High Dimensional Linear GMM

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Abstract

This paper proposes a desparsified GMM estimator for estimating high-dimensional regression models allowing for, but not requiring, many more endogenous regressors than observations. We provide finite sample upper bounds on the estimation error of our estimator and show how asymptotically uniformly valid inference can be conducted in the presence of conditionally heteroskedastic error terms. We do not require the projection of the endogenous variables onto the linear span of the instruments to be sparse; that is we do not impose the instruments to be sparse for our inferential procedure to be asymptotically valid. Furthermore, the variables of the model are not required to be sub-gaussian and we also explain how our results carry over to the classic dynamic linear panel data model. Simulations show that our estimator has a low mean square error and does well in terms of size and power of the tests constructed based on the estimator.

Keywords: GMM, Desparsification, Uniform inference, High-dimensional models, Linear regression, Dynamic panel data.

1 Introduction

GMM is one of the cornerstones of modern econometrics. It has been used to link economic theories to estimation of structural parameters as well as testing. It has also infused other fields such as finance, marketing and accounting. The popularity of GMM stems furthermore from its role in instrumental variable estimation in the presence of endogenous regressors.

Until recently, the validity of GMM based inference had only been established in asymptotic regimes with a fixed number of instruments and endogenous regressors as sample size tended to infinity. For example, Caner (2009) proposed a Bridge type of penalty on the GMM estimator with a fixed number of parameters and analyzed its model selection properties. Caner (2009) proposed a Bridge type of penalty. Furthermore,
Caner and Fan (2015) used an adaptive Lasso type penalty to select instruments — again this was done in a setting with a fixed number of instruments.

The setting of an increasing number of endogenous variables was analyzed by Caner and Zhang (2014). These authors considered the adaptive elastic net penalty and studied estimation and variable selection consistency. Next, Caner et al. (2018) recently proposed an adaptive elastic net based estimator which can simultaneously select the model and the valid instruments while at the same time estimating the structural parameters. However, the asymptotic framework in all of the above papers is pointwise and the sample size is always larger than the number of instruments (albeit this is allowed to diverge with the sample size).

In a seminal paper, Belloni et al. (2012) proposed a heteroskedasticity robust procedure for inference in IV estimation valid in settings with many more instruments than observations. Their results do not rely on the data being sub-gaussian making it very relevant for many economic applications. Gautier and Tsybakov (2014) also consider high-dimensional instrumental variable estimation. Their analysis relies on finding upper bounds on the sampling errors for the structural coefficients and making these feasible. Furthermore, Belloni et al. (2014) developed the first uniformly valid confidence intervals for the treatment coefficient in the presence of a high-dimensional vector of control variables.

Very recently, Zhu (2018) introduced new oracle inequalities for high-dimensional two-stage least squares estimators. Based on that work, Gold et al. (2018) have proposed a debiased version of a Lasso-based two-stage least squares estimator. Assuming sub-gaussian data, sparse instruments and homoskedastic errors, they develop oracle inequalities and asymptotically valid confidence intervals. This is a very creative way of handling an important problem in the IV literature. Simultaneously with Gold et al. (2018), Belloni et al. (2018) have proposed a new instrumental variable estimator for high dimensional models with sparse instruments. This estimator is based on empirical orthogonality conditions and can handle heteroskedastic data. Furthermore, their approach is very appealing as no tuning parameter selection is needed.

Our approach is based on debiasing a two-step Lasso-GMM estimator. Thus, our estimator is related to van de Geer et al. (2014) who proposed a desparsified Lasso estimator and established that the confidence intervals based on it are asymptotically uniformly valid. Simultaneously, similar advancements were made in the papers by Javanmard and Montanari (2014) and Zhang and Zhang (2014). Caner and Kock (2018) proposed debiasing the conservative Lasso in the context of a plain linear regression model without endogenous covariates and showed how it can be used to construct uniformly valid confidence intervals in the presence of heteroskedasticity. In addition, the asymptotic inference can simultaneously involve a large number of coefficients.

This paper proposes a high dimensional penalized GMM estimator where the number of instruments and explanatory variables are both allowed, yet not required, to be larger than the sample size. We do not impose sparsity on the instruments; that is, we do not require only a small subset of the instruments to be valid. While we develop the theory in the context of cross sectional data, we also explain how the theory is valid in dynamic panel data models upon taking first differences. The error terms are allowed
to be heteroskedastic conditionally on the instruments. Our approach does not impose the data to be sub-gaussian as we benefit from concentration inequalities by Chernozhukov et al. (2017). For debiasing our estimator we need an approximate inverse of a certain singular sample covariance matrix, cf Section 4.1. Our construction of this approximate inverse relies on the CLIME estimator of Cai et al. (2011). Uniformly valid confidence intervals for the debiased estimator are developed. The tuning parameter present is chosen by cross-validation. Finally, the finite sample properties of our estimator are investigated through simulations and we compare it to the estimator in Gold et al. (2018). In the presence of heteroskedasticity, our estimator performs very well in terms of size, power, length of the confidence interval and mean square error.

Section 2 introduces the model and estimator. Next, Section 3 lays out the used assumptions and and oracle inequality for the penalized two-step GMM estimator. Section 4 develops the approximate inverse used for debiasing the penalized two-step GMM estimator. Section 5 establishes the asymptotically uniform validity of our inference procedure and Section 6 explains the tuning parameter choice. Finally, the Monte Carlo simulations are contained in Section 7.

2 Notation, model and estimator

Prior to introducing the model and the ensuing inference problem we present notation used throughout the paper.

2.1 Notation

For any \( x \in \mathbb{R}^n \), let \( \| x \|_1, \| x \|_2 \) and \( \| x \|_\infty \) denote its \( l_1 \)-, \( l_2 \)-, and the \( l_\infty \)-norm, respectively. Also, we shall let \( \| x \|_{l_0} \) be the \( l_0 \)-“norm” counting the number of non-zero entries in \( x \). For an \( m \times n \) matrix \( A \), we define \( \| A \|_\infty = \max_{1 \leq i \leq m, 1 \leq j \leq n} |A_{ij}| \). \( \| A \|_{l_\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^n |A_{ij}| \) denotes the induced \( l_\infty \)-norm of \( A \). Similarly, \( \| A \|_{l_1} = \max_{1 \leq j \leq n} \sum_{i=1}^m |A_{ij}| \) denotes the induced \( l_1 \)-norm. For any symmetric matrix \( B \), let \( \text{Eigmin}(B) \) and \( \text{Eigmax}(B) \) denote the smallest and largest eigenvalues of \( B \), respectively. For \( S \subseteq \{1, \ldots, n\} \) we let \( x_S \) be the modification of \( x \) that places zeros in all entries of \( x \) whose index does not belong to \( S \). \( |S| \) denotes the cardinality of \( S \). For any \( n \times n \) matrix \( C \) let \( C_S \) denote the \( |S| \times |S| \) submatrix of \( C \) consisting only of the rows and columns indexed by \( S \). \( \text{diag}(x) \) denotes the diagonal matrix having \( x_j \) as its \( j \) diagonal element. \( e_j \) will denote the \( j \)th canonical basis vector for \( \mathbb{R}^n \). \( \xrightarrow{d} \) indicates convergence in distribution.

2.2 The Model

We consider the linear model

\[
Y = X\beta_0 + u, \tag{1}
\]

where \( X \) is the \( n \times p \) matrix of potentially endogenous explanatory variables and \( u \) is an \( n \times 1 \) vector of error terms. \( \beta_0 \) is the \( p \times 1 \) population vector of coefficients which we shall assume to be sparse. Thus, \( Y \)
is $n \times 1$. However, the location of the non-zero coefficients is unknown. Let $S_0 = \{j : \beta_{0j} \neq 0\}$ denote the set of relevant regressors and $s_0 = |S_0|$ their cardinality. In this paper we study the high-dimensional case where $p$ is much greater than $n$ but our results actually only require $n \to \infty$ — thus $p \leq n$ is covered as well. All regressors are allowed to be endogenous but are not required to be. In particular this means that upon taking first differences the classic linear dynamic panel data model can be cast in our framework. We provide more details on this in Section 9.1 of the supplementary appendix.

We assume that $q$ instruments are available and let $Z$ denote the $n \times q$ matrix of instruments. Exogenous variables can instrument themselves as usual. The regime under investigation is $q \geq p > n$ where there are many instruments and regressors compared to the sample size. However, our results can easily be adapted to any regime of orderings and growth rates of $p$, $q$ and $n$ (as long as we have at our disposal at least as many instruments as endogenous variables, i.e. $q \geq p$). Letting $X_i$ and $Z_i$ denote the $i$th row of $X$ and $Z$, respectively, $i = 1, \ldots, n$, written as column vectors, we assume that

$$EZ_iu_i = 0,$$

for all $i = 1, \ldots, n$ amounting to the the instruments being uncorrelated with the error terms.

The goal of this paper is to construct valid tests and confidence intervals for the entries of $\beta_0$. We do not impose that the columns of $A$ are sparse in a first step equation of the type $X = ZA + \epsilon$ (put differently, the $L_2$-projection of the covariates on the linear span of the instruments is not assumed to be sparse) and also allow $u_i$ to be heteroskedastic conditionally on $Z_i$ for each $i = 1, \ldots, n$. In addition, we do not impose the random variables in the model (1) to be sub-gaussian.

Based on (2), we propose the following penalized first-step Lasso GMM estimator.

$$\hat{\beta}_F = \arg\min_{\beta \in \mathbb{R}^p} \left( (Y - X\beta)'ZZ'(Y - X\beta) \frac{n^2}{n^2q} + 2\lambda_n \|\beta\|_1 \right),$$

where $\lambda_n$ is a positive tuning parameter sequence defined in (A.12) in the appendix. While we shall later see that this estimator is consistent under suitable regularity conditions, the main focus of this paper is a generalization of the classic GMM estimator to the high-dimensional setting, allowing for a $q \times q$ weight matrix $\hat{W}_d = \text{diag}(1/\hat{\sigma}_1^2, \ldots, 1/\hat{\sigma}_q^2)$ with $\hat{\sigma}_i^2 = \frac{\sum_{i=1}^n Z_i^2 \hat{u}_i^2}{n}$ and $\hat{u}_i = Y_i - X'\hat{\beta}_F$. This two-step Lasso GMM estimator is defined as

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left( (Y - X\beta)' Z\hat{W}_d Z'(Y - X\beta) \frac{n}{q} + 2\lambda_n^* \|\beta\|_1 \right).$$

For the two-step GMM estimator we shall use $\lambda_n^*$ as defined in (A.46). While the exact form of $\lambda_n^*$ is rather involved we note that under Assumption 2 below one has that $\lambda_n^* = O(\sqrt{\ln q/n})$.

**Remark 1.** Although we focus on the case of a diagonal weight matrix $\hat{W}_d$, it is worth mentioning that our results can be shown to remain valid in case of a general weight matrix $\hat{W}$ if there exists (a sequence of)

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1For details on arbitrary growth rates of $p$, $q$ and $n$ we refer to Remark 2 in the beginning of the appendix.
non-random matrices $W$ such that
\[ \| \hat{W} - W \|_{l_\infty} = o_p(1) \quad \text{and} \quad \| W \|_{l_\infty} \leq C < \infty. \]

for some universal $C > 0$. However, since $W$ is $q \times q$, assuming it to have uniformly bounded $l_\infty$-norm is restrictive. Thus, even though $\| \hat{W} - W \|_{l_\infty} = o_p(1)$ and $\| W \|_{l_\infty} \leq C < \infty$ can be relaxed at the expense of strengthening some of our other assumptions, we shall focus on the case of a diagonal weight matrix as this is enough to handle conditionally heteroskedastic error terms.

Note also that the classic choice of weighting matrix in low-dimensional GMM, $\hat{W} = [n^{-1} \sum_{i=1}^n Z_i Z_i' \hat{u}_i^2]^{-1}$, is not applicable since it is not well-defined for $q > n$ due to the reduced rank of $n^{-1} \sum_{i=1}^n Z_i Z_i' \hat{u}_i^2$.

### 3 Assumptions and oracle inequalities

Throughout we assume that $X_i, Z_i$ and $u_i$ are independently and identically distributed across $i = 1, \ldots, n$. Before stating our first assumption, we introduce the following notation. First, let
\[ \Sigma_{xz} = E X_1 Z_1', \]
and, with $\sigma_i^2 = E Z_i^2 u_i^2$, $l = 1, \ldots, q$, set
\[ W_d = \text{diag}(1/\sigma_1^2, \ldots, 1/\sigma_q^2). \]

Next, define the population adaptive restricted eigenvalue of $\Sigma_{xz} W_d \Sigma_{xz}'$
\[ \phi_{\Sigma_{xz}W_d \Sigma_{xz}'}^2(s) = \min \left\{ \frac{\delta'(\Sigma_{xz} W_d \Sigma_{xz}') \delta}{q \| \delta \|_2^2} : \delta \in \mathbb{R}^p \setminus \{0\}, \| \delta \|_1 \leq 3 \sqrt{s} \| \delta \|_2, |S| \leq s \right\} \]
(7)
which is the relevant extension of the classic adaptive restricted eigenvalue from the linear regression model with exogenous regressors (which only involves $E(X_i X_i')$). Thus, verifying that the sample counterpart of (7) is bounded away from zero, which is an important step in establishing the oracle inequalities in Theorem 1 below, becomes more challenging than in the classic setting, cf. Lemma S.3 in the Appendix.

**Assumption 1.** Assume that $EZ_i u_i = 0$ for $i = 1, \ldots, n$. Furthermore, $\max_{1 \leq j \leq p} E \| X_{1j} \|^{r_z}, \max_{1 \leq i \leq q} E \| Z_i \|^{r_z}$, and $E |u_1|^{r_x}$ are uniformly bounded from above (across $n$) for $r_z, r_x, r_u \geq 4$. Assume that $\phi_{\Sigma_{xz}W_d \Sigma_{xz}'}^2(s_0)$ and $\min_{1 \leq i \leq q} \sigma_i^2 = \min_{1 \leq i \leq q} E Z_i^2 u_i^2$ are bounded away from zero uniformly in $n$.

Note that Assumption 1 does not impose sub-gaussianity of the random variables. Assumption 1 is used to establish the oracle inequality in Theorem 1 below which in turn plays an important role for proving the asymptotic gaussianity of the (properly centered and scaled) desparsified two-step GMM estimator. Furthermore, Assumption 1 does not require $\Sigma_{xz} W_d \Sigma_{xz}' / q$ to be full rank. Thus, we allow for some of the instruments to be weakly correlated with the explanatory variables.

