ALEKSANDROV PROJECTION PROBLEM FOR CONVEX LATTICE SETS

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Abstract. Let $K$ and $L$ be origin-symmetric convex integer polytopes in $\mathbb{R}^n$. We study a discrete analogue of the Aleksandrov projection problem. If for every $u \in \mathbb{Z}^n$, the sets $(K \cap \mathbb{Z}^n)|u^\perp$ and $(L \cap \mathbb{Z}^n)|u^\perp$ have the same number of points, is then $K = L$? We give a positive answer to this problem in $\mathbb{Z}^2$ under an additional hypothesis that $(2K \cap \mathbb{Z}^2)|u^\perp$ and $(2L \cap \mathbb{Z}^2)|u^\perp$ have the same number of points.

1. Introduction

Let $K$ be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with nonempty interior. We say that $K$ is origin-symmetric if $K = -K$, where $tK := \{tx : x \in K\}, t \in \mathbb{R}$. In 1937, Aleksandrov proved the following result [1]:

Theorem 1.1. Let $K, L \subset \mathbb{R}^n$ be origin-symmetric convex bodies. If

$$\text{vol}_{n-1}(K|u^\perp) = \text{vol}_{n-1}(L|u^\perp)$$

for every $u \in S^{n-1}$, then $K = L$.

Here $u^\perp := \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$.

Gardner, Gronchi, and Zong suggested a discrete version of the Aleksandrov projection problem (see [2]). We say $A$ is a convex lattice set if $\text{conv}(A) \cap \mathbb{Z}^n = A$, where $\text{conv}(A)$ is the convex hull of $A$.

Problem 1.2. Let $A, B \subset \mathbb{Z}^n$ be origin-symmetric convex lattice sets. If $|A|u^\perp| = |B|u^\perp|$ for every $u \in \mathbb{Z}^n$, is it true that $A = B$?

Here, $|A|u^\perp|$ is the cardinality of $A|u^\perp$. Since the convex hull of a convex lattice set is a convex integer polytope, i.e. a polytope all of whose vertices are in $\mathbb{Z}^n$, it would be convenient to restate the problem as follows. Let $K, L \subset \mathbb{R}^n$ be origin-symmetric convex integer polytopes. If $|(K \cap \mathbb{Z}^n)|u^\perp| = |(L \cap \mathbb{Z}^n)|u^\perp|$ for every $u \in \mathbb{Z}^n$, is it true that $K = L$?

In [2], the authors gave a negative answer to Problem 1.2 in $\mathbb{Z}^2$. However, it is not known whether there are other counterexamples. Zhou [6] and Xiong [4] showed that these counterexamples are unique in some special classes. For higher dimensions, this problem is still open. Some work on related problems has been done in [3]. Since the answer is negative in

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2. NING ZHANG

dimension 2, Gardner, Gronchi, and Zong asked if it is possible to impose reasonable additional conditions to make the answer affirmative. In this paper, we obtain a positive answer to Problem 1.2 in \( \mathbb{Z}^2 \) under an additional hypothesis.

Before we state the theorem, some definition should be introduced (see \[1\] and \[5\]). Let \( K \) be a convex body in \( \mathbb{R}^n \). The support function of \( K \) in the direction \( u \) is

\[
h_K(u) := \sup \{ \langle u, x \rangle : x \in K \}.
\]

The width function of \( K \) in the direction \( u \) is

\[
w_K(u) := h_K(u) + h_K(-u).
\]

If \( K \) is a convex integer polytope, then we denote

\[
D_1 K := \{ u \in \mathbb{Z}^n : \exists x_1, x_2 \in K \cap \mathbb{Z}^n, u \parallel x_1x_2 \}.
\]

For a directed segment \( u \) with the initial point \((p_1, \ldots, p_n) \in \mathbb{Z}^n\) and the end point \((q_1, \ldots, q_n) \in \mathbb{Z}^n\), let

\[
\hat{u} := \left( \frac{q_1 - p_1}{d}, \ldots, \frac{q_n - p_n}{d} \right)
\]

denote the primitive vector in the direction \( u \), where \( d = \gcd(q_1 - p_1, \ldots, q_n - p_n) \).

