ENERGY QUANTIZATION FOR YAMABE’S PROBLEM IN
CONFORMAL DIMENSION

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Abstract. Riviére [11] proved an energy quantization for Yang-Mills fields defined on $n$-dimensional Riemannian manifolds, when $n$ is larger than the critical dimension 4. More precisely, he proved that the defect measure of a weakly converging sequence of Yang-Mills fields is quantized, provided the $W^{2,1}$ norm of their curvature is uniformly bounded. In the present paper, we prove a similar quantization phenomenon for the nonlinear elliptic equation

$$-\Delta u = |u|^{4/(n-2)},$$

in a subset $\Omega$ of $\mathbb{R}^n$.

1. Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $n \geq 3$. We consider the equation

$$-\Delta u = |u|^{4/(n-2)} \quad \text{in} \quad \Omega \quad (1.1)$$

We will say that $u$ is a weak solution of (1.1) in $\Omega$, if, for all $\Phi \in C^\infty(\Omega)$ with compact support in $\Omega$, we have

$$-\int_{\Omega} \Delta \Phi(x) u(x) dx = \int_{\Omega} \Phi(x) u(x) |u(x)|^{4/(n-2)} dx \quad (1.2)$$

If in addition $u$ satisfies

$$\int_{\Omega} \left[ \frac{\partial u}{\partial x_i} \frac{\partial \Phi^j}{\partial x_j} - \frac{1}{2} |\nabla u|^2 \frac{\partial \Phi^i}{\partial x_i} + \frac{n-2}{2n} u^{2n/(n-2)} \frac{\partial \Phi^i}{\partial x_i} \right] dx = 0 \quad (1.3)$$

for any $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^n) \in C^\infty(\Omega)$ with compact support in $\Omega$, we say that $u$ is stationary. In other words, a weak solution $u$ in $H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$ of (1.1) is stationary if the functional $E$ defined by

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{n-2}{2n} \int_{\Omega} |u|^{2n/(n-2)}$$

is stationary with respect to domain variations, i.e.

$$\frac{d}{dt}(E(u_t))|_{t=0} = 0$$

where $u_t(x) = u(x + t\Phi)$. It is easy to verify that a smooth solution is stationary.

In this paper we prove a monotonicity formula for stationary weak solution $u$ in $H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$ of (1.1) by a similar idea as in [10]. More precisely we have the following result.

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Lemma 1.1. Suppose that $u \in L^{2n/(n-2)}(\Omega) \cap H^1(\Omega)$ is a stationary weak solution of (1.1). Consider the function

$$E_u(x,r) = \int_{B(x,r)} |u|^{2n/(n-2)} \, dy + \frac{d}{dr} \int_{\partial B(x,r)} u^2 \, ds + r^{-1} \int_{B(x,r)} u^2 \, ds.$$ 

Then $r \mapsto E_u(x,r)$ is positive, nondecreasing and continuous.

This monotonicity formula together with ideas which go back to the work of Schoen [12], allowed to prove the following result.

Theorem 1.2. There exists $\varepsilon > 0$ and $r_0 > 0$ depend only on $n$ such that, for any smooth solution $u \in H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$ of (1.1), we have: For any $x_0 \in \Omega$, if

$$\int_{B(x_0,r_0)} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \leq \varepsilon,$$

then

$$\|u\|_{L^\infty(B_{x_0}(r_0))} \leq \frac{C(\varepsilon)}{r^{(n-2)/2}} \quad \text{for any } r < r_0,$$

where $B_{x_0}(r_0)$ is the ball centered at $x_0$ with radius $r_0$, and $C(\varepsilon)$ to $0$ as $\varepsilon \to 0$.

Zongming Guo and Jiay Li [5] studied sequences of smooth solutions of (1.1) having uniformly bounded energy, they proved the following result.

Theorem 1.3. Let $u_i$ be a sequence of smooth solutions of (1.1) such that

$$\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{2n/(n-2)}(\Omega)}$$

is bounded. Let $u_\infty$ be the weak limit of $u_i$ in $H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$. Then $u_\infty$ is smooth and satisfies equation (1.1) outside a closed singular subset $\Sigma$ of $\Omega$. Moreover, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\Sigma = \cap_{0 < r < r_0} \{ x \in \Omega : \liminf_{i \to \infty} E_{u_i}(x,r) \geq \varepsilon_0 \}.$$ 

We define the sequence of Radon measures

$$\eta_i := \left( \frac{1}{2} |\nabla u_i|^2 + \frac{n-2}{2n} |u_i|^{2n/(n-2)} \right) \, dx$$

Assumption that the sequence $(\|\nabla u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{2n/(n-2)}(\Omega)})$ is bounded, and up to a subsequences, we can assume that $\eta_i \to \eta$ in the sense of measures as $i \to \infty$. Namely, for any continuous function $\phi$ with compact support in $\Omega$

$$\lim_{i \to \infty} \int_\Omega \phi \, d\eta_i = \int_\Omega \phi \, d\eta.$$ 

Fatou’s Lemma then implies that we can decompose

$$\eta = \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) \, dx + \nu$$

where $\nu$ is a nonnegative Radon measure. Moreover, we prove that $\nu$ satisfies the following lemma.

Lemma 1.4. Let $\delta > 0$ such that $B_\delta \subset \Omega$. Then we have

(i) $\Sigma \subset spt(\nu)$

(ii) There exists a measurable, upper-semi-continuous function $\Theta$ such that

$$\nu(x) = \Theta(x) \mathcal{H}^0|\Sigma, \quad \text{for } x \in \Sigma.$$
Moreover, there exists some constants $c$ and $C > 0$ (only depending on $n$ and $\Omega$) such that
\[
c \varepsilon_0 < \Theta(x) < C \mathcal{H}^0 - \text{a.e. in } \Sigma
\]
where $H^0|\Sigma$ is the restriction to $\Sigma$ of the Hausdorff measure and $\Theta$ is a measurable function on $\Sigma$.

The main question we would like to address in the present paper concerns the multiplicity $\Theta$ of the defect measure which has been defined above. More precisely, we have proved the following theorem.