While Assumption 1 imposes restrictions for each $n \in \mathbb{N}$, the following assumption only imposes asymptotic restrictions. It restricts the growth rate of the moments of certain maxima as well as the number of non-zero entries of $\beta_0$, i.e. $s_0$. Prior to stating the assumption, we introduce the following maxima.
Definition 1.  
\[ M_1 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq q} |Z_{ij}u_i|, \]
\[ M_2 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} |Z_{ij}X_{ij} - EZ_{il}X_{ij}|. \]
\[ M_3 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} |Z_{ij}^2u_i^2 - EZ_{il}u_l^2|. \]
\[ M_4 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} |Z_{ij}^2u_iX_{ij} - EZ_{il}u_lX_{il}|. \]
\[ M_5 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} \max_{1 \leq k \leq p} \max_{1 \leq l \leq q} |Z_{ij}^2X_{ij}X_{il} - EZ_{il}X_{il}X_{il}|. \]

Assumption 2. (i).  
\[ \frac{s_0^2}{\ln q} \sqrt{\frac{\ln q}{n}} \rightarrow 0. \]

(ii).  
\[ \sqrt{\frac{\ln q}{n}} \max\left(\frac{EM_1^2}{C^2}, \frac{(EM_2)^2}{C^2}, \frac{(EM_3)^2}{C^2}, \frac{(EM_4)^2}{C^2}, \frac{(EM_5)^2}{C^2}\right) \rightarrow 0. \]

Assumption 2 restricts the number of non-zero entries of \( \beta_0 \). No sparsity is assumed on the instruments. However, the dimensionality of the model, as measured by the number of instruments \( q \), does influence how fast \( s_0 \) can increase. Part (ii) is similar to assumptions made in Chetverikov et al. (2017) in the context of establishing the validity of cross validation to choose the tuning parameter in the context of a linear regression model with exogenous regressors. Essentially, Assumption 2 (ii) restricts the growth rate of the second moments of the maxima \( M_1, \ldots, M_5 \).

Theorem 1. Under Assumptions 1 and 2, with \( r_x \geq 12, r_x \geq 6, r_u \geq 6 \), we have with probability at least \( 1 - \frac{21}{q^2} - \frac{K(5EM_1^2 + 10EM_2^2 + 2EM_3^2 + 2EM_4^2 + 2EM_5^2)}{n \ln q} \)

(i).  
\[ \| \hat{\beta} - \beta_0 \|_1 \leq \frac{24\lambda_n^* s_0}{\phi_{s_0}^2(s_0)} \]
for \( n \) sufficiently large. The above bound is valid uniformly over \( \mathcal{B}_{t_0}(s_0) = \{ \| \beta_0 \|_{t_0} \leq s_0 \} \).

(ii). Furthermore, \( \lambda_n^* = O(\sqrt{\ln q/n}) \) and the probability of (i) being valid tends to one.

While we mainly use Theorem 1 as a stepping stone towards testing hypotheses about the elements of \( \beta_0 \), it may be of interest in its own right as it guarantees that the two-step GMM estimator estimates \( \beta_0 \) precisely. In particular, we see that \( \| \hat{\beta} - \beta_0 \|_1 = O_p(s_0 \sqrt{\ln q/n}) \) allowing \( q \) to increase very quickly in \( n \) without sacrificing consistency of \( \hat{\beta} \) since the number of instruments only enters the upper bound on the \( l_1 \) estimation error through its logarithm.

Lemma 4.4 of Gold et al. (2017) and the line immediately below it provide an upper bound on the \( l_1 \) estimation error of their estimator which is of order \( O_p(s_0^2 \ln q/n + s_0 s_A \sqrt{\ln q/n}) \) where \( s_A < q \) is the number of relevant instruments and the remaining quantities are as in the present paper. Recall that we do not impose any restrictions on the number of relevant instruments.

Having established that the two-step GMM estimator estimates \( \beta_0 \) precisely, we turn towards desparsifying it in order to construct tests and confidence intervals.
4 Desparsification

4.1 The desparsified two-step GMM estimator

We now introduce the desparsified two-step GMM estimator that we use to construct tests and confidence intervals. To that end, consider the Karush-Kuhn-Tucker first order conditions for the problem in (4)

\[-X'Z\hat{W}_d Z'(Y - X\hat{\beta}) + \lambda_n^* \hat{\kappa} = 0,\]

(8)

where \(\|\hat{\kappa}\|_\infty \leq 1,\) and \(\hat{\kappa}_j = sgn(\hat{\beta}_j)\) when \(\hat{\beta}_j \neq 0,\) for \(j = 1, ..., p.\) Since \(Y = X\beta_0 + u,\)

\[\left[\begin{array}{c}
X' \hat{W}_d Z' X \\
q_n
\end{array}\right] (\hat{\beta} - \beta_0) + \lambda_n^* \hat{\kappa} = X' \hat{W}_d Z' u.
\]

(9)

Next, since \(\hat{\Sigma} := \left[\begin{array}{c}
X' \hat{W}_d Z' X \\
q_n
\end{array}\right]\) is of reduced rank, it is not possible to left-multiply by its inverse in the above display in order to isolate \(\sqrt{n}(\hat{\beta} - \beta_0).\) Instead, we construct an approximate inverse, \(\hat{\Gamma},\) of \(\hat{\Sigma}\) and control the error resulting from this approximation. We shall be explicit about the construction of \(\hat{\Gamma}\) in the sequel (cf Section 4.2) but first highlight which properties it must have in order to conduct asymptotically valid inference based on it. Left multiply (9) by \(\hat{\Gamma}\) to obtain

\[\hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' X \\
q_n
\end{array}\right] (\hat{\beta} - \beta_0) + \hat{\Gamma}\lambda_n^* \hat{\kappa} = \hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' u \\
q_n
\end{array}\right] - \Delta \sqrt{n}.
\]

(10)

Add \((\hat{\beta} - \beta_0)\) to both sides of (10) and rearrange to get

\[(\hat{\beta} - \beta_0) + \hat{\Gamma}\lambda_n^* \hat{\kappa} = \hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' u \\
q_n
\end{array}\right] - \Delta \sqrt{n}.
\]

(11)

Upon defining \(\Delta = \sqrt{n}\left(\hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' X \\
q_n
\end{array}\right] - I_p\right) (\hat{\beta} - \beta_0),\) which can be interpreted as the approximation error due using an approximate inverse of \(\left[\begin{array}{c}
X' \hat{W}_d Z' X \\
q_n
\end{array}\right]\) instead of an exact inverse, (11) can also be written as

\[(\hat{\beta} - \beta_0) + \hat{\Gamma}\lambda_n^* \hat{\kappa} = \hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' u \\
q_n
\end{array}\right] - \frac{\Delta}{\sqrt{n}}.
\]

(12)

Thus,

\[\hat{\beta} = \beta_0 - \hat{\Gamma}\lambda_n^* \hat{\kappa} + \hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' u \\
q_n
\end{array}\right] - \frac{\Delta}{\sqrt{n}}\]

where \(\hat{\Gamma}\lambda_n^* \hat{\kappa}\) is the shrinkage bias introduced to \(\hat{\beta}\) due to penalization in (4). By removing this we define the two-step desparsified GMM estimator

\[\hat{b} = \hat{\beta} + \hat{\Gamma}\lambda_n^* \hat{\kappa} = \beta_0 + \hat{\Gamma}\left[\begin{array}{c}
X' \hat{W}_d Z' u \\
q_n
\end{array}\right] - \frac{\Delta}{\sqrt{n}}.
\]

(13)

Note that by (8) one can calculate \(\hat{b}\) in terms of observable quantities as

\[\hat{\beta} + \hat{\Gamma}\frac{X' \hat{W}_d Z' (Y - X\hat{\beta})}{n}.
\]
Thus, to conduct inference on the $j$th component of $\beta_0$, we consider
\[
\sqrt{n}(\hat{b}_j - \beta_{0j}) = \sqrt{n}e_j'(\hat{b} - \beta_0) = \hat{\Gamma}_j \left( \frac{X'Z}{n} \right) - \Delta_j
\]
where $\hat{\Gamma}_j$ denotes the $j$th row of $\hat{\Gamma}$. Hence, in order to conduct asymptotically valid gaussian inference, it suffices to establish a central limit theorem for $\hat{\Gamma}_j \left( \frac{X'Z}{n} \right)$ as well as asymptotic negligibility of $\Delta_j$.

To achieve these two goals we need to construct an approximate inverse $\hat{\Gamma}$ and develop its properties. The subsequent section is concerned with these issues.

### 4.2 Constructing $\hat{\Gamma}$

In Section 4.1 we assumed the existence of an approximate inverse $\hat{\Gamma}$ of $\hat{\Sigma} = \frac{X'Z}{n}$. We now turn towards the construction of $\hat{\Gamma}$. Our construction builds on the CLIME estimator of Cai et al. (2011) (which was further refined in Gold et al. (2018)). We establish how our estimator can provide a valid approximate inverse allowing for conditional heteroskedasticity. First, define
\[
\Sigma = \Sigma_{xz}W_d\Sigma_{xz}',
\]
as well as its inverse $\Gamma = \Sigma_{xz}^{-1}$ which is guaranteed to exist by Assumption 4 below.

Our proposed approximate inverse for $\hat{\Sigma}$ is found by the following variant of the CLIME estimator: The $j$th row of $\hat{\Gamma}$, denoted $\hat{\Gamma}_j$, is found as
\[
\hat{\Gamma}_j = \text{argmin}_{a \in \mathbb{R}^p} \|a\|_1 \text{ s.t. } \|a\hat{\Sigma} - e_j'\|_\infty \leq \mu,
\]
where $\mu > 0$ and the dependence of $\hat{\Gamma}_j$ on $\mu$ is suppressed. The exact expression for $\mu$ is involved and given in the statement of Lemma A.8 in the appendix which also establishes that $\mu = O(m_{1\Gamma}s_{0\Gamma}^{1/2}) = o(1)$, cf (A.71). Throughout, we shall assume that for some $0 \leq f < 1$, $m_{1\Gamma} > 0$ and $s_{1\Gamma} > 0$,
\[
\Gamma \in U(m_{1\Gamma}, f, s_{1\Gamma}) := \left\{ A \in \mathbb{R}^{p \times p} : A > 0, \|A\|_{i_1} \leq m_{1\Gamma}, \max_{1 \leq j \leq p} \sum_{k=1}^p |A_{jk}|^f \leq s_{1\Gamma} \right\},
\]
where $m_{1\Gamma}$ and $s_{1\Gamma}$ regulate the sparsity of the matrices in $U(m_{1\Gamma}, f, s_{1\Gamma})$ and their potential dependence on $n$ is suppressed (in particular we allow $s_{1\Gamma}, m_{1\Gamma} \to \infty$). Note that $f = 0$ amounts to assuming that $\Gamma$ has exactly sparse rows. Furthermore, Lemma A.8 in the appendix shows that with probability converging to one $\|\Gamma \Sigma - I_p\|_\infty \leq \mu$ implying that the problem in (15) is well-defined (with probability approaching one) since $\Gamma$ satisfies the constraint. This is noteworthy since $\Gamma$ is not required to be strictly sparse.

**Assumption 3.** (i) $\Gamma \in U(m_{1\Gamma}, f, s_{1\Gamma})$.

(ii) $m_{1\Gamma}s_0^{1/2} = o(1)$.

Assumption 3 (i) restricts the structure of $\Gamma$ by imposing it to belong to $U(m_{1\Gamma}, f, s_{1\Gamma})$. Note that for $0 < f < 1$, $\Gamma$ is not required to be (exactly) sparse as opposed to much previous work. Part (ii) restricts the growth rates of $m_{1\Gamma}, s_0$ and $q$. 


5 Testing and uniformly valid confidence intervals

In this section we show how to conduct asymptotically valid gaussian inference on each entry of $\beta_0$. It is merely a technical exercise to extend this to joint inference, by eg Wald-type tests, on any fixed and finite number of elements of $\beta_0$. At the price of more technicalities and more stringent assumptions we also conjecture that it is possible to conduct joint inference on a subvector of $\beta_0$ of slowly increasing dimension. We refer to Caner and Kock (2018) for details in the case of the conservative Lasso applied to the high-dimensional plain linear regression model and do not pursue these extensions further here. To conduct inference on $\beta_{0j}$ we consider the studentized version of (14):

$$t_{W_d} = \frac{n^{1/2}c_j'(\hat{b} - \beta_0)}{\sqrt{e_j'\hat{\Gamma}V'Ve_j}},$$

where

$$\hat{V} = \left( \frac{X'Z W_d \hat{\Sigma}_Z W_d Z'X}{n} \right)$$

and $\hat{\Sigma}_Z = \frac{1}{n} \sum_{i=1}^{n} Z_i Z'_i \hat{u}_i^2$.

and $\hat{u} = Y - X\hat{\beta}_F$ are the first step Lasso-GMM residuals.

To state the next assumption, define the $q \times q$ matrix $\Sigma_{zu} = EZ_i Z'_i u_i^2$ as well as the $p \times p$ matrices $V_1 = \Sigma_{xz} W_d \Sigma_{zu} W_d \Sigma'_{xz}$ and $V_d = \frac{1}{q} V_1$. Finally, let

$$M_0 = \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} |X_{ij}u_i - EX_{ij}u_i|,$$

$$M_7 = \max_{1 \leq i \leq n} \max_{1 \leq m \leq q} |Z_{im} u_i^2 - EZ_{im} Z_{im} u_i^2|.$$

In order to establish the asymptotic normality of $\sqrt{n}(\hat{b}_j - \beta_{0j})$ when $\hat{\Gamma}$ is the CLIME estimator in (15), we impose the following assumptions.

**Assumption 4.** (i). $m_{T}^{r_u/2}/n^{r_u/4-1} \rightarrow 0$ and

$$s_T(m_{T}\mu)^{1-f} \sqrt{\ln q} = O(\frac{s_T m_{T}^{2-2f} s_0^{1-f} (\ln q)^{1-f/2}}{n^{(1-f)/2}}) = o(1).$$

In addition, $\frac{m_{T}^{s_0^{1-f}/n \ln q}}{\sqrt{n}} = o(1)$.

(ii) Eigmax($V_d$) and Eigmax($\Sigma$) are bounded from above. Eigmin($V_d$) and Eigmin($\Sigma$) are bounded away from zero.

(iii). $\sqrt{\ln q} \max[(EM^2_0)^{1/2}, (EM^2_1)^{1/2}] \rightarrow 0$.

(iv). $r_x > 12$ and $m_{T}^{s_0^{1-f/2} q^{1/r_x} n^{2/r_x} \sqrt{\ln q}} \rightarrow 0$.

Assumption 4 governs the permissible growth rates of the number of instruments, $q$, the sparsity imposed on $\Gamma$ via $s_T$ and $m_{T}$, as well as the number of non-zero entries in $\beta_0$, $s_0$. For example, assuming that $f = 1/2$, $r_u = 10$ and $r_x = 16$ along with $s_T = O(\ln n)$, $m_{T} = O(\ln n)$, $s_0 = n^{1/10}$, $q = 2n$ is in accordance with Assumption 4. Assumption 4(ii) is a standard assumption on population matrices. Note that the requirement
$E_{\text{gmin}}(\Sigma)$ being bounded away from zero implies the adaptive restricted eigenvalue being bounded away from zero (as required in Assumption 1). The considered largest and smallest eigenvalues can be allowed to be unbounded and approach zero, respectively, at the expense of strengthening other assumptions. Thus, instruments that are weakly correlated with the explanatory variables can be allowed for. Part (iii) is similar to assumptions imposed in Chetverikov et al. (2017), cf also the discussion of Assumption 2 above.

The following theorem establishes the validity of asymptotically gaussian inference for the desparsified two-step GMM estimator. Recall that $B_{0}(s_{0}) = \{ \| \beta_{0} \|_{l_{0}} \leq s_{0} \}$.

**Theorem 2.** Let $j \in \{1, \ldots, p\}$. Then, under Assumptions 1, 2, 3 and 4 with $r_{z} > 12, r_{x} \geq 6, r_{u} > 8$

(i).

\[
\frac{n^{1/2}(\hat{b}_{j} - \beta_{0j})}{\sqrt{e_{j}^{\prime} \hat{\Gamma} \hat{\Gamma}^{\prime} e_{j}}} \overset{d}{\to} N(0, 1) \quad \text{for every } \beta_{0} \in B_{0}(s_{0})
\]

(ii).

\[
\sup_{\beta \in B_{0}(s_{0})} |e_{j}^{\prime} \hat{\Gamma} \hat{\Gamma}^{\prime} e_{j} - e_{j}^{\prime} \Gamma \Gamma^{\prime} e_{j}| = o_{p}(1),
\]

Part (i) of Theorem 2 establishes asymptotic normality of $\hat{b}_{j}$ for every $j \in \{1, \ldots, p\}$ for every sequence of $\beta_{0} \in B_{0}(s_{0})$. This actually implies that the weak convergence to the standard normal distribution is uniform as made clear in part (i) of Theorem 3 below.

