Unusual eigenvalue spectrum and relaxation in the Lévy
Ornstein-Uhlenbeck process

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Abstract

We consider the rates of relaxation of a particle in a harmonic well, subject to Lévy noise characterized by its Lévy index $\mu$. Using the propagator for this Lévy Ornstein-Uhlenbeck process (LOUP), we show that the eigenvalue spectrum of the associated Fokker-Planck operator has the form $(n+m\mu)\nu$ where $\nu$ is the force constant characterizing the well, and $n, m \in \mathbb{N}$. If $\mu$ is irrational, the eigenvalues are all non-degenerate, but rational $\mu$ can lead to degeneracy. The maximum degeneracy is shown to be two. The left eigenfunctions of the fractional Fokker-Planck operator are very simple while the right eigenfunctions may be obtained from the lowest eigenfunction by a combination of two different step-up operators. Further, we find that the acceptable eigenfunctions should have the asymptotic behavior $|x|^{-n_1+n_2 \mu}$ as $|x| \to \infty$, with $n_1$ and $n_2$ being positive integers, though this condition alone is not enough to identify them uniquely. We also assert that the rates of relaxation of LOUP are determined only by the eigenvalues of the associated fractional Fokker-Planck operator and do not depend on the initial state.
Diffusion processes are ubiquitous in nature. In general, the relaxation of a system under diffusion is multi-exponential. The exponents are determined by the eigenvalues of the Fokker-Planck (FP) operator associated with the diffusion. The timescales involved are thus an intrinsic property of the system, independent of the initial condition. For normal diffusion of a particle in a potential $V(x)$, the associated FP operator is non-Hermitian \[1\]. However, one can make a similarity transformation to get a Hermitian operator, analogous to the Hamiltonian operator for a quantum system. This shows that the eigenvalues of the operator are all real and further, by imposing physically motivated boundary conditions (vanishing at infinity and square integrability), one finds the eigenvalue spectrum of the operator. This spectrum determines the relaxation characteristics of the system completely. It has been recently argued by Toenjes et al. \[2\] that this traditional wisdom may not hold in general. They suggested \[2\] that “initial distributions which are not mapped to square integrable functions by the similarity transformation, cannot be expanded in terms of the eigenfunctions of the corresponding Hamiltonian operator and will therefore relax at rates that may not be given by the Hermitian spectrum”. This has been referred to as non-spectral relaxation. Further, it was also suggested that “the smallest non-spectral rate can be smaller than the smallest spectral relaxation rate and thus, it will dominate the relaxation behavior over the whole time range”. This has been argued \[2\] to happen even for the simplest of processes, viz. the Ornstein-Uhlenbeck process (OUP).

In the recent past, a number of investigations have focussed on processes driven by Lévy noise\[3, 4\]. The result of such driving is anomalous diffusion, having the displacement scaling like $(\text{time})^{1/\mu}$ with $0 < \mu \leq 2$. The process is governed by a fractional Fokker-Planck equation \[5–7\], which is much more difficult to analyze. The properties of such operators are not well understood, as there are very few results on them. Laskin \[8\] introduced a generalization of quantum mechanics referred to as fractional quantum mechanics which has similar operators and the eigenvalue spectra and eigenfunctions of some operators have been investigated \[9, 10\]. Toenjes et al. \[2\] discuss the case of a particle in a harmonic well, subject to Lévy noise. The resultant Lévy Ornstein-Uhlenbeck process (LOUP) is one of the simplest of such processes and is governed by the corresponding fractional Fokker-Planck equation, given in equations (2) and (3). They perform a similarity transformation, converting it into a simple operator, analogous to the one for the usual Ornstein-Uhlenbeck process, impose boundary conditions appropriate for the normal OUP and find eigenvalues
similar to that of OUP, having the values \( n\nu/2 \) (\( \nu \) is the force constant) with \( n \in \mathbb{N} \). Also, they find explicit solution for the LOUP for an initial condition having a long tail behaving like \( |x|^{-\alpha -1} \), and find relaxation with exponents determined by \( \nu(n + m\alpha + l\mu) \) with \( l, m, n \in \mathbb{N} \).

