Delaunay-like Triangulation of Smooth Orientable Submanifolds by $\ell_1$-Norm Minimization

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Abstract

In this paper, we study the shape reconstruction problem, when the shape we wish to reconstruct is an orientable smooth $d$-dimensional submanifold of the Euclidean space. Assuming we have as input a simplicial complex $K$ that approximates the submanifold (such as the Čech complex or the Rips complex), we recast the problem of reconstructing the submanifold from $K$ as a $\ell_1$-norm minimization problem in which the optimization variable is a $d$-chain of $K$ over the field $\mathbb{R}$. Providing that $K$ satisfies certain reasonable conditions, we prove that the considered minimization problem has a unique solution which triangulates the submanifold and coincides with the flat Delaunay complex introduced and studied in a companion paper [4]. Since the objective is a weighted $\ell_1$-norm and the contraints are linear, the triangulation process can thus be implemented by linear programming.

1 Introduction

In many practical situations, the shape of interest is only known through a finite set of data points. Given as input these points, it is then natural to try to construct a triangulation of the shape, that is, a set of simplices whose union is homeomorphic to the shape. This paper focuses on one particular instance of this problem, where the shape we wish to reconstruct is a smooth orientable $d$-dimensional submanifold of the Euclidean space. We show that, under appropriate conditions, a triangulation of that submanifold can be expressed as the solution of a weighted $\ell_1$-norm minimization problem under linear constraints. This formulation gives rise to new algorithms for the triangulation of manifolds, in particular when the manifolds have large codimensions.

Variational formulation of Delaunay triangulations. Our work is based on the observation that when we consider a point cloud $P$ in $\mathbb{R}^d$, its Delaunay complex can be expressed as the solution of a particular $\ell_p$-norm minimization problem. This fact is best explained by lifting the point set $P$ vertically onto the paraboloid $\mathcal{P} \subseteq \mathbb{R}^{d+1}$ whose equation is $x_{d+1} = \sum_{i=1}^{d} x_i^2$. It is well-known that the Delaunay complex of $P$ is isomorphic to the boundary complex of the lower convex hull of the lifted points $\hat{P}$.

Starting from this equivalence, Chen has observed in [20] that the Delaunay complex of $P$ minimizes the $\ell_p$-norm of the difference between two functions over all triangulations $T$ of $P$. The graph of the first function is the lifted triangulation $\hat{T}$ and the graph of the second one is the paraboloid $\mathcal{P}$. This variational formulation has been successfully exploited in [2, 13, 21] for

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the generation of Optimal Delaunay Triangulations. When $p = 1$, the $\ell_p$-norm associated to $T$ is what we call in this paper the Delaunay energy of $T$ and, can be interpreted as the volume enclosed between the lifted triangulation $\hat{T}$ and the paraboloid $\mathcal{P}$.

**Our contribution.** While it seems difficult to extend the lifting construction when points of $P$ sample a $d$-dimensional submanifold of $\mathbb{R}^N$, our main result is to show that nonetheless, the induced variational formulation can still be transposed.

Consider a set of points $P$ that sample a $d$-dimensional submanifold $\mathcal{M}$. When searching for a triangulation of $\mathcal{M}$ given as input $P$, it seems reasonable to restrict the search within a simplicial complex $K$ built from $P$ (such as for instance the Čech complex or the Rips complex of $P$). A first crucial ingredient in our work is to embed the triangulations that one can build with the simplices of $K$ inside the vector space formed by simplicial $d$-cycles of $K$ over the field $\mathbb{R}$. In spirit, this is similar to what is done in the theory of minimal surfaces, when oriented surfaces are considered as particular elements of a much larger set, namely the space of currents [33], that enjoys the nice property of being a vector space. A second crucial ingredient is to define the Delaunay energy of each $d$-cycle as a weighted $\ell_1$-norm and search for $d$-cycles that minimize the Delaunay energy, in other words, search for $d$-cycles that are weighted $\ell_1$-minima. The celebrated sparsity of $\ell_1$-minima manifests itself in our context by the fact that the supports of such minima are sparse, in other words they are non-zero only on a small subset of simplices of $K$.

Going back to the case of points in the Euclidean space, if one minimizes the Delaunay energy in the larger set of simplicial chains with real coefficients and under adequate boundary constraints, one obtains a particular chain with coefficients in $\{0, 1\}$ whose simplices, roughly speaking, do not “overlap”. The support of that chain, that is the set of simplices with coefficient 1, coincides with the Delaunay triangulation. The proof is quite direct and relies on the geometric interpretation provided by the lifting construction [22, 39].

We show that, when transposing this to the case of points $P$ that sample a $d$-dimensional submanifold $\mathcal{M}$, minimizing the Delaunay energy provides indeed a triangulation of $\mathcal{M}$. The proof requires us to introduce a more elaborate construction, the Delloc complex of $P$, as a tool to describe the solution. The $d$-simplices of that complex possess exactly the property that we need for our analysis. In a companion paper [4] we show that the Delloc complex indeed provides a triangulation of the manifold, assuming the set of sample points $P$ to be sufficiently dense, safe, and not too noisy. Incidentally, the Delloc complex coincides with the flat Delaunay complex introduced in our companion paper [1] and is akin to the tangential Delaunay complex introduced and studied in [6, 7]. When the manifold is sufficiently densely sampled by the data points, all three constructions are locally isomorphic to a (weighted) Delaunay triangulation computed in a local tangent space to the manifold. Intuitively, this indicates that the Delaunay energy should locally reach a minimum for all three constructions and, therefore ought to be also a global minimum. Actually, turning this intuitive reasoning into a correct proof turns out to be more tricky than it appears and is the main purpose of the present paper. In particular, we need to globally compare the Delaunay energy of the cycle carrying the Delloc complex with that of alternate $d$-cycles, and this requires us to carefully distribute the Delaunay energy along barycentric coordinates (see Section 7).

For the purpose of the proof, it is convenient to first consider a rather artificial problem, where, besides the sample $P$, the manifold $\mathcal{M}$ is known. At the end of the paper, we show how to turn this problem into a more realistic one that takes as input only the sample of the unknown manifold, and is correct assuming that reasonable sampling conditions are met.

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1 Or relative $d$-cycles when the considered domain has a boundary.
**Algorithms.** Several authors, with computational topology or topological data analysis motivations, have considered the computation of \( \ell_1 \)-minimum homology representative cycles, \([17, 12, 23, 14, 24]\), generally for integers or integers modulo \( p \) coefficients. Note that an alternative algorithm to the one proposed in this paper could be to return such a minimal sparse representative. Indeed, when the data points sample sufficiently densely and accurately the manifold compared to the reach of the manifold, one could – in theory – take either the Čech complex or the Vietoris-Rips complex as the complex \( K \), since it is known that by choosing the scale parameter of these complexes carefully, they are guaranteed to have the same homotopy type as \( M \) \([16, 15, 5, 37, 31]\). Recall that, when \( M \) is orientable and connected, its \( d \)-homology group with real coefficients is one-dimensional, and a normalized generator of it is called the manifold fundamental class. Hence, when \( K \) and \( M \) are homotopy equivalent, the \( d \)-homology group of \( K \) is also one-dimensional. It follows that extracting any non-boundary cycle of \( K \) (using standard linear algebra operations on the boundary operators \( \partial_d \) and \( \partial_{d+1} \) of \( K \)) provides a \( d \)-cycle \( \gamma_0 \) which is, up to a multiplicative constant, a representative of a generator of the fundamental class of \( M \). The alternate algorithm could then search, among chains homologous to \( \gamma_0 \), for the one with the minimal Delaunay energy. Adapting the proof presented in this paper, one would probably be able to establish that the solution of the corresponding linear optimization problem is a chain which carries the Delloc complex. While elegant in theory, the size required for storing the \((d+1)\)-skeleton of the Čech or Vietoris-Rips complex may be prohibitive in practice.

Instead, we describe in the paper a slightly different and simpler algorithm that only requires the milder condition on \( K \) to be a simplicial complex large enough to contain the Delloc complex, at the cost of adding a certain form of normalization constraint on the solution. While we do not yet explore practical efficient algorithms in this paper, the minimization of a \( \ell_1 \)-norm under linear constraints in \( \mathbb{R}^n \), where \( n \) is the number of \( d \)-simplices in the considered simplicial complex \( K \), can be turned into a linear optimization problem in the standard form through slack variables, and can be addressed by standard linear programming techniques such as the simplex algorithm.

**Outline.** Section 2 presents generalities on sets, simplicial complexes and chains. Section 3 recalls basic facts about Delaunay complexes. In Section 4 we describe a weighted \( \ell_1 \)-norm minimization problem over the set of \( d \)-chains of \( K \) and state conditions under which the solution of this minimization problem provides indeed a triangulation of \( M \). In Section 5 we introduce the Delloc complex as a variant of the Delaunay complex. In Section 8 we prove that the \( d \)-chain supporting the Delloc complex is the unique solution to our minimization problem. For this, we rely on a technical lemma established in Section 6 and a result on power distances established in Section 7 that enables us to compare the cost of candidate solutions. In Section 9 we discuss some practical aspects.

## 2 Preliminaries

In this section, we review the necessary background and explain some of our terms.

### 2.1 Subsets and submanifolds

Given a subset \( A \subseteq \mathbb{R}^N \), the affine space spanned by \( A \) is denoted by \( \text{aff} A \) and the convex hull of \( A \) by \( \text{conv} A \). Recall that the \( r \)-tubular neighborhood of \( A \) is the set of points \( A^{\oplus r} = \{ x \in \mathbb{R}^N \mid d(x, A) \leq r \} \). The medial axis of \( A \), denoted as \( \text{axis}(A) \), is the set of points in \( \mathbb{R}^N \) that have at least two closest points in \( A \). The projection map \( \pi_A : \mathbb{R}^N \setminus \text{axis}(A) \to A \) associates to each point \( x \) its unique closest point in \( A \). The reach of \( A \) is the infimum of distances between \( A \) and its medial axis, and is denoted as reach \( A \). By definition, the projection map \( \pi_A \) is well-defined on
boundary operator

The value (or the coordinate) assigned to the oriented vertices of oriented $d$-chain $\gamma$.

Let $W$ be a weight function which assigns a non-negative weight in the space denoted by $\Sigma$. If it can be enclosed in a ball of radius $\rho$. Recall that the angle between two vector spaces $V_0$ and $V_1$ is defined as $\angle(V_0, V_1) = \max_{v_0 \in V_0} \min_{v_1 \in V_1} \angle v_0, v_1$. The angle between two affine spaces $A_0$ and $A_1$ whose corresponding vector spaces are $V_0$ and $V_1$ is $\angle(A_0, A_1) = \angle(V_0, V_1)$. Observe that the definition is not symmetric, unless the two affine spaces $A_0$ and $A_1$ have same dimension.

2.2 Simplicial complexes and faithful reconstructions

In this section, we review some background notation on algebraic topology and refer the reader to [35] for a detailed introduction to the topic.

All simplices and simplicial complexes that we consider in the paper are abstract. We recall that an abstract simplicial complex is a collection $K$ of finite non-empty sets with the property that if $\sigma$ belongs to $K$, so does every non-empty subset of $\sigma$. Each element $\sigma$ of $K$ is called an abstract simplex and its dimension is one less than its cardinality, $\dim(\sigma) = \text{card}(\sigma) - 1$. Each abstract simplex $\sigma \subseteq \mathbb{R}^N$ is naturally associated to a geometric simplex defined as $\text{conv}(\sigma)$. The dimension of $\text{conv}(\sigma)$ is the dimension of the affine space $\text{aff}(\sigma)$, and cannot be larger than the dimension of the abstract simplex $\sigma$. When the dimension of the geometric simplex $\text{conv}(\sigma)$ coincides with that of the abstract simplex $\sigma$, we say that $\sigma$ is non-degenerate. Equivalently, the vertices of $\sigma$ form an affinely independent set of points. We shall say that a simplicial complex $K$ is geometrically realized (or embedded) if (1) $\dim(\sigma) = \dim(\text{aff}(\sigma))$ for all $\sigma \in K$, and (2) $\text{conv}(\sigma \cap \beta) = \text{conv} \sigma \cap \text{conv} \beta$ for all $\alpha, \beta \in K$.

Given a set of simplices $\Sigma$ with vertices in $\mathbb{R}^N$ (not necessarily forming a simplicial complex), we let $\Sigma^{[d]}$ designate the $d$-simplices of $\Sigma$. We define the underlying space of $\Sigma$ as the subset of $\mathbb{R}^N$ covered by the relative interior of the geometric simplices associated to the abstract simplices in $\Sigma$, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \text{relint}(\text{conv}(\sigma))$. The closure of $\Sigma$ is the smallest simplicial complex that contains $\Sigma$.

2.3 Chains and weighted norms

In this section, we recall some standard notation concerning chains. Chains play an important role in this work as they provide a tool to embed the discrete set of candidate solutions (triangulations of $\mathcal{M}$ in $K$) into a larger continuous space. Consider an abstract simplicial complex $K$ and assume that each simplex $\sigma$ in $K$ is given an arbitrary orientation. A $d$-chain of $K$ with coefficients in $\mathbb{R}$ is a formal sum $\gamma = \sum_{\sigma} \gamma(\sigma) \sigma$, where $\sigma$ ranges over all $d$-simplices of $K$ and $\gamma(\sigma) \in \mathbb{R}$ is the value (or the coordinate) assigned to the oriented $d$-simplex $\sigma$ with the rule that if $\sigma$ and $\sigma'$ are two different orientations of the same simplex, then $\sigma = -\sigma'$. The set of such $d$-chains is a vector space denoted by $C_d(K, \mathbb{R})$. Recall that the $\ell_1$-norm of $\gamma$ is defined by $||\gamma||_1 = \sum_{\sigma} |\gamma(\sigma)|$. Let $W$ be a weight function which assigns a non-negative weight $W(\sigma)$ to each $d$-simplex $\sigma$ of $K$. The $W$-weighted $\ell_1$-norm of $\gamma$ is expressed as $||\gamma||_{1,W} = \sum_{\sigma} W(\sigma) |\gamma(\sigma)|$. We shall say that a chain $\gamma$ is carried by a subcomplex $D$ of $K$ if $\gamma$ has value 0 on every simplex that is not in $D$. The support of $\gamma$ is the set of simplices on which $\gamma$ has a non-zero value. It is denoted by $\text{Supp}\gamma$. The boundary operator is a homomorphism $\partial : C_d(K, \mathbb{R}) \rightarrow C_{d-1}(K, \mathbb{R})$ that associates to each oriented $d$-simplex $\sigma = [v_0, \ldots, v_p]$ the $(d-1)$-chain:

$$\partial \sigma = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p],$$
where the symbol \( \hat{v}_i \) means that the vertex \( v_i \) has been deleted from the sequence of vertices forming \( \sigma \).

3 Background on Delaunay complexes

In this section, we recall basic facts about Delaunay complexes. We give a geometric characterization in Section 3.1 before giving a variational characterization in Section 3.2. We then introduce Delaunay weights in Section 3.3 that will be useful in the next section. Delaunay complexes are known to provide triangulations of finite point sets. Before we start, let us recall what it means for a simplicial complex \( T \) to be a triangulation of a finite point set \( Q \).

**Definition 1** (Triangulation). Let \( Q \) be a finite point set of \( \mathbb{R}^N \). A triangulation \( T \) of \( Q \) designates a simplicial complex with vertex set \( Q \) which is geometrically realized and whose underlying space \( |T| \) is \( \text{conv } Q \).

3.1 A geometric characterization

We say that \( \sigma \subseteq Q \) is a Delaunay simplex of \( Q \) if there exists an \((N - 1)\)-sphere \( S \) that circumscribes \( \sigma \) and such that no points of \( Q \) belong to the open ball whose boundary is \( S \). The set of Delaunay simplices form an (abstract) simplicial complex called the Delaunay complex of \( Q \) and denoted as \( \text{Del}(Q) \).

**Definition 2** (General position). Let \( d = \dim(\text{aff } Q) \). We say that \( Q \subseteq \mathbb{R}^N \) is in general position if no \( d + 2 \) points of \( Q \) lie on a common \((d - 1)\)-dimensional sphere.

The following theorem is a classical result.

**Theorem 3.** When \( Q \) is in general position, \( \text{Del}(Q) \) is a triangulation of \( Q \).

3.2 A variational characterization

The Delaunay complex of \( Q \) optimizes many functionals over the set of triangulations of \( Q \) \([11, 36, 38]\), one of them being the Delaunay energy that we shall now define \([19]\).

In preparation for this, we recall a famous result which says that building a Delaunay complex in \( \mathbb{R}^N \) is topologically equivalent to building a lower convex hull in \( \mathbb{R}^{N+1} \). For simplicity, we shall identify each point \( x \in \mathbb{R}^N \) with a point \( (x, 0) \) in \( \mathbb{R}^{N+1} \). Consider the paraboloid \( \mathcal{P} \subseteq \mathbb{R}^{N+1} \) defined as the graph of the function \( || \cdot ||^2 : \mathbb{R}^N \rightarrow \mathbb{R}, x \mapsto ||x||^2 \), where \( || \cdot || \) designates the Euclidean norm. For each point \( x \in \mathbb{R}^N \), its vertical projection onto \( \mathcal{P} \) is the point \( \hat{x} = (x, ||x||^2) \in \mathbb{R}^{N+1} \), which we call the lifted image of \( x \). Similarly, the lifted image of \( Q \subseteq \mathbb{R}^N \) is \( \hat{Q} = \{ \hat{q} : q \in Q \} \). Recall that the lower convex hull of \( \hat{Q} \) is the portion of \( \text{conv } \hat{Q} \) visible to a viewer standing at \( x_{d+1} = -\infty \). A classical result says that for all \( \sigma \subseteq Q \), we have the following equivalence: \( \sigma \) is a Delaunay simplex of \( Q \) if and only if \( \text{conv } \hat{\sigma} \) is contained in the lower convex hull of \( \hat{Q} \) \([26]\).

We are now ready to define the Delaunay energy of any triangulation \( T \) of \( Q \). Let \( d = \dim(\text{aff } Q) \). Given a triangulation \( T \) of \( Q \), the Delaunay energy \( E_{\text{del}}(T) \) of \( T \) is defined as the \((d + 1)\)-volume between the \( d \)-manifold \( |\hat{T}| = \bigcup_{\sigma \in T} \text{conv } \hat{\sigma} \) and the paraboloid \( \mathcal{P} \). Let us give a formula for this \((d + 1)\)-volume. Consider a point \( x \in \text{conv } Q \). By construction, \( x \) belongs to at least one geometric \( d \)-simplex \( \text{conv } \sigma \) for some \( \sigma \in T \). Erect an infinite vertical half-line going up from \( x \). This half-line intersects the paraboloid \( \mathcal{P} \) at point \( \hat{x} \) and the lifted geometric \( d \)-simplex \( \text{conv } \hat{\sigma} \) at point \( x^*_\sigma \). We have

\[
E_{\text{del}}(T) = \sum_{\sigma} \int_{x \in \text{conv } \sigma} ||\hat{x} - x^*_\sigma|| \, dx,
\]
where \( \sigma \) ranges over all \( d \)-simplices of \( T \). Let us recall a well-known result [36, 20], that is a direct consequence of the lifting construction:

**Theorem 4.** When \( Q \) is in general position, the triangulation of \( Q \) that minimizes the Delaunay energy is unique and equals \( \text{Del}(Q) \).

### 3.3 Delaunay weights

To each abstract \( d \)-simplex \( \alpha \in \mathbb{R}^N \) we assign a non-negative real number that we call the Delaunay weight of \( \alpha \). This allows us to reformulate the Delaunay energy as a sum of Delaunay weights. Let \( \alpha \subseteq \mathbb{R}^N \) be an abstract \( d \)-simplex. If \( \alpha \) is non-degenerate, it makes sense to define \( S(\alpha) \) as the smallest \((N - 1)\)-sphere that circumscribes \( \alpha \). Let \( Z(\alpha) \) and \( R(\alpha) \) denote the center and radius of \( S(\alpha) \), respectively. We recall that the power distance of point \( x \in \mathbb{R}^N \) from \( S(\alpha) \) is

\[
\text{Power}_\alpha(x) = \|x - Z(\alpha)\| - R(\alpha)^2.
\]

**Definition 5 (Delaunay weight).** The Delaunay weight of an abstract \( d \)-simplex \( \alpha \) is:

\[
\omega(\alpha) = \begin{cases} 
-\int_{x \in \text{conv } \alpha} \text{Power}_\alpha(x) \, dx & \text{if } \alpha \text{ is non-degenerate}, \\
0 & \text{otherwise}.
\end{cases}
\]

Easy computations show that \( \text{Power}_\alpha(x) = -\|\hat{x} - x_\alpha^*\| \); see for instance [20]. Hence, writing \( d = \dim(\alpha) \), we see that \( \omega(\alpha) \) represents the \((d + 1)\)-volume between the lifted geometric simplex \( \text{conv } \hat{\sigma} \) and the paraboloid \( \mathcal{P} \). Therefore, if \( Q \) designates a finite point set of \( \mathbb{R}^N \) and we let \( d = \dim \text{aff } Q \), then the Delaunay energy of any triangulation \( T \) of \( Q \) can be expressed as

\[
E_{\text{del}}(T) = \sum_\alpha \omega(\alpha),
\]

where \( \alpha \) ranges over all \( d \)-simplices of \( T \). Below, we give a closed expression for the Delaunay weight due to Chen and Holst in [18]. For completeness, we provide a proof. Writing \( \text{vol}(\alpha) \) for the \( d \)-dimensional volume of \( \text{conv } \alpha \), we have:

**Lemma 6 ([18]).** The weight of the abstract \( d \)-simplex \( \alpha = \{a_0, \ldots, a_d\} \) is

\[
\omega(\alpha) = \frac{1}{(d + 1)(d + 2)} \text{vol}(\alpha) \left[ \sum_{0 \leq i < j \leq d} \|a_i - a_j\|^2 \right].
\]

**Proof.** Let \( \alpha = \{a_0, a_1, \ldots, a_d\} \subseteq \mathbb{R}^N \). If \( \alpha \) is degenerate, then \( \text{vol}(\alpha) = 0 \) and the result is clear. Suppose that \( \alpha \) is non-degenerate and recall that the standard simplex is

\[
\Delta_d = \{ \lambda \in \mathbb{R}^d \mid \sum_{i=1}^d \lambda_i \leq 1; \lambda_i \geq 0, i = 1, 2, \ldots, d \}.
\]

We introduce the map \( \psi : \mathbb{R}^d \to \mathbb{R}^d \), defined by \( \psi(\lambda) = a_0 + \sum_{i=1}^d \lambda_i (a_i - a_0) \), which establishes a one-to-one correspondence between the points \( \lambda \) of the standard simplex \( \Delta_d \) and the points \( x = \psi(\lambda) \) of \( \text{conv } \alpha \). Making the change of variable \( x = \psi(\lambda) \to \lambda \), we get that:

\[
w(\alpha) = \int_{\lambda \in \Delta_d} \text{Power}_\alpha(\psi(\lambda)) \cdot |\det(D\psi)(\lambda)| \, d\lambda.
\]
We obtain that \( \det(D\psi)(\lambda) = d! \text{vol}(\alpha) \). Observing that \( \psi(\lambda) \) has (normalized) barycentric coordinates \( (1 - \sum_{i=1}^{d} \lambda_i, \lambda_1, \lambda_2, \ldots, \lambda_d) \) and applying Lemma \([19]\) with \( z = a_0 \), we can write:

\[
\text{Power}_\alpha(\psi(\lambda)) = - \left( \sum_{i=1}^{d} \lambda_i \|a_i - a_0\|^2 \right) + \|\psi(\lambda) - a_0\|^2,
\]

and thus obtain (after plugging in the expression of \( \psi(\lambda) \))

\[
w(\alpha) = d! \text{vol}(\alpha) \int_{\lambda \in \Delta_d} \left[ \sum_{i=1}^{d} \lambda_i \|a_i - a_0\|^2 - \sum_{i=1}^{d} \lambda_i (a_i - a_0) \right]^2 d\lambda.
\]

We then use a formula for integrating a homogeneous polynomial on the standard simplex that may be found in \([11]\):

\[
\int_{\lambda \in \Delta_d} \lambda_1^{\eta_1} \cdots \lambda_d^{\eta_d} d\lambda = \frac{\eta_1! \cdots \eta_d!}{(d + \sum_i \eta_i)!}.
\]

We obtain that

\[
w(\alpha) = \frac{1}{(d + 1)(d + 2)} \text{vol}(\alpha) \left[ d \sum_{i=1}^{d} \|a_i - a_0\|^2 - 2 \sum_{1 \leq i < j \leq d} (a_i - a_0) \cdot (a_j - a_0) \right].
\]

Observing that \( \|a_i - a_0\|^2 + \|a_j - a_0\|^2 - 2(a_i - a_0) \cdot (a_j - a_0) = \|a_i - a_j\|^2 \), we can further rearrange the above formula to get the result. \( \Box \)

The expression of the Delaunay weight given in Lemma \([6]\) shows that two simplices that are isometric have the same Delaunay weight. Hence, a Delaunay energy can be straightforwardly associated to any “soup” \( \Sigma \) of \( d \)-simplices living in \( \mathbb{R}^N \) by setting \( E(\Sigma) = \sum_{\sigma \in \Sigma} \omega(\sigma) \). It is then tempting to ask what would happen if one minimizes this energy over all soups \( \Sigma \) of \( d \)-simplices whose vertices sample a \( d \)-dimensional submanifold \( \mathcal{M} \) and whose union is homeomorphic to that submanifold. As is, the problem is non-convex. We shall transform it into a convex problem in the next section.

### 4 Variational formulation for submanifold reconstruction

Let us first introduce the concept of faithful reconstruction which encapsulates what we mean by a “desirable” reconstruction of a subset \( \mathcal{M} \) of \( \mathbb{R}^N \):

**Definition 7** (Faithful reconstruction). Consider a subset \( \mathcal{M} \subseteq \mathbb{R}^N \) whose reach is positive, and a simplicial complex \( D \) with vertex set in \( \mathbb{R}^N \). We say that \( D \) reconstructs \( \mathcal{M} \) faithfully (or is a faithful reconstruction of \( \mathcal{M} \)) if the following three conditions hold:

- **Embedding**: \( D \) is geometrically realized;
- **Closeness**: \( |D| \) is contained in the \( r \)-tubular neighborhood of \( \mathcal{M} \) for some \( 0 \leq r < \text{reach} \mathcal{M} \);
- **Homeomorphism**: The restriction of \( \pi_M : \mathbb{R}^N \setminus \text{axis}(\mathcal{M}) \to \mathcal{M} \) to \( |D| \) is a homeomorphism.

