SHARP P-BOUNDS FOR MAXIMAL OPERATORS ON FINITE GRAPHS

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Abstract. Let $G = (V, E)$ be a finite graph and $M_G$ be the centered Hardy-Littlewood maximal operator defined there. We find the optimal value $C_{G,p}$ such that the inequality

$\text{Var}_p(M_Gf) \leq C_{G,p} \text{Var}_p(f)$

holds for every $f : V \to \mathbb{R}$, where $\text{Var}_p$ stands for the $p$-variation, when: (i) $G = K_n$ (complete graph) and $p \in \left[\frac{\log(4)}{\log(6)}, \infty\right)$ or $G = K_4$ and $p \in (0, \infty)$; (ii) $G = S_n$ (star graph) and $1 \geq p \geq \frac{1}{2}$, $p \in (0, \frac{1}{2})$ and $n \geq C(p)$ or $G = S_3$ and $p \in (1, \infty)$. We also find value of the norm $\|M_G\|_2$ when: (i) $G = K_n$ and $n \geq 3$; (ii) $G = S_n$ and $n \geq 3$.

1. Introduction

1.1. A brief historical overview and background. The study of maximal operators is a central theme in analysis. Since the beginning of the past century many properties of these operators have been useful in several areas of mathematics. In general, properties related with the behavior of the norm of these operators have been the main interest of study, until the work of Kinnunen [11] where he observed that it was possible to prove the boundedness of the map

$f \to Mf$

from $W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$, when $p > 1$, (where $M$ stands for the centered Hardy-Littlewood maximal function). He also showed meaningful applications of this property. This work was the first to study maximal operators at a derivative level. Since then many authors followed this path and proved several results concerning these derivative level questions in a broad class of contexts and for several kinds of maximal operators, see for instance [1, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 20].

An interesting framework of study is the following. Let $G = (V, E)$ be a graph with $f : V \to \mathbb{R}$ a real valued function. We define the Hardy-Littlewood maximal function of $f$ along $G$ at the point $e \in V$ by

$M_Gf(e) := \max_{r \geq 0} \frac{1}{|B(e, r)|} \sum_{m \in B(e, r)} |f(m)|,$

(1.1)

where $B(e, r) = \{m \in V; d_G(e, m) \leq r\}$, where $d_G$ is the metric induced by the edges of $G$ (that is, the distance between two vertices is the number of edges in a shortest path connecting them). A more general version of this, is the so called fractional maximal function defined by

$M_{\alpha,G}f(e) := \max_{r \geq 0} \frac{1}{|B(e, r)|^{1-\alpha}} \sum_{m \in B(e, r)} |f(m)|$
for all $\alpha \in (0, 1]$. Both operators have uncentered versions defined by

$$\tilde{M}_{\alpha,G} f(e) = \max_{B(v,r) \ni e} \frac{1}{|B(v,r)|^{1-\alpha}} \sum_{m \in B(v,r)} |f(m)|$$

for the fractional one, and $\tilde{M}_G = \tilde{M}_{0,G}$ for the classical one. In this paper we study the regularity properties of these objects acting on $l^p$–spaces and bounded $p$–variation spaces. We focus on the classical maximal function defined in $(1.1)$.

Discrete properties for maximal operators have caught significant attention along the last years, whether in a discrete context (see, for instance $[6, 17, 18, 23]$) or as an intermediary step towards the solution of a continuous problem (see, for instance $[19]$). The most natural context for the discrete version of the derivative level questions mentioned above is the following, given $p \in (0, \infty)$ we define the $p$-variation of a function $f : V \to \mathbb{R}$ as follows

$$\text{Var}_p f := \left( \frac{1}{2} \sum_n \sum_{d_G(n,m)=1} |f(n) - f(m)|^p \right)^{1/p}.$$

The first work to address a result concerning the derivative level (in this case, the variation) of a maximal operator in a discrete setting was $[2]$, where they, among other things, found sharp constant for the 1-variation of the uncentered Hardy-Littlewood maximal operator $\tilde{M}_Z$, where in $Z$ we take the usual distance. That is, they proved that for every $f : Z \to \mathbb{R}$, we have

$$\text{Var}_1(\tilde{M}_Z f) \leq \text{Var}_1(f),$$

and that the constant in front of $\text{Var}_1 f$ (1 in this case) is sharp. In the following we use the notation $\text{Var}_1 =: \text{Var}.$

Other kind of graphs were studied by Soria and Tradacete in $[21]$, where sharp $l^p$–bounds for maximal operators on finite graphs were first obtained. Later, some other regularity properties of maximal functions on graphs were studied by those authors in $[22]$. More recently, bounds for the $p$–variation of the maximal functions on finite graphs were established by Liu and Xue in $[16]$. Finding optimal bounds for both the $l^p$–norm of the maximal functions and the $p$-variation of the maximal functions acting on finite graphs is a very interesting and challenging problem. In this paper we make progress on this kind of problem.

1.2. Conjectures and results for the $p$-variation in finite graphs. For a given graph $G = (V, E)$ and $0 < p < \infty$, we define

$$C_{G,p} := \sup_{f: V \to \mathbb{R}; \text{Var}_p f > 0} \frac{\text{Var}_p M_G f}{\text{Var}_p f}.$$

Liu and Xue ($[16]$) obtained optimal results for $n = 3$ and for the general case $n > 3$ they found some bounds and posed some interesting conjectures. More precisely, they proved that if $G$ is the complete graph with $n$ vertices $K_n$ or the star graph with $n$ vertices $S_n$, then

$$1 - \frac{1}{n} \leq C_{G,p} \leq 1$$

for $0 < p < \infty$, and for $n = 3$ the lower bound becomes an equality. Moreover, Liu and Xue posed the following conjectures [See $[16]$, Conjecture 1].

Conjecture A (for the complete graph $K_n$): For every $n \geq 2$ and $p \in (0, \infty)$ we have

$$C_{K_n,p} = 1 - \frac{1}{n}.$$
In this paper we give a positive answer to this conjecture for all $p \geq \frac{\log 4}{\log 6} \approx 0.77$. This range is certainly not optimal and is an interesting problem to try to extend it. Also, we prove the conjecture for every $0 < p < 1$ when $n = 4$. That is the content of our Theorem 1.

**Theorem 1** (Complete graph). Let $0 < p \leq \infty$ and $K_n = (V, E)$ a complete graph with $n$ vertices $(a_1, a_2, \ldots, a_n)$. Then

(i) If $p \geq 1$, then
\[ C_{K_n, p} = 1 - \frac{1}{n} \]

(ii) If $0 < p < 1$ and $n = 4$, or $n \geq 3$ and $1 > p \geq \frac{\log(4)}{\log(6)} \approx 0.77$, then
\[ C_{K_n, p} = 1 - \frac{1}{n}. \]

Moreover, in both cases the function $\delta_{a_2}$ is an extremizer.

We notice that given the different behavior of the function $x \rightarrow x^p$ when $p \geq 1$ and $p < 1$ very contrasting techniques are needed in each case. Also, we observe that proving (2.10) in a larger range implies a proof of the second assertion of Theorem 1(ii) in the same range. This is the case because the remaining of the proof is independent of the condition $p \geq \frac{\log(4)}{\log(6)}$.

The second conjecture that they posed is the following.

**Conjecture B** (for the star graph $S_n$): For any $n \geq 2$ and $p \in (0, 1]$ we have
\[ C_{S_n, p} = 1 - \frac{1}{n}. \]

In this case we prove that, in fact, this equality is not true for $p > 1$. In fact, for $n = 3$, we find some bounds different to the ones conjectured in that case. However, we give a positive answer to this conjecture when $1/2 \leq p \leq 1$ for all $n \geq 2$. Moreover, we give a positive answer to the conjecture when $0 < p < 1/2$ if $n$ is sufficiently large, this is the content of our Theorem 2.

**Theorem 2** (Star graph). Let $S_n = (V, E)$ be a star graph with $n$ vertices $(a_1, a_2, \ldots, a_n)$, with center at $a_1$. Then, the following hold.

(i) For all $1 < p \leq \infty$ we have that
\[ C_{S_n, p} = \frac{(1 + 2p/(p-1))((p-1)/p)}{3}. \]  

(ii) If $p = 1$, then
\[ C_{S_n, p} = 1 - \frac{1}{n}. \]  

(iii) If $n = 4$ and $0 < p < 1$, or $n \geq 5$ and $1/2 \leq p \leq 1$, then
\[ C_{S_n, p} = 1 - \frac{1}{n}. \]  

Moreover, (1.4) holds for every $\frac{1}{2} > p > 0$ when $n \geq C(p)$, for some finite constant $C(p)$ depending only on $p$.

The range $(\frac{1}{2}, 1)$ in (iii) is certainly not optimal, to find improvements on this range is an interesting problem.

