UNIFORM DECISION PROBLEMS AND ABSTRACT PROPERTIES OF SMALL OVERLAP MONOIDS

MARK KAMBITES
School of Mathematics, University of Manchester, Manchester M13 9PL, England.

Mark.Kambites@manchester.ac.uk

Abstract. We study the way in which the abstract structure of a small overlap monoid is reflected in, and may be algorithmically deduced from, a small overlap presentation. We show that every $C(2)$ monoid admits an essentially canonical $C(2)$ presentation; by counting canonical presentations we obtain asymptotic estimates for the number of non-isomorphic monoids admitting $a$-generator, $k$-relation presentations of a given length. We demonstrate an algorithm to transform an arbitrary presentation for a $C(m)$ monoid ($m$ at least 2) into a canonical $C(m)$ presentation, and a solution to the isomorphism problem for $C(2)$ presentations. We also find a simple combinatorial condition on a $C(4)$ presentation which is necessary and sufficient for the monoid presented to be left cancellative. We apply this to obtain algorithms to decide if a given $C(4)$ monoid is left cancellative, right cancellative or cancellative, and to show that cancellativity properties are asymptotically visible in the sense of generic-case complexity.

1. Introduction

Small overlap conditions are natural combinatorial conditions on semigroup and monoid presentations, which serve to limit the complexity of derivation sequences between equivalent words. First studied by Remmers \[12\] \[13\] in the 1970's, they are the natural semigroup-theoretic analogue of the small cancellation conditions extensively used in combinatorial and geometric group theory \[10\]. Like word hyperbolicity for groups \[11\], small overlap properties are generic for monoid presentations with a fixed number of generators and relations, in the sense that a randomly chosen $a$-generator, $k$-relation presentation of size $n$ satisfies any given small overlap condition with probability which approaches 1 as $n$ increases \[6\].

Remmers' original work showed that monoids with presentations satisfying the condition $C(3)$ have decidable word problem; an accessible account of this and related results can be found in \[4\]. Recent research of the author \[7\] \[8\] \[9\] has shown that monoids with presentations satisfying the slightly stronger condition $C(4)$ have linear time solvable word problem, and a regular language of linear-time computable normal forms, and are also rational.
This paper is devoted to the question of how abstract algebraic properties (by which we mean isomorphism invariants) of small overlap monoids are reflected in, and can be algorithmically deduced from, their presentations. Section 2 briefly recalls some key definitions and results from the theory of small overlap presentations. In Section 3 we consider isomorphisms between small overlap monoids. We show that every isomorphism type of small overlap monoids has a canonical small overlap presentation, and that this presentation can be effectively computed from any other presentation for the monoid. It follows that the isomorphism problem for small overlap monoids is decidable. By counting canonical presentations, we obtain asymptotic estimates for the number of isomorphism types of $a$-generator $k$-relation semigroups of a given size, as a function of the positive integers $a$ and $k$.

In Section 4 we turn our attention to cancellativity properties. By applying results from [7, 8], we are able to give elementary combinatorial characterisations of those $C(4)$ presentations which present left cancellative, right cancellative and cancellative monoids. Since these properties of presentations can be easily tested, it follows that one can efficiently decide whether a given $C(4)$ presentation presents a left, right or two-sided cancellative monoid. We deduce also that left cancellativity, right cancellativity and cancellativity are asymptotically visible properties of finite presentations, in the sense that the proportion of $A$-generated, $k$-relation monoid presentations of size $n$ which present monoids with any of these properties converges to a limit strictly between 0 and 1, as $n$ increases.

2. Small Overlap Monoids

In this section we recap the definitions and some key results concerning small overlap monoids. We begin by recalling the basic definitions of combinatorial semigroup theory, chiefly in order to fix notation and terminology. Let $A$ be an alphabet, or set of symbols. A word over $A$ is a finite sequence of zero or more symbols from $A$. The free monoid $A^*$ is the set of all words over $A$, with multiplication defined by concatenation of sequences. The unique empty word containing no symbols is denoted $\varepsilon$; it forms the identity element in $A^*$. A monoid presentation consists of a pair $\langle A \mid R \rangle$ where $A$ is an alphabet, and $R$ is a binary relation on the free monoid $A^*$. The presentation is called finite if both $A$ and $R$ are finite. The elements of the binary relation $R$, which are pairs from $A^* \times A^*$, are rather confusingly called the relations of the presentation. A relation with both sides the same (that is, one of the form $(u, u)$ for some $u \in A^*$) is termed trivial.

The maximum relation length of the presentation is the length of the longest word appearing as one side of a relation in $R$ (or 0 if there are no relations), while the sum relation length is the total length of all the words forming sides of relations in $R$.

We say that a word $u \in A^*$ is obtained from a word $v \in A^*$ by an application of a relation from $R$ if $u = pxq$ and $v = pyq$ for some words $p, q, x, y \in A^*$ such that $(x, y) \in R$ or $(y, x) \in R$. We define a binary relation $\equiv_R$ (called just $\equiv$ where there is no ambiguity as to the presentation)
on \( \mathcal{A}^* \) by \( u \equiv_{\mathcal{A}} v \) if \( v \) can be obtained by \( u \) by a finite sequence of zero or more applications of relations from \( \mathcal{A} \). In fact \( \equiv_{\mathcal{A}} \) is a congruence on \( \mathcal{A}^* \), that is, an equivalence relation compatible with the multiplication. We denote by \([u]_{\mathcal{A}}\) (or just \([u]\)) the equivalence class of the word \( u \in \mathcal{A}^* \). The equivalence classes form a monoid with multiplication well-defined by

\[
[u]_{\mathcal{A}}[v]_{\mathcal{A}} = [uv]_{\mathcal{A}};
\]

this is called the **monoid presented by the presentation**.

We say that a word \( p \) is a possible prefix of \( u \) if there exists a (possibly empty) word \( w \) with \( pw \equiv u \), that is, if the element represented by \( u \) lies in the right ideal generated by the element represented by \( p \). The empty word is denoted \( \epsilon \).

A relation word is a word which occurs as one side of a relation in the presentation. A piece is a word in the generators which occurs as a factor in sides of two distinct relation words, or in two different (possibly overlapping) places within one side of a relation word. By convention, the empty word is always a piece. We say that a presentation is \( C(n) \), where \( n \) is a positive integer, if no relation word can be written as the product of strictly fewer than \( n \) pieces. Thus for each \( n \), \( C(n+1) \) is a stronger condition than \( C(n) \).

Notice that it is permissible for the same relation word to appear twice or more in a \( C(n) \) presentation, since by doing so it does not become a factor of two distinct relation words. We say that a presentation is strongly \( C(n) \) if it is \( C(n) \) and additionally has no repeated relation words. The condition we have called \( C(n) \) is that used in [4, 12, 13] and was called weakly \( C(n) \) in [7], while the condition we have called strongly \( C(n) \) was called \( C(n) \) in [8, 9, 6]. See [7] for detailed explanation of the relationship between the two definitions.

