On the integral closure of ideals

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Abstract. Among the several types of closures of an ideal $I$ that have been defined and studied in the past decades, the integral closure $\overline{I}$ has a central place being one of the earliest and most relevant. Despite this role, it is often a difficult challenge to describe it concretely once the generators of $I$ are known. Our aim in this note is to show that in a broad class of ideals their radicals play a fundamental role in testing for integral closedness, and in case $I \neq \overline{I}$, $\sqrt{I}$ is still helpful in finding some fresh new elements in $I \setminus \overline{I}$. Among the classes of ideals under consideration are: complete intersection ideals of codimension two, generic complete intersection ideals, and generically Gorenstein ideals.

1. Introduction

Let $R$ denote a Noetherian ring and $I$ one of its ideals. The integral closure of $I$ is the ideal $\overline{I}$ of all elements of $R$ that satisfy an equation of the form

$$X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n = 0, \quad a_i \in I.$$ 

The literature is very sparse on methods to find this ideal from $I$, except when $R$ is a ring of polynomials over a field and $I$ is an ideal generated by monomials; $\overline{I}$ is then the monomial ideal defined by the integral convex hull of the exponent vectors of $I$ (see [7, p. 140]). Already here the problem has computationally a non-elementary solution. Thus the general problem can be likened to solving a nonlinear integer programming question—quite a tall order.

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By comparison, the task of computing the radical, \( \sqrt{I} \), of the ideal \( I \) is simpler as one is only required to describe the solutions in \( R \) of equations of the form

\[ X^m - b = 0, \quad b \in I. \]

We always have

\[ I \subseteq \mathcal{T} \subseteq \sqrt{I}, \]

but in general these ideals differ from one another. A twin problem, but much simpler to approach, consists in deciding the equality \( I = \mathcal{T} \). Such ideals are called integrally closed or complete.

There are two basic ways to approach these questions. A theoretical way of testing whether or not an element \( z \) belongs to the integral closure of an ideal is based on the so called determinant trick: an element \( z \in R \) is integral over \( I \) if and only if there exists a finitely generated faithful \( R \)-module \( M \) such that \( zM \subseteq IM \). In fact, for any such \( M \) one has that

\[ IM : M \subseteq I. \]

In particular, if the ideal \( I \) is integrally closed then

\[ IM : M = I, \quad (1.1) \]

for any finitely generated faithful \( R \)-module \( M \). The issue is to find appropriate modules for a given ideal \( I \). Some of the natural choices we are going to employ are the powers of the ideal and of its radical, Jacobian ideals, and modules of syzygies.

Another crude theoretical approach is through the theory of the Rees algebra of \( I \)

\[ \mathcal{R} = R[It] = R + It + It^2 + \cdots \subset R[t], \]

and one looks for its integral closure in \( R[t] \),

\[ R + \mathcal{T}t + \mathcal{T}^2t^2 + \cdots \subset R[t]. \]

This is obviously wasteful of resources since the integral closure of all the powers of \( I \) will be computed. Nevertheless, there are some instances when this path can be profitably taken. First, in some special cases, it turns out that \( \mathcal{T}^n = (\mathcal{T})^n \) for all \( n \geq 1 \). Then, notions typical of the theory of Rees algebras can be brought in. For example, we say that \( J \subseteq I \) is a reduction of \( I \) if \( I^{r+1} = JI^r \) for some non-negative integer \( r \). A reduction is said to be minimal if it is minimal with respect to containment. It is known that an ideal has the same integral closure as any of its reductions (see \([18]\)). Appropriately we can then view all these algebras as modules over the Rees algebra \( R[It] \).

Our goals can be framed in terms of the following two issues:
• development of effective integral closedness criteria, i.e., tests for the equality $I = T$;
• development of the means to find the integral closure when the ideal $I$ fails to pass the above tests.

Our results do not meet this ambitious program, except for some classes of ideals. In Sections 2 and 3, we give a very direct validation of the equality $I = T$ in two cases of interest: (i) $I$ is a generic complete intersection, or (ii) $I$ is a generically Gorenstein ideal. In these cases we provide a ‘closed formula’ (see Theorem 2.1 and Theorem 3.9) which says that $I$ is integrally closed precisely when

$$\sqrt{I} = IL : L^2,$$

where $L = I : \sqrt{I}$. We also show (see Theorem 3.1) that the equality

$$I^2 : I = I$$

gives a practical way of testing whether or not Gorenstein ideals of codimension three are generically complete intersections.

We recall that an ideal $I$ satisfies a property generically if $I_p$ satisfies that property for every minimal prime $p$ of $I$. We also say that an ideal $I$ in a local ring $R$ is a complete intersection of Goto-type if $R$ is regular of dimension $d$ and $I = (x_1, \ldots, x_{d-1}, x_d')$, where $x_1, \ldots, x_d$ is a regular system of parameters (see [10, Theorem 1.1]). The import of these special complete intersections is that they are exactly the complete intersections which are integrally closed.

