NEW YAMABE-TYPE FLOW IN A COMPACT RIEMANNIAN MANIFOLD

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ABSTRACT. In this paper, we set up a new Yamabe type flow on a compact Riemannian manifold \((M,g)\) of dimension \(n \geq 3\). Let \(\psi(x)\) be any smooth function on \(M\). Let \(p = \frac{n+2}{n} + 2\) and \(c_n = \frac{(n-1)n}{n^2}\). We study the Yamabe-type flow \(u(t)\) satisfying
\[
  u_t = u^{1-p} (c_n \Delta u - \psi(x)u) + r(t)u, \text{ in } M \times (0,T), \ T > 0
\]
with
\[
  r(t) = \frac{\int_M (c_n |\nabla u|^2 + \psi(x)u^2)dv}{\int_M u^{p+1}},
\]
which preserves the \(L^{p+1}(M)\)-norm and we can show that for any initial metric \(u_0 > 0\), the flow exists globally. We also show that in some cases, the global solution converges to a smooth solution to the equation
\[
  c_n \Delta u - \psi(x)u + r(\infty)u^p = 0, \ \text{on } M
\]
and our result may be considered as a generalization of the result of T.Aubin, Proposition in p.131 in [1].

1. INTRODUCTION

Nonlocal evolution equations arise naturally from geometry. The most famous one is the normalized Ricci flow preserving the volume [15], introduced by R.Hamilton in 1982. The evolutions of planar curves preserving the length or area enclosed [14] [10] [19] are in this category. To solve the Yamabe problem from the viewpoint of evolution equation, Hamilton has also proposed the normalized Yamabe flow to approaching a Yamabe metric on a closed manifold. In this paper, we introduce a Yamabe type flow (which preserves the \(L^{2n/(n-2)}(M)\)-norm, see below for the definition) on a compact Riemannian manifold and study its global existence and convergence in some cases. We point out that some arguments in [25] and [6] about Yamabe flow can be used to handle such a general norm-preserving flow. We also notice that many heat flow methods may be introduced to functionals related to Yamabe problem on \((M,g)\) ([11] [5], [3], [18]).
We now introduce a new Yamabe-type flow on a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\). Let 
\[
p = \frac{n+2}{n-2}, \quad c_n = \frac{4(n-1)}{n-2},
\]
and let \(R(g)\) be the scalar curvature. Assume that \(\psi\) is a given smooth function in \(M\). The Yamabe-type flow \(u = u(t)\) is defined such that \(u(t)\) satisfies the evolution equation

\[
(1) \quad u_t = u^1 - p (c_n \Delta u - \psi(x) u) + r(t) u, \quad \text{in } M \times (0, T), \quad T > 0
\]

where

\[
r(t) = r_\psi(t) := \int_M (c_n |\nabla u|^2 + \psi(x) u^2) dv / \int_M u^{n+1}
\]

with initial data \(u(0) > 0\). The local in time solution of this problem \((1)\) is by now standard \([20]\) and can be obtained by the fixed point method or the method such as the implicit function theorem. This flow preserves the norm of the evolving function \(u(t)\),

\[
\int_M u(t)^{n+1} dv = \int_M u(0)^{n+1} dv
\]

which may be assumed to be one for simplicity. In fact, we have

\[
\frac{1}{p+1} \frac{d}{dt} \int_M u^{n+1} dv = \int_M u^n u_t
\]

\[
= \int_M u (c_n \Delta u - \psi(x) u) + r(t) \int_M u^{n+1} = 0.
\]

Hence,

\[
\int_M u(t)^{n+1} dv = \int_M u(0)^{n+1} dv = 1.
\]

Since, for \(u = u(t)\),

\[
r(t) = \int_M (c_n |\nabla u|^2 + \psi(x) u^2) dv, \quad \int_M u(t)^{n+1} = 1,
\]

we have

\[
\frac{1}{2} \frac{d}{dt} r(t) = \int_M c_n (\nabla u, \nabla u_t) + \psi(x) u u_t
\]

\[
= \int_M (-c_n \Delta u + \psi(x) u) u_t
\]

\[
= \int_M u^n (-\frac{u_t}{u} + r(t)) u_t
\]

\[
= - \int_M u^n \frac{u_t^2}{u} \leq 0.
\]

So, \(r(t)\) is non-increasing in \(t\) along the flow.

