ANALYSIS OF COMPLEXITY OF PRIMAL-DUAL INTERIOR-POINT ALGORITHMS BASED ON A NEW KERNEL FUNCTION FOR LINEAR OPTIMIZATION

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Abstract. Kernel functions play an important role in defining new search directions for primal-dual interior-point algorithm. In this paper, a new kernel function which its barrier term is integral type is proposed. We study the properties of the new kernel function, and give a primal-dual interior-point algorithm for solving linear optimization based on the new kernel function. Polynomial complexity of algorithm is analyzed. The iteration bounds both for large-update and for small-update methods are obtained, respectively. The iteration bound for small-update method is the best known complexity bound.

1. Introduction. Interior-point algorithm was presented by Karmarkar [6] in 1984 to solve linear optimization problems. After then, linear optimization problems became an active area of research. In 1986, Renegar [9] proposed trace the path interior-point algorithm. Then, Roos [10] proved that the essence of the Karmarkar's interior-point algorithm is a method of classical logarithmic barrier function. Nesterov [7] presented a primal-dual interior-point algorithm with polynomial time in 1994. In 2002, Peng [8] proposed a class of polynomial primal-dual interior-point algorithms, and then, several different primal-dual interior-point algorithms for linear optimization problems were proposed by Bai, et al, respectively. Readers may refer to [1–5]. In these primal-dual interior-point algorithms, Kernel functions play an important role in determining the search directions of primal-dual interior-point algorithms for solving linear optimization problems. In order to analyze the complexity bound of the algorithm, Bai, et al. introduced eligible kernel function which is required to satisfy several desirable properties , and gave a scheme that iteration bounds for both small- and large-update methods can be obtained for any eligible kernel function [1].

In [1], Bai, et al. introduced a new kernel function

$$
\psi(t) = \frac{t^2}{2} - 1 - \int_1^t e^{\xi-1}d\xi, t > 0,
$$

(1)
and they showed that the iteration bounds both for large-update and for small-update methods are $O(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon})$ and $O(\sqrt{n} \log \frac{n}{\epsilon})$, respectively. In [2], Bai, et al. introduced a parameter in the above kernel function, the iteration bound was improved by changing the parameter. We put forward a new kernel function inspired by the above kernel function

$$
\psi(t) = \frac{t^2 - 1}{2} - \int_1^t (2 - \xi)e^{\xi-1}d\xi, t > 0.
$$

Based on the new kernel function, a primal-dual interior-point algorithms for solving linear optimization problems is proposed. Then we analyze the iteration bounds both for large-update and for small-update methods by using the scheme proposed in [1]. The iteration bound for small-update method is the best known bound. Although the iteration bound for large-update method is worse than the best known bound, if we introduce a parameter in the new kernel function like literature [2], the best known bound can be obtained by choosing parameter. For convenience, we only study the new kernel function without parameter.

The remainder of the paper is organized as follows: in Section 2, we introduce the central path, the search directions, and generic primal-dual interior-point algorithm for linear optimization problems. In Section 3, we introduce kernel function and the properties of kernel function, and verify the new kernel function is eligible. We analyze iteration bound for the corresponding algorithm in Section 4. At last, a short conclusion is given in Section 5.

2. Primal-dual interior-point algorithm.

2.1. The central path. We consider the following pair of primal and dual linear optimization problems:

\[(LP) \quad \min \{c^T x : Ax = b, x \geq 0\}, \quad (3)\]

\[(LD) \quad \max \{b^T y : A^T y + s = c, s \geq 0\}, \quad (4)\]

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Finding an optimal solution of \((LP)\) and \((LD)\) is equivalent to solve the following system of optimality conditions:

\[
\begin{aligned}
Ax &= b, x \geq 0, \\
A^T y + s &= c, s \geq 0, \\
xs &= 0,
\end{aligned}
\]

where $xs$ denotes the componentwise product of the vectors $x$ and $s$. The third equation in (5) is so-called the complementarily condition for \((LP)\) and \((LD)\). In order to ensure that the equations (5) has a unique solution, we assume that the row rank of the constraint matrix $A$ is $m$, i.e. $\text{rank}(A) = m$, and both \((LP)\) and \((LD)\) satisfy the interior-point condition \((IPC)\), i.e. there exists a solution $(x^0, s^0, y^0)$ such that

\[
Ax^0 = b, x^0 \geq 0 \text{ and } A^T y^0 + s = c, s \geq 0.
\]

