Axial anomalies in gauge theory by exact renormalization group method

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Abstract

The global chiral symmetry of a $SU(2)$ gauge theory is studied in the framework of renormalization group (RG). The theory is defined by the RG flow equations in the infrared cutoff $\Lambda$ and the boundary conditions for the relevant couplings. The physical theory is obtained at $\Lambda = 0$. In our approach the symmetry is implemented by choosing the boundary conditions for the relevant couplings not at the ultraviolet point $\Lambda = \Lambda_0 \rightarrow \infty$ but at the physical value $\Lambda = 0$. As an illustration, we compute the triangle axial anomalies.

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In a gauge theory the introduction of a momentum cutoff seems to conflict with the local symmetry and gives rise to the fine tuning problem\[1\], \[2\]. A solution of this problem for a Yang-Mills (YM) theory has been proposed in\[3\], by using an approach based on the renormalization group (RG) method\[4\]-\[7\]. In this formulation one works directly in four space-time dimensions and thus its extension to theories with chiral symmetry is straightforward. This is an advantage with respect to dimensional regularization\[8\], in which one has to define $\gamma_5$ in complex space-time dimensions.

In this paper we study a $SU(2)$ non-Abelian gauge theory with two fermions. Neglecting the fermion masses this theory has the global $U(1)_V \times U(1)_A \times SU(2)_V \times SU(2)_A$ symmetry with four classically conserved currents. This symmetry gives rise to Ward identities for the vertices with current insertions. By using the RG method we study, in perturbation theory, these Ward identities and compute the corresponding anomalies\[9\] for the three point vertices, which require only a one loop calculation.

According to the RG procedure we follow, the needed subtractions are generated by imposing the usual physical conditions on all the relevant parameters of the vertex considered. The question now is how to choose the physical condition for a vertex with current insertions. For each vertex one has a given number of parameters and Ward identities. One then selects the relevant parameters in such a way to satisfy some of the Ward identities. If there are more Ward identities than parameters, then the remaining ones are predicted and typically anomalous.

Before recalling the RG method and the corresponding loop expansion, we describe the theory, its symmetries and Ward identities. We consider the model in which the BRS action in the Feynman gauge is

$$ S_{\text{BRS}} = \int d^4x \left\{ -\frac{1}{4} (F_{\mu\nu} \cdot F_{\mu\nu}) - \frac{1}{2} (\partial_\mu W_\mu)^2 + w_\mu \cdot D_\mu c - \frac{1}{2} \psi \cdot c \wedge c ight. $$
$$ + \left. \sum_{f=1}^2 \left( \bar{\psi}_f (i \mathcal{D} - m_f) \psi_f - \bar{\lambda}_f c \cdot t \psi_f - \bar{\psi}_f c \cdot t \lambda_f \right) \right\}, $$

where

$$ F_{\mu\nu}^a(x) = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g (W_\mu \wedge W_\nu)^a, \quad (W_\mu \wedge W_\nu)^a = \epsilon^{abc} W_\mu^b W_\nu^c, $$
$$ F_{\mu\nu} F_{\mu\nu} = F_{\mu\nu}^a F_{\mu\nu}^a, \quad D_\mu c = \partial_\mu c + g W_\mu \wedge c, \quad D_\mu \psi_f = (\partial_\mu + g W_\mu \cdot t) \psi_f, \quad w_\mu = \frac{1}{g} u_\mu + \partial_\mu \bar{c}, $$

$f$ is the flavour index and $t^a$ are $SU(2)$ matrices in the fundamental representation. This action is BRS invariant and we have added the sources $u_\mu^a$, $v^a$, $\lambda_f$ and $\bar{\lambda}_f$ for the BRS transformations\[10\] of $W_\mu^a$, $c^a$, $\bar{\psi}_f$ and $\psi_f$ respectively. For $m_f \to 0$ this action has the additional global $U(1)_V \times U(1)_A \times SU(2)_V \times SU(2)_A$ symmetry in the flavour space, with the four classically conserved currents

$$ j_A^\alpha = \left\{ \bar{\psi} \gamma_\mu \psi, \quad \bar{\psi} \gamma_\mu \gamma_5 \psi, \quad \bar{\psi} \gamma_\mu \tau^\alpha \psi, \quad \bar{\psi} \gamma_\mu \gamma_5 \tau^\alpha \psi \right\}. $$

We denoted by $\tau^\alpha$ the three $SU(2)$ matrices acting on the two dimensional flavour space, in order to avoid confusion with the colour matrices $t^a$. The index $A$ indicates the flavour
symmetry group, namely \( A = \{1, 2, 3\alpha, 4\alpha\} \). At the quantum level the properties of these currents are studied by introducing the sources \( \zeta^A_{\mu} \)

\[
S_{BRS}^\zeta = S_{BRS} + \int d^4x \zeta^A_{\mu} j^A_{\mu} .
\]  

(1)

From \( S_{BRS}^\zeta \) we can construct the effective action \( \Gamma[\Phi] \), where \( \Phi(x) \) denotes all the fields and sources, namely \( \Phi = (W^a_{\mu}, \psi, \bar{\psi}, c^a, \bar{c}^a, u^a, \bar{u}^a, \lambda, \bar{\lambda}, \zeta^A) \). This functional satisfies the Slavnov-Taylor identities

\[
\int d^4x \left\{ \frac{1}{g} \frac{\delta \Gamma'}{\delta u^a_{\mu}} + \frac{\delta \Gamma'}{\delta \bar{c}^a} - \frac{\delta \Gamma'}{\delta \bar{v}_{\mu}} + \frac{\delta \Gamma'}{\delta \psi} \delta A_{\mu} + \frac{\delta \Gamma'}{\delta \bar{\psi}} \delta A_{\mu} \right\} = 0 ,
\]  

