TAIL VARIATIONAL PRINCIPLE AND ASYMPTOTIC $h$-EXPANSIVENESS FOR AMENABLE GROUP ACTIONS

BY

TOMASZ DOWNAROWICZ

Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Wybrzeże Wyspińskiego 27, 50-370 Wrocław, Poland
e-mail: downar@pwr.edu.pl

AND

GUOHUA ZHANG

School of Mathematical Sciences and Shanghai Center for Mathematical Sciences
Fudan University, Shanghai 200433, China
e-mail: chiaths.zhang@gmail.com

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ABSTRACT

In this paper we prove the tail variational principle for actions of countable amenable groups. This allows us to extend some characterizations of asymptotic $h$-expansiveness from $\mathbb{Z}$-actions to actions of countable amenable groups.

1. Motivation

The initial goal of this note was to give a characterization of asymptotically $h$-expansive dynamical systems with actions of countable amenable groups in terms of the existence of a principal symbolic extension, just as it holds for actions of the integers (see [6]). It seemed that all needed ingredients are already available. The theory of symbolic extensions for countable amenable group actions as well as the notion of entropy structure have been recently developed in [11] (for a brief exposition on the symbolic extension theory in topological dynamics see

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also a recent survey [10] and the references therein). The item linking these two topics, the tail variational principle for such actions (analogous to that known for $\mathbb{Z}$-actions from [6]) was given in [23]. But we found that the proof of the tail variational principle in [23] has a gap (the details will be explained in Section 8). This forced us to deliver our own proof, and due to some unexpected subtleties, this proof grew to become the main part of this paper. We resolve the subtleties by providing a topological analog of the Pinsker formula, known for measure-preserving actions. Other than that, our proof relies on two already existing results on the interplay between topological and measure-theoretic dynamics of countable amenable group actions: the variational principle for topological relative entropy [5, Theorem 13.3] (another proof can be found in [21, Theorem 5.1]) and a characterization of the topological tail entropy in terms of selfjoinings of the action [22, Theorem 3.1]. Eventually, we were able to achieve our initial goal and characterize asymptotically $h$-expansive $G$-actions.

2. Introduction and statements of the main results

Throughout this paper, we focus on countable amenable groups $G$, where by “countable” we always mean “infinite countable”. Since the topology on $G$ plays no role, we can as well assume that it is discrete. By $\mathcal{F}_G$ we denote the collection of all finite nonempty subsets of $G$. Amenability of $G$ is equivalent to the existence of a Følner sequence, i.e., a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{F}_G$ such that

$$\lim_{n \to \infty} \frac{|F_n \cap gF_n|}{|F_n|} = 1,$$

for any $g \in G$, where $|\cdot|$ denotes the cardinality of a set.

Let $X$ be a compact metric space and let $\text{Hom}(X)$ denote the group of all homeomorphisms $\phi : X \to X$. By an action (more precisely, a topological action) of $G$ on $X$ we mean a homomorphism from $G$ into $\text{Hom}(X)$, i.e., an assignment $g \mapsto \phi_g$ such that $\phi_{gg'} = \phi_g \circ \phi_{g'}$ for every $g, g' \in G$. It follows automatically that $\phi_e = \text{Id}$ (the unit $e$ of $G$ acts by identity) and $\phi_{g^{-1}} = (\phi_g)^{-1}$ for every $g \in G$. Such an action will be denoted by $(X, G)$. Although the group may act on the same space in many different ways, we will usually consider just one such action, hence this notation should not lead to a confusion. An action $(X, G)$ is also called a $G$-action or, when the group is understood, a topological dynamical system or just a system. From now on, to reduce the multitude of symbols used in this paper, we will write $g(x)$ in place of $\phi_g(x)$. The same
applies to subsets $A \subset X$: $g(A)$ will replace $\phi_g(A) = \{\phi_g(x) : x \in A\}$. A Borel measurable set $A \subset X$ is called invariant if $g(A) = A$ for every $g \in G$. For a Borel probability measure $\mu$ on $X$ and $g \in G$, by $g(\mu)$ we will denote the measure defined by $g(\mu)(A) = \mu(g^{-1}(A))$ for all Borel subsets $A \subset X$. A measure $\mu$ is invariant if $g(\mu) = \mu$ for all $g \in G$. The collection of invariant Borel probability measures will be denoted by $M(X, G)$. By amenability, this collection is nonempty, and for general reasons it is convex and weakly-star compact.

The main result of this paper is Theorem 2.1 referred to as the tail variational principle, which extends the tail variational principle known for $\mathbb{Z}$-actions (see [6]). The meaning of $h^*(X, G)$, $u_1$ and $\theta_k$ is a straightforward adaptation of the analogous terms for $\mathbb{Z}$-actions (see [6]) and their definitions will be provided in the next section. For now, it suffices to say that $h^*(X, G)$ is the topological tail entropy introduced (for $\mathbb{Z}$-actions) by M. Misiurewicz under the name of topological conditional entropy (see [18]), while $u_1$ and $\theta_k$ are special functions on $M(X, G)$ associated to the so-called entropy structure of the action, introduced (again, for $\mathbb{Z}$-actions) by M. Boyle and T. Downarowicz in [1] (see also [6] for a more detailed exposition on entropy structures).

**Theorem 2.1:** Let $(X, G)$ be a topological action of a countable amenable group, with finite entropy. Then

$$h^*(X, G) = \max\{u_1(\mu) : \mu \in M(X, G)\} = \lim_{k \to \infty} \sup\{\theta_k(\mu) : \mu \in M(X, G)\}.$$}

The second equality is elementary. Roughly speaking, it is just swapping the limit of a decreasing sequence of functions with the supremum over the domain. In general, such a swapping leads to an inequality, but here we have equality due to upper semicontinuity of the involved functions (see, e.g., [1, Proposition 2.4]). The nontrivial part is, of course, the first equality. It is essential that $h^*(X, G)$ is defined exclusively in terms of topological dynamics, while both expressions on the right evidently refer to invariant measures. So the above variational principle (just like the majority of variational principles) describes the interplay between topological and measure-theoretic dynamics. For actions of $\mathbb{Z}$ this equality was first proved in [6] and an alternative proof was given in [2]. The proofs do not pass directly to actions of amenable groups, as both of them use some machinery typical for $\mathbb{Z}$-actions (such as Krengel’s subadditive ergodic theorem, or Lindenstrauss’ theorem on the small boundary property of certain systems).
A well-oriented reader will notice that the tail variational principle already exists in the literature, in [23]. But we found that the proof in that paper contains a gap. It seems that Y. Zhou attempted to prove Shearer’s inequality using plain subadditivity (which is not possible; see Section 8 for a more detailed explanation). There is a subtlety in the notion of conditional entropy: there are two possible versions of how this notion can be defined; in one case the conditioning object is invariant, in the other it is not. It is thus important to distinguish between them, so we will call one of them “conditional”, and the other “relative”. The two are connected by the so-called Pinsker formula, which is known in the measure-theoretic case, while the topological version needs to be created from scratch. And this is what we do in the first place. Our proof relies on “amenable tools” such as the tilings.

