GLOBAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL RANDOM VISCOUS NONLINEAR WAVE EQUATIONS

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Abstract. We study well-posedness of viscous nonlinear wave equations (vNLW) on the two-dimensional torus with a stochastic forcing or randomized initial data. In particular, we prove pathwise global well-posedness of the stochastic defocusing vNLW with an additive stochastic forcing $D^\alpha \xi$, where $\alpha < \frac{1}{2}$ and $\xi$ denotes the space-time white noise. We also study the deterministic vNLW with randomized initial data and prove its almost sure global well-posedness in the defocusing case.

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1. Introduction

1.1. Viscous nonlinear wave equations. In this paper, we consider the following nonlinear wave equation (NLW) on the two-dimensional torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$, augmented by viscous effects:

\[
\begin{align*}
\partial_t^2 u + (1 - \Delta) u + D \partial_t u + |u|^{p-1} u &= \phi \xi \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1),
\end{align*}
\]

where $p > 1$, $D = |\nabla| = \sqrt{-\Delta}$, and $\xi$ denotes the (Gaussian) space-time white noise on $\mathbb{R}_+ \times T^2$. Here, $\phi$ is a (possibly unbounded) operator on $L^2(T^2)$. In particular, we are interested in the following two cases.

(i) $\phi = D^{\alpha}$ for some suitable range of $\alpha < \frac{1}{2}$. In this case, the equation (1.1) reduces to the following stochastic viscous nonlinear wave equation (S\text{vNLW}):

\[
\begin{align*}
\partial_t^2 u + (1 - \Delta) u + D \partial_t u + |u|^{p-1} u &= D^{\alpha} \xi \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1).
\end{align*}
\]

(ii) $\phi = 0$. In this case, the equation (1.1) reduces to the viscous nonlinear wave equation (vNLW) and we study the well-posedness issue with randomized initial data:

\[
\begin{align*}
\partial_t^2 u + (1 - \Delta) u + D \partial_t u + |u|^{p-1} u &= 0 \\
(u, \partial_t u)|_{t=0} &= (u_0^\omega, u_1^\omega).
\end{align*}
\]

Here, $(u_0^\omega, u_1^\omega)$ denotes an appropriate randomization of a given pair $(u_0, u_1)$ of deterministic functions on $T^2$. See Subsection 1.3.

Our main goal in this paper is (i) to prove pathwise global well-posedness of (1.2) for $\alpha \leq \alpha_p$ and (ii) to prove almost sure global well-posedness of (1.3). As we see below, after the first order expansion (see (1.11) and (1.17)), the proof of pathwise global well-posedness of (1.2) and the proof of almost sure global well-posedness of (1.3) are reduced to establishing a certain energy bound and thus both of the problems can be treated essentially in a unified manner. See Sections 5 and 6.

In [29], Kuan-Čanić proposed the following viscous NLW on $\mathbb{R}^2$:

\[
\partial_t^2 u - \Delta u + 2\mu D \partial_t u = F(u),
\]

where $\mu > 0$ and $F(u)$ is a general external forcing. This equation typically shows up in fluid-structure interaction problems, such as the interaction between a stretched membrane and a viscous fluid. The viscosity term $2\mu D \partial_t u$ in (1.4) comes from the Dirichlet-Neumann operator typically arising in fluid-structure interaction problems in three dimensions. See [29] and [31] for the derivation of (1.4). It is easy to see that, when $\mu \geq 1$, the equation (1.4) is purely parabolic (see [31] [34]). On the other hand, when $0 < \mu < 1$, the viscous NLW (1.4) exhibits an interesting mixture of dispersive effects and parabolic smoothing. Since the precise value of $0 < \mu < 1$ does not play an important role, we simply set $\mu = \frac{1}{2}$.

In addition, we consider a defocusing power-type nonlinearity of the form

\[
F(u) = -|u|^{p-1} u,
\]

for positive real numbers $p > 1$. This power-type nonlinearity has been studied extensively for nonlinear dispersive equations (see, for example, [52]). With $\mu = \frac{1}{2}$ and
$F(u) = -|u|^{p-1} u$, the general form of vNLW (1.4) becomes the following version of vNLW:

$$\partial_t^2 u - \Delta u + D\partial_t u + |u|^{p-1} u = 0$$  \hspace{1cm} (1.5)

We now turn our attention to the analytical aspects of vNLW (1.5). Note that as in the case of the usual NLW:

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0$$  \hspace{1cm} (1.6)

the viscous NLW (1.5) on $\mathbb{R}^2$ enjoys the following scaling symmetry: $u(t, x) \rightarrow u_\lambda(t, x) := \lambda^{2/p - 1} u(\lambda t, \lambda x)$. Namely, if $u$ is a solution to (1.5), then $u_\lambda$ is also a solution to (1.5) for any $\lambda > 0$ with rescaled initial data. This scaling symmetry induces the scaling critical Sobolev regularity $s_{\text{scaling}}$ on $\mathbb{R}^2$ given by

$$s_{\text{scaling}} = 1 - \frac{2}{p - 1}$$

such that under this scaling symmetry, the homogeneous Sobolev norm on $\mathbb{R}^2$ remains invariant. While there is no scaling symmetry on $\mathbb{T}^2$, the scaling critical regularity $s_{\text{scaling}}$ still plays an important role in studying nonlinear partial differential equations in the periodic setting, especially for dispersive equations. Namely, in both periodic and non-periodic settings, a dispersive equation is usually well-posed in $H^s$ for $s > s_{\text{scaling}}$ and is usually ill-posed in $H^s$ for $s < s_{\text{scaling}}$. On the one hand, there is a good local well-posedness theory for dispersive equations above the scaling regularities (see [4, 41, 50] for the references therein). Moreover, we show in this paper that vNLW (1.5) is locally well-posed in $H^s(\mathbb{T}^2)$ for all $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$), where $s_{\text{crit}}$ is defined by

$$s_{\text{crit}} := \max(s_{\text{scaling}}, 0) = \max\left(1 - \frac{2}{p - 1}, 0\right),$$  \hspace{1cm} (1.7)

for a given $p > 1$. See Appendix A. Here, the second regularity restriction 0 is required to make sense of powers of $u$. On the other hand, many dispersive equations are known to be ill-posed below the scaling critical regularity. Among these ill-posedness results, many of them are in the form of norm inflation (see [15, 10, 13, 28, 41, 47, 14, 48, 53, 43, 19]), which is a stronger notion of ill-posedness. More precisely, norm inflation of the wave equation in $L^s(\mathbb{R}^d) = H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, for example, states the following: for any $\varepsilon > 0$, there exists a solution $u$ to (1.6) and $t \in (0, \varepsilon)$ such that

$$\|(u(0), \partial_t u(0))\|_{H^s} < \varepsilon \quad \text{but} \quad \|(u(t), \partial_t u(t))\|_{H^s} > \frac{1}{\varepsilon}.$$  

In [29], Kuan-Čanić proved norm inflation for vNLW (1.5) in $H^s(\mathbb{R}^d)$ for $0 < s < s_{\text{scaling}}$ and any odd integers $p \geq 3$. Moreover, they pointed out that the viscous term has the potential to slow down the growth of the $H^s$ norm, i.e. to slow down the speed of the norm inflation. For details, see [29]. Also, it is of interest to see if norm inflation for vNLW holds in negative Sobolev spaces. See [18].

In recent years, there is a growing interest in incorporating non-deterministic points of view in studying well-posedness of nonlinear dispersive equations below certain regularity thresholds. In particular, by considering randomized initial data (see [9, 10, 16, 35, 12, 3, 4, 49, 42, 5]), one can prove well-posedness results for nonlinear dispersive equations below certain critical regularities. See also a survey paper [6] for this method. In [29], Kuan-Čanić proved almost sure local well-posedness of the quintic vNLW (1.5) (with $p = 5$).
for an appropriate randomization of a given pair \((u_0, u_1)\) in \(\mathcal{H}^s(\mathbb{R}^2)\) for \(s > -\frac{1}{q}\). In \cite{31}, Kuan-Oh improved this result and proved almost sure local well-posedness of the quintic vNLW \((1.5)\) (with \(p = 5\)) in \(\mathcal{H}^s(\mathbb{R}^2)\) for \(s > -\frac{1}{5}\). Furthermore, they proved almost sure global well-posedness of the defocusing quintic vNLW \((1.5)\) for the same range of \(s > -\frac{1}{5}\).

We point out that the result in \cite{31} easily extends to the case \(1 < p < 5\), yielding almost sure global well-posedness of \((1.5)\) in \(\mathcal{H}^s(\mathbb{R}^2)\) for \(s > -\frac{1}{p}\). One of our main goals in this paper is to extend the result in \cite{31} to the super-quintic case \(p > 5\). In this paper, we study vNLW on \(\mathbb{T}^2\) so that we have a uniform treatment together with the stochastic vNLW \((1.2)\).

We point out, however, that our result on the deterministic vNLW \((1.3)\) with randomized initial data easily extends to vNLW on \(\mathbb{R}^2\). See Remark \ref{rem:1.8}.

Let us now turn our attention to the viscous NLW with a stochastic forcing. In \cite{30}, Kuan-ˇCani´c studied the stochastic viscous wave equation with a multiplicative noise on \(\mathbb{R}^d\), \(d = 1, 2\):

\[
\partial_t^2 u - \Delta u + D\partial_t u = f(u)\xi,
\]

where \(f\) is Lipschitz and \(\xi\) is the (Gaussian) space-time white noise on \(\mathbb{R}_+ \times \mathbb{R}^2\). In \cite{34}, T. Oh and the author studied (the renormalized version of) SvNLW \((1.2)\) with \(\alpha = \frac{1}{2}\). When \(\alpha = \frac{1}{2}\), the solution is not a function but is only a distribution and thus a renormalization on the nonlinearity is required to give a proper meaning to the dynamics (which in particular forces us to consider \(|u|^{p-1}u\) only for \(p \in 2\mathbb{N} + 1\) or \(u^k\) for an integer \(k \geq 2\)). See \cite{34} for details. In the cubic case, we proved pathwise global well-posedness. For an odd integer \(p \geq 5\), we also used an invariant measure argument to prove almost sure global well-posedness with suitable random initial data. In the stochastic setting, our goal in this paper is to investigate further well-posedness of SvNLW \((1.2)\) with an additive forcing \(D^\alpha\xi\) and, in particular, prove pathwise global well-posedness for any \(p > 1\), where the range of \(\alpha < \frac{1}{2}\) depends on the degree \(p > 1\) of the nonlinearity.

1.2. SvNLW with an additive stochastic forcing. Let us first consider SvNLW \((1.2)\). We say that \(u\) is a solution to SvNLW \((1.2)\) if \(u\) satisfies the following Duhamel formulation of \((1.2)\):

\[
u(t) = V(t)(u_0, u_1) - \int_0^t S(t-t')(|u|^p-1u)(t')dt' + \Psi.
\]

Here, \(V(t)\) is the linear propagator defined by

\[
V(t)(u_0, u_1) = e^{-\frac{D}{2}t}\left(\cos(t\|D\|) + \frac{D}{2\|D\|}\sin(t\|D\|)\right)u_0 + e^{-\frac{D}{2}t}\sin(t\|D\|)u_1
\]

and \(S(t)\) is defined by

\[
S(t) = e^{-\frac{D}{2}t}\sin(t\|D\|)/\|D\|,\]

where

\[
\|D\| = \sqrt{1 - \frac{3}{4}\Delta}.
\]

\footnote{Strictly speaking, almost sure global well-posedness holds for the noise \(\sqrt{2D^\frac{3}{2}}\xi\), which makes the Gibbs measure for the standard NLW invariant under the SvNLW dynamics. For pathwise global well-posedness, a precise coefficient in front of the noise \(D^\frac{3}{2}\xi\) does not play any role.}
and $\Psi$ denotes the stochastic convolution defined by
\[
\Psi := \Psi_\alpha = \int_0^t S(t-t')D^\alpha \xi(dt').
\] (1.10)

A standard argument shows that $\Psi$ belongs to $C(\mathbb{R}_+; W^{\frac{1}{2}-\alpha-\varepsilon, \infty}(\mathbb{T}^2))$ almost surely, where $\varepsilon > 0$ is sufficiently small; see Lemma 2.6 below. In particular, when $\alpha < \frac{1}{2}$, $\Psi$ is a well-defined function on $\mathbb{R}_+ \times \mathbb{T}^2$.

We first state a local well-posedness result for $\text{SvNLW (1.2)}$.

**Theorem 1.1.** Let $p > 1$ and $\alpha < \frac{1}{2}$. Define $q$, $r$, and $\sigma$ as follows.

(i) When $1 < p < 2$, set $q = 2 + \delta$, $r = \frac{4+2\delta}{1+\delta}$, and $\sigma = 0$, for some sufficiently small $\delta > 0$.

(ii) When $p \geq 2$, set $q = p + \delta$, $r = 2p$, and $\sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$ for some arbitrary $\delta > 0$.

Let $s \geq \sigma$. Then, $\text{SvNLW (1.2)}$ is pathwise locally well-posed in $H^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in H^s(\mathbb{T}^2)$, there exists $T = T_\omega(u_0, u_1)$ (which is positive almost surely) and a unique solution $u$ to $\text{SvNLW (1.2)}$ with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ in the class
\[
\Psi + C([0, T]; H^s(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)).
\]

We present the proof of Theorem 1.1 in Section 3. The proof of Theorem 1.1 is based on the following first order expansion ([37, 9, 17]):
\[
u = v + \Psi,
\] (1.11)

where the residual term $v$ satisfies the following equation:
\[
\begin{cases}
\partial_t^2 v + (1 - \Delta) v + D\partial_t v + |v + \Psi|^{p-1}(v + \Psi) = 0 \\
(v, \partial_t v)|_{t=0} = (u_0, u_1).
\end{cases}
\] (1.12)

See Proposition 3.1 for the pathwise local well-posedness result at the level of the residual term $v$ using the homogeneous Strichartz estimates for the viscous wave equation (Lemma 2.9).

The main idea of the proof of pathwise local well-posedness of $\text{SvNLW (1.2)}$ comes from [31]. Note that the nonlinearity $|u|^{p-1} u$ in $\text{SvNLW (1.2)}$ is not necessarily algebraic for general $p > 1$, which creates a difficulty for obtaining the difference estimate when applying the contraction argument. To deal with this issue, we apply the idea from Oh-Okamoto-Pocovnicu [42] using the fundamental theorem of calculus.

**Remark 1.2.** (i) Using the same argument, the proof of Theorem 1.1 works for both the defocusing case (with the nonlinearity $|u|^{p-1} u$) and the focusing case (with the nonlinearity $-|u|^{p-1} u$, i.e. with the negative sign). The same comment applies to Theorem 1.5.

The proof of Theorem 1.1 also works for $\text{SvNLW with nonlinearity } u^k$, where $k \geq 2$ is an integer. In fact, a simple argument based on Sobolev’s inequality can be applied to prove local well-posedness of $\text{SvNLW with nonlinearity } u^k$ in the class $\Psi + C([0, T]; H^s(\mathbb{T}^2))$ for $s \geq 1$. See, for example, Proposition 3.1 in [34].

(ii) We point out that Theorem 1.1 also applies if we have $-\Delta$ instead of $1 - \Delta$ in (1.2) by using an essentially identical proof. The same comment applies to Theorems 1.3, 1.5 and 1.7.
exists a time interval \( I \) with the Schauder estimate (Lemma 2.8) along with Theorem 1.1. See Section 5 for details.

We now turn our attention to pathwise global well-posedness of SvNLW (1.2), and we restrict our attention to the defocusing case. Our pathwise global well-posedness result reads as follows.

**Theorem 1.3.** Let \( p > 1 \) and \( \alpha < \min\left(\frac{1}{2}, \frac{2}{p-1} - \frac{1}{2}\right) \). Let \( \sigma = \max\left(0, 1 - \frac{1}{p+\delta} - \frac{1}{p}\right) \) for some arbitrary \( \delta > 0 \) and let \( s \geq \sigma \). Then, SvNLW (1.2) is pathwise globally well-posed in \( H^s(\mathbb{T}^2) \). More precisely, given any \((u_0, u_1) \in H^s(\mathbb{T}^2)\), there exists a unique global-in-time solution \( u \) to (1.2) with \((u, \partial_t u)|_{t=0} = (u_0, u_1)\) in the class

\[
\Psi + C(\mathbb{R}_+; H^q(\mathbb{T}^2)) \cap L^r(I(t_0); L^\sigma(\mathbb{T}^2)),
\]

where \( q, r \geq 2 \) are as in Theorem 1.1.

