Dependence of the Brillouin precursor form on the initial signal rise time

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Abstract

Propagation of a Brillouin precursor in a Lorentz dispersive medium is considered. The precursor is excited by a sine modulated initial signal, with its envelope described by a hyperbolic tangent function. The purpose of the paper is to show how the rate of growth of the initial signal affects the form of the Brillouin precursor. Uniform asymptotic approach, pertinent to coalescing saddle points, is applied in the analysis. The results are illustrated with numerical examples.

Key words: Lorentz medium, dispersive propagation, Brillouin precursor, uniform asymptotic expansions

1 Introduction

It is well known that if very fast, rapidly oscillating signals propagate in a real medium, they undergo the dispersion phenomenon. Various frequency components of a signal propagate with different phase velocities, and they are differently dumped. As a result, the shape of the signal is distorted during propagation the signal in the medium. Naturally, this phenomenon is practically important only at very short times and very high frequencies (of the order of $10^{12}$ Hz and above in the assumed model). In now classical works Sommerfeld [1] and Brillouin [2, 3] have shown that in the Lorentz model of a dispersive medium, apart of the main signal two small precursors are formed. In the asymptotic description of the total field these precursors are interpreted as contributions to the field resulting from two different pairs of saddle points. For the Sommerfeld precursor pertinent simple saddle points vary outside some disc in a complex frequency plane. As the space-time
coordinate $\theta$, to be defined later, takes the initial value equal unity, those points merge at infinity to form one saddle point of infinite order. As $\theta$ grows up to infinity, they separate into two simple saddle points that move symmetrically with respect to the imaginary axis towards corresponding branch points located in the left and the right half-plane, respectively. In the case of Brillouin precursor, two other simple saddle points vary inside a smaller disc. As the coordinate $\theta$ grows from unity, they move toward each other along the imaginary axis, coalesce into one saddle point of the second order on the axis, and then again split into simple saddle points that depart from the axis and move, symmetrically with respect to this axis, towards corresponding branch points in the left and the right half-plane, respectively. The location of the saddle points affects local oscillations and dumping of the precursor. It depends on the space-time coordinate $\theta$ and is governed by the saddle point equation.

In this paper we confine our attention to the Brillouin precursor, also called a second precursor (as opposed to the first, Sommerfeld precursor). Fundamental work on this precursor is due to Brillouin [2, 3]. Because of limitations of asymptotic methods then available (now referred to as non-uniform methods), Brillouin could not correctly describe the precursor’s dynamics for values of $\theta$ corresponding to the coalescence of simple saddle points into one saddle point of a higher order. With the development of advanced, uniform asymptotic techniques, complete description of the precursor now got feasible (Kelbert and Sazonov [4], and Oughstun and Sherman [5]). In the latter monograph, in addition to the delta function pulse, the unit step-function modulated signal and the rectangular modulated signal, the authors also studied an initial signal with finite rate of growth. In their model, however, the envelope of the initial signal is described by everywhere smooth function of time, tending to zero as time goes to minus infinity ([5], Sec. 4.3.4). In the present paper we consider more realistic excitation which is caused by an abruptly switched modulated sine signal, vanishing identically for time $t < 0$ and being non-zero for $t > 0$. At $t = 0$ the derivative of the signal’s envelope suffers a step discontinuity. As $t$ increases, the envelope grows with a finite speed, asymptotically tending to its maximum value. In the following sections we construct uniform asymptotic representation for the Brillouin precursor resulting from this sort of excitation, and show how the speed of growth in the initial signal affects the form of the precursor. We also illustrate the results with numerical examples.
2 Formulation of the problem

We consider a one dimensional electromagnetic problem of propagation in a Lorentz medium. The medium is characterized by the frequency-dependent complex index of refraction

\[ n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right)^{1/2}, \] (1)

where \( b \) is so called plasma frequency of the medium, \( \delta \) is a damping constant and \( \omega_0 \) is a characteristic frequency.

