The Cantor locus is the unique hyperbolic component, in the moduli space of quadratic rational maps \( \text{rat}_2 \), consisting of maps with totally disconnected Julia sets. Whereas the Cantor locus is well understood, its boundary and dynamical relation to the boundary is not. In this paper we explore the boundary of the Cantor locus near parabolic boundary points. We do this by constructing a local parametrization, which fixes the dynamical positions of the critical values, relative to the parabolic basin of the quadratic polynomial \( P_{\omega_{p/q}}(z) = \omega_{p/q}z + z^2, \omega_{p/q} = e^{2\pi ip/q} \). We characterize which pairs, of dynamical positions and parabolic boundary points, give rise to bounded and unbounded sequences in \( \text{rat}_2 \) respectively. Moreover, we show that this approach controls the dynamical positions of the critical values, for any limit point in the bounded case, and for any limit point of a rescaled sequence of \( q \)th iterates in the unbounded case.
3 Stars in quadratic attracting basins

4 A model for $R^λ$

5 Proof of Proposition 1 and Theorems 1 and 2

1 Introduction

A rational map is *hyperbolic* if and only if the orbit of every critical point converges to some attracting periodic orbit. The subset of moduli space consisting of hyperbolic maps is open, its connected components are called *hyperbolic components*. In the moduli space $rat_2$, of all quadratic rational maps up to Möbius conjugacy, any hyperbolic component, where maps have an attracting fixed point, is unbounded. Epstein shows [E2] that hyperbolic components with two distinct attracting cycles have compact closure in $rat_2$ if and only if neither attractor is a fixed point. One of the tools for this is a detailed understanding of unbounded sequences in $rat_2$, which have a fixed point eigenvalue tending to a $q$th root of unity. In particular [E2] gives algebraic conditions under which a sequence of maps $G_{λ_k, A_k}(z) = \frac{1}{λ_k}(z + A_k + \frac{1}{z})$, so that $λ_k$ tends to a $q$th root of unity and $[G_{λ_k, A_k}]$ diverges to infinity in $rat_2$, has a $q$th iterate that converges, locally uniformly on $C^*$, to a quadratic rational map. We refer to this as rescaling.

In $rat_2$ there is a unique hyperbolic component consisting of maps with totally disconnected Julia sets. This means the Julia sets are homeomorphic to the standard Cantor set. This component has various names, such as the *hyperbolic escape locus* or the *shift locus*, here we call it the *Cantor locus*. This component is well-understood: it consists of maps, where both critical orbits converge to the same attracting fixed point, the dynamics on the Julia set is conjugate to the one-sided shift on two symbols, and the Cantor locus is homeomorphic to $D \times (C \setminus \bar{D})$. [GK], [M1].

In this paper we explore the boundary of the Cantor locus near parabolic boundary points. The largely algebraic conditions in [E2] do not give enough information in this situation. Instead we construct a local parametrization (for details see Proposition 1) well adapted to the dynamical situation near such points. The parametrization fixes the dynamical position of the critical values, relative to the parabolic basin of a quadratic polynomial with a parabolic fixed point of the given eigenvalue. Using this parameter as a
steering mechanism, we study sequences in the Cantor locus, for which the dynamical positions of the critical values are fixed by this parametrization and a fixed point eigenvalue tends to a \( q \)th root of unity. We characterize which pairs, of dynamical positions and parabolic boundary points, give rise to bounded and unbounded sequences in \( \text{rat}_2 \) respectively. Moreover, we show that this approach controls the dynamical positions of the critical values, for any limit point in the bounded case, and for any limit point of a rescaled sequence of \( q \)th iterates in the unbounded case.

According to Milnor [M1], \( \text{rat}_2 \) is isomorphic to \( \mathbb{C}^2 \), with a preferred affine structure in which the loci

\[
\text{Per}_1(\lambda) = \{[f] \in \text{rat}_2 : \text{some fixed point of } f \text{ has eigenvalue } \lambda\}
\]

are lines. A conjugacy class \([f] \in \text{rat}_2\) is uniquely determined by the three fixed point eigenvalues \( \lambda, \mu, \nu \) of \( f \). For any ordered pair \((\lambda, \sigma)\) of complex numbers, there is a unique conjugacy class \( \Gamma_{\lambda,\sigma} \) consisting of maps such that some fixed point has eigenvalue \( \lambda \) and such that the product of the eigenvalues of the two other fixed points is \( \sigma \). The map

\[
\mathbb{C}^2 \ni (\lambda, \sigma) \mapsto \Gamma_{\lambda,\sigma} \in \text{rat}_2
\]

is generically three-to-one. For fixed \( \lambda \in \mathbb{C} \) the induced map

\[
\mathbb{C} \ni \sigma \mapsto \Gamma_{\lambda,\sigma} \in \text{Per}_1(\lambda)
\]

is an isomorphism, in fact it is a natural affine parametrization of \( \text{Per}_1(\lambda) \) [M1] Remark 6.9]. For \( \text{Per}_1(0) \) this isomorphism is given by

\[
\sigma \mapsto [Q_{\sigma}^4], \text{ where } Q_c(z) = z^2 + c.
\]

Let \( J(f) \) denote the Julia set of the rational map \( f \). Let \( M \) denote the Mandelbrot set

\[
M = \{c \in \mathbb{C} : J(Q_c) \text{ is connected}\}
\]

and let \( \mathcal{C} \) denote the connectedness locus for \( \text{rat}_2 \)

\[
\mathcal{C} = \{(\lambda,\sigma) \in \mathbb{C}^2 : \text{ any map in } \Gamma_{\lambda,\sigma} \text{ has connected Julia set}\}.
\]

For each \( \lambda \in \mathbb{D} \cup \{1\} \) let

\[
M^\lambda = \{\sigma \in \mathbb{C} : (\lambda,\sigma) \in \mathcal{C}\}.
\]
For $\lambda \in S^1$ we will use the notation $\omega_\theta = e^{2\pi i \theta}$ for any value of $\theta$, with the convention that $\omega_{p/q} = e^{2\pi ip/q}$ always denotes a primitive $q$-th root of unity.

Milnor shows [M1] that the Julia set for a quadratic rational map is either connected or totally disconnected, and that it is totally disconnected if and only if $\lambda \in \mathbb{D} \cup \{1\}$ and the two critical points are in the same Fatou component. This means that for $\lambda \in S^1 \setminus \{1\}$ the entire line $\text{Per}_1(\lambda)$ belongs to the connectedness locus $C$, and the above definition of $M^\lambda$ is unsuitable to that generalization. For $\lambda \in \mathbb{D} \cup \bigcup_{p/q} \{\omega_{p/q}\}$ we consider instead the following subset of $\mathbb{C}$:

- $R^\lambda$ consisting of all $\sigma \in \mathbb{C}$ such that both critical orbits of any $f \in \Gamma_{\lambda,\sigma}$ converge to the fixed point with eigenvalue $\lambda$.

We also call $R^\lambda$ the relatedness locus in $\text{Per}_1(\lambda)$. For $\lambda \in \mathbb{D}$, $R^\lambda = \mathbb{C} \setminus M^\lambda$ and for $\lambda = 1$, $R^1$ is the complement of $M^1$, together with the set of parameters $\sigma \in \mathbb{C}$ so that every $f \in \Gamma_{1,\sigma}$ has a critical value which is pre-fixed to the parabolic fixed point of eigenvalue 1.

Let furthermore $\mathcal{R}$ denote the Cantor locus, we see that

- $\mathcal{R} = \{(\lambda, \sigma) : \lambda \in \mathbb{D}$ and $\sigma \in R^\lambda\}$, equivalently $\mathcal{R}$ is the set of all $(\lambda, \sigma) \in \mathbb{C}^2$ so that both critical orbits of any $f \in \Gamma_{\lambda,\sigma}$ converge to a common attracting fixed point.

Yet another candidate for a name for this hyperbolic component is then the relatedness locus. We will however use the name Cantor locus.

It follows from the theory of polynomial-like maps that there is a holomorphic motion $\Psi : \mathbb{D} \times M \to \mathbb{C} \cap (\mathbb{D} \times \mathbb{C})$ of the form $\Psi(\lambda, c) = (\lambda, \Psi^\lambda(c))$. In [EU] sequences $(\lambda_k, c_k)$ which give rise to bounded sequences $(\lambda_k, \Psi^{\lambda_k}(c_k)) \in \mathcal{C}$ as $\lambda_k$ tends to $\omega_\theta \in S^1$ are characterized, focusing on sequences $c_k$ which lie in the closure of some hyperbolic component.

The objective of this paper is similar; to characterize sequences $(\lambda_k, \sigma_k) \in \mathcal{R}$, where $\lambda_k \to \omega_{p/q}$ subhorocyclicly and with $\sigma_k$ corresponding to given dynamical positions of the critical values. We say that a sequence $\lambda_k \in \mathbb{D}$ converges to $\omega \in S^1$ subhorocyclicly if $\Im(\omega \lambda_k) = o \left( \sqrt{1 - |\lambda_k|^2} \right)$. More geometrically, this condition asserts that $\lambda_k$ is eventually contained in any $\mathbb{D}$-horodisc at $\omega$. 
The holomorphic motion mentioned above is of little use for this application: although $\Psi^\lambda$ extends to all of $\mathbb{C}$, it carries no dynamical information on $\mathbb{C} \setminus M$. Instead we define a new map in Proposition 1 below, which parametrizes parts of $\mathcal{R}$, by $\lambda$ and the position of the critical values of a map $f \in \Gamma_{\lambda,\sigma}$, relative to the basin of the polynomial $P_{\omega_{p/q}}$.

The polynomial $P_{\omega_{p/q}}$. Consider the polynomial $P_{\omega_{p/q}} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, given by:

$$P_{\omega_{p/q}}(z) = \omega_{p/q}z + z^2.$$  

For the discussion in this paper we fix a $p/q$, with $(p,q) = 1$ and $q \geq 2$, and in order to ease the notation we will everywhere suppress the dependence on $p/q$. In particular, from now on $P$ will be used in place of $P_{\omega_{p/q}}$.

$P$ has a parabolic fixed point at 0 with eigenvalue $\omega_{p/q}$ and a critical point at $-\omega_{p/q}/2$, with critical value $-\omega_{p/q}^2/4$. Let $A$ denote the parabolic basin of 0, and $\phi : A \rightarrow \mathbb{C}$ an extended Fatou coordinate; a surjective, holomorphic map, which satisfies the functional equation:

$$\phi \circ P = 1/q + \phi.$$  

The Fatou coordinate will be normalized by setting $\phi(-\omega_{p/q}/2) = 0$. The parabolic basin $A$ has a $q$-cycle of components, whose closures meet only at the parabolic fixed point 0. These $q$ components perform a $p/q$ rotation around the fixed point 0, under iteration by $P$. They will be indexed, counterclockwise with respect to their cyclic ordering, $B^0, ..., B^{q-1}$, where $B^0$ is the component containing the critical point $-\omega_{p/q}/2$.

Removing the parabolic fixed point 0, separates the closure of the basin, $\overline{A}$, into $q$ connected components, we will let $S^p$ denote the component containing the critical value $-\omega_{p/q}^2/4$, then $B^p \subset S^p$.

For $\lambda \in \mathbb{D} \setminus [0, -\omega_{p/q}]$, define the number $m(\lambda) = \frac{2\pi \sin \theta}{q|L|}$, where $L = q\log(\lambda \omega_{-p/q})$ (Log denotes the principal branch of log) and $\theta$ is the angle between $L$ and $2\pi i$, see also section 3. For $M > \frac{1}{q^2}$, let

$$D_M = \{ \lambda \in \mathbb{D} : m(\lambda) \geq M \},$$

it is the $\mathbb{D}^*$ horodisk $\exp(\mathbb{D}(-r + 2\pi ip/q, r))$ at $\omega_{p/q}$, where $r = \frac{\pi}{q^2 M} < \pi$. Let

$$\Xi_M = \{ z \in A : |\Im(\phi(z))| < M/2 \} \cup \bigcup_{n \geq 0} P^n(0).$$
Let \( T^j \) denote the component of \((\phi^j)^{-1}(\{z \in \mathbb{R} : z \geq \frac{k(j)}{q}\})\) containing 0, where \( k(j) \) is the representative of \( j/p \in \mathbb{Z}_q \) in \( \{2, \ldots, q + 1\} \) and \( \phi^j \) is the restriction of \( \phi \) to \( B^j \). Let \( T = \bigcup_{j \in \mathbb{Z}_q} T^j \) and \( \Xi_M^* = \Xi_M \setminus (T \cup \{\frac{-\omega^2}{4}\}) \).

**Proposition 1** (and Definition of \( \Phi \)). For every \( M > \frac{1}{q^2} \), and every \( p/q \), \((p,q) = 1\), \(q \geq 2\) there exists a local parametrization \( \Phi : D_M \times \Xi_M^* \to D_M \times \mathbb{C} \) of the form \( (\lambda, x) \mapsto (\lambda, \Phi^x(\lambda)) \), which is a continuous, horizontally analytic map into \( \mathcal{R} \), in particular:

- \( \Phi^\lambda(\Xi_M^*) \subset \mathcal{R}^\lambda \) for every \( \lambda \in D_M \)
- \( \Phi^x \) is analytic in \( D_M \) for every \( x \in \Xi_M^* \) (horizontal analyticity)
- \( \Phi^\lambda \) is continuous in \( \Xi_M^* \) and analytic in \( \text{int}(\Xi_M^*) \) for every \( \lambda \in D_M \).
- \( \Phi \) is continuous in \( D_M \times \Xi_M \).

For \( x \in \Xi_M^* \) and \( \lambda \in D_M \) let \( \sigma = \sigma(\lambda, x) := \Phi^\lambda(x) \in \mathcal{R}^\lambda \) and let \( v_1 \) and \( v_2 \) denote the critical values of \( g_{\lambda, \sigma} \in \Gamma_{\lambda, \sigma} \).

