SEVERI DEGREES IN COGENUS 3

J. HARRIS AND R. PANDHARIPANDE

0. Introduction

0.1. Summary. Denote by $\mathcal{P}(\lceil \rceil)$ the linear system of degree $d$ curves in the complex projective plane $\mathbb{P}^2$. $\mathcal{P}(\lceil \rceil)$ is a projective space of dimension $\left( \frac{(d+2)}{2} \right) - 1$. Let $\mathcal{N}(\lceil \rceil) \subset \mathcal{P}(\lceil \rceil)$ be the subset corresponding to reduced, nodal curves with exactly $n$ nodes. $\mathcal{N}(\lceil \rceil)$ is empty unless $0 \leq n \leq \left( \frac{d}{2} \right)$. Points of $\mathcal{N}(\lceil \rceil)$ may correspond to reducible curves. If nonempty, $\mathcal{N}(\lceil \rceil)$ is a quasi-projective subvariety of pure codimension $n$ in $\mathcal{P}(\lceil \rceil)$. Let $\overline{\mathcal{N}}(n, d)$ denote the closure of $\mathcal{N}(\lceil \rceil)$ in $\mathcal{P}(\lceil \rceil)$. In this paper, formulas for the degree of $\overline{\mathcal{N}}(n, d)$ for $n = 1, 2, 3$ are computed:

\begin{align*}
  f_1(d) &= 3(d - 1)^2, \quad (1) \\
  f_2(d) &= \frac{3}{2}(d - 1)(d - 2)(3d^2 - 3d - 11), \quad (2) \\
  f_3(d) &= \frac{9}{2}d^5 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525, \quad (3)
\end{align*}

$\forall d \geq 1$, $\deg(\overline{\mathcal{N}}(1, d)) = f_1(d)$

$\forall d \geq 3$, $\deg(\overline{\mathcal{N}}(2, d)) = f_2(d)$

$\forall d \geq 3$, $\deg(\overline{\mathcal{N}}(3, d)) = f_3(d)$.

These formulas are classical. The computation presented here is new. The method involves the geometry of the Hilbert scheme of points in $\mathbb{P}^2$ and the Bott residue formula (following the technique developed in [E-S]). The most successful methods for obtaining cogenus formulas

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appear in [V]. Via a sophisticated singularity analysis, I. Vainsencher obtains the above results and the following further cogenus formulas:

\[
\begin{align*}
f_4(d) &= \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^6 - 642d^5 - 2529d^4 + \frac{37881}{8}d^3 + \frac{18057}{4}d - 8865, \\
f_5(d) &= \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5 + \frac{128859}{8}d^4 + 59097d^3 - 3528381d^2 - 946929d + 153513, \\
f_6(d) &= \frac{81}{80}d^{12} - \frac{243}{20}d^{11} - \frac{81}{20}d^{10} + \frac{8667}{16}d^9 - \frac{9297}{8}d^8 - \frac{47727}{5}d^7 + \frac{2458629}{80}d^6 + \frac{3243249}{40}d^5 - \frac{6577679}{20}d^4 - \frac{25387481}{80}d^3 + \frac{6352577}{4}d^2 + \frac{8290623}{20}d - 2699706.
\end{align*}
\]

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0.2. The Method. For each \( n \geq 1 \), let \( H(n) \) be the Hilbert scheme of length \( n \) subschemes of \( \mathbb{P}^3 \). \( H(n) \) is a nonsingular variety of dimension \( 2n \) with generic element corresponding to a subscheme of \( n \) distinct points of \( \mathbb{P}^3 \). There exists a rational map

\[
\psi_n : H(n) \to H(3n)
\]

given by squaring the ideal sheaf. For \( 1 \leq n \leq 3 \), we will consider resolutions

\[
\overline{\psi}_n : X(n) \to H(3n)
\]

of \( \psi_n \). It is easily checked that \( \psi_1 \) and \( \psi_2 \) are everywhere defined. Let \( X(1) = H(1), \overline{\psi}_1 = \psi_1 \) and \( X(2) = H(2), \overline{\psi}_2 = \psi_2 \). Let \( F \subset H(3) \) be the locus of length 3 subschemes isomorphic to \( \mathbb{C}[\{x, y\}] / (x^2, x y, y^2) \). \( F \) is a nonsingular subvariety abstractly isomorphic to \( \mathbb{P}^3 \). Since \((x^2, xy, y^2)\) is an ideal of length 10, \( \psi_3 \) is not defined on \( F \). Let \( X(3) \) be the blow-up of \( H(3) \) along \( F \). In section (1), it is shown that \( \psi_3 \) is defined on \( H(3) \setminus F \) and extends to a morphism

\[
\overline{\psi}_3 : X(3) \to H(9).
\]
The degrees of $\mathcal{N}(1, d)$, $\mathcal{N}(2, d)$, and $\mathcal{N}(3, d)$ will be expressed as Chern classes of certain tautological bundles over $X(1)$, $X(2)$, and $X(3)$ respectively.

