COMBINATORIAL HODGE THEORY FOR EQUITABLE KIDNEY PAIRED DONATION

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Abstract. Kidney Paired Donation (KPD) is a system whereby incompatible patient-donor pairs (PD pairs) are entered into a pool to find compatible cyclic kidney exchanges where each pair gives and receives a kidney. The donation allocation decision problem for a KPD pool has traditionally been viewed within an economic theory and integer-programming framework. While previous allocation schema work well to donate the maximum number of kidneys at a specific time, certain subgroups of patients are rarely matched in such an exchange. Consequently, these methods lead to systematic inequity in the exchange, where many patients are rejected a kidney repeatedly. Our goal is to investigate inequity within the distribution of kidney allocation among patients, and to present an algorithm which minimizes allocation disparities. The method presented is inspired by cohomology and describes the cyclic structure in a kidney exchange efficiently; this structure is then used to search for an equitable kidney allocation. Another key result of our approach is a score function defined on PD pairs which measures cycle disparity within a KPD pool; i.e., this function measures the relative chance for each PD pair to take part in the kidney exchange if cycles are chosen uniformly. Specifically, we show that PD pairs with underdemanded donors or highly sensitized patients have lower scores than typical PD pairs. Furthermore, our results demonstrate that PD pair score and the chance to obtain a kidney are positively correlated when allocation is done by utility-optimal integer programming methods. In contrast, the chance to obtain a kidney through our method is independent of score, and thus unbiased in this regard.

1. Introduction. Organ donation procedures have improved in past decades, yet the availability of organs remains scarce [3]. Live kidney donation offers an extended supply of organs and is often superior to a cadaveric kidney [19]. Many ideas have been discussed about how to encourage and optimize live organ donation in an ethical fashion to help alleviate the shortage of kidneys [14], [1], [12], [6].

If incompatible with their intended recipient, a donor can still represent the patient in a kidney paired donation (KPD) exchange. In the simplest case, two patient-donor (PD) pairs are matched together, each donor giving to the other
pair’s patient. Larger exchange cycles, such as the three PD pairs shown in Fig 1 are also possible. Incompatibility arises for many reasons, most commonly due to blood-type or HLA (Human Leukocyte Antigens) sensitivity [19]. In particular, blood type AB donors must donate to rare blood type AB patients, yielding a lower chance for such underdemanded pairs to obtain a kidney. While blood type is a permanent characteristic, HLA sensitivity can arise from receiving blood transfusions, pregnancy, and past transplantations as well as other, less common, conditions [19]. HLA sensitivity is measured as calculated panel reactive antibody (CPRA) percentage, with CPRA > 80% patients labeled as highly sensitized. Precisely, CPRA is the expected chance of HLA sensitivity to a randomly chosen donor nationwide, based on the results from a panel of HLA markers.

![Figure 1. A three-way kidney exchange cycle. The PD pairs are numbered 1, 2, and 3 with P for patient and D for donor indicating an individual’s role. Potential donations are indicated by red arrows from the donor to patient.](image)

Existing KPD allocation models maximize a utility function whose values are given for each possible kidney donation. Donation utilities can be given by a constant, the expected quality adjusted life years (QALY) for each donation, and/or the probability of a successful transplant. The Organ Procurement and Transplant Network (OPTN) has three main ethical concerns for its operations: utility, justice, and respect for persons. In [1], they describe a just allocation as “fairness in the pattern of distribution of the benefits and burdens of an organ procurement and allocation program”; we refer to this quality as equity in this paper. Ideally, an equitable distribution of organs does not depend on the difficulty in finding a compatible kidney. Despite the inherent challenge in this goal, an important equity goal is to mitigate powerful bias against any group, such as highly sensitized patients, minorities, or under demanded pairs.

To rigorously define and analyze inequity, consider the allocation chance conditioned by a function $g$ defined on PD pairs. For a (random) KPD pool, consider a randomly chosen PD pair $x$, and let $f$ indicate kidney allocation. Then, the function $\hat{h}(t) = \mathbb{P}[f(x) = 1|g(x) = t]$ describes the conditional probability of kidney allocation ($f = 1$) on the values of some score function $g$ dependent on some continuous threshold $t$; indeed, linear trends of the probability $\hat{h}(t)$ indicate systematic allocation bias according to $g$.

