The canonical projection associated to certain possibly infinite generalized iterated function system as a fixed point

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Abstract. In this paper, influenced by the ideas from A. Mihail, The canonical projection between the shift space of an IIFS and its attractor as a fixed point, Fixed Point Theory Appl., 2015, Paper No. 75, 15 p., we associate to every generalized iterated function system $\mathcal{F}$ (of order $m$) an operator $H_{\mathcal{F}} : C^m \to C$, where $C$ stands for the space of continuous functions from the shift space on the metric space corresponding to the system. We provide sufficient conditions (on the constitutive functions of $\mathcal{F}$) for the operator $H_{\mathcal{F}}$ to be continuous, contraction, $\varphi$-contraction, Meir-Keeler or contractive. We also give sufficient condition under which $H_{\mathcal{F}}$ has a unique fixed point $\pi_0$. Moreover, we prove that, under these circumstances, the closer of the image of $\pi_0$ is the attractor of $\mathcal{F}$ and that $\pi_0$ is the canonical projection associated to $\mathcal{F}$. In this way we give a partial answer to the open problem raised on the last paragraph of the above mentioned Mihail’s paper.

Key words and phrases: possibly infinite generalized iterated function system, canonical projection, attractor, fixed point, $\varphi$-contraction, Meir-Keeler function

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1. Introduction

As part of the effort to generalize the concept of iterated function system introduced by J. Hutchinson (see [3]), R. Miculescu and A. Mihail (see [6] and [8]) proposed the concept of generalized iterated function system. More precisely, given $m \in \mathbb{N}$ and a metric space $(X, d)$, a generalized iterated function system (for short a GIFS) of order $m$ is a finite family of functions $f_1, \ldots, f_n : X^m \to X$ satisfying certain contractive conditions. They proved that there exists a unique attractor of a GIFS and studied some of its properties. F. Strobin (see [14]) proved that, for any $m \geq 2$, there exists a Cantor subset of the plane which is an attractor of some GIFS of order $m$, but is not an attractor of a GIFS of order $m - 1$. This shows that GIFSs are real generalizations of iterated function systems. Certain algorithms generating images of attractors of GIFSs could be found in [4]. Let us list some extensions of
the concept of GIFS: a) D. Dumitru (see [1] and [2]) investigated generalized iterated function systems consisting of Meir-Keeler functions; b) F. Strobin and J. Swaczyna (see [15]) extended the concept of GIFS to the more general setting of $\varphi$-contractions; c) N. Secelean (see [13]) studied countable iterated function systems consisting of generalized contraction mappings on the product space $X^I$ into $X$, where $I \subseteq \mathbb{N}$; d) E. Oliveira and F. Strobin (see [11]) defined the notion of generalized iterated fuzzy function system. Moreover, the Hutchinson measure associated with a generalized iterated function system was studied in [7] (for GIFS with probabilities), in [5] (for generalized iterated function systems with place dependent probabilities) and in [12].

The canonical projection associated to an iterated function system is a crucial tool in the study of topological properties of the attractor of such a system. A significant position from the point of view of this paper is occupied by [10]. More precisely it is proved there that for a possibly infinite iterated function system, in two cases (namely: a) the constitutive functions of the system are uniformly Meir-Keeler; b) the metric space associated to the system is compact and the system consists of a finite number of contractive functions), the canonical projection between the shift space of the system and its attractor can be viewed as a fixed point.

The concept of code space for GIFSs was introduced by A. Mihail (see [9]) and reformulated by F. Strobin and J. Swaczyna (see [16]) in order to treat the problem of connectedness of the attractor of a GIFS.

In this paper, inspired by the ideas from [10], we associate to a generalized iterated function system $\mathcal{F}$ (of order $m$) an operator $H_{\mathcal{F}} : \mathcal{C}^m \to \mathcal{C}$, where $\mathcal{C}$ stands for the space of continuous functions from the shift space on the metric space corresponding to the system. In section 3, we provide sufficient conditions (on the constitutive functions of $\mathcal{F}$) for the operator $H_{\mathcal{F}}$ to be continuous, contraction, $\varphi$-contraction, Meir-Keeler or contractive. In section 4, we give sufficient condition under which $H_{\mathcal{F}}$ has a unique fixed point $\pi_0$ (see Theorem 4.1). Moreover, we prove that, under these conditions, the closer of the image of $\pi_0$ is the attractor of $\mathcal{F}$ (see Theorem 4.2) and that $\pi_0$ is the canonical projection associated to $\mathcal{F}$ (see Theorem 4.3). Our results can be considered as a partial answer to the open problem raised at the end of [10].

2. Preliminaries

A. The Hausdorff-Pompeiu metric
For a metric space \((X, d)\), the function \(h : \mathcal{B}(X) \times \mathcal{B}(X) \rightarrow [0, \infty)\) given by
\[
h(A, B) = \max\{d(A, B), d(B, A)\},
\]
for every \(A, B \in \mathcal{B}(X)\), where \(d(A, B) = \sup_{x \in A, y \in B} d(x, y)\), is a metric on \(\mathcal{B}(X)\) which is called the Hausdorff-Pompeiu metric. Here, by \(\mathcal{B}(X)\) we mean the set of all non-empty, closed and bounded subsets of \(X\). In the sequel by \(\mathcal{K}(X)\) we mean the set of all non-empty compact subsets of \(X\).

**B. The metric space \((X^m, d_{\max})\)**

For a metric space \((X, d)\) and \(m \in \mathbb{N}^*\), we endow the Cartesian product \(X^m\) with the maximum metric \(d_{\max}\) defined by
\[
d_{\max}((x_1, \ldots, x_m), (y_1, \ldots, y_m)) = \max\{d(x_1, y_1), \ldots, d(x_m, y_m)\},
\]
for all \((x_1, \ldots, x_m), (y_1, \ldots, y_m) \in X^m\).

For a metric space \((X, d)\) and \(m \in \mathbb{N}^*\), we define inductively the spaces \(X_1, X_2, \ldots, X_k, \ldots\) in the following way:
\[
X_1 = X \times X \times \ldots \times X = X^m
\]
m times

and
\[
X_{k+1} = X_k \times X_k \times \ldots \times X_k
\]
m times

for every \(k \in \mathbb{N}^*\).

We endow \(X_k\) with the maximum metric for every \(k \in \mathbb{N}^*\). Let us to lay stress upon the fact that \(X_k\) is isometric to \(X^{mk}\) with the maximum metric for every \(k \in \mathbb{N}^*\).

