On the tractability of the maximum independent set problem

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Abstract

The maximum independent set problem is a classical NP-complete problem in graph theory and has important practical applications in many domains. In this paper we show, in a partially non-constructive way, the existence of an exact polynomial-time algorithm for this problem. We outline the algorithm in pseudo-code style. Then we prove its exactness and efficiency by analysis.

Keywords: Independent set, independence number, induced subgraph, algorithm, polynomial time.

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1. Introduction

The independence number $\alpha(G)$ of a graph $G$ is the size of the largest independent set of $G$. An independent set of size $\alpha(G)$ is called a maximum independent set of $G$. The maximum independent set (MIS) problem asks for a maximum independent set of a given undirected graph. The MIS is an NP-complete [7, 15] and is computationally equivalent to the minimum vertex cover problem and the maximum clique problem. These are NP-hard problems [11] and so it is believed that no polynomial-time exact algorithms to solve any of these can be found. Still, it is worthwhile to attempt algorithms for any of these because they have important applications in fields such as bioinformatics, coding theory, computer vision, document clustering, image processing, pattern recognition and social networking [3, 10, 17, 18, 22].

There are exact algorithms, approximation algorithms and heuristics for the MIS. The latter two types constitute the class of non-exact MIS algorithms. Informative surveys of algorithms for the MIS along with many references are in [24]. Exact algorithms report $\alpha(G)$ ($G$ being the input graph, or problem instance) and at least one maximum independent set of $G$. A few outstanding exact algorithms are by Beigel [2], Bougreouis et al [4], Jian [14],
Robson [19] and Tarjan and Trojanowski [21]. But all the known exact algorithms for the MIS consume exponential time, and so are not fast in solving practical instances of large sizes.

Non-exact algorithms can run faster than exact ones and can return maximal (though not necessarily maximum) independent sets of large sizes in large graphs. A range of non-exact algorithms for the MIS are discussed in [17]. Approximation algorithms come with a provable guarantee that the optimal solution is always within a multiplicative factor of the reported solution whereas heuristics have no such guarantees. Non-exact algorithms can be of interest in practical applications even though they carry no guarantee of performance. Compact discussions on a few such applications are in [3, 17]. But such algorithms do not conclusively say anything on the gap between the optimal solution and the reported one.

Though non-exact MIS algorithms are accompanied by experimental reports and discussions on their performances, virtually no analysis is given in support of such reports [6]. Consequently, for any such algorithm, there seems to be a considerable gap between its reported capabilities and its worst-case performance.

This paper outlines an algorithm (named αMAX) for the MIS and follows it up with analysis that culminates in proving the exactness and the polynomial-time efficiency of the algorithm.

In section 2 we give relevant definitions and notation from graph theory. In section 3 we establish preliminary results that are essential to our proposed algorithm αMAX. In section 4 we outline the αMAX in pseudo-code style. In section 5 we give theory relevant to the running of the αMAX and prove the its exactness. In section 6 we show our algorithm is of polynomial-time complexity.

2. Definitions and notation

The definitions and notation given in this section are mainly from [1, 9, 13, 20]. Let $V$ be a non-empty finite set. The cardinality (or, size) of $V$ is denoted by $|V|$, and is the number of elements in $V$. The power set of $V$ is denoted by $2^V$, and is the set of all the subsets of $V$ including the empty set $\emptyset$. The set of all non-empty subsets of $V$ is denoted by $2^V^*$ – that is, $2^{V^*} = 2^V - \{\emptyset\}$.

A simple undirected graph is an ordered pair $G = (V, E)$ where $V$ is a non-empty finite set and $E \subset 2^{V^*}$ such that (i) $\bigcup_{X \in E} X \subseteq V$ and (ii) $|X| \leq 2$ for each $X \in E$. The sets $V$ and $E$ are, respectively, the vertex set and the edge set of $G$. Each element of $V$ is a vertex of $G$ and each member of $E$ is an edge of $G$. The integers $|V|$ and $|E|$ are, respectively, the order (= the number of vertices) and the number of edges of $G$. The order of $G$ may also be denoted by $|G|$. A loop is an edge $X$ with $|X| = 1$. $G$ is loop-free if $|X| = 2$ for each $X \in E$.

Throughout this paper, the term graph will mean a simple undirected loop-free graph. If $G = (V, E)$, then the expressions $x \in V$ and $x \in G$ will both mean $x$ is a vertex of $G$. Similarly, both $\{x, y\} \in E$ and $\{x, y\} \in G$ will mean $\{x, y\}$ is an edge of $G$. For the rest of this section, $G = (V, E)$ is assumed.
Two distinct vertices \( x \) and \( y \) of \( G \) are adjacent in \( G \) if \( \{x, y\} \in E \). If \( \{x, y\} \in E \) then \( x \) and \( y \) are the end points (or, ends) of this edge. If \( x \) and \( y \) are adjacent in \( G \) then each of \( x \) and \( y \) is a neighbour of the other in \( G \). For \( x \in V \), the set \( N(x) \) consisting of all the neighbours of \( x \) in \( G \) is the neighbourhood of \( x \) in \( G \). The degree of \( x \) in \( G \) is denoted by \( dx \) or by \( dx(G) \), and is the number of vertices of \( G \) that are neighbours of \( x \) - that is, \( dx = |N(x)| \). A vertex \( y \) of \( G \) is isolated in \( G \) if \( dy = 0 \). \( G \) is null if \( dx = 0 \) for every \( x \in G \). \( G \) is complete if all of its vertices are pairwise adjacent - that is, \( dx = n - 1 \) for each \( x \in G \) where \( n = |G| \).

