Abstract

We consider the integer points in a unimodular cone $K$ ordered by a lexicographic rule defined by a lattice basis. To each integer point $x$ in $K$ we associate a family of inequalities (lex-cuts) that defines the convex hull of the integer points in $K$ that are not lexicographically smaller than $x$. The family of lex-cuts contains the Chvátal–Gomory cuts, but does not contain and is not contained in the family of split cuts. This provides a finite cutting plane method to solve the integer program $\min \{c x : x \in S \cap \mathbb{Z}^n\}$, where $S \subset \mathbb{R}^n$ is a compact set and $c \in \mathbb{Z}^n$. We analyze the number of iterations of our algorithm.

1 Introduction

The area of nonlinear integer programming is rich in applications but quite challenging from a computational point of view. We refer to the recent articles \cite{5,7} for comprehensive surveys on these topics. The tools that are mainly used are sophisticated techniques that exploit relaxations, constraint enforcement (e.g., cutting planes) and convexification of the feasible set. Reformulations in an extended space and cutting planes for nonlinear integer programs have been investigated and proposed for some time, see e.g. \cite{8,12,17}. This line of research mostly provides a nontrivial extension of the theory of disjunctive programming to the nonlinear case. To the best of our knowledge, these results are obtained under some restrictive conditions: typically, convexity of the feasible set $S$, or $S \subseteq \{0,1\}^n$ (these cases cover some important areas of application).

In this paper we focus on linear inequalities that we use as cuts. As the convex hull of $S \cap \mathbb{Z}^n$ is a polytope when $S \subseteq \mathbb{R}^n$ is compact, a finite number of linear inequalities suffices for its characterization, and only $n$ such inequalities determine an optimal point. Furthermore, some relaxations are polyhedral: most notably, Dadush, Dey and Vielma \cite{10} proved that if $S$ is a compact and convex set, then its Chvátal closure is a polytope (whereas this is not the case for the split closure of $S$ \cite{9}).

However, nonlinear inequalities are fundamental in the characterization of the convex hull of some nonlinear sets that strengthen the original formulation. For instance, Burer and Kilimç-Karzan \cite{6}, extending several results (see, e.g., \cite{1,3,4,15}), show that the convex hull of the intersection of a second-order-cone representable set and a single homogeneous quadratic inequality can be described by adding a single nonlinear inequality, defining an additional second-order-cone representable set.

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In this paper we present a finite cutting plane algorithm for problems of the form

$$\min\{cx : x \in S \cap \mathbb{Z}^n\}, \quad (1.1)$$

where $S$ is a compact subset of $\mathbb{R}^n$ (not necessarily convex or connected) and $c \in \mathbb{Z}^n$. This algorithm uses a new family of cutting planes which includes the Chvátal–Gomory cuts, but neither it contains nor is contained in the family of split cuts. Furthermore, these cuts define a natural but nontrivial polyhedral relaxation of $S \cap \mathbb{Z}^n$.

The cutting planes employed in our algorithm are obtained as follows. We consider the integer points in a unimodular cone $K$, ordered by a lexicographic rule, associated with a lattice basis. To each integer point $x$ in $K$, we associate a family of inequalities (lex-cuts) that defines the convex hull of the integer points in $K$ that are not lexicographically smaller than $x$.

Our algorithm recursively solves optimization problems of the form

$$\min\{cx : x \in S \cap P\},$$

where $P$ is a polyhedron, and we assume that an algorithm for problems of this type is available as a black box. Note that when $S$ is a convex set, this is a convex program that is (in principle) efficiently solvable. To the best of our knowledge, our work represents the first attempt to define a finite cutting plane algorithm for the general problem $(1.1)$ with $S$ compact.

Deriving a finite cutting plane algorithm that uses a well defined family of inequalities does not seem to be straightforward. The oldest and most notable example is Gomory’s finite cutting plane algorithm for bounded integer programs based on fractional cuts [13, 14]. Balas, Ceria and Cornuéjols [2] give a finite cutting plane algorithm for mixed 0/1 problems based on lift-and-project cuts. In those algorithms, as well as in the method proposed here, crucial to the detection of a cutting plane is the computation of a lexicographically optimal solution.

2 Lexicographic orderings and lex-cuts

A lattice basis of $\mathbb{Z}^n$ is a set of $n$ linearly independent vectors $c^1, \ldots, c^n \in \mathbb{Z}^n$ such that for every $v \in \mathbb{Z}^n$ we have that $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ in the unique expression $v = \sum_{i=1}^n \lambda_i c_i$.

The lex-cuts that we introduce in this paper are defined for a given lattice basis of $\mathbb{Z}^n$. To simplify the presentation, we first work with the standard basis and then extend the results to general lattice bases.

We will use standard notions in the theory of polyhedra, for which we refer the reader to [16].

2.1 Standard basis

We consider the lexicographic ordering $\prec$ associated with the standard basis $e_1, \ldots, e_n$: given $x^1, x^2 \in \mathbb{R}^n$, $x^1 \prec x^2$ if and only if $x_1 \neq x_2$ and $x^1_1 < x^2_1$, where $i$ is the smallest index for which $x^1_i \neq x^2_i$. We use $\preceq, \succeq, \succ$ with the obvious meaning.

We consider the cone $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$. Given $\bar{x} \in K \cap \mathbb{Z}^n$, we define

$$Q(\bar{x}) := \text{conv}\{x \in K \cap \mathbb{Z}^n : x \succeq \bar{x}\},$$

where “conv” denotes the convex hull operator.

Given $\bar{x} \in K \setminus \{0\}$, we define the leading index $\ell(\bar{x})$ as the largest index $i$ such that $\bar{x}_i > 0$. 

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Lemma 2.1. Fix \( \bar{x} \in K \cap \mathbb{Z}^n \). The convex set \( Q(\bar{x}) \) has precisely the following extreme points \( v^1, \ldots, v^{\ell(\bar{x})} \): for \( k = 1, \ldots, \ell(\bar{x}) - 1 \), \( v^k \) has entries

\[
\begin{align*}
  v^k_i &= \bar{x}_i, & i = 1, \ldots, k - 1 \\
  v^k_k &= \bar{x}_k + 1 \\
  v^k_i &= 0, & i = k + 1, \ldots, n,
\end{align*}
\]

and \( v^{\ell(\bar{x})} = \bar{x} \). Furthermore, the recession cone of \( Q(\bar{x}) \) is \( K \). In particular, \( Q(\bar{x}) \) is a full-dimensional polyhedron.

