Universality and scaling laws in the interdependent network model with healing

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Cascading failures may lead to dramatic collapse in interdependent networks, where the breakdown takes place as a discontinuity of the order parameter. However, this is a hybrid transition, meaning that, besides this first order character, the transition shows scaling too. Recently we showed that there are two sets of exponents describing respectively the order parameter and the cascade statistics, which are connected by a scaling law. Here we study the question of universality of these exponents in the interdependent network model with healing. The healing is controlled by a parameter, which at some value suppresses the discontinuity of the order parameter. In this model we find two universality classes: Below the critical healing value the exponents agree with those of the original model, while above this value the model displays trivial scaling.

I. INTRODUCTION

Coupled infrastructural networks are extremely vulnerable \cite{1}. Buldyrev et al. introduced the concept of interdependent networks \cite{1} in order to elucidate the mechanism behind this observation. This cascading failure (CF) model exhibits rich behavior, among others it shows a hybrid phase transition (HPT), where the order parameter has both a jump critical scaling. The original model consists of two layers of networks, the nodes of which are connected by so called random dependency links in a one to one manner. Whenever a node fails, its dependent pair will also be eliminated. This model has been extended in several ways, e.g. dependency links were limited to finite range \cite{2} to capture the cost one must pay for long-range connections, multiple layers were considered \cite{3,4} to model a variety of interconnected infrastructures and dynamical healing was introduced \cite{5} that describes the efforts spent on repairing the network on longer timescales. Although the original model exhibits HPT, it was shown that tuning parameters, such as the range of dependency links or the healing probability, allows for eliminating the jump in the order parameter, i.e., changing the transition from hybrid to second order \cite{2,5}.

Recently it was found that the hybrid transition can be characterized by two sets of exponents, one for order parameter and another one for the statistics of finite cascades \cite{6}. These are related by a scaling law, which connects the exponent of the order parameter to that of the first moment of the cascade size distribution. Calculations were carried out on the square lattice and the Erdős–Rényi network (corresponding to the mean field case). A very efficient algorithm is needed to calculate these critical parameters \cite{7}. This is probably the reason, why little effort has been devoted to the problem of universality in such systems.

In this paper we study an extensions of the original CF model. First, we extend the simulation algorithm to interdependent networks with healing. Next, we identify two universality classes separated the critical healing probability: One bearing a hybrid phase transition and a continuous one described by trivial scaling exponents. Finally, we argue that networks close to the critical healing probability are mixtures of network realizations from below and above the critical healing therefore their behavior is ambiguous and not well characterized by scaling exponents.

II. THE CF MODEL

The interdependent network model \cite{1} is built of two topologically identical starting networks $A$ and $B$ with usual connectivity links. Here we consider square lattice networks with nearest-neighbor connectivity links within each layer. The layers are connected by random dependency links producing a one-to-one mapping of the nodes of the two layers. Dependency link means that if a node is removed, its dependent is removed too. Only mutually connected nodes are viable that is the dependents of a single connected component form a single connected component in the other layer. Due to this restriction, the removal of a node may result in fragmentation of connected components which are referred to as failures cascading between the layers. Then the robustness of the network is studied, i.e., one layer is subject to a random attack in which $1-p$ fraction of the original nodes is removed gradually and the finally remaining size of the largest mutually connected component is measured.

A. CF model with healing

The cascading failure model with healing \cite{5} is a dynamic version of the CF model. The nodes of one network layer are targeted in a random order. In each step the next node in the ordering is “targeted” and a successful “attack” is carried out if the node is part of the currently
largest connected component. (The fraction of targeted nodes is $1 - q$ while the fraction of attacked nodes is $1 - p$, it follows from the definition that $1 - q > 1 - p$, therefore $p > q$.) All of the connectivity links of the targeted node are removed. After that the following dynamics is applied to relax the network:

1. Make a list of the nodes that lost any links. For each node in the list take each pair of its previous neighbors that survived the removal and connect them with independent probability $w$ if there is no link between them and their dependents are in a connected component in the other layer.

2. Then, in the other network, remove those links of dependent counterparts that run between nodes that are no longer connected.

