Fields topology and observables

J.Manjavidze, A.Sissakian
JINR, Dubna, Russia

Abstract

The paper contains successive description of the strong-coupling perturbation theory. Formal realization of the idea is based on observation that the path-integrals measure for absorption part of amplitudes $\rho$ is Diracian ($\delta$-like). New form of the perturbation theory achieved mapping the quantum dynamics in the space $W_G$ of (action, angle) type collective variables. It is shown that the transformed perturbation theory contributions are accumulated on the boundary $\partial W_G$, i.e. are sensible to the topology of factor space $W_G$ and, therefore, to the theory symmetry group $G$. The abilities of our perturbation theory are illustrated by examples from quantum mechanics and field theory. Considering the Coulomb potential $1/|x|$ the total reduction of quantum degrees of freedom is demonstrated mapping the dynamics in the (angle, angular momentum, Runge-Lentz vector) space. To solve the (1+1)-dimensional sin-Gordon model the theory is considered in the space (coordinates, momenta) of solitons. It is shown the total reduction of quantum degrees of freedom and, in result, there is not multiple production of particles. This result we interpret as the $S$-matrix form of confinement. The scalar $O(4,2)$-invariant field theoretical model is quantized in the $W_O = O(4,2)/O(4) \times O(2)$ factor manifold. It is shown that the corresponding $\rho$ is nontrivial because of the scale invariance breaking.

1 Introduction.

The mostly intriguing phenomena in modern particles physics is unobservability of color charge in a free state. The underling vector fields theory posses the conformal $O(4,2)$ symmetry supported by the non-Abelian gauge symmetry. Absence of quantitative theory of the confinement leads to enormous number of speculations both in particles physics and in cosmology, e.g. [1]. To all appearance the theory of this phenomena may have influence on the ordinary statistics also.

Solution of the problem of confinement rest on absence of workable perturbation theory. The mostly powerful WKB expansion (or the stationary phase method in the functional integral terms) is noneffective if the fields topology is nontrivial [2] since the dynamics of perturbations of such fields is rather complicated. Beside that, to describe the renormalized vacuum of highly symmetrical modern systems, i.e. non-Abelian gauge fields, the infinite number of order parameters may be essential to find elements of $S$-matrix [3].

- The main goal of this paper is to present the perturbation theory to handle discussed problem. Our perturbation theory is nothing new if the topology of interacting fields is trivial, but is extremely effective for nontrivial topology case. Actually we offer the successive approach to the strong-coupling perturbation theory. Few examples from quantum mechanics (the Coulomb problem is solved exactly) and ((1+1)-dimensional sin-Gordon, scalar $O(4,2)$- invariant) field theories will be examined to illustrate abilities of the approach.

The aim of this review paper is to show

(i) The origin of desired perturbation theory.

Formally the approach based on the remark [3] that the path-integral representation just of observables is defined on the Dirac measure

$$\prod_x du(x)\delta(\partial_v^2 u + v'(u) - j),$$

1Permanent address: Inst. of Physics, Tbilisi, Georgia. E-mail: joseph@nu.jinr.ru
2There is many quantitative attempts to estimate the set of needed polynomial over fields order parameters to describe the spectra of classical hadrons. The today situation in this field is described in [2]. There is also an attempt to introduce highly nonlinear over fields order (or disorder) parameter. See also [4], where the phenomenological aspects of last one are discussed.
where \( j(x) \) is the random, defined on Gauss measure, source of quantum perturbations. We would show the mechanism of unitary, i.e. the total probability conserving, canonical mapping, see Sec.5, of the functional measure on the factor space \( W_G = \mathcal{G}/\tilde{\mathcal{G}} \), where \( \mathcal{G} \) is the theory symmetry group and \( \tilde{\mathcal{G}} \) is the symmetry of classical fields \( u_c = u_c(x; \xi, \eta) \). In other words, we would like to describe the quantum dynamics in the space of field parameters \( (\xi, \eta) (t) \). For instance, see Sec.9, \( W_G \) is the space of coordinates \( \xi \) and momenta \( \eta \) of solitons. The coordinate transformations \( x_\mu \rightarrow x'_\mu = X_\mu(x) \) will be described also, see Sec.6. In this case the field parameters would define the inner geometry of space with definite metrics, see also Secs.6,9.

(ii) **The structure of perturbation theory in the \( \mathcal{G}/\tilde{\mathcal{G}} \) space and as it can be applied.**

We want to warn that a reader may meet new for quantum theories phenomena in the developed strong-coupling perturbation theory\(^3\). By the same reason some conclusion would be seem unusual. It will be shown, see Secs.5,7,8,9, that the quantum corrections of the transformed theory are accumulated on the boundaries (bifurcation manifolds \(^3\)) \( \partial W_G \) of the factor space, i.e. are defined mainly by its topology. This and \( \delta \)- likeness of measure for observables justify the title of the paper.

The main quantitative consequence would be the observation that the quantum corrections may disappeared (totally or partly) on \( \partial W_G \). I.e., the problem of quantum corrections we reduce up to definition of intersection \( \{ \partial u_c \} \cap \partial W_G \) of the defining interactions set \( \{ \partial u_c(\xi, \eta) \} \) with the boundary \( \partial W_G \). This is the new phenomena of reduction of the quantum corrections to the quasiclassical approximation:

(iii) **The range of validity of described approach.**

We would discuss the general form of generating functional of observables, see Secs.2,3, to eliminate doubts in generality of the approach. This question seems important since we would formulate formalism in terms of observables. The Dirac measure of functional-integral representation of observables unambiguously defines the complete set of states of the interacting fields. This allows to construct the perturbation theory in arbitrary useful terms. But, performing transformation, it is impossible come back to the theory in terms of amplitudes since the transformation mixes various degrees of freedom and there is not factorization of \( \rho \) on the product of two amplitudes. We are obliged in result to deal with the observables, or theirs generating functional \( \rho \). This was the reason why the question of generality of our approach was considered in Secs.3,4.

**•** The paper is organized as follows.

- Sec.2. The overview of main ideas and results will be described qualitatively. We hope that this additional introductory section will help to read this large paper.
- Sec.3. The generating functional of observables \( \rho \) will be introduced. It will be shown that \( \rho \) can be used both for generation of observables in particles physics and for description of classical statistical media.
- Sec.4. The differential measure for \( \rho \) is derived and it is shown that the approach is applicable iff the theory is infrared stable. The Wigner functions formalism would by applied for this purpose.
- Sec.5. The main properties of new perturbation theory in the invariant subspace (factor manifold) would be shown considering simplest quantum-mechanical example.
- Sec.6. The main purpose of this section is to show that the transformed theory can not be deduced by naive transformation of integrands of initial path integral for amplitudes. This question is important searching the functional measure on the group manifold.
- Sec.7. We consider the Coulomb problem to show explicitly the role of additional reduction for quantum systems. This phenomena will be used describing the scalar \( O(4,2) \)-invariant field theory.
- Sec.8. The sin-Gordon model is simplest integrable field-theoretical models. By this reason this model will be considered to calculate \( \rho \) using the mapping on the invariant subspace. The explicit calculation leads to trivial \( \rho = 1 \), as the consequence of sin-Gordon solitons stability (or, in other words, of the \( \mathfrak{sl}(2,C) \) Kac-Moody algebra unbrokeness)
- Sec.9. The \( O(4,2) \) model will be considered to demonstrate explicitly the mapping of the 4-dimensional non-integrable field theory on the factor manifold. It will be shown that \( O(4,2) \) symmetry is broken up to \( O(4) \times O(2) \)

\(^3\)The probably noncompleat description of today experience in this field will be given in Sec.2.
symmetry. This phenomena of the scale invariance breaking gives the nontrivial \( \rho \), in contrast to the sin-Gordon model.

Sec.10. We observe the main details of the approach.

For sake of convenience we extract through the text of this paper the main steps as the statements\(^4\). The cumbersome calculations are shifted in the Appendices.

2 Introductory review.

It seems useful, noting large volume of the paper, to give previously the qualitative review of its content. The quantitative realizations, considering simplest characteristic examples, are given in subsequent sections.

- As was underlined above, the particularity of the approach consist in direct calculation of the observables leaving behind calculation of amplitudes. By this reason the useful functional-integral representation for observables will be discussed, specially concentrating attention on the universality of this description.

So, we will find the order parameters substitute natural for \( S \)-matrix theory. The parameter \( \Gamma(q; u) \) directly connected with external particles energy, momentum, spin, polarization, charge, etc., and sensitive to symmetry properties of the interacting fields system will be introduced\(^5\). For sake of simplicity we will consider in this paper \( u(x) \) as the real scalar field. The generalization would be evident.

In used \( S \)-matrix approach the expectation value of \( \Gamma(q; u) \) will define the asymptotic states density. So, the \( m \)-into \( n \)-particles transition (nonnormalized) probability \( \rho_{nm} \) would have a symmetrical form in terms of \( \Gamma(q; u) \):

\[
\rho_{nm}(q_1, ..., q_n; q'_1, ..., q'_m) = < \prod_{k=1}^{n} |\Gamma(q_k; u)|^2 \prod_{k=1}^{m} |\Gamma(q'_k; u)|^2 >_u .
\] (2.1)

Here \( q'(q) \) are the in(out)-going particles momenta and \( < >_u \) means averaging over field \( u(x) \).

By definition, \( \rho_{2n}(q'_1, q'_2; q_1, ..., q_n) \) can be used as the generator of events modeling the accelerator experiments \([10, 11]\). For instance,

\[
\frac{1}{J(s)} \sum_{n} \int_{s} \rho_{2n}(q_1, ..., q_n; q'_1, q'_2) = \sigma_{tot}(s),
\]

where \( \sigma_{tot} \) is the total cross section (\( J \) is the ordinary normalization factor). In this expression integrations are performed with constraint \( s = (q'_1 + q'_2)^2 \).

Beside that, the ordinary big partition function of statistical system can be expressed through

\[
\sum_{n,m} \int_{(\beta_i, z_i; \beta_f, z_f)} \rho_{nm}(q_1, ..., q_n; q'_1, ..., q'_m),
\] (2.2)

where \( \rho_{nm} \) is defined by (2.1). The summation and integration are performed with constraints that the mean energy of particles in initial(final) state is \( 1/\beta_i(1/\beta_f) \). So, \( 1/\beta \) has the meaning of temperature and \( z_i(z_f) \) is the activity for initial(final) state. We investigate in what conditions such description of statistics is rightful. It will be seen that the Bogolyubov’s relaxation of correlations near equilibrium is the only condition \([11]\).

It will be shown that this definition of statistics has the right classical limit iff the theory is infrared stable, i.e. iff the 4-dimensional scale of quantum fluctuations \( L_q \) is much smaller then the scale of statistical ones \( L_s \). In this case

\(^4\)Some of them are evident and the proof of others is given with a ‘physical’ level of strictness.

\(^5\)Following trivial analogy with ferromagnetic may be useful. So, the external magnetic field \( \mathcal{H} \sim \mu \), if \( \mu \) is the magnetics order parameter, and the phase transition means that \( \mu \neq 0 \). Offered \( \Gamma(q; u) \) has the same meaning as \( \mathcal{H} \).
one can deduce from \( \rho_{nm} \) the classical phase space \((R, q)\) distribution. The Wigner-functions \( W(R, q) \) formalism will be applied, see also \[13\] to show the origin of \( L_q << L_s \) condition. The details one can find in Secs.4.

- We will find that \( \Gamma(q; u) \) is the function of external particles momentum \( q \) and is the linear functional of \( u(x) \):

\[
\Gamma(q; u) = - \int dx e^{iqx} \frac{\delta S_0(u)}{\delta u(x)} = \int dx e^{iqx} (\partial^2 + m^2) u(x), \quad q^2 = m^2,
\]

for the mass \( m \) field. Here \( S_0(u) \) is the free part of the action. This parameter presents the momentum distribution of the interacting field \( u(x) \) on the remote hypersurface \( \sigma_ \infty \) if \( u(x) \) is the regular function. Note, the operator \((\partial^2 + m^2)\) cancels the mass-shell states of \( u(x) \).

Note, the definition (2.1) is rightful, i.e. \( q \) is the external particles momentum and the condition \( q^2 = m^2 \) is hold, iff the theory is infrared stable. It should be underlined, see Sec.4, that this condition is ‘seen’ if the \((R, q)\) distribution in the classical limit is considered.

The construction (2.3) means, because of the Klein-Gordon operator and the external states should be mass-shell by definition \[13\], the solution \( \rho_{nm} = 0 \) is possible for particular topology (compactness and analytic properties) of quantum field \( u(x) \). So, \( \Gamma(q; u) \) carry following remarkable properties:
- it directly defines the observables,
- is defined by the topology of \( u(x) \),
- is the linear functional of the actions symmetry group element \( u(x) \).

Note, the space-time topology of \( u(x, t) \) becomes important calculating integral (2.3) by parts. This procedure is available iff \( u(x, t) \) is the regular function. But the quantum fields are always singular since Green functions are the distributions (see Sec.8). Therefore, the solution \( \Gamma(q; u) = 0 \) is valid iff the quasiclassical approximation is exact. Just this situation is realized in the soliton sector of sin-Gordon model.

- It is evident that if \( \rho_{nm} = 0, n, m \neq 0 \), the field \( u(x) \) is confined since is not measurable as the external field. This is the \( S \)-matrix interpretation of the ‘confinement’ phenomena. Just this formulation of the confinement we would like to develop in future. Note, \( \rho_{nm} \neq 0 \) testify not only the broken symmetry but this expectation value is not zero in absence of any symmetry also.

Despite this ambiguity \( \Gamma(q; u) \) carry the definite properties of the order parameter since the opposite solution \( \rho_{nm} = 0 \) can be the dynamical display of an unbroken symmetry only\[14\], i.e. of the nontrivial topology of interacting fields, as the consequence of unbroken symmetry.

Note ‘sparingness’ of the \( S \)-matrix description. The external states of the \( S \)-matrix approach should be expandable on the Fock basis and belong to the mass shell. So, we can ask only – ‘is this particle (or multiparticle state), with given properties, observable in the free state’. We hope to construct the perturbation theory which is able to give the answer on this question unambiguously for arbitrary Lagrangian in arbitrary space-time dimension.

- Our hope based on the particularity of offered below quantization method. It consist in introduction of the unitarity condition in the path-integral formalism \[8\]. It will be seen that the dynamical display of unitarity condition is the local equilibrium of all forces in the quantum system. In other words, the total probability conservation principle is the quantum analogy of the classical D’Alembert’s variational principle \[10\]. This dynamical scheme

\[\text{There is also the ‘marginal distribution functions’ formalism \[14\] as the attempt of universal description both quantum and statistical systems. It will be seen that the Wigner-functions approach is natural for \( S \)-matrix \[13\] and gives the evident physical interpretation of transition from quantum to classical physics.}\]

\[\text{The} \ S \text{-matrix would be introduced phenomenologically, postulating the ordinary Lehmann-Symanzik-Zimmermann (LSZ) reduction formulae, see eq.\[13\]. So, the formal constraints, e.g. the Haag theorem, would not be taken into account.}\]
seems interesting if one remained above mentioned problem with description of modern systems vacuum, it allows to perform calculations without going into details of the renormalized vacuum structure.

So, it will be shown (see also §8) that the unitarity condition leads to \(\delta\)-like functional measure \(DM(u)\) for \(\rho_{nm}\):

\[
DM(u) = \prod_x du(x)\delta \left( \frac{\delta S(u)}{\delta u(x)} - j(x) \right),
\]

where \(j(x)\) is the provoking quantum excitations force and \(S(u)\) is the classical action. It is evident that this measure close logically the formalism since it defines a complete set of contributions for given classical Lagrangian in the sector of real-time fields.

One should assume that \(j(x)\) switched on adiabatically (in this case we expand contributions in the vicinity of \(j = 0\) for effective use of this definition of measure. Otherwise we should know \(j(x)\) exactly, including it into Lagrangian as the external field. The measure would remain \(\delta\)-like in last case also. \textbf{Note}, \(j(x)\) can be introduced adiabatically even if the dynamical symmetry breaking is expected \(^9\).

So, the measure (2.4) allows to conclude that the \textit{sum of all (including trivial), strict, regular, real-time solutions of the classical equation}

\[
\left( \frac{\delta S(u)}{\delta u(x)} \right) = 0.
\]

defines the complete set of contributions.

\textbf{Note}, eq. (2.5) reflects the ordinary Hamilton variational principle if the quantum perturbations switched on adiabatically. We would show therefore that the WKB expansion is in agreement with unitarity condition. It is the well known result. But the measure (2.4) contains following new information:

\begin{itemize}
  \item \textbf{a.} Only \textit{strict} solutions of eq. (2.5) should be taken into account. This ‘rigidness’ of the formalism means absence of pseudo-solutions (similar to multi-instanton, or multi-kink) contribution.
  \item \textbf{b.} \(\rho_{nm}\) is described by the \textit{sum} of all solutions of eq. (2.5), independently from theirs ‘nearness’ in the functional space;
  \item \textbf{c.} \(\rho_{nm}\) did not contain the interference terms from various topologically nonequivalent contributions. This displays the orthogonality of corresponding Hilbert spaces;
  \item \textbf{d.} The measure (2.4) includes \(j(x)\) as the external source. Its fluctuation disturb the solutions of eq. (2.5) and \textit{vice versa} since the measure (2.4) is strict;
  \item \textbf{e.} In the frame of above adiabaticity condition the field disturbed by \(j(x)\) belongs to the same manifold (topology class) as the classical field defined by (2.5);\(^9\)
\end{itemize}

One must underline that the measure (2.4) is derived for \textit{real-time} processes only, i.e. is not valid for tunneling ones. By this reason above conclusions should be taken carefully. The corresponding selection rule will be given below in Sec.5.

\textbf{Note}, the \(\delta\)-likeness of measure essential to find eq. (2.3). This will be shown in Sec.4.

\textbf{Note} also that some information is lost calculating probabilities only. We will show as the approach can be generalized to calculate the imaginary part of the elastic scattering amplitudes. Then, using dispersion relation one can restore the total amplitudes.

\begin{itemize}
  \item We start discussion of the approach, risking to loose generality, by simplest quantum-mechanical examples of particle motion in the potential hole \(v(x)\) with one non-degenerate minimum at \(x = 0\). We will calculate the probability \(\rho = \rho(E)\) to find the bound state with energy \(E\).
\end{itemize}

\(^8\)In the mathematical literature such measure is known as the ‘Dirac measure’.

\(^9\)There are in modern physics the remarkable attempts to construct a geometrical approach to quantum mechanics \(^8\)\cite{15} and field theory \cite{19}. Our approach, based on the dynamical equilibrium, see (2.4), between classical and quantum forces, contains evident geometrical interpretation and it will be widely used, see below. \textbf{Note}, our interpretation have deal with the classical phase-space flows. By this reason, in contrast with above mentioned approaches, the finite dimensional manifolds only, as in classical mechanics, would arise.
It will be seen that this experience is useful for quantization of nonlinear waves also. Indeed, introducing the convenient variables (collective coordinates) one can reduce the quantum soliton-like excitations problem to quantum-mechanical one. This idea was considered previously by many authors, e.g. [21, 22].

Quantitatively the problem looks as follows. It is not difficult to describe the one-particles dynamics in the quasiclassical approximation since corresponding equation for trajectory $x_c$ always can be solved. But, beyond this approximation, to use the ordinary WKB expansion of path integrals, one should solve the equation for Green function $G(t, t')$:

$$ (\partial^2 + v''(x_c))G(t, t') = \delta(t - t'). $$

(2.6)

Just eq. (2.4) offers a difficulty: it is impossible to find an exact solution of this equation since $x_c = x_c(t)$ is the non-trivial function and, therefore, $G(t, t')$ is not translationally invariant describing propagation of a ‘particle’ in the time-dependent potential $v''(x_c)$. We avoid this difficulty introducing the convenient dynamical variables.

The main formal difficulty, see e.g. [23], of this program consisting in transformation of the path-integral measure was solved in [24]. The phase space path-integrals differential measure $DM(x, p)$ for $\rho$ has the form (see (2.4)):

$$ DM(x, p) \sim \prod_t dx(t) dp(t) \delta(\dot{x} + \partial H_j / \partial p) \delta(\dot{p} - \partial H_j / \partial x), $$

(2.7)

where the Hamiltonian

$$ H_j = \frac{1}{2} \dot{x}^2 + v(x) - jx $$

includes the energy of quantum fluctuations $jx$, with the provoking quantum excitations force $j = j(t)$. Then the dynamical equilibrium between ordinary mechanical forces (kinetic $\ddot{p}(t)$ plus potential $v'(x)$) and quantum force ($j(t)$) determined by $\delta$-like measure (2.7) allows to perform an arbitrary transformation of quantum measure caused by transformation of classical forces, i.e. of $x$ and $p$ (see d.).

We will use this property introducing the ‘motion’ on the cotangent bundle $T^*G = (\theta, h)$, where $h$ is the bundle parameter and $\theta$ is the coordinate on it. For definiteness, let $h$ be the conserved on the classical trajectory energy and $\dot{\theta}$ is the conjugate to $h$ ‘time’. (The transformation to action-angle variables will be described also.) The mapping $(x, p) \rightarrow (\theta, h)$ is canonical and the corresponding equations of motion on the cotangent bundle should have the form:

$$ \dot{\theta} = + \frac{\partial h_j}{\partial \theta} = 1 - j \frac{\partial x_c}{\partial h}, $$

$$ \dot{h} = - \frac{\partial h_j}{\partial \theta} = + j \frac{\partial x_c}{\partial h}, $$

(2.8)

where

$$ h_j(\theta, h) = h - j x_c(\theta, h) $$

is the transformed Hamiltonian and $x_c(\theta, h)$ is the classical trajectory in the $(\theta, h)$ terms. The Green function of the eq. (2.8) $g(t, t')$ is translationally invariant since classically (at $j = 0$) the cotangent bundle is the time-independent manifold.

Above example shows that the mapping is constructive if the bundle parameters are generators of (sub)group violated by $x_c$. Corresponding phase space is the invariant subspace (see also footnote 8).

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10 One can find $G(t, t')$ perturbatively, expanding $v''(x_c)$ over $x_c$. However, it is too hard to handle the resulting double-parametric perturbation theory.

11 Number of problems of quantum mechanics was solved using also the ‘time sliced’ method [24]. This approach presents the path integral as the finite product of well defined ordinary integrals and, therefore, allows perform arbitrary space and space-time transformations. But transformed ‘effective’ Lagrangian gains additional term $\sim h^2$. Last one crucially depends from the way as the ‘slicing’ was performed. This phenomena considerably complicate calculations and the general solution of this problem is unknown for us. It is evident that this method is effective if the quantum corrections $\sim h$ play no role. Such models are well known. For instance, the Coulomb model in quantum mechanics, the sine-Gordon model in field theory, where the bound-state energies are exactly quasiclassical.
For some problems the mapping on the curved space with nontrivial metric tensor $g_{\mu\nu}(x)$ is useful, see, e.g., [26]. We will find the quantum mechanical Dirac measure directly in the curved space with nontrivial $g_{\mu\nu}(x)$.

We examine the factorizability of transformed measure for $\rho$ considering transformation of the Dirac measure to cylindrical coordinates. We will consider the general coordinate transformation $x_\mu \to x'_\mu = X_\mu(y)$ starting from flat space, where the metric tensor $g_{\mu\nu}$ is trivial. The Dirac measure for theory with Lagrangian

$$L = \frac{1}{2} g_{\mu\nu}(y) \dot{y}^\mu \dot{y}^\nu - v(y),$$

(2.9)

where $g_{\mu\nu}(y)$ is the nontrivial function is calculated also in Sec.6.

- It will be shown that the procedure of averaging $<\rho_u$ in (2.1) carried out by the integro-differential operator

$$\hat{\rho} = e^{-i\tilde{K}(j,e)} \int DM(x)e^{-iU(x,e)}.$$

(2.10)

It should be underlined that this representation is strict and is valid for arbitrary Lagrange theory of arbitrary dimensions. It can be considered also as the ‘unitary definition’ of the functional integral.

The operator $\hat{\rho}$ contains three element. The Dirac measure $DM$, the functional $U(x,e)$ and the operator $\tilde{K}(j,e)$. The functional $U(x,e) = O(e^3)$ describes interaction and is simply calculable from $S(x)$. The operator $\tilde{K}(j,e)$ contains the product $\dot{j} \cdot \dot{e}$ of the functional derivatives over $j$ and $e$, i.e. is Gaussian. The expansion over $K$ gives perturbation series. At the very end of calculations one should take the auxiliary variables $j$, $e$ equal to zero. Note, the structure of operator $\hat{\rho}$ is form-invariant concerning transformation of variables ($x$ in the configuration space, $(x,p)$ in the phase space).