As stated in Theorem 1.3 when \( 1 < p \leq 3 \), we have the condition \( \alpha < \frac{1}{2} \); when \( p > 3 \), we have the condition \( \alpha < \frac{2}{p-1} - \frac{1}{2} \). As \( p \to \infty \), the condition for \( \alpha \) becomes \( \alpha \leq -\frac{1}{2} \). Note that when \( 1 < p < 5 \), we can prove pathwise global well-posedness of SvNLW (1.2) with the space-time white noise (i.e. \( \alpha = 0 \)).

We prove Theorem 1.3 by studying (1.12) for the residual term \( v \) in Section 5. From the proof of Theorem 1.1 we see that pathwise global well-posedness follows once we control the \( H^1 \)-norm of \( \tilde{v}(t) := (v(t), \partial_t v(t)) \). For this purpose, we study the evolution of the energy

\[
E(\tilde{v}) = \frac{1}{2} \int_{\mathbb{T}^2} (v^2 + |\nabla v|^2) dx + \frac{1}{2} \int_{\mathbb{T}^2} (\partial_t v)^2 + \frac{1}{p+1} \int_{\mathbb{T}^2} |v|^{p+1} dx,
\]

which is conserved under the (deterministic) usual NLW:

\[
\partial_t^2 u + (1 - \Delta)u + |u|^{p-1}u = 0.
\]

Note that for our problem, we proceed with the first order expansion (1.11), where the residual term \( v = \Psi - u \) only satisfies (1.12). In this case, the energy \( E(\tilde{v}) \) is not conserved under the equation (1.12) because of the perturbative term \( |v + \Psi|^{p-1}(v + \Psi) - |v|^{p-1}v \). For our problem, we first follow the globalization argument by Burq-Tzvetkov [11] and establish an exponential growth bound on \( E(\tilde{v}) \), which works in the sub-cubic case \( 1 < p \leq 3 \). For the super-cubic case \( p > 3 \), this argument no longer works due to the high homogeneity of the non-linearity. When \( 3 < p \leq 5 \), we use an integration by parts trick introduced by Oh-Pocovnicu [44]. In the super-quintic case \( p > 5 \), we use a trick involving the Taylor expansion, where the idea comes from Latocca [32].

One important prerequisite for studying the evolution of the energy \( E(\tilde{v}) \) is that the local-in-time solution \( \tilde{v} \) lies in \( H^1(\mathbb{T}^2) \), which is not guaranteed by the pathwise local well-posedness result (Theorem 1.1) as it is written. Nonetheless, due to the dissipative nature of the equation, we show that \( \tilde{v}(t) \) indeed belongs to \( H^1(\mathbb{T}^2) \) for any \( t > 0 \) by using the Schauder estimate (Lemma 2.8) along with Theorem 1.1. See Section 5 for details.
Remark 1.4. In [34], T. Oh and the author studied SvNLW (1.2) with $\alpha = \frac{1}{2}$. In this case, due to $\alpha = \frac{1}{2}$, the stochastic term $\Psi$ defined in (1.10) turns out to be merely a distribution, so that we studied a renormalized version of (1.2) and proved pathwise global well-posedness in the cubic case. Because of the singular nature of the stochastic convolution in this setting, the standard Gronwall argument does not work, and so we used a Yudovich-type argument to bound the corresponding energy.

In the same paper, we also proved almost sure global well-posedness of (1.2) with $\alpha < \frac{1}{2}$ to bound the corresponding energy. However, the argument only works for $\alpha = \frac{1}{2}$, so it does not apply to our problem with $\alpha < \frac{1}{2}$ in this paper. Instead, in this paper, we establish pathwise global well-posedness of SvNLW (1.2).

1.3. Deterministic vNLW with randomized initial data. In this paper, we also consider the deterministic vNLW (1.3) with a randomization $(u_0^\omega, u_1^\omega)$ of the initial data $(u_0, u_1)$. To construct $(u_0^\omega, u_1^\omega)$, we first write $u_0$ and $u_1$ in terms of Fourier series:

$$u_j(x) = \sum_{n \in \mathbb{Z}^2} \hat{u}_j(n)e^{inx}, \quad j = 0, 1,$$

where $\hat{u}_j(-n) = \overline{\hat{u}_j(n)}$ for all $n \in \mathbb{Z}^2$, $j = 0, 1$. For $j = 0, 1$, we let $\{g_{n,j}\}_{n \in \mathbb{Z}^2}$ be a sequence of mean zero complex-valued random variables such that $g_{-n,j} = \overline{g_{n,j}}$ for all $n \in \mathbb{Z}^2$, $j = 0, 1$. Moreover, we assume that $g_{0,j}$ is real-valued and $\{g_{0,j} \in \mathbb{R}, g_{n,j} \in \mathbb{C} \}_{n \in \mathcal{I}, j = 0, 1}$ are independent. Here, $\mathcal{I}$ is the index set defined by

$$\mathcal{I} := (\mathbb{Z}_+ \times \{0\}) \cup (\mathbb{Z} \times \mathbb{Z}_+),$$

so that $\mathbb{Z}^2 = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$. Then, we can define the randomization $(u_0^\omega, u_1^\omega)$ of $(u_0, u_1)$ by

$$(u_0^\omega, u_1^\omega) := \left( \sum_{n \in \mathbb{Z}^2} g_{n,0}(\omega)\hat{u}_0(n)e^{inx}, \sum_{n \in \mathbb{Z}^2} g_{n,1}(\omega)\hat{u}_1(n)e^{inx} \right).$$

We also assume that there exists a constant $c > 0$ such that, on the probability distributions $\mu_{n,j}$ of $g_{n,j}$, we have

$$\int e^{\gamma \cdot x} d\mu_{n,j}(x) \leq e^{c|\gamma|^2}, \quad j = 0, 1,$$

for all $\gamma \in \mathbb{R}^2$ when $n \in \mathbb{Z}^2 \setminus \{0\}$ and all $\gamma \in \mathbb{R}$ when $n = 0$. Note that for standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions, (1.10) is satisfied.

We say that $u$ is a solution to the deterministic vNLW (1.3) with randomized initial data $(u_0^\omega, u_1^\omega)$ if $u$ satisfies the following Duhamel formulation of (1.3):

$$u(t) = V(t)(u_0^\omega, u_1^\omega) - \int_0^t S(t-t')(|u|^{p-1}u)(t')dt',$$

where $V(t)$ and $S(t)$ are as defined in (1.8) and (1.9), respectively.

We now state an almost sure local well-posedness result for the deterministic vNLW (1.3) with randomized initial data. As mentioned in Remark 1.2, the following local well-posedness result applies to both the defocusing case and the focusing case.

\[Here and after, we drop the factor 2\pi for simplicity.\]
Theorem 1.5. Let \( p > 1 \) and \(-\frac{1}{p} < s \leq 0\). Given \((u_0, u_1) \in \mathcal{H}^s(T^2)\), let \((u_0^\omega, u_1^\omega)\) be the randomization defined in (1.14), which satisfies (1.10). Then, the deterministic vNLW (1.3) is almost surely locally well-posed with respect to the randomization \((u_0^\omega, u_1^\omega)\) as initial data. More precisely, we define \( q, r, \) and \( \sigma \) as follows.

(i) When \( 1 < p < 2 \), set \( q = 2 + \delta, r = \frac{4 + 2\delta}{1 + \delta}, \) and \( \sigma = 0 \), for some sufficiently small \( \delta > 0 \).

(ii) When \( p \geq 2 \), set \( q = p + \delta, r = 2p \), and \( \sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p} \), for some sufficiently small \( \delta > 0 \).

Then, there exist constants \( C, c, \gamma > 0 \) such that for each sufficiently small \( 0 < T \ll 1 \), there exists a set \( \Omega_T \subset \Omega \) with the following properties:

(i) \( P(\Omega_T) < C \exp\left(-\frac{c}{T^{\gamma} \| (u_0, u_1) \|_{\mathcal{H}^s}^2}\right) \).

(ii) For each \( \omega \in \Omega_T \), there exists a unique solution \( u = u^\omega \) to (1.3) on \([0, T]\) with \((u, \partial_t u)_{t=0} = (u_0^\omega, u_1^\omega)\) in the class

\[ V(t)(u_0^\omega, u_1^\omega) + C([0, T]; H^\alpha(T^2)) \cap L^q([0, T]; L^r(T^2)). \]

To prove Theorem 1.5, we use a similar approach as in the case of SvNLW (1.2). We consider the following first order expansion (1.17):

\[ u = v + z, \]

where

\[ z(t) = z^\omega(t) = V(t)(u_0^\omega, u_1^\omega) \]

is the solution of the linear stochastic viscous wave equation with initial data \((u_0^\omega, u_1^\omega)\):

\[
\begin{aligned}
\begin{cases}
\partial_t^2 z + (1 - \Delta) z + D \partial_t z = 0 \\
(z, \partial_t z)_{t=0} = (u_0^\omega, u_1^\omega),
\end{cases}
\end{aligned}
\]

and the residual term \( v \) satisfies the following equation:

\[ \partial_t^2 v + (1 - \Delta) v + D \partial_t v + |v + z|^{p-1}(v + z) = 0. \]  

The regularity properties of \( z \) are established in Subsection 2.4. Once we have established the probabilistic estimates of \( z \), the local well-posedness result for the residual term \( v \) follows from similar steps as in the proof of Theorem 1.1. See Section 4 for details. The regularity condition \( s > -\frac{1}{p} \) is sharp in our approach; see Remark 4.6.

Remark 1.6. We can also prove a probabilistic continuous dependence of the solution to (1.3), which is a notion introduced by Burq-Tzvetkov [12]. Namely, given \( s > -\frac{1}{p} \) and \((u_0, u_1) \in \mathcal{H}^s\), for any \( \varepsilon > 0 \) and \( 0 < p < 1 \), there exists an event \( A = A(u_0, u_1, \varepsilon) \) and \( \delta = \delta(\varepsilon, T, p) > 0 \) such that \( P(A) > p \), and for each \( \omega \in A \), the local-in-time solution \( u \) to (1.3) exists and satisfies \( \|u\|_{C_T \mathcal{H}^s} < \varepsilon \) whenever \( \|(u_0, u_1)\|_{\mathcal{H}^s} < \delta \). See also [19, 20]. The proof of this probabilistic continuous dependence follows essentially from the proof of local well-posedness of (1.3) in Section 4. Thus, we omit details.

Lastly, we turn our attention to almost sure global well-posedness of the deterministic vNLW (1.3) with randomized initial data, and we restrict our attention to the defocusing case.
Theorem 1.7. Let $p > 1$ and $-\frac{1}{p} < s \leq 0$. Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, let $(u_0^\omega, u_1^\omega)$ be the randomization defined in (1.15), which satisfies (1.16). Then, the deterministic vNLW (1.3) is almost surely globally well-posed with respect to the randomization $(u_0^\omega, u_1^\omega)$ as initial data. More precisely, by letting $\sigma = \max(0, 1 - \frac{1}{p+\delta} - \frac{1}{p})$ for some sufficiently small $\delta > 0$, there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that, for each $\omega \in \Omega_0$, there exists a unique global-in-time solution $u = u^\omega$ to (1.3) with $(u, \partial_t u)_{t=0} = (u_0^\omega, u_1^\omega)$ in the class $V(t)(u_0^\omega, u_1^\omega) + C(\mathbb{R}_+; H^\sigma(\mathbb{T}^2))$.

In Theorem 1.7, the uniqueness holds in the following sense. For any $t_0 \in \mathbb{R}_+$, there exists a time interval $I^\omega(t_0) \ni t_0$ such that the solution $u$ to (1.3) is unique in

$$V(t)(u_0^\omega, u_1^\omega) + C(I^\omega(t_0); H^\sigma(\mathbb{T}^2)) \cap L^q(I^\omega(t_0); L^r(\mathbb{T}^2)),$$

where $q, r \geq 2$ are as in Theorem 1.5.

As in the case of Theorem 1.5, after we establish the probabilistic estimates of $z$, the proof of Theorem 1.7 follows from similar steps as that of Theorem 1.3 i.e. the pathwise global well-posedness result for SvNLW (1.2). See Section 6 for details. Again, as in the case of almost sure local well-posedness, the regularity condition $s > -\frac{1}{p}$ is sharp in our approach.

Remark 1.8. (i) The global well-posedness result for the deterministic vNLW (1.3) with randomized initial data can be easily extended to $\mathbb{R}^2$. On $\mathbb{R}^2$, one needs to replace the randomization in (1.15) with the Wiener randomization (see [35, 35, 3, 4, 35]). After establishing similar probabilistic estimates as in Lemma 2.14 and Lemma 2.17 with respect to the Wiener randomization (see [35, 49, 41, 45]), one can then prove global well-posedness of the deterministic vNLW (1.3) with randomized initial data on $\mathbb{R}^2$ by applying essentially the same steps as in the proof of Theorem 1.5 and Theorem 1.7.

(ii) In general, we can combine Theorem 1.3 and Theorem 1.7 to prove global well-posedness of the following SvNLW with additive noise and randomized initial data:

$$\begin{cases} 
\partial_t^2 u + (1 - \Delta) u + D\partial_t u + |u|^{p-1} u = D^\alpha \xi \\
(u, \partial_t u)_{t=0} = (u_0^\omega, u_1^\omega),
\end{cases}$$

where $p > 1$, $\alpha < \min\left(\frac{1}{2}, \frac{2}{p-1} - \frac{1}{2}\right)$, and $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for $-\frac{1}{p} < s \leq 1$. For simplicity, we only discuss them separately in this paper.

2. Preliminary lemmas

In this section, we discuss some notations and lemmas that are necessary for proving our well-posedness results.

We use $A \lesssim B$ to denote $A \leq CB$ for some constant $C > 0$, and we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Also, we use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. In addition, we use short-hand notations to work with space-time function spaces. For example, $C_T H^s_x = C([0, T]; H^s(\mathbb{T}^d))$. 
2.1. Sobolev spaces and Besov spaces. Let $s \in \mathbb{R}$. We denote $H^s(\mathbb{T}^d)$ as the $L^2$-based Sobolev space with the norm:

$$\|u\|_{H^s(\mathbb{T}^d)} = \|\langle \cdot \rangle^s \hat{u}(\cdot)\|_{L^2(\mathbb{T}^d)},$$

where $\hat{u}(n)$ is the Fourier coefficient of $u$ and $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$. We then define $\mathcal{H}^s(\mathbb{T}^d)$ as

$$\mathcal{H}^s(\mathbb{T}^d) = H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d).$$

Also, we denote $W^{s,p}(\mathbb{T}^d)$ as the $L^p$-based Sobolev space with the norm:

$$\|u\|_{W^{s,p}(\mathbb{T}^d)} = \|\mathcal{F}^{-1}(\langle \cdot \rangle^s \hat{u}(\cdot))\|_{L^p(\mathbb{T}^d)},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{T}^d$. When $p = 2$, we have $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

Let $\varphi : \mathbb{R} \to [0,1]$ be a bump function such that $\varphi \in C_c([-\frac{8}{5}, \frac{8}{5}])$ and $\varphi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}^d$, we define $\varphi_0(\xi) = \varphi(|\xi|)$ and

$$\varphi_j(\xi) = \varphi(\frac{\xi}{2^j}) - \varphi(\frac{\xi}{2^{j+1}})$$

for $j \in \mathbb{Z}_+$. Note that

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \varphi_j(\xi) = 1 \quad (2.1)$$

for any $\xi \in \mathbb{R}^d$. For $j \in \mathbb{Z}_{\geq 0}$, we define the Littlewood-Paley projector $P_j$ as

$$P_j u = \mathcal{F}^{-1}(\varphi_j \hat{u}).$$

Due to (2.1), we have

$$u = \sum_{j=0}^{\infty} P_j u. \quad (2.2)$$

We also recall the definition of Besov spaces $B^s_{p,q}(\mathbb{T}^d)$ equipped with the norm:

$$\|u\|_{B^s_{p,q}(\mathbb{T}^d)} = \|2^{sj}\|\mathcal{F}^{-1}(P_j u)\|_{L^p(\mathbb{T}^d)}\|_{l^q(\mathbb{Z}_{\geq 0})}.$$ Note that $H^s(\mathbb{T}^d) = B^s_{2,2}(\mathbb{T}^d)$.