**Conjecture C** (boundedness and continuity): Let $0 < p, q \leq \infty$ and $0 \leq \alpha < 1$. The operator $M_{\alpha,G}$ is bounded
and continuous from $BV_p(V)$ to $BV_q(V)$, where $BV_p(V) := \{ f : V \to \mathbb{R} \}$ is endowed with $\| f \|_{BV_p(V)} := \text{Var}_p f$ (note that $\| \cdot \|_{BV_p(V)}$ depends on $G$).

We prove that with a slight modification this affirmation is true. We also prove that a modification is strictly required. That is the content of our next theorem.

**Theorem 4.** Let $G_n = (V, E)$ be a graph with $n$ vertices $(a_1, a_2, \ldots, a_n)$. The following statements hold.

(i) **[Boundedness]** Let $\alpha \in [0, 1)$. For all $0 < p, q \leq \infty$ there exists a constant $C(n, p, q) > 0$ such that

$$\text{Var}_q M_{\alpha, G_n} f \leq C(n, p, q) \text{Var}_p f. \tag{1.5}$$

for all functions $f : V \to \mathbb{R}$.

(ii) **[Continuity]** Let $0 < p, q \leq \infty$. Consider a sequence of functions $f_j : V \to \mathbb{R}$ such that $\| f_j - f_0 \|_{BV_p(V)} \to 0$ as $j \to \infty$.

1. Assuming that $\lim_{j \to \infty} \min_{x \in V} |f(x) - f_j(x)| = 0$. Then

$$\text{Var}_q (M_{\alpha, G_n} f - M_{\alpha, G_n} f_j) \to 0 \text{ as } j \to \infty. \tag{1.6}$$

2. $(1.6)$ could fail to be true without the extra assumption that $\lim_{j \to \infty} \min_{x \in V} |f(x) - f_j(x)| = 0$.

1.3. **Optimal $l^p$ bounds for maximal operators on finite graphs.** We are also interested in the $l^p$ norm of $M_G$ when acting on finite graphs. That is, to find the exact value of the expression

$$\sup_{f : V \to \mathbb{R}, f \neq 0} \frac{\| M_G f \|_p}{\| f \|_p} =: \| M_G \|_p,$$

where $\| g \|_p := (\sum_{e \in V} |g(e)|^p)^{\frac{1}{p}}$, for $g : V \to \mathbb{R}$.

These norms were first treated by Soria and Tradacete, who found $\| M_G \|_p$ when $G = S_n$ and $G = K_n$, where $p \in (0, 1)$ (see [21] Proposition 2.7] and [21] Theorem 3.1). Their results rely strongly in Jensen’s inequality for the function $x \to x^p$ where $p \leq 1$, so those methods are not available when $p > 1$. In fact, they claimed that this problem was difficult when $p > 1$ (see [21] Remark 2.8]). The following inequality was proved by Soria and Tradacete [See [21], Proposition 2.7]

$$\left(1 + \frac{n - 1}{n^2}\right)^{1/2} \leq \| M_{K_n} \|_2 \leq \left(1 + \frac{n - 1}{n}\right)^{1/2}.$$ 

Our next result is a formula for the precise value of $\| M_{K_n} \|_2$ for $n \geq 2$. We also find extremizers for all $n \geq 2$. Moreover, we prove that $\| M_{K_{3n}} \|_2 = \| M_{K_3} \|_2$, for all $n \geq 2$. We list these results as follows.

**Theorem 4.** Let $K_n = (V, E)$ be the complete graph with $n$ vertices $V = \{ a_1, a_2, \ldots, a_n \}$. Then we have

$$\| M_{K_n} \|_2 = \max_{k \in \{ \frac{n}{3}, \ldots, \frac{n}{2} \}} \left(1 - \frac{k}{2n} + \frac{(4kn - 3k^2)^{1/2}}{2n}\right)^{1/2}.$$

In particular, we have.

**Corollary 5.** If $n = 3m$ for some $m \in \mathbb{N}$, then $\| M_{K_{3n}} \|_2 = (\frac{4}{3})^{1/2}$. For $n = 2$ we have $\| M_{K_2} \|_2 = (\frac{3}{2})^{1/2}$. 

Similarly, the following inequality was also proved by Soria and Tradacete [See [21], Proposition 3.4]

$$\left(1 + \frac{n - 1}{4}\right)^{1/2} \leq \| M_{S_n} \|_2 \leq \left(\frac{n + 5}{2}\right)^{1/2}.$$
Our next result is a formula for the precise value of $\|M_{S_n}\|_2$. Moreover, we find some extremizers.

**Theorem 6.** Let $n \geq 4$ and $S_n = (V, E)$ be the star graph with $n$ vertices $V = \{a_1, a_2, a_3, \ldots, a_n\}$ and center at $a_1$. Then, the following holds.

$$\|M_{S_n}\|_2 = \left(1 + \frac{n - 4}{8} + \frac{(n^2 + 8n)^{1/2}}{8}\right)^{1/2}. \quad (1.7)$$

**Remark 7.** It was observed by Soria and Tradecete that in the case $n = 2$ the optimal constant is $\left[3 + 5^{1/2}\right]^{1/2}/2$. [See remark 2.8 in [21]], this coincides with our formula (1.7).

2. **Proof of optimal bounds for the $p$-variation of maximal functions.**

We start by proving our results on $K_n$.

2.1. **Optimal bounds for the $p$-variation on $K_n$: Proof of Theorem 1.** For every result listed in Theorem 1 we can see that, taking $f = \delta_{a_1}$ in the definition of $C_{K_n,p}$, we have the following.

$$C_{K_n,p} \geq 1 - \frac{1}{n}. \quad (2.1)$$

**Proof of Theorem 1 (i).** We assume without loss of generality that $f$ is non-negative. Let

$$m := m_n := \sum_{i=1}^{n} f(a_i) / n,$$

and for all $k \in \{1, 2, \ldots, n-1\}$ we define

$$m_k = \sum_{i=1}^{k} f(a_i) / k.$$

Reordering if necessary, we can assume without loss of generality that

$$f(a_n) \geq f(a_{n-1}) \geq \cdots \geq f(a_r) \geq m > f(a_{r-1}) \geq \cdots \geq f(a_1),$$

thus we have that

$$M_{K_n} f(a_i) = f(a_i) \forall i \geq r \quad \text{and} \quad M_{K_n} f(a_i) = m \forall i < r.$$ Observe that $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_{n-1} \leq m$. Therefore

$$(\text{Var}_p M_{K_n} f)^p \leq (n-1)(f(a_n) - m)^p + (n-2)(f(a_{n-1}) - m)^p + \cdots + (r-1)(f(a_r) - m)^p \leq (n-1)(f(a_n) - m)^p + (n-2)(f(a_{n-1}) - m_{n-1})^p + \cdots + (r-1)(f(a_r) - m_r)^p. \quad (2.2)$$

Then, we note that by H"older inequality

$$f(a_i) - m_i \leq \sum_{t=1}^{i} |f(a_t) - f(a_i)| \leq \left(\sum_{t=1}^{i} |f(a_t) - f(a_i)|^p / i\right)^{1/p} (i-1)^{1/p'}.$$
Combining the two previous estimatives we obtain

\[
(\text{Var}_p M_{K_n} f)^p \leq (n - 1)(f(a_n) - m)^p + (n - 2)(f(a_{n-1}) - m_{n-1})^p \\
\cdots + (r - 1)(f(a_r) - m_r)^p \\
\leq (n - 1) \left( \frac{\sum_{t=1}^{n-1} |f(a_n) - f(a_t)|^p}{n^p} \right)^{p/p'} (n - 1)^{p/p'} \\
+ (n - 2) \left( \frac{\sum_{t=1}^{n-2} |f(a_{n-1}) - f(a_t)|^p}{(n - 1)^p} \right)^{p/p'} (n - 2)^{p/p'} \\
\cdots + (r - 1) \left( \frac{\sum_{t=1}^{r-1} |f(a_r) - f(a_t)|^p}{r^p} \right)^{p/p'} (r - 1)^{p/p'} \\
\leq \left( \frac{n - 1}{n} \right)^p \sum_{t=1}^{n-1} |f(a_n) - f(a_t)|^p \\
+ \left( \frac{n - 2}{n - 1} \right)^p \sum_{t=1}^{n-2} |f(a_{n-1}) - f(a_t)|^p \\
\cdots + \left( \frac{r - 1}{r} \right)^p \sum_{t=1}^{r-1} |f(a_r) - f(a_t)|^p \\
\leq \left( \frac{n - 1}{n} \right)^p (\text{Var}_p f)^p.
\]

From where we conclude (2.1) in this case. Concluding the proof of this assertion of Theorem 1.

\[\square\]

**Proof of Theorem 1 (ii).** We keep the notation of the previous proof and the assumption that

\[f(a_n) \geq \ldots f(a_r) \geq m \geq \ldots f(a_1)\]

We assume for the remaining of the proof that \(0 < p < 1\). The simplest case of the theorem (that holds in full generality for \(0 < p < 1\)), that is when \(r = n\), can be proved directly by

\[(n - 1)|f(a_n) - m|^p \leq (n - 1) \left| \frac{n - 1}{n} (f(a_n) - f(a_1)) \right|^p \\
\leq \left( \frac{n - 1}{n} \right)^p (|f(a_n) - f(a_1)|^p + \sum_{i=2}^{n-1} (|f(a_n) - f(a_i)|^p + |f(a_i) - f(a_1)|^p) \\
\leq \left( \frac{n - 1}{n} \right)^p (\text{Var}_p f)^p,
\]

where, in the second inequality, we used that if \(a, b \geq 0\) then \((a + b)^p \leq a^p + b^p\). So, in the following we assume that \(r < n\).

