A NOTE ON LOCAL PROPERTIES IN PRODUCTS

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ABSTRACT. We give conditions under which a product of topological spaces satisfies some local property. The conditions are necessary and sufficient when the corresponding global property is preserved under finite products. Further examples include local sequential compactness, local Lindelöfness, the local Menger property.

1. INTRODUCTION

Conditions under which a product of topological spaces satisfies some local property have long been known in many particular instances. Results with a general flavor appeared in Preuß [P, Section 5.3] and in Hoffmann [H1, Theorem 2.2 and Remark 2.4(b)]. Notice that the terminology used by the above authors sometimes differs from the one we shall use. The results from [P, H1] have been improved, together with significant examples and applications, in Brandhorst [B] and Brandhorst and Erné [BE]. We refer to [BE] for historical remarks and examples; in particular, about how the definitions and the results generalize classical cases in special situations. Here we give a complete characterization of those spaces which are local relative to some class closed under finite products. We also deal with some classes which are not even closed under finite product.

Let $\mathcal{T}$ be a class of topological spaces. Members of $\mathcal{T}$ will be called $\mathcal{T}$-spaces. For each class $\mathcal{T}$, three local notions are defined; see [BE, H1, P] and Hoshina [H2]. A topological space $X$ is a local $\mathcal{T}$-space (resp., a basic $\mathcal{T}$-space) if, for every point $x \in X$ and every neighborhood $U$ of $x$, there is a neighborhood (resp., an open neighborhood) $V$ of $x$ such that $V \subseteq U$ and $V \in \mathcal{T}$. We sometimes say that $X$ is $\mathcal{T}$-local, instead of saying that $X$ is a local $\mathcal{T}$-space, and similarly for $\mathcal{T}$-basic. Under the Axiom of Choice (AC) a space is $\mathcal{T}$-local if and only if every point of $x$ has a neighborhood base consisting of $\mathcal{T}$-subspaces. Here a $\mathcal{T}$-subspace is a subspace which belongs to $\mathcal{T}$; similarly, a $\mathcal{T}$-neighborhood of some

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point is a neighborhood of that point which belongs to $\mathcal{T}$. Again using AC, a space is $\mathcal{T}$-basic if and only if it has an open base consisting of $\mathcal{T}$-subspaces. However, we shall try to avoid the use of AC as much as possible; see Remark 6.5. As a rougher notion, a *local* $\mathcal{T}$-space is a space such that each point has at least one neighborhood which is a $\mathcal{T}$-space. In many cases, especially assuming some separation axiom, localness and local$_1$ness coincide, and sometimes all the three above local notions coincide, but sometimes not. See Lemma 1.1(d), where another local property frequently equivalent to $\mathcal{T}$-localness shall be mentioned.

Characterizations of basic and local $\mathcal{T}$-spaces appear in [H1, P], in the case when $\mathcal{T}$ is closed with respect both to products and to images of surjective continuous functions. A characterization under weaker conditions appears in [BE, Theorem 2.4]. In the quoted theorem $\mathcal{T}$ has to be closed under finite products, and a further condition has to be satisfied. We prove here a more general statement which applies to *every* class $\mathcal{T}$ which is closed under finite products. The proof is perhaps simpler. Then a characterization is given for certain $\mathcal{T}$ which are not even closed under finite products. Examples include local sequential compactness and local Lindelöfness. Moreover, we show that the assumption that $\mathcal{T}$ is closed with respect to images of surjective continuous functions can be considerably weakened. We reformulate many results in such a way that, seemingly, the Axiom of Choice is not needed. No separation axiom is used, either, unless explicitly stated otherwise. All products under consideration are endowed with the Tychonoff topology, the coarsest topology making all the projections continue. Most results would change dramatically, when considering the box topology, or intermediate topologies.

The next lemma is trivial, we shall use it (especially clause (f)) in many examples, generally without explicit mention. $T_3$ means *regular* and *Hausdorff*, while we do not assume that regular implies Hausdorff.

**Lemma 1.1.** Let $\mathcal{T}$ be a class of topological spaces.

(a) $\mathcal{T}$-basic implies $\mathcal{T}$-local and $\mathcal{T}$-local implies $\mathcal{T}$-local$_1$.

(b) A $\mathcal{T}$-space is $\mathcal{T}$-local$_1$. Hence, for every class $\mathcal{T}$, $\mathcal{T}$-local and $\mathcal{T}$-local$_1$ are equivalent if and only if every $\mathcal{T}$-space is $\mathcal{T}$-local.