(2)

where \( \Gamma' = \Gamma + \frac{1}{2} \int d^4x (\partial_{\mu} W^a_{\mu})^2 \). On the other side, the Ward identities associated to the chiral symmetry transformations are in general anomalous. Namely, defining the functional

\[
\mathcal{W}^A = \int d^4x \left\{ -i \partial_\mu \frac{\delta \Gamma}{\delta \zeta^A_{\mu}} + \frac{\delta \Gamma}{\delta \psi} T^A \psi + \bar{\psi} T^A \frac{\delta \Gamma}{\delta \bar{\psi}} + f^{ABC} c^B \frac{\delta \Gamma}{\delta \zeta^C} \right\} ,
\]

(3)

we have

\[
\mathcal{W}^A = A^A + \mathcal{O}(m_f) ,
\]

where the \( \mathcal{O}(m_f) \) corrections are present only for \( A = 2 \) or \( 4 \alpha \) (i.e. for \( U(1)_A \) or \( SU(2)_A \)) and for \( A = 3\alpha \) (i.e. for \( SU(2)_V \)) if \( m_f \) is not independent of \( f \). In \( \mathbb{H} \) we have set \( T^A = \{1, \gamma_5, \tau^\alpha, \gamma_5 \tau^\alpha\} \), \( T^A = \{1, -\gamma_5, \tau^\alpha, -\gamma_5 \tau^\alpha\} \) and \( f^{ABC} \) is different from zero only if \( (A, B, C) = (3\alpha, 3\beta, 3\gamma), (3\alpha, 4\beta, 4\gamma), (4\alpha, 3\beta, 4\gamma) \) or \( (4\alpha, 4\beta, 3\gamma) \). In these cases \( f^{ABC} = -i\epsilon^{\alpha\beta\gamma} \). As well known the anomaly \( A^A \) can be set to zero for \( A = 1 \) and 3 (i.e. for \( U(1)_V \) and \( SU(2)_V \)). For \( A = U(1)_A \) or \( SU(2)_A \) there are some anomalous contributions.

The physical parameters (masses, wave function renormalization and coupling constants) are given by fixing at some normalization point the vertex functions with non-negative dimension and possibly their derivatives. A similar thing has to be done for the vertices involving the currents \( j^A_{\mu} \). Also for these one has to identify the relevant parameters (with non-negative dimension) and the irrelevant vertices (with negative dimension). The relevant parameters are present in the following contribution to the effective action

\[
\Gamma^\zeta[W, \bar{\psi}, \psi, \zeta] = \int d^4x \left\{ \zeta^A_{\mu} [\bar{\psi} O^A_{\mu} (\bar{\psi})^A \psi + O^{Aab}_{\mu\nu} W^a_{\nu} W^b_{\mu} + O^{Aabc}_{\mu\nu\rho} W^a_{\nu} W^b_{\rho} W^c_{\mu}] + \zeta^A_{\mu} \zeta^B_{\nu} [\frac{1}{2} O^{AB}_{\mu\nu} + O^{Aab}_{\mu\rho\sigma} W^a_{\rho} W^b_{\sigma} + \zeta^A_{\mu} \zeta^B_{\nu} \zeta^C_{\rho} O^{ABC}_{\mu\rho\sigma} + \zeta^A_{\mu} \zeta^B_{\nu} \zeta^C_{\rho} \zeta^D_{\sigma} O^{ABCD}_{\mu\rho\sigma}] \right\} ,
\]

(4)

which contains only vertices \( O \) with non-negative dimension. We define the relevant parameters only for the three point vertices, since they give the basic anomalies. The decomposition for the other vertices in \( \mathbb{H} \) can be done in an analogous way. We will work in the momentum space. We denote the fermion momenta by the letter \( k \), the gluon momenta by \( q \) and the current momenta by \( p \). For simplicity we fix the subtraction point at vanishing momenta. This is possible for \( m_f \neq 0 \). In general one can use a symmetric subtraction point (3SP), defined by \( \tilde{p}_i \tilde{p}_j = \frac{4}{7} (3\delta_{ij} - 1) \), with some complications in the definition of the vertices.
1) The vertex $O^{Aab}_{\mu\nu\rho}$ is present only for $A = 2$ (i.e. $U(1)_A$), due to charge conjugation invariance, and we have

$$O^{Aab}_{\mu\nu\rho}(p, q, q') = \delta^{A2} \delta^{ab} \left\{ \epsilon_{\mu\nu\rho\sigma} (q - q')_\sigma [\rho_5 + \Sigma_5(p, q, q')] + \Sigma^5_{\mu\nu\rho}(p, q, q') \right\},$$

$$\Sigma_5(0, 0, 0) = 0.$$  

From this condition $\Sigma_5(p, q, q')$ is irrelevant (after factorizing a momentum factor, this vertex has negative dimension). In $\Sigma^5_{\mu\nu\rho}(p, q, q')$ the Lorentz indices are carried by the momentum, then this vertex is irrelevant. The ST identities (2) imply for this vertex the transversality of the gluons

$$q_\nu O^{Aab}_{\mu\nu\rho}(p, q, q') = q'_\rho O^{Aab}_{\mu\nu\rho}(p, q, q') = 0.$$  

From (3) one has to consider also the vertex

$$W^{Aab}_{\mu\nu}(p, q, q') = p_\mu O^{Aab}_{\mu\nu\rho}(p, q, q'),$$

which, as known, contains an anomaly.

2) For the $O^{ABC}_{\mu\nu\rho}$ vertex we have to distinguish five types of contributions, corresponding to the different values of the indices $ABC$ which give a nonvanishing result.

