A former action of $G$ equivariant map $G$ [Has17] that when $g$ moment map, where $g$ symplectic manifold is a real symplectic vector space in each case, yields a representation of $g$ moment map on $g$ by and $G$ underlying $C$ Mp in indefinite orthogonal group, symplectic vector space, moment map, canonical quantization, Howe duality, to find the $K$-type formula, the Gelfand-Kirillov dimension and the Bernstein degree of them for non-negative integers $m$. The $K$-type formula for $m = 0$ shows that it is nothing but the $(g, K)$-module of the minimal representation of $O(p, q)$. One finds that the Gelfand-Kirillov dimension is equal to $p + q - 3$ not only for $m = 0$ but for any $m$ satisfying $m + 3 \leq (p + q)/2$ when $p, q \geq 2$ and $p + q$ is even, and that the Bernstein degree for $m$ is equal to $(m + 1)$ times that for $m = 0$.

1. Introduction

Let $G$ be a Lie group with $g_0$ its Lie algebra and $g$ the complexification of $g_0$. An action of $G$ on a symplectic manifold $(M, \omega)$ is called symplectic if $g^* \omega = \omega$ for all $g \in G$, and a symplectic action is called Hamiltonian if there exists a smooth $G$-equivariant map $\mu : M \to g_0^*$ satisfying the condition (2.3) below, which is called a moment map, where $g_0^*$ is the dual vector space of $g_0$. We are concerned with the cases where the symplectic manifold is a real symplectic vector space $(W, \omega)$. It was shown in [Has17] that when $G = \text{Sp}(n, \mathbb{R})$, $U(p, q)$ and $O^*(2n)$, the canonical quantization of the moment map on $W = \mathbb{R}^{2n}$, $(\mathbb{C}^{p+q})_{\mathbb{R}}$ and $(\mathbb{C}^{2n})_{\mathbb{R}}$, with a choice of a Lagrangian subspace in each case, yields a representation of $\mathfrak{g}$ that is the differentiation of the oscillator (or Segal-Shale-Weil) representation of $\text{Mp}(n, \mathbb{R})$, $U(p, q)$ and $O^*(2n)$ respectively, where $\text{Mp}(n, \mathbb{R})$ is the metaplectic group, i.e., the double cover of $\text{Sp}(n, \mathbb{R})$.

In this paper, we consider the case where $W = (\mathbb{C}^{p+q})_{\mathbb{R}}$, the real vector space underlying $\mathbb{C}^{p+q}$:

$$W = \{z = x + iy \mid x, y \in \mathbb{R}^{p+q}\},$$

which we regard as a symplectic vector space equipped with a symplectic form $\omega$ given by

$$\omega(z, w) = \text{Im}(z^* I_{p,q} w) \quad (z, w \in W),$$

and $G = O(p, q)$, the indefinite orthogonal group defined by

$$O(p, q) = \{g \in \text{GL}_{p+q}(\mathbb{R}) \mid g I_{p,q} g = I_{p,q}\}$$

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with $I_{p,q} = \begin{bmatrix} 1_p & 0_q \\ 0_p & -1_q \end{bmatrix}$. The action of $G = \text{O}(p,q)$ on $W$ defined by matrix multiplication is symplectic and Hamiltonian. The $\text{O}(p,q)$-case we consider here is closely related to the $\text{U}(p,q)$-case mentioned above. In fact, the symplectic vector space $(W, \omega)$ for $\text{O}(p,q)$ is identical to the one for $\text{U}(p,q)$, and the action of $\text{O}(p,q)$ on $W$ is the restriction of the action of $\text{U}(p,q)$ induced from the canonical embedding of $\text{O}(p,q)$ into $\text{U}(p,q)$. Furthermore, the moment map for the $\text{O}(p,q)$-case is the real part of the one for the $\text{U}(p,q)$-case.

The canonical quantization of the moment map $\mu : W = (\mathbb{C}^{p+q})_\mathbb{R} \to \mathfrak{g}_0^*$ for $G = \text{O}(p,q)$, with a choice of a Lagrangian subspace $V$ of $W$, yields a representation $\pi$ of $\mathfrak{g}$ as in the cases mentioned above, which is shown to be a partial Fourier transformation of the representation $\pi^\sharp$ of $\mathfrak{g}$ obtained by differentiating the left regular representation of $G$ on $C^\infty(V)$. Note that if we restrict the operator $\pi^\sharp(X)$, $X \in \mathfrak{g}$, to a subspace consisting of homogeneous functions on $V$ with respect to the multiplicative group $\mathbb{R}_{>0}$, then the restricted representation is the degenerate principal series of $G$ obtained by inducing up a one-dimensional representation of a parabolic subgroup of $G$ (see [HT93]).

In the influential paper [How89], Howe showed that one can treat the classical invariant theory from a unified viewpoint — the dual pair. We focus our attention on the dual pair $(\text{O}(p,q), \text{SL}_2(\mathbb{R}))$, both components of which are non-compact, and apply the representation theory of $\text{SL}_2$ to cut out irreducible $(\mathfrak{g}, K)$-modules, which we denote by $M^+(m)$ and $M^-(m)$, $m = 0, 1, 2, \ldots$, in this paper, where $M^+(m)$ (resp. $M^-(m)$) consists of all highest (resp. lowest) weight vectors with respect to the $\text{SL}_2$-action (see Definition 4.1 below for details). We will see that such weight vectors are given in terms of harmonic polynomials and the Bessel functions of the first kind. Both $M^\pm(m)$ correspond to the $(m+1)$-dimensional irreducible representation of $\text{SL}_2$ under the Howe duality, and in fact are isomorphic to each other. They were originally considered in [RS80] without the condition of finite-dimensionality. Note that $M^+(0) = M^-(0)$ by definition.

In the cases of the oscillator representations mentioned above, i.e., when $G = \text{Sp}(n, \mathbb{R})$, $\text{U}(p,q)$ and $\text{O}^*(2n)$, we note that the counterpart $G'$ of $G$ for the dual pair $(G, G')$ is compact, hence, all its irreducible representations are finite-dimensional. Furthermore, the oscillator representations give examples of the minimal representations (we refer to [KM11] and the references therein for the definition of the minimal representation). When $G = \text{O}(p,q)$, its minimal representation is discussed e.g. in [Kos90, BZ91, ZH97, KO03, KM11].

The main result of this paper is the $K$-type formula of $M^\pm(m)$ for non-negative integers $m$ satisfying

$$m + 3 \leq \frac{p + q}{2},$$

from which one can show that $M^\pm(m)$ are irreducible $(\mathfrak{g}, K)$-modules for $p, q \geq 2$ with $p + q$ even (Theorem 4.6). The fact that the elements of $M^\pm(m)$ are described in terms of the Bessel function plays a rôle in the proof of our main result. The $K$-type formula of $M^+(0) = M^-(0)$, which corresponds to the one-dimensional trivial representation of $\text{SL}_2$, shows that it is nothing but the underlying $(\mathfrak{g}, K)$-module of the minimal representation of $\text{O}(p,q)$. We will see that the Gelfand-Kirillov dimension of $M^\pm(m)$ is equal to $p + q - 3$. 

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not only for $m = 0$ but for any non-negative integer $m$ satisfying (11). Meanwhile, the Bernstein degree of $M^+(m)$ is $(m + 1)$ times that of $M^+(0) = M^-(0)$ (Corollary 4.8).

The rest of this paper is organized as follows. In §2, we compute the moment map $\mu$ on $W$ for $G = O(p, q)$, and construct the representation $\pi$ of $\mathfrak{g}$ via canonical quantization of $\mu$. Then we show that $\pi$ is a partial Fourier transform of the differential representation of the left regular representation of $G$ on $C^\infty(V)$. In §3, we give an $sl_2$-action that commutes with $\pi$, and find both highest weight vectors and lowest weight vectors with respect to the $sl_2$-action. We remark that such weight vectors are given in terms of the Bessel functions of the first kind. In §4, we introduce $(\mathfrak{g}, K)$-modules $M^\pm(m)$ and prove that $M^+(m)$ and $M^-(m)$ are isomorphic to each other for any non-negative integer $m$. Then we find the $K$-type formula of $M^\pm(m)$ for $m$ satisfying (11) and show that they are irreducible. As a corollary, we obtain the Gelfand-Kirillov dimension and the Bernstein degree of $M^\pm(m)$.

**Notation.** Let $\mathbb{N}$ denote the set of non-negative integers $\{0, 1, 2, \ldots\}$, and $[p]$ the set $\{1, 2, \ldots, p\}$. For the sake of simplicity, we write $\bar{i} := p + i$ for $i \in [q]$. Finally, for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote the rising and the falling factorials by

$$(\alpha)_n := \prod_{i=1}^{n} (\alpha + i - 1) \quad \text{and} \quad (\alpha)_{n}^{-} := \prod_{i=1}^{n} (\alpha - i + 1),$$

respectively.

