Transition from degeneracy to coalescence: theorem and applications

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Exceptional point (EP) is exclusive for non-Hermitian system and distinct from that at a degeneracy point (DP), supporting intriguing dynamics, which can be utilized to probe quantum phase transition and prepare eigenstates in a Hermitian many-body system. In this work, we investigate the transition from DP for a Hermitian system to EP driven by non-Hermitian terms. We present a theorem on the existence of transition between DP and EP for a general system. The obtained EP is robust to the strength of non-Hermitian terms. We illustrate the theorem by an exactly solvable quasi-one dimensional model, which allows the existence of transition between fully degeneracy and exceptional spectra driven by non-Hermitian tunnelings in real and $k$ spaces, respectively. We also study the EP dynamics for generating coalescing edge modes in Su-Schrieffer-Heeger-like models. This finding reveals the ubiquitous connection between DP and EP.

I. INTRODUCTION

The theoretical and experimental studies on the non-Hermitian system indicate that the interplay of lattice geometry and non-Hermitian elements, such as imaginary on-site potential and asymmetry tunneling, can induce exotic quantum dynamics, which never happens in a Hermitian system. Intuitively, these phenomena are known to arise from the appearance of complex eigen energy within symmetry broken region, leading to the explosion of the Dirac probability of the eigenstate. However, it follows from a peculiar feature of the non-Hermitian system, exceptional point (EP) dynamics, without the need of symmetry breaking. The EP in a non-Hermitian system occurs when two or more eigenstates coalesce, and usually associates with the non-Hermitian phase transition. In a parity-time ($\mathcal{PT}$) symmetric non-Hermitian system (or other similar systems), the $\mathcal{PT}$-symmetry of eigenstates spontaneously breaks at the EP, which determines the exact $\mathcal{PT}$-symmetric phase and the broken $\mathcal{PT}$-symmetric phase in this system. The EP plays a pivotal role in intriguing dynamics and applications including asymmetric mode switching, unidirectional lasing, and enhanced optical sensing. Recently, EP dynamics is employed to engineer a target quantum state and probe quantum phase transitions. The mechanism of such a scheme is setting the target state as coalescing state of a non-Hermitian Hamiltonian. In general, a familiar target state as the central resource of quantum information processing is always an eigenstate of Hermitian system, such as topological state, many-particle entangled state. It, therefore, requires the coalescing state is Hermitian-related and robust to the perturbation.

The aim of this paper is to provide a method for setting a target quantum state to be a coalescing state. As well known, an EP is exclusive for non-Hermitian system and is distinct from a degeneracy point (DP). We will show that the transition from DP for a Hermitian system to EP can be realized by proper non-Hermitian terms. This theorem indicates that, for a general Hermitian system with a DP, one of degeneracy eigenstates can be set as a coalescing state of a non-Hermitian system. This allows the scheme to prepare a target quantum state based on the EP dynamics. To illustrate the theorem, we investigate an exactly solvable quasi-one dimensional model, which supports the transition between fully degeneracy and exceptional spectra. It is shown that such a model in the thermodynamic limit is equivalent to a two-coupled ring with power-law decay long-range hopping term. As its application, we also study the EP dynamics for generating coalescing edge modes in Su-Schrieffer-Heeger-like models. This finding provides a way to construct a non-Hermitian system with EP based on a Hermitian system with DP. The advantage of this method is that the coalescing state is the eigenstate of a Hermitian system and is robust to the strength of the non-Hermitian term. We expect our results benefit to experimental research.

This paper is organized as follows. In Sec. II we present a theorem for a general system to create robust EP from DP. In Sec. III and IV we illustrate the theorem from two examples. In Sec. VI we propose a dynamic scheme to prepare edge modes as application of the theorem. Finally, we provide a summary in Sec. VII.

II. THEOREM ON ROBUST EP FROM DP

A general system at EP is obtained by tuning an imaginary parameter, such as imaginary potential or flux, to switch real energy levels to complex ones. The critical value of the parameter is usually the solution of a transcendental equation, and the EP system is sensitive to the imaginary parameter. Then is a little tough to set an EP system precisely in practice. The main aim of this paper is to answer the questions of whether a robust EP system can be obtained. In this section, we present a theorem on establishing EP based on DP of a Hermitian system. We will show that the obtained EP is not sensitive to the strength of the non-Hermitian parameters.

