Accumulated Random Distances in High Dimensions – Ways of Calculation

Eliahu Levy
Department of Mathematics
Technion – Israel Institute of Technology
Technion City, Haifa 3200003, Israel
(eliahu@math.technion.ac.il)

March 24, 2020

Abstract
In this note we refine and improve some of the calculations in our 2019 article with Yair Censor (Applied Mathematics and Optimization, accepted for publication) where an analysis of the superiorization method is made via the principle of concentration of measure. Some paragraphs there are repeated here for the sake of completeness. Yet, for the case of accumulating ‘steps’ on the sphere, reference to distances as done there is replaced by reference to the angles, which makes simpler expressions. The treatment here of the action of a random transformation is also rather ‘cleaner’. For some standard deviations, precise inequalities are here obtained rather than just $O$-expressions. Some further settings are mentioned, of no direct interest as per the latter article, showing results that similar calculations yield.

1 Concentration of Measure in High Dimensional Euclidean Spaces

Concentration of (Probability) Measure is the phenomenon that probability is highly concentrated near one value, thus near the expectation.
A classical example is The Law of Large Numbers, stating the concentration of measure of the sum (or average) of many i.i.d. stochastic variables.

We will focus on peculiar concentration of measure phenomena in \( N \)-dimensional Euclidean spaces when \( N \) is large.

To see, as an example, why Euclidean spaces \( \mathbb{R}^N \) of high dimension \( N \) should lead to concentration of measure phenomena, consider the unit sphere in \( \mathbb{R}^N \).

‘Partitioning’ it into layers in parallel hyperplanes orthogonal to some fixed unit vector, the layer distanced \( t \) from the central hyperplane through 0 is a sphere of one less dimension with radius \( \sqrt{1-t^2} \). Its \( N-2 \)-‘area’ will be proportional to the \( N-2 \) power of its radius, hence to \((1-t^2)^{(N-2)/2}\).

And for a vector uniformly distributed on the sphere, the probability for distance \( t \) from the central hyperplane will have density proportional to that ‘\( N-2 \)-area’ multiplied by \((1-t^2)^{-1/2}\) (– the latter due to the need to project \( dt \) onto the sphere), thus to \((1-t^2)^{(N-3)/2}\).

\( N \) being ‘big’, if \( 1-t^2 \) is even slightly less than 1, the ‘big’ exponent \( N-3 \) will make \((1-t^2)^{(N-3)/2}\) small. Indeed, \((1-t^2)^{(N-3)/2}\), when not negligible, is approximately \( e^{-Nt^2/2} \), significant only when \( t = O(1/\sqrt{N}) \).

So we have the somewhat strange-sounding fact (later we’ll also derive that differently):

For two vectors in \( \mathbb{R}^N \) with independent uniformly distributed directions, it is highly unlikely that the angle between them is far from \( 90^\circ \). In fact, the deviation from \( 90^\circ \) is with high probability \( O(1/\sqrt{N}) \).

2 Using the Normal Distribution

Yet a main vehicle in studying \( \mathbb{R}^N \) is the Normal Distribution. (Indeed, as we saw above, our \((1-t^2)^{(N-3)/2}\) above ‘ended’ as approximately ‘normal’, as the Central Limit Theorem would make us expect.)

The standard normal distribution on \( \mathbb{R} \), \( \mathcal{N}(0,1) \), has distribution

\[ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx. \]

Thus in the Euclidean \( \mathbb{R}^N \), i.e. endowed with \( \ell^2 \) norm, letting the \( N \) coordinates to be independent identically distributed \( \sim \mathcal{N}(0,1) \) will give the distribution on \( \mathbb{R}^N \),

\[ (2\pi)^{-N/2} e^{-\frac{1}{2}\|x\|^2} \, dx_1 \ldots \, dx_N. \]

hence invariant under all linear orthogonal transformations of \( \mathbb{R}^N \).
In particular any projections of it on \( N \) orthogonal unit vectors are also \( N \) independent \( \sim \mathcal{N}(0, 1) \).

To see what one may obtain thus, for our above \( x = (x_1, \ldots, x_n) \) with coordinates i.i.d. \( \sim \mathcal{N}(0, 1) \), we have, by the orthogonal invariance,

**For any fixed unit vector** \( u \in \mathbb{R}^N \), \( u \cdot x \) is \( \sim \mathcal{N}(0, 1) \)

On the other hand, \( \|x\|^2 = x_1^2 + \ldots + x_N^2 \). These squares are still i.i.d, therefore, as always with sums of i.i.d., \( \|x\|^2 \) has variance \( N \) times the variance of the square of a \( \sim \mathcal{N}(0, 1) \), hence has standard deviation \( \sqrt{N} \) times the standard deviation of the square a \( \sim \mathcal{N}(0, 1) \) – some number independent of \( N \).

And, of course, the expectation of \( \|x\|^2 \) is \( N \) (because for a single coordinate it is the variance of a \( \sim \mathcal{N}(0, 1) \) itself = 1).

So \( \|x\|^2 \) *is highly concentrated near its expectation* \( N \) – with relative error \( O(1/\sqrt{N}) \), hence \( \|x\| \) is *highly concentrated near* \( \sqrt{N} \).

Thus, speaking somewhat loosely, for the above \( x \) with i.i.d. \( \sim \mathcal{N}(0, 1) \) coordinates, \( \left(\frac{1}{\sqrt{N}}\right) x \) will rarely deviate from 1.

And of course, by the orthogonal invariance, \( x/\|x\| \) must is distributed uniformly on the unit sphere. *So we have here a way to get this uniform distribution.*

But \( x/\|x\| \) is rarely much different from \( \left(\frac{1}{\sqrt{N}}\right) x \), thus the latter (Gaussian!) distribution may also serve as an approximation to the uniform distribution on the unit sphere – the unit point ‘wanders’ along the radius, but ‘rarely more than \( O(1/\sqrt{N}) \).’

And we can deduce the above fact about the angle between vectors:

The Gaussian \( \left(\frac{1}{\sqrt{N}}\right) x \) differs from \( x/\|x\| \), which gives the uniform distribution on the unit sphere, just by a relative \( O(1/\sqrt{N}) \) ‘perturbation’ of the vector.

The cosine of its angle with a fixed unit vector \( u \) is \( \left(\frac{1}{\sqrt{N}}\right) (u \cdot x) \) and \( (u \cdot x) \) is \( \sim \mathcal{N}(0, 1) \), making that cosine (= the sine of the difference with 90\(^\circ\)) approximately \( \sim \left(\frac{1}{\sqrt{N}}\right) \mathcal{N}(0, 1) \).

### 3 The norm of the sum of vectors with given norms

As a further example of the use of the normal distribution, suppose we are given \( M \) vectors \( y_1, y_2, \ldots, y_M \) of known norms \( d_1, d_2, \ldots, d_M \) in a high-dimensional \( E^N \). What should we expect the norm of their sum to be?
That can be answered: take the direction of each of them distributed uniformly on \( S^{N-1} \), even conditioned on fixed valued for the others. In other words, take them independent, each with direction distributed uniformly. This can be constructed by taking random \( M \) vectors in \( E^N \) (that is, a random \( M \times N \) matrix), with entries i.i.d. \( \sim \mathcal{N} \), dividing them by \( \sqrt{N} \), then by their norm (now highly concentrated near 1) and then multiplying them (i.e. the columns of the matrix), by \( d_1, d_2, \ldots, d_M \), respectively.

Then the sum \( \sum_{i=1}^{M} y_i \), if we ignore the division by the norm, is \( 1/\sqrt{N} \) times the random matrix applied to the vector \( (d_1, d_2, \ldots, d_M) \).