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs. Then \( G_1 \) is isomorphic to \( G_2 \) if there is a bijective [20] map \( f : V_1 \to V_2 \) with the following property: for each pair \( x \) and \( y \) of vertices of \( G_1 \), \( \{x, y\} \in E_1 \) if and only if \( \{f(x), f(y)\} \in E_2 \).

A subgraph of \( G \) is a graph \( J = (W, F) \) such that: (i) \( W \subseteq V \), (ii) \( F \subseteq E \) and (iii) each edge in \( J \) has the same end points in \( J \) as in \( G \). \( J \) is a proper subgraph of \( G \) if either \( W \neq V \) or \( E \neq F \). If \( A \subseteq V \) then the subgraph induced by \( A \) is the subgraph \( G[A] = (A, E[A]) \) where \( E[A] \) is the set of all those edges \( \{x, y\} \in E \) such that \( x \in A \) and \( y \in A \). In particular, if \( a \in V \) then the subgraph induced by \( V - \{a\} \) will be denoted by \( G - a \).

Let \( S \subseteq V, S \neq \emptyset \). \( S \) is an independent set of \( G \) if no two vertices of \( S \) are adjacent; that is, \( S \) is an independent set in \( G \) if the subgraph \( G[S] \) is null. \( S \) is a maximal independent set of \( G \) if \( S \) is not a proper subset of any independent set of \( G \). \( S \) is a maximum independent set of \( G \) if (i) \( S \) is an independent set of \( G \) and (ii) \( |S| \geq |B| \) for every independent set \( B \) of \( G \).

A given graph has a maximum independent set though such a set is not necessarily unique. If \( S_1 \) and \( S_2 \) are maximum independent sets of \( G \) then \(|S_1| = |S_2| \). If \( S \) is a maximum independent set of \( G \) then the positive integer \( |S| \) is the independence number \( \alpha(G) \) of \( G \). Obviously \( \alpha(G) = |G| \) if \( G \) is null and \( \alpha(G) = 1 \) if \( G \) is complete.

3. Preliminaries

Throughout this section, \( G = (V, E) \) and \(|G| \geq 2 \) are assumed.

**Proposition 3.1.** Let \( W \) be a nonempty proper subset of \( V \). If \( S \) is an independent set of \( G \) and \( S \subseteq W \) then \( S \) is an independent set of the induced subgraph \( G[W] \).

**Proof.** Let \( x \) and \( y \) be distinct vertices in \( S \). If \( \{x, y\} \) is an edge in \( G[W] \) then obviously \( S \) is not an independent set of \( G \). ■

**Corollary 3.2.** Let \( a \in G \) and \( S \) be an independent set of \( G \) such that \( a \notin S \). Then \( S \) is an independent set of \( G - a \).

**Proof.** The conclusion follows from the facts that the graph \( G - a \) is the induced subgraph \( G[W] \) and \( S \subseteq W \), where \( W = V - \{a\} \). ■

**Proposition 3.3.** Let \( a \in G \). If \( S \) is an independent set of \( G - a \) then \( S \) is an independent set of \( G \).
Proposition 3.11. \(\text{(ii)}\) Follows from:

independent set of size \(S\) so maximum independent set of \(G - a\). ■

Corollary 3.4. \(\alpha(G) \geq \alpha(G - a)\) for every \(a \in V\).

Proposition 3.5. Suppose \(x\) and \(y\) are any two distinct adjacent vertices of \(G\). Then either \(\alpha(G) = \alpha(G - x)\) or \(\alpha(G) = \alpha(G - y)\).

Proof. Let \(\alpha(G) = r\), \(\alpha(G - x) = p\) and \(\alpha(G - y) = q\). By corollary 3.4, \(r \geq p\) and \(r \geq q\). We assert that either \(r = p\) or \(r = q\). Let \(S\) be any maximum independent set of \(G\). Then \(|S| = r\). Since \(x\) and \(y\) are adjacent in \(G\), either \(x \not\in S\) or \(y \not\in S\). Here we invoke the corollary 3.2. If \(x \notin S\) then \(S\) is an independent set of \(G - x\), from which \(|S| \leq p\). If \(y \notin S\) then \(S\) is an independent set of \(G - y\), from which \(|S| \leq q\). So either \(r \leq p\) or \(r \leq q\), whence either \(r = p\) or \(r = q\). ■

Corollary 3.6. Suppose \(G\) is not null. Then \(\alpha(G) = \alpha(G - a)\) for some vertex \(a\) of \(G\).

Corollary 3.7. Let \(r_1 \cup r_2\) denote the larger of the two real numbers \(r_1\) and \(r_2\). If \(x\) and \(y\) are adjacent in \(G\) then \(\alpha(G) = \alpha(G - x) \cup \alpha(G - y)\).

Corollary 3.8. If \(G\) is not null then for some vertex \(a\) of \(G\), every maximum independent set of \(G - a\) is also a maximum independent set of \(G\).

Proposition 3.9. Let \(S\) be an independent set of \(G\). Then \(S\) is a maximal independent set if and only if to each \(x \in V - S\) there exists \(y \in S\) such that \(x\) and \(y\) are adjacent.

Proof. \((\rightarrow)\) Assume \(S\) is a maximal independent set of \(G\). Let \(x \in V - S\) be given. If \(x\) were adjacent to no vertex in \(S\) then \(S \cup \{x\}\) would be an independent set of \(G\), contradicting the maximality of \(S\).

