Critical exponents of the driven elastic string in a disordered medium

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We analyze the harmonic elastic string driven through a continuous random potential above the depinning threshold. The velocity exponent $\beta = 0.33(2)$ is calculated. We observe a crossover in the roughness exponent $\zeta$ from the critical value 1.26 to the asymptotic (large force) value of 0.5. We calculate directly the velocity correlation function and the corresponding correlation length exponent $\nu = 1.29(5)$, which obeys the scaling relation $\nu = 1/(2 - \zeta)$, and agrees with $\nu_{FS}$, the finite-size-scaling exponent of fluctuations in the critical force. The velocity correlation function is non-universal at short distances.

Driven elastic manifolds in disordered media model the physics of systems as diverse as charge density waves$^1$, interfaces in disordered magnets$^2$, contact lines of liquid menisci on rough substrates$^3$, vortices in type-II superconductors$^4$ and crack propagation in solids$^5$.

An elastic manifold, driven through disorder by an external force, undergoes a dynamic phase transition$^6$ that arises from competition between the driving force and the pinning energy due to the disorder, mediated by the elasticity of the manifold. Analogous to an equilibrium phase transition, the driving force acts as the control parameter, and the average center-of-mass velocity $v$ of the manifold acts as order parameter. Two phases of respectively zero and non-zero order parameter are separated by the critical force $f_c$. At forces below this depinning threshold $f_c$, the disorder ‘pins’ the manifold and for long enough times, the velocity of the manifold is zero. Above $f_c$, the manifold continues to advance in avalanches. When the threshold is approached from above ($f \to f_c^+$), the mean velocity tends to zero, and typical length, width and duration of the avalanches diverge. This critical divergence is characterized by two independent scaling exponents.

Much effort$^1$,$^2$,$^3$,$^4$,$^10$ has been spent on calculating the universal exponents, particularly for driven elastic manifolds in the limit of quasi-static motion. In this limit, inertial terms are neglected and the force acting on the manifold is assumed independent of velocity. The net force on the manifold comprises a constant driving force $f$, a position-dependent random force $\eta$ and an elastic restoring force. Choosing a harmonic short range elastic force leads to the following equation of motion for the manifold $h$ at zero temperature

$$\partial_t h(x,t) = f + \eta[x,h(x,t)] + \partial_x^2 h(x,t).$$  \hspace{1cm} (1)

In general, the manifold is represented as a single-valued function $h(x,t)$ defined over a $D$-dimensional transversal space $x$, moving in a $D + 1$ dimensional disorder. In this paper, we consider one discrete dimension $D = 1$ with periodic boundary conditions for the string of length $L$ ($x + L$ is identified with $x$) and for the disorder of lateral extension $M$ ($h + M$ is identified with $h$, up to a winding term). The continuous disorder of unit strength and unit correlation range is constructed like in$^11$.

In what follows, we numerically investigate the dynamics of the $1 + 1$-dimensional string above the depinning threshold. Convergence of the dynamical solution is ensured by exploiting the particular analytical structure$^12$,$^13$ of the equation of motion: it has a unique periodic solution for each disorder sample in the $t \to \infty$ limit. Having found the periodic solution, we are thus certain to have reached the asymptotic regime and to have shaken off all influence of arbitrary initial conditions. The asymptotic periodic solution is constructed to desired precision using a continuous integration routine for eq.\hspace{1cm}1. Averaging observables over one period and over disorder samples, we calculate the velocity exponent $\beta$ and describe in detail how the correlation length diverges as $f \to f_c^+$. In the thermodynamic limit, the string velocity obeys

![Graph](image-url)
a power law: \( v \sim (f - f_c)^\beta \) for \( f \rightarrow f_c \). On finite systems, the critical force \( f_{c,\text{smpl}} \) fluctuates from sample to sample, putting a limit on how small the control parameter \( f - (f_{c,\text{smpl}}) \) can be made without introducing undesired corrections to scaling relations; a limit from which previous numerical calculations suffer.

