Orthogonal Die Random Measures, Primes, and Applications

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September 23, 2020

Abstract

We show that the sequence of thinned uniform random counting measures converges weakly to the Poisson random measure. We call such measures ‘orthogonal dice’ due to their natural connection with certain counting problems and link the sizes of orthogonal dice with the primes by constructing the largest known prime orthogonal die. We give many examples of possible use of the construct of orthogonal dice in various areas of applications including gambling, cosmology, random matrices, approximation theory and circuits. The gambling example reveals for instance that fair six-sided die numbered \(\{1,2,3,4,5,6\}\) generate negative covariance in point representations of hands across players, in contrast to seven-sided dice numbered \(\{1,2,3,4,5,6,7\}\) that generate zero covariance and are orthogonal. The cosmology example suggests that a ‘supermassive’ orthogonal die underlies the galaxy point patterns of the Universe. The random matrix application reveals that the variance of the spectral gap of the Gaussian orthogonal ensemble exhibits non-monotone scaling with thinning for the Dirac (empirical) random measure, in contrast to the orthogonal dice where the spectral gap variance is monotone. Finally, the approximation application shows that the discrete Legendre polynomials converge to the Charlier polynomials in \(L^p\) for integers \(p \geq 1\) whereas the electronic circuit/shot noise application identifies electric current as the Ornstein-Uhlenbeck process driven by an orthogonal die random measure.

**Keywords**: Discrete uniform distribution, Poisson distribution, uniform convergence, random counting measure, weak convergence, thinning / restriction, primes
1 Introduction

Random counting measures, or point processes, are the central objects of this note. Random counting measures have numerous uses in statistics and probability. We study a broad class of random measures called the mixed binomial process (Kallenberg, 2017), which includes the Poisson and Dirac random measures, among many others. The Poisson random measure is a fundamental mixed binomial process, among other things it is additive—it is independent in disjoint subspaces (Cinlar, 2011). In this note we show that the family of random counting measures based on the discrete uniform distribution converges weakly to Poisson with thinning and certain ranging over the support parameters, and we construct an infinite family of such random measures that are decorrelated that we call the orthogonal dice whose sizes are linked to the primes.

The paper is organized as follows. In Section 2 we lay out the background. In Section 3 we show convergence of discrete uniform to Poisson. In Section 4 we identify a collection of discrete uniform random measures satisfying Poisson convergence with vanishing covariance in disjoint subspaces that we call ‘orthogonal dice.’ In Section 5 we visualize the family tree of relationships among the orthogonal dice, Poisson, and other related random measures. In Section 6 we discuss possible applications in different areas of the modern sciences, from card games to astronomy and cosmology to random matrices to approximation to shot noise.

2 Background

We give the necessary background by describing various processes. We start with the most general process considered in this article—the mixed binomial process. All processes considered in this article are mixed binomial processes.

Let \((E, \mathcal{E})\) be a measure space and let \(\nu\) be a probability measure on it. Let \(X = \{X_i\}\) be an independency (collection) of (iid) \(E\) valued random variables with law \(\nu\). Let \(K \sim \kappa\) be a \(\mathbb{N}_{\geq 0}\)-valued random variable independent of \(X\) with mean \(c > 0\) and variance \(\delta^2 \geq 0\). The mixed binomial process is identified to the pair of deterministic probability measures \(N = (\kappa, \nu)\) on \((E, \mathcal{E})\) through stone throwing construction (Cinlar, 2011; Bastian and Rempala, 2020; Kallenberg, 2017) (STC) as

\[
N(A) = \int_E N(dx) \mathbb{1}_A(x) \equiv \sum_{i=1}^K \mathbb{1}_A(X_i) \quad \text{for} \quad A \in \mathcal{E}
\] (1)

where \(\mathbb{1}_A\) is a set function. Because \(\mathbb{1}_A(x) = \delta_x(A)\) where \(\delta_x\) is the Dirac measure sitting at \(x \in E\), STC of the mixed binomial process may be concisely written as

\[
N = \sum_{i=1}^K \delta_{X_i}
\]
with independency \( \{K, X_1, X_2, \cdots \} \). Recall that the law of \( N \) is uniquely determined by the Laplace functional \( L \)

\[
L(f) = \mathbb{E} e^{-Nf} = \psi(ve^{-f}) \quad \text{for} \quad f \in \mathcal{E}_+ \tag{2}
\]

where \( \psi \) is the probability generating function (pgf) of \( K \).

We denote \( \mathcal{E}_+ \) the set of non-negative \( \mathcal{E} \)-measurable functions. For \( f \in \mathcal{E}_+ \) we have mean and variance of the random variable \( Nf \)

\[
\mathbb{E}Nf = cvf \tag{3}
\]

\[
\text{Var} Nf = cvf^2 + (\delta^2 - c)(vf)^2 \tag{4}
\]

For arbitrary \( f, g \in \mathcal{E}_+ \), we have covariance

\[
\text{Cov}(Nf, Ng) = cv(fg) + (\delta^2 - c)vfvg \tag{5}
\]

The moments of \( Nf \) (if they exist) can be attained from the Laplace functional

\[
\mathbb{E}(Nf)^n = (-1)^n \lim_{q \downarrow 0} \frac{\partial^n}{\partial q^n} L(qf) \quad \text{for} \quad n \in \mathbb{N}_{>0}
\]

Consider subspace \( A \subseteq E \) with \( \nu(A) = a > 0 \). The restriction (‘thinning’) of \( N \) to \( A \) is a trace random measure defined as \( N_A(B) = N(A \cap B) \) on the space \((E \cap A, \mathcal{E}_A)\) where \( \mathcal{E}_A = \{A \cap B : B \in \mathcal{E}\} \). It is indicated as \( N_A = (N[A], \nu_A) \) where \( \nu_A(B) = \nu(A \cap B)/\nu(A) \).

The law of \( N_A \) is encoded by its Laplace functional. The Laplace functional of \( N_A = (N[A], \nu_A) \) for \( A \subseteq E \) with \( \nu(A) = a > 0 \) is given by

\[
L_A(f) = \psi_A(\nu_Ae^{-f}) = \psi(a\nu_Ae^{-f} + 1 - a) \quad \text{for} \quad f \in \mathcal{E}_+ \tag{6}
\]

where \( \psi_A \) is pgf of \( N[A] \) and \( \psi \) is pgf of \( K = N[E] \). The mass function of \( N[A] \) is given by

\[
\mathbb{P}(N[A] = k) = \psi_A^{(k)}(0)/k! \quad \text{for} \quad k \geq 0 \tag{7}
\]

where \( \psi_A \). The moments of \( N[A] \) (if they exist) can also be attained through the moment generating function, \( \phi_A(t) = \psi_A(e^t) \), where

\[
\varphi_n(\kappa_A) = \mathbb{E}(N[A])^n = \phi_A^{(n)}(0) \quad \text{for} \quad n \in \mathbb{N}_{>0} \tag{8}
\]

gives the \( n \)-th moment.

Recall that two any positive naturals \( a \) and \( b \) are coprime if their greatest common divisor is 1, indicated by \( \gcd(a, b) = 1 \). Said another way, they share no common prime factors.
3 Convergence

The following is the first of three foundational results (theorems) of this note: the probability generating function of the discrete uniform distribution uniformly converges to Poisson while ranging over degenerating limiting support and thinning, the Poisson limit theorem (PLT).

**Theorem 1 (PLT).** Let \( \psi_{m,n}(t) \) be the probability generating function for a discrete uniform distribution supported on the set of consecutive integers \( m, \ldots, n \). Also, for any \( a \in [0,1] \) let

\[
\psi_{m,n}^a(t) = \psi_{m,n}(at + 1 - a).
\]

If \( m/n \to 1 \) and \( na \to b > 0 \) as \( m,n \to \infty \), \( a \to 0 \), then

\[
\sup_{t \in [0,1]} |\psi_{m,n}^a(t) - z_b(t)| \to 0
\]

where \( z_b \) is the probability generating function of a random variable \( \text{Poisson}(b) \).