This does not coincide with the eigenvalue spectrum of the operator found by the similarity transformation and hence, the authors argue that non-spectral relaxation is the rule rather than the exception for such processes.

In this letter, we point out that the Green’s function for the operator in Eq. \( \text{(2)} \) is enough to propagate any arbitrary initial condition for any value of \( \mu \), including \( \mu = 2 \), which is the usual OUP. We use the expression for the propagator that is already known \([3, 5, 11, 12]\) and find the exponents that are involved in the time evolution. They are of the form \( \nu(n + m\mu) \), thus showing that the eigenvalue spectrum of the FP operator for the LOUP to be characterized, in general, by two “quantum numbers” \( n \) and \( m \in \mathbb{N} \) (and not one, as one would normally expect). If \( \mu = 1 \) or \( 2 \), then the spectrum coincides with that for the OUP, but the degeneracies are different for \( \mu = 1 \). Also, we identify the left and right eigenfunctions of the operator and arrive at a generalization of the expansion of the propagator for OUP in terms of the Hermite polynomials \([1]\). Further, we give operators that can be used to generate the right eigenfunctions from the lowest possible eigenfunction, similar to the step-up operators of quantum mechanics. We also discuss the boundary conditions that when imposed on the solutions would lead naturally to the correct identification of these eigenfunctions and eigenvalues.

We consider the motion of a particle governed by the equation

\[
\frac{dx}{dt} = -\frac{V'(x)}{m\gamma} + \eta(t),
\]

where \( V(x) \) is the potential that the particle is subjected to, and \( m, \gamma \) are the mass and the friction coefficient. \( \eta(t) \) is the Lévy noise, best described by its characteristic functional

\[
\langle e^{i\int_0^T dt \eta(t)p(t)} \rangle \quad \text{which is equal to } e^{-D\int_0^T dt |p(t)|^\mu}. \]

The Fokker-Planck equation for the Lévy driven processes is \([5, 13]\)

\[
\frac{\partial P(x,t)}{\partial t} = \left\{-D \left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\mu}{2}} + \frac{\partial}{\partial x} \frac{V'(x)}{m\gamma} \right\} P(x,t). \tag{1}
\]

The usual Smoluchowski equation is a special case of this and is obtained when one puts \( \mu = 2 \). Taking the potential to be of the form \( V(x) = \lambda x^2/2 \), and changing over to a new variable \( D^{1/\mu} x \to x \) and putting \( \lambda/(m\gamma) = \nu \), we can write this as
\[
\frac{\partial P(x,t)}{\partial t} = -\hat{H}_\mu P(x,t),
\]
with
\[
\hat{H}_\mu = -\left\{ -\left( -\frac{\partial^2}{\partial x^2} \right)^{\frac{\mu}{2}} + \nu \frac{\partial}{\partial x} \right\}.
\]

The relaxation of the system governed by equations (2) and (3) was studied by Toenjes et. al [2]. We briefly summarize their main arguments. The relaxation is governed by the eigenvalues \( \lambda \) of the operator determined by \( \hat{H}_\mu \psi = \lambda \psi \), if the initial state satisfies the acceptability conditions (see below). Interestingly, one can find solutions of \( \hat{H}_\mu \psi = \lambda \psi \) for any \( \lambda \), but when one imposes acceptability conditions on the \( x \) dependence of \( \psi \), only certain discrete values \( \lambda_n \) and the associated eigenfunctions \( \psi_n \) are allowed. In particular, when \( \mu = 2 \), the acceptability condition imposed is \( \psi(x)e^{\nu x^2/4} \to 0 \) as \( x \to \pm \infty \), which leads to \( \lambda_n = \nu n \), with \( n \in \mathbb{N} \). According to them, the relaxation of the system, from any initial state \( P_0(x_i) \), which can be expanded in terms of these eigenfunctions as
\[
P_0(x_i) = \sum_n c_n \psi_n(x_i) e^{\nu x_i^2/4},
\]
is determined by these eigenvalues. In the above, \( c_n \) are the expansion coefficients. This means that the slowest relaxation would correspond to \( \nu \) (i.e., \( n = 1 \)). According to the paper, if the initial condition cannot be expanded as in Eq. (4), then the relaxation can be much slower.

