Hidden $U(1)_{Y}$ Ward-Takahashi identities in the spontaneously broken Abelian Higgs model and the decoupling of certain heavy particles in its simple extensions

Bryan W. Lynn$^{1,2,3}$, Glenn D. Starkman$^{1}$ and Raymond Stora$^{4,5,1}$

1 ISO/CERCA/Department of Physics, Case Western Reserve University, Cleveland, OH 44106-7079
2 University College London, London WC1E 6BT, UK
3 Department of Physics, University of Wisconsin, Madison, WI 53706-1390
4 Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland and
5 Laboratoire d’Annecy-le-Vieux de Physique Théorique (LAPTH), F-74941 Annecy-le-Vieux Cedex, France

The spontaneously broken $U(1)_{Y}$-hypercharge Abelian Higgs model (AHM) (i.e. the spontaneous symmetry breaking (SSB) gauge theory of a complex scalar $\phi = \frac{1}{\sqrt{2}}(H + i\pi)$ and a vector $A^{\mu}$) has, in Lorenz gauge, a massless pseudo-scalar $\pi$. Its physical states have a conserved $U(1)_{Y}$ global current (but no conserved charge), and a Goldstone Theorem (GT), $\pi$ becomes a Nambu-Goldstone boson (NGB), with only derivative couplings and a shift symmetry.

Since Slavnov-Taylor identities guarantee that on-shell T-matrix elements of physical states are independent of local $U(1)_{Y}$ gauge transformations (even though these break the Lagrangian’s BRST symmetry), we observe that they are therefore also independent of anomaly-free $U(1)_{Y}$ rigid/global transformations. We derive 2 towers of $\phi$-sector $U(1)_{Y}$ Ward-Takahashi Identities (WTI) which give: relations among Green’s functions; relations among off-shell T-matrix elements; powerful constraints on the dynamics of the $\phi$-sector. We prove Adler self-consistency relations for the $U(1)_{Y}$ gauge theory (i.e, beyond those usual for a global theory), which guarantee infra-red finiteness for on-shell $\phi$-sector T-matrix elements in Lorenz gauge: one of these is the GT.

All ultra-violet quadratic divergences (UVQD) contribute only to $m_{\phi}^{2}$, a finite pseudo-NGB mass-squared, which appears in intermediate steps of the calculations. The Goldstone theorem then enforces $m_{\phi}^{2} = 0$ exactly, so that all UVQD contributions (i.e. the only dangerous relevant AHM operators), originating in loops containing virtual $A^{\mu}; \phi$ and ghosts $\bar{\omega}, \omega$ vanish. The NGB $\pi$ decouples from the observable particle spectrum in the usual way, when the observable vector particle $B_{\mu} \equiv A_{\mu} + \frac{\pi}{m_{\phi}} \partial_{\mu}\pi$ absorbs it, as if it were a gauge transformation. Our $U(1)_{Y}$ WTI are then “hidden” from observable particle physics.

While the AHM has only one scale $m_{BEH}$, and so there are no ratios of scales to require fine-tuning, our regularization-scheme-independent, WTI-driven results are unchanged by the addition of certain heavy $U(1)_{Y}$ matter representations $(M_{\text{Heavy}}^{0} \mid |q^{2}|, (H)^{2} \sim m_{\text{W}}^{2})$, because the extended rigid $U(1)_{Y}$ WTI and Goldstone theorem cause all UVQD, log-divergent and finite relevant operators ($m_{\text{Heavy}}$) to vanish. We prove 5 heavy-particle SSB decoupling theorems, illustrating them with two explicit examples: a singlet $S_{2} \gg m_{\text{W}}^{2}$ real scalar field $S$ with discrete $Z_{2}$ symmetry and VEV $(S) = 0$; and a singlet right-handed Type I See-saw Majorana neutrino $\nu_{R}$ with $M_{\nu_{R}} \gg m_{\text{W}}^{2}$, and all loops containing virtual gauge bosons, fermions, scalars and ghosts, we prove that certain heavy degrees of freedom decouple completely from the $U(1)_{Y}$ low-energy effective SSB AHM Lagrangian, contributing only irrelevant operators after quartic-coupling renormalization. We also display a non-decoupling exception: heavy Type 1 See-saw $\nu_{R}^{\text{Majorana}}$ cannot completely decouple, but becomes invisible in practice.

The NGB $\pi$ decouples as usual, hiding the WTI. But our “Hidden SSB $U(1)_{Y}$ WTI,” and their embedded shift symmetry $\tilde{\pi} \rightarrow \tilde{\pi} + (H)\theta$, have protected the low-energy SSB AHM theory (i.e. its observable particle spectrum and dynamics) from loop contributions of heavy particles! Gauge-independent observable weak-scale $(m_{BEH,\text{pot}}^{2} \mid (H)^{2} \sim m_{\text{W}}^{2})$ are therefore “Goldstone Exceptionally Natural”, not fine-tuned in the Abelian Higgs model with these judicious extensions.

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I. INTRODUCTION

How can weak-scale $m_{\text{W}}^{2} \sim (100\mathrm{GeV})^{2}$ spontaneously broken gauge theories protect themselves against quantum loop corrections involving very heavy matter particles $M_{\text{Heavy}}^{2} \gg m_{\text{W}}^{2}$? The current consensus in the theoretical physics community is that, without the imposition of further new symmetry, they cannot: the scale of spontaneous symmetry breaking (SSB) will “naturally” rise from $m_{\text{W}}^{2}$ to $M_{\text{Heavy}}^{2}$ and only there will it be quantum-mechanically stable. Alternately, one is forced to fine-tune the theory, cancelling $\mathcal{O}(M_{\text{Heavy}}^{2})$ quantum loop contributions against similarly large bare counter-terms order-by-order to obtain weak-scale physical quantities such as the mass $m_{BEH} \sim \mathcal{O}(100\mathrm{GeV})$ of the Brout-Englert-Higgs (BEH) boson $H$ and similar
magnitude weak gauge boson masses.

We refer to theories requiring such cancellation among bare-Lagrangian terms as “bare fine-tuned” (Bare-FT).1 This indicates the theory’s unsuitability as a UV-complete model of particle physics. The Standard Model (SM) is regarded as having this “BEH fine-tuning problem” (hereafter FTP) and an important goal of Beyond the Standard Model (BSM) theories is to avoid the problem—typically by adding new symmetries, such as supersymmetry. This paper takes another step forward toward a non-BSM proposal, based on the Goldstone theorem,2 that may resolve the perceived crisis due to tension between LHC8 data and simple BSM solutions of the FTP. We do so by explaining how spontaneously broken gauge theories can avoid the quantum instability responsible for the FTP, and demonstrate that there is a wide class of heavy-particle \( M_{\text{Heavy}} \gg m_{\text{Weak}} \) matter representations from which low-energy weak-scale physics is protected in such theories, fortified as they are by the Goldstone theorem.

We show here that, in the SSB Abelian Higgs Model, a tower of Ward-Takahashi Identities (WTI) relates all relevant-operator contributions to AHM physical-scalar-sector physical observables to one another, and the Goldstone Theorem then enforces a strong version of no-FT by causing all such contributions to vanish. It does so through its insistence that the Nambu-Goldstone Boson (NGB) mass-squared vanishes exactly.3 This is regardless of the fact that the NGB is not a physical degree of freedom, but is absorbed ("eaten") by the Goldstone theorem.4

1 Our favorite fine-tuned theory is that of a real scalar \( S \), with \((S \rightarrow -S) \) \( Z_2 \) symmetry:

\[
L_S = \frac{1}{2} (\partial_{\mu} S)^2 - \frac{1}{2} \mu_S^2 S^2 - \frac{\lambda_S^2}{4} S^4 \quad \text{(1)}
\]

Its symmetric \( \mu_S^2 > 0 \) Wigner mode is Bare-FT. It is also Bare-FT in its \( \mu_S^2 < 0 \) SSB mode, which spontaneously breaks the \( Z_2 \) symmetry, because (crucially, as we see below) it has no Goldstone theorem or associated Nambu-Goldstone Bosons.

2 This BEH fine-tuning problem is separate from the Hierarchy problem—the philosophical question of why the ~100GeV scale of weak interactions is orders of magnitude smaller than that of gravity ~2 \times 10^{19} \text{GeV}—but they are often confused or conflated.

3 In June 2011 one of us introduced the idea of the “Goldstone Exception” (though not the term) for the SM, and showed that the ultra-violet quadratic divergences (UVQD) of the SM did not yield an ultraviolet BEH-FT problem. A December 2011 pedagogical companion paper3 simplified UVQDs in the context of the global Gell-Mann-Lévy model4 and named that concept. We defined “Goldstone Exceptional Naturalness” and showed that the Goldstone theorem protects the weak-scale global SSB SO(2) Schwinger model,5 (i.e. against 1-loop relevant operators \( M_{\text{Heavy}} \gg m_{\text{Weak}} \) which arise from virtual heavy particles) by two explicit 1-loop examples: a real singlet scalar \( S \) and a singlet Majorana neutrino \( \nu_R \) with \( M_{\nu_R}^2, M_{\text{Weak}}^2 \gg |q_{\nu_R}^2|, (H)^2 \).

Ref17 pushed those heavy-particle decoupling (and no-BEH-FT) results, for \( \langle S \rangle = 0 \), to all-loop-orders, using 2 towers of recursive \( SU(2)_L \times U(1)_Y \) WTI, while including the virtual effects of the lightest generation of SM quarks and leptons.

4 The literature provides various definitions/criteria for naturalness, with increasing levels of suppression of fine-tuning. G. ’t Hooft put forward a definition of what it means for parameters of theories to be naturally small3 [23]: “At any energy scale \( \mu \), a [dimensionless] physical parameter or a set of physical parameters \( \alpha(\mu) \) is allowed to be very small only if the replacement \( \alpha(\mu) = 0 \) would increase the symmetry of the system.”
cut off at a short-distance finite Euclidean UV scale, $\Lambda$, never taking the $A^2 \to \infty$ limit. Although that cut-off can be imagined to be near the Planck scale $\Lambda \approx M_{Pl}$, quantum gravitational loops are not included.

The reader should note that this paper concerns stability and protection of the Abelian Higgs model against UVQD (and, in Section III, finite relevant) operators. It does not address any of the other, more usual, stability issues of the Standard Model (cf. the discussion in, for example, [31], and references therein).

The structure of this paper is as follows:

Section II concerns the correct renormalization of the spontaneously broken AHM in Lorenz gauge. We treat the AHM in isolation, as a stand-alone flat-space weak-scale quantum field theory, not embedded or integrated into any higher-scale “Beyond-AHM” physics.

- Subsection II A defines the Abelian Higgs model
- Subsection II B builds a conserved rigid/global AHM current in Lorenz gauge, and reminds [32] us that the rigid/global $U(1)_Y$ charge is not conserved, even for the physical states.
- Subsection II C constructs the $\phi$-sector effective Lagrangian from those WTIs which govern 1-(h, $\pi$) Scalar-Particle-Irreducible (1-$\phi$-I) connected amputated Green’s functions.
- Subsection II D further constrains the $\phi$-sector effective Lagrangian with those WTIs which govern 1-(h, $\pi$)-Reducible (1-$\phi$-R) $\phi$-sector connected amputated on-shell T-Matrix elements, especially the Goldstone theorem.

Section III extends our AHM results to include the all-loop-orders virtual contributions of certain $M^2_{\text{heavy}} \gg m^2_{\text{Weak}}$ heavy $U(1)_Y$ matter representations (which might arise in certain Beyond-AHM models).

- Subsection III A constructs the effective Lagrangian for the extended-AHM, and proves 3 decoupling theorems.
- Subsection III B gives an example of complete heavy-physics decoupling without fine-tuning: a virtual singlet right-handed Type I See-saw Majorana neutrino $\nu_R$ with $M^2_{\nu_R} \gg m^2_{\text{Weak}}$.

Section IV reminds [32] us that the NGB $\tilde{\pi}$ disappears from the observable particle spectrum of the extended-AHM, carrying with it any fine-tuning problem due to heavy $M^2_{\text{heavy}} \gg m^2_{\text{Weak}}$ Beyond-AHM particles.

- Subsection IV A reminds [32] us that the SSB extended-AHM’s observable particle spectrum excludes the NGB $\tilde{\pi}$, precisely because it is a spontaneously broken $U(1)_Y$ gauge theory.
- Subsection IV B derives our 4th and 5th decoupling theorems, and shows complete decoupling, due to SSB, of heavy $M^2_{\text{heavy}} \gg m^2_{\text{Weak}}$ particles.

Section V discusses the exacting mathematical rigor which would have fully satisfied Raymond Stora.

Section VI reminds us that historically (with an important exception) the decoupling of heavy particles is the usual experience of physics.

Appendix A gives detailed derivation, to be used here and in [32], of the $U(1)_Y$ WTIs governing the $\phi$-sector of the AHM. Our renormalized WTIs include all contributions from virtual transverse gauge bosons: $\phi$-scalars; ghosts; $A^\mu; h; \pi; \tilde{\omega}; \omega; \Phi; \psi$. respectively.

Appendix B gives detailed derivation, to be used here and in [32], of $U(1)_Y$ (h, $\pi$)-sector WTIs, which now include the all-loop-orders contributions of certain additional $U(1)_Y$ matter representations: spin $S = 0$ scalars $\Phi$, and $S = \frac{1}{2}$ fermions $\psi$. They include all contributions from virtual transverse gauge bosons; ghosts, scalars; and fermions: $A^\mu; h; \pi; \tilde{\omega}; \omega; \Phi; \psi$. respectively.

II. THE ABELIAN HIGGS MODEL (AHM) IN LORENZ GAUGE

A. The Abelian Higgs model in Lorenz gauge

The BRST-invariant [33–35] Lagrangian of the $U(1)_Y$ AHM gauge theory may be written, in Lorenz gauge, in terms of exact renormalized fields: a transverse vector $A_\mu$, a complex scalar $\phi$, a ghost $\omega$, and an anti-ghost $\bar{\omega}$:

$$L_{\text{AHM}} = L_{\text{GaugeInvariant}} + L_{\text{GaugeFix;Lorenz}} + L_{\text{Ghost;Lorenz}}$$

where

$$L_{\text{GaugeInvariant}} = |D_\mu \phi|^2 - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - V(\phi^\dagger \phi)$$

The reader should note that this paper concerns stability and protection of the Abelian Higgs model against UVQD (and, in Section III, finite relevant) operators. It does not address any of the other, more usual, stability issues of the Standard Model (cf. the discussion in, for example, [31], and references therein).
with

\[ D_\mu \phi = (\partial_\mu - i e Y_\phi A_\mu) \phi \]
\[ A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]
\[ V_{\text{AHM}} = \mu_\phi^2 \left( \phi^\dagger \phi \right) + \lambda_\phi^2 \left( \phi^\dagger \phi \right)^2 \]  
(4)

and

\[ \phi = \frac{1}{\sqrt{2}} (H + i \pi); \quad H = \langle H \rangle + h; \quad Y_\phi = -1. \]  
(5)

Also

\[ L_{\text{Lorenz}}^{\text{Gauge Fix; Lorenz}} = -\lim_{\xi \to 0} \frac{1}{2 \xi} \left( \partial_\mu A^\mu \right)^2 \]
\[ L_{\text{Lorenz}}^{\text{Ghost; Lorenz}} = \tilde{\omega}(-\partial^2) \omega. \]  
(6)

The complex scalar \( \phi \) is manifestly renormalizable in the linear representation [3]. We shall see below that \( m_\pi^2 = e^{2Y_\phi^2} \langle H \rangle^2 \).

This paper distinguishes carefully between the local BRST-invariant \( U(1)_Y \) Lagrangian [2], and its 3 physical modes [36-40]: symmetric Wigner mode, classically scale-invariant point, and physical Goldstone mode.

1) Symmetric Wigner mode \( \langle H \rangle = 0, m_\pi^2 = 0, m_A^2 = m_{\text{BEH}}^2 = \mu_\phi^2 \neq 0 \):

This is QED with massless photons and massive charged scalars. All UVQD \( \sim \Lambda^2 \) and finite relevant operators in the AHM are absorbed into the pseudo-Nambu-Goldstone boson mass-squared \( m_\pi^2 \neq 0 \). That pseudo-NGB mass, which survives in Wigner mode (but not in Goldstone mode), is the result of intermediate steps in the calculations, and has been a source of confusion and controversy surrounding the BEH Fine-Tuning Problem. The final step in Wigner mode calculations, i.e. setting \( \langle H \rangle = 0 \), would support such FT in \( m_\pi^2 \). Since \( m_{\text{BEH}}^2 = m_A^2 \) in the symmetric Wigner mode, any FTP would be passed onto the scalar mass. Wigner mode would be regarded as FT by the standards of GEN.

Thankfully, Nature is not in Wigner mode! Further analysis and renormalization of the Wigner mode lies outside the scope of this paper.

2) Classically scale-invariant point \( \langle H \rangle = 0, m_\pi^2 = 0, m_A^2 = m_{\text{BEH}}^2 = 0 \): Analysis of the quantum scale-invariant point is outside the scope of this paper.

3) Spontaneously broken Goldstone mode \( \langle H \rangle \neq 0, m_\pi^2 = c^2 \langle H \rangle^2 \neq 0, m_A^2 = 0, m_{\text{BEH}}^2 \neq 0 \):

The “famous” Abelian Higgs model, with its Nambu-Goldstone boson (NGB) “eaten” by the Brout-Englert-Higgs mechanism (and, as we will see, WTI governed by the Goldstone theorem) is actually only the SSB “Goldstone mode” of the BRST-invariant local Lagrangian [2], and is the subject of this paper.

We work in Lorenz gauge for many reasons:

- After a subtlety concerning their mixing, \( \pi \) and \( A^\mu \) are orthonormal species. A term \( \sim A_\mu \partial^\mu \pi \) arises from \( \mid D_\mu \phi \mid^2 \) after SSB in [2]. A term \( \pi \partial^\mu A^\mu \) is shown to vanish for physical states in [36-40]. The resultant surface term \( \partial^\mu (\pi A^\mu) \) vanishes (for physical states) because \( A_\mu \) is massive,

- Only in the SSB Goldstone mode of the BRST-invariant Lagrangian [2], and only after first renormalizing in the linear \( \phi \) representation, does the renormalized Kibble \( \phi \) unitary representation

\[ \phi = \frac{1}{\sqrt{2}} (H + i \pi) \equiv \frac{1}{\sqrt{2}} \tilde{H} e^{-iY_\phi \pi/\langle H \rangle} \]
\[ H = \langle H \rangle + h; \quad \tilde{H} = \langle H \rangle + \tilde{h} \]
\[ \tilde{\pi} \equiv \langle H \rangle \theta \]  
(7)

make sense. Here the \( \phi \)-hypercharge \( Y_\phi \neq -1 \).

- We will prove an all-loop-orders Goldstone theorem [31, A27] which forces the \( \pi \) mass-squared \( m_\pi^2 = 0 \).

- We use “pion-pole dominance” (i.e. \( m_\pi^2 = 0 \)) arguments to derive \( U(1)_Y \) SSB WTI’s [33, A22, A30].

- We prove [32] with \( U(1)_Y \) WTI that, in SSB Goldstone mode, \( \tilde{\pi} \) in (7) is a Nambu-Goldstone boson (NGB), and that the resultant SSB gauge theory has a “shift symmetry” \( \tilde{\pi} \to \tilde{\pi} + \langle H \rangle \theta \) for constant \( \theta \).

Analysis is done in terms of the exact renormalized interacting fields, which asymptotically become the in/out states, i.e. free fields for physical S-Matrix elements.