We then recall the definition of paraproducts introduced by Bony [7]. For details, see [1, 23]. For given functions $u$ and $v$ on $\mathbb{T}^d$ of regularities $s_1$ and $s_2$, respectively. By (2.2), we can write the product $uv$ as

$$uv = u \otimes u + u \otimes v + u \otimes v$$

$$:= \sum_{j<k-2} P_j u P_k v + \sum_{|j-k| \leq 2} P_j u P_k v + \sum_{k<j-2} P_j u P_k v.$$ The term $u \otimes v$ (and the term $u \otimes v$) is called the paraproduct of $v$ by $u$ (and the paraproduct of $u$ by $v$, respectively), and it is well defined as a distribution of regularity $\min(s_2, s_1 + s_2)$ (and $\min(s_1, s_1 + s_2)$, respectively). The term $u \otimes v$ is called the resonant product of $u$ and $v$, and it is well defined in general only if $s_1 + s_2 > 0$.

With these definitions in hand, we recall some basic properties of Besov spaces.
Lemma 2.1. (i) Let \( s_1, s_2 \in \mathbb{R} \) and \( 1 \leq p, p_1, p_2, q, 1 \leq \infty \) which satisfies \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, we have
\[
\| u \odot v \|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{L^{p_1}(\mathbb{T}^d)} \| v \|_{B^s_{p_2,q}(\mathbb{T}^d)}. \tag{2.3}
\]
When \( s_1 + s_2 > 0 \), we have
\[
\| u \odot v \|_{B^{s_1+s_2}_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{B^{s_1}_{p_1,q}(\mathbb{T}^d)} \| v \|_{B^{s_2}_{p_2,q}(\mathbb{T}^d)}. \tag{2.4}
\]
(ii) Let \( s_1 < s_2 \) and \( 1 \leq p, q \leq \infty \). Then, we have
\[
\| u \|_{B^{s_1}_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{W^{s_2,p}(\mathbb{T}^d)}. \tag{2.5}
\]
In particular, when \( q = \infty \), we have
\[
\| u \|_{B^{s_1}_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{W^{s_1,p}(\mathbb{T}^d)}. \tag{2.6}
\]

See [11, 39] for the proofs of (2.3) and (2.4) in the \( \mathbb{R}^d \) setting, which can be easily extended to the \( \mathbb{T}^d \) setting. The embedding (2.5) follows from the \( L^p \) boundedness of \( P_j \) and the \( L^p \)-boundedness of \( \{2^{(s_1-s_2)j}\}_{j \in \mathbb{Z}_{\geq 0}} \), and the embedding (2.6) follows easily from the \( L^p \) boundedness of \( P_j \).

Using (2.3) and (2.4), we get the following product estimate.

Corollary 2.2. Let \( s > 0 \), \( 1 \leq p, q \leq \infty \) and \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \) satisfying
\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{p}.
\]
Then,
\[
\| uv \|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{B^{s_1}_{p/q_1}(\mathbb{T}^d)} \| v \|_{L^{q_1}(\mathbb{T}^d)} + \| u \|_{L^{q_2}(\mathbb{T}^d)} \| v \|_{B^{s_2}_{q/q_2}(\mathbb{T}^d)}. \]

Next, we recall the following chain rule estimates.

Lemma 2.3. Let \( u \) be a smooth function on \( \mathbb{T}^d \), \( s \in (0,1) \), \( r \geq 2 \). Let \( F \) denote the function \( F(u) = |u|^r u \) or \( F(u) = |u|^r \).

(i) Let \( 1 < p, p_1 < \infty \) and \( 1 \leq p_2 \leq \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, we have
\[
\| F(u) \|_{W^{s,r}(\mathbb{T}^d)} \lesssim \| u \|_{W^{s_1,p_1}(\mathbb{T}^d)} \| u \|_{L^{p_2}(\mathbb{T}^d)}^{r-1}. \tag{2.7}
\]
(ii) Let \( 1 \leq p, q \leq \infty \) and \( 1 \leq p_1, p_2 \leq \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, we have
\[
\| F(u) \|_{B^s_{p,q}(\mathbb{T}^d)} \lesssim \| u \|_{B^{s_1}_{p_1,q}(\mathbb{T}^d)} \| u \|_{L^{p_2}(\mathbb{T}^d)}^{r-1}. \tag{2.8}
\]

The estimate (2.7) follows immediately from the fractional chain rule on \( \mathbb{T}^d \) in [21]. For the proof of (2.8), see, for example, Lemma 3.5 in [32] in the \( \mathbb{R}^d \) setting, which can be easily extended to the \( \mathbb{T}^d \) setting.

Lastly, we recall the following Gagliardo-Nirenberg interpolation inequality.

Lemma 2.4. Let \( p_1, p_2 \in (1, \infty) \) and \( s_1, s_2 > 0 \). Let \( p > 1 \) and \( \theta \in (0,1) \) satisfying
\[
-\frac{s_1}{d} + \frac{1}{p} = (1 - \theta) \left( \frac{1}{p_1} - \frac{s_2}{d} \right) + \frac{\theta}{p_2} \quad \text{and} \quad s_1 \leq (1 - \theta)s_2.
\]
Then, for \( u \in W^{s_2,p_1}(\mathbb{T}^d) \cap L^{p_2}(\mathbb{T}^d) \), we have
\[
\| u \|_{W^{s_1,p}(\mathbb{T}^d)} \lesssim \| u \|_{W^{s_2,p_1}(\mathbb{T}^d)}^{1-\theta} \| u \|_{L^{p_2}(\mathbb{T}^d)}^{\theta}. \]
This inequality follows from a direct application of Sobolev’s inequality on $\mathbb{T}^d$ (see [2]) and then interpolation.

2.2. On the stochastic term. In this subsection, we aim to establish the regularity properties of the stochastic term $\Psi$ defined in (1.10). To achieve this goal, we need the following lemma. Given $h \in \mathbb{R}$ and a function $f$, we define $\delta_h$ as the difference operator

$$\delta_h f(t) = f(t + h) - f(t). \quad (2.9)$$

**Lemma 2.5.** Let $\{X_N\}_{N \in \mathbb{N}}$ and $X : \mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)$ be Wiener integrals.

(i) Let $t \in \mathbb{R}_+$. If there exists $s_0 \in \mathbb{R}$ such that

$$\mathbb{E}[|\hat{X}(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$, then we have $X(t) \in W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\hat{X}_N(n, t) - \hat{X}(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ and $N \geq 1$, then $X_N(t)$ converges to $X(t)$ in $W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely.

(ii) Let $T > 0$ and suppose that (i) holds on $[0, T]$. If there exists $\sigma \in (0, 1)$ such that

$$\mathbb{E}[|\hat{\delta}_h \hat{X}(n, t)|^2] \lesssim \langle n \rangle^{-d-2s_0+\sigma}|h|^\sigma,$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, and $h \in [-1, 1]$, then we have $X \in C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\sigma}{2}$, almost surely. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}[|\hat{\delta}_h \hat{X}_N(n, t) - \hat{\delta}_h \hat{X}(n, t)|^2] \lesssim N^{-\gamma} \langle n \rangle^{-d-2s_0+\sigma}|h|^\sigma,$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, $h \in [-1, 1]$, and $N \geq 1$, then $X_N$ converges to $X$ in $C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\sigma}{2}$, almost surely.

For the proof of Lemma 2.5, see Proposition 3.6 in [40] and Appendix in [43].

We now go back to the stochastic object $\Psi$ initially defined in (1.10), which can be written as the following stochastic convolution:

$$\Psi(t) = \Psi_\alpha(t) = \int_0^t S(t - t') D^\alpha dW(t'), \quad (2.10)$$

where $S(t)$ is as defined in (1.9) and $W$ is a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:

$$W(t) = \sum_{n \in \mathbb{Z}^2} B_n(t) e_n.$$ 

Here, $e_n(x) = e^{inx}$ and $\{B_n\}_{n \in \mathbb{Z}^2}$ is defined by $B_n(t) = \langle \xi, 1_{[0,t]} \cdot e_n \rangle_{t,x}$, where $\xi$ is the space-time white noise and $\langle \cdot, \cdot \rangle_{t,x}$ denotes the duality pairing on $\mathbb{R}_+ \times \mathbb{T}^2$. As a result of this definition, $\{B_n\}_{n \in \mathbb{Z}^2}$ is a family of independent complex-valued Brownian motions conditioned so that $B_{-n} = \overline{B_n}$ for any $n \in \mathbb{Z}^2$. Note that for any $n \in \mathbb{Z}^2$, we have

$$\text{Var}(B_n(t)) = \mathbb{E}[\langle \xi, 1_{[0,t]} \cdot e_n \rangle_{t,x} \overline{\langle \xi, 1_{[0,t]} \cdot e_n \rangle_{t,x}}] = \|1_{[0,t]} \cdot e_n\|_{L^2_{t,x}}^2 = t.$$ 

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3We impose $h \geq -t$ such that $t + h \geq 0$. 

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We now establish the regularity of the stochastic convolution $\Psi$ in the following lemma. Given $N \in \mathbb{N}$, we denote $\Psi_N = \pi_N \Psi$ as the truncated stochastic convolution, where $\pi_N$ is the frequency cutoff onto the spatial frequencies $\{|n| \leq N\}$.

**Lemma 2.6.** For any $\varepsilon > 0$ and $T > 0$, $\Psi_N$ converges to $\Psi$ in $C([0, T]; W^{1-2\alpha - \varepsilon, \infty}(\mathbb{T}^2))$ almost surely. In particular, we have

$$\Psi \in C([0, T]; W^{1/2 - \alpha - \varepsilon, \infty}(\mathbb{T}^2))$$

almost surely.

**Proof.** Let $t \geq 0$. From (2.10) and the Itô isometry, we can compute

$$\mathbb{E}[|\tilde{\Psi}(t, n)|^2] = \int_0^t e^{-(t-t')|n|} \left[ \frac{\sin((t-t')|n|)}{|n|} \right]^2 |n|^{2\alpha} dt' \sim \frac{1}{|n|^{3-2\alpha}}$$

for any $n \in \mathbb{Z}^2 \setminus \{0\}$. Hence from Lemma 2.5 (i), we have $\Psi(t) \in W^{1/2 - \alpha - \varepsilon, \infty}(\mathbb{T}^2)$ almost surely for any $\varepsilon > 0$.

Let $0 \leq t_2 \leq t_1$. From (2.10) and the Itô isometry, we compute

$$\mathbb{E}[|\tilde{\Psi}(t_1, n) - \tilde{\Psi}(t_2, n)|^2] = \int_0^{t_1} e^{-(t_1-t')|n|} \left[ \frac{\sin((t_1-t')|n|)}{|n|} \right]^2 |n|^{2\alpha} dt' + \int_0^{t_2} \left( e^{-(t_1-t')|n|/2} \frac{\sin((t_1-t')|n|)}{|n|} - e^{-(t_2-t')|n|/2} \frac{\sin((t_2-t')|n|)}{|n|} \right)^2 |n|^{2\alpha} dt'$$

$$=: I_1 + I_2.$$

For $I_1$, a direct computation gives

$$I_1 \leq \frac{|t_1 - t_2|}{|n|^{2-2\alpha}}$$

and

$$I_1 \leq \frac{1}{|n|^{2-2\alpha}} \int_0^{t_1-t_2} e^{-t'|n|} dt' \leq \frac{1}{|n|^{3-2\alpha}},$$

and so we can interpolate these two inequalities to obtain the bound

$$I_1 \leq \frac{|t_1 - t_2|^\sigma}{|n|^{3-2\alpha-\sigma}}, \quad (2.12)$$

for some $\sigma \in [0, 1]$. For $I_2$, by the mean value theorem, we have

$$I_2 \leq \frac{\langle n \rangle^{\sigma} |t_1 - t_2|^\sigma}{|n|^{3-2\alpha}} \int_0^{t_2} \left( e^{-\frac{(t_1-t')|n|}{2}} + e^{-\frac{(t_2-t')|n|}{2}} \right)^{2-\sigma} dt' \lesssim \frac{|t_1 - t_2|^\sigma}{|n|^{3-2\alpha-\sigma}}, \quad (2.13)$$

for some $\sigma \in [0, 1]$. Combining (2.11), (2.12), and (2.13), we obtain the bound

$$\mathbb{E}[|\tilde{\Psi}(t_1, n) - \tilde{\Psi}(t_2, n)|^2] \lesssim \frac{|t_1 - t_2|^\sigma}{|n|^{3-2\alpha-\sigma}}$$

for any $n \in \mathbb{Z}^2 \setminus \{0\}$, $0 \leq t_2 \leq t_1$ with $t_1 - t_2 \leq 1$, and $\sigma \in [0, 1]$. Thus, from Lemma 2.5 (ii), we have $\Psi(t) \in C(\mathbb{R}_+; W^{1/2 - \alpha - \varepsilon, \infty}(\mathbb{T}^2))$ almost surely for any $\varepsilon > 0$.

Proceeding as above, we get

$$\mathbb{E}[|\tilde{\Psi}_M(t, n) - \tilde{\Psi}_N(t, n)|^2] \leq C \mathbf{1}_{|n| > N} \cdot \langle n \rangle^{-3 + 2\alpha} \leq CN^{-\gamma} \langle n \rangle^{-3 + 2\alpha + \gamma}$$

for any $n \in \mathbb{Z}^2$, $M \geq N \geq 1$, and $\gamma \geq 0$. Similarly, with $\delta_\delta$ defined in (2.4), we have

$$\mathbb{E}[|\delta_\delta \tilde{\Psi}_M(t, n) - \delta_\delta \tilde{\Psi}_N(t, n)|^2] \leq C \mathbf{1}_{|n| > N} \cdot \langle n \rangle^{-3+2\alpha+\sigma} \langle |h| \rangle^{\sigma} \leq CN^{-\gamma} \langle n \rangle^{-3 + 2\alpha + \sigma + \gamma} \langle |h| \rangle^{\sigma}$$
lemma 2.9. given $T > 0$ and $\varepsilon > 0$, $\Psi_N$ converges to $\Psi$ in $C([0,T]; W^{-\frac{\alpha}{2} - \varepsilon, \alpha}(\mathbb{T}^2))$ almost surely.

Remark 2.7. One can use an integration by parts to write
\[
\tilde{\Psi}(t,n) = -\int_0^t B_n(t') \frac{d}{ds}\bigg|_{s=t'} \left( e^{-\frac{|n|^2}{2}(t-s)} \sin((t-s)[n]) |n|^\alpha \right) dt'
\]
almost surely, which allows us to compute that
\[
\partial_t \tilde{\Psi}(t,n) = \int_0^t \left( -\frac{|n|}{2[n]} e^{-\frac{|n|^2}{2}(t-t')} \sin((t-t')[n]) + e^{-\frac{|n|^2}{2}(t-t')} \cos((t-t')[n]) \right) |n|^\alpha dB_n(t')
\]
almost surely. Using a similar argument as in the proof of Lemma 2.6, we have $\partial_t \tilde{\Psi} \in C([0,T]; W^{-\frac{\alpha}{2} - \varepsilon, \alpha}(\mathbb{T}^2))$ almost surely. This will be useful in the proof of pathwise global well-posedness of $\text{SvNLW}$ (1.2) in Subsection 5.2.

2.3. Linear estimates. In this subsection, we show some relevant linear estimates and the Strichartz estimates that are used to prove our well-posedness results.

Let
\[
P(t) = e^{-\frac{D^2 t}{4}}
\]
be the Poisson kernel with a parameter $\frac{1}{2}$, which appears in the viscous wave linear propagator $V(t)$ defined in (1.8). We first recall the following Schauder-type estimate for the Poisson kernel $P(t)$. For a proof, see Lemma 2.3 in [34].

Lemma 2.8. Let $1 \leq p \leq q \leq \infty$ and $\beta \geq 0$. Then, we have
\[
\|D^\beta P(t)\phi\|_{L^q(\mathbb{T}^d)} \lesssim t^{-\beta - d(\frac{1}{p} - \frac{1}{q})} \|\phi\|_{L^p(\mathbb{T}^d)}
\]
for any $0 < t \leq 1$.

