A stability analysis on unsteady mixed convection stagnation-point flow over a moving plate along the flow impingement direction

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Abstract. The unsteady mixed convection stagnation-point flow over a moving plate along the flow impingement direction is numerically studied. The governing partial differentiation equations are transformed into ordinary differential equations using a similarity transformation, and then solved numerically by a shooting technique method. Dual solutions are observed in a certain range of opposing flow and regarding on these numerical solutions, we prepared a stability analysis to identify which solution is stable between non-unique solutions by bvp4c solver in Matlab. Further we obtain numerical results which enable us to discuss the features of the respective solutions.

1. Introduction

Nowadays, the boundary layer flow and heat transfer induced by a continuously moving surface are one of favorable interests in many manufacturing and industrial processes, such as in wire drawing, extrusion, hot rolling, just to name a few. Sakiadis [1] was the first researcher to consider the moving surface over a boundary layer flow problem. However, we noticed that most of studies nowadays considered steady state problem, where the velocity and other properties, e.g. pressure, are not depending upon time at every point. Besides, most engineers considered steady state because of its behavior in easily control. Thus, the problem of unsteady flow should not be neglected and it is much more important, because all boundary layer problems that occur in real-world practices are depending on time. Devi et al. [2] studied the problem of an unsteady mixed convection stagnation point flow in stagnation region adjacent to a vertical surface. The unsteady mixed convection stagnation point flow with a magnetic field adjacent to a vertical plate had been extensively studied by Takhar et al. [3]. Zhong and Fang [4] had studied the problem of time-dependent velocity on unsteady plane and axisymmetric stagnation point flow of an incompressible viscous fluid. Meanwhile, Fang [5] investigated the unsteady forced convection stagnation point flow over a moving wall and he reported that the buoyancy effects
were incorporated with the governing equations.

This present work has been undertaken to consider a stability analysis as it is closely associated with numerical errors. The method of finding stability between dual solutions is important as it provides a way to determine which solution is stable. This analysis is still limited because of it is consider as a new concept but several studies of stability analysis had been reported excellently by Merkin [6] and Weidman et al. [7]. In this present study, we investigate a stability analysis on unsteady mixed convection stagnation-point flow over a plate moving along flow impingement direction as studied by Saleh et al. [8]. The governing partial differential equations are reduced into a set of ordinary differential equations, and the ODEs are numerically solved with bvp4c solver. Later, a stability analysis is performed to determine which solution is physically realizable and stable.

2. Mathematical formulation

Consider the unsteady mixed convection flow of an incompressible and viscous fluid in the vicinity of the stagnation-point flow. Under these conditions along with the Boussinesq approximation, the equations of governing problem are as follows, see Saleh et al. [8]:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + m \left( \frac{1}{x} \frac{\partial}{\partial x} - \frac{u}{x^2} \right) \right) + g \beta (T - T_{\infty})
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + m \frac{\partial v}{x} \right)
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \epsilon \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

where \(t\) is time, \(u\) and \(v\) are velocity components along \(x\) and \(y\) axes, respectively, \(T\) is fluid temperature, \(g\) is gravitational acceleration, \(m = 0\) is plane flow and \(m = 1\) is axisymmetric flow. The boundary conditions are:

\[
t < 0 : \quad u = 0, \quad v = v_w, \quad T = T_{\infty} \quad \text{for any } x, y
\]

\[
t \geq 0 : \quad u = 0, \quad v = v_w, \quad T = T_w \quad \text{at } y = 0
\]

\[
u \rightarrow \infty, \quad T \rightarrow T_{\infty} \quad \text{as } y \rightarrow \infty
\]

We now introduce the following similarity variables:

\[
\psi = \sqrt{\frac{cv}{1 - at}} x^{1 + m} f(\eta), \quad T = T_{\infty} + (T_w - T_{\infty}) \theta(\eta), \quad \eta = \sqrt{\frac{c}{v(1 - at)}} y
\]

where \(\psi\) is stream function and \(\eta\) is similarity variable. The relations of \(u\) and \(v\) can be introduced as \(u = \partial \psi / \partial y\) and \(v = -\partial \psi / \partial x\) which satisfies equation (1). The pressure term as in equation (3) and we also assume \(T_w\), are given by:

\[
-\frac{1}{\rho} \frac{\partial p}{\partial y} = \left( 1 + \frac{a}{c} \right) \frac{c^2 x}{(1 - at)^2}, \quad T_w = T_{\infty} + \frac{bx}{(1 - at)^2}
\]