In Section 4, we then go on studying the following question: When $I$ integrally closed implies $I$ normal. Two cases are considered: the integral closure of complete intersections of codimension 2 in Cohen–Macaulay local rings of arbitrary dimensions such that $R_p$ has rational singularities in codimension two (see Theorem 4.1) and perfect Gorenstein ideals of codimension three in polynomials rings over fields of characteristic zero and with certain restrictions on the (local) number of generators (see Theorem 4.9). This makes the computational approach through Rees algebras more accessible even though not inexpensive (see Remark 4.2, Example 4.3, and Remark 4.4).

2. Complete intersections

As in the case of computation of radicals (see [8]), complete intersection ideals are the simplest case of study. A careful examination of [10], [3], and [4] shows that they deal with complementary aspects of integrality for
such ideals: While the first paper is concerned with integrally closed complete intersections, the remaining two require that the ideal be not integrally closed in their hypotheses.

We are going to frame these two local aspects together into a global integral closedness criterion that, when it fails, will still provide fresh elements which are integral over the ideal.

**Theorem 2.1.** Let $I$ be a height unmixed ideal in a Cohen–Macaulay ring $R$. Suppose that $I$ is generically a complete intersection. Then $I$ is integrally closed if and only if

$$\sqrt{I} = IL^2,$$

(2.1)

where $L = I : \sqrt{I}$.

**Proof.** Let us assume first that $I$ is an integrally closed ideal. In order to show that (2.1) holds it will suffice to prove it for any $p \in \operatorname{Ass}(R/\sqrt{I}) = \operatorname{Ass}(R/I)$, given that $\sqrt{I} \subseteq IL^2$, by the definition of $L$. If $p \in \operatorname{Ass}(R/I)$, the strict inclusion $(\sqrt{I})_p = pR_p \subsetneq I_pL_p$: $L^2_p = I_pL_p$. But $I_p$ is integrally closed, which forces the equality $L_p = I_p$: a contradiction.

Conversely, suppose that (2.1) holds. Since $I$ is unmixed, to show that $I = I : \sqrt{I}$ it suffices to check it at each minimal prime $p$ of $I$. First, if for such prime $R_p$ is not a regular local ring, the ideal $L_p$ satisfies $I_pL_p = L^2_p$ by [4], which violates (2.1). Let $p \in \operatorname{Ass}(R/I)$, so that $I_p = (f_1, f_2, \ldots, f_g)$, $g = \height(I)$. If two of the $f_i$’s belong to $p^2R_p$, by [6, Theorem 2.1], one has that the ideal $L_p$ has reduction number 1 with respect to $I_p$, i.e., $L^2_p = I_pL_p$, which again would contradict (2.1). Thus at most one of the $f_i$’s lies in the $p^2R_p$. This means that $I_p$ satisfies the conditions of [10] and is therefore integrally closed.

**Remark 2.2.** To check the hypotheses on the ideal $I$ one can proceed as follows (see [7,24]). Let $I$ be an ideal of codimension $m$.

(a) Suppose that $R$ is (locally) a Gorenstein ring and let $J = (f_1, \ldots, f_m)$ be a subideal of $I$ of codimension $m$. Then $I$ is height unmixed if and only if

$$I = J : (J : I).$$

(b) Let $R$ be a Cohen–Macaulay ring and let

$$R^p \xrightarrow{\varphi} R^q \rightarrow I \rightarrow 0$$

be a presentation of $I$. Then $I$ is generically a complete intersection if and only if the (Fitting ideal) ideal $I_{q-m}(\varphi)$ generated by the minors of $\varphi$ of order $q - m$ has height at least $m + 1$. 

If $I$ is not integrally closed, elements that go into the test of Theorem 2.1 can be used to produce new elements in its integral closure. This occurs as follows.

**Corollary 2.3.** Let $p_1, \ldots, p_n$ be the minimal prime ideals of $I$ listed in such a way that $I_p$ is integrally closed for $p = p_i$ for $i \leq s$, but not at the other primes. Set

$$A = \sqrt{I} = p_1 \cap \cdots \cap p_n, \quad B = p_1 \cap \cdots \cap p_s, \quad C = p_{s+1} \cap \cdots \cap p_n.$$ 

Let $L = I : \sqrt{I}$. Then

$$B = IL : L^2 \quad \text{and} \quad C = A : B.$$ 

If $I$ is not integrally closed, that is if $B \neq A$, then

$$H = I : C \neq I \quad \text{and} \quad H^2 = IH.$$ 

Note that one can arrange the computation in a manner that does not require $\sqrt{I}$ directly. The following formulation mimics some of the radical formulas of [8].

**Theorem 2.4.** Let $R$ be a ring of polynomials over a field of characteristic zero and let $I$ be a height unmixed ideal. Let $J$ denote the Jacobian ideal of $I$. If $I$ is integrally closed then

$$IJ : J = I.$$ 

The converse holds if $I$ is generically a complete intersection.