Next, we turn our attention to the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{T}^d$. Due to the boundedness of $V(t)$ in $\mathcal{H}^s(\mathbb{T}^d)$, we mainly need to show the following Strichartz estimate for the linear propagator $V(t)$ defined in (1.8).

Lemma 2.9. Given $s \geq 0$, suppose that $2 < q \leq \infty$, $2 \leq r \leq \infty$ satisfy the following scaling condition:
\[
\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s.
\]
Then, we have
\[
\|V(t)(\phi_0, \phi_1)\|_{L^q([0,T]; L^r(\mathbb{T}^d))} \lesssim \|(\phi_0, \phi_1)\|_{\mathcal{H}^s(\mathbb{T}^d)},
\]
for all $0 < T \leq 1$.

Proof. For the proof, we use the $TT^*$ method. We first consider the case when $s = 0$. Let
\[
V_1(t) = e^{-\frac{D}{2} t} \cos(t[D]), \quad V_2(t) = e^{-\frac{D}{2} t} \frac{D}{2[D]} \sin(t[D]),
\]
so that
\[
V(t)(\phi_0, \phi_1) = V_1(t)\phi_0 + V_2(t)\phi_0 + S(t)\phi_1.
\]

4We impose $h \geq -t$ so that $t + h \geq 0.$
Let $L : L^2(\mathbb{T}^d) \to L^q_T L^r_x(\mathbb{T}^d)$ be the linear operator given by $L \phi = V_1(t) \phi$. Note that $L^*$ is the linear operator given by

$$L^* f = \int_0^T V_1(t') f(t') dt'$$

for any space-time Schwartz function $f$. By Minkowski’s integral inequality, the Schauder estimate (Lemma 2.8) twice, the scaling condition (2.15), and the Hardy-Littlewood-Sobolev inequality, we have

$$\|L L^* f\|_{L^q_T L^r_x} \leq \left\| \int_0^T \left| e^{-\frac{D}{4}(t+t')} \cos(t[D]) \cos(t'[D]) f(t') \right|_{L^q_x} dt' \right\|_{L^q_T}$$

$$\lesssim \left\| \int_0^T \left| \frac{1}{|t-t'|^{2/4 + 1/4}} \right| f(t') \|_{L^q_x} dt' \right\|_{L^q_T}$$

$$\lesssim \left\| \int_0^\infty \frac{1}{|t-t'|^{2/4}} \left| 1_{[0,T]} f(t') \right|_{L^q_x} dt' \right\|_{L^q_T}$$

$$\lesssim \|f\|_{L^q_T L^r_x}.$$

Thus, by a standard duality argument, we obtain

$$\|V_1(t) \phi_0\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|\phi_0\|_{L^2(\mathbb{T}^d)}.$$

By using similar arguments, we obtain

$$\|V_2(t) \phi_0\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|\phi_0\|_{L^2}, \quad \|S(t) \phi_1\|_{L^q_T L^r_x} \lesssim \|\phi_0\|_{H^{-1}(\mathbb{T}^d)},$$

so that we have

$$\|V(t)(\phi_0, \phi_1)\|_{L^q_T L^r_x(\mathbb{T}^d)} \lesssim \|\phi_0\|_{H^0(\mathbb{T}^d)}, \quad (2.17)$$

When $s > 0$, by Sobolev’s inequality, the scaling condition (2.15), and (2.17), we obtain

$$\|V(t)(\phi_0, \phi_1)\|_{L^q_T L^r_x} \lesssim \|V(t)(\nabla)^s \phi_0, (\nabla)^s \phi_1\|_{L^q_T L^{r/d + 4/4}}^{1/(\frac{d}{4} + 1)}$$

$$\lesssim \|(\nabla)^s \phi_0, (\nabla)^s \phi_1\|_{H^s},$$

as desired.

Remark 2.10. (i) Compared to the Strichartz estimates for the usual linear wave equations ([22][33][27][24]), the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{T}^d$ hold for a larger class of pairs $(q, r)$, which is due to the dissipative effects of the viscosity term.

(ii) In [29], Kuan-Čanić proved the Strichartz estimates for the homogeneous linear viscous wave equation on $\mathbb{R}^d$. They used the method from Keel-Tao [27], so that their result requires $(q, r)$ to be $\sigma$-admissible for some $\sigma > 0$, i.e. $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{2}{q} + \frac{2\sigma}{r} \leq \sigma.$$
We point out that the \( TT^* \) method we use in the proof also works on \( \mathbb{R}^d \) and does not have this \( \sigma \)-admissible restriction on \( q \) and \( r \). However, our proof works only for \( s \geq 0 \).

We complete this subsection by establishing several inhomogeneous linear estimates.

**Lemma 2.11.** Let \( p \geq 2 \) and let \( S(t) \) be as in (1.9). Then, given \( \delta > 0 \), we have

\[
\left\| \int_0^t S(t-t')F(t')dt' \right\|_{L^{p+\delta}([0,T];L^2_2(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2(\mathbb{T}^2))}
\]

(2.18)

for any \( 0 < T \leq 1 \).

**Proof.** We let

\[
s = 1 - \frac{1}{p + \delta} - \frac{2}{2p} = 1 - \frac{1}{p + \delta} - \frac{1}{p},
\]

so that \((p+\delta,2p,s)\) satisfies the scaling condition in Lemma 2.9. By Minkowski’s integral inequality and Lemma 2.9, we obtain

\[
\left\| \int_0^t S(t-t')F(t')dt' \right\|_{L^{p+\delta}_p L^{2p}_2} \lesssim \int_0^T \|1_{[0,t]}(t')S(t-t')F(t')\|_{L^{p+\delta}_p L^{2p}_2}dt'
\]

\[
\lesssim \int_0^T \|F(t')\|_{H^{s-1}_x}dt'
\]

\[
\lesssim \|F\|_{L^p L^2_x},
\]

so that (2.18) follows. \( \square \)

**Lemma 2.12.** Let \( S(t) \) be as in (1.9). Then, given \( s \leq 1 \), we have

\[
\left\| \int_0^t S(t-t')F(t')dt' \right\|_{C([0,T];H^s_x(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2_2(\mathbb{T}^2))},
\]

(2.19)

\[
\left\| \partial_t \int_0^t S(t-t')F(t')dt' \right\|_{C([0,T];H^{s-1}_x(\mathbb{T}^2))} \lesssim \|F\|_{L^1([0,T];L^2_2(\mathbb{T}^2))}
\]

(2.20)

for any \( 0 < T \leq 1 \).

**Proof.** The estimate (2.19) follows from (1.9) and Minkowski’s integral inequality. The estimate (2.20) follows similarly by noting that

\[
\partial_t \int_0^t S(t-t')F(t')dt' = \int_0^t \partial_t S(t-t')F(t')dt',
\]

where

\[
\partial_t S(t) = e^{-\frac{\partial_t}{2t}}\left(\cos(t[\mathbb{D}]) - \frac{D}{2[D]}\sin(t[\mathbb{D}])\right).
\]

\( \square \)
2.4. Probabilistic estimates. In this subsection, we show several probabilistic Strichartz estimates of the random linear solution \( z = V(t)(u_0, u_1) \) defined in \((\mathbf{1.15})\), where \((u_0, u_1)\) defined in \((\mathbf{1.15})\) is the randomization of the initial data \((u_0, u_1)\). We also consider probabilistic estimates of \( \tilde{z} \) defined by

\[
\tilde{z} = \tilde{z}^\omega := (\nabla)^{-1} \partial_t z
\]

\[
= -\frac{D}{2\langle\nabla\rangle} e^{-\frac{D}{2}t} \left( \cos(t\|D\|) + \frac{D}{2\|D\|} \sin(t\|D\|) \right) u_0^\omega
\]

\[
+ \frac{1}{\langle\nabla\rangle} e^{-\frac{D}{2}t} \left( -[D] \sin(t\|D\|) + \frac{D}{2} \cos(t\|D\|) \right) u_1^\omega
\]

\[
- \frac{D}{2\langle\nabla\rangle} e^{-\frac{D}{2}t} \sin(t\|D\|) u_0^\omega + \frac{1}{\langle\nabla\rangle} e^{-\frac{D}{2}t} \cos(t\|D\|) u_1^\omega.
\]

We first recall the following probabilistic estimate. For the proof, see \([10]\).

Lemma 2.13. Let \( \{g_n\}_{n \in \mathbb{Z}^2} \) be a sequence of mean zero complex-valued random variables such that \( g_n = \overline{g}_n \) for all \( n \in \mathbb{Z}^2 \). Furthermore, \( g_0 \) is real-valued and \( \{g_n, \Re g_n, \Im g_n\}_{n \in \mathcal{I}} \) are independent, where \( \mathcal{I} \) is the index set defined in \((\mathbf{1.14})\). Moreover, assume that \((\mathbf{1.16})\) holds. Then, there exists \( C > 0 \) such that

\[
\left\| \sum_{n \in \mathbb{Z}^2} g_n(\omega)c_n \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_n\|_{l_2(\mathbb{Z}^2)}
\]

for any finite \( p \geq 2 \) and any sequence \( \{c_n\} \in l^2(\mathbb{Z}^2) \) that satisfies \( c_{-n} = \overline{c}_n \) for all \( n \in \mathbb{Z}^2 \).

We now present the following probabilistic estimates.

Lemma 2.14. Let \( 1 \leq q < \infty \) and \( \beta \geq 0 \), satisfying \( q\beta < 1 \). Let \( z^* = z \) or \( \tilde{z} \) defined in \((\mathbf{1.18})\) or \((\mathbf{2.21})\), respectively.

(i) When \( 2 \leq r < \infty \), there exist \( C, c > 0 \) depending only on \( q \) and \( r \) such that

\[
P\left( \|D^\beta z^*\|_{L^q([0,T];L^r_2(\mathbb{T}^2))} > \lambda \right) \leq C \exp\left( -\frac{c \lambda^2}{T^{\frac{4}{q} - 2\beta} \|(u_0, u_1)\|_{H^0(\mathbb{T}^2)}^2} \right),
\]

for any \( 0 < T \leq 1 \) and \( \lambda > 0 \).

(ii) When \( r = \infty \), given any \( \varepsilon > 0 \), there exist \( C, c > 0 \) depending only on \( q \) and \( \varepsilon \) such that

\[
P\left( \|D^\beta z^*\|_{L^q([0,T];L^\infty_2(\mathbb{T}^2))} > \lambda \right) \leq C \exp\left( -\frac{c \lambda^2}{T^{\frac{4}{q} - 2\beta} \|(u_0, u_1)\|_{H^\varepsilon(\mathbb{T}^2)}^2} \right),
\]

for any \( 0 < T \leq 1 \) and \( \lambda > 0 \).

Remark 2.15. Let \( 1 \leq q < \infty \) and \( \beta_0 \geq 0 \), satisfying \( q\beta_0 < 1 \). By applying Lemma 2.14 with \( \beta = 0 \) and \( \beta = \beta_0 \), when \( 2 \leq r < \infty \), we obtain

\[
P\left( \|\langle D\rangle^{\beta_0} z^*\|_{L^q([0,T];L^r_2(\mathbb{T}^2))} > \lambda \right) \leq C \exp\left( -\frac{c \lambda^2}{T^{\frac{4}{q} - 2\beta_0} \|(u_0, u_1)\|_{H^0(\mathbb{T}^2)}^2} \right),
\]

for any \( 0 < T \leq 1 \) and \( \lambda > 0 \). Also, given any \( \varepsilon > 0 \), we obtain

\[
P\left( \|\langle D\rangle^{\beta_0} z^*\|_{L^q([0,T];L^\infty_2(\mathbb{T}^2))} > \lambda \right) \leq C \exp\left( -\frac{c \lambda^2}{T^{\frac{4}{q} - 2\beta_0} \|(u_0, u_1)\|_{H^\varepsilon(\mathbb{T}^2)}^2} \right),
\]
for any $0 < T \leq 1$ and $\lambda > 0$.

**Proof.** (i) For $p \geq \max(q,r)$, by Lemma 2.13, we have

$$\left\| D^\gamma e^{-\frac{i}{2}t} \cos(t[D])u_0^\varphi \right\|_{L^q_x L^r_t} \lesssim \left\| t^{-\beta} \right\|_{L^p_x} \left\| \cos(t[D])u_0^\varphi \right\|_{L^q_x L^r_t} \lesssim \sqrt{p} \left\| t^{-\beta} \right\|_{L^p_x} \left\| \cos(t[D])u_0^\varphi \right\|_{L^q_x L^r_t} \lesssim \sqrt{p} \left\| t^{-\beta} \right\|_{L^q_x L^r_t} \left\| u_0 \right\|_{L^2} \sim \sqrt{p} T^\frac{1}{q} \left\| u_0 \right\|_{L^2}.$$ 

Thus, the above step implies that we have

$$\left\| D^\gamma z^* \right\|_{L^q_x L^r_t} \lesssim \sqrt{p} T^\frac{1}{q} \left\| u_0, u_1 \right\|_{H^0},$$

and so the tail estimate (2.22) follows from Chebyshev’s inequality.

(ii) Given $\epsilon > 0$, we choose $\tilde{\tau} \gg 1$ such that $\epsilon \tilde{\tau} > 2$. By Sobolev embedding theorem, we have

$$\left\| z^* \right\|_{L^q_x L^r_t} \lesssim \left\| (\nabla)^{\tilde{\tau}} z^* \right\|_{L^q_x L^r_t}.$$ 

Then, the tail estimate (2.23) follows from a similar argument as in (i). \qed

We now show a probabilistic estimate involving the $L^\infty_T$ norm. Let $V_+(t)$ and $V_-(t)$ be the linear propagators on $\mathbb{T}^2$ defined by

$$V_\pm(t) := e^{-\frac{i}{2}t \pm i[D]^t}.$$ 

We also define

$$U_\pm(t) := e^{\pm i[D]^t}$$ 

so that

$$V_\pm(t) = P(t) \circ U_\pm(t),$$

where $P(t)$ is the Poisson kernel defined in (2.14). As in (1.15), given $\phi \in H^s(\mathbb{T}^2)$, we define its randomization $\phi^\omega$ on $\mathbb{T}^2$ by

$$\phi^\omega(x) := \sum_{n \in \mathbb{Z}^2} g_{n,0}(\omega) \hat{\phi}(n)e^{i\omega \cdot x}.$$ 

Then, we have the following probabilistic estimate.

**Lemma 2.16.** Let $j \in \mathbb{Z}_+$, $2 \leq r \leq \infty$, $\beta_1 \geq 0$, and $\beta_2 \in \mathbb{R}$. Then, given any $\epsilon > 0$, there exist $C, c > 0$ depending only on $r$ and $\epsilon$ such that

$$P(\left\| D^{\beta_1[D]^\beta_2} V_\pm(0) \right\|_{L^\infty([j,j+1]; L^r_2(\mathbb{T}^2))} > \lambda) \leq C \exp\left( -c \frac{\lambda^2}{j^{-2\beta_1}\|\phi\|^2_{H^{\beta_2+}(\mathbb{T}^2)}} \right),$$

for any $\lambda > 0$. 

Proof. By the Schauder estimate (Lemma 2.8), we have
\[ \| D^{\beta_1} \left[ D^{\beta_2} V_\pm \phi^\omega \right] \|_{L^\infty_T ([j,j+1]; L^r_{x})} \lesssim \| D^{\beta_2} U_\pm \phi^\omega \|_{L^r_T ([j,j+1])} \]
\[ \leq j^{-\beta_1} \| D^{\beta_2} U_\pm \phi^\omega \|_{L^r_T ([j,j+1]; L^r_{x})}. \]
The rest of the argument is analogous to that of Lemma 3.4 in [11], which can be easily extended to the $\mathbb{T}^2$ setting. Thus, we omit the rest of the proof. 

By using subadditivity and Lemma 2.16 we obtain the following result.

**Lemma 2.17.** Let $T \gg 1 \geq T_0 > 0$, $2 \leq r \leq \infty$, and $\beta \geq 0$. Let $z^* = z$ or $\tilde{z}$ defined in (1.18) or (2.21), respectively. Then, given any $\varepsilon > 0$, there exist $C, c > 0$ depending only on $r$ and $\varepsilon$ such that
\[ P(\| D^{\beta_2} z^* \|_{L^\infty_T ([T_0,T]; L^r_{x}(\mathbb{T}^2))} > \lambda) \leq C T \exp \left( - c \frac{\lambda^2}{T_0^{-2\beta} \| (u_0, u_1) \|_{H^s(\mathbb{T}^2)}^2} \right), \]
for any $\lambda > 0$.