(c) If $\mathcal{T}$ is open-hereditary, then all the three local properties coincide, in particular any $\mathcal{T}$-space is $\mathcal{T}$-basic and $\mathcal{T}$-local.

(d) If $\mathcal{T}$ is closed-hereditary, then, in a regular topological space, $\mathcal{T}$-localness and $\mathcal{T}$-local$_1$ness are equivalent, and are also equivalent to the following.
(L) For every point $x$ and every neighborhood $U$ of $x$, there is an open neighborhood $V$ of $x$ such that $V \subseteq U$ and $V \in \mathcal{T}$.

(e) In particular, if $\mathcal{T}$ is closed-hereditary, then a regular $\mathcal{T}$-space is $\mathcal{T}$-local.

(f) In both (d) and (e) above, the assumption that $\mathcal{T}$ is closed-hereditary can be weakened to $\mathcal{T}$ being hereditary with respect to regular closed subsets.

2. A weak assumption

We shall use almost everywhere the following assumption (W).

(W) $\mathcal{T}$ is a class of topological spaces which satisfies the following properties.

(W1) $\mathcal{T}$ is closed under homeomorphic images.

(W2) Whenever $A, B$ are arbitrary topological spaces, $a \in A$, $b \in B$ and $T \subseteq A \times B$ is a $\mathcal{T}$-neighborhood of $(a, b)$, then there is $S \subseteq A$ which is a $\mathcal{T}$-neighborhood of $a$.

(W3) Whenever $A, B$ are nonempty topological spaces, $B'$ is a nonempty open subset of $B$, $A \times B' \subseteq T \subseteq A \times B$ and $T \in \mathcal{T}$, then $A \in \mathcal{T}$.

Notice that, in particular, (W3) implies the following weaker property.

(W3') If $A \times B \in \mathcal{T}$, for some nonempty $A, B$, then $A \in \mathcal{T}$.

In particular (provided that $\mathcal{T}$ contains at least one nonempty space), (W1) and (W3') imply that every one-element space is a $\mathcal{T}$-space. If $\mathcal{T}$ is closed under images of continuous surjections, then (W) is verified. Indeed, under that assumption, (W1) is trivial and, as far as (W2) and (W3) are concerned, it is enough to consider $\pi_1(T)$, where $\pi_1$ is the canonical projection onto the first factor. In another direction, (W) is verified also in case $\mathcal{T}$ is hereditary and closed under homeomorphic images. In this case it is enough to consider $T \cap (A \times \{b\})$, where, to get (W3), we pick any $b \in B'$. If we are working in the context of $T_1$ spaces (i.e., all spaces in (W2) and (W3) are assumed to be $T_1$) then it is enough to assume that $\mathcal{T}$ is closed-hereditary and closed under homeomorphic images. In particular, in the context of $T_1$ spaces, the class $\mathcal{T}$ of all normal spaces satisfies (W). Hence Property (W) seems to be definitely very Weak. A few more conditions implying (W) shall be discussed near the end of this note.

If $\prod_{i \in I} X_i$ is a product of topological spaces and $J \subseteq I$, then $\prod_{i \in J} X_i$ is called a subproduct of $\prod_{i \in I} X_i$. If $I \setminus J$ is finite, then (with a slight abuse of terminology) we shall call $\prod_{i \in J} X_i$ a cofinite subproduct of $\prod_{i \in I} X_i$. 
Lemma 2.1. Suppose that $X$ is a nonempty product of topological spaces and $\mathcal{T}$ is a class of topological spaces closed under homeomorphic images.

(a) If $\mathcal{T}$ satisfies (W2) and $X$ is $\mathcal{T}$-local, then all factors and all subproducts are $\mathcal{T}$-local.

(b) If $\mathcal{T}$ satisfies (W3), $X$ contains a set $T \in \mathcal{T}$ and $T$ contains some nonempty open set, then some cofinite subproduct belongs to $\mathcal{T}$.

(c) If $\mathcal{T}$ is closed under finite products, $\mathcal{T}$ satisfies (W3'), $X \in \mathcal{T}$ and all factors of $X$ are $\mathcal{T}$-local, then $X$ is $\mathcal{T}$-local.

(d) If $\mathcal{T}$ is closed under finite products, then the product of two local $\mathcal{T}$-spaces is still $\mathcal{T}$-local.