Equity and utility are in competition, as shown in [7] which boosts kidney allocation to highly sensitized patients and tracks the loss of utility. The approach used in [7], called weighted fairness (WF), must specify a priori a disadvantaged group of PD-pairs. The chosen group is given bonus utility per allocation, and then the new utility function is maximized via integer programming methods. In contrast, we present an algorithm which considers each donation’s potential exchange cycles
together, thereby creating a new description of the KPD pool with global cycle information summarized as a new utility value for every donation. Moreover, our algorithm *a posteriori* produces a score for each PD pair which quantitatively describes their relative advantage within the original KPD pool. As a hybrid approach, we also implement WF with bonus utilities determined from this score function.

A KPD pool is mathematically formulated as a kidney exchange graph (KEG) with PD pair vertices and edges drawn from donors to compatible patients. In this context, donation utility is a function on the edges of the KEG. Optimized match [17] finds the max-utility pairwise matching of PD pairs using the blossom algorithm of [9]. While fast, optimized match fails to include exchange cycles with more than 2 patients, limiting its applicability. Indeed, [16] and [5] both reveal max-utility increase when allowing larger exchange cycles, especially for smaller or highly sensitized KPD pools. The top trading cycles and chains (TTCC) algorithm [15] solves KPD through a greedy housing allocation algorithm. TTCC is fast, allocates larger exchange cycles, and can handle chains which start at an altruistic donor. In [20], optimal kidney allocation is solved as an integer programming problem, utilizing the random maximal matching (rCM) algorithm to find cycle allocations with maximal cardinality. The weighted fairness (WF) extends the integer programming solution by optimizing edge sum-utility with vertex bonuses or penalties. Throughout this paper, we compare our method with TTCC, rCM, and WF because (1) TTCC is greedy in a fashion similar to our algorithm and (2) rCM and WF are variants of the *optimal* integer programming method.

None of the aforementioned algorithms directly analyze the equity in a kidney exchange. While a few studies such as [7] make strides to help particular groups, none seeks to quantify or eliminate general disparity in a kidney exchange as ours does. Moreover, our algorithm is shown to be competitive with the maximal cardinality of rCM as well as the maximal utility of WF. The speed and scaling of our algorithm is faster than the traditional integer programming description of the KEG problem.

1.1. **Overview.** The rest of the paper is organized as follows. We begin by introducing, motivating, and describing our algorithm (Hodge Cycle) in Section 2. Precisely, Section 2.1 describes some of the technical details behind simplicial cohomology with definitions, examples, and concluding with the Helmholtz decomposition theorem (Theorem 2.1). We then explain how Helmholtz decomposition aids in finding exchange cycles. Section 2.2 outlines and discusses the Hodge Cycle algorithm in detail. In Section 3 we discuss our results, starting with motivating examples in 3.1. The first gives a concrete and simple example of Helmholtz decomposition, while the second demonstrates Hodge Cycle’s behavior. Section 3.2 lays out the methods used to generate our synthetic kidney exchange graphs. Section 3.3 uses Theorem 2.1 to produce a meaningful score which measures advantage within our KEGs and connects low scores to previously known disadvantaged groups (High CPRA and underdemanded pairs). Section 3.3 finishes by directly demonstrating a positive correlation between combined scores and the chance to obtain a kidney from TTCC, rCM, and WF, with little correlation between the scores and the chance to obtain a kidney from Hodge Cycle. Section 3.4 tests the speed and efficacy of Hodge Cycle and compares our method’s allocation cardinality and overall utility with TTCC, rCM, and WF directly. Finally, we conclude and discuss our method in Section 4.
2. Methodology. We describe a kidney paired donation (KPD) pool as a directed graph, \( G = (V,E) \) called a kidney exchange graph (KEG). Each vertex in \( G \) is either a patient or a donor. Edges drawn from patient to donor indicate that the donor represents the patient in the exchange as a friend or family member. Edges drawn from donor to patient indicate that the donor’s kidney is compatible with the patient. Details for generation of KEGs and edge utilities are given in Section 3.2.

In this formulation, every kidney exchange cycle is described by an oriented cycle in the KEG; i.e., a cycle where the end of every edge touches the beginning of the next. Each PD pair can take part in at most one exchange, so Exchange cycles cannot share a vertices. Thus, the initial goal is to find a collection of vertex-disjoint exchange cycles maximizing the sum utility of its constituent edges. However, our primary goal is to achieve equity; i.e., to give equal allocation chance regardless of PD pair characteristics. As a result, our method must achieve suboptimal utility, though it is desireable to minimize the loss of efficiency.