Part (ii) of Theorem 2 provides a uniformly consistent estimator of the asymptotic variance of $n^{1/2}(\hat{b}_{j} - \beta_{0j})$. This is valid even for conditionally heteroskedastic $u_{i}$, $i = 1, \ldots, n$ and even though the dimension $(p \times p)$ of the involved matrices diverges with the sample size.

Next, we show that the confidence bands resulting from Theorem 2 have asymptotically uniformly correct coverage over $B_{0}(s_{0})$. Furthermore, the bands contract uniformly at the optimal $\sqrt{n}$ rate. Let $\Phi(\cdot)$ denote the cdf of the standard normal distribution and let $z_{1-\alpha/2}$ be its $1 - \alpha/2$ quantile. For brevity, let $\hat{\sigma}_{bj} = \sqrt{e_{j}^{\prime} \hat{\Gamma} \hat{\Gamma}^{\prime} e_{j}}$ while $\text{diam}([a, b]) = b - a$ denotes the length of the interval $[a, b]$ in the real line.

**Theorem 3.** Let $j \in \{1, \ldots, p\}$. Then, under Assumptions 1, 2, 3 and 4 with $r_{z} > 12, r_{x} \geq 6, r_{u} > 8$

(i).

\[
\sup_{t \in \mathbb{R}} \sup_{\beta_{0} \in B_{0}(s_{0})} \left| P \left( \frac{n^{1/2}(\hat{b}_{j} - \beta_{0j})}{\sqrt{e_{j}^{\prime} \hat{\Gamma} \hat{\Gamma}^{\prime} e_{j}}} \leq t \right) - \Phi(t) \right| \to 0.
\]

(ii).

\[
\lim_{n \to \infty} \inf_{\beta_{0} \in B_{0}(s_{0})} P \left( \beta_{0j} \in [\hat{b}_{j} - z_{1-\alpha/2} \hat{\sigma}_{bj} / \sqrt{n^{1/2}}, \hat{b}_{j} + z_{1-\alpha/2} \hat{\sigma}_{bj} / \sqrt{n^{1/2}}] \right) = 1 - \alpha.
\]

(iii).

\[
\sup_{\beta_{0} \in B_{0}(s_{0})} \text{diam} \left( [\hat{b}_{j} - z_{1-\alpha/2} \hat{\sigma}_{bj} / \sqrt{n^{1/2}}, \hat{b}_{j} + z_{1-\alpha/2} \hat{\sigma}_{bj} / \sqrt{n^{1/2}}] \right) = O_{p} \left( \frac{1}{n^{1/2}} \right).
\]
Part (i) of Theorem 3 asserts the uniform convergence to the normal distribution of the properly centered and scaled $\hat{b}_j$. Part (ii) is a consequence of (i) and yields the asymptotic uniform validity of confidence intervals based on $\hat{b}_j$. Finally, (iii) asserts that the confidence intervals contract at rate $\sqrt{n}$ uniformly over $B_{l_0}(s_0)$. We also stress that the above results do not rely on a $\beta_{\min}$-type condition requiring the non-zero entries of $\beta_0$ to be bounded away from zero.

6 Tuning Parameter Choice

In this section we explain how we choose the tuning parameter sequences $\lambda_n$ and $\lambda_n^*$. We use cross validation since this has recently been shown by Chetverikov et al. (2017) to result in Lasso estimators with guaranteed low finite sample estimation and prediction error in the context of high-dimensional linear regression models. While the theoretical guarantees are for the linear regression model without endogenous regressors and sub-gaussian error terms we still use cross validation here and are content to leave the big task of establishing theoretical guarantees of cross validated two-step desparsified Lasso GMM in the presence of endogenous regressors for future work.

The exact implementation of the cross validation used is as follows: Let $W \in \{I_q, \hat{W}_d\}$ (indicating whether the first step or second step Lasso GMM estimator is used) and $K \in \mathbb{N}$ be the number of cross validation folds. Assuming, for simplicity, that $n/K$ is an integer we let $I_k = \{\frac{k-1}{K} n + 1, \frac{k}{K} n\}$, $k = 1, ..., K$ be a partition of $\{1, ..., n\}$ consisting of “consecutive” sets. Fix a $\lambda \in \Lambda_n \subseteq \mathbb{R}$ where $\Lambda_n$ is the candidate set of tuning parameters. With $n$ being the cardinality of the $I_k$ define

$$\hat{\beta}_{-k}(\lambda) = \arg\min_{b \in \mathbb{R}^p} \left\{ \left[ \frac{1}{n - n_k} \sum_{i \in I_k} Z_i(Y_i - X_i'b) \right]' \frac{1}{q} \left[ \frac{1}{n - n_k} \sum_{i \in I_k} Z_i(Y_i - X_i'b) \right] + \lambda \|b\|_1 \right\}$$

for $k = 1, ..., K$ and choose $\lambda$ as

$$\hat{\lambda}_{CV} = \arg\min_{\lambda \in \Lambda_n} \sum_{k=1}^K \left[ \sum_{i \in I_k} Z_i(Y_i - X_i'\hat{\beta}_{-k}(\lambda)) \right]' \frac{1}{q} \left[ \sum_{i \in I_k} Z_i(Y_i - X_i'\hat{\beta}_{-k}(\lambda)) \right].$$

The concrete choices of $K$ and $\Lambda_n$ are given in Section 7.

7 Monte Carlo

In this section we investigate the finite sample properties of the desparsified two-step GMM Lasso (DGMM) estimator and compare it to the desparsified two stage least squares (D2SLS) estimator of Gold et al. (2018). All designs are repeated $B = 100$ times as the procedures are computationally demanding. Before discussing the results, we explain how the data was generated and the performance measures used to compare DGMM to D2SLS.
7.1 Implementation details

The implementation of the D2SLS of Gold et al. (2018) is inspired by the publicly available code at https://github.com/LedererLab/HDIV/blob/master/src/estimation.r. We use five fold cross validation, $K = 5$, to select $\lambda_n$ and $\lambda_n^*$. $\Lambda_N$ is chosen by the glmnet package in R. As in Gold et al. (2018) we choose $\mu_j = 1.2 \cdot \inf_{a \in \mathbb{R}^p} \| a \Sigma - e'_j \|_\infty$, for $(a$ is a row vector) $j = 1, 2, ..., p$ and the minimization problem is solved by the MOSEK optimizer for R, R-MOSEK (2017).

7.2 Design 1

This design is inspired by the heteroskedastic design in Caner et al. (2018). We choose $Z_i \sim N_q(0, \Omega)$ with $\Omega_{j,k} = \rho_{|j-k|}$ for $\rho_{uz} = 0.25$ and set $X_i = \pi' Z_i + v_i$ where, for $\kappa_{i,k}$ being a $k \times 1$ vector of ones, we choose the $q \times p$ matrix $\pi = (2 + 2 \rho_{q/2}) - 1/2 (\kappa_2 \otimes I_{q/2})$. Thus, $p = q/2$. Furthermore, $Y_i = X_i' \theta_0 + u_i$ with $\theta_0 = (1, 1, 0'_{p-8}, 0.5, 0'_{5})'$ and $0_k$ being a $k \times 1$ vector of zeros. Thus, $\theta_0$ has three non-zero entries. The following notation is introduced to define the joint distribution of $v_i$ and $u_i$: Let $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2}, \epsilon_{i3}) \sim N_{p+2}(0, I_{p+2})$ where $\epsilon_{i1}$ and $\epsilon_{i2}$ are scalars and $\epsilon_{i3}$ is $p \times 1$.

$$\hat{u}_i = \sqrt{\rho_{uv}} \epsilon_{i1} + \sqrt{1 - \rho_{uv}} \epsilon_{i2} \quad \text{and} \quad v_i = \sqrt{\rho_{uv}} \epsilon_{i1} \kappa_{p} + \sqrt{1 - \rho_{uv}} \epsilon_{i3}$$

The following combinations of $n, p$ and $q$ are considered:

$$(n, p, q) \in \{(50, 50, 100), (75, 50, 100), (75, 10, 20), (75, 100, 200), (150, 100, 200), (150, 10, 20), (150, 200, 400), (300, 200, 400), (300, 10, 20)\}.$$

Note that these designs are in three categories: i) many moments/instruments and variables; $q > p > n$, ii) many moments/instruments; $q > n \geq p$ and iii) standard asymptotics; $n > q > p$. In the Tables 1-3 the results for these three settings can be found in columns i), ii) and iii), respectively.

7.3 Design 2

Everything is as in Design 1 except for $\pi = 1_{q,p}/q$ where $\pi = 1_{q,p}$ denotes a $q \times p$ matrix of ones. Thus, all instruments are (weakly) relevant.
7.4 Design 3

Everything is as in Design 1 except for \( \pi = (0.25 \cdot 1_{p,q/4}, 0_{p,3/4-q})' \) where \( 1_{p,q/4} \) is a \( p \times q/4 \) matrix of ones and \( 0_{p,3/4-q} \) is a \( p \times 3/4 \cdot q \) matrix of zeros.

7.5 Performance measures

The performance of D2SLS and DGMM are measured along the following dimensions.

1. Size: The size of the test in (16) is gauged by considered a test on \( \beta_{0,2} \) as in applied work interest often centers on a single coefficient (of the policy variable). The null hypotheses is always that this coefficient equals the true value assigned to (here the true value is always one). The nominal size of the test is 5%.

2. Power: To gauge the power of the test we test whether \( \beta_{0,j} \) equals its assigned value plus 1/2 in Design 1. In Designs 2 and 3 we test whether \( \beta_{0,j} \) equals its assigned value (which is 1) plus 1.5. The difference in alternatives is merely to obtain non-trivial power comparisons (i.e. to avoid either the power of all tests being (very close to) zero or (very close to) one).

3. Coverage rate: Let \( \hat{C}_j(\alpha) = \left[ \hat{b}_j - z_{1-\alpha/2} \frac{\hat{\sigma}_{b_j}}{\hat{n}^{1/2}}, \hat{b}_j + z_{1-\alpha/2} \frac{\hat{\sigma}_{b_j}}{\hat{n}^{1/2}} \right], j = 1, \ldots, p \) be the confidence intervals from Theorem 3. We calculate the average coverage rate across all \( p \) entries of \( \beta_0 \) and \( B = 100 \) Monte Carlo replications. We use \( \alpha = 0.05 \) throughout.

4. Length of confidence interval: We report the average length of the confidence intervals from Theorem 3 over all \( p \) entries of \( \beta_0 \) and \( B = 100 \) Monte Carlo replications.

5. MSE: We calculate the mean square error of \( \hat{b} \) across all \( B = 100 \) Monte Carlo replications

7.6 Results of simulations

In this section we report the results of our simulation study.

7.6.1 Design 1

Table 1 contains the results of Design 1. Our DGMM procedure is oversized (size above 5%) in 2 out of the 9 panels while the D2SLS is oversized in 1 out of 9 panels. In general, both procedures tend to be slightly undersized, however. Our DGMM procedure is non-inferior in terms of power in 7 out of 9 panels and achieves power advantages of up to 38%-point. Both procedures always have at least 95% coverage but the intervals produced by the DGMM procedure are more narrow in 7 out of 9 panels. Thus, the intervals are more informative. Finally, the MSE of the DGMM estimator is lower than the one of the D2SLS estimator in 8 out 9 panels; sometimes by more than a factor 10.
7.6.2 Design 2

Table 2 contains the results of Design 2. Recall that this is a setting where \( \pi \) is not sparse. While our DGMM estimator is oversized in 2 out 9 panels, the D2SLS is oversized in 7 out of 9 panels with sizes of up to 75%. Despite having generally lower size, the DGMM procedure has higher power than the D2SLS procedure in 8 out of 9 panels. Furthermore, the DGMM procedure does not exhibit undercoverage in terms of its confidence intervals for any of the 9 panels while D2SLS undercovers in 4 out 9 panels. The higher coverage of DGMM does not come at the price of longer confidence intervals as DGMM confidence intervals are always narrower than the ones stemming from D2SLS. Finally, the MSE is always lower for DGMM.

7.6.3 Design 3

Table 3 contains the results of Design 3. This design strikes a middle ground between Designs 1 and 2 in terms of the sparsity of \( \pi \). The tests based on DGMM and D2SLS are both oversized in 1 out of 9 panels. However, the former procedure results in more powerful tests than the latter in 6 out 9 panels. The largest power advantage of DGMM over D2SLS is 66%-point while the largest advantage of D2SLS over DGMM is 18%-point. The DGMM procedure always has at least 95% coverage while this is the case for 8 out of 9 panels for the D2SLS procedure. However, the DGMM procedure has a tendency to overcover. This tendency is less pronounced for the D2SLS procedure. Despite this fact, the confidence intervals resulting from the DGMM procedure are shorter than the ones stemming from the D2SLS procedure in 5 out of 9 panels. The DGMM procedure always has lower MSE than D2SLS.

8 Conclusion

This paper proposes a desparsified GMM estimator for estimating high-dimensional linear models with more endogenous variables than the sample size. The inference based on the estimator is shown to asymptotically uniformly valid even in the presence of conditionally heteroskedastic error terms. We do not impose the variables of the model to be sub-gaussian nor do we impose sparsity on the instruments. Finally, our results are shown to apply also to linear dynamic panel data models. Future work includes investigating the effect of the presence of (many) invalid instruments and potential remedies to this.

Appendix

This appendix consists of three parts. The first part is related to Theorem 1. The second part is related to estimation of the precision matrix. The third part considers the asymptotic properties of the new desparsified high dimensional GMM estimator.