**C. The Mihail-Strobin&Swaczyna generalized code space**

The notion of code space associated to a generalized iterated function system was introduced by A. Mihail (see [9]). A different but equivalent concept which can be easier handled is due to F. Strobin and J. Swaczyna (see [16]).

Given \(m \in \mathbb{N}^*\) and a set \(I\), we define inductively the sets \(\Omega_1, \Omega_2, \ldots, \Omega_k, \ldots\) in the following way:
\[ \Omega_1 = I \]

and

\[ \Omega_{k+1} = \Omega_k \times \Omega_k \times \ldots \times \Omega_k \]

for every \( k \in \mathbb{N}^* \).

We also consider the sets

\[ \Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_k \times \ldots \]

and

\[ _k\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_k, \]

where \( k \in \mathbb{N}^* \).

For \( i \in \{1, 2, \ldots, m\} \), \( k \in \mathbb{N} \), \( k \geq 2 \) and \( \alpha = \alpha_1^i \alpha_2^i \ldots \alpha_m^i \in _k\Omega \), where \( \alpha^2 = \alpha_1^2 \alpha_2^2 \ldots \alpha_m^2 \in \Omega_2 \), \ldots, \( \alpha^k = \alpha_1^k \alpha_2^k \ldots \alpha_m^k \in \Omega_k \), we consider

\[ \alpha(i) = \alpha_1^i \alpha_2^i \ldots \alpha_{i-1}^i \alpha_i^k \in _{k-1}\Omega. \]

For \( \alpha \in \Omega \) and \( i \in \{1, 2, \ldots, m\} \), we define \( \alpha(i) \) in a similar manner.

**Definition 1.1.** \( \Omega \) is called the Mihail-Strobin&Swaczyna generalized code space.

**Remark 1.2.** Endowed with the metric \( d \) given by

\[ d(\alpha, \beta) = \sum_{k \in \mathbb{N}} C^k d(\alpha^k, \beta^k), \]

for every \( \alpha = \alpha_1^1 \alpha_1^2 \ldots \alpha_i^{i+1} \ldots, \beta = \beta_1^1 \beta_2^1 \ldots \beta_i^{i+1} \ldots \in \Omega \), where \( d(\alpha^k, \beta^k) = \begin{cases} 1, & \alpha^k \neq \beta^k \\ 0, & \alpha^k = \beta^k \end{cases} \) and \( C \in (0, 1) \), \( (\Omega, d) \) becomes a complete metric space.

**Remark 1.3.** If \( I \) is finite, then the metric space \( (\Omega, d) \) is compact.

**D. Generalized possibly infinite iterated function systems**

**Definition 1.4.** A generalized possibly infinite iterated function system of order \( m \in \mathbb{N}^* \) is a pair \( F = ((X, d), (f_i)_{i \in I}) \), where \( (X, d) \) is a metric space, \( f_i : X^m \to X \) is continuous for every \( i \in I \) and the family of functions
\[ (f_i)_{i \in I} \text{ is bounded } (i.e. \bigcup_{i \in I} f_i(B) \text{ is bounded for each bounded subset } B \text{ of } X^m) \]

The function \( \mathcal{F}_F : (\mathcal{B}(X))^m \to \mathcal{B}(X) \), given by
\[
\mathcal{F}_F(B_1, ..., B_m) = \bigcup_{i \in I} f_i(B_1, ..., B_m),
\]
for all \((B_1, ..., B_m) \in (\mathcal{B}(X))^m\), is called the fractal operator associated to \( F \).

If \( \mathcal{F}_F \) has a unique fixed point, then it is called the attractor of \( F \) and it is denoted by \( A_F \).

If the set \( I \) is finite, then \( \mathcal{F}_F((\mathcal{K}(X))^m) \subseteq \mathcal{K}(X) \) and we make the convention to still denote the function \((B_1, ..., B_m) \in (\mathcal{K}(X))^m \to \mathcal{F}_F(B_1, ..., B_m) \in \mathcal{K}(X) \) by \( \mathcal{F}_F \). In this case \( A_F \in \mathcal{K}(X) \).

For a generalized possibly infinite iterated function system \( \mathcal{F} = ((X, d), (f_i)_{i \in I}) \) of order \( m \), we define inductively a family of functions \( \{f_\alpha : \Omega \to X \mid \alpha \in k\Omega\} \) for every \( k \in \mathbb{N}^* \) in the following way:

i) For \( k = 1 \), the family is \((f_i)_{i \in I}\).

ii) If the functions \( f_\alpha \), where \( \alpha \in k\Omega \), have been defined, then, we define
\[
f_\alpha(x_1, x_2, ..., x_m) = f_{\alpha^1}(f_{\alpha(1)}(x_1), ..., f_{\alpha(m)}(x_m))
\]
for every \( \alpha = \alpha^1 \alpha^2 ... \alpha^{k+1} \in k+1\Omega \), where \( \alpha^1 \in \Omega_1 \), \( \alpha^2 \in \Omega_2 \), ..., \( \alpha^k \in \Omega_k \), \( \alpha^{k+1} \in \Omega_{k+1} \), \((x_1, x_2, ..., x_m) \in X_{k+1} = X_k \times X_k \times ... \times X_k \).

Note that the above introduced families of functions are natural generalizations of compositions of functions since if \( m = 1 \), then \( k\Omega = I_k \) and if \( \omega = \omega^1 \omega^2 ... \omega^k \in k\Omega \), then \( f_\omega = f_{\omega^1} \circ ... \circ f_{\omega^k} \).

E. The operator \( H_F \) associated to a generalized possibly infinite iterated function system

For a generalized possibly infinite iterated function system \( \mathcal{F} = ((X, d), (f_i)_{i \in I}) \) of order \( m \), we consider the operator \( H_F : \mathcal{C}^m \to \mathcal{C} \) given by
\[
H_F(g_1, ..., g_m)(\alpha) = f_{\alpha^1}(g_{\alpha(1)}(1), ..., g_m(\alpha(m))),
\]
for every \( g_1, ..., g_m \in \mathcal{C} \) and every \( \alpha = \alpha^1 \alpha^2 ... \alpha^k \in \Omega \), \( \alpha^k \in \Omega_k \) for every \( k \in \mathbb{N}^* \), where the metric space \( (\mathcal{C}, d_u) \) is described by
\[
\mathcal{C} = \{ f : \Omega \to X \mid f \text{ is continuous and bounded} \} \]
and

\[ d_u(f, g) = \sup_{\alpha \in \Omega} d(f(\alpha), g(\alpha)) \]

for every \( f, g \in \mathcal{C} \).