\((\leftarrow)\) Assume \(S\) is not maximal. Then for some \(x \in V - S\), \(S \cup \{x\}\) is an independent set. So \(x\) is adjacent to no vertex in \(S\). ■

Proposition 3/10. Let \(W \subset V\) and \(W \neq V\). Let \(J = (W, F)\) be the subgraph of \(G\) induced by \(W\). Let \(x \in V - W\) such that \(x\) is not adjacent to any vertex of \(J\). Let \(H\) be the subgraph induced by \(W \cup \{x\}\). Then: (i) \(\alpha(H) = \alpha(J) + 1\) and (ii) \(M \cup \{x\}\) is a maximum independent set of \(H\) whenever \(M\) is a maximum independent set of \(J\).

Proof. (i) Obviously \(x\) is not adjacent to any vertex of \(H\). Let \(\alpha(J) = p\) and \(M\) be any maximum independent set of \(J\). Then \(|M| = p\), \(x \notin M\) and \(M \cup \{x\}\) is an independent set of \(H\). So \(\alpha(H) \geq p + 1\). If \(H\) had an independent set, say \(A\), of size \(p + 2\) then \(H - x\) would have an independent set of size \(p + 1\). This, in view of \(H - x\) being isomorphic to \(J\), contradicts \(\alpha(J) = p\). This proves (i).

(ii) Follows from: \(x \notin M\), \(M \cup \{x\}\) is an independent set of \(H\) and \(|M \cup \{x\}| = p + 1\). ■

Proposition 3.11. Upto isomorphism, there is only one graph \(G = (V, E)\) such that \(|V| = 3\) and \(|E| = 1\).
Proof. Let \( V = \{x, y, z\} \) and \( E \) consist of the one edge \( \{x, y\} \). Let \( J = (W, F) \) be any graph with \(|W| = 3\) and \(|F| = 1\). Name this edge \( \{a, b\} \) and write \( W = \{a, b, c\} \). The map \( h: G \rightarrow J \) defined by \( h = \{(x, a), (y, b), (z, c)\} \) is an isomorphism of \( G \) onto \( J \). ■

**Proposition 3.12.** Upto isomorphism, there is only one graph \( G = (V, E) \) such that \(|V| = 3\) and \(|E| = 2\).

Proof. In any such graph, two of the three vertices have degree 1 each and the other vertex has degree 2. Then the two edges share a vertex. So let \( V = \{x, y, z\} \) and \( E \) consist of the two edges \( \{x, y\} \) and \( \{x, z\} \). Let \( J = (W, F) \) be any graph with \(|W| = 3\) and \(|F| = 2\). Name these two edges \( \{a, b\} \) and \( \{a, c\} \) and write \( W = \{a, b, c\} \). The map \( h: G \rightarrow J \) defined by \( h = \{(x, a), (y, b), (z, c)\} \) is an isomorphism of \( G \) onto \( J \). ■

4. The proposed algorithm \( \alpha \text{MAX} \) (pseudo-code)

The principal idea behind this algorithm is proposition 3.5: if \( G \) is not null then \( \alpha(G) = \alpha(J) \) for some proper subgraph \( J \) of \( G \).

**Input:** The vertex set \( V \) and the edge set \( E \) of a graph \( G = (V, E) \).

**Pre-processing**

(i) Compute the adjacency list of \( G \)

(ii) Compute the adjacency matrix of \( G \) and go to the main algorithm

**The main algorithm**

BEGIN with (1)

(1) (i) Compute \( n = |V| \) and \( e = |E| \);

(ii) order the vertices of \( G \) as \( x_1, \ldots, x_n \) where \( dx_j \geq dx_{j+1} \) for \( j = 1, \ldots, n - 1 \);

(iii) Ver = \([x_1, \ldots, x_n]\); go to (2)

(2) If \( e = n(n - 1) / 2 \) then output: (i) \( \alpha(G) = 1 \) and (ii) the maximum independent sets of \( G \) are \( \{x_1\} \) through \( \{x_n\} \) and END; else go to (3)

(3) If \( e = 0 \) then output: (i) \( \alpha(G) = n \) and (ii) the only maximum independent set of \( G \) is \( V \) and END; else go to (4)

(4) MIS (STORED) = \( \{x_1\} \); \( \alpha(\text{STORED}) = 1 \) and \( r = 2 \); go to (5)

(5) \( W = [\text{Ver}(1), \ldots, \text{Ver}(r)] \); go to (6)

(6) If \( \text{Ver}(r) \) is not adjacent to any other element of \( W \) then

\( \alpha(\text{STORED}) \leftarrow \alpha(\text{STORED}) + 1 \) and
MIS(STORED) arrow MIS(STORED) ∪ \{Ver(r)\}

and go to (7); else go to (8)

(7) \(r \leftarrow r + 1\) and go to (22)

(8) Let \(m\) be the largest index \((\leq r - 1)\) such that \(x_m\) is adjacent to Ver\((r)\); go to (9)

(9) Let \(S = [W(1), \ldots, W(m - 1), W(m + 1), \ldots, W(r)]\); go to (10)

(10) \(j = 1;\) go to (11)

(11) \(IP = S;\) \(OP = \phi;\) go to (12)

(12) \(Lead = IP(1);\) \(OP \leftarrow [OP, Lead];\) Rev\(IP = \phi;\) go to (13)

(13) \(k = |IP|;\) go to (14)

(14) if \(k > 1\) got to (15), else go to (16);

(15) for \(a = 2\) to \(k\)

  if \(IP(a)\) is not adjacent to \(Lead\) then \(RevIP \leftarrow [RevIP, IP(a)]\)

  else end;

end for; go to (16)

(16) if \(RevIP = \phi\) then go to (18) else go to (17);

(17) \(IP \leftarrow RevIP\) and go to (12)

(18) \(\alpha(CURRENT) = |OP|;\) go to (19)

