A Polynomial Time Algorithm to Compute Geodesics in CAT(0) Cubical Complexes

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Abstract
This paper presents the first polynomial time algorithm to compute geodesics in a CAT(0) cubical complex in general dimension. The algorithm is a simple iterative method to update breakpoints of a path joining two points using Owen and Provan’s algorithm (IEEE/ACM Trans Comput Biol Bioinform 8(1):2–13, 2011) as a subroutine. Our algorithm is applicable to any path in any CAT(0) space in which geodesics between two close points can be computed, not limited to CAT(0) cubical complexes.

Keywords Geodesic · CAT(0) Space · Cubical complex

1 Introduction
Computing a shortest path in a polyhedral domain in Euclidean space is a fundamental and important algorithmic problem, which is intensively studied in computational geometry [18]. This problem is relatively easy to solve in the two-dimensional case; it can generally be reduced to a discrete graph searching problem where some combinatorial approaches can be applied. In three or more dimensions, however, the problem becomes much harder; it is not even discrete. In fact, it was proved by Canny and Reif [10] that the shortest path problem in a polyhedral domain is NP-hard. Mitchell and Sharir [19] have shown that the problem of finding a shortest obstacle-avoiding path is NP-hard even for the case of a region with obstacles that are disjoint axis-aligned boxes. On the other hand, there are some cases where one can obtain polynomial time complexity. For instance, it was shown by Sharir [27] that a shortest obstacle-avoiding path among \( k \) disjoint convex polyhedra having altogether \( n \) vertices, can be found in...
$n^{O(k)}$ time, which implies that this problem is polynomially solvable if $k$ is a small constant.

What determines the tractability of the shortest path problem in geometric domains? One promising answer to this challenging question is *global non-positive curvature*, or CAT(0) property [16]. CAT(0) spaces are metric spaces in which geodesic triangles are “not thicker” than those in the Euclidean plane, and enjoy various fascinating properties generalizing those in Euclidean and hyperbolic spaces. As Ghrist and LaValle [14] observed, no NP-hard example in [19] is a CAT(0) space. One of the significant properties of CAT(0) spaces is the uniqueness of geodesics: Every pair of points can be joined by a unique geodesic. Computational and algorithmic theory on CAT(0) spaces is itself a challenging research field [3].

One of fundamental and familiar CAT(0) spaces is a CAT(0) cubical complex. A cubical complex is a polyhedral complex where each cell is isometric to a unit cube of some dimension and the intersection of any two cells is empty or a single face. Gromov [16] gave a purely combinatorial characterization of cubical complexes of non-positive curvature as cubical complexes in which the link of each vertex is a flag simplicial complex. Chepoi [11] and Roller [25] established that the 1-skeletons of CAT(0) cubical complexes are exactly *median graphs*, i.e., graphs in which any three vertices admit a unique median vertex. It is also shown by Barthélémy and Constantin [7] that median graphs are exactly the domains of event structures [21]. These nice combinatorial characterizations are one of the main reasons why CAT(0) cubical complexes frequently appear in mathematics, for instance, in geometric group theory [25,26], metric graph theory [4], concurrency theory in computer science [21], theory of reconfigurable systems [1,15], and phylogenetics [8]. Median graphs have been used in phylogeny and human genetics [5].

There exist several polynomial time algorithms to find shortest paths in some CAT(0) cubical complexes. A noteworthy example is for a *tree space*, introduced by Billera et al. [8] as a continuous space of phylogenetic trees. This space is shown to be CAT(0), and consequently provides a powerful tool for comparing two phylogenetic trees through the unique geodesic. Owen and Provan [22,23] gave a polynomial time algorithm for finding geodesics in tree spaces, which was generalized by Miller et al. [17] to CAT(0) *orthant spaces*, i.e., complexes of Euclidean orthants that are CAT(0). Chepoi and Maftuleac [12] gave an efficient polynomial time algorithm to compute geodesics in a two-dimensional CAT(0) cubical complex. These meaningful polynomiality results naturally lead to a question: What about arbitrary CAT(0) cubical complexes?

Ardila et al. [2] gave a combinatorial description of CAT(0) cubical complexes, employing a poset endowed with an additional relation, called a *poset with inconsistent pairs (PIP)*. This can be viewed as a generalization of Birkhoff’s theorem that gives a compact representation of distributive lattices by posets. In fact, they showed that there is a bijection between CAT(0) cubical complexes and PIPs. (Through the above-mentioned equivalence between CAT(0) cubical complexes and median graphs, this can be viewed as a rediscovery of the result of Barthélémy and Constantin [7], who found a bijection between PIPs and pointed median graphs.) This relationship enables us to express an input CAT(0) cubical complex as a PIP: For a poset with inconsistent pairs $P$, the corresponding CAT(0) cubical complex $K_P$ is realized as a subcomplex of
the $|P|$-dimensional cube $[0, 1]^P$ in which the cells of $K_P$ are specified by structures of $P$. Adopting this embedding as an input, they provided the first algorithm to compute geodesics in an arbitrary CAT(0) cubical complex. Their algorithm is based on an iterative method to update a sequence of cubes that may contain the geodesic, where at each iteration it solves a touring problem using second order cone programming [24]. They also showed that the touring problem for general CAT(0) cubical complexes has intrinsic algebraic complexity, and geodesics can have breakpoints whose coordinates have nonsolvable Galois group. This implies that there is no exact simple formula for the geodesic and therefore in general, one can only obtain an approximate one. Unfortunately, even if the touring problem could be solved exactly, it is not known whether or not their algorithm is a polynomial one; that is, no polynomial time algorithm has been known for the shortest path problem in a CAT(0) cubical complex in general dimension.

**Main result.** In this paper, we present the first polynomial time algorithm to compute geodesics in a CAT(0) cubical complex in general dimension, answering the open question suggested by previous works; namely we show that:

Given a CAT(0) cubical complex $K$ represented by a poset with inconsistent pairs $P$ and two points $p, q$ in $K$, one can find a path joining $p$ and $q$ of length at most $d(p, q) + \epsilon$ in time polynomial in $|P|$ and $\log(1/\epsilon)$.

The algorithm is quite simple, not depending on any involved techniques such as semidefinite programming. To put it briefly, our algorithm first gives a polygonal path joining $p$ and $q$ with a fixed number ($n$, say) of breakpoints, and then iteratively updates the breakpoints of the path until it becomes a desired one. To update them, we compute the midpoints of the two close breakpoints by using Miller, Owen and Provan’s algorithm. The resulting number of iterations is bounded by a polynomial in $n$. Key tools that lead to this bound are linear algebraic techniques and the convexity of the metric of CAT(0) spaces, rather than inherent properties of cubical complexes. Due to its simplicity, our algorithm is applicable to any initial path in any CAT(0) space where geodesics between two close points can be found, not limited to CAT(0) cubical complexes. We believe that our result will be an important step toward developing computational geometry in CAT(0) spaces.

**Application.** A reconfigurable system [1,15] is a collection of states which change according to local and reversible moves that affect global positions of the system. Examples include robot motion planning, non-collision particles moving around a graph, and protein folding; see [15]. Abrams et al. [1,15] considered a continuous space of all possible positions of a reconfigurable system, called a state complex. Any state complex is a non-positively curved cubical complex [15], and it becomes CAT(0) in many situations. In the robotics literature, geodesics (in the $l_2$-metric) in the CAT(0) state complex correspond to the motion planning to get the robot from one position to another one with minimal power consumption. Our algorithm enables us to find such an optimal movement of the robot in polynomial time.