Proof. (a) Let $X = \prod_{i \in I} X_i$ be nonempty and $\mathcal{T}$-local and let $\emptyset \neq J \subseteq I$. Since $\prod_{i \in J} X_i$ is nonempty, then also $Y = \prod_{i \in J} X_i$ is nonempty. We have to show that $Y$ is $\mathcal{T}$-local (we allow $|J| = 1$, so this case takes into account factors). Let $H = I \setminus J$. If $H = \emptyset$, there is nothing to prove, hence we can suppose that $H \neq \emptyset$. Let $B = \prod_{i \in H} X_i$. Notice that $X$ is homeomorphic to $Y \times B$, hence $Y \times B$ is $\mathcal{T}$-local, since $\mathcal{T}$ is closed under homeomorphisms.

Let $a \in Y$ and suppose that $A$ is a neighborhood of $a$ in $Y$. Since $X = \prod_{i \in I} X_i$ is nonempty, then also $B = \prod_{i \in H} X_i$ is nonempty; pick $b \in B$. Now $A \times B$ is a neighborhood of $(a, b)$ in $Y \times B$, which is $\mathcal{T}$-local, hence there is $T \subseteq A \times B$ which is a $\mathcal{T}$-neighborhood of $(a, b)$. By (W2), there is $S \subseteq A$ which is a $\mathcal{T}$-neighborhood of $a$. The above argument works for every $a \in Y$ and every neighborhood $A$ of $a$, hence we get that $Y$ is $\mathcal{T}$-local.

(b) Let $X = \prod_{i \in I} X_i$, hence $T$ contains a basic nonempty open set of the form $\prod_{i \in J} Y_i$, where each $Y_i$ is open in $X_i$, and $Y_i = X_i$, for every $i \in J = I \setminus F$, with $F$ finite. If $F = \emptyset$ then $T = X$ and we are done, so suppose that $F \neq \emptyset$. Take $A = \prod_{i \in J} X_i$ and $B = \prod_{i \in F} X_i$. Since $\mathcal{T}$ is closed under homeomorphisms, we lose no generality if we identify $X$ with $A \times B$. Taking $B' = \prod_{i \in F} Y_i$, we have that $A \times B' \subseteq T \subseteq A \times B$, hence we can apply (W3) to get that the cofinite subproduct $A$ belongs to $\mathcal{T}$.

(c) If $x = (x_i)_{i \in I} \in X$ and $U$ is a neighborhood of $x$, then $U$ contains a basic open set of the form $\prod_{i \in I} U_i$, where $x_i \in U_i$ for every $i \in I$, and $U_i = X_i$, for all indices except perhaps for indices in a finite set $F$. By (W3') and closure under homeomorphisms, $C = \prod_{i \in I \setminus F} X_i \in \mathcal{T}$. Since each factor is $\mathcal{T}$-local, then, for every $i \in F$, $x_i$ has a $\mathcal{T}$-neighborhood $V_i \subseteq U_i$. Let $V_i = X_i$ for $i \notin F$. Then $\prod_{i \in I} V_i$ is a neighborhood of $x$ contained in $U$. Since $\prod_{i \in I} V_i$ is homeomorphic
to the finite product $C \times \prod_{i \in F} V_i$ and since $\mathcal{T}$ is closed under finite products and homeomorphisms, then $\prod_{i \in I} V_i \in \mathcal{T}$, hence $\prod_{i \in I} V_i$ is a neighborhood of $x$ as requested.

(d) is similar and easier. □

Notice that (a) in Lemma 2.1 holds also in case we give to $\prod_{i \in I} X_i$ the box topology, but this is not necessarily the case for (b) and (c).

### 3. Properties closed under products

**Theorem 3.1.** Suppose that $X$ is a nonempty product and $\mathcal{T}$ is a class of topological spaces closed under finite products and satisfying (W). Then the following conditions are equivalent (conditions marked with an asterisk are equivalent under the further assumption that every $\mathcal{T}$-space is $\mathcal{T}$-local).

1. $X$ is $\mathcal{T}$-local.
2. Each factor is $\mathcal{T}$-local and some cofinite subproduct is a $\mathcal{T}$-space.
3. *Some cofinite subproduct is a $\mathcal{T}$-space and each of the remaining factors are $\mathcal{T}$-local.*

If, in addition, $\mathcal{T}$ is closed under arbitrary products, then the preceding conditions are also equivalent to the following ones.

4. Every countable subproduct is $\mathcal{T}$-local.
5. Each factor is $\mathcal{T}$-local and all but a finite number of factors are $\mathcal{T}$-spaces.
6. *All but a finite number of factors are $\mathcal{T}$-spaces and the remaining factors are $\mathcal{T}$-local.*

**Proof.**

(1) $\Rightarrow$ (2) follows from Lemma 2.1(a)(b).