The perturbation theory with measure (2.3) on the cotangent bundles has unusual properties [27, 28] and few proposition concerning perturbation theory in the invariant subspaces will be offered. The main of them is possibility to split the ‘Lagrange’ source $j(t) \to j_{W_G}$, see (2.8), onto set of sources of each independent degree of freedom of the invariant subspace. For instance, the splitting

$$j(t) \to (j_\theta(t), j_h(t))$$

(2.11)

will be demonstrated. By this way the actually Hamilton’s description is achieved in the invariant subspace. Note, the transformation $[2.11]$ induce

$$e(t) \to (e_\theta(t), e_h(t))$$

(2.12)

leaving a structure of $U(x,e)$ unchanged. This remarkable property allows to describe interactions between various degrees of freedom in the invariant subspace since in result

$$\dot{j} \cdot \dot{e} \to \dot{j}_h \dot{e}_h + \dot{\theta} \dot{\theta}.$$

(2.13)

We will consider also the transformation to the cylindrical coordinates $(r, \varphi)$. In this case the Cartesian sources $(x_1, x_2) \to (j_r, j_\varphi)$ (and $(x_1, x_2) \to (e_r, e_\varphi)$).

Note, the probability $\rho \sim <\text{in}|\text{out}> <\text{in}|\text{out}>^*$ describes the ‘closed-path’ motion by definition. This means that the corresponding classical action $S_\varphi = \oint_{\varphi} p dx - H dt$, where $\varphi$ is composed from two $(\text{in} \to \text{out})$ and $(\text{out} \to \text{in})$ independent paths, is the Poincare invariant, i.e. $S_\varphi = S_{\varphi'}$, if the phase-space flows through the closed path $\varphi$ and $\varphi'$ coincide. We will see that new perturbation theory describes such variations of $\varphi$ that it enclose different phase-space tubes since the quantum perturbations are unable to change the phase-space ‘liquids’ density.

So, resulting perturbation theory describes fluctuations of phase-space flow, induced by $(\delta \theta(t), \delta h(t))$, through the elementary cells $\delta x_c \wedge \delta q_c$, where $\delta q_c$ is the tangent to $x_c$ vector and $\delta x_c \equiv e$ is the virtual variation from $x_c$. For instance,

$$\delta x_c \wedge \delta q_c = e_\theta \frac{\partial x_c}{\partial \theta} - e_h \frac{\partial x_c}{\partial \theta}$$

(2.14)

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in terms of local coordinates \((\theta, h)\). Each term of the perturbation series is proportional to derivatives of the nondegenerate 2-form \(d\omega = \delta x_c \wedge \delta p_c\) over \(\theta\) and/or \(h\).

Then the equality

\[ d\omega_i = (\delta x_c \wedge \delta p_c)_i = 0 \]  

(2.15)

means that there is not quantum flow, i.e. quantum corrections, in the \(i\)-th direction of the invariant subspace. Note, eq. (2.15) is hold if one of elements in the product \((\delta x_c)_i\) or \((\delta q_c)_i\) is equal to zero if the two-form \(d\omega\) is not degenerate in the ordinary sense. This will allow to consider not only the canonical transformations of the path-integral measure.

Secondly, each order of perturbation theory can be reduced to the sum of total derivative over global coordinate of the invariant subspace. So, if \((\theta_0, h_0)\) are the initial data for classical trajectory in the \((\theta, h)\) space then the quantum corrections to the quasiclassical approximation

\[ \rho^q = \int d\theta_0 dh_0 \{ \frac{\partial \rho_0}{\partial \theta_0} + \frac{\partial \rho_0}{\partial h_0} \} = \int_{\partial W_G} d\rho^q. \]

(2.16)

This property is important analyzing integrability of quantum systems since brings forward the topology of contributions, i.e. the boundary properties of classical trajectory \(x_c(\theta, h)\), or, in other words, the topology of factor manifold.

Note, the measure of integration in (2.16) over zero modes degrees of freedom \(\{\theta_0\) in this case\} was defined as the result of mapping on the invariant subspace, i.e. without introduction of the Faddeev-Popov anzats.

Above example shows that it is necessary to have the universal method of mapping of quantum dynamics on the useful manifolds. Let us introduce helpful notations. So, it is well known that the mapping

\[ J : T \to W, \]

(2.17)

where \(T\) is the 2\(N\)-dimensional phase space and \(W\) is a linear space, solves the mechanical problem exactly iff

\[ J = \bigotimes_1^N J_i, \]

(2.18)

where \(J_i\) are the first integrals in involution, e.g. \[29\] (the formalism of reduction (2.17) in classical mechanics is described also in \[30\]).

In other words, if \(J_i = J_i(x, p), \ i = 1, 2, ..., N,\) are the first integrals, \(\dot{J}_i = 0,\) in involution, \(\{J_i, J_j\} = 0,\) following to Liouville-Arnold theorem, the canonical transformation \((x, p) \to (J, Q)\) solves the mechanical problem. Then the \((x, p)\) flow is defined by the 2\(N\) system of coupled algebraic equations \(\eta = J(x, p), \ \xi = Q(x, p).\)

The mapping (2.17) introduces integral manifold \(J_\omega = J^{-1}(\omega)\) in such a way that the classical phase space flow with given \(\eta \in J_\omega\) belongs to \(J_\omega\) completely \((\eta = h_0\) in the above example\). Our methodological idea assumes quantization of the \(J_\omega\) manifold instead of flow in \(T\).

This becomes possible noting that the quantum trajectory should belong to \(J_\omega\). Indeed, in the above example the trajectory \(x_c = x_c(\theta, h)\) was defined by parameters \((\theta, h)\) completely, where \((\theta, h)\) are defined by eqs. (2.8). Note, this important property based on the \(\delta\)-likeness of measure (see a.) and assumption that \(j\) is switched on adiabatically.

The ‘direct’ mapping (2.17) used above, see \[10\], assumes that \(J\) is known. But this approach to the general problem of mapping of quantum dynamics seems inconvenient having in mind the nonlinear waves quantization, when the number of degrees of freedom \(N = \infty,\) or if the transformation is not canonical (or if we do not know is the transformation canonical). We will consider by this reason the ‘inverse’ approach starting from assumption that the classical flow is known. Then, since (i) if the flow belongs to \(J_\omega\) completely and (ii) if \(J_\omega\) presents the manifold \[16\], we would be able to introduce motion in \(W\).

The manifold \(J_\omega\) is invariant relatively to some subgroup \(G_\omega\) \[23\] in accordance to topological class of classical flow.\footnote{This could help to avoid the problems with Gribov’s ambiguities quantizing the non-Abelian Yang-Mills theory. Above suggestion means that the gauge phase will be considered as a dynamical degree of freedom, we suppose to quantize the Yang-Mills theory in the \(W_G \times \Omega_g\) space, where \(\Omega_g\) is the gage orbit.} This introduces the \(J_\omega\) classification and summation over all \((\text{homotopy})\) classes should be performed.

\footnote{By this reason \(W\) is the invariant subspace.}
Note, the classes are separated by the boundary bifurcation lines in $W$. Then, noting (2.16), the quantum dynamics is defined on this bifurcation lines.

If the quantum perturbations switched on adiabatically then the homotopy group should stay unbroken. It is the ordinary statement for quantum mechanics.

Therefore, we trying to describe in this paper the transition from quantum dynamics in the trajectories space to the dynamics in the space of parameters of classical trajectories. In the above example just $(\theta, h)$ were such parameters.

This step seems nontrivial since separates the problem of searching of solutions of eq. (2.5) and the problem of quantization in corresponding vicinity. Then, noting that the quantum corrections are accumulated on the boundary of factor manifold $\partial W$, we can estimate, at least qualitatively, the contributions since definition of $W$ is the algebraic problem, see Sec.9.

We would try to construct the mapping using the base of symplectic geometry. This is possible having in mind discussion of eq. (2.13). Indeed, starting from assumption that $(\xi, \eta)$ form the symplectic space (of arbitrary dimension) we would find the measure projecting this space on $W$. This step takes into account the symmetry structure of the $W$ space (the unnecessary ‘empty’ degrees of freedom are canceled by the normalization condition).

- We will calculate the bound state energies in the Coulomb potential to illustrate this idea. This popular problem was considered by many authors, using various methods, see, e.g., [31]. The path-integral solution of this problem was offered in [32] using the time-sliced method.

We will restrict ourselves by the plane problem. Corresponding phase space $T = (p, l, r, \phi)$ is 4-dimensional.

The classical flaw of this problem can be parametrized by the angular momentum $l$, corresponding angle $\phi$ and by the normalized on total Hamiltonian Runge-Lentz vectors length $n$. So, we will consider the mapping ($p$ is the radial momentum in the cylindrical coordinates):

$$J_{l,n} : (p,l,r,\phi) \rightarrow (l,n,\phi) \quad (2.19)$$

to construct the perturbation theory in the $W_C = (l,n,\phi)$ space. I.e. $W$ is not the symplectic space. Nevertheless we start from the symplectic space adding to $n$ the auxiliary canonical variable $\xi$.

Following to above definition, our transformed perturbation theory describes the quantum flow through the oriented cell, see e.g. (2.14), in the invariant subspace. Then, following to (2.19), $d\omega_n = (\delta x_\alpha \wedge \delta p_\alpha)_n = 0$ (since $\partial p_\alpha / \partial \xi \equiv 0$) and there is not quantum flow corresponding to $n$. The Coulomb problem would be considered to demonstrate just this partial reduction. We will find that the bound-state energies are pure quasiclassical since the quantum corrections are died out on the $\partial W_C = \partial(\phi, l)$ boundary.

- We would consider the field theories with nontrivial topology to verify the effectiveness of our formalism. For instance, we may consider creation of the free particles in the theory with Lagrangian

$$L = \frac{1}{2} (\partial_\mu u)^2 + \frac{m^2}{\lambda^2} [\cos(\lambda u) - 1]. \quad (2.20)$$

It is well known that this field model possess the soliton excitations in the (1+1) dimension.

Formally nothing prevents to linearize partly our problem considering the Lagrangian

$$L = \frac{1}{2} (\partial_\mu u)^2 - m^2 u^2 + \frac{m^2}{\lambda^2} [\cos(\lambda u) - 1 + \frac{\lambda^2}{2} u^2]$$

$$\lambda_{\text{Dirac}} \delta \text{ measure noting that the canonical transformation is a one-parametric group of diffeomorphisms.}$$

$$\partial W_C \text{ would not have the symplectic structure for Coulomb potential. It decomposed on the direct product } T O^* \times R^1 \text{ of the two-dimensional } (\phi, l) \text{ phase space and ordinary one-dimensional subspace } R^1. \text{ Last one correspond to } n. \text{ The quantum dynamics is realized in the } (\phi, l) \text{ phase space only. Note, this conclusion is in agreement with quantum uncertainty principle: if the dynamical variable has not conjugate pare in the Poisson brackets sense, then it stay being c-number in the quantization scheme. The integral over } \xi \text{ is canceled by normalization.}$$

$$\text{See also footnote 3.}$$
\[ S_0(u) - V(u) \]  
(2.21)

to describe particles creation (and absorption). The last term describes interactions. Corresponding ‘order parameter’ is

\[ \Gamma(q; u) = \int dxdte^{iqx}(\partial^2 + m^2)u(x,t), \quad q^2 = m^2. \]  
(2.22)

We will calculate the expectation value:

\[ \rho_{10}(q) \equiv \rho_2(q) = \langle \Gamma(q; u) \rangle_u. \]

Just the procedure of averaging would be the object of our interest. By definition, \( \rho_2 \) is the probability to find one mass-shell particle. Certainly, \( \rho_2(q) = 0 \) on the sourceless vacuum but, generally speaking, \( \rho_2(q) \neq 0 \) in a field (soliton) with nonzero energy density.

It will be shown by explicit calculations that

\[ \rho_2(q) = 0, \quad \text{if} \quad \vec{q} \neq 0, \]  
(2.23)

as the consequence of unbroken \( \tilde{sl}(2, C) \) Kac-Moody algebra on which the solitons of theory (2.20) are live, see e.g. [33] and references cited therein. The solution (2.23) seems important since it can be interpreted as the explicit demonstration of field \( u(x,t) \) confinement\(^{17}\). It will be shown that (2.23) is the consequence of previously developed proposition that the quasiclassical approximation is exact for sin-Gordon model\(^{35}\).

The same as for Coulomb problem reduction procedure will be applied. But the reduction procedure of the sin-Gordon Hamiltonian system with symmetry \( G \) looks like canonical transformation\(^{33}\), in opposite to Coulomb problem, i.e. \( W \) would have the symplectic structure. \textit{Note}, considered mapping assumes the infinite number of degrees of freedom reduction since it brings the quantum-field model, defined in the infinite dimensional phase space \( T \), up to the quantum-mechanical one, defined in the finite dimensional phase space \( TG^* \) (see footnote 8). We demonstrate this nontrivial procedure explicitly.

So, we would show the way as the field-theoretical problem may be reduced to the quantum-mechanical one. We would consider \( \eta \) as the ‘particles’ (solitons) generalized momentum and would introduce \( \xi \) as the conjugate to \( \eta \) coordinate. The \( N \)-soliton \( 2N \)-dimensional phase space (cotangent manifold) \( TG^* \) with local coordinates \((\xi, \eta)\) on it has natural symplectic structure, and \( DM(TG^*) = D^N M(\xi, \eta) \) in practical calculations,

\[ D^N M(\xi, \eta) = \prod_i \delta(\xi - \frac{\partial h_j}{\partial \eta})\delta(\eta + \frac{\partial h_j}{\partial \xi}), \]  
(2.24)

where \( h_j(\xi, \eta) \) is the transformed total Hamiltonian and \((\xi, \eta)\) are the \( N \) dimensional vectors. The summation over \( N \) is assumed.

The quantum corrections to quasiclassical approximation of transformed theory are simply calculable since the bundle parameters \( \eta \) are conserved in the classical limit. This is the particularity of solitons dynamics (solitons momenta are conserved quantities). One can consider the developed formalism as the path-integral version of nonlinear waves (solitons in our case) quantum theory (the canonical quantization of sin-Gordon model in the soliton sector was described also in [22]).

Noting that all solutions of model (2.20) are known\(^{36}\) we will find \( DM(TG^*) \) considering the mapping as an ordinary transformation to useful variables. This idea will be realized in the following way. We can introduce following formal parametrization of the field\(^{18}\)

\[ u(x,t) = u_c(x; \xi, \eta), \]  
(2.25)

\(^{17}\)This means also the solitons stability under quantum perturbations. The same conclusion was established for integrable model\(^{14}\) using the ‘1/\(N\) expansion’.

\(^{18}\)The noncovariant notations \((x, t)\) are useful since our perturbation theory is Lorentz-noncovariant.
where \((\xi, \eta)(t)\) are the \(N\)-dimensional vectors. It is evident that \(u_c(x; \xi, \eta)\) is the solution of incident equations iff it obey the Poisson brackets \((p\) is the conjugate to \(u\) momentum):

\[
\{h_j, u\} = \frac{\delta H_j}{\delta p}, \quad \{h_j, p\} = -\frac{\delta H_j}{\delta u},
\]

(2.26)

at \(u = u_c, \ p = p_c\) and \((\xi, \eta)\) obey the equations:

\[
\dot{\xi} = \frac{\partial h_j}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial h_j}{\partial \xi}.
\]

(2.27)

Note, last equations are the arguments of \(\delta\)-functions in the transformed measure (2.24).

\[\bullet\] The \(O(4,2)\)-invariant quantum field theory model with Lagrangian

\[
L = \frac{1}{2}(\partial_\mu u)^2 - \frac{1}{4}gu^4,
\]

(2.28)

where \(u(x, t)\) is the real scalar field, beside its simplicity, is important since the ansatz for Yang-Mills potential

\[
B^{\alpha}_{\mu}(x, t) = \eta^{\alpha}_{\mu\nu}\partial_{\nu}\ln u(x, t)
\]

(2.29)

reduces the Yang-Mills equations to the single one for the field \(u\):

\[
\partial^2_\mu u + gu^3 = 0, \quad g > 0.
\]

(2.30)

By this reason the theory (2.28) oftenly considered as a step toward Yang-Mills theory, see also [38].

We will calculate the expectation value

\[
\rho_2(q) = \langle |\Gamma(q, u)|^2 \rangle_u,
\]

(2.31)

where for given theory

\[
\Gamma(q, u) = \int d^3xdte^{iqx}\partial^2_\mu u(x, t), \quad q^2 = 0,
\]

(2.32)

Following to above experience, the \(W\) space is the factor manifold \((G/G)\) of the \(G = O(4,2)\) group and \(G\) is the compact 19 subgroup of \(G\) group. We will consider the highest \(O(4)\times O(2)\) subgroup 20.

So, let \(\phi(x, t)\) be the \(O(4)\times O(2)\)-invariant solution of eq.(2.30). Then the generators of translations \(T_0\) and special conformal transformations \(K_0\) would define the coordinates of the \(W_O\) space 21.

The classical phase flow in \(W\) is simple:

\[
\dot{\xi}_0 = \omega = \text{ const.}, \quad \dot{\eta}_0 = 0
\]

(2.33)

The developed perturbation theory describes the random fluctuations of \(W\).

Following reduction procedure will be demonstrated. The functional differential measure \(DM(u, p)\) of the initial phase space \(V^\infty\) will be mapped on the space \(W^8\) under the phase flow \((\phi, \pi)\), where \(\pi = \dot{\phi}\). This reduction would introduce the phase space with nondegenerate 2-form \(\delta\phi \wedge \delta\pi\), where, as was explained above, \(\delta\phi\) is the virtual deviation of \(u\) from the true trajectory \(\phi\) and \(\delta\pi\) is a tangent to \(\phi\) vector of the transformed theory. This symplectic structure

19Otherwise the classical solution would be singular.

20Another solutions of (2.30) must be taken into consideration also. So, strictly speaking, considered in this paper contribution presents the particular field-theoretical realization only.

21The \(O(4)\times O(2)\) subgroup of \(O(4,2)\) group is 7 parametric. So the factor manifold is 8 dimensional.
allows perform further reduction (see discussion of eq.(2.15)): $W^8 \rightarrow W_O = T^* \bar{G}_1 \times R^5$, where $R^5$ is the 5-dimensional zero modes manifold and $T^* \bar{G}_1$ is the 4-dimensional phase space with one constraint\(^{22}\).

All quantum dynamics is confined in $T^* \bar{G}_1$:

$$\dot{\xi}_i(t) = \omega_i(\eta) + j_{\xi_i}, \quad \dot{\eta}_i(t) = j_{\eta_i}, \quad i = 1, 2,$$

with one constraint on the boundary conditions:

$$\xi_1(0) = \xi_2(0).$$

The sources $j_{\xi_i}, j_{\eta_i}$ introduced for description of the $T^* \bar{G}_1$ subspace quantum deformations.

Inserting into (2.32) the $O(4) \times O(2)$-invariant solution of eq. (2.30)\(^{38, 39}\) $\phi(x, t)$ we find that

$$\Gamma(q, \phi) = 0, \quad q_\mu \neq 0,$$

since $\phi(x, t) = 0$ if $(x, t)$ belong to remote hypersurface $\sigma_\infty$. This example shows that the confinement condition $\rho_2(q) = 0$ is hold in the quasiclassical approximation.

We investigate the influence of quantum corrections, i.e. the question, can they alter the quasiclassical solution (2.36). It will be shown that quantum corrections broke the scale invariance and by this reason $\rho_2(q) \neq 0$ for theory (2.28) on the $O(4) \times O(2)$ -invariant solution.

Note, this example shows that the classical field $\phi$ can broke the symmetry, i.e. provoke the phase transition, in contrast to kink-like excitations.

### 3 Generating functional.

For sake of completeness it seems useful to introduce from the very beginning of this review paper the generating functional of observables (cross sections, created particles spectra, etc.) $\rho$. It seems also interesting to investigate in a what conditions this quantity has the right classical limit since in statistics $\rho$ would have the meaning of big partition function (see also\(^{11}\)). In result we get to the universal description of wide spectrum of physical phenomena.

This statement becomes evident remembering the microcanonical approach in thermodynamics. We would like to in this review paper that (i) the distinction between thermodynamics and $S$-matrix field theory consist in the choice of boundary conditions and (ii) the problem of thermal description is restricted by quantum uncertainty principle only.

So, we accommodate the economical thermodynamical description using the microcanonical approach to work with many-particles system. We will be seen that this and the canonical Gibbs descriptions (in the real-time formulation\(^{40}\)) are coincide iff the system is in equilibrium. The attempt to extend our thermal description on the quantum media (nonequilibrium as well) leads to the well known in statistics Bogolyubov condition on the particles correlation functions\(^{41}\). Just discussion of this condition allows to show in a what sense the $S$-matrix approach has the right classical limit.

One should draw attention also on the following factorization property natural for LSZ reduction formulae\(^{3.2}\). So, the generating functional may be written as the product:

$$\rho(\beta, z) = e^{-N(\beta, z; \phi)} \rho_0(\phi).$$

The differential over auxiliary field $\phi(x)$ operator $N(\beta, z; \phi)$ is the functional of ‘activity’ $z = z(q)$ (to ‘mark’ the momentum $q$ of external particles) and of temperature $T = 1/\beta$ (to define the mean value of the asymptotic states energy). The physical meaning of this quantities will be discussed.

\(^{22}\)This picture reminds the Coulomb problem, see footnote 7.
The functional $\rho_0(\phi)$ describes interaction of fields and is the vacuum-in-vacuum transition probability in presence of external (auxiliary) field $\phi(x)$. One can say that the operator $e^{-N(\beta,z;\phi)}$ maps the interacting fields system, described by $\rho_0(\phi)$, on the (external) states marked by given $\beta$ and $z$.

This factorization property will have important consequences since it assumes that external environment do not influence on the spectrum of quantum excitations. Just in frame of this assumption we would construct the perturbation theory.

So, at the very beginning we would show that

$S1$. Thermodynamics has the $S$-matrix interpretation.

The starting point of our calculations is following definition of $n$-into-$m$ particles transition amplitudes $a_{nm}$ in the momentum representation [11]:

$$a_{nm}(q';q) = \prod_{k=1}^{m} \hat{\phi}(q'_k) \prod_{k=1}^{n} \hat{\phi}^*(q_k)Z(\phi),$$

(3.2)

where $q'_k (q_k)$ are the incoming (outgoing) particles momenta. The energy-momentum conservation $\delta$-function must be extracted from $a_{nm}$. The ‘hat’ symbol means variation over corresponding quantity. For instance,

$$\hat{\phi}(q) \equiv \int dx e^{-iqx} \delta(\phi(x)) \equiv \int dx e^{-iqx} \hat{\phi}(x).$$

(3.3)

Note, $\hat{\phi}(q)$ acts as the annihilation operator of the incoming particle and $\hat{\phi}^*(q)$ as the creation one. The vacuum-in-vacuum transition amplitude in the background field $\phi$ is

$$Z(\phi) = \int Du e^{iS_0(u)-iV(u+\phi)},$$

(3.4)

where $S_0$ is the free part of action:

$$S_0(u) = \frac{1}{2} \int_{C_+} dx((\partial_\mu u)^2 - m^2 u^2)$$

(3.5)

and $V$ describes the interactions:

$$V(u) = \int_{C_+} dx v(u).$$

(3.6)

The time integrals in (3.4) are defined on Mills time contour [37]:

$$C_+: t \to t + i\varepsilon, \quad \varepsilon \to +0, \quad -\infty \leq t \leq +\infty.$$  (3.7)

This guaranties convergence of the path integral. At the end of calculations one must put the auxiliary field $\phi$ equal to zero.

Let us calculate now the probability

$$\tilde{\rho}_{nm}(P) = \frac{1}{n!m!} \int dw_m(q') dw_n(q)$$

$$\times \delta(P - \sum_{k=1}^{m} q'_k) \delta(P - \sum_{k=1}^{n} q_k)|a_{nm}|^2,$$

(3.8)

where

$$dw_m(q) = \prod_{k=1}^{m} \frac{dq_k}{(2\pi)^3 2\epsilon(q_k)}, \quad \epsilon(q) = (q^2 + m^2)^{1/2}$$

13
is the Lorentz-invariant phase-space element. Note, introduction of ‘probability’ $\tilde{\rho}$ leads to the doubling of degrees of freedom.

