Propagation of Gevrey regularity for solutions of the Boltzmann equation for Maxwellian molecules

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Abstract

We prove that Gevrey regularity is propagated by the Boltzmann equation with Maxwellian molecules, with or without angular cut-off. The proof relies on the Wild expansion of the solution to the equation and on the characterization of Gevrey regularity by the Fourier transform.

1 Introduction

This paper deals with a propagation property for the solution of the following Cauchy problem for the spatially homogenous Boltzmann equation for Maxwellian molecules

\[
\begin{cases}
\partial_t f(v,t) = Q(f,f)(v,t) \\
f(v,0) = f_0(v).
\end{cases}
\]

Here, \(f(v,t) : \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}\) is the probability density of a gas which depends only on the velocity \(v \in \mathbb{R}^3\) at the time \(t \geq 0\) and \(Q\) is the quadratic Boltzmann collision operator in the case of Maxwellian molecules:

\[
Q(f,f)(v,t) = \int_{w \in \mathbb{R}^3} \int_{n \in S^2} (f(v_*,t)f(w_*,t) - f(v,t)f(w,t)) b \left( \frac{v-w}{|v-w|} \cdot n \right) dn dw.
\]
Due to the physical assumptions that the gas evolves through binary, elastic collisions which are localized both in space and time, the relations between the velocities \( (v_*,w_*) \) of two particles before the collision and \( (v,w) \) after it are the following:

\[
\begin{align*}
v_* &= \frac{v + w}{2} + \frac{|v - w|}{2} n, \\
w_* &= \frac{v + w}{2} - \frac{|v - w|}{2} n,
\end{align*}
\]

where \( n \) is a vector in \( S^2 \), the unit sphere in \( \mathbb{R}^3 \), and parametrizes all the possible pre-collisional velocities.

The collision kernel \( b \), which is supposed to be nonnegative, is the function which selects in which way the pre-collisional velocities contribute to produce particles with velocity \( v \) after the collision and is supposed (this is precisely the assumption of Maxwellian molecules) to depend only on the cosine of the deviation angle \( \theta \), namely

\[ \cos \theta = \frac{v - w}{|v - w|} \cdot n. \]

Finally, we will make the so-called non cut-off assumption, which means that \( b \notin L^1_{\text{loc}}([-1,1]) \) and, more precisely, we shall consider

\[ b(\cos \theta) \sim \frac{1}{(1 - \cos \theta)^{\frac{5}{4}}} \quad \theta \to 0. \]

From a physical point of view, that means that the gas molecules repel each other with a force proportional to the fifth power of their distance and a great contribution to the integral collision term is given by the grazing collisions \( (\theta \sim 0) \). The assumption that the collision kernel \( b \) is instead integrable on \([-1,1]\] is called a cut-off assumption. For more information about Boltzmann equation and its physical meaning, the reader can consult for instance the review article by Villani [Vil02].

Due to the singularity of the collision kernel at the origin, the integral term (1) is not meaningful if \( f \) is not smooth and so it is convenient to consider the weak form of the Boltzmann equation: for \( \varphi \in C^\infty_c(\mathbb{R}^3) \),

\[
\int_{v \in \mathbb{R}^3} \partial_t f(v,t) \varphi(v) \, dv = \int_{v \in \mathbb{R}^3} Q(f,f)(v,t) \varphi(v) \, dv = \int_{v \in \mathbb{R}^3} \int_{w \in \mathbb{R}^3} \int_{n \in S^2} f(v,t)f(w,t) \left( \varphi(v_*) - \varphi(v) \right) b \left( \frac{v - w}{|v - w|} \cdot n \right) \, dn \, dw \, dv,
\]

or even, with another point of simplification, in the Fourier variable,

\[
\partial_t \hat{f}(\xi,t) = \int_{n \in S^2} \left( \hat{f}(\xi^+,t)\hat{f}(\xi^-,t) - \hat{f}(\xi,t)\hat{f}(0,t) \right) b \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn = \hat{Q}(\hat{f},\hat{f})(\xi,t),
\]

as was firstly done by Bobylev (see for instance [Bob88]). Here we have used the standard notations

\[ \xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2} n, \quad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2} n. \]
The first results about the non cut-off case for the weak equation go back to Arkeryd [Ark81], in a more general setting. In [PT96], A. Pulvirenti and Toscani reformulated the existence theory starting from the equation in the Fourier variable both for the cut-off and non cut-off cases. We briefly recall their method and their result, because they will be useful in the following. The classical approach is to find a solution of equation (2) through a limiting process on the solutions of a sequence of cut-off approximating problems in the following way. Let us consider the following sequence of bounded functions obtained by cutting out the singularity of $b$ at the origin

$$\tilde{b}_l = \min(b, l), \quad l \in \mathbb{N},$$

and let

$$b_l^* = \int_{n \in S^2} \tilde{b}_l \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn.$$  

Then, define

$$\beta_l = \frac{\tilde{b}_l}{b_l^*},$$

so that

$$\int_{n \in S^2} \beta_l \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn = 1,$$

and then consider the sequence of Cauchy problems

$$\begin{cases}
\frac{\partial}{\partial \tau} \varphi_l(\xi, \tau) = \int_{n \in S^2} \left( \varphi_l(\xi^+, \tau) \varphi_l(\xi^-, \tau) - \varphi_l(\xi, \tau) \varphi_l(0, \tau) \right) \beta_l \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn,
\varphi_l(\xi, 0) = \hat{f}_0(\xi).
\end{cases}$$

A. Pulvirenti and Toscani proved first the existence and uniqueness of a solution $\varphi_l$ of the Cauchy problems (6). Then, letting

$$\hat{f}_l(\xi, t) := \varphi_l(\xi, b_l^* t),$$

they proved the convergence in a suitable setting of a subsequence of $\hat{f}_l$ to a solution of the Cauchy problem for the non cut-off equation. More precisely, the result is as follows:

**Theorem 1 (A. Pulvirenti, Toscani [PT96])**

We consider an initial datum $f_0 \geq 0$ satisfying the following assumptions:

$$\int_{\mathbb{R}^3} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) v_i \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} f_0(v)|v|^2 \, dv = 3,$$

$$\int_{\mathbb{R}^3} f_0(v) \log f_0(v) \, dv < \infty,$$

and the following Cauchy problem:

$$\begin{cases}
\frac{\partial}{\partial \tau} \hat{f}(\xi, t) = \int_{n \in S^2} \left( \hat{f}(\xi^+, t) \hat{f}(\xi^-, t) - \hat{f}(\xi, t) \hat{f}(0, t) \right) b \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn, \quad t > 0, \\
\hat{f}(\xi, 0) = \hat{f}_0(\xi)
\end{cases}$$

3
where $b$ is a nonnegative function of $L^1_{\text{loc}}([-1,1])$ satisfying $b(\cos \theta) = O \left( \frac{1}{(1 - \cos \theta)^{5/4}} \right)$, $\theta \to 0$.

