HS-integral normal mixed Cayley graphs

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Abstract

If the eigenvalues of a mixed graph’s Hermitian-adjacency matrix of the second kind are integers, the graph is called HS-integral. The set \( S \) for which a normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral is provided.

Keywords. integral graphs; HS-integral mixed graph; Eisenstein integral mixed graph; normal mixed Cayley graph.

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1 Introduction

A mixed graph \( G \) is a pair \( (V(G), E(G)) \), where \( V(G) \) and \( E(G) \) are the vertex set and the edge set of \( G \), respectively. Here \( E(G) \subseteq V(G) \times V(G) \setminus \{(u, u) \mid u \in V(G)\} \). If \( G \) is a mixed graph, then \( (u, v) \in E(G) \) need not imply that \( (v, u) \in E(G) \). An edge \( (u, v) \) of a mixed graph \( G \) is called undirected if both \( (u, v) \) and \( (v, u) \) belong to \( E(G) \). An edge \( (u, v) \) of a mixed graph \( G \) is called directed if \( (u, v) \in E(G) \) but \( (v, u) \notin E(G) \). A mixed graph can have both undirected and directed edges. A mixed graph \( G \) is said to be a simple graph if all the edges of \( G \) are undirected. A mixed graph \( G \) is said to be an oriented graph if all the edges of \( G \) are directed.

For a mixed graph \( G \) on \( n \) vertices, its \((0, 1)\)-adjacency matrix and Hermitian-adjacency matrix of the second kind are denoted by \( A(G) = (a_{uv})_{n \times n} \) and \( H(G) = (h_{uv})_{n \times n} \), respectively, where

\[
a_{uv} = \begin{cases} 
1 & \text{if } (u, v) \in E \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
h_{uv} = \begin{cases} 
1 & \text{if } (u, v) \in E \text{ and } (v, u) \in E \\
\frac{1 + i\sqrt{3}}{2} & \text{if } (u, v) \in E \text{ and } (v, u) \notin E \\
\frac{1 - i\sqrt{3}}{2} & \text{if } (u, v) \notin E \text{ and } (v, u) \in E \\
0 & \text{otherwise.}
\end{cases}
\]
The Hermitian-adjacency matrix of the second kind was introduced by Bojan Mohar [26].

Let $G$ be a mixed graph. By an HS-eigenvalue of $G$, we mean an eigenvalue of $H(G)$. By an eigenvalue of $G$, we mean an eigenvalue of $A(G)$. Similarly, the HS-spectrum of $G$, denoted $Sp_H(G)$, is the multi-set of the HS-eigenvalues of $G$, and the spectrum of $G$, denoted $Sp(G)$, is the multi-set of the eigenvalues of $G$. Note that the Hermitian-adjacency matrix of the second kind of a mixed graph is a Hermitian matrix, and so its HS-eigenvalues are real numbers. However, if a mixed graph $G$ contains at least one directed edge, then $A(G)$ is non-symmetric. Accordingly, the eigenvalues of $G$ need not be real numbers. The matrix obtained by replacing $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$ by $i$ and $-i$, respectively, in $H(G)$, is called the Hermitian adjacency matrix of $G$. Hermitian adjacency matrix of mixed graphs was introduced in $[13, 24]$. A mixed graph is called $H$-integral if the eigenvalues of its Hermitian adjacency matrix are integers. A mixed graph $G$ is said to be HS-integral if all the HS-eigenvalues of $G$ are integers. A mixed graph $G$ is said to be Eisenstein integral if all the eigenvalues of $G$ are Eisenstein integers. Note that complex numbers of the form $a + b\omega_3$, where $a, b \in \mathbb{Z}, \omega_3 = \frac{-1+i\sqrt{3}}{2}$, are called Eisenstein integers. An HS-integral simple graph is called an integral graph. Note that $A(G) = H(G)$ for a simple graph $G$. Therefore in case of a simple graph $G$, the terms H-eigenvalue, HS-spectrum and HS-integrality of $G$ are the same with that of eigenvalue, spectrum and integrality of $G$, respectively.

Integrality of simple graphs have been extensively studied in the past. Integral graphs were first defined by Harary and Schwenk [14] in 1974 and proposed a classification of integral graphs. See [4] for a survey on integral graphs. Watanabe and Schwenk [31, 32] proved several interesting results on integral trees in 1979. Csikvari [10] constructed integral trees with arbitrary large diameters in 2010. Further research on integral trees can be found in [7, 6, 29, 30]. In 2009, Ahmadi et al. [2] proved that only a fraction of $2^{-\Omega(n)}$ of the graphs on $n$ vertices have an integral spectrum. Bussemaker et al. [8] proved that there are exactly 13 connected cubic integral graphs. Stevanović [28] studied the 4-regular integral graphs avoiding $\pm 3$ in the spectrum, and Lepović et al. [23] proved that there are 93 non-regular, bipartite integral graphs with maximum degree four. In 2017, Guo et. al. [13] found all possible mixed graphs on $n$ vertices with spectrum $\{-n + 1, -1, -1, ..., -1\}$.

Throughout the paper, we consider $\Gamma$ to be a finite group. Let $S$ be a subset, not containing the identity element, of $\Gamma$. The set $S$ is said to be symmetric (resp. skew-symmetric) if $S$ is closed under inverse (resp. $a^{-1} \notin S$ for all $a \in S$). Define $\overline{S} = \{u \in S : u^{-1} \notin S\}$. Clearly, $S \setminus \overline{S}$ is symmetric and $\overline{S}$ is skew-symmetric. The mixed Cayley graph $G = Cay(\Gamma, S)$ is a mixed graph, where $V(G) = \Gamma$ and $E(G) = \{(a, b) : a, b \in \Gamma, ba^{-1} \in S\}$. If $S$ is symmetric then $G$ is a simple Cayley graph. If $S$ is skew-symmetric then $G$ is an oriented Cayley graph. A mixed Cayley graph $Cay(\Gamma, S)$ is called normal if $S$ is the union of some conjugacy classes of the group $\Gamma$.

