we have the following 4D equations of motion
\[ (i\gamma_{\mu}\frac{\partial}{\partial x_{\mu}} - m_{el}) \psi_{el} = \mathcal{J}_{el}; \quad (i\gamma_{\mu}\frac{\partial}{\partial x_{\mu}} - m_{muon}) \psi_{muon} = \mathcal{J}_{muon}; \quad (9.1a) \]

\[ \mathcal{J}_{el} = (-e\gamma_{\mu}A_{\mu} + \frac{g^2 - g'^2}{2\sqrt{g^2 + g'^2}} \gamma_{\mu}Z_{\mu} \frac{1 + \gamma_{5}}{2} + g'\gamma_{\mu}Z_{\mu} \frac{1 - \gamma_{5}}{2}) \psi_{el} + \frac{g}{\sqrt{2}} \gamma_{\mu}W_{\mu} \frac{1 + \gamma_{5}}{2} \nu_{el}; \quad (9.1b) \]

\[ \mathcal{J}_{muon} = (-e\gamma_{\mu}A_{\mu} + \frac{g^2 - g'^2}{2\sqrt{g^2 + g'^2}} \gamma_{\mu}Z_{\mu} \frac{1 + \gamma_{5}}{2} + g'\gamma_{\mu}Z_{\mu} \frac{1 - \gamma_{5}}{2}) \psi_{muon} + \frac{g}{\sqrt{2}} \gamma_{\mu}W_{\mu} \frac{1 + \gamma_{5}}{2} \nu_{muon}; \quad (9.1c) \]

where \( \nu_{el} \) and \( \nu_{muon} \) denote the corresponding neutrino fields, \( W_{\mu} \) and \( Z_{\mu} \) stands for the charged and neutral vector fields, \( A_{\mu} \) is the photon field

\[ g = -\frac{e}{\sin\theta}; \quad g' = -\frac{e}{\cos\theta}; \quad \sin^2\theta = 0.222 \pm 0.011 \quad (9.2) \]

For derivation of (9.1a,b,c) was used the 4D gauge transformation of the neutrino-electron and neutrino-muon doublets according to (21.3.12)-(21.3.20) in [35].

\[ \sum_{\alpha} A_{\alpha}^{\mu}(x) t_{\alpha} + yB_{\mu} = \exp(\Lambda(x)) \frac{\partial}{\partial x_{\mu}} \exp(\Lambda(x)). \quad (9.1d) \]

\[ W_{\mu} = \frac{1}{\sqrt{2}} (A_{1}^{\mu} + iA_{2}^{\mu}); \quad Z_{\mu} = \cos\theta A_{3}^{\mu} + \sin\theta B_{\mu}; \quad A_{\mu} = -\sin\theta A_{3}^{\mu} + \cos\theta B_{\mu}; \quad (9.1e) \]

The direct 5D extension of the 4D Dirac equations (9.1a) can be obtained in the same way as (7.9) using the 4D gauge transformations (7.8a,b) with the common parameter \( x_{5} \)

\[ \int d^{4}ye^{i\Lambda(x-y,x_{5})} \left[ (i\gamma_{\mu}\frac{\partial}{\partial x_{\mu}} - m_{el}) \psi_{el}(x - y, x_{5}) - \bar{\eta}_{el}(x - y, x_{5}) \right] \mathcal{P}_{+}(y, x_{5}) = 0. \quad (9.3a) \]

\[ \int d^{4}ye^{i\Lambda(x-y,x_{5})} \left[ (i\gamma_{\mu}\frac{\partial}{\partial x_{\mu}} - m_{muon}) \psi_{muon}(x - y, x_{5}) - \bar{\eta}_{muon}(x - y, x_{5}) \right] \mathcal{P}_{+}(y, x_{5}) = 0, \quad (9.3b) \]

where \( \Lambda(x, x_{5} = 0) = \Lambda(x) \) For \( x_{5} = 0 \) the 5D fields and sources in (9.3a,b) are reduced into the 4D fields and sources