Proof. Consider the program

\[
\min \{gx : x \in Q(\bar{x})\},
\]

where \( g \in \mathbb{R}^n \). It is immediate to see that the value of (2.1) is finite if and only if \( g \geq 0 \). Hence the recession cone of \( Q(\bar{x}) \) is contained in \( K \). Since the vectors of the standard basis \( e_1, \ldots, e_n \) are clearly contained in the recession cone of \( Q(\bar{x}) \), we conclude that the recession cone of \( Q(\bar{x}) \) is \( K \).

Assume that \( g \geq 0 \). Define \( X := \{x \in K \cap \mathbb{Z}^n : x \geq \bar{x}\} \). We say that a point \( x \in X \) is minimal if there is no \( y \in X \setminus \{x\} \) such that \( y \leq x \) (where this notation does not indicate the lexicographic ordering, but means that all entries of \( y \) are at most as large as those of \( x \)). Since \( X \) is a set of nonnegative integer points, by Dickson's lemma \([11, \text{Lemma A}]\) \( X \) has finitely many minimal points. As \( g \geq 0 \), this implies that the minimum of \( gx \) over \( X \) exists and is one of the minimal points, say \( x^* \). Clearly \( x^* \) is also an optimal solution of (2.1).

Assume \( x^* \neq \bar{x} \) and let \( k \) be the lowest index such that \( x^*_k > \bar{x}_k \). Since \( g \geq 0 \) and \( x^* \in \mathbb{Z}^n \) is an optimal solution of (2.1), we have that \( x^*_k = \bar{x}_k + 1 \) and \( x^*_j = 0, j = k + 1, \ldots, n \). This proves the characterization of the extreme points.

Since the number of extreme points is finite and \( K \) is a full-dimensional polyhedral cone, \( Q(\bar{x}) \) is a full-dimensional polyhedron. \(\square\)

Let \( \bar{x} \in K \) be given. For every \( k \in \{1, \ldots, n\} \) and \( i \in \{1, \ldots, k\} \) we define

\[
d^k_i := \begin{cases} 
  1 & \text{if } i = k \\
  \bar{x}_k & \text{if } i = k - 1, \\
  \bar{x}_k \prod_{j=i+1}^{k-1} (\bar{x}_j + 1) & \text{if } i \leq k - 2.
\end{cases}
\]

(Note that the \( d^k_i \)'s depend on the choice of \( \bar{x} \), but we omit the dependence on \( \bar{x} \) to keep notation simpler: this will never generate any ambiguity.)

For every \( k \in \{1, \ldots, n\} \), the \( k \)-th lex-cut associated with \( \bar{x} \) is the inequality

\[
\sum_{i=1}^{k} d^k_i x_i \geq \sum_{i=1}^{k} d^k_i \bar{x}_i.
\]

(2.2)

Note that when \( \bar{x}_k = 0 \), (2.2) is the inequality \( x_k \geq 0 \).

Lemma 2.2. Let \( \bar{x} \in K \cap \mathbb{Z}^n \). For every \( k \in \{1, \ldots, n\} \), the \( k \)-th lex-cut \( \text{(2.2)} \) associated with \( \bar{x} \) is satisfied by every \( x \in Q(\bar{x}) \).
Proof. Let \( x \in K \cap \mathbb{Z}^n \) be such that \( x \geq \bar{x} \) and let \( t \) be the smallest index such that \( x_t \neq \bar{x}_t \) (with \( t = n + 1 \) if \( x = \bar{x} \)). If \( t \geq k \), then \( x_i = \bar{x}_i \) for \( i = 1, \ldots, k - 1 \) and \( x_k \geq \bar{x}_k \), and thus \( x \) satisfies (2.2). If, on the contrary, \( t < k \), then \( x_i = \bar{x}_i \) for \( i = 1, \ldots, t - 1 \) and \( x_t \geq \bar{x}_t + 1 \), and thus the slack of inequality (2.2) is

\[
\sum_{i=1}^{k} d_i^k (x_i - \bar{x}_i) = d_t^k - \sum_{i=t+1}^{k} d_i^k \bar{x}_i = d_t^k + \sum_{i=t+1}^{k-1} (d_i^k - d_{i-1}^k) - \bar{x}_k = d_{k-1}^k - \bar{x}_k \geq 0.
\]

In the inequality we used the fact that \( x_i \geq 0 \) for every \( i \). The first equality follows from the fact that \( d_{k-1}^k = d_i^k (\bar{x}_i + 1) \) for \( i < k - 1 \), and thus \( d_i^k \bar{x}_i = d_i^k - d_i^k \) for \( i < k - 1 \). This shows that (2.2) is satisfied by every \( x \in K \cap \mathbb{Z}^n \) such that \( x \geq \bar{x} \), and thus by every point in \( Q(\bar{x}) \).

**Theorem 2.3.** If \( \bar{x} \in K \cap \mathbb{Z}^n \setminus \{0\} \), then the lex-cuts (2.2) for \( k = 1, \ldots, n \) and the inequalities \( x_i \geq 0 \) for \( i = 1, \ldots, n \) provide a description of the polyhedron \( Q(\bar{x}) \).

**Proof.** As \( K \) is the recession cone of \( Q(\bar{x}) \) (Lemma 2.1) and \( Q(\bar{x}) \subseteq K \), it follows that every facet inducing inequality for \( Q(\bar{x}) \) (indeed every valid inequality) is of the type

\[
\sum_{i=1}^{n} a_i x_i \geq a_0
\]

where \( a_i \geq 0, i = 0, \ldots, n \).

Given \( k \in \{1, \ldots, n\} \), we let \( Q_k(\bar{x}) \subseteq \mathbb{R}^k \) denote the orthogonal projection of \( Q(\bar{x}) \) onto the first \( k \) variables, and we define \( \bar{x}_{[k]} := (x_1, \ldots, x_k) \). It follows from the definition of lexicographic ordering that \( Q_k(\bar{x}) = Q(\bar{x}_{[k]}) \).