If every element of a group is conjugate to its inverse, the group is said to be ambivalent. All symmetric groups $S_n$ are ambivalent, as is widely known. The alternating groups $A_n$ are ambivalent only
for \( n \in \{1, 2, 5, 6, 10, 14\} \). Due to the fact that, all normal mixed Cayley graphs over an ambivalent group are undirected. Thus, in order to examine the integrality of normal mixed Cayley graph with at least one directed edge, we must look at non-ambivalent groups. A basic example of a non-ambivalent group is the Abelian group.

In 1982, Bridge and Mena [5] introduced a characterization of integral Cayley graphs over abelian groups. Later on, same characterization was rediscovered by Wasin So [27] for cyclic groups in 2005. In 2009, Abdollahi and Vatandoost [1] proved that there are exactly seven connected cubic integral Cayley graphs. On the same year, Klotz and Sander [21] proved that if Cayley graph \( \text{Cay}(\Gamma, S) \) over an abelian group \( \Gamma \) is integral then \( S \) belongs to the Boolean algebra \( \mathcal{B}(\Gamma) \) generated by the subgroups of \( \Gamma \). Moreover, they conjectured that the converse is also true, which was proved by Alperin and Peterson [3]. In 2014, Cheng et al. [22] proved that normal Cayley graphs over symmetric groups are integral. In 2017, Lu et al. [25] gave necessary and sufficient condition for the integrality of Cayley graphs over the dihedral group \( D_n = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle \). In particular, they completely determined all integral Cayley graphs over the dihedral group \( D_p \) for a prime \( p \). In 2019, Cheng et al. [9] obtained several simple sufficient conditions for the integrality of Cayley graphs over the dicyclic group \( T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle \). In particular, they also completely determined all integral Cayley graphs over the dicyclic group \( T_{4p} \) for a prime \( p \). In [20], [19], [16], [18] and [17], we addressed a characterization of H-integral, HS-integral, Gaussian integral, and Eisenstein integral of mixed Cayley graphs over cyclic group and abelian group. In 2014, Godsil et al. [12] characterised integral normal Cayley graphs. We have also offered a characterisation of H-integral normal mixed Cayley graphs in [16]. We characterise the HS-integrality of normal mixed Cayley graphs in terms of their connection set in this study.

The following is how this document is structured. Some preliminary notions and results are addressed in Section 2. We express the HS-eigenvalues of a normal mixed Cayley graph, in particular, as the sum of the HS-eigenvalues of a simple Cayley graph and an oriented Cayley graph. We characterise the HS-integrality of normal oriented Cayley graphs in terms of symbol set in section 3. Then, in 4, we extend this characterisation to normal mixed Cayley graphs.

2 Preliminaries

Let \( \Gamma \) be a finite group and \( g, h \in \Gamma \). If \( g = x^{-1}hx \) for some \( x \in \Gamma \) then we say that \( h \) is a conjugate of \( g \). The set of all conjugates of \( g \) in \( \Gamma \) is called the conjugacy class of \( g \) and is denoted by \( \text{Cl}(g) \). Define \( C_{\Gamma}(g) \) to be the set of all elements of \( \Gamma \) that commute with \( g \). We denote the group algebra of \( \Gamma \) over the field \( \mathbb{Q}(\omega_3) \) by \( \mathbb{Q}(\omega_3)\Gamma \). That is, \( \mathbb{Q}(\omega_3)\Gamma \) is the set of all sums \( \sum_{g \in \Gamma} c_g g \), where \( c_g \in \mathbb{Q}(\omega_3) \) and we assume \( 1.g = g \) to have \( \Gamma \subseteq \mathbb{Q}(\omega_3)\Gamma \).
A representation of a finite group $\Gamma$ is a homomorphism $\rho : \Gamma \to GL(V)$ for some finite-dimensional vector space $V$ over the complex field $\mathbb{C}$, where $GL(V)$ denotes the group of automorphisms of $V$. The dimension of $V$ is called the degree of $\rho$. Two representations $\rho_1$ and $\rho_2$ of $\Gamma$ on $V_1$ and $V_2$, respectively, are equivalent if there is an isomorphism $T : V_1 \to V_2$ such that $T \rho_1(g) = \rho_2(g)T$ for all $g \in \Gamma$.

Let $\rho : \Gamma \to GL(V)$ be a representation. The character $\chi_{\rho} : \Gamma \to \mathbb{C}$ of $\rho$ is defined by setting $\chi_{\rho}(g) = Tr(\rho(g))$ for $g \in \Gamma$, where $Tr(\rho(g))$ is the trace of the matrix of $\rho(g)$ with respect to a basis of $V$. By the degree of $\chi_{\rho}$, we mean the degree of $\rho$, which is simply $\chi_{\rho}(1)$. If $W$ is a $\rho(g)$-invariant subspace of $V$ for each $g \in \Gamma$, then we call $W$ a $\rho(\Gamma)$-invariant subspace of $V$. If the $\rho(\Gamma)$-invariant subspaces of $V$ are $\{0\}$ and $V$ only, then we call $\rho$ an irreducible representation of $\Gamma$, and the corresponding character $\chi_{\rho}$ an irreducible character of $\Gamma$.

For a group $\Gamma$, we denote by $\text{IRR}(\Gamma)$ and $\text{Irr}(\Gamma)$ the complete set of pairwise non-equivalent irreducible representations of $\Gamma$ and the complete set of pairwise non-equivalent irreducible characters of $\Gamma$, respectively. Note that $\text{IRR}(\Gamma)$ is not necessarily unique, whereas $\text{Irr}(\Gamma)$ is unique.