2.1. Combinatorial Hodge theory. Our algorithm uses the 1-cohomology of a KEG to describe its cycle structure without listing the unique cycles. A brief exposition on cohomology related to our work is briefly discussed; however, the interested reader may refer to [8] and [11] for further details.

We consider a KEG \( G \) with vertices \( V \) and edges \( E \) and denote the space of (linear) functions on the vertices by \( C^0(G) \). We call a function \( f \in C^0(G) \) a score function, since it assigns a value to each patient and donor in the KEG. Similarly, we denote the space of functions on oriented edges of \( G \) by \( C^1(G) \). A function, or edge flow, \( h \in C^1(G) \) can fully describe a KPD pool by determining donation utilities. One can consider an edge flow \( h \) as an antisymmetric function on pairs of vertices; e.g., if \( h(c,a) = 2 \) on the edge from \( c \) to \( a \), then \( h(a,c) = -2 \) on the oppositely oriented edge.

Denoting the combinatorial gradient \( \partial_0 \), we obtain the following, called a cochain complex for \( G \):

\[
\{0\} \rightarrow C^0(G) \xrightarrow{\partial_0} C^1(G) \xrightarrow{0} \{0\}.
\]

In particular, the adjoin (matrix transpose) of the gradient, \( \partial_0^* : C^1(G) \rightarrow C^0(G) \), is the combinatorial divergence. For functions \( f \in C^0(G) \) on vertices and \( h \in C^1(G) \) on edges, the gradient and divergence are defined as follows:

\[
[\partial_0 f](v,w) = f(w) - f(v) \quad (1)
\]

\[
[\partial_0^* h](v) = \sum_{w \in V} h(v,w). \quad (2)
\]

According to Eq (2) and antisymmetry, \( \partial_0^* \) sums outgoing edges and subtracts incoming edges on an edge flow, measuring the outward flow of \( h \) at the input vertex. Taking \( f = \partial_0^*(h) \) defines a new function on the vertices. For example, consider the edge flow in Fig 2, then \( f(a) = 5 - 2 - 7 = -4 \) and \( f(b) = 11 + 3 - 5 = 9 \).

\[\text{Figure 2. A graph with vertices a through e and an edge flow } h \text{ defined on it via weights. For example, } h(a,b) = 5 \text{ and } h(b,e) = 11.\]
This setup enables us to define the 1-cohomology group of the graph $G$ via group modulus as: $H^1(G) = \ker(\partial_1)/\im(\partial_0) = \{x + \im(\partial_0) : x \in \ker(\partial_1)\}$. The elements of $H^1(G)$ are equivalence classes, where $x, y \in \ker(\partial_1)$ are equivalent when $x - y \in \im(\partial_0)$. Each generator of $H^1(G)$ describes a cycle of $G$ and so $H^1(G)$ reflects its cycle structure. While this compact description of exchange cycles shows promise for cycle allocation, the modular description of $H^1(G)$ is problematic. So, we appeal to Helmholtz decomposition [11]:

**Theorem 2.1. (Helmholtz Decomposition)** Define the Helmholtzian (1-Laplacian) we appeal to Helmholtz decomposition [11]:

$\Delta_1 : C^1(G) \to C^1(G)$ as $\Delta_1 = \partial_0 \circ \partial_0^*$. The set of edge flows $C^1(G)$ on a graph $G$ can be expressed via the orthogonal sum:

$$C^1(G) = \im(\partial_0) \oplus \ker(\Delta_1).$$

Moreover,

$$\ker(\Delta_1) = \ker(\partial_1) \cap \ker(\partial_0^*) \cong H^1(G)$$

via the canonical surjection $\phi(x) = x + \im(\partial_0)$.

As stated, Theorem 2.1 enables the decomposition of KEG edge utilities $U \in C^1(G)$ into two orthogonal pieces

$$U = V + W,$$

where $V \in \ker(\Delta_1) \cong H^1(G)$ captures the globally cyclic portion of $U$ and $W \in \im(\partial_0)$ corresponds to a consistent scoring $R \in C^0(G)$ on the vertices (differences do not depend on a path) with

$$W = \partial_0 R.$$  \hfill (4)

In the following, we leverage the orthogonality of the portions $V$ and $W$ of the utilities $U$ to describe an algorithm which yields equitable kidney allocation.