**Proof.** We only consider the converse. To show that $I$ is integrally closed it suffices to show that its primary components are integrally closed. Localizing at minimal primes of $I$ reduces the question to Theorem 2.1. But for any such prime $p$, $I_p$ is the generic socle of $I_p$. \hfill \Box

**Example 2.5.** The converse of Theorem 2.4 does not hold without the assumption of $I$ being generically a complete intersection. For example, let $R$ be a regular local ring of dimension 2 and maximal ideal $m = (x, y)$. Let $I$ be the Northcott ideal $(x^2, xy, y^5)$. Its Jacobian ideal is $J = (xy, x^2y^3, y^8)$. It can be checked that the condition $IJ : J = I$ is satisfied but $I$ is not integrally closed. Indeed, the element $xy^3 \notin I$ belongs to the integral closure of $I$ as it satisfies the monic equation $X^2 - y(x^2y^3) = 0$. It can actually be checked that $I = (x^2, xy^3, y^5)$. 


Remark 2.6. When applying these methods where $L$ is either $I$: $\sqrt{I}$, or the ordinary Jacobian ideal, the following comment may be helpful. If the test fails, that is if

$$IL: L = I' \neq I,$$

we could replace $I$ by $I'$ if the latter is height unmixed. Indeed it has the same radical as $I$ so that its generic socle would be

$$L' = I': \sqrt{I},$$

and we would test for

$$I'L': L' = I',$$

and so on. There may be difficulties to this process. For instance, it was an old question of Krull whether the integral closure of primary ideals are still primary. One of the authors (see [12]) gave a counterexample in characteristic two but the characteristic zero case is still open.

3. Gorenstein ideals

The first result in this section describes, for a special class of ideals, another practical way of telling when an ideal is generically a complete intersection. More precisely, Theorem 3.1 shows that, in a Gorenstein ring $R$, a perfect Gorenstein ideal $I$ of codimension three is generically a complete intersection if and only if $I^2: I = I$. An immediate consequence of Theorem 2.1 and Theorem 3.1 is that for any such $I$ the following conditions are equivalent:

(a) $I$ is an integrally closed ideal;
(b) $I^2: I = I$ and $\sqrt{I} = IL: L^2$, where $L = I: \sqrt{I}$.

The rest of the section will be devoted to prove a generalization of this fact. To be more precise, it will be shown in Theorem 3.3 that for a generically Gorenstein ideal $I$ the condition $\sqrt{I} = IL: L^2$, where $L = I: \sqrt{I}$, is necessary and sufficient to guarantee that $I$ be integrally closed, without restrictions on its height. Some regularity assumptions on the ring $R$ are required as pointed out in Example 3.10.

**Theorem 3.1.** Let $R$ be a Gorenstein ring and let $I$ be a perfect Gorenstein ideal of codimension three. Then $I$ is generically a complete intersection if and only if

$$I^2: I = I.$$

**Proof:** According to [11], for these ideals the conormal module $I/I^2$ is Cohen–Macaulay and its associated primes are the minimal primes of $I$. This means that we may localize at those primes and reduce the question to
On the integral closure of ideals

the case of a local ring \( R \) of dimension 3, from which it follows that if \( I \) is a complete intersection, then the asserted equality holds.

Conversely, suppose that \( I^2 : I = I \). This means that \( I/I^2 \) is a faithful module of the Artinian, Gorenstein local ring \( R/I \) and therefore there is an embedding \( R/I \rightarrow I/I^2 \), which leads to a decomposition

\[
I/I^2 \simeq R/I \oplus M,
\]

since \( R/I \) is self-injective. We may assume that the image of \( R/I \) in \( I/I^2 \) has a lift \( f \) which is a regular element of \( I \). As in [23, Proof of Lemma 2], this leads to a decomposition

\[
I/fI \simeq (f)/fI \oplus I/(f) \simeq R/I \oplus I/(f).
\]

This equality implies that as an \( R/(f) \)-module, \( I/(f) \) is a perfect ideal (of codimension two), whose first Betti number is the same as the second Betti number of \( I \). Thus \( I/(f) \) is a perfect, Gorenstein ideal of codimension two and therefore it is a complete intersection, which means that \( I \) is also a complete intersection.

Corollary 3.2 generalizes a similar result contained in [19, Corollary 2.7], as the ideals under considerations are only supposed to be integrally closed and not normal.

**Corollary 3.2.** Let \( R \) be a Gorenstein ring and let \( I \) be a perfect Gorenstein ideal of codimension three. If \( I \) is integrally closed then it is generically a complete intersection.

**Proof.** As \( I \) is integrally closed one has, in particular, that \( I^2 : I = I \). The assertion then follows from Theorem 3.1.

