HOMOLOGY OF GENERALIZED STEINBERG VARIETIES
AND WEYL GROUP INVARIANTS

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ABSTRACT. Let $G$ be a complex, connected, reductive algebraic group. In this paper we show analogues of the computations by Borho and MacPherson of the invariants and anti-invariants of the cohomology of the Springer fibres of the cone of nilpotent elements, $N$, of $\text{Lie}(G)$ for the Steinberg variety $Z$ of triples.

Using a general specialization argument we show that for a parabolic subgroup $W_P \times W_Q$ of $W \times W$ the space of $W_P \times W_Q$-invariants and the space of $W_P \times W_Q$-anti-invariants of $H^4_\ast(Z)$ are isomorphic to the top Borel-Moore homology groups of certain generalized Steinberg varieties introduced by Douglass and Röhrle (2004).

The rational group algebra of the Weyl group $W$ of $G$ is isomorphic to the opposite of the top Borel-Moore homology $H^4_\ast(Z)$ of $Z$, where $2n = \dim N$.

Suppose $W_P \times W_Q$ is a parabolic subgroup of $W \times W$. We show that the space of $W_P \times W_Q$-invariants of $H^4_\ast(Z)$ is $\epsilon_P QW \epsilon_P$, where $\epsilon_P$ is the idempotent in the group algebra of $W_P$ affording the trivial representation of $W_P$ and $\epsilon_Q$ is defined similarly. We also show that the space of $W_P \times W_Q$-anti-invariants of $H^4_\ast(Z)$ is $\epsilon_Q QW \epsilon_P$, where $\epsilon_P$ is the idempotent in the group algebra of $W_P$ affording the sign representation of $W_P$ and $\epsilon_Q$ is defined similarly.

1. Introduction

Suppose $G$ is a complex, reductive algebraic group and $B$ is the variety of Borel subgroups of $G$. Then $B$ is a smooth, projective variety. Let $T$ be a maximal torus in $G$ and choose a Borel subgroup, $B$, of $G$ containing $T$. Let $W = N_G(T)/T$ be the Weyl group of $(G, T)$. Then $W$ acts on $G/T$ on the right, the natural projection $G/T \to G/B$ has the structure of a vector bundle, and the varieties $G/B$ and $B$ are isomorphic. Thus, $W$ acts on the singular cohomology with rational coefficients of $B$ via the isomorphisms $H^\ast(B) \cong H^\ast(G/B) \cong H^\ast(G/T)$.

Now suppose $P$ is a parabolic subgroup of $G$ containing $B$ and $P$ is the variety of $G$-conjugates of $P$. Then $P$ is again a smooth, projective variety, and it is a classical result that $H^\ast(P)$ is isomorphic to the space of $W_P$-invariants in $H^\ast(B)$ where $W_P = N_P(T)/T$ is the Weyl group of $(P, T)$ (see [9]).
Borho and MacPherson have generalized this result to fixed point subvarieties of $B$ as follows. Let $g$ be the Lie algebra of $G$ and $N$ the cone of nilpotent elements in $g$. There is a moment map, $\mu_0: T^*B \to N$, where $T^*B$ is the cotangent bundle of $B$. For $x$ in $N$, set $B_x = \mu_0^{-1}(x)$. The variety $B_x$ may be identified with the variety of all Borel subgroups of $G$, whose Lie algebra contains $x$. The varieties $B_x$ vary from a point, when $x$ is regular, to $B$, when $x = 0$. The moment map factors as $\mu_0 = \eta_0 \circ \xi_0$, where $\xi_0^{-1}(x)$ may be identified with the variety of all subgroups in $P$ whose Lie algebra contains $x$. There is also a moment map $\mu_0^P$ from the cotangent bundle of $P$ to $N$, and $(\mu_0^P)^{-1}(x)$ may be identified with the variety of all subgroups in $P$ whose Lie algebras contain $x$ in their nilradical. Set $P^0_x = (\mu_0^P)^{-1}(x)$.

Springer [17] has defined an action of $W$ on $H^*(B_x)$, and Borho and MacPherson [4] have shown that if $W$ acts on $H^*(B_x)$ by the tensor product of Springer’s action with the sign representation, then:

1. $H^*(P_x)$ is isomorphic to the space of $W_P$-invariants in $H^*(B_x)$.
2. $H^*(P^0_x)$ is isomorphic to the subspace of $H^*(B_x)$ on which $W_P$ acts as the sign representation.

In a different direction, the Steinberg variety of $G$ is the fibred product $T^*B \times_N T^*B$ which may be identified with the closed subvariety

$$Z = \{ (x, B', B'') \in N \times B \times B \mid x \in \text{Lie}(B') \cap \text{Lie}(B'') \}$$

of $N \times B \times B$. Kazhdan and Lusztig [12] have defined an action of $W \times W$ on $H_*(Z)$, the rational, Borel-Moore homology of $Z$, and they showed that the representation of $W \times W$ on the top-dimensional homology group of $Z$, $H_{4n}(Z)$, where $n = \dim B$, is equivalent to the two-sided regular representation of $W$.

Tanisaki [19] and, more recently, Chriss and Ginzburg [4] have strengthened the connection between $H_*(Z)$ and $W$ by defining a $Q$-algebra structure on $H_*(Z)$ so that $H_i(Z) \cdot H_j(Z) \subseteq H_{i+j-4n}(Z)$ and $H_{4n}(Z)^{op}$ is isomorphic to the group algebra $Q[W]$.

In this paper we prove analogs of (1.1) and (1.2) for the Steinberg variety.

Suppose $Q$ is a parabolic subgroup of $G$ containing $B$ (a special case is when $Q = P$), $W_Q$ is the Weyl group of $(Q, T)$, and $Q$ is the conjugacy class of parabolic subgroups that contains $Q$. In [5] we defined generalized Steinberg varieties

$$X^{P, Q} = \{ (x, P', Q') \in N \times P \times Q \mid x \in \text{Lie}(P') \cap \text{Lie}(Q') \}$$

and

$$Y^{P, Q} = \{ (x, P', Q') \in N \times P \times Q \mid x \in \text{Lie}(U_{P'}) \cap \text{Lie}(U_{Q'}) \},$$

where $U_{P'}$ and $U_{Q'}$ are the unipotent radicals of $P'$ and $Q'$, respectively. It was shown in [5] that $X^{P, Q}$ is purely $2n$-dimensional and $Y^{P, Q}$ is purely $(2n-f)$-dimensional, where $f = \dim P/B + \dim Q/B$.

The first analogs of (1.1) and (1.2) are:

1. $H_{4n}(X^{P, Q})$ is isomorphic to the space of $W_P \times W_Q$-invariants in $H_{4n}(Z)$.
2. $H_{4n-2f}(Y^{P, Q})$ is isomorphic to the subspace of $H_{4n}(Z)$ on which $W_P \times W_Q$ acts as the sign representation.

We prove both of these statements in this paper.
More generally we consider the following statements:

(1.1') $H_\bullet(X^{P,Q})$ is isomorphic to the space of $W_P \times W_Q$-invariants in $H_\bullet(Z)$.

(1.2') $H_\bullet(Y^{P,Q})$ is isomorphic to the subspace of $H_\bullet(Z)$ on which $W_P \times W_Q$ acts as the sign representation.

In §3 we prove a general specialization result, in the spirit of [3], which has (1.1') as a special case. Obviously (1.1') follows immediately from (1.1''). It seems likely that (1.2') is true, but our proof of (1.2') uses dimension computations from [4] that are not available for $H_i(Y^{P,Q})$ for $i < 4n - 2f$.

In §4 we prove a general equivariance result in the spirit of [4]. A special case of this result is that there is a $W \times W$-equivariant isomorphism

$$\Ext^{4n-\bullet}_N(R(\mu_0),L\mu_0,\mathcal{U}_Q) \cong H_{4n}(Z)^{\text{op}}.$$  

It has been shown by Borho and MacPherson [2] that the $\mathbb{Q}$-algebras $\mathbb{Q}W$ and $\End_N(R(\mu_0),\mathcal{U}_Q)$ are isomorphic, and Chriss and Ginzburg [4, §8.6] have shown that

$$\Ext^{4n-\bullet}_N(R(\mu_0),L\mu_0,\mathcal{U}_Q) \cong H_{4n}(Z)^{\text{op}}.$$  

Thus, taking $\bullet = 4n$ we get $W \times W$-equivariant, $\mathbb{Q}$-algebra isomorphisms

$$\mathbb{Q}W \cong \End_N(R(\mu_0),\mathcal{U}_Q) \cong H_{4n}(Z)^{\text{op}},$$

where $W \times W$ acts on $\mathbb{Q}W$ by $(w, w') \cdot v = w'vw^{-1}$ for $w$ and $w'$ in $W$ and for $v$ in $\mathbb{Q}W$.

Using the isomorphism between $\mathbb{Q}W$ and $H_{4n}(Z)^{\text{op}}$ we may formulate (1.1') and (1.2') in terms of the group algebra of $W$:

(1.1'') If $e_P$ is the primitive idempotent in $\mathbb{Q}W_P$ corresponding to the trivial representation of $W_P$ and $e_Q$ is defined similarly, then $H_{4n}(X^{P,Q})$ is isomorphic to the subspace $e_Q\mathbb{Q}We_P$ of $\mathbb{Q}W$.

(1.2'') If $e_P$ is the primitive idempotent in $\mathbb{Q}W_P$ corresponding to the sign representation of $W_P$ and $e_Q$ is defined similarly, then $H_{4n-2f}(Y^{P,Q})$ is isomorphic to the subspace $e_Q\mathbb{Q}W\epsilon_P$ of $\mathbb{Q}W$.

In [5] we defined generalized Steinberg varieties $X_{c,d}^{P,Q}$. Statements (1.1'') and (1.2'') together with computations in some special cases suggest that the Borel-Moore homology of a general $X_{c,d}^{P,Q}$ is given as follows.

A generalized Steinberg variety, $X_{c,d}^{P,Q}$, depends on a pair of nilpotent adjoint orbits in $\text{Lie}(P/U_P)$ and $\text{Lie}(Q/U_Q)$, respectively. We will not recall the precise definition here but instead refer the interested reader to [5]. In turn, these nilpotent orbits determine irreducible representations of $W_P$ and $W_Q$, say $\rho_c$ and $\rho_d$ respectively, corresponding to the trivial representation of the component groups of the orbits via the Springer correspondence as defined in [2]. Let $e_c$ and $e_d$ denote primitive idempotents in $\mathbb{Q}W_P$ and $\mathbb{Q}W_Q$ affording $\rho_c$ and $\rho_d$, respectively. In [5, Corollary 2.6] we have given a sharp upper bound, $\delta_{c,d}^{P,Q}$, for the dimension of $X_{c,d}^{P,Q}$. 
We conjecture that
\[
H_{2e_{c,d}}(X_{e_{c,d}}^P, Q) \text{ is isomorphic to } e_d QW e_c.
\]

More generally, we conjecture that
\[
H_\bullet(X_{e_{c,d}}^P, Q) \text{ is isomorphic to } e_d H_\bullet(Z) e_c \text{ where we consider } e_c \text{ and } e_d \text{ in } H_\bullet(Z) \text{ via the isomorphism } QW \cong H_{4n}(Z)^{op}.
\]

In much of this paper (§2 – §4 and the Appendix) we are concerned with general sheaf theory. Most of our conclusions about the Borel-Moore homology of generalized Steinberg varieties are straightforward applications of more general results. The main theorems, which are described briefly below, are the specialization results, Theorem 3.1.2 and Corollary 3.5.2 and the equivariance results discussed in §4.1. We hope these general results will have applications outside the realm of generalized Steinberg varieties.

Our computation of the Borel-Moore homology of \(X^P, Q\) and \(Y^P, Q\) is given in §5. Although the results depend on facts proved in §3 and §4, this section may be read independently of the other sections.

The rest of this paper is organized as follows.

In §2 we fix notation and collect some sheaf-theoretic results that are used in subsequent sections for which we could not find a suitable reference.

In §3 we give an axiomatic approach to a specialization result which allows us to identify a direct image map in Borel-Moore homology with the averaging map for a group action. The basic idea goes back to Lusztig [14] and Borho-MacPherson [3]. A result which is similar in spirit, but which is in a sense dual to our result, and does not apply to Borel-Moore homology, has been used by Spaltenstein in [16]. Statement (1.1′′) is a straightforward consequence of the main result in this section, Theorem 3.1.2.

In §4 we continue the axiomatic approach from §3 and prove an equivariance result for two-sided group actions that is key for our application to generalized Steinberg varieties. The crucial result is Theorem 4.4.1 which, when applied to the Steinberg variety, implies that there is a \(W \times W\)-equivariant isomorphism between \(\text{Ext}_N^{4n-\bullet}(R(\mu_0)_! Q_{T-B}, R(\mu_0)_! Q_{T-B})\) and \(H_\bullet(Z)\). This result is similar in spirit to the results in [4, §8.6].

In §5 we specialize the results in the previous sections to the case of generalized Steinberg varieties and prove (1.1′′), (1.2′), (1.1″), and (1.2″).

In the Appendix, we prove two results about the natural transformation \(\xi^* \rightarrow \xi^{[2l]}\) for a morphism \(\xi: X \rightarrow Y\), where \(l = \dim Y - \dim X\). These results are needed in the proof of Theorem 4.4.1.

For simplicity, in this paper we have chosen to work with complex algebraic groups and Borel-Moore homology, but our arguments are essentially categorical and make sense in the setting of algebraic groups over arbitrary algebraically closed fields and \(l\)-adic cohomology.

2. Preliminaries

2.1. First, we fix some assumptions and notation that will be used throughout the rest of this paper. The reader is urged to skim this section quickly to become familiar with the notation and refer back to the results used in the sequel when necessary. The main references for sheaf-theoretic notation and results used in this
paper are the article [1] by Borel (with the collaboration of N. Spaltenstein) and
the book [11] by Kashiwara and Shapira.

The topological spaces we consider are complex algebraic varieties endowed with
their Euclidean topologies, although many arguments apply as well to pseudoman-
ifolds as defined in [8, §1.1].

The “dimension” of a space always means its dimension as a complex algebraic
variety.

If \( X \) is a variety, then \( D(X) \) denotes the derived category of the category of
sheaves of \( \mathbb{Q} \)-vector spaces on \( X \), \( D^b(X) \) denotes the full subcategory of \( D(X) \)
consisting of complexes with bounded cohomology, and \( D^b_c(X) \) denotes the full
subcategory of \( D^b(X) \) consisting of complexes with constructible cohomology.

For complexes \( A \) and \( B \) in \( D(X) \), \( \text{Ext}^j(A, B) \) is defined to be \( H^j(\mathcal{R}\text{Hom}(A, B)) \),
and it is shown in [1, §5.17] that \( \text{Ext}^j(A, B) = \text{Hom}_{D(X)}(A, B[j]) \). Define

\[
\text{Ext}^j_X(A, B) = \text{Hom}_{D(X)}(A, B[j]).
\]

Since \( D^b_c(X) \) is a full subcategory of \( D(X) \), if \( A \) and \( B \) are complexes in \( D^b_c(X) \),
then \( \text{Hom}_{D^b_c(X)}(A, B) = \text{Hom}_{D(X)}(A, B) \). To simplify the notation, we denote both
of these spaces by \( \text{Hom}_X(A, B) \). Also, we denote the complex \( A \overset{L}{\otimes} B \) simply by
\( A \otimes B \).

The constant sheaf on \( X \), considered as a complex concentrated in degree 0, is
denoted by \( \mathbb{Q}_X \), and the dualizing complex of \( X \) is denoted by \( \mathbb{D}_X \).

If \( A \) is a complex of sheaves of \( \mathbb{Q} \)-vector spaces on \( X \), then \( A^\vee = \mathcal{R}\text{Hom}(A, \mathbb{D}_X) \)
denotes the Verdier dual of \( A \). There is a canonical isomorphism between \( \mathbb{D}_X \) and \( \mathbb{Q}_X^\vee \) that we denote by \( dc_X \), so

\[
dc_X : \mathbb{D}_X \cong \mathbb{Q}_X^\vee.
\]

If \( f : A \to B \) is a morphism in \( D(X) \), and \( C \) is a complex in \( D(X) \), then \( f \)
duces natural morphisms in \( D(X) \),

\[
f^\sharp : \mathcal{R}\text{Hom}(B, C) \longrightarrow \mathcal{R}\text{Hom}(A, C) \quad \text{and} \quad f_\sharp : \mathcal{R}\text{Hom}(C, A) \longrightarrow \mathcal{R}\text{Hom}(C, B).
\]

In the special case when \( C = \mathbb{D}_X \), we have \( \mathcal{R}\text{Hom}(A, C) = A^\vee \) and \( \mathcal{R}\text{Hom}(B, C) = B^\vee \). We usually write \( f^\vee \) instead of \( f^\sharp \) in this case, so \( f^\vee : B^\vee \to A^\vee \) is the Verdier dual of \( f \).

Similarly, \( f \) induces natural linear transformations

\[
f^\sharp : \text{Ext}^*_X(B, C) \longrightarrow \text{Ext}^*_X(A, C) \quad \text{and} \quad f_\sharp : \text{Ext}^*_X(C, A) \longrightarrow \text{Ext}^*_X(C, B).
\]

The \( j \)th Borel-Moore homology group of a locally compact, Hausdorff topological
space, \( X \), has several equivalent definitions (see [4, §2.6]). In this paper we use the
canonical isomorphisms

\[
H^{-j}(X, \mathbb{D}_X) \cong H^{-j}(X, \mathcal{R}\text{Hom}(\mathbb{Q}_X, \mathbb{D}_X)) \cong \text{Ext}^{-j}_X(\mathbb{Q}_X, \mathbb{D}_X),
\]

where \( H^{-j}(X, \mathbb{D}_X) \) is the hypercohomology of \( X \) with coefficients in \( \mathbb{D}_X \), and we
\textit{define} the \( j \)th Borel-Moore homology group of \( X \) by

\[
H_j(X) = \text{Ext}^{-j}_X(\mathbb{Q}_X, \mathbb{D}_X).
\]
2. Now suppose that $\xi: X \to Y$ is a morphism of varieties. Then $\xi$ determines natural isomorphisms
\[ \phi_\xi: RHom(R\xi A, B) \longrightarrow R\xi_s RHom(A, \xi^1 B) \]
and
\[ \text{nat}_\xi: \xi^1 RHom(B, C) \longrightarrow RHom(\xi^* B, \xi^* C) \]
for $A$ in $D(X)$ and $B$ and $C$ in $D(Y)$.

There are canonical isomorphisms
\[ \alpha_\xi: \xi^* Q_X \longrightarrow Q_X \quad \text{and} \quad \beta_\xi: D_X \longrightarrow \xi^! D_Y \]
in the category of sheaves on $X$ and $D^b_{\xi}(X)$, respectively. It is straightforward to check that $\alpha_\xi$ and $\beta_\xi$ have the following properties:

(2.2.1) The maps $\beta_\xi: D_X \to \xi^! D_Y$ and
\[ (\beta_\xi)_2: RHom(\xi^* Q_Y, D_X) \longrightarrow RHom(\xi^* Q_Y, \xi^! D_Y) \]
are related by $(\beta_\xi)_2 \circ \alpha_\xi \circ dc_X = \text{nat}_\xi \circ \xi^!(dc_Y) \circ \beta_\xi$ where $dc_X$ and $dc_Y$ are as in [2.1]

(2.2.2) If $\eta: Y \to Z$ is another morphism of varieties, then $\alpha_{\eta \xi} = \alpha_\xi \circ \xi^*(\alpha_\eta)$ and $\beta_{\eta \xi} = \xi^!(\beta_\eta) \circ \beta_\xi$.