But the distribution of the random matrix is invariant with respect to any transformation which is orthogonal with respect to the Hilbert-Schmidt norm – the square root of the sum of squares of the entries (i.e., \( \| T \|_{HS} := \sqrt{\text{tr}(T' \cdot T)} = \sqrt{\text{tr}(T \cdot T')} \)), \( T' \) denoting the transpose and \( \text{tr} \) standing for the trace, see, e.g., [2].

In particular, the distribution of the sum is the same as that of \( 1/\sqrt{N} \) times \( \sqrt{d_1^2 + d_2^2 + \cdots + d_M^2} \) times the random matrix applied to \( (1, 0, \ldots, 0) \), which is, of course, distributed with independent \( \sim \mathcal{N} \) entries, thus, with norm concentrated near \( \sqrt{N} \). (With relative deviation \( O(1/\sqrt{N}) \).) This leads to the following conclusion.

**Conclusion 1** For \( M \) vectors \( y_1, y_2, \ldots, y_M \) of known norms \( d_1, d_2, \ldots, d_M \), in \( E^N \), (and taking their directions as random, distributed independently uniformly), we have that \( \| \sum_{i=1}^{M} y_i \| \) is near \( \sqrt{d_1^2 + d_2^2 + \cdots + d_M^2} \) with almost full probability (With relative deviation \( O(1/\sqrt{N}) \).)

We shall return to that in \( \S 6 \).

### 4 The accumulation of given distances on the unit sphere

Now ask a similar question on the unit sphere: what should we expect the angle of a chord made of \( M \) chords, the differences between consecutive elements in a sequence of points on the unit sphere \( S^{N-1} \subset E^N \), of given angles \( \theta_1, \theta_2, \ldots, \theta_M \).

Note that here the angles are the natural ‘distances’. Indeed they naturally measure the distance on geodesics – great circles.

Denote by \( \omega_{N-1} \) the normalized to be probability (i.e., of total mass 1) uniform measure on \( S^{N-1} \).

**Remark 2** By symmetry, for \( x = (x_1, x_2, \ldots, x_N) \in S^{N-1} \), \( \int x_k^2 d\omega_{N-1} \) is the same for all \( k \). Of course, their sum is \( \int 1 d\omega_{N-1} = 1 \). Therefore,

\[
\int x_k^2 d\omega_{N-1} = \frac{1}{N}, \quad k = 1, 2, \ldots, N. \tag{1}
\]
Hence, for a polynomial of degree $\leq 2$ on $E^n$:

$$p(x) = \langle Qx, x \rangle + 2\langle a, x \rangle + \gamma,$$  \hspace{1cm} (2)

where $Q$ is a symmetric $N \times N$ matrix, $a \in E^N$ and $\gamma \in E$, we will have

$$\int p(x) \, d\omega_{N-1} = \frac{1}{N} \text{tr } Q + \gamma.$$ \hspace{1cm} (3)

Now, for some fixed $0 \leq \theta \leq \pi$, the set of points in $S^{N-1}$ of angular distance $\theta$ from some fixed vector $u \in S^{N-1}$ is the $(N-2)$-sphere $\subset S^{N-1}$, $\Sigma(u, \theta)$ given by

$$\Sigma(u, \theta) := \cos \theta \cdot u + \sin \theta \cdot S^{N-2} u^\perp,$$ \hspace{1cm} (4)

where $S^{N-2} u^\perp$ stands for the unit sphere in the hyperplane perpendicular to $u$.

In our scenario, one performs a Markov chain, see, e.g., [1]. Starting from a point $u_0$ on $S^{N-1}$, move to a point $u_1 \in \Sigma(u_0, \theta_1)$ uniformly distributed there. Then, from that $u_1$, to a point $u_2 \in \Sigma(u_1, \theta_2)$ uniformly distributed there, and so on, until one ends with $u_M$. We would like to find the expected cosine of the angle between $u_0$ and $u_M$, namely $E[\langle u_M, u_0 \rangle]$.

If we denote by $\mathcal{L}_\theta$ the operator mapping a function $p$ on $S^{N-1}$ to the function whose value at a vector $u \in S^{N-1}$ is the average of $p$ on $\Sigma(u, \theta)$, then $\mathcal{L}_{\theta_k}(p)$ evaluated at $u$ is the expectation of $p$ at the point to which $u$ moved in the $k$-th step above. Hence, in the above Markov chain, the expectation of $p(u_M)$ is

$$\left(\mathcal{L}_{\theta_M} \mathcal{L}_{\theta_{M-1}} \cdots \mathcal{L}_{\theta_1}(p)\right)(u_0).$$ \hspace{1cm} (5)

Thus, what we are interested in is

$$\mathbb{E}[\langle u_M, u_0 \rangle] = \mathcal{L}_{\theta_M} \mathcal{L}_{\theta_{M-1}} \cdots \mathcal{L}_{\theta_1}(\langle x, u_0 \rangle) \bigg|_{x=u_0}.$$ \hspace{1cm} (6)

So, let us calculate $\mathcal{L}_\theta(p)$ for polynomials of degree $\leq 2$ as in (2). In performing the calculation, assume $u = (1, 0, \ldots, 0)$. For $x = (x_1, x_2, \ldots, x_N) \in E^N$ write $y = (x_2, x_3, \ldots, x_N) \in E^{N-1}$. In (2) write $a = (a_1, b)$ where $b = (a_2, a_3, \ldots, a_N) \in E^{N-1}$ and

$$Q = \begin{pmatrix} \eta & c^t \\ c & Q_1 \end{pmatrix},$$ \hspace{1cm} (7)

where $Q_1$ is a symmetric $(N-1) \times (N-1)$ matrix, $c \in E^{N-1}$ and $\eta \in E$. Note that for our $u = (1, 0, \ldots, 0)$, $a_1 = \langle a, u \rangle$, $\eta = \langle Qu, u \rangle$ and $\text{tr } Q_1 = \text{tr } Q - \eta = \text{tr } Q - \langle Qu, u \rangle$.