We will need the well-known Pick’s theorem (see \[3\]). Let \( K \subset \mathbb{R}^2 \) be a convex integer polygon. Then

\[
\text{vol}_2(K) = |K \cap \mathbb{Z}^2| - \frac{1}{2} |\partial K \cap \mathbb{Z}^2| - 1,
\]

where \( \partial K \) is the boundary of \( K \).

We are now ready to state our main result.

**Theorem 1.3.** Let \( K, L \subset \mathbb{R}^2 \) be origin-symmetric convex integer polygons. If

\[
|(K \cap \mathbb{Z}^2)|u_{\perp}^1| = |(L \cap \mathbb{Z}^2)|u_{\perp}^1|
\]

and

\[
|(2K \cap \mathbb{Z}^2)|u_{\perp}^1| = |(2L \cap \mathbb{Z}^2)|u_{\perp}^1|
\]

for all \( u \in \mathbb{Z}^2 \), then \( K = L \).

**Remark 1.4.** It will be clear from the proof that we do not need projections in all directions, only in directions parallel to the edges of \( K \) and \( L \), and one more direction \( \xi \in \mathbb{Z}^2 \) \( \setminus (D_1 K \cup D_1 L) \).

2. PROOF OF THEOREM 1.3

**Theorem 2.1.** Let \( K \) be an origin-symmetric convex integer polygon in \( \mathbb{R}^2 \) with edges \( \{e_i\}_{i=1}^{2n} \), where \( e_i \) and \( e_{n+i} \) are symmetric with respect to the origin. Then

\[
|(K \cap \mathbb{Z}^2)|e_i| = |\hat{e}_i|w_K(e_i^\perp) + 1, \text{ for } 1 \leq i \leq n,
\]

where \( |\hat{e}_i| \) is the length of the primitive vector parallel to \( e_i \). Here and below, \( w_K(u_{\perp}^1) \) means the width in the direction perpendicular to \( u \).

We will first prove the theorem in a simple case.
Lemma 2.2. Let $K \subset \mathbb{R}^2$ be a parallelogramm with edges $\{e_i\}_{1 \leq i \leq 4}$, where $e_1 \parallel e_3$ and $e_2 \parallel e_4$. Then
\[ |(K \cap \mathbb{Z}^2)|e_i^+ = |\hat{e}_i|w_K(e_i^+) + 1, \text{ for } i = 1, 2. \]

Proof. Consider the point lattice $\Lambda$ generated by $\hat{e}_1$ and $\hat{e}_2$ and the quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$. Set $l(e_1)$ to be the line passing through the origin and parallel to $e_1$. If $x \in K \cap \Lambda$, then
\[ |(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2|. \]
If $x \in (K \cap \mathbb{Z}^2) \setminus \Lambda$, then $\pi((x + l(e_1)) \cap (K \cap \mathbb{Z}^2))$ contains only one point; otherwise, $x \in \Lambda$. Thus,
\[ |(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2| - 1. \]
One can see that,
\[ |(K \cap \Lambda)|e_i^+ = |e_2 \cap \mathbb{Z}^2|. \]
Furthermore when projecting $(K \cap \mathbb{Z}^2) \setminus \Lambda$ onto $e_i^+$, each point in the projection has $|e_1 \cap \mathbb{Z}^2| - 1$ preimages. Thus,
\[ |((K \cap \mathbb{Z}^2) \setminus \Lambda)|e_i^+ = \frac{|(K \cap \mathbb{Z}^2)| - |e_1 \cap \mathbb{Z}^2||e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1}; \]
hence,
\begin{align*}
|(K \cap \mathbb{Z}^2)|e_i^+ &= \frac{|(K \cap \mathbb{Z}^2)| - |e_1 \cap \mathbb{Z}^2||e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1} + |e_2 \cap \mathbb{Z}^2| \\
&= \frac{|(K \cap \mathbb{Z}^2)| - |e_2 \cap \mathbb{Z}^2|}{|e_1 \cap \mathbb{Z}^2| - 1} \\
&= \frac{\text{vol}_2(K) + |e_1 \cap \mathbb{Z}^2| - 1}{|e_1 \cap \mathbb{Z}^2| - 1} \quad \text{(by Pick’s theorem)} \\
&= \frac{|\hat{e}_1||e_1 \cap \mathbb{Z}^2| - 1|w_K(e_i^+) + |e_1 \cap \mathbb{Z}^2| - 1}{|e_1 \cap \mathbb{Z}^2| - 1} \\
&= |\hat{e}_1|w_K(e_i^+) + 1.
\end{align*}