\[ \psi_{el}(x) = \psi_{el}(x, x_{5} = 0); \quad \psi_{muon}(x) = \psi_{muon}(x, x_{5} = 0); \quad (9.3c) \]

\[ \bar{\eta}_{el}(x, x_{5} = 0) = \mathcal{J}_{el}(x); \quad \bar{\eta}_{muon}(x, x_{5} = 0) = \mathcal{J}_{muon}(x), \quad (9.3d) \]

where
\[ \eta_{\text{el}}(x, x_5) = \frac{-e \gamma_\mu A^\mu + g^2 - g'^2}{2 \sqrt{g^2 + g'^2}} \gamma_\mu Z^\mu \frac{1 + \gamma_5}{2} \psi_{\text{el}} + \frac{g}{\sqrt{2}} \gamma_\mu W^\mu \frac{1 + \gamma_5}{2} \nu_{\text{el}}, \]

\[ \eta_{\text{muon}}(x, x_5) = \frac{-e \gamma_\mu A^\mu + g^2 - g'^2}{2 \sqrt{g^2 + g'^2}} \gamma_\mu Z^\mu \frac{1 + \gamma_5}{2} + g' \gamma_\mu Z^\mu \frac{1 - \gamma_5}{2} \psi_{\text{muon}} + \frac{g}{\sqrt{2}} \gamma_\mu W^\mu \frac{1 + \gamma_5}{2} \nu_{\text{muon}}, \]

(9.4a)

(9.4b)

It is easy to check, that for \( x_5 = 0 \) equation (9.3a,b) transforms into (9.1a) due to \( P_+(x, x_5 = 0) = \delta(x) \).

In order to place the Fourier conjugate of the 5D sources (9.4a,b) onto hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) we shall use the following convolution formula

\[ \eta_{\text{el}}(x, x_5) = \int d^4 y \bar{\eta}_{\text{el}}(x - y, x_5) P_+(y, x_5); \quad \eta_{\text{muon}}(x, x_5) = \int d^4 y \bar{\eta}_{\text{muon}}(x - y, x_5) P_+(y, x_5). \]

(9.5a)

The exact form of \( P_+ \) (3.6a,b) allows to rewrite (9.5a) as

\[ \psi_{\text{el}} \text{ and } \psi_{\text{muon}} \text{ satisfy the condition } (7.1). \]

Therefore, there are two different ways for construction of the 5D fields \( \psi_{\text{el}} \) and \( \psi_{\text{muon}} \):

1. If the electron and muon fields are completely independent, then the constrain (7.1) can be reproduced through doubling of the electron and muon fields separately, i.e. through \( (\psi_{\text{el}})_\pm \) and \( (\psi_{\text{muon}})_\pm \), where \( (\psi_{\text{el}})_- \) and \( (\psi_{\text{muon}})_- \) corresponds to the “electron” and “muon” with the negative or imaginary mass. This doubling of the electron and muon states can be realized via the appropriate constrains (8.5) and the equations of motion (7.9).

2. If the 5D fields \( \psi_{\text{el}} \) and \( \psi_{\text{muon}} \) consists of the same parts, then in (7.1) and (7.9) \( \psi_{\text{el}} \equiv \psi_+ \) and \( \psi_{\text{muon}} \equiv \psi_- \) and for their 4D reductions we have

\[ \Psi_{\text{el}}(x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \left[ \sum_{N=I,III} \Psi_N(q) + \sum_{N=II,IV} \Psi_N(q) \right], \]

(9.6a)

\[ \Psi_{\text{muon}}(x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \left[ \sum_{N=I,III} \Psi_N(q) - \sum_{N=II,IV} \Psi_N(q) \right]. \]

(9.6b)

The equations (9.6a,b) unify the 4D electron and muon Heisenberg fields which satisfies the 4D equation of motion (9.1a). Despite of the mixing of the \( \Psi_{\text{el}}(x) \) and \( \Psi_{\text{muon}}(x) \) in (9.6a,b) the 4D equations of motion (9.1a) are the same as in the standard model [35], where the electron-muon coupling is strongly suppressed. Therefore, the perturbation series constructed in the framework of the Weinberg-Salam \( SU(2) \times U(1) \) theory and
the perturbation series based on the equation of motion (9.1a,b,c) with the mixed fields (9.6a,b) coincides.

