Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution

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Abstract

We show that the consequences of an introduction of a manifest time-dependence in a pseudo-Hermitian Hamiltonian $H = H(t)$ are by far less drastic than suggested by A. Mostafazadeh in Phys. Lett. B 650 (2007) 208 (arXiv:0706.1872v2 [quant-ph]). In particular, the unitarity of the evolution does not necessitate the time-independence of the metric $\eta_+ = \eta_+(t)$.

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1 Introduction

In his letter [1], Ali Mostafazadeh arrives at a very surprising assertion that a given time-dependent pseudo-Hermitian Hamiltonian operator \( H(t) \) defines a consistent and unitary quantum evolution if and only if it is quasi-stationary, i.e., if and only if it is \( \eta_+ \)-pseudo-Hermitian with respect to a time-independent metric operator \( \eta_+ \). In our present critical comment on this influential letter (used, by his author, i.a., in an extremely interesting recent discussion on the paradox of quantum brachistochrone [2]) we shall re-analyze the text and show that it relies on certain assumptions which need not be satisfied in general. In this sense we shall oppose Mostafazadeh’s conclusions and claim that the time evolution of many quantum systems can remain unitary even if their time-dependent pseudo-Hermitian Hamiltonian operators \( H(t) \) are left non-quasi-stationary.

In an introductory part of our argument (section 2) we briefly review the terminology and summarize some basic concepts and definitions. In the subsequent section 3 we address “the heart of the matter” and show why the quasistationarity of \( H(t) \) as postulated in [1] (and having even some practical relevance, say, in laser physics [3]) is not a necessary condition of the unitarity of the evolution. On an elementary two-by-two matrix example we also demonstrate that the assumption of the quasistationarity (i.e., of the time-independence of the metric) is extremely counterintuitive. Section 4 finally summarizes briefly the message of our present comment.

2 Quasi-Hermitian Hamiltonians

Scholtz et al [4] were probably the first physicists (in fact, nuclear physicists) who discovered that whenever the standard quantization recipe happens to produce a prohibitively complicated version of a realistic Hamiltonian operator \( h = h^\dagger \) in a usual Hilbert space, it is still possible to try to simplify the underlying Schrödinger equation by its mapping into another space. Thus, typically, a complicated fermionic \( h \) has been studied as isospectral to its simpler bosonic partner \( H \) while one maps \( \mathcal{H}^{(\text{ref})} \to \mathcal{H}_{\text{phys}} \) via a mere redefinition of the inner product, \( \langle \cdot | \cdot \rangle \to \langle \cdot | \cdot \rangle^+ := \langle \cdot | \eta_+ \cdot \rangle \). In this context, Eq. Nr. (2) of ref. [1] giving \( \langle \cdot | \cdot \rangle \to \langle \cdot | \cdot \rangle^+ := \langle \cdot | \eta_+ \cdot \rangle \) just updates the notation of ref. [4]. An alternative update of this type is being used by Bender et al [5] who factorize the metric \( \eta_+ = \mathcal{C} \mathcal{P} \) (where \( \mathcal{P} \) is parity) and make it unique via an artificial (or, if you wish, “physical”) constraint \( \mathcal{C}^2 = I \) imposed upon their “charge” operator \( \mathcal{C} \). In our recent comment [6] inspired by Solombrino [7] and admitting generalized \( \mathcal{P} \neq \mathcal{P}^\dagger \) we showed that in such a generalization

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1based, say, on the principle of correspondence

2\( \mathcal{H}^{(\text{ref})} = L_2(\mathbb{R}^n) \) or a similar “reference” Hilbert space endowed with the standard inner product \( \langle \cdot | \cdot \rangle \)
one has either to re-write $\eta_+ = C\mathcal{P}^\dagger$ or, alternatively, to introduce quasiparity $\mathcal{Q}$ and factorize $\eta_+ = \mathcal{P}\mathcal{Q}$. Being exposed to this long menu of alternatives we often recommend the abbreviation $\eta_+ := \Theta$ which replaces the original symbol $\mathcal{T}$ (too much reminiscent of the time reversal operator of ref. [8]) simply by its “Greek-alphabetic” version.