**Lemma 2.1.** Let $\Gamma$ be a finite group, $\text{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$ and $j \in \{1, ..., h\}$. Then there exists $k \in \{1, ..., h\}$ such that $\chi_k = \chi_j$, where $\chi_k : \Gamma \to \mathbb{C}$ such that $\chi_k(x) = \chi_j(x)$ for all $x \in \Gamma$.

Let $\Gamma$ be a finite group and $\alpha : \Gamma \to \mathbb{C}$ be a function. The Cayley color digraph of $\Gamma$ with connection function $\alpha$, denoted by $\text{Cay}(\Gamma, \alpha)$, is defined to be the graph (with loops) having vertex set $\Gamma$ and edge set $\{(x, y) : x, y \in \Gamma\}$ such that each edge $(x, y)$ is colored by $\alpha(yx^{-1})$. The adjacency matrix of $\text{Cay}(\Gamma, \alpha)$ is defined to be the matrix whose rows and columns are indexed by the elements of $\Gamma$, and the $(x, y)$-entry is equal to $\alpha(yx^{-1})$. The eigenvalues of $\text{Cay}(\Gamma, \alpha)$ are simply the eigenvalues of its adjacency matrix.

In the special case when $\alpha : \Gamma \to \{0, 1, \omega_6, \omega_6^5\}$ with $\alpha(g) = \overline{\alpha(g^{-1})}$ and the set $S = \{g : \alpha(g) = 1 \text{ or } \omega_6\}$ satisfying $1 \not\in S$ are chosen, the adjacency matrix of $\text{Cay}(\Gamma, \alpha)$ is same with the Hermitian-adjacency matrix of the second kind of the mixed graph $\text{Cay}(\Gamma, S)$. Similarly, if $\alpha : \Gamma \to \{0, 1\}$ and the set $S = \{g : \alpha(g) = 1\}$ satisfying $1 \not\in S$ are chosen, then the adjacency matrix of $\text{Cay}(\Gamma, \alpha)$ is same with the $(0, 1)$-adjacency matrix of the mixed graph $\text{Cay}(\Gamma, S)$.

**Theorem 2.2.** [11] Let $\alpha : \Gamma \to \mathbb{C}$ be a class function and $\text{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$. Then the spectrum of the Cayley color digraph $\text{Cay}(\Gamma, \alpha)$ can be arranged as $\{[\lambda_1]^{d_1}, ..., [\lambda_h]^{d_h}\}$, where

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{x \in \Gamma} \alpha(x) \chi_j(x) \quad \text{and} \quad d_j = \chi_j(1) \quad \text{for all } j = 1, ..., h.$$

**Lemma 2.3.** Let $\text{Cay}(\Gamma, S)$ be a normal mixed Cayley graph and $\text{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$. Then the HS-spectrum of $\text{Cay}(\Gamma, S)$ can be arranged as $\{[\gamma_1]^{d_1}, ..., [\gamma_h]^{d_h}\}$, where $\gamma_j = \lambda_j + \mu_j$,

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in \overline{S} \cap S} \chi_j(s) \quad \text{and} \quad \mu_j = \frac{1}{\chi_j(1)} \sum_{s \in \overline{S}} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) \quad \text{for all } j = 1, ..., h.$$
Proof. Let \( \alpha : \Gamma \rightarrow \{0, 1, \omega_0^6, \omega_0^5\} \) be a function such that
\[
\alpha(s) = \begin{cases} 
1 & \text{if } s \in S \setminus \overline{S} \\
\omega_6 & \text{if } s \in \overline{S} \\
\omega_0^5 & \text{if } s \in \overline{S}^{-1} \\
0 & \text{otherwise.}
\end{cases}
\]
Since \( S \) is a union of some conjugacy classes of \( \Gamma \), so \( \alpha \) is a class function. By Theorem 2.2,
\[
\gamma_j = \frac{1}{\chi_j(1)} \left( \sum_{s \in S \setminus \overline{S}} \chi_j(s) + \sum_{s \in \overline{S}} \omega_6 \chi_j(s) + \sum_{s \in \overline{S}^{-1}} \omega_0^5 \chi_j(s) \right),
\]
and the result follows. \( \square \)

As special cases of Lemma 2.3, we have the following two corollaries.

**Corollary 2.3.1.** Let \( \text{Cay}(\Gamma, S) \) be a normal simple Cayley graph and \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). Then the spectrum (or HS-spectrum) of \( \text{Cay}(\Gamma, S) \) can be arranged as \( \{[\lambda_1]^d, \ldots, [\lambda_h]^d\} \), where
\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \overline{S}} \chi_j(s) \text{ and } d_j = \chi_j(1) \text{ for all } j = 1, \ldots, h.
\]

**Corollary 2.3.2.** Let \( \text{Cay}(\Gamma, S) \) be a normal oriented Cayley graph and \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). Then the HS-spectrum of \( \text{Cay}(\Gamma, S) \) can be arranged as \( \{[\mu_1]^d, \ldots, [\mu_h]^d\} \), where
\[
\mu_j = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_0^5 \chi_j(s^{-1})) \text{ and } d_j = \chi_j(1) \text{ for all } j = 1, \ldots, h.
\]

Let \( n \geq 2 \) be a positive integer. For a divisor \( d \) of \( n \), define \( G_n(d) = \{k : 1 \leq k \leq n-1, \gcd(k, n) = d\} \). It is clear that \( G_n(d) = dG_{n/d}(1) \).

Alperin and Peterson [3] considered a Boolean algebra generated by a class of subgroups of a group in order to determine the integrality of simple Cayley graphs over abelian groups. Suppose \( \mathcal{F}_\Gamma \) is the family of all subgroups of \( \Gamma \). The Boolean algebra \( \mathbb{B}(\Gamma) \) generated by \( \mathcal{F}_\Gamma \) is the set whose elements are obtained by arbitrary finite intersections, unions, and complements of the elements in the family \( \mathcal{F}_\Gamma \). The minimal non-empty elements of this algebra are called \textit{atoms}. Thus each element of \( \mathbb{B}(\Gamma) \) is the union of some atoms. Consider the equivalence relation \( \sim \) on \( \Gamma \) such that \( x \sim y \) if and only if \( y = x^k \) for some \( k \in G_m(1) \), where \( m = \text{ord}(x) \).