Then, there exists a nonnegative solution $f \in C^1([0, +\infty), L^1(\mathbb{R}^3))$ to eq. (7) satisfying for all $t > 0$:

$$
\int_{\mathbb{R}^3} f(v,t) \, dv = 1, \quad \int_{\mathbb{R}^3} f(v,t) \, v_i \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} f(v,t) |v|^2 \, dv = 3,
$$

$$
\int_{\mathbb{R}^3} f(v,t) |\log f(v,t)| \, dv < \infty.
$$

Moreover, for all $t > 0$, the Fourier transform $\hat{f}(\cdot, t)$ of the solution is obtained as the (uniform on compact sets) limit of a subsequence of the functions $\varphi l(\cdot, b^* t) \in C^1([0, +\infty), C_b(\mathbb{R}^3))$, solutions of the cut-off Cauchy problems (6), which have the following explicit representation (called Wild’s expansion):

$$
\varphi l(\xi, \tau) = e^{-\tau} \sum_{k=0}^{\infty} \varphi l^{(k)}(\xi)(1 - e^{-\tau})^k,
$$

where

$$
\varphi l^{(0)}(\xi) = \hat{f}_0(\xi),
\varphi l^{(k+1)}(\xi) = \frac{1}{k + 1} \sum_{j=0}^{k} \int_{n \in S^2} \varphi l^{(j)}(\xi^+) \varphi l^{(k-j)}(\xi^-) \beta_l \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn.
$$

In [TV99], Toscani and Villani proved that under the hypotheses of Theorem 1, the solution for the non cut-off equation is indeed unique, whereas the uniqueness of the solution for the cut-off problem (6) was already known.

The question whether an extra property satisfied by the initial datum $f_0$ propagates along the solution has been already addressed concerning Sobolev or Lebesgue regularity. In [DM05], Desvillettes and Mouhot proved the uniform propagation of $L^p$ moments for both the cut-off and non cut-off equations (Cf. also [Gus86], [Gus88], [MV04] for earlier works on the propagation of $L^p$ regularity). In [CGT99], Carlen, Gabetta and Toscani proved for the cut-off equation that also all the $H^s$ Sobolev norms remain uniformly bounded if they exist initially (Cf. also [MV04] for related results in the case of hard potentials and hard spheres). When the non cut-off equation is considered, the same is true, and moreover the $H^s$ norms are immediately created (Cf. [Des95, Des03]). In [Uka84], Ukai proved for both the cut-off and non cut-off equations that a regularity property of Gevrey type satisfied by the initial datum keeps on being satisfied at least for a finite time by the solution. We shall come back to this result later.

Note finally that many papers address the important question of the propagation of the behavior of the solution with respect to large $v$ (that is, propagation of moments, evolution of Maxwellian tales, etc.). We do not investigate in this direction in this work.

This paper is devoted to the discussion of the following question: if the initial datum $f_0$ satisfies the upper bound

$$
|\hat{f}_0(\xi)| \leq K_1 e^{-K_2|\xi|^s}, \quad \xi \in \mathbb{R}^3, \quad K_1 \geq 1, \quad K_2 > 0, \quad s > 0,
$$

then...
does the solution (of the cut-off or non cut-off Boltzmann equation with Maxwellian molecules) keep on satisfying the same property? The answer is positive, provided that \( s \in [0, 2] \) and we allow the constants \( K_1, K_2 \) to be different from those of the initial datum.

More precisely, the result we are going to prove is the following:

**Theorem 2**

Let \( f_0 \) be a nonnegative function satisfying

\[
\int_{\mathbb{R}^3} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} f_0(v)|v|^2 \, dv = 3,
\]

\[
\int_{\mathbb{R}^3} f_0(v) \log f_0(v) \, dv < \infty.
\]

If \( f_0 \) is such that

\[
\sup_{|\xi| \leq R_0} |\hat{f}(\xi)| e^{K_2 \psi(|\xi|^2)} \leq K_1,
\]

for some \( K_1 \geq 1, \) \( K_2 > 0, \) and for some concave function \( \psi : [0, +\infty) \to [0, +\infty), \) such that \( \psi(0) = 0, \) \( \psi(r) \leq r \) for \( r \) large enough and \( \psi(r) \to +\infty \) for \( r \to +\infty, \) then there exist \( R_0 > 0, K > 0 \) such that the unique solution of the Cauchy problem (7) with \( f_0 \) as initial datum satisfies

\[
\sup_{|\xi| < R_0} |\hat{f}(\xi, t)| e^{K|\xi|^2} \leq 1, \quad t \geq 0,
\]

\[
\sup_{|\xi| \geq R_0} |\hat{f}(\xi, t)| e^{K\psi(|\xi|^2)} \leq 1, \quad t \geq 0.
\]

Denoting by \( G^\nu(\mathbb{R}^3) \) the space of Gevrey functions and by \( G^\nu_0(\mathbb{R}^3) \) the space of Gevrey functions with compact support (we shall recall in Section 4 their definition), we are able to deduce from the previous result the propagation along the solution of a Gevrey-type regularity satisfied by the initial datum.

**Corollary 3**

Let \( f_0 \) be a nonnegative function satisfying

\[
\int_{\mathbb{R}^3} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} f_0(v)|v|^2 \, dv = 3,
\]

\[
\int_{\mathbb{R}^3} f_0(v) \log f_0(v) \, dv < \infty. \quad (8)
\]

i) If \( \nu > 1 \) and \( f_0 \in G^\nu_0(\mathbb{R}^3), \) then the solution \( f(\cdot, t) \) of the Cauchy problem (7) is in \( G^\nu(\mathbb{R}^3), \) uniformly for all \( t \geq 0. \)

ii) If \( \nu \geq 1, \) \( f_0 \in G^\nu(\mathbb{R}^3) \cap S'(\mathbb{R}^3), \) and moreover satisfies \( \sup_{\xi \in \mathbb{R}^3} |\hat{f}_0(\xi)| \leq K_1 e^{-K_2|\xi|^2} \)

for \( K_1 \geq 1, K_2 > 0, \) then the solution \( f(\cdot, t) \) of the Cauchy problem (7) is in \( G^\nu(\mathbb{R}^3), \)

uniformly for all \( t \geq 0. \)

The plan of the paper is the following:
– in Section 2, we shall present the result in a simpler form and for the so-called Kac model
(which is 1-dimensional and describes radially symmetric solutions of the Boltzmann equation);
– in Section 3, we shall generalize the result both to Boltzmann equation and to more
general bounds on the initial datum;
– finally, in Section 4, we shall recall the main definitions of Gevrey functions, and we
shall state the propagation result of a Gevrey-type regularity.