Proof of Theorem 2.1. Without loss of generality, we only need to compute $|(K \cap \mathbb{Z}^2)|e_i^+|$. Create a convex lattice set with convex hull being a parallelogramm with edges $e_1$ and $e_{n+1}$, denoted by $\mathcal{D}$. Note that, for any $x \in (K \cap \mathbb{Z}^2) \setminus \mathcal{D}$, $x + l(e_1) \cap \mathcal{D} \neq \emptyset$. Thus, there exists $m \in \mathbb{Z}$, such that $x \in \mathcal{D} + me_1$, which implies $x - me_1 \in \mathcal{D} \cap \mathbb{Z}^2$. Therefore, by Lemma 2.2
\[ |(K \cap \mathbb{Z}^2)|e_i^+| = |(\mathcal{D} \cap \mathbb{Z}^2)|e_i^+| = |\hat{e}_1|w_K(e_i^+) + 1. \]

Theorem 2.1 implies that if $K$ and $L$ have parallel edges, then there is a uniqueness in Problem 1.2.
Lemma 2.3. Let $K$ be an origin-symmetric convex integer polygon in $\mathbb{Z}^2$. Let $u \in D_1K$. If $2(|(K \cap \mathbb{Z}^2)|u^+| - 1) = |(2K \cap \mathbb{Z}^2)|u^+| - 1$, then

$$|(K \cap \mathbb{Z}^2)|u^+| = |\hat{u}|w_K(u^+) + 1.$$  

Proof. Let $u \in D_1K$. If $u$ is parallel to one of the edges of $K$, then, by Theorem 2.1

$$|(K \cap \mathbb{Z}^2)|u^+| = |\hat{u}|w_K(u^+) + 1;$$  

if not, consider the pair of points $(x_1, x_2) \in \{(x, y) \in K \times K : xy \parallel \hat{u}\}$ such that

$$\text{dist}(O, \overline{x_1x_2}) = \max_{\{(x, y) \in K \times K : xy \parallel \hat{u}\}} \text{dist}(O, \overline{xy}).$$

Here, we denoted by $\text{dist}(O, A) = \inf_{x \in A} \|x - O\|_2$, the distance between $O$ and a set $A$. The set $\{(x, y) \in K \times K : xy \parallel \hat{u}\}$ is not empty, since $u \in D_1K$.

Thus, the lines passing through $x_1, x_2$ and $-x_1, -x_2$ divide $\mathbb{R}^2$ into three parts $E_1, E_2$, and $E_3$, where $O \in E_2$ and $E_1, E_3$ are reflections of each other with respect to $O$.

Note that, $E_2 \cap K \cap \mathbb{Z}^2$ is a convex lattice set and $x_1, x_2, -x_1, -x_2$ lie on two parallel edges of $E_2 \cap K$. (Here, $E_2 \cap K$ can be a segment.) Then, by Theorem 2.1 we have

$$|(E_2 \cap K \cap \mathbb{Z}^2)|u^+| = |\hat{u}|w_{E_2\cap K}(u^+) + 1$$

and set $|(E_1 \cap K \cap \mathbb{Z}^2)|u^+| = |(E_3 \cap K \cap \mathbb{Z}^2)|u^+| = m$. We have

