Cohomological Induction over $\mathbb{Q}$

and

Frobenius-Schur indicators for $(\mathfrak{g}, K)$-modules

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Abstract

In this paper we lay the foundations of a general theory of $(\mathfrak{g}, K)$-modules over any field of characteristic 0. After introducing appropriate notions of rationality, we prove a fundamental Homological Base Change Theorem, which has important consequences for the existence of rational models of Harish-Chandra modules. We investigate geometric properties of Harish-Chandra modules and set up a theory of cohomological induction over any field of characteristic 0. Furthermore we discuss Frobenius-Schur indicators for $(\mathfrak{g}, K)$-modules, which are particularly useful in the study of descent in imaginary quadratic extensions. As an application of our theory we prove that, maybe somewhat surprisingly, cohomological representations of $GL_n(k \otimes_{\mathbb{Q}} R), k$ a number field, are defined over the field of rationality, which agrees with the field of definition of the infinitesimal character. This is an archimedean analog of a well known result of Clozel in the non-archimedean case and proves the existence of rational structures on regular algebraic automorphic representations.

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Introduction

Since their introduction by Harish-Chandra in the 50ies [10, 11, 12], Harish-Chandra modules have played a fundamental role in the representation theory of real reductive groups. Their influence has been substantial: Harish-Chandra’s theory allowed Langlands to give the first instance of his famous Langlands classification [23], a cornerstone of his ambitious Langlands program. Its implications, in particular the interplay between arithmetics and automorphic representations, are far reaching and many aspects are yet to be explored.

By Langlands’ philosophy and generalized Taniyama-Shimura type modularity conjectures, we expect a close relationship between motives over number fields and certain automorphic representations. To each motive $M$ we may, assuming Grothendieck’s Standard Conjectures, attach a field of coefficients $E$ in a natural way, which is a number field. This field plays a fundamental role in arithmetic questions: The coefficients of the $L$-function attached to $M$ all lie in $E$, and a famous conjecture of Deligne [6] predicts that for a motive $M$ defined over a number field $E/Q$, the a priori complex special values of its $L$-function, appropriately normalized, also lie in $E$.

Now if we believe that we may attach to $M$ an automorphic representation $\pi_M$ of $GL(n)$, it is not obvious how the inherently transcendental object $\pi_M$ reflects these rationality properties. In [5] Clozel went a long way to single out
the class of (regular) algebraic automorphic representations of $GL(n)$ as the candidates which should correspond to absolutely irreducible motives of rank $n$ (cf. loc. cit. Conjecture 4.16). Clozel went on to show that the finite part of a regular algebraic representation $\pi$ over a number field $F$ is defined over a number field $Q(\pi)$ (cf. loc. cit. Proposition 3.16), which in particular implies that the coefficients of the standard $L$-function of $\pi$ all lie in $Q(\pi)$. Well known results on special values of automorphic $L$-functions show the validity of the analog of Deligne’s Conjecture with $Q(\pi)$ replacing $E$.

An essential feature of Clozel’s rational structure is that it is optimal: It exists over the field of rationality of the finite part of $\pi$ (cf. Proposition 3.1 of loc. cit.). In his treatment Clozel circumvents the problem of rationality of the archimedean part of $\pi$ by studying an ad hoc action of the Galois group on the infinity type. This is legitimate as a well known result of Matsushima relates the occurrence of an automorphic representation in Betti cohomology to its type at infinity, and Clozel’s rational structure originates in the cohomology of arithmetic groups.

Thus, in order to complete Clozel’s picture, we are naturally led to the following question: Is the infinite part $\pi_\infty$ defined over $Q(\pi)$? Or equivalently: Is a non-degenerate irreducible admissible $GL_n(F \otimes Q R)$-module with non-trival Lie-algebra cohomology defined over its field of rationality?

We eventually answer this question positively in Theorem 7.1. In particular we show that in this case the field of rationality agrees with the field of definition of the infinitesimal character. We thereby prove the existence of a global $Q(\pi)$-rational structure on every regular algebraic representation $\pi$.

Recently it has become more and more apparent, that global rational structures on automorphic representations play a fundamental role in the study of special values of automorphic $L$-functions (cf. [13], [9], [22], [17]). The theory we develop here is powerful enough to answer the analog questions for any reductive group $G$ over any number field $F$. In general we expect a direct relationship between the size of the fields of definition of the archimedean representations and the size of the corresponding $L$-packets.

Another motivation for our theory is the author's algebraic character theory [16]. In light of this theory, a natural question to ask is whether algebraic characters (and hence even Harish-Chandra’s global characters) respect rationality, or more fundamentally, if over arbitrary fields of characteristic 0 there is a theory of Harish-Chandra modules which allows for a definition of algebraic characters. We show that this is indeed the case (cf. Section 4.3).

To answer these questions of rationality we need to set up an appropriate theory of $(g, K)$-modules over general fields of characteristic 0. In the first three sections we lay the foundation of such a theory. After introducing the abstract notion of a pair in this setting and introducing the appropriate categories of modules for pairs, we set up the appropriate homological machinery which allows us to define and study related rationality questions. Our first main result is the Homological Base Change Theorem (cf. Theorem 2.1), which has many important consequences in the sequel.

We define rational Zuckerman functors and show that they commute with
base change, and in particular are Galois equivariant. We also sketch a rational algebraic character theory, generalizing the algebraic characters defined by the author in [16], which is Galois equivariant as well (cf. Theorem 4.3 and Proposition 4.4).

This setup yields rational models of Harish-Chandra modules of interest over $\mathbb{Q}$ resp. certain number fields. In order to push the theory further, we introduce rational translation functors and generalize the classical notion Frobenius-Schur indicators $\mathbb{R}$ to $(g, K)$-modules. They are particularly useful in the study of rationality properties of infinitesimally unitary $(g, K)$-modules, as in this case they are closely related to the existence of symmetric resp. antisymmetric invariant bilinear forms. In particular we show that the Frobenius-Schur indicator of a complex infinitesimally unitary irreducible $(g, K)$-module agrees with the Frobenius-Schur indicator of the sum of its minimal $K^0$-types, thereby reducing an infinite-dimensional problem for $(g, K)$ to a finite-dimensional one for $K$.

In order to study translation functors, we first investigate the Galois action on central characters (cf. Propositions 1.4 and 1.5). It is no surprise that in terms of Harish-Chandra’s parametrization of characters by weights, this action is the same as the one studied by Borel and Tits in [2]. This then allows us to define rational translation functors accordingly. As their definition is straightforward, we only treat them implicitly and use them in the proof of Theorem 7.1.

As an application of our theory we show in Theorem 7.1 that for $G = \text{Res}_{F/\mathbb{Q}} \text{GL}_n$, where $F$ is any number field, all cohomological $(g, K)$-modules are defined over their field of definition of the coefficients in cohomology. To be more precise, we deduce from $G$ a reductive pair $(g, K)$ over $\mathbb{Q}$, and prove that if $M$ is an absolutely irreducible $G$-module defined over a number field $E$, then the infinitesimally unitary irreducible $(g, K)$-module $V_C$ satisfying

$$H^\bullet(g, GK^0; V_C \otimes M) \neq 0,$$

has a unique model $V_E$ over $E$, where $GK$ denotes the product of $K$ with the center of $G$. In particular we find global models of regular algebraic automorphic representations over their fields of rationality as defined by Clozel in [5].

Our proof proceeds in three steps. First we treat the case of the trivial representation $M = C$. Then $V_C$ is cohomologically induced from a $\theta$-stable Borel subalgebra $q \subseteq g \otimes Q(i)$, which yields a model of $V_C$ over $V_{Q(i)}$. Now the latter is defined over $Q$ if and only if $V_C$ is defined over $R$. As $V_C$ is infinitesimally unitary and selfdual, our theory of Frobenius-Schur indicators of Section 5 reduces this problem to that of the minimal $K$-type of $V_C$. It is a consequence of well known classical results that this $K$-type is defined over $R$, a result that we recall in Section 6. We conclude that $V_C$ is indeed defined over $Q$. With this result at hand, the general case easily follows via the application of translation functors.
As already indicated, we do not develop a full blown theory of \((g, K)\)-modules over any field here, as this would certainly yield to a monograph of size comparable to [18]. Therefore we do not discuss rational Hecke algebras, and consequently omit the treatment of Bernstein functors and hard duality. However we will discuss these topics in the future.

Further motivation for our work comes from related work of Michael Harris and his joint work with Harald Grobner, as well as work of Günter Harder and A. Raghuram.

The beginnings of a rational theory of Harish-Chandra modules had already been sketched in the author’s manuscript [15]. Previously Michael Harris sketched a variant of Beilinson-Bernstein Localization over \(\mathbb{Q}\) and discussed applications to periods of automorphic representations in [13]. Harris emphasizes the Galois-equivariant viewpoint, i.e. he considers modules over \(\mathbb{Q}\) together with a Galois action, and Beilinson-Bernstein Localization already puts strict finiteness conditions on the modules under consideration. Our results show that this is not a serious loss, as finite length modules satisfy those conditions automatically (cf. Theorem 3.7).

In the preprint [9], Harris and Grobner observed a weaker variant of Theorem 7.1 for \(\text{GL}(n, F \otimes \mathbb{Q} \mathbb{R})\) for imaginary quadratic fields \(F\), which is an essential ingredient in their work.

In their recent investigation of special values of Rankin-Selberg \(L\)-functions, Harder and Raghuram implicitly compare rational structures on Harish-Chandra modules in [22]. In this context Harder observed in [21], independently from us, that cohomological modules for \(\text{GL}(n, \mathbb{R})\) are defined over \(\mathbb{Q}\), which is a special case of Theorem 7.1. Harder even went further and gave models over \(\mathbb{Z}\). He first gives an ad hoc construction of models for \(\text{GL}(2, \mathbb{R})\)-modules and then invokes an explicit algebraic variant of parabolic induction in order to produce models for \(\text{GL}(n, \mathbb{R})\)-modules without reference to ambient categories of modules.

Harder pursues rationality and integrality properties of intertwining operators, which yield applications in [22]. In this context questions of integrality of inverses of intertwining operators boil down to combinatorial identities [20], which have been taken up by Don Zagier in [33] in a special but non-trivial case.

It is interesting to see that the well known frameworks, parabolic induction, cohomological induction and Beilinson-Bernstein localization, all have purely algebraic variants and it seems clear that rational models of Harish-Chandra modules will allow for more applications to special values of \(L\)-functions.

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**Notation and Conventions**

Throughout the paper all fields we consider are subfields of \(\mathbb{C}\). This condition may be dropped, but is not a serious restriction in light of an appropriate
Lefschetz Principle (see also [25]). We denote fields by \( F, F', \ldots \), linear algebraic groups by \( B, G, K, \ldots \), their Lie algebras by the corresponding gothic letters \( \mathfrak{b}, \mathfrak{g}, \mathfrak{t}, \ldots \), and by \( G' \), resp. \( \mathfrak{g}' \) their base change in an extension \( F'/F \). Reductive linear algebraic groups are not assumed to be connected. \( U(\mathfrak{g}) \) denotes the universal enveloping algebra of a Lie algebra \( \mathfrak{g} \) over \( F \) and \( Z(\mathfrak{g}) \) its center, which is defined over \( F \), cf. Proposition 1.4. For an algebraic group or a real Lie group, the superscript \(^0\) denotes the connected component of the identity.

1 Rational pairs and modules

Let \( F \) be a field of characteristic 0. A pair \((a, B)\) over \( F \) consists of a reductive linear algebraic group \( B \) over \( F \) (not necessarily connected), and a Lie algebra \( a \) over \( F \), and the following additional data:

(i) A Lie algebra monomorphism

\[
\iota_B : \mathfrak{b} := \text{Lie}(B) \to a.
\]

(ii) An action of \( B \) on \( a \), whose differential is the action of \( \mathfrak{b} \) on \( a \).

We consider the category \( \mathcal{C}_{\text{id}}(B) \) of finite-dimensional rational \( B \)-modules over \( F \). This is an \( F \)-linear tensor category, with a natural forgetful functor

\[
\mathcal{F}^B : \mathcal{C}_{\text{id}}(B) \to \mathcal{C}_{\text{id}}(1),
\]

which turns \((\mathcal{C}_{\text{id}}(B), \mathcal{F}^B)\) into a Tannakian category, with Tannakian dual canonically isomorphic to \( B \).

We remark that by (i) and (ii), \( a \) is a Lie algebra object in \( \mathcal{C}_{\text{id}}(B) \), i.e. we have a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{a} \otimes_F a & \xrightarrow{[\cdot, \cdot]} & a \\
\Delta & \uparrow & \uparrow \\
a & \longrightarrow & 0
\end{array}
\]

in \( \mathcal{C}(B) \), and a similar one reflecting the Jacobi identity.