Then, for $p$ as in (2),

$$p(x) = \eta x_1^2 + 2x_1 \langle c, y \rangle + \langle Q_1 y, y \rangle + 2a_1 x_1 + 2\langle b, y \rangle + \gamma.$$ \hspace{1cm} (8)
Hence, taking account of \( (\mathcal{L}_\theta p)(u) = (\mathcal{L}_\theta p)(1, 0, \ldots, 0) \)
\[
= \cos^2 \theta \cdot \eta + \frac{1}{N-1} \sin^2 \theta \, \text{tr} \, Q_1 + 2 \cos \theta \, a_1 + \gamma = \\
= \cos^2 \theta \langle Qu, u \rangle + \frac{1}{N-1} \sin^2 \theta \left( \text{tr} \, Q - \langle Qu, u \rangle \right) + 2 \cos \theta \langle a, u \rangle + \gamma = \\
= \left( \cos^2 \theta - \frac{1}{N-1} \sin^2 \theta \right) \langle Qu, u \rangle + \frac{1}{N-1} \sin^2 \theta \, \text{tr} \, Q + 2 \cos \theta \langle a, u \rangle + \gamma.
\]
which, by symmetry, will hold for any \( u \in S^{N-1} \). In particular, we find, as should be expected, that
\[
\int (\mathcal{L}_\theta(p))(x) \, d\omega_{N-1} = \frac{1}{N} \left( \cos^2 \theta - \frac{1}{N-1} \sin^2 \theta \right) \, \text{tr} \, Q + \frac{1}{N-1} \sin^2 \theta \, \text{tr} \, Q + \gamma = \frac{1}{N} \, \text{tr} \, Q + \gamma = \int p(x) \, d\omega_{N-1}.
\]
We are interested, for some fixed \( u \in S^{N-1} \), in
\[
p(x) = \langle u, x \rangle.
\] (9)
Then there is no \( Q \) term, so one has
\[
\left( \mathcal{L}_\theta(\langle u, x \rangle) \right)(u) = \cos \theta \cdot \langle u, x \rangle.
\] (10)
Consequently,
\[
\mathbb{E} \left[ \langle u_M, u_0 \rangle \right] = \left( \mathcal{L}_{\theta M} \mathcal{L}_{\theta_{M-1}} \cdots \mathcal{L}_{\theta_1} (\langle x, u_0 \rangle) \right) \bigg|_{x=u_0} = \\
= \prod_{i=1}^{M} \cos \theta_i \cdot (\langle u_0, x \rangle) \bigg|_{x=u_0} = \prod_{i=1}^{M} \cos \theta_i.
\]
We also assess the standard deviation, which is
\[
\sigma = \sqrt{\left( \mathcal{L}_{\theta M} \mathcal{L}_{\theta_{M-1}} \cdots \mathcal{L}_{\theta_1} (\langle u_0, x \rangle^2) \bigg|_{x=u_0} - \left( \prod_{i=1}^{M} \cos \theta_i \right)^2 \right)}.
\] (11)
Here \( p(x) = \langle a, x \rangle^2 \), so there is only the \( Q \) term with \( Q(x) := \langle a, x \rangle^2 \). Then \( \text{tr} \, Q = \|a\|^2 \), and we find
\[
\left( \mathcal{L}_\theta(\langle a, x \rangle^2) \right)(u) = \\
= \left( \cos^2 \theta - \frac{1}{N-1} \sin^2 \theta \right) \langle a, u \rangle^2 + \frac{1}{N-1} \sin^2 \theta \|a\|^2.
\]
Consequently, for $a = u_0$ (note $\| u_0 \|^2 = 1$),

$$
\sigma^2 = (L_{\theta_M} \cdots L_{\theta_1}) (\langle u_0, x \rangle^2)_{x = u_0} - \left( \prod_{i=1}^{M} \cos \theta_i \right)^2 = 
$$

$$
= - \left( \prod_{i=1}^{M} \cos \theta_i \right)^2 + \prod_{i=1}^{M} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
$$

$$
+ \frac{1}{N-1} \left[ \sin^2 \theta_1 + \sin^2 \theta_2 \left( \cos^2 \theta_1 - \frac{1}{N-1} \sin^2 \theta_1 \right) 
+ \sin^2 \theta_3 \left( \cos^2 \theta_2 + \frac{1}{N-1} \sin^2 \theta_2 \right) \left( \cos^2 \theta_1 - \frac{1}{N-1} \sin^2 \theta_1 \right) 
+ \cdots + \sin^2 \theta_M \prod_{i=1}^{M-1} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) \right], 
$$

But,

$$
\prod_{i=1}^{M} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
$$

$$
= - \frac{1}{N-1} \sin^2 \theta_M \prod_{i=1}^{M-1} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
$$

$$
- \cos^2 \theta_M \cdot \frac{1}{N-1} \sin^2 \theta_{M-1} \cdot \prod_{i=1}^{M-2} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
$$

$$
- \cos^2 \theta_M \cos^2 \theta_{M-1} \cdot \frac{1}{N-1} \sin^2 \theta_{M-2} \cdot \prod_{i=1}^{M-3} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
$$

$$
- \cdots - \prod_{i=3}^{M} \cos^2 \theta_i \cdot \frac{1}{N-1} \sin^2 \theta_2 \left( \cos^2 \theta_1 - \frac{1}{N-1} \sin^2 \theta_1 \right) 
$$

$$
- \prod_{i=2}^{M} \cos^2 \theta_i \cdot \frac{1}{N-1} \sin^2 \theta_1 + \prod_{i=1}^{M} \cos^2 \theta_i. 
$$

So we find,

$$
\sigma^2 = \frac{1}{N-1} \left[ \sin^2 \theta_{M-1} \left( 1 - \cos^2 \theta_M \right) \prod_{i=1}^{M-2} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
+ \sin^2 \theta_{M-2} \left( 1 - \cos^2 \theta_M \cos^2 \theta_{M-1} \right) \prod_{i=1}^{M-3} \left( \cos^2 \theta_i - \frac{1}{N-1} \sin^2 \theta_i \right) 
+ \cdots + \sin^2 \theta_2 \left( 1 - \prod_{i=3}^{M} \cos^2 \theta_i \right) \left( \cos^2 \theta_1 - \frac{1}{N-1} \sin^2 \theta_1 \right) 
+ \sin^2 \theta_1 \left( 1 - \prod_{i=2}^{M} \cos^2 \theta_i \right) \right]. 
$$
So $\sigma \leq \frac{1}{\sqrt{N-1}} \|\sin^2 \theta_1, \sin^2 \theta_2, \ldots, \sin \theta_{M-1}\|_2$.

**Conclusion 3** The cosine of angle $\theta$ of a chord made by $M$ chords of given angles $\theta_1, \theta_2, \ldots, \theta_M$, between consecutive elements in a sequence of points on the unit sphere $S^{N-1} \subset E^N$, modeled by the above Markov chain, is with almost full probability, near

$$\prod_{i=1}^{M} \cos \theta_i. \quad (12)$$

(With deviation $O\left((1/\sqrt{N}) \cdot \|\sin \theta_1, \sin \theta_2, \ldots, \sin \theta_{M-1}\|_2\right)$.)

**5 What does (12) tell us?**

Firstly, we conclude, rather surprisingly, that if all $\theta_i$ were acute angles – between 0 and $90^\circ$, i.e. with nonnegative cosines, then invariably also the accumulated angle is expected to be acute, no matter what $M$ – the number of $\theta_i$’s is!

Secondly, since multiplying by a cosine always does not increase the absolute value, the product in (12), giving the expected cosine of the accumulated angle, would tend to be small, meaning angle near $90^\circ$.

Moreover, if one of the $\theta_1$ was $90^\circ$, the resulting expected $\theta$ is invariably $90^\circ$!

Speaking somewhat floridly: $90^\circ$ turns out to be both an impassible barrier and a forceful attractor!

Yet the wonder might subside if one recalls what we had found before: that anyhow it is highly unlikely for an angle to be far from $90^\circ$, thus ‘one should expect any ‘nudge’ to push it into that dominating realm and if already there to remain there’.

Note also that for a tiny area on a big sphere, (12) agrees with the ‘flat’ case §3: then the $\theta$’s are small, $\cos \theta \sim 1 - \frac{1}{2} \theta^2$ and multiplying these corresponds approximately to adding the $\theta^2$’s.

**6 As an Aside, Let’s do the Markov Chain also for the ‘Flat’ Case**

We inquire about the norm of the sum $y$ of $M$ vectors $y_1, y_2, \ldots, y_M$ of known norms $d_1, d_2, \ldots, d_M$ in the Euclidean $\mathbb{R}^N$. Their directions taken as distributed uniformly and independently.

Then, as before, the sum $y$ is the result of a Markov chain: Starting from $x_0 = 0$, move to a point $x_1 \in \Sigma(u_0, d_1)$ ($\Sigma(x, d)$ here denoting the sphere with center $x$ and radius $d$), uniformly distributed there, then from that $x_1$.
to a point \( x_2 \in \Sigma(x_1, d_2) \) uniformly distributed there, and so on, until one ends with \( y = x_M \). And we wish to find \( \mathbb{E}[\|x_M - x_0\|^2] \).