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2 The Kac equation

In this section, we present our result in the simpler case of the Kac equation. This equation, in
its cut-off or non cut-off version, is obtained when one considers radially symmetric solutions
of the homogenous Boltzmann equation for Maxwellian molecules. It reads:

$$\partial_t f(v,t) = \int_{w \in \mathbb{R}} \int_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} (f(\tilde{v}_s, t)f(\tilde{w}_s, t) - f(v, t)f(w, t)) \, b(\theta) \, d\theta \, dw = \tilde{Q}(f, f)(v, t).$$

Here, \( f(v, t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) is the probability density of a gas of one dimensional particles
which depends only on the velocity \( v \in \mathbb{R} \) at the time \( t \geq 0 \), and which evolves through
collisions which conserve energy but not momentum. The relations between the velocities
(\( \tilde{v}_s, \tilde{w}_s \)) of two particles before the collision and \( (v, w) \) after it are the following

\[
\begin{align*}
\tilde{v}_s &= v \cos \theta + w \sin \theta, \\
\tilde{w}_s &= v \sin \theta - w \cos \theta.
\end{align*}
\]

We shall make the following non cut-off assumption on the collision kernel \( b \):

\[
b(\theta) = O_{\theta \to 0} \left( \frac{\cos \theta}{|\sin \theta|^\gamma} \right), \quad \gamma \in [1, 3[.
\]

Actually, this kind of assumption for the Kac equation was introduced by Desvillettes in
[Des95] whereas, in the original equation, \( b(\theta) \) is a strictly positive constant. In the same way
as for the Boltzmann equation, it is useful to consider the Cauchy problem in the Fourier
variable

\[
\begin{align*}
\partial_t \hat{f}(\xi, t) &= \int_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left( \hat{f}(\xi \cos \theta, t)\hat{f}(\xi \sin \theta, t) - \hat{f}(\xi, t)\hat{f}(0, t) \right) b(\theta) \, d\theta = \hat{Q}(f, f)(\xi, t), \\
\hat{f}(\xi, 0) &= \hat{f}_0(\xi),
\end{align*}
\]

where the even initial datum \( f_0 \geq 0 \) satisfies the assumptions:

\[
\int_{\mathbb{R}} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}} v^2 f_0(v) \, dv = 1, \quad \int_{\mathbb{R}} f_0(v) \, \log f_0(v) \, dv < \infty.
\]
The Kac equation shares with the homogenous Boltzmann equation for Maxwellian molecules the existence and uniqueness theory for the solutions. By considering the sequence of cut-off approximating problems

\[
\begin{aligned}
\partial_t \varphi_l(\xi, \tau) &= \int_{|\theta| < \frac{\pi}{2}} (\varphi_l(\xi \cos \theta, \tau) \varphi_l(\xi \sin \theta, \tau) - \varphi_l(\xi, \tau) \varphi_l(0, \tau)) \beta_l(\theta) \, d\theta, \\
\varphi_l(\xi, 0) &= \hat{f}_0(\xi),
\end{aligned}
\]

(11)

where each $\beta_l(\theta)$ is a bounded function defined as in (3), (4) and (5), it is possible to prove that each Cauchy problem (11) has a unique solution $\hat{\varphi}_l$, which has the following explicit representation, called Wild’s expansion:

\[
\varphi_l(\xi, \tau) = e^{-\tau} \sum_{n=0}^{\infty} \varphi_l^{(n)}(\xi)(1 - e^{-\tau})^n,
\]

where

\[
\varphi_l^{(0)}(\xi) = \hat{f}_0(\xi),
\]

\[
\varphi_l^{(n+1)}(\xi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{|\theta| < \frac{\pi}{2}} \varphi_l^{(j)}(\xi \cos \theta) \varphi_l^{(n-j)}(\xi \sin \theta) \beta_l(\theta) \, d\theta.
\]

Finally, letting

\[
\hat{f}_l(\xi, t) := \varphi_l(\xi, b_l^s t),
\]

it is possible to establish the (uniform on compact sets) convergence of a subsequence of $\hat{f}_l$ to a solution $\hat{f}$ of the Cauchy problem for the (non necessarily cut-off) equation (9).

Let us suppose now that the initial datum $f_0$ satisfies the extra property:

\[
|\hat{f}_0(\xi)| \leq e^{-K|\xi|^s}, \quad \xi \in \mathbb{R}, \quad K > 0, \quad s \in (0, 2].
\]

Thanks to the representation of the solution of the cut-off equation (11) in Wild’s expansion, it is straightforward to prove that the solution itself satisfies the same upper bound. Indeed, by a direct computation, we have that $|\varphi_l^{(1)}(\xi)| \leq e^{-K|\xi|^s}$, since

\[
\varphi_l^{(1)}(\xi) e^{K|\xi|^s} = \int_{\theta} e^{K|\xi|^s - K|\xi \cos \theta|^s - K|\xi \sin \theta|^s} \hat{f}_0(\xi \cos \theta) e^{K|\xi \cos \theta|^s} \hat{f}_0(\xi \sin \theta) e^{K|\xi \sin \theta|^s} \beta_l(\theta) \, d\theta,
\]

and $1 - |\cos \theta|^s - |\sin \theta|^s \leq 0$, for $s \in (0, 2]$. Hence by an immediate iteration argument, the same inequality holds for any $\varphi_l^{(n)}(\xi)$, and finally for the solution of the cut-off equation for any $t \geq 0$. Passing to the limit when $l \to +\infty$ in the estimate $\varphi_l(\xi, b_l^s t) \leq e^{-K|\xi|^s}$, we see that the inequality also holds for the solution of the (non necessarily cut-off) equation (9).

Due to the non-linearity of the collision operator, if we now consider the weaker assumption

\[
|\hat{f}_0(\xi)| \leq K_1 e^{-K_2|\xi|^s}, \quad \xi \in \mathbb{R}, \quad K_1 > 1, \quad K_2 > 0, \quad s \in (0, 2],
\]

the same argument allows to prove that the solution of each cut-off equation (11) satisfies the same upper bound, but only for a finite interval of time. In this case, by letting $l$ go to infinity, the interval of time where the estimate is true can reduce to nothing.

In spite of this, we prove in this section that the condition $|\hat{f}_0(\xi)| \leq K_1 e^{-K_2|\xi|^s}$ propagates (though possibly with different constants $K_1$ and $K_2$) along the solution of the (non necessarily cut-off) Kac equation.
2.1 Some preliminary properties of initial data

In this section we emphasize some useful properties satisfied by any even, nonnegative function $g$ such that $\int_{\mathbb{R}} g(v) \, dv = 1$ and $\int_{\mathbb{R}} v^2 g(v) \, dv = 1$.

