The Kardar-Parisi-Zhang exponents for the $2 + 1$ dimensions

Márcio S. Gomes-Filho$^a$, André L. A. Penna$^a$, Fernando A. Oliveira$^b$

$^a$Instituto de Física, Universidade de Brasília, Brasília-DF, Brazil.
$^b$Instituto de Física, Universidade Federal da Bahia, Campus Universitário da Federação, Rua Barão de Jeremoabo s/n, 40170-115, Salvador-BA, Brazil.

Abstract
The Kardar-Parisi-Zhang (KPZ) equation has been connected to a large number of important stochastic processes in physics, chemistry and growth phenomena, ranging from classical to quantum physics. The central quest in this field is the search for ever more precise universal growth exponents. Notably, exact growth exponents are only known for $1 + 1$ dimensions. In this work, we present physical and geometric analytical methods that directly associate these exponents to the fractal dimension of the rough interface. Based on this, we determine the growth exponents for the $2 + 1$ dimensions, which are in agreement with the results of thin films experiments and precise simulations. We also make a first step towards a solution in $d + 1$ dimensions, where our results suggest the inexistence of an upper critical dimension.

Keywords: KPZ equation, Growth phenomena, KPZ exponents, Universality

1. Introduction

In most physical systems, the growth process occurs when particles, or aggregates of particles, reach a surface via diffusion, an injection beam or some kind of deposition process. To investigate the growth, we follow the height $h(x,t)$, where $t$ is the time and is the position in a space of dimension $d$. Since $h(x,t)$ has scaling properties different from $x$, we say that $(h(x,t), x)$ forms a $d + 1$ dimensional space. Field equations have been proposed for the $h(x,t)$ dynamics such as the Kardar-Parisi-Zhang (KPZ) equation $^1$:

$$\frac{\partial h(x,t)}{\partial t} = v\nabla^2 h(x,t) + \frac{\lambda}{2} (\nabla h(x,t))^2 + \eta(x,t),$$

where the Gaussian white noise, $\eta(x,t)$, has zero mean $\langle \eta(x,t) \rangle = 0$ and variance

$$\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta^{(d)}(x-x')\delta(t-t').$$

The coupling parameter $g = DL^2/\nu^3$ connects the KPZ coefficients, being $\nu$ (surface tension) associated with the Laplacian smoothing mechanism and $\lambda$ related to the tilt mechanism.

A large number of phenomena $^2$ $^3$ $^4$ $^5$ $^6$ $^7$ can be understood by defining a few physical quantities such as the average height $\langle h \rangle$ and the roughness or surface width

$$w(l,t)^2 = \langle h^2(t) \rangle - \langle h(t) \rangle^2,$$

where $l$ is the sample size. We are interested in physical systems in which the roughness grows with time and afterwards saturated with a maximum value $w_s$, i.e. $^2$:

$$w(l,t) = \begin{cases} \epsilon t^\beta, & \text{if } t \ll t_x, \\ w_s \propto l^\alpha, & \text{if } t \gg t_x, \end{cases}$$

being $t_x \propto l^\gamma$ and the exponents related by:

$$z = \begin{cases} \alpha, & \text{if } z \ll 2, \\ \beta, & \text{if } z \gg 2, \end{cases}$$

and by Galilean invariance $^1$:

$$\alpha + z = 2.$$  

The KPZ approach describes and connects a wide range of experiments $^3$ $^4$ $^5$ $^6$ $^7$ $^8$ $^9$ $^{10}$ $^{11}$ $^{12}$ $^{13}$ $^{14}$ $^{15}$ $^{16}$ $^{17}$ $^{18}$ $^{19}$ $^{20}$ $^{21}$ $^{22}$ $^{23}$ and models $^{24}$ $^{25}$ $^{26}$ $^{27}$ $^{28}$. Note that most of these processes are interconnected, as an example, the single step (SS) model $^{22}$ $^{23}$ $^{24}$ $^{25}$ $^{26}$ is connected with the asymmetric simple exclusion process $^{25}$, the six-vertex model $^{22}$ $^{27}$, and the kinetic Ising model $^{22}$ $^{28}$. Recently, quantum versions of KPZ equation have been formulated $^{29}$ $^{30}$ $^{31}$ $^{32}$. Despite all effort, we are still far from a satisfactory theory for the KPZ equation, what makes it one of the toughest problems in modern mathematical physics, and probably one of the most important problem in nonequilibrium statistical physics, for reviews see $^{24}$ $^{25}$ $^{33}$ $^{34}$ $^{35}$ $^{36}$. Thus, two main questions are still open:

1. What is the probability of height distributions?
2. What are the exponents $\alpha$, $\beta$ and $z$?

Up to now these questions have been answered exactly
only in 1 + 1 dimensions. The first due to the succession of
researches \[1,13,14,37,38\] and the second by the original
KPZ work \[1\]. In this work we propose to obtain the ex-
ponents for a 2 + 1 dimensional system from first principle,
precise simulations and analysis of data in the literature.

2. Theory

Let us start considering that if we know one of the exponents (α, β, z),
we can determine the others from the relations \[5\] and \[6\], in a such way that we
will concentrate in the determination of the roughness exponent α. In
the search for the KPZ exponents several analytical methods,
such as scaling relation and renormalization group (RG)
approaches have been tried. Up to now we can resume that as:

1. Scaling fails for all dimensions.
2. RG works only for 1 + 1 dimensions \[1\]. It fails for
d all \(d > 1\).
3. Field theoretical methods yield exponents that are
not precise \[39\].

By the “failure” of these approaches, we mean that they
were not able to give precise exponents. However, some
calculations using RG produces very useful results. For
example, Canet et al. \[40,41\] used nonperturbative RG and
obtained the only complete analytical approach yielding a
qualitatively correct phase/flow diagram to date. We shall
consider what is particularly valid in these approaches and
consider all KPZ parameters, we rewrite Eq. \[9\] as:

\[
w_s = \left( \frac{D}{24 \nu} \right)^{\alpha} \Psi^{\frac{\alpha}{2} - 1},
\]

(10)

where \(\Psi\) is a dimensionless number. Nevertheless,
dimensional analysis does not give us the nondimensional quan-
tities, it is convenient to keep Eq. \[9\] in the above form.
First, we shall distinguish dimensional analysis from scaling,
for example \(w_s\) scales as \(t^\alpha\), however its physical
dimension is the same as the height \(h\), \([w_s] = [h] = [L]\), i.e
in experiments they are measured in units of length, as it
must be from definition \[3\]. For example, in experiments of
cadmium telluride thin growing on Si (001) surfaces \[9\].
While the substrates have size 10\(\mu\)m \(\times\) 10\(nm\), the \(h \sim \) \(nm\)
and \(w \sim nm\), which means that both quantities scale dif-
fent, but their physical unit remains in length units.

1+1 dimensions

In order to prove the above relation, let us first perform
the dimensional analysis in 1 + 1 dimensions. The physical
dimensions involved are \([\nu] = [L^2][T^{-1}], [D] = [L^3][T^{-1}]\)
and \([\lambda] = [L][T^{-1}]\) where \([T]\) is the time dimension. Consid-
ering all KPZ parameters, we rewrite Eq. \[9\] as:

\[
w_s = \left( \frac{al}{24 \nu} \right)^{\alpha} \Psi^{\frac{\alpha}{2} - 1},
\]

(11)

where

\[
A = \frac{D}{2\nu}.
\]

Since \(w_s\) must be time independent, we have \(\phi = -(\phi_1 + \phi_2)\) and

\[
\Psi(\lambda) = \left( \frac{D}{\nu} \right)^{\phi_1-1} \left( \frac{\lambda}{\nu} \right)^{\phi_2}. \tag{12}
\]