**Lemma 2.4.** [3] The equivalence classes of \( \sim \) are the atoms of \( \mathbb{B}(\Gamma) \).

For \( x \in \Gamma \), let \( [x] \) denote the equivalence class of \( x \) with respect to the relation \( \sim \). Also, let \( \langle x \rangle \) denote the cyclic group generated by \( x \).

**Lemma 2.5.** [3] The atoms of the Boolean algebra \( \mathbb{B}(\Gamma) \) are the sets \( [x] = \{y : \langle y \rangle = \langle x \rangle\} \).
By Lemma 2.5, each element of $\mathbb{B}(\Gamma)$ is a union of some sets of the form $[x] = \{ y : \langle y \rangle = \langle x \rangle \}$. Thus, for all $S \in \mathbb{B}(\Gamma)$, we have $S = [x_1] \cup \ldots \cup [x_k]$ for some $x_1, \ldots, x_k \in \Gamma$.

**Theorem 2.6.** \cite{12} Let $\text{Cay}(\Gamma, S)$ be a normal simple Cayley graph. Then $\text{Cay}(\Gamma, S)$ is integral if and only if $S \in \mathbb{B}(\Gamma)$.

Let $n \equiv 0 \pmod{3}$. For a divisor $d$ of $\frac{n}{3}$ and $r \in \{1, 2\}$, define

$$G_{n,3}(d) = \{dk : k \equiv r \pmod{3}, \gcd(dk, n) = d\}.$$  

It is easy to see that $G_n(d) = G_{n,3}(d) \cup G_{n,3}(d)$ is a disjoint union and $G_{n,3}(d) = dG_{n,3}(1)$ for $r = 1, 2$.

Define $\Gamma(3)$ to be the set of all $x \in \Gamma$ satisfying $\text{ord}(x) \equiv 0 \pmod{3}$. Define an equivalence relation $\approx$ on $\Gamma(3)$ such that $x \approx y$ if and only if $y = x^k$ for some $k \in G_{m,3}(1)$, where $m = \text{ord}(x)$. Observe that if $x, y \in \Gamma(3)$ and $x \approx y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 7 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(3)$ and $x \sim y$ but $x \not\approx y$. For $x \in \Gamma(3)$, let $\langle \langle x \rangle \rangle$ denote the equivalence class of $x$ with respect to the relation $\approx$.

### 3 HS-integral normal oriented Cayley graphs

Let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let $E$ be the matrix of size $h \times n$, whose rows are indexed by $1, \ldots, h$ and columns are indexed by the elements of $\Gamma$ such that $E_{j, g} = \chi_j(g)$. Note that $EE^* = nI_h$ and the rank of $E$ is $h$, where $E^*$ is the conjugate transpose of $E$. Let $w_m = \exp\left(\frac{2\pi i}{m}\right)$ be a primitive $m$-th root of unity.

It is well known that the Galois group $Gal(\mathbb{Q}(w_m)/\mathbb{Q}) = \{\sigma_r : r \in G_m(1)\}$, where $\sigma_r(w_m) = w_{m}^r$. Assume that $m \equiv 0 \pmod{3}$. Since $\mathbb{Q}(\omega_3, w_m) = \mathbb{Q}(w_m)$, the Galois group $Gal(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3))$ is a subgroup of $Gal(\mathbb{Q}(w_m)/\mathbb{Q})$. Therefore, $Gal(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3))$ contains all those automorphisms from $Gal(\mathbb{Q}(w_m)/\mathbb{Q})$ which fix $\omega_3$. Note that $G_m(1) = G_{m,3}(1) \cup G_{m,3}(2)$ is a disjoint union. Using $\sigma_r(\omega_3) = \omega_3$ for all $r \in G_{m,3}(1)$ and $\sigma_r(\omega_3) = \omega_3^2$ for all $r \in G_{m,3}(2)$, we get

$$\text{Gal}(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3)) = \text{Gal}(\mathbb{Q}(w_m)/\mathbb{Q}(\omega_3)) = \{\sigma_r : r \in G_{m,3}(1)\}.$$  

Now assume that $m \not\equiv 0 \pmod{3}$. Then

$$[\mathbb{Q}(\omega_3, w_m) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, w_m) : \mathbb{Q}(w_m)] \times [\mathbb{Q}(w_m) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = [\mathbb{Q}(w_m) : \mathbb{Q}] = \varphi(m).$$

Here $\varphi$ denotes the Euler $\varphi$-function. Thus the field $\mathbb{Q}(\omega_3, w_m)$ is a Galois extension of $\mathbb{Q}(\omega_3)$ of degree $\varphi(m)$. Any automorphism of the field $\mathbb{Q}(\omega_3, w_m)$ is uniquely determined by its action on $w_m$. Hence $Gal(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3)) = \{\tau_r : r \in G_m(1)\}$, where $\tau_r(w_m) = w_m^r$ and $\tau_r(\omega_3) = \omega_3$. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of $z$ and $\Re(z)$ (resp. $\Im(z)$) denote the real part (resp. imaginary part) of $z$.  

