Emission of scalar particles from cylindrical black holes

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Abstract: We study quantum tunneling of scalar particles from black strings. For this purpose we apply WKB approximation and Hamilton-Jacobi method to solve the Klein-Gordon equation for outgoing trajectories. We find the tunneling probability of outgoing charged and uncharged scalars from the event horizon of black strings, and hence the Hawking temperature for these black configurations.
1. Introduction

Black holes are objects in this universe with such a strong gravitational field that even light cannot escape from them. The important breakthrough in the field of black hole physics occurred when Stephen Hawking showed that quantum mechanically black holes emit radiations [1, 2]. Due to the strong gravitational field and vacuum fluctuations at the event horizon of the black hole, virtual particles-anti particles are created. Here we can have three types of scenarios: (a) both the particles fall into the hole, (b) both of them escape from the event horizon, and (c) one particle falls into the hole while the other escapes. The particle that escapes appears as the Hawking radiation. The negative energy particle that falls into the black hole reduces the mass, charge and the angular momentum of the black hole. As a result, the black hole shrinks. This particle must go into the black hole to conserve energy.

After Hawking’s discovery these thermal radiations have been studied for different black bodies. There are different methods to derive Hawking radiation and Hawking temperature. These can be studied, for example, by calculating the Bogoliubov transformation [1, 3] between the initial and final states of ingoing and outgoing radiation. The Wick rotation method [4, 5] is also used for investigating Hawking radiation. Recently, the black hole tunneling method ([6]- [20]), anomaly method [21] and the technique of dimensional reduction [22] have been used to investigate Hawking radiation and Hawking temperature. The radiation spectrum from black holes contains all types of particles including scalar particles [16, 17]. Here, we have used the black hole tunneling method to derive the tunneling probability of scalar particles from black strings ([23]- [26]). In order to do this we solve the Klein-Gordon equation by using WKB approximation and complex path integration. As a result we obtain Hawking temperature also.

2. Black strings

The Einstein field equations have a large number of solutions. Here, we discuss some special solutions of these equations, which are exact with negative cosmological constant, called black strings or cylindrical black holes.

A four dimensional metric with $g_{\mu \nu}$ $(\mu, \nu = 0, 1, 2, 3)$ is given by [25]

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = g_{mn} dx^m dx^n + e^{-4\phi} dz^2,$$  \hspace{1cm} (2.1)

where $g_{mn}$ and $\phi$ are metric functions, $m, n = 0, 1, 2, \ x^\mu = (t, r, \theta, z)$ and $z$ is the Killing coordinate. We will write a cylindrically symmetric metric by taking the $\theta$
coordinate also in Killing direction from Eq. (2.1). We consider the Einstein-Hilbert action in four dimensions with a negative cosmological constant in the presence of an electromagnetic field. The total action is given by

\[ S + S_{em} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}. \]  

(2.2)

Here, \( S \) is the Einstein-Hilbert action in four dimensions, \( S_{em} \) is the action for electromagnetic field, \( R \) is the Ricci scalar, \( g \) the determinant of the metric tensor, \( \Lambda \) the cosmological constant, \( G \) the gravitational constant, and the Maxwell tensor \( F_{\mu\nu} \) is given by

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  

(2.3)

where \( A_\nu \) is vector potential and is given by \( A_\nu = -h(r)\delta^0_\nu \), \( h(r) \) being an arbitrary function of the radial coordinate \( r \). Here, we take the solution of the Einstein-Maxwell equations with cylindrical symmetry. The line element for static charged black string with negative cosmological constant in the presence of electromagnetic field becomes [25, 26]

\[ ds^2 = -(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}) dt^2 + (\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2})^{-1} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \]  

(2.4)

where

\[ \alpha^2 = \frac{-1}{3} \Lambda, \]  

(5.5)

\[ b = 4GM, \]  

(2.6)

\[ c^2 = 4GQ^2, \]  

(2.7)

\[ h(r) = \frac{2Q}{\alpha r} + \text{const.}, \]  

(2.8)

\[ -\infty < t < \infty, 0 \leq r < \infty, -\infty < z < \infty, 0 \leq \theta \leq 2\pi. \]  

(2.9)

Here, \( Q \) is the linear charge density per unit length of the \( z \) line and \( M \) is mass per unit length of the \( z \) line of black string. The event (outer) horizon can be found by putting \( \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} = 0 \) and is given by [25]

\[ r = r_+ = \frac{b \sqrt{s} + \sqrt{2s^2 - 4p^2 - s}}{2\alpha}, \]  