Following the terminology introduced by M. Misiurewicz for $\mathbb{Z}$-actions, we will call a system $(X, G)$ asymptotically $h$-expansive if $h^*(X, G) = 0$. Because of the (fairly obvious from the definition) inequality $h^*(X, G) \leq h_{\top}(X, G)$, any system with zero topological entropy is asymptotically $h$-expansive. It is also relatively easy to see that any $G$-subshift with a finite alphabet is asymptotically $h$-expansive as well. But the class of asymptotically $h$-expansive systems is much richer. For example, it is known that an algebraic $\mathbb{Z}$-action has finite topological entropy if and only if it is asymptotically $h$-expansive (which is due to Misiurewicz, for details see [18, Section 7]). This result was extended to algebraic actions $(X, G)$ of countable amenable groups $G$ in [4, Theorem 7.1]. J. Buzzi provided another prominent class of asymptotically $h$-expansive $\mathbb{Z}$-actions, namely those generated by $C^\infty$ diffeomorphisms on compact smooth manifolds (see [3]). If $G$ is a countable amenable group containing $\mathbb{Z}$ as a subgroup of infinite index, then any action $(X, G)$ by $C^1$ maps on a compact smooth manifold (here we allow the manifold to have different dimensions for different connected components, even including zero dimension) has zero topological entropy (see for example [17, Lemma 5.7]), and so is trivially asymptotically $h$-expansive.

For $\mathbb{Z}$-actions there are several conditions equivalent to asymptotic $h$-expansiveness (see, e.g., [6]), one of which is the existence of a principal symbolic extension, i.e., a topological extension in the form of a subshift with a finite

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1 An algebraic action of a discrete group $\Gamma$ is a homomorphism from $\Gamma$ to the group of continuous automorphisms of a compact abelian group $X$. 
alphabet, which preserves the entropy of all invariant measures. So, a natural question arises: is the same true for actions of countable amenable groups? The notion of a subshift, as well as that of a principal extension is in this context perfectly understandable, so the positive answer would shed light, in particular, on algebraic actions with finite entropy (such a question was actually raised by Hanfeng Li [16]).

As a consequence of Theorem 2.1 and the theory of symbolic extensions for amenable group actions developed in [11], we are able to provide a quasi-positive answer to this question, with the same restrictions as apply to the entire theory of symbolic extensions for actions of amenable groups (explanations are provided right after the formulation).

**Theorem 2.2:** Let $(X, G)$ be a topological action of a countable amenable group. Then the following conditions are equivalent:

1. $(X, G)$ is asymptotically $h$-expansive.
2. The entropy structure of $(X, G)$ converges uniformly to the entropy function.
3. $(X, G)$ admits a principal quasi-symbolic extension.
4. For any $\varepsilon > 0$ the action admits a quasi-symbolic extension with topological relative entropy at most $\varepsilon$.

Furthermore, if $G$ is either residually finite or enjoys the comparison property then the quasi-symbolic extensions in the above statements can be replaced by symbolic extensions.

We explain, that by a topological extension of a system $(X, G)$ we mean another system $(Y, G)$ and continuous surjection $\pi : Y \to X$ (called the factor map) which is equivariant, i.e., satisfies the condition $\pi(g(y)) = g(\pi(y))$ for each $y \in Y$ and $g \in G$. An extension $(Y, S)$ is quasi-symbolic if it is a topological joining of a subshift with a specific zero-dimensional system depending only on the group $G$. This extra system is called a tiling system and although it has topological entropy zero (hence is trivially asymptotically $h$-expansive), it is in general not known whether it admits a symbolic extension at all (let alone principal). This open problem is equivalent to another: does every countable amenable group enjoy the so-called comparison property (we refer the reader to [11] for details). This equivalence explains the last relaxation in the theorem. It has been proved as [11, Theorem 6.33] that every subexponential
group enjoys the comparison property (note that every subexponential group is amenable). Examples of subexponential groups are: Abelian, nilpotent and virtually nilpotent groups (they have polynomial growth), and the Grigorchuk group, whose growth is subexponential but superpolynomial. In particular, our theorem implies that any algebraic action by any subexponential group $G$ has finite entropy if and only if it admits a principal symbolic extension. Similar relaxation for residually finite amenable groups follows by a different argument, which can also be found in [11].

3. Preliminaries on entropy

Let $(X, G)$ be a topological action. We denote by $\mathcal{B}_X$ the collection of all Borel subsets of $X$. By a cover of $X$ we mean a family of Borel sets whose union is $X$. A partition of $X$ is a cover whose elements are pairwise disjoint. Let us denote the set of all covers by $\mathcal{C}_X$, the set of all finite open covers by $\mathcal{C}_X^0$, the set of all finite closed covers by $\mathcal{C}_X^c$, and the set of all finite partitions by $\mathcal{P}_X$. If $X$ is totally disconnected, then $\mathcal{P}_X \cap \mathcal{C}_X^0 \cap \mathcal{C}_X^c$ is nonempty and its members are called clopen partitions. Note that for any $U \in \mathcal{C}_X$ and $g \in G$, the family

$$g^{-1}(U) = \{g^{-1}(U) : U \in U\}$$

is also a cover, and, by continuity of the action, if $U$ is either open, closed, or a partition, so is $g^{-1}(U)$. Let $U, W \in \mathcal{C}_X$. The family

$$U \vee W = \{U \cap W : U \in U, W \in W\}$$

is a cover called the join of $U$ and $W$. Obviously, the families $\mathcal{P}_X$, $\mathcal{C}_X^0$ and $\mathcal{C}_X^c$ are closed under the join operation. The definition of the join extends naturally to any finite or even countable collection of covers, however a countable join of finite covers is no longer finite. Note the obvious fact, that a countable join of closed covers is a closed cover (which can be uncountable). For $U \in \mathcal{C}_X$ and $F \subset G$ ($F$ finite or countable), we set

$$U^F = \bigvee_{g \in F} g^{-1}U.$$ 

By convention we also let $U^\emptyset$ be the trivial cover $\{X\}$.

If each element of a cover $U$ is contained in some element of another cover $W$ then we say that $U$ is finer than $W$ (and write $U \succeq W$).
For two covers $\mathcal{U}, \mathcal{W} \in \mathfrak{C}_X$ we let

$$N(\mathcal{U}, \mathcal{W}) = \max_{W \in \mathcal{W}} \min_{U' \subset \mathcal{U}, W \subset \bigcup U'} \{|U'| : U' \subset U, W \subset \bigcup U'\}$$

(the minimal integer $N$ such that every element of $\mathcal{W}$ can be covered by $N$ elements of $\mathcal{U}$), and we set the \textbf{conditional counting entropy} of $\mathcal{U}$ given $\mathcal{W}$ to be

$$H(\mathcal{U}|\mathcal{W}) = \log N(\mathcal{U}, \mathcal{W}).$$

3.1. \textsc{Topological notions.} We shall now recall the key notions of topological entropy and topological tail entropy of a topological $G$-action. They are straightforward adaptations of the Adler–Conheim–McAndrew notion of topological entropy and Misiurewicz’ topological conditional entropy [18], respectively (the term tail entropy was introduced later). We remark that topological tail entropy for actions of countable amenable groups has been addressed in [4, 22] and also several equivalent definitions are given in [23].