As before, denote by \( L_d \) the operator mapping a function \( p \) on \( \mathbb{R}^N \) to the function whose value at a vector \( x \in \mathbb{R}^N \) is the average of \( p \) on \( \Sigma(u, \xi) \), then, in our Markov chain, the expectation of \( p(x_M) \) will be

\[
(L_{d_M} L_{d_{M-1}} \cdots L_{d_1}(p))(x_0).
\]

We are interested in

\[
\mathbb{E}[\|x_M - x_0\|^2] = L_{d_M} L_{d_{M-1}} \cdots L_{d_1}(\|x - x_0\|^2) \Big|_{x=x_0},
\]

and will also be concerned with the standard deviation.

(This necessitates finding \( L_d(p) \) for \( p \) polynomials of degree \( \leq 4 \). While for general such \( p \) the integrals of monomials over \( S^{N-1} \) such as those calculated in appendix (23) might be used, in our case these will not be needed.)

But note that for \( a = (a_1, 0, \ldots, 0) \),

\[
\int \langle a, x \rangle^2 d\omega_{N-1} = \int a_1^2 x_1^2 d\omega_{N-1}, \quad \text{i.e.,}
\]

\[
\int \langle a, x \rangle^2 d\omega_{N-1} = \frac{1}{N} a_1^2 = \frac{1}{N} \|a\|^2,
\]

which by symmetry holds for any \( a \).

Now, the value of \( L_d(p) \) at \( x_0 \), the average of \( p \) over \( \Sigma(x_0, d) \) – the sphere of radius \( d \) around \( x_0 \), will be the average of \( p(x - x_0) \) over \( \Sigma(0, d) \) – the sphere of radius \( d \) around the origin.

In our case \( p(x) := \|x\|^2 \), and

\[
p(x - x_0) = \|x\|^2 + \|x_0\|^2 - 2\langle x_0, x \rangle.
\]

whose average on \( \Sigma(0, d) \) is \( d^2 + \|x_0\|^2 \). So we have

\[
(L_d(\|x\|^2)) \big|_{x=x_0} = \|x_0\|^2 + d^2.
\]

Therefore

\[
\mathbb{E}[\|x_M\|^2] = L_{d_M} L_{d_{M-1}} \cdots L_{d_1}(\|x\|^2) \big|_{x=0} = (\|x\|^2 + (d_1^2 + d_2^2 + \ldots d_M^2)) \big|_{x=0} = d_1^2 + d_2^2 + \ldots d_M^2.
\]

As for the standard deviation, it is given by

\[
\sigma^2 = \mathbb{E}[\|x_M\|^4] - (d_1^2 + d_2^2 + \ldots d_M^2)^2 = L_{d_M} L_{d_{M-1}} \cdots L_{d_1}(\|x\|^4) \big|_{x=0} - (d_1^2 + d_2^2 + \ldots d_M^2)^2
\]
For \( p(x) := \|x_M\|^4 \) we have

\[
p(x - x_0) = \left( \|x\|^2 + \|x_0\|^2 - 2\langle x_0, x \rangle \right)^2 = \\
\|x\|^4 + \|x_0\|^4 + 4\langle x_0, x \rangle^2 + 2\|x_0\|^2 \cdot \|x\|^2 - 4\|x_0\|^2 \langle x_0, x \rangle - 4\|x\|^2 \cdot \langle x_0, x \rangle.
\]

Its average on \( \Sigma(0, d) \), which is \( (\mathcal{L}_d(p)) (x_0) \), will be

\[
d^4 + \|x_0\|^4 + \frac{4}{N} d^2 \|x_0\|^2 + 2d^2 \|x_0\|^2 \\
= \|x_0\|^4 + \left( 2 + \frac{4}{N} \right) d^2 \|x_0\|^2 + d^4.
\]

Thus,

\[
\|x_0\|^4 \rightarrow_{\mathcal{L}_{d_1}} d_1^4 + \left( 2 + \frac{4}{N} \right) d_2^2 \|x\|^2 + \|x\|^4 \\
\rightarrow_{\mathcal{L}_{d_2}} d_1^4 + d_2^4 + \left( 2 + \frac{4}{N} \right) d_2^2 (d_2^2 + \|x\|^2) + \left( 2 + \frac{4}{N} \right) d_2^2 \|x\|^2 + \|x\|^4 \\
\rightarrow_{\mathcal{L}_{d_3}} d_1^4 + d_2^4 + d_3^4 + \left( 2 + \frac{4}{N} \right) d_3^2 (d_3^2 + \|x\|^2) \\
+ \left( 2 + \frac{4}{N} \right) d_2^2 (d_3^2 + \|x\|^2) + \left( 2 + \frac{4}{N} \right) d_3^2 \|x\|^2 + \|x\|^4 \rightarrow_{\mathcal{L}_{d_3}} \cdots \\
\rightarrow_{\mathcal{L}_{d_M}} d_1^4 + d_2^4 + \cdots + d_M^4 + \left( 2 + \frac{4}{N} \right) \sum_{i<j} d_i^2 d_j^2 \\
+ \left( 2 + \frac{4}{N} \right) (d_1^2 + \cdots + d_M^2) \|x\|^2 + \|x\|^4.
\]

And

\[
\sigma^2 = \mathbb{E} \left[ \|x_M\|^4 \right] - \left( d_1^2 + d_2^2 + \ldots + d_M^2 \right)^2 = \\
\mathcal{L}_{d_M} \mathcal{L}_{d_{M-1}} \cdots \mathcal{L}_{d_1} (\|x\|^2) \bigg|_{x=0} - \left( d_1^2 + d_2^2 + \ldots + d_M^2 \right)^2 = \\
- (d_1^2 + d_2^2 + \ldots + d_M^2)^2 + d_1^4 + d_2^4 + \cdots + d_M^4 + \left( 2 + \frac{4}{N} \right) \sum_{i<j} d_i^2 d_j^2 \\
= 4 \frac{1}{N} \sum_{i<j} d_i^2 d_j^2.
\]

Thus, for any \( \alpha \geq 0 \),

\[
\sigma^2 = 4 \frac{1}{N} \sum_{i<j} d_i^2 d_j^2 \leq 4 \frac{1}{N} \sum_{i<j} \left( d_i^2 d_j^2 + \alpha (d_i^2 - d_j^2)^2 \right) = \\
4 \frac{N-1}{N} \alpha \left( d_1^2 + d_2^2 + \ldots + d_N^2 \right)^2 - 8 \frac{N-1}{N} \alpha \sum_{i<j} d_i^2 d_j^2 + 4 \frac{1}{N} (1 - 2\alpha) \sum_{i<j} d_i^2 d_j^2 = \\
4 \frac{N-1}{N} \alpha \left( d_1^2 + d_2^2 + \ldots + d_N^2 \right)^2 + \left( \frac{4}{N} - 8\alpha \right) \sum_{i<j} d_i^2 d_j^2
\]
So take $\alpha = \frac{1}{2N}$ which will annul the last term, to find

$$\sigma \leq \frac{\sqrt{2(N-1)}}{N} \left( d_1^2 + d_2^2 + \ldots + d_N^2 \right),$$

in accordance with Conclusion 1.

7 The action of a ‘Random’ Symmetric linear operator With Given Eigenvalues in a High-Dimensional Space

Consider an $N \times N$ symmetric matrix $A$ with given eigenvalues $s_1, s_2, \ldots, s_N$. This means that

$$A = U^{-1} \text{diag}(s_1, s_2, \ldots, s_N) U$$

where $U$ is orthogonal, which we take random with uniform distribution.