**Lemma 4**

Let $g$ be a nonnegative, even function, satisfying

$$\int_{\mathbb{R}} g(v) \, dv = 1, \quad \int_{\mathbb{R}} g(v) v^2 \, dv = 1.$$ 

Then, there exist $\rho > 0$ and $\tilde{K} > 0$ such that for all $|\xi| \leq \rho$:

$$|\hat{g}(\xi)| \leq e^{-\tilde{K}|\xi|^2}.$$ 

**Proof:** We observe that under the hypotheses of the lemma, $\hat{g}$ is of class $C^2$ and satisfies the following property: $\hat{g}(0) = 1$, $\hat{g}'(0) = 0$, and $\hat{g}''(0) = -1$. Using a Taylor expansion of $\hat{g}$ at order 2, we obtain $\hat{g}(\xi) = 1 - \frac{1}{2}\xi^2 + o(\xi^2)$ when $\xi \to 0$. Then, the estimate of the lemma holds for any $\tilde{K} \in [0, \frac{1}{2})$. $\square$

Then, we prove the:

**Lemma 5**

Let $g \geq 0$ such that $\int_{\mathbb{R}} g(v) \, dv = 1$. Then, for all $r > 0$, there exist $C_r \in (0, \frac{1}{2})$ and $\tilde{C}_r \in (0, \frac{1}{2})$ such that

$$\int_{\mathbb{R}} g(v) \sin^2 \left( \frac{v \xi}{2} \right) \, dv \geq C_r, \quad |\xi| > r,$$

$$\int_{\mathbb{R}} g(v) \cos^2 \left( \frac{v \xi}{2} \right) \, dv \geq \tilde{C}_r, \quad |\xi| > r.$$

**Proof:** We only prove the first inequality, since the second one can be proven in exactly the same way.

Thanks to Lebesgue’s dominated convergence theorem and thanks to the absolute continuity of the measure $\nu(E) := \int_E g(v) \, dv$ with respect to the Lebesgue measure, there exist $R > 0$ and $\delta > 0$ such that for all measurable set $A \subset \mathbb{R}$ such that $|A| \leq \delta$, we have

$$\int_{A \cap B(0,R)} g(v) \, dv \geq \frac{1}{2}. \tag{12}$$

Let $\xi \in \mathbb{R}$ be fixed. For $\mu \in [0, \pi/2]$, we define

$$K_{\mu,R} := \left\{ v \in \mathbb{R}, |v| \leq R \text{ and } \exists k \in \mathbb{Z}, \left| \frac{v \xi}{2} - k\pi \right| \leq \mu \right\}.$$ 

It is clear that

$$|K_{\mu,R}| \leq \left( \frac{|\xi|R}{\pi} + 1 \right) \frac{4\mu}{|\xi|} = 4\mu \left( \frac{R}{\pi} + \frac{1}{|\xi|} \right)$$

so that, when $|\xi| \geq r$ we have $|K_{\mu,R}| \leq 4\mu \left( \frac{R}{\pi} + \frac{1}{r} \right)$. When $\mu = \frac{\delta}{4\left( \frac{R}{\pi} + \frac{1}{r} \right)}$, we see thanks to (12) that

$$\int_{K_{\mu,R} \cap B(0,R)} g(v) \, dv \geq \frac{1}{2}.$$
We can therefore conclude that
\[
\int_{\mathbb{R}} g(v) \sin^2 \left( \frac{v \xi}{2} \right) dv \geq \int_{K^c_{\mu,R} \cap B(0,R)} g(v) \sin^2 \left( \frac{v \xi}{2} \right) dv \geq \frac{1}{2} \sin^2 \mu := C_r.
\]

\[\square\]

2.2 The propagation theorem

We are now in position to state the theorem.

Theorem 6

Let \( f_0 \) be a nonnegative, even function, satisfying
\[
\int_{\mathbb{R}} f_0(v) dv = 1, \quad \int_{\mathbb{R}} f_0(v) v^2 dv = 1, \quad \int_{\mathbb{R}} |f_0(v)| \log f_0(v) dv < \infty.
\]

We suppose that \( f_0 \) is such that
\[
\sup_{\xi \in \mathbb{R}} |\hat{f}_0(\xi)| e^{K_2 |\xi|^s} \leq K_1
\]
for some \( K_1 \geq 1, K_2 > 0, \) and \( 0 < s \leq 2 \). Then there exist \( R_0 > 0, K > 0 \) such that the unique solution of the Cauchy problem (9) satisfies:
\[
\sup_{|\xi| < R_0} |\hat{f}(\xi,t) e^{K|\xi|^s}| \leq 1, \quad t \geq 0,
\]
\[
\sup_{|\xi| \geq R_0} |\hat{f}(\xi,t) e^{K|\xi|^s}| \leq 1, \quad t \geq 0.
\]
(13)

We begin by proving the following proposition.

Proposition 7

Let \( g \) be a nonnegative, even function, satisfying
\[
\int_{\mathbb{R}} g(v) dv = 1, \quad \int_{\mathbb{R}} g(v) v^2 dv = 1.
\]

Let us suppose moreover that, for given \( s \in (0,2] \), \( K_1 > 1 \) and \( K_2 > 0 \), \( g \) satisfies the following bound:
\[
|\hat{g}(\xi)| \leq K_1 e^{-K_2 |\xi|^s}, \quad \xi \in \mathbb{R}.
\]

Then, there exists \( \eta > 0 \) such that for all \( R > \eta \), there exists \( K > 0 \) (depending on \( R \)) such that
\[
|\hat{g}(\xi)| \leq \begin{cases} e^{-K|\xi|^2}, & |\xi| < R, \\ e^{-K|\xi|^s}, & |\xi| \geq R. \end{cases}
\]
Proof: We have already proven in Lemma 4 that \( |\hat{g}(\xi)| \leq e^{-K|\xi|^2} \) for \( |\xi| \leq \rho \), where \( \tilde{K} \) and \( \rho \) are suitably chosen. Now, let \( \eta = \left( \frac{\log \tilde{K}}{K_2} \right)^{1/2} \). For every \( R > \eta \), we can find \( 0 < K_3 < K_2 \) such that
\[
K_1 e^{-K_3|\xi|^2} \leq e^{-K_3|\xi|^2}, \quad |\xi| \geq R,
\]
so that
\[
|\hat{g}(\xi)| \leq e^{-K_3|\xi|^2}, \quad |\xi| \geq R.
\]
It is now enough to find \( K_4 > 0 \) such that
\[
|\hat{g}(\xi)| \leq e^{-K_4|\xi|^2}, \quad \rho < |\xi| < R.
\]