In this notation, our attention will solely be paid here to the quasi-Hermitian Hamiltonians [4] which obey the rule given also in ref. [1] as Eq. Nr. (3),

$$H^\dagger = \Theta H \Theta^{-1}.$$  \hfill (1)

From the same source we shall also recall the subsequent Eq. Nr. (4) re-written here, in a compactified notation with $\omega = \Theta^{1/2} = \eta_+^{1/2} = \omega^\dagger$, as

$$h = \omega H \omega^{-1}.$$  \hfill (2)

This is a similarity transformation between the auxiliary Hermitian $h = h^\dagger$ (acting in $\mathcal{H}^{(aux)}$) and the quasi-Hermitian physical $H \neq H^\dagger$ (acting in $\mathcal{H}^{phys}$). This mapping is unitary (cf. footnote Nr. 5 in [1]).

### 2.1 A two-by-two matrix example

In Eqs. Nr. (17) and (18) of ref. [1] a complex two-by-two matrix Hamiltonian has been chosen for illustrative purposes. Once we omit a trivial overall shift $q \in \mathbb{R}$ of its spectrum $E_1 = q + E$ and $E_2 = q - E$ we have

$$H_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = E \begin{pmatrix} \cos \theta & e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & -\cos \theta \end{pmatrix},$$  \hfill (3)

with the real scale factor $E := \sqrt{a^2 + bc} \in [0, \infty)$ and with the two complex angles $\theta, \varphi \in \mathbb{C}$ (as in [1] one could set $\Re(\theta) \in [0, \pi]$ and $\Re(\varphi) \in [0, 2\pi]$ where symbols $\Re(\cdot)$ and $\Im(\cdot)$ denote the real- and imaginary-part functions, respectively). Closed formulae for eigenvectors are also available and define the positive-definite metric (cf. Eqs. Nr. (19), (20) and (21) of [1]).

For our present purposes it will be sufficient to consider just a real-matrix subfamily of eq. (3) with $q = \Re(\theta) = \Im(\varphi) = 0$ and with $\varphi = \pi/2$. This establishes the correspondence between formulae of [1] and their special cases in [9]. In the resulting reduced, one-parametric family of Hamiltonians we have the purely imaginary $\theta$s so that we may set $E = 1/\cos \theta = \sin \alpha$ and arrive at the toy Hamiltonian defined in terms of a single real variable $\alpha \in (0, \pi/2)$,

$$H_{00} = \begin{pmatrix} 1 & \cos \alpha \\ -\cos \alpha & -1 \end{pmatrix}.$$  \hfill (4)
The eligible metrics remain two-parametric and have a compact form
\[ \Theta = \Theta(H_{00}) = Z \begin{pmatrix} 1 + \sin \alpha \sin \gamma & -\cos \alpha \\ -\cos \alpha & 1 - \sin \alpha \sin \gamma \end{pmatrix}. \] (5)

Although the scale factor \( Z \in \mathbb{R} \) itself can be understood as less relevant in the time-independent case \([10]\), it is necessary to pick up and fix a suitable value of the real angle \( \gamma \in [0, \pi/2) \). Its ambiguity is an unpleasant problem \([11]\). Fortunately, the solution is easy for the finite-dimensional Hamiltonians \( H \) where one simply requires the validity of the quasi-Hermiticity condition for some other operators of observables \( \mathcal{O} = \mathcal{O}_n \), \( n = 1, 2, \ldots, N \).

As long as one has \( N = 1 \) in our two-dimensional real model \((4)\), we may set
\[ \mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] (7)
(with, for simplicity, real elements) and convert eq. (6) in the single constraint
\[ (d - a) \cos \alpha = (b - c) + (b + c) \sin \alpha \sin \gamma. \] (8)

This enables us to fix \( \gamma = \gamma(a, b, c, d) \) whenever we choose \( b \neq -c \) in our auxiliary non-Hamiltonian observable \( \mathcal{O} \). In particular, for illustration purposes we may select
\[ \mathcal{O}_{00} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad b = \frac{1}{2} e^{-\varrho}, \quad c = \frac{1}{2} e^\varrho, \quad \varrho \in (0, \infty). \] (9)

with real eigenvalues = ±1 (cf. also \([12]\) in this respect) and with eqs. (6) or (8) reduced to the single elementary relation
\[ \tanh \varrho = \sin \alpha \sin \gamma \] (10)
with an easy re-insertion in eq. (5).

2.2 Bases in any number of dimensions

The change of name \( \eta_+ \rightarrow \Theta \) emphasizes that we restrict attention to the quasi-Hermitian models where there exists the positive-definite metric \( \Theta \). In comparison, the Mostafazadeh’s selection of a

\[ \text{i.e., equivalently, where the energies are real and observable} \]
broader class of pseudo-Hermitian $H$s in \[1\] seems formal, especially when just the unitarity of the quantum evolution is concerned and studied\[4\]. In a way emphasized in the context of laser optics \[3\] we may/shall simply ignore the existence of a pseudometric (denoted by symbol $\eta$ entering Eq. Nr. (1) in \[1\]) and assume the reality of the spectrum.