We say that a relation word \( R \) is a complement of a relation word \( R \) if there are relation words \( R = R_1, R_2, \ldots, R_n = T \) such that either \((R_i, R_{i+1})\) or \((R_{i+1}, R_i)\) is a relation in the presentation for \( 1 \leq i < n \). We say that \( T \) is a proper complement of \( R \) if, in addition, \( T \neq R \). Abusing notation and terminology slightly, if \( R = XRYRZ \) and \( T = XRYTZ \), then we write \( \overline{R} = X_{\overline{R}}, \overline{XYR} = X_{\overline{R}}Y_{\overline{R}} \) and so forth. We say that \( \overline{R} \) is a complement of \( R \), and \( \overline{XYR} \) is a complement of \( XRY \).

Now let \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) be a \( C(3) \) presentation. Recall that a relation prefix of a word is a prefix which admits a (necessarily unique, as a consequence of the small overlap condition) factorisation of the form \( aXY \) where \( X \) and \( Y \) are the maximal piece prefix and middle word respectively of some relation word \( XXY \). An overlap prefix (of length \( n \)) of a word \( u \) is a relation prefix which admits an (again necessarily unique) factorisation of the form \( bX_1Y'_1X_2Y'_2 \ldots X_nY_n \) where

- \( n \geq 1 \);
- \( bX_1Y'_1X_2Y'_2 \ldots X_nY_n \) has no factor of the form \( X_0Y_0 \), where \( X_0 \) and \( Y_0 \) are the maximal piece prefix and middle word respectively of some relation word, beginning before the end of the prefix \( b \);
- for each \( 1 \leq i \leq n \), \( R_i = X_iY_iZ_i \) is a relation word with \( X_i \) and \( Z_i \) the maximal piece prefix and suffix respectively; and
- for each \( 1 \leq i < n \), \( Y'_i \) is a proper, non-empty prefix of \( Y_i \).
Notice that if a word has a relation prefix, then the shortest such must be an overlap prefix. A relation prefix $aXY$ of a word $u$ is called clean if $u$ does not have a prefix

$$aXY'X_1Y_1$$

where $X_1$ and $Y_1$ are the maximal piece prefix and middle word respectively of some relation word, and $Y'$ is a proper, non-empty prefix of $Y$.

We recall some key technical results about $C(n)$ presentations.

**Proposition 2.1** ([7] Proposition 3). Suppose a word $u$ has an overlap prefix $wXY$ and that $u = wXYu''$. Then $u \equiv v$ if and only if $v = uv'$ where $v' \equiv XYu''$.

**Lemma 2.2** ([7] Lemma 3). Let $\langle A \mid R \rangle$ be a $C(4)$ presentation. Suppose $u = XYu'$ where $XY$ is a clean overlap prefix of $u$. Then $u \equiv v$ if and only if one of the following mutually exclusive conditions holds:

1. $u = XY\hat{Z}u''$ and $v = XYZv''$ and $\hat{Z}u'' \equiv \hat{Z}v''$ for some complement $\hat{Z}$ of $Z$;
2. $u = XY\hat{Z}u''$, $v = XYv'$, and $Z$ fails to be a prefix of at least one of $u'$ and $v'$, and $u' \equiv v'$;
3. $u = XY\hat{Z}u''$, $v = X\hat{Y}Zu''$ for some uniquely determined proper complement $X\hat{Y}Z$ of $XYZ$, and $\hat{Z}u'' \equiv \hat{Z}v''$ for some complement $\hat{Z}$ of $Z$;
4. $u = XY\hat{Z}u''$, $v = X\hat{Y}Zu''$ for some uniquely determined proper complement $X\hat{Y}Z$ of $XYZ$ but $Z$ is not a prefix of $u'$ and $u' \equiv \hat{Z}v''$;
5. $u = XY\hat{Z}u''$, $v = XX\hat{Y}v'$ for some uniquely determined proper complement $XX\hat{Y}$ of $XYZ$, but $\hat{Z}$ is not a prefix of $v'$ and $\hat{Z}u'' \equiv v'$;
6. $u = XY\hat{Z}u''$, $v = XX\hat{Y}v'$ for some uniquely determined proper complement $XX\hat{Y}$ of $XYZ$, $Z$ is not a prefix of $u'$ and $\hat{Z}$ is not a prefix of $v'$, but $Z = \hat{Z}z$, $\hat{Z} = z\hat{Z}$, $u' = \hat{Z}v''$, $v' = zv''$ where $u'' \equiv v''$ and $z$ is the maximal common suffix of $Z$ and $\hat{Z}$, $z$ is non-empty, and $z$ is a possible prefix of $u''$.

**Proposition 2.3** ([7] Corollary 1). Let $\langle A \mid R \rangle$ be a $C(3)$ presentation. If a word $u$ has no clean overlap prefix, then it contains no relation word as a factor, and so if $u \equiv v$ then $u = v$.

### 3. Minimal Presentations and Isomorphisms

Our main aim in this section is to show that every small overlap monoid has a canonical small overlap presentation, and that this presentation can be effectively computed from any other presentation for the monoid. From this we are able to demonstrate a solution to the isomorphism problem for monoid presentations satisfying the condition $C(2)$. Some of the essential ideas in this section were prefigured in work of Jackson [5], although the main results and applications given here are new.

We begin by introducing some terminology. A presentation $\langle A \mid R \rangle$ is called an equivalence presentation if the set $R$ of relations is an equivalence relation on the set of relation words. The equivalence closure of a presentation $\langle A \mid R \rangle$ is the presentation $\langle A \mid I \rangle$ where $I$ is the reflexive, symmetric, transitive closure of $R$. Clearly, $\langle A \mid I \rangle$ presents the same
monoid as \( \langle A \mid R \rangle \). Notice also that, since we require the set of relations to be an equivalence relation only on the set of relation words, and not on the whole free monoid \( A^* \), the equivalence closure of a finite presentation is still finite, and can be easily computed. It has the same set of relation words as the original presentation. In particular, it satisfies \( C(m) \) for any \( m \geq 1 \) if and only if the original presentation satisfies \( C(m) \).

Recall that if \( \langle A \mid R \rangle \) is a monoid presentation then a generator \( a \in A \) is called redundant if it is equivalent to a product of (zero or more) other generators, or equivalently, if the monoid presented by the presentation is generated by the subset \( A \setminus \{a\} \). We call a presentation generator-minimal if it has no redundant generators. Notice in particular that, in a generator-minimal presentation, no two generators represent the same element. A non-identity element \( s \) of the monoid presented is called indecomposable (or by some authors, an atom) if whenever \( x, y \) are elements of the monoid such that \( xy = s \) we have \( x = 1 \) or \( y = 1 \). A generator \( a \in A \) is called indecomposable if \([a]_R\) is indecomposable. Notice that a non-identity indecomposable element of a monoid must belong to every generating set for the monoid.

**Proposition 3.1.** Let \( \langle A \mid R \rangle \) be a \( C(2) \) presentation and \( u \) a relation word. Then for any \( v \in A^* \) we have \( u \equiv v \) if and only if there exists \( n \geq 1 \) and words \( u = u_1, u_2, \ldots, u_n = v \) such that for \( 1 \leq i < n \) either \((u_i, u_{i+1}) \in R\) or \((u_{i+1}, u_i) \in R\).