We consider a general non-Hermitian Hamiltonian in the form

$$H = H_0 + H', \quad (1)$$
which can be separated into two parts, a Hermitian and non-Hermitian ones, i.e.,
\[ H_0 = \langle H_0 \rangle, \text{but } H' \neq \langle H' \rangle. \] (2)
Here \( H_0 \) has two-fold degenerate eigenstates \( |A\rangle \) and \( |B\rangle \), i.e.,
\[ H_0 |A\rangle = 0, H_0 |B\rangle = 0, \] (3)
where we take zero degenerate eigen energy for the sake of simplicity. If the non-Hermitian term \( H' \) satisfies
\[ H' |A\rangle = 0, \langle H' \rangle |B\rangle = 0, \] (4)
we have
\[ H |A\rangle = 0, H |B\rangle = 0. \] (5)
Two states \( |A\rangle \) and \( |B\rangle \) are mutually biorthogonal conjugate and \( \langle B|A \rangle \) is the biorthogonal norm of them. The vanishing norm indicates that state \( |A\rangle \) is coalescing state of \( H \). So when two Hamiltonians \( H_0 \) and \( H' \) have a common zero-energy state \( |A\rangle \) \( \langle B| \) \( \rangle \), one can say that Hamiltonian \( H \) get an EP, which is robust to the strength of \( H' \) varies.

The theorem is given here without specific reference to the detailed form of the Hamiltonian. It should work for the nonrelativistic and relativistic, continuous and discrete Hamiltonians. Applying it to a tight-binding model, we can find some detailed signatures of the degenerate eigen states. Considering the conditions, \( H' |A\rangle = 0 \) and \( \langle H' \rangle |B\rangle = 0 \), the simplest example for such a \( H' \) is unidirectional hopping term, i.e., \( \kappa a_i^\dagger a_j \) \( (i \neq j) \), where \( a_i \) and \( a_j \) are fermion or boson operators. Note that the above conditions can be satisfied if \( a_j |A\rangle = 0 \) and \( a_i |B\rangle = 0 \). It means that two states \( |A\rangle \) and \( |B\rangle \) have nodal points at \( j \) and \( i \) respectively. In addition, the existence of EP is independent of the nonzero value of \( \kappa \). This rigorous conclusion has important implications in the design of quantum devices to prepare a targeted quantum state at will. In the following sections, we will present serval illustrative examples to demonstrate the theorem.

III. ZERO POINT IN REAL SPACE

Consider a uniform \( 2N \)-site ring system with the Hamiltonian
\[ H_0 = \sum_{j=1}^{2N} (c_j^\dagger c_{j+1} + \text{H.c.}) = \sum_k k |c_k \rangle \langle c_k |, \] (6)
where
\[ c_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} e^{ikj} c_j^\dagger, \] (7)
with the wave vector \( k = \pi n/N, n = 1, 2, ..., 2N \). We note that there are \( N-1 \) pairs of degenerate eigenstates. The aim of this section is to answer the question of what kind of \( H' \) can result in the transition from a DP to EP. A single-particle eigenstate has the form
\[ |k\rangle = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} e^{ikj} |j\rangle, \] (8)
which has no nodal point for any \( k \). Here \( |j\rangle = c_j^\dagger |0\rangle \) is the position state, where \( |0\rangle \) is the vacuum state. However, we can construct two-fold degenerate eigenstates in the form
\[ |\psi^+_k\rangle = \frac{1}{\sqrt{2}} (|k\rangle \pm e^{i2k \theta} |−k\rangle), \] (9)
with \( \langle \psi^+_k |\psi^-_k \rangle = 0 \). The additional factor \( e^{i2k \theta} \) leads to a nodal point at both site, i.e.,
\[ \langle l_0 |\psi^-_k \rangle = 0. \] (10)
It easy to check that for another state \( |\psi^+_k \rangle \), there is a nodal point at the \( (l_0 + r) \) th site, i.e.,
\[ \langle l_0 + r |\psi^+_k \rangle = 0, \] (11)
when the following condition is satisfied
\[ \cos(kr) = 0, \] (12)
which requires \( k \) to be specific values
\[ k = \frac{(2m + 1) \pi}{2r}, m = 0, 1, 2, \ldots \] (13)
Then non-Hermitian term \( H' \) is in the form
\[ H' = \kappa c_{l_0}^\dagger c_{l_0 + r}. \] (14)