The most important issue for fine-tuning (and heavy particle decoupling) is the classification and disposal of relevant operators, in this case the \( \pi, h \) and \( A_\mu \) inverse propagators (together with tadpoles). Define the exact renormalized pseudo-scalar propagator in terms of a massless \( \pi \), the Källén-Lehmann [36, 31] spectral density \( \rho_{\text{AHM}}^\pi \), and wavefunction renormalization \( Z_{\text{AHM}}^\phi \). In Lorenz gauge:

\[ \Delta_{\text{AHM}}^\pi(q^2) = -i(2\pi)^2 \langle 0 | T \{ \pi(y)\pi(0) \} | 0 \rangle \text{Fourier} \]
\[ = \frac{1}{q^2 + i\epsilon} + \int dm^2 \rho_{\text{AHM}}^\pi(m^2) \]
\[ \left[ Z_{\text{AHM}}^\phi \right]^{-1} = 1 + \int dm^2 \rho_{\text{BEH}}^\pi(m^2) \]  
(8)

Define also the BEH scalar propagator in terms of a BEH scalar pole and the (subtracted) spectral density \( \rho_{\text{BEH}} \), and the same wavefunction renormalization. We
assume \( h \) decays weakly, and resembles a resonance:

\[
\Delta_{AHM}^{\text{BEH}}(q^2) = -i(2\pi)^2 \langle 0 | T[h(x)h(0)] | 0 \rangle \text{ Fourier Transform}
\]

\[
= \frac{1}{q^2 - m_{\text{BEH}}^2 + i\epsilon} + \int \frac{d^2 \rho_{\text{AHM}}(m^2)}{q^2 - m^2 + i\epsilon}
\]

\[
\left[ Z_{AHM}^\phi \right]^{-1} = 1 + \int \frac{d^2 \rho_{\text{BEH}}(m^2)}{q^2 - m^2 + i\epsilon}
\]

\[
\int d^2 \rho_{\text{AHM}}(m^2) = \int d^2 \rho_{\text{BEH}}(m^2)
\]

Although \( m_{\text{BEH}}^2 \) may (or may not!) be FT, the spectral density parts of the propagators

\[
\Delta_{AHM}^{\pi,\text{Spectral}}(q^2) = \int d^2 \rho_{\text{AHM}}(m^2) \frac{1}{q^2 - m^2 + i\epsilon}
\]

\[
\Delta_{AHM}^{\text{BEH, Spectral}}(q^2) = \int d^2 \rho_{\text{BEH}}(m^2) \frac{1}{q^2 - m^2 + i\epsilon}
\]

are certainly not fine-tuned. Dimensional analysis of the wavefunction renormalizations shows that the contribution of a state of mass/energy \( \sim M_{\text{Heavy}} \) to the spectral densities \( \rho_{\text{AHM}}^\pi(M_{\text{Heavy}}^2), \rho_{\text{BEH}}(M_{\text{Heavy}}^2) \sim \frac{1}{M_{\text{Heavy}}} \), and to \( \Delta_{AHM}^{\pi,\text{Spectral}}, \Delta_{BEH,\text{Spectral}} \) only irrelevant terms \( \sim \frac{1}{M_{\text{Heavy}}} \). The finite Euclidean cut-off contributes only irrelevant terms \( \sim \frac{1}{\Lambda} \).

B. Rigid/global \( U(1)_Y \) WTI and conserved rigid/global current, for the physical states of the SSB AHM, in Lorenz gauge. Rigid/global \( U(1)_Y \) Charge is not conserved!

In their seminal work, E. Kraus and K. Sibold identified, in the Abelian Higgs model, an anomaly-free “deformed” rigid/global \( U(1)_Y \) symmetry as a rigid subset of that anomaly-free deformed local/gauge symmetry.

The SSB case is tricky: neither the usual gauge/local symmetry, nor even its restriction to a rigid/global symmetry (where the gauge transformations are taken to be independent of position), commute with global BRST symmetry. Only deformed versions of them do.

Kraus and Sibold then constructed deformed Ward-Takahashi Identities (WTI), allowing them to demonstrate (with appropriate normalization conditions) proof of all-loop-orders renormalizability and unitarity for the SSB Abelian Higgs model. Because their renormalization relies only on deformed \( U(1)_Y \) WTI, Kraus and Sibold’s results are independent of regularization scheme, for any acceptable scheme (i.e. if one exists).

Nevertheless, Slavnov-Taylor identities prove that the on-shell S-Matrix elements of “physical particles” (i.e. spin \( S = 0 \) scalars \( h, \pi \), and \( S = 1 \) transverse gauge bosons \( A_\mu \), but not fermionic ghosts \( (\bar{\phi}, \phi) \) are independent of the usual undeformed anomaly-free \( U(1)_Y \) local/gauge transformations, even though these break the Lagrangian’s BRST symmetry.

We therefore observed in [32] that SSB S-Matrix elements are therefore also independent of anomaly-free undeformed \( U(1)_Y \) global/rigid transformations, resulting in a “new” global/rigid current and appropriate undeformed \( U(1)_Y \) Ward-Takahashi identities. All this is done without reference to the unbroken Wigner mode and scale-invariant point.

We are interested in rigid-symmetric relations among 1-\((h, \pi)\)-Irreducible (1-\(\phi\)-I) connected amputated Green’s functions \( \Gamma_{i,j} \), and among 1-\((h, \pi)\)-Reducible (1-\(\phi\)-R) connected amputated transition-matrix (T-Matrix) elements \( T_{i,j} \), with external \( \phi \) scalars. It is convenient to use the powerful old tools (e.g. canonical quantization) from Quantum Field Theory (QFT), a name coined by Ergin Sezgin.

We focus on the rigid/global AHM current \( J_\mu^{AHM} \) which transforms as an axial-vector

\[
J_\mu^{AHM} = \pi \partial^\mu H - H\partial^\mu \pi - eA^\mu (\pi^2 + H^2)
\]

Rigid/global transformations of the fields are as usual: from the equal-time commutators \( [\Lambda, T] \)

\[
\delta H(t, \vec{y}) = -i\theta \int d^3 z [J_\mu^{AHM}(t, \vec{z}), H(t, \vec{y})]
\]

\[
= -\theta \int d^3 z \pi(t, \vec{z}) \delta^3(\vec{z} - \vec{y})
\]

\[
\delta \pi(t, \vec{y}) = -\theta \int d^3 z [J_\mu^{AHM}(t, \vec{z}), \pi(t, \vec{y})] = \theta \int d^3 z H(t, \vec{z}) \delta^3(\vec{z} - \vec{y}) = H(t, \vec{y}) \theta
\]

\[
\delta A^\mu(t, \vec{y}) = 0
\]

\[
\delta \omega(t, \vec{y}) = 0
\]

so \( J_\mu^{AHM}(t, \vec{z}) \) serves as a “proper” local current, for commutator purposes.

In contrast, [32] showed that, in Lorenz gauge, \( U(1)_Y \) AHM (and therefore also extended \( U(1)_Y \) AHM) has no
proper global charge $Q(t) \equiv \int d^3 z J^\mu_{AHM}(t, \vec{z})$, because $\frac{d}{dt} Q(t) \neq 0$. See Eqn. (17) below.

The classical equations of motion reveal a crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian [2], the classical current (10) is not conserved. In Lorenz gauge

$$\partial_\mu J^\mu_{AHM} = -H m A F_A$$
$$m A = e^\mu(H)$$
$$F_A = \partial_\beta A^\beta$$

(12)

with $F_A$ the gauge fixing condition.

But the the global $U(1)_Y$ current (10) is conserved by the physical states [32], and therefore still qualifies as a “real current”. Strict quantum constraints are imposed, which force the relativistically-covariant theory of gauge bosons to propagate only its true number of quantum spin $S = 1$ degrees of freedom: these constraints are, in the modern literature, implemented by use of spin $S = 0$ fermionic Fadeev-Popov ghosts ($\tilde{\omega}, \omega$).

The physical states and their time-ordered products, but not the BRST-invariant Lagrangian then obey G. ’t Hooft’s [48] Lorentz gauge gauge-fixing condition interpreted as follows for the $\phi$-sector connected time-ordered product

$$\langle 0 | T \left[ \left( \partial_\beta A^\beta(z) \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(13)

Here we have $N$ external renormalized scalars $h = H - \langle H \rangle$ (coordinates $x$, momenta $p$), and $M$ external ($CP = -1$) renormalized pseudo-scalars $\pi$ (coordinates $y$, momenta $q$).

Eqn. (13) restores conservation of the rigid/global $U(1)_Y$ current for $\phi$-sector connected time-ordered products [32]

$$\langle 0 | T \left[ \left( \partial_\mu J^\mu_{AHM}(z) \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(14)

It is in this “physical” connected-time-ordered-product sense that the rigid global $U(1)_Y$ “physical current” is conserved: the physical states, but not the BRST-invariant Lagrangian obey the physical current conservation equation (14). It is this “physical conserved current” which generates our $U(1)_Y$ WTI [32].

Appendix A derives 2 towers of quantum $U(1)_Y$ WTIs, which exhaust the information content of (14), and severely constrain the dynamics (i.e. the connected time-ordered products) of the $\phi$-sector physical states of the SSB AHM.

We might have hoped to build a conserved charge by restricting it to physical connected time-ordered products

$$\langle 0 | T \left[ \left( \frac{d}{dt} Q_{AHM}(t) \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = \left[ \int_2 \zeta^{2-\text{surface}} \right. \langle 0 | T \left[ \left( J_{AHM}(t, \vec{z}) \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}}$$

(15)

where we have used Stokes theorem, and $\zeta^{2-\text{surface}}$ is a unit vector normal to the 2-surface. The time-ordered product constrains the 2-surface to lie on-or-inside the light-cone.

At a given point on the surface of a large enough 3-volume $\int d^3 z$ (i.e. the volume of all space), which lies on-or-inside the light cone, all fields on the $\zeta^{2-\text{surface}}$ are asymptotic in-states and out-states; are properly quantized as free fields; with each field species orthogonal to the others; and evaluated at equal times, so that time-ordering is unnecessary.

But (15) does not vanish [32] because, after SSB, a specific term in $J^\mu_{AHM}$ in (10)

$$\int_{\text{LightCone} \rightarrow \infty} dz \zeta^{2\text{LightCone}} \left[ \langle 0 | T \left[ \left( - \langle \partial_\mu J^\mu_{AHM} \rangle \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle \right.$$  

(16)

does not vanish. $\pi$ is massless (in Lorenz gauge) in the SSB AHM, capable of carrying (along the light-cone) long-ranged pseudo-scalar forces out to the very ends of the light-cone ($\zeta^{2\text{LightCone}} \rightarrow \infty$), but not inside it.

Eqns. (15–16) then show that the spontaneously broken $U(1)_Y$ AHM charge is not conserved, even for connected time-ordered products [32], in Lorenz gauge

$$\langle 0 | T \left[ \left( \frac{d}{dt} Q_{U(1);AHM}(t) \right) \times h(x_1) \ldots h(x_N) \pi_{i_1}(y_1) \ldots \pi_{i_M}(y_M) \right] | 0 \rangle_{\text{Connected}} \neq 0$$

(17)

dashing all further hope.

The simplest, most powerful, and most general proof of the Goldstone theorem [18, 19, 52] requires a conserved charge $\frac{d}{dt} Q = 0$, so that proof fails for spontaneously broken gauge theories. This (of course) is a very famous result [49, 52], and allows the spontaneously broken AHM to generate a mass-gap $m_A$ for the vector $A^\mu$, and avoid massless particles in its observable physical spectrum. This is true, even in Lorenz gauge, where there
is a Goldstone theorem, $\pi$ is massless, and $\tilde{\pi}$ is a NGB.

Massless $\pi$ is the basis of our pion-pole-dominance-based $U(1)_Y$ WTI, derived in Appendix A, which give: relations among $1$-\(\phi\)-R connected amputated $\phi$-sector T-Matrix elements $T_{N,M}$; relations among $1$-\(\phi\)-I connected amputated $\phi$-sector Green's functions $\Gamma_{N,M}$; 1-soft-pion theorems; infra-red finite for $m^2_{\pi} = 0$; and a Goldstone theorem.

### C. Construction of the scalar-sector effective Lagrangian from those $U(1)_Y$ WTI which govern connected amputated $1$-\(\phi\)-I Green's functions

In Appendix A, we derive $U(1)_Y$ “pion-pole-dominance” $1$-\(\phi\)-R connected amputated T-Matrix WTI for the SSB AHM. Their solution is a tower of identities:

$$\Gamma_0(q, -q) \equiv \left[ \Delta_{\pi}(q^2) \right]^{-1}$$

$$\Gamma_2(q, -q) \equiv \left[ \Delta_{BEH}(q^2) \right]^{-1}$$

We can now form the $\phi$-sector effective Lagrangian in Lorenz gauge. All perturbative quantum loop corrections, to all-loop-orders and including all UVQD, log-divergent and finite contributions, are included in the $\phi$-sector effective Lagrangian: 1-\(\phi\)I Green’s functions $\Gamma_{N,M}(p_1,...p_N;q_1,...q_M)$; wavefunction renormalizations; renormalized $\phi$-scalar propagators; the BEH VEV $\langle \phi \rangle$; all gauge boson and ghost propagators. This includes the full all-loop-orders renormalization of the AHM $\phi$-sector, originating in quantum loops containing transverse virtual gauge bosons, $\phi$-scalars and ghosts: $A^\mu; h; \pi; \bar{\omega}; \omega$, respectively. Because they arise entirely from global $U(1)_Y$ WTI, our results are independent of regularization-scheme.

We want to classify operators arising in AHM loops, and separate the finite operators from the divergent ones. We focus on finite operators, as well as quadratic and logarithmically divergent operators, which may cause a fine-tuning problem.

There are 3 classes of finite operators, which cannot generate fine-tuning in the AHM:

- Finite $O^{1/\Lambda^2}_{AHM}$ vanish as $m_{Weak}/\Lambda^2 \to 0$;
- $O^{Dim > 4; \text{Light}}_{AHM}$ are finite dimension $Dim > 4$ operators, where only the light degrees of freedom $A^\mu; h; \pi; \bar{\omega}; \omega$ contribute to all-loop-orders renormalization;
- $O^{Dim \leq 4; \text{NonAnalytic}}_{AHM}$ are finite dimension $Dim \leq 4$ operators which are non-analytic in momenta or in a renormalization scale $\mu^2$ (e.g. finite renormalization-group logarithms).

All such operators will be ignored.

$$O^{\text{Ignore}}_{AHM} = O^{1/\Lambda^2; \text{Irrelevant}}_{AHM} + O^{Dim > 4; \text{Light}}_{AHM} + O^{Dim \leq 4; \text{NonAnalytic}}_{AHM}$$

Such finite operators appear throughout the $U(1)_Y$ Ward-Takahashi IDs.

---

7 The rigid $U(1)_Y$ WTI for the $U(1)_Y$ AHM gauge theory are a generalization of the classic work of B.W. Lee, who constructed 2 all-loop-orders renormalized towers of WTI’s for the global $SU(2)_L \times SU(2)_R$ Gell-Mann Levy (GML) model with Partially Conserved Axial-vector Currents (PCAC). We replace GML’s strongly-interacting Linear Sigma Model ($\Sigma$M) with a weakly-interacting BEH $\Sigma$M, with explicit PCAC breaking $= 0$. Replace $\sigma \to H, \bar{\sigma} \to \pi, m_\sigma \to m_{BEH}, f_\sigma \to \langle H \rangle$, and add local gauge group $U(1)_Y$. This generates a set of global $U(1)_Y$ WTI governing relations among weak-interaction 1-\(\phi\)-R T-Matrix elements $T_{N,M}$. A solution-set of those $U(1)_Y$ WTI then govern relations among $U(1)_Y$ 1-\(\phi\)-I Green’s functions $\Gamma_{N,M}$. As observed by Lee for GML, one of those on-shell T-Matrix WTI is the Goldstone theorem. Appendix A includes, in Table 1, translation between the WTI proofs in this paper (a gauge theory) and in B.W. Lee (a global theory).

8 In the Standard Model, there are finite operators that arise entirely from SM degrees of freedom, which are crucially important for computing experimental observables. The most familiar are the standard 1-loop high precision Standard Model predictions for the top-quark from Z-pole physics in 1984 and the $W^\pm$ mass in 1980, as well as the 2-loop BEH mass from Z-pole physics and the $W^\pm$ mass: those precisely predicted the experimental discovery-masses of the top quark at FNAL and BEH scalar at CERN. But the $U(1)_Y$ analogy of such finite operators are not the point of this paper.
• $N + M \geq 5$ is $\mathcal{O}_{\text{AHM}}^{1/N^2;\text{irrelevant}}$ and $\mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light}$;

• The left hand side of (18) for $N + M = 4$ is also $\mathcal{O}_{\text{AHM}}^{1/N^2;\text{irrelevant}}$ and $\mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light}$;

• $N + M \leq 4$ operators $\mathcal{O}_{\text{AHM}}^{\text{Dim}<4};\text{NonAnalytic}$ appear in (18).

Finally, there are $N + M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor-series, and ignore $\mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light}$ and $\mathcal{O}_{\text{AHM}}^{1/N^2;\text{irrelevant}}$ terms in that series.

The all-loop-orders renormalized scalar-sector effective Lagrangian is then formed for $(h, \pi)$ with $\text{CP}=(-1, -1)$

\[
\mathcal{L}^\text{Eff:Wigner,SI,Goldstone}_{\text{L}\text{AHM}:\phi,Lorenz} = \Gamma_1(0;h) + \frac{1}{2!}\Gamma_2,0(p,-p)h^2
\]

\[
+ \frac{1}{2!}\Gamma_0,2(q,-q)\pi^2 + \frac{1}{3!}\Gamma_3(0000;h^3
\]

\[
+ \frac{1}{2!}\Gamma_1(000;h^2\pi^2
\]

\[
+ \frac{1}{2!}\Gamma_2,0(000;h^2\pi^2
\]

\[
+ \frac{1}{2!}\Gamma_4(000;\pi^4 + \mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light})
\]

(21)

The Ward-Takahashi IDs (18) for Greens functions severely constrain the effective Lagrangian (21).

• WTI $N = 0, M = 1$

\[
\Gamma_{1,0}(0;h) = \langle H \rangle \Gamma_{0,2}(0;0)
\]

(22)

since no momentum can run into the tadpoles.

• WTI $N = 1, M = 1$

\[
\Gamma_{2,0}(q,-q) - \Gamma_{0,2}(q,-q)
\]

\[
\Gamma_{1,0}(0;0) + \mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light}
\]

\[
\Gamma_{2,0}(0;0) = \Gamma_{0,2}(0;0) + \langle H \rangle \Gamma_{1,2}(0;0)
\]

\[
+ \mathcal{O}_{\text{AHM}}^{\text{Dim}>4};\text{Light}
\]

(24)

9 In previous papers on $SU(2)_L \times SU(2)_R$ Gell-Mann-Lévý [13], we have written this $N = 1, M = 1$ WTI as a mass relation between the BEH scalar and the pseudo-Nambu-Goldstone boson $\pi$ pseudo-scalar. In the Källén-Lehmann representation

\[
m_{\text{BEH}}^2 = m_\pi^2 + 2\lambda_\phi^2\langle H \rangle^2
\]

\[
m_\pi^2 = \left[\frac{1}{m_{\pi,\text{Pole}}} + \int \frac{dm^2 P_{\pi}(m^2)}{m^2}\right]^{-1}
\]

\[
m_{\text{BEH}}^2 = \left[\frac{1}{m_{\text{BEH,Pol}}} + \int \frac{dm^2 P_{\text{BEH}}(m^2)}{m^2}\right]^{-1}
\]

10 The inclusive Gell-Mann Lévý effective potential derived [13] from B.W. Lee’s WTI [36] reduces to the three different effective potentials of the global $SU(2)_L \times SU(2)_R$ Schwinger model [16]: Schwinger Wigner mode ($\langle H \rangle = 0$, $m_\pi^2 = m_\phi^2 \neq 0$); Schwinger Scale-Invariant point ($\langle H \rangle = 0$, $m_\pi^2 = m_\phi^2 \neq 0$); or Schwinger Goldstone mode ($\langle H \rangle \neq 0$, $m_\pi^2 = 0; m_{\text{BEH}}^2 \neq 0$).
The T-matrix vanishes as one of the pion momenta goes to zero (i.e. 1-soft-pion theorems), provided all other physical scalar particles are on mass-shell. Eqn. (33) also “asserts the absence of infrared (IR) divergences in the scalar-sector (of AHM) Goldstone mode (in Lorenz gauge). Although individual Feynman diagrams are IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit” [39].

