EXCHANGE GRAPHS OF CLUSTER ALGEBRAS HAVE THE 
NON-LEAVING-FACE PROPERTY

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Dedicated to Professor Jie Xiao on the Occasion of his 60th Birthday

Abstract. The claim in the title is proved.

1. Introduction

Generalized associahedra were introduced by Fomin and Zelevinsky in connection to 
ccluster algebras of finite type. Ceballos and Pilaud [CP16] showed that all type B-C-D 
associahedra have the non-leaving-face property, namely, any geodesic connecting two 
vertices in the graph of the polytope stays in the minimal face containing both. In fact, this 
property was already proven by Sleator, Tarjan and Thurston [STT88] for associahedra of 
type A, before the name was coined in [CP16]. For a finite Coxeter system \((W, S)\), Williams [Wil17] established the non-leaving-face property for \(W\)-permutation and \(W\)-associahedra. However, it is known that there are examples related to the associahedron which do not satisfy the non-leaving-face property (cf. [CP16]).

An important combinatorial invariant of a cluster algebra is its exchange graph. The 
ababstract exchange graph was introduced in [BY13], which unifies various exchange graphs 
arising from representation theory of finite-dimensional algebras, marked surfaces, cluster 
algebras and so on. An abstract exchange graph has the structure of a generalized poly-
tope and then the non-leaving-face property can be formulated in this general situation. 
In [BZ19], the non-leaving-face property for exchange graphs arising from unpunctured 
marked surfaces was established.

The main contribution of this note is Theorem 3.11, in which we prove that for any 
cluster algebra, the exchange graph has the non-leaving-face property.

2. Background on cluster algebras

Fix a positive integer \(n\). Let \(\mathcal{F}\) be the field of rational functions in \(n\) indeterminates 
with coefficients in \(\mathbb{Q}\).

First of all, a labeled seed is a pair \((x, B)\), where

- \(x = (x_1, \ldots, x_n)\) is an \(n\)-tuple of elements of \(\mathcal{F}\) forming a free generating set of \(\mathcal{F}\), 
  that is, \(x_1, \ldots, x_n\) are algebraically independent and \(\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_n)\).
- \(B = (b_{ij})\) is an \(n \times n\) integer matrix which is skew-symmetrizable, that is, there 
  exists a positive integer diagonal matrix \(S\) such that \(SB\) is skew-symmetric.

We have the terminology:

- \(x\) is the (labeled) cluster of this seed;
- \(x_1, \ldots, x_n\) are its cluster variables;
- \(B\) is the exchange matrix.

As in [FZ02], let \(k \in \{1, \ldots, n\}\). The seed mutation \(\mu_k\) in direction \(k\) transforms \((x, B)\) 
into a new labeled seed \((x', B') := \mu_k(x, B)\) defined as follows:
• The entries of $B' = (b'_{ij})$ are given by

$$
(2.1) \quad b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k, \\
 b_{ij} + \lceil b_{ik} \rceil_+ b_{kj} + b_{ik} [-b_{kj}]_+ & \text{otherwise},
\end{cases}
$$

where we use the notation $[b]_+ = \max(b, 0)$ for an integer $b$.

• The cluster variables $x' = (x'_1, \ldots, x'_n)$ are given by

$$
(2.2) \quad x'_j = \begin{cases} 
\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+} & \text{if } j = k, \\
x_j & \text{otherwise}.
\end{cases}
$$

Let $T_n$ be the $n$-regular tree whose edges are labeled by the numbers $1, \ldots, n$ such that the $n$ edges emanating from each vertex have different labels. We write $t \xrightarrow{k} t'$ to indicate that vertices $t, t' \in T_n$ are joined by an edge labeled by $k$.

A seed pattern is defined by assigning a labeled seed $(x_t, B_t)$ to every vertex $t \in T_n$, so that the seeds assigned to the end points of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation in direction $k$. Let

$$
\mathcal{X}_A := \bigcup_{t \in T_n} x_t
$$

be the set of all cluster variables appearing in its seeds. The cluster algebra $A$ is the $\mathbb{Q}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}_A$.