6
Theorem 3.1. Let $\Gamma$ be a finite group, $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, and $x = \sum_{g \in \Gamma} c_g g$ an element in $\mathbb{Q}(\omega_3)\Gamma$. Then $\chi_j(x)$ is rational for each $1 \leq j \leq h$ if and only if the following conditions hold:

(i) $\sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^{-1})} c_s$ for all $g \in \Gamma$,

(ii) $\sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s$ for all $g_1, g_2 \in \Gamma(3)$ and $g_1 \approx g_2$,

(iii) $\Re \left( \sum_{s \in \text{Cl}(g)} c_s \right) = \Re \left( \sum_{s \in \text{Cl}(g^{-1})} c_s \right)$ for all $g \in \Gamma \setminus \Gamma(3)$, and

(iv) $\Im \left( \sum_{s \in \text{Cl}(g)} c_s \right) = 0$ for all $g \in \Gamma \setminus \Gamma(3)$.

Proof. Let $L$ be a set of representatives of the conjugacy classes in $\Gamma$. Since characters are class functions, we have

$$
\chi_j(x) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g) \text{ for all } 1 \leq j \leq h. 
$$

(1)

Assume that $\chi_j(x)$ is rational for each $1 \leq j \leq h$. Let $g_1, g_2 \in \Gamma(3)$, $g_1 \approx g_2$ and $m = \text{ord}(g_1)$. Then there is $r \in G^1_{m,3}(1)$ and $\sigma_r \in \text{Gal}(\mathbb{Q}(w_m)/\mathbb{Q}(\omega_3))$ such that $g_2 = g_1^r$ and $\sigma_r(w_m) = w_m^r$. Note that $\sigma_r(\chi_j(g_1)) = \chi_j(g_1^r)$ for all $1 \leq j \leq h$. For $t \in \Gamma$, let $\theta_t = \sum_{j=1}^{h} \chi_j(t) \overline{\chi}_j$, where $\overline{\chi}_j(g) = \overline{\chi}_j(g)$ for all $g \in \Gamma$. By orthogonality of characters, we have

$$
\theta_t(u) = \begin{cases} 
|C_{\Gamma}(t)| & \text{if } u \text{ and } t \text{ are conjugates to each other} \\
0 & \text{otherwise.} 
\end{cases}
$$

So $\theta_t(x) = |C_{\Gamma}(t)| \sum_{s \in \text{Cl}(t)} c_s \in \mathbb{Q}(\omega_3)$ implies that $\sigma_r(\theta_t(x)) = \theta_t(x)$. By assumption, $\sigma_r(\chi_j(x)) = \chi_j(x)$ for all $j = 1, \ldots, h$. Thus

$$
|C_{\Gamma}(g_1)| \sum_{s \in \text{Cl}(g_1)} c_s = \theta_{g_1}(x) = \sigma_r(\theta_{g_1}(x)) = \sum_{j=1}^{h} \sigma_r(\chi_j(g_1)) \sigma_r(\overline{\chi}_j(x))
$$

$$
= \sum_{j=1}^{h} \chi_j(g_1^r) \overline{\chi}_j(x)
$$

$$
= \theta_{g_1^r}(x) = \theta_{g_2}(x) = |C_{\Gamma}(g_2)| \sum_{s \in \text{Cl}(g_2)} c_s. \tag{2}
$$

Since $g_1 \approx g_2$, we have $C_{\Gamma}(g_1) = C_{\Gamma}(g_2)$. So Equation (2) implies that $\sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s$. Hence condition (ii) holds.
Again
\[
0 = \chi_j(x) - \overline{\chi_j(x)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g) - \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} \overline{c_s} \right) \chi_j(g)
\]
\[
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g) - \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} \overline{c_s} \right) \chi_j(g^{-1})
\]
\[
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \overline{c_s} \right) \chi_j(g),
\]
and so
\[
\sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \overline{c_s} \right) \begin{bmatrix}
\chi_1(g) \\
\chi_2(g) \\
\vdots \\
\chi_h(g)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]
(3)

Since characters are class functions and the rank of \(E\) is \(h\), the columns of \(E\) corresponding to the elements of \(L\) are linearly independent. Thus by Equation (3),
\[
\left( \sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \overline{c_s} \right) = 0 \quad \text{for all } g \in L,
\]
and so condition (i) holds.

Let \(g \in \Gamma \setminus \Gamma(3)\) and \(m = \text{ord}(g)\). Then there exists \(\tau_{m-1} \in \text{Gal}(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3))\) such that \(\tau_{m-1}(w_m) = w_m^{m^{-1}}\). Note that \(\tau_{m-1}(\chi_j(g)) = \chi_j(g^{m^{-1}})\) for all \(1 \leq j \leq h\). Now
\[
|C_\Gamma(g)| \sum_{s \in \text{Cl}(g)} c_s = \theta_g(x) = \tau_{m-1}(\theta_g(x)) = \sum_{j=1}^{h} \tau_{m-1}(\chi_j(g)) \tau_{m-1}(\overline{\chi_j(x)})
\]
\[
= \sum_{j=1}^{h} \chi_j(g^{m^{-1}}) \overline{\chi_j(x)}
\]
\[
= \theta_g(x) = \theta_{g^{-1}}(x) = \theta_g^{-1}(x) = |C_\Gamma(g^{-1})| \sum_{s \in \text{Cl}(g^{-1})} c_s.
\]
(4)

Since \(C_\Gamma(g) = C_\Gamma(g^{-1})\), Equation (4) implies that
\[
\sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^{-1})} c_s.
\]
This, together with condition (i), condition (iii) and (iv) hold.