Inserting (3.2) into (3.8) we find:

$$\tilde{\rho}_{nm}(P) = (-1)^{n+m} \frac{1}{n!m!} N_m(P, \hat{\phi}) N_n^*(P, \hat{\phi}) \rho_0(\phi),$$

where

$$N_m(P, \hat{\phi}) = \int d\omega_m(q') \delta(P - \sum_{k=1}^m q'_k)$$

$$\times \int \prod_{k=1}^m dx'_k dy'_k e^{-i(q'_k - y'_k)\hat{\phi}_-(y'_k)\hat{\phi}_+(x'_k)}$$

and

$$\rho_0(\phi) = Z(\phi_+) Z^*(-\phi_-).$$

Introducing new coordinates:

$$x_k = R_k + r_k/2, \quad y_k = R_k - r_k/2,$$

$$x'_k = R'_k + r'_k/2, \quad y'_k = R'_k - r'_k/2$$

we come naturally from (3.3) to definition of the Wigner functions [13]:

$$\tilde{\rho}_{nm}(P) = \frac{1}{n!m!} \int d\omega_m(q') d\omega_n(q)$$

$$\times \delta(P - \sum_{k=1}^m q'_k) \delta(P - \sum_{k=1}^n q_k)$$

$$\times \int \prod_{k=1}^n dR_k \prod_{k=1}^m dR'_k W_{mn}(q, R; q', R'),$$

where

$$W_{nm}(q, R; q', R') = (-1)^{n+m} \prod_{k=1}^m N_+(q'_k, R_k; \hat{\phi})$$

$$\times \prod_{k=1}^m N_-(q'_k, R'_k; \hat{\phi}) \rho_0(\phi),$$

and

$$N_\pm(R, q; \phi) \equiv \int d\rho e^{i\rho r} \hat{\phi}_\pm(R + r/2) \hat{\phi}_-^*(R - r/2)$$

will be considered as the particles number operator, local in phase space $(R, q)$.

The Wigner function has formal meaning in quantum case (it is not positively definite), but in classical limit it has the meaning of ordinary in statistics phase-space distribution function. It obey the Liouville equation [13] conserving the phase space volume. We will use this function searching the classical limit of our $S$-matrix formalism.
We would consider $d\Gamma_n = |a_{mn}|d\omega_n(q)$ as the differential measure of final state and $d\Gamma_m = |a_{mn}|d\omega_m(q')$ of initial one. Two $\delta$-functions of energy-momentum shells was introduced separately in (3.8) to distinguish the initial and final states. The Fourier transform of this $\delta$-functions in (3.13) gives new quantity:

$$\tilde{\rho}_{mn}(P) = \int \frac{d\alpha_i}{(2\pi)^4} \frac{d\alpha_f}{(2\pi)^4} e^{iP(\alpha_i + \alpha_f)} \rho_{mn}(\alpha)$$

(3.16)

where

$$\rho_{mn}(\alpha) = \frac{1}{n!m!} \int d\omega_m(q')d\omega_n(q)$$

$$\times \prod_{k=1}^{m} dR'_k e^{-i\alpha_{q'_k}} \prod_{k=1}^{n} dR_k e^{-i\alpha_{q_k}} W_{mn}(q, R; q', R').$$

(3.17)

Inserting (3.14) into (3.17) we find:

$$\rho_{mn}(\alpha) = (-1)^{n+m} \frac{1}{n!m!} N_{n}^m(\alpha_i; \hat{\phi}) N_{m}^n(\alpha_f; \hat{\phi}) \rho_0(\phi),$$

(3.18)

where

$$N_{\pm}(\alpha; \hat{\phi}) \equiv \int dRd\omega_1(q)e^{-i\alpha q}$$

$$\times \int dr e^{iqr}\hat{\phi}_{\pm}(R+r/2)\hat{\phi}_{\mp}(R-r/2)$$

$$\equiv \int dRd\omega_1(q)e^{-i\alpha q} N_{\pm}(R, q; \hat{\phi}).$$

(3.19)

It is natural to introduce the generating functional weighing the operator $N_{\pm}(R, q; \hat{\phi})$ by the arbitrary ‘good’ function $z(R, q)$:

$$N_{\pm}(\alpha, z; \hat{\phi}) \equiv \int dRd\omega_1(q)e^{-i\alpha q} z(R, q) N_{\pm}(R, q; \hat{\phi}).$$

(3.20)

In result, summation over all $n, m$ gives the generating functional (3.1):

$$\rho(\alpha, z) = e^{-N_+(\alpha_i,z;\hat{\phi})-N_-(\alpha_f,z;\hat{\phi})} \rho_0(\phi)$$

$$\equiv e^{-N(\alpha,z;\hat{\phi})} \rho_0(\phi).$$

(3.21)

It is the generating functional of Wigner functions in the temperature representation [12]. At the same time, if $z = z(q)$ then (3.21) is the Fourier transform of expectation values (2.1) generating functional [13]. Indeed, in this case

$$N_{\pm}(\alpha, z; \hat{\phi}) \equiv \int d\omega_1(q)e^{-i\alpha q} z(q) \hat{\phi}_{\pm}(q)\hat{\phi}_{\mp}^{*}(q).$$

(3.22)

This representation is suitable for quantum case.

Let us consider the ‘spectral representation’ (3.22). It leads to the following generating functional ($z = 1$ is chosen for simplicity):

$$\rho(\alpha) = e^{i\int dxdx'\hat{\phi}_+(x)G_+(x-x',\alpha_2)\hat{\phi}_-(x')}$$

$$\times e^{-i\int dxdx'\hat{\phi}_-(x)G_-(x-x',\alpha_1)\hat{\phi}_+(x')} Z(\phi_+)Z^{*}(\phi_-),$$

(3.23)
where \( G_{+-} \) and \( G_{-+} \) are the positive and negative frequency correlation functions:

\[
G_{+-}(x - x', \alpha) = -i \int d\omega(q) e^{iq(x - x' - \alpha)} \tag{3.24}
\]

describes the process of particles creation at the time moment \( x_0 \) and its absorption at \( x_0' \), \( x_0 > x_0' \), and \( \alpha \) is the center of mass (CM) 4-coordinate. The function

\[
G_{-+}(x - x', \alpha) = i \int d\omega(q) e^{-iq(x - x' + \alpha)} \tag{3.25}
\]

describes the opposite process, \( x_0 < x_0' \). These functions obey the homogeneous equations:

\[
(\partial^2 + m^2) G_{+-} = (\partial^2 + m^2) G_{-+} = 0 \tag{3.26}
\]

since the ‘propagation’ of mass-shell particles is described.

Let us suppose that \( Z(\phi) \) may be computed perturbatively. Following transformation, suitable for the arbitrary nonsingular at origin functional, would be useful:

\[
e^{-iV(\phi)} = e^{-i \int dx j(x) \phi'(x) e^{i \int dx j(x) \phi(x)} e^{-iV(\phi')} = e^{\int dx \phi(x) \phi'(x) e^{-iV(\phi')} = e^{-iV(-\hat{j}) e^{i \int dx j(x) \phi(x)}, \tag{3.27}}
\]

where \( \hat{j} \) and \( \hat{\phi} \) are variational derivatives over corresponding quantities. At the end of calculations the auxiliary variables \( j, \phi' \) should be taken equal to zero.

Using first equality in (3.27) we find that

\[
Z(\hat{\phi}) = e^{-i \int dx \hat{j}(x) \hat{\phi'}(x) e^{i \int dx j(x) \phi(x)} e^{-iV(\phi+)} \times e^{\int dx dx' j(x) G_{++}(x-x') j(x'), \tag{3.28}}
\]

where \( G_{++} \) is the causal Green function:

\[
(\partial^2 + m^2) x G_{++}(x - y) = \delta(x - y). \tag{3.29}
\]

Inserting (3.28) into (3.23) after simple manipulations with differential operators, see (3.27), we find the expression:

\[
\rho(\alpha) = e^{-iV(-\hat{j}+)+iV(-\hat{j} -)} \times e^{\int dx dx' j_i(x) G_{ik}(x-x') j_k(x'), \tag{3.30}}
\]

where

\[
G_{11}(x - x') = -G_{++}(x - x'),
G_{12}(x - x') = G_{+-}(x - x', \alpha_1),
G_{21}(x - x') = -G_{-+}(x - x', \alpha_2),
G_{22} = G_{--} = (G_{++})^\ast, \tag{3.31}
\]

\( G_{--} \) is the anticausal Green function.
The structure of generating functional (3.30) is the same as in the real-time finite-temperature field theories, e.g. [43]. The difference is only in definition of Green functions.

The Green functions \( G_{ij} \) were defined on the time contours \( C_{\pm} \) in the complex time plane (\( C_- = C_+^* \)). This definition of the time contours coincide with Keldysh’ time contour [44]. The expression (3.30) was written in the compact matrix form [45].

Note, the doubling of degrees of freedom is \textit{unavoidable} since Green functions \( G_{ij} \) are singular on the light cone. But it will be seen below that one can shift the time contour on the real axis if the perturbation theory is constructed in the invariant subspaces.

Considering the system with large number of particles we can simplify calculations choosing the CM frame \( P = (P_0 = E, \vec{0}) \). It is useful also [46] to rotate the contours of integration over \( \alpha_0 \): \( \alpha_0 = -i\beta, \Im \beta = 0 \). In result, omitting unnecessary constant, we will consider \( \rho = \rho(\beta, z) \). Note, \( \beta \) is conjugate to particles energy, i.e. \( 1/\beta \) has the meaning of temperature.

So, we construct the two-temperature theory (for initial and final states separately). In such theory with two temperatures the Kubo-Martin-Schwinger (KMS) [47, 48] periodic boundary conditions applicability is not evident. Note, KMS boundary condition play the crucial role in Gibbs thermodynamics since the temperature in it is introduced just by this condition, e.g. [43], see (3.42).

Let us consider the dynamical origin of KMS condition, see also [11]. By definition, the path integrals

\[
\rho_0(\phi_{\pm}) = \int Du_+ D u_- e^{iS_0(u_+)} - iS_0(u_-) \times e^{-iV(u_+ + \phi_+)} + iV(u_- - \phi_-),
\]

(3.32)

should describe the \textit{closed path motion} in the space of fields \( u \). The equality:

\[
\int_{\sigma_{\infty}} d\sigma \mu u_+ \partial^\mu u_+ = \int_{\sigma_{\infty}} d\sigma \mu u_- \partial^\mu u_-.
\]

(3.33)

takes into account this boundary condition. So, \( \rho(\beta, z) \) is defined on the periodic (in the \( u \) space) trajectories by definition.

Mostly general solution of eq.(3.33) means that the fields \( u_+ \) and \( u_- \) (and theirs first derivatives \( \partial_\mu u_{\pm} \)) must coincide on the boundary hypersurface \( \sigma_{\infty} \):

\[
u_\pm(\sigma_{\infty}) = u(\sigma_{\infty}),
\]

(3.34)

where, by definition, \( u(\sigma_{\infty}) \) is an arbitrary, ‘turning-point’, field. The value of \( u(\sigma_{\infty}) \) specify the environment of the system. This boundary condition guaranties absence of surface terms up to ‘non-integrable’ term [3]. Last one can arise if the topology of interacting fields is nontrivial (see e.g. Sec.5).

The simplest (minimal) choice of \( u(\sigma_{\infty}) \neq 0 \) assumes that the system under consideration is surrounded by black-body radiation. One should underline also that this choice of boundary condition is not unique: one can consider another organization of the environment, e.g. the external flow can be consist of the correlated particles as it happens in the heavy ion collisions (the nucleons in ion may be considered as the quasi-free particles).

Let us suppose that on the infinitely far hypersurface \( \sigma_{\infty} \) there are only free, mass-shell, particles. This assumption is natural in the \( S \)-matrix framework [13]. In this paper we will assume also that there are not any special correlations among background particles.

In this framework our derivation is the same as in [49]. By this reason we restrict ourselves mentioning only the main quantitative points.

Calculating the product \( a_{n,m} a_{n,m}^* \) we describe a process of particles creation and theirs further adsorption. In the vacuum case two \textit{time ordered process} of particles creation and absorption were taken into account. In presence of
the background particles this time-ordered picture is slurring over since the possibility to absorb particles before its creation appears. Taking new possibilities into account,

\[ \rho_{\alpha\beta} = e^{iN(\beta, z; \hat{\phi})} \rho_0(\hat{\phi}), \]  

(3.35)

where \( \rho_0(\hat{\phi}) \) is the same generating functional, see (3.32).

The operator \( N(\beta, z; \hat{\phi}), i, j = +, -, \) describes the external particles environment. It can be expanded over the activity operator \( \hat{\phi}_i^*(q) \hat{\phi}_j(q) \). We can leave only the first nontrivial term:

\[ N(\beta, z; \hat{\phi}) = \int d\omega_1(q) \hat{\phi}_i^*(q) n_{ij}(\beta, z) \hat{\phi}_j(q), \]  

(3.36)

since no special correlation among background particles should be expected. Following to our interpretation of \( \hat{\phi}_i^* \hat{\phi}_j \) we conclude that \( n_{ij} \) is the mean multiplicity (occupation number) of background particles. In (3.36) the normalization condition:

\[ N(0) = 0 \]  

(3.37)

was used and summation over all \( i, j \) was assumed. In the vacuum case only the combinations \( i \neq j \) are present.

Having background particles flow it is important to note that to each vertex of in-going in \( a_{nm} \) particle we must adjust the factor \( e^{-i \alpha_1 q/2} \) and for each out-going particle we have correspondingly \( e^{-i \alpha_2 q/2} \), see (3.17).

So, the product \( e^{-i \alpha_1 q/2} e^{-i \alpha_2 q/2} \) can be interpreted as the probability factor of the one-particle (creation + annihilation) process. The \( n \)-particles (creation + annihilation) process’ probability is the simple product of these factors if there is not the special correlations among background particles. This interpretation is evident in the CM frame \( \alpha_k = (-i \beta_k, 0) \). After this preliminaries it is not too hard to find \( \bar{n}_{ij} \) (see Appendix A). Corresponding generating functional has the standard form:

\[ \rho_{\beta}(j_\pm) = \exp\{ -iV(-i \hat{j}_+^\dagger) + iV(-i \hat{j}_-) \} \times \]  

\[ \times \exp \left\{ \frac{i}{2} \int dx dx' j_i(x) G_{ij}(x - x', \beta) j_j(x') \right\} \]  

(3.38)

where the summation over repeated indexes is assumed.

Inserting (3.38) in the equation of state (3.43) we can find that \( \beta_1 = \beta_2 = \beta(E) \). If \( \beta(E) \) is a ‘good’ parameter then \( G_{ij}(x - x'; \beta) \) coincide with the Green functions of the real-time finite-temperature field theory and the KMS boundary condition:

\[ G_{+-}(t - t') = G_{-+}(t - t' - i\beta), \]  

\[ G_{-+}(t - t') = G_{+-}(t - t' + i\beta), \]  

(3.39)

is restored. The eq.(3.39) deduced from (3.38) by direct calculations. It is known that the KMS boundary condition without fail leads to the equilibrium fluctuation-dissipation conditions [70] (see also [71]).

The energy and momentum in our approach are still locally conserved quantities since an amplitude \( a_{nm} \) is translational invariant. So, we can perform evident in the S-matrix theory transformation:

\[ \alpha_1 \sum q_k = (\alpha_1 - \alpha_2) \sum q_k + \sigma_1 \sum q_k \rightarrow \]  

\[ (\alpha_1 - \sigma_1) \sum q_k + \sigma_1 P \]  

(3.40)

since 4-momenta are conserved. The choice of \( \sigma_1 \) defines the reference frame. This degree of freedom of the theory was considered in [12] [49]. It gives the rule as the time contour can be shifted.
Using this rule it is not hard to present $\rho(\beta)$ in the form of one path integral defined on the ‘closed time-path’ contour $C$:

$$
\rho(\beta) = \int D\omega e^{iS_C(u)},
$$

where the action $S_C$ is defined on the Mills time contour $C$. It starts at time $t_i$ goes to right, at $t = t_f$ it turns and end at $t_i - i\beta$. The temperature is introduced through the KMS boundary condition

$$
u(t_i) = u(t_i - i\beta).
$$

Note, one can find (3.38) from (3.41) considering $t_i \to -\infty$ and $t_f \to +\infty$ iff the interactions are disappeared on $\sigma_\infty$. In the ordinary perturbation theory this condition is hold. But the symmetries (hidden as well) may survive the ‘interactions’.

The contour $C$ unavoidably contains both along the real and imaginary axis parts. It is not clear by this reason how the nontrivial topology requirements can be applied for such time contour. In contrast with it the representation (3.38) is free from the imaginary parts of time contour.

- The temperature was introduced as the parameter conjugate to created particles energies. By this reason the uncertainty principle restrictions should be taken into account. We would like to show that

**S2. The thermal $S$-matrix description can be used for infrared stable field theory.**

Considering the Fourier-transformed probability $\rho(\alpha, z)$ as the observable quantity the phase-space boundaries are not fixed exactly, i.e. the 4-vector $P$ can be defined with some accuracy only if $\alpha_i$ are fixed, and vice versa. It is the ordinary quantum uncertainty condition. In the particles physics namely the 4-vector $P$ is defined by experiment. Let us find the condition when both $P$ and $\beta$ may be the well defined quantities, i.e. may be used for description of $\rho(\alpha, z)$. This is necessary if we want to use the temperature formalism in particles physics also.

Note, in statistical physics such formulation of problem has no meaning since the interaction with thermostat is assumed. In result of this interaction the energy of system is not conserved, i.e. the systems word line belong to the energy surface.

The stationary phase condition for integrals (3.16) gives the equations of (final and initial) states familiar for microcanonical description. We will chose the CM frame $P = (E, \vec{0})$ when $\alpha_{i(f)} = (-i\beta_{i(f)}, \vec{0})$. We can interpret $1/\beta_{i(f)}$ as the temperature of initial (final) state. The corresponding equations of state have the form:

$$
E = -\frac{\partial}{\partial \beta_{i(f)}} \ln \rho(\beta, z).
$$

But one can not define $\dot{\rho}(E)$ correctly even knowing the solutions $\beta_{i(f)}(E)$ of eqs.(3.43) if $\beta_{i(f)}(E)$ are not a ‘good’ parameters, i.e. iff the fluctuations in a vicinity of them are not Gaussian. This condition would be considered as the definition of equilibrium.

Indeed, to calculate the integrals over $\beta_{i(f)}$ the expansion near solutions $\beta_{i(f)}(E)$ of eqs.(3.43) should be examined. This leads to asymptotic series with coefficients

$$
\sim \int \prod_{i} d\omega_1(q_i) \ll \varepsilon(q_1), \varepsilon(q_2), ..., \varepsilon(q_s) \gg = \int D_s(q_1, q_2, ..., q_s),
$$

\footnote{It can be thin if the interaction with thermostat is weak.}
since \( \ln \rho(\beta, z) \) is the essentially nonlinear function of \( \beta \). So, the fluctuations near \( \beta_{(f)}(E) \) are defined by the value of s-particle inclusive energy spectra \( <\varepsilon(q_1),\varepsilon(q_2),\ldots,\varepsilon(q_s)> \) familiar in particles physics. The analysis shows that it is enough to have the factorization \( \int \prod_{i=1}^{s} d\omega(q_i) <\varepsilon(q_1),\varepsilon(q_2),\ldots,\varepsilon(q_s)> \sim \prod_{i=1}^{s} \int d\omega(q_i) <\varepsilon(q_i)> \) (3.45) for correct estimation of this asymptotic series. It must be underlined that this is the unique solution of the thermal descriptions problem.

Discussed factorization is the well known Bogolyubov’s condition for ‘truncated’ description of nonequilibrium media \[11\], when s- particle distribution functions \( D_s \), \( s > 1 \), is expressed through \( D_1 \) through the relation (3.45).

Considering the general problem of particles creation it is hard to expect that the constant \( \beta_{(f)}(E) \) is a ‘good’ parameter, i.e. that the factorization conditions (3.45) are hold. Nevertheless there is a possibility to have the above factorization property in the restricted space-time domains of size \( L \). It is the so called ‘kinetic’ phase of the process when the memory of initial state was disappeared, the ‘fast’ fluctuations was averaged over and we can consider the long- range ‘slow’ fluctuations only.

In this ‘kinetic’ phase one can use the ‘local equilibrium’ hypothesis in frame of which \( \beta_{(f)}(E) \rightarrow \beta_{(f)}(R, E) \), where \( R \in L_c \) and \( L_c \) is the dimension of the measurement cell\[24\]. It is natural to take \( L_c << L \), (3.46) where \( L \) is the characteristic thermal fluctuations dimension. It is assumed that \( \beta_{(f)}(R, E) = \text{const.} \) if \( R \in L_c \). In the equilibrium \( L \rightarrow \infty \).

By definition, \( 1/\beta_{(f)}(R, E) = \bar{\varepsilon}(R) \) is the mean energy of particles in the cell with dimension \( L_c \). The fluctuations in \( \bar{\varepsilon}(R) \) vicinity should be Gaussian.

The quantum uncertainty principle dictates also the condition \[13\]:

\[ L_c >> L_q, \]  

(3.47)

where \( L_q \) is the characteristic scale of quantum fluctuations (\( L_q \sim 1/m \) for massive theories).

The ‘infrared unstable’ situation means that

\[ L_q >> L. \]  

(3.48)

One should underline that \( L \) defines the scale of thermodynamics fluctuations and, by this reason, the inequality (3.48) points to (unphysical) instability in the infrared domain.

So, if conditions (3.46, 3.47) are hold we can use the Wigner functions to describe the phase-space distributions, i.e. the formalism has right classical limit in this case.

To introduce the scales \( L, L_c \) into formalism we can divide the \( R 4 \)-space on the cells of \( L_c \) dimension \[11\]:

\[ \int dR = \sum_r \int_{L_c} dR, \]  

(3.49)

where \( r \) can be considered as the cells 4-coordinate. Assuming that the inequality (3.46) hold we can assume that \( \beta = \beta(r), \quad z = z(q, r) \), are the constants at least on the \( L_c \) scale. With this definitions

\[ \hat{N}_{\pm}(\beta, z; \phi) \equiv \sum_r \int d\omega_1(q) e^{-\beta(r)(\varepsilon(q)+\mu(r,q))} \]

\[24\text{Note, we always can divide the external particles measuring device on cells since the in free state are measured. Such description of nonstationary media seems favorable in comparison with traditional one, e.g. [53].} \]
\[ \times \int_{L_c} dR \tilde{N}_\pm(R, q; \phi) , \]  
(3.50)

where

\[ \mu(r, q) \equiv \frac{1}{\beta(r)} \ln z(r, q) \]

is the local chemical potential.

**bullet** On the more early pre-kinetic stages no thermodynamical shortened description can be applied and the pure quantum description (in terms of momenta only) should be used. For this purpose one should expand \( \rho(\alpha, z) \) over operators \( N_\pm \) and the integrations over \( \alpha_i, \alpha_f \) gives ordinary energy-momentum conservation \( \delta \)-functions, i.e. defines the system on the infinitely thin energy sheet.

## 4 Unitarity condition

Purpose of this section is to show how the \( S \)-matrix unitarity condition can be introduced into the path-integral formalism to find measure (2.4) [8]. We will start from the quantum-mechanical example to do the calculations more evident.

The unitarity condition for the \( S \)-matrix \( S S^+ = I \) presents the infinite set of nonlinear equalities:

\[ iAA^* = A - A^* , \]  
(4.1)

where \( A \) is the amplitude, \( S = I + iA \). Expressing the amplitude by the path integral one can see that the l.h.s. of this equality offers the double integral and, at the same time, the r.h.s. is the linear combination of integrals. Let us consider what this linearization of product \( AA^* \) gives.

Using the spectral representation of one-particle amplitude:

\[ A_1(x_1, x_2; E) = \sum_n \frac{\Psi_n^*(x_2)\Psi_n(x_1)}{E - E_n + i\varepsilon} , \quad \varepsilon \to +0 , \]  
(4.2)

let us calculate

\[ \rho_1(E) = \int dx_1 dx_2 A_1(x_1, x_2; E)A_1^*(x_1, x_2; E) . \]  
(4.3)

The integration over end points \( x_1 \) and \( x_2 \) is performed for sake of simplicity only. Using ortho-normalizability of the wave functions \( \Psi_n(x) \) we find that

\[ \rho_1(E) = \sum_n \frac{1}{E - E_n + i\varepsilon}^2 = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n) . \]  
(4.4)

Certainly, the last equality is nothing new but it is important to note that \( \rho_1(E) \equiv 0 \) for all \( E \neq E_n \), i.e. that all unnecessary contributions with \( E \neq E_n \) were canceled by difference in the r.h.s. of eq.(4.1). We will put this phenomena in the basis of the approach.

We will build the perturbation theory for \( R(E) \) using the path-integral definition of amplitudes [8]. It leads to loss of some information since the amplitudes can be restored in such formulation with the phase accuracy only. Yet, it is sufficient for calculation of the energy spectrum. We would consider this quantity to demonstrate following statement:

---

25 Note, in this definition the amplitude is dimensionless. In ordinary definitions the energy-momentum conservation \( \delta \)-functions are extracted from amplitudes.
S3. The unitarity condition unambiguously determines contributions in the path integrals for \( \rho_1(E) \).