(2.10)

where

\[ s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left(\frac{4p^2}{3}\right)^3} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left(\frac{4p^2}{3}\right)^3} \right)^{\frac{1}{3}}, \]  

\[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left(\frac{4p^2}{3}\right)^3}, \]  

\[ \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left(\frac{4p^2}{3}\right)^3}, \]  

\[ \sqrt{2s^2 - 4p^2 - s}, \]  

\[ 2\alpha. \]
\[ p^2 = \frac{c^2}{b^2}. \]

If we put \( Q = 0 \) in Eq. (2.4) we obtain the simplest case of black strings, which contains only one parameter, which is the mass of black string. The line element for this case is given by

\[ ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)dt^2 + \left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)^{-1}dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2. \]  
\[ (2.11) \]

The outer horizon for this black string can be found by putting \( \alpha^2 r^2 - \frac{b}{\alpha r} = 0 \), which gives

\[ r = r_+ = \frac{b}{\alpha}. \]  
\[ (2.12) \]

3. Quantum tunneling of scalar particles from black strings

In order to work out the tunneling probability of scalar particles from the event horizon of black strings we use the Klein-Gordon equation for a scalar field \( \Psi \) given by

\[ g^{\mu\nu} \partial_\mu \partial_\nu \Psi - \frac{m^2}{\hbar^2} \Psi = 0. \]  
\[ (3.1) \]

Using WKB approximation, we assume an ansatz of the form

\[ \Psi(t, r, \theta, z) = e^{\left(\frac{i}{\hbar} I(t,r,\theta,z) + O(h)\right)}, \]  
\[ (3.2) \]

for Eq. (3.1) which can be written as

\[ g^{00} \partial_t \partial_t \Psi + g^{11} \partial_r \partial_r \Psi + g^{22} \partial_\theta \partial_\theta \Psi + g^{33} \partial_z \partial_z \Psi - \frac{m^2}{\hbar^2} \Psi = 0. \]  
\[ (3.3) \]

Now by using Eq. (3.2) in Eq. (3.3) and evaluating term by term in the highest order of \( \hbar \) and dividing by the exponential term and multiplying by \( \hbar^2 \), we get

\[ 0 = -(\alpha^2 r^2 - \frac{b}{\alpha r})^{-1}(\partial_t I)^2 + (\alpha^2 r^2 - \frac{b}{\alpha r})(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{\alpha^2 r^2}(\partial_z I)^2. \]  
\[ (3.4) \]

Keeping in view the Killing fields, \( \partial_t, \partial_\theta \) and \( \partial_z \), of the background spacetime, we separate the variables and consider a solution for Eq. (3.4) of the form

\[ I(t, r, \theta, z) = -Et + W(r) + J_1 \theta + J_2 z + K, \]  
\[ (3.5) \]
where \( E, J_1, J_2 \) and \( K \) are constants. Here, we are only considering the radial trajectories. Using Eq. (3.5) in Eq. (3.4) yields after simplification

\[
W'(r) = \pm \sqrt{-\frac{g^{00}}{g^{11}}} \left( E^2 + \frac{g^{22}}{g^{00}}(J_1)^2 + \frac{g^{33}}{g^{00}}(J_2)^2 + \frac{1}{g^{00}} m^2 \right). \tag{3.6}
\]

Integrating this and substituting the values of \( g^{\mu\nu} \) gives,

\[
W(r) = \pm \int \frac{\sqrt{E^2 - f(r) \left( m^2 + \frac{(J_1)^2 + \frac{(J_2)^2}{\alpha^2}}{r^2} \right)}}{f(r)} \, dr, \tag{3.7}
\]

where

\[
f(r) = \alpha^2 r^2 - \frac{b}{\alpha r}. \tag{3.8}
\]

We have to integrate Eq. (3.7) around the pole at the event horizon, \( r_+ = \frac{b^+}{\alpha} \). We use the residue theory for semi circle yielding

\[
W_\pm(r) = \pm \frac{\pi i}{f'(r_+)} \sqrt{E^2 - f(r_+) \left( m^2 + \frac{(J_1)^2 + \frac{(J_2)^2}{\alpha^2}}{r^2_+} \right)}. \tag{3.9}
\]

As \( f(r_+) = 0 \), the above equation reduces to

\[
W_\pm(r) = \pm \frac{\pi i E}{f'(r_+)}, \tag{3.10}
\]

which implies that

\[
ImW_\pm(r) = \pm \frac{\pi E}{f'(r_+)}, \tag{3.11}
\]

where

\[
f'(r_+) = 2\alpha^2 r_+ + \frac{b}{\alpha r^2_+} = 3\alpha b^+. \tag{3.12}
\]

