A GEOMETRIC CONSTRUCTION FOR INVARIANT JET DIFFERENTIALS

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1. Introduction

The action of the reparametrization group $G_k$, consisting of $k$-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, on the bundle $J_k = J_k T^* X$ of $k$-jets at 0 of germs of holomorphic curves $f : \mathbb{C} \to X$ in a complex manifold $X$ has been a focus of investigation since the work of Demailly [5] which built on that of Green and Griffiths [13]. Here $G_k$ is a non-reductive complex algebraic group which is the semi-direct product $G_k = U_k \rtimes \mathbb{C}^*$ of its unipotent radical $U_k$ with $\mathbb{C}^*$; it has the form

$$G_k \cong \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & \cdots \\ 0 & 0 & \alpha_1^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C} \right\}$$

where the entries above the leading diagonal are polynomials in $\alpha_1, \ldots, \alpha_k$, and $U_k$ is the subgroup consisting of matrices of this form with $\alpha_1 = 1$. The bundle of Demailly-Semple jet differentials of order $k$ over $X$ has fibre at $x \in X$ given by the algebra $O((J_k)_x)^{G_k}$ of $U_k$-invariant polynomial functions on the fibre $(J_k)_x = (J_k T^* X)_x$ of $J_k T^* X$. More generally following [25] we can replace $\mathbb{C}$ with $\mathbb{C}^p$ for $p \geq 1$ and consider the bundle $J_{k,p} T^* X$ of $k$-jets at 0 of holomorphic maps $f : \mathbb{C}^p \to X$ and the reparametrization group $G_{k,p}$ consisting of $k$-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$; then $G_{k,p}$ is the semi-direct product of its unipotent radical $U_{k,p}$ and the complex reductive group $GL(p)$, while its subgroup $G'_{k,p} = U_{k,p} \rtimes SL(p)$ (which equals $U_{k,p}$ when $p = 1$) fits into an exact sequence $1 \to G'_{k,p} \to G_{k,p} \to \mathbb{C}^* \to 1$. The generalized Demailly-Semple algebra is then $O((J_{k,p})_x)^{G'_{k,p}}$.

The Demailly-Semple algebras $O(J_k)^{U_k}$ and their generalizations have been studied for a long time. The invariant jet differentials play a crucial role in the strategy devised by Green, Griffiths [13], Bloch [4], Demailly [5, 6], Siu [28, 29, 30] and others to prove Kobayashi’s 1970 hyperbolicity conjecture [19] and the related conjecture of Green and Griffiths in the special case of hypersurfaces in projective space. This strategy has been recently used successfully by Diverio, Merker and Rousseau in [7] and then by the first

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author in [1] to give effective lower bounds for the degrees of generic hypersurfaces in \( \mathbb{P}_n \) for which the Green-Griffiths conjecture holds.

In particular it has been a long-standing problem to determine whether the algebras of invariants \( O((J_{k,p}))_{k,p}^G \) and bi-invariants \( O((J_{k,p}))_{k,p}^{G'\times U_{n,x}} \) (where \( U_{n,x} \) is a maximal unipotent subgroup of \( GL(T,X) \cong GL(n) \)) are finitely generated as graded complex algebras, and if so to provide explicit finite generating sets. In [20] Merker showed that when \( p = 1 \) and both \( k \) and \( n = \dim X \) are small then these algebras are finitely generated, and for \( p = 1 \) and all \( k \) and \( n \) he provided an algorithm which produces finite sets of generators when they exist. In this paper we will describe methods inspired by [2] and the approach of [9] to non-reductive geometric invariant theory (GIT) to prove the finite generation of \( O((J_{k,p}))_{k,p}^G \) for all \( n \) and \( k \geq 2 \) (from which the finite generation of the corresponding bi-invariants follows). We will also give a geometric description of a finite set of generators for \( O((J_{k,p}))_{k,p}^G \), and a geometric description of the associated affine variety

\[
SL(k)/U_k = \text{Spec}(O(SL(k))^{U_k})
\]

which leads to a geometric description of the affine variety

\[
(J_{k,p})_{k} = \text{Spec}(O((J_{k,p}))_{k})^{U_k})
\]

as a GIT quotient

\[
((J_{k,p})_{k} \times (SL(k)/U_k))/SL(k)
\]

by the reductive group \( SL(k) \), in the sense of classical geometric invariant theory [23]. Similarly we expect that if \( p > 1 \) and \( k \) is sufficiently large (depending on \( p \)) then \( G'_{k,p} \) is a subgroup of \( SL(\text{sym}^\leq k(p)) \), where

\[
\text{sym}^\leq k(p) = \sum_{i=1}^{k} \dim \text{Sym}^i \mathbb{C}^p,
\]

such that the algebra \( O^{G'}(\text{sym}^\leq k(p))^{G'_{k,p}} \) is finitely generated, and thus that the algebra \( O((J_{k,p}))_{k,p}^{G'} \) is also finitely generated, and we have a geometric description of the associated affine variety

\[
(J_{k,p})_{k} / G'_{k,p}.
\]

The layout of this paper is as follows. §2 reviews the reparametrization groups \( G_k \) and \( G_{k,p} \) and their actions on jet bundles and jet differentials over a complex manifold \( X \). Next §3 reviews some of the results of [9] on non-reductive geometric invariant theory. In §4 we recall from [2] a geometric description of the quotients by \( U_k \) and \( G_k \) of open subsets of \( (J_k)_{k} \), and in §5 this is used to find explicit affine and projective embeddings of these quotients and explicit embeddings of \( SL(k)/U_k \). In §6 we see that the complement of \( SL(k)/U_k \) in its closure for a suitable embedding in an affine space has codimension at least two. In §7 we conclude that \( U_k \) is a Grosshans subgroup of
SL(k) when $k \geq 2$, so that $O(SL(k))^{U_k}$ and $O(J_k)^{U_k}$ are finitely generated, and provide a geometric description of a finite set of generators of $O(SL(k))^{U_k}$. Finally §8 and §9 discuss how to extend the results of §6 and §7 to the action of $G_{k,p}$ on the jet bundle $J_{k,p} \to X$ of $k$-jets of germs of holomorphic maps from $\mathbb{C}^p$ to $X$ for $p > 1$.

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## 2. Jets of curves and jet differentials

Let $X$ be a complex $n$-dimensional manifold and let $k$ be a positive integer. Green and Griffiths in [13] introduced the bundle $J_k \to X$ of $k$-jets of germs of parametrized curves in $X$; its fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if the derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open neighbourhood $\Omega \subset X$ around $x$, the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order $k$ at $t = 0$ of $\mathbb{C}^n$-valued maps

$$f = (f_1, f_2, \ldots, f_n)$$

on open neighbourhoods of 0 in $\mathbb{C}$. Thus in these coordinates the fibre is

$$J_{k,x} = \{(f'(0), \ldots, f^{(k)}(0)/k!)\} = (\mathbb{C}^n)^k,$$

which we identify with $\mathbb{C}^{nk}$. Note, however, that $J_k$ is not a vector bundle over $X$, since the transition functions are polynomial, but not linear.

Let $G_k$ be the group of $k$-jets at the origin of local reparametrizations of $(\mathbb{C}, 0)$

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^n, \alpha_2, \ldots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms $t^j$ for $j > k$. This group acts fibrewise on $J_k$ by substitution. A short computation shows that this is a linear action on the fibre:

$$f \circ \varphi(t) = f'(0) \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k) + \frac{f''(0)}{2!} \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k)^2 + \ldots$$

$$+ \frac{f^{(k)}(0)}{k!} \cdot (\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k)^k \pmod{t^{k+1}}$$
so the linear action of $\varphi$ on the $k$-jet $(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$
(1) \quad (f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\
0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1} + \cdots + \alpha_{k-1}\alpha_1 \\
0 & 0 & \alpha_3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \cdots \\
0 & 0 & 0 & \cdots & \alpha_1^k
\end{pmatrix}
$$

where the matrix has general entry

$$(G_k)_{i,j} = \sum_{s_1 \geq 1, \ldots, s_i \geq 1, \ s_1 + \cdots + s_i = j} \alpha_{s_1} \cdots \alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$
(2) \quad 1 \to U_k \to G_k \to \mathbb{C}^* \to 1,
$$

where $G_k \to \mathbb{C}^*$ is the morphism $\varphi \to \varphi'(0) = \alpha_1$ in the notation used above, and

$$
G_k = U_k \rtimes \mathbb{C}^*
$$

is a semi-direct product. With the above identification, $\mathbb{C}^*$ is the subgroup of $G_k$ consisting of diagonal matrices satisfying $\alpha_2 = \ldots = \alpha_k = 0$ and $U_k$ is the unipotent radical of $G_k$, consisting of matrices of the form above with $\alpha_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on $k$-jets is thus described by

$$
\lambda \cdot (f''(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) = (\lambda f'(0), \lambda^2 f''(0)/2!, \ldots, \lambda^k f^{(k)}(0)/k!)
$$

Let $E^n_{k,m}$ denote the vector space of complex valued polynomial functions

$$
Q(u_1, u_2, \ldots, u_k)
$$

of $u_1 = (u_{1,1}, \ldots, u_{1,n}), \ldots, u_k = (u_{k,1}, \ldots, u_{k,n})$ of weighted degree $m$ with respect to this $\mathbb{C}^*$ action, where $u_i = f^{(i)}(0)/i!$; that is, such that

$$
Q(\lambda u_1, \lambda^2 u_2, \ldots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \ldots, u_k).
$$

Thus elements of $E^n_{k,m}$ have the form

$$
Q(u_1, u_2, \ldots, u_k) = \sum_{|i_1| + 2|i_2| + \cdots + k|l_k| = m} u_1^{i_1} u_2^{i_2} \cdots u_k^{i_k},
$$

where $i_1 = (i_{1,1}, \ldots, i_{1,n}), \ldots, i_k = (i_{k,1}, \ldots, i_{k,n})$ are multi-indices of length $n$. There is an induced action of $G_k$ on the algebra $\bigoplus_{m \geq 0} E^n_{k,m}$. Following Demailly (see [5]), we denote by $E^n_{k,m}$ (or $E_{k,m}$) the Demailly-Semple bundle whose fibre at $x$ consists of the $U_k$-invariant polynomials on the fibre of $J_k$ at $x$ of weighted degree $m$, i.e those which satisfy

$$
Q((f \circ \varphi)'(0), (f \circ \varphi)''(0)/2!, \ldots, (f \circ \varphi)^{(k)}(0)/k!) = \varphi'(0)^m \cdot Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!),
$$

where $\alpha_i = \alpha_{i,j}$ for $j = 1, \ldots, n$.
and we let \( E^n_k = \oplus_mE^n_{k,m} \) denote the Demailly-Semple bundle of graded algebras of invariants.

We can also consider higher dimensional holomorphic surfaces in \( X \), and therefore we fix a parameter \( 1 \leq p \leq n \), and study germs of maps \( \mathbb{C}^p \to X \).

Again we fix the degree \( k \) of our map, and introduce the bundle \( J^k_{p,x} \) of \( k \)-jets of maps \( \mathbb{C}^p \to X \). The fibre over \( x \in X \) is the set of equivalence classes of germs of \( \mathbb{C} \)-valued maps \( f : (\mathbb{C}^p, 0) \to (X, x) \), with the equivalence relation \( f \sim g \) if and only if all derivatives \( f^{(j)}(0) = g^{(j)}(0) \) are equal for \( 0 \leq j \leq k \).

We need a description of the fibre \( J^k_{p,x} \) in terms of local coordinates as in the case when \( p = 1 \). Let \((z_1, \ldots, z_n)\) be local holomorphic coordinates on an open neighbourhood \( \Omega \subset X \) around \( x \), and let \((u_1, \ldots, u_p)\) be local coordinates on \( \mathbb{C}^p \). The elements of the fibre \( J^k_{p,x} \) are \( \mathbb{C}^n \)-valued maps

\[
f = (f_1, f_2, \ldots, f_n)
\]
on \( \mathbb{C}^p \), and two maps represent the same jet if their Taylor expansions around \( z = 0 \)

\[
f(z) = x + zf'(0) + \frac{z^2}{2!}f''(0) + \ldots + \frac{z^k}{k!}f^{(k)}(0) + O(z^{k+1})
\]

coincide up to order \( k \). Note that here \( f^{(j)}(0) \in \text{Hom} (\text{Sym}^j \mathbb{C}^p, \mathbb{C}^p) \) and in these coordinates the fibre is a finite-dimensional vector space

\[
J_{k,p} = \left\{(f'(0), \ldots, f^{(k)}(0)/k!)\right\} \cong \mathbb{C}^{n(k+1)}/k! \cong \mathbb{C}^{n(k+1)/k!}.
\]

Let \( G_{k,p} \) be the group of \( k \)-jets of germs of biholomorphisms of \( (\mathbb{C}^p, 0) \). Elements of \( G_{k,p} \) are represented by holomorphic maps

\[
(3) \quad u \to \varphi(u) = \Phi_1 u + \Phi_2 u^2 + \ldots + \Phi_k u^k = \sum_{i_1\ldots i_p} a_{i_1\ldots i_p} u_{i_1}^{i_1} \ldots u_{i_p}^{i_p}, \quad \Phi_1 \text{ is non-degenerate}
\]

where \( \Phi_i \in \text{Hom} (\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p) \). The group \( G_{k,p} \) admits a natural fibrewise right action on \( J_{k,p} \), by reparametrizing the \( k \)-jets of holomorphic \( p \)-discs. A computation similar to that in [2] shows that

\[
f \circ \varphi(u) = f'(0)\Phi_1 u + (f'(0)\Phi_2 + \frac{f''(0)}{2!}\Phi_1^2)u^2 + \ldots + \sum_{i_1+\ldots+i_p=d} \frac{f^{(i)}(0)}{i!}\Phi_{i_1} \ldots \Phi_{i_p} u^i.
\]

This defines a linear action of \( G_{k,p} \) on the fibres \( J_{k,p,x} \) of \( J_{k,p} \) with the matrix representation given by
where

- \( \Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p) \) is a \( p \times \text{dim}(\text{Sym}^i \mathbb{C}^p) \)-matrix, the \( i \)th degree component of the map \( \Phi \), which is represented by a map \( (\mathbb{C}^p)^{\otimes i} \to \mathbb{C}^p \);
- \( \Phi_{i_1} \ldots \Phi_{i_k} \) is the matrix of the map \( \text{Sym}^{i_1 + \ldots + i_k}(\mathbb{C}^p) \to \text{Sym}^i \mathbb{C}^p \), which is represented by
  \[
  \sum_{(\sigma \in S_p)} \Phi_{i_1} \otimes \ldots \otimes \Phi_{i_k} : (\mathbb{C}^p)^{\otimes i_1} \otimes \ldots \otimes (\mathbb{C}^p)^{\otimes i_k} \to (\mathbb{C}^p)^{\otimes i};
  \]
- the \((l, m)\) block of \( G_{k,p} \) is \( \sum_{i_1 + \ldots + i_m = l} \Phi_{i_1} \ldots \Phi_{i_m} \). The entries in these boxes are indexed by pairs \( (\tau, \mu) \) where \( \tau \in \binom{p+l-1}{l-1}, \mu \in \binom{p+m-1}{m-1} \) correspond to bases of \( \text{Sym}^i(\mathbb{C}^p) \) and \( \text{Sym}^m(\mathbb{C}^p) \).

**Example 2.1.** For \( p = 2, k = 3 \), using the standard basis \( \{e_i, e_j, e_i e_j : 1 \leq i \leq j \leq 2\} \) of \( (\mathbb{C}^2)^{\otimes 3} \), we get the following \( 9 \times 9 \) matrix for a general element of \( G_{3,2} \):

\[
\begin{pmatrix}
\alpha_{10} & \alpha_{01} & \alpha_{20} & \alpha_{11} & \alpha_{02} & \alpha_{10} & \alpha_{21} & \alpha_{12} & \alpha_{03} \\
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{10} & \beta_{21} & \beta_{12} & \beta_{03} \\
0 & 0 & \alpha_{10} & \alpha_{01} & \alpha_{20} & \alpha_{11} & \alpha_{02} & \alpha_{10} & \alpha_{21} \\
0 & 0 & \alpha_{20} & \alpha_{10} & \alpha_{01} & \alpha_{21} & \alpha_{10} & \alpha_{02} & \alpha_{21} \\
0 & 0 & \alpha_{30} & \alpha_{20} & \alpha_{10} & \alpha_{01} & \alpha_{31} & \alpha_{21} & \alpha_{10} \\
0 & 0 & \alpha_{40} & \alpha_{30} & \alpha_{20} & \alpha_{10} & \alpha_{31} & \alpha_{21} & \alpha_{10} \\
0 & 0 & \alpha_{50} & \alpha_{40} & \alpha_{30} & \alpha_{20} & \alpha_{31} & \alpha_{21} & \alpha_{10} \\
0 & 0 & \alpha_{60} & \alpha_{50} & \alpha_{40} & \alpha_{30} & \alpha_{31} & \alpha_{21} & \alpha_{10} \\
0 & 0 & \alpha_{70} & \alpha_{60} & \alpha_{50} & \alpha_{40} & \alpha_{30} & \alpha_{31} & \alpha_{21} \\
0 & 0 & \alpha_{80} & \alpha_{70} & \alpha_{60} & \alpha_{50} & \alpha_{40} & \alpha_{30} & \alpha_{21} \\
0 & 0 & \alpha_{90} & \alpha_{80} & \alpha_{70} & \alpha_{60} & \alpha_{50} & \alpha_{40} & \alpha_{30} \end{pmatrix}
\]

where

\[
P = \alpha_{10} \beta_{11} + \alpha_{11} \beta_{10} + \alpha_{20} \beta_{01} + \alpha_{01} \beta_{20} \quad \text{and} \quad Q = \alpha_{01} \beta_{11} + \alpha_{11} \beta_{01} + \alpha_{02} \beta_{10} + \alpha_{10} \beta_{02}.
\]

This is a subgroup of the standard parabolic \( P_{2,3,4} \subset GL(9) \). The diagonal blocks are the representations \( \text{Sym}^i \mathbb{C}^2 \) for \( i = 1, 2, 3 \) of \( GL(2) \), where \( \mathbb{C}^2 \) is the standard representation of \( GL(2) \).