The common structure of the interacted Heisenberg fields $\Psi_{el}(x)$ and $\Psi_{muon}(x)$ in (9.6a,b) is important for the theories beyond Weinberg-Salam model. In particular, within the non-perturbative formulation the functional integrals with the 4D electron and muon fields $\int D(\Psi_{el})D(\Psi_{muon})[\ldots]$ and $\int D(\Psi_{muon})D(\Psi_{muon})[\ldots]$ are strongly correlated due to (9.6a,b).

Doubling of the electron and muon fields (9.6a,b) can be examined within the 5D invariant time theories [8, 9] which allow to construct the independent 5D fields $\Upsilon_{el}(x, x_5)$ and $\Upsilon_{muon}(x, x_5)$. If the parts of the $\Upsilon_{el}(q, q_5^2)$ and $\Upsilon_{muon}(q, q_5^2)$ are not the same as in (9.6a,b), then they are doubled and according to the present approach appear the fields $\Upsilon_{el}^\pm(x, x_5)$ and $\Upsilon_{muon}^\pm(x, x_5)$ with the negative or imaginary mass.

Nowadays unification of the different lepton families is performed within the 5D grand unification models [5]. In these models the fifth dimension is the indivisible part of the particle interaction, the space-time is not asymptotically flat, in the extra dimensions enter other than the gravitation fields and is argued the breakdown of the gauge coupling unification. The present formulation can be used for the 4D projections’ of the corresponding 5D electron and muon fields $\Upsilon_{el}$ and $\Upsilon_{muon}$. For this aim we put the Fourier conjugate of the complete 5D fields $\Upsilon_{el}$ and $\Upsilon_{muon}$ on the hyperboloids $q^2 \pm q_5^2 = \pm M^2$ using the replacement of the integrals $\int d^4qdq_5^2\{\ldots\}$ with $\int d^4qdq_5^2\left[\delta(q^2 + q_5^2 - M^2)\{\ldots\} \pm \delta(q^2 - q_5^2 + M^2)\{\ldots\}\right]$. Then in analogy with (I.4a,b) and (I.7) we get the 5D fields $(\psi_{el})\pm(x, x_5)$ and $(\psi_{muon})\pm(x, x_5)$. In the low energy region, where the asymptotically flat space is assumed, $(\psi_{el})\pm(x, x_5 = 0)$ and $(\psi_{muon})\pm(x, x_5 = 0)$ determine the 4D fields $(\Psi_{el})\pm(x)$ and $(\Psi_{muon})\pm(x)$. The gauge invariant 4D fields $(\Psi_{el})_+(x)$ and $(\Psi_{el})_-(x)$ consist from the same four parts as well as $(\Psi_{muon})_+(x)$ and $(\Psi_{muon})_-(x)$. Thus in addition to $\Psi_{el}(x)$ $\Psi_{muon}(x)$ we get two other fields $(\Psi_{el})_-(x)$ and $(\Psi_{muon})_-(x)$ with the negative or imaginary mass.

10. Gauge transformation as generalized translation

The generalized translation of the four-momentum $q^\mu = q^\mu - eA^\mu(q)$ in the equation of motion can be performed through the gauge transformations. In particular, the 4D and 5D gauge transformations (7.11a,b) and (7.8a,b) generates the corresponding translations of the four momentum $q^\mu$ and $i\partial/\partial x^\mu$

\begin{align}
q^\mu' &= q^\mu - eA^\mu_\pm(q, q_5^2) \iff i\frac{\partial}{\partial x^\mu} = -eA^\mu_\pm(x, x_5) \\
q_5' &= q_5 - eA^\mu_\pm(q, q_5^2) \iff i\frac{\partial}{\partial x^5} = -eA^\mu_\pm(x, x_5)
\end{align}

(10.1a)

with the 5D fields $A^\mu_\pm$, $A^5_\pm$, $\psi_\pm$ and $\psi'_\pm$.