The probabilities of crossing the horizon from inside to outside and outside to inside are given by [8, 10]

\[
P_{\text{emission}} \propto \exp \left( \frac{-2}{\hbar} ImI \right) = \exp \left( \frac{-2}{\hbar} (ImW_+ + ImK) \right), \tag{3.13}
\]

\[
P_{\text{absorption}} \propto \exp \left( \frac{-2}{\hbar} ImI \right) = \exp \left( \frac{-2}{\hbar} (ImW_- + ImK) \right). \tag{3.14}
\]
We know that the probability of any incoming particles crossing the horizons and entering the black hole is one, so it is necessary to set

\[ \text{Im}K = -\text{Im}W, \quad (3.15) \]

in the above equations. From Eq. (3.10), we have

\[ W_+ = -W_. \quad (3.16) \]

This means that the probability of a particle tunneling from inside to outside the horizon is

\[ \Gamma = \exp\left(-\frac{4}{\hbar} \text{Im}W_+\right). \quad (3.17) \]

Thus, by using Eq. (3.11), (by choosing \( \hbar = 1 \)), we get

\[ \Gamma = \exp\left(-\frac{4\pi E}{f'(r_+)}\right). \quad (3.18) \]

This is the probability of the outgoing scalar particle from the event horizon, \( r = r_+ \). We can find the Hawking temperature, by comparing Eq. (3.18) with the Boltzmann factor \([8, 10], \Gamma = \exp(-\beta E)\), where \( E \) is the energy of the particle and \( \beta \) is the inverse of Hawking temperature. Thus we get

\[ T_H = \frac{f'(r_+)}{4\pi}, \quad (3.19) \]

or

\[ T_H = \frac{3\alpha b^{\frac{1}{3}}}{4\pi}. \quad (3.20) \]

4. Quantum tunneling of scalar particles from charged black strings

To study the quantum tunneling from charged black strings (Eq. (2.4)) for scalar field \( \Psi \), we use the charged Klein-Gordon equation

\[ \frac{1}{\sqrt{-g}} \left( \partial_\mu - \frac{iq}{\hbar} A_\mu \right) \left( \sqrt{-g} g^{\mu\nu} (\partial_\nu - \frac{iq}{\hbar} A_\nu) \Psi \right) - \frac{m^2}{\hbar^2} \Psi = 0. \quad (4.1) \]

Proceeding as before, we apply WKB approximation and assume the field of the form given in Eq. (3.2). Substituting this in Eq. (4.1) and keeping terms only in the leading order of \( \hbar \) and dividing by exponential term and multiplying by \( \hbar^2 \) gives

\[ g^\mu(\partial_\mu I + qh(r))^2 + g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2 + m^2 = 0. \quad (4.2) \]
Again assuming a solution of the type of Eq. (3.5) for the above equation and solving for \( W(r) \) we get

\[
W_\pm(r) = \pm \int \frac{\sqrt{(-E + qh(r))^2 - f(r) \left(m^2 + \frac{(J_2)^2 + (J_2)^2}{r^2}\right)}}{f(r)} dr, \quad (4.3)
\]

where

\[
f(r) = \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}. \quad (4.4)
\]

Using the complex integration techniques, the integral around the simple pole at the event horizon given by Eq. (2.10) yields

\[
W_\pm(r) = \pm \frac{\pi i (-E + qh(r_+))}{f'(r_+)}. \quad (4.5)
\]

This implies that

\[
ImW_\pm(r) = \pm \frac{\pi (-E + qh(r_+))}{f'(r_+)}, \quad (4.6)
\]

where

\[
f'(r_+) = 2\alpha^2 r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2 r_+^3}, \quad (4.7)
\]

\[
h(r_+) = \frac{2Q}{\alpha r_+}. \quad (4.8)
\]

Thus the tunneling probability of scalar particles from the charged black string comes out to be

\[
\Gamma = \exp \left(-\frac{4\pi (-E + qh(r_+))}{hf'(r_+)}\right). \quad (4.9)
\]

From this, we can find the Hawking temperature by comparing this with the Boltzmann factor of particle energy

\[
T_H = \frac{f'(r_+)}{4\pi}, \quad (4.10)
\]

where \( f'(r_+) \) is given in Eq. (4.7). So

\[
T_H = \frac{1}{4\pi} \left(2\alpha^2 r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2 r_+^3}\right), \quad (4.11)
\]

which is consistent with the literature [20, 27].
5. Conclusion

In this paper we have studied Hawking radiation of scalar particles from uncharged and charged black strings. By using Hamilton-Jacobi method we have solved the charged and uncharged Klein-Gordon equations. In order to do this we have employed WKB approximation to Klein-Gordon equation to derive the tunneling probability of outgoing particles. At the end, by comparing with the Boltzmann factor of energy for the particles, we have derived the Hawking temperature for these black configurations. These results are found to be consistent with the literature. If we put $Q = 0$ in Eq. (4.11), the temperature reduces to that for the uncharged case in Eq. (3.20).

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