We are able to determine the exact sample-dependent depinning threshold \( f_{c,\text{smpl}} \) thanks to a recent algorithm [14]. The sample critical force itself shows non-negligible fluctuations of the order of \( \sigma_{f_c} \approx L^{-1/\nu_{rs}} \). The algorithm also finds the final critical configuration \( h_c \) and the roughness exponent \( \zeta \) at depinning [11].

Knowing the critical force \( f_{c,\text{smpl}} \) of each sample, we plot the time- and disorder-averaged velocity \( v \) against \( f - f_{c,\text{smpl}} \). Thus we eliminate the statistical noise due to the fluctuating critical force, and obtain extraordinarily clean data. Furthermore, the control parameter \( f - f_{c,\text{smpl}} \) can be made arbitrarily small, which is not possible when using \( f - (f_{c,\text{smpl}}) \).

The mean velocity on a finite sample shows three different regimes (see figure 1). For very small \( f - f_{c,\text{smpl}} \), the motion of the entire string is correlated, and it behaves effectively like a single particle \( v \sim (f - f_{c,\text{smpl}})^{1/2} \). At intermediate forces, the string is correlated on length-scales \( \xi \) and moves collectively \( v \sim (f - f_{c,\text{smpl}})^{\beta} \). At high forces the motion is essentially uncorrelated \( v \sim f \).

At \( f = f_{c,\text{smpl}} \), the finite dynamical system (eq. 1) (in the \( t \rightarrow \infty \) limit) undergoes a saddle node bifurcation between a static (pinned) and a periodic (depinned) solution; the global velocity minimum changes from stable to unstable. For very small positive \( f - f_{c,\text{smpl}} \), the string spends the major part of its time-period passing through the velocity minimum, and negligibly little time completing its orbit. The motion through the minimum is dominated by the mode \( h \) with the largest eigenvalue in a linear stability analysis around the critical configuration \( h_c \). The mode \( h \) moves at a velocity equal to \( f - f_{c,\text{smpl}} \) plus quadratic and higher corrections in \( h \). Hence we can express the time spent inside the minimum as:

\[
\frac{dT}{df} = \int_{f_f}^{f_c} \frac{dh}{f - f_{c,\text{smpl}}} = T_c(f - f_{c,\text{smpl}})^{-1/2},
\]

for first order in \( f - f_{c,\text{smpl}} \). The remaining time \( T_o \) spent outside the velocity minimum depends only weakly on \( f - f_{c,\text{smpl}} \). This yields for the velocity as a function of force:

\[
M/v = T = T_o + T_c(f - f_{c,\text{smpl}})^{-1/2}.
\]

(2)

As shown in figure 1, this function — combining the effective single-particle exponent one half \( v \sim (f - f_{c,\text{smpl}})^{1/2} \) for very small forces with the saturation at slightly larger forces — fits the data perfectly.

At intermediate forces, the dynamics leave the single-particle regime and enter the regime of critical collective motion. The cross-over takes place when the dynamic correlation length \( \xi \), which diverges as \( \xi \sim (f - f_c)^{-\nu} \), equals the system size \( L \) [15]. It follows that the difference between the corresponding cross-over force and the sample critical force scales as \( L^{-1/\nu} \). This scaling is quite slow because of the small exponent \( 1/\nu \approx 0.7 \), as our data confirms. The other cross-over, between the critical region and the linear regime, is independent of system size.

Between the two cross-overs lies the window of collective behavior, small and slowly growing with \( L \). Only strings of length \( L \geq 512 \) allow to see any significant evidence for critical behavior, and even for \( L = 2048 \) (see inset of figure 1) the window is less than two decades. These pronounced finite-sample-size effects limiting the critical window (also observed in the CDW model [14]) could obscure critical behavior in experimental situations. For example, in an experiment of a liquid-solid contact line advancing on a disordered substrate [17] the sample size, i.e. the capillary length acting as upper cutoff, is \( L \approx 200 \) in units of the disorder correlation length. For these small samples we expect the window of collective behavior to be hardly noticeable and finite-size effects to dominate.