On going over to Fourier space, with \( \mathcal{P}(p,t) = \int dx \, P(x,t)e^{ipx} \), the Eq. (2) becomes
\[
\frac{\partial \mathcal{P}(p,t)}{\partial t} = \hat{H}_\mu \mathcal{P}(p,t),
\]
with
\[
\hat{H}_\mu = |p|^{\mu} + \nu p \frac{\partial}{\partial p}.
\]

In the case with \( \mu \neq 2 \), Toenjes et. al. [2] use the transformation \( p = |\kappa|^{2/\mu} \text{sign}(\kappa) \), under which the operator \( \hat{H}_\mu \) gets transformed to \( \kappa^2 + \frac{\nu}{2} \kappa \frac{\partial}{\partial \kappa} \). As this operator is similar to the one for OUP, they impose boundary conditions appropriate for OUP and get the eigenvalues \( n\nu / 2 \), with \( n \in \mathbb{N} \). For cases where the initial distribution has a long tail, Toenjes et. al. claim to solve the time evolution exactly and find that it contains time scales determined not by \( n\nu / 2 \), but by the numbers \( \nu(n + n\mu + l\alpha) \), where \( \alpha \) is determined by the characteristics of the initial distribution and \( l, m, n \in \mathbb{N} \). This is what the authors refer to as non-spectral.
relaxation. Note that according to their paper, the long term relaxation is not determined by the lowest non-zero eigenvalue of the operator $\hat{H}_\mu$.

We now give our analysis of the problem. The equation [5] can be solved by the method of characteristics, for any initial condition $P(p,0) = P_0(p)$ to get the solution at a final time $T$ as

$$P(p,T) = P_0(pe^{-2\nu T})e^{-|p|^{\mu}(1-e^{-2\nu T})/(2\mu
u)}.$$  

(7)

Writing $P_0(p) = \int dx_i e^{ipx}P_0(x_i)$, we can express the position space probability distribution at the final time $T$ as

$$P(x_f,T) = \int dx_i G(x_f,T|x_i,0) P_0(x_i)$$

with

$$G(x_f,T|x_i,0) = \int \frac{dp}{2\pi} e^{-|p|^{\mu}(1-e^{-\mu T})/(\mu\nu)+ip(x_f-x_i)e^{-\nu T)}$$

$$= \left( \frac{(\mu\nu)^{1/\mu}}{(1-e^{-\mu T})^{1/\mu}} \right) L_\mu \left( \frac{(\mu\nu)^{1/\mu}(x_f-x_i)e^{-\nu T}}{(1-e^{-\mu T})^{1/\mu}} \right),$$ \hspace{1cm} (8)

where $L_\mu(x)$ is the Lévy stable distribution defined by

$$L_\mu(x) = \frac{1}{\pi\mu} \int_{-\infty}^{\infty} dp e^{ipx-|p|^{\mu}}.$$ \hspace{1cm} (9)

If $\mu > 1$, this may be evaluated as the series

$$L_\mu(x) = \frac{1}{\pi\mu} \sum_{n=0}^{\infty} (-1)^n x^{2n} \frac{\Gamma(\frac{2n+1}{\mu})}{\Gamma(2n+1)}.$$ \hspace{1cm} (10)

We have recently developed a path integral approach to Lévy flights which leads to exactly this result [12]. The above analysis shows that irrespective of what the initial distribution is, the time development of the system is determined only by the propagator and hence the time scales of relaxation uniquely determined. Note that this conclusion is valid for all values of $\mu$ and is therefore applicable to the usual Brownian motion too, contrary to the conclusion in [2]. It is possible to expand the propagator $G(x_f,T|x_i,0)$ in terms of the left eigenfunctions $\tilde{\psi}_n(x)$ and the right eigenfunctions $\psi_n(x)$ of the operator $\hat{H}_\mu$ as

$$G(x_f,T|x_i,0) = \left\langle x_f \left| e^{-T\hat{H}_\mu} \right| x_i \right\rangle = \sum_n \tilde{\psi}_n(x_i)\psi_n(x_f)e^{-\lambda_n T}.$$ \hspace{1cm} (11)