Let $A$ act on a fixed unit vector $u$. One gets a distribution for $T u$. Note that for a fixed orthogonal $V$ which fixes $u$, replacing the random $U$ by $UV^{-1}$ does not change the distribution, but replaces $T u$ by $VT u$. Therefore the distribution of $T u$ is invariant under action by any orthogonal $V$ which fixes $u$.

This means that that distribution will be determined if we know the distribution of $\|Tu\|$ and $\cos \angle(Tu, u) = \langle Tu, u \rangle/\|Tu\|\|u\|$.

As for $\|Tu\|$, it is $\|USUu\|$ where $S := \text{diag}(s_1, s_2, \ldots, s_N)$, $Uu$ uniformly distributed on the unit sphere.

By Section 2 that would be almost as $S$ applied to $(1/\sqrt{N})x$, $x$ with coordinates i.i.d. $\sim \mathcal{N}$, which is, of course, a vector with independent coordinates but the $j$-th coordinate distributed as $(1/\sqrt{N})s_j$ times $\mathcal{N}$.

And, similarly to what we had in Section 2 the square of the norm of $S \cdot (1/\sqrt{N})x$, which is $(1/N) \sum_{j=1}^{N} s_j^2 x_j^2$ has mean

$$(1/N) \sum_{j=1}^{N} s_j^2 = \left( \| (s_1, s_2, \ldots, s_N) \|_2^{(s)} \right)^2,$$

around which it is concentrated – its standard deviation being

$$\sigma_0 \cdot \sqrt{1/N^2} \sum_{j=1}^{N} s_j^4 = (1/\sqrt{N})\sigma_0 \cdot \left( \| (s_1, s_2, \ldots, s_N) \|_4^{(s)} \right)^2,$$

where $\sigma_0$ is the standard deviation for $x^2$ when $x \sim \mathcal{N}$, namely,

$$\sigma_0 = \sqrt{1/2\pi} \int (x^2 - 1)^2 \exp(-\frac{1}{2}x^2) \, dx = \sqrt{2}.$$
By the relative deviation is, thus, expected, with almost full probability, to be $O(1/\sqrt{N})$.

Note that since $A = U^{-1} \cdot \text{diag}(s_1, s_2, \ldots, s_N) \cdot U$, the value around which that norm $\|Tu\|$ is concentrated would be

$$\|(s_1, s_2, \ldots, s_N)\|_2^\pi = (1/\sqrt{N})\|S\|_{HS} = (1/\sqrt{N})\|A\|_{HS}. \quad (18)$$

As for $\langle Tu, u \rangle$, it is $\langle SUu, Uu \rangle$, $Uu$ distributed uniformly. And, as we did above, replace $Uu$ by $(1/\sqrt{N})x$, $x$ with coordinates i.i.d. $\sim \mathcal{N}$,

$$\langle Sx, x \rangle = (1/\sqrt{N})\sum_{j=1}^{N} s_j x_j^2, \quad (19)$$

which has mean $(1/N)\sum_{j=1}^{N} s_j = (1/N)\text{tr} A$ and $(1/\sqrt{N})\sigma\|(s_1, \ldots, s_N)\|_2^\pi$ is its standard deviation. Of course, if $A$ is positive semidefinite then the $s_\ell \geq 0$ and the above mean is $\|(s_1, s_2, \ldots, s_N)\|_1^\pi$. This leads to the following conclusion.

**Conclusion 4** An $N \times N$ symmetric matrix $A$ with given eigenvalues $s_1, s_2, \ldots, s_N$, acting on a high-dimensional $E^N$, would be expected to multiply the norm of a fixed vector $u$, with almost full probability, by

$$\|(s_1, s_2, \ldots, s_N)\|_2^\pi = (1/\sqrt{N})\|A\|_{HS}, \quad (20)$$

up to a relative deviation $O(1/\sqrt{N})$, while the cosine of the angle between $v$ and $Tv$ is, with almost full probability, near (with deviation $O(1/\sqrt{N})$)

$$\frac{(1/N)\text{tr} A}{(1/\sqrt{N})\|A\|_{HS}} = \frac{(1/N)\text{tr} A}{\|(s_1, s_2, \ldots, s_N)\|_2^\pi}, \quad (21)$$

which, if $A$ is positive-semidefinite, is equal to

$$\frac{\|(s_1, s_2, \ldots, s_N)\|_1^\pi}{\|(s_1, s_2, \ldots, s_N)\|_2^\pi}. \quad (22)$$

Otherwise the distribution of $Au$ is invariant w.r.t. rotations in the hyper-plane orthogonal to $u$.

For a product $A_M A_{M-1} \cdots A_1$, of a sequence of symmetric operators $A_i$ with given eigenvalues $(s_1, s_2, \ldots, s_N)$ $A = U' \cdot \text{diag}(s_1, s_2, \ldots, s_N)U$ with the $U$ independently uniformly distributed on the orthogonal group, employ the Markov chain in §4 with independent uniform distributions on the spheres, and one has by Conclusion 3.

**Conclusion 5** For a product $A_M A_{M-1} \cdots A_1$, of a sequence of symmetric operators $A_i$ with given eigenvalues $(s_1, s_2, \ldots, s_N)$, the cosine of the angle
between $v$ and $A_M A_{M-1} \cdots A_1 v$, is, with almost full probability, near (with deviation $O(\sqrt{M}/\sqrt{N})$)

$$\prod_{i=1}^{M} \frac{(1/N) \text{tr} A_i}{(1/\sqrt{N}) \|A_i\|_{HS}} = \prod_{i=1}^{M} \frac{(1/N) \text{tr} A_i}{\|\langle s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)} \rangle\|_{2}},$$

which, if for all $i$, $A_i$ is positive semidefinite, is equal to

$$\prod_{i=1}^{M} \frac{\|\langle s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)} \rangle\|_{(\pi)}}{\|\langle s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)} \rangle\|_{2}},$$

(24)

**Remark 6** Note that if the $A_i$ are positive semidefinite, the value (24) around which the square of the cosine is concentrated, is nonnegative, that is, the angle between the vectors is $\leq 90^\circ$.

8 Recap: The ‘Kappa’-Calculations in the Superiorization Article [3]—Somewhat ‘Neater’ Formulas

For the accumulated terms $(1 + d \cdot \kappa)^{-1}$ along the path (in what follows we denote by $k$ indices along the path, i.e., $k \in \text{path}$) denote the eigenvalues (here, also the singular values) of the encountered $\kappa_k$ (the curvature operator in the hyperplane $H$), i.e., the relevant principal curvatures, by $(\kappa_{\ell}^{(k)})$, for $\ell = 1, 2, \ldots, N - 1$. Then those of $(1 + d \cdot \kappa)^{-1}$ are $\left((1 + d_k \cdot \kappa_{\ell}^{(k)})^{-1}\right)$, so that, by Conclusion 4 and using the $\| \cdot \|_\pi$ norm of Appendix A below, for $(N - 1)$-dimensional vectors, their product is expected to multiply the norm of the vector they act upon by

$$\prod_{k \in \text{path}} \left\|\left((1 + d_k \cdot \kappa_{\ell}^{(k)})^{-1}\right)\right\|_{2}^{\text{(\pi)}},$$

(25)

still with relative deviation of the order of at most $O(\sqrt{n - i}/\sqrt{N})$.