3. Repeat 1 to 2 on all layers until no more nodes are removed.

The creation of new links in Step 1 can be interpreted as the effort of finding new partner to replace the failed ones. These healing links change the original topology of the network [5]. After the network is relaxed, one proceeds with the next step in which the next node on the list is targeted. The procedure is pursued until the connectivity links cease to exist. The principal quantity of interest is the order parameter, which is the relative size $m$ of the giant component as compared to the original size of the network. For small healing probability $w < w_c \approx 0.355$ the model exhibits a first-order (hybrid) phase transition while for $w > w_c$ the phase transition is of second-order [5].

### III. SIMULATION METHOD

Phase transitions are often accompanied by scaling laws but the numerical test of the relevant quantities for the CF model had been a challenging task. Recently, however, efficient algorithms have been developed, that allow for large scale simulation of the mutually connected components (MCC) in the CF model with computational time of $O(N^{1.2})$ [7, 8].

The implementation of [7] is based on the idea that only connectivity links within MCCs are kept and other links are inactivated. Consider a node removal (and the removal of its links as a consequence) in the layer $A$ that splits up a component. Then the new MCCs are to be found. The split is propagated to layer $B$ by inactivating all the links that bridged the newly split components in layer $B$. Of course this might trigger further splits that must be propagated back to layer $A$ and so on. This algorithm becomes very efficient using a proper graph data structure.

The underlying fully dynamic graph algorithm (see [9]) can account for connectivity in a single layer and this accounting is efficient both for adding and removing edges. On each edge removal or inactivation it detects if the component in the layer is split. Then one can query the size of the new components and optionally the members of any of these or even whether two nodes belong to the same component, all of this very efficiently.

The application of algorithm [2] to the healing problem needs some effort. Notably, the algorithm must integrate the step of adding healing links efficiently which is not obvious. In the following we generalize the algorithm to efficiently simulate interdependent networks with healing.

In the cascading failure model only the largest mutually connected component (GMCC) is of interest. By deleting links the GMCC and the smaller mutually connected components get fragmented into smaller components. However, adding healing links might save a component from the fragmentation. Therefore the deletions are not to be propagated immediately.

While the split of a component is easily propagated, the inverse, notably adding a link between two components in one layer can be computationally expensive. Adding a link requires a search to find which of the previously inactivated links need to be reactivated to find the GMCC. This search would possibly involve many small MCCs. The design challenge lies in rewriting the steps of the previous healing algorithm in such a way that healing links are added within components only. As a consequence it is assured that manipulation avoid reuniting components, i.e., components are always left intact or split. This choice is justified in the following.

To fulfill the above constraints we propose an algorithm that buffers the links to be deleted while it adds the healing links before actually deleting or inactivating any links in the given layer, see the Algorithm in the Appendix.

With the generalized algorithm, we can investigate critical properties of the hybrid percolation transition of the CF model with healing thoroughly. We measure various critical exponents including susceptibility and correlation size that were missing in previous studies for the healing-enabled version of the two-dimensional (2D) lattice interdependent networks.

### IV. RESULTS ON THE CF MODEL WITH HEALING

In the study of the healing we choose a 2D embedding topology with two $N = L \times L$ square lattices, both with periodic boundary conditions. Each node in one layer has a one-to-one partner node in the other layer.

The number of externally removed nodes is controlled. We define the control parameter $p$ in the view of the one-by-one removal, i.e., in each time step a still functional random node is externally attacked and removed. The attack is eventually followed by a cascade of failures. At the end only the largest mutually connected component is considered functional. The fraction of original nodes attacked externally is denoted by $1 - p$. 


behavior in the sense that the standard deviation is inversely proportional to the square root of the number of nodes in accordance with the central limit theorem. Near $w_c$ we could not disclose the true value of $\bar{\nu}_{m}$ because of the following reason. Trying to simulate a system with a specific $w$ near $w_c$ results in an ensemble of systems mixing scaling behaviors below and above $w_c$. This mixing has an extra contribution, in addition to and dominating over the sample variance. The large $\bar{\nu}_{m}$ indicates that the “unintentional” mixing part depends less on system size, at least for the system sizes that are accessible even with our efficient algorithm. The extra variance necessarily makes the distribution of the critical point wider as a consequence, it also makes it rather difficult and unreliable to extrapolate the critical point for the infinite system. In fact, the scaling breaks down and the determination of the critical exponents is hampered by the above effect in this regime.