Let \( \mathcal{M} \) designate the shape we wish to reconstruct and assume that \( \mathcal{M} \) is a compact orientable \( C^2 \) \( d \)-dimensional submanifold of \( \mathbb{R}^N \) for some \( d < N \). Let \( P \) be a finite point set that samples \( \mathcal{M} \). Suppose furthermore that we have at our disposal a simplicial complex \( K \) whose vertices are the points of \( P \). The complex \( K \) can be thought of as some rough approximation of \( \mathcal{M} \). Details on how to derive \( K \) from \( P \) are given at the end of the section. Given as input \( K \), we
explain in this section how to reconstruct faithfully $\mathcal{M}$ by solving a convex optimization problem. More precisely, in Section 4.1 we encode subcomplexes of $K$ as $d$-chains and prove that faithful reconstructions are encoded as cycles. In Section 4.2 we describe a convex optimization problem on the $d$-chains of $K$. In Section 4.3 we introduce notations and definitions to describe the quality of the sample $P$ of $\mathcal{M}$. In Section 4.4 we state conditions under which our convex optimization problem provides indeed a faithful reconstruction of $\mathcal{M}$.

Throughout the paper, we let $\mathcal{R} = \text{reach} \, \mathcal{M}$ and note that our assumptions on $\mathcal{M}$ imply that $\mathcal{R}$ is positive and finite \cite{27}. For any point $m \in \mathcal{M}$, we denote by $T_m \mathcal{M}$ the vector tangent space to $\mathcal{M}$ at $m$. The corresponding affine tangent subspace of $\mathbb{R}^N$ is denoted as $T_m \mathcal{M} = x + T_x \mathcal{M}$.

### 4.1 Encoding faithful reconstructions as cycles

In this section, we associate to each subcomplex $D$ of $K$ a $d$-chain and show that if $D$ is a faithful reconstruction of $\mathcal{M}$, then the $d$-chain associated to $D$ is a cycle under weak conditions on $K$. Let us assume that $\mathcal{M}$ together with all $d$-simplices of $K$ have received an (arbitrary) orientation. We define the angular deviation of a simplex $\sigma$ relatively to $\mathcal{M}$ as

$$\text{angularDeviation}_{\mathcal{M}}(\sigma) = \max_{m \in \pi_D(\text{conv} \, \sigma)} \angle(\text{aff} \, \sigma, T_m \mathcal{M}).$$

For each $d$-simplex $\alpha \in K$ such that $\text{angularDeviation}_{\mathcal{M}}(\alpha) < \frac{\pi}{2}$, we define the sign of $\alpha$ with respect to $\mathcal{M}$ as follows:

$$\text{sign}_{\mathcal{M}}(\alpha) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ is consistent with that of } \mathcal{M}, \\ -1 & \text{otherwise.} \end{cases}$$

We refer the reader to Appendix 2 for a formal definition of consistency and more details. We associate to any subcomplex $D \subseteq K$ the $d$-chain code$_D$ of $K$ whose coordinates are:

$$\text{code}_D(\alpha) = \begin{cases} \text{sign}_{\mathcal{M}}(\alpha) & \text{if } \alpha \in D^{[d]}, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 8.** Let $r, \rho \geq 0$ such that $\rho < \sqrt{2} (\mathcal{R} - r)$. Let $K$ be a simplicial complex such that $|K| \subseteq \mathcal{M}^{\rho+r}$ and whose $d$-simplices are $p$-small and have an angular deviation smaller than $\frac{\pi}{4}$ relatively to $\mathcal{M}$. If the subcomplex $D \subseteq K$ is a faithful reconstruction of $\mathcal{M}$, then code$_D$ is a cycle.

**Proof.** We first prove that for all simplices $\sigma \in D$ and all points $m \in \pi_D(\text{conv} \, \sigma)$, we have that

$$\pi_D(\text{conv} \, \sigma) \subseteq B \left( m, \sin \left( \frac{\pi}{4} \right) \text{reach} \, \mathcal{M} \right) \circ.$$  \hspace{1cm} (1)

Indeed, consider $x, x' \in \text{conv} \, \sigma$. Suppose that $m = \pi_D(x)$ and let $m' = \pi_D(x')$. We know from \cite{27} page 435 that for $0 \leq r < \text{reach} \, \mathcal{M}$, the projection map $\pi_D$ is $\left( \frac{\mathcal{R}}{\mathcal{R} - r} \right)$-Lipschitz for points at distance less than $r$ from $\mathcal{M}$. It follows that

$$\|m - m'\| \leq \frac{\mathcal{R}}{\mathcal{R} - r} \|x - x'\| \leq \frac{2\mathcal{R}}{\mathcal{R} - r} \mathcal{R} < \frac{\sqrt{2}}{2} \mathcal{R}$$

and Inclusion \cite{1} follows.

Given a simplicial complex $L$ and a point $x \in \mathbb{R}^N$, we define the **star of $x$ in $L$** as the set of simplices $\text{St}(x, L) = \{ \sigma \in L \mid x \in \text{conv} \, \sigma \}$. Since $D$ is a faithful reconstruction of $\mathcal{M}$, $|D|$ is a $d$-manifold. Hence, each $(d-1)$-simplex $\tau \in D$ has exactly two $d$-cofaces $\sigma_1$ and $\sigma_2$. Consider a
point $x$ in the relative interior of $\tau$ and its projection $m = \pi_M(x)$ onto $M$. The star of $x$ in $D$ consists of the two $d$-simplices $\sigma_1$ and $\sigma_2$ and the common $(d-1)$-face $\tau$. It follows that the set $\pi T_m M(St(x,D))$ possesses exactly two $d$-simplices $\sigma'_1 = \pi T_m M(\sigma_1)$ and $\sigma'_2 = \pi T_m M(\sigma_2)$, and one $(d-1)$-simplex $\tau' = \pi T_m M(\tau)$. As we project the $d$-simplex $\sigma_i = [u_0, \ldots, u_d]$, we preserve the vertex ordering, that is, we let $\sigma'_i = [\pi T_m M(u_0), \pi T_m M(u_1), \ldots, \pi T_m M(u_d)]$. Let us give to $T_m M$ an orientation that is consistent with that of $M$. Inclusion $\{1\}$ allows to apply Lemma $\{39\}$ in Appendix $\{\}$ each $d$-simplex $\sigma'_i$ has the same orientation with respect to $T_m M$ than that of $\sigma_i$ with respect to $M$. Let $s_i = \text{sign} \pi T_m M(\sigma'_i) = \text{sign} \pi M(\sigma_i)$.

We claim that the two geometric $d$-cofaces $\text{conv} \sigma'_1$ and $\text{conv} \sigma'_2$ of $\text{conv} \tau'$ have disjoint interiors. Indeed, let us denote by $U_x$, $U_x^1$, and $U_x^2$ small open neighborhoods of $x$ in $|D|$, $\text{aff} \sigma_1$ and $\text{aff} \sigma_2$, respectively. Suppose that $U_x$ is sufficiently small so that it is contained in $|St(x,D)|$.

Note that, for $i = 1, 2$, the map $\pi T_m M \circ \pi M|_{U_x^2}$ is differentiable and the map $\pi T_m M|_{U_x^2}$ is affine. Both maps have equal differential maps at $x$, that is:

$$D_x \left( \pi T_m M \circ \pi M|_{U_x^2} \right) = D_x \left( \pi T_m M|_{U_x^2} \right). \quad (2)$$

Let $T_i^+$ denote the set of all vectors parallel to $\text{aff} \sigma_i$ and pointing inside $\text{conv} \sigma_i$ after translation at $x$. This set forms a closed half-space in the vector tangent space to $\text{aff} \sigma_i$. Since $\pi T_m M|_{U_x^2}$ is affine, it coincides, up to a constant, with its differential at $x$ and using Equation $(2)$, we get that

$$\text{conv} \sigma'_i \subseteq x + D_x \left( \pi T_m M|_{U_x^2} \right)(T_i^+) = x + D_x \left( \pi T_m M \circ \pi M|_{U_x^2} \right)(T_i^+). \quad (3)$$

Observe that the map $\pi T_m M \circ \pi M|_{U_x^2}$, being the composition of two injective functions, is injective. It follows that $D_x \left( \pi T_m M \circ \pi M|_{U_x^2} \right)(T_i^+)$ and $D_x \left( \pi T_m M \circ \pi M|_{U_x^2} \right)(T_2^+)$ are two half-spaces in the vector space $T_m M$ with disjoint interiors. Using Equation $(3)$, we obtain that $\text{conv} \sigma'_1$ and $\text{conv} \sigma'_2$ of $\text{conv} \tau'$ also have disjoint interiors, as claimed.

It follows that $\partial(s_1 \sigma'_1 + s_2 \sigma'_2)$ is on $\tau'$, and consequently $\partial(s_1 \sigma_1 + s_2 \sigma_2)$ is on $\tau$. We have shown that $\partial \text{code}_D = 0$.

4.2 Least $\ell_1$-norm problem

Let $\omega$ be the weight function which assigns to each $d$-simplex $\alpha$ of $K$ its Delaunay weight $\omega(\alpha)$ introduced in Section $\{2\}$. We define the Delaunay energy of the chain $\gamma \in C_d(K, \mathbb{R})$ to be its $\omega$-weighted $\ell_1$-norm:

$$E_{\text{del}}(\gamma) = \|\gamma\|_{1,\omega} = \sum_\alpha \omega(\alpha) \cdot |\gamma(\alpha)| = \sum_\alpha \left( \int_{x \in \text{conv} \alpha} -\text{Power}_\alpha(x) \, dx \right) \cdot |\gamma(\alpha)|,$$

where $\alpha$ ranges over all $d$-simplices of $K$. Given a $d$-manifold $\mathcal{A}$, a point $a \in \mathcal{A}$, a set of simplices $\Sigma \subseteq K$, we assign to each $d$-chain $\gamma$ of $K$ the real number:

$$\text{load}_{a, \mathcal{A}, \Sigma}(\gamma) = \sum_{\sigma \in \Sigma^{[d]}} \gamma(\sigma) \text{sign}_{\mathcal{A}}(\sigma) \mathbf{1}_{\sigma \in \text{conv} \sigma}(a)$$

and call it the load of $\gamma$ on $\mathcal{A}$ at $a$ restricted to $\Sigma$. Letting $m_0$ be a generic$^2$ point on $M$, we are interested in the following optimization problem over the set of chains in $C_d(K, \mathbb{R})$:

$^2$Generic in the sense that it is not in the projection on $M$ of the convex hull of any $(d-1)$-simplex of $K$.
minimize \( \gamma E_{\text{del}}(\gamma) \)
subject to \( \partial \gamma = 0 \), \( \text{load}_{m_0,M,K}(\gamma) = 1 \)

Problem (⋆) is a least-norm problem whose constraint functions \( \partial \) and \( \text{load}_{m_0,M,K} \) are clearly linear. Problem (⋆) is thus a convex optimization problem and as such is solvable by linear programming. The first constraint \( \partial \gamma = 0 \) expresses the fact that we are searching for \( d \)-cycles. The second constraint \( \text{load}_{m_0,M,K}(\gamma) = 1 \) can be thought of as a kind of normalization of \( \gamma \). It forbids the zero chain to belong to the feasible set and we shall see that, under the assumptions of our main theorem, it forces the solution to take its coordinate values in \( \{0,+1,-1\} \).

In Problem (⋆), besides the simplicial complex \( K \) that we shall see how to build from \( P \), the knowledge of the manifold \( M \) seems to be required as well for expressing the normalization constraint. In Section 9.1, we discuss how to transform Problem (⋆) into an equivalent problem that does not refer to \( M \) anymore.

### 4.3 Geometric conditions

Recall that our goal is to give conditions under which a solution to Problem (⋆) provides a faithful reconstruction of \( M \). To express the conditions that we need, let us introduce some notations and definitions.

**Definition 9 (Dense sample).** We say that \( P \) is an \( \varepsilon \)-dense sample of \( M \) if for every point \( m \in M \), there is a point \( p \in P \) with \( \|p - m\| \leq \varepsilon \) or, equivalently, if \( M \subseteq P^\varepsilon \).

**Definition 10 (Accurate sample).** We say that \( P \) is a \( \delta \)-accurate sample of \( M \) if for every point \( p \in P \), there is a point \( m \in M \) with \( \|p - m\| \leq \delta \) or, equivalently, if \( P \subseteq M^{\pm \delta} \).

The separation of a point set \( P \) is
\[
\text{separation}(P) = \min_{p \neq q \in P} \|p - q\|.
\]

We recall that the height of a simplex \( \sigma \) is
\[
\text{height}(\sigma) = \min_{v \in \sigma} d(v, \text{aff}(\sigma \setminus \{v\})).
\]

The height of \( \sigma \) vanishes if and only if \( \sigma \) is degenerate. The protection of a simplex \( \sigma \) relatively to a point set \( Q \) is
\[
\text{protection}(\sigma, Q) = \min_{q \in \pi_{\text{aff}} \sigma(Q/\sigma)} d(q, S(\sigma)).
\]

We stress that our definition of a simplex protection differs slightly from the one in [8, 7]. We now associate to a finite point set \( P \) and a scale \( \rho \) three quantities that describe the quality of the pair \( (P, \rho) \) at dimension \( d \):
\[
\text{height}(P, \rho) = \min_{\sigma} \text{height}(\sigma),
\]
\[
\text{angularDeviation}_M(P, \rho) = \max_{\sigma} \text{angularDeviation}_M(\sigma),
\]
\[
\text{protection}(P, \rho) = \min_{\sigma} \text{protection}(\sigma, P \cap B(c_\sigma, \rho)),
\]

where the two minima and the maximum are over all \( \rho \)-small \( d \)-simplices \( \sigma \subseteq P \). Observe that assuming \( \text{height}(P, \rho) > 0 \) is equivalent to assuming that all \( \rho \)-small \( d \)-simplices of \( P \) are non-degenerate.
Definition 11 (Safety condition). Let $\varepsilon$, $\delta$, and $\rho$ be non-negative real numbers. The safety condition on $(P, \varepsilon, \delta)$ at scale $\rho$ is the existence of a real number $\theta \in \left[0, \frac{\pi}{6}\right]$ such that:

\[
\text{angularDeviation}_M(P, \rho) \leq \frac{\theta}{2} - \arcsin\left(\frac{\rho + \delta}{R}\right),
\]

\[
\text{separation}(P) > 8(\delta \theta + \rho \theta^2) + 6\delta + \frac{2\rho^2}{R},
\]

\[
\text{protection}(P, 3\rho) > 8(\delta \theta + \rho \theta^2) \left(1 + \frac{4\varepsilon}{\text{height}(P, \rho)}\right).
\]

Roughly speaking, assuming that safety condition at scale $\rho$ enforces $\rho$-small $d$-simplices of $P$ to make a sufficiently small angle relatively to $M$. It also enforces $P$ to be both sufficiently separated and protected at scale $3\rho$. As explained in the companion paper [4], the safety condition on $(P, \varepsilon, \delta)$ can be met by considering a $\frac{\varepsilon}{\rho^2}$-dense $(\frac{\pi}{6})$-accurate point set $P$ and perturbing it as described in [4].

4.4 Main theorem

In the statement of our main theorem, there is a constant $\Omega(\Delta_d)$ that depends only upon the dimension $d$ and whose definition is given in the proof of Lemma 24. Recall that the Čech complex of $P$ at scale $r$, denoted as $\mathcal{C}(P, r)$, is the set of simplices of $P$ that are $r$-small.

Theorem 12 (Faithful reconstruction by a variational approach). Let $M$ be a compact orientable $C^2$ $d$-dimensional submanifold of $\mathbb{R}^N$ for some $d < N$. Let $\varepsilon$, $\delta$, and $\rho$ be non-negative real numbers such that $\delta \leq \varepsilon$ and $16\varepsilon \leq \rho < \frac{R}{4}$. Let $\Theta = \text{angularDeviation}_M(P, \rho)$ and assume that $\Theta \leq \frac{\pi}{6}$. Set

\[
J = \frac{(R + \rho)^d}{(R - \rho)^d} \left(\cos \Theta\right)^{\min\{d, N-d\}} - 1.
\]

Let $P$ be a $\delta$-accurate $\varepsilon$-dense sample of $M$ such that $\text{height}(P, \rho) > 0$. Suppose that the safety condition on $(P, \varepsilon, \delta)$ is satisfied at scale $\rho$. Suppose furthermore that

\[
\text{protection}(P, 3\rho)^2 + \text{protection}(P, 3\rho) \text{separation}(P) > \\
\max \left\{10\rho \Theta(\varepsilon + \rho \Theta), \frac{4J(1 + J)}{(d + 2)(d - 1)! \Omega(\Delta_d) \rho^2}\right\}.
\]

Consider a simplicial complex $K$ such that

\[
\text{Del}(P) \cap \mathcal{C}(P, \varepsilon) \subseteq K \subseteq \mathcal{C}(P, \rho).
\]

Then Problem \(\star\) has a unique solution and the closure of the support of that solution is a faithful reconstruction of $M$.

Observe that our main theorem does not require $K$ to be geometrically realized nor to retain the homotopy type of $M$. One may ask about the feasibility of realizing the assumptions of Theorem 12. In Section 9.2, we explain how to apply Moser Tardos Algorithm ([34] and [7, Section 5.3.4]) as a perturbation scheme to enforce both the safety condition and Condition (4) required by Theorem 12.
Choosing the simplicial complex $\mathcal{K}$. The Rips complex of $P$ at scale $r$, denoted as $\mathcal{R}(P,r)$, consists of all simplices of $P$ with diameter at most $2r$. It is a more easily-computed version of the Čech complex. We stress that our main theorem applies to any simplicial complex $\mathcal{K}$ such that $\text{Del}(P) \cap \mathcal{C}(P,\rho) \subseteq \mathcal{K} \subseteq \mathcal{C}(P,\rho)$. Since $\mathcal{C}(P,r) \subseteq \mathcal{R}(P,r) \subseteq \mathcal{C}(P,\sqrt{2}r)$ for all $r \geq 0$, it applies in particular to any $K = \mathcal{R}(P,r)$ with $\varepsilon \leq r \leq \frac{\rho}{\sqrt{2}}$. This choice of $K$ is well-suited for applications in high dimensional spaces, while choosing $K = \text{Del}(P) \cap \mathcal{C}(P,\rho)$ for any $\varepsilon \leq r \leq \rho$ may be more suited for applications in low dimensional spaces.

5 Delloc complexes

In this section, we define the Delloc complex. We then recall a key result established in the companion paper [4]: when the Delloc complex is computed over a finite point set $P$ that samples some $d$-dimensional submanifold of $\mathbb{R}^N$, it provides a faithful reconstruction of that submanifold. Incidentally, under the right assumptions, the Delloc complex coincides with the flat Delaunay complex [4] and the tangential Delaunay complex [6, 7]. Since all the results in this paper are based on the property for a simplex to belong to the Delloc complex, we find it more enlightening to formulate the results of this paper using the Delloc complex.

Definition. Afterwards, $P$ designates a finite set of points in $\mathbb{R}^N$, $d$ designates an integer in $[0,N)$ and $\rho \geq 0$ designates a scale parameter.

Definition 13 (Delloc complex). We say that a simplex $\sigma$ is delloc in $P$ at scale $\rho$ if

$$\sigma \in \text{Del}(\pi_{\text{aff}}(P \cap B(c_\sigma, \rho)))$$

where $c_\sigma$ denotes the center of the smallest $N$-ball enclosing $\sigma$. The $d$-dimensional Delloc complex of $P$ at scale $\rho$, denoted by Delloc$_d(P,\rho)$, is the set of $d$-simplices that are delloc in $P$ at scale $\rho$ together with all their faces.

Remark 14. It is easy to see that if $2R(\sigma) \leq \rho$, then the smallest circumsphere $S(\sigma)$ of $\sigma$ is contained in $B(c_\sigma, \rho)$. It follows that a delloc simplex $\sigma$ in $P$ at scale $\rho$ is also a Gabriel simplex of $P$, by which we mean that $S(\sigma)$ does not enclose any point of $P$ in its interior. In particular, when $2R(\sigma) \leq \rho$, then $\sigma$ is a Delaunay simplex of $P$.

Key result. We now recall a key result established in the companion paper [4] and which gives condition under which the Delloc complex of $P$ is a faithful reconstruction of $\mathcal{M}$.

Theorem 15 (Faithful reconstruction by a geometric approach). Let $\varepsilon$, $\delta$, and $\rho$ be non-negative real numbers such that $\delta \leq \varepsilon$ and $16\varepsilon \leq \rho < \frac{\rho}{4}$. Let $P$ be a $\delta$-accurate $\varepsilon$-dense sample of $\mathcal{M}$ such that $\text{height}(P,\rho) > 0$. Suppose that the safety condition on $(P,\varepsilon,\delta)$ is satisfied at scale $\rho$. Then, Delloc$_d(P,\rho)$ is a faithful reconstruction of $\mathcal{M}$. Furthermore, for all $d$-simplices $\sigma \in \text{Delloc}_d(P,\rho)$, we have $R(\sigma) \leq \varepsilon$.

Remark 16. Under the assumptions of Theorem 15, Remark 14 implies that the Delloc complex of $P$ at scale $\rho$ is a subset of the Delaunay complex of $P$, that is, Delloc$_d(P,\rho) \subseteq \text{Del}(P)$. It follows that under the assumptions of Theorem 15,

$$\text{Delloc}_d(P,\rho) \subseteq \text{Del}(P) \cap \mathcal{C}(P,\varepsilon)$$

and therefore any simplicial complex $\mathcal{K}$ that satisfies the assumptions of Theorem 12 contains Delloc$_d(P,\rho)$.
6 Technical lemma

The proof of Theorem 12 relies on a technical lemma which we now state and prove.

**Lemma 17.** Let \( \mathcal{D} \) be a \( d \)-dimensional submanifold (with or without boundary) of \( \mathbb{R}^N \) and let \( K \) be a simplicial complex with vertices in \( \mathbb{R}^N \). Assume that there is a map \( \varphi : |K| \to \mathcal{D} \). Suppose that for each \( d \)-simplex \( \alpha \in K \), we have two positive weights \( W(\alpha) \geq W_{\min}(\alpha) \) and that there exists a map \( f : \mathcal{D} \to \mathbb{R} \) such that \( W_{\min}(\alpha) = \int_{\varphi(\conv \alpha)} f \). Consider the \( d \)-chain \( \gamma_{\min} \) on \( K \) defined by

\[
\gamma_{\min}(\alpha) = \begin{cases} 1 & \text{if } W_{\min}(\alpha) = W(\alpha), \\ 0 & \text{otherwise}. \end{cases}
\]

Suppose that \( \sum_{\alpha \in K'[d]} \gamma_{\min}(\alpha)1_{\varphi(\conv \alpha)}(x) = 1 \), for almost all \( x \in \mathcal{D} \). Then the \( \ell_1 \)-like norm \( \|\gamma\|_{1,W} \) attains its minimum over all \( d \)-chains \( \gamma \) such that

\[
\sum_{\alpha \in K'[d]} \gamma(\alpha)1_{\varphi(\conv \alpha)}(x) = 1, \quad \text{for almost all } x \in \mathcal{D}
\]

if and only if \( \gamma = \gamma_{\min} \).

**Proof.** We write \( \tilde{\alpha} = \varphi(\conv \alpha) \) throughout the proof for a shorter notation. We prove the lemma by showing that for all \( d \)-chains \( \gamma \) on \( K \) that satisfy constraint (6), we have:

\[
\|\gamma\|_{1,W} \geq \|\gamma\|_{1,W_{\min}} \geq \int_{\mathcal{D}} f = \|\gamma_{\min}\|_{1,W_{\min}} = \|\gamma_{\min}\|_{1,W},
\]

with the first inequality being an equality if and only if \( \gamma = \gamma_{\min} \). Clearly, \( \|\gamma\|_{1,W} \geq \|\gamma\|_{1,W_{\min}} \) because \( W(\alpha) \geq W_{\min}(\alpha) \). To obtain the second inequality, recall that we have assumed \( \sum_{\alpha} \gamma(\alpha)1_{\tilde{\alpha}}(x) = 1 \) almost everywhere in \( \mathcal{D} \). We use this to write that:

\[
\|\gamma\|_{1,W_{\min}} \geq \sum_{\alpha} \gamma(\alpha) \int_{\tilde{\alpha}} f = \sum_{\alpha} \gamma(\alpha) \int_{\mathcal{D}} f \cdot 1_{\tilde{\alpha}} = \int_{\mathcal{D}} f \cdot \sum_{\alpha} \gamma(\alpha)1_{\tilde{\alpha}} = \int_{\mathcal{D}} f,
\]

where sums are over all \( d \)-simplices \( \alpha \) in \( K \). Setting \( \gamma = \gamma_{\min} \) in (8), we observe that the inequality in (8) becomes an equality because none of the coefficients of \( \gamma_{\min} \) are negative by construction. It follows that \( \int_{\mathcal{D}} f = \|\gamma_{\min}\|_{1,W_{\min}} \). Finally, \( \|\gamma_{\min}\|_{1,W_{\min}} = \|\gamma_{\min}\|_{1,W} \) because \( \gamma_{\min} \) has been defined so that for all simplices \( \alpha \) in its support, \( W_{\min}(\alpha) = W(\alpha) \). We have thus established (7). Suppose now that \( \gamma \neq \gamma_{\min} \) and let us prove that \( \|\gamma\|_{1,W} > \|\gamma\|_{1,W_{\min}} \), or equivalently that

\[
\sum_{\alpha \in \text{Supp } \gamma} |\gamma(\alpha)| (W(\alpha) - W_{\min}(\alpha)) > 0.
\]

Since none of the terms in the above sum are negative, it suffices to show that there exists at least one simplex \( \alpha \in \text{Supp } \gamma \) for which \( W(\alpha) > W_{\min}(\alpha) \). By contradiction, assume that for all \( \alpha \in \text{Supp } \gamma \), \( W(\alpha) = W_{\min}(\alpha) \). By construction, we thus have the implication: \( \gamma(\alpha) \neq 0 \implies \gamma_{\min}(\alpha) = 1 \), and therefore \( \text{Supp } \gamma \subseteq \text{Supp } \gamma_{\min} \). But, since \( \sum_{\alpha} \gamma_{\min}(\alpha)1_{\tilde{\alpha}}(x) = 1 \) for almost all \( x \in \mathcal{D} \) and coefficients of \( \gamma_{\min} \) are either 0 or 1, it follows that for almost all \( x \in \mathcal{D} \), point \( x \) is covered by a unique \( d \)-simplex in the support of \( \gamma_{\min} \). Hence, the simplices in \( \text{Supp } \gamma_{\min} \) have pairwise disjoint interiors while their union covers \( \mathcal{D} \). Since \( \sum_{\alpha} \gamma(\alpha)1_{\tilde{\alpha}}(x) = 1 \) for almost all \( x \in \mathcal{D} \), the simplices in \( \text{Supp } \gamma \) must also cover \( \mathcal{D} \) while using only a subset of simplices in \( \text{Supp } \gamma_{\min} \). The only possibility is that \( \gamma = \gamma_{\min} \), yielding a contradiction. \( \square \)
7 Comparing power distances

The goal of this section is to relate the two maps \( \text{Power}_\alpha(x) \) and \( \text{Power}_\beta(y) \) for two \( d \)-simplices \( \alpha \in \text{Del}_d(P, \rho) \) and \( \beta \subseteq P \), and for two points \( x \in \text{conv} \alpha \) and \( y \in \text{conv} \beta \), such that \( \pi_M(x) = \pi_M(y) \). The main result of the section is stated in the following lemma and proved at the end of the section. We recall that given a non-degenerate simplex \( \alpha \) and a point \( x \in \text{aff} \sigma \), the (normalized) barycentric coordinates of \( x \) relatively to the simplex \( \alpha \) are real numbers \( \{ \lambda_a \}_{a \in \alpha} \) such that \( x = \sum_{a \in \alpha} \lambda_a a \) and \( \sum_{a \in \alpha} \lambda_a = 1 \). We write

\[
\text{BarycentricCoord}_a^\alpha(x) = \lambda_a
\]

**Lemma 18.** Let \( \varepsilon, \delta, \rho \geq 0 \) such that \( 0 \leq 2\varepsilon \leq \rho \), and \( 16\delta \leq \rho \leq \frac{\rho}{3} \). Suppose that \( P \subseteq M^{\varepsilon, \delta} \). Let \( p = \text{projection}(P, 3\rho) \), \( s = \text{separation}(P) \), and \( \Theta = \text{angularDeviation}_M(P, \rho) \). Assume that \( \Theta \leq \frac{\pi}{6} \) and

\[
10p \Theta (\varepsilon + \rho \Theta) < p^2 + ps.
\]

Then, for every non-degenerate \( \varepsilon \)-small \( d \)-simplex \( \alpha \in \text{Del}_d(P, \rho) \), every non-degenerate \( \rho \)-small \( d \)-simplex \( \beta \subseteq P \), every \( x \in \text{conv} \alpha \), and every \( y \in \text{conv} \beta \) such that \( \pi_M(x) = \pi_M(y) \):

\[
- \text{Power}_\beta(y) + \text{Power}_\alpha(x) \geq \frac{1}{2} \left( p^2 + ps \right) \sum_{b \in \beta \setminus \alpha} \text{BarycentricCoord}_b^\beta(y).
\]

To prove the lemma, we need a few auxiliary results. We start by recalling a useful expression of the power distance of a point \( x \) from the circumsphere \( S(\alpha) \) of \( \alpha \) when \( x \) is an affine combination of the vertices of \( \alpha \).