(19) If \(\alpha(CURRENT) > \alpha(STORED)\)

then \(\alpha(STORED) \leftarrow \alpha(CURRENT)\) and MIS(STORED) \(\leftarrow OP\)

else end; go to (20)

(20) \(j \leftarrow j + 1;\) go to (21)

(21) If \(j \leq |S|\) then \(S \leftarrow [S(2), \ldots, S(r - 1), S(1)]\) and go to (11)

  else \(r \leftarrow r + 1\) and go to (22)

(22) if \(r \leq n\) go to (5) else go to (23)

(23) **OUTPUT:** (i) \(\alpha(G) = \alpha(STORED)\) and

(ii) MIS(STORED) is a maximum independent set of \(G\)

END
5. Proof of the exactness of the algorithm $\alpha$MAX

A few definitions and notations are needed to prove the feasibility and the exactness of the $\alpha$MAX. Let $X$ be a non-empty finite set and $|X| = n$. An ordered set (or, an ordered r-set) over $X$ is a $1 \times r$ matrix $P = [x_1, \ldots, x_r]$ where (i) $0 \leq r \leq n$, (ii) $x_j \in X$ for $j = 1$ through $r$ and (iii) the entries of $P$ are all distinct elements of $X$. The set $X$ is the base set for $P$. The integer $r$ is the cardinality (or, size) of $P$, and is also denoted by $|P|$. If $r = 0$ then $P$ is the empty ordered set over $X$. An ordered $r$-set will be referred to as an ordered set unless the mention of $r$ is warranted. An ordered set over $X$ will be referred to as an ordered set if $X$ is understood from the context. The empty ordered set will be called the empty set and will be denoted by $\phi$. If $P = [x_1, \ldots, x_r]$, then the element $x_j$, where $1 \leq j \leq r$, is the $j$th element of $P$, and is denoted by $P(j)$. If $x \in X$ then $x \in P$ if and only if $x = x_j$ for some $j$ with $1 \leq j \leq |P|$.

Let $P_1$ and $P_2$ be two ordered sets over $X$. Write $P_1 = [x_1, \ldots, x_k]$ and $P_2 = [y_1, \ldots, y_r]$. Then $P_1 = P_2$ if and only if (i) $k = r$ and (ii) $x_j = y_j$ for $j = 1, \ldots, k$. If $P = [x_1, \ldots, x_r]$ is an ordered set over $X$, $y \in X$ and $y \notin P$ then the augmentation of $P$ by $y$ (or, the $y$-augmentation of $P$) is denoted by $[P, y]$ and is defined to be the ordered set $[x_1, \ldots, x_r, y]$.

In the pseudo-code of section 4, the notation $A(m)$, where $m$ is a positive integer, will mean the $m$th element of the ordered set $A$ (over $V$). For instance, $W(m - 1)$ (in step (9)) is the $(m - 1)$th element of the ordered set $W$, and $IP(1)$ (in step (12)) is the first element of the ordered set $IP$.

**Proposition 5.1.** In the pseudo-code of the $\alpha$MAX (section 4), the sets $Ver$, $W$, $S$, $IP$, $OP$ and $RevIP$ are all ordered sets over the vertex set $V$ of the input graph $G$.

**Proof.** Steps (1)(iii), (5), (9), (11), (12) and (15) show the conclusion for Ver, $W$, $S$, $IP$, $OP$ and $RevIP$, respectively. ■

**Proposition 5.2.** The algorithm $\alpha$MAX terminates in a finite number of computations.

**Proof.** The pre-processing phase of the $\alpha$MAX terminates in finitely many computations because $V$ and $E$ are finite sets. Let $n$ be the order of the input graph $G$. If $G$ is null or complete, then the running of the $\alpha$MAX begins with step (1) and terminates with either step (2) or step (3). Each of these steps executes only a finite number of computations. So suppose $G$ is neither null nor complete.

Steps (4) through (9) clearly involve only finitely many computations. So do (20) through (23).

The first iteration begins when $r = 2$ and the last iteration begins when $r = n$. For each $r = 2, \ldots, n$, the first sub-iteration begins when $j = 1$ and the last sub-iteration begins when $j = |S| = r - 1$. For each $j$, the loop in (15) is executed $k - 1$ times where $k = |IP|$. But then by (15) and (17), the value of $k$ decreases by at least 1 every time the control returns to (12), and so $k = 1$ happens in a finite number of computations. Hence (15) is done only finitely many times. Also, the steps (11) through (14) and (16) through (19) all depend on $|IP|$. Since $RevIP = \phi$
must happen when \( k = 1 \), it follows that (11) through (14) and (16) through (19) all involve only a finite number of computations.

Consequently, the \( \alpha \text{MAX} \) executes only finitely many computations for each \( r = 2, \ldots, n \).

Next, by (21) it is clear that \( r = n + 1 \) happens after finitely many iterations. Finally, by (22) and (23), it is clear that the algorithm terminates when \( r = n + 1 \). ■

The ordered set \( \text{OP} \) at the end of a sub-iteration will be referred to as an ordered output set, shortened to OOS.

**Proposition 5.3.** Let \( \text{OP} = [y_1, \ldots, y_k] \) be an OOS. Then for each \( j = 1, \ldots, k - 1 \), the vertex \( y_j \) is adjacent to none of \( y_j + 1 \) through \( y_k \).

