A Convergent ADMM Framework for Efficient Neural Network Training

Junxiang Wang, Hongyi Li, and Liang Zhao Senior Member, IEEE

Abstract—As a well-known optimization framework, the Alternating Direction Method of Multipliers (ADMM) has achieved tremendous success in many classification and regression applications. Recently, it has attracted the attention of deep learning researchers and is considered to be a potential substitute to Gradient Descent (GD). However, as an emerging domain, several challenges remain unsolved, including 1) The lack of global convergence guarantees, 2) Slow convergence towards solutions, and 3) Cubic time complexity with regard to feature dimensions. In this paper, we propose a novel optimization framework to solve a general neural network training problem via ADMM (diADMM) to address these challenges simultaneously. Specifically, the parameters in each layer are updated backward and then forward so that parameter information in each layer is exchanged efficiently. When the diADMM is applied to specific architectures, the time complexity of subproblems is reduced from cubic to quadratic via a dedicated algorithm design utilizing quadratic approximations and backtracking techniques. Last but not least, we provide the first proof of convergence to a critical point sublinearly for an ADMM-type method (diADMM) under mild conditions. Experiments on seven benchmark datasets demonstrate the convergence, efficiency, and effectiveness of our proposed diADMM algorithm.

Index Terms—Alternating Direction Method of Multipliers, Multi-Layer Perceptron, Graph Convolutional Networks, Convergence, Quadratic Approximation, Backtracking

1 INTRODUCTION

The last two decades have witnessed the rapid development of deep learning techniques. Deep learning models have numerous advantages over traditional machine learning models, one of which is rich expressiveness: deep learning methods consist of non-linear modules, and hence have a powerful capacity to express different levels of representations [1]. Because deep learning methods have a wide range of large-scale applications ranging from computer vision to graph learning, they entail efficient and accurate optimizers to reach solutions within time limits.

Deep learning models are usually trained by the backpropagation algorithm, which is achieved by the Gradient Descent (GD) and many of its variants. They are state-of-the-art optimizers because of simplicity and efficiency. However, many drawbacks of GD put up a barrier to its wide applications. For example, it suffers from the gradient vanishing problem, where the error signal diminishes as the gradient is backpropagated; another example, it is also sensitive to the poor conditioning problem, namely, a small magnitude of input change can lead to a dramatic variation of the gradient. Recently, a well-known optimization framework, the Alternating Direction Method of Multipliers (ADMM) is being considered as an alternative to GD for training deep learning models. The principle of the ADMM is to partition a problem into multiple subproblems, each of which usually has an analytic solution. It has achieved great success in many conventional machine learning problems [2]. The ADMM has many potential advantages when solving deep learning problems: it scales linearly as a model is trained in parallel across cores; it serves as a gradient-free optimizer and hence avoids the gradient vanishing and the poor conditioning problems [3].

Even though the ADMM seems promising, there are still several challenges at must be overcome: 1. The lack of global convergence guarantees. Although many empirical experiments have shown that ADMM converges in deep learning applications, the underlying theory governing this convergence behavior remains mysterious. This is because a typical deep learning problem consists of a combination of linear and nonlinear mappings, causing optimization problems to be highly nonconvex. This means that traditional proof techniques cannot be directly applied. 2. Slow convergence towards solutions. Although ADMM is a powerful optimization framework that can be applied to large-scale deep learning applications, it usually converges slowly to high accuracy, even for simple examples [2]. It is often the case that ADMM becomes trapped in a modest solution. 3. Cubic time complexity with regard to feature dimensions. The implementation of the ADMM is very time-consuming for real-world datasets. Experiments conducted by Taylor et al. found that ADMM required more than 7000 cores to train a neural network with just 300 neurons [3]. This computational bottleneck mainly originates from the matrix inversion required to update the weight parameters. Computing an inverse matrix needs further subiterations, and its time complexity is approximately $O(n^3)$, where $n$ is a feature dimension [2].

In order to deal with these difficulties simultaneously, in this paper we propose a novel optimization framework for a deep learning Alternating Direction Method of Multipliers (diADMM) algorithm. For a general neural network training problem, our proposed diADMM algorithm updates param-
eters first in a backward direction and then forwards. This update approach propagates parameter information across the whole network and accelerates the convergence empirically. Then we apply the dIAADM algorithm to two specific architectures, namely, Multi-Layer Perceptron (MLP) and Graph Convolutional Network (GCN). When updating parameters of the proposed dIAADM, we avoid the operation of matrix inversion using the quadratic approximation and backtracking techniques, reducing the time complexity from $O(n^3)$ to $O(n^2)$. Finally, to the best of our knowledge, we provide the first convergence proof of the ADMM-based method (dIAADM) to a critical point with a sublinear convergence rate. The assumption conditions are mild enough for many common loss functions (e.g. cross-entropy loss and square loss) and activation functions (e.g. Rectified Linear Unit (ReLU) and leaky ReLU) to satisfy. Our proposed framework and convergence proofs are highly flexible for MLP models and GCN models, as well as being easily extendable to other popular network architectures such as Convolutional Neural Networks [3] and Recurrent Neural Networks [3]. Our contributions in this paper include:

- We present a novel and efficient dIAADM algorithm to handle a general neural network training problem. The new dIAADM updates parameters in a backward-forward fashion to speed up convergence empirically. We apply the proposed dIAADM algorithm to two specific architectures, MLP and GCN models.
- We propose the use of quadratic approximation and backtracking techniques to avoid the need for matrix inversion as well as to reduce the computational cost for large-scale datasets. The time complexity of subproblems in dIAADM is reduced from $O(n^3)$ to $O(n^2)$.
- We investigate several attractive convergence properties of the proposed dIAADM. The convergence assumptions are very mild to ensure that most deep learning applications satisfy our assumptions. The proposed dIAADM is guaranteed to converge to a critical point whatever the initialization is, when the hyperparameter is sufficiently large. We also analyze the new algorithm’s sublinear convergence rate.
- We conduct extensive experiments on MLP models and GCN models on seven benchmark datasets to validate our proposed dIAADM algorithm. The results show that the proposed dIAADM algorithm not only is convergent and efficient, but also performs better than most existing state-of-the-art algorithms, including GD and its variants.

The rest of this paper is organized as follows. In Section 2, we summarize recent research related to this topic. In Section 3, we present the new dIAADM algorithm, the quadratic approximation, and the backtracking techniques utilized. In Section 4, we introduce the main convergence results for the dIAADM algorithm. The results of extensive experiments conducted to show the convergence, efficiency, and effectiveness of our proposed new dIAADM algorithm are presented in Section 5 and Section 6 concludes this paper by summarizing the research.

2 RELATED WORK

Previous literature related to this research includes gradient descent and alternating minimization for deep learning models, and ADMM for nonconvex problems.

Gradient descent for deep learning models: The GD algorithm and its variants play a dominant role in the research conducted by the deep learning optimization community. The famous back-propagation algorithm was firstly introduced by Rumelhart et al. to train the neural network effectively [3]. Since the superior performance exhibited by AlexNet [4] in 2012, deep learning has attracted a great deal of researchers’ attention and many new optimizers based on GD have been proposed to accelerate the convergence process, including the use of Polyak momentum [7], as well as research on the Nesterov momentum and initialization by Sutskever et al. [8]. Adam is the most popular method because it is computationally efficient and requires little tuning [9]. Other well-known methods that incorporate adaptive learning rates include AdaGrad [10], RMSProp [11], AMSGrad [12], AdaBound [13], and generalized Adam-type optimizers [14]. These adaptive optimizers, however, suffer from worse generalization than GD, as verified by various studies [15], [16]. Many theoretical investigations have been conducted to explain such generalization gaps by exploring their convergence properties [17], [18], [19], [20], [21]. Moreover, their convergence assumptions do not apply to deep learning problems, which often require non-differentiable activation functions such as the ReLU.

Alternating minimization for deep learning: During the last decade, some alternating minimization methods have been studied to apply the Alternating Direction Method of Multipliers (ADMM) [3], Block Coordinate Descent (BCD) [22], [23] and auxiliary coordinates (MAC) [24] to replace a nested neural network with a constrained problem without nesting. These methods also avoid gradient vanishing problems and allow for non-differentiable activation functions such as ReLU as well as allowing for complex non-smooth regularization and the constraints that are increasingly important for deep neural architectures. More follow-up works have extended previous works to specific architectures [25], [26], or to achieve parallel or efficient computation [27], [28], [29], [30].

ADMM for nonconvex problems: The excellent performance achieved by ADMM over a range-wide of convex problems has attracted the attention of many researchers, who have now begun to investigate the behavior of ADMM on nonconvex problems and made significant advances. For example, Wang et al. proposed an ADMM to solve multi-convex problems with a convergence guarantee [31], while Wang et al. presented convergence conditions for a coupled objective function that is nonconvex and nonsmooth [32]. Chen et al. discussed the use of ADMM to solve problems with quadratic coupling terms [33] and Wang et al. studied the behavior of the ADMM for problems with nonlinear equality constraints [34]. Other papers improved nonconvex ADMM via Anderson acceleration [35], [36], linearization [37], or overrelaxation [38]. Even though ADMM has been proposed to solve deep learning applications [3], [39], there remains a lack of theoretical convergence analysis for the application of ADMM to such problems.
Fig. 1. The dlADMM framework to train a general neural network: update parameter backward and then forward.
$V^{k+1}$ denote forward updates of the diADMM for the $l$-th layer in the $(k+1)$-th iteration. $B^{k+1} = \{B_{l}^{k+1}\}_{l=1}^{L}$, $V^{k+1} = \{V_{l}^{k+1}\}_{l=1}^{L}$. Algorithm 1 shows the procedure of the proposed diADMM. Specifically, Lines 4-5 and Lines 8-9 update variables backward and forward via solving subproblems, respectively. Lines 11 and 12 calculate the primal residual and the dual variable $U^{k+1}$, respectively. In the next two sections, we discuss how to apply the proposed diADMM algorithm to two well-known architectures, and solve subproblems efficiently.

Algorithm 1 the diADMM Algorithm to Solve Problem 2

Require: $Y$, $V_{0}$, $\rho$, $\Psi$.
Ensure: $B_{l}(l = 1, \cdots, L)$, $V_{l}(l = 1, \cdots, L)$.
1: Initialize $k = 0$.
2: while $B^{k+1}$, $V^{k+1}$ not converged do
3: for $l = 1$ to $L$ do
4: Update $V^{k+1}_{l}$ via argmin$_{l} L\{B^{k+1}_{l}\}_{i=1}^{l}\left\{\sum_{i=1}^{L} V^{k+1}_{l} - V_{l} - (\nabla V^{k+1}_{l})_{i=1}^{L} - \psi_{l}\right\} V_{l}$.
5: Update $B^{k+1}_{l}$ via argmin$_{l} L\{B^{k+1}_{l}\}_{i=1}^{l}\left\{\sum_{i=1}^{L} B^{k+1}_{l} - B_{l} - (\nabla B^{k+1}_{l})_{i=1}^{L} - \phi_{l}\right\} B_{l}$.
6: end for
7: for $l = 1$ to $L$ do
8: Update $B^{k+1}_{l}$ via argmin$_{l} L\{B^{k+1}_{l}\}_{i=1}^{l}\left\{\sum_{i=1}^{L} B^{k+1}_{l} - B_{l} - (\nabla B^{k+1}_{l})_{i=1}^{L} - \phi_{l}\right\} B_{l}$.
9: Update $V^{k+1}_{l}$ via argmin$_{l} L\{B^{k+1}_{l}\}_{i=1}^{l}\left\{\sum_{i=1}^{L} V^{k+1}_{l} - V_{l} - (\nabla V^{k+1}_{l})_{i=1}^{L} - \psi_{l}\right\} V_{l}$.
10: end for
11: $\nu^{k+1} = \nu^{k} + 1 / \nu^{k}$.
12: $U^{k+1} = U^{k} + \rho \nu^{k+1}$.
13: $k \leftarrow k + 1$.
14: end while
15: Output $B_{l}$, $V_{l}$.