**Lemma 19.** Let \( \alpha \subseteq \mathbb{R}^N \) be a non-degenerate simplex. For every \( z \in \mathbb{R}^N \)

\[
\text{Power}_\alpha(x) = \|x - z\|^2 - \sum_{a \in \alpha} \text{BarycentricCoord}_a^\alpha(x) \|a - z\|^2.
\]

**Proof.** Recall that \( \text{Power}_\alpha(x) = \|x - Z(\alpha)\|^2 - R(\alpha)^2 \). Let \( z \in \mathbb{R}^N \). On one hand, we have that

\[
\|x - Z(\alpha)\|^2 = \|x - z\|^2 + 2(x - z) \cdot (z - Z(\alpha)) + \|z - Z(\alpha)\|^2.
\]

On the other hand, writing \( \lambda_a = \text{BarycentricCoord}_a^\alpha(x) \), we have that

\[
R(\alpha)^2 = \sum_{a \in \alpha} \lambda_a \|Z(\alpha) - a\|^2 = \sum_{a \in \alpha} \lambda_a \left[ \|Z(\alpha) - z\|^2 + 2(Z(\alpha) - z) \cdot (z - a) + \|z - a\|^2 \right] = \|Z(\alpha) - z\|^2 + 2(Z(\alpha) - z) \cdot (z - x) + \sum_{a \in \alpha} \lambda_a \|z - a\|^2.
\]

Subtracting the above expressions of \( \|x - Z(\alpha)\|^2 \) and \( R(\alpha)^2 \) yields the result. \( \square \)

**Lemma 20.** Let \( \alpha \) and \( \beta \) be two non-degenerate abstract \( d \)-simplices in \( \mathbb{R}^N \) such that \( \alpha \in \text{Del}(\pi_{\text{aff}}(\alpha \cup \beta)) \). Let \( p = \text{projection}(\alpha, \beta) \). Then for every \( y \in \text{conv} \beta \), we have

\[
\text{Power}_\beta(y) \leq \text{Power}_\alpha(\pi_{\text{aff}}(\alpha)(y)) - (p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \text{BarycentricCoord}_b^\beta(y).
\]
that $\beta$ then for every $x \in \text{conv} \alpha$ and $y \in \text{conv} \beta$ with $\pi_{\mathcal{M}}(x) = \pi_{\mathcal{M}}(y)$, we have

$$\text{Power}_{\beta}(y) \leq \text{Power}_{\alpha}(x) - \frac{1}{2}(p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \text{BarycentricCoord}_{\beta}^{\alpha}(y).$$

Figure 1: Notation for the proof of Lemma 20

**Proof.** See Figure 1. Let $Z(\alpha)$ be the radius of the $(d-1)$-dimensional circumsphere of $\alpha$. Clearly, $\|a - Z(\alpha)\| = R(\alpha)$ for all $a \in \alpha$. Since $\alpha \in \text{Del}(\pi_{\text{aff} \alpha}(\alpha \cup \beta))$ and $p = \text{protection}(\alpha, \beta)$, we get:

$$(R(\alpha) + p)^2 \leq \|\pi_{\text{aff} \alpha}(b) - Z(\alpha)\|^2, \quad \text{for all } b \in \beta \setminus \alpha,$$

$$R(\alpha)^2 = \|\pi_{\text{aff} \alpha}(b) - Z(\alpha)\|^2, \quad \text{for all } b \in \beta \cap \alpha.$$

Let $\mu_b = \text{BarycentricCoord}_{\beta}^{\alpha}(y)$ and note that $\mu_b \geq 0$. Multiplying both sides of each equation above by $\mu_b$ and summing over all $b \in \beta$, we obtain:

$$R(\alpha)^2 + (p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \mu_b \leq \sum_{b \in \beta} \mu_b \|\pi_{\text{aff} \alpha}(b) - Z(\alpha)\|^2. \quad (9)$$

For short, write $y' = \pi_{\text{aff} \alpha}(y)$ and $\beta' = \pi_{\text{aff} \alpha}(\beta)$. Noting that $y' = \sum_{b \in \beta} \mu_b b'$ and applying Lemma 19 with $z = Z(\alpha)$, we get that

$$\text{Power}_{\beta'}(y') = \|y' - Z(\alpha)\|^2 - \sum_{b \in \beta} \mu_b \|\pi_{\text{aff} \alpha}(b) - Z(\alpha)\|^2.$$

Subtracting $\|y' - Z(\alpha)\|^2$ from both sides of (9) and using the above expression, we obtain

$$- \text{Power}_{\alpha}(y') + (p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \mu_b \leq - \text{Power}_{\beta'}(y').$$

Applying Lemma 19 again, with $Z = y'$ and $Z = y$ respectively, we get that:

$$- \text{Power}_{\beta'}(y') = \sum_{b \in \beta} \mu_b \|\pi_{\text{aff} \alpha}(b) - \pi_{\text{aff} \alpha}(y)\|^2 \leq \sum_{b \in \beta} \mu_b \|b - y\|^2 = - \text{Power}_{\beta}(y),$$

which concludes the proof. \qed

**Lemma 21.** Let $\alpha$ and $\beta$ be two non-degenerate abstract $d$-simplices in $\mathbb{R}^N$ such that $\alpha \in \text{Del}(\pi_{\text{aff} \alpha}(\alpha \cup \beta))$. Let $p = \text{protection}(\alpha, \beta)$ and $\Theta = \text{angularDeviation}_{\mathcal{M}}(\alpha)$. Suppose that both $\text{conv} \alpha$ and $\text{conv} \beta$ are contained in the $\left(\frac{\pi}{4}\right)$-tubular neighborhood of $\mathcal{M}$. Suppose furthermore that $\beta$ is $\rho$-small. If $\Theta \leq \frac{\pi}{6}$ and

$$5\rho \sin(\Theta) \left(2R(\alpha) + \frac{\rho}{2} \sin(\Theta)\right) < p^2 + 2pR(\alpha),$$

then for every $x \in \text{conv} \alpha$ and every $y \in \text{conv} \beta$ with $\pi_{\mathcal{M}}(x) = \pi_{\mathcal{M}}(y)$, we have

$$\text{Power}_{\beta}(y) \leq \text{Power}_{\alpha}(x) - \frac{1}{2}(p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \text{BarycentricCoord}_{\beta}^{\alpha}(y).$$
Proof. Consider a point \( x \in \text{conv} \alpha \) and a point \( y \in \text{conv} \beta \) with \( \pi_M(x) = \pi_M(y) \). We distinguish two cases depending on whether \( y \) belongs to \( \text{conv}(\alpha \cap \beta) \) or not.

First, assume that \( y \in \text{conv}(\alpha \cap \beta) \). In that case, we claim that the only possibility is that \( x = y \). Indeed, assume for a contradiction that this is not the case. Then, we would have two distinct points \( x \neq y \) of \( \text{conv} \alpha \) that share the same projection onto \( M \), showing that \( \angle(\text{aff} \alpha, T_{\pi_M(x)} M) = \frac{\pi}{2} \) for some \( x \in \text{conv} \alpha \) and contradicting our assumption that \( \Theta < \frac{\pi}{6} \). Hence, \( x = y \in \text{conv}(\alpha \cap \beta) \). We claim that furthermore \( \text{Power}_\alpha(x) = \text{Power}_\beta(y) \). Indeed, Lemma 19 implies that when \( x \) is an affine combination of points in \( \alpha \), that is, when \( x = \sum_{a \in \alpha} \lambda_a a \) with \( \sum_{a} \lambda_a = 1 \), then \( \text{Power}_\alpha(x) = -\sum_{a \in \alpha} \lambda_a \| x - a \|^2 \). In particular, if \( x \) belongs to the convex hull of a face of \( \alpha \), the expression of the power distance depends only upon the vertices of that face. It follows that

\[
\text{Power}_\alpha(x) = \text{Power}_{\alpha \cap \beta}(x) = \text{Power}_{\alpha \cap \beta}(y) = \text{Power}_\beta(y).
\]

Since \( y \in \text{conv}(\alpha \cap \beta) \), we have \( \sum_{b \in \beta \setminus \alpha} \text{BarycentricCoord}^\beta_b(y) = 0 \) and combining this with the above equality, we get the desired inequality.

Second, assume that \( y \in \text{conv} \beta \setminus \text{conv}(\alpha \cap \beta) \); see Figure 2. Write \( \mu_b = \text{BarycentricCoord}^\beta_b(y) \) and note that \( \mu_b \geq 0 \). Letting \( y' = \pi_{\text{aff} \alpha}(y) \), we know by Lemma 20 that:

\[
\text{Power}_\beta(y) \leq \text{Power}_\alpha(y') - (p^2 + 2pR(\alpha)) \sum_{b \in \beta \setminus \alpha} \mu_b. \tag{10}
\]

Because \( y \notin \text{conv}(\alpha \cap \beta) \), we have \( \sum_{b \in \beta \setminus \alpha} \mu_b \neq 1 \) and therefore \( \sum_{b \in \beta \setminus \alpha} \mu_b \neq 0 \). First, suppose that \( y = x \). In that case, \( y' = x \) and the result follows immediately. Second, suppose that \( y \neq x \). We claim that in that case we also have \( y \neq y' \). Indeed, if we were to have that \( y = y' \), then both \( x \) and \( y \) would belong to \( \text{aff} \alpha \) and since \( \pi_M(x) = \pi_M(y) \), this would mean that \( \angle(\text{aff} \alpha, T_{\pi_M(x)} M) = \frac{\pi}{2} \) for some \( x \in \text{conv} \alpha \), contradicting our assumption that \( \Theta < \frac{\pi}{6} \). Thus, \( x \neq y \) and \( y \neq y' \), and we can define the angle \( \theta = \angle xy y' \). Noting that \( \theta \leq \angle(\text{aff} \alpha, T_{\pi_M(x)} M) \leq \Theta \) and \( \| x - y' \| = \| x - y \| \sin \theta \), and recalling that \( Z(\alpha) \) is the radius of the \((d - 1)\)-dimensional circumsphere of \( \alpha \), we have:

\[
\text{Power}_\alpha(y') - \text{Power}_\alpha(x) = \| y' - Z(\alpha) \|^2 - \| x - Z(\alpha) \|^2 = (y' - x) \cdot (x + y' - 2Z(\alpha)) \\
\leq \| x - y' \| \cdot ((\| x - Z(\alpha) \| + \| y' - Z(\alpha) \|) \\
\leq \| x - y' \| \cdot (2\| x - Z(\alpha) \| + \| x - y' \|) \\
\leq \| x - y \| \sin(\theta) (2R(\alpha) + \| x - y \| \sin(\theta)) \tag{11}
\]

Figure 2: Right: Notation for the proof of Lemma 21.
Writing $m = \pi_M(x) = \pi_M(y)$, we have $\|x - y\| \leq \|x - m\| + \|m - y\| \leq \frac{\rho}{2}$. Summing up Inequalities (10) and (11), we get

$$\text{Power}_\beta(y) - \text{Power}_\alpha(x) \leq - \left( p^2 + 2pR(\alpha) \right) \sum_{b \in \beta \setminus \alpha} \mu_b + \| x - y \| \sin(\Theta) \left( 2R(\alpha) + \frac{\rho}{2} \sin(\Theta) \right).$$

To establish the lemma in the second case, it suffices to show that $2B < A$, that is,

$$2\|x - y\| \sin \Theta \cdot \left( 2R(\alpha) + \frac{\rho}{2} \sin \Theta \right) < \left( p^2 + 2pR(\alpha) \right) \sum_{b \in \beta \setminus \alpha} \mu_b. \tag{12}$$

We consider two subcases:

Subcase 1: $\alpha \cap b = \emptyset$. In that case, $\sum_{b \in \beta \setminus \alpha} \mu_b = 1$, and because $2\|x - y\| \leq 4\rho \leq 5\rho$, one can see that (12) follows from our assumptions.

Subcase 2: $\alpha \cap b \neq \emptyset$. In that case, we know that there exists a point $u \in \text{conv}(\beta \setminus \alpha)$ and a point $v \in \text{conv}(\beta \setminus \alpha)$ such that $y = \sum_{b \in \beta \cap \alpha} \mu_b u + \sum_{b \in \beta \setminus \alpha} \mu_b v$; see Figure 2. Furthermore, letting $v' = \pi_{\text{aff}}(v)$ we have

$$\sum_{b \in \beta \setminus \alpha} \mu_b \left( \frac{y - u}{\| v - u \|} \right) + \sum_{b \in \beta \cap \alpha} \mu_b \left( \frac{y - u}{\| v - u \|} \right) \geq \left( \frac{x - y}{\text{Diam}(\beta)} \right) \geq \left( \frac{x - y}{\cos \sigma} \right) \geq \frac{\sqrt{3}}{4\rho} \cdot \| x - y \|.$$ 

Again, (12) follows from our assumptions.

The next lemma says that if a subset $\sigma \subseteq \mathbb{R}^N$ is sufficiently small and sufficiently close to a subset $A \subseteq \mathbb{R}^N$ compare to the reach of $A$, then the convex hull of $\sigma$ is not too far away from $A$.

**Lemma 22.** Let $16\delta \leq \rho \leq \frac{\text{reach} A}{3}$. If the subset $\sigma \subseteq A^{\pm \delta}$ is $\rho$-small, then $\text{conv} \sigma \subseteq A^{\pm \frac{\rho}{4}}$.

**Proof.** Let $\mathcal{R} = \text{reach} A$. Applying Lemma 14 in [3], we get that $\text{conv} \sigma \subseteq A^{\pm \mathcal{R}}$ for $r = \mathcal{R} - \sqrt{(\mathcal{R} - \delta)^2 - \rho^2}$. Since $\delta \leq \frac{\rho}{10}$, we deduce that $\frac{\rho}{10} \leq 1 - \sqrt{\left(1 - \frac{\rho}{10}\right)^2 - \frac{\rho^2}{4}}$ and since for all $0 \leq t \leq \frac{1}{3}$ we have $1 - \sqrt{(1 - t)^2 - t^2} \leq \frac{1}{4}$, we obtain the result.

We are now ready to prove Lemma 18.

**Proof of Lemma 18.** Let $\alpha$ be a non-degenerate $\varepsilon$-small $\varepsilon$-simplex of Delloc$_d(P, \rho)$. Because $\alpha \in \text{Delloc}_d(P, \rho)$, we have that $\alpha \in \text{Del}(\pi_{\text{aff}}(P \cap B(c_{\sigma}, \rho)))$, and because $\alpha$ is $\varepsilon$-small, we have that $B(Z(\alpha), R(\alpha)) \subseteq B(c_{\sigma}, R(\alpha))$, and consequently $\alpha \in \text{Del}(\pi_{\text{aff}}(P \cap B(c_{\sigma}, 3\rho)))$.

Let $\beta$ be a non-degenerate $\rho$-small $\rho$-simplex of $P$. Assume that $\pi_M(\alpha) \cap \pi_M(\beta) \neq \emptyset$ and let us show that $\beta \subseteq P \cap B(c_{\sigma}, 3\rho)$. Suppose that $x \in \text{conv} \alpha$ and $y \in \text{conv} \beta$ share the same projection $m$ onto $\mathcal{M}$, that is, $m = \pi_M(x) = \pi_M(y)$. Since both $\alpha$ and $\beta$ are $\rho$-small, Lemma 22 implies that both convex $\alpha$ and convex $\beta$ are contained in the $(\frac{\rho}{4})$-tube neighborhood of $\mathcal{M}$ and in particular $\| x - y \| \leq \| x - m \| + \| m - y \| \leq \frac{\rho}{4} + \frac{\rho}{4} \leq \frac{\rho}{2}$. For all vertices $b \in \beta$, we thus have

$$\| c_{\alpha} - b \| \leq \| c\alpha_2 - x \| + \| x - y \| + \| y - b \| \leq \varepsilon + \frac{\rho}{2} + 2\rho \leq 3\rho,$$

showing that $\beta \subseteq P \cap B(c_{\sigma}, 3\rho)$. Hence, we get that $\alpha \in \text{Del}(\pi_{\text{aff}}(\alpha \cup \beta))$ and can easily see that $p = \text{protection}(P, 3\rho) \leq \tilde{p} = \text{protection}(\alpha, \beta)$. Let $\Theta = \text{angularDeviation}(\mathcal{M}, \alpha) \leq \Theta$. To apply Lemma 21 we need to verify that

$$5\rho \sin(\Theta) \left( 2R(\alpha) + \frac{\rho}{2} \sin(\Theta) \right) < \tilde{p}^2 + 2\tilde{p}R(\alpha).$$

17
Since \( \frac{1}{2} \leq R(\alpha) \leq \varepsilon \) and \( \sin t \leq t \) for all \( t \geq 0 \), this follows from:

\[
10\rho \Theta (\varepsilon + \rho \Theta) < p^2 + ps,
\]

which is a consequence of our hypotheses.

\[\square\]

8 Proving the main result

Suppose that \( K \) is a simplicial complex with vertex set \( P \). Write \( D = \text{Deltocd}(P, \rho) \), \( D = |D| \) and \( K = |K| \) for short. In this section, we prove our main theorem by applying Lemma [17]. This requires us to define two maps \( \varphi : K \rightarrow D \) and \( f : D \rightarrow \mathbb{R} \), two weights \( W(\alpha) \) and \( W_{\text{min}}(\alpha) \) for each \( d \)-simplex \( \alpha \in K \), and to check that these maps and weights satisfy the requirements of Lemma [17]. For each \( \alpha \in K \), let \( W(\alpha) = \omega(\alpha) \) be the Delaunay weight of \( \alpha \). To be able to define \( \varphi, f, \) and \( W_{\text{min}} \), we assume that the following conditions are met:

1. \( D \) is a faithful reconstruction of \( M \);
2. For every \( d \)-simplex \( \sigma \subseteq K \), the map \( \pi_M|_{\text{conv} \sigma} \) is well-defined and injective.

These conditions are easily derived from the assumptions of the main theorem. We are now ready to introduce additional notation. Consider a subset \( X \subseteq \mathbb{R}^N \) and suppose that the map \( \pi_M|_X \) is well-defined and injective. Then it is possible to define a bijective map \( \pi_X : X \rightarrow \pi_M(X) \). Because \( D \) is a faithful reconstruction of \( M \), the map \( \pi_{D \rightarrow M} \) is well-defined and bijective. Similarly, for every \( d \)-simplex \( \alpha \in K \), the map \( \pi_{\text{conv} \sigma \rightarrow M} \) is well-defined and bijective. We now introduce the map \( \varphi : K \rightarrow D \) defined by \( \varphi = [\pi_{D \rightarrow M}]^{-1} \circ \pi_M \) and let \( f : D \rightarrow \mathbb{R} \) be the map defined by:

\[
f(x) = \min_{\sigma} \left(-\text{Power}_\alpha([\pi_{\text{conv} \sigma \rightarrow M}]^{-1} \circ \pi_M(x))\right),
\]

where the minimum is taken over all \( d \)-simplices \( \sigma \in K \) such that \( x \in \varphi(\text{conv} \sigma) \). Note that \( f(x) \) can be defined equivalently as the minimum of \(-\text{Power}_\beta(y)\) over all \( d \)-simplices \( \beta \in K \) and all points \( y \in \text{conv} \beta \) such that \( \pi_M(x) = \pi_M(y) \). Given a \( d \)-simplex \( \sigma \in K \), we associate to \( \sigma \) the weight:

\[
W_{\text{min}}(\sigma) = \int_{x \in \varphi(\text{conv} \sigma)} f(x) \, dx.
\]

Lemma 23. Under the assumptions of Theorem [12]

- For every \( d \)-simplex \( \alpha \in D \) and every point \( x \in \text{conv} \alpha \), we have \( f(x) = -\text{Power}_\alpha(x) \).

- For every \( d \)-simplex \( \alpha \in D \), we have \( W_{\text{min}}(\alpha) = W(\alpha) \).

**Proof.** Consider a \( d \)-simplex \( \alpha \in D \), a \( d \)-simplex \( \beta \in K \), \( x \in \text{conv} \alpha \) and \( y \in \text{conv} \beta \) such that \( \pi_M(x) = \pi_M(y) \). Applying Lemma [18], we obtain that \( \text{Power}_\beta(y) \leq \text{Power}_\alpha(x) \) or equivalently \( \text{Power}_\beta([\pi_{\text{conv} \beta \rightarrow M}]^{-1} \circ \pi_M(x)) \leq \text{Power}_\alpha(x) \) and therefore \( f(x) = -\text{Power}_\alpha(x) \). To establish the second item of the lemma, notice that for all \( \alpha \in D \), the restriction of \( \varphi \) to \( \text{conv} \alpha \) is the identity function, \( \varphi|_{\text{conv} \alpha} = \text{Id} \) and therefore \( \varphi(\text{conv} \alpha) = \text{conv} \alpha \). Since we have just established that \( f(x) = -\text{Power}_\alpha(x) \), we get that

\[
W_{\text{min}}(\alpha) = \int_{x \in \varphi(\text{conv} \alpha)} f(x) \, dx = \int_{x \in \text{conv} \alpha} -\text{Power}_\alpha(x) \, dx = \omega(\alpha) = W(\alpha),
\]

which concludes the proof. \[\square\]
Lemma 24. Under the assumptions of Theorem 13 for every $d$-simplex $\beta \in K \setminus D$, we have $W_{\min}(\beta) < W(\beta)$.