In general the linear group \( G_{k,p} \) is generated along its first \( p \) rows; that is, the parameters in the first \( p \) rows are independent, and all the remaining entries are polynomials in these parameters. The assumption on the parameters is that the determinant of the
The parameters in the \((1, m)\) block are indexed by a basis of Sym\(^m(C^p) \times C^p\), so they are of the form \(a^l\), where \(l \in (p+m-1)\) is an \(m\)-tuple and \(1 \leq l \leq p\). An easy computation shows that:

**Proposition 2.2.** The polynomial in the \((l, m)\) block and entry indexed by

\[
\tau = (\tau[1], \ldots, \tau[l]) \in \binom{p+l-1}{l-1}
\]

and \(\nu \in \binom{p+m-1}{m-1}\) is

\[
(G_{k,p})_{\tau,\nu} = \sum_{\nu_1 + \cdots + \nu_l = \nu} a^{(\nu_1)}_{\nu_1} a^{(\nu_2)}_{\nu_2} \cdots a^{(\nu_l)}_{\nu_l}
\]

(6)

Note that \(G_{k,p}\) is an extension of its unipotent radical \(\mathbb{U}_{k,p}\) by \(GL(p)\); that is, we have an exact sequence

\[1 \to \mathbb{U}_{k,p} \to G_{k,p} \to GL(p) \to 1,\]

and \(G_{k,p}\) is the semi-direct product \(\mathbb{U}_{k,p} \rtimes GL(p)\). Here \(G_{k,p}\) has dimension \(p \times \text{sym}^k(p)\) where \(\text{sym}^k(p) = \dim(\otimes_{i=1}^k \text{Sym}^i(C^p))\), and is a subgroup of the standard parabolic subgroup \(P_{p,\text{sym}^1(p), \ldots, \text{sym}^k(p)}\) of \(GL(\text{sym}^k(p))\) where \(\text{sym}^i(p) = \dim(\text{Sym}^i(C^p))\). We define \(G'_{k,p}\) to be the subgroup of \(G_{k,p}\) which is the semi-direct product

\[G'_{k,p} = \mathbb{U}_{k,p} \rtimes SL(p)\]

(so that \(G'_{k,p} = G_{k,p}\) when \(p = 1\)) fitting into the exact sequence

\[1 \to \mathbb{U}_{k,p} \to G'_{k,p} \to SL(p) \to 1.\]

The action of the maximal torus \((C^*)^p \subset GL(p)\) of the Levi subgroup of \(G_{k,p}\) is

\[\lambda_1, \ldots, \lambda_p \cdot f^{(i)} = (\lambda_1^{\partial f_i} \ldots \lambda_p^{\partial f_i}) \cdot f^{(i)}\]

(7)

We introduce the *Green-Griffiths* vector bundle \(E_{k,p,m}^{GG} \to X\), whose fibres are complex-valued polynomials

\[Q(f^{(0)}, f^{(2)!}, \ldots, f^{(k)}) = \lambda_1 \cdot \lambda_p Q(f^{(0)}, f^{(2)!}, \ldots, f^{(k)})\]

on the fibres of \(J_{k,p}\), having weighted degree \((m, \ldots, m)\) with respect to the action (7) of \((C^*)^p\). That is, for \(Q \in E_{k,p,m}^{GG}\)

\[Q(\lambda f^{(0)}, \lambda f^{(2)!}, \ldots, \lambda f^{(k)}) = \lambda_1^m \cdots \lambda_p^m Q(f^{(0)}, f^{(2)!}, \ldots, f^{(k)})\]

for all \(\lambda \in \mathbb{C}^p\) and \((f^{(0)}, f^{(2)!}, \ldots, f^{(k)}) \in J_{k,p,m}\).
**Definition 2.3.** The generalized Demailly-Semple bundle $E_{k,p,m} \to X$ over $X$ has fibre consisting of the $G_{k,p}$-invariant jet differentials of order $k$ and weighted degree $(m, \ldots, m)$; that is, the complex-valued polynomials $Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$ on the fibres of $J_{k,p}$ which transform under any reparametrization $\phi \in G_{k,p}$ of $(\mathbb{C}^p, 0)$ as 

$$Q(f \circ \phi) = (J_{\phi})^m Q(f) \circ \phi,$$

where $J_{\phi} = \det \Phi_1$ denotes the Jacobian of $\phi$ at 0. The generalized Demailly-Semple bundle of algebras $E_{k,p} = \oplus_{m \geq 0} E_{k,p,m}$ is the associated graded algebra of $G_{k,p}$-invariants, whose fibre at $x \in X$ is the generalized Demailly-Semple algebra $O((J_{k,p})_x)^{G_{k,p}}$.

The determination of a suitable generating set for the invariant jet differentials when $p = 1$ is important in the longstanding strategy to prove the Green-Griess conjecture. It has been suggested in a series of papers [13, 5, 27, 20, 7, 21] that the Schur decomposition of the Demailly-Semple algebra, together with good estimates of the higher Betti numbers of the Schur bundles and an asymptotic estimation of the Euler characteristic, should result in a positive lower bound for the global sections of the Demailly-Semple jet differential bundle.

3. Geometric invariant theory

Suppose now that $Y$ is a complex quasi-projective variety on which a linear algebraic group $G$ acts. For geometric invariant theory (GIT) we need a linearization of the action; that is, a line bundle $L$ on $Y$ and a lift $L$ of the action of $G$ to $L$. Usually $L$ is ample, and hence (as it makes no difference for GIT if we replace $L$ with $L \otimes k$ for any integer $k > 0$) we can assume that for some projective embedding $Y \subseteq \mathbb{P}^n$ the action of $G$ on $Y$ extends to an action on $\mathbb{P}^n$ given by a representation $\rho : G \to GL(n + 1)$, and take for $L$ the hyperplane line bundle on $\mathbb{P}^n$.

For classical GIT developed by Mumford [23] (cf. also [8, 22, 24, 26]) we require the complex algebraic group $G$ to be reductive. Let $Y$ be a projective complex variety with an action of a complex reductive group $G$ and linearization $L$ with respect to an ample line bundle $L$ on $Y$. Then $y \in Y$ is semistable for this linear action if there exists some $m > 0$ and $f \in H^0(Y, L^\otimes m)^G$ not vanishing at $y$, and $y$ is stable if also the action of $G$ on the open subset

$$Y_f := \{ x \in Y \mid f(x) \neq 0 \}$$

is closed with all stabilizers finite. $Y_{ss}$ has a projective categorical quotient $Y_{ss} \to Y/G$, which restricts on the set of stable points to a geometric quotient $Y^s \to Y^s/G$ (see [23, Theorem 1.10]). The morphism $Y_{ss} \to Y/G$ is surjective, and identifies $x, y \in Y_{ss}$ if and only if the closures of the $G$-orbits of $x$ and $y$ meet in $Y_{ss}$; moreover each point in $Y/G$ is represented by a unique closed $G$-orbit in $Y_{ss}$. There is an induced action of $G$ on the homogeneous coordinate ring

$$\hat{O}_L(Y) = \bigoplus_{k \geq 0} H^0(Y, L^\otimes k)$$
of \( Y \). The subring \( \hat{\mathcal{O}}_L(Y)^G \) consisting of the elements of \( \hat{\mathcal{O}}_L(Y) \) left invariant by \( G \) is a finitely generated graded complex algebra because \( G \) is reductive, and the GIT quotient \( Y//G \) is the projective variety \( \text{Proj}(\hat{\mathcal{O}}_L(Y)^G) \) \cite{23}. The subsets \( Y^{ss} \) and \( Y^s \) of \( Y \) are characterized by the following properties (see \cite[Chapter 2]{23} or \cite{24}).

**Proposition 3.1.** (Hilbert-Mumford criteria) (i) A point \( x \in Y \) is semistable (respectively stable) for the action of \( G \) on \( Y \) if and only if for every \( g \in G \) the point \( gx \) is semistable (respectively stable) for the action of a fixed maximal torus of \( G \).

(ii) A point \( x \in Y \) with homogeneous coordinates \([x_0 : \ldots : x_n]\) in some coordinate system on \( \mathbb{P}^n \) is semistable (respectively stable) for the action of a maximal torus of \( G \) acting diagonally on \( \mathbb{P}^n \) with weights \( \alpha_0, \ldots, \alpha_n \) if and only if the convex hull

\[
\text{Conv}\{\alpha_i : x_i \neq 0\}
\]

contains 0 (respectively contains 0 in its interior).

Similarly if a complex reductive group \( G \) acts linearly on an affine variety \( Y \) then we have a GIT quotient

\[
Y//G = \text{Spec}(\mathcal{O}(Y)^G)
\]

which is the affine variety associated to the finitely generated algebra \( \mathcal{O}(Y)^G \) of \( G \)-invariant regular functions on \( Y \). In this case \( Y^{ss} = Y \) and the inclusion \( \mathcal{O}(Y)^G \hookrightarrow \mathcal{O}(Y) \) induces a morphism of affine varieties \( Y \to Y//G \).

Now suppose that \( H \) is any complex linear algebraic group, with unipotent radical \( U \leq H \) (so that \( R = H/U \) is reductive and \( H \) is isomorphic to the semi-direct product \( U \rtimes R \)), acting linearly on a complex projective variety \( Y \) with respect to an ample line bundle \( L \). Then \( \text{Proj}(\hat{\mathcal{O}}_L(Y)^H) \) is not in general well-defined as a projective variety, since the ring of invariants

\[
\hat{\mathcal{O}}_L(Y)^H = \bigoplus_{k \geq 0} H^0(Y, L^\otimes k)^H
\]

is not necessarily finitely generated as a graded complex algebra, and so it is not obvious how GIT might be generalised to this situation (cf. \cite{9, 11, 10, 14, 15, 18}). However in some cases it is known that \( \hat{\mathcal{O}}_L(Y)^U \) is finitely generated, which implies that

\[
\hat{\mathcal{O}}_L(Y)^H = \left( \bigoplus_{k \geq 0} H^0(Y, L^\otimes k)^U \right)^{H/U}
\]

is finitely generated and hence the **enveloping quotient** in the sense of \cite{9} is given by the associated projective variety

\[
Y//H = \text{Proj}(\hat{\mathcal{O}}_L(Y)^H).
\]

Similarly if \( Y \) is affine and \( H \) acts linearly on \( Y \) with \( \mathcal{O}(Y)^H \) finitely generated, then we have the enveloping quotient

\[
Y//H = \text{Spec}(\mathcal{O}(Y)^H).
\]
There is a morphism

\[ q : Y^{ss} \to Y//H, \]

from an open subset \( Y^{ss} \) of \( Y \) (where \( Y^{ss} = Y \) when \( Y \) is affine), which restricts to a geometric quotient

\[ q : Y^s \to Y^s/H \]

for an open subset \( Y^s \subset Y^{ss} \). However in contrast with the reductive case, the morphism \( q : Y^{ss} \to Y//H \) is not in general surjective; indeed the image of \( q \) is not in general a subvariety of \( Y//H \), but is only a constructible subset.

If there is a complex reductive group \( G \) containing the unipotent radical \( U \) of \( H \) such that the algebra \( O(G)^U \) is finitely generated and the action of \( U \) on \( Y \) extends to a linear action of \( G \), then

\[ O(Y)^U \cong (O(Y) \otimes O(G)^U)^G \]

is finitely generated and hence so is

\[ O(Y)^H = (O(Y)^U)^{H/U} \]

(or if \( Y \) is projective with an ample linearisation \( L \) then \( \hat{O}_L(Y)^U \) is finitely generated and hence so is \( \hat{O}_L(Y)^H \)). In this situation we say that \( U \) is a Grosshans subgroup of \( G \) (cf. [16, 17]). Then geometrically \( G/U \) is a quasi-affine variety with \( O(G/U) \cong O(G)^U \), and it has a canonical affine embedding as an open subvariety of the affine variety

\[ G//U = \text{Spec}(O(G)^U) \]

with complement of codimension at least two. Moreover if a linear action of \( U \) on an affine variety \( Y \) extends to a linear action of \( G \) then

\[ Y//U \cong (Y \times G//U)//G \]

(and a corresponding result is true if \( Y \) is projective). Conversely if we can find an embedding of \( G/U \) as an open subvariety of an affine variety \( Z \) with complement of codimension at least two, then

\[ O(G)^U \cong O(Z) \]

is finitely generated and \( G//U \cong Z \).

Suppose that \( U \) is a unipotent group with a reductive group \( R \) of automorphisms of \( U \) given by a homomorphism \( \phi : R \to \text{Aut}(U) \) such that \( R \) contains a central one-parameter subgroup \( \lambda : \mathbb{C}^* \to R \) for which the weights of the induced \( \mathbb{C}^* \) action on the Lie algebra \( u \) of \( U \) are all nonzero. Then we can form the semi-direct product

\[ \hat{U} = \mathbb{C}^* \ltimes U \subseteq R \ltimes U \]

given by \( \mathbb{C}^* \times U \) with group multiplication

\[ (z_1, u_1)(z_2, u_2) = (z_1z_2, (\lambda(z_2^{-1})(u_1))u_2). \]

The groups \( \mathbb{G}_k = U_k \ltimes \mathbb{C}^* \) and \( \mathbb{G}_{k,p} = U_{k,p} \ltimes \text{GL}(p) \) which act on the fibres of the jet bundles \( J_k \) and \( J_{k,p} \) are of this form. We will use this structure to study the Demailly-Semple algebras of invariant jet differentials \( E^n_k \) and \( E^n_{k,p} \) and prove
Theorem 3.2. The fibres $O((J_k)_x)^{U_k}$ and $O((J_{k,p})_x)^{\mathbb{G}_k,p}$ of the bundles $E^G_k$ and $E^G_{k,p}$ are finitely generated graded complex algebras.

Thus we have non-reductive GIT quotients
$$(J_k)_x//U_k = \text{Spec}(O((J_k)_x)^{U_k})$$
and
$$(J_{k,p})_x//\mathbb{G}_{k,p} = \text{Spec}(O((J_{k,p})_x)^{\mathbb{G}_{k,p}})$$
and we would like to understand them geometrically. There is a crucial difference here from the case of reductive group actions, even though the invariants are finitely generated: when $H$ is a non-reductive group we cannot describe $Y//H$ geometrically as $Y^{ss}$ modulo some equivalence relation. Instead our aim is to use methods inspired by [2] to study these geometric invariant theoretic quotients and the associated algebras of invariants.

Here a crucial ingredient would be to find an open subset $W$ of $(J_{k,p})_x$ with a geometric quotient $W/\mathbb{G}_{k,p}$ embedded as an open subset of an affine variety $Z$ such that the complement of $W/\mathbb{G}_{k,p}$ in $Z$ has (complex) codimension at least two, and the complement of $W$ in $(J_{k,p})_x$ has codimension at least two. For then we would have
$$O((J_{k,p})_x) = O(W)$$
and
$$O((J_{k,p})_x)^{\mathbb{G}_{k,p}} = O(W)^{\mathbb{G}_{k,p}} = O(W/\mathbb{G}_{k,p}) = O(Z),$$
and it follows that $O((J_{k,p})_x)^{\mathbb{G}_{k,p}}$ is finitely generated since $Z$ is affine, and that
$$Z = \text{Spec}(O(Z)) = \text{Spec}(O((J_{k,p})_x)^{\mathbb{G}_{k,p}}) = ((J_{k,p})_x)//\mathbb{G}_{k,p}.$$  
Similarly if we can find a complex reductive group $G$ containing $\mathbb{G}_{k,p}$ as a subgroup, and an embedding of $G/\mathbb{G}_{k,p}$ as an open subset of an affine variety $Z$ with complement of codimension at least two, then $O(G)^{\mathbb{G}_{k,p}}$ is finitely generated. It follows as above that if $Y$ is any affine variety on which $G$ acts linearly then
$$O(Y)^{\mathbb{G}_{k,p}} \cong (O(Y) \otimes O(G)^{\mathbb{G}_{k,p}})^G$$
is finitely generated, and hence so is $O(Y)^{\mathbb{G}_{k,p}} = (O(Y)^{\mathbb{G}_{k,p}})^C$, and similarly $\hat{O}_L(Y)^{\mathbb{G}_{k,p}}$ and $\hat{O}_L(Y)^{\mathbb{G}_{k,p}}$ are finitely generated if $Y$ is any projective variety with an ample line bundle $L$ on which $G$ acts linearly.

We can use the ideas of [2] to look for suitable affine varieties $Z$ as above, and in particular to prove

Theorem 3.3. $\mathbb{G}_{k,p}'$ is a subgroup of the special linear group $\text{SL}(\text{sym}^{\leq k} p)$ where
$$\text{sym}^{\leq k} p = \sum_{i=1}^{k} \text{dim Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1}$$
such that the algebra of invariants $O(\text{SL}(\text{sym}^k p))^{G_k}$ is finitely generated, and every linear action of $G_{k,p}$ or $G_{k}$ on an affine or projective variety (with an ample linearisation) which extends to a linear action of $\text{GL}(\text{sym}^k p)$ has finitely generated invariants.

Theorem 3.2 is an immediate consequence of this theorem, since the action of $G_{k,p}$ on $(J_{k,p})_s$ extends to an action of the general linear group $\text{GL}(\text{sym}^k p)$. Moreover we will find a geometric description of

$$\text{SL}(\text{sym}^k p)//G'_{k,p} \cong \text{Spec}(O(\text{SL}(\text{sym}^k p))^{G_{k,p}})$$

and thus a geometric description of

$$(J_{k,p})_s//G'_{k,p} \cong ((J_{k,p})_s \times \text{SL}(\text{sym}^k p)//G'_{k,p})//\text{SL}(\text{sym}^k p).$$

4. A description via test curves

In [2] the action of $G_k$ on jet bundles is studied using an idea coming from global singularity theory. The construction goes as follows.

If $u, v$ are positive integers, let $J_k(u, v)$ denote the vector space of $k$-jets of holomorphic maps $(\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \ldots, k$.

With this notation, the fibres of $J_k$ are isomorphic to $J_k(1, n)$, and the group $G_k$ is simply $J_k(1, 1)$ with the composition action on itself.