2.3. Let $\delta: X \to X \times X$ be the diagonal embedding and let $p$ and $q$ denote the projections of $X \times X$ on the first and second factors, respectively. In [1, Theorem 10.25] it is shown that there is a natural isomorphism
\[ \lambda: A^\vee \boxtimes B \longrightarrow RHom(p^* A, q^1 B) \]
in $D^b_{\xi}(X \times X)$. It then follows that $\text{nat}_\delta \circ \delta^!(\lambda)$ is a natural isomorphism between $\delta^!(A^\vee \boxtimes B)$ and $RHom(A, B)$.

**Proposition 2.3.1.** Suppose $A$ and $B$ are in $D^b_{\xi}(X)$, $u: A \to A$ is an endomorphism of $A$, and $v: B \to B$ is an endomorphism of $B$. Then the diagram
\[ \begin{array}{ccc}
A^\vee \boxtimes B & \xrightarrow{u^\vee \boxtimes v} & A^\vee \boxtimes B \\
\downarrow \lambda & & \downarrow \lambda \\
RHom(p^* A, q^1 B) & \xrightarrow{(p^* u)^{1} \circ (q^1 v)} & RHom(p^* A, q^1 B)
\end{array} \]
commutes.

**Proof.** By definition $A^\vee \boxtimes B = p^* RHom(A, D_X) \otimes q^* B$ and $u^\vee \boxtimes v = p^*(u^2) \otimes q^* v$.

In the special case when $A$ is the constant sheaf, the isomorphism $\lambda$ may be identified with a natural isomorphism $\lambda': p^* D_X \otimes q^* B \to q^1 B$ as in [1, ¶10.24]. Then for an arbitrary $A$, the isomorphism $\lambda$ is defined as the composition $\lambda'_2 \circ h_2 \circ h_1$, where $h_1$ and $h_2$ are the natural maps
\[ h_1: p^* RHom(A, D_X) \otimes q^* B \longrightarrow RHom(p^* A, p^* D_X) \otimes q^* B \]
and
\[ h_2: RHom(p^* A, p^* D_X) \otimes q^* B \longrightarrow RHom(p^* A, q^* B). \]
It is straightforward to check that
\[ h_1 \circ (p^* (u^*) \otimes q^* v) = ((p^* u)^*) \otimes q^* v) \circ h_1 \]
and
\[ h_2 \circ ((p^* u)^*) \otimes q^* v) = ((p^* u)^2 \circ (id \otimes q^* v)_2) \circ h_2. \]
Moreover, it follows from the naturality of \( \lambda' \) that
\[ \lambda'_2 \circ ((p^* u)^2 \circ (id \otimes q^* v)_2) = ((p^* u)^2 \circ (q^1 v)_2) \circ \lambda'_2. \]
Therefore \( \lambda \circ (u^\vee \boxtimes v) = ((p^* u)^2 \circ (q^1 v)_2) \circ \lambda, \) as desired. \( \square \)

**Corollary 2.3.2.** With the preceding notation, the diagram

\[
\begin{array}{ccc}
\delta^! (A^\vee \boxtimes B) & \overset{\delta^! (u^\vee \boxtimes v)}{\longrightarrow} & \delta^! (A^\vee \boxtimes B) \\
\downarrow \text{nats} \circ \delta^! (\lambda) & & \downarrow \text{nats} \circ \delta^! (\lambda) \\
\text{RHom}(A, B) & \overset{u^* \circ v^*}{\longrightarrow} & \text{RHom}(A, B)
\end{array}
\]

commutes.

**Proof.** We have just seen that \( \lambda \circ (u^\vee \boxtimes v) = ((p^* u)^2 \circ (q^1 v)_2) \circ \lambda, \) so
\[ \delta^! (\lambda) \circ \delta^! (u^\vee \boxtimes v) = \delta^! (\delta^* (p^* u)^2 \circ (q^1 v)_2) \circ \delta^! (\lambda). \]
It is straightforward to check that
\[ \text{nats} \circ \delta^! (\delta^* (p^* u)^2 \circ (q^1 v)_2) = \text{nats} \circ \delta^! (\delta^* (p^* u)^2 \circ (q^1 v)_2), \]
so
\[ \text{nats} \circ \delta^! (\lambda) \circ \delta^! (u^\vee \boxtimes v) = \text{nats} \circ \delta^! (\delta^* (p^* u)^2 \circ (q^1 v)_2) \circ \delta^! (\lambda) = (u^* \circ v^* \circ \text{nats}) \circ \delta^! (\lambda). \]
This proves the corollary. \( \square \)

It is shown in [11] §2.6 that for \( A, B, \) and \( C \) in \( D(X) \) there is a natural isomorphism \( \text{Hom}_X(C \otimes A, B) \cong \text{Hom}_X(C, \text{RHom}(A, B)). \) It follows that there is an isomorphism of graded vector spaces \( \text{Ext}^\bullet_X(C \otimes A, B) \cong \text{Ext}^\bullet_X(C, \text{RHom}(A, B)). \)

Taking \( C = \mathbb{Q}_X \) and using the canonical isomorphism \( \mathbb{Q}_X \otimes A \cong A, \) we get a natural isomorphism of graded vector spaces
\[ \text{can} : \text{Ext}^\bullet_X(A, B) \cong \text{Ext}^\bullet_X(\mathbb{Q}_X, \text{RHom}(A, B)). \]

The next proposition follows from the naturality of \( \text{can}. \)

**Proposition 2.3.3.** Suppose \( A \) and \( B \) are in \( D(X), \) \( u : A \to A \) is an endomorphism of \( A, \) and \( v : B \to B \) is an endomorphism of \( B. \) Then the diagram

\[
\begin{array}{ccc}
\text{Ext}^\bullet_X(A, B) & \overset{\text{can}}{\longrightarrow} & \text{Ext}^\bullet_X(\mathbb{Q}_X, \text{RHom}(A, B)) \\
\downarrow \text{u}^* \circ v^* & & \downarrow (u^* \circ v^*)_2 \\
\text{Ext}^\bullet_X(A, B) & \overset{\text{can}}{\longrightarrow} & \text{Ext}^\bullet_X(\mathbb{Q}_X, \text{RHom}(A, B))
\end{array}
\]

commutes.
2.4. As in \(2.5\), \(\xi: X \to Y\) is a morphism of varieties. The functors \(\xi^*\) and \(R\xi_*\) form an adjoint pair. We denote by
\[
\Psi\xi: \text{Hom}_X(\xi^*B, A) \xrightarrow{\cong} \text{Hom}_Y(B, R\xi_*A)
\]
the adjunction mapping for \(A\) in \(D(X)\) and \(B\) in \(D(Y)\) and by \(\chi^\xi\) the unit of the adjunction. Although \(\chi^\xi\) is a natural transformation, \(\chi^\xi_B: B \to R\xi_*\xi^*B\), in order to simplify the notation we omit the subscript and just write \(\chi^\xi\) instead of \(\chi^\xi_B\). The appropriate subscript is always uniquely determined by the context, and so this should cause no confusion.

Similarly, the functors \(R\xi^*\) and \(\xi^!\) form an adjoint pair. We denote by
\[
\Phi\xi: \text{Hom}_Y(R\xi^*A, B) \xrightarrow{\cong} \text{Hom}_X(A, \xi^!B)
\]
the adjunction mapping and by \(\epsilon^\xi\) the counit of the adjunction.

We need the following identities for morphisms \(f: R\xi^*A \to B\) and \(k: B \to B'\) in \(D(Y)\) and \(g: A \to \xi^!B\) and \(h: A' \to A\) in \(D(X)\) (see \([13\ IV.1]\)):
\[
\begin{align*}
(2.4.1) & \quad \epsilon^\xi = \Phi\xi^{-1}(id), \quad \Phi\xi^{-1}(g) = \epsilon^\xi \circ R\xi(g), \\
(2.4.2) & \quad \Phi\xi(f \circ R\xi(h)) = \Phi\xi(f) \circ h, \quad \Phi\xi(k \circ f) = \xi^!(k) \circ \Phi\xi(f), \\
(2.4.3) & \quad \Phi\xi^{-1}(g \circ h) = \Phi\xi^{-1}(g) \circ R\xi(h), \quad \Phi\xi^{-1}(\xi^!(k) \circ g) = k \circ \Phi\xi^{-1}(g).
\end{align*}
\]

Verdier duality defines contravariant automorphisms of the subcategories \(D^b(X)\) and \(D^b(Y)\) of \(D(X)\) and \(D(Y)\), respectively. In these subcategories we can use standard identities for Verdier duality in \([11\ §10]\) to express \(\Phi\xi\) and \(\epsilon^\xi\) in terms of \(\Psi\xi\) and \(\chi^\xi\) as follows.

Suppose \(A\) is in \(D^b(X)\), \(B\) is in \(D^b(Y)\), and \(f\) is in \(\text{Hom}_Y(R\xi^*A, B)\). Then \(\Psi\xi^{-1}(f^\vee)^\vee\) is in \(\text{Hom}_X(A, \xi^!B)\). Clearly, \(f \to \Psi\xi^{-1}(f^\vee)^\vee\) is natural in \(A\) and \(B\) and so we may define \(\Phi\xi\) by \(\Phi\xi(f) = \Psi\xi^{-1}(f^\vee)^\vee\).

Similarly, taking the Verdier dual of \(\chi^\xi_B: B \to R\xi_*\xi^*B\) we get \((\chi^\xi_B)^\vee: R\xi^*\xi^!B^\vee \to B^\vee\) and we conclude that \((\chi^\xi_B)^\vee = \epsilon^\xi_{B^\vee}^\vee\).

2.5. Next, consider a cartesian square
\[
\begin{array}{ccc}
X' & \xrightarrow{\eta} & X \\
\downarrow{\xi} & & \downarrow{\xi} \\
Y' & \xrightarrow{j} & Y
\end{array}
\]
where \(\xi\) and \(\eta\) are proper morphisms. Then \(\Psi\xi^{-1}(R\xi_*\xi^!(\chi^i)): j^*R\xi_* \to R\eta_*i^*\) is a natural equivalence of functors from \(D(X)\) to \(D(Y)\). Restricting to \(D^b(X)\) and \(D^b(Y)\) and taking the Verdier dual we conclude that \(\Psi\xi^{-1}(R\xi_*\xi^!(\chi^i))^\vee: R\eta^!i^\vee \to j^!R\xi^!\) is a natural equivalence. It follows from the discussion in \([2.4]\) above that
\[
\Psi\xi^{-1}(R\xi_*\xi^!(\chi^i))^\vee = \Phi\xi_j((R\xi_*\xi^!(\chi^i))^\vee) = \Phi\xi_j(R\xi^!(\chi^!(\xi^i))^\vee) = \Phi\xi_j((R\xi^!(\xi^i))^\vee).
\]
Define
\[
bc_{\eta,i}: j^! \circ R\xi^! \xrightarrow{\cong} R\eta^! \circ i^\vee \quad \text{by} \quad bc_{\eta,i} = \Phi\xi_j(R\xi^!(\xi^i))^\vee.
\]
Then \(bc_{\eta,i}\) is a natural equivalence and \(bc_{\eta,i}^{-1} = \Phi\xi_j(R\xi^!(\xi^i))^{-1}\).
Lemma 2.5.2. Suppose that in diagram (2.5.1) the maps $i$ and $j$ are open embeddings. Then, for $A$ in $D^b_c(X)$ and $B$ in $D^b_c(Y)$, the diagram

$$
j^! R\rho\mathcal{H}om(A, \xi^! B) \xrightarrow{j^!(\phi^{-1}_\eta)} j^! R\rho\mathcal{H}om(R\xi_i A, B) \xrightarrow{\text{nat}_j} R\rho\mathcal{H}om(j^* R\xi_i A, j^! B)
$$

commutes in $D^b_c(Y')$, where $bc = bc_{\eta,i}$.

Proof. Since $i$ and $j$ are open embeddings, we have $i^! = i^*$ and $j^! = j^*$, so the statement of the lemma makes sense and is easily proved for sheaves on $X$ and $Y$. The result then follows using standard arguments for derived functors. □

2.6. If $U$ is a smooth, open, dense subvariety of $X$, and $L$ is a local system on $U$, then we denote the intersection complex, as in [1], middle perversity, by $IC(X, L)$. It is a complex of sheaves in $D^b_c(X)$. It is shown in [8, Theorem 3.5] that $IC$ defines a fully faithful functor from the category of local systems on $U$ to $D^b_c(X)$.

Notice that if we start with a complex, $A$, on an open, dense subvariety of $X$ with $H^p(A) = 0$ for $p \neq 0$, then we may construct a complex $IC(X, A)$ as in [1] §2.2] starting with $A$. The complexes $IC(X, A)$ and $IC(X, H^0(A))$ are isomorphic in $D^b_c(X)$.

3. Specialization

3.1. In this section we axiomatize a specialization argument that allows us to compute invariants in Borel-Moore homology. There are various schemes that allow one to use generic information to prove (co)-homological results about special fibres, or more generally closed subvarieties (see [7, 4, 15]). Our approach, which is based on an idea of Lusztig in [14] that was generalized by Borho and MacPherson [3], is to use intersection complexes of local systems on open, dense subvarieties of a variety, $N$, to obtain information about the Borel-Moore homology groups of a closed subvariety, $N_0$, of $N$.

We start with what we call the “basic commutative diagram” of morphisms of complex, algebraic varieties consisting of cartesian squares:

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\eta_0} & P_0 & \xrightarrow{\xi_0} & N_0 \\
\downarrow j_M & & \downarrow j_P & & \downarrow j_N \\
M & \xrightarrow{\eta} & P & \xrightarrow{\xi} & N \\
\downarrow i_M & & \downarrow i_P & & \downarrow i_N \\
M_r & \xrightarrow{\eta_r} & P_r & \xrightarrow{\xi_r} & N_r
\end{array}
$$

Define

$$\mu = \xi_0 \eta_0, \quad \mu_r = \xi_r \eta_r, \quad \text{and} \quad \mu_0 = \xi_0 \eta_0.$$

We assume that this basic commutative diagram has the following properties:

D1 The varieties $M, P$, and $N$ are purely $d$-dimensional.

D2 The varieties $M, P$, and $N_r$ are rational homology manifolds.
Theorem 3.1.2. The morphisms $\xi$ and $\mu$ are surjective, proper morphisms that are small (see [8, §6.2]) in the sense that for all $r > 0$,
\[ \dim \{ z \in N \mid \dim \xi^{-1}(z) \geq r \} < \dim N - 2r \]
and
\[ \dim \{ z \in N \mid \dim \mu^{-1}(z) \geq r \} < \dim N - 2r. \]

D3 The morphisms $\xi$ and $\mu$ are small maps to identify the intersection complexes with higher direct images of constant sheaves. Thus we obtain a sheaf-theoretic version of Theorem 3.1.2 for complexes of sheaves in $D^b_c(N)$. In §5.3 we complete the proof of the theorem by restricting to $N_0$, applying Ext$_{N_0}(\mathbb{Q}_{N_0}, \mathcal{J}^\ast_{N_0}(\cdot))$, and showing that the induced map in Borel-Moore homology is $(\eta_h)_\ast$. Since we are concerned not only with complexes of sheaves, but also the precise maps between them, most of the work involved is in keeping track of morphisms as we apply the various functors.

D4 The morphisms $i_M$, $i_P$, and $i_N$ are open embeddings.

D5 The morphisms $j_M$, $j_P$, and $j_N$ are closed embeddings.

D6 A finite group, $\Sigma$, acts on $M_r$ on the right so that $N_r \cong M_r/\Sigma$ and $\mu_r$ may be identified with the orbit map.

D7 There is a subgroup, $\Sigma'$, of $\Sigma$, so that $P_r \cong M_r/\Sigma'$ and $\eta_r$ may be identified with the orbit map.

Since $\eta$ and $\xi$ are proper morphisms and the squares in the basic commutative diagram are cartesian, it follows that all the horizontal maps in the basic commutative diagram are proper morphisms and that $\mu$, $\mu_r$, and $\mu_0$ are proper morphisms. Thus, if $f$ is any of the morphisms in the basic commutative diagram except $i_M$, $i_P$, or $i_N$, then $RF_s = RF_f$. Since $i_M$, $i_P$, and $i_N$ are open embeddings, we have $i_M^! = i_M^\ast$, $i_P^! = i_P^\ast$, and $i_N^! = i_N^\ast$. Finally, since $\eta_r$, $\xi_r$, and $\mu_r$ are finite covering maps, we have $\eta_r^\ast = \eta_r^\ast$, $\xi_r^\ast = \xi_r^\ast$, $\mu_r^\ast = \mu_r^\ast$, $R(\eta_r^\ast) = (\eta_r)_!$, $R(\xi_r^\ast) = (\xi_r)_!$, and $R(\mu_r^\ast) = (\mu_r)_!$.

In this section we prove the following theorem.

**Theorem 3.1.2.** The group $\Sigma$ acts on $H_\bullet(M_0)$ and there is an isomorphism $h' \colon H_\bullet(P_0) \cong H_\bullet(M_0)^{\Sigma'}$ so that if $Av \colon H_\bullet(M_0) \to H_\bullet(M_0)^{\Sigma'}$ is the averaging map, then the diagram

\[
\begin{array}{ccc}
H_\bullet(M_0) & \xrightarrow{(\eta_h)_\ast} & H_\bullet(P_0) \\
\downarrow{Av} & & \downarrow{h'} \\
H_\bullet(M_0)^{\Sigma'} & & \\
\end{array}
\]

of graded vector spaces commutes.

The idea of the argument is a standard one and is given in the next three subsections. In §3.2 we prove Proposition 3.2.1 the analog of Theorem 3.1.2 for local systems on $M_r$, $P_r$, and $N_r$. In §3.3 we apply IC and use the fact that $\xi$ and $\mu$ are small maps to identify the intersection complexes with higher direct images of constant sheaves. Thus we obtain a sheaf-theoretic version of Theorem 3.1.2 for complexes of sheaves in $D^b_c(N)$. In §5.3 we complete the proof of the theorem by restricting to $N_0$, applying Ext$_{N_0}(\mathbb{Q}_{N_0}, \mathcal{J}^\ast_{N_0}(\cdot))$, and showing that the induced map in Borel-Moore homology is $(\eta_h)_\ast$. Since we are concerned not only with complexes of sheaves, but also the precise maps between them, most of the work involved is in keeping track of morphisms as we apply the various functors.