**Proof.** Since the Poisson variable is uniquely defined by its moments, it suffices to show convergence of all the moments, in order to argue weak convergence from which the above convergence of pgfs will follow (see e.g. Billingsleys book [Billingsley, 1995]). Further it suffices to show the convergence of all factorial moments. The factorial moments of the thinned uniform variable \( X \) (i.e., the one with pgf \( \psi_{m,n}^a(t) \)) are given by

\[
E[X(X-1) \cdots (X-k+1)] = \mu_k(m,n,a) = a^k D_k[\psi_{m,n}^a(t)]_{t=1}
\]

where \( D_k[\cdot] \) denotes the \( k \)-th derivative. Note that

\[
\psi_{m,n}(t) = (n-m+1)^{-1} \sum_{i=m}^{n} i^k
\]

and

\[
D_k[\psi_{m,n}(t)]_{t=1} = a^k \frac{n(n-1) \cdots (n-k+1) \cdots m(m-1) \cdots (m-k+1)}{n-m+1}
\]

that satisfies

\[
a^k m^k \leq D_k[\psi_{m,n}^a(t)]_{t=1} \leq a^k n^k
\]

which implies in view of the assumptions that

\[
D_k[\psi_{m,n}^a(t)]_{t=1} \to b^k.
\]

But we note that \( \{b^k\}_{k=1}^\infty \) is a sequence of factorial moments of a Poisson random variable with mean \( b \). \( \square \)
The second foundational result which immediately follows is weak convergence of the random counting measures.

**Theorem 2 (Random measure PLT).** The sequence of thinned discrete uniform random measures \((N_{m,n}^a)\) converges in distribution (converges weakly) to the Poisson random measure \(N\), that is,

\[
\lim_{m,n \to \infty} \lim_{a \to 0} \mathbb{E}e^{-N_{m,n}^a f} = \mathbb{E}e^{-Nf} \quad \text{for } f \in \mathcal{E}_+
\]

**Proof.** The result follows from the unique determination of law by the Laplace functional \([2]\) and Theorem \([1]\). \(\square\)

### 4 Construction

In this section we construct an explicit family of discrete uniform random measures that satisfy the hypotheses of Theorem \([1]\). This involves the idea of ‘orthogonality’ in Section \([4.1]\) where mean equals variance of the counting distribution, and application in Section \([4.2]\).

#### 4.1 Orthogonality

We define orthogonality for the mixed binomial process as a property encoded by \(\kappa\) where the mean equals the variance.

**Definition 1 (Orthogonality).** The mixed binomial process \(N = (\kappa, \nu)\) is orthogonal if \(c = \delta^2\), which implies for arbitrary \(f, g \in \mathcal{E}_+\) that \(\text{Cov}(Nf, Ng) = c \nu(fg)\).

The following result relates orthogonality to vanishing covariance for disjoint functions.

**Proposition 1 (Disjointedness).** Let \(N = (\kappa, \nu)\) be an orthogonal mixed binomial process. Then \(\text{Cov}(Nf, Ng) = 0\) iff \(f, g \in \mathcal{E}_+\) are disjoint.

**Proof.** Necessity follows from the definition of a disjoint function— \(f(x)g(x) = 0\) for all \(x \in \mathcal{E}\)—so \(\nu(0) = 0\). Sufficiency follows from \(c > 0\), so \(c \nu(fg) = 0\) implies \(\nu(fg) = 0\), which is true only for the zero functions \(f\) or \(g\) with \(fg = 0\), so \(f\) and \(g\) are disjoint. \(\square\)

The canonical disjoint functions are set functions based on disjoint sets, i.e. for \(f = I_A\), \(g = I_B\), and \(A \cap B = \emptyset\), we have \(fg = I_{A \cap B} = 0\).

#### 4.2 Discrete Uniform

The set of permissible supports for the discrete uniform distribution is defined as

\[A_0 \equiv \{(m, n) : \text{integers } 0 \leq m \leq n \text{ except } m = n = 0\}\]
Consider the discrete (rectangular) uniform family of distributions \( \mathcal{K} \)

\[
\mathcal{K} = \{ \kappa_{mn} = \text{Uniform}\{m, m+1 \ldots, n-1, n\} : (m, n) \in A_0 \}
\]  

(9)

where \( \kappa_{mn} \) has mean \( c = (m+n)/2 > 0 \) and variance \( \delta^2 = ((n-m+1)^2 - 1)/12 \geq 0 \). This may be thought of in terms of rolling fair dice. The number of sides of the dice is equal to \( n - m + 1 \) (its ‘size’). Consider the uniform random measure \( N = (\kappa_{mn}, \nu) \) on \( (E, \mathcal{E}) \) based on \( \kappa_{mn} \in \mathcal{K} \). The pdf of \( K \) is given by

\[
P(K = k) = \frac{1}{n - m + 1} \quad \text{for} \quad k \in \{m, m+1, \ldots, n-1, n\}
\]

and pgf is

\[
\psi_{m,n}(t) = \frac{t^m - t^{n+1}}{(n - m + 1)(1 - t)}
\]  

(10)

**Remark 1** (Degeneracy). The degenerate member \( \kappa_{mn} \in \mathcal{K} \) is \( m = n \geq 1 \) and corresponds to the Dirac measure \( \kappa_{mm} = \delta_m \) with \( \delta^2 = 0 \). \( N = (\delta_m, \nu) \) is the binomial process with law \( N(A) \sim \text{Binomial}(m, \nu(A)) \) for every \( A \subseteq E \). For \( \nu(A) = 1 \), this is \( \delta_m = \text{Binomial}(m,1) \).

We define orthogonal dice and random measures.

**Definition 2** (Orthogonal die). We call \( \kappa_{mn} \) an orthogonal die if \( N = (\kappa_{mn}, \nu) \) is orthogonal. In turn we call \( N \) an orthogonal die random measure.

For \( m = 0 \) and arbitrary \( f, g \in \mathcal{E}_+ \), we have covariance

\[
\text{Cov}(Nf, Ng) = \frac{n}{2} \left( \nu(fg) + \frac{(n - 4)}{6} \nu f \nu g \right)
\]

which is orthogonal for \( n = 4 \). This is our first orthogonal die.

**Proposition 2** (Orthogonal die on \( \{0, 1, 2, 3, 4\} \)). \( \kappa_{mn} \) on \( \{0, 1 \ldots, n-1, n\} \) is an orthogonal die for \( n = 4 \).

Note that we if change the support of \( \kappa_{mn} \) to \( \{1, \ldots, n\} \), then we have covariance

\[
\text{Cov}(Nf, Ng) = \frac{n + 1}{2} \left( \nu(fg) + \frac{(n - 7)}{6} \nu f \nu g \right)
\]

so \( N \) is orthogonal for \( n = 7 \), i.e. seven-sided dice.

**Proposition 3** (Orthogonal die on \( \{1, 2, 3, 4, 5, 6, 7\} \)). \( \kappa_{mn} \) on \( \{1, 2 \ldots, n-1, n\} \) is an orthogonal die for \( n = 7 \).
We find there are infinite such orthogonal dice.

**Theorem 3** (Existence and uniqueness of orthogonal rectangular dice). Orthogonal dice $\kappa_{mn}$ with support $\{m, m+1 \ldots, n-1, n\}$ for integers $0 \leq m < n$ are completely enumerated by the collection

$$S = \{\kappa_{mn} : (m, n) \in A\} \subset \mathcal{K}$$

where

$$A = \{(m, n) : k = 1, 2, 4, 5, 7, 8, \ldots, m = (k^2 - 1)/3, n = 2k + m + 2\}$$

with $|S| = \infty$ and $m/n \to 1$ as $m, n \to \infty$. Moreover, for each $\kappa_{mn} \in S$, the integer $n - m + 1$ is a product of one or more primes each having value equal to or greater than five.