For instance, for \( r = 1 \), we have \( k = \pi/2 \), and for \( r = 2 \), we have \( k = \pi/4 \) and \( k = 3\pi/4 \). It means that one can take \( H' = \kappa c_{l_0}^\dagger c_{l_0 + 1} \) to acquire the coalescing state \( |\psi^+_{3\pi/4}\rangle \), while taking \( H' = \kappa c_{l_0}^\dagger c_{l_0 + 2}, \) obtain \( |\psi^+_{3\pi/4}\rangle \). In Fig. 4, we schematically illustrate a finite system with several kinds of non-Hermitian hopping terms and plot the corresponding energy levels. It indicates that for the finite system, there are some DPs that cannot be transmitted to EPs by a single asymmetric hopping term.

IV. ZERO POINT IN \( k \)-SPACE

In the previous section, we have shown that a single asymmetric hopping term for a Hermitian ring can drive the transition between DP and EP for some specific but not any levels. A natural question is what kind of \( H' \) can result in the transition from a DP to EP for any given \( k \). Actually, this can be done by simply taking
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FIG. 1. Schematic illustration of the levels of a Hermitian 12-site ring, which show the effect of an additional single unidirectional hopping (red arrow) crossing two sites. Two degenerate levels are indicated by a pair of blue segments, while a coalescing level is in red. The parameter for red arrow is $\kappa = 0.5$. We can see that a single asymmetric hopping term for a Hermitian ring can drive the transition between DP and EP for some specific but not any levels.

(a) (b) (c)

FIG. 2. (a) The schematic diagram of Hamiltonian in Eq. (16). The black straight lines denote the real long-range couplings. The colored lines with bidirectional arrows denote the pure imaginary couplings, and we adopt different colors and opacity to distinguish the imaginary couplings. The dark red lines represent the coupling where the sum of the site’s position is equal to 5 or 3, and we further distinguish 5 and 3 by the reduced transparency. The same goes for the green lines standing for 23, 21, 19, the dark green lines for 11, 9, 7, and the red lines for 17, 15, 13. (b) We reshape the schematic diagram of (a) to be a ladder system. (c) The schematic illustration of Hamiltonian in Eq. (18). The black ladder represents $H_0$, the colored lines with bidirectional arrows denote $H'$. Ignoring the hoppings on the boundary, the intralayer long-range hoppings for (b) and (c) have the same linking method but different strengths.

\[ H'_k = Jc_k^\dagger c_{-k}, \]
which contains all range unidirectional hopping terms. In the above example, $H' = \kappa c^\dagger c_j$, we do not restrict $c^\dagger$ and $c_j$ to be the operators in real space. For example, a fermionic EP Hamiltonian can be $H' = \kappa c^\dagger k_j c_{k_2}$. The corresponding coalescing state is $c^\dagger_k |0\rangle_k_1 |0\rangle_k_2$, while the auxiliary state is $c_j^\dagger |0\rangle_k |0\rangle_k_2$. Now we investigate an example to demonstrate the application of the result above. We consider a system with the non-Hermitian term

\[ H' = \sum_k H'_k = \sum_{k>0} c_k^\dagger c_{-k}. \]

In the case of $k \neq 0$ and $\pi$, both $H' |k\rangle = 0$ and $(H')^\dagger | -k\rangle = 0$ always hold, which indicates $H'$ drives all DPs into EPs. Straightforward derivation leads to

\[
H'_k \approx \frac{1}{N} \sum_{l+j=odd(l>l)} i \cot \left[ \frac{(l+j)\pi}{N} \right] \left( c_{-j}^\dagger c_j + H.c. \right) + \frac{1}{2} \sum_{l+j=N,2N} c_{-j}^\dagger c_j.
\]