**Remark 1.5.**

i) If \((X, d)\) is complete, then \( \mathcal{C} \) is complete.

ii) \( H_F \) is well defined, i.e. \( H_F(g_1, \ldots, g_m) \) is continuous and bounded for all \( g_1, \ldots, g_m \in \mathcal{C} \). Indeed, on one hand, the continuity follows from the following facts: \( \Omega = \bigcup_{i \in I} \Omega^i \), where \( \Omega^i = \{ \alpha = \alpha^1 \alpha^2 \ldots \alpha^i \alpha^{i+1} \ldots \in \Omega \mid \alpha^1 = i \} \), and the restriction of \( H_F(g_1, \ldots, g_m) \) to the open set \( \Omega^i \) is continuous for every \( i \in I \). On the other hand, the boundedness follows from the boundedness of the family of functions \((f_i)_{i \in I}\), the boundedness of the functions \( g_1, \ldots, g_m \) and from the fact that

\[
H_F(g_1, \ldots, g_m)(\Omega) = H_F(g_1, \ldots, g_m)(\bigcup_{i \in I} \Omega^i) =
\]

\[
= \bigcup_{i \in I} H_F(g_1, \ldots, g_m)(\Omega^i) = \bigcup_{i \in I} f_i(g_1(\Omega), \ldots, g_m(\Omega)).
\]

**F. Some classes of functions** \( f : X^m \to X \) and their fixed points

Given a set \( X, m \in \mathbb{N}^* \) and a function \( f : X^m \to X \), we define inductively a family of functions \( f^{[k]} : X^{m^k} \to X, \ k \in \mathbb{N}^* \), in the following way:

i) \( f^{[1]} = f \)

ii) assuming that we have defined \( f^{[k]} \), then

\[
f^{[k+1]}(x_1, \ldots, x_m) = f(f^{[k]}(x_1), \ldots, f^{[k]}(x_m)),
\]

for every \((x_1, \ldots, x_m) \in X^{m^k} \times \ldots \times X^{m^k} = X^{m^{k+1}} = X_{k+1} \).

Note that for \( m = 1 \), we have \( f^{[k]} = f \circ \ldots \circ f \) \( k \) times.

**Definition 1.6.** Given a set \( X \) and \( m \in \mathbb{N}^* \), an element \( x \) of \( X \) is called a fixed point of a function \( f : X^m \to X \) if

\[
f(x, \ldots, x) = x.
\]
Definition 1.7. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if it satisfies the following properties:

i) it is nondecreasing;

ii) it is right-continuous;

iii) $\varphi(t) < t$ for every $t > 0$.

Definition 1.8. Given a metric space $(X, d)$, $m \in \mathbb{N}^*$ and a comparison function $\varphi$, a function $f : X^m \to X$ is called a $\varphi$-contraction if

$$d(f(x), f(y)) \leq \varphi(d_{\text{max}}(x, y)),$$

for all $x, y \in X^m$.

Definition 1.9. Given a metric space $(X, d)$ and $m \in \mathbb{N}^*$, a function $f : X^m \to X$ is called Meir-Keeler if for every $\varepsilon > 0$ there exists $\delta_{\varepsilon} > 0$ such that

$$d(f(x), f(y)) < \varepsilon,$$

for all $x, y \in X^m$ having the property that $d_{\text{max}}(x, y) < \varepsilon + \delta_{\varepsilon}$.

Definition 1.10. Given a metric space $(X, d)$ and $m \in \mathbb{N}^*$, a function $f : X^m \to X$ is called contractive if

$$d(f(x), f(y)) < d_{\text{max}}(x, y),$$

for all $x, y \in X^m$, $x \neq y$.

Theorem 1.11 (see Theorem 19 from [2]). Given a complete metric space $(X, d)$ and $m \in \mathbb{N}^*$, for each Meir-Keeler function $f : X^m \to X$ there exists a unique fixed point $x_0$ of $f$ and the sequence $(f^n(x, ..., x))_{n \in \mathbb{N}^*}$ converges to $x_0$ for every $x \in X$.

The following definition is inspired by Definition 2.3 from [10] and Definition 17 from [2].

Definition 1.12. Given a metric space $(X, d)$ and $m \in \mathbb{N}^*$, a family of functions $f_i : X^m \to X$, $i \in I$, is called uniformly Meir-Keeler if for every $\varepsilon > 0$ there exist $\delta_{\varepsilon}, \lambda_{\varepsilon} > 0$ such that

$$d(f_i(x), f_i(y)) < \varepsilon - \lambda_{\varepsilon},$$
for all $i \in I$ and all $x, y \in X^m$ having the property that $d_{\text{max}}(x, y) < \varepsilon + \delta$.

G. Special classes of generalized iterated function systems

**Theorem 1.13** (see Theorem 3.11 from [15]). For each generalized possibly infinite iterated function system $F = ((X, d), (f_i)_{i \in I})$, where $(X, d)$ is a complete metric space, $I$ is finite and all the functions $f_i$ are $\varphi$-contractions for some comparison function $\varphi$, there exists a unique set $A_F \subseteq K(X)$ such that $F(F(A_F), ..., A_F) = A_F$ (i.e. $F$ has attractor).

**Remark 1.14.**

i) In the framework of the above theorem, the set $\bigcap_{k \in \mathbb{N}} f_{\alpha^1...\alpha^k}(A_F)$ consists on a single element denoted by $x_\alpha$ for every $\alpha = \alpha^1...\alpha^i... \in \Omega$. The function $\pi : \Omega \to X$ given by $\pi(\alpha) = x_\alpha$, for every $\alpha \in \Omega$, is called the canonical projection associated to $F$. For the properties of this function see Theorem 3.7, Corollary 3.9 and Theorem 3.11 from [16].

ii) The same line of reasoning used in [15] and [16] leads to the following conclusion: Each generalized possibly infinite iterated function system $F = ((X, d), (f_i)_{i \in I})$, where $(X, d)$ is a complete metric space and all the functions $f_i$ are $\varphi$-contractions for some comparison function $\varphi$, has attractor, i.e. there exists a unique set $A_F \subseteq \mathcal{B}(X)$ such that $F(F(A_F), ..., A_F) = A_F$. The function $\pi : \Omega \to X$, described by $\{\pi(\alpha)\} = \bigcap_{k \in \mathbb{N}} f_{\alpha^1...\alpha^k}(A_F)$ for every $\alpha \in \Omega$, which is called the canonical projection associated to $F$, has the property that $A_F = \overline{\pi(\Omega)}$.