**Proof.** One implication is immediate from the definitions. For the converse, observe that since the presentation satisfies \( C(2) \), no relation word contains another relation word as a proper factor. It follows that the only relations which can be applied to the word \( u \) are relations of the form \((u, u')\) or \((u', u)\). But now \( u' \) is also a relation word, so a simple inductive argument shows that any rewriting sequence taking \( u \) to \( v \) must have the given form. \( \square \)

**Proposition 3.2.** Let \( \langle A \mid R \rangle \) be a \( C(2) \) presentation and \( u \) a relation word. Then for any proper factor \( u' \) of \( u \) and any \( v \in A^* \) we have \( u' \equiv v \) if and only if \( u' = v \).

**Proof.** Since the presentation satisfies \( C(2) \), no relation word contains another relation word as a proper factor. It follows that \( u' \) contains no relation word as a factor, and so no relation can be applied to it. \( \square \)

**Proposition 3.3.** Let \( \langle A \mid R \rangle \) be a monoid presentation satisfying \( C(2) \), and let \( a \in A \). Then \( a \) is either indecomposable or redundant.

**Proof.** Since the presentation satisfies \( C(1) \), the empty word is not a relation word. If the presentation contains no non-trivial relations of the form \((w, a)\) or \((a, w)\) then no relations are applicable to \( a \), and so \( a \) is indecomposable. On the other hand, suppose the presentation does contain a non-trivial relation of the form \((w, a)\) or \((a, w)\). Then \( a \) is a relation word, which since the presentation satisfies \( C(2) \) means that \( a \) is not a piece, and so \( a \) cannot feature in the word \( w \). Thus, the given relation can be used to rewrite \( a \) as a product of the other generators, which means that \( a \) is redundant. \( \square \)
**Proposition 3.4.** Let \( m \geq 2 \). Every \( C(m) \) monoid has a generator-minimal \( C(m) \) equivalence presentation, which can be effectively computed starting from any \( C(m) \) presentation for the monoid.

**Proof.** Let \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) be a \( C(m) \) presentation for \( M \), and suppose that this presentation is not generator-minimal. Let \( a \in \mathcal{A} \) be a redundant generator. Clearly, some non-trivial relation in \( \mathcal{R} \) can be applied to the word \( a \); since the presentation satisfies \( C(1) \) neither side of this relation can be the empty word, so this relation must have the form \((a, w)\) or \((w, a)\) for some \( w \in \mathcal{A}^+ \) with \( w \neq a \).

Let \( \hat{a} \) denote the set of all words \( w \) such that \((a, w)\) or \((w, a)\) is a relation and \( w \neq a \). Since the presentation satisfies \( C(2) \) and \( a \) is a relation word, \( a \) cannot be a factor of any \( w \in \hat{a} \). Let

\[
\mathcal{B} = \mathcal{A} \setminus \{a\}
\]

and

\[
\mathcal{I} = (\mathcal{B} \setminus \{(a, w), (w, a) \mid w \in \mathcal{A}^*\}) \cup \{(u, v) \mid u, v \in \hat{a}\}.
\]

Consider now the presentation \( \langle \mathcal{B} \mid \mathcal{I} \rangle \). Since every relation word in \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) is also a relation word in \( \langle \mathcal{A} \mid \mathcal{R} \rangle \), it is clear that \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) is a \( C(m) \) presentation with strictly fewer generators than \( \langle \mathcal{A} \mid \mathcal{R} \rangle \). We claim that \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) is a presentation for the same monoid as \( \langle \mathcal{A} \mid \mathcal{R} \rangle \). Indeed, let \( \iota : \mathcal{B} \rightarrow \mathcal{A} \) be the inclusion map. Then \( \iota \) extends to a morphism \( \hat{\sigma} : \mathcal{B}^* \rightarrow \mathcal{A}^* \) of free monoids, and since every relation in \( \mathcal{I} \) is satisfied in \( \langle \mathcal{A} \mid \mathcal{R} \rangle \), this induces a morphism \( \sigma : \langle \mathcal{B} \mid \mathcal{I} \rangle \rightarrow \langle \mathcal{A} \mid \mathcal{R} \rangle \) well-defined by \( \sigma([w]_{\mathcal{I}}) = [\hat{\sigma}(w)]_{\mathcal{R}} \). Now the image of \( \sigma \) contains \([w]_{\mathcal{R}} = [a]_{\mathcal{R}} \) and also \([b]_{\mathcal{R}} \) for every \( b \in \mathcal{B} = \mathcal{A} \setminus \{a\} \), so \( \sigma \) must be surjective. Now suppose \( u, v \in \mathcal{B}^* \) are such that \( \sigma([u]_{\mathcal{I}}) = \sigma([v]_{\mathcal{I}}) \). Then \( \hat{\sigma}(u) \equiv_{\mathcal{A}} \hat{\sigma}(v) \), so there is a sequence of words

\[
u = \hat{\sigma}(u) = y_0, y_1, \ldots, y_n = \hat{\sigma}(v) = v \in \mathcal{A}^*\]

such that each \( y_{i+1} \) can be obtained from \( y_i \) by an application of a single relation in \( \mathcal{B} \). For each \( i \), let \( z_i \) be the word obtained from \( y_i \) by replacing any occurrences of the letter \( a \) with the word \( w \). Since \( a \) is not a proper factor of any relation word in \( \langle \mathcal{A} \mid \mathcal{B} \rangle \), it follows from the definition of \( \mathcal{I} \) that each \( y_{i+1} \) can be obtained from \( y_i \) by an application of a single relation from \( \mathcal{I} \). But \( y_0 = u \) and \( y_n = v \), so we deduce that \([u]_{\mathcal{I}} = [v]_{\mathcal{I}} \). Thus, \( \sigma \) is injective, and so is an isomorphism.

We have shown that given a non-generator-minimal \( C(m) \) presentation, there always exists a \( C(m) \) presentation for the same monoid with strictly fewer generators; it follows that a \( C(m) \) presentation with the fewest possible number of generators must be generator-minimal. Moreover, given a non-generator-minimal \( C(m) \) presentation, one may identify a redundant generator by seeking a relation of the form \((a, w)\) or \((w, a)\) for some \( a \in \mathcal{A} \) and \( w \in \mathcal{A}^* \), and then use the approach described above to effectively compute a \( C(m) \) presentation with strictly fewer generators. By iteration, one may thus compute a generator-minimal \( C(m) \) presentation for the same monoid.

Finally, to obtain a generator-minimal \( C(m) \) equivalence presentation, it suffices to take a generator-minimal \( C(m) \) presentation and compute the equivalence closure of the set of relations. By our remarks at the beginning
of the section, this is a $C(m)$ equivalence presentation for the same monoid, and is clearly still generator-minimal since it has the same generating set. □

**Corollary 3.5.** In any monoid admitting a $C(2)$ presentation, the set of non-identity indecomposable elements forms a generating set which is contained in every generating set for the monoid.