**Organization.** The rest of this paper is organized as follows. In Sect. 2, we devise an algorithm to compute geodesics in general CAT(0) spaces. In Sect. 3, we present a
polynomial time algorithm to compute geodesics in CAT(0) cubical complexes, using the result of Sect. 2. Sections 2.1, 3.1, 3.2, 3.3 and 3.4 are preliminary sections, where CAT(0) metric spaces, CAT(0) cubical complexes, median graphs, PIPs and CAT(0) orthant spaces are introduced.

2 Computing Geodesics in CAT(0) Spaces

In this section we devise an algorithm to compute geodesics in general CAT(0) spaces, not limited to CAT(0) cubical complexes.

2.1 CAT(0) Space

Let \((X, d)\) be a metric space. A geodesic joining two points \(x, y \in X\) is a map \(\gamma : [0, 1] \to X\) such that \(\gamma(0) = x, \gamma(1) = y\) and \(d(\gamma(s), \gamma(t)) = d(x, y) |s - t|\) for all \(s, t \in [0, 1]\). The image of \(\gamma\) is called a geodesic segment joining \(x\) and \(y\). A metric space \(X\) is called (uniquely) geodesic if every pair of points \(x, y \in X\) is joined by a (unique) geodesic.

For any triple of points \(x_1, x_2, x_3\) in a metric space \((X, d)\), there exists a triple of points \(\bar{x}_1, \bar{x}_2, \bar{x}_3\) in the Euclidean plane \(\mathbb{E}^2\) such that \(d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j)\) for \(i, j \in \{1, 2, 3\}\). The Euclidean triangle whose vertices are \(\bar{x}_1, \bar{x}_2\) and \(\bar{x}_3\) is called a comparison triangle for \(x_1, x_2, x_3\). (Note that such a triangle is unique up to isometry.) A geodesic metric space \((X, d)\) is called a CAT(0) space if for any \(x_1, x_2, x_3 \in X\) and any \(p\) belonging to a geodesic segment joining \(x_1\) and \(x_2\), the inequality \(d(x_3, p) \leq d_{\mathbb{E}^2}(\bar{x}_3, \bar{p})\) holds, where \(\bar{p}\) is the unique point in \(\mathbb{E}^2\) satisfying \(d(\bar{x}_i, \bar{p}) = d_{\mathbb{E}^2}(\bar{x}_i, p)\) for \(i = 1, 2\). See Fig. 1.

This simple definition yields various significant properties of CAT(0) spaces; see [9] for details. One of the most basic properties of CAT(0) spaces is the convexity of the metric. A geodesic metric space \((X, d)\) is said to be Busemann convex if for any two geodesics \(\alpha, \beta : [0, 1] \to X\), the function \(f : [0, 1] \to \mathbb{R}\) given by \(f(t) := d(\alpha(t), \beta(t))\) is convex, i.e., \(d(\alpha(t), \beta(t)) \leq (1 - t) d(\alpha(0), \beta(0)) + t d(\alpha(1), \beta(1))\) for any \(t \in [0, 1]\).

Lemma 2.1 [9, Prop. II.2.2] Every CAT(0) space is Busemann convex.

A Busemann convex space \(X\) is uniquely geodesic. Indeed, for any two geodesics \(\alpha, \beta : [0, 1] \to X\) with \(\alpha(0) = \beta(0)\) and \(\alpha(1) = \beta(1)\), one can easily see that \(\alpha\) and...
β coincide, since \(d(α(t), β(t)) ≤ (1−t) d(α(0), β(0)) + td(α(1), β(1)) = 0\) for all \(t ∈ [0, 1]\). This implies

**Theorem 2.2** [9, Prop. II.1.4] Every CAT(0) space is uniquely geodesic.

### 2.2 Algorithm

Let \(X\) be a CAT(0) space. We shall refer to an element \(x\) in the product space \(X^{n+1}\) as a chain, and write \(x_{i−1}\) to denote the \(i\)-th component of \(x\), i.e., \(x = (x_0, x_1, \ldots, x_n)\). For any chain \(x ∈ X^{n+1}\), we define the length of \(x\) by

\[\ell(x) := \sum_{i=0}^{n−1} d(x_i, x_{i+1}).\]

Suppose that we can compute geodesics in \(X\) locally; namely we are given an oracle to perform the following operation for some \(D > 0\):

**Oracle 1** Given two points \(p, q ∈ X\) with \(d(p, q) ≤ D\), compute the geodesic joining \(p\) and \(q\) in arbitrary precision.

Under this situation, we consider the problem of finding an approximate geodesic between two points \(p, q ∈ X\) when given some path between \(p\) and \(q\); namely we consider the following problem:

**Problem 2.3** Given two points \(p, q ∈ X\), a chain \(x ∈ X^{n+1}\) with \(x_0 = p\) and \(x_n = q\), and a positive parameter \(ε > 0\), find a chain \(y ∈ X^{n+1}\) such that \(y_0 = p\), \(y_n = q\) and \(\ell(y) ≤ d(p, q) + ε\).

Our algorithm for this problem is based on the iterative local modifications of a given chain \(x\). All we have to do is to compute an approximate midpoint of two points in \(X\). Since \(X\) is uniquely geodesic, every pair of points \(p, q ∈ X\) has a unique midpoint \(w\) satisfying \(2d(w, p) = 2d(w, q) = d(p, q)\). For a nonnegative real number \(δ ≥ 0\), a \(δ\)-midpoint of \(p\) and \(q\) is a point \(w′ ∈ X\) satisfying \(d(w′, w) ≤ δ\), where \(w\) is the midpoint of \(p\) and \(q\).

**Definition 2.4** (\(δ\)-halved chain) Let \(δ\) be a nonnegative real number and \(x ∈ X^{n+1}\) be a chain. A chain \(z ∈ X^{n+1}\) is called a \(δ\)-halved chain of \(x\) if it satisfies the following:

\[z_0 = x_n, z_n = x_0\] and \(z_{n−i}\) is a \(δ\)-midpoint of \(z_{n−i+1}\) and \(x_i\) for \(i = 1, 2, \ldots, n−1\).

For an integer \(k ≥ 0\), we say that \(x^{(k)}\) is a \(k\)-th \(δ\)-halved chain of \(x\) if there exists a sequence \(\{x^{(j)}\}_{j=0}^{k}\) of chains in \(X^{n+1}\) such that \(x^{(0)} = x\) and \(x^{(j)}\) is a \(δ\)-halved chain of \(x^{(j−1)}\) for \(j = 1, 2, \ldots, k\).

Our algorithm can be described as follows.

To put it briefly, the algorithm just finds a \(k\)-th \(δ\)-halved chain of a given chain \(x\) for some large \(k\) and small \(δ\); see Fig. 2 for an illustration. In the algorithm the local
Algorithm 1

**Input.** Two points $p, q \in X$, a chain $x \in X^{n+1}$ with $x_0 = p$ and $x_n = q$, and parameters $\epsilon > 0$, $\delta \geq 0$.

1. Set $x^{(0)} := x$.
2. For $j = 0, 1, 2, \ldots$, do the following:
   
   2-1. Set $z_0 := x^{(j)}_n$ and $z_n := x^{(j)}_0$.
   
   2-2. For $i = 1, 2, \ldots, n-1$, do the following:

   Compute a $\delta$-midpoint $w$ of $z_{n-i+1}$ and $x^{(j)}_i$, and set $z_{n-i} := w$. \hfill (2.1)

   2-3. Set $x^{(j+1)} := (z_0, z_1, \ldots, z_n)$.