The category of finite-dimensional \((a, B)\)-modules is defined as the category \( \mathcal{C}_{\text{id}}(a, B) \) of \( a \)-module objects in \( \mathcal{C}_{\text{id}}(B) \). Considering the category \( \mathcal{C}(B) \) of ind-objects in \( \mathcal{C}_{\text{id}}(B) \), we define mutatis mutandis the category of \((a, B)\)-modules \( \mathcal{C}(a, B) \) as the category of \( a \)-module objects in \( \mathcal{C}(B) \). All these categories are equipped with natural forgetful faithful exact functors

\[
\mathcal{C}(a, B) \to \mathcal{C}(1)
\]

into the category of \( F \)-vector spaces.
1.1 Base change

Let $F'/F$ be a field extension. Then every pair $(a, B)$ over $F$ gives rise to a pair

$$(a', B') := (a, B) \otimes_F F' := (a \otimes_F F', B \times_{\text{Spec } F} \text{Spec } F')$$

over $F'$. Similarly every $(a, B)$-module $X$ gives rise to an $(a', B')$-module $X \otimes_F F'$, and this construction extends to an exact faithful functor

$$- \otimes_F F' : \mathcal{C}(a, B) \to \mathcal{C}(a', B'),$$

which is left adjoint to the exact faithful forgetful functor

$$\cdot|_F : \mathcal{C}(a', B') \to \mathcal{C}(a, B).$$

**Proposition 1.1.** For any two $(a, B)$-modules $X, Y$ we have a natural isomorphism

$$\text{Hom}_{(a, B)}(X, Y) \otimes_F F' \cong \text{Hom}_{(a', B')}(X \otimes_F F', Y \otimes_F F').$$

**Proof.** We first observe that we have a natural isomorphism

$$\text{Hom}_F(X, Y) \otimes_F F' \cong \text{Hom}_{F'}(X \otimes_F F', Y \otimes_F F').$$

As taking $B$-invariants resp. $B'$-invariants commutes with base change, we conclude that

$$\text{Hom}_B(X, Y) \otimes_F F' \cong \text{Hom}_{B'}(X \otimes_F F', Y \otimes_F F').$$

Cutting out $a$- resp. $a'$-linear maps is a linear operation, and therefore commutes with base change as well. This concludes the proof. $\square$

We will generalize Proposition 1.1 to arbitrary Ext groups in Section 2 (cf. Corollary 2.2).

1.2 Restriction of scalars

If $F'/F$ is finite, and if $(a', B')$ is a pair over $F'$, we may define the restriction of scalars

$$(a'', B'') := \text{Res}_{F'/F}(a', B') := (\text{Res}_{F'/F} a', \text{Res}_{F'/F} B'),$$

where on the right hand side $\text{Res}_{F'/F}$ denotes restriction of scalars à la Weil 31. We have similarly a functor

$$\text{Res}_{F'/F} : \mathcal{C}(a', B') \to \mathcal{C}(a'', B'').$$

naively given by sending an $(a', B')$-module $X'$ to $X'$ considered as an $F$-vector space $\text{Res}_{F'/F} X'$. We have the straightforward

**Proposition 1.2.** The functor $\text{Res}_{F'/F}$ is fully faithful.
Proof. It is well known that
\[ \text{Res}_{F'/F} : C_{\text{id}}(B') \to C_{\text{id}}(B'') \]
is fully faithful. This naturally extends to the categories of ind-objects, and as
the image of \( a' \) under \( \text{Res}_{F'/F} \) is \( a'' \), the claim follows.

In the sequel, depending on the context, we also consider \( \text{Res}_{F'/F} \) also as a
functor
\[ \text{Res}_{F'/F} : C_{\text{id}}(a', B') \to C_{\text{id}}(a, B), \]
i.e. we implicitly compose the restriction of scalars with the forgetful functor
along the unit map \( (a, B) \to (a'', B'') \) of the adjunction.

1.3 Associated pairs
We depart from an \( F \)-rational pair \( (a, B) \). Let \( \sigma : F \to C \) be an embedding and
denote
\[ (a', B') := (a, B) \times_{F, \sigma} C \]
the corresponding base change. Then we consider \( B'(C) \) as a real Lie group
and fix a maximal compact subgroup \( K' \). It is unique up to conjugation by an
element of \( B'(C) \). The inclusion
\[ K' \to B'(C) \]
induces an equivalence
\[ C_{\text{id}}(B \otimes_{F, \sigma} C) \to C_{\text{id}}(K') \]
of the categories of finite-dimensional representations. Then \( (a', K') \) constitutes a classical pair that we call associated to \( (a, B) \) (with respect to \( \sigma \)).

For associated pairs we have

**Proposition 1.3.** The category \( C(a', B') \) of \( C \)-rational modules is naturally
equivalent to the category \( C(a', K') \) of classical \( (a', K') \)-modules. This equiva-
lence induces an equivalence of the corresponding categories of finite-dimensional
modules.

**Proof.** First observe that in the case \( B = 1 \) both categories agree and are
given by the category of \( a' \)-modules. The general case reduces to this case by
the observation that the corresponding subcategories of the categories of \( a' \)-
modules that are cut out by a possibly nontrivial \( B' \) resp. \( K' \) are the same,
thanks to the equivalence (1).
1.4 Reductive pairs

A pair \((g, K)\) over \(F\) is reductive if furthermore \(g\) is reductive, \(K\) is the space of fixed points under an \(F\)-linear involution \(\theta\) of \(g\), and we are given a non-degenerate invariant bilinear form \(\langle \cdot, \cdot \rangle : g \to F\).

On the categories of reductive pairs and their modules we have natural base change and restriction of scalars functors as before, that we denote by \(- \otimes_F F'\) and \(\text{Res}_{F'/F} -\), where the restriction of scalars on pairs is only defined for finite extensions.

By the classification of reductive algebraic groups we know that each reductive pair \((g, K)\) that corresponds to a linear reductive real Lie group \(G\), has an \(F\)-rational model \((g_F, K_F)\) over a number field \(F \subseteq \mathbb{C}\), i.e. \(F/\mathbb{Q}\) is finite. However in general such a model is not unique, and the resulting notion of rationality for modules depends on the choice of model.

We call a parabolic subalgebra \(q \subseteq g\), defined over \(F\), \(F\)-germane if it has a Levi decomposition \(l + u\) over \(F\) with a \(\theta\)-stable Levi factor \(l\). Write \(\tilde{L} \cap K\) for the maximal subgroup of \(K\) whose Lie algebra is contained in \(l\). We assume it to be defined over \(F\) and set \(L \cap K := N_{\tilde{L} \cap K}(q) \cap N_{\tilde{L} \cap K}(\theta q)\). Then \((l, L \cap K)\) is another reductive pair over \(F\). In the sequel we implicitly assume Levi factors of \(F\)-germane parabolic subalgebras to be always of this specific form.

1.5 Galois actions

Let \(F'/F\) be a not necessarily finite Galois extension with Galois group \(\text{Gal}(F'/F)\). If \((a, B)\) is a pair over \(F\) with base change \((a', B')\) to \(F'\), \((\epsilon, D) \subseteq (aF', B_{F'})\) a subpair over \(F'\) and \(\tau \in \text{Gal}(F'/F)\), we have the Galois twists

\[ \epsilon^\tau := \tau(\epsilon) \subseteq a' = a \otimes_F F' \]

and similarly the subgroup

\[ D^\tau := \tau(D) \subseteq B' = B \otimes_F F' \]

Then we have the subpair

\[ (\epsilon, D)^\tau := (\epsilon^\tau, D^\tau) \subseteq (a', B'). \]

We remark that mutatis mutandis \(\tau\) acts on the universal enveloping algebra \(U(a')\).

**Proposition 1.4.** Let \(F'\) be an algebraic closure of \(F\), \((a, B)\) a pair with base change \((a', B')\) to \(F'\). Write \(Z(a')\) for the center of the universal enveloping algebra \(U(a')\). Then \(Z(a')\) is defined over \(F\), i.e.

\[ Z(a') = Z(a) \otimes_F F'. \]
Proof. An element $a \in U(a')$ lies in $Z(a')$ if and only if one (hence all) its Galois twists $a^\tau$ lie in $Z(a')$ as well, $\tau \in \text{Gal}(F'/F)$. Hence by Galois descent for vector subspaces of $U(a')$, $Z(a')$ is defined over $F$. \qed 

Let $F'/F$ be a Galois extension as before and let $X$ be an $(a', B')$-module. Then for any $\tau \in \text{Gal}(F'/F)$ we have on 

$$X^\tau := X \otimes_{F', \tau} F'$$

a natural $F'$-linear action of $a$, induced by the action of $a$ on $X$. This action extends uniquely to an action of 

$$a' = a \otimes_F F'.$$

Similarly we have a unique action of $B'$ on $X^\tau$, extending the natural $F'$-linear action of $B$ on $X$. If $X^\tau$ is defined over $F$, this action coincides with the usual Galois-twisted action.

We remark that unless $X$ is defined over $F$, $X^\tau$ is only defined up to unique isomorphism. To be more precise, the twisted module $X^\tau$ comes with a natural $\sigma$-linear isomorphism 

$$\iota_{\tau} : X \to X^\tau.$$ 

Then $(X^\tau, \iota_{\tau})$ is unique up to unique isomorphism.

In their investigation of rationality questions of rational representations of reductive groups, Borel and Tits introduced in [2, Section 6] the following Galois action on weights. Let $g$ be a reductive Lie algebra defined over $F$, $F'/F$ a Galois extension, over which the base change $g'$ of $g$ to $F'$ splits, and $h \subseteq g'$ a split Cartan subalgebra. Fix a positive system $\Delta^+ \subseteq \Delta(g', h)$, giving rise to a Borel subalgebra $b = h + n$ with nilpotent radical $n$. We write $W(g, h)$ for the corresponding Weyl group and $X(h)$ for the space of weights of $h$, and $\rho \in X(h)$ for the half sum of positive roots. Then for every $\sigma \in F'$ we find a unique inner automorphism $\alpha \in \text{Inn}(g') \subseteq \text{Aut}(g)$ sending $h^\sigma$ to $h$ and $b^\sigma$ to $b$. Then a weight $\lambda \in X(h)$ is sent via $\tau$ to $\Delta(\tau)(\lambda)$, which is characterized by the property 

$$\Delta(\tau)(\lambda(h^\sigma)) = \lambda(h)^\tau, \quad h \in h.$$ 

This action is well defined, i.e. independent of the choice of $\alpha$, and sends dominant weights to dominant weights. More concretely, if $\lambda \in X(h)$ is the highest weight of an irreducible finite-dimensional $g'$-module $V(\lambda)$, then $V(\lambda)^\tau$ is irreducible of highest weight $\Delta(\tau)(\lambda)$. Observe that $\Delta(\rho) = \rho$.

We remark that for $h \in U(h)^{W(g', h)}$, also 

$$\alpha(h^\tau) \in U(h)^{W(g', h)},$$

and this element is independent of the choice of $\alpha$. Indeed, if $\alpha' \in \text{Inn}(g')$ sends $h^\sigma$ to $h$ and $n^\sigma$ to $n$, then the element 

$$\alpha' \circ \alpha^{-1} \in N(h)$$

sends the positive system $\Delta^+$ to $\Delta^+$, thus lies in the centralizer of $h$. 

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Proposition 1.5. If $V$ is a quasi-simple $g'$-module on which $Z(g')$ acts via the infinitesimal character $\lambda + \rho$, then $V^\tau$ has infinitesimal character
$$\Delta \tau(\lambda + \rho) = \Delta \tau(\lambda) + \rho.$$

Proof. In light of Proposition 1.4, the Galois action on infinitesimal characters
$$\eta : Z(g') \to F',$$
is given by
$$\eta^\tau = \tau \circ \eta \circ \tau^{-1}.$$
The same formula applies to the action of $g$ on $V^\tau$. Recall the definition of the Harish-Chandra map
$$\gamma : Z(g') \to U(h')^{W(g', h)}.$$
It is given by the composition of the projection
$$p_n : U(g') = U(h) \oplus (n^-U(g') + U(g')n) \to U(h),$$
with the algebra map
$$\rho_n : U(h) \to U(h),$$
which is induced by the map
$$\mathfrak{h} \to \mathfrak{h}, \quad h \mapsto h - \rho(h) \cdot 1_{U(h)}.$$
Let $\tau \in \text{Gal}(F'/F)$ and where $\alpha \in \text{Inn}(g)$ as before. Then
$$\alpha(n^\tau) = n,$$
and the same formula holds for $n^-$. For any $g \in U(g')$ with decomposition
$$g = p_n(g) + r_n(g), \quad r_n(g) \in (n^-U(g') + U(g')n),$$
we deduce the decomposition
$$\alpha(g^\tau) = \alpha(p_n(g)^\tau) + \alpha(r_n(g)^\tau).$$
Hence, by (3),
$$p_n(\alpha(g^\tau)) = \alpha(p_n(g)^\tau).$$
Similarly we deduce from relation (2), the relation
$$\rho(\alpha(h^\tau)) = \Delta \tau(\rho)(\alpha(h^\tau)) = \rho(h)^\tau,$$
and
$$\alpha(1_{U(h)}) = 1_{U(\alpha(h^\tau))} = 1_{U(h)},$$
that for any $h \in \mathfrak{h},$
$$\rho_n(\alpha(h^\tau)) = \alpha(h^\tau) - \rho(h)^\tau \cdot 1_{U(h)}.$$n conclusion we obtain for any $z \in Z(g)$,
$$\gamma(\alpha(z^\tau)) = \alpha(p_n(z)^\tau) - \rho(h)^\tau \cdot 1_{U(h)}.$$Let $\eta : Z(g') \to F'$ be the infinitesimal character parametrized by $\lambda \in \mathfrak{h}^*/W(g', h)$. Applying $\lambda$ to the identity (4) proves the claim. \square
2 Homological Base Change Theorems

Let \((a, B)\) be any \(F\)-rational pair. As \(B\) is reductive, the category \(C_{\text{id}}(B)\) of finite-dimensional rational representations of \(B\) is semisimple, hence all objects in \(C_{\text{id}}(B)\) are injective and projective, and the same remains valid in the ind-category \(C(B)\).