### 2. Moment Map and its Quantization

Let $G$ be the indefinite orthogonal group $O(p, q)$, which we realize by

$$O(p, q) = \{g \in GL_{p+q}(\mathbb{R}) \mid g I_{p,q}g^{-1} = I_{p,q}\}$$

with $I_{p,q} = \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix}$. Let $K$ be a maximal compact subgroup of $G$ given by

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in G \mid a \in O(p), d \in O(q) \right\} \simeq O(p) \times O(q).$$

We denote the Lie algebra of $K$ and its complexification by $k_0$ and $\mathfrak{k}$ respectively.

Let $\{X^{\pm}_{ij}\}$ be a basis for $\mathfrak{g}_0 = \mathfrak{o}(p, q)$ given by

$$X^{\pm}_{i,j} = E_{i,j} - E_{j,i} \quad (i, j \in [p])$$

$$X^{\pm\pm}_{i,j} = E_{i,j} - E_{j,i} \quad (i, j \in [q])$$

(2.1)

which also forms a basis for $\mathfrak{g} = \mathfrak{o}_{p+q}$, the complexification of $\mathfrak{g}_0 = \mathfrak{o}(p, q)$. We often identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the invariant bilinear form $B$ given by

$$B(X, Y) = \frac{1}{2} \text{tr}(XY) \quad (X, Y \in \mathfrak{g}),$$

where $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$. Finally, let $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of $\mathfrak{g}$ with

$$\mathfrak{t} = \sum_{i,j \in [p]} \mathbb{C}X^{+}_{i,j} \oplus \sum_{i,j \in [q]} \mathbb{C}X^{+\pm}_{i,j}, \quad \mathfrak{p} = \sum_{i \in [p], j \in [q]} \mathbb{C}X^{-}_{i,j}.$$
Let $W$ be the real vector space $(\mathbb{C}^{p+q})_\mathbb{R}$ underlying the complex vector space $\mathbb{C}^{p+q}$:

$$W = \{ z = x + iy \mid x = i(x_1, \ldots, x_{p+q}), y = i(y_1, \ldots, y_{p+q}) \in \mathbb{R}^{p+q} \},$$

which is equipped with a symplectic form $\omega$ given by

$$\omega(z, w) = \text{Im}(z^* I_{p,q} w) \quad (z, w \in W) \quad (2.2)$$

Then $G$ acts on $(W, \omega)$ symplectically via $z \mapsto g z$ (matrix multiplication) for $z \in W$ and $g \in G$. Furthermore, the action of $G$ on $(W, \omega)$ is Hamiltonian, i.e., there exists a moment map $\mu : W \to g_0^*$, whose definition we briefly recall: if, in general, a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ symplectically, a smooth $G$-equivariant map $\mu : M \to g_0^*$ that satisfies

$$d\langle \mu, X \rangle = i(X_M)\omega \quad \text{for all } X \in g_0.$$ 

(2.3)

is called a moment map, where $i$ stands for the contraction and $X_M$ denotes the vector field on $M$ given by

$$X_M(p) = \frac{d}{dt} \bigg|_{t=0} \exp(-tX) p \quad (p \in M).$$

Under the identification that $e_i := (0, \ldots, 1, \ldots, 0) \leftrightarrow \partial_{x_i}$ and $i e_i \leftrightarrow \partial_{y_i}$ for $i = 1, 2, \ldots, p + q$, the symplectic form $\omega$ given in (2.2) can be rewritten as

$$\omega = \sum_{i=1}^{p+q} \epsilon_i \, dx_i \wedge dy_i$$

with $\epsilon_i = 1$ for $i \in [p]$ and $\epsilon_{p+i} = -1$ for $i \in [q]$.

**Proposition 2.1.** The action of $G = O(p, q)$ on $(W, \omega)$ is Hamiltonian, and the moment map $\mu : W \to g_0^* \simeq g_0$ is given by

$$\mu(z) = -\frac{i}{2} \left( zz^* - i(zz^*) \right) I_{p,q}$$

$$= (-x'y + y'x) I_{p,q}$$

$$= \begin{bmatrix} -x'iy'y' + y''x' + x'y'y'' - y'y'' \xi & x'y'y'' - y'y'' \xi' \\ -x''iy'y' + y''x' + x'y'y'' - y'y'' \xi' & x'y'y'' - y'y'' \xi \end{bmatrix}$$

for $z = x + iy \in W$ with $x = (x', x'')$, $y = (y', y'') \in \mathbb{R}^{p+q}$ and $x', y', x'', y'' \in \mathbb{R}^q$.

**Proof.** See e.g. [CG97, Proposition 1.4.6].

**Remark 2.2.** Recall that the moment map $\mu_U : W \to u(p, q)^*$ for the action of $U(p, q)$ on $(W, \omega)$ is given by

$$\mu_U(z) = -i z z^* I_{p,q} \quad (z = x + iy \in W)$$

where we identify $u(p, q)^*$ with $u(p, q)$ via the invariant bilinear form $B$ given by $B(X, Y) = (1/2) \text{tr}(XY)$. Therefore, the moment map $\mu$ in the proposition is related to $\mu_U$ by

$$\mu(z) = \frac{\mu_U(z) + \overline{\mu_U(z)}}{2}.$$ 

Namely, one has $\mu = \text{Re } \mu_U$. 


We define a Poisson bracket by
\[ \{f, g\} = \omega(\xi_f, \xi_g), \]
where \(\xi_f\) denotes the Hamiltonian vector field on \(W\) corresponding to \(f \in C^\infty(W)\), i.e. the vector field that satisfies \(i(\xi_f)\omega = df\). Then the Poisson bracket among the coordinate functions are given by
\[ \{x_i, y_j\} = -\delta_{i,j}e_i, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0 \]
for \(i, j = 1, 2, \ldots, p + q\). The Dirac’s quantization conditions requires that
\[ \{f_1, f_2\} = f_3 \implies \{\hat{f}_1, \hat{f}_2\} = -i\hbar \hat{f}_3 \]
for \(f_i \in C^\infty(W)\) (see e.g. [Woo92]). Thus, we quantize the coordinate functions as follows:
\[ \hat{x}_i = x_i, \quad \hat{y}_i = -i\hbar \partial_{x_i}, \quad (i = 1, \ldots, p), \]
\[ \hat{x}_j = -i\hbar \partial_{y_j}, \quad \hat{y}_j = y_j, \quad (j = 1, \ldots, q), \]
where \(\partial_{x_i}\) and \(\partial_{y_j}\) denote \(\partial / \partial x_i\) and \(\partial / \partial y_j\) respectively. In what follows, we set \(\hbar = 1\) for brevity.

The quantization (2.4) corresponds to a Lagrangian subspace \(V\) of \(W\) given by
\[ V = \langle e_1, \ldots, e_p, i e_1, \ldots, i e_q \rangle_{\mathbb{R}} \]
in the sense that the quantized operators are realized in \(\mathbb{P}^{\mathbb{D}}(V)\), the ring of polynomial coefficient differential operators on \(V\). Therefore, the quantized moment map \(\hat{\mu}\) is given by
\[ \hat{\mu} = (-\hat{x}'\hat{y} + \hat{y}'\hat{x})_{I_{p,q}} = \begin{bmatrix} i(x'\partial_{x'} - \partial_{y'}x') & x'y'' + \partial_{y'}y'' \\ \partial_{x'''}y' + y'''x' & i(y'''\partial_{x'''} - \partial_{y'''}y''' ) \end{bmatrix}, \]
where
\[ \hat{x}' = i'(\hat{x}_1, \ldots, \hat{x}_{p+q}) = i'(x', -i\partial_{y''}), \]
\[ \hat{y}' = i'(\hat{y}_1, \ldots, \hat{y}_{p+q}) = i(-i\partial_{x'}, y''), \]
and
\[ x' = i'(x_1, \ldots, x_p), \quad \partial_{x'} = i'(\partial_{x_1}, \ldots, \partial_{x_p}), \]
\[ y'' = i'(y_1, \ldots, y_{p+q}), \quad \partial_{y''} = i'(\partial_{y_1}, \ldots, \partial_{y_{p+q}}). \]

Note that \(x_1, \ldots, x_p, y_1, \ldots, y_{p+q}\) are considered to be the coordinate functions on \(V\) with respect to the basis \(e_1, \ldots, e_p, i e_1, \ldots, i e_{p+q}\).

