The Automorphism Group of a Self Dual Binary [72,36,16] Code Does Not Contain $Z_4$

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Abstract

It has been proven in a series of works that the order of the automorphism group of a binary [72,36,16] code does not exceed five. We obtain a parametrization of all self-dual binary codes of length 72 with automorphism of order 4 which can be extremal. We use extensive computations in MAGMA and on a supercomputer to show that an extremal binary code of length 72 does not have an element of order 4.

Keywords: Automorphism, extremal code, self dual code.

1 Introduction

A binary self-dual code of length $n$ is doubly-even if the weight of every code vector is a multiple of four. An upper bound $d \leq 4\lfloor n/24 \rfloor + 4$ for the minimum weight $d$ of such code is given in [15]. Self-dual codes achieving this bound are called extremal. The extremal codes of length divisible by 24 are of particular interest. There are unique extremal codes of length 24 and 48 [18], [12]. It is not known if an extremal code of length 72 exists [21].

Let $C$ be a binary extremal self-dual doubly-even code of length 72. One possible way of looking for such code is by assuming that $C$ has nontrivial automorphisms. The automorphism group of $C$ is studied in a series of works. The possible prime divisors of the group are determined in [9]. The prime divisors bigger than 5 are eliminated in [19], [20], [14], and [10]. The automorphism group of $C$ and its possible order is studied in [7], [24], [8], [22], [17], [2], [4], and [3]. As a result it is known that the order of the automorphism group of $C$ is at most 5.

In this note we search for a code $C$ which allows the cyclic group of degree 4, $Z_4$, as an automorphism group. The subcode of $C$ fixed by an automorphism of order 2 corresponds to a self dual [36,18,8] binary code. All such codes are completely classified [1]. Up to equivalence their number is 41. We prove that only three of them correspond to $C$, namely the codes $C_4$, $C_{12}$, and $C_{19}$ from the
Munemasa’s online database of binary self-dual codes [16]. Using some recent results on the module structure of $C$ [17], we obtain that a generator matrix of $C$ depends on 72 binary parameters. Each of these $3(2^{72})$ matrices generates a doubly even self-dual code of length 72 with cyclic automorphism group of order 4. We further decrease the search space by using affine transformations which preserve the fixed subcode that determines $C_4$, $C_{12}$, and $C_{19}$, correspondingly. These transformations act on a certain 27 dimensional subcode of $C$ which matrix depends only on the first 36 parameters. Using computer algebra system Magma [5] and a desktop computer we split these subcodes into orbits and compute the minimum weight of a representative of each orbit. The total number of orbits of weight 16 subcodes is 1558954. Each of them determines $2^{36}$ generator matrices for $C$ which is still a formidable search space. These 1558954($2^{36}$) binary matrices were traversed and checked on the Janus supercomputer at the University of Colorado. Each matrix generated a vector of weight less than 16. Thus we have the following result.

**Theorem 1** The automorphism group of a binary self-dual [72,36,16] code does not have an element of order four.

## 2 Parametrization of $C$

In this section $C \leq \mathbb{F}_2^{72}$ is a self-dual binary code of length 72 with minimum weight 16 and automorphism $g$ of order 4. As $g^2$ has order 2, it does not have fixed points [6]. Hence, $g$ is free of fixed points and we may assume that

$$g = (1, 2, 3, 4)(5, 6, 7, 8) \cdots (69, 70, 71, 72).$$

(1)

It is known [17] that $C$ is a free module over the group ring $R = \mathbb{F}_2 \langle g \rangle$. As the dimension of $R$ as vector space over $\mathbb{F}_2$ is 4, the free $R$-module $C$ has rank 9.

Let

$$v = (v_{1,0}, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,0}, v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{18,0}, v_{18,1}, v_{18,2}, v_{18,3})$$

be a vector from $\mathbb{F}_2^{72}$. We define the maps $\mu : \mathbb{F}_2^{72} \rightarrow R^{18}$ and $\mu' : \mathbb{F}_2^{72} \rightarrow R^{18}$ by

$$\mu(v) = \left(\sum_{i=0}^{3} v_{1,i}g^i, \sum_{i=0}^{3} v_{2,i}g^i, \ldots, \sum_{i=0}^{3} v_{18,i}g^i\right),$$

$$\mu'(v) = \left(\sum_{i=0}^{3} v_{1,i}g^{-i}, \sum_{i=0}^{3} v_{2,i}g^{-i}, \ldots, \sum_{i=0}^{3} v_{18,i}g^{-i}\right).$$

Thus $\mu(C)$ is a code of length 18 over the ring $R$ and has rank 9.