The forgetful functor \(\mathcal{F}_B^{a,B}\) sending \((a, B)\)-modules to \(B\)-modules has a left adjoint
\[
\text{Ind}^{a,B}_B : M \mapsto U(a) \otimes_{U(b)} M \quad (5)
\]
and a right adjoint
\[
\text{Pro}^{a,B}_B : M \mapsto \text{Hom}_B(U(a), M)_{B-\text{finite}}. \quad (6)
\]
Then \(\text{Ind}^{a,B}_B\) sends projectives to projectives and \(\text{Pro}^{a,B}_B\) sends injectives to injectives and both functors commute with base change. As all \(B\)-modules are injective and projective, we see that \(C(a, B)\) has enough injectives and enough projectives. Therefore standard methods from homological algebra apply.

2.1 The General Homological Base Change Theorem

We have the fundamental

**Theorem 2.1** (Homological Base Change). Let \((a, B)\) and \((c, D)\) be two pairs over \(F\), and let
\[
\mathcal{F} : C(a, B) \rightarrow C(c, D)
\]
be a left (resp. right) exact functor. Consider for a map of fields \(\tau : F \rightarrow F'\) another left (resp. right) exact functor
\[
\mathcal{F}' : C(a \otimes_{F,\tau} F', B \otimes_{F,\tau} F') \rightarrow C(c \otimes_{F,\tau} F', D \otimes_{F,\tau} F')
\]
which extends \(\mathcal{F}\), i.e. there is a natural isomorphism
\[
\iota : \mathcal{F}(-) \otimes_{F,\tau} F' \rightarrow \mathcal{F}' \circ (- \otimes_{F,\tau} F').
\]
Then the natural isomorphism \(\iota\) extends naturally to natural isomorphisms
\[
(R^q\mathcal{F})(-)) \otimes_{F,\tau} F' \rightarrow (R^q\mathcal{F}') \circ (- \otimes_{F,\tau} F')
\]
resp.
\[
(L_q\mathcal{F})(-)) \otimes_{F,\tau} F' \rightarrow (L_q\mathcal{F}') \circ (- \otimes_{F,\tau} F')
\]
for all \(q\), which are compatible with the associated long exact sequences.

**Proof.** The right (resp. left) derived functors of \(\mathcal{F}\) and \(\mathcal{F}'\) may be computed via resolutions computed inductively with standard injectives (resp. projectives). As the construction of resolutions by standard injectives (resp. projectives) via the functors (5) and (6) commutes with base change to \(F'\) along \(\tau\), the claim follows. 

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**Corollary 2.2.** For any two \((a, B)\)-modules \(X\) and \(Y\) we have in every degree \(q\) natural isomorphisms

\[
\text{Ext}^q_{a,B}(X, Y) \otimes_F F' \cong \text{Ext}^q_{a',B'}(X \otimes_F F', Y \otimes_F F').
\]

In particular a short exact sequence

\[
0 \to X \to Y \to Z \to 0
\]

of \((a, B)\)-modules splits over \(F\) if and only if the short exact sequence

\[
0 \to X \otimes_F F' \to Y \otimes_F F' \to Z \otimes_F F' \to 0
\]

of \((a', B')\)-modules splits over \(F'\).

### 2.2 Equivariant Homology and Cohomology

As before let \((a, B)\) be a pair over \(F\). Assume that \((c, D)\) is a subpair which is normalized by another subpair \((\tilde{a}, \tilde{B})\) of \((a, B)\) with the property that

\[
a = \tilde{a} + c.
\]

We obtain a functor

\[
H^0(c, D; -) : \mathcal{C}(a, B) \to \mathcal{C}(\tilde{a}, \tilde{B}),
\]

sending a module to its \((c, D)\)-invariant subspace, on which \((\tilde{a}, \tilde{B})\) acts naturally.

\(H^0(c, D; -)\) is left exact and the higher right derived functors

\[
H^q(c, D; -) := R^qH^0(c, D; -) : \mathcal{C}(a, B) \to \mathcal{C}(\tilde{a}, \tilde{B})
\]

are the \(F\)-rational \(c, D\)-cohomology and may, thanks to (7), be computed via the usual standard complex

\[
\text{Hom}_D(\bigwedge c/\partial, X)
\]

of \((\tilde{a}, \tilde{B})\)-modules.

Dually we may define and explicitly compute \(F\)-rational \(c, D\)-homology as the left derived functors of the coinvariant functor

\[
H_0(c, D; -) : \mathcal{C}(a, B) \to \mathcal{C}(\tilde{a}, \tilde{B}),
\]

sending a module to its \((c, D)\)-coinvariant subspace, on which \((\tilde{a}, \tilde{B})\) acts naturally.

\(H_0(c, D; -)\) is right exact and the higher left derived functors

\[
H_q(c, D; -) := L^qH_0(c, D; -) : \mathcal{C}(a, B) \to \mathcal{C}(\tilde{a}, \tilde{B})
\]

are the \(F\)-rational \(c, D\)-homology and may, thanks to (8), be computed via the usual standard complex

\[
\left(\bigwedge (c/\partial) \otimes_F X\right)^D
\]

of \((\tilde{a}, \tilde{B})\)-modules.
of \((\tilde{a}, \tilde{B})\)-modules.

These homology and cohomology theories satisfy the usual Poincaré duality relations, Künneth formalism, and give rise to \(F\)-rational Hochschild-Serre spectral sequences [14].

**Proposition 2.3.** For any \(F\)-rational \((\tilde{a}, B)\)-module \(M\) the cohomology \(H^q(\zeta, D; X)\) is \(F\)-rational and for any map \(\tau : F \to F'\) of fields we have a natural isomorphism

\[
H^q(\zeta, D; X) \otimes_{F, \tau} F' \to H^q(\tilde{u}_\tau, X \otimes_{F, \tau} F')
\]

of \((\tilde{a}, B) \otimes_{F, \tau} F'\)-modules. The same statement is true for \(\zeta, D\)-homology, and the duality maps and Hochschild-Serre spectral sequences respect the rational structure.

**Proof.** This is obvious from the \(F\)-resp. \(F'\)-rational standard complexes computing cohomology and homology, and also a consequence of Theorem 2.1. □

### 2.3 Equivariant Cohomological Induction

To define \(F\)-rational cohomological induction, we adapt Zuckerman’s original construction as in [29, Chapter 6]. Assume we are given an \(F\)-rational reductive subgroup \(C \subseteq B\). We start with an \((\tilde{a}, C)\)-module \(M\) and set

\[
\tilde{\Gamma}_0(M) := \{ m \in M | \dim_F U(b) \cdot m < \infty \}.
\]

and

\[
\Gamma_0(M) := \{ m \in \tilde{\Gamma}_0(M) | \text{the } b\text{-representation } U(b) \cdot m \text{ lifts to } B^0 \}.
\]

As in the analytic case the obstruction for a lift to exist is the (algebraic) fundamental group of \(B^0\). In particular there is no rationality obstruction, as a representation \(N\) of \(b\) lifts to \(B^0\) if and only if it does so after base change to one (and hence any) extension of \(F\). It is easy to see that \(\Gamma_0(M)\) is an \((\tilde{a}, B^0)\)-module.

We define the space of \(B^0\)-finite vectors in \(M\) as

\[
\Gamma_1(M) := \{ m \in \Gamma_0(M) | \text{the actions of } C \text{ (on } M) \text{ and } C \cap B^0 \subseteq B^0 \text{ agree on } m \}.
\]

This is an \((\tilde{a}, C \cdot B^0)\)-module. Finally the space of \(B\)-finite vectors in \(M\) is

\[
\Gamma(M) := \text{Pro}_{\tilde{a}, C \cdot B^0}(\Gamma_1(M)).
\]

Remark that this is not a subspace of \(M\) in general. By Frobenius reciprocity it comes with a natural map \(\Gamma(M) \to M\).

The functor \(\Gamma\) is a right adjoint to the forgetful functor along

\[
i : (\tilde{a}, C) \to (\tilde{a}, B)
\]
and hence sends injectives to injectives. We obtain the higher Zuckerman fun-
c tors as the right derived functors

\[ \Gamma^q := R^q \Gamma : \mathcal{C}(a, C) \to \mathcal{C}(a, B). \]

As in the classical case we can show

**Proposition 2.4.** In every degree \( q \) we have a commutative square

\[
\begin{array}{ccc}
\mathcal{C}(a, C) & \xrightarrow{\Gamma^q} & \mathcal{C}(a, B) \\
F_{a,C} & \downarrow & \downarrow F_{a,B} \\
\mathcal{C}(b, C) & \xrightarrow{\Gamma^q} & \mathcal{C}(b, B)
\end{array}
\]

Strictly speaking the commutativity only holds up to natural isomorp hism. The natural isomorphisms are unique as it turns out that \( \Gamma(\cdot) \) is in both cases the right adjoint of the classical forgetful functor along the corresponding inclusion of pairs. We will not go into this as the commutativity may also easily deduced via base change from the classical setting, thanks to Proposition 1.1. However, for the sake readability, we decided to ignore such higher categorical aspects in the sequel.

**Proof.** For \( q = 0 \) the commutativity is obvious from the explicit construction of the functor \( \Gamma \). For \( q > 0 \) this follows from the standard argument that the forgetful functors have an exact left adjoint given by induction along the Lie algebras and hence carry injectives to injectives. Furthermore they are exact, which means that the Grothendieck spectral sequences for the two compositions both degenerate. Therefore the edge morphisms of said spectral sequence yield the commutativity, this proves the claim. \( \square \)

**Theorem 2.5.** For any \( F \)-rational \( (a, C) \)-module \( M \) and each degree \( q \), the Zuckerman functor \( \Gamma^q(M) \) for the inclusion \( C \to B \) is an \( F \)-rational \( (a, B) \)-module, and for any map \( \tau : F \to F' \) of fields we have a natural isomorphism

\[ \Gamma^q(X) \otimes_{F,\tau} F' \to \Gamma^q(X \otimes_{F,\tau} F') \]

of \( (a, B) \otimes_{F,\tau} F' \)-modules. Furthermore these isomorphisms are functorial in \( a, B, C, \) and respect the long exact sequences associated to \( \Gamma \).

**Proof.** By the Homological Base Change Theorem 2.1 it suffices to show that the rational Zuckerman functor \( \Gamma \) commutes with base change. This follows easily from the above construction of \( \Gamma \). \( \square \)

We remark that by Theorem 2.5 the functors \( \Gamma^q \) satisfy the usual properties, i.e. they vanish for \( q > \dim_F b/c \), we have a Hochschild-Serre spectral sequence for \( B \)-types, for parabolic cohomological induction the effect on infinitesimal characters is the same as in the classical setting, etc.
3 Geometric properties of \((g, K)\)-modules

A natural question to ask is which properties of Harish-Chandra modules are geometric, in the sense that a the property of an \((a, B)\)-module \(M\) holds over \(F\), if and only if it holds over one (and hence any) extension \(F'\) of \(F\). We will see in this section that many classical properties, i.e. admissibility, \(Z(g)\)-finiteness, and finite length are geometric properties. In order to to control finite length in extensions we need to digress to Quillen’s generalization of Dixmier’s variant of Schur’s Lemma.

3.1 Quillen’s Lemma and Fields of Definition

Let again \(F'/F\) be an extension and the pair \((a', B')\) be the extension of scalars to \(F'\) of a pair \((a, B)\) over \(F\). We say that a \((a', B')\)-module \(X'\) over \(F'\) is defined over \(F\), if there is an \((a, B)\)-module \(X\) satisfying

\[
F' \otimes_F X \cong X'
\]

We say that the pair \((a, B)\) satisfies condition \((Q)\), if we find a finite subgroup \(B_0 \subseteq B\) such that the map

\[
B_0 \to \pi_0(B \otimes_F \bar{F})
\]

is surjective, where \(\bar{F}\) denotes an algebraic closure of \(F\).

For example condition \((Q)\) is satisfied if \((a, B)\) is a reductive pair coming from a connected reductive linear algebraic group \(G\).

For later use we first remark Quillen’s generalization of Schur’s Lemma in

**Proposition 3.1** (Quillen [26]). If \((a, B)\) satisfies condition \((Q)\) and if \(X\) is an irreducible \((a, B)\)-module, then \(\operatorname{End}_{a,B}(X)\) is an algebraic division algebra over \(F\).

An algebraic division algebra over \(F\) is a division algebra \(A\) over \(F\), all of whose elements are algebraic over \(F\), i.e. for each \(a \in A\), \(F(a)/F\) is a (necessarily finite) algebraic extension.