**Proof.** By Proposition 3.4, such a monoid has a generator-minimal $C(2)$ presentation, that is, one in which no generators are redundant. By Proposition 3.3, the generators in this presentation must be indecomposable, which shows that the set of all non-identity indecomposable elements forms a generating set. Finally, we have already observed that every non-identity indecomposable element must lie in every generating set for the monoid. □

We now introduce some more terminology. Let $\langle A \mid R \rangle$ and $\langle B \mid S \rangle$ be presentations. An isomorphism $\hat{\sigma} : A^* \to B^*$ of free monoids is called an *inclusion* of $\langle A \mid R \rangle$ into $\langle B \mid S \rangle$ if for every relation $(u, v) \in R$ there is a relation $(\hat{\sigma}(u), \hat{\sigma}(v)) \in S$.

We say that $\langle A \mid R \rangle$ is a *sub-presentation* of $\langle B \mid S \rangle$ if there exists an inclusion of $\langle A \mid R \rangle$ into $\langle B \mid S \rangle$. An *isomorphism* between two presentations is an inclusion whose inverse is also an inclusion, and two presentations are called *isomorphic* if there is an isomorphism between them. Recall that, since a free monoid has a unique free generating set, isomorphisms between free monoids are in a natural one-one correspondence with bijections between the corresponding free generating sets. While conceptually inclusions should be thought of as isomorphisms between free monoids, computationally it is more practical to work with the corresponding bijections. Notice that it is a completely routine matter to determine, given two finite presentations and a bijection between the alphabets, whether the bijection extends to an inclusion (and hence also whether it extends to an isomorphism).

The following theorem says, informally, that any generator-minimal equivalence presentation contains a copy of every generator-minimal $C(2)$ presentation for the same monoid.

**Theorem 3.6.** Let $\langle A \mid R \rangle$ be a generator-minimal $C(2)$ presentation and $\langle B \mid S \rangle$ be any generator-minimal equivalence presentation. Let $\sigma : \langle A \mid R \rangle \to \langle B \mid S \rangle$ be an isomorphism between the monoids presented. Then there is a unique morphism of free monoids $\hat{\sigma} : A^* \to B^*$ such that $[\hat{\sigma}(a)]_S = \sigma([a]_R)$ for all $a \in A$, and this morphism is an inclusion of $\langle A \mid R \rangle$ into $\langle B \mid S \rangle$.

**Proof.** Since $\langle A \mid R \rangle$ is generator-minimal, the formal generators must represent distinct elements of the monoid and, by Proposition 3.3 those elements are indecomposable. Since $\sigma$ is an isomorphism, we deduce that the images of these generators are indecomposable, which by Corollary 3.5 means that they lie in the set of elements represented by the formal generators in $B$. It follows that we can define $\hat{\sigma} : A^* \to B^*$ by choosing for each $a \in A$ an element $\hat{\sigma}(a) \in B$ such that $[\hat{\sigma}(a)]_S = \sigma([a]_R)$, and then
extending to a morphism of \( \mathcal{A}^* \) using the universal property of free monoids. Moreover, since \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) is generator-minimal, the generators in \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) represent distinct elements, so these choices are unique and \( \hat{\sigma} \) is the unique function with the given property.

Now \( \sigma \) is injective, so \( \hat{\sigma} \) separates \( \mathcal{A} \), and since \( \mathcal{B} \) is free it follows that \( \hat{\sigma} \) is injective. Moreover, \( \sigma \) is an isomorphism, so the images of the generating set for \( \langle \mathcal{A} \mid \mathcal{B} \rangle \) comprise a generating set for \( \langle \mathcal{B} \mid \mathcal{I} \rangle \). Since the rewrite is non-trivial, which by the property of \( \hat{\sigma} \) we have that \( \hat{\sigma}(w) \equiv_{\mathcal{B}} \hat{\sigma}(y) \) for every \( w \in \mathcal{B} \).

Moreover, since \( \hat{\sigma} \) is an isomorphism, so the images of the generating set for \( \langle \mathcal{A} \mid \mathcal{B} \rangle \) comprise a generating set for \( \langle \mathcal{B} \mid \mathcal{I} \rangle \). By the property of \( \hat{\sigma} \) described above we certainly have that \( \hat{\sigma}(x) \equiv_{\mathcal{B}} \hat{\sigma}(y) \) in the monoid \( \langle \mathcal{B} \mid \mathcal{I} \rangle \). This means that we can choose a sequence of words

\[ u_0 = \hat{\sigma}(x), u_1, u_2, \ldots, u_n = \hat{\sigma}(y) \in \mathcal{B}^* \]

such that each \( u_{i+1} \) can be obtained from \( u_i \) by an application of a relation from \( \mathcal{I} \). This by definition means that for \( 1 \leq i < n \) we may choose \( p_i, q_i, q'_i, r_i \in \mathcal{B}^* \) such that \( u_i = p_iq_ir_i, u_{i+1} = p_iq'_ir_i, q_i \neq q'_i \) and (using the fact that \( \mathcal{I} \) is symmetric) also \( (q_i, q'_i) \in \mathcal{I} \).

For each \( i \), let \( w_i = \hat{\sigma}^{-1}(u_i) \in \mathcal{A}^* \). Then using again the property above of \( \hat{\sigma} \) we have that \( w_1, \ldots, w_n \) are all equivalent in \( \langle \mathcal{A} \mid \mathcal{B} \rangle \) to the relation word \( w_0 = \hat{\sigma}^{-1}(u_0) = x \). By Proposition 3.3, this means in particular that every \( u_i \) is a relation word in the presentation \( \langle \mathcal{A} \mid \mathcal{B} \rangle \).

We claim now that \( u_i = q_i \) for every \( i \). Indeed, suppose for a contradiction that there is some \( i \) for which \( q_i \neq u_i \). We know that \( q_i \equiv_{\mathcal{B}} q'_i \) and \( q_i \neq q'_i \) (since the rewrite is non-trivial), which by the property of \( \hat{\sigma} \) above means that \( \hat{\sigma}^{-1}(q_i) \equiv_{\mathcal{B}} \hat{\sigma}^{-1}(q'_i) \) and \( \hat{\sigma}^{-1}(q_i) \neq \hat{\sigma}^{-1}(q'_i) \). But \( \hat{\sigma}^{-1}(q_i) \) is a proper factor of \( u_i = \hat{\sigma}^{-1}(u_i) \), which is a relation word, so by Proposition 3.2 we must have \( \hat{\sigma}^{-1}(q_i) = \hat{\sigma}^{-1}(q'_i) \) and hence \( q_i = q'_i \) which gives the desired contradiction and completes the proof of the claim that \( u_i = q_i \) for every \( i \).

It follows immediately that \( q'_i = u_{i+1} = q_{i+1} \). But now \( (u_i, u_{i+1}) = (q_i, q'_i) \in \mathcal{I} \) for every \( i \), which since \( \mathcal{I} \) is transitive, means that \( (u_0, u_n) \in \mathcal{I} \) as required.

**Corollary 3.7.** Two generator-minimal \( C(2) \) equivalence presentations present isomorphic monoids if and only if they are isomorphic.