**Remark 3.3.** A similar argument as the one in the proof of Theorem 3.1 will show that if \( (R, m) \) is a local ring of dimension \( d \geq 3 \) and \( I \) is an \( m \)-primary ideal that is perfect and Gorenstein then \( I \) is a complete intersection if and only if

\[
d-2 \bigwedge I/I^2
\]

is a faithful \( R/I \)-module.

**Remark 3.4.** From a computational point of view, the two methods (the one involving the Fitting ideals of \( I \) and the one described in Theorem 3.1) are essentially equivalent.

The next two lemmas are crucial in the proof of Theorem 3.9. The first of them is about the existence of a dual basis.
Lemma 3.5. Let \((R, \mathfrak{m})\) be a Noetherian local ring with embedding dimension \(n\) at least two. Let \(I\) be an \(\mathfrak{m}\)-primary irreducible ideal contained in \(\mathfrak{m}^2\). If \(\mathfrak{m} = (x_1, \ldots, x_n)\) then there exist \(y_1, \ldots, y_n\) such that for all \(1 \leq i, j \leq n\)
\[
x_iy_j \equiv \delta_{ij}\Delta \mod I,
\]
where \(\Delta\) is a lift in \(R\) of a socle generator of \(R/I\) and \(\delta_{ij}\) denotes Kronecker’s delta.

Proof. Note that \(I \subset (x_1, \ldots, x_j^2, \ldots, x_n)\), as \(I\) is contained in \(\mathfrak{m}^2\). For any \(1 \leq j \leq n\) it will be enough to find \(y_j \in I\) : \((x_1, \ldots, x_j^2, \ldots, x_n)\) such that \(y_j \not\in I\) : \((x_1, \ldots, x_n)\). If not, one must have \(I\) : \((x_1, \ldots, x_j^2, \ldots, x_n) = I\) : \((x_1, \ldots, x_n)\) and hence \((x_1, \ldots, x_j^2, \ldots, x_n) = (x_1, \ldots, x_n)\), as \(\text{Hom}(\_ , R/I)\) is a self-dualizing functor; a contradiction.

The element \(y_j\) has the property that \(x_iy_j \in I\) for any \(i \neq j\) and \(x_jy_j \in I\) : \((x_1, \ldots, x_n)\). Hence we can write \(y_jx_j = a_j + g_j\Delta\) with \(a_j \in I\) and \(g_j \neq 0\). However, \(g_j \not\in (x_1, \ldots, x_n)\) as otherwise \(y_jx_j \in I\). But then \(g_j\) is an invertible element, so that \(g_j^{-1}y_j\) will have all the required properties as in (3.1).

Lemma 3.6. Let \((R, \mathfrak{m})\) be a Noetherian local ring with embedding dimension \(n\) at least two. Suppose that \(I\) is an \(\mathfrak{m}\)-primary ideal contained in \(\mathfrak{m}^2\) such that \(R/I\) is Gorenstein. Letting \(L = I:\mathfrak{m}\) then the following conditions hold:

(a) \(L\) has reduction number one with respect to \(I\), i.e., \(L^2 = IL\);
(b) \(I\mathfrak{m} = L\mathfrak{m}\).

Proof. (a) As \(L = (I, \Delta)\) one only needs to show that \(\Delta^2 \in IL\). If we let \(m = (x_1, \ldots, x_n)\), by Lemma 3.5 we can find \(y_1, \ldots, y_n\) and \(a_1, \ldots, a_n \in I\) such that \(\Delta = x_1y_1 + a_1\) for \(1 \leq i \leq n\). As \(n \geq 2\), we can write
\[
\Delta^2 = (x_1y_1 + a_1)(x_2y_2 + a_2) = x_1y_1x_2y_2 + x_1y_1a_2 + a_1x_2y_2 + a_1a_2
= (x_1y_2)(x_2y_1) + (x_1y_1)a_2 + (x_2y_2)a_1 + a_1a_2.
\]
Note that each term in the last sum belongs to the ideal \(I(I, \Delta) = I^2 + I\Delta = IL\).

(b) We only need to show the inclusion \(Lm \subseteq Im\), or better \(\Delta \in Im : \mathfrak{m}\), as \(L = (I, \Delta)\). But for any \(1 \leq i \leq n\) pick \(j \neq i\) and write \(\Delta = x_jy_j + a_j\) with \(a_j \in I\); hence
\[
x_i\Delta = x_i(x_jy_j + a_j) = x_j(x_iy_j) + x_i a_j \in \mathfrak{m}I
\]
as desired.

Theorem 3.7. Let \((R, \mathfrak{m})\) be a Noetherian local ring and let \(I\) be an \(\mathfrak{m}\)-primary ideal such that \(R/I\) is Gorenstein. Then either
(a) there exists a minimal generating set $x_1, \ldots, x_n$ of the maximal ideal $m$ such that $I = (x_1, \ldots, x_{n-1}, x_n)$, or
(b) $L^2 = IL$, where $L = I : m$.