Any electromagnetic field in the medium satisfies the Maxwell equations

\[
\nabla \times E(\mathbf{r}, t) - \frac{1}{c} \frac{\partial H(\mathbf{r}, t)}{\partial t} = 0, \\
\n\nabla \times H(\mathbf{r}, t) - \frac{1}{c} \frac{\partial E(\mathbf{r}, t)}{\partial t} = 0, \\
\nD(\mathbf{r}, t) = \int_{-\infty}^{t} \tilde{\epsilon}(t-\tau) E(\mathbf{r}, \tau) d\tau, \\
B(\mathbf{r}, t) = \mu H(\mathbf{r}, t),
\]

where \( \tilde{\epsilon}(t) \) is a real function and \( \mu \) is a real constant (hereafter assumed to be equal 1). By Fourier transforming the equations with respect to \( t \) and assuming that the fields depend on one spatial coordinate \( z \) only, we obtain the following equations for transforms of the respected fields

\[
\hat{\mathbf{z}} \times \mathbf{H}(z, \omega) = -\frac{i\omega}{c} \epsilon(z, \omega), \\
\hat{\mathbf{z}} \times \mathbf{E}(z, \omega) = \frac{i\omega}{c} \mu \mathcal{H}(z, \omega),
\]

where \( \hat{\mathbf{z}} \) is the unit vector directed along \( z \)-axis and \( \epsilon(z, \omega) = n^2(\omega)/(c^2\mu) \) is the Fourier transform of \( \tilde{\epsilon}(t) \). It then follows that \( \hat{\mathbf{z}}, \mathcal{E} \) and \( \mathcal{H} \) are mutually perpendicular. Moreover, if \( \mathcal{E} \) is known then \( \mathcal{H} \) is also known, and vice versa. It is also true for the electromagnetic field components, which are the inverse Fourier transforms of \( \mathcal{E} \) and \( \mathcal{H} \). Therefore, the knowledge of the electric (magnetic) field is sufficient to determine the full electromagnetic field. To make the calculations as simple as possible, it is advisable that the \( x \) (or \( y \)) axis be directed to coincide with the electric or magnetic field.

Assume that in the plane \( z = 0 \) an electromagnetic signal is turned on at the moment \( t = 0 \). For \( t > 0 \) it oscillates with a fixed frequency \( \omega_c \) and its envelope is described by a hyperbolic tangent function. Suppose the selected Cartesian component (say \( x \)-component) of one of these fields in the plane \( z = 0 \) is given by

\[
A(0, t) = \begin{cases} 
0 & t < 0 \\
\tanh \beta t \sin \omega_c t & t \geq 0,
\end{cases}
\] (2)

The parameter \( \beta \) determines how fast the envelope of the signal grows.
This initial electromagnetic disturbance excites a signal $A(z, t)$ outside the plane $z = 0$. In what follows we will be interested in the field propagating in the half-space $z > 0$. The problem under investigation can be classified as a mixed, initial-boundary value problem for the Maxwell equations.

The exact solution for this specific form of the initial signal $A(0, t)$ is described by the contour integral

$$A(z, t) = \int_{C} g(\omega) e^{i \phi(\omega, \theta)} d\omega,$$

defined in the complex frequency plane $\omega$. Here,

$$g(\omega) = \frac{1}{4\pi} \left[ i \beta B \left( \frac{-i(\omega - \omega_c)}{2\beta} \right) + \frac{1}{\omega - \omega_c} - \frac{i}{\beta} B \left( \frac{-i(\omega + \omega_c)}{2\beta} \right) - \frac{1}{\omega + \omega_c} \right],$$

the complex phase function $\phi(\omega, \theta)$ is given by

$$\phi(\omega, \theta) = i \frac{c}{z} [\tilde{k}(\omega)z - \omega t] = i\omega[n(\omega) - \theta],$$

and $B(s)$ is the beta function [7] defined via the psi function as

$$B(s) = \frac{1}{2} \left[ \psi \left( \frac{s + 1}{2} \right) - \psi \left( \frac{s}{2} \right) \right].$$

$B(s)$ is a Fourier transform of the initial signal envelope $\tanh \beta t$. The dimensionless parameter

$$\theta = \frac{ct}{z}$$

defines a space-time point $(z, t)$ in the field, and $c$ is the speed of light in vacuum. The contour $C$ is the line $\omega = \omega' + ia$, where $a$ is a constant greater than the abscissa of absolute convergence for the function in square brackets in (4) and $\omega'$ ranges from negative to positive infinity.