**Theorem 1.15** (see Theorem 32 from [2]). For each generalized possibly infinite iterated function system $F = ((X, d), (f_i)_{i \in I})$, where $(X, d)$ is a complete metric space and the family of functions $(f_i)_{i \in I}$ is uniformly Meir-Keeler, there exists a unique set $A_F \subseteq \mathcal{B}(X)$ such that $F(F(A_F), ..., A_F) = A_F$ (i.e. $F$ has attractor).

**Remark 1.16.** The considerations from Remark 1.14 are also valid in the framework of Theorem 1.15. The arguments supporting this Remark are almost the same with the ones used for Remark 1.14. The only fact which needs a special attention is the justification of the following equality:

$$\lim_{n \to \infty} \sup_{\alpha \in n\Omega} \text{diam} \left( f_\alpha(A_F), ..., A_F \right) = 0,$$ (1)
Let us suppose, by reductio ad absurdum, that $l > l$. Consequently, we obtain the following contradiction:

$$\alpha$$ for every $\alpha$ in $\mathbb{F}$ system such that $\lim_{n \to \infty} f_{\alpha}^n(x) = 0$. Let us note that $d_n \in \mathbb{N}^*$ is a decreasing sequence of positive real numbers (since $f_{\alpha^1 \alpha^2 \cdots \alpha^n}(A_F, \ldots, A_F) \subset f_{\alpha^1 \alpha^2 \cdots \alpha^n}(A_F, \ldots, A_F)$ for every $\alpha^1 \in \Omega_1, \alpha^2 \in \Omega_2, \ldots, \alpha^n \in \Omega_n, \alpha^{n+1} \in \Omega_{n+1}$, $n \in \mathbb{N}^*$), so there exists $l \geq 0$ such that $\lim_{n \to \infty} d_n = l$. The justification of (1) is done if we prove that $l = 0$. Let us suppose, by reductio ad absurdum, that $l > 0$. Then, in view of the fact that the family of functions $(f_i)_{i \in I}$ is uniformly Meir-Keeler, there exist $\delta_l, \lambda_l > 0$ such that

$$d(f_i(x), f_i(y)) < l - \lambda_l,$$

for all $i \in I$ and all $x, y \in X^m$ having the property that $d_{\max}(x, y) < l + \delta_l$. Now, based on the fact that $\lim_{n \to \infty} d_n = l$, let us choose $n_0 \in \mathbb{N}^*$ such that $d_{n_0} < l + \delta_l$ and note that

$$d(f_{\alpha}(x_1, \ldots, x_m), f_{\alpha}(y_1, \ldots, y_m)) =$$

$$= d(f_{\alpha^1}(f_{\alpha}(x_1), \ldots, f_{\alpha}(x_m)), f_{\alpha^1}(f_{\alpha}(y_1), \ldots, f_{\alpha}(y_m))) \leq l - \lambda_l,$$

since

$$d_{\max}(f_{\alpha}(x_1), \ldots, f_{\alpha}(x_m)), f_{\alpha}(y_1), \ldots, f_{\alpha}(y_m)) =$$

$$= \max\{d(f_{\alpha}(x_1), f_{\alpha}(y_1)), \ldots, d(f_{\alpha}(x_m), f_{\alpha}(y_m))\} \leq$$

$$\leq \max\{\text{diam } f_{\alpha}(A_F, \ldots, A_F), \ldots, \text{diam } f_{\alpha}(A_F, \ldots, A_F)\} \leq d_{n_0} < l + \delta_l,$$

for every $\alpha = \alpha^1 \alpha^2 \cdots \alpha^{n_0} \alpha^{n_0+1} \in \Omega_{n_0+1}$, where $\alpha^1 \in \Omega_1, \alpha^2 \in \Omega_2, \ldots, \alpha^{n_0} \in \Omega_{n_0}, \alpha^{n_0+1} \in \Omega_{n_0+1}$, and every $x_1, \ldots, x_m, y_1, \ldots, y_m \in (A_F)^{m_{n_0}}$. Consequently, we obtain the following contradiction: $l \leq d_{n_0+1} \leq l - \lambda_l < l$.

3. The properties of the operator $H_F$

In this section we present some results which give sufficient conditions (on the constitutive functions of a generalized possibly infinite iterated function system $F$) for the operator $H_F$ to be continuous, generalized contraction, generalized $\varphi$-contraction, Meir-Keeler or contractive.

**Proposition 3.1.** For every generalized possibly infinite iterated function system $F = ((X, d), (f_i)_{i \in I})$ of order $m$ such that the family of functions
(f_i)_{i \in I} is uniformly equicontinuous (i.e. for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $i \in I$ and every $x, y \in X^m$ having the property that $d_{\text{max}}(x, y) < \delta_\varepsilon$ we have $d(f_i(x), f_i(y)) < \varepsilon$), the operator $H_F$ is continuous.

Proof. We are going to prove that for each sequence $(g_n)_{n \in \mathbb{N}}$ of elements from $C^m$ and $g \in C^m$ such that $\lim_{n \to \infty} g_n = g$, we have $\lim_{n \to \infty} H_F(g_n) = H_F(g)$. Let us suppose that $g_n = (g^1_n, \ldots, g^m_n)$ and $g = (g^1, \ldots, g^m)$, where $g^1_n, \ldots, g^m_n, g^1, \ldots, g^m \in \mathcal{C}$.

Let us fix $\varepsilon > 0$.

Since the family of functions $(f_i)_{i \in I}$ is uniformly equicontinuous, there exists $\delta_\varepsilon > 0$ such that

$$d(f_i(x), f_i(y)) < \varepsilon,$$

for every $i \in I$ and every $x, y \in X^m$ having the property that $d_{\text{max}}(x, y) < \delta_\varepsilon$.