**Theorem 1.5.** Let $\nu$ be the defect measure of the sequence $(|\nabla u_i|^2 + |u_i|^{2n/(n-2)})dx$ defined above. Then $\nu$ is quantized. That is, for a.e $x \in \Sigma$,
\[
\Theta(x) = \sum_{j=1}^{N_x} \|
abla v_{x,j}\|_{L^2(\Omega)}^2 + \|v_{x,j}\|_{L^{2n/(n-2)}(\Omega)}^{2n/(n-2)}(1.4)
\]
where $N_x$ is a positive integer and where the functions $v_{x,j}$ are solutions of $\Delta v + \frac{v^{n+2}}{n} = 0$ which are defined on $\mathbb{R}^n$, issued from $(u_i)$ and that concentrate at $x$ as $i \to \infty$.

The sentence “issued from $(u_i)$ and that concentrate at $x$ as $i \to \infty$” means that there are sequences of conformal maps $\psi_i^j$, a finite family of balls $(B_{l_i,j})$ such that the pulled back function
\[
\tilde{u}_{i,j} = (\psi_i^j)^* u_i
\]
satisfies
\[
\tilde{u}_{i,j} \to v_j \text{ strongly in } L^2(\mathbb{R}^n \setminus \cup_l B_{l_i,j}), \quad \nabla \tilde{u}_{i,j} \to \nabla v_j \text{ strongly in } L^2(\mathbb{R}^n \setminus \cup_l B_{l_i,j})
\]

In the context of Yang-Mills fields in dimension $n \geq 4$ a similar concentration result has been proven by Rivi`ere [11]. More precisely, Rivi`ere has shown that, if $(A_i)_i$ is a sequence of Yang-Mills connections such that $(\|\nabla A_i \nabla F(A_i)\|_{L^1(B_1)})_i$ is bounded, then the corresponding defect measure $\nu = \Theta H^{n-4}|\Sigma$ of a sequence of smooth Yang-Mills connections is quantized.

The proof of Theorem 1.5 uses technics introduced by Lin and Rivi`ere in their study of Ginzburg-Landau vortices [10] and also the technics developed by Rivi`ere in [5]. These technics use as an essential tool the Lorentz spaces, more specifically the $L^2,\infty - L^{2,1}$ duality [14].

This paper is organized in the following way: In Section 2 we establish first a monotonicity formula for smooth solutions of problem (1.1) which allows us to prove an $\varepsilon$-regularity Theorem. Then, we prove Theorem 1.5 and Lemma 1.4. While Section 3 is devoted to the proof of our main result, Theorem 1.5.

### 2. A MONOTONICITY INEQUALITY

In this section, we establish a monotonicity formula for smooth solutions of problem (1.1). Using Pohozaev identity: Multiplying (1.1) by $x_i \frac{\partial u}{\partial x_i}$ (summation over $i$ is understood) and integrating over $B(x,r)$, the ball centered at $x$ of radius $r$, we obtain
\[
- \int_{B(x,r)} x_i \frac{\partial u}{\partial x_i} \Delta u \, dy = \int_{B(x,r)} x_i \frac{\partial u}{\partial x_i} u |u|^{1/(n-2)} \, dy
\]
By Green formula, we get
\[
\begin{align*}
\frac{n-2}{2} \int_{B(x,r)} |u|^{2n/(n-2)} dy - \frac{n-2}{2} \int_{B(x,r)} |\nabla u|^2 dy \\
- \frac{n-2}{2} \int_{\partial B(x,r)} |u|^{2n/(n-2)} ds \quad + \quad \frac{1}{2} \int_{\partial B(x,r)} |\nabla u|^2 ds \\
= r \int_{\partial B(x,r)} \frac{\partial u}{\partial r}^2 dy 
\end{align*}
\]  
(2.1)

On the other hand, multiplying (1.1) by \(u\) and integrating over \(B(x,r)\), we get
\[
\int_{B(x,r)} |\nabla u|^2 dy - \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds = \int_{B(x,r)} |u|^{2n/(n-2)} dy
\]  
(2.2)

Deriving (2.2) with respect to \(r\), we obtain
\[
\int_{\partial B(x,r)} |\nabla u|^2 dy - \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds = \int_{\partial B(x,r)} |u|^{2n/(n-2)} dy
\]  
(2.3)

Combining (2.1), (2.2) and (2.3), we get
\[
-\frac{r}{n} \int_{\partial B(x,r)} |u|^{2n/(n-2)} ds \\
= \frac{1}{2} \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds - r \int_{\partial B(x,r)} \frac{\partial u}{\partial r}^2 dy + r^{-1} u \frac{\partial u}{\partial r} ds.
\]

Moreover, we have that
\[
\frac{d^2}{dr^2} \left( \int_{\partial B(x,r)} u^2 ds \right) = \frac{d}{dr} \left( 2 \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds + \frac{n-1}{r} \int_{\partial B(x,r)} u^2 ds \right) \\
= (n-1) \left[ \frac{2}{r} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds + \left( \frac{n-1}{r^2} - \frac{1}{r^2} \right) \int_{\partial B(x,r)} u^2 ds \right] \\
+ 2 \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds \\
= \frac{n-1}{r} \left[ 2 \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds + \frac{n-2}{r} \int_{\partial B(x,r)} u^2 ds \right] \\
+ \frac{2}{r} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds.
\]

Hence
\[
\frac{1}{n} \frac{d}{dr} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d^2}{dr^2} \int_{\partial B(x,r)} u^2 ds \\
= \int_{\partial B(x,r)} \left( \frac{\partial u}{\partial r}^2 + \frac{2n-3}{2r} u \frac{\partial u}{\partial r} + \frac{(n-1)(n-2)}{4} r^{-2} u^2 \right) ds.
\]
Moreover,
\[
\frac{d}{dr} \left( \frac{1}{r} \int_{\partial B(x,r)} u^2 ds \right) = -\frac{1}{r^2} \int_{\partial B(x,r)} u^2 ds + \frac{2}{r} \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds + \frac{n-1}{r^2} \int_{\partial B(x,r)} u^2 ds
\]
\[
= \frac{n-2}{r^2} \int_{\partial B(x,r)} u^2 ds + \frac{2}{r} \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds.
\]