One can construct $A^\mu_\pm$ and $A^5_\pm$ starting from the 6D gauge translations

\begin{align}
\kappa_C' &= \kappa_C - ea_C(\kappa) \iff i\frac{\partial}{\partial \xi C} = -e_a\xi(\xi) \\
\kappa_C &= \kappa_C - ea_C(\kappa) \iff i\frac{\partial}{\partial \xi C} = -e_a\xi(\xi)
\end{align}

(10.2a)
\[ \psi'(\xi) = \exp \left( i e \lambda(\xi) \right) \psi(\xi); \quad a_C(\xi) = \exp \left( -i e \lambda(\xi) \right) \frac{\partial}{\partial \xi} \exp \left( i e \lambda(\xi) \right), \quad (10.2b) \]

where \( C \equiv \mu; 5, 6 = 0, 1, 2, 3; 5, 6 \) and the Fourier conjugate of \( \psi'(\xi), \psi(\xi), \lambda(\xi) \) and \( a_C(\xi) \) are not placed on the cone \( \kappa_C \kappa^C = 0 \).

According to invariance of the 6D cone \( \kappa_C \kappa^C = 0 \) we have

\[ \kappa_C \kappa^C = \kappa_C \kappa^C = \left( \kappa_C - e a_C(\kappa) \right) \left( \kappa^C - e a^C(\kappa) \right) = 0, \quad (10.3) \]

where \( \kappa_+ \) is invariant under the translations as it is indicated in (1.2a). The invariance of \( \kappa_+ \) under the 4D translations (10.2a) requires that \( a_5(\kappa) = -a_6(\kappa) \). Substituting this condition in (10.3) we get

\[ a_5(\kappa) = \frac{1}{2M} \left( -a_\nu(\kappa) \kappa^\nu - \kappa^\nu a_\nu(\kappa) + e a_\nu(\kappa) a^\nu(\kappa) \right); \quad \frac{\kappa'_+}{\kappa_+} = \frac{\kappa_-}{\kappa_+} - e \frac{a_5(\kappa)}{\kappa_+}. \quad (10.4) \]

In the considered formulation the 4D reduction of the 6D fields \( a_\mu(\kappa) \) and \( \psi(\kappa) \) based on the intermediate projections of the 6D cone \( \kappa_C \kappa^C = 0 \) into 5D hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) (1.3a,b) for the domains I, III and II, IV of the Table 1. The corresponding 5D reductions of the 6D fields \( a_\mu(\kappa) \) and \( \psi(\kappa) \) are

\[ \Upsilon(q, q_5^2) = \frac{M^2}{2} \int k_+^3 d\kappa_+ \theta(\kappa_+) \psi(q, q_5^2, \kappa_+), \quad (10.5) \]

where \( \Upsilon(q, q_5^2) \) is Fourier conjugate of \( \Upsilon(x, x_5) \) in (7.2) and

\[ \tilde{A}_\mu(q, q_5^2) = \frac{M^2}{2} \int k_+^3 d\kappa_+ \theta(\kappa_+) a_\mu(q, q_5^2, \kappa_+); \quad \tilde{A}_5(q, q_5^2) = \frac{M^2}{2} \int k_+^3 d\kappa_+ \theta(\kappa_+) a_5(q, q_5^2, \kappa_+). \quad (10.6) \]

The 5D gauge fields \( A_\pm^\mu(x, x_5) \) which Fourier conjugate are placed on the hyperboloids \( q_5^2 \pm q_5^2 = \pm M^2 \) are constructed through \( \tilde{A}_\mu \)

\[ A_\pm^\mu(x, x_5) = \int d^5 y \tilde{A}_\mu(x - y, x_5 - y_5) \mathcal{P}_\pm(y, y_5), \quad (10.7a) \]

\[ A_5^5(x, x_5) = \int d^5 y \tilde{A}_5(x - y, x_5 - y_5) \mathcal{P}_\pm(y, y_5), \quad (10.7b) \]

The relationship between the fifth gauge field \( A_5^5(10.7b) \) and \( A_\pm^\mu \) (10.7a) is given in (7.13) and (7.14a,b).