When extrapolating data to the thermodynamic limit, we have to carefully choose the lateral sample size: \( M \) has to be of the order of the typical width \( W \) of a string of length \( L \) at depinning. When increasing \( L, W \) scales as \( L^\zeta, \zeta \) being the roughness exponent. Hence the lateral sample size \( M \) has to scale as \( L^\zeta \), too. Otherwise, if \( M \) is scaled with an exponent \( \zeta' < \zeta \), the periodic sample is too short, the string wraps around it, and generates correlations which mix in properties of the charge density wave model (CDW, \( M \sim 1 \)), itself member of a different universality class [18]. If on the other hand \( M \) scales with \( \zeta' > \zeta \), the sample is too long and contains \( M/W \sim L^{\zeta' - \zeta} \) independent critical configurations of size \( L \times W \). Each of these configurations has a slightly different local critical force, and the critical force of the entire sample is given by their maximum, and not their mean. Consequently the critical force of a sample of size \( M \gg W \) overestimates \( f_c \), even in the thermodynamic limit.

We therefore use a scaling of \( M \sim L^\zeta \), with strings of length \( L = 512, ..., 2048 \), when analyzing the mean velocity. Inside the window of critical collective behavior, we find \( \beta = 0.33(2) \) (see inset of figure 1). This value is consistent with 0.2(2) from a two-loop functional renormalization [4], and with 0.35(5) from a simulation of an advancing magnetic domain wall in a 2D random Ising system [14]. It is larger than previous estimates on continuous systems 0.22(2) [20] and on automaton models 0.25(3) [11]. In these studies, however, \( v \) was plotted against \( f - (f_{c,\text{smpl}}) \), and \( \beta \) was probably measured partly inside the critical window and partly inside the finite-size-effect region where the velocity saturates. This naturally leads to \( \beta \) being underestimated.

At the depinning threshold the typical size \( \xi \) of avalanches diverges. Whereas previous work did not succeed in accessing \( \xi \) directly, we are able to compute this
FIG. 2: Connected velocity-velocity correlations $C_v(x)$ for different values of $f - f_c$, ($L = 512, M = 20000$, 700 samples). Inset: correlation length $\xi$ as obtained from a fit to $C_v(x) = C_0 x^{-\kappa} e^{-x/\xi}$.

The inset of figure 2 shows the correlation length $\xi$ as obtained from a fit to $C_v(x) = C_0 x^{-\kappa} e^{-x/\xi}$.

The non-universal behavior of the velocity correlation function at small distances might be linked to the fact that in one dimension the harmonic model’s roughness exponent is greater than one: The average distance between neighboring points grows without bound, implying that any real string described by this model would eventually break [22]. Preliminary data on a model with higher than harmonic elasticity [23] does indicate universal velocity correlations, suggesting that the harmonic short range model is indeed exceptional.

An avalanche of typical length $\xi$ has a typical width $w$ which scales like $w \sim \xi^{\zeta}$. As a second method to estimate the roughness exponent $\zeta$, we study the time- and disorder-averaged structure factor $S(q)$, which behaves like $1/q^{1+2\zeta}$, when defined as $S(q) = \langle h(q) h(-q) \rangle_c$ with $h(q) = \sum_i e^{iqx} h(x)$.

At large driving forces $f \to \infty$, the roughness exponent of the harmonic elastic string is $\zeta = 1/2$. The noise due to the disorder becomes equivalent to thermal noise, which can be seen from expanding the disorder term $\eta$ in eq. [4] of motion – transformed to the center of mass reference frame – in powers of $v^{-1} \approx f^{-1}$, yielding a $\delta(t)$ correlated noise to first order. At depinning, the roughness exponent takes on the value $\zeta = 1.26(1)$ [11].