To understand this flow well, we introduce the pseudo-scalar curvature

\[
(2) \quad R_\psi = u^{-p} (-c_n \Delta u + \psi(x) u).
\]
Then the equation (1) can be written as

\[ u_t = u(-R^g + r(t)), \quad \text{in } M \times (0, T), \quad T > 0. \]

We remark that one may study the flow

\[ u_t = -R^g u \]

on any complete non-compact Riemannian manifold of dimension \( n \geq 0 \).

For the evolution problem (1) on the compact Riemannian manifold \((M, g)\) of dimension \( n \geq 0 \), we can show that for any initial data \( u(0) > 0 \), the flow exists globally.

**Theorem 1.** For any initial data \( u(0) > 0 \), the Yamabe-type flow \((u(t))\) to (1) above exists globally and

\[ r_\infty = \lim_{t \to \infty} r(t) = r(\infty) \]

exists.

One may give a proof of this result using similar arguments as in section 4 in [25] or as in [6], which is a local in natural argument for the solution \( u \) to Yamabe flow. Here, we prefer to give a direct proof to control the norm growth of pseudo-scalar curvature along the flow. In the case when \((M, g)\) is a closed surface, we may also introduce the \( \psi \)-Gauss flow flow. Let

\[ g(t) = e^{2u(t)} g, \quad \text{where } u(t) : M \to \mathbb{R} \text{ is a smooth function, and let } \psi : M \to \mathbb{R} \text{ be a given smooth function. The } \psi \text{-Gauss flow is defined by} \]

\[ e^{2u} u_t = \Delta u - \psi(x) + r(t) e^{2u}, \quad \text{in } M \times (0, T), \quad T > 0 \]

where

\[ r(t) := r_\psi(t) := \frac{\int_M K^\psi dv}{\int_M e^{2u}} \]

where \( K^\psi = e^{-2u}(-\Delta u + \psi(x)) \) with initial data \( u(0) > 0 \). By similar method, we know that there is a global flow for (4). Interesting questions are to find similar results to Chang-Yang [8], [9], and Ding-Liu [11]. Related Yamabe type flow with boundary data may also be studied.

We can also get the convergence result of the flow \((g(t))\) to (1) as in the Yamabe-scalar negative and zero cases. So we may define the Yamabe-type invariant below. Define, for \( p = \frac{n+2}{n-2} \), for \( u \in H^1(M); u \neq 0 \),

\[ E(u) = \frac{\int_M (c_n |\nabla u|^2 + \psi u^2) dv}{(\int_M |u|^{p+1} dv)^{2/(p+1)}} \]

and

\[ Y_\psi(M) = \inf \{ E(u); u \in H^1(M); u \neq 0 \} \]

which is called the Yamabe-type invariant of \( M \). We denote for \( M = S^n \) and \( \psi = n(n-1) \)

\[ Y(S^n) = \inf \{ E(u); u \in H^1(M); u \neq 0 \} \]

for the Yamabe constant on \( S^n \). Using Aubin’s argument (see p.131 in [1], see also [21] and [12]), we know that \( Y_\psi(M) \leq Y(S^n) \). In [2], Aubin proved
that if \( n \geq 4 \) and \( \psi(x) < R(g)(x) \) somewhere, then there is a minimizer for \( Y_\psi(M) \). Of course, one may use the argument of Brezis-Nirenberg [7] to know \( Y_\psi(M) < Y(S^n) \) provided \( \psi < n(n-1) \) for \( M^n = S^n \) with \( n \geq 4 \).

We define
\[
\lambda_1(\psi) = \inf_{\{u \in H^1(M) : u \neq 0\}} \frac{\int_M (c_n |\nabla u|^2 + \psi u^2) dv}{\int_M |u|^2 dv}.
\]
Then it is standard to know that there is a positive function \( u_1 \) such that
\[
\lambda_1(\psi) = \int_M (c_n |\nabla u_1|^2 + \psi u_1^2) dv, \quad \int_M u_1^2 dv = 1,
\]
and
\[-c_n \Delta u_1 + \psi(x)u_1 = \lambda_1(\psi)u_1, \quad \text{in} \ M.\] We remark that for \( \lambda_1(\psi) \geq 0 \), we have \( Y_\psi(M) \geq 0 \). In fact, for ant \( u \neq 0 \),
\[
\frac{\int_M (c_n |\nabla u|^2 + \psi u^2) dv}{\left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}} = \frac{\int_M (c_n |\nabla u|^2 + \psi u^2) dv}{\int_M u^2 dv} \frac{\int_M u^2 dv}{\left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}} \\
\geq \lambda_1(M) \frac{\int_M u_1^2 dv}{\left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}} \geq 0.
\]
For \( \lambda_1(\psi) < 0 \), we have \( Y_\psi(M) < 0 \). In fact, taking \( u = u_1 \) above, we have
\[
Y_\psi(M) \leq \frac{\int_M (c_n |\nabla u|^2 + \psi u^2) dv}{\int_M u^2 dv} \frac{\int_M u^2 dv}{\left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}} \\
= \lambda_1(M) \frac{\int_M u^2 dv}{\left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}} < 0.
\]