Proof. We need some notation. Given $\alpha$ and $\beta$ in $K$, we write $\text{conv} \{ \alpha \}$ for the set of points $y \in \text{conv} \beta$ for which there exists a point $x \in \text{conv} \alpha$ such that $\pi_M(x) = \pi_M(y)$. We define the map $\varphi_{\beta \to \alpha} : \text{conv} \{ \alpha \} \to \text{conv} \beta$ as $\varphi_{\beta \to \alpha}(y) = [\pi_{\text{conv} \alpha \to M}]^{-1} \circ \pi_{\text{conv} \beta \to M}(y)$. Note that $\varphi_{\beta \to \alpha}$ is invertible and its inverse is $\varphi_{\alpha \to \beta}$. Also, note that $J$ in Theorem 12 has been chosen precisely so that one can apply Lemma 40 in Appendix D and guarantee that $|\det(D\varphi_{\beta \to \alpha}(y))| \in \left[ \frac{1}{1 + J}, 1 + J \right]$ for all $\alpha, \beta \in K$ and all $y \in \text{conv} \{ \alpha \}$. Consider a $d$-simplex $\beta \in K \setminus D$. By Lemma 23, $f(x) = -\text{Power}_\alpha(x)$ and therefore:

$$W_{\min}(\beta) = \sum_{\alpha \in D|\beta} \int_{x \in \text{conv} \{ \alpha \}} -\text{Power}_\alpha(x) \, dx.$$  

For any convex combination $y$ of points in $\beta$, let $\{ \mu^\beta_b(y) \}_{b \in \beta}$ designate the family of non-negative real numbers summing up to 1 such that $y = \sum_{b \in \beta} \mu^\beta_b(y)b$. Plugging in the upper bound on $-\text{Power}_\alpha(x)$ provided by Lemma 18 letting

$$c = \frac{1}{2} \left( p^2 + ps \right),$$

and making the change of variable $x = \varphi_{\beta \to \alpha}(y)$, we upper bound $W_{\min}(\beta)$ as follows:

$$W_{\min}(\beta) \leq \sum_{\alpha \in D|\beta} \int_{x \in \text{conv} \{ \alpha \}} \left[ -\text{Power}_\beta(\varphi_{\alpha \to \beta}(x)) - c \sum_{b \in \beta \setminus \alpha} \mu^\beta_b(\varphi_{\alpha \to \beta}(x)) \right] \, dx$$

$$= \sum_{\alpha \in D|\beta} \int_{y \in \text{conv} \{ \alpha \}} \left[ -\text{Power}_\beta(y) - c \sum_{b \in \beta \setminus \alpha} \mu^\beta_b(y) \right] |\det(D\varphi_{\alpha \to \beta}(y))| \, dy$$

$$\leq (1 + J)W(\beta) - (1 + J)^{-1}c \sum_{\alpha \in D|\beta} \int_{y \in \text{conv} \{ \alpha \}} \sum_{b \in \beta \setminus \alpha} \mu^\beta_b(y) \, dy.$$

A key observation is that, because $\beta \neq \alpha$, then $\beta \setminus \alpha \neq \emptyset$. Therefore the sum $\sum_{b \in \beta \setminus \alpha} \mu^\beta_b(y)$ does not vanish and is always lower bounded by $\inf_{b \in \beta} \mu^\beta_b(y)$. Associating the quantity

$$\Omega(\beta) = \int_{y \in \text{conv} \beta} \inf_{b \in \beta} \mu^\beta_b(y) \, dy,$$

to $\beta$ we thus obtain that $W_{\min}(\beta) \leq (1 + J)W(\beta) - (1 + J)^{-1}c \Omega(\beta)$. Hence, $W_{\min}(\beta) < W(\beta)$ as long as

$$JW(\beta) < (1 + J)^{-1}c \Omega(\beta).$$

(15)

Using a change of variable, it is not too difficult to show that $\Omega(\beta) = d! \text{vol}(\beta)\Omega(\Delta_d)$, where $\Delta_d = \{ \lambda \in \mathbb{R}^d \mid \sum_{i=1}^d \lambda_i \leq 1; \lambda_i \geq 0, i = 1, 2, \ldots, d \}$ represents the standard $d$-simplex. Remark that $\Omega(\Delta_d)$ is a constant that depends only upon the dimension $d$ and is thus universal. Plugging in $\Omega(\beta) = d! \text{vol}(\beta)\Omega(\Delta_d)$ on the right side of (15), and the expression of $W(\beta) = \omega(\beta)$ given by Lemma 8 on the left side of (15), and recalling that $\beta$ is $\rho$-small, we find that condition (15) is implied by the following condition:

$$J \rho^2 < (1 + J)^{-1}\frac{(d + 2)(d - 1)!}{4} \left( \frac{p^2 + ps}{\Omega(\Delta_d)} \right),$$

which we have assumed to hold.

\[ \square \]
Proof of Theorem 12. We start with pointing out that Problem \(\star\) is invariant under change of orientation of \(d\)-simplices in \(K\) and thus we may assume that every \(d\)-simplex \(\alpha\) in \(K\) has an orientation that is consistent with that of \(\mathcal{M}\), that is, \(\text{sign}_{\mathcal{M}}(\alpha) = 1\) for all \(\alpha \in K^{[d]}\). Let \(D = \text{Delloc}_d(P, \rho)\), \(D = |D|\) and \(K = |K|\). Theorem 15 ensures that \(D\) is a \(d\)-manifold and \(\pi_{\mathcal{M}} : D \rightarrow \mathcal{M}\) is a homeomorphism. Define \(\varphi : K \rightarrow D\), \(f : D \rightarrow \mathbb{R}, W\), and \(W_{\text{min}}\) as explained at the beginning of the section. Consider the \(d\)-chain \(\gamma_{\text{min}}\) on \(K\):

\[
\gamma_{\text{min}}(\alpha) = \begin{cases} 
1 & \text{if } W_{\text{min}}(\alpha) = W(\alpha), \\
0 & \text{otherwise}.
\end{cases}
\]

By Lemma 23 and Lemma 24 the following property holds: for all \(\alpha \in K\), \(W_{\text{min}}(\alpha) = W(\alpha)\) if and only if \(\alpha\) is a \(d\)-simplex of \(D\). It follows that \(\gamma_{\text{min}} = \text{code}_D\). Furthermore, we have \(\sum_{\alpha \in K^{[d]}} \gamma_{\text{min}}(\alpha) \mathbf{1}_{\varphi(\text{conv } \alpha)}(x) = \sum_{\alpha \in D^{[d]}} \mathbf{1}_{\text{conv } \alpha}(x) = 1\) for almost all \(x \in D\). Recalling that \(W = \omega\) and therefore \(\|\gamma\|_{1, W} = E_{\text{del}}(\gamma)\), and applying Lemma 17 we deduce that \(\gamma_{\text{min}} = \text{code}_D\) is the unique solution to the following optimization problem over the set of chains in \(C_d(K, \mathbb{R})\):

\[
\begin{align*}
\text{minimize} & \quad E_{\text{del}}(\gamma) \\
\text{subject to} & \quad \sum_{\alpha \in K^{[d]}} \gamma(\alpha) \text{sign}_{\mathcal{M}}(\alpha) \mathbf{1}_{\varphi(\text{conv } \alpha)}(x) = 1, \text{ for almost all } x \in D \quad (\star\star)
\end{align*}
\]

We now claim that the feasible set of Problem (\(\star\star\)) contains the feasible set of Problem (\(\star\)). Indeed, consider a \(d\)-chain \(\gamma\) that satisfies the constraints of Problem (\(\star\)), that is, such that

\[
\begin{align*}
\partial \gamma &= 0, \\
\sum_{\alpha \in K^{[d]}} \gamma(\alpha) \text{sign}_{\mathcal{M}}(\alpha) \mathbf{1}_{\pi_{\mathcal{M}}(\text{conv } \alpha)}(m_0) &= 1.
\end{align*}
\]

Then, by Lemma [5] in Appendix F we obtain that \(\gamma\) also satisfies the following constraint:

\[
\sum_{\alpha \in K^{[d]}} \gamma(\alpha) \text{sign}_{\mathcal{M}}(\alpha) \mathbf{1}_{\pi_{\mathcal{M}}(\text{conv } \alpha)}(m) = 1, \text{ for almost all } m \in \mathcal{M},
\]

which is equivalent to the constraint of Problem (\(\star\star\)). Since the unique solution to Problem (\(\star\star\)) is \(\text{code}_D\). Theorem 15 guarantees that \(D = \text{Delloc}_d(P, \rho)\) is a faithful reconstruction of \(\mathcal{M}\). By Lemma 8, \(\text{code}_D\) is thus a cycle. Hence, the unique solution \(\text{code}_D\) to Problem (\(\star\star\)) also satisfies the constraints of Problem (\(\star\)) and, because the feasible set of Problem (\(\star\star\)) contains the feasible set of Problem (\(\star\)), \(\text{code}_D\) is also the unique solution to Problem (\(\star\)).

\(\blacksquare\)

9 Practical aspects

In this section, we discuss practical aspects.

9.1 Transforming the problem into a realistic algorithm

Besides the complex \(K\) that one can build from \(P\), Problem (\(\star\)) seems to require the knowledge of \(\mathcal{M}\) for expressing the normalization constraint \(\text{load}_{m_0, \mathcal{M}, K}(\gamma) = 1\). What we call a realistic algorithm is an algorithm that takes only the point set \(P\) as input. In this section, we explain how to transform Problem (\(\star\)) into an equivalent problem that does not refer to \(\mathcal{M}\) anymore, thus
providing a realistic algorithm. Roughly, we simply replace the constraint \( \text{load}_{m_0, M, K}(\gamma) = 1 \) by a constraint of the form \( \text{load}_{p_0, \Pi, \Sigma}(\gamma) = 1 \), where \( p_0 \in P \), \( \Pi \) is a \( d \)-flat that "roughly approximates" \( M \) near \( p_0 \) and \( \Sigma \) are simplices of \( K \) "close" to \( p_0 \). Lemma 25 (see below) makes this idea precise. Given a point \( x \in \mathbb{R}^N \) and \( r \geq 0 \), let us introduce the subset of \( K \):

\[
K[x, r] = \{ \sigma \in K \mid \text{conv} \sigma \cap B(x, r) \neq \emptyset \}.
\]

Note that \( K[x, r] \) is not necessarily a simplicial complex.

**Lemma 25.** Suppose \( 0 \leq \rho \leq \frac{2 \sigma}{25} \). Consider a point \( x \in M \cap \rho \) and a \( d \)-dimensional affine space \( \Pi \) passing through \( x \). Suppose that \( \angle(\Pi, T_{\pi_M(x)}M) \leq \frac{\pi}{8} \) and that the orientation of \( \Pi \) is consistent with that of \( T_{\pi_M(x)}M \). Then, Problem (R) is equivalent to the problem obtained by replacing the constraint \( \text{load}_{m_0, M, K}(\gamma) = 1 \) with the constraint \( \text{load}_{x, \Pi, K[x, 4\rho]}(\gamma) = 1 \).

**Proof.** This is a direct consequence of Lemma 52 in Appendix F.

Observe that the conditions on the \( d \)-flat \( \Pi \) in the above lemma are rather mild. Indeed, we only require \( \Pi \) to pass through a point \( x \) such that \( d(x, M) \leq \frac{2 \sigma}{25} \) and \( \angle(\Pi, T_{\pi_M(x)}M) \leq \frac{\pi}{8} \). Hence, \( \Pi \) only needs to be what we could call a rough approximation of \( M \) near \( x \). In practice, we may take for \( x \) any point \( p_0 \in P \) and for \( \Pi \) the \( d \)-dimensional affine space \( T_{p_0}(P, \rho) \) passing through \( p_0 \) and parallel to the \( d \)-dimensional vector space \( V_{p_0}(P, \rho) \) defined as follows: \( V_{p_0}(P, \rho) \) is spanned by the eigenvectors associated to the \( d \) largest eigenvalues of the inertia tensor of \( P \cap B(p_0, \rho) \) − \( c \), where \( c \) is the center of mass of \( P \cap B(p_0, \rho) \). By Lemma 54 in Appendix G, for \( \frac{\sigma}{R} \) small enough and \( \varepsilon < \frac{\rho}{10} \), we have

\[
\angle(T_{p_0}(P, \rho), T_{\pi_M(p_0)}M) \leq \frac{\pi}{8}.
\]

See Section G for more details. Hence, the assumptions of the above lemma hold for \( x = p_0 \) and \( \Pi = T_{p_0}(P, \rho) \). This shows that the normalization constraint in Problem (R) can be replaced by a constraint whose definition depends only upon the point set \( P \), thus providing a realistic algorithm.

### 9.2 Perturbing the data set for ensuring the safety conditions

In this section, we assume that \( P_0 \) is a \( \delta_0 \)-accurate \( \epsilon_0 \)-dense sample of \( M \) and perturbe it to obtain a point set \( P \) that satisfies the assumptions of our main theorem. For this, we use the Moser Tardos Algorithm [34] as a perturbation scheme in the spirit of what is done in [7], Section 5.3.4.

The perturbation scheme is parametrized with real numbers \( \rho \geq 0 \), \( \rho_{\text{pert}} \geq 0 \), \( \text{Height}_{\text{min}} > 0 \), and \( \text{Prot}_{\text{min}} > 0 \). To describe it, we need some notations and terminology. Let \( T_{p_0} = T_{p_0}(P_0, 3\rho) \) be the \( d \)-dimensional affine space passing through \( p_0 \) and parallel to the \( d \)-dimensional vector space \( V_{p_0}(P_0, 3\rho) \). To each point \( p_0 \in P_0 \), we associate a perturbed point \( p \in P \), computed by applying a sequence of elementary operations called reset. Precisely, given a point \( p \in P \) associated to the point \( p_0 \in P_0 \), the reset of \( p \) is the operation that consists in drawing a point \( q \) uniformly at random in \( T_{p_0} \cap B(p_0, \rho_{\text{pert}}) \) and assigning \( q \) to \( p \). Finally, we call any of the two situations below a bad event:

- **Violation of the height condition:** There exists a \( \rho \)-small \( d \)-simplex \( \sigma \subseteq P \) such that \( \text{height}(\sigma) < \text{Height}_{\text{min}} \).
- **Violation of the protection condition:** There exists a pair \((p, \sigma)\) made of a point \( p \in P \) and a \( d \)-simplex \( \sigma \subseteq P \setminus \{p\} \) such that \( p \in B(c_{\sigma}, 3\rho) \) and \( \text{protection}(\sigma, \{p\}) \leq \text{Prot}_{\text{min}} \).
In both situations, we associate to the bad event $E$ a set of points called the points \textit{correlated} to $E$. In the first situation, the points correlated to $E$ are the $d+1$ vertices of $\sigma$ and in the second situation, they are the $d+2$ points of $\{p\} \cup \sigma$.

### Moser-Tardos Algorithm:

1. For each $p_0 \in P_0$, compute the $d$-dimensional affine space $T^*_p$.
2. For each point $p \in P$, reset $p$.
3. \textbf{WHILE} (some bad event $E$ occurs):
   - For each point $p$ correlated to $E$, reset $p$.
   - \textbf{END WHILE}
4. Return $P$.

Roughly speaking, in our context, the Moser Tardos Algorithm reassigns new coordinates to any point $p \in P$ that is correlated to a bad event as long as a bad event occurs. A beautiful result from [34] tells us that if bad events are mostly independent from one another and have each a sufficiently small probability to occur, then the Moser-Tardos Algorithm terminates and does so in a number of steps that is expected to be linear in the size of $P_0$. Precisely, suppose that each bad event is independent of all but at most $\Gamma$ of the other bad events and the probability of a bad event is at most $\bar{\omega}$. Then, the result in [34] tells us that the Moser-Tardos algorithm terminates with expected time $O(2P_0)$ whenever

$$\bar{\omega} \leq \frac{1}{e(\Gamma+1)}, \quad (16)$$

where $e$ is the base of natural logarithms. Using this result, one can establish the following lemma, the proof of which is beyond the scope of this paper.

**Lemma 26.** Let $\varepsilon_0 \geq 0$, $\eta_0 > 0$, and $\rho = C_{\text{ste}}\varepsilon_0$, where $C_{\text{ste}} \geq 32$. Let $\delta_0 = \frac{\varepsilon^2}{\kappa}$, $r_{\text{pert.}} = \frac{\eta_0\varepsilon_0}{20\rho}$, $\varepsilon = \frac{21}{20}\varepsilon_0$, and $\delta = 2\delta_0$. There are positive constants $c_0$, $c_1$, and $c_2$ that depend only upon $\eta_0$, $C_{\text{ste}}$, and $d$ such that if $\rho < c_0$ then, given a point set $P_0$ such that $\mathcal{M} \subseteq (P_0)^{\varepsilon_0}$, $P_0 \subseteq \mathcal{M}^{\varepsilon_0}$, and separation($P_0$) $> \eta_0\varepsilon_0$, the point set $P$ obtained after resetting each of its points satisfies $\mathcal{M} \subseteq P^{\varepsilon_0}$, $P \subseteq \mathcal{M}^{\varepsilon_0}$, and separation($P$) $> \frac{9}{10}\eta_0\varepsilon_0$. Moreover, whenever we apply the Moser-Tardos Algorithm with Height$_{\text{min}} = c_1(\rho^{\frac{1}{3}})^{\frac{1}{2}}\rho$ and Prot$_{\text{min}} = c_2(\rho^{\frac{1}{3}})^{\frac{1}{2}}\rho$, the algorithm terminates with expected time $O(2P_0)$ and returns a point set $P$ that is a $\delta$-accurate $\varepsilon$-dense sample of $\mathcal{M}$ and that satisfies the assumptions of Theorem 12.

We only sketch the proof of Lemma 26 below.

**Sketch of proof.** The proof consists in applying the Moser Tardos theorem [34]. In other words, we show that Condition [16] holds, for a well-chosen upper bound $\bar{\omega}$ on the probability of each bad event and a well-chosen upper bound $\Gamma$ on the number of bad events to which each bad event is dependent upon. Upper bounds $\bar{\omega}$ and $\Gamma$ are obtained by adapting the proof of a similar simpler result presented in the appendix of [4]. The intuition is that thanks to Lemma 54 in Appendix 3, one can compute from the sample $P_0$ a local approximation $T_{P_0}(P_0, 3\rho)$ of a local tangent space with accuracy $O(\frac{a}{\kappa})$. It follows that, if $\frac{a}{\kappa}$ is small enough, the volume, in $\Pi_{\rho \in P_0}T_{P_0}(P_0, 3\rho)$, for which a height or protection condition is violated, can be made arbitrary small, and Condition [16] required for Moser-Tardos algorithm to terminate will be met.

When the Moser-Tardos algorithm terminates, we thus have two positive constants $c_1$ and $c_2$ such that

$$\text{height}(P, \rho) > c_1(\frac{\rho}{\kappa})^{\frac{1}{3}}\rho, \quad (17)$$

$$\text{protection}(P, 3\rho) > c_2(\frac{\rho}{\kappa})^{\frac{1}{3}}\rho, \quad (18)$$
For short, write $p = \text{protection}(P, 3\rho)$, $s = \text{separation}(P)$, and $\Theta = \text{angularDeviation}_{\mathcal{M}}(P, \rho)$. Let us check that the safety assumptions of Theorem 12 are then satisfied. For this, we need to show that one can find $\theta \in [0, \frac{\pi}{6}]$ such that:

\begin{align*}
\Theta &\leq \frac{\theta}{2} - \arcsin\left(\frac{\rho + \delta}{R}\right), \\
 s &> 8(\delta \theta + \rho \theta^2) + 6\delta + \frac{2\rho^2}{R}, \\
p &> 8(\delta \theta + \rho \theta^2)\left(1 + \frac{4d\varepsilon}{\text{height}(P, \rho)}\right), \\
p^2 + ps &> \max\left\{10\rho \Theta (\varepsilon + \rho \Theta), \frac{4J(1 + J)}{(d + 2)(d - 1)! \Omega(\Delta_d)} \rho^2\right\},
\end{align*}

where

$$J = \frac{(R + \rho)^d}{(R - \rho)^d (\cos \Theta)^{\min\{d,N-d\}}} - 1.$$  

By Lemma 38 in Appendix B we obtain that

$$\Theta \leq \arcsin\left(\frac{2d}{\text{height}(P, \rho)} \left(3\rho^2 \frac{\rho}{R} + \delta\right)\right).$$

Hence, since there exists a positive constant $c_1$ such that $\text{height}(P, \rho) > c_1 \left(\frac{\rho}{R}\right)^{\frac{1}{3}}$, we deduce that there exists a positive constant $c_3$ such that for $\frac{\rho}{R}$ small enough we have

$$\Theta \leq c_3 \left(\frac{\rho}{R}\right)^{\frac{2}{3}}.$$

Let $\theta = 3\Theta$ and observe that for $\frac{\rho}{R}$ small enough, $\theta \in [0, \frac{\pi}{6}]$. With this choice of $\theta$ and using $s > \frac{m}{\text{cost}} \rho$, $p > c_2 \left(\frac{\rho}{R}\right)^{\frac{1}{3}} \rho$, $\varepsilon = \frac{21}{20\text{cost}} \rho$, $\delta = \frac{2\rho^2}{R}$, and $\text{height}(P, \rho) > c_1 \left(\frac{\rho}{R}\right)^{\frac{1}{3}}$, it is easy to check that for $\frac{\rho}{R}$ small enough, Inequalities (19), (20), (21), and (22) hold and therefore the safety assumptions of Theorem 12 are met. \qed

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A Angle between affine spaces

In this appendix, we start by recalling how the angle between two vector spaces and two affine spaces are defined, following for this [30]. The appendix presents classical results (see [30] and also the Wikipedia page untitled "Angles between flats"), except for Lemma 30 which provides an explicit expression of the path of the orthonormal frame.

Definition 27. The angle between two vector subspaces \( V_1 \) and \( V_2 \) of same Euclidean space is defined as:

\[
\angle V_1, V_2 = \sup_{v_1 \in V_1} \inf_{v_2 \in V_2} \left( \frac{\angle v_1, v_2}{\|v_1\| = 1, \|v_2\| = 1} \right) = \max_{v_1 \in V_1} \min_{v_2 \in V_2} \angle v_1, v_2 \tag{23}
\]

The angle between affine subspaces \( A_1 \) and \( A_2 \) is defined as the angle between their associated vector spaces.

One gets trivially the equivalent definition:

\[
\angle V_1, V_2 = \inf \left\{ \theta \geq 0, \forall v_1 \in V_1 \setminus \{0\}, \exists v_2 \in V_2 \setminus \{0\}, \angle v_1, v_2 \leq \theta \right\}. \tag{24}
\]

Since when \( \dim V_1 = \dim V_2 \) there is an isometry (mirror symmetry) that swaps \( V_1 \) and \( V_2 \) and preserves angles, we get the following implication:

\[
\dim V_1 = \dim V_2 \implies \angle V_1, V_2 = \angle V_2, V_1,
\]

and one gets from (24) and the triangular inequality on angles between vectors that:

\[
\angle V_1, V_3 \leq \angle V_1, V_2 + \angle V_2, V_3.
\]

Lemma 28 (Minimal angle corresponds to orthogonal projection). Let \( V \subseteq \mathbb{R}^N \) be a vector subspace and \( \pi_V \) the orthogonal projection on \( V \). Let \( v' \in \mathbb{R}^N \) a vector such that \( \|v'\| = 1 \), \( \pi_V(v') \neq 0 \) and \( \theta = \min_{v^* \in V, \|v^*\| = 1} \angle v^*, v' \). Then

\[
\arg \min_{v^* \in V, \|v^*\| = 1} \angle v^*, v' = \frac{1}{\cos \theta} \pi_V(v').
\]

Proof. One has by definition of \( \pi_V \):

\[
\pi_V(v') = \arg \min_{v^* \in V} (v^* - v')^2 \tag{25}
\]

and since \( \|v'\| = 1 \) and \( \pi_V(v') - v' \) is orthogonal to \( V \), one has \((\pi_V(v') - v')^2 = (\sin \angle \pi_V(v'), v')^2\). Also, for any vector \( v^* \neq 0 \), one has \((\sin \angle v^*, v')^2 = \min_\lambda (\lambda v^* - v')^2 \) so that (25) implies that \( \pi_V(v') \) (as well as all its positively collinear vectors) minimises \( v^* \rightarrow \sin^2 \angle v^*, v' \) in \( V \). It follows that \( \pi_V(v') \) is collinear to \( \arg \min_{v^* \in V, \|v^*\| = 1} \angle v^*, v' \) and since its norm is \( \cos \theta \) we get the result.

If \( V \) is a vector subspace of \( \mathbb{R}^N \), and \( \pi_V \) the orthogonal projection on \( V \) then it is well known that:

- \( \pi_V \) is self-adjoint and therefore its matrix in any orthonormal frame is symmetric.
- \( \pi_V \circ \pi_V = \pi_V \)
the kernel of $\pi_V$ is the vector space normal to $V$ and its restriction to $V$ is the identity.

Let $V$ and $V'$ be two $d$-dimensional vector subspaces of $\mathbb{R}^N$ such that $\theta = \angle V', V < \pi/2$ and let $\pi_V$ and $\pi_{V'}$ be their corresponding orthogonal projections. Thanks to Lemma 28 and since the cosinus function is decreasing on $[0, \pi/2]$, one has:

$$\cos \theta = \min_{v' \in V', ||v'||=1} ||\pi_V(v')||$$

So that, since $v' \in V' \Rightarrow v' = \pi_{V'}(v')$:

$$(\cos \theta)^2 = \min_{v' \in V', ||v'||=1} \langle \pi_V \circ \pi_{V'}(v') | \pi_V \circ \pi_{V'}(v') \rangle$$

Denoting $M_V$ and $M_{V'}$ the respective symmetric matrix of $\pi_V$ and $\pi_{V'}$ in some orthonormal basis and since $M_V M_{V'} = M_{V'}$, we obtain:

$$(\cos \theta)^2 = \min_{v' \in V', ||v'||=1} \langle M_{V'} M_{V'} v' | M_V M_{V'} v' \rangle = \min_{v' \in V', ||v'||=1} v'^t M_{V'} M_V M_{V'} v' = \min_{v' \in V', ||v'||=1} v'^t M_{V'}^t M_{V'} M_{V'} v' = \min_{v' \in V', ||v'||=1} v'^t M_{V'}^t M_{V'} v'$$

Since $M_{V'} = M_{V'}$ and $v' \in V'$, we obtain that $M_V v' = v'$ and therefore

$$(\cos \theta)^2 = \min_{v' \in V', ||v'||=1} v'^t M_{V'} M_{V'} v'$$.  

Let $A_{V'} : V \rightarrow V'$ be the restriction of $M_{V'}$ to $V$ and $A_V : V' \rightarrow V$ the restriction of $M_V$ to $V'$. One has:

$$(\cos \theta)^2 = \min_{v' \in V', ||v'||=1} v'^t A_{V'} A_V v'$$

Since $M_{V'} M_V M_{V'} : \mathbb{R}^N \rightarrow V' \subseteq \mathbb{R}^N$ is self-adjoint, so is its restriction $C' = A_{V'} A_V : V' \rightarrow V'$. It follows that $C'$ has $d$ (counting multiplicities) real eigenvalues, associated to $d$ eigenvectors of $C'$ making an orthogonal basis of $V'$, and $(26)$ is the Rayleigh quotient of $C'$ which gives that the smallest eigenvalue of $C'$ is $(\cos \theta)^2$. Since $\theta < \pi/2$ we have that all eigenvalues of $C'$ are positive. In particular, $C'$ is invertible.

It follows that $A_V$ and $A_{V'}$ have rank $d$ and are also invertible so that $C = A_V A_{V'} : V \rightarrow V$ is also invertible. If $v_i'$ is an eigenvector of $C'$ with eigenvalue $\lambda_i$, then $C' v_i' = A_{V'} A_V v_i' = \lambda_i v_i'$ and:

$$A_V A_{V'} v_i' = \lambda_i A_V v_i'$$

Since $A_V$ is invertible, $A_V v_i' \neq 0$ and $(27)$ says that $v_i = A_V v_i'$ is an eigenvector of $C = A_V A_{V'}$ with eigenvalue $\lambda_i$.

$$C v_i = A_V A_{V'} v_i = \lambda_i v_i$$

Also, since $A_V$ and $A_{V'}$ have their $L^2$ operator norms upper bounded by 1, so is the operator norm of $C$ and $C'$. We have shown that:

**Lemma 29.** The orthogonal projection $A_{V'} : V' \rightarrow V$ sends an orthogonal basis of $V'$ made of eigenvectors of $C' = A_{V'} A_V$ to an orthogonal basis of $V$ made of eigenvectors of $C = A_V A_{V'}$ with the same eigenvalues. These eigenvalues are included in $[(\cos \theta)^2, 1]$ with the smallest one being equal to $(\cos \theta)^2$.  

28
Lemma 30 (Rotation between two vector spaces). Let $V$ and $V'$ be $d$-dimensional vector subspaces of Euclidean space such that the angle $\theta = \angle V, V'$ satisfies:

$$0 < \angle V, V' < \frac{\pi}{2}$$

and $d' = d - \dim(V \cap V')$. Then there is an orthonormal basis $v_1, \ldots, v_{d'}, v'_1, \ldots, v'_{d'}, w_1, \ldots, w_{d-d'}$ and a sequence of angles $\theta_1 \geq \theta_2 \geq \ldots, \theta_{d'} > 0$ such that $\theta_1 = \theta$,

$$v_1, \ldots, v_{d'}, w_1, \ldots, w_{d-d'}$$

is a basis of $V$ and:

$$\cos \theta_1 v_1 + \sin \theta_1 v'_1, \ldots, \cos \theta_{d'} v_{d'} + \sin \theta_{d'} v'_{d'}, w_1, \ldots, w_{d-d'}$$

is a basis of $V'$.

Proof. We first claim that, for $v \in V$ one has:

$$Cv = v \iff A_{V'}v = v \iff v \in V \cap V'$$

(28)

Indeed, if $v \in V \cap V'$ one has trivially $A_{V'}v = v$ and $Cv = A_VA_{V'}v = A_{V'}v = v$. In the other direction, if $CV = A_{V'}v = v$, since the operator norm of $A_{V'}$ is 1, one must have $\|A_{V'}v\| = \|v\|$ and, since the operator norm of $A_{V'}$ is 1, one must have $\|v\| = \|A_{V'}A_{V'}v\| = \|A_{V'}v\|$. Therefore one has $\|A_{V'}v\| = \|v\|$. But since $\|A_{V'}v\| = \|v\| \cos \angle v, A_{V'}v$ we get $\cos \angle v, A_{V'}v = 1$ and

$$\angle v, A_{V'}v = 0.$$ 

This with $\|A_{V'}v\| = \|v\|$ gives $A_{V'}v = v \in V \cap V'$.