The $N = 0, M = 1$ case of [33] is the Goldstone theorem (A27) itself [66]:

\[
(H) T_{0,2}(0;0) = 0
\]

Since the 2-point $T_{0,2}$ is already 1-\phi-I, we may write the Goldstone theorem as a further constraint on the 1-\phi-I Greens function

\[
\langle H \rangle \Gamma_{0,2} ;00 \rangle = \langle H \rangle \left[ \Delta_\pi (0) \right]^{-1} = 0
\]

Another crucial effect of the Goldstone theorem, together with the $N = 0, M = 1$ $U(1)_Y$ Ward-Takahashi Greens function identity [18], is to automatically eliminate tadpoles in (21)

\[
\Gamma_{1,0}(0) = \langle H \rangle \Gamma_{0,2} ;00 \rangle = 0
\]

so that separate tadpole renormalization is un-necessary. We re-write the effective potential (30) but now including the constraint from the Goldstone theorem [34] 35:

\[
L_{\text{Eff}; \text{Goldstone}}^{\text{AHM}; \phi; \text{Lorenz}} = L_{\text{Kinetic}; \text{Eff}; \text{Goldstone}}^{\text{AHM}; \phi; \text{Lorenz}} - V_{\text{Eff}; \text{Goldstone}}^{\text{AHM}; \phi; \text{Lorenz}} + \mathcal{O}^{\text{AHM}; \text{Ignore}}
\]

\[
V_{\text{Eff}; \text{Goldstone}}^{\text{AHM}; \phi; \text{Lorenz}} = \lambda_\phi \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle \hbar \right]^2
\]

and wavefunction renormalization

\[
\Gamma_{0,2} (q,-q) - \Gamma_{0,2} (0;0) = q^2 + \mathcal{O}^{\text{AHM}; \text{Ignore}}
\]

so the \phi-sector Goldstone mode effective coordinate space Lagrangian becomes

\[
L_{\text{Eff}; \text{Goldstone}}^{\text{AHM}; \phi; \text{Lorenz}} = \varrho_\phi \left[ \theta \phi - \frac{\lambda_\phi}{2} \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle \hbar \right]^2 + \mathcal{O}^{\text{AHM}; \text{Ignore}}
\]

Eqn. (39) is the \phi-sector effective Lagrangian of the spontaneously broken Abelian Higgs model, in Lorenz gauge, contrained by the Goldstone theorem: 11

Imagine we suspected that $\pi$ is not all-loop-orders massless in Lorenz gauge SSB AHM, and simply/naively wrote a mass-squared $m_\pi^{2, \text{Pole}}$ (which we even imagined to be fine-tuned!) into the $\pi$ inverse-propagator

\[
\left[ \Delta_\pi (0) \right]^{-1} \equiv -m_\pi^{2, \text{Pole}} = m_\pi^{2, \text{Pole}} \left[ 1 + m_\pi^{2, \text{Pole}} \int \frac{d^3 p}{(2\pi)^3} \frac{\rho_\pi (m^2)}{m^2} \right]^{-1}
\]

But the Goldstone theorem (35) insists instead that

\[
\langle H \rangle \left[ \Delta_\pi (0) \right]^{-1} \equiv -m_\pi^{2} = \langle H \rangle \Gamma_{0,2};00 \rangle = 0
\]

The $\pi$-pole mass vanishes exactly, and is GEN not FT.

\[
m_\pi^{2, \text{Pole}} = m_\pi^{2} \left[ 1 - m_\pi^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{\rho_\pi (m^2)}{m^2} \right]^{-1} = 0
\]
• It includes all divergent $O(\Lambda^2), O(\ln \Lambda^2)$ and finite terms which arise, to all perturbative loop-orders in the full $U(1)_Y$ gauge theory, due to virtual transverse gauge bosons, $\phi$ scalars and ghosts; $A^\mu; \pi; \tilde{\omega}, \omega$ respectively.

• It obeys the Goldstone theorem \([34,35]\) and all other $U(1)_Y$ Ward-Takahashi Green's function and T-Matrix identities;

• It is minimized at $\langle H = \langle H \rangle, \pi = 0 \rangle$; and obeys stationarity \([30]\) of that true minimum;

• It preserves the theory’s renormalizability and unitarity, which require that wavefunction renormalization, $\langle H \rangle_{\text{Bare}} = [Z^\phi_{\text{AHM}}]^{1/2} \langle H \rangle$ \([17,40,41]\), forbid UVQD, relevant, or any other dimension-2 operator corrections to $\langle H \rangle$;

• The Goldstone theorem \([34]\) has caused all relevant operators in the spontaneously broken Abelian Higgs model to vanish!

In order to make manifest that $\tilde{\pi}$ is a true NGB \([47,69]\) in Lorenz gauge, re-write \((39)\) in the unitary Kibble representation. \([31,69]\) with $Y_\phi = -1$ the $\phi$ hypercharge. In coordinate space

$$\phi = \frac{1}{\sqrt{2}} \hat{H} e^{-iY_\phi \tilde{\pi}/\langle H \rangle}$$

$L^\text{Eff;Goldstone}_{\text{AHM;\phi;Lorenz}} = \frac{1}{2} \left( \partial_\mu \hat{H} \right)^2 + \frac{1}{2} \langle \hat{H} \rangle^2 \left( \partial_\mu \tilde{\pi} \right)^2$

$$- \frac{\lambda^2_\phi}{4} \left[ \hat{H}^2 - \langle H \rangle^2 \right]^2 + O^\text{AHM}_{\text{Ignore}}$$

$$= \frac{1}{2} \left( \partial_\mu \hat{h} \right)^2 + \frac{1}{2} \left( 1 + \frac{\hat{h}}{\langle H \rangle} \right)^2 \left( \partial_\mu \tilde{\pi} \right)^2$$

$$- \frac{\lambda^2_{\phi}}{4} \left[ \hat{h}^2 + \langle H \rangle \hat{h} \right]^2 + O^\text{AHM}_{\text{Ignore}}$$

(43)

shows that $\tilde{\pi}$ has only derivative couplings and, for constant $\theta$, a shift symmetry

$$\tilde{\pi} \rightarrow \tilde{\pi} + \langle H \rangle \theta$$

(44)

The Green’s function Ward-Takahashi ID \([18]\) for $N = 1, M = 1$, constrained by the Goldstone theorem \([35]\), relates the BEH mass to the coefficient of the $h^2 \pi^2$ vertex

$$\Gamma_{2,0}(00; \cdot \cdot) = \langle H \rangle \Gamma_{1,2}(0; 00)$$

(45)

Therefore, the BEH mass-squared in \([43]\)

$$m^2_{\text{BEH}} = 2\lambda^2_\phi \langle H \rangle^2$$

(46)

arises entirely from SSB, as does (together with its AHM decays) the gauge-independent observable resonance pole-mass-squared

$$m^2_{\text{BEH; Polec}} = 2\lambda^2_\phi \langle H \rangle^2 \left[ 1 - 2\lambda^2_\phi \langle H \rangle^2 \int dm^2 \rho^\text{BEH}_{\text{AHM;\phi}}(m^2) \right]^{-1} + O^\text{AHM}_{\phi;\text{Ignore}}$$

(47)

where the spectral density $\rho^\text{BEH}_{\text{AHM;\phi}}$ is displayed in the Kibble representation.

Since weak scale $\langle H \rangle = [Z^\phi_{\text{AHM}}]^{-\frac{1}{2}} \langle H \rangle_{\text{Bare}}$ and dimensionless $\lambda^2_{\phi}$, absorb no relevant operators, and are therefore not FT, the $\phi$-sector of the SSB AHM gauge theory is Goldstone Exceptionally Natural, with far more powerful suppression of fine-tuning than G. ’t Hooft’s naturalness criteria \([23]\) would demand.

III. EXTENDED-AHM: $U(1)_Y$ WTI CAUSE CERTAIN HEAVY MATTER REPRESENTATIONS TO DECOUPLE FROM THE LOW-ENERGY $\phi$-SECTOR EFFECTIVE LAGRANGIAN

If the Euclidean cutoff $\Lambda^2$ were a true proxy for very heavy $M^2_{\text{heavy}} = 0$ scalars $\Phi$, and $S = \frac{1}{2}$ spin fermions $\psi$, we would already be in a position to comment on their de-coupling. Unfortunately, although the literature seems to cite such proxy, it is simply not true. In order to prove theorems which reveal symmetry-driven results in gauge theories, one must keep all of the terms arising from all Feynman graphs; i.e. not just “a selection of interesting terms from a representative subset of Feynman graphs” (Ergin Sezgin’s dictum).

A. $\phi$-sector effective Lagrangian for the extended-AHM

1) $1$-$\phi$-I connected amputated $\phi$-sector Green’s functions $\Gamma^\text{Extended}_{N,M}$: In Appendix \([44]\) we derive a tower of recursive $U(1)_Y$ WTI \([B18]\) which govern connected $1$-$\phi$-I Green’s functions for the extended-AHM:

$$\langle H \rangle \Gamma^\text{Extended}_{N,M+1}(p_1 \cdots p_N; 0 q_1 \cdots q_M)$$

$$= \sum_{m=1}^{M} \Gamma^\text{Extended}_{N+1,M-1}(q_m p_1 \cdots p_N; q_1 \cdots \hat{q}_m \cdots q_M)$$

$$- \sum_{n=1}^{N} \Gamma^\text{Extended}_{N-1,M+1}(p_1 \cdots \hat{p}_n \cdots p_N; p_n q_1 \cdots q_M)$$

(48)

valid for $N, M \geq 0$.

$\Gamma^\text{Extended}_{N,M}$ includes all the loop-orders renormalization of the $\phi$-sector SSB extended-AHM, including virtual transverse gauge bosons, $\phi$-scalars, ghosts new scalars, and new fermions: $A^\mu; \pi; \tilde{\omega}, \omega; \Phi; \psi$; respectively.

There are $4$ classes of $\text{finite}$ operators in the full SSB extended-AHM gauge theory, which can generate neither fine-tuning nor “non-decoupling” of heavy particles

• Finite $O^\text{1/Lambda^2;irrelevant}_{E-\text{AHM;\phi}}$ vanish as $m^2_{\text{Weak}}/\Lambda^2 \rightarrow 0$ or $M^2_{\text{Heavy}}/\Lambda^2 \rightarrow 0$;

• Finite $O^\text{Dim>4;Light}_{E-\text{AHM;\phi}}$ are dimension $\text{Dim} > 4$ operators, where only the light degrees of freedom,
including ghosts ($\hat{\omega}, \omega$) and ($\Phi_{\text{Light}}, \psi_{\text{Light}}$), contribute to all-loop-orders renormalization.

- $O^{\text{Dim} \leq 4, \text{NonAnalytic}; \phi}_{E \text{- AHM}; \phi}$ are finite dimension operators, which are non-analytic in momenta or in a renormalization scale $\mu^2$, where only the light degrees of freedom $A^i, h, \pi; \hat{\omega}, \omega; \Phi_{\text{Light}}; \psi_{\text{Light}}$ contribute to all-loop-orders renormalization.

- $O^{1/M^2_{\text{heavy}}; \text{Irrelevant}}_{E \text{- AHM}; \phi}$ vanish as $m^2_{\text{Weak}}/M^2_{\text{Heavy}} \rightarrow 0$.

In addition, $O^{\text{Dim} \leq 4, \text{NonAnalytic}; \text{Heavy}}_{E \text{- AHM}; \phi}$ are finite dimension operators, which are non-analytic in momenta or in a renormalization scale $\mu^2$, where the heavy degrees of freedom $\Phi_{\text{Heavy}}; \psi_{\text{Heavy}}$ contribute to all-loop-orders renormalization. Analysis of these operators lies outside the scope of this paper.

All such operators will be ignored

$$O^{\text{Ignore}}_{E \text{- AHM}; \phi} = O^{1/\Lambda^2; \text{Irrelevant}}_{E \text{- AHM}; \phi} + O^{\text{Dim} > 4; \text{Light}}_{E \text{- AHM}; \phi} + O^{\text{Dim} \leq 4; \text{NonAnalytic}; \text{Light}}_{E \text{- AHM}; \phi} + O^{\text{Dim} \leq 4; \text{NonAnalytic}; \text{Heavy}}_{E \text{- AHM}; \phi} + O^{1/M^2_{\text{heavy}}; \text{Irrelevant}}_{E \text{- AHM}; \phi}$$  \hspace{1cm} (49)

Such finite operators appear throughout the extended $U(1)_Y$ Ward-Takahashi IDs [48]

- $N + M \geq 5$ is $O^{1/\Lambda^2; \text{Irrelevant}}_{E \text{- AHM}; \phi}, O^{\text{Dim} > 4; \text{Light}}_{E \text{- AHM}; \phi}$ and $O^{1/M^2_{\text{heavy}}; \text{Irrelevant}}_{E \text{- AHM}; \phi}$;

- The left hand side of [48] for $N + M = 4$ is also $O^{1/\Lambda^2; \text{Irrelevant}}_{E \text{- AHM}; \phi}, O^{\text{Dim} > 4; \text{Light}}_{E \text{- AHM}; \phi}$ and $O^{1/M^2_{\text{heavy}}; \text{Irrelevant}}_{E \text{- AHM}; \phi}$;

- $N + M \leq 4$ operators $O^{\text{Dim} \leq 4; \text{NonAnalytic}; \text{Light}}_{E \text{- AHM}; \phi}$ also appear in [48].

Finally, there are $N + M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor-series, and ignore $O^{\text{Dim} > 4; \text{Light}}_{E \text{- AHM}; \phi}, O^{1/\Lambda^2; \text{Irrelevant}}_{E \text{- AHM}; \phi}$ and $O^{1/M^2_{\text{heavy}}; \text{Irrelevant}}_{E \text{- AHM}; \phi}$ in that series.

The all-loop-orders renormalized $\phi$-sector effective momentum-space Lagrangian for extended-AHM is then formed for $(h, \bar{\pi})$ external particles with CP=$(1, -1)$

$$\Gamma^{\text{Eff}; \text{Wigner,SI,Goldstone}}_{E \text{- AHM}; \phi} = \Gamma^{\text{Extended}}_{1,0} (0; ) h$$

\begin{align*}
&+ \frac{1}{2!} \Gamma^{\text{Extended}}_{2,0} (p, -p; ) h^2 \\
&+ \frac{1}{2!} \Gamma^{\text{Extended}}_{0,2} (; q, -q) \pi^2 + \frac{1}{3!} \Gamma^{\text{Extended}}_{5,0} (000; ) h^3 \\
&+ \frac{1}{2!} \Gamma^{\text{Extended}}_{1,2} (00; 00) h^2 \pi^2 + \frac{1}{4!} \Gamma^{\text{Extended}}_{4,0} (0000; ) h^4 \\
&+ \frac{1}{4!} \Gamma^{\text{Extended}}_{2,2} (00; 00) h^2 \pi^2 + \frac{1}{4!} \Gamma^{\text{Extended}}_{0,4} (0000; \pi^4 + O^{E \text{- AHM}}_{\text{Ignore}} \end{align*}

The $U(1)_Y$ Ward-Takahashi IDs [48] severely constrain the effective Lagrangian of the extended-AHM.

- WTI $N = 0, M = 1$

$$\Gamma^{\text{Extended}}_{1,0} (0; ) = \langle H \rangle \Gamma^{\text{Extended}}_{0,2} (00; )$$  \hspace{1cm} (51)

since no momentum can run into the tadpoles.

- WTI $N = 1, M = 1$

$$\Gamma^{\text{Extended}}_{2,0} (−q, q; ) − \Gamma^{\text{Extended}}_{0,2} (q, −q)$$

\begin{align*}
&= \langle H \rangle \Gamma^{\text{Extended}}_{1,2} (−q, q; 0) \\
&= \langle H \rangle \Gamma^{\text{Extended}}_{1,2} (00; 00) + O^{\text{Ignore}}_{E \text{- AHM}; \phi}
\end{align*}

$$\Gamma^{\text{Extended}}_{2,0} (00; ) = \Gamma^{\text{Extended}}_{0,2} (00; ) + \langle H \rangle \Gamma^{\text{Extended}}_{1,2} (00; 00) + O^{\text{Ignore}}_{E \text{- AHM}; \phi}$$  \hspace{1cm} (52)

- WTI $N = 2, M = 1$

$$\langle H \rangle \Gamma^{\text{Extended}}_{2,2} (00; 00) = \Gamma^{\text{Extended}}_{3,0} (000; )$$

\begin{align*}
&− 2 \Gamma^{\text{Extended}}_{1,2} (00; 00) \\
&= \Gamma^{\text{Extended}}_{3,0} (000; ) − 2 \Gamma^{\text{Extended}}_{1,2} (00; 00) \\
&− \Gamma^{\text{Extended}}_{1,2} (00; 00) + \langle H \rangle \Gamma^{\text{Extended}}_{2,2} (00; 00)
\end{align*}

- WTI $N = 0, M = 3$

$$\langle H \rangle \Gamma^{\text{Extended}}_{0,4} (; 0000) = 3 \Gamma^{\text{Extended}}_{1,2} (00; 00)$$  \hspace{1cm} (54)

- WTI $N = 1, M = 3$

$$0 = 3 \Gamma^{\text{Extended}}_{2,2} (00; 00) − \Gamma^{\text{Extended}}_{0,4} (0000; )$$  \hspace{1cm} (55)

- WTI $N = 3, M = 1$

$$0 = \Gamma^{\text{Extended}}_{4,0} (0000; ) − 3 \Gamma^{\text{Extended}}_{2,2} (00; 00)$$  \hspace{1cm} (56)

- The quartic coupling constant is defined in terms of a 4-point 1-SPI connected amputated GF

$$\Gamma^{\text{Extended}}_{0,4} (0000; ) \equiv −6 \lambda^2_{\phi}$$  \hspace{1cm} (57)
The all-loop-orders renormalized \( \phi \)-sector effective Lagrangian \([50]\), severely constrained only by the \( U(1)_Y \) WTI governing connected amputated Green's functions \([48]\), may be written
\[
L_{\text{Eff,Wigner,SI,Goldstone}}^{E-AHM:\phi} = L_{\text{Kinetic}}^{E-AHM:\phi} - L_{\text{Wigner,SI,Goldstone}}^{E-AHM:\phi} + O^{\text{Ignore}}_{E-AHM:\phi}
\]
\[
= \frac{1}{2} \left( \Gamma^{\text{Extended}}_{0,2}(; p, -p) - \Gamma^{\text{Extended}}_{0,2}(; 00) \right) h^2
+ \frac{1}{2} \left( \Gamma^{\text{Extended}}_{0,2}(; q, -q) - \Gamma^{\text{Extended}}_{0,2}(; 00) \right) \pi^2
\]
\[
V_{\text{Wigner,SI,Goldstone}}^{E-AHM:\phi} = m_\phi^2 \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle h \right]^2 + \lambda_\phi^2 \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle h \right]^2
\]
with finite non-trivial wavefunction renormalization
\[
\Gamma^{\text{Extended}}_{0,2}(; q, -q) - \Gamma^{\text{Extended}}_{0,2}(; 00) \sim q^2.
\]

The \( \phi \)-sector effective Lagrangian \([58]\) for the extended-AHM has in-sufficient boundary conditions to distinguish between the 3 modes of the BRST-invariant Lagrangian \( L_{E-AHM}^{\phi} \)\(^{12}\). The effective potential \( V_{\text{Wigner,SI,Goldstone}}^{E-AHM:\phi} \) in \([58]\) becomes in various limits:

\(^{12}\) It is instructive, and dangerous and famously worrisome, to ignore vacuum energy and re-write the potential in \([58]\) as:
\[
V_{\text{Eff,Wigner,SI,Goldstone}}^{E-AHM:\phi} = \lambda_\phi^2 \left[ \phi^4 - \frac{1}{2} \left( \langle H \rangle^2 - \frac{m_\phi^2}{3} \right) \right]^2
\]

A so-called BEH fine-tuning (FT) problem arises when one mistakenly minimizes \( V_{\text{Eff,Wigner,SI,Goldstone}}^{E-AHM:\phi} \) in \([58]\), while ignoring the crucial constraint by the Goldstone theorem (see Subsection II D: “... unless otherwise constrained by Ward identities”). We can further constrain the allowed terms in the \( \phi \)-sector effective extended-AHM Lagrangian with those \( U(1)_Y \) Ward-Takahashi identities which govern 1-\( \phi \)-R T-Matrix elements.

In Appendix B\(^{13}\) we derive 3 such identities governing 1-\( \phi \)-R connected amputated T-Matrix elements \( T_{N,M}^{\text{Extended}} \) in the \( \phi \)-sector of the extended-AHM.

Wigner mode is therefore quantum-loop unstable, because the heavy scale cannot decouple from the weak scale! Eqn. \([62]\) is the basis of the so-called “BEH fine-tuning problem”, and the motivation for much BSM physics. By GEN standards, weak-scale \( m_{\phi,\text{BEH}} \) in Wigner mode is indeed fine-tuned.

The Scale-Invariant point’s FT properties are beyond the scope of this paper.

**Spontaneously broken Goldstone mode**, where \( \langle H \rangle \neq 0 \): In obedience to the Goldstone theorem \([67, 15, 61]\) below, the bare counter-term \( \mu_{\phi,\text{Bare}}^2 \) in \([61]\) is defined by
\[
- \langle H \rangle \Gamma_{0,2}(; 00) = - \langle H \rangle T_{0,2}(; 00) = \langle H \rangle m_\phi^2 = 0
\]
We show below that, for constant \( \theta \), the zero-value in \([63]\) is protected by the NGB shift symmetry
\[
\hat{\pi} \rightarrow \hat{\pi} + \langle H \rangle \theta
\]
In addition, minimization of \([60]\) violates stationarity of the true minimum at \( \langle H \rangle \)\(^{10}\), and destroys the theory's renormalizability and unitarity, which require that dimensionless wavefunction renormalization \( \langle H \rangle_{\text{Bare}} = \left[ Z_{\phi}^{E-AHM} \right]^{1/2} \langle H \rangle \) contain no relevant operators \([12, 40, 41]\). The crucial observation is that, in obedience to the Goldstone theorem, \( \text{Renormalized}(\langle H \rangle_{\text{Bare}}^2) \neq \langle H \rangle_{\text{FT}}^2 \) in Lorenz gauge SSB extended-AHM.