Now we prove the assertion for \(n = 4\).

**Case** \(n = 4\). Since the case \(r = 4\) was already solved, we have two cases left. First we treat the case \(r = 3\).

**Case** \(r = 3\). We have the following inequality.

\[
\left( \frac{3}{4} \right)^p (|f(a_4) - f(a_3)|^p + |f(a_3) - f(a_2)|^p + |f(a_2) - f(a_1)|^p) \geq |f(a_4) - f(a_3)|^p + |f(a_3) - f(a_2)|^p. \quad (2.3)
\]
Step 1: Proving (2.3). In order to prove this, we write \( f(a_3) - f(a_2) = x \) and \( f(a_4) - f(a_3) = y \), then
\[
m = \frac{f(a_1) + 3f(a_2) + 2x + y}{4}
\]
and
\[
\frac{f(a_1) + 3f(a_2) + 2x + y}{4} \leq f(a_2) + x \implies f(a_1) + y \leq f(a_2) + 2x,
\]
also
\[
m \geq f(a_2) \implies f(a_2) \leq f(a_1) + 2x + y.
\]

Then
\[
\left( \frac{3}{4} \right)^p (|f(a_4) - f(a_3)|^p + |f(a_3) - f(a_2)|^p + |f(a_2) - f(a_1)|^p) = \left( \frac{3}{4} \right)^p (y^p + (f(a_2) - f(a_1))^p + x^p),
\]
Consider first the case where \( f(a_2) - f(a_1) + 2x \leq 4y \). Here, by Karamata’s inequality we have
\[
(3y)^p + (f(a_2) - f(a_1) + 2x)^p \geq (4y)^p + (f(a_2) - f(a_1) + 2x - y)^p,
\]
then since \((3x)^p + (3(f(a_2) - f(a_1)))^p \geq (f(a_2) - f(a_1) + 2x)^p \) we have
\[
(3y)^p + (3x)^p + (3(f(a_2) - f(a_1)))^p \geq (4y)^p + (f(a_2) - f(a_1) + 2x - y)^p,
\]
from where (2.4) follows.

In the other case, where \( f(a_2) - f(a_1) + 2x \geq 4y \), considering that
\[
(4^p - 3^p) \left( \frac{f(a_2) - f(a_1)}{4} + \frac{x}{2} \right)^p + (f(a_2) - f(a_1) + 2x)^p \geq (4^p - 3^p)(y^p) + (f(a_2) - f(a_1) + 2x - y)^p,
\]
we have that (2.4) follows by
\[
(3x)^p + (3(f(a_2) - f(a_1)))^p \geq \left( \frac{4^p - 3^p}{4^p} + 1 \right) (f(a_2) - f(a_1))^p + \left( \frac{4^p - 3^p}{2^p} + 2^p \right) x^p
\]
\[
\geq (4^p - 3^p) \left( \frac{f(a_2) - f(a_1)}{4} + \frac{x}{2} \right)^p + (f(a_2) - f(a_1) + 2x)^p,
\]
where in the first inequality we use that \( 3^p \geq \frac{4^p - 3^p}{4^p} + 1 \) and \( 3^p \geq \frac{4^p - 3^p}{2^p} + 2^p \), both consequences of the inequality
\[
6^p + 3^p = e^{\log(6)p} + e^{\log(3)p} \geq 2e^{\log(18)p} = 2(4)^p.
\]
In the second inequality we use
\[
(4^p - 3^p) \left( \frac{f(a_2) - f(a_1)}{4} + \frac{x}{2} \right)^p \leq \frac{4^p - 3^p}{4^p} (f(a_2) - f(a_1))^p + \frac{4^p - 3^p}{2^p} x^p
\]
and
\[
(f(a_2) - f(a_1) + 2x)^p \leq (f(a_2) - f(a_1))^p + 2^px^p.
\]
In the following, we also need the inequality
\[
\left( \frac{3}{4} \right)^p (|f(a_4) - f(a_2)|^p + |f(a_3) - f(a_1)|^p) \geq |f(a_4) - m|^p + |f(a_3) - m|^p.
\] (2.5)
Step 2: Proving (2.5). We have that (2.5) is equivalent to

\[(3x + 3y)^p + (3x + 3(f(a_2) - f(a_1)))^p \geq (f(a_2) - f(a_1) + 2x + 3y)^p + (f(a_2) - f(a_1) + 2x - y)^p.\]

Here we distinguish among two cases, the first when \(x + 4y \geq f(a_2) - f(a_1)\). Here, by the concavity of the function \(x \to x^p\), since

\[4x + 2(f(a_2) - f(a_1)) + 4y \geq 3x + 3y \geq 2x + f(a_2) - f(a_1) - y,\]

we have

\[(3x + 3y)^p + (3x + 3(f(a_2) - f(a_1)))^p \geq (4x + 2(f(a_2) - f(a_1)) + 4y)^p + (2x + (f(a_2) - f(a_1)) - y)^p \geq ((f(a_2) - f(a_1)) + 2x + 3y)^p + (f(a_2) - f(a_1) + 2x - y)^p,\]

from where (2.5) follows. In the other case, where \(x + 4y \leq f(a_2) - f(a_1)\), we can prove that

\[(f(a_2) - f(a_1) + 2x + 2y)^p + (f(a_2) - f(a_1) + 2x)^p \geq (f(a_2) - f(a_1) + 2x + 3y)^p + (f(a_2) - f(a_1) + 2x - y)^p, \tag{2.6}\]

by Karamata’s inequality. Also, since \(y \leq \frac{f(a_2) - f(a_1)}{4}\) and \(2x + y \geq f(a_2) - f(a_1)\), we have \(x \geq \frac{3(f(a_2) - f(a_1))}{8}\), therefore we obtain

\[
\log(3x) + \log(3x + 3(f(a_2) - f(a_1))) \geq \log \left( \frac{3}{2} (f(a_2) - f(a_1)) + 2x \right) + \log (f(a_2) - f(a_1) + 2x). \tag{2.7}
\]

Let us observe that

\[
\log(3x) \leq \log (f(a_2) - f(a_1) + 2x) \leq \log \left( \frac{3}{2} (f(a_2) - f(a_1)) + 2x \right),
\]

let us take then \(v := \log (f(a_2) - f(a_1) + 2x) + \log (3/2 (f(a_2) - f(a_1)) + 2x) - \log (3x)\), by Karamata’s inequality (now applied to the convex function \(x \to e^{px}\) and (2.7) we have

\[
e^{p \log(v)} \leq e^{p \log(3x)} + e^{p \log(3x + 3(f(a_2) - f(a_1)))}
\]

and therefore:

\[
(3x + 3y)^p + (3x + 3(f(a_2) - f(a_1)))^p \geq (3x)^p + (3x + 3(f(a_2) - f(a_1)))^p = e^{p \log(3x)} + e^{p \log(3x + 3(f(a_2) - f(a_1)))}
\]

\[
\geq e^{p \log(3/2 (f(a_2) - f(a_1)) + 2x)} + e^{p \log(f(a_2) - f(a_1) + 2x)}
\]

\[
= (3/2 (f(a_2) - f(a_1)) + 2x)^p + (f(a_2) - f(a_1) + 2x)^p
\]

\[
\geq (f(a_2) - f(a_1) + 2x + 2y)^p + (f(a_2) - f(a_1) + 2x)^p,
\]

from where we obtain, by combining this with (2.6), the desired inequality (2.4).

Step 3: Conclusion case \(r = 3\). The case \(r = 3\) then follows by combining (2.5), (2.3) and the inequality \(\frac{3}{4} (f(a_4) - f(a_1)) \geq f(a_4) - m\).