By Conclusion 5 they are expected to rotate the direction of the vector, i.e., shift the normalized vector, by an angle with cosine

$$\prod_{k \in \text{path}} \left\|\left((1 + d_k \cdot \kappa_{\ell}^{(k)})^{-1}\right)\right\|_{1}^{\text{(\pi)}},$$

(26)

with relative deviation of the order of at most $O(\sqrt{n - i}/\sqrt{N})$. 

13
Observe that \( \| \cdot \|_1^{(\pi)} \leq \| \cdot \|_2^{(\pi)} \) (cf. Appendix A) and, by Remark 6, the value of (22) is always \( \leq 1 \), meaning angle of rotation \( \leq 90^\circ \). Indeed, in many cases it will be much less than \( 90^\circ \). For example, for vectors \( ((1 + d_k \cdot \kappa_{\ell}^{(k)})^{-1}) \) with equal (resp. almost equal) entries (in our case – either ‘spherical’ curvature or when the \( d_k \cdot \kappa \) are small), the \( \| \cdot \|_2^{(\pi)} \) norm will be equal (resp. almost equal) to the \( \| \cdot \|_1^{(\pi)} \) norm, hence the terms in the product in (26) will be near 1.

Both (25) and (26) refer to the \((N - 1)\)-dimensional vectors \( v = ((1 + d_k \cdot \kappa_{\ell}^{(k)})^{-1})_\ell \), having entries in \((0, 1)\). In (25), which controls how much the norm was reduced, we have the product \( v \cdot k \) of \((1 + d_k \cdot \kappa_{\ell})\), having entries in \((0, 1)\), one has

\[
\| v \|_2^{(\pi)} = \frac{1}{N - 1} \sum_{\ell=1}^{N-1} v_\ell^2 = 1 - \frac{1}{N - 1} \sum_{\ell=1}^{N-1} (1 - v_\ell^2)
\]

\[
= 1 - \frac{1}{N - 1} \sum_{\ell=1}^{N-1} (1 - v_\ell)(1 + v_\ell) \geq 1 - 2 \frac{1}{N - 1} \sum_{\ell=1}^{N-1} (1 - v_\ell)
\]

\[
= 2 \frac{1}{N - 1} \sum_{\ell=1}^{N-1} v_\ell - 1 = 2\| v \|_1^{(\pi)} - 1.
\]

Hence, \( \| v \|_1^{(\pi)} \leq \frac{1}{2} (\| v \|_2^{(\pi)} + 1) \), which completes the proof. ■

As a consequence of this proposition we have,

\[
\frac{\| v \|_1^{(\pi)}}{\| v \|_2^{(\pi)}} \geq \frac{(\| v \|_2^{(\pi)})^2}{\| v \|_2^{(\pi)} - \| v \|_2^{(\pi)}},
\]

\[
\frac{\| v \|_1^{(\pi)}}{\| v \|_2^{(\pi)}} \leq \frac{1}{2} \left( \| v \|_2^{(\pi)} + \frac{1}{\| v \|_2^{(\pi)}} \right).
\]

So, there is here a ‘balancing effect’ – if the angle of rotation becomes close to \( 90^\circ \) in (26), then the norm will be reduced considerably in (25). Thus, when \( i \) is such that \( d_i \) times a ‘typical’ curvature \( \kappa \) (loosely, the ratio between \( d \) and a ‘typical’ radius of the \( C_i \)) is still considerably larger than
1 (maybe while in the early columns of the superiorization matrix with \(i\) small), then, by (25), the cascade of DP will reduce the norm hugely, hence, anyway applying \(\nabla \phi\) then will give a negligible result.

On the other hand, when we reach a stage where \(d_i, d_{i+1}, \ldots, d_n\) are small, both the possible rotation and the distance traveled are controlled. But of course, then the decrease of the \(\beta_k\) should also be taken into account. For big \(i\), thus small \(\beta_i\), the contribution might again be negligible. This shows that the main contribution in (8) of [3] seems to come from intermediate terms.

As said above, the angle of rotation, both by the \(\alpha\) and by the \(\kappa\) seems to be controlled, as long as the number of steps \(n\) does not approach the vector space dimension \(N\). If conditions are imposed on the target function \(\phi\) then point (3) above could also be tackled, in view of the preceding paragraph, bringing our analysis closer to conclusion.

A The probability \(L^p\) norms of vectors

For a vector \(x \in E^N\), and \(1 \leq p < \infty\), denote by \(\| \cdot \|_p^{(\pi)}\) (\(\pi\) stands for ‘probability space’) its \(L^p\) norm when the set of indices \(\{1, 2, \ldots, N\}\) is made into a uniform probability space, giving each index a weight \(1/N\), namely

\[
\|x\|_p^{(\pi)} := \left(\frac{1}{N} \sum_{j=1}^{N} |x_j|^p \right)^{1/p}, \tag{28}
\]

see, e.g., [4]. As with any probability measure, always \(\| \cdot \|_p^{(\pi)}\) increases with \(p\).

For \(x_1, x_2, \ldots, x_N\) i.i.d. \(\sim \mathcal{N}\), \((\|x\|_p^{(\pi)})^p\) is an average: its expectation \(\mathbb{E}\) will be the same as the expectation of \(|x|^p\) for \(x\) a scalar distributed \(\sim \mathcal{N}\):

\[
\mathbb{E} [|x|^p] = \frac{1}{\sqrt{2\pi}} \int |x|^p \exp\left(-\frac{1}{2}x^2\right) dx, \tag{29}
\]

but its standard deviation will be \(1/\sqrt{N}\) that of \(|x|^p\) for a scalar \(\sim \mathcal{N}\):

\[
\frac{1}{\sqrt{N}} \frac{1}{\sqrt{2\pi}} \int (|x|^p - \mathbb{E} [|y|^p])^2 \exp\left(-\frac{1}{2}x^2\right) dx. \tag{30}
\]

Thus, \(\|x\|_p^{(\pi)}\) is highly concentrated around the, not depending on \(N\), \((\mathbb{E} [|x|^p])^{1/p}\) with degree of concentration \(O(1/\sqrt{N})\).

One may conclude, loosely speaking, that in any case, these \(\| \cdot \|_p^{(\pi)}\) norms, having not depending on \(N\) means, are expected to be \(O(1)\), for all \(N\).
B  The integrals over $S^{N-1} \subset \mathbb{R}^N$ of monomials of degree $\leq 4$ of $x_1, x_2, \ldots, x_N$

Clearly these are 0 if some power of some $x_i$ is odd. The integral of $x_i^2$ is $\frac{1}{N}$ \([\text{II}].\) So we are left with $\int x_i^4$ and $\int x_i^2 x_j^2, i \neq j$, both, by symmetry, the same for all relevant $i$ and $j$.

To calculate these, note that

$$1 = \int (x_1^2 + \cdots + x_N^2)^2 \, d\omega_{N-1} = N \int x_1^4 + N(N-1) \int x_i^2 x_j^2 \quad (*) .$$

A rotation by $45^\circ$ in the $x_1, x_2$ plane transforms $x_1 \to (1/\sqrt{2})(x_1 + x_2)$ and $x_2 \to (1/\sqrt{2})(x_1 - x_2)$,

hence $x_1 x_2 \to \frac{1}{2}(x_1^2 - x_2^2)$ thus $x_i^2 x_j^2 \to \frac{1}{4}(x_i^2 - x_j^2)^2$. Therefore

$$\int x_1^2 x_2^2 = \frac{1}{4} \int (x_1^2 - x_2^2)^2 = \frac{1}{4} \left( 2 \int x_1^4 - 2 \int x_i^2 x_j^2 \right) .$$

Consequently $\int x_1^4 = 3 \int x_1 x_2^2$. And finally, by $(*)$,

$$\int x_1^2 x_2^2 \, d\omega_{N-1} = \frac{1}{N(N+2)}, \quad \int x_1^4 \, d\omega_{N-1} = \frac{3}{N(N+2)} .$$

C  In the Hyperbolic Space

Having discussed the sphere, it would be illuminating to do the same for the hyperbolic space, proceeding with much analogy.