Thus, we get:

\[
\alpha = \frac{1}{2 + \phi}, \tag{13}
\]

with \(\phi = (\phi_1 - 1) - \phi_2\). Note that in order to have Eq. \[10\]
equal to Eq. \[9\] we must have \(\Psi = 1\), which gives \(\phi_1 = 1\)
and \(\phi_2 = 0\), which implies in \(\phi = 0\), and one recovers the
Krug’s result \[8\] with \(\alpha = 1/2\).

d+1 dimensions

Before we continue our analysis, let us remember that
the fluctuation relation Eq. \[2\] works for \(d = 1\) \[11,15,13\],
however it does not work properly for higher dimensions.
The violation of the FDT is well-known in the literature,
for example, in KPZ \[1,13,14\] for \(d > 1\), in structural
glass \[15\] and in ballistic diffusion \[16,17,18,19,20\]. Note that all parameters \(w_s, a\) and \(\nu\) have a well fixed time
and space dimension, the only one that change with the
space dimension $d$ is the fluctuating or noise parameter $D$ and, therefore, we expect some violation of fluctuations relations, such as [2].

In growth we may have a simple answer for that, consider for example the SS model, which is defined in such way that the height difference between two neighboring heights $\eta = h_i - h_j$ is just $\eta = \pm 1$. Now, let us consider a hypercube of side $L$ and volume $V = L^d$. Thus, we can select a site $i$ and compare its height with that of its neighbors $j$, applying the following rules:

1. At time $t$, randomly choose a site $i \in V$;
2. If $h_i(t)$ is a minimum, then $h_i(t + \Delta t) = h_i(t) + 2$, with probability $p$;
3. If $h_i(t)$ is a maximum, then $h_i(t + \Delta t) = h_i(t) - 2$, with probability $q$. 

Note that in (1) we have a white noise in the $d + 1$ space. However, due to rules (2) and (3) only a part of the noise will be really effective. An analogy is to throw a beam of light on a surface. For a flat surface we get the same beam reflected. However, if the surface is rough, the reflected beam will be completely modified. Thus, we must consider not the applied noise, but the noise selected by the rough surface (the system’s response), which can have different properties, such as intensity and dimension.

Second and more strong reason to change the dimension of the response is the widely known fact that the surface has a fractal dimension [2, 32], as already mentioned above, and as exhibit for example in SiO$_2$ films [3] or in the rough interface generated for the 2 + 1 SS model [26]. To illustrate that, we show in Fig. 1 the fractal geometry for 2 + 1 dimensions and the method described to obtain the fractal dimension.

Now, under this condition the noise intensity at the interface will have the dimension $[D] = [L^{d+1}]$ [T$^{-1}$]. Thus, the dimensional analysis yields now:

$$\alpha = \frac{1}{d_f + 1 + \phi}, \quad (14)$$

with

$$\phi = (\phi_1 - 1)d_f - \phi_2. \quad (15)$$

**Universality**

For $d + 1$ dimensions, as the parameters $\nu$ and $\lambda$ are not universal, i.e. they change with the model, in the particular case of the SS model, they are function of the probability $p$. If each model has different values of $\phi$ that results in different values of $\alpha$, which contradicts the KPZ universality. Thus all models in the KPZ universality classe must have $\phi = 0$. On the other hand, there is a less restrictive solution with $\Psi \neq 1$ that preserves universality, i.e. $\phi = 0$, with

$$\phi_1 = 1 + \phi_2/d_f, \quad (16)$$

for that we do not need to know what the exponent $\phi_2$ is, and the parameter $\Psi$ is just a number, which may change from model to model. Consequently, without contradicting universality $\Psi \neq 1$ is not out of the cards. Therefore, it follows that the dimensional analysis with universality, $\phi = 0$, determines the roughness exponent as:

$$\alpha = \begin{cases} 1/2, & \text{if } d = 1 \\ 1/(1+d_f), & \text{if } d \geq 2. \end{cases} \quad (17)$$

Thus a RG approach for $d + 1$ dimensions must consider the fractality of the interface[1].

However, for $d > 2$, the available simulation data in the literature for the exponents are very rough, and we shall focus our analysis in $2 + 1$ dimensions.