It is well known that the \((IPC)\) can be assumed without loss of generality. In fact, we may assume that $x^0 = s^0 = e$. The basic idea of primal-dual interior-point methods \((IPMS)\) is to replace the complementarily condition in (5) by the parameterized
equation $x \bar{s} = \mu e$, where $\mu$ is a positive number, and $e$ denotes the all-one vector. Therefore, we consider the following system:

$$
\begin{cases}
Ax = b, x \geq 0, \\
A^T y + s = c, s \geq 0, \\
xs = \mu e.
\end{cases}
$$

(7)

Under the assumptions of the IPC and $\text{rank}(A) = m$, the system (7) has a unique solution for any $\mu > 0$. It is denoted as $(x(\mu), y(\mu), s(\mu))$. We call $x(\mu)$ the $\mu$-center of $(LP)$ and $(y(\mu), s(\mu))$ the $\mu$-center of $(LD)$. When $\mu$ runs through all positive real numbers, $(x(\mu), y(\mu), s(\mu))$ gives a homotopy curve. It is called the central path of $(LP)$ and $(LD)$. If $\mu$ tends to zero, then the limit point of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for the $(LP)$ and the $(LD)$.

2.2. The search directions for $LP$. Let $x > 0$ and $(y, s)$ with $s > 0$ be strictly feasible for $(LP)$ and $(LD)$, respectively. For any $\mu > 0$, we define

$$
v = \sqrt{\frac{xs}{\mu}}. \tag{8}
$$

It is obvious that $v = e$ if and only if $x$ and $(y, s)$ are the $\mu$-center of $(LP)$ and $(LD)$, respectively. For any kernel function $\psi(t)$ the corresponding primal-dual barrier function $\Phi(x, s, \mu)$ is defined by

$$
\Phi(x, s, \mu) = \Psi(v) = \sum_{i=1}^{n} \psi(v_i). \tag{9}
$$

Note that $\psi(t)$ is a strictly convex differentiable nonnegative function and minimal at $t = 1$, with $\psi(1) = 0$. Thus, $\Phi(x, s, \mu)$ is nonnegative, and zero if and only if $v = e$. Therefore, the value of $\Phi(x, s, \mu)$ can be considered as a measure for the closeness of $x$ and $(y, s)$ to the $\mu$-center of $(LP)$ and $(LD)$.

Let

$$
V = \text{diag}(v), \ X = \text{diag}(x) \text{ and } \bar{A} = \frac{1}{\mu} AV^{-1} X.
$$

By solving the following equations, we can obtain $d_x$, $\Delta y$, and $d_s$

$$
\begin{cases}
\bar{A}d_x = 0, \\
\frac{1}{\mu} A^T \Delta y + d_s = 0, \\
d_x + d_s = -\nabla \psi(v).
\end{cases}
$$

(10)

We define

$$
\Delta x = \frac{xd_x}{v} \text{ and } \Delta s = \frac{sd_s}{v}.
$$

Then $\Delta x$, $\Delta y$, and $\Delta s$ are the search directions in the $x$-space, $y$-space, and $s$-space, respectively. By taking a step along these search directions, with the step size $\alpha \in (0, 1)$ defined by some line search rules, we can construct a new triple $(x_+, y_+, s_+)$ according to

$$
x_+ = x + \alpha \Delta x, y_+ = y + \alpha \Delta y, \text{ and } s_+ = s + \alpha \Delta s. \tag{11}
$$
2.3. **Generic primal-dual interior-point algorithm for LP.** We assume that a proximity parameter $\tau$ and a barrier update parameter $\theta$ are given with $\tau > 0$ and $0 < \theta < 1$. If $\Psi(v) \leq \tau$ then we start a new outer iteration by performing a $\mu$-update, otherwise we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of $\mu$ and apply (11) to get new iterates. The generic form of the algorithm is shown in Figure 1.