We obtain
\[
\frac{d}{dr} \left[ \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy \right] = \frac{1}{n} \int_{\partial B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{B(x,r)} u^2 ds - \frac{1}{n} \frac{1}{r} \int_{\partial B(x,r)} u^2 ds
\]
\[
= \int_{\partial B(x,r)} \left( \frac{\partial u}{\partial r} \right)^2 + (n-2) r^{-1} \frac{\partial u}{\partial r} + \frac{(n-2)^2}{n} r^{-2} u^2 ds
\]
\[
= \frac{n}{\partial B(x,r)} \left( \frac{\partial u}{\partial r} + \frac{n-2}{2} r^{-1} u \right)^2 ds \geq 0
\]

We conclude that
\[
E_u(x,r) = \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{B(x,r)} u^2 ds + \frac{1}{n} \frac{1}{r} \int_{\partial B(x,r)} u^2 ds \quad (2.4)
\]
is a nondecreasing function of \( r \). Using the fact that
\[
\int_{\partial B(x,r)} |u|^{2n/(n-2)} dy - \int_{\partial B(x,r)} |\nabla u|^2 dy = - \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds,
\]
one can easily get
\[
E_u(x,r) = \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{1}{n} \frac{1}{r} \int_{\partial B(x,r)} u^2 ds
\]
\[
= \frac{n}{2} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{2n} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds
\]
\[
+ \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{1}{4} r^{-1} \int_{\partial B(x,r)} u^2 ds
\]
\[
= \frac{1}{2} \int_{B(x,r)} |\nabla u|^2 dy - \frac{1}{2} \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds - \frac{n-2}{2n} \int_{B(x,r)} |u|^{2n/(n-2)} dy
\]
\[
+ \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{1}{4} r^{-1} \int_{\partial B(x,r)} u^2 ds
\]
\[
= \frac{1}{2} \int_{B(x,r)} (|\nabla u|^2 - \frac{n-2}{2n} |u|^{2n/(n-2)}) dy + \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds
\]
\[
- \frac{1}{4} r^{-1} \int_{\partial B(x,r)} u^2 ds - \frac{1}{2} \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds.
\]

We obtain an equivalent formulation of \( E_u(x,r) \)
\[
E_u(x,r) = \frac{1}{2} \int_{B(x,r)} (|\nabla u|^2 - \frac{n-2}{2n} |u|^{2n/(n-2)}) dy + \frac{n-2}{4} r^{-1} \int_{\partial B(x,r)} u^2 ds \quad (2.5)
\]

Moreover, using the fact that
\[
\frac{d}{dr} \int_{\partial B(x,r)} u^2 ds = 2 \int_{\partial B(x,r)} \frac{\partial u}{\partial r} ds + \frac{n-1}{r} \int_{\partial B(x,r)} u^2
\]
we obtain
\[
\frac{1}{r} \int_{\partial B(x,r)} u^2 \, ds = \frac{1}{n-1} \frac{d}{dr} \frac{1}{n-1} \int_{\partial B(x,r)} u^2 \, ds - \frac{2}{n-1} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds
\]
\[
= \frac{1}{n-1} \frac{d}{dr} \int_{\partial B(x,r)} u^2 \, ds
\]
\[
+ \frac{2}{n-1} \left[ \int_{B(x,r)} |u|^{2n/(n-2)} \, dy - \int_{B(x,r)} |\nabla u|^2 \, dy \right]
\]

Then \( E_u(x, r) \) can also be written
\[
E_u(x, r) = \frac{1}{2(n-1)} \int_{B(x,r)} (|\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)}) \, dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x,r)} u^2 \, ds.
\]

Proof of Lemma 1.1. To prove that \( (x, r) \mapsto E_u(x, r) \) is continuous it suffices to prove that
\[
(x, r) \mapsto \int_{\partial B(x,r)} u^2 \, ds
\]
is continuous with respect to \( x \) and \( r \). We have
\[
\int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds = \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |u|^{2n/(n-2)} \, dy
\]
Thus \( (x, r) \mapsto \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \) is continuous, and this allows to get the conclusion.

Now, to prove that \( E_u \) is positive, we proceed by contradiction. If the result is not true, then there would exists \( x \in \Omega \) and \( R > 0 \) such that \( E_u(x, R) < 0 \). For almost every \( y \) in some neighborhood of \( x \), we have
\[
\lim_{r \to 0} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds = 0
\]
integrating \( E_u(x, r) \) over the interval \([0, R]\) and using the fact that \( r \mapsto E_u(x, r) \) is increasing, we obtain
\[
\int_0^R E_u(y, r) \, dr = \frac{1}{2(n-1)} \int_0^R \frac{d}{dr} \int_{B(y,r)} (|\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)}) \, dx
\]
\[
+ \frac{n-2}{4(n-1)} \int_{\partial B(y,R)} u^2 \, ds
\]
\[
\leq RE_u(y, R) < 0
\]
which is not possible. This proves Lemma 1.1.

\[
\square
\]

Lemma 2.1. There exist \( r_0 > 0 \) and some constant \( c > 0 \), depending only on \( n \), such that
\[
\int_{B(x,r)} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy < cE_u(x, r)
\]
for any \( r < r_0/2 \).
Proof. Using the fact that \((x, r) \mapsto E_u(x, r)\) is nondecreasing, we have
\[
 r E_u(x, r) \geq \int_0^r E_u(x, s) \, ds = \frac{1}{2n-2} \int_0^r ds \int_{B(x,s)} (|\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)}) \, dy + \frac{n-2}{4(n-1)} \int_0^r ds \int_{\partial B(x,s)} u^2 \, d\sigma \\
 \geq \frac{1}{2(n-1)} \frac{n-2}{n} \int_0^r ds \int_{B(x,s)} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy \geq C(n) \frac{r}{2} \int_{B(x,\frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy
\]
where \(C(n)\) is a positive constant depending only on \(n\). This gives the desired result. \(\square\)

As a consequence of Lemma 2.1, we have the following result.