Lemmas A.1-A.5 establish results that are used in the proof of Theorem 1 and parts of other proofs. Occasionally, we allow constants such as \( C \) and \( K \) to change from line to line and display to display. Except for in Lemmas A.1 and A.2 \( \kappa_n = \ln q \) throughout this appendix, cf also the remark prior to Assumption A.1.
Table 1:

| Design 1 | Design 2 | Design 3 | Design 4 |
|----------|----------|----------|----------|
| n = 50, p = 50, q = 100 | n = 75, p = 10, q = 20 |
| D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 17% | 21% | 2% | 17% | 4% | 2% |
| Power | 7% | 32% | 19% | 47% | 47% | 46% |
| Coverage | 0.9670 | 0.9574 | 0.9771 | 0.9992 | 0.9710 | 0.9820 |
| Length | 34.8783 | 8.9940 | 1.8287 | 0.9578 | 0.9629 | 0.9786 |
| MSE | 701.667 | 1.9198 | 0.1985 | 0.0315 | 0.0545 | 0.0333 |
| n = 75, p = 100, q = 200 | n = 150, p = 10, q = 20 |
| D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 0% | 0% | 1% | 0% | 2% | 3% |
| Power | 40% | 78% | 36% | 70% | 95% | 82% |
| Coverage | 0.9741 | 0.9897 | 0.9640 | 0.9957 | 0.9720 | 0.9630 |
| Length | 1.6973 | 0.7616 | 1.1276 | 0.8289 | 0.5705 | 0.6255 |
| MSE | 0.1615 | 0.0160 | 0.0169 | 0.0188 | 0.0169 | 0.0188 |
| n = 150, p = 200, q = 400 | n = 300, p = 10, q = 20 |
| D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 2% | 0% | 4% | 0% | 1% | 3% |
| Power | 62% | 97% | 83% | 97% | 100% | 100% |
| Coverage | 0.9709 | 0.9963 | 0.9537 | 0.9973 | 0.9650 | 0.9710 |
| Length | 0.9611 | 0.5521 | 0.6792 | 0.5750 | 0.3531 | 0.3941 |
| MSE | 0.0477 | 0.0080 | 0.0289 | 0.0080 | 0.0065 | 0.0037 |

8.1 Two auxiliary lemmas

We first provide concentration inequalities for maxima of centered iid sums. These are taken directly from Lemmas E.1 and E.2 of Chernozhukov et al. (2017) specialized to our iid setting to simplify the used conditions slightly. They can be found in Lemmas A.1 and A.2. To set the stage assume that $F_i = (F_{i1}, ..., F_{ij}, ..., F_{id})' \in \mathbb{R}^d$ and that the vectors are iid across $i = 1, ..., n$. Define

$$
\max_{1 \leq j \leq d} \left| \sum_{i=1}^{n} (F_{ij} - EF_{ij}) \right| = n \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j|,
$$

where $\hat{\mu}_j = n^{-1} \sum_{i=1}^{n} F_{ij}$, $\mu_j = EF_{ij}$. Next, define

$$
M_F = \max_{1 \leq i \leq n} \max_{1 \leq j \leq d} |F_{ij} - EF_{ij}|,
$$

and

$$
\sigma_F^2 = \max_{1 \leq j \leq d} \sum_{i=1}^{n} \mathbb{E} \left| F_{ij} - EF_{ij} \right|^2 = n \max_{1 \leq j \leq d} \text{var}(F_{ij}).
$$
n = 50, p = 50, q = 100  |  n = 75, p = 50, q = 100  |  n = 75, p = 10, q = 20

| Design 2 | D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
|----------|-------|-------|-------|-------|-------|-------|
| Size     | 56%   | 21%   | 57%   | 16%   | 32%   | 1%    |
| Power    | 59%   | 36%   | 66%   | 93%   | 47%   | 66%   |
| Coverage | 0.9572| 0.9634| 0.9576| 0.9692| 0.9040| 0.9890|
| Length   | 51.8943| 14.0387| 57.6295| 1.3173| 10.6126| 2.3726|
| MSE      | 9354.21| 26.4915| 10472.80| 0.0670| 46.2285| 0.2014|

n = 75, p = 100, q = 200  |  n = 150, p = 100, q = 200  |  n = 150, p = 10, q = 20

| Design 2 | D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
|----------|-------|-------|-------|-------|-------|-------|
| Size     | 61%   | 5%    | 74%   | 0%    | 9%    | 0%    |
| Power    | 63%   | 100%  | 76%   | 100%  | 33%   | 85%   |
| Coverage | 0.9523| 0.9848| 0.9362| 0.9930| 0.9580| 0.9890|
| Length   | 73.0592| 0.9154| 72.2950| 1.0717| 5.5107| 2.2228|
| MSE      | 30676.17| 0.0303| 323884.4| 0.0339| 15.1127| 0.1712|

n = 150, p = 200, q = 400  |  n = 300, p = 200, q = 400  |  n = 300, p = 10, q = 20

| Design 2 | D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
|----------|-------|-------|-------|-------|-------|-------|
| Size     | 66%   | 2%    | 74%   | 0%    | 1%    | 0%    |
| Power    | 67%   | 100%  | 75%   | 100%  | 63%   | 83%   |
| Coverage | 0.8230| 0.9921| 0.8126| 0.9947| 0.9830| 0.9820|
| Length   | 106.34| 0.6914| 145.48| 0.7522| 2.9207| 2.0897|
| MSE      | 110527.2| 0.0150| 137180.5| 0.0172| 0.5160| 0.1907|

Table 2:

From Lemma E.1 of Chernozhukov et al. (2017) one has that there exists a universal constant $K > 0$ such that

$$nE_{\max_{1\leq j\leq d}} |\hat{\mu}_j - \mu_j| \leq K\sqrt{n \max_{1\leq j\leq d} \text{var}(F_{ij})} \sqrt{\ln d} + \sqrt{EM_F^2 \ln d},$$

which implies

$$E_{\max_{1\leq j\leq d}} |\hat{\mu}_j - \mu_j| \leq K\sqrt{\max_{1\leq j\leq d} \text{var}(F_{ij})} \ln d + \sqrt{EM_F^2 \ln d}. \tag{A.1}$$

Next, Lemma E.2(ii) of Chernozhukov et al. (2017) states that for all $\eta > 0, t > 0, \gamma \geq 1$

$$P\left[\max_{1\leq j\leq d} |\hat{\mu}_j - \mu_j| \geq 2E_{\max_{1\leq j\leq d}} |\hat{\mu}_j - \mu_j| + \frac{t}{n}\right] = P\left[n \max_{1\leq j\leq d} |\hat{\mu}_j - \mu_j| \geq (1 + \eta)nE_{\max_{1\leq j\leq d}} |\hat{\mu}_j - \mu_j| + t\right]$$

$$\leq \exp\left(-t^2/3\sigma_F^2\right) + K\frac{EM_F^2}{t\gamma}. \tag{A.2}$$

Remark 2. For ease of reference, we state the versions of (A.1) and (A.2) appropriate for our purpose as a lemma. In the rest of the paper we shall use $\eta = 1$ and $\gamma = 2$. Furthermore, for the purpose of proving
| Design 3-Semi-Sparse No of Instruments | 
|-------------------------------------|
| \(n = 50, p = 50, q = 100\) | \(n = 75, p = 50, q = 100\) | \(n = 75, p = 10, q = 20\) |
| D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 8% | 10% | 1% | 0% | 1% | 1% |
| Power | 13% | 19% | 45% | 71% | 29% | 68% |
| Coverage | 0.9404 | 0.9802 | 0.9646 | 0.9956 | 0.9760 | 0.9860 |
| Length | 32.0919 | 14.1922 | 3.5628 | 2.4496 | 11.3496 | 2.3582 |
| MSE | 550.36 | 4.3972 | 0.9067 | 0.0912 | 53.7831 | 0.2082 |

| \(n = 75, p = 100, q = 200\) | \(n = 150, p = 100, q = 200\) | \(n = 150, p = 10, q = 20\) |
| D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 2% | 0% | 4% | 0% | 0% | 0% |
| Power | 99% | 81% | 78% | 81% | 18% | 83% |
| Coverage | 0.9514 | 0.9995 | 0.9706 | 0.9999 | 0.9740 | 0.9870 |
| Length | 1.0409 | 2.2356 | 1.9492 | 2.2473 | 9.0591 | 2.2711 |
| MSE | 0.0715 | 0.0372 | 0.2123 | 0.0451 | 16.0593 | 0.1971 |

| \(n = 150, p = 200, q = 400\) | \(n = 300, p = 200, q = 400\) | \(n = 300, p = 10, q = 20\) |
| D2SLS | D2GMM | D2SLS | D2GMM | D2SLS | D2GMM |
| Size | 4% | 0% | 3% | 0% | 3% | 2% |
| Power | 100% | 92% | 98% | 93% | 19% | 85% |
| Coverage | 0.9543 | 1.00 | 0.9701 | 1.00 | 0.9660 | 0.9730 |
| Length | 0.6188 | 2.2290 | 1.1898 | 2.1920 | 7.3244 | 2.0423 |
| MSE | 0.0266 | 0.0192 | 0.0716 | 0.0223 | 8.6713 | 0.2151 |

Table 3:

Our theorems in the case of \(q \geq p > n\), we introduce the sequence \(\kappa_n\). This sequence will be chosen to equal \(\ln(q)\). At the expense of a bit more involved notation and slightly altered assumptions, one can also set \(\kappa_n = \max(\ln q, \ln n)\) in order to handle all possible regimes/orderings of \(p, q\) and \(n\). The following lemma is stated for a maximum over \(d\) terms, where often in the sequel we will have \(d = q\).

**Assumption A.1.** Assume \(F_i\) are iid random \(d \times 1\) vectors across \(i = 1, ..., n\) with \(\max_{1 \leq j \leq d} \text{var}(F_{ij})\) bounded from above. Finally, let \(\kappa_n = \ln d\).

**Lemma A.1. Under Assumption A.1**

(i).

\[
E \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| \leq K \left[ \sqrt{\frac{\max_{1 \leq j \leq d} \text{var}(F_{ij}) \ln d}{n}} + \sqrt{\frac{EM_F}{n} \ln d} \right] \leq K \left[ \sqrt{\frac{\ln d}{n} + \sqrt{\frac{EM_F}{n} \ln d}} \right],
\]
(ii) Set $t = t_n = (n\kappa_n)^{1/2} = (n \ln d)^{1/2}$. There exist constants $C, K > 0$ such that
\[
P \left[ \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| \geq 2E \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| + \left( \frac{\kappa_n}{n} \right)^{1/2} \right] \leq \exp(-C\kappa_n) + K \frac{EM_k^2}{n\kappa_n} = \frac{1}{d^C} + K \frac{EM_k^2}{n \ln d}.
\]

For the purpose of obtaining asymptotic results, we introduce the following assumption.

**Assumption A.2.**
\[
\frac{(EM_k^2)^{1/2} \sqrt{\ln d}}{\sqrt{n}} \to 0.
\]

**Lemma A.2.** Under Assumptions A.1 A.2

(i).
\[
P \left( \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| \leq 2E \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| + \left( \frac{\kappa_n}{n} \right)^{1/2} \right) = 1.
\]

(ii).
\[
E \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| = O\left( \sqrt{\ln d/n} \right).
\]

(iii). Thus,
\[
\max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| = O_p\left( 2E \max_{1 \leq j \leq d} |\hat{\mu}_j - \mu_j| + \left( \frac{\kappa_n}{n} \right)^{1/2} \right) = O_p\left( \sqrt{\ln \kappa_n/n} \right) = O_p\left( \sqrt{\ln d/n} \right)
\]

In the sequel we use the above two lemmata with $\kappa_n = \ln q$.

### 8.2 Some useful events

To establish the desired oracle inequality for the estimation error of our estimator we need to bound certain moments. Let $M_1, M_2$ be as defined before Assumption 2. Define
\[
\mathcal{A}_1 = \{ \| Z'u/n \|_\infty \leq t_1/2 \},
\]
where
\[
t_1 = 2K \left[ C \frac{\sqrt{\ln q}}{\sqrt{n}} + \frac{\sqrt{EM_k^2 \ln q}}{n} \right] + \sqrt{\frac{\kappa_n}{n}}.
\]
for some $C > 0$ made precise below. Next, define the set
\[
\mathcal{A}_2 = \{ \| \frac{Z'X}{n} \|_\infty \leq t_2 \},
\]
with $t_2 = t_3 + C$, where
\[
t_3 = 2K \left[ \frac{C \sqrt{\ln pq}}{\sqrt{n}} + \frac{\sqrt{EM_k^2 \ln (pq)}}{n} \right] + \sqrt{\frac{\kappa_n}{n}}.
\]
Now we provide probabilities on the bounds and asymptotic rates.

**Lemma A.3.** (i) Under Assumption 1,
\[
P(\mathcal{A}_1) \geq 1 - \exp(-C\kappa_n) - \frac{KEM_k^2}{n\kappa_n},
\]
where $C > 0$ is the constant from Lemma A.1
(ii). Adding Assumption 2 to (i)

\[ \| \frac{Z' u}{n} \|_\infty = O_p(\frac{\sqrt{\ln q}}{\sqrt{n}}). \]

(iii). Under Assumption 1,

\[ P(A_2) \geq 1 - \exp(-C\kappa n) - \frac{KEM^2}{n\kappa n}, \]

(iv). Adding Assumption 2 to (iii)

\[ \| \frac{Z' X_n}{n} - E[Z' X_n] \|_\infty = O_p(\frac{\sqrt{\ln q}}{\sqrt{n}}). \]

\[ \| \frac{Z' X_n}{n} \|_\infty = O_p(1). \]

**Proof of Lemma A.3.** (i)-(ii). First, note that by Lemma A.1, replacing \( F_i \) with \( Z_i u_i \) which is a \( q \times 1 \) vector, under Assumption 1, \( A_1 \) has probability at least \( 1 - \exp(-C\kappa n) - KEM^2 1^{2n} \), Adding Assumption 2, via Lemma A.2,

\[ \| \frac{Z' u}{n} \|_\infty = O_p(\frac{\sqrt{\ln q}}{\sqrt{n}}). \]

(A.7)

(iii)-(iv). Next, consider \( \| \frac{Z' X_n}{n} \|_\infty \). By Lemma A.1

\[ \max_{1 \leq l \leq q} \max_{1 \leq j \leq p} \frac{\sum_{i=1}^n [Z_{il} X_{ij} - EZ_{il} X_{ij}]}{n} > t_3 \leq \exp(-C\kappa n) + \frac{KEM^2}{n\kappa n}, \]

(A.8)

In conjunction with Assumption 2 (A.8) implies, via Lemma A.2, and using \( p \leq q \)

\[ \max_{1 \leq l \leq q} \max_{1 \leq j \leq p} n^{-1} \sum_{i=1}^n [Z_{il} X_{ij} - EZ_{il} X_{ij}] = O_p(\sqrt{\frac{\ln(pq)}{n}}) = O_p(\sqrt{\frac{\ln q}{n}}). \]

(A.9)

Also, by Assumption 1 and Cauchy-Schwarz inequality

\[ \max_{1 \leq l \leq q} \max_{1 \leq j \leq p} |EZ_{il} X_{ij}| = O(1). \]

(A.10)

Combining (A.9) with (A.10) we have that \( A_2 \) occurs with probability at least \( 1 - \exp(-C\kappa n) - \frac{KEM^2}{n\kappa n} \)

**8.3 Oracle inequality for the first step estimator**

Lemmata A.4 and A.5 below are needed for the proof of Theorem 1. Define the norm \( \|x\|_n = (x'x/n)^{1/2} = \|x\|_2/n^{1/2} \) on \( \mathbb{R}^q \). One can thus write

\[ \hat{\beta}_F = \arg\min_{\beta \in \mathbb{R}^p} \left[ \|Z'(Y - X\hat{\beta}_F)\|_n^2 + 2\lambda_n \|\beta\|_1 \right]. \]

Define also the sample covariance between regressors and instruments:

\[ \hat{\Sigma}_{xz} = \frac{X'Z}{n}. \]

With this notation in place we can introduce the concept of empirical adaptive restricted eigenvalue in GMM:

\[ \hat{\phi}^2_{\hat{\Sigma}_{xz}}(s) = \min \left\{ \frac{\delta' \hat{\Sigma}_{xz} \delta}{q\|\delta\|_2^2} : \delta \in \mathbb{R}^p \setminus \{0\}, \|\delta_{S^c}\|_1 \leq 3\sqrt{s}\|\delta_S\|_2, |S| \leq s \right\}. \]

(A.11)
We also define the population adaptive restricted eigenvalue for the first step GMM: $\phi_{Zzz}^2(s)$, as (7) evaluated at $W_d = I_q$. In the sequel we shall choose

$$\lambda_n = t_1t_2 = \left[2K\left(\frac{C\sqrt{\ln q}}{\sqrt{n}} + \sqrt{EM^2_q\ln q} + \sqrt{\frac{\kappa_n}{n}}\right) + 2K\left(\frac{C\sqrt{\ln pq}}{\sqrt{n}} + \sqrt{EM^2_p\ln(pq)} + \sqrt{\frac{\kappa_n}{n}} + C\right)\right]$$

(A.12)

and note that under Assumption 2, $\lambda_n = O(\sqrt{\ln q/n})$.