### 2.2. Hodge Cycle algorithm.

We describe the utility on a KEG as an edge flow $U$. Theorem 2.1 allows us to decompose the utilities as per Eq (3). Algorithm 1 begins by projecting $U$ onto $\ker(\Delta_1)$ to find $V$. In practice, solve the graph Laplacian equation to find a scoring $R$ with $W = \partial_0 R$, then take $V = U - W$. This is the most expensive step of the algorithm and scales as $O(n^2)$, where $n = (\# \text{ of PD pairs})$. The number of cycles scales linearly, so the overall runtime scales as $O(n^3)$.

Since $\ker(\Delta_1) \cong H^1(G)$, the new utilities in $V$ can be expressed as a sum of representing cycles $C_i$ with weights $\alpha_i$ indicating the relative contribution of each cycle: $V = \sum_i \alpha_i C_i$. Thus, $V$ on each edge $e \in E$ represents the total contribution of its resident cycles: $V(e) = \sum_{\{i : e \in C_i\}} \alpha_i$. While the choice of cycles $C_i$ is non-unique, the summary $V$ is well-defined by Theorem 2.1 and organizes cycle weights without the identification all possible cycles. Indeed, such a cycle count is a daunting (exponential) computational task. By using $V$ in place of $U$, Hodge Cycle compares potential exchange cycles instead of single donations. We define the PD pair scores as the values of $-W = V - U$ evaluated on PD pair edges, wherein $U(e) = 1$. Since $V$ and $-W$ differ by a constant on PD pair edges, PD pair score estimates the relative number (and utility) of potential exchange cycles to which a PD pair belongs, compared to the average. In this fashion, the new KEG with utilities from $V$ represents a kidney exchange which is indifferent to PD pair score. It is important, however, that as an *a priori* assigned score, $R$ cannot measure allocation inequity.
Despite the drastic change of viewpoint, the sum utility of any cycle is the same in \( U \) and \( V \), since \( W = U - V \) is a gradient and thus sums to 0 over any cycle. Thus, any by-cycle optimization performed on both \( U \) and \( V \), such as integer programming methods, will yield identical results. For these reasons, Hodge Cycle uses a greedy algorithm that *locally* seeks to optimize the new utilities on \( V \) by comparing individual donation edges, keeping in mind that edges in \( V \) contain non-local information about the original utilities \( U \). In our results, we ultimately verify the expectation that using \( V \) to locally optimize yields an equitable donation allocation.

The greedy algorithm typically begins with the highest utility edge in \( V \). Since \( V \) is divergence-free, there is always a positive utility outward edge for any vertex belonging to at least one cycle. As a result, the algorithm *always* closes a cycle. After a vertex is visited twice, a cycle is found and it must be tested that the cycle is oriented in \( U \); Any non-oriented "bad" cycle will follow a donation edge backwards and thus attempts to force a patient to donate a kidney! In this event, the starting edge for the next iteration is randomized to prevent a repeat. In practice, most cycles found are oriented in \( U \); moreover, the costly projection step need only be redone after an oriented cycle is found. After an oriented cycle is found, the donations are allocated and the cycle’s vertices and incident edges are removed from the KEG and \( U \), their donations now in use. The algorithm restarts with a new projection \( V \) for the smaller \( U \) to iteratively find a collection of exchange cycles.

Our implementation is summarized in Algorithm 1. MATLAB code is available on the author’s website at [https://sites.google.com/site/joshmikemath/code](https://sites.google.com/site/joshmikemath/code).

```
Input: KEG, G, with utility edge flow, U.
Output: Cycle collection (kidney allocation).

\( HM = \Delta_1 \) on \( G \) (Helmholtzian matrix)
\( V = \text{Proj}_{\text{ker}(HM)}U \) (Cyclic portion)
BadCycle = False

while \( V \neq 0 \) or not max iteration
    if BadCycle:
        \( v(1) \) = random vertex in \( G \).
    else
        \( v(1) \) = tail of max utility edge in \( V \).
    end
    Cycle = \emptyset, \( k = 1 \)
    \( e = \text{max utility edge with tail} v(k) \)
    \( v(k + 1) = \text{target of} e, k = k + 1 \),
    if \( v(k) = v(j) \) for \( j \leq k \):
        Cycle = \( v(j) \) through \( v(k - 1) \)
    end
    end
    if Cycle is oriented:
        BadCycle = False
        Record & remove Cycle from \( G \) and \( U \)
        \( HM = \Delta_1 \) on \( G \)
        \( V = \text{Proj}_{\text{ker}(HM)}U \)
    else
        BadCycle = True
    end
end

Algorithm 1: Hodge Cycle
```
3. Results. In all of our results we implement simple improvements to the greedy algorithm which require negligible local computation. First, we increase the depth of the greedy algorithm, weighting the next donation utilities by half and ignoring representation utilities. Second, we change utility for donations that finish a cycle to discourage large exchange cycles, which are challenging or unrealistic to actually perform.