**Proof.** It suffices to remark that an irreducible \((a, B)\)-module \(X\) remains irreducible after replacing \(B\) by an appropriate finite subgroup \(B_0\) satisfying \([11]\). Therefore \(X\) is an irreducible module over the convolution algebra \(U(a) \ast B_0\). Quillen’s result in [26] applies to this case and proves the claim. \(\square\)

**Corollary 3.2.** Let \((g, K)\) be a reductive pair over \(F\) satisfying condition \((Q)\). If \(X\) is an irreducible \((g, K)\)-module, then \(Z(g)\) acts on \(X\) via a finite-dimensional quotient. In particular \(X\) is \(Z(g)\)-finite, and so is every \((g, K)\)-module of finite length.

**Proof.** Fix an extension \(F'/F\) over which \(g\) splits. Over this extension the center \(Z(g) \otimes_F F'\) of \(U(g \otimes_F F')\) (cf. Proposition\([1,4]\)) is noetherian, hence \(Z(g)\) is noetherian. Therefore its image in \(\operatorname{End}_{g,K}(X)\) is finitely generated, and thus
finite-dimensional by Proposition 3.1. Therefore $Z(g)$ acts on $X$ via a finite-dimensional quotient and the corollary follows.

**Corollary 3.3.** If $X$ is an irreducible $(a, B)$-module over $F = \mathbb{C}$, $(a, B)$ satisfying condition (Q), then $\text{End}_{a, B}(X) = \mathbb{C}$. If $X$ is an irreducible $(a, B)$-module over $F = \mathbb{R}$, $(a, B)$ satisfying condition (Q), then

$$\text{End}_{a, B}(X) = \begin{cases} \mathbb{R}, \\ \mathbb{C}, \\ H. \end{cases}$$

In general, for a Galois extension $F'/F$, models of $(a', B')$-modules $X'$ over $F$ need not be unique. The existence of non-isomorphic models is equivalent to the existence of non-trivial 1-cocycles of $\text{Gal}(F'/F)$ with coefficients in the group $\text{Aut}_{(a', B')}(X')$. At least for absolutely irreducible modules satisfying Quillen’s Lemma, Hilbert’s Satz 90 guarantees the uniqueness.

**Proposition 3.4.** Let $F'/F$ be a Galois extension $F$, assume that $X'$ is an $(a', B')$-module satisfying

$$\text{End}_{a', B'}(X) = F', \quad (12)$$

and $X$ is a model of $X'$ over $F$. Then $X$ is unique up to isomorphism.

**Proof.** The proof proceeds mutatis mutandis as in [2, p. 741].

### 3.2 Geometric properties

**Definition 3.5.** An $(a, B)$-module $X$ is *admissible* if for each finite-dimensional $B$-module $V$

$$\dim_F \text{Hom}_B(V, X) < \infty. \quad (13)$$

Admissibility is a *geometric* property in the following sense.

**Proposition 3.6.** Let $X$ be an $(a, B)$-module. Then the following are equivalent:

(i) $X$ is an admissible $(a, B)$-module.

(ii) For some extension $F'/F$, $X \otimes_F F'$ is an admissible $(a, B) \otimes_F F'$-module.

(iii) For every extension $F'/F$, $X \otimes_F F'$ is an admissible $(a, B) \otimes_F F'$-module.

**Proof.** Let us show that (i) implies (iii). Let $F'/F$ be any extension. Then by the theory of reductive algebraic groups we know that there is a finite subextension $F''/F$ with the property that the scalar extension functor $- \otimes_{F''} F'$ induces a faithful essentially surjective functor

$$C_{\text{id}}(B'') \to C_{\text{id}}(B'),$$

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i.e. all finite-dimensional representations of $B$ which are defined over $F'$ are already defined over $F''$. This observation naturally extends to the ind-categories

$$\mathcal{C}(B'') \rightarrow \mathcal{C}(B').$$

In light of Proposition 1.1 this reduces our considerations to the cases where $F'/F$ is finite-dimensional.

Now assume (i), and let $V$ be any finite-dimensional $B'$-module over a finite extension $F'/F$. Then

$$\text{Hom}_{B'}(V, X \otimes_F F') = \text{Hom}_B(\text{Res}_{F'/F} V, X).$$

As $\text{Res}_{F'/F} V$ is finite-dimensional over $F$, (iii) follows.

As (iii) implies (ii), we are left to show that (ii) implies (i). So assume that $X \otimes_F F'$ is an admissible $(\mathfrak{a}', B')$-module for an extension $F'/F$. Let $V$ be a finite-dimensional $B$-module. Then by Proposition 1.3 we have

$$\dim_F(V, X) = \dim_{F'}(V \otimes_F F', X \otimes_F F') < \infty.$$ 

This concludes the proof.

**Theorem 3.7.** Let $(\mathfrak{g}, K)$ be a reductive pair over $F$ satisfying condition (Q), and let $X$ be an $F$-rational $(\mathfrak{g}, K)$-module. Then the following are equivalent:

(i) $X$ is of finite length.

(ii) $X$ is $Z(\mathfrak{g})$-finite and admissible.

Proof. That (ii) implies (i) reduces to the case $F = C$ by Proposition 3.6, as $Z(\mathfrak{g})$-finiteness is a geometric property as well. By Proposition 1.3 the case $F = C$ follows from the classical case, where the statement is well known.

The implication (i) to (ii) reduces to the classical case as follows. Let $X$ be an irreducible $(\mathfrak{g}, K)$-module over $F$. Then $X$ is $Z(\mathfrak{g})$-finite by Corollary 3.2, and also finitely generated. The latter two properties are stable under base change, and to prove the admissibility of $X$, we may by Proposition 1.3 assume that $F = C$, in which case, by Proposition 1.3, the result is well known (cf. [24]).

**Corollary 3.8.** Let $X$ be an $F$-rational $(\mathfrak{g}, K)$-module for a reductive pair $(\mathfrak{g}, K)$ satisfying condition (Q). Then the following are equivalent:

(i) $X$ is of finite length.

(ii) For some extension $F'/F$, $X \otimes_F F'$ is of finite length.

(iii) For every extension $F'/F$, $X \otimes_F F'$ is of finite length.

Proof. By Theorem 3.7 and Proposition 3.6 it suffices to observe that $Z(\mathfrak{g})$-finiteness is a geometric property as well, which is obvious.
3.3 u-cohomology and constructible parabolic subalgebras

Proposition 3.9. If \( q \) is a \( \theta \)-stable \( F \)-germane parabolic subalgebra of a reductive \( F \)-rational pair \((g, K)\), with Levi decomposition \( q = l + u \), then for all degrees \( q \), the functors

\[
H^q(u; -) \quad \text{and} \quad H_q(u; -),
\]

preserve admissibility, \( Z(g) \)-finiteness and if \((g, K)\) satisfies condition (Q), then also finite length.

Proof. The preservation of \( Z(g) \)-finiteness is proven as in the classical case, which holds in fact for any parabolic subalgebra. The preservation of admissibility also follows mutatis mutandis as in the classical case. With Theorem 3.7 we conclude that \( u \)-cohomology preserves finite length.

Assume that the field \( F \) has a real place, i.e. we have

\[
\text{Hom}(F, R) \neq 0,
\]

and that \((g, K)\) gives rise to a classical reductive pair \((g_R, K_R)\) after extension of scalars along an embedding \( i_R : F \to R \). Let \( F'/F \) be an extension, and \((g', K')\) the usual extension of scalars in the extension \( F'/F \). We call an \( F' \)-germane parabolic subalgebra \( q' \subseteq g' \) \( F \)-constructible, if there exists a sequence of \( F' \)-germane parabolic subalgebras

\[
q' = q'_0 \subseteq q'_1 \subseteq \cdots \subsetneq q'_l = g'
\]

with the following property: Inside the Levi pair \((L'_i+1, L'_{i+1} \cap K')\) of the associated parabolic \( q'_i \), the parabolic \( q'_i \cap L'_{i+1} \) is \( \theta \)-stable or defined over \( F \). This notion generalizes the notion of constructibility introduced in [16]. In the case \( F = R \) and \( F' = C \) the two notions coincide.

The motivation for this notion is

Proposition 3.10. Assume that \((g, K)\) is a reductive pair over a field \( F \subseteq R \) and that \( q' \subseteq g' \) is \( F' \)-constructible with Levi decomposition \( q' = l' + u' \). Then for all degrees \( q \), the functors

\[
H^q(u'; -) \quad \text{and} \quad H_q(u'; -),
\]

preserve finite length.

Proof. Assume we are given a sequence as in (14) satisfying the defining property of \( F' \)-constructibility. By the Hochschild-Serre spectral sequence for \( u' \)-(co)homology it is enough to show that finite length is preserved in the case where \( q' \) is \( \theta \)-stable or defined over \( F \). In the \( \theta \)-stable case the claim follows from Proposition 3.9. The other case reduces by Corollary 3.8 to the case of a real parabolic \( q' \) over \( F = R \) after base change along \( i_R \), where the statement is well known.
3.4 Restrictions of irreducibles

Proposition 3.11. Assume $F'/F$ to be a finite Galois extension. Let $X'$ be an irreducible $(a', B')$-module, then as an $(a, B)$-module, Res$_{F'/F} X'$ decomposes into a finite direct sum of irreducible $(a, B)$-modules. The number of summands is bounded by the degree $[F' : F]$.

Proof. We write $X$ for the $(a, B)$-module Res$_{F'/F} X'$. Consider the $(a', B')$-module $X'' := X \otimes_F F'$. The identity

$$F' \otimes_F F' = \bigoplus_{\sigma \in \text{Gal}(F'/F)} F' \otimes_{F', \sigma} F'$$

shows that $X''$ decomposes into a finite direct sum

$$X'' = X' \otimes_{F'} F' \otimes_F F' = \bigoplus_{\sigma \in \text{Gal}(F'/F)} X' \otimes_{F', \sigma} F', \quad (15)$$

of irreducible $(a', B')$-modules

$$F'_{\sigma} := X' \otimes_{F', \sigma} F'.$$

Therefore, by Corollary 2.2, each short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$$

of $(a, B)$-modules splits, as it must split over $F'$ by (15), and we have isomorphisms

$$Y \otimes_F F' = \bigoplus_{\sigma \in \Xi} F'_{\sigma}$$

and

$$Z \otimes_F F' = \bigoplus_{\sigma \notin \Xi} F'_{\sigma}$$

for some unique subset $\Xi \subseteq \text{Gal}(F'/F)$. Inductively we conclude that $X$ decomposes into a finite sum of irreducible $(a, B)$-modules as claimed. \hfill \Box

Corollary 3.12. Assume $F'/F$ to be a finite extension. Let $X'$ be an absolutely irreducible $(a', B')$-module. Then, as an $(a, B)$-module, Res$_{F'/F} X'$ decomposes into a finite direct sum of irreducible $(a, B)$-modules.

Proof. As $X'$ is absolutely irreducible, we may replace $F'$ without loss of generality by its normal hull over $F$. Then the claim follows from Proposition 3.11. \hfill \Box

4 Rationality

In this section we investigate rationality questions from different perspectives. We first consider how rational models of irreducible $a$-modules may be ‘generated’. Then we consider rationality of cohomologically induced modules and dually the rationality of algebraic characters.
4.1 Rationality of $g$-modules

Let $F'/F$ is an extension, possibly infinite, and the pair $(a', B')$ is the extension of scalars to $F$ of a pair $(a, B)$ over $F$. We consider an $a'$-module $M'$ and its restriction of scalars $M_0 := \text{Res}_{F'/F} M'$ as an $a$-module. As sets we have a natural identification $M_0 = M'$. In this subsection we investigate the following question.

Given a non-zero vector $m \in M'$, we may consider the submodule

$$U(a) \cdot m \subseteq \text{Res}_{F'/F}(M')$$

as an $a$-module over $F$. The question arises when is this an $F$-model for $M'$, or more generally when the induced map

$$(U(a) \cdot m) \otimes_F F' \to M'$$

over $F'$ is a monomorphism. Even if $M'$ has a model over $F$, this map doesn’t need to be a monomorphism.

Assume $M'$ to be simple as $a'$-module. Then for any $0 \neq m \in M'$ we have a short exact sequence

$$0 \longrightarrow m_m \longrightarrow U(a') \overset{p_m}{\longrightarrow} M' \longrightarrow 0$$

of $U(a')$-modules, where the map $p_m$ is given by

$$g \mapsto g \cdot m,$$

the first map being the inclusion of the annihilator ideal $m_m$ of $m$ into $U(a')$. Then $m_m$ is a maximal left ideal.

To each $0 \neq m, m' \in M'$ we find $a, a' \in U(a')$ with

$$a'm = m'$$

and

$$am' = m.$$

We conclude that

$$m_m a' = m_m,$$

and

$$m_m a = m_m'.$$

**Proposition 4.1.** The following three statements are equivalent:

(a) $M'$ is defined over $F$,

(b) for some $0 \neq m \in M'$, $m_m$ is defined over $F$ with respect to the canonical rational structure $U(a) \subseteq U(a')$,

(c) for some $0 \neq m \in M'$, $U(a) \cdot m$ is an $F$-model of $M'$.
Proof. Assume $M'$ is defined over $F$, i.e. we find an $\mathfrak{a}$-module $M$ together with a map $i : M \to \text{Res}_{F'/F} M'$, which after base change of $M$ to $F'$ induces an isomorphism

$$F' \otimes_F M \cong M'$$

of $\mathfrak{a}'$-modules. Note that $M$ must be a simple $\mathfrak{a}$-module. Pick any $0 \neq m \in M$, and consider its image under $i$. Then we have a natural isomorphism

$$F' \otimes_F m_m \cong m_i(m),$$

which proves that (a) implies (b).