**Proof.** It is obvious that isomorphic presentations present isomorphic monoids.

Conversely, let \( \langle \mathcal{A} \mid \mathcal{B} \rangle \) and \( \langle \mathcal{B} \mid \mathcal{I} \rangle \) be generator-minimal \( C(2) \) equivalence presentations, and let \( \sigma_1 : \langle \mathcal{A} \mid \mathcal{B} \rangle \rightarrow \langle \mathcal{B} \mid \mathcal{I} \rangle \) be an isomorphism between the monoids presented, and \( \sigma_2 : \langle \mathcal{B} \mid \mathcal{I} \rangle \rightarrow \langle \mathcal{A} \mid \mathcal{B} \rangle \) be its inverse. Let \( \hat{\sigma}_1 : \mathcal{A}^* \rightarrow \mathcal{B}^* \) and \( \hat{\sigma}_2 : \mathcal{B}^* \rightarrow \mathcal{A}^* \) be the inclusions given by Theorem 3.6. To show that the presentations are isomorphic, it will suffice to show that the inclusions \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) are mutually inverse. Now we know from Theorem 3.6 that \( \hat{\sigma}_2 \) is the unique isomorphism from \( \mathcal{B}^* \) to \( \mathcal{A}^* \) satisfying

\[ [\hat{\sigma}_2(b)]_\mathcal{B} = \sigma_2([b]_\mathcal{I}) \]

for all \( b \in \mathcal{B} \) so it will suffice to show that \( \hat{\sigma}_1^{-1} \) also satisfies this condition.
By the properties of $\hat{\sigma}_1$ guaranteed by Theorem 3.6 we have

$$[\hat{\sigma}_1(a)]_{\mathcal{A}} = \sigma_1([a]_{\mathcal{A}})$$

for all $a \in \mathcal{A}$. Applying $\sigma^{-1}_2 = \sigma_2$ to both sides we obtain

$$\sigma_2([\hat{\sigma}_1(a)]_{\mathcal{A}}) = \sigma_2(\sigma_1([a]_{\mathcal{A}})) = [a]_{\mathcal{A}}.$$ 

Now for any $b \in \mathcal{B}$, letting $a = \hat{\sigma}_1^{-1}(b)$ in this expression yields

$$\sigma_2([b]_{\mathcal{A}}) = [\hat{\sigma}_1^{-1}(b)]_{\mathcal{A}}$$

as required. □

As a straightforward consequence, we obtain the fact that the isomorphism problem for $C(2)$ monoids is algorithmically solvable.

**Theorem 3.8.** There is an algorithm which, given as input two $C(2)$ presentations, decides whether the monoids presented are isomorphic.

**Proof.** Given two $C(2)$ presentations, by Proposition 3.4, we may compute generator-minimal $C(2)$ equivalence presentations for the same monoids. Now by Corollary 3.7 it suffices to check if the resulting presentations are themselves isomorphic. This can clearly be done, for example by enumerating all bijections between the generating sets and checking if any extend to isomorphisms. □

The following theorem says that a small overlap presentation without redundant generators will always satisfy the strongest small overlap conditions satisfied by any presentation for the same monoid.

**Theorem 3.9.** Let $m$ be a positive integer, and $\langle \mathcal{A} \mid R \rangle$ be a generator-minimal $C(2)$ presentation for a monoid which admits a $C(m)$ presentation. Then $\langle \mathcal{A} \mid R \rangle$ is a $C(m)$ presentation.

**Proof.** If $m \leq 2$ then the claim is trivial, so suppose $m \geq 3$.

Suppose for a contradiction that $\langle \mathcal{A} \mid R \rangle$ does not satisfy $C(m)$. Then there is a relation word $r \in \mathcal{A}^*$ which can be written as a product of strictly fewer than $m$ pieces of $\langle \mathcal{A} \mid R \rangle$, say $r = r_1r_2\ldots r_j$ where $r_1, \ldots, r_j \in \mathcal{A}^*$ are pieces and $j < m$.

Now the monoid presented by $\langle \mathcal{A} \mid R \rangle$ admits a $C(m)$ presentation, so by Proposition 3.4 it admits a generator-minimal $C(m)$ equivalence presentation. Let $\langle \mathcal{B} \mid S \rangle$ be such a presentation and $\sigma : \langle \mathcal{A} \mid R \rangle \to \langle \mathcal{B} \mid S \rangle$ an isomorphism. Let $\hat{\sigma} : \mathcal{A}^* \to \mathcal{B}^*$ be the corresponding morphism of free monoids given by Theorem 3.6. Since $\hat{\sigma}$ is an inclusion it follows easily that $\hat{\sigma}(r)$ is a relation word in $\langle \mathcal{B} \mid S \rangle$, and $\hat{\sigma}(r_1), \ldots, \hat{\sigma}(r_j)$ are pieces of $\langle \mathcal{B} \mid S \rangle$. But since $\hat{\sigma}$ is a morphism, we have

$$\hat{\sigma}(r) = \hat{\sigma}(r_1)\hat{\sigma}(r_2)\ldots \hat{\sigma}(r_j)$$

which contradicts the assumption that $\langle \mathcal{B} \mid S \rangle$ satisfies the condition $C(m)$. □

We also obtain a corresponding statement for strong small overlap conditions.
Corollary 3.10. Let \( m \) be a positive integer, and \( \langle A \mid R \rangle \) be a generator-minimal strongly \( C(2) \) presentation for a monoid which admits a strongly \( C(m) \) presentation. Then \( \langle A \mid R \rangle \) is a strongly \( C(m) \) presentation.

Proof. By Theorem 3.10 \( \langle A \mid R \rangle \) is a \( C(m) \) presentation. Since it is strongly \( C(2) \) it contains no repeated relation words, which means it is strongly \( C(m) \). \( \square \)

Corollary 3.11. There is an algorithm which, given as input a monoid presentation, finds a generator-minimal \( C(m) \) equivalence presentation for the same monoid with \( m \geq 2 \) as high as possible, provided such a presentation exists (and may not terminate otherwise).

Proof. If the presentation satisfies \( C(m) \) for some \( m \geq 2 \) then by definition it satisfies \( C(2) \), and by Proposition 3.4 the monoid presented admits a generator-minimal \( C(2) \) equivalence presentation. By Theorem 3.9 any such presentation must also satisfy \( C(m) \), so it suffices to find such a presentation. It is well known and easy to show (using for example the theory of Tietze transformations [2, Section 7.2]) that there is an algorithm which, given as input any monoid presentation, recursively enumerates all presentations isomorphic to it. So it suffices to do this, checking if each is a generator-minimal \( C(2) \) equivalence presentation, until we find one which is. \( \square \)

Corollary 3.12. Let \( m \) be a positive integer. Then there is no algorithm to decide, given as input a monoid presentation, whether the monoid presented admits a \( C(m) \) presentation.