Case \(r = 2\). Here, we have that \(m \leq f(a_2) \to 2x + y \leq f(a_2) - f(a_1)\). We prove first the inequality

\[
|f(a_4) - f(a_3)|^p + |f(a_3) - f(a_2)|^p + |f(a_2) - m|^p \leq \left( \frac{3}{4} \right)^p (|f(a_4) - f(a_3)|^p
\]

\[
+ |f(a_3) - f(a_2)|^p + |f(a_2) - f(a_1)|^p), \tag{2.8}
\]

where (2.8) follows by combining (2.5), (2.3) and the inequality \(\frac{3}{4} (f(a_4) - f(a_1)) \geq f(a_4) - m\).
Step 1: Proving (2.8). Our desired inequality (2.8) is equivalent to

\[(4y)^p + (4x)^p + (f(a_2) - f(a_1) - 2x - y)^p \leq (3y)^p + (3x)^p + (3(f(a_2) - f(a_1)))^p.\]

We observe that

\[(4y)^p + (4x)^p + (f(a_2) - f(a_1) - 2x - y)^p \leq (4y)^p + (4x)^p + (f(a_2) - f(a_1))^p,\]

also, since \(x \leq \frac{f(a_2) - f(a_1)}{2}\) and \(y \leq f(a_2) - f(a_1)\), we have

\[(4^p - 3^p)(x^p + y^p) \leq (4^p - 3^p) \left( \frac{1}{2} + 1 \right) (f(a_2) - f(a_1))^p \leq (3^p - 1)(f(a_2) - f(a_1))^p,\]

because \(4^p + 8^p + 2^p \leq 2(6)^p + 3^p\) by Jensen inequality, from where it follows (2.8). Also, we have that

\[[(f(a_4) - f(a_2))^p + (f(a_3) - f(a_1))^p \left( \frac{3}{4} \right)^p \geq (|f(a_4) - f(a_2)|^p + |f(a_3) - m|^p),\]  \hspace{1cm} (2.9)

Step 2: Proving (2.9). We have that (2.9) is equivalent to

\[(4x + 4y)^p + (f(a_2) - f(a_1) - 2x - y)^p \leq (3x + 3y)^p + (3(x + f(a_2) - f(a_1)))^p,\]

but, since \(4^p + 2^p \leq 2(3)^p\) and \(x + y \leq f(a_2) - f(a_1)\), we have

\[(4^p - 3^p)(x^p + y^p) + (2(f(a_2) - f(a_1)))^p \leq (3(f(a_2) - f(a_1)))^p \leq (3(f(a_3) - f(a_1)))^p,\]

from where (2.9) follows by observing that \(2x \leq f(a_2) - f(a_1)\).

Step 3: Conclusion of \(r = 2\) and \(n = 4\). The case \(r = 2\), and thus our result in \(n = 4\) follows by combining (2.8), (2.9) and the inequality \(f(a_4) - m \leq \left( \frac{3}{4} \right)(f(a_4) - f(a_1))\). We conclude this part.

Now we prove our assertion for general \(n\) and \(p \in [\frac{\log(4)}{\log(6)}, 1]\).

Case \(n \geq 5\) and \(1 > p \geq \frac{\log(4)}{\log(6)}\). We write \(x_i := f(a_i) - f(a_r)\) for \(i = n, \ldots, r + 1, u = f(a_r) - m, y_i = f(a_r) - f(a_i)\) for \(i = r - 1, \ldots, 1\). We have then that \(\sum x_i + nu = \sum y_i\). One key step is to prove the following.

\[\sum_{i=r+1}^{n} x_i^p + (r - 1)u^p \leq \left( 1 - \frac{1}{n} \right)^p \left( \sum_{i=r+1}^{n} x_i^p + \sum_{i=1}^{r-1} y_i^p \right).\]  \hspace{1cm} (2.10)

In order to prove that let us see that, by Karamata’s inequality (since \(y_i \geq u\))

\[\sum_{i=1}^{r-1} y_i^p \geq (r - 2)u^p + \left( \sum_{i=1}^{r-1} y_i - (r - 2)u \right)^p = (r - 2)(u)^p + \left[ (n - r + 2)u + \sum_{i=r+1}^{n} x_i \right]^p,\]

also, by Jensen’s inequality we have \(((n - r + 2)u + \sum_{i=r+1}^{n} x_i)^p \geq 2^{p-1}((n - r + 2)u)^p + (\sum_{i=r+1}^{n} x_i)^p).\]

Therefore,

\[\left( 1 - \frac{1}{n} \right)^p \left( \sum_{i=r+1}^{n} x_i^p + \sum_{i=1}^{r-1} y_i^p \right) \geq \left( 1 - \frac{1}{n} \right)^p \left( \sum_{i=r+1}^{n} x_i^p + (r - 2)u^p + 2^{p-1}((n - r + 2)u)^p + 2^{p-1}(\sum_{i=r+1}^{n} x_i)^p \right),\]
so, in order to get (2.10) is enough (since \((\sum x_i)^p \geq (n-r)^{p-1}(\sum x_i^p)\) by Jensen)

\[
1 \leq \left(1 - \frac{1}{n}\right)^p (1 + 2^{p-1}(n-r)^{p-1})
\]

and

\[
r - 1 \leq \left(1 - \frac{1}{n}\right)^p (r - 2 + 2^{p-1}(n - r + 2)^p).
\]

In order to prove (2.11) is enough (since \((\sum x_i)^p \geq (n-r)^{p-1}(\sum x_i^p)\) by Jensen)

\[
1 \leq \left(\frac{2}{3}\right)^p (1 + 2^{p-1})
\]

and that is equivalent to \(2(3)^p \leq 2^p 3^p\), an elementary fact. Now, to prove (2.12), we observe that (since \(p \geq \frac{\log(4)}{\log(n)}\)),

\[
2^{p-1}(n - r + 2)^p \geq 2^{p-1}(3)^p \geq 2,
\]

and therefore

\[
\left(1 - \frac{1}{n}\right)^p (r - 2 + 2^{p-1}(n - r + 2)^p) \geq \left(1 - \frac{1}{n}\right)^p (r) \geq \left(1 - \frac{1}{n}\right) r \geq (r - 1),
\]

from where (2.10) follows.

Now we do an inductive argument. Assume that inequality (2.1) holds for every \(f \in K_{n-1}\), whenever \(p \geq \frac{\log(4)}{\log(n)}\) (it holds for \(n = 3, 4\)). Then, if \(b_1, \ldots, b_{n-1}\) are the vertex of the \(K_{n-1}\) graph, defining \(\tilde{f}\) as \(\tilde{f}(b_i) = f(a_{i+1})\) for \(i = r, \ldots, n - 1\) and \(\tilde{f}(b_i) = f(a_i)\) for \(i = 1, \ldots, r - 1\). We write \(\tilde{m} = \frac{\sum_{i,j} \tilde{f}(i,j)}{n-1}\), is clear that \(\tilde{m} \leq m\). Then, by the inductive hypothesis, we have

\[
\sum_{i,j \in \{r+1, \ldots, n\}} |f(a_i) - f(a_j)|^p + (r-1) \sum_{i=r+1}^n |f(a_i) - \tilde{m}|^p \leq \left(1 - \frac{1}{n-1}\right)^p \left( \sum_{i,j \in \{1, \ldots, r-1, r+1, \ldots, n\}} |f(a_i) - f(a_j)|^p \right),
\]

therefore (since \(f(a_i) - m \leq f(a_i) - \tilde{m}\) for \(i = r + 1, \ldots, n\)) we have

\[
\sum_{i,j \in \{r+1, \ldots, n\}} |f(a_i) - f(a_j)|^p + (r-1) \sum_{i=r+1}^n |f(a_i) - m|^p \leq \left(1 - \frac{1}{n}\right)^p \left( \sum_{i,j \in \{1, \ldots, r-1, r+1, \ldots, n\}} |f(a_i) - f(a_j)|^p \right).
\]

Combining this with (2.10) we conclude

\[
\sum_{i,j \in \{r, \ldots, n\}} |f(a_i) - f(a_j)|^p + (r-1) \sum_{i=r}^n |f(a_i) - m|^p \leq \left(1 - \frac{1}{n}\right)^p \left( \sum_{i,j \in \{1, \ldots, n\}} |f(a_i) - f(a_j)|^p \right),
\]

that is equivalent to (2.1) in this case. This concludes the proof of our theorem.

\[\Box\]

2.2. Optimal bounds for the \(p\)-variation on \(S_n\): Proof of Theorem 2 Now we deal with the problems related to the \(p\)-variation of the maximal operator in \(S_n\).

Proof of Theorem 2 (i). We assume without loss of generality that \(f\) is non-negative. We analyse three different cases. Case 1: \(f(a_1) \geq \max\{f(a_2), f(a_3)\}\).

In this case we have that \(M_{S_3} f(a_1) = f(a_1)\), then
(\text{Var}_p M_{S_3} f)^p \leq \left( f(a_1) - \frac{f(a_1) + f(a_2)}{2} \right)^p + \left( f(a_1) - \frac{f(a_1) + f(a_3)}{2} \right)^p \\
\leq \frac{1}{2p}(\text{Var}_p f)^p.