Conversely, assume that all the given conditions hold. Let \(n = \text{ord}(\Gamma)\). Then we have the following two cases:

**Case 1:** Assume that \(n \equiv 0 \pmod{3}\). Let \(\sigma_k \in \text{Gal}(\mathbb{Q}(\omega_3, w_n)/\mathbb{Q}(\omega_3))\). Then \(\sigma_k(w_n) = w_n^k\) and \(k \in G_{n,3}^1(1)\), and so \(\sigma_k(\chi_j(g)) = \chi_j(g^k)\) for all \(1 \leq j \leq h\). Thus
\[
\sigma_k(\chi_j(x)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \sigma_k(\chi_j(g))
\]
\[
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g^k).
\]
(5)
Since \( g \approx g^k \), by condition (ii) we have \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^k)} c_s \). From Equation (5), we get

\[
\sigma_k(\chi_j(x)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g^k)} c_s \right) \chi_j(g^k) = \chi_j(x). \tag{6}
\]

The second equality in (6) holds, because \( \{g^k : g \in L\} \) is also a set of representatives of conjugacy classes of \( \Gamma \). Now \( \sigma_k(\chi_j(x)) = \chi_j(x) \) for all \( k \in G_{n,3}(1) \) implies that \( \chi_j(x) \in \mathbb{Q}(\omega_3) \).

**Case 2:** Assume that \( n \not\equiv 0 \pmod{3} \). Same as Case 1, we get \( \tau_r(\chi_j(x)) = \chi_j(x) \) for all \( r \in G_n(1) \), where \( \tau_r \in \text{Gal}(\mathbb{Q}(\omega_3, w_m)/\mathbb{Q}(\omega_3)) \). Thus \( \chi_j(x) \in \mathbb{Q}(\omega_3) \).

In both cases, we get \( \chi_j(x) \in \mathbb{Q}(\omega_3) \). Taking complex conjugates in Equation (1),

\[
\overline{\chi_j(x)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \overline{\chi_j(g)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g^{-1})
\]

\[
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g^{-1})} c_s \right) \chi_j(g^{-1})
\]

\[
= \chi_j(x). \tag{7}
\]

Thus Equation (7) implies that \( \chi_j(x) \in \mathbb{Q} \). \( \square \)

For \( \Gamma(3) \neq \emptyset \), define \( \mathcal{E}(\Gamma) \) to be the set of all skew-symmetric subsets \( S \) of \( \Gamma \) such that \( S = \langle x_1 \rangle \cup \ldots \cup \langle x_k \rangle \) for some \( x_1, \ldots, x_k \in \Gamma(3) \) and \( S = \text{Cl}(y_1) \cup \ldots \cup \text{Cl}(y_l) \) for some \( y_1, \ldots, y_l \in \Gamma \). For \( \Gamma(3) = \emptyset \), define \( \mathcal{E}(\Gamma) = \{\emptyset\} \). We can replace condition (ii) of Theorem 3.1 by \( \sum_{s \in \text{Cl}(x)} c_s = \sum_{s \in \text{Cl}(y)} c_s \) for all \( x, y \in \langle g \rangle \) and \( g \in \Gamma(3) \).

**Theorem 3.2.** Let \( \text{Cay}(\Gamma, S) \) be a normal oriented Cayley graph. Then \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( S \in \mathcal{E}(\Gamma) \).

**Proof.** Let \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \) and \( x = \sum_{g \in \Gamma} c_g g \), where

\[
c_g = \begin{cases} 
-\omega_3^2 & \text{if } g \in S \\
-\omega_3 & \text{if } g \in S^{-1} \\
0 & \text{otherwise}.
\end{cases}
\]

Observe that \( \chi_j(x) = \sum_{s \in S} (-\omega_3^2 \chi_j(s) - \omega_3 \chi_j(s^{-1})) \), so that \( \chi_j(x) \) is an HS-eigenvalue of \( \text{Cay}(\Gamma, S) \). Assume that the normal oriented Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral. Here \( S \) is the union of some conjugacy classes of \( \Gamma \). Then \( \chi_j(x) \) is an integer for each \( 1 \leq j \leq h \), and therefore all the four conditions of Theorem 3.1 hold. By the fourth condition of Theorem 3.1, we get \( \mathfrak{Im} \left( \sum_{s \in \text{Cl}(g)} c_s \right) = 0 \) for all \( g \in \Gamma \setminus \Gamma(3) \). If \( g \in S \), then \( \text{Cl}(g) \subseteq S \), and so by the definition of \( c_g \), we get \( \mathfrak{Im} \left( \sum_{s \in \text{Cl}(g)} c_s \right) = \frac{|\text{Cl}(g)|}{|\Gamma(n)|} \neq 0 \). Thus

\[
\mathfrak{Im} \left( \sum_{s \in \text{Cl}(g)} c_s \right) = \frac{|\text{Cl}(g)|}{|\Gamma(n)|} \neq 0.
\]
$S \cap (\Gamma \setminus \Gamma(3)) = \emptyset$, that is, $S \subseteq \Gamma(3)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(3)$ and $g_1 \approx g_2$. By the second condition of Theorem 3.1, we get $0 < \sum_{s \in Cl(g_1)} c_s = \sum_{s \in Cl(g_2)} c_s$, which implies that $g_2 \in S$. Thus $g_1 \in S$ gives $\langle g_1 \rangle \subseteq S$. Hence $S \in E(\Gamma)$.

Conversely, assume that $S \in E(\Gamma)$. Let $\text{Cay}(\Gamma, S)$ be a normal oriented Cayley graph, so that $S$ is a union of some conjugacy classes of $\Gamma$. Let $S = \langle x_1 \rangle \cup \cdots \cup \langle x_r \rangle = Cl(y_1) \cup \cdots \cup Cl(y_k) \subseteq \Gamma(4)$ for some $x_1, \ldots, x_r, y_1, \ldots, y_k \in \Gamma(4)$. Then $S^{-1} = \langle x_1^{-1} \rangle \cup \cdots \cup \langle x_r^{-1} \rangle = Cl(y_1^{-1}) \cup \cdots \cup Cl(y_k^{-1}) \subseteq \Gamma(4)$. Now for $g_1, g_2 \in \Gamma(4)$, if $g_1 \approx g_2$ then $Cl(g_1), Cl(g_2) \subseteq S$ or $Cl(g_1), Cl(g_2) \subseteq S^{-1}$ or $Cl(g_1), Cl(g_2) \subseteq (S \cup S^{-1})^c$. Note that $|Cl(g_1)| = |Cl(g_2)|$. For all the cases, using the definition of $c_g$, we find