Let $U(n) \hookrightarrow H(n) \times \mathbb{P}^d$ be the universal subscheme over $H(n)$. For $1 \leq n \leq 3$, let

$$Y(n) = X(n) \times_{H(3n)} U(3n) \hookrightarrow X(n) \times \mathbb{P}^d.$$  

Let $\pi_n$, $\rho_n$ be the projections from $Y(n)$ to $X(n)$, $\mathbb{P}^d$ respectively. For pairs $(n, d)$ where $1 \leq n \leq 3$ and $\mathcal{N}(\setminus, \lceil) \neq \emptyset$, let

$$E(n, d) = \pi_n \ast \rho_n^*(\mathcal{O}_{\mathbb{P}^d}(d)).$$

$E(n, d)$ is easily seen to be a rank $3n$ vector bundle on $X(n)$. We claim

$$c_2(n)(E(n, d)) = \text{degree}(\mathcal{N}(n, d)).$$

A sketch of the argument is as follows. Let $1 \leq n \leq 3$ and let $d$ be such that $\mathcal{N}(\setminus, \lceil) \neq \emptyset$. Since $\mathcal{N}(n, d)$ is of codimension $n$ in $\mathcal{P}(\lceil)$, the degree is the cardinality of a generic $n$-plane slice. An $n$-plane, $L \subset \mathcal{P}(\lceil)$, is equivalent to an $n + 1$-dimensional linear subspace of $L \subset H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(d))$. $L$ canonically yields an $n + 1$ dimensional subspace $\mathcal{T} \subset H^0(X(n), E(n, d))$. A generic point $\xi \in X(n)$ corresponds to $n$ points of $\mathbb{P}^d$. It is checked that $\mathcal{T}$ drops rank at $\xi$ if and only if there exists an element of $\mathcal{L}$ corresponding to a plane curve singular at the $n$ points of $\xi$. It is further checked, for generic $\mathcal{L}$, the singular plane curve must be reduced, nodal with exactly $n$ nodes. The nongeneric points of $X(n)$ make no contribution to the degeneracy locus. The degree of $\mathcal{N}(n, d)$ is thus equal to the cardinality of the degeneracy locus of $n + 1$ sections of $E(n, d)$. The latter is the $c_2(n)(E(n, d))$. Section (2) contains the full argument. In principle, this approach may be attempted for $n \geq 4$. Explicit resolutions of $\psi_n$ are needed. For $n \geq 4$, a correction term to $c_2(n)(E(n, d))$ for small $d$ is required to account for the contribution of nonreduced curves to the degeneracy locus.

It remains to compute $c_2(n)(E(n, d))$. There is a diagonal torus action on $\mathbb{P}^d$ which can be lifted to $X(n)$ and $E(n, d)$. The Bott residue formula expresses the desired Chern class in terms of the differential data of the torus action at fixed points. This approach yields the degree formulas. It should be mentioned there are more direct ways of obtaining formulas (1) and (2). Since the residue calculations for formula (3) contain those required for (1) and (2), we present a unified approach. The explicit residue computations are presented in section (3).
1. \( \psi_n \) And \( \overline{\psi}_n \) For \( n = 1, 2, 3 \)

Consider the universal subschemes \( U(n) \hookrightarrow H(n) \times \mathbb{P}^k \). Let \( U^2(n) \) be subscheme of \( H(n) \times \mathbb{P}^k \) defined by the square of the ideal of \( U(n) \). Let \( G(n) \subset H(n) \) denote the generic locus corresponding to subschemes of \( n \) distinct points of \( \mathbb{P}^k \). \( U^2(n) \) is flat over \( G(n) \) of degree \( 3n \). Therefore there is map

\[
\psi_n : G(n) \to H(3n).
\]

Certainly \( G(1) = H(1) \). For \( n = 2 \), the complement of \( G(2) \) in \( H(2) \) consists of linear double points (with ideals isomorphic to \( (x, y^2) \subset \mathbb{C}[\langle, \rangle] \)). Since \( (x, y^2)^2 = (x^2, xy^2, y^4) \) has length 6, \( U^2(2) \) is flat over \( H(2) \). The map \( \psi_2 \) extends to \( H(2) \).

For \( n = 3 \), the situation is more complex. By the results for \( n = 2 \), \( \psi_3 \) extends to all of \( H(3) \) with the possible exception of the the triple points. The isomorphism classes of length 3 subschemes of \( \mathbb{C}^k \) supported at a point are given by the following ideals:

(i.) \( (x, y^3) \subset \mathbb{C}[\langle, \rangle] \).
(ii.) \( (x + y^2, x^2, xy) \subset \mathbb{C}[\langle, \rangle] \).
(iii.) \( (x^2, xy, y^2) \subset \mathbb{C}[\langle, \rangle] \).

It is easy to check the squares of the ideals of type (i) and (ii) have length 9. \( (x^2, xy, y^2)^2 = (x^4, x^3y, x^2y^2, xy^3, y^4) \) has length 10. Let \( F \hookrightarrow H(3) \) be the nonsingular subscheme corresponding to the points of type (iii). \( U^2(3) \) is flat over \( H(3) \setminus F \). Therefore \( \psi_3 \) extends to \( \overline{\psi}_3 : H(3) \setminus F \to H(9) \).