If one persisted in regarding \( \mu_{\phi,\text{Bare}}^2 \) or \( \langle H \rangle^2 \) as FT, they would have to be FT to \( \sim 10^{24} \) to satisfy \([63]\), not just a mere \( \sim 10^{20} \). That of course is the nature of symmetry: e.g. \([64]\). SSB Goldstone mode extended-AHM is Goldstone Exceptionally Natural, not fine-tuned.
Adler self-consistency conditions (originally written for the global SU(2) \(L \times SU(2)_R\) Gell-Mann-Lévy model with PCAC\[^7\] \[^68\] constrain the extended-AHM gauge theory's effective \(\phi\)-sector Lagrangian in Lorenz gauge \[^B10\]:

\[
\langle H \rangle T_{N,M+1}^{\text{Extended}} (p_1\ldots p_N; 0q_1\ldots q_M) = 0 \tag{66}
\]

The extended-AHM T-matrix vanishes as one of the pion momenta goes to zero (i.e. 1-soft-pion theorems), provided all other physical scalar particles are on mass-shell. Eqn. (66) also "asserts the absence of infrared (IR) divergences in the \(\phi\)-sector extended-AHM) Goldstone mode (in Lorenz gauge). Although individual Feynman diagrams are IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the 1-soft-pion limit \[^B36\].

- The \(N = 0, M = 1\) case of (66) is the Goldstone theorem \[^B15\] itself \[^32\] \[^30\]:

\[
\langle H \rangle T_{0,2}^{\text{Extended}} (0) = 0 \tag{67}
\]

- Define \(T_{N,M+1}^{\text{Extended},\text{Internal}}\) as the 1-\(\phi\)-R \(\phi\)-sector T-Matrix with one soft \(\pi(q_a = 0)\) attached to an external-leg as in Figure 1. Now separate

\[
T_{N,M+1}^{\text{Extended},\text{External}} (p_1\ldots p_N; 0q_1\ldots q_M)
= T_{N,M+1}^{\text{Extended},\text{External}} (p_1\ldots p_N; 0q_1\ldots q_M)
+ T_{N,M+1}^{\text{Extended,Internal}} (p_1\ldots p_N; 0q_1\ldots q_M) \tag{68}
\]

Appendix \[^B\] \[^B17\] proves that

\[
\langle H \rangle T_{N,M+1}^{\text{Extended},\text{Internal}} (p_1\ldots p_N; 0q_1\ldots q_M) = 0
= \sum_{m=0}^{N} T_{N+1,M+1}^{\text{Extended}} (q_m p_1\ldots p_N; 0q_1\ldots q_m)
+ \sum_{n=1}^{M} T_{N+1,M+1}^{\text{Extended}} (p_1\ldots p_N; q_n q_1\ldots q_m) \tag{69}
\]

The \(U(1)_y\) WITs \[^48\] \[^B18\] governing 1-\(\phi\)-1 connected amputated Greens functions \(T_{N,M}^{\text{Extended}}\) are solutions to \(^{69\text{B17}}\).

We re-write the Extended-AHM effective \(\phi\)-sector Lagrangian \[^B58\] but now include the constraint from the Goldstone theorem \[^67\] \[^B15\]:

\[
\begin{align*}
L_{E-AHM;\phi}^{\text{Eff;Goldstone}} &= L_{\text{Kinetic;E-AHM};\phi}^{\text{Eff;Goldstone}} + O_{E-AHM;\phi}^{\text{Eff;Goldstone}} \\
V_{E-AHM;\phi}^{\text{Eff;Goldstone}} &= \lambda_{\phi}^{\text{Eff;Goldstone}} \left[ h^2 + \pi^2 + \langle H \rangle \hbar \right]^2 \tag{70}
\end{align*}
\]

and wavefunction renormalization

\[
\Gamma_0,2^{\text{Extended}} (q, -q) - \Gamma_0,2^{\text{Extended}} (00) = q^2 + O_{\text{Eff;Goldstone}}^{\text{ignore;}} \tag{71}
\]

A crucial effect of the Goldstone theorem, together with the \(N = 0, M = 1\) Ward-Takahashi Greens function identity \[^48\] is to automatically eliminate tadpoles in \[^70\]:

\[
\Gamma_0,0^{\text{Extended}} (00) = \langle H \rangle \Gamma_0,2^{\text{Extended}} (00) = 0 \tag{72}
\]

so that separate tadpole renormalization is unnecessary.

We form the effective Goldstone mode \(\phi\)-sector Lagrangian in coordinate space \[^B23\]:

\[
\begin{align*}
L_{E-AHM;\phi}^{\text{Eff;Goldstone}} &= |\partial_\phi \phi|^2 - V_{E-AHM;\phi}^{\text{Eff;Goldstone}} + O_{\text{ignore;E-AHM;\phi}}^{\text{Eff;Goldstone}} \\
V_{E-AHM;\phi}^{\text{Eff;Goldstone}} &= \lambda_{\phi}^{\text{Eff;Goldstone}} \left[ h^2 + \pi^2 + \langle H \rangle \hbar \right]^2 \tag{73}
\end{align*}
\]

Eqn. (73) is the \(\phi\)-sector effective Lagrangian of the spontaneously broken extended-AHM in Lorenz gauge:

- It obeys the Goldstone theorem \[^67\] \[^B15\] and all other \(U(1)_y\) WTI \[^48\] \[^60\] \[^61\] \[^69\] \[^B10\] \[^B15\] \[^B17\] \[^B18\];

- It is minimized at \((H = \langle H \rangle, \pi = 0)\), and obeys stationarity \[^40\] of that true minimum:

\[^13\] It is not lost on the authors that, since we derived it from connected amputated Greens functions (where all vacuum energy and disconnected vacuum bubbles are absorbed into an overall phase, which cancels exactly in the S-matrix \[^40\] \[^41\]), the vacuum energy in \(V_{E-AHM;\phi}^{\text{Eff;Goldstone}}\) in (73) is exactly zero.
• It preserves the theory’s renormalizability and unitarity, which require that wavefunction renormalization, \( \langle H \rangle_{\text{Bare}} = \left[ Z_{E_{-AHM}}^\phi \right]^{1/2} \langle H \rangle \) [17 40 41], forbid any relevant operator corrections to \( \langle H \rangle \).

• It includes all divergent \( \mathcal{O}(\Lambda^2), \mathcal{O}(\ln \Lambda^2) \) and finite terms which arise, to all perturbative loop-orders in the full \( U(1)_Y \) theory, due to virtual transverse gauge bosons, AHM scalars, ghosts, new scalars, and new fermions \( A^\mu, h, \pi; \bar{\omega}, \omega; \Phi; \psi \); respectively.

• The Goldstone theorem [67 B15] has caused all relevant operators in [73] to vanish!

iii) Decoupling of heavy matter representations:
We take all of the new scalars \( \Phi \) and fermions \( \psi \) to be very heavy. For Beyond-AHM scalar(s)
\[
L_{\text{Beyond-AHM}}: \Phi = \left( \partial_\mu \Phi \right)^2 - V_\Phi - V_{\Phi\Phi}
\]
\[
V_\Phi = M_\Phi^2 \Phi^\dagger \Phi + \lambda_\Phi^2 \left( \Phi^\dagger \Phi \right)^2
\]
\[
V_{\Phi\Phi} = \lambda_{\Phi\Phi}^2 \left( \Phi^\dagger \Phi \right) \left( \Phi^\dagger \Phi \right)
\]
(74)
we take
\[
M_\Phi^2 \sim M_{\text{Heavy}}^2
\]
\[
\gg \left( |q^2|, m_A^2, m_{\text{BEH}}^2 \right) \sim m_{\text{Weak}}^2 \sim (100 \text{GeV})^2
\]
(75)
with \( q_\Phi \) typical for a studied low-energy process. For pedagogical simplicity, we have chosen a single \( \Phi \) representation with \( U(1)_Y \) hypercharge \( Y_\Phi = Y_\psi = -1 \), but the analysis is easily extended to other and multiple \( U(1)_Y \) Beyond-AHM representations.

For Beyond-AHM fermion(s)
\[
L_{\text{Beyond-AHM}}^\text{GlobalInvariant:} \Psi = i \bar{\Psi}_L \gamma^\mu \partial_\mu \Psi_L + i \bar{\Psi}_R \gamma^\mu \partial_\mu \Psi_R
\]
\[
+ L_{\text{Yukawa}}^\text{Beyond-AHM:} \Psi_L + L_{\text{Majorana}}^\text{Beyond-AHM:} \Psi_R
\]
(76)
\[
\left[ L_{\text{Yukawa}}^\text{Beyond-AHM:} \Psi_L = - \left( y_\psi \bar{\Psi}_L \Phi \Psi_L + y_\psi \bar{\Psi}_L \bar{\Phi} \Psi_L \right)
\right.
\]
\[
- \left( y_\psi \bar{\Psi}_R \Phi \Psi_R + y_\psi \bar{\Psi}_R \bar{\Phi} \Psi_R \right)
\]
\[
\left. - \left( y_\psi \bar{\Psi}_L \Phi \Psi_L + y_\psi \bar{\Psi}_L \bar{\Phi} \Psi_L \right) \right)
\]
\[
L_{\text{Majorana}}^\text{Beyond-AHM:} \Psi_R = - \frac{1}{2} M_{\psi L} \left( \bar{\psi}_{Weyl L} \psi_{Weyl L} + \bar{\psi}_{Weyl L} \psi_{Weyl L} \right)
\]
\[
- \frac{1}{2} M_{\psi R} \left( \bar{\psi}_{Weyl R} \psi_{Weyl R} + \bar{\psi}_{Weyl R} \psi_{Weyl R} \right)
\]
with fermion \( U(1)_Y \) hypercharges chosen so that the axial anomaly is zero. To remain perturbative, we keep the Yukawa couplings \( y_\psi \), \( y_\psi \bar{\Psi}_L \sim 1 \), but take the Majorana masses-squared
\[
M_{\psi L}^2, M_{\psi R}^2 \sim M_{\text{Heavy}}^2
\]
\[
\gg \left( |q^2|, m_A^2, m_{\text{BEH}}^2 \right) \sim m_{\text{Weak}}^2
\]
(77)
We keep all Yukawas and masses real for pedagogical simplicity.

Some comments are in order:

• We have ignored finite \( \mathcal{O}(\Lambda^2)_{\text{Irrelevant}} \) which decouple and vanish as \( m_{\text{Weak}}^2 / M_{\text{Heavy}}^2 \to 0 \).

• Among the terms included in (75) are finite relevant operators dependent on the heavy matter representations:
\[
M_{\text{Heavy}}^2, M_{\text{Heavy}}^2 \ln \left( M_{\text{Heavy}}^2 \right),
\]
\[
M_{\text{Heavy}}^2 \ln \left( m_{\text{Weak}}^2 \right), m_{\text{Weak}}^2 \ln \left( M_{\text{Heavy}}^2 \right)
\]
(78)
but they have vanished because of the Goldstone theorem [67 B15]. That fact is one of the central results of this paper.

• Marginal operators \( \sim \ln \left( M_{\text{Heavy}}^2 \right) \) have been absorbed in (75); i.e. in the renormalization of gauge-independent observables (i.e. the quartic-coupling constant \( \lambda_\phi^2 \) and the BEH VEV \( \langle H \rangle \)), and in unobservable wavefunction renormalization (71).

No trace of \( M_{\text{Heavy}}^2 \)-scale \( \Phi, \psi \) survives in (73)! All the heavy Beyond-AHM matter representations have completely decoupled.

iv) 1st decoupling theorem: SSB \( \phi \)-sector connected amputated 1-\( \phi \)-R T-Matrices.
\[
T_{N,M}^{\text{Extended}} \xrightarrow{m_{\text{Weak}}^2 / M_{\text{Heavy}}^2 \to 0} T_{N,M}
\]
(79)
become equal in the limit \( m_{\text{Weak}}^2 / M_{\text{Heavy}}^2 \to 0 \).

v) 2nd decoupling theorem: SSB \( \phi \)-sector connected amputated 1-\( \phi \)-I Green’s functions.
\[
\Gamma_{N,M}^{\text{Extended}} \xrightarrow{m_{\text{Weak}}^2 / M_{\text{Heavy}}^2 \to 0} \Gamma_{N,M}
\]
(80)
become equal in the limit \( m_{\text{Weak}}^2 / M_{\text{Heavy}}^2 \to 0 \).

vi) 3rd decoupling theorem: SSB \( \phi \)-sector BEH pole-mass-squared. The \( N = 1, M = 1 \) connected amputated Green’s function \( U(1)_Y \) WTI [48], augmented by the Goldstone theorem [67] reads
\[
\Gamma_{2,0}^{\text{Extended}}(0; 0) = \langle H \rangle \Gamma_{2,0}^{\text{Extended}}(0; 0)
\]
\[
= -2 \lambda_\phi^2 \langle H \rangle^2
\]
(81)
shows that the BEH pole-mass-squared arises entirely from SSB. Define
\[
\Delta_{E-AHM}^{BEH}(q^2) = \frac{1}{q^2 - m_{BEH;\text{Pole}}^2 + i\epsilon}
\]
\[
+ \int dm^2 \rho_{E-AHM}^{BEH}(m^2)
\]
\[
q^2 - m^2 + i\epsilon
\]
(82)
m^2_{BEH;\text{Pole}} is the gauge-independent observable BEH resonance pole-mass-squared. We now show that it is

14 We take \( \mathcal{O}(\Lambda^2)_{\text{Irrelevant}} \to 0 \) so to unencumber our notation.
not-FT. Since, in analogy with (10), the spectral density \( \rho_{BEH}^{EAHM}(m^2) \) is not FT

\[
\rho_{BEH}^{E-AHM}(m^2) = \rho_{AHM}(m^2) + \mathcal{O}(1/M_{AHM}^2)\text{Irrelevant}
\]

\[
\Gamma_{2,0}^{\text{Extended}(00)} \equiv \left[ B_{E-AHM}^{\text{BEH}}(0) \right]^{-1}
= -2\lambda^2_\phi (H)^2
= -m_{BEH:Pole}^2 \left[ 1 + m_{BEH:Pole}^2 \int dm^2 \rho_{AHM}(m^2) \right]^{-1}
+ \mathcal{O}(1/M_{AHM}^2)\text{Irrelevant}
\]

We have

\[
m_{BEH:Pole}^2 = 2\lambda^2_\phi (H)^2 \left[ 1 - 2\lambda^2_\phi (H)^2 \right] \int dm^2 \rho_{AHM}(m^2) \right]^{-1}
+ \mathcal{O}(1/M_{AHM}^2)\text{Irrelevant}
\]

Because \( \lambda^2_\phi, Z^2_{\text{ExtendedAHM}} \) are dimensionless, \( \lambda^2_\phi \) and

\[
\langle H \rangle = \left[ Z^\phi_{\text{ExtendedAHM}} \right]^{-1/2} \langle H \rangle_{\text{Bare}}
\]

absorbs no relevant operators and are therefore not FT, Eqn. (84) shows that the BEH pole mass-squared \( m_{BEH:Pole}^2 \) becomes equal in the limit \( m_{Weak}^2/M_{Heavy}^2 \rightarrow 0 \). We call the “SSB BEH-Mass Decoupling Theorem”. By dimensional analysis, heavy \( \Phi, \psi \) decouple from the \( \pi \) spectral functions

\[
\Delta_{E-AHM}^{\pi: \text{Spectral}} (q^2) = \Delta_{AHM}^{\pi: \text{Spectral}} (q^2) + \mathcal{O}(1/M_{AHM}^2)
\]

The SSB extended-AHM \( \phi \)-sector is therefore Goldstone Exceptionally Natural, with far more suppression of fine-tuning than G. ’t Hooft’s no-FT criteria would demand.

**B. Decoupling of gauge singlet** \( M_2^2 \gg m_{Weak}^2 \) **real scalar field** \( S \) with discrete \( Z_2 \) symmetry and \( \langle S \rangle = 0 \)

For the heavy scalar we consider a \( U(1)_Y \) gauge singlet real scalar \( S \), with \( (S \rightarrow -S) Z_2 \) symmetry, \( M_2^2 \gg m_{Weak}^2 \), and \( \langle S \rangle = 0 \). We add to the renormalized theory

\[
L_{S} = \frac{1}{2} (\partial^\mu S)^2 - V_{\phi S}
\]

\[
V_{\phi S} = \frac{1}{2} M_2^2 S^2 + \frac{\lambda^2_\phi}{4} S^4 + \frac{1}{2} \lambda^2_{\phi S} S^2 \left[ \phi^\dagger \phi - \frac{1}{2} (H)^2 \right]
\]

\[
\phi^\dagger \phi - \frac{1}{2} (H)^2 = \frac{\mu^2 + \eta^2}{2} + \langle H \rangle h
\]

Since \( S \) is a gauge singlet, it is also a rigid/global singlet. Its \( U(1)_Y \) hypercharge, transformation and current

\[
Y_S = 0; \quad \delta S(t, y) = 0
\]

\[
m_{BeyondAHM}^2 = 0
\]

therefore satisfy all of the decoupling criteria in Appendix [B]

- Since it is massive, \( S \) cannot carry information to the surface \( \mathbb{R}^3 \)-surface \( \rightarrow \infty \) of the (all-space-time) 4-volume \( \int d^4 z \), and so satisfies \( [B8] \):

- The equal-time commutators satisfy \( [B6] \)

\[
\delta(z_0 - y_0) \left[ J^{\mu S}_{BeyondAHM} (z), H(y) \right] = 0
\]

\[
\delta(z_0 - y_0) \left[ J^{0 S}_{BeyondAHM} (z), \pi(y) \right] = 0
\]

- The classical equation of motion

\[
\partial_\mu \left( J^{\mu S}_{BeyondAHM} + J^{\mu AHM} \right) = \frac{\partial J^{\mu S}}{\partial \phi} \frac{\partial A^\beta}{\partial \phi}
\]

restores conservation of the rigid/global \( U(1)_Y \) extended current for \( \phi \)-sector physical states, and satisfies \( [B5] \)

\[
\langle 0 | T \left[ \partial_\mu \left( J^{\mu S}_{BeyondAHM} + J^{\mu AHM} \right) \right] (z) \times h(x_1) ... h(x_N) \pi(y_1) ... \pi(y_M) \rangle | 0 \rangle_{\text{Connected}} \rightarrow 0
\]

- The zero VEV \( \langle S \rangle = 0 \) satisfies \( [B7] \)

The \( U(1)_Y \) WTI governing the extended \( \phi \)-sector transition matrix \( T_{N,M}^{\text{Extended}} \) are therefore true, namely: the extended Adler self-consistency conditions \([66][B10]\), together with their proof of infra-red finiteness in the presence of massless NGB; the extended Goldstone theorem \([67][B15]\); the extended 1-soft-\( \pi \) theorems \([69][B17]\): The extended \( U(1)_Y \) WTI \([48][B18]\) governing connected amputated \( \phi \)-sector Green’s functions \( \Gamma_{N,M}^{\text{Extended}} \) are also true. The 3 decoupling theorems \([79][80][88]\) therefore follow, so that no trace of \( M_2^2 \sim M_{Heavy}^2 \) scalar \( S \) survives the \( m_{Weak}^2/M_{Heavy}^2 \) limit: i.e. it has completely decoupled! The \( \phi \)-sector connected amputated T-Matrices and Green’s functions \( \Gamma_{N,M}^{\text{Extended}} \) become equal in the limit \( m_{Weak}^2/M_{Heavy}^2 \rightarrow 0 \). \( [15] \)

\footnote{\( m_{BEH:Pole}^2 \) decoupling is in exact disagreement with \([28][30]\).}
C. The lightest generation of Standard Model quarks and leptons, augmented by a right-handed Dirac mass $m^\mu_{\text{Dirac}}$; Gauged hypercharge and global colors

These 16 spin $S = \frac{1}{2}$ fermions: $u^c_L; d^c_L; u^c_R; d^c_R; e^c_L, \nu_L; e^c_R; \nu_R$; with global $SU(3)$ colors $c =$ red, white, blue, and gauged $U(1)_Y$ hypercharge, are regarded here as extended-AHM matter representations. Baryon and lepton-number conserving Dirac masses-squared arise entirely from SSB and are very light: $m_{\text{Dirac}}^2, m_{\text{Dirac}}^2 \ll m_{\text{Weak}}^2$. The so-extended $U(1)_Y$ AHM gauge theory has zero axial-anomaly because quark/lepton AHM quantum numbers are chosen to be their SM hypercharges (including $Y_{BR} = 0$).

i) Beyond-AHM Dirac quarks:

\[
L_{\text{Global Invariant Beyond AHM}} = L_{\text{Kinetic Beyond AHM}} + L_{\text{Yukawa Beyond AHM}} \tag{94}
\]

\[
L_{\text{Kinetic Beyond AHM}} = \sum_{r,w,b} \sum_{u,d} \left( \bar{q}^c_L \gamma^\mu \partial_\mu q^c_R + \bar{q}^c_R \gamma^\mu \partial_\mu q^c_R \right)
\]

\[
L_{\text{Yukawa Beyond AHM}} = -\sum_{r,w,b} \sum_{u,d} y_q \left( \bar{q}^c_L \phi q^c_R + \bar{q}^c_R \phi^\dagger q^c_L \right)
\]

The $U(1)_Y$ quark current and transformation properties are

\[
J^\mu_{\text{Dirac Beyond AHM};q} = -\sum_{r,w,b} \sum_{u,d} \left( \bar{q}^c_L \gamma^\mu \partial_\mu q^c_R + \bar{q}^c_R \gamma^\mu \partial_\mu q^c_R \right) \times \left( Y_{qL} \bar{q}^c_L \gamma^\mu q^c_R + Y_{qR} \bar{q}^c_R \gamma^\mu q^c_R \right)
\]

\[
\delta q^c_L(t,\vec{x}) = -i Y_{qL} \bar{q}^c_L(t,\vec{x}) \theta
\]

\[
\delta q^c_R(t,\vec{x}) = -i Y_{qR} \bar{q}^c_R(t,\vec{x}) \theta
\]

\[
Y_{UL} = \frac{1}{3}; Y_{DL} = \frac{1}{3}; Y_{UR} = \frac{4}{3}; Y_{DR} = -\frac{2}{3} \tag{95}
\]

ii) Beyond-AHM Dirac leptons:

\[
L_{\text{Global Invariant Beyond AHM;}} = L_{\text{Kinetic Beyond AHM;}} + L_{\text{Yukawa Beyond AHM;}} \tag{96}
\]

\[
L_{\text{Kinetic Beyond AHM;}} = \sum_{\nu,e} \left( \bar{\nu}_L \gamma^\mu \partial_\mu \nu_R + \bar{\nu}_R \gamma^\mu \partial_\mu \nu_R \right)
\]

\[
L_{\text{Yukawa Beyond AHM;}} = -\sum_{\nu,e} y_{\nu} \left( \bar{\nu}_L \phi \nu_R + \bar{\nu}_R \phi^\dagger \nu_L \right)
\]

The lepton $U(1)_Y$ current and transformation properties are

\[
J^\mu_{\text{Dirac Beyond AHM;}} = -\sum_{\nu,e} \left( \bar{\nu}_L \bar{\nu}_L \gamma^\mu \nu_R + \bar{\nu}_R \bar{\nu}_R \gamma^\mu \nu_R \right)
\]

\[
\delta_{L}(t,\vec{x}) = -i \bar{\nu}_L \nu_L(t,\vec{x}) \theta
\]

\[
\delta_{R}(t,\vec{x}) = -i \bar{\nu}_R \nu_R(t,\vec{x}) \theta
\]

\[
Y_{UL} = -1; Y_{EL} = -1; Y_{UR} = 0; Y_{ER} = -2 \tag{97}
\]

With these Standard Model quark and lepton hypercharges $Y$, our $U(1)_Y$ WTI have zero axial anomaly.