The following theorem states that if we start with a sufficiently “dense” chain $x \in X^{n+1}$ and a sufficiently small $\delta$, then Algorithm 1 solves Problem 2.3, with $O(\text{poly}(n, \log(D/\epsilon)))$ calls of Oracle 1.

**Theorem 2.5** Let $p, q \in X$ be given two points, $x \in X^{n+1}$ be a given chain with $x_0 = p$ and $x_n = q$, and $\epsilon > 0$, $0 \leq \delta \leq \epsilon/(16n^3)$ be parameters.

(i) For $j \geq n^2 \log(4n \cdot \ell(x)/\epsilon)$, one has $\ell(x^{(j)}) \leq d(p, q) + \epsilon$.

(ii) If $\text{gap}(x) \leq D/2 - \epsilon$ for some $D > 0$, then for all $j \geq 0$ and for $i = 1, 2, \ldots, n-1$, one has $d(z_{n-i+1}, x^{(j)}_i) \leq D$ in (2.1).

In particular, for $\text{gap}(x) \leq D/2 - \epsilon$, one can find a chain $y \in X^{n+1}$ such that $y_0 = p$, $y_n = q$ and $\ell(y) \leq d(p, q) + \epsilon$, with $O(n^3 \log(nD/\epsilon))$ calls of Oracle 1.

Fig. 2 Illustration of Algorithm 1
Example 2.6 We give an example of CAT(0) spaces to which our algorithm is applicable. A $B_2$-complex is a two dimensional piecewise Euclidean complex in which each 2-cell is isomorphic to an isosceles right triangle with short side of length one [13]. A CAT(0) $B_2$-complex is called a folder complex [11]; see Fig. 3 for an example. One can show that for a folder complex $F$, computing the geodesic between two points $p, q \in F$ with $d(p, q) \leq 1$ can be reduced to an easy calculation on a subcomplex of $F$ having a few cells. This implies that our algorithm enables us to find geodesics between two points $p, q$ in a folder complex $F$ in time bounded by a polynomial in the size of $F$ if some initial path between $p$ and $q$ is given.

2.3 Analysis

For any chain $x \in X^{n+1}$, we define the reference chain $\hat{x} \in X^{n+1}$ of $x$ as follows: $\hat{x}_0 := x_0$ and $\hat{x}_i := \gamma((i + 1)/(n + 1))$ for $i = 1, 2, \ldots, n$, where $\gamma : [0, 1] \to X$ is the geodesic with $\gamma(0) = x_0$ and $\gamma(1) = x_n$. Reference chains are designed not to be equally spaced but to have a double gap in the beginning so that the analysis of the algorithm will be easier. Note that the reference chain $\hat{x}$ of $x$ is determined just by its end components $x_0, x_n$, and therefore for any chain $x$ and any even $\delta$-halved chain $x(2k)$ of $x$ their reference chains are identical: $\hat{x}(2k) = \hat{x}$. A key observation that leads to Theorem 2.5 is that: For any chain $x \in X^{n+1}$ and any $k$-th $\delta$-halved chain $x^{(k)}$ of $x$ with $k$ sufficiently large and $\delta$ sufficiently small, the “distance” between $x^{(k)}$ and its reference chain $\hat{x}^{(k)}$ is small enough for its length $\ell(x^{(k)})$ to approximate well $d(x_0, x_n)$; moreover, the value of such a $k$ can be bounded by a polynomial in $n$. The next lemma states this fact.

Lemma 2.7 Let $x \in X^{n+1}$. Any $k$-th $\delta$-halved chain $x^{(k)}$ of $x$ satisfies

$$d(x_i^{(k)}, \hat{x}_i^{(k)}) \leq \frac{5}{4} \ell(x) e^{-k/n^2} + 3n^2 \delta$$

for $i = 1, 2, \ldots, n - 1$, where $e$ is the base of the natural logarithm.

Proof Let $\{x^{(j)}\}_{j \geq 0}$ be a sequence of chains in $X^{n+1}$ such that $x^{(0)} = x$ and $x^{(j)}$ is a $\delta$-halved chain of $x^{(j-1)}$ for $j \geq 1$. Fix an integer $1 \leq i \leq n - 1$ and an integer $k \geq 0$. Note that by definition $x_i^{(k+1)}$ is a $\delta$-midpoint of $x_{i+1}^{(k+1)}$ and $x_{n-i}^{(k)}$ and that $\hat{x}_i^{(k+1)}$ is the
midpoint of $\hat{x}_{i+1}^{(k)}$ and $\hat{x}_{n-i}^{(k)}$. Hence, by Lemma 2.1 and the triangle inequality, we have

$$2d(x_{i+1}^{(k)}, \hat{x}_{i+1}^{(k)}) \leq 2d(w, \hat{x}_{i}^{(k)}) + 2\delta$$

$$\leq d(x_{i+1}^{(k)}, \hat{x}_{i+1}^{(k)}) + d(x_{n-i}^{(k)}, \hat{x}_{n-i}^{(k)}) + 2\delta,$$

(2.2)

where $w$ is the midpoint of $x_{i+1}^{(k)}$ and $x_{n-i}^{(k)}$. See Fig. 4 for intuition.

Let $v^{(k)}$ be a column vector of dimension $n-1$ whose $i$-th entry equals $d(x_{i}^{(k)}, \hat{x}_{i}^{(k)})$ for $i = 1, 2, \ldots, n-1$. Let $J$ be a square matrix of order $n-1$ whose $(i, j)$ entry equals 1 if $i + j = n$ and 0 otherwise. Let $K$ be a square matrix of order $n-1$ whose $(i, j)$ entry equals 1 if $j = i + 1$ and 0 otherwise. Then, by (2.2) we have

$$2v^{(k+1)} \leq K v^{(k+1)} + J v^{(k)} + 2\delta \mathbf{1}$$

(2.3)

for each $k \geq 0$, where $\mathbf{1}$ is a column vector with all entries equal to 1. Let $A_{n-1}$ be a square matrix of order $n-1$ whose $(i, j)$ entry equals $(1/2)^{n+1-i-j}$ if $i + j \leq n$ and 0 otherwise. Then one can easily verify that $(2I - K)A_{n-1} = I$, i.e., $(2I - K)^{-1} = A_{n-1}J$. Hence by (2.3) we have

$$v^{(k+1)} \leq A_{n-1}J^2 v^{(k)} + A_{n-1}J(2\delta \mathbf{1}) \leq A_{n-1}v^{(k)} + 2\delta \mathbf{1}$$

(2.4)

for each $k \geq 0$.

We show that

$$v^{(k)} \leq \left(\frac{5}{4} \ell(x) e^{-k/n^2} + 3n^2\delta\right) \mathbf{1}$$

(2.5)

for any integer $k \geq 0$. The inequality (2.4) inductively yields that $v^{(k)} \leq (A_{n-1})^k v^{(0)} + 2\delta(I + A_{n-1} + \cdots + (A_{n-1})^{k-1}) \mathbf{1} \leq \ell(x)(A_{n-1})^k \mathbf{1} + 2\delta(I - A_{n-1})^{-1} \mathbf{1}$. Here, the inequality $v^{(0)} \leq \ell(x) \mathbf{1}$ comes from the triangle inequality. Indeed, we have

$$d(x_i, \hat{x}_i) \leq \min\left\{d(x_0, \hat{x}_i) + \sum_{j=0}^{i-1} d(x_j, x_{j+1}), \ d(\hat{x}_i, x_n) + \sum_{j=i}^{n-1} d(x_j, x_{j+1})\right\}$$

$$\leq \frac{d(x_0, x_n) + \ell(x)}{2} \leq \ell(x)$$

Fig. 4 Illustration of the proof of Lemma 2.7. The chain $x^{(j)}$ is a $j$-th $\delta$-halved chain of $x$ for $j = k, k + 1$. 