Since $\lim_{n \to \infty} g_n = g$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_{\text{max}}(g_n, g) = \max\{d_u(g^1_n, g^1), \ldots, d_u(g^m_n, g^m)\} < \delta_\varepsilon,$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Consequently

$$d(g^i_n(\alpha), g^i(\alpha)) \leq \sup_{\alpha \in \Omega} d(g^i_n(\alpha), g^i(\alpha)) = d_u(g^i_n, g^i) \leq d_{\text{max}}(g_n, g) < \delta_\varepsilon,$$

for all $i \in \{1, \ldots, m\}$, $\alpha \in \Omega$ and $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Therefore

$$d_{\text{max}}((g^1_n(\alpha(1)), \ldots, g^m_n(\alpha(m))), (g^1(\alpha(1)), \ldots, g^m(\alpha(m)))) < \delta_\varepsilon,$$

for all $\alpha \in \Omega$ and $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Based on (1) and (2) we deduce that

$$d(f_{\alpha_1}(g^1_n(\alpha(1)), \ldots, g^m_n(\alpha(m))), f_{\alpha_2}(g^1(\alpha(1)), \ldots, g^m(\alpha(m)))) < \varepsilon,$$

for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$ and all $\alpha = \alpha^1\alpha^2\ldots\alpha^k\alpha^{k+1}\ldots \in \Omega$, where $\alpha^k \in \Omega_k$ for every $k \in \mathbb{N}$. The last inequality takes the following shape:

$$d(H_F(g^1_n, \ldots, g^m_n)(\alpha), H_F(g^1, \ldots, g^m)(\alpha)) < \varepsilon,$$

for all $\alpha \in \Omega$ and $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Hence

$$d_u(H_F(g^1_n, \ldots, g^m_n), H_F(g^1, \ldots, g^m)) =$$

$$= \sup_{\alpha \in \Omega} d(H_F(g^1_n, \ldots, g^m_n)(\alpha), H_F(g^1, \ldots, g^m)(\alpha)) \leq \varepsilon,$$
for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, i.e. $\lim_{n \to \infty} H_\mathcal{F}(g_n) = H_\mathcal{F}(g)$. □

**Proposition 3.2.** For every generalized possibly infinite iterated function system $\mathcal{F} = ((X, d), (f_i)_{i \in I})$ of order $m$ we have $\text{lip}(H_\mathcal{F}) \leq \sup_{i \in I} \text{lip}(f_i)$. In particular, if $\sup_{i \in I} \text{lip}(f_i) < 1$, then the operator $H_\mathcal{F}$ is a contraction.

**Proof.** With the notation $\sup_{i \in I} \text{lip}(f_i) \equiv C$, we have

$$d_u(H_\mathcal{F}(g), H_\mathcal{F}(h)) = \sup_{\alpha \in \Omega} d(H_\mathcal{F}(g^1, ..., g^m)(\alpha), H_\mathcal{F}(h^1, ..., h^m)(\alpha)) =$$

$$= \sup_{\alpha = \alpha_1 \alpha_2 ... \alpha_k \alpha^{k+1} \in \Omega} d(f_{\alpha^1}(g^1(\alpha(1)), ..., g^m(\alpha(m))), f_{\alpha^1}(h^1(\alpha(1)), ..., h^m(\alpha(m)))) \leq$$

$$\leq C \sup_{\alpha \in \Omega} d_{\max}(g^1(\alpha(1)), ..., g^m(\alpha(m))), (h^1(\alpha(1)), ..., h^m(\alpha(m)))) =$$

$$= C \sup_{\alpha \in \Omega} \max\{d(g^1(\alpha(1)), h^1(\alpha(1))), ..., d(g^m(\alpha(m)), h^m(\alpha(m)))\} \leq$$

$$\leq C \max\{\sup_{\alpha \in \Omega} d(g^1(\alpha), h^1(\alpha)), ..., \sup_{\alpha \in \Omega} d(g^m(\alpha), h^m(\alpha))\} =$$

$$= C \max\{d_u(g^1, h^1), ..., d_u(g^m, h^m)\} = C \ d_{\max}(g, h),$$

for every $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in \mathcal{C}^m$. □

**Proposition 3.3.** For every comparison function $\varphi$ and every generalized possibly infinite iterated function system $\mathcal{F} = ((X, d), (f_i)_{i \in I})$ of order $m$ such that all the functions $f_i$ are $\varphi$-contractions, the operator $H_\mathcal{F}$ is a $\varphi$-contraction.

**Proof.** We have

$$d_u(H_\mathcal{F}(g), H_\mathcal{F}(h)) = \sup_{\alpha \in \Omega} d(H_\mathcal{F}(g^1, ..., g^m)(\alpha), H_\mathcal{F}(h^1, ..., h^m)(\alpha)) =$$

$$= \sup_{\alpha = \alpha_1 \alpha_2 ... \alpha_k \alpha^{k+1} \in \Omega} d(f_{\alpha^1}(g^1(\alpha(1)), ..., g^m(\alpha(m))), f_{\alpha^1}(h^1(\alpha(1)), ..., h^m(\alpha(m)))) \leq$$

$$\leq \sup_{\alpha \in \Omega} \varphi(d_{\max}(g^1(\alpha(1)), ..., g^m(\alpha(m))), (h^1(\alpha(1)), ..., h^m(\alpha(m)))) =$$

$$= \sup_{\alpha \in \Omega} \varphi(\max\{d(g^1(\alpha(1)), h^1(\alpha(1))), ..., d(g^m(\alpha(m)), h^m(\alpha(m)))\}) \leq$$

$$\leq \varphi(\max\{\sup_{\alpha \in \Omega} d(g^1(\alpha), h^1(\alpha)), ..., \sup_{\alpha \in \Omega} d(g^m(\alpha), h^m(\alpha))\}) =$$

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then for all $i$

$$\delta_g(\alpha)$$

for all $d$ such that $d \alpha$ for all $d$ we obtained that $d \alpha$ where $\varepsilon > 0$ be fixed, but arbitrarily chosen.

Consequently, as $d \alpha$ ($\varepsilon$ for $\varepsilon > 0$) and all $d \alpha$ where $\varepsilon > 0$ be fixed, but arbitrarily chosen.

Since the family of functions $(f_i)_{i \in I}$ is uniformly Meir-Keeler, there exist $\delta_\varepsilon, \lambda_\varepsilon > 0$ such that

$$d(f_i(x), f_i(y)) < \varepsilon - \lambda_\varepsilon, \quad (1)$$

for all $i \in I$ and all $x, y \in X^m$ having the property that $d_{\text{max}}(x, y) < \varepsilon + \delta_\varepsilon$.