(a) For $(A, B, C) = (1, 3\beta, 4\gamma)$ we have $c^{ABC} = \delta^{3\gamma}$

$$O^{ABC}_{\mu\nu\rho}(p, p', p'') = c^{ABC} \left\{ \epsilon_{\mu\nu\rho\sigma} p'_\sigma [\rho_6^{ABC} + \Sigma_6^{ABC}(p, p', p'')] + \epsilon_{\mu\nu\rho\sigma} p''_\sigma [\rho_7^{ABC} + \Sigma_7^{ABC}(p, p', p'')] 
+ \Sigma^{ABC}_{\mu\nu\rho}(p, p', p'') \right\},$$

$$\Sigma_6(0, 0, 0) = \Sigma_7(0, 0, 0) = 0.$$  

For $(A, B, C) = (2, 1, 1), (2, 3\beta, 3\gamma)$ and $(2, 4\beta, 4\gamma)$, we have the same decomposition with $\rho_6 = -\rho_7$ and $\Sigma_6 = -\Sigma_7$, due to the symmetry under the exchange of $B$ and $C$, and with $c^{ABC} = 1, \delta^{3\gamma}$ and $\delta^{3\gamma}$, respectively.

(b) For $(A, B, C) = (2, 2, 2)$ the coefficient proportional to one momentum vanishes for symmetry reasons and we have an irrelevant vertex

$$O^{ABC}_{\mu\nu\rho}(p, p', p'') = \Sigma^{ABC}_{\mu\nu\rho}(p, p', p'').$$

(c) For $(A, B, C) = (3\alpha, 3\beta, 3\gamma)$ we have

$$O^{ABC}_{\mu\nu\rho}(p, p', p'') = \epsilon^{\alpha\beta\gamma} \left\{ [g_{\mu\nu}(p' - p)_\rho + g_{\nu\rho}(p'' - p')_\mu + g_{\rho\mu}(p - p'')_\nu] [\rho_8 + \Sigma_8(p, p', p'')] + \Sigma^{ABC}_{\mu\nu\rho}(p, p', p'') \right\},$$

$$\Sigma_8(0, 0, 0) = 0.$$  

(d) For $(A, B, C) = (3\alpha, 4\beta, 4\gamma)$ we have

$$O^{ABC}_{\mu\nu\rho}(p, p', p'') = \epsilon^{\alpha\beta\gamma} \left\{ g_{\nu\rho}(p'' - p')_\mu [\rho_9 + \Sigma_9(p, p', p'')] + (g_{\mu\nu}p_\rho - g_{\mu\rho}p_\nu) 
\times [\rho_{10} + \Sigma_{10}(p, p', p'')] + (g_{\mu\nu}p'_\rho - g_{\mu\rho}p'_\nu) [\rho_{11} + \Sigma_{11}(p, p', p'')] + \Sigma^{ABC}_{\mu\nu\rho}(p, p', p'') \right\},$$
Table 1: Parameters and Ward identities for the three-point vertices

| Vertices   | Parameters | Relations | Predictions |
|------------|------------|-----------|-------------|
| $O^{\mu\nu\rho}_{Aab}$ | $A = 2$ | $U(1)_V$ | $SU(2)_A$ | $U(1)_A$ |
| $ABC = 211$ | 1 | 1 | 1 | $U(1)_A$ |
| $ABC = 233$ | 1 | 1 | 1 | $U(1)_A$ |
| $ABC = 244$ | 1 | 1 | 1 | $U(1)_A$ |
| $ABC = 413$ | 2 | 1 | 1 | $SU(2)_A$ |
| $ABC = 222$ | - | 1 | - | $U(1)_A$ |
| $ABC = 333$ | 1 | - | - | - |
| $ABC = 344$ | 3 | 1 | 1 | - |

From (3) the vertex of $W^A$ which contains $O^{ABC}_{\mu\nu\rho}$ is

$$W^{ABC}_{\nu\rho}(p,p',p'') = p_\mu O^{ABC}_{\mu\nu\rho}(p,p',p'') + f^{ABC}_{\bar{X}C} O^{\bar{X}C}_{\nu\rho}(p'') + f^{ACX}_{\bar{X}B} O^{\bar{X}B}_{\nu\rho}(p').$$

As known, the axial identities are anomalous in the cases (a) and (b). The situation for the three point vertices is summarized in tab. 1, where we report for each vertex the number of relevant parameters and the relations given by ST or Ward identities.

We have now to fix the values of these 10 relevant parameters. First of all we must require the ST identities (5), which involve only the vertex $O^{Aab}_{\mu\nu\rho}$. This implies $\rho_5 = 0$. Then, following the usual procedure, we require as many Ward identities $W^A = 0$ as possible. However, as shown in the table, one has for the three point vertices more relations than relevant parameters. Therefore some of the relations $W^A = 0$ can not be satisfied. One usually prefers to satisfy the vector Ward identities. In this case one obtains that all the 10 couplings $\rho$ are zero. This choice is given in tab. 1, which shows that the Ward identities which should be evaluated and that may contain anomalies are

$$W^{ABC}_{\nu\rho}(p,p',p''), \quad ABC = 211, 233, 244, 413, 222,$$

$$W^{Aab}_{\nu\rho}(p,k,k'), \quad A = 2.$$

In the next section we show how this procedure of extracting the relevant couplings naturally emerges in the framework of exact RG.