Our goal is twofold. First, we shall seek an asymptotic formula for the second (Brillouin) precursor that results from the excitation $A(0, t)$. In other words, we shall find near saddle points contribution to the uniform asymptotic expansion of the total field $A(z, t)$. Second, we shall examine how the speed parameter $\beta$ in (2) affects the form of the Brillouin precursor.

### 3 Asymptotic representation for the second precursor

Our derivation of the asymptotic formula for the Brillouin precursor is based on the technique developed by Chester et al. [8] for two simple saddle points
coalescing into one saddle point of the second order. The technique is also conveniently described in [9] and [10].

The locations in the complex $\omega$-plane of the saddle points in (3) are determined from the saddle point equation

$$n(\omega) + \omega n'(\omega) - \theta = 0.$$  \hspace{1cm} (8)

At these points the first derivative $\phi'_\omega(\omega, \theta)$ of the phase function vanishes. We are interested in the near saddle points, varying in the domain $|\omega| < \sqrt{\omega_0^2 + \delta^2}$. As $\theta$ increases from 1 to a value denoted by $\theta_1$, the near saddle points $\omega_1$ and $\omega_2$ approach each other along the imaginary axis from below and from above, respectively ([5]). They coalesce to form a second order saddle point at $\theta = \theta_1$. Finally, as $\theta$ tends to infinity they depart from the axis and symmetrically approach the points $\omega_{1,2} = \pm \omega_0 - i \delta$ in the right and in the left complex $\omega$ half plane, respectively. If $n(\omega)$ is eliminated from (8) then the equation can be represented in the form of an eighth degree polynomial in $\omega$ on its left hand side, and zero on its right hand side. It does not seem to be possible to solve the equation exactly. In what follows we shall employ the solution to (8) which was obtained numerically. Alternatively, a simple approximate solution found in [11] could be used here at the expense of accuracy in resulting numerical examples.

The first step in the procedure is to change the integration variable $t$ in (3) to a new variable $s$, so that the map $s(\omega)$ in some disk $D$ containing the saddle points $\omega_{\pm}$ (but not any other saddle points) is conformal and at the same time the exponent takes the simplest, polynomial form

$$\phi(\omega, \theta) = \rho + \gamma^2 s - \frac{s^3}{3} \equiv \tau(s, \theta).$$  \hspace{1cm} (9)

Notice that $\tau(s, \theta)$ has two simple saddle points $s = \pm \gamma$ that can coalesce into one saddle point $s = 0$ of the second order, corresponding to $\omega = \omega_s$. From

$$\dot{\omega}(s) = \frac{\gamma^2 - s^2}{\phi'_\omega(\omega, \theta)}$$  \hspace{1cm} (10)

we infer that for $s(\omega)$ to be conformal, $s = \gamma$ should correspond to $\omega = \omega_1$, and $s = -\gamma$ should correspond to $\omega = \omega_2$. Then,

$$\dot{\omega}(\pm \gamma) = \sqrt{\frac{-2\gamma}{\phi''(\omega_{1,2})}},$$  \hspace{1cm} (11)

where $\phi''(\omega_{1,2})$ is a short notation for $\phi''(\omega_{1,2}, \theta)$. In case the saddle points $\omega = \omega_{1,2}$ merge to form one saddle point of the second order $\omega = \omega_s$, one has
\( \phi''(\omega_s) = 0 \), and the relevant formula for \( \dot{\omega}_s \) is

\[
\dot{\omega}(0) = \left[ \frac{-2}{\phi''(\omega_s)} \right]^{1/3}.
\] (12)

