Impartial Games: A Challenge for Reinforcement Learning

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Abstract

While AlphaZero-style reinforcement learning (RL) algorithms excel in various board games, in this paper we show that they face challenges in impartial games where players share pieces. We present a concrete example of a game - namely the children’s game of Nim - and other impartial games that seem to be a stumbling block for AlphaZero-style and similar self-play reinforcement learning algorithms.

Our work is built on the challenges posed by the intricacies of data distribution on the ability of neural networks to learn parity functions, exacerbated by the noisy labels issue. Our findings are consistent with recent studies showing that AlphaZero-style algorithms are vulnerable to adversarial attacks and adversarial perturbations, showing the difficulty of learning to master the games in all legal states.

We show that Nim can be learned on small boards, but the learning progress of AlphaZero-style algorithms dramatically slows down when the board size increases. Intuitively, the difference between impartial games like Nim and partisan games like Chess and Go can be explained by the fact that if a small part of the board is covered for impartial games it is typically not possible to predict whether the position is won or lost as there is often zero correlation between the visible part of a partly blanked-out position and its correct evaluation. This situation starkly contrasts partisan games where a partly blanked-out board position typically provides abundant or at least non-trifle information about the value of the fully uncovered position.

1 Introduction

The AlphaZero algorithm and its variations have demonstrated exceptional capabilities by generating remarkably high-quality play in complex strategy games including chess, Go, and shogi against the best human players and the best available computer software on these games Silver et al. (2017, 2018); Schrittwieser et al. (2020); Sadler and Regan (2019). AlphaZero’s self-play algorithm was trained on powerful computer hardware and achieved superhuman performance in less than 24 hours. However, despite the groundbreaking advances of AlphaZero-style algorithms in complex games, it has certain limitations. For instance, KataGo, a superhuman AlphaZero-style Go-playing agent, is vulnerable to adversarial attacks Wang et al. (2023) and adversarial perturbations Lan et al. (2022), and is blind to some states in the sense that the best move on them is completely neglected Wu (2019), suggesting that they did not learn to master the games on all legal states and can be deceived into making fatal mistakes.

Many games require implicitly learning a parity function to identify the winning move. This applies to some positions in Go and chess. The parity function is especially prominent in determining the winning strategy in the Nim game. The Sprague–Grundy theorem states that every impartial game is equivalent to a nim-heap Berlekamp et al. (2001). Thus, Nim plays a central role in the theory of impartial games.

It is well-known that learning the parity function from a uniform distribution is a challenge for neural networks Abbe et al. (2023); Shalev-Shwartz et al. (2017); Daniely and Malach (2020); Cornacchia and Mossel (2023); Raz (2018). However, the game positions encountered by a self-play RL agent do not present a uniform distribution. In the optimal scenario, they could establish a learning environment akin to a curriculum learning setting where at the beginning of the self-play, the NNs are
presented with easy positions and as the training progresses complex positions arise later when the agent explores the game more. But Zhou and Riis (2023) has empirically shown, through a supervised learning setting simulating the noisy label effect in self-play reinforcement learning, that the noisy label problem in the self-play RL algorithms is another adverse factor impeding the ability of NNs to model a parity function to the extent where when the number of wrong labels accounts for more than 5 percent on bitstrings of length 100, their model cannot learn a parity function better than random guessing.

Along with theoretical considerations related to the statistical neutrality of the parity function Thornton (1996), we conjectured that AlphaZero-style algorithm without any addition of special customised tricks and features in practice would not be able to master Nim and consequently numerous other impartial games on sufficiently large boards. Abstracting away the limitations of the algorithms, much of the challenge stems from the complexities and intricacies of modelling parity functions with neural networks.

To gain deeper insights into the challenges faced by RL agents in mastering Nim-like games, we revisited AlphaZero-style algorithms and their variations, particularly within the context of chess. Our focus was on chess positions where the resolution of straightforward parity-related problems was crucial for accurately identifying optimal moves. This intensive exploration led to an in-depth discussion and analysis of numerous theoretical and practical issues associated with impartial games and parity problems, culminating in the development and in-depth analysis of an AlphaZero-style algorithm tailored for Nim from the champion and expert perspective, two measures of the performance of a self-play RL agent we propose in this paper.

Our results show that AlphaZero-style algorithms appear to encounter difficulty in effectively learning to play Nim, particularly when the board size is sufficiently large. Although from both human and computational complexity perspectives, Nim is a game of low complexity, possessing a simple and well-defined optimal strategy. On specific boards or towards the end of the game, the algorithm has enough resources to play well by essentially using an exhaustive search. However, there were some larger nontrivial board positions where the policy network failed to provide any valuable information to guide the Monte Carlo Three Search (MCTS) algorithm, and the value network failed to evaluate the board positions any better than random guessing. These positions were referred to as blind spots of the RL agents where the best action is completely missed Lan et al. (2022); Wu (2019).

In games like chess, AlphaZero’s Policy-Value neural networks (PV-NN) are not perfect, which is typically not a serious issue as it is compensated by the MCTS guided by them. It turns out that imperfection is not necessarily a problem on some Nim boards since the policy network can help the winning player force the game into positions that do not require accurate evaluations. However, we speculate that there is no such advantage on sufficiently large Nim boards as the inability of PV-NN to model a parity function leads to two difficulties:

1. The policy network is unable to identify high-quality moves better than random guessing
2. The value network is unable to evaluate a position better than random guessing

According to (1), the policy network essentially cannot learn to select relevant candidate moves necessary to identify winning or high-quality moves. According to (2), even if the search can look many moves ahead, any evaluation is not reliable as it, in general, does not provide helpful information.

This can also be intuitively explained. Suppose a small part of a position in an impartial game is blanked out. In that case, the prediction from PV-NN can typically not perform better than pure random guessing. As a comparison, the general predictions of PV-NN might be wrong if we cover part of a Go, chess or shogi board. However, in general, the visible part of their board contains information positively correlated with the correct evaluation of the full board. Whereas, for impartial games like Nim, any board covering where a small but unknown number of counters is covered makes it impossible to evaluate the resulting position correctly. It is impossible to predict whether the position is won or lost, as there is zero correlation between the visible part of a partly blanked-out position and its correct evaluation. This heuristic argument shows that a small noise level can potentially erase the correlations needed to bootstrap the positive feedback mechanism of an AlphaZero-style algorithm.

We have distilled our primary contributions as follows:

- We identified a distinct category of strategic games that require nontrivial modifications to existing AlphaZero-style algorithms or current RL methodologies.
• Our research indicates that the challenges RL algorithms face with impartial games are even more severe than those suggested by mere parity-related issues. These complexities compound, significantly reducing the self-play RL algorithm’s effectiveness, and restricting its proficiency in Nim to handling fewer than 10 heaps, a stark contrast to the anticipated capability with 50+ heaps.
• Given two levels of solving a two-player game namely weak solved and strong solved, we introduce two definitive levels of mastery, champion and expert for evaluating an RL agent’s performance. These are essential metrics for a nuanced assessment of a self-play RL agent’s capabilities.
• We implemented an AlphaZero-style algorithm for the Nim game and examined its performance from the champion and expert perspective. Our results suggest that mere adjustments of the hyperparameters are predominantly ineffective in mitigating the learning challenges.
• We sketched potential directions for expanding the AlphaZero paradigm, establishing a foundation for the creation of innovative algorithmic approaches.