1. RG formulation

The exact RG formulation for this theory can be obtained by generalizing the method used for the $SU(2)$ Yang-Mills case in ref. [3]. For completeness we recall here the procedure and the relevant new steps.

i) Cutoff effective action. The “cutoff effective action” $\Gamma[\Phi; \Lambda, \Lambda_0]$, is obtained by putting an infrared (IR) cutoff $\Lambda$ and an ultraviolet (UV) cutoff $\Lambda_0$ for all propagators in the
vertices of the physical effective action $\Gamma[\Phi]$. Namely one sets to zero each propagator if its frequency is lower than $\Lambda$ or larger than $\Lambda_0$. Then $\Gamma[\Phi; \Lambda = 0, \Lambda_0 \to \infty]$ is just the physical effective action and, at this point, the ST identities and the Ward identities, anomalous or not, should be satisfied.

ii) Evolution equation. The RG flow in the IR cutoff $\Lambda$ is more easily written for the interacting part $\Pi[\Phi; \Lambda, \Lambda_0]$, i.e. the cutoff effective action $\Gamma[\Phi; \Lambda, \Lambda_0]$ minus the free inverse cutoff propagator contributions. The evolution equation has the form

$$\Lambda \frac{d}{d\Lambda} \Pi[\Phi; \Lambda, \Lambda_0] = I[\Phi; \Lambda, \Lambda_0],$$

where the functional $I[\Phi; \Lambda, \Lambda_0]$ is given (non linearly) in terms of $\Gamma[\Phi; \Lambda, \Lambda_0]$. This equation is obtained by observing that $\Lambda$ enters only as a cutoff in all internal propagators and is pictorially given in fig. 1. The expression of $I[\Phi; \Lambda, \Lambda_0]$ can be obtained form the one given in ref. [3] for the pure YM case by adding the fermion field contribution. As shown in fig. 1 it is constructed in terms of vertices of $\Gamma[\Phi; \Lambda, \Lambda_0]$. This equation is suitable for the loop expansion, due to the fact that the $q$-integration in the r.h.s. adds a loop.

iii) Relevant couplings. The evolution equation (7) allows one to compute the cutoff effective action once the boundary conditions are given. As one expects, the boundary conditions for the various vertices depend on dimensional counting. Therefore one must distinguish irrelevant vertices, which have negative mass dimension, and relevant couplings, i.e. couplings with non-negative dimension. In order to identify these relevant parameters (masses, wave function constants and couplings), one has to consider all the vertices of $\Gamma[\Phi; \Lambda, \Lambda_0]$ with non-negative dimension. There are 10 of such vertices involving only the fields and sources $W_\mu, \psi, \bar{\psi}, c, w_\mu, v, \bar{\lambda}$ and $\lambda$. The extraction of the relevant couplings for the Yang-Mills sector is done in ref. [3], and for the fermion sector in the appendix. For the part involving the sources $\zeta^A_\mu$, one has to consider the 7 vertices $O$ in (4), which now depend on $\Lambda$. From them one extracts the relevant parameters $\rho_i(\Lambda)$ and the irrelevant vertices $\Sigma_i(p, \cdots; \Lambda)$, as previously done. After this one divides the effective action in two parts, $\Pi = \Pi_{rel} + \Gamma_{irr}$, where $\Pi_{rel}$ is given by a polynomial in $\Phi$ with these relevant couplings as coefficients. The irrelevant part $\Gamma_{irr}$ of the cutoff effective action contains the vertices with negative dimension and the vertices $\Sigma_i$.

iv) Boundary conditions. The boundary conditions for the evolution equation (7) are fixed
as follows: (a) Due to dimensional reason, the irrelevant part of the effective action is fixed to vanish at \( \Lambda = \Lambda_0 \to \infty \), namely we impose the boundary condition

\[
\Gamma_{\text{irr}}[\Phi; \Lambda = \Lambda_0, \Lambda_0] = 0 ; \tag{8}
\]

(b) The boundary conditions on the relevant part of the effective action are given at \( \Lambda = 0 \) and fix the physical parameters. As discussed in ref. [3], they are

\[
\Pi_{\text{rel}}[\Phi; 0, \Lambda_0] = S_{\text{int}}^{\text{BRS}} + \int d^4x \left\{ \frac{\rho_{AA}(0)}{8} \left[ 2(W_\mu \cdot W_\nu)^2 + (W_\mu \cdot W_\mu)^2 \right] + \frac{\rho_{\text{vec}}(0)}{2} v \cdot c \wedge c \right\} , \tag{9}
\]

where \( S_{\text{BRS}}^{\text{int}} \) is obtained from (1) after having subtracted the free inverse propagators. The two additional couplings \( \rho_{AA}(0) \) and \( \rho_{\text{vec}}(0) \) are due to a four gluon quantum interaction with a group structure not present in \( S_{\text{BRS}}^{\text{int}} \) and quantum corrections to the BRS variation of \( c \). Notice that one does not have a corresponding correction for the contribution of the BRS variation of \( W_\mu \) since the effective action depends on the combination \( w_\mu = \frac{1}{g} u_\mu + \partial_\mu c \). The presence of these additional terms with couplings \( \rho_{AA}(0) \) and \( \rho_{\text{vec}}(0) \) is directly associated to the fact that the ST identities are nonlinear and mix relevant and irrelevant part of the action. Since \( S_{\text{BRS}}^{\text{int}} \) satisfies by itself ST identities, it follows that the two couplings must be fixed in terms of irrelevant vertices evaluated at appropriated normalization points. This is explained in detail in [3]. As shown in the appendix, in the quark sector there is no such mixing.