A version of the following lemma for automorphisms of odd prime orders is proved in [23].

**Lemma 2** Let $u$ and $v$ be vectors from $\mathbb{F}_2^{72}$. The vector $u$ is orthogonal to the vectors $g^i v$ for $i = 0, 1, 2, 3$ if and only if $\mu(u)$ and $\mu'(v)$ are orthogonal with respect to the usual inner product in $R^{18}$.

2
Let $h = 1 + g$. Then $h^2 = 1 + g^2$, $h^3 = 1 + g + g^2 + g^3$, $h^4 = 1 + g^4 = 0$, and $1$, $h$, $h^2$, $h^3$ is a basis of $R$ over $\mathbb{F}_2$. Thus, $R = \mathbb{F}_2 + \mathbb{F}_2 h + \mathbb{F}_2 h^2 + \mathbb{F}_2 h^3$. The next obvious lemma is included for convenience in referencing.

**Lemma 3** The $\mathbb{F}_2$-linear transformation on the ring $R$ defined by $g^i \mapsto g^{-i}$, for $i = 1$, 2, 3, maps the basis elements 1, $h$, $h^2$, $h^3$ to 1, $h + h^2 + h^3$, $h^2$, $h^3$, correspondingly.

In the next theorem we obtain further restrictions on the $R$-code $\mu(C)$.

**Theorem 4** Let $C$ be a self-dual doubly-even binary code of length 72 with minimum weight 16 and automorphism $g$ of order 4 given in (7). Let $h = 1 + g$. Up to a column permutation, $\mu(C)$ is a code of length 18 over the ring $R$ with a generator matrix $[I + B_1 h + B_2 h^2 + B_3 h^3 | A]$ where $I$ is the identity matrix and $B_1$, $B_2$, $B_3$, and $A$ are binary square matrices of order 9 satisfying the following requirements:

(i) $A$ is orthogonal ($A^T A = I$ where $A^T$ is the transpose of $A$),
(ii) $B_1$ is symmetric ($B_1^T = B_1$),
(iii) $B_2 + B_2^T = B_1^2 + B_1$,
(iv) $B_3 + B_3^T = B_2 B_1 + B_1 B_2 + B_1^3 + B_1$.

If $B_1$, $B_2$, $B_3$, and $A$ satisfy the above four conditions, then the $\mu$-preimage of an $R$-code with a generator matrix $[I + B_1 h + B_2 h^2 + B_3 h^3 | A]$ is a binary self-dual code of length 72.

**Proof.** As we know $\mu(C)$ is a free $R$ module of rank 9 [17]. Since $h^4 = 0$, $1 + \mathbb{F}_2 h + \mathbb{F}_2 h^2 + \mathbb{F}_2 h^3$ is the set of all units of $R$. It follows that, up to a permutation of columns, the code $\mu(C)$ has a generator matrix

$$[I \mid A + A_1 h + A_2 h^2 + A_3 h^3]$$

where $I$ is the identity matrix of order 9 and $A$, $A_1$, $A_2$, $A_3$ are binary square matrices of order 9. Lemma 2 and Lemma 3 imply

$$I + (A + A_1 h + A_2 h^2 + A_3 h^3) (A^T + A_1^T (h + h^2 + h^3) + A_2^T h^2 + A_3^T h^3) = 0.$$ 

As 1, $h$, $h^2$, $h^3$ are linearly independent over $\mathbb{F}_2$, we obtain $I + AA^T = 0$. Thus $A$ is orthogonal.

Using this we can replace (2) with

$$[I + B_1 h + B_2 h^2 + B_3 h^3 | A].$$

Additional applications of Lemma 2 and Lemma 3 on the matrix (3) give parts (ii), (iii), and (iv). The proof of the second part of the Theorem is straightforward.

Since the matrix $B_2 + B_2^T$ from condition (iii) has zero diagonal, we obtain the following corollary.

**Corollary 5** The matrix $B_1$ from Theorem 4 is such that $B_1^2 + B_1$ has zero diagonal.
Any $R$-code with a generator matrix satisfying the requirements of Theorem 4 corresponds under the map $\mu^{-1}$ to a binary self-dual code of length 72 which is not necessarily doubly-even. Since an extremal code of length 72 is doubly-even, the search space can be restricted. It is done in the next theorem which provides a necessary and sufficient condition for a code to be doubly-even.

We use the following well known Lemma (see [13], page 8) in the proof of Theorem 7.