(2) $\Rightarrow$ (1) By Lemma 2.1(c), the cofinite subproduct given by (2) is $\mathcal{T}$-local, and then $X$ is homeomorphic to a finite product of $\mathcal{T}$-local spaces, hence $\mathcal{T}$-local, by Lemma 2.1(d).

(2) $\Rightarrow$ (3) is trivial.

If (3) holds and every $\mathcal{T}$-space is $\mathcal{T}$-local, then the cofinite subproduct given by (3) is $\mathcal{T}$-local, hence every factor is $\mathcal{T}$-local, by Lemma 2.1(a). Thus (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (5) follows by (W3′) and (W1); (5) $\Rightarrow$ (2) is immediate from the additional assumption. Hence, under the additional assumption, (1), (2) and (5) are equivalent.

(1) $\Rightarrow$ (4) follows from Lemma 2.1(a).

If (4) holds, then, again by 2.1(a), all factors are $\mathcal{T}$-local. Suppose by contradiction that (4) holds and (5) fails, thus there are infinitely many factors which are not $\mathcal{T}$-spaces. Choose a countable subfamily.
By (4), the subproduct of the members of such a family is $T$-local. Applying the already proved implication $(1) \Rightarrow (5)$ to this countable subproduct, we get that all but finitely many members of the subfamily are $T$-spaces, a contradiction.

The equivalence of (5) and (6) is immediate from the assumption that every $T$-space is $T$-local. □

Notice that the equivalence of (1) and (2) above improves [BE, Theorem 2.4]. This is because the assumptions in [BE, Theorem 2.4] imply that $T$ is closed under finite products, and, under the same assumptions, the last conclusion in [BE, Theorem 2.4] is equivalent to the product having a cofinite subproduct in $T$.

The versatility of Theorem 3.1 and the broad range of validity of Property (W) are shown by the samples presented in the next two corollaries. In some cases the results are well-known. Further examples can be found in [BE]; in some cases the results here are slightly more general. Following [BE], if $\kappa$ is an infinite cardinal, we denote by $T_{\kappa}$ the class of all spaces which can be obtained as the union of $< \kappa$ many $T$-spaces. Notice that if $T$ is closed under finite products, then $T_{\kappa}$ is closed under finite products, too.

**Corollary 3.2.** A nonempty product of topological spaces is locally Hausdorff if and only if all but finitely many factors are Hausdorff and all the remaining factors are locally Hausdorff. The same holds when “Hausdorff” is replaced by any one of the following: $T_3$, regular, Tychonoff.

If we work in the context of regular spaces, the same applies to compact, sequentially pseudocompact, bounded, $\lambda$-bounded, $D$-compact, $D$-feebly compact (for some given ultrafilter $D$). Here and below we can also consider the conjunction of any set of the above properties, in particular, simultaneous $D$-compactness, for $D$ belonging to a given set of ultrafilters.

Without assuming separation axioms, a nonempty product of topological spaces is locally $D$-compact if and only if all factors are locally $D$-compact and all but finitely many factors are $D$-compact. The same applies when “$D$-compact” is replaced by any of the above mentioned properties, as well as by connected, path-connected, $H$-closed.

Relative to any of the above properties a nonempty product is local if and only if every countable subproduct is local.

If $\kappa$ is an infinite cardinal, a nonempty product is locally $\kappa$-sequentially compact if and only if all factors are locally $\kappa$-sequentially compact and some cofinite subproduct is $\kappa$-sequentially compact. The same applies
when “κ-sequentially compact” is replaced by $T_\kappa$ (if $T$ is closed under finite products and $T_\kappa$ satisfies (W)), or “of cardinality $< \kappa$”.

Notice that, for example, a Hausdorff compact space is locally compact, but this is not necessarily true without assuming the Hausdorff property. Hence, in case we assume no separation axiom, we get only the weaker statements in the third paragraph of Corollary 3.2. In most cases the Hausdorff property is not enough and regularity is needed. As an example, if some space is $D$-feebly compact, then the closure of every open set is $D$-feebly compact, that is, $D$-feebly compactness is hereditary with respect to regular closed sets. Hence, by Lemma 1.1(f), a regular $D$-feebly compact space is locally $D$-feebly compact, but, again, this is not necessarily the case, without assuming some separation axiom. Notice that in the context of Tychonoff spaces, $D$-feebly compact spaces are usually called $D$-pseudocompact.