In Fig. 2(a), we schematically illustrate a finite system with above $H'$. We note that the non-Hermitian hopping strength $i \cot (l+j)\pi/N$ is large as $l+j$ close to 0, $N$, and $2N$. For $l+j = N+1$, we have $l-j = N+1-2j$, which indicates long range hopping. For example, taking
\[ j = N/4, \text{ we have } l = 3N/4 + 1. \text{ Then the hopping spacing is } l - j = N/2 + 1. \text{ However, if we reshape the ring as a ladder, such long-range hopping terms become short-range as shown in Fig. 2(b). In large } N \text{ limit, considering the case with } l + j = nN + \Delta, \text{ with } \Delta \ll N \left( \Delta = \pm 1, \pm 3, \ldots, \text{ and } n = 0, 1, 2, \ldots, \right), \text{ we can simplify the coupling constant} \]
\[
\frac{1}{N} \cot \left( \frac{(l + j)\pi}{N} \right) = \frac{1}{N} \cot \left( \frac{\Delta \pi}{N} \right) \approx \frac{1}{\pi \Delta}. \tag{17}
\]

This example indicates that for the finite system, \( N - 1 \) pairs of degenerate levels can be switched into coalescing \( N - 1 \) levels by \( H' \).

The underlying mechanism of realizing the transition from degeneracy spectrum to coalescing spectrum can be revealed by the following model. \textbf{Inspired by Eq. (17),} we consider a two coupled uniform chains system, which has a two-leg ladder structure. Figure 2(c) sketches the geometry of the system, in which the hopping amplitudes in each leg are uniform and the hopping strengths are distance-dependent. Such a ladder system is a bipartite lattice system, consisting of two sub-lattices \( A \) and \( B \). We write down the Hamiltonian for the system in the form
\[
H_0 = \sum_{j=1}^{N} \left( a_j^\dagger b_j + a_j^\dagger a_{j+1} + b_j^\dagger b_{j+1} + \text{H.c.} \right), \tag{18}
\]
\[
H' = \frac{iJ}{2} \sum_{j=1}^{N} \sum_{n=1}^{N} \frac{1}{2n-1} \left( a_j^\dagger b_{j+2n-1} - b_j^\dagger a_{j+2n-1} + \text{H.c.} \right), \tag{19}
\]
where \( a_j^\dagger \) and \( b_j^\dagger \) are the creation operators of fermion or boson at the \( l \)th site of sub-lattice \( A \) and \( B \), respectively. We take a periodic boundary condition, by setting \( a_{N+1}^\dagger = a_1^\dagger \) and \( b_{N+1}^\dagger = b_1^\dagger \). Taking the transformation
\[
\begin{cases}
  a_k = \frac{1}{\sqrt{N}} \sum_j e^{ikj} a_j \\
  b_k = \frac{1}{\sqrt{N}} \sum_j e^{ikj} b_j,
\end{cases}
\]
we have
\[
H = \sum_k H_k = \sum_k \left( a_k^\dagger b_k^\dagger \right) h_k \left( a_k b_k \right), \tag{20}
\]
where the wave vector \( k = \pi (2n - N) / N \), \( n = 0, 1, \ldots, N - 1 \). The Bloch Hamiltonian is
\[
h_k = \begin{pmatrix}
  0 & 1 - \Delta_k \\
  1 + \Delta_k & 0
\end{pmatrix} + 2 \cos k. \tag{21}
\]
where
\[
\Delta_k = J \sum_{n=1}^{N} \sin \left( (2n - 1)k / 2n - 1 \right). \tag{22}
\]

The Hamiltonian \( H \) can be easily diagonalized since \( [H_k, H_{k'}] = 0 \), and the spectrum is
\[
\varepsilon_k = 2 \cos k \pm \sqrt{1 - (\Delta_k)^2}. \tag{23}
\]

We note that when the summation in \( \Delta_k \) covers to infinity, \( \Delta_k \) is a step function
\[
\Delta_k = \frac{J \pi}{4} \text{sgn}(k) \tag{24}
\]
according to Fourier analysis. Then the spectrum becomes
\[
\varepsilon_k = \pm \sqrt{1 - (J \pi/4)^2}, \tag{25}
\]
which vanishes at \( J = 4/\pi \) and the EP spectrum appears.