Proof. It is well known (see for example [2, Corollary 7.3.8]) that there is no algorithm to decide, given a monoid presentation, whether the monoid presented is trivial. The trivial monoid admits the empty presentation, which is a \( C(m) \) presentation for every \( m \). If there were an algorithm to decide whether a given monoid admits a \( C(m) \) presentation then one could decide whether the monoid was trivial, by first checking if it admits a \( C(m) \) presentation. If not then it cannot be trivial. If so then by Corollary 3.11 we could compute a \( C(m) \) presentation for the monoid, and by Theorem 3.8 we could check if it is isomorphic to the trivial monoid. \( \square \)

Using Theorem 3.8 and techniques from [6], we can obtain asymptotics for the number of non-isomorphic semigroups admitting \( a \)-generator, \( k \)-relation presentations of a certain length. Recall that for functions \( f, g : \mathbb{N} \to \mathbb{N} \) we say that \( f(n) \) is \( O(g(n)) \) [respectively, \( \Omega(g(n)), \theta(g(n)) \)] if \( f(n) \) is bounded above [respectively, below, above and below] by a linear function of \( g(n) \) with positive coefficients.

Theorem 3.13. Let \( a \) and \( k \) be fixed positive integers. Then there are \( \Theta(a^n n^{2k-1}) \) distinct isomorphism types of semigroups admitting \( a \)-generator, \( k \)-relation presentations of sum relation length \( n \).

Proof. The proof utilises some arguments from [6]; since this result is not the main purpose of the present paper we refrain from repeating these in detail and instead refer the reader to [6] for a more detailed explanation whether appropriate.
For ease of counting, we will first consider ordered presentations, by which we mean a pairs \( \langle A | R \rangle \) where \( A \) is an alphabet and \( R \) is a (finite or infinite) sequence of pairs of words over \( A \). As observed in [6], an ordered \( A \)-generated, \( k \)-relation monoid presentation of sum relation length \( n \) is uniquely determined by its shape (the sequence of lengths of relation words) together with the concatenation in order of the relation words (a word of length \( n \)).

Recall (from for example [1, Theorem 5.2]) that the number of weak compositions of \( s \) into \( r \) (that is, ordered sequences of \( r \) non-negative integers summing to \( s \)) is given by

\[
C'_r(s) = \frac{(s + r - 1)!}{s!(r - 1)!}.
\]

Note that if \( r \) is fixed and \( s \) remains variable then \( C'_r(s) \) is a polynomial of degree \( r - 1 \) in \( s \).

Let \( f : \mathbb{N} \to \mathbb{R} \) be a function. The total number of shapes of length \( n \) is \( C'_{2k}(n) \). As shown in the proof of [6, Lemma 3.4], the number of shapes featuring a block of size \( f(n) \) or less is bounded above by

\[
2k(f(n) + 1)C'_{2k-1}(n)
\]

so the number of such shapes not featuring such a block is bounded below by

\[
C'_{2k}(n) - 2k(f(n) + 1)C'_{2k-1}(n) = \frac{(n + 2k - 2)!}{n!(2k - 2)!} - 2k(f(n) + 1)\frac{(n + 2k - 2)!}{n!(2k - 2)!}
= \frac{(n + 2k - 2)!}{n!(2k - 2)!} \left( \frac{n + 2k - 2}{2k - 1} - 2k(f(n) + 1) \right)
= C'_{2k-1}(n) \left( \frac{n}{2k - 1} - 2k(f(n) + 1) + 1 \right)
= nC'_{2k-1}(n) \left( \frac{1}{2k - 1} - 2k(f(n) + 1) + 1 \right)
\]

Now fix an alphabet \( A \) of size \( a \). By [6, Lemma 3.2] there are at most \( n^2a^{a-f(n)} \) distinct words of length \( n \) which contain a repeated factor of length \( f(n) \), and so there are at least

\[
a^n - n^2a^{a-f(n)} = a^n \left( 1 - \frac{n^2}{a^{f(n)}} \right)
\]

words of length \( n \) which do not contain such a repeated factor. Combining such a word with a shape with no blocks of length \( f(n) \) or less clearly yields a presentation in which no relation word appears as a factor of any other relation word, that is a strongly \( C(2) \) presentation. Dividing by \( k! \) to allow for reordering the \( k \) relations, we have at least

\[
\frac{1}{k!} a^n \left( 1 - \frac{n^2}{a^{f(n)}} \right) C'_{2k-1}(n) n \left( \frac{1}{2k - 1} - 2k(f(n) + 1) + 1 \right)
\]

\( A \)-generated \( k \)-relation generator-minimal strongly \( C(2) \) unordered presentations of length \( n \).

Since a presentation presents the same monoid as its equivalence closure it follows by Corollary 3.7 that two such presentations present isomorphic
monoids if and only their equivalence closures are isomorphic presentations. Since a strongly $C(2)$ presentation must be transitive and irreflexive, the only difference between two strongly $C(2)$ presentations with the same equivalence closure is in the order of each relation, so at most $2^k$ distinct strongly $C(2)$ presentations have the same equivalence closure. And since an isomorphism of presentations over $A$ is determined by a bijection on $A$, each such equivalence closure is isomorphic to at most $a!$ others. Thus, at most $2^k a!$ distinct strongly $C(2)$ presentations present isomorphic monoids, so the strongly $C(2)$ presentations found above must present at least

$$\frac{1}{a! k! 2^k} a^n \left(1 - \frac{n^2}{a^n(n)}\right) C'_2(n) \frac{1}{n} (1 - \frac{1}{2k - 1} - \frac{2k(3 \log_a n + 1) + 1}{n})$$

distinct isomorphism types of monoids. Setting $f(n) = 3 \log_a n$ this becomes at least

$$\frac{1}{a! k! 2^k} a^n \left(1 - \frac{1}{n}\right) C'_2(n) \frac{1}{n} (1 - \frac{1}{2k - 1} - \frac{2k(3 \log_a n + 1) + 1}{n})$$

distinct isomorphism types. Now the factor

$$\frac{1}{a! k! 2^k}$$

is a positive constant, while the factors

$$\left(1 - \frac{1}{n}\right) \text{ and } \left(1 - \frac{1}{2k - 1} - \frac{2k(3 \log_a n + 1) + 1}{n}\right)$$

are eventually bounded below by positive constants and $C'_2(n)$ is a polynomial of degree $2k - 2$ in $n$. Thus, the number of distinct isomorphism types is $\Omega(a^n n^{2k-1})$.

Finally, the number of isomorphism types of semigroups admitting $a$-generator, $k$-relation presentations of length $n$ is clearly bounded above by the number of $k$-generator presentations over a fixed alphabet of size $a$. This is bounded above by the number of words of length $n$ times the number of weak compositions of $n$ into $2k$, that is $a^n C'_2(n)$. Since $C'_2(n)$ is a polynomial of degree $2k - 1$ in $n$, it follows that the number of presentations, and hence the number of isomorphism types, is $O(a^n n^{2k-1})$. □

A closer analysis of the combinatorics in the proof of Theorem 3.13 would of course yield more precise information about the number of isomorphism classes, as well as asymptotics applicable when $a$ and $k$ are permitted to vary.