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for \( i = 1, 2, \ldots, n-1 \). In Lemma 2.8 below, we prove \((I - A_{n-1})^{-1} \mathbf{1} \leq (5(n-1)^2/4) \mathbf{1}\) (for \( n - 1 \geq 2 \)). This yields that \((I - A_{n-1})^{-1} \mathbf{1} \leq (3/2)n^2 \mathbf{1}\) for \( n \geq 2 \). Also, we prove \((A_{n-1})^k \mathbf{1} \leq (5/4) e^{-k/(n-1)^2} \mathbf{1}\) (for \( n - 1 \geq 2 \)) in Lemma 2.8. This implies that \((A_{n-1})^k \mathbf{1} \leq (5/4) e^{-k/n^2} \mathbf{1}\) for \( n \geq 2 \). This proves (2.5) and therefore completes the proof of the lemma. \( \square \)

Let us now prove Theorem 2.5.

**Proof of Theorem 2.5** We may assume that \( n \geq 2 \). We first show (i). If \( \delta \leq \epsilon/(16n^3) \) and \( j \geq n^2 \log(4n \cdot \ell(x)/\epsilon) \), then by Lemma 2.7, any \( j \)-th \( \delta \)-halved chain \( x^{(j)} \) of \( x \) satisfies \( d(x_i^{(j)}, \hat{x}_i^{(j)}) \leq 5\epsilon/(16n) + 3\epsilon/(16n) = \epsilon/(2n) \) for \( i = 1, 2, \ldots, n-1 \). Hence one has

\[
d(x_i^{(j)}, x_{i+1}^{(j)}) \leq d(x_i^{(j)}, \hat{x}_i^{(j)}) + d(\hat{x}_i^{(j)}, \hat{x}_{i+1}^{(j)}) + d(\hat{x}_{i+1}^{(j)}, x_{i+1}^{(j)}) \leq d(\hat{x}_i^{(j)}, \hat{x}_{i+1}^{(j)}) + \frac{\epsilon}{n} \tag{2.6}
\]

for \( i = 0, 1, \ldots, n - 1 \). This implies that \( \ell(x^{(j)}) = \sum_{i=0}^{n-1} d(x_i^{(j)}, x_{i+1}^{(j)}) \leq \sum_{i=0}^{n-1} (d(\hat{x}_i^{(j)}, \hat{x}_{i+1}^{(j)}) + \epsilon/n) = d(x_0, x_n) + \epsilon = d(p, q) + \epsilon \), and therefore completes the proof of (i).

To prove (ii), we first show

\[
d(z_{n-i+1}^{(j)}, x_i^{(j)}) \leq \text{gap}(x^{(j)}) + 2\delta \quad (i = 1, 2, \ldots, n; \ j \geq 0), \tag{2.7}
\]

by induction on \( i \). The case \( i = 1 \) being trivial, suppose that \( i \geq 2 \). Since \( z_{n-i+1}^{(j)} \) is a \( \delta \)-midpoint of \( z_{n-i+2}^{(j)} \) and \( x_i^{(j)} \), the triangle inequality and the induction yield

\[
d(z_{n-i+1}^{(j)}, x_i^{(j)}) \leq \delta + d(z_{n-i+2}^{(j)}, x_{i-1}^{(j)})/2 + d(x_{i-1}^{(j)}, x_i^{(j)}) \leq \delta + (\text{gap}(x^{(j)})/2 + \delta) + \text{gap}(x^{(j)})/2 = \text{gap}(x^{(j)}) + 2\delta,
\]

which completes the induction. See Fig. 5 for intuition.

We claim that

\[
\text{gap}(x^{(j+1)}) \leq \text{gap}(x^{(j)}) + 4\delta \tag{2.8}
\]
for \( j \geq 0 \). The case \( i = n \) in (2.7) implies that \( d(z_1, z_0) = d(z_1, x_n^{(j)}) \leq \text{gap}(x^{(j)}) + 2\delta \); on the other hand, by the triangle inequality and (2.7), one has \( d(z_{n-i+1}, z_{n-i}) \leq d(z_{n-i+1}, x_i^{(j)})/2 + \delta \leq \text{gap}(x^{(j)})/2 + 2\delta \) for \( i = 1, 2, \ldots, n - 1 \). Thus, one has \( \text{gap}(x^{(j+1)}) \leq \max \{ \text{gap}(x^{(j)}) + 2\delta, 2(\text{gap}(x^{(j)})/2 + 2\delta) \} = \text{gap}(x^{(j)}) + 4\delta \). This proves (2.8).

The inequality (2.7) implies that in order to prove (ii) it suffices to show that \( \text{gap}(x^{(j)}) + 2\delta \leq D \) for all \( j \geq 0 \). Suppose that \( \delta \leq \epsilon/(16n^3) \). We consider two cases.

Case 1: \( j \leq n^2 \log(4n \cdot \ell(x)/\epsilon) \). Note that \( \ell(x) \leq n \cdot \text{gap}(x) \). Also notice that \( \text{gap}(x^{(j)}) \leq \text{gap}(x) + 4j\delta \) from (2.8). No matter how roughly one estimates an upper bound of \( 4j\delta \), one can get

\[
4j\delta \leq 4 \cdot \frac{\epsilon}{16n^2} \cdot n^2 \log \frac{4n^2 \cdot \text{gap}(x)}{\epsilon} = \frac{\epsilon}{4n} \left( \log \frac{\text{gap}(x)}{\epsilon} + 2 \log 2n \right) \leq \frac{\text{gap}(x) + 2\epsilon}{4ne} + \frac{\epsilon}{e},
\]

where the last inequality comes from the fact that \( \log t \leq t/e \) for any \( t > 0 \). It is easy to see that since \( \text{gap}(x) \leq D/2 - \epsilon \) and \( n \geq 2 \), we obtain \( \text{gap}(x^{(j)}) + 2\delta \leq \text{gap}(x) \cdot \text{gap}(x)/(4ne) + \epsilon/e + \epsilon/(8n^3) \leq D \).

Case 2: \( j \geq n^2 \log(4n \cdot \ell(x)/\epsilon) \). Note that \( d(x_0, x_n)/(n + 1) \leq \text{gap}(x)/2 \). Hence by (2.6) and the definition of the reference chains we have \( d(x_0^{(j)}, x_1^{(j)}) \leq \text{gap}(x) + \epsilon/n \) and \( d(x_i^{(j)}, x_{i+1}^{(j)}) \leq \text{gap}(x)/2 + \epsilon/n \) for \( i = 1, 2, \ldots, n - 1 \), which implies that

\[
\text{gap}(x^{(j)}) \leq \max \left\{ \text{gap}(x) + \frac{\epsilon}{n}, 2 \left( \frac{\text{gap}(x)}{2} + \frac{\epsilon}{n} \right) \right\} = \text{gap}(x) + \frac{2\epsilon}{n}.
\]

It is easy to see that since \( \text{gap}(x) \leq D/2 - \epsilon \) and \( n \geq 2 \), we obtain \( \text{gap}(x^{(j)}) + 2\delta \leq \text{gap}(x) + 2\epsilon/n + \epsilon/(8n^3) \leq D \). This completes the proof of (ii).