**Figure 1.** Generic primal-dual interior-point algorithm

3. **Kernel function and the properties of kernel function.** In this section, we first introduce the concepts of kernel function and barrier function. Then, we define a new kernel function and discuss its properties.

3.1. **Definitions of kernel function and barrier function.**

**Definition 3.1.** Let $\psi : R_{++} \to R_+$ be twice differentiable. The function $\psi$ is called a kernel function if it satisfies the following conditions:

\[
\begin{align*}
(i) & \quad \psi(t) = \psi'(t) = 0, \\
(ii) & \quad \lim_{t \to \infty} = \lim_{t \to 0} = \infty, \\
(iii) & \quad \psi''(t) > 0, t > 0,
\end{align*}
\]

where $R_{++}$ and $R_+$ are the set of positive real numbers and the set of nonnegative real numbers, respectively.
**Definition 3.2.** \( \Psi : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+ \) is called a barrier function if it is defined as follows:

\[
\Psi(v) = \sum_{i=1}^{n} \psi(v_i), v_i > 0.
\]

**Definition 3.3.** Let \( \psi : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+ \) be \( r \) times differentiable. The kernel function \( \psi(t) \) is called eligible if it satisfies the following conditions:

\[
t\psi''(t) + \psi'(t) > 0, t < 1, \quad (15)
\]

\[
t\psi''(t) - \psi'(t) > 0, t > 1, \quad (16)
\]

\[
\psi''(t) < 0, t > 0, \quad (17)
\]

\[
2\psi''(t) - \psi'(t)\psi'''(t) > 0, t < 1. \quad (18)
\]

For any eligible kernel function, iteration bounds for both small- and large-update methods can be obtained by using the scheme presented in [6]. The scheme is given in Figure 2.

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**Figure 2.** Scheme for analyzing the algorithm
3.2. Properties of the new kernel function. A new kernel function is given as follows:

\[
\psi(t) = \frac{t^2 - 1}{2} - \int_1^t (2 - \xi)e^{\frac{1}{\xi - 1}}\,d\xi, \quad t > 0.
\]  

(19)

The first, second and third-order derive of \(\psi(t)\) are

\[
\psi'(t) = t - (2 - t)e^{\frac{1}{t - 1}},
\]  

(20)

\[
\psi''(t) = 1 + e^{\frac{1}{t - 1}}\left(\frac{t^2 - t + 2}{t^2}\right),
\]  

(21)

\[
\psi'''(t) = -e^{\frac{1}{t - 1}}\left(2t^4 + 3t^3\right).
\]  

(22)

It is easy to verify that \(\psi(t)\) satisfies the definition of kernel function, that is, (12), (13) and (14) hold for the new kernel function.

Next, we will verify that (15), (16), (17) and (18) are satisfied for the new kernel function.

For \(t \in (0, 1)\), we have

\[
t\psi''(t) + \psi'(t) = 2t + \frac{2t^2 - 3t + 2}{t}e^{\frac{1}{t - 1}} = 2t + \frac{2(t - \frac{3}{4})^2 + \frac{7}{4}t}{t}e^{\frac{1}{t - 1}} > 0.
\]

This proves that condition (15) is satisfied. For all \(t > 0\), we can obtain

\[
t\psi''(t) - \psi'(t) = \left(\frac{1}{t} + 1\right)e^{\frac{1}{t - 1}} > 0.
\]

This shows that the condition (16) holds. By (22), it is easy to verify that the condition (17) is also satisfied. Finally, since

\[
2\psi''(t)^2 - \psi'(t)\psi'''(t) = 2 + e^{\frac{1}{t - 1}}\left(\frac{4t^2 - 4t + 8}{t^2} + \frac{2}{t^3} + \frac{3}{t^2}\right)
\]

\[
+ \left(e^{\frac{1}{t - 1}}\right)^2 \frac{2t^4 - 4t^3 + 13t^2 - 12t + 4}{t^4}
\]

\[
= 2 + e^{\frac{1}{t - 1}}\left(\frac{4(t - \frac{1}{2})^2}{t^2} + \frac{2}{t^3} + \frac{3}{t^2}\right)
\]

\[
+ \left(e^{\frac{1}{t - 1}}\right)^2 \frac{2t^2(t - 1)^2 + 11(t - \frac{6}{11})^2 + \frac{8}{11}}{t^4} > 0.
\]

\(\psi(t)\) also satisfies condition (18).