4. Cancellativity

In this section we investigate the conditions under which the monoid presented by a $C(4)$ presentation is left cancellative, right cancellative or (two-sided) cancellative. It transpires that these properties can be characterised by very simple and natural conditions on the presentation, and from this it follows that one can check in linear time whether a given $C(4)$ monoid has any of these properties. Interestingly, it follows also that cancellativity properties are asymptotically visible properties of finite monoid presentations, in the sense that the probability that an $A$-generated, $k$-relation presentation
of length selected uniformly at random presents a (left, right or two-sided) cancellative monoid converges to a value strictly between 0 and 1, as the size of the presentation increases.

We begin with a technical definition. Let \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) be a C(4) presentation. We define a function \( \rho : \mathcal{A}^* \to \mathbb{Z} \) as follows. For \( w \in \mathcal{A}^* \) define \( \rho(w) = -1 \) if \( w \) has no clean overlap prefix; otherwise we define \( \rho(w) \) to be the length of the suffix of \( w \) following the clean overlap prefix. Notice that \( \rho \) takes integer values greater than or equal to \(-1\); we shall use it as an induction parameter.

**Lemma 4.1.** If \( w = XY Z w' \) where \( XY \) is a clean overlap prefix and \( p \) is a piece then \( \rho(pw') < \rho(w) \)

**Proof.** By definition we have \( \rho(w) = |Zw'| \geq 0 \). If \( pw' \) has no clean overlap prefix then again by definition we have \( \rho(pw') = -1 \) and we are done. Otherwise, suppose \( pw' \) has a clean overlap prefix \( aX'Y' \), say \( pw' = aX'Y'w'' \). Then \( \rho(pw') = |w''| \). Now \( X' \) is the maximum piece prefix of a relation word \( X'Y'Z' \), so \( X'Y' \) is not a piece. In particular \( X'Y' \) cannot be a factor of \( p \), so we must have that \( aX'Y' \) is strictly longer than \( p \), and so \( |w''| < |w'| \). Hence,

\[
\rho(pw') = |w''| < |w'| \leq |Zw'| = \rho(w)
\]
as required. \( \square \)

We are now ready to prove our main result of this section, which characterises C(4) equivalence presentations of left cancellative monoids.

**Theorem 4.2.** Let \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) be a C(4) equivalence presentation. Then the monoid presented is left cancellative if and only if \( \mathcal{R} \) contains no relation of the form \( (ar, as) \) where \( a \in \mathcal{A} \) and \( r, s \in \mathcal{A}^* \) with \( r \neq s \).

**Proof.** Suppose first that \( \mathcal{R} \) contains a relation \( (ar, as) \) for some letter \( a \in \mathcal{A} \) and words \( r, s \in \mathcal{A}^* \) with \( r \neq s \). Since the presentation satisfies C(2) and \( r \) is a proper factor of a relation word, it follows by Proposition 3.2 that \( r \neq s \). But by definition we have \( ar \equiv as \), so the monoid presented is not left cancellative.

Conversely, suppose \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) contains no relations of the given form, and suppose for a contradiction that the monoid presented by \( \langle \mathcal{A} \mid \mathcal{R} \rangle \) is not left cancellative. Then the generators are not left cancellable, so there exists a generator \( a \in \mathcal{A} \) and words \( u, v \in \mathcal{A}^* \) such that \( au \equiv av \) but \( u \neq v \). Let \( a, u \) and \( v \) be chosen such that these conditions hold and \( \rho(au) \) is as small as possible.

First, note that if \( au \) has no clean overlap prefix then by Proposition 2.3 we have \( au = av \) as words, so that \( u = v \) which gives a contradiction. Next, suppose that \( au \) has a clean overlap prefix \( wXY \) with \( w \) non-empty, say \( au = wXY u'' \). Then by Proposition 2.1 we have \( av = uv' \) where \( v' \equiv XY u'' \). But now \( w = au' \) for some word \( w \) and we have \( u = w'XY u'' \equiv w'v' = v \), again giving a contradiction.

There remains only the case that \( au \) has a clean overlap prefix of the form \( XY \). Then one of the six mutually exclusive conditions of Lemma 2.2 hold in respect of \( au \) and \( av \). Now conditions (3)-(6) are impossible because \( au \) and \( av \) begin with the same letter, while \( XY \) and the complement \( XY \).
cannot. Indeed, if they did begin with the same letter then so would the
distinct relation words $XY Z$ and $YXYZ$. But by symmetry and transitivity of
the presentation we have $(XY Z, XYZ) \in R$, so this would contradict the
assumption on the presentation.

If condition (2) holds we have $au = XY u'$, $av = XY v'$ where $u' \equiv v'$. But
$Y$ is non-empty, so we may write $XY = ar$ for some $r \in \mathcal{A}^*$, whereupon
$u = ru' \equiv ru' = v$ again giving a contradiction.

If condition (1) holds then $au = XY Zu''$ and $av = XY Zv''$ and $Zu'' \equiv Zv''$ for
some complement $Z$ of $Z$. We claim that the existence of any piece $q$ such that $qv'' \equiv qv''$ implies that $u'' \equiv v''$; the proof of this claim is by
induction on the length of $q$. For the base case, if $q$ is the empty word then
certainly we have $u'' = qv'' \equiv qv'' = v''$. Now suppose for induction that $q$
is not the empty word, that $qv'' \equiv qv''$ and that the claim holds for shorter
$q$. Write $q = bq'$ where $b \in \mathcal{A}$ and $q' \in \mathcal{A}^*$. Then by Lemma 4.1 we have
\[
\rho(bq'u'') = \rho(qu'') < \rho(XYZu'') = \rho(au)
\]
so by the minimality assumption on $a$, $u$ and $v$ we have $q'u'' \equiv q''v''$, which
by the inductive hypothesis implies that $u'' \equiv v''$. This completes the proof
of the claim. Now returning to the main proof, setting $q = Z$ we deduce
that $u'' \equiv v''$, and an argument similar to that in case (2) completes the
proof that $u \equiv v$, once again establishing the required contradiction.

The combinatorial condition given by Theorem 4.2 can clearly be checked
in polynomial time (more precisely, linear time in the RAM model of com-
putation) so we have:

**Corollary 4.3.** There is an algorithm which, given as input a $C(4)$ equiva-
rence presentation, decides in polynomial time whether the monoid presented
is left cancellative, right cancellative and/or cancellative.

**Corollary 4.4.** Let $\langle \mathcal{A} \mid I \rangle$ be a strongly $C(4)$ presentation. Then the
monoid presented is left cancellative if and only if $R$ contains no relation of
the form $(ar, as)$ where $a \in \mathcal{A}$ and $r, s \in \mathcal{A}^*$.

**Proof.** Since a strongly $C(4)$ presentation contains no repeated relation
words it is already transitive. Hence its equivalence closure $\langle \mathcal{A} \mid I \rangle$ can be
obtained simply by adding all relations of the form $(u, u)$, $(v, v)$ and $(v, u)$
where $(u, v)$ is a relation in $R$. Now $\langle \mathcal{A} \mid I \rangle$ presents the same monoid
as $\langle \mathcal{A} \mid R \rangle$, so by Theorem 4.2 $\langle \mathcal{A} \mid R \rangle$ is left cancellative exactly if $I$
contains no relation of the form $(ar, as)$ with $a \in \mathcal{A}$ and $r \neq s$. Clearly, this
is true exactly if $R$ contains no relation of the form $(ar, as)$ with $a \in \mathcal{A}$ and
$r, s \in \mathcal{A}^*$.