If we fix local coordinates $z_1, \ldots, z_u$ at $0 \in \mathbb{C}^u$ we can again identify the $k$-jet of $f$, using derivatives at the origin, with $(f^{(0)}(0), f^{(0)}(0)/2!, \ldots, f^{(k)}(0)/k!)$, where $f^{(j)}(0) \in \text{Hom}(\text{Sym}^j \mathbb{C}^u, \mathbb{C}^v)$. This way we get an identification

$$J_k(u, v) = \oplus_{j=1}^k \text{Hom}(\text{Sym}^j \mathbb{C}^u, \mathbb{C}^v).$$

We can compose map-jets via substitution and elimination of terms of degree greater than $k$; this leads to the composition maps

$$J_k(v, w) \times J_k(u, v) \rightarrow J_k(u, w), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1 \text{modulo terms of degree } > k.$$

When $k = 1$, $J_1(u, v)$ may be identified with $u$-by-$v$ matrices, and (8) reduces to multiplication of matrices.

The $k$-jet of a curve $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^u, 0)$ is simply an element of $J_k(1, n)$. We call such a curve $\varphi$ regular if $\varphi'(0) \neq 0$. Let us introduce the notation $J_k^{\text{reg}}(1, n)$ for the set of regular curves:

$$J_k^{\text{reg}}(1, n) = \{ \gamma \in J_k(1, n); \gamma'(0) \neq 0 \}.$$

Note that if $n > 1$ then the complement of $J_k^{\text{reg}}(1, n)$ in $J_k(1, n)$ has codimension at least two. Let $N \geq n$ be any integer and define

$$\Upsilon_n = \{ \Psi \in J_k(n, N) : \exists \gamma \in J_k^{\text{reg}}(1, n) : \Psi \circ \gamma = 0 \}$$

to be the set of those $k$-jets which take at least one regular curve to zero. By definition, $\Upsilon_n$ is the image of the closed subvariety of $J_k(n, N) \times J_k^{\text{reg}}(1, n)$ defined by the algebraic
equations \( \Psi \circ \gamma = 0 \), under the projection to the first factor. If \( \Psi \circ \gamma = 0 \), we call \( \gamma \) a test curve of \( \Psi \).

This term originally comes from global singularity theory, where this is called the test curve model of \( A_k \)-singularities. In global singularity theory singularities of polynomial maps \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0) \) are classified by their local algebras, and

\[
\Sigma_k = \{ f \in J_k(n, m) : \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m \rangle = \mathbb{C}[t]/t^{k+1} \}
\]

is called a Morin singularity, or \( A_k \)-singularity. The test curve model of Gaffney \cite{12} tells us that

\[
\Sigma_k = \overline{T_k}
\]

in \( J_k(n, m) \).

A basic but crucial observation is the following. If \( \gamma \) is a test curve of \( \Psi \in \Upsilon_k \), and \( \varphi \in J_k^{\text{reg}}(1, 1) = G_k \) is a holomorphic reparametrization of \( \mathbb{C} \), then \( \gamma \circ \varphi \) is, again, a test curve of \( \Psi \):

\[
\mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N
\]

\( \Psi \circ \gamma = 0 \Rightarrow \Psi \circ (\gamma \circ \varphi) = 0. \)

In fact, we get all test curves of \( \Psi \) in this way from a single \( \gamma \) if the following open dense property holds: the linear part of \( \Psi \) has 1-dimensional kernel. Before stating this more precisely in Proposition 4.3 below, let us write down the equation \( \Psi \circ \gamma = 0 \) in coordinates in an illustrative case. Let \( \gamma = (\gamma', \gamma'', \ldots, \gamma(k)) \in J_k^{\text{reg}}(1, n) \) and \( \Psi = (\Psi', \Psi'', \ldots, \Psi(k)) \in J_k(n, N) \) be the \( k \)-jets. Using the chain rule, the equation \( \Psi \circ \gamma = 0 \) reads as follows for \( k = 4 \):

\[
\Psi'(\gamma') = 0, \\
\frac{1}{2!} \Psi'(\gamma'') + \Psi''(\gamma', \gamma') = 0, \\
\frac{1}{3!} \Psi'(\gamma''') + \frac{3}{2!} \Psi''(\gamma', \gamma'') + \Psi'''(\gamma', \gamma', \gamma') = 0, \\
\frac{1}{4!} \Psi'(\gamma'''') + \frac{3}{3!} \Psi''(\gamma'', \gamma'') + \frac{3}{2!} \Psi'''(\gamma', \gamma'', \gamma'') + \Psi''''(\gamma'', \gamma', \gamma', \gamma') = 0.
\]

**Definition 4.1.** To simplify our formulas we introduce the following notation for a partition \( \tau = [i_1 \ldots i_l] \) of the integer \( i_1 + \ldots + i_l \):

- the length: \( |\tau| = l \),
- the sum: \( \sum \tau = i_1 + \ldots + i_l \),
- the number of permutations: \( \text{perm}(\tau) \) is the number of different sequences consisting of the numbers \( i_1, \ldots, i_l \) (e.g. \( \text{perm}([1, 1, 1, 3]) = 4 \)),
- \( \gamma_\tau = \prod_{j=1}^{l} \gamma^{(i_j)} \in \text{Sym}^l \mathbb{C}^n \) and \( \Psi(\gamma_\tau) = \Psi(\gamma^{(i_1)}, \ldots, \gamma^{(i_l)}) \in \mathbb{C}^N \).

**Lemma 4.2.** Let \( \gamma = (\gamma', \gamma'', \ldots, \gamma(k)) \in J_k^{\text{reg}}(1, n) \) and \( \Psi = (\Psi', \Psi'', \ldots, \Psi(k)) \in J_k(n, N) \) be \( k \)-jets. Then the equation \( \Psi \circ \gamma = 0 \) is equivalent to the following system of \( k \) linear
equations with values in $\mathbb{C}^N$:

$$\sum_{\tau \in \Pi[m]} \frac{\text{perm}(\tau)}{\prod_{i \in \tau} i!} \Psi(\gamma_\tau) = 0, \quad m = 1, 2, \ldots, k,$$

where $\Pi[m]$ denotes the set of all partitions of $m$.

For a given $\gamma \in J_k^{reg}(1, n)$ let $S_\gamma$ denote the set of solutions of (11); that is,

$$S_\gamma = \{ \Psi \in J_k(n, N); \Psi \circ \gamma = 0 \}.$$

The equations (11) are linear in $\Psi$, hence $S_\gamma \subset J_k(n, N)$ is a linear subspace of codimension $kN$. Moreover, the following holds:

**Proposition 4.3.** ([2], Proposition 4.4)

(i) For $\gamma \in J_k^{reg}(1, n)$, the set of solutions $S_\gamma \subset J_k(n, N)$ is a linear subspace of codimension $kN$.

(ii) Set

$$J_k^0(n, N) = \{ \Psi \in J_k(n, N) | \dim \ker(\Psi') = 1 \}.$$

For any $\gamma \in J_k^{reg}(1, n)$, the subset $S_\gamma \cap J_k^0(n, N)$ of $S_\gamma$ is dense.

(iii) If $\Psi \in J_k^0(n, N)$, then $\Psi$ belongs to at most one of the spaces $S_\gamma$. More precisely, if $\gamma_1, \gamma_2 \in J_k^{reg}(1, n)$, $\Psi \in J_k^0(n, N)$ and $\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0$,

then there exists $\varphi \in J_k^{reg}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

(iv) Given $\gamma_1, \gamma_2 \in J_k^{reg}(1, n)$, we have $S_{\gamma_1} = S_{\gamma_2}$ if and only if there is some $\varphi \in J_k^{reg}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

By the second part of Proposition 4.3 we have a well-defined map

$$\nu : J_k^{reg}(1, n) \rightarrow \text{Grass}(\text{codim} = kN, J_k(n, N)), \quad \gamma \mapsto S_\gamma$$

to the Grassmannian of codimension-$kN$ subspaces in $J_k(n, N)$. From the last part of Proposition 4.3 it follows that:

**Proposition 4.4.** ([2]) $\nu$ is $\mathbb{G}_k$-invariant on the $J_k^{reg}(1, 1)$-orbits, and the induced map on the orbits

$$\bar{\nu} : J_k^{reg}(1, n)/\mathbb{G}_k \rightarrow \text{Grass}(\text{codim} = kN, J_k(n, N))$$

is injective.
5. Embedding into the flag of equations

In this section we will recast the embedding (12) of \( J_k^{reg}(1, n) / \mathbb{G}_k \) given by Proposition 4.3 into a more useful form, still following [2]. Let us rewrite the linear system \( \Psi \circ \gamma = 0 \) associated to \( \gamma \in J_k^{reg}(1, n) \) in a dual form. The system is based on the standard composition map (8):

\[
J_k(n, N) \times J_k(1, n) \rightarrow J_k(1, N),
\]

which, via the identification via tensoring with \( \mathbb{C}^N \), may be written out explicitly as follows

\[
\text{passing to linear duals, we may rewrite this correspondence in the form}
\]

\[
J_k(n, 1) \times J_k(1, n) \rightarrow J_k(1, 1)
\]

via tensoring with \( \mathbb{C}^N \). Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

\[
\phi : J_k(1, n) \rightarrow \text{Hom}(J_k(1, 1)^*, J_k(n, 1)^*).
\]

If \( \gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k(1, n) = (\mathbb{C}^n)^k \) is the \( k \)-jet of a curve, we can put \( \gamma'^j \in \mathbb{C}^n \) into the \( j \)th column of an \( n \times k \) matrix, and

- identify \( J_k(1, n) \) with Hom(\( \mathbb{C}^k, \mathbb{C}^n \));
- identify \( J_k(n, 1)^* \) with Sym\( \mathbb{C}^n \);
- identify \( J_k(1, 1)^* \) with \( \mathbb{C}^k \).

Using these identifications, we can recast the map \( \phi \) in (13) as

\[
\phi : \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \rightarrow \text{Hom}(\mathbb{C}^k, \text{Sym}^k \mathbb{C}^n),
\]

which may be written out explicitly as follows

\[
(\gamma', \gamma'', \ldots, \gamma^{(k)}) \mapsto \left( \gamma'^j, \gamma'^j + \left( \sum_{i_1 + i_2 + \cdots + i_d = d} \frac{1}{i_1! \cdots i_d!} \gamma'^{i_1 j_1} \gamma'^{i_2 j_2} \cdots \gamma'^{i_d j_d} \right) \right).
\]

The set of solutions \( S_\gamma \) is the linear subspace orthogonal to the image of \( \phi_k(\gamma', \ldots, \gamma^{(k)}) \) tensored by \( \mathbb{C}^N \), that is,

\[
S_\gamma = \text{im}(\phi_k(\gamma))^\perp \otimes \mathbb{C}^N \subset J_k(n, N).
\]

Consequently, it is straightforward to take \( N = 1 \) and define

\[
S_\gamma = \text{im}(\phi_k(\gamma)) \in \text{Grass}(k, \text{Sym}^k \mathbb{C}^n).
\]

Moreover, let \( B_k \subset GL(k) \) denote the Borel subgroup consisting of upper triangular matrices and let

\[
\text{Flag}_k(\mathbb{C}^n) = \text{Hom}(\mathbb{C}^k, \text{Sym}^k \mathbb{C}^n) / B_k = \{0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n, \text{dim } F_l = l\}
\]

denote the full flag of \( k \)-dimensional subspaces of \( \text{Sym}^k \mathbb{C}^n \). In addition to (15) we can analogously define

\[
\mathcal{T}_\gamma = (\text{im}(\phi(\gamma^1)) \subset \text{im}(\phi(\gamma^2)) \subset \cdots \subset \text{im}(\phi(\gamma^d))) \in \text{Flag}_k(\text{Sym}^k \mathbb{C}^n).
\]

Using these definitions Proposition 4.3 implies the following version of Proposition 4.4, which does not contain the parameter \( N \).
Proposition 5.1. The map $\phi$ in (14) is a $G_k$-invariant algebraic morphism
$$\phi: J_k^{\text{reg}}(1, n) \to \text{Hom}(\mathbb{C}^k, \text{Sym}^{\leq k} \mathbb{C}^n),$$
which induces
- an injective map on the $G_k$-orbits to the Grassmannian:
  $$\phi^{\text{Gr}}: J_k^{\text{reg}}(1, n)/G_k \hookrightarrow \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n),$$
defined by $\phi^{\text{Gr}}(\gamma) = S_{\gamma};$
- an injective map on the $G_k$-orbits to the flag manifold:
  $$\phi^{\text{Flag}}: J_k^{\text{reg}}(1, n)/G_k \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n),$$
defined by $\phi^{\text{Flag}}(\gamma) = F_{\gamma}.$

In addition,
$$\phi^{\text{Gr}} = \phi^{\text{Flag}} \circ \pi_k,$$
where $\pi_k : \text{Flag}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \to \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ is the projection to the $k$-dimensional subspace.

Composing $\phi^{\text{Gr}}$ with the Plücker embedding
$$\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n),$$
we get an embedding
(17) $$\phi^{\text{Proj}}: J_k^{\text{reg}}(1, n)/G_k \hookrightarrow \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^n)).$$
The image
$$\phi^{\text{Gr}}(J_k^{\text{reg}}(1, n))/G_k \subset \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$$
is a $GL(n)$-orbit in $\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$, and therefore a nonsingular quasi-projective variety. Its closure is, however, a highly singular subvariety of $\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$, which when $k \leq n$ is a finite union of $GL(n)$ orbits.

Definition 5.2. Recall that we can identify $J_k(1, n)$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and then
$$J_k^{\text{reg}}(1, n) = \{\rho \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \rho(e_1) \neq 0\}.$$ 

Let
$$J_k^{\text{nondeg}}(1, n) = \{\rho \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \text{rank}\rho = \max\{k, n\}\},$$
and let
$$X_{n, k} = \phi^{\text{Proj}}(J_k^{\text{nondeg}}(1, n)), \ Y_{n, k} = \phi^{\text{Proj}}(J_k^{\text{reg}}(1, n)),$$
so that if $n \leq k$ then
$$X_{n, k} \subset Y_{n, k} \subset \text{Grass}(n, \text{Sym}^{\leq k} \mathbb{C}^n) \subset \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^n)).$$

It is clear that $J_k^{\text{nondeg}}(1, n)$ is an open subset of $J_k^{\text{reg}}(1, n)$. If we identify the elements of $J_k(1, n)$ with $n \times k$ matrices whose columns are the derivatives of the map germs $f = (f', \ldots, f^{(n)}) : \mathbb{C} \to \mathbb{C}^n$, then $J_k^{\text{nondeg}}(1, n)$ is the set of such matrices of maximal rank and $J_k^{\text{reg}}(1, n)$ consists of the matrices with nonzero first column.
Definition 5.3. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$; then

$$\{e_{i_1, i_2, \ldots, i_s} = e_{i_1} \cdot \ldots \cdot e_{i_s} : 1 \leq i_1 \leq \ldots \leq i_s \leq n, 1 \leq s \leq k\}$$

is a basis of $\text{Sym}^k \mathbb{C}^n$, and

$$\{e_{e_1} \wedge \ldots \wedge e_{e_n} : e_i \in \Pi_{\leq n}\}$$

is a basis of $\mathbb{P}(\wedge^n (\text{Sym}^k \mathbb{C}^n))$, where

$$\Pi_{\leq n} = \{(i_1, i_2, \ldots, i_s) : 1 \leq i_1 \leq \ldots \leq i_s \leq n, 1 \leq s \leq k\}.$$

The corresponding coordinates of $x \in \text{Sym}^k \mathbb{C}^n$ will be denoted by $x_{e_1, e_2, \ldots, e_n}$. Let $A_{n,k} \subset \mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^n))$ consist of the points whose projection to $\wedge^k (\mathbb{C}^n)$ is nonzero. This is the subset where $x_{1,2,\ldots,n} \neq 0$ for some $1 \leq i_1 \leq \ldots \leq i_k \leq n$.

Remark 5.4. If $n = k$ then $A_{n,n} \subset \mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^n))$ is the affine chart where $x_{1,2,\ldots,n} \neq 0$.

Let us take a closer look at the space $\text{Grass}(n, \text{Sym}^k \mathbb{C}^n)$, which has an induced $\text{GL}(n)$ action coming from the $\text{GL}(n)$ action on $\text{Sym}^k \mathbb{C}^n$. Since $\phi^{\text{proj}}$ is a $\text{GL}(n)$-equivariant embedding, we conclude that

Lemma 5.5. 

(i) For $k \leq n$ $X_{n,k}$ is the $\text{GL}(n)$ orbit of

$$(18) \quad z = \phi^{\text{proj}}(e_1, \ldots, e_k) = [e_1 \wedge (e_2 + e_2^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_k = k} e_{i_1} \ldots e_{i_k})]$$

in $\mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^n))$. For arbitrary $g \in \text{GL}(n)$ with column vectors $v_1, \ldots, v_n$ the action is given by

$$g \cdot z = \phi^{\text{proj}}(g) = \phi^{\text{proj}}(v_1, \ldots, v_n) = [v_1 \wedge (v_2 + v_2^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_k = n} v_{i_1} \ldots v_{i_k})].$$

(ii) For $k \leq n$ $Y_{n,k}$ is a finite union of $\text{GL}(n)$ orbits.

(iii) For $k > n$ the images $X_{n,k}$ and $Y_{n,k}$ are $\text{GL}(n)$-invariant quasi-projective varieties with no dense $\text{GL}(n)$ orbit.

Lemma 5.6. If $k \leq n$ then

(i) $A_{n,k}$ is invariant under the $\text{GL}(n)$ action on $\mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^n))$.

(ii) $X_{n,k} \subset A_{n,k}$; however, $Y_{n,k} \nsubseteq A_{n,k}$.

Proof. To prove the first part take a lift

$$\tilde{z} = z^1 \oplus z^2 \in \text{Hom} (\mathbb{C}^n, \text{Sym}^k \mathbb{C}^n)$$

of $z \in \text{Grass}(n, \text{Sym}^k \mathbb{C}^n)$, where

$$z^1 \in \text{Hom} (\mathbb{C}^n, \mathbb{C}^n) \text{ and } z^2 \in \text{Hom} (\mathbb{C}^n, \oplus_{i=2}^n \text{Sym}^i (\mathbb{C}^n))$$

Then $z \in A_{n,k}$ if and only if $x_{1,2,\ldots,n}(z) = \det(z^1) \neq 0$, which is preserved by the $\text{GL}(n)$ action. For the second part note that for $(v_1, \ldots, v_k) \in j_{\text{nondeg}}^k (1, n)$ we have $v_1 \wedge \ldots \wedge v_k \neq 0$ so by definition $\phi^{\text{proj}}(v_1, \ldots, v_k) \in A_{n,k}$. On the other hand

$$\phi^{\text{proj}}(e_1, 0, \ldots, 0) = e_1 \wedge e_1^2 \wedge \ldots \wedge e_1^k \in Y_{n,k} \setminus A_{n,k}.$$
When \( k = n \) we have

**Lemma 5.7.** \( X_{k,k} \cong \text{GL}(k)/\mathbb{G}_k \) is embedded in the affine space \( A_{k,k} \subset \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^k) \) as the \( \text{GL}(k) \) orbit of \([e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_s = k} e_{i_1} \ldots e_{i_s})]\).