The paper is structured as follows. Section 2 introduces the Nim game and shows that any impartial games can be boiled down to the Nim. In Section 3, we revisit AlphaZero and Leela Chess Zero (LCZero) and look behind some of the impressive moves these chess programs have produced. The main point of this analysis is to identify the strengths and weaknesses of LCZero. LCZero’s defects are often hidden and mainly appear behind the scenes and emerge when we look under the hood. However, we will show that one of the weaknesses is parity-related issues impacting PV-NN. In Section 4, we introduce two levels of mastery of a self-play RL agent and illustrate the distinction with one analogy and one special Nim example we examined in our experiments. Section 5 presents an overview of the AlphaZero-style algorithms and our re-implementation. We analyse the performance of the trained algorithm from the champion and expert perspective respectively, and then examine the moves made on some Nim board positions, demonstrating the challenges the AlphaZero-style algorithm faces in becoming an expert agent on large boards. We conclude the paper with general remarks, conjectures and directions for further research in Section 6.

2 Impartial games and Nim

The class of impartial games is an important subclass of combinatorial games Berlekamp et al. (2001, 2002, 2003, 2004). Impartial games include Take-and-break, Subtraction, Heap, and Poset games. It also includes some mathematical games, including Nim, Sprout, Treblecross, Cutcake, Guiles, Wyt queens, Kayles, Grundy’s game, Quarto, Cram, Chomp, Subtract a square, and Notakto Berlekamp et al. (2001, 2002). Many impartial games have multiple variants.

An impartial game is a two-player game in which players take turns to make moves, and the actions available from a given position do not rely on whose turn it is. In other words, the legal moves in impartial games are identical for both players. A player loses if they cannot make a move on their turn (i.e. a player wins if they move to a position from which no action is possible). In impartial games, any position can be classified into either a losing or winning position, where the player to move has no winning move or has at least one winning move, respectively.

Nim is exemplary of impartial games and is often classified as a mathematical game because it has a well-defined mathematical solution. Nim is played by two players who take turns removing counters from a given row of heaps Bouton (1901); Nowakowski (1998). A player must remove at least one counter on each turn and may remove any number of counters, provided they are all on the same heap. The game’s goal is to be the player who removes the last counter i.e. leaves an empty board to the opponent. The initial board of Nim can be represented as an array of numbers:

$$[n_1, n_2, ..., n_k]$$

where $n_k \in \{1, 3, \ldots, 2k - 1\}$. The maximum number of legal moves in Nim on the initial board $[n_1, n_2, ..., n_k]$ is $\sum_j n_j$ as each player needs to remove at least one counter for each move. A position in a game on that board can be represented as an array

$$[v_1, v_2, ..., v_k]$$

1The code for the experiments in this paper is publicly available at: https://github.com/sagebei/Impartial-Games-a-Challlenge-for-Reinforcement-Learning
where $v_j \leq n_j$. A Nim position is commonly specified without reference to a board; however, we always specify the initial position of the board since the algorithm requires a fixed board size, and each self-play game starts with the initial position of the board. Fig. 1 demonstrates an example of an initial Nim board, an intermediate board position during play and the final board position leading to the end of the game.

Figure 1: The initial board consists of a series of heaps (aka rows or piles) of counters (aka lines or matches). The left graph shows that the initial board is $[n_1, n_2, ..., n_k] = [1, 3, 5, 7, 9]$. The two players take turns removing counters, resulting in one of the positions in the gameplay that is $[v_1, v_2, ..., v_k] = [1, 2, 4, 4, 3]$, as shown in the middle graph. In the usual game version, the player who removes the last counter(s) wins, as shown in the right graph where all the counters are cleared.

For any Nim position, it is mathematically easy to determine which player will win and which winning moves are available to a given player, if any. The value (won or lost) of a Nim position can be determined by calculating the binary digital sum of the number of counters in the heaps, i.e., the sum in binary, neglecting all carry from one digit to another Bouton (1901). Within combinatorial game theory, this is commonly referred to as the nim-sum. The complexity of this calculation was shown to be linear time Fraenkel (2004) and with the use of logarithmic space memory Calabro (2006). Thus, deciding whether a given Nim position is won or lost is computationally easy.

Sprague-Grundy theorem states that every finite impartial game is equivalent to a one-heap game of nim. Ascribing to this, our results on the Nim game naturally extend to our impartial games. Each board position in impartial games has a nimber value called the Sprague-Grundy value. For every position, the nimber $G(s)$ is defined as

$$G(s) = \text{mex}(\{G(s') : s' \in N(s)\})$$

where $s' \in N(s)$ denotes all the state $s'$ that can be reached by from $s$ in one legal play, and the output of a mex function is the minimum excluded value from a set, which is the least non-negative integer not in the set Beling and Rogalski (2020). A position with Sprague-Grundy value $G(s)$ is equivalent to Nim heap with $G(s)$ counters. A position is a losing position when its Sprague-Grundy value is 0.

The analysis of impartial games is closely linked to the nim-sum, which is linked to the parity function. Thus, the parity function plays implicitly or explicitly a central role in the theory of impartial games as such games often can mimic Nim or parts of Nim. To illustrate this, consider the impartial game called sprout Berlekamp et al. (2001); Gardner (1967) invented by John Conway and Michael Paterson.

Positions in Sprout typically have nimber values 0, 1, 2 and 3 Berlekamp et al. (2003). The position in Figure 2a has nimber value of 3. The Sprout position in Figure 2b also has nimber value of 3, but it has been modified to become an "isolated land" that cannot interact with anything on the outside. A Sprout starting position consisting of $n$ copies of the gadget in Figure 2b can mimic any Nim position reached from a starting position with $n$ heaps of 3 counters.
Figure 2: Nim played on a board $[3, 3, 3, \ldots, 3]$ with $n$ heaps, is equivalent to Sprout with $n$ copies of the Sprout position (gadget). (see the diagram on p599 in Berlekamp et al. (2003) for details).

Unlike Nim, some impartial games cannot be solved by a simple calculation. This follows from the fact that the complexity of some impartial games is PSPACE complete. From the perspective of algorithms, it turns out that many impartial games are as hard as Nim, but some impartial games like Node Kayles and Geography are much harder as they are PSPACE-complete Schaefer (1978). The subclass of combinatorial games of so-called partisan games can occasionally pose issues similar to that of impartial games e.g. Dawson’s chess that, in effect, is an impartial game in disguise Berlekamp et al. (2001).

So far, algorithms for impartial games have used handcrafted programs that use ideas akin to the alpha-beta search but are specially designed for the mex operation VIENNOT (2007). Beling and Rogalski (2020) proposed a novel method that prunes the search tree according to the node values calculated by the mex function on short impartial games, like Nim, Chomp, and Cram, but this approach does not, in general, scale to large board sizes. These conventional programs that utilise that mex operation and the Sprague-Grundy values could be used as benchmarks to test whether new RL-based algorithms can outperform conventional algorithms.

3 Revisiting AlphaZero and LCZero

In this section, we illustrate that a distinct set of board positions whose best move hinges on employing the parity function poses challenges for AlphaZero-style algorithms. We demonstrate that by showing LCZero struggles in some chess positions whose winning move is determined by the parity function.

AlphaZero has not been made generally available: however, some self-play RL agents in the open-sourced projects such as LCZero for chess, Leela Zero and KataGo Wu (2019) for Go, and AobaZero for shogi, have in essence replicated AlphaZero Cazenave et al. (2020). These projects outsourced the demanding computational task of training the game-playing agents to communities of computer strategy game enthusiasts. The participants would typically lend the project computer power directly by contributing hardware resources or running a script on the Google Colab cloud platform. Recently, a variant of AlphaZero has been proposed and published as open-source Wu (2019). The open-source Go project Elf is also worth mentioning Tian et al. (2019).