Finally, in §3.5 we discuss a two variable version of Theorem 3.1.2. Here $M$, $P$, and $N$ are replaced by $M \times M$, $P \times Q$, and $N \times N$, respectively, $M_0$ and $P_0$ are replaced by the fibre products $Z = (M \times M) \times_{N \times N} N_0$ and $X = (P \times Q) \times_{N \times N} N_0$, respectively, and $j_N$ is replaced by $\delta j_N : N_0 \to N \times N$, where $\delta$ is the diagonal map. In the application we are mainly interested in (see §5.1.2), $M \times M = \mathfrak{g} \times \mathfrak{g}$, $Z$ is the Steinberg variety of $G$, and $X$ is the generalized Steinberg variety $X^{P_0, Q}$. 
As we have observed above, all the horizontal maps in the basic commutative
diagram are proper, so direct image and direct image with proper support are the
same functors for these maps. Direct image with proper support is better adapted
to Borel-Moore homology, so the following argument is phrased in terms of direct
image with proper support.

3.2. First, $\mu_r$ may be identified with the orbit map from $M_r$ to $M_r/\Sigma$, and so $\Sigma$
acts as automorphisms on the local system $(\mu_r)!Q_{M_r}$ on $N_r$. Similarly, $\Sigma'$
acts as automorphisms on the local system $(\eta_r)!Q_{M_r}$ on $P_r$.

Next, local systems on $N_r$ form an abelian category, so we may consider the
$\Sigma'$-invariants of the local system $(\mu_r)!Q_{M_r}$. Let

$$Av: (\mu_r)!Q_{M_r} \rightarrow ((\mu_r)!Q_{M_r})^{\Sigma'}$$

denote the projection onto the local system of $\Sigma'$-invariants given by averaging over
$\Sigma'$.

Finally, recall from §2.2 that $\alpha_{\eta_r} : \eta_r^*Q_{P_r} \rightarrow Q_{M_r}$ is the natural isomorphism.
Since $\eta_r^* = \text{Id}$, we may consider $\alpha_{\eta_r}$ as a map from $\eta_r^*Q_{P_r}$ to $Q_{M_r}$, and so we may apply $\Phi_{\eta_r}^{-1}$ to $\alpha_{\eta_r}$ and get a map from $(\eta_r)!Q_{M_r}$ to $Q_{P_r}$. Define

$$\gamma_r : (\eta_r)!Q_{M_r} \rightarrow Q_{P_r} \quad \text{by} \quad \gamma_r = \Phi_{\eta_r}^{-1}(\alpha_{\eta_r}) = \epsilon_{\eta_r} \circ (\eta_r)!((\alpha_{\eta_r})^{-1}).$$

The following proposition is easily proved either directly or by using the corre-
spondence between local systems and representations of fundamental groups.

**Proposition 3.2.1.** There is an isomorphism $h_r : (\xi_r)!Q_{P_r} \cong ((\mu_r)!Q_{M_r})^{\Sigma'}$ so that the diagram

$$
\begin{array}{ccc}
(\mu_r)!Q_{M_r} & \xrightarrow{(\xi_r)!((\gamma_r)}) & (\xi_r)!Q_{P_r} \\
Av & \cong & h_r \\
& ((\mu_r)!Q_{M_r})^{\Sigma'}
\end{array}
$$

do not commute.

3.3. In this subsection we prove the following proposition, the analog of Proposition
3.2.1 for $M, P,$ and $N$.

**Proposition 3.3.1.** There is a map $\gamma : R\mu_M Q_M \rightarrow Q_P$ and an isomorphism $h : R\xi_M Q_P \rightarrow (R\mu_M Q_M)^{\Sigma'}$ so that the diagram

$$
\begin{array}{ccc}
R\mu_M Q_M & \xrightarrow{R\xi_M(\gamma)} & R\xi_M Q_P \\
Av & \cong & h \\
& (R\mu_M Q_M)^{\Sigma'}
\end{array}
$$

do not commute.

We can apply the functor $IC(N, \cdot)$ to the diagram of local systems in Proposition
3.2.1 and obtain a commutative triangle of complexes in $D^b_c(N)$. Since the functor
$IC(N, \cdot)$ takes its values in an abelian category of perverse sheaves on $N$ and is an
additive functor by construction, we may consider $\text{IC}(N, \cdot)$ as an additive functor between abelian categories. It follows that $\Sigma$ acts on $\text{IC}(N, (\mu_r); Q_{M_r})$, that

$$\text{IC}(N, ((\mu_r); Q_{M_r}) \cong \text{IC}(N, (\mu_r); Q_{M_r})$$

and that if $\text{Av}$ is the averaging map, the diagram

$$\begin{array}{ccc}
\text{IC}(N, (\mu_r); Q_{M_r}) & \xrightarrow{\text{IC}(N, (\xi_r); (\gamma_r))} & \text{IC}(N, (\xi_r); Q_{P_r}) \\
\text{Av} & \searrow & \swarrow \text{IC}(h_r) \\
\text{IC}(N, (\mu_r); Q_{M_r}) & \xrightarrow{\Sigma} & \text{IC}(N, (\xi_r); Q_{M_r}) \\
\end{array}$$

of complexes in $D^b_c(N)$ commutes.

Since $\xi$ and $\mu$ are small maps, it follows from the axioms characterizing intersection complexes (see [1 §4.13]) that $\text{IC}(N, (\mu_r); Q_{M_r})$ and $\text{IC}(N, (\xi_r); Q_{P_r})$ are isomorphic in $D^b_c(N)$ to the direct images $R\mu_! Q_M$ and $R\xi_! Q_P$, respectively. Moreover, since the $\Sigma$-action on $R\mu_! Q_M$ comes from transport of structure from $(\mu_r); Q_{M_r}$, it follows that there are isomorphisms, $\overline{\mu}$, $\overline{\xi}$, and $h$, so that if $g = \overline{\xi}^{-1} \circ \text{IC}(N, (\xi_r); (\gamma_r)) \circ \overline{\mu}$, then the diagram

$$\begin{array}{ccc}
\text{IC}(N, (\mu_r); Q_{M_r}) & \xrightarrow{\text{IC}(N, (\xi_r); (\gamma_r))} & \text{IC}(N, (\xi_r); Q_{P_r}) \\
\overline{\mu} & \searrow & \swarrow \overline{\xi} \\
R\mu_! Q_M & \xrightarrow{g} & R\xi_! Q_P \\
\end{array}$$

in $D^b_c(N)$ commutes. We can apply the functor $\text{Ext}_{N_0}^\bullet (Q_{N_0}, j^! (\cdot))$ to the bottom triangle in (3.3.2) and obtain a commutative triangle of Ext-groups that are isomorphic to the Borel-Moore homology groups in the statement of Theorem 3.1.2.

In order to show that the resulting horizontal map is indeed the direct image map in Borel-Moore homology induced by $\eta_0$, we need to choose the isomorphisms $\overline{\mu}$ and $\overline{\xi}$ appropriately and identify the map $g$ in (3.3.2). This is accomplished in the next lemma and the following corollary.

Since $P$ is a purely $d$-dimensional, rational homology manifold, we have $\mathbb{D}_P \cong Q_P[2d]$ in $D^b_c(P)$. We denote by $\nu_P$ a fixed isomorphism, $\nu_P : \mathbb{D}_P \to Q_P[2d]$ in $D^b_c(P)$.

Now $i^!_M \mathbb{D}_M[-2d]$ and $i^!_M Q_M$ are in $D^b_c(M_r)$ and $i^*_M (\alpha_\eta) \circ i^*_M (\eta^! (\nu_P) \circ \beta_\eta)$ is an isomorphism between them, so $i^!_M \mathbb{D}_M[-2d]$ is in fact a local system on $M_r$. Notice that $\eta^! (\nu_P) : \eta^! \mathbb{D}_P[-2d] \to \eta^! Q_P$ and $\alpha_\eta : \eta^* Q_P \to Q_M$, so the composition $\alpha_\eta \circ \eta^! (\nu_P)$ is not defined. However,

$$i^!_M \eta^! = (\eta i^*_M)^! = (i^*_P \eta_P)^! = \eta^* i^*_P = \eta^* = i^!_M \eta^*,$$

so the composition $i^!_M (\alpha_\eta) \circ i^!_M (\eta^! (\nu_P))$ is defined.

By [1 Lemma 4.11] there is a unique isomorphism of local systems on $M$ that restricts to $i^!_M (\alpha_\eta) \circ i^!_M (\eta^! (\nu_P) \circ \beta_\eta)$. The statement in [1] assumes that $M$ is a manifold, but the argument applies when $M$ is a variety that is a rational homology
manifold. Denote this isomorphism by $\nu_P^*: D_{M}[-2d] \to \mathbb{Q}_M$ and 

$$i_M^*(\nu_P^*) = i_M^*(\alpha_\eta) \circ i_M^*(\eta^i(\nu_p) \circ \beta_\eta). \quad (3.3.3)$$

Define $\gamma: R\eta Q_M \to \mathbb{Q}_P$ by

$$\gamma = \nu_P \circ \Phi^{-1}_\eta(\beta_\eta \circ (\nu_M^*)^{-1}) = \nu_P \circ \epsilon^\eta \circ R\eta(\beta_\eta \circ (\nu_M^*)^{-1}) = \Phi^{-1}_\eta(\eta^i(\nu_p) \circ \beta_\eta \circ (\nu_M^*)^{-1}).$$

**Lemma 3.3.4.** The diagram

$$\begin{array}{ccc}
\iota^N_R M & \xrightarrow{\iota^N_R \xi(\gamma)} & \iota^N_R P \\
(\mu_r)(\alpha_i) \circ b_{c_{\xi,i}} & \downarrow & (\mu_r)(\alpha_i) \circ b_{c_{\xi,i}} \\
(\xi_r)(\alpha_i) \circ b_{c_{\xi,i}} & \xrightarrow{(\xi_r)(\gamma_r)} & (\xi_r) ! P_r \\
\end{array}$$

of complexes in $D_c^b(N_r)$ commutes.

**Proof:** Since $b_{c_{\xi,i}} = (\xi_r)(bc_{\eta,i}) \circ b_{c_{\xi,i}}$, we need to show that

$$(\xi_r)(\alpha_i) \circ b_{c_{\xi,i}} \circ \iota^N_R(\gamma) = (\xi_r)(\gamma_r) \circ (\mu_r)(\alpha_i) \circ (\xi_r)(bc_{\eta,i}).$$

Using the naturality of the base change morphism $bc_{\xi,i}$, we see that it is enough to show that

$$(\xi_r)(\alpha_i) \circ (\xi_r) \iota^N_R(\gamma) = (\xi_r)(\gamma_r) \circ (\mu_r)(\alpha_i) \circ (\xi_r)(bc_{\eta,i}).$$

Since $\gamma_r = \epsilon^\eta \circ (\eta_r)(\alpha_\eta^{-1})$, it’s enough to show that

$$\alpha_i \circ \iota^N_R(\gamma) = \epsilon^\eta \circ (\eta_r)(\alpha_\eta^{-1} \circ \alpha_i).$$

Equivalently, it’s enough to show that

$$\iota^N_R(\gamma) \circ b_{c_{\eta,i}}^{-1} = \alpha_i^{-1} \circ \epsilon^\eta \circ (\eta_r)(\alpha_\eta^{-1} \circ \alpha_i).$$

Finally, $\eta \iota_M = i_p \eta_r$, and so $\Phi_i \Phi_\eta = \Phi_\eta \Phi_i$ and hence $\Phi_i \Phi^{-1}_\eta = \Phi^{-1}_\eta \Phi_i$. Therefore:

\[
\begin{align*}
\iota^N_R(\gamma) & \circ b_{c_{\eta,i}}^{-1} \\
& = \iota^N_R(\gamma) \circ b_{c_{\eta,i}}^{-1} \\
& = \iota^N_R(\gamma) \circ \epsilon^\eta \circ (\eta_r)(\alpha_\eta^{-1} \circ \alpha_i \\
& = \epsilon^\eta \circ (\eta_r)(\alpha_\eta^{-1} \circ \alpha_i) \circ (\eta_r)(\alpha_\eta^{-1} \circ \alpha_i). \quad \text{(by 2.2.1)}
\end{align*}
\]

This completes the proof of the lemma. □
Corollary 3.3.5. There are isomorphisms,
\[ \overline{\pi} : R\mu Q_M \longrightarrow IC(N, (\mu_r)_! Q_{M_r}) \quad \text{and} \quad \overline{\xi} : R\xi Q_P \longrightarrow IC(N, (\xi_r)_! Q_{P_r}), \]
so that the diagram
\[ \begin{array}{ccc}
R\mu Q_M & \xrightarrow{R\xi(\gamma)} & R\xi Q_P \\
\overline{\pi} \downarrow & & \downarrow \overline{\xi} \\
IC(N, (\mu_r)_! Q_{M_r}) & \xrightarrow{IC((\xi_r)_! (\gamma_r))} & IC(N, (\xi_r)_! Q_{P_r})
\end{array} \]
of complexes in \( D^b_c(N) \) commutes.

Proof. We have already observed that since \( \xi \) and \( \mu \) are small maps, the direct images, \( R\xi Q_P \) and \( R\mu Q_M \), are isomorphic in \( D^b_c(N) \) to \( IC(N, \xi_! Q_{P_r}) \) and to \( IC(N, \mu_! Q_{M_r}) \), respectively. Thus, \( R\xi Q_P \) and \( R\mu Q_M \) are in the image of \( IC \). It is shown in [8 Theorem 3.5] that on the image of \( IC \), the composition \( IC(N, \cdot) \circ i^*_N \) is naturally equivalent to the identity, so there are isomorphisms,
\[ ic_\mu : R\mu Q_M \xrightarrow{=} IC(N, i^*_N R\mu Q_M) \quad \text{and} \quad ic_\xi : R\xi Q_P \xrightarrow{=} IC(N, i^*_N R(\xi_r)_! Q_{P_r}) \]
in \( D(N) \) with \( i^*_N (ic_\mu) = id \) and \( i^*_N (ic_\xi) = id \). Since \( IC \) is fully faithful, it follows that the diagram
\[ \begin{array}{ccc}
R\mu Q_M & \xrightarrow{R\xi(\gamma)} & R\xi Q_P \\
ic_\mu \downarrow & & \downarrow ic_\xi \\
IC(N, i^*_N R\mu Q_M) & \xrightarrow{IC((\xi_r)_! (\gamma_r))} & IC(N, i^*_N R(\xi_r)_! Q_{P_r})
\end{array} \]
commutes.

If we apply \( IC \) to the commutative diagram in the lemma we get a commutative diagram
\[ \begin{array}{ccc}
IC(N, i^*_N R\mu Q_M) & \xrightarrow{IC((\xi_r)_! (\gamma_r))} & IC(N, i^*_N R(\xi_r)_! Q_{P_r}) \\
IC((\mu_r)_! (\alpha_{i_r}) \circ bc_{\mu_r} \cdot i_M) \downarrow & & \downarrow IC((\xi_r)_! (\alpha_{i_r}) \circ bc_{\xi_r} \cdot i_P) \\
IC(N, (\mu_r)_! Q_{M_r}) & \xrightarrow{IC((\xi_r)_! (\gamma_r))} & IC(N, (\xi_r)_! Q_{P_r})
\end{array} \]

Therefore, if we define
\[ \overline{\pi} = IC((\mu_r)_! (\alpha_{i_r}) \circ bc_{\mu_r} \cdot i_M) \circ ic_\mu \quad \text{and} \quad \overline{\xi} = IC((\xi_r)_! (\alpha_{i_r}) \circ bc_{\xi_r} \cdot i_P) \circ ic_\xi, \]
the corollary follows. \( \square \)

Since \( \overline{\pi} : R\mu Q_M \rightarrow IC(N, (\mu_r)_! Q_{M_r}) \) is an isomorphism, it follows that \( \Sigma \) acts on \( R\mu Q_M \) by transport of structure and that \( \overline{\pi} \) induces an isomorphism between \( \Sigma^r \)-invariants, say \( \overline{\pi}^r : (R\mu Q_M)^{\Sigma^r} \rightarrow IC(N, (\mu_r)_! Q_{M_r})^{\Sigma^r} \), which commutes with the respective averaging maps.
Now consider the diagram

\[
\begin{array}{ccccccccc}
R_{\mu}(Q_M) & \xrightarrow{\mathbf{IC}(N, (\mu_r)Q_M, J)} & \mathbb{IC}(N, (\mu_r)Q_M) & \xrightarrow{\mathbf{IC}(h_r)} & \mathbb{IC}(N, (\xi_r)Q_M) & \xrightarrow{h} & R_{\xi}(Q_M) \\
\downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} \\
(R_{\mu}(Q_M))' & & \mathbb{IC}(N, (\mu_r)Q_M) & & \mathbb{IC}(N, (\mu_r)Q_M) & & (R_{\mu}(Q_M))' \\
\end{array}
\]

If \(h\) is defined by \(h = (\pi')^{-1} \circ \mathbf{IC}(h_r) \circ \xi\), then the diagram commutes. By Corollary \[3.3.3\], the composition across the top row is just \(R_{\xi}(\gamma)\), and so tracing around the outside of the diagram we see that \(h \circ R_{\xi}(\gamma) = \mathbf{Av}\). This completes the proof of Proposition \[3.3.1\].

3.4. In this subsection, we complete the proof of Theorem \[3.1.2\].

**Lemma 3.4.1.** There are isomorphisms of graded vector spaces,

\[
J': H_{2d-\bullet}(M_0) \xrightarrow{\simeq} \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M))
\]

and

\[
J'_1: H_{2d-\bullet}(P_0) \xrightarrow{\simeq} \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\xi}(Q_P)),
\]

so that the diagram

\[
\begin{array}{ccc}
H_{2d-\bullet}(M_0) & \xrightarrow{(\eta_0)} & H_{2d-\bullet}(P_0) \\
\downarrow{J'} & & \downarrow{J'_1} \\
\mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M)) & \xrightarrow{(j_N^1R_{\xi}(\gamma))_\sharp} & \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\xi}(Q_P)) \\
\downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} \\
\mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M))^\Sigma' & \xrightarrow{(j_N^1(h))_\sharp} & \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M))^\Sigma' \\
\end{array}
\]

commutes.

Assuming for a moment that the lemma has been proved, we complete the proof of Theorem \[3.1.2\] using the argument at the end of \[3.3\] as follows.

Since \(J'\) is an isomorphism, \(\Sigma\) acts on \(H_{2d-\bullet}(M_0)\) by transport of structure and \(J'\) induces an isomorphism between \(\Sigma'\)-invariants, say \(J\), which commutes with the respective averaging maps.

Now consider the diagram

\[
\begin{array}{cccccc}
H_{2d-\bullet}(M_0) & \xrightarrow{J'} & E_1 & \xrightarrow{(j_N^1R_{\xi}(\gamma))_\sharp} & E_2 & \xrightarrow{J'_1} & H_{2d-\bullet}(P_0) \\
\downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} & & \downarrow{\mathbf{Av}} \\
H_{2d-\bullet}(M_0)^\Sigma' & \xrightarrow{(j_N^1(h))_\sharp} & E_3 & \xrightarrow{h'} & H_{2d-\bullet}(M_0)^{\Sigma'}
\end{array}
\]

where

\[
E_1 = \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M)), \quad E_2 = \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\xi}(Q_P)), \quad \text{and} \quad E_3 = \mathbf{Ext}^\bullet_{N_0}(Q_{N_0}, j_N^1R_{\mu}(Q_M))^\Sigma'.
\]
If $h'$ is defined by $h' = (J)^{-1} \circ (j^1_N(h))_2 \circ J'_i$, then the diagram commutes. By Lemma 3.4.1, the composition across the top row is $(\eta_0)_\ast$, and so tracing around the outside of the diagram we see that $h' \circ (\eta_0)_\ast = Av$. This proves Theorem 3.1.2.