3. **Local well-posedness of SvNLW**

In this section, we prove Theorem 1.1 i.e. the pathwise local well-posedness result for SvNLW (1.2). As mentioned in Subsection 1.2 we consider the following vNLW:
\[
\begin{align*}
\partial_t^2 v + (1 - \Delta) v + D \partial_t v + F(v + \Psi) &= 0, \\
(v, \partial_t v)|_{t=0} &= (u_0, u_1)
\end{align*}
\]
for given initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, $F(u) = |u|^{p-1}u$, and $\Psi$ is the stochastic convolution defined in (1.10). By Lemma 2.6 we can fix a good $\omega \in \Omega$ such that $\Psi = \Psi(\omega) \in C([0,T]; W^{1,\alpha-\varepsilon,\infty}(\mathbb{T}^2))$ for $\alpha < \frac{1}{2}$ and sufficiently small $\varepsilon > 0$, so that (3.1) becomes a deterministic equation. Then, we prove the following pathwise local well-posedness of (3.1).

**Proposition 3.1.** Let $p > 1$ and $\alpha < \frac{1}{2}$. Define $q$, $r$, and $\sigma$ as follows.

(i) When $1 < p < 2$, set $q = 1 + \delta, r = \frac{4+2\delta}{1+\delta}$, and $\sigma = 0$, for some sufficiently small $\delta > 0$.

(ii) When $p \geq 2$, set $q = p + \delta, r = 2p, and \sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$, for some arbitrary $\delta > 0$.

Let $s \geq \sigma$. Then, (3.1) is pathwise locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$. More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists $0 < T = T_{\omega}(u_0, u_1) \leq 1$ and a unique solution $\tilde{v} = (v, \partial_t v)$ to (3.1) in the class
\[ (v, \partial_t v) \in C([0,T]; \mathcal{H}^\sigma(\mathbb{T}^2)) \]\nand
\[ v \in L^q([0,T]; L^r(\mathbb{T}^2)). \]

Note that Theorem 1.1 follows immediately from Proposition 3.1. The main idea of the proof of Proposition 3.1 comes from [31].

**Proof.** We first consider the case when $p \geq 2$. We write (3.1) in the Duhamel formulation:
\[
v(t) = \Gamma(v) := V(t)(u_0, u_1) - \int_0^t S(t-t')F(v + \Psi)(t')dt',
\]
\[ (v, \partial_t v)|_{t=0} = (u_0, u_1) \]
where $V(t)$ and $S(t)$ are as defined in (1.8) and (1.9), respectively. Let \( \Gamma(v) = (\Gamma(v), \partial_t \Gamma(v)) \) and \( \bar{v} = (v, \partial_t v) \). Given $0 < T \leq 1$, we define the space \( \mathcal{X}(T) \) as
\[
\mathcal{X}^\sigma(T) = \mathcal{X}_1^\sigma(T) \times \mathcal{X}_2^\sigma(T),
\]
where
\[
\mathcal{X}_1^\sigma(T) := C([0, T]; H^\sigma(\mathbb{T}^2)) \cap L^{p+\delta}(0, T; L^{2p}(\mathbb{T}^2)),
\]
\[
\mathcal{X}_2^\sigma(T) := C([0, T]; H^{\sigma-1}(\mathbb{T}^2)).
\]
Here, $\delta > 0$ is arbitrary and $\sigma = 1 - \frac{1}{p+\delta} - \frac{1}{p}$. Note that this choice of $\sigma$ along with the $L^{p+\delta}_T L^{2p}_x$ norm satisfies the scaling condition in Lemma 2.9. Our goal is to show that $\Gamma$ is a contraction on a ball in $\mathcal{X}^\sigma(T)$ for some $0 < T \leq 1$.

By (3.2), Lemma 2.9, (1.8), Lemma 2.11, and Sobolev’s inequality with the fact that $|\mathbb{T}^2| = 1$, we have
\[
\|\Gamma(v)\|_{\mathcal{X}^\sigma(T)} \lesssim \|(u_0, v_1)\|_{H^\sigma} + \|v + \Psi\|^P_{L^1_T L^2_x} 
\lesssim \|(u_0, v_1)\|_{H^\sigma} + T^\theta \left( \|v\|^P_{L^{p+\delta}_T L^{2p}_x} + \|\Psi\|^P_{L^{p+\delta}_T L^{2p}_x} \right) 
\lesssim \|(u_0, v_1)\|_{H^\sigma} + T^\theta \left( \|v\|^P_{\mathcal{X}^\sigma(T)} + \|\Psi\|^P_{C_T W^{\frac{1}{2} \alpha - \varepsilon, \infty}_x} \right)
\]
for some $\theta > 0$ and sufficiently small $\varepsilon > 0$.

For the difference estimate, we use the idea from Oh-Okamoto-Pocovnicu [32]. Noticing that $F'(u) = p |u|^{p-1}$, we use (3.2), Lemma 2.9, Lemma 2.11, Lemma 2.12, and the fundamental theorem of calculus, Minkowski’s integral inequality, Hölder’s inequality, and Sobolev’s inequality to obtain
\[
\|\Gamma(v_1) - \Gamma(v_2)\|_{\mathcal{X}^\sigma(T)} \lesssim \|F(v + \Psi) - F(v + \Psi)\|_{L^1_T L^2_x} 
= \left\| \int_0^1 F'(w + \Psi + \tau(v - w))(v - w) d\tau \right\|_{L^1_T L^2_x} 
\leq \int_0^1 \|w + \Psi + \tau(v - w)\|^P_{L^{p+\delta}_T L^{2p}_x} \|v - w\|_{L^{p+\delta}_T L^{2p}_x} d\tau 
\lesssim T^\theta \left( \|v\|^{p-1}_{L^{p+\delta}_T L^{2p}_x} + \|w\|^{p-1}_{L^{p+\delta}_T L^{2p}_x} + \|\Psi\|^{p-1}_{L^{p+\delta}_T L^{2p}_x} \right) \|v - w\|_{L^{p+\delta}_T L^{2p}_x} 
\lesssim T^\theta \left( \|v\|^{p-1}_{\mathcal{X}^\sigma(T)} + \|w\|^{p-1}_{\mathcal{X}^\sigma(T)} + \|\Psi\|^{p-1}_{C_T W^{\frac{1}{2} \alpha - \varepsilon, \infty}_x} \right) \|\bar{v} - \bar{w}\|_{\mathcal{X}^\sigma(T)}
\]
for some $\theta > 0$ and sufficiently small $\varepsilon > 0$.

Thus, by choosing $T = T_\omega(\|(u_0, u_1)\|_{H^\sigma}) > 0$ small enough, we obtain that $\Gamma$ is a contraction on the ball $B_R \subset \mathcal{X}^\sigma(T)$ of radius $R \sim 1 + \|(u_0, u_1)\|_{H^\sigma}$. Note that at this point, the uniqueness of the solution $v$ only holds in the ball $B_R$, but we can use a standard continuity argument to extend the uniqueness of $v$ to the entire $\mathcal{X}^\sigma(T)$.

For the case when $1 < p < 2$, we may have $p + \delta < 2$, so that Lemma 2.9 may not work for the $L^{p+\delta}_T L^{2p}_x$ norm. Instead, we consider the $L^q_T L^r_x$ norm with $q = 2 + \delta$ and $r = \frac{4 + 2\delta}{1 + \delta}$, where $\delta > 0$ is small enough so that $r$ is close enough to 4. We also set $\sigma = 0$, so that this choice of $\sigma$ along with this $L^q_T L^r_x$ norm satisfies the scaling condition in Lemma 2.9. Note that we also need to modify the definition of $\mathcal{X}_1^\sigma(T)$ using this $L^q_T L^r_x$ norm. We
then modify (3.4) as follows. By (3.2), Lemma 2.9 (1.8), Lemma 2.11, Lemma 2.12, and Sobolev’s inequality, we have

\[ \|\tilde{F}(v)\|_{\mathcal{X}_0(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^0} + \|v + \Psi\|_{L^p_t L^2_x} \]

\[ \lesssim \|(u_0, u_1)\|_{\mathcal{H}^0} + T^\theta \left( \|v\|_{L^{2+\delta}_t L^2_x}^{p-2} + \|\Psi\|_{L^{p+\delta}_t L^2_x}^{p} \right) \]

\[ \lesssim \|(u_0, u_1)\|_{\mathcal{H}^0} + T^\theta \left( \|v\|_{\mathcal{X}_0(T)}^p + \|\Psi\|_{C_T W^{1,\infty}_x}^p \right) \]

for some \( \theta > 0 \). Here, we can ensure that \( 2p < r = \frac{4+2\delta}{1+\delta} \) for any \( 1 < p < 2 \) by choosing \( \delta = \delta(p) > 0 \) small enough. A similar modification can be applied to (3.5) to obtain a difference estimate, which then allows us to close the contraction argument.

\[ \square \]

**Remark 3.2.** We point out that the local well-posedness result of vNLW (3.1) can be improved using the inhomogeneous Strichartz estimates. In particular, we can show that (3.1) is locally well-posed in \( \mathcal{H}^s(\mathbb{T}^2) \) as long as \( s \geq s_{\text{crit}} \) (with a strict inequality when \( p = 3 \)), where \( s_{\text{crit}} \) is the critical regularity as defined in (1.7). For details, see Theorem A.6 and Remark A.7.

### 4. Almost sure local well-posedness of vNLW with randomized initial data

In this section, we study Theorem 1.5 i.e. the almost sure local well-posedness result for the deterministic vNLW (1.3) with randomized initial data. In this setting, we study the equation (1.20) with general deterministic initial data \((v_0, v_1)\):

\[
\begin{align*}
\partial_t^2 v + (1 - \Delta) v + D \partial_t v + F(v + z) &= 0 \\
(v, \partial_t v)|_{t=0} &= (v_0, v_1),
\end{align*}
\]

where \( F(u) = |u|^{p-1} u \). Here, we recall from (1.18) that \( z = V(t)(u_0^\omega, u_1^\omega) \), where \( V(t) \) is as defined in (1.8) and \((u_0^\omega, u_1^\omega)\) defined in (1.15) is the randomization of the deterministic initial data \((u_0, u_1) \in \mathcal{H}^\sigma(\mathbb{T}^2)\).

**Proposition 4.1.** Let \( p > 1 \) and \( -\frac{1}{p} < s \leq 0 \). Define \( q, r, \) and \( \sigma \) as follows.

(i) When \( 1 < p < 2 \), set \( q = 2 + \delta \), \( r = \frac{4+2\delta}{1+\delta} \), and \( \sigma = 0 \), for some sufficiently small \( \delta > 0 \).

(ii) When \( p \geq 2 \), set \( q = p + \delta \), \( r = 2p \), and \( \sigma = 1 - \frac{1}{p} - \frac{1}{p+\delta} \), for some sufficiently small \( \delta > 0 \).

Fix \((v_0, v_1) \in \mathcal{H}^\sigma(\mathbb{T}^2)\). Then, there exist constants \( C, c, \gamma > 0 \) and \( 0 < T_0 \ll 1 \) such that for each \( 0 < T \leq T_0 \), there exists a set \( \Omega_T \subset \Omega \) with the following properties:

(i) The following exponential probability bound holds:

\[ P(\Omega_T^c) < C \exp \left( -\frac{c}{T^\gamma \|(u_0, u_1)\|_{\mathcal{H}^0}^2} \right). \]

(ii) For each \( \omega \in \Omega_T \), there exists a unique solution \( \tilde{v} = (v, \partial_t v) \) to (1.1) in the class \( (v, \partial_t v) \in C([0,T]; \mathcal{H}^\sigma(\mathbb{T}^2)) \) and \( v \in L^4([0,T]; L^4(\mathbb{T}^2)) \).

Note that Theorem 1.5 follows immediately from Proposition 4.1 by taking \((v_0, v_1) = (0,0)\) in (1.1). Again, the main idea of the proof of Proposition 4.1 comes from [31].
Proof. Fix $s$ that satisfies $-\frac{1}{p} < s \leq 0$. Note that we can fix a choice of $\delta > 0$ sufficiently small such that $s > -\frac{1}{p+\delta}$. We define $\Omega_T \subset \Omega$ by

$$\Omega_T = \{ \omega \in \Omega : \|z\|_{L^{p+\delta}([0,T];L^p)} \leq 1 \}.$$  

(4.3)

Then, by Lemma 2.14 we obtain the probability bound (4.2) with $\gamma = \frac{2}{p+\delta} + 2s > 0$ for any $0 < T \leq 1$.

We write (4.1) in the Duhamel formulation:

$$v(t) = \Gamma(v) := V(t)(v_0, v_1) - \int_0^t S(t-t')F(v+z)(t')dt',$$

where $V(t)$ and $S(t)$ are as defined in (1.8) and (1.9), respectively. Let $\hat{\Gamma}(v) = (\Gamma(v), \partial_t \Gamma(v))$ and $\hat{v} = (v, \partial_t v)$. Given $0 < T \leq 1$, we define the space $Z(T)$ as

$$Z(T) = Z_1(T) \times Z_2(T),$$

where

$$Z_1(T) = C([0,T];H^\sigma(T^2)) \cap L^q([0,T];L^r(T^2)),$$

$$Z_2(T) = C([0,T];H^\sigma(T^2)).$$

Here, $q$, $r$, and $\sigma$ are as defined in the statement of the proposition. It remains to show that there exists $0 < T_0 \ll 1$ such that $\hat{\Gamma}$ is a contraction on a ball in $Z(T)$ for any $0 < T \leq T_0$ and for any $\omega \in \Omega_T$, where $\Omega_T$ is as defined in (4.3).

The remaining steps are similar to those in the proof of Proposition 3.1 so we omit details. \hfill \Box

Remark 4.2. (i) The regularity condition $s > -\frac{1}{p}$ seems to be sharp in our current approach based on the first order expansion (1.17). Measuring the $p$th power term in $L^1$ in time requires us to measure $z$ in $L^p$ in time. In view of Lemma 2.14 we need essentially $L^p$ integrability of $t^s$, which requires $s > -\frac{1}{p}$. Note that the restriction $s > -\frac{1}{p}$ comes from the temporal integrability and has nothing to do with the spatial integrability.

(ii) For further improvement over Proposition 4.1 one may investigate higher order expansions. See, for example, [5, 46].

5. Global well-posedness of SvNLW

In this section, we aim to prove Theorem 1.3, i.e. pathwise global well-posedness of SvNLW (1.2). As mentioned in Subsection 1.2 we prove Theorem 1.3 by studying the equation (1.12) for $v$ with $(v, \partial_t v)|_{t=0} = (u_0, u_1)$, for given initial data $(u_0, u_1) \in \mathcal{H}^s(T^2)$ of (1.2).

Fix an arbitrary $T \geq 1$. In view of Proposition 3.1 in order to show well-posedness of (3.1) on $[0, T]$, it suffices to show that the $\mathcal{H}^\sigma$-norm of the solution $\tilde{v}(t) = (v(t), \partial_t v(t))$ to (3.1) remains finite on $[0, T]$, where $\sigma$ is as defined in Proposition 3.1. This will allow us to iteratively apply the pathwise local well-posedness result in Proposition 3.1.

In fact, we show that the solution $\tilde{v}(t)$ belongs to $\mathcal{H}^1(T^2)$. Let $0 < t \leq 1$. From Lemma 2.8 we have

$$\|V(t)(u_0, u_1)\|_{\mathcal{H}^1} \lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{\mathcal{H}^\sigma}.  \hspace{2cm} (5.1)$$

Then, let $0 < T_0 \leq 1$ be the local existence time as in the proof of Proposition 3.1. Thus, given $s \geq \sigma$, by (5.2), (5.1), Lemma 2.12, Hölder’s inequality, and Sobolev’s inequality, we have that for $0 < t \leq T_0$,

$$
\|\tilde{v}(t)\|_{H^s} \lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{H^s} + \|(v + \Psi)^p\|_{L^1_t L^2_x} \lesssim (1 + t^{-1+\sigma})\|(u_0, u_1)\|_{H^s} + T_0^\theta \left(\|v\|_{L^2_t L^r_x}^p + \|\Psi\|_{C_{T_0} W^{\frac{1}{p}, \infty}_{2-\alpha, \infty}}^p\right),
$$

(5.2)

where $\delta > 0$, $\varepsilon > 0$ are sufficiently small, $\theta > 0$, and $q, r$ are as defined in the statement of Proposition 5.1. Here, due to Lemma 2.6, we can fix a good $\omega \in \Omega$ such that $\Psi = \Psi(\omega) \in C([0, T_0]; W^{\frac{1}{p}, \infty}_{2-\alpha, \infty}(\mathbb{T}^2))$ for $\alpha < \frac{1}{2}$ and sufficiently small $\varepsilon > 0$, so that we know from (5.2) that $\|\tilde{v}(t)\|_{H^s} < \infty$. A standard argument then shows that $\tilde{v} \in C((0, T_0]; H^1(\mathbb{T}^2))$. Thus, our main goal is to control the $H^1$-norm of $\tilde{v}(t)$ on $[0, T]$ by bounding the energy $E(\tilde{v})$ defined in (1.13).