AlphaZero and LCZero often evaluate positions differently from human players or traditionally hand-coded chess engines. The revolutionary and outstanding evaluations and moves they made have taken the chess world by storm. This section focuses on a highly trained version of LCZero. In general, LCZero, with access to suitable computational resources that the programs need to run Monte Carlo Tree Search (MCTS) simulations, can replicate the moves made by AlphaZero chess. LCZero uses the neural network composed of 30 blocks × 384 filters which are beyond the initial $20 \times 256$ architecture of AlphaZero, and LCZero additionally employs the Squeeze-and-Excitation layers to the residual block and supports endgame tablebases Maharaj et al. (2021), enabling it to surpass the original AlphaZero algorithm possibly. The older versions of LCZero running on a $20 \times 256$ architecture are somewhat weaker than the later versions running on the $30 \times 384$ architecture.

We also gained access to open-source Stockfish 14, which, like Stockfish 8, was initially developed by Tord Romstad, Marco Costalba, and Joona Kiiski, and has been further extended and maintained by the Stockfish community. Unlike Stockfish 8, stockfish 14 also has an NNUE (Efficiently Updatable

\(^2\)http://www.yss-aya.com/aobazero/index_e.html

\(^3\)version: Last T60: 611246 (384 × 30)
Neural Network) Nasu (2018) for evaluating board positions. Thus, stockfish 14 is more powerful as it equips traditionally handcrafted chess engines with neural networks. Game playing strength is usually measured by Elo rating (see section 5.2.1 for the detailed description), but there has been an attempt to measure playing strength based on the quality of the played chess moves rather than on the results Regan and Haworth (2011). On the chess engine rating list rating (August 7, 2021) the latest iteration of Stockfish 14 has 3555, Stockfish 8 has 3375 and LCZero 3336. LCZero, running on hardware similar to AlphaZero against a special fixed time (1 minute) for each move version of Stockfish 8, has been shown to lead to a similar dramatic achievement as AlphaZero.

Although the source code of the original AlphaZero is not available, the detailed evaluation of its moves on some chess positions has been made public Sadler and Regan (2019). We took one of the board positions and used it to compare the evaluations made by AlphaZero and LCZero. This board position is demonstrated in Fig. 3a where the best move is $\Delta d5$. The policy network of a highly trained LCZero using the net T60: 611246 (384x30) predicts that $d5$ is the move to give the most attention, which can be seen from the policy network’s prior probabilities in Fig. 3b where $d5$ is the move with the highest probability (23.13%). AlphaZero had a different training history, so the prior probabilities differed, but the policy network of AlphaZero agreed that $d5$ is a promising move, although it did not consider $d5$ the best and instead preferred $\Delta d3$. But as shown in Fig. 3c, after investigating 64 and 256 nodes 4, AlphaZero is highly optimistic about the move $d5$. While LCZero is somewhat less confident, they both agree that $d5$ is the best move. Additionally, on this board position, the value networks of AlphaZero and LCZero both agree that the white player is in a better situation. Their confidence grows as more nodes are evaluated by the value network, as shown in Fig. 3d.

In Fig. 3d, with more MCTS simulations, AlphaZero discovered various defensive resources, so its win probability began to drop. The win probability for a position $s$ is calculated by $(0.5 + v/2)%$ where the $v$ is a scalar output from the value network with position $s$ as its input. When more nodes were examined, the win probabilities climbed up again. Eventually, after 4194304 nodes, the two engines had almost identical evaluations with win probabilities of 73.5% resp. 73.1%. In general, LCZero plays very similarly to AlphaZero, and it is fair to conclude that AlphaZero’s capability in Chess has been replicated by LCZero. The quality of evaluation of AlphaZero for chess is discussed in great detail in Sadler and Regan (2019). The moves selected by LCZero without search (i.e. the moves were selected greedily according to the evaluation from the PV-NN) play remarkably well and are occasionally able to win games against strong chess players.

When we compare a move made by a human with the one selected by LCZero, the human typically looks at far fewer positions, but human pattern recognition sometimes spots features that LCZero’s PV-NN misses. A human grandmaster maximally considers a few hundred moves, i.e. significantly fewer than the ones considered by LCZero, which considers dramatically fewer moves than conventional chess programs. As an example to illustrate human superiority and understand limitations that become relevant for judging impartial games, consider the position in Fig. 4. Any decent chess player can perceive intuitively - without any calculation - that white has a won position leading to mate (1. $\text{Qe}8+$, $\text{Ke}8$ and 2. $\text{Qf}7+$ mate). But the policy network first led the algorithm slightly astray by prioritising the move $\text{Qf}7$ and caused it to spend time investigating the positions arising after 1. $\text{Qf}7$. It is important to stress that LCZero found the winning sequence in less than a millisecond. The point is that neither the value nor the policy network, in general, evaluates all positions accurately. This fact is crucial for understanding the limitations of impartial games where even the slightest change or perturbation in position completely wipes out any positive correlation between the PV-NN evaluations and the correct ones. Lan et al. (2022) also found that the KataGo agent does not possess the general knowledge that applies to all the legal states and that it is not immune to adversarial perturbation where even though the perturbation added to a board position did not alter its best move and the win probability, KataGo can be misled to make bad moves.

The next example in Fig. 5 illustrates the difficulty for AlphaZero-style algorithms in handling parity-related problems. On the right part of the position, any player who takes a move is bound to lose the game. The situation on the left side of the board is equivalent to the Nimposition $[3, 2]$ consisting of two heaps with three and two counters, respectively. In this Nim position, only removing one or two counters from a heap is permitted. This version of Nim is sometimes referred to as bogus Nim. The winning move is removing one counter from the heap with three counters. Analogously, the winning move for the white is making the move $\Delta c3$-c4. But LCZero’s policy network suggests $\Delta a2$-a4,

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4 More specifically, the number of new nodes visited, which corresponds to the number of MCTS simulations
(a) A chess board position where the white player is about to make a move.

(b) Prior probabilities from the policy network of AlphaZero and LCZero for the top moves

| Move | ♗d3 | ♖f3 | ♖c6 | ♖d5 | ♖g2 | ♖f3 | ♖h1 | ♖c4 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
| Prior prob (AlphaZero) | 29.77% | 18.82% | 16.15% | 10.21% | 4.75% | 3.5% | 4.75% | 1.2% |
| Prior prob (LCZero) | 6.01% | 12.36% | 16.27% | 23.13% | 1.74% | 3.73% | 1.41% | 8.68% |

(c) The win probabilities from the value networks of AlphaZero and LCZero after visiting 64 and 256 nodes (that is equivalent to running 64 and 256 MCTS simulations.)

| Move | ♗d3 | ♖f3 | ♖c6 | ♖d5 | ♖g2 | ♖f3 | ♖h1 |
|------|-----|-----|-----|-----|-----|-----|-----|
| Win prob (AlphaZero 64 nodes) | 60.1% | 64.5% | 77.3% | 87.1% | 61.6% | 67.3% | 61.6% |
| Win prob (AlphaZero 256 nodes) | 60.1% | 64.5% | 77.7% | 83.1% | 61.6% | 67.3% | 61.6% |
| Win prob (LCZero 64 nodes) | 62.8% | 62.8% | 71.2% | 71.6% | 55.7% | 55.7% | 55.7% |
| Win prob (LCZero 256 nodes) | 59.0% | 62.2% | 63.3% | 67.8% | 50.0% | 58.2% | 50.0% |

(d) Comparison after running a given number of MCTS simulations. The move d5 was considered to be the best move in all cases.

![Chessboard diagram]

Figure 3: On this chess board position, LCZero and AlphaZero promptly agree that d5 is the best move although their evaluations from PV-NN differ slightly.

disastrously leading to a losing position. Like in the previous example, it is important to stress that LCZero finds the winning sequence almost immediately, and again, the point is that the PV-NN, in general, cannot evaluate positions accurately on certain board positions.
Figure 4: White has a forced win in two moves where he sacrifices the queen before mating with his knight. It is not surprising that LCZero’s evaluation with no search cannot conclude that white has a forced win. Only after 11 nodes did the program jump away from investigating lines beginning with $\text{Qf7}$, and switch to investigate the winning move $\text{Qg8}$. And only after visiting 15 nodes did the program find the forced mate.