- For each \( x \in \Xi_M^*, \lambda \in D_M, \) and for \( M \leq m \leq m(\lambda) \), there exists a dynamical conjugacy \( \Psi_{\lambda, \sigma} : V_m(x) \to \mathring{\mathbb{C}} \) between \( P \) and \( g_{\lambda, \sigma} \),

\[
\Psi_{\lambda, \sigma} \circ P = g_{\lambda, \sigma} \circ \Psi_{\lambda, \sigma},
\]
whose domain \( V_m(x) \) is a connected, forward invariant subset of \( \Xi_m \), so that \( x, -\omega_{p/q}^2/4 \in V_m(x) \). The map \( \Psi_{\lambda, \sigma} \) is a homeomorphism, holomorphic on the interior of \( V_m(x) \), and satisfies

\[
\Psi_{\lambda, \sigma}(\{x, -\omega_{p/q}^2/4\}) = \{v_1, v_2\}.
\]

For precise definitions of \( \Phi^\lambda : \Xi_M^* \to \mathbb{C} \), its dynamical counterpart \( \Psi_{\lambda, \sigma} \) and its domain \( V_m(x) \), see section 5 and the proof of Proposition 1. We remark that the continuity of \( \Phi \) follows from the horizontal analyticity and the continuity of \( \Phi^\lambda \) in much the same way as continuity of holomorphic motions is proven in the \( \lambda \)-lemma (cf. [MSS]). We also remark that \( \Phi \) is not an embedding of the full domain into \( \mathcal{R} \), however making alterations to \( \Xi_M \) the map \( \Phi \) could be made to be an embedding into \( \mathcal{R} \). Since it is not needed for the present application that \( \Phi \) is an embedding, we will leave out the technicalities here.
Theorem 1. Let \( x \in \Xi_M^* \), and let \( \lambda_k \in D_M \) be a sequence converging to \( \omega_{p/q} \) subhorocyclicly. Set \( \sigma_k = \Phi^{\lambda_k}(x) \). If \( x \notin S^p \), then

A. \( \sigma_k \) is bounded

B. for any limit point \( \sigma \) of \( (\sigma_k) \): \( \sigma \in \mathcal{R}^{\omega_{p/q}} \) and there is a homeomorphism \( \Psi : V_m(x) \rightarrow \hat{\mathbb{C}} \), holomorphic on the interior of \( V_m(x) \), and such that

\[
\Psi \circ P = f \circ \Psi \quad \text{and} \quad \Psi(\{x, -\omega_{p/q}/4\}) = \{v_1, v_2\},
\]

for any representative \( f \in \Gamma_{\omega_{p/q}, \sigma} \) with critical values \( v_1 \) and \( v_2 \).

In fact, because of the way we control the position of the critical values in the limit, \( \sigma \) is independent of the sequence \( \lambda_k \), as shown in [U1] and [U2], which implies that \( \Phi^{\lambda_k} \) converges to a local parametrization of \( \mathcal{R}^{\omega_{p/q}} \).

Theorem 2. Let \( x \in \Xi_M^* \), and let \( \lambda_k \in D_M \) be a sequence converging to \( \omega_{p/q} \) subhorocyclicly. Set \( \sigma_k = \Phi^{\lambda_k}(x) \). If \( x \in S^p \), then \( \sigma_k \rightarrow \infty \).

Moreover, choosing representatives \( G_{\lambda_k, A_k}(z) = \frac{1}{\lambda_k}(z + A_k + \frac{1}{z}) \) where \( A_k^2 = (\lambda_k - 2)^2 - \sigma_k\lambda_k^2 \), so that \( G_{\lambda_k, A_k} \in \Gamma_{\lambda_k, \sigma_k} \), and where \( \Psi_k(-\omega_{p/q}/2) = 1 \) (which fixes a choice of \( A_k \)):

A. \( G_{\lambda_k, A_k} \rightarrow \infty \) locally uniformly on \( \mathbb{C}^* \)

B. \( G_{\lambda_k, A_k}^0 \rightarrow G_T \), \( G_T(z) = z + T + \frac{1}{z} \) locally uniformly on \( \mathbb{C}^* \), where \( T \in \mathbb{C} \) and \( \sigma = 1 - T^2 \in \mathcal{R}^1 \).

If \( x \in B^p \), then the sequence of conjugacies \( \Psi_{\lambda_k, \sigma_k} \) converges locally uniformly on \( V_M(x) \cap B^0 \) to a univalent map \( \Psi_\infty \), satisfying

\[
\Psi_\infty \circ P^q = G_T \circ \Psi_\infty,
\]

\[
\Psi_\infty(P^{q-1}(x)) = G_T(-1) \quad \text{and} \quad \Psi_\infty(P^q(-\omega_{p/q}/2)) = G_T(1),
\]

and both critical points \( \pm 1 \) of \( G_T \) are in the same component of the parabolic basin of \( \infty \), i.e. \( \sigma \in \mathbb{C} \setminus M^1 \).

If \( x \in S^p \setminus B^p \) then there exists \( n \geq 1 \) such that \( G_T(-1) = 0 \), so that \( \sigma \in \mathcal{R}^1 \cap M^1 \).

There are three main tools for the proofs. The behaviour of sequences that diverge to infinity in \( \text{rat}_2 \) treated in [E2] and summarized for our use in section 2. A construction called stars in attracting basins, introduced in [P].

We define the construction in section 3 and extend some of the results from [P] to the case of non-simply connected quadratic basins. Lastly, a model for \( \mathcal{R}_\lambda \), which is modified from a model introduced in [GK], given in section 4.
2 Preliminaries

2.1 Normal forms

For polynomials with an attracting fixed point we use the notation 
\[ P_\lambda(z) = \lambda z + z^2, \lambda \in \mathbb{D}^*. \]

\( P_\lambda \) has an attracting fixed point at 0 with eigenvalue \( \lambda \) and a critical point at \( -\frac{\lambda}{2} \), with critical value \( -\frac{\lambda^2}{4} \).

Each class \( \Gamma_{\lambda,\sigma} \) in \( \text{Per}_1(\lambda) \) has a representative of the form
\[ G_{\lambda,A}(z) = \frac{1}{\lambda} \left( z + A + \frac{1}{z} \right). \]

(2)

Maps \( G_{\lambda,A} \) have critical points \( \pm 1 \) and a fixed point at \( \infty \) with eigenvalue \( \lambda \).

Note that \([G_{\lambda,A}] = [G_{\lambda,-A}] = \Gamma_{\lambda,\sigma}\) where
\[ \sigma = \frac{(\lambda - 2)^2 - A^2}{\lambda^2}. \]

(3)

For the case \( \lambda = 1 \) we use the notation:
\[ G_T(z) = G_{1,T}(z) = z + T + \frac{1}{z}, \quad G_T \in \Gamma_{1,\sigma}, \]

(4)

where \( \sigma = 1 - T^2 \). Maps \( G_T \) have critical points \( \pm 1 \) and a double fixed point at \( \infty \) with eigenvalue 1.

2.2 Divergence to infinity in \( \text{rat}_2 \)

We summarize here some results from [E2] and [M1] concerning the dynamical behavior of sequences that diverge to infinity in \( \text{rat}_2 \). For the formulation given here, see also [K] and the discussion in [EU].

A sequence is bounded in \( \text{rat}_2 \) if and only if the corresponding sequence of fixed point eigenvalues, \( \{\lambda_k, \mu_k, \nu_k\} \) is bounded in \( \mathbb{C} \). The well-known holomorphix index formula, which for \( \lambda \neq 1 \neq \mu \) can be written
\[ \nu = \frac{2 - \lambda - \mu}{1 - \lambda \mu}, \]

shows that if \( \lambda_k, \mu_k \) tend to \( \lambda, \frac{1}{\lambda} \), for some \( \lambda \in \mathbb{C} \setminus \{1\} \), then \( \sigma_k = \mu_k \nu_k \to \infty \).

For the other direction, any unbounded sequence in \( \text{rat}_2 \) has a subsequence, such that the fixed point eigenvalues tend to some unordered triple \( \lambda, \frac{1}{\lambda}, \infty \), where \( \lambda \in \mathbb{C} \). Here we consider sequences \( \Gamma_{\lambda_k,\sigma_k} \) where \( \lambda_k \to \omega_{p/q} \).
Lemma 1 ([E2]). Let $f_k$ be a sequence of quadratic rational maps, with fixed point eigenvalues $\lambda_k, \mu_k, \nu_k$ tending to $\omega_{p/q}, \bar{\omega}_{p/q}, \infty$, $q \geq 2$. If $f_k = G_{\lambda_k,A_k}$ then

$$f^l_k \to \infty \quad \text{locally uniformly on } \mathbb{C}^* \text{ for } 1 \leq l < q$$

and on passing to a subsequence if necessary

$$f^q_k \to G_T \quad \text{locally uniformly on } \mathbb{C}^* \text{ for some } T \in \hat{\mathbb{C}},$$

where we use the convention $G_{\infty} = \infty$. Moreover, if $T \in \mathbb{C}$, and $\lambda_k \neq \omega_{p/q}$, $\mu_k \neq \bar{\omega}_{p/q}$, then there are $q$-cycles $\langle z \rangle_k$ with eigenvalues $\rho_k \to 1 - T^2$, while for any other $q$-cycles $\langle \hat{z} \rangle_k \neq \langle z \rangle_k$ the eigenvalues tend to infinity.

3 Stars in quadratic attracting basins

Stars for attracting basins in general were introduced by Petersen in [P], where the case of simply connected quadratic basins is studied. Here the objective is quadratic rational maps who have attracting basins with two critical points, which implies that the basin is infinitely connected. We will give an overview of the construction and properties of stars, and refer to [P] for more background.

Let $g_{\lambda, \sigma} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quadratic rational map $g_{\lambda, \sigma} \in \Gamma_{\lambda, \sigma}$, for some $\lambda \in \mathbb{D}^*$ and some $\sigma \in \mathbb{C}$, with the attracting fixed point of eigenvalue $\lambda$ at $z_0$, so that $g_{\lambda, \sigma}(z_0) = z_0$ and $g'_{\lambda, \sigma}(z_0) = \lambda$. Let $A_{\lambda, \sigma}$ denote the attracting basin of $z_0$. Such a map has an extended linearizing coordinate $\phi_{\lambda, \sigma} : A_{\lambda, \sigma} \to \mathbb{C}$, a holomorphic map so that $\phi_{\lambda, \sigma}(z_0) = 0$ and $\phi_{\lambda, \sigma} \circ g_{\lambda, \sigma} = \lambda \cdot \phi_{\lambda, \sigma}$. The linearizing coordinate is univalent on some neighborhood of the fixed point.

If $\sigma \in M^\lambda$, then there is only one critical point in $A_{\lambda, \sigma}$, which will be denoted $c_1$. If $\sigma \in R^\lambda$, then there are two critical points in $A_{\lambda, \sigma}$, denoted $c_1$ and $c_2$. The corresponding critical values are denoted $v_1$ and $v_2$. In our notation $g_{\lambda, \sigma}$ denotes a general quadratic rational map and could be a polynomial ($\sigma = 0$). When there is need to emphasize that the map in question is a polynomial, we will use the notation introduced in section [2]: we choose the specific representative $P_{\lambda} \in \Gamma_{\lambda, 0}$, and use $A_{\lambda}$ and $\phi_{\lambda}$ to denote the attracting basin and linearizing coordinate respectively. In all cases (if necessary after an appropriate choice of $c_1$) the linearizing coordinate is normalized by setting $\phi_{\lambda, \sigma}(c_1) = 1$. 
3.1 Definition of the \((\log \lambda, p/q)\)-star

Given \(\lambda \in \mathbb{C}^* \setminus S^1\) and an irreducible rational number \(p/q\), \((p, q) = 1\), we define the vector \(L = L(p/q) = q \log(\lambda e^{-i2\pi p/q}) = q \log \lambda - p2\pi i\); here \(\text{Log} : \mathbb{C} \setminus [-\infty, 0] \to \mathbb{C}\) is the principal branch of \(\log\) and \(\log \lambda\) is the appropriate choice of logarithm of \(\lambda\). We define also the number

\[
 r_\lambda = \frac{|L|}{2q \sin \theta},
\]

where \(\theta\) is the angle between \(L\) and \(2\pi i\). The number \(r_\lambda\) is the radius of the circle through \(\log \lambda\), and tangent to the imaginary axis at \(2\pi ip/q\). For the following \(L\) and \(p/q\) are fixed.

Let \(c_i, i \in I\), denote the critical points in the basin, which are not in the backwards orbit of \(z_0\), where \(I = \{1, 2\}\) if \(\sigma \in \mathcal{R}^\lambda\) and neither critical point is in the backwards orbit of \(z_0\), and \(I = \{1\}\) otherwise. Let \(\zeta_i\) denote a logarithm of \(\phi_{\lambda,\sigma}(c_i)\). The logarithm \(\zeta_1\) of \(\phi_{\lambda,\sigma}(c_1) = 1\) can be chosen to be 0. Consider the lattices:

\[
 \Lambda_i = \{n \log \lambda + m2\pi i + \zeta_i : n, m \in \mathbb{Z}\}
\]

\[
 \Lambda = \bigcup_{i \in I} \Lambda_i.
\]

The image of the lattice \(\Lambda_i\) under the exponential function contains the non-zero critical values of \(\phi_{\lambda,\sigma}\), which come from pre-images of the critical point \(c_i\), \(\{\lambda^{-n}\phi_{\lambda,\sigma}(c_i) : n \geq 0\}\), whence the image of \(\Lambda\) contains all non-zero critical values of \(\phi_{\lambda,\sigma}\).