The relation between \( \lambda_1(M) \) and \( Y_\psi(M) \) can be given below. Since, by the Holder inequality, we have
\[
\int_M |u|^2 dv \leq (Vol(M))^\frac{p+1}{p} \left(\int_M |u|^{p+1} dv\right)^{2/(p+1)}.
\]
We then have
\[
Y_\psi(M) \leq (Vol(M))^{-\frac{p+1}{p}} \lambda_1(\psi),
\]
for \( \lambda_1(\psi) \geq 0 \) and
\[
Y_\psi(M) \geq (Vol(M))^{-\frac{p+1}{p}} \lambda_1(\psi),
\]
for \( \lambda_1(\psi) < 0 \). Note that if \( \psi(x) \leq 0 \) on \( M \), then \( \lambda_1(M) < 0 \). In fact, we may take \( u = 1/\sqrt{\text{vol}(M)} \). Then, \( \lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \psi dv < 0. \)

Our main result is below.

**Theorem 2.** Assume \( 0 < Y_\psi(M) < Y(S^n) \) and assume, for the initial metric \( g_0 = u_0^{4/(n-2)} g \) with \( u_0 > 0 \) on \( M \), \( E(u_0) \leq Y(S^n) \). Then along the
Yamabe-type flow \((u(t))\) to \([1]\), we have a convergent subsequence \(u(t_j) \to u_\infty > 0\), \(t_j \to \infty\), and \(u_\infty\) is a smooth function satisfies

\[-c_n \Delta u + \psi(x)u = r_\infty u^p, \text{ in } M, \int_M u^{p+1} = 1.\]

Our result may be considered as a generalization of Proposition in p.131 in \([1]\). We remark that with more detailed analysis (see \([4]\)), one may obtain similar result to Theorem 1.1 in \([6]\) and we leave this open for interested readers for pleasure. Whether the Yamabe-type invariant on \((M,g)\) can be achieved by some smooth function \(u > 0\) in \(M\), generally speaking, is still an open problem and may be discussed in latter chances.

Assume that \(\lambda_1(\psi) < 0\) and \(\psi(x) < 0\) on \(M\), we can show that the flow converges at time infinity.

**Theorem 3.** Assume that \(\lambda_1(\psi) < 0\) and \(\psi(x) < 0\) on \(M\). The Yamabe-type flow \((u(t))\) converges to a metric of constant pseudo-scalar curvature at \(t = \infty\).

We remark that the results may be extended to the case when \(p > 1\). In fact, assuming that \(\lambda_1(\psi) = 0\) and \(\psi(x) = 0\) on \(M\), we have the following result.

**Theorem 4.** Assume that \(\lambda_1(\psi) = 0\) and \(\psi(x) = 0\) on \(M\). Fix any \(p > 1\). The Yamabe-type flow \((u(t))\) satisfying

\[
\frac{u_t}{u} = u^{-p}c_n \Delta u + r(t), \text{ in } M \times (0,T), \ T > 0
\]

with

\[r(t) = \int_M c_n |\nabla u|^2 dv / \int_M u^{p+1}\]

exists globally and converges to a positive constant at \(t = \infty\).

Via a use of bubble analysis, we can handle more complicated case as in \([6]\), since the proof is lengthy, we prefer to present it elsewhere.

The plan of this note is below. The proof of Theorem \([1]\) will be given in section \([2]\) below. Using Struwe’s compactness result, we prove Theorem \([2]\) in section \([3]\) The proofs of Theorem \([3]\) and Theorem \([4]\) will be given in section \([4]\).

**2. Global existence of Yamabe-type flows**

We treat the difficulty case when \(\psi(x) > 0\) and \(r(0) > 0\) and the other case can be handled in the proof of Theorem \([3]\) below. We first establish the following result. Note that \(Y_\psi(M) > 0\) when \(r > 0\).