**Lemma A.4.** Under Assumptions 1 and 2, for universal positive constants $K, C$, for $n$ sufficiently large one has with probability at least $1 - 3\exp(-C\kappa_n) - K\frac{EM^2_q + EM^2_p}{\kappa_n n}$

(i). 

$$\|\hat{\lambda}_F - \lambda_0\|_1 \leq \frac{24\lambda_n s_0}{\phi_{Zzz}^2(s_0)}.$$ 

(ii). The result in (i) holds with probability approaching one and we have $\lambda_n = O(\sqrt{\ln q/n})$ as seen in (A.19) below. (i) is valid uniformly over $B_{l_0}(s_0) = \{\|\beta_0\|_{l_0} \leq s_0\}$.

**Proof of Lemma A.4.** (i). Since

$$\|Z'(Y - X\hat{\beta}_F)\|_n^2 = \frac{1}{n} \left( (Y - X\hat{\beta}_F)Z' nq (Y - X\hat{\beta}_F) \right),$$

the minimizing property of $\hat{\beta}_F$ implies that

$$\|Z'(Y - X\hat{\beta}_F)\|_n^2 + 2\lambda_n \sum_{j=1}^p |\hat{\beta}_{F,j}| \leq \|Z'(Y - X\beta_0)\|_n^2 + 2\lambda_n \sum_{j=1}^p |\beta_{0,j}|.$$ 

(A.13)

Next use that $Y = X\beta_0 + u$ and simplify to get

$$\|Z'X(\hat{\beta}_F - \beta_0)\|_n^2 + 2\lambda_n \sum_{j=1}^p |\hat{\beta}_{F,j}| \leq 2\frac{u'Z}{nq} \frac{Z'X}{nq} (\hat{\beta}_F - \beta_0) + 2\lambda_n \sum_{j=1}^p |\beta_{0,j}|.$$ 

(A.14)

Consider the first term on the right side of (A.14), and denote the $l$th row of $Z'X$ by $(Z'X)_l$, $l = 1, 2, ..., q$:

$$2\frac{u'Z}{nq} \frac{Z'X}{nq} (\hat{\beta}_F - \beta_0) \leq 2\frac{u'Z}{nq} \frac{Z'X}{nq} (\hat{\beta}_F - \beta_0)\|_1 \leq 2\frac{u'Z}{nq} \frac{Z'X}{nq} (\hat{\beta}_F - \beta_0)\|_1 \leq 2\frac{u'Z}{nq} \frac{Z'X}{nq} (\hat{\beta}_F - \beta_0)\|_1,$$

(A.15)

(A.16)

where we use Hölder’s inequality in (A.15) and (S.1) in (A.16).

Assume that $A_1 \cap A_2$ occurs (we shall later provide a lower bound on the probability of this). By (A.3)(A.5), in (A.17) we have on $A_1 \cap A_2$

$$2\frac{u'Z}{nq} \frac{Z'X}{nq} \left( \max_{1 \leq l \leq q} \frac{(Z'X)_l}{nq} \right) \|\hat{\beta}_F - \beta_0\|_1 \leq \lambda_n \|\hat{\beta}_F - \beta_0\|_1.$$ 

(A.17)

We note that by Assumption 2, Lemma A.2, (A.7) and Lemma A.3 (iv)

$$\lambda_n = O(\sqrt{\frac{\ln q}{n}}).$$ 

(A.19)
In combination with (A.14) we get:

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + 2\lambda_n \sum_{j=1}^{p} |\hat{\beta}_{F,j}| \leq \lambda_n \|\hat{\beta}_F - \beta_0\|_1 + 2\lambda_n \sum_{j=1}^{p} |\beta_{0,j}|. \tag{A.20}
$$

Next, use that $$\|\hat{\beta}_F\| = \|\hat{\beta}_{F,S_0}\|_1 + \|\hat{\beta}_{F,S_0^c}\|_1$$ on the second term on the left side of (A.20)

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + 2\lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq \lambda_n \|\hat{\beta}_F - \beta_0\|_1 + 2\lambda_n \sum_{j \in S_0} |\hat{\beta}_{F,j} - \beta_{0,j}|, \tag{A.21}
$$

where we used the reverse triangle inequality to get the last term on the right side of (A.21) and $$\|\beta_{0,S_0^c}\|_1 = 0$$. Using that $$\|\hat{\beta}_F - \beta_0\|_1 = \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_1 + \|\hat{\beta}_{F,S_0^c}\|_1$$ on the first right hand side term in (A.21) yields

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + \lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq 3\lambda_n \sum_{j \in S_0} |\hat{\beta}_{F,j} - \beta_{0,j}|. \tag{A.22}
$$

Furthermore, using that $$\|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_1 \leq \sqrt{n_0} \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_2$$ in (A.22)

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + \lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq 3\lambda_n \sqrt{n_0} \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_2. \tag{A.23}
$$

We see that the restricted set condition in (A.11) is satisfied by ignoring the first term on the right side of (A.23) and dividing each side by $$\lambda_n$$

$$
\|\hat{\beta}_{F,S_0^c}\|_1 \leq 3\sqrt{n_0} \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_2.
$$

Thus, the empirical adaptive restricted eigenvalue condition in (A.11) can be used in (A.23)

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + \lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq 3\lambda_n \sqrt{n_0} \frac{\|Z'X(\hat{\beta}_F - \beta_0)\|_n}{\hat{\phi}_{\Sigma_X}(s_0)}. \tag{A.24}
$$

Next, use $$3uv \leq u^2/2 + 9v^2/2$$ with $$u = \|Z'X(\hat{\beta}_F - \beta_0)\|_n$$ and $$v = \lambda_n \sqrt{n_0} / \hat{\phi}_{\Sigma_X}(s_0)$$ to get

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + \lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq \frac{\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n}{2} + \frac{9}{2} \lambda_n \frac{\lambda_n^2 s_0}{\hat{\phi}_{\Sigma_X}(s_0)}. \tag{A.25}
$$

Multiply each side of (A.25) by 2 and simplify to get

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + 2\lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq \frac{9\lambda_n^2 s_0}{\hat{\phi}_{\Sigma_X}(s_0)}. \tag{A.26}
$$

Next, assume that $$\mathcal{A}_3 = \{\hat{\phi}_{\Sigma_{xz}}(s_0) \geq \hat{\phi}_{\Sigma_{xx}}(s_0)/2\}$$ such that we are working on $$\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$$. Thus,

$$
\|Z'X(\hat{\beta}_F - \beta_0)\|^2_n + 2\lambda_n \sum_{j \in S_0^c} |\hat{\beta}_{F,j}| \leq \frac{18\lambda_n^2 s_0}{\hat{\phi}_{\Sigma_{xz}}(s_0)}. \tag{A.27}
$$

So we have the oracle inequality on set $$\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$$.

To get the l1 error bound ignore the first term in (A.23) above and add $$\lambda_n \|\hat{\beta}_{S_0} - \beta_{0,S_0}\|_1$$ to both sides to get

$$
\lambda_n \|\hat{\beta}_F - \beta_0\|_1 \leq \lambda_n \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_1 + 3\lambda_n \sqrt{n_0} \|\hat{\beta}_{F,S_0^c} - \beta_{0,S_0^c}\|_2. \tag{A.28}
$$

Then use $$\|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_1 \leq \sqrt{n_0} \|\hat{\beta}_{F,S_0} - \beta_{0,S_0}\|_2$$ that as well as the empirical adaptive restricted eigenvalue condition for GMM in (A.11)

$$
\lambda_n \|\hat{\beta}_F - \beta_0\|_1 \leq 4\lambda_n \frac{\sqrt{n_0} \|Z'X(\hat{\beta}_F - \beta_0)\|_n}{\hat{\phi}_{\Sigma_X}(s_0)}. \tag{A.29}
$$
Next, as $A_3$ is assumed to occur and using the prediction norm upper bound established in (A.27) results in

$$
\|\hat{\beta}_F - \beta_0\|_1 \leq \frac{24\lambda_n s_0}{\phi_{L_\infty}(s_0)}.
$$

(A.30)

Thus, in total

$$
1 - P(A_1^c) - P(A_2^c) - P(A_3^c) = 1 - 3 \exp(-C\kappa_n) - \frac{K(EM_1^2 + 2EM_2^2)}{n\kappa_n} \to 1,
$$

by Assumption 2 where the convergence to 1 establishes ii).

\[\square\]

8.4 Controlling $\min_{1 \leq i \leq q} \hat{\sigma}_i^2$ for $\hat{W}_d$

**Lemma A.5.** Under Assumptions 1 and 2 as well as $r_z \geq 12, r_x \geq 6, r_\nu \geq 8$, we have that for $n$ sufficiently large there exists a $C > 0$ such that

$$
\min_{1 \leq i \leq q} \hat{\sigma}_i^2 \geq \min_{1 \leq i \leq q} \sigma_i^2/2,
$$

with probability at least $1 - 9 \exp(-C\kappa_n) - \frac{K[2EM_1^2 + 4EM_2^2 + EM_3^2 + EM_4^2 + EM_5^2]}{n\kappa_n} \to 1$. The result is valid uniformly over $B_{\alpha} = \{\|\beta_0\|_0 \leq s_0\}$.

**Remark 3.** In the course of the proof of Lemma A.5 we actually establish that under the assumptions of said lemma,

$$
P\left(\max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2| \geq c_n\right) \leq 9 \exp(-C\kappa_n) + \frac{K[2EM_1^2 + 4EM_2^2 + EM_3^2 + EM_4^2 + EM_5^2]}{n\kappa_n} \to 0
$$

for a sequence $c_n \to 0$ (where $c_n$ is defined precisely in (A.33)). To be precise, the proof reveals that $c_n = O(s_0\sqrt{\ln q/n})$.

**Proof of Lemma A.5.** First, note that

$$
\min_{1 \leq i \leq q} \hat{\sigma}_i^2 \geq \min_{1 \leq i \leq q} \sigma_i^2 - \max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2|.
$$

(A.31)

We start by upper bounding $\max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2|$. To this end, note that we can write

$$
\hat{\sigma}_i^2 = \frac{1}{n} \sum_{i=1}^n Z_{ii}^2 \hat{u}_i^2 = \frac{1}{n} \sum_{i=1}^n Z_{ii}^2 |u_i - X'_i(\hat{\beta}_F - \beta_0)|^2.
$$

Then, by the triangle inequality

$$
\max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2| \leq \max_{1 \leq i \leq q} \frac{1}{n} \sum_{i=1}^n (Z_{ii}^2 u_i^2 - EZ_{ii}^2 u_i^2) + \max_{1 \leq i \leq q} \frac{2}{n} \sum_{i=1}^n Z_{ii}^2 |u_i| |X'_i(\hat{\beta}_F - \beta_0)|
$$

\[+ \max_{1 \leq i \leq q} |(\hat{\beta}_F - \beta_0)| \sum_{i=1}^n \frac{Z_{ii}^2 |X'_i|}{n} (\hat{\beta}_F - \beta_0).\]

(A.32)

Define the following events in order to upper bound the right hand side of A.32

$$
B_1 = \{ \max_{1 \leq i \leq q} |n^{-1} \sum_{i=1}^n (Z_{ii}^2 u_i^2 - EZ_{ii}^2 u_i^2)| \leq t_4 \}.
$$

$$
B_2 = \{ \max_{1 \leq i \leq q} \max_{1 \leq j \leq p} |n^{-1} \sum_{i=1}^n Z_{ii}^2 u_i X_{ij}| \leq t_5 \}.
$$

$$
B_3 = \{ \max_{1 \leq i \leq q} \max_{1 \leq j \leq p} \max_{1 \leq k \leq p} |n^{-1} \sum_{i=1}^n Z_{ii}^2 X_{ij} X_{ik}| \leq t_6 \}.
$$

22
We analyze the first term on the right side of (A.35). Next by Lemma A.1, \[ C[t_4 + t_5 \lambda_n s_0 + t_0 \lambda_n^2 s_0^2] =: c_n \leq \min_{1 \leq t \leq q} \sigma^2_t / 2 \] (A.33)

Consider \( B_1 \) first. By Lemma A.1, with \( F_{il} = Z_{il}^3 u_i^2 \) and \( d = q \) via Assumption 1, and \( r_z \geq 8, r_u \geq 8 \) as well as the Cauchy-Schwarz inequality

\[
P \left( \max_{1 \leq t \leq q} \left| \sum_{i=1}^n \left( Z_{il}^3 u_i^2 - EZ_{il}^3 u_i^2 \right) \right| > t_4 \right) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n}, \quad (A.34)
\]

with \( t_4 = 2K[C_U \frac{\ln q}{n} + \sqrt{EM^2 \ln q} + \sqrt{\frac{\kappa_n}{n}}] \). Note that by Assumption 2 one has \( t_4 \to 0 \). Next consider the second term on the right side of (A.32). By Hölder’s inequality

\[
\max_{1 \leq t \leq q} \left| \frac{1}{n} \sum_{i=1}^n Z_{il}^3 u_i X_i (\beta_F - \beta_0) \right| \leq \max_{1 \leq t \leq q} \left| \frac{1}{n} \sum_{i=1}^n Z_{il}^3 u_i X_i \right| \| \beta_F - \beta_0 \|_1.
\] (A.35)

We analyze the first term on the right side of (A.35). Next by Lemma A.1

\[
P \left( \max_{1 \leq t \leq q} \max_{1 \leq j \leq p} \left| \sum_{i=1}^n \left( Z_{il}^3 u_i X_{ij} - EZ_{il}^3 u_i X_{ij} \right) \right| > t_5^* \right) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n}, \quad (A.36)
\]

where

\[ t_5^* = 2K[C_U \frac{\ln q}{n} + \sqrt{EM^2 \ln pq} + \sqrt{\frac{\kappa_n}{n}}] \to 0. \] (A.37)

by Assumption 2. By Assumption 1 with \( r_z \geq 12, r_u \geq 8, r_x \geq 6 \)

\[
E|Z_{il}^3 u_i X_{ij}|^2 = E|Z_{il}^3 u_i X_{ij}|^2 \leq \begin{cases} E|Z_{il}|^{12/3} |E|u_i|^3 |E|X_{ij}|^3 |^{2/3} \leq |E|Z_{il}|^{12/3} |E|u_i|^5 |E|X_{ij}|^6 |^{1/3} \leq C < \infty. \end{cases} \quad (A.38)
\]

such that

\[
\max_{1 \leq t \leq q} \max_{1 \leq j \leq p} \left| E(Z_{il}^3 u_i X_{ij}) \right| \leq C. \quad (A.39)
\]