Let \(\widetilde{\tau}_i^j\) denote lines parallel with \(L\) through the lattice \(\Lambda_i\):

\[
 \widetilde{\tau}_i^j := tL + \frac{j}{q}2\pi i + \zeta_i \quad \text{for } t \in \mathbb{R}, j \in \mathbb{Z} \text{ and } i \in I.
\]

Moreover set

\[
 \widetilde{\tau}_i = \bigcup_{j \in \mathbb{Z}} \widetilde{\tau}_i^j \quad \text{and} \quad \widetilde{\tau} = \bigcup_{i \in I} \widetilde{\tau}_i,
\]

where lines in \(\widetilde{\tau}\) are named \(\widetilde{\tau}_i^j\) according to their horizontal ordering and so that \(\widetilde{\tau}_0 = \widetilde{\tau}_i^0\). Note that when \(I = \{1, 2\}\), \(\widetilde{\tau}_i^j \neq \widetilde{\tau}_i^{j+1}\) in general, since each \(\widetilde{\tau}_i^j\) is situated between \(\widetilde{\tau}_i^{j'}\) and \(\widetilde{\tau}_i^{j'+1}\) for some \(j'\) and vice versa, and that \(\widetilde{\tau} = \widetilde{\tau}_1 = \widetilde{\tau}_2\) if and only if \(\zeta_2 \in \widetilde{\tau}_1\).
The collection \( \tilde{\tau} \) projects by the exponential map to \( k \) \( q \)-cycles, under multiplication by \( \lambda \), of logarithmic spirals, where \( k = 1 \) if \( I = \{1\} \) or \( \zeta_2 \in \tilde{\tau}_1 \) and \( k = 2 \) otherwise. For \( j \in \mathbb{Z}_q \) let \( \hat{\gamma}_j = \exp(\tilde{\gamma}_j) \). Each \( q \)-cycle \( \hat{\tau}_i = \exp(\tilde{\tau}_i) \) of logarithmic spirals is \( p/q \) rotated under the map \( z \mapsto \lambda z \). All non-zero critical values of \( \phi_{\lambda, \sigma} \) are contained in \( \hat{\tau} = \exp(\tilde{\tau}) \).

The collection of lines \( \tilde{\gamma} \) bound open strips \( \tilde{U}_j \), \( j \in \mathbb{Z} \), which are named according to their horizontal ordering and so that \( \tilde{U}^0 \) is bounded below by \( \tilde{\tau}_0 \). The strips are invariant under the translation \( z \mapsto z + L \), and each \( \tilde{U}_j \) is mapped to \( \tilde{U}_{j+ \mathbb{Z}} \) under the translation \( z \mapsto z + \log \lambda \).

Let \( \tilde{\gamma} = \bigcup_{j \in \mathbb{Z}} \tilde{\gamma}_j \) denote a collection of straight lines \( \tilde{\gamma}_j \subset \tilde{U}_j \) parallel to \( L \), and so that \( \tilde{\gamma}_j \) is mapped to \( \tilde{\gamma}_{j+ \mathbb{Z}} \) by \( z \mapsto z + \log \lambda \). In most cases it is convenient to choose \( \tilde{\gamma}_j \) as central straight lines in \( \tilde{U}_j \). Each curve \( \tilde{\gamma}_j \) is invariant under \( z \mapsto z + L \).

The collection \( \tilde{\gamma} \) projects to \( k \) \( q \)-cycles of logarithmic spirals by the exponential map, and the strips \( \tilde{U}_j \) to \( k \) \( q \)-cycles of "strips". Each \( q \)-cycle of logarithmic spirals or "strips" is \( p/q \) rotated under the map \( z \mapsto \lambda z \).
The \((\log \lambda, p/q)\)-star of \(A_{\lambda,\sigma}\) for \(g_{\lambda,\sigma}\). For \(j \in \mathbb{Z}_{kq}\), let \(U^j \subset A_{\lambda,\sigma}\) be the unique connected component of \(\phi^{-1}_{\lambda,\sigma}(\exp(\tilde{U}^j))\) with \(z_0\) on the boundary. In \([\mathcal{P}]\), the \((\log \lambda, p/q)\)-star of \(A_{\lambda,\sigma}\) for \(g_{\lambda,\sigma}\), here denoted \(\Sigma_{\lambda,\sigma}\), is defined as the following open subset of \(A_{\lambda,\sigma}\):

\[
\Sigma_{\lambda,\sigma} = \text{int}(\bigcup_{j \in \mathbb{Z}_{kq}} U^j).
\]

To ease notation, and since \(p/q\) is fixed, we will suppress the dependence on \(p/q\) and let \(\Sigma_{\lambda}\) denote the \((\log \lambda, p/q)\)-star of \(A_{\lambda}\) for \(P_{\lambda}\) and \(\Sigma_{\lambda,\sigma}\) the \((\log \lambda, p/q)\)-star of \(A_{\lambda,\sigma}\) for \(g_{\lambda,\sigma}\).

For each \(j \in \mathbb{Z}_{kq}\):

- the set \(U^j\) is called the \(j\)th strip of the star,
- a curve \(\gamma^j = \phi^{-1}_{\lambda,\sigma}(\exp(\tilde{\gamma}^j)) \cap U^j\) is called a \(j\)th wire of the star,
- the set \(U^j \setminus A_{\lambda,\sigma}\) is called the \(j\)th tip of the star.

For \(j \in \mathbb{Z}_q\) let \(\tau^j_i = \phi^{-1}_{\lambda,\sigma}(\tilde{\tau}^j_i) \cap \Sigma_{\lambda,\sigma}\), then

- the set \(\tau = \bigcup_{j \in \mathbb{Z}_q, 1 \leq i \leq k} \tau^j_i \cup \{z_0\}\) is called the twig of the star. It has \(kq\) branches.

A \(q\)-cycle of wires will be denoted \(\gamma\) (or \(\gamma_i\) when there is need to distinguish between several cycles of wires). Sometimes we will need to consider a truncated twig, or truncated branches of the twig, in this case we will use the parametrization from equation \([\mathcal{P}]\), so that for example \(\tau^j_i|_{t \geq K} = \phi^{-1}_{\lambda,\sigma}(\exp(\tilde{\tau}^j_i|_{t \geq K}))\).

### 3.2 Properties of the star

Note that \(g_{\lambda,\sigma}\) maps the star univalently onto itself. The collection of strips \(U^j\) and wires \(\gamma^j\) are each grouped in \(kq\)-orbits under \(g_{\lambda,\sigma}\), each orbit performing a \(p/q\) rotation around \(z_0\): \(g_{\lambda,\sigma}(U^j) = U^{(j+kp) \mod kq}\) and \(g_{\lambda,\sigma}(\gamma^j) = \gamma^{(j+kp) \mod kq}\). Furthermore, for each \(j \in \mathbb{Z}_{kq}\):

- the log-linearizing coordinate \(\log \circ \phi : U^j \to \tilde{U}^j\) is biholomorphic, and conjugates \(g^q_{\lambda,\sigma}\) to \(z \mapsto z + L\), whence
- the strip \(U^j\) and the wire \(\gamma^j\) is invariant under \(g^q_{\lambda,\sigma}\).
Essential wires. If \( k = 1 \) then the star has \( q \) strips, one \( q \)-cycle of wires and the twig has \( q \) branches. If \( k = 2 \) then the star has \( 2q \) strips, two \( q \)-cycles of wires and the twig has \( 2q \) branches, however in this case if one of the critical values, say \( v_2 \), is not in the star \( (v_2 \notin \Sigma_{\lambda,\sigma}) \) then branches \( \tau_2 = \bigcup_{j \in \mathbb{Z}_q} \tau^j_2 \) contain no critical points of \( \phi_{\lambda,\sigma} \), whence the branches have the same properties as wires. In this case any wire in \( \gamma_2 \) can be continuously deformed to a wire in \( \gamma_1 \) and vice versa, and we will consider two strips separated by a branch from \( \tau_2 \) as one strip. This means we will consider the star as having \( q \) strips, one \( q \)-cycle of wires and \( q \) branches of the twig, and we will say that the star has one essential \( q \)-cycle of wires (and \( q \) essential strips). In case the star has one \( q \)-cycle of wires, it will also be called essential.

- For \( k = 1 \), for each \( U^j \), \( j \in \mathbb{Z}_q \), the quotient \( U^j/g^q_{\lambda,\sigma} \) is conformally isomorphic to the cylinder \( C = \tilde{U}^j/(z \mapsto z + L) \), whence it has modulus
  \[
  \mod(U^j/g^q_{\lambda,\sigma}) = \mod(C) = \frac{2\pi \sin \theta}{q|L|},
  \]
  where \( \theta \) is the angle between \( L \) and \( 2\pi i \).

- If \( k = 2 \) and \( v_2 \notin \Sigma_{\lambda,\sigma} \), then for essential strips \( U^j \), \( j \in \mathbb{Z}_q \), the same holds:
  \[
  \mod(U^j/g^q_{\lambda,\sigma}) = \frac{2\pi \sin \theta}{q|L|}.
  \]

- If \( k = 2 \) and \( v_1, v_2 \in \Sigma_{\lambda,\sigma} \), let \( h \) denote the least distance between \( \tilde{\tau}_1/L \) and \( \tilde{\tau}_2/L \) (we will also call it the distance between \( \tau_1 \) and \( \tau_2 \) measured in the normalized log-linearizing coordinate \( \log \circ \phi_{\lambda,\sigma} \)), whence \( h \leq \frac{\pi \sin \theta}{q|L|} \).

In this case, strips \( U^j_1 \) belonging to one cycle of wires, say \( \gamma_1 \), will have modulus:
\[
\mod(U^j_1/g^q_{\lambda,\sigma}) = \frac{2\pi \sin \theta}{q|L|} - h \geq \frac{\pi \sin \theta}{q|L|},
\]
and strips \( U^j_2 \) belonging to the other cycle of wires \( \gamma_2 \) will have modulus:
\[
\mod(U^j_2/g^q_{\lambda,\sigma}) = h \leq \frac{\pi \sin \theta}{q|L|}.
\]

In the last case, there is some \( j \in \mathbb{Z}_q \) so that the distance between \( \tau^p_2 \) and \( \tau^j_1 \) is equal to \( h \), we will say that \( \tau^p_2 \) is bound to \( \tau^j_1 \).
A fat $q$-cycle of wires. An essential $q$-cycle of wires is called fat if for corresponding strips: $\text{mod}(U_j/g_{\lambda,\sigma}^q) \geq \frac{\pi \sin \theta}{q|L|}$. Note that $g_{\lambda,\sigma}$ always has one fat $q$-cycle of wires, and two if and only if $v_1, v_2 \in \Sigma_{\lambda,\sigma}$ and $h = \frac{\pi \sin \theta}{q|L|}$.

Landing of wires. Let $\gamma$ denote a $q$-cycle of wires of the star. We will say that the wires land, if the corresponding tips of the star are one point sets.

Lemma 2. Let $g_{\lambda,\sigma} \in \Gamma_{\lambda,\sigma}$, with $\lambda \in \mathbb{D}^*$ and $\sigma \in \mathbb{C}$. Then any $q$-cycle of wires $\gamma$ in the $(\log \lambda, p/q)$-star of $A_{\lambda,\sigma}$ lands. Moreover, the wires in the cycle either land together on a fixed point of $g_{\lambda,\sigma}$, or they land separately on a $q$-cycle of $g_{\lambda,\sigma}$.

Wires and critical value sector for the $p/q$-star $\Sigma_{\lambda}$ for $P_{\lambda}$. Let $\gamma_{\lambda}$ denote the $q$-cycle of wires for the $p/q$-star $\Sigma_{\lambda}$ for $P_{\lambda}$. The Julia set of $P_{\lambda}$ is a quasi-circle, in particular a Jordan curve, so $\gamma_{\lambda}$ necessarily lands on a $q$-cycle of $P_{\lambda}$. The $q$-cycle of wires $\gamma_{\lambda}$ separates the basin $A_{\lambda}$ into $q$ simply connected components, the component that contains the critical value $-\lambda^2/4$ will be called the critical value sector given by the $p/q$-star, and denoted $S_{p_{\lambda}}$.

For the purpose of this paper we will use the basin $A_{\lambda}$ and the $p/q$-star $\Sigma_{\lambda}$ for $P_{\lambda}$ as a model for the dynamics of maps $g_{\lambda,\sigma} \in \Gamma_{\lambda,\sigma}$, $\lambda \in \mathbb{D}^*$ and $\sigma \in \mathcal{R}_{\lambda}$, as well as a model for the locus $\mathcal{R}_{\lambda}$. To this end, dynamical conjugacies between $g_{\lambda,\sigma}$ and $P_{\lambda}$ will be introduced next.

3.3 Dynamical conjugacies

Recall that $\phi_{\lambda} : A_{\lambda} \to \mathbb{C}$ denotes the extended linearizing coordinate for $P_{\lambda}$, normalized so that $\phi_{\lambda}(\frac{-\lambda}{2}) = 1$. Let $U_{\lambda}$ denote the component of the pre-image by $\phi_{\lambda}$ of the disc $|z| < |\lambda|$, containing 0. Consider the map $z \mapsto \lambda^2/z$, it is an involution in the circle $|z| = |\lambda|$, with fixed points $\lambda$ and $-\lambda$. Its conjugate by the linearizing coordinate $I_{\lambda} := \phi_{\lambda}^{-1} \circ \frac{\lambda^2}{\phi_{\lambda}}$ is an involution in $\partial U_{\lambda}$, on suitable domains, with fixed points $-\lambda^2/4$ and $\phi_{\lambda}^{-1}(-\lambda)$.