**Proof.** Assume that (a) does not hold. This remains true after completing and further it suffices to prove (b) after completion. We can then write $R$ as a homomorphic image of a regular local ring $S$ and it will suffice to prove (b) for the pullback of $I$. Notice that the pullback of $I$ to $S$ cannot satisfy (a). Henceforth we may assume that $R$ is regular and that $I$ does not satisfy (a). Let $t$ be the vector space dimension of $(I + m^2)/m^2$ and choose $x_1, \ldots, x_t$ in $I$ whose images span this vector space. Replace $R$ by $R/(x_1, \ldots, x_t)$ and $I$ by $I/(x_1, \ldots, x_t)$. We may then assume that $I$ is in $m^2$. If the dimension of $R$ is at least 2 then Lemma 3.6 yields (b). If not $R$ is either a field or a discrete valuation ring and in this case $I$ satisfies (a), a contradiction.

**Remark 3.8.** Under the assumptions of Theorem 3.7 if we also assume that $R$ is regular then we can restate (a) as

(a') $I$ is a complete intersection of Goto-type.

**Theorem 3.9.** Let $I$ be a height unmixed ideal in a ring $R$. Suppose that $I$ is a generically Gorenstein ideal and that $R_p$ is a regular local ring for all prime ideals $p$ minimal over $I$. Then $I$ is integrally closed if and only if

$$\sqrt{I} = IL: L^2,$$

(3.2)

where $L = I : \sqrt{I}$. If either of these equivalent conditions hold, $I$ is generically a complete intersection of Goto-type.

**Proof.** Let us assume first that $I$ is an integrally closed ideal. In order to show that (3.2) holds it will suffice to prove it for any $p \in \text{Ass}(R/\sqrt{I}) = \text{Ass}(R/I)$, given that $\sqrt{I} \subseteq IL: L^2$, by the definition of $L$. If $p \in \text{Ass}(R/I)$, the strict inclusion $(\sqrt{I})_p = pR_p \subset I_pL_p: L^2_p$ implies $L^2_p = I_pL_p$. But $I_p$ is integrally closed, which forces the equality $L_p = I_p$: a contradiction.

Conversely, suppose that (3.2) holds. Since $I$ is unmixed, to show that $I = \overline{I}$ it suffices to check it at each minimal prime $p$ of $I$. Without loss of generality we may assume that $R$ is regular and $R/I$ is zero dimensional Gorenstein. Then by Theorem 3.7 either $L^2 = IL$ or $I$ is a complete intersection of Goto-type. The first possibility cannot happen by (3.2) so $I$ must be a complete intersection of Goto-type and is therefore integrally closed. \[\square\]

**Example 3.10.** The regularity assumptions in Theorem 3.9 are required. For example, let $S = k[X, Y, Z]$ be a polynomial ring in 3 variables over a field $k$. Let $R = S/(X^4 + Y^4 + Z^4)$ and let $x, y, \text{ and } z$ denote the image of $X, Y, \text{ and } Z$ in $R$, respectively. $R$ is easily seen to be a two dimensional Gorenstein ring.
Consider the $R$-ideal $I = (x, y, z^2)$. Its lift $(X, Y, Z^2)$ back in $S$ is a complete intersection of Goto-type, hence it is integrally closed. However, $I$ itself is not an integrally closed $R$-ideal as $L = (x, y, z)$ is integral over $I$. However, note that $(x, y, z)^2 \neq I(x, y, z)$.

**Corollary 3.11.** With the same assumptions as in Theorem 3.9, if in addition $I$ is an integrally closed ideal, in the linkage class of a complete intersection with $R/I$ Gorenstein then $I^2$ is integrally closed as well.

**Proof.** Since $R/I$ is Gorenstein and in the linkage class of a complete intersection $R/I^2$ is Cohen–Macaulay, and in particular unmixed (see [3]). To prove $I^2$ is integrally closed it then suffices to prove it locally at its minimal primes. After localizing at any such prime $I$ is a complete intersection of Goto-type. But then all powers of $I$ are also generically complete intersections of Goto-type.

**Corollary 3.12.** Let $I$ be an unmixed ideal of a regular local ring $R$ and suppose that $I$ is Gorenstein. Then $I$ is integrally closed and it is generically a complete intersection of Goto-type.

**Proof.** It is enough to show the equality $I = \overline{I}$ at the minimal associated primes of $I$. Thus, after localizing, we may assume $R/I$ to be a zero dimensional Gorenstein ring. By Theorem 3.9 $I$ is generically a complete intersection ideal of Goto-type. In particular $I$ cannot have proper reductions. As $I$ is always a reduction of its integral closure, this forces the equality $I = \overline{I}$, hence the claim.

We conclude the section by pointing out that condition (3.3) in Theorem 3.9 is not sufficient to guarantee the integral closedness in the case of ideals of type greater than or equal to 2.