We now fix a vertex $s \in T_n$ and a skew-symmetrizable integer $n \times n$ matrix $B = (b_{ij}) = B_s$. To $s$ and $B$ we associate a family of integer vectors $c_{j,s} \in \mathbb{Z}^n$ (c-vectors); here $j = 1, \ldots, n$ and $t \in T_n$. According to [FZ07], the vectors $c_{j,s}$ can be defined by the initial conditions

$$
c_{j,s} = e_j \quad (j = 1, \ldots, n)
$$

together with the recurrence relations

$$
(2.3) \quad c_{j,t'} = \begin{cases} 
-c_{j,t} & \text{if } j = k, \\
c_{j,t} + \lceil b_{kj} \rceil_+ c_{k,t} + b_{kj} [-c_{k,t}]_+ & \text{if } j \neq k.
\end{cases}
$$

for any $t \xrightarrow{k} t'$ in $T_n$. Here $e_1, \ldots, e_n$ are unit vectors in $\mathbb{Z}^n$ and for an integer matrix $C$, we write $\lceil C \rceil_+$ for the matrix obtained from $C$ by applying the operation $c \mapsto \lceil c \rceil_+$ to all entries of $C$. The $n \times n$ integer matrix

$$
C_{t}^{B_s} = (c_{1,t}^{B_s}, \ldots, c_{1,t}^{B_s})
$$

is called the $C$-matrix at vertex $t$ with respect to $s$.

A non-zero integer vector $c \in \mathbb{Z}^n$ is sign-coherent if all entries are either non-negative or non-positive. The following statement was proven in [GHKK18].

**Theorem 2.1** (sign-coherence of c-vectors). For each vertex $t \in T_n$, the matrix $C_{t}^{B_s}$ is invertible over $\mathbb{Z}$ and each column vector is sign-coherent.

For a matrix $C \in \text{M}_n(\mathbb{Z})$ and an index $k$, we will denote the matrix obtained from $C$ by replacing all entries outside of the $k$-th row (resp. column) with zeros $C^{k*}$ (resp. $C^{*k}$). As noted in [NZ12], the operations $C \mapsto \lceil C \rceil_+$ and $C \mapsto C^{k*}$ commute with each other, making the notation $\lceil C \rceil_+$ (and $C^{*k}$) unambiguous. So the sign-coherence of c-vectors implies that for every $k$, there is the sign $\epsilon_k(C_{t}^{B_s}) = \pm 1$ such that $[-\epsilon_k(C_{t}^{B_s})c_{1,t}^{B_s}]_+ = 0$. For each $1 \leq k \leq n$, let $J_k$ be the diagonal matrix obtained from the identity matrix by replacing the $(k, k)$-entry with $-1$. The following statement was proven in [NZ12 Proposition 1.4].
**Proposition 2.2.** Let \( s \xrightarrow{k} s' \) be an edge of \( T_n \). Then for any vertex \( t \in T_n \)
\[
C_t^{B_t; s'} = (J_k + [-\epsilon_k (C_s^{-B_t; t} B_{s'}') B_{s'}] \epsilon_k + C_t^{B_t; s}).
\]
where \( C_s^{-B_t; t} \) is the \( C \)-matrix at \( s \) with respect to \( t \) by assigning \( -B_t \) to \( t \).

3. **Non-Leaving-Face property of exchange graphs**

3.1. **Background on exchange graphs and non-leaving-face property.** As in [BY13], let \( V \) be a non-empty set with a reflexive and symmetric relation \( R \). Two elements \( x \) and \( y \) of \( V \) are compatible if \( (x, y) \in R \). Assume \((V, R)\) satisfies the following conditions:
- All maximal subsets of pairwise compatible elements are finite and have the same cardinality, say \( n \). We refer the subsets clusters;
- Any subset of \( n - 1 \) pairwise compatible elements is contained in precisely two clusters.

The *exchange graph* \( G \) of \((V, R)\) is defined to be the graph whose vertices are the clusters and where two vertices are joined by an edge if and only if their intersection has cardinality \( n - 1 \). A face \( F \) of \( G \) is a full subgraph of \( G \) such that
- there is a subset \( U \) of pairwise compatible elements;
- the vertices of \( F \) are precisely the clusters containing \( U \) as a subset.

Clearly, \( F \) is uniquely determined by \( U \) and we denote it by \( F_U \). The inclusion of sets induces a partial order on faces. Namely, we say \( F_U \leq F_V \) if \( V \subseteq U \).

For two cluster \( v_1 \) and \( v_2 \), we write for \( v_1 \rightarrow v_2 \) to indicate that they are linked by an edge. A path
\[
v = v_1 \rightarrow v_2 \cdots \rightarrow v_m = w
\]
from \( v \) to \( w \) is a geodesic connecting vertices \( v \) and \( w \), if the length of the path is minimal.

We remark that the graph \( G \) may not be connected.

We now recall non-leaving-face property from [CP16].

**Definition 3.1.** If any geodesic connecting two vertices in \( G \) lies in the minimal face containing both, then we say that \( G \) has **non-leaving-face** property.