**Theorem 2.3.** For \(X \in \mathfrak{g}\), set \(\pi(X) := i(\hat{\mu}, X)\). Then \(\pi : \mathfrak{g} \to \mathbb{P}^{\mathbb{D}}(V)\) is a Lie algebra homomorphism. In terms of the basis (2.1), it is given by
\[ \pi(X) = \begin{cases} -x_j\partial_{x_i} + x_i\partial_{x_j} & \text{if } X = X^+_{i,j} ; \\ -y_j\partial_{y_i} + y_i\partial_{y_j} & \text{if } X = X^+_{i,j} ; \\ i(x_jy_i + x_iy_j) & \text{if } X = X^-_{i,j} . \end{cases} \]

**Proof.** This is proved in the same manner as [Has17, Theorem 2.3] (or, one can verify the commutation relations by direct calculation). \(\square\)
There is another canonical quantization that corresponds to the same Lagrangian subspace $V$ of $W$ as given in (2.5). Namely, if we quantize the coordinate functions as
\[
\hat{x}_i = x_i, \quad \hat{y}_i = -i \partial_{x_i}, \quad (i = 1, \ldots, p),
\]
\[
\hat{x}_j = y_j, \quad \hat{y}_j = i \partial_{y_j}, \quad (j = 1, \ldots, q).
\]
then the quantized moment map, which we denote by $\hat{\mu}^q$, is given by
\[
\hat{\mu}^q = (-\hat{x}'\hat{y} + \hat{y}'\hat{x}) I_{p,q} = i \begin{bmatrix}
x'\partial_{x'} - \partial_{x'}'x' & x'\partial_{y''} + \partial_{x'}'y'' \\
y''\partial_{x'} + \partial_{y''} 'x' & y''\partial_{y''} - \partial_{y''}'y''
\end{bmatrix},
\]
where
\[
\hat{x} = \hat{x}'(\hat{x}_1, \ldots, \hat{x}_{p+q}) = \hat{x}'(x', y'),
\]
\[
\hat{y} = \hat{y}'(\hat{y}_1, \ldots, \hat{y}_{p+q}) = \hat{y}'(\partial_{x'}x', \partial_{y'}y').
\]
Hence one obtains a representation $\pi^q : g \to \mathcal{P}\mathcal{D}(V)$ if one sets $\pi^q(X) := i \langle \hat{\mu}^q, X \rangle$ for $X \in g$. It is given in terms of the basis (2.1) by
\[
\pi^q(X) = \begin{cases}
-x_j \partial_{x_j} + x_j \partial_{x_j} & \text{if } X = X_{ij}^+; \\
-y_j \partial_{y_j} + y_j \partial_{y_j} & \text{if } X = X_{ij}^-; \\
-(x_j \partial_{y_j} + y_j \partial_{x_j}) & \text{if } X = X_{ij};
\end{cases}
\]
(2.8)

Remark 2.4. (i) Comparing (2.7) with (2.4), one sees that $\pi^q$ is related to $\pi$ through the partial Fourier transform on $\mathbb{R}^{p+q}$ with respect to the variables $y_1, \ldots, y_q$. In fact, if we denote the dual variable of $y_j$ by $\eta_j$, $j = 1, 2, \ldots, q$, then $\pi$ and $\pi^q$ interchange with each other under the correspondence
\[
-x_j \partial_{y_j} \longleftrightarrow \eta_j, \quad y_j \longleftrightarrow i \partial_{\eta_j}, \quad (j = 1, \ldots, q);
\]
the former operators $-i \partial_{y_j}$ and $\eta_j$ are the realizations of $\hat{x}_j$, while the latter operators $y_j$ and $i \partial_{\eta_j}$ are the realizations of $\hat{y}_j$.

(ii) Recall that one can obtain $\pi^q$ by differentiating the left regular representation of $G = \text{O}(p, q)$ on $C^\infty(V)$, the space of complex-valued smooth functions on $V$, where $G$ acts on $V$ by matrix multiplication under the identification of $V$ with $\mathbb{R}^{p+q}$ given by $t'(x', i y'') \leftrightarrow t'(x', y'')$ (see e.g. [RS80, HT93]). As one can see from (2.6) and (2.8), $\pi^q(X)$ coincides with $\pi(X)$ for all $X \in \mathfrak{f}$. Thus, the action $\pi$ restricted to $\mathfrak{f}_0$ lifts to the action of $K$ on $C^\infty(V)$.

3. Dual Pair $(\text{O}(p, q), \mathfrak{sl}_2(\mathbb{R}))$

Henceforth, let us denote $x' = \hat{x}_1, \ldots, \hat{x}_p$ and $y'' = \hat{y}_1, \ldots, \hat{y}_q$ by
\[
x = \hat{x}_1, \ldots, \hat{x}_p \quad \text{and} \quad y = \hat{y}_1, \ldots, \hat{y}_q
\]
respectively for the sake of simplicity if there exists no risk of confusion. Namely, we regard $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$ as the canonical coordinate functions on $\mathbb{R}^p$ and on $\mathbb{R}^q$ respectively.
If we denote the Casimir elements of \( \mathfrak{g} \) by \( \Omega_q \), then the corresponding Casimir operator is given by

\[
\pi(\Omega_q) = (E_x - E_y)^2 + (p - q)(E_x - E_y) - 2(E_x + E_y) - \left( r_x^2 r_y^2 + r_x^2 \Delta_x + r_y^2 \Delta_y + \Delta_x \Delta_y \right) - pq,
\]

where

\[
E_x = \sum_{i \in [p]} x_i \partial_{x_i}, \quad r_x^2 = \sum_{i \in [p]} x_i^2, \quad \Delta_x = \sum_{i \in [p]} \partial_{x_i}^2,
\]

\[
E_y = \sum_{j \in [q]} y_j \partial_{y_j}, \quad r_y^2 = \sum_{j \in [q]} y_j^2, \quad \Delta_y = \sum_{j \in [q]} \partial_{y_j}^2.
\]

Now, taking account of the fact that our realization of the representation operators of \( \mathfrak{g} \) given in (2.6) is a partial Fourier transform of the ones given in [RS80, HT93] as we mentioned in Remark 2.4 (i) above, we define elements \( H, X^+, X^- \) of \( \mathcal{P}D(V) \) by

\[
H = -E_x - \frac{p}{2} + E_y + \frac{q}{2}, \quad X^+ = -\frac{1}{2}(\Delta_x + r_x^2), \quad X^- = \frac{1}{2}(r_x^2 + \Delta_y).
\]

Then, it is immediate to see that the commutation relations among them are given by

\[
[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = H.
\]

**Proposition 3.1.** Let \( \mathfrak{g}' := \mathbb{C}\text{-span} \{H, X^+, X^-\} \). Then \( \mathfrak{g}' \) is a Lie subalgebra of \( \mathcal{P}D(V)^g \) isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \), where \( \mathcal{P}D(V)^g \) denotes the commutant of \( \mathfrak{g} \) in \( \mathcal{P}D(V) \).

**Proof.** Note that \( \pi(X_{i,j}^+) \), \( i, j \in [p] \), span the Lie subalgebra isomorphic to \( \mathfrak{o}_p \) commuting with \( E_x, \Delta_x \) and \( r_x^2 \), and that \( \pi(X_{i,j}^-) \), \( i, j \in [q] \), span the Lie subalgebra isomorphic to \( \mathfrak{o}_q \) commuting with \( E_y, \Delta_y \) and \( r_y^2 \). Hence, it remains to show that each \( \pi(X_{i,j}^-) \) commutes with \( H, X^+ \) and \( X^- \) given in (3.3).

We will only show here that \( [\pi(X_{i,j}^-), X^+] = 0 \). The other cases can be shown similarly. Now, one sees

\[
-2i \left[ \pi(X_{i,j}^-), X^+ \right] = [x_i y_j + \partial_{x_i} \partial_{y_j}, -\Delta_x - r_x^2]
\]

\[
= \sum_{k=1}^p \left[ \partial_{x_k}^2, x_i \right] y_j - \sum_{l=1}^q \partial_{y_l} \left[ \partial_{y_l}, y_i^2 \right]
\]

\[
= \sum_{k=1}^p 2 \delta_{k,i} \partial_{x_k} y_j - \sum_{l=1}^q 2 \partial_{y_l} \delta_{j,l} y_l
\]

\[
= 2 \partial_{x_i} y_j - 2 \partial_{x_i} y_j = 0.
\]

This completes the proof. \( \square \)

If one denotes the Casimir element of \( \mathfrak{g}' \) by \( \Omega_{\mathfrak{g}'} \), then the corresponding Casimir operator that is defined by

\[
\pi(\Omega_{\mathfrak{g}'}) = H^2 + 2(X^+ X^- + X^- X^+)
\]

\[
= H^2 - 2H + 4X^+ X^-
\]

\[
= H^2 - 2H + 4X^+ X^+
\]
is concretely written in terms of the operators given by (3.2) as follows:

\[
\pi(\Omega_{\nu}) = (E_x - E_y)^2 + (p - q)(E_x - E_y) - 2(E_x + E_y) \\
- \left( r_x^2 r_y^2 + r_x^2 \Delta_x + r_y^2 \Delta_y + \Delta_x \Delta_y \right) + \frac{1}{4}(p - q)^2 - (p + q).
\]

(3.4)

It follows from (3.1) and (3.4) that

\[
\pi(\Omega_{\nu}) = \pi(\Omega_{\nu'}) - \frac{1}{4}(p + q)^2 + (p + q)
\]

(see [How79, RS80]).