Let $\mathcal{U}, \mathcal{W} \in \mathfrak{C}_X$ and on $\mathfrak{S}_G$ consider the nonnegative function $F \mapsto H(\mathcal{U}^F|\mathcal{W}^F)$. It is crucial, that the set $F$ appears here in both “exponents”. Due to this, it is relatively easy to check that this function is $G$-invariant and subadditive, i.e.,

$$H(\mathcal{U}^{F_1 \cup F_2}|\mathcal{W}^{F_1 \cup F_2}) \leq H(\mathcal{U}^{F_1}|\mathcal{W}^{F_1}) + H(\mathcal{U}^{F_2}|\mathcal{W}^{F_2}) \quad (F_1, F_2 \in \mathfrak{S}_G)$$

(for details see [4, Lemma 5.2]). We define the \textbf{topological conditional entropy} of the cover $\mathcal{U}$ given the cover $\mathcal{W}$ as the following limit

$$h_G(\mathcal{U}|\mathcal{W}) = \lim_{n \to \infty} \frac{1}{|F_n|} H(\mathcal{U}^{F_n}|\mathcal{W}^{F_n}),$$

where $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence in $G$. The Ornstein–Weiss Lemma (see e.g. [13, 1.3.1]) guarantees that the above limit exists and does not depend on the choice of the Følner sequence. Next, one defines a series of related notions:

$$h_G(\mathcal{U}) = h_G(\mathcal{U}|\{X\}),$$

$$h_G(X|\mathcal{W}) = \sup_{U \in \mathfrak{C}_X^X} h_G(U|\mathcal{W}),$$

$$h_G(X) = h_{\text{top}}(X, G) = h_G(X|\{X\}) = \sup_{U \in \mathfrak{C}_X^X} h_G(U),$$

$$h^*(X, G) = \inf_{\mathcal{W} \in \mathfrak{C}_X^X} h_G(X|\mathcal{W}).$$

They are called, respectively, \textbf{topological entropy of the cover $\mathcal{U}$}, \textbf{topological conditional entropy} (of the action) \textbf{given the cover $\mathcal{W}$}, \textbf{topological entropy} (of the action), and \textbf{topological tail entropy} (of the action).
Directly by the definitions, we have \( h^*(X, G) \leq h_{\text{top}}(X, G) \). It is also not hard to see that if \( h_{\text{top}}(X, G) = \infty \) then also \( h^*(X, G) = \infty \). Following [18], we introduce another key notion of this paper. We remark that the adaptation to amenable group actions has already been addressed in [4].

**Definition 3.1:** A topological action \((X, G)\) is **asymptotically h-expansive** if \( h^*(X, G) = 0 \).

We now review a family of notions very similar to the previous ones, in which we replace the conditioning cover \( \mathcal{W}^{F_n} \) by the invariant (and usually uncountable) cover \( \mathcal{W}^G \). In order to distinguish the two approaches, we will now use the term “relative” rather than “conditional”.

**Definition 3.2:** Let \((F_n)_{n \in \mathbb{N}}\) be a Følner sequence of \( G \) and let \( \mathcal{U} \in \mathcal{C}_X^G \) while \( \mathcal{W} \) is any cover of \( X \). The **topological relative entropy** of the cover \( \mathcal{U} \) given the cover \( \mathcal{W} \) is the limit

\[
\bar{h}_G(\mathcal{U}\|\mathcal{W}) = \lim_{n \to \infty} \frac{1}{|F_n|} H(\mathcal{U}^{F_n}\|\mathcal{W}^G).
\]

The function \( F \mapsto H(\mathcal{U}^F\|\mathcal{W}^G) \), is obviously subadditive and invariant on \( \mathcal{F}_G \). Thus, by the already mentioned Ornstein–Weiss Lemma, the limit in the definition exists and does not depend on the Følner sequence.

A special case of the above relative entropy occurs when the cover \( \mathcal{W} \) is a closed upper semicontinuous partition.\(^2\) Then \( \mathcal{W}^G \) defines a closed invariant equivalence relation in \( X \). It is well known that such a relation determines a topological factor of the system \((X, G)\), say \((Z, G)\). In this case we will denote \( \bar{h}_G(\mathcal{U}\|\mathcal{W}) \) by \( \bar{h}_G(\mathcal{U}\|Z) \) and call it the **relative entropy** of the cover \( \mathcal{U} \) given the factor \((Z, G)\). Conversely, given a topological factor \((Z, G)\) of \((X, G)\) (via a topological factor map \( \pi : X \to Z \)), the partition \( \mathcal{W} \) of \( X \) into the fibers of \( \pi \) is closed, upper semicontinuous and invariant (i.e., \( \mathcal{W}^G = \mathcal{W} \)). In this case, we have

\[
\bar{h}_G(\mathcal{U}\|Z) = \bar{h}_G(\mathcal{U}\|\mathcal{W}).
\]

Finally, given a topological factor \((Z, G)\) of \((X, G)\) we define the **topological relative entropy** of the system \((X, G)\) given the factor \((Z, G)\), as

\[
\bar{h}_G(X\|Z) = \sup_{\mathcal{U} \in \mathcal{C}_X^G} \bar{h}_G(\mathcal{U}\|Z).
\]

\(^2\) A closed partition is upper semicontinuous if and only if it defines a closed equivalence relation.
It is elementary to see that if \((X, G)\) and \((Y, G)\) are subshifts and \(U\) and \(W\) are the clopen “zero-coordinate” partitions (or “one-symbol” partitions), then
\[
\bar{h}_G(X|Z) = \bar{h}_G(U|W).
\]

3.2. Measure-theoretic notions. We now pass to the review of measure-theoretic notions of entropy. We consider a measure-preserving \(G\)-action \((X, \Sigma_X, \mu, G)\), that is, \((X, \Sigma_X, \mu)\) is a probability space on which \(G\) acts by measure-preserving automorphisms. For example, if \((X, G)\) is a topological action and \(\mu \in \mathcal{M}(X, G)\) then \((X, \mathcal{B}_X, \mu, G)\) is a measure-preserving \(G\)-action.

Let \(P \in \mathfrak{P}_X\). The Shannon entropy of \(P\) equals
\[
H_\mu(P) = -\sum_{P \in \mathfrak{P}} \mu(P) \log(\mu(P)).
\]

Shannon entropy satisfies strong subadditivity, i.e.,
\[
H_\mu(P^{F_1 \cup F_2}) \leq H_\mu(P^{F_1}) + H_\mu(P^{F_2}) - H_\mu(P^{F_1 \cap F_2}) \quad (F_1, F_2 \in \mathfrak{F}_G).
\]
The dynamical entropy of \(P\) with respect to \(\mu\) is defined as
\[
h_\mu(P, G) = \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(P^{F_n}) = \inf_{F \in \mathfrak{F}_G} \frac{1}{|F|} H_\mu(P^F),
\]
where \((F_n)_{n \in \mathbb{N}}\) is a Følner sequence in \(G\), while the latter equality is derived using strong subadditivity (the derivation can be found, e.g., in [8]). In particular, the dynamical entropy of a partition (and hence also the Kolmogorov–Sinai entropy defined below) does not depend on the choice of the Følner sequence. The Kolmogorov–Sinai entropy of the measure-preserving \(G\)-action \((X, \Sigma_X, \mu, G)\) is defined as
\[
h_\mu(X, G) = \sup_{P \in \mathfrak{P}_X} h_\mu(P, G).
\]

Next, consider two partitions \(P, Q \in \mathfrak{P}_X\). One of the simplest ways of defining the conditional Shannon entropy of \(P\) given \(Q\) is
\[
H_\mu(P|Q) = H_\mu(P \lor Q) - H_\mu(Q).
\]
For a partition \(P \in \mathfrak{P}_X\) and a sub-sigma-algebra \(\Sigma \subset \Sigma_X\), the conditional Shannon entropy of \(P\) given \(\Sigma\) can be defined (in one of several equivalent ways) as
\[
H_\mu(P|\Sigma) = \inf_{Q \in \mathfrak{P}_X, Q \subset \Sigma} H_\mu(P|Q).
\]
Now, going back to two partitions $P, Q \in \mathfrak{P}_X$, one defines the **conditional entropy** of the process generated by $P$ given the process generated by $Q$ as

$$h_\mu(P, G|Q) = \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(P^{F_n}|Q^{F_n}).$$

We point out that the conditional entropy $H_\mu(P^{F_n}|Q^{F_n})$ viewed as a function of $\mathfrak{F}_G$ is subadditive, but not strongly subadditive (see Example 8.2). Thus, although the existence and independence on the Følner sequence of the above limit is guaranteed by the aforementioned Ornstein–Weiss Lemma, it does not follow directly that the above limit equals the corresponding infimum over $\mathfrak{F}_G$. It actually is, but the proof requires a stronger tool—the Pinsker formula (see below).