For the following computation, we need to work with the smooth solution $(v_N, \partial_t v_N)$ to the truncated equation with initial data $(\pi_N v_0, \pi_N v_1)$, where $\pi_N$ is the frequency truncation onto the frequencies $\{|n| \leq N\}$. After establishing an upper bound for $E(\tilde{v}(t))$ with the implicit constant independent of $N$, we can take $N \to \infty$ by using Proposition 3.1 (specifically, the continuous dependence of a solution on the initial data). Here, we omit details and work with $(v, \partial_t v)$ instead for simplicity. See, for example, [44] for a standard argument.

5.1. Case $1 < p \leq 3$. In this case, we follow the globalization argument by Burq-Tzvetkov [11]. For simplicity of notation, we set $E(t) = E(\tilde{v}(t))$.

Given $T > 0$, we fix $0 < t < T$. By (1.13) and (1.12), we have

$$
\partial_t E(t) = \int_{\mathbb{T}^2} \partial_t v (\partial_t^2 v + (1 - \Delta) v + |v|^{p-1} v) \, dx 
\leq - \int_{\mathbb{T}^2} \partial_t v (|v + \Psi|^p - |v|^{p-1} v) \, dx.
$$

(5.3)

Let $F(u) = |u|^{p-1} u$, so that we can compute $F'(u) = p|u|^{p-1}$. Thus, by the fundamental theorem of calculus, we have

$$
|v + \Psi|^p - |v|^{p-1} v = F(v + \Psi) - F(v) = \Psi \int_0^1 F'(v + \tau \Psi) \, d\tau \lesssim |\Psi||v|^{p-1} + |\Psi|^p.
$$

(5.4)

Combining (5.3) and (5.4) and then applying the Cauchy-Schwartz inequality, we obtain

$$
\partial_t E(t) \lesssim \|\Psi\|_{L^\infty} \int_{\mathbb{T}^2} |\partial_t v||v|^{p-1} \, dx + \int_{\mathbb{T}^2} |\partial_t v||\Psi|^p \, dx
\lesssim \|\Psi\|_{L^\infty} \left(\int_{\mathbb{T}^2} (\partial_t v)^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} |v|^{2(p-1)} \, dx\right)^{\frac{1}{2}} + \|\Psi\|_{L^\infty}^{2p} \left(\int_{\mathbb{T}^2} (\partial_t v)^2 \, dx\right)^{\frac{1}{2}} \lesssim C(\Psi) E(t),
$$

(5.5)
as long as $2(p - 1) \leq p + 1$, or equivalently, $p \leq 3$. By Gronwall’s inequality on (5.5), we get

$$E(t) \lesssim e^{C(\Psi)t}$$

for any $0 < t \leq T$.

5.2. Case $3 < p \leq 5$. In this case, we follow the idea introduced by Oh-Pocovnicu [44]. See also [51, 36, 38] for similar arguments. In this setting, we let $\alpha < \frac{2}{p - 1} - \frac{1}{2}$, the reason of which will become clear in the following steps.

By (1.13), (1.12), and Taylor’s theorem, we have

$$\partial_t E(t) = \int_{\mathbb{T}^2} \partial_t v (\partial_t^2 v + (1 - \Delta)v + |v|^{p-1}v) \, dx$$

$$= -\int_{\mathbb{T}^2} \partial_t v (|v + \Psi|^{p-1}(v + \Psi) - |v|^{p-1}v) \, dx - \int_{\mathbb{T}^2} (D^2 \partial_t v)^2 \, dx$$

$$\leq -p \int_{\mathbb{T}^2} \partial_t v \cdot |v|^{p-1}v \, dx - \frac{p(p-1)}{2} \int_{\mathbb{T}^2} \partial_t v \cdot |v + \theta \Psi|^{p-3}(v + \theta \Psi) \Psi^2 \, dx$$

$$=: A_1 + A_2,$$

where $\theta \in (0, 1)$. To estimate $A_2$, by the Cauchy-Schwartz inequality and Cauchy’s inequality, we have

$$|A_2| \lesssim \int_{\mathbb{T}^2} |\partial_t v| (|v|^{p-2} \Psi^2 + \Psi^p) \, dx$$

$$\lesssim \left( \int_{\mathbb{T}^2} (\partial_t v)^2 \, dx \right)^{1/2} \left( \|\Psi\|_{L^\infty}^{1/2} \int_{\mathbb{T}^2} |v|^{2(p-2)} \, dx + \|\Psi\|_{L^{2p}}^{p} \right)^{1/2}$$

$$\lesssim (1 + \|\Psi\|_{L^\infty}^{1/2}) E(t) + \|\Psi\|_{L^{2p}}^{p},$$

where in the last inequality, we need $2(p - 2) \leq p + 1$, which is equivalent to $p \leq 5$. To estimate $A_1$, for $0 < t_1 \leq t_2 \leq T$, by integration by parts and Young’s inequality, we have

$$\int_{t_1}^{t_2} A_1 dt' = -\int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t (|v|^{p-1}v) \Psi \, dx dt'$$

$$= -\int_{\mathbb{T}^2} |v(t_2)|^{p-1}v(t_2) \Psi(t_2) \, dx + \int_{\mathbb{T}^2} |v(t_1)|^{p-1}v(t_1) \Psi(t_1) \, dx$$

$$+ \int_{t_1}^{t_2} \int_{\mathbb{T}^2} |v|^{p-1}v(\partial_t \Psi) \, dx dt'$$

$$\lesssim \varepsilon \|v(t_2)\|_{L^{p+1}}^{p+1} + \frac{1}{\varepsilon} \|\Psi(t_2)\|_{L^{p+1}}^{p+1} + \|v(t_1)\|_{L^{p+1}}^{p+1} + \|\Psi(t_1)\|_{L^{p+1}}^{p+1}$$

$$+ \int_{t_1}^{t_2} \int_{\mathbb{T}^2} |v|^{p-1}v(\partial_t \Psi) \, dx dt',$$
where $0 < \varepsilon < 1$. We see in Remark 2.7 that $\partial_t \Psi \in C([0, T]; W^{-\frac{1}{2} - \alpha - \infty}(\mathbb{T}^2))$. By duality, Hölder’s inequality, Lemma 2.3 (i), and Lemma 2.4, we obtain

$$
\int_{t_1}^{t_2} \int_{\mathbb{T}^2} |v|^{p-1} v(\partial_t \Psi) dx dt' \\
= \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \langle \nabla \rangle^{\frac{1}{2} + \alpha} (|v|^{p-1} v) \langle \nabla \rangle^{-\frac{1}{2} - \alpha} (\partial_t \Psi) dx dt' \\
\lesssim \int_{t_1}^{t_2} \|v\|_{L^{p+1}_x} \|\langle \nabla \rangle^{\frac{1}{2} + \alpha} v'(t')\|_2 \|\langle \nabla \rangle^{-\frac{1}{2} - \alpha} (\partial_t \Psi)(t')\|_{L^\infty} dt' \\
\lesssim \|\partial_t \Psi\|_{C_T W^{\frac{1}{2} - \alpha, \infty}_x} \int_{t_1}^{t_2} E(t') \|v\|_{L^{2}_x} \|\langle \nabla \rangle v\|_{L^{p+1}_x} dt' \\
\lesssim \|\partial_t \Psi\|_{C_T W^{\frac{1}{2} - \alpha, \infty}_x} \int_{t_1}^{t_2} E(t') dt',
$$

(5.9)

where we require that

$$
\frac{1}{2} + \alpha = \frac{2}{p-1} - \frac{1}{2},
$$

which is equivalent to $\alpha < \frac{2}{p-1} - \frac{1}{2}$. By combining (5.6), (5.7), (5.8), and (5.9), we have

$$
E(t_2) \leq (1 + C_1(\Psi)) \int_{t_1}^{t_2} E(t') dt' + C_2(\Psi, v(t_1)).
$$

By Gronwall’s inequality, we get

$$
E(t) \lesssim e^{C(\Psi) t}
$$

for any $0 < t \leq T$.

5.3. Case $p > 5$. In this case, we follow the idea by Latocca [22]. In this setting, we also let $\alpha < \frac{2}{p-1} - \frac{1}{2}$.

We need the following lemma to close the energy estimates in the Gronwall argument. We define $\beta_p := \left[ \frac{p-3}{2} \right], F(u) := |u|^{p-1} u$, and $s_p := \frac{p-3}{p-1}$.

**Lemma 5.1.** For any $0 < t \leq T$ and every integer $1 \leq k \leq \beta_p$, we have

$$
\left| \int_{\mathbb{T}^2} F^{(k-1)}(v(t)) \Psi(t) v^{k-1} \partial_t \Psi(t) dx \right| \\
\lesssim g(\|\Psi\|_{L^\infty([0, T]; X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L^\infty([0, T]; Y)}) (1 + E(t)),
$$

where $g$ is a polynomial with positive coefficients, and

$$
X := L^\infty(\mathbb{T}^2) \cap B^{1-s_p}_{s_p+1}(\mathbb{T}^2) \quad \text{and} \quad Y := L^\infty(\mathbb{T}^2) \cap B^{s_p}_{\infty, 1}(\mathbb{T}^2).
$$

Note that given $\alpha < \frac{2}{p-1} - \frac{1}{2}$, by Lemma 2.6, Remark 2.7, and Lemma 2.1 (ii), we have

$$
g(\|\Psi\|_{L^\infty([0, T]; X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L^\infty([0, T]; Y)}) < \infty
$$

almost surely.

Let us first assume Lemma 5.1 and work on the energy bound. As in the case when $p > 3$, we can compute that for $0 < t \leq T$,

$$
\partial_t E(t) \leq - \int_{\mathbb{T}^2} \partial_t v (F(v + \Psi) - F(v)) dx.
$$

(5.10)
For our convenience we compute that for $k \in \mathbb{Z}_+$,
\[
F^{(k)}(u) = \begin{cases} 
C_{p,k}|u|^{p-k-1}u & \text{for } k \text{ even}, \\
C_{p,k}|u|^{p-k} & \text{for } k \text{ odd}.
\end{cases}
\]

By Taylor’s formula at the point $v(t, x)$ with integral remainder up to the order $\beta_p = \lceil \frac{p-3}{2} \rceil$, we have
\[
F(v + \Psi) - F(v) = \sum_{k=1}^{\beta_p} \frac{1}{k!} F^{(k)}(v) \Psi^k + \int_v^{v+\Psi} \frac{F(\beta_p+1)(\tau)}{\beta_p!} (v + \Psi - \tau)^{\beta_p} d\tau.
\]

Let $0 < t_1 \leq t_2 \leq T$. By integrating (5.10) from $t_1$ to $t_2$, we can write
\[
E(t_2) \leq E(t_1) + \sum_{k=1}^{\beta_p} C_k I_k + C_p R,
\]

where
\[
I_k := - \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t v F^{(k)}(v) \Psi^k dxdt' \quad \text{for } 1 \leq k \leq \beta_p,
\]
\[
R := - \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t v F^{(\beta_p+1)}(\tau)(v + \Psi - \tau)^{\beta_p} d\tau dxdt'.
\]

We first estimate $R$. Note that for $\tau \in [v, v + \Psi]$, we have
\[
|F^{(\beta_p+1)}(\tau)| \lesssim |v|^{p-\beta_p-1} + |\Psi|^{p-\beta_p-1}.
\]

Thus, by Hölder’s inequality and Young’s inequality, we have
\[
R \lesssim \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t v |v|^{p-\beta_p-1} |\Psi|^{\beta_p+1} + |\Psi|^p dxdt'
\]
\[
\lesssim \int_{t_1}^{t_2} \|\partial_t v(t')\|_{L_x^p} \|v(t')\|_{L_x^{p+1}}^{p-\beta_p-1} \|\Psi(t')\|_{L_x^{p+1}}^{\beta_p+1} dt' + \int_{t_1}^{t_2} \|\partial_t v(t')\|_{L_x^p}^2 dt'
\]
\[
+ \|\Psi\|_{L_x^{2p}}^{2p} + \left(1 + \|\Psi\|_{L_x^{p+1}}^{\beta_p+1} \right) \int_{t_1}^{t_2} \max \left\{ E(t'), E(t')^{1 + \frac{p-\beta_p-1}{p+1}} \right\} dt',
\]

where $r_p$ satisfies $\frac{1}{2} + \frac{p-\beta_p-1}{p+1} + \frac{1}{r_p} = 1$. Since $\beta_p = \lceil \frac{p-3}{2} \rceil \geq \frac{p-3}{2}$, we have $\frac{p-\beta_p-1}{p+1} \leq \frac{1}{2}$, so that
\[
R \lesssim \|\Psi\|_{L_x^{2p} L_x^p}^{2p} + \left(1 + \|\Psi\|_{L_x^{p+1}}^{\beta_p+1} \right) \int_{t_1}^{t_2} (1 + E(t')) dt'.
\]
We now estimate $I_k$. By Fubini’s theorem and integration by parts in time, we have

$$|I_k| = -\int_{T^2} \int_{t_1}^{t_2} \partial_t (F^{(k-1)}(v)) \Psi^k dt' dx
\leq \int_{T^2} F^{(k-1)}(v(t_2)) \Psi^k(t_2) dx + \int_{T^2} F^{(k-1)}(v(t_1)) \Psi^k(t_1) dx
+ k \int_{T^2} \int_{t_1}^{t_2} F^{(k-1)}(v(t')) \Psi(t')^{k-1} \partial_t \Psi(t') dt' dx,$$  

(5.13)

$$\lesssim \int_{T^2} |v(t_2)|^{p-k+1} |\Psi(t_2)|^k + |v(t_1)|^{p-k+1} |\Psi(t_1)|^k dx
+ \int_{t_1}^{t_2} \int_{T^2} F^{(k-1)}(v(t')) \Psi(t')^{k-1} \partial_t \Psi(t') dx dt',$$

$$= J_k + K_k.$$

To handle $J_k$, by Hölder’s inequality and Young’s inequality, we obtain

$$J_k \leq E(t_2) \|\Psi(t_2)\|_{L_{x}^{p+1}}^{p} + E(t_1) \|\Psi(t_1)\|_{L_{x}^{p+1}}^{p}
\leq \varepsilon E(t_2) + C_1 E(t_1) + C_2 \|\Psi\|_{L_{x}^{p+1}}^{p+1},$$

(5.14)

where $0 < \varepsilon < 1$. To deal with $K_k$, by Lemma 5.1,

$$K_k \lesssim g(\|\Psi\|_{L_{x}^{\infty}(\{0,T\};X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L_{x}^{\infty}(\{0,T\};Y)}) \left(1 + \int_{t_1}^{t_2} E(t') dt'\right).$$

(5.15)

By combining (5.11), (5.12), (5.13), (5.14), (5.15), we obtain

$$E(t_2) \lesssim \left(1 + \|\Psi\|_{L_{x}^{p+1}}^{p} + g(\|\Psi\|_{L_{x}^{\infty}(\{0,T\};X)}, \|\langle \nabla \rangle^{-1} \partial_t \Psi\|_{L_{x}^{\infty}(\{0,T\};Y)})\right)
\times \left(1 + \int_{t_1}^{t_2} E(t') dt'\right) + \|\Psi\|_{L_{x}^{p+1}}^{p+1}.$$  

We can then use Gronwall’s inequality to get the desired bound.