Thus we have shown that \(\psi(t)\) is eligible.

In [6] the authors gave the Lemma 3.4 as follows:

**Lemma 3.4.** Let \(\varrho : [0, \infty) \to [1, \infty)\) be the inverse function of \(\psi(t)\) for \(t > 1\). Then we have

\[
\sqrt{1 + 2s} \leq \varrho(s) \leq 1 + \sqrt{2s}.
\]  

(23)

**Lemma 3.5.** Let \(\rho : [0, \infty) \to (0, 1]\) be the inverse function of \(-\frac{1}{2}\psi'(t)\) to the interval \((0, 1]\). Then we have

\[
\rho(s) \geq \frac{1}{\log(1 + 2s) + 1}.
\]  

(24)
Proof. Let \( s = -\frac{1}{2} \psi'(t) \), then \( t(2 - t)e^{\frac{t}{1 - e^{t}}} = -2s, t \in (0, 1] \). Since \( t(2 - t)e^{\frac{t}{1 - e^{t}}} \leq 1 - e^{\frac{t}{1 - e^{t}}} \), we have \( 2s + 1 \geq e^{\frac{t}{1 - e^{t}}} \). Thus we have

\[
t = \rho(s) \geq \frac{1}{\log(1 + 2s) + 1}.
\]

\[
\square
\]

4. Analysis of the algorithm. In this section, we derive the iteration bounds both for large- and small-update methods of primal-dual interior-point algorithms based on a new kernel function for linear optimization according to the scheme in Figure 2.

At the start of inner iteration we have \( \Psi(v) \geq \tau \), we assume that \( \tau \geq 1 \), and that \( \tau \) is large enough to ensure that \( \delta(v) \geq 1 \), we assume throughout that the threshold parameter \( \tau \) in the algorithm satisfies \( \tau \geq 2 \).

Step 1. Solve the equation \(-\frac{1}{2} \psi'(t) = s\) to get \( \rho(s), t \in (0, 1] \). If the equation is hard to solve, derive a lower bound for \( \rho(s) \).

This lower bound for \( \rho(s) \) is given in (24).

Step 2. Letting \( f(\alpha) \) denote the decrement of \( \Psi(v) \) during an inner iteration, compute an upper bound for \( f(\tilde{\alpha}) \) in terms of \( \delta = \delta(v) \), where \( \tilde{\alpha} \) is the default step size. Using (21) and (24) and that \( \psi''(t) \) is monotonically decreasing, it follows that

\[
f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi'(\rho(2\delta))} \leq \frac{\delta^2}{1 + e^{\rho(2\delta) - 1} - \frac{1}{\rho(2\delta) + 1}} = \frac{\delta^2}{1 + 2(1 + 4\delta)(1 + \log(1 + 4\delta))^2}.
\]

Step 3. Solve the equation \( \psi(t) = s \) to get \( \varrho(s), t \geq 1 \). If the equation is hard to solve, derive lower and upper bounds for \( \varrho(s) \).

These bounds are given in (23).