**2+1 dimensions**

The 2+1 dimensions is the most important one, besides being our real world, the growth phenomena are associated to surface science, also to the development of new technological devices, such as thin films. There are experiments and more simulation results available and we can get more precise exponents than for $3 + 1$, for example. For the EW equation the fractal dimension is $d_f = 2$, in this case from Eq. [7] we get $\alpha = 0$, in agreement with the scaling relation $\alpha = (2 - d)/2 = 0$ [2], but in strong disagreement with the relation (17). However, a null exponent in phase transition and also in diffusive process [34] does not mean a $w_s$ independent of $l$, but rather a logarithmic behavior, $w_s \propto \ln (l)$, being this behavior recently observed for EW model [29]. In this case, the power law behavior suggested in [9] is not valid, and we are only concerned here with situations in which the power law behavior holds.

---

1 Although it is not our objective here to do a full RG for KPZ, we have done a first draft of this RG approach with a fractal noise, within one loop expansion. The integrals becomes more complicate, but the contributions due to corrections in $\lambda$ sums up zero as in the KPZ classical work [1]. Thus, the Galilean invariance [1] is still valid.
Now we return to the KPZ equation, where the equations (21) and (7) for \( d = 2 \) yield:

\[
\alpha = \frac{3 - \sqrt{5}}{2}; \quad \beta = \sqrt{5} - 2; \quad z = \frac{1 + \sqrt{5}}{2},
\]

which gives \( d_f = 1.61803 \ldots \) and \( \alpha = 0.381966011 \ldots \). The exponents above are our major results.

3. Simulations Results

In order to have some precise numerical results, let us consider the etching model \([10, 17, 20]\) for 2 + 1 dimensions. This model belongs to the KPZ universality class \([21]\). First, we obtain the fractal dimension, \( d_f \), from the boxing counting method \([2]\), as it is depicted in Fig. (1). The results were averaged over the number of sites and also over different numerical experiments. The number of experiments is given by \( N_e = (2^d/l)^{5/2} \), and a single experiment for \( l \geq 2^9 \), in such way that a large sample needs a small number of experiments.

We show in Fig 2 (a) \( d_f \) as function of the time for a square lattice of size \( l = 2^{11} \). After convergence, we get the time average, which is represented by the vertical black line, to obtain \( d(l, t \to \infty) = d_f \equiv d_f(l) \), for each value of \( l \). In (b), the semi-log plot of the \( d_f \) as function of the size \( l \). In order to correct the finite size effects, we adjust the points to \( d_f(l) = d_f - c/l \) and obtain \( d_f = 1.612(2) \), which inserted into Eq. (17) yields \( \alpha = 0.3828(3) \).

It is important to note that these values are very close to the analytical results \( d_f = 1.61803 \) and \( \alpha = 0.381966011 \ldots \). Already in Fig. 2 (c), we exhibit \( \alpha \) versus time \( t \) for a square lattice of size \( l = 2^{10} \) obtained from the correlation function:

\[
C(r, t) = \langle [h(\vec{x} + r, t) - h(\vec{x}, t)]^2 \rangle \propto r^{2 \alpha},
\]

where \( r \) is the modulus of the vector \( \vec{r} \) with \( r < \xi \), being \( \xi \) the correlation length \([42]\). In this sense, we consider the first 9 neighbors along the principal axes, the results were averaged over 1000 different simulations and \( \alpha(t) \) was found by fitting the correlation curve. \( \alpha \) increases rapidly with time and afterwards equilibrated with its values fluctuating around the mean, \( \bar{\alpha} = 0.38211(1) \), where it was estimated in the range of \( 1.2 \leq t \leq 2 \) (solid line), being this value very close to our analytical values (dashed line), as it is depicted in the inset of Fig. 2 (c).

A remarkable result was found for the SS model for a system with \( p = 1 \) and size \( l = 2048 \) in which the results were averaged over the number of sites, 3 experiments and again over time to obtain \( \bar{\alpha} = 0.381955(60) \), as shown in Fig. 3.