Let \( V \subset \mathbb{P}^k \) be a coordinate affine chart. \( V \) is a two dimensional complex vector space. Let

\[
A = \bigoplus_{k=0}^{\infty} \text{Sym}^k(V^*)
\]

be the affine coordinate ring of \( V \). Let \( m \subset A \) be the maximal ideal corresponding to the point \( 0 \in V \). The ideal \( m^2 \) is of type (iii). Let \([A/m^2] \in H(3)\) denote the Hilbert point corresponding to \( A/m^2 \). The tangent space to \( H(3) \) at \([A/m^2]\) is canonically isomorphic the module of \( A \)-homomorphisms \( \text{Hom}_A(m^2/m^4, A/m^2) \). It is easily seen there are canonical isomorphisms

\[
\text{Hom}_A(m^2/m^4, A/m^2) \cong \text{Hom}_\mathbb{C}(m^2/m^3, m/m^2) \cong \text{Hom}_\mathbb{C}(\text{Sym}^2(V^*), V^*).
\]
V is canonically identified with the space of invariant vector fields on V. Therefore, there is a canonical map

\[ \mu : V \rightarrow \text{Hom}_\mathbb{C}(\text{Sym}^2(V^*), V^*) \]
given by differentiation of functions. Certainly, \([A/m^2] \in F\). The tangent space to F at \([A/m^2]\) is canonically isomorphic to V. The map \(\mu\) is the differential of the inclusion of F in \(H(3)\) at \([A/m^2]\). Consider the exact sequence

\[ 0 \rightarrow K \rightarrow \text{Sym}^2(\text{Sym}^2(V^*)) \rightarrow \text{Sym}^4(V^*) \rightarrow 0 \]
given by multiplication. K is a one dimension \(\mathbb{C}\)-vector space. There is a canonical map \(\nu\) and an exact sequence:

\[ 0 \rightarrow V \xrightarrow{\mu} \text{Hom}_\mathbb{C}(\text{Sym}^2(V^*), V^*) \xrightarrow{\nu} \text{Hom}_\mathbb{C}(K, \text{Sym}^3(V^*)) \rightarrow 0. \]  

(4)

Briefly, an element of \(\gamma \in \text{Hom}_\mathbb{C}(\text{Sym}^2(V^*), V^*)\) yields a map \(\text{id} + \gamma : \text{Sym}^2(V^*) \rightarrow A\). Multiplication of \(\text{id} + \gamma\) induces a map \(\text{Sym}^2(\text{Sym}^2(V^*)) \rightarrow A\). The latter map takes K to \(\text{Sym}^3(V^*)\). The exactness of (4) is a simple exercise.

Let \(X(3)\) be the blow up of \(H(3)\) along F. By sequence (4), there is a natural correspondence between the fiber of the projective normal bundle of F in \(H(3)\) at \([A/m^2]\) and the projective space \(P(\mathbb{H} \times \bowtie C(\mathbb{K}, S \bowtie \bowtie (V^*))\)). The map \(\psi_3\) can be extended to the projective normal bundle of F by mapping an element \([\xi] \in P(\mathbb{H} \times \bowtie C(\mathbb{K}, S \bowtie \bowtie (V^*))\)) to the ideal of length 9 given by \((m^4, \text{image}(\xi)) \subset A\). We have defined a map

\[ \overline{\psi}_3 : X(3) \rightarrow H(9). \]

It is not hard to check that \(\overline{\psi}_3\) is an algebraic morphism.

2. \(E(n, d)\), \textbf{Degeneracy Loci, and degree}(\(\mathcal{N}(n, d)\)).

Let \(n = 1, 2, \text{ or } 3\). Let \(d\) be an integer such that \(\mathcal{N}([, \overline{\gamma}]\) is nonempty. Following the notation of section (0.2),

\[ E(n, d) = \pi_\ast n \ast (\mathcal{O}_{\mathbb{P}^d}(d)). \]

Let \(\mathcal{L} \subset \mathcal{P}([\overline{\gamma}]\) be an \(n\)-plane corresponding to an \(n + 1\) dimensional subspace \(L \subset H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(d))\). Let \(\overline{\mathcal{L}}\) denote the naturally induced \(n + 1\) dimensional subspace of \(H^0(X(n), E(n, d))\). For any \(\xi \in X(n)\), let \(\overline{\mathcal{T}}_{\xi} \subset E(n, d)_{\xi}\) be the subspace of the fiber generated by \(L\). Let \(I_{\xi} \subset \mathcal{O}_{\mathbb{P}^d}\) denote the ideal sheaf of subscheme corresponding to \(\overline{\psi}_n(\xi)\).
Lemma 1. Let $[\xi] \in X(n)$. Then, $\dim(\mathcal{L}_\xi) < n + 1$ if and only if there exists a nonzero $l \in L$ such that $l \in H^0(\mathbb{P}^\mathcal{P}, \mathbb{I}_\xi())$.