**Proof.** Suppose there were \( y_p, y_t \in \text{OP} \) (\( 1 \leq p < t \leq k \)) with \( y_p \) adjacent to \( y_t \). Since \( y_p \in \text{OP} \), it happened by dint of (12) that Lead = \( y_p \) at some point of this iteration. Then when (15) is executed in this iteration, \( y_t \) would not be included in RevIP. Then, by (16) and (17), \( y_t \) would not be in IP at any subsequent point in this iteration, and so \( y_t \) would not be in \( \text{OP} \) at the end of this iteration. But this patently conflicts with \( y_t \in \text{OP} \). ■

**Proposition 5.4.** Let \( \text{OP}= [y_1, \ldots, y_k] \) be an OOS. If \( x \in V \) and \( x \notin \text{OP} \), then \( x \in N(y_j) \) for some \( y_j \) in \( \text{OP} \).

**Proof.** Clearly \( x \) was not Lead at any point - else by (12), \( x \) would be in \( \text{OP} \). Next, if \( x \in V - N(y) \) for every \( y \in \text{OP} \) then \( x \) would be in RevIP when Lead = \( y_p \) happened with each \( p \in \{1, \ldots, k\} \). Consequently, \( x \) would have been included in RevIP when Lead = \( y_k \). Since \( \text{OP} = [y_1, \ldots, y_k] \) implies the iteration has no further vertex candidates for Lead, it follows that RevIP = \( \phi \) when Lead = \( y_k \). But here a patent contradiction arises when Lead = \( y_k \) - namely, RevIP = \( \phi \) and \( x \in \text{RevIP} \) at once. ■

**Corollary 5.5.** Let \( \text{OP} = [y_1, \ldots, y_k] \) be an OOS at the end of a sub-iteration. If \( x \in V \) and \( x \notin \text{OP} \) then for some \( z \in \text{OP} \), \( x \) was not augmented to RevIP when Lead = \( z \) during this sub-iteration.

**Proposition 5.6.** Let \( \text{OP}= [y_1, \ldots, y_k] \) be an OOS. Then then the set \( M = \{ y_1, \ldots, y_k \} \) is a maximal independent set of \( G \).

**Proof.** If \( y_i \) and \( y_j \) are in \( \text{OP} \) where \( i \neq j \), then \( y_i \) cannot be adjacent to \( y_j \) by proposition 5.3. So \( M \) is an independent set of \( G \). Next, let \( x \in V - M \) be given. Since \( x \notin \text{OP} \), \( x \) was not augmented to RevIP when Lead = \( z \) for some vertex \( z \) during the concerned sub-iteration (by corollary 5.5). Clearly, then, \( z \) and \( x \) are neighbours. Further, since Lead = \( z \) during this sub-iteration, we have that \( z \in \text{OP} \) at the end of this sub-iteration (that has \( \text{OP} \) as the OOS), whence \( z = y_p \) for some \( p \in \{1, \ldots, k\} \). So \( x \) is adjacent to some element of \( M \). Then by proposition 3.9, \( M \) is a maximal independent set of \( G \). ■

**Proposition 5.7.** If \( G = (V, E) \) is complete or null, then the \( \alpha \text{MAX} \) returns \( \alpha(G) \) and a maximum independent set of \( G \).
Proof. Let \( n = |V| \) and \( e = |E| \). If \( G \) is complete then \( e = n(n-1)/2 \). The algorithm checks this to be true - in step (2) - and reports: (i) \( \alpha(G) = 1 \) and (ii) the maximum independent sets of \( G \) are: \( \{x_1\}, \ldots \) and \( \{x_n\} \).

If \( G \) is null, then \( e = 0 \). The algorithm checks this to be true - in step (3) - and reports: (i) \( \alpha(G) = n \) and (ii) \( V \) is the maximum independent set of \( G \). ■

**Proposition 5.8.** Let \( G = (V, E) \) be of order 3. The \( \alpha \)MAX returns the independence number of \( G \) and a maximum independent set of \( G \).

Proof. Throughout this proof, let \( V = \{x, y, z\} \). Also, in the proof, \((*) \rightarrow (**)) \) indicates the passing the control from the computation represented by \((*)\) to the logically next one represented by \((**)\) in the \( \alpha \)MAX.

Suppose \( G \) is null. The \( \alpha \)MAX runs on \( G \) as follows:

\[
n = 3 \rightarrow e = 0 \rightarrow \text{Ver} = [x, y, z] \rightarrow \alpha(G) = 3 \text{ and } \{x, y, z\} \text{ is the only maximum independent set of } G \{x, y, z\} \rightarrow \text{END}.
\]

Suppose \( G \) is not null. Then \( G \) has at least two adjacent vertices. Also, \( G \) at least one edge and at most three edges. These are covered in the following three cases.

**Case 1:** \( G \) has exactly one edge - say, \( \{x, y\} \). Up to isomorphism, there is only one such graph \( G \) (by proposition 3.6). The \( \alpha \)MAX runs on \( G \) as follows:

\[
\text{BEGIN } \rightarrow n = 3 \rightarrow e = 1 \rightarrow \text{Ver} = [x, y, z] \rightarrow e \neq 0 \rightarrow e \neq n(n-1)/2 \rightarrow \text{MIS (STORED)} = \{x\} \rightarrow \alpha(\text{STORED}) = 1 \rightarrow r = 2 \rightarrow W = [x, y] \rightarrow S = [y] \rightarrow j = 1 \rightarrow \text{IP} = [y] \rightarrow \text{OP} = \phi \rightarrow \text{Lead} = y \rightarrow \text{OP} = [y] \rightarrow \text{RevIP} = \phi \rightarrow k = 1 \rightarrow \alpha(\text{CURRENT}) = 1 \rightarrow j = 2 \rightarrow r = 3 \rightarrow W = [x, y, z] \rightarrow \alpha(\text{STORED}) = 2 \rightarrow \text{MIS(STORED)} = \{x, z\} \rightarrow r = 4 \rightarrow \text{OUTPUT: } \alpha(G) = 2 \text{ and } \{x, z\} \text{ is a maximum independent set of } G \rightarrow \text{END}
\]