**Proof.** Recall that for arbitrary $f, g \in \mathcal{E}_+$, we have covariance $\text{Cov}(Nf, Ng) = \nu(fg) + (\delta^2 - c)\nu f \nu g$. Orthogonality requires $c = \delta^2$, so we solve $\delta^2 - c = 0$ for $n$, giving solutions $n = m + 2 \pm 2\sqrt{3m + 1}$, where in view of the hypotheses of Theorem 1, we have $m/n \to 1$ as $m, n \to \infty$. We take the positive root so that $m < n$ and denote the solution $n = h(m)$. $n$ is an integer whenever $3m + 1 = k^2$ for some integer $k \geq 1$. Then this is equivalent to $m = (k^2 - 1)/3$ being an integer for integer $k$, which is the case for all integers $k \geq 1$ except multiples of three. To see this, we recall that the integers $k^2$ and $k^2 - 1$ are coprime, so if the factorization of $k^2$ contains 3, then the factorization of $k^2 - 1$ does not; conversely, and relevant for our case, if the factorization of $k^2 - 1$ contains 3, then the factorization of $k^2$ does not. $k^2$ has the same prime factors as $k$, so the factorization of $k$ cannot contain three. Hence $k$ takes values of all positive integers except multiples of three, $k \in \mathbb{N}_{>0} \setminus 3\mathbb{N}_{>0}$. We put $n = h(m) = m + 2 + 2\sqrt{3m + 1} = m + 2k + 2$. Notice that $n - m + 1 = 2k + 3$ enumerates all odd integers starting with five except multiples of three. Therefore the factorization of $n - m + 1$ may contain any prime except 2 or 3, i.e. the primes starting with 5. \hfill \Box

The first 15 orthogonal dice are shown below in Table 1.
Table 1: First 15 orthogonal dice of $S$

| $m$ | $n$ | $c$ | $n - m + 1$ |
|-----|-----|-----|-------------|
| 0   | 4   | 2   | 5           |
| 1   | 7   | 4   | 7           |
| 5   | 15  | 10  | 11          |
| 8   | 20  | 14  | 13          |
| 16  | 32  | 24  | 17          |
| 21  | 39  | 30  | 19          |
| 33  | 55  | 44  | 23          |
| 40  | 64  | 52  | 25          |
| 56  | 84  | 70  | 29          |
| 65  | 95  | 80  | 31          |
| 85  | 119 | 102 | 35          |
| 96  | 132 | 114 | 37          |
| 120 | 160 | 140 | 41          |
| 133 | 175 | 154 | 43          |
| 161 | 207 | 184 | 47          |

Remark 2 (Poisson convergence). As stated in Theorem 3, the family of discrete uniform support parameters $A$ satisfies the hypothesis $m/n \to 1$ as $m, n \to \infty$ of Theorem 2 so by Theorem 3 ranging over this family and thinning retrieves the Poisson law. An illustration of this convergence is depicted in Figure 1 in the next section.

Remark 3 (Index and canonical parameter). The set $S$ is indexed by $I = \mathbb{N}_{>0} \setminus 3\mathbb{N}_{>0}$. This is the canonical parameter of the orthogonal die. The index $k \in I$ has position $\lceil \frac{2}{3}k \rceil$.

Remark 4 (Asymptotics). The number of sides $n - m + 1 = 2k + 3$ scales linearly $O(k)$, whereas the mean $c = \frac{1}{3}(k+1)(k+2)$ scales quadratically $O(k^2)$. Hence as $k$ gets large we have $n - m + 1 = 2k + 3 \ll \frac{1}{3}(k+1)(k+2) = c$.

Remark 5 (Nearest die). The mean is given by $c(k) = \frac{1}{3}(k+1)(k+2)$ for $k \in I$. The orthogonal die $\kappa_{mn}$ with mean closest to some given mean $c^*$ is located at index

$$k^* = \arg \min_{k \in I} |c(k) - c^*|$$

The following is a consequence of Theorem 3 and gives a convolution interpretation of $\kappa_{mn} \in S$. 

8
Corollary 1 (Convolution distribution). For every $\kappa_{mn} \in \mathcal{S}$ and $f \in (\{m, \cdots, n\})_+$

$$\kappa_{mn} f = (\delta_{(m+n)/2} * \xi_{mn}) f$$

where $\delta_{(m+n)/2} = \text{Dirac}((m + n)/2)$, $\xi_{mn} = \text{Uniform}\{-\frac{n-m}{2}, \cdots, \frac{n-m}{2}\}$ and $*$ is convolution.

**Proof.** Use Theorem 3 to get $\mathcal{S}$. Consider $\kappa_{mn} \in \mathcal{S}$. Both $(m+n)/2$ and $(n-m)/2$ are integers, so $\delta_{(m+n)/2}$ and $\xi_{mn} = \text{Uniform}\{-\frac{n-m}{2}, \cdots, \frac{n-m}{2}\}$ are well-defined $\mathbb{Z}$ valued probability measures. Then for $f \in (\{m, \cdots, n\})_+$

$$\kappa_{mn} f = \sum_{z \in \{m, \cdots, n\}} \kappa_{mn}\{z\} f(z)$$

$$= \sum_{x \in \{(m+n)/2\}} \delta_{(m+n)/2}\{x\} \sum_{y \in \{-\frac{n-m}{2}, \cdots, \frac{n-m}{2}\}} \xi_{mn}\{y\} f(x+y)$$

$$= (\delta_{(m+n)/2} * \xi_{mn}) f$$

defines $\kappa_{mn}$ in terms of the convolution $\delta_{(m+n)/2} * \xi_{mn}$. Note that $(m+n)/2 - (n-m)/2 = m$ and $(m+n)/2 + (n-m)/2 = n$. \hfill \Box

Remark 6 (Orthogonal die decomposition). Corollary 1 can be cast in terms of random variables. Let $\kappa_{mn} \in \mathcal{S}$. For $K \sim \kappa_{mn}$, we have decomposition

$$K = C + L$$

where $C \sim \delta_{(m+n)/2}$ and $L = K - C \sim \xi_{mn}$. Hence an orthogonal die random variable may be interpreted as the sum of degenerate and mean-zero uniform random variables.

The following is another consequence of Theorem 3 providing a way of characterizing the construction of $\mathcal{S}$ in terms of prime numbers.

Corollary 2 (Prime construction). Let the integer $p$ be a product of one or more primes, where each prime is equal to or greater than five. Then there exists an orthogonal die $\kappa_{mn} \in \mathcal{S}$ with $p$ sides, canonical parameter $k = (p-3)/2 \in \mathcal{I}$, support parameters

$$m = \frac{1}{12}(p - 5)(p - 1)$$

$$n = \frac{1}{12}(p + 7)(p - 1)$$

mean

$$c = \frac{1}{12}(p^2 - 1)$$

and position $\lceil \frac{2}{3}k \rceil = \lceil \frac{1}{3}p \rceil - 1$. 

9
Proof. Use Theorem 3 to get $S$. As stated there, $p = n - m + 1$, and this is unique for each orthogonal die (the orthogonal die are indexed by products of powers of primes greater than or equal to five). Thus for each such $p$, there exists an orthogonal die $\kappa_{mn}$. To get the parameters $m$ and $n$, we set up and solve the system of equations based on $c = \delta^2$ for $m$ and $n$:

$$\frac{m + n}{2} = \frac{1}{12}(p^2 - 1)$$

$$p = n - m + 1$$

Set $n = p + m - 1$ from the second equation and plug into the LHS of the first equation and solve for $m$. Similarly, set $m = n - p + 1$ and solve for $n$. The index follows from $p = n - m + 1 = 2k + 3$. \hfill \square

**Remark 7** (Prime dice). Let $p \geq 5$ be prime. Then by Corollary 2 each such prime number is uniquely identified to an orthogonal die. These are the prime dice. If these primes are taken from some family, then we carry the name over to the corresponding family of orthogonal dice, i.e. Mersenne primes and their orthogonal dice.

**Remark 8** (Largest known prime die). $p = 2^{82,589,933} - 1$ is the largest known prime with 24,862,048 digits and is Mersenne. Then by Corollary 2 the corresponding orthogonal die $\kappa_{mn}$ has $p$ sides, parameters

$$m = \frac{1}{12}(2^{82,589,933} - 6)(2^{82,589,933} - 2)$$

$$n = \frac{1}{12}(2^{82,589,933} + 6)(2^{82,589,933} - 2)$$

and mean

$$c = \frac{1}{12}2^{82,589,933}(2^{82,589,933} - 2)$$

It has canonical parameter

$$k = 2^{82,589,932} - 2 \in I$$

at position $\lfloor \frac{2}{3}k \rfloor = \lfloor \frac{1}{3}(2^{82,589,933} - 4) \rfloor$.

In the following Corollary we have a modest consequence of Theorem 3 for number theory that gives a new formula for counting positive naturals coprime to 2 and 3 that are less than or equal to a certain natural.