\[ V. \text{ EMERGING EDGE MODES} \]

In the above sections, the obtained coalescing states are all extended states, which have real wave vectors. In this section, we focus on the transition from DP to EP where the coalescing bound states appear. A coalescing bound state is a local state and lives at an energy gap, and then can be a stable target state of the time evolution at EP. In the following, we at first present two examples of coalescing edge states in SSH-like models. Then study the dynamical preparation of a 2D edge state.

\[ A. \text{ SSH chain} \]

We start our investigation by considering a SSH chain with single unidirectional hopping across two ends, with the Hamiltonian in the form
\[
H_0 = \frac{1}{2} \sum_{l=1}^{2N-1} \left[ 1 + (-1)^l \delta \right] c_l^\dagger c_{l+1} + \text{H.c.},
\]
\[
H' = \kappa c_{2N}^\dagger c_{2N}. \tag{26}
\]
It is a bipartite lattice, i.e., it has two sublattices \( A, B \) such that each site on lattice \( A \) has its nearest neighbors on sublattice \( B \), and vice versa. The Hermitian system \( H_0 \) is the prototype of a topologically nontrivial band insulator with a symmetry protected topological phase. In recent years, it has been attracted much attention and extensive studies have been demonstrated. The degenerate zero modes take the role of topological invariant in the infinite \( N \) limit and are explicitly expressed as
\[
\begin{aligned}
|L\rangle &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \frac{\delta + 1}{\delta + 1} \right)^{j-1} c_{2j-1} \left| 0 \right> \\
|R\rangle &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \frac{-\delta + 1}{\delta + 1} \right)^{N-j} c_{2j} \left| 0 \right>, \tag{27}
\end{aligned}
\]
for \( \delta > 0 \) where the normalization factor is \( \Omega = \left\{ 1 - \left[ (\delta - 1) / (\delta + 1) \right]^{2N} \right\} / \left\{ 1 - \left[ (\delta - 1) / (\delta + 1) \right]^2 \right\} \). It is easy to check that
\[
c_{2N} |L\rangle = c_{1} |R\rangle = 0, \tag{28}
\]
FIG. 3. (a) Schematic illustration of SSH chain. The uni-
directional hopping $\kappa$ pointing from the head to the tail of
the SSH chain is denoted by a red arrow. Two degenerate
global hopping $\kappa$ pointing from the head to the tail of
the SSH chain is denoted by a red arrow. Two degenerate
each other. (b) Schematic illustration of SSH cylinder. The details
couplings are shown in the inset, wherein the unidirectional
coupling is indicated by the red arrow.

which ensues that

$$H'|L\rangle = (H')^\dagger|R\rangle = 0. \quad (29)$$

According to the theorem aforementioned, $|L\rangle$ is a coa-
lescing edge state. In previous works, it has been shown that,
two bound states appear in the bulk of a chain when
in this case, it is not surprising that a single bound state $|L\rangle$
can appear in the bulk of a finite size but sufficient long
chain. The profiles of the edge modes are schematically
illustrated in Fig. 3 (a). In contrast, when the unidi-
rectional hopping replaces a weak bond, there is no zero
mode appears. Therefore, the topological feature can be
demonstrated by adding a non-Hermitian impurity, as a
generalization of the bulk-edge correspondence.

B. SSH cylinder

A similar situation can happen in 2D system by adding
a local non-Hermitian impurity. In the following we
present an example of 2D system, which is an extended
2D SSH cylinder. The 2D SSH cylinder consists of $M$
(even) chains which are uniformly coupled,

$$H_{sc}^0 = \sum_{j=1}^{M/2} \sum_{l=1}^{N} \left[ 1 + (-1)^l \delta \right] c_{j,l}^\dagger c_{j,l+1}^\dagger$$

$$+ J \sum_{j=1}^{M/2} \sum_{l=1}^{N} c_{j,l}^\dagger c_{j,l+1,l}^\dagger + H.c., \quad (30)$$

and a perturbation on the boundary,

$$H_{sc}' = \kappa c_{1,1}^\dagger c_{M,1}, \quad (31)$$

where $j$ ($l$) is the index of row (column). Therefore, the
Hamiltonian of the SSH cylinder reads $H_{sc} = H_{sc}^0 + H_{sc}'$.