$$\sum_{s \in Cl(g_1)} c_s = \sum_{s \in Cl(g_2)} c_s.$$  

Thus condition (ii) holds. Again for $g \in \Gamma$, $Cl(g) \subseteq S$ if and only if $Cl(g^{-1}) \subseteq S^{-1}$. Therefore

$$\sum_{s \in Cl(g)} c_g = \sum_{s \in Cl(g^{-1})} c_g,$$

and so condition (i) holds. Further, if $g \not\in \Gamma(4)$, then $Cl(g) \cap (S \cup S^{-1}) = Cl(g^{-1}) \cap (S \cup S^{-1}) = \emptyset$, and so $\sum_{s \in Cl(g)} c_s = 0 = \sum_{s \in Cl(g^{-1})} c_s$, therefore condition (iii) and (iv) hold. Thus all four conditions of Theorem 3.1 are satisfied by $x$, and therefore $\chi_j(x)$ is rational for each $1 \leq j \leq h$. Consequently, the HS-eigenvalue $\mu_j = \frac{\chi_j(x)}{\chi_j(1)}$ of $\text{Cay}(\Gamma, S)$ is a rational algebraic integer, and hence an integer for each $1 \leq j \leq h$. \hfill \Box

**Example 3.1.** Consider the alternating group $A_4$ and $S = \{(1, 2, 3), (1, 2), (1, 3, 2), (1, 3, 4), (2, 3, 4), (2, 3, 4, 1)\}$. Note that all the conjugacy classes of $A_4$ are $\{I\}, \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}, \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}$ and $\{(1, 3, 2), (2, 4, 3, 1)\}$. The normal oriented graph $\text{Cay}(A_4, S)$ is regular graph of eight degree. We see that $S = \langle (1, 2, 3) \rangle \cup \langle (2, 4, 3) \rangle \cup \langle (3, 4, 1) \rangle = Cl((1, 2, 3))$. Therefore $S \in E(\Gamma)$. From [15], the character table of $A_n$ is given in Table 1, where $\text{Irr}(\Gamma) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Further, using Lemma 2.3, the HS-spectrum of $\text{Cay}(A_4, S)$ is $\{[\mu_1]^1, [\mu_2]^1, [\mu_3]^1, [\mu_4]^9\}$, where $\mu_1 = 4(\omega_6 + \omega_9^2)$, $\mu_2 = 4(\omega_6\omega_3 + \omega_9^3\omega_2^3)$, $\mu_3 = 4(\omega_6\omega^2_3 + \omega_9^5\omega_3)$ and $\mu_4 = 0$. Thus the HS-spectrum of $\text{Cay}(A_4, S)$ is $\{[4]^1, [-8]^1, [4]^1, [0]^9\}$, therefore it is HS-integral.
4 HS-integral normal mixed Cayley graphs

**Lemma 4.1.** Let $S$ be a skew-symmetric, union of some conjugacy classes of the finite group $\Gamma$, $\operatorname{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$, and $t(\neq 0) \in \mathbb{Q}$. If \( \frac{1}{\chi_j(1)} \sum_{s \in S} t(\sqrt{3}i\chi_j(s) - \sqrt{3}i\chi_j(s^{-1})) \) is integer for each $1 \leq j \leq h$ then $S \in \mathcal{E}(\Gamma)$.

**Proof.** Let $x = \sum_{g \in \Gamma} c_g g \in \mathbb{Q}(\omega_3)\Gamma$, where

\[
c_g = \begin{cases} \frac{it\sqrt{3}}{3} & \text{if } g \in S \\ -\frac{it\sqrt{3}}{3} & \text{if } g \in S^{-1} \\ 0 & \text{otherwise} \end{cases}
\]

Assume that $\chi_j(x) = \frac{1}{\chi_j(1)} \sum_{s \in S} t(\sqrt{3}i\chi_j(s) - \sqrt{3}i\chi_j(s^{-1}))$ is integer for each $1 \leq j \leq h$. Then all the four conditions of Theorem 3.1 hold. By the fourth condition of Theorem 3.1, we get $\exists \left( \sum_{s \in \Cl(g)} c_s \right) = 0$ for all $g \in \Gamma \setminus \Gamma(3)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(3)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(3)$ and $g_1 \approx g_2$. The second condition of Theorem 3.1 gives $\sum_{s \in \Cl(g_1)} c_s = \sum_{s \in \Cl(g_2)} c_s$ and it is equal to $\pm it\sqrt{3}\Cl(g_1)$. Which implies that $g_2 \in S$. Thus $g_1 \in S$ implies $\langle g_1 \rangle \subseteq S$. Hence $S \in \mathcal{E}(\Gamma)$. \qed

**Lemma 4.2.** Let $S$ be a skew-symmetric, union of some conjugacy classes of the finite group $\Gamma$, $\operatorname{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$, and $t(\neq 0) \in \mathbb{Q}$. If \( \frac{1}{\chi_j(1)} \sum_{s \in S} t(\sqrt{3}i\chi_j(s) - \sqrt{3}i\chi_j(s^{-1})) \) is integer for each $1 \leq j \leq h$ then \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is integer for each $1 \leq j \leq h$.