**Corollary 3** (Counting naturals). Let $\#(n)$ be the number of positive naturals less than or equal to integer $n$ that are coprime to 2 and 3

$$\#(n) = \sum_{j \geq 1} \mathbb{1}(j \leq n, \gcd(j, 2) = \gcd(j, 3) = 1) = n - \lfloor n/2 \rfloor - \lfloor n/3 \rfloor + \lfloor n/6 \rfloor$$

with $\lim_{n \to \infty} \frac{1}{n} \#(n) = 1/3$. Then for all naturals $n \geq 5$

$$\#(n) = \lfloor n/3 \rfloor - \mathbb{1}(n - 4 \equiv 0 \pmod{6}).$$
Proof. Use Theorem 3 to get $\mathcal{S}$. First consider $n = p$ for $p \geq 5$ composed of primes greater than or equal to 5. We have $p = 2k + 3$, so $k = (p - 3)/2$ is the index of the orthogonal die. As remarked, $[2/3k] = [p/3 - 1] = [p/3] - 1$. This is the number of naturals greater than or equal to 5, coprime to 2 and 3, and bounded by $p$. For consideration of the naturals with starting 1, we add 1 for the number 1 that is less than 5 and coprime to 2 and 3 to give us $\#(p) = [p/3]$ for this $p$. Now consider all $n \geq 5$. This includes numbers whose factorizations contain 2 and 3. Put $f(n) = [n/3]$. For $n = p$ of Theorem 3 we have $f(n) = \#(n)$, so we focus on $p$ whose factorization contains 2 and/or 3. We have

$$f(n) - \#(n) = [n/3] - n + [n/2] + [n/3] - [n/6]$$

which is positive for $n = 4 + 6j$ for all $j \geq 1$, taking value 1, and zero otherwise. Therefore the general formula for $n \geq 5$ is

$$\#(n) = f(n) - \mathbb{1}(n - 4 \equiv 0 \text{ (mod 6)})$$

We verify the limit: the limit of the first term $\frac{1}{n}f(n)$ is $\frac{1}{n}$ and the limit of the second term $\frac{1}{n}\mathbb{1}(n - 4 \equiv 0 \text{ (mod 6)})$ is zero, so the limit of $\frac{1}{n}\#(n)$ is $\frac{1}{3}$.

Remark 9 (Complexity). Corollary 3 says that $\#(n)$ may be attained at the cost of a single division operation, a single application of ceiling function, two addition operations, and a single modulo operation (five operations total), in contrast to the cost of three divisions, three floor functions, and three additions for direct calculation (nine operations total).

Yet another consequence of Theorem 3 is a measurable disjoint partition of $\mathcal{K}$ into infinite orthogonal, positively correlated, and negatively correlated components.

Corollary 4 (Disjoint partition of $\mathcal{K}$). $\mathcal{K}$ may be partitioned into three disjoint infinite subsets

$$\mathcal{K} = \mathcal{S} \cup \mathcal{C}_+ \cup \mathcal{C}_-$$

where $\mathcal{S}$ and

$$\mathcal{C}_+ = \{\kappa_{mn}: \text{integers } m \geq 0, n > m + 2 + 2\sqrt{3m + 1}\}$$

$$\mathcal{C}_- = \{\kappa_{mn}: \text{integers } m \geq 0, n < m + 2 + 2\sqrt{3m + 1} \text{ except } m = n = 0\}$$

The notation $\mathcal{C}_-$ and $\mathcal{C}_+$ indicates the sign of the covariance of the corresponding random measures $N = (\kappa_{mn}, \nu)$ for disjoint functions. Elements of $\mathcal{C}_-$ are called negative dice, and elements of $\mathcal{C}_+$ are called positive dice.

Proof. Use Theorem 3 to get $\mathcal{S}$, where $\delta^2 - c = 0$ for $n = m + 2 + 2\sqrt{3m + 1}$ for the $m$ there. Then for every integer $m \geq 0$ we have $\delta^2 - c < 0$ for integers $n < m + 2 + 2\sqrt{3m + 1}$, except for $m = n = 0$, and $\delta^2 - c > 0$ for integers $n > m + 2 + 2\sqrt{3m + 1}$. \qed
Now that we have identified $S$, we give statistical properties of $S$. Towards this, the following proposition gives the mean, variance, and covariance of orthogonal die random measures.

**Proposition 4** (Mean, variance, and covariance of $N$). Let $N = (\kappa, \nu)$ be an orthogonal die random measure on $(E, \mathcal{E})$. Then the mean and variance of $Nf$ for $f \in \mathcal{E}_+$ are given by

\[
\mathbb{E}Nf = cvf \quad (12)
\]
\[
\text{Var}Nf = cvf^2 \quad (13)
\]

and are known as Campbell’s formulas. The covariance of $Nf$ and $Ng$ for arbitrary $f, g \in \mathcal{E}_+$ is

\[
\text{Cov}(Nf, Ng) = cv(fg) \quad (14)
\]

**Proof.** These follow from the formulas of the mixed binomial process and the definition of orthogonality. □

We give the mean, variance, and covariance of restrictions.

**Proposition 5** (Mean, variance, and covariance of restrictions of $N$). Let $N_A = (N_{1A}, \nu_A)$ be a restricted orthogonal die. Then $N_{1A}f$ for $f \in \mathcal{E}_+$ has mean and variance

\[
\mathbb{E}N_{1A}f = acv_{1A}f \quad (15)
\]
\[
\text{Var}N_{1A}f = acv_{1A}f^2 \quad (16)
\]

and $N_{1A}f$ and $N_{1Ag}$ for arbitrary $f, g \in \mathcal{E}_+$ have covariance

\[
\text{Cov}(N_{1A}f, N_{1Ag}) = acv_{1A}(fg) \quad (17)
\]

**Proof.** The formulas for the mean, variance, and covariance follow from the formulas of the mixed binomial process using the relation $a \nu_{1A}f = \nu_{1A}f$ and the definition of orthogonality. □

**Remark 10** (No distributional closure of dice). *The law of the restriction is not uniform for $0 < a < 1$, i.e. uniform random measures are not closed under thinning.*

**Remark 11** (Mass function of $N_{1A}$). Per [7], the mass function of $N_{1A}$ is given by $P(N_{1A} = k) = \psi_A^{(k)}(0)/k!$ for $k \leq n$ where $\psi_A$ [6] with $\text{supp}(N_{1A}) = \{0, 1, \cdots, n\}$ for $A \subset E$ and $\text{supp}(N_{1E}) = \{m, \cdots, n\}$.

### 4.3 Illustration

In Figure 1 we show the mass functions of Poisson and a sequence of $a$-thinned orthogonal dice with mean 114.
Figure 1: Convergence: Mass functions of Poisson and a sequence of $a$-thinned orthogonal dice with mean 114

5 Family tree

We show the relationships among fundamental mixed binomial processes in Figure 2 relative to the orthogonal dice.
Figure 2: Relationships among certain mixed binomial processes (green is orthogonal, blue is negative covariance)

6 Applications

We discuss some applications of random measures based on orthogonal dice. The first is for card games where \( \nu \) is atomic; the second is for astronomy and cosmology; the third is for random matrices, the fourth is for discrete polynomial approximation, and the fifth is for shot noise, all where \( \nu \) is diffuse.
6.1 Card games

Consider $N = (\kappa_{mn}, \nu)$ on $(E, \mathcal{E})$ where $\kappa_{mn} \in \mathcal{K}$. We define $(E, \mathcal{E})$ as a game space, such as a deck of cards. Let $|E| = 52$ for a standard 52 card deck. Realizations of $K \sim \kappa_{mn}$ are rolls of fair dice, which return a uniform number of draws $X = \{X_i : i = 1, \ldots, K\}$, made with replacement from a deck of cards with law $\nu$. $N(A)$ is the number of cards in subspace $A \subseteq E$, such as in a “hand” of cards. For disjoint hands $A$ and $B$ and $m = 0$, the covariance of $N(A)$ and $N(B)$ changes sign as a function of $n$. For the four-sided dice ($n = 3$), the covariance is negative for disjoint hands, $-\frac{1}{6} \nu f \nu g$. The five-sided dice ($n = 4$) have uncorrelated counts for disjoint hands. The six-sided dice ($n = 5$) have positive covariance for disjoint hands, $+\frac{1}{6} \nu f \nu g$.