Lemma 2.2. Assume that there exist \(x_0\) and \(r_0 > 0\) such that \(E_u(x_0, r_0) \leq \varepsilon\) then
\[
\int_{B(x,r)} (|\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)}) \, dy \leq C\varepsilon \quad \forall \ 0 < r < 2r_0
\]
where \(C\) is a positive constant depending only on \(n\).

Proof. Let \(x_0\) and \(r_0\) be such that \(E_u(x_0, r_0) \leq \varepsilon\) and let \(0 < r < r_0\), then for all \(x \in B(x_0, \frac{r}{2})\) we have
\[
B(x, \frac{r}{2}) \subset B(x_0, r) \subset B(x_0, r_0)
\]
Thus
\[
E_u(x_0, r_0) \geq \frac{n-2}{2n(n-1)} \int_{B(x_0, \frac{r}{2})} |u|^{2n/(n-2)} \, dy + \frac{1}{2(n-1)} \int_{B(x_0, \frac{r}{2})} |\nabla u|^2 \, dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0, r)} u^2 \, d\sigma \\
\geq \frac{1}{2(n-1)} \int_{B(x_0, \frac{r}{2})} (|u|^{2n/(n-2)} + |\nabla u|^2) \, dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0, r)} u^2 \, d\sigma
\]
Integrating between 0 and \(r\), we obtain
\[
r E_u(x, r_0) \geq \frac{1}{2(n-1)} \int_0^r ds \int_{B(x, \frac{s}{2})} (|u|^{2n/(n-2)} + |\nabla u|^2) \, dy + \frac{n-2}{4(n-1)} \int_{\partial B(x_0, r)} u^2 \, d\sigma \\
\geq \frac{1}{2(n-1)} \int_0^r ds \int_{B(x, \frac{s}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy \\
\geq \frac{1}{2(n-1)} \int_0^r \int_{\frac{r}{2}}^s (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy \\
\geq \frac{1}{2(n-1)} \frac{r}{2} \int_{B(x, \frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy.
\]
Then
\[ E_u(x_0, r_0) \geq \frac{1}{4(n-1)} \int_{B(x_0, \frac{r_0}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy, \]

thus
\[ \int_{B(x, r)} (|\nabla u|^2 + |u|^{2n/(n-2)}) \, dy \leq C \varepsilon \quad \forall r < 2r_0. \]

This proves the desired result. \(\Box\)

**Proof of Theorem 1.2** Without loss of generality, we can assume that \(x_0 = 0\) and we denote by \(B_{r_0}\) the ball of radius \(r_0\) centered at \(x_0 = 0\).

We use the idea of Schoen [12]. For \(r < r_0\), we define
\[ F(y) = \left( \frac{r}{2} - |y| \right)^{(n-2)/2} u(y) \]

Clearly \(F\) is continuous over \(B_{\frac{r_0}{2}}\), then there exist \(y_0 \in B_{\frac{r_0}{2}}\) such that
\[ F(y_0) = \max_{y \in B_{\frac{r_0}{2}}} (\frac{r}{2} - |y|)^{(n-2)/2} u(y) = (\frac{r}{2} - |y_0|)^{(n-2)/2} u(y_0) \]

Let \(0 < \sigma < \frac{r_0}{2}\), for all \(y \in B_\sigma\), we have
\[ u(y) \leq \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} u(y_0) \]

Then
\[ \sup_{y \in B_\sigma} u(y) \leq \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} \sup_{y \in B_{\frac{r_0}{2}}} u(y) \]

where \(\sigma_0 = |y_0|\). Let \(y_1 \in B_{\sigma_0}\) be such that
\[ u(y_1) = \sup_{y \in B_{\sigma_0}} u(y) \]

We claim that
\[ u(y_1) \leq \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} 2^{(n-2)/2}. \]

Indeed, on the contrary case, we get
\[ (u(y_1))^{-2/(n-2)} \leq \frac{1}{2} \left( \frac{r}{2} - |y_0| \right)^{2(n-2)/2} \]

Let \(\mu = (u(y_1))^{-2/(n-2)}\). We have
\[ B_\mu(y_1) \subset B_{\frac{x_0 + \frac{r_0}{2}}{2}} \]

\(|z - y_1| < \mu \text{ take } |z| < \frac{\hat{\mu} + |y_0|}{2}\). Hence
\[ \sup_{y \in B_\mu(y_1)} u(y) \leq \left( \frac{\hat{\mu}}{2} - |y_0| \right)^{(n-2)/2} u(y_1) = 2^{(n-2)/2} u(y_1) \]

Let \(v(x) = \mu^{(n-2)/2} u(\mu x + y_1)\). Easy computations shows that \(v\) satisfies
\[ \Delta v^{2n/(n-2)} = \frac{2n}{n-2} \left[ \frac{n+2}{n-2} v^{4/(n-2)} |\nabla v|^2 + v^\frac{n+2}{n-2} \Delta v \right] \]
\[ \geq \frac{2n}{n-2} v^\frac{n+2}{n-2} \Delta v = -\frac{2n}{n-2} v^\frac{n+2}{n-2} \]
On the other hand
\[
v^{2n/(n-2)}(0) = \mu \frac{2n}{n-2} u^{n/(n-2)}(y_1) = 1.
\]
Moreover, we have
\[
\sup_{B_1} v(x) = \mu^{(n-2)/2} \sup_{B_1} u(\mu x + y_1)
\]
\[
= \mu^{(n-2)/2} \sup_{B_\mu(y_1)} u(x)
\]
\[
\leq \mu^{(n-2)/2} 2^{(n-2)/2} u(y_1) = 2^{(n-2)/2}.
\]
Then \(\sup_{B_1} v^{2n/(n-2)} \leq 2^n\). Therefore,
\[
-\Delta v^{2n/(n-2)} \leq C(n) v^{2n/(n-2)}.
\]
We conclude that
\[
1 = v^{2n/(n-2)}(0) \leq C \int_{B_1} v^{2n/(n-2)}(x) dx = C \mu^n \int_{B_\mu} u^{2n/(n-2)}(x) dx \leq C \varepsilon.
\]
For \(\varepsilon\) sufficiently small, we derive a contradiction. It follows that
\[
\sup_{B_\frac{r}{2}} u(y) \leq \left(\frac{r}{2} - |y_0|\right)^{(n-2)/2} \cdot \frac{2^{(n-2)/2}}{(r/2 - |y_0|)^{(n-2)/2}} = \frac{2^{(n-2)/2}}{(r - |y|)^{(n-2)/2}}.
\]
For \(|y| < r/4\), we have
\[
\sup_{B_\frac{r}{2}} u(y) \leq C(n)/v^{(n-2)/2}
\]
This in turns proves the Theorem □