3.2 The Application on the MLP Problem

In this section, we discuss how to apply the proposed diADMM algorithm on the MLP problem. Table 2 lists important notations of the MLP model. A typical MLP model is defined by multiple linear mappings and nonlinear activation functions. A linear mapping for the $l$-th layer is composed of a weight matrix $W_{l} \in \mathbb{R}^{n_{l} \times n_{l-1}}$ and an intercept vector $b_{l} \in \mathbb{R}^{n_{l}}$, where $n_{l}$ is the number of neurons for the $l$-th layer; a nonlinear mapping for the $l$-th layer is defined by a continuous activation function $f_{l}(\bullet)$. Given an input $a_{l-1} \in \mathbb{R}^{n_{l-1}}$ from the $(l-1)$-th layer, the $l$-th layer outputs $a_{l} = f_{l}(W_{l}a_{l-1} + b_{l})$. Obviously, $a_{l-1}$ is nested in $a_{l} = f_{l}(\bullet)$. By introducing an auxiliary variable $z_{l}$, the task of training a MLP problem is formulated mathematically as follows:

Problem 3 (MLP Training Problem).
\[
\min_{W_{l}, b_{l}, z_{l}, a_{l}} \sum_{l=1}^{L} \Omega_{l}(W_{l}), s.t. z_{l} = W_{l}a_{l-1} + b_{l}(l = 1, \cdots, L), a_{l} = f_{l}(z_{l})(l = 1, \cdots, L - 1).
\]

In Problem 3 $a_{0} = x \in \mathbb{R}^{n_{0}}$ is the input of the MLP model where $n_{0}$ is the number of feature dimensions, and $y$ is a predefined label vector. $R(z_{L}; y)$ is a risk function for the $L$-th layer, which is convex, continuous, and proper, and $\Omega_{l}(W_{l})$ is a regularization term for the $l$-th layer, which is also convex, continuous, and proper. Similar to the problem relaxation in the previous section, Problem 3 is relaxed to Problem 4 as follows:

Problem 4.
\[
\min_{W_{l}, b_{l}, z_{l}, a_{l}} R(z_{L}; y) + \sum_{l=1}^{L} \Omega_{l}(W_{l}), s.t. z_{l} = W_{l}a_{l-1} + b_{l}(l = 1, \cdots, L), a_{l} = f_{l}(z_{l})(l = 1, \cdots, L - 1).
\]

The subproblem is transformed into the following form after it is replaced by Equation (1).
\[
\min_{\bar{W}_{l+1}, \bar{b}_{l+1}, \bar{z}_{l+1}, \bar{a}_{l}} \phi(\bar{W}_{l+1}, \bar{b}_{l+1}, \bar{z}_{l+1}, \bar{a}_{l}) = R(\bar{z}_{L}; y) + \frac{\nu}{2} \sum_{l=1}^{L} \left\|z_{l} - W_{l}a_{l-1} - b_{l}\right\|_{2}^{2} + \left\|a_{l} - f_{l}(z_{l})\right\|_{2}^{2}, \quad s.t. z_{L} = W_{L}a_{L-1} + b_{L}.
\]

where $\bar{W} = \{W_{l}\}_{l=1}^{L}$, $\bar{b} = \{b_{l}\}_{l=1}^{L}$, $\bar{z} = \{z_{l}\}_{l=1}^{L}$, $\bar{a} = \{a_{l}\}_{l=1}^{L}$ and $\nu > 0$ is a tuning parameter. The augmented Lagrangian function of Problem 4 is shown as follows:
\[
L_{\rho}(\bar{W}, \bar{b}, \bar{z}, \bar{a}, \rho, \psi) = R(\bar{z}_{L}; y) + \sum_{l=1}^{L} \Omega_{l}(W_{l}) + \frac{\nu}{2} \sum_{l=1}^{L} \left\|z_{l} - W_{l}a_{l-1} - b_{l}\right\|_{2}^{2} + \left\|a_{l} - f_{l}(z_{l})\right\|_{2}^{2}, \quad s.t. z_{L} = W_{L}a_{L-1} + b_{L}.
\]
\textbf{Algorithm 2} the dLADMM Algorithm to Solve Problem \cite{4}

\textbf{Require:} $y, a_0 = a, \rho, \nu.
\textbf{Ensure:}$ $a(l = 1, \ldots, L - 1, W_l(l = 1, \ldots, L), b_l(l = 1, \ldots, L)$, $z_l(l = 1, \ldots, L)$.

1: Initialize $k = 0$.
2: while $W^{k+1}, k^{k+1}, a^{k+1}$, $\phi^{k+1}$ not converged do
3: for $l = 1$ to $L$ do
4: if $l < L$ then
5: Update $\tau^{k+1}_l$ in Equation (3).
6: else
7: Update $\tau^{k+1}_l$ in Equation (3).
8: end if
9: end for
10: Update $W^{k+1}$ in Equation (9).
11: end for
12: Output $W, b, z, a$.

\[ L_l(a_l; \tau^{k+1}_l) \] as a quadratic approximation of $\phi$ at $a^k_l$, which is mathematically reformulated as follows:

\[ L_l(a_l; \tau^{k+1}_l) = \phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) + (\nabla a^k_l \phi)^T (W^{k+1}_l + 1, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) (a_l - a^k_l) + \|a_l - a^k_l\|^2 / 2. \]

where $\tau^{k+1}_l > 0$ is a parameter vector, $\phi$ denotes the Hadamard product (the elementwise product), and $a^{k+1}_l$ denotes $a$ to the Hadamard power of $b$ and $\|\cdot\|_1$ is the $\ell_1$ norm. $\nabla a^k_l \phi$ is the gradient of $\tau_l$ at $a^k_l$. Obviously, $L_l(a_l; \tau^{k+1}_l) = \min_{a_l} L_l(a_l; \tau^{k+1}_l)$. Rather than minimizing the original problem in Equation (2), we instead solve the following problem:

\[ a^{k+1}_l \leftarrow \arg \min_{a_l} L_l(a_l; \tau^{k+1}_l). \tag{3} \]

Because $L_l(a_l; \tau^{k+1}_l)$ is a quadratic function with respect to $a_l$, the solution can be obtained by

\[ a^{k+1}_l \leftarrow a^k_l - \nabla a^k_l \phi / \tau^{k+1}_l. \]

given a suitable $\tau^{k+1}_l$. Now the main focus is how to choose $\tau^{k+1}_l$. Algorithm \cite{8} shows the backtracking algorithm utilized to find a suitable $\tau^{k+1}_l$. Lines 2-5 implement a while loop until the condition $\phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) \leq L_l(a^{k+1}_l; \tau^{k+1}_l)$ is satisfied. As $\tau^{k+1}_l$ becomes larger and larger, $a^{k+1}_l$ is close to $a^k_l$, and $a^k_l$ satisfies the loop condition, which precludes the possibility of the infinite loop. The time complexity of Algorithm \cite{8} is $O(n^2)$, where $n$ is the number of features or neurons.

\textbf{Algorithm 3} The Backtracking Algorithm to update $\tau^{k+1}_l$

\textbf{Require:} $W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k, \rho$, some constant $\eta > 1$.

\textbf{Ensure:} $\tau^{k+1}_l$.

1: Pick up $\tau$ and $\eta$.
2: while $\phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) \leq L_l(a^{k+1}_l; \tau^{k+1}_l)$ do
3: $\tau \leftarrow \tau / \eta$.
4: end while
5: Output $\tau^{k+1}_l \leftarrow \tau$.

2. Update $\tau^{k+1}_l (l = 1, \ldots, L)$ are updated as follows:

\[ \tau^{k+1}_l \leftarrow \arg \min_{\tau_l} L_{\rho}(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) \]

which is equivalent to the following forms: for $\tau^{k+1}_l (l = 1, \ldots, L - 1)$,

\[ \tau^{k+1}_l \leftarrow \arg \min_{\tau_l} L_{\rho}(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k), \]

and for $\tau^{k+1}_L$,

\[ \tau^{k+1}_L \leftarrow \arg \min_{\tau_L} L_{\rho}(W, b^k, z^k, a^k, u^k) \]

\[ + R(z_L; y). \tag{5} \]

Equation (4) is highly nonconvex because the nonlinear activation function $f(z_l)$ is contained in $\phi(\bullet)$. For common activation functions such as the ReLU and leaky ReLU, Equation (4) has a closed-form solution; for other activation functions like sigmoid and hyperbolic tangent (tanh), a lookup table is recommended \cite{3}.

Equation (5) is a convex problem because $\phi(\bullet)$ and $R(\bullet)$ are convex with regard to $z_l$. Therefore, Equation (5) can be solved by Fast Iterative Soft-Thresholding Algorithm (FISTA) \cite{40}.

3. Update $b^k_l (l = 1, \ldots, L)$ are updated as follows:

\[ b^{k+1}_l \leftarrow \arg \min_{b_l} L_{\rho}(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k), \]

which is equivalent to the following form:

\[ b^{k+1}_l \leftarrow \arg \min_{b_l} L_{\rho}(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) \]

Similarly to the update of $\tau^{k+1}_l$, we define $\bar{U}_l(b_l; B)$ as a quadratic approximation of $\phi(\bullet)$ at $b^k_l$, which is formulated mathematically as follows \cite{40}:

\[ \bar{U}_l(b_l; B) = \phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) + (\nabla b^k_l \phi)^T (W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, a^{k+1}_l, u^k) (b_l - b^k_l) + (B / 2) \|b_l - b^k_l\|_2^2, \]

where $B > 0$ is a parameter. Here $\bar{B} \geq \nu$ for $l = 1, \ldots, L - 1$ and $B \geq \rho$ for $l = L$ are required for the convergence
The Backtracking Algorithm to update quadratic approximation is mathematically reformulated as

\[
\begin{align*}
\beta_i^{k+1} &\leftarrow \arg \min_{\nu_i} \mathcal{U}_i(b_i; \nu) \quad (l = 1, \ldots, L - 1). \\
\beta_L^{k+1} &\leftarrow \arg \min_{\nu_L} \mathcal{U}_L(b_L; \rho).
\end{align*}
\]

Equation (6) is a convex problem and has a closed-form solution as follows:

\[
\begin{align*}
\beta_i^{k+1} &= \nu_i - \nabla b_i \phi / \nu_i, \quad (l = 1, \ldots, L - 1) \\
\beta_L^{k+1} &= \beta_L^{k+1} - \nabla b_L \phi / \rho.
\end{align*}
\]

4. Update \( \overline{w}_i^{k+1} \)

The variables \( \overline{w}_i^{k+1} \) are updated as follows:

\[
\overline{w}_i^{k+1} = \arg \min_{\nu_i} \mathcal{L}_p(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{\overline{w}_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]

which is equivalent to the following form:

\[
\overline{w}_i^{k+1} = \arg \min_{\nu_i} \phi(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{\overline{w}_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) + \Omega(W_i).
\]

Due to the same challenge in updating \( \pi_i^{k+1} \), we define \( P_i(W_i; \theta_i^{k+1}) \) as a quadratic approximation of \( \phi \) at \( w_i^{k} \). The quadratic approximation is mathematically reformulated as follows [40]:

\[
P_i(W_i; \theta_i^{k+1}) = \phi(\overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k)
+ (\nabla w_i \phi)^T (\overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) - W_i - W_i^k + \|\theta_i^{k+1} - \Omega(W_i)\|_2/2,
\]

where \( \theta_i^{k+1} > 0 \) is a parameter vector, which is chosen by the Algorithm 4. Instead of minimizing the Equation 8, we minimize the following:

\[
\overline{w}_i^{k+1} = \arg \min_{\nu_i} \overline{p}_i(W_i; \theta_i^{k+1}) + \Omega(W_i).
\]

Equation (9) is convex and hence can be solved exactly. If \( \Omega_i \) is either an \( \ell_1 \) or an \( \ell_2 \) regularization term, Equation (9) has a closed-form solution.

Algorithm 4 The Backtracking Algorithm to update \( w_i^{k+1} \)

Require: \( \overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k, \rho \), some constant \( \tau > 1 \).
Ensure: \( \pi_i^{k+1}, \pi_L^{k+1} \).
1: Pick up \( \pi_i \) and \( \zeta = W_i - W_i^k \).
2: while \( \pi_i + (\nabla w_i \phi)^T (\overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) > \Omega(W_i) \) do
3: \( \pi_i \leftarrow \tau \pi_i, \zeta \leftarrow \tau \zeta \).
4: Solve \( \zeta \) by Equation 9.
5: end while
6: Output \( \pi_i^{k+1} \).
7: Output \( \pi_L^{k+1} \).

5. Update \( w_i^{k+1} \)

The variables \( w_i^{k+1} \) are updated as follows:

\[
w_i^{k+1} = \arg \min_{\nu_i} \mathcal{L}_p(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{w_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]

which is equivalent to

\[
w_i^{k+1} = \arg \min_{\nu_i} \phi(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{w_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) + \Omega(W_i).
\]

Similarly, we define \( P_i(W_i; \theta_i^{k+1}) \) as a quadratic approximation of \( \phi \) at \( w_i^{k+1} \). The quadratic approximation is then mathematically reformulated as follows [40]:

\[
P_i(W_i; \theta_i^{k+1}) = \phi(\overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k)
+ (\nabla w_i \phi)^T (\overline{w}_i^{k+1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) - (W_i - W_i^k) + \|\theta_i^{k+1} - \Omega(W_i - W_i^k)\|_2/2,
\]

where \( \theta_i^{k+1} > 0 \) is a parameter vector. Instead of minimizing the Equation 10, we minimize the following:

\[
w_i^{k+1} = \arg \min_{\nu_i} P_i(W_i; \theta_i^{k+1}) + \Omega(W_i).
\]

The choice of \( \theta_i^{k+1} \) is discussed in the supplementary materials.