Note that our random variables $X$, cards, are iid. Many card games are based on sampling without replacement, where the $X$ are not independent (they are correlated). However, these two sampling schemes converge under certain conditions: with $\nu$ as sampling without replacement, we have convergence to the multinomial distribution: for disjoint partition $\{A, \ldots, B\}$ of $E$ for $i + \ldots + j = k$ as $|E| \to \infty$.

$$
\mathbb{P}(N(A) = i, \ldots, N(B) = j | k) = \frac{|A|^i \cdots |B|^j}{|E|^k} \rightarrow_{|E| \to \infty} \frac{k!}{i! \cdots j!} \nu(A)^i \cdots \nu(B)^j
$$

We partition by suits $\{\spade, \heartsuit, \diamondsuit, \clubsuit\}$, where $\nu(\spade) = \nu(\heartsuit) = \nu(\diamondsuit) = \nu(\clubsuit) = 1/4$ and $|\spade| = |\heartsuit| = |\diamondsuit| = |\clubsuit| = 13$.

For example, consider a draw of $k = 7$ cards. We have $\mathbb{P}(N(\spade) = 2, N(\heartsuit) = 2, N(\diamondsuit) = 2, N(\clubsuit) = 1 | k = 7) \approx 0.0461128$ for sampling without replacement, whereas this probability is approximately 0.0384521 for sampling with replacement. For $k = 1$, there is no approximation and the probabilities coincide. As $k$ increases, the approximation worsens.

Suppose each card $x$ is marked with a value $h(x)$ in $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$, where $Q(x, \cdot) = \delta_{h(x)}(\cdot)$. 

\[ \text{15} \]
Consider the random measure $M = (\kappa_{mn}, \nu \times Q)$ on $(E \times \mathbb{R}_+, \mathcal{E} \otimes \mathcal{B}_{\mathbb{R}_+})$. If we take 

$$h(x) = \begin{cases} 
13 & \text{if } x = \text{Ace} \\
12 & \text{if } x = \text{King} \\
11 & \text{if } x = \text{Queen} \\
10 & \text{if } x = \text{Jack} \\
9 & \text{if } x = 10 \\
8 & \text{if } x = 9 \\
7 & \text{if } x = 8 \\
6 & \text{if } x = 7 \\
5 & \text{if } x = 6 \\
4 & \text{if } x = 5 \\
3 & \text{if } x = 4 \\
2 & \text{if } x = 3 \\
1 & \text{if } x = 2 
\end{cases}$$

and take $f \in (\mathcal{E} \otimes \mathcal{B}_{\mathbb{R}_+})_+$ as $f(x, y) = y$ then the random variable $Mf$ has mean $12$ and variance $13$.

$$\mathbb{E}Mf = c \int_E \nu(dx) \int_{\mathbb{R}_+} Q(x, dy)y = c \sum_{x \in E} \nu\{x\} h(x) = 7c$$

$$\text{Var}Mf = c \int_E \nu(dx) \int_{\mathbb{R}_+} Q(x, dy)y^2 = c \sum_{x \in E} \nu\{x\} h^2(x) = 63c$$

Consider

$$f_A(x, y) = \mathbb{I}_A(x)y$$

for each suit $A$. Each function is identified to a suit, forming random variables $Mf_{\spadesuit}, Mf_{\heartsuit}, Mf_{\diamondsuit}, Mf_{\clubsuit}$, representing points. Noting $\nu\mathbb{I}_A = a \nu_A$, we have for $A \in \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ mean

16
\[ EM_f_A = c(\nu \times Q)f_A = c \int_A \nu(dx) \int_{\mathbb{R}^+} Q(x, dy)y \]
\[ = \frac{1}{4} c \sum_{x \in A} \nu_A\{x\} h(x) \]
\[ = \frac{7}{4} c \]
\[ \text{Var}M_f_A = c(\nu \times Q)f_A^2 = c \int_A \nu(dx) \int_{\mathbb{R}^+} Q(x, dy)y^2 \]
\[ = \frac{1}{4} c \sum_{x \in A} \nu_A\{x\} h^2(x) \]
\[ = \frac{63}{4} c \]

A game can be played as follows. This example is simple and there are infinite possible games to be had. Let us suppose there are four players identified to suits ♠, ♥, ♦, ♣. For each \( \omega \in \Omega \), \( K(\omega) \sim \kappa_{mn} \) is drawn with mean \( c \). The number \( K(\omega) \) is not known by the players; however, the value \( c \) is known by the players. The random variables \( M_{\omega}f_A \) for \( A \in \{♠, ♥, ♦, ♣\} \) are known by respective players, each player knowing only their random variable with mean \( \frac{7}{4} c \). For \( \omega \in \Omega \), player \( A \) guesses the value of \( I\{M_{\omega}f_B \geq \frac{7}{4} c\} \) for \( B \neq A \), that is, they make three boolean guesses. This is equivalent to examining the sign of \( M_{\omega}f_A - \frac{7}{4} c \) and using this to guess the sign of \( M_{\omega}f_B - \frac{7}{4} c \). The player(s) with the most correct guesses for \( \omega \in \Omega \) wins. For \( \kappa_{mn} \in \mathcal{S} \), the random variables are mutually orthogonal. However, for \( \kappa_{mn} \in \mathcal{K} \setminus \mathcal{S} \), the variables are correlated, either negatively or positively. Hence each player can use correlation of non-orthogonal dice in disjoint subspaces to improve their chances of winning.

To examine covariance, consider an orthogonal die \( \kappa_{mn} \in \mathcal{S} \), for \( m = 1 \) and \( n = 7 \). For \( m = 1 \), we compare to \( n = 6 \), the six-sided die, and \( n = 8 \), the eight-sided die, neither belonging to \( \mathcal{S} \), with negative and positive covariance respectively for disjoint \( f_A \) and \( f_B \). Put \( \mu = \nu \times Q \). For \( n = 6 \) we have \( \text{Cov}(M_f_A, M_f_B) = -\frac{7}{144} \mu f_A f_B = -\frac{343}{192} \) and for \( n = 8 \) we have \( \text{Cov}(M_f_A, M_f_B) = +\frac{3}{4} \mu f_A f_B = +\frac{147}{64} \).

The typical roulette wheel has 37 sides, numbered from 0 to 36 (Epstein, 2010). Note that \( \mu_{mn} \in \mathcal{S} \) for \( m = 0 \) and \( n = 4 \). For \( n = 36 \), we have mean \( c = 18 \) and \( \text{Cov}(M_f_A, M_f_B) = 96 \mu f_A f_B = 294 \). With \( \text{Var}M_f_A = 18 \times 63/4 + 96 \times 7 \times 7/4/4 = 1155/2 \), this gives correlation approximately 51%. The orthogonal die with 37 sides is supported on \( \{96, \ldots, 132\} \) with mean \( c = 114 \). The histogram distribution of \( (K_A, K_B) \) for \( A = ♠ \) and \( B = ♦ \) is shown in Figure 4 for \( 10^5 \) samples. The positive correlation for support \( \{0, \ldots, 36\} \) is evident, as is the decorrelation for support \( \{96, \ldots, 132\} \) with orthogonal Gaussian or gamma appearance.
Figure 3: Density plot of $K_A$ vs $K_B$ and $Mf_A$ vs $Mf_B$, $A = ♠$ and $B = ♦$ for two 37-sided dice

(a) $m = 0, n = 36$

(b) $m = 96, n = 132$ (orthogonal)

(c) $m = 0, n = 36$

(d) $m = 96, n = 132$ (orthogonal)
Figure 4: Density plot of $K_A$ vs $K_B$ and $Mf_A$ vs $Mf_B$, $A = ♠$ and $B = ♦$ for correlated dice
6.2 Space and astronomy, cosmology

We explore a couple applications of the uniform random measure to concepts of space, including to gravitational potential and to galaxy point patterns.

6.2.1 Gravitational potential

Consider orthogonal die random measure \( N = (\kappa_{mn}, \nu) \) on the location space \((E, \mathcal{E}) \subset (\mathbb{R}^3, \mathcal{B}_{\mathbb{R}^3})\). Let \( X = \{X_i\} \) be the collection of (star) locations in \((E, \mathcal{E})\) with law \(\nu\), here a Borel distribution. Suppose each (star at) location \(X_i\) is marked with a mass \(Y_i\) in \((F, \mathcal{F}) = ((0, \infty), \mathcal{B}_{(0, \infty)})\) according to law \(\pi\) with mean \(b\) and variance \(d^2\), i.e. \(Y_i \sim \pi\). Put \(\mu = \nu \times \pi\). Then \((X, Y)\) forms the random measure \(M = (\kappa_{mn}, \mu)\) on \((E \times F, \mathcal{E} \otimes \mathcal{F})\).