For certain properties, some slightly more refined results can be obtained. Local sequential compactness shall be dealt with in the next section.

**Corollary 3.3.** (a) A nonempty product is locally metrizable if and only if all but countably many factors are one-element, all but finitely many factors are metrizable and the remaining factors are locally metrizable. In particular, a nonempty product is locally metrizable if and only if each subproduct by $\leq \omega_1$ factors is locally metrizable.

(b) A nonempty product is locally finite if and only if all but a finite number of factors are one-element spaces and the remaining factors are locally finite. A nonempty product is locally finite if and only if each countable subproduct is locally finite.

The same applies when “finite” is replaced by either “countable”, or “of cardinality $< \kappa$”, if $\omega \leq \kappa \leq 2^\omega$ (of course, this adds nothing, if the Continuum Hypothesis holds).

**4. Local sequential compactness**

We first present another corollary of Theorem 3.1. It deals with the general situation in which a product belongs to $T$ if and only if all subproducts by a small number of factors belong to $T$.

**Corollary 4.1.** Suppose that $T$ is a class of topological spaces closed under finite products, $T$ satisfies (W) and there is some cardinal $\kappa > \omega$ such that a product belongs to $T$ if and only if every subproduct by $< \kappa$ factors belongs to $T$.

If $X = \prod_{i \in I} X_i$ is a nonempty product, then the following conditions are equivalent.
(I) $X$ is $\mathcal{T}$-local.
(II) Every subproduct by $< \kappa$ factors is $\mathcal{T}$-local.

Proof. (I) $\Rightarrow$ (II) follows from Lemma 2.1(a).
We shall show that (II) implies Condition (2) in Theorem 3.1. If (II) holds, then all factors are $\mathcal{T}$-local, again by Lemma 2.1(a). Arguing as in the last part of the proof of Theorem 3.1 and since $\kappa$ is uncountable, we get that all but a finite number of factors are $\mathcal{T}$-spaces. Let $J$ be the set of those factors which are in $\mathcal{T}$. By assumption, any subproduct $\prod_{i \in H} X_i$ of $X$ such that $|H| < \kappa$ is $\mathcal{T}$-local, in particular, this happens if $H \subseteq J$. By Theorem 3.1(1) $\Rightarrow$ (2) applied to the product $\prod_{i \in H} X_i$, we get that $\prod_{i \in H'} X_i$ is a $\mathcal{T}$-space, for some $H'$ cofinite in $H$. If $H \subseteq J$, then $X_i$ is a $\mathcal{T}$-space, for $i \in H \setminus H'$, hence, since, by assumption, $\mathcal{T}$ is closed under finite products, $\prod_{i \in H} X_i$ is a $\mathcal{T}$-space. Since this happens for every $H \subseteq J$ such that $|H| < \kappa$, we get from the assumption on $\mathcal{T}$ that $\prod_{i \in J} X_i$ belongs to $\mathcal{T}$. Thus 3.1(2) holds. □

Corollary 4.2. Let $X = \prod_{i \in I} X_i$ be a nonempty product. Then the following conditions are equivalent.

(1) $X$ is locally sequentially compact;
(2) each factor is locally sequentially compact and some cofinite subproduct is sequentially compact;
(3) each factor is locally sequentially compact and there is a cofinite $J \subseteq I$ such that whenever $J' \subseteq J$ and $|J'| \leq s$, then $\prod_{i \in J'} X_i$ is sequentially compact;
(4) all subproducts by $\leq s$ factors are locally sequentially compact;
(5) ($h = s$) all factors are locally sequentially compact, all but a finite number of factors are sequentially compact and the set of factors with a nonconverging sequence has cardinality $< s$.
(6) ($h = s$, for $T_1$ spaces) all factors are locally sequentially compact, all but a finite number of factors are sequentially compact, and the set of factors with more than one point has cardinality $< s$.
(7) ($h = s$, for $T_3$ spaces) the set of factors with more than one point has cardinality $< s$, all but a finite number of factors are sequentially compact, and the remaining factors are locally sequentially compact.

Proof. In [L, Corollary 6.4] we have proved that a product is sequentially compact if and only if all subproducts by $\leq s$ factors are sequentially compact. See [L] for the definition of $s$, $h$ and further references.