Step 4. Derive a lower bound for \( \delta = \delta(v) \) in terms of \( \Psi(v) \) by using \( \delta(v) \geq \frac{1}{2} \psi'(\rho(\Psi(v))) \). Due to the expression for \( \psi'(t) \) in (20) and the lower bound for \( \rho(s) \) in (23), we may write

\[
\delta(v) \geq \frac{1}{2} \psi'(\rho(\Psi(v))) = \frac{1}{2} \left( \sqrt{1 + 2\Psi(v)} - \left( 2 - \sqrt{1 + 2\Psi(v)} \right) e^{\frac{1}{\Psi(v) + 1}} \right).
\]

Since \( \Psi \geq \tau \geq 2 \), one has \(-2 - \sqrt{1 + 2\Psi(v)} > 0\). Using \( e^x \geq 1 + x \), we have

\[
\delta(v) \geq \frac{1}{2} \left( \sqrt{1 + 2\Psi(v)} - \left( 2 - \sqrt{1 + 2\Psi(v)} \right) \frac{1}{\sqrt{1 + 2\Psi(v)}} \right)
\geq \frac{1}{2} \left( \sqrt{1 + 2\Psi(v)} - \frac{2}{\sqrt{1 + 2\Psi(v)}} + 1 \right)
\geq \frac{2\Psi(v)}{\sqrt{1 + 2\Psi(v)}} \geq \frac{2\Psi(v)}{\sqrt{3\Psi(v)}} \geq \frac{2}{\sqrt{3}} \Psi(v).
\]

By (26), one has

\[
\delta \geq 1 \text{ and } 2\delta \geq \frac{\sqrt{3}}{2} \delta \geq \sqrt{\Psi}.
\]

Step 5. Using the results of Steps 2, 3 and 4 find positive constants \( \kappa \) and \( \gamma \), with \( \gamma \in (0, 1] \), such that \( f(\tilde{\alpha}) \leq -\kappa \Psi^{1-\gamma} \).
Since $\Psi_0 \geq \Psi$, from (25) and (27), we can obtain

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{1 + 2(1 + 4\delta)(1 + \log(1 + 4\delta))^2}$$

$$\leq -\frac{\delta + 2(\delta + 4\delta)(1 + \log(1 + 4\delta))^2}{2(1 + \log(1 + 2\Psi))^2}$$

$$\leq -\frac{\sqrt{\Psi}}{2(1 + \log(1 + 2\Psi))^2}$$

Thus it follows that $f(\tilde{\alpha}) \leq -\kappa \Psi^{1-\gamma}$ by taking the following parameters:

$$\kappa = \frac{1}{22(1 + \log(1 + 2\Psi_0))^2} \quad \text{and} \quad \gamma = \frac{1}{2}$$

Step 6. Calculate the uniform upper bound $\Psi_0$ for $\Psi(v)$ from

$$\Psi_0 \leq L_\psi(n, \theta, \tau) = n\psi\left(\frac{\phi(\frac{n}{n})}{\sqrt{1 - \theta}}\right).$$

By (23), we have

$$n\psi\left(\frac{\phi(\frac{n}{n})}{\sqrt{1 - \theta}}\right) \leq n\psi\left(1 + \sqrt{\frac{2\pi}{n}}\right).$$

Since $\psi(t) \leq \frac{t^2 - 1}{2}$ for $t > 1$, one has

$$\Psi_0 \leq n\left(\frac{1 + \sqrt{\frac{2\pi}{n}}}{\sqrt{1 - \theta}} - 1\right) = \frac{2\tau + 2\sqrt{2n\tau} + n\theta}{2(1 - \theta)}.$$ 

Step 7. Derive an upper bound for the total number of iterations from

$$\frac{\Psi_0^n}{\theta\kappa^\gamma} \log \frac{n}{\epsilon}.$$ 

From (29) and (31), we can obtain the iteration bound as follows:

$$\frac{44(1 + \log(1 + 2\sqrt{\Psi_0}))^2}{\theta} \sqrt{\frac{2\tau + 2\sqrt{2n\tau} + n\theta}{2(1 - \theta)}} \log \frac{\epsilon}{n}.$$ 

Step 8. In order to get an iteration bound for large-update method, let $\tau = O(n)$ and $\theta = \Theta(1)$, then we have

$$\psi_0 \leq \frac{2\tau + 2\sqrt{2n\tau} + n\theta}{2(1 - \theta)} = O(n),$$

and

$$\log(1 + \sqrt{\psi_0}) = O(\log n).$$ 

Next, we apply to $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$ to calculate an iteration bound for small-update method.