Since \( g \) is an even function, we have
\[
\hat{g}(\xi) = \int_{\mathbb{R}} g(v) e^{-i\xi v} \, dv = \int_{\mathbb{R}} g(v) \left( \frac{e^{-i\xi v} + e^{i\xi v}}{2} \right) \, dv = \int_{\mathbb{R}} g(v) \cos(\xi v) \, dv.
\]
Then, \( \hat{g} \) is real and \( |\hat{g}(\xi)| \leq 1 \) for all \( \xi \in \mathbb{R} \). Moreover
\[
1 - \hat{g}(\xi) = \int_{\mathbb{R}} g(v)(1 - \cos(\xi v)) \, dv = 2 \int_{\mathbb{R}} g(v) \sin^2 \left( \frac{\xi v}{2} \right) \, dv,
\]
\[
\hat{g}(\xi) + 1 = \int_{\mathbb{R}} g(v)(1 + \cos(\xi v)) \, dv = 2 \int_{\mathbb{R}} g(v) \cos^2 \left( \frac{\xi v}{2} \right) \, dv.
\]
According to Lemma 5, we know that
\[
1 - \hat{g}(\xi) \geq 2C_\rho, \quad |\xi| > \rho,
\]
\[
\hat{g}(\xi) + 1 \geq 2\tilde{C}_\rho, \quad |\xi| > \rho.
\]
Therefore,
\[
|\hat{g}(\xi)| \leq 1 - \min(2C_\rho, 2\tilde{C}_\rho), \quad |\xi| > \rho.
\]
Now, there exists \( K_4 > 0 \) such that
\[
1 - \min(2C_\rho, 2\tilde{C}_\rho) \leq e^{-K_4R^2},
\]
which implies
\[
|\hat{g}(\xi)| \leq 1 - \min(2C_\rho, 2\tilde{C}_\rho) \leq e^{-K_4|\xi|^2} \leq e^{-K_4|\xi|^2}, \quad \rho < |\xi| \leq R.
\]
We can conclude letting \( K = \min(\tilde{K}, K_3, K_4) \).

Proof of Theorem 8 the cut off case. Thanks to Proposition 7, there exists \( \eta > 0 \) such that for any \( R_0 > \eta \), there exists a strictly positive \( K \) such that the initial datum \( \hat{f}_0 \) satisfies
\[
\begin{align*}
\sup_{|\xi| < R_0} |\hat{f}_0(\xi)| e^{K|\xi|^2} &\leq 1, \\
\sup_{|\xi| \geq R_0} |\hat{f}_0(\xi)| e^{K|\xi|^2} &\leq 1.
\end{align*}
\]
In order to prove the theorem for the cut-off case, it is enough to establish that any $\varphi_l^{(n)}$ in Wild’s sums satisfies \( |\xi| < R_0 \). Let us define

$$H(|\xi|) = \begin{cases} K|\xi|^2, & |\xi| < R_0, \\ K|\xi|^s, & |\xi| \geq R_0. \end{cases}$$

Condition (11) on the initial datum $f_0$ reads therefore

$$\sup_{\xi \in \mathbb{R}} |\hat{f}_0(\xi)| e^{H(|\xi|)} \leq 1.$$ 

Then

$$\left| e^{H(|\xi|)} \varphi_l^{(1)}(\xi) \right| \leq \int_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \left| e^{H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|)} \right| \left| \hat{f}_0(\xi \sin \theta)| e^{H(|\xi \cos \theta|)} \right| \hat{f}_0(\xi \cos \theta) |\beta_l(\theta)| d\theta$$

$$\leq \int_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} e^{H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|)} |\beta_l(\theta)| d\theta.$$ 

Since $\int_0 \beta_l(\theta) d\theta = 1$, we end the estimate by proving that $H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|) \leq 0$ for $\xi \in \mathbb{R}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ if $R_0 \geq 1$. Thanks to the symmetries of the function $H$ with respect to $\theta$, we can restrict ourselves to the interval $[0, \frac{\pi}{2}]$. Now, when $|\xi| < R_0$, we have

$$H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|) = K|\xi|^2(1 - (\sin \theta)^2 - (\cos \theta)^2) = 0.$$ 

If $|\xi| \geq R_0$ and $|\xi \sin \theta| \geq R_0$, $|\xi \cos \theta| \geq R_0$, then

$$H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|) = K|\xi|^s(1 - (\sin \theta)^s - (\cos \theta)^s) \leq 0$$

for $0 < s \leq 2$. Whenever $|\xi| \geq R_0$ and $|\xi \sin \theta| < R_0$, $|\xi \cos \theta| < R_0$ we have

$$H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|) = K (|\xi|^s - |\xi|^2 ((\sin \theta)^2 + (\cos \theta)^2)) = K (|\xi|^s - |\xi|^2).$$

If we choose $R_0 \geq 1$, we can conclude since $|\xi| \geq R_0$ that

$$|\xi|^s - |\xi|^2 \leq 0.$$ 

If now $|\xi| \geq R_0$ and $|\xi \sin \theta| < R_0$, $|\xi \cos \theta| \geq R_0$, we have

$$H(|\xi|) - H(|\xi \sin \theta|) - H(|\xi \cos \theta|) = K (|\xi|^s - |\xi|^2 (\sin \theta)^2 - |\xi|^s (\cos \theta)^s)$$

$$\leq K (|\xi|^s - |\xi|^s (\sin \theta)^2 - |\xi|^s (\cos \theta)^2) = 0.$$ 

Note that since $0 \leq \theta \leq \pi/4$, there is no other case to treat.

**The non cut-off case.** As we have recalled in the introduction of section 2, the solution of (10) in the non cut-off case is obtained as the limit of a subsequence of the solutions of the Cauchy problems (11). Since the estimate on $\varphi_l(\xi, \tau)$ holds true (for any $\tau \geq 0$), the same is valid for $\hat{f}_l(\xi, t)$ and hence for $\hat{f}(\xi, t)$.

**Remark:** In Theorem 6, the hypothesis that $f_0$ is even could be replaced by the weaker hypothesis that $\int_{\mathbb{R}} f_0(v) v \, dv = 0$. Since the Kac equation comes from the Boltzmann equation when one considers radially symmetric solutions, it is however natural to study only even initial data.
3 Propagation for the Boltzmann equation

We would like to extend to the solution of the Boltzmann equation \(2\) the results proven in the previous section for the solution of the Kac equation. Two kinds of extensions are in order: first, we have to pass from the one-dimensional to the three-dimensional setting; second we would like to state the result considering not only functions like \(e^{-|\xi|^s}\), but also like \(e^{-\psi(|\xi|^2)}\), where \(\psi\) is a suitable concave function. We now begin by restating the lemmas of the previous section in three dimensions. We shall only indicate the major modifications in the proofs.