The dependence of the order parameter on $p$ and $w$ is shown in Figure 2. We define the value of $w_c$ as the smallest $w$ for which we observe a continuous phase transition and we find $w_c = 0.355$. This value agrees well with the $w$ where $\bar{\nu}_{m}$ has a sharp maximum. The scaling $1 - m(p,w) = a(w) - a(w) m \left( 1 - \frac{w_c}{w} \right)$ that is asymptotically satisfied in the $w \rightarrow 0$ limit [3] is confirmed by the new measurements. $a(w) = (1 - p_c(0) - \Delta p_c(w))/(1 - p_c(0))$ where $\Delta p_c(w) \equiv p_c(w) - p_c(0) \propto w^\delta$ and the exponent value is $\delta = 1.006 \pm 0.009$.

The critical control parameter value and the size of the breakdown (see Figure 3) are extrapolated by finite size scaling, see Figure 8 in the Appendix. For $w > w_c$ the size of the breakdown $m_0 = 0$ indicating that macroscopic cascades do not occur. During the process when eliminating nodes one by one small cascades may occur; the smallest cascades cease to exist only at $w = 1$, see Figure 3.

The scaling exponent $\beta_m$ quantitatively describes the order parameter in the scaling domain of the hybrid phase transition and is defined according to [1] as $(m(p) - m_0) \propto (p - p_c)^\beta_m$. Without healing, the previously obtained and analytically proved $\beta_m = 0.5$ [4] is reproduced and it holds up to very close to $w_c$, see Figure 4. This value seems to be universal for $0 \leq w < w_c$.

These results suggest that there exist two universality classes. One is characterized by the hybrid transition with $\beta_m = 0.5$ and $\bar{\nu}_m \approx 2.2$. The other universality class has a vanishing giant component at breakdown and is characterized by exponents $\bar{\nu}_m \approx 2$ and $\beta_m = 1$. For the latter, see Figs. 1 and 2 as well as, section IV C

The exponent $\gamma_m$ is defined by the scaling of the susceptibility $\chi \equiv N \langle (m^2) - \langle m \rangle^2 \rangle \propto (p - p_c)^{-\gamma_m}$. Unfortunately, we were able to calculate the $\gamma_m$ values only with rather large error bars. They scatter between $1.41 \pm 0.15$ and $1.33 \pm 0.15$ (see Figure 5). We conclude that they do not contradict the assumption of universality. Due to lack of data we are unable to measure $\gamma_m$ in the region $w > w_c$. Later we will present an argument that the exponent $\gamma_m$ should be 0 in this region.

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**Figure 1.** (color online) The scaling exponent $\bar{\nu}_m$ takes two distinct values: $\bar{\nu}_m = 2.2 \pm 0.1$ for $w < w_c$ and $\bar{\nu}_m = 2.0 \pm 0.1$ for $w > w_c$, indicating two universality classes separated by the critical value of the healing parameter $w_c$. In the latter regime the system exhibits trivial scaling in the sense that the variance is inversely proportional to the number of nodes in the system. Near $w_c$ the sample variance is dominated by the stochastic mixing of systems that are above and below their perceived critical point. This makes the measurement of $\bar{\nu}_m$ unreliable.

We simulated system sizes $N = 256^2$, $512^2$, $1024^2$, and $2048^2$, with 384 network configurations for the largest system and increasingly more for smaller systems ($\#$ configurations $\approx 1.6 \cdot 10^9/N$).

**A. Critical behavior of the GMCC**

The order parameter $m(p)$ is defined as the size of the GMCC per node, which shows the typical behavior of the order parameter at a hybrid phase transition:

$$m(p) = \begin{cases} 0, & \text{for } p < p_c, \\ m_0 + u(p - p_c)^\beta_m, & \text{for } p \geq p_c. \end{cases}$$

(1)

The critical values $m_0$ and $p_c$ have been published [3] with great accuracy for the square lattice without healing ($w = 0$). We used this case as a benchmark test. We calculated the critical values for various $w$ parameter settings using our algorithm.