On the other hand, the function $H_\mu(P^{F_n}|Q^G)$, where $Q^G$ is the invariant sub-sigma-algebra generated by all partitions $g^{-1}(Q)$, $g \in G$, is strongly subadditive, which allows one to define the **relative entropy** of the process generated by $P$ given the process generated by $Q$, as

$$\bar{h}_\mu(P, G|Q) = \lim_{n \to \infty} \frac{1}{|F_n|} H_\mu(P^{F_n}|Q^G) = \inf_{F \in \mathfrak{F}_G} \frac{1}{|F|} H_\mu(P^{F}|Q^G).$$

The Pinsker formula is responsible for the equality between conditional and relative entropies, as follows:

$$h_\mu(P, G|Q) = \bar{h}_\mu(P, G|Q)$$

(this version of the Pinsker formula for $G$-actions can be found as [20, Theorem 4.4] and also as [12, Lemma 1.1]). This is the reason, why we can forget about distinction between these two measure-theoretic notions.

Now consider a measure-theoretic factor map $\pi : Y \to X$ between two measure-preserving $G$-actions $(Y, \Sigma_Y, \nu, G)$ and $(X, \Sigma_X, \mu, G)$ (equivalently, one can consider just an invariant sub-sigma-algebra $\Sigma_X \subset \Sigma_Y$ and $\mu$ equal to the restriction of $\nu$ to $\Sigma_X$). In this case we define the **conditional entropy** of $\nu$ given $X$ (equivalently, given $\Sigma_X$), as follows:

$$h_\nu(Y, G|X) = \sup_{P \in \mathfrak{P}_Y} \inf_{Q \in \mathfrak{P}_X} h_\nu(P, G|Q),$$

where $P$ ranges over all finite measurable partitions of $Y$, while $Q$ ranges over all finite measurable partitions of $X$ lifted to $Y$ (equivalently, $Q \subset \Sigma_X$). If $h_\mu(X, G) < \infty$ then $h_\nu(Y, G|X)$ is simply the difference

$$h_\nu(Y, G) - h_\mu(X, G).$$
Because of this “difference formula”, the infimum over $Q$ and supremum over $P$ in (3.1) can be swapped.

The notion provided below was coined by Ledrappier [15] (he did it for $\mathbb{Z}$-actions, but the extension to $G$-actions requires no modification).

Definition 3.3: Let $\pi : Y \to X$ be a topological factor map between two topological actions $(Y, G)$ and $(X, G)$. If, for every $\nu \in \mathcal{M}(Y, G)$,

$$h_\nu(Y, G|X) = 0$$

then $(Y, G)$ (together with the factor map $\pi$) is called a principal extension of $(X, G)$.

By the standard variational principle, it follows that principal extension preserves topological entropy. On the other hand, preservation of topological entropy is a strictly weaker condition.

3.3. Entropy structure. By a structure on a compact metric space $\mathcal{M}$ we will understand any nondecreasing sequence of commonly bounded nonnegative functions on $\mathcal{M}$, $\mathcal{F} = (f_k)_{k \geq 0}$ with $f_0 \equiv 0$. Clearly, the pointwise limit function $f = \lim_k f_k$ exists and is nonnegative and bounded. Two structures $\mathcal{F} = (f_k)_{k \geq 0}$ and $\mathcal{F}' = (f'_k)_{k \geq 0}$ are said to be uniformly equivalent if

$$\forall \varepsilon > 0, k_0 \geq 0 \exists k \geq 0 \ (f'_k > f_{k_0} - \varepsilon \text{ and } f_k > f'_{k_0} - \varepsilon).$$

Notice the obvious fact that uniformly equivalent structures have a common limit function.

Entropy structure of a topological $G$-action is defined in [11] in exactly the same manner as it is done for $G = \mathbb{Z}$ in [1]. We present here only one of several equivalent definitions. By [14, Theorem 3.2], any topological action $(X, G)$ has a principal zero-dimensional extension $(X', G)$. This allows one to define entropy structure in two steps, the first one being the definition for actions on zero-dimensional spaces, while the second is an easy generalization relying on principal zero-dimensional extensions.

Definition 3.4: Let $(X, G)$ be a topological action on a zero-dimensional space and assume that $h_{\text{top}}(X, G) < \infty$. Let $(\mathcal{P}_k)_{k \in \mathbb{N}}$ be a refining sequence of finite clopen partitions, meaning that $\mathcal{P}_{k+1} \succ \mathcal{P}_k$ for each $k \in \mathbb{N}$ and

$$\lim_k \text{diam}(\mathcal{P}_k) = 0,$$
where $\text{diam}(P_k)$ stands for the maximal diameter of an atom of $P_k$. For every $\mu \in \mathcal{M}(X,G)$ we set

$$h_k(\mu) = h_\mu(P_k,G).$$

Then the sequence $\mathcal{H} = (h_k)_{k \geq 0}$ on $\mathcal{M}(X,G)$ is a structure called an **entropy structure** of $(X,G)$.

It is obvious that the nondecreasing limit $\lim_{k} h_k$ equals the entropy function $h$ on $\mathcal{M}(X,G)$, given by

$$h(\mu) = h_\mu(X,G).$$

It is known that entropy structures arising from different refining sequences of finite clopen partitions are uniformly equivalent (see [1] or [6])). Thus, entropy structure can be defined as the corresponding uniform equivalence class and in this understanding it does not depend on the refining sequence of finite clopen partitions (which is equivalent to saying that it is a conjugacy invariant within the class of zero-dimensional systems).

**Definition 3.5:** Let $(X,G)$ be a topological action with finite entropy. We define the **entropy structure** of $(X,G)$ as any structure $\mathcal{H} = (h_k)_{k \geq 0}$ on $\mathcal{M}(X,G)$, such that for any principal zero-dimensional extension $\pi' : X' \to X$, and any entropy structure $\mathcal{H}' = (h_{k}')_{k \geq 0}$ on $\mathcal{M}(X',G)$, the structure $\mathcal{H} = (h_k)_{k \geq 0}$ lifted against $\pi'$ (i.e., the sequence $(h_k \circ \pi')_{k \in \mathbb{N}}$) is uniformly equivalent to $\mathcal{H}'$.

The existence of such an entropy structure (which is by no means obvious) is proved for $\mathbb{Z}$-actions in [1] (see also [6]), while the adaptation to actions of amenable groups can be found in [11].