In short we denote by \(R = R_\psi\). Recall

\[r - R = \frac{u_t}{u}.\]

By \([2]\) we know that

\[-R = u^{-p}(c_n \Delta u - \psi(x)u).\]
Taking the time derivative on both sides, we have
\[
\frac{d}{dt}(-R) = -pu^{-p-1}u_t(c_n\Delta u - \psi(x)u) + u^{-p}(c_n\Delta u_t - \psi(x)u_t)
\]
\[
= p\frac{u_t}{u}R + u^{-p}(c_n\Delta[(r - R)u] - \psi(x)(r - R)u]
\]
\[
= p(r - R)R + u^{-p}(c_n\Delta[(r - R)u] - \psi(x)(r - R)u)]
\]
which can be written as
\[
\frac{\partial}{\partial t}R = u^{-p}(c_n\Delta[(r - r)u] - \psi(x)((r - r)u)] + p(R - r)R.
\]
Define
\[
L^u v = u^{-p}(c_n\Delta[vu] - \psi(x)[vu]),
\]
and then we have
\[
\frac{\partial}{\partial t}R = -L^u(r - R) + p(R - r)R.
\]
Then we may use the maximum principle to obtain

**Proposition 5.** We have the pseudo-scalar curvature lower bound

\[
\inf_M R_\psi(t) \geq \min\left\{ \inf_M R_\psi(0), 0 \right\}
\]
along the Yamabe-type flow.

**Proof.** Note that at the minimum point of \( R(t) \), we have Recall that
\[
\frac{\partial}{\partial t}R \geq u^{-p}((R - r)c_n\Delta u - \psi(x)((r - r)u)] + p(R - r)R.
\]
Recall that
\[
R = u^{-p}(-\Delta u + \psi(x)u).
\]
Then we have
\[
\frac{\partial}{\partial t}R \geq (R - r)(-R) + p(R - r)R = (p - 1)R(R - r).
\]
We may write it as
\[
\frac{\partial}{\partial t}\left(R \exp((p - 1) \int_0^t (r - R))\right) \geq 0,
\]
which implies that
\[
R \exp((p - 1) \int_0^t (r - R)) \geq \inf_M R(0).
\]
Then we have the conclusion. \( \square \)

Similarly, at the maximum point of \( R \), we have
\[
\frac{\partial}{\partial t}R \leq (R - r)(-R) + p(R - r)R = (p - 1)R(R - r),
\]
which implies that
\[ R \exp((p-1) \int_0^t (r - R)) \leq \sup_M R(0), \]
which is useful in the case when \( \sup_M R(0) \leq 0. \)

To get better estimate, we now choose \( \sigma \geq 1 \) such that
\[ (8) \quad \sigma = \max \left\{ \sup_M (1 - R(0)), 1 \right\}. \]

Then applying the maximum principle again, we have
\[ (9) \quad R \psi(t) + \sigma \geq 1 \text{ for all } t \geq 0. \]

At any finite time interval, we have Harnack inequality for the flow in the sense below.

**Lemma 6.**

\[ (10) \quad \sup_M u(t) \leq C(T) \]

and
\[ (11) \quad \inf_M u(t) \geq c(T) > 0. \]

**Proof.** Note that
\[ (12) \quad u^{-1} \frac{\partial}{\partial t} u(t) = - (R \psi(t) - r(t)) \leq (r(0) + \sigma) \]

Thus,
\[ (13) \quad \sup_M u(t) \leq C(T) \]

for \( t \in [0,T] \). Hence, for
\[ (14) \quad P := \psi(x) + \sigma \left( \sup_{0 \leq t \leq T} \frac{u(t)}{M} \right)^{p-1}, \]

we have
\[ -c_n \Delta u(t) + Pu(t) \]
\[ \geq -c_n \Delta u(t) + \psi(x)u(t) + \sigma u(t)^{\frac{n+2}{n-2}} \]
\[ = (R \psi(t) + \sigma) u(t)^p \geq 0 \]
for all \( t \in [0,T] \). Then using a Moser iteration argument (or by Cor. A.5 in [6]) we have
\[ \int_M u(t) \leq C(T) \inf_M u(t). \]

Then by the volume constrain condition, we have
\[ (16) \quad 1 \leq C(T) \inf_M u(t) \left( \sup_M u(t) \right)^p. \]
for all \( t \in [0, T] \). Since \( \sup_M u(t) \leq C(T) \), we get the conclusion.