It remains to prove Lemma 3.4.1.

First, we apply the functor $\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N(\cdot))$ to the diagram in Proposition 3.3.1 and obtain a commutative triangle of graded vector spaces. Since the functor $\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N(\cdot))$ restricted to the abelian category of perverse sheaves in which $\text{IC}(N, \cdot)$ takes its values is an additive functor between abelian categories, it follows that $\Sigma$ acts on $\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu Q_M)$, that $\Sigma$ acts on $\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu Q_M)^{\Sigma'}$, and that if $Av$ is the averaging map, the diagram of graded vector spaces

$$
\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu Q_M) \xrightarrow{(j^1_N R\xi(\gamma))_2} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\xi Q_P) \xleftarrow{\text{Av}} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu Q_M)^{\Sigma'}
$$

commutes.

Next, recall that $\gamma = \nu_\eta \circ \epsilon^H \circ R\eta(\beta_\eta \circ (\nu^P_M)^{-1})$, so using 2.4.1 we get

$$
\nu_\eta \circ \Phi^{-1}_\eta(\beta_\eta) = \gamma \circ R\eta(\nu^P_M): R\eta \mathbb{D}_M[-2d] \longrightarrow Q_P.
$$

Applying $j^1_N R\xi$ we get $j^1_N R\xi(\nu_\eta) \circ j^1_N R\xi(\Phi^{-1}_{\eta}(\beta_\eta)) = j^1_N R\xi(\gamma) \circ j^1_N R\mu (\nu^P_M)$. This shows that the diagram

$$
\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu \mathbb{D}_M[-2d]) \xrightarrow{(j^1_N R\xi(\Phi^{-1}_{\eta}(\beta_\eta)))_2} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\xi \mathbb{D}_P[-2d]) \xleftarrow{(j^1_N R\mu(\nu^P_M)))_2} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu Q_M)
$$

commutes.

Finally, we show that there are isomorphisms

$$
J: H_\ast(M_0) \xrightarrow{\cong} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu \mathbb{D}_M)
$$

and

$$
J_1: H_\ast(P_0) \xrightarrow{\cong} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\xi \mathbb{D}_P)
$$

so that the diagram

$$
\text{Ext}^{\bullet}_{M_0}(Q_{M_0}, \mathbb{D}_{M_0}) \xrightarrow{(\eta_0)_\ast} \text{Ext}^{\bullet}_{P_0}(Q_{P_0}, \mathbb{D}_{P_0}) \xrightarrow{J} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu \mathbb{D}_M) \xleftarrow{(j^1_N R\xi(\Phi^{-1}_{\eta}(\beta_\eta)))_2} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\mu \mathbb{D}_M) \xrightarrow{(j^1_N R\xi(\beta_\eta)))_2} \text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j^1_N R\xi \mathbb{D}_P)
$$

commutes. Once this has been done, set $J' = (j^1_N R\mu (\nu^P_M))_2 \circ J$ and $J'_i = (j^1_N R\xi(\nu_\eta))_2 \circ J_1$. Then $J'_i \circ (\eta_0)_\ast = (j^1_N R\xi(\gamma))_2 \circ J'$, and so the diagram in the statement of Lemma 3.4.1 commutes as claimed.
Recall that since $\eta_0$ is a proper map, it induces a map in Borel-Moore homology. If $\Psi_{\eta_0}$ is the adjunction of the adjoint pair $(\eta_0^*,\,(R\eta_0)_*)$, then $(\eta_0)_*$ is the composition

$$H_{\cdot\cdot}(M_0) = \text{Ext}^*_{M_0}(Q_{M_0},D_{M_0})$$

$$\cong \text{Ext}^*_{M_0}(\eta_0^*Q_{P_0},D_{M_0}) \quad \text{by } \alpha^\sharp_{\eta_0}$$

$$\cong \text{Ext}^*_{P_0}(Q_{P_0},(R\eta_0)_*D_{M_0}) \quad \text{by } \Psi_{\eta_0}$$

$$\cong \text{Ext}^*_{P_0}(Q_{P_0},R(\eta_0)_*\eta_0^*D_{P_0}) \quad \text{by } (R(\eta_0)_*(\beta_{\eta_0}))^\sharp$$

$$\rightarrow \text{Ext}^*_{P_0}(Q_{P_0},D_{P_0}) \quad \text{by } (\epsilon^\gamma_0)^\sharp$$

$$= H_{\cdot\cdot}(P_0),$$

so

$$(\eta_0)_* = (\epsilon^\gamma_0 \circ R(\eta_0)_*(\beta_{\eta_0}))^\sharp \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0} = \Phi_{\eta_0}^{-1}(\beta_{\eta_0})^\sharp \circ \Psi_{\eta_0} \circ \alpha^\sharp_{\eta_0}.$$
Assume for a moment that (** \circ (\dagger) = (\dagger\dagger) \circ (*) \) and define

\[ J = (\dagger)^{-1} \circ \Psi_{\eta_0} \circ \alpha_{\eta_0}^2 : \text{Ext}^*_M(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}) \longrightarrow \text{Ext}^*_N(\mathbb{Q}_{N_0}, j_1^i R\mu \mathbb{D}_M) \]

and

\[ J_1 = (\dagger\dagger)^{-1} : \text{Ext}^*_N(\mathbb{Q}_{N_0}, j_1^i R\xi_1 \mathbb{D}_P) \longrightarrow \text{Ext}^*_M(\mathbb{Q}_{M_0}, \mathbb{D}_{M_0}). \]

Then \( J \) and \( J_1 \) are isomorphisms and

\[ J_1 \circ (\eta_0)_* = (\dagger\dagger)^{-1} \circ (\eta_0)_* = (j_1^i R\xi_1 (\Phi_{-1}(\beta\eta)))_2 \circ ((\dagger\dagger)^{-1} \circ \Psi_{\eta_0} \circ \alpha_{\eta_0}^2) = (j_1^i R\xi_1 (\Phi_{-1}(\beta\eta)))_2 \circ J, \]

so diagram (3.4.2) commutes as claimed.

It remains to show (** \circ (\dagger) = (\dagger\dagger) \circ (*). Suppose \( h \) is in \( \text{Ext}^*_N(\mathbb{Q}_{N_0}, j_1^i R\mu \mathbb{D}_M) \).

Then

\[ ((** \circ (\dagger))(h) = \Phi_{\eta_0}^{-1}(\beta\eta_0) \circ \Psi_{\xi_0}^{-1} \left( R(\mu_0)!((\beta_j)^{-1}) \circ R(\xi_0)! ((\beta_j)^{-1}) \circ (bc_{\xi_0,j_1}) \circ h \right) \circ \alpha_{\xi_0}^{-1} = \Psi_{\xi_0}^{-1} \left( R(\xi_0)!((\beta_j)^{-1}) \circ R(\eta_0)^! ((\beta_j)^{-1}) \circ (bc_{\xi_0,j_1}) \circ h \right) \circ \alpha_{\xi_0}^{-1}. \]

On the other hand, using the naturality of the base change \( bc_{\xi_0,j_1} \), we have

\[ ((\dagger\dagger \circ (*))(h) = \Psi_{\xi_0}^{-1} \left( R(\xi_0)!((\beta_j)^{-1}) \circ (j_1^p \Phi_{\eta}^{-1}(\beta\eta)) \circ h \right) \circ \alpha_{\xi_0}^{-1} = \Psi_{\xi_0}^{-1} \left( R(\xi_0)!((\beta_j)^{-1}) \circ (j_1^p \Phi_{\eta}^{-1}(\beta\eta)) \circ h \right) \circ \alpha_{\xi_0}^{-1} = \Psi_{\xi_0}^{-1} \left( R(\xi_0)!((\beta_j)^{-1}) \circ (j_1^p \Phi_{\eta}^{-1}(\beta\eta)) \circ h, h \right) \circ \alpha_{\xi_0}^{-1}, \]

so it is enough to show that

\[ \Phi_{-1}^{-1}(\beta\eta_0) \circ R(\eta_0)!((\beta_j)^{-1}) \circ (bc_{\xi_0,j_1}) = (\beta_j)^{-1} \circ (j_1^p (\Phi_{\eta}^{-1}(\beta\eta))). \]

This last equality is easily proved by a computation similar to the computation in the proof of Lemma 3.3.4 using the definition of \( bc_{\eta,j_1} \) from (2.2.5) the identities (2.4.1), (2.4.2), and (2.4.3); the equality \( \Phi_{j_1} \Phi_{\eta}^{-1} = \Phi_{\eta_0} \Phi_{j_1} \); and (2.2.2). We omit the details. This completes the proof of Lemma 3.4.1 and Theorem 3.1.2.

3.5. From now on we denote \( \eta \) by \( \eta^p \) and \( \xi^p \), respectively.

In this subsection we consider the case when we have two factorizations of \( \mu, \mu = \xi^p \circ \eta^p = \xi^q \circ \eta^q \), and the spaces \( M \) and \( N \) in the basic commutative diagram (3.1.1) are replaced by \( M \times M \) and \( N \times N \), respectively. So, suppose that \( Q \) is a purely \( d \)-dimensional, rational homology manifold and that in addition to the assumptions already made concerning the basic commutative diagram, the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\eta^Q} & Q \\
\downarrow{M_r} & & \downarrow{Q_r} \\
M_r & \xrightarrow{\eta^Q} & Q_r \\
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\xi^Q} & N \\
\downarrow{\xi^Q} & & \downarrow{\xi^Q} \\
& \xrightarrow{N_r} & N_r \\
\end{array}
\]

satisfies conditions D1, D2, D3, D4, and D7 with \( P \) replaced by \( Q \) and \( \Sigma' \) replaced by a possibly different subgroup, \( \Sigma'' \), of \( \Sigma \).
The group morphisms of graded vector spaces from $4$, 4.1. In this section we continue the analysis of diagram (3.5.1) and consider isomorphism of graded vector spaces commutes.

This action induces an action of $\Sigma$ on $H_\bullet(Z)$ by functoriality and transport of structure via the isomorphism $J': H_\bullet(Z) \rightarrow \text{Ext}_{N_0}^{\eta_\bullet}(\mathbb{Q}_{N_0}, j_N^! \delta^1 R(\mu \times \mu))|_{Q_{M \times M}}$.

There is an isomorphism $h': H_\bullet(X) \rightarrow H_\bullet(Z)^{\Sigma' \times \Sigma''}$ so that if $\Lambda V: H_\bullet(Z) \rightarrow H_\bullet(Z)^{\Sigma' \times \Sigma''}$ is the averaging map, then the diagram

$$
\begin{array}{ccc}
H_\bullet(Z) & \xrightarrow{\eta^*} & H_\bullet(X) \\
\Lambda V \downarrow & & \searrow h' \\
H_\bullet(Z)^{\Sigma' \times \Sigma''} & & \\
\end{array}
$$

of graded vector spaces commutes.

4. Equivariance

4.1. In this section we continue the analysis of diagram (3.5.1) and consider isomorphisms of graded vector spaces from $\mathbb{Q}$ [§8.6]:

$$
H_\bullet(Z) \xrightarrow{J} \text{Ext}_{N_0}^\bullet(\mathbb{Q}_{N_0}, j_N^! \delta^1 R(\mu \times \mu)|_{D_{M \times M}})$$

$$\xrightarrow{K} \text{Ext}_{N_0}^{4n-\bullet}(R(\mu_0)|_{Q_{M_0}}, R(\mu_0)|_{Q_{M_0}}),$$

where $\dim N_0 = 2n$, $J$ is as in §3.4, and $K$ is defined below. Notice that

$$\text{End}_{N_0}(R(\mu_0)|_{Q_{M_0}}) = \text{Ext}_{N_0}^{0}(R(\mu_0)|_{Q_{M_0}}, R(\mu_0)|_{Q_{M_0}}) \cong H_{4n}(Z).$$

Recall that $\dim M = \dim N = d$ and define $l = \text{codim}_N N_0 = d - 2n$. From now on, we assume that $M_0$ and $N_0$ are purely $2n$-dimensional, rational, homology manifolds. We also assume that the fibred products $X$ and $Z$ in §3.5 are purely $2n$-dimensional varieties.

The graded vector space $\text{Ext}_{N_0}^{4n-\bullet}(R(\mu_0)|_{Q_{M_0}}, R(\mu_0)|_{Q_{M_0}})$ is a graded $\mathbb{Q}$-algebra, and the composition $K \circ J$ can be used to give $H_\bullet(Z)$ a $\mathbb{Q}$-algebra structure with $H_l(Z) \cdot H_j(Z) \subseteq H_{l+j-4n}(Z)$. 

Let $\delta: N \rightarrow N \times N$ be the diagonal embedding. Then $\delta j_N: N_0 \rightarrow N \times N$ is a closed embedding. Define $X$ to be the fibred product $(P \times Q) \times_{N \times N} N_0$ and define $Z$ to be the fibred product $(M \times M) \times_{N \times N} N_0$. It follows immediately from the definition that a cartesian product of two small morphisms is again a small morphism. Therefore, modifying the notation as indicated, the diagram

$$(3.5.1) \quad \begin{array}{ccc} Z & \xrightarrow{\eta_0} & X \\
\downarrow j_Z & & \downarrow j_X \\
M \times M & \xrightarrow{\eta_p \times \eta_Q} & P \times Q \\
\uparrow M_r \times M_r & & \uparrow P_r \times Q_r \\
M_r \times M_r & \xrightarrow{\eta_p \times \eta_Q} & N_r \times N_r \\
\end{array}$$

satisfies conditions $D1 - D7$ in (3.5.1).

We have the following corollary to Theorem (3.1.2).

**Corollary 3.5.2.** The group $\Sigma \times \Sigma$ acts on the local system $(\mu_r \times \mu_r)|_{Q_{M_r \times M_r}}$. This action induces an action of $\Sigma \times \Sigma$ on $R(\mu \times \mu)|_{Q_{M \times M}}$ and hence an action of $\Sigma \times \Sigma$ on $H_\bullet(Z)$ by functoriality and transport of structure via the isomorphism $J': H_\bullet(Z) \rightarrow \text{Ext}_{N_0}^{\eta_\bullet}(\mathbb{Q}_{N_0}, j_N^! \delta^1 R(\mu \times \mu))|_{Q_{M \times M}}$.

There is an isomorphism $h': H_\bullet(X) \rightarrow H_\bullet(Z)^{\Sigma' \times \Sigma''}$ so that if $\Lambda V: H_\bullet(Z) \rightarrow H_\bullet(Z)^{\Sigma' \times \Sigma''}$ is the averaging map, then the diagram

$$
\begin{array}{ccc}
H_\bullet(Z) & \xrightarrow{\eta^*} & H_\bullet(X) \\
\Lambda V \downarrow & & \searrow h' \\
H_\bullet(Z)^{\Sigma' \times \Sigma''} & & \\
\end{array}
$$

of graded vector spaces commutes.
Since the multiplication in $\text{Ext}_0^{4n-\bullet}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0})$ is composition, we have

$$\text{Ext}_0^{4n-\bullet}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0}) \cong H_{\bullet}(Z)^{\text{op}}.$$  

We saw in §3.3 that $\Sigma$ acts on $R\mu_0\mathbb{Q}_M$. This action induces a degree-preserving action of $\Sigma \times \Sigma$ on $\text{Ext}_0^{4n-\bullet}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0})$. On the other hand, as in §3.5, $\Sigma \times \Sigma$ acts on $R(\mu \times \mu);\mathbb{Q}_{M \times M}$. This action induces a degree-preserving $\Sigma \times \Sigma$-action on $\text{Ext}_0^{4n-\bullet}(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu);\mathbb{D}_{M \times M}).$

In this section we show that the isomorphisms $J$ and $K$ are $\Sigma \times \Sigma$-equivariant. It then follows that if $\Sigma \times \Sigma$ acts on the group algebra $\mathbb{Q}\Sigma$ in the usual way, then there are $\Sigma \times \Sigma$-equivariant, $\mathbb{Q}$-algebra homomorphisms

$$\mathbb{Q}\Sigma \longrightarrow \text{End}_{N_0}(R(\mu_0);\mathbb{Q}_{M_0}) \longrightarrow H_0(Z)^{\text{op}}.$$  

In §4.2 we describe the $\Sigma \times \Sigma$-action on $\text{Ext}_0^{4n-\bullet}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0})$. In §4.3 we describe the $\Sigma \times \Sigma$-action on $\text{Ext}_0^{\bullet}(\mathbb{Q}_{N_0}, j_N^! \delta^! R(\mu \times \mu);\mathbb{D}_{M \times M})$ and observe that $J$ is $\Sigma \times \Sigma$-equivariant. In §4.4 we define the map $K$, and in §4.5–§4.8 we show that $K$ is $\Sigma \times \Sigma$-equivariant.

4.2. We first consider the $\Sigma \times \Sigma$-action on $\text{Ext}_0^{\bullet}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0})$. Returning to our original basic commutative diagram (3.1.1), $\Sigma$ acts on the direct image, $(\mu_\tau);\mathbb{Q}_{M_\tau}$. This action induces a $\Sigma$-algebra homomorphism

$$L_\tau: \mathbb{Q}\Sigma \longrightarrow \text{End}_{N_\tau}((\mu_\tau);\mathbb{Q}_{M_\tau}).$$

Applying IC and transporting the action via the isomorphism

$$\mathbb{F}: \text{Hom}_N(N, (\mu_\tau);\mathbb{Q}_{M_\tau}) \longrightarrow \mathbb{Q}\Sigma$$

from Corollary 3.3.5 gives rise to a $\Sigma$-algebra homomorphism

$$L: \mathbb{Q}\Sigma \longrightarrow \text{End}_N(R\mu_0\mathbb{Q}_M)$$

with $L(\sigma) = \mathbb{F}^{-1} \circ \text{IC}(L_\tau(\sigma)) \circ \mathbb{F}$.

Since $L$ is a ring homomorphism, we get an action of $\Sigma \times \Sigma$ on $\text{End}_N(R\mu_0\mathbb{Q}_M)$ with

$$(\sigma, \sigma') \cdot f = L(\sigma') \circ f \circ L(\sigma^{-1})$$

for $f$ in $\text{End}_N(R\mu_0\mathbb{Q}_M)$.

Clearly, if $\Sigma \times \Sigma$ acts on $\mathbb{Q}_\Sigma$ by $(\sigma, \sigma') \cdot x = \sigma' x \sigma^{-1}$, then $L$ is $\Sigma \times \Sigma$-equivariant.