6. Update \( b_i^{k+1} \)

The variables \( b_i^{k+1} \) are updated as follows:

\[
b_i^{k+1} = \arg \min_{\nu_i} \mathcal{L}_p(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{b_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]

which is equivalent to the following formulation:

\[
b_i^{k+1} = \arg \min_{\nu_i} \phi(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{b_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]

\[
U_i(b_i; B) = \phi(\overline{w}_i^{k+1}, b_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k)
+ \nabla w_i \phi(\overline{w}_i^{k+1}, b_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k) + (B/2)\|b_i - \beta_i^{k+1}\|_2^2,
\]

where \( B > 0 \) is a parameter. We set \( B = \nu \) for \( l = 1, \ldots, L - 1 \) and \( B = \rho \) for \( l = L \), and solve the resulting subproblems as follows:

\[
b_i^{k+1} = \arg \min_{\nu_i} U_i(b_i; \nu) \quad (l = 1, \ldots, L - 1).
\]

\[
b_L^{k+1} = \arg \min_{\nu_L} U_L(b_L; \rho).
\]

7. Update \( z_i^{k+1} \)

The variables \( z_i^{k+1} \) are updated as follows:

\[
z_i^{k+1} = \arg \min_{\nu_i} \mathcal{L}_p(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{z_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]

which is equivalent to the following forms for \( z_i(l = 1, \ldots, L - 1) \):

\[
z_i^{k+1} = \arg \min_{\nu_i} \phi(\{w_i^{k+1}\}_{i=1}^{L-1}, W_i, \{z_i^{k+1}\}_{i=1}^{L-1}, \beta_i^{k+1}, z_i^{k+1}, a_i^{k+1}, u_k),
\]
and for $z_L$:
\[
z^{k+1}_L \leftarrow \arg\min_{z_L} \phi(W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, \{a_i^{k+1}\}_{i=1}^{L-1}, z_{L-1}, a_k^{k+1}, u^k)
+ R(z_L; y).
\]

Solving Equations (14) and (15) proceeds exactly the same as solving Equations (4) and (5), respectively.

8. Update $a_i^{k+1}$
The variables $a_i^{k+1}$ $(l = 1, \cdots, L - 1)$ are updated as follows:
\[
a_i^{k+1} \leftarrow \arg\min_{a_i} L_{\rho}(W^{k+1}_i, b^{k+1}_i, z^{k+1}_i, \{a_i^{k+1}\}_{i=1}^{L-1}, a_i,
\{\tau^{k+1}_l\}_{i=1}^{L-1}, z^{k+1}_L, \{a_1^0\}_l, a_k^{k+1}, u^k),
\]

which is equivalent to the following form:
\[
a_i^{k+1} \leftarrow \arg\min_{a_i} \phi(W^{k+1}_i, b^{k+1}_i, z^{k+1}_i, \{a_i^{k+1}\}_{i=1}^{L-1}, a_i,
\{\tau^{k+1}_l\}_{i=1}^{L-1}, z^{k+1}_L, a_k^{k+1}, u^k).
\]

$Q_l(a_l; \tau^{k+1}_l)$ is defined as the quadratic approximation of $\phi$ at $a_l^{k+1}$ as follows:
\[
Q_l(a_l; \tau^{k+1}_l) = \phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, \{a_i^{k+1}\}_{i=1}^{L-1}, a_l, u_l)
+ \nabla a_l \phi(W^{k+1}_l, b^{k+1}_l, z^{k+1}_l, \{a_i^{k+1}\}_{i=1}^{L-1}, a_l, u_l)
(\alpha - \alpha^{k+1}_l) + \frac{1}{2}(\alpha - \alpha^{k+1}_l) \phi / \tau^{k+1}_l,
\]

and we can solve the following problem instead:
\[
a_i^{k+1} \leftarrow \arg\min_{a_i} Q_l(a_l; \tau^{k+1}_l).
\]

where $\tau^{k+1}_l > 0$ is a parameter vector. The solution to Equation (16) can be obtained by $a_i^{k+1} = a_i^{k+1} - \nabla a_i \phi / \tau^{k+1}_l$.
To choice of an appropriate $\tau^{k+1}_l$ is shown in the supplementary materials.

### 3.3 The Application on the GCN Problem

**Algorithm 5 The dlADMM Algorithm to Solve Problem 5**

**Require:** $\gamma, Z_0, \mu, \rho$

**Ensure:** $Z_l, W_l, l = 1, \cdots, L$.

1. Initialize: $k = 0$.
2. while $Z_0^k, W_0^k$ not converged do
3. for $l = 1$ to $L - 1$ do
4. if $l < L$ then
5. Update $Z_l^k$ in Equation (17).
6. else
7. Update $Z_l^k$ in Equation (18).
8. end if
9. Update $W_{l+1}^k$ in Equation (19).
10. end for
11. for $l = 1$ to $L$ do
12. Update $W_{l+1}^k$ in Equation (20).
13. if $l < L$ then
14. Update $Z_l^k$ in Equation (22).
15. else
16. Update $Z_l^k$ in Equation (21).
17. end if
18. end for
19. Update $k + 1 \leftarrow Z_{k+1}^k - A Z_{k+1} W_{k+1}^k$.
20. Update $U_{k+1}^k \leftarrow U_k^k + \rho e_{k+1}$.
21. end while

In this section, we apply the dlADMM algorithm to the GCN problem, which is one of the state-of-the-art architectures in Graph Neural Networks (GNNs). Important notations are listed in Table 3. We consider the semi-supervised node classification task, in which the input of GCN is an undirected and unweighted graph $G = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V}$ a set of $N$ nodes and $\mathcal{E}$ a set of edges between them.
We apply similar quadratic approximation techniques as the diADMM algorithm in detail.

1. Update $\mathbf{Z}_{l}^{k+1}$

The variable $\mathbf{Z}_{l}^{k+1}$ is updated as follows:

$$\mathbf{Z}_{l}^{k+1} \leftarrow \arg\min_{\mathbf{Z}_{l}} \mathcal{L}_{P_{l}}(\mathbf{W}_{l+1}^{k+1}, \{Z_{l}^{k+1}\}_{i=1}^{L_{l+1}}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}) = \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}),$$

where $l = 1, \ldots, L - 1$. We still use the quadratic approximation as follows:

$$\mathbf{Z}_{l}^{k+1} \leftarrow \arg\min_{\mathbf{Z}_{l}} \mathcal{L}_{Q_{l}}(\mathbf{W}_{l+1}^{k+1}, \mathbf{U}_{l}^{k}),$$

(17)

where $\mathcal{L}_{Q_{l}}(\mathbf{Z}_{l}; \mathbf{U}_{l}^{k}) = \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}) + \langle \mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle$, and $\mathbf{U}_{l}^{k} > 0$ is a parameter to be chosen via the backtracking technique, it should satisfy $\mathcal{L}_{Q_{l}}(\mathbf{Z}_{l}; \mathbf{U}_{l}^{k}) \geq \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k})$. The solution is

$$\mathbf{Z}_{l}^{k+1} \leftarrow \mathbf{Z}_{l}^{k} - \mathbf{U}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

For $l = L$, we have

$$\mathbf{Z}_{L}^{k+1} \leftarrow \arg\min_{\mathbf{Z}_{L}} \mathcal{L}_{P_{L}}(\mathbf{W}_{L+1}^{k+1}, \{Z_{L}^{k+1}\}_{i=1}^{L_{L+1}}, \mathbf{Z}_{L}^{k+1}, \mathbf{U}_{L}^{k}) = \mathcal{R}(\mathbf{Z}, \mathbf{U}) = \psi(\mathbf{W}_{L+1}^{k+1}, \{Z_{L}^{k+1}\}_{i=1}^{L_{L+1}}, \mathbf{Z}_{L}^{k+1}, \mathbf{U}_{L}^{k}).$$

(18)

We can solve Equation (18) via FISTA [40].

2. Update $\mathbf{W}_{l}^{k+1}$

The variable $\mathbf{W}_{l}^{k+1}$ is updated as follows:

$$\mathbf{W}_{l}^{k+1} \leftarrow \arg\min_{\mathbf{W}_{l}} \mathcal{L}_{P_{l}}(\mathbf{W}_{l+1}^{k+1}, \mathbf{W}_{l}, \{\mathbf{W}_{l}^{k+1}\}_{i=1}^{L_{l+1}}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}) = \psi(\mathbf{W}_{l+1}^{k+1}, \{\mathbf{W}_{l}^{k+1}\}_{i=1}^{L_{l+1}}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

We apply similar quadratic approximation techniques as follows:

$$\mathbf{W}_{l}^{k+1} \leftarrow \arg\min_{\mathbf{W}_{l}} \mathcal{L}_{P_{l}}(\mathbf{W}_{l+1}^{k+1}, \mathbf{U}_{l}^{k+1}),$$

(19)

where $\mathcal{L}_{P_{l}}(\mathbf{W}_{l}; \mathbf{U}_{l}^{k+1}) = \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}) + \langle \mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle$, and $\mathbf{U}_{l}^{k+1}$ is a parameter to be chosen via the backtracking technique. It should satisfy:

$$\mathcal{L}_{P_{l}}(\mathbf{W}_{l}; \mathbf{U}_{l}^{k+1}) \geq \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

The solution to Equation (19) is:

$$\mathbf{W}_{l}^{k+1} \leftarrow \mathbf{W}_{l}^{k} - \mathbf{U}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

(20)

3. Update $\mathbf{U}_{l}^{k+1}$

The variable $\mathbf{U}_{l}^{k+1}$ is updated as follows:

$$\mathbf{U}_{l}^{k+1} \leftarrow \arg\min_{\mathbf{U}_{l}} \mathcal{L}_{P_{l}}(\mathbf{W}_{l+1}^{k+1}, \mathbf{W}_{l}, \{\mathbf{W}_{l}^{k+1}\}_{i=1}^{L_{l+1}}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}) = \psi(\mathbf{W}_{l+1}^{k+1}, \{\mathbf{W}_{l}^{k+1}\}_{i=1}^{L_{l+1}}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

(21)

Similarly, we define $P_{l}(\mathbf{W}_{l}; \mathbf{U}_{l}^{k+1})$ as the quadratic approximation of $\psi$ at $\mathbf{W}_{l}^{k+1}$ that is formulated as below:

$$P_{l}(\mathbf{W}_{l}; \mathbf{U}_{l}^{k+1}) = \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}) + \langle \mathbf{W}_{l+1}^{k+1}, \mathbf{W}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle + \langle \mathbf{U}_{l+1}^{k+1}, \mathbf{W}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle + \langle \mathbf{U}_{l+1}^{k+1}, \mathbf{W}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle + \langle \mathbf{U}_{l+1}^{k+1}, \mathbf{U}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle + \langle \mathbf{U}_{l+1}^{k+1}, \mathbf{U}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k} \rangle$$

where $\mathbf{U}_{l}^{k+1} > 0$ is a parameter to be chosen via the backtracking technique. It should satisfy:

$$P_{l}(\mathbf{W}_{l}; \mathbf{U}_{l}^{k+1}) \geq \psi(\mathbf{W}_{l+1}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}).$$

The solution to Equation (20) is:

$$\mathbf{W}_{l+1}^{k+1} \leftarrow \mathbf{W}_{l+1}^{k+1} - \mathbf{U}_{l+1}^{k+1}, \mathbf{W}_{l}^{k+1}, \mathbf{Z}_{l}^{k+1}, \mathbf{U}_{l}^{k}).$$

(21)

We can solve Equation (21) by FISTA [40].

4. Convergence Analysis

In this section, the theoretical convergence of the proposed diADMM algorithm is analyzed. For the sake of simplicity, we only prove its convergence in the MLP problem. Similar convergence results can be derived for the GCN problem. Before we formally present the convergence results of the diADMM algorithms, Section 4.1 presents necessary assumptions to guarantee the convergence of diADMM. In Section 4.2, we prove the convergence of the diADMM algorithm to a critical point with a sublinear convergence rate $o(1/k)$.
4.1 Assumptions

Assumption 1 (Closed-form Solutions). There exist activation functions \( a_i = f_i(z_i) \) such that Equations \( 4 \) and \( 14 \) have closed form solutions \( z_i^{k+1} = h_i(W_i^{k+1}, b_i^k, a_i^{k-1}) \) and \( z_k^{k+1} = h(W_k^{k+1}, b_k^k, a_k^{k-1}) \), respectively, where \( h(\bullet) \) are continuous functions.