By now it is standard to set the global existence of the flow. Using the result from [24] (see also [16]), we have the result below.

**Proposition 7.** For \( T > 0 \), there exist \( 0 < \alpha < 1 \) and a constant \( C(T) \) such that

\[
|u(x_1, t_1) - u(x_2, t_2)| \leq C(T) \left( (t_1 - t_2)^\alpha + d(x_1, x_2)^\alpha \right)
\]

for all \( x_1, x_2 \in M \) and any \( t_1, t_2 \in [0, T] \) with \( 0 < t_2 - t_1 < 1 \).

**Proof.** Set \( v = u^p \). The Yamabe type flow equation (1) on a compact sub-domain \( D \subset M \) can be written as the divergence form that

\[
\partial_t v = c_n \text{div}(v^{1/p-1}\nabla v) - p\psi(x)v^{1/p-1}v + pr(t)v.
\]

One can see that the structure conditions (2.1) and (1.2-1.3) in [24] are satisfied. Then we can invoke Theorem 4.2 in [24] to get the locally uniformly Holder estimate in \( D \) for solutions \( u \) up to the initial time \( t = 0 \). Namely, for any \( x_0 \in M, r > 0, \) and \( T > 0, \) there exists uniform positive constants \( \beta = \beta(B_r(x_0), \sup B_r(x_0)u(0)) \in (0, 1), \) and \( C(B_r(x_0), \beta, u(0), T) \) such that for any \( j \geq 1 \) with \( B_r(x_0) \subset D, \) there holds

\[
|u|_{C^{\beta, \beta/2}(B_{r/2}(x_0) \times [0, T_0])} \leq C(B_r(x_0), \sup B_r(x_0)u(0), T).
\]

As always, we have used the Holder spaces \( C^{\alpha/2}(B_{r/2}(x_0) \times [0, T]) \) in parabolic distance defined by \( g + dt^2 \). Using the covering argument we can extend the estimate above to whole parabolic region \( M \times [0, T] \). □

We now prove Theorem 1 which is the global existence result of the flow for any initial data.

**Proof.** Fix any \( T > 0 \). With the understanding of Proposition 7 we may use the standard regularity theory for parabolic equations (see [17], Theorem 5 on p. 64 or the book [16]) to conclude that all higher order derivatives of the solution \( u \) are uniformly bounded on every fixed time interval \([0, T]\). Hence, we can extend the flow beyond \( T \) and then the flow exists for all time. This then completes the proof of Theorem 1. □

3. P.S. Sequence of Yamabe-type flows and proof of Theorem 2

To understand the asymptotic behavior of the Yamabe type flow (11), we need some integral estimates about the scalar curvature \( R \). Then we consider the P.S. Sequence of the functional \( E(u) \). Note that once we have the P.S. sequence along the flow, we may invoke the by now standard argument, that is, Struwe’s compactness result [23] to get the partial compactness of the sequence.
Let $\tilde{R} = R - r$. Then
\[ \frac{1}{2} \frac{d}{dt} \tilde{R}^2 = \tilde{R} \dot{R} = \tilde{R} L^u(\tilde{R}) + pR \tilde{R}^2 - \dot{R} r. \]

Recall that for $g(t) = u \frac{4}{n-2} g$ for some fixed background metric $g$ and $p = \frac{n+2}{n-2}$, we have
\[ R_\psi(t) = u^{-p}(-c_n \Delta_{g_0} u + \psi(x) u). \]

Let $g(t) = u(t) \frac{4}{n-2} g$. The Yamabe-type flow (11) may be written as
\[ \partial_t u^p = c_n \Delta_{g_0} u - \psi(x) u + r_\psi(t) u^p = -R_\psi(t) u^p + r_\psi(t) u^p. \]
We may denote by $\Delta = \Delta_{g_0}$ for simplicity. Then
\[ \partial_t u^{p+1} = -(p+1) \tilde{R} u^{p+1}. \]

We now compute
\[ \partial_t \int_M [\tilde{R}^2 u^{p+1}] = \int_M [\tilde{R} L^u(\tilde{R}) + pR \tilde{R}^2 - \dot{R} r] u^{p+1} - (p+1) \tilde{R}^3 u^{p+1}. \]
Using $\int \tilde{R} u^{p+1} = 1$, we have
\[ \partial_t \int_M [\tilde{R}^2 u^{p+1}] = \int_M [c_n \tilde{R} u \Delta(\tilde{R} u) - \psi(\tilde{R} u)^2] - \int_M \tilde{R}^3 u^{p+1} + pr \tilde{R}^2 u^{p+1}, \]
which is
\[ = - \int_M [c_n |\nabla(\tilde{R} u)|^2 + \psi(\tilde{R} u)^2] - \int_M \tilde{R}^3 u^{p+1} + pr \int_M \tilde{R}^2 u^{p+1}. \]