Lemma (1) is a tautological statement. We abuse notation slightly to let $G(n) \subset X(n)$ denote the locus of points $\xi \in X(n)$ such that $\psi_n(\xi)$ corresponds to a subscheme of $n$ distinct points of $\mathbb{P}^\mathcal{P}$.

Lemma 2. For generic $\mathcal{L}$, the following holds:

$$\forall \xi \in X(n) \setminus G(n), \quad \dim(\mathcal{L}_\xi) = n + 1.$$ 

Proof. For $n = 1$, the result is vacuous.

For $n = 2$, $d$ must be at least 3 to ensure $\mathcal{N}(\varepsilon, [\cdot]) \neq \emptyset$. $X(2) \setminus G(2)$ consists of the 3 dimensional locus of linear double points (with ideals isomorphic to $(x, y^2) \subset \mathbb{C}[\wedge, \wedge]$). The squared ideal $(x^2, xy^2, y^4)$ imposes 6 conditions on linear series of degree $d \geq 3$. Hence, the locus of elements $[l] \in \mathcal{P}(\mathcal{P})$ such that $l \in H^0(\mathbb{P}^\mathcal{P}, \mathbb{I}_\xi())$ for some $\xi \in X(2) \setminus G(2)$ is of codimension at least $6 - 3 = 3$. A generic 2-plane has empty intersection with this locus.

For $n = 3$, $d$ must again be at least 3 to ensure $\mathcal{N}(\exists, [\cdot]) \neq \emptyset$. $X(3) \setminus G(3)$ consists of three loci:

(a.) The projective normal bundle of $F$, $\mathbb{P}(\mathcal{N}_F)$.
(b.) The quasi-projective locus, $B$, corresponding to ideals of type (i) and (ii) of section (I).
(c.) The quasi-projective locus, $C$, corresponding to a point $p$ union a linear double point supported at $q \neq p$.

The ideal $I_\xi$ for any $\xi$ in $\mathbb{P}(\mathcal{N}_F)$ certainly imposes 9 conditions on linear series of degree $d \geq 3$. The dimensions of $\mathbb{P}(\mathcal{N}_F)$ is 5. The locus of elements $[l] \in \mathcal{P}(\mathcal{P})$ such that $l \in H^0(\mathbb{P}^\mathcal{P}, \mathbb{I}_\xi())$ for some $\xi \in \mathbb{P}(\mathcal{N}_F)$ is codimension at least $9 - 5 = 4 > 3$.

$B$ consists of a 3 dimensional locus of ideals of type (i) and a 4 dimensional locus of ideals of type (ii). Ideals of type (i) and (ii) are easily seen to impose 7 and 9 conditions respectively on linear series of degree $d \geq 3$. We see $7 - 3 = 4 > 3$ and $9 - 4 = 5 > 3$.

If $J$ is the ideal of a linear double point supported at $q \in \mathbb{P}^\mathcal{P}$, $J$ imposes 6 conditions on linear series of degree $d \geq 3$. The generic element of $H^0(\mathbb{P}^\mathcal{P}, \mathcal{J}())$ is singular only at $q$. The condition that $l \in H^0(\mathbb{P}^\mathcal{P}, \mathcal{J}())$
be singular at some point in $\mathbb{P}^k \setminus \{0\}$ is therefore a codimension 1 condition on $H^0(\mathbb{P}^k, \mathcal{I}(\ell))$. The locus of elements $[l] \in \mathcal{P}(\ell)$ such that $l \in H^0(\mathbb{P}^k, \mathbb{I}_\xi(\ell))$ for some $\xi \in T$ is therefore of codimension at least $7 - 3 = 4 > 3$. A generic 3-plane in $\mathcal{P}(\ell)$ avoids the loci corresponding to subsets (a), (b), and (c) of $\mathcal{X}(3) \setminus G(3)$.

\[ \square \]

**Lemma 3.** Let $\text{Sing}(n, d) \subset \mathcal{P}(\ell)$ be the quasi-projective locus of reduced curves with at least $n$ singular points. Then $\text{Sing}(n, d) \subset \overline{\mathcal{N}}(n, d)$.

(This result holds for all $n$.)

**Proof.** Let $[C] \in \text{Sing}(n, d)$. We must show $[C] \in \overline{\mathcal{N}}(n, d)$. Since $\overline{\mathcal{N}}(m, d) \subset \overline{\mathcal{N}}(n, d)$ for $m > n$, we can reduce to the case were $C$ has exactly $n$ nodes. Let $\sum$ be the $n$ singular points of $C$. The projective tangent space to $\text{Sing}(n, d)$ at $[C]$ is given by the linear system of degree $d$ curves passing through $\sum$. By a study of the adjoint conditions ([ACGH], p.60), $\sum$ imposes $n$ independent conditions on degree $d$ curves. It also follows from the adjoint analysis that any subideal of $I(\sum)$ of index 1 imposes independent conditions on degree $d$ curves. Since the condition of having exactly $n$ nodes is open in $\text{Sing}(n, d)$, $\text{Sing}(n, d)$ is nonsingular of codimension $n$ at $[C]$.