In what follows, we denote by \( \mathcal{H}^k(\mathbb{R}^n) \) the space of homogeneous harmonic polynomials on \( \mathbb{R}^n \) of degree \( k \). It is well known that \( \mathcal{H}^k(\mathbb{R}^n) \) is an irreducible \( O(n) \)-module and its dimension is given by

\[
\dim \mathcal{H}^k(\mathbb{R}^n) = \binom{k + n - 1}{n - 1} - \binom{k + n - 3}{n - 1} = \frac{(k + n - 3)!}{k!(n - 2)!} (2k + n - 2)
\]

if \( n \geq 2 \) and \( k \in \mathbb{N} \), where \( \binom{n}{k} \) denotes the binomial coefficient. Note that it can be further rewritten as

\[
\dim \mathcal{H}^k(\mathbb{R}^n) = \frac{2(k + n/2 - 1)}{(n - 2)!} (k + 1)(k + 2) \cdots (k + n - 3). \tag{3.5}
\]

Now, we will find a highest weight vector with respect to the \( g' \)-action (3.3), i.e. a function \( f \) on \( V \) which satisfies

\[
Hf = \lambda f \quad \text{and} \quad X^+ f = 0 \tag{3.6}
\]

for some \( \lambda \in \mathbb{C} \). Taking account of the fact that the algebra of polynomial functions on \( V \), say \( \mathcal{P}(V) \), can be written as

\[
\mathcal{P}(V) = \mathbb{C}[x_1, \ldots, x_p] \otimes \mathbb{C}[y_1, \ldots, y_q]
\]

\[
= \bigoplus_{k=0}^{\infty} \left( \mathbb{C}[r_x^2] \otimes \mathcal{H}^k(\mathbb{R}^p) \right) \otimes \bigoplus_{l=0}^{\infty} \left( \mathbb{C}[r_y^2] \otimes \mathcal{H}^l(\mathbb{R}^q) \right)
\]

\[
= \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[r_x^2, r_y^2],
\]

we will seek for a function that satisfies (3.6) of the form

\[
f(x, y) = h_1(x) h_2(y) \phi(r_x^2, r_y^2), \tag{3.7}
\]

where \( h_1 \in \mathcal{H}^k(\mathbb{R}^p), h_2 \in \mathcal{H}^l(\mathbb{R}^q) \), and \( \phi(s, t) \in \mathbb{C}[[s, t]] \) (Caution: we do not assume that \( \phi \) is a polynomial). Namely, our function \( f \) on \( V \) lives in the space \( \tilde{E} \) defined by

\[
\tilde{E} := \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[r_x^2, r_y^2] \] (algebraic direct sum). \tag{3.8}
Recall that the action $\pi$ of $\mathfrak{t}_0$ lifts to the action of $K$ on $\tilde{\mathcal{E}}$ as we mentioned in Remark 2.4 (ii), which we denote by the same letter $\pi$.

**Lemma 3.2.** Let $\Delta = \sum_{i=1}^n \partial_i^2$ and $r^2 = \sum_{i=1}^n x_i^2$. For $h = h(x_1, \ldots, x_n)$ a homogeneous harmonic polynomial on $\mathbb{R}^n$ of degree $d$ and for $\varphi(u)$ a smooth function in a single variable $u$, we have

$$\Delta(h\varphi(r^2)) = (4d + 2n)h\varphi''(r^2) + 4r^2 h\varphi''(r^2).$$

**Proof.** Since $\partial_i \varphi(r^2) = 2x_i \varphi'(r^2)$ and $\partial_i^2 \varphi(r^2) = 2\varphi'(r^2) + 4x_i^2 \varphi''(r^2)$, one obtains

$$\Delta \varphi(r^2) = 2n \varphi'(r^2) + 4r^2 \varphi''(r^2).$$

Thus,

$$\Delta(h\varphi(r^2)) = \sum_{i=1}^n \left( \partial_i^2 h \cdot \varphi(r^2) + 2\partial_i h \cdot \partial_i \varphi(r^2) + h \cdot \partial_i^2 \varphi(r^2) \right)$$

$$= 4dh\varphi'(r^2) + h\varphi''(r^2)$$

$$= 4dh\varphi'(r^2) + h \left( 2n \varphi'(r^2) + 4r^2 \varphi''(r^2) \right)$$

$$= (4d + 2n)h\varphi'(r^2) + 4r^2 h\varphi''(r^2).$$

\qed

For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ (resp. $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$) given, we define its shifted degree by $\kappa_+(h_1) := k + p/2$ (resp. $\kappa_-(h_2) := l + q/2$), which we denote just by $\kappa_+$ (resp. $\kappa_-$) if there is no risk of confusion.

It follows from Lemma 3.2 that if $f$ is of the form in (3.7) then

$$X^+ f = -\frac{1}{2}(\Delta_x(h_1h_2\phi) + r^2 h_1h_2\phi)$$

$$= -2h_1h_2 \left( r_2 \left( \partial^2_x \phi(r_2^2, r_2^2) + \kappa_+ \partial_x \phi(r_2^2, r_2^2) + r_2^2 \phi(r_2^2, r_2^2) \right) \right),$$

which shows that $f = h_1(x)h_2(y)\phi(r_2^2, r_2^2)$ satisfies $X^+ f = 0$ if and only if $\phi$ is a solution to a differential equation

$$s \partial^2_x \phi + \kappa_+ \partial_x \phi + t \phi = 0 \quad (3.9)$$

with $\kappa_+ = \kappa_+(h_1) = k + p/2$. Solving the differential equation (3.9) by power series, one obtains that

$$\phi(s, t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\kappa_+)_n} \left( \frac{st}{4} \right)^n, \quad (3.10)$$

where $a_0$ is an arbitrary formal power series in $t$. Note that if one defines a power series $\Psi_\alpha$ by

$$\Psi_\alpha(u) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\alpha)_n} u^n = 1 - \frac{u}{\alpha} + \frac{u^2}{2! \alpha(\alpha + 1)} - \frac{u^3}{3! \alpha(\alpha + 1)(\alpha + 2)} + \cdots \quad (3.11)$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$, then it converges on the whole $\mathbb{C}$ and is a unique solution to a differential equation

$$u \Psi''_\alpha(u) + \alpha \Psi'_\alpha(u) + \Psi_\alpha(u) = 0 \quad (3.12)$$

that satisfies the initial condition $\Psi_\alpha(0) = 1$. 


In the sequel, we set
\[ \psi_a^{(n)} := \Psi_a^{(n)}(r^2_s/4) \quad (n \in \mathbb{N}) \] (3.13)
for brevity, where \( \Psi_a^{(n)}(u) \) denotes the \( n \)-th derivative of \( \Psi_a(u) \) in \( u \).

If, in addition, \( f \) satisfies that \( Hf = \lambda f \) for some \( \lambda \in \mathbb{C} \), then the factor \( a_0 \) in (3.10) is equal to \( t^\mu \) up to a constant multiple with \( \mu_\pm = (1/2)(\lambda + \kappa_+ - \kappa_-) \in \mathbb{N} \), \( \kappa_+ = \kappa_+(h_1) \) and \( \kappa_- = \kappa_-(h_2) \).

Thus, for \( h_1 \in \mathcal{H}^k(\mathbb{R}^p) \) and \( h_2 \in \mathcal{H}^l(\mathbb{R}^q) \) given, a highest weight vector \( f = f(x, y) \) with respect to the \( g' \)-action, i.e. a function that satisfies (3.6) is given by
\[ f = h_1(x)h_2(y)r^2_y\psi_{\kappa_+} \]
with
\[ \lambda = -\kappa_+ + \kappa_- + 2\mu_- \quad (\mu_- \in \mathbb{N}) \quad (3.14) \]
Similarly, for \( h_1 \in \mathcal{H}^k(\mathbb{R}^p) \) and \( h_2 \in \mathcal{H}^l(\mathbb{R}^q) \) given, one can show that a lowest weight vector \( f = f(x, y) \) of the form (3.7) with respect to the \( g' \)-action, i.e. a function that satisfies
\[ Hf = \lambda f \quad \text{and} \quad X^- f = 0 \]
for some \( \lambda \in \mathbb{C} \), is given by
\[ f(x, y) = h_1(x)h_2(y)r^2_x\psi_{\kappa_-} \]
with
\[ \lambda = -\kappa_+ + \kappa_- - 2\mu_+ \quad (\mu_+ \in \mathbb{N}) \quad (3.15) \]
Let us summarize the above argument in the following.

**Proposition 3.3.** Given \( h_1 \in \mathcal{H}^k(\mathbb{R}^p) \) and \( h_2 \in \mathcal{H}^l(\mathbb{R}^q) \), let \( f = f(x, y) \) be a function of the form given in (3.7).

1. If \( f \) is a highest weight vector satisfying (3.6) with respect to the \( g' \)-action, then it is given by
   \[ f(x, y) = h_1(x)h_2(y)r^2_y\psi_{\kappa_+} \]
   with \( \lambda = -\kappa_+ + \kappa_- + 2\mu_- \). Moreover, such a function is unique up to a constant multiple.
2. If \( f \) is a lowest weight vector satisfying (3.15) with respect to the \( g' \)-action, then it is given by
   \[ f(x, y) = h_1(x)h_2(y)r^2_x\psi_{\kappa_-} \]
   with \( \lambda = -\kappa_+ + \kappa_- - 2\mu_+ \). Moreover, such a function is unique up to a constant multiple.