We now prove applicability of our $U(1)_Y$ WTI: i.e. for connected amputated $\phi$-sector Greens functions $\Gamma^{\text{Extended}}_{N,M}$ and T-Matrix elements $T^{\text{Extended}}_{N,M}$,

- The equal-time quantum commutators satisfy $[B6]$

\[
\delta (z_0, y_0) \left[ J^\mu_{\text{Dirac Beyond AHM};q}(y), H(y) \right] = 0
\]

\[
\delta (z_0, y_0) \left[ J^\mu_{\text{Dirac Beyond AHM};q}(y), \pi(y) \right] = 0
\]

\[
\delta (z_0, y_0) \left[ J^\mu_{\text{Dirac Beyond AHM;}}(y), H(y) \right] = 0
\]

\[
\delta (z_0, y_0) \left[ J^\mu_{\text{Dirac Beyond AHM;}}(y), \pi(y) \right] = 0 \tag{98}
\]

- The classical equation of motion

\[
\partial_{\mu} \left( J^\mu_{\text{Dirac Beyond AHM;}} + J^\mu_{\text{Dirac Beyond AHM;}} + J^\mu_{\text{AHM}} \right) = -\epsilon (H) \partial_{\beta} A^\beta \tag{99}
\]

restores conservation of the rigid/global $U(1)_Y$ extended current for $\phi$-sector physical states, and satisfies $[B5]$

\[
\langle 0 | T \left( J^\mu_{\text{Dirac Beyond AHM;}} + J^\mu_{\text{Dirac Beyond AHM;}} + J^\mu_{\text{AHM}} \right)(z) \times h(x_1)...h(x_N)\pi_{L}(y_1)...\pi_{L}(y_M) | 0 \rangle_{\text{Connected}} = 0 \tag{100}
\]

- Dirac quark surface terms: Since they are here taken massive $m_u = \frac{1}{\sqrt{2}} y_u \langle H \rangle$ and $m_d = \frac{1}{\sqrt{2}} y_d \langle H \rangle$ (i.e. in deference to the SM with its massive strongly-interacting hadronic pion), and we need only connected graphs, the light quarks $u,d$ cannot carry information to the surface $z^3$-surface $\to \infty$ of the (all-space-time) 4-volume $\int d^4 z$, and so satisfy $[B8]$.

In contrast, massless quarks could carry (on the light-cone) $U(1)_Y$ information to the $z^3$-surface $\to \infty$, and would violate $[B8]$ and so destroy the spirit, results and essence of our $U(1)_Y$-WTI-based no-FT and heavy particle decoupling results here in Section [11]. Still, from the harm they do our global $U(1)_Y$ WTI, we worry that the infrared structure of massless quarks might also harm $U(1)_Y$ local Slavnov-Taylor identities. $[16]$

---

16 It is amusing to elevate $U(1)_Y$ WTI s to a “Principle of Nature”, so as to give them predictive power. Imagine we impose, on quark-extended AHM, an extra gauge/local $SU(3)_{\text{Color}}$, an extra rigid/global $SU(2)_L$ on left-handed $(u_L, d_L)$ quarks, and an extra rigid/global $SU(2)_R$ on right-handed $(u_R, d_R)$ quarks. The structure group would be

\[
U(1)_Y \otimes SU(3)_{\text{Color_Local}} \otimes SU(2)_L_{\text{Global}} \otimes SU(2)_R_{\text{Global}} \tag{101}
\]
• Charged Dirac lepton surface terms: Since it is massive \( m_e = \frac{1}{\sqrt{2}} y_e \langle H \rangle \), and we need only connected graphs, the electron \( e \) cannot carry information to the surface \( z^3_{\text{surface}} \to \infty \) of the (all-space-time) 4-volume \( \int d^4 z \), and so satisfies \( (B5) \).

• Dirac neutrino surface terms: Since it is here taken massive \( m_\nu^{\text{Dirac}} = \frac{1}{\sqrt{2}} y_\nu \langle H \rangle \) (i.e. in deference to observed SM neutrino mixing), and we need only connected graphs, the light neutrino \( \nu_L \) cannot carry information to the surface \( z^3_{\text{surface}} \to \infty \) of the (all-space-time) 4-volume \( \int d^4 z \), and so satisfies \( (B5) \). In contrast, a massless Goldstone could carry (on the light-cone) \( U(1)_Y \) information to the surface \( z^3_{\text{surface}} \to \infty \), and would violate \( (B5) \), and so destroy the spirit, results and essence of our \( U(1)_Y \)-WTI-based no-FT and heavy particle decoupling results here in Section \( \text{III} \). Still, from the harm it does our global \( U(1)_Y \) WTI, we worry that the infra-red structure of a massless neutrino might also harm \( U(1)_Y \) local Slavnov-Taylor identities.

Having satisfied all of the criteria in Appendix \( \text{B} \) the \( U(1)_Y \) WTI governing the extended connected amputated \( \phi \)-sector \( T_{N,M}^{\text{Extendedq}} \) are therefore true, namely: the extended Adler self-consistency conditions \( (66\text{B10}) \), together with their proof of infra-red finiteness in the presence of massless NGB; the extended Goldstone theorem \( (67\text{B15}) \); the extended 1-soft-\( \pi \) theorems \( (69\text{B17}) \): The extended \( U(1)_Y \) WTI \( (48\text{B18}) \) governing connected amputated \( \phi \)-sector Green's functions \( T_{N,M}^{\text{Extendedq}} \) are also true.

The result would add to the quark-extended AHM a Chiral Perturbation Theory \( (\chi PT) \) \( (17) \) with massive neutrinos and protons, and 3 massless hadronic pions: there is no QED electric charge in this toy model. Now impose our SSB \( U(1)_Y \) WTI, which require and demand non-zero quark masses \( m_u = m_d \neq 0 \). The result is \( \chi PT \), but now broken (in just the right way, after further breaking \( m_u \neq m_d \)) to give strong-interaction hadronic pions their masses. Would we then claim that spontaneously broken \( U(1)_Y \) WTI predict the breaking of hadronic SU(2)\( _L \times SU(2)_R \) \( \chi PT \)?

Imagine we are able to extend this work to the Standard Model itself \( (21\text{B2}) \). With its local/gauge group SU(3)\( _\text{Color} \times SU(2)_L \times U(1)_Y \), we would build 3 sets of rigid/global WTI: unbroken SU(3)\( _\text{Color} \): unbroken electromagnetic \( U(1)\_\text{QED} \); and spontaneously broken SU(2)\( _L \). It is then amusing to elevate such rigid/global WTI to a “Principle of Nature”, so as to give them predictive power for actual experiments and observations. The SU(3)\( _\text{Color} \) and \( U(1)\_\text{QED} \) WTI are \textbf{unbroken vector-current ID}s, and will not yield information analogous with that of SSB extended-AHM here. But the axial-vector current inside the SSB SU(2)\( _L \) WTI will require and demand a non-zero SSB Dirac mass for each and every one of the weak-interaction eigenstates \( m_\nu^{\text{Dirac}}, m_e^{\text{Dirac}}, m_\pi^{\text{Dirac}} \neq 0 \). The observable PNMS mixing matrix would then rotate those mass-eigenstates \( m_\nu^{\text{Dirac}}, m_e^{\text{Dirac}}, m_\pi^{\text{Dirac}} \neq 0 \). Would we then claim that spontaneously broken SU(2)\( _L \) WTI predict neutrino oscillations? To make possible connection with Nature, although current experimental neutrino mixing data cannot rule out an exactly-zero mass for the lightest neutrino \( (17) \), the mathematical self-consistency of SU(2)\( _L \) WTI would!

\[ \text{D. } \nu\text{AHM: (in practice) decoupling of gauge singlet right-handed Type I See-saw Majorana neutrino } \nu_R \text{ with } M_{\nu_R}^2 \gg m_{\text{BEH}}^2 \]

For the heavy fermion we consider a \( U(1)_Y \) gauge-singlet right-handed Majorana neutrino \( \nu_R \), with \( M_{\nu_R}^2 \gg m_{\nu_R}^{\text{Dirac}} \) involved in a Type \( 1 \) See-Saw with a left-handed neutrino \( \nu_L \). Yukawa coupling \( y_\nu \) and resulting Dirac mass \( m_\nu^{\text{Dirac}} = y_\nu \langle H \rangle / \sqrt{2} \).

We add to the renormalized theory in Subsection \( \text{III}\text{C} \) a Majorana mass

\[ L_{\text{Majorana}} = -\frac{1}{2} M_{\nu_R} \left( \bar{\nu}_R^{\text{Weyl}} \gamma_5 \nu_R^{\text{Weyl}} + \bar{\nu}_R^{\text{Weyl}} \gamma_5 \nu_R^{\text{Weyl}} \right) \]

Since \( \nu_R \) is a gauge singlet, it is also a rigid/global singlet. Its hypercharge, \( U(1)_Y \) transformation and current

\[ Y_{\nu_R} = 0; \quad \partial \nu_{\text{Majorana}} \gamma_\mu = 0 \]

therefore satisfy all of the de-coupling criteria in Appendix \( \text{B} \)

• Since it has a Dirac mass, \( \nu_L \) cannot carry information to the surface \( z^3_{\text{surface}} \to \infty \) of the (all-space-time) 4-volume \( \int d^4 z \), and so satisfies \( (B5) \).

• The equal-time quantum commutators satisfy \( (B6) \)

\[ \delta(z_0 - y_0) \left( J_{\nu_R, \text{Majorana}}^{\text{BeyondAHM};\nu_R}(z), H(y) \right) = 0 \]

\[ \delta(z_0 - y_0) \left( J_{\nu_R, \text{Majorana}}^{\text{BeyondAHM};\nu_R}(z), \pi(y) \right) = 0 \]

• The classical equation of motion

\[ \partial_\mu \left( J_{\nu_R, \text{Majorana}}^{\text{BeyondAHM};\nu_R} + J_{\nu_R, \text{Dirac}}^{\text{BeyondAHM};\mu} \right) + J_{\nu_R, \text{Dirac}}^{\text{BeyondAHM};\mu} + J_{\nu_R, \text{Dirac}}^{\text{BeyondAHM};\mu} \right) \]

\[ = -e \delta(H) \partial_\beta A^\beta \]

restores conservation of the rigid/global \( U(1)_Y \) extended current for \( \phi \)-sector physical states, and satisfies \( (B5) \)

\[ \langle 0| T \left[ \partial_\mu \left( J_{\nu_R, \text{Majorana}}^{\text{BeyondAHM};\nu_R} + J_{\nu_R, \text{Dirac}}^{\text{BeyondAHM};\mu} \right) + J_{\nu_R, \text{Dirac}}^{\text{BeyondAHM};\mu} \right) \]

\[ \times h(x_1) \ldots h(x_N) \pi_{i_1}(y_1) \ldots \pi_{i_M}(y_M) \left| 0 \right\rangle_{\text{Connected}} = 0 \]

Having satisfied all of the criteria in Appendix \( \text{B} \) the \( U(1)_Y \) WTI governing the extended connected amputated \( \phi \)-sector \( T_{N,M}^{\text{Extendedq}} \) are therefore true, namely:
the extended Adler self-consistency conditions \cite{66, 10}, together with their proof of infra-red finiteness in the presence of massless NGB; the extended Goldstone theorem \cite{67, 15}; the extended 1-soft-\(\pi\) theorems \cite{69, 17}. The extended \(U(1)\)_Y WTI \cite{48, 18}, governing connected amputated \(\phi\)-sector Green’s functions \(T^{Extended}_{N,M}\) are also true.

The 3 decoupling theorems \cite{79, 80, 86} therefore follow, but there is a “non-decoupling subtlety.” Recall that the vanishing of the \(\nu\) surface terms requires a non-zero neutrino Dirac mass

\[
m^{Dirac}_\nu = \frac{1}{\sqrt{2}} y_{\nu}(H) \neq 0 \tag{107}\]

The light and heavy Type I See-saw \(\nu\) masses are

\[
m_{\text{Light}} \sim m^{2;Dirac}_\nu / M_{\nu R} \]

\[
m_{\text{Heavy}} \sim M_{\nu R} \tag{108}\]

But, in obedience to our proof of \(U(1)\)_Y WTI, \(m_{\text{Light}}\) must not vanish. Therefore Type I See-saw \(\nu\)s do not allow the \(M_{\nu R} \rightarrow \infty\) limit! For the decoupling theorems, we instead imagine huge, but finite \(M_{\nu R}\), with

\[
1 \gg m^{2;Dirac}_\nu / M_{\nu R} \neq 0 \tag{109}\]

No practical trace of the \(M_{\nu R}^2 \sim M_{\text{Heavy}}^2\) right-handed neutrino \(\nu_R\) survives \(1 \gg m^{2;\text{Weak}} / M_{\text{Heavy}}^2 \neq 0\): i.e. it has become practically invisible! The \(\phi\)-sector connected amputated T-Matrices and Green’s functions, and the BEH pole masses-squared

\[
T^{Extended; u,d; \nu_L, \nu_R}_{N,M} \stackrel{1 \gg m^{2;\text{Weak}} / M_{\nu R}^2 \neq 0}{\longrightarrow} T^{Extended; u,d; \nu_L, \nu_R; \text{Invisible}}_{N,M} \tag{110}\]

\[
\Gamma^{Extended; u,d; \nu_L, \nu_R}_{N,M} \stackrel{1 \gg m^{2;\text{Weak}} / M_{\nu R}^2 \neq 0}{\longrightarrow} \Gamma^{Extended; u,d; \nu_L, \nu_R; \text{Invisible}}_{N,M} \tag{111}\]

\[
m^{2;\text{BEH; Pole:} \phi}_{\nu_R} \stackrel{1 \gg m^{2;\text{Weak}} / M_{\nu R}^2 \neq 0}{\longrightarrow} m^{2;\text{BEH; Pole:} \phi}_{\nu_R; \text{Invisible}} \tag{112}\]

become, to high approximation, equal in practice, for \(1 \gg m^{2;\text{Weak}} / M_{\text{Heavy}}^2 \neq 0\).\footnote{\(m^{2;\text{AHM}}_{\text{BEH; Pole}}\) decoupling is in exact disagreement with \cite{28, 30}.}

Still, our \(U(1)\)_Y WTI’s insist that in principle a very heavy Majorana mass \(M_{\nu R}\) cannot completely decouple, and may still have some measureable or observational effect.

IV. SSB EXTENDED-AHM’S PHYSICAL PARTICLE SPECTRUM EXCLUDES THE NGB \(\tilde{\pi}\): DECOUPLING OF HEAVY PARTICLES;

G.S. Guralnik, C.R. Hagan and T.W.B. Kibble \cite{31} first showed in the spontaneously broken Abelian Higgs model that, although there are no massless particles in the \(\mathcal{A}^0 = 0, \bar{\mathcal{A}} = 0\) “radiation gauge”, there is a Goldstone theorem, and a true massless NGB, in the covariant \(\partial_{\mu} A^\mu = 0\) Lorenz gauge. T.W.B. Kibble then showed \cite{52} that the results of experimental measurements are nevertheless the same in radiation and Lorenz gauges, and that the spectrum and dynamics of the observable particle states are gauge-independent.

A. SSB extended-AHM’s physical particle spectrum excludes the NGB \(\tilde{\pi}\) whose S-Matrix elements all vanish \cite{32}

We repeat here (and include further detail) the discussion from the companion Letter \cite{32} in order that this paper be pedagogically self-contained, clear and complete.