It follows from (28) that the eigenspace of $C$ corresponding to the eigenvalue 1 coincides with $V \cap V'$. We sort the eigenvalues of $C$ in increasing order (see Lemma 29).

$$(\cos \theta)^2 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d,$$

with $\lambda_{d+1} = \ldots = \lambda_d = 1$ for $d' = d - \dim(V \cap V')$.

For any $k$, $1 \leq k \leq d'$, we define $v_k$ as an unit eigenvector associated\(^3\) to the eigenvalue $\lambda_k$.

For any $l$, $1 \leq l \leq d - d'$ we define $w_l \in V \cap V'$ such that $w_1, \ldots, w_{d-d'}$ is an orthonormal basis of $V \cap V'$, where $w_l$ is associated to the eigenvalue $\lambda_{d+l} = 1$ by (28). Then $(v_1, \ldots, v_{d'}, w_1, \ldots, w_{d-d'})$ is an orthonormal basis of $V$ and, from (28), $(w_l)_{d \leq l \leq d'}$ is an orthonormal of $V \cap V'$.

For $1 \leq k \leq d'$ define $\theta_k$ as $\theta_k = \angle v_k, A_{V'}v_k$. We have seen that $\theta_1 = \theta$ and one has $0 < \theta_k \leq \theta < \pi/2$ and:

$$\theta_k = \angle v_k, A_{V'}v_k = \angle v_k, A_{V'}v_k = \angle C v_k, A_{V'}v_k = \angle A_V(A_{V'}v_k), A_{V'}v_k$$

It follows that:

$$\lambda_k \|v_k\| = \|C v_k\| = \cos \theta_k \|A_{V'}v_k\| = \cos^2 \theta_k \|v_k\|$$

So that $(\cos \theta_k)^2 = \lambda_k$.

We define now for $1 \leq k \leq d'$:

$$v'_k = \frac{A_{V'}v_k - (v_k, A_{V'}v_k)v_k}{\|A_{V'}v_k - (v_k, A_{V'}v_k)v_k\|}$$

where the denominator is no zero since $\theta_k > 0$. One has by construction that $\|v'_k\| = 1$, $v'_k$ is orthogonal to $v_k$ and:

$$\frac{A_{V'}v_k}{\|A_{V'}v_k\|} = \cos \theta_k v_k + \sin \theta_k v'_k$$

(29)

\(^3\)For multiple eigenvalues we choose orthogonal unit vectors as the eigenspace basis.
Since \( \left( \frac{A_{i\cdots d}v_k}{\|A_{i\cdots d}v_k\|} \right)_{1 \leq k \leq d'} \) are unit eigenvectors of \( C' \) they form an orthonormal basis of \( V' \). In order to complete the proof, it remains to prove that, for \( k' \neq k \), \( v_k \) is orthogonal to \( v_{k'}' \). One has from (29), using that \( A_V A_{i\cdots d}v_k = \cos^2 \theta_k v_k \) and \( \|A_{i\cdots d}v_k\| = \cos \theta_k \):

\[
\sin \theta_k v_k' = \frac{A_V A_{i\cdots d}v_k}{\|A_V A_{i\cdots d}v_k\|} - \cos \theta_k v_k = \frac{A_V A_{i\cdots d}v_k}{\|A_V A_{i\cdots d}v_k\|} - \cos \theta_k \frac{A_V (A_{i\cdots d}v_k)}{\cos^2 \theta_k} = \frac{A_V A_{i\cdots d}v_k}{\|A_V A_{i\cdots d}v_k\|} - A_V \left( \frac{A_V A_{i\cdots d}v_k}{\|A_V A_{i\cdots d}v_k\|} \right)
\]

Recall that \( A_V \) is the orthogonal projection on \( V \) and the last equality shows then that \( v_k' \) is orthogonal to \( V \) and therefore orthogonal to any \( v_{k'} \) for \( 1 \leq k' \leq d' \).

\[ \square \]

**B \( C^2 \)-submanifold of Euclidean space**

We recall the definition of a smooth submanifold; see for instance [https://maths-people.anu.edu.au/~andrews/DG/DG_chap3.pdf](https://maths-people.anu.edu.au/~andrews/DG/DG_chap3.pdf)

**Definition 31.** A subset \( M \) of \( \mathbb{R}^N \) is a \( d \)-dimensional \( C^2 \) submanifold if for every point \( x \) in \( M \) there exists a neighborhood \( V \) of \( x \) in \( \mathbb{R}^N \), an open set \( U \subseteq \mathbb{R}^d \) and a \( C^2 \) map \( \xi : U \to \mathbb{R}^N \) such that \( \xi \) is a homeomorphism onto \( M \cap V \), and the differential \( D_y \xi \) is injective for every \( y \in U \).

A \( d \)-dimensional \( C^2 \) submanifold is a \( C^2 \) manifold topologically embedded in \( \mathbb{R}^N \) but the converse is not true in general. For instance, the square is not a submanifold of the plane. Yet, it is a \( C^\infty \) manifold topologically embedded in the plane. Precisely, the square can be defined as the image of the circle \( S^1 \) (which is a \( C^\infty \) manifold) by a \( C^\infty \) map that is non-regular (i.e. whose derivative is non-injective) precisely at the four corners of the square.

A compact \( C^2 \) submanifold has positive reach [27]. Moreover one has (see for example Lemma 4 and following paragraph in [10] or Proposition 2.3 in [1]) that:

**Lemma 32** (The inverse of the reach bounds the curvature). If \( M \subseteq \mathbb{R}^N \) is a \( C^2 \) \( d \)-dimensional submanifold with reach \( \text{reach}(M) > 0 \) then \( 1/\text{reach}(M) \) bounds the (absolute values of the ) principal curvatures at \( m \in M \) in the direction \( v \) for any vector \( v \) in the space normal to \( M \) at \( m \). In particular \( 1/\text{reach}(M) \) bounds the principal curvatures when \( M \) has codimension 1.

The following lemma, due to Federer, bounds the distance of a point \( q \in M \) to the tangent space at a point \( p \in M \). It holds for any set with positive reach and in particular for \( C^2 \) submanifolds.

**Lemma 33** (Distance to tangent space, Theorem 4.8(7) of [27]). Let \( p, q \in M \subseteq \mathbb{R}^N \) such that \( \|p - q\| < \text{reach}(M) \). We have

\[
\sin \angle ([pq], T_p M) \leq \frac{\|p - q\|}{2 \text{reach}(M)}, \tag{30}
\]

and

\[
d(q, T_p M) \leq \frac{\|p - q\|^2}{2 \text{reach}(M)}. \tag{31}
\]

Next lemma bounds the angle variation for \( C^2 \) manifolds (a slightly weaker condition is given for \( C^{1,1} \) manifolds in the same paper):

30
Lemma 34 (Corollary 3 in [10]). For any \( p, q \in \mathcal{M} \), we have
\[
\sin \left( \frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2} \right) \leq \frac{\|p - q\|}{2 \text{ reach}(\mathcal{M})}.
\]

Using Lemmas 33 and 34 we can show that the projection on a tangent space defines a local chart for \( \mathcal{M} \). Indeed, if \( m \in \mathcal{M} \) then for any \( p, q \in \mathcal{M} \cap B(m, \text{sin}(\pi/4) \text{ reach } \mathcal{M}) \), (30) gives:
\[
\sin \angle([pq], T_p\mathcal{M}) \leq \frac{\|p - q\|}{2 \text{ reach}(\mathcal{M})} \leq \frac{2 \text{sin}(\pi/4) \text{ reach } \mathcal{M}}{2 \text{ reach}(\mathcal{M})} = \text{sin}(\pi/4)
\]
So that:
\[
\angle([pq], T_p\mathcal{M}) \leq \pi/4,
\]
and Lemma 34 gives:
\[
\sin \left( \frac{\angle(T_p\mathcal{M}, T_m\mathcal{M})}{2} \right) \leq \frac{\|p - m\|}{2 \text{ reach}(\mathcal{M})} \leq \frac{\text{sin}(\pi/4) \text{ reach } \mathcal{M}}{2 \text{ reach}(\mathcal{M})} = \frac{\text{sin}(\pi/4)}{2}
\]
so that:
\[
\angle(T_p\mathcal{M}, T_m\mathcal{M}) \leq 2 \text{arcsin} \left( \frac{\text{sin}(\pi/4)}{2} \right) < \pi/4
\]
Summing with (32) gives:
\[
\angle([pq], T_m\mathcal{M}) < \pi/2
\]
and this in turn implies:
\[
p \neq q \Rightarrow \pi_{T_m\mathcal{M}}(p) \neq \pi_{T_m\mathcal{M}}(q)
\]
We have shown that the restriction of \( \pi_{T_m\mathcal{M}} \) to \( \mathcal{M} \cap B(m, \text{sin}(\pi/4) \text{ reach } \mathcal{M})^\circ \) is injective, and, by Invariance of Domain Theorem ([13]), it is an homeomorphism on its image, which gives us:

Lemma 35 (Projection on tangent space is a local chart). If \( \mathcal{M} \) is a compact \( C^2 \) submanifold of Euclidean space, and \( m \in \mathcal{M} \) then, identifying \( T_m\mathcal{M} \) with \( \mathbb{R}^d \) through a given frame, the restriction of \( \pi_{T_m\mathcal{M}} \) to \( \mathcal{M} \cap B(m, \text{sin}(\pi/4) \text{ reach } \mathcal{M})^\circ \) is a local chart of \( \mathcal{M} \). Moreover, for any \( p \in \mathcal{M} \cap B(m, \text{sin}(\pi/4) \text{ reach } \mathcal{M})^\circ \), one has:
\[
\angle(T_p\mathcal{M}, T_m\mathcal{M}) < \pi/4
\]

Following [9, Notation 1.1], we call Thickness of a \( k \)-simplex:
\[
t(\sigma) = \frac{h}{kL}
\]
where \( h \) is the smallest altitude of \( \sigma \) and \( L \) the length of the longest edge. Then we have (adaptation of [40, Section IV.15], proven in [7, Lemma 8.11]):

Lemma 36 (Angle between simplices tangent space). If \( \mathcal{M} \) is a compact \( C^2 \) submanifold of Euclidean space, \( \sigma \) a \( d \)-simplex with vertices in \( \mathcal{M} \), and \( p \) a vertex of \( \sigma \): then:
\[
\angle \text{aff } \sigma, T_p\mathcal{M} \leq \frac{L}{t(\sigma) \text{ reach } \mathcal{M}} = \frac{dL^2}{h \text{ reach } \mathcal{M}}
\]
We shall also need the Whitney angle bound established in [8].
Applying Lemma 33, we get that

\[ \sin \angle(\text{aff } \sigma, H) \leq \frac{2t \dim(\sigma)}{\text{height}(\sigma)} \]

Building on these results, we derive yet another bound between the affine space spanned by a simplex and a nearby tangent space.

**Lemma 38.** Consider a non-degenerate \( \rho \)-small simplex \( \sigma \subseteq \mathcal{M}^{\oplus \delta} \) with \( 16\delta \leq \rho \leq \frac{R}{4} \). Let \( x \in \text{conv } \sigma \). Then,

\[ \angle(\text{aff } \sigma, T_{\pi_M(x)} \mathcal{M}) \leq \arcsin \left( \frac{2 \dim(\sigma)}{\text{height}(\sigma)} \left( \frac{3 \rho^2}{R} + \delta \right) \right). \]

**Proof.** Let \( v \in \sigma \). Write \( v^* = \pi_M(v) \) and \( x^* = \pi_M(x) \). We know from [27, page 435] that for \( 0 \leq h < \text{reach } \mathcal{M} \), the projection map \( \pi_M \) onto \( \mathcal{M} \) is \( \left( \frac{R}{R-h} \right) \)-Lipschitz for points at distance less than \( h \) from \( \mathcal{M} \). By Lemma 22, both \( x \) and \( v \) belong to \( \mathcal{M}^{\oplus \delta} \) for \( h = \frac{R}{4} \), we thus have

\[ \|v^* - x^*\| \leq \frac{R}{R - \frac{R}{4}} \times \|v - x\| \leq \frac{R}{R - \frac{R}{3\times4}} \times 2\rho = \frac{24}{11} \rho \leq \sqrt{6}\rho \]

Applying Lemma 33 we get that

\[ d(v, T_{x^*} \mathcal{M}) \leq d(v^*, T_{x^*} \mathcal{M}) + \|v - v^*\| \leq \frac{\|v^* - x^*\|^2}{2R} + \delta \leq \frac{3\rho^2}{R} + \delta. \]

Hence, \( \sigma \subseteq (T_{x^*} \mathcal{M})^{\oplus t} \) for \( t = \frac{3\rho^2}{R} + \delta \) and applying Whitney angle bound (Lemma 37), we conclude that

\[ \sin \angle(\text{aff } \sigma, T_{x^*} \mathcal{M}) \leq \frac{2 \dim(\sigma)}{\text{height}(\sigma)} \left( \frac{3 \rho^2}{R} + \delta \right). \]

**C Smooth local parametrization of the normal bundle**

Let \( \mathcal{M} \subseteq \mathbb{R}^N \) be a \( C^2 \) \( d \)-dimensional submanifold with reach greater than \( \rho > 0 \). If \( x \in \mathcal{M} \), the vector spaces \( T_x \mathcal{M} \) and \( N_x \mathcal{M} \) are the respective tangent and normal spaces to \( \mathcal{M} \) at \( x \). To avoid confusion, we denote by \( T_x \mathcal{M} \) and \( N_x \mathcal{M} \) the corresponding affine subspaces of ambient \( \mathbb{R}^N \): \( T_x \mathcal{M} = x + T_x \mathcal{M} \) and \( N_x \mathcal{M} = x + N_x \mathcal{M} \).

We denote by \( \mathcal{M}^{\oplus \rho} \) the \( \rho \)-tubular neighborhood of \( \mathcal{M} \). The normal bundle of \( \mathcal{M} \) is denoted \( N_\mathcal{M} \) and defined as:

\[ N_\mathcal{M} \overset{\text{def.}}{=} \bigoplus_{x \in \mathcal{M}} N_x \mathcal{M} \]

where \( \bigoplus \) is the disjoint union. The normal bundle restricted to radius \( \rho \) is denoted \( N_\rho \mathcal{M} \):

\[ N_\rho \mathcal{M} \overset{\text{def.}}{=} \bigoplus_{x \in \mathcal{M}} \{(x, v) \in N_x \mathcal{M}, \|v\| \leq \rho\} \]

We know from [27, Item (13) of Theorem 4.8] that, since \( \rho < \text{reach } \mathcal{M} \), the map \( \Psi : \mathcal{M}^{\oplus \rho} \to N_\rho \mathcal{M} \subseteq \mathbb{R}^N \times \mathbb{R}^N \) defined by:

\[ \Psi(y) = (\pi_M(y), y - \pi_M(y)) \]

is a Lipschitz homeomorphism between \( \mathcal{M}^{\oplus \rho} \) and \( N_\rho \mathcal{M} \) whose inverse \( \sigma : N_\rho \mathcal{M} \to \mathcal{M}^{\oplus \rho} \) defined as

\[ \sigma(x, v) = x + v \] (33)
is Lipschitz as well. We have \cite{[28]} that inside \( \mathcal{M}^{\bowtie \rho} \) the distance function \( d_{\mathcal{M}} \) is \( C^2 \) and the projection \( \pi_\mathcal{M} \) is \( C^1 \). It follows that \( \Psi \) is \( C^1 \) and since its inverse is Lipschitz, its Jacobian cannot be singular. It follows that \( \Psi \) is a \( C^1 \) diffeomorphims between \( \mathcal{M}^{\bowtie \rho} \) and \( N^\rho \mathcal{M} \).

For \( x \in \mathcal{M} \) we consider an open neighborhood \( U_0 \) of 0 in \( \mathbb{R}^d \) and a \( C^2 \) injective map \( \xi : U_0 \to \mathcal{M} \), regular in the sense that differential \( \left( \frac{d \xi}{du} \right)(u) \) has rank \( d \) for any \( u \in U_0 \), and such that \( \xi(0) = x \). Let us choose \( \xi \) such that, moreover, \( \left( \frac{d \xi}{du} \right|_{u=0} \right)_{j=1,d} \) forms an orthonormal basis of \( T_x \mathcal{M} \).

Also, we consider, as in the proof of Lemma 6.3 of \cite{[32]}, a set of \( (N-d) \) \( C^1 \)-smooth vector valued maps \( (w_k)_{k=1\ldots N-d} \), where \( w_k : U_0 \to \mathbb{R}^N \) and such that, for any \( u \in U_0 \), \( (w_k(u))_{k=1\ldots N-d} \) is an orthonormal basis of \( N_{\xi(u)} \mathcal{M} \).

As done by J. Milnor in \cite{[32]} paragraph 6, proof of Lemma 6.3], \( \xi \) and \( (w_k)_{k=1\ldots N-d} \) defines a local trivialization of the normal bundle \( N_{\mathcal{M}} \), that is a chart \( \phi \) of \( N_{\mathcal{M}} \) in the neighborhood of \( (x,0) \) where the parameter \( (u^1,\ldots,u^d, t^1,\ldots,t^{N-d}) \in U_0 \times \mathbb{R}^{N-d}(0,\rho) \) corresponds to the point:

\[
\phi(u,t) = \phi \left( u^1,\ldots,u^d, t^1,\ldots,t^{N-d} \right) = \left( \xi(u^1,\ldots,u^d), \sum_{k=1}^{N-d} t^k w_k(u^1,\ldots,u^d) \right) \in N^\rho \mathcal{M}
\]

and

\[
\sigma \circ \phi(u,t) = \xi(u^1,\ldots,u^d) + \sum_{k=1}^{N-d} t^k w_k(u^1,\ldots,u^d) \in \mathcal{M}^{\bowtie \rho} \subset \mathbb{R}^N
\]

Derivating \cite{[34]} at \( u = 0 \) gives:

\[
\frac{d}{du} \bigg|_{u=0} \sigma \circ \phi(u,t) = \frac{d \xi}{du} \bigg|_{u=0} + \sum_{k=1}^{N-d} t^k \frac{d w_k}{du} \bigg|_{u=0} = \frac{d \xi}{du} \bigg|_{u=0} + \sum_{k=1}^{N-d} t^k \frac{d w_k}{du} \bigg|_{u=0}
\]

In order to express the derivative of \( \sigma : N^\rho \mathcal{M} \to \mathcal{M}^{\bowtie \rho} \) at point \( (x,0) \in N^\rho \mathcal{M} \) we need, besides the chart \( \phi \) of a neighborhood of \( (x,v) \), a chart \( \hat{\phi} \) of a neighborhood of \( \sigma(x,v) = x + v \) in euclidean space. A natural choice for \( \hat{\phi} \) is

\[
\hat{\phi}(y) = \hat{\phi}(y^1,\ldots,y^N) = x + v + \sum_{k=1}^{d} y^k \frac{d \xi}{du^k} \bigg|_{u=0} + \sum_{k=1}^{N-d} y^{d+k} w_k
\]

Observe that, since \( \left( \frac{d \xi}{du^k} \bigg|_{u=0} \right)_{k=1}^d \), \( \left( \frac{d \xi}{du^k} \bigg|_{u=0} \right)_{k=N+1}^{N+d} \) is an orthonormal basis, seeing \( N^\rho \mathcal{M} \) and \( \mathcal{M}^{\bowtie \rho} \) as Riemannian manifolds, the metric tensor associated to the charts \( \phi \) and \( \hat{\phi} \) respectively at \( (x,v) \) and \( \sigma(x,v) = x + v \) are the identity matrix.

One has:

\[
\hat{D}_{y=0} \hat{\phi} = \left( \frac{d \xi}{du^1} \bigg|_{u=0} \ldots, \frac{d \xi}{du^d} \bigg|_{u=0}, w_1,\ldots,w_{N-d} \right)
\]

and to columns of \cite{[37]} are unitary and pairwise orthogonal, taking the inner product of \cite{[35]} and \cite{[36]} with the columns of \( \hat{D}_{y=0} \hat{\phi} \), namely \( \left( \frac{d \xi}{du^j} \bigg|_{u=0} \right)_{j=1,d} \) and \( (w_k)_{k=1\ldots N-d} \), gives an expression of the Jacobian of \( \sigma : N^\rho \mathcal{M} \to \mathcal{M}^{\bowtie \rho} \subset \mathbb{R}^N \) at the point \( \phi(0,t) \) in the charts \( \phi \) and \( \hat{\phi} \) as:

\[
\left( \frac{d \xi}{du^j} \bigg|_{u=0}, \frac{d \xi}{du^j} \bigg|_{u=0} \right) + \sum_{k=1}^{N-d} t^k \left( \frac{dw_k}{du^k} \bigg|_{u=0}, \frac{d \xi}{du^j} \bigg|_{u=0} \right) \sum_{k=1}^{N-d} t^k \left( \frac{dw_k}{du^k} \bigg|_{u=0}, w_l \right)
\]

\cite{[38]}
Where in both $N^\rho M \subseteq \mathbb{R}^N$ and $M^{\oplus \rho}$ the tangent space is decomposed as direct sum of the tangential and orthogonal fibers.

Denoting by $I$ the first fundamental form of $M$:

$$I = \left\langle \frac{d\xi}{du^i}, \frac{d\xi}{du^j} \right\rangle.$$

Since we have chosen $\xi$ in such a way that $\left( \frac{d\xi}{du}|_{u=0} \right)_{j=1,d}$ is an orthonormal basis of $T_xM$, we have at $x = \xi(0)$ that $I = I$.

For a unit vector $v \in N_xM$ with $v = \sum_{k=1}^{N-d} t^k v_k$ we can call, following J. Milnor again, the second fundamental form in the direction $v$:

$$\Pi_v = \sum_{k=1}^{N-d} t^k \left\langle \frac{dw_k}{du^i}, \frac{d\xi}{du^j} \right\rangle.$$

We also denote the “torsion” term by $T_v$:

$$T_v = \sum_{k=1}^{N-d} t^k \left\langle \frac{dw_k}{du^i}, w_l \right\rangle.$$

In fact we claim that, without loss of generality, one can choose the maps $(w_k)_{k=1,N-d}$ in such a way that $T_v = 0$. Indeed, since, for any $u \in U_0$, $\langle w_k(u), w_l(u) \rangle = \delta_{k,l}$ where $\delta_{k,l}$, the Kronecker delta, is constant, one has:

$$0 = \frac{d}{du^i} \langle w_k, w_l \rangle = \left\langle \frac{dw_k}{du^i}, w_l \right\rangle + \left\langle w_k, \frac{dw_l}{du^i} \right\rangle$$

in other words, $\left\langle \frac{dw_k}{du^i}, w_l \right\rangle$ is antisymmetric and can then be seen as an infinitesimal rotation, i.e., formally, an element of the Lie algebra of $SO(N-d)$. It results that if we replace $w$ by $w'$ defined by :

$$w'(u^1, \ldots, u^d) = \exp \left( -\sum_{i=1}^{d} u_i \left( \frac{dw_k}{du^i} \Big|_{u=0} , w_l \right) \right) w(u^1, \ldots, u^d),$$

we get:

$$\left\langle \frac{dw'_k}{du^i} \Big|_{u=0} , w_l \right\rangle = 0.$$

Since the exp term is a rotation depending smoothly ($C^\infty$) on the $u_i$ and is applied to the basis $w$, $w$ can be replaced by $w'$ which proves the claim.

We have that for $\tau \in [0, \rho]$, at the point $(x, \tau v) \in N^\rho M$ with $\sigma(x, \tau v) = x + \tau v \in M^{\oplus \rho}$ the Jacobian (38) can be expressed as, with the assumption $T_v = 0$:

$$\begin{pmatrix} \mathbb{I} + \tau \Pi_v & T_v \\ 0 & \mathbb{I} \end{pmatrix} = \begin{pmatrix} \mathbb{I} + \tau \Pi_v & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{39}$$

Thanks to Lemma 32, $\|\Pi_v\|_2$, the operator $L^2$ norm of $\Pi_v$, which is the maximal extrinsic curvature of $M$, is upper bounded by the inverse of the reach of $M$:

$$\forall v \in N_xM, \|v\| = 1, \|\Pi_v\|_2 \leq \frac{1}{\text{reach } M} \tag{40}$$
Since in (39) one has $\tau \leq \rho < \text{reach } \mathcal{M}$, we get that $\|\tau \mathbb{I}_{v}\|_2 \leq \frac{\tau}{\text{reach } \mathcal{M}} < 1$ and $\mathbb{I} + \tau \mathbb{I}_{v}$, the upper left bloc in (39), is invertible. It follows that matrix (39) is invertible:

$$
D_{v=0} \left( \phi^{-1} \circ \Psi \circ \hat{\phi} \right) = \begin{pmatrix} \mathbb{I} + \tau \mathbb{I}_{v} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbb{I} + \tau \mathbb{I}_{v})^{-1} & 0 \\ 0 & 1 \end{pmatrix}
$$

(41)

By local inversion Theorem, (41) gives us the Jacobian of $\Psi = \sigma^{-1}$ at $x + \tau v \in \mathcal{M}^{\oplus \rho}$. In particular, the first row of (41):

$$
(\mathbb{I} + \tau \mathbb{I}_{v})^{-1} 0
$$

(42)

is the Jacobian of $\xi^{-1} \circ \pi_{\mathcal{M}} \circ \hat{\phi}$ at $x + \tau v \in \mathcal{M}^{\oplus \rho}$.

## D Restriction of $\pi_{\mathcal{M}}$ to a d-dimensional affine space in a neighborhood of $x + \tau v \in \mathcal{M}^{\oplus \rho}$

Let $\Pi \subseteq \mathbb{R}^d$ denote a $d$-dimensional affine space that contains the point $y = x + \tau v \in \mathcal{M}^{\oplus \rho}$ for $x \in \mathcal{M}$, $v \in N_x \mathcal{M}$ a unit vector and $\tau < \rho$. If $(y, e_1, \ldots, e_d)$ is an orthonormal frame of $\Pi$ centered at $y$. This frame defines a parametrization $\mathbb{R}^d \to \Pi$ defined by

$$
(z^1, \ldots, z^d) \mapsto y + \sum_{i=1}^d z^i e_i.
$$

(43)

Each vector $e_i$ can be decomposed uniquely as a sum $e_i = e^T_i + e^N_i$ where $e^T_i \in T_x \mathcal{M}$ and $e^N_i \in N_x \mathcal{M}$. From the definition (23) one has:

$$
\min_{z \in \mathbb{R}^d, \|z\|=1} \left\| \sum_{i=1}^d z^i e^T_i \right\| = \min_{z \in \mathbb{R}^d, \|z\|=1} \left\| \pi_{T_x \mathcal{M}} \left( \sum_{i=1}^d z^i e_i \right) \right\| = \cos \angle \Pi, T_x \mathcal{M}
$$

(44)

Using (42) we get:

$$
\frac{d}{dz^j} \big|_{z=0} \xi^{-1} \circ \pi_{\mathcal{M}} \left( y + \sum_{i=1}^d z^i e_i \right) = (\mathbb{I} + \tau \mathbb{I}_{v})^{-1} e^T_j
$$

(45)

Since $\tau \leq \rho$ and the eigenvalues of $\Pi_{e}$ are the principal curvatures, by Lemma 32 they are upper bounded by $\mathcal{R}^{-1}$, the inverse of the reach of $\mathcal{M}$. The eigenvalues of the symmetric matrix $(\mathbb{I} + \tau \mathbb{I}_{v})^{-1}$ are therefore included in:

$$
\begin{bmatrix}
\frac{\mathcal{R}}{\mathcal{R} + \rho} & \frac{\mathcal{R}}{\mathcal{R} - \rho}
\end{bmatrix}
$$

and therefore its determinant included by:

$$
\left[ \left( \frac{\mathcal{R}}{\mathcal{R} + \rho} \right)^d, \left( \frac{\mathcal{R}}{\mathcal{R} - \rho} \right)^d \right]
$$

From (44) we have that the determinant of

$$
(z^1, \ldots, z^d) \mapsto \pi_{T_x \mathcal{M}} \left( y + \sum_{i=1}^d z^i e_i \right) = \sum_{i=1}^d z^i e^T_i
$$

is
Then one can defines an homomorphism

\[ \left( \cos \angle \Pi, T_x \mathcal{M} \right)^d, 1 ] \]

In fact, a finer analysis exploiting Lemma 30 allows to improve this bound to:

\[ \left( \cos \angle \Pi, T_x \mathcal{M} \right)^{\min(d, N - d)} \]

since, when \( N < 2d \), \( T_x \mathcal{M} \) and the vector space associated to \( \Pi \) have a common subspace of dimension at least \( 2d - N \).