For a certain enhancement of clarity we shall employ here an amended Dirac’s notation \[14\] and denote the Mostafazadeh’s specific, mutually biorthogonal eigenvectors $|\psi_n\rangle$ and $|\phi_n\rangle$ as the single-ketted $|E_n\rangle$ and double-ketted $|E_n\rangle\rangle$, respectively. This enables us to omit the redundant Greek letters and to rewrite Eqs. Nr. (5) of \[1\] in the more transparent form

\[
H |E_n\rangle = E_n |E_n\rangle, \quad H^\dagger |E_n\rangle\rangle = E_n |E_n\rangle\rangle .
\]

This emphasizes the difference between $H$ and $H^\dagger$ and the asymmetry between the two symbols since $|E_n\rangle\rangle \sim \Theta |E_n\rangle$ where, by assumption, $\Theta \neq I$ is nontrivial \[3\].

One immediately concludes that the symbol $\langle a | b \rangle$ of ref. \[1\] should in fact be re-read, in our present notation, as the overlap $\langle a | b \rangle$ of the two states in the self-dual $\mathcal{H}_{\text{phys}}$ where the self-duality is nontrivial, $\Theta \neq I$. In the language of physics one could treat the kets $|E_n\rangle$ simply as “elements” of $\mathcal{H}_{\text{phys}}$ while the ketkets $|E_n\rangle\rangle$ should be understood as linear functionals in the same Hilbert space.

In the biorthogonal-basis representation of this space $\mathcal{H}_{\text{phys}} = \mathcal{H}(\Theta)$ with $\Theta \neq I$, our present modification of the notation enables us to make the respective biorthogonality and completeness relations (cf. Eqs. Nr. (6) and (7) of \[1\]) more explicit and much more transparent,

\[
\langle a | b \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |E_n\rangle \langle a | b |E_n\rangle = I .
\]

One of the important merits of this notation convention is that it emphasizes that there exists a sequence of numbers $\kappa_n \in \mathbb{C} \setminus \{0\}$ which are arbitrary free parameters. Their existence is characteristic for biorthogonal bases as it reflects the freedom\[5\] of a change of the normalization of their individual elements (cf. Eqs. Nr. (19) and (20) in \[1\] or related remarks in our recent preprint \[15\]). Precisely these parameters also enter the general formula

\[
\Theta = \Theta^{(\kappa)} = \sum_{n=0}^{\infty} |E_n\rangle\rangle \frac{1}{\kappa_n \kappa_n^*} \langle a | b \rangle .
\]

\[4\]as emphasized in \[1\], all the quantum models become unphysical for all the non-quasi-Hermitian pseudo-Hermitian $H$s. Beyond the domain of quantum theory, these operators still find applications, pars pro toto, in classical magnetohydrodynamics \[13\].

\[5\]not existing in orthogonal bases
which assigns a menu of eligible metric operators to a given $H$. In this sense, one simply assumes that in Eq. Nr. (7) of [1] we fix the choice of the basis and of the metric at once. Only then we are allowed to set all $\kappa_n = 1$ in (13).

3 Evolution in time

3.1 The amendment of the evolution law

Let us start from a remark that, originally and paradoxically, the evolution law based on Eqs. Nr. (10) of ref. [1] attracted our attention not so much by its central position and by its key relevance for the flow of the argumentation in ref. [1] but rather by a certain innocent-looking apparent inconsistency of the notation where in the elements $| \psi(t) \rangle$ and $| \phi(t) \rangle$ of a Hilbert space the kets were omitted. Suddenly (e.g., in the footnote Nr. 5 in ref. [1]), the “evolving state vectors” were written and presented as the mere unbracketed functions $\psi(t)$ and $\phi(t)$, respectively.