**Lemma 6** If $u$ and $v$ are binary vectors of the same length, then
\[
wt(u + v) \equiv wt(u) + wt(v) + 2(u, v) \mod 4
\]
where $(u, v)$ is the inner product.

**Theorem 7** Let $D$ be an $R$-code with a generator matrix $[I + B_1 h + B_2 h^2 + B_3 h^3 \mid A]$ satisfying the requirements (i), (ii), (iii), and (iv) of Theorem 4. The corresponding binary code $\mu^{-1}(D)$ of length 72 is doubly-even if and only if
\[
B_3[i, i] \equiv \frac{1 + wt(A[i])}{2} + B_2[i, i] + \sum_{j=1}^{9} (B_1[i, j] + 1)B_2[i, j] \mod 2
\]
for $i = 1, 2, \ldots, 9$ where $A[i]$ is row $i$ of the matrix $A$ and $B_2[i, j]$ is the entry in row $i$ column $j$ of the matrix $B_2$.

**Proof.** Since $h^4 = 1$, the matrices $[Ih + B_1 h^3 + B_2 h^5 \mid Ah], [Ih^2 + B_1 h^3 + Ah^2], \text{and } [Ih^3 + Ah^3]$ generate subcodes of $D$. Replacing $h$ with $1 + g$ and collecting the terms with respect to the powers of $g$, we obtain the matrix
\[
\begin{bmatrix}
I + B_1 + B_2 + B_3 & B_1 + B_3 & B_2 + B_3 & B_3 & A & 0 & 0 & 0 \\
I + B_1 + B_2 & I + B_2 & B_1 + B_2 & B_2 & A & 0 & 0 \\
I + B_1 & B_1 & I + B_1 & B_1 & A & 0 & A & A \\
I & I & I & A & A & A & A
\end{bmatrix}
\]
which generates a code equivalent to $\mu^{-1}(D)$. Hence the code is self-dual by Theorem 4.

Let $w_i$ be the weight of row $i$ of matrix (4). Thus $w_i$ is even for $i = 1, 2, \ldots, 36$. The code is doubly-even if and only if $w_i$ is a multiple of 4. For $i = 1, 2, \ldots, 9$ we have
\[
w_i = wt(I[i]) + B_1[i] + B_2[i] + B_3[i] + wt(B_1[i] + B_3[i]) + wt(B_2[i] + B_3[i]) + wt(A[i]).
\]
Applying Lemma 6 several times, we obtain
\[
w_i \equiv wt(I[i]) + wt(A[i]) + 2wt(B_1[i]) + 2wt(B_2[i]) + 2(I[i], B_1[i]) + 2(I[i], B_2[i]) + 2(I[i], B_3[i]) + 2(B_1[i], B_2[i]) \mod 4.
\]
Since $A$ is orthogonal, $\text{wt}(A[i])$ is odd and

$$
\frac{w_i}{2} = \frac{1 + \text{wt}(A[i])}{2} + \text{wt}(B_1[i]) + \text{wt}(B_2[i]) + B_1[i, i] + B_2[i, i] + B_3[i, i] + (B_1[i], B_2[i]) \mod 2
$$

As $\text{wt}(B_1[i]) \equiv B_1B_1^T[i, i] \mod 2$ and $B_1$ is symmetric, we have

$$
B_1[i, i] + \text{wt}(B_1[i]) \equiv B_1[i, i] + B_2[i, i] \mod 2.
$$

Now Corollary 3 gives $B_1[i, i] + \text{wt}(B_1[i]) \equiv 0 \mod 2$. Hence, $w_i$ is a multiple of 4 if and only if

$$
B_3[i, i] \equiv \frac{1 + \text{wt}(A[i])}{2} + B_2[i, i] + \text{wt}(B_2[i]) + (B_1[i], B_2[i]) \mod 2.
$$

For $i = 10, 11, \ldots, 36$ we check similarly that $w_i$ is always a multiple of 4. 