It is clear that properly understood entropy structure is a uniform equivalence class and in this understanding it is a conjugacy invariant. Nonetheless, it is technically much more convenient to work with entropy structures understood as individual sequences of functions $(h_k)_{k \geq 0}$. It was proved in [11] that an entropy structure of a zero-dimensional topological $G$-action consists of affine functions $h_k$ such that $h_k - h_{k-1}$ is upper semicontinuous for every $k \in \mathbb{N}$ (in particular, each $h_k$ is upper semicontinuous). In such case we will say that $\mathcal{H}$ is an **affine structure with upper semicontinuous differences**. In [11] (following [1] and [6]) it is shown that any topological action $(X,G)$ with finite entropy has an entropy structure which is affine and has upper semicontinuous differences.
For any bounded from above function $f$ defined on a compact metric space $\mathcal{M}$ we denote by $\tilde{f}$ the upper semicontinuous envelope of $f$, i.e., the function

$$\tilde{f}(x) = \max\{f(x), \limsup_{x' \to x} f(x')\}.$$ 

In fact, $\tilde{f}$ equals the pointwise infimum of all continuous functions dominating $f$ and it is the smallest upper semicontinuous function dominating $f$.

Now let $\mathcal{H} = (h_k)_{k \geq 0}$ be an entropy structure of a topological action $(X, G)$ with finite entropy and let $h$ be the entropy function on $\mathcal{M}(X, G)$. We set

$$\theta_k(\mu) = h(\mu) - h_k(\mu) \quad \text{and} \quad u_1(\mu) = \lim_{k \to \infty} \tilde{\theta}_k(\mu) \quad (\mu \in \mathcal{M}(X, G)).$$

Note that the functions $\theta_k$ are equal to the conditional entropies $h_\mu(X, G|X_k)$, where $(X_k, G)$ is the symbolic factor $(X, G)$ generated by the clopen partition $\mathcal{P}_k$, and they converge pointwise to zero. So, $u_1$ relies only on the defects of upper semicontinuity of the conditional entropy functions $\theta_k$. The function $u_1$ is sometimes called the tail entropy function (on invariant measures), although this term is also used for a slightly different function. The reason why it is denoted by $u_1$ is that this is the first of a transfinite sequence of functions $(u_\alpha)_{\alpha < \omega_1}$ responsible for “higher order defects”. For more details the reader is referred to either [1], [6] or [7] in case of $\mathbb{Z}$-actions, or to [11] for $G$-actions (as a matter of fact, the acting countable amenable group plays no role in this context).

### 4. The topological Pinsker formula

The aim of this section is to prove a “topological Pinsker formula”, i.e., equality between topological conditional and topological relative entropies. This formula will be needed in the next section, in the proof of the conditional variational principle for zero-dimensional systems, which in turn will allow us to prove the tail variational principle for such actions. We begin with a lemma.

**Lemma 4.1:** Let $\mathcal{U} \in \mathfrak{C}_X$ and $\{\mathcal{V}_n : n \in \mathbb{N}\} \subset \mathfrak{C}_X^c$. Let $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ and, for each $n \in \mathbb{N}$, let $\mathcal{W}_n = \bigcup_{i=1}^{n} \mathcal{V}_i$. Then there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$, we have

$$N(\mathcal{U}, \mathcal{W}) = N(\mathcal{U}, \mathcal{W}_n).$$
Proof. Consider the compact metric space $V = \prod_{n \in \mathbb{N}} V_n$, where each $V_n$ is viewed as a discrete space containing exactly $|V_n|$ points. This space consists of formal sequences $v = (V_n)_{n \in \mathbb{N}}$ such that $V_n \in V_n$ for each $n \in \mathbb{N}$, and for every such element $v$ we let

$$W_n(v) = \bigcap_{i=1}^{n} V_i \quad \text{and} \quad W(v) = \bigcap_{n \in \mathbb{N}} V_n.$$  

Clearly the families $\{W_n(v) : v \in V\}$ and $\{W(v) : v \in V\}$ coincide with $W_n$ and $W$, respectively. On $V$ we define the following functions:

$$f_n(v) = \min \left\{ |U'| : U' \subset U, W_n(v) \subset \bigcup U' \right\} \quad (n \in \mathbb{N}),$$

$$f_{\infty}(v) = \min \left\{ |U'| : U' \subset U, W(v) \subset \bigcup U' \right\},$$

and then we have the following obvious identities:

$$N(U, W_n) = \max \{f_n(v) : v \in V\} \quad (n \in \mathbb{N}),$$

$$N(U, W) = \max \{f_{\infty}(v) : v \in V\}.$$  

Clearly, the sequence of functions $(f_n)_{n \in \mathbb{N}}$ is nonincreasing and dominates $f_{\infty}$. Further, the functions $f_n$ are by definition constant on the clopen cylinders in $V$ consisting of sequences $v$ which agree on the initial $n$ positions. This implies that the functions $f_n$ are continuous. Once we show that the functions $f_n$ converge pointwise to $f_{\infty}$, the proof will be ended, because in such case, by the already mentioned elementary “exchange of suprema and infima” statement (see, e.g., [1, Proposition 2.4 ]), we will have

$$\max \{f_{\infty}(v) : v \in V\} = \max_{v \in V} \inf_{n} f_n(v) = \inf_{v \in V} \max_{n} f_n(v).$$

Since the values of $f_n$ are integers, the latter infimum must be attained for some $n_0$.

The pointwise convergence of the functions $f_n$ to $f_{\infty}$ is proved as follows. For $v \in V$, let $U'$ be a subfamily of $U$ which covers $W(v)$, and assume it has the minimal cardinality $f_{\infty}(v)$. Since the sets $W_n(v)$ are compact and their decreasing intersection is $W(v)$, it is clear that for some $n$, $W_n(v)$ is covered by the same (open) family $U'$. This shows that $f_n(v) \leq f_{\infty}(v)$ for some $n$, while the converse inequality is trivial.  

We are able to prove the topological Pinsker formula in case $U$ is open and $W$ is closed:
Theorem 4.2: Let $(F_n)_{n \in \mathbb{N}}$ be a Følner sequence of $G$ and $U \in \mathcal{C}_X$, $\mathcal{W} \in \mathcal{C}_X$. Then

$$h_G(U|\mathcal{W}) = \bar{h}_G(U|\mathcal{W}).$$

Remark 4.3: For $\mathbb{Z}$-actions and the standard Følner sequence $F_n = \{0, 1, \ldots, n-1\}$ the above equality follows quickly from Lemma 4.1. The key ingredient is that both conditional counting entropies, $H(U^F|\mathcal{W}^F)$ and $H(U^F|\mathcal{W}^G)$, are subadditive on $\mathfrak{F}$, which for $\mathbb{Z}$-actions is sufficient for the “infimum rule” to hold: the involved limits on both sides are in fact infima. For more general $G$-actions, subadditivity is insufficient for the infimum rule; strong subadditivity (or at least Shearer’s inequality, see e.g. [8]) is required. The conditional counting entropy is not strongly subadditive (even for clopen partitions, see Example 8.2) and thus the proof is more complicated.