**Case 2:** \( G \) has exactly two edges. Let \( E \) consist of the edges, say, \( \{x, y\} \) and \( \{x, z\} \). Up to isomorphism, there is only one such graph \( G \) (by proposition 3.7). The \( \alpha \)MAX runs on \( G \) as follows:

\[
\text{BEGIN } \rightarrow n = 3 \rightarrow e = 1 \rightarrow \text{Ver} = [x, y, z] \rightarrow e \neq 0 \rightarrow e \neq n(n-1)/2 \rightarrow \text{MIS (STORED)} = \{x\} \rightarrow \alpha(\text{STORED}) = 1 \rightarrow r = 2 \rightarrow W = [x, y] \rightarrow S = [y] \rightarrow j = 1 \rightarrow \text{IP} = [y] \rightarrow \text{OP} = \phi \rightarrow \text{Lead} = y \rightarrow \text{OP} = [y] \rightarrow \text{RevIP} = \phi \rightarrow k = 1 \rightarrow \alpha(\text{CURRENT}) = 1 \rightarrow j = 2 \rightarrow r = 3 \rightarrow W = [x, y, z] \rightarrow S = [y, z] \rightarrow j = 1 \rightarrow \text{IP} = [y, z] \rightarrow \text{OP} = \phi \rightarrow \text{Lead} = y \rightarrow \text{OP} = [y] \rightarrow \text{RevIP} = \phi \rightarrow k = 2 \rightarrow \text{RevIP} = [z] \rightarrow \text{IP} = [z] \rightarrow \text{Lead} = z \rightarrow \text{OP} = [y, z] \rightarrow \text{RevIP} = \phi \rightarrow k = 1 \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow \alpha(\text{STORED}) = 2 \rightarrow \text{MIS(STORED)} = \{y, z\} \rightarrow j = 2 \rightarrow S = [z, y] \rightarrow \text{IP} = [z, y] \rightarrow \text{OP} = \phi \rightarrow \text{Lead} = z \rightarrow \text{OP} = [z] \rightarrow \text{RevIP} = \phi \rightarrow k = 2 \rightarrow \text{RevIP} = [y] \rightarrow \text{IP} = [y] \rightarrow \text{Lead} = y \rightarrow \text{OP} = [z, y] \rightarrow \text{RevIP} = \phi \rightarrow k = 1 \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 3 \rightarrow r = 4 \rightarrow \text{OUTPUT: } \alpha(G) = 2 \text{ and } \{y, z\} \text{ is a maximum independent set of } G \rightarrow \text{END}
\]

**Case 3:** \( G \) has exactly three edges. The \( \alpha \)MAX runs on \( G \) as follows:
BEGIN → n = 3 → e = 3 → e = n (n − 1) /2 → α(G) = 1 → the maximum independent sets of G are \{x\}, \{y\} and \{z\} → END

Thus, when G has order 3, the αMAX returns α(G) and a maximum independent set of G. ■  

Proposition 5.9. Let G = (V, E) be of order n. Then there is a linear ordering of vertices of G for which the αMAX returns (i) α(G) and (ii) a maximum independent set of G by returning α(J) and a maximum independent set of J for some subgraph J of G such that α(G) = α(J). In other words, the αMAX converges on a desired output, which is a pair (α(G), S) where S is a maximum independent set of G.

Proof. Let the proposition be called P(n). The proof is by induction on n. The proof for n = 3 was established by proposition 5.7.

Assume P(k) is true. Next, suppose n = k + 1. If G is null then by proposition 5.6 the αMAX returns (i) α(G) = k + 1 and (ii) V is the only maximum independent set of G. If G is complete then, again by proposition 5.6, the algorithm returns (i) α(G) = 1 and (ii) the only maximum independent sets of G are the singleton subsets of V. In both these cases the subgraph J in the statement of the proposition is G, and any ordering of the vertices can be taken as the required ordering. So P(k + 1) is true when G null or complete.

Suppose G neither complete nor null. Then G has two vertices - say, a and b - that are neighbours in G. Clearly G − a and G − b have order k. By the induction hypothesis, there is a linear vertex ordering in G − a and one in G − b for which the αMAX returns the following:

(i) α(G − a) = r₁ (say), (ii) a maximum independent set S₁ of G − a, (iii) α(G − b) = r₂ (say) and (iv) a maximum independent set S₂ of G − b. Here |S₁| = r₁ and |S₂| = r₂.

Let r = r₁ ∨ r₂. By corollary 2 to proposition 3.3, α(G) = r. Further, either S₁ or S₂ is an independent set of size r - and hence a maximum independent set of G. But then the αMAX has returned r and an independent set S (= S₁ or S₂) that are respectively α(G) and a maximum independent set of G. The required subgraph J in the proposition is either G − a (if r = r₁) or G − b (if r = r₂). The required ordering of vertices of G is the one in G − a (if r = r₁) or the one in G − b (if r = r₂). This completes the induction. ■

6. The worst-case time complexity of the αMAX

The worst-case time complexity of the αMAX is discussed using the asymptotic growth rate function O (big oh) [8, 16, 23]. The term time complexity will mean the worst-case one, throughout this section. The function O has the following k-sum property [16] that will be invoked at the end of this section.

k-sum property: Let k be a fixed positive integer (that is, k does not depend on the input size n in the algorithm under discussion). If f₁ (j = 1, . . ., k) and h are functions such that fᵢ = O(h) for all j = 1, . . . , k then f₁ + . . . + fₖ = O(h).
Primitive computational steps. Throughout this section, by the phrases “(*) is bounded by time $O(n^d)$” and “(*) takes $O(n^d)$ time,” we will mean that there are absolute constants $c > 0$ and $d > 0$ so that on every input graph of order $n$, the running time of the process in the place of (*) is bounded by $cn^d$ primitive computational steps ([16], chapter 2). The following are the primitive computational steps in the αMAX:

(p-c 1) Assigning a value to a variable;
(p-c 2) placing a new element at the end of a list of elements;
(p-c 3) reading an element from a list; and
(p-c 4) any of the four fundamental operation on real numbers.