As $\hat{H}_\mu$ is not a Hermitian operator, the eigenfunctions are not necessarily orthogonal. It is obvious that if we can expand the right hand side of Eq. [8] as a series in exponentials
involving $T$, then we would be able to find the eigenvalues and eigenfunctions of the operator $\hat{H}_\mu$. We can expand the right hand side of Eq. (8) using Eq. (10), and then perform two more expansions using the binomial theorem to obtain

$$G(x_f, T|x_i, 0) = \sum_{n=0}^{\infty} \frac{e^{-n\nu T}(-x_i)^n}{\Gamma(n+1)} \sum_{m=0}^{\infty} \frac{e^{-m\mu T}}{\Gamma(m+1)} H_{n,m}(x_f).$$

In the above,

$$H_{n,m}(x_f) = \frac{1}{\mu \pi} \sum_{k=0}^{\infty} \frac{(-1)^{(n+[n])/2}+k}{(n+2k)!} \Gamma \left( \frac{n+[n]+2k+1}{\mu} + m \right) x_f^{(2k+[n])},$$

where $[n] = n \text{ modulo } 2$. From Eq. (12), it is clear that the eigenvalues of $\hat{H}_\mu(x)$ have the form $(n + m\mu)\nu$ with $n$ and $m$ belonging to $\mathbb{N}$. Thus we have a very interesting situation that if $\mu$ is irrational, the eigenvalues are characterized by two ‘quantum’ numbers $n$ and $m$, unlike the usual situation where there is only one quantum number. Further, for such values of $\mu$, the left eigenfunction is simply $x_i^n / \Gamma(n+1)$ while the right eigenfunction is given by $H_{n,m}(x_f)$. If $\mu$ is a rational number, the expressions are a little bit more complex. Eq. (12) is the generalization of the classic expansion of the propagator for Ornstein-Uhlenbeck process to the LOUP. The eigenvalue spectrum for LOUP is shown in Fig. 1 as a function of $\mu$.

It is possible to express the eigenfunctions in an interesting way. If one allows $T \to \infty$, $G(x, T|x_i, 0)$ approaches the steady equilibrium state and becomes

$$\psi_0(x) = (\mu \nu)^{1/\mu} L_\mu \left( (\mu \nu)^{1/\mu} x \right).$$

This is the lowest eigenfunction of the operator $\hat{H}_\mu$, having the eigenvalue zero. It is easy to prove the following commutation relations:

$$\left[ \frac{\partial}{\partial x}, \hat{H}_\mu \right] = -\nu \frac{\partial}{\partial x},$$

and

$$\left[ \left( \frac{-\partial^2}{\partial x^2} \right)^{\mu/2}, \hat{H}_\mu \right] = -\mu \nu \left( \frac{-\partial^2}{\partial x^2} \right)^{\mu/2}.$$

The above imply that if $\psi$ is an eigenfunction of $\hat{H}_\mu$ with an eigenvalue $\varepsilon$, then $\frac{\partial \psi}{\partial x}$ and $\left( \frac{-\partial^2}{\partial x^2} \right)^{\mu/2} \psi$, too are eigenfunctions with eigenvalues $\varepsilon + \nu$ and $\varepsilon + \mu \nu$ respectively. It follows that the right eigenfunction $H_{n,m}(x) \propto \left( \frac{-\partial^2}{\partial x^2} \right)^{m/2} \frac{\partial^n}{\partial x^n} \psi_0(x)$. The asymptotic behavior as $x \to \pm \infty$ is

$$H_{n,m}(x) \sim \frac{1}{|x|^{n+1+\mu}} \text{ when } m = 0$$

as $x \to \pm \infty$.
FIG. 1: Eigenvalues $n + m\mu$ plotted as functions of $\mu$ for $\mu \in [0, 2]$, for $n = 0, 1, 2 \ldots 6$ and $m = 0, 1, 2 \ldots 6$. The spectrum has only non-degenerate eigenvalues if $\mu$ is irrational. Intersections of two or more lines occur when $\mu$ is rational and can lead to points where there is degeneracy. However, the maximum degeneracy is only two, even though more than two lines may intersect at the same point. At $\mu = 2$, all eigenvalues are non-degenerate, and when $\mu = 1$, all eigenvalues except the lowest are doubly degenerate.

and

$$H_{n,m}(x) \sim \frac{1}{|x|^{n+1+m\mu}} \text{ when } m \neq 0$$

(17)

However, one can find eigenfunctions other than these, as shown below. This is easily done in the momentum space.