Therefore, the determinant of the differential of \( (z^1, \ldots, z^d) \mapsto \xi^{-1} \circ \pi_M \left( y + \sum_{i=1}^d z^i e_i \right) \) is bounded by:

\[
\det \left( \frac{d}{dz^2} \right)_{z=0} \xi^{-1} \circ \pi_M \left( y + \sum_{i=1}^d z^i e_i \right) \in \left[ \left( \frac{R}{R + \rho} \right)^d \left( \cos \angle \Pi, T_x \mathcal{M} \right)^{\min(d, N - d)}, \left( \frac{R}{R - \rho} \right)^d \cos \angle \Pi, T_x \mathcal{M} \right]
\]

If \( \angle \Pi, T_x \mathcal{M} < \pi/2 \), it follows from (44) that the set of vectors \( (e_i^T)_{i=1}^d \) spans \( T_x \mathcal{M} \). Since \( (1 + \tau \Pi_i)^{-1} \) has full rank \( d \) we get that \( \frac{d}{dz^2} \right)_{z=0} \xi^{-1} \circ \pi_M \left( y + \sum_{i=1}^d z^i e_i \right) \) has full rank. We have proven that:

**Lemma 39.** Let \( \Pi \subseteq \mathbb{R}^N \) denote a \( d \)-dimensional affine space that contains the point \( y \in \mathcal{M}^{\oplus p} \). If \( \angle \Pi, T_{\pi_M(y)} \mathcal{M} < \pi/2 \), then, in some open neighborhood \( U_y \) of \( y \) in \( \Pi \), \( \pi_M|_{U_y} \) is a \( C^1 \)-diffeomorphism on its image.

We consider now \( \Pi_1, \Pi_2 \subseteq \mathbb{R}^N \), two \( d \)-dimensional affine space, where, for \( i = 1, 2 \) the affine space \( \Pi_i \) contains the point \( y_i = x + \tau_i v_i \in \mathcal{M}^{\oplus p} \) for \( x \in \mathcal{M} \), \( v_i \in N_x \mathcal{M} \) a unit vector and \( \tau_i < \rho \). We assume that, for \( i = 1, 2 \), one has \( \angle \Pi_i, T_{\pi_M(y_i)} \mathcal{M} < \pi/3 \).

By Lemma 39, the projection on \( \mathcal{M} \) restricted to some neighborhood of \( y_i \) in \( \Pi_i \), is an homeomorphism. For \( i = 1, 2 \), let \( U_i \) be an open neighborhood of \( y_i \) in \( \Pi_i \) such that \( \pi_M|_{U_i} \) is an homeomorphism on its image and \( U_i \subseteq \mathcal{M}^{\oplus p} \). Assume moreover that \( \pi_M(U_1) = \pi_M(U_2) \). Then one can defines an homomorphism \( \varphi_{1 \rightarrow 2} : U_1 \rightarrow U_2 \) as:

\[
\varphi_{1 \rightarrow 2} \overset{\text{def.}}{=} \left( \pi_M|_{U_2} \right)^{-1} \circ \pi_M|_{U_1}
\]

One has, for a chart \( \xi : U_0 \rightarrow \mathcal{M} \), where \( U_0 \subseteq \mathbb{R}^d \), and is such that \( \xi(U_0) \supset \pi_M(U_1) = \pi_M(U_2) \), that:

\[
\varphi_{1 \rightarrow 2} = \left( \pi_M|_{U_2} \right)^{-1} \circ \pi_M|_{U_1} = \left( \pi_M|_{U_2} \right)^{-1} \circ \xi \circ \xi^{-1} \circ \pi_M|_{U_1}
\]

Choosing some coordinate systems for \( \Pi_1 \) and \( \Pi_2 \) defined by respective orthonormal frame \((y_1, e_{11}, \ldots, e_{1d}) \) and \((y_2, e_{21}, \ldots, e_{2d}) \), as in (43), we denote by \( D\varphi_{1 \rightarrow 2} \) the matrix of the derivative of \( \varphi_{1 \rightarrow 2} \) in these coordinate systems we have:

\[
D\varphi_{1 \rightarrow 2}(y) = \left( \frac{d}{dz^2} \right)_{z=0} \xi^{-1} \circ \pi_M \left( y + \sum_{i=1}^d z^i e_{2i} \right) \left( \frac{d}{dz^2} \right)_{z=0} \xi^{-1} \circ \pi_M \left( y_1 + \sum_{i=1}^d z^i e_{1i} \right)
\]
the point
the coordinates associated to an orthonormal frame, is bounded by:
with the manifold (Definition 44). Finally, we provide conditions under which the property for a
where, as
In other words:
To sum up, we have proven:
Lemma 40. Let \( \Pi_1, \Pi_2 \subseteq \mathbb{R}^N \), be two \( d \)-dimensional affine spaces, where, for \( i = 1, 2 \) \( \Pi_i \) contains
the point \( y_i \in \mathcal{M}^{\oplus \rho} \) with \( \pi_M(y_i) = x \in M \). For \( i = 1, 2 \), let \( U_i \subseteq \Pi_i \cap \mathcal{M}^{\oplus \rho} \) be an open subset of
\( \Pi_i \) such that \( \pi_M(U_1) = \pi_M(U_2) \) and, for \( i = 1, 2 \), \( \forall \, z \in U_i, \angle M \cap \mathcal{M}(z) \cap M < \theta < \pi/3 \).
Then, the Jacobian, of the map \( \varphi_1 \rightarrow_2 \overset{\text{def.}}{=} \left( \pi_M|_{U_2} \right)^{-1} \circ \pi_M|_{U_1} \), taking as chart for \( U_1 \) and \( U_2 \)
the coordinates associated to an orthonormal frame, is bounded by:
\[
|\det D\varphi_1 \rightarrow_2(y)| \in \left[ \frac{(R - \rho)^d}{(R + \rho)^d} \left( \cos \theta \right)^{\min(d,N-d)} \frac{R^d}{(R - \rho)^d} \left( \cos \theta \right)^{\min(d,N-d)} \right] \tag{47}
\]
Remark 41. The bound \( \text{47} \) can be expressed as
\[
|\det D\varphi_1 \rightarrow_2(y)| \in \left[ (1 + J)^{-1}, 1 + J \right] \tag{48}
\]
with
\[
J = \left( \frac{(R + \rho)^d}{(R - \rho)^d} \left( \cos \theta \right)^{\min(d,N-d)} - 1 \right) \tag{49}
\]
where, as \( \frac{\rho}{R} \to 0 \), one has:
\[
J = 0 \left( \frac{\rho}{R} \right) \tag{49}
\]
E Transfering orientation
In this section, we start by recalling what it means for a manifold to be orientable. Given a
manifold with a prescribed orientation, we then explain how to orient a simplex consistently with the manifold (Definition 44). Finally, we provide conditions under which the property for a simplex to be consistently oriented with a manifold is preserved under projection onto a nearby tangent space (Lemma 49).
Definition 42 (Manifold orientation). An orientation of a \( C^1 \)-manifold \( M \) consists of an atlas
\( \{ (U_i \subseteq M, \psi_i : U_i \to \mathbb{R}^d) \}_{i \in I} \) such that:
\[
U_i \cap U_j \neq \emptyset \implies \forall m \in U_i \cap U_j, \det D (\psi_j \circ \psi_i^{-1}) (\psi_i(m)) > 0.
\]
\( M \) is said to be orientable if such an atlas exists.
Definition 43 (Chart consistent with the manifold orientation). If \( U \) is an open subset of the
oriented manifold \( M \), a local chart \( \psi : U \to \mathbb{R}^d \) is called consistent with the orientation of \( M \)
defined by the atlas \( \{ (U_i, \psi_i) \}_{i \in I} \) if the following implication holds:
\[
U_i \cap U \neq \emptyset \implies \forall m \in U_i \cap U, \det D (\psi \circ \psi_i^{-1}) (\psi_i(m)) > 0.
\]
Let us associate to each oriented non-degenerate abstract \( d \)-simplex \( \sigma = [u_0, u_1, \ldots, u_d] \subseteq \mathbb{R}^N \) the linear map \( \psi_\sigma \) defined by

\[
\psi_\sigma : \begin{cases}
\text{aff } \sigma & \mapsto \mathbb{R}^d \\
x & \mapsto (t_1, \ldots, t^d),
\end{cases}
\]

where \( t_1, \ldots, t^d \) are the coordinates of \( x \) in the frame \((u_0, u_1 - u_0, \ldots, u_d - u_0)\), that is, real numbers such that \( x = u_0 + \sum_{i=1}^{d} t^i(u_i - u_0) \). We note that the orientation of \( \sigma \) induces a natural orientation of the affine subspace \( \text{aff } \sigma \) defined by the atlas formed of a unique chart \( \{(\text{aff } \sigma, \psi_\sigma)\} \).

Let us now define what we mean for a simplex to have an orientation consistent with that of a manifold.

**Definition 44** (Consistent orientation between a simplex and a manifold). Let \( \mathcal{M} \subseteq \mathbb{R}^N \) be an orientable manifold, whose reach is greater than \( \rho > 0 \). Let \( \sigma \subseteq \mathbb{R}^N \) be an oriented non-degenerate \( d \)-simplex such that \( \text{conv } \sigma \subseteq \mathcal{M}^{\text{aff}} \), and suppose that

\[
\max_{x \in \text{conv } \sigma} \angle(\text{aff } \sigma, \mathbf{T}_{\mathcal{M}(x)} \mathcal{M}) < \frac{\pi}{2}.
\]

We say that the orientation of \( \sigma \) is consistent with a given orientation of \( \mathcal{M} \) if there exists a point \( x \in \text{conv } \sigma \) and an open neighborhood \( U_x \) of \( x \) in \( \text{aff } \sigma \) such that the chart

\[
\psi_\sigma \circ (\mathbf{M}_{U_x})^{-1} : \mathcal{M}(U_x) \subseteq \mathcal{M} \rightarrow \psi_\sigma(U_x) \subseteq \mathbb{R}^d
\]

is consistent with the orientation of \( \mathcal{M} \).

**Remark 45.** By Lemma \[39\], the chart defined in \((50)\) is indeed a valid \( C^1 \)-chart for \( \mathcal{M} \).

**Remark 46.** It is easy to see that the above definition does not depend upon the choice of \( x \) inside \( \text{conv } \sigma \). Indeed, suppose that the orientation of \( \mathcal{M} \) is defined by the atlas \( \{(U_i, \psi_i)\}_{i \in I} \). Consider a point \( m \) such that \( m \in U_i \cap \pi_\mathcal{M}(U_x) \neq \emptyset \) for some \( i \in I \). The orientation consistency between \( \sigma \) and \( \mathcal{M} \) is determined by the sign of

\[
\det \mathbf{D} \left( \psi_\sigma \circ (\mathbf{M}_{U_x})^{-1} \circ (\psi_i)^{-1} \right)(\psi_i(m)).
\]

Thanks to Lemma \[39\], the above determinant does not vanish as \( x \) moves in \( \text{conv } \sigma \), and, since \( \text{conv } \sigma \) is connected and the determinant is continuous, its sign is constant over \( \text{conv } \sigma \).

Given two \( d \)-dimensional vector subspaces \( V \) and \( V' \) of \( \mathbb{R}^N \) such that \( \angle(V, V') < \frac{\pi}{2} \), Lemma \[30\] implicitly gives an expression of the matrix of the orthogonal projection \( \pi_{V'}|_V \) on \( V' \) restricted to \( V \) in an orthonormal basis. In particular, it says that the matrix of \( \pi_{V'}|_V \) has a full rank and therefore is invertible. Observe that if \( B \) is a basis of \( V \) that defines an orientation of \( V \), then \( \pi_{V'}|_V(B) \) is a basis of \( V' \) that induces an orientation of \( V' \). This allows us to transfer the orientation of \( V \) to \( V' \) as follows.

**Definition 47** (Consistent orientation between vector spaces). Let \( V \) and \( V' \) be two \( d \)-dimensional vector subspaces of \( \mathbb{R}^N \). Let \( B \) (resp. \( B' \)) be a basis of \( V \) (resp. \( V' \)). We say that \((V', B') \) has a consistent orientation through projection with \((V, B)\) if the determinant of the matrix of \( \pi_{V'}|_V \) with respect to the bases \( B \) and \( B' \) is positive.

As seen in the proof of Lemma \[30\], the map \( \pi_V|_{V'} \circ \pi_{V'}|_V : V \rightarrow V \) admits a positive definite matrix in an orthonormal basis and therefore preserves the orientation. It follows that the relation “has a consistent orientation through projection with” is symmetric and could have been
defined either by saying (as in the definition above) that the determinant of the matrix of $\pi V|_V$ with respect to bases $B$ and $B'$ is positive, or by saying that the determinant of the matrix of $\pi V|_{V'}$ with respect to bases $B'$ and $B$ is positive.

However, the relation “has a consistent orientation through projection with” is not transitive in general. For example, if $\Delta_1, \Delta_2, \Delta_3$ are three 1-dimensional vector spaces in $\mathbb{R}^2$, each oriented by basis $B_1 = \{(1, 0)\}$, basis $B_2 = \{(\cos \frac{\pi}{3}, \sin \frac{\pi}{3})\}$, and basis $B_3 = \{(\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3})\}$ respectively, then $(\Delta_1, B_1)$ has a consistent orientation through projection with $(\Delta_2, B_2)$, as do $(\Delta_2, B_2)$ with $(\Delta_3, B_3)$, but not $(\Delta_1, B_1)$ with $(\Delta_3, B_3)$. However, one has the following lemma, useful for the proof of Lemma 49.

**Lemma 48** (Making consistent orientation between vector spaces transitive). Let $V$, $V_1$, $V_2$, and $V_3$ be $d$-dimensional vector subspaces of $\mathbb{R}^N$ such that:

$$\angle(V, V_i) < \frac{\pi}{4}, \text{ for } i \in \{1, 2, 3\}.$$  

Then, in any basis $B_1$ of $V_1$ the matrix of the linear map from $V_1$ to $V_1$:

$$\pi V_1|_{V_3} \circ \pi V_3|_{V_2} \circ \pi V_2|_{V_1}$$

has a positive determinant. Equivalently, assuming $B_1$, $B_2$, and $B_3$ are bases defining orientation of $V_1$, $V_2$, and $V_3$ respectively, we have the following: If $(V_i, B_i)$ has a consistent orientation through projection with $(V_2, B_2)$ and $(V_2, B_2)$ has a consistent orientation through projection with $(V_3, B_3)$, then so does $(V_1, B_1)$ with $(V_3, B_3)$.

**Proof.** The lemma holds trivially when $V_1 = V_2 = V_3 = V$. In this case, the matrix associated to $\pi V_1|_{V_3} \circ \pi V_3|_{V_2} \circ \pi V_2|_{V_1}$ is the identity and its determinant is 1. Using Lemma 30, it is easy to build a basis $B_i(t)$ of a $d$-dimensional vector space $V_i(t)$ for each $i \in \{1, 2, 3\}$, parametrized by $t \in [0, 1]$ and continuous as a function of $t$, such that $B_i(0)$ is a basis of $V$, $B_i(1)$ a basis of $V_i$, and $\angle(V_i(t)), V_j(t)) < \frac{\pi}{4}$ for all $t \in [0, 1]$. In this condition, the determinant of:

$$\pi V_1(t)|_{V_3(t)} \circ \pi V_3(t)|_{V_2(t)} \circ \pi V_2(t)|_{V_1(t)}$$

is a continuous function of $t$ which equals 1 when $t = 0$. Since for any $i, j \in \{1, 2, 3\}$ one has $\angle(V_i(t), V_j(t)) < \frac{\pi}{4}$, each projection has a full rank and therefore so does their composition. Thus, the determinant does not vanish and must remain positive for all $t \in [0, 1]$.  

Consider a compact $C^2$ $d$-dimensional submanifold $\mathcal{M}$ of the Euclidean space $\mathbb{R}^N$ whose reach is positive. For $m \in \mathcal{M}$, write

$$\mathcal{M}_m = \mathcal{M} \cap B \left( m, \sin \left( \frac{\pi}{4} \text{ reach } \mathcal{M} \right) \right).$$

Suppose that the orientation of the tangent space $T_m\mathcal{M}$ is defined by the chart $\psi : T_m\mathcal{M} \rightarrow \mathbb{R}^d$. By Lemma 33, the restriction of $\psi \circ \pi T_m\mathcal{M}$ to $\mathcal{M}_m$ is a local chart of $\mathcal{M}$. We say that the orientation of the tangent space $T_m\mathcal{M}$ is **consistent** with the orientation of $\mathcal{M}$ if the local chart $\psi \circ \pi T_m\mathcal{M} : \mathcal{M}_m \rightarrow \mathbb{R}^d$ is consistent with the orientation of $\mathcal{M}$.

**Lemma 49** (Projection on a tangent space gives the orientation). Let $\mathcal{M}$ be a compact, orientable, $C^2$ $d$-submanifold of the Euclidean space $\mathbb{R}^N$ with a reach greater than $\rho > 0$. Let $m \in \mathcal{M}$ and assume that $T_m\mathcal{M}$ has orientation that is consistent with that of $\mathcal{M}$. Let $\sigma = \{v_0, \ldots, v_d\}$ be an oriented, non-degenerate $d$-simplex such that $\text{conv } \sigma \subseteq \mathcal{M}^{\mathbb{R}^\rho}$, $\pi_{\mathcal{M}}(\text{conv } \sigma) \subseteq B \left( m, \sin(\frac{\pi}{4} \text{ reach } \mathcal{M}) \right)$,
and $\angle(\operatorname{aff} \sigma, T_m \mathcal{M}) < \frac{\pi}{4}$. Then the orientation of $\sigma$ is consistent with the orientation of $\mathcal{M}$ if and only if the orientation of

$$\pi_{T_m \mathcal{M}}(\sigma) = [\pi_{T_m \mathcal{M}}(v_0), \ldots, \pi_{T_m \mathcal{M}}(v_d)]$$

is consistent with the orientation of $T_m \mathcal{M}$.

Proof. For $x \in \text{conv} \sigma$, let us denote by $\tilde{x} = \pi_{\mathcal{M}}(x)$ the projection of $x$ on $\mathcal{M}$. Since $x \in \text{conv} \sigma$, we have that $\tilde{x} \in \mathcal{M} \cap B(m, \sin(\frac{\pi}{4})$ reach $\mathcal{M}$) and applying Lemma 35 one obtains that:

$$\angle(T_m \mathcal{M}, T_{\tilde{x}} \mathcal{M}) < \frac{\pi}{4}.$$  

Using the above inequality and our assumption that $\angle(\operatorname{aff} \sigma, T_m \mathcal{M}) < \frac{\pi}{4}$, we can apply Lemma 18 with vector spaces $V = V_1 = T_m \mathcal{M}$, $V_2 = \operatorname{aff} \sigma$, and $V_3 = T_{\tilde{x}} \mathcal{M}$. It follows that we can choose orthonormal bases for $T_m \mathcal{M}$, aff $\sigma$, and $T_{\tilde{x}} \mathcal{M}$ which have a consistent orientation through projection. These bases define respective frames of $T_m \mathcal{M}$, aff $\sigma$, and $T_{\tilde{x}} \mathcal{M}$ centered at $m$, $x$, and $\tilde{x}$, respectively. These frames define coordinate systems, or charts, for $T_m \mathcal{M}$, aff $\sigma$, and $T_{\tilde{x}} \mathcal{M}$. We can always choose those charts so that the induced orientation of $T_m \mathcal{M}$ is consistent with the orientation of $\mathcal{M}$.

Suppose that the orientation of $\pi_{T_m \mathcal{M}}(\sigma) = [\pi_{T_m \mathcal{M}}(v_0), \ldots, \pi_{T_m \mathcal{M}}(v_d)]$ is consistent with the orientation of $T_m \mathcal{M}$ and let us prove that the orientation of $\sigma = [v_0, \ldots, v_d]$ is consistent with the orientation of $\mathcal{M}$. By definition, we thus need to show that for an open neighborhood $U_x$ of $x$ in aff $\sigma$, we have

$$\det D\left(\psi \circ (\pi_{\mathcal{M}|U_x})^{-1} \circ (\psi \circ \pi_{T_m \mathcal{M}|M_m})^{-1} \left(\psi \circ \pi_{T_m \mathcal{M}}(\tilde{x})\right)\right) > 0$$  

(51)

Without loss of generality, we may assume $\psi = \text{Id}$, identifying $T_m \mathcal{M}$ with $\mathbb{R}^d$. Because of our choice of frames for $T_m \mathcal{M}$, aff $\sigma$, and $T_{\tilde{x}} \mathcal{M}$, we obtain immediately that the orientation of $\sigma$ is consistent with the orientation of aff $\sigma$. Thus, det $D\psi(y) > 0$ for all $y \in \operatorname{aff} \sigma$ and (51) is equivalent to

$$\det D\left((\pi_{\mathcal{M}|U_x})^{-1} \circ (\pi_{T_m \mathcal{M}|M_m})^{-1} \left(\pi_{T_m \mathcal{M}}(\tilde{x})\right)\right) > 0.$$  

(52)

which in turn is equivalent to

$$\det D\left(\pi_{T_m \mathcal{M}|M_m} \circ \pi_{\mathcal{M}|U_x}\right)(x) > 0.$$  

(53)

Since $\mathcal{M}$ and $T_{\tilde{x}} \mathcal{M}$ are tangent at $\tilde{x}$, we obtain, by using the chart $\pi_{T_{\tilde{x}} \mathcal{M}|M_{\tilde{x}}}$ for $\mathcal{M}$ in a neighborhood $M_{\tilde{x}}$ of $\tilde{x}$ in $\mathcal{M}$:

$$D\left(\pi_{T_{\tilde{x}} \mathcal{M}|M_{\tilde{x}}}\right)(\tilde{x}) = \mathbb{I}.$$  

(54)

Since (15) assumes the projection on $T_{\tilde{x}} \mathcal{M}$ equipped with the orthonormal basis $(e_1, \ldots, e_d)$ of $T_{\tilde{x}} \mathcal{M}$ as a chart for $\mathcal{M}$ in a neighborhood $M_{\tilde{x}}$ of $\tilde{x}$, it gives us an expression of the derivative of $\pi_{T_{\tilde{x}} \mathcal{M}|M_{\tilde{x}}} \circ \pi_{\mathcal{M}|U_x}$:

$$D\left(\pi_{T_{\tilde{x}} \mathcal{M}|M_{\tilde{x}}} \circ \pi_{\mathcal{M}|U_x}\right)(x) = (\mathbb{I} + \lambda \Pi_{\mathcal{M}}(\tilde{x}))^{-1} e_j^T = (\mathbb{I} + \lambda \Pi_{\mathcal{M}}(\tilde{x}))^{-1} \pi_{T_{\tilde{x}} \mathcal{M}|U_x},$$

where $v$ is a unit vector such that $x = \tilde{x} + \lambda v$, and $\pi_{T_{\tilde{x}} \mathcal{M}|U_x}$ is the differential of $\pi_{T_{\tilde{x}} \mathcal{M}|U_x}$. Then, using (54):

$$D\left(\pi_{\mathcal{M}|U_x}\right)(x) = (\mathbb{I} + \lambda \Pi_{\mathcal{M}}(\tilde{x}))^{-1} \pi_{T_{\tilde{x}} \mathcal{M}|U_x}.$$  

(55)
Using (54), that is the tangent space at \( \tilde{x} \) to either \( M_m \) or \( M_{\tilde{x}} \) coincides, with the same chart, with the tangent space at \( \tilde{x} \) to \( T_{\tilde{x}}M \), we get:

\[
D \left( \pi T_m M|_{M_m} \right) (\tilde{x}) = D \left( \pi T_m M|_{T_{\tilde{x}}M} \right) (\tilde{x}) = \pi T_m M|_{T_{\tilde{x}}M}^1
\]

since \( \pi T_m M|_{T_{\tilde{x}}M} \) is the derivative of \( \pi T_m M|_{T_{\tilde{x}}M} \). Combining (55) and (56), we obtain that (53) holds if and only of the determinant of

\[
\pi T_m M|_{T_{\tilde{x}}M} \left( I + \lambda II_v(\tilde{x}) \right)^{-1} \pi T_{\tilde{x}} M|_{U_x}
\]

is positive. This holds since each of the matrices associated to the three linear maps has a positive determinant: indeed, it holds for the two projections by the choice of coordinate systems, and for \( (I + \lambda II_v(\tilde{x}))^{-1} \) because, since \( \lambda \leq \rho < \text{reach} M \) and \( ||\lambda II_v(\tilde{x})||_2 < 1 \), the matrix \( I + \lambda II_v(\tilde{x}) \) is symmetric and positive definite.

For the reverse implication, we establish the contraposition. Precisely, letting \( \sigma' \) be the simplex sharing with \( \sigma \) the same set of vertices but having the opposite orientation, we show that if \( \sigma' \) has orientation consistent through projection on \( T_m M \), then \( \sigma' \) has its orientation consistent with \( M \). But, this statement is a consequence of the direct proposition that we have just proved, in which we replace \( \sigma \) with \( \sigma' \).

\[\Box\]

**F Establishing practical conditions**

In this section, we prove formally what is intuitively quite obvious. Since, in Problem (4), we consider only cycles \( \gamma \) (that is, chains such that \( \partial \gamma = 0 \)), the normalization condition (load_m,M,K(\gamma) = 1) is, in some sense, “stable”. Indeed, a small change in \( m_0 \) or in the projection direction, may only cross a \((d - 1)\)-simplex, which, since \( \partial \gamma = 0 \) and the small angular change preserves the orientation, will not change the resulting “load”.