Figure 3 b) shows the schematic illustration of SSH cylin-
der, the red arrow in the inset represents the perturbation. $H_{sc}^0$ has four degenerate local zero modes, and two of
which localizes at the left boundary

$$\begin{align*}
|L_o\rangle &= \frac{1}{\sqrt{\Omega}} \sum_{j=1}^{N} \sum_{l=1}^{2N} \left( \delta^j \right)_{l-1} \left( \delta^l \right)^{j-1} (H_{2ij}^0 |2j,2l-1\rangle |0\rangle, \quad (32)
|L_o\rangle &= \frac{1}{\sqrt{\Omega}} \sum_{j=1}^{N} \sum_{l=1}^{2N} \left( \delta^j \right)_{l-1} \left( \delta^l \right)^{j-1} (H_{2ij}^0 |2j,2l-1\rangle |0\rangle
\end{align*}$$

where

$$\Omega = M/2 \{1 - [(\delta + 1)^j + (\delta + 1)^l]/\{1 - [(\delta - 1)/(\delta + 1)^2]\}.$$ It is not hard to check that

$$H_{sc}' |L_o\rangle = 0, (H_{sc}')^\dagger |L_o\rangle = 0, \quad (33)$$

which means $|L_o\rangle$ is a coalescing edge state of $H_{sc}$, and
$|R_{e(o)}\rangle$ remains degenerate states, $H |R_{e(o)}\rangle = 0$.

C. Dynamical preparation of edge state

It seems that DP and EP systems are totally two differ-
et ones. Taking $H' \rightarrow \kappa H'$ to impose a strength on the
non-Hermitian term, one can investigate the effect of $H'$
on the system quantitatively. Intuitively, a small change
of $\kappa$ from zero can result in a drastic change. However, in
the following, we will show that there exists a continuous
crossover between them. We measure the signature of the
system by detecting the dynamics of the observable,
such as the time evolution of particle probability. As the
application of the theorem, the relationship between
time and particle probability reminds a new approach to
prepare edge modes.

We consider a 2-site system as the simplest example
for the obtained theorem, which has the Hamiltonian

$$H_{2s} = \kappa c_1^\dagger c_2 + \varepsilon_0 \left( c_1^\dagger c_1 + c_2^\dagger c_2 \right). \quad (34)$$

The time evolution operator has the form

$$U(t) = \exp(-iH_{2s}t)$$

$$= \exp \left[ -i\varepsilon_0 \left( c_1^\dagger c_1 + c_2^\dagger c_2 \right) t \right] \exp(-i\kappa c_1^\dagger c_2 t). \quad (35)$$
Then for an initial state with fixed particle number, i.e.,
\[ |\psi\rangle = |c_1^\dagger 0\rangle \]
while the time evolution of the auxiliary state \( |\psi_a\rangle = c_2^\dagger |0\rangle \) is
\[ U(t) |\psi_a\rangle = \exp (-i\varepsilon_0 t) (|\psi_a\rangle - i\kappa t |\psi_a\rangle) , \tag{40} \]
where we have used the identity \((c_1^\dagger c_2)^2 = 0\). The difference between DP and EP is obvious when \(\kappa t \gg 1\). However, within the time scale \(t \ll 1/\kappa\), the dynamics under the DP and EP has no difference. It indicates that the crossover from DP to EP is continuous. Similarly, it has been shown that a non-Hermitian system around EP exhibits some peculiar critical dynamics as EP.\(^{26}\)