Proof. Since $N(U^F, \mathcal{W}^F) \geq N(U^F, \mathcal{W}^G)$ for each $F \in \mathfrak{F}_G$, the inequality “$\geq$” in the theorem is obvious and it remains to show that for a suitable Følner sequence $(F_n)_{n \in \mathbb{N}}$, every $\varepsilon > 0$ and large enough $n$, one has

$$\frac{1}{|F_n|} H(U^{F_n}|\mathcal{W}^{F_n}) \leq \bar{h}_G(U|\mathcal{W}) + 2\varepsilon. \quad (4.1)$$

We now invoke the theory of tilings. It follows from [9, Theorem 5.2] that there exists a Følner sequence $(F_n)_{n \in \mathbb{N}}$ of sets containing the unit $e$, which breaks into countably many portions

$$\{F_1, F_2, \ldots, F_{n_1}\}, \{F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_2}\}, \{F_{n_2+1}, F_{n_2+2}, \ldots, F_{n_3}\}, \ldots,$$

such that every set $F_n$ with $n > n_{i+1}$ is a disjoint union of shifted sets from the $i$th portion:

$$F_n = \bigcup_{k=n_i+1}^{n_{i+1}} \bigcup_{c \in C_{k,n}} F_k c \quad (4.2)$$

(that is, such finite sets $C_{k,n} \subset F_n$ exist). Now, fix any $\varepsilon > 0$. There exists an $i \in \mathbb{N}$ such that

$$\frac{1}{|F_k|} H(U^{F_k}|\mathcal{W}^G) < \bar{h}_G(U|\mathcal{W}) + \varepsilon, \quad \text{for each } k \in \{n_i + 1, \ldots, n_{i+1}\}. \vspace{1cm}$$

\(^3\) The quasitilings introduced by Ornstein and Weiss in [19] should also suffice, but tilings are simply more convenient to use.
For each $k$ as above we apply Lemma 4.1 to $U^F_k$ and get a set $B_k \in \mathcal{F}_G$ (we can assume that $e \in B_k$), such that $N(U^F_k, W^G) = N(U^F_k, W^B_k)$. Let

$$B = \bigcup_{k=n_i+1}^{n_{i+1}} B_k$$

(clearly, $B \in \mathcal{F}_G$). Then we have

$$N(U^F_k, W^G) = N(U^F_k, W^B)$$

for all $k \in \{n_i+1, \ldots, n_{i+1}\}$, in particular

(4.3) \hspace{1cm} H(U^F_k | W^B) < |F_k| (\bar{h}_G(U | W) + \varepsilon).

For any $\delta > 0$ and large enough $n \in \mathbb{N}$, the set $F_n$ (henceforth abbreviated as $F$) is $(B, \delta)$-invariant (i.e., $\frac{|BF_n \Delta F_n|}{|F_n|} < \delta$ where $\Delta$ denotes the symmetric difference of sets), and it splits as the disjoint union (4.2).

Let $F_B$ denote the $B$-core of $F$, that is

$$F_B = \{ g \in F : Bg \subset F \}.$$ 

For $\delta$ sufficiently small, the $(B, \delta)$-invariance of $F$ implies that

(4.4) \hspace{1cm} |F \setminus F_B| < \frac{\varepsilon}{D \cdot (1 + \log |U|)} |F|,

where

$$D = (n_{i+1} - n_i) \max\{|F_k| : k = n_i + 1, \ldots, n_{i+1}\}.$$ 

We have, by subadditivity,

(4.5) \hspace{1cm} \frac{1}{|F|} H(U^F | W^F) \leq \frac{1}{|F|} \sum_{k=n_i+1}^{n_{i+1}} \sum_{c \in C_{k,n}} H(U^{F_{k,c}} | W^F).

For each $k \in \{n_i+1, \ldots, n_{i+1}\}$ we have (using (4.3) and (4.4) to get the last inequality)

$$\sum_{c \in C_{k,n}} H(U^{F_{k,c}} | W^F) = \sum_{c \in C_{k,n} \setminus F_B} H(U^{F_k} | W^{F^{-1}}) + \sum_{c \in C_{k,n} \cap F_B} H(U^{F_k} | W^{F^{-1}})$$

$$\leq |F \setminus F_B| \cdot |F_k| \cdot \log |U| + \sum_{c \in C_{k,n} \cap F_B} H(U^{F_k} | W^B)$$

$$\leq \frac{\varepsilon}{n_{i+1} - n_i} |F| + |C_{k,n}| \cdot |F_k| (\bar{h}_G(U | W) + \varepsilon).$$
Therefore, the right-hand side (and thus also the left hand side) of (4.5) is dominated by
\[
\frac{1}{|F|} \sum_{k=n_i+1}^{n_i+1} \left( \frac{\varepsilon}{n_i+1-n_i} |F| + |C_{k,n}| \cdot |F_k| (\tilde{h}_G(U|W) + \varepsilon) \right) = \tilde{h}_G(U|W) + 2\varepsilon
\]
(we have used the equality \( \sum_{k=n_i+1}^{n_i+1} |C_{k,n}| \cdot |F_k| = |F| \), which follows from the disjoint union (4.2)). We have just shown (4.1), which ends the proof. 

5. Conditional variational principle for zero-dimensional systems

Our main Theorems 2.1 and 2.2 depend on the following variational principle for topological conditional entropy in zero-dimensional systems. By the topological Pinsker formula (Theorem 4.2), the topological conditional entropy of a clopen disjoint cover given another clopen disjoint cover equals the topological relative entropy given a topological factor. From here we can use some existing results.

**Proposition 5.1:** Let \((X,G)\) be a topological action and \(\mathcal{P}, \mathcal{Q} \in \mathfrak{P}_X\). Assume that \(\mathcal{P} \supseteq \mathcal{Q}\), and that \(\mathcal{P}\) (and hence \(\mathcal{Q}\)) is clopen (i.e., \(\mathcal{P}, \mathcal{Q} \in \mathfrak{P}_X \cap \mathcal{C}_X \cap \mathcal{C}_X^c\)). Then
\[
h_G(\mathcal{P}|\mathcal{Q}) = \max \{ h_\mu(\mathcal{P},G) - h_\mu(\mathcal{Q},G) : \mu \in \mathcal{M}(X,G) \}.
\]

**Proof.** Since \(\mathcal{P}\) is a finite clopen partition of \(X\), it generates naturally a symbolic topological factor \((X_\mathcal{P},G)\) of \((X,G)\) with the alphabet \(\mathcal{P}\) via the **itinerary map**
\[
\pi_\mathcal{P}(x) = (x_g)_{g \in G},
\]
where, for each \(g \in G\), we set \(x_g = \mathcal{P} \in \mathcal{P} \iff g(x) \in \mathcal{P}\). The action of \(G\) on such itineraries is the standard shift: \(h((x_g)_{g \in G}) = (x_{gh})_{g \in G}\)  \((h \in G)\). Likewise, \(\mathcal{Q}\) generates a symbolic \(G\)-action \((X_\mathcal{Q},G)\) with the alphabet \(\mathcal{Q}\) and there is a natural topological factor map from \((X_\mathcal{P},G)\) onto \((X_\mathcal{Q},G)\) which coincides with the following **one-block code** \(\pi_{\mathcal{P},\mathcal{Q}}\):
\[
\pi_{\mathcal{P},\mathcal{Q}}((x_g)_{g \in G}) = (y_g)_{g \in G},
\]
where, for each \(g \in G\), \(y_g\) is the unique element of \(\mathcal{Q}\) which contains \(x_g\).