We check first whether the simulation results are in agreement with previous findings. The critical exponent $\bar{\nu}_m$ is defined via finite size behavior of the interdependent network as $\sigma \propto N^{-1/\bar{\nu}_m}$ where $\sigma$ stands for the standard deviation of the breakdown point. (For sake of simplicity, we incorporated the dimensionality $d = 2$ into the definition $\bar{\nu}_m \equiv \nu_m d = 2\nu_m$.) In case of $w = 0$ the exponent was previously measured $\bar{\nu}_m = 2.2 \pm 0.2$ [4].

For the network with healing the above value for $\bar{\nu}_m$ persists (see Figure 1) until approaching $w_c$. Above $w_c$ it gets stabilized at $\bar{\nu}_m \approx 2$ indicative of trivial scaling.
Figure 2. (color online) Critical behavior of the order parameter as obtained as an average over systems of size $N = 2048^2$. The solid curves are listed from right to left, dashed lines of the same color represent following $w$ in between using equidistant steps. Large symbols mark the breakdown point extrapolated for infinite systems using finite size scaling. (a) An overview, (b) zoom-in to show that for $w_c \geq 0.355$ the phase transition is of second order.

B. Critical behavior of avalanches

There is another set of critical exponents $\beta_a$, $\gamma_a$, $\sigma_a$, $\alpha$ and $\nu_a$ describing the statistics of avalanches. They can be evaluated only for $w < w_c$ where the number of avalanches is sufficient and avalanches are governed by scaling laws. The size $s$ of the avalanche is the number of nodes failing related to a single external attack. The finite avalanches are those that happen before the breakdown. The exponent $\gamma_a$ is defined with the average size of the finite cascades $\langle s_{\text{finite}} \rangle \propto \Delta p^{-\gamma_a}$ for which $1 - \alpha = \gamma_a$ holds [6]. This relationship is confirmed reasonably well for $w \leq 0.1$ where sufficient data is at our disposal, see Figure 6. For $w > w_c$ we would need even larger samples then studied to have sufficient statistics. Near $w_c$, however, the previously described mixing of realizations of network states from both continuous and discontinuous transitions makes our estimate for $p_c$ less reliable. As a consequence the distance $p - p_c$ from the critical point is less reliable too making it difficult to measure critical exponents [10].

The other exponents are defined as $p_s|_{N=\infty} \propto \Delta p^{-(\gamma_a - 1)/\sigma_a}$ and $p_s|_{\Delta p=0} \propto N^{(\gamma_a - 1)/\nu_a}$. The measurements support our hypothesis of a single universality class for avalanche-related exponents below $w_c$, see Figure 7. The avalanche-related exponents are meaningless above $w_c$ therefore we do not analyze them.

C. Behavior near $w = 1$

We have confirmed numerically that there is a critical value $w_c$ for the healing above which macroscopic cascades disappear and the network tends to get more and more connected. Here we focus on the critical behavior close to $w = 1$ and we prove that $\beta = 1$ for this case.

In a square lattice between any two nodes $U$ and $V$
Figure 4. (color online) The exponent $\beta_m$ characterizes the change of the order parameter above the jump $m_0$ in the critical regime. $\beta_m = 0.5 \pm 0.05$ for $w < w_c$ which confirms the theory for a universality class, and reproduces $\beta_m = 0.5$ for $w = 0$, as proved in [6]. For perfect healing ($w = 1$) the value $\beta_m = 1$.

Figure 5. (color online) The exponent $\gamma_m$ describes the scaling of the susceptibility close to the critical point. Due to lack of data we are unable to measure $\gamma_m$ in the region $w > w_c$.

Figure 6. (color online) The exponent $\gamma_a$ shows that the average avalanche size increases as the system gets close to the breakdown. Applying healing above $w_c$ stops most avalanches, $\gamma_a = 0$. Simulation near $w_c$ get a mixture of systems above and below the critical healing due to finite size effects therefore our measurement of $\gamma_a$ gets imprecise.

there exist initially at least two disjoint paths that only have $U$ and $V$ in common. Whenever we externally remove a node (different from $U$ and $V$) at least one of those paths remains intact. As a consequence all the remaining nodes remain attached to the giant component. Thanks to the perfect healing ($w = 1$) all possible bridges over the removed node are formed. This way the eventually cut paths between $U$ and $V$ are re-established and again there will be at least two disjoint paths between any two nodes. Correspondingly, in the case of interdependent layers, the removal of one node causes only the removal of its interdependent counterpart and no avalanches are induced: $m_0 = p_c = 0, m \equiv p$, therefore $\beta_m = 1, \chi_m \equiv 0$ and it does not make any sense to calculate $\gamma_m$ or the exponents related to avalanches.