This assumption can be commonly used activation functions such as ReLU and leaky ReLU. For example, for the ReLU function \( a_i = \max(z_i, 0) \), Equation \( 14 \) has the following solution:

\[
z_i^{k+1} = \begin{cases} 
\min(W_i^{k+1} a_i^{k-1} + b_i^k, 0) & z_i^{k+1} \leq 0 \\
\max((W_i^{k+1} a_i^{k-1} + b_i^k + a_i^{k-1})/2, 0) & z_i^{k+1} > 0 
\end{cases}
\]

Assumption 2 (Objective Function). \( F(W, b, z, a) \) is coercive over the nonempty set \( G = \{ (W, b, z, a) : z_L - W_L a_{L-1} - b_L = 0 \} \). In other words, \( F(W, b, z, a) \to -\infty \) if \( (W, b, z, a) \in G \) and \( ||(W, b, z, a)|| \to \infty \). Moreover, \( R(z; y) \) is Lipschitz differentiable with Lipschitz constant \( \geq 0 \).

The Assumption 2 is mild enough for most common loss functions to satisfy. For example, the cross-entropy and square loss are Lipschitz differentiable.

4.2 Key Properties

We present the main convergence result of the proposed diADMM algorithm in this section. Specifically, as long as Assumptions \([1, 2] \) hold, then Properties \([3] \) are satisfied, which are important to prove the global convergence of the proposed diADMM algorithm. The proof details are included in the supplementary materials.

Property 1 (Boundness). If \( \rho > 2H \), then \( \{W^k, b^k, z^k, a^k, u^k\} \) is bounded, and \( L_\rho(W^k, b^k, z^k, a^k, u^k) \) is lower bounded.

Property 2 (Sufficient Descent). If \( \rho > 2H \) so that \( C_1 = \rho/2 - H^2/2 > 0 \), then there exists \( C_2 = \min(\nu/2, C_1, \{\vartheta_i^{k+1}\}_{i=1}^L, \{\vartheta_i^{k+1}\}_{i=1}^L, \{\tau_i^{k+1}\}_{i=1}^L, \{\tau_i^{k+1}\}_{i=1}^L) \) such that

\[
L_\rho(W^k, b^k, z^k, a^k, u^k) \geq C_2 \sum_{i=1}^L (||W_i^{k+1} - W_i^k||^2 + ||W_i^{k+1} - W_i^k||^2 + ||b_i^{k+1} - b_i^k||^2 + ||a_i^{k+1} - a_i^k||^2 + ||z_i^{k+1} - z_i^k||^2 + ||z_i^{k+1} - z_i^k||^2 + ||z_i^{k+1} - z_i^k||^2 + ||z_i^{k+1} - z_i^k||^2).
\]

Property 3 (Subgradient Bound). There exist a constant \( C > 0 \) and \( g \in \partial L_\rho(W^k, b^k, z^k, a^k, u^k) \) such that

\[
||g|| \leq C(||W^k - W_i^{k+1}|| + ||b^k - b_i^{k+1}|| + ||z^k - z_i^{k+1}|| + ||a^k - a_i^{k+1}|| + ||z^k - z_i^{k+1}||).
\]

Property 3 ensures that the subgradient of the objective function is bounded by variables. The proof of Property 3 requires Property 1 and the proof is elaborated in the supplementary materials. Now the global convergence of the diADMM algorithm is presented. The following theorem states that Properties \([1, 2] \) are guaranteed.

Theorem 1. For any \( \rho > 2H \), if Assumptions \([1, 2] \) are satisfied, then Properties \([3] \) hold.

The next theorem presents the global convergence of the diADMM algorithm.

Theorem 2 (Global Convergence). If \( \rho > 2H \), then for the variables \( (W, b, z, u) \) in Problem \( 4 \) starting from any \( (W^0, b^0, z^0, a^0, u^0) \), it has at least a limit point \( (W^*, b^*, z^*, a^*, u^*) \), and any limit point \( (W^*, b^*, z^*, a^*, u^*) \) is a critical point of Problem \( 4 \). That is, \( 0 \in \partial L_\rho(W^*, b^*, z^*, a^*, u^*) \).

Proof. Because \( \{W^k, b^k, z^k, a^k, u^k\} \) is bounded, there exists a subsequence \( \{W^k, b^k, z^k, a^k, u^k\} \) such that \( \{W^k, b^k, z^k, a^k, u^k\} \to (W^*, b^*, z^*, a^*, u^*) \) and \( \{W^k, b^k, z^k, a^k, u^k\} \) is non-increasing and lower bounded and hence converges. By Property 2 we prove that \( ||W^k - W^*|| \to 0 \), \( ||b^k - b^*|| \to 0 \), \( ||a^k - a^*|| \to 0 \), \( ||z^k - z^*|| \to 0 \), \( ||b^k - b^*|| \to 0 \), \( ||a^k - a^*|| \to 0 \), \( ||z^k - z^*|| \to 0 \), and \( ||z^k - z^*|| \to 0 \), as \( k \to \infty \). Therefore \( ||W^k - W^*|| \to 0 \), \( ||b^k - b^*|| \to 0 \), and \( ||z^k - z^*|| \to 0 \), as \( k \to \infty \). Moreover, from Assumption the we know that \( z^k \to z^* \) and \( z^k \to z^* \) as \( k \to \infty \). Therefore, \( z^k \to z^* \). In other words, the limit point \( (W^*, b^*, z^*, a^*, u^*) \) is a critical point of \( L_\rho \) defined in Equation \( 1 \).

Theorem 2 shows that our diADMM algorithm converges globally for sufficiently large \( \rho \), which is consistent with previous literature \([24, 43] \). The next theorem shows that the diADMM converges globally with a sublinear convergence rate \( o(1/k) \).

Theorem 3 (Convergence Rate). For a sequence \( (W^k, b^k, z^k, a^k, u^k) \), define \( c_k = \min_{i \leq k} \sum_{i=1}^L (||W_i^k - W_i^k||^2 + ||b_i^k - b_i^k||^2 + ||a_i^k - a_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2) + \sum_{i=1}^L (||W_i^k - W_i^k||^2 + ||b_i^k - b_i^k||^2 + ||a_i^k - a_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2 + ||z_i^k - z_i^k||^2), then the convergence rate of \( c_k \) is \( o(1/k) \).

Proof. The proof of this theorem is included in the supplementary materials.
5 Experiments

In this section, the proposed diADMM is evaluated on MLP models and GCN models using seven benchmark datasets. Effectiveness, efficiency, and convergence of the proposed diADMM algorithm are compared with state-of-the-art optimizers. The experiments on MLP models were conducted on a 64-bit machine with Intel(R) Xeon processor and GTX1080Ti GPU, and the experiments on GCN models were conducted on a 64-bit machine with Xeon processor and NVIDIA Quadro RTX 5000 GPU.

5.1 Experiments on MLP Models

5.1.1 Experiment Setup

For MLP models, two benchmark datasets were used for performance evaluation: MNIST [44] and Fashion MNIST [45]. The MNIST dataset has ten classes of handwritten-digit images, which was firstly introduced by LeCun et al. in 1998 [44]. Unlike the MNIST dataset, the Fashion MNIST dataset has ten classes of assortment images on the website of Zalando, which is Europe’s largest online fashion platform [45]. Their statistics are available in Table 4. We set up a network architecture which contained two hidden layers with 1,000 neurons each. The ReLU was used for the activation function for both network structures. The loss function was set as the deterministic cross-entropy loss. \( \nu \) and \( \rho \) were both set as \( 10^{-6} \). The number of iteration (i.e. epoch) was set to 200.

GD and its variants and ADMM serve as comparison methods. GD-based methods trained models in a full-batch fashion. All parameters were chosen based on the optimal training accuracy. All comparison methods are outlined as follows:

1. Gradient Descent (GD) [46]. The GD updates parameters based on their gradients. The learning rate of GD was set to \( 10^{-6} \).

2. Adaptive gradient (Adagrad) [10]. Rather than fixing the learning rate, Adagrad adapts the learning rate to some hyperparameter. The learning rate was set to \( 10^{-3} \).

3. Adaptive learning rate (Adadelta) [47]. The Adadelta is proposed to overcome the sensitivity to hyperparameter selection. The learning rate was set to 0.1.

4. Adaptive momentum estimation (Adam) [9]. The Adam estimates the first and second moments in order to correct the biased gradient. The learning rate of Adam was set to \( 10^{-3} \).

5. Alternating Direction Method of Multipliers (ADMM) [3]. ADMM is a powerful convex optimization method because it can split an objective function into a series of subproblems, which are coordinated to get global solutions. It is scalable to large-scale datasets and supports parallel computations. The \( \rho \) of ADMM was set to 1.

![Fig. 2. Convergence curves of the proposed diADMM on two datasets when \( \rho = 1 \).](image)

![Fig. 3. Divergence curves of the proposed diADMM on two datasets when \( \rho = 10^{-6} \).](image)

Table 5: The relationship between running time per epoch (in second) and the number of neurons for each layer as well as value of \( \rho \) when the training size was fixed: generally, the running time increased as the number of neurons and the value of \( \rho \) became larger.

| Dataset          | Feature# | Class# | Training Sample# | Test Sample# |
|------------------|----------|--------|------------------|--------------|
| MNIST            | 784      | 10     | 55,000           | 10,000       |
| Fashion MNIST    | 784      | 10     | 60,000           | 10,000       |

![Table 4: Statistics of two benchmark datasets on MLP models.](image)
Performance: Figure 4 shows the test accuracy of our proposed diADMM and all comparison methods on two datasets. Overall, our proposed diADMM outperforms most of them on both training accuracy and test accuracy on two datasets. Specifically, the curves of our proposed diADMM soar to 0.8 at the early stage and then rise steadily towards more than 0.9. The curves of the most GD-related methods, such as GD, Adadelta, and Adagrad, climb more slowly than our proposed diADMM. The curves of the proposed ADMM also rocket to around 0.8, but decrease slightly since then. Only the Adam performs better than the proposed diADMM marginally by around 4%.

Efficiency Analysis: Finally, the relationship between running time per epoch of our proposed diADMM and three potential factors, namely, the value of $\rho$, the size of training samples, and the number of neurons was explored. The running time was calculated by the average of 200 iterations. The running time per epoch of our proposed diADMM and three ADMM also rocket to around 0.8, but decrease slightly since then. Only the Adam performs better than the proposed diADMM marginally by around 4%.

5.2 Experiments on GCN Models

5.2.1 Experiment Setup

For GCN models, five benchmark datasets were used for performance evaluation, which are shown in Table 6. All of them are outlined as follows: 1. Cora [48]: The Cora dataset consists of 2708 scientific publications classified into one of seven classes. 2. PubMed [48]: PubMed comprises 30M+ citations for biomedical literature that have been collected from sources such as MEDLINE, life science journals, and published online e-books. 3. Citeseer [48]: The Citeseer dataset was collected from the Tagged.com social network website. The original task on the dataset is to identify (i.e., classify) the spammer users based on their relational and non-relational features. 4. Coauthor CS and Coauthor Physics [49]: They are co-authorship graphs based on the Microsoft Academic Graph from the KDD Cup 2016 challenge 3. Here, nodes are authors, that are connected by an edge if they co-authored a paper.

We set up a GCN model which contained one hidden layer with 128 neurons. We choose this shallow model because the performance of GCN models deteriorates as they go deeper due to the over-smoothing problem [50]. The ReLU was used for the activation function. The loss function was set as the cross-entropy loss. $\nu$ and $\rho$ were both set to $10^{-3}$ based on the optimal training accuracy. The number of epoch was set to 500.

GD [46], Adagrad [10], Adadelta [47], and Adam [9] are utilized to compare performance. The full batch dataset was used for training models. All parameters were chosen by the accuracy of the training dataset, and the learning rates for GD, Adagrad, Adadelta and Adam were set to 0.1, 0.001, 0.001 and 0.01, respectively.

5.2.2 Experimental Results

The experimental results on GCN models are analyzed in this section. Convergence: Figures 5 and 6 show convergence curves...
and divergence curves of the proposed dADMM on three datasets when $\rho = 1$ and $\rho = 10^{-3}$, respectively. When $\rho = 1$ (i.e. in Figure 5), the objectives tumble down monotonously and smoothly, and the residuals converge to 0, while the objective and the residual on the Cora dataset fluctuate when $\rho = 10^{-3}$ (i.e. in Figure 6). Specifically, in Figure 6(a), the objective on the Cora dataset keeps decreasing, whereas those on the PubMed and Citeseer datasets converge within 100 epochs. The residuals on three datasets converge to 0 before 100 epochs, as demonstrated in Figure 5(b). In Figure 6(a), the objective on the Cora dataset increases with the increase of epochs, while it decreases on the other two datasets. For the residuals, Figure 6(b) shows that the fluctuation on the Cora dataset is more drastic than that on the PubMed and Citeseer datasets.