Model the $N - 1$-dimensional hyperbolic space as the subset $\mathbf{H}^{N-1}$, of an $N$-dimensional ‘Minkowski’ space with signature $(+, -, \cdots , -)$, consisting of ‘future’ vectors (in the sense of Relativity Theory) with norm 1, i.e. the half of the hyperboloid $x_1^2 - x_2^2 - \cdots - x_N^2 = 1$ that lies in the half-space $x_1 > 0$.

Denote by $\langle \cdot, \cdot \rangle_\mathbf{H}$ the ‘Minkowski’ inner product $\langle x, y \rangle_\mathbf{H} := x_1 y_1 - x_2 y_2 - \cdots - x_N y_N$, thus $\|x\|_\mathbf{H}^2 = \langle x, x \rangle_\mathbf{H} = x_1^2 - x_2^2 - \cdots - x_N^2$, while keeping the notation $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$.

The ‘distance’ between vectors $u, v \in \mathbf{H}^{N-1}$, i.e. the distance along a straight line – a geodesic, is the ‘hyperbolic arc’ $\xi$ defined by $\cosh \xi = \langle u, v \rangle_\mathbf{H}$.

And as above we wish to find what should we expect the accumulated hy-arc of a path made of $M$ steps of given hy-arcs, $\xi_1, \xi_2, \ldots, \xi_M$, to be.

Here again, for some fixed $\xi \geq 0$, the set of points in $\mathbf{H}^{N-1}$ of hy-arc $\xi$ from some fixed vector $u \in \mathbf{H}^{N-1}$ is an $(N-2)$-sphere $\subset \mathbf{H}^{N-1}$, $\Sigma(u, \xi)$ given by

$$\Sigma(u, \xi) := \cosh \xi \cdot u + \sinh \xi \cdot S^{N-2} \, u ,$$

(31)
where $S^{N-2}_u$ stands for the unit sphere in the hyperplane perpendicular to $u$. (on which the induced geometry is negative-squared-norm Euclidean, not hyperbolic!)

Again, one performs a Markov chain. Start from a point $u_0$ on $H^{N-1}$, move to a point $u_1 \in \Sigma(u_0, \xi_1)$ uniformly distributed there, then from that $u_1$ to a point $u_2 \in \Sigma(u_1, \xi_2)$ uniformly distributed there, and so on, until one ends with $u_M$. We would like to find the expected hyperbolic cosine of the hy-arc between $u_0$ and $u_M$, namely $\mathbb{E}[\langle u_M, u_0 \rangle_H]$.

And again, denote by $L_\xi$ the operator mapping a function $p$ on $S^{N-1}$ to the function whose value at a vector $u \in H^{N-1}$ is the average of $p$ on $\Sigma(u, \xi)$, then $L_\xi(p)$ evaluated at $u$ is the expectation of $p$ at the point to which $u$ moved in the $k$-th step above. Hence, in the above Markov chain, the expectation of $p(u_M)$ is

$$
\left( L_{\xi_M} L_{\xi_{M-1}} \cdots L_{\xi_1}(p) \right)(u_0).
$$

(32)

Thus, what we are interested in is

$$
\mathbb{E}[\langle u_M, u_0 \rangle_H] = L_{\xi_M} L_{\xi_{M-1}} \cdots L_{\xi_1}(\langle x, u_0 \rangle_H) \bigg|_{x = u_0}.
$$

(33)

So, let us calculate $L_\xi(p)$ for polynomials of degree $\leq 2$ like in (2).

**Care in working with an invariant trace**

Here we seek invariance w.r.t. the ‘Lorentz group’ – linear transformations that preserve the ‘Minkowski’ norm – so we must be careful in speaking about trace when referring to a quadratic form.

Indeed, let

$$
p(x) = \langle Q x, x \rangle + 2 \langle a, x \rangle_H + \gamma.
$$

An invariant trace $\text{tr}_H Q$ here would be a linear mapping from the bilinear forms to the scalars, which maps a bilinear form of rank 1

$$
(x, y) \mapsto \langle x, a \rangle_H \cdot \langle y, b \rangle_H
$$

(34)

to $\langle a, b \rangle_H$.

In trying to write (34) as $\langle Q x, y \rangle$, denote

$$
\tilde{x} := (x_2, x_3, \ldots, x_N) \in E^{N-1}.
$$

then $x = (x_1, \tilde{x})$, and $\langle x, y \rangle = x_1 y_1 - \langle \tilde{x}, \tilde{y} \rangle$. Our bilinear form (34) will be

$$
(x, y) \mapsto (a_1 x_1 - \langle \tilde{a}, \tilde{x} \rangle) \cdot (b_1 y_1 - \langle \tilde{a}, \tilde{y} \rangle) = \langle Q x, y \rangle
$$

where

$$
Q = \begin{pmatrix}
a_1 b_1 & -b_1 \tilde{a}' \\
-a_1 \tilde{b} & \tilde{b} \cdot \tilde{a}'
\end{pmatrix}.
$$

(35)
And our $\text{tr}_H$ should map it to
\[
\langle a, b \rangle_H = a_1 b_1 - \langle \tilde{a}, \tilde{b} \rangle
\]
Which mandates
\[
\text{tr}_H \left( \begin{array}{cc}
q_{11} & c' \\
d & Q_1
\end{array} \right) := q_{11} - \text{tr}Q_1.
\] (36)

Returning to calculating $\mathcal{L}_\xi(p)$ for
\[
p(x) = \langle Qx, x \rangle + 2\langle a, x \rangle_H + \gamma,
\]
i.e. its value at some $u \in H^{N-1}$, we may again, in performing the calculation, assume $u = (1, 0, \ldots, 0)$, and $Q$ as in (36).

Note that for our $u = (1, 0, \ldots, 0)$, $a_1 = \langle a, u \rangle_H$, $q_{11} = \langle Qu, u \rangle_H - \text{tr}_H Q$. And we have
\[
p(x) = \langle Qx, x \rangle + 2\langle a, x \rangle_H + \gamma
\]
\[
= q_{11}x_1^2 + \langle c + d, \tilde{x} \rangle + \langle Q_1 \tilde{x}, \tilde{x} \rangle + 2a_1 x_1 - 2\langle \tilde{a}, \tilde{x} \rangle + \gamma
\]
Hence, taking account of (31) for $u = (1, 0, \ldots, 0)$, and using (1),
\[
(\mathcal{L}_\xi p)(u) = (\mathcal{L}_\xi p)(1, 0, \ldots, 0) =
\]
\[
= \cosh^2 \xi \cdot q_{11} + \frac{1}{N-1} \sinh^2 \xi \cdot \text{tr}Q_1 + 2 \cosh \xi \cdot a_1 + \gamma =
\]
\[
= \cosh^2 \xi \cdot \langle Qu, u \rangle + \frac{1}{N-1} \sinh^2 \xi \cdot (\langle Qu, u - \text{tr}_H Q \rangle) + 2 \cosh \xi \cdot \langle a, u \rangle_H + \gamma
\]
\[
= \left( \cosh^2 \xi + \frac{1}{N-1} \sinh^2 \xi \right) \langle Qu, u \rangle - \frac{1}{N-1} \sinh^2 \xi \cdot \text{tr}_H Q + 2 \cosh \xi \cdot \langle a, u \rangle_H + \gamma.
\]
which, by symmetry, will hold for any $u \in H^{N-1}$.