The following definition is useful to investigate the non-leaving-face property (cf. [STT88, CP16, Wil17, BZ19]).

**Definition 3.2.** Let \( G \) be an exchange graph and \( F \) a face of \( G \). We say a map \( P : G \rightarrow F \) is a projection if the following conditions satisfied
- (P1) \( P(v) \) is a vertex in \( F \) for each vertex \( v \in G \);
- (P2) \( P(v) = v \) whenever \( v \in F \);
- (P3) \( P \) sends edges in \( G \) to edges or vertices in \( F \). Namely, if \( v \rightarrow w \) is an edge, then either \( P(v) = P(w) \) or \( P(v) \rightarrow P(w) \) is an edge of \( F \);
- (P4) if \( v \rightarrow w \) is an edge in \( G \) such that \( v \in F \) but \( w \not\in F \), then \( P(v) = P(w) \).

The following result is obvious.

**Lemma 3.3.** Let \( G \) be an exchange graph. If there exists a projection for each face \( F \), then \( G \) has non-leaving-face property.

We retain all the notation and conventions of the preceding section. For a cluster \( \mathbf{x}_t \), we denote by
\[
[\mathbf{x}_t] = \{x_{1:t}, \ldots, x_{n,t}\}
\]
the set of cluster variables belonging to \( \mathbf{x}_t \), and we refer to \( [\mathbf{x}_t] \) the non-labelled cluster.

We introduce the following relation \( R_A \) on \( \mathcal{X}_A \):
- two cluster variables \( z_1 \) and \( z_2 \) have the relation \( R_A \) if they belong to a same non-labelled cluster.
According to [CL20, Theorem 6.2, Corollary 7.5], \((\mathcal{X}_A, R_A)\) meets the requirements at the beginning of this section. Hence we have the exchange graph of \((\mathcal{X}_A, R_A)\).

**Definition 3.4.** The exchange graph \(G_A\) of a cluster algebra \(A\) is defined to be the exchange graph of \((\mathcal{X}_A, R_A)\). Namely, it has non-labelled clusters as vertex set and two non-labelled clusters are joined by an edge if and only if their intersection has cardinality \(n - 1\).

**Remark 3.5.** (1) The above definition is different from [FZ02, Definition 4.2] but is equivalent;
(2) This note is devoted to studying all cluster algebras and their exchange graphs. According to [CL20, Proposition 6.1], the exchange graph of a cluster algebra is independent of the choice of its coefficients. Considering the cluster algebra defined in the previous section is sufficient for us.

### 3.2. Main result.
In this section we prove that the exchange graph of any cluster algebra has the non-leaving-face property. Our main idea is to use Bongartz completion to construct the projection needed in Lemma 3.3.

**Definition 3.6 ([CGY21]).** Let \(U\) be a subset of some non-labelled cluster of \(A\). A non-labelled cluster \(B_U(s) := [x_t]\) is called the Bongartz completion of \(U\) with respect to the vertex \(s\) if

- \(U\) is a subset of \([x_t]\);
- The \(i\)-th \(c\)-vector \(c^{B_U}_{t,i; s}\) is a non-negative vector for any \(i\) such that \(x_{i,t} \notin U\).

The following statement was proved recently in [CGY21, Theorem 4.14].

**Theorem 3.7** (existence and uniqueness of Bongartz completion). Let \(U\) be a subset of some non-labelled cluster of \(A\). The Bongartz completion \(B_U(s)\) of \(U\) with respect to the vertex \(s\) exists and is unique.

Let’s first work with the root vertex \(s \in T_n\).

**Lemma 3.8.** If \(U \subseteq [x_s]\), then \(B_U(s) = [x_s]\).

**Proof.** Without loss of generality, we may assume that \(U = \{x_1, \ldots, x_{l,s}\}\). By definition, it suffices to show that the \(i\)-th \(c\)-vector \(c^{B_U}_{t,i; s}\) is a non-negative vector for any \(l + 1 \leq i \leq n\). But this is obvious, since \(C^{B_U}_{s; i} = I_n\).

**Lemma 3.9.** Let \(s \xrightarrow{k} s'\) be an edge connected to the root vertex \(s\). If \(U \subseteq [x_s]\) and \(U \nsubseteq [x_{s'}]\), then \(B_U(s') = [x_s]\).