In addition, in Fig. 3 we show \( w_\nu \) versus \( l \) for the 2+1 etching model. We use the value \( A = 3.629(9) \) from \([51]\) to obtain \( \bar{\alpha} = 0.3815(2) \). From the etching model \([21]\) using \( \frac{D}{\nu} = 3.62(3) \) thus Eq. (11) yields \( \Psi = 1.00(2) \). It should be noted that this result confirms not only the exponent but

\[\begin{array}{c}
\alpha = 0.38211(1) \\
\alpha^* = (3 - \sqrt{5})/2
\end{array}\]

Figure 2: Etching model in 2 + 1 dimensions: (a) \( d_f \) against time \( t \) (in units of \( t_x \)), (b) the semi-log plot of the \( d_f \) in function of the size \( l \). The adjusted curve was obtained from the fit of the function \( d_f(l) = d_f - c/l \). (c) \( \alpha \) as a function of time \( t \) (in units of \( t_x \)) obtained from the correlation function \([19]\), being the dashed line the analytical result, Eq. (18).
We shall analyze carefully the existing experimental results. First, on measuring \( \alpha \), or computing it using (19) we must know in what regime we are. For example, one should always remember that in the growth phenomena each time unit corresponds to a deposition layer. Consequently, thin film will not achieve the saturation regime of Fig. (3), where \( x \sim t \). In addition, we must distinguish between local and global value of \( \alpha \). Accurate experiments give \( \alpha = 3.629(9) \) [5] and \( \alpha = 3.81966011\ldots \), Eq. (18). For small \( l \) we see some finite size effects.

also the factors within Eq. (10), the second part, \( \Psi = 1 \), up to now only for the etching model. However, the major point here is that the factor \( \Psi \) does not alter our dimensional analysis, therefore the exponents are independent of it.

We have found the simulations results for different models (etching and SS models) in a good agreement with the analytical results. In the following section we shall compare our analytical results with experimental/computational results from the literature.

4. Literature data

| Authors | \( \alpha \) | \( \delta \) | Model |
|---------|-------------|-------------|-------|
| PW(A)   | 3 - \( \sqrt{5} \) | 0           | Universal |
| PW(b)   | 0.3828(3) | -0.008(3)  | Etching |
| PW(c)   | 0.3821(1) | -0.00014(1) | Etching |
| PW(d)   | 0.3815(2) | 0.005(2)   | Etching |
| PW(e)   | 0.381555(60) | 0.000011(60) | SS |
| [12]    | 0.387(4)  | -0.005(4)  | RSOS |
| [12]    | 0.386(6)  | -0.004(6)  | SS |
| [12]    | 0.387(7)  | 0.001(7)   | Etching |
| [12]    | 0.387(13) | -0.005(13) | RSOSC |
| [12]    | 0.374(20) | 0.008(26)  | SSS |
| [12]    | 0.379(9)  | 0.003(9)   | Eden (001) |
| [12]    | 0.386(8)  | -0.004(8)  | Eden (111) |
| [12]    | 0.383(8)  | -0.001(8)  | Etching |
| [12]    | 0.38(8)   | -0.002(8)  | RSOS |
| [52]    | 0.3869(4) | -0.0049(4) | RSOS |
| [53]    | 0.3889(3) | -0.0069(3) | Octahedron |
| [53]    | 0.377(15) | 0.005(15)  | DLC |
| [53]    | 0.388(1)  | -0.006(1)  | BCSOS |
| [55]    | 0.385(4)  | -0.003(4)  | DPRM |
| [55]    | 0.38(1)   | -0.002(1)  | BCSOS |
| [57]    | 0.38(1)   | -0.002(1)  | Mode-coupling-KPZ |

Table 1: Values of \( \alpha \) and deviation \( \delta \). Here \( \delta = \alpha^* - \alpha \), measures the deviation from the analytical value \( \alpha^* = 3 - \sqrt{5} / 2 \). PW(A) stands for Present Work Analytical results. PW(b) by determination of the fractal dimension, Fig. (2b); PW(c) using the correlation function, Eq. (19), Fig. (2c); PW(d) from Fig. (2f), and PW(e) from the SS model, Fig. (3). Observe that the same method [12] can yield slightly different results, mainly fluctuations, for different models. The average of all values gives \( \overline{\alpha} = 0.3828 \pm 0.0037 \) and \( \delta = -0.0008 \pm 0.0037 \).