\textit{Case 2: } f(a_1) \leq \min\{f(a_2), f(a_3)\} . We assume without loss of generality that \( f(a_1) \leq f(a_3) \leq f(a_2) \). Then

\begin{align*}
(\text{Var}_p M_{S_3} f)^p &= \left( f(a_2) - \frac{f(a_1) + f(a_2) + f(a_3)}{3} \right)^p + \left( f(a_3) - \frac{f(a_1) + f(a_2) + f(a_3)}{3} \right)^p \\
&= \left( f(a_2) - \frac{f(a_1) + f(a_2) + f(a_3)}{3} \right)^p + \left( f(a_3) - \frac{f(a_1) - (f(a_2) - f(a_3))}{3} \right)^p \\
&= \left( \frac{2(f(a_2) - f(a_1)) - (f(a_3) - f(a_1))}{3} \right)^p + \left( \frac{2(f(a_3) - f(a_1)) - (f(a_2) - f(a_1))}{3} \right)^p \\
&\leq \left( \frac{2(f(a_2) - f(a_1)) - (f(a_3) - f(a_1))}{3} \right)^p + \left( \frac{2(f(a_3) - f(a_1)) - (f(a_2) - f(a_1))}{3} \right)^p \\
&\leq \frac{1 + 2'^p/p'}{3p}(\text{Var}_p f)^p.
\end{align*}

Where we have used the fact that \( p > 1 \) in the fourth line and the final step follows by Hölder’s inequality.

\textit{Case 3: } \min\{f(a_2), f(a_3)\} < f(a_1) < \max\{f(a_2), f(a_3)\} . We assume without loss of generality that \( f(a_3) < f(a_1) < f(a_2) \). Then, since \( p > 1 \), by Holder inequality we have

\begin{align*}
(\text{Var}_p M_{S_3} f)^p &= (f(a_2) - M_{S_3} f(a_1))^p + \left( M_{S_3} f(a_1) - \frac{f(a_1) + f(a_2) + f(a_3)}{3} \right)^p \\
&\leq \left( f(a_2) - \frac{f(a_1) + f(a_2) + f(a_3)}{3} \right)^p \\
&= \left( \frac{2(f(a_2) - f(a_1)) + (f(a_1) - f(a_3))}{3} \right)^p \\
&\leq \frac{(1 + 2'^p/p')}{3p}(\text{Var}_p f)^p.
\end{align*}

This conclude the proof of

\[ C_{S_3, p} \leq \frac{(1 + 2^{p/(p-1)})(p-1)/p}{3}. \]

in (1.2). Finally, we observe that

\[ C_{S_3, p} \geq \frac{(1 + 2^{p/(p-1)})(p-1)/p}{3}. \]  

(2.13)

For that we consider the function \( f : V \to \mathbb{R} \) defined by

\[ f(a_3) = 2, f(a_1) = 3 \text{ and } f(a_2) = 3 + 2^{1/p}. \]
Then, \( \text{Var}_p f = (1 + 2^{\frac{1}{p'}})^{\frac{1}{p'}} \). Moreover,
\[
M_{S_3} f(a_2) = f(a_2) = 3 + 2^{\frac{1}{p'}} \quad \text{and} \quad M_{S_3} f(a_3) = M_{S_3} f(a_1) = \frac{2 + 3 + 2^{\frac{1}{p'}}}{3}.
\]
Thus
\[
\text{Var}_p M_{S_3} f = M_{S_3} f(a_2) - M_{S_3} f(a_1) = \frac{1 + 2^{\frac{1}{p'}}}{3}.
\]
Therefore
\[
\frac{\text{Var}_p M_{S_3} f}{\text{Var}_p f} = \frac{(1 + 2^{\frac{1}{p'}})^{\frac{1}{p'}}}{3}.
\]
So, we obtain (2.13) and thus (1.2).

The proof of the previous result provides an example where the value
\[
\sup_{f: V \rightarrow \mathbb{R}; \text{Var}_p f > 0} \frac{\text{Var}_p M_{S_n} f}{\text{Var}_p f}
\]
is not attained by any Dirac delta. This is a sign of the complexity of this problem when \( p > 1 \), since is not clear how the extremizers should behave for \( n > 3 \).

In the case \( p = 2 \), an interesting example is the following: let \( S_n = (V, E) \) as in the Theorem consider the function \( f: V \rightarrow \mathbb{R} \) defined by
\[
f(a_1) = n, \quad f(a_2) = n + (n - 1), \quad \text{and} \quad f(a_i) = n - 1 \quad \text{for all} \quad 3 \leq i \leq n.
\]
In this case
\[
M_{S_n} f(a_2) = n + (n - 1) \quad \text{and} \quad M_{S_n} f(a_i) = n + \frac{1}{n} \quad \text{for all} \quad i \neq 2.
\]
Then
\[
\frac{\text{Var}_2 M_{S_n} f}{\text{Var}_2 f} = \frac{n - 1 - \frac{1}{n}}{(n - 1)^2 + (n - 2)^{1/2}} = \frac{[(n - 1)^2 + (n - 2)]^{1/2}}{n} > \frac{n - 1}{n}.
\]
This provides further evidence to the fact that in general the extremizers on \( S_n \) are different when \( p > 1 \) than when \( p \leq 1 \).

Now we deal with the next assertion of our theorem. Taking \( f = \delta_{a_2} \) on the definition of \( C_{S_n, p} \) we have that
\[
C_{S_n, p} \geq 1 - \frac{1}{n}.
\]
In the following we prove the inequality
\[
C_{S_n, p} \leq 1 - \frac{1}{n}, \quad (2.14)
\]
from where both assertion follows. This inequality is equivalent to
\[
\text{Var}_p M_{S_n} f \leq (1 - \frac{1}{n}) \text{Var}_p f, \quad (2.15)
\]
for all functions \( f: V \rightarrow \mathbb{R} \).

Proof of Theorem (ii). We assume without loss of generality that \( f \) is non-negative. Let
\[
m = \frac{1}{n} \sum_{i=1}^{n} f(a_i).
\]
Then
\[
\text{Var}_p M_{S_n} f = \sum_{i=2}^{n} |M_{S_n} f(a_i) - M_{S_n} f(a_1)|
\]
\[\sum_{M_{u_n}(a_i) > M_{u_n}(f(a_1))} M_{s_n} f(a_i) - M_{s_n} f(a_1) + \sum_{M_{u_n}(f(a_1)) > M_{u_n}(f(a_1))} M_{s_n} f(a_1) - M_{s_n} f(a_i)\]
\[\leq \sum_{M_{u_n}(f(a_1)) > M_{u_n}(f(a_1))} f(a_1) - m + \sum_{M_{u_n}(f(a_1)) > M_{u_n}(f(a_1))} f(a_1) - m\]
\[= \sum_{M_{u_n}(f(a_1)) > M_{u_n}(f(a_1))} \left[ \frac{n-1}{n} (f(a_1) - f(a_1)) + \sum_{j \neq 1} \frac{f(a_1) - f(a_j)}{n} \right] + \sum_{M_{u_n}(f(a_1)) > M_{u_n}(f(a_1))} \frac{n}{n} f(a_1) - f(a_k)\]
\[\leq \frac{n-1}{n} \text{Var} f,\]
from where (2.11) follows and therefore our result. \(\square\)

**Proof of Theorem 2 (iii).** We write \(f(a_2) \geq \cdots \geq f(a_r) \geq m > f(a_{r+1}) \geq \cdots \geq f(a_n)\). We distinguish among two cases, the first being \(f(a_1) \leq m\).

**Case 1:** \(f(a_1) \leq m\). In this case it is enough to prove inequality (2.15) when \(f(a_i) < f(a_1)\) for \(i > r\). In fact, if (2.15) fails for some \(f\) with \(f(a_i) > f(a_1)\) and \(i > r\), it also fails for the function \(\tilde{f}\) defined by \(\tilde{f}(e) = f(e)\) for every \(e \notin \{a_2, a_1\}\), \(\tilde{f}(a_i) = 2f(a_1) - f(a_i)\) and \(\tilde{f}(a_2) = f(a_2) + f(a_i) - f(a_i)\). This holds because
\[(1 - \frac{1}{n})^p f(a_2) - f(a_1) < f(a_2) - m^p \geq (1 - \frac{1}{n})^p f(a_2) - f(a_1)^p - (f(a_2) - m)^p,\]
and this is the case because
\[|\tilde{f}(a_2) - \tilde{f}(a_1)|^p - |f(a_2) - f(a_1)|^p \leq |\tilde{f}(a_2) - \tilde{f}(a_1)|^p - |f(a_2) - m|^p,\]
inequality that follows because \(\tilde{f}(a_1) = f(a_1)\), the concavity of the function \(x \to x^p\) (and thus the function \(x \to (x + c)^p - x^p\) is decreasing for \(x, c > 0\)) and the fact that \(f(a_1) \leq m\). Iterating the previous argument we get the desired reduction. We write \(f(a_i) - m = x_i\) for \(i = 2, \ldots, r\); \(m - f(a_1) = u\) and \(y_i = f(a_1) - f(a_i)\) for \(i = r + 1, \ldots, n\). Observe that given our reduction we have \(y_i \geq 0\). We observe that
\[\sum_{i=2}^{r} x_i = u + \sum_{i=r+1}^{n} (u + y_i),\]
from where we obtain \(u \leq \frac{\sum_{i=2}^{r} x_i}{n-r+1}\). Also, let us observe that (2.15) is equivalent in this case to
\[\sum_{i=2}^{r} |x_i| \leq \left(1 - \frac{1}{n}\right)^p \left(\sum_{i=2}^{r} |x_i + u|^p + \sum_{i=r+1}^{n} |y_i|^p\right),\]
but since \(\sum_{i=r+1}^{n} |y_i|^p \geq \sum_{i=r+1}^{n} |x_i - (n-r+1)u|^p\), then, we observe that the function \(g(u) := \sum_{i=2}^{r} |x_i + u|^p + \sum_{i=r+2}^{r} x_i - (n-r+1)u|^p\), for \(u \in [0, \sum_{i=2}^{r} x_i] \) is concave, therefore \(g \geq \min\{g(\sum_{i=2}^{r} x_i / (n-r+1)), g(0)\}\), so, it is enough to prove that
\[\sum_{i=2}^{r} |x_i| \leq \left(1 - \frac{1}{n}\right)^p \left(\sum_{i=2}^{r} |x_i|^p + \sum_{i=2}^{r} |x_i|^p\right),\] (2.16)
and
\[ \sum_{i=2}^{r} |x_i|^p \leq \left( 1 - \frac{1}{n} \right)^p \left( \sum_{i=2}^{r} |x_i + \sum_{i=2}^{r} x_i^{n-r+1} |^p \right), \quad (2.17) \]