### 6. Affine embeddings of \( \text{SL}(k)/U_k \)

In the last section we embedded \( \text{GL}(k)/\mathbb{G}_k \) in the affine space \( A_{k,k} \subset \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^k) \) as the \( \text{GL}(k) \) orbit of

\[
[e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_s = k} e_{i_1} \ldots e_{i_s})] \in \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k)).
\]

Equivalently we have

\[
\text{SL}(k)/\text{SL}(k) \cap \mathbb{G}_k = \text{SL}(k)/U_k \rtimes F_k
\]

embedded in \( \wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k) \) as the \( \text{SL}(k) \) orbit of

\[
p_k = e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_s = k} e_{i_1} \ldots e_{i_s}),
\]

where \( \text{SL}(k) \cap \mathbb{G}_k \) is the semi-direct product \( U_k \rtimes F_k \) of \( U_k \) by the finite group \( F_k \) of \( \ell_k \)th roots of unity in \( \mathbb{C} \) for \( \ell_k = 1 + \ldots + k = \binom{k+1}{2} \), embedded in \( \text{SL}(k) \) as

\[
\epsilon \mapsto \begin{pmatrix} e & 0 & \ldots & 0 \\ 0 & e^2 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & e^k \end{pmatrix} \in \text{SL}(k).
\]

In this section we will look for affine embeddings of \( \text{SL}(k)/U_k \) in spaces of the form

\[
W_{k,K} = \wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}
\]

for suitable \( K \) and study their closures.

**Lemma 6.1.** Let \( K = M(1+2+\ldots+k)+1 = \binom{k+1}{2}M + 1 \) where \( M \in \mathbb{N} \). Then the point

\[
p_k \otimes e_1^{\otimes K} \in \wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}
\]

where

\[
p_k = e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_s = k} e_{i_1} \ldots e_{i_s}) \in \wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k)
\]

has stabiliser \( U_k \) in \( \text{SL}(k) \).
Proof. By Proposition \ref{prop:Stabilizer} the stabiliser of
\[ [p_k] \in \mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^k)) \cong \mathbb{P}(\wedge^k (\text{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}e_1)^\otimes K) \subseteq \mathbb{P}(W_{k,K}) \]
in $GL(k)$ is $G_k = U_k \rtimes \mathbb{C}^*$, so the stabiliser of
\[ p_k \otimes e_1^\otimes K \in \wedge^k (\text{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}e_1)^\otimes K \]
is contained in $G_k$. Moreover by the proof of Proposition \ref{prop:Stabilizer} the stabiliser of
\[ p_k \otimes e_1^\otimes K \]contains $U_k$. Finally
\[
\begin{pmatrix}
  z & 0 & \ldots & 0 \\
  0 & z^2 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \ldots & z^k \\
\end{pmatrix} \in \mathbb{C}^* \subseteq G_k
\]acts on $p_k \otimes e_1^\otimes K$ as multiplication by
\[ z^{1+2+\cdots+k+K} = z^{(M+1)(1+2+\cdots+k)+1} \]
and has determinant 1 if and only if $z^{1+2+\cdots+k} = 1$, so it lies in $SL(k)$ and fixes $p_k \otimes e_1^\otimes K$ if and only if $z = 1$.\qed

We will prove

\begin{theorem}
If $k \geq 4$ and $K = M(1+2+\ldots+k)+1$ where $M \in \mathbb{N}$ is sufficiently large, then the orbit of $p_k \otimes e_1^\otimes K$ where
\[ p_k = e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge \left( \sum_{i_1+\ldots+i_s=k} e_{i_1} \ldots e_{i_s} \right) \in \wedge^k (\text{Sym}^k \mathbb{C}^k) \]
under the natural action of $SL(k)$ on
\[ W_{k,K} = \wedge^k (\text{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}e_1)^\otimes K \]
is isomorphic to $SL(k)/U_k$, and its complement in its closure $\overline{\text{SL}(k)(p_k \otimes e_1^\otimes K)}$ in $W_{k,K}$ has codimension at least two.
\end{theorem}

This theorem has an immediate corollary.

\begin{corollary}
If $k \geq 2$ then $U_k$ is a Grosshans subgroup of $SL(k)$, so that every linear action of $U_k$ which extends to a linear action of $SL(k)$ has finitely generated invariants.
\end{corollary}

Proof. This follows directly from Theorem \ref{thm:MainTheorem} when $k \geq 4$. When $k = 2$ and $k = 3$ it is already known (cf. \cite{Gor2}).\qed

The remainder of this section will be devoted to proving Theorem \ref{thm:MainTheorem}.

It follows directly from Lemma \ref{lem:Stabilizer} that the $SL(k)$-orbit of $p_k \otimes e_1^\otimes K$ in $W_{k,K} = \wedge^k (\text{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}e_1)^\otimes K$ is isomorphic to $SL(k)/U_k$.\hfill \square
Recall that

\[
\mathbb{U}_k = \left\{ \begin{pmatrix}
1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\
0 & 1 & 2\alpha_2 & \cdots & 2\alpha_{k-1} + \cdots \\
0 & 0 & 1 & \cdots & 3\alpha_{k-2} + \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & (k-1)\alpha_2 \\
0 & 0 & \cdots & 1 & 1
\end{pmatrix} : \alpha_2, \ldots, \alpha_k \in \mathbb{C} \right\}
\]

so that \( \mathbb{U}_k \) is generated along its last column as well as along its first row.

Let \( B_k \subset SL(k) \) denote the standard Borel subgroup of \( SL(k) \) which stabilises the filtration \( \mathbb{C} e_1 \subset \mathbb{C} e_1 \oplus \mathbb{C} e_2 \subset \cdots \oplus \mathbb{C}^k \). Then \( B_k = B_{k-1} \cdot \mathbb{U}_k \) where the Borel subgroup \( B_{k-1} \) of \( GL(k-1) = GL(\mathbb{C} e_1 \oplus \mathbb{C} e_2 \oplus \cdots \oplus \mathbb{C} e_{k-1}) \) is embedded diagonally in \( SL(k) \) via

\[
A \mapsto \begin{pmatrix}
A & 0 \\
0 & (\det A)^{-1}
\end{pmatrix}.
\]

Since \( \mathbb{U}_k \) stabilises \( p_k \) and \( e_1 \) we have

\[
B_k(p_k \otimes e_1^{\otimes k}) = B_{k-1}(p_k \otimes e_1^{\otimes k}),
\]

and since \( SL(k)/B_k \) is projective we have

\[
SL(k)(p_k \otimes e_1^{\otimes k}) = B_k(p_k \otimes e_1^{\otimes k}) = B_{k-1}(p_k \otimes e_1^{\otimes k}).
\]

Since the closure \( SL(k)(p_k \otimes e_1^{\otimes k}) \) of the \( SL(k) \)-orbit of \( p_k \otimes e_1^{\otimes k} \) in \( W_{k,k} \) is the union of finitely many \( SL(k) \)-orbits, to prove Theorem 6.2 it suffices to prove

**Lemma 6.4.** Suppose that \( k \geq 4 \) and \( a \) and \( b \) are strictly positive integers with \( b/a \) large enough and that \( x \) lies in the closure in

\[
(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}
\]

of the orbit \( B_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) \) of \( p_k^{\otimes a} \otimes e_1^{\otimes b} \) under the natural action of the Borel subgroup \( B_k \) of \( SL(k) \). Then either \( x \in B_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) \) or the stabiliser of \( x \) in \( SL(k) \) has dimension at least \( k + 1 \).

We will split the proof of this lemma into two parts. Let \( T_k \) denote the standard maximal torus of \( SL(k) \) consisting of the diagonal matrices in \( SL(k) \). Lemma 6.4 follows immediately from Lemmas 6.3 and 6.6 below.

**Lemma 6.5.** Suppose that \( k \geq 4 \) and \( a \) and \( b \) are strictly positive integers with \( b/a \) large enough and that \( x \) lies in the closure \( T_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) \) in

\[
(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}
\]

of the orbit \( T_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) \) of \( p_k^{\otimes a} \otimes e_1^{\otimes b} \) under the natural action of the maximal torus \( T_k \) of \( SL(k) \). Then either \( x \in T_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) \) or the stabiliser of \( x \) in \( SL(k) \) has dimension at least \( k + 1 \).
Lemma 6.6. Suppose that $k \geq 2$ and $a$ and $b$ are strictly positive integers and that $x$ lies in the closure in

$$(\wedge^k (\text{Sym}^{\otimes k} \mathbb{C}^k))^\otimes \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the Borel subgroup $B_k$ of $\text{SL}(k)$. Then either $x \in B_k T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of $x$ in $\text{SL}(k)$ has dimension at least $k + 1$.

We will start with the proof of Lemma [6.6].

Proof. We have

$$x \in B_k(p_k^{\otimes a} \otimes e_1^{\otimes b}) = B_{k-1}(p_k^{\otimes a} \otimes e_1^{\otimes b})$$

as above, so there is a sequence of matrices

$$b^{(m)} = \begin{pmatrix}
    b_{11}^{(m)} & b_{12}^{(m)} & \ldots & b_{1k-1}^{(m)} & 0 \\
    0 & b_{22}^{(m)} & \ldots & b_{2k-1}^{(m)} & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & 0 & b_{kk}^{(m)}
\end{pmatrix} \in B_{k-1} \subset \text{SL}(k)$$

such that $b^{(m)}(p_k^{\otimes a} \otimes e_1^{\otimes b}) \to x$ as $m \to \infty$. Now expanding the wedge product in the definition of $p_k$ we get

$$b^{(m)}(p_k^{\otimes a}) = (e_1 \wedge \ldots \wedge e_n + \ldots + (b_{11}^{(m)})^{1+2+\ldots+k} e_1 \otimes e_1^2 \otimes \ldots \otimes e_1^k)^{\otimes a}$$

while

$$b^{(m)}(e_1^{\otimes b}) = (b_{11}^{(m)})^b e_1^{\otimes b},$$

so by considering the coefficient of $(e_1 \wedge \ldots \wedge e_n)^{\otimes a} \otimes e_1^{\otimes b}$ we see that $(b_{11}^{(m)})^b$ tends to a limit in $\mathbb{C}$ as $m \to \infty$. Thus, by replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$b_{11}^{(m)} \to b_{11}^{(\infty)} \in \mathbb{C}$$

as $m \to \infty$.

First suppose that $k = 2$. Then $\text{Sym}^{\otimes k} \mathbb{C}^k = \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2$ and

$$(\wedge^k (\text{Sym}^{\otimes k} \mathbb{C}^k))^\otimes \otimes (\mathbb{C}^k)^{\otimes b} = (\wedge^2 (\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2))^\otimes \otimes (\mathbb{C}^2)^{\otimes b}$$

and

$$p_k = e_1 \wedge (e_2 + e_1^2),$$

so if

$$b^{(m)} = \begin{pmatrix} b_{11}^{(m)} & b_{12}^{(m)} \\ 0 & b_{22}^{(m)} \end{pmatrix} \in \text{SL}(2)$$

then $b_{11}^{(m)} b_{22}^{(m)} = 1$ and

$$b^{(m)}(p_2^{\otimes a} \otimes e_1^{\otimes b}) = (b_{11}^{(m)})^b (e_1 \wedge (e_2 + (b_{11}^{(m)})^3 e_1^3))^{\otimes a} \otimes e_1^{\otimes b}$$

$$\to x = (b_{11}^{(\infty)})^b (e_1 \wedge (e_2 + (b_{11}^{(\infty)})^3 e_1^3))^{\otimes a} \otimes e_1^{\otimes b}$$
as \( m \to \infty \). If \( b_{11}^{(\infty)} \neq 0 \) then \( x \in \text{SL}(2)((p_{2}^{\oplus a} \otimes e_{1}^{\oplus b}) \), while if \( b_{11}^{(\infty)} = 0 \) then \( x = 0 \) is fixed by \( \text{SL}(2) \) which has dimension \( 3 = k + 1 \).

Now suppose that \( k > 2 \), and assume first that \( b_{11}^{(\infty)} \neq 0 \). We have that

\[
b^{(m)}(p_{k}^{\oplus a} \otimes e_{1}^{\oplus b}) = (b_{11}^{(m)})^{b}(b^{(m)}p_{k})^{\otimes a} \otimes e_{1}^{\oplus b} \to x
\]

and \( b_{11}^{(m)} \to b_{11}^{(\infty)} \in \mathbb{C} \setminus \{0\} \) as \( m \to \infty \), so by replacing the sequence \((b^{(m)})\) with a subsequence if necessary, we can assume that

\[
(b_{11}^{(m)})^{b(a)}b^{(m)}p_{k} \to p_{k}^{\infty} \in \wedge^{k}(\text{Sym}^{\leq k}\mathbb{C}^{k})
\]

as \( m \to \infty \), where

\[
 b^{(m)}b^{(m)} = b_{11}^{(m)}e_{1} \wedge (b_{22}^{(m)}e_{2} + (b_{11}^{(m)})^{2}e_{1}^{2}) \wedge \ldots \wedge (b_{ii}^{(m)}e_{i} + b_{i-1,i}^{(m)}e_{i-1} + \ldots
 \]

\[
 \ldots + b_{ii}^{(m)}e_{i} + \sum_{s=2}^{i-1} \sum_{i_{i_{1} + \ldots + i_{s}}} (b_{i_{i_{1}}i_{1}}^{(m)}e_{i_{1}} + \ldots + b_{i_{i_{s}}i_{s}}^{(m)}e_{i_{s}}) \ldots (b_{ii_{s}}^{(m)}e_{i_{s}} + \ldots + b_{i_{i_{s}}i_{s}}^{(m)}e_{i_{s}})(b_{11}^{(m)})^{i_{s}}e_{1}^{i_{s}} \ldots
\]

Looking at the coefficient of

\[
e_{1} \wedge e_{1}^{2} \wedge \ldots \wedge e_{j}^{i-1} \wedge e_{j} \wedge e_{j+1}^{i+1} \wedge \ldots \wedge e_{1}^{k}
\]

when \( 1 \leq j \leq i \leq k \), we see that

\[
(b_{11}^{(m)})^{1+2+\ldots+(i-1)+(i+1)+\ldots+k}b_{ji}^{(m)}
\]

tends to a limit in \( \mathbb{C} \) as \( m \to \infty \), and so since \( b_{11}^{(\infty)} \neq 0 \)

\[
b_{ji}^{(m)} \to b_{ji}^{(\infty)} \in \mathbb{C}.
\]

Also \( b_{11}^{(m)}b_{22}^{(m)} \ldots b_{kk}^{(m)} = 1 \) for all \( m \), so \( b_{11}^{(\infty)}b_{22}^{(\infty)} \ldots b_{kk}^{(\infty)} = 1 \), so \( b^{(m)} \to b^{(\infty)} \in \text{SL}(k) \). Therefore

\[
x = b^{(\infty)}(p_{k}^{\oplus a} \otimes e_{1}^{\oplus b})
\]

lies in the orbit of \( p_{k}^{\oplus a} \otimes e_{1}^{\oplus b} \) as required.

So it remains to consider the case when \( b_{11}^{(\infty)} = 0 \). If \( p_{k}^{\infty} = 0 \) then its stabiliser is \( \text{SL}(k) \) which has dimension \( k^{2} - 1 \geq k + 1 \), so we can assume that \( p_{k}^{\infty} \neq 0 \). Recall that then

\[
(b_{11}^{(m)})^{b(a)}b^{(m)}p_{k} \to p_{k}^{\infty} \in \wedge^{k}(\text{Sym}^{\leq k}\mathbb{C}^{k})
\]

and

\[
[b^{(m)}p_{k}] \to [p_{k}^{\infty}] \in \mathcal{P}(\wedge^{k}(\text{Sym}^{\leq k}\mathbb{C}^{k}))
\]

as \( m \to \infty \), where

\[
b^{(m)}p_{k} = b_{11}^{(m)}e_{1} \wedge (b_{22}^{(m)}e_{2} + (b_{11}^{(m)})^{2}e_{1}^{2}) \wedge \ldots \wedge (b_{ii}^{(m)}e_{i} + b_{i-1,i}^{(m)}e_{i-1} + \ldots
 \]

\[
 \ldots + b_{ii}^{(m)}e_{i} + \sum_{s=2}^{i-1} \sum_{i_{i_{1} + \ldots + i_{s}}} (b_{i_{i_{1}}i_{1}}^{(m)}e_{i_{1}} + \ldots + b_{i_{i_{s}}i_{s}}^{(m)}e_{i_{s}}) \ldots (b_{ii_{s}}^{(m)}e_{i_{s}} + \ldots + b_{i_{i_{s}}i_{s}}^{(m)}e_{i_{s}})(b_{11}^{(m)})^{i_{s}}e_{1}^{i_{s}} \ldots
\]
By replacing the sequence \( b^{(m)} \) with a subsequence if necessary, we can assume that
\[
[b_{i_1}^{(m)} e_i + b_{i_1}^{(m)} e_{i-1} + \ldots + b_{i_1}^{(m)} e_1] \to [c_{i_1}^{(\infty)} e_i + c_{i_1}^{(\infty)} e_{i-1} + \ldots + c_{i_1}^{(\infty)} e_1] \in \mathbb{P}(\mathbb{C}^k)
\]
as \( m \to \infty \) for \( 2 \leq i \leq k \), which implies that
\[
((b_{i_1}^{(m)} e_i + \ldots + b_{i_1}^{(m)} e_1) \ldots (b_{i_1}^{(m)} e_i + \ldots + b_{i_1}^{(m)} e_1)) \to
\]
\[
(c_{i_1}^{(\infty)} e_i + \ldots + c_{i_1}^{(\infty)} e_1) \ldots (c_{i_1}^{(\infty)} e_i + \ldots + c_{i_1}^{(\infty)} e_1) \in \mathbb{P}(\text{Sym}^i \mathbb{C}^k)
\]
whenever \( i_1 + \ldots + i_s = i \in \{2, \ldots, k\} \), and hence that
\[
p_k^{(\infty)} \in \wedge^k (\text{Sym}^l D)
\]
where \( D \) is the span in \( \mathbb{C}^k \) of
\[
\{ e_1 \} \cup \{ e_i \} + c_{i_1}^{(\infty)} e_{i_1} + \ldots + c_{i_1}^{(\infty)} e_1 : 2 \leq i \leq k \}.
\]
Moreover since \( b^{(m)} \in B_{k-1} \) we have \( b^{(m)}_{jk} = 0 \) if \( j < k \) so
\[
[c_{kk}^{(\infty)} e_k + c_{k-1k}^{(\infty)} e_{k-1} + \ldots + c_{1k}^{(\infty)} e_1] = [e_k]
\]
so \( e_k \in D \).