Here \( \kappa_+ = \kappa_+(h_1) = k + p/2 \), \( \kappa_- = \kappa_-(h_2) = l + q/2 \), \( \mu_+, \mu_- \in \mathbb{N} \), and \( \psi_{\kappa_\pm} \) is an element of \( \mathbb{C}[r^2_x, r^2_y] \) given by (3.13) with \( \alpha = \kappa_\pm \) and \( n = 0 \).

Taking account of the discussion so far, let us introduce the subspace \( \mathcal{E} \) of \( \bar{\mathcal{E}} \) by
\[ \mathcal{E} := \mathbb{C}\text{-span} \left\{ h_1(x)h_2(y)\rho^a_x\rho^b_y\psi_a \in \bar{\mathcal{E}} \mid h_1 \in \mathcal{H}^k(\mathbb{R}^p), h_2 \in \mathcal{H}(\mathbb{R}^q), a, b \in \mathbb{N}, a, b \in \mathbb{C} \setminus (-\mathbb{N}) \right\} \]
Then one will find that \( \mathcal{E} \) is stable under the action of \( (\mathfrak{g}, K) \) as well as that of \( g' \) (see Propositions 3.5 and 3.6 below).
Remark 3.4. (i) The function $Ψ_α$ given in (3.11) can be written in terms of the Bessel function $J_ν$ of the first kind of order $ν$

$$J_ν(t) = \sum_{n=0}^{∞} \frac{(-1)^n}{n!Γ(n + ν + 1)} \left( \frac{t}{2} \right)^{ν+2n}$$

that solves the Bessel’s differential equation

$$\frac{d^2w}{dt^2} + \frac{1}{t} \frac{dw}{dt} + \left( 1 - \frac{ν^2}{t^2} \right)w = 0 \quad (3.17)$$

(see e.g. [WW27]). Namely, one has

$$Ψ_α(u) = Γ(α) u^{-(α-1)/2} J_{α-1}(2u^{1/2}). \quad (3.18)$$

Therefore,

$$ψ_α = Γ(α) \frac{r_x r_y}{2} (α-1) J_{α-1}(r_x r_y).$$

Note that (3.12) corresponds to (3.17) under (3.18).

(ii) Recall that our representation $π$ is related to $π^♯$ via the partial Fourier transform with respect to $y_1, \ldots, y_q$, as we mentioned in Remark 2.4 (i). Namely, one can obtain $π^♯$ by replacing $-i \partial_{y_j}$ and $y_j$ in $π$ by $η_j$ and $i \partial_{η_j}$, $j = 1, \ldots, q$, respectively. Under this correspondence, one finds that $H = -E_x - p/2 + E_y + q/2$ and $X^+ = -\frac{1}{2}(Δ_x + r_y^2)$ transforms, up to constant multiples, into the shifted degree operator $E_{p,q}$ and the d’Alembertian $∂_{p,q}$ on $\mathbb{R}^{p+q}$ that are given by

$$E_{p,q} = \sum_{i=1}^{p} x_i \partial_{x_i} + \sum_{j=1}^{q} η_j \partial_{η_j} + \frac{p - q}{2},$$

$$∂_{p,q} = \sum_{i=1}^{p} \partial_{x_i}^2 - \sum_{j=1}^{q} \partial_{η_j}^2,$$

respectively. Therefore, the highest weight vector $f$ satisfying $H f = λ f$ for some $λ \in \mathbb{C}$ and $X^+ f = 0$ corresponds to a homogeneous solution $f$ to the equation $∂_{p,q} f = 0$.

Note that $Ψ^{(n)}_α$ is equal to $Ψ_{α+n}$ up to a constant multiple. In fact, differentiating both sides of (3.12) $n$ times, one obtains

$$uΨ^{(n+2)}_α(u) + (α + n)Ψ^{(n+1)}_α(u) + Ψ^{(n)}_α(u) = 0. \quad (3.19)$$

Since $Ψ_{α+n}$ is a unique solution to (3.12) with $α$ replaced by $α + n$ that satisfies $Ψ_{α+n}(0) = 1$, it follows that $Ψ^{(n)}_α = (-1)^n/(α_n)Ψ_{α+n}$. Thus, one obtains

$$ψ^{(n)}_α = \frac{(-1)^n}{(α_n)} ψ_{α+n} \quad (n \in \mathbb{N}). \quad (3.20)$$

In what follows, we set $ρ_X := r_x^2/2$ and $ρ_Y := r_y^2/2$ for economy of space. Then, it follows from (3.19) and (3.20) that

$$ρ_X ρ_Y ψ_{α+2} = α(α + 1)(ψ_{α+1} - ψ_α) \quad (3.21)$$
for \( \alpha \in \mathbb{C} \setminus (-\mathbb{N}) \). Furthermore, setting \( \rho := r^2/2 \), one can rewrite (3.2) as
\[
\frac{1}{2} \Delta(h\varphi(\rho)) = \left(d + \frac{n}{2}\right) h\varphi'(\rho) + h\rho\varphi''(\rho),
\]
where \( \rho, \Delta, r^2 \) and \( \varphi \) are as in Lemma 3.2.

**Proposition 3.5.** For \( f = h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha \in \mathcal{E} \), one has
\[
H(h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha) = (-\kappa_+ + \kappa_- - 2a + 2b)h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha, \tag{3.23}
\]
\[
X^+(h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha) = h_1 h_2 \left(-a(\kappa_+ + a - 1)\rho_\alpha^a \rho_\beta^b \psi_\alpha + \frac{\kappa_+ + 2a - \alpha}{\alpha} \rho_\alpha^a \rho_\beta^{b+1} \psi_{\alpha+1}\right), \tag{3.24}
\]
\[
X^-(h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha) = h_1 h_2 \left(b(\kappa_- - b - 1)\rho_\alpha^a \rho_\beta^{b+1} \psi_\alpha - \frac{\kappa_+ + 2b - \alpha}{\alpha} \rho_\alpha^a \rho_\beta^{a+1} \psi_{\alpha+1}\right). \tag{3.25}
\]
In particular, the \( g' \)-action preserves the \( K \)-type of each element of \( \mathcal{E} \).

**Proof.** It is immediate to show (3.23), and we will only show (3.24) here; the other case (3.25) can be shown similarly.

Set \( \varphi(u) := u^a \Psi_\alpha(\rho_\beta u) \), one sees
\[
\varphi'(u) = au^{-1} \Psi_\alpha(\rho_\beta u) + u^a \rho_\beta \Psi_\alpha'(\rho_\beta u),
\]
\[
\varphi''(u) = a(a - 1)u^{-2} \Psi_\alpha(\rho_\beta u) + 2au^{-1} \rho_\beta \Psi_\alpha'(\rho_\beta u) + u^a \rho_\beta^2 \Psi_\alpha'(\rho_\beta u). \]

Hence it follows from (3.22)
\[
\frac{1}{2} \Delta_x(h_1 \rho_\alpha^a \psi_\alpha) = a(\kappa_+ + a - 1)h_1 \rho_\alpha^{a-1} \psi_\alpha + (\kappa_+ + 2a)h_1 \rho_\alpha^{a+1} \rho_\beta^2 \psi_\alpha' = h_1 \left(a(\kappa_+ + a - 1)\rho_\alpha^{a-1} \psi_\alpha + (\kappa_+ + 2a - \alpha)\rho_\alpha^a \rho_\beta \psi_\alpha' - \rho_\alpha^a \rho_\beta^2 \psi_\alpha'\right)
\]
since \( \rho_\alpha \rho_\beta \psi_\alpha' = -a \psi_\alpha' - \psi_\alpha \). Therefore, one obtains that
\[
X^+(h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha) = -\frac{1}{2}(\Delta_x + 2\rho_\alpha)(h_1 h_2 \rho_\alpha^a \rho_\beta^b \psi_\alpha) = -a(\kappa_+ + a - 1)h_1 h_2 \rho_\alpha^{a-1} \rho_\beta^b \psi_\alpha - (\kappa_+ + 2a - \alpha)h_1 h_2 \rho_\alpha^a \rho_\beta^{b+1} \psi_\alpha',
\]
which, by (3.20), equals the right-hand side of (3.24). This completes the proof. \( \Box \)

We conclude this section by calculating the action of \( \mathfrak{p} \) on \( \mathcal{E} \), i.e. \( \pi(X_{ij}^{-})f \) for \( X_{ij}^{-} \in \mathfrak{p} \) and \( f \in \mathcal{E} \).