$$|(K \cap \mathbb{Z}^2)|u^+| = 2m + |\hat{u}|w_{E_2\cap K}(u^+) - 1.$$  

On the other hand, $|(2E_2 \cap 2K \cap \mathbb{Z}^2)|u^+| = 2|\hat{u}|w_{E_2\cap K}(u^+) + 1$. Moreover, a line $l$ parallel to $u$ divides $2E_1 \cap 2K$ into two parts of equal width in the direction perpendicular to $u$, denoted by $E_{11} \cap 2K$ and $E_{12} \cap 2K$, where $\text{dist}(O, E_{11}) > \text{dist}(O, E_{12})$.

Note that there exists a pair of points $y_1, y_2 \in l \cap 2K \cap \mathbb{Z}^2$. To see this, pick a point $z$ from $E_1 \cap K \cap \mathbb{Z}^2$ such that $w_{[-z, z]}(u^+) = w_K(u^+)$, where $[-z, z]$ is the segment connecting $-z$ and $z$. Then $2z, 2x_1, 2x_2 \in 2K \cap \mathbb{Z}^2$ implies $y_1 = z + x_1, y_2 = z + x_2 \in 2K \cap l$.

Now we obtain $E_{11} \cap 2K \supset E_{11} \cap K + z$. To see this, assume $E_{11} \cap K = \{x \in \mathbb{R}^2 : \langle x, v_i \rangle \leq a_i\}$ with $x_1, x_2 \in \{x \in \mathbb{R}^2 : \langle x, v_i \rangle = a_i\}$, then $\langle z, v_i \rangle = a_i - w_{E_{11} \cap K}(v_i)$ and $\langle u, v_i \rangle = 0$. Thus for any $x \in E_{11} \cap K$, $\langle x + z, v_i \rangle \leq 2a_i$ and $\langle x + z, v_i \rangle \leq 2a_i - w_{E_{11} \cap K}(v_i)$, implying $x + z \in 2(E_{11} \cap 2K) \cap E_{11} = E_{11} \cap 2K$.

Since $E_{12} \cap 2K$ contains a parallelogram $\square$ with vertices $2x_1, x_1 + x_2, y_1, y_2$, we have

$$|(E_{11} \cap 2K \cap \mathbb{Z}^2)|u^+| \geq |(E_{11} \cap K \cap \mathbb{Z}^2)|u^+| = m$$

and

$$|(E_{12} \cap 2K \cap \mathbb{Z}^2)|u^+| = |\hat{u}|w_{\square}(u^+) + 1 = |\hat{u}|w_{E_{11} \cap K}(u^+) + 1.$$  

Hence,

$$|(2E_1 \cap 2K \cap \mathbb{Z}^2)|u^+| \geq m + |\hat{u}|w_{E_{11} \cap K}(u^+).$$

Therefore,

$$|(2K \cap \mathbb{Z}^2)|u^+| \geq 2(m + |\hat{u}|w_{E_{11} \cap K}(u^+)) + 2|\hat{u}|w_{E_{12} \cap K}(u^+) - 1.$$  

By the assumption, we have

$$2(2m + |\hat{u}|w_{E_{12} \cap K}(u^+) - 2) = 2(|(K \cap \mathbb{Z}^2)|u^+| - 1) = |(2K \cap \mathbb{Z}^2)|u^+| - 1 \geq 2(m + |\hat{u}|w_{E_{11} \cap K}(u^+)) + 2|\hat{u}|w_{E_{12} \cap K}(u^+) - 1,$$
which implies

\[ m \geq |\hat{u}|w_{E_1 \cap K}(u^\perp) + 1. \]

On the other hand, \( m \leq |\hat{u}|w_{E_1 \cap K}(u^\perp) + 1, \) by constructing a large parallelogramm containing \( E_1 \cap K, \) that has two edges parallel to \( u \) and whose width perpendicular to \( u \) is \( w_{E_1 \cap K}(u^\perp); \) thus,

\[ m = |\hat{u}|w_{E_1 \cap K}(u^\perp) + 1. \]

The conclusion follows. \( \square \)

**Definition 2.4.** Let \( K^n \) be the collection of all origin-symmetric convex bodies in \( \mathbb{R}^n. \) Define an operator \( \uplus : K^n \times K^n \rightarrow K^n, \) satisfying

\[ A \uplus B := \text{conv}(A \cup B). \]

One can easily prove the following properties.