**Example 3.13.** Let $R$ be a regular local ring of dimension 2 and maximal ideal $m = (x, y)$. The Northcott ideal $I = (x^2, xy^4, y^5)$ has type $t = 2$ and satisfies the condition $IL : L^2 = m = \sqrt{I}$, with $L = (x^2, xy^3, y^4)$. However $I$ is not integrally closed, as we already observed in Example 2.5.

4. Integral closedness and normality

There are instances when it is more appropriate to prove that an ideal is normal to avail ourselves of the Jacobian criterion.
Codimension two complete intersections

The simplest case for tackling the question of computing the integral closure of ideals is certainly that of a codimension two complete intersection. Several known facts come together to make the approach via Rees algebras amenable. The methods to find the integral closure of affine domains become an option (see [24]).

**Theorem 4.1.** Let $R$ be a Cohen–Macaulay local ring and suppose that $R_p$ has rational singularities for all codimension two prime ideals $p$. If $J = (a, b)$ is a complete intersection of codimension two, then $\mathcal{J}^n = (\mathcal{J})^n$ for all $n \geq 1$.

**Proof.** For all $n \geq 1$ one has that

$$(a, b)^{n-1} J \subseteq (\mathcal{J})^n \subseteq \mathcal{J}^n,$$

hence it will be enough to show that the equality

$$(a, b)^{n-1} J = \mathcal{J}^n \quad (4.1)$$

holds for all $n \geq 1$.

We claim that $R/(a, b)^{n-1} J$ is unmixed of height two. First of all, observe that

$$(a, b)^{n-1}/(a, b)^{n-1} J \simeq (a, b)^{n-1}/(a, b)^n \otimes R/\mathcal{J} \simeq (R/(a, b))^n \otimes R/\mathcal{J} \simeq (R/\mathcal{J})^n,$$

as $(a, b)^{n-1}/(a, b)^n$ is isomorphic to the degree $n-1$ component of a polynomial ring in two variables with coefficients in $R/(a, b)$. On the other hand, one can consider the short exact sequence

$$0 \rightarrow (a, b)^{n-1}/(a, b)^{n-1} J \longrightarrow R/(a, b)^{n-1} J \longrightarrow R/(a, b)^n \rightarrow 0.$$

The claim now follows from the fact that both $R/\mathcal{J}$ and $R/(a, b)^{n-1}$ are unmixed (for the first fact see [23]; Theorem 2.12] and also [13], Theorem 4.1).

Finally, it is enough to prove (4.1) at the associated primes of $R/(a, b)^{n-1} J$. In this case, however, the result is true because of our assumption that $R_p$ has rational singularities for all codimension two prime ideals (see [24]).

**Remark 4.2.** As mentioned earlier, an alternative approach to the computation of the integral closure of an ideal $I$ is through the construction of the integral closure of its Rees algebra $R[I]$. If $I = (a_1, \ldots, a_n)$ then one can represent $R[I]$ as the quotient $R[T_1, \ldots, T_n]/\Omega$, where $\Omega$ is the kernel of the map that sends $T_i$ to $a_i$. If $R[I]$ is an affine domain over a field of characteristic zero and $\text{Jac}$ denotes its Jacobian ideal, then the ring

$$\text{Hom}_{R[I]}(\text{Jac}^{-1}, \text{Jac}^{-1})$$
is guaranteed to be larger than $R[It]$ if the ring is not already normal (see [24, Chapter 6]). It also turns out that

\[ \text{Hom}_{R[It]}(\text{Jac}^{-1}, \text{Jac}^{-1}) = \text{Hom}_{R[It]}((\text{Jac}^{-1})^{-1}, (\text{Jac}^{-1})^{-1}) = (\text{Jac Jac}^{-1})^{-1}, \]

the latter being the inverse of the so called trace ideal of Jac. Finally, to get the equations for $(\text{Jac Jac}^{-1})^{-1}$ one then uses again [24, Algorithm 6.2.1].

Naturally, this process can be repeated several times until the integral closure of $R[It]$ has been reached. The degree one component of the final output gives the desired integral closure of $I$.

**Example 4.3.** Let $k$ be a field of characteristic zero and let $J \subseteq R = k[x,y]$ be the codimension two complete intersection

\[ J = (x^3 + y^6, xy^3 - y^5). \]

Using the methods outlined in Remark 4.2, it turns out that the integral closure of $J$ can be obtained in three steps. After the first step has been completed one obtains the ideal

\[ J_1 = (x^3 + y^6, xy^3 - y^5, y^8); \]

after a second iteration one gets

\[ J_2 = (x^3 + y^6, xy^3 - y^5, x^2y^2 - y^6, y^7); \]

finally, at the end of the last step, one has that the integral closure of $J$ is given by the ideal

\[ J_3 = \overline{J} = (xy^3 - y^5, y^6, x^3, x^2y^2). \]

Note that $\overline{J}$ is also a normal ideal.