By using correspondence \( \omega_{1,2} \leftrightarrow \pm \gamma \) in (9) one finds that \( \gamma^3 \) and \( \rho \) are equal to

\[
\frac{4\gamma^3}{3} = \phi(\omega_1) - \phi(\omega_2),
\] (13)

\[
\rho = \frac{1}{2} [\phi(\omega_1) + \phi(\omega_2)].
\] (14)

The equation (13) for \( \gamma \) has three complex roots. Only one root corresponds to a regular branch of the transformation (9) leading to the conformal map \( s(\omega) \). To find the proper value of \( \gamma \) we first note from (12) that \( \arg \dot{\omega}(0) \) can take one of the three values: \( \pi/6, 5\pi/6 \) or \( -\pi/2 \) corresponding to three different branches of the transformation (9). It can be readily verified that for \( \theta < \theta_s \) both \( \phi(\omega_1) \) and \( \phi(\omega_2) \) are real valued and \( \phi(\omega_1) > 0, \phi(\omega_2) < 0 \). Then it follows from (13) that \( \gamma^3 > 0 \). On the other hand, if \( \theta > \theta_s \) then \( \omega_1 = -\omega_2^* \). This implies that in the present case \( \phi(\omega_1) = [\phi(\omega_2)]^* \), and similarly \( \phi''(\omega_1) = [\phi''(\omega_2)]^* \). It is now seen that RHS of (9) equals \(-2i \text{ Im } \phi(\omega_2)\), where \( \text{Im } \phi(\omega_2) < 0 \). Hence for \( \theta > \theta_s \), \( \arg \gamma^3 = \pi/2 \). We now take advantage of the fact that \( \dot{\omega}(s) \) as given by (11) tends in the limit to (12) as \( s \to 0 \). Because \( \arg \phi(\omega_1) = 0 \) and \( \arg \phi(\omega_2) = \pi \) for \( \theta < \theta_s \), and \( \arg \phi(\omega_1) \to -\pi/2 \) and \( \arg \phi(\omega_2) \to \pi/2 \) as \( \theta \to \theta_s^+ \), we conclude that \( \arg \gamma = 0 \) for \( \theta < \theta_s \) and \(-\pi/2 \) for \( \theta > \theta_s \), i.e.

\[
\gamma = \left[ \frac{3}{4} [\phi(\omega_1) - \phi(\omega_2)] \right]^{1/3} e^{i\alpha},
\] (15)

where \( \alpha = 0, -\pi/2 \) if \( \theta < \theta_s \) or \( \theta > \theta_s \), respectively.

With the new variable of integration the integral (3) can be written down in the form

\[
A(z, t) = \int_{C_1 \cap \hat{D}} G(s, \theta) e^{\lambda \tau(s, \theta)} ds + \mathcal{E},
\] (16)

where \( \lambda = z/c \) and

\[
G(s, \theta) = g[\omega(s)] \dot{\omega}(s).
\] (17)

The contour \( C_1 \) is an infinite arc in the left \( s \) complex half-plane, symmetrical with respect to the real axis, running upwards and having rays determined by the angles \(-i2\pi/3 \) and \( i2\pi/3 \) as its asymptotes. The domain \( \hat{D} \) is the image of \( D \) under (9). The term \( \mathcal{E} \), standing for the integral of \( G(s, \theta) \exp \lambda \tau(s, \theta) \)
defined over the parts of $C_1$ outside $\hat{D}$, is exponentially smaller than $A(z, t)$ itself.

We now represent $G(s, \theta)$ in the canonical form

$$G(s, \theta) = c_0 + c_1 s + (s^2 - \gamma^2)H(s, \theta).$$

(18)

Provided the function $H(s, \theta)$ is regular, the last term in (18) vanishes at the saddle points $s = \pm \gamma$, and its contribution to the asymptotic expansion is smaller than that from the first two terms. Indeed, it can be shown that integration by parts of the last term leads to an integral of similar form as (11) multiplied by $\lambda^{-1}$.