Figure 5: A chess position that mimics a Nim position $[3, 2]$. Neither black or white would like to move on the right-hand side of the board. A simple analysis concludes that white has to play $\text{Da3c4}$, which is a winning move. Any other move leads to a loss. However, the LCZero’s policy network gave the move $\text{Da2-a4}$ a prior probability of 50.4% and suggested that the second most promising move is $\text{Da3-c4}$, which scores 20.7%. Notice that the value network already judged the position after $\text{Da4}$ as favourable for white (which we found quite impressive). Though it is a stochastic process, LCZero’s MCTS needed only to investigate a few nodes before it considered $\text{Da3-c4}$, which it then liked immediately.

Despite that LCZero is equipped with rather incredible PV-NN, in general, it cannot accurately evaluate positions related to parity issues, including "zugzwang", "waiting moves", and "triangle manoeuvres", etc. which are common themes in chess. This is not an issue for AlphaZero or LCZero, as the issues are effectively handled by the MCTS. In actual gameplay, the parity problem or other potential issues do not manifest because the MCTS can compensate.

However, some examples show that the policy network occasionally might be blind to crucial key moves that cannot be rediscovered by MCTS. For example, consider the board position in Fig. 6 that occurred in a game between Stockfish 12 and LCZero. The white player can force checkmate in 5 moves, starting with $\text{Ra2}$. However, the policy network gave the move of $\text{Ra5+}$ the highest prior probability. The serious problem is that LCZero’s policy network failed to guide the MCTS properly, and after more than 1 million nodes LCZero were unable to find the winning sequence. The failure to find the winning sequence is partly a drawback of using MCTS. In contrast, alpha-beta search, almost universally used in conventional chess engines, typically finds the forced mate instantly.
Figure 6: LCZero chess played Bf5 which is a blunder. LCZero failed to realize that white has a forced mate in 5 moves. In the diagram position, the highly trained LCZero failed to find (even after having looked at more than a million nodes) the forced mate: 1.\texttt{Rc2+}, \texttt{Kb5} 2.\texttt{Ra5+!!} 2.- \texttt{Ka5} 3. \texttt{Ra2+}, \texttt{Kb5} 4.\texttt{a4+}, and now 4.-\texttt{c5} 5. \texttt{c2+} mate, or 4.-\texttt{a5} or 4.-\texttt{a6} followed by 5.\texttt{c6+} mate.

4 Two levels of mastery

Assuming both players always make optimal moves, a two-player game is considered to be solved if its outcome can be predicted from any position Allis et al. (1994). A solution can be classified into one of the three levels. An ultra-weak solution concerns whether the first player gains an advantage from taking the move from the initial position Van Den Herik et al. (2002). As a self-play RL agent needs to make moves from the positions that can arise during gameplay, the ultra-weak solution is not relevant to our discussion. We consider the other two levels described in table 1.

Table 1: Two levels of solving a two-player game

| Level            | Description                                                                 |
|------------------|------------------------------------------------------------------------------|
| Weakly solved    | The algorithm has learned to make a sequence of good moves against any opponent in actual gameplay that lead to winning the game when played from the initial position. |
| Strongly solved  | The algorithm has learned to make optimal moves in all possible game positions arising from play from the initial position. |

Given these two levels on which a game can solved, we propose two different ways to assess the extent to which an RL algorithm has learned to master a game, namely champion and expert distinguished by their level of mastery of the game, as shown in table 2.

Table 2: Two measures of mastery

| Measure | Description                                                                 |
|---------|------------------------------------------------------------------------------|
| Champion| A player’s ability to get optimal results against any opponent when the game is started from the initial position. |
| Expert  | A player’s ability to play the optimal move in any position that can arise by legal play from the initial position. |

This distinction is vital. The question proposed in Lan et al. (2022) as to whether an RL agent has learned the general knowledge applicable to any legal game state concerns its ability to become an expert. The champion might have developed a skill to steer the game into its comfort zone where it masters it. The expert agent always makes the best move in any position. But in some games, becoming a champion without becoming an expert is impossible. It is outside the scope of this paper.
to systematically prove this claim. Intuitively, this can be explained by the inability of the winning side to control the game enough to steer the game into territory it is familiar with. A more rigorous and theoretical analysis of this issue is left open.

There is a special discipline called problem chess, where the task is to solve composed chess problems artificially created, rather than problems arising from actual competitive play. LCZero is not trained to solve artificial problems, and it is unsurprising that the program has difficulty dealing with them. AlphaZero’s successes in chess, shogi and Go lay on its ability to learn to play and win games. The task was not to solve problem positions, i.e. find good moves in artificial situations. Thus, AlphaZero-style algorithms’ mastery of a game is measured by a champion notion instead of applying an expert measure.

The champion and expert concepts will be discussed with experimental results in Section 5. We also use an analogy and a crafted Nim board below to expound the distinction between them.

Example 1: Imagine a fighting game where one of the agents can force the opponent to fight on either the savanna or in the jungle. A champion might be an expert in savanna fight, but pretty hopeless in the jungle (or visa versa). Regardless, the champion can win by compelling the combat into the savanna. To be an expert, the agent needs to master both savanna and jungle fights to win in any situation it could encounter.

Here is another simple but rather extreme example of the relevance of the two notions from the game of Nim.

Example 2: Let \( n \in \mathbb{N} \) and consider the Nim board \([2, 1, 1, \ldots, 1]\) with one heap with 2 counters, and \( n \) heaps with 1 counter where \( n \) is a large number, for instance, 100. It is likely that an agent can become a champion in this game after some training as it can rapidly learn that if there are two counters in the first heap, it has to remove either 1 or 2 counters from that pile. Thus with only relatively little self-play training, the agent becomes an optimal agent of type 1, i.e. a champion that essentially learns and memorises the initial two "opening" moves. But the task of selecting the best move in a general position \([2, v_2, v_3, \ldots, v_n]\) with \( v_j \in \{0, 1\}, j \in \{2, 3, \ldots, n\} \) is equivalent to evaluating the parity of \([v_2, v_3, \ldots, v_n]\). Thus due to the intricacies of neural networks to model the parity function, especially without any serious incentive to learn to do so, a champion agent for this Nim board is not expected to become an expert agent.

5 Reinforcement Learning for Nim

There exist some open-source implementations of AlphaZero-style algorithms on GitHub \(^5\) and some RL libraries also offer fast implementations \(^6\). But we needed some additional functionalities that are not provided in any of them, for instance, calculating the Elo rating of the agent being trained against its ancestors, evaluating the accuracy of output of the policy network as the training progresses against the winning move derived from the nim-sum, etc. This prompted us to implement a version of AlphaZero-style self-play algorithm in PyTorch \(^5\) for neural network training and Ray \(^6\) for running simulations in parallel. We made considerable efforts to ensure our implementation replicates the original AlphaZero algorithms at granular details as described in Silver et al. (2017, 2018). However, necessary changes to neural network architectures tailored to the Nim were made, which will be specified in the section 5.1.

We are aware that a re-implementation of a method that does not perform well on a new domain does not mean that the method is flawed or will not work on the new domain. An extremely in-depth ablation over many hyperparameters, including neural network architecture and representations, would need to be conducted. However, our aim is not to prove empirically that AlphaZero-style algorithms cannot master the game but to illustrate the challenges and difficulties it encounters. One of the chess positions we consider in Fig. 5 mirrors a rudimentary parity (nim) position. Our analysis revealed that even the highly trained LCZero, which for all practical purposes emulates AlphaZero, failed to assess the position accurately. We will show in this section that in Nim where the winning strategy closely links to the parity function, this issue is more pronounced.