Thus, with \( t_5 = t_5^* + C = O(1) \) by Assumption 2 we have that \( P(B_2) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n} \). Using this in (A.35) together with the upper bound on \( \| \beta_F - \beta_0 \|_1 \) in Lemma A.4 yields

\[
\max_{1 \leq t \leq q} \left( \frac{1}{n} \sum_{i=1}^n Z_{il}^3 u_i X_i (\beta_F - \beta_0) \right) \leq C[t_5 \lambda_n s_0] = C[C + t_5^*] \lambda_n s_0 \to 0, \quad (A.40)
\]

with probability at least \( 1 - 4\exp(-C\kappa_n) - \frac{K(EM^2 + EME^2 + EM^2)}{n\kappa_n} \) and where the convergence to 0 is by Assumption 2. We now turn to \( B_3 \). By Assumption 1, and similar analysis in (A.38) gives

\[
P \left( \max_{1 \leq t \leq q} \max_{1 \leq j \leq p} \max_{1 \leq k \leq p} \left| \frac{1}{n} \sum_{i=1}^n \left( Z_{il}^3 X_{ij} X_{ik} - EZ_{il}^3 X_{ij} X_{ik} \right) \right| > t_6^* \right) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n}, \quad (A.41)
\]

where

\[ t_6^* = 2K[C_U \frac{\ln p^2 q}{n} + \sqrt{EM^2 \ln p^2 q} + \sqrt{\frac{\kappa_n}{n}}] \to 0. \quad (A.42)\]
Thus, with probability at least \( \exp(-C\kappa_n) \), we have that 
\[
\frac{1}{n} \sum_{i=1}^{n} Z_{it}^2 X_i X'_i \leq t_6 \quad \text{where} \quad t_6 = t_6^* + C = O(1) \quad \text{by Assumption 2. Thus, using also Lemma A.4 one has}
\]
\[
\max_{1 \leq l \leq q} \left| \hat{\beta}_l - \beta_0 \right| \leq \left\| \hat{\beta}_l - \beta_0 \right\|_2 \left( \max_{1 \leq l \leq q} \max_{1 \leq j \leq p} t_6 \right) \leq C t_6 \lambda_n^2 s_0^2 \to 0,
\]
with probability at least \( 1 - 4 \exp(-C\kappa_n) = - \frac{KEM^2}{n\kappa_n} \) and where the convergence to zero is by Assumption 2. The above results are valid uniformly over \( t_0 \) ball: \( B_{t_0} = \{ \| \beta_0 \|_2 \leq s_0 \} \). This can be seen by (A.40) and (A.44) since the dependence on \( \beta_0 \) in the bounds is through \( s_0 \) only. \( \square \)

8.5 Proof of Theorem 1

For the purpose of proving Theorem 1 below we introduce the empirical version of the adaptive restricted eigenvalue condition for GMM at \( s = s_0 \):

\[
\phi^2_{\Sigma_{n+1}, n} (s_0) = \min \left\{ \left\| \frac{\hat{W}_n}{n} \delta \right\|_{2}^2 : \delta \in \mathbb{R}^p \setminus \{0\}, \| \delta S_0 \|_1 \leq 3\sqrt{s_0} \| \delta S_0 \|_2, \| S_0 \| \leq s_0 \right\}
\]

\[
= \min \left\{ \left\| \frac{\hat{W}_n^{1/2}}{n} \delta X \right\|_{2}^2 : \delta \in \mathbb{R}^p \setminus \{0\}, \| \delta S_0 \|_1 \leq 3\sqrt{s_0} \| \delta S_0 \|_2, \| S_0 \| \leq s_0 \right\}.
\]  

(A.45)

Furthermore, we shall choose \( \lambda_n^* \) as follows:

\[
\lambda_n^* = \frac{2}{\min_{1 \leq l \leq q} \sigma^2_{q}} = \frac{2}{\min_{1 \leq l \leq q} \sigma^2_{q}} \left[ 2K \left[ \frac{C \sqrt{\ln q}}{\sqrt{n}} + \frac{\sqrt{EM^2 \ln q}}{n} \right] + \sqrt{\frac{\kappa_n}{n}} \right] \left[ 2K \left[ \frac{C \sqrt{\ln pq}}{\sqrt{n}} + \frac{\sqrt{EM^2 \ln (pq)}}{n} \right] + \sqrt{\frac{\kappa_n}{n}} + C \right].
\]  

(A.46)

where we used the definition of \( \lambda_n \) in (A.12). Recall that \( C \) and \( K \) are universal constants guaranteed to exist by Lemma A.1. Note that under Assumptions 1 and 2, one has that \( \lambda_n^* = O(\sqrt{\ln q/n}) \).

Proof of Theorem 1. i) The proof is very similar to the one of Lemma A.4 above. Thus, we only point out the differences. There are four main differences. The first one is the set up of instruments, the second one is the noise term, and the third one is the empirical adaptive restricted eigenvalue condition, the fourth one is the tuning parameter. We will show how each component changes the proof. First, the instrument matrix is transformed from \( Z \) to \( \hat{Z} = Z \hat{W}_d^{1/2} \), which is again a \( n \times q \) matrix but

\[
\hat{Z} = \left[ Z_1, \ldots, Z_l, \ldots, Z_q \right] \begin{bmatrix} 1/\hat{\sigma}_1 & \cdots & 0 \\ \vdots & 1/\hat{\sigma}_l & \vdots \\ 0 & \cdots & 1/\hat{\sigma}_q \end{bmatrix}.
\]  

(A.47)

\( \hat{W}_d^{1/2} \) is a diagonal matrix and \( \hat{\sigma}^2 = n^{-1} \sum_{i=1}^{n} Z_{it}^2 \hat{u}_i^2 \), with \( \hat{u}_i = Y_i - X'_i \hat{\beta}_F \). Note that, \( \| \hat{Z}'(Y - X \hat{\beta}) \|_n^2 = \frac{(Y - X \hat{\beta})'Z \hat{W}_d Z'(Y - X \hat{\beta})}{n \times q} \). Using the definition of \( \hat{\beta} \) in (4) yields

\[
\| \hat{Z}'(Y - X \hat{\beta}) \|_n^2 + 2\lambda_n^* \sum_{j=1}^{p} |\hat{\beta}_j| \leq \| \hat{Z}'(Y - X \beta_0) \|_n^2 + 2\lambda_n^* \sum_{j=1}^{p} |\beta_{0,j}|.
\]  

(A.48)
After (A.48) we continue as in (A.13)-(A.17) with (A.47) and remembering that \((\tilde{W}_d Z'X)_l\) is \(l\)th row of \(\tilde{W}_d Z'X\).

\[
\|\tilde{Z}'X(\tilde{\beta} - \beta_0)\|_n^2 + 2\lambda_n^* \sum_{j=1}^p |\tilde{\beta}_j| \leq 2 \left| \frac{u^T \tilde{Z}'X(\tilde{\beta} - \beta_0)}{n} \right| + 2\lambda_n^* \sum_{j=1}^p |\beta_{0,j}| \\
\leq 2 \left| \frac{u^T \tilde{Z}'X(\tilde{\beta} - \beta_0)}{n} \right| + 2\lambda_n^* \sum_{j=1}^p |\beta_{0,j}|
\]

(A.49)

\[
\leq 2 \left| \frac{u^T \tilde{Z}'X(\tilde{\beta} - \beta_0)}{n} \right| + 2\lambda_n^* \sum_{j=1}^p |\beta_{0,j}|
\]

(A.50)

where H"older's inequality is used for the second inequality, Lemma S.1(i) for the third inequality, simple manipulations for the fourth one, and equation (S.3) for the fifth inequality.

So the difference from (A.17) is the presence of \(1/\min_{1\leq i\leq q} \tilde{\sigma}_l^2\) in (A.50). By Lemma A.5 on the set \(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \cap \mathcal{B}_4\) for \(n\) sufficiently large

\[
\frac{1}{\min_{1\leq i\leq q} \tilde{\sigma}_l^2} \leq \frac{2}{\min_{1\leq i\leq q} \tilde{\sigma}_l^2},
\]

(A.51)

We proceed again as in (A.3)-(A.5) to get (on the set \(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \cap \mathcal{B}_4\))

\[
2 \left| \frac{u^T \tilde{Z}'X}{n} \right| \left[ \frac{\|Z'X\|_n}{\min_{1\leq i\leq q} \tilde{\sigma}_l^2} \right] \|\tilde{\beta} - \beta_0\|_1 \frac{1}{\min_{1\leq i\leq q} \tilde{\sigma}_l^2} \leq \lambda_n^* \|\tilde{\beta} - \beta_0\|_1,
\]

(A.52)

where

\[
\lambda_n^* = \frac{2}{\min_{1\leq i\leq q} \tilde{\sigma}_l^2}.
\]

(A.53)

Note that by Assumption 2, we have \(\lambda_n^* = O(\lambda_n) = O\left(\sqrt{\frac{\ln q}{n}}\right) = o(1)\). Next, proceed as in (A.20)-(A.23) (replacing \(Z\) by \(\tilde{Z}\) and \(\tilde{\beta}\) by \(\tilde{\beta}\))

\[
\|\tilde{Z}'X(\tilde{\beta} - \beta_0)\|_n^2 + \lambda_n^* \sum_{j=1}^p |\tilde{\beta}_j| \leq 3\lambda_n^* \sqrt{s_0} \|\tilde{\beta}_{S_0} - \beta_{0,S_0}\|_2.
\]

(A.54)

Ignoring the first term in (A.54), the restricted set condition is satisfied for the eigenvalue condition. Use (A.45) and proceed as in (A.24)-(A.25) to get

\[
\|\tilde{Z}'X(\tilde{\beta} - \beta_0)\|_n^2 + 2\lambda_n^* \sum_{j=1}^p |\tilde{\beta}_j| \leq \frac{9(\lambda_n^*)^2 S_0}{\phi_{\Sigma_{x=x_0}}^2(s_0)}.
\]

Now proceed as in the proof of Lemma A.4

\[
\|\tilde{Z}'X(\tilde{\beta} - \beta_0)\|_n^2 + 2\lambda_n^* \sum_{j=1}^p |\tilde{\beta}_j| \leq \frac{18(\lambda_n^*)^2 S_0}{\phi_{\Sigma_{x=x_0}}^2(s_0)}.
\]

(A.55)
where we used the fact that we are also on $A_4 = \{\phi_{s_{xzw}}^2(s_0) \geq \phi_{s_{xzw}}^2(s_0)/2\}$.

We now turn to upper bounds on the $l_1$ estimation error. Instead of (A.29) we have
\[
\|\hat{\beta} - \beta_0\|_1 \leq 4\sqrt{s_0}\|Z'X(\hat{\beta} - \beta_0)\|_n \cdot \phi_{s_{xzw}}^2(s_0).
\]
Using (A.55) on the right side of the above equation yields
\[
\|\hat{\beta} - \beta_0\|_1 \leq \frac{24\lambda^*s_0\phi_{s_{xzw}}(s_0)}{\phi_{s_{xzw}}^2(s_0)},
\]
where we used the fact that we are on $A_4 = \{\phi_{s_{xzw}}^2(s_0) \geq \phi_{s_{xzw}}^2(s_0)/2\}$. These upper bounds are valid uniformly over $B_{l_0} = \{\|\beta_0\|_{l_0} \leq s_0\}$. Note that the upper bounds are valid on the event $A_1 \cap A_2 \cap A_4 \cap B_1 \cap B_2 \cap B_3 \cap B_4$. We lower bound the probability of $B_1 \cap B_2 \cap B_3 \cap B_4$ by Lemma A.5. The probability of $A_1 \cap A_2 \cap A_4$ is lower bounded by Lemma A.3 and Lemma S.3.

ii) By Assumption 2 the probability of $A_1 \cap A_2 \cap A_4 \cap B_1 \cap B_2 \cap B_3 \cap B_4$ can then be shown to tend to 1.

8.6 Properties of the CLIME estimator $\hat{\Gamma}$

We next establish three lemmata on the properties of the CLIME estimator. The first two lemmata are adapted from Gold et al. (2018) and applied to our case. We provide the proofs of them so that it is easy to establish the third lemma we develop for GMM case. Prior to the first lemma, define the event
\[
T_\Gamma(\mu) = \{\|\hat{\Gamma}\hat{\Sigma} - I_p\|_\infty \leq \mu\}.
\]

**Lemma A.6.** Assume that $\|\Gamma\|_{l_1}$ is bounded from above by $m_\Gamma < \infty$. Suppose that the rows of $\hat{\Gamma}$, which are denoted $\hat{\Gamma}_j$, are obtained by the CLIME program in section 4.2. Then, on the set $T_\Gamma(\mu)$, for each $j = 1, 2, \ldots, p$
\[
\|\hat{\Gamma}_j - \Gamma_j\|_\infty \leq 2m_\Gamma \mu.
\]

**Remark 4.** As the result is for a fixed sample size, one can choose a different $m_\Gamma$ for each $n$. We shall utilize this in the sequel.

**Proof of Lemma A.6.** By $\Gamma = \Sigma^{-1}$, adding and subtracting $\hat{\Gamma}\hat{\Sigma}$ in the second equality
\[
\hat{\Gamma} - \Gamma = (\hat{\Gamma}\Sigma - I_p)\Gamma = (\hat{\Gamma}\hat{\Sigma} + \hat{\Gamma}(\Sigma - \hat{\Sigma})) - I_p)\Gamma
\]
\[
= [\hat{\Gamma}\hat{\Sigma} - I_p]\Gamma + \hat{\Gamma}(\Sigma - \hat{\Sigma})\Gamma
\]
\[
= [\hat{\Gamma}\hat{\Sigma} - I_p]\Gamma + \hat{\Gamma}(I_p - \hat{\Sigma}\Gamma)
\]
Next, by the definition of the CLIME program in section 4.2 $\|\hat{\Gamma}\hat{\Sigma} - I_p\|_\infty \leq \mu$, and using that we are on $T_\Gamma(\mu)$
\[
\|\hat{\Gamma} - \Gamma\|_\infty \leq \|\hat{\Gamma}\hat{\Sigma} - I_p\|_\infty + \|\hat{\Gamma}(I_p - \hat{\Sigma}\Gamma)\|_\infty
\]
\[
\leq \|\hat{\Gamma}\hat{\Sigma} - I_p\|_\infty \|\Gamma\|_{l_1} + \|\Gamma\|_{l_\infty} \|I_p - \hat{\Sigma}\Gamma\|_\infty
\]
\[
\leq 2m_\Gamma \mu,
\]
where we used dual norm inequality on p.44 of van de Geer (2016), (S.3) and that $\|\hat{\Gamma}\|_{l_\infty} \leq \|\Gamma\|_{l_\infty}$ on $T_\Gamma(\mu)$. Furthermore, since $\Gamma$ is symmetric, we have $\|\Gamma\|_{l_\infty} = \|\Gamma\|_{l_1} \leq m_\Gamma$. \qed
Recall from Section 4.2 that for $f \in [0,1)$

$$U(m_\Gamma, f, s_\Gamma) = \{ A \in \mathbb{R}^{p\times p} : A > 0, \|A\|_{t_1} \leq m_\Gamma, \max_{1 \leq j \leq p} \sum_{k=1}^p |A_{jk}|^f \leq s_\Gamma \}.$$  

The next Lemma can be proved by using Lemma A.6 and adapting the proof of equation (27) on p.604-605 of Cai et al. (2011) to our purpose and its proof therefore omitted. Equation (27) is the proof of equation (14) in Cai et al. (2011). For the purpose of the next lemma, define the constant $c_f = 1 + 2^{1-f} + 3^{1-f}$.