v) Loop expansion. By using the boundary conditions (8) and (9), the evolution equation (7) can be written as an integral equation

\[
\Pi[\Phi; \Lambda, \Lambda_0] = \Pi_{\text{rel}}[\Phi; 0, \Lambda_0] + \int_0^\Lambda d\lambda \frac{d\lambda}{\lambda} I_{\text{rel}}[\Phi; \lambda, \Lambda_0] + \int_0^{\Lambda_0} d\lambda \frac{d\lambda}{\lambda} \left\{ I[\Phi; \lambda, \Lambda_0] - I_{\text{rel}}[\Phi; \lambda, \Lambda_0] \right\} , \tag{10}
\]

where the relevant part \( I_{\text{rel}} \) of the functional \( I \), defined in (8), is identified in the same way as \( \Pi_{\text{rel}} \). The iterative solution of this equation gives the usual loop expansion. In the last term of (10) the functional \( I_{\text{rel}} \) gives the subtractions needed to have a finite \( \Lambda_0 \to \infty \) limit. The first loop calculation for the pure YM sector are reported in [3], together with the analysis of some ST identities in the physical limit (\( \Lambda = 0 \) and \( \Lambda_0 \to \infty \)). In the next section we perform the one loop calculations for the current sector and show how the vector Ward identities are satisfied in the physical limit, while the axial ones are anomalous.

2. One loop calculations

In order to show that this formulation is suitable for dealing with chiral symmetries (global in this case) we compute the anomalies. The calculation is simple and similar to the one performed by Taylor expansion of Feynman diagram integrands.

The one loop calculations are deduced from the iterative solution of (10). Since we are interested in the physical vertices, we set \( \Lambda = 0 \). At zero loop one has \( \Gamma[\Phi] = S_{\text{BRS}}^{\text{int}} \). As shown in [3], the one loop contribution \( \Gamma^{(1)}(p_1 \cdots p_n) \) of a general \( n \)-point vertex, is obtained as follows. First one computes \( \Gamma'(p_1 \cdots p_n, \Lambda_0) \), which is the usual one loop Feynman graph, in which all internal propagators have an UV cutoff \( \Lambda_0 \). If the vertex \( \Gamma(p_1 \cdots p_n) \) has negative dimension, the loop integration is convergent and one has \( \Gamma^{(1)}(p_1 \cdots p_n) = \Gamma'(p_1 \cdots p_n, \Lambda_0 \to \infty) \). If \( \Gamma(p_1 \cdots p_n) \) has non-negative dimension, one must impose the
physical conditions on the relevant parameters. This generates the necessary subtractions and the limit \( \Lambda_0 \to \infty \) can be taken.

We now show that the vector Ward identities are satisfied if we fix all the \( \rho(0) = 0 \) in the current sector. Moreover, one finds the correct values of anomalies for the axial Ward identities. We limit our analysis to the three point functions. For simplicity we set \( m_f = m \) independent of \( f \).

We consider first the case which gives the axial anomaly, namely \( O^{ABC}_{\mu\nu\rho} \) with \( (A, B, C) = (2, 1, 1) \). The one loop graph is

\[
O'_{\mu\nu\rho}(p, p', p'') = i \int_\ell \frac{K_{\Lambda_0}(\ell)}{\ell - m} \gamma^\mu \gamma^\nu \frac{K_{\Lambda_0}(\ell - p' - p'')}{\ell - p' - p'' - m} \gamma^\rho \frac{K_{\Lambda_0}(\ell - p')}{\ell - p' - m} \gamma^\nu
\]

with \( K_{\Lambda_0}(\ell) \) rapidly vanishing for \( \ell^2 \geq \Lambda_0^2 \) and \( \int_\ell = \int \frac{d^4 \ell}{(2\pi)^4} \). Imposing the condition \( \rho_0(0) = 0 \) we have

\[
O^{211}_{\mu\nu\rho}(p, p', p'') = O'_ {\mu\nu\rho}(p, p', p'') - (p' - p'')_\sigma \frac{\partial}{\partial \bar{p}_\sigma}O'_{\mu\nu\rho}(\bar{p}, p', p'')|_{\bar{p}' = \bar{p}'' = 0}.
\]

We now proceed in evaluating the Ward identities for this vertex. The calculation follows the usual steps, in particular we must take into account the surface terms coming from differences of cutoff functions. We find

\[
p'_\nu O'_{\mu\nu\rho}(p, p', p'') = -p''_\nu O'_{\mu\nu\rho}(p, p', p'') = \frac{1}{6\pi^2} \epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + O(\frac{1}{\Lambda_0^2}),
\]

\[
p_\mu O'_{\mu\nu\rho}(p, p', p'') = \frac{1}{6\pi^2} \epsilon_{\alpha\beta}\epsilon_{\mu\nu\rho} + O(\frac{1}{\Lambda_0^2}).
\]

From this we obtain

\[
\frac{\partial}{\partial \bar{p}^\sigma}O'_{\mu\nu\rho}(\bar{p}, p', p'')|_{\bar{p}' = \bar{p}'' = 0} = -\frac{1}{6\pi^2} \epsilon_{\mu\nu\rho\sigma} + O(\frac{1}{\Lambda_0^2}).
\]

Thus we have that the vector Ward identities are satisfied in the limit \( \Lambda_0 \to \infty \), while, as well known, at one loop the axial Ward identity develops an anomalous term

\[
p'_\nu O^{211}_{\mu\nu\rho}(p, p', p'') = p''_\nu O^{211}_{\mu\nu\rho}(p, p', p'') = 0, \quad p_\mu O^{211}_{\mu\nu\rho}(p, p', p'') = \frac{1}{2\pi^2} \epsilon_{\alpha\nu\rho}\epsilon_{\gamma\delta} + O(\frac{1}{\Lambda_0^2}).
\]