Now, since \(\mathcal{P}\) is (in particular) an open cover and \(\mathcal{Q}\) is (in particular) a closed cover, we can apply our topological Pinsker formula (Theorem 4.2), which reads
\[
h_G(\mathcal{P}|\mathcal{Q}) = \tilde{h}_G(\mathcal{P}|\mathcal{Q}).
\]
Because $\mathcal{P}$ and $\mathcal{Q}$ are clopen partitions and they are topological generators of the systems $(X_\mathcal{P}, G)$ and $(X_\mathcal{Q}, G)$, respectively, the right-hand side equals the topological relative entropy $\bar{h}_G(X_\mathcal{P}|X_\mathcal{Q})$ of the topological factor map $\pi_{\mathcal{P}, \mathcal{Q}}$ for which the variational principle is already known, see [5, Theorem 13.3] (another proof can be found in [21, Theorem 5.1]):

$$\bar{h}_G(X_\mathcal{P}|X_\mathcal{Q}) = \sup \{ h_\nu(X_\mathcal{P}, G|X_\mathcal{Q}) : \nu \in \mathcal{M}(X_\mathcal{P}, G) \}.$$ 

Since $\mathcal{P}$ and $\mathcal{Q}$ are measure-theoretic generators in $(X_\mathcal{P}, G)$ and $(X_\mathcal{Q}, G)$ for any invariant measure in the respective systems, we have, for each $\nu \in \mathcal{M}(X_\mathcal{P}, G)$, the equality

$$h_\nu(X_\mathcal{P}, G|X_\mathcal{Q}) = h_\mu(\mathcal{P}, G|\mathcal{Q}) = h_\mu(\mathcal{P}, G) - h_\mu(\mathcal{Q}, G),$$

where $\mu$ is any invariant measure on $X$ satisfying $\nu = \pi_\mathcal{P}(\mu)$. Thus, we obtain the desired equality

$$h_G(\mathcal{P}|\mathcal{Q}) = \max \{ h_\mu(\mathcal{P}, G) - h_\mu(\mathcal{Q}, G) : \mu \in \mathcal{M}(X, G) \},$$

where the supremum is replaced by the maximum due to upper semicontinuity of the function $\mu \mapsto h_\mu(\mathcal{P}, G|\mathcal{Q})$ for finite clopen partitions.

6. Preservation of topological tail entropy

In this section we show that topological tail entropy of a topological $G$-action is preserved by principal extensions. This fact for $\mathbb{Z}$-actions is well known and it was proved by F. Ledrappier, see [15, Theorem 3]. It also follows a posteriori from the tail variational principle and the (discussed earlier) preservation of entropy structure by principal extensions. But since we intend to use it in the derivation of the tail variational principle, we need an independent proof. The shortest proof of Proposition 6.1 that we were able to come up with relies heavily on the already mentioned variational principle for topological relative entropy ([5, Theorem 13.3] or [21, Theorem 5.1]) and, in the most crucial place, on a result from [22]. We use these results to jump several times between topological and measure-theoretic notions. Clearly, we do not claim any credit for this Proposition, and also we allow ourselves to be a bit sketchy.

**Proposition 6.1**: Let $\pi : Y \to X$ be a topological factor map between two topological actions $(Y, G)$ and $(X, G)$. If $\pi$ is a principal extension then

$$h^*(Y, G) = h^*(X, G).$$
Proof. Since $\pi$ is a principal extension, it preserves topological entropy. As we have already remarked, infinite topological entropy implies infinite topological tail entropy for both actions. It remains to continue assuming that $h_{\text{top}}(X, G) < \infty$.

Consider the product action $(X \times X, G)$ (given by $g(x_1, x_2) = (g(x_1), g(x_2))$), and its projection to the first coordinate $\pi_1 : X \times X \to X$ (which is a topological factor map). Then by [22, Theorem 3.1] one has

$$h^*(X, G) = \max\{(\tilde{h}_\mu(X \times X, G|X) - h_\mu(X \times X, G|X)) : \mu \in \mathcal{M}(X \times X, G)\},$$

and an analogous formula holds for $(Y \times Y, G)$ factoring onto $(Y, G)$. For a real-valued function $f$ on a compact space we call the difference $\tilde{f} - f$ the defect function (meaning the defect of upper semi-continuity). So, in the large round parentheses above we see the defect function of the function $\mu \mapsto h_\mu(X \times X, G|X)$ on $\mathcal{M}(X \times X, G)$. We need to show that these defect functions on $X \times X$ and $Y \times Y$ have the same maxima. Since $\pi : Y \to X$ is a principal extension, the variational principle for topological relative entropy of a topological factor map between topological $G$-actions implies that the topological relative entropy associated with the topological factor map $\pi : Y \to X$ equals zero. Further, the topological relative entropy associated with the topological factor map $\pi \times \pi : Y \times Y \to X \times X$ is at most the sum of two zeros (this can be readily seen as it suffices to consider product covers of $Y \times Y$), so it equals zero as well. Applying the same relative variational principle “backwards”, we deduce that $\pi \times \pi$ is a principal extension\footnote{It is probably possible to prove this fact directly using only measure-theoretic entropy, but we failed to find a short argument of this kind.}. This implies that the function $\nu \mapsto h_\nu(Y \times Y, G|Y)$ on $\mathcal{M}(Y \times Y, G)$ equals the function $\mu \mapsto h_\mu(X \times X, G|X)$ on $\mathcal{M}(X \times X, G)$ lifted against (i.e., composed with) $\pi \times \pi$. It takes an elementary exercise in topology to see that in the context of a continuous surjection between two compact metric spaces, the operation of taking the upper semicontinuous envelope of a function commutes with the operation of lifting. The same is obviously true for the defect operation, and this observation ends the proof. \[\blacksquare\]
7. Proofs of the main theorems

Now we are ready to prove our main Theorems 2.1 and 2.2. For the reader’s convenience we state them again before each proof.

**Theorem 2.1:** Let \((X,G)\) be a topological action of a countable amenable group, with finite entropy. Then

\[
h^*(X,G) = \max\{u_1(\mu) : \mu \in \mathcal{M}(X,G)\} = \lim_{k \to \infty} \sup \{\theta_k(\mu) : \mu \in \mathcal{M}(X,G)\}.
\]

**Proof.** As we have already mentioned, the second equality is just swapping the limit of a decreasing sequence of upper semicontinuous functions with the maximum over the domain, see, e.g., [1, Proposition 2.4 ] and then applying the obvious fact that once the maximum is inside, we can skip the upper semicontinuous envelope marks (and replace maximum by supremum). It remains to prove that

\[
h^*(X,G) = \lim_{k \to \infty} \sup \{\theta_k(\mu) : \mu \in \mathcal{M}(X,G)\}.
\]

At first we assume that \(X\) is zero-dimensional, and that the entropy structure of \((X,G)\),

\[
\mathcal{H} = (h_k)_{k \geq 0},
\]

is determined by a refining sequence \((\mathcal{P}_k)_{k \in \mathbb{N}}\) of finite clopen partitions. In this case it is easy to see that both the supremum and infimum in the definition of \(h^*(G,X)\) are realized along the sequence \(\mathcal{P}_k\) (viewed as open covers), i.e., we can write

\[
h^*(X,G) = \inf_{k \in \mathbb{N}} \sup_{m \in \mathbb{N}} h_G(\mathcal{P}_m|\mathcal{P}_k).
\]