**Performance:** Table 8 shows the test accuracy of the proposed dADMM and all comparison methods on five datasets. They were averaged by 10 random initializations. Overall, our proposed dADMM outperforms others marginally on four out of five datasets. Specifically, it performs 0.76 on the PubMed dataset, 1% better than Adam, which is the best of all comparison methods. For other datasets such as Coauthor CS and Coauthor Physics, while all methods perform well, our proposed dADMM achieves marginally better performance by about 0.4%. For the Citeseer: from 16 to 256 neurons

| neuron | 16 | 32 | 64 | 128 | 256 |
|--------|----|----|----|-----|-----|
| $\rho$ | 10^{-6} | 0.8083 | 0.8187 | 0.8247 | 0.8457 | 0.8448 |
| $\rho$ | 10^{-9} | 0.9120 | 0.9846 | 0.9921 | 0.9331 | 0.9547 |
| $\rho$ | 10^{-4} | 1.0560 | 1.0862 | 1.0894 | 1.0832 | 1.0784 |
| $\rho$ | 10^{-3} | 1.1179 | 1.1366 | 1.1564 | 1.1760 | 1.2041 |
| $\rho$ | 10^{-2} | 1.1749 | 1.2078 | 1.1609 | 1.2149 | 1.1972 |

Cora: from 16 to 256 neurons

| neuron | 16 | 32 | 64 | 128 | 256 |
|--------|----|----|----|-----|-----|
| $\rho$ | 10^{-6} | 0.1848 | 0.1756 | 0.1710 | 0.1806 | 0.1860 |
| $\rho$ | 10^{-9} | 0.2082 | 0.2056 | 0.2081 | 0.2121 | 0.2191 |
| $\rho$ | 10^{-4} | 0.2483 | 0.2439 | 0.2374 | 0.2495 | 0.2338 |
| $\rho$ | 10^{-3} | 0.2836 | 0.2791 | 0.2780 | 0.2877 | 0.2944 |
| $\rho$ | 10^{-2} | 0.2964 | 0.2964 | 0.3058 | 0.3210 | 0.3335 |

Coauthor CS: from 16 to 256 neurons

| neuron | 16 | 32 | 64 | 128 | 256 |
|--------|----|----|----|-----|-----|
| $\rho$ | 10^{-6} | 1.1163 | 1.0684 | 1.0826 | 1.1041 | 1.1182 |
| $\rho$ | 10^{-9} | 1.2663 | 1.2495 | 1.1517 | 1.1334 | 1.1328 |
| $\rho$ | 10^{-4} | 1.2480 | 1.2380 | 1.1963 | 1.2013 | 1.2165 |
| $\rho$ | 10^{-3} | 1.2401 | 1.2445 | 1.3293 | 1.2728 | 1.3161 |
| $\rho$ | 10^{-2} | 1.2108 | 1.1952 | 1.3281 | 1.2728 | 1.3147 |

Coauthor Physics: from 16 to 256 neurons

| neuron | 16 | 32 | 64 | 128 | 256 |
|--------|----|----|----|-----|-----|
| $\rho$ | 10^{-6} | 0.8890 | 0.8926 | 0.9154 | 0.8955 | 0.9585 |
| $\rho$ | 10^{-9} | 0.9929 | 0.9945 | 1.0649 | 1.0766 | 1.1329 |
| $\rho$ | 10^{-4} | 1.1891 | 1.1159 | 1.2319 | 1.1663 | 1.2309 |
| $\rho$ | 10^{-3} | 1.2210 | 1.1574 | 1.2737 | 1.2491 | 1.3408 |
| $\rho$ | 10^{-2} | 1.2052 | 1.1653 | 1.2782 | 1.2573 | 1.5391 |

The relationship between running time per epoch (in second) and the number of neurons for each layer as well as value of $\rho$ generally, the running time increases with the increase of the value of $\rho$.

| neuron | 16 | 32 | 64 | 128 | 256 |
|--------|----|----|----|-----|-----|
| $\rho$ | 10^{-6} | 0.5889 | 0.5926 | 0.6154 | 0.5955 | 0.6685 |
| $\rho$ | 10^{-9} | 0.9929 | 0.9945 | 1.0649 | 1.0766 | 1.1329 |
| $\rho$ | 10^{-4} | 1.1891 | 1.1159 | 1.2319 | 1.1663 | 1.2309 |
| $\rho$ | 10^{-3} | 1.2210 | 1.1574 | 1.2737 | 1.2491 | 1.3408 |
| $\rho$ | 10^{-2} | 1.2052 | 1.1653 | 1.2782 | 1.2573 | 1.5391 |

### Table 8

Performance on five datasets averaged by 10 initializations: the proposed dADMM performs the best on most of them.
seer dataset, GD is the best optimizer, followed by the proposed diADMM, which is 0.4% inferior in performance. The GD performs differently on five datasets: it is inferior to others on the PubMed dataset, but performs competitively on the Cora dataset. Compared with other optimizers, the Adadelta performs poorly in general: its performance on the Cora, PubMed, and Citeseer datasets is 9%, 7% and 5% inferior to that of the proposed diADMM, respectively. Last but not least, the standard deviations of all methods are small, which suggests that they are resistant to random noises.

**Efficiency Analysis:** Next, we explore the relationship between running time per epoch of our proposed diADMM and two potential factors, namely, the value of $\rho$, and the number of neurons. The running time was calculated by the average of 500 epochs.

The running time on five datasets is shown in Table 9. The number of neurons for each layer ranged from 16 to 256. The value of $\rho$ ranged from $10^{-6}$ to $10^{-2}$, with being multiplied by 10 each time. Generally, the running time increases in proportion to the value of $\rho$. For example, the running time per epoch rises from 0.8 to 1.17 when the number of neurons is 16 on the Citeseer dataset, and it climbs from 0.167 to 0.28 on the Cora dataset correspondingly. Moreover, we also find that the running time increases with the increase of the number of neurons on some datasets. As an instance, when $\rho = 10^{-2}$, the running time grows from 1.21 to 1.54 as the number of neurons increases from 16 to 256. However, the running time per epoch remains constant on other datasets. As an example, the running time keeps around 0.18 when $\rho = 10^{-6}$ on the Cora and PubMed datasets.

**Memory Analysis:** Finally, we investigate the memory usages of our proposed diADMM on all datasets, which is shown in Figure 7. A blue bar and a green bar denote 16 neurons and 256 neurons, respectively. The Coauthor-Physics consumes the most memory of all datasets. It occupies more than 2500MB, whereas the Cora dataset uses less than 100MB. Compare Figure 7 with Table 7, we find that the used memories are approximately in proportion to the number of nodes on five datasets. Moreover, the effect of the number of neurons is slim on the memory usages: the memory used on the same dataset is almost the same for 16 neurons and 256 neurons. Therefore, we infer datasets, instead of model architectures, are the main component of memory usages.

6 CONCLUSION

Alternating Direction Method of Multipliers (ADMM) is a good alternative to Gradient Descent (GD) for deep learning problems. In this paper, we propose a novel deep learning Alternating Direction Method of Multipliers (diADMM) to address some previously mentioned challenges. Firstly, the diADMM updates parameters from backward to forward in order to transmit parameter information more efficiently. The time complexity is successfully reduced from $O(n^3)$ to $O(n^2)$ by iterative quadratic approximations and backtracking. Finally, the diADMM is guaranteed to converge to a critical point under mild conditions with a sublinear convergence rate $o(1/k)$. Experiments on MLP and GCN models on seven benchmark datasets demonstrate not only the convergence and efficiency of our proposed diADMM algorithm, but also its outstanding performance against comparison methods.

**References**

[1] Y. LeCun, Y. Bengio, and G. Hinton, “Deep learning,” nature, vol. 521, no. 7553, p. 436, 2015.

[2] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” Foundations and Trends® in Machine Learning, vol. 3, no. 1, pp. 1–122, 2011.

[3] G. Taylor, R. Burmeister, Z. Xu, B. Singh, A. Patel, and T. Goldstein, “Training neural networks without gradients: A scalable admm approach,” in International Conference on Machine Learning, 2016, pp. 2722–2731.

[4] A. Krizhevsky, I. Sutskever, and G. E. Hinton, “Imagenet classification with deep convolutional neural networks,” in Advances in neural information processing systems, 2012, pp. 1097–1105.

[5] T. Mikolov, M. Karafiát, L. Burget, J. Černocký, and S. Khudanpur, “Recurrent neural network based language model,” in Eleventh Annual Conference of the International Speech Communication Association, 2010.

[6] D. E. Rumelhart, G. E. Hinton, and R. J. Williams, “Learning representations by back-propagating errors,” nature, vol. 323, no. 6088, p. 533, 1986.

[7] B. T. Polyak, “Some methods of speeding up the convergence of iteration methods,” USSR Computational Mathematics and Mathematical Physics, vol. 4, no. 5, pp. 1–17, 1964.

[8] I. Sutskever, J. Martens, G. Dahl, and G. Hinton, “On the importance of initialization and momentum in deep learning,” in International Conference on machine learning, 2013, pp. 1139–1147.

[9] D. P. Kingma and J. Ba, “Adam: A method for stochastic optimization,” Proceedings of the 3rd International Conference on Learning Representations (ICLR), 2015.

[10] J. Duchi, E. Hazan, and Y. Singer, “Adaptive subgradient methods for online learning and stochastic optimization,” Journal of Machine Learning Research, vol. 12, no. Jul, pp. 2121–2159, 2011.

[11] T. Tieleman and G. Hinton, “Divide the gradient by a running average of its recent magnitude: coursera: Neural networks for machine learning,” Technical Report, Tech. Rep.

[12] S. J. Reddi, S. Kale, and S. Kumar, “On the convergence of adam and beyond,” in International Conference on Learning Representations, 2018.

[13] L. Luo, Y. Xiong, and Y. Liu, “Adaptive gradient methods with dynamic bound of learning rate,” in International Conference on Learning Representations, 2019.

[14] X. Chen, S. Liu, R. Sun, and M. Hong, “On the convergence of a class of adam-type algorithms for non-convex optimization,” in International Conference on Learning Representations, 2019.

[15] N. S. Keskar and R. Socher, “Improving generalization performance by switching from adam to sgd,” arXiv preprint arXiv:1712.07628, 2017.

[16] A. C. Wilson, R. Roelofs, M. Stern, N. Srebro, and B. Recht, “The marginal value of adaptive gradient methods in machine learning,” in NIPS, 2017, pp. 4151–4161.

[17] H. He, G. Huang, and Y. Yuan, “Asymmetric valleys: Beyond sharp and flat local minima,” in Advances in Neural Information Processing Systems, vol. 32, pp. 2553–2564, 2019.

[18] P. Izmailov, A. Wilson, D. Podoprikhin, D. Vetrov, and T. Garipov, “Averaging weights leads to wider optima and better generalization,” in 34th Conference on Uncertainty in Artificial Intelligence 2018, UAI 2018, 2018, pp. 876–885.

[19] N. S. Keskar, J. Nocedal, P. T. P. Tang, M. Mudigere, and M. Smelyanskiy, “On large-batch training for deep learning: Generalization gap and sharp minima,” in 5th International Conference on Learning Representations, ICLR 2017, 2017.

[20] H. Li, Z. Xu, G. Taylor, C. Studer, and T. Goldstein, “Visualizing the loss landscape of neural nets,” in Proceedings of the 32nd International Conference on Neural Information Processing Systems, 2018, pp. 6391–6401.

[21] P. Zhou, J. Feng, C. Ma, C. Xiong, S. C. H. Hoi et al., “Towards theoretically understanding why sgd generalizes better than adam in deep learning,” Advances in Neural Information Processing Systems, vol. 33, 2020.
[22] J. Zeng, T.-K. Lau, S. Lin, and Y. Yao, “Global convergence of block coordinate descent in deep learning,” in Proceedings of the 36th International Conference on Machine Learning, ser. Proceedings of Machine Learning Research, K. Chaudhuri and R. Salakhutdininov, Eds., vol. 97, Long Beach, California, USA: PMLR, 09–15 Jun 2019, pp. 7313–7323.

[23] A. Choromanska, B. Cowen, S. Kumaravel, R. Luss, M. Rigotti, I. Rish, P. Diachille, V. Gurov, B. Kingsbury, R. Tejwani et al., “Beyond backprop: Online alternating minimization with auxiliary variables,” in International Conference on Machine Learning. PMLR, 2019, pp. 1193–1202.