The 4D gauge field \( A_\pm^\mu(x) = A_\pm^\mu(x, 0) \) are determined through \( A_\pm^\mu(q, q_5^2) \) in the domains \( N = I, II, III, IV \) as

\[ A_\pm^\mu(x) = \int \frac{d^4 q}{(2\pi)^4} \exp (-i q x) \left[ A_7^\mu(q) \pm A_I^\mu(q) + A_{II}^\mu(q) \pm A_{IV}^\mu(q) \right]. \quad (10.8) \]
The convolution formula (10.7) can be represented in the 4D form

\[ A^\mu_\pm(x, x_5) = \int d^4y A^\mu_\pm(x - y) P_+ (y, x_5). \]  \hspace{1cm} (10.9)

This representation of \( A^\mu_\pm(x, x_5) \) indicates that the 5D and 4D fields \( A^\mu_\pm(x, x_5) \) and \( A^\mu(x) \) satisfy the same 4D equations of motion. On the other hand \( A^\mu_\pm(x, x_5) \) consists of the parts placed on the hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) and satisfy the similar to (3.1) 5D conditions

\[ \frac{\partial^2 A^\mu_\pm(x, x_5)}{\partial x^\mu \partial x_\mu} + \left( \frac{\partial^2}{\partial x^5 \partial x_5} + M^2 \right) A^\mu_\pm(x, x_5) = 0, \] \hspace{1cm} (10.10)

The consistency condition of (10.10) and the corresponding equation of motion have the same form as (4.10) for the scalar field. Thus the present 5D formulation allows to construct simultaneously the 4D fields \( A^\mu_\pm(x) \) and \( A^\mu(x) \) which consists of the same parts \( A^\mu_\pm, A^\mu_{II}, A^\mu_{III}, A^\mu_{IV} \) and \( A^\mu_\pm(x) \) and \( A^\mu(x) \) have the same quantum numbers, but the different sources. For the photon field \( A^\mu_\pm(x) \) the role of \( A^\mu_\pm \) can play the Z-boson.

It must be noted that the gauge transformations can be performed also for the neutral (uncharged) particles. Within the nonlinear \( \sigma \) model \([16, 36]\) for the triplet of the neutral auxiliary pion fields \( \pi^\alpha (\alpha = 1, 2, 3, \pi^\pm = 1/2(\pi^1 \pm i\pi^2); \pi^0 \equiv \pi^3) \) is replaced with the interpolating pion field

\[ \pi^\alpha(x) = U(x) \chi^\alpha(x), \] \hspace{1cm} (10.11)

where in \([16, 36]\) \( U(x) = \left( 1 + \chi^2(x)/4f_\pi^2 \right)^{-1} \) \( f_\pi = 93 \text{ MeV} \) is the pion decay constant and \( \chi^2 = \sum_{\alpha=1}^{3} \chi^\alpha \chi^\alpha \). The replacement (10.11) generates the following transformations

\[ \frac{\partial}{\partial x_\mu} \pi^\alpha = U \left[ \frac{\partial}{\partial x_\mu} + D^\mu \right] \chi^\alpha; \hspace{1cm} D^\mu = U^{-1} \frac{\partial}{\partial x_\mu} U \] \hspace{1cm} (10.12)

which presents the extension of (7.11a,b) for the neutral fields. The transformations (10.11) and (10.12) generalise also the gauge transformations (1.11) for the neutral fields.