In the limit \( m \to 0 \) we can write the form of the physical vertex \( O^{211}_{\mu\nu\rho} \) in terms of the dimensionless function \( \Sigma^{211}_6 \), defined in (11)

\[
O^{211}_{\mu\nu\rho}(p, p', p'') = \frac{1}{2\pi^2} \epsilon_{\alpha\nu\rho\sigma} \frac{p'_\alpha p'_\beta p'\gamma}{p^2} + \left[ \epsilon_{\mu\nu\rho\sigma}(p' - p'')_\sigma + \epsilon_{\mu\nu\rho\sigma} \frac{p'_\alpha p'_\beta p'\gamma}{p^2} \right] \Sigma^{211}_6(p, p', p'').
\]

Notice that, as discussed by [11], this amplitude is not a pure pole in \( p^2 \).
The case of $O_{\mu\nu}^{Aab}$ is analogous to the previous one. In fact by choosing $\rho_5(0) = 0$, we have
\[ O_{\mu\nu}^{Aab}(p, p', p'') = -g^2 \delta^{A2} \text{tr}(t^a t^b) O_{\mu\nu}^{211}(p, p', p'') \, . \]
Thus the axial Ward identity is anomalous.

Also in the two cases $(A, B, C) = (2, 3\alpha, 3\beta)$ and $(A, B, C) = (2, 4\alpha, 4\beta)$, setting $\rho_{6\alpha}^{ABC}(0) = 0$, we fulfil the vector Ward identities but we find an anomalous axial contribution. We have
\[ O_{\mu\nu}^{ABC}(p, p', p'') = \text{tr}(\tau^\alpha \tau^\beta) O_{\mu\nu}^{211}(p, p', p'') \, . \]

When $(A, B, C) = (1, 3\alpha, 4\beta)$, apart from a factor $\text{tr}(\tau^\alpha \tau^\beta)$, the unsubtracted vertex is given by $O'_{\mu\nu}(p, p', p'')$. The one loop vertex is then $(\rho_6(0) = \rho_7(0) = 0)$
\[ O_{\mu\nu}^{13a4\beta}(p, p', p'') = \text{tr}(\tau^\alpha \tau^\beta) \{ O'_{\mu\nu}(p, p', p'') - \left( p'_\sigma \frac{\partial}{\partial p'_\sigma} + p''_\sigma \frac{\partial}{\partial p''_\sigma} \right) O'_{\mu\nu}(p, p', p'') \}|_{p''=0} \, . \]

From (11) one finds
\[ \frac{\partial}{\partial p'_\sigma} O'_{\mu\nu}(p, p', p'')|_{p''=0} = \frac{\partial}{\partial p''_\sigma} O'_{\mu\nu}(p, p', p'')|_{p''=0} = -\frac{1}{6\pi^2} \epsilon_{\mu\nu\rho\sigma} + O\left(\frac{1}{\Lambda^2} \right) \, . \]
Thus for this vertex we have the same result as in (12).

If $(A, B, C) = (2, 2, 2)$, since no subtraction is needed we have that $O_{\mu\nu}^{222} = O'_{\mu\nu}$ and find from (11) that all the axial identities are anomalous.

In the case $(A, B, C) = (3\alpha, 4\beta, 4\gamma)$ the Ward identities $W^A = 0$ are
\[ p_\mu O_{\mu\nu}^{3a4\beta4\gamma}(p, p', p'') = i\epsilon^{\alpha\beta\gamma'} O_{\nu\nu}^{4\beta4\gamma}(p'') + i\epsilon^{\alpha\gamma\beta'} O_{\nu\nu}^{4\gamma4\beta}(p') \, , \]
\[ p'_\mu O_{\mu\nu}^{3a4\beta4\gamma}(p, p', p'') = i\epsilon^{\beta\gamma\alpha'} O_{\mu\nu}^{3\alpha'\gamma}(p) + i\epsilon^{\beta\alpha\gamma'} O_{\mu\nu}^{4\gamma4\beta}(p'') \, . \]
These imply $\rho_9 = \rho_{10} = \rho_{11} = 0$ at $\Lambda = 0$. The Ward identities for this vertex are analogous to the three gluon vertex ST identity and we will not verify it.

In the case $(A, B, C) = (3\alpha, 3\beta, 3\gamma)$ the Ward identity $W^A = 0$ is
\[ p_\mu O_{\mu\nu}^{3\alpha3\beta3\gamma}(p, p', p'') = i\epsilon^{\alpha\gamma\beta'} O_{\nu\nu}^{3\beta3\gamma}(p') + i\epsilon^{\alpha\beta\gamma'} O_{\nu\nu}^{3\gamma3\beta}(p'') \, . \]
This implies $\rho_8(0) = 0$. At one loop this vertex and the corresponding Ward identity are equal to the previous case $(A, B, C) = (3\alpha, 4\beta, 4\gamma)$.

In this paper we analyzed a RG formulation of a non-Abelian gauge theory with global chiral symmetry. In dimensional regularization such a study is technically difficult due to the noninvariant counterterms induced by the definition of $\gamma_5$ in complex dimensions. This problem is known as the fine tuning problem.

When studying the RG evolution equation for a local gauge theory, one meets the same problem, since the momentum cutoff is a symmetry-breaking regulator. In a previous paper we showed how this difficulty can be avoided by imposing the physical boundary conditions on the relevant part of the cutoff effective action. In this way the bare couplings are automatically fine-tuned.
Since in the RG framework one works directly in four dimensions, it should be straightforward to extend our procedure to local chiral symmetry. This is confirmed by the analysis of the global chiral symmetry we have made in this paper. This symmetry is implemented when one gives the boundary conditions for the relevant part of the cutoff effective action in the current sector. In sect. 2 we performed the one loop calculations, analyzed in this formulation the axial Ward identities for the three current vertices and re-obtained the anomalies given by their well-known values. The one loop calculation of the chiral Ward identity involving the vertex $O_{\mu}^{\bar{\psi}\psi}(k)$, performed in the appendix, also shows that we indeed avoid the complicated fine tuning of the counterterms (see for instance ref. [8]).