Assume (b), and let $0 \neq m \in M'$ such that $m_m$ is defined over $F$, i.e.

$$m := m_m \cap U(\mathfrak{a}),$$

is an $F$-model of $m_m$. Then $m$ is the annihilator of $m \in U(\mathfrak{a}) \cdot m \subseteq M'$. Therefore $U(\mathfrak{a}) \cdot m$ is a model of $M'$ over $F$. This shows the implication (b) $\Rightarrow$ (c).

Assuming (c), (a) follows trivially.

We remark that properties (b) resp. (c) can not be expected for all $0 \neq m \in M'$ in general. Let us assume $\mathfrak{a} = \mathfrak{g}$ to be reductive of positive rank, with base change to $F'$ denoted by $\mathfrak{g}'$ as usual. Consider the Verma module

$$V' = U(\mathfrak{g}') \otimes_{U(\mathfrak{q}')} \lambda,$$

where $\mathfrak{q}' \subseteq \mathfrak{g}'$ is a Borel subalgebra defined over $F$, and $\lambda$ is a regular character of the fixed Levi $\mathfrak{l}'$, also defined over $F$. Then obviously for every non-zero $1_\lambda \in \lambda$, $U(\mathfrak{g}') \cdot (1 \otimes 1_\lambda)$ is an $F$-model of $V'$, hence $V'$ is defined over $F$. We denote the corresponding models of algebras and spaces with $\mathfrak{g}, \mathfrak{q}, \mathfrak{l}$ and $V$ accordingly.

We may find a non-zero $u \in \mathfrak{u}^-$ in the opposite of the radical $\mathfrak{u}' \subseteq \mathfrak{q}'$ with the property that there is a $u' \in \mathfrak{u}^+$ over $F$, such that $\lambda([u, u'])$ is not an element of $F$.

Then the vector $v_0 := (u + 1) \otimes 1_\lambda$ generates a $\mathfrak{g}$-submodule $V_0 \subseteq \text{Res}_{F'/F} V'$ which is easily seen to contain the two $F$-linearly independent vectors $1 \otimes 1$ and $\lambda([u', u]) \cdot (1 \otimes 1)$. Therefore, in this case, $V_0$ is not an $F$-model of $V'$.

We may assume for simplicity that $u$ is root vector for $\mathfrak{l}$ with weight $\beta \in \mathfrak{l}'$. Then in this example we have

$$m_{v_0} = \{ g \in U(\mathfrak{g}) \mid g \cdot u \otimes 1_\lambda = -g \otimes 1_\lambda \}.$$ 

By the Poincaré-Birkhoff-Witt Theorem we may find for any $0 \neq g \in U(\mathfrak{g})$ non-zero elements $g_u \in U(\mathfrak{u}), g_l \in U(\mathfrak{l})$, and $g_u^- \in U(\mathfrak{u}^-)$ such that

$$g = g_u^- \cdot g_l \cdot g_u.$$

We treat two cases separately:

$$g_u = g_u'' + g_u' + 1,$$

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where \( g_{u''} \in U(u) \cdot u \cdot u \), and \( g_{u'} \in U(u) \cdot u \),

\[
g_u = g_{u''} + g_{u'} \in U(u) \cdot u,
\]

with the same assumptions on the two summands.

In the first case, \( g \in m_{v_0} \) if and only if

\[
g - g_{u} \cdot g_{l} \cdot u \otimes 1_{\lambda} + (\lambda(g_{l}) \cdot g_{u} \otimes 1_{\lambda} = 0.
\]

This equation is equivalent to

\[
(\lambda + \beta)(g_{l}) \cdot g_{u} \cdot u \otimes 1_{\lambda} + (\lambda(g_{l}[g_{u'}]) \cdot g_{u} \otimes 1_{\lambda} = 0,
\]

which in turn is equivalent to

\[
(\lambda + \beta)(g_{l}) = \lambda(g_{l}) \cdot \lambda([g_{u''}, u] + 1) = 0,
\]

by reasoning by degree.

In the second case the same reasoning yields the equivalent condition

\[
\lambda(g_{l}[g_{u''}, u]) \cdot g_{u''} \otimes 1_{\lambda} = 0,
\]

in other words

\[
\lambda(g_{l}[g_{u''}, u]) = 0.
\]

(17)

We see that the space of solutions to condition (17) and the vanishing of the first term of (16) are defined over \( k \). Whereas the space of solutions to the vanishing of the second term in (16) is the union of two varieties, one defined over \( k \), and another one which is not. This also illustrates how (b) in Proposition 4.1 fails here: \( m_{v_0} \cap U(g) \) is too small, as no \( g_{u'} \in u_k \) can contribute to the vanishing of the second term in (16).

4.2 Rational models of Harish-Chandra modules

As a consequence of Theorem 2.1 we obtain

**Proposition 4.2.** Each cohomologically induced \((g, K)\)-module has a model over the field of definition of its inducing data.

In particular if the parabolic subpair \((q, L \cap K)\) of an \( F \)-rational pair \((g, K)\), and the inducing module \( Z \) are all rational over \( F \subseteq C \), then so are the induced modules \( R^q(Z) \) for all \( q \), where

\[
R^q(Z) = R^q\Gamma(U(g) \otimes_{U(q)} Z_q),
\]

\( Z_q \) denoting the \((q, L \cap K)\)-module obtained from \( Z \otimes \wedge^{\dim u} u \) via pullback along the projection \( q \to l \), and \( \Gamma \) denotes the Zuckerman functor for the inclusion \((g, L \cap K) \to (g, K)\).

In particular by [30], every \((g, K)\)-module underlying a discrete series representation, or more generally of any unitary representation with non-trivial \((g, K)\)-cohomology and \( \overline{Q} \)-rational infinitesimal character has a model over a number field.

The classical character formulae, as for example in the latter case given in loc. cit., are rational over the same field of definition as well if interpreted in our theory as explained in the next section.
4.3 Rational Algebraic Characters

In the context of the theory developed here, it is possible to construct an abstract theory of algebraic characters over any field \( F \) of characteristic 0 following the axiomatic treatment given in [16]. Our results about geometric properties of modules may be used to produce non-trivial instances of this theory. To give a concrete example, we sketch the case of finite length modules here. The case of discretely decomposables discussed in [16] generalizes to arbitrary base fields of characteristic 0 along the same lines as well.

We depart from a reductive pair \((g, K)\) over a field \( F \) and fix an \( F' \)-constructible parabolic subalgebra \( q' \subseteq g' \) over an extension \( F'/F \) with Levi decomposition \( q' = l' + u' \). By Proposition 3.10 we have for each \( q \in \mathbb{Z} \) a well defined functor

\[
H^q(u', -) : \mathcal{C}_{fl}(g', K') \to \mathcal{C}_{fl}(l', L' \cap K')
\]
on the corresponding categories of finite length modules over \( F' \). We remark that by Corollary 3.8 the categories of finite length modules are essentially small. Hence, by the long exact sequence of cohomology, the Euler characteristic of these functors gives rise to a group homomorphism

\[
H_q' : \mathcal{K}_{fl}(g', K') \to \mathcal{K}_{fl}(l', L' \cap K'),
\]

\[
[X] \mapsto \sum_{q \in \mathbb{Z}} (-1)^q [H^q(u', X)]
\]
of the corresponding Grothendieck groups. Here the bracket \([\cdot]\) denotes the class associated to a module. We define the Weyl denominator relative to \( q' \) as

\[
W_{q'} := H_q'(1_{g', K'}),
\]

where \( 1_{g', K'} \) denotes the trivial \((g', K')\)-module.

We know that the category \( \mathcal{C}_{fl}(l', L' \cap K') \) of finite length modules is closed under tensor products with finite-dimensional representations, again by Corollary 3.8 as this is well known over \( \mathbb{C} \), cf. [19]. In particular \( \mathcal{K}_{fl}(l', L' \cap K') \) is naturally a module over the commutative ring \( \mathcal{K}_{fd}(l', L' \cap K') \) of finite-dimensional modules, the (scalar)multiplication stemming from the tensor product. The relative Weyl denominator \( W_{q'} \) lies in the latter ring and we may therefore consider the module-theoretic localization

\[
C_q'(l', L' \cap K') := \mathcal{K}_{fl}(l', L' \cap K')[W_{q'}^{-1}].
\]

Now the \( q' \)-character map is by definition the map

\[
c_q : \mathcal{K}_{fl}(g', K') \to C_q'(l', L' \cap K'),
\]

\[
[X] \mapsto \frac{H_q'(X)}{W_{q'}}.
\]
The generalization of Theorem 1.4 of loc. cit. in our context is
Theorem 4.3. The map $c_{q'}$ has the following properties:

(i) The map $c_{q'}$ is additive, i.e. for all $X, Y \in K_{fl}(g', K')$ we have
$$c_{q'}(X + Y) = c_{q'}(X) + c_{q'}(Y).$$

(ii) The map $c_{q'}$ is multiplicative, i.e. for all $X \in K_{fl}(g', K')$ and $Y \in K_{fd}(g', K')$ we have
$$c_{q'}(X \cdot Y) = c_{q'}(X) \cdot c_{q'}(Y),$$
and
$$c_{q'}(1_{g', K'}) = 1_{1, L \cap K'}.$$

(iii) If $q$ is $\theta$-stable then $c_{q'}$ respects admissible duals, i.e. for all $X \in K_{fl}(g', K')$ we have
$$c_{q'}(X^\vee) = c_{q'}(X)^\vee,$$
where $X^\vee$ denotes the $K$-finite dual.

(iv) If for $X \in K_{fl}(g', K')$ its restriction lies in $K_{fl}(l', L' \cap K')$, then we have the formal identity
$$c_{q'}(X) = [X]$$
in $C_{q'}(l', L' \cap K')$.

Proof. The additivity is clear. The multiplicativity is proven mutatis mutandis as in Theorem 1.4 in [16]. This also applies to (iv).

To prove (iii), we recall the notion of an $(u'$-admissible) pair of categories having duality. In our situation we consider the pair of categories of finite length modules for $(g', K')$ and $(l', L' \cap K')$ respectively. By definition this pair has duality if for each finite length $(g', K')$-module $X$ in each degree $q$ the three modules
$$H^q(u'; X^\vee), \ H^q(u'; X^*), \ H^q(u'; X^*/X^\vee),$$
all lie in $C_{fl}(l', L' \cap K')$, where the superscripts $\cdot^\vee$ and $\cdot^*$ denote the $K'$- resp. $(L' \cap K')$-finite duals, and furthermore the Euler characteristic
$$\sum_q (-1)^q [H^q(u'; X^*/X^\vee)] = 0$$
vanishes. Once this is established in our situation, the proof of the analogous statement of Theorem 1.4 in loc. cit. goes through word by word.

To see that the two categories of finite length modules form a pair with duality, it suffices in our context to see that the natural map
$$H^q(u'; X^\vee) \to H^q(u'; X^*)$$
is an isomorphism, which is equivalent to the vanishing of $H^q(u'; X^*/X^\vee)$. This may be proved as in Proposition 1.2 in loc. cit., which is stated for admissible modules, but the proof given in loc. cit. works mutatis mutandis also in the finite length case for arbitrary $F'$.  \[\square\]
As in Proposition 1.6 in [16] we see that our algebraic characters over $F'$ are transitive, and similarly we see as in Proposition 1.7 of loc. cit. that our characters are compatible with restrictions, which generalizes statement (iv) of Theorem 1.8. The compatibility of $F'$-rational characters with translation functors as discussed in section 2 of loc. cit. remains valid as well.

Given a map of subfields $\tau : F' \to F''$ of $C$, which induces the identity on $F$, we may consider the exact base change functors

$$C_\tau(\mathfrak{g}', K') \to C_\tau(\mathfrak{g}'', K''),$$

for $? \in \{\text{fd}, \text{fl}\}$, and similarly

$$C_\tau(\mathfrak{l}', L' \cap K') \to C_\tau(\mathfrak{l}'', L'' \cap K''),$$

where double prime denotes the base change to $F''$ along $\tau$. These induce maps

$$- \otimes_{F', \tau} F'' : K_\tau(\mathfrak{g}', K') \to K_\tau(\mathfrak{g}'', K''),$$

on the level of Grothendieck groups, and mutatis mutandis for the modules over the Levi factor of $\mathfrak{q}'$. These maps are additive, multiplicative, and respect duals in the sense of Theorem 4.3.