Let $bc^*: j_N^* R\mu_1 \longrightarrow R(\mu_0);j_N^* R(\mu_1)$ be as in §2.5. Then $R(\mu_0);(\alpha_{j_M}) \circ bc^*$ is an isomorphism between $j_N^* R\mu_1\mathbb{Q}_M$ and $R(\mu_0);\mathbb{Q}_{M_0}$. We define

$$L_0: \mathbb{Q}\Sigma \longrightarrow \text{End}_{N_0}(R(\mu_0);\mathbb{Q}_{M_0})$$

by

$$L_0(\sigma) = R(\mu_0);(\alpha_{j_M}) \circ bc^* \circ j_N^* L(\sigma) \circ (bc^*)^{-1} \circ R(\mu_0);(\alpha_{j_M})^{-1}.$$  

Since $\text{Ext}_{N_0}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0}) \cong \text{Hom}_{N_0}(R(\mu_0);\mathbb{Q}_{M_0}, R(\mu_0);\mathbb{Q}_{M_0}[j])$ is naturally an $\text{End}_{N_0}(R(\mu_0);\mathbb{Q}_{M_0})$-bimodule, we may define an action of $\Sigma \times \Sigma$ on
the graded vector space $\text{Ext}^\bullet_{N_0}(R(\mu_0), \mathbb{Q}_{M_0}, R(\mu_0), \mathbb{Q}_{M_0})$ by

$$(\sigma, \sigma') \cdot g = L_0(\sigma') \circ g \circ L_0(\sigma^{-1})$$

for $\sigma$ and $\sigma'$ in $\Sigma$ and $g$ in $\text{Ext}^\bullet_{N_0}(R(\mu_0), \mathbb{Q}_{M_0}, R(\mu_0), \mathbb{Q}_{M_0})$.

4.3. Next we consider the $\Sigma \times \Sigma$-action on $\text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{D}_{M \times M})$. Since $M$ is a rational homology manifold, so is $M \times M$, and we denote by $\nu_{M \times M}$ a fixed isomorphism, $\nu_{M \times M} : \mathbb{D}_{M \times M} \to \mathbb{Q}_{M \times M}$.

As in §3.4 and §3.5, $\Sigma \times \Sigma$ acts as automorphisms on $R(\mu \times \mu)! \mathbb{Q}_{M \times M}$, and we transport the group action on $R(\mu \times \mu)! \mathbb{Q}_{M \times M}$ to an action on $R(\mu \times \mu)! \mathbb{D}_{M \times M}$ using $R(\mu \times \mu)! (\nu_{M \times M})$. The group actions induce ring homomorphisms

$$L_2 : \mathbb{Q}(\Sigma \times \Sigma) \to \text{End}_{\mathbb{N} \times \mathbb{N}}(R(\mu \times \mu)! \mathbb{D}_{M \times M})$$

and

$$L'_2 : \mathbb{Q}(\Sigma \times \Sigma) \to \text{End}_{\mathbb{N} \times \mathbb{N}}(R(\mu \times \mu)! \mathbb{D}_{M \times M})$$

where $L_2$ and $L'_2$ are related by

$$L'_2(\sigma, \sigma') = R(\mu \times \mu)! (\nu_{M \times M}^{-1}) \circ L_2(\sigma, \sigma') \circ R(\mu \times \mu)! (\nu_{M \times M}).$$

Notice that $L'_2$ depends on the choice of the orientation $\nu_{M \times M}$.

Applying $\text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! (\cdot))$ to $R(\mu \times \mu)! \mathbb{D}_{M \times M}$ and using $L'_2$ we get an action of $\Sigma \times \Sigma$ on $\text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{D}_{M \times M})$ with

$$(\sigma, \sigma') \cdot f = (j_{N_0}^! \delta^! L'_2(\sigma, \sigma'))_\sharp(f) = j_{N_0}^! \delta^! L'_2(\sigma, \sigma') \circ f$$

for $f$ in $\text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{D}_{M \times M})$.

As in §3.4 and §3.5, the $\Sigma \times \Sigma$-action on $\text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{Q}_{M \times M})$ induces an action of $\Sigma \times \Sigma$ on $H^\bullet(Z)$ by transport of structure using the isomorphism

$$J' = (j_{N_0}^! \delta^! R(\mu \times \mu)! (\nu_{M \times M}))_\sharp \circ J : H_{4d-\bullet}(Z) \to \text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{Q}_{M \times M}).$$

It follows from the definitions that $(j_{N_0}^! \delta^! R(\mu \times \mu)! (\nu_{M \times M}))_\sharp$ is $\Sigma \times \Sigma$-equivariant. This proves the following proposition.

**Proposition 4.3.1.** The isomorphism

$$J : H^\bullet(Z) \to \text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{D}_{M \times M})$$

is $\Sigma \times \Sigma$-equivariant.

4.4. Define

$$K : \text{Ext}^\bullet_{N_0}(\mathbb{Q}_{N_0}, j_{N_0}^! \delta^! R(\mu \times \mu)! \mathbb{D}_{M \times M}) \to \text{Ext}^{\bullet+4n}_{N_0}(R(\mu_0)! \mathbb{Q}_{M_0}, R(\mu_0)! \mathbb{Q}_{M_0})$$

by...
to be the composition

\[
\Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, j_N^j \delta R(\mu \times \mu) : \mathbb{D}_M \times \mathbb{M} \right) \\
\xrightarrow{j_N^j \delta(k')_1} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, j_N^j \delta (R\mu_\mathbb{D}_M \boxtimes R\mu_\mathbb{D}_M) \right) \\
\xrightarrow{j_N^j \delta(c^{-1} \boxtimes id)_1} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, j_N^j \delta ((R\mu_\mathcal{Q}_M)^\vee \boxtimes R\mu_\mathbb{D}_M) \right) \\
\xrightarrow{j_N^j \delta(\lambda)_1} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, R\text{Hom}(p^* R\mu_\mathcal{Q}_M, q^* R\mu_\mathbb{D}_M) \right) \\
\xrightarrow{(\text{nat}_{\delta_n})_*} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, R\text{Hom}(j_N^j R\mu_\mathcal{Q}_M, j_N^j R\mu_\mathbb{D}_M) \right) \\
\xrightarrow{(\text{can}^{-1})_{\text{obj}}} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, (R\mu_0)_! \mathcal{Q}_{M_0}, (R\mu_0)_! \mathcal{Q}_{M_0} \right)
\]

where the notation is as follows:

- \( k' : R(\mu \times \mu)_! \mathbb{D}_M \times \mathbb{M} \rightarrow R\mu_\mathbb{D}_M \boxtimes R\mu_\mathbb{D}_M \) is the Künneth isomorphism (recall that \( \mu \) is proper).
- \( c = R\mu_!(\text{dc}_M^{-1} \circ (\beta_\mu^{-1})_{\mathbb{D}_M}) \circ \phi_\mu : (R\mu_\mathcal{Q}_M)^\vee \rightarrow R\mu_\mathbb{D}_M \) where \( \text{dc}_M \) is as in \( [2.1] \) and \( \beta_\mu \) and \( \phi_\mu \) are as in \( [2.2] \). Notice that \( c \) is an isomorphism in \( D_c^b(N) \), so \( j_N^j \delta(c^{-1} \boxtimes id)_J \) makes sense.
- \( \lambda, \text{nat}_\delta, \) and \( \text{can} \) are as in \( [2] \).
- \( a = R(\mu_0)_!(\alpha_{\mathcal{Q}_M}) \circ \text{bc} : j_N^j R\mu_\mathcal{Q}_M \rightarrow R(\mu_0)_! \mathcal{Q}_{M_0} \) (see \( [4.1] \)).
- \( b = R(\mu_0)_!(\nu_{M_0} \circ \beta_{\mathcal{M}_0}^{-1}) \circ \text{bc} : j_N^j R\mu_\mathcal{D}_M \rightarrow R(\mu_0)_! \mathcal{Q}_{M_0} \), where \( \text{bc} : j_N^j R\mu_\mathbb{D}_M \rightarrow R(\mu_0)_! \mathcal{D}_{M_0} \) is as in \( [2.2] \), \( \beta_{\mathcal{M}_0} \) is as in \( [2.2] \) and \( \nu_{M_0} : \mathbb{D}_{M_0} \rightarrow \mathcal{Q}_{M_0}[4n] \) is an isomorphism in \( D_c^b(M_0) \) (recall that \( M_0 \) is a rational homology manifold).

Since \( K \) is a composition of isomorphisms of graded vector spaces, it follows that \( K \) is an isomorphism of graded vector spaces that increases the grading by \( 4n \).

**Theorem 4.4.1.** The isomorphism

\[
K : \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, j_N^j \delta R(\mu \times \mu)_! \mathbb{D}_M \times \mathbb{M} \right) \xrightarrow{\text{can}^{-1}} \Ext_{N_0}^* \left( \mathbb{Q}_{N_0}, (R\mu_0)_! \mathcal{Q}_{M_0}, (R\mu_0)_! \mathcal{Q}_{M_0} \right)
\]

is \( \Sigma \times \Sigma \)-equivariant.

To prove the theorem we show that

\[
j_N^j \delta(k')_1, j_N^j \delta(c^{-1} \boxtimes id)_1, \text{can}^{-1} \circ (\text{nat}_{\delta_N} \circ j_N^j \delta(\lambda))_1, \text{and} (a^{-1})^2 \circ b^1\]

are \( \Sigma \times \Sigma \)-equivariant in \( \Sect \), \( \Sect \), \( \Sect \), and \( \Sect \), respectively.

4.5. In the situation of \( [4.5] \) we have two factorizations of \( \mu : \mu = \xi P \eta P = \xi Q \eta Q \).

Let \( \nu^P_\mathbb{D}_M \) and \( \nu^Q_\mathbb{D}_M \) be two isomorphisms, \( \mathbb{D}_M \xrightarrow{\sim} \mathcal{Q}_M[2d] \). Then \( \nu^P_\mathbb{D}_M \boxtimes \nu^Q_\mathbb{D}_M : \mathbb{D}_M \boxtimes \mathbb{D}_M \rightarrow \mathcal{Q}_M \boxtimes \mathcal{Q}_M[4d] \) is an isomorphism in \( D_c^b(M \times M) \). The superscripts \( P \) and \( Q \) do not necessarily have anything to do with \( P \) and \( Q \), but are convenient for distinguishing between the factors.
Using the orientations $\nu^p$ and $\nu^q$ we can define $\mathbb{Q}$-algebra homomorphisms
\[
L'_p : \mathbb{Q}\Sigma \longrightarrow \text{End}_N(R\mu_1\mathbb{D}_M) \quad \text{and} \quad L'_q : \mathbb{Q}\Sigma \longrightarrow \text{End}_N(R\mu_1\mathbb{D}_M)
\]
by $L'_p(\sigma) = R\mu_1(\nu^p)^{-1} \circ L(\sigma) \circ R\mu_1(\nu^p)$ and $L'_q(\sigma) = R\mu_1(\nu^q)^{-1} \circ L(\sigma) \circ R\mu_1(\nu^q)$, respectively.

In the following, we always assume that $\nu_{M \times M}$ is chosen so that
\[
\nu_{M \times M} = (k'')^{-1} \circ (\nu^p_1 \boxtimes \nu^q_1) \circ k',
\]
where
\[
k' : R(\mu \times \mu)_1\mathbb{D}_{M \times M} \longrightarrow R\mu_1\mathbb{D}_M \boxtimes R\mu_1\mathbb{D}_M
\]
and
\[
k'' : R(\mu \times \mu)_1\mathbb{Q}_{M \times M} \longrightarrow R\mu_1\mathbb{Q}_M \boxtimes R\mu_1\mathbb{Q}_M
\]
are Künneth isomorphisms.

The next lemma follows from the naturality of $k'$.

**Lemma 4.5.1.** For $\sigma$ and $\sigma'$ in $\Sigma$, the diagram

\[
\begin{array}{ccc}
R(\mu \times \mu)_1\mathbb{D}_{M \times M} & \xrightarrow{k'} & R\mu_1\mathbb{D}_M \boxtimes R\mu_1\mathbb{D}_M \\
L'_p(\sigma, \sigma') \downarrow & & \downarrow L'_p(\sigma) \boxtimes L'_q(\sigma') \\
R(\mu \times \mu)_1\mathbb{Q}_{M \times M} & \xrightarrow{k'} & R\mu_1\mathbb{Q}_M \boxtimes R\mu_1\mathbb{Q}_M \\
\end{array}
\]

commutes.

The lemma shows that if $\Sigma \times \Sigma$ acts on $\text{Ext}_{N_0}^\bullet (\mathbb{Q}_{\mathbb{Q}_0}, j_N^! \delta^! (R\mu_1\mathbb{D}_M \boxtimes R\mu_1\mathbb{D}_M))$ by
\[
(\sigma, \sigma') \cdot f = j_N^! \delta^! (L'_p(\sigma) \boxtimes L'_q(\sigma')) \circ f,
\]
then $j_N^! \delta^! (k')_2$ is $\Sigma \times \Sigma$-equivariant.

4.6. Here we show that if $\Sigma \times \Sigma$ acts on $\text{Ext}_{N_0}^\bullet (\mathbb{Q}_{\mathbb{Q}_0}, j_N^! \delta^! ((R\mu_1\mathbb{Q}_M)^\vee \boxtimes R\mu_1\mathbb{D}_M))$ by
\[
(\sigma, \sigma') \cdot f = j_N^! \delta^! (L(\sigma^{-1})^\vee \boxtimes L'_q(\sigma')) \circ f,
\]
then $j_N^! \delta^! (c^{-1} \boxtimes \text{id})_2$ is $\Sigma \times \Sigma$-equivariant. In order to do this, it is enough to show that $c : (R\mu_1\mathbb{Q}_M)^\vee \rightarrow R\mu_1\mathbb{D}_M$ intertwines $L(\sigma^{-1})^\vee$ and $L'_p(\sigma)$ for $\sigma$ in $\Sigma$.

In the rest of this subsection, we denote $\nu^p_1$ and $L'_p$ simply by $\nu^p$ and $L'$, respectively.

It is shown in [1] Theorem 9.8] that the Verdier dual of the intersection complex of a local system is, up to a shift, the intersection complex of the dual local system. Also, in the equivalence between local systems and representations of the fundamental group, the dual of a local system corresponds to the contragredient representation, and the direct image of local systems corresponds to the induced representation. On the representation theory side, we are considering permutation representations, which are obviously equivalent to their contragredients, so it is natural to expect that for $\sigma$ in $\Sigma$, the Verdier dual of $\sigma$ acting on $(R\mu_1\mathbb{Q}_M)^\vee$ may be identified with $\sigma^{-1}$ acting on $(R\mu_1\mathbb{Q}_M)$. This is indeed the case, and the next proposition gives the precise formulation we need.
Proposition 4.6.1. If \( c = R\mu_!(d\mu^{-1}_M \circ (\beta^{-1}_\mu)_L) \circ \phi_\mu \), then the diagram

\[
(R\mu_!(Q_M))^\vee \xrightarrow{L(\sigma^{-1})^\vee} (R\mu_!Q_M)^\vee \\
\downarrow c \downarrow c \\
R\mu_!D_M \xrightarrow{L'(\sigma)} R\mu_!D_M
\]

of isomorphisms of complexes in \( D^b_c(N) \) commutes for every \( \sigma \) in \( \Sigma \).

Proof. It follows from [1, Theorem 9.8] that there is a unique isomorphism

\[ vd: IC(N, (\mu_\tau)_!Q_{M_\tau})^\vee[-2d] \rightarrow IC(N, ((\mu_\tau)_!Q_{M_\tau})^\vee[-2d]) \]

with the property that \( i_\tau^*(vd) = (\beta^{-1}_N)_L \circ \text{nat} \).

Define \( \nu_{M_\tau} = \alpha_{i_M} \circ i_M^1(\nu_M) \circ \beta_M \); so \( \nu_{M_\tau} : D_{M_\tau} \rightarrow Q_{M_\tau}[2d] \) is an isomorphism.

Now consider the "cube"

\[
\begin{array}{ccc}
   & IC(N, (\mu_\tau)_!Q_{M_\tau})^\vee[-2d] & \\
   \downarrow x & & \downarrow x \\
   IC(N, (\mu_\tau)_!Q_{M_\tau}) & IC(L_\tau(\sigma)) & IC(N, (\mu_\tau)_!Q_{M_\tau}) \\
   \downarrow \nu & \downarrow \nu & \downarrow \nu \\
   (R\mu_!Q_M)^\vee[-2d] & (R\mu_!Q_M)^\vee[-2d] & (R\mu_!Q_M)^\vee[-2d] \\
   \downarrow \nu^{-1} & \downarrow \nu^{-1} & \downarrow \nu^{-1} \\
   R\mu_!Q_M & R\mu_!Q_M & R\mu_!Q_M
\end{array}
\]

where \( x = IC((\mu_\tau)_!(\nu_M \circ dc^{-1}_M \circ (\beta^{-1}_\mu)_L) \circ \phi_\mu) \circ vd \), \( y = R\mu_!(\nu_M) \circ c \), and \( \phi_\mu \) is as in §2.2. It follows from the definitions of \( y \) and \( L' \) that it is enough to show that the front face commutes. We show that all faces besides the front face commute, and so the front face must commute also.
The top and bottom faces of (4.6.2) commute by definition and the left and right faces are equal, so we need to show that the back face and the left face commute.

To show that the left face of (4.6.2) commutes we need to show that the diagram

(4.6.3) \[
\begin{array}{ccc}
IC(N, (\mu_r)\mathbb{Q}_{M_r})[-2d] & \xrightarrow{\quad \nu_d \quad} & (R\mu_!\mathbb{Q}_M)[-2d] \\
\downarrow R\mu_!((\beta_{\mu}^{-1})_*\circ \phi_{\mu}) & & R\mu_!((\beta_{\mu}^{-1})_*\circ \phi_{\mu}) \\
IC(N, ((\mu_r)\mathbb{Q}_{M_r})[-2d]) & \xrightarrow{\quad IC((\mu_r)((\beta_{\mu}^{-1})_1)\circ \phi_{\mu}) \quad} & R\mu_!\mathbb{D}_M[-2d] \\
IC(N, (\mu_r)\mathbb{Q}_{M_r})[-2d]) & \xrightarrow{\quad IC((\mu_r)((\beta_{\mu}^{-1})_1)\circ \phi_{\mu}) \quad} & R\mu_!(\nu_{M_r}) \\
IC(N, (\mu_r)\mathbb{Q}_{M_r}) & \xrightarrow{\quad \overline{\nu}^{-1} \quad} & R\mu_!\mathbb{Q}_M
\end{array}
\]

commutes.

For the rest of this proof, set $bc = bc_{\mu_r,i_{M_r}}$.

As in the proof of Corollary 3.3.5 since we have

\[
R\mu_!\mathbb{Q}_M[-2d] \cong IC(N, (\mu_r)\mathbb{Q}_{M_r} \mathbb{Q}_{M_r}[-2d])
\]

and

\[
R\mu_!\mathbb{D}_M[-2d] \cong IC(N, (\mu_r)\mathbb{Q}_{M_r} \mathbb{Q}_{M_r}[-2d]),
\]

there are isomorphisms

\[
\text{ic}_c: R\mu_!\mathbb{Q}_{M_r}[-2d] \xrightarrow{\quad} IC(N, (\mu_r)\mathbb{Q}_{M_r} \mathbb{Q}_{M_r}[-2d])
\]

and

\[
\text{ic}_d: R\mu_!\mathbb{D}_M[-2d] \xrightarrow{\quad} IC(N, (\mu_r)\mathbb{Q}_{M_r} \mathbb{Q}_{M_r}[-2d])
\]

in $D(N)$ with $i_N^*(\text{ic}_c) = id$ and $i_N^*(\text{ic}_d) = id$. Define

\[
z = \text{ic}_c^{-1} \circ IC(bc^{-1} \circ (\mu_r)((\text{nat}_{i_{M_r}})^{-1}_* \circ (\beta_{i_{M_r}})_2 \circ \alpha_{i_{M_r}}^{-1}))), \quad w = \text{ic}_d^{-1} \circ IC(bc^{-1} \circ (\mu_r)((\beta_{i_{M_r}})))
\]

and recall that $\overline{\nu}^{-1} = ic_{\mu, i_{M_r}}^{-1} \circ IC(bc^{-1} \circ (\mu_r)((\alpha_{i_{M_r}})^{-1}))$.