**Lemma A.7.** Suppose that the conditions of Lemma A.6 hold and that $\Gamma \in U(m_\Gamma, f, s_\Gamma)$. Then, for every $j \in \{1, \ldots, p\}$

$$\|\hat{\Gamma}_j - \Gamma_j\|_1 \leq 2c_f(2m_\Gamma \mu)^{1-f}s_\Gamma,$$

$$\|\hat{\Gamma}_j - \Gamma_j\|_2 \leq 2c_f(2m_\Gamma \mu)^{1-f}s_\Gamma,$$

The proof in Cai et al. (2011) also holds for non-symmetric matrices. Now we lower bound the probability of $T_\Gamma(\mu)$. To that end define

$$c_{1n} = \frac{c_n}{(\min_{1 \leq t \leq q} \sigma_t^2 - c_n) \min_{1 \leq t \leq q} \sigma_t^2}, \quad (A.57)$$

where $c_n$ is defined in Lemma A.5. Also recall from (A.8).

$$t_3 = 2K[C \sqrt{\ln(pq)} + EM^2 \ln(pq)] + \sqrt{\frac{\kappa_n}{n}}, \quad (A.58)$$

The following inequality, using the notation of Lemma S.1, will be useful.

$$\|BFA\|_\infty \leq \|B\|_\infty \|FA\|_{t_1} q \leq q \|B\|_\infty \|F\|_{t_\infty} \|A\|_\infty \quad (A.59)$$

where we used the dual norm inequality for the first inequality from p.44 of van de Geer (2016) and for the second inequality we used Lemma S.1(vi). We can now introduce the following new lemma for GMM in high dimensional models.

**Lemma A.8.** Under Assumptions 1, 2 and 3 one has

$$P[\|\hat{\Sigma} - I_p\|_\infty > \mu] \leq 10 \exp(-C\kappa_n) + \frac{K[2EM_1^2 + 5EM_2^2 + EM_2^2 + EM_3^2 + EM_1^2]}{n\kappa_n} \to 0,$$

where

$$\mu = m_\Gamma \{(t_3)^2 c_{1n} + 2Ct_3 c_{1n} + C(t_3)^2 + 2Ct_3 + Cc_{1n}\} \to 0.$$

**Proof of Lemma A.8.** We start by noting that $\Gamma \hat{\Sigma} = I_p$ such that

$$\|\Gamma \hat{\Sigma} - I_p\|_\infty = \|\hat{\Sigma} - \Sigma\|_\infty \leq \|\Gamma\|_{t_\infty} \|\hat{\Sigma} - \Sigma\|_\infty = \|\Gamma\|_{t_1} \|\hat{\Sigma} - \Sigma\|_\infty. \quad (A.60)$$

where we used (S.3) and $\Gamma$ being symmetric. From the definitions of $\hat{\Sigma}$ and $\Sigma$, by simple algebra and the fact that the max norm of a transpose of a matrix is equal to max norm of a matrix:

\[
\|\hat{\Sigma} - \Sigma\|_\infty \leq \frac{1}{q} \|\left(\frac{X'Z}{n} - \Sigma_{xz}\right)(W_d - W_d)\left(\frac{Z'X}{n} - \Sigma'_{xz}\right)\|_\infty \quad (A.61)
\]

\[
+ \frac{2}{q} \|\left(\frac{X'Z}{n} - \Sigma_{xz}\right)(W_d - W_d)\Sigma'_{xz}\|_\infty \quad (A.62)
\]

\[
+ \frac{1}{q} \|\left(\frac{X'Z}{n} - \Sigma_{xz}\right)(W_d)\left(\frac{Z'X}{n} - \Sigma'_{xz}\right)\|_\infty \quad (A.63)
\]

\[
+ \frac{2}{q} \|\left(\frac{X'Z}{n} - \Sigma_{xz}\right)(W_d)(\Sigma'_{xz})\|_\infty \quad (A.64)
\]

\[
+ \frac{1}{q} \|(\Sigma_{xz})(W_d - W_d)(\Sigma'_{xz})\|_\infty \quad (A.65)
\]
Before analyzing the individual terms in the above display note that if \( \max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2| \leq c_n \) (an event whose probability we can control by Lemma A.5, see in particular (A.33)) then

\[
\|\hat{W}_d - W_d\|_\infty = \max_{1 \leq i \leq q} \left| \frac{1}{\hat{\sigma}_i^2} - \frac{1}{\sigma_i^2} \right| \leq \frac{\max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2|}{\min_{1 \leq i \leq q} \sigma_i^2} \min_{1 \leq i \leq q} \sigma_i^2 \leq \frac{c_n}{(\min_{1 \leq i \leq q} \sigma_i^2) \min_{1 \leq i \leq q} \sigma_i^2} = c_{1n}.
\]

Assume furthermore that the following event occurs (the probability of which can be controlled by (A.8))

\[
\left\{ \|XZ/n - \Sigma_{xz}\|_\infty \leq t_3 \right\}.
\]

Using (A.59) we can upper bound (A.61) as follows on \( \{ \max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2| \leq c_n \} \cap \{ \|XZ/n - \Sigma_{xz}\|_\infty \leq t_3 \}
\]

\[
\frac{1}{q} \| (XZ/n - \Sigma_{xz}) (\hat{W}_d - W_d) (Z^nX - \Sigma'_{xz}) \|_\infty \leq \| (XZ/n - \Sigma_{xz}) \| (\hat{W}_d - W_d) \|_\infty (\|Z^nX - \Sigma'_{xz}\|_\infty \leq (t_3)^2 c_{1n}.
\]

Consider (A.62). We have \( \|\Sigma'_{xz}\|_\infty \leq C < \infty \) by (A.10). By the same arguments as the ones that lead to (A.66) we have

\[
\frac{2}{q} \| (XZ/n - \Sigma_{xz}) (\hat{W}_d - W_d) \Sigma'_{xz} \|_\infty \leq 2 C t_3 c_{1n}.
\]

Consider (A.63). Note that \( \|W_d\|_\infty = 1/\min_{1 \leq i \leq q} \sigma_i^2 \). Using the same arguments as in (A.66) yields

\[
\frac{1}{q} \| (XZ/n - \Sigma_{xz}) (W_d) (Z^nX - \Sigma'_{xz}) \|_\infty \leq (t_3)^2/ \min_{1 \leq i \leq q} \sigma_i^2.
\]

Consider (A.64) and (A.65). By the same analysis as the one that lead to (A.66) one gets

\[
\frac{2}{q} \| (XZ/n - \Sigma_{xz}) (W_d) (\Sigma'_{xz}) \|_\infty \leq 2 C t_3/ \min_{1 \leq i \leq q} \sigma_i^2.
\]

\[
\frac{1}{q} \| (\Sigma_{xz}) (\hat{W}_d - W_d) (\Sigma'_{xz}) \|_\infty \leq C^2 c_{1n}.
\]

Combine all constants \( C, C^2, (\min_{1 \leq i \leq q} \sigma_i^2) \) as \( C \). Then set

\[
\mu = m_T [(t_3)^2 c_{1n} + 2 C t_3 c_{1n} + C (t_3)^2 + 2 C t_3 + C c_{1n}].
\]

Thus, we have that

\[
P[\|\Sigma - I_p\|_\infty > \mu] \leq P[\|XZ/n - \Sigma_{xz}\|_\infty > t_3] + P[\max_{1 \leq i \leq q} |\hat{\sigma}_i^2 - \sigma_i^2| > c_n]
\]

\[
\leq \exp(-c_n) + \frac{K E M_2^2}{n \kappa_n}
\]

\[
+ 9 \exp(-c_n) + \frac{K [2 E M_2^2 + 4 E M_2^2 + E M_2^2 + E M_2^2 + E M_2^2]}{n \kappa_n}
\]

\[
\to 0,
\]

by (A.8) and (A.33) and the comment just above the latter as well as Assumption 2 for the convergence to zero. It remains to be argued that \( \mu \to 0 \). Using Assumption 2, we see that by \( p \leq q \),

\[
\sqrt{n} \ln(pq) = \left( \frac{\sqrt{E M_2^2} \sqrt{\ln(pq)}}{n^{1/2}} \right) \frac{\sqrt{\ln(pq)}}{n^{1/2}} = o(1) \frac{\sqrt{\ln(q)}}{n^{1/2}}.
\]

Thus, \( t_3 \) as given in (A.58) is \( O(\sqrt{n q}) \). Furthermore, \( c_n \) as given in (A.33) is \( O(s_0 \sqrt{n q}) \) implying that the same is the case for \( c_{1n} \). Therefore,

\[
\mu = O(m_T c_{1n}) = O(m_T s_0 \sqrt{n q}) = o(1).
\]

(71)

where the last assertion is by Assumption 3.
The following lemma combines Lemmas A.7 and A.8.

**Lemma A.9.** Under Assumptions 1, 2 and 3(i)-(ii), by using the program in section 4.2 to get $\hat{\Gamma}$

(i).
\[
\max_{1 \leq j \leq p} \| \hat{\Gamma}_j - \Gamma_j \|_1 = O_p((m_\Gamma \mu)^{1-f})s_\Gamma.
\]

(ii).
\[
\max_{1 \leq j \leq p} \| \hat{\Gamma}_j - \Gamma_j \|_2 = O_p((m_\Gamma \mu)^{1-f})s_\Gamma.
\]

### 8.7 Proof of Theorem 2

**Proof of Theorem 2.** We prove that $t_{W_d}$ is asymptotically standard normal. This will be done in case of a diagonal weight $W_d$. The case of general symmetric positive definite weight will be discussed afterwards. We divide the proof into several steps.

First, decompose $t_{W_d}$:
\[
t_{W_d} = t_{W_{d1}} + t_{W_{d2}},
\]

where
\[
t_{W_{d1}} = e_j' \hat{\Gamma} \left( \frac{X'Z W_d Z'u_n}{n \sigma^2} \right) \sqrt{e_j' \hat{\Gamma} V_1 \hat{\Gamma} e_j}.
\]
\[
t_{W_{d2}} = - \frac{e_j' \Delta}{\sqrt{e_j' \hat{\Gamma} V_1 \hat{\Gamma} e_j}}.
\]

**Step 1.**
In the first step, we introduce an infeasible $t_{W_{d}^*}$ (it is infeasible since since $V_1 = q^{-2}V_1$) and show that it is asymptotically standard normal.

\[
t_{W_{d}^*} = e_j' \Gamma_\Sigma x W_d Z'u_n/n^{1/2} \sqrt{e_j' \Gamma_\Sigma x W_d Z'u_n/n^{1/2}}
\]

where
\[
V_1 = \Sigma_{xz} W_d \Sigma_{zu} W_d \Sigma_{xz}'.
\]
Recall also that $\Sigma_{zu} = EZ_i Z'u_i^2$, and $\Sigma_{xz} = EX_i Z'_i$.

To establish that $t_{W_d}$ is standard normal, we verify the conditions for Lyapounov’s central limit theorem. First, note that
\[
E \left[ \frac{e_j' \Gamma_\Sigma x W_d \sum_{i=1}^n Z_i u_i/n^{1/2}}{\sqrt{e_j' \Gamma_\Sigma x W_d \sum_{i=1}^n Z_i u_i/n^{1/2}}} \right] = 0
\]
since $EZ_i u_i = 0$ by exogeneity of the instruments. Next,
\[
E \left[ \frac{e_j' \Gamma_\Sigma x W_d \sum_{i=1}^n Z_i u_i/n^{1/2}}{\sqrt{e_j' \Gamma_\Sigma x W_d \sum_{i=1}^n Z_i u_i/n^{1/2}}} \right]^2 = 1,
\]
where we used that
\[E \left[ \frac{\sum_{i=1}^n Z_i u_i}{n^{1/2}} \right] \left[ \frac{\sum_{i=1}^n Z_i u_i}{n^{1/2}} \right]' = EZ_i Z'u_i^2 = \Sigma_{zu}.
\]

Next, we want to show
\[
\frac{1}{(e_j' \Gamma_\Sigma x W_d \sum_{i=1}^n Z_i u_i/n^{1/2})^{r_u/2}} \sum_{i=1}^n E |e_j' \Gamma_\Sigma x W_d Z_i u_i/n^{1/2}|^{r_u/2} \to 0.
\]
First, since $\Gamma_j$ is the $j$th row vector in $\Gamma$ in section 4.2, with $\|\Gamma\|_{t_1} \leq m_\Gamma$, and $\Gamma$ being symmetric we get
\[
\max_{1 \leq j \leq p} \|\Gamma e_j\|_1 = \max_{1 \leq j \leq p} \|e_j'\Gamma\|_1 = \|\Gamma\|_{t_\infty} = \|\Gamma\|_{t_1} \leq m_\Gamma. \tag{A.72}
\]
We see that for every $i \in \{1, ..., n\}$
\[
E|e_j'\Gamma \Sigma x Z_d Z_i u_i/n^{1/2}|^{r_u/2} \leq E \left[\|e_j'\Gamma\|_1\|\Sigma x Z_d Z_i u_i/n^{1/2}\|^{r_u/2}\right] \\
\leq E\{m_\Gamma \|\Sigma x Z_d Z_i u_i/\sqrt{n}\|_1\}^{r_u/2} \\
\leq \left[m_\Gamma \|\Sigma x \|_\infty \|Z_d Z_i u_i/\sqrt{n}\|^{r_u/2}/\sqrt{n}\right]^{r_u/2} E\left[\sum_{i=1}^q |Z_d Z_i u_i|\right]^{r_u/2} \tag{A.73}
\]
\[
\leq O\left(\frac{m_\Gamma^{r_u/2} q^{r_u/2}}{n^{r_u-4}}\right), \tag{A.74}
\]
where we used Hölder’s inequality for the first estimate, and Jensen’s inequality as well as $W_d$ being diagonal for the third. For the other estimates we used Assumption 1, (A.10) and $\min_{1 \leq i \leq q} \sigma_i^2 > 0$ with $\max_{1 \leq i \leq q} E|Z_d Z_i u_i|^{r_u/2} \leq C < \infty$ by the Cauchy-Schwarz inequality with $r_u, r_z > 8$. Therefore,
\[
\sum_{i=1}^n E|e_j'\Gamma \Sigma x Z_d Z_i u_i/n^{1/2}|^{r_u/2} = O\left(\frac{m_\Gamma^{r_u/2} q^{r_u/2}}{n^{r_u-4}}\right). \tag{A.75}
\]
Next recalling that $V_d q^2 = V_1$ we get
\[
[e_j'\Gamma V_1 \Gamma e_j]^{r_u/4} \geq [Eigmin(V_1)\|\Gamma e_j\|_2^2]^{r_u/4} \geq [Eigmin(V_1) Eigmin(\Gamma)^2\|\Gamma e_j\|_2^2]^{r_u/4} \\
= q^{r_u/2} [Eigmin(V_1)]^{1/4} \frac{1}{Eigmax(\Sigma)^{2}} \leq 0, \tag{A.76}
\]
where by Assumption 4(ii), we have that $Eigmin(V_1)$ is bounded away from zero and $Eigmax(\Sigma)$ is bounded from above. Thus, by dividing (A.75) with (A.76), the Lyapunov conditions are seen to be satisfied by Assumption 4(i). Therefore, $t_{W_{di}} \rightarrow N(0, 1)$.

**Step 2.** Here we show $t_{W_{di}} - t_{W_{d_1}} = o_p(1)$.

We do so by showing that the numerators as well as denominators of $t_{W_{di}} - t_{W_{d_1}}$ are asymptotically equivalent and by arguing that the denominators are bounded away from 0 in probability.