The eigenvalue equation in the momentum space

$$\left(|p|^\mu + \nu p \frac{\partial}{\partial p}\right) \psi_{\lambda}(p) = \nu \lambda \psi_{\lambda}(p),$$

(18)

has the solution

$$\psi_{\lambda}(p) = e^{-|p|^\mu/(\mu \nu)} p^\rho |p|^\sigma,$$

(19)

for any real positive numbers $\rho$ and $\sigma$ such that $\lambda = \rho + \sigma$, as may be easily verified by direct substitution. However all these eigenvalues do not appear in the expansion given in Eq. (12). Clearly, the ones do that appear are selected by the acceptability criterion on the
eigenfunction. In order to see what the condition is, we take the inverse transform of $\psi_\lambda(p)$ given in Eq. (19), which may be written as

$$\psi_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \ e^{-|p|^\mu/(\mu\nu) + ipx} |p|^\lambda \ (\text{sign} p)^\rho.$$  

The above integral may be rearranged to get

$$\psi_\lambda(x) = \frac{1 + e^{i\pi \rho}}{2\pi} C_\lambda(x) + \frac{1 - e^{i\pi \rho}}{2\pi} S_\lambda(x)$$  

(20)

with

$$C_\lambda(x) = \int_0^\infty dp \ e^{-p^\mu/(\mu\nu)} p^\lambda \cos(px)$$

(21)

$$S_\lambda(x) = \int_0^\infty dp \ e^{-p^\mu/(\mu\nu)} p^\lambda \sin(px).$$  

(22)

We identify that $C_\lambda(x)$ and $S_\lambda(x)$ are the two linearly independent solutions in the position space for the eigenvalue equation (Eq. (18)) having the same eigenvalue $\nu \lambda$. We now look at the asymptotic behavior of these two solutions. For this we use the identity

$$\cos(px) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} ds \ \frac{\Gamma(s)}{\Gamma(1-s/2)\Gamma(1+s/2)} (px)^{-s},$$

where $c > 0$ (note the the right hand side of the equation is a Mellin-Barnes integral) in the expression for $C_\lambda(x)$ and then perform the integral over $p$, to get

$$C_\lambda(x) = \frac{(\mu
u)^{\lambda/\mu}}{2i\mu} \int_{c-i\infty}^{c+i\infty} ds \ \frac{\Gamma(s)\Gamma(1+s/2)}{\Gamma(1+s/2)\Gamma(1+s/2)} \ ((\mu
u)^{1/\mu} x)^{-s}. \quad (23)$$

If one closes the contour on the left hand side using a semi-circle of radius $R \to \infty$, one gets a series expansion in terms of $x^n$, while if one closes the contour on the right hand side, one gets an asymptotic expansion in inverse powers of $x$. This asymptotic expansion is

$$C_\lambda(x) = \frac{(\mu
u)^{\lambda/\mu}}{\mu} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\lambda + k \mu + 1)}{\Gamma(k + 1)} \cos \left( \frac{\pi}{2} (\lambda + k \mu + 1) \right) \ ((\mu
u)^{1/\mu} x)^{-(\lambda+k\mu+1)}. \quad (24)$$

$S_\lambda(x)$ can be evaluated in an exactly similar fashion to be

$$S_\lambda(x) = \frac{(\mu
u)^{\lambda/\mu}}{\mu} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\lambda + k \mu + 1)}{\Gamma(k + 1)} \sin \left( \frac{\pi}{2} (\lambda + k \mu + 1) \right) \ ((\mu
u)^{1/\mu} x)^{-(\lambda+k\mu+1)}. \quad (25)$$

From these solutions, it seems as if any non-negative real value of $\lambda$ is an eigenvalue for the problem. However, by comparing the asymptotic behavior of the right eigenfunction in Eq.
and Eq. (17) and that of $C_\lambda(x)$ and $S_\lambda(x)$, we conclude that only if $\lambda = n + m\mu$, where $n, m$ are non-negative integers, would one have the correct asymptotic behavior. Interestingly even imposing this condition is not enough to restrict eigenfunctions and eigenvalues to those found from the Green’s function, because for any $\lambda$ we still seem to be having two solutions, viz. $C_\lambda(x)$ and $S_\lambda(x)$. We now discuss how to identify the correct eigenfunctions which appear in the expansion of Eq. (11).