To determine $c_0$ and $c_1$ we substitute $s = \pm \gamma$ in (18) and thus find

$$c_0 = \frac{G(\gamma, \theta) + G(-\gamma, \theta)}{2},$$

$$c_1 = \frac{G(\gamma, \theta) - G(-\gamma, \theta)}{2\gamma}.$$  

(19)  

(20)

By using (18) and (9) in (11), and extending the integration contour in the resulting integrals to $C_1$, we find that the leading term of the asymptotic expansion of $A(z, t)$ as $\lambda \to \infty$ is given by

$$A(z, t) \sim 2\pi i e^{\lambda \rho(\theta)} \left( \lambda^{-1/3} c_0(\theta) \text{Ai}[\lambda^{2/3}\gamma(\theta)^2] + \lambda^{-2/3} c_1(\theta) \text{Ai}'[\lambda^{2/3}\gamma(\theta)^2] \right).$$

(21)

It is defined through the Airy function and its derivative, as given by ([10])

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_{C_1} e^{sx-s^3/3} ds$$

$$\text{Ai}'(x) = \frac{1}{2\pi i} \int_{C_1} s e^{sx-s^3/3} ds.$$  

(22)

Plots of both functions for real $x$ are shown in Fig. 1.

The expansion holds for any $\gamma(\theta)$, including $\gamma = 0$. This special case corresponds to coalescing of the two simple saddle points $\tau = \pm \gamma$ into one saddle point of the second order. In other words the expansion is uniform in $\gamma$, and hence in $\theta$. It is seen that for $\gamma = 0$, i.e. for $\theta = \theta_s$, the algebraic order of $A(z, t)$ in $\lambda$ is $\lambda^{-1/3}$. This behavior is characteristic of an integral with a saddle point of the second order.

For $\gamma$ well separated from zero the Airy function and its derivative can be replaced by their asymptotic expansions ([12])

$$\text{Ai}(x) \sim e^{-2x^{3/2}/3} \left[ \frac{1}{2\sqrt{\pi} x^{1/4}} + O(x^{-3/4}) \right],$$

$$\text{Ai}'(x) \sim e^{-2x^{3/2}/3} \left[ -\frac{x^{1/4}}{2\sqrt{\pi}} + O(x^{-3/4}) \right].$$

(23)

(24)
as \( x \to \infty \), and

\[
\text{Ai}(x) \sim \frac{1}{\sqrt{\pi}(-x)^{1/4}} \left\{ \sin \left[ \frac{2}{3}(-x)^{3/2} + \frac{\pi}{4} \right] (1 + O \left[ (-x)^{-2} \right]) \right. \\
+ \left. O \left[ (-x)^{-3/2} \right] \right\},
\]

(25)

\[
\text{Ai}'(x) \sim -\frac{(-x)^{1/4}}{\sqrt{\pi}} \left\{ \cos \left[ \frac{2}{3}(-x)^{3/2} + \frac{\pi}{4} \right] (1 + O \left[ (-x)^{-2} \right]) \right. \\
+ \left. O \left[ (-x)^{-3/2} \right] \right\}
\]

as \( x \to -\infty \). By using these expansions in (21) we arrive at the following non-uniform asymptotic representation of the precursor

\[
A(z, t) \sim e^{\lambda \phi(\omega_2)} \left( \frac{-2\pi}{\lambda \phi''(\omega_2)} \right)^{1/2} g(\omega_2)
\]

(27)

if \( \theta < \theta_s \), and

\[
A(z, t) \sim e^{\lambda \phi(\omega_1)} \left( \frac{-2\pi}{\lambda \phi''(\omega_1)} \right)^{1/2} g(\omega_1) + e^{\lambda \phi(\omega_2)} \left( \frac{-2\pi}{\lambda \phi''(\omega_2)} \right)^{1/2} g(\omega_2)
\]

(28)

if \( \theta > \theta_s \).