This statement looks like a tautology since \( e^{iS(x)} \), where \( S(x) \) is the quantum-mechanical action, is the unitary operator which shifts a system along the trajectory.\(^{26}\) I.e. the unitarity is already included in the path integrals. But the general path-integral solution contains unnecessary degrees of freedom (unobservable states with \( E \neq E_n \) in our example). We would define the quantum measure \( DM \) in such a way that the condition of absence of unnecessary contributions in the final (measurable) result be loaded from the very beginning. Just in this sense the unitarity looks like the necessary and sufficient condition unambiguously determining the complete set of contributions. Solution is simple: one should find, as it follows from (4.4), the linear path-integral representation for \( \rho_1(E) \) to introduce this condition into the formalism.

Indeed, to see the integral form of our approach, let us use the proper-time representation:

\[
A_1(x_1, x_2; E) = \sum_n \Psi_n(x_1)\Psi_n^*(x_2) i \int_0^\infty dT e^{i(E-E_n+i\varepsilon)T} \tag{4.5}
\]

and insert it into (4.3):

\[
\rho_1(E) = \sum_n \int_0^\infty dT_+ dT_- e^{-(T_++T_-)\varepsilon} e^{i(E-E_n)(T_+-T_-)}. \tag{4.6}
\]

We will introduce new time variables instead of \( T_\pm \): \( T_\pm = T \pm \tau \), \( \tau \leq T \), \( 0 \leq T \leq \infty \). But we can put \( |\tau| \leq \infty \) since \( T \sim 1/\varepsilon \rightarrow \infty \) is essential in integral over \( T \). In result,

\[
\rho_1(E) = 2\pi \sum_n \int_0^\infty dT e^{-2\varepsilon T} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi} e^{2i(E-E_n)\tau}. \tag{4.8}
\]

In the last integral all contributions with \( E \neq E_n \) are canceled. Note that the product of amplitudes \( AA^* \) was 'linearized' after introduction of 'virtual' time \( \tau = (T_+ - T_-)/2 \). The physical meaning of such variables will be discussed, see also \[54\].

We will consider following path-integral:

\[
A_1(x_1, x_2; E) = i \int_0^\infty dTe^{iET} \int_{x_1}^{x_2=x(T)} \int_{x_1(0)}^{x(0)} Dxe^{iS_{C_+}(x)}, \tag{4.9}
\]

where \( C_\pm \) is the Mills complex time contour \[57\]. Calculating the probability to find a particle with energy \( E \) (\( Im \ E \) will not be mentioned for sake of simplicity) we have:

\[
\rho_1(E) = \int dx_1 dx_2 |A|^2 = \int_0^\infty \int_{x_+ (T_+)=x_-(T_-)}^{x_+(T_+)=x_-(T_+)} dT_+ dT_- e^{i(E(T_+-T_-))}

\times \int_{x_+(0)=x_-(0)}^{x_+(T_+)=x_-(T_-)} D_{C_+} x_+ D_{C_-} x_-

\times e^{iS_{C_+}(x_+)-iS_{C_-}(x_-)}. \tag{4.10}
\]

where \( C_-(T) = C_+^*(T) \). Note that the total action in (4.10) \( (S_{C_+}(T_+)(x_+)-S_{C_-}(T_-)(x_-)) \) describes the closed-path motion by definition.

\(^{26}\)It is well known that this unitary transformation is the analogy of tangent transformations of classical mechanics \[54\].
New independent time variables $T$ and $\tau$ will be used again, see \((4.7)\). We will introduce also the mean trajectory $x(t) = (x_+(t) + x_-(t))/2$ and the deviation $e(t)$ from it: $x_\pm(t) = x(t) \pm e(t)$. Note that one can do surely this linear transformations in the path integrals.

We will consider $e(t)$ and $\tau$ as the fluctuating, virtual, quantities and calculate the integrals over them perturbatively. In the zero order over $e$ and $\tau$, i.e. in the quasiclassical approximation, $x$ is the classical path and $T$ is the total time of classical motion.

The boundary conditions (see \((4.10)\)) states the closed-path motion and therefore we have the boundary conditions for $e(t)$ only:

\[
e(0) = e(T) = 0 \quad (4.11)
\]

Note the uniqueness of this solution if the integral over $\tau$ is calculated perturbatively.

Extracting the linear over $e$ and $\tau$ terms from the closed-path action $\left(S_{C_+(T^+)}(x_+) - S_{C_-(T^-)}(x_-)\right)$ and expanding over $e$ and $\tau$ the remainder terms:

\[
-\hat{H}_T(x; \tau) = \left(S_{C_+(T^+)}(x) - S_{C_-(T^-)}(x)\right) + 2\tau H_T(x), \quad (4.12)
\]

where $H_T(x)$ is the Hamiltonian at the time moment $T$, and

\[
-U_T(x, e) = \left(S_{C_+(T)}(x + e) - S_{C_-(T)}(x - e)\right) + 2\Re \int_{C_+(T)} dt (\ddot{x} + v'(x)) e
\]

we find that

\[
\rho_1(E) = 2\pi \int_0^\infty dT e^{-i\hat{K}(\omega, \tau; j, e)} \\
\times \int DM(x) e^{-i\hat{H}_T(x; \tau) - iU_T(x, e)}. \quad (4.14)
\]

Note the necessity of boundary condition \((4.11)\) to find \((4.14)\). It allows to split the expansions over $\tau$ and $e$.

The expansion over differential operators:

\[
\hat{K}(\omega, \tau; j, e) = \frac{1}{2} \left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial \tau} + \Re \int_{C_+(T)} dt \frac{\delta}{\delta j(t)} \frac{\delta}{\delta e(t)} \right)
\]

will generate the perturbation series. We will assume that it exist at least in Borel sense.

In \((4.14)\) the functional measure

\[
DM(x) = \delta(E + \omega - H_T(x)) \prod_t dx(t) \delta(\ddot{x} + v'(x) - j)
\]

unambiguously defines the complete set of contributions in the path integral. The functional $\delta$-function is defined as follows:

\[
\prod_t \delta(\ddot{x} + v'(x) - j) = (2\pi)^2 \int_{e(0)}^{e(T)} \prod_t \frac{de(t)}{\pi} e^{-2\Re \int_{C_+} dt e(\ddot{x} + v'(x) - j)}
\]

\[
(4.17)
\]

The physical meaning of this $\delta$-function is following. We can consider $(\ddot{x} + v'(x) - j)$ as the total force and $e(t)$ as the virtual deviation from true trajectory $x(t)$. In classical mechanics the virtual work must be equal to zero:
\( (\dot{x} + v'(x) - j)e(t) = 0 \) (d’Alembert) since the motion is time reversible. From this evident dynamical principle one can find the ‘classical’ equation of motion:

\[
\ddot{x} + v'(x) = j,
\]

(4.18) since \( e(t) \) is arbitrary.

In quantum theories the virtual work usually is not equal to zero, i.e. the quantum motion is not time reversible since the quantum corrections can shift the energy levels. But integration over \( e(t) \), with boundary conditions (1.11), leads to the same result. So, in quantum theories the unitarity condition [54] play the same role as the d’Alembert’s variational principle in classical mechanics. We can conclude, the unitarity condition as the dynamical principle establishes the time – local equilibrium between classical (r.h.s. of (4.18)) and quantum (l.h.s. of (4.18)) forces.

• We would like to show that

\section*{S4. The functional measure of \( \rho(\beta, z) \) is \( \delta \)-like.}

Let us consider now the integrals

\[
\rho_0(\hat{\phi}) = \int Du_+ Du_- e^{iS_0(u_+) - iU(u_+ + \phi_+)} \times e^{-iS_0(u_-) + iV(u_- - \phi_-)},
\]

(4.19)

where \( S_0, V \) were defined in (7.4), (7.5) and \((u_-, \phi_-)\) are defined on the complex conjugate time contour \( C_- \). \textit{Note}, the fields \( \phi_\pm \) carry all external information and this integrals should be calculated with ‘closed-path’ boundary condition, see (3.33).

Instead of two independent fields \( u_+ \) and \( u_- \) we will use new ones [8]:

\[
u(x)_\pm = u(x) \pm \varphi(x)
\]

(4.20)

with ‘closed-path’ boundary condition:

\[
\int_{\sigma_\infty} dx \varphi(x) \partial^\mu u(x) = 0,
\]

(4.21)

where \( \sigma_\infty \) is the remote hypersurface. We will choose following solution of (4.21):

\[
\varphi(x)|_{x \in \sigma_\infty} = 0
\]

(4.22)

With this boundary condition the total action \((S_0(u_+) - V(u_+)) - S_0(u_-) + V(u_-)\) describes the closed-path motion with turning-point field \( u(x)|_{x \in \sigma_\infty} \). The integration over it is assumed. The physical meaning of this ‘minimal’ boundary condition in the \( S \)-matrix approach was described in Sec.3 [11].

We will consider \( \varphi \) as a pure quantum field expanding (4.11) over them. Introducing the auxiliary field \( \phi(x, t) \):

\[
\phi(x, t \in C_\pm) = \hat{\phi}_\pm(x, t \in C_\pm)
\]

and introducing the variational derivative by equality:

\[
\frac{\delta \phi(x, t \in C_\pm)}{\delta \phi(x', t' \in C_j)} = \delta_{ij} \delta(x - x') \delta(t - t'), \quad i, j = +, -.
\]

we can write \((\text{Im} \alpha_{i(f)} < 0)\):

\[
N(\alpha, z; \hat{\phi})_\pm = \int d\omega(q) \int_{C_+} dx \int_{C_-} dy \hat{\phi}(x) \hat{\phi}(y) \times e^{-ip\omega(\phi)} z_{i(f)} e^{i\alpha_{i(f)}(x-y)}.
\]

(4.23)
Note, introducing the Wigner coordinates, see (3.12), \( R = (x + y)/2 \) and \( r = x - y \) we find taking into account (4.23) that \( 3R = 0 \) and \( 3r \sim \varepsilon \) since \( C_{-} = C_{+}^{*} \). Therefore, \( \alpha_{(f)}(R) \) and \( z_{(f)}(q, R) \) stay real on the complex time contours.

Using this notations let us extract in the exponents (4.19) the linear over \((\phi + \varphi)\) term:

\[
V(u + (\phi + \varphi)) - V(u - (\phi + \varphi)) = U(u, \phi + \varphi)
+ 2R \int_{C_{+}} dx(\phi(x) + \varphi(x))v'(u),
\]

(4.24)

and

\[
S_{0}(u + \varphi) - S_{0}(u - \varphi) = s_{0}(u)
- 2iR \int_{C_{+}} dx \varphi(x)(\partial^{2}_{\mu} + m^{2})u(x).
\]

(4.25)

where

\[
2R \int_{C_{+}} = \int_{C_{+}} + \int_{C_{-}}.
\]

Note, generally speaking, \( s_{0}(u) \neq 0 \) if the topology of field \( u(x) \) is nontrivial. The reason of this phenomena was demonstrated in [16] on the quantum-mechanical examples. It has the meaning of ‘nonintegrable’ term.

The expansion over \((\phi + \varphi)\) can be written in the form, see (3.27):

\[
e^{-iU(u,\phi+\varphi)} = e^{\hat{\phi}Re \int_{C_{+}} dx j(x) \hat{\varphi}'(x)}
\times e^{i2R \int_{C_{+}} dx d\phi(x) \hat{\phi}(x) (\partial^{2}_{\mu} + m^{2})u(x)} e^{-iU(u,\varphi')},
\]

(4.26)

where \( j(x), \hat{\varphi}'(x) \) are the variational derivatives. The auxiliary variables \((j, \varphi')\) must be taken equal to zero at the very end of calculations.

In result,

\[
\rho_{0}(\phi) = e^{\hat{\phi}Re \int_{C_{+}} dx j(x) \hat{\varphi}(x)} \int DM(u) e^{-is_{0}(u) - iU(u,\varphi)}
\times e^{i2R \int_{C_{+}} dx (j(x) - v'(u))\hat{\phi}(x)}
\times \prod_{x} \delta(\partial^{2}_{\mu}u + m^{2}u + v'(u) - j),
\]

(4.27)

where the functional \( \delta \)-function was defined by the equality:

\[
\prod_{x} \delta(\partial^{2}_{\mu}u + m^{2}u + v'(u) - j) = \int D\varphi e^{-2iR \int_{C_{+}} dx (\partial^{2}_{\mu}u + m^{2}u + v'(u) - j)\varphi(x)}
\]

(4.28)

The eq.(4.27) can be rewritten in the equivalent form:

\[
\rho_{0}(\phi) = e^{-iK(j,\varphi)} \int DM(u) e^{-is_{0}(u) - iU(u,\varphi)}
\times e^{i2Re \int_{C_{+}} dx \phi(x)(\partial^{2}_{\mu} + m^{2})u(x)}
\]

(4.29)
because of the \( \delta \)-functional measure

\[
DM(u) = \prod_x du(x) \delta(\partial^2 u + m^2 u + v'(u) - j),
\]

(4.30)

with

\[
\hat{K}(j, \varphi) = \frac{1}{2} \Re \int_{C_+} dx \hat{j}(x) \hat{\varphi}(x).
\]

(4.31)

Not at the end that the contour \( C_+ \) in (4.23) and (4.31) can not be shifted on the real time axis.

- It is easy to show now having definition (4.30) that

\[
S_5. \text{The expectation values } \rho_{nm} \text{ has the (2.1) form iff the field theory is infrared stable.}
\]

The action of operator \( N(\beta, z; \hat{\phi}) \) maps the interacting fields system on the physical states. Last ones are ‘marked’ by \( z_i(f) \) and \( \beta_i(f) \). The operator exponent is the linear functional over \( \phi \) and this allows easily find the result of

\[
\rho(\beta, z) = e^{-i\hat{K}} \int DM(u)e^{iA(u) - iU(u;\varphi)}e^{N(\beta, z; u)},
\]

(4.32)

where

\[
N(\beta, z; u) = n(\beta_i, z_i; u) + n^*(\beta_f, z_f; u)
\]

and

\[
n(\beta, z; \phi) = \sum_r \int d\omega(q) dk \delta_{L_c}(k)e^{-\beta(r)(\varepsilon(q) + \mu(q,r))} \times \Gamma(q + k, \phi) \Gamma^*(q - k, \phi).
\]

(4.33)

if the thermodynamical parameters \( \beta \) and \( z \) are local quantities. In (4.33)

\[
\delta_{L_c}(k) = \int_{L_c} \frac{dR}{(2\pi)^4} e^{ikR}
\]

was introduced (for 4-dimensional theory). Here \( L_c \) is the space-time dimension where \( \beta_i(f)(R) \) and \( z_i(f)(q, R) \) can be considered as the constants. If \( L_c << L \) then \( \delta_{L_c}(k) \) can be replaced on the usual \( \delta \)-function \( \delta(k) \) and, therefore, in this limit:

\[
N(\beta, z; u) = \int d\omega(q) dr \sigma(r)e^{-\beta(r)(\varepsilon(q) + \mu(q,r))} |\Gamma(q, u)|^2
\]

(4.34)

Integration over \( r \) means summation over cell coordinates, where the factor \( \sigma(r) \) is the measure of this replacement. The translational invariance gives \( \sigma(r) = 1 \). In this expression

\[
\Gamma(q, u) = \int_{C_+} dx e^{iqx}(\partial^2 + m^2)u(x)
\]

(4.35)

is the function of external particles momentum \( q \) only.

The considered limit is hold if the theory is ‘infrared stable’. In opposite case the eq.(4.33) must be used because of arbitrary-range quantum fluctuations. Note, in this case the quantum description is not hold also. This is typical unphysical instability which may arise if the ground state is unstable.
Note that
\[ \sum_{nm} \int dP \tilde{\rho}_{nm}(P) = \Delta A_0, \]  
(4.36)
where \( \tilde{\rho}_{nm}(P) \) was defined in (3.8), is the absorption part of vacuum-to-vacuum amplitude \( A_0 = \langle \text{vac}|\text{vac} \rangle \). We found in previous section the functional measure for \( \Delta A_0 \) using the unitarity condition. It was enough to know this quantity to reconstruct the real-time finite-temperature field theory which is the analytic continuation of the ordinary in statistics Matsubara approach [55]. It was shown also the way as our microcanonical approach can be used for nonequilibrium media description and the condition when our formalism has the right classical limit. For this purpose the Wigner functions was used.

At the end, without evident calculations, we would like to note that

S6. The absorption part of the elastic amplitude \( A_2(q_1, q_2; q_1', q_2') \) is defined by the expression:
\[
\Delta A_2(q_1, q_2; q_1', q_2'; \alpha_f, z_f) = \prod_{i=1}^{2} \hat{\phi}(q_i)\hat{\phi}^*(q'_i)e^{-N(\alpha_f,z_f;\hat{\phi})e^{-i\hat{K}(j,\varphi)}}
\]
\[
\times \int DM(u)e^{i\phi(\alpha) - iU(\alpha,\varphi)} e^{2i\mathbb{R}\int_{C^+} dx \phi(x) (\partial_x^2 + m^2) u(x)},
\]  
(4.37)

We leave in this expression the \((\alpha, z)\) dependence to show as the observables can be described.

This formula is important since having \( \Delta A_2 \) and using dispersion relations one can find the total amplitude \( A_2(q_1, q_2; q_1', q_2') \) which is the main quantity in particles physics.

5  Perturbation theory

Now let us consider representation (4.14). It is not hard to show that

S7. Eq. (4.14) restores the perturbation theory of stationary phase method.

For this purpose it is enough to consider the ordinary integral:
\[
A(a, b) = \int_{-\infty}^{+\infty} dx \frac{e^{i(ax^2 + bx^3)}}{(2\pi)^{1/2}},
\]  
(5.1)
with \( \Im a \to +0 \) and \( b > 0 \). Computing the ‘probability’ \( \rho = |A|^2 \) we find:
\[
\rho(a, b) = e^{\frac{b}{a} i e} \int_{-\infty}^{+\infty} dx e^{-2(x^2 + e^2) \Im a} e^{2i(\hat{X} x^3)} x \delta(Re ax + bx^2 + j).
\]  
(5.2)

The ‘hat’ symbol means the derivative over corresponding quantity: \( \hat{X} \equiv \partial/\partial X \). One should put the auxiliary variables \((j, e)\) equal to zero at the very end of calculations.

Performing the trivial transformation \( e \to ie, \hat{e} \to -i\hat{e} \) of auxiliary variable we find at the limit \( \Im a = 0 \) that the contribution of \( x = 0 \) extremum (minimum) gives expression:
\[
\rho(a, b) = \frac{1}{a} e^{\frac{b}{a} i e} (1 - 4bj/a^2)^{-1/2} e^{2i(\hat{X} x^3)}
\]  
(5.3)
and the expansion of operator exponent gives the asymptotic series:

\[ \rho(a, b) = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{(6n-1)!!}{n!} \left(\frac{2b^4}{3a^6}\right)^n. \] (5.4)

This series is convergent in Borel sense.

Eq.(5.2) can be considered as the definition of integral (5.1). By this reason one may put \( \Im a = 0 \) from the very beginning. We will use this property.

Let us calculate now \( \rho \) using the stationary phase method. Contribution from the minimum \( x = 0 \) gives (\( \Im a = 0 \)):

\[ A(a, b) = e^{-ij\hat{x} - \frac{2}{3}a^3} e^{\frac{i}{a}} e^{\frac{2i}{3}b^3} e^{\frac{4i}{3}a^2b^2}. \]

The corresponding 'probability' is

\[ \rho(a, b) = \frac{1}{a} e^{-\frac{1}{2}j^2 e^{\frac{2i}{3}e^3} e^{\frac{2i}{3}a^2b^2}} \] (5.5)

This expression does not coincide with (5.3) but it leads to the same asymptotic series (5.4).

To find the representation (5.5) from (5.3) the transformation, see (3.12),

\[ \delta(\Re a x + j) = e^{-\frac{i}{2}j^2 e^{\frac{2i}{3}e^3} e^{\frac{4i}{3}a^2b^2} \delta(\Re a x + j')} \] (5.6)

can be applied. Indeed, inserting this equality into (5.3) we find (5.5). The transformation (5.6) becomes evident from the Fourier transformation of \( \delta \)-function. Eq.(5.6) reflects the freedom in choice of the perturbation theory in vicinity of topologically-equivalent trajectories in functional space.

- The solution \( x_j(t) \) of eq.(4.18) we would search expanding it over \( j(t) \):

\[ x_j(t) = x_c(t) + \int dt_1 G(t, t_1) j(t_1) + ... \]

This is sufficient since \( j(t) \) is the auxiliary variable, i.e. we assume that \( j \) is switched on adiabatically. In this decomposition \( x_c(t) \) is the strict solution of unperturbed equation \( \ddot{x} + v'(x) = 0 \) and \( G(t, t') \) must obey eq.(2.6).

Note that the functional \( \delta \)-function in (4.17) does not contain the end-point values of time \( t = 0 \) and \( t = T \). This means that the initial conditions to the eq.(4.18) are arbitrary and the integration over them is assumed because of our definition of \( \rho \).

The \( \delta \)-likeness of measure allows to conclude:

S8. All strict regular solutions (including trivial) of classical (unperturbed by \( j \)) equation(s) of motion must be taken into account.

We must consider only 'strict' solutions because of strict cancellation of needless contributions when the \( \delta \)-likeness of measure is derived. The \( \delta \)-likeness of measure means that the probability \( \rho(E) \) should contain a sum over all discussed solutions. This is the main distinction of our unitary method of quantization from stationary phase method: even having few solutions there is not interference terms in the sum over them in \( \rho \).

Note that the interference terms are absent independently from solutions 'nearness' in the functional space. This reflects the orthogonality of Hilbert spaces builded on the various \( x_c \) and is the consequence of unitarity condition.

The solutions must be regular since the singular \( x_c \) gives zero contribution on \( \delta \)-like measure.

- We offer following selection rule to define what contribution is significant on the sum over various orbits \( x_c \):

...
S9. In the sum over topologically nonequivalent trajectories one should leave the contribution defined in the highest factor manifold \( G/\bar{G} \) if \( G \) is the group violated by given \( x_c \) and \( \bar{G} \) is the \( x_c \)-invariance subgroup of \( G \) group.\(^{27}\)

Indeed, summation over all solutions of classical equation of motion means necessity to take into account all topologically-equivalent orbits \( x_c \), also, i.e. means integration over parameters of factor manifold \( G/\bar{G} \). This naturally introduces integration over zero-mode degrees of freedom. The corresponding measure will be defined by mapping on the factor manifold \( G/\bar{G} \), i.e. without usage of the Faddeev-Popov ansatz.

It is evident that in the sum over contributions of various \( x_c \) we must leave largest, i.e. the term with maximal number of zero modes. This selection rule \(^{28}\) presents our definition of the vacuum.

• The \( \delta \)-like measure defines the real-time motion only and is not applicable for tunnelling processes since reflects the (space-)time-local equilibrium of all forces. For instance, it excludes the ‘kink’ contributions for potential \( v(x) \sim (x^2 - a^2)^2 \).

This orbits belongs to the bifurcation line, using the terminology of Smale \(^{25}\), i.e. the ‘kink’ contributions should be added to the contributions defined by our \( \delta \)-like measure. Then, following to our selection rule, we should leave those contribution(s) which are proportional to the highest zero-modes volume. So, our definition of measure is rightful if the real-time contributions factor manifold have the largest volume.\(^{28}\)

The explicit investigation of this condition is the nontrivial task in spite of its seeming simplicity (the volume of \( G/\bar{G} \) is defined by classical solution only). Actually we should know (i) all classical orbits and (ii) show that \( G/\bar{G} \) is stable. So, above selection rule gives the classification of mostly probable contributions only.

It is evident that\(^{29}\)

S10. The measure (4.16) admits the canonical transformations. This evidently follows from \( \delta \)- likeness of measure. The phase space differential measure has the form:

\[
DM(x,p) = \delta(E + \omega - H_T(x)) \times \prod_t dx dp \delta(\dot{x} - \frac{\partial H_j}{\partial p}) \delta(\dot{p} + \frac{\partial H_j}{\partial x}),
\]

(5.7)

where

\[
H_j = \frac{1}{2}p^2 + v(x) - jx
\]

(5.8)

is the total Hamiltonian which is time dependent through \( j(t) \).