Our results in this section also have a consequence for the theory of generic
properties in monoids and semigroups [6]. In the terminology of generic-case
complexity, the following theorem says that left, right and two-sided cancel-
vativity are asymptotically visible properties of $A$-generated, $k$-relation
monoids. Such properties are of interest since they seem to be rather rare,
with most abstract properties of groups and semigroups tending to be ei-
ther generic or negligible amongst finitely presented examples. To avoid
defining large amounts of terminology for a single use, we state the result in
elementary combinatorial terms.
Theorem 4.5. Let $\mathcal{A}$ be an alphabet with $|\mathcal{A}| \geq 2$. Then the proportion $\mathcal{A}$-generated, $k$-relation monoid presentations of sum relation length $n$ (or of maximum relation length $n$) which present left cancellative [right cancellative] monoids approaches
\[
\left(\frac{|\mathcal{A}| - 1}{|\mathcal{A}|}\right)^k
\]
as $n$ tends in $\infty$. The proportion of $\mathcal{A}$-generated, $k$-relation monoid presentations of sum relation length $n$ (or of maximum relation length $n$) which present cancellative monoids approaches
\[
\left(\frac{|\mathcal{A}| - 1}{|\mathcal{A}|}\right)^{2k}
\]
as $n$ tends in $\infty$. In particular, for $|\mathcal{A}| \geq 2$, left [right] cancellativity is an asymptotically visible property of $\mathcal{A}$-generated $k$-relation presentations.

Proof. As with the proof of Theorem 3.13, we make use of some results and arguments from [6]. Rather than repeating these at length, we instead refer the interested reader to that paper.

By [6, Theorem 3.5] (or [6, Theorem 3.8] when the maximum relation length is considered), the proportion of ordered presentations which are not strongly $C(4)$ approaches 0 as $n$ increases. Corollary 4.4 tells us that a strongly $C(4)$ presentation presents a left [right] cancellative monoid exactly if it contains no relation of the form $ar = as$ [$ra = sa$] where $a \in \mathcal{A}$ and $r, s \in \mathcal{A}^*$, so to find the limit of the proportion of left [right] cancellative monoid, it suffices to compute the proportion of presentations satisfying this condition.

As described in the proof of Theorem 3.13 an ordered presentation is uniquely determined by its shape (a weak composition of $n$ into $2k$) and the concatenation in order of its relation words (a word over $\mathcal{A}$ of length $n$). Now for any shape not featuring a block of length 0, the proportion of words which do not yield two relation words beginning [ending] with the same letter is clearly exactly $(|\mathcal{A}| - 1)/|\mathcal{A}|)^k$. Similarly, for any shape not featuring a block of length 0 or 1 (so that the first and last positions of each relation word are distinct), the proportion of words which do not yield two relation words beginning with the same letter or ending with the same letter is exactly $(|\mathcal{A}| - 1)/|\mathcal{A}|)^{2k}$.

Now by [6, Lemma 3.4] (or [6, Lemma 3.6] when the maximum relation length is considered), the proportion of $\mathcal{A}$-generated, $k$-relation ordered presentations of length $n$ whose shape features a relation word of length 0 or 1 tends to zero as $n$ tends to infinity. It follows that we may ignore presentations with these shapes, and conclude that the proportion of $\mathcal{A}$-generated, $k$-relation ordered presentations of sum or relation length $n$ which are left cancellative, right cancellative and cancellative approaches
\[
\left(\frac{|\mathcal{A}| - 1}{|\mathcal{A}|}\right)^k, \left(\frac{|\mathcal{A}| - 1}{|\mathcal{A}|}\right)^k \text{ and } \left(\frac{|\mathcal{A}| - 1}{|\mathcal{A}|}\right)^{2k}
\]
respectively as $n$ tends to $\infty$.

Finally, we turn our attention to unordered presentations. By [6, Theorem 4.10] the proportion of unordered presentations of length $n$ which fail
to be strongly $C(4)$ approaches 0 at $n$ tends to $\infty$, so it suffices to compute the proportion of strongly $C(4)$ unordered presentations which present left cancellative [right cancellative, cancellative] monoids. However, since a strongly $C(4)$ presentation contains no repeated relation words, each $k$-relation unordered strongly $C(4)$ presentation corresponds to precisely $k!$ distinct ordered strongly $C(4)$ presentations. Since moreover every ordered strongly $C(4)$ presentation arises in this way, it follows that the required proportions are the same for unordered presentations as for ordered presentations.

In fact the argument in the last paragraph of the proof of Theorem 4.5 establishes the following fact, which generalises [6, Theorem 4.10].

**Theorem 4.6.** Let $\mathcal{C}$ be any class of abstract monoids, $\mathcal{A}$ an alphabet with $|\mathcal{A}| \geq 2$ and $k$ a positive integer. Then the proportion of $\mathcal{A}$-generated $k$-relation ordered monoid presentations of sum [maximum] relation length $n$ which present monoids in $\mathcal{C}$ converges with the proportion of $\mathcal{A}$-generated $k$-relation monoid presentations of sum [maximum] relation length $n$ which present monoids in $\mathcal{C}$ as $n$ tends to $\infty$.

**Acknowledgements**

This research was supported by an RCUK Academic Fellowship. The author would like to thank S. W. Margolis and N. Ruskuc for posing some of the questions considered in this paper.

**References**

[1] M. Bóna. *A walk through combinatorics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
[2] R. V. Book and F. Otto. *String-rewriting Systems*. Springer-Verlag, 1993.
[3] A. Duncan and R. H. Gilman. Word hyperbolic semigroups. *Math. Proc. Cambridge Philos. Soc.*, 136(3):513–524, 2004.
[4] P. M. Higgins. *Techniques of semigroup theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992. With a foreword by G. B. Preston.
[5] D. A. Jackson. On the invariance of small overlap hypotheses. *Semigroup Forum*, 43:299–304, 1991.
[6] M. Kambites. Generic complexity of finitely presented monoids and semigroups. *Computational Complexity* (to appear).
[7] M. Kambites. A note on the definition of small overlap monoids. arXiv:0910.4511v1 [math.GR], 2009.
[8] M. Kambites. Small overlap monoids I: the word problem. *J. Algebra*, 321:2187–2205, 2009.
[9] M. Kambites. Small overlap monoids II: automatic structures and normal forms. *J. Algebra*, 321:2302–2316, 2009.
[10] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory*. Springer-Verlag, 1977.
[11] A. Yu. Ol’shanski˘ı. Almost every group is hyperbolic. *Internat. J. Algebra Comput.*, 2(1):1–17, 1992.
[12] J. H. Remmers. *Some algorithmic problems for semigroups: a geometric approach*. PhD thesis, University of Michigan, 1971.
[13] J. H. Remmers. On the geometry of semigroup presentations. *Adv. in Math.*, 36(3):283–296, 1980.
[14] J. Sakarovitch. *Easy multiplications I. The realm of Kleene’s theorem*. *Inform. and Comput.*, 74:173–197, 1987.