This formulation provides a systematic cutoff procedure which generates higher order corrections. This is done by solving iteratively the integral equation (10). This equation is in principle nonperturbative, although only iterative solution has been considered here. This is another advantage respect to dimensional regularization, which is perturbative in nature and, for chiral theories, difficult to extend to higher orders.

We have benefited greatly from discussions with A. Bassetto, C. Becchi, R. Soldati, M. Testa and M. Tonin.

Appendix

In this appendix we analyze the fermion sector. First of all we decompose the vertices involving the quarks into relevant and irrelevant parts.

1) The contribution to $\Gamma[\Phi; \Lambda, \Lambda_0]$ from the quark propagator, $\bar{\psi}_f(-k)\Gamma^{(\bar{\psi}\psi)}(k; \Lambda)\psi_f(k)$, can be written as

$$\Gamma^{(\bar{\psi}\psi)}(k; \Lambda) = (k - m_f)K_{\Lambda\Lambda_0}^{-1}(k) + \sigma_{m_{\psi}}(\Lambda) + \sigma_{\psi}(\Lambda)(k - m_f) + \Sigma^{(\bar{\psi}\psi)}(k; \Lambda),$$

$$\Sigma^{(\bar{\psi}\psi)}(k; \Lambda) = 0 \quad \text{at} \quad k = m_f, \quad \frac{\partial}{\partial k_{\mu}}\Sigma^{(\bar{\psi}\psi)}(k; \Lambda) = 0 \quad \text{at} \quad k^2 = \mu^2.$$

2) For the quark-gluon contribution, $\bar{\psi}(k)\Gamma^a_{\mu}(k, k', q; \Lambda)\psi(k')W^a_{\mu}(q)$, we define

$$\Gamma^a_{\mu}(k, k', q; \Lambda) = \gamma_{\mu}t^a\sigma_g(\Lambda) + \Sigma_{\mu}^{(\bar{\psi}\psi W)}a(k, k', q; \Lambda), \quad \Sigma_{\mu}^{(\bar{\psi}\psi W)}a(k, -k, 0; \Lambda)|_{k^2 = \mu^2} = 0.$$

3) For the contribution $\bar{\lambda}(p)\Gamma^{(\bar{\lambda}\psi)c}(p, k, q; \Lambda)\psi(k)c^a(q)$, involving the source $\bar{\lambda}$, we define

$$\Gamma^{(\bar{\lambda}\psi)c}(p, k, q; \Lambda) = t^a\sigma_{\lambda}(\Lambda) + \Sigma_{\mu}^{(\bar{\lambda}\psi)c}(p, k, q; \Lambda), \quad \Sigma_{\mu}^{(\bar{\lambda}\psi)c}(p, k, q; \Lambda)|_{3SP} = 0,$$

and analogously for $\Gamma^{(\bar{\lambda}\psi)c}$a. All the $\sigma$’s are relevant couplings while the $\Sigma$’s are irrelevant vertices. The contribution of the above vertices to the relevant part of the cutoff effective action is then $(\sigma_i = \sigma_i(\Lambda))$

$$\Pi_{rel}^F[\Phi; \Lambda] = \int d^4x \left\{ \bar{\psi}_f \left[ \sigma_{m_{\psi}} + \sigma_{\psi}(im\partial - m_f) \right] \psi_f + \sigma_{\bar{\psi}_f}\bar{\psi}_f W \cdot t \psi_f + \sigma_{\lambda_{f\lambda}}(t \psi_f + \bar{\psi}_f c \cdot t \lambda_f) \right\}.$$

The boundary conditions for the five parameters $\sigma$ are fixed according to eq. (9). The fact that the coupling $\sigma_{\lambda}$ does not receive corrections can be seen by considering the ST identity for the quark-gluon vertex at $\Lambda = 0$

$$\Gamma^a_{\mu}(k, k', q)\Gamma^{(ac)}_{\mu}(q) + \Gamma^{(\bar{\psi}\lambda)c}(k, k', q)\Gamma^{(\bar{\psi}\bar{\psi})}(k') - \Gamma^{(\bar{\lambda}\psi)c}(k, k', q)\Gamma^{(\bar{\psi}\psi)}(-k) = 0.$$
By evaluating this identity at $3SP$, one finds that all the irrelevant contributions vanish, and the coupling $\sigma_\lambda(0)$ is fixed to be its tree level value ($\sigma_\lambda(0) = 1$).

We now examine the only relevant vertex which couples fermions and currents, namely the first vertex in (4). We have

$$O_\mu^{(\bar{\psi}\psi)A}(p, k, k') = \left\{ \gamma_\mu (1 + \rho^A_\mu) + \Sigma^A_\mu (p, k, k') \right\} T^A, \quad \Sigma^A_\mu (0, \vec{k}, -\vec{k})|_{\vec{k}^2 = \mu^2} = 0.$$  

From (3), the vertex of the functional $\mathcal{W}^A$ which contains $O_\mu^{(\bar{\psi}\psi)A}$ is

$$\mathcal{W}^{(\bar{\psi}\psi)A}(p, k, k') = p_\mu O^{(\bar{\psi}\psi)A}_\mu (p, k, k') - \Gamma^{(\bar{\psi}\psi)}(-k) T^A + T^A \Gamma^{(\bar{\psi}\psi)}(k').$$

We now show how this identity $\mathcal{W}^A = 0$ is satisfied in the physical limit at one loop order. Thus we set $\Lambda = 0$ and, for simplicity, $m_f = m$.