We can introduce new pare \((\theta, h)\) instead of \((x, p)\) inserting

\[
1 = \int D\theta Dh \prod_t \delta(h - \frac{1}{2}p^2 - v(x)) \delta(\theta - \int x dx (2(h-v(x)))^{-1/2}).
\]

(5.9)

It is important that both differential measures in (5.9) and (5.7) are \( \delta \)-like. This allows to change the order of integration surely and firstly integrate over \((x, p)\). Calculating result one can use the \( \delta \)-functions of (5.7). In this case the \( \delta \)-functions of (5.5) will define the constraints. But if we will use the \( \delta \)-functions of (5.9) the mapping \((x, p) \rightarrow (\theta, h)\) is performed and the remaining \( \delta \)-functions would define motion in the factor space. We conclude that our transformation takes into account the constraints since both ways must give the same result.

\(^{27}\)Note, \( G \) may be wider then the actions invariance group.

\(^{28}\)One can say in this case that the imaginary-time contributions are realized on zero measure.
We find by explicit calculations that:

\[ DM(\theta, h) = \delta(E + \omega - h(T)) \prod_t \delta(\dot{\theta} - \frac{\partial h_j}{\partial \theta}) \delta(\dot{h} + \frac{\partial h_j}{\partial \theta}), \]  

(5.10)

since considered transformation is canonical, \( \{h(x, p), \theta(x, p)\} = 1 \), where

\[ h_j(\theta, h) = h - jx_c(\theta, h) \]  

(5.11)

is the transformed Hamiltonian and \( x_c(\theta, h) \) is the classical trajectory parametrized by \( h \) and \( \theta \).

- The \((\theta, h)\) parametrized solution obey the equation:

\[ \frac{\partial x_c(\theta, h)}{\partial \theta} = p_c(\theta, h), \]

see (5.9). One should underline that only the non-trivial solution of this equation is considered performing mapping \((x, p) \rightarrow (\theta, h)\). It is evident that for trivial solution \( x_c = 0, p_c = 0 \) such transformation is impossible since the corresponding cotangent manifold is empty.

The transformed perturbation theory presents expansion over \( 1/g \) if \( x_c \sim 1/g \), where \( g \) is the interaction constant. So, we wish construct the perturbation theory in the ‘strong coupling’ limit. But one should remind also that all solutions must be taken into account. This means that the perturbation theory for \( \rho(E) \) contains simultaneously both series over \( g \) (from trivial solution \( x_c = 0 \)) and over \( 1/g \), i.e. the sum of week-coupling and strong-coupling expansions. According to our selection rule we should leave largest among then.

On the cotangent bundle we must solve following equations of motion:

\[ \dot{h} = j \frac{\partial x_c}{\partial \theta}, \quad \dot{\theta} = 1 - j \frac{\partial x_c}{\partial h} \]  

(5.12)

and they have a simple structure:

\[ S11. \text{ The Green function on the cotangent bundle is simple } \Theta\text{-function.} \]

Indeed, expanding solutions of eqs.(5.12) over \( j \) in the zero order we find \( \theta_0 = t_0 + t \) and \( h_0 = \text{const} \). The first order gives equation for Green function \( g(t, t') \):

\[ \dot{g}(t, t') = \delta(t - t'). \]  

(5.13)

The solution of this equation introduces the time ‘irreversibility’:

\[ g(t, t') = \Theta(t - t'), \]  

(5.14)

in opposite to causal particles propagator \( G(t, t') \)[29]. But, as will be seen below, see S10, the perturbation theory with Green function (5.14) is time reversible. Note also, that the solution (5.14) is the unique and is the direct consequence of usual in the quantum theories \( i\varepsilon\)-prescription.

The uncertainty is contained in the boundary value \( g(0) \). We will see that \( g(0) = 0 \) excludes some quantum corrections. By this reason one should consider \( g(0) \neq 0 \). We will assume that

\[ g(0) = 1 \]  

(5.15)

[29] Last one contains sum of retarded and advanced parts.
since this boundary condition to eq. (5.13) is natural for local theories. We will use also following formal equalities:

\[ g(t, t')g(t', t) = 0, \quad 1 = g(t, t') + g(t', t) \]  

(5.16)

considering \( g(t, t') \) as the distribution.

- It is important to note that \( \text{Im} \ g(t) = 0 \) on the real time axis. This allows to conclude that

**S12. The perturbation theory on the \((h, \theta)\) bundle can be constructed on the real-time axis.**

Indeed, the \( i\varepsilon \)-prescription is not necessary since, as was mentioned above in S2, the \( \delta \)-functional measure defines a complete set of contributions. But for more confidence one may introduce the \( i\varepsilon \)-prescription and, extracting the \( \delta \)-function in the measure, one can put \( \varepsilon = 0 \) if the contributions are regular at this limit.

One can point out the examples when \( \varepsilon = 0 \) is the singular point.

(a) The Green function \( G(t, t') \) is singular at \( \varepsilon = 0 \). The \( i\varepsilon \)-prescription introduces the wave damping in this case.

(b) The terms of perturbation theory are singular at \( \varepsilon = 0 \) even if the Green functions are regular. This singularities are connected with light-cone singularities of the real-time theories.

(c) There is the tunneling phenomena. The \( i\varepsilon \)-prescription is necessary to define a theory in the turning points (it is the usual WKB prescription).

- Note now that \( \partial x_c / \partial \theta \) and \( \partial x_c / \partial h \) in the r.h.s. of (5.12) can be considered as the sources. This allows to offer the statement:

**S13. The mapping on the cotangent bundle splits ‘Lagrange’ quantum force \( j \) on a set of quantum forces individual to each independent degree of freedom, i.e. to each independent local coordinate of the cotangent manifold.**

Indeed, the simple algebra gives (see Appendix B):

\[
R(E) = 2\pi \int_0^\infty dTe^{\frac{\pi}{\hbar} \hat{\omega}}
\times e^{\frac{1}{\hbar} \int_0^T dt \left( \hat{j}_h(t)\hat{e}_h(t) + \hat{j}_\theta(t)\hat{e}_\theta(t) \right)}
\times \int DhD\theta e^{-iH(x_c, \tau) - iV_T(x_c, e_c)} \delta(E + \omega - h(T))
\times \prod_t \delta(\dot{h} - j_h)\delta(\dot{\theta} - 1 - j_\theta) \quad (5.17)
\]

Therefore, according to splitting \( j \to (j_h, j_\theta) \) we must change \( e \to e_c \), where

\[
e_c = e_h \frac{\partial x_c}{\partial \theta} - e_\theta \frac{\partial x_c}{\partial h} \equiv (e_h \hat{\theta} - e_\theta \hat{h})x_c. \quad (5.18)
\]

carry the symplectic structure of Hamilton’s equations of motion, see (5.10), i.e. \( e_c \) is the invariant of canonical transformations. This quantity describes the flow \( \delta_h x_c \wedge \delta_\theta p_c \) generated by quantum perturbations through the bundles elementary cell.

Hiding the \( x_c(t) \) dependence in \( e_c \) we had solve the problem of the functional determinants and simplify the equation of motion as much as possible:

\[
DM(h, \theta) = \delta(E + \omega - h(T) - h'(T))
\times \prod_t dh(t)d\theta(t)\delta(h(t))\delta(\dot{\theta}(t) - 1) \quad (5.19)
\]
and the perturbations generating operator

\[
\hat{K} = \frac{1}{2}(\hat{\omega} \hat{\tau} + \int_0^T dt_1 dt_2 \Theta(t_1 - t_2)(\hat{e}_h(t_1)\hat{h}'(t_2) + \hat{e}_\theta(t_1)\hat{\theta}'(t_2))
\]

(5.20)

In \( U_T(x, e) \) we must change \( h \to (h + h') \) and \( \theta \to (\theta + \theta') \).

Noting that

\[
\int \prod_t dX(t) \delta(\dot{X}(t)) = \int dX(0) = \int dX_0
\]

we see that the measure (5.19) coincide with the measure of ordinary integrals over \( h_0 \) and \( t_0 \). Last one defines the volume of translational mode.

- Let us consider motion in the action-angle phase space. Corresponding perturbations generating operator has the form:

\[
\hat{K} = \frac{1}{2} \int_0^T dt dt' \Theta(t' - t)(\hat{I}(t)\hat{e}_I(t') + \hat{\phi}(t)\hat{e}_\phi(t'))
\]

\[
\equiv \hat{K}_I + \hat{K}_\phi.
\]

(5.21)

The result of integration using last \( \delta \)-function is

\[
R(E) = 2\pi \int_0^\infty dT e^{-i\hat{K}} \int_0^{2\pi} d\phi_0 \oint \frac{e^{-iU_T(x, e)}}{\omega(E)},
\]

(5.22)

where

\[
\omega = \partial h(I_0)/\partial I_0
\]

and \( I_0 = I_0(E) \) is defined by the algebraic equation:

\[
E = h(I).
\]

The classical trajectory

\[
x_c(t) = x_c(I_0(E) + I(t) - I(T), \phi_0 + \hat{\omega}t + \phi(t)),
\]

(5.23)

where

\[
\hat{\omega} = \frac{1}{t} \int dt' g(t, t')\omega(I_0 + I(t')).
\]

The interaction ‘potential’ \( V_T \) depends from

\[
e_c = e_\phi \partial x_c/\partial I - e_I \partial x_c/\partial \phi.
\]

(5.24)

One can note that eq.(5.31) contains unnecessary contributions. Indeed, action of the operator

\[
\int_0^T dt dt' \Theta(t - t')\hat{e}_I(t')\hat{I}(t')
\]

on \( \hat{H}(x_c; \tau) \), defined in (4.24), leads to the time integrals with zero integration range:

\[
\int_0^T dt \Theta(T - t)\Theta(t - T) = 0.
\]
This simplification was used in (5.21) and (5.22).

The operator \( \hat{K} \) is linear over \( \hat{e}_\phi \), \( \hat{e}_I \). The result of its action can be written in the form:

\[
R(E) = 2\pi \int_0^\infty dT \int_0^{2\pi} \frac{d\phi}{\omega(E)} : e^{-iU_T(x_c, \dot{x}_c/2)} :,
\]  

(5.25)

where

\[
\hat{e}_c = \hat{j}_\phi \frac{\partial x_c}{\partial I} - \hat{j}_I \frac{\partial x_c}{\partial \phi} = (\hat{j}_\phi \hat{I} - \hat{j}_I \hat{\phi}) x_c
\]

(5.26)

since

\[
\hat{j}_X(t) = \int_0^T dt' \theta(t - t') \hat{X}(t'), \quad X = \phi, I.
\]  

(5.27)

The colons in (5.25) means 'normal product': the differential operators must stay to the left of all functions in expansion over commutator

\[
\{ \hat{\phi}(t'), \hat{I}(t) \} = \hat{\phi}(t') \hat{I}(t) - \hat{I}(t') \hat{\phi}(t).
\]

Now we are ready to offer the important statement:

S14. If the eqs.(5.14, 5.15, 5.16) are hold then each term of perturbation theory in the invariant subspace can be represented as the sum of total derivatives over the subspace coordinates.

This statement directly follows from definition of perturbation generating operator \( \hat{K} \) on the cotangent bundle (5.20) and of translationally invariance of the cotangent manifold in the classical approximation. The proof of this statement is given in Appendix C.

We can conclude, contributions are defined by boundary values of classical trajectory \( x_c \) in the invariant subspace since the integration over \( X_0 \) is assumed, see (5.22), and since contributions are the total derivatives over \( X_0 \).

- One can observe following new phenomena:

S15. The quantum corrections to angular variables are canceled if the classical motion is periodic.

If \( x_c \) is the periodic function:

\[
x_c(I_0(E) + I(t) - I(T), (\phi_0 + 2\pi) + \hat{\Omega} t + \phi(t)) = x_c(I_0(E) + I(t) - I(T), \phi_0 + \hat{\Omega} t + \phi(t)).
\]  

(5.28)

this statement is elementary consequence of S10 and is the result of averaging over \( \phi_0 \), see eq.(5.22).

We would like to note now that, generally speaking,

S15. The quantum corrections to angular variables are canceled if the classical motion is periodic.

If \( x_c \) is the periodic function:

\[
x_c(I_0(E) + I(t) - I(T), (\phi_0 + 2\pi) + \hat{\Omega} t + \phi(t)) = x_c(I_0(E) + I(t) - I(T), \phi_0 + \hat{\Omega} t + \phi(t)).
\]  

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\]  

(5.28)

this statement is elementary consequence of S10 and is the result of averaging over \( \phi_0 \), see eq.(5.22).

We would like to note now that, generally speaking,

S16. The transformed measure can not be deduced from direct transformations of path integral (4.9).

Let us consider now the coordinate transformations. For instance, the two dimensional model with potential \( v = v((x_1^2 + x_2^2)^{1/2}) \) is simplified considering it in the cylindrical coordinates \( x_1 = r \cos \phi, \quad x_2 = r \sin \phi \). Note, this transformation is not canonical.

Starting from flat space with trivial metric tensor \( g_{\mu
u} \) and inserting

\[
1 = \int Dr D\phi \prod_t \delta(r - \sqrt{x_1^2 + x_2^2}) \delta(\phi - \arctan x_2/x_1)
\]  

(5.29)
we find the measure in the cylindrical coordinates:

\[
D^{(2)}M(r, \phi) = \delta(E + \omega - H_T(r, \phi)) \times \prod_t drd\phi r^2(t) \delta(\vec{r} - \vec{\phi}^2 r + v'(r) - j_r) \delta(\dot{\phi}r^2) - rj_\phi),
\]  

(5.30)

where \( v'(r) = \partial v(r) / \partial r \) and \( j_r, j_\phi \) are the components of \( \vec{j} \) in the cylindrical coordinates.

The perturbation generating operator has the form:

\[
\hat{K}(j, e) = \frac{1}{2}(\hat{\omega}r + Re \int_{C_p} dt(\hat{j}_r(t)e_r(t) + \hat{j}_\phi(t)e_\phi(t)))
\]  

(5.31)

and in \( U_T(x, \vec{e}) \) we must change \( \vec{e}_c \) on \( \vec{e}_c \) with components

\[
e_{c,1} = e_r \cos \phi - re_\phi \sin \phi, \quad e_{c,2} = e_r \sin \phi + re_\phi \cos \phi.
\]  

(5.32)

Note, \( e_\phi \) was arise in product with \( r \). To find (5.31) and (5.32) one can use (3.27).

The transformation looks quite classically but the measure (5.30) and perturbation generating operator (5.31) can not be derived by naive coordinate transformation of initial path integral for amplitude. This becomes evident noting that transformed representation for \( \rho_1(E) \) can not be written in the product form \( AA^\ast \) of two functional integrals.

Indeed, \( \rho_1(E) \) has not the factorization property because of the mixing of various quantum degrees of freedom when the transformation was performed. This is seen explicitly calculating \( \rho_1(E) \) with the measure (5.30) and the perturbations generating operator (5.31):

\[
\rho_1(E) = 4\pi \int_0^\infty dTde^{2iT} \times \int \prod_t r^2drd\phi \frac{de_r de_\phi}{\pi^2}e^{iST_{+}\left(x+e\right)-iST_{-}\left(x-e\right)}
\]

\[
\times e^{-2\omega t} \int dt e^{i\delta S_T(r, \phi)} e^{-i\delta S_T(r, \phi)} e^{-i\delta S_T\left(x, \phi\right)},
\]  

(5.33)

where the action of \( \exp\{-i\hat{K}\} \) was performed. It is assumed that all quantities in this expression are written in the cylindrical coordinates.

Introducing the ‘main’ variables \( r_\pm = r \pm e_r, \phi_\pm = \phi \pm e_\phi \) and \( T_\pm = T \pm t \) we can see easily that (5.33) can not be factorised onto product of two path integrals. For instance,

\[
\prod_t r^2 dr d\phi de_r de_\phi = \prod_t (r_+ + r_-) \frac{dr_+ d\phi_+ dr_- d\phi_-}{2\pi} \neq \prod_t \frac{dr_+ d\phi_+}{2\pi} \prod_t \frac{dr_- d\phi_-}{2\pi}
\]  

(5.34)

\[
\text{Note, we can introduce also the motion in the phase space with Hamiltonian}
\]

\[
H_j = \frac{1}{2}\vec{p}^2 + \frac{l^2}{2r^2} + v(r) - j_r r - j_\phi \phi.
\]

The Dirac measure becomes four dimensional:

\[
D^{(4)}M(r, \phi, p, l) = \delta(E + \omega - H_T) \times \prod_t drd\phi dpdt \delta(\dot{\phi})
\]

\[
\times \delta(\dot{r} - \frac{\partial H_j}{\partial p}) \delta(\dot{\phi} - \frac{\partial H_j}{\partial l}) \delta(\dot{l} + \frac{\partial H_j}{\partial \phi}).
\]  

(5.35)
Note absence of the coefficient $\prod r^2(t)$ in this expression.

It is interesting also to find the measure starting from the curved space with the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu}(y) \dot{y}^\mu \dot{y}^\nu - v(y)$$

(5.36)

It is enough to consider the kinetic term only since, to find the Dirac measure, we should extract the odd over $e$ terms from the ‘closed-path’ action $S_F(y + e) - S_F(y - e)$. This procedure is ‘trivial’ for potential term. The lowest over $e^\lambda$ part of the kinetic term have the form:

$$2\{g_{\lambda\mu}\dot{y}^\mu + \Gamma_{\lambda,\mu\nu}\dot{y}^\mu \dot{y}^\nu\}.$$  

(5.37)

Therefore, the quasiclassical approximation is restored.

To find the quantum corrections we should linearise at least the $O(e^3)$ term in the exponent

$$\exp\{\Re \int dt g_{\lambda,\mu\nu} e^\lambda \dot{e}^\mu \dot{e}^\nu\}.$$  

(5.38)

using (3.27). This is possible noting that

$$e^\mu(t') \dot{e}^\mu(t') \dot{e}^\nu(t) = e^\mu(t') \delta_{\mu\nu} \delta(t - t') = \dot{e}^\nu \delta(t - t').$$

In result,

$$DM(y) = \sqrt{|g(y + e)||g(y - e)|} \prod_\lambda \prod_t dy_\lambda \delta(g_{\lambda\mu}\dot{y}^\mu

+ \Gamma_{\lambda,\mu\nu}\dot{y}^\mu \dot{y}^\nu + v_\lambda(y) - j_\lambda).$$

(5.38)

where $v_\lambda(y) = \partial_\lambda v(y)$ and $\Gamma_{\lambda,\mu\nu}$ is the Christoffel index. The perturbations generating operator $\hat{K}$ and the weight functional $V(y; e)$ have the standard form.

• Note, above consideration shows that the mechanical systems quantization in the space of nontrivial topology crucially depends from the way as the metrics is introduced.

6 H-atom problem

We will calculate the integral:

$$\rho_1(E) = \int_0^\infty dTe^{-i\hat{K}(j,e)} \int DM(p, l, r, \varphi)e^{-iU_T(r,e)},$$

(6.1)

where $\rho_1(E)$ is the probability to find a particle with energy $E$, i.e. we should find \[8\] that normalized on the zero-modes volume

$$\rho_1(E) = \pi \sum_n \delta(E - E_n),$$

(6.2)

where $E_n$ are the bound states energies. For $H$-atom problem $E_n \leq 0$. This condition would define considered homotopy class.

Expansion over operator

$$\hat{K}(j,e) = \frac{1}{2} \int_0^T dt(\dot{j}_r \dot{\hat{e}}_r + \dot{j}_\varphi \dot{\hat{e}}_\varphi), \quad \hat{X}(t) \equiv \delta/\delta X(t),$$

(6.3)
generates the perturbation series. It will be seen that in our case we may omit the question of perturbation theories convergence.

The differential measure
\[
DM(p, l, r, \varphi) = \delta(E - H_0) \\
\times \prod_t dr(t) dp(t) dl(t) d\varphi(t) \\
\times \delta(\dot{r} - \frac{\partial H_j}{\partial p}) \delta(\dot{p} + \frac{\partial H_j}{\partial r}) \delta(\dot{l} + \frac{\partial H_j}{\partial \varphi})
\]
with total Hamiltonian \(H_0 = H_j|_{j=0}\)
\[
H_j = \frac{1}{2} p^2 - \frac{l^2}{2r^2} - \frac{1}{r} - j_r r - j_\varphi \varphi
\]
allows perform arbitrary transformations because of its \(\delta\)-likeness. The functional
\[
U_T(r, e) = -s_0(r) + \\
+ \int_0^T dt \left[ \frac{1}{((r + e_r)^2 + r^2 e_\varphi^2)^{1/2}} \\
- \frac{1}{((r - e_r)^2 + r^2 e_\varphi^2)^{1/2}} \right] + 2 \frac{e_r}{r}
\]
describes the interaction between various quantum modes and \(s_0(r)\) defines the nonintegrable phase factor \(\delta\). The quantization of this factor determines the bound state energy (see below). Such factor will appear if the phase of amplitude can not be fixed (as, for instance, in the Aharonov-Bohm case). Note that the Hamiltonian (6.5) contains the energy of radial \(j_r r\) and angular \(j_\varphi \varphi\) excitation independently.

We would like to offer following general method of mapping. It is important start from the assumption that the invariant subspace has symplectic structure of cotangent manifold \(T^*G\) and its further possible reduction to linear subspace \(W\) \((\text{dim}(T^*G) \geq \text{dim}(W))\) would be realized as the reduction of quantum degrees of freedom.

Therefore, the first step of mapping consist in demonstration that

\[S17. \text{The classical trajectories belong to } T^*G \text{ completely.}\]

Let
\[
\Delta = \int \prod_t d^2\xi d^2\eta \delta(r - r_c(\xi, \eta)) \delta(p - p_c(\xi, \eta)) \\
\times \delta(l - l_c(\xi, \eta)) \delta(\varphi - \varphi_c(\xi, \eta))
\]
be the functional of known functions \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\), where \((\xi, \eta)\) are two-vectors. It is assumed that one can find such functions \((\xi, \eta)(t)\) at given \((r, p, \varphi, l)(t)\) that the functional determinant
\[
\Delta_c = \int \prod_t d^2\xi d^2\eta \delta(\frac{\partial r_c}{\partial \xi} \cdot \bar{\xi} + \frac{\partial r_c}{\partial \eta} \cdot \bar{\eta}) \\
\times \delta(\frac{\partial p_c}{\partial \xi} \cdot \bar{\xi} + \frac{\partial p_c}{\partial \eta} \cdot \bar{\eta}) \delta(\frac{\partial \varphi_c}{\partial \xi} \cdot \bar{\xi} + \frac{\partial \varphi_c}{\partial \eta} \cdot \bar{\eta}) \\
\times \delta(\frac{\partial l_c}{\partial \xi} \cdot \bar{\xi} + \frac{\partial l_c}{\partial \eta} \cdot \bar{\eta}) \neq 0.
\]

\[30\] This reminds the classic reduction phenomena since the ‘quantum reduction’ arise if additional degrees of freedom of \(T^*G\) are the irrelevant variables for classical flow.
Note that this is the condition for \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\) only since one can choose \((r, p, \varphi, l)(t)\) in eq. (6.7) in an arbitrary useful way.

To perform the mapping we should insert

\[
1 = \Delta/\Delta_c
\]

into (6.1) and integrate over \(r(t), p(t), \varphi(t)\) and \(l(t)\). In result of simple calculations (see Appendix D) we find that

\[
DM(\xi, \eta) = \delta(E - H_0) \prod_t d^2\xi d^2\eta \delta^2(\xi - \frac{\partial h_1}{\partial \eta}) \delta^2(\eta + \frac{\partial h_1}{\partial \xi}), \tag{6.9}
\]

It is the desired result of transformation of the measure for given ‘generating’ functions \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\). In this case the ‘Hamiltonian’ \(h_j(\xi, \eta)\) is defined by four equations (D.3). But there is another possibility. Let us assume that

\[
h_j(\xi, \eta) = H_j(r_c, p_c, \varphi_c, l_c) \tag{6.10}
\]

and the functions \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\) are unknown. Then eqs. (D.3) are the equations for this functions. It is not hard to see that the eqs. (D.3) simultaneously with equations given by \(\delta\)-functions in (6.9) are equivalent of incident equations if the equality \((6.10)\) is hold. So, incident dynamical problem was divided on two parts. First one defines the trajectory in the \(W\) space through eqs. (D.3). Second one defines the dynamics, i.e. the time dependence, through the equations in arguments of \(\delta\)-functions in the measure.

- Therefore, we should consider \(r_c, p_c, \varphi_c, l_c\) as the solutions in the \(\xi, \eta\) parametrization. The desired parametrization of classical orbits has the form (one can find it in arbitrary textbook of classical mechanics):

\[
\begin{align*}
r_c &= \frac{\eta_1^2(\eta_1^2 + \eta_2^2)^{1/2}}{(\eta_1^2 + \eta_2^2)^{1/2} + \eta_2 \cos \xi_1}, \quad \varphi_c = \xi_1, \\
p_c &= \frac{\eta_2 \sin \xi_1}{\eta_1(\eta_1^2 + \eta_2^2)^{1/2}}, \quad l_c = \eta_1. \tag{6.11}
\end{align*}
\]

At the same time,

\[
h_j = \frac{1}{2(\eta_1^2 + \eta_2^2)^{1/2}} - j_r r_c - j_\varphi \xi_1 - h(\eta) - j_r r_c - j_\varphi \xi_1. \tag{6.12}
\]

Note that \(\xi_2\) is the irrelevant variable for classical flow (6.11). This conclusion hides the assumption that the space is flat and homogeneous. So, the external field would violate our solution.