(1) $\Leftrightarrow$ (2) is a particular case of the corresponding equivalence in Theorem 3.1.
(2) ⇔ (3) follows from [L, Corollary 6.4].
(1) ⇔ (4) follows from [L, Corollary 6.4] and Corollary 4.1 with
κ = s +.
In [L, Corollary 6.6] we have proved that if h = s, then a product is
sequentially compact if and only if all factors are sequentially compac
t and the set of factors with a nonconverging sequence has cardinality
< s. This implies (2) ⇔ (5).
(5) ⇔ (6) follows from the fact that a T_1 space in which every se-
uence converges is necessarily a one-point space.
(6) ⇔ (7) follows from the fact that a T_3 sequentially compact space
is locally sequentially compact. □

5. SOME CLASSES WHICH ARE NOT CLOSED UNDER PRODUCTS

In order to work with classes which are not necessarily closed under
products, we shall consider the following property of some class T.

(S) There are a class S of topological spaces and an infinite cardinal
κ such that a nonempty product \( \prod_{i \in I} X_i \) belongs to T if and
only if I can be written as a disjoint union \( I = J \cup K \) in such
a way that \(|J| < \kappa\), \( \prod_{i \in J} X_i \) is a T-space and \( \prod_{i \in K} X_i \) is an
S-space. We also require that S is closed under homeomorphic
images and under taking cofinite subproducts.

In the above condition we allow both \( J = \emptyset \) and \( K = \emptyset \). This is
consistent, since if T satisfies (W3'), then any one-element space is a
T-space. Moreover, “S being closed under cofinite subproduct” can
be interpreted in a sense that it implies that any one-element space
belongs to S. In particular, (S) implies that every S-space is a T-space
and, more generally, that the product of a T-space with an S-space is
a T-space. Hence also the product of a T-space with finitely many
S-spaces is a T-space. If not otherwise mentioned, we do not require
that S satisfy any special further property.

However, we should mention that if S satisfies the additional as-
sumption that a nonempty product belongs to S if and only if each
factor belongs to S then a nonempty product belongs to T if and only
if every subproduct by \( \leq \kappa \) factors belongs to T. Indeed, if the latter is
the case, we cannot have \( \kappa \)-many factors failing to be S-spaces, hence
the product is a T-space, by (S).

Theorem 5.1. Suppose that T is a class of topological spaces and T
satisfies (W) and (S), as given by S and κ. If \( X = \prod_{i \in I} X_i \) is a
nonempty product, then the following conditions are equivalent.
(1) \( X \) is T-local.
Both the following conditions hold.

(a) All subproducts of $X$ by $< \kappa$ factors are $T$-local, and

(b) the index set $I$ can be partitioned into two disjoint subsets as $I = H \cup K$ in such a way that $|H| < \kappa$ and $\prod_{i \in K} X_i$ is an $S$-space.

The index set $I$ can be partitioned into two disjoint subsets as $I = H \cup K$ in such a way that $|H| < \kappa$, $\prod_{i \in K} X_i$ is an $S$-space and $\prod_{i \in H \cup F} X_i$ is $T$-local, for every finite $F \subseteq I$.

If $S$ satisfies the additional assumption that a nonempty product belongs to $S$ if and only if each factor belongs to $S$, then the preceding conditions (1)-(3) are equivalent to the following.

(4) All subproducts by $\leq \kappa$ factors are $T$-local.

Proof. If (1) holds, then each subproduct is a $T$-space by Lemma 2.1(a), hence (2)(a) holds. Moreover, by Lemma 2.1(b), some cofinite subproduct is a $T$-space, hence (2)(b) follows from (S), since if $F$ is finite and $|J| < \kappa$ then $|J \cup F| < \kappa$, $\kappa$ being infinite.

(2) $\Rightarrow$ (3) is trivial.

Suppose that (3) holds, $x = (x_i)_{i \in I} \in X$ and $U$ is a neighborhood of $x$. Thus $U$ contains a basic neighborhood of the form $\prod_{i \in I} U_i$, where $U_i = X_i$, except for those $i$ in some finite set $F \subseteq I$. If $H$ and $K$ are given by (3), then, by the last requirement in (S), $\prod_{i \in K \setminus F} X_i$ is an $S$-space. By (3), the subproduct $X' = \prod_{i \in H \cup F} X_i$ is $T$-local. Consider the neighborhood $U' = \prod_{i \in H \cup F} U_i$ of $x' = (x_i)_{i \in H \cup F}$ in $X'$. Since $X'$ is $T$-local, we get some $T \in T$ such that $x' \in T \subseteq U'$. By (S), $T \times \prod_{i \in K \setminus F} X_i$ is a $T$-space and, modulo the natural homeomorphism, it is a neighborhood of $x$ contained in $U$. Hence we have proved that $X$ is $T$-local, that is (1) holds.