Now $\mathfrak{q}$ itself gives rise to a parabolic subalgebra $\mathfrak{q}'' \subseteq \mathfrak{g}''$, which is easily seen to be constructible, and we have the relation

$$W_{\mathfrak{q}'} \otimes_{F', \tau} F'' = W_{\mathfrak{q}''}.$$

By the very definition of $u$- resp. $u''$-cohomology via standard complexes (or the Homological Base Change Theorem), we see that

$$( - \otimes_{F', \tau} F'' ) \circ H_{\mathfrak{q}'} = H_{\mathfrak{q}''}(- \otimes_{F', \tau} F'').$$

As base change commutes with tensor products, we just proved

**Proposition 4.4.** For each constructible subalgebra $\mathfrak{q}' \subseteq \mathfrak{g}'$ and each $F$-linear map of fields $\tau : F' \to F'' \subseteq C$, $\mathfrak{q}'' = \mathfrak{q}' \otimes_{F', \tau} F''$ is constructible again and we have the commuting square

$$\begin{array}{ccc}
K_{\mathfrak{q}'}(\mathfrak{g}', K') & \xrightarrow{- \otimes_{F', \tau} F''} & K_{\mathfrak{q}''}(\mathfrak{g}'', K'') \\
\downarrow c_{\mathfrak{q}'} & & \downarrow c_{\mathfrak{q}''} \\
C_{\mathfrak{q}'}(\mathfrak{l}', L' \cap K') & \xrightarrow{- \otimes_{F', \tau} F''} & C_{\mathfrak{q}''}(\mathfrak{g}'', K'')
\end{array}$$

5 Frobenius-Schur indicators

In this section we introduce Frobenius-Schur indicators for complex $(\mathfrak{a}, B)$-modules and discuss their relation to the descent problem inside the Galois extension $C/R$. In the infinitesimally unitary case they are particularly useful, as in this context the indicator classifies irreducibles and their invariant bilinear forms accordingly. The situation turns out to be completely analogous to the classical case treated by Frobenius and Schur in [S].
5.1 The general notion

In this section we write \((a, B)\) for a pair over \(\mathbb{R}\) satisfying condition \((Q)\), and write \((a_\mathbb{C}, B_\mathbb{C})\) for its base change to \(\mathbb{C}\). Let \(X\) be an irreducible \((a_\mathbb{C}, B_\mathbb{C})\)-module, and denote by \(X_\mathbb{R} := \text{Res}_{\mathbb{C}/\mathbb{R}} X\) its restriction of scalars to \(\mathbb{R}\), which we consider as an \((a, B)\)-module (over \(\mathbb{R}\)). We write \(\tau : \mathbb{C} \to \mathbb{C}\) for complex conjugation and set

\[
\overline{X} := X \otimes_{\mathbb{C}, \tau} \mathbb{C}.
\]

We remark that the action of \((a_\mathbb{C}, B_\mathbb{C})\) on \(\overline{X}\) depends on the real form \((a, B)\). To be more precise, consider \(X\) as a complex representation of \((a, B)\), i.e. we have an \(\mathbb{R}\)-linear map

\[
\iota : a \to \text{End}_\mathbb{C}(X)
\]

decomposition. For \(B\) we proceed mutatis mutandis.

Then we have for the complexification of \(X_\mathbb{R}\) the decomposition

\[
X_\mathbb{C} := X_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = X \oplus \overline{X},
\]

as \((a_\mathbb{C}, B_\mathbb{C})\)-module.

Now \(X_\mathbb{R}\) is either irreducible or, by Proposition 3.11, decomposes into a direct sum

\[
X_\mathbb{R} \cong X_1 \oplus X_2
\]

of two irreducible \((a, B)\)-modules, this decomposition being compatible with the decomposition \((18)\).

**Proposition 5.1.** The irreducible \((a_\mathbb{C}, B_\mathbb{C})\)-module \(X\) is defined over \(\mathbb{R}\) if and only if \(X_\mathbb{R}\) is reducible. In this case \(X_\mathbb{R}\) decomposes into two isomorphic copies of the same module, i.e. \(X_1 \cong X_2\), where \(X_1\) is absolutely irreducible, \(\text{End}_{a, B}(X_1) = \mathbb{R}\) and \(\text{End}_{a, B}(X_\mathbb{R}) = \mathbb{R}^{2 \times 2}\), furthermore \(X \cong \overline{X}\).

**Proof.** Assume \(X\) is defined over \(\mathbb{R}\), i.e. there exists an \((a, B)\)-module \(X_0\) with

\[
X \cong X_0 \otimes_\mathbb{R} \mathbb{C}.
\]

Then, as \((a, B)\)-modules, we have

\[
X_\mathbb{R} = X_0 \otimes_\mathbb{R} \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C},
\]

which clearly decomposes into two copies of the irreducible module \(X_0\). With Proposition 1.4 we conclude that \(\text{End}_{a, B}(X_0) = \mathbb{R}\), because \(\text{End}_{a_\mathbb{C}, B_\mathbb{C}}(X) = \mathbb{C}\) by Corollary 3.3, hence the claim about endomorphism rings follows.
As $X_1 \otimes_R \mathbb{C}$ is a submodule of $X \mathbb{C}$, which decomposes according to (18) into a sum of two irreducibles, we conclude that $X_1 \otimes_R \mathbb{C}$ is irreducible, hence absolutely irreducible.

Now assume $X_R$ to be reducible. Then we claim that $X_1$ is a real model of $X$. Indeed, consider the complexification map

$$X_1 \otimes_R \mathbb{C} \rightarrow X$$

(20)
of $(\mathfrak{a}_C, B_C)$-modules. It is clearly an epimorphism as $X$ is irreducible and the image of $X_1$ in $X$ is non-zero by definition.

Again by the same argument as above, we conclude that, as a submodule of (18), the module $X_1 \otimes_R \mathbb{C}$ must be irreducible. Hence the non-zero map (20) is an isomorphism and the claim follows.

**Proposition 5.2.** If $X_R$ is an irreducible $(\mathfrak{a}, B)$-module, then either $X \cong X$ and $\text{End}_{\mathfrak{a}, B}(X_R) = \mathbb{H}$, or $\text{End}_{\mathfrak{a}, B}(X_R) = \mathbb{C}$ otherwise.

**Proof.** We first observe that by Corollary 3.3

$$\mathbb{C} = \text{End}_{\mathfrak{a}_C, B_C}(X) \subseteq \text{End}_{\mathfrak{a}, B}(X_R),$$

which by the case $F = \mathbb{R}$ of Corollary 3.3 tells us that only the two claimed endomorphism rings are possible. Now by Proposition 3.11 and the proof of Proposition 3.11 we know that

$$\text{End}_{\mathfrak{a}, B}(X_R) \otimes_R \mathbb{C} = \text{End}_{\mathfrak{a}_C, B_C}(X_R \otimes \mathbb{C}) = \text{End}_{\mathfrak{a}_C, B_C}(X \oplus X),$$

which proves the claim.

We define the Frobenius-Schur indicator $FS(X)$ of $X$ accordingly as

$$FS(X) = 1 \iff \text{End}_{\mathfrak{a}, B}(X_R) = \mathbb{R}^{2 \times 2},$$

$$FS(X) = 0 \iff \text{End}_{\mathfrak{a}, B}(X_R) = \mathbb{C},$$

$$FS(X) = -1 \iff \text{End}_{\mathfrak{a}, B}(X_R) = \mathbb{H}.$$ As usual we refer to the real, complex, and quaternionic case respectively.

### 5.2 Frobenius-Schur indicators in the infinitesimally unitary case

In this section we show that even in the infinitesimally unitary setting the previously defined Frobenius-Schur indicators behave the same way as the do classically in loc. cit.

From now on we assume $X$ to be an irreducible admissible $(\mathfrak{a}_C, B_C)$-module which is infinitesimally unitary. For us a Hermitian form is a linear equivariant map

$$h : X \otimes_C X \rightarrow \mathbb{C}.$$  

(21)
satisfying the usual condition of conjugate symmetry. $X$ is infinitesimally unitary if the form $h$ is positive definite.
**Theorem 5.3.** Let $X$ be an irreducible admissible infinitesimally unitary $(\mathfrak{a}_C, B_C)$-module. Then $\text{FS}(X) \neq 0$ if and only if $X$ carries a non-zero invariant bilinear form. In this case $X$ is real resp. complex if and only if the bilinear form may be chosen symmetric resp. anti-symmetric.

**Proof.** The Hermitian form $h$ corresponds uniquely to a non-zero $(\mathfrak{a}_C, B_C)$-linear map

$$h^\vee : X \to X^\vee,$$

$$x \mapsto h(- \otimes x).$$

As $X$ is reflexive, $h^\vee$ is an isomorphism and the complexification $X_C$ of $X_R$ becomes, as an $(\mathfrak{a}_C, B_C)$-module, isomorphic to

$$X_C \cong X \oplus X^\vee.$$  

Therefore, saying that $X$ is real or quaternionic is, by Propositions 5.1 and 5.2, equivalent to saying that $X$ is self-dual, which is the same to say that $X$ carries a non-zero invariant bilinear form. This proves the first part of the theorem.

As to the second part, our proof is a mere categorical reinterpretation of the classical proof in [8]. Let us introduce some notation. As we assume $\text{FS}(X) \neq 0$, we may fix a complex linear isomorphism

$$\iota : X \to X$$

of $(\mathfrak{a}_C, B_C)$-modules. Write

$$\kappa : X \to \overline{X}, \ v \mapsto v \otimes 1$$

for the natural conjugate linear isomorphism of vector spaces.

Let furthermore

$$\varepsilon : X \otimes_C \overline{X} \to X \otimes_C \overline{X},$$

$$v \otimes w \mapsto \kappa^{-1}(w) \otimes \kappa(v)$$

be the conjugate linear transposition of the arguments of $h$. Then the symmetry of $h$ is reflected in the identity

$$h \circ \varepsilon = \tau \circ h,$$

where $\tau : C \to C$ denotes complex conjugation as before.

Now with the symmetry property (24) of $h$, we verify that

$$h' := \tau \circ h \circ (\kappa \otimes \kappa)$$

is another positive definite Hermitian form on $X$. Therefore, by Schur’s Lemma, we find an $\eta \in \mathbb{R}_{>0}$ satisfying

$$h' = \eta \cdot h.$$
If we replace \( \iota \) by \( \lambda \cdot \iota, \lambda \in \mathbb{C}^\times \), we obtain

\[
\tau \circ h \circ ((\lambda \iota) \kappa \otimes \kappa(\lambda \iota)) = (|\lambda|^2 \cdot \eta) \cdot h.
\]

Thus we may assume without loss of generality that \( \eta = 1 \), i.e.

\[
h = \tau \circ h \circ (\iota \kappa \otimes \kappa \iota).
\] (25)

Again by Schur’s Lemma, the isomorphism \((\iota \kappa)^2 : X \to X\) equals

\[
(\iota \kappa)^2 = \lambda \cdot 1_X
\] (26)

for a constant \( \lambda \in \mathbb{C}^\times \), and the relation

\[
(\iota \kappa)^2 \circ \iota = \iota \circ (\kappa \iota)^2
\]

implies that likewise that

\[
(\kappa \iota)^2 = \lambda \cdot 1_{X^*}.
\]

We conclude with (25) that

\[
\lambda^2 \cdot h = h \circ ((\iota \kappa)^2 \otimes (\kappa \iota)^2)
= \tau \circ \tau \circ h \circ (\iota \kappa \otimes \kappa \iota) \circ (\iota \kappa \otimes \kappa \iota)
= \tau \circ h \circ (\iota \kappa \otimes \kappa \iota)
= h,
\]

hence \( \lambda \in \{ \pm 1 \} \).

Now consider the invariant bilinear form

\[
b := h \circ (1_X \otimes \kappa \iota \kappa)
\]

on \( X \). As \( h \) is non-degenerate, so is \( b \). A direct calculation, exploiting relations (25) and (26), yields

\[
b(w \otimes v) = \lambda \cdot h(\iota \kappa \iota \kappa(w) \otimes \kappa \iota \kappa(v))
= \lambda \cdot h(\kappa^{-1}(\iota \kappa \iota \kappa(w)) \otimes \kappa(\iota \kappa(v)))
= \lambda \cdot (\iota \kappa \circ h(\iota \kappa(v) \otimes \kappa \iota \kappa \iota \kappa(w)))
= \lambda \cdot h(v \otimes \kappa \iota \kappa(w))
= \lambda \cdot h(v \otimes w).
\]

Hence the explicit relation

\[
b \circ \delta = \lambda \cdot b,
\] (27)

where

\[
\delta : V \otimes V \to V \otimes V, \quad v \otimes w \mapsto w \otimes v.
\]

In particular \( b \) is symmetric if and only if \( \lambda = 1 \).

If \( X \) is of real type, and \( X_0 \) is a real model of \( X \), we may choose \( \iota \) as a real multiple of the map

\[
\iota' : v \otimes \tau(c) \mapsto v \otimes c,
\]

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where $v \in X_0$ and $c \in \mathbb{C}$. We conclude that for

$$0 \neq v = v_0 \otimes 1 \in X_0 \otimes_{\mathbb{R}} \mathbb{C},$$

we obtain

$$(\iota \kappa)^2(v) = v,$$

and thus $\lambda = 1$ in this case.