For the records, despite the fact that the original setting for the problem is a polynomial ring in 2 variables over a field of characteristic zero, overall we had to make use of additional 18 variables: quite a waste!

**Remark 4.4.** In the case of the integral closure of complete intersections, it has to be pointed out that some of the initial iterations of the process described in Remark 4.2 may be avoided by using the results of [5, 20].

If the complete intersection $J$ is primary to the maximal ideal $m$ of a Gorenstein ring $R$ and $J \subseteq m^s$ but $J \not\subseteq m^{s+1}$ then one has an increasing sequence of ideals $I_k = J_m: m^k$ satisfying $I_k^2 = JI_k$ for $k = 1, \ldots, s$ if $\dim(R) \geq 3$ or for $k = 1, \ldots, s - 1$ if $R$ is a regular local ring and $\dim(R) = 2$. In particular this says that the $I_k$’s are contained in the integral closure of $J$. Hence, instead of computing the integral closure of $R[It]$ one may want to start directly from $R[It]$ (or $R[I_{s-1}t]$ if $R$ is a regular local ring and $\dim(R) = 2$).
If $J$ is not primary to the maximal ideal one may initially use instead the sequence of ideals $I_k = J$: $(\sqrt{J})^k = J$: $(\sqrt{J})^k$, provided that at each localization at the associated primes of $J$ the conditions in [5, 20] are satisfied so that $I_k^2 = JI_k$ for all $k$'s in the appropriate range.

In the specific case of the complete intersection $J$ of Example 4.3 one can easily compute $I_1 = J: m$ and $I_2 = J: m^2$ and check that $J_1 = I_1$ and $J_2 = I_2$. Hence, starting directly from a presentation of $R/I_2$ a single iteration of the process as in Remark 4.2 gives the integral closure of $J$.

**Remark 4.5.** In [9] D. Eisenbud and B. Sturmfels developed a nice theory about binomial ideals. A natural question that one can raise on this matter asks: Is the integral closure of a binomial ideal still a binomial ideal? Despite what Example 4.3 and similar other examples would suggest, the answer is in general negative. We give below an example in characteristic zero. We use the fact that an ideal of a polynomial ring over a field is binomial if and only if some (equivalently, every) reduced Gröbner basis for the ideal consists of binomials (see [9, Corollary 1.2]).

**Example 4.6.** Let $R = k[x, y, z, w]$ be a ring with characteristic zero and consider the ideal

$$I = (x^2 - xy, -xy + y^2, z^2 - zw, -zw + w^2).$$

It turns out that a primary decomposition for $I$ is given by

$$I = (x - y, z - w) \cap (x - y, z^2, zw, w^2) \cap (z - w, x^2, xy, y^2) \cap (x, y^2, z, w^2).$$

Hence the integral closure of $I$ is nothing but

$$\overline{I} = (x - y, z - w) \cap (x - y, z^2, zw, w^2) \cap (z - w, x^2, xy, y^2)$$

$$= (x^2 - xy, -xy + y^2, z^2 - zw, -zw + w^2, xz - yz - xw + yw).$$

It can be checked that these generators for $\overline{I}$ are a Gröbner basis and so it is not a binomial ideal. The last fact can be easily checked if one chooses the order $x > z > y > w$ and then shows that the $S$-resultant of each pair of the given generating set of $\overline{I}$ reduces to zero modulo that same generating set (Buchberger criterion).

**Remark 4.7.** It has to be remarked that with the following linear change of variables

$$X = x - y \quad Y = y \quad Z = z - w \quad W = w$$

the ideal $I$ of Example 4.6 becomes the following monomial ideal

$$(X^2, -YX, Z^2, -WZ)$$

so that its integral closure is again a monomial ideal, namely

$$(X^2, -YX, Z^2, -WZ, XZ).$$
Gorenstein ideals

We now address the normality for certain classes of perfect Gorenstein ideals.

Example 4.8. Let $k$ be a field of characteristic zero and $I \subset k[x,y,z,w]$ the ideal generated by the Pfaffians of the five by five skew-symmetric matrix

$$
\varphi = \begin{pmatrix}
0 & x & y & z & w \\
-x & 0 & x & y & z \\
-y & -x & 0 & x & y \\
-z & -y & -x & 0 & z \\
-w & -z & -y & -z & 0
\end{pmatrix}.
$$

One has that

$$I = (x^2 - y^2 + xz, xy - yz + xw, xz - z^2 + yw, xw, y^2 - 2xz)$$

is a perfect Gorenstein ideal of codimension three. One can show that

$$\sqrt{I} = (x - z, zw, yw, y^2 - 2z^2), \quad L = (y, x, z^2),$$

and that the conditions specified in part (b) at the beginning of Section 3 are satisfied. Thus, $I$ is an integrally closed ideal. As a consequence of Corollary 4.10 below, $I$ is also a normal ideal.