If $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in C^m$ are such that $d_{\text{max}}(g, f) < \varepsilon + \delta_\varepsilon$, then

$$d_{\text{max}}((g^1(\alpha(1)), ..., g^m(\alpha(m))), (h^1(\alpha(1)), ..., h^m(\alpha(m)))) =$$

$$= \max\{d(g^1(\alpha(1)), h^1(\alpha(1))), ..., d(g^m(\alpha(m)), h^m(\alpha(m)))\} \leq$$

$$\leq \max\{d_u(g^1, h^1), ..., d_u(g^m, h^m)\} = d_{\text{max}}(g, f) < \varepsilon + \delta_\varepsilon$$

for all $\alpha \in \Omega$. Then, taking into account (1), we get

$$d(f_{\alpha^1}(g^1(\alpha(1)), ..., g^m(\alpha(m))), f_{\alpha^1}(h^1(\alpha(1)), ..., h^m(\alpha(m)))) < \varepsilon - \lambda_\varepsilon, \quad (2)$$

for all $\alpha = \alpha^1 \alpha^2 ... \alpha^k \alpha^{k+1} ... \in \Omega$ and all $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in C^m$ such that $d_{\text{max}}(g, f) < \varepsilon + \delta_\varepsilon$. Hence

$$\sup_{\alpha \in \Omega} d_u(H_{\mathcal{F}}(g^1, ..., g^m)(\alpha), H_{\mathcal{F}}(h^1, ..., h^m)(\alpha)) =$$

$$= \sup_{\alpha \in \Omega} d(f_{\alpha^1}(g^1(\alpha(1)), ..., g^m(\alpha(m))), f_{\alpha^1}(h^1(\alpha(1)), ..., h^m(\alpha(m)))) \leq$$

$$\leq \varepsilon - \lambda_\varepsilon < \varepsilon,$$

for all $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in C^m$ such that $d_{\text{max}}(g, f) < \varepsilon + \delta_\varepsilon$, where $\alpha = \alpha^1 \alpha^2 ... \alpha^k \alpha^{k+1} ...$.

Consequently, as

$$d_u(H_{\mathcal{F}}(g), H_{\mathcal{F}}(h)) = \sup_{\alpha \in \Omega} d_u(H_{\mathcal{F}}(g^1, ..., g^m)(\alpha), H_{\mathcal{F}}(h^1, ..., h^m)(\alpha)),$$

we obtained that $d_u(H_{\mathcal{F}}(g), H_{\mathcal{F}}(h)) < \varepsilon$ for every $g, h \in C^m$ such that $d_{\text{max}}(g, f) < \varepsilon + \delta_\varepsilon$, i.e. $H_{\mathcal{F}}$ is Meir-Keeler. □
Proposition 3.5. For every generalized possibly infinite iterated function system $\mathcal{F} = ((X,d),(f_i)_{i \in I})$ of order $m$ such that $I$ is finite and all the functions $f_i$ are Meir-Keeler, the operator $H_{\mathcal{F}}$ is Meir-Keeler.

Proof. Let $\varepsilon > 0$ be a fixed, but arbitrarily chosen.

Since all the functions $f_i$ are Meir-Keeler, there exist $\delta, \lambda > 0$ such that

$$d(f_i(x), f_i(y)) < \varepsilon,$$

for all $i \in I$ and all $x, y \in X^m$ having the property that $d_{\max}(x, y) < \varepsilon + \delta$.

If $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in C^m$ are such that $d_{\max}(g, f) < \varepsilon + \delta$, then

$$d_{\max}(g^1(\alpha(1)), ..., g^m(\alpha(m))), (h^1(\alpha(1)), ..., h^m(\alpha(m)))) =$$

$$\max \{d(g^1(\alpha(1)), h^1(\alpha(1))), ..., d(g^m(\alpha(m)), h^m(\alpha(m)))\} \leq$$

$$\max \{d_u(g^1, h^1), ..., d_u(g^m, h^m) = d_{\max}(g, f) < \varepsilon + \delta,$$

for every $\alpha \in \Omega$. Then, taking into account (1), we get

$$d(f_{\alpha^1}(g^1(\alpha(1)), ..., g^m(\alpha(m))), f_{\alpha^1}(h^1(\alpha(1)), ..., h^m(\alpha(m)))) =$$

$$\varepsilon < \varepsilon,$$

for every $\alpha = \alpha^1 \alpha^2 ... \alpha^k \alpha^{k+1} \ldots \in \Omega$ and every $g = (g^1, ..., g^m), h = (h^1, ..., h^m) \in C^m$ such that $d_{\max}(g, f) < \varepsilon + \delta$. As the metric space $(\Omega, d)$ is compact (see Remark 1.3 and take into account that $I$ is finite), there exists $\alpha_0 \in \Omega$ such that

$$d_u(H_{\mathcal{F}}(g), H_{\mathcal{F}}(h)) = \sup_{\alpha \in \Omega} d(H_{\mathcal{F}}(g)(\alpha), H_{\mathcal{F}}(h)(\alpha)) = d(H_{\mathcal{F}}(g)(\alpha_0), H_{\mathcal{F}}(h)(\alpha_0)).$$

In view of (2) we conclude that $d_u(H_{\mathcal{F}}(g), H_{\mathcal{F}}(h)) < \varepsilon$ for every $g, h \in C^m$ such that $d_{\max}(g, f) < \varepsilon + \delta$, i.e. $H_{\mathcal{F}}$ is Meir-Keeler. \(\square\)

Proposition 3.6. For every comparison function $\varphi$ and every generalized possibly infinite iterated function system $\mathcal{F} = ((X,d),(f_i)_{i \in I})$ of order $m$ such that all the functions $f_i$ are $\varphi$-contractions, the operator $H_{\mathcal{F}}$ is Meir-Keeler.

Proof. Let us suppose that the family of functions $(f_i)_{i \in I}$ is not uniformly Meir-Keeler. Then there exists $\varepsilon_0 > 0$ with the property that for every $\delta, \lambda > 0$ there exist $x_{\delta, \lambda}, y_{\delta, \lambda} \in X^m$ and $i_0 \in I$ such that

$$d_{\max}(x_{\delta, \lambda}, y_{\delta, \lambda}) < \varepsilon_0 + \delta. \quad (1)$$
and
\[ d(f_{i_0}(x_{\delta,\lambda}), f_{i_0}(y_{\delta,\lambda})) \geq \varepsilon_0 - \lambda. \] (2)

Consequently we get
\[ \varepsilon_0 - \lambda \leq d(f_{i_0}(x_{\delta,\lambda}), f_{i_0}(y_{\delta,\lambda})) \leq \varphi(d_{\text{max}}(x_{\delta,\lambda}, y_{\delta,\lambda})) \leq \varphi(\varepsilon_0 + \delta), \] (1)

so
\[ \varepsilon_0 - \lambda \leq \varphi(\varepsilon_0 + \delta), \] (3)

for every \( \delta, \lambda > 0 \). Based on the right continuity of \( \varphi \) (see ii) from Definition 1.7, by passing to limit in (1) as \( \delta, \lambda \to 0 \), we get the contradiction \( \varepsilon_0 \leq \varphi(\varepsilon_0) < \varepsilon_0 \).