Further recall that $\phi_{\lambda,\sigma} : A_{\lambda,\sigma} \to \mathbb{C}$ denotes the extended linearizing coordinate for $g_{\lambda,\sigma}$. Since $\sigma \in \mathcal{R}_{\lambda}$, $g_{\lambda,\sigma}$ has two critical values in the attracting basin, denoted $v_1$ and $v_2$. There are now two cases:

There is only one critical value of $g_{\lambda,\sigma}$ in the $p/q$-star $\Sigma_{\lambda,\sigma}$. Then the critical value in $\Sigma_{\lambda,\sigma}$ is denoted $v_1$ and $\phi_{\lambda,\sigma}$ is normalized by sending the
corresponding critical point to 1, $\phi_{\lambda,\sigma}(c_1) = 1$. The map

$$\eta_{\lambda,\sigma} := \phi_{\lambda}^{-1} \circ \phi_{\lambda,\sigma} : \Sigma_{\lambda,\sigma} \to \Sigma_{\lambda},$$

is a conformal isomorphism, with $\eta_{\lambda,\sigma}(z_0) = 0$, $\eta_{\lambda,\sigma}(v_1) = -\frac{\lambda^2}{4}$ and:

$$\eta_{\lambda,\sigma} \circ g_{\lambda,\sigma} = P_{\lambda} \circ \eta_{\lambda,\sigma}. \quad (7)$$

The conjugacy $\eta_{\lambda,\sigma}$ can be extended by means of the functional equation (7) to a conformal conjugacy, $\eta_{\lambda,\sigma} : V_{\lambda,\sigma} \to V_{\lambda}$, which coincides with $\eta_{\lambda,\sigma}$ on $\Sigma_{\lambda,\sigma}$, where $V_{\lambda,\sigma} = g_{\lambda,\sigma}^{-n}(\Sigma_{\lambda,\sigma}) \subset A_{\lambda,\sigma}$ and $V_{\lambda} = P_{\lambda}^{-n}(\Sigma_{\lambda}) \subset A_{\lambda}$ for $n \geq 1$ so that $v_2 \in V_{\lambda,\sigma}$ and $c_2 \notin V_{\lambda,\sigma}$.

Both critical values of $g_{\lambda,\sigma}$ are in the $p/q$-star $\Sigma_{\lambda,\sigma}$. First assume that $(\tau_\sigma)_1 \neq (\tau_\sigma)_2$. In this case $\phi_{\lambda,\sigma}$ is normalized by choosing one of the corresponding critical points, call it $c_1$, and letting $\phi_{\lambda,\sigma}(c_1) = 1$. The map

$$\eta_{\lambda,\sigma} := \phi_{\lambda}^{-1} \circ \phi_{\lambda,\sigma} : \Sigma_{\lambda,\sigma} \to \Sigma_{\lambda} \setminus \bigcup_{j \in \mathbb{Z}_q} (\tau_\sigma)_{\frac{j}{q}} \{ i \leq -k(j)/q \},$$

where $k(j) = -j/p \in \mathbb{Z}_q$, is a conformal isomorphism, with $\eta_{\lambda,\sigma}(z_0) = 0$, $\eta_{\lambda,\sigma}(v_1) = -\frac{\lambda^2}{4}$ and it obeys the functional equation (7). Here the domain and range of $\eta_{\lambda,\sigma}$ correspond to $q$ twice-slit strips in the log-linearizing coordinates, namely slit along the collection of lines $\tilde{\tau}_1 = (\tilde{\tau}_\sigma)_1$ and $(\tilde{\tau}_\sigma)_2$, at 0, $\zeta_2$ respectively and their backwards orbits under $z \mapsto z + \log \lambda$.

Let $x := \eta_{\lambda,\sigma}(v_2)$, so that $\eta_{\lambda,\sigma}(\{v_1, v_2\}) = \{-\frac{\lambda^2}{4}, x\}$. Choosing the other critical point for the normalization of $\phi_{\lambda,\sigma}$ instead, corresponds to changing the normalization by post-composing $\phi_{\lambda,\sigma}$ with $z \mapsto z^{\frac{\lambda}{\phi_\lambda(x)}}$. With this new normalization $\eta_{\lambda,\sigma}(\{v_1, v_2\}) = \{-\frac{\lambda^2}{4}, I_\lambda(x)\}$.

The case $(\tau_\sigma)_1 = (\tau_\sigma)_2$ is simpler. Here $c_1$ is chosen to be the critical point so that $|\phi_{\lambda,\sigma}(c_1)| \leq |\phi_{\lambda,\sigma}(c_2)|$ for any normalization, and $\phi_{\lambda,\sigma}$ is then normalized by $\phi_{\lambda,\sigma}(c_1) = 1$. The map

$$\eta_{\lambda,\sigma} := \phi_{\lambda}^{-1} \circ \phi_{\lambda,\sigma} : \Sigma_{\lambda,\sigma} \to \Sigma_{\lambda},$$

is a conformal isomorphism, with $\eta_{\lambda,\sigma}(z_0) = 0$, $\eta_{\lambda,\sigma}(v_1) = -\frac{\lambda^2}{4}$, $x := \eta_{\lambda,\sigma}(v_2) \in \Sigma_{\lambda}$ and it obeys the functional equation (7).

In the case where this choice of $c_1$ is not unique (i.e. $|\phi_{\lambda,\sigma}(c_1)| = |\phi_{\lambda,\sigma}(c_2)|$) a different choice of normalization, as before corresponds to moving the image of $v_2$ under the involution $I_\lambda$ so that $\eta_{\lambda,\sigma}(\{v_1, v_2\}) = \{-\frac{\lambda^2}{4}, I_\lambda(x)\}$. 

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For the remaining discussion, in all cases we will let $V_{\lambda,\sigma}$ and $V_{\lambda}$ denote the domain and range respectively of $\eta_{\lambda,\sigma}$, so that

$$\eta_{\lambda,\sigma} : V_{\lambda,\sigma} \to V_{\lambda}$$

(8)
denotes the dynamical conjugacy between $g_{\lambda,\sigma}$ and $P_{\lambda}$, with domain and range constructed as described above. Note that $V_{\lambda,\sigma}$ and $V_{\lambda}$ are open, simply connected and forward invariant for $g_{\lambda,\sigma}$ and $P_{\lambda}$ respectively.

### 3.4 Characterization of landing points of wires

Let $\gamma_{\lambda}$ denote the $q$-cycle of wires for the $p/q$-star for $P_{\lambda}$ and $\gamma_{\lambda,\sigma}$ a $q$-cycle of wires for the $p/q$-star for $g_{\lambda,\sigma}$. Let $V_{\lambda,\sigma}$ denote the domain of the conjugacy $\eta_{\lambda,\sigma}$ as defined in section 3.3, let $c_1$ denote the critical point of $g_{\lambda,\sigma}$ chosen for the normalization of $\eta_{\lambda,\sigma}$ so that $\eta_{\lambda,\sigma}(c_1) = -\lambda/2$ and let $v_2 = g_{\lambda,\sigma}(c_2)$ denote the other critical value.

A $q$-cycle of wires $\gamma_{\lambda,\sigma}$ separates $V_{\lambda,\sigma}$ into $q$ simply connected components, we will say that a cycle of wires $\gamma_{\lambda,\sigma}$ is non-separating if $v_1$ and $v_2$ are in the same component of $V_{\lambda,\sigma} \setminus \gamma_{\lambda,\sigma}$.

Now either $v_2 \notin \Sigma_{\lambda,\sigma}$, whence there is one essential cycle of wires $\gamma_{\lambda,\sigma}$, which is then automatically fat. In this case $\eta_{\lambda,\sigma}(\gamma_{\lambda,\sigma}) = \gamma_{\lambda}$ and $\gamma_{\lambda,\sigma}$ is non-separating if and only if $\eta_{\lambda,\sigma}(v_2) \in S^p_{\lambda}$. Or else $v_2 \in \Sigma_{\lambda,\sigma}$, in which case there exists a fat non-separating cycle of wires if and only if $\tau_{2,p}$ is bound to $\tau_{1,p}$ with $h \leq \frac{\pi \sin \theta}{q|L|}$, which in turn is equivalent to $\eta_{\lambda,\sigma}(v_2) \in S^p_{\lambda}$. All in all this discussion shows the following:

**Lemma 3.** There exists a fat non-separating $q$-cycle of wires $\gamma_{\lambda,\sigma}$ of $g_{\lambda,\sigma}$ if and only if $\eta_{\lambda,\sigma}(v_2) \in S^p_{\lambda}$.

**Lemma 4.** Let $g_{\lambda,\sigma} \in \Gamma_{\lambda,\sigma}$, with $\lambda \in \mathbb{D}^*$ and $\sigma \in \mathcal{R}_{\lambda}$ and let $v_2$ denote the critical value of $g_{\lambda,\sigma}$ so that $\eta_{\lambda,\sigma}(v_2) \neq \frac{-\lambda^2}{4}$, as chosen above. If

1. $\eta_{\lambda,\sigma}(v_2) \in S^p_{\lambda}$ then there exists a fat $q$-cycle of wires $\gamma_{\lambda,\sigma}$ of $g_{\lambda,\sigma}$, which lands on a repelling fixed point $\zeta$ of $g_{\lambda,\sigma}$.

2. $\eta_{\lambda,\sigma}(v_2) \notin S^p_{\lambda}$ then any fat $q$-cycle of wires $\gamma_{\lambda,\sigma}$ of $g_{\lambda,\sigma}$ lands on some repelling $q$-cycle $\langle \zeta \rangle$ of $g_{\lambda,\sigma}$.
The proof will use the Denjoy-Wolff Theorem, which we recall here:

**Denjoy-Wolff Theorem.** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map, not conjugate to $z \mapsto e^{\theta}z$, then there exists $p \in \overline{\mathbb{D}}$ such that $f^n(z) \to p$, as $n \to \infty$, for all $z \in \mathbb{D}$. In fact, the convergence is uniform on compact subsets of $\mathbb{D}$.

**Proof of Lemma** $\Box$ To ease notation, and since $\lambda$ is fixed here, we will during the proof use the convention $\gamma = \gamma_\lambda$ and $\gamma_\sigma = \gamma_{\lambda,\sigma}$. By Lemma 2 any $q$-cycle of wires lands, either on a fixed point or a $q$-cycle of $g_{\lambda,\sigma}$. Note that since $\sigma \in R^\lambda$, all landing points of wires are necessarily repelling periodic points, by the Fatou-Shishikura inequality.

For case 2, first assume that a fat $q$-cycle of wires $\gamma_\sigma$ of $g_{\lambda,\sigma}$ lands on a fixed point $\zeta$, then $\hat{\mathbb{C}} \setminus \gamma_\sigma$ consists of $q$ topological disks, denoted $D^0, \ldots, D^{q-1}$ named according to their cyclic ordering and so that the critical point $c_1$ is in $D^0$, then the critical value $v_1 = g_{\lambda,\sigma}(c_1)$ is in $D^p$, since the wires are $p/q$ rotated by $g_{\lambda,\sigma}$. The pre-image $\gamma'_\sigma$, $\gamma'_\sigma \cap \gamma_\sigma = \emptyset$, of $\gamma_\sigma$ by $g_{\lambda,\sigma}$ lands on a pre-image of $\zeta$, whence $\hat{\mathbb{C}} \setminus g_{\lambda,\sigma}^{-1}(\gamma_\sigma) = \hat{\mathbb{C}} \setminus (\gamma_\sigma \cup \gamma'_\sigma)$ consists of $2(q-1)$ topological disks and one annular component $A(\gamma_\sigma, \gamma'_\sigma) \subset D^0$, containing $c_1$. Since $g_{\lambda,\sigma} : A(\gamma_\sigma, \gamma'_\sigma) \to D^p$ is a proper, branched covering of degree 2, it follows from the Riemann-Hurwitz Formula, that the number of branch points in $A(\gamma_\sigma, \gamma'_\sigma)$ is 2, whence $v_2$ is also in $D^p$. But then $\gamma_\sigma$ is non-separating, and it follows that $\eta_{\lambda,\sigma}(v_2) \in \overline{S^p_\lambda}$ by Lemma 3 above.

Now for case 1, assume that $\eta_{\lambda,\sigma}(v_2) \in \overline{S^p_\lambda}$, then a fat non-separating $q$-cycle of wires $\gamma_\sigma$ of $g_{\lambda,\sigma}$ exists according to Lemma 3. We alter the domain $V_{\lambda,\sigma}$ of the conjugacy $\eta_{\lambda,\sigma}$ slightly: choose $R > 0$ so that $v_1, v_2 \in \phi_{\lambda,\sigma}^{-1}(\mathbb{D}_R)$ and so that the new domain and range

$$V_{\lambda,\sigma}(R) := V_{\lambda,\sigma} \cap \phi_{\lambda,\sigma}^{-1}(\mathbb{D}_R) \quad \text{and} \quad V_{\lambda}(R) := V_{\lambda} \cap \phi_{\lambda}^{-1}(\mathbb{D}_R)$$

are still simply connected and $c_2 \notin \overline{V_{\lambda,\sigma}(R)}$. Note that this alteration of the domain corresponds to removing the ends of the strips of the star, and all their pre-images, so that $V_{\lambda,\sigma}(R)$ is compactly contained in the basin $A_{\lambda,\sigma}$. The new domain $V_{\lambda,\sigma}(R)$ is still forward invariant under $g_{\lambda,\sigma}$ and $v_1, v_2 \in V_{\lambda,\sigma}(R)$. Since $c_2 \notin \overline{V_{\lambda,\sigma}(R)}$, the conjugacy $\eta_{\lambda,\sigma}$ extends as an isomorphism $\eta_{\lambda,\sigma} : V_{\lambda,\sigma}(R) \to V_{\lambda}(R)$ and $\overline{V_{\lambda,\sigma}(R)}$ is simply connected and forward invariant under $g_{\lambda,\sigma}$. Hence the complement $W = \hat{\mathbb{C}} \setminus V_{\lambda,\sigma}(R)$ is open, simply connected and forward invariant under branches of the inverse $g_{\lambda,\sigma}^{-1}$.

Since $v_1, v_2 \in V_{\lambda,\sigma}(R)$, it follows from the Riemann-Hurwitz Formula that the pre-image $g_{\lambda,\sigma}^{-1}(V_{\lambda,\sigma}(R))$ is doubly connected, whence the complement
\( \hat{\mathbb{C}} \setminus g_{\lambda,\sigma}^{-1}(V_{\lambda,\sigma}(R)) = g_{\lambda,\sigma}^{-1}(W) \subset W \) consists of two open, simply connected components, \( W_1, W_2 \subset W \), each mapped isomorphically to \( W \) by \( g_{\lambda,\sigma} \). Note that the ends and landing points of the cycle of wires \( \gamma_\sigma \) must be contained in \( g_{\lambda,\sigma}^{-1}(W) \) by construction of the domain \( V_{\lambda,\sigma}(R) \). Moreover:

**Claim.** All landing points of \( \gamma_\sigma \) are in the same component of \( g_{\lambda,\sigma}^{-1}(W) \), this component will be denoted \( W_1 \).