Let $f(x, y)$ be the equation of a plane curve singularity at the origin $(0, 0)$. Consider first order, equisingular deformations of the type $f(x, y) + \epsilon \cdot g(x, y)$. It is a fact ([fact]) that such $g(x, y)$ generate the the maximal ideal of $(0, 0)$ if and only if the singularity of $f(x, y)$ at the origin is a node.

Suppose $[C]$ is not nodal. The equisingular deformations of $[C]$ correspond at most to the linear system of degree $d$ curves passing through a subideal of $I(\sum)$ of index 1. Hence the these equisingular deformations are of codimension at least $n + 1$. Therefore, the generic member of each component must be nodal.

Finally, a simple dimension analysis yields:

**Lemma 4.** Let $\text{Nonred}(d) \subset \mathcal{P}(\ell)$ be the closed locus of nonreduced curves. For all $d$, the codimension of $\text{Nonred}(d)$ is greater than 2. For $d \geq 3$, the codimension of $\text{Nonred}(d)$ is greater than 3.
Let $\mathcal{L} \subset \mathcal{P}(\ell)$ be a generic $n$-plane. By Lemma (3) the following sequence of inclusions hold:

$\mathcal{N}(\ell, \ell) \subset \mathcal{S}(\ell, \ell) \subset \mathcal{N}(\ell, \ell)$.

Therefore, we obtain

$\mathcal{L} \cap \mathcal{N}(\ell, \ell) = \mathcal{L} \cap \mathcal{S}(\ell, \ell) = \mathcal{L} \cap \mathcal{N}(\ell, \ell)$.

(5)

The intersection is transverse and,

$\mid \mathcal{L} \cap \mathcal{N}(\ell, \ell) \mid = \mathcal{N}(\ell, \ell)$.

The degeneracy locus $D$ of $\mathcal{L}$ is the subscheme where $\mathcal{L}$ drops rank. By Lemma (2), $D$ is supported in $G(n)$. If $n = 1, 2, 3$ and $d$ is such that $\mathcal{N}(\ell, \ell) \neq \emptyset$, $E(n,d)$ is generated on $G(n)$ by sections induced from $H^0(\mathbb{P}^e, \mathcal{O}_{\mathbb{P}^e}(\ell))$. (Note $E(n,d)$ need not be generated by these sections on $X(n)$.) Therefore, the canonical sections yield a map

$\lambda_n(d) : G(n) \rightarrow Grassmanian(n,d)$.

The degeneracy locus $D$ on $G(n)$ is expressed as the $\lambda_n(d)$-intersection with a Schubert class determined by $\mathcal{L}$. By Kleiman’s result on intersections in homogeneous spaces ([H], p.273), $D$ is a reduced dimension zero subscheme of $G(n)$ for generic $\mathcal{L}$. If $\xi \in D \subset G(n)$ is a point, by Lemma (4) there is a curve $[\ell] \in \mathcal{L}$ singular at the $n$ points of $\mathbb{P}^e$ corresponding to $\xi$. $[\ell]$ must be reduced by Lemma (4). $[\ell]$ therefore corresponds to a point of the intersection (3) and must be nodal with exactly $n$ nodes. $[\ell]$ must be unique or else there would be a pencil of curves of $\mathcal{L}$ singular at the $n$ points corresponding to $\xi$. We have defined a set map $\tau : D \rightarrow \mathcal{L} \cap \mathcal{N}(\ell, \ell)$. Since $D$ can be recovered from the nodes of $[\ell]$, $\tau$ is injective. By Lemma (5), $\tau$ is surjective. Hence $|D| = |\mathcal{L} \cap \mathcal{N}(\ell, \ell)|$.

By the Thom-Porteous formula, $|D| = c_{2n}(E(n,d))$.

Proposition 1. For $n = 1, 2, 3$, degree$(\mathcal{N}(n,d)) = c_{2n}(E(n,d))$.

3. The Bott Residue Computation

3.1. The Formula. We first state the form of the Bott Residue Formula ([B]) that will be used. Let $M$ be a nonsingular variety of dimension $m$ with an algebraic $\mathbb{C}^*$-action. Let $q \in M$ be a fixed point of the $\mathbb{C}^*$-action. The differential of the action naturally induces a $\mathbb{C}^*$-representation on the tangent space $T_q(M)$. Let $\alpha_1(q), \ldots, \alpha_m(q)$ be the $m$ weights of the $\mathbb{C}^*$-representation on $T_q(M)$. Let $Q \subset M$ be the fixed point set. Assume
(1.) $Q$ is discrete.
(2.) $\forall q \in Q$ and $\forall j$, $\alpha_j(q) \neq 0$.