respectively, where \(b\) is temperature characteristic, which \(b > 0\) describes heated surface and \(b < 0\) describes cooled surface. By substituting equation (6) into equations (2) and (4), we finally get:
\[
\frac{d^2 g}{d\eta^2} + (1 + m) g f' - \alpha \left( \frac{1}{2} (g + f')' \right) + \lambda \theta + 1 = 0
\] (8)

\[
\frac{1}{Pr} \frac{d^2 G}{d\eta^2} + (1 + m) f' \theta' - f \theta' - \alpha \left( \frac{1}{2} \eta \theta' + 2 \theta \right) = 0
\] (9)

alongside boundary conditions as follows:

\[
f(1) = \frac{a}{2 + 2m}, f'(1) = 0, \theta(1) = 1 \quad \text{at} \quad \eta = 0,
\]

\[
f'(\eta) \to 1, \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty
\] (10)

where \( \alpha = \frac{a}{c} \) is unsteadiness parameter, \( Pr = \frac{\nu}{\epsilon} \) is Prandtl number and \( \lambda \) is mixed convection parameter. It should be mentioning that \( \lambda = 0 \) corresponds to forced convection, \( \lambda < 0 \) is opposing flow (cooled surface) and \( \lambda > 0 \) is assisting flow (heated surface). This present work is only focusing on opposing flow.

### 3. Stability analysis

In this present work, we found that there are dual solutions exist for certain range of \( \lambda \) and it is interesting to say that we can performed a stability analysis to identify which solution is stable. Following Weidman et al. [7], we now introduced new dimensionless variable \( \tau \), where \( \tau \) initiates with an initial value problem and consistent with the question of which solutions will be physically realizable. With the introduction of \( \tau \) and equation (6), we have:

\[
\psi = \sqrt{\frac{cv}{1 - at}} y, \quad T = T_\infty + (T_w - T_\infty) \theta(\eta, \tau), \quad \eta = \sqrt{\frac{c}{v(1 - at)}} y,
\]

\[
\tau = \frac{ct}{1 - at}
\] (11)

Substituting equation (11) into equations (2) and (4) and linearizing, we finally get:

\[
\frac{\partial^3 f}{\partial \eta^3} + (1 + m) f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} - \alpha \left( \frac{1}{2} \eta \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial f}{\partial \eta} - 1 \right) + \lambda \theta + 1 - \frac{\partial^2 f}{\partial \eta \partial \tau} = 0
\] (12)

\[
\frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} + (1 + m) f \frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta} \theta - \alpha \left( \frac{1}{2} \eta \frac{\partial \theta}{\partial \eta} + 2 \theta \right) - \frac{\partial \theta}{\partial \tau} = 0
\] (13)

subject to the boundary conditions:

\[
f(1, \tau) = \frac{a}{2 + 2m}, \quad \frac{\partial f}{\partial \eta}(1, \tau) = 0, \quad \theta(1, \tau) = 1 \quad \text{at} \quad \eta = 0,
\]

\[
\frac{\partial f}{\partial \eta}(\eta, \tau) \to 1, \theta(\eta, \tau) \to 0 \quad \text{as} \quad \eta \to \infty
\] (14)

The stability of dual solutions is determined by adopting the analysis suggested by Weidman et al. [7]:

\[
f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} F(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma \tau} G(\eta, \tau)
\] (15)

where \( \gamma \) is unknown eigenvalue and it describes the growth or decay of the disturbance rate. The initial grow of disturbance will appears if the smallest eigenvalue is negative, and it can concluded as unstable, and if smallest eigenvalue is positive, there is initial decay and the flow is considered stable. Substituting equation (15) into equations (12) and (13), we finally get:
\[ F_0''' + (1 + m)f_0F_0'' + f_0''F_0 - F_0'(2f_0' - \gamma) - \alpha \left( \frac{1}{2} \eta F_0'' + F_0 + \frac{1}{2} \eta f_0'' + f_0' - 1 \right) + \frac{\lambda}{Pr} \left( G_0'' + \theta_0 + \frac{1}{1 + m} \left( f_0'G_0' + F_0'\theta_0 + f_0\theta_0' \right) - \frac{1}{2} \eta G_0' + \frac{1}{2} \eta \theta_0' + 2\theta_0 \right) \]

\[ \lambda(G_0 + \theta_0) + 1 = 0 \quad (16) \]

along with the boundary conditions:

\[ F_0(1) = 0, \quad F_0'(1) = 0, \quad G_0(1) = 0, \quad F_0'(\eta) \to 0, \quad G_0(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \quad (18) \]

The stability of steady flow solutions is determined by the smallest eigenvalue, \( \gamma \). The range of possible eigenvalues can be determined by relaxing the boundary conditions on \( F_0(\eta) \) or \( G_0(\eta) \) as suggested by Harris et al. [9].