Hence the family of functions \((f_i)_{i \in I}\) is uniformly Meir-Keeler and Proposition 3.4 assures us that \( H_F \) is Meir-Keeler. \( \square \)

**Proposition 3.7.** For every generalized possibly infinite iterated function system \( F = ((X,d),(f_i)_{i \in I}) \) of order \( m \) such that \( \sup_{i \in I} \text{lip}(f_i) < 1 \), the operator \( H_F \) is Meir-Keeler.

**Proof.** Indeed, we note that all the functions \( f_i \) are \( \varphi_0 \)-contractions, where the comparison function \( \varphi_0 : [0, \infty) \to [0, \infty) \) is given by \( \varphi_0(t) = (\sup_{i \in I} \text{lip}(f_i))t \) for every \( t \in [0, \infty) \). Therefore, taking into account Proposition 3.6, we conclude that \( H_F \) is Meir-Keeler. \( \square \)

**Proposition 3.8.** For every generalized possibly infinite iterated function system \( F = ((X,d),(f_i)_{i \in I}) \) of order \( m \) such that \( I \) is finite and all the functions \( f_i \) are contractive, the operator \( H_F \) is contractive.

**Proof.** For every \( g = (g^1, \ldots, g^m), h = (h^1, \ldots, h^m) \in \mathcal{C}^m, g \neq h, \) as the metric space \((\Omega, d)\) is compact (see Remark 1.3 and take into account that \( I \) is finite), there exists \( \beta \in \Omega \) such that

\[ d_u(H_F(g), H_F(h)) = \sup_{\alpha \in \Omega} d(H_F(g)(\alpha), H_F(h)(\alpha)) = d(H_F(g)(\beta), H_F(h)(\beta)). \]

Then
\[ d_u(H_F(g), H_F(h)) = \]
\[ = d(f_{\beta_1}(g^1(\beta(1)), \ldots, g^m(\beta(m))), f_{\beta_1}(h^1(\beta(1)), \ldots, h^m(\beta(m)))) \]
\[ < d_{\text{max}}((g^1(\beta(1)), \ldots, g^m(\beta(m))), (h^1(\beta(1)), \ldots, h^m(\beta(m)))) \]

< \varepsilon_0.
\[
\max \{d(g^1(\beta(1)), h^1(\beta(1))), ..., d(g^m(\beta(m)), h^m(\beta(m)))\} \leq \\
\max \{d_u(g^1, h^1), ..., d_u(g^m, h^m) = d_{\text{max}}(g, f),
\]
for all \(g, h \in \mathcal{C}^m, g \neq h\), where \(\beta = \beta^1 \beta^2 ... \beta^k \in \Omega\), i.e. \(H_F\) is contractive.

\[\square\]

4. The main results

**Theorem 4.1.** For a generalized possibly infinite iterated function system \(F = ((X, d), (f_i)_{i \in I})\) there exists a unique \(\pi_0 \in \mathcal{C}\) such that:

a) \(H_F(\pi_0, ..., \pi_0) = \pi_0\);

b) \(\lim_{n \to \infty} H^n_F(f, ..., f) = \pi_0\),

for every \(f \in \mathcal{C}\), provided that one of the following conditions is satisfied:

i) there exists a comparison function \(\varphi\) such that all the functions \(f_i\) are \(\varphi\)-contractions (in particular this happens if \(\sup_{i \in I} \text{lip}(f_i) < 1\));

ii) the family of functions \((f_i)_{i \in I}\) is uniformly Meir-Keeler;

iii) \(I\) is finite and all the functions \(f_i\) are Meir-Keeler.

**Proof.** Just use Proposition 3.4, Proposition 3.5, Proposition 3.6 and Theorem 1.11. \(\square\)

**Proposition 4.2.** In the framework of the previous theorem, we have \(\overline{\pi_0(\Omega)} = A_F\).

**Proof.** In view of Theorem 4.1, we have

\[
\overline{\pi_0(\Omega)} \overset{\text{Theorem 4.1, a)}}{=} H_F(\pi_0, ..., \pi_0)(\Omega) = \bigcup_{\alpha \in \Omega} \{H_F(\pi_0, ..., \pi_0)(\alpha)\} = \\
\bigcup_{\alpha = \alpha^1 \alpha^2 ... \alpha^k \alpha^{k+1} \in \Omega} \{f_{\alpha^1}(\pi_0(\alpha(1)), ..., \pi_0(\alpha(m)))\} = \\
\bigcup_{i \in I} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \overset{(*)}{=} \bigcup_{i \in I} f_i(\overline{\pi_0(\Omega)}, ..., \overline{\pi_0(\Omega)}) = \\
F_F(\overline{\pi_0(\Omega)}, ..., \overline{\pi_0(\Omega)}),
\]
i.e.

\[
F_F(\overline{\pi_0(\Omega)}, ..., \overline{\pi_0(\Omega)}) = \overline{\pi_0(\Omega)},
\]

(1)
where the equality \((\ast)\) is justified in the following way: we have
\[
f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq f_i(\pi_0(\Omega), ..., \pi_0(\Omega)),
\]
so
\[
\bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq \bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq \bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega))
\]
and therefore
\[
\bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq \bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq \bigcup_{i \in \Omega} f_i(\pi_0(\Omega), ..., \pi_0(\Omega)).
\]

From Remark 1.14, ii) and Theorem 1.15, as \(\pi_0(\Omega) \in \mathcal{B}(X)\), using relation (1), we conclude that \(\pi_0(\Omega) = A_F\).

**Proposition 4.3.** In the framework of the previous theorem, \(\pi_0\) is the canonical projection associated to \(\mathcal{F}\).

**Proof.** For \(i \in I\), let us consider the function \(F_i : \Omega^m \to \Omega\) described in the following way:
\[
F_i(\beta_1, ..., \beta_m) = \alpha_1\alpha_2...\alpha_p...
\]
where if, \(\beta_k = \beta_1^k\beta_2^k...\beta_p^k \in \Omega\), with \(\beta_p^k \in \Omega_p\), \(k \in \{1, ..., m\}\), we have \(\alpha_1 = i \in \Omega_1\), \(\alpha_2 = \beta_1^1\beta_2^1...\beta_1^m \in \Omega_2\), ..., \(\alpha_{p+1} = \beta_1^p\beta_2^p...\beta_p^m \in \Omega_{p+1}\), ...