[24] M. Carreira-Perpinan and W. Wang, “Distributed optimization of deeply nested systems,” in Artificial Intelligence and Statistics, 2014, pp. 10–19.

[25] S. Lu, N. Khan, I. Y. Akhalwaya, R. Riegel, L. Horesh, and A. Gray, “Training logical neural networks by primal–dual methods for neuro-symbolic reasoning,” in ICASSP 2021-2021 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2021, pp. 5559–5563.

[26] Y. Tang, Z. Kan, D. Sun, L. Qiao, J. Xiao, Z. Lai, and D. Li, “Admimrn: Training rn with stable convergence via an efficient admm approach,” in Machine Learning and Knowledge Discovery in Databases (ECML-PKDD) 2020, F. Hutter, K. Kersting, J. Lijffijt, and I. Valera, Eds. Cham: Springer International Publishing, 2021, pp. 3–18.

[27] L. Guan, Z. Yang, D. Li, and X. Lu, “pdladmm: An admm-based framework for parallel deep learning training with efficiency,” Neurocomputing, vol. 435, pp. 264–272, 2021.

[28] S. Khorram, X. Fu, M. H. Danesh, Z. Qi, and L. Fuxin, “Stochastic block-admm for training deep networks,” arXiv preprint arXiv:2105.00339, 2021.

[29] L. Qiao, T. Sun, H. Pan, and D. Li, “Inertial proximal deep learning alternating minimization for efficient neutral network training,” in ICASSP 2021-2021 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2021, pp. 3895–3899.

[30] J. Zhang, Z. Chai, Y. Cheng, and L. Zhao, “Toward model parallelism for deep neural network based on gradient-free admm framework,” in 2020 IEEE International Conference on Data Mining (ICDM). IEEE, 2020, pp. 591–600.

[31] J. Wang and L. Zhao, “Multi-convex inequality-constrained alternating direction method of multipliers,” arXiv preprint arXiv:1902.10882, 2019.

[32] Y. Wang, W. Yin, and J. Zeng, “Global convergence of admm in nonconvex nonsmooth optimization,” Journal of Scientific Computing, pp. 1–35, 2015.

[33] C. Chen, M. Li, X. Liu, and Y. Ye, “Extended admm and bcd for nonseparable convex minimization models with quadratic coupling terms: convergence analysis and insights,” Mathematical Programming, pp. 1–41, 2015.

[34] J. Wang and L. Zhao, “Nonconvex generalization of alternating direction method of multipliers for nonlinear equality constrained problems,” Results in Control and Optimization, p. 100009, 2021.

[35] W. Ouyang, Y. Peng, Y. Yao, J. Zhang, and B. Deng, “Anderson acceleration for nonconvex admm based on douglas-rachford splitting,” in Computer Graphics Forum, vol. 39, no. 5. Wiley Online Library, 2020, pp. 221–239.

[36] J. Zhang, Y. Peng, W. Ouyang, and B. Deng, “Accelerating admm for efficient simulation and optimization,” ACM Transactions on Graphics (TOG), vol. 38, no. 6, pp. 1–21, 2019.

[37] Q. Liu, X. Shen, and Y. Gu, “Linearized admm for nonconvex nonsmooth optimization with convergence analysis,” IEEE Access, vol. 7, pp. 76 131–76 144, 2019.

[38] A. Themelis and P. Patrinos, “Douglas–rachford splitting and admm for nonconvex optimization: Tight convergence results,” SIAM Journal on Optimization, vol. 30, no. 1, pp. 149–181, 2020.

[39] Y. Gao, L. Zhao, L. Wu, Y. Ye, H. Xiong, and C. Yang, “Incomplete label mult-task deep learning for spatiotemporal event subtype forecasting,” 2019.

[40] A. Beck and M. Teboulle, “A fast iterative shrinkage-thresholding algorithm for linear inverse problems,” SIAM journal on imaging sciences, vol. 2, no. 1, pp. 183–202, 2009.

[41] Z. Wu, P. Pan, F. Chen, G. Long, C. Zhang, and P. S. Yu, “A comprehensive survey on graph neural networks,” IEEE Transactions on Neural Networks and Learning Systems, vol. 32, no. 1, pp. 4–24, 2021.

[42] R. T. Rockafellar and R. J.-B. Wets, Variational analysis. Springer Science & Business Media, 2009, vol. 317.
Supplementary Materials

APPENDIX

ALGORITHMS TO UPDATE $W_{l}^{k+1}$ AND $a_{l}^{k+1}$

The algorithms to update $W_{l}^{k+1}$ and $a_{l}^{k+1}$ are described in the Algorithms 6 and 7 respectively.

Algorithm 6 The Backtracking Algorithm to update $W_{l}^{k+1}$

Require: $W_{l-1}^{k+1}, b_{l-1}^{k+1}, z_{l-1}^{k+1}, a_{l-1}^{k+1}, u^{k}, \rho$, some constant $\gamma > 1$.
Ensure: $W_{l}^{k+1}$.

1: Pick up $\alpha$ and $\zeta = W_{l}^{k+1} - \nabla W_{l}^{k+1} \phi / \alpha$.
2: while $\phi(W_{l}^{k+1}) - \nabla W_{l}^{k+1} \phi + \theta_{l}^{k+1} \circ (W_{l}^{k+1} - W_{l}) + \zeta > P_{l}(\zeta; \alpha)$ do
3: $\alpha \leftarrow \alpha \gamma$.
4: Solve $\zeta$ by Equation (11).
5: end while
6: Output $\theta_{l}^{k+1} \leftarrow \alpha$.
7: Output $W_{l}^{k+1} \leftarrow \zeta$.

Algorithm 7 The Backtracking Algorithm to update $a_{l}^{k+1}$

Require: $W_{l}^{k+1}, b_{l}^{k+1}, z_{l}^{k+1}, a_{l-1}^{k+1}, u^{k}, \rho$, some constant $\eta > 1$.
Ensure: $a_{l}^{k+1}$.

1: Pick up $t$ and $\beta = a_{l}^{k+1} - \nabla a_{l}^{k+1} \phi / t$.
2: while $\phi(W_{l}^{k+1}) - \nabla W_{l}^{k+1} \phi + \theta_{l}^{k+1} \circ (W_{l}^{k+1} - W_{l}) + \zeta > P_{l}(\zeta; t)$ do
3: $t \leftarrow t \eta$.
4: $\beta \leftarrow \eta^{k} - \nabla a_{l}^{k+1} \phi / t$.
5: end while
6: Output $\tau_{l}^{k+1} \leftarrow t$.
7: Output $a_{l}^{k+1} \leftarrow \beta$.

LEMMAS FOR THE PROOFS OF PROPERTIES

The following several lemmas are preliminary results.

Lemma 1. Equation (9) holds if and only if there exists $\pi \in \partial \Omega_{t}(W_{l}^{k+1})$, the subgradient of $\Omega_{t}(W_{l}^{k+1})$ such that

$$\nabla W_{l}^{k} \phi(W_{l}^{k+1}) - \nabla W_{l}^{k+1} \phi + \theta_{l}^{k+1} \circ W_{l}^{k+1} - W_{l} = 0$$

Likewise, Equation (11) holds if and only if there exists $s \in \partial \Omega_{t}(W_{l}^{k+1})$, the subgradient of $\Omega_{t}(W_{l}^{k+1})$ such that

$$\nabla W_{l}^{k+1} \phi(W_{l}^{k+1}) - \nabla W_{l}^{k+1} \phi + \theta_{l}^{k+1} \circ W_{l}^{k+1} - W_{l} = s = 0$$

Proof. These can be obtained by directly applying the optimality conditions of Equation (9) and Equation (11), respectively.

Lemma 2. $\nabla_{z_{L}} R(z_{L}; y) + u^{k} = 0$ for all $k \in \mathbb{N}$.

Proof. The optimality condition of $z_{L}$ in Equation (15) gives rise to

$$\nabla_{z_{L}} R(z_{L}; y) + \rho(z_{L}^{k} - W_{L} u_{L}^{k} - b_{L}^{k}) + u^{k-1} = 0$$

Because $u^{k} = u^{k-1} + \rho(z_{L}^{k} - W_{L} u_{L}^{k} - b_{L}^{k})$, then we have $\nabla_{z_{L}} R(z_{L}; y) + u^{k} = 0$.

Lemma 3. It holds that $\forall z_{L,1}, z_{L,2} \in \mathbb{R}^{nL}$,

$$R(z_{L,1}; y) \leq R(z_{L,2}; y) + \nabla_{z_{L,2}} R^{T}(z_{L,2}; y)(z_{L,1} - z_{L,2}) + (H/2) \|z_{L,1} - z_{L,2}\|^{2}$$

$$- R(z_{L,1}; y) \leq -R(z_{L,2}; y) - \nabla_{z_{L,2}} R^{T}(z_{L,2}; y)(z_{L,1} - z_{L,2}) + (H/2) \|z_{L,1} - z_{L,2}\|^{2}$$

Proof. Because $R(z_{L}; y)$ is Lipschitz differentiable by Assumption 2 so is $-R(z_{L}; y)$. Therefore, this lemma is proven exactly as same as Lemma 2.1 in [40].
Lemma 4. For Equations (37) and (12), if \( T, B \geq \nu \), then the following inequalities hold:

\[
\mathcal{U}_l(b_l^{k+1} ; B) \geq \phi(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k) \quad (25)
\]

\[
\mathcal{U}_l(b_l^{k+1} ; B) \geq \phi(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k) \quad (26)
\]

Proof. Because \( \phi(W, b, z, a, u) \) is Lipschitz differentiable with respect to \( b \) with Lipschitz coefficient \( \nu \) (the definition of Lipschitz differentiability can be found in [40]), we directly apply Lemma 2.1 in [40] to \( \phi \) to obtain Equations (25) and (26), respectively. \( \square \)

Lemma 5. It holds that for \( \forall k \in \mathbb{N}, \)

\[
\mathcal{L}_m(W^{k+1}, b_m^{k+1}, z_m^{k+1}, a_m^{k+1}, u^k) = \mathcal{L}_m(W_m^k, b_m^{k+1}, z_m^{k+1}, a_m^{k+1}, u^k)
\]

We introduce Equation (39) into Equation (38) to obtain Equation (29).

Secondly, we focus on Inequality (33). The stopping criterion of Algorithm 6 shows that

\[
\mathcal{U}_l(b_l^{k+1} ; B) = \phi(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k)
\]

For Equations

Because differentiability can be found in [40], we directly apply Lemma 2.1 in [40] to \( \phi \) to obtain Equations (25) and (26), respectively. We only prove Inequality (29), (33), (34) and (36). This is because Inequalities (27), (32) and (35) follow the routine of Inequality (33), Inequalities (30), (31) and (38) follow the routine of Inequality (34), and Inequality (28) follows the routine of Inequality (36).

Firstly, we focus on Inequality (29).

\[
\mathcal{L}_o(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k) = \mathcal{L}_o(W_l^k, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k)
\]

where the second equality follows from the cosine rule \( \|z_l^{k+1} - W_l^k a_l^{k+1} - b_l^{k+1}\|_2^2 = \|z_l^{k+1} - W_l^k a_l^{k+1} - b_l^{k+1}\|_2^2 \).

According to the optimality condition of Equation (5), we have \( \nabla z_l^{k+1} R(z_l^{k+1} ; y) + u^k + \rho(z_l^{k+1} - W_l^k a_l^{k+1} - b_l^{k+1}) = 0 \). Because \( R(z_l ; y) \) is convex and differentiable with regard to \( z_l \), its gradient is also its gradient. According to the definition of subgradient, we have

\[
R(z_l ; y) \geq R(z_l^{k+1} ; y) + \nabla z_l^{k+1} R(z_l^{k+1} ; y) (z_l - z_l^{k+1}) = R(z_l^{k+1} ; y) - (u^k + \rho(z_l^{k+1} - W_l^k a_l^{k+1} - b_l^{k+1}))^T (z_l - z_l^{k+1})
\]

We introduce Equation (39) into Equation (38) to obtain Equation (29).

Secondly, we focus on Inequality (33). The stopping criterion of Algorithm 6 shows that

\[
\phi(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k) \leq P(W_l^{k+1} ; \theta_l^{k+1})
\]

Because \( \Omega_{W_l}(W_l) \) is convex, according to the definition of subgradient, we have

\[
\Omega_l(W_l^{k+1}) \geq \Omega_l(W_l^{k+1}) + s^T (W_l^{k+1} - W_l^{k+1}),
\]

where \( s \) is defined in the premise of Lemma 3. Therefore, we have

\[
\mathcal{L}_o(W_l^{k+1}, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k) - \mathcal{L}_o(W_l^k, b_l^{k+1}, z_l^{k+1}, a_l^{k+1}, u^k)
\]

where \( \theta_l^{k+1} \) is defined in the premise of Lemma 4.