3.1. Motivating examples. First, we present the process of Helmholtz decomposition (Theorem 2.1) for utilities $U$ on a small graph (Fig 3a). We readily identify basis cycles $C_1$ and $C_2$ and thus present $\ker(\Delta_1)$ as $\{M(C_1) + N(C_2) : M, N \in \mathbb{R}\}$ (Fig 3b). To find the cyclic portion, $V$, of $U$, we can use Gramm-Schmidt to find an orthonormal basis for $\ker(\Delta_1)$ and project onto this basis (Fig 3c). Alternatively, minimizing the distance $\|W\| = \|U - V\|$ leads to the graph Laplacian equation used in practice. By Theorem 2.1, $W = U - V$ is the gradient portion of $U$ (Fig 3d). The gradient portion is used to find the scoring $R$ so that $W = \partial_0 R$. One arbitrarily defines the score for a single vertex and then follows the edges of $W$ to find the other scores (Fig 3e). The scoring function is then shifted to obtain a mean of 0 to allow better comparison among KEGs (Fig 3f).

![Figure 3. An edge flow and the process used to Hodge decompose it. (a) Original edge flow $U$. (b) Space of cyclic flows, $\ker(\Delta_1)$. (c) The globally cyclic portion $V$ minimizing $\|U - V\|$. (d) The gradient portion, $W = U - V$. (e) A scoring made by assigning 0 to one vertex value. (f) A new scoring with mean 0.](image-url)

Next, Fig 4 presents a more complex graph including various utility values. This example exhibits differences in greedy algorithm behavior caused by the cyclic projection. For example, the cyclic portion in Fig 4b has lesser utilities for shortcut donation edges. Allocating such donations isolates a PD pair from the rest of the KEG, leading to suboptimal allocation. For example, the edges with circled utilities in Fig 4a and 4b change from 1 to very low values of 0.21 and 0.30 after projecting. Fig 4c depicts the cyclic portion after the first exchange cycle is allocated, $V_1$. $V_1$ separates into two basis cycles: one cycle has utility 0.66 and the other has utility 0.38; edges belonging to both cycles are weighted as the sum, 1.04. In $U$, the greedy algorithm would select the smaller basis cycle. This choice hinges on one pair of donations (weights boxed in Fig 4a and 4c), comparing utilities of $1 < 0.8$ in $U$ or $0.38 > 0.66$ in $V_1$.

3.2. Generated data. Here refer to Table 1 for our choice of parameters. The generated KEGs account for variation in blood type and HLA (Human Leukocyte Antigen) sensitivity. The data are initialized as an array of patient donor pairs, where typical KEGs have between 50 and 200 PD pairs. Each patient and donor is randomly assigned a blood type according to multinomial distributions (Table 1, top). Each patient is also given a random CPRA (HLA sensitivity) value. First, the
Figure 4. The progression of Hodge Cycle on a tiny KEG. Green nodes are patients while red nodes are donors. Recorded cycles are dark and blue. Removed edges are unlabeled. (a) Initial KEG, $U$. (b) Initial cyclic KEG, $V_0$. (c) One cycle allocated, $V_1$. (d) Two cycles allocated (empty).

A patient is labeled as either low (0-10%), medium (10-80%), or high (80-100%) CPRA according to a fixed multinomial distribution (Table 1, right); then, a specific CPRA value is chosen from the matching (low, med, high) continuous uniform distribution.

Most of our simulations are paired: one with US proportions and the other with uniform proportions (Table 1), allowing us to analyze algorithm performance in a realistic setting as well as gauge sensitivity and robustness. The US proportions reflect the US average blood type proportions for waiting list patients and the whole population, respectively for patients and donors. Assigning blood groups with equal probability allows us to investigate underdemanded pairs without the effects of AB rarity. Simulations which compare Hodge Cycle (HC) to Top Trading Cycles and Chains (TTCC), random (maximal) Cardinality Matching (rCM), and/or weighted fairness (WF) use precisely the same KEGs for direct comparison.

