Precise large deviations of aggregate claims in a size-dependent renewal risk model with stopping time claim-number process

Shuo Zhang¹, Dehui Wang¹* and Shihang Yu²*

¹Correspondence: wangdh@jlu.edu.cn, qqhrsh@163.com
²Institute of Mathematics, Jilin University, Changchun, 130012, China

1 School of Science, Qiqihar University, Qiqihar, 161006, China

Abstract
In this paper, we consider a size-dependent renewal risk model with stopping time claim-number process. In this model, we do not make any assumption on the dependence structure of claim sizes and inter-arrival times. We study large deviations of the aggregate amount of claims. For the subexponential heavy-tailed case, we obtain a precise large-deviation formula; our method substantially relies on a martingale for the structure of our models.

MSC: 60F10; 91B30; 60K05

Keywords: aggregate claims; stopping time; dependent renewal risk model; precise large deviations

1 Introduction
Consider the following renewal risk model. Let \( \{X_k, k \in \mathbb{N}\} \) and \( \{\theta_k, k \in \mathbb{N}\} \) be claim sizes and inter-arrival times, respectively. Assume that \((X_k, \theta_k), k \in \mathbb{N}, \) form a sequence of independent and identically distributed (i.i.d.) copies of a generic random pair \((X, \theta)\) with marginal distribution functions \( F = 1 - F \) on \([0, \infty)\) and \( G \) on \([0, \infty), \) with dependent components \( X \) and \( \theta. \) The claim arrival times are \( \tau_k = \sum_{i=1}^{k} \theta_i, k \in \mathbb{N}, \) with \( \tau_0 = 0. \) The number of claims is defined by

\[
N_t^* = \inf \{k \in \mathbb{N} : \tau_k \geq t\},
\]

(1.1)

then \( N_t^* \) is a stopping time. In this way, the aggregate amount of claims over the \([0, t]\) is of the form

\[
S_t^* = \sum_{k=1}^{N_t^*} X_k, \quad S_0^* = X_0 = 0, \quad t \geq 0.
\]

(1.2)

Note that \( N_t^* = \sup \{k \in \mathbb{N} : \tau_k \leq t\} \equiv N_t \) if \( \tau_k = t, \) whereas \( N_t^* = N_t + 1 \) if \( \tau_k \neq t. \)

We study large deviations of \( S_t^* \) in (1.2). We only consider the case of heavy-tailed claim-size distributions. One of the most important classes of heavy-tailed distributions is the...
class \( S \) of subexponential distributions. By definition, a distribution \( F \) on \([0, \infty)\) is subexponential if 
\[
F(x) = 1 - F(x) \quad \text{for all } x \geq 0
\]
and the relation 
\[
\lim_{x \to \infty} \frac{F^n(x)}{F(x)} = n
\]
holds for all \( n \geq 2 \), where \( F^n \) denotes the \( n \)-fold convolution of \( F \). Clearly, (1.3) implies 
\[
\lim_{x \to \infty} \frac{P(X_1 + X_2 + \cdots + X_n > x)}{P(\max\{X_1, \ldots, X_n\} > x)} = 1,
\]
where \( X_1, X_2, \ldots \) is a sequence of i.i.d. r.v.'s with the common distribution function (d.f.) \( F \). See Embrechts et al. [1] for a nice review of subexponential distributions in the context of insurance and finance.

Our main result is given now.

**Theorem 1.1** Consider the aggregate amount of claims \( S_\tau^* \) in (1.2), assume that \( F \in S \), \( E[X] = \mu \in (0, \infty) \) and \( E[\theta] = 1/\lambda \in (0, \infty) \). Then, for arbitrarily given \( \gamma > 0 \), we have uniformly for all \( x \geq \gamma t \)
\[
P(S_\tau^* - \mu \lambda t > x) \sim \lambda t F(x), \quad t \to \infty.
\]

A non-standard renewal risk model with dependent components \( X \) and \( \theta \), which was firstly proposed by Albrecher and Teugels [2] and further studied by Boudreault et al. [3], Cossette et al. [4], Badescu et al. [5], and among many others. Recently, Asimit and Badescu [6] introduced a general dependence structure for \((X, \theta)\), via the conditional tail probability of \( X \) given \( \theta \); see also Li et al. [7]. In particular, Chen and Yuen [8] considered that there is a random variable \( \tilde{\theta} \) such that
\[
\Pr(\theta > t | X > x) \leq \Pr(\tilde{\theta} > t)
\]
holds for \( t \geq 0 \) and large enough \( x \), they studied the large deviations of the aggregate amount of \( C \) heavy-tailed claims, where \( C \subset S \) (see Embrechts et al. [1]).