for (2.16) we observe that \[ \sum_{i=2}^{r} |x_i|^p \geq \max_{i=2\ldots r} |x_i|^p \geq \frac{\sum_{i=2}^{r} |x_i|^p}{r} \], so
\[ \left( 1 - \frac{1}{n} \right)^p \left( \sum_{i=2}^{r} |x_i|^p \right) \geq \left( \frac{\sum_{i=2}^{r} |x_i|^p}{r} \right)^p \geq \left( \sum_{i=2}^{r} |x_i|^p \right)^p, \]

concluding this inequality.

For (2.17), we notice that \[ x_i + \sum_{i=2}^{r} x_i \geq x_i(1 + \frac{1}{n-r+1}) \], so, since \( (1 - \frac{1}{n})^p(1 + \frac{1}{n-r+1}) \geq 1 \) for \( r \geq 2 \), we conclude this inequality, and therefore this case. Notice that this argument holds for every \( p \in (0, 1) \).

Case 2: \( f(a_1) > m \). Here, we observe that if \( f(a_2) \leq f(a_1) \) then \( |M_{S_n} f(a_1) - M_{S_n} f(a_i)| \leq \frac{|f(a_1) - f(a_i)|}{r} \)
for all \( i \) and thus (2.15) follows in this case. So we can assume that \( f(a_2) > f(a_1) \). Let us take \( k \) such that \( f(a_2) \geq f(a_3) \geq \ldots f(a_k) \geq f(a_1) \geq f(a_{k+1}) \), and \( s \) is the minimum such that \( f(a_1) + f(a_{k+1}) \geq 2m \). Let us write \( u = f(a_1) - m \), \( f(a_1) - f(a_i) = x_i \) for \( i = 2, \ldots, k \) and \( y_i = f(a_1) - f(a_i) \) for \( i = k + 1, \ldots, n \). We observe that \( \sum_{i=2}^{k} x_i + nu = \sum_{i=k+1}^{n} y_i \). Then (2.16) is equivalent to
\[ \sum_{i=2}^{k} \left( x_i + nu \right), \quad (2.18) \]

It is useful to solve first the case \( k = n - 1 \). In this case we observe that \( y_n = \sum_{i=2}^{k} x_i + nu \), here \( \sum_{i=2}^{k} x_i + nu \leq (1 - 1/n)^p(\sum_{i=2}^{k} x_i^{p} + (\sum_{i=2}^{k} x_i + nu)^p) \), from where we conclude this case. This claim follows by
\[ \sum_{i=2}^{k} (nx_i) + (nu)^p \leq \sum_{i=2}^{k} (nx_i) + \sum_{i=2}^{k} x_i + nu \]
\[ (n-1)x_i + (\sum_{i=2}^{k} x_i + nu)(n-1) \],

since \( (n-1)^p - 1 \geq (n^p - (n-1)^p) \), by Jensen’s inequality. So, we assume in the following that \( k \leq n - 2 \).

We observe that \( u \leq \frac{nu}{2} \) for \( i = s + 1 \),…,\( n \), and thus
\[ \sum_{i=2}^{k} (x_i)^p + \sum_{i=k+1}^{n} \left( \frac{y_i}{2} \right)^p + \sum_{s+1}^{n} u^p \leq \sum_{i=2}^{k} (x_i)^p + \sum_{i=k+1}^{n} \left( \frac{y_i}{2} \right)^p, \]

therefore (2.18) would follow if
\[ \sum_{i=2}^{k} (x_i)^p \left( 1 - \left( 1 - \frac{1}{n} \right)^p \right) \leq \left( 1 - \frac{1}{n} \right)^p \left( \frac{1}{2} \right)^p \]
\[ \sum_{i=k+1}^{n} y_i^p \geq (\sum_{i=k+1}^{n} y_i)^p \geq (\sum_{i=2}^{k} x_i)^p \geq (k-1)^{p-1}(\sum_{i=2}^{k} x_i^p) \], so, we need \( (k-1)^{1-p}(1 - (1 - \frac{1}{n})^p) \leq (1 - \frac{1}{n})^p - \frac{1}{2p} \). Since \( k-1 \leq n-3 \) is enough
\[ (n-3)^{1-p} \left( 1 - \left( 1 - \frac{1}{n} \right)^p \right) \leq \left( 1 - \frac{1}{n} \right)^p - \frac{1}{2p}, \quad (2.19) \]
but that is equivalent to
\((n - 3)^{1-p}(n^p - (n - 1)^p) \leq (n - 1)^p - \left(\frac{n}{2}\right)^p\),

the, is enough to prove
\((n - 3)^{1-p}p(n - 1)^{p-1} \leq (n - 1)^p - \left(\frac{n}{2}\right)^p\), \hspace{1cm} (2.20)

or, the stronger bound, \(p \leq (n - 1)^p - \left(\frac{n}{2}\right)^p\). Fixed \(p\), is possible to observe that this last inequality holds for \(n\) big enough. Therefore, we conclude the last statement of Theorem 2 (iii). Now we assume that \(1 > p \geq \frac{1}{2}\).

First observe that for \(n \geq 6\) we have that \(p \leq (n - 1)^p - \left(\frac{n}{2}\right)^p\), in fact \(g(n) = (n - 1)^p - \left(\frac{n}{2}\right)^p\) is increasing for \(n \geq 6\) and \(p > 0\). So, we need to prove \(p \leq 5^p - 3^p\), but \(g(p) = 5^p - 3^p - p\) is increasing for \(p \geq \frac{1}{2}\) and \(\sqrt{5} - \sqrt{3} - \frac{1}{2} \geq 0\). Then, considering [16] Theorem 1.4, the only cases left are \(n = 4\) and \(n = 5\). For \(n = 4\), considering (2.20), we just need
\[\left(\frac{1}{3}\right)^{1-p} p \leq 3^p - 2^p,\]
or, equivalently, \(p \leq 3 - 3\left(\frac{4}{3}\right)^p\), but \(g(p) = 3 - 3\left(\frac{4}{3}\right)^p - p\) is concave in \((0, 1)\), so, since \(g(0) = 0 = g(1)\), we conclude in this case. Notice that this argument holds for every \(1 > p > 0\), and therefore the case \(n = 4\) is completed.

Finally, for \(n = 5\), we just need
\[\left(\frac{1}{2}\right)^{1-p} p \leq 4^p - \left(\frac{5}{2}\right)^p,\]
or equivalently
\[p \leq 2^p - \left(\frac{5}{4}\right)^p,\]
but \(g(p) = 2^p - \left(\frac{5}{4}\right)^p - \frac{p}{2}\) is increasing for \(p \geq \frac{1}{2}\), then since \(\sqrt{2} - \sqrt{\frac{5}{4}} - \frac{1}{2} \geq 0\) we conclude this case. Since we finished the analysis of cases, we conclude the proof of the theorem. \(\square\)

Remark 8. It is possible, in fact, to prove (2.19) for every \(0 < p < 1\) when \(n = 5\), thus proving Theorem 2 (iii) for every \(0 < p < 1\) in this case. We omit the details for simplicity.

2.3. Qualitative results: Proof of Theorem 3 In the last part of this section we prove our versions of the qualitative results conjectured in Conjecture C.