For any value of $\lambda$, we can write down the different possible ways in which the eigenvalue can have the form $n + m\mu$. We now consider two separate possibilities:

1. $\mu$ is irrational: Then for any $\lambda = n + m\mu$, the values $(n, m)$ are unique. Further, from the Eq. (21) and (22), it is clear that if $n$ is even, then $C_n(x)$ behaves like $|x|^{-n-\mu-1}$, and $S_n(x) \sim |x|^{-n-1}$ and hence, only $C_n(x)$ is acceptable. The function $C_{n+m\mu}(x)$ is obtained from $C_n(x)$ by the application of $\left(-\frac{\partial^2}{\partial x^2}\right)^{\mu/2}$ $m$ times and therefore becomes the acceptable eigenfunction for $\lambda = n + m\mu$ where $\mu$ is even as $C_n(x)$ itself is acceptable. $S_{n+m\mu}(x)$ is an unacceptable solution when $n$ is even since $S_n(x)$ itself is not. One can make a similar argument when $n$ is odd.

2. $\mu$ is rational: Then $\mu$ can be written as $p/q$, where $p$ and $q$ are integers having no common factors. Then one can have degeneracy if $m$ is an integral multiple of $q$ equal to $kq$. The associated eigenvalue is $(n + kp)\nu$. Then all the states having quantum numbers $(n + kp, 0), (n + (k - 1)p, q), (n + (k - 2)p, 2q) \ldots (n, kq)$ will have the same eigenvalue and hence, one expects degeneracy. However, if $n$ and $p$ even, then all these possibilities lead to the even eigenfunction, viz. $C_{n+kp}(x)$ and the level will be non-degenerate. On the other hand, if either $n$ or $p$ is odd then one of the states will be $S_{n+kp}(x)$ and the other will be $C_{n+kp}(x)$. Thus one would have degeneracy in this case, and the degeneracy would be two.

Another interesting observation about the degeneracies of states has emerged out this analysis. If we consider a particular state say $\lambda = 4$ for $\mu = 1$ in Fig. 1 we see that five lines intersect at this point, seeming to suggest that the degeneracy of this state is 5. However, from the previous arguments the degeneracy of this state can be found to be 2. Of the five states which intersect at $\lambda = 4$, three of them become exactly identical to $C_4(x)$ and two of them to $S_4(x)$ and therefore, are not independent states. This sudden reduction in the number of eigenstates as one goes from $\mu = 1 - \epsilon$ to $\mu = 1$ to where $\epsilon$ is infinitesimally small
is not a problem as they are not constrained to be orthogonal, as is the case in quantum mechanics. This can also be treated as an evidence for the lack of a similarity transformation for the LOUP operator converting it to a Hermitian operator, as proposed by Toenjes et al. If such a transformation existed, it seems impossible for the eigenfunctions to undergo this sudden reduction in their number.

In conclusion, we have shown that the spectrum of the Fokker-Planck operator for LOUP determines completely the relaxation behavior of the system. Using the spectral expansion of the propagator, we have found the left and right eigenfunctions of the operator. The eigenvalue spectrum is found to be of the form \( n + m\mu \), characterized by the two quantum numbers \( n, m \in \mathbb{N} \). For irrational values of \( \mu \) the spectrum is non-degenerate, while for rational \( \mu \) there could be degeneracies with the maximum degeneracy being two. We also find that any acceptable eigenfunction of the operator should satisfy the condition that as \( |x| \to \infty \), the functions should behave like \( |x|^{-(n_1 + \mu n_2)} \), where \( n_1 \) and \( n_2 \) are positive integers, though this condition alone is not enough uniquely identify the eigenstates.

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