Noting that the derivatives over \(\xi_2\) are equal to zero[31] we find that

\[
DM(\xi, \eta) = \delta(E - h(T)) \prod_t d^2\xi d^2\eta \delta(\xi_1 - \omega_1 + j_r \frac{r_c}{\partial \eta_1})
\]

\[
\times \delta(\xi_2 - \omega_2 + j_r \frac{r_c}{\partial \eta_2}) \delta(\eta_1 - j_r \frac{\partial r_c}{\partial \xi_1} - j_\varphi) \delta(\eta_2), \tag{6.13}
\]

where

\[
\omega_i = \frac{\partial h}{\partial \eta_i} \tag{6.14}
\]

are the conserved in classical limit \(j_r = j_\varphi = 0\) ‘velocities’ in the \(W\) space.

Now we can show the example of reduction of quantum degrees of freedom. We can conclude that

\[S18. \text{The dynamical variable stay the c-number if it has not the canonically conjugate pare.}\]

[31]To have the condition \((6.8)\) we should assume that \(\partial r_c/\partial \xi_2 \sim \varepsilon \neq 0\). We put \(\varepsilon = 0\) at the end of transformation.
We see from (6.13) that the length of Runge-Lentz vector is not perturbed by the quantum forces \(j_r\) and \(j_\varphi\). To investigate the consequence of this fact it is useful to project these forces on the axis of \(W\) space. This means splitting of \(j_r\), \(j_\varphi\) on \(j_\xi\), \(j_\eta\). The equality

\[
\prod_t \delta(\dot{\xi}_1 - \omega_1 + j_r \frac{r_c}{\partial \eta_1}) = e^{\frac{j_r}{2} \int_0^T dt \dot{\xi}_1} \prod_t \delta(\dot{\xi}_1 - \omega_1 + j_\xi_1)
\]

becomes evident if the Fourier representation of \(\delta\)-function is used, S13 (see also [16]). The same transformation of arguments of other \(\delta\)-functions in (6.13) can be applied. Then, noting that the last \(\delta\)-function in (6.13) is source-free, we find the same representation as (6.1) but with

\[
\hat{K}(j, e) = \int_0^T dt (\dot{j}_\xi \hat{e}_\xi + \dot{j}_\eta \hat{e}_\eta),
\]

where the operators \(\hat{j}\) are defined by the equality:

\[
\dot{j}_X(t) = \int_0^T dt' \theta(t - t') \dot{X}(t')
\]

and \(\theta(t - t')\) is the Green function of our perturbation theory [16].

We should change also

\[
e_r \rightarrow e_c = e_m \frac{\partial r_c}{\partial \xi_1} - e_\xi \frac{\partial r_c}{\partial \eta_1} - e_\eta \frac{\partial r_c}{\partial \eta_2}, \quad e_\varphi \rightarrow e_\xi,
\]

in the eq.(6.17). The differential measure takes the simplest form:

\[
DM(\xi, \eta) = \delta(E - h(T)) \times \prod_t d^2\xi d^2\eta \delta(\dot{\xi}_1 - \omega_1 - j_\xi_1) \delta(\dot{\xi}_2 - \omega_2 - j_\xi_2) \times \delta(\dot{\eta}_1 - j_\eta_1) \delta(\dot{\eta}_2).
\]

Note now that the \(\xi, \eta\) variables are contained in \(r_c\) only:

\[
r_c = r_c(\xi_1, \eta_1, \eta_2).
\]

This means that the action of the operator \(\hat{j}_\xi_2\) gives identical to zero contributions into perturbation theory series. And, since \(\hat{e}_\xi_2\) and \(\hat{j}_\xi_2\) are conjugate operators, see (6.18), we can put

\[
j_\xi_2 = e_\xi_2 = 0.
\]

This conclusion ends the reduction:

\[
\hat{K}(j, e) = \int_0^T dt (\dot{j}_\xi \hat{e}_\xi + \dot{j}_\eta \hat{e}_\eta),
\]

\[
e_c = e_m \frac{\partial r_c}{\partial \xi_1} - e_\xi \frac{\partial r_c}{\partial \eta_1}.
\]

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The measure has the form:

$$DM(\xi, \eta) = \delta(E - h(T))d\xi_2d\eta_2$$

$$\times \prod_t d\xi_t d\eta_t \delta(\dot{\xi}_t - \omega_1 - j_{\xi_t}) \delta(\dot{\eta}_t - j_{\eta_t})$$  \hspace{1cm} (6.22)

since $U_T = U_T(r_C, e, \xi_1)$ is $\xi_2$ independent and

$$\int \prod_t dX(t) \delta(\dot{X}) = \int dX(0).$$

- One can see from (6.22) that the reduction can not solve the H-atom problem completely: there are nontrivial corrections to the orbital degrees of freedom $\xi_1, \eta_1$. By this reason we should consider the expansion over $\hat{K}$. In result we will see that

$S19$. The quantum corrections may give zero contributions if the interactions disappeared on the bifurcation line.

Using last $\delta$-functions in (6.22) we find, see also [16] (normalizing $\rho_1(E)$ on the integral over $\xi_2$):

$$\rho(E) = \int_0^\infty dTe^{-i\hat{K}(j, e)} \int dMe^{-iU_T(r_c, e)},$$  \hspace{1cm} (6.23)

where

$$dM = \frac{d\xi_1 d\eta_1}{\omega_2(E)}.$$  \hspace{1cm} (6.24)

The operator $\hat{K}(j, e)$ was defined in (6.20) and

$$U_T(r, e) = -s_0(r) + \int_0^T dt \left[ \frac{1}{((r_C + e_c)^2 + r_C^2 c^2/\xi_1)^{1/2}} \right.$$

$$- \frac{1}{((r_C - e_c)^2 + r_C^2 c^2/\xi_1)^{1/2} + 2 e_c/r_c} \Bigg]$$  \hspace{1cm} (6.25)

with $e_c$ defined in (6.21) and

$$r_c(t) = r_c(\eta_1 + \eta(t), \bar{\eta}_2(E, T), \xi_1 + \omega_1(t) + \xi(t)),$$

$$E \equiv h(\eta_1 + \eta(T), \bar{\eta}_2),$$  \hspace{1cm} (6.26)

where $\bar{\eta}_2(E, T)$ is the solution of equation $E = h$.

The integration range over $\xi_1$ and $\eta_1$ is as follows:

$$\partial W_C : 0 \leq \xi_1 \leq 2\pi, \quad -\infty \leq \eta_1 \leq +\infty.$$  \hspace{1cm} (6.27)

First inequality defines the principal domain of the angular variable $\varphi$ and second ones take into account the clockwise and anticlockwise motions of particle on the Kepler orbits, $|\eta_1| = \infty$ is the bifurcation line.

We can write:

$$\rho(E) = \int_0^\infty dT \int dM : e^{-iV(r_c, \bar{e})} :$$  \hspace{1cm} (6.28)

since the operator $\hat{K}$ is linear over $\dot{\xi}_1, \dot{\eta}_1$. The colons means ‘normal product’ with operators staying to the left of functions and $V(r_c, \bar{e})$ is the functional of operators:

$$2i\dot{e}_c = \dot{\eta}_1 \frac{\partial r_c}{\partial \xi_1} - \dot{\xi}_1 \frac{\partial r_c}{\partial \eta_1},$$

$$2i\dot{\xi}_1 = \dot{\eta}_1.$$  \hspace{1cm} (6.29)
Expanding \( U_T(r_c, \dot{e}) \) over \( \dot{e}_c \) and \( \dot{e}_{\eta_1} \) we find:

\[
U_T(r_c, \dot{e}) = -s_0(r_c) + 2 \sum_{n+m \geq 1} C_{n,m} \int_0^T \frac{e^{2n+1} e^{m \dot{e}_{\eta_1}}}{r_c^{2n+2}} dt,
\]

where \( C_{n,m} \) are the numerical coefficients. We see that the interaction part presents expansion over \( 1/r_c \) and, therefore, the expansion over \( V \) generates an expansion over \( 1/r_c \).

In result,

\[
\rho(E) = \int_0^\infty dT \int dM \{ e^{i s_0(r_c)} + B_{\xi_1}(\xi_1, \eta_1) + B_{\eta_1}(\xi_1, \eta_1) \}. \tag{6.31}
\]

The first term is the pure quasiclassical contribution and last ones are the quantum corrections. They can be written as the total derivatives:

\[
B_{\xi_1} = \frac{\partial}{\partial \xi_1} b_{\xi_1}, \quad B_{\eta_1} = \frac{\partial}{\partial \eta_1} b_{\eta_1}. \tag{6.32}
\]

This means that the mean value of quantum corrections in the \( \xi_1 \) direction are equal to zero:

\[
\int_0^{2\pi} d\xi_1 \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1) = 0 \quad \tag{6.33}
\]

since \( r_c \) is the closed trajectory independently from initial conditions, see S15.

In the \( \eta_1 \) direction the motion is classical:

\[
\int_{-\infty}^{+\infty} d\eta_1 \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1) = 0 \tag{6.34}
\]

since (i) \( b_{\eta_1} \) is the series over \( 1/r_c^2 \) and (ii) \( r_c \to \infty \) when \( |\eta_1| \to \infty \). Therefore,

\[
\rho(E) = \int_0^\infty dT \int dM e^{i s_0(r_c)}. \tag{6.35}
\]

This is the desired result.

- Noting that

\[
s_0(r_c) = kS_1(E), \quad k = \pm 1, \pm 2, ...,
\]

where \( S_1(E) \) is the action over one classical period \( T_1 \):

\[
\frac{\partial S_1(E)}{\partial E} = T_1(E),
\]

and using the identity \[8\]:

\[
\sum_{-\infty}^{+\infty} e^{i n S_1(E)} = 2\pi \sum_{-\infty}^{+\infty} \delta(S_1(E) - 2\pi n),
\]

we find normalizing on zero-modes volume, that

\[
\rho(E) = \pi \sum_n \delta(E + 1/2n^2). \tag{6.36}
\]
Our aim is to calculate the integral:
\[ \rho_2(q) = e^{-iK(j,e)} \int DM(u,p)|\Gamma(q;u)|^2 e^{\imath \nu_0(u) - \imath U(u,e)}, \]  
(7.1)
where \( \Gamma(q;u) \) was defined in (2.3). The time integrals would be defined on the Mills time contour \([37]\) to avoid the mass-shell singularities of the perturbation theory.

In this expression the expansion over operator
\[ \hat{K}(j,e) = \frac{1}{2} \Re \int_{C_+} dx dt \frac{\delta}{\delta j(x,t)} \frac{\delta}{\delta e(x,t)} \]
(7.2)
generates the perturbation series. We will assume that this series exist. The variational derivatives in (7.2) are defined as follows:
\[ \frac{\delta \phi(x,t \in C_i)}{\delta \phi(x',t' \in C_j)} = \delta_{ij} \delta(x-x') \delta(t-t'), \quad i, j = +, -, \]

The auxiliary variables \((j, e)\) must be taken equal to zero at the very end of calculations.

The functionals \(U(u,e)\) and \(s_0(u)\) are defined by the equalities:
\[ U(u,e) = (V(u+e) - V(u-e)) \]
\[ -2\Re \int_{C_+} dx dt e(x,t) v(u), \]
\[ s_0(u) = (S_0(u+e) - S_0(u-e)) \]
\[ +2\Re \int_{C_+} dx dt e(x,t) (\partial^2 + m^2) u(x,t), \]
(7.3)
where \(S_0(u)\) corresponds to the free part of Lagrangian (2.20) and \(V(u)\) describes interactions. The quantity \(s_0(u)\) is not equal to zero since soliton configurations have nontrivial topological charge (see also \([8]\)).

Considering motion in the phase space \((u, p)\) the measure \(DM(u,p)\) has the form:
\[ DM(u,p) = \prod_{x,t} du(x,t) dp(x,t) \times \delta(\hat{u} - \delta H_j(u,p) / \delta p) \delta(p + \delta H_j(u,p) / \delta u) \]
(7.4)
with the total Hamiltonian
\[ H_j(u,p) = \int dx \left\{ \frac{1}{2} p^2 + \frac{1}{2} (\partial_x u)^2 - \frac{m^2}{\lambda^2} [\cos(\lambda u) - 1] - j u \right\}. \]
(7.5)

The problem will be considered assuming that \(u(x,t)\) belongs to Schwarz space:
\[ u(x,t)|_{|x| = \infty} = 0 \pmod{2\pi / \lambda}. \]
(7.6)
This means that \(u(x,t)\) tends to zero \((\pmod{2\pi / \lambda})\) at \(|x| \to \infty\) faster then any power of \(1/|x|\).
The measure \(\ref{7.4}\) allows to perform arbitrary transformations. But, as was explained in Sec., the canonical transformations are essential since they conserve the form of equations of motion and help to define the quantum degrees of freedom in the \(W\) space, see \(\mathcal{S}\).

Hence, assuming that this transformation exist \(\ref{36}\), one may propose that

\[ S21. \text{The } N\text{-solitons functional measure in the } W \text{ space has the form:} \]

\[ D^N \mathcal{M}(\xi, \eta) = \prod_t d^N \xi(t) d^N \eta(t) \times \delta(\dot{\xi} - \frac{\partial h_j(\xi, \eta)}{\partial \eta(t)}) \delta(\dot{\eta} + \frac{\partial h_j(\xi, \eta)}{\partial \xi(t)}), \]  

(7.7)

where \(h_j\) is the transformed Hamiltonian:

\[ h_j(\xi, \eta) = h_N(\eta) - \int dx j(x, t) u_N(x; \xi, \eta) \]  

(7.8)

and \(u_N(x; \xi, \eta)\) is the \(N\)-soliton configuration parametrized by \((\xi, \eta)\).

The proof of eq.(7.7) is the same as for considered above Coulomb problem. But the case of \((1+1)\)-dimensional space needs additional explanations. First of all one must introduce

\[ \Delta(u, p) = \int \prod_t d^N \xi(t) d^N \eta(t) \times \prod_{x,t} \delta(u(x, t) - u_c(x; \xi, \eta)) \delta(p(x, t) - p_c(x; \xi, \eta)) \]  

(7.9)

This distribution is infinite. But this infinity is formal because of disappearance of the determinant of transformation \(\Delta(u_c, p_c)\) in the final result iff the Poisson brackets:

\[ \{ u_c(x, t), h_j \} = \frac{\delta H_j}{\delta p_c(x, t)}, \quad \{ p_c(x, t), h_j \} = -\frac{\delta H_j}{\delta u_c(x, t)}. \]  

(7.10)

or, using definition \(\ref{7.8}\), iff

\[ \{ u_c(x; \xi, \eta), u_c(y; \xi, \eta) \} = \{ p_c(x; \xi, \eta), p_c(y; \xi, \eta) \} = 0, \quad \{ u_c(x; \xi, \eta), p_c(y; \xi, \eta) \} = \delta(x - y) \]  

(7.11)

are hold. For more confidence one can introduce the cells in the \(x\)-space \(\mathcal{R}\). Eqs. (7.11) are the necessary and sufficient conditions for considered mapping

\[ J : \{ u, p \}(x, t) \rightarrow \{ \xi, \eta \}(t) \]  

(7.12)

with local coordinates \((\xi, \eta)\) defined by the equations:

\[ \dot{\xi} = \frac{\partial h_j}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial h_j}{\partial \xi}. \]  

(7.13)

Eqs.(7.11, 7.13) must be considered simultaneously.
The eqs. (7.10) fulfilled for arbitrary \( j(x, t) \). Therefore, the quantum perturbations should not alter the Poisson brackets algebra. In our terms this means that the quantum force \( j(x, t) \) excites the \((\xi, \eta)\) manifold only, leaving the topology of classical trajectory \((u, p)\) unchanged. So, since the complete set of canonical coordinates \((\xi, \eta)\) for \( \text{sin-Gordon model} \) is known, see e.g. [36], we can use them immediately.

The classical Hamiltonian \( h \) is the sum:

\[
h(\eta) = \int dp \sigma(r) \sqrt{r^2 + m^2} + \sum_{i=1}^{N} h(\eta_i),
\]

where \( \sigma(r) \) is the continuous spectrum and \( h(\eta) \) is the soliton energy. Note absence of interaction energy among solitons.

New degrees of freedom \((\xi, \eta)(t)\) must obey the equations (7.15):

\[
\begin{align*}
\dot{\xi}_i &= \omega(\eta_i) - \int dx j(x, t) \frac{\partial u_N(x; \xi, \eta)}{\partial \eta_i}, \\
\dot{\eta}_i &= \int dx j(x, t) \frac{\partial u_N(\xi, \eta)}{\partial \xi_i}, \quad \omega(\eta) \equiv \frac{\partial h(\eta)}{\partial \eta}.
\end{align*}
\]

Hence the source of quantum perturbations are proportional to the time-local tangent vectors \( \partial u_N(x; \xi, \eta) / \partial \eta_i \) and \( \partial u_N(x; \xi, \eta) / \partial \xi_i \) to the soliton configurations. It suggests the idea to split the ‘Lagrange’ sources: \( j(x, t) \to (\dot{\xi}, \dot{\eta}) \).

This leads to new weight functional \( U(u_N; e, e) \) and new perturbations generating operator \( \hat{K}(e, e; \dot{\xi}, \dot{\eta}) \).

In result:

\[
\begin{align*}
\rho_2(q) &= \sum_{N} e^{-i\hat{K}(e, e; \dot{\xi}, \dot{\eta})} \\
&\times \int D^N M(\xi, \eta) e^{i\eta_0(u_N)} e^{-iU(u_N; e, e)} |\Gamma(q; u_N)|^2
\end{align*}
\]

where, using vector notations,

\[
\hat{K}(e, e; \dot{\xi}, \dot{\eta}) = \frac{1}{2} \text{Re} \int_{C^+} dt \{ \dot{\xi}(t) \cdot \dot{\xi}(t) + \dot{\eta}(t) \cdot \dot{\eta}(t) \}.
\]

The measure takes the form:

\[
\begin{align*}
D^N M(\xi, \eta) &= \prod_{i=1}^{N} \prod_{t} d\xi_i(t) d\eta_i(t) \\
&\times \delta(\dot{\xi}_i - \omega(\eta_i) - j_\xi(i)(t)) \delta(\dot{\eta}_i - j_\eta(i)(t))
\end{align*}
\]

The effective potential

\[
U(u_N; e, e) = -\frac{2m^2}{\lambda^2} \int dx dt \sin \lambda u_N (\sin \lambda e - \lambda e)
\]

with

\[
e(x, t) = e(x) - e(x) \cdot \frac{\partial u_N(x; \xi, \eta)}{\partial \eta(t)} - e(x) \cdot \frac{\partial u_N(x; \xi, \eta)}{\partial \xi(t)}.
\]
Performing the shifts:

\[ \xi_i(t) \rightarrow \xi_i(t) + \int dt' g(t-t') j_{\xi,i}(t') \equiv \xi_i(t) + \xi_i'(t), \]

\[ \eta_i(t) \rightarrow \eta_i(t) + \int dt' g(t-t') j_{\eta,i}(t') \equiv \eta_i(t) + \eta_i'(t), \]

we can get the Green function \( g(t-t') \) into the operator exponent:

\[ \hat{K}(\epsilon_\xi, \epsilon_\eta; j_\xi, j_\eta) = \frac{1}{2} \int dt dt' \Theta(t-t') \{ \tilde{\epsilon}_i'(t') \cdot \hat{\epsilon}_\xi(t) + \tilde{\eta}_i'(t') \cdot \hat{\epsilon}_\eta(t) \}. \] (7.22)

since the Green function \( g(t-t') \) of eqs.(7.15) is the step function:

\[ g(t-t') = \Theta(t-t') \] (7.23)

Its imaginary part is equal to zero for real times and this allows to shift \( C_\pm \) to the real-time axis. Note the Lorentz noncovariantness of our perturbation theory with Green function (7.23).

In result:

\[ D^N_M(\xi, \eta) = \prod_{i=1}^N \prod_t d\xi_i(t) d\eta_i(t) \delta(\dot{\xi}_i - \omega(\eta + \eta')) \delta(\dot{\eta}_i) \] (7.24)

with

\[ u_N = u_N(x; \xi + \xi', \eta + \eta'). \] (7.25)

The equations:

\[ \dot{\xi}_i = \omega(\eta_i + \eta_i') \] (7.26)

are trivially integrable. In quantum case \( \eta_i' \neq 0 \) this equation describes motion in the nonhomogeneous and anisotropic manifold. So, the expansion over \( (\hat{\xi}', \hat{\epsilon}_\xi, \hat{\eta}', \hat{\epsilon}_\eta) \) generates the local in time fluctuations of \( W \) manifold. The weight of this fluctuations is defined by \( U(u_N; \epsilon_\xi, \epsilon_\eta) \).

Using the definition:

\[ \int Dx \delta(\dot{x}) = \int dx(0) = \int dx_0 \]

functional integrals are reduced to the ordinary integrals over initial data \( (\xi, \eta)_0 \). This integrals define zero modes volume. Note once more that the zero-modes measure was defined without Faddeev-Popov anzats.

The proof of (2.23) we would divide on two parts. First of all we would consider the quasiclassical approximation and then we will show that this approximation is exact.

This strategy is necessary since it seems to important to show the role of quantum corrections noting that for all physically acceptable field theories \( \rho_{nm} = 0 \) in the quasiclassical approximation. We would like to show that

**S22. The exactness of quasiclassical approximation is the necessary and sufficient condition to have (2.23) in the model (2.20, 7.6).**

Note, this statement is not evident from the unitarity condition.

---

32It is natural to assume that the fields should tend to zero at \( \sigma_\infty \).
The $N$-soliton solution $u_N$ depends from $2N$ parameters. Half of them $N$ can be considered as the position of solitons and other $N$ as the solitons momentum. Generally at $|t| \to \infty$ the $u_N$ solution decomposed on the single solitons $u_s$ and on the double soliton bound states $u_b$:

$$u_N(x, t) = \sum_{j=1}^{n_1} u_{s,j}(x, t) + \sum_{k=1}^{n_2} u_{b,k}(x, t) + O(e^{-|t|})$$

Note that this asymptotic is achieved if $\xi_i \to \infty$ or $\eta_i \to \infty$. Last one defines the bifurcation line of our model. So, the one soliton $u_s$ and two-soliton bound state $u_b$ would be the main elements of our formalism. Its $(\xi, \eta)$ parametrizations confirmed to eqs. (7.10, 7.11) have the form:

$$u_s(x; \xi, \eta) = -\frac{4}{\lambda} \arctan\{\exp(mx \cosh \beta \eta - \xi}\}, \quad \beta = \frac{\lambda^2}{8}$$

and

$$u_b(x; \xi, \eta) = -\frac{4}{\lambda} \arctan\{\tan \frac{\beta \eta_2}{2} \frac{mx \sinh \frac{\beta \eta_2}{2}}{\lambda} - \xi_2\}.$$  

(7.28)

The $(\xi, \eta)$ parametrization of solitons individual energies $h(\eta)$ takes the form:

$$h_s(\eta) = \frac{m}{\beta} \cosh \beta \eta, \quad h_b(\eta) = \frac{2m}{\beta} \cosh \frac{\beta \eta_1}{2} \sin \frac{\beta \eta_2}{2} \geq 0.$$  

(7.28)

The bound-states energy $h_b$ depends from $\eta_2$ and $\eta_1$. First one defines inner motion of two bounded solitons and second one the bound states center of mass motion. Correspondingly we will call this parameters as the internal and external ones. Note that the inner motion is periodic, see (7.28).