Consider point \(z \in E\). Consider the function \(f_z \in (\mathcal{E} \otimes \mathcal{F})_+\) defined as

\[
f_z(x, y) = \frac{\mathcal{G} y}{\|x - z\|}
\]

where \(\cdot\) is the Euclidean norm and \(\mathcal{G}\) is the gravitational constant. The function \(-f_z(x, y)\) gives the contribution of a mass to gravitational potential at point \(z\). Hence the negative gravitational potential is

\[
Z_z = -V(z) = Mf_z = \int_{E \times F} M(dx, dy) f_z(x, y) = \sum_{i=1}^{K} \frac{\mathcal{G} Y_i}{\|X_i - z\|}.
\]

The mean (12) and variance (13) are given by

\[
\mathbb{E}Z_z = c\mu f_z = c\mathcal{G} \int_E \nu(dx) \frac{1}{\|x - z\|} \int_F \pi(dy) y = bc\mathcal{G} \int_E \nu(dx) \frac{1}{\|x - z\|}
\]

\[
\text{Var}Z_z = c\mu f_z^2 = c\mathcal{G}^2 \int_E \nu(dx) \frac{1}{\|x - z\|^2} \int_F \pi(dy) y^2 = (b^2 + d^2)c\mathcal{G}^2 \int_E \nu(dx) \frac{1}{\|x - z\|^2}
\]

For distinct locations \(w, z \in E\) we have covariance (14)

\[
\text{Cov}(Z_w, Z_z) = c\mu(f_w f_z) = (b^2 + d^2)c\mathcal{G}^2 \int_E \nu(dx) \frac{1}{\|x - w\||x - z\|}
\]

If \(\nu\) is absolutely continuous with respect to the Lebesgue, \(\nu(dx) = \rho(x)dx\) for some density function \(\rho\), then

\[
\mathbb{E}Z_z = bc\mathcal{G} \int_E \frac{\rho(x)}{\|x - z\|} dx
\]

We denote the mass measure \(\tilde{\nu} = bc\nu\). The expected potential \(\mathbb{E}V(z) = -\mathbb{E}Z_z\) may also be recovered using Poisson’s equation

\[
\triangle \mathbb{E}V = 4\pi \mathcal{G} \tilde{\nu}
\]
where $\Delta = \nabla^2$ is the Laplace operator. Let

$$g(x) = 4\pi \mathcal{G} b c \rho(x) \quad \text{for} \quad x \in E$$

The solution $\mathbb{E}V$ may be attained using the Green’s function

$$G(x - z) = -\frac{1}{4\pi \|x - z\|}$$

so that we have

$$\mathbb{E}V(z) = (G * g)(z) = \int_E G(x - z)g(x)dx = -c\mu f_z = -c\int_{E \times F} \mu(dx, dy) f_z(x, y)$$

where $*$ is the convolution operator. This relation gives another interpretation of the expected value as a convolution kernel.

For example, consider the Milky Way galaxy $(E, \mathcal{E})$, which contains $\sim 250$ billion stars (Blanton et al., 2017). The first element of $\mathcal{S}$ with mean $c$ greater than this has support $\{m, m + 1, \ldots, n - 1, n\}$ with parameters

\[
\begin{align*}
  m &= 249,999,189,525 \\
  n &= 250,000,921,575 \\
  c &= 250,000,055,550 \\
  n - m + 1 &= 1,732,051
\end{align*}
\]

The number of sides $n - m + 1 = 1,732,051$ is the 130,347th prime and $n - m + 1 \ll c$ consistent with Remark 4 on asymptotics. $\nu$ encodes a barred spiral galaxy. The mass distribution $\pi$ is assumed to be log-normal with mean $b = 4$ solar masses, so we have the constant $bc = 1,000,000,222,200 \simeq 10^{12}$.

### 6.2.2 Does a supermassive orthogonal die underlie the Universe?

In the previous application, we suggested an orthogonal die random measure for modeling the counts and locations of stars in the Milky Way galaxy. This demonstrates that no matter how many points there are, there are always nearby orthogonal dice.

Spatial point processes are commonly encountered in astronomy (Babu and Feigelson, 1996), especially the Poisson. Moreover, the cosmological principle states that the spatial distribution of matter at large scales is homogeneous and isotropic, suggesting a homogeneous Poisson random measure for large scale structure. Indeed, Neyman and Scott developed the Poisson cluster process, which is built using the Poisson process, to describe the observed clustered point pattern of galaxies (Neyman and Scott, 1958).

The indistinguishability of $a$-thinned orthogonal dice from Poisson random measures as $a \to 0$ and as $k \to \infty$ is suggestive: if there were a supermassive orthogonal die underlying
the Universe, could we detect it? To unpack this, note that the diameter of the observable universe from Earth is approximately $d = 8.8 \times 10^{26}$ meters. Estimates for the size of the entire universe, assuming it is finite, range from $250d$ \cite{Vardanyan2011} to $10^{10,122}$ megaparsecs \cite{Page2007}. Assuming constant matter density, this means the size of $d$ is either small or infinitesimally small. Said another way, we interact with the Universe’s putative random measure $N = (\kappa, \nu)$ on $(E, \mathcal{E})$ only through restrictions $N|_A$ to subspaces $A$ with small mass $0 < \nu(A) = a \ll 1$, and the mean number of points of the random measure is very large. Hence, while it is supposed the Poisson random measure is involved in the structure of the Universe, we argue that if the total size of a realization is bounded, then an orthogonal die is instead involved. As $(a, k) \to (0, \infty)$, the orthogonal die converges to Poisson. Because for this setting we are well into this regime, it seems that the ability to statistically test the Poisson assumption is very limited. In view of this backdrop, we suggest a possibility that the “big bang” may be identified to some (random) large prime, which indexes an orthogonal die random measure, which random measure through the stone throwing construction conveys the spatiotemporal structure of the point patterns of the Universe.

6.3 Random matrices

Consider $(E, \mathcal{E})$ as the space of symmetric $\mathbb{R}$-valued $n \times n$ matrices. Each $n \times n$ matrix is equivalent to a vector of length $n(n+1)/2$, so we take $E = \mathbb{R}^{n(n+1)/2}$ and $\mathcal{E} = \mathcal{B}_{\mathbb{R}^{n(n+1)/2}}$. Let $\nu(dx) = \prod_{i \leq j} \nu_{ij}(dx_{ij})$ where $\nu_{ii} = \text{Gaussian}(0, 2)$ and $\nu_{ij} = \text{Gaussian}(0, 1)$ for $i < j$. This is the Gaussian orthogonal ensemble (GOE)

$$\nu(dx) = e^{-\frac{1}{4}(\sum_i x_{ii}^2 + 2 \sum_{i < j} x_{ij}^2)} \prod_{1 \leq i \leq j \leq n} dx_{ij}$$

We interpret $X \sim \nu$ both as a matrix

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

and as a vector $X = (x_{11}, x_{12}, \cdots, x_{n-1,n}, x_{nn})$. In matrix notation, we have

$$\nu(dX) = e^{-\frac{1}{4} \text{tr} X^2} \frac{1}{2^{n/2}(2\pi)^{n(n+1)/4}} dX$$

The GOE is so-called because for each matrix $X \sim \nu$, we have $OXO^\top \sim \nu$ for all orthogonal matrices $O$ ($OO^\top = O^\top O = I$).
Let $N = (\kappa_{mn}, \nu)$ be an orthogonal die random measure on $(E, \mathcal{E})$ formed by independence of random matrices $X = \{X_i : i = 1, \ldots, K\}$. Let $n = 2$ and $f \in \mathcal{E}^+$ be the spectral gap

$$f(\begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}) = f((x_{11}, x_{12}, x_{22})) = \sqrt{(x_{11} - x_{22})^2 + 4x_{12}^2}$$