\(^5\)GitHub. https://github.com/suragnair/alpha-zero-general
\(^6\)GitHub. https://github.com/junxiaosong/AlphaZero_Gomoku
5.1 Implementing an AlphaZero-style algorithm for Nim

The AlphaZero algorithm starts training from a clean slate with no specific game knowledge besides knowing the rules of the games Silver et al. (2018). Policy network, value network and MCTS are three pillars of the AlphaZero algorithm. The policy network outputs a probability distribution over all the actions \( \mathcal{A} \). \( P(s,a) \) stands for the prior probability associated with action \( a \in \mathcal{A} \) at state \( s \). The probabilities of the illegal actions are set to zero, and the remaining probabilities are re-normalized so that the summation of all the probabilities remains one. The policy network narrows down the search to the actions with a high probability of leading to a win position, reducing the search tree’s breath. The value network outputs a scalar value \( v \in [-1, 1] \), estimating the expected outcome from position \( s \) if following the actions suggested by the policy network. A higher value of \( v \) indicates the current player taking a move at position \( s \) has a higher chance of winning, and vice versa. The value of the leaf node is predicted by the value network, without which it can only be known at the end of the game, where it is 1 when the current play won or -1 when lost. An effective value network thus can reduce the search tree’s depth.

The target for training the value network comes from the actual outcome of the self-play. If the agent wins the game, all the positions it encountered are marked as winning states, and vice versa. Because of that, the labels for the board positions are likely to be wrong, especially at the early stage of training where the PV-NN are not well-trained. These noisy labels pose a challenge for NN to model a parity function on long bitstrings Zhou and Riis (2023). Thus this challenge extends to the PV-NN in evaluating the large Nim board positions.

LSTM is one of the neural networks that is capable of modelling a perfect parity function, so we used it to construct the PV-NN. The policy and value networks share the LSTM layer but use two separate heads: the policy head and the value head. We did the experiments on Nim of 5, 6 and 7 heaps and tweaked the network architectures and configuration to adapt to different board sizes. The shared layers of the policy and value network are one LSTM layer with a hidden size of 128 for all three Nim games. It is natural to increase the number of layers as the board size grows, however, we found that a larger model is detrimental to the performance and tends to destabilize the training process. The MCTS starts with the root node of the search tree corresponding to the current state the player is taking action on. Each node represents a state encountered in the game and each edge is an action. The tree is constructed in the course of running a predefined number of simulations, each of which starts with the root node and ends with a leaf node, following a sequence of actions selected using the Formula 2.

We ran a predefined simulation for each move, and during training, we collected 100 episodes of interaction data in the form of \((s, \pi, r)\) to train the policy network with cross-entropy loss and to train the value network with mean square error (MSE).

\[
a_t = \arg \max_a (Q(s,a) + U(s,a))
\]

(2)

where \( Q(s,a) \) is the averaged action value across simulations calculated by

\[
Q(s,a) = \frac{1}{N(s,a)} \sum_{s'|s,a \rightarrow s'} V(s')
\]

(3)

in which \( N(s,a) \) is a counter that records the number of times action \( a \) has been taken from state \( s \), \( s'|s,a \rightarrow s' \) denotes that action \( a \) is taken at state \( s \). This simulation terminates at state \( s' \), and \( V(s') \) represents the value of the end state of a simulation from the perspective of the current player, obtained from either the value network if \( s' \) is an intermediate state or the game outcome if it is a terminal state. The \( U(s,a) \) is calculated using

\[
U(s,a) = c_{put} P(s,a) \frac{\sum_b N(s,b)}{1 + N(s,a)}
\]

(4)

\footnote{See the python scripts in the reinforcement learning folder in our GitHub repository for the implementations of all the formula used in this section.}
where the $c_{\text{put}}$ is a constant controlling the level of exploration. AlphaGo Silver et al. (2016), AlphaGo Zero Silver et al. (2017) and AlphaZero Silver et al. (2018) all leave $c_{\text{put}}$ unspecified. We found that $c_{\text{put}}$ value significantly affects the performance of AlphaZero on Nim because setting it too low discourages the exploration, while setting it too high weights down the action value, impairing the effectiveness of the search depth. Tian et al. (2019) found setting $c_{\text{put}} = 1.5$ yields satisfactory results, but this value, along with other sensible ones like $c_{\text{put}} = \{1, 1.5, 2, 3\}$ works poorly for Nim. We thus adopted another formula Schrittwieser et al. (2020) to calculate the $U(s, a)$, as shown below.

$$U(s, a) = P(s, a) \cdot \frac{\sqrt{\sum_b N(s, b)}}{1 + N(s, a)} \left( c_1 + \log \left( \frac{\sum_b N(s, b) + c_2 + 1}{c_2} \right) \right)$$

(5)

where $c_1 = 0.25$ and $c_2 = 19652$. To further encourage the exploration, Dirichlet noise was added to the prior probability $P(s, a)$ of the root node where the search begins. The Dirichlet noise is indispensable as it ensures that the search tree is widely branched, thus avoiding always visiting the moves with high prior probability.

$$P(s, a) \gets (1 - \epsilon) \cdot P(s, a) + \epsilon \cdot \eta_a$$

(6)

where $\eta_a$ is sampled from the Dirichlet distribution $\text{Dir}(\alpha)$ in which $\alpha$ is set to 0.35. $\epsilon$ is constant set to 0.25 during training. But it is set to 0 during the evaluation to negate the effect of the Dirichlet noise. The values of $\alpha$ used for chess, shogi and Go are 0.3, 0.15, and 0.003, respectively. The alpha value should be in inverse proportion to the approximate number of legal moves at given positions, as the average number of legal moves in the Nim we run experiments on is less than that of chess, we opted for a higher $\alpha$ value 0.35. Although in theory $\alpha$ should be set to 0.5, in practice, setting $\alpha$ to 0.35 yields a better outcome. The left arrow denotes that the prior probability is reassigned to the value on the right.

$$U(s, a) \propto \frac{P(s, a)}{1 + N(s, a)}$$

(7)

As shown in the equation 7, the $U(s, a)$ is proportional to the prior probability $P(s, a)$. The visit count $N(s, a)$ in the denominator relatively enlarges the prior probability on nodes being visited with less frequency to boost exploration. At each state $s$, the action selection is jointly determined by the action value $Q(s, a)$, the visit count $N(s, a)$ and the prior probability $P(s, a)$ obtained from the policy network. The action with a lower visit count, higher prior probability and higher value has a better chance of being chosen. Thus, when the policy network fails to assign a higher probability to winning moves, the search is directed to the nodes with less chance of leading to a winning state, making the search less effective. The problem also exists for the value network that fails to estimate the correct moves, the search is directed to the nodes with less chance of leading to a winning state, making the chance of being chosen. Thus, when the policy network fails to assign a higher probability to winning network. The action with a lower visit count, higher prior probability and higher value has a better action value.

After the simulations are finished, the search returns a probability distribution over all the actions according to the formula 8, from which an action is sampled to take at state $s$.

$$\pi(a|s) = \frac{N(s, a)^{1/\tau}}{\sum_b N(s, b)^{1/\tau}}$$

(8)

where $\tau$ is the temperature that changes according to the number of moves made during gameplay, we call the $\pi(a|s)$ posterior probabilities related to the prior probabilities as they are derived from MCTS simulation partially guided by prior probability from the policy network. The posterior probabilities are the target for training the policy network. For the first 3 moves, we set the temperature to $\tau = 1$ so that the chance of each action being sampled is proportional to its visit count. For the rest of the game’s moves, the temperature is set to $\tau = 0$, so the chosen action is always the one with the highest visit count. Note that the temperature is set to $\tau = 0$ during the evaluation phase for all the moves.