Our almost exaggerated attention paid to the notation helped us to reveal that the interpretation of these kets is in fact deeply ambiguous. Our point can immediately be clarified in a better notation where one recollects that in [1], both $\psi(t)$ and $\phi(t)$ were presented as elements of $\mathcal{H}_{\text{phys}}$ (cf. footnote Nr. 5 in [1] once more). This is in conflict with the fact that the formalism is already presented using a biorthogonal basis (cf. Eqs. Nr. (5), (6) and (7) in [1] or our remark made in the sequel of eq. (11) above). One concludes that although just one state is prepared at an initial time $t = 0$, it can be represented in the two different forms of a superposition over our basis. For the purposes of our forthcoming analysis this suggests the following change of the denotation of the symbols entering Eq. Nr. (10) in [1],

$$
\psi(0) = \sum_{n=0}^{\infty} | E_n(0) \rangle c_n(0) := | \psi(0) \rangle , \quad \phi(0) = \sum_{n=0}^{\infty} | E_n(0) \rangle \rangle d_n(0) := | \phi(0) \rangle \rangle .
$$

Moreover, once we abolished the Mostafazadeh’s “obligatory” assignment of letters (with his $\psi_n$ meaning our states $| E_n \rangle$ and with his $\phi_n$ meaning the corresponding linear functionals $| E_n \rangle \rangle$), we may also replace his time-dependent symbol $\langle \psi(t), \phi(t) \rangle$ by its present equivalent $\langle \langle \psi(t) | \psi(t) \rangle$. Its form properly emphasizes that we consider the single time-dependent and evolving physical state $\psi(t)$ possessing the two mathematically slightly different, “left and right” or “brabra and ket” or “functional and vector” representants $| \psi(t) \rangle$ and $| \psi(t) \rangle \rangle$, respectively.

On this background we may turn attention to $t > 0$ and consider the symbol $\langle a, b \rangle$ of ref. [1] with $a$ replaced by the evolving “left ketket” and with $b$ representing the evolving “right ket”. In such an arrangement it is obvious that for $H \neq H(t)$, the time evolution of both of these different
representations of the same state (denoted, in order to avoid confusion, by the new symbol $\Phi$) must be controlled by the different operators, viz., by $H$ and $H^\dagger$, respectively. This observation simply discourages us to postulate the evolution law in the oversimplified form of Eqs. Nr. (10) of [1]. These equations must be replaced by a more flexible double ansatz of ref. [16], with the different time-evolution operators acting to the right and to the left, respectively,
\[
|\Phi(t)\rangle = U_R(t) |\Phi(0)\rangle, \quad \langle\Phi(t)| = \langle\Phi(0)| U_L(t) .
\]

In the next step let us recollect, once more, the pullback relations mentioned in the footnote Nr. 5 of [1] and having the compact form of a definition (2) of an isospectral Hamiltonian $h = h^\dagger$ acting in $\mathcal{H}^{(aux)}$. Under a specific normalization, our biorthogonal basis in $\mathcal{H}_{phys} = \mathcal{H}^{(\Theta)}$ may be assumed adapted, as we already agreed, to a fixed $\Theta$ in such a way that $\kappa_n = 1$ at all $n$ in eq. (13) (cf. Eq. Nr. (7) in [1]). In this way we re-derive the spectral-representation formula
\[
h = \sum_{n=0}^{\infty} |\chi_n\rangle E_n \langle \chi_n | = h^\dagger, \quad |\chi_n\rangle = \omega |E_n\rangle = \omega^{-1} |E_n\rangle .
\]

The textbook wisdom becomes applicable and we can immediately deduce the time-evolution law in $\mathcal{H}^{(aux)}$. Thus, at any time-dependence in $h = h(t)$ the evolution starts from the state
\[
|\chi(0)\rangle = \sum_{n=0}^{\infty} \omega(0) |E_n(0)\rangle c_n(0) = \omega(0) |\psi(0)\rangle \equiv \sum_{n=0}^{\infty} \omega^{-1}(0) |E_n(0)\rangle d_n(0) = \omega^{-1}(0) |\phi(0)\rangle
\]
prepared in $\mathcal{H}^{(aux)}$ at $t = 0$. There is no problem with writing down its descendant existing at $t > 0$,
\[
|\chi(t)\rangle = u(t) |\chi(0)\rangle.
\]

Here, the evolution operator is determined by the standard Schrödinger equation,
\[
i\partial_t u = h(t) u(t), \quad u(0) = I.
\]

It is easy to deduce that
\[
|\Phi(t)\rangle = \omega^{-1}(t) |\chi(t)\rangle = U_R(t) |\Phi(0)\rangle, \quad |\Phi(t)\rangle = \omega(t) |\chi(t)\rangle = U_L^\dagger(t) |\Phi(0)\rangle
\]