Now we determine the possible matrices $B_1$ and $A$ for the code $C$. We use the automorphism $g^2 = (1, 3)(2, 4) \cdots (71, 73)(72, 74)$ to define two mappings. Let $C(g^2) = \{ c \in C \mid g^2(c) = c \}$ be the fixed code of $g^2$. The first mapping is $\pi : C(g^2) \rightarrow \mathbb{F}_2^{36}$ defined by

$$
(c_1, c_2, c_1, c_2, \ldots, c_{35}, c_{36}, c_{35}, c_{36}) \mapsto (c_1, c_2, \ldots, c_{36}). \quad (5)
$$

The second mapping is $\Phi : C \rightarrow \mathbb{F}_2^{36}$ defined by

$$
(c_1, c_2, \ldots, c_{72}) \mapsto (c_1 + c_3, c_2 + c_4, \ldots, c_{70} + c_{72}).
$$

It is known that $\pi(C(g^2)) = \Phi(C)$ is a self-dual [36,18,8] binary code. An application of $\Phi$ is the same as identifying coordinate positions 1 and 3, 2 and 4, and so on. This identification makes $g^2$ trivial and $\Phi(C)$ becomes a module over the quotient ring $R/\langle g^2 \rangle \cong \mathbb{F}_2 + \overline{h}\mathbb{F}_2$ where $\overline{h}$ is the coset $h \langle h^2 \rangle$ and $\overline{h^2} = 0$. Thus, $\Phi(C)$ is generated by the matrix $[I + B_1\overline{h}]$ over the ring $\mathbb{F}_2 + \overline{h}\mathbb{F}_2$ and has automorphism $\overline{g} = (1, 2)(3, 4) \cdots (34, 36)$ as a binary code. There are 41 inequivalent self-dual [36,18,8] binary codes [11]. For each of these 41 codes we find the conjugacy classes of automorphisms of order 2 without fixed points and select a representative. For each pair of code and orbit representative we compute the matrix $B_1$. Only for three of the pairs the requirement of Corollary 5 are met. They come from the codes $C_4$, $C_{12}$, and $C_{19}$ from the Munemasa’s online database of binary self-dual codes [16]. We reorder the coordinates of the three codes in such way that $\overline{g} = (1, 2)(3, 4) \cdots (34, 36)$ is the automorphism of order 2 in each of the three pairs. This way we obtain the following lemma.

**Lemma 8** Let $C$ be a self-dual doubly-even binary code of length 72 with minimum weight 16 and automorphism $g$ of order 4 given in [11]. Up to equivalence, the matrices $B_1$ and $A$ from Theorem 4 can be selected as follows:
Each matrix $B_{1}^{(j)}$, $j = 1, 2, 3,$ has zero diagonal. The generator matrix (3) depends on $B_2$ and $B_3$. Multiplying the columns of the matrix (3) by $g^2 = 1 + h^2$ as needed we can make the diagonal of $B_2$ to be zero without changing the entries of $B_{1}^{(j)}$ and $A_{1}^{(j)}$. Condition (iii) of Theorem 4 determines the entries below the diagonal of $B_2$. Hence, $B_2$ depends on 36 parameters. Condition (iv) of Theorem 4 and Theorem 7 determine the entries on and below the diagonal of $B_3$. Thus, $B_3$ depends on 36 parameters. We obtain the following result.

**Corollary 9** An extremal self-dual code of length 72 with an automorphism of order 4 is equivalent to one of the 3 $(2^{72})$ codes determined by the matrix (3), Theorem 4, Theorem 7, and Lemma 8.
3 Further Reduction of the Search Space

Let \( P = \mathbb{F}_2[x_1, x_2, \ldots, x_{72}] \) be the the polynomial ring of the indeterminates \( x_1, x_2, \ldots, x_{72} \) over the binary field \( \mathbb{F}_2 \). Corollary \( \text{[9]} \) shows that the matrices \( B_2 \) and \( B_3 \) are determined by the entries above the diagonal:

\[
\begin{align*}
B_2[1, 2] &= x_1, \quad B_2[1, 3] = x_2, \ldots, \quad B_2[1, 9] = x_9, \\
B_2[2, 3] &= x_9, \ldots, \quad B_2[2, 9] = x_{15}, \ldots, \quad B_2[8, 9] = x_{36}, \\
B_3[1, 2] &= x_{37}, \quad B_3[1, 3] = x_{38}, \ldots, \quad B_3[1, 9] = x_{44}, \\
B_3[2, 3] &= x_{45}, \ldots, \quad B_3[2, 9] = x_{51}, \ldots, \quad B_3[8, 9] = x_{72}.
\end{align*}
\]

As the matrices \( B^{(j)}_1 \) and \( A^{(j)} \), \( j = 1, 2, 3 \), are determined in Lemma \( \text{[8]} \) for any selection of binary values for \( x_1, x_2, \ldots, x_{72} \) the matrix \( \text{[9]} \) determines a doubly even self-dual code \( C \) of length 72. The [36,18,8] code \( \Phi(C) \) has a generator matrix

\[
\begin{bmatrix}
I_2 \otimes I + J_2 \otimes B^{(j)}_1 & I_2 \otimes A^{(j)}
\end{bmatrix}
\]

where \( \otimes \) denotes the Kronecker product, \( I_2 \) and \( J_2 \) are the identity and the all-one matrices of order 2, correspondingly. Let \( G_j \) be the automorphism group of this code, \( j = 1, 2, 3 \). We know that \( \tau \in G_j \).