Assume now that $\lambda = 1$. We consider $\iota \kappa$ as an endomorphism of $X_\mathbb{R}$. As it is conjugate complex linear, it is not a multiple of the identity. Hence by (26) its minimal polynomial is given by

$$\xi^2 - 1 = (\xi - 1) \cdot (\xi + 1).$$

Therefore the endomorphism

$$1_{X_\mathbb{R}} - \iota \kappa \in \text{End}_{a,B}(X_\mathbb{R})$$

has a kernel satisfying

$$0 \neq \ker(1_{X_\mathbb{R}} - \iota \kappa) \subseteq X_\mathbb{R},$$

which implies that $X_\mathbb{R}$ is reducible, and with Proposition [5.1] we conclude that $X$ is real.

We saw that $\text{FS}(X) = \lambda$, which by (27) concludes the proof. \qed

**Corollary 5.4.** Let $(\mathfrak{g}, K)$ be a reductive pair over $\mathbb{R}$ satisfying condition (Q), and let $X$ be an irreducible infinitesimally unitary $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module with $\text{FS}(X) \neq 0$. Write $X_0$ for the sum of the minimal $K_\mathbb{C}$-types in $X$ and assume that $X_0$ is an irreducible $K_\mathbb{C}$-module. Then $X$ is defined over $\mathbb{R}$ if and only if $X_0$ is defined over $\mathbb{R}$ as a $K_\mathbb{C}$-module.

**Proof.** By our assumptions $X_0$ is an irreducible infinitesimally unitary $K_\mathbb{C}$-module, satisfying $X_0 \cong X$, hence $\text{FS}(X_0) \neq 0$. Therefore $X_0$ satisfies the hypotheses of Theorem [5.3] and is defined over $\mathbb{R}$ if and only if it carries a symmetric bilinear form. However this is the case if and only if $X$ carries a symmetric bilinear form, concluding the proof. \qed

### 6 Tits’ Theorem and real forms of $K$-types

In this section we recall known results of rationality of representations of the groups $O(n)$ and $U(n)$ respectively. In the former case the representation theory over fields of characteristic 0 is well known, cf. [32, Chapter V, Section B]. In this section we give a quick discussion of the relevant results over $\mathbb{R}$ using a Theorem of Tits. We freely use well known results about the root systems of types $A_n$, $B_n$, and $D_n$, and also about the classical groups attached to them, as discussed in detail for example in [4, Chapitres VI et IX].
6.1 Tits’ Theorem

We recall Tits’ Theorem on the Frobenius-Schur indicators of compact groups. In this subsection we assume \( g = k \) and \( K_C \) to be semi-simple and connected, and consider the root space decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha.
\]

We fix a positive system \( \Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h}) \), and a basis \( \Delta_0 = \{\alpha_1, \ldots, \alpha_r\} \) of simple roots. We write \( h_\alpha \in \mathfrak{h} \) for the coroot corresponding to \( \alpha \). We fix accordingly standard generators \( x_i \in \mathfrak{g}_{\alpha_i} \) and \( y_i \in \mathfrak{g}_{-\alpha_i} \), satisfying

\[
[x_i, y_i] = h_{\alpha_i},
\]

and set

\[
\rho^\vee := \frac{1}{2} \sum_{\alpha \in \Delta^+} h_\alpha
\]

the half sum of positive coroots. It is well known that there are uniquely determined complex numbers \( a_1, \ldots, a_r \in \mathbb{C} \) with the property that the three elements

\[
E := \sum_{i=1}^r x_i,
\]

\[
H := 2 \rho^\vee,
\]

\[
F := \sum_{i=1}^r a_i y_i,
\]

are standard generators of a so-called principal three-dimensional subalgebra \( p \subseteq \mathfrak{g} \), i.e. \( E, H, F \) are standard generators of a copy of \( \mathfrak{sl}_2 \subseteq \mathfrak{g} \) with distinguished properties, unique up to conjugation. We write

\[
w_0 \in W(\mathfrak{g}, \mathfrak{h})
\]

for the long Weyl element sending \( \Delta^+ \) to \( -\Delta^+ \).

With this notation we have the classical

**Theorem 6.1** (Tits [28, Proposition on p. 212]). *Assume that \( K(\mathbb{R}) \) is compact and connected, and let \( X \) be an irreducible representation of \( K_C \) of highest weight \( \lambda \). Then \( \text{FS}(X) \neq 0 \) if and only if \( \lambda + w_0 \lambda = 0 \), and in this case

\[
\text{FS}(X) = (-1)^{\lambda(2\rho^\vee)}.
\]

*Proof.* The Theorem is usually stated for semisimple groups. Let us indicate how the semisimple case implies the general reductive case.

We write \( K' \subseteq K \) for the derived group. Then \( K'(\mathbb{R}) \) is a semisimple compact group. Fix a maximal torus \( T' \subseteq K' \) and a maximal torus \( T \subseteq K \) containing \( T' \). Then we have a natural map

\[
p : X^*(T) \to X^*(T'),
\]

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and the quantity $\lambda(2\rho^\vee)$ depends only on the image of $\lambda$ under $p$.

Now the selfduality condition for $\lambda$ with respect to $K$ is stronger than the selfduality condition for $p(\lambda)$ with respect to $K'$. In particular the former implies, that the center of $K$ acts trivially on $X$, and thus the Frobenius-Schur indicator of $X$ as a representation of $K$ agrees with the indicator of $X$ as a representation of $K'$.

For a proof in the semisimple case we refer the reader to [28], [4, Chapitre IX], and also to [1].

6.2 The case of orthogonal groups

For $n \geq 1$, we choose a maximal torus $T \subseteq \text{SO}(2n+1)$. Then the root system $\Delta \subseteq X^*(T)$ of $\text{SO}(2n+1)$ is of type $B_n$, and we identify

$$X^*(T) \otimes \mathbb{Z} \mathbb{R} = \mathbb{R}^n,$$

by means of a standard orthonormal basis $e_1, \ldots, e_n$. We assume that in our notation the root system is given by

$$\Delta = \{ \pm e_i \pm e_j \mid i < j \} \cup \{ \pm e_i \},$$

Our choice of simple roots in this notation being

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n.$$

The orthogonal group $\text{O}(2n+1)$ has no non-trivial outer automorphisms and is a direct product

$$\text{O}(2n+1) = \text{SO}(2n+1) \times \{ \pm 1 \},$$

and thus irreducible $\text{O}(2n+1)$-modules $W^\pm(\lambda)$ are indexed by a sign $\pm$ and analytically integral dominant weights

$$\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_1, \ldots, \lambda_n \in \mathbb{Z},$$

for $\text{SO}(2n+1)$, satisfying the dominance condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Consequently the Frobenius-Schur indicator of such a module coincides with the Frobenius-Schur indicator of the irreducible $\text{SO}(2n+1)$-module $V(\lambda)$ with highest weight $\lambda$.

In the even case we consider for $n \geq 2$ the root system

$$\Delta \subseteq X^*(T) \subseteq X^*(T) \otimes \mathbb{Z} \mathbb{R} = \mathbb{R}^n,$$

of $\text{SO}(2n)$ for a maximal torus $T \subseteq \text{SO}(2n)$, with the same standard basis as above. It is of type $D_n$, and we have

$$\Delta = \{ \pm e_i \pm e_j \mid i < j \}.$$
We fix the simple roots as
\[ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n. \]
The outer automorphism group of \( O(2n) \) is of order two and the orthogonal group \( O(2n) \) is the unique non-split semidirect product
\[ O(2n) = SO(2n) \rtimes \{\pm 1\}. \]
By classical Mackey Theory there are two distinct cases. Given an analytically integral dominant weight
\[ \lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_1, \ldots, \lambda_n \in \mathbb{Z}, \]
of \( SO(2n) \), i.e. satisfying
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|, \]
the induced representation
\[ W(\lambda) := \text{Ind}_{SO(2n)}^{O(2n)} V(\lambda) \]
is irreducible whenever \( \lambda_n \neq 0 \), and its isomorphism class depends only on the tuple \( (\lambda_1, \lambda_2, \ldots, |\lambda_n|) \). In the case \( \lambda_n = 0 \), the module \( \text{Ind}_{SO(2n)}^{O(2n)} V(\lambda) \) decomposes into two non-isomorphic irreducible \( O(2n) \)-representations \( W^{\pm}(\lambda) \), which differ by a sign twist.

We have the following classical results.

**Proposition 6.2.** For \( n \geq 1 \), for every dominant weight \( \lambda \) for \( SO(2n + 1) \) and every sign \( \pm \), the irreducible \( O(2n + 1) \)-module \( W^{\pm}(\lambda) \) is real.

**Proof.** This result is classical and well known. We sketch a proof for the convenience of the reader.

We first remark that as the root system \( B_n \) has no non-trivial automorphism, \( -w_0 \) acts as the identity on the weights, hence the condition
\[ \lambda + w_0 \lambda = 0, \]
is automatic in this case.

Denote by \( \langle \cdot, \cdot \rangle \) the standard scalar product on \( \mathbb{R}^n \), i.e. \( (e_i, e_j) = \delta_{ij} \) (Kronecker delta). We use it to identify \( \mathbb{R}^n \) with its dual. Then for each \( \alpha \in \Delta \), its dual root \( \alpha^\vee \) is given by
\[ \alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}. \quad (28) \]
In particular we have for the positive roots
\[ (e_i \pm e_j)^\vee = e_i \pm e_j, \]
and
\[ e_i^\vee = 2e_i. \]
Then we conclude that, in the notation of Section 6,
\[ 2\rho^\vee = 2n \cdot e_1 + 2(n-1)e_2 + \cdots + 2e_n, \]
and in particular,
\[ \lambda(2\rho^\vee) = (\lambda, 2\rho^\vee) \in 2\mathbb{Z}. \]
By Tits’ Theorem this shows that the Frobenius-Schur indicator of the irreducible \( \text{SO}(2n+1) \)-module \( V(\lambda) \) is always 1. As \( \text{O}(2n+1) \) is the direct product of \( \text{SO}(2n+1) \) and \( \{\pm 1\} \), this concludes the proof.

**Proposition 6.3.** All complex irreducible representations of \( \text{O}(2) \) are real.

**Proof.** We sketch a proof for the convenience of the reader. We set
\[ \tau := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
and for \( \varphi \in \mathbb{R} \)
\[ D(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}. \]
Then the map
\[ \mathbb{R} \to \text{O}(2), \ \varphi \mapsto D(\varphi) \]
induces an isomorphism of groups
\[ \mathbb{R}/2\pi\mathbb{Z} \cong \text{SO}(2), \]
and furthermore
\[ \text{O}(2) = \text{SO}(2) \cup \text{SO}(2)\tau, \]
and
\[ \tau D(\varphi) \tau^{-1} = D(\varphi)^{-1} = D(-\varphi). \] \tag{29}
For each integer \( n \in \mathbb{Z} \) we consider
\[ V_n := \mathbb{R}^2 \]
with the action
\[ (D(\varphi)\tau^\varepsilon) \ast_n v := D(\varphi)^n \cdot \tau^\varepsilon \cdot v, \]
where \( \varepsilon \in \{0,1\} \) and on the right hand side ‘\( \ast_n \)’ denotes matrix multiplication. By \eqref{29} we see that this is a representation of \( \text{O}(2) \). Obviously \( V_n \) must be irreducible whenever \( n \neq 0 \) and decomposes into \( 1 \oplus \text{sgn} \) for \( n = 0 \). By the discussion in Section 6.2 we conclude that the \( V_n, n \neq 0 \), resp. \( 1 \) and \( \text{sgn} \) provide real forms of all irreducible complex representations of \( \text{O}(2) \).

We remark that the models of the representations given in proof 6.3 are eventually defined over \( \mathbb{Q} \) if we choose the classical \( \mathbb{Q} \)-structure on \( \text{O}(2) \).
Proposition 6.4. For \( n \geq 2 \), and every regular dominant weight \( \lambda \) for \( \text{SO}(2n) \) satisfying
\[
\lambda + w_0 \lambda = 0,
\] (30)
the irreducible \( \text{SO}(2n) \)-module \( V(\lambda) \) is real. Condition (30) is automatic for \( n \) even. Moreover for all \( n \geq 2 \) and all regular dominant \( \lambda \) (regardless of (30)), the irreducible \( \text{O}(2n) \)-module \( W(\lambda) \) is real.

Proof. The proof proceeds along the same lines as the proof of Proposition 6.2 with the following modifications.

That condition (30) is automatic for even \( n \) follows again from the observation that for our choice of \( \Delta^+ \) the longest Weyl element \( w_0 \) sends \( e_i \) to \(-e_i\) in this case. As the Weyl group action may only negate an even number of signs, this argument does not apply for the case of an odd \( n \).

In the analogous notation we obtain
\[
2\rho^\vee = 2(n-1) \cdot e_1 + 2(n-2) e_2 + \cdots + 2 e_{n-1},
\]
which allows for the same conclusion in the case \( \text{SO}(2n) \). This also settles the case \( \text{O}(2n) \) for even \( n \).