**Proof.** Assume that \( \frac{1}{\chi_j(1)} \sum_{s \in S} t(\sqrt{3}i\chi_j(s) - \sqrt{3}i\chi_j(s^{-1})) \) is integer for each $1 \leq j \leq h$. By Lemma 4.1 we have $S \in \mathcal{E}(\Gamma)$, and so $S = \langle x_1 \rangle \cup ... \cup \langle x_k \rangle$ for some $x_1, ..., x_k \in \Gamma(3)$. Therefore, we get $S \cup S^{-1} = [x_1] \cup ... \cup [x_k] \in \mathcal{B}(\Gamma)$. Thus by Theorem 2.6, $\operatorname{Cay}(\Gamma, S \cup S^{-1})$ is integral, or equivalent to say, \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is integer for each $1 \leq j \leq h$. \qed

**Lemma 4.3.** Let $\operatorname{Cay}(\Gamma, S)$ be a normal mixed Cayley graph. Then $\operatorname{Cay}(\Gamma, S)$ is HS-integral if and only if $\operatorname{Cay}(\Gamma, S \setminus \overline{S})$ and $\operatorname{Cay}(\Gamma, \overline{S})$ are HS-integral.

**Proof.** Let $\operatorname{Irr}(\Gamma) = \{\chi_1, ..., \chi_h\}$. By Lemma 2.3, the HS-spectrum of $\operatorname{Cay}(\Gamma, S)$ is $\{[\gamma_1]^{d_1}, ..., [\gamma_h]^{d_h}\}$, where $\gamma_j = \lambda_j + \mu_j$,

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in \Delta \setminus \overline{S}} \chi_j(s), \quad \mu_j = \frac{1}{\chi_j(1)} \sum_{s \in \overline{S}} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) \quad \text{and} \quad d_j = \chi_j(1) \text{ for all } j = 1, ..., h.
\]

Note that $\{[\lambda_1]^{d_1}, ..., [\lambda_h]^{d_h}\}$ is spectrum of $\operatorname{Cay}(\Gamma, S \setminus \overline{S})$ and $\{[\mu_1]^{d_1}, ..., [\mu_h]^{d_h}\}$ is HS-spectrum of $\operatorname{Cay}(\Gamma, \overline{S})$.  

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Assume that mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral. Let \( j \in \{1, \ldots, h\} \). By Lemma 2.1, there exist \( k \in \{1, \ldots, h\} \) such that \( \chi_k = \sqrt[3]{\gamma_j} \). Then \( \lambda_j = \lambda_k \) and \( \chi_j(1) = \chi_k(1) \), therefore

\[
\gamma_j - \gamma_k = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) - \frac{1}{\chi_k(1)} \sum_{s \in S} (\omega_6 \chi_k(s) + \omega_6^5 \chi_k(s^{-1}))
\]

\[
= \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) - \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s^{-1}) + \omega_6^5 \chi_j(s))
\]

\[
= \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) - \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) - \omega_6^5 \chi_j(s^{-1}))
\]

\[
= \frac{1}{\chi_j(1)} \sum_{s \in S} \sqrt[3]{\gamma_j} \chi_j(s) - \sqrt[3]{\gamma_j} \chi_j(s^{-1})
\]

By assumption \( \gamma_j, \gamma_k \in \mathbb{Z} \), and so \( \frac{1}{\chi_j(1)} \sum_{s \in S} \sqrt[3]{\gamma_j} \chi_j(s) - \sqrt[3]{\gamma_j} \chi_j(s^{-1}) \in \mathbb{Z} \) for all \( 1 \leq j \leq h \). By Lemma 4.2, we get \( \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \in \mathbb{Z} \) for all \( 1 \leq j \leq h \). Since

\[
\mu_j = \frac{1}{2 \chi_j(1)} \sum_{s \in S} \chi_j(s) + \frac{1}{2 \chi_j(1)} \sum_{s \in S} \left( \sqrt[3]{\gamma_j} \chi_j(s) - \sqrt[3]{\gamma_j} \chi_j(s^{-1}) \right)
\]

then \( \mu_j \) is a rational algebraic integer, and hence it is integer for each \( 1 \leq j \leq h \). Thus \( \text{Circ}(\mathbb{Z}_n, \overline{S}) \) is HS-integral. Now we have \( \gamma_j, \mu_j \in \mathbb{Z} \), and so \( \lambda_j = \gamma_j - \mu_j \in \mathbb{Z} \) for each \( 1 \leq j \leq h \). Hence \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral.

Conversely, assume that \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral and \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral. Then Lemma 2.3 implies that \( \text{Cay}(\Gamma, S) \) is integral.

\[ \square \]

**Theorem 4.4.** Let \( \text{Cay}(\Gamma, S) \) be a normal mixed Cayley graph. Then \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( S \setminus \overline{S} \in \mathbb{B}(\Gamma) \) and \( \overline{S} \in \mathbb{B}(\Gamma) \).

**Proof.** By Lemma 4.3, \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral and \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral. Now the proof follows from Theorem 2.6 and Theorem 3.2.

\[ \square \]

**Example 4.1.** Consider the alternating group \( A_4 \) and

\[ S = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\} \]

The underlying graph of the normal mixed graph \( \text{Cay}(A_4, S) \) is a complete graph. Observe that \( \overline{S} = \{(1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\} \) and \( S \setminus \overline{S} = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\} \). The character table of \( A_n \) is given in Table 1, where \( \text{Irr}(\Gamma) = \{\chi_1, \chi_2, \chi_3, \chi_4\} \). Further, using Lemma 2.3, the HS-spectrum of \( \text{Cay}(A_4, S) \) is \( \{[\gamma_1]^1, [\gamma_2]^1, [\gamma_3]^1, [\gamma_4]^1\} \), where \( \gamma_1 = 3 + 4(\omega_6 + \omega_6^5) \), \( \gamma_2 = 3 + 4(\omega_6\omega_3 + \omega_6^5\omega_3^2) \), \( \gamma_3 = 3 + 4(\omega_6\omega_3^2 + \omega_6^5\omega_3) \) and \( \gamma_4 = -1 \). Thus the HS-spectrum of \( \text{Cay}(A_4, S) \) is \( \{[\gamma_1]^1, [-5]^1, [\gamma_1]^1, [-1]^1\} \), therefore it is HS-integral.
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