**Proof of Lemma 1.4**. We keep the above notations. To show (i), suppose \(x_0 \in B_1 \setminus \Sigma\), then there exists \(r_1 > 0\) such that
\[
\liminf_{i \to \infty} E_{u_i}(x_0, r_1) < \varepsilon_0.
\]
Then, we may find a sequence \(n_j \to \infty\) as \(j \to \infty\) such that
\[
\sup_{n_j} E_{u_{n_j}}(x_0, r_1) < \varepsilon_0.
\]
We deduce from the \(\varepsilon\)-regularity Theorem (Theorem 1.2) that
\[
\sup_{n_j} \sup_{x \in B_{\frac{r}{2n}}(x_0)} |u_{n_j}| \leq \frac{C}{r^{(n-2)/2}},
\]
for some constant \(C\) depending only on \(n\). Then
\[
\lim_{n_j} u_{n_j} \to u \quad \text{in} \quad C^1(B_{\frac{r}{8n}}(x_0))
\]
a similar argument allows to show that
\[
\nabla u_{n_j} \to \nabla u \quad \text{in} \quad C^1(B_{\frac{r}{4n}}(x_0))
\]
Then
\[
\mu_{n_j} := \left(\frac{1}{2} |\nabla u_{n_j}|^2 + \frac{n-2}{2n} u_{n_j}^{2n/(n-2)}\right) dx \to \left(\frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} u^{2n/(n-2)}\right) dx
\]
as radon measure. Hence \(\nu = 0\) on \(B_{\frac{r}{4n}}(x_0)\) i.e \(x_0 \notin \text{supp}(\nu)\) and then we deduce that \(\text{supp}(\nu) \subset \Sigma\).
To show (ii), let us first recall some properties of the function $E_u(x, r)$ that has been defined above:

- For all $x \in \Omega$, there exists $r_0 > 0$ and a constant $C > 0$ such that
  \[
  \int_{B(x,r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) \leq CE_u(x,r_0) \quad \forall r < \frac{r_0}{2}.
  \]
  This is explained in the proof of Lemma 1.1.

- Using the fact that $E_u(x, r)$ is increasing on $r$ together with the fact that
  \[
  \lim_{r \searrow 0} E_u(x, r) = 0 \quad \mathcal{H}^0 \text{-a.e. } x \in \Omega
  \]
  we deduce that for $\mathcal{H}^0$-a.e. $x \in \Sigma$, $\lim_{r \searrow 0} \int_{B(x,r)} \nu$ exists. and the density $\Theta(\eta, .)$ defined by
  \[
  \Theta(\eta, x) := \lim_{r \searrow 0} \eta(B_r(x)) (2.6)
  \]
  exists for every $x \in \Omega$. Moreover, for $\mathcal{H}^0$-a.e. $x \in \Omega$, $\Theta_u(x) = 0$, where
  \[
  \Theta_u(x) := \lim_{r \searrow 0} \int_{B(x,r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) dy. (2.7)
  \]
  Now, for $r$ sufficiently small and $i$ sufficiently large
  \[
  \int_{B(x,r)} \frac{1}{2} |\nabla u_i|^2 + \frac{n-2}{2n} |u_i|^{2n/(n-2)} \leq CE_{u_i}(x,r) \leq C(\Lambda, \Omega) (2.8)
  \]
  where $\Lambda$ is given above and $C(\Lambda, \Omega)$ is a constant depending only on $\Lambda$ and $\Omega$. Hence
  \[
  \eta(B(x, r)) \leq C(\Lambda, \Omega) \quad \text{for } x \in B^n_1 \quad (2.9)
  \]
  In particular, this implies that $\eta|\Sigma$ is absolutely continuous with respect to $\mathcal{H}^0|\Sigma$. Applying Radon-Nikodym’s Theorem [3], we conclude that
  \[
  \eta|\Sigma = \Theta(x)|\mathcal{H}^0|\Sigma \quad \text{for } \mathcal{H}^0\text{-a.e. } x \in \Sigma (2.10)
  \]
  Using 2.8 we conclude that
  \[
  \nu(x) = \Theta(x)|\mathcal{H}^0|\Sigma \quad (2.11)
  \]
  for a $\mathcal{H}^0$-a.e. $x \in \Sigma$ (recall that $\eta = (\frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)}) \ dx + \nu$ and supp($\nu$) $\subset \Sigma$).

For any $y \in B^n_1$ and any sufficiently small $\lambda > 0$, we define the scaled measure $\eta_{y, \lambda}$ by
\[
\eta_{y, \lambda}(x) := \eta(y + \lambda x) \quad (2.12)
\]
We have the following lemma.

**Lemma 2.3.** Assume that $(\lambda_j)_j$ satisfies $\lim_{j \to \infty} \lambda_j = 0$. Then, there exist a subsequence $(\lambda_j')_j$ and a Radon measure $\chi$ defined on $\Omega$, such that $\eta_{y, \lambda_j'} \rightharpoonup \chi$ in the sense of measures.

**Proof.** For each $i \in \mathbb{N}$, we define the scaled function $u_{i,y, \lambda}$ by
\[
u(x) = \Theta(x)|\mathcal{H}^0|\Sigma \quad (2.11)
\]
for a $\mathcal{H}^0$-a.e. $x \in \Sigma$ (recall that $\eta = (\frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)}) \ dx + \nu$ and supp($\nu$) $\subset \Sigma$).