For a homogeneous polynomial \( P = P(x_1, \ldots, x_n) \) on \( \mathbb{R}^n \) of degree \( d \), set
\[
P^\dagger := P - \frac{r^2}{4(d + n/2 - 2)} \Delta P,
\]
where \( \Delta = \sum_{i=1}^n \partial_{x_i}^2 \) and \( r^2 = \sum_{i=1}^n x_i^2 \). Note that if \( \Delta^2 P = 0 \) then \( P^\dagger \) is harmonic by Lemma 3.2 and that if \( h = h(x_1, \ldots, x_n) \) is harmonic then \( \Delta(x_i h) = 2 \partial_{x_i} h \) and \( \Delta^2(x_i h) = 0 \).
Proposition 3.6. For \( f = h_1 h_2 \rho_x^a \rho_y^b \psi_a \in \mathcal{E} \), one has
\[
-i \pi (X_{ij}) (h_1 h_2 \rho_x^a \rho_y^b \psi_a) = (\partial_{x_i} h_1)(\partial_{y_j} h_2) \rho_x^a \rho_y^b \left( \frac{\kappa_+ a + b - a}{\kappa_- b + a} \psi_{a+1} + \frac{b(\kappa_+ a + b - a)}{\kappa_- b + a} \rho_y^b \psi_a \right) + (x_i h_1)^{\dagger} (y_j h_2) \left( \frac{\kappa_+ a + b - a}{\kappa_- b + a} \rho_x^a \rho_y^b \psi_a + \frac{b(\kappa_+ a + b - a)}{\kappa_- b + a} \rho_y^b \psi_a \right).
\]

(3.26)

Proof. Since \( \partial_{x_i} \psi_a = \rho_y x_i \psi_a' \) and \( \partial_{y_j} \psi_a = \rho_x y_j \psi_a' \), one obtains
\[
-i \pi (X_{ij}) f = (\partial_{x_i} \partial_{y_j} (x_i h_1)(y_j h_2)) \rho_x^a \rho_y^b \psi_a = (\partial_{x_i} h_1)(\partial_{y_j} h_2) \rho_x^a \rho_y^b \psi_a + \rho_x^a \rho_y^b \psi_a' + \rho_x^a \rho_y^b \psi_a'.
\]

(3.27)

Now, by definition, one has
\[
x_i h_1 = (x_i h_1)^{\dagger} + \frac{\rho_x}{\kappa_- - 1} \partial_{x_i} h_1 \text{ and } y_j h_2 = (y_j h_2)^{\dagger} + \frac{\rho_y}{\kappa_- - 1} \partial_{y_j} h_2.
\]

(3.28)

Substituting (3.28) into (3.27), and using the relation (3.20) and (3.21), one sees that the coefficient of \( (\partial_{x_i} h_1)(\partial_{y_j} h_2) \) in (3.27) equals the one of \( (\partial_{x_i} h_1)(\partial_{y_j} h_2) \) in the right-hand side of (3.26). One can verify that each coefficients of \( (\partial_{x_i} h_1)(y_j h_2)^{\dagger} \), \( (x_i h_1)^{\dagger} (\partial_{y_j} h_2) \) and \( (x_i h_1)(y_j h_2)^{\dagger} \) in (3.27) equals the one of the corresponding terms in (3.26) similarly. This completes the proof. \( \square \)

4. \((g, K)\)-module associated with finite-dimensional \(\mathfrak{sl}_2\)-module

If \( f \in \mathcal{E} \) satisfies \( H f = \lambda f \), \( X^+ f = 0 \) and \( (X^-)^{m+1} f = 0 \) (resp. \( H f = \lambda f \), \( X^- f = 0 \) and \( (X^+)^{m+1} f = 0 \)) for some \( m \in \mathbb{N} \), then it follows from the representation theory of \( g' = \mathfrak{sl}_2 \) that \( \lambda = m \) (resp. \( \lambda = -m \)). Thus, we introduce \((g, K)\)-modules associated with the finite-dimensional \(\mathfrak{sl}_2\)-module \( F_m \) of dimension \( m + 1 \) as follows, which are the main objects of this paper.

Definition 4.1. Given \( m \in \mathbb{N} \), we define \((g, K)\)-modules \( M^\pm(m) \) by
\[
M^+(m) := \left\{ f \in \mathcal{E} \mid H f = m f, \quad X^+ f = 0, \quad (X^-)^j f \neq 0 \quad (1 \leq j \leq m), \quad (X^-)^{m+1} f = 0 \right\},
\]
\[
M^-(m) := \left\{ f \in \mathcal{E} \mid H f = -m f, \quad X^- f = 0, \quad (X^+)^j f \neq 0 \quad (1 \leq j \leq m), \quad (X^+)^{m+1} f = 0 \right\}.
\]

The modules \( M^\pm(m) \) were originally introduced in [RS80] without the condition of finite dimensionality. Note that \( M^+ (0) \) is identical to \( M^- (0) \) by definition and that both \( M^\pm(m) \) should correspond to the \( \mathfrak{sl}_2 \)-module \( F_m \) under the Howe duality (cf. [How89]).
If $M^+(m) \neq \{0\}$ (resp. $M^-(m) \neq \{0\}$), then one sees that $p \equiv q \mod 2$; for, if one takes a non-zero $f = h_1 h_2 \rho_2^{\mu_+} \psi_{k_+} \in M^+(m)$ (resp. $f = h_1 h_2 \rho_2^{\mu_-} \psi_{k_-} \in M^-(m)$) with $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $\mu_\pm \in \mathbb{N}$, then

$$\pm m = -\kappa_+ + \kappa_- \pm 2\mu_\pm = -k + l - \frac{p - q}{2} \pm 2\mu_\pm \in \mathbb{Z}$$

by (3.14) (resp. (3.16)). Hence one obtains $(p - q)/2 \in \mathbb{Z}$. Therefore, we assume that $p \equiv q \mod 2$ in the rest of this paper.

**Lemma 4.2.** For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $m \in \mathbb{N}$, let

$$v^+ = h_1 h_2 \rho_2^{\mu_+} \psi_{k_+} \in M^+(m) \quad \text{and} \quad v^- = h_1 h_2 \rho_2^{\mu_-} \psi_{k_-} \in M^-(m),$$

where $\mu_+, \mu_- \in \mathbb{N}$ such that $\mu_+ + \mu_- = m$. Then the $g'$-module generated by $v^+$ coincides with the one generated by $v^-:$

$$\langle v^+ \rangle_{g'} = \langle v^- \rangle_{g'}.$$

**Proof.** Both $v^+$ and $(X^+)^m v^-$ (resp. $v^-$ and $(X^-)^m v^+$) are elements of $E \subset \tilde{E}$ that are highest (resp. lowest) weight vectors of weight $m$ (resp. $-m$) under $g'$-action. Namely, they are solutions in $\tilde{E}$ to the differential equation

$$H f = \pm m f \quad \text{and} \quad X^\pm f = 0.$$

As we mentioned Proposition 3.3, they are respectively equal to each other up to a constant multiple. This completes the proof. \qed

**Proposition 4.3.** For $m \in \mathbb{N}$, $M^+(m)$ and $M^-(m)$ are isomorphic to each other.

**Proof.** For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, set

$$v^+ = h_1 h_2 \rho_2^{\mu_+} \psi_{k_+} \in M^+(m) \quad \text{and} \quad v^- = h_1 h_2 \rho_2^{\mu_-} \psi_{k_-} \in M^-(m).$$

If $\mu_+ + \mu_- = m$, then $(X^+)^m v^-$ is equal to $v^+$ up to a constant multiple as in Lemma 4.2, and thus, $(X^+)^m (X^-)^m v^+$ is equal to $v^+$ up to a constant multiple. Moreover, this constant is non-zero and is independent of the $K$-type of $v^+$. In fact, using the relations (3.3), one has

$$H(X^-)^i = (X^-)^i H + [H, (X^-)^i]$$

$$= (X^-)^i H + [H, X^-](X^-)^{i-1} + X^- [H, X^-] (X^-)^{i-2} + \cdots + (X^-)^{i-1} [H, X^-]$$

$$= (X^-)^i H - 2i(X^-)^i$$

for $i \in \mathbb{N}$, hence,

$$(X^+)^i (X^-)^j v^+$$

$$= (X^+)^{j-1} \left( (X^-)^j X^+ + [X^+, (X^-)^j] \right) v^+$$

$$= (X^+)^{j-1} \left( H(X^-)^{j-1} + X^- H(X^-)^{j-2} + \cdots + (X^-)^{j-2} HX^- + (X^-)^{j-1} H \right) v^+$$

$$= j(m - j + 1) (X^+)^{j-1} (X^-)^j v^+$$

$$= j(m - j + 1) \cdot (j - 1) (m - j + 2) (X^+)^{j-2} (X^-)^{j-2} v^+$$

$$= \cdots$$

$$= j!(m)^j v^+$$
Proof. Let Lemma 4.5. To show for the other coefficients is trivial and omitted.

for \( j \in \mathbb{N} \) since \( X^+ v^+ = 0 \). In particular, \((X^+)^m(X^-)^m v^+ = (m!)^2 v^+ \). Therefore,

\[
(X^-)^m : M^+(m) \longrightarrow M^-(m)
\]

provides an isomorphism of \((g,K)\)-module. This completes the proof. \( \Box \)

Now we prepare two lemmas to prove our main result. Note that Lemma 4.4 below is just a special case of Proposition 3.6. However, we state it separately to highlight the rôle of the relation \((3.21)\).