Proof of Theorem 3 (i). We assume without loss of generality that \(f\) is non-negative. Also, in the following we assume that \(G\) is connected, since the general case follows from there. Given \(u, v \in G_n := \{a_1, a_2, \ldots, a_n\}\), such that \(M_{\alpha,G_n} f(u) > M_{\alpha,G_n} f(v)\), we observe that there exists \(k \leq n - 1\) such that
\[M_{\alpha,G_n} f(u) = \frac{k^{\alpha}}{|B(u, k)|} \sum_{a_i \in B(u, k)} f(a_i),\]
then
\[M_{\alpha,G_n} f(u) - M_{\alpha,G_n} f(v) \leq \frac{k^{\alpha}}{|B(u, k)|} \sum_{a_i \in B(u, k)} f(a_i) - \frac{n^{\alpha}}{n} \sum_{i=1}^{n} f(a_i)\]
\[\leq n^{\alpha} \left[\frac{1}{|B(u, k)|} \sum_{a_i \in B(u, k)} f(a_i) - \frac{1}{n} \sum_{i=1}^{n} f(a_i)\right]^{15}\]
\[ \leq n^\alpha(f(x) - f(y)) \]
\[ \leq n^\alpha(n - 1)^{\max(1 - \frac{1}{p}, 0)} \Var_p f \]

Where, in the third line \( x \in G_n \) is choose such that \( f(x) := \max\{f(a_i); a_i \in B(u, k)\} \) and \( y \in G_n \) is choose such that \( f(y) := \min\{f(a_i); a_i \in G_n\} \). In the fourth line we used Hölder inequality.

Therefore

\[
\Var_q M_\alpha, G_n = \left( \frac{1}{2} \sum_{u \in G_n} \sum_{v \in N_{G_n}(u)} |M_\alpha, G_n f(u) - M_\alpha, G_n f(v)|^q \right)^{1/q} 
\leq \left( \frac{n(n - 1)}{2} \right)^{1/q} n^\alpha(n - 1)^{\max(1 - \frac{1}{p}, 0)} \Var_p f 
= C(n, p, q) \Var_p f.
\]

\[ \square \]

**Proof of Theorem** 3 (ii). We start observing that for all \( j \geq 1 \)
\[
\|f - f_j\|_{l^\infty(G_n)} = \max_{y \in V} |f(y) - f_j(y)| - \min_{x \in V} |f(x) - f_j(x)| + \min_{x \in V} |f(x) - f_j(x)| 
\leq \Var(f - f_j) + \min_{x \in V} |f(x) - f_j(x)| 
\leq n^{\max(1 - 1/p, 0)} \Var_p(f - f_j) + \min_{x \in V} |f(x) - f_j(x)|.
\]

Then, assuming that \( \lim_{j \to \infty} \min_{x \in V} |f(x) - f_j(x)| = 0 \), we have that
\[
\|f - f_j\|_{l^\infty(G_n)} \to 0 \text{ as } j \to \infty.
\]

Moreover, for any \( u, v \in G_n \) we have that
\[
M_\alpha, G_n f(u) - M_\alpha, G_n f_j(u) - [M_\alpha, G_n f(v) - M_\alpha, G_n f_j(v)] \leq M_\alpha, G_n(f - f_j)(u) + M_\alpha, G_n(f - f_j)(v) 
\leq 2\|f - f_j\|_{l^1(G_n)} 
\leq 2n\|f - f_j\|_{l^\infty(G_n)} \to 0 \text{ as } j \to \infty.
\]

Therefore
\[
\Var_q(M_\alpha, G_n f - M_\alpha, G_n f_j) \leq \left( \frac{n(n - 1)}{2} \right)^{1/q} 2n\|f - f_j\|_{l^\infty(G_n)} \to 0 \text{ as } j \to \infty.
\]

Finally, we observe that without the assumption that \( \lim_{j \to \infty} \min_{x \in V} |f(x) - f_j(x)| = 0 \) the continuity property could fail, with this purpose in mind consider the following situation: Let \( G_n = S_n \) the star graph with \( n \) vertices \( V = \{a_1, a_2, \ldots, a_n\} \) and center at \( a_1 \), for simplicity we take \( \alpha = 0 \) and \( p = q = 1 \). We define the function \( f \) by \( f(a_1) = 2 \) and \( f(a_i) = 1 \) for all \( i \neq 1 \) thus \( M_{S_n}, f(a_1) = 2 \) and \( M_{S_n}, f(a_i) = 3/2 \) for all \( i \neq 1 \). Then, we consider the sequence of functions \((f_j)_{j \in \mathbb{N}}\) defined by \( f_j(a_1) = f(a_1) - 3 \) for all \( a_i \in V \) and for all \( j \in \mathbb{N} \). Then \( \Var(f - f_j) = 0 \) for all \( j \in \mathbb{N} \), moreover \( M_{S_n}, f_j(a_1) = \frac{1+2(n-1)}{n} \) and \( M_{S_n}, f_j(a_2) = 2 \) for all \( i \neq 1 \). Therefore
\[
\Var(M_{S_n} f - M_{S_n} f_j) \geq M_{S_n} f(a_1) - M_{S_n} f_j(a_1) - [M_{S_n} f(a_2) - M_{S_n} f_j(a_2)] 
= 2 - \frac{1 + 2(n - 1)}{n} - [3/2 - 2]
\]
\[ = \frac{1}{n} + \frac{1}{2} \text{ for all } j \in \mathbb{N}. \]

Then \( \text{Var}(M_{S_n}f - M_{S_n}f_j) \to 0 \) as \( j \to \infty. \)

\[ \square \]

3. PROOF OF OPTIMAL BOUNDS FOR THE 2-NORM OF MAXIMAL FUNCTIONS

In this subsection we prove our results concerning the values \( \|M_G\|_2 \) for our graphs of interest.

3.1. 2-norm of the maximal operator in \( K_n \): Proof of Theorem 4 and Corollary 5. We start by proving that Corollary 5 follows by Theorem 4.

Proof of Corollary 5. The inequality

\[ \|M_{K_n}f\|_2 \leq \left( \frac{4}{3} \right)^{1/2} \|f\|_2 \]

follows from the Theorem 4 since \( k = n/3 \) in the right hand side. On the other hand, we consider the following example: we define \( f : V \to \mathbb{R} \) by

\[ f(a_i) = 4 \text{ for all } 1 \leq i \leq \frac{n}{3} \text{ and } f(a_i) = 1 \text{ for all } \frac{n}{3} + 1 \leq i \leq n. \]

Then, in this case we have

\[ M_{K_n}f(a_i) = 4 \text{ for all } 1 \leq i \leq \frac{n}{3} \text{ and } M_{K_n}f(a_i) = 2 \text{ for all } \frac{n}{3} + 1 \leq i \leq n. \]

Therefore

\[ \|M_{K_n}f\|_2 = \left( \frac{16n^2 + 4\cdot2n}{3n^2} \right)^{1/2} \|f\|_2 = \left( \frac{4}{3} \right)^{1/2} \|f\|_2. \]

\[ \square \]

Now we prove our bound that holds for \( K_n \) for every \( n \geq 2 \).

Proof of Theorem 4. We assume without loss of generality that \( f \) is no-negative. Consider the case

\[ f(a_1) \geq f(a_2) \geq \cdots \geq f(a_k) \geq m \geq f(a_{k+1}) \geq \cdots \geq f(a_n). \]

Then, in this case

\[ M_{K_n}f(a_i) = f(a_i) \text{ for all } 1 \leq i \leq k, \text{ and } M_{K_n}f(a_i) = m \text{ for all } k+1 \leq i \leq n. \]

Therefore

\[ \|M_{K_n}f\|_2^2 = \sum_{i=1}^{k} f(a_i)^2 + (n-k)m^2 \]

\[ = \left( 1 + \frac{n-k}{n^2} \right) \sum_{i=1}^{k} f(a_i)^2 + \frac{n-k}{n^2} \sum_{i=k+1}^{n} f(a_i)^2 \]

\[ + \frac{2(n-k)}{n^2} \sum_{1 \leq i < j \leq k} f(a_i)f(a_j) + \frac{2(n-k)}{n^2} \sum_{k+1 \leq i < j \leq n} f(a_i)f(a_j) \]
where $A_k := 1 + \frac{(n-k)x}{n^2}$ and $B_k := \frac{(n-k)y}{n^2}$. Observe that $A_k - B_k = \frac{3nk - 2k^2}{n^2}$ and by the AM-GM inequality

$$
\|M_{K_n} f\|_2^2 \leq A_k \sum_{i=1}^{k} f(a_i)^2 + B_k \sum_{i=k+1}^{n} f(a_i)^2 + \frac{2(n-k)}{n^2} \sum_{1 \leq i \leq k, k+1 \leq j \leq n} f(a_i) f(a_j),
$$

for all $0 < x, y$ such that $xy = (n-k)^2$. Then, we choose $x, y$ such that

$$
A_k + \frac{(n-k)x}{n^2} = B_k + \frac{ky}{n^2}.
$$

So, $x$ is the positive solution for the equation

$$
(3nk - 2k^2)x + (n-k)x^2 = k(n-k)^2.
$$

More precisely

$$
x := \frac{-(3nk - 2k^2) + (4kn^3 - 3n^2k^2)^{1/2}}{2(n-k)}.
$$

Therefore, combining (3.1) and (3.2) we obtain

$$
\|M_{K_n} f\|_2^2 \leq \max_{k \in [1, n-1]} \left( A_k + \frac{(n-k)x}{n^2} \right) \sum_{i=1}^{n} f(a_i)^2
$$

$$
= \max_{k \in [1, n-1]} \left( 1 + \frac{(n-k)x}{n^2} + \frac{(4kn^3 - 3n^2k^2)^{1/2} - (3nk - 2k^2)}{2n^2} \right) \sum_{i=1}^{n} f(a_i)^2
$$

$$
= \max_{k \in [1, n-1]} \left( 1 - \frac{k}{2n} + \frac{(4kn - 3k^2)^{1/2}}{2n} \right) \sum_{i=1}^{n} f(a_i)^2.
$$

Then, we consider the function $g : [1, n-1] \to \mathbb{R}$ defined by $g(t) := -t + (4tn - 3t^2)^{1/2}$. 