The BRST-invariant Lagrangian for the extended-AHM in Lorenz gauge is

\[
\mathcal{L}^{\text{Lorenz}_L}_{\text{Extended-AHM}} = \mathcal{L}^{\text{Lorenz}}_{\text{AHM}} + \mathcal{L}^{\text{GaugeInvariant}}_{\text{Extended-AHM}; \phi} + \mathcal{L}^{\text{GaugeInvariant}}_{\text{Extended-AHM}; \psi} \tag{112}\]

with \(\mathcal{L}^{\text{Lorenz}}_{\text{AHM}}\) in \(\cite{2}\), \(\mathcal{L}^{\text{GaugeInvariant}}_{\text{Extended-AHM}; \phi}\) in \(\cite{74}\), and \(\mathcal{L}^{\text{GaugeInvariant}}_{\text{Extended-AHM}; \psi}\) in \(\cite{76}\).

i) Lagrangian governing dynamics of observable particles: We now identify the observable particle spectrum of Lorenz gauge extended-AHM by re-writing \(\mathcal{L}^{\text{Lorenz}}_{\text{Extended-AHM}}\) in terms of a new gauge field

\[
B_\mu \equiv A_\mu + \frac{1}{e(H)} \partial_\mu \tilde{\pi} \tag{113}\]

and transforming to the Kibble representation \cite{31}

- Gauge field

\[
A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu
\]

\[
= \partial_\mu B_\nu - \partial_\nu B_\mu \equiv B_{\mu\nu} \tag{114}\]

- AHM scalar

\[
\tilde{\pi} = \langle H \rangle \vartheta
\]

\[
\phi = \frac{1}{\sqrt{2}} \tilde{\mathcal{H}} e^{-iY_\phi \vartheta}; \quad \tilde{\mathcal{H}} = \tilde{h} + \langle H \rangle
\]

\[
D_\mu \phi = \frac{1}{\sqrt{2}} \left[ \partial_\mu - ieY_\phi A_\mu \right] \tilde{\mathcal{H}} e^{-iY_\phi \vartheta}
\]

\[
= \frac{1}{\sqrt{2}} \left[ \partial_\mu \tilde{H} - ieY_\phi \tilde{H} \left( A_\mu + \frac{1}{e} \partial_\mu \vartheta \right) \right] e^{-iY_\phi \vartheta}
\]

\[
= \frac{1}{\sqrt{2}} \left[ \partial_\mu \tilde{H} - ieY_\phi \tilde{H} B_\mu \right] e^{-iY_\phi \vartheta} \tag{115}\]
Beyond-AHM scalar

\[ \Phi = \Phi e^{-iY_\Phi \theta} \]
\[ \langle \Phi \rangle = 0 \]
\[ D_\mu \Phi = \left[ \partial_\mu - ieY_\Phi A_\mu \right] \Phi e^{-iY_\Phi \theta} \]
\[ = \left[ \partial_\mu \Phi - ieY_\Phi \Phi \left( A_\mu + \frac{1}{e} \partial_\mu \theta \right) \right] e^{-iY_\Phi \theta} \]
\[ = \partial_\mu \Phi - ieY_\Phi \Phi B_\mu e^{-iY_\Phi \theta} \]

\[ (115) \]

Beyond-AHM fermion(s)

\[ \psi = \tilde{\psi} e^{-iY_\psi \theta} \]
\[ D_\mu \psi = \left[ \partial_\mu - ieY_\psi A_\mu \right] \tilde{\psi} e^{-iY_\psi \theta} \]
\[ = \left[ \partial_\mu \tilde{\psi} - ieY_\psi \tilde{\psi} \left( A_\mu + \frac{1}{e} \partial_\mu \theta \right) \right] e^{-iY_\psi \theta} \]
\[ = \partial_\mu \tilde{\psi} - ieY_\psi \tilde{\psi} B_\mu e^{-iY_\psi \theta} \]

\[ (116) \]

The extended-AHM Lagrangian, which governs the spectrum and dynamics of particle physics is

\[ L_{E-AHM}^{\text{ParticlePhysics}} \left( B_\mu; \tilde{H}; \tilde{\Phi}; \tilde{\psi} \right) \]
\[ = L_{\text{Lorenz}}^{\text{AHM}; \tilde{H}; B_\mu; \tilde{\Phi}; \tilde{\psi}} \]
\[ + L_{\text{GaugeInvariant; BeyondAHM}; \Phi} + L_{\text{GaugeInvariant; BeyondAHM}; \tilde{\psi}} \]

\[ (117) \]

where the spin \( S = 1 \) field \( B_\mu \)

\[ L_{\text{Lorenz}}^{\text{AHM}; \tilde{H}; B_\mu; \tilde{\Phi}; \tilde{\psi}} = L_{\text{GaugeInvariant; AHM}; \tilde{H}; B_\mu; \tilde{\Phi}; \tilde{\psi}} \]
\[ + L_{\text{GaugeFix; Lorenz; BeyondAHM}; B_\mu; \tilde{\Phi}; \tilde{\psi}} \]
\[ + L_{\text{GhostLorenz; BeyondAHM}; B_\mu; \tilde{\Phi}; \tilde{\psi}} \]
\[ V_{\text{AHM}} = \frac{1}{4} \lambda_\Phi \left( \tilde{H}^2 - \langle H \rangle^2 \right) \]

\[ (118) \]

For the Beyond-AHM scalar

\[ L_{\text{GaugeInvariant; BeyondAHM}; \Phi} = \left| D_\mu \Phi \right|^2 - V_\Phi - V_{\Phi} \]
\[ (119) \]

while for Beyond-AHM fermions

\[ L_{\text{GaugeInvariant; BeyondAHM}; \tilde{\psi}} = i \tilde{\psi} L D_\mu \tilde{\psi} + i \tilde{\psi} R D_\mu \tilde{\psi} \]
\[ + L_{\text{Yukawa; BeyondAHM}; \tilde{\psi}} + L_{\text{Majorana; BeyondAHM}; \tilde{\psi}} \]
\[ D_\mu \tilde{\psi}_L = \left[ \partial_\mu - ieY_\psi B_\mu \right] \tilde{\psi}_L \]
\[ D_\mu \tilde{\psi}_R = \left[ \partial_\mu - ieY_\psi B_\mu \right] \tilde{\psi}_R \]

\[ L_{\text{Yukawa; BeyondAHM; \tilde{\psi}}} = -\frac{1}{\sqrt{2}} \bar{\tilde{\psi}} \left( \tilde{L} \tilde{\psi}_L + \tilde{\psi}_L \tilde{\psi}_L \right) \tilde{H} \]
\[ - \bar{\tilde{\psi}} \left( \tilde{L} \tilde{\psi}_R + \tilde{\psi}_R \tilde{\psi}_R \right) \tilde{H} \]
\[ L_{\text{Majorana; BeyondAHM; \tilde{\psi}}} = \frac{1}{2} M_{\psi} \left( \tilde{W} \tilde{\psi}_L \tilde{W} \tilde{\psi}_L + \tilde{\psi}_L \tilde{W} \tilde{\psi}_L \tilde{W} \tilde{\psi}_L \right) \]
\[ - \frac{1}{2} M_{\psi} \left( \tilde{W} \tilde{\psi}_R \tilde{W} \tilde{\psi}_R + \tilde{\psi}_R \tilde{W} \tilde{\psi}_R \tilde{W} \tilde{\psi}_R \right) \]

\[ ii) \text{ 4th decoupling theorem: extended-AHM } B_\mu \text{ pole-mass} \ [75] \text{. The } B_\mu \text{ mass-squared in } (118) \text{ arises entirely from SSB} \]

\[ m_{B_\mu}^2 = m_{A_\mu}^2 = e^2 \langle H \rangle^2 \]

\[ (121) \]

Dimensional analysis shows that the contribution of a state of mass/energy \( \sim M_{\text{heavy}} \) to the spectral function \( \Delta_{E-AHM} \) gives terms \( \sim 1/M_{\text{heavy}}^2 \), so that

\[ \Delta_{E-AHM}(q^2) = \Delta_{AHM}(q^2) + O(1/M_{\text{heavy}}^2) \]

\[ \Delta_{AHM}(q^2) = \frac{1}{q^2 - m_{B,Pole}^2 + i\epsilon} + \int dm^2 \rho_{AHM}(m^2) \frac{1}{q^2 - m^2 + i\epsilon} \]

\[ Z_{E-AHM}^B = Z_{AHM}^B + O(1/M_{\text{heavy}}^2) \]

\[ (122) \]

Therefore the gauge-independent \( B_\mu \) observable pole-mass-squared

\[ \left[ \Delta_{E-AHM}(0) \right]^{-1} = -m_B^2 = -e^2 \langle H \rangle^2 \]

\[ m_{B,Pole}^2 = e^2 \langle H \rangle^2 \left[ 1 - e^2 \langle H \rangle^2 \int dm^2 \rho_{AHM}(m^2) \right]^{-1} \]

\[ + O(1/M_{\text{heavy}}^2) \]

(123)

\[ \text{with Kibble representation } \rho_{AHM}^B \text{ are Goldstone Exceptionally Natural, not fine-tuned.} \]

\[ iii) \text{ Decoupling of NGB } \pi, \text{ particle spectrum and dynamics} [32] \text{. It is crucial for SSB gauge theories [51][52] to remember the additional gauge-fixing term inside (111)} \]

\[ L_{\text{Lorenz; E-AHM}} = L_{\text{ParticlePhysics}} \]
\[ - \lim_{\xi \to 0} \frac{1}{2\xi} \left( \frac{1}{e \langle H \rangle} \partial^2 \pi \right) \left( \frac{1}{e \langle H \rangle} \partial^2 \pi - 2\partial_\mu B_\mu \right) \]

\[ (124) \]

The Lagrangian (124) is guaranteed to generate all of the results in Sections II and III and Appendices A and B. In practice, this is done via the manifestly renormalizeable extended-AHM Lagrangian (124).
G. Guralnik, T. Hagan and T.W.B. Kibble \cite{51}, and T.W.B. Kibble \cite{52}, showed that, in the Kibble representation in Lorenz gauge, the $U(1)_Y$ AHM quantum states factorize. In the analogous $U(1)_Y$ extended-AHM, and in the $m^2_{W_{\text{weak}}}/M^2_{\text{heavy}} \to 0$ limit the analogous $U(1)_Y$ E-AHM also factorizes \cite{32}

$$
\left| \Psi \left( A^\mu; \phi; \bar{\psi}, \omega; \Phi \psi \right) \right\rangle \to \left| \Psi^{\text{Particles}} \left( B^\mu; \bar{H} \right) \right\rangle (125)
$$

$$
\left| \Psi^{\text{Ghost}} \left( \bar{\omega}, \omega \right) \right\rangle \left| \Psi^{\text{Goldstone}} \left( \tilde{\pi} \right) \right\rangle \left| \Psi^{\text{E-AHM}} \left( \Phi; \bar{\psi} \right) \right\rangle
$$

With $\partial^2 \omega = 0; \partial^2 \bar{\omega} = 0$, the ghosts $\omega$ and $\bar{\omega}$ are free and massless and co-decouple in Lorenz gauge. Furthermore, the final gauge-fixing condition $\partial^2 \tilde{\pi} = 0$ in (124) forces the NGB $\tilde{\pi}$ to also be a free massless particle, which completely decouples from, and disappears from, the observable particle spectrum and its dynamics \cite{51} \cite{52}, and whose states factorize as in (125).

In the $m^2_{W_{\text{weak}}}/M^2_{\text{heavy}} \to 0$ limit, all physical measurements and observations are then entirely predicted by the AHM Lagrangian (118) and its states in (125)

$$
L^\text{E-AHM: } B^\mu \left( \bar{H}; B^\mu; \bar{\psi}, \omega \right) ; \left| \Psi^{\text{Particle Physics}} \left( B^\mu; \bar{H}; \bar{\omega}, \omega \right) \right\rangle (126)
$$

What has become of our SSB $U(1)_Y$ Ward-Takahashi identities? Although the NGB $\tilde{\pi}$ has de-coupled, it still governs the SSB dynamics and particle spectrum of (126) \cite{32}: it is simply hidden from explicit view. But that decoupling NGB still causes powerful hidden constraints on (126) to arise from its hidden shift symmetry \cite{32}

$$
\tilde{\pi} \to \tilde{\pi} + \langle H \rangle \theta
$$

for constant $\theta$.

Our SSB $U(1)_Y$ WTI’s, and all of the results of Section II, Section III, Appendix A, and Appendix B are also hidden but still in force: connected amputated Green’s functions $\Gamma_{\nu, \gamma} \left( T^\text{E-AHM} \right)$; connected amputated T-Matrix elements $T^\text{E-AHM} \left( \tilde{\pi} \right)$; Adler self-consistency conditions \cite{10,11}; together with their proof of IR finiteness; Goldstone theorem \cite{16}; 1-soft-$\pi$ theorems \cite{10,11}; decoupling theorems for Green’s functions and T-Matrix elements \cite{79,80}; and the decoupling theorem for the BEH pole-mass-squared $m^2_{\text{BEH:Pole}}$ \cite{81}. These still govern the SSB dynamics and particle spectrum of (126): they are simply hidden from explicit view. We call them “Hidden $U(1)_Y$ Ward-Takahashi identities of the SSB Abelian Higgs model”.

B. SSB causes decoupling of heavy $M^2_{\text{heavy}} \gg m^2_{\text{weak}}$ particles. This fact is hidden, from the observable particle spectrum of the $U(1)_Y$ extended-AHM and its dynamics, by the decoupling of the NGB $\tilde{\pi}$

We now take all of the new scalars $\hat{\Phi}$ and fermions $\hat{\psi}$ in the extended-AHM to be very heavy, and are only interested in low-energy processes:

$$
M^2_{\hat{\Phi}}; M^2_{\hat{\psi}} \sim M^2_{\text{heavy}} \gg m^2_{\text{weak}} \tag{127}
$$

$$
q_\mu \approx m_{\text{weak}}
$$

where $q_\mu$ is a typical momentum transfer. In the limit $m^2_{W_{\text{weak}}}/M^2_{\text{heavy}} \to 0$ the effective Lagrangian of the spontaneously broken extended-AHM gauge theory \cite{75}

$$
L^\text{Eff: SSB}_{\text{E-AHM}} \left( k^\mu; B^\mu; \bar{H}; \hat{\Phi}; \hat{\psi} \right) \to L^\text{Eff: SSB}_{\text{AHM}} \left( k^\mu; B^\mu; \bar{H} \right) + O \left( m^2_{W_{\text{weak}}}/M^2_{\text{heavy}} \right) \tag{128}
$$

with the possible exception that the dimension $\text{Dim} = 2$ operator $\propto \mu^2$ in $V_{\text{AHM}}$ in (12).

$$
V_{\text{AHM}} = \mu^2 \left( \phi^4 \right) + \lambda^2 \left( \phi^4 \right) \tag{129}
$$

has caused a fine-tuning problem, and raised the BEH massed-squared to the heavy scale: $m^2_{\text{BEH}} \sim M^2_{\text{heavy}}$. We now show that this is not the case.

1) 5th decoupling theorem: SSB Abelian Higgs model: Eqn. (129) lies entirely within the $\phi$-sector of the extended theory, and is therefore subject to all of the results of Sections II and III, and Appendices A and B. Therefore we know, instead, that the BEH pole-mass-squared (84) arises entirely from SSB and (un-extended) AHM decays. We also know that

$$
V^\text{Eff}_{E-AHM} = \lambda^2 \left( \phi^4 \right) - \frac{1}{2} \langle H \rangle^2 \tag{130}
$$

$$
\sim O^\text{Ignore}_{E-AHM}
$$

$$
\sim \lambda^2 \left( \phi^4 \right) - \frac{1}{2} \langle H \rangle^2 \tag{131}
$$

- In (84,130) finite $O^\text{Ignore}_{E-AHM}$ decouple and vanish as $m^2_{W_{\text{weak}}}/M^2_{\text{heavy}} \to 0$.

- Among the terms included in (130) are finite relevant operators dependent on the heavy matter representations:

$$
M^2_{\text{heavy}}; M^2_{\text{heavy}} \ln \left( M^2_{\text{heavy}} \right), \tag{132}
$$

$$
M^2_{\text{heavy}} \ln \left( m^2_{\text{weak}} \right), m^2_{\text{weak}} \ln \left( M^2_{\text{heavy}} \right) \tag{133}
$$

but the Goldstone theorem \cite{76} has made them vanish! That fact is a central point of this paper.

- Marginal operators $\sim \ln \left( M^2_{\text{heavy}} \right)$ have been absorbed in (130); i.e. in the renormalization of gauge-independent observables (i.e. the quartic-coupling constant $\lambda^2$ and the BEH VEV $\langle H \rangle$), and in the un-observable wavefunction renormalization $Z^\phi_{E-AHM}$ \cite{71}.
Therefore, no trace of $M_{\text{Heavy}}$-scale $\Phi, \psi$ survives in \[130\]! All the heavy Beyond-AHM matter representations have completely decoupled, and the two SSB gauge theories

$$ E - \text{AHM} \xrightarrow{m_{\text{Weak}}^2/M_{\text{Heavy}}^2 \to 0} \text{AHM} \quad (132) $$

become equivalent in the limit $m_{\text{Weak}}^2/M_{\text{Heavy}}^2 \to 0$, a central result of this paper.

ii) Gauge-independence of our results \[32\]: S.-H. Henry Tye and Y. Vtorov-Karevsky \[74\] proved that, displayed in the Kibble representation, $\lambda^2_{\phi} \langle H \rangle^2$ and the AHM effective potential,

$$ V_{\text{Eff}} = \frac{\lambda^2_{\phi}}{4} \langle \hat{h}^2 + 2\langle H \rangle \hat{h} \rangle^2 + O_{\text{AHM}}^{\text{ignore}} \quad (133) $$

are all-loop-orders gauge-independent. The renormalized gauge-coupling-constant-squared at zero momentum $e^2 \equiv e^2(0)$ is gauge-independent. With our 5 decoupling theorems \[79,80,84,123,132\], so are $\lambda^2_{\phi} \langle H \rangle^2$ and $V_{E, \text{AHM}}$ \[130\], and the $B_p$ pole-mass-squared \[123\]. These all appear in the decoupled particle physics \[126\] of extended-AHM.

iii) Observable gauge-independent BEH mass: Slavnov-Taylor identities guarantee that the on-shell T-Matrix-element definition of the experimentally observable BEH mass-squared

$$ T^\text{extended}_{2,0}(p, -p) |_{p^2 = m_{\text{BEH}, \text{Experimental}}^2} = 0 \quad (134) $$

gauge-independent to all-loop-orders. But in Lorentz gauge we have the specific instance

$$ T^\text{extended}_{2,0}(p, -p) |_{p^2 = m_{\text{BEH}, \text{Pole}}^2} = \left[ \Delta_{E, \text{AHM}} \right]^{-1} \frac{1}{p^2 - m_{\text{BEH}, \text{Pole}}^2 + i\epsilon} + \int \frac{d^2 \rho_{\text{BEH}}(m^2)}{p^2 - m^2 + i\epsilon} \left[ \int \frac{d^2 \rho_{\text{BEH}}(m^2)}{m^2 - i\epsilon} \right]^{-1} = 0 \quad (135) $$

with $\rho_{\text{AHM}}$ in the Kibble representation. Therefore the experimentally observable BEH mass, in extended-AHM, is the BEH pole-mass

$$ m_{\text{BEH}, \text{Experimental}}^2 = m_{\text{BEH}, \text{Pole}}^2 = 2\lambda^2_{\phi} \langle H \rangle^2 \left[ 1 - 2\lambda^2_{\phi} \langle H \rangle^2 \right] \int \frac{d^2 \rho_{\text{BEH}}(m^2)}{m^2 - i\epsilon} + O_{E_{\text{AHM}; \phi}}^{1/4} \quad (136) $$

and is gauge-independent to all-loop-orders.

V. BWL & GDS: NOTRE VISION À TRAVERS LE PRISME DE LA RIGUERÉ MATHÉMATIQUE QU’IMPOSAIT RAYMOND STORA

Raymond Stora regarded Vintage-QFT as incomplete, fuzzy in its definitions, and primitive in technology. For example, he worried whether the off-shell T-Matrix could be mathematically rigourously defined to exist in Lorentz gauge: e.g. without running into some infra-red (IR) subtlety. This, even though we prove here the IR finiteness of the $\phi$-sector on-shell T-Matrix.

Although he agreed on the correctness of the results presented here, Raymond might complain that we fall short of a strict mathematically rigourous proof (i.e. according to his exacting mathematical standards). He reminded us that much has been learned about QFT, via modern path integrals, in the recent ~ 45 years. In the time up to his passing, he was intent on improving this work by focussing on 3 issues:

- Properly defining and proving the Lorenz gauge results presented here, but with modern path integrals;
- Tracking our central results (i.e. no-FT and heavy-particle decoupling) directly to SSB, via BRST methods, in an arbitrary manifestly IR finite ’t Hooft $R_\xi$ gauge: i.e. proving to his satisfaction that they are not an artifact of Lorenz gauge;
- Tracking our central results directly to those Slavnov-Taylor IDS governing the SSB Goldstone mode of the BRST-invariant extended-AHM Lagrangian.

Any errors, wrong-headedness, mis-understanding, or mis-representation appearing in this paper are solely the fault of BWL and GDS.

VI. CONCLUSION: HISTORICALLY, COMPLETE DECOUPLING OF HEAVY INVISIBLE PARTICLES IS THE USUAL PHYSICS EXPERIENCE

We showed, in Sections \[11,15,19\] and Appendix \[15\] that the low-energy weak-scale effective SSB extended-AHM Lagrangian is protected, by new “hidden” rigid/global SSB $U(1)_Y$ WTI’s and a hidden Goldstone theorem, against contributions from certain heavy $M_{\text{Heavy}}^2 \gg m_{\text{Weak}}$ Beyond-AHM particles. Five decoupling theorems \[79,80,84,123,132\] govern certain heavy particles $\Phi, \psi$. Renormalized gauge-independent observable $\langle \langle H \rangle^2, m_{\text{BEH}, \text{Pole}}^2 \rangle$ are therefore not fine-tuned, but instead Goldstone Exceptionally Natural, with far more powerful suppression of fine-tuning than G. ’t Hooft’s naturalness criteria \[24\] would demand.

What is remarkable is that heavy particle decoupling and Goldstone Exceptional Naturalness are obscured/hidden from the physical particle spectrum \[126\] and its dynamics. The decoupling of the NGB $\pi$ has famously spared the AHM an observable massless particle \[49,51\]. But it has also hidden, from that physical particle spectrum and dynamics, our $U(1)_Y$ WTI \[48,66,67\], \[69,110,115,117,118\], and their severe constraints on the effective low-energy extended-AHM Lagrangian. In
particular, the **weak-scale** extended-AHM SSB gauge theory has a **hidden** $U(1)_Y$ **shift symmetry**, for constant $\theta$:

$$\tilde{\pi} \rightarrow \tilde{\pi} + \langle H \rangle \theta$$  \hspace{1cm} (137)

which has protected it from any Brout-Englert-Higgs fine-tuning problem, and caused the complete \cite{29} decoupling of certain heavy $M_{\text{Heavy}}^2 \gg m_{\text{Weak}}^2 U(1)_Y$ matter-particles.