Therefore the facet inducing inequalities of \( Q_k(\bar{x}) \) are the facet inducing inequalities of \( Q(\bar{x}) \) such that \( a_j = 0 \) for \( j = k + 1, \ldots, n \). (This can be seen, e.g., as a consequence of the method of Fourier-Motzkin to compute projections.)

As the theorem trivially holds for \( Q_1(\bar{x}) \), to prove the result by induction on \( n \) it suffices to characterize the facets with \( a_n > 0 \). As the only facet inducing inequality with \( a_n > 0 \) and \( a_0 = 0 \) is \( x_n \geq 0 \), from now on we consider a facet inducing inequality (2.3) with \( a_n > 0 \) and \( a_0 > 0 \).

Assume first that \( \bar{x}_n = 0 \). Then by Lemma 2.1 we have that \( Q(\bar{x}) = Q_{n-1}(\bar{x}) \times \{x_n \in \mathbb{R} : x_n \geq 0\} \) and we are done by induction. Therefore we assume \( \bar{x}_n > 0 \). Recall that, by Lemma 2.1 \( Q(\bar{x}) \) has \( n \) vertices, \( v^1, \ldots, v^n = \bar{x} \).

**Claim 1:** \( \bar{x} \) satisfies (2.3) at equality.

Since \( v^k_n = 0 \) for \( k = 1, \ldots, n - 1 \), if \( \bar{x} \) does not satisfy (2.3) at equality, the inequality

\[
\sum_{i=1}^{n-1} a_i x_i + (a_n - \varepsilon) x_n \geq a_0
\]

is valid for \( Q(\bar{x}) \) for some \( \varepsilon > 0 \). Since (2.3) is the sum of \( \varepsilon x_n \geq 0 \) and the above inequality, and these inequalities are not multiples of each other as \( a_0 > 0 \), (2.3) does not induce a facet of \( Q(\bar{x}) \). This proves Claim 1.

**Claim 2:** \( a_k > 0 \) for \( k = 1, \ldots, n \).
By Claim 1 we have that

\[ \sum_{i=1}^{n} a_i \bar{x}_i = a_0. \]

Pick \( k \in \{1, \ldots, n-1\} \). Since \( v^k \) satisfies (2.3), we have that

\[ \sum_{i=1}^{k} a_i \bar{x}_i + a_k \geq a_0. \]

Subtracting the above equation to this inequality, we obtain

\[ a_k \geq \sum_{i=k+1}^{n} a_i \bar{x}_i > 0, \]

where the inequality follows because \( a_i \geq 0 \) for \( i = 1, \ldots, n \) and \( a_n \bar{x}_n > 0 \). This proves Claim 2.

Claim 2 shows that if \( x'' \neq x' \), \( x'' \geq x' \) and \( x' \) satisfies (2.3), then \( x'' \) cannot satisfy (2.3) at equality. Therefore, as \( Q(\bar{x}) \) is a full dimensional polyhedron and (2.3) induces a facet, this inequality must be satisfied at equality by \( v^1, \ldots, v^n \). This implies that, up to positive scaling, (2.3) is

\[ \sum_{i=1}^{n} d_i^0 x_i \geq \sum_{i=1}^{n} d_i^0 \bar{x}_i. \]

**Remark 2.4.** In the description given by Theorem 2.3, for every \( k \) such that \( \bar{x}_k = 0 \) the \( k \)-th lex-cut is redundant, as it is the inequality \( x_k \geq 0 \). Furthermore, if \( \bar{x}_1 > 0 \) then also the inequality \( x_1 \geq 0 \) is redundant, as it is dominated by the first lex-cut (which is \( x_1 \geq \bar{x}_1 \)). It can be verified that the remaining inequalities provide an irredundant description of \( Q(\bar{x}) \).

### 2.2 General lattice bases

Let \( \{e^1, \ldots, e^n\} \) be a lattice basis of \( \mathbb{Z}^n \). Then the \( n \times n \) matrix \( C \) whose rows are \( e^1, \ldots, e^n \) is unimodular, i.e., it is an integer matrix with determinant 1 or \(-1\). The unimodular transformation \( x \mapsto Cx \) and its inverse map integer points to integer points. By applying the transformation \( x \mapsto Cx \), the results of the previous subsection can be immediately extended to the lattice basis \( \{e^1, \ldots, e^n\} \).

In particular, the lexicographic ordering defined by the lattice basis is as follows: given \( x^1, x^2 \in \mathbb{R}^n \), we have \( x^1 < x^2 \) if and only if \( x_1 \neq x_2 \) and \( c^i x^1 < c^i x^2 \), where \( i \) is the smallest index for which \( c^i x^1 \neq c^i x^2 \).

The unimodular cone \( K \) is defined as \( K := \{ x \in \mathbb{R}^n : c^i x \geq 0, i = 1, \ldots, n \} \) and, for \( \bar{x} \in K \cap \mathbb{Z}^n \), \( Q(\bar{x}) := \text{conv}\{ x \in K \cap \mathbb{Z}^n : x \succeq \bar{x} \} \).

The leading index \( \ell(\bar{x}) \), for \( \bar{x} \in K \setminus \{0\} \), is the largest index \( i \) such that \( c^i \bar{x} > 0 \). Lemma 2.1 now reads as follows.

**Lemma 2.5.** Fix \( \bar{x} \in K \cap \mathbb{Z}^n \). The convex set \( Q(\bar{x}) \) has precisely the following extreme points \( v^1, \ldots, v^{\ell(\bar{x})} \): for \( k = 1, \ldots, \ell(\bar{x}) - 1 \), \( v^k \) is the unique point satisfying

\[ c^i v^k = c^i \bar{x}, \quad i = 1, \ldots, k - 1 \]

\[ c^k v^k = c^k \bar{x} + 1 \]
\[ c^i v^k = 0, \quad i = k + 1, \ldots, n, \]

and \( v^{(\bar{x})} = \bar{x} \). Furthermore, the recession cone of \( Q(\bar{x}) \) is \( K \). In particular, \( Q(\bar{x}) \) is a full-dimensional polyhedron.