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**Proof.** For each $i \in \mathbb{N}$, we define the scaled function $u_{i,y, \lambda}$ by
\[
u(x) = \Theta(x)|\mathcal{H}^0|\Sigma \quad (2.11)
\]
for a $\mathcal{H}^0$-a.e. $x \in \Sigma$ (recall that $\eta = (\frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)}) \ dx + \nu$ and supp($\nu$) $\subset \Sigma$).
In addition, for any \( r > 0 \) sufficiently small, we have
\[
\int_{B_r(0)} \left( \frac{1}{2} \left| \nabla u_{i,y,\lambda} \right|^2 + \frac{n-2}{2n} |u_{i,y,\lambda}|^{2n/(n-2)} \right) \, dx = \int_{B_{2r}(y)} \left( \frac{1}{2} \left| \nabla u_i \right|^2 + \frac{n-2}{2n} |u_i|^{2n/(n-2)} \right) \, dx \leq C(\Lambda, \Omega). \tag{2.14}
\]
Finally for fixed \( \lambda \),
\[
\left( \frac{1}{2} \left| \nabla u_{i,y,\lambda} \right|^2 + \frac{n-2}{2n} |u_{i,y,\lambda}|^{2n/(n-2)} \right)(x) \, dx = \lambda^n \left( \frac{1}{2} \left| \nabla u_i \right|^2 - \frac{n-2}{2n} |u_i|^{2n/(n-2)} \right)(\lambda x + y) \, dx \to \eta(\lambda x + y) = \eta_{y,\lambda}(x)
\]
in the sense of measures as \( i \to \infty \). On the other hand letting \( i \) tends to infinity in (2.14), we conclude that for any \( r > 0 \)
\[
\eta_{y,\lambda}(B_r(0)) \leq C(\Omega, \Lambda). \tag{2.15}
\]
Hence, we may find a subsequence \( \{\lambda'_j\} \) of \( \{\lambda_j\} \) and a Radon measure \( \chi \) such that \( \eta_{y,\lambda'_j} \) converge weakly to \( \chi \) as Radon measure on \( \Omega \). Then
\[
\lim_{j \to \infty} \lim_{i \to \infty} \int \left( \frac{1}{2} \left| \nabla u_{i,y,\lambda'_j} \right|^2 + \frac{n-2}{2n} |u_{i,y,\lambda'_j}|^{2n/(n-2)} \right) \, dx = \lim_{j \to \infty} \eta_{y,\lambda'_j}(x) = \chi
\]
Using a diagonal subsequence argument, we may find a subsequence \( i_j \to \infty \), such that
\[
\lim_{j \to \infty} \int \left( \frac{1}{2} \left| \nabla u_{i,y,\lambda'_j} \right|^2 + \frac{n-2}{2n} |u_{i,y,\lambda'_j}|^{2n/(n-2)} \right) \, dx = \chi
\]
This proves the Lemma. \( \square \)

**Remark 2.4.** Observe that
\[
\chi(B_r(0)) = \lim_{j \to \infty} \eta_{y,\lambda'_j}(B_r(0)) = \lim_{j \to \infty} \eta(B_{\lambda'_j r}(y)) = \Theta(\eta, y)
\]
In particular, we deduce that \( \chi(B_r(0)) \) is independent of \( r \).

### 3. Proof of Theorem 1.5

The idea of the proof comes from Riviè re [11] in the context of Yang-Mills Fields. To simplify notation and since the result is local, we assume that \( \Omega \) is the unit ball \( B^n \) of \( \mathbb{R}^n \). Let \( (u_k) \) be a sequence of smooth solutions of (1.1) such that
\[
\left( \|u_k\|_{H^1(\Omega)} + \|u_k\|_{L^{2n/(n-2)}(\Omega)} \right)
\]
is bounded and let \( \nu \) be the defect measure defined above. We claim that for \( \delta > 0 \), we have
\[
\lim_{k \to \infty} \sup_{y \in B_1(x_0)} \int_{B_2(y_0)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \geq \varepsilon(n) \tag{3.1}
\]
where \( \varepsilon(n) \) is given by Theorem 1.5. Indeed if (3.1) would not hold, we have for \( \delta > 0 \) and \( k \in \mathbb{N} \) large enough
\[
\sup_{y \in B_1(x_0)} \int_{B_2(y_0)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \leq \varepsilon(n)
\]
and by Theorem 1.2 we have
\[ \|\nabla u_k\|_{L^\infty(B_{\frac{1}{2}}(y))} \leq C(\varepsilon)/r^{n/2} \]
This contradict the concentration phenomenon and the claim is proved. We then conclude that there exists sequences \( \delta_k \to 0 \) as \( k \to \infty \) and \( (y_k) \subset B_1(x_0) \) such that
\[
\int_{B_{\delta_k}(y_k)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \, dx = \sup_{y \in B_1(x_0)} \int_{B_{\delta_k}(y_0)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \, dx \\
= \frac{\varepsilon(n)}{2}.
\] (3.2)

In other words, \( y_k \) is located at a bubble of characteristic size \( \delta_k \). More precisely, if one introduces the function
\[ \tilde{u}_k(x) = \delta_k^{(n-2)/2} u_k(\delta_k x + y_k); \]
we have, up to a subsequence, that
\[ \tilde{u}_k \to u_\infty \text{ in } C_\text{loc}^\infty(\mathbb{R}^n) \text{ as } k \to \infty, \]
\[ \nabla \tilde{u}_k \to \nabla u_\infty \text{ in } C_\text{loc}^\infty(\mathbb{R}^n) \text{ as } k \to \infty. \]
Therefore,
\[ -\Delta u_\infty = |u_\infty|^{4/(n-2)} \text{ in } \mathbb{R}^n. \]
This is the first bubble we detect. On the other hand, we have clearly that
\[
\int_{\mathbb{R}^n} \left( |u_\infty|^{2n/(n-2)} + |\nabla u_\infty|^2 \right) \, dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_{R\delta_k}(y_k)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \, dx.
\] (3.3)

Indeed:
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_{R\delta_k}(y_k)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) \, dx \\
= \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R(0)} \left( |u_k|^{2n/(n-2)} + |\nabla(u_k)|^2 \right) (\delta_k x + y_k) \delta_k^n \, dx \\
= \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R(0)} \left( |\delta_k^{\frac{2n}{n-2}} \tilde{u}_k(x)|^{2n/(n-2)} + |\delta_k^{\frac{2n-n}{n-2}} \delta_k^{-1} \nabla \tilde{u}_k(x)|^2 \right) \delta_k^n \, dx \\
= \lim_{R \to \infty} \int_{B_R(0)} \left( |u_\infty(x)|^{2n/(n-2)} + |\nabla u_\infty(x)|^2 \right) \, dx \\
= \int_{\mathbb{R}^n} \left( |u_\infty(x)|^{2n/(n-2)} + |\nabla u_\infty(x)|^2 \right) \, dx.
\]