**Proposition 2.5.** Let \( A, B \in K^n. \) Then

\[ h_{A \uplus B}(u) = \max\{h_A(u), h_B(u)\} \quad \text{and} \quad w_{A \uplus B}(u) = \max\{w_A(u), w_B(u)\}. \]

**Lemma 2.6.** Let \( K \) and \( L \) be origin-symmetric convex polygons in \( \mathbb{R}^2. \) If \( w_K(u^\perp) = w_L(u^\perp) \) for all \( u \in E_K \cup E_L, \) then \( K = L. \) Here, \( E_K \) is the collection of all directions parallel to the edges of \( K. \)

**Proof.** Clearly, \( K \subseteq K \uplus L \) and \( w_K(u^\perp) = w_{K \uplus L}(u^\perp), \) for all \( u \in E_K. \) Assume \( K \not\subseteq K \uplus L. \) Then there exists a point \( \{x\} \in K \uplus L, \) but \( \{x\} \notin K. \) Thus we can find a direction \( \eta \in E_K, \) such that, \( 2\langle x, \eta \rangle = \langle x - (-x), \eta \rangle > w_K(\eta^\perp), \) implying \( w_K(\eta^\perp) < w_{[-x,x] \uplus K}(\eta^\perp). \) On the other hand, since \([−x,x] \uplus K \subseteq K \uplus L, \) we have

\[ w_{[-x,x] \uplus K}(\eta^\perp) \leq w_{K \uplus L}(\eta^\perp) = w_K(\eta^\perp), \]

Contradiction. \( \square \)

**Proof of Theorem 1.3.** Here, we use the weaker condition mentioned in Remark 1.4. Note that, \( |(K \cap \mathbb{Z}^2)|u^\perp| < |K \cap \mathbb{Z}^2|, \) if \( u \in D_1K; \) but \( |(K \cap \mathbb{Z}^2)|u^\perp| = |K \cap \mathbb{Z}^2|, \) if \( u \in \mathbb{Z}^2 \setminus D_1K. \) For any \( u \in E_K, \) we have \( u \in D_1L; \) otherwise,

\[ |(L \cap \mathbb{Z}^2)|u^\perp| = |L \cap \mathbb{Z}^2| = |(L \cap \mathbb{Z}^2)|\xi^\perp| = |(K \cap \mathbb{Z}^2)|\xi^\perp| = |K \cap \mathbb{Z}^2| > |(K \cap \mathbb{Z}^2)|u^\perp| \]

for some \( \xi \in \mathbb{Z}^2 \setminus (D_1K \cup D_1L). \) Then by Lemma 2.3 we have

\[ |(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp) + 1 \quad \text{and} \quad |(2K \cap \mathbb{Z}^2)|u^\perp| = 2|\hat{u}|w_K(u^\perp) + 1. \]

By the assumption,

\[ |(2L \cap \mathbb{Z}^2)|u^\perp| - 1 = |(2K \cap \mathbb{Z}^2)|u^\perp| - 1 = 2|\hat{u}|w_K(u^\perp) \]

\[ = 2((K \cap \mathbb{Z}^2)|u^\perp| - 1) = 2(|(L \cap \mathbb{Z}^2)|u^\perp| - 1). \]

Applying Lemma 2.3

\[ |(L \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_L(u^\perp) + 1 = |(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp) + 1. \]
Therefore,
\[ w_L(u^\perp) = w_K(u^\perp), \]
for any \( u \in E_K \). Similarly, we can show \( w_L(u^\perp) = w_K(u^\perp) \), for any \( u \in E_L \). Then the conclusion follows from Lemma 2.6.

\[ \square \]

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