The independency is $(X, Y, Z) = \{(X_i, Y_i, Z_i)\}$. Then the random variable

$$Nf = \int_E N(dx, dy, dz)f((x, y, z)) = \sum_{i=1}^{K} f((X_i, Y_i, Z_i))$$

has mean (12) and variance (13)

$$\mathbb{E} Nf = c \nu f = \sqrt{2\pi c} \simeq 2.50663c$$

$$\mathbb{V}ar Nf = c \nu f^2 = 8c$$

We can attain an orthogonal die random measure on the spectral gap space $(\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+})$ as the image random measure $L = N \circ f^{-1} = (\kappa_{mn}, \nu \circ f^{-1})$. The image measure $\mu = \nu \circ f^{-1}$ has density

$$\mu(dy) = \frac{y}{4} e^{-\frac{y^2}{8}} dy$$

with mean $\sqrt{2\pi}$ and variance $8 - 2\pi$ (equivalently obtained using $\nu f$ and $\nu f^2$). This is known as Wigner’s surmise [Livan et al., 2017]. $\mu$ may be sampled as follows: for $U \sim \text{Uniform}(0, 1)$, then $Y = 2\sqrt{2 \log(\frac{1}{1-U})} \sim \mu$. For $g \in (\mathcal{B}_{\mathbb{R}^+})^+$ we have

$$\mathbb{E} Lg = c \mu g$$

$$\mathbb{V}ar Lg = c \mu g^2$$

Consider disjoint $A, B \subset E$ and put $f_A((x, y, z)) = \mathbb{I}_A((x, y, z))f((x, y, z))$ and similarly $f_B$ so $f_A$ and $f_B$ are disjoint. Then $\text{Cov}(Nf_A, Nf_B) = 0$. For example, let $A = (0, \infty) \times \mathbb{R} \times (-\infty, 0)$ and $B = (-\infty, 0) \times \mathbb{R} \times (0, \infty)$ with $\nu(A) = \nu(B) = 1/4$. We have for $D \in \{A, B\}$ mean (15) and variance (16) of $N_D f = N f_D$

$$\mathbb{E} N_D f = ac \nu_D f \simeq \frac{1}{4} c 2.98373 = 0.745931c$$

$$\mathbb{V}ar N_D f = ac \nu_D f^2 \simeq \frac{1}{4} c 10.5465 = 2.63662c$$

Notice that $\nu_D f > \nu f$ and $\nu_D f^2 > \nu f^2$.

For comparison, consider the Dirac (degenerate) random measure $M = (\kappa_{cc}, \nu)$ on $(E, \mathcal{E})$. This has the same mean, minimum variance ($\delta^2 = 0$)

$$\mathbb{V}ar Mf = c \mathbb{V}ar f = (8 - 2\pi)c \simeq 1.71681c$$

23
restriction to \( D \in \{A,B\} \)

\[
\Var M_D f = ac \nu_D f^2 - ca^2(\nu_D f)^2 \simeq 2.0802c
\]

and negative covariance \( \Cov(M f_A, M f_B) = -cr f_A \nu f_B \simeq -0.556414c \). Notice that the variance of the restriction increases. This is due to the non-linearity of \( f \) and the non-orthogonal structure of \( M \).

To summarize these results, we define \( \mathcal{A}_r = (r, \infty) \times \mathbb{R} \times (-\infty, -\frac{r}{2}) \).

We have

\[
a_r = \nu(A_r) = \frac{1}{4}(1 - \text{erf}(\frac{r}{2}))^2
\]

In Figure 5, we show \( \frac{1}{r} \Var N_{A_r} f \) and \( a_r \) as functions of \( r \) for the Dirac and orthogonal dice random measures. The Dirac measure is non-monotone (Figure 5a), increasing then decreasing, whereas the orthogonal die is monotone (Figure 5b), decreasing. Figure 5c shows \( a_r \) as a function of \( r \), decreasing.
Figure 5: Plots of $\frac{1}{c} \text{Var} (N_{A_f})$ and $a_r$ as functions of $r$.

### 6.4 Approximators

The discrete distributions $\{\kappa\}$ of this paper—uniform and Poisson—having finite moments at all orders, generate unique sets of pairwise orthogonal polynomials $P(\kappa) = \{P_k\}_{k \geq 0}$, where $\deg P_k = k$. We apply the convergence result of the previous sections to the corresponding orthogonal polynomial systems of orthogonal dice and Poisson to establish a convergence result in Theorem 4.

The orthogonal polynomials can be constructed through the Gram-Schmidt process applied to the monomials $\{x^k\}_{k \geq 0}$ (a basis $\{B_k\}_{k \geq 0}$) with respect to the inner product $\langle \cdot, \cdot \rangle_{\kappa}$. Alternatively the orthogonal polynomials can be constructed from the moments of $\kappa$. The orthogonality property is

$$\kappa(P_i P_j) = \langle P_i, P_j \rangle_{\kappa} = \sum_{x \geq 0} \kappa\{x\} P_i(x) P_j(x) = 0 \text{ for } i \neq j$$
Recall that the $p$-norm $\|f\|_{L^p(\kappa)}$ of function $f$ for integer $p \geq 1$ is defined as

$$\|f\|_{L^p(\kappa)} = \left( \sum_{x \geq 0} \kappa(x) |f(x)|^p \right)^{1/p}$$

We have the fact that polynomials belong to $L^p(\kappa)$ for all integers $p \geq 1$ if $\kappa$ has finite moments at all orders, that is, $|\varphi_n(\kappa)| < \infty$ for all $n \geq 0$.

**Proposition 6** (Polynomials in $L^p(\kappa)$). Let $\kappa$ have finite moments at all orders and let $P$ be an arbitrary polynomial on $\mathbb{N}_{\geq 0}$. Then $P \in L^p(\kappa)$ for integers $p \geq 1$.

**Proof.** Let $\deg P = k$ and consider the $p$-norm $\|\cdot\|_{L^p(\kappa)}$. Because $|P(x)|^p = |P(x)^p|$, we define polynomial $Q(x) \equiv P(x)^p$ with $\deg Q = kp < \infty$. Because $Q(x) = \sum_{j \leq kp} a_j x^j$ and $\kappa$ has finite moments at all orders, then we have

$$\kappa Q \leq \sum_{j \leq kp} |a_j| |\varphi_j(\kappa)| < \infty$$

and therefore $P \in L^p(\kappa)$ for integers $p \geq 1$. \qed

Consider the thinned measure $\kappa_A$ for some $A \subseteq E$ with $\nu(A) = a > 0$. The polynomials of thinned Poisson measures are directly attained from the unthinned polynomials, i.e. if $P(\kappa_\theta)$ are the Charlier polynomials for Poisson $\kappa_\theta$, then $P(\kappa_{a\theta})$ are the Charlier polynomials for Poisson $\kappa_{a\theta}$. The discrete orthogonal polynomials of $\kappa_A$ for orthogonal dice are attained as described above using Gram-Schmidt or directly using the moments. We call the polynomials of $\kappa_A$ the $a$-thinned discrete Legendre polynomials.

We define a distance function $d_\kappa$ between polynomials $P$ and $Q$ in terms of $\|\cdot\|_{L^p(\kappa)}$

$$d_\kappa(P, Q) = \|P - Q\|_{L^p(\kappa)} \quad (19)$$

We use Theorems 1 and 3 to establish a convergence result for these families of orthogonal polynomials with respect to this distance function.

**Theorem 4** (Orthogonal polynomial convergence). The family of discrete Legendre polynomials corresponding to the orthogonal dice converges to the Charlier (Poisson) polynomials in $L^p$ for integers $p \geq 1$ with thinning and degenerating limiting support.

**Proof.** Consider polynomials $\mathcal{P}(\kappa_A) = \{P^A_k\}_{k \geq 0}$ and $\mathcal{P}(\kappa^*_A)$ generated by the pair of convergent measures: orthogonal dice (Theorem 3) $\kappa_A$ and Poisson $\kappa^*_A$. By Proposition 6 we have $\mathcal{P}(\kappa_A) \subset L^p(\kappa_A)$ and $\mathcal{P}(\kappa^*_A) \subset L^p(\kappa^*_A)$. Now consider the distance function $d_{\kappa^*_A}(P_i^A, Q_i^A)$ \quad (19) for $P_i^A \in \mathcal{P}(\kappa_A)$ and $Q_i^A \in \mathcal{P}(\kappa^*_A)$ for integers $i \geq 0$ and $p \geq 1$. The $d_{\kappa^*_A}(P_i^A, Q_i^A)$ is a continuous function of the thinning parameter $a$ and support parameters $m, n$ for $i \geq 0$. Using this and Theorem 1 by the continuous mapping theorem we conclude $\lim_{\kappa_A \to \kappa^*_A} d_{\kappa^*_A}(P_i^A, Q_i^A) = d_{\kappa^*_A}(Q_i^A, Q_i^A) = 0$ for all integers $i \geq 0$ and $p \geq 1$ as $a \to 0$ and $m, n \to \infty$. That is, the polynomials converge $\mathcal{P}(\kappa_A) \to \mathcal{P}(\kappa^*_A)$ in $L^p$ for integers $p \geq 1$ with thinning and degenerating limiting support. \qed
Remark 12 (Exact $P_1$ and $P_2$). Note that, because the first two moments of orthogonal dice and Poisson are identical, the first and second order orthogonal polynomials of dice and Poisson are identical. The third and higher moments of the dice and Poisson differ, so their third and higher-order polynomials differ.