From (i) and (ii), we can show that if \( \text{gap}(x) \leq D/2 - \epsilon \), then one can find a chain \( y \in X^{n+1} \) satisfying \( y_0 = p, y_n = q \) and \( \ell(y) \leq d(p, q) + \epsilon \), with \( O(n^3 \log(nD/\epsilon)) \) oracle calls. Indeed, for \( k := \lceil n^2 \log(4n \cdot \ell(x)/\epsilon) \rceil \), one can find a \( k \)-th \( \delta \)-halved chain \( x^{(k)} \) of \( x \) with \( O(nk) = O(n^3 \log(nD/\epsilon)) \) oracle calls, from (ii); its length \( \ell(x^{(k)}) \) is at most \( d(p, q) + \epsilon \), from (i).

We end this section by showing the result used in the proof of Lemma 2.7. Let \( A_n \) be an \( n \times n \) matrix whose \( (i, j) \)-entry is defined by

\[
(A_n)_{ij} := \begin{cases} 
(1/2)^{n+2-i-j} & \text{if } i + j \leq n + 1, \\
0 & \text{otherwise}
\end{cases} \quad (2.9)
\]

for \( i, j = 1, 2, \ldots, n \). Since \( A_n \) is a nonnegative matrix, its spectral radius \( \rho(A_n) \) is at most the maximum row sum of \( A_n \), which immediately yields that \( \rho(A_n) \leq 1 - (1/2)^n \). This inequality, however, is not tight unless \( n = 1 \). In fact, one can obtain a more useful upper bound on \( \rho(A_n) \).

**Lemma 2.8** Let \( n > 1 \) be an integer, and let \( A_n \) be an \( n \times n \) matrix defined by (2.9). Then its spectral radius \( \rho(A_n) \) is at most \( 1 - 1/n^2 \). In addition, one has \( (I - A_n)^{-1} \leq (5n^2/4) \mathbf{1} \) and \( (A_n)^k \mathbf{1} \leq (5/4) e^{-k/n^2} \mathbf{1} \) for any integer \( k \geq 0 \).
Proof Let $u$ be a positive column vector of dimension $n$ whose $k$-th entry is defined by $u_k := k(n - k) + n^2$ for $k = 1, 2, \ldots, n$. By the Collatz–Wielandt inequality, in order to show $\rho(A_n) \leq 1 - 1/n^2$ it suffices to prove that $A_n u \leq (1 - 1/n^2) u$. The $k$-th entry of the vector $A_n u$ is

$$(A_n u)_k = \sum_{j=1}^{n+1-k} \frac{u_j}{2^{n+2-k-j}} = \frac{1}{2^{n+2-k}} \sum_{j=1}^{n+1-k} 2^j (-j^2 + nj + n^2).$$

Hence, using the general formulas

$$\sum_{j=1}^{m} j \cdot 2^j = 2 + 2^{m+1} (m - 1) \quad \text{and} \quad \sum_{j=1}^{m} j^2 \cdot 2^j = -6 + 2^{m+1} ((m - 1)^2 + 2),$$

we have

$$(A_n u)_k = u_k - 2 - \frac{n^2 - n - 3}{2^{n+1-k}}.$$ 

It is easy to see that for $n \geq 2$ and $1 \leq k \leq n$ one has

$$\frac{u_k}{n^2} = 1 + \frac{k(n - k)}{n^2} \leq \frac{5}{4} \leq \left(2 - \frac{1}{2^{n+1-k}}\right) + \frac{(n - 2)(n + 1)}{2^{n+1-k}},$$

which implies that

$$\frac{u_k}{n^2} \leq 2 + \frac{n^2 - n - 3}{2^{n+1-k}} \quad (k = 1, 2, \ldots, n).$$

This completes the proof of the inequality $A_n u \leq (1 - 1/n^2) u$.

Let us show the latter part of the lemma. Note that $\bar{1} \leq (1/n^2) u \leq (5/4) \bar{1}$. Since $(1/n^2) u \leq (I - A_n) u$ and $(I - A_n)^{-1}$ is a nonnegative matrix (as $\rho(A_n) < 1$), we have $(I - A_n)^{-1} \bar{1} \leq (1/n^2)(I - A_n)^{-1} u \leq u \leq (5n^2/4) \bar{1}$.

Since $A_n u \leq (1 - 1/n^2) u \leq e^{-1/n^2} u$, we have $(A_n)^k u \leq e^{-k/n^2} u$ for any integer $k \geq 0$. Hence, $(A_n)^k \bar{1} \leq (1/n^2)(A_n)^k u \leq (1/n^2) e^{-k/n^2} u \leq (5/4) e^{-k/n^2} \bar{1}$.

Remark 2.9 In proving Theorem 2.5, we utilized only the convexity of the metric of $X$. Hence our algorithm works even when $X$ is a Busemann convex space.

3 Computing Geodesics in CAT(0) Cubical Complexes

In this section we give an algorithm to compute geodesics in CAT(0) cubical complexes, with an aim of the result of the preceding section. In Sects. 3.1, 3.2, 3.3 and 3.4, we recall CAT(0) cubical complexes, median graphs, PIPs and CAT(0) orthant spaces. Section 3.5 is devoted to proving our main theorem.
3.1 CAT(0) Cubical Complex

A cubical complex $K$ is a polyhedral complex where each $k$-dimensional cell is isometric to the unit cube $[0, 1]^k$ and the intersection of any two cells is empty or a single face. The underlying graph of $K$ is the graph $G(K) = (V(K), E(K))$, where $V(K)$ denotes the set of vertices (0-dimensional faces) of $K$ and $E(K)$ denotes the set of edges (1-dimensional faces) of $K$.

A cubical complex $K$ has an intrinsic metric induced by the $l_2$-metric on each cell. For two points $p, q \in K$, a string in $K$ from $p$ to $q$ is a sequence of points $p = x_0, x_1, \ldots, x_m = q$ in $K$ such that for each $i = 0, 1, \ldots, m - 1$ there exists a cell $C_i$ containing $x_i$ and $x_{i+1}$, and its length is defined to be $\sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is measured inside $C_i$ by the $l_2$-metric. The distance between two points $p, q \in K$ is defined to be the infimum of the lengths of strings from $p$ to $q$.

Gromov [16] gave a combinatorial criterion which allows us to easily decide whether or not a cubical complex $K$ is nonpositively curved. The link of a vertex $v$ of $K$ is the abstract simplicial complex whose vertices are the edges of $K$ containing $v$ and where $k$ edges $e_1, \ldots, e_k$ span a simplex if and only if they are contained in a common $k$-dimensional cell of $K$. An abstract simplicial complex $L$ is called flag if any set $S$ of vertices is a simplex of $L$ whenever every 2-element subset of $S$ spans a simplex.

**Theorem 3.1** (Gromov [16]) A cubical complex $K$ is CAT(0) if and only if $K$ is simply connected and the link of each vertex is flag.