**Lemma 8**

Let \(g: \mathbb{R}^3 \to \mathbb{R}\) be a nonnegative function satisfying

\[
\int_{\mathbb{R}^3} g(v) \, dv = 1, \quad \int_{\mathbb{R}^3} g(v) \, v_i \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} g(v) |v|^2 \, dv = 3.
\]

Then there exist \(\rho > 0\) and \(\tilde{K} > 0\) such that for all \(|\xi| \leq \rho\):

\[
|\hat{g}(\xi)| \leq e^{-\tilde{K}|\xi|^2}.
\]

**Proof:** We observe that the result of the lemma is not changed when \(g\) is replaced by \(g \circ R\), where \(R\) is any rotation of \(\mathbb{R}^3\). As a consequence, we can suppose that the symmetric matrix \((\int_{\mathbb{R}^3} g(v) \, v_i \, v_j \, dv)_{i,j \in \{1, 2, 3\}}\) is diagonal. Moreover, since \(g \in L^1(\mathbb{R}^3)\) and \(\int_{v \in \mathbb{R}^3} g(v) \, |v|^2 \, dv = 3\), we see that \(\int_{v \in \mathbb{R}^3} g(v) \, v_i^2 \, dv > 0\) for \(i = 1, 2, 3\).

Then, \(\hat{g}(\xi) = 1 - \sum_{j=1}^3 \lambda_j \xi_j^2 + o_{\xi \to 0}(|\xi|^2)\), with \(\lambda_j > 0\) for \(j = 1, 2, 3\), and we conclude like in Lemma 4. \(\square\)

**Lemma 9**

Let \(g: \mathbb{R}^3 \to \mathbb{R}\) be a nonnegative function satisfying \(\int_{\mathbb{R}^3} g(v) \, dv = 1\). Then, for all \(r > 0\), there exists \(C_r \in (0, \frac{1}{2})\) such that for all \(\theta \in \mathbb{R}\),

\[
\int_{\mathbb{R}^3} g(v) \sin^2 \left(\frac{v \cdot \xi + \theta}{2}\right) \, dv \geq C_r, \quad |\xi| > r.
\]

**Proof:** The proof follows the same lines as that of Lemma 5. Let \(\xi \in \mathbb{R}^3\) be fixed. We start by choosing in \(\mathbb{R}^3\) an orthogonal system in which the unitary vector along the z-axis is \(\frac{\xi}{|\xi|}\). As in the one-dimensional case, there exist \(R > 0\) and \(\delta > 0\) such that for all measurable set \(A \subset \mathbb{R}^3\) such that \(|A| < \delta\), we have

\[
\int_{\mathbb{A} \cap Q(0, R)} g(v) \, dv \geq \frac{1}{2},
\]

where \(Q(0, R)\) is the cube centered at the origin:

\[
Q(0, R) = \{v \in \mathbb{R}^3 : |v_i| \leq R, \, i = 1, 2, 3\}.
\]

Now, for \(\mu \in [0, \frac{\pi}{2}]\), we define

\[
K_{\mu,R,\theta} := \left\{v \in \mathbb{R}^3, v \in Q(0, R) \text{ and } \exists k \in \mathbb{Z}, \left|\frac{v \cdot \xi + \theta}{2} - k\pi\right| \leq \mu\right\}.
\]
Thanks to the choice of the coordinate system, we have

\[ K_{\mu,R,\theta} = \left\{ v \in \mathbb{R}^3, |v_i| \leq R, i = 1, 2, 3 \text{ and } \exists k \in \mathbb{Z}, \left| \frac{v_3|\xi| + \theta}{2} - k\pi \right| \leq \mu \right\}. \]

So, it is easy to see that

\[ |K_{\mu,R,\theta}| \leq \left( \frac{|\xi|R}{\pi} + 1 \right) \frac{4\mu}{|\xi|} R^2 = \frac{4\mu R^2}{\pi} \left( \frac{R}{\pi} + \frac{1}{|\xi|} \right), \]

and we can conclude as in the one-dimensional case. \(\square\)

We are now in position to prove the main theorem of our paper, namely Theorem 2. The proof of this theorem relies on the following proposition:

**Proposition 10**

Let \( g \) be a nonnegative function satisfying

\[ \int_{\mathbb{R}^3} g(v) \, dv = 1, \quad \int_{\mathbb{R}^3} g(v) v_i \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}^3} g(v) |v|^2 \, dv = 3. \]

Let us suppose moreover that

\[ |\hat{g}(\xi)| \leq K_1 e^{-K_2 \varphi(|\xi|)}, \quad \xi \in \mathbb{R}^3, \]

where \( K_1 \geq 1, K_2 > 0 \) and \( \varphi : [0, +\infty) \to [0, +\infty) \) satisfies \( \lim_{t \to +\infty} \varphi(t) = +\infty \). Then, there exists \( \eta > 0 \) such that for all \( R > \eta \), there exists \( K > 0 \) (depending on \( R \)) such that

\[ |\hat{g}(\xi)| \leq \begin{cases} e^{-K_1|\xi|^2}, & |\xi| < R, \\ e^{-K_2 \varphi(|\xi|)}, & |\xi| \geq R. \end{cases} \]

**Proof:** The proof of this proposition is only a slight modification of that of Proposition 4. We only point out the few differences.

First, we explain how to find \( \eta > 0 \): we can fix \( K_3 \in (0, K_2) \) and let \( \eta = \eta(K_3) \) be a positive constant such that \( \varphi(|\xi|) \geq \frac{1}{K_2 - K_3} \log K_1 \) for every \( |\xi| \geq \eta \); then of course for every \( R \geq \eta \),

\[ K_1 e^{-K_2 \varphi(|\xi|)} \leq e^{-K_3 \varphi(|\xi|)}, \quad |\xi| \geq R. \]

Second, we observe that for all \( \xi \in \mathbb{R}^3 \) it is possible to find \( \theta \in \mathbb{R} \) (depending on \( \xi \)) such that

\[ |\hat{g}(\xi)| = \hat{g}(\xi) e^{i\theta} = \int_{\mathbb{R}^3} g(v) \cos (\xi \cdot v + \theta) \, dv. \]

So, we have

\[ 1 - |\hat{g}(\xi)| = 2 \int_{\mathbb{R}^3} g(v) \sin^2 \left( \frac{\xi \cdot v + \theta}{2} \right) \, dv. \]

Thanks to Lemma 9, there exists \( C_\rho \) such that

\[ 2 \int_{\mathbb{R}^3} g(v) \sin^2 \left( \frac{\xi \cdot v + \theta}{2} \right) \, dv \geq 2 C_\rho, \quad |\xi| > \rho. \]
Then, we can conclude as in the proof of Proposition 7.

**Proof of Theorem 2** As we did for Kac equation, for each of the Cauchy problems, we write the solution \( \varphi_l \) under the form of a Wild’s expansion.