Remark 13 (Third-order polynomials of orthogonal dice and Poisson). We illustrate the convergence of orthogonal polynomials of a-thinned orthogonal dice to the Charlier polynomials, where $\kappa_A$ is the die and $\kappa_A^*$ is Poisson, for third-order polynomials. Recall that per Remark 12, only one moment of the first three (the third) is different. We consider mean 114. For the dice, this corresponds to index $k = 17$ with $a = 1$. Then, we increase $k$ to $l$ and for each $l$ we set $a$ such that the mean is unchanged, i.e. $a(l) = c(17)/c(l)$. We take $l \in \{17, 19, 22, 25, 31\}$.

Figure 6: Plot of third-order orthogonal polynomial $P_3(x)$ for Poisson (Charlier) and a sequence of $a$-thinned orthogonal die ($P_3(x|a,k)$ : $a$ decreasing, $k$ increasing), all generated by measures with mean 114, plotted with respect to $\kappa \sim \text{Poisson}(114)$, i.e. visualizing $P_3(x)\kappa\{x\}$

### 6.5 Shot noise

Consider orthogonal die random measure $N = (\kappa_{mn}, \nu)$ on $(E, \mathcal{E}) = ([0,T], \mathcal{B}_{[0,T]})$, with $X = \{X_i : i = 1, \ldots, K\}$ a collection of iid arrival times of electrons at an anode and $\nu = \text{Uniform}[0,T]$. Each electron produces a current with intensity $g(u)$ after $u$ time units, where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is Borel, integrable, and decays rapidly to zero. The currents are additive. Let $f_t(s) = \mathbb{1}_{[0,t]}(s)g(t-s)$ so the current’s intensity at time $t$ is

$$Z_t = Nf_t = \int_{[0,t]} N(ds)g(t-s) = \sum_{i=1}^K g(t - X_i)\mathbb{1}_{[0,t]}(X_i) \quad \text{for} \quad t \in [0,T]$$
where \( Z_0 = 0 \) with mean (12) and variance (13)

\[
\mathbb{E}Z_t = \mathbb{E}N f_t = cvf_t = \frac{c}{T} \int_{[0,t]} ds g(t-s)
\]

\[
\text{Var}Z_t = \text{Var}N f_t = cvf_t^2 = \frac{c}{T} \int_{[0,t]} ds g^2(t-s)
\]

The covariance (14) is given by

\[
\text{Cov}(Z_s, Z_t) = \text{Cov}(N f_s, N f_t) = cv(f_s f_t) = \frac{c}{T} \int_{[0,s \wedge t]} du g(t-u)g(s-u)
\]

For example put \( g(u) = ae^{-bu} \) for \( a, b > 0 \). Then we have mean and variance

\[
\mathbb{E}Z_t = \frac{ca}{bT} (1 - e^{-bt})
\]

\[
\text{Var}Z_t = \frac{ca^2}{2bT} (1 - e^{-2bt})
\]

and covariance

\[
\text{Cov}(Z_s, Z_t) = \frac{ca^2}{2bT} (e^{-b|t-s|} - e^{-b(s+t)})
\]

Note that \( \mathbb{E}Z_0 = \text{Var}Z_0 = 0 \) and \( \mathbb{E}Z_t \) and \( \text{Var}Z_t \) converge exponentially fast to stationary solution (equilibrium) with rates \( b \) and \( 2b \) for the mean and variance respectively. Therefore \( (Z_t)_{t \in [0,T]} \) describes the current’s intensity going from “off” at \( t = 0 \) to “on” for \( t > 0 \) with exponentially fast response.

Notice for this \( g \) that

\[
\frac{d}{dt}Z_t = \int_{[0,t]} N(ds)g'(t-s) + g(0)N
\]

\[
= -bZ_t + aN
\]

Hence \( (Z_t) \) is an Ornstein-Uhlenbeck process driven by an orthogonal die random measure

\[
dZ_t = -bZ_t dt + aN(dt) \text{ for } t > 0
\]

This is a linear stochastic differential equation. Written another way, this is

\[
Z_t = -b \int_0^t Z_s ds + aN([0, t])
\]

These results are similar to those of the classic Poisson shot-noise model discussed in Eliazar and Klafter (2005), here where the support of \( K \) is bounded and \( g \) is deterministic.
7 Discussion and Conclusions

In the foundational result of Theorem 1 we show that the pgf of the discrete uniform distribution uniformly converges to the pgf of Poisson when one ranges over the discrete uniform distribution in such a way that its limiting support degenerates and if one additionally thins the pgf. We use this to establish weak convergence of the random measures in Theorem 2. To the best of our knowledge, these results are novel. The discrete uniform distribution has bounded parameterized support, unlike the Poisson with unbounded support. Thus these results relate two distinct objects in a powerful way. In Theorem 3 we identify an infinite family of orthogonal dice that satisfy Theorem 1 and describe their link to the primes. The appearance of the primes is another novelty, whereby the sizes of the orthogonal dice are coprime to 2 and 3. Thus these results connect the orthogonal die family of random measures, the Poisson family of random measures, and the primes.

Uniform counting distributions show up often in games of chance with fair dice, where each side of the dice corresponds to a count of points, moves, etc. Hence we refer to these distributions as ‘dice.’ The first example shows that canonical six-sided dice with labels $1 - 6$ generate negatively correlated random variables in disjoint subspaces and that the orthogonal die on $\{1, 2, \ldots, n - 1, n\}$ has seven sides ($n = 7$) with labels $1 - 7$. We suggest that seven-sided dice, being orthogonal, can ensure fairness in count or point-based games by eliminating correlation. In the example card game with six-sided dice, any player knows that the point representation of their hand is (slightly) negatively correlated to the point representations of the other players’ hands. A dramatic difference in correlation is seen with the 37-sided roulette wheel, for which there exists an orthogonal die. Here, labeling $0 - 36$ generates strong positive correlation in disjoint subspaces, whereas labeling $96 - 132$ is orthogonal.

Orthogonal dice can be used to construct more evolved processes. We illustrate how gravitational potential and shot noise are retrieved from the action of the random measure on a suitable test function, and we identify a candidate random measure for the Milky Way galaxy.

Orthogonal dice share formulas with the Poisson family for mean, variance, and covariance, although the Poisson family is stronger by conferring independence to disjoint subspaces. Another point of distinction is in thinning. The Poisson random measure (and more generally the Poisson-type random measures (Bastian and Rempala, 2020)) is closed under restriction to subspaces: in all subspaces counts are Poisson(-type) random variables. As remarked, every restriction of an orthogonal dice into a subspace is not uniform, i.e. not a die, and all restrictions form orthogonal random measures. Poisson and discrete uniform are statistically (information theoretically) and computationally indistinguishable in small subspaces with large mean by Theorems 1 and 2. This has interesting implications, such as for understanding and detecting the structure of the Universe, explored in an application.

We also suggest orthogonal dice to be a competitor to the family of Dirac measures $\{\delta_c\}$ in the sense that each Dirac measure $\delta_c = \kappa_{cc} \in \mathcal{K}$ has fixed and bounded support as the
degenerate member, with mean \( c \) and negative covariance in disjoint subspaces. The nearest element \( \kappa_{mn} \in S \) has bounded support and is orthogonal. For large mean \( c \), the number of sides is small in comparison and the distribution \( \kappa_{mn} \in S \) “looks” Dirac. The example on random matrices shows surprising behavior of the Dirac measure on restrictions, where the variance of the restriction is non-monotone. Per Corollary 1, orthogonal dice can be interpreted as noisy Dirac measures.

The family tree in Figure 2 shows that the orthogonal dice, bounded and orthogonal, are intermediaries between the Dirac measure, bounded and non-orthogonal, and the Poisson measure, unbounded and orthogonal.
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