Performing last integration we find:

$$\rho_2(q) = \sum_{N} \int \prod_{i=1}^{N}(d\xi_0 d\eta_0) e^{-iK e^{is_0(u_N)}} \times e^{-iU(u_N; \xi_0, \eta_0)} |\Gamma(q; u_N)|^2$$

(7.29)

where

$$u_N = u_N(\eta_0 + \eta', \xi_0 + \omega(t) + \xi').$$

(7.30)

and

$$\omega(t) = \int dt' \theta(t - t')\omega(\eta_0 + \eta')(t').$$

(7.31)

In the quasiclassical approximation $\xi' = \eta' = 0$ we have:

$$u_N = u_N(x; \eta_0, \xi_0 + \omega(\eta_0)t).$$

(7.32)

Note now that if the surface term

$$\int dx \partial_t e^{i\eta x u_N} = 0$$

(7.33)
then
\[ \int d^2x e^{iqx} (\partial^2 + m^2)u_N(x, t) = - (q^2 - m^2) \int d^2x e^{iqx} u_N(x, t) = 0 \]  
(7.34)
since \( q^2 \) belongs to mass shell by definition. The condition (7.33) is satisfied for all \( q_\mu \neq 0 \) since \( u_N \) belong to Schwarz space (the periodic boundary condition for \( u(x, t) \) do not alter this conclusion). Therefore, in the quasiclassical approximation (2.23) is hold.

Expanding the operator exponent in (7.29) we find that action of operators \( \hat{\xi}', \hat{\eta}' \) create terms
\[ \sim \int d^2x e^{iqx} \theta(t - t') (\partial^2 + m^2)u_N(x, t) \neq 0 \]  
(7.35)
This ends our proof.

• Now we will show that

\( S23. \) The quasiclassical approximation is exact in the soliton sector of sin-Gordon model. The structure of the perturbation theory is readily seen in the ‘normal-product’ form:
\[ \rho_2(q) = \sum_N \int \prod_{i=1}^N \{ d\xi_0 d\eta_0 \} i \times e^{-iU(u_N; \hat{j}/2i)} e^{i\epsilon_0(u_N)} |\Gamma(q; u_N)|^2 ; \]  
(7.36)
where
\[ \dot{j} = \partial u_N / \partial \eta - \partial u_N / \partial \xi = \Omega \partial u_N / \partial X \]  
(7.37)
and
\[ \hat{j}_X = \int dt' \Theta(t - t') \hat{X}(t') \]  
(7.38)
with \( 2N \)-dimensional vector \( X = (\xi, \eta) \). In eq. (7.37) \( \Omega \) is the ordinary symplectic matrix.

The colons in (7.36) mean that the operator \( \hat{j} \) should stay to the left of all functions in the perturbation theory expansion over it. The structure (7.37) shows that each order over \( \hat{j}_X \) is proportional at least to the first order derivative of \( u_N \) over conjugate to \( X_i \) variable.

The expansion of (7.36) over \( \hat{j}_X \) can be written using \( S10 \) of \( [16] \) in the form (omitting the quasiclassical approximation):
\[ \rho_2(q) = \sum_N \int \prod_{i=1}^N \{ d\xi_0 d\eta_0 \} i \{ \sum_{i=1}^{2n} \frac{\partial}{\partial X_0} P_X_i(u_N) \} \]  
(7.39)
where \( P_X_i(u_N) \) is the infinite sum of ‘time-ordered’ polynomials (see [16]) over \( u_N \) and its derivatives. The explicit form of \( P_X_i(u_N) \) is complicated since the interaction potential is nonpolynomial. But it is enough to know, see (7.37), that
\[ P_X_i(u_N) \sim \Omega_{ij} \partial u_N / \partial X_{0j} \]  
(7.40)

Therefore,
\[ \rho_2(q) = 0 \]  
(7.41)
since (i) each term in (7.39) is the total derivative, (ii) we have (7.40) and (iii) \( u_N \) belongs to Schwarz space.

- We can conclude that the equality (7.41) is not hold iff
  \[
  \frac{\partial u_N}{\partial X_0} \neq 0 \quad \text{at} \quad X_0 \in \alpha W_s,
  \]
  (7.42)
where the boundary \( \partial W_s \) is the bifurcation line of the invariant subspace.

In our consideration, in accordance to our selection rule, the continuous spectrum contributions are absent since they are realized on zero measure \( \sim \{ \text{volume of } W \}^{-1} \).

8 Differential measure on \( T^* \overline{G}_1 \times \mathbb{R}^5 \)

The general properties of quantum dynamics of the massless scalar fields in the
\[
W_O = O(4, 2)/O(4) \times O(2)
\]
factor space would be described. We start calculations from the integral:
\[
\rho_2(q) = e^{-i \hat{K}(j, e)} \int DM(u) |\Gamma(q, u)|^2 e^{-iU(u, e)},
\]
(8.1)
where the Dirac measure
\[
DM(u) = \prod_{x,t} du(x,t) \delta(\partial^2 u(x,t) + gu^3(x,t) - j(x,t))
\]
(8.2)
defines a complete set of contributions and
\[
\Gamma(q,u) = \int d^3xdte^{iqx} \partial^2 u(x,t).
\]
(8.3)
for this theory. The expansion of operator exponent \( e^{-i \hat{K}} \),
\[
\hat{K}(j, e) = \frac{1}{2} \int d^3 xdt \hat{j}(x,t) \hat{e}(x,t),
\]
(8.4)
means the expansion in vicinity of zero of auxiliary variables \( j \) and \( e \). This allows to start from the unperturbated equation
\[
\partial^2 u + gu^3 = 0
\]
(8.5)
finding the contributions into integral (8.1).

The Green function is singular on the light cone and to avoid this singularity the \( i\varepsilon \)-prescription should be used. This imply continuation of (8.1) into the space of complex fields. Nevertheless we simplify notations considering
\[
U(u, e) = 2g\Re \int_{C_+} d^3xdtu(x)e^3(x)
\]
(8.6)
to describe the interactions since the mapping into the \( W_O \) space would be considered.
We will take into account following solution of (8.5) (see [38] and references cited therein):

\[
\phi(x) = \left(\frac{4}{g\eta_1^2}\right)^{1/2}\{(1 + \frac{(x-x_0)^2}{\eta_1^2}) + (2\frac{\eta_2\mu(x-x_0)^\mu}{\eta_1^2})\}^{-1/2} = O(1/\sqrt{g}),
\]

(8.7)

where \(x_0, l\) are the 4-vectors. Note, this solution of eq.(8.5) is \(O(4) \times O(2)\) invariant [38]. It is assumed in (8.7) that \(l_\mu l^\mu = +1/\eta_2^2 \geq 0, \quad \vec{l}_2^2 = 1\),

(8.8)

i.e. \(1/\eta_2^2\) is the time scale. Then \(\phi\) is the 8 parametric function.

The \(W_O\) space we define by the conditions:

\[
\infty \geq \eta_1^2 \geq 0, \quad |\eta_2| \leq \infty, \quad |x_{0,\mu}| \leq \infty
\]

(8.9)

Note, \(\phi\) becomes imaginary if \(\eta_1^2\) is negative. By this reason the region \(\eta_1^2 < 0\) would not be considered. Therefore,

\[
\partial W_O : = \{\eta_1 = 0, +\infty, \quad \eta_2 = \pm \infty, \quad x_{0,\mu} = \pm \infty\}.
\]

(8.10)

Note, The integrals over \(\phi(x)\) becomes singular at \(\eta_1 = 0\). To avoid this unphysical singularities the \(i\varepsilon\)-prescription was preserved, see (8.6).

Note, the naive insertion of (8.7) into (8.3) gives \(\rho_2^\eta(q) = 0\) if \(q_\mu \neq 0\). It will be shown that the quantum corrections gets to \(\rho_2 \neq 0\).

The parameter \(\eta_1\) defines the scale of \(\phi\). Considered solution (8.7) has following asymptotics:

\[
\phi(x) \sim \frac{\eta_1}{|x|} \quad \text{at} \quad \eta_1/|x| \to 0
\]

(8.11)

and

\[
\phi(x) \sim \frac{1}{\eta_1} \quad \text{at} \quad \eta_1/|x| \to \infty.
\]

(8.12)

We will see that the asymptotics (8.11) leads to divergences in the perturbation series. Following to estimations (8.11,8.12) \(\eta_1 = \infty\) defines the ‘ultraviolet’ region over \(x\) and the ‘infrared’ region over \(x\) correspond to \(\eta_1 = 0\). Therefore, our perturbation theory would contain the ultraviolet divergences and be the infrared stable.

The quantum corrections are defined on the boundary \(\partial W_O\), see S14, and \(\rho_2(q) \neq 0\) if

\[
\{\varphi\} \cap \partial W_O \neq \emptyset.
\]

(8.13)

Noting (8.7) and (8.10) one may expect that \(\rho_2(q) \neq 0\) since

\[
\{\varphi\} \cap \inf \partial_{\eta_1} W_O \neq \emptyset.
\]

(8.14)

in contrast to Coulomb problem and sin-Gordon model, where the intersection of ‘fields’ set with boundaries was empty.

Note, other directions in the \(W_O\) space did not give the contributions since

\[
\{\varphi\} \cap \partial_{\eta_2} W_O = \emptyset, \quad \{\varphi\} \cap \sup \partial_{\eta_1} W_O = \emptyset, \quad \{\varphi\} \cap \partial_{x_{0,\mu}} W_O = \emptyset.
\]

(8.15)
The expected nontrivial result $\rho_2(q) \neq 0$ we interpret as a consequence of broken scale symmetry since its appearance is connected with non-emptyness of the $\partial_{\eta_1} W_0$ boundary. Certainly, the more careful analysis is needed for this conclusion.

With definition (8.9) the classical fields energy:

$$h_c(\eta) = \int d^3x \{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\vec{\partial} \phi)^2 + \frac{1}{4} g \phi^4 \}$$

$$= \frac{1}{\sqrt{|\eta_1|}} h(\eta_2) \geq 0 \quad (8.16)$$

is the well defined conserved quantity. Note, the classical fields energy $h_c \to \infty$ at $\eta_1 \to 0$, but the renormalization procedure would ‘hide’ this divergences. Note, the singularity at $\eta_1 = 0$ is the point of bifurcation.

Other properties of $\phi$ one can find in [38].

• In the phase space the measure (8.2) has the form:

$$DM(u, p) = \prod_x du(t) dp(t) \delta(\dot{u} - \frac{\delta H_j}{\delta p}) \delta(\dot{p} + \frac{\delta H_j}{\delta u}) \quad (8.17)$$

where the total Hamiltonian

$$H_j(u, p) = \int d^3x \{ \frac{1}{2} p^2 + \frac{1}{2} (\vec{\partial} u)^2 + \frac{1}{4} g u^4 - j u \} \quad (8.18)$$

includes the energy of quantum excitations $j u$.

We want to show that $S24$. The mapping on the $W_0$ manifold gives

$$DM(\xi, \eta) = d^3x_0 x_0 d^3\delta(\vec{l}^2 - 1) dt_0 \delta(\xi_1(0) - \xi_2(0))$$

$$\times \prod_t d^2\xi(t) d^2\eta(t) \delta^2(\dot{\eta} - \frac{\partial h_j}{\partial \eta}) \delta^2(\dot{\xi} + \frac{\partial h_j}{\partial \xi}) \quad (8.19)$$

if $u = u_c$, where ($l_0 = \pm \sqrt{1 + 1/\eta_2^2}$)

$$u_c(\vec{x}; \xi, \eta) = \frac{2\eta_1 \omega^2_2}{\sqrt{g}} (\omega_2^2 (\eta_1^2 + \xi_1 - \omega_2^2 (\vec{x} - \vec{x}_0)^2)^2$$

$$+ 4 \omega_1^2 \eta^2_1 l_0 (\eta_2 - \eta_2 \omega_2 \vec{l} \cdot (\vec{x} - \vec{x}_0))^2)^{-1/2}, \quad (8.20)$$

Here $h_j$ is the transformed total Hamiltonian:

$$h_j(\xi, \eta) = H_j(u_c, p_c) = h_c(\eta) - \int d^3x j u_c, \quad p_c = \dot{u}_c. \quad (8.21)$$

and the ‘velocities’

$$\omega_i(\eta_1, \eta_2) = \frac{\partial}{\partial \eta_i} h_c(\eta_1, \eta_2). \quad (8.22)$$

was introduced.

The function $u_c$, defined in (8.20), is the solution of incident eq. (8.19) if $(\xi, \eta)$ obey the equations

$$\dot{\xi}_i = \frac{\partial h_j}{\partial \eta_i}, \quad \dot{\eta}_i = - \frac{\partial h_j}{\partial \xi_i}, \quad i = 1, 2, \quad (8.23)$$
and the boundary condition
\[ \xi_1(0) = \xi_2(0) \]  
(8.24)
is applied. One can check this easily inserting into (8.20) the solutions of eq.(8.23). The same is seen solving the equation
\[ \{u_c, h_j\} = \frac{\delta H_j(u_c, p_c)}{\delta p_c}, \quad p_c = \dot{u}_c, \]  
(8.25)
where \( \{,\} \) is the Poisson bracket in the \((\xi, \eta)\) phase space.

The proof of S24 repeats calculation of measures for Coulomb problem and sin-Gordon model. But it must be shown to find (8.20) how the constraints can be included into formalism.

Let us consider for this purpose
\[
\Delta F(u, p) = \int \prod_t d\xi(t) d\eta(t) 
\times \prod_{x,t} \delta(u(x, t) - u_c(x; \xi, \eta)) \delta(p(x, t) - p_c(x; \xi, \eta)) 
\times F(\xi, \eta) \]  
(8.26)
with some known functions \((u_c, p_c)\) of two \( n \)-dimensional vectors \((\xi, \eta)\). The functional \( F(\xi, \eta) \) was introduced to take into account the constraints. We will specify it below.

It is evident that \( \Delta F \sim \prod_x \delta\)-functions, i.e. is the distribution. So, it must be defined on the ‘good’ (Schwartz) support. This means that the continuum of equations
\[ u(x, t) = u_c(x; \xi, \eta), \quad p(x, t) = p_c(x; \xi, \eta) \]  
should have, independent from \( x \), \( 2n \) nontrivial solutions for \((\xi, \eta)\). Then, expanding arguments of \( \delta \)-functions near this solutions \((\xi_c, \eta_c)\) we find:
\[
\Delta F(u, p) = F(\xi, \eta) \int \prod_t d\tilde{\xi} d\tilde{\eta} 
\times \prod_{x,t} \delta(\frac{\partial u_c}{\partial \xi} \cdot \tilde{\eta} + \frac{\partial u_c}{\partial \eta} \cdot \tilde{\eta}) \delta(\frac{\partial p_c}{\partial \xi} \cdot \xi + \frac{\partial p_c}{\partial \eta} \cdot \eta). \]  
(8.27)
Deriving this equality we assume that the functions \((u_c, p_c)\) obey the condition
\[ \det(\frac{\partial u_c}{\partial \xi}, \frac{\partial p_c}{\partial \eta}) \neq 0. \]  
(8.28)
This means that \( \Delta F \sim \prod_x \delta(0) \). To regularize this quantity one can divide the \( \tilde{x} \) space onto \( n \) cells \[36\]. One can consider also the \( \delta \)-functions of (8.27) as the limit of Gauss distribution functions,
\[ \delta(\ldots) = \lim_{\varepsilon \to 0} \delta(\ldots) \]  
and put \( \varepsilon = 0 \) at the very end of calculations. Note, \((\xi, \eta)\) in (8.27) are arbitrary functions of \( t \).

Inserting
\[
1 = \frac{1}{\Delta F(u, p)} \int \prod_t d\xi(t) d\eta(t) 
\times \prod_{x,t} \delta(u - u_c) \delta(p - p_c) F(\xi, \eta) \]  
(8.29)
where $\Delta_F(u, p)$ was defined in (8.27), and integrating over $(u, p)$ we find the measure:

$$DM(\xi, \eta) = \prod_t d\xi(t)d\eta(t) \frac{F(\xi, \eta)}{F(\xi_0, \eta_0)} \delta(\dot{\xi} - \frac{\partial h}{\partial \eta})\delta(\dot{\eta} - \frac{\partial h}{\partial \xi})$$

if $u_c, p_c$ are defined by equations:

$$\{u_c, h_j\} = \frac{\delta H}{\delta p_c}, \quad \{p_c, h_j\} = -\frac{\delta H}{\delta u_c}$$

and $h_j$ is the total transformed Hamiltonian.

It is important to note that the constraint term

$$(F(\xi, \eta)/F(\xi_0, \eta_0)) \equiv (F/F_c)$$

was factorized in (8.30).

- The last step based on the identity [16]:

$$\prod_t \delta(X - j_x) = e^{-\frac{i}{2} \int dt \hat{j}_\xi \hat{e}_\xi e^{2i \int dt e c x j_x} \delta(X - j_x)}.$$ 

Therefore,

$$DM = e^{-\frac{i}{2} \int dt \hat{j}_\xi \hat{e}_\xi e^{2i \int dt e c x j_x} \delta(X - j_x)} \times \prod_t d^n \xi d^n \eta \delta(n) \langle \dot{\xi} - \frac{\partial h_c}{\partial \eta} - j_\xi, \dot{\eta} - \frac{\partial h_c}{\partial \xi} - j_\eta \rangle,$$

where

$$\delta u_c \wedge \delta p_c = e_\xi(t) \cdot \frac{\partial u_c}{\partial \eta(t)} - e_\eta(t) \cdot \frac{\partial u_c}{\partial \xi(t)}$$

reflects the symplectic structure of the transformed phase space.

We can calculate action of the operator $\exp\{-i\hat{K}(j, e)\}$ and in result, extracting new perturbations generating operator $\exp\{-i\hat{K}_c(j, e)\}$, where

$$\hat{K}_c(j, e) = \frac{1}{2} \int dt (\hat{j}_\xi \cdot \hat{e}_\xi + \hat{j}_\eta \cdot \hat{e}_\eta),$$

we find the measure of transformed theory

$$DM(X, Y) = \prod_t d^n \xi d^n \eta \frac{F}{F_c} \times \delta(\dot{\xi} - \frac{\partial h_c}{\partial \eta} - j_\xi, \dot{\eta} - \frac{\partial h_c}{\partial \xi} - j_\eta).$$

At the same time we should change in (8.6)

$$e(x, t) \rightarrow e_c(x, t) = \delta u_c \wedge \delta p_c.$$ 

The resulting theory describes perturbations in the $(\xi, \eta)$ phase space.

To adopt the general definitions (8.34-8.36) to our concrete problem note that our solution extracts two generators, $T_0$ and $K_0$. Therefore, for invariant subspace definition we will choose, in accordance with (8.19), $\xi_i, \eta_i, i = 1, 2$. Other coordinates can be chosen arbitrarily. For instance,

$$\xi_{2+i} = x_{0i}, \eta_{2+i} = 0, \xi_{5+i} = l_i, \eta_{5+i} = 0, i = 1, 2, 3.$$
and other \((\xi, \eta) = 0\). This means that \(u_c = u_c(\vec{x} - \vec{x}_0; \xi, \eta, \vec{l})\) is \(\eta_i, \ i = 3, 4, ...\), independent. Then

\[
\mathcal{K}_t = \int dt (\dot{\xi} \cdot \dot{\xi} + \dot{\eta} \cdot \dot{\eta})
\]

since there is not canonically conjugate pair for \(\xi_i, \ i = 3, 4, ...\). Taking into account the definition:

\[
\prod_t dX(t) \delta(\dot{X}) = dX(0) \equiv dX_0.
\]

corresponding measure take the form:

\[
DM = d^3x_0d^3l \prod_t d^2\xi d^2\eta \frac{F}{F_c} \delta^2(\dot{\xi} - \omega - j\xi) \delta^2(\dot{\eta} - j\eta)
\]

with

\[
e_c(x, t) = e\xi(t) \cdot \frac{\partial u_c(x, t)}{\partial \eta(t)} - e\eta \cdot \frac{\partial u_c(x, t)}{\partial \xi(t)}.
\]

So, the invariant subspace \(T^*\bar{G}\) only, with local coordinates \((\xi, \eta)\), is influenced by quantum perturbations.

The measure \((8.38)\) contains 10 degrees of freedom. But only 8 among them are independent. So, we should shrink the space with measure \((8.38)\) on two units. For this purpose we would use extra factor \(F(\xi, \eta)\) in \((8.26)\) choosing, for instance,

\[
F(\xi, \eta) = \delta(\sum_i \xi_{5+i}^2 - 1)\delta(\xi_1(0) - \xi_2(0))
\]

Then (see Appendix E)

\[
\frac{F}{F_c} = \delta(\bar{t}^2 - 1)\delta(\xi_1(0) - \xi_2(0))
\]

We would consider \(|\xi_i| \leq \infty\). Therefore,

\[
\{u_c\} \cap \partial\bar{W}_O = \emptyset.
\]

This completes splitting of \(W_O\) onto \(T^*\bar{G}_1 \times M_5\) invariant subspace. Note, the \(x\) dependence is practically disappeared and the reduced problem looks like quantum mechanical one. This property of our approach reflects the Lorentz-noncovariantness of developed perturbation theory.

One can use the perturbation theory in terms of the initial ‘Lagrange’ source \(j\) and conjugate to it auxiliary field \(e\). In this case the formalism is manifestly Lorentz-covariant. But this formulation is ‘rough’, it unables to take into account the topology of the \(W\) space.

- Further main results are the consequence of the statement that each term of the perturbation theory can be written as the total derivative over \((\xi, \eta)\).

We found that

\[
\rho(q) = e^{-\mathcal{K}_t(\bar{t}, e)} \int DM|\Gamma(q, u_c)|^2 e^{-iV(u_c, e_c)},
\]

where \(\mathcal{K}_t\) was defined in \((8.37)\), \(DM\) in \((8.38)\), \((8.41)\), \(V\) in \((8.6)\) with \(e_t\) defined in \((8.39)\) and the function \(u_c\) is given in \((8.20)\).

Last integrations over \((\xi, \eta)\) in \((8.42)\) are trivial. To perform them we can use the shift:

\[
\xi \rightarrow \xi_c = \xi_0 + \omega + \xi, \quad \eta \rightarrow \eta_c = \eta_0 + \eta,
\]

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where \((\xi, \eta)\) in the r.h.s. are solutions of equations:

\[
\dot{\xi} = j_\xi, \quad \dot{\eta} = j_\eta.
\] (8.44)

The Green function of this equations \(g(t - t')\) is ordinary step function \(\theta(t - t')\), see S.

The shift (8.43) gives the nonlocal operators \(\hat{j}_\gamma\):

\[
\hat{j}_\gamma(t) = \int dt' g(t - t') \hat{\gamma}(t'),
\] (8.45)

where \(\hat{\gamma} = (\hat{\xi}, \hat{\eta})\) is the four-vector and

\[
\hat{K}_t = \frac{1}{2} \int dt dt' \theta(t - t') \hat{e}(t) \cdot \hat{\gamma}(t')
\] (8.46)

with four-vector \(\hat{e} = (\hat{e}_\xi, \hat{e}_\eta)\).

The remaining integrations over \((\xi_0, \eta_0)\) are ordinary integrals with measure

\[
dM = d^3x_0 d^3l d^2\xi_0 d^2\eta_0 \delta(\xi_{c1}(0) - \xi_{c2}(0)) \delta(\vec{l}^2 - 1)
\] (8.47)

Introducing \(\exp{-i\hat{K}}\) into the integral:

\[
\rho(q) = \int dMe^{\hat{K}} |\Gamma(q; u_c)|^2 e^{-\hat{U}(u_c; q_c)}
\] (8.48)

(the trivial rotation \(e \rightarrow -ie, \hat{e} \rightarrow i\hat{e}\) was performed) we can calculate action of the operator \(\exp{-\hat{K}}\) at \(\gamma = 0\).

The operator \(\hat{K}\) is linear over \(\hat{e}\). Therefore,

\[
\rho(q) = \int dM : e^{\hat{U}(u_c; j)/4} |\Gamma(q; u_c)|^2 :
\] (8.49)

where

\[
\hat{U}(u_c, j) = 2g\Re \int_{C_+} dx dt (\delta \hat{u}_c \wedge \delta p_c)^3 u_c
\] (8.50)

where

\[
\delta \hat{u}_c \wedge \delta p_c = \hat{j}_\gamma \wedge \delta p_c
\] (8.51)

and \(\hat{j}_\gamma\) was defined in (8.45). The colons mean normal product when the operators stay to the left of functions.

Calculating the integral over \(x\) in (8.50) we reduce our field-theoretical problem to the quantum-mechanical one with complicated non-polynomial potential of interactions. We plan to consider this perturbation series for more realistic Yang-Mills theory.

We conclude this section noting that, using \(S\) and \(S\), that \(\rho(q) \neq 0\) since

\[
\{u_c\} \cap \inf \partial_n W_O \neq \emptyset.
\]

9 Instead of conclusion

Described approach is based on three ‘whales’. They are (i) the definition of observables in quantum theories as the modulo square of amplitudes, (ii) the description of quantum processes as the transformation induced by unitary
operator \( \exp\{iS(x)\} \), where \( S(x) \) is the classical action and (iii) the unitarity condition which determines connection between quantum dynamics and classical measurement. Less principal assumption, usually taken ‘by treaty’, that the quantum perturbations are switched on adiabatically was used also.