(41)
Finally, we focus on Equation (36). This follows directly from the optimality of $z_{i+1}$ in Equation (14).

Lemma 6. If $\rho > 2H$ so that $C_1 = \rho/2 - H/2 - H^2/\rho > 0$, then it holds that
\[
L_\rho(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, d^{k+1}, u^{k+1}) - L_\rho(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, d^{k+1}, u^{k+1}) \\
\geq C_1 \|z_{L+1}^k - \pi_L^{k+1}\|^2 - (H^2/\rho) \|\pi_L^{k+1} - z_{L+1}^k\|^2.
\]

Proof. We know that $z_L$ is Lipschitz differentiable, and so are

\[
(L_\rho(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, a^{k+1}, u^{k+1}) - L_\rho(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, a^{k+1}, u^{k+1})) \\
= \phi(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, a^{k+1}, u^{k+1}) - \phi(W^{k+1}, b^{k+1}, z_{L+1}^{k+1}, a^{k+1}, u^{k+1}) \\
\geq (\nu/2)\|b^{k+1} - b^k\|_2^2. \quad \text{(Lemma 3)}
\]

We choose $\rho > 2H$ to make $C_1 > 0$. □

**Proof of Theorem 4**

Proving Theorem 1 is equal to proving jointly Theorem 2 and 3, which are elaborated in the following.

Theorem 4. Given that Assumptions 1 and 2 hold, the dADMM satisfies Property 2.

Proof. There exists $z_L$ such that $z_L - W_b^k a_{L-1}^k - b_L^k = 0$. By Assumption 2, we have
\[
F(W^k, b^k, \{z_L^{i-1}\}_{i=1}^L, z_L^L, a^k) \geq \min S > -\infty
\]
where $S = \{F(W, b, z, a) : z_L - W_L a_{L-1} - b_L = 0\}$. Then we have
\[
L_\rho(W^k, b^k, z^L, a^k, u^k) \\
= F(W^k, b^k, z^L, a^k) + (u^k)^T (z_L^L - W_L a_{L-1}^k - b_L^k) + (\nu/2)\|L_\rho(W^k, b^k, z^L, a^k, u^k)\|^2
\]
\[
= F(W^k, b^k, z^L, a^k) + (u^k)^T (z_L^L - z_L^L) + (\nu/2)\|L_\rho(W^k, b^k, z^L, a^k, u^k)\|^2
\]
\[
= F(W^k, b^k, z^L, a^k) + \nabla_{z^L} F(z^L; y) (z_L^L - z_L^L) + (\nu/2)\|L_\rho(W^k, b^k, z^L, a^k, u^k)\|^2
\]
\[
= \sum_{i=1}^L \Omega_i(W^k) + (\nu/2)\sum_{i=1}^{L-1} \|z_L^i - W_L a_{L-1}^i - b_L^i\|_2^2 + (\nu/2)\|z_L^L - W_L a_{L-1}^L - b_L^L\|_2^2
\]
\[
\geq \sum_{i=1}^L \Omega_i(W^k) + (\nu/2)\sum_{i=1}^{L-1} \|z_L^i - W_L a_{L-1}^i - b_L^i\|_2^2 + (\nu/2)\|z_L^L - W_L a_{L-1}^L - b_L^L\|_2^2
\]
\[
\geq \sum_{i=1}^L \Omega_i(W^k) + (\nu/2)\sum_{i=1}^{L-1} \|z_L^i - W_L a_{L-1}^i - b_L^i\|_2^2 + (\nu/2)\|z_L^L - W_L a_{L-1}^L - b_L^L\|_2^2
\]
\[
\geq \sum_{i=1}^L \Omega_i(W^k) + (\nu/2)\sum_{i=1}^{L-1} \|z_L^i - W_L a_{L-1}^i - b_L^i\|_2^2 + (\nu/2)\|z_L^L - W_L a_{L-1}^L - b_L^L\|_2^2
\]

It concludes from Lemma 5 and Lemma 6 that $L_\rho(W^k, b^k, z^L, a^k, u^k)$ is upper bounded by $L_\rho(W^k, b^k, z^L, a^k, u^k)$ and so are
\[
\sum_{i=1}^L \Omega_i(W^k) + (\nu/2)\sum_{i=1}^{L-1} \|z_L^i - W_L a_{L-1}^i - b_L^i\|_2^2 + (\nu/2)\|z_L^L - W_L a_{L-1}^L - b_L^L\|_2^2
\]

By Assumption 2, $L_\rho(W^k, b^k, z^L, a^k, u^k)$ is bounded. By Lemma 2, it is obvious that $u^k$ is bounded as well. □

Theorem 5. Given that Assumptions 1 and 2 hold, the dADMM satisfies Property 2.

Proof. This follows directly from Lemma 3 and Lemma 4. □

Theorem 6. Given that Assumptions 1 and 2 hold, the dADMM satisfies Property 3.

Proof. We know that
\[
\partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}) \\
= (\partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}), \partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}), \partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}), \partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}), \partial L_\rho(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^{k+1}))
\]
where \(\partial_{W_{k+1}} L_{\rho} = \{\partial_{W_{k+1}} L_{\rho}\}_{k=1}^{L-1}\), \(\nabla_{k+1} L_{\rho} = (\nabla_{b_{k+1}} L_{\rho})_{k=1}^{L-1}\), \(\partial_{k+1} L_{\rho} = \{\partial_{a_{k+1}} L_{\rho}\}_{k=1}^{L-1}\), and \(\nabla_{a_{k+1}} L_{\rho} = (\nabla_{a_{k+1}} L_{\rho})_{k=1}^{L-1}\). To prove Property \(\mathcal{L}\) we need to give an upper bound of \(\partial_{W_{k+1}} L_{\rho}, \nabla_{k+1} L_{\rho}, \partial_{k+1} L_{\rho}, \nabla_{a_{k+1}} L_{\rho}\), and \(\nabla_{a_{k+1}} L_{\rho}\) by a linear combination of \(\|W_{k+1} - W_{\rho}\|, \|b_{k+1} - b_{\rho}\|, \|z_{k+1} - z_{\rho}\|, \|a_{k+1} - a_{\rho}\|\), and \(\|z_{k+1} - z_{\rho}\|\).

For \(W_{k+1}^T(l < L)\),

\[
\partial_{W_{k+1}} L_{\rho} = \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k+1}) \\
= \partial_{W_{k+1}} (W_{k+1}) + \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) \\
= \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k}) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) - \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
= \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k}) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) - \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
+ \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) - \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) \\
\]

Because

\[
\| - \theta_{k+1}^l (W_{k+1} - W_{\rho}) + \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) - \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) \| \\
= \| - \theta_{k+1}^l (W_{k+1} - W_{\rho}) + \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) - \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) \| \\
\leq \| \theta_{k+1}^l (W_{k+1} - W_{\rho}) \| + \nu(\|W_{k+1} - W_{\rho}\|) a_{k+1}^T + \nu(b_{k+1} - b_{\rho}) \|a_{k+1}\| + \nu(z_{k+1} - z_{\rho}) \|a_{k+1}\| \\
\| - \theta_{k+1}^l (W_{k+1} - W_{\rho}) + \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) - \nu(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1}(a_{k+1})^T) \| \\
\]

and the optimality condition of Equation \(\mathcal{L}\) yields

\[
0 \in \partial_{W_{k+1}} (W_{k+1} + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k})) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
\]

Because \(a_{k+1}\) is bounded by Property \(\mathcal{L}\) \(\|\partial_{W_{k+1}} L_{\rho}\|\) can be upper bounded by a linear combination of \(\|W_{k+1} - W_{\rho}\|, \|b_{k+1} - b_{\rho}\|\) and \(\|z_{k+1} - z_{\rho}\|\).

For \(W_{k+1}^T\),

\[
\partial_{W_{k+1}} L_{\rho} = \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k}) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
= \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k}) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) - \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
= \partial_{W_{k+1}} (W_{k+1}) + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k}) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) - \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
+ \rho(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T - \rho(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T) \\
\]

Because

\[
\| - \theta_{k+1}^l (W_{k+1} - W_{\rho}) + \rho(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T - \rho(W_{k+1} a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T) \| \\
= \| - \theta_{k+1}^l (W_{k+1} - W_{\rho}) + \rho(W_{k+1} - W_{\rho}) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T - \rho(W_{k+1} - W_{\rho}) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T \| \\
\leq \| \theta_{k+1}^l (W_{k+1} - W_{\rho}) \| + \rho(\|W_{k+1} - W_{\rho}\|) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T + \rho(\|W_{k+1} - W_{\rho}\|) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T \| \\
+ \|\rho(\|W_{k+1} - W_{\rho}\|) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T \| \|\theta_{k+1}^l (W_{k+1} - W_{\rho}) \| + \rho(\|W_{k+1} - W_{\rho}\|) a_{k+1} + b_{k+1} - z_{k+1} - u_{k} - \rho(a_{k+1})^T \| \\
\]

and the optimality condition of Equation \(\mathcal{L}\) yields

\[
0 \in \partial_{W_{k+1}} (W_{k+1} + \nabla_{W_{k+1}} \phi(W_{k+1}, b_{k+1}, z_{k+1}, a_{k+1}, u_{k})) + \theta_{k+1}^l (W_{k+1} - W_{\rho}) \\
\]

Because \(a_{k+1}\) is bounded by Property \(\mathcal{L}\) \(\|\partial_{W_{k+1}} L_{\rho}\|\) can be upper bounded by a linear combination of \(\|W_{k+1} - W_{\rho}\|, \|b_{k+1} - b_{\rho}\|\) and \(\|z_{k+1} - z_{\rho}\|\).
For $b^{k+1}_L (L < L)$,

$$
\nabla_{b^{k+1}_L} \rho = \nabla_{b^{k+1}_L} (W^{k+1}, b^{k+1}_L, z^{k+1}, a^{k+1}, u^{k+1})
$$

$$
= \nu(W^{k+1}_L, a^{k+1}_L, b^{k+1}_L - z^{k+1}_L)
$$

$$
= \nabla_{z^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1}) + \nu(b^{k+1}_L - z^{k+1}_L - \nabla_{b^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

The optimality condition of Equation (12) yields

$$
0 \in \nabla_{z^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1}) + \nu(b^{k+1}_L - z^{k+1}_L)
$$

Therefore, $\| \nabla_{b^{k+1}_L} \rho \|$ is linearly independent on $\| z^{k+1}_L - z^{k+1}_L \|$. For $b^{k+1}_L$,

$$
\nabla_{b^{k+1}_L} \rho = \nabla_{b^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

$$
= \nabla_{b^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1}) + \nu(b^{k+1}_L - b^{k+1}_L - \nabla_{b^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

Because

$$
\| \nabla_{b^{k+1}_L} \rho \| \leq \| z^{k+1}_L - z^{k+1}_L \| + \| u^k - u^{k+1} \| (\text{triangle inequality})
$$

and the optimality condition of Equation (13) yields

$$
0 \in \nabla_{b^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1}) + \nu(b^{k+1}_L - b^{k+1}_L)
$$

Therefore, $\| \nabla_{b^{k+1}_L} \rho \|$ is upper bounded by a combination of $\| z^{k+1}_L - z^{k+1}_L \|$ and $\| z^{k+1}_L - z^{k+1}_L \|$.

For $z^{k+1}_L (L < L)$,

$$
\partial_{z^{k+1}_L} \rho = \partial_{z^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

$$
= \nu(z^{k+1}_L - W^{k+1}_L, a^{k+1}_L, b^{k+1}_L - \nabla_{z^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

because $z^{k+1}_L$ is bounded and $f_{i}$ is continuous and hence $f_{i}(z^{k+1}_L)$ is bounded, and the optimality condition of Equation (14) yields

$$
0 \in \partial_{z^{k+1}_L} (W^{k+1}_L, b^{k+1}_L, z^{k+1}_L, a^{k+1}_L, u^{k+1})
$$

Therefore, $\| \partial_{z^{k+1}_L} \rho \|$ is upper bounded by $\| a^{k+1}_L - a^{k+1}_L \|$.
The optimality condition of Equation (15) yields

\[ 0 \in \nabla_{z_{k+1}} R(z_{k+1}; y) + \nabla_{z_{k+1}} \phi(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^k) \]

and

\[ \| u^{k+1} - u^k \| = \| \nabla_{z_{k+1}} R(z_{k+1}; y) - \nabla_{z_{k+1}} R(z_k; y) \| \leq H \| z_{k+1} - z_k \| \]

(Cauchy-Schwarz inequality, \( R(z_k; y) \) is Lipschitz differentiable)

Therefore \( \| \nabla_{z_{k+1}, L} \| \) is upper bounded by \( \| z_{k+1} - z_k \| \).