Since $\psi \geq 0$,
\[ \partial_t \int_M [\tilde{R}^2 u^{p+1}] \leq - \int_M \tilde{R}^3 u^{p+1} + pr \int_M \tilde{R}^2 u^{p+1}. \]
Let
\[ f(t) = \int_M [\tilde{R}^2 u^{p+1}] := \int_M [\tilde{R}^2 u^{p+1}] dv_{g_0}. \]
By
\[ f(t) = \int_M [\tilde{R}^2 u^{p+1}] \leq \left( \int_M [\tilde{R}^3 u^{p+1}] \right)^{2/3}, \]
we know that
\[ f'(t) \leq -f(t)^{3/2} + pf(t). \]
Recall that
\[ \frac{d}{dt} r(t) = -\frac{n-2}{2} f(t). \]
Then for any $t > 0$,
\[ \int_t^\infty f(s) ds < \infty. \]
Hence,
\[ \lim_{t \to \infty} f(t) = 0. \]
By this and the differential inequality of $f$, we obtain that as $t \to \infty$, $f(t) \to 0$. Define
\[ r_\infty = \lim_{t \to \infty} r(t). \]
Then we have
\[ \lim_{t \to \infty} \int_M |R - r_\infty|^2 u^{p+1} dv_{g_0} = 0. \]

We now let \( t_k \to \infty \), \( u_k = u(t_k) \), and \( g_k = g(t_k) = u_k^{4/(n-2)} g_0 \). Then

\[ \int_M dv_{g_k} = \int_M u_k^{2n/(n-2)} dv_{g_0} = 1, \]
and
\[ \int_M |R_{g_k} - r_\infty|^{2n/(n+2)} dv_{g_k} \to 0, \]

that is, as \( t_k \to \infty \),

\[ \int_M |c_n \Delta u_k - \psi(x) u_k + r_\infty u_k^{\frac{n+2}{2}}|^{2n/(n+2)} dv_{g_0} \to 0. \]

We then apply Struwe’s compactness result [23] (see Theorem 3.1 in [13] for the detailed proof) to conclude the following result.

**Proposition 8.** Let \( u_k \) be as above with (19) and (20). After passing to a subsequence, we may find a non-negative integer \( m \), a non-negative smooth function \( u_\infty \) and a sequence of \( m \)-tuplets \( (x_{i,k}, \epsilon_{i,k}) \) with the following properties

(i) The limiting function \( u_\infty \) satisfies
\[ c_n \Delta u_\infty - \psi(x) u_\infty + r_\infty u_\infty^{\frac{n+2}{2}} = 0. \]

(ii) For \( i \neq j \), we have, as \( k \to \infty \),
\[ \frac{\epsilon_{j,k}}{\epsilon_{i,k}} + \epsilon_{i,k} + \frac{d(x_{i,k}, x_{j,k})^2}{\epsilon_{i,k} \epsilon_{i,k}} \to \infty \]

(iii) We have as \( t_k \to \infty \),
\[ \|u_k - u_\infty - \sum_{i=1}^m w_{(x_{i,k}, \epsilon_{i,k})}\|_{H^1(M)} \to 0, \]
where
\[ w_{(x_{i,k}, \epsilon_{i,k})}(z) = \eta_{x_{i,k}}(z) \left( \frac{4n(n-1)}{r_\infty} \right)^{n-2/4} \left( \epsilon_{i,k}^2 + d(x_{i,k}, z)^2 \right)^{-n/4} \]

with \( \eta_{x_{i,k}}(z) \) is the cut-off function defined inside of the ball of the radius \( 2\delta \) smaller than the injectivity radius on \( M \), namely,
\[ \eta_{x_{i,k}}(z) = \eta_{\delta}(exp_{x_{i,k}}^{-1}(z)) \]

where \( \eta_{\delta} \in C_0^\infty(B_0(2\delta)) \), \( B_0(2\delta) \subset \mathbb{R}^n \) the ball with center \( 0 \) and with radius \( 2\delta > 0 \).