But such heavy-particle decoupling is historically (i.e. except for high-precision electro-weak $S,T$ and $U$ \cite{31,54}) the usual physics experience, at each energy scale, as experiments probed smaller and smaller distances. After all, Willis Lamb did not need to know the top quark or BEH mass in order to interpret theoretically the experimentally observed $O(m_c \alpha^2 \ln \alpha)$ splitting in the spectrum of hydrogen.

Such heavy-particle decoupling may be the reason why the Standard Model \cite{71,72}, viewed as an effective low-energy weak-scale theory, is the most experimentally and observationally successful and accurate theory of Nature known to humans, i.e. when augmented by classical General Relativity and neutrino mixing: that “Core Theory” \cite{77} has no known experimental or observational counter-examples.

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Appendix A: \( U(1)_Y \) Ward-Takahashi identities in the SSB Abelian Higgs Model

In this Appendix A, we present the full detailed derivations of our \( U(1)_Y \) WTI for the SSB AHM. For pedagogical completeness, we reproduce all details of the entire argument.

- This paper is interested in building the effective \( \phi \) potential, and showing no-fine-tuning of the BEH mass.
- A companion Letter [32] focusses instead on the WTI themselves, showing that AHM physics has more symmetry than the AHM effective Lagrangian.

We focus on the rigid/global current \( J_{A H M}^\mu \) of the Abelian Higgs model, the spontaneously broken gauge theory of a complex scalar \( \phi = \frac{1}{\sqrt{2}}(H + i\pi) = \frac{1}{\sqrt{2}}\tilde{H}e^{i\beta/(H)} \), and a massive \( U(1)_Y \) gauge field \( A_\mu \).

\[
J_{A H M}^\mu = \pi \partial^\mu H - H \partial^\mu \pi - eA^\mu \left( \pi^2 + H^2 \right) \tag{A1}
\]

The classical equations of motion reveal the crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian, the classical axial-vector current (A1) is not conserved. In Lorenz gauge

\[
\partial_\nu J_{A H M}^{\nu} = -H m_A F_A \\
m_A = e(H) \\
F_A = \partial_\beta A^\beta \tag{A2}
\]

with \( F_A \) the gauge fixing function. Still, the physical states \( A_\mu; h, \pi \) of the theory (but not the BRST-invariant...
Lagrangian) obey $F_A = 0$. In Lorentz gauge, $A_a$ is transverse and $\pi$ is a massless Nambu-Goldstone Boson (NGB).

The purpose of this Appendix A is to derive a tower of quantum $U(1)_Y$ Ward-Takahashi identities, which exhausts the information content of (A2), and severely constrains the dynamics (i.e. the connected time-ordered products) of the physical states of the spontaneously broken Abelian Higgs model.

1) **We study a total differential of a certain connected time-ordered product**

$$\partial_\mu \langle 0 | T \left[ J^\mu_{AHM}(z) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}}$$

(A3)

written in terms of the physical states of the complex scalar field $\phi$. Here we have $N$ external renormalized scalars $h = H - \langle H \rangle$ (coordinates $x$, momenta $p$), and $M$ external $(CP = -1)$ renormalized pseudo-scalars $\pi$ (coordinates $y$, momenta $q$).

2) **Conservation of the global $U(1)_Y$ current for the physical states**: Strict quantum constraints are imposed, which force the relativistically-covariant theory of gauge bosons to propagate only its true number of quantum spin $S = 1$ degrees of freedom: these constraints are implemented by use of spin $S = 0$ fermionic Fadeev-Popov ghosts ($\bar{\omega}, \omega$) and, in Landau gauge, $S = 0$ massless $\pi$. Physical states and their connected time-ordered products, but not the BRST-invariant Lagrangian, obey (A8) the gauge-fixing condition $F_A = \partial_\beta A^\beta = 0$ in Landau gauge

$$\langle 0 | T \left[ \partial_\beta A^\beta(z) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(A4)

which restores conservation of the rigid/global $U(1)_Y$ current for physical states

$$\langle 0 | T \left[ \partial_\mu J^\mu_{AHM}(z) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(A5)

It is in this “time-ordered-product” sense that the “physical” rigid global $U(1)_Y$ current $J^\mu_{AHM}$ is “conserved”, and it is this conserved current which generates 2 towers of quantum $U(1)_Y$ WTI. These WTI severely constrain the dynamics of the $\phi$-sector.

3) **Vintage QFT and canonical quantization**: Equal-time commutators are imposed on the exact renormalized fields, yielding equal-time quantum commutators at space-time points $y, z$.

$$\delta(z_0 - y_0) \left[ J^0_{AHM}(z), H(y) \right] = -i\pi(x) \delta^4(z - y)$$

$$\delta(z_0 - y_0) \left[ J^0_{AHM}(z), \pi(y) \right] = i\pi(y) \delta^4(z - y)$$

(A7)

Field normalization follows from the non-trivial commutators

$$\delta(z_0 - y_0) \left[ \partial^0 H(z), H(y) \right] = -i\delta^4(z - y)$$

$$\delta(z_0 - y_0) \left[ \partial^0 \pi(z), \pi(y) \right] = -i\delta^4(z - y)$$

(A8)

4) **Certain surface integrals vanish**: As appropriate to our study of the massless $\pi$, we use pion pole dominance to derive 1-soft-pion theorems, and form the surface integral

$$\lim_{\delta \to 0} \int d^4 z \bar{\rho}_\mu \langle 0 | T \left[ \left( J^\mu_{AHM} + \langle H \rangle \right) \partial^\mu \pi(z) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}}$$

(A9)

$$\int_{3-surface} d^3 z \bar{\rho}_\mu \langle 0 | T \left[ \left( J^\mu_{AHM} + \langle H \rangle \right) \partial^\mu \pi(z) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(A10)

where we have used Stokes theorem, and $\bar{\rho}_\mu$ is a unit vector normal to the $3-surface$. The time-ordered-product constrains the $3-surface$ to lie on, or inside, the light-cone.

At a given point on the surface of a large enough 4-volume $\int d^4 z$ (i.e. the volume of all space-time): all fields are asymptotic in-states and out-states, properly quantized as free fields, with each field species orthogonal to the others, and they are evaluated at equal times, making time-ordering un-necessary at (A10). Input the global AHM current (A11) to (A9), using $\partial_\mu (H) = 0$.

$$\int_{3-surface} d^3 z \bar{\rho}_\mu \langle 0 | T \left[ \left( \pi \partial^0 h - h \partial^0 \pi - e A^\mu (\pi^2 + H^2) \right) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle_{\text{Connected}} = 0$$

(A10)

The surface integral (A10) vanishes because both $(h, A^\mu)$ are massive in the spontaneously broken $U(1)_Y$ AHM, with $(m^2_{BEH} \neq 0, m_A^2 = e^2 \langle H \rangle^2)$ respectively. Propagators connecting $(h, A^\mu)$, from points
on $z^{3-\text{surface}} \rightarrow \infty$ to the localized interaction points $(x_1...x_N; y_1...y_M)$, must stay inside the light-cone, die off exponentially with mass, and are incapable of carrying information that far.

It is very important for “pion-pole-dominance” and this paper, that this argument fails for the remaining term in $J^{\mu}_{AHM}$ in (A1):

$$\int_{2-\text{surface} \rightarrow \infty} d^2z \sum_{\mu} \langle 0| T \left( -\langle H \rangle \partial^\mu \pi(z) \right) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M) \rangle |0\rangle_{\text{Connected}} \neq 0$$  \hspace{1cm} (A11)

$\pi$ is massless in the SSB AHM, capable of carrying (along the light-cone) long-lived pseudo-scalar forces out to the 2-surface ($z^{2-\text{surface}} \rightarrow \infty$): i.e. the very ends of the light-cone (but not inside it). That massless-ness is the basis of our pion-pole-dominance-based 1-soft-pion theorems (A18), infra-red finiteness for $m^2_{\pi} = 0$ (A22), and a Goldstone theorem (A27).

5) Master equation: Using (A5,A8) in (A3) to form the right-hand-side, and (A10) in (A8) to form the left-hand-side, we write the master equation

$$-\langle H \rangle \partial^\mu \langle 0| T \left[ (\partial^\mu \pi(z)) \times h(x_1)\ldots h(x_N)\pi(y_1)\ldots\pi(y_M) \right] |0\rangle_{\text{Connected}}$$  \hspace{1cm} (A12)

where the “hatted” fields $\hat{h}(x_n)$ and $\pi(\hat{y}_m)$ are to be removed. We have also thrown away a sum of $M$ terms, proportional to $\langle H \rangle$, which corresponds entirely to disconnected graphs.

6) $\phi$-sector connected amplitudes: Connected momentum-space amplitudes, with $N$ external BEHs and $M$ external $\pi$s, are defined in terms of $\phi$-sector connected time-ordered products

$$iG_{N,M}(p_1...p_N; q_1...q_M)(2\pi)^4\delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$$  \hspace{1cm} (A13)

$$= \prod_{n=1}^{N} \int d^4x_ne^{ip_nx_n} \prod_{m=1}^{M} \int d^4y_me^{iq_my_m}$$

The master eqn. (A12) can then be re-written

$$i\langle H \rangle k^2 G_{N,M+1}(p_1...p_N; kq_1...q_M)$$  \hspace{1cm} (A14)

$$= \sum_{n=1}^{N} G_{N-1,M+1}(p_1...\hat{p}_n...p_N; (k + p_n)q_1...q_M)$$

$$- \sum_{m=1}^{M} G_{N+1,M-1}((k + q_m)p_1...p_N; q_1...\hat{q}_m...q_M)$$

with the “hatted” momenta ($\hat{p}_n, \hat{q}_m$) removed in (A14), and an overall momentum conservation factor of $(2\pi)^4\delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$.

7) $\phi$-propagators: Special cases of (A13) are the BEH and $\pi$ propagators

$$iG_{2,0}(p_1, -p_1) = i \int \frac{d^4p_2}{(2\pi)^4} G_{2,0}(p_1, p_2)$$

$$= \int d^4x_1 e^{ip_1x_1} \langle 0| T \left[ h(x_1)h(0) \right] |0\rangle$$

with an overall momentum conservation factor of $(2\pi)^4\delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$.

8) $\phi$-sector connected amputated 1-$(h, \pi)$-Reducible (1-$\phi$-R) transition matrix (T-matrix): With an overall momentum conservation factor $(2\pi)^4\delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$, the $\phi$-sector connected amplitudes are related to $\phi$-sector connected amputated T-matrix elements

$$G_{N,M}(p_1...p_N; q_1...q_M)$$  \hspace{1cm} (A16)

$$\equiv \prod_{n=1}^{N} \left[ i\Delta_{BEH}(p_n^2) \right] \prod_{m=1}^{M} \left[ i\Delta_{\pi}(q_m^2) \right] T_{N,M}(p_1...p_N; q_1...q_M)$$

so that the master equation (A12) can be written

$$i\langle H \rangle k^2 \left[ i\Delta_{\pi}(k^2) \right] T_{N,M+1}(p_1...p_N; kq_1...q_M)$$

$$= \sum_{n=1}^{N} T_{N-1,M+1}(p_1...\hat{p}_n...p_N; (k + p_n)q_1...q_M)$$

$$\times \left[ i\Delta_{\pi}((k + p_n)^2) \right] \left[ i\Delta_{BEH}(p_n^2) \right]^{-1}$$

$$- \sum_{m=1}^{M} T_{N+1,M-1}((k + q_m)p_1...p_N; q_1...\hat{q}_m...q_M)$$

$$\times \left[ i\Delta_{BEH}((k + q_m)^2) \right] \left[ i\Delta_{\pi}(q_m^2) \right]^{-1}$$  \hspace{1cm} (A17)

with the “hatted” momenta ($\hat{p}_n, \hat{q}_m$) removed in (A17), and an overall momentum conservation factor of $(2\pi)^4\delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$.
9) “Pion pole dominance” and “1-soft-pion theorems” for the T-matrix: Consider the “soft-pion limit”

$$\lim_{k_i \to 0} k_i^2 \Delta_e(k_i^2) = 1$$  \hspace{1cm} (A18)

where the $\pi$ is hypothesized to be all-loop-orders massless, and written in the Källén-Lehmann representation [41] with spectral density $\rho_{AHM}$

$$\Delta_e(k_i^2) = \frac{1}{k_i^2 + i\epsilon} + \int \frac{d^2 \rho_{AHM}^s(m^2)}{k_i^2 - m^2 + i\epsilon}.$$  \hspace{1cm} (A19)

The master equation (A12) then becomes

$$-\langle H \rangle T_{N+1,M+1}(p_1, ..., p_N; q_1, ..., q_M) = \sum_{n=1}^N T_{N-1,M+1}(p_1, ..., p_n, ..., p_N; q_1, ..., q_M)$$

$$\times [i\Delta_e(p_n^2)] [i\Delta_{BEH}(p_n^2)]^{-1}$$

$$- \sum_{m=1}^M T_{N+1,M-1}(q_m, p_1, ..., p_n, ..., p_N, q_1, ..., q_m, ..., q_M)$$

$$\times [i\Delta_{BEH}(q_m^2)] [i\Delta_e(q_m^2)]^{-1}$$  \hspace{1cm} (A20)

in the 1-soft-pion limit. As usual the “hatted” momenta ($p_n, q_m$) and associated fields are removed [A20], and an overall momentum conservation factor $(2\pi)^4 \delta^4(\sum_{n=1}^N p_n + \sum_{m=1}^M q_m)$ applied.

The set of 1-soft-pion theorems [A20] have the form

$$\langle H \rangle T_{N, M+1} \sim T_{N-1, M+1} - T_{N+1, M-1}$$  \hspace{1cm} (A21)

relating, by the addition of a zero-momentum pion, an $N + 1$-point function to $N$ + $M$-point functions.

10) The Adler self-consistency relations (but now for a gauge theory) rather than global SU(2)$_L \times$ SU(2)$_R$ [67, 68] are gotten by putting the remainder of the [A20] particles on mass-shell

$$\langle H \rangle T_{N, M+1}(p_1, ..., p_N; q_1, ..., q_M)$$

$$\times (2\pi)^4 \delta^4(\sum_{n=1}^N p_n + \sum_{m=1}^M q_m)$$

$$\rho_{1^2=2^2=3^2=m_{BEH}}(q_1^2=q_2^2=...=q_M^2=0) = 0$$  \hspace{1cm} (A22)

which guarantees the infra-red (IR) finiteness of the $\phi$-sector on-shell T-matrix in the SSB AHM gauge theory in Lorenz gauge, with massless $\pi$ in the 1-soft-pion limit. These “1-soft-pion” theorems [67, 68] force the T-matrix to vanish as one of the pion momenta goes to zero, provided all other physical scalar particles are on-mass-shell. Eqn. (A22) asserts the absence of infrared divergences in the physical-scalar sector in Goldstone mode. “Although individual Feynman diagrams may be IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit [69].”

11) 1-($h, \pi$) Reducibility (1-$\phi$-R) and 1-($h, \pi$) Irreducibility (1-$\phi$-I): With some exceptions, the $\phi$-sector connected amputated transition matrix $T_{N, M}$ can be cut apart by cutting an internal $h$ or $\pi$ line, and are designated 1-$\phi$-R. In contrast, the $\phi$-sector connected amputated Green’s functions $\Gamma_{N, M}$ are defined to be 1-$\phi$-I: i.e. they cannot be cut apart by cutting an internal $h$ or $\pi$ line.

$$T_{N, M} = \Gamma_{N, M} + (1 - \phi - R)$$  \hspace{1cm} (A23)

Both $T_{N, M}$ and $\Gamma_{N, M}$ are 1-($A_{\mu}$)-Reducible (1-$A_{\mu}$-R): i.e. they can be cut apart by cutting an internal transverse-vector $A_{\mu}$ gauge-particle line.

12) $\phi$-sector 2-point functions, propagators and a 3-point vertex: The special 2-point functions $T_{0, 2}(q, q - q)$ and $T_{2, 0}(p, p - p)$, and the 3-point vertex $T_{1, 2}(q, 0, -q)$, are 1-$\phi$-I (i.e. they are not 1-$\phi$-R), and are therefore equal to the corresponding 1-$\phi$-I connected amputated Green’s functions. The 2-point functions

$$T_{2, 0}(p, -p) = T_{0, 2}(q, q - q) = \Gamma_{BEH}(p^2)=1$$

$$T_{0, 2}(q, q - q) = \Gamma_{BEH}(q^2)=1$$  \hspace{1cm} (A24)

are related to the (1$h$, 2$\pi$) 3-point $h\pi^2$ vertex

$$T_{1, 2}(p, q - p - q) = \Gamma_{1, 2}(p, h - p - q)$$  \hspace{1cm} (A25)

by a 1-soft-pion theorem [A20]

$$\langle H \rangle T_{1, 2} = \langle H \rangle T_{2, 0} - T_{2, 0} - T_{0, 2}$$

$$= \langle H \rangle T_{1, 2} = \langle H \rangle T_{0, 2} = 0$$  \hspace{1cm} (A26)

13) The Goldstone theorem, in the spontaneously broken AHM in Lorenz gauge, is a special case of that SSB gauge theory’s Adler self-consistency relations [A22]

$$\langle H \rangle T_{0, 2}(0, 0) = 0$$

$$\langle H \rangle T_{0, 2}(0, 0) = 0$$

$$\langle H \rangle \Delta_e(0) = 0$$  \hspace{1cm} (A27)

proving that $\pi$ is massless. That all-loop-orders renormalized massless-ness is protected/guaranteed by the global $U(1)_y$ symmetry of the physical states of the gauge theory after spontaneous symmetry breaking.

14) $T_{N, M}^{External} \phi$-sector T-Matrix with one soft $\sigma(q_\sigma = 0)$ attached to an external-leg: Figure [4]
shows that

\[
\langle H \rangle T_{N,M+1}^{\text{External}}(p_1...p_N;0q_1...q_M) = \sum_{n=1}^{N} \left[ i(H)\Gamma_{1,2}(p_n, 0, -p_n) \left[ i\Delta_\pi(p_n^2) \right] \right] \\
\times T_{N-1,M+1}(p_1...\hat{p}_n...p_N; p_n q_1...q_M) + \sum_{m=1}^{M} \left[ i(H)\Gamma_{1,2}(q_m, 0, -q_m) \left[ i\Delta_\pi(q_m^2) \right] \right] \\
\times T_{N+1,M-1}(q_m p_1...p_N; q_1...\hat{q}_m...q_M) \\
= \sum_{n=1}^{N} \left( 1 - \left[ i\Delta_\pi(p_n^2) \right] \left[ i\Delta_\pi(p_n^2) \right]^{-1} \right) \\
\times T_{N-1,M+1}(p_1...\hat{p}_n...p_N; p_n q_1...q_M) - \sum_{m=1}^{M} \left( 1 - \left[ i\Delta_\pi(q_m^2) \right] \left[ i\Delta_\pi(q_m^2) \right]^{-1} \right) \\
\times T_{N+1,M-1}(q_m p_1...p_N; q_1...\hat{q}_m...q_M)
\]  

(A28)

where we used (A26). Now separate

\[
T_{N,M+1}(p_1...p_N;0q_1...q_M) = T_{N,M+1}^{\text{External}}(p_1...p_N;0q_1...q_M) \\
+ T_{N,M+1}^{\text{Internal}}(p_1...p_N;0q_1...q_M)
\]

(A29)

so that

\[
\langle H \rangle T_{N,M+1}^{\text{Internal}}(p_1...p_N;0q_1...q_M) = \sum_{m=1}^{M} T_{N+1,M-1}(q_m p_1...p_N; q_1...\hat{q}_m...q_M) \\
- \sum_{n=1}^{N} T_{N-1,M+1}(p_1...\hat{p}_n...p_N; p_n q_1...q_M)
\]

(A30)

15) Recursive U(1)_Y WTI for 1-(h, \pi)-Irreducible (1-\Phi-I) connected amputated Green’s functions \( \Gamma_{N,M} \): Removing the 1-(h, \pi)-Reducible (1-\Phi-R) graphs from both sides of \( \text{[A30]} \) yields the recursive identity

\[
\langle H \rangle \Gamma_{N,M+1}(p_1...p_N;0q_1...q_M) = \sum_{m=1}^{M} \Gamma_{N+1,M-1}(q_m p_1...p_N; q_1...\hat{q}_m...q_M) \\
- \sum_{n=1}^{N} \Gamma_{N-1,M+1}(p_1...\hat{p}_n...p_N; p_n q_1...q_M)
\]

(A31)

B.W. Lee [36] gave an inductive proof for the corresponding recursive \( SU(2)_L \times SU(2)_R \) WTI in the global Gell-Mann Levy model with PCAC [34]. Specifically, he proved that, given the global \( SU(2)_L \times SU(2)_R \) analogy of \( \text{[A30]} \), the global \( SU(2)_L \times SU(2)_R \) analogy of \( \text{[A31]} \) follows. This he did by examination of the explicit reducibility/irreducibility of the various Feynman graphs involved.