We are interested, for some fixed $u \in H^{N-1}$, in
\[
p(x) = \langle u, x \rangle_H.
\] (37)
Then there is no $Q$ term, so one has
\[
(\mathcal{L}_\xi (\langle u, x \rangle_H))(u) = \cosh \xi \cdot \langle u, x \rangle_H.
\] (38)
Consequently,
\[
\mathbb{E} \left[ \langle u_M, u_0 \rangle_H \right] = \left( \mathcal{L}_{\xi_M} \mathcal{L}_{\xi_{M-1}} \cdots \mathcal{L}_{\xi_1} (\langle x, u_0 \rangle_H) \right) |_{x=u_0} =
\]
\[
= \prod_{i=1}^M \cosh \xi_i \cdot (\langle u_0, x \rangle_H) |_{x=u_0} = \prod_{i=1}^M \cosh \xi_i.
\]
And we also assess the standard deviation, which is
\[
\sigma = \sqrt{\left( \mathcal{L}_{\xi_M} \mathcal{L}_{\xi_{M-1}} \cdots \mathcal{L}_{\xi_1} (\langle u_0, x \rangle_H^2) \right) |_{x=u_0} - \left( \prod_{i=1}^M \cosh \xi_i \right)^2}.
\] (39)
Here $p(x)$ is of the form $\langle c, x \rangle_{\mathcal{H}}^2$, and by symmetry we assume $c = (c_1, 0, \ldots, 0)$ making $p(x) = c_1^2 x_1^2$. So there is only the $Q$ term with $\langle Q, x \rangle := c_1^2 x_1^2$. Then $\text{tr}_\mathcal{H} Q = c_1^2 = \|c\|_{\mathcal{H}}^2$, which by symmetry holds for general $c$. And we find

$$
\left( \mathcal{L}_\xi (\langle c, x \rangle_{\mathcal{H}}^2) \right)(u) = \left( \cosh^2 \xi + \frac{1}{N - 1} \sinh^2 \xi \right) \langle c, u \rangle_{\mathcal{H}}^2 - \frac{1}{N - 1} \sinh^2 \xi \cdot \|c\|_{\mathcal{H}}^2.
$$

Consequently, for $c = u_0$ (note $\|u_0\|_{\mathcal{H}}^2 = 1$),

$$
\sigma^2 = \left( \mathcal{L}_{\xi_M} \mathcal{L}_{\xi_{M-1}} \cdots \mathcal{L}_{\xi_1} \right) (\langle u_0, x \rangle^2) \Big|_{x = u_0} - \left( \prod_{i=1}^M \cosh \xi_i \right)^2 = 
$$

$$
= - \left( \prod_{i=1}^M \cosh \xi_i \right)^2 + \prod_{i=1}^M \left( \cosh^2 \xi_i + \frac{1}{N - 1} \sinh^2 \xi_i \right)
$$

$$
- \frac{1}{N - 1} \left[ \sinh^2 \xi_1 + \sinh^2 \xi_2 \left( \cosh^2 \xi_1 + \frac{1}{N - 1} \sinh^2 \xi_1 \right) \right.
$$

$$
+ \sinh^2 \xi_3 \left( \cosh^2 \xi_2 + \frac{1}{N - 1} \sinh^2 \xi_2 \right) \left( \cosh^2 \xi_1 + \frac{1}{N - 1} \sinh^2 \xi_1 \right)
$$

$$
+ \cdots + \sinh^2 \xi_M \cdot \prod_{i=1}^{M-1} \left( \cosh^2 \xi_i + \frac{1}{N - 1} \sinh^2 \xi_i \right),
$$

$$
= - \left( \prod_{i=1}^M \cosh \xi_i \right)^2 + \sum_{S \subseteq \{1, \ldots, M\}} \left( \frac{N}{N - 1} \right)^{\# S} \prod_{i \in S} \sinh^2 \xi_i
$$

$$
- \frac{1}{N - 1} \sum_{\emptyset \neq S \subseteq \{1, \ldots, M\}} \left( \frac{N}{N - 1} \right)^{\# S - 1} \prod_{i \in S} \sinh^2 \xi_i
$$

$$
= - \prod_{i=1}^M \left( 1 + \sinh^2 \xi_i \right) + 1 + \sum_{\emptyset \neq S \subseteq \{1, \ldots, M\}} \left( \frac{N}{N - 1} \right)^{\# S - 1} \prod_{i \in S} \sinh^2 \xi_i
$$

$$
= \sum_{S \subseteq \{1, \ldots, M\}, \# S \geq 2} \left( \frac{N}{N - 1} \right)^{\# S - 1} - 1 \prod_{i \in S} \sinh^2 \xi_i.
$$
Thus
\[ \sigma^2 \leq \sum_{S \subset \{1, \ldots, M\}} \left( \left( \frac{N}{N-1} \right)^{M-1} - 1 \right) \prod_{i \in S} \sinh^2 \xi_i \]
\[ = \left( \left( \frac{N}{N-1} \right)^{M-1} - 1 \right) \prod_{i=1}^{M} (1 + \sinh^2 \xi_i) = \left( \left( \frac{N}{N-1} \right)^{M-1} - 1 \right) \prod_{i=1}^{M} \cosh^2 \xi_i. \]

\[ \sigma \leq \sqrt{\left( \left( \frac{N}{N-1} \right)^{M-1} - 1 \right) \prod_{i=1}^{M} \cosh \xi_i}. \]

**Conclusion 8** In the hyperbolic space \( H^{N-1} \), the hyperbolic cosine of the ‘hyperbolic arc’ \( \xi \) made by \( M \) moves of given ‘hyperbolic arcs’ \( \xi_1, \xi_2, \ldots, \xi_M \), modeled by the above Markov chain, is with almost full probability, near

\[ \prod_{i=1}^{M} \cosh \xi_i. \]  \hspace{1cm} (40)

**With relative deviation** \( O \left( \sqrt{M}/\sqrt{N} \right) \).

**What does conclusion (8) tell us?** Firstly, as with the sphere \( \mathbb{S}^N \), (40) agrees with the ‘flat’ case \( \mathbb{R}^N \) when the \( \xi \)'s are small: here \( \cosh \xi \sim 1 + \frac{1}{2} \xi^2 \) and again multiplying these corresponds approximately to adding the \( \xi^2 \)'s.

But when the \( \xi \) are large, \( \cosh \xi \sim \frac{1}{2} e^\xi \) and multiplying these corresponds approximately to adding the \( \xi \)'s themselves, and subtracting \( \log_2 M \).

So, in the hyperbolic space, for large \( \xi \) ‘the distances combine as if they were all about on the same line’.

**References**

[1] E. Behrends, *Introduction to Markov Chains*, Springer, 2000.

[2] J. Bell, Trace class operators and Hilbert-Schmidt operators, Technical report, April 18, 2016, 26pp. Available on Semantic Scholar at https://www.semanticscholar.org/.

[3] Y. Censor, E. Levy, An analysis of the superiorization method via the principle of concentration of measure, *Applied Mathematics and Optimization*, accepted for publication, (2019).

It can be found as item [162] on:
http://math.haifa.ac.il/yair/censor-recent-pubs.html#bottom

The paper contains a brief introduction to the superiorization methodology and its history.

[4] D. Song and A. Gupta, \( L_p \)-norm uniform distribution, *Proceedings of the American Mathematical Society* 125, (1997), 595–601.