The policy network is the lighthouse that guides the search to the moves with higher chances of winning the game. If it fails to work as intended, the search is misguided, immensely impacting the effectiveness of the search to the extent that when the search space is sufficiently large, the MCTS is equivalent to or worse than brute force search if the policy is skewed towards losing moves. In addition, improving the policy network relies completely on the heuristics from the search. Thus, if
the policy network is weak, a vicious cycle is formed where the poor policy misleads the search and the ineffective search leads to poor improvement of the policy. Danihelka et al. (2021) came up with a policy improvement algorithm in Gumbel AlphaZero, which ensures the heuristic improves the policy network consistently. However, in that approach, the target of the policy improvement entails the approximation from the value network (see Section 4: Learning an Improved Policy in Danihelka et al. (2021) for the details). Due to the parity-related problems that challenge neural networks, they still might not be better than the original AlphaZero-style algorithm in mastering Nim with a sufficiently large board size.

5.2 Experiment setup and results

In this section, the experiment results, along with the detailed configurations of the experiments, are presented. The performance of the trained agent is analysed in depth from the champion and expert perspectives.

In our implementation, the PV-NN share one LSTM layer but have separate heads. They were trained simultaneously on the experiences (rollout data) collected during the self-play. Besides the numbers of nodes in the heads being different to adapt to different action spaces, our implementations for the Nim with 5, 6 and 7 heaps employ the same network architecture. We choose a large number of simulations for each move and increase it as the number of heaps in the Nim grows, not only because more simulations lead to better heuristics but they could offset the effect of varying hyperparameters to which the algorithm is sensitive. During both training and evaluation, each move ran \( s = \{50, 60, 100\} \) number of simulations on Nim of \( h = \{5, 6, 7\} \) heaps, respectively. The simulations ran on 8 CPUs in parallel. A large number of simulations on 7 heaps Nim was used because we intended to eliminate the possible negative impact on the algorithm’s performance brought by an insufficient number of simulations, although this incurred hefty computational costs. We used the version of Nim described in section 2 where each heap contains an odd number of counters. For instance, the initial board of 5 heaps of nim, as shown in Fig. 1, consists of \([1, 3, 5, 7, 9]\). The counters are represented by unitary numbers, 1 denoting the counters on the heaps and 0 denoting counters that have been removed. Counters on different heaps are separated by -1.

Every board position of 5 heaps Nim comprises a bitstring of length 29. The board positions for 6 and 7 heaps consist of length 41 and 55 bitstrings. The state spaces of 5, 6, and 7 heaps Nim are 3840, 46080 and 645120 and the action spaces of them are 25, 36, and 49, respectively. On average, the number of moves, if taken randomly by two players, for the 5 heaps Nim is 10, 13 for 6 heaps and 16 for 7 heaps.

5.2.1 Champion measure by Elo rating

Elo rating, commonly used in chess, is an approach to ranking the players for multiplayer games in terms of their competitiveness. It is a champion measure in terms of its ability to defeat other players. AlphaZero adopted the Elo rating score to evaluate its performance against other algorithms or programs like AlphaGo Zero, AlphaGo Lee and Stockfish. Unfortunately, no existing Nim agent with an associated Elo rating reflecting its competitiveness is available. Thus, we opted for a variation of self-play Elo rating Tian et al. (2019) as an approach to measure the relative strength of an agent and monitor the training progress, in which the relative strength of an agent is evaluated in comparison with all its ancestors whose rating is updated every time a new trained agent joins in to ensure that the rating reflects their competitiveness.

In our self-play Elo rating system, every new agent is assigned an initial score of 1000. At the end of each training iteration, the trained agent is saved into a reservoir of the agents that have been trained preceding it, and its Elo rating is calculated against all the trained agents hoarded in the system. The agent being trained is denoted as Player A and its opponent who is one of its predecessors as Player B. Both of them have an expected score representing their probability of winning the match, calculated by this formula for Player A:

\[
E_A = \frac{1}{1 + 10^{(R_B - R_A)/400}} \tag{9}
\]

Analogously, the expected score for player B is calculated by

\[
E_B = \frac{1}{1 + 10^{(R_A - R_B)/400}} \tag{10}
\]
There is only one of two possible outcomes of the match for player A, being either won or lost. For Nim, the drawing situation does not exist. If player A wins the game, its Elo rating is updated by

\[ R_A = R_A + K(1 - E_A) \]  

where K is called the K-factor. The K value is usually set to 16 for masters (strong players) and 32 for novices (weak players). In our setting, the K value for the players who have engaged in the tournaments beyond 20 times is set to 32, and otherwise 16 due to the consideration that the more times a player engages in matches, the more accurately the Elo ratings reflect its strength and hence the higher K value should be. The updated Elo rating for player B is

\[ R_B = R_B + K(0 - E_B) \]  

This approach is self-contained, not relying on any external program. This also reveals one drawback of this method. The Elo rating only measures the performance of the RL agent against its predecessors, meaning that it should not be compared with the rating of the agents outside the group. However, the ratings can be used as an indicator of the agents’ performance and a baseline for future research.

We monitored the self-play Elo rating of the AlphaZero agent on Nim of 5, 6 and 7 heaps. As shown in Fig. 7, the self-play Elo rating of the trained agent grows as the training progresses, indicating it is getting more competitive. The AlphaZero agent for 5 heaps Nim grew rapidly since the inception of training. In comparison, the growth of the agent on Nim of 6 is relatively slow and that of the Nim of 7 heaps is stagnant after 420 iterations, showing that while the agent is more competitive and on the path towards becoming a champion, there seems like a ceiling of its competitiveness that is hard to crack. This bottleneck, as shown in the next section, is caused by the inability of the PV-NN to precisely evaluate Nim board positions.

![Figure 7: The self-play Elo rating score of the agent being trained on Nim of 5, 6 and 7 heaps, respectively, calculated at the end of every training epoch against all the agents saved in the pool. Calculating the Elo rating during training takes a huge amount of time. Our program ran roughly 200 hours with the above-mentioned configurations to obtain the results for the 7 heaps nim.](image_url)

5.2.2 Expert measure by the accuracy of PV-NN

In addition to the champion measure by the Elo-rating, to examine the performance of our agent from the expert perspective, we employed two accuracy measures on PV-NN. The action probability distribution yielded by the policy network should be biased towards the moves that have a higher chance of leading to winning the game. In Nim, the winning moves can be calculated by nim-sum, according to which the accuracy of the policy network is evaluated by comparing the accuracy of the most probable moves against the winning moves. A similar policy measure was used in Danihelka et al. (2021) as Policy Top 1 Accuracy where the top 1 refers to the move with the highest probability.

We used the random policy as a baseline to compare with the AlphaZero policy. As shown in Fig. 8, the AlphaZero policy surpasses the random policy by a large margin on the 5 heaps nim, but the advantage diminishes on larger board sizes. While according to the Policy Top 1 accuracy, the AlphaZero policy on 7 heaps Nim is tantamount to the random policy, it possesses more knowledge than the random policy as it could guide the MCTS.
Fig. 10. It is salient that both value and policy networks can gradually fit into the targets, empowering accuracy on the board positions of 6 heaps Nim exceeds 60 per cent, but that on the ones of the 7 positions of 5 heaps Nim rises constantly and reaches 90 per cent at 500 training iterations. The board positions. The value network on 7 heaps Nim barely outperformed random guessing.