We now provide the proof of Lemma 5.1

Proof of Lemma 5.1. Recall that $s_p = \frac{p-3}{p-1}$. We first consider the case when $k \geq 2$. By the Fourier-Plancherel theorem, we have

$$\left| \int_{T^2} F^{(k-1)}(v(t)) \Psi(t)^{k-1} \partial_t \Psi(t) dx \right|
= \left| \sum_{j' = -1}^{1} \sum_{j \geq 0} \int_{T^2} P_j(F^{(k-1)}(v(t)) \Psi(t)^{k-1}) P_{j+j'}(\partial_t \Psi(t)) dx \right|
\lesssim \sum_{j > 2} \int_{T^2} |P_j(F^{(k-1)}(v(t)) \Psi(t)^{k-1})| |P_j(\partial_t \Psi(t))| dx
+ \sum_{j = 0}^{2} \int_{T^2} |P_j(F^{(k-1)}(v(t)) \Psi(t)^{k-1})| |P_j(\partial_t \Psi(t))| dx$$  

(5.16)

$$=: I_1 + I_2.$$
Let $r_k := (k-1)(p+1)$. To estimate $I_2$, by Hölder’s inequality, Bernstein’s inequality, and Young’s inequality, we have

$$I_2 \lesssim \|\tilde{\Psi}(t)\|_{L^{p+1}_x}^{k-1} \|v(t)\|_{L^{p-k+1}_x}^{p-k+1} \sum_{j=0}^{2} \|P_j \partial_t \tilde{\Psi}(t)\|_{L^\infty_x}$$

$$\lesssim \|\tilde{\Psi}(t)\|_{L^{p+1}_x}^{k-1} \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}(t)\|_{L^\infty_x} E(t)^{\frac{p-k+1}{p+1}}$$

$$\lesssim E(t) + \|\tilde{\Psi}\|_{L^\infty_x}^{r_k} \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}\|_{L^\infty_x}^{\frac{p+1}{p+1}}.$$

(5.17)

It remains to estimate $I_1$. By Hölder’s inequality, Bernstein’s inequality, and then Hölder’s inequality for series,

$$I_1 \lesssim \sum_{j>2} 2^{j(1-s_p)} \|P_j(F^{(k-1)}(v(t))\tilde{\Psi}(t))^{k-1}\|_{L^1_x} 2^{jsp} \|P_j(\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}(t))\|_{L^\infty_x}$$

$$\leq \|F^{(k-1)}(v(t))\tilde{\Psi}(t)^{k-1}\|_{B^{1-s_p}_1} \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}\|_{B^{s_p}_{\infty}}.$$  

Then, by Corollary 2.2, we have

$$\|F^{(k-1)}(v(t))\tilde{\Psi}(t)^{k-1}\|_{B^{1-s_p}_1} \lesssim \|F^{(k-1)}(v(t))\|_{B^{1-s_p}_1} \|\tilde{\Psi}(t)^{k-1}\|_{L^{p+1}_x}$$

$$+ \|v(t)^{p-k+1}\|_{L^{p-k+1}_x} \|\tilde{\Psi}(t)^{k-1}\|_{B^{1-s_p}_{p+1} \infty}$$

$$\lesssim \|F^{(k-1)}(v(t))\|_{B^{1-s_p}_1} \|\tilde{\Psi}(t)^{k-1}\|_{L^{p+1}_x} + E(t)^{\frac{p-k+1}{p+1}} \|\tilde{\Psi}(t)^{k-1}\|_{B^{1-s_p}_{p+1, \infty}},$$

(5.18)

where $p_k$ satisfies $\frac{1}{p_k} + \frac{p-k+1}{p+1} = 1$. By Lemma 2.3 (ii), we have

$$\|\tilde{\Psi}(t)^{k-1}\|_{B^{1-s_p}_{p+1, \infty}} \lesssim \|\tilde{\Psi}(t)\|_{B^{1-s_p}_{p+1, \infty}} \|\tilde{\Psi}(t)\|_{L^\infty}$$

$$\lesssim \|\tilde{\Psi}(t)\|_{B^{1-s_p}_{p+1, \infty}} \|\tilde{\Psi}(t)\|_{B^{1-s_p}_{p+1, \infty}}^{k-2}.$$  

(5.19)

By Lemma 2.3 (ii), Lemma 2.1 (ii), and Lemma 2.4, we have

$$\|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{p+1, \infty}} \lesssim \|v(t)\|_{B^{1-s_p}_{p+1, \infty}} \|v(t)^{p-k}\|_{L^{p+1}_x}$$

$$\lesssim \|v(t)\|_{W^{1-s_p, \frac{p+1}{p+1}}(\mathbb{R}^n)} E(t)^{\frac{p-k}{p+1}}$$

$$\lesssim \|\langle \nabla \rangle v(t)\|_{L^{1-\beta}_x} \|v(t)\|_{L^{p+1}_x}^{\frac{p-k}{p+1}} E(t)^{\frac{p-k}{p+1}},$$

where $\beta \in [0, s_p]$ satisfies $\frac{2}{p+1} = \frac{1-s_p}{2} + \frac{\beta}{p+1}$, and so $\beta = \frac{p-3}{p+1} = s_p$. Thus, we obtain

$$\|F^{(k-1)}(v(t))\|_{B^{1-s_p}_{p+1, \infty}} \lesssim E(t)^{\frac{1-\beta}{2} + \frac{\beta}{p+1} + \frac{p-k}{p+1}} = E(t)^{\frac{2}{p+1} + \frac{p-k}{p+1}} \lesssim 1 + E(t).$$  

(5.20)

By combining (5.18), (5.19), and (5.20), we obtain the desired bound for $I_1$.

For the case when $k = 1$, after (5.16), we have the estimate $I_2 \lesssim E(t) + \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}\|_{L^\infty_x} L^\infty_x$.

For the term $I_1$, by the estimate in (5.20), we have

$$I_1 \lesssim \|F(v(t))\|_{B^{1-s_p}_{\infty}} \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}\|_{B^{s_p}_{\infty, 1}} \lesssim \|\langle \nabla \rangle^{-1} \partial_t \tilde{\Psi}\|_{B^{s_p}_{\infty, 1}} (1 + E(t)),$$

as desired. □
6. Almost sure global well-posedness of vNLW with randomized initial data

In this section, we study Theorem 1.7 i.e. the almost sure global well-posedness of the deterministic vNLW (1.3) with randomized initial data. Due to the Borel-Cantelli lemma, it suffices to show the following “almost” almost sure global well-posedness result (see [16, 4]).

**Proposition 6.1.** Let $p > 1$ and $-\frac{1}{p} < s \leq 0$. Given initial data $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, let $(u_0^\omega, u_1^\omega)$ be the randomization of $(u_0, u_1)$ defined in (1.15). Then, given any $T, \varepsilon > 0$, there exists $\Omega_{T,\varepsilon} \subset \Omega$ such that

(i) $P(\Omega_{T,\varepsilon}^c) < \varepsilon$,

(ii) For each $\omega \in \Omega_{T,\varepsilon}$, there exists a (unique) solution $u$ to (1.3) on $[0, T]$ with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$.

The argument for proving Proposition 6.1 is similar to that of Theorem 1.3. The only difference for proving Proposition 6.1 is that, instead of dealing with the stochastic object $\Psi$ (as defined in (2.10)), we need to work with the random linear solution $z(t) = V(t)(u_0^\omega, u_1^\omega)$. Thanks to Lemma 2.14 and Lemma 2.17 we are able to bound $z$ in Strichartz spaces $L_t^q L_x^r$ for $1 \leq q, r \leq \infty$ outside an exceptional set of small probability. Note that in order to apply Lemma 2.17 we need to consider well-posedness in the time interval $[0, T]$ (instead of $[0, T_0]$), where $0 < T_0 \leq 1$ is the local existence time as in Proposition 4.1.

We refer the readers to Section 5 for the rest of the details for proving Proposition 6.1.

**Appendix A. On local well-posedness of subcritical vNLW**

In this appendix, we aim to show that the deterministic viscous NLW is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ with $s \geq s_{\text{crit}}$, where we recall that $s_{\text{crit}}$ is defined by

$$s_{\text{crit}} := \max \left(1 - \frac{2}{p - 1}, 0\right). \quad (A.1)$$

More precisely, we prove local well-posedness of the following subcritical vNLW:

$$\begin{cases}
\partial_t^2 u + (1 - \Delta)u + D\partial_t u \pm |u|^{p-1}u = 0 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1),
\end{cases} \quad (A.2)$$

where $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$ and $s \geq s_{\text{crit}}$ (with a strict inequality when $p = 3$). To achieve this, we will need the inhomogeneous Strichartz estimates for the linear viscous wave equation on $\mathbb{T}^2$.

**A.1. The inhomogeneous Strichartz estimates.** In this subsection, we prove the Strichartz estimates for the inhomogeneous linear viscous wave equation on $\mathbb{T}^d$. To achieve this, we first establish the following estimate for the linear operator $S(t)$ defined in (1.9).

**Lemma A.1.** Let $1 \leq p \leq 2 \leq q \leq \infty$. Then, we have

$$\|S(t)\phi\|_{L^q(\mathbb{T}^d)} \lesssim t^{1-\frac{d(p-1)}{2}}\|\phi\|_{L^p(\mathbb{T}^d)}$$

for any $0 < t \leq 1$. 

Proof. By (1.9) and applying the Schauder estimate (Lemma 2.8) twice, we obtain
\begin{align*}
\|S(t)\phi\|_{L^q(T^d)} &= \left\|e^{-\frac{D}{2}t}\sin(\frac{D}{2}t)\phi\right\|_{L^q(T^d)} \\
&\lesssim t^{-d\left(\frac{1}{2} - \frac{1}{q}\right)}\left\|e^{-\frac{D}{2}t}\phi\right\|_{L^2(T^d)} \\
&\lesssim t^{1 - d\left(\frac{1}{2} - \frac{1}{q}\right)}\left\|\phi\right\|_{L^q(T^d)},
\end{align*}
as desired.

We now establish the Strichartz estimates for the inhomogeneous linear viscous wave equation on \( T^d \). We say that \( u \) is a solution to the following inhomogeneous linear viscous wave equation:
\begin{equation}
\begin{aligned}
\partial^2_t u + (1 - \Delta)u + D\partial_t u &= f \\
(u, \partial_t u)|_{t=0} &= (\phi_0, \phi_1),
\end{aligned}
\tag{A.3}
\end{equation}
if \( u \) satisfies the following Duhamel formulation:
\[ u(t) = V(t)(\phi_0, \phi_1) + \int_0^t S(t - t')f(t')dt', \]
where \( V(t) \) and \( S(t) \) are as defined in (1.8) and (1.9), respectively.

**Lemma A.2.** Given \( s \geq 0 \), suppose that \( 1 < \tilde{q} \leq 2 < q < \infty \), \( 1 \leq \tilde{r} \leq 2 \leq r \leq \infty \) satisfy the following scaling condition:
\begin{equation}
\frac{1}{q} + \frac{d}{r} = \frac{d}{\tilde{r}} - s = \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} - 2.
\tag{A.4}
\end{equation}
Then, a solution \( u \) to the inhomogeneous linear viscous wave equation (A.3) satisfies the following inequality:
\begin{equation}
\|(u, \partial_t u)\|_{C_T H^s_x(T^d)} + \|u\|_{L^q_x L^r_t(T^d)} \lesssim \|(\phi_0, \phi_1)\|_{H^s(T^d)} + \|f\|_{L^\tilde{q}_x L^\tilde{r}_t(T^d)},
\end{equation}
for all \( 0 < T \leq 1 \).

**Proof.** By (1.8), we have
\[ \|(V(t)(\phi_0, \phi_1), \partial_t V(t)(\phi_0, \phi_1))\|_{C_T H^s_x(T^d)} \lesssim \|(\phi_0, \phi_1)\|_{H^s(T^d)}. \tag{A.6} \]
By Lemma 2.9 we have
\[ \|V(t)(\phi_0, \phi_1)\|_{L^q_x L^r_t(T^d)} \lesssim \|(\phi_0, \phi_1)\|_{H^s(T^d)}. \tag{A.7} \]
We then use Lemma 3.5 in [29] (which is in the \( \mathbb{R}^d \) setting, but the proof also works in the \( T^d \) setting with Lemma A.1 in hand) to obtain
\[ \left\|\int_0^t S(t - t')f(t')dt'\right\|_{L^q_x L^r_t(T^d)} \lesssim \|f\|_{L^\tilde{q}_x L^\tilde{r}_t(T^d)}. \tag{A.8} \]
It remains to show
\[ \left\|\int_0^t S(t - t')f(t')dt'\right\|_{C_T H^s_x(T^d)} \lesssim \|f\|_{L^\tilde{q}_x L^\tilde{r}_t(T^d)}. \tag{A.9} \]
and
\[
\left\| \partial_t \int_0^t S(t-t')f(t')dt' \right\|_{C_T H_{n}^{s-1}(T^d)} \lesssim \|f\|_{L_{x}^q(L_{t}^r(T^d))},
\]  
(A.10)
so that (A.5) follows from (A.6), (A.7), (A.8), (A.9), and (A.10).

To show that the inequality (A.9) holds, we use the Littlewood-Paley decomposition as in Lemma 3.6 in [29]. In view of the proof of Lemma 3.6 in [29], we know that it suffices to show (A.9) for all \( f \) such that \( \widehat{f} \) is supported in \( \{ n \in \mathbb{Z}^d : 2^{-j-1} \leq |n| \leq 2^{j+1} \} \) for all \( j \in \mathbb{Z}_+ \) (the case for \( \{ n \in \mathbb{Z}^d : 0 \leq |n| \leq 2 \} \) follows in a similar manner) with the underlying constant independent of \( j \). Fix \( 0 < t < T \). By Minkowski’s integral inequality, Hölder’s inequality in \( n \), Hausdorff-Young inequality, Hölder’s inequality in \( t' \) (along with the fact that the number of lattice points inside a ball of radius \( R \) in \( \mathbb{R}^d \) is \( O(R^d) \)), and a change of variable, we have
\[
\left\| \int_0^t S(t-t')f(t')dt' \right\|_{C_T H_{n}^{s}} \lesssim \int_0^t \left( \sum_{n \in \mathbb{Z}^d} |n|^{2s} e^{-\frac{|n|}{2}(t-t') \sin((t-t')|n|)} \left| \widehat{f}(t',n) \right|^2 \right)^{1/2} dt' \\
\lesssim 2^{j+1} \int_0^t (t-t') e^{2^{j-2}(t-t')} \left( \sum_{n \in \mathbb{Z}^d} \left| \widehat{f}(t',n) \right|^2 \right)^{1/2} dt' \\
\lesssim 2^{j+1} \int_0^t (t-t') e^{2^{j-2}(t-t')} \left( 2^{(j+1)\delta} \left\| \widehat{f}(t',n) \right\|_{\ell^2} \right) dt' \\
\lesssim 2^{j+1} (s + \frac{\delta}{p} - \frac{\delta}{q}) \left( \int_0^t (t-t') e^{2^{j-2}(t-t')} dt' \right)^{1/2} \left\| f \right\|_{L_{x}^q(L_{t}^r(T^d))} \\
\lesssim 2^{j+1} (s + \frac{\delta}{p} - \frac{\delta}{q}) \left\| f \right\|_{L_{x}^q(L_{t}^r(T^d))}.
\]

By using the second equality in the scaling condition (A.4), we obtain the desired inequality with the underlying constant independent of \( j \), and so the inequality (A.9) follows. The inequality (A.10) follows in a similar manner.

**Remark A.3.** As in the case of the homogeneous Strichartz estimates (Lemma 2.9), the Strichartz estimates for the inhomogeneous linear viscous wave equation on \( T^d \) also hold for a larger class of pairs \( (q,r) \) and \( (\tilde{q},\tilde{r}) \) compared to the Strichartz estimates for the usual linear wave equations [22, 33, 27, 24]. Again, this is due to the dissipative effects of the viscosity term. Note that this is also true on \( \mathbb{R}^d \) (see [29]).

We complete this subsection by making the following observation. Recall that we are considering the viscous NLW on \( T^2 \) with nonlinearity \( |u|^{p-1}u \) for \( p > 1 \). Suppose that we can find pairs \( (q,r) \) and \( (\tilde{q},\tilde{r}) \) satisfying the scaling condition (A.4) such that
\[
q > p\tilde{q} \quad \text{and} \quad r \geq p\tilde{r}.
\]
Then, by Hölder’s inequality and the fact that \( |T^2| = 1 \), we have
\[
\left\| |u|^{p-1}u \right\|_{L_{x}^q(L_{t}^r(T^d))} \leq T^{\frac{1}{\tilde{q}} - \frac{1}{q}} \left\| u \right\|_{L_{x}^q(T^d)}^{p}.
\]
Note that the power of $T$ is positive when $q > \tilde{p}\tilde{q}$. The following lemma shows that there exist such pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$.