Thus (1)-(3) are equivalent.

(1) $\Rightarrow$ (4) follows again by Lemma 2.1(a).

We shall conclude the proof by showing that (4) implies (2), under the additional assumption. The implication (4) $\Rightarrow$ (2)(a) is trivial. In order to show (2)(b), in view of the additional hypothesis, it is enough to show that the set of all factors which are not $S$-spaces has cardinality $< \kappa$. Suppose by contradiction that $J \subseteq I$, $|J| = \kappa$ and $X_i \not\in S$, for every $i \in J$. By (4), the subproduct $\prod_{i \in J} X_i$ is $T$-local, but then we get a contradiction by applying (1) $\Rightarrow$ (2)(b) to that subproduct. □

If $\prod_{i \in I} X_i$ is a product of topological spaces and $J \subseteq I$, we shall say, again with some abuse of terminology, that a product $\prod_{i \in H} X_i$ is a finite superproduct of $(X_i)_{i \in J}$ if $H = J \cup F$, for some finite $F \subseteq I$.  

Corollary 5.2. Suppose that \( n < \omega \) and \( X \) is a nonempty product. Then the following conditions are equivalent.

1. \( X \) is locally finally \( \omega_n \)-compact.
2. All but \( < \omega_n \) factors are compact, and any finite superproduct of the set of noncompact factors is locally finally \( \omega_n \)-compact.
3. Every subproduct by \( \leq \omega_n \) factors is locally finally \( \omega_n \)-compact.
4. (for \( T_2 \) spaces) All but \( < \omega_n \) factors are compact, and the product of the noncompact factors is locally finally \( \omega_n \)-compact.

If \( \lambda \) is a strong limit cardinal with \( \text{cf} \lambda \geq \omega_n \), then all the above conditions hold when final \( \omega_n \)-compactness is everywhere replaced by \([\omega_n, \lambda]\)-compactness and compactness is replaced by initial \( \lambda \)-compactness (but the separation assumption in (4) should be \( T_3 \)).

Proof. Immediate from Theorem 5.1 and [L, Theorems 4.1 and 4.3]. □

Notice that \( \omega_1 \)-final compactness is the same as Lindelöfness. Since the product of countably many copies of \( \omega \) with the discrete topology is Lindelöf and locally Lindelöf, but the product of uncountably many copies of \( \omega \) is not Lindelöf (hence not locally Lindelöf, either), we get that “\( \leq \omega_1 \)” in Condition (3) above cannot be improved to “\( < \omega_1 \)”.

However, we do not know whether Corollary 5.2 can be improved, say, in the case of Lindelöfness, to the following. A product is locally Lindelöf if and only if all but countably many factors are compact, all but finitely many factors are Lindelöf and every finite subproduct is locally Lindelöf. We expect the above statement to be false, in general.

Again applying Theorem 5.1 in this case together with [L, Corollary 5.3 and Propositions 5.1 and 5.2], we get the following.

Corollary 5.3. If \( X \) is a nonempty product, then the following conditions are equivalent.

1. \( X \) is locally Menger.
2. All but countably many factors are compact, and any finite superproduct of the set of non Menger factors is locally Menger.
3. Every subproduct by \( \leq \omega_1 \) factors is locally Menger.
4. (for \( T_2 \) spaces) All but countably many factors are Menger, and the product of the non Menger factors is locally Menger.

All the above conditions hold when Menger is everywhere replaced by either the Rothberger property, or the Rothberger property for countable covers, and compactness by supercompactness.

6. Further remarks

All the above arguments, with the obvious modifications, can be applied also to the “basic” and the “local1” case.
Proposition 6.1. Lemma 2.1, Theorems 3.1 and 5.1 and Corollary 4.1 hold with “local” replaced everywhere by either “basic” or “local1”, except that in the “basic” case Condition (W2) should be replaced everywhere by the following Condition (W2O), and (W) should be modified accordingly, that is, we should consider (W_O), the conjunction of (W1), (W2O) and (W3).

(W2O) Whenever \( A, B \) are topological spaces, \( a \in A, b \in B \) and \( T \subseteq A \times B \) is an open \( T \)-neighborhood of \((a,b)\), then there is \( S \subseteq A \) which is an open \( T \)-neighborhood of \( a \).