In fact, condition (2) is a consequence of condition (1). Suppose $E$ is an algebraic vector bundle of rank $r$ on $M$ with an equivariant $\mathbb{C}^*$-action. For each $q \in Q$, there is a $\mathbb{C}^*$-representation on $E_q$. Let $\beta_1(q), \ldots, \beta_r(q)$ be the weights of this $\mathbb{C}^*$-representation. Finally, let $\sigma_{i,j}(x_1, x_2, \ldots, x_j)$ be the $i^{th}$ elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_j$. If $i > j$, $\sigma_{i,j} = 0$. The Bott Residue Formula expresses $c_m(E)$ in terms of the $\mathbb{C}^*$-weights at the fixed points:

$$c_m(E) = \sum_{q \in Q} \frac{\sigma_{m,r}(\beta_1(q), \ldots, \beta_r(q))}{\sigma_{m,m}(\alpha_1(q), \ldots, \alpha_m(q))}.$$ 

Since $\sigma_{m,m}$ is the product monomial and $\alpha_j(q) \neq 0$, the right hand side is well defined.

### 3.2. Torus Actions

Let $Z$ be a 3 dimensional $\mathbb{C}$-vector with basis $\mathfrak{z} = (z_0, z_1, z_2)$. Let $\mathfrak{w} = (w_0, w_1, w_2)$ be a triple of integral weights. Let $\lambda(\mathfrak{w})$ be the $\mathbb{C}^*$-representation with weights $(w_0, w_1, w_2)$ diagonal with respect to $\mathfrak{z}$. Let $\mathbb{P}^d = \mathbb{P}(\mathbb{Z})$. The representation $\lambda(\mathfrak{w})$ induces a $\mathbb{C}^*$-action on $\mathbb{P}(\mathbb{Z})$. There is an induced $\mathbb{C}^*$-action on $H(n)$, $X(n)$, and $Y(n)$. Recall the natural isomorphism $H^0(\mathbb{P}(\mathbb{Z}), \mathcal{O}_{\mathbb{P}(\mathbb{Z})}(d)) \cong Sym^d(\lambda^*(\mathfrak{w}))$. There is a canonical equivariant lifting of $\lambda(\mathfrak{w})$ to $\mathcal{O}_{\mathbb{P}(\mathbb{Z})}(d)$ such that the induced representation on global sections is $Sym^d(\lambda^*(\mathfrak{w}))$. The canonical equivariant lifting of $\lambda(\mathfrak{w})$ to $\mathcal{O}_{\mathbb{P}(\mathbb{Z})}(d)$ induces an equivariant $\mathbb{C}^*$-action on $E(n, d) = \pi_{n,*}\rho^*(\mathcal{O}_{\mathbb{P}(\mathbb{Z})}(d))$ over $X(n)$.

### 3.3. The Case $n = 1$. $X(1) = H(1) = \mathbb{P}(\mathbb{Z})$.

There are 3 fixed points for distinct weights $\mathfrak{w} = (w_0, w_1, w_2)$. Analysis of the fixed point $[z_0]$ yields:

$$\alpha_1([z_0]) = w_1 - w_0$$
$$\alpha_2([z_0]) = w_2 - w_0$$

To simplify notation, let $Z_0, Z_1, Z_2$ be a basis of $Z^*$ dual to $\mathfrak{z}$. To calculate the action on $E(1, d)$, observe that $\mathfrak{w}_1([z_0])$ is the subscheme given by the ideal $(Z_1^2, Z_1Z_2, Z_2^2)$. Therefore the sections $Z_0^d, Z_0^{d-1}Z_1, Z_0^{d-1}Z_2$ generate the fibre of $E(1, d)$ at $[z_0]$. We obtain:

$$\beta_1([z_0]) = -dw_0$$
$$\beta_2([z_0]) = -(d-1)w_0 - w_1$$
$$\beta_3([z_0]) = -(d-1)w_0 - w_2$$
The analysis for \([z_1]\) and \([z_2]\) is similar. The Bott Residue Formula now yields \(c_2(E(1, d)) = 3(d - 1)^2\). Since the Chern class does not depend upon the weights, it is simplest to fix values \(\overline{w} = (0, 1, 2)\) when using the residue formula.

3.4. **The Case** \(n = 2\). \(X(2) = H(2)\). There are 9 fixed points for distinct weights \(\overline{w} = (w_0, w_1, w_2)\). Three fixed points correspond to the subschemes:

\[
[z_0] \cup [z_1], \quad [z_0] \cup [z_2], \quad [z_1] \cup [z_2].
\]

For these points, the analysis of the \(n = 1\) suffices to yield the \(\alpha\) and \(\beta\)-weights. There are 6 fixed points given by the subschemes \(D_{ij} = \mathcal{O}_{P(Z)}/(Z_i^2, Z_j)\) for ordered pairs \(1 \leq i \neq j \leq 3\). We carry out the analysis at the fixed point \([D_{1,2}]\). In the affine open \(Z_0 \neq 0\), let \(I = ((Z_1/Z_0)^2, (Z_2/Z_0)).\) Let \(A = \mathbb{C}[(Z_{w'}/Z_{w'}), (Z_{w'}/Z_{w'})]\). The tangent space to \(H(2)\) at \([D_{1,2}]\) is canonically isomorphic to \(\text{Hom}_A(I/I^2, A/I)\). \(I/I^2\) is the free \(A/I\) module with generator \((Z_1/Z_0)^2\) and \((Z_2/Z_0).\) \(A/I\) is generated by 1 and \((Z_1/Z_0).\) Since we know the \(\mathbb{C}^*\)-action on basis elements of \(I/I^2\) and \(A/I\), we obtain:

\[
\begin{align*}
\alpha_1([D_{1,2}]) &= 2w_1 - 2w_0 \\
\alpha_2([D_{1,2}]) &= w_1 - w_0 \\
\alpha_3([D_{1,2}]) &= w_2 - w_0 \\
\alpha_4([D_{1,2}]) &= w_2 - w_1 \\
\end{align*}
\]