Since all the complexes in (4.6.3) are in the image of IC, it is enough to show that (4.6.3) commutes after applying $i_N^*$. 
First, it follows from the definition of $\nu_{M_r}$ and the naturality of $bc$ that

$$i_N^* (\Pi^{-1} \circ IC((\mu_r)(\nu_{M_r}))) = bc^{-1} \circ (\mu_r)(\alpha_{M}^{-1}) \circ (\mu_r)(\nu_{M_r})$$

$$= i_N^* R\mu_!(\nu_{M_r}) \circ bc^{-1} \circ (\mu_r)(\beta_{M})$$

$$= i_N^* (R\mu_!(\nu_{M_r}) \circ w).$$

Second, it follows from (2.2.1) applied to $i_N$, and the naturality of $bc$ that

$$i_N^* (R\mu_!(dc_M^{-1} \circ z) = i_N^* R\mu_!(dc_M^{-1} \circ bc^{-1} \circ (\mu_r)(\beta_{M}) \circ \alpha_{M}^{-1})$$

$$= bc^{-1} \circ (\mu_r)(\beta_{M}) \circ dc_M^{-1}$$

$$= i_N^* (w \circ IC((\mu_r)(dc_M^{-1}))).$$

Lastly, it follows from the naturality of $\nu_{M_r}$, $\nu_{i_M}$, $bc$, $\phi_{\mu}$, and $\phi_{\nu}$, Lemma (2.2.2), and the equality $i_N^*(vd) = (\beta_{M}^{-1})_z \circ \nu_{i_M}$, that

$$i_N^* (R\mu_!((\beta_{M}^{-1})_z) \circ \phi_{\mu} \circ \Pi^{-1})$$

$$= i_N^* (R\mu_!((\beta_{M}^{-1})_z) \circ \phi_{\mu} \circ \nu_{i_M}^{-1} \circ (\mu_r)(\alpha_{i_M}^{-1}) \circ bc \circ \phi_{\nu} \circ \nu_{i_M}^{-1})$$

$$= bc^{-1} \circ (\mu_r)(\nu_{i_M}^{-1} \circ (\beta_{M}) \circ \alpha_{i_M}^{-1} \circ (\beta_{M}) \circ \phi_{\nu} \circ \nu_{i_M}^{-1})$$

$$= i_N^* (z \circ IC((\mu_r)(\nu_{i_M}^{-1} \circ (\beta_{M}) \circ \phi_{\nu} \circ \nu_{i_M}^{-1})).$$

Finally, consider the back face of diagram (4.6.2). It follows from the uniqueness of $vd$ that $vd \circ IC(L_r(\sigma^{-1}) \circ vd$. Thus, too show that the back face commutes, it is enough to show that

$$(\mu_r)(\nu_{M_r} \circ dc_M^{-1} \circ (\beta_{M}^{-1})_z) \circ \phi_{\mu_r} \circ L_r(\sigma^{-1}) \circ vd = L_r(\sigma) \circ (\mu_r)(\nu_{M_r} \circ dc_M^{-1} \circ (\beta_{M}^{-1})_z) \circ \phi_{\mu_r}.$$

In other words, we need to show that the diagram of local systems

$$\begin{array}{ccc}
((\mu_r)(Q_{M_r})^\vee [-2d] & L_r(\sigma^{-1}) & ((\mu_r)(Q_{M_r})^\vee [-2d] \\
((\mu_r)(\nu_{M_r} \circ dc_M^{-1})) \circ \phi_{\mu_r} & (\mu_r)(\nu_{M_r} \circ dc_M^{-1})) \circ \phi_{\mu_r} & (\mu_r)(\nu_{M_r} \circ dc_M^{-1})) \circ \phi_{\mu_r} \\
(\mu_r)(Q_{M_r}^\vee [-2d] & (\mu_r)(Q_{M_r}^\vee [-2d] & (\mu_r)(Q_{M_r}^\vee [-2d] \\
((\mu_r)(\nu_{M_r} \circ dc_M^{-1})) & ((\mu_r)(\nu_{M_r} \circ dc_M^{-1})) & ((\mu_r)(\nu_{M_r} \circ dc_M^{-1})) \\
(\mu_r)(\nu_{M_r}) & (\mu_r)(\nu_{M_r}) & (\mu_r)(\nu_{M_r}) \\
(\mu_r)(Q_{M_r}) & (\mu_r)(Q_{M_r}) & (\mu_r)(Q_{M_r}) \\
L_r(\sigma) & L_r(\sigma) & L_r(\sigma)
\end{array}$$

commutes.

Using the fact that $((\mu_r)(Q_{M_r})^\vee [-2d]$ is isomorphic to the dual local system, $((\mu_r)(Q_{M_r})^\vee$, and $(\mu_r)(Q_{M_r}^\vee [-2d]$ is isomorphic to $(\mu_r)(Q_{M_r}^\vee$, since $M_r$ and $N_r$ are
rational homology manifolds, it is straightforward to show that for $x$ in $N_r$, the
diagram obtained from diagram (4.6.4) by taking the stalk at $x$ commutes. It
follows that diagram (4.6.4) commutes as desired. □

4.7. In this subsection we show that if $\Sigma \times \Sigma$ acts on $\text{Ext}^{\bullet}_{N_0}(j_N^* R\mu|Q_M, j_N^* R\mu|D_M)$
by
\[(\sigma, \sigma') \cdot f = j_N^* L_Q(\sigma') \circ f \circ j_N^* L(\sigma^{-1}) = (j_N^* L(\sigma^{-1})^2 \circ j_N^* L_Q(\sigma')^2) (f),\]
then $\text{can}^{-1} \circ (\text{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_2$ is $\Sigma \times \Sigma$-equivariant.

Suppose $\sigma$ and $\sigma'$ are in $\Sigma$ and $f$ is in $\text{Ext}^{\bullet}_{N_0}(Q_{N_0}, j_N^! \delta^! ((R\mu|Q_M)^\vee \otimes R\mu|D_M))$. Then,
setting $u = L(\sigma^{-1})$, $v = L_Q(\sigma')$ and using $\text{nat}_{\delta j_N} = \text{nat}_{j_N} \circ j_N^!(\text{nat}_{\delta})$, Corollary [2.3.2] the naturality of $\text{nat}_{j_N}$, and Proposition [2.3.3]
we have
\[
\text{can}^{-1} \circ (\text{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_2 ((\sigma, \sigma') \cdot f) \\
= \text{can}^{-1} \left(\text{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda \circ (u^\vee \otimes v)) \circ f\right) \\
= \text{can}^{-1} \left(\text{nat}_{j_N} \circ j_N^! \left(\text{nat}_{\delta} \circ \delta^!(\lambda \circ (u^\vee \otimes v))\right) \circ f\right) \\
= \text{can}^{-1} \left(\text{nat}_{j_N} \circ j_N^! (u^\delta \circ v_\delta \circ \text{nat}_{\delta} \circ \delta^!(\lambda)) \circ f\right) \\
= \text{can}^{-1} \left(j_N^* (u)^\delta \circ j_N^* (v)_2 \circ \text{nat}_{j_N} \circ j_N^! (\text{nat}_{\delta} \circ \delta^!(\lambda)) \circ f\right) \\
= \text{can}^{-1} \left(j_N^* (u)^\delta \circ j_N^* (v)_2 \circ \text{can}^{-1} \left(\text{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda) \circ f\right)\right) \\
= (\sigma, \sigma') \cdot \left(\text{can}^{-1} \circ (\text{nat}_{\delta j_N} \circ j_N^! \delta^!(\lambda))_2 (f)\right).
\]

4.8. To complete the proof of Theorem [4.4.1] we need to show that
\[
\text{Ext}^{\bullet}_{N_0}(j_N^* R\mu|Q_M, j_N^* R\mu|D_M) \xrightarrow{(a^{-1})^1 \circ b_2} \text{Ext}^{\bullet+4n}_{N_0}(R(\mu_0)|Q_{M_0}, R(\mu_0)|Q_{M_0})
\]
is $\Sigma \times \Sigma$-equivariant where $\Sigma \times \Sigma$ acts on the domain as in §4.7. For this, it is
easy to show that
\[
b \circ j_N^* L_Q(\sigma') \circ f \circ j_N^* L(\sigma^{-1}) \circ a^{-1} = L_0(\sigma') \circ b \circ f \circ a^{-1} \circ L_0(\sigma^{-1})
\]
for $\sigma$ and $\sigma'$ in $\Sigma$ and $f$ in $\text{Ext}^{\bullet}_{N_0}(j_N^* R\mu|Q_M, j_N^* R\mu|D_M)$.

Recall from [4.4.2] and [4.4.3] that $a = R(\mu_0)! (\alpha_{j_M}) \circ bc^*$ and
\[
L_0(\sigma^{-1}) = R(\mu_0)! (\alpha_{j_M}) \circ bc^* \circ j_N^* L(\sigma^{-1}) \circ (R(\mu_0)! (\alpha_{j_M}) \circ bc^* )^{-1} = a \circ j_N^* L(\sigma^{-1}) \circ a^{-1}.
\]

Therefore, to show that $(a^{-1})^2 \circ b_2$ is $\Sigma \times \Sigma$-equivariant, it is enough to show that
\[
b \circ j_N^* L_Q(\sigma') = L_0(\sigma') \circ b.
\]
Recall from [4.4.3] that $b = R(\mu_0)! (\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ bc^!$, so we need to show that
\[
R(\mu_0)! (\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ bc^! \circ j_N^* (R(\mu_0)! (\nu_{M_0}^Q)^{-1} \circ L(\sigma) \circ R(\mu_0)! (\nu_{M_0}^Q))
\]
\[
= R(\mu_0)! (\alpha_{j_M}) \circ bc^* \circ j_N^* L(\sigma) \circ (bc^*)^{-1} \circ R(\mu_0)! (\alpha_{j_M}^{-1}) \circ R(\mu_0)! (\nu_{M_0} \circ \beta_{j_M}^{-1}) \circ bc^!
\]
for $\sigma$ in $\Sigma$. 

Setting $\nu_M = \nu_M^Q$, and using the naturality of $bc^i$, it follows that it is enough to show that

$$j_N^i L(\sigma) \circ (bc^i)^{-1} \circ R(\mu_0); (j_N^i(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ bc^i = (bc^i)^{-1} \circ R(\mu_0); (j_N^i(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ bc^i \circ j_N^i L(\sigma).$$

Set $\tau = (bc^i)^{-1} \circ R(\mu_0); (j_N^i(\nu_M) \circ \beta_{j_M} \circ \nu_{M_0}^{-1} \circ \alpha_{j_M}) \circ bc^i$. Then $\tau$ is an isomorphism in $D^*(N_0)$, $\tau; j_N^i R\mu \mathbb{Q}_M \to j_N^i R\mu \mathbb{Q}_M[2l]$, where $l = \text{codim}_N N_0 = \text{codim}_M M_0$, and we need to show that

$$(4.8.1) j_N^i L(\sigma) \circ \tau = \tau \circ j_N^i L(\sigma).$$

We prove the following proposition in the Appendix.

**Proposition 4.8.2.** There is a natural transformation, $\rho^i_N; j_N^i \to j_N^i[2l]$, so that $\tau = \rho^i_R \mu \mathbb{Q}_M$.

Given the truth of the proposition, it follows from the naturality of $\rho^i_N$ that $j_N^i(g) \circ \tau = \tau \circ j_N^i(g)$ for $g$ in $\text{End}_N(R\mu \mathbb{Q}_M)$, and so in particular (4.8.1) holds for $\sigma$ in $\Sigma$. This completes the proof of Theorem 4.4.1.

### 5. Generalized Steinberg varieties

5.1. In this section we apply the results of §3 and §4 to generalized Steinberg varieties.

We start with the following incarnation of the basic commutative diagram (3.1.1) as in [3]:

$$(5.1.1) \xymatrix{ \tilde{N} \ar[r]^{\eta^p} \ar[d]_{j_N} & \tilde{N}_p \ar[r]^{\xi^p} \ar[d]_{j_N^p} & N \ar[d]_{j_N} \\
\tilde{g} \ar[r]^{\eta^p} & \tilde{g}_p \ar[r]^{\xi^p} & g }$$

The notation is as follows:

- $G$ is a connected, reductive, complex algebraic group with Lie algebra $\mathfrak{g}$, $\mathcal{N}$ is the cone of nilpotent elements in $\mathfrak{g}$, and $\mathfrak{g}_{rs}$ is the open subvariety of regular semisimple elements in $\mathfrak{g}$.
- $\mathcal{P}$ is a conjugacy class of parabolic subgroups of $G$.
- $\mathfrak{g} = \{ (x,B) \in \mathfrak{g} \times \mathcal{B} \mid x \in \text{Lie}(B) \}$, where $\mathcal{B}$ is the variety of Borel subgroups of $G$ and $\mathfrak{g}_p = \{ (x,P) \in \mathfrak{g} \times \mathcal{P} \mid x \in \text{Lie}(P) \}$.
- The maps $\eta^p$ are defined by $\eta^p(x,B) = (x,P)$ where $P$ is the unique subgroup in $\mathcal{P}$ that contains $B$.
- The maps $\xi^p$ are projection on the first factor.
Theorem 5.1.3. If $H_*$ is given by the $W \times W$-action induced from the $W \times W$-action on $(\mu_\text{reg}, \mu_\text{reg}) : \mathbb{g}_\text{rs} \times \mathbb{g}_\text{rs}$, then there is an isomorphism of vector spaces,

\[
\eta^P = \xi^P \circ \eta^P \quad \text{is the projection on the first factor.}
\]

\[
\eta^P = \xi^P \circ \eta^P = \{ (x, B) \in \mathbb{g}_\text{rs} \times B \mid x \in \text{Lie}(B) \} \quad \text{and } \eta^Q = \xi^Q \circ \xi^Q \circ \eta^Q = \{ (x, P) \in \mathbb{g}_\text{rs} \times P \mid x \in \text{Lie}(P) \}.
\]

\[
N = \mu^{-1}(\mathbb{N}) = \{ (x, B) \in \mathbb{N} \times B \mid x \in \text{Lie}(B) \} \quad \text{and } N^P = \xi^P \circ \mu^P = \{ (x, P) \in N \times P \mid x \in \text{Lie}(P) \}.
\]

In accordance with the notation above, we also assume that $\dim G = \dim \mathbb{g} = d$, $\dim \mathbb{N} = 2n$, and $l = d - 2n = \text{codim}_G \mathbb{N}$.

It is shown in [3] that diagram (5.1.1) has properties D1 – D7 of the basic commutative diagram. For the convenience of the reader, we recall the group action involved in properties D6 and D7.

Fix a maximal torus, $T$, and a Borel subgroup, $B$, of $G$ with $T \subseteq B$. Define $t = \text{Lie}(T)$ and $t_\text{reg} = t \cap \mathbb{g}_\text{rs}$, so $t_\text{reg}$ is the set of regular semisimple elements in $t$. Let $W = N_G(T)/T$ be the Weyl group of $(G, T)$. Then $W$ acts on $t_{\text{reg}} \times G/T$ on the right by $(t, gT) \cdot w = (\text{Ad}(w)^{-1} t, gwT)$ for $w$ in $W$, $t$ in $t_\text{reg}$, and $g$ in $G$. It is well-known and easy to check that the rule $(t, gT) \mapsto (\text{Ad}(g)t, gBg^{-1})$ defines an isomorphism of varieties $t_{\text{reg}} \times G/T \cong \mathbb{g}_\text{rs}$, and we use this isomorphism to transport the $W$-action from $t_{\text{reg}} \times G/T$ to $\mathbb{g}_\text{rs}$. It is also well-known and easy to prove that the projection on the first factor, from $\mathbb{g}_\text{rs}$ to $\mathbb{g}_\text{rs}$, is an orbit map for the right $W$-action on $\mathbb{g}_\text{rs}$. Thus, diagram (5.1.1) has property D6.

Next, let $P$ be the subgroup in $P$ with $B \subseteq P$ and set $W_P = N_P(T)/T$, the Weyl group of $P$, so $W_P$ is a subgroup of $W$. It is straightforward to check that $\eta^P|_{\mathbb{g}_\text{rs}}$ is an orbit map for the action of $W_P$ on $\mathbb{g}_\text{rs}$. Thus, diagram (5.1.1) has property D7.

If $Q$ is a second conjugacy class of parabolic subgroups of $G$, then the two variable version of diagram (5.1.1), as in §3.5, is the following:

\[
\begin{array}{c}
\bigcirc \quad Z \quad \eta \quad \xi \quad \delta \circ \mathbb{N} \quad \bigcirc \\
\downarrow \quad \eta^P \times \eta^Q \quad \downarrow \quad \xi^P \times \xi^Q \quad \downarrow \quad \mathbb{g} \times \mathbb{g} \\
\mathbb{g}_\text{rs} \times \mathbb{g}_\text{rs} \quad \bigcirc \\
\bigcirc \downarrow \quad \mathbb{g}_\text{rs} \times \mathbb{g}_\text{rs} \quad \downarrow \quad \mathbb{g}_\text{rs} \times \mathbb{g}_\text{rs} \\
\end{array}
\]

Since $(\mathbb{g} \times \mathbb{g}) \times_{\mathbb{g} \times \mathbb{g}} \mathbb{N} = \{ ((x, B'), (x, B'')) \mid x \in \text{Lie}(B') \cap \text{Lie}(B'') \}$, we may identify $Z$ with the Steinberg variety of $G$. Then $j_z : Z \rightarrow \mathbb{g} \times \mathbb{g}$ by $j_z(x, B', B'') = ((x, B'), (x, B''))$.

Also, since $(\mathbb{g}^P \times \mathbb{g}^Q) \times_{\mathbb{g} \times \mathbb{g}} \mathbb{N} = \{ ((x, P'), (x, Q')) \mid x \in \text{Lie}(P') \cap \text{Lie}(Q') \}$, we may identify $X$ with the generalized Steinberg variety $X^P \times \mathbb{Q}$ from §1. Then $j_x : X^P \times \mathbb{Q} \rightarrow \mathbb{g} \times \mathbb{g}$ by $j_x(x, P', Q') = ((x, P'), (x, Q'))$.

Applying Corollary 5.5.2, we have our first main result.