We now comment on the approaches used in this work. First, in Theorem 1.1, we do not make an assumption on the dependence structure of \((X, \theta)\). The existing results usually require a conditional tail probability of \( X \) given \( \theta \), e.g., Chen and Yuen [8] made the assumption (1.6), to say the least. Second, we extend the asymptotic behavior of the large deviations of \( S_\tau^* \) to the case of \( S \) heavy-tailed claims. Finally, we construct a martingale to prove our result.

The rest of the paper is organized as follows. Section 2 recalls various preliminaries and prepares a few lemmas. Section 3 presents the proof of the main result. We end the paper with conclusions in Section 4.

### 2 Preliminaries

Throughout this paper, for two positive functions \( a(\cdot) \) and \( b(\cdot) \), we write \( a(x) \lesssim b(x) \) if \( \limsup_{x \to \infty} a(x)/b(x) \leq 1 \), write \( a(x) \gtrsim b(x) \) if \( \liminf_{x \to \infty} a(x)/b(x) \geq 1 \), and write \( a(x) \sim b(x) \) if both. Very often we equip limit relationships with certain uniformity, which is crucial for our purpose. For instance, for two positive bivariate functions \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \), we
say that $a(\cdot, \cdot) \sim b(\cdot, \cdot)$ holds uniformly for $x \in \Delta \neq \emptyset$ if
\[
\lim_{t \to \infty} \sup_{x \in \Delta} \left| \frac{a(x; t)}{b(x; t)} - 1 \right| = 0.
\]

Clearly, the asymptotic relation $a(\cdot, \cdot) \sim b(\cdot, \cdot)$ holds uniformly for $x \in \Delta$ if and only if
\[
\lim_{t \to \infty} \sup_{x \in \Delta} \frac{a(x; t)}{b(x; t)} \leq 1 \quad \text{and} \quad \lim_{t \to \infty} \inf_{x \in \Delta} \frac{a(x; t)}{b(x; t)} \geq 1.
\]

To obtain our desired results, we need to mention the following useful lemma.

**Lemma 2.1** Consider the renewal counting process $N^*_t$ in (1.1). Under the assumption $E[\theta] = 1/\lambda < \infty$. Then, for any $p \geq 1$, we have
\[
E[N^*_t]^p \sim (\lambda t)^p, \quad \text{as } t \to \infty. \tag{2.1}
\]

**Proof** First, for arbitrarily $\varepsilon > 0$, by definition of $N^*$ and $E[\theta] = 1/\lambda$, we have
\[
P\left\{N^*_t > (\lambda + \varepsilon)t\right\} = P\left\{\sum_{j=1}^{(\lambda+\varepsilon)t} \theta_j \leq t\right\}
\]
\[
= P\left\{1 / (\lambda + \varepsilon)t \sum_{j=1}^{(\lambda+\varepsilon)t} \theta_j \leq 1 / \lambda + \varepsilon\right\}
\]
\[
\to 0. \tag{2.2}
\]

Similarly,
\[
P\left\{N^*_t < (\lambda - \varepsilon)t\right\} \to 0. \tag{2.3}
\]

Combining (2.2) and (2.3), we have
\[
\frac{N^*_t}{t} \xrightarrow{p} \lambda. \tag{2.4}
\]

It remains to show
\[
\sup_{t \geq 1} E\left(\frac{N^*_t}{t}\right)^p < \infty, \quad \text{for } \forall p \geq 1.
\]