**Proof.** Without loss of generality, we may assume that \(U = \{x_1, \ldots, x_{l,s}\}\). By definition, it suffices to show that the \(i\)-th \(c\)-vector \(c^{B_U}_{i; s'}\) is a non-negative vector for any \(l + 1 \leq i \leq n\). The condition that \(U \nsubseteq [x_{s'}]\) can be stated as

\[1 \leq k \leq l < i.\]

Remembering the recursion formula (2.3) and \(C^{B_U}_{s,i; s'} = I_n\), we have the following equality

\[c^{B_U}_{t,i; s'} = c^{B_U}_{i; s'} + [b_{ki,s} + c^{B_U}_{k,i; s'} + b_{ki,s'} - c^{B_U}_{k,i; s'}]_{+} = e_i + [b_{ki,s'}]_{+} + e_k,

which is non-negative. As a consequence, \(B_U(s') = [x_s]\).

**Lemma 3.10.** Let \(t \xrightarrow{k} t'\) be an edge of \(T_n\). Then either \(B_U(t) = B_U(t')\) or \(|B_U(t) \cap B_U(t')| = n - 1\).

**Proof.** Without loss of generality, we may assume that \(B_U(t) = [x_v]\) where \(v\) is a vertex in \(T_n\) and \(U = \{x_1, \ldots, x_{l,t}\}\). Let us denote the matrix as \(C^{B_U}_{v,t} = (c_{ij})\). Since \(B_U(t) = [x_v]\), it follows that \(c_{ij} \geq 0\) whenever \(l + 1 \leq j \leq n\). Using Proposition 2.2, we obtain

\[C^{B_U}_{v,t'} = (J_k + [-c_k(C^{B_w}_{t,v})B^w_{1,t}]_{+}^*)C^{B_U}_{v,t}.


More precisely,

\[
C^{B_U; t'}_v = \begin{bmatrix}
1 \\
\vdots \\
[c]_+ & \cdots & -1 & \cdots & [c]_+ \\
\vdots \\
1
\end{bmatrix}
\begin{bmatrix}
c_{11} & \cdots & c_{1k} & \cdots & c_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
c_{k1} & \cdots & c_{kk} & \cdots & c_{kn} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
c_{n1} & \cdots & c_{nk} & \cdots & c_{nn}
\end{bmatrix}
\]

Let \( j \in [l + 1, n] \). First we note that if there exists an \( i \neq k \) such that \( c_{ij} \neq 0 \) (hence \( c_{ij} > 0 \)), then the sign-coherence of \( c \)-vectors implies that the \( j \)-th \( c \)-vector \( c^{B_U; t'}_j \) is a non-negative vector. Since \( C^{B_U; t'}_v \) is invertible over \( \mathbb{Z} \), we deduce that there is at most one \( j \in [l + 1, n] \) such that for any \( i \neq k \), \( c_{ij} = 0 \).

If no such \( j \) exists, then the last \( n - l \) \( c \)-vectors are all non-negative. Hence

\[ B_U(t) = [x_i] = B_U(t'). \]

Otherwise, without loss of generality, we may assume that \( c_{in} = 0 \) for all \( i \neq k \). In this case, \( c_{kn} = 1 \) for the reason that \( C^{B_U; t'}_v \) is invertible over \( \mathbb{Z} \). Now we work with \( v - n \cdot v' \).

Applying (2.3), we conclude that the last \( n - l \) column vectors of \( C^{B_U; t'}_v \) are non-negative. That is \([x_{ij}'] = B_U(t')\) and hence \(|B_U(t) \cap B_U(t')| = n - 1\). \( \square \)

We are now ready to state and prove the main result of this note.

**Theorem 3.11.** Let \( A \) be a cluster algebra. The exchange graph \( G_A \) has the non-leaving-face property.

**Proof.** Let \( U \) be a subset of a non-labelled cluster. Let \( F_U \) be the face of \( G_A \) determined by \( U \). Denote by \( v(G_A) \) (resp. \( v(F_U) \)) the set of vertices of \( G_A \) (resp. \( F_U \)). Theorem 3.7 induces a well-defined map

\[ P_U : v(G_A) \to v(F_U). \]

\[ [x_i] \mapsto B_U(t) \]

Lemma 3.8, 3.9 and 3.10 imply that \( P_U \) extends to a projection \( P_U : G_A \to F_U \). Now the result follows from Lemma 3.3. \( \square \)

**Remark 3.12.** Instead of using Bongartz completion via \( c \)-vectors, Theorem 3.11 can also be achieved by studying Bongartz co-completion via \( g \)-vectors [Cao21]. In particular, the \( g \)-vector versions of Lemma 3.8, 3.9 and 3.10 can be obtained from Remark 5.5, Theorem 5.6, Corollary 5.10 and Theorem 5.11 of [Cao21] along with the \( G \)-systems of cluster algebras.

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