**Lemma 4.4.** Let \( h_1 \in \mathfrak{H}^l(\mathbb{R}^p) \) and \( h_2 \in \mathfrak{H}^l(\mathbb{R}^q) \), and set \( \kappa_+ = k + p/2, \kappa_- = l + q/2 \).

1. For a highest weight vector \( f = h_1 h_2 \rho_{\mu^+}^\mu \psi_{\kappa_+}, \pi(X_{ij}^-)f \) is given by

\[
-\imath \pi(X_{ij}^-)(h_1 h_2 \rho_{\mu^+}^\mu \psi_{\kappa_+}) = \frac{\kappa_- + \mu_+ - 1}{\kappa_- - 1} \frac{\mu_+ - (\partial_{\psi_i} h_1)(\partial_{\psi_j} h_2)^+ \rho_{\mu^+}^\mu \psi_{\kappa_-}}{\kappa_+ (\kappa_- - 1)} + \frac{\kappa_- - \kappa_+ - \mu_+}{\kappa_+ (\kappa_- - 1)} (x_i h_1)^+ \partial_{\psi_j} h_2 \rho_{\mu^+}^\mu \psi_{\kappa_-} + 1 \quad (4.1)
\]

2. For a lowest weight vector \( f = h_1 h_2 \rho_{\mu^+}^\mu \psi_{\kappa_-}, \pi(X_{ij}^-)f \) is given by

\[
-\imath \pi(X_{ij}^-)(h_1 h_2 \rho_{\mu^+}^\mu \psi_{\kappa_-}) = \frac{\kappa_- + \mu_+ - 1}{\kappa_- - 1} \frac{\mu_+ - (\partial_{\psi_i} h_1)(\partial_{\psi_j} h_2)^+ \rho_{\mu^+}^\mu \psi_{\kappa_-}}{\kappa_+ (\kappa_- - 1)} + \frac{\kappa_- - \kappa_+ - \mu_+}{\kappa_+ (\kappa_- - 1)} (x_i h_1)^+ \partial_{\psi_j} h_2 \rho_{\mu^+}^\mu \psi_{\kappa_-} + 1 \quad (4.2)
\]

**Proof.** We only show \((4.1)\) here. The other formula \((4.2)\) can be shown similarly.

Set \( a = 0, b = \mu_- \) and \( \alpha = \kappa_+ \) in \((3.26)\). Then, using the relation \((3.21)\) in this case, i.e.

\[
\rho_x \rho_y \psi_{\kappa+i} = \kappa_+ (\kappa_+ - 1) (\psi_{\kappa_+} - \psi_{\kappa_-}),
\]

one sees that the coefficient of \( (\partial_{\psi_i} h_1)(\partial_{\psi_j} h_2)^+ \) equals

\[
- \frac{\mu_-}{\kappa_+ (\kappa_- - 1)} \rho_{\mu^+}^\mu \psi_{\kappa_-} + 1 + \mu_- \rho_{\mu^+}^\mu \psi_{\kappa_+} = \mu_- \rho_{\mu^+}^\mu \psi_{\kappa_-} + 1.
\]

To show for the other coefficients is trivial and omitted. \( \Box \)

**Lemma 4.5.** Let \( h_1 \in \mathfrak{H}^l(\mathbb{R}^p) \) and \( h_2 \in \mathfrak{H}^l(\mathbb{R}^q) \), and set \( \kappa_+ = k + p/2, \kappa_- = l + q/2 \).

1. For a highest weight vector \( f = h_1 h_2 \rho_{\mu^+}^\mu \psi_{\kappa_+} \), of weight \( \lambda = -\kappa_+ + \kappa_- + 2 \mu_- \), one has

\[
(X^-)^\nu f = h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} (-\lambda + \nu - 1)_j^{-1} (\mu_-)_{\nu-i}^{-1} \rho_{\mu^+}^\mu \psi_{\kappa_+} \quad (4.3)
\]

for \( \nu = 0, 1, 2, \ldots \).
Theorem 4.6. Assume that $\mu, \nu \in \mathbb{N}$, one has

$$\sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + v - 1)_i^j (\mu + 1 - 1)^j_{v-j}}{(\kappa)_i} \rho_{\mu, v-j} \rho_{\mu, v-j} \psi_{\kappa + v-j}$$

for $\nu = 0, 1, 2, \ldots$

Proof. We only show (4.4) by induction on $\nu$ here. The other case (4.3) can be shown similarly.

It is trivial if $\nu = 0$, and it is nothing but Proposition [3.5] if $\nu = 1$. Assume that it is true for $\nu \geq 1$, and apply $X^+$ to the both sides of (4.4). Then, one sees that the right-hand side equals

$$(-1)^{\nu} h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + v - 1)_i^j (\mu + 1 - 1)^j_{v-j}}{(\kappa)_i} \times \left(-\frac{(\lambda + v + i)(\mu + v + i - 1)}{\lambda + i - \nu - j - 1} \frac{\lambda + v - j - 1}{\kappa + j - 1} \right).$$

The coefficient of $(-1)^{\nu} h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + v - 1)_i^j (\mu + 1 - 1)^j_{v-j}}{(\kappa)_i} \rho_{\mu, v-j} \psi_{\kappa + v-j}$ in (4.5), $j = 0, 1, \ldots, \nu + 1$, equals

$$\binom{\nu}{j} \frac{(\lambda + v - 1)_i^j (\mu + 1 - 1)^j_{v-j}}{(\kappa)_i} \times \left(-\frac{(\lambda + v - j - 1)}{\kappa + j - 1} \right).$$

The following is our main result.

Theorem 4.6. Assume that $p \geq 1$, $q \geq 1$ and $p + q \in 2\mathbb{N}$. Let $m \in \mathbb{N}$ be a non-negative integer satisfying $m + 3 \leq (p + q)/2$. Then one has the following.

1. The $K$-type formula of $M^x(m)$ is given by

$$M^x(m)_{|K} \cong \bigoplus_{k, l \geq 0} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q),$$

where $\Lambda_m = \{m, -m + 2, -m + 4, \ldots, m - 2, m\}$, the set of $H$-weights of $F_m$;

2. Suppose further that $p, q \geq 2$. Then $M^x(m)$ are irreducible $(\mathfrak{g}, K)$-modules.
Proof. It suffices to show the theorem for $M^+(m)$. Let $f = h_1 h_2 \rho_{\mu, \psi_{\kappa_+}} \neq 0$ be an element of $M^+(m)$, where $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$. Then by Lemma [4.5], one obtains

$$(X^-)^{m+1} f = h_1 h_2 \sum_{i=0}^{m+1} \binom{m+1}{i} (0)^{-} (\mu_{-})_{m+1-1}^i (\kappa_+ - \mu_+ - 1)_{m+1}^{-} \rho_x^i \rho_y^{\mu_+ - m - i} \psi_{\kappa_+ + i}$$

Thus, $(X^-)^{m+1} f = 0$ implies that $(\mu_{-})_{m+1}^0 = 0$ or $(\kappa_+ - \mu_+ - 1)_{m+1}^0 = 0$. Namely,

$$\mu_{-} = 0, 1, \ldots, m, \quad \text{or}$$

$$\mu_{-} = -\kappa_+ + 1, -\kappa_+ + 2, \ldots, -\kappa_+ + m + 1.$$  (4.7)

The assumption that $m + 3 \leq (p+q)/2$, however, implies that (4.8) is impossible; if it held, then it would follow from (3.14) that

$$m = -\kappa_+ + \kappa_- + 2\mu_+$$  (4.9)

for $i = 1, 2, \ldots, m + 1$, hence that

$$\kappa_+ + \kappa_- = -m + 2i \leq m + 2,$$

which contradicts $\kappa_+ + \kappa_- \geq (p+q)/2 \geq m + 3$. Therefore, it follows from (4.7) that

$$k - l + \frac{p-q}{2} = \kappa_+ - \kappa_- = -m + 2\mu_+ \in \Lambda_m,$$

which proves (1).