For \( \bar{x} \in K, k \in \{1, \ldots, n\} \) and \( i \in \{1, \ldots, k\} \), the definition of the \( d^k_i \)'s is as follows:

\[
d^k_i := \begin{cases} 
1 & \text{if } i = k \\
\bar{x}^k & \text{if } i = k - 1, \\
\bar{x}^k \prod_{j=i+1}^{k-1} (c^j \bar{x} + 1), & \text{if } i \leq k - 2.
\end{cases}
\]

(2.4)

The \( k \)-th lex-cut associated with \( \bar{x} \) is the inequality

\[
\sum_{i=1}^{k} d^k_i c^i x \geq \sum_{i=1}^{k} d^k_i c^i \bar{x}.
\]

(2.5)

Lemma 2.2 still holds, while Theorem 2.3 reads as follows:

**Proposition 2.6.** If \( \bar{x} \in K \cap \mathbb{Z}^n \setminus \{0\} \), then the lex-cuts (2.2) for \( k = 1, \ldots, n \) and the inequalities \( c^i x \geq 0 \) for \( i = 1, \ldots, n \) provide a description of the polyhedron \( Q(\bar{x}) \).

### 3 An application to Nonlinear Integer Programming

Let \( \mathcal{S} \) be a family of compact (not necessarily connected or convex) subsets of \( \mathbb{R}^n \) with the following property:

- \( \text{if } S \in \mathcal{S} \) and \( H \) is a closed halfspace in \( \mathbb{R}^n \), then \( S \cap H \in \mathcal{S} \).

**Linear optimization** over \( \mathcal{S} \) is the following problem: given \( S \in \mathcal{S} \) and \( c \in \mathbb{Z}^n \), determine an optimal solution to the problem \( \min \{cx : x \in S\} \) or certify that \( S = \emptyset \). (Since \( S \) is compact, either \( S = \emptyset \) or the minimum is well defined.)

**Integer linear optimization** over \( \mathcal{S} \) is defined similarly, but the feasible region is \( S \cap \mathbb{Z}^n \), the set of integer points in \( S \).

We prove that an oracle for solving linear optimization over \( \mathcal{S} \) suffices to design a finite cutting plane algorithm that solves integer linear optimization over \( \mathcal{S} \).

We now make this statement more precise. Given a subset \( S \) of \( \mathbb{R}^n \) and \( c \in \mathbb{Z}^n \), let \( \bar{x} \in S \) be an optimal solution of the program \( \min \{cx : x \in S\} \). A cutting plane is a linear inequality that is valid for \( S \cap \mathbb{Z}^n \) and is violated by \( \bar{x} \) (which in this case is in \( S \setminus \mathbb{Z}^n \)).

A (pure) cutting plane algorithm for integer linear optimization over \( \mathcal{S} \) is an iterative procedure of the following type:

- Let \( S \in \mathcal{S} \) and \( c \in \mathbb{Z}^n \) be given.
- If \( S = \emptyset \), then \( S \cap \mathbb{Z}^n = \emptyset \). Otherwise, find an optimal solution \( \bar{x} \) of \( \min \{cx : x \in S\} \).
- If \( \bar{x} \in S \cap \mathbb{Z}^n \), stop: \( \bar{x} \) is an optimal solution to \( \min \{cx : x \in S \cap \mathbb{Z}^n\} \). Otherwise, detect a cutting plane and let \( H \) denote the corresponding half-space. Replace \( S \) with \( S \cap H \) and iterate.
Assume without loss of generality that the objective function vector \( c \) has relatively prime entries. Then there exists a lattice basis \( \{c^1, \ldots, c^n\} \) of \( \mathbb{Z}^n \) such that \( c^1 = c \). The optimal solution \( \bar{x} \) of \( \min\{cx : x \in S\} \) found by our algorithm will be a \textit{lexicographically minimum} or \( \text{lex-min} \) solution in \( S \) with respect to the lattice basis: i.e., \( \bar{x} < x \) for every \( x \in S \setminus \{\bar{x}\} \). The \text{lex-min} vector \( \bar{x} \) in \( S \) can be computed by solving the following programs:

- \( c^1\bar{x} = \min\{c^1x : x \in S\} \);
- \( c^2\bar{x} = \min\{c^2x : x \in S, c^1x = c^1\bar{x}\} \);
- \( c^3\bar{x} = \min\{c^3x : x \in S, c^1x = c^1\bar{x}, c^2x = c^2\bar{x}\} \);
- \ldots
- \( c^n\bar{x} = \min\{c^n x : x \in S, c^1x = c^1\bar{x}, \ldots, c^{n-1}x = c^{n-1}\bar{x}\} \).

Since \( S \) is nonempty and compact, the above minima are well-defined and can be computed by applying the oracle \( n \) times. Furthermore these conditions uniquely define \( \bar{x} \).

Algorithm 1 describes the procedure in detail, in particular how to find \( \bar{x} \) and a cutting plane whenever \( \bar{x} \not\in S \cap \mathbb{Z}^n \). Note that since \( S \) is compact, numbers \( \ell_1^*, \ldots, \ell_n^* \) (as defined in Algorithm 1) exist and can be determined by querying the linear optimization oracle \( 2n \) times. Moreover, as \( \{c^1, \ldots, c^n\} \) is a lattice basis of \( \mathbb{Z}^n \), an index \( k \) as in step 4 always exists when \( \bar{x} \not\in \mathbb{Z}^n \).

**Algorithm 1:** Resolution of integer linear optimization over \( S 

**Input:** \( S \in S \) with \( S \neq \emptyset \), \( c \in \mathbb{Z}^n \setminus \{0\} \) with relatively prime entries, and a lattice basis \( \{c^1, \ldots, c^n\} \) of \( \mathbb{Z}^n \) with \( c^1 = c \).

**Output:** an optimal integer solution \( \bar{x} \) for the problem \( \min\{cx : x \in S\} \) or a certificate that \( S \cap \mathbb{Z}^n = \emptyset \).