At zero loop we have $O^{(\bar{\psi}\psi)A}_\mu = \gamma^A_\mu T^A$ and $\Gamma^{(\bar{\psi}\psi)}(k) = \vec{k} - m$, thus the Ward identity $\mathcal{W}^{(\bar{\psi}\psi)A} = 0$ is trivially satisfied in the limit $m \to 0$ at this order.

(a) One loop calculation of $\Gamma^{(\bar{\psi}\psi)}(k) = \vec{k} - m + \Pi(k)$. The one loop graph is ($C_F = \frac{3}{4}$)

$$\Pi'(k) = -ig^2 C_F \int \frac{1}{\ell^2} \gamma_\rho \frac{1}{\vec{k} + \vec{\ell} - m} \gamma_\rho K_{\lambda_0}(\ell) K_{\lambda_0}(k + \ell).$$

Imposing the physical renormalization conditions, we find

$$\Pi(k) = \Pi'(k) - \Pi'(\vec{k}) - (k_\mu - \overline{k}_\mu) \partial_\mu \Pi'(k'), \quad \vec{k} = m, \quad \vec{k}'^2 = \mu^2.$$  

Because of these subtractions, we can take the limit $\Lambda_0 \to \infty$. The one loop fermion propagator is then

$$\Pi(k) = -ig^2 C_F \int \frac{1}{\ell^2} \left\{ \gamma_\rho \left( \frac{1}{\vec{k} + \vec{\ell} - m} - \frac{1}{\vec{k} + \vec{\ell} - m} \right) \gamma_\rho + (k_\mu - \overline{k}_\mu) \chi_\mu(\vec{k'}, \ell) \right\},$$

where

$$\chi_\mu(p, p', \ell) \equiv \gamma_\rho \frac{1}{\vec{p} + \vec{\ell} - m} \gamma_\rho \frac{1}{\vec{p}' - \vec{\ell} - m} \gamma_\rho.$$  

(b) One loop calculation of $O^{(\bar{\psi}\psi)A}_\mu$. The one loop graph is

$$O'^A_\mu(p, k, k') = ig^2 C_F \int \frac{1}{\ell^2} \chi_\mu^A(k', -k, \ell) K_{\lambda_0}(\ell) K_{\lambda_0}(\ell + k') K_{\lambda_0}(\ell - k),$$

with

$$\chi_\mu^A(p, p', \ell) \equiv \gamma_\rho \frac{1}{\vec{p} + \vec{\ell} - m} \gamma_\rho T^A \frac{1}{\vec{p}' - \vec{\ell} - m} \gamma_\rho.$$  

Imposing the condition $\rho^A_\mu(0) = 0$ we find, in the limit $\Lambda_0 \to \infty$, the one loop contribution to $O^{(\bar{\psi}\psi)A}_\mu$

$$O^{(\bar{\psi}\psi)A}_\mu(p, k, k') = O'^A_\mu(p, k, k') - O'^A_\mu(0, k', -k')$$

$$= ig^2 C_F \int \frac{1}{\ell^2} \left[ \chi_\mu^A(k', -k, \ell) - \chi_\mu^A(-k', -\overline{k}' , \ell) \right].$$

Using

$$p_\mu \chi_\mu^A(k', -k, \ell) = \gamma_\rho \left( \frac{1}{\vec{p} + \vec{\ell} - m} T^A - T^A \frac{1}{\vec{p}' - \vec{\ell} - m} \right) \gamma_\rho + \mathcal{O}(m),$$

where the $\mathcal{O}(m)$ term is present if $A = 2, 4\alpha$. Then one automatically satisfies at one loop level the Ward identity $\mathcal{W}^{(\bar{\psi}\psi)A} = 0$ in the limit $m \to 0$. 

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References

[1] C. Becchi, in Elementary particles, field theory and statistical mechanics, eds. M. Bonini, G. Marchesini and E. Onofri, Parma University 1993.

[2] A. Borrelli, L. Maiani, G. C. Rossi, R. Sisto and M. Testa, Nucl. Phys. B333 (1990) 335.

[3] M. Bonini, M. D’Attanasio and G. Marchesini, Renormalization group flow for SU(2) Yang-Mills theory and gauge invariance, Nucl. Phys. B to be published.

[4] K.G. Wilson, Phys. Rev. B 4 (1971) 3174,3184; K.G. Wilson and J.G. Kogut, Phys. Rep. 12 (1974) 75.

[5] J. Polchinski, Nucl. Phys. B231 (1984) 269; G. Gallavotti, Rev. Mod. Phys. 57 (1985) 471.

[6] G. Keller and C. Kopper, Phys. Lett. 273B (1991) 323; G. Keller, C. Kopper and M. Salmhofer, Helv. Phys. Acta 65 (1992) 32; C. Wetterich, Phys. Lett. 301B (1993) 90; M. Reuter and C. Wetterich, preprint DESY-93-152; U. Ellwanger, Heidelberg preprint HD-THEP-94-2.

[7] M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B409 (1993) 441, Ward identities and Wilson renormalization group for QED, Nucl. Phys. B to be published.

[8] for a recent review and a list of references see for example J. Collins, Renormalization, Cambridge University Press.

[9] S. Adler, In Lectures on Elementary particles and quantum field theory, eds. S. Deser et al. (M.I.T., Cambridge, Mass.)

[10] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. 42 (1975) 127, Ann. Phys. (NY) 98 (1976) 287.

[11] A. Andrianov, A. Bassetto and R. Soldati, Phys. Rev. D 44 (1991) 2602.

[12] M. Tonin, Nucl. Phys. B (Proc. Suppl.) 29B, C (1992) 137 and references therein.