11. Summary

The one-to-one relationship between the 4D and 5D fields and their equations of motion, established by present article, based on the equivalence of the conformal transformations of the four momentum \( q_\mu \) (1.1a)-(1.1e) and the 6D rotations on the cone \( \kappa_A \kappa^A = 0 \) (1.2a)-(1.2d) and its 5D projections on the two invariant forms \( q^2 \pm q_5^2 = \pm M^2 \) of the \( O(2, 3) \) and \( O(1, 4) \) subgroups of the conformal group \( O(2, 4) \). The 6D cone \( \kappa_A \kappa^A = 0 \) and its 5D projections \( q^2 \pm q_5^2 = \pm M^2 \) are invariant under the 6D rotations and the corresponding 4D conformal transformations. Consequently, the 4D projection of the 6D field \( \varsigma(\kappa) \) with the intermediate projection on the two hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) determine the two 5D fields \( \varphi_1(x, x_5) \) (1.4a) and \( \varphi_2(x, x_5) \) (1.4b). The Fourier conjugate of these 5D fields are placed on the hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) before and after conformal transformations. Consequently the 5D fields \( \varphi_+ = \varphi_1 + \varphi_2 \) and \( \varphi_- = \varphi_1 - \varphi_2 \) are defined on the whole domain of \(-\infty < q^2 < \infty\) and reproduce the 4D fields \( \Phi_{\pm}(x) = \varphi_{\pm}(x, x_5 = 0) \).
In addition the fields $\varphi_{\pm}(x, x_5)$ satisfy the 5D constrains (3.1) that have the form of the coupled sourceless 5D equations. The 5D equation of motion for $\varphi_{\pm}$ (4.1a) and their 4D reductions (4.1b) are embedded into these sourceless coupled 5D conditions (3.1). The consistency conditions of the 5D equation of motion (4.1a) and the 5D constrains (3.1) generate the constrains for $\partial \varphi_{\pm}/\partial x_5$. The similar sourceless 5D coupling condition (7.1) are valid for the interacting fermion fields $\psi_+$ and $\psi_-$. 

This scheme can be used for the 5D extension of the 4D models and for the 4D reductions of the 5D formulations. The parts of the 5D field $\phi(q, q_5^2 = M^2 - q^2)$ and $\phi(q, q_5^2 = M^2 + q^2)$ unambiguously determine the Fourier conjugate of the 4D fields $\Phi_+(x)$ and $\Phi_-(x)$ (I.10)-(I.11). The same expressions determine also the Fourier conjugate of the 5D fields $\varphi_1$ (I.4a), $\varphi_2$ (I.4b) and $\varphi_{\pm}$ (I.7). These parts of the single 5D field $\phi(q, q_5^2)$ determine unambiguously the two 4D fields $\Phi_{\pm}(x)$ with the same quantum numbers, but with the different masses and sources. And vice versa, starting from the 4D field $\Phi_+(x)$ one can construct $\Phi_-(x)$ and the parts of the 5D field $\phi(q, q_5^2 = M^2 - q^2)$ and $\phi(q, q_5^2 = M^2 + q^2)$. This doubling of the 4D fields is result the intermediate projection of the 6D field placed on the 6D cone $\kappa_A \kappa^A = 0$ into the two 5D fields $\varphi_{\pm}$ embedded into two invariant forms $q^2 \pm q_5^2 = \pm M^2$. The considered intermediate 5D projections take into account the symmetry under the inversion $q_\mu = -M^2 q_\mu/q^2$ and reflection $q^2 = -q^2$ between the domains of these forms. As it is mentioned in the last two paragraphs of Sect. 2, the stereographic and other 5D projections of the 6D cone $\kappa_A \kappa^A = 0$ can be reproduced through the considered projections on the hyperboloids $q^2 \pm q_5^2 = \pm M^2$.