1. Compute \( \ell_i^* := \min\{c^i x : x \in S\} \) and \( \ell_i := \lceil \ell_i^* \rceil \) for \( 1 \leq i \leq n \), and apply a translation so that \( \ell_i = 0 \) for \( 1 \leq i \leq n \). Let \( K := \{x \in \mathbb{R}^n : c^i x \geq 0, \ i = 1, \ldots, n\} \) and replace \( S \) with \( S \cap K \).
2. If \( S = \emptyset \), stop: the given problem is infeasible.
3. Else, compute the \text{lex-min} solution \( \bar{x} \) in \( S \) with respect to \( \{c^1, \ldots, c^n\} \).
4. If \( \bar{x} \in \mathbb{Z}^n \), return \( \bar{x} \).
5. Else, let \( k \) be the smallest index such that \( c^k\bar{x} \not\in \mathbb{Z} \) and compute

\[
d_i^k := \begin{cases} 1 & \text{if } i = k \\ \lceil c^k\bar{x} \rceil & \text{if } i = k - 1, \\ \lceil c^k\bar{x} \rceil \prod_{j=i+1}^{k-1} (c^j\bar{x} + 1) & \text{if } i \leq k - 2. \end{cases}
\]

Replace \( S \) with \( S \cap H \), where \( H \) is the halfspace defined by the inequality

\[
\sum_{i=1}^{k} d_i^k c^i x \geq \sum_{i=1}^{k-1} d_i^k c^i \bar{x} + d_k^k \lceil c^k\bar{x} \rceil \quad (3.1)
\]

and go to step 2.
Given $x \in K$, let $x^\uparrow$ be the lex-min vector in $K \cap \mathbb{Z}^n$ such that $x \preceq x^\uparrow$. Obviously $x = x^\uparrow$ if and only if $x \in \mathbb{Z}^n$. If $x \not\in \mathbb{Z}^n$, let $k$ be the smallest index such that $c^k x \not\in \mathbb{Z}$. It is immediate to see that $x^\uparrow$ is the unique point satisfying the following conditions:

$$c^i x^\uparrow = c^i x, \; i < k; \; c^k x^\uparrow = \lceil c^k x \rceil; \; c^i x^\uparrow = 0, \; i > k. \quad (3.2)$$

**Proposition 3.1.** Inequality (3.1) defines a cutting plane. Algorithm 4 terminates after a finite number of iterations.

*Proof.* By (3.2), inequality (3.1) is the $k$-th lex-cut (2.5) associated with $\bar{x}^\uparrow$. Since, after the preprocessing of step 1, $S \subseteq K$ and $\bar{x}$ is the lex-min point in $S$, $\bar{x} \preceq \bar{x}^\uparrow < x'$ for every $x' \in S \cap \mathbb{Z}^n \setminus \{\bar{x}\}$. Thus $S \cap \mathbb{Z}^n \subseteq Q(\bar{x}^\uparrow)$ and by Lemma 2.2 inequality (3.1) is valid for $S \cap \mathbb{Z}^n$. As $c^k \bar{x} \not\in \mathbb{Z}$ and $d^k > 0$, the inequality is violated by $\bar{x}$. This shows that (3.1) defines a cutting plane.

As different iterations of the algorithm use cuts (3.1) associated with lexicographically increasing vectors in $S \cap \mathbb{Z}^n$, and $S$ is bounded, the number of iterations of the algorithm is finite. □

### 4 Comparison with Gomory and split cuts

Given a set $S$, a Chvátal–Gomory inequality for $S$ is a linear inequality of the form $g x \geq \lceil \gamma \rceil$ for some $g \in \mathbb{Z}^n$ and $\gamma \in \mathbb{R}$ such that the inequality $g x \geq \gamma$ is valid for $S$. We call $g x \geq \lceil \gamma \rceil$ a proper Chvátal–Gomory inequality if $g x \geq \lceil \gamma \rceil$ is violated by at least one point in $S$.

**Proposition 4.1.** Given $S \subseteq \mathbb{R}^n$, every proper Chvátal–Gomory inequality for $S$ is an inequality of the type (3.1) for some lattice basis $\{c^1, \ldots, c^n\}$ of $\mathbb{Z}^n$.

*Proof.* Let $g x \geq \lceil \gamma \rceil$ be a proper Chvátal–Gomory inequality for $S$. Without loss of generality, we assume that the entries of $g$ are relatively prime integers. Let $\bar{x}$ be the lex-min solution found at the first iteration of Algorithm 4 with respect to some lattice basis $\{c^1, \ldots, c^n\}$, with $c^1 = g$. Since $g x \geq \lceil \gamma \rceil$ is a proper Chvátal–Gomory inequality for $S$, we have $\gamma \leq g \bar{x} < \lceil \gamma \rceil$. In particular, $g \bar{x} \not\in \mathbb{Z}$. Then the corresponding cut of the type (3.1) is (equivalent to) $g x \geq \lceil g \bar{x} \rceil = \lceil \gamma \rceil$. □

The converse of the above proposition is false; this will follow from a stronger result.

A linear inequality is a split cut for $S$ if there exist $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$ such that the inequality is valid for both $\{x \in S : \pi x \leq \pi_0\}$ and $\{x \in S : \pi x \geq \pi_0 + 1\}$. It is known that every Chvátal–Gomory inequality is a split cut but not vice versa.

The next result shows that our family of cuts is not included in and does not include the family of split cuts. Combined with the previous proposition, this implies that our family of cuts strictly contains the Chvátal–Gomory inequalities.

**Proposition 4.2.** There exist a bounded polyhedron $S$ and a split cut for $S$ that cannot be obtained as (and is not implied by) an inequality of the type (3.1). Conversely, there exist a bounded polyhedron $S$ and an inequality of the type (3.1) that is not a split cut for $S$.

*Proof.* Let $S \subseteq \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(1/2,-1)$. (See Figure 4 to follow the proof.) The inequality $x_2 \geq 0$ is a split cut for $S$, as it is valid for both sets $\{x \in S : x_1 \leq 0\}$ and $\{x \in S : x_1 \geq 1\}$. Note that after the application of the cut,
the continuous relaxation becomes the segment with endpoints \((0,0)\) and \((1,0)\), which is the convex hull of the integer points in \(S\).

Assume that the cut \(x_2 \geq 0\) can be obtained via an iteration of Algorithm \(\mathcal{P}\) for some lattice basis \(\{c^1, c^2\}\) and the corresponding bounds \(\ell_1, \ell_2 \in \mathbb{Z}\). In the following, we will write \(c^1 = (c^1_1, c^1_2)\) and \(c^2 = (c^2_1, c^2_2)\).