Equation (40) indicates that the time evolution of the coalescing state linearly depends on time \(t\), so the evolved state is approximately equal to \(|\psi_c\rangle\) for the relatively large time scale. When \(|\psi_c\rangle\) is a localized coalescing state, the dynamical preparation of the robust edge state is as follows. Consider the time evolution driven by Hamiltonian \(H_{sc}\), the initial state is
\[ |\psi(0)\rangle = \frac{1}{\sqrt{M/2}} \sum_{j=1}^{M/2} c_{2j-1,1}^\dagger |0\rangle , \tag{41} \]
which is just \(|L_0\rangle\) when \(\delta\) infinitely approaches 1. Figure 4(a) exhibits numerical simulations of \(|\psi(t)\rangle\), the system size is \(N = 100, M = 20\). We only show the region \(N \leq 30\) because the probability is almost zero in the other region. Figure 4(b) exhibits the fidelity
\[ F(t) = \langle L_0 |\psi(t)\rangle \tag{42} \]
to evaluate the closeness of evolved state \(|\psi(t)\rangle\) and the coalescing edge state \(|L_0\rangle\). In Fig. 4(a), we show the probability distribution of \(|\psi(0)\rangle\) in the real space, the fidelity is almost zero. Before \(t = 100\), the probability distribution of \(|\psi(t)\rangle\) presents as an unstable stripe, and the fidelity has a small value. At \(t = 200\), the stripe tends to be stable and looks like the edge state, the fidelity reaches to around 0.9. At \(t = 800\), we get a stable stripe, and the fidelity is close to 1 which indicates that \(|\psi(t)\rangle\) is the coalescing edge state. Our numerical results show that EP dynamics based scheme has better efficiency. The setting of system parameters is expected to provide guidance for the experiment.

\section{VI. SUMMARY}

In summary, we have developed a theory for a class of non-Hermitian Hamiltonian which supports robust EPs. Such Hamiltonian consists two separated parts, Hermitian and non-Hermitian ones. The Hermitian Hamiltonian has degenerate eigenstates, which coalesce into a single state by the non-Hermitian part. The most fascinating and important feature of such systems is that the EP is not sensitive to the strength of the non-Hermitian perturbation. As examples, we have investigate three types

\[ \text{FIG. 4. (a) Snapshots of the probability distribution at various time moments for initial excitation in Eq. (31). The probability is normalized at any moments. The system size is } M = 20, N = 100. \text{ Only the limited region within } N = 30 \text{ is shown because the intensity is almost vanishing outside this area. The other parameters are } \delta = 0.1, J = 1, \text{ and } \kappa = 0.5. \text{ (b) The fidelity between the evolved state } |\psi(t)\rangle \text{ and the coalescing edge state } |L_0\rangle. \text{ The units of time is } 1/J. \]

which drives the time evolution for an initial state \(|\psi(0)\rangle\)
\[ |\psi(t)\rangle = U(t) |\psi(0)\rangle . \tag{36} \]

(i) In the case of \(\kappa = 0\), we have
\[ |\psi(t)\rangle = \exp \left[ -i\varepsilon_0 \left( c_1^\dagger c_1 + c_2^\dagger c_2 \right) t \right] |\psi(0)\rangle . \tag{37} \]

Then for an initial state with fixed particle number, i.e, \(\left( c_1^\dagger c_1 + c_2^\dagger c_2 \right) |\psi(0)\rangle = n |\psi(0)\rangle \) with \(n = 0, 1, \text{ and } 2\), \(U(t)\) only contributes a phase factor to \(|\psi(0)\rangle\).

(ii) In the case of \(\kappa \neq 0\), we have
\[ \exp(-i\kappa c_1^\dagger c_2 t) = 1 - i\kappa c_1^\dagger c_2 t . \tag{38} \]

Then the time evolution of the coalescing state \( |\psi_c\rangle = c_1^\dagger |0\rangle \) is
\[ U(t) |\psi_c\rangle = \exp (-i\varepsilon_0 t) |\psi_c\rangle , \tag{39} \]
of systems: (i) uniform ring system with a single asymmetric hopping term, in which several pairs of degenerate states become coalescing states, (ii) uniform ladder system with long-range power-law decaying imaginary hopping terms, in which the degenerate spectrum becomes coalescing spectrum, (iii) SSH-like system in a nontrivial topological phase with a single asymmetric hopping term, in which the degenerate edge state becomes coalescing edge state. We also demonstrate the application of the EP dynamics based on numerical simulation. It is shown that the 2D coalescing edge state can be generated by a local initial state. Our findings offer a method for the efficient construction of a robust EP system and are expected to be necessary and insightful for quantum engineering.

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