Note that \( b^{(m)} \in B_{k-1} \) and \( B_{k-1} \) normalises the maximal unipotent subgroup \( U_k \) of \( B_k \) which contains the stabiliser \( U_k \) of \( p_k \). Therefore for each \( m \) there is a \((k-1)\)-dimensional subgroup of \( U_k \) which stabilises \( b^{(m)} p_k \), and it follows that there is a \((k-1)\)-dimensional subgroup \( U_k^{\infty} \) of \( U_k \) which stabilises \( p_k^{(\infty)} \). In addition by [3] Theorem 6.4 if \( p_k^{(\infty)} \) does not lie in \( \text{SL}(k) p_k \) then it is stabilised by a nontrivial one-parameter subgroup \( \lambda^{(\infty)} : \mathbb{C}^* \to \text{SL}(k) \) of \( \text{SL}(k) \). Moreover if \( D \neq \mathbb{C}^k \) then there is some \( j \in \{2, \ldots, k-1\} \) such that \( e_j \) is not in \( D \), and then there is an automorphism of \( \mathbb{C}^k \) which fixes every element of \( D \) and sends \( e_j \) to \( e_j + e_k \). This automorphism is independent of \( U_k^{\infty} \) (since \( U_k^{\infty} \subseteq U_k \)) and the one-parameter subgroup \( \lambda^{(\infty)} \) of \( \text{SL}(k) \) fixing \( p_k^{(\infty)} \), so the stabiliser of \( p_k^{(\infty)} \) in \( \text{SL}(k) \) has dimension at least
\[
\dim U_k^{\infty} + 2 = k + 1.
\]
Thus we can assume that \( D = \mathbb{C}^k \), and hence \( c_{ii}^{(\infty)} \neq 0 \) for \( 2 \leq i \leq k \), so that
\[
b_{ii}^{(m)} \to c_{ii}^{(\infty)} \in \mathbb{C}
\]
as \( m \to \infty \). Then by applying an element of \( B_{k-1} \) to \( p_k^{(\infty)} \) we can assume that
\[
[c_{ii}^{(\infty)} e_i + c_{i-1i}^{(\infty)} e_{i-1} + \ldots + c_{1i}^{(\infty)} e_1] = [e_i]
\]
or equivalently that
\[
[b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \ldots + b_{1i}^{(m)} e_1] \to [e_i]
\]
as \( m \to \infty \) for \( 2 \leq i \leq k \), and hence that
\[
[(b_{ii}^{(m)} e_i + \ldots + b_{i_1}^{(m)} e_1) \ldots (b_{i_1}^{(m)} e_i + \ldots + b_{1i}^{(m)} e_1)] \to [e_{i_1} \ldots e_i] \in \mathbb{P}(\text{Sym}^i \mathbb{C}^k)
\]
whenever \( i_1 + \cdots + i_s = i \in \{2, \ldots , k\} \). Now by again replacing the sequence \((b^{(m)})\) with a subsequence if necessary, we can assume that

\[
[b^{(m)}_{i_1} e_1 + b^{(m)}_{i_2} e_2 + \cdots + b^{(m)}_{i_s} e_s + \sum_{s=2}^{i-1} \sum_{i_1 + \cdots + i_s = i} (b^{(m)}_{i_1} e_1 + \cdots + b^{(m)}_{i_s} e_s)] \to [d^\infty_i] \in \mathbb{P}(\text{Sym}^{\leq k} \mathbb{C}^k)
\]

where

\[
d^\infty_i = \gamma_i^{(\infty)} e_1 + \sum_{s=2}^{i} \gamma_i^{(\infty)} e_s \in \text{Sym}^{\leq k} \mathbb{C}^s \setminus \{0\}
\]

for some \( \gamma_i^{(\infty)} \in \mathbb{C} \). In addition \( \{d^\infty_i : 1 \leq i \leq k\} \) is linearly independent so that

\[
[p_k^\infty] = [d^\infty_1 \wedge \cdots \wedge d^\infty_k] \in \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^k))
\]

and

\[
p_k^\infty = \lim_{m \to \infty} t^{(m)} p_k
\]

where \( t^{(m)} \) is the diagonal matrix with entries \( b^{(m)}_{11}, \ldots , b^{(m)}_{kk} \).

Thus we can assume that \( p_k^\infty \in T_k p_k \) where \( T_k \) is the standard maximal torus in \( \text{SL}(k) \), which completes the proof of Lemma 6.6. \( \square \)

It therefore remains to prove Lemma 6.5. We can continue with the notation above and use the following standard result:

**Lemma 6.7.** Let \( T \) be an algebraic torus acting on the projective variety \( Z \), and \( z \in Z \). Then \( y \in Tz \) if and only if there is \( \tau \in T \), and a one-parameter subgroup \( \lambda : \mathbb{C} \to T \) such that \( \tau y \in \lambda(\mathbb{C}) z \).

Hence we may assume without loss of generality that there is a one-parameter subgroup

\[
t \mapsto \lambda(t) = \begin{pmatrix}
  t^{i_1} & 0 & \cdots & 0 \\
  0 & t^{i_2} & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots \\
  0 & \cdots & \cdots & t^{i_k}
\end{pmatrix}
\]

of \( \text{SL}(k) \) such that \( \lambda_1 > 0 \) and \( t^{\lambda_h / a} \lambda(t) p_k \to p_k^\infty \) as \( t \to 0 \). Therefore

\[
p_k^\infty = \lim_{t \to 0} t^{\lambda_h / a} e_1 \wedge (e_2 + t^{2i_1 - i_2} e_1) \wedge \cdots \wedge (e_k + \sum_{s=2}^{k} \sum_{i_1 + \cdots + i_s = i} i_1 + \cdots + i_s = k t^{i_1 + \cdots + i_s} e_i) = \lambda_1 + \cdots + \lambda_k = 0.
\]

We are assuming that \( p_k^\infty \neq 0 \) so

\[
[p_k^\infty] = \lim_{t \to 0} [e_1 \wedge (e_2 + t^{2i_1 - i_2} e_1) \wedge \cdots \wedge (e_k + \sum_{s=2}^{k} \sum_{i_1 + \cdots + i_s = i} i_1 + \cdots + i_s = k t^{i_1 + \cdots + i_s} e_i)].
\]

If \( \lambda_{i_1} + \cdots + \lambda_{i_j} < \lambda_j \) for some \( j \in \{2, \ldots , k-1\} \) and \( s \geq 2 \) and \( i_1, \ldots , i_s \geq 1 \) such that \( i_1 + \cdots + i_s = j \), then \( [p_k^\infty] \) is independent of \( e_j \) and so as above the stabiliser of \( p_k^\infty \) in \( \text{SL}(k) \) has dimension at least \( k + 1 \). So we can assume that

\[
(20) \quad \lambda_{i_1} + \cdots + \lambda_{i_s} \geq \lambda_j
\]
for any \( j \in \{2, \ldots, k - 1\} \) and \( s \geq 2 \) and \( i_1, \ldots, i_s \geq 1 \) such that \( i_1 + \cdots + i_s = j \), and in particular that \( \lambda_j \leq j \lambda_1 \) for each \( j \in \{2, \ldots, k - 1\} \). Let
\[
\rho_j = j \lambda_1 - \lambda_j
\]
for \( j \in \{1, \ldots, k - 1\} \); then \( \rho_1 = 0 \) and \( \rho_j \geq 0 \) and
\[
\rho_1 + \cdots + \rho_i \leq \rho_j
\]
for any \( j \in \{2, \ldots, k - 1\} \) and \( s \geq 2 \) and \( i_1, \ldots, i_s \geq 1 \) such that \( i_1 + \cdots + i_s = j \). In addition looking at the coefficient of
\[
e_1 \land e_2 \land \cdots \land e_{k-1} \land e_{i_1} \cdots e_{i_s}
\]
where \( i_1 + \cdots + i_s = k \), we find that
\[
0 \leq \lambda_1 b/a + \lambda_i + \cdots + \lambda_i - \lambda_k = \lambda_1 (b/a + k(k + 1)/2) - (\rho_1 + \cdots + \rho_i + \rho_2 + \cdots + \rho_{k-1}),
\]
and since \( p_k^\infty \neq 0 \) there is some \( i_1, \ldots, i_s \) with \( i_1 + \cdots + i_s = k \) and
\[
\lambda_1 b/a + \lambda_i + \cdots + \lambda_i = \lambda_k
\]
or equivalently
\[
\lambda_1 (b/a + k(k + 1)/2) = \rho_1 + \cdots + \rho_i + \rho_2 + \cdots + \rho_{k-1}.
\]
Thus
\[
p_k^\infty = \lim_{r \to 0} e_1 \land (e_2 + t^{2 \lambda_1 - \lambda_2} e_1^2) \land \cdots
\]
\[
\cdots \land (e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1+\cdots+i_s=k-1} r^{\lambda_1+\cdots+\lambda_i-\lambda_k} e_{i_1} \cdots e_{i_s}) \land (r^{\lambda_1 b/a} \sum_{s=2}^{k} \sum_{i_1+\cdots+i_s=k} r^{\lambda_1+\cdots+\lambda_i} e_{i_1} \cdots e_{i_s})
\]
\[
= e_1 \land \cdots \land (e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1+\cdots+i_s=k-1} e_{i_1} \cdots e_{i_s}) \land \left( \sum_{s=2}^{k} \sum_{i_1+\cdots+i_s=k} e_{i_1} \cdots e_{i_s} \right)
\]
is independent of \( e_k \) and hence is fixed by the automorphisms of \( \mathbb{C}^k \) which fix \( e_1, \ldots, e_{k-1} \) and send \( e_k \) to \( e_k + e_j \) for \( j \in \{1, \ldots, k - 1\} \), as well as by the one-parameter subgroup
\[
\lambda(t) = \begin{pmatrix}
  r^{\lambda_1} & 0 & \cdots & 0 \\
  0 & r^{\lambda_2} & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & r^{\lambda_k}
\end{pmatrix}
\]
of \( T_k \). Thus to complete the proof of Lemma 6.5 and hence of Theorem 6.2, it suffices to find an additional one-dimensional stabiliser, which will be done in the rest of this section.
Letting
\[ z = [p_k] = [e_1 \wedge (e_2 + e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_k = k} e_{i_1} \ldots e_{i_k})] \]
as at (18) we have
\[ \lambda(t)z = [r^{11}e_1 \wedge (r^{12}e_2 + r^{21}e_1^2) \wedge \ldots \wedge (\sum_{i_1 + \ldots + i_k = k} r^{i_1+\ldots+i_k}e_{i_1} \ldots e_{i_k})] = \]
\[ = [r^{11}+\ldots+k_\sigma(e_1 \wedge \ldots \wedge e_k) + r^{11+2k_1+\ldots+k_\sigma}(e_1 \wedge e_1^2 \wedge e_3 \wedge \ldots \wedge e_k) + \ldots]. \]
The generic term in this expression is
\[ r^{k_1+\ldots+k_\sigma}(e_{e_1} \wedge \ldots \wedge e_{e_k}), \Sigma(e_i) = i \]
where
\[(24) \lambda_t = \sum_{i \in \tau} \lambda_i \text{ and } e_\tau = \Pi_{i \in \tau} e_i \text{ if } \tau = (i_1, \ldots, i_3).\]

**Definition 6.8.** For any one-parameter subgroup \( \lambda \) as above let
- \( m_\lambda = \min \{ e_1, \ldots, e_k \} (\lambda_{e_1} + \lambda_{e_2} + \ldots \lambda_{e_k}) \),
- \( \mu_\lambda = \sum_{1 \leq \Sigma \leq k, \mu = m_\lambda} e_\Sigma \),
- \( m_\lambda[i] = \min \{ e_{\mu = m_\lambda} \} \text{ for } 1 \leq i \leq k \),
- \( \mu_\lambda[i] = \sum_{\Sigma \leq i, \mu = m_\lambda[i]} e_\Sigma \).

Let \( O_1 \) denote the \( SL(k) \)-orbit of \( z_3 \).

It is clear that the one-parameter subgroup \( \lambda(t) = (t, t^2, \ldots, t^k) \) stabilises \( z \), where \( z \) is defined as at (18), and therefore \( z = z_3 \) and its \( SL(k) \)-orbit is equal to its \( GL(k) \)-orbit.

We need a more precise description of the orbit structure of the closure of the orbit \( O_0 = O_1 \). Since \( \lambda_i = i\lambda_1 \) for \( i = 1, \ldots, k \), for \( \lambda \neq \lambda \) we have a smallest index \( \sigma \in \{2, \ldots, k\} \) with \( \lambda_\sigma \neq \sigma \lambda_1 \).

**Definition 6.9.** We call \( \sigma = Head(\lambda) \) the head of \( \lambda = (\lambda_1, \ldots, \lambda_n) \) if
\[ \lambda_i = i\lambda_1 \text{ for } i < \sigma \text{ and } \lambda_\sigma \neq \sigma \lambda_1. \]
If \( \lambda_\sigma < \sigma \lambda_1 \text{ then we call } \lambda \text{ regular } \); otherwise we call \( \lambda \) degenerate.

We will say that a one-parameter subgroup \( \lambda \) is *maximal* if the closure of the orbit \( GL(k) \cdot z_3 \) is a maximal boundary component of the closure of the orbit of \( z \).

**Definition 6.10.** Fix \( 0 < \varepsilon < 1 \) and \( 2 \leq \sigma \leq k \). Let \( \lambda^\sigma = (\lambda_1^\sigma, \ldots, \lambda_n^\sigma) \) and \( \mu^\sigma = (\mu_1^\sigma, \ldots, \mu_k^\sigma) \) be the following one-parameter subgroups of \( GL(k) \):
\[(25) \lambda_i^\sigma = i - \lfloor \frac{i}{\sigma} \rfloor \varepsilon \text{ for } 1 \leq i \leq k, \]
\[(26) \mu_i^\sigma = \begin{cases} i \text{ for } i \neq \sigma, i \leq k, \\ \sigma + \varepsilon \text{ for } i = \sigma. \end{cases} \]
It is easy to see that \( \text{Head}(\lambda^r) = \text{Head}(\mu^r) = \sigma \), and \( \lambda^r \) is regular, whereas \( \mu^r \) is degenerate.

**Definition 6.11.** Let \( \lambda \) be a 1-parameter subgroup. We call
\[
\# \{ i : z_\lambda[i] = e_i \}
\]
the toral dimension of the limit point \( z_\lambda \).

**Lemma 6.12.** If the \( \text{SL}(k) \)-orbit of \( p_k^\infty \) has codimension one in \( \text{SL}(k) p_k \), then \( [p_k^\infty] \) lies in the orbit of one of \( z_{\lambda_1}, \ldots, z_{\lambda^k} \) or \( z_{\mu_1}, \ldots, z_{\mu^{k-1}} \).

**Proof.** We can assume that \( [p_k^\infty] = z_\lambda \) for some one-parameter subgroup \( \lambda \). First suppose that \( \lambda \) is a regular one-parameter subgroup with \( \text{Head}(\lambda) = \sigma \) and \( [p_k^\infty] = z_\lambda \). Without loss of generality we can assume that \( \lambda_i = i \) for \( i < \sigma \) and \( \lambda_\sigma = \sigma - \epsilon \).

We will call \( d(i) = \lfloor i/\sigma \rfloor \) the defect of \( i \) and \( d(\tau) = d(i_1) + \ldots + d(i_s) \) the defect of \( \tau = (i_1, \ldots, i_s) \), so that when \( i \leq \sigma \) we have \( d(i) \epsilon = \rho_i \) as defined at (21). Since \( \lambda_{(\sigma,j,\ldots,\sigma)}(\tau) = j + m(\sigma - \epsilon) \) for \( 1 \leq j \leq \sigma - 1, m \geq 0 \),

we have
\[
(27) \quad m_\lambda[i] \leq i - d(i) \epsilon \quad \text{for} \quad 1 \leq i \leq k.
\]

If \( \lambda_s < s - d(s) \epsilon \) for \( s > i \) and \( s \) is the smallest index with this property then \( m_\lambda[s] = \lambda_s \) and \( z_\lambda[s] = e_s \), so
\[
z_\lambda[1] = e_1, z_\lambda[\sigma] = e_\sigma, z_\lambda[s] = e_s,
\]
while \( z_\lambda \) is independent of \( e_k \) by (23), so \( [p_k^\infty] \) is fixed by a three-dimensional torus in \( \text{SL}(k) \) and thus \( p_k^\infty \) is fixed by a two-dimensional torus in \( \text{SL}(k) \) as well as a unipotent subgroup of dimension \( k - 1 \). So we can assume that \( \lambda_i \geq i - d(i) \epsilon \) for \( 1 \leq i \leq k \), and therefore
\[
m_\lambda[i] = i - d(i) \epsilon \quad \text{for} \quad 1 \leq i \leq k.
\]

So
\[
(28) \quad e_\tau \notin z_\lambda[i] \quad \text{if} \quad d(\tau) > d(i).
\]

On the other hand the distinguished 1-parameter subgroup \( \lambda^r \) is defined as \( \lambda^r_i = i - d(i) \epsilon \), and therefore
\[
z_{\lambda^r}[i] = \sum_{\Sigma(\tau) = i, d(\tau) = d(i)} e_\tau.
\]

Comparing (28) and (29) we conclude
\[
z_\lambda[i] \subset z_{\lambda^r}[i] \quad \text{for} \quad 1 \leq i \leq k.
\]
Now let $\mu$ be a degenerate 1-parameter subgroup with $\text{Head}(\mu) = \sigma$. Without loss of generality we can assume again that
\[ \mu_i = i \text{ for } i < \sigma \text{ and } \mu_\sigma = \sigma + \varepsilon. \]
Since
\[ \mu_{(1, \ldots, 1)} = i \text{ for } 1 \leq i \leq k \]
we have
\[ (30) \]
Again, $\mu _t < s$ cannot happen for $s > \sigma$ since in that case $z_\mu [s] = e_s$ would hold and the codimension of $\text{SL}(k)p_\infty^e$ would be at least two. So $\mu_s \geq s$ and therefore $\mu_r \geq \Sigma(r)$ with strict inequality if $\sigma \in \tau$. Therefore
\[ (31) \]
On the other hand $\mu^\kappa$ satisfies equality in (30), and
\[ (32) \]
Comparing (31) and (32) we get
\[ z_\mu [i] \subset z_\mu^\kappa [i] \text{ for } 1 \leq i \leq k, \]
and so it remains to consider the possibility that $[p_\infty^e] = z_\mu^\kappa$. But by (22) there is some $i_1, \ldots, i_s$ with $i_1 + \cdots + i_s = k$ and
\[ \lambda_1 b/a + \lambda_{i_1} + \cdots + \lambda_{i_s} = \lambda_k \]
and hence $\lambda_k > \lambda_{i_1} + \cdots \lambda_{i_s}$. Thus $[p_\infty^e]$ cannot be equal to $z_\mu^\kappa$ because the coefficient of $e_1 \wedge e_1^2 \wedge \cdots \wedge e_k^k$ is nonzero for $z_\mu^\kappa$ but zero for $[p_\infty^e]$, and the result follows. \hfill \Box

We summarize our information about the maximal boundary components in

**Proposition 6.13.** We have $z_\mu^\kappa = \bigwedge_{i=1}^k z_{\mu^\kappa} [i]$, where $z_{\mu^\kappa} [i] = \oplus_{\Sigma(r)=i,d(r)=d(i)} e_r$, and $z_\mu^\kappa = \bigwedge_{i=1}^k z_{\mu^\kappa} [i]$ where $z_{\mu^\kappa} [i] = \oplus_{\Sigma(r)=i,d(r) \neq d(i)} e_r$.