For \( a_t^{k+1} (t < L - 1) \),

\[ \nabla_{a_t^{k+1}} \phi(W^{k+1}, b^{k+1}, z^{k+1}, a^{k+1}, u^k) \]

\[ = \nu(W_t^{k+1}) \nabla_{a_t^{k+1}} (W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1}) + \nu(a_t^{k+1} - f_t(z_t^{k+1})) \]

\[ = \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ = \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) - \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) - \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ = \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) - \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) - \nu(W_t^{k+1}) \nabla_{a_t^{k+1}} (W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1}) \]

\[ - \nu(\pi_t^{k+1} - f_t(z_t^{k+1})) + \nu(W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1}) + \nu(a_t^{k+1} - f_t(z_t^{k+1})) \]

Because

\[ \| - \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) - \nu(W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1}) - \nu(\pi_t^{k+1} - f_t(z_t^{k+1})) \]

\[ + \nu(W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1}) + \nu(a_t^{k+1} - f_t(z_t^{k+1})) \]

\[ \leq \| \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) \| + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \]

\[ + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \] (triangle inequality)

Then we need to show that \( \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \) is therefore upper bounded by \( \| W_t^{k+1} - W_t^{k+1} \|, \| b_t^{k+1} - b_t^{k+1} \|, \| z_t^{k+1} - z_t^{k+1} \|, \) and \( \| a_t^{k+1} - a_t^{k+1} \| \).

Because \( \| W_t^{k+1} \| \) and \( \| W_t^{k+1} \| \) are upper bounded, \( \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \) is therefore upper bounded by a combination of \( \| W_t^{k+1} - W_t^{k+1} \| \) and \( \| b_t^{k+1} - b_t^{k+1} \| \).

Similarly, \( \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \) is upper bounded by a combination of \( \| W_t^{k+1} - W_t^{k+1} \| \) and \( \| z_t^{k+1} - z_t^{k+1} \| \).

Because \( \| W_t^{k+1} \| \), \( \| W_t^{k+1} \| \) and \( \| a_t^{k+1} \| \) are upper bounded, \( \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \) is therefore upper bounded by a combination of \( \| W_t^{k+1} - W_t^{k+1} \| \) and \( \| a_t^{k+1} - a_t^{k+1} \| \).

For \( a_t^{k+1} \),

\[ \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ = \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) \]

\[ = \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) + \nabla_{a_t^{k+1}} \phi(W_t^{k+1}, b_t^{k+1}, z_t^{k+1}, a_t^{k+1}, u^k) - \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) + \rho(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \]

\[ - \nu(\pi_t^{k+1} - f_t(z_t^{k+1})) + \rho(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \]

\[ + \nu(a_t^{k+1} - f_t(z_t^{k+1})) ]

\[ \leq \| \tau_t^{k+1} o (a_t^{k+1} - \pi_t^{k+1}) \| + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \]

\[ + \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \] (triangle inequality)

Because \( \| W_t^{k+1} \| \), \( \| W_t^{k+1} \| \) and \( \| a_t^{k+1} \| \) are upper bounded, \( \nu(\| W_t^{k+1} a_t^{k+1} + b_t^{k+1} - z_t^{k+1} \|) \) is therefore upper bounded by a combination of \( \| W_t^{k+1} - W_t^{k+1} \| \) and \( \| a_t^{k+1} - a_t^{k+1} \| \).
Because
\[ \parallel - \tau_{k+1} \circ (a_{k+1} - \tilde{a}_{k+1}) - \rho(\tilde{W}_{k+1})^T (\tilde{W}_{k+1})^{-1} + b_{k+1} - z_{k+1} - u_k / \rho - \nu(\tilde{a}_{k+1} - f_{L-1}(z_{k+1})) \]
\[ + \rho(\tilde{W}_{k+1})^T (W_{k+1}) a_{k+1} + b_{k+1} - z_{k+1} - u_k / \rho + \nu(a_{k+1} - f_{L-1}(z_{k+1})) \parallel \]
\[ \leq \parallel \tau_{k+1} \circ (a_{k+1} - \tilde{a}_{k+1}) \parallel + \parallel (\tilde{W}_{k+1})^T b_{k+1} - (\tilde{W}_{k+1})^T b_{k+1} \parallel \]
\[ + \rho(\tilde{W}_{k+1})^T z_{k+1} - (\tilde{W}_{k+1})^T z_{k+1} \parallel + \parallel (W_{k+1})^T u_{k+1} - (\tilde{W}_{k+1})^T u_{k+1} \parallel \]
\[ + \rho(\tilde{W}_{k+1})^T W_{k+1} a_{k+1} - (\tilde{W}_{k+1})^T \tilde{W}_{k+1} a_{k+1} \parallel \] (triangle inequality)

Then we need to show that \( \rho(\tilde{W}_{k+1})^T b_{k+1} - (\tilde{W}_{k+1})^T b_{k+1} \parallel \), \( \rho(\tilde{W}_{k+1})^T z_{k+1} - (\tilde{W}_{k+1})^T z_{k+1} \parallel \), \( (W_{k+1})^T u_{k+1} - (\tilde{W}_{k+1})^T u_{k+1} \parallel \) and \( \rho(\tilde{W}_{k+1})^T W_{k+1} a_{k+1} - (\tilde{W}_{k+1})^T \tilde{W}_{k+1} a_{k+1} \parallel \) are upper bounded by \( \parallel W_{k+1} - \tilde{W}_{k+1} \parallel, \parallel b_{k+1} - b_{k+1} \parallel, \parallel z_{k+1} - z_{k+1} \parallel, \parallel a_{k+1} - \tilde{a}_{k+1} \parallel \) and \( \parallel z_{k+1} - z_{k+1} \parallel \).

\[ \rho(\tilde{W}_{k+1})^T b_{k+1} - (\tilde{W}_{k+1})^T b_{k+1} \parallel = \rho(\tilde{W}_{k+1})^T b_{k+1} - (\tilde{W}_{k+1})^T b_{k+1} \]
\[ + (W_{k+1})^T u_{k+1} - (\tilde{W}_{k+1})^T u_{k+1} \parallel \]
\[ \leq \rho(\tilde{W}_{k+1})^T || b_{k+1} || + || (W_{k+1})^T || u_{k+1} || - || (\tilde{W}_{k+1})^T || u_{k+1} || \]

(triangle inequality, Cauchy-Schwarz inequality)

Because \( || b_{k+1} || \) and \( || W_{k+1} \parallel \) are upper bounded, \( \rho(\tilde{W}_{k+1})^T b_{k+1} - (\tilde{W}_{k+1})^T b_{k+1} \parallel \) is therefore upper bounded by a combination of \( || W_{k+1} - \tilde{W}_{k+1} \parallel \) and \( || b_{k+1} - b_{k+1} \parallel \).

Similarly, \( \rho(\tilde{W}_{k+1})^T z_{k+1} - (\tilde{W}_{k+1})^T z_{k+1} \parallel \) is upper bounded by a combination of \( || W_{k+1} - \tilde{W}_{k+1} \parallel \) and \( || z_{k+1} - z_{k+1} \parallel \).

\[ || W_{k+1} || \parallel \] and \( || u_{k+1} \parallel \) are bounded, \( || W_{k+1} \parallel || z_{k+1} - z_{k+1} || \) and \( || W_{k+1} - \tilde{W}_{k+1} \parallel \).

\[ \rho(\tilde{W}_{k+1})^T W_{k+1} a_{k+1} - (\tilde{W}_{k+1})^T \tilde{W}_{k+1} a_{k+1} \parallel \]
\[ \leq \rho(\tilde{W}_{k+1})^T || a_{k+1} || + || (W_{k+1})^T || a_{k+1} \parallel - || (\tilde{W}_{k+1})^T || a_{k+1} \parallel \]

(Lemma 2)

\( (R(z_{k+1}; y) \parallel \) is Lipschitz differentiable)

Because \( || W_{k+1} || \parallel \) and \( || z_{k+1} \parallel \) are upper bounded, \( || W_{k+1} \parallel || z_{k+1} - z_{k+1} \parallel \) is therefore upper bounded by a combination of \( || W_{k+1} - \tilde{W}_{k+1} \parallel \) and \( || u_{k+1} - \tilde{a}_{k+1} \parallel \).

For \( u_{k+1} \),
\[ \nabla u_{k+1} L_{\rho} = \nabla u_{k+1} \phi(W_{k+1}) + b_{k+1} + (z_{k+1} - a_{k+1} + u_{k+1}) \]
\[ = z_{k+1} - W_{k+1} u_{k+1} - b_{k+1} \]
\[ = (1 / \rho)(u_{k+1} - u_k) \]
\[ = (1 / \rho)(\nabla z_{k+1} R(z_{k+1}; y) - \nabla z_{k+1} R(z_{k+1}; y)) \]

(Lemma 2)

Because
\[ || (1 / \rho)(\nabla z_{k+1} R(z_{k+1}; y) - \nabla z_{k+1} R(z_{k+1}; y)) \parallel \leq (H / \rho) \parallel z_{k+1} - z_{k+1} \parallel \]
(R(z_{k+1}; y) is Lipschitz differentiable)

Therefore, \( \nabla u_{k+1} L_{\rho} \parallel \) is upper bounded by \( || z_{k+1} - z_{k+1} \parallel \).
Proof of Theorem 3

Proof. To prove this theorem, we will first show that $c_k$ satisfies two conditions: (1) $c_k \geq c_{k+1}$, (2) $\sum_{k=0}^{\infty} c_k$ is bounded. We then conclude the convergence rate of $o(1/k)$ based on these two conditions. Specifically, first, we have

$$c_k = \min_{0 \leq l \leq k} \left( \sum_{l=1}^{L} \left( \|W_{l+1}^k - W_l^k\|^2 + \|W_{l+1}^k - W_l^k\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 \right) + \sum_{l=1}^{L-1} \left( \|\tilde{a}_l - a_l\|^2 + \|a_{l+1} - \pi_{l+1}^k\|^2 \right) + \|z_{l+1}^k - z_l^k\|^2 + \|z_{l+1}^k - z_l^k\|^2 \right) \geq \min_{0 \leq l \leq k+1} \left( \sum_{l=1}^{L} \left( \|W_{l+1}^k - W_l^k\|^2 + \|W_{l+1}^k - W_l^k\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 \right) + \sum_{l=1}^{L-1} \left( \|\tilde{a}_l - a_l\|^2 + \|a_{l+1} - \pi_{l+1}^k\|^2 \right) + \|z_{l+1}^k - z_l^k\|^2 + \|z_{l+1}^k - z_l^k\|^2 \right) = c_{k+1}$$

Therefore $c_k$ satisfies the first condition. Second,

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} \min_{0 \leq l \leq k} \left( \sum_{l=1}^{L} \left( \|W_{l+1}^k - W_l^k\|^2 + \|W_{l+1}^k - W_l^k\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 \right) + \sum_{l=1}^{L-1} \left( \|\tilde{a}_l - a_l\|^2 + \|a_{l+1} - \pi_{l+1}^k\|^2 \right) + \|z_{l+1}^k - z_l^k\|^2 + \|z_{l+1}^k - z_l^k\|^2 \right) \leq \sum_{k=0}^{\infty} \left( \sum_{l=1}^{L} \left( \|W_{l+1}^k - W_l^k\|^2 + \|W_{l+1}^k - W_l^k\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 + \|\tilde{b}_{l+1} - \tilde{b}_l\|^2 \right) + \sum_{l=1}^{L-1} \left( \|\tilde{a}_l - a_l\|^2 + \|a_{l+1} - \pi_{l+1}^k\|^2 \right) + \|z_{l+1}^k - z_l^k\|^2 + \|z_{l+1}^k - z_l^k\|^2 \right) \leq (L_0(W^0, b^0, \tilde{z}, \pi^0, u^0) - L_0(W^*, b^*, \tilde{z}^*, \pi^*, u^*)) / C_3 \text{(Property 2)}$$

So $\sum_{k=0}^{\infty} c_k$ is bounded and $c_k$ satisfies the second condition. Finally, it has been proved that the sufficient conditions of convergence rate $o(1/k)$ are: (1) $c_k \geq c_{k+1}$, and (2) $\sum_{k=0}^{\infty} c_k$ is bounded, and (3) $c_k \geq 0$ (Lemma 1.2 in [51]). Since we have proved the first two conditions and the third one $c_k \geq 0$ is obvious, the convergence rate of $o(1/k)$ is proven. \(\square\)