The boundary conditions of the 5D fields $\varphi_{\pm}$ (I.8) and $\psi_{\pm}$ (7.4) at $x_5 = 0$ can be extended for an arbitrary value $x_5 = t_5$ if one replaces $x_5$ by $x_5 - t_5$ in the definitions of the 5D fields (I.4a,b) and (i.7). In particular, $t_5 = \sqrt{x_0^2 - x^2}$ in the formulation within the relativistic invariant time theories [8, 9].

The common parts of the 4D interacting fields $\Phi_+(q)$ and $\Phi_-(q)$ in (I.10)-(I.11) allow to separate the $4! = 24$ different fields with the same quantum numbers and with the different masses and sources. This unification scheme of the interacting Heisenberg fields can be applied for description of the nucleons and resonances in the P11 states $N(1440), N(1710), ...$ [37] based on an additional dynamical mechanism for the generation of the resonances. Similarly, one can combine the interacting Heisenberg fields of the pions and the resonances with the same quantum numbers $\pi(1300)$.

The particle states with the same quantum numbers and the different masses and sources can be constructed within the various 5D relativistic invariant time theories [8, 9]. The different 5D fields $\phi_+(q, q_5^2)$ and $\phi_-(q, q_5^2)$ in these theories do not have the common parts $\phi_+(q, q_5^2 = M^2 - q^2)$ and $\phi_-(q, q_5^2 = M^2 + q^2)$. Nevertheless the present formulation requires doubling of the 5D fields, i.e. instead of the $\phi_+(x, x_5)$ and $\phi_-(x, x_5)$ we get $\phi_{\pm}(x, x_5)$ and $\phi_{\pm}(x, x_5)$, where $\phi_{\pm}(x, x_5)$ and $\phi_{\pm}(x, x_5)$ must have the negative or imaginary masses. The relationships between the different masses and sources for the 5D fields $\phi_{\pm}(x, x_5)$ and $\phi_{\pm}(x, x_5)$ are determined by constrains for the $\partial(\phi_{\pm})/\partial x_5$ and $\partial(\phi_{\pm})/\partial x_5$. Thus the present approach allows to get the 4D projections of the 5D fields from [8, 9] for $x_5 = 0$.

It must be noted, that if the 4D fields $\Phi_{\pm}(x)$ are determined through the parts of the
5D fields $\phi_+(q, q_5^2 = M^2 \mp q^2)$ and $\phi_-(q, q_5^2 = M^2 \mp q^2)$, then the Fourier conjugate of the observed 4D fields $\Phi_\pm(q)$ can have the jump singularities at $q^2 = 0, \pm M^2$.

Presently the different lepton families including the electron and muon fields $\Upsilon_{el}$ and $\Upsilon_{\mu}$ are constructed within the 5D grand unification theories [5, 6]. In this formulation the 5D electron and muon fields $\Upsilon_{el}(q, q_5^2 = M^2 \mp q^2)$ and $\Upsilon_{\mu}(q, q_5^2 = M^2 \mp q^2)$ are independent. The present scheme of the 4D reduction of the 5D fields allows to get the 4D projections of these 5D fields with the doubling of the 4D electron and muon fields $(\Psi_{el})_\pm(x)$ and $(\Psi_{\mu})_\pm(x)$, where $(\Psi_{el})_-$ and $(\Psi_{\mu})_-$, have the negative or imaginary masses. These 4D projections of the 5D fields restore the 4D gauge invariance, because the gauge transformations are considered as the generalized 4D translation in the momentum space (see Sect. 10) and translations of the four momentum preserve invariance of the 6D cone $\kappa A_k A^A = 0$ and their projections on the 5D invariant forms.

Other kind of the unification of the 4D electron and muon fields in this approach is considered in the Section 9 for the Standard Model [35], where $\Phi_{el}(q)$ and $\Phi_{\mu}(q)$ consist of the same parts (9.6a,b) of the complete 5D field $\Upsilon(q, q_5^2 = M^2 \mp q^2)$. The coupling between the electron and muon fields are strongly suppressed in the Standard Model. Therefore, the common structure of the electron and muon fields can not be observed in the perturbation series of the corresponding 4D equations.

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