**Proof of claim.** The cycle of wires \( \gamma_\sigma \) is non-separating, so \( v_1 \) and \( v_2 \) can be connected by a curve \( \ell \) in \( V_{\lambda,\sigma}(R) \), which does not cross \( \gamma_\sigma \). For simplicity, consider a continuous curve that does not self-intersect, then the preimage is a simple closed curve in \( g_{\lambda,\sigma}^{-1}(V_{\lambda,\sigma}(R)) \) containing both critical points \( c_1 \) and \( c_2 \), and separating \( \hat{\mathbb{C}} \) into two components, each compactly containing a component of \( g_{\lambda,\sigma}^{-1}(W) \). Since the curve \( \ell \) does not cross \( \gamma_\sigma \), neither does its pre-image, whence all landing points of \( \gamma_\sigma \) are in the same component of \( g_{\lambda,\sigma}^{-1}(W) \). \( \square \)

Let \( f_1 \) denote the branch of the inverse of \( g_{\lambda,\sigma} \) that maps \( W \) to \( W_1 \), i.e. \( f_1 : W \xrightarrow{\cong} W_1 \subset W \). Since all landing points of \( \gamma_\sigma \) are in \( W_1 \), they constitute an attracting periodic cycle for \( f_1 \). Let \( \psi : W \to \mathbb{D} \) denote a Riemann map for \( W \), and consider the conjugate map \( f_\psi = \psi \circ f_1 \circ \psi^{-1} : \mathbb{D} \to \mathbb{D} \). Since \( W_1 \) is a proper subset of \( \mathbb{D} \), it follows that \( f_\psi(\mathbb{D}) \not\subset \mathbb{D} \), whence \( f_\psi \) is not conjugate to a rotation. But then \( f_1 \) can have no periodic cycles in \( W \) of period \( n > 1 \), since that would correspond to periodic cycles for \( f_\psi \) in \( \mathbb{D} \) of period \( n > 1 \), in contradiction to the Denjoy-Wolff Theorem. So the wires in \( \gamma_\sigma \) land together on an attracting fixed point \( \zeta \) of \( f_1 \), hence a repelling fixed point \( \zeta \) of \( g_{\lambda,\sigma} \). \( \square \)

### 3.5 Modulus estimates and subhorocyclic convergence

Recall that \( r_\lambda \) is the radius of the circle through \( \log \lambda \), and tangent to the imaginary axis at \( 2\pi ip/q \) (cf. equation 5). Similarly, let \( r_\mu \) denote the radius of the circle through \( \log \mu \), and tangent to the imaginary axis at \( -2\pi ip/q \), and \( r_\rho \) the radius of the circle through \( \log \rho \), and tangent to the imaginary axis at 0.

The following Lemma is an extension of Theorem B from [P] to the case of non-simply connected basins:
Lemma 5. Let \( g_{\lambda,\sigma} \in \Gamma_{\lambda,\sigma} \), with \( \lambda \in \mathbb{D}^* \) and \( \sigma \in \mathcal{R}^\lambda \) and let \( v_2 \) denote the critical value of \( g_{\lambda,\sigma} \) so that \( \eta_{\lambda,\sigma}(v_2) \neq -\frac{\lambda^2}{4} \), and \( h \) the distance between \( \tau_1 \) and \( \tau_2 \) measured in the normalized log-linearizing coordinate, as defined in section 3.2. Recall that \( k \) is the number of \( q \)-cycles of wires in the star \( \Sigma_{\lambda,\sigma} \).

1. For \( \eta_{\lambda,\sigma}(v_2) \in \overline{S^p_{\lambda}} \), let \( \zeta \) denote the repelling fixed point of \( g_{\lambda,\sigma} \) from Lemma 4 and \( \mu \) its eigenvalue. Then,
   (a) if \( k = 1 \) or \( v_2 \notin \Sigma_{\lambda,\sigma} \), then \( r_\mu \leq r_\lambda \).
   (b) if \( k = 2 \) and \( v_1, v_2 \in \Sigma_{\lambda,\sigma} \), then \( \frac{\pi}{q r_\mu} \geq \frac{\pi}{q r_\lambda} - h q \).

2. For a map \( g_{\lambda,\sigma} \), so that \( \eta_{\lambda,\sigma}(v_2) \notin \overline{S^p_{\lambda}} \), let \( \langle z \rangle \) denote the repelling \( q \)-cycle of \( g_{\lambda,\sigma} \) from Lemma 4 and \( \rho \) its eigenvalue. Then,
   (a) if \( k = 1 \) or \( v_2 \notin \Sigma_{\lambda,\sigma} \), then \( r_\rho \leq q^2 r_\lambda \).
   (b) if \( k = 2 \) and \( v_1, v_2 \in \Sigma_{\lambda,\sigma} \), then \( \frac{\pi}{q^2 r_\rho} \geq \frac{\pi}{q^2 r_\lambda} - h \).

Note that 1.a and 2.a also hold in the case of simply connected basins, \( \sigma \in \mathcal{M}^\lambda \). The essential distinction is whether there is one or two critical values in the \( p/q \)-star \( \Sigma_{\lambda,\sigma} \). The proof is an application of Bers inequality. We recall it here, for the benefit of the reader, inspired by the presentation in [M2].

Consider a flat torus \( T = \mathbb{C}/\Lambda \), where \( \Lambda \subset \mathbb{C} \) is a two-dimensional lattice, equipped with the induced Euclidean metric from \( \mathbb{C} \). Let \( A \subset T \) be an embedded annulus. The central curve of \( A \) lifts by the universal covering map \( \mathbb{C} \rightarrow T \), to a curve segment which joins a point \( z_0 \in \mathbb{C} \) to a translate \( z_0 + w \) by the lattice element \( w \in \Lambda \). The lattice element \( w \) is called the winding number of \( A \) in \( T \), and \( A \subset T \) is called an essentially embedded annulus if \( w \neq 0 \).

Bers Inequality. If the flat torus \( T = \mathbb{C}/\Lambda \) contains several pairwise disjoint, essentially embedded annuli \( A_i \), then the annuli have the same winding number \( w \in \Lambda \), and

\[
\sum \text{mod}(A_i) \leq \frac{\text{area}(T)}{|w|^2}.
\]

Proof of Lemma 5. For each of the cases, we consider quotient tori for the dynamics of \( g_{q_{\lambda,\sigma}} \), at the relevant (fixed or) periodic points. Since wires and strips of the star \( \Sigma_{\lambda,\sigma} \) are forward invariant under \( g_{q_{\lambda,\sigma}} \), wires which land at
the periodic point, descend to closed curves and strips descend to essentially embedded annuli in these tori. The moduli of these annuli depend on the position of $\eta_{\lambda,\sigma}(v_2)$ through $h$ and on the eigenvalue $\lambda$.

For case 1. consider the quotient torus $T_\mu$, for the dynamics of $g_{\lambda,\sigma}^q$, at the fixed point $\zeta$ with eigenvalue $\mu$ for $g_{\lambda,\sigma}$. The torus $T_\mu$ is conformally isomorphic to $\mathbb{C}/\Lambda$, where $\Lambda$ is the lattice spanned by $2\pi i$ and $M = q \log \mu + p2\pi i$, where log is an appropriate choice of logarithm, so that $|q \log \mu + p2\pi i|$ is minimal. The spanning element $M$ is chosen in this way, so as to correspond to the homotopy class of the projection of the $q$-cycle of wires $\gamma_{\lambda,\sigma}$ on $T_\mu$, whence the strips $U_j^\mu$ of the star $\Sigma_{\lambda,\sigma}$ descend to essentially embedded annuli in $T_\mu$ with winding number $w = M$.

For case 2. consider similarly the quotient torus $T_\rho$, for the dynamics of $g_{\lambda,\sigma}^q$, at one of the periodic points of $g_{\lambda,\sigma}$ in the $q$-cycle $\langle z \rangle$. The torus $T_\rho$ is conformally isomorphic to $\mathbb{C}/\Lambda$, where $\Lambda$ here is the lattice spanned by $2\pi i$ and $R = \text{Log}(\rho) + n2\pi i$, where $n \in \mathbb{Z}$ is chosen so that the strip $U_j^\rho$ of $\Sigma_{\lambda,\sigma}$, which lands on this periodic point, descends to an essentially embedded annulus with winding number $w = R$ in $T_\rho$.

Recall that in case 1.(a) and 2.(a) the star has $q$ essential strips, whence each annulus $U_j^\mu / g_{\lambda,\sigma}^q$, $j \in \mathbb{Z}_q$, has modulus

$$\text{mod}(U_j^\mu / g_{\lambda,\sigma}^q) = \frac{2\pi \sin(\theta_L)}{q |L|},$$

where $\theta_L$ is the angle between $L$ and $2\pi i$.

In case 1.(a) Bers inequality yields

$$\sum_{j \in \mathbb{Z}_q} \text{mod}(U_j^\mu / g_{\lambda,\sigma}^q) \leq \frac{\text{area}(T)}{|M|^2}.$$  

Rewriting this in terms of the horocyclic radii $r_\mu$ and $r_\lambda$ proves case 1.(a):

$$\frac{\pi}{q r_\lambda} = \frac{2\pi \sin(\theta_L)}{|L|} = \sum_{j \in \mathbb{Z}_q} \text{mod}(U_j^\mu / g_{\lambda,\sigma}^q) \leq \frac{\text{area}(T)}{|M|^2} = \frac{2\pi \sin(\theta_M)}{|M|} = \frac{\pi}{q r_\mu},$$

where $\theta_M$ is the angle between $M$ and $2\pi i$.

Similarly in case 2.(a) Bers inequality yields

$$\frac{\pi}{q^2 r_\lambda} = \frac{2\pi \sin(\theta_L)}{q |L|} = \text{mod}(U_j^\mu / g_{\lambda,\sigma}^q) \leq \frac{\text{area}(T_\rho)}{|R|^2} = \frac{2\pi \sin(\theta_R)}{|R|} = \frac{\pi}{r_\rho},$$

where $\theta_R$ is the angle between $R$ and $2\pi i$. 

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where \( \theta_R \) is the angle between \( R \) and \( 2pi \).

In case 1.(b) and 2.(b) the star has \( 2q \) essential strips, grouped into \( 2q \)-cycles, with each \( q \)-cycle containing a cycle of wires. Let \( U^i_1 \) denote the strips containing the fat cycle of wires \( \gamma_{\lambda,\sigma} \).

In this case the annuli \( U^j_1/g^q_{\lambda,\sigma} \) have moduli

\[
\text{mod}(U^j_1/g^q_{\lambda,\sigma}) = \frac{2\pi \sin(\theta_L)}{q|L|} - h.
\]

In case 1.(b) the fat cycle of wires lands on the fixed point \( \zeta \) with eigenvalue \( \mu \), and Bers inequality yields:

\[
\frac{\pi}{q r_\lambda} - h q = \frac{2\pi \sin(\theta_L)}{|L|} - h q = \sum_{j \in \mathbb{Z}_q} \text{mod}(U^j_1/g^q_{\lambda,\sigma}) \leq \frac{\text{area}(T)}{|M|^2} = \frac{2\pi \sin(\theta_M)}{|M|} = \frac{\pi}{q r_\mu}.
\]

In case 2.(b) the fat cycle of wires lands on the \( q \)-cycle \( \langle z \rangle \) with eigenvalue \( \rho \) and Bers inequality yields

\[
\frac{\pi}{q^2 r_\lambda} - h = \frac{2\pi \sin(\theta_L)}{q |L|} - h = \text{mod}(U^j_1/g^q_{\lambda,\sigma}) \leq \frac{\text{area}(T_\rho)}{|R|^2} = \frac{2\pi \sin(\theta_R)}{|R|} = \frac{\pi}{r_\rho}.
\]

**Proposition 2.** Let \( g_k = g_{\lambda_k,\sigma_k} \in \Gamma_{\lambda_k,\sigma_k} \), with \( \lambda_k \in \mathbb{D}^* \) and \( \sigma_k \in R^{\lambda_k} \) and let \( v_2 \) denote the critical value of \( g_k \) so that \( \eta_{\lambda_k,\sigma_k}(v_2) \neq \frac{\lambda^2_k}{4} \). Let \( \lambda_k \rightarrow \omega_{p/q} \) subhorocyclicly.

1. For a sequence \( g_k \), so that \( \eta_{\lambda_k,\sigma_k}(v_2) \in \overline{S^p_{\lambda_k}} \), let \( \zeta_k \) denote the repelling fixed point of \( g_k \) from Lemma 5.1, and \( \mu_k \) its eigenvalue. Then \( \mu_k \rightarrow \omega_{-p/q} \) subhorocyclicly.

2. For a sequence \( g_k \), so that \( \eta_{\lambda_k,\sigma_k}(v_k) \notin \overline{S^p_{\lambda_k}} \), let \( \langle z \rangle_k \) denote the repelling \( q \)-cycle \( \langle z \rangle \) of \( g_k \) from Lemma 5.2, and \( \rho_k \) its eigenvalue. Then \( \rho_k \rightarrow 1 \) subhorocyclicly.

**Proof.** Since \( h_k \leq \frac{\pi}{2q^2 r_{\lambda_k}} \) by definition, and

\[
\lambda_k \rightarrow \omega_{p/q} \text{ subhorocyclicly} \iff r_{\lambda_k} \rightarrow 0, \quad \mu_k \rightarrow \omega_{-p/q} \text{ subhorocyclicly} \iff r_{\mu_k} \rightarrow 0, \quad \rho_k \rightarrow 1 \text{ subhorocyclicly} \iff r_{\rho_k} \rightarrow 0,
\]

the statements follow immediately from Lemma 5. \( \square \)
We introduce a model for $R_\lambda$, which is a re-formulation of a model introduced by Goldberg and Keen in [GK]. Their formulation is quite different from ours, since they work with marked critical points, so that they study critically marked slices $\text{Per}_1(\lambda)_{\text{cm}}$, which are twofold branched coverings of $\text{Per}_1(\lambda)$, parametrized by the normal form $G_{\lambda,A}$. (See also [M1] for a discussion of the different versions of moduli space and slices herein.)