**Remark 6.14.** Since the one-parameter subgroup $\tilde{\lambda}(t) = (t, t^2, \ldots, t^k)$ of $\text{GL}(k)$ stabilises $T_kz$, it follows from Lemma [6.12] that it is enough to prove the codimension-at-least-two property we require only for the one-parameter subgroups $\tilde{\lambda}^\sigma$ (for $2 \leq s \leq k$) and $\tilde{\mu}^\sigma$ (for $2 \leq s \leq k - 1$) of $\text{SL}(k)$ given by
\[ \tilde{\lambda}^\sigma(t) = (\lambda^\sigma(t) \tilde{\lambda}(t)^{\mu^\sigma})^{\mu^\sigma} \]
and
\[ \tilde{\mu}^\sigma(t) = (\mu^\sigma(t) \tilde{\lambda}(t)^{\lambda^\sigma})^{\lambda^\sigma} \]
for suitable $q_\sigma, r_\sigma \in \mathbb{Q}$ and $n_\sigma, m_\sigma \in \mathbb{Z}$. But we observed at (20) that the property is satisfied by a one-parameter subgroup $\lambda$ of $\text{SL}(k)$ if $\lambda_{i_1} + \cdots + \lambda_{i_s} < \lambda_j$ for any $j \in \sigma \setminus \tau$.\hfill \Box
[2, \ldots, k - 1] such that \( i_1 + \cdots + i_s = j \), so it is enough to consider the one-parameter subgroups \( \lambda^s \) for \( 2 \leq s \leq k \).

### 6.1. The limit of the stabilisers.

In order to prove Lemma 6.5 it now suffices by Remark 6.14 to find a \( k \)-dimensional unipotent subgroup of the stabiliser \( G_{x^\sigma} \) of \( z_{x^\sigma} \) in \( \text{GL}(k) \) for each \( \sigma \) when \( z_{x^\sigma} = \{ p_{1}^{\sigma} \} \), since we know that \( p_{k}^{\sigma} \) is fixed by a one-parameter subgroup of the maximal torus \( T_k \) of \( \text{SL}(k) \), and any unipotent group which stabilises \( z_{x^\sigma} = \{ p_{k}^{\sigma} \} \) also stabilises \( p_{k}^{\sigma} \).

In this subsection we will study the limits \( \lim_{\varepsilon \to 0} G_{x^\sigma(\varepsilon)z} \) of the stabiliser groups for the one-parameter subgroups \( \lambda^\sigma \) for \( 2 \leq \sigma \leq k \), and use this to prove Lemma 6.5, which together with Lemma 6.6 will complete the proof of Theorem 6.2.

**Proposition 6.15.** \( G^\sigma = \lim_{\varepsilon \to 0} G_{x^\sigma(\varepsilon)z} \subset \text{GL}(k) \) is a \( k \)-dimensional subgroup of \( G_{x^\sigma} \) which contains a \( k - 1 \)-dimensional subgroup of the maximal unipotent subgroup \( U_k \) of \( \text{SL}(k) \).

**Proof.** Consider the stabilizer

\[
G_{x^\sigma(\varepsilon)z} = \lambda^\sigma(t)^{-1} G_{x} \lambda^\sigma(t).
\]

Recall that

\[
G_{x} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & 2\alpha_1\alpha_{n-1} + \cdots \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^d \end{pmatrix} \right\}
\]

where the polynomial in the \((i, j)\) entry is

\[
p_{i,j}(\alpha) = \sum_{a_1+a_2+\cdots+a_i=j} \alpha_{a_1}\alpha_{a_2}\cdots\alpha_{a_i}.
\]

Therefore, the \((i, j)\) entry of the stabilizer of \( \lambda^\sigma(t)z \) is

\[
(G_{x^\sigma(\varepsilon)z})_{i,j} = t^{\lambda_{j-1}^\sigma - \lambda_{j}^\sigma} p_{i,j}(\alpha)
\]

If \( \varepsilon \) is small enough then \( \lambda_1^\sigma < \lambda_2^\sigma < \cdots < \lambda_k^\sigma \), and we define the positive number

\[
n_i^\sigma = \max_{1 \leq j \leq n-i+1} (\lambda_{j+1}^\sigma - \lambda_j^\sigma), \quad i = 1, \ldots, k.
\]

Note that by definition \( n_i^\sigma = 0 \) for all \( \sigma \).

**Lemma 6.16.** Under the substitution

\[
\beta_i^\sigma = t^{-n_i^\sigma} \alpha_i^\sigma
\]

we have

\[
G_{x^\sigma(\varepsilon)z}(\beta_1, \ldots, \beta_k) \in \text{GL}(\mathbb{C}[\beta_1, \ldots, \beta_k][t]),
\]

so the entries are polynomials in \( t \) with coefficients in \( \mathbb{C}[\beta_1, \ldots, \beta_k] \).
Proof. Compute the substitution as follows:

\[(35) \quad G_{x^r(t)x}, _{i,j} = t^{x^r_j - x^r_i} \sum_{a_1 + a_2 + \ldots + a_i = j} \alpha_{a_1} \alpha_{a_2} \ldots \alpha_{a_i} = \]

\[(36) \quad = \sum_{a_1 + \ldots + a_i = j} t^{x^r_j - x^r_i} t^{\alpha_{a_1} + \alpha_{a_2} + \ldots + \alpha_{a_i}} \beta_{a_1} \beta_{a_2} \ldots \beta_{a_i}.\]

By definition

\[n_{a_1}^r \geq \lambda_{r+1,i-1}^r - \lambda_i^r; \quad n_{a_2}^r \geq \lambda_{r+2,i-2}^r - \lambda_i^r; \quad \ldots; \quad n_{a_i}^r \geq \lambda_{r+i,i-i}^r - \lambda_i^r.\]

Adding up these inequalities and using \(a_1 + \ldots + a_i = j\) we get an alternating sum on the left cancelling up to

\[n_{a_1}^r + \ldots + n_{a_i}^r \geq \lambda_j^r - \lambda_i^r.\]

Substituting this into \((35)\) we get

\[(37) \quad (G_{x^r(t)x}, _{i,j} = \sum_{a_1 + \ldots + a_i = j} t^{x^r_j - x^r_i} t^{\alpha_{a_1} + \alpha_{a_2} + \ldots + \alpha_{a_i}} \beta_{a_1} \beta_{a_2} \ldots \beta_{a_i} \in \mathbb{C}[\beta_1, \ldots, \beta_k][t].\]

This proves Lemma 6.16. \(\square\)

As a corollary we get the existence of

\[G^r = \lim_{t \to 0} G_{x^r(t)x}(\beta_1, \ldots, \beta_k) \in GL(\mathbb{C}[\beta_1, \ldots, \beta_k]).\]

To prove that \(\dim G^r = k\) and complete the proof of Proposition 6.15 for \(1 \leq i \leq k\) choose \(\theta(i)\) such that

\[(38) \quad n_i^r = \lambda_{\theta(i) + i - 1}^r - \lambda_{\theta(i)}^r\]

holds. Then

\[(39) \quad \eta_{\theta(i), \theta(i) + i - 1}(\beta_1, \ldots, \beta_k) = \sum_{a_1 + \ldots + a_i = \theta(i) + i - 1} t^{n_{a_1}^r + \ldots + n_{a_i}^r} \beta_{a_1} \ldots \beta_{a_i}\]

so

\[(40) \quad (G^r)_{\theta(i), \theta(i) + i - 1} = \lim_{t \to 0} t^{-n_i^r} \eta_{\theta(i), \theta(i) + i - 1}(\beta_1, \ldots, \beta_k) = \lim_{t \to 0} (t^{n_i^r} \beta_{\theta(i)}^{n_i^r - 1} \beta_i + \ldots) = \beta_{\theta(i)}^{n_i^r - 1} \beta_i + q_{\theta(i), \theta(i) + i - 1}\]

where

\[q_{\theta(i), \theta(i) + i - 1} \in \mathbb{C}[\beta_1, \ldots, \beta_k][t].\]

It follows that the elements \(\frac{\partial}{\partial t} G^r(t(e_i + e_i)) \in \text{Lie}(G^r)\) are independent, where \(t(e_1 + e_i) = (t, 0, \ldots, 0, t, 0, \ldots, 0)\) with the \(t\)'s are in the 1st and \(i\)th position if \(i > 1\) but interpreted as \((2t, 0, \ldots, 0)\) if \(i = 1\). This completes the proof of Proposition 6.15. \(\square\)
Moreover, we can also choose \( G \) where
\[
-i \frac{\beta}{\beta_i^{p_i}} \quad (46)
\]
Indeed, since \( T \) is unipotent but not \( T \) is unipotent if \( \xi \neq 0 \).

**Case 1:** \( \sigma = k \).

*Proof.* Let \( T_\xi \in GL(k) \) denote the transformation
\[
T_\xi(e_i) = e_i \quad \text{for} \ i \neq k - 1 \ ; \ T_\xi(e_{k-1}) = e_{k-1} + \xi e_k \quad \text{for} \ \xi \in \mathbb{C}
\]
Since \( e_{k-1} \) does not occur just in \( z_{k'}[k-1] \), \( T_\xi \) stabilises \( p_k^{\infty} \). This gives us a subgroup of \( SL(k) \) of dimension at least \( k+1 \) which stabilises \( p_k^{\infty} \), because \( T_\xi \) is unipotent but not upper triangular if \( \xi \neq 0 \). \( \square \)

**Case 2:** \( \sigma < k \) and \( k \neq -1 \) mod \( \sigma \).

*Proof.* Let \( T \) be the transformation
\[
T(e_i) = e_i \quad \text{for} \ i \neq k \ ; \ T(e_k) = e_k + \xi e_\sigma.
\]
Since \( e_k \) occurs only in \( z_{k'}[k] \), and \( z_{k'}[\sigma] = \sigma \), we have
\[
T \cdot z_{k'} = z_{k'}(e_1, \ldots, e_{k-1}, e_k + \xi e_\sigma) = \\
= z_{k'}[1] \wedge \ldots \wedge z_{k'}[\sigma - 1] \wedge e_\sigma \wedge z_{k'}[\sigma + 1] \wedge \ldots \wedge z_{k'}[k] + \\
+ \xi \cdot z_{k'}[1] \wedge \ldots \wedge z_{k'}[\sigma - 1] \wedge e_\sigma \wedge z_{k'}[\sigma + 1] \wedge \ldots \wedge z_{k'}[k-1] \wedge e_\sigma = z_{k'},
\]
so \( T \in G_{k'} \).

It is slightly harder task to show that \( T \notin G' = \lim_{t \to 0} G_{k'}(t) \). First, we compute \( n_1 \) for \( i = k - \sigma \). We claim that for \( k \neq -1 \) mod \( \sigma \)
\[
n_{k-\sigma+1} = \lambda_{k} - \lambda_{\sigma} = \lambda_{k-\sigma+1} - \lambda_1.
\]
Indeed,
\[
\lambda_{j+k-\sigma+1} - \lambda_j = \ldots \leq \lambda_{k} - \lambda_{\sigma} = \lambda_{k-\sigma+1} - \lambda_1
\]
This means that we can choose \( \theta(k - \sigma + 1) = \sigma \) in (38) and substitute into (40)
\[
(G')_{\sigma,k} = \beta_{1}^{\sigma-1} \beta_{k-\sigma+1} + q_{s,k}(\beta_1, \ldots, \beta_k),
\]
where \( q_{s,k}(\beta_1, \ldots, \beta_k) \) is a polynomial, whose monomials \( \beta_1^{p_i} \ldots \beta_k^{p_{k'}} \) satisfy
\[
i_1 b_1 + \ldots + i_{\sigma} b_{\sigma} = k.
\]
Moreover, we can also choose \( \theta(k - \sigma + 1) = 1 \), by (43), and then (40) gives us
\[
(G')_{1,k-\sigma+1} = \beta_{k-\sigma+1}.
\]
Suppose now that $T \in G^\sigma$, that is
\begin{equation}
T = G^\sigma(\beta_1, \ldots, \beta_k) \text{ for some } \beta_1 \in \mathbb{C}^*, \beta_2, \ldots, \beta_k \in \mathbb{C}.
\end{equation}
Let $(T)_{i,j}$ denote the $(i, j)$ entry of $T$. Then
\begin{equation}
(T)_{i,k} = \zeta^i, \quad (T)_{j,i} = 0 \text{ for } i \neq j, \quad (T)_{i,i} = 1.
\end{equation}
Comparing the $(1, 1)$ and $(1, k - \sigma + 1)$ entries of $T$ and $G^\sigma$ we get
\begin{equation}
\beta_1 = 1, \beta_{k-\sigma+1} = 0.
\end{equation}
Choose $\theta(i)$ for $i = 2, \ldots, k$ as in (38) and let $\theta(k - \sigma + 1) = \sigma$. Since all off-diagonal entries of $T$ but the $(\sigma, k)$ are zero, (47) forces the following equations
\begin{equation}
\beta_i + q_{\theta(i), \theta(i)+1} = 0 \quad \text{for } i \neq k - \sigma + 1,
\end{equation}
\begin{equation}
\beta_{k-\sigma+1} + q_{\sigma, k} = \zeta.
\end{equation}
By (48), these are $k - 1$ polynomial equations in $k - 2$ variables, and the Jacobian at 0 is the origin, so we have finitely many solutions near the origin. Therefore, for some $\zeta$, it follows that $T$ is not in $G^\sigma$.

**Case 3:** $\sigma < k$ and $d = -1 \mod \sigma$.

*Proof.* This case works very similarly to the previous one. Suppose $k - 1 > \sigma$, that is, if $k = c\sigma - 1$ where $c \geq 2$ (this holds because $k \geq \sigma$), the condition is that $c\sigma - 2 > \sigma$, which is true for all $k \geq 4$.

Let $T$ be the transformation
\begin{equation}
T(e_i) = e_i \text{ for } i \neq k, k - 1; \quad T(e_{k-1}) = e_{k-1} + \zeta e_{\sigma}; \quad T(e_k) = e_k + \zeta e_{\sigma}
\end{equation}
First we check again that $T \in G_{k,\sigma}$. We have
\begin{align*}
z_{\sigma}[\sigma] &= e_{\sigma}; \\
z_{\sigma}[\sigma + 1] &= e_{\sigma + 1} + e_1 e_{\sigma}; \\
z_{\sigma}[k] &= e_k + \sum_{j=1}^{k-1} e_j e_{k-j}.
\end{align*}
An easy computation shows that
\begin{equation}
T \cdot z_{\sigma} = z_{\sigma}(e_1, \ldots, e_{k-2}, e_{k-1} + \zeta e_{\sigma}, e_k + \zeta e_{\sigma+1}) = \\
= z_{\sigma}[1] \land \ldots \land z_{\sigma}[k-2] \land (z_{\sigma}[k-1] + \zeta z_{\sigma}[\sigma]) \land (z_{\sigma}[k] + \zeta z_{\sigma}[\sigma + 1]) = \\
= z_{\sigma}[1] \land \ldots \land z_{\sigma}[k] = z_{\sigma}.
\end{equation}
Now we prove that $T \not\in G^\sigma$ in a similar way to the second case above. Since $k - 1 \neq -1 \mod \sigma$ we can substitute $k - 1$ instead of $k$ in (43):
\begin{equation}
n_{k-\sigma} = \lambda_{k-1}^\sigma - \lambda_k^\sigma = \lambda_{k-\sigma}^\sigma - \lambda_1^\sigma.
\end{equation}
Moreover, we also get the extra equation
\begin{equation}
\sigma_{k-\sigma} = \lambda^\sigma_k - \lambda^\sigma_{k+1},
\end{equation}
and similarly to (44) and (46) it follows that
\begin{align}
(G^\sigma)_{\sigma,k-1} &= \beta_{\sigma}^{\sigma-1}\beta_{k-\sigma} + q_{\sigma,k-1}(\beta_1, \ldots, \beta_k); \\
(G^\sigma)_{\sigma+1,k} &= \beta_{\sigma+1}^\sigma \beta_{k-\sigma} + q_{\sigma+1,k}(\beta_1, \ldots, \beta_k); \\
(G^\sigma)_{1,k-\sigma} &= \beta_{k-\sigma}.
\end{align}
Since $T$ differs from the identity matrix only by the entries
\begin{equation}
(T)_{\sigma,k-1} = (T)_{\sigma+1,k} = \zeta,
\end{equation}
the equality
\begin{equation}
T = G^\sigma(\beta_1, \ldots, \beta_k)
\end{equation}
forces $\beta_{k-\sigma} = 0, \beta_1 = 1$ and the analogue of (49) , (50):
\begin{align}
\beta_i + q_{\sigma(i),\sigma(i)+1-1} &= 0 \text{ for } i \neq k - \sigma \\
\beta_{k-\sigma} + q_{\sigma,k-1} &= \zeta \\
\beta_{k-\sigma} + q_{\sigma+1,k} &= \zeta
\end{align}
which are, again, $k + 1$ nondegenerate polynomial equations in $k - 1$ variables, such that for some $\zeta$ there is no solution.