- To use all above fundamental principles the formalism in terms of observables only was considered. We tryed to show that such approach is sufficiently general being able to describe a wide spectrum of experiments (without claim on general philosophy). Our approach should be considered as the useful technical trick (probably not unique) helping to calculate the observables if the nontrivial topologies should be taken into account and the corresponding physical vacuum is so complicated that its definition is hopeless task.

Following points of the formalism should be picked out. Firstly, wishing to count quantum perturbations of fields we generalize on quantum case the postulate of classical mechanics that the initial conditions \((\xi_0, \eta_0)\) determines the classical trajectory \(\varphi\) completely, \(\varphi = \varphi(\vec{x}, t; \xi_0, \eta_0)\). In considered examples the set of this parameters form the manifold \(W_G\). Then we introduce the dynamics in the \(W_G\) space noting that the quantum motion in the \(W_G\) space should describe all excitations of \(\varphi\) since the complet set of field states \(\{\varphi\} \in W_G\). The unitarity of used mapping guaranteed by \(\delta\)-likeness of measure. This, new for quantization schemes trick, significantly simplify perturbation theory.

Secondly, we discovered that the motion in definite directions of the \(W_G\) space is exactly classical. This is the new phenomena in quantum theories. We found that \(W_G\) may decoupled on direct product of \(T^*G\) subspace and \(R^n\). The quantum dynamics is realized in the symplectic \(T^*G\) subspace of the \(W_G\) space and motion in \(R^n\) is exactly classical. Actually we found the generalization of the canonical quantization scheme.

Thirdly, the analyses of the perturbation theory allows to show that the quantum corrections are accumulated on the boundary \(\partial W_G = \partial T^*G\). Then, if \(\{\partial\varphi\} \cap \partial T^*G = \emptyset\) the quasiclassical approximation is exact. We would assume this phenomena as a basis of the confinement since in this case the particle creations generating functional is trivial. This formulation seems selfconsistent because of \(\delta\)-likeness of measure and if the above selection rule taken into account.

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A Appendix
In the CM frame we have:

\[
n_{++}(q_0) = n_{--}(q_0) = \sum_{n=0}^{\infty} ne^{-\frac{\beta_1 + \beta_2}{2}|q_0|^n} = \frac{1}{\sum_{n=0}^{\infty} e^{-\frac{\beta_1 + \beta_2}{2}|q_0|^n}} = n(|q_0|\frac{\beta_1 + \beta_2}{2}).
\]

(A.1)

Computing \(n_{ij}\) for \(i \neq j\) we must take into account that we have one more particle:

\[
n_{+-}(q_0) = \theta(q_0) \sum_{n=1}^{\infty} ne^{-\frac{\beta_1 + \beta_2}{2}|q_0|^n} + \Theta(q_0) \sum_{n=0}^{\infty} ne^{-\frac{\beta_1 + \beta_2}{2}|q_0|^n}
\]

\[
= \Theta(q_0)(1 + \tilde{n}(q_0\beta_1)) + \Theta(-q_0)\tilde{n}(-q_0\beta_1)
\]

(A.2)
and

\[ n_{+}(q_0) = \Theta(q_0)\tilde{n}(q_0\beta_2) + \Theta(-q_0)(1 + \tilde{n}(-q_0\beta_2)). \tag{A.3} \]

Using (A.1-A.3) we find the Green functions \((z = 1)\):

\[ G_{i,j}(x - x', \beta) = \int \frac{d^4 q}{(2\pi)^4} e^{iq(x-x')} \tilde{G}_{ij}(q, \beta) \tag{A.4} \]

where

\[ i\tilde{G}_{i,j}(q, \beta) = \begin{pmatrix} \tilde{n}(\beta_1 + \beta_2 |q_0|) & \tilde{n}(\beta_2 |q_0|) \alpha(\beta_2) \\ \tilde{n}(\beta_1 |q_0|) \alpha(-\beta_1) & \tilde{n}(\beta_1 + \beta_2 |q_0|) \end{pmatrix} \tag{A.5} \]

and

\[ a_\pm(\beta) = -e^{\frac{\beta}{2} |q_0| \pm q_0}. \tag{A.6} \]

**B Appendix**

To show the splitting mechanism let us consider the action of the perturbation - generating operators:

\[ e^{-iU_T(x_c, e_c)} \times \prod_t \delta(\dot{h} - j \frac{\partial x_c}{\partial \theta}) \delta(\dot{\theta} - 1 + j \frac{\partial x_c}{\partial h}) = \]

\[ = \int De_h De_\theta e^{2i\Re \int_{x_c} dt_j(\dot{h}(t)) e^{-iu_T(x_c, e_c)}}, \tag{B.1} \]

where

\[ e_c = e_h \frac{\partial x_c}{\partial \theta} - e_\theta \frac{\partial x_c}{\partial h} = (e_h \dot{\theta} - e_\theta \dot{h}) x_c. \tag{B.2} \]

The integrals over \((e_h, e_\theta)\) will be calculated perturbatively:

\[ e^{-iU_T(x_c, e_c)} = \sum_{n_h, n_\theta = 0}^{\infty} \frac{1}{n_h! n_\theta!} \]

\[ \times \int \prod_{k=1}^{n_h} (dt_k e_h(t_k)) \prod_{k=1}^{n_\theta} (dt'_k e_\theta(t'_k)) \]

\[ \times P_{n_h, n_\theta}(x_c, t_1, \ldots, t_{n_h}, t'_1, \ldots, t_{n_\theta}), \tag{B.3} \]

where

\[ P_{n_h, n_\theta}(x_c, t_1, \ldots, t_{n_h}, t'_1, \ldots, t_{n_\theta}; x'_c) = \]

\[ \times \prod_{k=1}^{n_h} \tilde{e}'_h(t_k) \prod_{k=1}^{n_\theta} \tilde{e}'_\theta(t'_k) e^{-iU_T(x_c, e'_c)} \tag{B.4} \]
with $c_e' \equiv e_c(c'_e, e'_\theta)$ and the derivatives in this equality are calculated at $e'_h = 0, e'_\theta = 0$. At the same time,

$$
\prod_{k=1}^{m_h} e_h(t_k) \prod_{k=1}^{m_\theta} e_\theta(t'_k) = \prod_{k=1}^{m_h} (i\dot{j}_h(t_k)) \prod_{k=1}^{m_\theta} (i\dot{j}_\theta(t'_k))
$$

$$
\times e^{-2i\Re \int_{C_+} dt (j_h(t)e_h(t) + j_\theta(t)e_\theta(t))}.
$$

(B.5)

The limit $(j_h, j_\theta) = 0$ is assumed. Inserting (B.4), (B.5) into (B.1) we find new representation for $R(E)$:

$$
\rho(E) = 2\pi \int_0^\infty dTe^{2i\Re \int_{C_+} dt (j_h(t)e_h(t) + j_\theta(t)e_\theta(t))}
$$

$$
\times D\theta e^{-i\hat{R}(x_c, \tau) - iV_T(x_c, e_c)}
$$

$$
\times \delta(E + \omega - h(T)) \prod_t \delta(\dot{h} - j_h) \delta(\dot{\theta} - 1 - j_\theta)
$$

(B.6)

in which the ‘energy’ and the ‘time’ quantum degrees of freedom are splitting.

C Appendix

By definition $U_T$ is the odd over $\hat{e}_c$ local functional:

$$
V_T(x_c, \hat{e}_c) = 2 \int_0^T \sum_{n=1}^\infty (\hat{e}_c(t)/2i)^{2n+1} v_n(x_c),
$$

(C.1)

where $v_n(x_c)$ is some function of $x_c$. Inserting (5.26) we find:

$$
: e^{-iV_T(x_c, \hat{e}_c)} : = \prod_{n=1}^\infty \prod_{k=0}^{2n+1} : e^{-iV_{k,n}(\hat{j}, x_c)} :,
$$

(C.2)

where

$$
V_{k,n}(\hat{j}, x_c) = \int_0^T dt (\hat{j}_\phi(t))^{2n-k+1} (\hat{j}_1(t))^k b_{k,n}(x_c).
$$

(C.3)

Explicit form of the function $b_{k,n}(x_c)$ is not important.

Using definition (5.27) it easy to find:

$$
\hat{j}(t_1)b_{k,n}(x_c(t_2)) = \Theta(t_1 - t_2)\partial b_{k,n}(x_c)/\partial X_0
$$

since $x_c = x_c(X(t) + X_0)$, see (5.23), or

$$
\hat{j}_{X,1} b_2 = \Theta_{12} \partial X_0 b_2
$$

(C.4)

since indices $(k, n)$ are not important.

Let as start consideration from the first term with $k = 0$. Then expanding $V_{0,n}$ we describe the angular quantum fluctuations only. Noting that $\partial X_0$ and $\hat{j}$ commute we can consider lowest orders over $\hat{j}$. The typical term of this expansion is (omitting index $\phi$)

$$
\hat{j}_1\hat{j}_2 \cdots \hat{j}_n b_1 b_2 \cdots b_m.
$$

(C.5)
It is enough to show that this quantity is the total derivative over $\phi_0$. The number $m$ counts an order of perturbation, i.e. in $m$-th order we have $(\hat{V}_{0,n})^m$.

$m = 1$. In this approximation we have, see (C.4),

$$\hat{j}_1 b_1 = \Theta_{11} \partial_0 b_1 = \partial_0 b_1 \neq 0.$$  \hspace{1cm} (C.6)

Here the definition (5.13) was used.

$m = 2$. This order is less trivial:

$$\hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21} b_1^2 b_2 + b_1^1 b_2^1 + \Theta_{12} b_1 b_2^2,$$  \hspace{1cm} (C.7)

where

$$b_i^n \equiv \partial_i^n b_i.$$  \hspace{1cm} (C.8)

Deriving (C.7) the first equality in (5.16) was used. At first glance (C.7) is not the total derivative. But inserting

$$1 = \Theta_{12} + \Theta_{21},$$

(see the second equality in (5.16)) we can symmetrize it:

$$\hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21} (b_1^2 b_2 + b_1^1 b_2^1) + \Theta_{12} (b_1 b_2^2 + b_1^1 b_2^1)
= \partial_0 (\Theta_{21} b_1^2 b_2 + \Theta_{12} b_1 b_2^1)$$  \hspace{1cm} (C.9)

since the explicit form of function $b$ is not important. So, the second order term can be reduced to the total derivative also. Note, that the contribution (C.9) contains the sum of all permutations. This shows the ‘time reversibility’ of the constructed perturbation theory.

Let us consider now expansion over $\hat{V}_{k,m}$, $k \neq 0$. The typical term in this case is

$$\hat{j}_1 \hat{j}_2 \cdots \hat{j}_l \hat{j}_{l+1} \hat{j}_{l+2} \cdots \hat{j}_m b_1 b_2 \cdots b_m, \quad 0 < l < m,$$  \hspace{1cm} (C.10)

where, for instance,

$$\hat{j}_k^i \equiv \hat{j}_i(t_k), \quad \hat{j}_k^\phi \equiv \hat{j}_\phi(t_k)$$

and

$$\hat{j}_1 b_2 = \Theta_{12} \partial_0^0 b_2$$  \hspace{1cm} (C.11)

instead of (C.4).

$m = 2$, $l = 1$. We have in this case:

$$\hat{j}_1 \hat{j}_2^i b_1 b_2 = \Theta_{21} (b_2 \partial_0^1 \partial_0^0 b_1 + (\partial_0^2 b_2) (\partial_0^1 \partial_0^0 b_1))
+ \Theta_{12} (b_1 \partial_0^1 \partial_0^0 b_2 + (\partial_0^2 b_2) (\partial_0^1 \partial_0^0 b_1))
= \partial_0^1 (\Theta_{21} b_2 \partial_0^0 b_1 + \Theta_{12} b_1 \partial_0^0 b_2)
+ \partial_0^2 (\Theta_{21} b_2 \partial_0^0 b_1 + \Theta_{12} b_1 \partial_0^1 \partial_0^0 b_2).$$  \hspace{1cm} (C.12)

Therefore, we have the total-derivative structure yet.

This important property of new perturbation theory is conserved in arbitrary order over $m$ and $l$ since the time-ordered structure does not depend from upper index of $\hat{j}$, see (C.11).
D Appendix

The resulting measure looks as follows:

\[
DM(\xi, \eta) = \frac{1}{\Delta_c} \delta(E - H_0) \prod_t d^2 \xi d^2 \eta \delta(r_c - \frac{\partial H_j}{\partial p_c}) \times \\
\delta(p_c + \frac{\partial H_j}{\partial r_c}) \delta(\dot{\varphi}_c - \frac{\partial H_j}{\partial \varphi_c}) \delta(l_c + \frac{\partial H_j}{\partial l_c}),
\]

(D.1)

Note that the parametrization \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\) was not specified.

A simple algebra gives:

\[
DM(\xi, \eta) = \delta(E - H_0) \prod_t d^2 \xi d^2 \eta \int \prod_t d^2 \tilde{\xi} d^2 \tilde{\eta} \\
\times \delta^2(\tilde{\xi} - (\dot{\xi} - \frac{\partial h_j}{\partial \eta}), \tilde{\eta} - (\dot{\eta} + \frac{\partial h_j}{\partial \xi})) \\
\times \delta(\frac{\partial r_c}{\partial \xi} \cdot \tilde{\xi} + \frac{\partial r_c}{\partial \eta} \cdot \tilde{\eta} + \{r_c, h_j\} - \frac{\partial H_j}{\partial p_c}) \\
\times \delta(\frac{\partial p_c}{\partial \xi} \cdot \tilde{\xi} + \frac{\partial p_c}{\partial \eta} \cdot \tilde{\eta} + \{p_c, h_j\} + \frac{\partial H_j}{\partial r_c}) \\
\times \delta(\frac{\partial \varphi_c}{\partial \xi} \cdot \tilde{\xi} + \frac{\partial \varphi_c}{\partial \eta} \cdot \tilde{\eta} + \{\varphi_c, h_j\} - \frac{\partial H_j}{\partial l_c}) \\
\times \delta(\frac{\partial l_c}{\partial \xi} \cdot \tilde{\xi} + \frac{\partial l_c}{\partial \eta} \cdot \tilde{\eta} + \{l_c, h_j\} + \frac{\partial H_j}{\partial \varphi_c}),
\]

(D.2)

The Poisson notation:

\[
\{X, h_j\} = \frac{\partial X}{\partial \xi} \frac{\partial h_j}{\partial \eta} - \frac{\partial X}{\partial \eta} \frac{\partial h_j}{\partial \xi}
\]

was introduced in (D.3).

We will define the ‘auxiliary’ quantity \(h_j\) by following equalities:

\[
\{r_c, h_j\} - \frac{\partial H_j}{\partial p_c} = 0, \quad \{p_c, h_j\} + \frac{\partial H_j}{\partial r_c} = 0, \\
\{\varphi_c, h_j\} - \frac{\partial H_j}{\partial l_c} = 0, \quad \{l_c, h_j\} + \frac{\partial H_j}{\partial \varphi_c} = 0.
\]

(D.3)

Then the functional determinant \(\Delta_c\) is canceled and

\[
DM(\xi, \eta) = \delta(E - H_0) \prod_t d^2 \xi d^2 \eta \delta^2(\tilde{\xi} - \frac{\partial h_i}{\partial \eta}) \delta^2(\tilde{\eta} + \frac{\partial h_i}{\partial \xi}),
\]

(D.4)

E Appendix

Let us consider

\[
\Delta_F(u, p) = \int \prod_t d\tilde{x}(t) d\eta(t) \prod_{\tilde{x}, t} \delta(u(\tilde{x}, t) - u_c(\tilde{x}; \xi, \eta)) \\
\times \delta(p(\tilde{x}, t) - p_c(\tilde{x}; \xi, \eta)) \mathcal{F}(\xi, \eta)
\]

(E.1)
more carefully. Let \((u_c, p_c)(x; \xi, \eta)\) are the known functions of \((\xi, \eta)(t)\) and let us assume that \((\xi, \eta)(t)\) are solutions of equations:

\[
\dot{\xi} = \partial h_j / \partial \eta, \quad \dot{\eta} = -\partial h_j / \partial \xi,
\]

with boundary condition:

\[
\xi_1(0) = x_2(0),
\]

where

\[
h_j(\xi, \eta) = H_j(u_c, p_c) = h_c(\eta) - \int d^3x j u_c, \quad p_c = \dot{u}_c.
\]

The boundary condition \((E.3)\) is necessary to have the 8 parametric solution of incident equation.

It is assumed in \((E.1)\) that one may find such functions \((u, p)(x, t)\) that \(\Delta F(u, p) \neq 0\). But we always can invert the problem saying that \((u, p)(x, t)\) are chosen in such a way that the condition \((E.3)\) is hold. So, assuming that \((\xi, \eta)c(t)\) obey the \((E.3)\) condition, we can put

\[
F(\xi_c, \eta_c) \equiv F_c = 1 \quad (E.5)
\]

To perform the mapping one should insert

\[
1 = \frac{1}{\Delta F_c} \int \prod_t d\xi(t) d\eta(t) \prod \delta(u - u_c) \delta(p - p_c) F(\xi, \eta), \quad (E.6)
\]

where \(\Delta F_c\) was defined in \((E.1)\) with condition \((E.3)\) and choosing

\[
F(\xi, \eta) = \delta(x_1(0) - x_2(0)) \delta(t^2 - 1) \quad (E.7)
\]

we come to \((8.40)\). Note that the ratio \((E.6)\) equal to one since \((u, p)c(x; \xi, \eta)\) are obey the equations \((8.24)\). The variables \((\xi, \eta)(t)\) should obey the eqs. \((E.2)\) and the boundary condition is fixed by \((E.7)\).

References

[1] F.Wilchek, Beyond the Standard Model: an Answer and Twenty Questions, hep-ph/9802400

[2] R.Rajaraman, Solitons and Instantons (North-Holland Publ. Comp., Amsterdam, New York, Oxford, 1982)

[3] M.Shifman, Lectures given at the 1997 Yukawa Int. Sem. ‘Non-Perturbative QCD - Structure of QCD Vacuum’, Kyoto, Dec. 2, 1997, hep-ph/9802214

[4] A.M.Polyakov, Phys. Lett., B72, 477 (1978); A.M.Polyakov, String Theory and Confinement, hep-th/9711002; L.McLeran, Rev. Mod. Phys., 58, 1021 (1986)

[5] B.Svetitsky and L.Yaffe, Nucl. Phys., B210 [FS6], 423 (1982);

[6] G.t’Hooft, Acta Phys. Austr. Suppl., XXII, 531 (1980)

[7] J.Ellis, Aspects of M theory and Phenomenology, hep-ph/9804440

[8] J.Manjavidze, Sov.Nucl.Phys., 45, 442 (1987)

[9] S.Smale, Inv. Math., 11:1, 45 (1970)

[10] J.Manjavidze and A.Sissakian, JINR Rapid Comm., 5/31, 5 (1988); J.Manjavidze and A.Sissakian, JINR Rapid Comm., 2/288, 13 (1988)
[11] J.Manjavidze, hep-ph/9802318, to be published in El. Part. and At. Nucl. (1999)

[12] E.Wigner, Phys. Rev., 40, 749 (1932); K.Hisimi, Proc. Phys. Math. Soc. Jap., 23, 264 (1940); R.J.Glauber, Phys. Rev. Lett., 10, 84 (1963); E.C.G.Sudarshan, Phys. Rev. Lett., 10, 177 (1963); R.E.Cahill and R.G.Glauber, Phys. Rev., 177, 1882 (1969)

[13] P.Carrusers and F.Zachariasen, Phys. Rev., D37, 950 (1986); P.Carrusers and F.Zachariasen, Rev. Mod. Phys., 55, 245 (1983)

[14] S.Mancini, V.I.Man’ko and P.Tombesi, Quant. Semicl. Opt., 7, 615 (1995); V.I.Man’ko, L.Rose and P.Vitale, Probability Representation in Quantum Field Theory, hep-th/9806166

[15] L.Landau and R.Peierls, Zs. Phys., 69, 56 (1931)

[16] J.Manjavidze, Perturbation theory in the invariant subspace, hep-th/9801188

[17] T.W.Kibble, Comm. Math. Phys., 65, 189 (1979)

[18] J.Anandan, Found. Phys., 21, 1265 (1991)

[19] N.P.Konopleva and V.N.Popov, Gauge fields (Harward Acad. Publ., Chur-London-New York, 1981)

[20] S.Coleman, The Uses of Instantons, (The Whys of Subnuclear Physics, Proc. of the 1977 Int. School of Subnucl. Phys., Eric, Italy, Ed. A. Zichichi, N.Y., Plenum, 1979)

[21] J.Goldstone and R.Jackiw, Phys.Rev., D11, 1486 (1975)

[22] V.E.Korepin anf L.D.Faddeev, Theor. Math. Phys., 25, 147 (1975)

[23] M.S.Marinov, Phys.Rep., 60, 1 (1980)

[24] C.Grosche, Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae (World Scint., Singapore, New Jersey, London, Hong Kong, 1995)

[25] S.Smale, Usp. Math. Nauk, XXXVII, 77 (1972)

[26] J.S.Dowker, Ann. Phys. (NY), 62, 361 (1971)

[27] J.Manjavidze, The Unitary Transformation of the Path-Integral Measure, quant-ph/9507003

[28] J.Manjavidze, On the Canselation of Quantum-Mechanical Corrections at the Periodic Motion, hep-ph/9509202

[29] V.I.Arnold,Mathematical Methods of Classical Mechanics,(Springer Verlag, New York, 1978)

[30] R.Abraham and J.E.Marsden, Foundations of Mechanics (Benjamin/ Cummings Publ. Comp., Reading, Mass., 1978)

[31] V.S.Popov, High Energy Physics and Elementary Particles Theory, (Naukova Dumka, Kiev, 1967)

[32] I.H.Duru and H.Kleinert, Phys. Lett., B84, 185 (1979)

[33] L.D.Faddeev in Solitons, 363 (Ed. by R.K.Bullough and P.J.Caydry, Springer-Verl., Heidelberg, New York, 1980)

[34] Zamolodchikov, A.B. and A.B.Zamolodchikov, Phys. Lett., 72B, 503 (1978)

[35] R.Dashen, B.Hasslacher and A.Neveu, Phys. Rev., D10, 3424(1975)
[36] L.D.Faddeev and L.A.Takhtajan, Sov. TMF, 21, 160 (1974); L.A.Takhtajan and L.D.Faddeev, Hamilton Approach in Solitons Theory (Moskow, Nauka, 1986)

[37] R.Mills, Propagators of Many-Particles Systems, (Gordon & Breach, 1969)

[38] A.Actor, Rev. Mod. Phys., 51, 461 (1979)

[39] V.DeAlfaro, S.Fubini and G.Furlan, Phys. Lett., B65, 163 (1976)

[40] N.P.Landsman and Ch.G.vanWeert, Phys. Rep., 145, 141 (1987)

[41] N.N.Bogolyubov, Studies in Statistical Mechanics, eds. J.deBoer and G.E.Uhlembeck (North-Holland, Amsterdam, 1962)

[42] E.Calsetta and B.Hu, Phys. Rev.D37, 2878 (1988)

[43] A.J.Niemi and G.Semenoff, Ann. Phys. (NY), 152 105 (1984)

[44] L.Keldysh, JETP (Sov. Phys.), 20, 1018 (1964)

[45] P.M.Bakshi and K.T.Mahanthappa, J.Math.Phys., 4, 1, (1961); ibid., 4, 12 (1961)

[46] E.Byukling and K.Kajantie, Particles Kinematics (John Wiley and Sons, London, 1973)

[47] M.Martin and J.Schwinger, Phys.Rev., 115, 342 (1959)

[48] R.Kubo, J.Phys.Soc.Japan, 12, 570 (1957)

[49] J.Schwinger, Particles, Sources and Fields Vol.1 (Addison-Wesley Pabl.Comp., 1970)

[50] R.Haag, N.Hugengoltz and M.Winnink, Commun.Math.Phys., 5, 5 (1967)

[51] H.Chu and H.Umezawa, Int.J.Mod.Phys., A9 2363 (1994)

[52] H.Matsumoto, Y.Nakano, H.Umetzava, F.Mancini and M.Marinaro, Prog. Theor. Phys., 70, 559(1983); H.Matsumoto, Y.Nakano and H.Umetzava, J.Math.Phys., 25, 3076 (1984)

[53] D.N.Zubarev, Nonequilibrium Statistical Thermodynamics (Consultants Bureau, NY, 1974)

[54] V.Fock, Vestnik LGU, 16, 442 (1959)

[55] T.Matsubara, Prog.Theor.Phys., 14, 351 (1955)

[56] S.Coleman, Phys.Rev., D15, 2929 (1977)

[57] D.Jordan, Phys.Rev., D33, 44 (1986)