First we describe the model space, which is constructed from the attracting basin $A_\lambda$ for the fixed point 0 of the polynomial $P_\lambda$. Recall from section 3.3 that $\phi_\lambda : A_\lambda \to \mathbb{C}$ denotes the extended linearizing coordinate for $P_\lambda$, normalized so that $\phi_\lambda(\frac{-\lambda^2}{2}) = 1$, $U_\lambda$ denotes the component of the pre-image by $\phi_\lambda$ of the disc $|\lambda|$ of radius $|\lambda|$, containing 0, and that $I_\lambda := \phi_\lambda^{-1} \circ \frac{\lambda^2}{\phi_\lambda}$ is an involution in $\partial U_\lambda$, on suitable domains. Note that $\phi_\lambda : U_\lambda \to \overline{D}_{|\lambda|}$ is an isomorphism and that the critical value $-\lambda^2/4$ of $P_\lambda$ is on $\partial U_\lambda$.

Let $X^\lambda = A_\lambda \setminus U_\lambda$. We will identify points on $\partial U_\lambda = \partial X^\lambda$ via the involution $I_\lambda : \partial U_\lambda \to \partial U_\lambda$ in the following way:

**Definition 1.** Two points $z_1, z_2 \in \partial U_\lambda$ are called equivalent modulo $\lambda$, written $z_1 \sim_\lambda z_2$, if $z_2 = I_\lambda(z_1)$.

The model space, denoted $\Delta^\lambda$, is defined as the quotient $\Delta^\lambda = X^\lambda / \sim_\lambda$. Clearly $\Delta^\lambda$ is a Riemann surface, isomorphic to the unit disk.

### 4.1 The $p/q$ model space

We give an alternative formulation of the model space, based on the $p/q$-star for the basin $A_\lambda$, which will be more convenient for our use. We again use the involution $I_\lambda$ to define an equivalence relation, this time on a larger domain. The equivalence relation will essentially be defined on the $p/q$-star $\Sigma_\lambda$, except that we only keep the part of its twig $\tau$ which is in $\Sigma_\lambda \setminus U_\lambda$, and the image of this part under the involution $I_\lambda$. To be precise, let $\bar{\tau}$ denote the twig $\tau$, for the $p/q$-star $\Sigma_\lambda$ for $A_\lambda$, truncated in the following way:

$$\bar{\tau} = \bigcup_{J=0}^{q-1} \tau^J \left| t \geq \frac{i(j)}{q} \right.,$$

where $i(j)$ is the representative of $j/p \in \mathbb{Z}_q$ in $\{2, \ldots, q + 1\}$. 

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Consider now the involution $I_\lambda : \Sigma \setminus \tilde{\tau} \to \Sigma \setminus \tilde{\tau}$ and define an equivalence relation in the following way:

**Definition 2.** Two points $z_1, z_2 \in \Sigma \setminus \tilde{\tau}$ are called equivalent modulo $(\lambda, p/q)$, written $z_1 \sim_{\lambda, p/q} z_2$, if $z_2 = I_\lambda(z_1)$.

Let $X^\lambda_{p/q} = A_\lambda \setminus \tilde{\tau}$, then the $p/q$-model space $\Delta^\lambda_{p/q}$, is defined as the quotient $\Delta^\lambda_{p/q} = X^\lambda_{p/q} / \sim_{\lambda, p/q}$. Let $\pi_{\lambda, p/q} : X^\lambda_{p/q} \to \Delta^\lambda_{p/q}$ denote the projection map. Clearly $\Delta^\lambda_{p/q}$ is a Riemann surface, isomorphic to the unit disk, moreover for every $p/q$ the canonical isomorphism $\Delta^\lambda \to \Delta^\lambda_{p/q}$ is induced by the injection of $X^\lambda$ into $X^\lambda_{p/q}$. Because of this isomorphism, we will use $\Delta^\lambda_{p/q}$ synonymously, and in accordance with our general convention for notation, suppress the dependence on $p/q$, so that $\pi_\lambda$ denotes the projection map $X^\lambda_{p/q} \to \Delta^\lambda$.

### 4.2 Definition of the map $\chi^\lambda : \mathcal{R}^\lambda \to \Delta^\lambda$

Let $c_1$ denote the critical point of $g_{\lambda, \sigma}$ chosen for the normalization of $\eta_{\lambda, \sigma}$ so that $\eta_{\lambda, \sigma}(c_1) = -\lambda/2$ and let $v_2 = g_{\lambda, \sigma}(c_2)$ denote the other critical value, so that $\eta_{\lambda, \sigma}(v_2) \neq -\lambda^2/4$, as defined in section 3.3.

**Definition 3.** For $\lambda \in \mathbb{D}^*$, $\sigma \in \mathcal{R}^\lambda$ let $g_{\lambda, \sigma} \in \Gamma_{\lambda, \sigma}$, and let $v_2$ denote the critical value of $g_{\lambda, \sigma}$ so that $\eta_{\lambda, \sigma}(v_2) \neq -\lambda^2/4$. Then the map $\chi^\lambda : \mathcal{R}^\lambda \to \Delta^\lambda$ is defined by:

$$\chi^\lambda(\sigma) = \pi_\lambda \circ \eta_{\lambda, \sigma}(v_2).$$

Note that in cases where there is an ambiguity in the choice of $c_1$, i.e. when both $v_1, v_2 \in \Sigma_{\lambda, \sigma}$, the different choices get identified under the involution $I_\lambda$ (cf. section 3.3), whence the map $\chi^\lambda$ is well-defined. In our setting the result of [GK] is the following:

**Proposition 3 ([GK] Lemma 3.1 and Thm. 3.3).** Let $\Delta^{\lambda*} = \Delta^{\lambda} \setminus \pi_\lambda(-\lambda^2/4)$. The map $\chi^\lambda : \mathcal{R}^\lambda \to \Delta^{\lambda*}$ is an isomorphism. The point $\pi_\lambda(-\lambda^2/4)$ corresponds to $\infty \in \text{Per}_1(\lambda) \cong \hat{\mathbb{C}}$, in the sense that $(\lambda, \sigma) \to (\lambda, \infty)$ if and only if $\chi^\lambda(\sigma) \to \pi_\lambda(-\lambda^2/4) \in \Delta^\lambda$.

The map $\eta_{\lambda, \sigma}$ is composed of linearizing coordinates, that vary analytically with the parameter $\lambda$, and $\Delta^\lambda$ is an analytically varying family of Riemann surfaces, whence the map $\chi^\lambda$ is analytic in $\lambda$. 

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5 Proof of Proposition 1, Theorems 1 and 2

The map \( \Phi \) will be defined through a dynamical conjugacy between the quadratic polynomials \( P \) and \( P_\lambda \), defined on a suitable subset of the parabolic basin.

Consider the \( p/q \)-star \( \Sigma_\lambda \) for \( A_\lambda \), normalized by \( \log \circ \phi_\lambda (-\lambda/2) = 0 \) so that \( 0 \in \tilde{\tau}^0 \). Choose the \( q \) wires of the star \( \gamma^j, j \in \mathbb{Z}_q \), so that the corresponding \( \tilde{\gamma}^j \) are central straight lines in \( \tilde{U}^j \). Two adjacent lines \( \tilde{\gamma}^j \) and \( \tilde{\gamma}^{j-1} \) bound a strip, which in the normalized log-linearizing coordinate \( \log \circ \phi_\lambda L \) becomes a horizontal strip of height \( m(\lambda) := \frac{2\pi \sin \theta}{qL} \), with \( \tilde{\tau}^j/L \) as central straight line.

Let \( C^j_m \) denote a slit strip of height \( m(\lambda) \), with the real axis as the central straight line, cut along the negative real axis, starting from the first point corresponding to a point in the backwards orbit of \(-\lambda/2\) under \( P_\lambda \), to be precise:

\[
C^j_m = \{ z \in \mathbb{C} : |\Im(z)| < m(\lambda)/2 \} \setminus \{ z \in \mathbb{R} : z \leq \frac{1}{q} + \frac{k(j)}{q} - 1 \},
\]

where \( k(j) = \frac{j-p}{p} \in \mathbb{Z}_q \).

The \( q \)-cycle of wires \( \gamma \) separates the \( p/q \)-star in \( q \) components, denote these \( \Omega^j_\lambda, j \in \mathbb{Z}_q \), so that \( \tau^j \in \Omega^j_\lambda \). The component \( \Omega^0_\lambda \) is isomorphic to the slit strip \( C^0_m \):

\[
\frac{\log \circ \phi_\lambda}{L} : \Omega^0_\lambda \cong C^0_m.
\]

Similarly, \( \Omega^j_\lambda \) is isomorphic to the slit strip \( C^j_m \).

On the other hand, the parabolic basin of \( 0 \) for \( P \) has \( q \) components, \( B^0, ..., B^{q-1} \). Let \( \Omega^0_m \) denote the component of \( \phi^{-1}(C^0_m) \) in \( B^0 \), with \(-\omega_{p/q}/2\) and the parabolic fixed point \( 0 \) on the boundary. It is a (double) tile in the so-called checkerboard-tiling of the basin, with a pair of sepals of height \( m(\lambda)/2 \) removed. Now:

\[
\Omega^0_m \cong C^0_m \cong \Omega^0_\lambda
\]

and the map

\[
h^0_\lambda := (\frac{\log \circ \phi_\lambda}{L})^{-1} \circ \phi : \Omega^0_m \to \Omega^0_\lambda
\]

is an isomorphism, and conjugates \( P^q \) to \( P^q_\lambda \). Analogously define for every \( j \in \mathbb{Z}_q \) holomorphic conjugacies

\[
h^j_\lambda : \Omega^j_m \to \Omega^j_\lambda.
\]
Each $h_j^j$ extends continuously to $h_j^j : \Omega m_j \cup \{0\} \to \Omega \lambda_j \cup \{0\}$, and we can paste the $q$ maps together to obtain

$$h_\lambda : \bigcup_{j \in \mathbb{Z}_q} \Omega m_j \cup \{0\} \to \bigcup_{j \in \mathbb{Z}_q} \Omega \lambda_j \cup \{0\},$$

which is a homeomorphism, a conjugacy of $P$ to $P_\lambda$, and holomorphic on $\bigcup_{j \in \mathbb{Z}_q} \Omega m_j$. Let $\Gamma = \phi^{-1}_\lambda(\exp(\tilde{\gamma})) = \{P_\lambda^n(\gamma) : n \in \mathbb{Z}, n \geq 0\}$ and as before

$$\Xi_M = \{z \in A : |\Im(\phi(z))| < m(\lambda)/2\} \cup \bigcup_{n \geq 0} P^{-n}(0).$$

By successive lifting of $h_\lambda \circ P$ to $P_\lambda$, each time choosing the lift that coincides with $h_\lambda$ on the previous domain, $h_\lambda$ extends to a homeomorphism:

$$h_\lambda : \Xi_M \to A_\lambda \setminus \Gamma,$$

which is holomorphic on $\text{int}(\Xi_M)$ and conjugates $P$ to $P_\lambda$. Moreover, for every $x \in \Xi_M$, $h_\lambda$ depends analytically on the parameter $\lambda \in D_M$.

**Proof of Proposition 1.** The map $\Phi$ is defined by letting for every $\lambda \in D_M$ $\Phi^\lambda = (\chi^\lambda)^{-1} \circ \pi_\lambda \circ h_\lambda : \Xi_M^* \to R^\lambda$. It follows immediately from the definition and properties of the maps $\chi^\lambda$, $\pi_\lambda$ and $h_\lambda$, that for every $\lambda \in D_M$, $\Phi^\lambda$ is continuous on its domain, holomorphic on the interior, and for every $x \in \Xi_M^*$, $\lambda \mapsto \Phi^\lambda(x)$ is analytic. The analyticity of $\lambda \mapsto \Phi^\lambda(x)$ is essential to the proof of continuity of $\Phi$ in $(\lambda, x)$: the proof is similar to the proof of the $\lambda$-lemma for holomorphic motions, cf. [MSS], with the exception that in our case $\Phi^\lambda$ is not injective on the full domain, which means the proof becomes a little more technical. The technicalities can be avoided by using the continuity of $\Phi^\lambda$ in $x$.

Claim: Let $x_n, x \in \Xi_M^*$, where $(x_n)$ is any sequence $x_n \to x$, then the sequence of functions $\Phi_{x_n} : D_M \to \mathbb{C}$ converges locally uniformly on $D_M$ to $\Phi_x : D_M \to \mathbb{C}$.

Proof of claim: Note that $\{0,1\} \cap R^\lambda = \emptyset$ for all $\lambda \in \mathbb{D}$, so that since $\Phi^\lambda(\Xi_M^*) \subset R^\lambda$ for all $\lambda \in D_M$ the maps $\Phi_{x_n}$ omit $\{0,1,\infty\}$. Then by Montel’s theorem they form a normal family, and a subsequence can be extracted, which converges locally uniformly to a limit function $f : D_M \to \mathbb{C}$ as $x_n \to x$. On the other hand, continuity of $\Phi$ in $x$ is equivalent to $\Phi_{x_n} \to \Phi_x$ pointwise on $D_M$, for any sequence $x_n \to x$, whence $f \equiv \Phi_x$, which proves the claim.
The continuity of \( \Phi \) in \((\lambda, x)\) now follows from the claim and the analyticity (hence continuity) of \( \Phi_x \) in \( \lambda \).