**Theorem 5.1.3.** If $H_* (Z)$ is given by the $W \times W$-action induced from the $W \times W$-action on $(\mu_\text{rs} \times \mu_\text{rs}) : \mathbb{g}_\text{rs} \times \mathbb{g}_\text{rs}$, then there is an isomorphism of vector spaces,
\( H_\bullet(X^P, \mathbb{Q}) \cong H_\bullet(Z)^{W_P \times W_Q} \), so that the diagram

\[
\begin{array}{ccc}
H_\bullet(Z) & \xrightarrow{\eta_*} & H_\bullet(X^P, \mathbb{Q}) \\
\downarrow{\text{Av}} & & \downarrow{\cong} \\
H_\bullet(Z)^{W_P \times W_Q} & & 
\end{array}
\]

commutes.

5.2. Now we consider the special case of Theorem 5.1.3 when \( \bullet = 2 \dim Z = 4n \) as in §4.1. Borho and MacPherson [2] have shown that the \( \mathbb{Q} \)-algebra homomorphism, \( \mathbb{Q}W \to \text{End}_\mathcal{N}(R(\mu_0); \mathbb{Q}_\mathcal{N}) \), from §4.2 is an isomorphism. Therefore, from (4.1.1) we get the result originally proved by Kazhdan and Lusztig [12] and strengthened by Chriss and Ginzburg [4].

**Theorem 5.2.1.** If \( W \times W \) acts on \( \mathbb{Q}W \) by \( (w, w') \cdot x = w'xw^{-1} \), then there are \( W \times W \)-equivariant isomorphisms

\[
\begin{align*}
\mathbb{Q}W & \xrightarrow{\cong} \text{End}_\mathcal{N}(R(\mu_0); \mathbb{Q}_\mathcal{N}) \xrightarrow{\cong} H_{4n}(Z)^{\text{op}}. \\
\end{align*}
\]

Recall that \( e_P \) denotes the primitive idempotent in \( \mathbb{Q}W_P \) corresponding to the trivial representation of \( W_P \). Since \( (\mathbb{Q}W)^{W_P \times W_Q} = e_Q \mathbb{Q}W e_P \), the next corollary follows immediately from Theorems 5.1.3 and 5.2.1.

**Corollary 5.2.2.** The \( W \times W \)-equivariant isomorphism \( \mathbb{Q}W \xrightarrow{\cong} H_{4n}(Z)^{\text{op}} \) in Theorem 5.2.1 induces an isomorphism between the subspace \( e_Q \mathbb{Q}W e_P \) of \( \mathbb{Q}W \) and \( H_{4n}(X^P, \mathbb{Q}) \), the top Borel-Moore homology group of the generalized Steinberg variety, \( X^P, \mathbb{Q} \).

\[
\begin{array}{ccc}
\mathbb{Q}W & \xrightarrow{\cong} & H_{4n}(Z)^{\text{op}} \\
\downarrow{\text{Av}} & & \downarrow{\eta_*} \\
e_Q \mathbb{Q}W e_P & \cong & H_{4n}(X^P, \mathbb{Q})
\end{array}
\]

5.3. In this subsection, we use Corollary 5.2.2 to compute the action of a simple reflection in \( W \) on \( H_{4n}(Z) \). What we prove is the analog for \( H_{4n}(Z) \) of the “easy” part of Hotta’s transformations for the action of a simple reflection in the cohomology of a Springer fibre. Our argument is inspired by Hotta’s argument in [10].

It is well-known that \( W \) indexes the \( G \)-orbits on \( B \times B \) and that if \( Z_w \) denotes the preimage of the orbit indexed by \( w \) in \( W \) under the projection of \( Z \) onto \( B \times B \) given by the projection on the second and third factors, then the dimension of \( Z_w \) is \( 2n \) and the irreducible components of \( Z \) are the closures of the \( Z_w \)’s (see [18]). Thus, if \( [Z_w] \) denotes the canonical class of \( Z_w \) in \( H_{4n}(Z) \), it follows that \( \{ [Z_w] \mid w \in W \} \) is a basis of \( H_{4n}(Z) \).

Recall that we have fixed a Borel subgroup, \( B \), of \( G \) containing \( T \). The choice of \( B \) determines a set of Coxeter generators of \( W \) and hence a length function and a partial order, the Bruhat order, on \( W \).
For the time being we fix a simple reflection, \( s \), in \( W \) and let \( \mathcal{P}_s \) denote the conjugacy class of minimal parabolic subgroups of \( G \) determined by \( s \). Then \( \mathcal{P}_s \) and \( \mathcal{B} \) are conjugacy classes of parabolic subgroups of \( G \) and we may consider \( \eta_* : H_{\text{un}}(Z) \to H_{\text{un}}(X_{\mathcal{P}_s \times \mathcal{B}}) \).

Let \( \mathcal{P}_s \) be the subgroup in \( \mathcal{P}_s \) that contains \( B \). It is shown in \([5, \S3]\) that if \( w \) is in \( W \), then \( \dim \eta_*(Z_w) = \dim Z_w \) if and only if \( w \) is minimal in its \((W_{\mathcal{P}_s}, W_B)\)-double coset. Since \( W_{\mathcal{P}_s} = \{ 1, s \} \) and \( W_B = \{ 1 \} \), it follows that \( w \) is minimal in its double coset if and only if \( sw > w \) in the Bruhat order. Therefore, if \( sw < w \) we have \( \eta_*((Z_w)) = 0 \). It follows that \( \dim \ker \eta_* \geq |W|/2 \).

On the other hand, by Corollary \([5,2.2]\) we may identify \( \eta_* \) with the averaging map onto the set of \( W_{\mathcal{P}_s} \times W_B \)-invariants in \( \mathbb{Q}W \). In this case, the averaging map from \( \mathbb{Q}W \) to \( \mathbb{Q}W_{W_{\mathcal{P}_s} \times W_B} \) is \( x \mapsto \frac{1}{2}(x + xs) \), and so its kernel is \( \{ x \in \mathbb{Q}W \mid xs = -x \} \) and has dimension equal \( |W|/2 \). Therefore, the kernel of \( \eta_* \) is the subspace \( \{ c \in H_{\text{un}}(Z) \mid s \cdot c = -c \} \), and it has dimension equal \( |W|/2 \). Since \( \ker \eta_* \) contains the linearly independent set \( \{ (Z_w) \mid sw < w \} \), it follows that \( \{ (Z_w) \mid sw < w \} \) is a basis of \( \ker \eta_* \). This proves the following theorem.

**Theorem 5.3.1.** If \( s \) is a simple reflection in \( W \), then \( \{ (Z_w) \mid sw < w \} \) is a basis of the subspace \( \{ c \in H_{\text{un}}(Z) \mid s \cdot c = -c \} \) of \( H_{\text{un}}(Z) \). In particular, if \( w \) is in \( W \) and \( s \) is a simple reflection, then \( s \cdot (Z_w) = -Z_w \) if and only if \( sw < w \) in the Bruhat order.

5.4. We now turn to computing the top Borel-Moore homology group of the generalized Steinberg variety \( Y^{\mathcal{P}, \mathcal{Q}} \). Recall that we have fixed parabolic subgroups, \( \mathcal{P} \) in \( \mathcal{P} \) and \( \mathcal{Q} \) in \( \mathcal{Q} \), with \( B \subseteq \mathcal{P} \cap \mathcal{Q} \). Then

\[
Y^{\mathcal{P}, \mathcal{Q}} = \{ (x, P', Q') \in \mathcal{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \text{Lie}(U_{P'}) \cap \text{Lie}(U_{Q'}) \} \subseteq X^{\mathcal{P}, \mathcal{Q}}
\]

and

\[
Z^{\mathcal{P}, \mathcal{Q}} = \eta^{-1}(Y^{\mathcal{P}, \mathcal{Q}}).
\]

Thus, we have a cartesian square

\[
\begin{array}{ccc}
Z^{\mathcal{P}, \mathcal{Q}} & \xrightarrow{j} & Z \\
\downarrow{\pi} & & \downarrow{\eta} \\
Y^{\mathcal{P}, \mathcal{Q}} & \xrightarrow{j} & X^{\mathcal{P}, \mathcal{Q}}
\end{array}
\]

where the horizontal arrows are inclusions and \( \pi \) is the restriction of \( \eta \) to \( Z^{\mathcal{P}, \mathcal{Q}} \).

It follows from the definitions that \( \pi \) is a fibre bundle with smooth fibres isomorphic to \( \mathcal{P}/B \times \mathcal{Q}/B \).

Define \( W^{\mathcal{P}, \mathcal{Q}} \) to be the set of maximal length \((W_{\mathcal{P}}, W_{\mathcal{Q}})\)-double coset representatives in \( W \), so \( W^{\mathcal{P}, \mathcal{Q}} \) indexes the \( G \)-orbits on \( \mathcal{P} \times \mathcal{Q} \).

It was shown in \([5, \S4]\) that if \( Y_w \) denotes the preimage of the orbit indexed by \( w \) in \( W^{\mathcal{P}, \mathcal{Q}} \) under the projection of \( Y^{\mathcal{P}, \mathcal{Q}} \) onto \( \mathcal{P} \times \mathcal{Q} \) given by the projection on the second and third factors, then the dimension of \( Y_w \) is \( \dim \mathcal{P} + \dim \mathcal{Q} \) and the irreducible components of \( Y^{\mathcal{P}, \mathcal{Q}} \) are the closures of the \( Y_w \)'s.
It was also shown in [3] §4 that \( \{ \overline{Z_w} \mid w \in W^{P,Q} \} \) is the set of irreducible components of \( Z^{P,Q} \). Clearly \( \overline{\eta}(Z_w) \subseteq Y_w \), and so since \( \overline{\eta} \) is proper, \( Z_w \) and \( Y_w \) are irreducible, and the fibres \( \overline{\eta} \) all have the same dimension, it follows that \( \overline{\eta}(Z_w) = Y_w \).

Since \( \overline{\eta} \) is a fibre bundle with smooth fibres, if \( f = \dim P/B + \dim Q/B \), then there is an inverse image map in Borel-Moore homology, \( \overline{\eta}: H_* (Y^{P,Q}) \to H_{*+2f}(Z^{P,Q}) \) (see [3] 8.3.31).

It is straightforward to check that if \( [Y_w] \) denotes the canonical class of \( Y_w \) in \( H_{4n-2f}(Y^{P,Q}) \), then \( \overline{\eta}([Y_w]) \) is a multiple of \( [Z_w] \) (see [4]). Since \( \dim H_{4n-2f}(Y^{P,Q}) = \dim H_{4n}(Z^{P,Q}) \) it follows that \( \overline{\eta} \) is injective.

Next, \( Z^{P,Q} \) is a closed subvariety of \( Z \), so if \( j \) denotes the inclusion, there is a direct image map in Borel-Moore homology, \( j_*: H_* (Z^{P,Q}) \to H_* (Z) \). It follows immediately that \( j_*(\overline{Z_w}) = \overline{Z_w} \) for \( w \in W^{P,Q} \) and that \( j_* \) is injective.

Combining the results in the last two paragraphs we have proven the next proposition.

**Proposition 5.4.1.** The mapping \( \overline{\eta}^*: H_{4n-2f}(Y^{P,Q}) \to H_{4n}(Z^{P,Q}) \) is an isomorphism of vector spaces, and the mapping \( j_*: H_{4n}(Z^{P,Q}) \to H_{4n}(Z) \) is injective with its image equal to the span of \( \{ \overline{Z_w} \mid w \in W^{P,Q} \} \).

5.5. We identify the image of \( H_{4n}(Z^{P,Q}) \) with its image in \( H_{4n}(Z) \). Then we see that \( H_{4n}(Z^{P,Q}) \) is the span of \( \{ \overline{Z_w} \mid w \in W^{P,Q} \} \) in \( H_{4n}(Z) \) and \( H_{4n}(Z^{P,Q}) = H_{4n-2f}(Y^{P,Q}) \). Define \( \overline{H}^{P,Q} \) to be the subspace of \( c \) in \( H_{4n}(Z) \) with the property that \( s \cdot c = -c \) and \( t \cdot c = -c \) for all simple reflections, \( s \) in \( W_P \) and \( t \) in \( W_Q \). It follows from Theorem 5.3.1 that \( H_{4n}(Z) \supseteq \overline{H}^{P,Q} \).

Recall that \( \epsilon_P \) and \( \epsilon_Q \) denote the primitive idempotents in \( W_P \) and \( W_Q \) corresponding to the sign representations of \( W_P \) and \( W_Q \), respectively. Then \( \dim \epsilon_Q W \epsilon_P = |W^{P,Q}| \) and \( \epsilon_Q W \epsilon_P \) is the set of all \( x \) in \( QW \) with the property that \( sx = -x \) and \( xt = -x \) for all simple reflections, \( s \) in \( W_Q \), and \( t \) in \( W_P \). It follows from Theorem 5.2.1 that under the isomorphism \( QW \cong H_{4n}(Z)^{op} \), the subspace \( \overline{H}^{P,Q} \) is the image of \( \epsilon_Q W \epsilon_P \). Therefore, \( \dim \overline{H}^{P,Q} = |W^{P,Q}| \) and hence \( \overline{H}^{P,Q} = H_{4n}(Z^{P,Q}) \). This proves the following theorem.

**Theorem 5.5.1.** The \( W \times W \)-equivariant isomorphism \( QW \cong H_{4n}(Z)^{op} \) in Theorem 5.2.1 induces an isomorphism between the subspace \( \epsilon_Q W \epsilon_P \) of \( QW \) and \( H_{4n-2f}(Y^{P,Q}) \), the top Borel-Moore homology group of the generalized Steinberg variety, \( Y^{P,Q} \),

\[
\begin{align*}
\epsilon_Q W \epsilon_P & \cong H_{4n-2f}(Y^{P,Q}) \\
QW & \cong H_{4n}(Z)^{op}
\end{align*}
\]

where the left vertical arrow is inclusion.

**Appendix A**

A.1. In this appendix we change notation slightly from [24]. For a morphism, \( \xi: X \to Y \), of complex, algebraic varieties, the units of the adjoint pairs \( (\xi^\star, \xi) \) and \( (\xi_1^\star, \xi_1) \) are denoted by \( \eta^\star_\xi \) and \( \eta^\star_1 \), respectively. Similarly, the counits are denoted by \( \epsilon_\xi \) and \( \epsilon_1 \), respectively.
Suppose $\xi: X \to Y$ is a morphism between complex, algebraic varieties that are rational homology manifolds. For a fixed choice of isomorphisms $\nu_X: \mathbb{D}_X \xrightarrow{\cong} \mathbb{Q}_X[2\dim X]$ and $\nu_Y: \mathbb{D}_Y \xrightarrow{\cong} \mathbb{Q}_Y[2\dim Y]$ there is a natural transformation $\rho^\xi: \xi^* \to \xi_![2l]$ where $l = \dim Y - \dim X$. For a complex $A$ in $D^b(X^\xi)$, $\rho^\xi = \rho^\xi_A$ is defined to be the composition

$$\xi^* A \xrightarrow{m_1^{-1}} \xi^* A \xrightarrow{\omega_\xi \otimes \text{id}} \xi^!(\xi_X \otimes \xi^* A)[2l]$$

where the notation is as follows:

- $m_1: \mathbb{Q}_X \otimes B \xrightarrow{\cong} B$ is the natural isomorphism for $B$ in $D(X)$.
- $\omega_\xi = \xi^!(\nu_Y) \circ \beta_\xi \circ \nu_X^{-1}: \mathbb{Q}_X \xrightarrow{\cong} \xi^!(\mathbb{Q}_Y[2l])$, so $\omega_\xi$ is an isomorphism in $D^b(X)$.
- $\eta_\xi$ and $\epsilon_\xi$ are as above.
- For $B$ in $D^b(X)$ and $C$ in $D^b(Y)$, $\text{pr}_\xi: R\xi(B \otimes C) \xrightarrow{\cong} R\xi(B \otimes \xi^*C)$ is the projection isomorphism.

Notice that $\rho^\xi$ is a natural transformation since each map in the definition of $\rho^\xi_A$ is natural in $A$.

Now consider a cartesian square

$$(A.1.1) \quad \begin{array}{c}
M_0 \xrightarrow{\mu_0} N_0 \\
\downarrow j_M \quad \quad \quad \downarrow j_N \\
M \xrightarrow{\mu} N
\end{array}$$

satisfying the following conditions:

- C1 The spaces are all complex, algebraic varieties that are rational homology manifolds.
- C2 The maps are all proper morphisms.
- C3 $j_M$ and $j_N$ are closed embeddings.
- C4 $\dim M_0 = \dim N_0 = 2n$, $\dim M = \dim N = d$, and $l = d - 2n$.

For a cartesian square as in $(A.1.1)$, we have base change isomorphisms

$$\text{bc}^*: j^*_N R\mu_{|j^*_M} \xrightarrow{\cong} R(\mu_0)|j^*_M$$

and

$$\text{bc}^!: j^!_N R\mu_{|j^!_M} \xrightarrow{\cong} R(\mu_0)|j^!_M$$

defined as in $(2.5)$.

We prove the following lemmas in the next two subsections.

**Lemma A.1.2.** If $X$ and $Y$ are complex, algebraic varieties that are rational homology manifolds and $\xi: X \to Y$ is a proper morphism, then $\rho^\xi_{\mathbb{Q}_Y} = \omega_\xi \circ \alpha_\xi$.

**Lemma A.1.3.** If $\nu_{M_0}$ is chosen appropriately, then in the cartesian square $(A.1.1)$ the morphisms $\rho^\xi_{\mathbb{Q}_M}$ and $\rho^\xi_{R\mu\mathbb{Q}_M}$ are related by

$$\text{bc}^! \circ \rho^\xi_{R\mu\mathbb{Q}_M} = R(\mu_0)|j^!_M \circ (\rho^\xi_{\mathbb{Q}_M}) \circ \text{bc}^*.$$
by
\[ \tau = (bc)^{-1} \circ R(\mu_0); (\nu_N^1 \circ \beta_M) \circ \nu_{M_0}^{-1} \circ \alpha_M \circ bc^* \]
where \( \nu_M : D_M \xrightarrow{\cong} Q_M[2d] \) and \( \nu_M : D_{M_0} \xrightarrow{\cong} Q_{M_0}[4n] \) are isomorphisms in \( D^b_L(M) \) and \( D^b_L(M_0) \), respectively.

Assuming Lemmas A.1.2 and A.1.3 have been proved we have
\[
\rho_{R\mu_0,J_M}^{j_1} = (bc)^{-1} \circ R(\mu_0); (\nu_N^1 \circ \beta_M) \circ \nu_{M_0}^{-1} \circ \alpha_M \circ bc^* = (bc)^{-1} \circ R(\mu_0); (\nu_N^1 \circ \beta_M) \circ \nu_{M_0}^{-1} \circ \alpha_M \circ bc^* = \tau.
\]
This proves Proposition 4.8.2.

A.2. In this subsection we prove Lemma A.1.2. Before doing so, we need some preliminary results.

If \( A \) is in \( D^b_L(X) \) we denote the canonical isomorphisms \( Q_X \otimes A \xrightarrow{\cong} A \) and \( A \otimes Q_X \xrightarrow{\cong} A \) by \( m_1 \) and \( m_2 \), respectively. When \( A = Q_X \) we set \( m = m_1 = m_2 \).

The proof of the next lemma is a straightforward computation using stalks and is omitted.