Indeed, for any $\delta > 0$,
\[
E\left(\frac{N^*_t}{t}\right)^p = p \int_0^\infty u^{p-1} P\left\{N^*_t > ut\right\} du
\]
\[
= p(\lambda + \delta)^p \int_0^\infty u^{p-1} P\left\{N^*_t > (\lambda + \delta)ut\right\} du
\]
\[
\leq p(\lambda + \delta)^p \int_0^\infty u^{p-1} \left\{\sum_{j=1}^{(\lambda+\delta)t} \theta_j \leq t\right\} du. \tag{2.5}
\]
For the case when \( u \geq 1 \) is an integer
\[
\Pr \left\{ \sum_{j=1}^{(\lambda + \delta)u} \theta_j \leq t \right\} \leq \Pr \left\{ \bigcap_{k=1}^{u} \left\{ \sum_{j=(k-1)(\lambda + \delta)+1}^{k(\lambda + \delta)} \theta_j \leq t \right\} \right\} = \left( \Pr \left\{ \sum_{j=1}^{(\lambda + \delta)t} \theta_j \leq t \right\} \right)^u.
\]

By the law of large numbers
\[
\Pr \left\{ \sum_{j=1}^{(\lambda + \delta)t} \theta_j \leq t \right\} \to 0 \quad (t \to \infty).
\]

So we have the bound, for any constant \( c > 0 \),
\[
\Pr \left\{ \sum_{j=1}^{(\lambda + \delta)t} \theta_j \leq t \right\} \leq e^{-ct}.
\]

Combining (2.5) and (2.6), uniformly for large \( u \) and \( t \),
\[
\mathbb{E} \left( \frac{N^*_{\infty}}{t} \right)^p \leq p(\lambda + \delta)^p \int_0^\infty u^{p-1} e^{-cu} \, du < \infty.
\]

By (2.4) and (2.7), we obtain Lemma 2.1. □

3 Proof of Theorem 1.1

By \( F \in \mathcal{S} \) and (1.4), we need only to prove
\[
\Pr \left\{ \max_{k \leq N^*_t} (X_k - \mu) > x \right\} \sim \lambda t \tilde{F}(x) \quad \text{for } t, x \to \infty.
\]

Write \( \xi_k = I_{(X_k - \mu) > x} \). First,
\[
\sum_{k=1}^{N^*_t} I_{(X_k - \mu) > x} \leq \sum_{k=1}^{N^*_t} \xi_k.
\]

By Wald’s equation
\[
\Pr \left\{ \max_{k \leq N^*_t} (X_k - \mu) > x \right\} \leq (\mathbb{E} N^*_t) \Pr \{ X > x + \mu \}.
\]

From Lemma 2.1 and \( \tilde{F}(x + \mu) \sim \tilde{F}(x) \) (see class \( \mathcal{S} \subset \mathcal{L} \) and definition of \( \tilde{L} \), Embrechts et al. [1]), we have
\[
\Pr \left\{ \max_{k \leq N^*_t} (X_k - \mu) > x \right\} \lesssim \lambda t \tilde{F}(x) \quad \text{for } t, x \to \infty.
\]

On the other hand,
\[
I_{(\max_{k \leq N^*_t} (X_k - \mu) > x)} \geq \sum_{k=1}^{N^*_t} \xi_k - \sum_{1 \leq j \leq N^*_t} \xi_j \xi_k.
\]
All we need is to bound the second term. Notice that

\[ M_n = \sum_{1 \leq j < k \leq n} \xi_j (\xi_k - \mathbb{E} \xi_k), \quad n = 1, 2, \ldots \]

is a martingale. By Doob’s stopping rule,

\[ \mathbb{E} M_{N^*_t} = \mathbb{E} M_1 = 0. \]

Or

\[ \mathbb{E} \left( \sum_{1 \leq j < k \leq N^*_t} \xi_j (\xi_k - \mathbb{E} \xi_k) \right) = 0. \]

Hence,

\[ \mathbb{E} \left( \sum_{1 \leq j < k \leq N^*_t} \xi_j \xi_k \right) = \mathbb{E} \xi \cdot \mathbb{E} \left( \sum_{1 \leq j < k \leq N^*_t} \xi_j \right) = \mathbb{E} \xi \cdot \mathbb{E} \left( N^*_t - \sum_{j=1}^{N^*_t-1} \xi_j (N^*_t - j) \right). \]

Write \( Z_n = \sum_{j=1}^n \xi_j \). Notice that

\[ \sum_{j=1}^{N^*_t-1} \xi_j (N^*_t - j) = \sum_{n=1}^{N^*_t-1} Z_n. \]