For the definition of the underlying dynamical conjugacy, let \( \sigma := \Phi^k(x) \in \mathcal{R}^\lambda \) and let \( v_1 \) and \( v_2 \) denote the critical values of \( g_{\lambda, \sigma} \in \Gamma_{\lambda, \sigma} \). According to section \( 3.3 \), the normalization of \( \eta_{\lambda, \sigma} \) can be chosen so that \( \eta_{\lambda, \sigma}(\{v_1, v_2\}) = \{\frac{-\lambda^2}{4}, h_\lambda(x)\} \). Let \( V_{\lambda, \sigma} \) and \( V_\lambda \) denote the domain and range of \( \eta_{\lambda, \sigma} \) as defined in section \( 3.3 \). Then \( \Psi_{\lambda, \sigma} \) is defined by setting \( V_m(x) := h_\lambda^{-1}(V_\lambda) \subset \Xi_m(\lambda) \) and

\[
\Psi_{\lambda, \sigma} = \eta_{\lambda, \sigma}^{-1} \circ h_\lambda : V_m(x) \to \hat{\mathbb{C}}.
\]

By the properties of \( \eta_{\lambda, \sigma} \) and \( h_\lambda \), and the choice of normalization for \( \eta_{\lambda, \sigma} \), it follows that \( \Psi_{\lambda, \sigma} \) is a homeomorphism onto \( V_{\lambda, \sigma} \), holomorphic on the interior of \( V_m(x) \), and

\[
\Psi_{\lambda, \sigma} \circ P = g_{\lambda, \sigma} \circ \Psi_{\lambda, \sigma} \quad \text{and} \quad \Psi_{\lambda, \sigma}(\{x, -\omega_2^p/q/4\}) = \{v_1, v_2\}.
\]

Let \( \Omega_m = \bigcup_{j \in \mathbb{Z}_q} \Omega^{j}_m \cup \{0\} \), where \( \Omega^{j}_m \) is the tile in \( B^j \) with a pair of sepals of height \( m/2 \) removed, as defined in the definition of \( h_\lambda \) above. Let \( C^j_m(x) \) denote the twice-slit strip

\[
C^j_m(x) = C^j_m \setminus \{z \in \mathbb{C} : \Re(z) = \Re(\phi(x)) \wedge \Re(z) \leq \Re(\phi(x)) + h(j)/q - 1\}
\]

where \( h(j) = \frac{j-j'}{p} \in \mathbb{Z}_q \) for \( j' \) so that \( x \in \Omega^{j'}_M \subset B^j' \). Further, let \( \Omega^{j}_m(x) \) denote the component of the pre-image of \( C^j_m(x) \) by the Fatou coordinate \( \phi \) contained in \( \Omega^{j}_m \):

\[
\Omega^{j}_m(x) = \phi^{-1}(C^j_m(x)) \cap \Omega^{j}_m.
\]

If \( x \notin \Omega_M \), then \( V_m(x) = P^{-n}(\Omega_m) \), where \( n \) is smallest so that \( x \in P^{-n}(\Omega_m) \). If \( x \in \Omega_M \), then \( V_m(x) = \bigcup_{j \in \mathbb{Z}_q} \Omega^{j}_m(x) \cup \{0\} \).

We will now consider sequences of functions \( g_k = g_{\lambda_k, \sigma_k} \in \Gamma_{\lambda_k, \sigma_k} \), where \( \lambda_k \in D_M \) and \( \sigma_k = \Phi^\lambda_k(x) \) with \( x \in \Xi^*_M \). The domain \( V_{m_k}(x) \) of the conjugacy \( \Psi_k = \Psi_{\lambda_k, \sigma_k} \) depends on \( x \) and \( \lambda_k \). For fixed \( x \) and \( \lambda_k \in D_M \), all conjugacies \( \Psi_k \) are defined at least on the domain \( V_M(x) \), and we use this domain as the common domain for the maps \( \Psi_k \) in the following.

Recall that \( B^0 \) denotes the component of the immediate basin of 0 for \( P \) containing the critical point \(-\omega_2^p/2\), let \( V^0 = V_M(x) \cap B^0 \) and \( V^0_k = \Psi_k(V^0) \subset \mathbb{C}^* \). The domains \( V^0 \) and \( V^0_k \) are isomorphic to disks, forward invariant under \( P^q \) and \( g^q_k \) respectively, and each \( \Psi_k \) is univalent on \( V^0 \), where it conjugates \( P^q \) to \( g^q_k \).
For maps $G_T$, for which the critical point 1 is in the parabolic basin of $\infty$, abbreviated $B_T = A_{1,1-T^2}$, let $B_T(1)$ denote the component of $B_T$, which contains 1.

**Lemma 6** (Dynamics at infinity). Let $x \in \Xi_M^\ast$, and let $\lambda_k \in D_M$ be a sequence converging to $\omega_{p/q}$ subhorocyclic. Set $\sigma_k = \Phi^\lambda_k(x)$ and let $G_k = G_{\lambda_k,A_k} \in \Gamma_{\lambda_k,\sigma_k}$, with $A_k$ so that $\sigma_k = \frac{(\lambda_k-2)^2-A_k^2}{\lambda_k^2}$ and $\Psi_k(-\omega_{p/q}/2) = 1$. Assume that $G_k^l \to \infty$ locally uniformly on $\mathbb{C}^\ast$ when $0 < l < q$, and $G_k^l \to G_T$, $T \in \hat{\mathbb{C}}$, locally uniformly on $\mathbb{C}^\ast$. Then $T \in \mathbb{C}$, $1 \in B_T$ and:

Either there is a smallest $n = mq+l > 0$ so $G_k^n(-1) \in V_k^0$, in which case:

A. if $0 < l < q$ then $G_k^{m'}(-1) = 0$ for some $m' < m$

B. if $l = 0$ and $m > 1$, then $G_k^{m'}(-1) = 0$ for some $m' < m$

C. if $l = 0$ and $m = 1$ then $x \in B^p$ and $G_T(-1) \in B_T(1)$.

Or, there is a smallest $n = mq+l > 0$ so that $G_k^n(-1) = 0$, in which case:

D. $G_k^{m'}(-1) = 0$ for some $m' \leq m$.

**Proof.** Let $\Psi_k = \Psi_{\lambda_k,\sigma_k} : V_M(x) \to \hat{\mathbb{C}}$ denote the dynamical conjugacy $[9]$. By assumption $\Psi_k(-\omega_{p/q}/2) = 1$, whence $\Psi_k(x) = G_k(-1)$.

By the properties of the conjugacy $\Psi_k$ and its domain $V_M(x)$ it follows that $G_k^q(1) = \Psi_k(P^q(-\omega_{p/q}/2)) \in V_k^0$, $G_k^{2q}(1) = \Psi_k(P^{2q}(-\omega_{p/q}/2)) \in V_k^0$ and each $\Psi_k$ is univalent on the domain $V_k^0$, where it conjugates $P^q$ to $G_k^q$.

It then follows from the general theory of univalent functions that the sequence of functions $(\Psi_k)_k$ converges uniformly on compact subsets on $V_k^0$ to a limiting function $\Psi_\infty : V_\infty \to \hat{\mathbb{C}}$, which is either univalent or constant. First, in both cases $2 + T = G_T(1) = \Psi_\infty(P^q(-\omega_{p/q}/2))$. Assuming $\Psi_\infty$ is constant, either $-\omega_{p/q}/2 \in V_k^0$ or the domain $V_\infty$ can be extended slightly to include $-\omega_{p/q}/2$. Since $\Psi_k(-\omega_{p/q}/2) = 1$, the assumption implies that $\Psi_\infty$ is the map $z \mapsto 1$, whence $T = -1$. But $\sigma_k \in \mathcal{R}^\lambda_s$, so by the Fatou–Shishikura inequality the only non-repelling cycle of $G_k$ is the attracting fixed point at $\infty$, whence $T = -1$ would be a contradiction to Lemma 1.

So $\Psi_\infty$ is univalent, whence $T \in \mathbb{C}$, since $2 + T = \Psi_\infty(P^q(-\omega_{p/q}/2)) \neq \Psi_\infty(P^{2q}(-\omega_{p/q}/2)) = 2 + 2T + \frac{1}{2+T}$. The univalence of $\Psi_\infty$ implies (from the equivalence between Riemann maps and pointed disks, cf [McM, Thm. 5.1]) that the sequence of pointed disks $(V_k^0,G_k^q(1))$ converges in the Carathéodory
topology to a pointed disk \((V_\infty, G_T(1))\). From [McM Thm. 5.3] if the distance \(d_k(G_k^q(1), w_k)\) is bounded uniformly in \(k\) in the hyperbolic metric \(d_k\) on \(V_k^0\), and \(w_k \to w\), then \(w \in V_\infty\). In particular, this implies that if there exists \(n > 0\) so that \(G_k^n(-1) \in V_k^0\) and \(G_k^n(-1)\) converges to some point \(w\), then \(w \in V_\infty\), since the hyperbolic distance

\[d_k(G_k^q(1), G_k^n(-1)) = d(P^q(-\omega_{p/q}/2), P^{n-1}(x))\]

for all \(k\), where \(d\) denotes the hyperbolic distance on \(V^0\). The commuting diagram \(\Psi_k \circ P^q = G_k^q \circ \Psi_k\) is preserved in the limit, whence \(\Psi_\infty \circ P^q = G_T \circ \Psi_\infty\), and \(V_\infty\) is a subset of the parabolic basin of \(G_T\), in particular \(1 \in V_\infty \subset B_T\).

If \(x \in \text{int}(V_M(x))\), then there exists a least \(n = qm + l > 0\) so that \(G_k^n(-1) \in V_k^0 \subset \mathbb{C}\).

Part A: If \(0 < l < q\) then \(G_k^n(-1) = G_k^l \circ (G_k^q)^m(-1) \in \mathbb{C}\), and \(G_k^n(-1) \not\to \infty\), again since the hyperbolic distance to \(G_k^q(1)\) is fixed;

\[d_k(G_k^q(1), G_k^n(-1)) = d(P^q(-\omega_{p/q}/2), P^{m-1}(x))\]

for all \(k\).

On the other hand \(G_k^l \to \infty\) locally uniformly on \(\mathbb{C}^*\), whence there is a smallest \(m' \leq m\) so that \(G_k^{qm'}(-1) \to 0 = G_T^{m'}(-1)\).

Part B: If \(l = 0\) and \(m > 1\) then there is a smallest \(m' < m\) so that \(G_k^{qm'}(-1) \to 0 = G_T^{m'}(-1)\). Assume not, then \(G_k^n(-1) = G_k^{qm}(-1) \to G_T^{m'}(-1) \in V_\infty\). Because of the properties of the domain \(V_M(x)\), and consequently of \(V_\infty\), the two pre-images of \(G_T^{m'}(-1)\) under \(G_T\) belong to \(V_\infty\); \(G_T^{-1}(\{G_T^{m'}(-1)\}) \in V_\infty\). One of these pre-images, call it \(y\), must be in the forward orbit of \(-1\), i.e., \(y = G_T^{m-1}(-1) \in V_\infty\). But then \(G_k^{qm}(m-1)-1 \in V_k^0\) for \(k\) sufficiently large, in contradiction to the assumption that \(n = qm\) was the first such iterate.

Part C: If \(l = 0\) and \(m = 1\) then \(G_k^q(-1) \in V_k^0\) (equivalently \(P^{q-1}(x) \in V^0\)). First note that if \(x \in B_p^0\) \((P^{-1}(\{x\}) \cap B^0 = \emptyset)\) then \(G_k^q(-1) \in V_k^0\), and again by the hyperbolic argument above, \(G_k^q(-1) \to G_T(-1) \in V_\infty \subset B_T(1)\).

To see that this is the only possibility, assume to the contrary that \(x \notin B_p\), whence \(P^{-1}(\{x\}) \cap B^0 = \emptyset\). In this case \(\Psi_k\), and consequently \(\Psi_\infty\), extends to a univalent conjugacy on all of \(B^0\). There are two distinct pre-images of \(P^{q-1}(x)\) under \(P^q\) in \(B^0\), which since \(\Psi_\infty\) is a univalent conjugacy on \(B^0\), and \(\Psi_\infty(P^{q-1}(x)) = G_T(-1)\), would imply that there are two distinct pre-images of \(G_T(-1)\) under \(G_T\), for a contradiction.
Part D: If \( x \in \partial V_M(x) \), then there exists a smallest \( n = qm + l > 0 \) so that \( G^n_k(-1) = 0 \), and the result is obtained in much the same way as before. If \( 0 < l < q \) then \( G_k^m(-1) = G_k^l \circ G_k^{qm}(-1) = 0 \), but on the other hand \( G_k^m(-1) \to 0 = G_T^m(-1) \). If \( l = 0 \), then either there is a smallest \( m' \leq m \) so that \( G_k^{qm}(-1) \to 0 = G_m^T(-1) \) or \( 0 = G_k^m(-1) = G_k^{qm}(-1) \to G_T^m(-1) \). If \( l = 0 \), then either there is a smallest \( m' < m \) so that \( G_k^{qm}(-1) \to 0 = G_m^T(-1) \) or \( 0 = G_k^m(-1) = G_k^{qm}(-1) \to G_T^m(-1) \), whence \( G_k^m(-1) = 0 \).

\[ \frac{\text{Proof of Theorem 1.}}{\text{Maps}} \]

\[ f_k = f_{\lambda_k,\sigma_k} \in \Gamma_{\lambda_k,\sigma_k} \] have attracting fixed points with eigenvalues \( \lambda_k \to \omega_{p/q} \) and by Proposition 2 repelling \( q \)-cycles \( \langle z \rangle_k \) with eigenvalues \( \rho_k \to 1 \). Let \( \Psi_k = \Psi_{\lambda_k,\sigma_k} : V_M(x) \to \hat{C} \) denote the dynamical conjugacy between \( f_k \) and \( P \) as defined in Proposition 1, restricted to \( V_M(x) \) so that all \( \Psi_k \) have the same domain.