Therefore, the normalization condition can be replaced by a constraint which does not refer to \( M \) anymore but refers only to a rough approximation \( II \) of \( M \).

**Remark 50.** Recall that the boundary of an oriented simplex \([v_0, \ldots, v_d]\) is defined as:

\[
\partial[v_0, \ldots, v_d] = \sum_{i=0}^{d} (-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_d]
\]

where \( \hat{v}_i \) means that vertex \( v_i \) is omitted, so that the simplex \( \tau = [v_0, \ldots, v_{d-1}] \) has the sign \((-1)^d \) in \( \partial[v_0, \ldots, v_d] \). It follows that for any \( d \)-coface \( \sigma' = \tau \cup \{v'_d\} \) of \( \tau \), \( \tau \) appears in \( \partial \sigma' \) with the same sign as in \( \partial \sigma \) if and only if \( \sigma' \) is oriented as \( [v_0, \ldots, v_{d-1}, v'_d] \).

**Lemma 51.** Let \( K \) be a simplicial complex with vertices in \( \mathbb{R}^N \) such that \( |K| \subseteq M^{\circ \rho} \) and suppose that the \( d \)-simplices of \( K \) are non-degenerate and have a diameter upper bounded by \( \rho \). We also assume that for all \( d \)-simplices \( \tau \in K \) and all points \( y \in \text{conv} \tau \), we have

\[
\angle_{\text{aff} \tau, T\pi_M(y)} < \frac{\pi}{2}
\]

(57)

Choose an orientation for \( M \) and assume that all \( d \)-simplices of \( K \) inherit this orientation. Then, for any \( d \)-cycle \( \gamma \) in \( K \), the map

\[
\sum_{\alpha} \gamma(\alpha)1_{\pi_M(\text{conv} \alpha)}
\]

is constant almost everywhere.
Proof. Given a set $\Sigma$ of $d$-simplices in $K$ and a $d$-chain $\gamma$ in $K$, we denote by $\gamma|_{\Sigma}$ the restriction of $\gamma$ to $\Sigma$. In other words, $\gamma|_{\Sigma}$ is the chain that coincides with $\gamma$ on $\Sigma$ and is zero elsewhere. We define the map $\pi_M^d \gamma : M \to \mathbb{R}$ as the regularization of $m \mapsto \sum_{\alpha} \gamma(\alpha) 1_{\pi_M(\text{conv } \alpha)}$:

$$\left(\pi_M^d \gamma \right)(m) \overset{\text{def}}{=} \sum_{\alpha} \gamma(\alpha) 1_{\pi_M(\text{conv } \alpha)}(m).$$

The notation $\pi_M^d \gamma$ is justified by the fact that $\pi_M^d \gamma$ is a linear map from the set of chains in $K$ to the set of piecewise-constant real valued functions on $M$ modulo equality almost everywhere. The regularized version of it is:

$$\left(\pi_M^d \gamma \right)(m) \overset{\text{def}}{=} \lim_{r \to 0^+} \frac{\int_{m' \in M \cap B(m,r)} \left(\pi_M^d \gamma \right)(m') d\mu_M}{\int_{m' \in M \cap B(m,r)} d\mu_M},$$

where, as usual, $B(m,r)$ designates the ambient ball with center $m$ and radius $r$, and $\mu_M$ is the $d$-volume on $M$. This regularization will allow us to conclude the proof by exploiting the continuity of this regularized function.

For a simplex $\sigma \in K$, we denote the set of $d$-simplices in the star of $\sigma$ in $K$ by $\text{St}(\sigma, K)$. We denote the $k$-skeleton of $K$ by $\text{Sk}^k(K)$. We start by proving three claims:

**Claim 1:** $\pi_M(\text{Sk}^{d-1}(K))$ has $\mu_M$-measure zero and $M \setminus \pi_M(\text{Sk}^{d-2}(K))$ is open and connected.

Since $\pi_M$ is $C^1$, the image $\pi_M(\sigma)$ of a $(d-1)$-simplex $\sigma$ has a zero $d$-Hausdorff measure. $\text{Sk}^{d-1}(K)$ is a finite union of such images. $\pi_M(\text{Sk}^{d-2}(K))$, as the image of a compact set by $\pi_M$, is a compact set and its complement is therefore open. Similarly, $\pi_M(\text{Sk}^{d-2}(K))$ is a finite union of smooth compact $(d-2)$-submanifolds of $M$. It follows that any intersection of $\pi_M(\text{Sk}^{d-2}(K))$ with a smooth curve is non generic. Therefore, for any two points $m_1, m_2 \in M \setminus \pi_M(\text{Sk}^{d-2}(K))$, there must exist a smooth curve in $M \setminus \pi_M(\text{Sk}^{d-2}(K))$ connecting $m_1$ to $m_2$. This proves Claim 1.

**Claim 2:** If $\tau \in K$ is a $d$-simplex and $m \in \pi_M(\tau^o)$, then $\pi_M^d \gamma|_{\text{St}(\tau, K)} = \pi_M^d \gamma|_{\text{St}(\tau, K)}$ is constant in a neighborhood of $m$ in $M$.

Indeed, $\text{St}(\tau, K) = \{\tau\}$ and for any $m' \in \pi_M(\tau^o)$ one has $\pi_M^d \gamma|_{\text{St}(\tau, K)}(m') = \gamma(\tau)$. Due to Lemma 39, $\pi_M|_{\tau^o}$ is an open map and there is a neighborhood $U_m$ of $m$ in $M$ such that $U_m \subseteq \pi_M(\tau^o)$. It follows that for any $m' \in U_m$, $\pi_M^d \gamma|_{\text{St}(\tau, K)}(m') = \gamma(\tau)$. This proves Claim 2.

Note that Claim 2 does not use the assumption that $\partial \gamma = 0$. Since, for any $d$-simplex $\tau$, the complement of $\pi_M(\tau)$ in $M$ is open in $M$, a consequence of Claim 2 is that $\pi_M^d \gamma$ and $\pi_M^d \gamma$ coincide on $M \setminus \pi_M(\text{Sk}^{d-1}(K))$.

**Claim 3:** If $\sigma \in K$ is a $(d-1)$-simplex and if $\partial \gamma = 0$ then for any $m \in \pi_M(\sigma^o)$, $\pi_M^d \gamma|_{\text{St}(\sigma, K)}$ is constant in a neighborhood of $m$ in $M$.

In order to prove the claim, pick one $d$-dimensional coface $\tau$ of $\sigma$ and let $y \in \sigma^o$ be such that $\pi_M(y) = m$. Thanks to Remark ??, one can use $\xi = \psi_\tau \circ \left(\pi_M|_{U_{y, \tau}}\right)^{-1}$ as a chart for $M$ in an open neighborhood $U_m$ of $m$ in $M$. $U_m$ can be chosen small enough to be included in $\pi_M(\text{St}(\sigma, K)^o)$ and $U_{\xi(m)} = \xi(U_m) \subseteq \mathbb{R}^d$ is an open neighborhood of $\xi(m)$.

Without loss of generality, assume that $\sigma = [v_0, \ldots, v_{d-1}]$ and that the orientation of $\tau$ defined by $\tau = [v_0, \ldots, v_{d-1}, v_d]$ is consistent with the given orientation of $M$.
From the definition of $\xi$ and $\psi_\tau$ in Definition \[44\] one has:

$$\xi(U_m \cap \pi_M(\sigma)) = U_{\xi(m)} \cap \left(\mathbb{R}^{d-1} \times \{0\}\right).$$

For a $d$-simplex $\tau' \in \text{St}(\sigma, K)$, since the restriction of $\xi \circ \pi_M$ to $\tau'$ is an homeomorphism, the boundary of $\xi(U_m \cap \pi_M(\tau'))$ in $U_{\xi(m)}$ has to coincide with $U_{\xi(m)} \cap (\mathbb{R}^{d-1} \times \{0\})$ and $\xi(U_m \cap \pi_M(\tau'))$ therefore equals either $U_{\xi(m)} \cap (\mathbb{R}^{d-1} \times \mathbb{R}^-)$ or $U_{\xi(m)} \cap (\mathbb{R}^{d-1} \times \mathbb{R}^+)$. Let us denote the vertex in $\tau' \setminus \sigma$ by $v'_d$ and consider the curve $y'(t) = y + t(v'_d - v_0)$, with $t \in [0, \lambda]$, $\lambda > 0$ small enough to have $y'(\{0, \lambda\}) \subseteq \tau^0$ and $\pi_M(y'(\{0, \lambda\})) \subseteq U_m$. If $\tau'$ is oriented as $[v_0, \ldots, v_{d-1}, v'_d]$ and if both $\tau$ and $\tau'$ have been given orientations consistent with $\mathcal{M}$, we have (Definition \[44\]):

$$\det D(\xi \circ \pi_M \circ \psi^{-1}_{\tau'})(\psi_\tau(y)) > 0,$$

which is equivalent to:

$$\det \begin{pmatrix} \xi \circ \pi_M(y) \cdot (v_1 - v_0), & \ldots, & D(\xi \circ \pi_M(y) \cdot (v_{d-1} - v_0), & D(\xi \circ \pi_M(y) \cdot (v'_d - v_0) \end{pmatrix} > 0. \quad (58)$$

But by definition of $\xi$ one has that for $i = 1, \ldots, d-1$, $D(\xi \circ \pi_M)(y) \cdot (v_i - v_0) \in \mathbb{R}^d$ equals $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 appears in position $i$ and thus (58) implies that the last coordinate of $D(\xi \circ \pi_M)(y) \cdot (v'_d - v_0)$ is positive, which implies that, for $t$ small enough, $\xi \circ \pi_M(y'(t)) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ under our assumption for $\tau'$ to be oriented as $[v_0, \ldots, v_{d-1}, v'_d]$. At the same time, we know from Remark \[50\] that $\sigma$ appears with the same sign in $\partial \tau'$ as in $\partial \tau$ if and only if $\tau'$ is oriented as $[v_0, \ldots, v_{d-1}, v'_d]$. Since the orientations of $\tau$ and $\tau'$ are consistent with the chosen orientation of $\mathcal{M}$, $\xi(U_m \cap \pi_M(\tau'))$ is therefore included in $H^- = \mathbb{R}^{d-1} \times \mathbb{R}^-$ or $H^+ = \mathbb{R}^{d-1} \times \mathbb{R}^+$ respectively depending on whether $\sigma$ appears negatively or positively in $\partial \tau$.

The set $\text{St}(\sigma, K)$ of $d$-cofaces of $\sigma$ can be decomposed in $\text{St}(\sigma, K)^-$ and $\text{St}(\sigma, K)^+$ such that:

$$\partial \tau(\sigma) = -1 \iff \tau \in \text{St}(\sigma, K)^- \quad \text{and} \quad \partial \tau(\sigma) = 1 \iff \tau \in \text{St}(\sigma, K)^+.\quad \text{Then:}\quad (59)$$

$$\partial \gamma = 0 \Rightarrow \left(\sum_{\tau \in \text{St}(\sigma, K)^+} \gamma(\tau)\right) - \left(\sum_{\tau \in \text{St}(\sigma, K)^-} \gamma(\tau)\right) = 0. \quad (59)$$

If $m^- \in \xi^{-1}_\sigma(H^- \cap \xi_\sigma \circ \pi_M(U_x) \setminus \Pi)$ then:

$$\pi^\sharp_M|_{\text{St}(\sigma, K)^-}(m^-) = \pi^\sharp_M|_{\text{St}(\sigma, K)^-}(m^-) = \sum_{\tau \in \text{St}(\sigma, K)^-} \gamma(\tau), \quad (60)$$

and if $m^+ \in \xi^{-1}_\sigma(H^+ \cap \xi_\sigma \circ \pi_M(U_x) \setminus \Pi)$ then:

$$\pi^\sharp_M|_{\text{St}(\sigma, K)^+}(m^+) = \pi^\sharp_M|_{\text{St}(\sigma, K)^+}(m^+) = \sum_{\tau \in \text{St}(\sigma, K)^+} \gamma(\tau). \quad (61)$$

Now (59), (60), and (61) give us:

$$\pi^\sharp_M|_{\text{St}(\sigma, K)^-}(m^+) - \pi^\sharp_M|_{\text{St}(\sigma, K)^-}(m^-) = 0,$$
which proves Claim 3.

One has, for any $m \in \mathcal{M}$:

$$\pi^1_{\mathcal{M}} \gamma(m) = \sum_{m \in \pi_{\mathcal{M}}(\sigma^0)} \pi^1_{\mathcal{M}} \gamma|_{\text{St}(\sigma, K)}(m). \quad (62)$$

If $m \in \mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K))$ then the simplices $\sigma$ such that $m \in \pi_{\mathcal{M}}(\sigma^0)$ that contribute to the sum (62) are of dimension either $d$ or $d - 1$. It follows from Claims 1, 2, and 3, that if $m \in \mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K))$ then $\pi^1_{\mathcal{M}} \gamma$ is constant in an open neighborhood of $m$. As a consequence, for $m_0 \in \mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K))$, the set

$$\{m \in \mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K)) \mid \pi^1_{\mathcal{M}} \gamma(m) = \pi^1_{\mathcal{M}} \gamma(m_0)\}$$

is open. Since $\mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K))$ is connected by Claim 1, it follows that $\psi_{\tau, K}$ is constant on $\mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-2}(K))$. Since $\psi_{\gamma, K}$ and $m \mapsto \sum_{\alpha \in K} \gamma(\alpha)1_{\pi_{\mathcal{M}}(\text{conv} \alpha)}$ coincide on $\mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-1}(K))$, and since, due to Claim 1, $\pi_{\mathcal{M}}(\text{Sk}^{d-1}(K))$ has a zero Lebesgue measure, we have shown that $\sum_{\alpha \in K} \gamma(\alpha)1_{\pi_{\mathcal{M}}(\text{conv} \alpha)}$ is constant almost everywhere. \qed

The next lemma is useful to derive a realistic algorithm in Section 9.1. Roughly, it says that the normalization constraint in Problem (P) and since, due to Claim 1, $\mathcal{M} \setminus \pi_{\mathcal{M}}(\text{Sk}^{d-1}(K))$ has a zero Lebesgue measure, we have shown that $\sum_{\alpha \in K} \gamma(\alpha)1_{\pi_{\mathcal{M}}(\text{conv} \alpha)}$ is constant almost everywhere.

**Lemma 52** (Practical normalization lemma). Suppose $0 \leq \rho \leq \frac{\pi}{2}$. Let $K$ be a simplicial complex with vertices in $\mathbb{R}^N$ such that $|K| \subseteq \mathcal{M}^{\oplus \rho}$ and suppose that the $d$-simplices of $K$ are non-degenerate and have a diameter upper bounded by $\rho$. Suppose furthermore that for all $d$-simplices $\tau \in K$ and all points $y \in \text{conv} \tau$, we have

$$\angle(\text{aff} \tau, T_{\pi_{\mathcal{M}}(y)} \mathcal{M}) < \frac{\pi}{2} \quad \text{(63)}$$

and that every $d$-simplex in $K$ inherits the orientation of the manifold $\mathcal{M}$. Consider a $d$-dimensional affine space $\Pi$ passing through a point $x \in \mathcal{M}^{\oplus \rho}$ such that

$$\angle(\Pi, T_{\pi_{\mathcal{M}}(x)} \mathcal{M}) < \frac{\pi}{2} \quad \text{(64)}$$

Assume that $\Pi$ is oriented consistently with $\mathcal{M}$. Let $K' = K[x, 4\rho]$ and suppose that the following conditions hold:

(i) $\forall \beta \in K'[d-1], \ x \notin \pi_{\Pi}(\text{conv} \beta)$;

(ii) $\forall \alpha \in K'[d], \ \angle \alpha, T_{\pi_{\mathcal{M}}(x)} \mathcal{M} < \frac{\pi}{2}$;

(iii) $\forall \alpha \in K'[d], \ \alpha$ inherits its orientation from $\Pi$.

Then, for all $d$-cycles $\gamma$ in $K$ and all points $m \in \mathcal{M} \setminus \pi_{\mathcal{M}}(|K'[d-1]|)$, we have

$$\sum_{\alpha \in K'[d]} \gamma(\alpha)1_{\pi_{\mathcal{M}}(\text{conv} \alpha)}(m) = \sum_{\alpha \in K'[d]} \gamma(\alpha)1_{\pi_{\Pi}(\text{conv} \alpha)}(x) \quad \text{(65)}$$
We distinguish two cases depending on whether $\Pi = \Pi_0$ or not.

**Case II = \Pi_0.** We first claim that:

\[
\left( \pi_M|_{\alpha \in \mathcal{A}} \right)^{-1} \left( \{ \pi_M(x) \} \right) = \pi_{\Pi_0}^{-1} \left( \{ x \} \right) \cap \mathcal{M}^{\oplus \rho} \cap B(x, 5\rho).
\] (66)

Indeed, the left hand member is equal to $\mathcal{N}_{\pi_M(x)} \cap B(\pi_M(x), \rho)$, where $\mathcal{N}_{\pi_M(x)} = \{ \pi_M(x) \} + N_{\pi_M(x)} \mathcal{M}$ is the $(N-d)$-dimensional affine subspace subspace of $\mathbb{R}^N$ orthogonal to $\mathcal{M}$ at $\pi_M(x)$. Notice that $\mathcal{N}_{\pi_M(x)} \mathcal{M}$ contains $x$ and coincides with $\pi_{\Pi_0}^{-1} \left( \{ x \} \right)$ — the affine space through $x$ orthogonal to $\Pi_0$. Since $B(\pi_M(x), \rho) \subseteq B(x, 5\rho)$, the left hand member is included in the right hand member. To get the reverse inclusion we only need to show that:

\[
\pi_{\Pi_0}^{-1} \left( \{ x \} \right) \cap \mathcal{M}^{\oplus \rho} \cap B(x, 5\rho) \subseteq B(\pi_M(x), \rho).
\]

For that observe that for $y \in \pi_{\Pi_0}^{-1} \left( \{ x \} \right) \cap \mathcal{M}^{\oplus \rho} \cap B(x, 5\rho)$ one has $y \in B(\pi_M(x), 6\rho)$ and, since $6\rho < R$ and $y \in \mathcal{N}_{\pi_M(x)} \mathcal{M}$, one has $\pi_M(y) = \pi_M(x)$ and $d(y, \mathcal{M}) = d(y, \pi_M(x))$. Thus, $y \in \mathcal{M}^{\oplus \rho}$ implies $y \in B(\pi_M(x), \rho)$. Equality (66) is proved. Let $y \in \pi_{\Pi_0} \left( \{ x \} \right) \cap \mathcal{M}^{\oplus \rho} \cap B(x, 5\rho)$ and assume that $x = \pi_{\Pi_0}(y)$. If $\conv \alpha \subseteq B(x, 5\rho)$, then $y \in \pi_{\Pi_0}^{-1} \left( \{ x \} \right) \cap \mathcal{M}^{\oplus \rho} \cap B(x, 5\rho)$ and (66) gives $\pi_M(y) = \pi_M(x)$. Then:

\[
\left( \conv \alpha \subseteq B(x, 5\rho) \text{ and } x \in \pi_{\Pi_0}(\conv \alpha) \right) \iff \pi_M(x) \in \pi_M(\conv \alpha).
\]

If we assume the diameter of simplices to be upper bounded by $\rho$, then $\conv \alpha \cap B(x, 4\rho) \neq \emptyset$ implies $\conv \alpha \subseteq B(x, 5\rho)$, and we get:

\[
\left( \conv \alpha \cap B(x, 4\rho) \neq \emptyset \text{ and } x \in \pi_{\Pi_0}(\conv \alpha) \right) \iff \pi_M(x) \in \pi_M(\conv \alpha).
\]

It results that for any $\alpha \in K$, $\pi_M(x) \in \pi_M(\conv \alpha)$ implies $\alpha \in K'$, and:

\[
\sum_{\alpha \in K\cap [d]} \gamma(\alpha) \mathbf{1}_{\pi_M(\conv \alpha)}(\pi_M(x)) = \sum_{\alpha \in K'\cap [d]} \gamma(\alpha) \mathbf{1}_{\Pi_0}(\conv \alpha)(x).
\] (67)

We assume the following generic condition:

\[
\pi_M(x) \in \mathcal{M} \setminus \pi_M \left( \left[ K^{[d-1]} \right] \right).
\] (68)

The condition is generic because, if it does not hold, a sufficiently small $C^2$-perturbation $\mathcal{M}'$ of $\mathcal{M}$ would satisfy it and still meet all the conditions of the lemma. Assuming the generic condition to hold, one can see using (67) that, in the particular case where $\Pi = \Pi_0$, the lemma is just another formulation of Lemma 30.

**Case II \neq \Pi_0.** From now on, we assume that $\Pi \neq \Pi_0$. Consider a differentiable path $\Gamma : [0, 1] \to (\mathbb{R}^N)^{d+1}$, with $\Gamma(t) \equiv (x, B(t))$ in the space of orthonormal $d$-dimensional frames in $\mathbb{R}^N$. Precisely, $\Gamma(t) = (x, B(t))$ is the orthonormal frame of a $d$-dimensional affine space $\Pi(t)$ containing $x$, with $\Pi(0) = \Pi_0$ and $\Pi(1) = \Pi$, and $B(t)$ is an orthonormal basis of the vector space associated to $\Pi(t)$. Lemma 30 allows us to give an explicit formulation for $B$. Since $\Pi \neq \Pi_0$, $\theta = \angle(\Pi_0, \Pi)$ satisfies $0 < \theta < \frac{\pi}{2}$, and both affine spaces contain $x$. Consider two $d$-dimensional vector spaces $V$ and $V'$ such that $\Pi_0 = x + V$ and $\Pi = x + V'$, respectively. Applying Lemma 30...
to $V$ and $V'$ and borrowing its notation, we define the parametrized family of orthonormal bases $\mathcal{B}(t)$ for $t \in [0,1]$ as:

$$
\mathcal{B}(t) \overset{\text{def.}}{=} (\mathcal{B}_1(t), \ldots, \mathcal{B}_d(t)) = (u_1(t), \ldots, u_d(t), w_1, \ldots, w_{d-d'})
$$

where $u_k(t) = \cos(t\theta_k)u_k + \sin(t\theta_k)v_k^t$. \hspace{1cm} (69)

For any $t \in [0,1]$, $\Gamma(t) = (x, \mathcal{B}(t))$ is an orthonormal frame of $\Pi(t)$. We want to follow the evolution of the function $\phi : [0,1] \to \mathbb{R}$ defined as:

$$
\phi(t) \overset{\text{def.}}{=} \sum_{\alpha \in K'[d]} \gamma(\alpha)1_{\pi_{\Pi(t)}(\text{conv } \alpha)}(x),
$$

or its regularization $\hat{\phi}$, defined similarly as in the proof of Lemma 51

$$
\hat{\phi}(t) \overset{\text{def.}}{=} \lim_{h \to 0} \frac{1}{\min(1,t+h) - \max(0,t-h)} \int_{\max(0,t-h)}^{\min(1,t+h)} \phi(s)ds.
$$

The proof then consists of showing that $\hat{\phi}(t)$ remains constant along the path $t \mapsto \Pi(t)$ and thus, since Equation 65 is satisfied for $\Pi_0 = \Pi(0)$ by (67), it will extend to $\Pi = \Pi(1)$.

The family of bases $\mathcal{B}(t) = (u_1(t), \ldots, u_d(t), w_1, \ldots, w_{d-d'})$, parametrized by $t \in [0,1]$, induces a smooth map $\psi : [0,1] \times \mathbb{R}^N \to \mathbb{R}^d$ where, for $y \in \mathbb{R}^N$, the components of $\psi(t, y) \in \mathbb{R}^d$ are the coordinates of $\pi_{\Pi(t)}(y) - x$ in the basis $\mathcal{B}(t)$:

$$
\psi(t, y) = \langle \mathcal{B}_k(t), \pi_{\Pi(t)}(y) - x \rangle = \langle \mathcal{B}_k(t), y - x \rangle.
$$

With this definition one has:

$$
\psi(t, y) = 0 \iff \pi_{\Pi(t)}(y) = x. \hspace{1cm} (70)
$$

If $\sigma \in K'[d-2]$, the set $\psi([0,1], \text{conv } \sigma)$ is included in a compact smooth $(d-1)$-manifold with boundary in $\mathbb{R}^d$. Since it corresponds to the finite union of complements of sets of codimension 1, the condition:

$$
0 \notin \psi([0,1], \bigcup_{\sigma \in K'[d-2]} \text{conv } \sigma)
$$

is generic. We now make the assumption that this generic condition holds, since, if it does not, it can be satisfied after an arbitrarily small perturbation of $t \mapsto \Gamma(t)$. One can easily check using (69) that for any $t \in [0,1]$, $\angle \Pi(t), \Pi_0 \leq \angle \Pi, \Pi_0$, and even if a small perturbation is required for ensuring the generic condition, we can assume that, from (64):

$$
\forall t \in [0,1], \angle \Pi(t), T_{\pi_M(x)} M = \angle \Pi(t), \Pi_0 < \frac{\pi}{8}. \hspace{1cm} (72)
$$

We will need the following claim:

$$
\forall t \in [0,1], \quad (\pi_{\Pi(t)})^{-1}(x) \cap M^\oplus \cap B(\pi_M(x), 6\rho) \subseteq B(\pi_M(x), 3\rho). \hspace{1cm} (73)
$$

Indeed, consider $y \in (\pi_{\Pi(t)})^{-1}(x) \cap M^\oplus \cap B(\pi_M(x), 6\rho)$. Then $\pi_M(y) \in B(\pi_M(x), 7\rho)$, and using Lemma 33 one has:

$$
\pi_M(y) \in (T_{\pi_M(x)} M)^{\oplus \frac{7\rho^2}{2\pi}} \subseteq (T_{\pi_M(x)} M)^{\oplus \rho},
$$
which, since \( d(y, \pi_M(y)) < \rho \), implies:

\[
y \in (T_{\pi_M(x)}M)^{\perp 2\rho}.
\]

Denote the orthogonal projection onto \( N_{\pi_M(x)}M \) by \( \pi N_{\pi_M(x)}M \). Then, since \( x \in N_{\pi_M(x)}M \cap (T_{\pi_M(x)}M)^{\perp \rho} \), one has with (74):

\[
\| \pi N_{\pi_M(x)}M(y) - \pi N_{\pi_M(x)}M(x) \| \leq 2\rho + \rho = 3\rho.
\]

From (72), one has that \( \angle(y-x), N_{\pi_M(x)}M < \frac{\pi}{8} \), which gives:

\[
d(y, N_{\pi_M(x)}M) \leq 3\rho \tan \frac{\pi}{8} < \frac{3}{2}\rho.
\]

Applying (74) again, we obtain

\[
d(y, \pi_M(x)) \leq \sqrt{d(y, T_{\pi_M(x)}M)^2 + d(y, N_{\pi_M(x)}M)^2} < \sqrt{4 + \frac{9}{4}}\rho < 3\rho,
\]

which proves Equation (73). Since \( B(x, 5\rho) \subseteq B(\pi_M(x), 6\rho) \) and \( B(\pi_M(x), 3\rho) \subseteq B(x, 4\rho) \), Equation (73) gives us:

\[
\forall t \in [0, 1], (\pi_{\Pi(t)})^{-1}(x) \cap M^{\perp \rho} \cap B(x, 5\rho) \subseteq B(x, 4\rho).
\]

(75)

Since \( K' \) is not a simplicial complex, it is convenient to introduce the smallest simplicial complex \( \overline{K}' \) containing \( K' \), in other words the set of all faces of simplices in \( K' \). In particular, \( \overline{K}' \) contains all \((d-1)\)-faces of the \(d\)-simplices in \( K' \). One has \( |\overline{K}'| \subseteq B(x, 5\rho) \) and, using (70), (73) yields:

\[
\sigma \in \overline{K}'^{d-1}, \quad \text{and} \quad 0 \in \psi([0, 1], \text{conv} \sigma) \Rightarrow \sigma \in K'^{d-1}.
\]

(76)

As shown below, the changes in \( t \mapsto \hat{\phi}(t) \) may happen only when \( \psi(t, \text{conv} \sigma) = 0 \), where \( \sigma \) is a \((d-1)\)-face of some \(d\)-simplex \( \alpha \in K'^{d} \), in other words \( \sigma \in \overline{K}'^{d-1} \). We need (76) to ensure that every such \( \sigma \) in fact belongs to \( K'^{d-1} \).