We have now proved Lemma 6.5 which together with Lemma 6.6 completes the proof of Theorem 6.2.

7. Geometric description of Demailly-Semple invariants

As an immediate consequence of Corollary 6.3, we can now prove Theorem 3.3 in the case when $p = 1$.

**Theorem 7.1.** If $k \geq 2$ then $G'_{\lambda} = G_k$ is a Grosshans subgroup of the special linear group $SL(k)$, so that $O(SL(k)^{G_k})^{SL(k)}$ is a finitely generated complex algebra and moreover every linear action of $U_k$ or $G_k$ on an affine or projective variety $Y$ (with respect to an ample linearisation) which extends to a linear action of $GL(k)$ has finitely generated invariants.

In particular we have the special case of Theorem 3.2 when $p = 1$.

**Theorem 7.2.** The fibre $O((J_k,)^{G_k})$ of the bundle $E^n_k$ is a finitely generated graded complex algebra.

**Proof.** We have
\begin{equation}
O((J_k,)^{G_k}) \cong (O((J_k,))^\otimes O(SL(k)^{G_k})^{SL(k)}
\end{equation}
which is finitely generated because $O(SL(k)^{G_k})^{SL(k)}$ is finitely generated and $SL(k)$ is reductive.

Theorem 6.2 also allows us to describe the algebra $O(SL(k))^{U_k}$. In §6 we constructed an embedding of $SL(k)/U_k$ in the affine space $\wedge^k(\operatorname{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}^k)^\otimes K$ for suitable large $K$, and in Theorem 6.2 we proved that the boundary components of the closure $SL(k)(p_k \otimes e_1^\otimes K)$ of its image have codimension at least two. Thus we obtain the following corollary of Theorem 6.2:

**Theorem 7.3.** (i) If $k \geq 4$ then the canonical affine completion

$$SL(k)/U_k = \operatorname{Spec}(O(SL(k))^{U_k})$$

of $SL(k)/U_k$ is isomorphic to the closure $SL(k)(p_k \otimes e_1^\otimes K)$ of the orbit $SL(k)(p_k \otimes e_1^\otimes K) \cong SL(k)/U_k$ of $p_k \otimes e_1^\otimes K$ in $\wedge^k(\operatorname{Sym}^k \mathbb{C}^k) \otimes (\mathbb{C}^k)^\otimes K$ where $K = M(1 + 2 + \cdots + k) + 1$ for any strictly positive integer $M$;

(ii) The algebra $O(SL(k))^{U_k}$ is generated by the Plücker coordinates on $\mathbb{P}(\wedge^k(\operatorname{Sym}^k \mathbb{C}^k))$, which can be expressed as

$$\{\Delta_{i_1, \ldots, i_s} : s \leq k\},$$

where $i_j$ denotes a multi-index corresponding to basis elements of $\operatorname{Sym}^k \mathbb{C}^k$, and $\Delta_{i_1, \ldots, i_s}$ is the corresponding minor of $\phi(f^1, \ldots, f^s) \in \operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^k \mathbb{C}^k)$, together with the coordinates $f_1^1, \ldots, f_s^k$ of $f$.

It follows immediately from this theorem that the non-reductive GIT quotient

$$(J_k)_x/\mathbb{U}_k = \operatorname{Spec}(O((J_k)_x)^{\mathbb{U}_k})$$

is isomorphic to the reductive GIT quotient

$$((J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K))/SL(k).$$

This can be identified with the quotient of the open subset $((J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K))^{ss}$ of $SL(k)$-semistable points of $(J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K)$ by the equivalence relation $\sim$ such that $y \sim z$ if and only if the closures of the $SL(k)$-orbits of $y$ and $z$ intersect in $((J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K))^{ss}$. Equivalently it can be identified with the closed $SL(k)$-orbits in $((J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K))^{ss}$. Since $SL(k)(p_k \otimes e_1^\otimes K)$ is the union of finitely many $SL(k)$-orbits, with stabilisers $H_{1} = \mathbb{U}_k, H_{2}, \ldots, H_{s}$, say, we can stratify $(J_k)_x/\mathbb{U}_k$ so that the stratum corresponding to $H_j$ is identified with the $H_j$-orbits in $(J_k)_x$, such that the corresponding $SL(k)$-orbit in $(J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K)$ is semistable and closed in $((J_k)_x \times SL(k)(p_k \otimes e_1^\otimes K))^{ss}$.

**Example 7.4.** When $k = 2$ we have

$$J_2^{reg}(1, 2) = \{(f_1^1, f_2^1, f_1^2, f_2^2) \in (\mathbb{C}^2)^2 : (f_1^1, f_2^1) \neq (0, 0)\}.$$
and fixing a basis \( \{e_1, e_2\} \) of \( \mathbb{C}^2 \) and the induced basis \( \{e_1, e_2, e'_1, e'_2\} \) of \( \mathbb{C}^2 \oplus \Sym^2 \mathbb{C}^2 \), the map \( \phi : J_2(1,2) = \Hom(\mathbb{C}^2, \mathbb{C}^2) \to \Hom(\mathbb{C}^2, \Sym^3 \mathbb{C}^2) \) of (14) is given by
\[
(f'_1, f'_2, f''_1, f''_2) \mapsto \begin{pmatrix}
\frac{1}{2!}f''_1 & \frac{1}{2!}f''_2 & (f'_1)^2 & f'_1 f'_2 & (f'_2)^2
\end{pmatrix}.
\]
The 2 × 2 minors of this 2 × 5 matrix are \((f'_1)^3\), \((f'_1)^2 f'_2\), \(f'_1 (f'_2)^2\), \((f'_2)^3\) and
\[
\Delta_{[1,2]} = f'_1 f'_2 - f''_1 f''_2.
\]

On \( SL(2) \) we have \( \Delta_{[1,2]} = 1 \) and the algebra of invariants \( O(SL(2))^{\mathbb{U}_2} \) is generated by \( f'_1 \) and \( f'_2 \), as expected since \( SL(2)/\mathbb{U}_2 \cong \mathbb{C}^2 \setminus \{0\} \) and its canonical affine completion \( SL(2)/\mathbb{U}_2 \) is \( \mathbb{C}^2 \).

**Example 7.5.** When \( k = 3 \) the finite generation of the Demailly-Semple algebra \( O((J_k)_3)^{\mathbb{U}_3} \) was proved by Rousseau in [27]. We have
\[
J_{3,k}^{[1,1]}(1,3) = \{(f'_1, f'_2, f'_3, f''_1, f''_2, f''_3, f''''_1, f''''_2) \in (\mathbb{C}^3)^8; (f'_1, f'_2, f'_3) \neq (0,0,0)\},
\]
and if we fix a basis \( \{e_1, e_2, e_3\} \) of \( \mathbb{C}^3 \) and the induced basis
\[
\{e_1, e_2, e_3, e'_1, e'_2, e'_3, e''_2, e''_3, \ldots \}
\]
of \( \mathbb{C}^3 \oplus \Sym^2 \mathbb{C}^3 \oplus \Sym^3 \mathbb{C}^3 \), the map \( \phi : \Hom(\mathbb{C}^3, \mathbb{C}^3) \to \Hom(\mathbb{C}^3, \Sym^3 \mathbb{C}^3) \) in (14) sends
\[
(f'_1, f'_2, f'_3, f''_1, f''_2, f''_3, f''''_1, f''''_2)
\]
to a \( 3 \times 19 \) matrix, whose first 9 columns (corresponding to \( \Sym^2 \mathbb{C}^3 \)) are
\[
\begin{pmatrix}
\frac{1}{2!}f''_1 & \frac{1}{2!}f''_2 & (f'_1)^2 & f'_1 f'_2 & (f'_2)^2 & f'_1 f'_3 & f'_2 f'_3 & (f'_3)^2
\end{pmatrix},
\]
and the remaining 10 columns (corresponding to \( \Sym^3 \mathbb{C}^3 \)) are
\[
\begin{pmatrix}
(f'_1)^3 & f'_1 (f'_2)^2 & (f'_1)^2 f'_2 & f'_1 f'_3 & (f'_2)^3 & f'_2 f'_3 & (f'_3)^2 & f'_3 f'_2 & f'_3 (f'_2)^2
\end{pmatrix}.
\]
The \( 3 \times 3 \) minors of this matrix together with \( f'_1, f'_2, f'_3 \) generate the algebra of invariants \( O(SL(3))^{\mathbb{U}_3} \).

8. **Generalized Demailly-Semple jet bundles**

The aim of this section is to extend the earlier constructions for \( p = 1 \) to generalized Demailly-Semple invariant jet differentials when \( p > 1 \).

Let \( X \) be a compact, complex manifold of dimension \( n \). We fix a parameter \( 1 \leq p \leq n \), and study the maps \( \mathbb{C}^p \to X \). Recall that as before we fix the degree \( k \) of the map, and introduce the bundle \( J_{k,p} \to X \) of \( k \)-jets of maps \( \mathbb{C}^p \to X \), so that the fibre over \( x \in X \) is the set of equivalence classes of germs of holomorphic maps \( f : (\mathbb{C}^p,0) \to (X,x) \), with the equivalence relation \( f \sim g \) if and only if all derivatives \( f^{(j)}(0) = g^{(j)}(0) \) are equal for
0 \leq j \leq k. Recall also that \( \mathcal{G}_{k,p} \) is the group of \( k \)-jets of germs of biholomorphisms of \((\mathbb{C}^p, 0)\), which has a natural fibrewise right action on \( J_{k,p} \) with the matrix representation given by

\[
G_{k,p} = \begin{pmatrix}
\Phi_1 & \Phi_2 & \cdots & \Phi_k \\
0 & \Phi_1 & \cdots & \Phi_2 \\
0 & 0 & \cdots & \Phi_1 \\
& & & \\
& & & \Phi_1
\end{pmatrix},
\]

for \( G_{k,p} \in \mathcal{G}_{p,k} \) where \( \Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p) \) and \( \det \Phi_1 \neq 0 \). Recall also that \( \mathcal{G}_{k,p} \) is generated along its first \( p \) rows, in the sense that the parameters in the first \( p \) rows are independent, and all the remaining entries are polynomials in these parameters. The parameters in the \((1, m)\) block are indexed by a basis of \( \text{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p \), so they are of the form \( \alpha_v \) where \( v \in (p+m-1 \atop m-1) \) is an \( m \)-tuple and \( 1 \leq l \leq p \), and the polynomial in the \((l, m)\) block and entry indexed by \( \tau = (\tau[1], \ldots, \tau[l]) \in (p+l-1 \atop l-1) \) and \( v \in (p+m-1 \atop m-1) \) is given by

\[
(G_{k,p})_{\tau,v} = \sum_{v_1 + \ldots + v_l = v} \alpha_v^{[1]} \alpha_v^{[2]} \ldots \alpha_v^{[l]}.
\]

Recall also that \( \mathcal{G}_{k,p} = \mathbb{U}_{k,p} \rtimes \text{GL}(p) \) is an extension of its unipotent radical \( \mathbb{U}_{k,p} \) by \( \text{GL}(p) \), and that the generalized Demaillly-Semple jet bundle \( E_{k,p,m} \to X \) of invariant jet differentials of order \( k \) and weighted degree \((m, \ldots, m)\) consists of the jet differentials which transform under any reparametrization \( \phi \in \mathcal{G}_{k,p} \) of \((\mathbb{C}^p, 0)\) as

\[
Q(f \circ \phi) = (J_{\phi})^m Q(f) \circ \phi,
\]

where \( J_{\phi} = \det \Phi_1 \) denotes the Jacobian of \( \phi \), so that \( E_{k,p} = \oplus_{m \geq 0} E_{k,p,m} \) is the graded algebra of \( \mathcal{G}_{k,p}' \)-invariants where \( \mathcal{G}_{k,p}' = \mathbb{U}_{k,p} \rtimes \text{SL}(p) \).

8.1. **Geometric description for** \( p > 1 \). As in the case when \( p = 1 \) our goal is to prove that \( \mathcal{G}_{k,p}' \) is a Grosshans subgroup of \( \text{SL}(\text{sym}^k(\mathbb{C})) \) where \( \text{sym}^k(p) = \sum_{i=1}^k \dim \text{Sym}^i \mathbb{C}^p \) by finding a suitable embedding of the quotient \( \text{SL}(\text{sym}^k(\mathbb{C}))/\mathcal{G}_{k,p}' \).

**Remark 8.1.** In [25] Pacienza and Rousseau generalize the inductive process given in [5] of constructing a smooth compactification of the Demaillly-Semple jet bundles. Using the concept of a directed manifold, they define a bundle \( X_{k,p} \to X \) with smooth fibres, and the effective locus \( Z_{k,p} \subset X_{k,p} \), and a holomorphic embedding \( J_{k,p}^{\text{reg}}/\mathcal{G}_{k,p} \hookrightarrow Z_{k,p} \) which identifies \( J_{k,p}^{\text{reg}}/\mathcal{G}_{k,p} \) with \( Z_{k,p} \rtimes X_{k,p}^{\text{reg}} \cap Z_{k,p} \), so that \( Z_{k,p} \) is a relative compactification of \( J_{k,p}/\mathcal{G}_{k,p} \). We choose a different approach, generalizing the test curve model, resulting in a holomorphic embedding of \( J_{k,p}/\mathcal{G}_{k,p} \) into a partial flag manifold and a different compactification, which is a singular subvariety of the partial flag manifold, such that the invariant jet differentials of degree divisible by \( \text{sym}^k(p) \) are given by polynomial expressions in the Plücker coordinates.
Fix $x \in X$ and an identification of $T_x X$ with $\mathbb{C}^n$; then let $J_k(p, n) = J_{k,p,n}$ as defined in §2. Let

$$J_k^{\text{reg}}(p, n) = \{ \gamma \in J_k(p, n) : \Gamma_1 \text{ is non-degenerate} \}$$

where $\gamma$ is represented by

$$u \mapsto \gamma(u) = \Gamma_1 u + \Gamma_2 u^2 + \ldots + \Gamma_k u^k$$

with $\Gamma_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$. Let $N \geq n$ be any integer and define

$$\Upsilon_{k,p} = \left\{ \Psi \in J_k(n, N) : \exists \gamma \in J_k^{\text{reg}}(p, n) : \Psi \circ \gamma = 0 \right\}.$$

**Remark 8.2.** The global singularity theory description of $\Upsilon_{k,p}$ is

$$\Upsilon_{k,p} = \left\{ p = (p_1, \ldots, p_N) \in J_k(n, N) : \mathbb{C}[z_1, \ldots, z_n]/\langle p_1, \ldots, p_N \rangle \cong \mathbb{C}[x, y]/\langle z_1, \ldots, z_n \rangle \right\}.$$

Note, again, as in the $p = 1$ case, that if $\gamma \in J_k^{\text{reg}}(p, n)$ is a test surface of $\Psi \in \Upsilon_{k,p}$, and $\varphi \in \mathcal{G}_k$ is a holomorphic reparametrization of $\mathbb{C}^p$, then $\gamma \circ \varphi$ is, again, a test surface of $\Psi$:

$$\begin{array}{ccccccccc}
\mathbb{C}^p & \xrightarrow{\varphi} & \mathbb{C}^p & \xrightarrow{\gamma} & \mathbb{C}^n & \xrightarrow{\Psi} & \mathbb{C}^N \\
\Psi \circ \gamma = 0 & \Rightarrow & \Psi \circ (\gamma \circ \varphi) = 0
\end{array}$$

**Example 8.3.** Let $k = 2$, $p = 2$ and let $\Psi(z) = \Psi' z + \Psi'' z^2$ for $z \in \mathbb{C}^n$, and

$$\gamma(u_1, u_2) = \gamma_{10} u_1 + \gamma_{01} u_2 + \gamma_{20} u_1^2 + \gamma_{11} u_1 u_2 + \gamma_{02} u_2^2, \gamma_{ij} \in \mathbb{C}^n.$$

Then $\Psi \circ \gamma = 0$ has the form

$$\begin{array}{ccccccccc}
\Psi'(\gamma_{10}) = 0 & ; & \Psi'(\gamma_{01}) = 0 \\
\Psi'(\gamma_{20}) + \Psi''(\gamma_{10}, \gamma_{10}) = 0 & ; & \Psi'(\gamma_{11}) + 2\Psi''(\gamma_{10}, \gamma_{01}) = 0 & ; & \Psi'(\gamma_{01}) + \Psi''(\gamma_{01}, \gamma_{01}) = 0.
\end{array}$$

We introduce

$$\mathcal{S}_\gamma = \{ \Psi \in J_k(n, N) : \Psi \circ \gamma = 0 \}$$

and the following analogue of $J_k^{\text{reg}}(1, n)$:

$$J_k^{\text{reg}}(n, N) = \{ \Psi \in J_k(n, N) : \dim \ker \Psi = p \}.$$

The proof of the following proposition is analogous to that of Proposition 4.7 in [2], and we omit the details. We use the notation

$$\text{sym}^l(p) = \text{dim}(\text{Sym}^l \mathbb{C}^p); \text{sym}^{\leq k}(p) = \text{dim}(\mathbb{C}^p \oplus \text{Sym}^2 \mathbb{C}^p \oplus \ldots \oplus \text{Sym}^k \mathbb{C}^p) = \sum_{i=1}^{k} \text{sym}^i p.$$

**Proposition 8.4.**

(i) If $\gamma \in J_k^{\text{reg}}(p, n)$ then $\mathcal{S}_\gamma \subset J_k(n, N)$ is a linear subspace of codimension $N \text{sym}^{\leq k}(p)$.