Assume first that we have only one bubble of characteristic \( \delta_k \). We have shown that
\[ \Theta = \lim_{k \to \infty} \int_{B_{\frac{1}{2}}(0)} \left( |\nabla u_k|^2 + |u_k|^{2n/(n-2)} \right) \, dx = \int_{\mathbb{R}^n} \left( |\nabla u_\infty|^2 + |u_\infty|^{2n/(n-2)} \right) \, dx, \] (3.4)
where $\Theta$ is defined above. It suffices to prove that
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_1^t(0) \setminus B_{R\delta_k}(y_k)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) \, dx = 0. \tag{3.5}
\]

In other words, there is no "neck" of energy which is quantized.

To simplify notation, we assume that $y_k = 0$. We claim that for any $\varepsilon > 0$ small enough, there exists $R > 0$ and $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ and $R\delta_k \leq r \leq \frac{1}{2}$, we have
\[
\int_{B_R^2(0) \setminus B_r(0)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) \, dx \leq \varepsilon \tag{3.6}
\]

Indeed, if is not the case, we may find $\varepsilon_0 > 0$, a subsequence $k' \to \infty$ (Still denoted $k$) and a sequence $r_k$ such that
\[
\int_{B_R^2(0) \setminus B_r(0)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) \, dx \geq \varepsilon_0, \tag{3.7}
\]

where $\Theta$ is defined above. It suffices to prove that for any $\varepsilon_0 > 0$, a subsequence $k' \to \infty$ (Still denoted $k$) and a sequence $r_k$ such that $r_k/\delta_k \to \infty$ as $k \to \infty$,
\[
\int_{B_R^2(0) \setminus B_r(0)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) \, dx \geq \varepsilon_0.
\]

Let $\alpha_k \to 0$ such that $r_k/\alpha_k = o(1)$ and $\alpha_k r_k/\delta_k \to \infty$ and let
\[
v_k(x) = r_k^{n-2}/u_k(r_k x)
\]

clearly $v_k$ satisfies
\[
-\Delta v_k = v_k |v_k|^{4/(n-2)} \quad \text{in } B_{2\alpha_k} \setminus B_{\alpha_k}
\]

Therefore,
\[
\int_{B_R^2(0) \setminus B_r(0)} \left( |v_k(x)|^{2n/(n-2)} + |\nabla v_k(x)|^2 \right) \, dx \geq \varepsilon(n)
\]

and then we have a second bubble. This contradicts our assumption.

We deduce from (3.7) and Theorem 1.2 that for any $\varepsilon < \varepsilon(n)$, there exist $R > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $|x| \geq R\delta_k$
\[
|\nabla u_k(x)| \leq C(\varepsilon)/|x|^{n/2}
\]

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then
\[
|\nabla u_k|^2(x) \leq C(\varepsilon)/|x|^n. \tag{3.8}
\]

We define $E^k_\lambda$ by
\[
E^k_\lambda = \text{meas} \{ x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda \}
\]

We have $E^k_\lambda \leq C(\varepsilon)/\lambda^2$; indeed
\[
\{ x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda \} \subset \{ x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2} \}
\]

and
\[
\text{meas} \left\{ x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2} \right\} \leq \frac{C(\varepsilon)}{\lambda^2}
\]

We deduce from (3.8) that
\[
\| \nabla u_k \|_{L^{2,\infty}(C_R \delta_k)} \leq C(\varepsilon) \tag{3.9}
\]

where $L^{2,\infty}$ is the Lorentz space defined in [14], the weak $L^2$ space, and $\| \cdot \|_{L^{2,\infty}}$ is the weak norm defined by
\[
\| f \|_{L^{2,\infty}} = \sup_{0 < t < \infty} t^{1/2} f^*(t)
\]
where $f^*$ is the nonincreasing rearrangement of $|f|$. Indeed
\[
\|\nabla u_k\|_{L^{2,\infty}(C_{B_Rk})} = \sup_{0<t<\infty} t^{1/2}(\nabla u_k)^*(t)
\]
by definition,\[
(\nabla u_k)^*(t) = \inf\{\lambda > 0 : E_{\lambda}^k \leq t\}
\]
For all $t > 0$ such that $C(\varepsilon) \leq t$, we have $E_{\lambda}^k \leq t$. Then
\[
\inf\{\lambda > 0 : E_{\lambda}^k \leq t\} \leq \inf\left\{\lambda > 0 : \frac{C(\varepsilon)}{\lambda^2} \leq t\right\}
\]
\[
\leq \inf\left\{\lambda > 0 : \frac{C(\varepsilon)^{1/2}}{t^{1/2}}\right\}
\]
\[
= \frac{(C(\varepsilon))^{1/2}}{t^{1/2}}
\]
Hence $t^{1/2}(\nabla u_k)^*(t) \leq C(\varepsilon)$ and so
\[
\|\nabla u_k\|_{L^{2,\infty}(C_{B_Rk})} \leq C(\varepsilon) \quad (3.10)
\]
We claim that the sequence $(\nabla u_k)$ is uniformly bounded in the Lorentz space $L^{2,1}(B^n_t)$ (see [14] for the definition). We prove this claim using an iteration proceeding; Indeed, the sequence $(u_k)$ is bounded in $L^{2\pi\tau}(B^n_t)$. Then
\[
\Delta u_k = -u_k|u_k|^{4/(n-2)}
\]
is bounded in $L^{2\pi\tau}(B^n_t)$ which implies by the elliptic regularity Theorem that the sequence $(u_k)$ is bounded in $W^{2,\pi\tau}(B^n_t)$. Using the imbedding Theorem for Sobolev spaces
\[
W^{m,p}(B^n_t) \subset W^{r,s}(B^n_t) \quad \text{if} \ m \geq r, \ p \geq s \ \text{and} \ m - \frac{n}{p} = r - \frac{n}{s}.
\]
In particular, $W^{2,\frac{2\pi\tau}{n-2}}(B^n_t)$ is continuously imbedded in $W^{1,2}(B^n_t)$. On the other hand by Proposition 4 in [14], we have
\[
W^{1,2}(B^n_t) \hookrightarrow L^{2\pi\tau,2}(B^n_t) = L^{\frac{2\pi\tau}{n-2},2}(B^n_t)
\]
continuously. We then deduce that
\[
\Delta u_k = -u_k|u_k|^{4/(n-2)}
\]
is bounded in $L^{2\pi\tau,2}(B^n_t)$. Here, we have used the following lemma.