As the consequence of Proposition 8 we know that

\[ r_\infty = (E(u_\infty)^{n/2} + mY(S^n)^{n/2})^{2/n}. \]

Once we have this result, we may easily prove Theorem 2.
Proof. First, we remark that $u_\infty$ is a smooth solution so that if it has a zero point in $M$, by the maximum principle we know that it is identically zero.

Second, by (21), along the flow (1) we have

$$Y_\psi(M) \leq r_\infty \leq r(t) < r(0) \leq Y(S^n).$$

Third, this then implies that $m = 0$. Then we have $u_\infty > 0$ on $M$, which is the limit of the flow with initial data $u_0$. Thus we have proved Theorem 2. □

4. Convergence part of Yamabe-type flows for negative or flat cases

In this section we prove Theorem 3. In the proof below, we may let $p > 1$ be any number and we assume $\lambda_1(\psi) < 0$.

We now prove Theorem 3.

Proof. Define the Yamabe-type quotient

$$Q(g(t)) = \frac{\int_M R_\psi(t)dv_t}{(\int_M u(t)^{p+1}dv)^{2/(p+1)}},$$

where $g(t) = u(t)^{p-1}g$ and we may let

$$dv_t = u(t)^{p+1}dv.$$

Along the Yamabe-type flow, we may assume that

$$\int_M dv_t = 1.$$

In this case we have

$$Q(g(t)) = r_\psi(t) = \int_M (c_n|\nabla u|^2 + \psi(x)u^2)dv / \int_M u^{p+1}dv,$$

i.e.,

$$Q(g(t)) = r_\psi(t) = \int_M (c_n|\nabla u|^2 + \psi(x)u^2)dv \geq \lambda_1(\psi) \int u^2dv.$$

Then by $\int u^2 \leq 1$, we know that

$$r_\psi(t) \geq \lambda_1(\psi)$$

for all $t > 0$.

Recall that $r_\psi(t)$ is decreasing in $t$. Hence, as expected, in case $\lambda_1(\psi) \leq 0$, it is not difficult to see that the factor $u(t)$ is uniformly bounded above and below and as $t \to \infty$, the flow converges to a metric of constant pseudo-scalar curvature. This will be done below.

Recall that $\psi(x) < 0$ on $M$. Then

$$\lambda_1(\psi) \leq r_\psi(t) \leq r_\psi(0).$$

Let $u_m(t) = \inf_M u(t)$. Then by (18) we have

$$\partial_t u_m^p \geq - \sup_M \psi(x)u_m + r_\psi(t)u_m^p.$$
Then

$$\partial_t u_m^p \geq - \sup_M \psi(x) u_m + \lambda_1(\psi) u_m^p.$$ 

This can be considered in two cases.

**Case 1.** If $$- \sup_M \psi(x) u_m + \lambda_1(\psi) u_m^p \geq 0$$, then

$$u_m^p(t) \geq u_m^p(0).$$

**Case 2.** If $$- \sup_M \psi(x) u_m + \lambda_1(\psi) u_m^p \leq 0$$, then we have

$$- \sup_M \psi(x) u_m \leq - \lambda_1(\psi) u_m^p.$$ 

and

$$u_m^{p-1}(t) \geq \sup_M \psi(x)/\lambda_1(\psi).$$

This implies that $$u_m(t)$$ is uniformly bounded away from zero that

$$u_m^{p-1}(t) \geq \min\{\sup_M \psi(x)/\lambda_1(\psi), u_m^{p-1}(0)\}. \quad (23)$$

Similarly for $$u_M(t) = \sup_M u(t)$$, we have

$$\partial_t u_M^p \leq - \inf_M \psi(x) u_M + r \psi(t) u_M^p, \quad (24)$$

and by $$r \psi(t) \leq r \psi(0)$$,

$$\partial_t u_M^p \leq - \inf_M \psi(x) u_M + r \psi(0) u_M^p.$$ 

We may write this differential inequality as

$$\frac{P}{P - 1} \partial_t u_M^{p-1} \leq - \inf_M \psi(x) + r \psi(0) u_M^{p-1}. \quad (22)$$

When $$- \inf_M \psi(x) + r \psi(0) u_M^{p-1} \leq 0$$, we have

$$\partial_t u_M^{p-1} \leq 0.$$ 

When $$- \inf_M \psi(x) + r \psi(0) u_M^{p-1} \geq 0$$, we have

$$u_M^{p-1} \leq \inf_M \psi(x)/r \psi(0).$$

Then we have

$$u_M^{p-1} \leq \max(\inf_M \psi(x)/r \psi(0), u_M^{p-1}(0)).$$

Then we can get the convergent of the flow $$g(t)$$ at $$t = \infty$$.