(ii) For any $\gamma \in J_k^{\text{reg}}(p, n)$, the subset $\mathcal{S}_\gamma \cap J_k^{\text{reg}}(n, N)$ of $\mathcal{S}_\gamma$ is dense.
(iii) If $\Psi \in J^0_k(n,N)$, then $\Psi$ belongs to at most one of the spaces $S_\gamma$. More precisely, if $\gamma_1, \gamma_2 \in J^0_k(p,n)$, $\Psi \in J^0_k(n,N)$ and $\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0$, then there exists $\varphi \in J^0_k(p,p)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

(iv) Given $\gamma_1, \gamma_2 \in J^0_k(1,n)$, we have $S_{\gamma_1} = S_{\gamma_2}$ if and only if there is some $\varphi \in J^0_k(1,1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

With the notation

$$T_{k,p} = \mathcal{T}_{k,p} \cap J^0_k(n,N),$$

we deduce from Proposition 8.4 the following

**Corollary 8.5.** $T^0_k$ is a dense subset of $T_{k,p}$, and $T^0_{k,p}$ has a fibration over the orbit space $J^0_k(p,n)/J^0_k(p,p) = J^0_k(p,n)/\mathcal{O}_{k,p}$ with linear fibres.

**Remark 8.6.** In fact, Proposition 8.4 says a bit more, namely that $T^0_k$ is fibrewise dense in $T_{k,p}$ over $J^0_k(p,n)/\mathcal{O}_{k,p}$, but we will not use this stronger statement.

By the first part of Proposition 8.4, the assignment $\gamma \to S_\gamma$ defines a map

$$\nu : J^0_k(p,n) \to \text{Grass}(kN, J_k(n,N))$$

which, by the fourth part, descends to the quotient

$$\tilde{\nu} : J^0_k(p,n)/\mathcal{O}_{k,p} \to \text{Grass}(kN, J_k(n,N))$$

(cf. Proposition 4.4). Next, we want to rewrite this embedding in terms of the identifications introduced in §5. So we

- identify $J_k(p,n)$ with $\text{Hom}(\text{Sym}^1 p \oplus \ldots \oplus \text{Sym}^p, \mathbb{C}^n) = \text{Hom}(\text{Sym}^1 \mathbb{C}^p, \mathbb{C}^n)$ where $\text{sym}^i p = \text{dim Sym}^i \mathbb{C}^p$ and $\text{sym}^k p = \sum_{j=1}^k \text{sym}^j p$;

- identify $J_k(n,1)^*$ with $\text{Sym}^{\text{sym}^k} \mathbb{C}^n = \oplus_{i=1}^k \text{Sym}^i \mathbb{C}^n$.

We think of an element $\nu$ of $\text{Hom}(\text{Sym}^1 \mathbb{C}^p, \mathbb{C}^n)$ as an $n \times \text{sym}^k p$ matrix, with column vectors in $\mathbb{C}^n$. These columns correspond to basis elements of $\mathbb{C}^1 \oplus \ldots \oplus \mathbb{C}^p$, and the columns in the $i$th component are indexed by $i$-tuples $1 \leq t_1 \leq t_2 \leq \ldots \leq t_i \leq p$, or equivalently by

$$(e_{t_1} + e_{t_2} + \ldots + e_{t_i}) \in \mathbb{Z}^p_0$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the $j$th place, giving us

$$\nu = (v_{010\ldots0}, v_{010\ldots0}, \ldots, v_{00\ldots0}) \in \text{Hom}(\text{Sym}^k \mathbb{C}^p, \mathbb{C}^n).$$

The elements of $J^0_k(p,n)$ correspond to matrices whose first $p$ columns are linearly independent. When $n \geq \text{sym}^k p$ there is a smaller dense open subset $J^\text{nondeg}_k(p,n) \subset J^0_k(p,n)$ consisting of the $n \times \text{sym}^k p$ matrices of rank $\text{sym}^k p$.

Define the following map, whose components correspond to the equations in (64):

$$\phi : \text{Hom}(\text{Sym}^k \mathbb{C}^p, \mathbb{C}^n) \to \text{Hom}(\mathbb{C}^{\text{sym}^k p}, \text{Sym}^{\text{sym}^k} \mathbb{C}^n)$$

$$(v_{010\ldots0}, v_{010\ldots0}, \ldots, v_{00\ldots0}) \mapsto (\ldots, \sum_{k_1+k_2+\ldots+k_t=n} v_{k_1} v_{k_2} \ldots v_{k_t})$$

where on the right hand side $s \in \mathbb{Z}^p_0$. 

Example 8.7. If $k = p = 2$ then $\phi$ is given by

$$\phi(v_{10}, v_{01}, v_{20}, v_{11}, v_{02}) = (v_{10}, v_{01}, v_{20} + v_{10}^2, v_{11} + 2v_{10}v_{01}, v_{02} + v_{01}^2).$$

Let $P_{k,p} \subset GL_{\text{sym}^k(p)}$ denote the standard parabolic subgroup with Levi subgroup $GL(\text{sym}^1 p) \times \cdots \times GL(\text{sym}^k p)$, where $\text{sym}^i p = \dim \text{Sym}^i \mathbb{C}^p$ and $\text{sym}^k p = \sum_{j=1}^k \text{sym}^j p$. Then (65) has the following reformulation, analogous to Proposition 5.1.

Proposition 8.8. The map $\phi$ in (66) is a $\mathbb{G}_{k,p}$-invariant algebraic morphism

$$\phi : J_k^{\text{reg}}(p, n) \to \text{Hom}(\mathbb{C}^{\text{sym}(p)}, \text{Sym}^{\text{sym}k} \mathbb{C}^n)$$

which induces an injective map $\phi^{\text{Grass}}$ on the $\mathbb{G}_{k,p}$-orbits:

$$\phi^{\text{Grass}} : J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \text{Grass}_{\text{sym}^k(p)}(\text{Sym}^{\text{sym}k} \mathbb{C}^n)$$

and

$$\phi^{\text{Flag}} : J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \text{Flag}_{\text{sym}^k(p)}(\text{Sym}^{\text{sym}k} \mathbb{C}^n) \hookrightarrow \text{Hom}(\mathbb{C}^{\text{sym}(p)}, \text{Sym}^{\text{sym}k} \mathbb{C}^n)/P_{k,p}.$$ 

Composition with the Plücker embedding gives

$$\phi^{\text{Proj}} = \text{Pluck} \circ \phi^{\text{Grass}} : J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\Lambda^{\text{sym}k(p)} \text{Sym}^{\text{sym}k} \mathbb{C}^n).$$

As in the case when $p = 1$, we introduce the following notation

$$X_{n,k,p} = \phi^{\text{Proj}}(J_k^{\text{reg}}(p, n)), \quad Y_{n,k,p} = \phi^{\text{Proj}}(J_k^{\text{reg}}(p, n)) \subset \mathbb{P}(\Lambda^{\text{sym}k(p)} \text{Sym}^{\text{sym}k} \mathbb{C}^n).$$

Definition 8.9. Let $n \geq \text{sym}^k(p) = \text{sym}^1(p) + \cdots + \text{sym}^k(p)$. Then the open subset of $\mathbb{P}(\Lambda^{\text{sym}k(p)} \text{Sym}^{\text{sym}k} \mathbb{C}^n)$ where the projection to $\Lambda^{\text{sym}k(p)} \mathbb{C}^n$ is nonzero is denoted by $A_{n,k,p}$.

Since $\phi^{\text{Grass}}$ and $\phi^{\text{Proj}}$ are $\text{GL}(n)$-equivariant, and for $n \geq \text{sym}^k(p)$ the action of $\text{GL}(n)$ is transitive on $\text{Hom}^{\text{nondeg}}(\text{Sym}^{\text{sym}k(p)}, \mathbb{C}^n)$, we have

Lemma 8.10. (i) If $n \geq \text{sym}^k(p)$ then $X_{n,k,p}$ is the $\text{GL}(n)$ orbit of

$$z = [\Lambda_{j_1 + \cdots + j_p \leq k} e_{j_1} \cdots e_{j_p}]$$

in $\mathbb{P}(\Lambda^{\text{sym}k(p)} \text{Sym}^{\text{sym}k} \mathbb{C}^n)$.

(ii) If $n \geq \text{sym}^k(p)$ then $X_{n,k,p}$ and $Y_{n,k,p}$ are finite unions of $\text{GL}(n)$ orbits.

(iii) For $k > n$ the images $X_{n,k,p}$ and $Y_{n,k,p}$ are $\text{GL}(n)$-invariant quasi-projective varieties, though they have no dense $\text{GL}(n)$ orbit.

Similar statements hold for the closure of the image in the Grassmannian

$$\text{Grass}_{\text{sym}^k(p)}(\text{Sym}^{\text{sym}k} \mathbb{C}^n)$$

(or equivalently in the projective space $\mathbb{P}(\Lambda^{\text{sym}^k(p)} \text{Sym}^{\text{sym}k} \mathbb{C}^n)$).

Lemma 8.11. Let $n \geq \text{sym}^k(\mathbb{C}^n)$; then
(i) $A_{n,k,p}$ is invariant under the $GL(n)$ action on $\mathbb{P}(\wedge^{\text{sym}^{l}k}(\text{Sym}^{l}C^n))$;
(ii) $X_{n,k,p} \subseteq A_{n,k,p}$, although $Y_{n,k,p} \not\subseteq A_{n,k,p}$;
(iii) $\overline{X}_{n,k,p}$ is the union of finitely many $GL(n)$-orbits.

9. Affine embeddings of $\text{SL}(\text{sym}^{l}k)/\mathbb{G}_{k,p}$

In this section we study the case when $n = \text{sym}^{l}k$ and so $GL(n) \subset J^{\text{reg}}_{k}(p,n)$. In the previous section we embedded $J^{\text{reg}}_{k}(p,n)/\mathbb{G}_{k,p}$ in the affine space $A_{n,k,p} \subset \mathbb{P}(\wedge^{n} \text{Sym}^{l}C^n)$, which can be restricted to $GL(n)$ to give us an embedding

$$GL(n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^{n} \text{Sym}^{l}C^n)$$

as the $GL(n)$ orbit of

$$[\ldots \wedge \sum_{[s]=\bar{s}} \sum_{s_1+s_2+\ldots+s_l=s} e_{s_1} e_{s_2} \ldots e_{s_l} \wedge \ldots].$$

Equivalently we have $SL(n)/(SL(n) \cap \mathbb{G}_{k,p}) = SL(n)/\mathbb{G}_{k,p}' \times F_{k,p}$ embedded in $\wedge^{k}(\text{Sym}^{l}C^n)$ as the $SL(k)$ orbit of

$$p_{k,p} = \ldots \wedge \sum_{[s]=\bar{s}} \sum_{s_1+s_2+\ldots+s_l=s} e_{s_1} e_{s_2} \ldots e_{s_l} \wedge \ldots,$$

where $SL(n) \cap \mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{G}_{k,p}' \times F_{k,p}$ of $\mathbb{G}_{k,p}'$ by the finite group $F_{k,p}$ of $l_{k,p}$th roots of unity in $C$ for $l_{k,p} = \sum_{i=1}^{k} \text{sym}^{i}p$. In analogy with §6 we can consider an embedding of $SL(n)/\mathbb{G}_{k,p}'$ in

$$\wedge^{n}(\text{Sym}^{l}C^n) \otimes (\wedge^{P}(C^n))^{\otimes K}$$

for suitable $K$ and its closure in this affine space. We expect the following result generalising Theorem 6.2.

**Conjecture 9.1.** Let $K = M(\sum_{i=1}^{k} \text{sym}^{i}p) + 1$ where $M \in \mathbb{N}$. Then the point

$$p_{k,p} \otimes (e_1 \wedge \ldots \wedge e_p)^{\otimes K} \in \wedge^{n}(\text{Sym}^{l}C^n) \otimes (\wedge^{P}(C^n))^{\otimes K}$$

where

$$p_{k,p} = \ldots \wedge \sum_{[s]=\bar{s}} \sum_{s_1+s_2+\ldots+s_l=s} e_{s_1} e_{s_2} \ldots e_{s_l} \wedge \ldots$$

has stabiliser $\mathbb{G}_{k,p}'$ in $SL(n)$, and the closure of its $SL(n)$ orbit

$$\text{SL}(n)(p_{k,p} \otimes (e_1 \wedge \ldots \wedge e_p)^{\otimes K})$$

is the union of the orbit of $p_{k,p} \otimes (e_1 \wedge \ldots \wedge e_p)^{\otimes K}$ and finitely many other $SL(n)$-orbits, all of which have codimension at least two if $k$ is large enough (depending on $p$) and $M$ is sufficiently large (depending on $k$ and $p$).
The proof of Conjecture 9.1 should be similar to that of Theorem 6.2, with the role of the Borel subgroup $B_k$ of $SL(k)$ played by the standard parabolic subgroup $P \subset SL(n)$ which stabilises the filtration

$$0 \subset \mathbb{C}^p = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_p \subset \mathbb{C}^p \oplus \text{Sym}^2 \mathbb{C}^p \subset \cdots \subset \mathbb{C}^p \oplus \cdots \oplus \text{Sym}^k \mathbb{C}^p = \mathbb{C}^n.$$ 

It follows immediately from Conjecture 9.1 that we would have

**Conjecture 9.2.** If $p \geq 1$ and $k$ is large enough (depending on $p$) then the reparametrisation group $G'_{k,p}$ is a subgroup of the special linear group $SL(\text{sym}^k \mathbb{C}^p)$, where

$$\text{sym}^k \mathbb{C}^p = \sum_{i=1}^{k} \dim \text{Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1},$$

such that the algebra of invariants

$$O(SL(\text{sym}^k \mathbb{C}^p))^{G'_{k,p}}$$

is finitely generated, so that every linear action of $G_{k,p}$ or $G'_{k,p}$ on an affine or projective variety (with respect to an ample linearisation) which extends to a linear action of $GL(\text{sym}^k \mathbb{C}^p)$ has finitely generated invariants.

In particular we would have

**Conjecture 9.3.** If $p \geq 1$ and $k$ is large enough (depending on $p$) then the fibres $O((J_{k,p})_x)^{G'_{k,p}}$ of the bundle $E^p_n$ are finitely generated graded complex algebras.

We would also obtain geometric descriptions of the associated affine varieties

$$\text{Spec}(O(SL(\text{sym}^k \mathbb{C}^p))^{G_{k,p}})$$

and $\text{Spec}(O((J_{k,p})_x)^{G'_{k,p}})$ generalising those in §7.

**References**

[1] G. Bérczi, Thom polynomials and the Green-Griffiths conjecture, [arXiv:1011.4710](http://arxiv.org/abs/1011.4710).
[2] G. Bérczi, A. Szenes, Thom polynomials of Morin singularities, [arXiv:math/0608285](http://arxiv.org/abs/math/0608285), Annals of Math., to appear.
[3] D. Birkes, Orbits of linear algebraic groups, Annals of Math. 93 (1971) 459-475.
[4] A. Bloch, Sur les systèmes de fonctions uniformes satisfaisant l’équation d’une variété algébrique dont l’irrégularité dépasse la dimension, J. de Math. 5 (1926), 19-66.
[5] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Proc. Sympos. Pure Math. 62 (1982), Amer. Math. Soc., Providence, RI, 1997, 285-360.
[6] J.-P. Demailly, J. El-Goul, Hyperbolicity of generic surfaces of high degree in projective 3-space, Amer. J. Math. 122 (2000), 515-546.
[7] S. Diverio, J. Merker, E. Rousseau, Effective algebraic degeneracy, Invent. Math. 180(2010) 161-223.
[8] I. Dolgachev, Lectures on invariant theory, London Mathematical Society Lecture Note Series 296, Cambridge University Press, 2003.
[9] B. Doran and F. Kirwan, Towards non-reductive geometric invariant theory, Pure Appl. Math. Q. 3 (2007), 61–105.
[10] A. Fauntleroy, Categorical quotients of certain algebraic group actions, Illinois Journal Math. 27 (1983), 115-124.

[11] A. Fauntleroy, Geometric invariant theory for general algebraic groups, Compositio Mathematica 55 (1985), 63-87.

[12] T. Gaffney, The Thom polynomial of \( P^{[1]} \), Singularities, Part 1, Proc. Sympos. Pure Math., 40, (1983), 399-408.

[13] M. Green, P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, The Chern Symposium 1979. (Proc. Intern. Sympos., Berkeley, California, 1979) 41-74, Springer, New York, 1980.

[14] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions, Proc. London Math. Soc. (3) 67 (1993) 75-105.

[15] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions II, Singularities (Oberwolfach 1996), 27-36, Progress in Math. 162, Birkhauser, Basel 1998.

[16] F. Grosshans, Algebraic homogeneous spaces and invariant theory, Lecture Notes in Mathematics, 1673, Springer-Verlag, Berlin, 1997.

[17] F. Grosshans, The invariants of unipotent radicals of parabolic subgroups, Invent. Math. 73 (1983), 1–9.

[18] F. Kirwan, Quotients by non-reductive algebraic group actions, ‘Moduli Spaces and Vector Bundles’, L. Brambila-Paz, S. Bradlow, O. Garcia-Prada, S. Ramanan (editors), London Mathematical Society Lecture Note Series 359, Cambridge University Press 2009.

[19] S. Kobayashi, Hyperbolic complex spaces, Grundlehren der Mathematischen Wissenschaften 318, Springer Verlag, Berlin, 1998.

[20] J. Merker, Applications of computational invariant theory to Kobayashi hyperbolicity and to Green-Griffiths algebraic degeneracy, Journal of Symbolic Computations, 45 (2010), 986-1074.

[21] J. Merker, Jets de Demailly-Semple d’ordres 4 et 5 en dimension 2, Int. J. Contemp. Math. Sciences, 3 (2008) no. 18. 861-933.

[22] S. Mukai, An introduction to invariants and moduli, Cambridge University Press 2003.

[23] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, 3rd edition, Springer, 1994.

[24] P. E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute Lecture Notes, Springer, 1978.

[25] G. Pacienza, E. Rousseau, Generalized Demailly-Semple jet bundles and holomorphic mappings into complex manifolds, [arXiv:0810.4911]

[26] V. Popov, E. Vinberg, Invariant theory, Algebraic geometry IV, Encyclopedia of Mathematical Sciences v. 55, 1994.

[27] E. Rousseau, Etude des jets de Demailly-Semple en dimension 3, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 2, 397-421.

[28] Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 439-448, Higher Ed. Press, Beijing, 2002.

[29] Y.-T. Siu, Hyperbolicity in complex geometry, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, 543-566.

[30] Y.-T. Siu, S.-K. Yeung, Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane, Invent. Math. 124, (1996), 573-618.