**Lemma 3.1.** If $f \in L^{p,q}(B^n_t)$ and $\alpha \in \mathbb{Q}^+$, then $f^\alpha \in L^{p',\frac{n}{\alpha}}(B^n_t)$.

**Proof.** In the case where $\alpha \in \mathbb{N}$, the result follows from the fact that
\[
f \in L^{a,b}(B^n_t) \quad \text{and} \quad g \in L^{c,d}(B^n_t) \Rightarrow fg \in L^{a',d'}(B^n_t),
\]
where $\frac{1}{q} = \frac{1}{a} + \frac{1}{b}$ and $\frac{1}{r} = \frac{1}{c} + \frac{1}{d}$ (see [2]). The general case is a consequence of the fact that the increasing rearrangement of the function $|f|^\beta$ is equal to the puissance $\beta$ of the increasing rearrangement of $|f|$ since $(f^\beta)^*$ is the only one function verifying\[
\text{meas}\{x \in \mathbb{R}^n : f^\beta(x) \geq \lambda\} = \text{meas}\{t > 0 : (f^\beta)^*(x) \geq \lambda\}
\]
This in turns proves Lemma 3.1. \qed
Now, using in [14, Theorem 8], we deduce from (3.7) that \((\nabla u_k)\) is uniformly bounded in the space \(L^{2/(n+2)}(B_1^n)\) = \(L^{2/(n+2)}(B_1^n)\). Hence \((u_k)\) is bounded in \(L^{2/(n+2)}(B_1^n)\). Then
\[
\Delta u_k = -u_k |u_k|^{4/(n-2)}
\]
is bounded in \(L^{2n/(2n-3p)}(B_1^n)\). Hence, again by [14, Theorem 8], the sequence \((\nabla u_k)\) is bounded in \(L^{2/(n+2)}(B_1^n)\) and by elliptic regularity Theorem
\[
\Delta u_k = -u_k |u_k|^{4/(n-2)}
\]
is bounded in \(L^{2/(n+2)}(B_1^n)\). We obtain after \(p\) iterations that
\[
\Delta u_k = -u_k |u_k|^{4/(n-2)}
\]
is bounded in \(L^{2n/(2n-3p)}(B_1^n)\). We choose \(p > 0\) such that \(6p > n\), we have in particular \(2(n-2)p/(n+2)p < 1\) which gives
\[
\Delta u_k = -u_k |u_k|^{4/(n-2)}
\]
is bounded in \(L^{2n/(2n-3p)}(B_1^n)\). Here we have used the fact that
\[
L^{p,q_1}(B_1^n) = L^{p,q_2}(B_1^n)\quad \text{if} \quad q_1 < q_2.
\]
We use also [14, Theorem 8] to deduce that \((\nabla u_k)\) is uniformly bounded in \(L^{2/(n+2)}(B_1^n)\). In particular, there exist a constant \(C > 0\) depending only on \(n\) such that
\[
\|\nabla u_k\|_{L^{2/(n+2)}(B_1^n)} \leq C.
\]
We deduce from (3.10), (3.11) together with the \(L^{2,1} - L^{2,\infty}\) duality that
\[
\|\nabla u_k\|_{L^2(B_1^n \setminus B_R\delta_k)} \leq \| \nabla u_k \|_{L^{2,1}(B_1^n \setminus B_R\delta_k)} \| \nabla u_k \|_{L^{2,\infty}(B_1^n \setminus B_R\delta_k)} \leq C(\varepsilon)
\]
for a constant \(C(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Now, we use the embedding \(H^1 \hookrightarrow L^{2n/(n-2)}\) continuously, we obtain
\[
\| u_k \|_{L^{2n/(n-2)}(B_1^n \setminus B_R\delta_k)} \leq C \| \nabla u_k \|_{L^2(B_1^n \setminus B_R\delta_k)} \leq C(\varepsilon) \to 0\quad \text{as} \quad \varepsilon \to 0.
\]
We deduce that
\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_1^n(0) \setminus B_R\delta_k(0)} (|u_k|^{2n/(n-2)} + |\nabla u_k|)(x) \, dx = 0
\]
This proves Theorem [13] in the case of one bubble.

The case of more than one bubble can be handled in a very similar way and we just give few details for \(m = 2\). The proof starts the same until (3.11) which cannot hold any more otherwise we would have had one bubble only as it is (3.11) holds. It remains to show that: for any \(\varepsilon \geq 0\), there are sufficiently large \(R > 0\) and a sequence \(r_i \to 0\) such that for any \(R\delta_i \leq r_i \leq 1/2,
\[
\lim_{R \to \infty} \lim_{i \to \infty} \int_{(0) \times B_{r_i} \setminus B_R\delta_i(0)} \left( \frac{1}{2} |\nabla v_i|^2 + \frac{n-2}{2n} |v_i|^{2n/(n-2)} \right) \, dx = 0,
\]
\[
\lim_{i \to \infty} \int_{(0) \times B_{r_i} \setminus B_{r_i}(0)} \left( \frac{1}{2} |\nabla v_i|^2 + \frac{n-2}{2n} |v_i|^{2n/(n-2)} \right) \, dx = 0
\]
where $v_i$ is defined by $v_i(y) = r_i^{(n-2)/2} u_i(r_i y), \ y \in \mathbb{R}^n$.

The proof of (3.12) can be done exactly as the proof of (3.4), the case of 2 bubbles is then proved. To prove the general case, for any number $m \geq 2$, one can follow exactly the same strategy.

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