Let us say that \( \mathcal{T} \) satisfies (C) if \( \mathcal{T} \) is closed under images of continuous surjection. As we mentioned, (C) implies (W). It is easy to see that if \( \mathcal{T} \) satisfies (C), then the image of a local \( \mathcal{T} \)-space under a continuous open map is still a local \( \mathcal{T} \)-space. In order to get the above conclusion, it is not enough to assume (W) in place of (C). E.g., the image of a Hausdorff space (hence locally Hausdorff) is not necessarily locally Hausdorff. The example is classical: take two disjoint copies of the unit real interval and pairwise identify the copies of 0, as well as the copies of \( 1/n \), for each \( n > 0 \).

However, there are conditions weaker than (C) which still imply that images of local \( \mathcal{T} \)-spaces under open continuous maps are \( \mathcal{T} \)-local.

\((C^{-})\) Whenever \( X \) is a topological space, \( T \subseteq X \) is a subspace, \( T \in \mathcal{T} \) and \( \pi : X \to Y \) is a continuous open surjection, then \( \pi(T) \in \mathcal{T} \).

\((C^{=}\mathcal{T})\) Whenever \( X \) is a topological space, \( T \subseteq X \) contains some open set of \( X \) and \( \pi : X \to Y \) is a continuous open surjection, then \( \pi(T) \in \mathcal{T} \).

\((C^{=}\mathcal{O})\) Whenever \( X \) is a topological space, \( x \in T \subseteq X \), \( T \in \mathcal{T} \) is a \( T \)-neighborhood of \( x \) in \( X \) and \( \pi : X \to Y \) is a continuous open surjection, then \( \pi(x) \) has some \( T \)-neighborhood.

Notice that (C) \( \Rightarrow \) (C-) \( \Rightarrow \) (C=) \( \Rightarrow \) (C=\mathcal{T}) \( \Rightarrow \) (W2) and (C=) \( \Rightarrow \) (W). Consider also the following property (C=\mathcal{O}), which implies (W2O).

\((C=\mathcal{O})\) Whenever \( X \) is a topological space, \( x \in T \subseteq X \), \( T \in \mathcal{T} \) is an open neighborhood of \( x \) in \( X \) and \( \pi : X \to Y \) is a continuous open surjection, then \( \pi(x) \) has some open neighborhood in \( \mathcal{T} \).

Lemma 6.2. If \( \mathcal{T} \) is a class of topological spaces satisfying (C=), then the image of any local (resp., local1) \( \mathcal{T} \)-space under a continuous open surjection is a local (resp., local1) \( \mathcal{T} \)-space.

If \( \mathcal{T} \) is a class of topological spaces satisfying (C=\mathcal{O}), then the image of any basic \( \mathcal{T} \)-space under a continuous open surjection is a basic \( \mathcal{T} \)-space.
Remark 6.3. We have usually worked in the class of arbitrary topological spaces, however, essentially all the above definitions and results can be considered as restricted to some special class, e.g., $T_1$, Hausdorff or Tychonoff spaces. Seemingly, we can allow also spaces with a richer structure, e.g., topological groups. We only need an ambient in which it makes sense to talk of (arbitrary) products, and, if there is more structure other than topology, the topological Tychonoff product agrees with the product of the structure. If we work in a specific ambient, say, of Hausdorff spaces, everything should be interpreted relative to that ambient; for example, in that context, a class $T$ is “closed under images of surjective continuous functions” if whenever $f : X \to Y$ is continuous and surjective, $X \in T$ and $X$ and $Y$ are Hausdorff, then $Y \in T$. For example, the class of Hausdorff compact spaces is closed under images of surjective continuous functions in the Hausdorff context, but not in the context of arbitrary topological spaces.

Remark 6.4. It seems that, whenever we use the assumption that $T$ is closed under finite products, we can do with the following weaker condition.

(FP) Whenever $A, B \in T$ and $x \in A \times B$, then $x$ has a neighborhood in $T$.

This remark applies, e.g., to Lemma 2.1(c)(d), Theorem 3.1(1)-(3) and Corollary 4.1. Notice that (FP) can be reformulated as “the product of two $T$-spaces is $T$-local\textsubscript{1}”. Notice also that if $T$ is such that every $T$-space is $T$-local, then (FP) is equivalent to the assertion that the product of two $T$-local spaces is $T$-local. We know no application of the above remarks, hence we have kept the statements in the simpler (but less general) form.

Remark 6.5. Concerning our use of the Axiom of Choice (AC), as the results are formulated, it seems unnecessary in the statements and proofs of Lemmas 1.1, 2.1, 6.2, Theorems 3.1 and 5.1 (except for 3.1(4) and 5.1(4)) and in the corresponding parts of Proposition 6.1. The use of AC seems to be essential in most examples and applications.

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