We claim that $$r \psi(t)$$ will eventually become negative, even if this may not be the case at the initial time. Assume that $$r \psi(t) \geq 0$$ for all time, then by (22) we have

$$\partial_t u_m^p \geq - \sup_M \psi(x) u_m > 0$$

and this implies that as $$t \to \infty$$,

$$u_m^p(t) \to \infty,$$

which is impossible since the volume of $$g(t)$$ is fixed. Hence, we may choose $$t_0 > 0$$ such that

$$r \psi(t_0) < 0.$$
and then
\[ r_\psi(t) \leq r_\psi(t_0) < 0 \]
for all \( t \geq t_0 \). By \( \text{(24)} \) we have for \( t \geq t_0 \),
\[ u_M^p(t) \leq \max\{u_M^p(t_0), \frac{1}{r_\psi(t_0)} \sup \psi(x)\}. \]
This together with \( \text{(23)} \) implies that \( u \) is uniformly bounded from above and away from zero. Then as in \( \text{(25)} \), we can show that \( g(t) \) converges smoothly at an exponential rate to a limit metric with negative pseudo-scalar curvature.

□

For comparison, we give an outline of the proof of global existence and convergence result, Theorem 4, in case for any \( p > 1 \) with \( \lambda_1(\psi) = 0 \) and \( \psi(x) = 0 \) on \( M \).

Proof. The global existence of the flow can be done as in the proof of Theorem 1. So we need only consider the convergence part. Note that in this case, we have \( r_\psi(t) \geq 0 \). If \( r_\psi(0) = 0 \), then we have \( r_\psi(t) = 0 \) for all \( t > 0 \) and \( r_\psi(t) = 0 \) for all time. Thus we may assume that \( r_\psi(0) > 0 \).

By \( \text{(22)} \) we have
\[ \frac{u_M^p(t)}{u_M^p(0)} \geq \exp(\int_0^t r_\psi(t)). \]
We need to get an uniform upper bound for \( u_M(t) = \sup_M u(t) \). Recall that
\[ \partial_t u_M^p \leq r_\psi(t) u_M^p. \]
We use the Gronwall inequality to get
\[ u_M^p(t)/u_M^p(0) \leq \exp(\int_0^t r_\psi(s)ds). \]
Then we have
\[ u_M^p(t)/u_M^p(0) \leq \exp(\int_0^t r_\psi(s)ds) \leq \frac{u_M^p(t)}{u_M^p(0)}. \]
This is the global Harnack inequality. Therefore, using \( \int_M u(t)^{p+1} dv = 1 \), we have the uniform smooth estimations for \( u \). Using \( \text{(25)} \) we know that \( r_\psi(t) \to 0 \) as \( t \to \infty \).

Multiplying \( \text{(18)} \) by \( u^{1-p} \Delta u \) and integrating, we have
\[ \frac{d}{dt} \int_M c_{n} |\nabla u|^2 dv + 2 \int_M c_{n} |\Delta u|^2 u^{1-p} dv = 2r_\psi(t) \int_M c_n |\nabla u|^2 dv \]
Using the inequality
\[ \int_M c_n |\Delta u|^2 dv \geq c \int_M |\nabla u|^2 dv \]
for some uniform constant $c > 0$ and the fact that $r_\psi(t) \to 0$ as $t \to \infty$ we have

\[
\int_M |\nabla u|^2 \, dv \leq Ce^{-ct}
\]

Integrating (27) in time variable we have for any $T > 0$,

\[
\int_T^\infty \int_M |\Delta u|^2 \, dv \leq Ce^{-cT}
\]

which says that

\[
\int_T^\infty \int_M R_\psi^2 \, dv \leq Ce^{-cT}
\]

Then for $t \in [T, T + 1)$,

\[
\int_M R_\psi^2 \, dv \leq Ce^{-ct}.
\]

Since $r_\psi(t)$ decreases we have

\[
r_\psi(t) \leq Ce^{-ct}
\]

for all $t > 0$. By (29) and Poincare inequality we know that $u^p$ converges to its average exponentially in the $L^2$ norm. It follows that $u(t)$ converges exponentially to some limit positive constant. □

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