When $w \lesssim 1$ avalanches might occur but they are rare and small. For example, to initiate a cascade at least a small region of $n$ nodes one needs to get separated from the connected component in one of the layers. To achieve this at least the perimeter of that region must be cut. In two dimensions the length of the perimeter is at least $\propto \sqrt{n}$. Cutting means that healing links are not allowed to form bridges. This happens with probability at most $(1 - w)^\sqrt{n}$. When separated, the propagation of the damage from the initial $n$ nodes may lead to a cascade of size $s \geq n$. As the dependency links have here unlimited range, the counterparts of the original nodes are far away from each other and it has small probability that their failure will lead to further separation of other components because such a separation must be prepared similarly. That is, for separating a single node in the 2D case at least three other connectivity links are needed to be cut previously without healing. This happens with probability smaller than $(1 - w)^3$. So the typical cascades are of small size and one iteration. This has been confirmed by simulations. This means that, when approaching the critical point $p_c \approx 0$, the network is so densely connected that cascades are prevented almost surely so $\beta_m = 1$ holds also in the vicinity of $w = 1$. It is tempting to conclude that this observation points toward the existence of universality for $w > w_c$. Assuming that the small avalanches are mostly independent their number per unit cell is determined by the central limit theorem, that is the fluctuation of their number is inversely proportional to the square root of the system size, hence $\bar{\nu}_m \approx 2$. 
eliminated the network is characterized by trivial scaling exponents in the sense that fluctuations follow the central limit theorem. The scaling relation $1 - \beta_m = \gamma_a$ holds reasonably well for $w < 0.1$.

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APPENDIX

Algorithm

1. Let $\mathcal{D}_A$ be the set of edges to be deleted from layer $A$.

2. Let $\mathcal{H}_A$ be the set of edges proposed as healing edges. This set is built as follows: take all the endpoints of the edges in $\mathcal{D}_A$. For each node $v$ among the endpoints list all possible pairs of the neighbors of $v$. Add the edge between each pair to $\mathcal{H}_A$ with independent probability $w$ if the edge connects two points whose dependents are in the same component in layer $B$ and the edge is not already in $\mathcal{H}_A$ nor does it exist in the network.

3. Create the edges $\mathcal{H}_A$ to the layer $A$. During the previous step, these edges were not yet added to the layer $A$ on purpose. Adding the edges in parallel with enumerating the nodes in $\mathcal{D}_A$ has unwanted side-effects that consist of nodes explored later encountering more healing links than nodes explored first. We want to avoid this and keep the algorithm independent of the order of enumeration.

4. Remove all edges in $\mathcal{D}_A$. Whenever an edge removal splits up a connected component in $A$ into two parts, the edges that run between the parts in layer $B$ are scheduled for deletion, add them to $\mathcal{D}_B$. (This step is the analogue to immediately inactivating edges in [2].)

5. If $\mathcal{D}_B$ is not empty, repeat the above steps swapping the roles $A \leftrightarrow B$ until no more edges are removed.

It is clear that the link creation step 3 is realized within the component before any deletion involving step 4; therefore the efficiency of the underlying dynamic graph algorithm is not degraded.

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Figure 8. (color online) (a) The transition point $p_c$ and (b) the size of the last avalanche $m_0$ as a function of the system size $N = L^2$ and the healing probability $w$. Based on this information the critical values for $L = \infty$ are extrapolated using standard finite size scaling, however, near $w_c$ the extrapolation might fail. The possible failure is due to the unreliable measurement of $\nu_m$, see Figure 3. The finite size scaling might be further hindered by a crossover meaning that for a small fixed $w$ the order of $m_N$ is increasing with $N$ but this gets reversed at $w \approx 0.34$.

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