Part A: Assume to the contrary that \( \sigma_k \) is unbounded, then, on passing to a subsequence if necessary, \( \sigma_k \to \infty \) and the fixed point eigenvalues \( \lambda_k, \mu_k, \nu_k \) tend to \( \omega_{p/q}, \omega_{-p/q}, \infty \). We can without loss of generality choose representatives \( f_k = G_{\lambda_k,A_k} \), where \( A_k \in \mathbb{C} \) so that \( \sigma_k = \frac{(\lambda_k - 2)^2 - A_k^2}{\lambda_k^2} \) and \( \Psi_k(-\omega_{p/q}/2) = 1 \), whence \( \Psi_k(x) = G_{\lambda_k,A_k}(-1) \). It then follows from Lemma 1, passing to a subsequence if necessary, that

\[ f_k^l \to \begin{cases} \infty & \text{for } 1 \leq l < q \\ G_0 & \text{for } l = q \end{cases} \quad \text{locally uniformly on } \mathbb{C}^* \text{ as } \lambda_k \to \omega_{p/q}. \]

Note that \( T \in \mathbb{C} \) because of Lemma 6, hence \( T = 0 \) because of the last part of Lemma 1 since there are repelling \( q \)-cycles \( \langle z \rangle_k \) with eigenvalues \( \rho_k \to 1 \).

But the map \( G_0 \) has a double parabolic fixed point at \( \infty \), its basin consists of two fixed components, each containing a critical point, 1 or -1, both having infinite forward orbit. So no iterate of -1 by \( G_0 \) is 0 or in the basin component containing 1, which is in contradiction to all of the possible cases in Lemma 6. Therefore \( \sigma_k \) is bounded.

Part B: from Part A the sequence \( (\sigma_k)_k \) is bounded, so a subsequence can be extracted for which \( (\lambda_k, \sigma_k) \to (\omega_{p/q}, \sigma) \) for some \( \sigma \in \mathbb{C} \). We prove that any such limit point \( \sigma \) has the property that \( \sigma \in \mathcal{R}_{\omega_{p/q}} \) and that the sequence of conjugacies \( \Psi_k \) converges to a map with the required properties.

Without loss of generality we can choose representatives, say \( f_k = G_{\lambda_k,A_k} \in \Gamma_{\lambda_k,\sigma_k} \) and \( f_\sigma = G_{\omega_{p/q},A} \in \Gamma_{\omega_{p/q},\sigma} \) (where \( A_k \) and \( A \) are chosen so that...
\( \sigma_k = \frac{(\lambda_k - 2)^2 - A_k^2}{A_k^2}, \quad \sigma = \frac{(\lambda - 2)^2 - A^2}{A^2} \) and \( \Psi_k(-\omega_{p/q}/2) = 1 \), whence \( \Psi_k(x) = G_{\lambda_k, A_k}(-1) \), so that \( f_k \) converges to \( f_\sigma \) uniformly on \( \hat{\mathbb{C}} \).

From the uniform convergence \( f_k \to f_\sigma \), the functional equation \([\text{9}]\) and the general theory of univalent maps, it follows that the sequence \( \Psi_k \) converges to a limiting map \( \Psi_\infty \), and that the convergence is locally uniform on each component of \( \text{int}(V_M(x)) \) and pointwise on \( \partial V_M(x) \). Moreover, the limiting map \( \Psi_\infty \) is injective, univalent on \( \text{int}(V_M(x)) \), and obeys the functional equation:

\[
\Psi_\infty \circ P = f_\sigma \circ \Psi_\infty \quad \text{on} \quad V_M(x). \tag{10}
\]

Let \( \Xi_{\sigma,M} = \{ z \in A_\sigma : |\Im(\phi_\sigma(z))| < M/2 \} \cup \bigcup_{n \geq 0} f_\sigma^{-n}(\infty) \), where \( A_\sigma \) is the parabolic basin at \( \infty \) for \( f_\sigma \) and \( \phi_\sigma \) its Fatou coordinate. Since \( V_M(x) \subset \Xi_M \) is forward invariant under \( P \), \( \Psi_\infty(V_M(x)) \subset \Xi_{\sigma,M} \), whence \( f_\sigma(-1) \in \Psi_\infty(V_M(x)) \subset \Xi_{\sigma,M} \), and \( \sigma \in \mathcal{R}^{\omega_{p/q}} \).

Let \( \mathcal{C}^j \) denote the slit plane \( \mathcal{C}^j = \mathbb{C} \setminus \{ z \in \mathbb{R} : z \leq \frac{j - p}{q} - 1 \} \), where \( k(j) = \frac{j - p}{q} \in \mathbb{Z}_q \). Note that \( C_m^j \subset \mathcal{C}^j \), where \( C_m^j \) is the slit strip used in the definition of \( h_\lambda \). Let \( \Omega^j \) denote the component of \( \phi^{-1}(\mathcal{C}^j) \) in \( B^j \), with the parabolic fixed point 0 on the boundary and containing points from the forward orbit of \(-\omega_{p/q}/2\) under \( P \). As mentioned in the definition of \( h_\lambda \), it is a (double) tile in the so-called checkerboard-tiling of the basin. Let \( \Omega = \bigcup_{j \in \mathbb{Z}_q} \Omega^j \cup \{0\} \).

Let \( \mathcal{C}^j(x) \) denote the twice-slit plane

\[
\mathcal{C}^j(x) = \mathcal{C}^j \setminus \{ z \in \mathbb{C} : \Im(z) = \Im(\phi(x)) \land \Re(z) \leq \Re(\phi(x)) + h(j)/q - 1 \},
\]

where \( h(j) = \frac{j - p}{q} \in \mathbb{Z}_q \) for \( j \) so that \( x \in \Omega^{j'} \subset B^{j'} \). Further, let \( \Omega^j(x) \) denote the component of \( \phi^{-1}(\mathcal{C}^j(x)) \) contained in \( \Omega^j \) where \( \Omega^j(x) = \phi^{-1}(\mathcal{C}^j(x)) \cap \Omega^j \).

If \( x \notin \Omega_M \), then \( V_m(x) = P^{-n}(\Omega_m) \), where \( n \) is smallest so that \( x \in P^{-n}(\Omega_m) \). If \( x \in \Omega_M \), then \( V_m(x) = \bigcup_{j \in \mathbb{Z}_q} \Omega_m^j(x) \cup \{0\} \).

The limiting map \( \Psi_\infty \) is in fact defined on the larger domain \( V(x) = P^{-n}(\Omega) \), where \( n \) is smallest so that \( x \in P^{-n}(\Omega) \) in case \( x \notin \Omega \) and \( V(x) = \bigcup_{j \in \mathbb{Z}_q} \Omega^j(x) \cup \{0\} \) in case \( x \in \Omega \). Namely, for each compact component in \( \text{int}(V(x)) \) there is a \( K > M \) so that \( \Psi_{\lambda,\sigma} \) converges uniformly to \( \Psi_\infty \) there, as \( \lambda \in D_K \) tends subhorocyclicly to \( \omega_{p/q} \). Note that \( V_M(x) = V(x) \cap \Xi_M \).

**Claim:** \( \Psi_\infty \) **is continuous.** It is again necessary to distinguish between two situations, according to whether or not \( x \in \Omega \) (equivalent to both critical
values of \( f_k \) in the \( p/q \) star \( \Sigma_{\lambda_k, \sigma_k} \). Note that as \( \Psi_\infty \) is univalent on \( \operatorname{int}(V(x)) \), it is only necessary to show continuity at points in \( \partial V(x) \cap V(x) \).

Claim for \( x \in \Omega \): \( \Psi_\infty \) is continuous on \( \{ V(x) \cap \phi^{-1}(\mathbb{H}_x) \} \cup \{ 0 \} \), where \( \mathbb{H}_x \) is any right halfplane containing both \( \phi(-\omega_{p/q}) = 1/q \) and \( \phi(x) \), and therefore their forward orbits.

In this case, each component \( \Omega^j_{m_k}(x) \) in the domain of \( \Psi_k \) is a pre-image by the Fatou coordinate \( \phi \) of the twice slit strip \( C^j_{m_k}(x) \subset \mathbb{C}^j(x) \) of height \( m_k = m(\lambda_k) = \frac{2\pi \sin \theta}{q|l_k|} \). As \( \lambda_k \) tends to \( \omega_{p/q} \) subhorocyclicly, \( m(\lambda_k) \to \infty \), whence \( \Psi_\infty = \phi^{-1}_D \circ \phi \) on \( \Omega^*_j(x) = \bigcup_{j \in \mathbb{Z}_q} \Omega^j(x) \), where \( \phi \) is a Fatou coordinate for \( f_* \).

Now consider a sequence \( z_n \in V(x) \cap \phi^{-1}(\mathbb{H}_x) \), so that \( z_n \to 0 \), then \( \phi \circ \Psi_\infty^{-1}(\phi(z_n)) = \phi(z_n) \to \infty \) in \( \mathbb{H}_x \), and since each \( \Omega^j(x) \) is forward invariant under \( P^s \), \( \Psi_\infty(\Omega^j(x)) \) is forward invariant under \( f_* \), and it follows that \( \Psi_\infty(z_n) \to \infty = \Psi_\infty(0) \).

Claim for \( x \notin \Omega \): \( \Psi_\infty \) is continuous on \( V(x) \). The argument for continuity at 0 is the same as in the previous case, except less care is needed concerning the domains. In this case, each of the tiles \( \Omega^j \subset V(x) \) in the domain of \( \Psi_\infty \) is a pre-image of the slit plane \( \mathbb{C}^j \), \( \phi(\Omega^j) = \mathbb{C}^j \), and \( \Psi_\infty = \phi^{-1}_D \circ \phi \) on \( \Omega^* = \bigcup_{j \in \mathbb{Z}_q} \Omega^j \), where \( \phi \) is a Fatou coordinate for \( f_* \).

Now consider a sequence \( z_n \in V(x) \), so that \( z_n \to 0 \), then the \( z_n \) are eventually in \( \Omega^* \) and

\[
\Omega^* \ni z_n \to 0 \Rightarrow \phi \circ \Psi_\infty^{-1}(\phi(z_n)) = \phi(z_n) \to \infty \text{ in } \Omega^j \Rightarrow \Psi_\infty(z_n) \to \infty = \Psi_\infty(0).
\]

For \( z \in P^{-n}(\{ 0 \}) \), \( z \neq 0 \), it follows from the continuity at 0 and the functional equation (10) that \( \Psi_\infty(z_n) \to z_0 \in P^{-n}(\{ \infty \}) \) as \( z_n \to z \) from within a component of \( V(x) \). To see that \( z_0 = \Psi_\infty(z) \), let \( D(z) \) be a sufficiently small disc around \( z \), so that the pre-image by \( \phi \) of the positive real axis has \( q \) components in \( V(x) \cap D(z) \), with \( z \) as the only common boundary point. Let \( Y \) denote the closure of this pre-image:

\[
Y = \overline{\phi^{-1}(\mathbb{R}_+) \cap V(x) \cap D(z)}
\]

Note that \( Y \) is closed, connected and that \( Y \setminus A = \{ z \} \). Let \( Y_k = \Psi_k(Y) \), which is closed and connected since \( \Psi_k \) is a homeomorphism, whence on passing to a subsequence if necessary, \( Y_k \) converges in the Hausdorff topology to a closed and connected \( Y_\infty \). It follows that for a sequence \( Y \ni z_n \to z \),

\[
Y_\infty \ni \Psi_\infty(z_n) \to \Psi_\infty(z), \text{ whence } z_0 = \Psi_\infty(z).
\]
Since $\Psi_\infty$ is continuous, it has all the properties required of the map $\Psi$ of Part B.

\textbf{Proof of Theorem 2.} Maps $f_k = f_{\lambda_k, \sigma_k} \in \Gamma_{\lambda_k, \sigma_k}$ have attracting fixed points with eigenvalues $\lambda_k \to \omega_{p/q}$ and by Proposition 21 the maps $f_k$ also have repelling fixed points with eigenvalues $\mu_k \to \omega_{-p/q}$, whence it follows from the discussion in section 2.2 that $\sigma_k \to \infty$.

Now consider representatives $G_k = G_{\lambda_k, A_k} \in \Gamma_{\lambda_k, \sigma_k}$, where $A_k \in \mathbb{C}$ so that $\sigma_k = \left(\frac{(\lambda_k - 2) - A_k^2}{M_k}\right)^2$. Let $\Psi_k = \Psi_{\lambda_k, \sigma_k} : V_M(x) \to \widehat{\mathbb{C}}$ denote the dynamical conjugacy between $G_k$ and $P$ from Proposition 1, restricted to $V_M(x)$ so that all $\Psi_k$ have the same domain. Without loss of generality, we choose representatives so that $\Psi_k(-\omega_{p/q}/2) = 1$, whence $\Psi_k(x) = G_{\lambda_k, A_k}(-1)$.

Part A: It follows from Lemma 1 that $G_k^l \to \infty$ locally uniformly on $\mathbb{C}^*$ for $0 < l < q$, in particular for $l = 1$.

Part B: It follows from Lemma 1 that $G_k^q \to G_T$ locally uniformly on $\mathbb{C}^*$, for some $T \in \widehat{\mathbb{C}}$, as $\lambda_k \to \omega_{p/q}$. Then it follows from Lemma 6 that $T \in \mathbb{C}$ and in each of the cases $\sigma = 1 - T^2 \in \mathcal{R}^1$. Moreover from Lemma 6 recall that the sequence $\Psi_k$ converges locally uniformly to a univalent map $\Psi_\infty : V^0 \to \mathbb{C}^*$, $V^0 = V_M(x) \cap B^p$, $\Psi_\infty$ is a conjugacy of $P^q$ to $G_T$ and $\Psi_\infty(V^0) = V_\infty \subset B_T$.

For $x \in B^p$, we are in case C of Lemma 6, whence

$$
\Psi_\infty(P^{q-1}(x)) = G_T(-1) \quad \text{and} \quad \Psi_\infty(P^q(-\omega_{p/q}/2)) = G_T(1),
$$

and both critical points $\pm 1$ of $G_T$ are in the same component of the parabolic basin of $\infty$, i.e $\sigma \in \mathbb{C} \setminus M^1$.

The case $x \in S^p \setminus B^p$ corresponds to case A, B or D of Lemma 6. In each case there exists $n \geq 1$ ($m'$ in the Lemma) such that $G_T(-1) = 0$, so that $\sigma \in \mathcal{R}^1 \cap M^1$.

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