In order to prove the bound for the solution, it is enough to prove it for every term \( \varphi_l^{(n)} \) in the sum. We define

\[
H(\xi) = \begin{cases} 
K|\xi|^2, & |\xi| < R_0, \\
K\psi(|\xi|^2), & |\xi| \geq R_0,
\end{cases}
\]

where \( R_0 \) will be chosen (large enough) later, and \( K \) is given by Proposition 10. Thanks to this proposition, the initial datum satisfies

\[
\sup_{\xi \in \mathbb{R}^3} |\hat{f}_0(\xi)| e^{H(|\xi|)} \leq 1.
\]

We recall the identities ([Des03], page 56)

\[
|\xi^+| = |\xi| \cos \frac{\theta}{2}, \\
|\xi^-| = |\xi| \sin \frac{\theta}{2}.
\]

For the first term \( \varphi_l^{(1)} \) we have:

\[
\left| \varphi_l^{(1)}(\xi) e^{H(|\xi|)} \right| = \left| \int_{S^2} e^{H(|\xi|) - H(|\xi^+|) - H(|\xi^-|)} \hat{f}_0(\xi^+) e^{H(|\xi^+|)} \hat{f}_0(\xi^-) e^{H(|\xi^-|)} \beta_l \left( \frac{\xi}{|\xi|} \cdot n \right) \, dn \right|
\]

\[
\leq \int_{\theta \in (0, \pi)} \int_{\varphi \in (0, 2\pi)} e^{H(|\xi|) - H(|\xi| \cos \frac{\theta}{2}) - H(|\xi| \sin \frac{\theta}{2})} \beta_l (\cos \theta \sin \theta \sin \varphi, \cos \theta \cos \varphi) \, d\theta \, d\varphi,
\]

where in the last integral, we have used the spherical coordinates with \( \frac{\xi}{|\xi|} \) as z-axis. Then, in order to establish \( \left| \varphi_l^{(1)}(\xi) e^{H(|\xi|)} \right| \leq 1 \), we show that for \( R_0 \) large enough,

\[
H(|\xi|) - H \left( |\xi| \cos \frac{\theta}{2} \right) - H \left( |\xi| \sin \frac{\theta}{2} \right) \leq 0, \quad \theta \in (0, \pi), \quad \xi \in \mathbb{R}^3.
\]

We denote \( \tilde{\theta} = \frac{\theta}{2} \). Thanks to the symmetries of the functions \( H(|\xi|) - H(|\xi \sin \tilde{\theta}|) - H(|\xi \cos \tilde{\theta}|) \) with respect to \( \theta \), we can restrict ourselves to the interval \( (0, \frac{\pi}{2}) \). Now, the case \( |\xi| < R_0 \) is the same as in Theorem 3. If \( |\xi| \geq R_0 \) and \( |\xi \sin \tilde{\theta}| \geq R_0 \), \( |\xi \cos \tilde{\theta}| \geq R_0 \), then

\[
H(|\xi|) - H(|\xi \sin \tilde{\theta}|) - H(|\xi \cos \tilde{\theta}|) = K(\psi(|\xi|^2) - \psi(|\xi \sin \tilde{\theta}|^2) - \psi(|\xi \cos \tilde{\theta}|^2)).
\]

Thanks to the concavity property of \( \psi \) and the fact that \( \psi(0) = 0 \), we have \( \psi(|\xi| (\sin \tilde{\theta})^2) \geq (\sin \tilde{\theta})^2 \psi(|\xi|^2) \) and \( \psi(|\xi|^2) \geq (\cos \tilde{\theta})^2 \psi(|\xi|^2) \). Hence

\[
\psi(|\xi|^2) - \psi(|\xi \sin \tilde{\theta}|^2) - \psi(|\xi \cos \tilde{\theta}|^2) \leq \psi(|\xi|^2) - (\sin \tilde{\theta})^2 \psi(|\xi|^2) - (\cos \tilde{\theta})^2 \psi(|\xi|^2) = 0.
\]

Whenever \( |\xi| \geq R_0 \) and \( |\xi \sin \tilde{\theta}| < R_0 \), \( |\xi \cos \tilde{\theta}| < R_0 \) we have

\[
H(|\xi|) - H(|\xi \sin \tilde{\theta}|) - H(|\xi \cos \tilde{\theta}|) = K(\psi(|\xi|^2) - |\xi|^2 (\sin \tilde{\theta})^2 + (\cos \tilde{\theta})^2)) = K(\psi(|\xi|^2) - |\xi|^2).
\]
If we choose $R_0$ large enough, thanks to the assumption that $\psi(r) \leq r$ for $r$ large enough, and since $|\xi| \geq R_0$, we can conclude

$$\psi(|\xi|^2) - |\xi|^2 \leq 0.$$ 

By using now the concavity property of $\psi$ and the fact that $\psi(r) \leq r$, if now $|\xi| \geq R_0$ and $|\xi \sin \tilde{\theta}| < R_0, |\xi \cos \tilde{\theta}| \geq R_0$ we have

$$H(|\xi|) - H(|\xi \sin \tilde{\theta}|) - H(|\xi \cos \tilde{\theta}|) = K(\psi(|\xi|^2) - |\xi|^2 (\sin \tilde{\theta})^2 - \psi(|\xi \cos \tilde{\theta}|^2))$$

$$\leq K(\psi(|\xi|^2) - \psi(|\xi|^2)(\sin \tilde{\theta})^2 - \psi(|\xi \cos \tilde{\theta}|^2))$$

$$\leq K\psi(|\xi|^2)(1 - (\sin \tilde{\theta})^2 - (\cos \tilde{\theta})^2) = 0.$$

We end the proof by first noticing that a simple induction shows the estimate $|\phi^{(n)}(\xi) e^{H(|\xi|)}| \leq 1$ when $n \geq 1$, and then we may pass to the limit when $l \to +\infty$ if necessary (that is, in the non cut-off case).

Remark:
We would like to point out that the assumption of concavity for the function $\psi(|\xi|^2)$ is not mandatory and that the argument exploited in the proof of Theorem 2 could work also in a more general framework.

By analysing the proof, one can see that, instead of assuming that $\psi$ is concave, it is in fact enough to assume that for some $R_0$ large enough,

$$\psi(\lambda^2|\xi|^2) \geq \lambda^2 \psi(|\xi|^2)$$

when $0 \leq \lambda \leq 1, \lambda |\xi| \geq R_0$. This is true for example when $\psi(t) = \frac{1}{2} \sqrt{t |\log t|}$.

4 Gevrey spaces

In this section, we translate the propagation result obtained in the previous sections in terms of Gevrey regularity for the solutions of Boltzmann equation. Let us begin by recalling the classical definitions of Gevrey functions and a useful characterisation of these functions through their Fourier transform. For more information, the interested reader can consult for instance the book by Rodino [Rod93], from where we have taken the following recalls.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\nu \geq 1$ be a fixed real number.