**Claim.**
\[
\pi_0(F_{\alpha_1\alpha_2...\alpha_k}(\Lambda_1, ..., \Lambda_m)) = f_{\alpha_1\alpha_2...\alpha_k}(\pi_0(\Lambda_1), ..., \pi_0(\Lambda_m)), \quad (1)
\]
for all \(k \in \mathbb{N}^*\), \(\alpha_1 \in I\), \(\alpha_2 \in \Omega^1\), ..., \(\alpha_k \in \Omega^k\) and \(\Lambda_1, ..., \Lambda_m \subseteq \Omega\).

**Justification.** We are going to use the mathematical induction method. Let us start by noting that
\[
\pi_0(\alpha) = H_F(\pi_0, ..., \pi_0)(\alpha) = f_{\alpha}(\pi_0(\alpha(1)), ..., \pi_0(\alpha(m))),
\]
for every \(\alpha = \alpha_1\alpha_2...\alpha_k... \in \Omega\), so
\[
\pi_0(F_{\alpha}(\Lambda_1, ..., \Lambda_m)) = f_{\alpha}(\pi_0(\Lambda_1), ..., \pi_0(\Lambda_m)), \quad (2)
\]
for all \(\alpha \in I\) and \(\Lambda_1, ..., \Lambda_m \subseteq \Omega\), i.e. (1) is valid for \(k = 1\).
Now we suppose that (1) is valid for \( k \) and we prove that is valid also for \( k + 1 \). Indeed, we have

\[
\pi_0(F_{\alpha_1^1\alpha_2^2...\alpha_k\alpha_{k+1}^1}(\Lambda_1, ..., \Lambda_{m_{k+1}})) =
\]

\[
= \pi_0(F_{\alpha_1^1}(F_{\alpha(1)}(\Lambda_1, ..., \Lambda_{m_1}), ..., F_{\alpha(m)}(\Lambda_{m_{k+1}-m_k+1}, ..., \Lambda_{m_{k+1}})))
\]

\[(2)\]

\[
= f_{\alpha_1^1}(\pi_0(F_{\alpha(1)}(\Lambda_1, ..., \Lambda_{m_1})), ..., \pi_0(F_{\alpha(m)}(\Lambda_{m_{k+1}-m_k+1}, ..., \Lambda_{m_{k+1}})))
\]

Claim for \( k \)

\[
= f_{\alpha_1^1}(f_{\alpha(1)}(\pi_0(\Lambda_1), ..., \pi_0(\Lambda_{m_1})), ..., f_{\alpha(m)}(\pi_0(\Lambda_{m_{k+1}-m_k+1}), ..., \pi_0(\Lambda_{m_{k+1}})))
\]

\[
= f_{\alpha_1^1}(\alpha_2^2...\alpha_k\alpha_{k+1}^1)(\pi_0(\Lambda_1), ..., \pi_0(\Lambda_{m_{k+1}}))
\]

where \( \alpha = \alpha_1^1\alpha_2^2...\alpha_k^k \), and the justification of the claim is done.

For \( \alpha = \alpha_1^1\alpha_2^2...\alpha_k^k \) arbitrarily chosen in \( \Omega \), where \( \alpha^k \in I \), we have

\[
\pi_0(\alpha) \in \pi_0(\bigcap_{k \in \mathbb{N}^*} F_{\alpha_1^1\alpha_2^2...\alpha_k^k}(\Omega, ..., \Omega)) \subseteq
\]

\[
\subseteq \pi_0(\bigcap_{k \in \mathbb{N}^*} F_{\alpha_1^1\alpha_2^2...\alpha_k^k}(\Omega, ..., \Omega)) \subseteq \bigcap_{k \in \mathbb{N}^*} \pi_0(F_{\alpha_1^1\alpha_2^2...\alpha_k^k}(\Omega, ..., \Omega))
\]

Claim

\[
= \bigcap_{k \in \mathbb{N}^*} f_{\alpha_1^1\alpha_2^2...\alpha_k^k}(\pi_0(\Omega), ..., \pi_0(\Omega)) \subseteq \bigcap_{k \in \mathbb{N}^*} f_{\alpha_1^1\alpha_2^2...\alpha_k^k}(\pi_0(\Omega), ..., \pi_0(\Omega))
\]

Proposition 4.2

\[
= \bigcap_{k \in \mathbb{N}^*} f_{\alpha_1^1\alpha_2^2...\alpha_k^k}(A_{\mathcal{F}}, ..., A_{\mathcal{F}}) \equiv 1.14 \{ \pi(\alpha) \},
\]

so \( \pi_0 = \pi \), i.e. \( \pi_0 \) is the canonical projection associated to \( \mathcal{F} \).

\[
\square
\]

5. An example

Following [2], we consider the generalized possibly infinite iterated function system \( \mathcal{F} = ((X, d), (f_n)_{n \in \mathbb{N}}) \) of order 2, where \( X = [0, 1] \), \( d \) is the euclidean metric and the functions \( f_n : [0, 1] \times [0, 1] \rightarrow [0, 1] \) are given by

\[
f_n(x, y) = \frac{1}{2^{n+2}}(x + y) + \frac{1}{2^{n+1}},
\]

for every \( n \in \mathbb{N} \), \( x, y \in [0, 1] \).

Note that:

i) the family \( (f_n)_{n \in \mathbb{N}} \) is uniformly Meir-Keeler since for every \( \varepsilon > 0 \) there exist \( \lambda_\varepsilon = \frac{\varepsilon}{9} > 0 \) and \( \delta_\varepsilon = \frac{\varepsilon}{3} > 0 \) such that \( |f_n(u) - f_n(v)| < \varepsilon - \lambda_\varepsilon \) for all \( u, v \in [0, 1] \times [0, 1] \) with the property that \( d_{\max}(u, v) < \varepsilon + \delta_\varepsilon \);
ii) the operator $H_F : C^2 \to C$ acts in the following way:

$$H_F(g_1, g_2)(\alpha) = \frac{1}{2^{n^1+2}}(g_1(\alpha^1_1 \alpha^1_2 \ldots \alpha^1_k \ldots) + g_2(\alpha^2_1 \alpha^2_2 \ldots \alpha^2_k \ldots)) + \frac{1}{2^{n^1+1}},$$

for every $g_1, g_2 \in C$ and every $\alpha = \alpha^1 \alpha^2 \ldots \alpha^k \ldots \in \Omega$, where $\alpha^k = \alpha^1_k \alpha^2_k \in \Omega_k$ with $\alpha^1_k, \alpha^2_k \in \Omega_{k-1}$.

iii) $A_F = [0, 1]$.

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