4. Results and discussions

The ordinary differential equations in equations (8) and (9), alongside boundary conditions in equation (10) were solved numerically by shooting technique method in Maple software together with bvp4c function implemented in Matlab. Since there are dual solutions in certain range of parameters, we perform a stability analysis by finding the smallest eigenvalue, \( \gamma \). If the smallest eigenvalue shows negative, thus the disturbance is on an initial growth and the flow is considered unstable; while if the smallest eigenvalue shows positive, there is an initial decay and the flow is considered stable. In that case, the first solution is physically significant and stable while the second solution is physically insignificant and unstable. Table 1 and Table 2 show the smallest eigenvalue \( \gamma \) for selected values of \( \alpha \) and \( m \), respectively, which shows that the first solution is positive and the second solution is negative. From these tables, the first solution is physically significant and stable while the second solution is physically insignificant and unstable.

The values of skin friction coefficient, \( f''(1) \), and thermal gradient, \( -\theta'(1) \), are presented in Figure 1 and 2, respectively, for different values of \( \alpha \) against \( \lambda \) when \( m = 0 \) (plane flow). In addition, Figure 3 and 4 illustrated the values of \( f''(1) \) and \( -\theta'(1) \) for various values of \( \alpha \) and \( \lambda \) when \( m = 1 \) (axysimmetric flow). From these four figures, the analysis confirmed that there are dual solutions exist for certain range of \( \alpha \).

In order to investigate the effects of \( \alpha \) on velocity and temperature profiles, we choose a constant value of \( Pr, m \) and \( \lambda \). Figure 5 shows the velocity profiles, \( f'(\eta) \), while Figure 6 illustrates the temperature profiles, \( \theta(\eta) \), for various values of \( \alpha \) when \( Pr = 0.71, m = 1.0 \) (axisymmetric flow), and \( \lambda = -0.1 \). From these figures, it is clearly seen that the boundary layer thickness is decreasing in both solutions, while the temperature increases alongside \( \alpha \) enhancement.
Table 1. Smallest eigenvalue $\gamma$ for several values of $\alpha$ when $m = 0.0$ (plane flow).

| $\alpha$ | first solution | second solution |
|----------|----------------|-----------------|
| -0.1     | 0.051766       | -0.044217       |
| -0.2     | 0.078349       | -0.078007       |
| -0.3     | 0.098676       | -0.125390       |
| -0.4     | 0.125527       | -0.145502       |
| -0.5     | 0.171194       | -0.192003       |

Table 2. Smallest eigenvalue $\gamma$ for several values of $\alpha$ when $m = 1.0$ (axisymmetric flow).

| $\alpha$ | first solution | second solution |
|----------|----------------|-----------------|
| -0.1     | 0.069940       | -0.059894       |
| -0.2     | 0.096853       | -0.081004       |
| -0.3     | 0.131560       | -0.127886       |
| -0.4     | 0.173443       | -0.152327       |
| -0.5     | 0.208140       | -0.183351       |

Figure 1. Skin friction coefficient, $f''(1)$, for several values of $\alpha$ against $\lambda$ when $m = 0.0$ and $Pr = 0.71$.

Figure 2. Temperature gradient, $-\theta'(1)$, for several values of $\alpha$ against $\lambda$ when $m = 0.0$ and $Pr = 0.71$.

5. Conclusions
A stability analysis on unsteady mixed convection stagnation-point flow over a moving plate along the flow impingement direction has been analyzed and discussed in this study. Numerical solutions of momentum and energy equations are given and we found that the involving parameters significantly affected the flow field and temperature distribution. This can be seen as the enhancement values of $\lambda$ increases the range of velocity parameter, $\alpha$, and there exists dual solutions in certain range of unsteadiness parameter. This happens because the effect of $\alpha$ on viscous incompressible liquid is to suppress the velocity field. We then stimulated a stability analysis to analyze which solution is stable. It has been observed that the first solution initiated decay while the second solution initiated growth. From our observation, we concluded that the first solution is stable and physically realizable than the second solution.
Figure 3. Skin friction coefficient, $f''(1)$, for several values of $\alpha$ against $\lambda$ when $m = 1.0$ and $Pr = 0.71$.

Figure 4. Temperature gradient, $-\theta'(1)$, for several values of $\alpha$ against $\lambda$ when $m = 1.0$ and $Pr = 0.71$.

Figure 5. Velocity profiles, $f'(\eta)$, for several values of $\alpha$ when $m = 1.0$, $\lambda = -0.1$ and $Pr = 0.71$.

Figure 6. Temperature profiles, $\theta(\eta)$, for several values of $\alpha$ when $m = 1.0$, $\lambda = -0.1$ and $Pr = 0.71$.

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