Definition 11
The class $G^\nu(\Omega)$ of Gevrey functions of order $\nu$ in $\Omega$ is the set of functions $f \in C^\infty(\Omega)$ satisfying the following property: for every compact subset $K$ of $\Omega$, there exists a positive constant $C = C(K)$ such that for all $l \in \mathbb{N}^n$ and all $x \in K$,

$$|\partial^l f(x)| \leq C^{l!+1}(l!)^\nu.$$  

(15)

Assumption (15) can be replaced by other equivalent assumptions, for example

$$|\partial^l f(x)| \leq RC^{l!}(l!)^\nu,$$

15
where $R$ and $C$ are two positive constants independent of $l$ and $x \in K$. It is easy to recognize that $G^1(\Omega) = A(\Omega)$, the space of all analytic functions in $\Omega$, and that for $\nu \leq \tau$, one has $G^\nu(\Omega) \subseteq G^\tau(\Omega)$. Moreover, it is interesting to underline the following inclusions:

$$A(\Omega) \subset \bigcap_{\nu > 1} G^\nu(\Omega), \quad \bigcup_{\nu \geq 1} G^\nu(\Omega) \subset C^\infty(\Omega),$$

which are strict in both cases. We also recall that the Gevrey class $G^\nu(\Omega)$ is closed under differentiation.

In what follows, we shall also need the following

**Definition 12**

Assume $\nu > 1$. We shall denote by $G^\nu_0(\Omega)$ the vector space of all $f \in G^\nu(\Omega)$ with compact support in $\Omega$.

The exclusion of $\nu = 1$ in the previous definition is mandatory, because there are no analytic test functions other than the zero function. As for the other values $\nu > 1$, one could wonder whether such compact supported functions do exist. An example in $\mathbb{R}$ is the following: let $r > 0$, $\nu > 1$, $d = \frac{1}{1 - \nu}$ and

$$\varphi(t) = \begin{cases} e^{-t^d} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

The function

$$f(x) = \varphi(x + r)\varphi(x - r)$$

is then in $G^\nu_0(\mathbb{R})$ (Rod93).

The result that we are going to use in order to relate our propagation result to Gevrey regularity is the following.

**Theorem 13** (Rod93, Theorem 1.6.1 page 31)

\begin{enumerate}
\item[i)] Let $\nu > 1$. If $\varphi \in G^\nu_0(\mathbb{R}^n)$, then there exist positive constants $C$ and $\varepsilon$ such that

$$|\hat{\varphi}(\xi)| \leq Ce^{-\varepsilon|\xi|^{\frac{1}{\nu}}}, \quad \xi \in \mathbb{R}^n. \tag{16}$$

\item[ii)] Let $\nu \geq 1$. If the Fourier transform of $\varphi \in S'(\mathbb{R}^n)$ satisfies (16), then $\varphi \in G^\nu(\mathbb{R}^n)$.\end{enumerate}  

We can therefore deduce Corollary 3 concerning the regularity of the solutions of Boltzmann equation.

**Proof of Corollary 3** The result is straightforward from Theorem 13 and Theorem 2. It is enough to notice that one can replace the uniform estimate

$$\sup_{|\xi| < R_0} |\hat{f}(\xi,t)| e^{K|\xi|^2} \leq 1, \quad t \geq 0,$$

$$\sup_{|\xi| \geq R_0} |\hat{f}(\xi,t)| e^{K|\xi|^2} \leq 1, \quad t \geq 0,$$

obtained in Theorem 2 by the following uniform estimate:

$$\sup_{\xi \in \mathbb{R}^3} |\hat{f}(\xi,t)| e^{K_2|\xi|^2} \leq K_1, \quad t \geq 0,$$
for \( \tilde{K}_1 \geq 1 \) and \( \tilde{K}_2 > 0 \) properly chosen. Letting now \( \nu = \frac{1}{2} \), one immediately gets the result. This ends the proof. \( \square \)

We end up this section by comparing the regularity result we have just obtained with the one obtained by Ukai in [Uka84]. In his work, he considered among others a Cauchy problem for the homogeneous Boltzmann equation for Maxwellian molecules both in the cut-off and non cut-off settings. He considered only initial data \( f_0 \) satisfying a strong regularity assumption: for \( \alpha \geq 0, \rho \geq 0 \) and \( \nu \geq 1 \) he supposed \( f_0 \in \gamma^\nu_{\alpha,\rho} \), where

\[
\gamma^\nu_{\alpha,\rho} = \{ g : \| g \|_{\alpha,\rho,\nu} = \sum_{l \in \mathbb{N}^3} \rho^{||l||} \sup_{v \in \mathbb{R}^3} e^{\alpha(1+|v|^2)^{1/2}} |\partial^l g(v)| < \infty \}.
\]

Comparing this space with the spaces in Definition 11, one can deduce that initial data in Ukai setting are indeed in the Gevrey space \( G^\nu(\mathbb{R}^3) \), but also decay very strongly at infinity together with all their derivatives. Ukai was able to prove by a fixed point argument that there exists a unique, local in time solution \( f \) belonging at every time to a functional space of the same kind as the initial datum but in which the indices change with \( t \). More precisely, he proved that there exist \( T > 0, \beta > 0, \sigma > 0 \) such that

\[
\| f(\cdot, t) \|_{\alpha - t\beta, \rho - t\sigma, \nu} \leq 2 \| f_0 \|_{\alpha, \rho, \nu}, \quad t \in [0, T].
\]

Since initial data which belong to \( G^\nu_0(\mathbb{R}^3) \) also belong to Ukai space, the question of comparing the two results is meaningful. Let us consider a nonnegative function \( f_0 \in G^\nu_0(\mathbb{R}^3) \). If we suppose moreover that \( f_0 \) satisfies assumptions [8], then by Theorem 1 we know that there is a solution \( f \in C^1([0, +\infty), L^1(\mathbb{R}^3)) \) which is unique by Toscani–Villani’s result. Since \( G^\nu_0(\mathbb{R}^3) \subset \gamma^\nu_{\alpha,\rho} \) for all \( \alpha \geq 0 \), we can deduce from Ukai result that for all \( \alpha \geq 0 \) there is a time \( T = T(\alpha) > 0 \) (possibly finite) such that this solution stays in the class [17] for \( t \in [0, T(\alpha)] \) but this space is not uniform in time (in addition to the fact that for all \( \alpha \) it is difficult to compare the life times \( T(\alpha) \)). Our result says instead that the solution stays for \( t \in [0, \infty) \) in the same Gevrey class as its initial datum, without any information about the decay at infinity and moreover that all the estimates on the Gevrey seminorms are uniform in time.

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