**Lemma A.2.1.** Suppose \( A \) and \( B \) are in \( D^b_L(X) \) and \( p_A : A \rightarrow Q_X \), and \( p_B : B \rightarrow Q_X \) are two morphisms in \( D^b_L(X) \). Then the diagrams

\[
\begin{array}{ccc}
R\xi(A) \otimes Q_Y & \xrightarrow{pr_X} & R\xi(A \otimes \xi^* Q_Y) \\
m_2 & & R\xi(id \otimes \alpha) \\
& & R\xi(A \otimes Q_X) \\
m_1 & & \end{array}
\]

and

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{id \otimes p_B} & A \otimes Q_X \\
p_A \otimes id & & m_2 \\
& & A \\
p_A & & m_1 \\
B & \xrightarrow{id \otimes p_B} & Q_X \\
p_B & & m \\
& & Q_X \\
& & \end{array}
\]

commute.

Let \( nat^\otimes : \xi^*(A \otimes B) \xrightarrow{\cong} \xi^* A \otimes \xi^* B \) denote the canonical isomorphism in \( D^b(X) \).

**Lemma A.2.2.** The diagram

\[
\begin{array}{ccc}
R\xi Q_Y \otimes Q_Y & \xrightarrow{pr_X} & R\xi(Q_Y \otimes \xi^* Q_Y) \\
c_\xi \otimes id & & R\xi(id \otimes \alpha) \\
& & R\xi(id \otimes \alpha) \\
Q_Y \otimes Q_Y & \xrightarrow{\Phi^{-1}(m_2)} & Q_Y \\
m & & m \\
& & \end{array}
\]

commutes.

**Proof.** Using the definition of \( \Phi^{-1} \) we have
\[
\Phi^{-1}(m_2) \circ R\xi(id \otimes \alpha) \circ pr_X = c_\xi \circ R\xi(m_2) \circ R\xi(id \otimes \alpha) \circ pr_X.
\]
Also, using the naturality of $\epsilon_1^1$ we have $m \circ (\epsilon_1^1 \otimes id) = \epsilon_1^1 \circ m_2$, so it is enough to show that

$$m_2 = R\xi_t(m_2) \circ R\xi_t(id \otimes \alpha_\xi) \circ pr_\xi.$$  \hfill (A.2.3)

Since $\xi$ is proper, we have $pr_\xi = \Psi_\xi((\epsilon_1^1 \otimes id) \circ \text{nat}_\xi^\otimes).$

The proof of (A.2.3) is a straightforward computation using the formula for $pr_\xi$ and Lemma A.2.2. We omit the details. \hfill \Box

We can now complete the proof of Lemma A.1.2. Recall that

$$\rho_{\xi_t}^V = \xi^1 \left( m_1 \circ \epsilon_1^1 \circ pr_\xi^{-1} \right) \circ \eta_1^1 \circ (\omega_\xi \otimes id) \circ m_1^{-1} = \Phi_\xi(m_1 \circ \epsilon_1^1 \circ pr_\xi^{-1}) \circ (\omega_\xi \otimes id) \circ m_1^{-1},$$

so to prove the lemma, we need to show that

$$\Phi_\xi(m_1 \circ \epsilon_1^1 \circ pr_\xi^{-1}) = \omega_\xi \circ \alpha_\xi \circ m_1 \circ (\omega_\xi^{-1} \otimes id).$$

Taking $A = \xi^!Q_Y[2]$, $p_A = \omega_\xi^{-1}$, $B = \xi^*Q_Y$, and $p_B = \alpha_\xi$ in Lemma A.2.1 we get

$$\alpha_\xi \circ m_1 \circ (\omega_\xi^{-1} \otimes id) = \omega_\xi^{-1} \circ m_2 \circ (id \otimes \alpha_\xi),$$

so it is enough to show that $\Phi_\xi(m_1 \circ \epsilon_1^1 \circ pr_\xi^{-1}) = m_2 \circ (id \otimes \alpha_\xi)$. This last equality follows immediately from Lemma A.2.2. This completes the proof of Lemma A.1.2.

A.3. In this subsection we prove Lemma A.1.3.

The proof is accomplished by showing that the diagrams (A.3.1) and (A.3.2) below are commutative. Then juxtaposing these diagrams and tracing around the outside gives the desired result.

It is easy to see that any unlabeled regions of diagrams (A.3.1) and (A.3.2) commute. The commutativity of the labeled regions is shown in the corresponding statements below.

To make the diagrams as clear as possible, we need to simplify the notation. First, for a morphism, $\xi: X \to Y$, we denote the derived functors $R\xi_*$ and $R\xi_!$ simply by $\xi_*$ and $\xi_!$, respectively. Second, we denote $j_N$, simply by $j$. Third, we label the maps in the diagrams using only the core maps or natural transformations involved. For example, we write $\alpha_{\mu_0}$ instead of $(\mu_0)!((\alpha_{\mu_0} \otimes id)$ and $bc^*$ instead of $j^!j_!(id \otimes bc^*)$.

If $\xi: X \to Y$, and $A$ and $B$ are complexes, then

$$pr^1: \xi_*A \otimes B \sim \xi_!(A \otimes \xi^*B) \quad \text{and} \quad pr^2: A \otimes \xi_!B \sim \xi_!(\xi^*A \otimes B).$$

With this notation we have $pr^2 = \xi^!(m_1 \circ (\epsilon_1^1 \otimes id) \circ (pr_1^1)^{-1}) \circ \eta_1^1 \circ (\omega_\xi \otimes id) \circ m_1^{-1}.$

Notice that if $\xi$ is proper, then

$$pr_1^\xi = \Psi_\xi((\epsilon_1^1 \otimes id) \circ \text{nat}_\xi^\otimes) \quad \text{and} \quad pr_2^\xi = \Psi_\xi((id \otimes \epsilon_1^1) \circ \text{nat}_\xi^\otimes).$$

For a cartesian square as in (A.1.1) we have a base change isomorphism,

$$\overset{\sim}{bc}: \mu_\ast R(j_N)_! \sim R(j_M)_! \mu_0^\ast,$$

defined by $\overset{\sim}{bc} = \Psi_{j_M} (\mu_0^\ast(\epsilon_N^1))$. Define $\sigma: \mu_0^\ast j_N^! \to j_M^! \mu_\ast$ by

$$\sigma = \Phi_{j_M} (\mu_\ast \epsilon_N^1) \circ \overset{\sim}{bc}^{-1}.$$  

Then $\sigma$ is a natural transformation.
(A.3.1)
The regions labeled (A.2.1) and (A.2.2) commute using the analog of the first rectangle in Lemma A.2.1 with $m_1$ instead of $m_2$, taking $A = j_M^1 \mu Q_N$.

**Lemma A.3.3.** The mapping $\sigma_\psi : m_1 j_\psi Q_N - j_\psi Q_N$ is an isomorphism in $D_l(M)$. 

\[ \begin{array}{ccc}
  j^1 (j^\psi Q_N \otimes j^* \mu Q_M) & \xrightarrow{pr^1} & j^1 (j_\psi j^\psi Q_N \otimes \mu Q_M) \\
  \downarrow \sigma & & \downarrow \sigma \\
  j^1 (j_\psi (j_\psi^1 Q_N \otimes \mu \mu Q_M)) & \xrightarrow{pr^1} & j^1 (j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M)) \\
  \downarrow \sigma & & \downarrow \sigma \\
  j^1 (j_\psi (j_\psi^1 Q_N \otimes \mu \mu Q_M)) & \xrightarrow{pr^1} & j^1 (j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M)) \\
  \downarrow \sigma & & \downarrow \sigma \\
  (\mu_0) j_\psi (j_\psi^1 Q_N \otimes j_\psi^1 Q_M) & \xrightarrow{pr^1} & (\mu_0) j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M) \\
  \downarrow \sigma & & \downarrow \sigma \\
  (\mu_0) j_\psi (j_\psi^1 Q_N \otimes j_\psi^1 Q_M) & \xrightarrow{pr^1} & (\mu_0) j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M) \\
  \downarrow \sigma & & \downarrow \sigma \\
  (\mu_0) j_\psi (j_\psi^1 Q_N \otimes j_\psi^1 Q_M) & \xrightarrow{pr^1} & (\mu_0) j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M) \\
  \downarrow \sigma & & \downarrow \sigma \\
  (\mu_0) j_\psi (j_\psi^1 Q_N \otimes j_\psi^1 Q_M) & \xrightarrow{pr^1} & (\mu_0) j_\psi (j_\psi j_\psi^1 Q_N \otimes Q_M) \\
  \end{array} \]
Proof. We have
\[
\sigma = \Phi_{j_M}(\mu^*(\epsilon_{j_N}^1)) \circ (\beta c) = j_M^!(\mu^*(\epsilon_{j_N}^1)) \circ j_M^!(\beta c) \circ \eta_{j_M}^1.
\]
Since \(j_M\) is a closed embedding, \(\eta_{j_M}^1\) is an isomorphism, so it is enough to show that \(j_M^!(\mu^*(\epsilon_{j_N}^1)) : j_M^!(\mu^*(\epsilon_{j_N}^1)) \otimes \mathbb{Q}_N \rightarrow j_M^!\mu^*\mathbb{Q}_N\) is an isomorphism. Since \(M\) and \(N\) are purely \(d\)-dimensional, rational homology manifolds and \(M_0\) and \(N_0\) are purely \(2n\)-dimensional, rational homology manifolds, it follows that \(j_M^!\mu^*(\epsilon_{j_N}^1) \otimes \mathbb{Q}_N\) and \(j_M^!\mu^*\mathbb{Q}_N\) are both isomorphic to \(\mathbb{Q}_M[-2l]\). It follows that \(j_M^!(\mu^*(\epsilon_{j_N}^1))\) is an isomorphism. □

Since \(\sigma_{\mathbb{Q}_N}\) is an isomorphism, the composition
\[
\beta j_M^{-1} \circ j_M^!(\nu_M^{-1} \circ \alpha_\mu) \circ \sigma \circ \mu_0^*(\omega_{j_N}^{-1}) \circ \alpha_{\mu}^{-1} : \mathbb{Q}_{M_0} \rightarrow D_{M_0}[-4n]
\]
is an isomorphism, so we may choose \(\nu_{M_0}\) so that
\[
\nu_{M_0}^{-1} = \beta j_M^{-1} \circ j_M^!(\nu_M^{-1} \circ \alpha_\mu) \circ \sigma_{\mathbb{Q}_N} \circ \mu_0^*(\omega_{j_N}^{-1}) \circ \alpha_{\mu}^{-1}.
\]
It then follows that
\[
(A.3.4) \quad \omega_{j_M} \circ \alpha_{\mu_0} = j_M^!(\alpha_\mu) \circ \sigma_{\mathbb{Q}_N} \circ \mu_0^*(\omega_{j_N}).
\]

As in (2.4.2) \((bc)^{-1} : \mu_0 \circ j_M^! \rightarrow j_N^!\mu_0\) by \((bc)^{-1} = \Phi_{j_N}(\mu_1^!(\epsilon_{j_M}^1))\). Thus, using (A.3.4) we have
\[
(bc)^{-1} \circ (\mu_0); (\eta_{j_M}^1) = \Phi_{j_N}(\mu_1^!(\epsilon_{j_M}^1)) \circ (\mu_0); (\eta_{j_M}^1)
\]
\[
= \Phi_{j_N}(\mu_1^!(\epsilon_{j_M}^1)) \circ \mu_0^!(\eta_{j_M}^1)
\]
\[
= \Phi_{j_N}(\mu_1^!(\Phi_{j_M}^{-1}(\eta_{j_M}^1)))
\]
\[
= \Phi_{j_N}(id)
\]
\[
= \eta_{j_N}^1.
\]
Therefore,
\[
(A.3.5) \quad bc \circ \eta_{j_N}^1 = (\mu_0); (\eta_{j_M}^1).
\]

Lemma A.3.6. The diagram
\[
\begin{array}{ccc}
R(j_N);j_N^!A \otimes R\mu_1B & \xrightarrow{pr_{j_N}^1} & R(j_N); (j_N^!A \otimes j_N^!R\mu_1B) \\
pr_{j_N}^2 \downarrow & & \downarrow R(j_N);(id \otimes bc^{-1}) \\
R\mu_1(\mu^*R(j_N);j_N^!A \otimes B) & \xrightarrow{R(j_N)!((id \otimes bc^{-1}))} & R(j_N)!; (j_N^!A \otimes R(\mu_0);j_N^*B) \\
R\mu_1(\beta c) \downarrow & & \downarrow R(j_N)!; (pr_{j_M}^1) \\
R\mu_1(R(j_N);j_N^!A \otimes B) & \xrightarrow{R\mu_1(pr_{j_M}^1)} & R(j_N)!; (\mu_0^*j_N^!A \otimes j_M^*B)
\end{array}
\]
commutes for \(A\) in \(D^b(N)\) and \(B\) in \(D^b(M)\).
Proof. First, using the formulas for $pr_{\mu}^1$ and $pr_\mu^2$ and the analogs of equation (2.4.2) for $\Psi_{J_N}$ and $\Psi_\mu$, we see that it is enough to show that

$$
\Psi_\mu \left( pr_{J_M}^1 \circ (\widetilde{bc} \otimes \epsilon_\mu^*) \circ \text{nat}^\mu \right) = \Psi_{J_N} \left( pr_{\mu_0}^2 \circ (\epsilon_N^* \otimes bc^*) \circ \text{nat}_{J_N} \right).
$$

Next, using the formulas for $pr_{J_M}^1$ and $pr_{\mu_0}^2$ and the analogs of equation (2.4.2) for $\Psi_{J_M}$ and $\Psi_{\mu_0}$, we see that it is enough to show that

$$
\Psi_\mu \Psi_{J_M} \left( (\epsilon_M^* \otimes id) \circ \text{nat}_{J_M} \circ j_M^* \left( (\widetilde{bc} \otimes \epsilon_\mu^*) \circ \text{nat}_\mu \right) \right)
= \Psi_{J_N} \Psi_{\mu_0} \left( (id \otimes \epsilon_\mu^*) \circ \text{nat}_{\mu_0} \circ \mu_0^* \left( (\epsilon_N^* \otimes bc^*) \circ \text{nat}_{J_N} \right) \right).
$$

Now using the fact that $\Psi_{fg} = \Psi_f \Psi_g$ and the naturality of $\text{nat}_\psi$ and $\text{nat}_\mu$, we see that it is enough to show that

$$
(\epsilon_M^* \otimes id) \circ (j_M^* \circ \widetilde{bc} \otimes j_M^* (\epsilon_\mu^*)) \circ \text{nat}_{J_M} \circ j_M^* (\text{nat}_\mu)
= (id \otimes \epsilon_\mu^*) \circ (\mu_0^* (\epsilon_N^*) \circ \mu_0^* (bc^*)) \circ \text{nat}_{\mu_0} \circ \mu_0^* (\text{nat}_{J_N}).
$$

Since $\text{nat}_\psi \circ g^* (\text{nat}_f) = \text{nat}_{fg}$, we only need to show that

$$
\epsilon_M^* \circ j_M^* (\widetilde{bc}) = \mu_0^* (\epsilon_N^*) \ \text{and} \ j_M^* (\epsilon_\mu^*) = \epsilon_M^* \circ \mu_0^* (bc^*)
$$

which is the same as

$$
\Psi_{J_M}^{-1} (\widetilde{bc}) = \mu_0^* (\epsilon_N^*) \ \text{and} \ j_M^* (\epsilon_\mu^*) = \Psi_{\mu_0}^{-1} (bc^*).
$$

These last two equations follow immediately from the definitions

$$
\widetilde{bc} = \Psi_{J_M} (\mu_0^* (\epsilon_N^*)) \ \text{and} \ bc^* = \Psi_{\mu_0} (j_M^* (\epsilon_\mu^*))
$$

above. □

Since $\Phi_{J_M} (\xi) = j_M^1 (\xi) \circ \eta_{J,M}^1$, using the naturality of $j_M^1$ and the fact that $\epsilon_M^1 \circ \eta_{J,M}^1 = id$, we have

$$
\epsilon_M^1 \circ j_M^1 (\sigma) \circ \widetilde{bc} = \epsilon_M^1 \circ j_M^1 (\mu^* (\epsilon_N) \circ \widetilde{bc}^{-1}) \circ \eta_{J,M}^1 \circ \widetilde{bc}
$$

(A.3.7)

$$
= \mu^* (\epsilon_N) \circ \widetilde{bc}^{-1} \circ \epsilon_M^1 \circ \eta_{J,M}^1 \circ \widetilde{bc}
$$

$$
= \mu^* (\epsilon_N).
$$

REFERENCES

1. A. Borel (ed.), Intersection cohomology (Bern, 1983), Progr. Math., vol. 50, Birkhäuser, Boston, 1984.
2. W. Borho and R. MacPherson, Représentations des groupes de Weyl et homologie d’intersection pour les variétés nilpotentes, C. R. Acad. Sci. Paris 292 (1981), no. 15, 707–710. MR618892 (82f:14002)
3. , Partial resolutions of nilpotent varieties. Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque 101 (1983), 23–74. MR737927 (85i:14087)
4. N. Chriss and V. Ginzburg, Representation theory and complex geometry, Birkhäuser, Boston, 1997. MR1433132 (98k:22021)
5. J. M. Douglass and G. Röhrle, The geometry of generalized Steinberg varieties, Adv. Math. 187 (2004), no. 2, 396–416. MR2078347 (2005i:20071)
6. W. Fulton, Intersection theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1984. MR732620 (85k:14004)
7. W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. 31 (1981), no. 243. MR609831 (83a:55015)

8. M. Goresky and R. MacPherson, *Intersection homology. II*, Invent. Math. 72 (1983), no. 1, 77–129. MR696691 (84h:57012)

9. H. Hiller, *Geometry of Coxeter groups*, Research Notes in Mathematics, vol. 54, Pitman (Advanced Publishing Program), Boston, Mass., 1982. MR649068 (83h:57045)

10. R. Hotta, *A local formula for Springer’s representation*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., North-Holland, Amsterdam, 1985, pp. 127–138. MR803332 (87b:20059)

11. M. Kashiwara and P. Shapira, *Sheaves on manifolds*, Springer-Verlag, 1990. MR1074006 (92a:58132)

12. D. Kazhdan and G. Lusztig, *A topological approach to Springer’s representations*, Adv. in Math. 38 (1980), 222–228. MR697198 (82f:20076)

13. S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York-Berlin, 1971. MR0354798 (50:7275)

14. G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. in Math. 42 (1981), 169–178. MR641425 (83c:20059)

15. W. Rossmann, *Picard-Lefschetz theory for the coadjoint quotient of a semisimple Lie algebra*, Invent. Math. 121 (1995), no. 3, 531–578. MR1353308 (96j:32052)

16. N. Spaltenstein, *On the reflection representation in Springer’s Theory*, Comment. Math. Helv. 66 (1991), no. 4, 618–636. MR1129801 (93a:20070)

17. T.A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. 36 (1976), 173–207. MR0442103 (56:491)

18. R. Steinberg, *On the desingularization of the unipotent variety*, Invent. Math. 36 (1976), 209–224. MR0430094 (55:3101)

19. T. Tanisaki, *Twisted differential operators and affine Weyl groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 2, 203–221. MR914019 (88m:22037)

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