Using Magma we determine the groups \( G_j \) and \( C_{G_j}(\bar{\tau}) \), the centralizer of \( \bar{\tau} \in G_j \). The order of \( C_{G_j}(\bar{\tau}) \) for \( j = 1, 2, 3 \) is 96, 384, and 96 and the number of generators is 4, 5, and 5, correspondingly. Clearly \( C_{G_j}(\bar{\tau}) \subseteq C_{S_{72}}(\bar{\tau}) \) which is isomorphic to the wreath product \( Z_2 \wr S_{18} \). On the other hand \( C_{S_{72}}(g) \) is isomorphic to \( Z_4 \wr S_{18} \). Any permutation from \( C_{S_{72}}(g) \) maps \( C \) to an equivalent code with automorphism \( g \).

Let \( \bar{\tau} \) be a generator of \( C_{G_j}(\bar{\tau}) \). We lift \( \bar{\tau} \) to \( \tau \in C_{S_{72}}(g) \) having the same permutation part from \( S_{18} \) as \( \bar{\tau} \) such that when \( \tau \) is applied to the matrix

\[
[I + B^{(j)}_1 h + B_2 h^2 + B_3 h^3 \mid A^{(j)}]
\] (6)

the matrices \( I, B^{(j)}_1 \), and \( A^{(j)} \) do not change and \( B_2 \) maps to a matrix \( B_2' \) with zero diagonal. The computations show that the entries above the diagonal of \( B_2 \) are \( X \ast T_{\tau} + v_{\tau} \), where \( X = (x_1, x_2, \ldots, x_{36}) \), \( T_{\tau} \) is a 36x36 nonsingular binary matrix and \( v_{\tau} \) is a binary vector of length 36. The affine transformations \( (T_{\tau}, v_{\tau}) \) generate an affine group \( K_j \) for \( j = 1, 2, 3 \), of order 12288, 49152, 12288, correspondingly. Every binary code given by \( \text{[6]} \) has a subcode of dimension 27 defined by the matrix

\[
[I h + B^{(j)}_1 h^2 + B_2 h^3 \mid A^{(j)} h]
\] (7)

We used computations with Magma on a desktop computer to obtain the next lemma.

**Lemma 10** The number of orbits of weight 16 codes \( \text{[7]} \) under the action of the group \( K_j \), for \( j = 1, 2, 3 \), is 501142, 131840, and 925972, correspondingly.
4 The Test of $1558954(2^{36})$ Codes

The number of orbits from Lemma 10 is $1558954$. For each orbit we select a representative and determine the corresponding matrix (7). As the matrix $B_3$ from (6) depends on 36 binary parameters, the search space contains $1558954(2^{36})$ codes. In order to speed up the computations we use a 36-bit binary reflected Gray code [11] to order the vectors of the 36 dimensional binary vector space in a sequence

$$u^{(0)}, u^{(1)}, u^{(2)}, \ldots, u^{(n-1)}$$

$n = 2^{36}$, such that consecutive vectors $u^{(i-1)}$ and $u^{(i)}$, $i = 1, 2, \ldots, n-1$, differ in exactly one bit in position, say $g(i)$. For a fixed matrix (7), the corresponding $2^{36}$ matrices (8) form a sequence

$$M^{(0)}, M^{(1)}, M^{(2)}, \ldots, M^{(n-1)}$$

such that $M^{(i)} - M^{(i-1)} = D^{(g(i))}$ is one of 36 predetermined mask matrices

$$D^{(1)}, D^{(2)}, \ldots, D^{(36)}.$$ 

Each of the mask matrices has at least 28 zero rows. As a result a low weight vector found in a code with a generator matrix from (8) often belongs to the next several codes.

We carried out these computations on the Janus supercomputer at the University of Colorado Denver. We wrote a computer code in C programming language that was highly optimized for both the task and the Janus hardware. The program employs some of the 64-bit single-cycle bitwise operations of the processors. We used about 6 million CPU core hours over a period of more than three months to find a vector of weight less than 16 in each of the $1558954(2^{36})$ codes from the search space. This completes the proof of the Theorem 1.

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