3.2 Median Graph

Let $G = (V, E)$ be a simple undirected graph. The distance $d_G(u, v)$ between two vertices $u$ and $v$ is the length (the number of edges) of a shortest path between $u$ and $v$. The interval $I_G(u, v)$ between $u$ and $v$ is the set of vertices $w \in V$ with $d_G(u, v) = d_G(u, w) + d_G(w, v)$. A vertex subset $U \subseteq V$ is said to be gated if for every vertex $v \in V$, there exists a unique vertex $v' \in U$, called the gate of $v$ in $U$, such that $v' \in I_G(u, v)$ for all $u \in U$. Every gated subset is convex, where a vertex subset $U \subseteq V$ is said to be convex if $I_G(u, v)$ is contained in $U$ for all $u, v \in U$.

A vertex subset $H \subseteq V$ is called a halfspace of $G$ if both $H$ and its complement $V \setminus H$ are convex. A graph $G$ is called a median graph if for all $u, v, w \in V$ the set $I_G(u, v) \cap I_G(v, w) \cap I_G(w, u)$ contains exactly one element, called the median of $u, v, w$. Median graphs are connected and bipartite. In median graphs $G$, every convex set $S$ of $G$ is gated. (Indeed, for each $v \in V$ one can take a vertex $v' \in S$ such that $I_G(v', v) \cap S = \{v'\}$. Then for any $u \in S$ the median $m$ of $u, v, v'$ should be $v'$, as $m \in I_G(u, v') \subseteq S$ and $m \in I_G(v', v)$. This implies that $v'$ is the gate of $v$ in $S$.) Thus, in median graphs gated sets and convex sets coincide. A median complex is a cubical complex derived from a median graph $G$ by replacing all cube-subgraphs of $G$ by solid cubes. It has been shown independently by Chepoi [11] and Roller [25] that median complexes and CAT(0) cubical complexes constitute the same objects.
**Theorem 3.2** (Chepoi [11], Roller [25]) The underlying graph of every CAT(0) cubical complex is a median graph, and conversely, every median complex is a CAT(0) cubical complex.

For a cubical complex $\mathcal{K}$ and any $S \subseteq V(\mathcal{K})$, we denote by $\mathcal{K}(S)$ the subcomplex of $\mathcal{K}$ induced by $S$. The following property of CAT(0) cubical complexes is particularly important for us.

**Theorem 3.3** [12, Prop. 1] Let $\mathcal{K}$ be a CAT(0) cubical complex. For any convex set $S$ of the underlying graph $G(\mathcal{K})$, the subcomplex $\mathcal{K}(S)$ induced by $S$ is convex in $\mathcal{K}$.

### 3.3 Poset with Inconsistent Pairs (PIP)

Barthélémy and Constantin [7] established a Birkhoff-type representation theorem for median semilattices, i.e., pointed median graphs, by employing a poset with an additional relation. This structure was rediscovered by Ardila et al. [2] in the context of CAT(0) cubical complexes. An antichain of a poset $P$ is a subset of $P$ that contains no two comparable elements. A subset $I$ of $P$ is called an order ideal of $P$ if $a \in I$ and $b \leq a$ imply $b \in I$. A poset $P$ is locally finite if every interval $[a, b] = \{c \in P | a \leq c \leq b\}$ is finite, and it has finite width if every antichain is finite.

**Definition 3.4** A poset with inconsistent pairs (or, briefly, a PIP) is a locally finite poset $P$ of finite width, endowed with a symmetric binary relation $\sim$ satisfying:

1. If $a \sim b$, then $a$ and $b$ are incomparable.
2. If $a \sim b$ and $a \preceq a'$, then $a' \sim b$.

A pair $\{a, b\}$ with $a \sim b$ is called an inconsistent pair. An order ideal of $P$ is called consistent if it contains no inconsistent pairs.

Conditions (1) and (2) are just the definition of event structures (with binary conflict) from concurrency theory.

For a CAT(0) cubical complex $\mathcal{K}$ and a vertex $v$ of $\mathcal{K}$, the pair $(\mathcal{K}, v)$ is called a rooted CAT(0) cubical complex. Given a poset with inconsistent pairs $P$, one can construct a cubical complex $\mathcal{K}_P$ as follows: The underlying graph $G(\mathcal{K}_P)$ is a graph $G_P$ whose vertices are consistent order ideals of $P$ and where two consistent order ideals $I, J$ are adjacent if and only if $|I \triangle J| = 1$; replace all cube-subgraphs (i.e., subgraphs isomorphic to cubes of some dimensions) of $G_P$ by solid cubes. See Fig. 6 for an example. In fact, the resulting cubical complex $\mathcal{K}_P$ is CAT(0), and moreover:

**Theorem 3.5** (Ardila et al. [2]) The map $P \mapsto \mathcal{K}_P$ is a bijection between posets with inconsistent pairs and rooted CAT(0) cubical complexes.

This bijection can also be derived from Theorem 3.2 and the result of Barthélémy and Constantin [7], who found a bijection between PIPs and pointed median graphs.

Given a poset with inconsistent pairs $P$, one can embed $\mathcal{K}_P$ into a unit cube in the Euclidean space as follows, which we call the standard embedding of $P$ [2]:

$$\mathcal{K}_P = \{(x_i)_{i \in P} \in [0, 1]^P \mid i < j \text{ and } x_i < 1 \Rightarrow x_j = 0, \text{ and } i \sim j \Rightarrow x_i x_j = 0\}.$$
Fig. 6 A poset with inconsistent pairs and the corresponding rooted CAT(0) cubical complex. Dotted lines represent minimal inconsistent pairs, where an inconsistent pair \( \{a, b\} \) is said to be minimal if there is no other inconsistent pair \( \{a', b'\} \) with \( a' \leq a \) and \( b' \leq b \).

For each pair \((I, M)\) of a consistent order ideal \( I \) of \( P \) and a subset \( M \subseteq I_{\text{max}} \), where \( I_{\text{max}} \) is the set of maximal elements of \( I \), the subspace

\[
C^I_M := \{ x \in K_P \mid i \in I \setminus M \Rightarrow x_i = 1, \text{ and } i \notin I \Rightarrow x_i = 0 \}
\]

corresponds to a unique \(|M|\)-dimensional cell of \( K_P \).

3.4 CAT(0) Orthant Space

Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers. Let \( L \) be an abstract simplicial complex on a finite set \( V \). The orthant space \( O(L) \) for \( L \) is a subspace of \(|V|\)-dimensional orthant \( \mathbb{R}_+^V \) constructed by taking the union of all subcones \( \{O_S \mid S \in L\} \) associated with simplices of \( L \), where \( O_S \) is defined by \( O_S := \mathbb{R}_+^S \times \{0\}^{V \setminus S} \) for each simplex \( S \in L \); namely, \( O(L) = \bigcup_{S \in L} \{x \in \mathbb{R}_+^V \mid x_v = 0 \text{ for each } v \notin S\} \), where \( x_v \) denotes the \( v \)-coordinate of \( x \). The distance between two points \( x, y \in O(L) \) is defined in a similar way as in the case of cubical complexes. An orthant space is a special instance of cubical complexes.

Theorem 3.6 (Gromov [16]) The orthant space \( O(L) \) for an abstract simplicial complex \( L \) is a CAT(0) space if and only if \( L \) is a flag complex.

A typical example of CAT(0) orthant spaces is a tree space [8]. Owen and Provan [22,23] gave a polynomial time algorithm to compute geodesics in tree spaces, which was generalized to CAT(0) orthant spaces by Miller et al. [17].

Theorem 3.7 [17,22,23] Let \( L \) be a flag abstract simplicial complex on a finite set \( V \) and \( O(L) \) be the CAT(0) orthant space for \( L \). Let \( x, y \in O(L) \), and let \( S_1 \) and \( S_2 \) be the inclusion-wise minimal simplices such that \( x \in O_{S_1} \) and \( y \in O_{S_2} \). Then one can find the explicit description of the geodesic joining \( x \) and \( y \) in \( O(|S_1| + |S_2|)^4) \) time.