After PD pair data is initialized, we draw edges. First, an edge is drawn from patient to donor for every PD pair with utility 1; then, each other donor-patient combination is checked against CPRA and blood type to potentially draw an edge from donor to patient. (i) Blood type compatible if the patient has all blood markers the donor has; (ii) a random percentage must be greater than the patient’s CPRA value. Each donation edge is then randomly assigned a utility value between $1/2$ and 1. A 200 PD pair KEG with US proportions has about 21,500 edges on average.

3.3. Measuring equity. By Theorem 2.1, the gradient portion $W$ creates a score which in context measures a PD pair’s advantage (higher number of cycles). Recall that the score $R$ is a function on the vertices of the KEG (see Eq. (4)), giving a value for each patient and each donor. Here we demonstrate that highly sensitized patients (CPRA > 80) have low patient scores and that underdemanded (type AB donor) pairs have high donor scores; both patterns lead to a smaller PD pair score ($-W$). PD pairs of either group can take part in fewer exchanges, leaving them less likely to receive a kidney in a purely utilitarian allocation.

We begin by generating KEGs with CPRA levels and blood types distributed according to US averages. Now we directly compare a patient’s assigned CPRA value to their score in Fig 5. These results show a negative, loosely logarithmic correlation between patient scores and CPRA, as well as an increase in score variability.
Table 1. (Top) Blood type proportions for generated KEGs. (Bottom) CPRA level proportions for generated KEGs. These are the multinomial probabilities used in practice. US averages from [2]. Specific CPRA is chosen uniformly within level range (0-10, 10-80, or 80-100%).

| Blood Type | US wait-list | US whole | Uniform |
|------------|--------------|----------|---------|
| O          | 48.6%        | 44%      | 25%     |
| A          | 32.7%        | 42%      | 25%     |
| B          | 14.9%        | 10%      | 25%     |
| AB         | 3.8%         | 4%       | 25%     |

| CPRA level | US wait-list | Uniform |
|------------|--------------|---------|
| Low        | 81.3%        | 10%     |
| Med        | 11%          | 70%     |
| High       | 7.7%         | 20%     |

as CPRA increases. In particular, patient scores plummet at high CPRA values, expressing the extreme difficulty for highly sensitized patients to find a compatible donor. Since AB patients are rare (about 4% of the US population), we expect AB donor pairs to have a disadvantage in obtaining an exchange cycle. Indeed, Fig 5 (right) shows AB donors (red x’s) with substantially higher donor scores.

Figure 5. (Left) Patient score and CPRA. (Right) Donor score and representing patient CPRA. A comparison of vertex scores with patient CPRA values in KEGs with US proportions (Table 1). Both plots represent 50 random KEGs with 100 PD pairs each and show every PD pair. Underdemanded PD pairs with AB donors are marked with red Xs.

We repeat the analysis in KEGs with uniform distribution of CPRA and blood type (Fig 6). Despite different KPD pool composition, the same logarithmic correlation between CPRA and patient score is present, demonstrating robustness of the scoring function. By making blood type uniform, the effect of AB blood type disparity is reduced; this is reflected in Fig 6 (right) by reduced variation in donor scores between AB donor pairs and the rest, as compared to US average proportions shown in Fig 5 (right). Lastly, note that blood type has little effect in Fig 6 (left),
while CPRA has little effect in Fig 6 (right), demonstrating that patient scores are largely independent of donor characteristics and conversely.

Figure 6. (Left) Patient score and CPRA. (Right) Donor score and representing patient CPRA. A comparison of vertex scores with patient CPRA values in KEGs with uniform proportions (Table 1). Both plots represent 10 random KEGs with 100 PD pairs each, and show every PD pair. Underdemanded PD pairs with AB donors are marked with red Xs.

To further understand the relationship between PD pair score and equity, we directly investigate the chance of obtaining a kidney from Hodge Cycle (HC), Top Trading Cycles and Chains (TTCC), random maximal matching (rCM), and weighted fairness (WF), as a function of score. TTCC and rCM are presented here as benchmarks and utilitarian methods. WF is presented as an alternative approach to attaining equity, using half of the PD pair score as the bonus weights in WF in place of indicator weights. PD pair score is defined as $-W$, or the difference between patient score and donor score, so that higher scored pairs belong to more exchange cycles. From these scorings we create probability density functions (pdfs) for (i) all PD pairs, and (ii)-(v) PD pairs allocated by HC, TTCC, rCM, and WF (Fig 7).