Let \( \sigma \in K'^{d-1} \) be such that \( 0 \in \psi([0, 1], \text{conv} \sigma) \), and denote the set of \(d\)-cofaces of \( \sigma \) in \( K' \) (respectively in \( K \)) by \( \text{St}(\sigma, K') \) (respectively \( \text{St}(\sigma, K) \)). Note that, if \( \tau \) is a \(d\)-coface of \( \sigma \) in \( K \), and since \( \text{conv} \sigma \subseteq \text{conv} \tau \), one has the implication \( \text{conv} \sigma \cap B(x, 4\rho) \neq \emptyset \Rightarrow \text{conv} \tau \cap B(x, 4\rho) \neq \emptyset \). Thus, \( \tau \in K' \) and we have:

\[
\sigma \in K'^{d-1} \implies \text{St}(\sigma, K') = \text{St}(\sigma, K).
\]

(77)

Remark 53. Thanks to (77), the cycle condition \( (\partial \gamma)(\sigma) = 0 \) is inherited on each \((d-1)\)-simplex \( \sigma \in K'^{d-1} \) by the restriction of \( \gamma \) to \( K' \). This is not true for the \((d-1)\)-simplices of \( \overline{K}'^{d-1} \setminus K'^{d-1} \). Thanks to (76) we only have to consider how \( \phi(t) \) evolves as \( t \) continuously increases from 0 to 1 on \((d-1)\)-simplices in \( K'^{d-1} \) and benefit from the cycle condition.

From Condition (ii) and (72) we get that:

\[
\forall t \in [0, 1], \tau \in K'^{d} \implies \angle \text{aff}(\tau), \Pi(t) < \frac{3\pi}{8}.
\]

(78)

As a consequence of Lemma (30), the restriction of \( \pi_{\Pi(t)} \) to \( \text{aff} \sigma \) is an affine bijection and in particular an homeomorphism. It sends the boundary of a \(d\)-simplex \( \text{conv} \alpha \) onto the boundary of the image of \( \text{conv} \alpha \): \( \pi_{\Pi(t)}(\partial \text{conv} \alpha) = \partial \pi_{\Pi(t)}(\text{conv} \alpha) \).
The fact that \(\psi\) is uniformly continuous (in fact \(C^\infty\) with a compact domain) also in \(t\), means that in particular:

\[
\forall \epsilon > 0, \exists \eta > 0, \forall y \in \text{aff } \sigma, \forall t, t^* \in [0, 1], \quad |t^* - t| < \eta \implies \|\psi(t^*, y) - \psi(t, y)\| < \epsilon.
\]

If 0 is not on the boundary of \(\psi(t, \text{ conv } \alpha)\) there is a \(\epsilon > 0\) such that either \(B(0, \epsilon) \subseteq \psi(t, \text{ conv } \alpha)\) or \(B(0, \epsilon) \cap \psi(t, \text{ conv } \alpha) = \emptyset\). It follows that:

\[
0 \in \psi(t, \text{ conv } \alpha) \implies \exists \eta > 0, 0 \in \bigcap_{t^* \in [t - \eta, t + \eta]} \psi(t^*, \text{ conv } \alpha), \quad (79)
\]

\[
0 \notin \psi(t, \text{ conv } \alpha) \implies \exists \eta > 0, 0 \notin \psi([t - \eta, t + \eta], \text{ conv } \alpha). \quad (80)
\]

We are now ready to track the evolution of \(t \mapsto \hat{\phi}(t)\). For that we consider two cases. First we consider the values of \(t\) such that \(x \notin \pi_{\Pi(t)} \left(\overline{K^{d-1}}\right)\), or in other words \(0 \notin \psi \left(t, \left[\overline{K^{d-1}}\right]\right)\). Let \(\alpha \in K^{d}[d]\). Since 0 is not on the boundary of \(\psi(t, \text{ conv } \alpha)\), one of the two cases (79) or (80) must occur. This implies that, for some \(\eta > 0\), \(t \mapsto \phi(t)\) is constant on \([t - \eta, t + \eta]\), which in turn implies that in this case \(\phi\) and \(\hat{\phi}\) coincide:

\[
0 \notin \psi \left(t, \left[\overline{K^{d-1}}\right]\right) \implies \phi(t) = \hat{\phi}(t).
\]

We now consider the second case, namely when \(t\) is such that \(x \in \pi_{\Pi(t)} \left(\overline{K^{d-1}}\right)\). According to (76), if \(x \in \pi_{\Pi(t)} (\text{ conv } \sigma)\) for some \(\sigma \in \overline{K^{d-1}}\), then \(\sigma \in K^{d}[d]\). Therefore (77) and Remark 53 applies.

We are interested in the possible change of value of \(\phi(t^*)\) when \(t^*\) belongs to a neighborhood of \(t\). Generically, if \(x \in \pi_{\Pi(t)} \left(\left[\overline{K^{d-1}}\right]\right)\), there is a unique \(\sigma \in K^{d}[d]\) such that \(x \in \pi_{\Pi(t)} (\text{ conv } \sigma)\). However, we do not need this generic condition, since \(\phi(t^*)\) can be expressed as the following sum for any \(t^* \in [0, 1]\). We consider the \(d\)-simplices \(\alpha\) separately, depending on whether the inverse image of 0 by \(\psi(t, \cdot)\) is (1) interior: \(0 \in \psi(t, \text{ conv } \alpha)\) (the first sum below), (2) on the boundary: \(0 \in \partial(\psi(t, \text{ conv } \alpha))\) (the second sum below), or (3) the complement: \(0 \notin \psi(t, \text{ conv } \alpha)\) (the third sum below):

\[
\forall t^* \in [0, 1], \phi(t^*) = \sum_{\alpha \in K^{d}[d]} \gamma(\alpha)
\]

\[
\sum_{\alpha \in K^{d}[d]} \gamma(\alpha) = \sum_{\alpha \in K^{d}[d]} \gamma(\alpha) + \sum_{0 \notin \psi(t^*, \text{ conv } \alpha)} \gamma(\alpha) + \sum_{0 \in \psi(t^*, \text{ conv } \alpha)} \gamma(\alpha).
\]

Thanks to (79) and (80) there is a \(\eta > 0\) such that, for any \(t^* \in [\max(0, t - \eta), \min(1, t + \eta)]\), \(0 \in \psi(t^*, \text{ conv } \alpha)\) always holds in the first sum and never occurs in the third one. Then:

\[
\forall t^* \in [\max(0, t - \eta), \min(1, t + \eta)],
\phi(t^*) = \sum_{\alpha \in K^{d}[d]} \gamma(\alpha) + \sum_{0 \in \partial(\psi(t, \text{ conv } \alpha))} \gamma(\alpha)
\]

\[
= \sum_{\alpha \in K^{d}[d]} \gamma(\alpha) + \sum_{\sigma \in \overline{K^{d-1}}} \sum_{0 \in \psi(t, \text{ conv } \sigma)} \gamma(\alpha). \quad (81)
\]
We have proven that

\[
\sigma_1 \neq \sigma_2 \quad \text{and} \quad 0 \in \psi(t, \text{conv} \sigma_1), \quad \text{and} \quad 0 \in \psi(t, \text{conv} \sigma_2) \Rightarrow \text{St}(\sigma_1, K') \cap \text{St}(\sigma_2, K') = \emptyset. \tag{82}
\]

The first sum does not depend on \(t^* \in [\max(0, t-\eta), \min(1, t+\eta)]\) and therefore remains constant in this interval, as in the first case. Thanks to (\ref{2}), there are several \((d-1)\)-simplices \(\sigma \in K'[d-1]\) such that \(0 \in \psi(t, \text{conv} \sigma)\) in the second sum, their stars are disjoint. It is then enough to study the variation of:

\[
t^* \mapsto \phi_\sigma(t^*) \overset{\text{def.}}{=} \sum_{\alpha \in \text{St}(\sigma, K')} \gamma(\alpha)
\]

for a single \(\sigma \in K'[d-1]\) such that \(0 \in \psi(t, \text{conv} \sigma)\), when \(t^*\) belongs to a neighborhood of \(t \in [0, 1]\).

According to Lemma \ref{35}, the projection \(\pi_{\Pi(0)}\) is a chart of \(M\) with a domain \(U_0 = M \cap B(\pi_M(x), \sin(\pi/4) \text{reach } M)\). Since \(M\) is orientable, we can orient \(M\) consistently with a given orientation of \(\pi_{\Pi(0)}\). The simplices in \([K'[d]] \subseteq \mathbb{B}(x, 5\rho)\) are sent into \(M \cap \mathbb{B}(x, 6\rho) \subseteq M \cap \mathbb{B}(\pi_M(x), 7\rho)\) by \(\pi_M\). Since, as assumed in the lemma, one has \(\rho < \frac{R}{2\pi}\), we get \(\pi_M([K'[d]]) \subseteq U_0\).

In other words, \(\psi(0, \cdot)\) restricted to \(U_0\) is a chart of \(M\) consistent with the orientation.

Let \(\tau\) be a \(d\)-simplex in \(K'\) oriented consistently with the orientation of \(M\), \(y \in \tau\), and \(U_y\) a neighborhood of \(y\) in \(\text{aff}(\tau)\). According to Lemma \ref{49}, the projection \(\psi(0, \cdot)(\tau)\) onto \(\Pi(0)\) is positively oriented with respect to the orientation of \(\Pi(0)\). Since, for \(y \in \tau\), the map \(t \mapsto \det D \left(\psi(t, \cdot) \big|_{\text{conv } \tau}\right)(y)\) is continuous and does not vanish by (\ref{78}), one has

\[
\det D \left(\psi(0, \cdot) \big|_{\text{conv } \tau}\right)(y) > 0 \Rightarrow \forall t \in [0, 1], \det D \left(\psi(t, \cdot) \big|_{\text{conv } \tau}\right)(y) > 0.
\]

We have proven that \(\psi(t, \cdot) \big|_{\text{conv } \tau}\) preserves the orientation of any \(d\)-simplex in \(K'\), which means that the projection \(\psi(t, \tau) = [\psi(t, v_0), \ldots, \psi(t, v_d)]\) of any \(d\)-simplex \(\tau = [v_0, \ldots, v_d]\) oriented consistently with \(M\) is positively oriented with respect to the orientation of \(\Pi(t)\).

Therefore, we can apply the same argument as for Claim 3 of the proof of Lemma \ref{51} to the \(d\)-simplices in the star of \(\sigma\): if \(x \in \pi_{\Pi(0)}(\text{conv } \sigma)\), in other words if for some \(y \in \text{conv } \sigma\) one has \(\psi(t, y) = 0\), and if \(H^-\) and \(H^+\) are the two half spaces in \(\mathbb{R}^d\) bounded by the hyperplane spanned by \(\psi(t, \sigma)\), then \(\alpha \in \text{St}(\sigma, K')\) appears positively (resp. negatively) in the boundary of a \(d\)-coface \(\alpha \in \text{St}(\sigma, K')\) if \(\psi(t, \alpha) \subseteq H^+\) (resp. \(\psi(t, \alpha) \subseteq H^-\)).

It follows that, for a point \(z\) in a neighborhood of \(V_0\) of \(0\),

\[
z \mapsto \sum_{\alpha \in \text{St}(\sigma, K')} \gamma(\alpha)
\]

has the same value in \(V_0 \cap (H^-)^\circ\) and in \(V_0 \cap (H^+)^\circ\). We consider the \(C^1\) map \(F : [0, 1] \times \text{conv } \sigma^o \to [0, 1] \times \mathbb{R}^d\) defined by:

\[
F(t^*, y^*) \overset{\text{def.}}{=} (t^*, \psi(t^*, y^*)�).
\]

We note that, in particular, \(F(t, y) = (t, 0)\).

By the Thom Transversality Theorem \ref{29} Chapter 2 the map \(F\) is generically transversal to the manifold \([0, 1] \times \{0\}\), which implies that \(F\) is regular at \((t, y)\), i.e., its derivative at \((t, y)\) has rank \(d\) and the vector \((1, 0) \in \mathbb{R} \times \mathbb{R}^d\) does not belong to the image of the derivative of \(F\) when \(F(t^*, y^*) \in (0, 1) \times \{0\}\). Since the transversality property on \(F\) is generic, if it does not hold, it will after an arbitrary small perturbation of \(F\).
It follows that the image of \( F \) in a neighborhood of \( F(t, y) \) is a smooth hyper-surface whose tangent space does not contain the vector \((1, 0) \in \mathbb{R} \times \mathbb{R}^d\) and therefore separates the two vectors \((t - \eta, 0)\) and \((t + \eta, 0)\) of \( \mathbb{R} \times \mathbb{R}^d\) for \( \eta > 0 \) small enough. In particular, (since \( K^{[d-1]} \) is finite) the set of all \( t' \) such that \( \psi(t', [K^{[d-1]}]) \) is made of isolated values.

Therefore, for some \( c > 0 \) and \( \alpha > 0 \), the complement of \( B((t, 0), \varepsilon^0) \cap F([t - \eta, t + \eta], \text{conv } \sigma^0) \) in \( B((t, 0), \varepsilon) \) has exactly two connected components, which are open and contain \( \{t\} \times (H^-)^0 \cap B((t, 0), \varepsilon) \) and \( \{t\} \times (H^+)^0 \cap B((t, 0), \varepsilon) \) respectively.

We know by (79) and (80) that the sum

\[
\sum_{\alpha \in \mathcal{S}(\sigma, \mathbb{R})} \gamma(\alpha)
\]

is locally constant in each connected component. Then, since it has same value in \( \{t\} \times (H^-)^0 \cap B((t, 0), \varepsilon) \) and \( \{t\} \times (H^+)^0 \cap B((t, 0), \varepsilon) \), it has same value in both connected components.

We have proven that \( \psi(\alpha, [t^*, \text{conv } \sigma]) \) has the same value for \( t^* \in (t - \eta, t) \) and \( t^* \in (t, t + \eta) \), and therefore its regularization \( \psi_\sigma \) is locally constant. This ends the proof of the lemma.

\[ \square \]

G Approximate tangent space computed by PCA

**Lemma 54.** Let \( 0 < 16 \varepsilon \leq \rho < \frac{R}{10} \) and suppose that \( P \subseteq \mathcal{M}^{\varepsilon} \) for some \( 0 \leq \delta < \frac{\epsilon^2}{16} \), \( \mathcal{M} \subseteq \mathbb{P}^{\varepsilon} \) and separation(\( P \)) > \( \eta \varepsilon \) for some \( \eta > 0 \). If, for any point \( p \in P \), \( c_p \) is the center of mass of \( P \cap B(p, \rho) \) and \( V_p \) the linear space spanned by the \( n \) eigenvectors corresponding to the \( n \) largest eigenvalues of the inertia tensor of \( (P \cap B(p, \rho)) - c_p \), then one has:

\[
\angle_{V_p} T_{\pi_M(p)} \mathcal{M} < \Xi_0(\eta, d) \frac{\rho}{R},
\]

where the function \( \Xi_0 \) is polynomial in \( \eta \) and exponential in \( d \).

**Proof.** Consider a frame centered at \( c_p \) with an orthonormal basis of \( \mathbb{R}^N \), whose \( d \) first vectors \( e_1, \ldots, e_d \) belong to \( T_{\pi_M(p)} \mathcal{M} \) and the \( N - d \) last vectors \( e_{d+1}, \ldots, e_N \) to the normal fiber \( N_{\pi_M(p)} \mathcal{M} \). Consider the symmetric \( N \times N \) normalized inertia tensor \( A \) of \( P \cap B(p, \rho) \) in this frame:

\[
A_{ij} \overset{\text{def.}}{=} \frac{1}{\mu P \cap B(p, \rho)} \sum_{p \in P \cap B(p, \rho)} \langle v_i, p - c_p \rangle \langle v_j, p - c_p \rangle.
\]

The symmetric matrix \( A \) decomposes into 4 blocs:

\[
A = \begin{pmatrix} A_{TT} & A_{TN} \\ A_{TN}^t & A_{NN} \end{pmatrix},
\]

where the tangential inertia \( A_{TT} \) is a \( d \times d \) symmetric define positive matrix. Because of the sampling conditions, we claim\(^4\) that there is a constant \( C_{TT} > 0 \) depending only on \( \eta \) and \( d \) such that the smallest eigenvalue of \( A_{TT} \) is at least \( C_{TT} \rho^2 \):

\[
\forall u \in \mathbb{R}^d, \|u\|_1 = 1 \Rightarrow u^t A_{TT} u \geq C_{TT} \rho^2.
\]

Observe that, by Lemma 33, the points in \( P \cap B(p, \rho) \) are at a distance less than \( \frac{(\rho + \delta)^2}{2\varepsilon} \) from the space \( \pi_M(p) + T_{\pi_M(p)} \mathcal{M} \), and therefore so is \( c_p \). Thus, the points in \( P \cap B(p, \rho) \) are at a

\[ ^4 \text{This claim has to be detailed if one wants to provide an explicit expression of the quantity } \Xi_0(\eta, d). \]
distance less than $2\frac{(\rho + \delta)^2}{2R} \leq 2\frac{\rho^2}{R}$ (assuming $(\rho + \delta)^2 \leq 2\rho^2$) from the space $c_p + T_{\pi_M(p)}M$. Then, there are constants $C_{TN}$ and $C_{NN}$ such that the operator norms of $A_{TN}$ and $A_{NN}$ induced by the Euclidean vector norm are upper bounded by:

$$\forall u, v \in \mathbb{R}^d, \|u\| = \|v\| = 1 \Rightarrow v^T A_{TN} u \leq C_{TN} \frac{\rho^2}{R} \rho,$$  \hspace{1cm} (85)

and:

$$\forall u \in \mathbb{R}^d, \|u\| = 1 \Rightarrow u^T A_{NN} u \leq C_{NN} \frac{\rho^2}{R} \rho.$$  \hspace{1cm} (86)

Let $v \in \mathbb{R}^N$ be a unit eigenvector of $A$ with an eigenvalue $\lambda$:

$$A v = \lambda v.$$  \hspace{1cm} (87)

Define $T = \mathbb{R}^d \times \{0\}^{N-d} \subseteq \mathbb{R}^N$ and $N = \{0\}^d \times \mathbb{R}^{N-d} \subseteq \mathbb{R}^N$, corresponding, in the space of coordinates, to $T_{\pi_M(p)}M$ and $N_{\pi_M(p)}M$ respectively.

Let $\theta$ be the angle between $v$ and $T$. There are unit vectors $v_T \in \mathbb{R}^d$ and $v_N \in \mathbb{R}^{N-d}$ such that:

$$v = ((\cos \theta) v_T, (\sin \theta) v_N)^T,$$

where for a matrix $u$, $u^T$ denotes the transpose of $u$. [87] can be rewritten as:

$$\begin{pmatrix} A_{TT} & A_{TN} \\ A_{TN}^T & A_{NN} \end{pmatrix} \begin{pmatrix} (\cos \theta) v_T \\ (\sin \theta) v_N \end{pmatrix} = \lambda \begin{pmatrix} (\cos \theta) v_T \\ (\sin \theta) v_N \end{pmatrix}.$$  \hspace{1cm} (88)

Equivalently:

$$\begin{align*}
(\cos \theta) A_{TT} v_T + (\sin \theta) A_{TN} v_N &= \lambda (\cos \theta) v_T, \\
(\cos \theta) A_{TN}^T v_T + (\sin \theta) A_{NN} v_N &= \lambda (\sin \theta) v_N.
\end{align*}$$  \hspace{1cm} (89) \hspace{1cm} (90)

Multiplying the two equations on the left hand side by $(\sin \theta) v_T^I$ and $(\cos \theta) v_N^I$ respectively, we get:

$$\begin{align*}
(\sin \theta)(\cos \theta) v_T^I A_{TT} v_T + (\sin \theta)^2 v_T^I A_{TN} v_N &= \lambda (\sin \theta)(\cos \theta), \\
(\cos \theta)^2 v_N^I A_{TN}^T v_T + (\cos \theta)(\sin \theta) v_N^I A_{NN} v_N &= \lambda (\cos \theta)(\sin \theta).
\end{align*}$$

Thus,

$$\begin{align*}
(\sin \theta)(\cos \theta) v_T^I A_{TT} v_T + (\sin \theta)^2 v_T^I A_{TN} v_N &= (\cos \theta)^2 v_N^I A_{TN}^T v_T + (\cos \theta)(\sin \theta) v_N^I A_{NN} v_N.
\end{align*}$$

Using [85] and [86], we get:

$$\begin{align*}
(\sin \theta)(\cos \theta) v_T^I A_{TT} v_T &\leq 2C_{TN} \frac{\rho^2}{R} \rho + C_{NN} \frac{\rho^2}{R} \rho, \\
(\sin \theta)(\cos \theta) &\leq 2 \frac{C_{TN} \rho}{C_{TT} R} + C_{NN} \frac{C_{NN} \rho^2}{C_{TT} R^2}.
\end{align*}$$

Using [84], we get:

$$\begin{align*}
(\sin \theta)(\cos \theta) &\leq 2 \frac{C_{TN} \rho}{C_{TT} R} + C_{NN} \frac{C_{NN} \rho^2}{C_{TT} R^2}.
\end{align*}$$

Using $\sin 2\theta = 2 \sin \theta \cos \theta$, we get:

$$\frac{1}{2} \sin 2\theta \leq 2 \frac{C_{TN} \rho}{C_{TT} R} + C_{NN} \frac{C_{NN} \rho^2}{C_{TT} R^2} = O \left( \frac{\rho}{R} \right).$$  \hspace{1cm} (91)
Thus,
\[
\theta \in [0, t] \cup \left[ \frac{\pi}{2} - t, \frac{\pi}{2} \right],
\]
with
\[
t = \frac{1}{2} \arcsin^2 \left( 2 \frac{C_{T N} \rho}{C_{T T} \bar{R}} + C_{N N} \frac{\rho^2}{C_{T T} \bar{R}^2} \right) = O \left( \frac{\rho}{\bar{R}} \right).
\]

This means that the eigenvectors of \( A \) form an angle less than \( O \left( \frac{\rho}{\bar{R}} \right) \) with either \( T \) or \( N \). For the non-generic situation of a multiple eigenvalue, one chooses arbitrarily the vectors of an orthogonal basis of the corresponding eigenspace. Since no more than \( d \) pairwise orthogonal vectors can make a small angle with the \( d \)-dimensional space \( T \), and the same holds for the \((N - d)\)-dimensional space \( T \), we know that \( d \) eigenvectors form an angle \( O \left( \frac{\rho}{\bar{R}} \right) \) with \( T \) and the \((N - d)\) others, an angle \( O \left( \frac{\rho}{\bar{R}} \right) \) with \( N \). Multiplying the left hand side of (89) by \( v_T^i \), and the left hand side of (90) by \( v_T^j \), we get:
\[
(\cos \theta)v_T^i A_{TT} v_T + (\sin \theta)v_T^i A_{TN} v_N = \lambda (\cos \theta),
\]
\[
(\cos \theta)v_T^j A_{TN} v_T + (\sin \theta)v_T^j A_{NN} v_N = \lambda (\sin \theta).
\]

When the angle between the eigenvector \( v \) and the space \( T \) in is \( O \left( \frac{\rho}{\bar{R}} \right) \), then \(|1 - \cos \theta| = O \left( \frac{\rho}{\bar{R}} \right)^2 \) and \(|\sin \theta| = O \left( \frac{\rho}{\bar{R}} \right) \). The first equation implies that \( \lambda \) approximately equals \( v_T^i A_{TT} v_T \geq C_{TT} \rho^2 \).

When the angle between the eigenvector \( v \) and the space \( N \) is in \( O \left( \frac{\rho}{\bar{R}} \right) \), the second equation implies that \( \lambda = O \left( \frac{\rho}{\bar{R}} \right)^2 \), which is smaller than \( C_{TT} \rho^2 \) for \( \frac{\rho}{\bar{R}} \) small enough.

So far we have proven that the \( d \) orthonormal eigenvectors \( v_1, \ldots, v_d \) corresponding to the \( d \) largest eigenvalues of \( A \) form an angle with \( T_{\pi, \mathcal{M}(p)} \mathcal{M} \) that is upper bounded by \( C \left( \frac{\rho}{\bar{R}} \right) \), for some constant \( C \) that depends only upon \( d \) and \( \eta \). For any unit vector \( u \), its angle with \( T_{\pi, \mathcal{M}(p)} \mathcal{M} \) satisfies:
\[
\sin \angle u, T_{\pi, \mathcal{M}(p)} \mathcal{M} = \| u - \pi T_{\pi, \mathcal{M}(p)} \mathcal{M} u \|.
\]

Any unit vector \( u = \sum_{i=1}^{d} a_i v_i \) in the \( d \)-space spanned by \( v_1, \ldots, v_d \) satisfies:
\[
\sin \angle u, T_{\pi, \mathcal{M}(p)} \mathcal{M} = \left\| \sum_{i=1}^{d} a_i v_i - \pi T_{\pi, \mathcal{M}(p)} \mathcal{M} \left( \sum_{i=1}^{d} a_i v_i \right) \right\|
\]
\[
= \left\| \sum_{i=1}^{d} a_i \left( v_i - \pi T_{\pi, \mathcal{M}(p)} \mathcal{M} v_i \right) \right\|
\]
\[
\leq \sum_{i=1}^{d} |a_i| \left\| v_i - \pi T_{\pi, \mathcal{M}(p)} \mathcal{M} v_i \right\|
\]
\[
\leq \sum_{i=1}^{d} |a_i| C \left( \frac{\rho}{\bar{R}} \right) \leq \sqrt{d} C \left( \frac{\rho}{\bar{R}} \right),
\]

since \( \sum_{i=1}^{d} a_i^2 = 1 \), we conclude that \( \sum_{i=1}^{d} |a_i| \leq \sqrt{d} \). \( \square \)