Recall that in Algorithm \(\mathcal{P}\) a translation is applied such that \(\ell_i = 0\) for every \(i\). However, in this proof it is convenient to work without applying the translation. It is easy to see that in this case the form of the lex-cut is still \((3.1)\), but now the \(d^k_i\) are defined as follows:

\[
d^k_i := \begin{cases} 
1 & \text{if } i = k \\
\left[ c^k_x - \ell_k \right] & \text{if } i = k - 1, \\
\left[ c^k_x - \ell_k \right] \prod_{j=i+1}^{k-1} (c^j_x + 1 - \ell_j), & \text{if } i \leq k - 2.
\end{cases}
\]

Since the point \((1/2, -1)\) is the only fractional vertex of \(S\), we must have \(\bar{x} = (1/2, -1)\), otherwise no cut is generated. Suppose \(k = 1\), i.e., \(c^1 \bar{x} \notin \mathbb{Z}\) (see step 5 of the algorithm). Then the inequality generated by the algorithm is equivalent to \(c^1 x \geq \lfloor c^1 \bar{x} \rfloor\). Since this inequality must be equivalent to \(x_2 \geq 0\) and the entries of \(c^1\) are relatively prime integers, we necessarily have \(c^1 = (0,1)\). But then \(c^1 \bar{x} = -1\), a contradiction to the assumption \(c^1 \bar{x} \notin \mathbb{Z}\).

Suppose now \(k = 2\), i.e., \(c^1 \bar{x} \in \mathbb{Z}\) and \(c^2 \bar{x} \notin \mathbb{Z}\). Then the inequality given by the algorithm is

\[
d^2_1 (c^1 x - c^1 \bar{x}) + c^2 x - \lfloor c^2 \bar{x} \rfloor \geq 0. \tag{4.1}
\]

We claim that \(c^1_1 \neq 0\). If this is not the case, then \(c^1_1 = 0\) and \(c^1_2 \neq 0\) (as \(\{c^1, c^2\}\) is a basis), and inequality \(4.1\) does not reduce to the desired cut \(x_2 \geq 0\), as the coefficient of \(x_1\) is \(d^2_1 c^1_1 + c^2_1 = c^2_1 \neq 0\). Thus \(c^1_1 \neq 0\). This implies that either the point \((0,-1)\) or the point \((1,-1)\) satisfies the strict inequality \(c^1 x > c^1 \bar{x}\). We assume that this holds for \(\hat{x} := (0,-1)\) (the other case is similar). Note that \(c^1 \hat{x} \geq c^1 \bar{x} + 1\), as \(c^1 \bar{x} \in \mathbb{Z}\) and \(c^1, \hat{x} \in \mathbb{Z}^2\). Furthermore, the slope of the line defined by the equation \(c^1 x = c^1 \bar{x}\) is positive.

If \(c^2 \hat{x} \geq \ell_2\), then \(\hat{x}\) satisfies inequality \(4.1\), as \(c^1 \hat{x} - c^1 \bar{x} \geq 1\) and \(c^2 \hat{x} - c^2 \bar{x} \geq \ell_2 - c^2 \bar{x} \geq -d^2_1\). Since the point \((1,0)\) also satisfies \(4.1\) (as it is an integer point in \(S\)), the middle point of \(\hat{x}\) and \((1,0)\) satisfies \(4.1\). However, the middle point is \((1/2, -1/2)\), which is in \(S\). This shows that in this case \((4.1)\) is not equivalent to \(x_2 \geq 0\).

Therefore we assume \(c^2 \hat{x} < \ell_2\). Since \(c^2 \hat{x} \geq \ell_2\), the line defined by the equation \(c^2 x = \ell_2\) intersects the line segments \([\hat{x}, \bar{x}]\) in a point distinct from \(\hat{x}\). Then, because \((0,0)\) satisfies the inequality \(c^2 x \geq \ell_2\) (as it is in \(S\)), the slope of the line defined by the equation \(c^2 x = \ell_2\) is negative. Furthermore, since \(c^2, \hat{x} \in \mathbb{Z}^2\), we have \(c^2 \hat{x} \leq \lfloor \ell_2 \rfloor\), and thus the line defined by the equation \(c^2 x = \lfloor \ell_2 \rfloor\) intersects \([\hat{x}, \bar{x}]\) in some point \(x^*\).

Now consider the system \(c^1 x = c^1 \bar{x}, c^2 x = \lfloor \ell_2 \rfloor\). Since the constraint matrix is unimodular (as \(\{c^1, c^2\}\) is a lattice basis of \(\mathbb{Z}^2\)) and the right-hand sides are integer, the unique solution to this system is an integer point. However, the first equation defines a line with positive slope containing \(\hat{x}\) and the second equation defines a line with negative slope containing \(x^*\). From this we see that the intersection of the two lines is a point satisfying \(0 < x_1 \leq 1/2\) and therefore cannot be an integer point, a contradiction. This concludes the proof that there is a split cut that cannot be obtained via an iteration of Algorithm \(\mathcal{P}\).

For the converse, let \(S \subseteq \mathbb{R}^2\) be the triangle with vertices \((0,3/2), (1/4,0)\) and \((1,0)\). If we take \(c^1, c^2\) to be the vectors in the standard basis of \(\mathbb{R}^2\), and \(\ell_1 = \ell_2 = 0\), then Algorithm \(\mathcal{P}\) yields the cut \(2x_1 + x_2 \geq 2\). Note that every point in \(S\) other than \((1,0)\) is cut off by
Figure 1: Illustration of the first part of the proof of Proposition 4.2. The inequality $x_2 \geq 0$ is a split cut for the shadowed triangle, but is not of the type (3.1).

This inequality. Thus, if the inequality $2x_1 + x_2 \geq 2$ is a split cut for $S$, then there exist $\pi \in \mathbb{Z}^2$ and $\pi_0 \in \mathbb{Z}$ such that $S$ is contained in the “strip” \{ $x \in \mathbb{R}^2 : \pi_0 \leq \pi x \leq \pi_0 + 1$ \}. Since $S$ contains a horizontal and a vertical segment of length $3/4$, this is possible only if the Euclidean distance between the lines \{ $x \in \mathbb{R}^2 : \pi x = \pi_0$ \} and \{ $x \in \mathbb{R}^2 : \pi x = \pi_0 + 1$ \} is at least $3\sqrt{2}/4$. Therefore $\|\pi\|^2 \leq \left(\frac{3\sqrt{2}}{3}\right)^2 = \frac{32}{9} < 4$. Since $\pi$ is an integer vector, we deduce that $\pi_1, \pi_2 \in \{0, 1, -1\}$. It can be verified that if $|\pi_1| = |\pi_2| = 1$ then $S$ is not contained in the strip. Therefore one entry of $\pi$ is 0 and the other is 1 or $-1$. It can be checked that the only strip of this type containing $S$ is \{ $x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1$ \}. However, the inequality $2x_1 + x_2 \geq 2$ is not valid for all the points in \{ $x \in S : x_1 \leq 0$ \} $\cup$ \{ $x \in S : x_1 \geq 1$ \}, as the point $(0, 3/2)$ is in this set but violates the inequality.