Figure 7. (Left) Score pdfs for US average blood type and CPRA. (Right) Scoring probability densities for uniform blood type and CPRA. Probability density functions for the score of a random PD pair, given allocation by HC, TTCC, rCM, or WF. Each plot represents 50 random KEGs with (left) 50 PD pairs each or (right) 100 PD pairs each. Pairs without an obtainable cycle have been removed from the score determination.

Finally, we divide each allocation pdf by the pdf for all PD pairs to obtain the proportion of allocated PD pairs as a function of their combined PD pair score, shown in Fig 8. Here we see that TTCC and rCM strongly prefer allocating kidneys
to higher scoring pairs, with the chance of allocation smoothly increasing with the score. This correlation demonstrates inequity in kidney allocation with specific bias against highly sensitized and underrepresented PD pairs. WF yields somewhat equitable allocations in the US proportioned KEGs, though it performs poorly on the lowest scoring PD pairs. Moreover, WF still shows marked bias in the uniformly proportioned KEGs. On the other hand, Hodge Cycle does not show particular bias to any PD pairs. Indeed, Fig 8 exhibits the primary objective of the Hodge Cycle algorithm: independence between kidney allocation and PD pair score. Note that many disadvantaged patients are not shown in these allocations because they had no possible exchange cycle in their entire KEG. In particular, 1.72% of all pairs are missing in Figs 7 and 8 (left) and 2.01% in Figs 7 and 8 (right).

**Figure 8.** (Left) US average blood type and CPRA. (Right) Uniform blood type and CPRA. Both plots show the chance to obtain a kidney via HC, TTCC, rCM, or WF as a function of PD pair scores (patient score minus donor score). The solid blue line indicates the range of occurring PD pair scores. These plots are likelihood ratios of the densities seen in Fig 7; specifically, these values are the conditional probabilities of being allocated a kidney given a particular score.

### 3.4. Hodge Cycle performance

We now analyze the efficiency and efficacy of Hodge Cycle (HC) by measuring the time elapsed, number of donations allocated, and the average utility in various KEGs. As before, we use Top Trading Cycles and Chains (TTCC), random maximal matching (rCM), and weighted fairness (WF) as benchmarks for the performance of our algorithm.

All timed simulations in Fig 9 were performed on a Sager laptop with an Intel Core i7 and 8 GB RAM. Due to the memory requirements of Hodge Cycle, the timed KEG sizes are typically under 15,000 edges.

Timed trials of HC yield the average times seen in Fig 9. This graph demonstrates that Hodge Cycle is very efficient on sparse KEGs, but runs slower on dense KEGs. As expected from the description of our algorithm, the time taken to find a full cycle configuration scales at $O(n^3)$ in the number of PD pairs, $n$. Fig 9 also shows that computation time scales linearly with patient sensitivity, and hence also the density of the graph. Computation time is feasible for regional scale KEGs; for example, a KPD pool with 1000 PD pairs should take about an hour to complete at the given speed.

Fig 10 compares the proportions of patients who obtain a kidney from HC, TTCC, rCM, and WF. In particular, we demonstrate that HC and WF consistently outperform TTCC while both also perform similarly to the cardinality optimal rCM. This small discrepancy shows HC and WF to be competitive, especially since rCM directly maximizes cardinality while ignoring equity.
Figure 9. (Left) KEGs with US proportions and varying numbers of PD pairs. (Right) KEGs with 150 PD pairs and varying patient sensitivity. Both plots describe the average time for Hodge Cycle to find an allocation with one standard deviation error bars. Non-high CPRA patients are split evenly between low (0-10%) and medium (10-80%) CPRA.

Figure 10. (Left) KEGs with US proportions and 50 PD pairs each. (Right) KEGs with uniform proportions and 100 PD pairs each. Both plots show cross comparison of the percentage of patients who were allocated kidneys by HC to TTCC, rCM, and WF algorithms. Each point represents a single randomly generated KEG and its solution with the marked method.