Inside the critical window, both these values for $\zeta$ show up in the structure factor, and characterize two different ranges of $q$ separated by the inverse of the correlation length: for large wave-vectors ($q \gtrsim 2\pi/\xi$) the structure...
factor displays the critical roughness, whereas at small wave-vectors \( q \lesssim \frac{2\pi}{L} \) it shows the thermal roughness. Figure 11 tracks the crossover between the two regimes as \( f \to f_c^\infty \), illustrating the diverging correlation length \( \xi \).

The corresponding correlation length exponent \( \nu \) is consistent with the value calculated from the velocity correlation function, but less precise.

In our calculation of the correlation function, the lateral size \( M \) must be chosen sufficiently large for the string not to notice the periodicity of the disorder — otherwise, the structure factor displays at small \( q \), in addition to the two regimes mentioned, the roughness of a CDW \( \zeta_{\text{CDW}} = 3/2 \) \( \text{(18)} \). This means that the smallest \( q \)-modes have to completely decorrelate on the scale \( M \). The mode \( q \) decorrelates in a time \( \sim q^{-z} \) with \( z \) the dynamic exponent. \( z \) not only controls the typical duration of the avalanches (of size \( \xi(f - f_c) \)), but also the decorrelation time of all length scales \( \text{(22)} \), in particular the string length \( L \). Note that inside the critical window \( L > \xi \). This implies that at a mean velocity of order unity, i.e. at the upper end of the critical window, the smallest mode needs a distance of roughly \( L^z \) to decorrelate. But \( z = 1.5 \) is larger than \( \xi \). We therefore have to choose an \( M \gg L^z \) in our analysis of \( S(q) \) and \( C_v(x) \), in order to eliminate auto-correlation effects in the largest length-scales.

On the contrary, in our initial analysis of the mean velocity (see figure 1), we were justified in adopting a scaling \( M \sim L^z \) because the effective string viscosity — which determines \( \beta \) — is caused by pinning on length scales below \( \xi \). The periodicity \( M \sim L^z \) thus does not influence the velocity exponent \( \beta \).

At very large \( M \), the sample critical force \( f_c^\text{smpl} \) becomes too large to be used as the critical force. We therefore calculate a mean critical force \( f_c^\infty \) from a finite-size-scaling ansatz: The average sample critical force is known to depend on the sample length \( L \) as \( f_c^\infty - f_c^\text{smpl}(L) \sim L^{-1/\nu \beta} \). The asymptotic value \( f_c^\infty = 1.913(2) \) is then used as a mean critical force when analyzing the velocity correlations and the structure factor.

When investigating the correlation length \( \xi \) directly by means of \( C_v(x) \) and \( S(q) \), we do not need to know \( f_c^\text{smpl} \) in order to avoid the cross-over from the critical window into the finite-size-effect region: We simply limit our analysis to sufficiently small values of \( \xi/L \). In addition, the smallest value of \( f - f_c^\infty \) analyzed is larger than typical fluctuations in \( f_c^\text{smpl} \), which are therefore negligible.

In conclusion, we analyze the quasi-static dynamics of the harmonic elastic string driven through a random potential, above the depinning transition. We calculate the exact periodic solution for each sample, and encounter large finite-sample-size effects, which will have repercussions in experimental situations. Knowing exactly the sample critical force enables us to identify the limited window of critical collective behavior and to determine the velocity exponent \( \beta = 0.33(2) \). We investigate the velocity correlation function, determine the correlation length exponent \( \nu = 1.29(5) \), and confirm that it both obeys the statistical tilt symmetry and agrees with the finite-size-scaling correlation length exponent. Surprisingly, we find a non-universal functional form for the velocity correlation function: the exponent \( \kappa \) describing the short distance behavior depends on the control parameter \( f - f_c \).

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[24] In fact, \( z \) governs the convergence of our numerical algorithm \( \text{(cf.} \text{22)} \).