We see from the above formulas that for \( \theta \) sufficiently distant from \( \theta_s \) (for Brillouin’s choice of medium parameters \( \theta_s \approx 1.5027 \)), the representation (21) reduces to a simple saddle point contribution from \( \omega_2 \) if \( \theta < \theta_s \), and to a sum of simple saddle point contributions from \( \omega_1 \) and \( \omega_2 \) if \( \theta > \theta_s \). In this
manner it is confirmed that the saddle point $\omega_1$ does not contribute when $\theta < \theta_s$. This is a direct consequence of the fact that the original contour of integration in (3) cannot be deformed to a descent path from imaginary $\omega = \omega_1$. The algebraic order of $A(z,t)$ in $\lambda$ is now $\lambda^{-1/2}$ because in this case separate simple saddle points contribute to the expansion. From (27) and (28) it is also seen that these formulas are non-applicable at $\theta = \theta_s$ (i.e. $\gamma = 0$), where $\phi''(\omega_{1,2}) = 0$. On the other hand the uniform expansion (21) remains valid for any $\theta$ (and $\gamma$). In particular it provides a smooth transition between the cases of small and large $|\gamma|$.

If $\theta > \theta_s$, then it can be readily seen that $g(\omega_1) = g^*(\omega_2)$, $\phi(\omega_1) = \phi^*(\omega_2)$, and similarly $\phi''(\omega_1) = \phi''^*(\omega_2)$. In this case (28) can be written down in a more compact form

$$ A(z, t) \sim 2\text{Re} \left[ e^{\lambda \phi(\omega_2)} \left( \frac{-2\pi}{\lambda \phi''(\omega_2)} \right)^{1/2} g(\omega_2) \right]. \quad (29) $$

Fig. 2 shows the dynamics of the Brillouin precursor in a Lorentz medium as given by its uniform representation (21) and non-uniform ones (27) and (28). Throughout this work the Brillouin’s choice of medium parameters

$$ b = \sqrt{20.0 \times 10^{16} \text{s}^{-1}}, \quad \omega_0 = 4.0 \times 10^{16} \text{s}^{-1}, \quad \delta = 0.28 \times 10^{16} \text{s}^{-1} \quad (30) $$

is assumed.

![Figure 2](image.png)

**Figure 2:** Uniform (solid line) and non-uniform (dashed line) representation of the Brillouin precursor in a Lorentz medium described by Brillouin’s parameters. Here, $\beta = 1.0 \times 10^{19} \text{s}^{-1}$, $\omega_c = 2.5 \times 10^{16} \text{s}^{-1}$ and $\lambda = 3.0 \times 10^{-15} \text{s}$.

For $\theta < \theta_s$ the function $\gamma(\theta)$ takes positive values and the precursor is described by a monotonically changing function. Adversely, for $\theta > \theta_s$ the
argument in both functions takes negative values which leads to oscillatory behavior of the precursor. This reflects the behavior of both Airy function and its derivative for positive and negative values of their argument (see Fig 1).

4 Dependence of the precursor on the rate parameter $\beta$

![Graph 1](image1)

![Graph 2](image2)

Figure 3: Dynamic behavior of the Brillouin precursor in the Lorentz medium described by Brillouin’s parameters obtained for: (top) $\beta = 2.0 \times 10^{14} s^{-1}$, and (bottom) $\beta = 2.0 \times 10^{17} s^{-1}$. Here, $\omega_c = 2.5 \times 10^{16} s^{-1}$ and $\lambda = 3.0 \times 10^{-15} s$.

An important question arises on how the rate parameter $\beta$ affects the form of the Brillouin precursor.
As the parameter $\beta$ in (21) increases starting from relatively small values, the shape of the precursor remains virtually unchanged while its magnitude grows. This tendency is no longer valid if $\beta$ enters a transitory interval. In that interval the shape of the precursor changes and its magnitude rapidly increases. Above transitory interval, further increase of $\beta$ leaves the shape and the magnitude of the precursor virtually constant. The form of Brillouin precursor for $\beta$ below $(2.0 \times 10^{15} s^{-1})$ and above $(2.0 \times 10^{17} s^{-1})$ the transitory interval is shown in Fig. 3.