Now we investigate the average utility of allocated donations. For each KEG, the average utility is the sum allocated donation utility divided by the allocation cardinality. Fig 11 compares the proportion of kidneys allocated to the average donation utility on a series of KEGs. Note that the average donation utility of a kidney exchange often suffers when many highly sensitized patients are involved in the allocation. Fig 11 shows that the average utility depends on the method and sparsity of the graph (as a result of patient CPRA) and not necessarily the percentage of patients saved, as the measurements within each group are uncorrelated. The Hodge Cycle and weighted fairness methods perform similarly in terms of average utility. rCM is not included in this analysis because the method does not use utilities and instead maximizes cardinality, and as expected its average donation utility was found to be approximately 0.75 in every case.

4. Discussion. We presented an algorithm focused on measuring and eliminating inequity on a KEG while simultaneously achieving near-optimal utility. When information is known about the relationships between the PD pairs in a KPD pool to create a KEG, Hodge Cycle measures each patient’s disadvantage and then systematically eliminates inequity in the KPD utility function; therefore, the scoring distributions in Figs 5, 6, and 8 demonstrate the benefits of testing compatibility in order to build the entire KEG before making allocation decisions. In particular,
the lack of bias achieved by Hodge Cycle could shorten wait times for sensitized patients who otherwise remain on dialysis while their chances deteriorate further. Ultimately, this supports and utilizes hospital protocols for obtaining HLA marker information from donors and HLA sensitivity information from patients. In general, an equitable allocation is desired by OPTN (Organ Procurement and Transplantation Network) and our algorithm can be used to explore and eliminate all types of bias, e.g., due to location, ethnicity, or blood type, so long as the associated variables are included in the creation of the KEG and its utilities. In particular, African Americans are known to have difficulties in obtaining kidneys [18] and liver transplantation suffers from serious geographical disparity [13].

The equitable allocations obtained by Hodge Cycle should be applicable in other arenas, including any scarce goods, non-monetary barter system. Indeed, the local, iterative nature easily allows multiple cycle allocations when vertices have more than 1 item and wish to partake in multiple exchanges.

Future study with Hodge Cycle will investigate the effects of an ongoing KPD pool with newcomers entering the pool over time in a more general framework bypassing any linearity assumption on the vertices of the graph. In particular, we will study the ongoing effects upon the population of highly sensitized patients. It is important to note that many of the highly sensitized patients were removed from Figs 5 and 6 since they were not a part of any cycles in their initial KEG, and consequently had no potential to obtain a kidney regardless of the allocation method. In typical time dependent models (such as in [21]) and in actual scenarios, these highly sensitized patients tend to collect in KPD pools due to their challenge in obtaining a kidney. We have already shown that Hodge Cycle has little to no bias against highly sensitized patients and therefore we expect that our algorithm will mitigate the growing collection of sensitization in KPDs.

Theoretically and experimentally, our algorithm scales at $O(n^3)$ on the number of PD pairs, $n$. We wish to speed the algorithm to enable use in larger KPD pools, since a national KPD pool could have upwards of 10,000 or more PD pairs and would be difficult for our algorithm as-is. Specifically, we will investigate using the 1-cohomology group to more efficiently construct a basis. Such a change in approach could also simplify parallelization and allow for more efficient memory

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**Figure 11.** (Left) KEGs with US proportions and 50 PD pairs each. (Right) KEGs with uniform proportions and 100 PD pairs each. Both plots compare average donation utility to number of patients allocated. Each point represents a single randomly generated KEG and its solution with the marked method. 0.75 is the expected average of all donation utilities in a KEG, since they are chosen uniformly between 0.5 and 1. The same KEGs are used for each solution method, and are the same as those used in Fig 10.
usage. The local methods used, including the use of greedy algorithm for starting and subsequent edge choices, need to be investigated further. It is unclear whether a better method exists, especially when trying to balance equity with utility or cardinality. One possibility is to randomly choose edges according with probabilities proportional to the (local) weights of \( V \).

The results of this paper were performed on data generated through populations averages from the literature (see Table 1 and [2]) due to the challenge in obtaining sufficiently detailed potential donation data, which is private and therefore has restricted access. While our data is built to reflect actual KPD pools, particularly the disadvantaged groups studied herein, such considerations are still limited. Besides using only a couple of the most important medical indicators, the different characteristics and relationships in an actual KEG may be highly interdependent. Indeed, our method has been shown to level the field for both underdemanded and highly sensitized PD pairs, and may be helpful to other groups as well.

Acknowledgments. The authors would like to thank one anonymous reviewer for their comments.

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