Experimental results

We shall analyze carefully the existing experimental results. First, on measuring \( \alpha \), or computing it using [19] we must know in what regime we are. For example, one should always remember that in the growth phenomena each time unit corresponds to a deposition layer. Consequently, thin film will not achieve the saturation regime of Fig. (3), where \( x \sim t \). In addition, we must distinguish between local and global value of \( \alpha \). Thus, a good exponent is obtained in experiments that measure the exponent \( z \). Accurate experiments give \( z = 1.6(2) \) [3], \( z = 1.6(1) \) [6], \( z = 1.61(5) \) [9], and \( z = 1.61(11) \) in agreement with our value of \( z = (3 - \sqrt{5}) / 2 = 1.61803\ldots \). As the final destination of any theory is decided by the experiments, we can say that so far the odds favor us.
Computational and theoretical results

Finally, in order to provide how much our analytical value, \( \alpha^* = (3 - \sqrt{5})/2 = 0.381966011, \ldots \), is close to \( \alpha \) values for some models in \( 2 + 1 \) dimensions, as listed in Table 1, we measure \( \delta = \alpha^* - \alpha \), which is the deviation from the analytical value. Thus, as we can see, all of these values are close to our analytical results and \( \delta \) oscillates around zero.

5. \( d +1 \) dimensions

Now we propose an extension of the Eq. (20) for \( d \geq 2 \) as
\[
\alpha = \bar{D} - \bar{D}_i = d - d_f.
\]
(20)
Since \( 0 < \alpha < 1 \), then \( d - 1 < d_f < d \). This is what we expect for the dimension of one fractal embedded in an integer space.

Therefore, for \( d+1 \) dimensions with \( d \geq 2 \), the Eq. (20) together with the Eq. (17) yields:
\[
d_f = \frac{d - 1 + \sqrt{\Delta}}{2}, \quad \alpha = \frac{d + 1 - \sqrt{\Delta}}{2}
\]
(21)
and
\[
z = \frac{3 - d + \sqrt{\Delta}}{2}, \quad \beta = \frac{d - \sqrt{\Delta}}{3 - 2d}
\]
(22)
with \( \Delta = (d + 1)^2 - 4 \). This simple relation gives us the fractal dimension and the exponents (\( \alpha, \beta, z \)) for all \( d \geq 2 \). This shows as well that there is no upper critical limit for the KPZ equation, since the exponents decay with \( d \) without showing any specific upper critical dimension. Unfortunately, for \( d \geq 2 \), the available simulations data is not as precise as that of 2 + 1.

6. Conclusion

In this work, our objective was to obtain the most accurate values of the growth exponents in \( 2 + 1 \) dimensions for the KPZ equation. In order to do that, we extend the Krug’s universal solution for \( 1 + 1 \) dimensions to \( d + 1 \) dimensions and we impose the fractality of the interface, by replacing the integer dimension \( d \) by the fractal dimension \( d_f \) of the roughness interface, from which we were able to determine the values of \( 2 + 1 \) Kardar-Parisi-Zhang exponents. The solution provides the golden ratio \( d_f = \frac{1 + \sqrt{5}}{2} \), which also appears in many natural phenomena, such as growth of vegetable and animals. Moreover, we show that our results are not only in good agreement with our numerical results, but also with the literature data for some models in \( 2 + 1 \) dimensions, which shows that the KPZ universality class holds. For \( 2 + 1 \) dimensions we believe that our proposal is fully justified. Considering that no approximation was done to obtain the exponents (18), we believe that these are good candidates to be the exact values. However, for \( d + 1 \) dimensions, with \( d > 2 \) there is a lack of reliable results, thus new theoretical and simulations results will be necessary for higher dimensions. Nevertheless, we have achieved important progress. First, we have explicit analytical results well confirmed by simulations for \( 2 + 1 \) dimensions. Better, it is in accordance with experimental results; second, notice that the exponents appear as irrational numbers and not as the ratio between integers as in the first guess (59, 60) and they are now directly related to the fractal dimension \( d_f \). This is more close to what has been obtained from numerical simulations. Finally, the discussions presented here open a new scenario for further investigation of different forms of growth both theoretical and numerical. For example, the RG approach in the fractal interface will probably originate new important results, as mentioned above, one loop expansion preserves the Galilean invariance (6). However, it deserves further developments. The attempt to obtain exact height fluctuations for the stationary KPZ equations, as well as for most of physics in \( 2 + 1 \) dimensions is still in its begin. These theoretical methods will benefit from the fixed points obtained by precise KPZ exponents, and from a fractal geometry that must be associated with them (34). We expect as well that new methods would confirm our results. This work opens a new horizon for KPZ research.