Therefore,

\[ \mathbb{E} \left( \sum_{j=1}^{N^*_t-1} \xi_j (N^*_t - j) \right) = \mathbb{E} \left( \sum_{n=1}^{N^*_t-1} Z_n \right) = \sum_{n=1}^{\infty} \mathbb{E}(Z_n 1_{\{N^*_t \geq n+1\}}) = \sum_{n=1}^{\infty} \mathbb{E}(Z_n 1_{\{\tau_n \leq t\}}). \quad (3.3) \]

For each \( n \),

\[ \mathbb{E}(Z_n 1_{\{\tau_n \leq t\}}) = \sum_{j=1}^n \mathbb{E}(\xi_j 1_{\{\tau_n \leq t\}}) \leq \sum_{j=1}^n \mathbb{E}(\xi_j 1_{\{\sum_{k=1}^{n-1} \theta_k \leq t\}}) = \sum_{j=1}^n \mathbb{E} \xi \cdot \mathbb{P} \left\{ \sum_{k=1}^n \theta_k \leq t \right\} = n \mathbb{E} \xi \cdot \mathbb{P} \{ N^*_t \geq n \}. \quad (3.4) \]
By (3.3) and (3.4), \( \exists c > 0 \) such that
\[
E\left( \sum_{j=1}^{N_t^* - 1} \xi_j (N_t^* - j) \right) \leq E \xi \cdot \sum_{n=1}^{\infty} n P\{N_t^* \geq n\} \leq c \cdot E \xi \cdot E(N_t^*)^2.
\]
Combining our computation and Lemma 2.1,
\[
E\left( \sum_{1 \leq j < k \leq N_t^*} \xi_j \xi_k \right) \leq c \cdot (E \xi)^2 \cdot E(N_t^*)^2 \leq c (\lambda t F(x))^2 = o(\lambda t F(x)).
\]
By (3.2), therefore,
\[
P\left\{ \max_{k \leq N_t^*} (X_k - \mu) > x \right\} \geq \lambda t F(x) \quad \text{for } t, x \to \infty.
\]
Hence, by (3.1) and (3.5), we have
\[
P\left\{ \max_{k \leq N_t^*} (X_k - \mu) > x \right\} \sim \lambda t F(x) \quad \text{for } t, x \to \infty.
\]

4 Conclusions
As was remarked by a few researchers in the area, precise large-deviation results of size-dependent renewal risk models are particularly useful for evaluating some risk measures such as the conditional tail expectation of the aggregate amount of claims from a large insurance portfolio. Finally, we would like to point out that equation (1.5) agrees with existing ones in the literature. This indicates that the aggregate amount of \( S_t^* \) defined by (1.1) does not affect the asymptotic behavior of the large deviations.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
This work is supported by National Natural Science Foundation of China (No. 11271155, 11371168, J1310022, 11501241), Jilin Province Natural Science Foundation (20130101066JC, 20150520053JH), and Science and Technology Research Program of Education Department in Jilin Province for the 12th Five-Year Plan (440020031139). Science and technology projects program fund of Qiqihar City (No. RIX-201513).

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 November 2016 Accepted: 23 March 2017 Published online: 21 April 2017

References
1. Embrechts, P, Kluppelberg, C, Mikoseh, T: Modeling Extremal Events for Insurance and Finance. Springer, Berlin (1997)
2. Albrecher, H, Teugels, J: Exponential behavior in the presence of dependence in risk theory. J. Appl. Probab. 43(1), 257-273 (2006)
3. Boudreault, M, Cossette, H, Landrau, D, Marceau, E: On a risk model with dependence between interclaim arrivals and claim sizes. Scand. Actuar. J. 5, 265-285 (2006)
4. Cossette, H, Marceau, E, Marri, F: On the compound Poisson risk model with dependence based on a generalized Farlie-Gumbel-Morgenstern copula. Insur. Math. Econ. 43(3), 444-455 (2008)
5. Badescu, AL, Cheung, ECK, Landraut, D: Dependent risk models with bivariate phase-type distributions. J. Appl. Probab. 46(1), 113-131 (2009)
6. Asimit, AV, Badescu, AL: Extremes on the discounted aggregate claims in a time dependent risk model. Scand. Actuar. J. 2, 93-104 (2010)
7. Li, J, Tang, Q, Wu, R: Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. Adv. Appl. Probab. 42, 1126-1146 (2010)
8. Chen, Y, Yuen, KC: Precise large deviations of aggregate claims in a size-dependent renewal risk model. Insur. Math. Econ. 51(2), 457-461 (2012)