5 Lexicographic enumeration and the number of iterations

Recall the notation $x^\dagger$ introduced in (3.2). We extend that definition to sets as follows: given $S \subseteq \mathbb{R}^n$, let $S^\dagger := \{ x^\dagger : x \in S \}$.

**Observation 5.1.** Given a nonempty set $S \in \mathcal{S}$, let $(\bar{x})$ be the sequence of points in $S$ computed at step 3 of Algorithm 1. Then the sequence $(\bar{x}^\dagger)$ is the lex-ordering of some distinct points in $S^\dagger$.

**Proof.** If $\bar{x}$ is a point computed at step 3 of Algorithm 1 then clearly $\bar{x}^\dagger \in S^\dagger$, as $\bar{x} \in S$. Thus we only have to show that if $\bar{x}$ and $\bar{x}$ are points computed at step 3 in two consecutive iterations (say iterations $q$ and $q + 1$), then $\bar{x}^\dagger < \bar{x}^\dagger$. Assume not. Then $\bar{x}^\dagger = \bar{x}^\dagger$ and therefore the cuts introduced at these two iterations would be exactly the same. But then the cut generated at iteration $q$ would already cut off $\bar{x}$, contradicting the fact that at iteration $q + 1$ the point computed at step 3 is $\bar{x}$. \hfill $\square$

The above observation shows that $|S^\dagger|$ is an upper bound on the number of cuts produced by Algorithm 1. We next construct a convex body containing no integer points for which this bound is exponential and tight.

**Proposition 5.2.** For every $n \in \mathbb{N}$, there is a convex subset $S$ of $[0, 1]^n$ (described by a single convex constraint plus variable bounds) on which Algorithm 1 computes $|S^\dagger| = 2^n - 1$ cuts.
Proof. We choose the standard basis \( \{ e^1, \ldots, e^n \} \) as lattice basis of \( \mathbb{Z}^n \). Let \( 1 \) be the point in \( \mathbb{R}^n \) with all entries equal to 1, and let \( \| \cdot \| \) denote the Euclidean norm. Define

\[ S := \left\{ x \in [0,1]^n : \left\| x - \frac{1}{2} \right\|^2 \leq \frac{n}{4} - \frac{3}{16} \right\}. \]

Note that \( S \cap \mathbb{Z}^n = \emptyset \) and \( \ell_i = 0 \) for \( i = 1, \ldots, n \). Furthermore, for every \( x \in \{0,1\}^n \setminus \{1\} \), \( S \) contains the point \( x' \) obtained from \( x \) by setting to \( \frac{1}{4} \) the entry with largest index that is 0. As \( S \subseteq [0,1]^n \), this shows that \( S^\uparrow = \{0,1\}^n \setminus \{0\} \), and thus \( |S^\uparrow| = 2^n - 1 \).

We now show that every point in \( S^\uparrow \) is of the form \( \bar{x}^\uparrow \) for some point \( \bar{x} \) found in step 3. Let \( \bar{x} \) be the point computed at some iteration of step 3. Then the cut (3.1) introduced at step 3 is a lex-cut associated with \( \bar{x}^\uparrow \). By Lemma 2.2, this inequality is satisfied by all \( x \in \{0,1\}^n \) such that \( x \preceq \bar{x}^\uparrow \). As (3.1) is an inequality with nonnegative coefficients, it is also satisfied by the point \( x' \) of \( S \) for every \( x \in \{0,1\}^n \) such that \( x \preceq \bar{x}^\uparrow \). This implies that, if we denote by \( \bar{x} \) the point computed in step 3 at the next iteration, \( \bar{x}^\uparrow \) is the lex-min point in \( \{0,1\}^n \) that is lexicographically larger than \( \bar{x}^\uparrow \). Thus every point in \( S^\uparrow \) is of the form \( \bar{x}^\uparrow \) for some point \( \bar{x} \) found in step 3. Together with Observation 5.1, this shows that precisely \( |S^\uparrow| \) cuts are needed to discover that \( S \) contains no integer points. \( \square \)

One may ask whether there exists an enumerative algorithm that achieves the same performance. Algorithm 2 is the best we could find.

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**Algorithm 2:** Resolution of integer linear optimization over \( S \) via lex-enumeration

**Input:** \( S \subseteq \mathbb{R}^n \), \( c \in \mathbb{Z}^n \setminus \{0\} \) with relatively prime entries and a lattice basis \( \{ c^1, \ldots, c^n \} \) of \( \mathbb{Z}^n \), with \( c^1 = c \).

**Output:** an optimal integer solution \( \bar{x} \) for the problem \( \min \{ cx : x \in S \} \) or a certificate that \( S \cap \mathbb{Z}^n = \emptyset \).

1. Translate \( S \) so that \( S \subseteq \{ x \in \mathbb{R}^n : c_i^j x \geq 0, \ i = 1, \ldots, n \} \). Set \( \alpha_1 := \cdots := \alpha_n := 0 \) and \( i^* := 1 \).
2. Let \( S^* := S \cap \{ x \in \mathbb{R}^n : c_i^j x = \alpha_i, \ i < i^* ; \ c_i^j x \geq \alpha_i, \ i \geq i^* \} \).
3. If \( S^* = \emptyset \):
   4. If \( i^* = 1 \), stop: \( S \cap \mathbb{Z}^n = \emptyset \).
   5. Else update \( i^* := i^* - 1 \), \( \alpha_i := \alpha_i + 1 \), \( \alpha_i := 0 \) for \( i > i^* \), and go to step 2.
6. Else
   7. Let \( \bar{x} \) be the lex-min point in \( S^* \).
   8. If \( \bar{x}^\uparrow \in S^* \), stop: \( \bar{x}^\uparrow \) is the lex-min point in \( S \cap \mathbb{Z}^n \).
   9. Else update \( i^* := n \), \( \alpha_i := c_i^\uparrow x \) for \( i = 1, \ldots, n \), and go to step 2.