The value network yields the estimated outcome of the game at given positions when following the move suggested by the policy network. In Nim, the won positions are the positions where the winning move exists. According to this property, we monitored the accuracy of the value network. All the possible board positions that could arise from the initial board position of 5 heaps Nim are evaluated, but due to the large state space of 6 and 7 heaps, the evaluation was conducted on 10000 randomly sampled board positions. The prediction from the value network is considered to be correct if the value network outputs a positive number on the won position and a negative number on the lost position. The accuracy of the value network is shown in Fig. 9. As the number of heaps increased, so did the state space and the board size, the value network was facing a rising challenge in precisely evaluating the board positions. The value network on 7 heaps Nim barely outperformed random guessing.

Figure 8: The accuracy of the policy network measured against the winning moves on Nim of 5, 6 and 7 heaps. The AlphaZero policy is superior to the random policy on a smaller board, but as the board size grows the accuracy of the policy drops drastically.

Figure 9: The accuracy of the value network on Nim of 5, 6 and 7 heaps. The accuracy on the board positions of 5 heaps Nim rises constantly and reaches 90 per cent at 500 training iterations. The accuracy on the board positions of 6 heaps Nim exceeds 60 per cent, but that on the ones of the 7 heaps Nim fluctuates near 50 per cent.

The policy network learns from the heuristics \( \pi(a|s) \) derived from the MCTS, as shown in Formula 8. The value network learns from the actual game output. To probe how policy and value networks learned, we kept track of the training loss for each of them during the training process, as shown in Fig. 10. It is salient that both value and policy networks can gradually fit into the targets, empowering the agent to be increasingly competitive. However, the difficulties for the policy network to digest the heuristic and for the value network to model the expected outcome of the game grow as the board size increases, which is a major problem that impedes the agent from becoming an expert.
Figure 10: The loss of the policy network (left) and the loss of the value network (right) on Nim of 5, 6 and 7 heaps. It occurs to both neural networks that the larger the board size grows, the harder it becomes for them to fit in the heuristic from the MCTS.

The gradually dropping loss of the PV-NN, coupled with the rising Elo rating of the agents, indicates the agent is learning to become a champion. However, the dropping accuracy of the PV-NN as the size of the board grows shows that it is challenging for the agent to become an expert. On 7 heaps nim, the PV-NN could memorize the heuristics from the MCTS and the actual outcome at the end of the game. But they merely enable the agent to become more competitive in comparison with its ancestors and cannot guide the MCTS effectively to form an improvement loop. This reflects our general claim that on large boards nim, impartial games are a challenge for reinforcement learning algorithms.

5.2.3 Analysis on some Nim positions

The graphs and analysis in the previous sections provide a high-level overview of the algorithm’s performance on Nim with different numbers of heaps. In this section, we will evaluate the performance of the algorithm on the initial board position along one of the intermediate positions from 5, 6 and 7 heaps nim, as we did in section 3 where the statistics of the algorithms on some chess positions by LCZero are analysed.

The analysis of the initial position of a 5 heaps Nim game with our trained PV-NN is shown in Fig. 11. All the values are calculated from the perspective of the player taking the move on these positions. Each move is represented by a letter and a digit where the letter is the heap label, and the digit denotes the number of counters the move removes from this heap. For instance, the move e9 removes 9 counters from the heap labelled as e. On this position, the policy network assigns 97.9% prior probability to the winning move e9, and the value network accurately estimates the resulting position of taking e9 is advantageous to the current player. However, this overconfidence, if not being well-grounded, could bring disastrous consequences that cannot be restored.

(a) 5 heaps: [1, 3, 5, 7, 9]

| Move  | e9   | a1   |
|-------|------|------|
| Winning Move | yes | no   |
| Prior Probability | 97.9% | 1.9% |
| Win Probability   | 99.5% | 5.0% |
| V-value           | 0.97  | -0.89 |

Figure 11: The policy and value network accurately evaluate the initial position of the 5 heap nim, assigning 97.9% prior probability and 99.5% winning probability to the winning move e9. The move with the second highest prior probability a1 is assigned with a 5.0% winning probability.
On an intermediate position similar to the initial position as shown in 12a, the search is misled by the policy network that is obsessed with the losing e8 move as the prior probability assigned to the move is 97.4%, which is overwhelmingly larger than that of other moves. This leads to the misfortune that the winning move is still rejected after 4 million simulations even though the value network predicts the correct position value after taking the move e8. This is worse than the observation in Shih et al. (2022) where some Tesuji problems they investigated need millions of simulations to find the right action.

| Move | e8 | b1 | a1 | d4 |
|------|----|----|----|----|
| Winning Move | no | no | no | no |
| Prior Probability | 97.4% | 0.7% | 0.4% | 0.2% |
| Win Probability | 0.14% | 5.6% | 12.5% | 9.8% |
| V-value | -0.99 | -0.88 | -0.74 | -0.80 |

(a) 5 heaps: [1, 3, 5, 5, 9]

(b) Evaluations from policy and value network for the 4 moves with the highest prior probabilities.

| Number of simulations |
|-----------------------|
| Winning Move | 64 | 256 | 1024 | 65536 | 4194304 |
| e7 | 1.56% | 0.39% | 0.09% | 0.0015% | 0.00021% |

(c) The posterior probability of the winning move e7 after different number of MCTS simulations

Figure 12: The only winning move available in this position is e7. The policy network offers a completely wrong estimation. The move e8, a losing move, is assigned with 97.4% prior probability. Albeit the value network predicts the resulting position of taking e8 is disadvantageous for the current player with high confidence, the algorithm fails to find out the winning move after more than 4 million simulations.

For the initial position of the 6 heaps Nim as shown in 13a, the policy network strongly promotes the winning move b2, and the value network also evaluates the resulting positions correctly with high confidence.

| Move | b2 | a1 | b1 | d6 |
|------|----|----|----|----|
| Winning Move | yes | no | no | no |
| Prior Probability | 92.8% | 4.0% | 0.64% | 0.63% |
| Win Probability | 78.0% | 14.3% | 4.76% | 2.17% |
| V-value | 0.56 | -0.71 | -0.90 | -0.95 |

(a) 6 heaps: [1,3,5,7,9,11]

(b) Evaluations from policy and value network for the 4 moves with the highest prior probabilities.

Figure 13: The policy and value network succeeds in predicting the winning move and estimating the resulting position. The probability assigned to the winning move b2 exceeds the second largest probability which is assigned to move a1 by a large margin.

There are also positions with 6 heaps where the PV-NN stumbled, one of which is shown in 14a. However, unlike the situation with a 5 heaps Nim position, as shown in Fig. 12, the prior probabilities assigned to these winning moves are not high enough to completely mislead the search. As shown in
the table 14c, the MCTS identifies the winning move f10 after 65536 simulations and the posterior probability of choosing it increases as more simulations are conducted.

| Move     | f3    | f2    | f4    | f9    |
|----------|-------|-------|-------|-------|
| Winning Move | no    | no    | no    | no    |
| Prior Probability | 47.4% | 12.6% | 10.1% | 7.5%  |
| Win Probability  | 0.18% | 1.03% | 0.03% | 81.2% |
| V Value         | -0.99 | -0.97 | -0.99 | 0.62  |

(a) 6 heaps: [1,3,3,5,4,10]  

(b) Evaluations from policy and value network for the 4 moves with the highest prior probabilities.

| Number of simulations |
|-----------------------|
| Winning Move         |
| 64                    | 256                | 1024                | 65536                | 4194304               |
| f10                  | 4.68%              | 1.17%               | 0.29%                | 53.7%                 | 99.1%                 |

(c) The posterior probability of the winning move f10 given different number of MCTS simulations

Figure 14: In this position, the only winning move is f10. However, the top 4 probabilities yielded by the policy network are all assigned to losing moves, causing the algorithm to fail to find the winning even after more than 1000 simulations. Fortunately, the value network predicts with high confidence that the resultant positions from moves f3, d2 and f4 favour the current player’s opponent, making the winning move f10 stand out after 65536 simulations and the confidence is boosted with more simulations.