Let us consider a closed subset $D_m \subset \mathbb{R}^2$ (with respect to the standard topology of $\mathbb{R}^2$) given by

$$D_m = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 \geq p/2, t_2 \geq q/2, |t_1 - t_2| \leq m\},$$  (4.10)

and the set of integral points of $D_m$ given by

$$\Sigma_m = \{(t_1, t_2) \in D_m \mid t_1 - p/2 \in \mathbb{N}, t_2 - q/2 \in \mathbb{N}, |t_1 - t_2| \leq \Lambda_m\}. $$  (4.11)

Note that the sum in the right-hand side of (4.6) can be written as the one with $(\kappa_+, \kappa_-)$ running over the set $\Sigma_m$. Now, applying (4.6) to $f = h_1 h_2 \rho_{\mu, \psi_{\kappa_+}} \in M^+(m)$, we denote the coefficient of

$$(\partial_{x_1} h_1)(\partial_{y_j} h_2), \quad (x_i h_1)(\partial_{y_j} h_2)^\dagger, \quad (\partial_{x_1} h_1)(y_j h_2)^\dagger \quad \text{and} \quad (x_i h_1)^\dagger(y_j h_2)^\dagger$$

in the right-hand side of (4.11) by $C_{--}$, $C_{+-}$, $C_{-+}$ and $C_{++}$ respectively, where $\mu_- = 0, 1, \ldots, m$. Namely,

$$C_{--} = \frac{\kappa_+ + \mu_- - 1}{\kappa_- - 1} \rho_y^{\mu_-} \psi_{\kappa_- - 1}, \quad C_{+-} = \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_- - 1},$$

$$C_{-+} = \frac{\kappa_+ + \mu_- - \kappa_-}{\kappa_+ (\kappa_- - 1)} \rho_y^{\mu_- + 1} \psi_{\kappa_- + 1}, \quad C_{++} = \frac{\kappa_+ + \mu_- - 1}{\kappa_+} \rho_y^{\mu_-} \psi_{\kappa_- + 1}.$$  (4.11)

(i) First, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ is an interior point of $D_m$. Then, one obtains $\mu_- = 1, 2, \ldots, m - 1$ by (4.7) and (4.9). In particular, $C_{--} \neq 0$. Now,
$C_{-} = 0$ would imply $\mu_- = -\kappa_- + 1$, which contradicts $m + 3 \leq (p + q)/2$ as we saw above. It also follows from (4.9) that $\kappa_- - \kappa_+ + \mu_- = m - \mu_-$, and $C_{++} \neq 0$. Finally, $C_{++} = 0$ would imply that $\kappa_+ + \kappa_- = m + 2$, which is absurd. Thus, all the coefficients in (4.1) never vanish.

(ii) Next, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ in the boundary of $D_m$. Then there are three sub-cases:

(ii-a) $\mu_- = 0$,

(ii-b) $\mu_- = m$,

(ii-c) $0 < \mu_- < m$ and $k = 0$ or $l = 0$.

In Case (ii-a), $C_{-} = 0$, and by the same reason as Case (i), $C_{--}$, $C_{+-}$ and $C_{++}$ are non-zero. In Case (ii-b), $C_{++} = 0$ since $\kappa_+ - \kappa_- = m$, and all the other coefficients are non-zero. In Case (ii-c), all the coefficients are non-zero, but $\partial_{x_1} h_1 = 0$ or $\partial_{y_2} h_2 = 0$.

Therefore, one can move from any $K$-type in $M^+(m)$ to any other $K$-type in $M^+(m)$ by applying $\pi(X_{i,j})$. This completes the proof of (2), and of the theorem. □

**Example 4.7.** Figure 1 below illustrates $D_m$ in (4.10) and $\Sigma_m$ in (4.11) in the case where $p = 14$, $q = 12$ and $m = 4$. The colored area and the dots sitting in the area indicates $D_m$ and $\Sigma_m$ respectively. Each $K$-type of $M^+(m)$ corresponds to a dot by the correspondence

\[
\mathcal{H}^l(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \longleftrightarrow (k, l) \quad \text{(or } (\kappa_+, \kappa_-)).
\]

Let us apply $\pi(X_{i,j})$ to an element $f$ of $M^+(m)$. Then, if the $K$-type of $f$ corresponds to a white dot $\circ$ in Fig. 1 one can move to any adjacent dots in the north-east, north-west, south-east, and south-west direction; if it corresponds to a black dot $\bullet$ in Fig. 1 one can move only to adjacent dots in the interior or in the boundary of $D_m$.

![Fig. 1. Applying $\pi(X_{i,j})$, one can move from $\circ$ to dots in NE, NW, SE and SW directions, while from $\bullet$, only to dots in the interior or in the boundary.](image-url)
Now, let us briefly recall the definitions of the Gelfand-Kirillov dimension and the Bernstein degree of a finitely generated $U(g)$-module $M$, where $U(g)$ denotes the universal enveloping algebra of $g$. Namely, we choose a finite-dimensional subspace $M_0$ so that $M = U(g)M_0$, and for each non-negative integer $n$, we set $M_n := U_n(g)M_0$ with $U_n(g)$ denoting the subspace of $U(g)$ spanned by products of at most $n$ elements of $g$. Then there exists a polynomial $\psi_M(t) \in \mathbb{Q}[t]$ of degree $d - 1$ such that

$$\psi_M(n) = \dim(M_n/M_{n-1})$$

for all sufficiently large $n$. Moreover, the leading term of $\psi_M$ is of the form

$$\frac{m}{(d - 1)!}t^{d-1}$$

for a positive integer $m$. We call $d$ the Gelfand-Kirillov dimension of $M$, and $m$ its Bernstein degree, which we denote by $\dim M$ and $\deg M$ respectively (see [Vog78] for more details).

**Corollary 4.8.** If $p, q \geq 2, p + q \in 2\mathbb{N}$ and $m + 3 \leq (p + q)/2$, then the Gelfand-Kirillov dimension and the Bernstein degree of $M^x(m)$ are given by

$$\dim M^x(m) = p + q - 3, \quad (4.12)$$

$$\deg M^x(m) = \frac{4(m + 1)(p + q - 4)!}{(p - 2)!(q - 2)!} \quad (4.13)$$

respectively.

**Proof.** Without loss of generality, one can assume that $p \geq q$. We will consider $M^+(m)$ here. Then, let $\ell(j)$ be a line in $\mathbb{R}^2$ given by

$$\ell(j) = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 + t_2 = j\}$$

with $j \in \mathbb{N}$, and set

$$c := \min\{j \in \mathbb{N} \mid \ell(j) \cap \Sigma_m = \Lambda_m\}. \quad (4.14)$$

As a generating $(K$-invariant) subspace of $M^+(m)$, we take

$$M_0 := \bigoplus_{(\kappa_+, \kappa_-) \in \Sigma_m, \kappa_+ + \kappa_- \leq c} \mathcal{H}^{\kappa_+}(\mathbb{R}^p) \otimes \mathcal{H}^{\kappa_-}(\mathbb{R}^q)\rho_{\kappa_+}^{\mu_-}$$

where, in each summand, $\mu_-$ is determined by $\mu_- = (1/2)(\kappa_+ - \kappa_- + m)$. If one sets $M_n := U_n(g)M_0$ ($M_{-1} := 0$), then it follows from (4.6) and (3.5) that

$$\dim(M_n/M_{n-1})$$

$$= \sum_{j=0}^{m} \dim(\mathcal{H}^{n+j}(\mathbb{R}^p) \otimes \mathcal{H}^{n-m-j+\frac{p+q}{2}}(\mathbb{R}^q))$$

$$= 4 \sum_{j=0}^{m} \frac{n + j + \frac{p}{2} - 1}{(p - 2)!} (n + j + 1)(n + j + 2) \cdots (n + j + p - 3)$$

$$\times \frac{n + m - j + \frac{p-q}{2} + \frac{q}{2} - 1}{(q - 2)!} (n + m - j + \frac{p-q}{2} + 1)(n + m - j + \frac{p-q}{2} + 2)$$

$$\cdots (n + m - j + \frac{p-q}{2} + q - 3)$$
\[
\frac{4(m + 1)}{(p - 2)!(q - 2)!} n^{p+q-4} + \text{(lower order terms in } n) \quad (4.16)
\]

for all \( n \in \mathbb{N} \), which implies (4.12). Furthermore, since the leading term of (4.16) can be rewritten as

\[
\frac{4(m + 1)}{(p - 2)!(q - 2)!} n^{p+q-4} = \frac{4(m + 1)(p + q - 4)!}{(p - 2)!(q - 2)!} \frac{n^{p+q-4}}{(p + q - 4)!},
\]

one obtains (4.13). This completes the proof. \( \square \)

Remark 4.9. One can show that the non-negative integer \( c \) in (4.14) is in fact equal to \( \max\{m+p, m+q\} \). It is well known that the Gelfand-Kirillov dimension of the minimal representation of \( \mathcal{O}_p,q \) is equal to \( p + q - 3 \) (cf. [KØ03, ZH97]). The \( K \)-type formula (4.6) for \( m = 0 \) in Theorem 4.6 tells us that \( M^+(0) = M^-(0) \) corresponds to the \((q,K)\)-module of the minimal representation of \( (p,q) \). However, as we have seen in Corollary 4.8, the Gelfand-Kirillov dimension of \( M^\pm(m) \) is equal to \( p + q - 3 \) not only for \( m = 0 \) but for any \( m \in \mathbb{N} \) satisfying \( m + 3 \leq (p + q)/2 \). The Bernstein degree can distinguish the minimal representation from the others.

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