---

The correctness of this algorithm is based on the following fact.

**Remark 5.3.** In Algorithm 2, if \( \bar{x} \) and \( \bar{x} \) denote the points computed at two consecutive executions of line 4 then \( \bar{x} \) is the lex-min point in \( S \) that is lexicographically larger than \( \bar{x}^\uparrow \).

Let \( C \) be the \( n \times n \) matrix whose rows are \( c^1, \ldots, c^n \) and let \( S \subseteq \mathbb{R}^n \) be given. For every \( \bar{x} \subseteq S^\uparrow \), let \( V(\bar{x}) \) be the set of the following \( n \) vectors: \( \alpha = C\bar{x} \) and, for every \( k \in \{1, \ldots, n-1\} \), the unique solution to the system \( \alpha_i = c_i^k \bar{x} \) for \( i < k \), \( \alpha_k = c_k^\uparrow \bar{x} + 1 \), \( \alpha_i = 0 \) for \( i > k \). Notice that by Lemma 2.5, \( V(\bar{x}) \) contains all vectors of the form \( Cx \) where \( x \) is a vertex of \( Q(\bar{x}) \).
Let $V(S) = \bigcup_{x \in S^\dagger} V(x)$. Notice that, given $\bar{x}, \bar{y} \in S^\dagger$, the set $V(\bar{x}) \cap V(\bar{y})$ may be nonempty.

**Proposition 5.4.** Given a set $S \in S$, let $(\alpha)$ be the sequence of vectors used to define the sequence of sets $(S^\star)$ in Step 3 of Algorithm 3

- If $S \cap \mathbb{Z}^n = \emptyset$, then $(\alpha)$ is the lex-ordering of all points in $V(S) \cup \{0\}$ with respect to the standard basis.
- If $S \cap \mathbb{Z}^n \neq \emptyset$, the sequence is truncated to the lex-min vector $\alpha$ (with respect to the standard basis) such that $C^{-1} \alpha \in S \cap \mathbb{Z}^n = S \cap S^\dagger$.

**Proof.** Clearly the sequence $(\alpha)$ starts with $\alpha = 0$ and is lexicographically increasing with respect to the standard basis.

Let $\alpha \neq 0$ be a vector used in step 3 at some iteration $q > 1$ and let $\bar{x}$ be the last point computed at line 7 before iteration $q$; say that $\bar{x}$ is computed at iteration $q' < q$. If $q' = q - 1$, then $\alpha = C\bar{x}^\dagger$ and therefore $\alpha \in V(S)$. If $q' = q - t$ for some $t > 1$, then line 5 is executed $t - 1$ times between iterations $q'$ and $q$. In this case, $\alpha$ is the vector defined by $\alpha_i = c^i \bar{x}^\dagger$ for $i \leq n - t$, $\alpha_{n-t+1} = c^{n-t+1} \bar{x}^\dagger + 1$, $\alpha_i = c^i \bar{x}^\dagger$ for $i \geq n - t + 2$, and therefore $\alpha \in V(S)$.

We now show that every point in $V(S)$ is in the sequence $(\alpha)$. By Remark 5.3, the sequence $(\alpha)$ contains all points of the form $Cx$ for $x \in S^\dagger$. Let $\alpha \in V(S)$, where $\alpha$ is not of the form $Cx$ for any $x \in S^\dagger$. Then there exist $\bar{x} \in S^\dagger$ and an index $k < n$ such that $\alpha_i = c^i \bar{x}$ for $i < k$, $\alpha_kx = c^k \bar{x} + 1$, and $\alpha_i = 0$ for $i > k$. Consider the last iteration of line 7 in which $\bar{x}$ satisfies $c^i \bar{x}^\dagger = c^i \bar{x}$ for $i \leq k$ (this definition makes sense because, as shown above, $\bar{x} = \bar{x}^\dagger$ at some iteration of line 7). The algorithm now sets $\alpha = C\bar{x}^\dagger$ and executes line 5 $k$ consecutive times. After this, we have $\alpha = C\bar{x}$. This shows that every point in $V(S)$ is in the sequence $(\alpha)$.

We remark that in the definition of $\alpha$ at line 9 we could impose the stronger condition $\alpha_n := c^n \bar{x}^\dagger + 1$. However, this would not change substantially the bounds on the number of iterations shown above. Moreover, when $S$ is convex the number of iterations is precisely the same in both cases.

By Observation 5.1 and Proposition 5.4, the number of iterations of Algorithm 1 and Algorithm 2 is upper-bounded by $|S^\dagger|$ and $|V(S)| + 1$, respectively. Note that the latter bound is always larger than the former. In particular, for the example in Proposition 5.2 we have $|V(S)| = 2^n + 2^{n-1} - 2$, thus in that case Algorithm 2 executes roughly 50% more iterations than Algorithm 1. However comparing the two algorithms by counting the number of iterations may not be “fair”, as the computational effort varies from iteration to iteration: for instance, the computation of a lex-min solution (line 3 of Algorithm 1 and line 7 of Algorithm 2) requires up to $n$ oracle calls, while the iterations of Algorithm 2 in which $S^\star$ is empty only require a single oracle call. Nonetheless the results on the number of iterations at least indicate that, from the theoretical point of view, Algorithm 1 tends to be more efficient than Algorithm 2.

### 6 Concluding remarks

An obvious variant of Algorithm 1 is the following: instead of being computed only once at the beginning of the procedure, the lower bounds $\ell_i$ can be updated at every iteration.
or whenever it seems convenient. It can be verified that the bounds of Observation 5.1 and Proposition 5.2 also hold for this variant of the algorithm: the proofs are the same.

In view of Observation 5.1 and Proposition 5.2 the cardinality of $S^\uparrow$ truncated to the lex-min point in $S^\uparrow \cap S$ plays a crucial role in the performance of Algorithm 1. This number is dependent on the choice of the lattice basis and its ordering. It is easy to see that different choices of the lattice basis (or different choices of the ordering of the elements of the same lattice basis) may result in a different number of iterations of the algorithm. However, this is not always the case: for instance, in the example in Proposition 5.2 Algorithm 1 would produce the same number of iterations regardless of the ordering of the standard basis.

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