Then, by Theorem 5.1, we obtain

\[
h^*(X,G) = \inf_{k \in \mathbb{N}} \sup_{m \in \mathbb{N}} \max\{h_m(\mu) - h_k(\mu) : \mu \in \mathcal{M}(X,G)\}
= \inf_{k \in \mathbb{N}} \sup_{m \in \mathbb{N}} \{\sup(h_m(\mu) - h_k(\mu)) : \mu \in \mathcal{M}(X,G)\}
= \inf_{k \in \mathbb{N}} \sup_{m \in \mathbb{N}} \{\theta_k(\mu) : \mu \in \mathcal{M}(X,G)\},
\]

and we are done.
Now consider a general topological $G$-action $(X, G)$. By [14, Theorem 3.2], this system has a principal zero-dimensional extension $(X', G)$. It is fairly obvious that the quantity
\[
\lim_{k \to \infty} \sup \{\theta_k(\mu) : \mu \in \mathcal{M}(X, G)\}
\]
does not depend on the choice of an individual entropy structure within the uniform equivalence class. This implies that the above quantity for $(X, G)$ is the same as for $(X', G)$. On the other hand, by Proposition 6.1, we also have
\[
h^*(X, G) = h^*(X', G).
\]
Since we have already proved the desired equality for zero-dimensional actions, this equality passes to $(X, G)$, and the proof is finished. \hfill \blacksquare

**Theorem 2.2:** Let $(X, G)$ be a topological action of a countable amenable group. Then the following conditions are equivalent:

1. $(X, G)$ is asymptotically $h$-expansive.
2. The entropy structure of $(X, G)$ converges uniformly to the entropy function.
3. $(X, G)$ admits a principal quasi-symbolic extension.
4. For any $\varepsilon > 0$ the action admits a quasi-symbolic extension with topological relative entropy at most $\varepsilon$.

Furthermore, if $G$ is either residually finite or enjoys the comparison property then the quasi-symbolic extensions in the above statements can be replaced by symbolic extensions.

**Proof.** The equivalence of (1)$\iff$(2) follows from the tail variational principle Theorem 2.1 and the obvious fact that the entropy structure converges uniformly to the entropy function if and only if the sequence $(\theta_k)_{k \in \mathbb{N}}$ converges uniformly to zero. The equivalence of (2)$\iff$(3) is just [11, Theorem 5.6].\footnote{We remark that in [11] asymptotic $h$-expansiveness of a topological $G$-action was defined as uniform convergence of the entropy structure.} The equivalence of (3)$\iff$(4) follows from the symbolic extension theory for $G$-actions [11] (roughly speaking, if the infimum of the entropy functions in all quasi-symbolic extensions is affine, then it is attained). Once the above equivalences are proved, the final statement follows directly from [11, Theorems 7.10 and 7.11]. \hfill \blacksquare
8. Final comments and an example

In [23], the author attempts to prove the tail variational principle directly (with the harder inequality valid only for essential partitions; as we already know, this would suffice for the general case as well). The problem occurs in the measure-theoretic considerations leading to the construction of an invariant measure realizing the supremum appearing in the theorem. On page 1528 the author writes:

$$\frac{1}{|F|} H_{\mu_F} (\mathcal{P}_l^F | \mathcal{P}_k^F) \leq \frac{1}{|E|} H_{\nu_F} (\mathcal{P}_l^E | \mathcal{P}_k^E) + \text{a small error term},$$

where $F$ and $E$ are finite subsets of $G$ and $F$ is $(E, \delta)$-invariant for a small $\delta$. Unfortunately, the presence of the crucial coefficient $\frac{1}{|E|}$ does not follow in any way from the preceding calculation. Note, that the inequality resembles Shearer’s inequality.\(^6\) In spite of the above mistake, this inequality would hold if the function $F \mapsto H_{\mu} (\mathcal{P}_l^F | \mathcal{P}_k^F)$ was strongly subadditive on $G$. But, as we show in Example 8.2 below, this is simply not true. Nonetheless, it is still possible that Shearer’s inequality holds despite the lack of strong subadditivity (there are examples of this kind; see, e.g., [8, Example 6.4]). In fact, the problem whether conditional Shannon entropy satisfies Shearer’s inequality seems to be open (we failed to prove it or disprove it). But it is certain, that this inequality cannot be proved using plain subadditivity, as it is attempted in [23], because Shearer’s inequality is strictly stronger than plain subadditivity (see, e.g., [8, Example 6.5]).

**Remark 8.1:** Example 8.2 uses the finite group $\mathbb{Z}_3$. An example with an infinite group is easily obtained by adapting Example 8.2 to $G = \mathbb{Z}_3 \times \mathbb{Z}$ with an action that does not depend on the second coordinate of $G$.

**Example 8.2:** Conditional Shannon entropy is not strongly subadditive. The same example shows also that conditional counting entropy between covers is not strongly subadditive even when the considered covers are finite clopen partitions.

\(^6\) A function $H$ on $\mathfrak{S}_G$ satisfies Shearer’s inequality (adapted to our context) if whenever $F = \bigcup_{i=1}^n E_i$ and each element of $F$ is contained in at least $k$ of the sets $E_i$, then

$$H(F) \leq \frac{1}{k} \sum_{i=1}^n H(E_i).$$
And so, let $G = \mathbb{Z}_3$ and let $(X, G)$ be the full shift on two symbols, i.e., $X = \{a, b\}^{\mathbb{Z}_3}$ with the shift action. This action is generated by one map $\sigma$ defined by

$$\sigma((x_n)_{n \in \mathbb{Z}_3}) = (x_{(n+1) \text{ mod } 3})_{n \in \mathbb{Z}_3}.$$ 

Let $\mu$ be the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure and let $P$ denote the “zero-coordinate” partition $P = \{[a], [b]\}$, where

$$[a] = \{(x_n)_{n \in \mathbb{Z}_3} \in X : x_0 = a\}$$

and $[b]$ is defined analogously (or as the complement of $[a]$). We also define

$$Q = P^{\{0,1\}} = P \lor \sigma^{-1}(P), \quad R = P^{\{1,2\}} = \sigma^{-1}(P) \lor \sigma^{-2}(P).$$

Consider the following three finite subsets of $\mathbb{Z}$:

$$D = \{0\}, \quad E = \{1\}, \quad F = \{2\}.$$ 

Clearly,

$$Q^E = P^{E \cup F} \quad \text{and} \quad R^E = P^{F \cup D},$$

and

$$Q^{D \cup E \cup F} = R^{D \cup E \cup F} = Q^{D \cup E} = R^{D \cup E} = Q^{E \cup F} = R^{E \cup F} = P^{D \cup E \cup F}.$$ 

Below we will just write $H(\cdot|\cdot)$ but the same applies to $H_\mu(\cdot|\cdot)$.

By a straightforward verification, we have

$$H(Q^E|R^E) = \log 2 > 0,$$

$$H(Q^{D \cup E \cup F}|R^{D \cup E \cup F}) = H(Q^{D \cup E}|R^{D \cup E}) = H(Q^{E \cup F}|R^{E \cup F}) = 0.$$ 

In particular, strong subadditivity fails, because of the strict inequality

$$H(Q^{D \cup E \cup F}|R^{D \cup E \cup F}) > H(Q^{D \cup E}|R^{D \cup E}) + H(Q^{E \cup F}|R^{E \cup F}) - H(Q^E|R^E),$$

while $D \cup E \cup F$ and $E$ are respectively the union and intersection of the sets $D \cup E$ and $E \cup F$.

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