Theorem 4.9. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in $d$ variables over a field $k$ of characteristic zero. Let $I \subset R$ be a Gorenstein ideal defined by the Pfaffians of a $n \times n$ skew–symmetric matrix $\varphi$ with linear entries. Suppose $n = d + 1$ and that $I$ is a complete intersection on the punctured spectrum. If $I$ is integrally closed it is also normal.

Proof. $I$ is an ideal of analytic spread $\ell = d$, of reduction number $d - 2$. Its Rees algebra $R[It]$ is also Cohen–Macaulay and has a presentation

$$R[It] = R[T_1, \ldots, T_n]/L,$$

with $L = (T \cdot \varphi, f)$, where $f$ is the equation of analytic dependence among the $n$ forms that generate $I$. (All of these facts can be traced to [22].)

To show that $I$ is normal, we estimate the codimension of the non-normal locus of $R[It]$ and show that it is at least 2. Since $R[It]$ is Cohen–Macaulay this will suffice to prove that $I$ is normal.

Let $\mathcal{N}_I$ be the ideal that defines the non-normal locus of $R[It]$. As $I$ is an integrally closed complete intersection on the punctured spectrum of $R$, we may assume that the ideal $(x) = (x_1, \ldots, x_d)$ is contained in $\mathcal{N}_I$. We now look at the size of the Jacobian ideal modulo $(x)$. $L$ is an ideal of height...
\[ n - 1 = d, \text{ and the corresponding } (n + d) \times (n + 1) \text{ Jacobian matrix has the form} \]

\[
\begin{pmatrix}
\begin{bmatrix}
B(\varphi)
\end{bmatrix}
& 0 \\
& \vdots \\
& 0 \\
\varphi & \frac{\partial f}{\partial T_1} \\
& \vdots \\
& \frac{\partial f}{\partial T_n}
\end{pmatrix}
\]

where \( B(\varphi) \) is the (Jacobian dual) matrix of the linear forms in the \( T_i \)'s such that \( T \cdot \varphi = x \cdot B(\varphi) \). Note that \( B(\varphi) \) is a \( d \times n \) matrix.

The Jacobian ideal contains the product of the minors of order \( d - 1 \) of \( B(\varphi) \) by the ideal generated by the partial derivatives of \( f \). Since \( f \) is an irreducible polynomial and \( k \) has characteristic zero, the latter has height at least 2. On the other hand, based on an argument from \[17\], Proposition 2.4, one has that height \( I_{d-1}(B(\varphi)) \geq 2 \). Hence we have an ideal of forms in the variables \( T_i \)'s to add to the ideal \( (x) \); both are contained in \( \mathcal{N} \), and from the previous calculations it follows that

\[
\text{height } \mathcal{N} \geq d + 2 = \text{height } \mathcal{L} + 2,
\]

as claimed.

**Corollary 4.10.** Let \( R = k[x_1, x_2, x_3, x_4] \) be a polynomial ring in 4 variables over a field \( k \) of characteristic zero. Let \( I \subset R \) be a Gorenstein ideal defined by the Pfaffians of a five by five skew–symmetric matrix \( \varphi \) with linear entries. If \( I \) is integrally closed it is also normal.

**Proof.** By Theorem 3.1, \( I \) is a complete intersection on the punctured spectrum, hence Theorem 4.9 applies. \[ \square \]

**Example 4.11.** The hypothesis that the entries of the matrix \( \varphi \) are linear forms is necessary. Let \( k \) be a field of characteristic zero and \( I \subset k[x, y, z, w] \) be the codimension three Gorenstein ideal \( I \) generated by the Pfaffians of the five by five skew-symmetric matrix \( \varphi \) (see \[24\], Example 8.3.3)

\[
\varphi = \begin{pmatrix}
0 & -x^2 & -y^2 & -z^2 & -w^2 \\
x^2 & 0 & -w^2 & -xy & -z^2 \\
y^2 & w^2 & 0 & -x^2 & -xy \\
z^2 & xy & x^2 & 0 & -y^2 \\
w^2 & z^2 & xy & y^2 & 0
\end{pmatrix},
\]
where $x, y, z, w$ are variables. It can be checked that $IL: L^2 = \sqrt{I}$ where $L = I: \sqrt{I}.$ Hence $I$ is integrally closed. On the other hand, this ideal is not normal, a fact that can be shown as follows. Since $I$ is generated by five forms $f_1, \ldots, f_5$ of the same degree, the Rees algebra of $I$ decomposes as

$$R[It] = k[f_1, \ldots, f_5] \oplus L.$$ 

To prove that $R[It]$ is not normal it suffices to show that $k[f_1, \ldots, f_5]$ is not normal. But $k[f_1, \ldots, f_5]$ can be easily seen to be an hypersurface ring and an application of the Jacobian criterion shows that it is not normal.

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