Explanation of this behavior lies in the properties of the coefficients $c_0$ and $c_1$ in (21), which are $\beta$-dependent. The coefficients, in turn, determine the weight with which Airy function and its derivative contribute to the precursor.

First, consider the case of $\theta < \theta_s$. In Fig. 4 the coefficients of, respectively, $\text{Ai}$ and $\text{Ai}'$ in the parentheses in (21) multiplied by $2\pi i$ are plotted against $\beta$. The value of $\theta$ is chosen to be slightly below $\theta_s$. For relatively small $\beta$ the term proportional to $\text{Ai}'(\cdot)$ dominates over the term proportional to $\text{Ai}(\cdot)$. In this case the ratio of both terms remains unchanged in a wide interval of $\beta$ variation. The magnitude of the precursor increases with $\beta$ growth up to the moment where the contribution from $\text{Ai}(\cdot)$ changes sign and rapidly grows until finally it settles down at a virtually constant level. At the same time the contribution from $\text{Ai}'(\cdot)$ decreases to another constant level and is very small compared to the other term. At this stage the shape and the magnitude of the precursor are approximately determined by the special form of the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Plots of $2\pi i \lambda^{-1/3} c_0(\theta; \beta)$ and $2\pi i \lambda^{-2/3} c_1(\theta; \beta)$ against the speed parameter $\beta$ at $\theta = 1.502$. Here, $\omega_c = 2.5 \times 10^{16} s^{-1}$ and $\lambda = 3.0 \times 10^{-15} s$.}
\end{figure}
function

\[ g(\omega) = \frac{1}{4\pi} \left( \frac{1}{\omega + \omega_c} - \frac{1}{\omega - \omega_c} \right) \]  

(31)

which appears in \( c_0 \) and \( c_1 \), which is a limiting case of \( g(\omega) \) as \( \beta \to \infty \), i.e. for the initial signal with a unit step function envelope.

Now consider the case of \( \theta > \theta_s \) where the precursor becomes oscillatory. The envelope of the oscillations can be conveniently approximated with the help of (29) by

\[ \tilde{A}(z, t; \beta) \approx 2e^{\lambda \text{Re}[\phi(\omega_2)]} \left| \left( \frac{-2\pi}{\lambda \phi''(\omega_2)} \right)^{1/2} g(\omega_2; \beta) \right|, \]  

(32)

provided \(|\gamma(\theta)|\) is sufficiently large.

![Figure 5: Dependence of the magnitude of the Brillouin precursor’s envelope on \( \beta \). Calculated at different values of \( \theta \). Here, \( \omega_c = 2.5 \times 10^{16}\text{s}^{-1} \) and \( \lambda = 3.0 \times 10^{-15} \).](image)

In Fig. 5 the magnitude of the precursor envelope is plotted against the parameter \( \beta \) for different values of \( \theta \). It is seen again that after fast growth of the envelope magnitude at relatively small values of \( \beta \), which occurs with approximately the same rate for all \( \theta \), the magnitude reaches a saturation level at higher values of \( \beta \). Since the saturation appears earlier at larger values of \( \theta \), the precursor envelope has a tendency to become narrower with growing \( \beta \). Additionally, one observes that with growing \( \beta \) the first extremum moves towards larger values of \( \theta \). This is a direct consequence of the fact that the first extremum of the Airy function is shifted towards negative values of its argument as compared to the first extremum of the derivative of the Airy function. It has an additional effect on narrowing the precursor shape.
5 Conclusions

In this paper we have derived the uniform and non-uniform asymptotic representations for the Brillouin precursor in a Lorentz medium, excited by an incident signal of finite rise time, and well defined, startup time. With the use of these representations we analyzed the effect of the speed parameter \( \beta \) on the form and magnitude of the precursor. The results obtained can be helpful e.g. in applications involving triggering devices that work with signal amplitudes close to the noise level. In this paper we did not consider the problem of smooth transition from Brillouin precursor to the main signal.

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