An interesting thing about this algorithm is that it solves as a subproblem a combinatorial optimization problem: the Maximum Weight Stable Set problem on a bipartite
graph whose color classes have at most $|S_1|, |S_2|$ vertices, respectively. We should note that the above explicit descriptions of geodesics are radical expressions. Computationally, for a point $p$ on a geodesic, one can compute a rational point $p' \in O(L)$ such that $d(p', p) \leq \delta$ and the number of bits required for each coordinate of $p'$ is bounded by $O(\log(|V|/\delta))$.

For a CAT(0) orthant space $O(L)$ and a real number $r > 0$, we call $O(L)|_{[0,r]} := O(L) \cap [0, r]^V$ a truncated CAT(0) orthant space. As a consequence of Theorem 3.7, one obtains the following:

**Theorem 3.8 [17]** Let $L$ be a flag abstract simplicial complex on a finite set $V$ and $O(L)|_{[0,r]}$ be a truncated CAT(0) orthant space for $L$. Then for any two points $x, y \in O(L)|_{[0,r]}$, one can find the explicit description of the geodesic joining $x$ and $y$ in $O(|V|^4)$ time.

In fact, a truncated CAT(0) orthant space $O(L)|_{[0,r]}$ is a convex subspace of $O(L)$.

### 3.5 Main Theorem

We now consider the following problem. It should be remarked that as stated in [2] there are no simple formulas for the breakpoints in geodesics in CAT(0) cubical complexes due to their algebraic complexity, and hence one can only compute them approximately. Computationally, we adopt the standard embedding as an input CAT(0) cubical complex.

**Problem 3.9** Given a poset with inconsistent pairs $P$, two points $p, q$ in the standard embedding $K_P$ of $P$, and a positive parameter $\epsilon > 0$, find a sequence of points $p = x_0, x_1, \ldots, x_{n-1}, x_n = q$ in $K_P$ with $\sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq d(p, q) + \epsilon$ and compute the geodesic joining $x_i$ and $x_{i+1}$ for $i = 0, 1, \ldots, n - 1$.

Our main result is the following theorem.

**Theorem 3.10** Problem 3.9 can be solved in $O(|P|^7 \log(|P|/\epsilon))$ time. Moreover, the number of bits required for each coordinate of points in $K_P$ occurring throughout the algorithm can be bounded by $O(\log(|P|/\epsilon))$.

The remainder of the section is devoted to proving this theorem. Let $m$ denote the number of elements of $P$ and let $D < 1$ be a positive constant close to 1 (e.g., set $D := 0.9$). Theorem 2.5 implies that in order to prove Theorem 3.10 it suffices to show that:

(a) Given two points $p, q \in K_P$, one can find a sequence of points $p = x_0, x_1, \ldots, x_{n-1}, x_n = q$ in $K_P$ such that $n = O(m)$ and $d(x_i, x_{i+1}) \leq D/4 - \epsilon$ for $i = 0, 1, \ldots, n - 1$.

(b) Given two points $p, q \in K_P$ with $d(p, q) \leq D$, one can compute the geodesic joining $p$ and $q$ in $O(m^4)$ time and find a $\delta$-midpoint $w$ of $p$ and $q$ such that each coordinate of $w$ requires only $O(\log(m/\delta))$ bits.

It is relatively easy to show (a), by considering a curve $c(p, q)$ issuing at $p$, going through an edge geodesic (a shortest path in the underlying graph of $K_P$) between
some vertices of cells containing \( p, q \), and ending at \( q \). Since such a curve \( c(p, q) \) has length at most \( O(m) \), dividing it into parts appropriately, one can get a desired sequence of points. This proves (a).

**Remark 3.11** One can easily find an edge geodesic between two vertices \( u \) and \( v \) of \( K_P \) under the PIP representation. Reroot the complex \( K_P \) at \( u \). In other words, construct a poset \( P' \) for which \( K_{P'} \cong K_P \) and \( u \) is the root of \( K_{P'} \); this construction is implicitly stated in [2]. Then the edge geodesic in \( K_{P'} \) from the root \( u = \emptyset \) to \( v = I \), where \( I \) is a consistent order ideal of \( P' \), can be found by considering a linear extension of the elements of \( I \).

To show (b), we need the following two results.

**Lemma 3.12** Let \( K \) be a CAT(0) cubical complex and \( v \) be a vertex of \( K \). Then the star \( St(v, K) \) of \( v \) in \( K \), i.e., the subcomplex spanned by all cells containing \( v \), is convex in \( K \).

**Proof** Let \( G^\Delta \) be the graph having the same vertex set as \( G = G(K) \), where two vertices are adjacent if and only if they belong to a common cube of \( G \). It is well known that every ball \( B(u, r) := \{ u' \in V(G^\Delta) \mid d_{G^\Delta}(u, u') \leq r \} \) of \( G^\Delta \) is a convex set in a median graph \( G \); see, e.g., [6, Prop. 2.6]. In particular, the ball \( B(v, 1) \) of \( G^\Delta \), which coincides with the vertex set of \( St(v, K) \), is convex in \( G \). Hence, by Theorem 3.3, \( St(v, K) \) is convex in \( K \) in the \( \ell_2 \)-metric. \( \Box \)

**Lemma 3.13** Let \( K \) be a CAT(0) cubical complex. Let \( p, q \) be two points in \( K \) with \( d(p, q) < 1 \) and \( R_1, R_2 \) be the minimal cells of \( K \) containing \( p, q \), respectively. Then \( R_1 \cap R_2 \neq \emptyset \).

We give a proof of Lemma 3.13 in Sect. 3.6. Using these lemmas, we show (b). Suppose that we are given two points \( p, q \in K_P \) with \( d(p, q) \leq D \). First notice that one can find in linear time the minimal cells \( R_1 \) and \( R_2 \) of \( K_P \) that contain \( p \) and \( q \), respectively, just by checking their coordinates. (Indeed, one has \( R_1 = C^I_M \) for \( I = \{ i \in P \mid p_i > 0 \} \) and \( M = \{ i \in P \mid 0 < p_i < 1 \} \).) Since \( d(p, q) \leq D < 1 \), from Lemma 3.13 we know that \( R_1 \cap R_2 \neq \emptyset \). Let \( v \) be a vertex of \( R_1 \cap R_2 \). Then \( p \) and \( q \) are contained in the star \( St(v, K_P) \) of \( v \). Since \( St(v, K_P) \) is convex in \( K_P \) by Lemma 3.12, we only have to compute the geodesic in \( St(v, K_P) \). Obviously, \( St(v, K_P) \) is a truncated CAT(0) orthant space, and hence one can compute the geodesic between \( p \) and \( q \) in \( St(v, K_P) \) in \( O(m^4) \) time, by Theorem 3.8. In addition, one can find a \( \delta \)-midpoint \( w \in St(v, K_P) \) of \( p \) and \( q \) such that each coordinate of \( w \) requires only \( O(\log(m/\delta)) \) bits. This implies (b) and therefore completes the proof of Theorem 3.10.