That proof also works for the U(1)_Y SSB AHM, thus establishing our tower of 1-\Phi-I connected amputated Green’s functions’ recursive U(1)_Y WTI \( \text{[A31]} \) for a local/gauge theory.

Rather than including the lengthy proof here, we paraphrase \( \text{[A31]} \) as follows: \( \text{[A26]} \) shows that \( \text{[A31]} \) is true for \( (N = 1, M = 1) \). Assume it is true for all \((n, m)\) such that \( n + m < N + M \). Consider \( \text{[A30]} \) for \( n = N, m = M \). The two classes of graphs contributing to \( T_{N,M+1}^{\text{Internal}}(p_1...p_N;0q_1...q_M) \) are displayed in Figure 2.

The top graphs in Figure 2 are 1-\Phi-R. For \( (n, m; n + m < N + M) \) we may use \( \text{[A31]} \), for those 1-\Phi-I Green’s functions \( \Gamma_{n,m} \) which contribute to \( \text{[A30]} \), to show that the contribution of 1-\Phi-R graphs to both sides of \( \text{[A30]} \) are identical.

The bottom graphs in Figure 2 are 1-\Phi-I and so already obey \( \text{[A31]} \).

16) Goldstone theorem makes tadpoles vanish:

\[
\langle 0 | h(x = 0) | 0 \rangle_{\text{Connected}} = i \left[ i\Delta_\pi(0) \right] \Gamma_{1,0}(0; )
\]

(A32)

but the \( N = 0, M = 1 \) case of \( \text{[A31]} \) reads

\[
\Gamma_{1,0}(0; ) = \langle H \rangle \Gamma_{0,2}(00; ) = 0
\]

(A33)

where we used \( \text{[A27]} \), so that tadpoles all vanish automatically, and separate tadpole renormalization is unnecessary. Since we can choose the origin of coordinates anywhere we like

\[
\langle 0 | h(x) | 0 \rangle_{\text{Connected}} = 0
\]

(A34)

17) Renormalized gauge-independent observable \( \langle H \rangle \).

\[
\langle 0 | H(x) | 0 \rangle_{\text{Connected}} = \langle 0 | h(x) | 0 \rangle_{\text{Connected}} + \langle H \rangle = \langle H \rangle
\]

\[
\partial_\mu \langle H \rangle = 0
\]

(A35)
18) Benjamin W. Lee’s 1970 Cargese summer school lectures’ [36] proof of $\phi$-sector WTI focusses on the global $SU(2)_L \times SU(2)_R$ Gell-Mann Levy theory and Partially Conserved Axial-vector Currents (PCAC). But it gives a detailed pedagogical account of the appearance of the Goldstone theorem and true massless Nambu-Goldstone bosons in global theories, and is recommended reading. We include a translation guide in Table 1.

Table 1: Derivation of Ward-Takahashi identities

| Property                     | This paper | B.W.Lee [36]          |
|------------------------------|------------|-----------------------|
| LagrangianInvariant structure group | BRST       | global group          |
| local/gauge group            | $U(1)_Y$   | $SU(2)_L \times SU(2)_R$ |
| rigid/global group           | $U(1)_Y$   | $SU(2)_L \times SU(2)_R$ |
| global currents              | $J^\mu_{AH,M}$ | $\bar{V}^\mu, \bar{A}^\mu$ |
| PCAC current divergence      | $-Hm_A\partial_\beta A^\beta$ | yes |
| $L_{Gauge\,Fixing}$ gauge ghosts $\bar{\omega}, \omega$ | Lorenz | $0; f_\pi m^2_\pi \bar{\pi}$ |
| conserved current            | physical states | Lagrangian           |
| physical states interaction  | $A_\mu, h, \pi, \Phi, \psi$ | $s, \bar{\pi}$ |
| fields                       | $A_\mu, H, \pi, \bar{\omega}, \omega$ | strong $\sigma, \bar{\pi}$ |
| BEH scalar                   | $h = H - \langle H \rangle$ | $s = \sigma - < \sigma >$ |
| VEV                          | $\langle H \rangle$ | $< \sigma >= v = f_\pi$ |
| particles in loops renormalization amplitudes | Physical&Ghosts | all-loop-orderss |
| ConnectedAmplitudes          | $G_{N,M}$ | all-loop-orders $G$ |
| NoPionPoleSingularity        | $\bar{H}$  | $H$                  |
| T-Matrix                     | $T_{N,M}$  | $T$                  |
| $1-\phi$                    | $h, \pi$   | $s, \bar{\pi}$ |
| $\phi$-sector $T_{N,M}$      | $1-\phi-R$ | $1-\phi-R$ |
| $\Gamma_{N,M}$              | $\Gamma_{N,M}$ | $\Gamma_{N,M}$ |
| connected $\Gamma_{N,M}$ amputated | $\Gamma_{N,M}$ | $\Gamma_{N,M}$ |
| connected $T_{N,M}$ amputated | $1-\phi-I$ | $1-\phi-I$ |
| External $\pi(q_\mu = 0)$    | $T_{\text{External}}^N_{N,M+1}$ | $T_1$ |
| Internal $\pi(q_\mu = 0)$   | $T_{\text{Internal}}^N_{N,M+1}$ | $T_2$ |
| BEH propagator               | $\Delta_{BEH}$ | $\Delta_\sigma$ |
| TransversePropagator         | $\Delta^\mu_\sigma$ | $\delta^{\sigma \nu} \Delta_\sigma$ |
| SSB                          | GoldstoneMode | GoldstoneMode |
| NGB after SSB                | $\tilde{\pi}$ | $\tilde{\pi}$ |
| Pion propagator              | $\Delta_\sigma$ | $\delta^{\sigma \nu} \Delta_\sigma$ |
| Goldstone theorem            | physical states | GoldstoneMode |
Appendix B: $U(1)^Y$ $(h, \pi)$-sector WTIs, which now include the all-loop-orders contributions of certain additional virtual $U(1)^Y$ matter representations $\Phi, \psi$ in the extended-Abelian Higgs Model (E-AHM)

In this Appendix [3] we present the full detailed derivations of our $U(1)^Y$ WTI for the SSB E-AHM. For pedagogical completeness, we reproduce all details of the entire argument.

- This paper is interested in building the effective E-AHM $\phi$ potential, no-fine-tuning of the BEH mass, and the decoupling in E-AHM of certain heavy matter particles from the effective low-energy AHM theory.

- A companion Letter [32] focusses instead on the WTI themselves, showing that E-AHM physics has more symmetry than the E-AHM effective Lagrangian.

We focus on the rigid/global extended-AHM current

$$J_{E-AHM}^\mu = J_{AHM}^\mu(A^\mu, \phi) + J_{BeyondAHM}^\mu(\Phi, \Psi) \tag{B1}$$

of the “extended Abelian Higgs model”, the spontaneously broken gauge theory of a complex spin $S = 0$ scalar $\phi = \sqrt{2}(H+\pi^T)$, a massive $U(1)^Y$ $S = 1$ transverse gauge field $A^\mu$, and certain $S = 0$ scalars $\Phi$ and $S = 1/2$ fermions $\psi$ originating in Beyond-AHM models.

The classical equations of motion reveal that, due to gauge-fixing terms in the BRST-invariant Lagrangian, the classical current (B1) is not conserved. In Lorenz gauge

$$\begin{align*}
\partial_\mu J_{E-AHM}^\mu &= -H_{\mu} F_A \\
m_A &= e(H) \\
F_A &= \partial_\beta A^\beta
\end{align*} \tag{B2}$$

with $F_A$ the gauge fixing function.

The purpose of this Appendix is to derive a tower of $U(1)^Y$ extended WTIs, which exhausts the information content of [B2], and severely constrains the dynamics (i.e. the connected time-ordered products) of the physical states of the SSB extended-AHM. We make use here of all of the results in Appendix A concerning $J_{AHM}^\mu$.

1) We study a certain total differential of a connected time-ordered product:

$$\partial_\mu \langle 0|T \left[ J_{E-AHM}^\mu(z) \times h(x_1) \cdots h(x_N) \pi(y_1) \cdots \pi(y_M) \right]|0\rangle_{\text{Connected}} \tag{B3}$$

written in terms of the physical states of the complex scalar $\phi$. Here we have $N$ external renormalized scalars $h = H - \langle H \rangle$ (coordinates $x$, momenta $p$), and $M$ external ($CP = -1$) renormalized pseudo-scalars $\pi$ (coordinates $y$, momenta $q$).

2) Conservation of the global $U(1)^Y$ current for the physical states: Strict quantum constraints are imposed, which force the relativistically-covariant theory of a massive transverse gauge boson to propagate only its true number of quantum spin $S = 1$ degrees of freedom. Physical states and their time-ordered products, but not the BRST-invariant Lagrangian, obey the gauge-fixing condition $F_A = \partial_\beta A^\beta = 0$ in Lorenz gauge [48].

$$\langle 0|T \left[ (\partial_\beta A^\beta(z)) \times h(x_1) \cdots h(x_N) \pi(y_1) \cdots \pi(y_M) \right]|0\rangle_{\text{Connected}} = 0 \tag{B4}$$

which restores conservation of the rigid/global $U(1)^Y$ extended current for physical states

$$\langle 0|T \left[ (\partial_\mu J_{E-AHM}^\mu(z)) \times h(x_1) \cdots h(x_N) \pi(y_1) \cdots \pi(y_M) \right]|0\rangle_{\text{Connected}} = 0 \tag{B5}$$

It is in this “time-ordered-product” sense that the rigid global extended $U(1)^Y$ current $J_{E-AHM}^\mu$ is conserved, and it is this conserved current which generates our tower of $U(1)^Y$ extended WTI. These extended WTI severely constrain the dynamics of $\phi$.

3) Vintage QFT and canonical quantization: Equal-time commutators are imposed on the exact renormalized Beyond-AHM fields, yielding equal-time quantum commutators at space-time points $y, z$.

$$\begin{align*}
\delta(z_0 - y_0) \left[ J_{BeyondAHM}^\mu(z), H(y) \right] &= 0 \\
\delta(z_0 - y_0) \left[ J_{BeyondAHM}^\mu(z), \pi(y) \right] &= 0 \tag{B6}
\end{align*}$$

Only certain $U(1)^Y$ matter particles $\Phi, \psi$ obey this condition.

- Renormalized $\langle H \rangle$ is defined to match the (unextended) AHM. Our extended $U(1)^Y$ WTI therefore require that all of the new spin $S = 0$ fields in $J_{BeyondAHM}^\mu$ have zero vacuum expectation value (VEV):

$$\langle \Phi_{BeyondAHM} \rangle = 0 \tag{B7}$$

Only certain $U(1)^Y$ matter particles $\Phi$ obey this condition.

4) Certain connected surface integrals must vanish: As appropriate to our study of massless $\pi$, we again use pion pole dominance to derive 1-soft-pion the-
orems, and require that the connected surface integral
\[
\lim_{k_\lambda \to -0} \int d^4 z e^{ikz} \partial_\mu \langle 0 | T \left[ (J^\mu_{\text{BeyondAHM}}(z)) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \\
= \int d^3 z \partial_\mu \langle 0 | T \left[ (J^\mu_{\text{BeyondAHM}}(z)) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \\
= \int_{3-\text{surface} \to -\infty} d^3 z \tilde{z}^3 \mu \times \langle 0 | T \left[ (J^\mu_{\text{BeyondAHM}}(z)) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \\
= 0 \quad \text{(B8)}
\]
where we have used Stokes theorem, and \(\tilde{z}^3\)-surface is a unit vector normal to the 3-surface. The time-ordered-constraint produces the 3-surface to lie on-or-inside the light-cone.

At a given point on the surface of a large enough 4-volume \(\int d^4 z\) (i.e. the volume of all space-time); all fields are asymptotic in-states and out-states; they are properly quantized as free fields; with each field species orthogonal to the others; and they are evaluated at equal times, making time-ordering unnecessary at \((\tilde{z}^3\text{-surface} \to -\infty)\).

Only certain \(U(1)_Y\) massive massive particles \(\Phi, \psi\) obey this condition.

5) Extended master equation: Using \[B5,B6\] in \(B3\) to form the right-hand-side, and \[B8\] in \(B3\) to form the left-hand-side, we write the extended master equation, which relates connected time-ordered products:
\[-(H) \partial_\mu \langle 0 | T \left[ (\partial^\mu \pi(z)) \times h(x_1) \ldots h(x_N) \pi(y_1) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \\
= \sum_{m=1}^M i\delta^4(z - y_m) \langle 0 | T \left[ h(z) h(x_1) \ldots h(x_N) \times \pi(y_1) \ldots \pi(y_m) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \]
\[-\sum_{n=1}^N i\delta^4(z - x_n) \langle 0 | T \left[ h(x_1) \ldots h(x_n) \ldots h(x_N) \times \pi(y_1) \ldots \pi(y_m) \ldots \pi(y_M) \right] | 0 \rangle \text{Connected} \quad \text{(B9)}
\]
where the “hatted” fields \(\widehat{h}(x_n)\) and \(\widehat{\pi}(y_m)\) are to be removed. We have also thrown away a sum of \(M\) terms, proportional to \((H)\), which corresponds entirely to disconnected graphs.

• \(U(1)_Y\) Ward-Takahashi identities for the \(\phi\)-sector of the extended-AHM: The extended master equation \[B9\] governing the \(\phi\)-sector of the extended-AHM, is identical to the master equation \[A12\] governing the \(\phi\)-sector of the (un-extended) AHM. This proves that, for each \(U(1)_Y\) WTI which is true in the AHM, an analogous \(U(1)_Y\) WTI is true for the extended-AHM. Appendix A proved \(U(1)_Y\) WTI relations among \(1-\phi\)-R \(\phi\)-sector T-Matrix elements \(T_{N,M}\), as well as \(U(1)_Y\) WTI relations among \(1-\phi\)-I \(\phi\)-sector Green’s functions \(\Gamma_{N,M}\), in the spontaneously broken AHM. Analogous \(U(1)_Y\) WTI relations among \(1-\phi\)-R \(\phi\)-sector T-Matrix elements \(T^{\text{Extended}}_{N,M}\), as well as analogous \(U(1)_Y\) WTI relations among \(1-\phi\)-I \(\phi\)-sector Green’s functions \(\Gamma^{\text{Extended}}_{N,M}\), are therefore here proved true for the spontaneously broken extended-AHM.

But there is one huge difference! The renormalization of our \(U(1)_Y\) WTI, governing \(\phi\)-sector \(T^{\text{Extended}}_{N,M}\) and \(\phi\)-sector \(\Gamma^{\text{Extended}}_{N,M}\), now includes the all-loop-orders contributions of virtual gauge bosons, \(\phi\)-scalars, ghosts, new Beyond-AHM scalars and new Beyond-AHM fermions: i.e. \(A^\mu; h, \pi, \bar{\pi}, \omega ; \Phi, \psi\) respectively.

10) Adler self-consistency relations, but now for the extended-AHM gauge theory:
\[\langle H \rangle T^{\text{Extended}}_{N,M+1+1} (p_1, p_N; 0| q_1, \ldots, q_M) \times (2\pi)^4 \delta^4 \left( \sum_{n=1}^N p_n + \sum_{m=1}^M q_m \right) q_1^2 = q_2^2 = \ldots = q_M^2 = 0 \quad \text{(B10)}
\] These prove the infra-red (IR) finiteness of the \(\phi\)-sector on-shell connected T-matrix in the extended-AHM gauge theory, with massless \(\pi\), in Lorenz gauge, in the 1-soft-pion limit.

11) 1-(\(h, \pi\)) Reducibility (1-\(\phi\)-R) and 1-(\(h, \pi\)) Irreducibility (1-\(\phi\)-I): With some exceptions, the extended \(\phi\)-sector connected amputated T-Matrix elements \(T^{\text{Extended}}_{N,M}\) can be cut apart by cutting an internal \(h\) or \(\pi\) line: they are designated 1-\(\phi\)-R. In contrast, the extended \(\phi\)-sector Green’s functions \(\Gamma^{\text{Extended}}_{N,M}\) are defined to be 1-\(\phi\)-I: i.e. they cannot be cut apart by cutting an internal \(h\) or \(\pi\) line.
\[T^{\text{Extended}}_{N,M} = \Gamma^{\text{Extended}}_{N,M} + (1 - \phi - R) \quad \text{(B11)}
\] As usual, both \(T^{\text{Extended}}_{N,M}\) and \(\Gamma^{\text{Extended}}_{N,M}\) are 1-(\(A^\mu\))-Reducible (1-A\(^\mu\)-R): i.e. they can be cut apart by cutting an internal transverse-vector \(A^\mu\) gauge-particle line.

But both \(T^{\text{Extended}}_{N,M}\) and \(\Gamma^{\text{Extended}}_{N,M}\) are also 1-\(\Phi\) Reducible (1-\(\Phi\)-R): i.e. they can be cut apart by cutting an internal \(\Phi\) scalar line.

12) \(\phi\)-sector 2-point functions, propagators and a 3-point vertex: The 2-point functions
\[T^{\text{Extended}}_{2,2} (p, -p) = \Gamma^{\text{Extended}}_{2,2} (p, -p) = \left[ \Delta_{\text{BEH}} (p^2) \right]^{-1} \quad \text{(B12)}
\] are related to the \((1h, 2\pi)\) 3-point \(\pi^2\) vertex
\[T^{\text{Extended}}_{1,2} (p, q, -p - q) = \Gamma^{\text{Extended}}_{1,2} (p, q, -p - q) \quad \text{(B13)}
\] by a 1-soft-pion theorem \[B18\]
\[\langle H \rangle T^{\text{Extended}}_{1,2} (q; 0, -q) = \left[ \Delta_{\text{BEH}} (q^2) \right]^{-1} - \left[ \Delta_{\pi} (q^2) \right]^{-1} \quad \text{(B14)}
\]
13) The Goldstone theorem, in Lorenz-gauge-extended-AHM is the \( N = 0, M = 1 \) case of (B10):

\[
\langle H \rangle T^{Extended}_{0,2}(0;0) = 0 \\
\langle H \rangle [\Delta_\pi(0)]^{-1} = 0 \\
\langle H \rangle \Gamma^{Extended}_{0,2}(0;0) = 0 \\
\]

proves that \( \pi \) is still massless in the extended-AHM, whose all-loop-orders renormalized massless-ness is protected/guaranteed by the global \( U(1)_Y \) symmetry of the physical states of the extended-AHM gauge theory after SSB.

14) \( T^{Extended; External}_{N,M+1} \) are the 1-\( \phi \)-R \( \phi \)-sector connected amputated T-Matrix elements, with one soft \( \pi(q_\mu = 0) \) attached to an external-leg, as shown in Figure 1. With the separation

\[
T^{Extended; Internal}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = T^{Extended; External}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) + T^{Extended; Internal}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) \\
\]

we have the recursive \( U(1)_Y \) T-Matrix WTI

\[
\langle H \rangle T^{Extended; Internal}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^{M} T^{Extended}_{N+1,M-1}(q_mp_1\ldots p_N;q_1\ldots \hat{q}_m\ldots q_M) - \sum_{n=1}^{N} T^{Extended}_{N-1,M+1}(p_1\ldots \hat{p}_n\ldots p_N;p_nq_1\ldots q_M) \\
\]

15) Recursive \( U(1)_Y \) WTIs for 1-\( \phi \)-I \( \phi \)-sector connected amputated extended Green’s functions \( \Gamma^{Extended}_{N,M} \) are a solution to (B17)

\[
\langle H \rangle \Gamma^{Extended}_{N,M+1}(p_1\ldots p_N;0q_1\ldots q_M) = \sum_{m=1}^{M} \Gamma^{Extended}_{N+1,M-1}(q_mp_1\ldots p_N;q_1\ldots \hat{q}_m\ldots q_M) - \sum_{n=1}^{N} \Gamma^{Extended}_{N-1,M+1}(p_1\ldots \hat{p}_n\ldots p_N;p_nq_1\ldots q_M) \\
\]

16) Goldstone theorem makes tadpoles vanish:

\[
\langle 0|\hat{h}(x=0)|0 \rangle_{Connected} = i\left[ i\Delta_{BEH}(0) \right] \Gamma^{Extended}_{1,0}(0;0) \\
\]

but the \( N = 0, M = 1 \) case of (B18) reads

\[
\Gamma^{Extended}_{1,0}(0;0) = \langle H \rangle \Gamma^{Extended}_{0,2}(0;0) = 0 \\
\]

where we have used (B15), so that tadpoles all vanish automatically, and separate tadpole renormalization is un-necessary. Since we can choose the origin of coordinates anywhere we like

\[
\langle 0|\hat{h}(x)|0 \rangle_{Connected} = 0 \\
\]

17) Renormalized gauge-independent observable \( \langle H \rangle \).

\[
\langle 0|H(x)|0 \rangle_{Connected} = \langle 0|\hat{h}(x)|0 \rangle_{Connected} + \langle H \rangle \\
\partial_\mu \langle H \rangle = 0 \\
\]