Acknowledgments

The authors are grateful for the many useful communications and advice from Prof. Janos Kertesz, Thiago A. Assis, Tiago J. Oliveira, and Francisco Alcaraz. We would also like to thank Mrs. Rayra S. S. Veloso for her kind review. This work was supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Grant No. CNPq-312497/2018-0 and the Fundação de Apoio a Pesquisa do Distrito Federal (FAPDF), Grant No. FAPDF-00193-0000120/2019-79.

References

[1] M. Kardar, G. Parisi, Y.-C. Zhang, Dynamic scaling of growing interfaces, Phys. Rev. Lett. 56 (1986) 889–892.
[2] A.-L. Barabási, H. E. Stanley, Fractal concepts in surface growth, Cambridge university press, 1995.
[3] J. Merikoski, J. Maunuksela, M. Myllys, J. Timonen, M. J. Alava, Temporal and spatial persistence of combustion fronts in paper, Phys. Rev. Lett. 90 (2) (2003) 024501.
[4] P. Le Doussal, S. N. Majumdar, A. Rosso, G. Schehr, Exact short-time height distribution in the one-dimensional kardar-parisi-zhang equation and edge fermions at high temperature, Phys. Rev. Lett. 117 (7) (2016) 070403.
[5] P. A. Orrillo, S. N. Santalla, R. Cuerno, L. Vázquez, S. B. Ribotta, L. M. Gassa, F. Mompean, R. C. Salvarezza, M. E. Vela, Morphological stabilization and kpe scaling by electro-chemically induced co-deposition of nanostructured niw alloy films, Sci. Rep. 7 (1) (2017) 1–12.
[6] F. Ojeda, R. Cuerno, R. Salvarezza, L. Vázquez, Dynamics of rough interfaces in chemical vapor deposition: Experiments and a model for silica films, Phys. Rev. Lett. 84 (2000) 3125–3128. doi:10.1103/PhysRevLett.84.3125.
URL https://link.aps.org/doi/10.1103/PhysRevLett.84.3125.
L. Chen, C. F. Lee, J. Toner, Mapping two-dimensional polar active fluids to two-dimensional soap and one-dimensional sandblasting, Nat. Commun. 7 (1) (2016) 1–10.

K. A. Takeuchi, Crossover from growing to stationary interfaces in the kardar-parisi-zhang class, Phys. Rev. Lett. 110 (21) (2013) 210604.

R. Almeida, S. Ferreira, T. Oliveira, F. A. Reis, Universal fluctuations in the growth of semiconductor thin films, Phys. Rev. B 89 (4) (2014) 045309.

R. A. Almeida, S. O. Ferreira, I. Ferraz, T. J. Oliveira, Initial pseudo-steady state & asymptotic kpc universality in semiconductor on polymer deposition, Sci. Rep. 7 (1) (2017) 1–10.

D. Fusco, M. Gralka, J. Kayser, A. Anderson, O. Hallatschek, Excess of mutational jackpot events in expanding populations revealed by spatial luria-delbruck experiments, Nat. Commun. 7 (2016) 12760.