Further, the term “instance” in this section will mean an input graph. For each instance, steps (1) through (4) of the αMAX are run once, and so is step (23). Steps (5) through (9) are run at most $n$ times each. Steps (10) through (22) are run at most $n^2$ times each. In the algorithm, the positive integer $r$ ranges from 2 to $n$ (the order of the input instance) and each value of $r$ corresponds to an iteration.

In the following analysis, the time complexity of steps (1) through (4) and (23) are for one instance and that of steps (8) through (22) are for an arbitrary iteration.

In step (1) of the pseudo-code (section 4), computing $|V|$ takes $O(n)$ time whereas computing $|E|$ takes $O(n^2)$ time. Next, computing the ordered set $Ver$ takes $O(n)$ time. Hence step (1) is bounded by time $O(n^2)$. Step (2) is bounded by time $O(n^2)$; so is (3).

If $G$ is neither complete nor null then the control goes to (4). The assignations $a(\text{STORED}) = 1$ and $\text{MIS(\text{STORED})} = x_1$ require time $O(1)$ each. Hence steps (1) through (4) are bounded by time $O(n^2)$.

Step (5) is bounded by $O(n)$ since it is a finite sequence of primitive computational step of the type (p-c 2) seen above. In step (6), checking if $Ver(r)$ is adjacent to any other element of $W$ can be done in $O(n)$ time using the adjacency matrix. The assignations seen in (6) are bounded by time $O(n)$. (7) obviously requires only constant time. Finding an element $x_m$ as in (8) is done using the adjacency matrix, and so (8) is bounded by $O(n^2)$. (9) is similar to (5) and so is bounded by $O(n)$. Obviously (10), (11) and (12) require only constant time. (13) is bounded by $O(n)$ since $|IP| \leq n$. (14) is done in constant time.

Whether $a$ is adjacent to $\text{Lead}$ is determined by checking the adjacency matrix for an edge connecting $\text{Lead}$ and $a$. This can be done in constant time. Further, $k \leq n - 1$. So each time the for loop in (15) is executed fully, there are at most $\sum(n - 1)$ computational steps, each of constant time. In each iteration, (15) is run at most $n$ times. Hence (15) is bounded by time $O(n^3)$.

In (16), the algorithm needs to access only the first element of $\text{RevIP}$ (if there is one). So (16) takes $O(1)$ time. (17) is bounded by time $O(n)$; so is (18). In (19), the logical operation requires constant time, as does the assignation $a(\text{STORED}) \leftarrow a(\text{CURRENT})$. The other
assignation \( \text{MIS(STORED)} = \text{OP} \) is bounded by time \( O(n) \). So (19) takes \( O(n) \) time. It is clear that (20) is done in \( O(1) \) time. Next, (21) is bounded by time \( O(n) \) whereas (22) and (23) are each bounded by \( O(1) \).

By the k-sum property, each iteration is bounded by time \( O(n^3) \). Since there are \( n - 1 \) iterations for each instance (corresponding to \( r = 2 \) through \( n \)), the time complexity of the \( \alpha\text{MAX} \) is \( O(n^4) \).

7. An example and some comments

Let \( G = (V, E) \) where \( V = \{1, 2, 3, 4, 5, 6, 7\} \). The adjacency list of \( G \) is:

(i) \( N(1) = \{2, 3, 4\} \), (ii) \( N(2) = \{1\} \), (iii) \( N(3) = \{1\} \), (iv) \( N(4) = \{1\} \), (v) \( N(5) = \{6, 7\} \),

(vi) \( N(6) = \{5, 7\} \) and (vii) \( N(7) = \{5, 6\} \), where \( N(x) \) denotes the neighbourhood (in \( G \)) of the vertex \( x \in G \).

The vertices of \( G \) in a non-ascending order of degrees are: 1, 5, 6, 7, 2, 3, 4. In the following steps, the numbering (1) through (9) is not connected with that in the pseudo-code of the \( \alpha\text{MAX} \). The arrows (arrow) indicate the sequence of computations.

BEGIN with (1)

(1) \( n = |V| = 7, e = |E| = 6 \) and \( \text{Ver} = [1, 5, 6, 7, 2, 3, 4] \). Go to (2).

(2) \( \alpha(\text{STORED}) = 1 \) and \( \text{MIS(STORED)} = \{1\} \). Go to (3).

(3) \( r = 2 \rightarrow W = [1, 5] \rightarrow \alpha(\text{STORED}) = 2 \rightarrow \text{MIS(STORED)} = \{1, 5\} \rightarrow r = 3 \). Go to (4).

(4) \( r = 3 \rightarrow W = [1, 5, 6] \rightarrow S = [1, 6] \rightarrow j = 1 \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 2 \rightarrow S = [6, 1] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 3 \). Go to (5).

(5) \( r = 4 \rightarrow W = [1, 5, 6, 7] \rightarrow S = [1, 5, 7] \rightarrow j = 1 \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 2 \rightarrow S = [5, 7, 1] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 3 \rightarrow S = [7, 1, 5] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 4 \). Go to (6).