**Lemma A.4.** Let $s_{\text{crit}}$ be as defined in (A.1). Given $s_{\text{crit}} < s < 1$, there exist $1 < \tilde{q} \leq 2 < q < \infty$, $1 \leq \tilde{r} \leq 2 \leq r \leq \infty$ satisfying the scaling condition (A.4) such that

$$q > \tilde{p}\tilde{q} \quad \text{and} \quad r \geq p\tilde{r}. \quad (A.11)$$

**Proof.** In view of Lemma 3.3 in [24], given $0 < s < 1$, we have

$$\min\left(\frac{q}{\tilde{q}}, \frac{r}{\tilde{r}}\right) \leq \frac{3 - s}{1 - s},$$

and the equality holds by taking, for example,

$$(q, r) = \left(\frac{3 - s}{1 - s}\delta, \frac{2}{1 - s - \frac{1 - s}{(3 - s)d}}\right) \quad \text{and} \quad (\tilde{q}, \tilde{r}) = \left(\delta, \frac{2}{3 - s - \frac{1}{\delta}}\right), \quad (A.12)$$

where $\delta = \delta(s) > 1$ is sufficiently close to 1. Moreover, we note that $\frac{3 - s}{1 - s} > p$ if and only if $s > 1 - \frac{2}{p-1}$. Thus, as long as $s_{\text{crit}} < s < 1$, there exist pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ that satisfy (A.11). \hfill \square

**Remark A.5.** In the case when $p > 3$ and $s = s_{\text{crit}} = 1 - \frac{2}{p-1} > 0$, we have

$$\min\left(\frac{q}{\tilde{q}}, \frac{r}{\tilde{r}}\right) \leq \frac{3 - s}{1 - s} = p,$$

so that we can only find pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ that satisfy $q = \tilde{p}\tilde{q}$ and $r = p\tilde{r}$ instead of $q > \tilde{p}\tilde{q}$ and $r \geq p\tilde{r}$. Such pairs do exist. One can take, for example, $(q, r)$ and $(\tilde{q}, \tilde{r})$ as in (A.12).

In the case when $1 < p \leq 3$ and $s = s_{\text{crit}} = 0$, there does not exist any pair $(\tilde{q}, \tilde{r})$ that satisfies $1 < \tilde{q} \leq 2$, $1 \leq \tilde{r} \leq 2$, and the scaling condition (A.4) (with $d = 2$) simultaneously. In this case, the inhomogeneous Strichartz estimates (Lemma A.2) no longer applies, so that an alternative approach is needed to deal with this case.

**A.2. Local well-posedness of subcritical vNLW.** In this subsection, we prove the following theorem for the local well-posedness result of vNLW (A.2).

**Theorem A.6.** Let $p > 1$ and let $s_{\text{crit}}$ be as in (A.1). Then, (A.2) is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$ for

(i) $p \neq 3$: $s \geq s_{\text{crit}}$ \quad or \quad (ii) $p = 3$: $s > s_{\text{crit}}$.

More precisely, given any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exists $0 < T = T(u_0, u_1) \leq 1$ and a unique solution $\bar{u} = (u, \partial_t u)$ to (A.2) in the class

$$(u, \partial_t u) \in \mathcal{C}(\mathbb{R}_+; \mathcal{H}^s(\mathbb{T}^2)) \quad \text{and} \quad u \in L^q([0, T]; L^r(\mathbb{T}^2),$$

for some suitable $q, r \geq 2$.

**Proof.** For the proof, we only consider the case $s < 1$. We first consider the case $s > s_{\text{crit}}$. We write (A.2) in the Duhamel formulation:

$$u(t) = \Gamma(u) := V(t)(u_0, u_1) - \int_0^t S(t - t')F(u(t'))dt', \quad (A.13)$$
where $F(u) = |u|^{p-1}u$, $V(t)$ is as defined in (1.8), and $S(t)$ is as defined in (1.9). Let $\bar{\Gamma}(u) = (\Gamma(u), \partial_t \Gamma(u))$ and $\bar{u} = (u, \partial_t u)$.

Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be as given in Lemma A.4 which guarantees that $q > p\tilde{q}$ and $r \geq p\tilde{r}$. Given $0 < T \leq 1$, we define the space $\mathcal{Y}(T)$ as

$$
\mathcal{Y}^s(T) = \mathcal{Y}^s_1(T) \times \mathcal{Y}^s_2(T),
$$

where

$$
\mathcal{Y}^s_1(T) := C([0, T]; H^s(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)),
$$

$$
\mathcal{Y}^s_2(T) := C([0, T]; H^{s-1}(\mathbb{T}^2)).
$$

Our goal is to show that $\bar{\Gamma}$ is a contraction on a ball in $\mathcal{Y}^s(T)$ for some $0 < T \leq 1$.

By (A.13), Lemma A.2, and Hölder’s inequality, we have

$$
\|\bar{\Gamma}(u)\|_{\mathcal{Y}^s(T)} \lesssim \|(u_0, u_1)\|_{H^s} + \|u\|_{L^q_1 L^r} + \|u\|_{L^q_1 L^r}
$$

$$
\lesssim \|(u_0, u_1)\|_{H^s} + T^\theta \|u\|_{L^q_1 L^r} + \|u\|_{L^q_1 L^r}
$$

(A.14)

for some $\theta > 0$.

For the difference estimate, we use the idea from Oh-Okamoto-Pocovnicu [32]. Noticing that $F'(u) = p|u|^{p-1}$, we use (A.13), Lemma A.2, the fundamental theorem of calculus, Minkowski’s integral inequality, and Hölder’s inequality to obtain

$$
\|\bar{\Gamma}(u) - \bar{\Gamma}(v)\|_{\mathcal{Y}^s(T)} \lesssim \|F(u) - F(v)\|_{L^q_1 L^r}
$$

$$
= \left\| \int_0^1 F'(v + \tau(u - v))(u - v) d\tau \right\|_{L^q_1 L^r}
$$

$$
\lesssim \int_0^1 \|v + \tau(u - v)\|^{p-1}_{L^q_1 L^r} \|u - v\|_{L^q_1 L^r} d\tau
$$

$$
\lesssim T^\theta \left( \|u\|^{p-1}_{L^q_1 L^r} + \|v\|^{p-1}_{L^q_1 L^r} \right) \|u - v\|_{L^q_1 L^r}
$$

$$
\lesssim T^\theta \left( \|\tilde{u}\|^{p-1}_{\mathcal{Y}^s(T)} + \|\tilde{v}\|^{p-1}_{\mathcal{Y}^s(T)} \right) \|\tilde{u} - \tilde{v}\|_{\mathcal{Y}^s(T)}.
$$

for some $\theta > 0$.

Thus, by choosing $T = T(||(u_0, u_1)||_{H^s}) > 0$ small enough, we obtain that $\bar{\Gamma}$ is a contraction on the ball $B_R \subset \mathcal{Y}^s(T)$ of radius $R \sim 1 + \|(u_0, u_1)||_{H^s}$.

In the case when $p > 3$ and $s = s_{crit} = 1 - \frac{2}{p-1} > 0$, we can only find pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ that satisfy $q = p\tilde{q}$ and $r = p\tilde{r}$ (see Remark A.5). In this case, we modify the argument as follows. By (A.13), Lemma A.2, and Hölder’s inequality, we obtain

$$
\|\Gamma(u)\|_{L^q_1 L^r} \lesssim \|V(t)(u_0, u_1)\|_{L^q_1 L^r} + \|u\|_{L^q_1 L^r}
$$

$$
\lesssim \|V(t)(u_0, u_1)\|_{L^q_1 L^r} + \|u\|_{L^q_1 L^r}
$$

for some $\theta > 0$ and sufficiently small $\varepsilon > 0$. A difference estimate on $\Gamma(u) - \Gamma(v)$ also holds by a similar computation. By the dominated convergence theorem, we have $\|u\|_{L^q_1 L^r} \to 0$ as $T \to 0$. Thus, we can choose $T = T(u_0, u_1) > 0$ sufficiently small such
that \( \|V(t)(u_0, u_1)\|_{L^3_t L^3_x} \leq \frac{\eta}{T} \) \( \ll 1 \), so that we can show that \( \Gamma \) is a contraction on the ball of radius \( \eta \) in \( L^q_T L^r_x \). Moreover, \((A.14)\) gives
\[
\|\vec{u}\|_{C_T H^s} = ||\vec{\Gamma}(u)||_{C_T H^s} \lesssim \|(u_0, u_1)\|_{H^s} + \|u\|^p_{L^p_t L^q_x} < \infty,
\]
so that \( \vec{u} = (u, \partial_t u) \in C_T H^s_x \).

In the case when \( 1 < p < 3 \) and \( s = s_{\text{crit}} = 0 \), we proceed as follows. Note that \( s = 0 \) along with the \( L^3_t L^3_x \) norm satisfies the scaling condition \((2.15)\) in Lemma \(2.9\). By \((A.13)\), Minkowski’s integral inequality, Lemma \(2.9\) Sobolev’s inequality, and Hölder’s inequality, we obtain
\[
\|\Gamma(u)\|_{L^3_t L^3_x} \lesssim \|V(t)(u_0, u_1)\|_{L^3_t L^3_x} + \int_0^T \|1_{[0, t]}(t')S(t-t')(|u|^{p-1}u)(t')\|_{L^3_t L^3_x} dt'
\]
\[
\lesssim \|(u_0, u_1)\|_{H^0} + \int_0^T \|(|u|^{p-1}u)(t')\|_{H^{-1}_x} dt'
\]
\[
\lesssim \|(u_0, u_1)\|_{H^0} + \|u\|^p_{L^p_t L^q_x}
\]
\[
\lesssim \|(u_0, u_1)\|_{H^0} + T^\theta \|\vec{u}\|_{L^3_t L^3_x}
\]
for some \( \theta > 0 \). Also, by \((1.8)\) and \((1.9)\), we easily obtain
\[
\|\vec{\Gamma}(u)\|_{C_T H^s_x} \lesssim \|(u_0, u_1)\|_{H^s} + T^\theta \|\vec{u}\|_{L^3_t L^3_x}.
\]
Similar difference estimates also hold, so that we can conclude using the standard contraction argument. This finishes the proof. \( \Box \)

We finish this appendix by stating several remarks.

Remark A.7. (i) At this point, we do not know how to prove local well-posedness for the cubic vNLW (with \( p = 3 \)) in \( L^2(\mathbb{T}^2) \), i.e. with \( s = s_{\text{crit}} = 0 \). It would be of interest to investigate if spaces of functions of bounded \( p \)-variation (i.e. \( U^p \)- and \( V^p \)-spaces) such as those in \([41, 42]\) can be applied to handle the cubic case.

(ii) A slight modification of the proof of Theorem A.6 yields local well-posedness of SvNLW \((1.2)\) in \( H^s(\mathbb{T}^2) \) for all \( s \geq s_{\text{crit}} \) (with a strict inequality when \( p = 3 \)), which improves the local well-posedness result for SvNLW \((1.2)\) in Theorem 1.1.

(iii) One can compare the local well-posedness result for vNLW \((A.2)\) in Theorem A.6 with the local well-posedness result for the usual NLW (see Remark 1.4 in \([24]\)):
\[
\partial_t^2 u - \Delta u \pm |u|^{p-1}u = 0.
\]
Note that vNLW enjoys a better local well-posedness result than does the usual NLW, thanks to the dissipative effects of the viscosity term.

(iv) Note that the global well-posedness result of SvNLW \((1.2)\) in Theorem 1.3 easily gives global well-posedness of vNLW \((A.2)\) in the class \( H^s(\mathbb{T}^2) \) for \( s \geq \max(0, 1 - \frac{1}{p + \delta} - \frac{1}{p}) \), where \( \delta > 0 \) is arbitrary. However, at this point, we do not know how to prove global well-posedness of vNLW \((A.2)\) in \( H^s(\mathbb{T}^2) \) for \( s_{\text{crit}} \leq s < \max(0, 1 - \frac{1}{p + \delta} - \frac{1}{p}) \). The main difficulty for this range of \( s \) is showing \( \vec{v}(t) \in H^1(\mathbb{T}^2) \) for all small enough \( t > 0 \), which is needed to guarantee the finiteness of the energy \( E(\vec{v}) \) defined in \((1.13)\).
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References

[1] H. Bahouri, J.Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.

[2] Á. Bényi, T. Oh, *The Sobolev inequality on the torus revisited*, Publ. Math. Debrecen, 83 (2013), no. 3, 359–374.

[3] Á. Bényi, T. Oh, O. Pocovnicu, *Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS*, Excursions in harmonic analysis. Vol. 4, 3–25, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2015.

[4] Á. Bényi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$*, Trans. Amer. Math. Soc. Ser. B, 2 (2015), 1–50.

[5] Á. Bényi, T. Oh, O. Pocovnicu, *Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^3$*, Trans. Amer. Math. Soc. Ser. B, 6 (2019), 114–160.

[6] Á. Bényi, T. Oh, O. Pocovnicu, On the probabilistic Cauchy theory for nonlinear dispersive PDEs, in Landscapes of Time-Frequency Analysis, Applied and Numerical Harmonic Analysis, pages 1–32. Birkhäuser/Springer, 2019.

[7] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup., 14 (1981), no. 2, 209–246.

[8] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys., 166 (1994), no. 1, 1–26.

[9] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys., 176 (1996), no. 2, 421–445.

[10] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math., 173 (2008), no. 3, 449–475.

[11] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. II. A global existence result*, Invent. Math., 173 (2008), no. 3, 477–496.

[12] N. Burq, N. Tzvetkov, *Probabilistic well-posedness for the cubic wave equation*, J. Eur. Math. Soc., 16 (2014), no. 1, 1–30.

[13] R. Carles, T. Kappeler, *Norm-inflation with infinite loss of regularity for periodic NLS equations in negative Sobolev spaces*, Bull. Soc. Math. France, 145 (2017), no. 4, 623–642.

[14] A. Choffrut, O. Pocovnicu, *Ill-posedness of the cubic nonlinear half-wave equation and other fractional NLS on the real line*, Int. Math. Res. Not. IMRN, (2018), no. 3, 699–738.

[15] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, arXiv:0311048 [math.AP] (2003).

[16] J. Colliander, T. Oh, *Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $l^2(\mathbb{T})$*, Duke Math. J., 161 (2012), no. 3, 367–414.

[17] G. Da Prato, A. Debussche, *Strong solutions to the stochastic quantization equations*, Ann. Probab., 31 (2003), no. 4, 1900–1916.

[18] P. de Roubin, M. Okamoto, *Norm inflation for the viscous wave equations in negative Sobolev spaces*, in preparation.

[19] J. Forlano, M. Okamoto, *A remark on norm inflation for nonlinear wave equations*, Dyn. Partial Differ. Equ., 17 (2020), no. 4, 361–381.

[20] P. Friz, N. Victoir, *Multidimensional stochastic processes as rough paths. Theory and applications*, Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010. xiv+656 pp.

[21] A.E. Gatto, *Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition*, J. Funct. Anal., 188 (2002), no. 1, 27–37.

[22] J. Ginibre, G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal., 133 (1995), 50–68.
[51] C. Sun, B. Xia, *Probabilistic well-posedness for supercritical wave equations with periodic boundary condition on dimension three*, Illinois Journal of Mathematics, 60 (2016), no. 2, 481–503.

[52] T. Tao, *Nonlinear dispersive equations: Local and global analysis*, CBMS Regional Conference Series in Mathematics, vol. 106, Amer. Math. Soc., Providence, RI, 2006. xvi+373 pp.

[53] N. Tzvetkov, *Random data wave equations*, in F. Flandoli, M. Gubinelli, and M. Hairer, editors, *Singular random dynamics*, volume 2253 of Lecture Notes in Mathematics, pages 221–313. Springer, 2019.

[54] T. Zhang, D. Fang, *Random data Cauchy theory for the generalized incompressible Navier-Stokes equations*, J. Math. Fluid Mech., 14 (2012), no. 2, 311–324.

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