By using Taylor’s theorem and (12), we have
\[ \psi(t) < \frac{\psi''(1)}{2}(t - 1)^2, t > 1. \] 

From (35) and (30), we obtain

\[ \Psi_0 \leq h\psi \left( \frac{1 + \sqrt{2\tau}}{\sqrt{1 - \theta}} \right) < \frac{1}{2} n\psi''(1) \left( \frac{1 + \sqrt{2\tau}}{\sqrt{1 - \theta}} - 1 \right)^2. \]

Since \( \psi''(1) = 3 \) and \( \sqrt{1 - \theta} \leq \theta \), one has

\[ \Psi_0 \leq \frac{3}{2} n \left( \frac{\theta + \sqrt{2\tau}}{\sqrt{1 - \theta}} \right)^2 = \frac{3}{2(1 - \theta)} (\sqrt{2\tau} + \theta \sqrt{n})^2. \] 

Using this upper bound for \( \Psi_0 \) in Step 7, we can get the following iteration bound:

\[ 44 \left( 1 + \log(1 + 2\sqrt{\Psi_0}) \right)^2 \sqrt{\frac{3}{2(1 - \theta)} (\sqrt{2\tau} + \theta \sqrt{n})} \frac{1}{\theta} \log \frac{\epsilon}{n}. \]

Since \( \tau = O(1) \) and \( \theta = \Theta(\frac{1}{\sqrt{n}}) \), we get that \( \Psi_0 = O(1) \) and the iteration bound for small-update methods is

\[ O\left( \sqrt{n \log \frac{\epsilon}{n}} \right). \]

It is the best known bound for small-update methods.

5. Conclusion. This paper presents a new kernel function which satisfies the definition of eligible kernel function. For this paper, it is a major contribution. Based on the new kernel function, we present a primal-dual interior-point method for linear optimization problems. According to the scheme introduced in [1], we derive the iteration bounds both for large- and small-update methods of the proposed method. The iteration bound for small-update method is the best known bound, but the iteration bound for large-update method does not meet the best known bound. If we introduce a parameter in the new kernel function, the best known iteration bound for large-update method can be obtain by choosing the appropriate parameter value. It will be the subject of future research.

REFERENCES

[1] Y. Q. Bai, M. El Ghami and C. Roos, A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization, SIAM Journal on Optimization, 15 (2004), 101–128.
[2] Y. Q. Bai, J. Guo and C. Roos, A new kernel function yielding the best known iteration bounds for primal-dual interior-point algorithms, Acta Mathematica Sinica English Series, 25 (2009), 2169–2178.
[3] Y. Q. Bai and C. Roos, A polynomial-time algorithm for linear optimization based on a new simple kernel function, Optimization Methods and Software, 18 (2003), 631–646.
[4] Y. Q. Bai, M. El Ghami and C. Roos, A new efficient large-update primal-dual interior-point method based on a finite barrier, SIAM Journal on Optimization, 13 (2002), 766–782.
[5] Y. Q. Bai, C. Roos and M. El Ghami, A primal-dual interior-point method for linear optimization based on a new proximity function, Optimization Methods and Software, 17 (2002), 985–1008.
[6] N. Karmarkar, A new polynomial-time algorithm for linear programming, Combinatorica, 4 (1984), 373–395.
[7] Y. Nesterov and A. Nemirovskii, *Interior-point polynomial methods in convex programming*, SIAM, Philadelphia, 1994.
[8] J. M. Peng, C. Roos and T. Terlaky, A new class of polynomial primal-dual methods for linear and semidefinite programming, *European Journal of Operational Research*, **143** (2002), 234–256.
[9] J. Renegar, A polynomial time algorithm based on Newton’s method for linear programming, *Mathematical Programming*, **40** (1988), 59–94.
[10] C. Roos and J. P. Vial, A polynomial method of approximate centers for linear programming, *Mathematical Programming*, **54** (1992), 295–305.

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