For \( n \) odd we may argue that the outer automorphism group of \( \text{SO}(2n) \) is realized by the component group \( \pi_0(\text{O}(2n)) \) via the conjugation action of \( \text{O}(2n) \) on \( \text{SO}(2n) \). On the Dynkin diagram the non-trivial outer automorphism \( \tau \) induces necessarily the automorphism \(-w_0 \) (for \( n = 4 \), this is the only nontrivial automorphism which descends to \( \text{SO}(2n) \)). Hence \( W(\lambda) \), as an \( \text{SO}(2n) \)-module, is the direct sum of \( V(\lambda) \) and its dual \( V(-w_0 \lambda) \), thus is a real \( \text{SO}(2n) \)-module. Now \( \tau \) stabilizes this real structure, hence the claim follows.

\[ \square \]

6.3 The case of unitary groups
For \( n \geq 2 \) the unitary group \( \text{U}(n) \) is of type \( A_{n-1} \). Again we denote the roots by
\[
\Delta = \{ e_i \pm e_j | i < j \},
\]
and consider the simple roots
\[
e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n.
\]
The analytically integral dominant weights are given by
\[
\lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_1, \ldots, \lambda_n \in \mathbb{Z}
\]
satisfying the dominance condition
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.
\]
We have for the contragredient dominant weight the relation
\[
-w_0 \lambda = -\lambda_n e_1 - \lambda_{n-1} e_2 - \cdots - \lambda_1 e_n.
\]
Proposition 6.5. For $n \geq 3$, for every dominant weight $\lambda$ for $U(n)$, the irreducible $U(n)$-module $V(\lambda)$ of highest weight $\lambda$ has non-zero Frobenius-Schur indicator if and only if
\[ \lambda + w_0 \lambda = 0. \]
In this case the module $V(\lambda)$ is always real.

Proof. Again we proceed as in the proof of Proposition 6.2. With the notation as before we conclude that
\[ 2 \rho^\vee = (n-1)e_1 + (n-3)e_2 + \cdots - (n-1)e_n, \]
again with respect to the same scalar product $(\cdot, \cdot)$ on $R^n$. Now the self-contragredient is reflected in the identity
\[ \lambda_i = -\lambda_{n+1-i}, \]
whence
\[ \lambda (2 \rho^\vee) = \begin{cases} 2(n-1)\lambda_1 + 2(n-3)\lambda_2 + \cdots + 4\lambda_2, & n \text{ odd}, \\ 2(n-1)\lambda_1 + 2(n-3)\lambda_2 + \cdots + 2\lambda_2, & n \text{ even}. \end{cases} \]
We conclude that
\[ \lambda (2 \rho^\vee) \in 2 \mathbb{Z}. \]
Therefore the claim follows for $U(n)$ again by Tits’ Theorem.

We remark that for $SU(n)$ and $n \equiv 2 \pmod{4}$, it is well known that selfduality of an irreducible does not imply that it is real. The ‘middle’ fundamental representation of $SU(n)$ is a counter example in this case.

7 Applications to cohomologically induced modules

In this section we determine the fields of definition of non-degenerate cohomological representations of $GL_n(k \otimes_{Q} R)$ for number fields $k$.

7.1 Cohomological induction

We depart from a reductive pair $(g, K)$ over $Q$ with Cartan involution $\theta$, also defined over $Q$. We assume that $K$ is split over an imaginary quadratic extension $F'/Q$ and write $(g', K')$ for its base change to $F'$.

Let $q'$ be a $\theta$-stable germane parabolic inside of $g'$ with Levi decomposition $q' = l' + u'$. Then $l'$ is defined over $Q$ and the complex conjugate of $u'$ is nothing but the opposite of $u'$, i.e.
\[ \overline{u'} = u'^{-}. \]
Consider the Cartan decomposition
\[ g = p \oplus k, \]
with base change
\[ g' = p' \oplus k', \]
to \( F' \). Then \( q' \cap k' \) is a parabolic subalgebra of \( k' \).

We define the middle degree
\[ S_{q'} := \dim_{F'} u' \cap p'. \]

Let \( Z \) be an \((l', L' \cap K')\)-module. As before we consider
\[ Z_{q'} := Z \otimes_{F'} \bigwedge^{\dim_{F'} u'} u' \]
as an \((q', L' \cap K')\)-module with trivial action of the radical. Then we have the cohomologically induced module
\[ \mathcal{R}^q(Z) = R^q \Gamma(U(g') \otimes_{U(q')} Z_{q'}), \]
where \( \Gamma(\cdot) \) denotes the rational Zuckerman functor from section 2.3 for the inclusion
\[ (g', L' \cap K') \to (g', K'). \]

Then the functor \( \mathcal{R}^q(\cdot) \) preserves infinitesimal characters in the sense that if \( L' \cap K' \) meets every (geometrically) connected component of \( K' \), and if \( Z \), as an \((l', L' \cap K')\)-module, has infinitesimal character \( \lambda \), then \( \mathcal{R}^q(Z) \) has infinitesimal character \( \lambda + \rho(u') \). This easily follows via base change from the classical case (cf. [18, Corollary 5.25] for example). Mutatis mutandis classical non-vanishing and irreducibility criteria generalize to our setting.

### 7.2 The bottom layer

Consider the commutative square of pairs
\[
\begin{array}{ccc}
(g', L' \cap K') & \longrightarrow & (g', K') \\
\uparrow & & \uparrow \\
(t', L' \cap K') & \longrightarrow & (t', K')
\end{array}
\]

By Proposition 2.4 the corresponding Zuckerman functors commute with the tautological forgetful functors and we obtain a natural map
\[
\beta : \mathcal{R}^q_K(Z \otimes_{F'} \bigwedge^\alpha u' \cap p') := R^q \Gamma_K(U(t') \otimes_{U(q')} (Z_{q'} \cap t' \otimes_{F'} \bigwedge^\alpha u' \cap p')) \to R^q \Gamma(U(g') \otimes_{U(q')} Z_{q'} = \mathcal{R}^q(Z),
\]
where the first Zuckerman functor \( \Gamma_K \) is associated to the inclusion of the compact pair, and the second one to the (possibly) non-compact one. Traditionally this map is called the ‘bottom layer map’, even though conceptually it is in modern terms defined as a projection rather than an inclusion.
7.3 Cohomological representations of GL(n)

Let $F$ be a number field, and $G := \text{Res}_{F/Q} GL(n)$. We have

$$G(\mathbb{R}) = \prod_{v \, \text{real}} GL_n(\mathbb{R}) \times \prod_{v \, \text{complex}} GL_n(\mathbb{C}),$$

where in the first product $v$ runs over the real places of $F$ and in the second $v$ runs through the complex places, i.e. pairs of complex conjugate embeddings $v, \overline{v} : F \to \mathbb{C}$.

The cohomological representations in the unitary duals of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ have been identified by Speh [27] and Enright [7] respectively. In particular the cohomological representations of $G(\mathbb{R})$ are well known.

By Vogan-Zuckerman [30] we know that those representations are cohomologically induced. More concretely, by our preceding discussion, we know the following.

Write $\mathfrak{g}$ for $\mathbb{Q}$-Lie algebra of $G$, and fix a Cartan involution $\theta$ defined over $\mathbb{Q}$, and a compatible $\mathbb{Q}$-form $K$ of the maximal compact subgroup $K_\theta \subseteq G(\mathbb{R})$ corresponding to $\theta$. In this section we prove

**Theorem 7.1.** Keep the notation as before. Let $M$ be an absolutely irreducible finite-dimensional $G$-module, which is defined over a number field $E$, then the non-degenerate infinitesimally unitary irreducible $(\mathfrak{g}, K)$-module $V_C$ satisfying

$$H^\bullet(\mathfrak{g}; GK^0; V_C \otimes M) \neq 0,$$

has a unique model $V_E$ over $E$, where $GK$ denotes the product of $K_\theta$ with the center of $G(\mathbb{R})$.

**Proof.** Using the notation of the previous section for the base change of the above data to $F'$, where we may assume $K$ to be split. We may even arrange $F' = \mathbb{Q}(i)$, $i = \sqrt{-1}$ in this case, but we won’t need this. We choose a $\theta$-stable Borel $q' \subseteq \mathfrak{g}'$ over $F'$ with Levi decomposition

$$q' = l' \oplus u'.$$

Then

$$(L' \cap K')(\mathbb{R}) = \prod_{v \, \text{real}} T_{R} \times \prod_{v \, \text{complex}} T_{C},$$

where $T_{C} \subseteq U(n)$ and $T_{R} \subseteq SO(n)$ are maximal tori, and

$$\text{det}(T_{R}) = \begin{cases} 1, & n \text{ even}, \\ \{\pm 1\}, & n \text{ odd}. \end{cases}$$

If we choose $Z$ to be the trivial $(l', L' \cap K')$-module, we know that the $(\mathfrak{g}', K')$-module $A' := \mathcal{R}^{\mathfrak{g}'}(Z)$ is an $F'$-model for the ‘standard module’ $A_{q'}(\mathbb{C})$, hence absolutely irreducible with non-trivial $(\mathfrak{g}, K)$-cohomology for trivial coefficients. Furthermore $A_{q'}(\mathbb{C})$ is infinitesimally unitary and selfdual.
By construction the bottom layer
\[ A_0' := \beta(\mathcal{R}^S u' (Z \otimes F') \bigcap u' \cap p') \subseteq \mathcal{R}^S u' (Z) = A' \]
is an irreducible $K'$-module, which is generated by an irreducible $(K')^0$-module of highest weight $2\rho(u' \cap p')$. In particular $A_0'$ is a self-dual $K'$-module. Its complexification is by Propositions 6.2, 6.3, 6.4, 6.5 a real representation. Therefore $A_0' \otimes F$ is defined over $\mathbb{R}$. We conclude with Corollary 5.4 that $A_{q'}(\mathbb{C})$ is defined over $\mathbb{R}$.

By the elementary base change theorem for Galois cohomology for the two extensions $\mathbb{C}/\mathbb{R}$ and $F'/\mathbb{Q}$, for each $n \in \mathbb{N} \cup \{\infty\}$, the short exact sequence
\[ 1 \longrightarrow \text{GL}_n \longrightarrow \text{PGL}_n \longrightarrow \text{PGL} \longrightarrow 1 \]
of group schemes gives rise to a commutative diagram
\[
\begin{array}{ccc}
H^1(\text{Gal}(\mathbb{C}/\mathbb{R}); \text{PGL}_n(\mathbb{C})) & \longrightarrow & H^2(\text{Gal}(\mathbb{C}/\mathbb{R}); \text{GL}_1(\mathbb{C})) \\
\delta_{\mathbb{R}} \downarrow & & \downarrow \\
H^1(\text{Gal}(F'/\mathbb{Q}); \text{PGL}_n(F')) & \longrightarrow & H^2(\text{Gal}(F'/\mathbb{Q}); \text{GL}_1(F'))
\end{array}
\]
with exact rows and in which all vertical maps are monomorphisms. Here we understand $\text{GL}_\infty$ as the automorphism group of an infinite-dimensional vector space with countable dimension, and
\[ \text{PGL}_\infty := \text{GL}_\infty / \text{GL}_1. \]

Now each $\sigma \in \text{Gal}(F'/\mathbb{Q})$ gives rise to an isomorphism
\[ \iota_\sigma : A' \longrightarrow A' \otimes_{F', \sigma} F', \]
unique up to automorphisms of $A'$, i.e. up to a scalar in $\text{GL}_1(F')$ by Schur’s Lemma, and we may associate to the collection $\iota_\sigma$, $\sigma \in \text{Gal}(F'/F)$ a unique class
\[ c_{A'} \in H^1(\text{Gal}(F'/\mathbb{Q}); \text{PGL}_n(F')). \]
By Galois descent for vector spaces $A'$ is defined over $\mathbb{Q}$ if and only if the image of this class under $\delta_{\mathbb{Q}}$ is trivial.

As $A_{q'}(\mathbb{C}) = A' \otimes_{F'} \mathbb{C}$ is defined over $\mathbb{R}$, the image of $c_{A' \otimes_{F', \sigma} \mathbb{C}}$ in $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}); \text{PGL}_n(\mathbb{C}))$ under $\delta_{\mathbb{R}}$ is trivial, and this coincides with the image of $\delta_{\mathbb{Q}}(c_{A'})$ in $H^2(\text{Gal}(\mathbb{C}/\mathbb{R}); \text{GL}_1(\mathbb{C}))$. Hence $A'$ is defined over $\mathbb{Q}$.

Now if $M$ is an absolutely irreducible $G$-module defined over a number field $F'$, we consider the translation functor
\[ T_M : X \mapsto \pi_{\eta_M}(X \otimes_{F'} M), \]
where $X$ is a $Z(\mathfrak{g} \otimes \mathbb{Q} F')$-finite $($$\mathfrak{g} \otimes \mathbb{Q} F'$, $K \otimes \mathbb{Q} F'$)$-module, and $\eta_M$ denotes the infinitesimal character of $M$ and $\pi_{\eta_M}$ is the projection to the $\eta_M$-isotypical component. We know that $\eta_M$ is defined over $F'$, so this functor is indeed well defined and obviously commutes with base change. Therefore $T_M(A \otimes_{\mathbb{Q}} F')$ is an $F'$-model of the standard module $A_{q'}(\lambda)$ with $(\mathfrak{g}, K)$-cohomology with non-trivial coefficients in $M'$. \[ \square \]
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