(6) \( r = 5 \rightarrow W = [1, 5, 6, 7, 2] \rightarrow S = [5, 6, 7, 2] \rightarrow j = 1 \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 2 \rightarrow S = [6, 7, 2, 5] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 3 \rightarrow S = [7, 2, 5, 6] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 4 \rightarrow S = [2, 5, 6, 7] \rightarrow \alpha(\text{CURRENT}) = 2 \rightarrow j = 5 \). Go to (7).

(7) \( r = 6 \rightarrow W = [1, 5, 6, 7, 2, 3] \rightarrow S = [5, 6, 7, 2, 3] \rightarrow j = 1 \rightarrow \alpha(\text{CURRENT}) = 3 \rightarrow \alpha(\text{STORED}) = 3 \rightarrow \text{MIS(STORED)} = \{5, 2, 3\} \rightarrow j = 2 \rightarrow S = [6, 7, 2, 3, 5] \rightarrow \alpha(\text{CURRENT}) = 3 \rightarrow j = 3 \rightarrow S = [7, 2, 3, 5, 6] \rightarrow \alpha(\text{CURRENT}) = 3 \rightarrow j = 4 \rightarrow S = [2, 3, 5, 6, 7] \rightarrow \alpha(\text{CURRENT}) = 3 \rightarrow j = 5 \rightarrow S = [3, 5, 6, 7, 2] \rightarrow \alpha(\text{CURRENT}) = 3 \rightarrow j = 6 \). Go to (8).
(8) \( r = 7 \rightarrow W = [1, 5, 6, 7, 2, 3, 4] \rightarrow S = [5, 6, 7, 2, 3, 4] \rightarrow j = 1 \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow \alpha(\text{STORED}) = 4 \rightarrow \text{MIS(STORED)} = \{5, 2, 3, 4\} \rightarrow j = 2 \rightarrow S = [6, 7, 2, 3, 4, 5] \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow j = 3 \rightarrow S = [7, 2, 3, 4, 5, 6] \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow j = 4 \rightarrow S = [2, 3, 4, 5, 6, 7] \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow j = 5 \rightarrow S = [3, 4, 5, 6, 7, 2] \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow j = 6 \rightarrow S = [4, 5, 6, 7, 2, 3] \rightarrow \alpha(\text{CURRENT}) = 4 \rightarrow j = 7. \) Go to (9).

(9) \( r = 8 \rightarrow \alpha(G) = 4 \) and \( \{5, 2, 3, 4\} \) is a maximum independent set of \( G \).

END

**Comments on propositions.** Propositions 3.1 and 3.3 are important to establish proposition 3.5. Proposition 3.5 proves that for a given graph \( G = (V, E) \) that is not null, there is a proper subgraph \( J \) such that \( \alpha(G) = \alpha(J) \). This is crucial to the \( \alpha\text{MAX} \) because the required subgraph \( J \) is either \( G - a \) or \( G - b \), where \( a \) and \( b \) are any two adjacent vertices of \( G \). Inherent in the algorithm is the existence of a linear ordering of the elements of some nonempty subset \( W \) of \( V \). The \( \alpha\text{MAX} \) returns an optimal solution under this linear ordering of the elements of \( W \). This is explicit in the proof of proposition 5.8 and implicit in the proof of proposition 5.9.

Proposition 5.9 proves that for a given graph \( G \), the \( \alpha\text{MAX} \) does return \( \alpha(G) \) and a maximum independent set of \( G \). Proposition 5.7 and 5.8 are essential preliminaries to proposition 5.9. The results in section 5 preceding proposition 5.7 prove the feasibility and some capabilities of the \( \alpha\text{MAX} \).

**The non-constructive facet of \( \alpha\text{MAX} \).** Since \( \alpha(G) \) is known for every graph \( G \) of order 3, the proof of proposition 5.8 (the base case of the induction) was straightforward. However, for arbitrary \( n \), it is reasoned (in the proof of proposition 5.9) that there exists an ordering of vertices of \( G \) for which \( \alpha\text{MAX} \) returns \( \alpha(G) \), but such an order is not constructed explicitly here. This linear vertex ordering is the only aspect of \( \alpha\text{MAX} \) that is not explicit, which is the reason we deem \( \alpha\text{MAX} \) partially non-constructive.

**8. Concluding remarks**

We have presented an exact polynomial-time algorithm for determining \( \alpha(G) \) of a given graph \( G \). The vertex cover number of a given graph \( G \) can be deduced from \( \alpha(G) \) whereas the clique number of \( G \) can be found by applying the \( \alpha\text{MAX} \) on the complement graph [18] of \( G \). We have not reported any experimentation with the \( \alpha\text{MAX} \) because we have analytically proved its exactness and efficiency (in sections 5 and 6, respectively).

To sum up: (i) there exists an exact polynomial-time algorithm for the MIS for all graphs of order 3 (proposition 5.8) and (ii) if this algorithm returns \( \alpha(G) \) for all graphs of \( G \) order \( k \) then the algorithm returns \( \alpha(G) \) for all graphs of \( G \) order \( k + 1 \) (proposition 5.9). Hence there exists an exact polynomial-time algorithm (the \( \alpha\text{MAX} \)) for the MIS for all graphs of order \( n \geq 3 \). Symbolically: (i) \( P(3) \) is true and (ii) \( P(k + 1) \) is true whenever \( P(k) \) is true; hence \( P(n) \) is true for all \( n \geq 3 \).

Thus, the MIS is tractable. For an account of tractability of problems, we refer the reader to [5, 8, 12, 16].
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