18
Observe that
\[ \max_{t \in [1, n/3]} g(t) = g \left( \frac{n}{3} \right). \]
Moreover, \( g \) is increasing in \([1, n/3]\) and decreasing in \([n/3, n - 1]\). Therefore
\[ \| M_{K_n} f \|^2 \leq \max_{k \in \left[ \frac{3}{n}, \frac{2}{n} \right]} \left( 1 - \frac{k}{2n} + \frac{(4kn - 3k^2)^{1/2}}{2n} \right) \| f \|^2. \quad (3.3) \]

Finally, observe that in order to have an equality in (3.3) it is enough to have equality in (3.1) and (3.2). Moreover, the equality in (3.1) is attained if and only if \( f(a_i) = f(a_1) = \gamma \) for all \( 1 \leq i \leq k \), and \( f(a_j) = f(a_{k+1}) = \eta \) for all \( k + 1 \leq j \leq n \), for some \( 0 < \eta < \gamma \). We can assume without loss of generality that \( \eta = 1 \). On the other hand, the equality in (3.2) is attained if and only if \( y^{1/2} = x^{1/2} \gamma = (n - k)^{1/2} \gamma^{1/2} \), or equivalently \( \gamma = \frac{n-k}{n} \). Therefore, in order to obtain an equality in (3.3) for \( k \in \{ \left\lfloor \frac{n}{3} \right\rfloor, \left\lceil \frac{n}{3} \right\rceil \} \) we consider the function \( g_k : V \to \mathbb{R} \) defined by
\[ g_k(a_i) = \gamma := \frac{2(n-k)^2}{(4kn^3 - 3n^2k^2)^{1/2} - (3nk - 2k^2)} \quad \text{for all} \quad 1 \leq i \leq k, \]
and \( g_k(a_j) = 1 \) for all \( k + 1 \leq j \leq n \). Then, by construction
\[ \| M_{K_n} \|_2 = \max_{k \in \left[ \frac{3}{n}, \frac{2}{n} \right]} \frac{\| M_{K_n} g_k \|_2}{\| g_k \|_2}, \]
this shows that our bound is optimal, moreover we have found extremizers. Observe that, in the particular case when \( n = 3k \), we obtain \( \gamma = 4 \) as in the Corollary. \( \square \)

3.2. 2-norm of the maximal operator in \( S_n \): Proof of Theorem

Now we prove our result concerning the 2-norm of our maximal operator on \( S_n \).

Proof of Theorem \( \square \) As usual we assume without loss of generality that \( f \) is non-negative and we denote by \( m \) the average of \( f \) along \( V \) i.e. \( m = \frac{\sum_{i=1}^{n} f(a_i)}{n} \). We observe that \( M_{S_n} f(a_1) = f(a_1) \) or \( M_{S_n} f(a_1) = m \). We study this two cases separately.

**Case 1:** \( M_{S_n} f(a_1) = f(a_1) \). Assume without loss of generality that \( M_{S_n} f(a_i) = f(a_i) \) for all \( 1 \leq i \leq k \), \( M_{S_n} f(a_i) = \frac{f(a_k) + f(a_{k+1})}{2} \) for all \( k + 1 \leq i \leq k + r \), and \( M_{S_n} f(a_i) = m \) for all \( k + r + 1 \leq i \leq n \). By Cauchy-Schwarz inequality we have
\[ m^2 \leq \frac{\sum_{i=1}^{n} f(a_i)^2}{n}. \]
Then
\[ \| M_{S_n} f \|^2 \leq \left( 1 + \frac{r}{4} \right) f(a_1)^2 + \frac{k}{4} \sum_{i=2}^{k} f(a_i)^2 + \frac{1}{4} \sum_{i=k+1}^{k+r} f(a_i)^2 + \frac{2}{4} \sum_{i=k+1}^{k+r} f(a_i) f(a_1) + \frac{s}{n} \sum_{i=1}^{n} f(a_i)^2 \]
\[ = \left( 1 + \frac{r}{4} + \frac{s}{n} \right) f(a_1)^2 + \left( 1 + \frac{s}{n} \right) \sum_{i=2}^{k} f(a_i)^2 + \left( \frac{1}{4} + \frac{s}{n} \right) \sum_{i=k+1}^{k+r} f(a_i)^2 \]
\[ + \frac{2}{4} \sum_{i=k+1}^{k+r} f(a_i) f(a_1) + \frac{s}{n} \sum_{i=k+r+1}^{n} f(a_i)^2. \]
where \( s := n - k - r \). Moreover, for all \( k + 1 \leq i \leq k + r \), we have that
\[
\frac{2}{4} f(a_i) f(a_1) \leq x f(a_1)^2 + y f(a_i)^2
\]
for all \( x, y > 0 \) such that \( xy \geq \frac{1}{16} \). We can choose \( x \) and \( y \) such that
\[
y - rx = 1 + \frac{r - 1}{4} \quad \text{and} \quad xy = \frac{1}{16}.
\]
or equivalently
\[
x := \frac{[(r + 9)(r + 1)]^{1/2} - (r + 3)}{8r}.
\]

Therefore, for all \( n \geq 4 \) we have
\[
\|M_{s_n} f\|^2 \leq \max_{\{k, r \in \mathbb{N} : 1 \leq k + r \leq n\}} \left( 1 + \frac{n - k - r}{n} + \frac{r}{4} + \frac{[(r + 9)(r + 1)]^{1/2} - (r + 3)}{8} \right) \|f\|^2
\]
\[
\leq \left( 1 + \frac{n - 1}{4} + \frac{(n^2 + 8n)^{1/2} - (n + 2)}{8} \right) \|f\|^2.
\]

**Case 2:** \( M_{s_n} f(a_1) = m \). In this case \( k \geq 2 \). Following the same strategy (and notation), for all \( n \geq 4 \) we obtain that
\[
\|M_{s_n} f\|^2 \leq \left( \frac{r}{4} + \frac{s + 1}{n} \right) f(a_1)^2 + \left( 1 + \frac{s + 1}{n} \right) \sum_{i=2}^{k} f(a_i)^2 + \left( \frac{1}{4} + \frac{s + 1}{n} \right) \sum_{i=k+1}^{k+r} f(a_i)^2
\]
\[
+ \frac{2}{4} \sum_{i=k+1}^{k+r} f(a_i) f(a_1) + \frac{s + 1}{n} \sum_{i=k+r+1}^{n} f(a_i)^2.
\]
\[
\leq \max_{\{k, r \in \mathbb{N} : 1 \leq k + r \leq n\}} \left( \frac{n - k - r + 1}{n} + \frac{r + 1}{4}, \frac{n - k - r + 1}{n} + 1 \right) \|f\|^2
\]
\[
= \max_{\{k, r \in \mathbb{N} : 1 \leq k + r \leq n\}} \left( \frac{n - k - r + 1}{n} + \frac{r + 1}{4}, \frac{n - 1}{n} + 1 \right) \|f\|^2.
\]
The inequality
\[
\|M_{s_n} f\|_2 \leq \left( 1 + \frac{n - 1}{4} + \frac{(n^2 + 8n)^{1/2} - (n + 2)}{8} \right)^{1/2} := C_n
\]
follows from these two estimates.

Finally, we observe that \( \|M_{s_n} \|_2 = C_n \). Consider the function \( g : V \to \mathbb{R} \) defined by \( g(a_i) = 1 \) for all \( 1 \leq i \leq n - 1 \) and \( g(a_0) = \gamma \), where we choose \( \gamma \) to be a positive real number larger than 1, such that \( \gamma \) is a solution for the quadratic equation
\[
aX^2 + bX + c := \left( C_n^2 - 1 - \frac{1}{2} \right) x^2 - \frac{1}{2} x + C_n^2 (n - 1) - \frac{1}{4} = 0.
\]
The existence of \( \gamma \) follows from the definition of \( C_n \), since we can see that \( b^2 - 4ac = 0 \) and \( c > 1 \). More precisely
\[
\gamma = -\frac{b}{2a} = \frac{2(n - 1)}{(n^2 + 8n)^{1/2} - (n + 2)}.
\]
For this particular function we have
\[
\frac{\|M_{s_n} g\|_2}{\|g\|_2} = \left( \frac{\gamma^2 + (n - 1)\left(\frac{2(n - 1)}{\gamma^2 + (n - 1)}\right)^2}{\gamma^2 + (n - 1)} \right)^{1/2} = C_n.
\]
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