T. J. Oliveira, S. G. Alves, S. C. Ferreira, Kardar-parisi-zhang universality class in (0 + 1) dimensions: Universal geometry-dependent distributions and finite-time corrections, Phys. Rev. E 87 (4) (2013) 040102(R).

P. Calabrese, P. Le Doussal, A. Rosso, Free-energy distribution of the directed polymer at high temperature, Europhys. Lett. 90 (2) (2010) 20002.

G. Amir, I. Corwin, J. Quastel, Probability distribution of the etching model in high dimensions, Journal of Physics A: Mathematical and Theoretical 48 (3) (2014) 035001.

E. A. Rodrigues, F. A. Oliveira, B. A. Mello, On the existence of an upper critical dimension for systems within the kpz universality class, Acta Physica Polonica B 46 (issue 6) (2015) 1231.

doi:10.5506/apphyspolb.46.1231

E. A. Rodrigues, B. A. Mello, F. A. Oliveira, Growth exponents of the etching model in high dimensions, Journal of Physics A: Mathematical and Theoretical 48 (3) (2014) 035001.

E. A. Rodrigues, F. A. Oliveira, B. A. Mello, On the existence of an upper critical dimension for systems within the kpz universality class, Acta Physica Polonica B 46 (issue 6) (2015) 1231.

doi:10.5506/apphyspolb.46.1231

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

doi:10.1103/physreve.94.042119

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.

W. S. Alves, E. A. Rodrigues, H. A. Fernandes, B. A. Mello, F. A. Oliveira, I. V. L. Costa, Analysis of etching at a solid-solid interface, Phys. Rev. E 94 (2016) 042119.

F. A. Reis, Universality in two-dimensional kardar-parisi-zhang growth, Phys. Rev. E 69 (2) (2004) 021610.
Oliveira, Khinchin theorem and anomalous diffusion, Phys. Rev. Lett. 101 (2008) 230602. [doi:10.1103/PhysRevLett.101.230602]

[51] I. Carrasco, K. Takeuchi, S. Ferreira, T. Oliveira, Interface fluctuations for deposition on enlarging flat substrates, New J. Phys. 16 (12) (2014) 123057.

[52] A. Pagnani, G. Parisi, Numerical estimate of the kardar-parisi-zhang universality class in (2+ 1) dimensions, Phys. Rev. E 92 (1) (2015) 010101.

[53] J. Kelling, G. Ódor, S. Gemming, Dynamical universality classes of simple growth and lattice gas models, J. Phys. A: Math. Theor. 51 (3) (2017) 035003.

[54] G. Ódor, B. Liedke, K.-H. Heinig, Mapping of (2+ 1)-dimensional kardar-parisi-zhang growth onto a driven lattice gas model of dimers, Phys. Rev. E 79 (2) (2009) 021125.

[55] T. Halpin-Healy, (2+ 1)-dimensional directed polymer in a random medium: Scaling phenomena and universal distributions, Phys. Rev. Lett. 109 (17) (2012) 170602.

[56] C.-S. Chin, M. den Nijs, Stationary-state skewness in two-dimensional kardar-parisi-zhang type growth, Phys. Rev. E 59 (3) (1999) 2633.

[57] F. Colaiori, M. A. Moore, Upper critical dimension, dynamic exponent, and scaling functions in the mode-coupling theory for the kardar-parisi-zhang equation, Phys. Rev. Lett. 86 (2001) 3946–3949.

[58] E. E. M. Luis, T. A. de Assis, S. C. Ferreira, R. F. S. Andrade, Local roughness exponent in the nonlinear molecular-beam-epitaxy universality class in one dimension, Phys. Rev. E 99 (2019) 022801. [doi:10.1103/PhysRevE.99.022801]

[59] D. Wolf, J. Kertesz, Surface width exponents for three-and four-dimensional eden growth, Europhys. Lett. 4 (6) (1987) 651.

[60] J. M. Kim, J. Kosterlitz, Growth in a restricted solid-on-solid model, Physical review letters 62 (19) (1989) 2289.