**Remark 3.14** As noted in [12, Prop. 2], the geodesic between \( p \) and \( q \) in \( K_P \) is contained in a subcomplex \( K_P(I(u, v)) \) of \( K_P \), where \( u \) and \( v \) are mutually furthest (in the graph \( G(K_P) \)) vertices of \( R_1 \) and \( R_2 \), and \( I(u, v) \) is the interval between \( u \) and \( v \) in \( G(K_P) \). Hence we may assume that the given PIP is one that represents \( K_P(I(u, v)) \) whose size is possibly less than \( m = |P| \).
3.6 Proof of Lemma 3.13

We end this paper by giving a proof of Lemma 3.13, which was used in proving Theorem 3.10. We start with some properties of halfspaces in a median graph $G = (V, E)$. For any edge $ab$ of $G$, define $H(a, b) := \{v \in V \mid d_G(a, v) < d_G(b, v)\}$ and $H(b, a) := \{v \in V \mid d_G(b, v) < d_G(a, v)\}$. Then $H(a, b)$ and $H(b, a)$ are complementary halfspaces of $G$ [20], i.e., $H(a, b)$ and $H(b, a)$ are convex in $G$ and $H(a, b) = V \setminus H(b, a)$. The boundary of a halfspace $H$ of $G$, denoted by $\partial H$, consists of all vertices of $H$ which have a neighbor in the complement $H' := V \setminus H$ of $H$. (Note that such a neighbor is unique for each vertex in $\partial H$, since $G$ is bipartite and $H, H'$ are convex.) Thus, one can define a bijection $\psi_H : \partial H \to \partial H'$ such that $v = \psi_H(u)$ if and only if there exists an edge $uv$ with $u \in \partial H$ and $v \in \partial H'$. It was shown by Mulder [20] that the boundaries $\partial H$ and $\partial H'$ of complementary halfspaces $H, H'$ induce convex subgraphs of $G$ with an isomorphism $\phi_H$. Hence $\partial H \cup \partial H'$ and $\partial H \cup H'$ are also convex sets of $G$.

We recall some basic properties of CAT(0) spaces. Let $X$ be a CAT(0) space, and let $Y$ be a complete closed convex subset of $X$. Then for every $x \in X$, there exists a unique point $\pi(x) \in Y$ such that $d(x, \pi(x)) = d(x, Y) := \inf_{y \in Y} d(x, y)$. The resulting map $\pi : X \to Y$ is called the orthogonal projection onto $Y$; see [9] for details.

**Proposition 3.15** Let $\mathcal{K}$ be a CAT(0) cubical complex. Let $C$ and $v$ be a cell and a vertex of $\mathcal{K}$, respectively. Then the gate of $v$ in $V(C)$ in the graph $G(\mathcal{K})$ coincides with the image $\pi(v)$ of $v$ under the orthogonal projection $\pi : \mathcal{K} \to C$ onto $C$.

**Proof** Let $v'$ be the gate of $v$ in $V(C)$ in the graph $G = G(\mathcal{K})$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of $v'$ in $V(C)$, where $k$ is the dimension of $C$. Let us write $H_i := H(v', v_i) = \{u \in V(G) \mid d_G(v', u) < d_G(v_i, u)\}$ for each $i = 1, 2, \ldots, k$. Note that each halfspace $H_i$ contains $v$, as $v' \in I_G(v_i, v)$.

We show $\pi(v) \in C \cap \mathcal{K}(H_i)$ for each $i = 1, 2, \ldots, k$. To see this, fix an arbitrary $i$ and set $H := H_i$ and $H' := V(G) \setminus H_i$. Let $\gamma$ be the geodesic in $\mathcal{K}$ with $\gamma(0) = v$ and $\gamma(1) = \pi(v)$. Then one can find a point $p := \gamma(s)$ on the geodesic for some $s \in [0, 1]$ that belongs to $\mathcal{K}(\partial H)$. As remarked above, $\partial H \cup \partial H'$ is convex in $G$, and hence $\mathcal{K}(\partial H \cup \partial H')$ is convex in $\mathcal{K}$ by Theorem 3.3. Note that $C \subseteq \mathcal{K}(\partial H \cup \partial H')$. This implies that the geodesic segment joining $p$ and $\pi(v)$ is contained in $\mathcal{K}(\partial H \cup \partial H')$. Note that $\mathcal{K}(\partial H \cup \partial H')$ is isometric to $\mathcal{K}(\partial H) \times [0, 1]$. Let $\psi : \mathcal{K}(\partial H \cup \partial H') \to \mathcal{K}(\partial H) \times [0, 1]$ be the isometry that sends $\mathcal{K}(\partial H)$ to $\mathcal{K}(\partial H) \times \{0\}$ and $\mathcal{K}(\partial H')$ to $\mathcal{K}(\partial H) \times \{1\}$. For each point $y$ in $\mathcal{K}(\partial H \cup \partial H')$, when writing its image $\psi(y)$ as $(y_1, y_2) \in \mathcal{K}(\partial H) \times [0, 1]$, we shall write $y_H$ to denote the point $\psi^{-1}((y_1, 0))$ in $\mathcal{K}(\partial H)$. Let $\gamma' : [0, 1] \to \mathcal{K}$ be the map obtained from $\gamma$ by resetting $\gamma'(t) := (\gamma(t))_H$ for all $t \in [s, 1]$; see Fig. 7 for intuition. Then $\gamma'$ is a continuous map in $\mathcal{K}$ joining $\gamma'(0) = v$ and $\gamma'(1) \in C \cap \mathcal{K}(H)$ whose length is at most the length of $\gamma$. This implies that $\pi(v)$ should belong to $C \cap \mathcal{K}(H)$. Thus, we obtain $\pi(v) \in C \cap \mathcal{K}(H_i)$ for each $i = 1, 2, \ldots, k$.

Note that the intersection of all $C \cap \mathcal{K}(H_i)$ is a singleton $\{v'\}$. This implies that $\pi(v) \in \{v'\}$ and completes the proof. □
Fig. 7 Illustration of the proof of Proposition 3.15. Taking some \( s \in [0, 1] \) such that \( p := \gamma(s) \) lies in \( K(\partial H) \), and “projecting” the point \( \gamma(t) \) onto \( K(\partial H) \) for all \( t \in [s, 1] \), one can get a path \( \gamma' \) joining \( v \) and some point on \( C \cap K(H) \) whose length is at most that of \( \gamma \).

Now let us show Lemma 3.13. Let \( p, q \) be two points in \( K \) with \( d(p, q) < 1 \) and \( R_1, R_2 \) be the minimal cells of \( K \) containing \( p, q \), respectively. Let \( u \in V(R_1) \) and \( v \in V(R_2) \) be vertices satisfying \( d_G(u, v) = \min\{d_G(u', v') \mid u' \in V(R_1), v' \in V(R_2)\} \) in the underlying graph \( G = G(K) \). It is easy to see that \( u \) is the gate of \( v \) in \( V(R_1) \) and \( v \) is the gate of \( u \) in \( V(R_2) \), in the graph \( G \). Hence by Proposition 3.15 we have \( \pi_1(v) = u \) and \( \pi_2(u) = v \), where \( \pi_i : K \to R_i \) is the orthogonal projection onto \( R_i \) for \( i = 1, 2 \). This implies that \( d(u, v) = d(R_1, R_2) := \inf\{d(x, y) \mid x \in R_1, y \in R_2\} \). Since \( d(R_1, R_2) \leq d(p, q) < 1 \), we have \( d(u, v) < 1 \). Hence \( u \) and \( v \) should be the same vertex of \( K \), and thus, we have \( R_1 \cap R_2 \neq \emptyset \). This completes the proof of Lemma 3.13.

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