\(\overline{\psi}_2([D_{1,2}])\) is the subscheme defined by \((Z_1^1, Z_1^2 Z_2, Z_2^2).\) Therefore the elements \(Z_0^d, Z_0^{d-1} Z_1, Z_0^{d-1} Z_2, Z_0^{d-2} Z_1^2, Z_0^{d-2} Z_1 Z_2,\) and \(Z_0^{d-3} Z_1^3\) yield a basis the fiber of \(E(2, d)\) over \([D_{1,2}]\). The \(\beta\)-weights are therefore:

\[
\begin{align*}
\beta_1([D_{1,2}]) &= -dw_0 \\
\beta_2([D_{1,2}]) &= -(d - 1)w_0 - w_1 \\
\beta_3([D_{1,2}]) &= -(d - 1)w_0 - w_2 \\
\beta_4([D_{1,2}]) &= -(d - 2)w_0 - 2w_1 \\
\beta_5([D_{1,2}]) &= -(d - 2)w_0 - w_1 - w_2 \\
\beta_6([D_{1,2}]) &= -(d - 3) - 3w_1 \\
\end{align*}
\]

The \(\alpha\) and \(\beta\)-weights at the other points \([D_{i,j}]\) are obtained by appropriate permutations of \(w_0, w_1\) and \(w_2\) in the above formulas. After some algebra (MAPLE was used at this point), the Bott Residue formula yields

\[
c_4(E(2, d)) = \frac{3}{2}(d - 1)(d - 2)(3d^2 - 3d - 11).
\]
3.5. **The Case n=3.** Consider the $\mathbb{C}^*$-action for weights $\overline{w}$ on $X(3)$. The analysis at nontriple points reduces to previous computations. There is one fixed point corresponding to the subscheme $[z_0] \cup [z_1] \cup [z_2]$. There are 12 fixed points of the type

$$[z_i] \cup D_{j,k}$$

where $\text{Supp}(D_{j,k}) \neq [z_i]$. The $\alpha$ and $\beta$-weights for these 1+12 points are easily obtained from the weights in the $n = 1$ and 2 cases.

Next consider fixed points points $\xi \in X(3) \setminus \mathbb{P}_F(N)$ where $\psi_3(\xi)$ is a triple point. If $(w_0, w_1, w_2)$ are distinct and no two weights sum to twice the third, then there are 6 such $\xi$. They are given by the subschemes $T_{i,j} = \mathcal{O}_{\mathbb{P}(\mathbb{Z})}/(Z^3_i, Z^3_j)$ for $1 \leq i \neq j \leq 3$. We carry out the weight analysis at the point $[T_{1,2}]$. Following the notation of section (3.4), let $A = \mathbb{C}[(Z_{\mu}/Z_{\mu}), (Z_{\mu}/Z_{\mu})]$ be the affine coordinate ring for $Z_0 \neq 0$. Let $I = ((Z_1/Z_0)^3, (Z_2/Z_0))$. Since $X(3)$ is isomorphic to $H(3)$ at $[T_{1,2}]$, the tangent space is given by $\text{Hom}_{A}(I/I^2, A/I)$. $I/I^2$ is seen to be a free $A/I$ module of rank 2 with basis $(Z_1/Z_0)^3$ and $(Z_2/Z_0)$. We hence obtain the six $\alpha$-weights:

$$3w_1 - 3w_0, \ 2w_1 - 2w_0, \ w_1 - w_0, \ w_2 - w_0, \ w_2 - w_1, \ w_2 + w_0 - 2w_1.$$  

Since $(Z^3_1, Z^3_2)^2 = (Z^6_1, Z^6_1 Z_2, Z^3_2^2)$, the fiber of $E(3, d)$ at $[T_{1,2}]$ is spanned by the (possibly rational) sections $Z_0, Z_0^{-1} Z_1, Z_0^{-2} Z_2, Z_0^{-2} Z_1 Z_2, Z_0^{-3} Z^3_1, Z_0^{-3} Z_1^2 Z_2, Z_0^{-4} Z_1^4, Z_0^{-5} Z_1^5$. Therefore, the nine $\beta$-weights are:

$$-dw_0 \quad -(d-1)w_0 - w_1 \quad -(d-1)w_0 - w_2$$
$$-(d-2)w_0 - 2w_1 \quad -(d-2)w_0 - w_1 - w_2 \quad -(d-3)w_0 - 3w_1$$
$$-(d-3)w_0 - 2w_1 - w_2 \quad -(d-4)w_0 - 4w_1 \quad -(d-5)w_0 - 5w_1$$

Again, the weights at the other $[T_{i,j}]$ are obtained by appropriate permutations of $\overline{w}$.