With the insertions of $|\Phi(0)\rangle = \omega^{-1}(0) |\chi(0)\rangle$ and $|\Phi(0)\rangle = \omega(0) |\chi(0)\rangle$ we may conclude that
\[
U_R(t) = \omega^{-1}(t) u(t) \omega(0), \quad U_L^\dagger(t) = \omega^\dagger(t) u(t) [\omega^{-1}(0)]^\dagger.
\]

This means, obviously, that
\[
\langle\Phi(t)| \Phi(t)\rangle = \langle\Phi(0) | U_L(t) U_R(t) |\Phi(0)\rangle = \langle\Phi(0) | \Phi(0)\rangle .
\]

We demonstrated that the evolution is unitary.
3.2 Two-by-two model made time-dependent

In a two-level model let us introduce time \( t \) not only in the single-parametric Hamiltonian, by admitting that \( \alpha = \alpha(t) \) in \( H_{00} \) of eq. (1), but also in the complementary observable \( O_{00} \), by allowing that \( \varrho = \varrho(t) \) in eq. (9). This gives us the basic methodical guidance for a replacement of eq. (2) by its time-dependent generalization

\[
h(t) = \omega(t) H(t) \omega^{-1}(t) .
\] (19)

In full analogy with the time-independent case [4] we imagine that the operators on both sides of eq. (19) should represent the same information about the dynamics of our system. This means that we are allowed to ignore the specific additional quasi-stationarity constraints deduced from the incorrect assumptions in [1]. One comes to this conclusion with a great relief since in our example the time-dependence encoded in \( \alpha(t) \) and \( \varrho(t) \) becomes immediately transferred, via eq. (10), to the metric \( \Theta \) in eqs. (5) and to its square root \( \omega \) in eq. (20).

One of the specific merits of our choice of the example is that the latter two matrices can still be written in closed form. Indeed, in terms of the independently variable \( T = \tanh \varrho = T(t) \) and \( C = \cos \alpha = C(t) \) we may define our metric \( \Theta = \Theta(t) \) as well as the matrix \( U = U(t) \) of its (unnormalized) eigenvectors by the elementary prescriptions

\[
\Theta = Z \begin{pmatrix} 1 + T & -C \\ -C & 1 - T \end{pmatrix}, \quad U = \begin{pmatrix} T + R & C \\ -C & T + R \end{pmatrix}.
\]

Using further the abbreviations \( R = \sqrt{T^2 + C^2} \) and

\[
S = \frac{2R}{C^2 + (T + R)^2} \times \frac{1}{\sqrt{1 - R + \sqrt{1 + R}}}
\]

it is entirely straightforward to derive

\[
\omega = \begin{pmatrix} \sqrt{1 - R + SC^2} & -SC(T + R) \\ -SC(T + R) & \sqrt{1 - R + S(T^2 + R^2)} \end{pmatrix} \quad (20)
\]

i.e., the closed form of the matrix of transformation entering eqs. (2) and (17).

4 Conclusions

It is obvious that once \( \text{both} \) a quasi-Hermitian Hamiltonian \( H \) and the associated observables \( O_n \) become \emph{manifestly and simultaneously} time-dependent, one encounters an entirely new situation
because the time-dependence of the system becomes, in general, transferred to the metric $\Theta$. This means that in principle, the evolution of the quantum system ceases to be dictated solely by the Hamiltonian.

In an opposite direction, once one makes a tacit assumption that the metric $\Theta(t)$ does not carry any independent information about the changes of dynamics with time, no trace of the variation of the associated observables $O_n = O_n(t)$ would be left observable. We may conclude that this type of assumption does not seem reasonable at all. Moreover, ref. [1] remains useful as showing how such an assumption would produce severe restrictions imposed upon $H(t)$ itself.

We re-analyzed the problem since we were really unpleasantly surprised by the drastic nature of the conditions of unitarity of the time evolution as deduced in [1]. Using the modified Dirac’s notation we re-derived and corrected the key Eqs. Nr. (10) of ref. [1].

Fortunately, the situation is clarified now. Our conclusion is that even for the quasi-Hermitian models there arise no problems with the unitarity of the evolution. Briefly it is possible to summarize that whenever we have $H = H(t)$ and $O_n = O_n(t)$, we have to search for an evolution equation which depends not only on the Hamiltonian $H(t)$ but also on the changes in time which are carried by the metric $\Theta = \Theta(t)$ itself. In another perspective, in contrast to the hypotheses formulated in ref. [1], there emerge no surprising and far reaching differences between the role and/or interpretation of the time-dependent and time-independent pseudo-Hermitian Hamiltonians.

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