The initial position of the 7 heaps Nim as shown in Fig. 15a has three winning moves available. In this position, the prior probabilities from the policy network on all the legal moves are almost equal. The value network evaluates all the resulting positions as having around 50% winning probability. These predictions contributed little to the search because the MCTS did not recognise the winning moves after more than 4 million simulations.

In Nim with 5, 6 and 7 heaps, there are positions where PV-NN succeed in evaluation and positions where they failed. The high accuracy of their predictions on the initial positions of small boards indicates that they tend to be more knowledgeable about the positions that they encounter with high frequency. But the confidence in prioritising the winning move decays, and the difficulty in accurately evaluating positions escalates as the board size (and state space) grows.

On relatively small boards, the parity issues of computing the correct nim-sum do not get involved as calculating the parity is feasible. However, mastering Nim on larger boards is dramatically more challenging because finding the right move and correctly evaluating large board positions entails solving parity-related issues using neural networks, and the huge state space forces PV-NN to generalize to unseen states, which, as we have argued, is difficult. As the data an RL agent learns from is garnered from its interaction with the environment, this difficulty also involves the unknown and unpredictable data distribution the RL agent learns from because the way the data are sampled plays a crucial role in neural networks learning to model a perfect parity function Daniely and Malach (2020); Cornacchia and Mossel (2023). Furthermore, the noisy label problem arising in self-play RL adds more difficulty for the PV-NN to model the parity function, as discussed in Zhou and Riis (2023). All these complexities pose a fundamental challenge to RL algorithms on their way to mastering Nim. Our results show that in a real self-play RL setting, the parity function presents a more substantial obstacle for an RL agent to be an expert on Nim game. As all the impartial games can be translated into the Nim game, the challenge naturally applies to other impartial games.
Figure 15: The 4 moves with the highest prior probabilities are losing moves, and the policy network does not particularly favour any of them. The evaluation from the value network on these positions is near 0, showing they all have a 50% winning probability.

6 Concluding remarks and conjectures

The AlphaZero-style learning paradigm can be seen as part of a larger ambition to build intelligent systems that can learn to solve complex tasks by themselves. As articulated by Demis Hassabis, the CEO of DeepMind, the ambition is to solve intelligence and then use it to solve everything else (Sadler and Regan 2019). While this ambition is truly inspiring, the results in this paper remind us that thinking in various strategy games varies fundamentally in nature. General AI will need to handle different modes of thinking.

From a human perspective, board games like chess and Go are somehow distinct from Nim and other impartial games. The former games have clear criteria for measuring good play. Progress in learning these games is typically incremental, and pattern recognition plays a central role. On the contrary, Nim can be mastered by humans but through an entirely different thought process and analysis. Learning Nim typically happens in non-incremental steps. It seems inconceivable that a human could learn to master Nim on a large board without having solved the game for any board size. Thus, when humans master nim, transfer learning and abstract generalisation play an important role.

AlphaZero has been truly groundbreaking, but new ideas are needed to expand reinforcement learning to games that, like Nim seem to require high-level non-associative thinking. For humans, counting is a straightforward process. Even young children understand that the number of objects is invariant, so recounting should in principle lead to the same number. In mathematics, this principle is called the pigeon-hole principle; however, as we explained, such basic counting principles require a kind of insight that seems challenging for AIs to develop solely on their own.

We acknowledge that learning parity from a uniform distribution is a well-known nuisance for neural networks, trivial to compute but loved by theoreticians to bang NN engineers over the head. The motivation of our work is not to bang anyone on their head but to understand how the AlphaZero paradigm can be expanded. The concept of self-play in model-based AI is one of the ways an agent can learn by exploring the environment with an effective search strategy. However, it is essential to understand the actual way self-play RL algorithms learn, which is why we looked carefully and closely at how the AlphaZero clone LCZero played chess. And it is why this paper on impartial games might help navigate further research on expanding AlphaZero-style learning.
Parity-related problems occur naturally for a large class of combinatorial games. However, we discovered that the difficulty of learning to master these games is much more dramatic than just learning parity. The problem is robust and tenacious, and fiddling with the algorithm’s hyperparameters does not seem to have much effect. We did break down the AlphaZero style algorithm and checked the various components separately. We even tested the parity issues with novel architectures that were not part of the original AlphaZero algorithms.

There are many factors impacting the performance of our AlphaZero style Nim algorithm. There is an unlimited number of settings so it is impossible to try all of them out. Proving the results rigorously seems well outside current theoretical techniques. From a philosophy of science perspective, one can always object that a specific pattern might fail at values not tested. If such objections were taken seriously, science would be impossible. Consider experiments suggesting some formula e.g. $F = ma$. It is always (logically) possible that the formula would seriously fail for values not tested. It is always possible to make this kind of objection to an experimental result. Experimental results should always seen as part of a wider theory and understanding that is aligned with experiments but, in principle, could be falsified Popper (1972).

We anticipated that Nim would be practically unlearnable by AlphaZero-style algorithms on boards with 50+ heaps. However, to our surprise, the practical upper bound of the learnability was much lower than we expected, as the algorithms experienced substantial limitations with seven heaps. Our work also shows there is an issue when attempting to apply NNs (e.g. Stockfish NNUE style NN) to guide the search in the new algorithms Nasu (2018); Maharaj et al. (2021) for impartial game due to the difficulty of the networks guiding the search.

Another point we would like to stress is that the difficulty of learning Nim on small boards is not even due to parity issues. The parity required to compute correct Nim sums etc has not kicked in on small boards as learning parity for small values of $n$, (e.g. $n = 7$ for 7 piles) is pretty feasible as our experiments showed. On the board $[1, 3, 5, 7, 9, 11, 13]$, we established that no positive feedback loop occurs, and the policy and value network essentially drift around without the ability to learn anything besides memorizing some heuristics derived from MCTS. Remarkably, at least with the available resources, this happened even though the state space is relatively small, and most states will be seen multiple times during training if all the positions are fully explored. On larger boards, where the state space exceeds any number of states that feasibly can be reached during training, the value and policy network needs to generalise to unseen positions. Failing to generalise adds additional noise to the learning as the evaluation on some positions becomes random and uncorrelated with correct values, preventing the positive feedback mechanism of RL from functioning properly. Added to this difficulty, on larger boards the difficulty of learning the parity function also kicks in an already very noisy situation.

AlphaZero employs a strategy that combines evaluation with search-guided calculation. However, some aspects of impartial games seem to require mathematical thinking guided by abstract, symbolic reasoning. Some PSPACE-complete impartial games, e.g. node kayles and geography, can mimic NP-hard problems which are intractable. Thus, any algorithm that could learn any of these games to perfection would be able to break current cryptography. However, other impartial games can be solved by mathematical reasoning. Thus, it is possible to express optimal strategies in simple mathematical terms, e.g. sprout that has been analysed to a deep level might eventually be solvable by AI with sufficient built-in reasoning abilities.

Despite the success of AlphaZero, our work indicates that fundamentally new ideas are needed for an AlphaZero-style approach to be successful for impartial games. When humans learn to master nim, they might scribble on paper, play around with small toy examples, form conjectures, etc. Applying AlphaZero style reinforcement learning in an extended setting that considers auxiliary actions external to the game, such as applying abstract transformations or reading and writing to external memory, might be possible. These meta-actions, analogous to the algorithm’s actions during simulations, are not directly linked to the move it makes but significantly boost its ability to plan forward. The results in this paper indicate that new ideas will be needed to make such ideas work.

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