Finally, consider the fixed points $\xi \in \mathbb{P}_F(N) \subset X(\mathcal{M})$. Let $m_i$ be the ideal of the point $[z_i]$. Let $F_i$ for $1 \leq i \leq 3$ be the subscheme $\mathcal{O}_{\mathbb{P}(\mathbb{Z})}/m_i^2$. $[F_i] \in F$. The points $[F_i]$ are the 3 fixed points of the $\mathbb{C}^*$-action on $F$. The fixed points of $\mathbb{P}_F(N)$ must lie in the fibers of $\mathbb{P}_F(N)$ over the $[F_i]$. We analyze the case $i = 1$. The fibered $\mathbb{P}^M$ of $\mathbb{P}_F(N)$ over $[F_1]$ was intrinsically described in section (3). We see

$$\mathbb{P}^M = \mathbb{P}(H \times \mathcal{C}(K, S \bowtie^M (V^*))) \cong \mathbb{P}(S \bowtie^M (V^*))$$

where $V^*$ has a basis given by $(Z_1/Z_0)$ and $(Z_2/Z_0)$ with induced $\mathbb{C}^*$-weights $-w_1 + w_0$ and $-w_2 + w_0$ respectively. For weights $\overline{w}$ such that no two sum to twice the third, the $\mathbb{C}^*$ action on the fibered $\mathbb{P}^M$ has 4
isolated fixed points. The total number of fixed points for the $\mathbb{C}^*$-action on $X(3)$ is $31 = 1 + 12 + 6 + 3 \cdot 4$.

The $\alpha$-weights at the 4 fixed points in the fibered $\mathbb{P}^{14}$ are obtained in the following manner. First the differential of the $\mathbb{C}^*$-action on $H(3)$ at $F_1$ is determined. Then the blow-up along the locus $F$ is examined. The sequence (4) of section (1) contains all the necessary data. The four fixed points in $\mathbb{P}(\mathcal{O}_\mathbb{P}(3))$ are $P_{r,s} = [(Z_1/Z_0)^r(Z_2/Z_0)^s]$ for non-negative integers $r, s$ with sum $r + s = 3$.

We tabulate the weight formulas for the four points $P_{r,s}$. First the six $\alpha$-weights:

- $P_{3,0} : w_1 - w_0 \quad w_2 - w_0 \quad 2w_2 - w_1 - w_0 \quad w_1 - w_2 \quad 2w_1 - 2w_2 \quad 3w_1 - 3w_2$
- $P_{2,1} : w_1 - w_0 \quad w_2 - w_0 \quad w_2 - w_1 \quad w_1 - w_2 \quad 2w_1 - 2w_2$
- $P_{1,2} : w_1 - w_0 \quad w_2 - w_0 \quad w_1 - w_0 \quad 2w_2 - 2w_1 \quad w_2 - w_1 \quad w_1 - w_2$
- $P_{0,3} : w_1 - w_0 \quad w_2 - w_0 \quad 2w_1 - w_2 - w_0 \quad 3w_2 - 3w_1 \quad 2w_2 - 2w_1 \quad w_2 - w_1$

The $\beta$-weights for the four point $P_{r,s}$ all include the the six weights:

- $-dw_0$
- $-(d-1)w_0 - w_1$
- $-(d-1)w_0 - w_2$
- $-(d-2)w_0 - 2w_1$
- $-(d-2)w_0 - w_1 - w_2$
- $-(d-2)w_0 - 2w_2$

The additional three $\beta$-weights at the points $P_{r,s}$ are:

- $P_{3,0} : -(d-3)w_0 - 2w_1 - w_2 \quad -(d-3)w_0 - w_1 - 2w_2 \quad -(d-3)w_0 - 3w_2$
- $P_{2,1} : -(d-3)w_0 - 3w_1 \quad -(d-3)w_0 - w_1 - 2w_2 \quad -(d-3)w_0 - 3w_2$
- $P_{1,2} : -(d-3)w_0 - 3w_1 \quad -(d-3)w_0 - 2w_1 - w_2 \quad -(d-3)w_0 - 3w_2$
- $P_{0,3} : -(d-3)w_0 - 3w_1 \quad -(d-3)w_0 - 2w_1 - w_2 \quad -(d-3)w_0 - w_1 - 2w_2$

As before, the $\alpha$ and $\beta$-weights at the fixed points on the other fibered $\mathbb{P}^{14}$'s can be obtained by permuting the weights $\bar{w}$ in the above formulas.

The Bott Residue Formula now yields:

$$c_6(E(3, d)) = \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525.$$  

The algebraic computation was done on MAPLE with weights $\bar{w} = (0, 1, 3)$.

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Department of Math, Harvard University, harris@math.harvard.edu

Department of Math, University of Chicago, rahul@math.uchicago.edu