**Step 2a.** Regarding the numerators note that
\[
|e_j'\Gamma \left(\frac{X'Z}{n}\right) \hat{W}_d Z'u - e_j'\Gamma \Sigma x Z_d Z' u| \\
\leq |e_j'\Gamma \left(\frac{X'Z}{n}\right) \hat{W}_d Z'u - e_j'\Gamma \left(\frac{X'Z}{n}\right) \hat{W}_d Z'u| \\
+ |e_j'\Gamma \Sigma x Z_d Z' u - e_j'\Gamma \Sigma x Z_d Z' u| \\
+ |e_j'\Gamma \Sigma x Z_d Z' u - e_j'\Gamma \Sigma x Z_d Z' u| \tag{A.77}
\]

Start with (A.77). By Hölder’s inequality
\[
|e_j'\Gamma \left(\frac{X'Z}{n}\right) \hat{W}_d Z'u - e_j'\Gamma \left(\frac{X'Z}{n}\right) \hat{W}_d Z'u| \leq \left[\|e_j'\Gamma\|_1\right] \left[\|\frac{X'Z}{n} \hat{W}_d Z'u\|_\infty\right]. \tag{A.80}
\]
Next,

$$
\| \frac{X'Z}{n} \hat{W}_d Z'u \|_\infty \leq \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{\sum_{i=1}^n |X_{ij}Z_{il}|}{n} \| \hat{W}_d Z'u \|_{n^{1/2} \infty} \\
\leq \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{\sum_{i=1}^n |X_{ij}Z_{il}|}{n} \left( \frac{1}{\min_{1 \leq l \leq q} \sigma_l^2} \right) \| Z'u \|_{n^{1/2} \infty}, \quad (A.81)
$$

where we used (S.2) for the first inequality and for the second inequality we used Example 5.6.5 (page 345) of Horn and Johnson (2013) and \( \| \hat{W}_d \|_{\infty} = \frac{1}{\min_{1 \leq l \leq q} \sigma_l^2} \). Use (A.81) in (A.80) to get

$$
|e'_j \Gamma \left( \frac{X'Z}{n} \right) \hat{W}_d \frac{Z'u}{q \ n^{1/2}} - e'_j \Gamma \left( \frac{X'Z}{n} \right) \hat{W}_d \frac{Z'u}{q \ n^{1/2}}| \\
\leq \left[ \| e'_j (\hat{\Gamma} - \Gamma) \|_1 \right] \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{\sum_{i=1}^n |X_{ij}Z_{il}|}{n} \left( \frac{1}{\min_{1 \leq l \leq q} \sigma_l^2} \right) \| Z'u \|_{n^{1/2} \infty} \\
= O_p \left( s_T (m\Gamma \mu)^{1-f} \right) [O_p(1)] O_p(1) \left( \sqrt{n} O_p \left( \frac{\ln q}{n^{1/2}} \right) \right) \\
= O_p \left( s_T (m\Gamma \mu)^{1-f} \sqrt{\ln q} \right) = o_p(1), \quad (A.82)
$$

where in the first equality we use Lemma A.9 for the first term on the right side, Lemma A.3(iv) for the second term, Lemma A.5 for the third term and Lemma A.3(ii) for the fourth term. The last equality follows by Assumption 4(i). Regarding (A.78) note first that

$$
|e'_j \Gamma \left( \frac{X'Z}{n} \right) \hat{W}_d \frac{Z'u}{q \ n^{1/2}} - e'_j \Gamma \Sigma_{zz} \hat{W}_d \frac{Z'u}{q \ n^{1/2}}| \\
\leq \| e'_j \Gamma \|_1 \left[ \| \frac{X'Z}{n} - \Sigma_{zz} \frac{X'Z}{q \ n^{1/2}} \|_\infty \right] \\
\leq \| \Gamma_j \|_1 \left[ \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{1}{n} \sum_{i=1}^n |X_{ij}Z_{il} - EX_{ij}Z_{il}| \right] \| \hat{W}_d \frac{Z'u}{n^{1/2} \infty} \\
\leq \| \Gamma_j \|_1 \left[ \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{1}{n} \sum_{i=1}^n |X_{ij}Z_{il} - EX_{ij}Z_{il}| \right] \| \hat{W}_d \|_{\infty} \| \frac{Z'u}{n^{1/2} \infty} \\
= \| \Gamma_j \|_1 \left[ \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{1}{n} \sum_{i=1}^n |X_{ij}Z_{il} - EX_{ij}Z_{il}| \right] \frac{1}{\min_{1 \leq l \leq q} \sigma_l^2} \| Z'u \|_{n^{1/2} \infty}, \quad (A.83)
$$

where we used Hölder’s inequality for the first estimate, (S.2) for the second inequality and for third inequality we used Example 5.6.5 of Horn and Johnson (2013). Observe that by Assumption 3(i) \( \max_{1 \leq j \leq p} \| \Gamma_j \|_1 = O(m\Gamma) \) and by Lemma A.3(iv), and \( p \leq q \) such that \( \ln(pq) \leq 2 \ln q \) and so

$$
\max_{1 \leq j \leq p} \max_{1 \leq l \leq q} \frac{1}{n} \sum_{i=1}^n |X_{ij}Z_{il} - EX_{ij}Z_{il}| = O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right). \quad .
$$

By Lemma A.5 and A.3(ii) we have

$$
\frac{1}{\min_{1 \leq l \leq q} \sigma_l^2} \| Z'u \|_{n^{1/2} \infty} = \sqrt{n} O_p(1) O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right) = O_p \left( \sqrt{\ln q} \right).
$$

Using the above two displays in (A.83) yields
To this end, consider the following three terms:

\[ |e_j' \tilde{\Gamma} \left( X'Z \right) \frac{W_d}{n} \tilde{W}_d Z'u - e_j' \tilde{\Gamma} \Sigma_{xz} \frac{W_d}{n} Z'u| \]

\[ = O(m_\Gamma)O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right) \]

\[ = O_p \left( \frac{m_\Gamma \ln q}{n^{1/2}} \right) = o_p(1), \quad (A.84) \]

by Assumption 4(i). Now consider (A.79):

\[ |e_j' \tilde{\Gamma} \Sigma_{xz} \frac{W_d}{n} Z'u - e_j' \tilde{\Gamma} \Sigma_{xz} \frac{W_d}{n} Z'u| \leq \|e_j' \tilde{\Gamma}\|_1 \|\Sigma_{xz} \frac{(W_d - W_d)}{n^{1/2}} Z'u \|_\infty \]

\[ \leq \|\Gamma_j\|_1 \max_{1 \leq j \leq p} \max_{1 \leq i \leq q} \frac{1}{n} \sum_{i=1}^n E|X_j Z_i| \|(|W_d - W_d) \frac{Z'u}{n^{1/2}} \|_\infty \]

\[ \leq \|\Gamma_j\|_1 \max_{1 \leq j \leq p} \max_{1 \leq i \leq q} \frac{1}{n} \sum_{i=1}^n E|X_j Z_i| \|(|W_d - W_d) \frac{Z'u}{n^{1/2}} \|_\infty \]

\[ = \|\Gamma_j\|_1 \max_{1 \leq j \leq p} \max_{1 \leq i \leq q} \frac{1}{n} \sum_{i=1}^n E|X_j Z_i| \max_{1 \leq i \leq q} \frac{1}{\sigma_i^2 - \sigma_i^2}| \|Z'u \|_n^{1/2} \]

\[ = O(m_\Gamma)O(1)O_p \left( \frac{\sqrt{\ln q s_0}}{\sqrt{n}} \right) \left[ n^{1/2}O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right) \right] \]

\[ = O_p \left( \frac{m_\Gamma s_0 \ln q}{n^{1/2}} \right) = o_p(1), \quad (A.85) \]

where we use Hölder’s inequality for the first estimate, (S.2) for the second and for third inequality we used Example 5.6.5 of Horn and Johnson (2013). Assumption 1, 3(i), Lemma A.3 as well as the following display are used as well. The last equality is obtained by Assumption 4(i). In (A.85) we used that

\[ \max_{1 \leq i \leq q} \left\{ \frac{1}{\sigma_i^2} - \frac{1}{\sigma_i^2} \right\} = \max_{1 \leq i \leq q} \left\{ \frac{\sigma_i^2 - \sigma_i^2}{\sigma_i^2 \sigma_i^2} \right\} \]

\[ \leq \frac{\max_{1 \leq i \leq q} |\sigma_i^2 - \sigma_i^2|}{\min_{1 \leq i \leq q} \sigma_i^2 \min_{1 \leq i \leq q} \sigma_i^2} \]

\[ = O_p \left( \frac{\sqrt{\ln q s_0}}{n^{1/2}} \right), \quad (A.86) \]

where we obtained the rate in the last equality from Remark 3.

Step 2b). Here we start analyzing the denominator of $t_{W_d}$. As an intermediate step define the infeasible estimator $\tilde{V}_d = (X'Z \tilde{W}_d \tilde{\Sigma}_{zu} \tilde{W}_d Z'X \tilde{W}_d Z'u^2)$ of $V_d$, where $\tilde{\Sigma}_{zu} = \frac{1}{n} \sum_{i=1}^n Z_i Z_i^2$. We show that

\[ |e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j - e_j' \tilde{\Gamma} \Gamma e_j| = o_p(1). \quad (A.87) \]

To this end, consider the following three terms:

\[ |e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j - e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j|, \quad (A.88) \]

\[ |e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j - e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j|, \quad (A.89) \]

\[ |e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j - e_j' \tilde{\Gamma} \tilde{V}_d \tilde{\Gamma} e_j|. \quad (A.90) \]
To establish (A.87) we show that the above three terms tend to zero in probability. We start with (A.88). Use Hölder’s inequality twice to get

\[ |e_j' \hat{\Gamma} \hat{d} \Gamma e_j - e_j' \hat{\Gamma} \hat{d} \Gamma e_j| \leq \|\hat{\Gamma}_d - \hat{\Gamma}_d\|_\infty \|\Gamma e_j\|^2. \]  

(A.91)

Then, in (A.91), by the definition of \( \hat{\Gamma}_d \) and \( \hat{\Gamma}_d \)

\[ \|\hat{\Gamma}_d - \hat{\Gamma}_d\|_\infty = \| \left( \frac{X'Z}{n} \right) \hat{W}_d \|_{\mathcal{F}_n} \leq \| \left( \frac{X'Z}{n} \right) \hat{W}_d \|_{\mathcal{F}_n} \]  

(A.92)

where we used Lemma S.1 (iii). Now

\[ \| \left( \frac{X'Z}{n} \right) \hat{W}_d \|_{\mathcal{F}_n} \leq \| \left( \frac{X'Z}{n} \right) \hat{W}_d \|_{\mathcal{F}_n} \]  

(A.93)

by Lemma S.1(iv). Furthermore,

\[ \| \left( \frac{X'Z}{n} \right) \|_{\mathcal{F}_n} = O_p(1), \]  

(A.94)

by Lemma A.3(iv). Next by Lemma A.5 and \( \hat{W}_d \) being diagonal

\[ \| \hat{W}_d \|_{\mathcal{F}_n} = \| \hat{W}_d \|_{\mathcal{F}_n} = O_p(1). \]  

(A.95)

Now insert (A.94) and (A.95) into (A.93) to conclude

\[ \| \left( \frac{X'Z}{n} \right) \hat{W}_d \|_{\mathcal{F}_n} = O_p(1). \]  

(A.96)

Recalling that \( \hat{u}_i = u_i - X_i'(\hat{\beta}_F - \beta_0) \) one gets

\[ \hat{\Sigma}_{zu} - \Sigma_{zu} = - \frac{2}{n} \sum_{i=1}^{n} Z_i Z_i' u_i X_i'(\hat{\beta}_F - \beta_0) + \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i'(\hat{\beta}_F - \beta_0)' X_i X_i'(\hat{\beta}_F - \beta_0). \]  

(A.97)

Consider the first term on the right side of (A.97):

\[ \max_{1 \leq i \leq n} \max_{1 \leq t \leq t_q} \max_{1 \leq m \leq q} \left| \sum_{i=1}^{n} Z_i Z_i' u_i X_i'(\hat{\beta}_F - \beta_0) \right| \leq \max_{1 \leq i \leq n} \max_{1 \leq t \leq t_q} \max_{1 \leq m \leq q} \left| Z_i Z_i' u_i X_i'(\hat{\beta}_F - \beta_0) \right| \]

(A.98)

where \( t_q = M q^{4/r_2} n^{2/r_2} \) for \( r_2 > 12 \). This shows that

\[ \max_{1 \leq i \leq n} \max_{1 \leq t \leq t_q} \max_{1 \leq m \leq q} |Z_i Z_i'| = O_p(q^{4/r_2} n^{2/r_2}). \]  

(A.99)

Next, by Lemmas A.1-A.2 and Assumption 4(iii)

\[ P \left( \max_{1 \leq i \leq p} \left| \sum_{i=1}^{n} X_i u_i - E X_i u_i \right| > t_8 \right) \leq \exp(-C \kappa_n) + \frac{E M^2}{n \kappa_n} = o(1), \]  

(A.100)
where \( t_s = O(\sqrt{\frac{\ln n}{n^{1/2}}}) \). Next, by Assumption 1 and the Cauchy-Schwarz inequality
\[
\max_{1 \leq j \leq p} |EX_{ij}u_i| = O(1).
\]
Combining the above two displays gives
\[
\|\frac{1}{n} u'X\|_\infty = O(1) + O_p(\frac{\sqrt{\ln p}}{n^{1/2}}) = O(1) + o_p(1),
\]
(A.100)
where the \( o_p(1) \) term is obtained by Assumption 2. Now use (A.99) and (A.100) in (A.98) together with Lemma A.4(ii) with Assumption 2 to get \( \lambda_n = O(\sqrt{\ln q}/\sqrt{n}) \),
\[
\left\| \frac{2}{n} \sum_{i=1}^{n} Z_i Z_i' u_i X_i'(\hat{\beta}_F - \beta_0) \right\|_\infty \leq O_p(q^{4/r_x}n^{2/r_x})(O(1) + o_p(1))O_p(\frac{\sqrt{\ln q}q_0}{n^{1/2}})
\]
\[
= O_p(\frac{q_0(\sqrt{\ln q})q^{4/r_x}n^{2/r_x}}{n^{1/2}}) = o_p(1),
\]
(A.101)
where Assumption 4(iv), \( q_0q^{4/r_x}n^{2/r_x}\sqrt{\ln q}/n^{1/2} = o(1) \) implies the last equality. Now analyze the following in (A.97)
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i' X_i'X_i(\hat{\beta}_F - \beta_0) \right\|_\infty \leq \max_{1 \leq i \leq n} \max_{1 \leq i \leq q} \max_{1 \leq m \leq q} |Z_{im}Z_{im}||\hat{\beta}_F - \beta_0||\frac{1}{n} \sum_{i=1}^{n} X_iX_i'(\hat{\beta}_F - \beta_0)
\]
\[
\leq \max_{1 \leq i \leq n} \max_{1 \leq i \leq q} \max_{1 \leq m \leq q} |Z_{im}||X_iX_i'\|_\infty ||\hat{\beta}_F - \beta_0||^2,
\]
where we used Hölder’s inequality twice for the last estimate.
By Lemmas A.1 and A.2
\[
\|\frac{1}{n} \sum_{i=1}^{n} [X_iX_i' - EX_iX_i']\|_\infty = O_p(\frac{\sqrt{\ln p}}{n^{1/2}}),
\]
and by the Cauchy-Schwarz inequality with \( r_x \geq 6 \) bounded moments.
\[
\|EX_iX_i'\|_\infty = O(1).
\]
By Assumption 2 the previous two displays imply
\[
\|\frac{X'X}{n}\|_\infty = O(1) + O_p(\frac{\sqrt{\ln p}}{n^{1/2}}) = O(1) + o_p(1).
\]
Then, using (A.99), the above display and Lemma A.4
\[
\left| \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i'(\hat{\beta}_F - \beta_0)'X_iX_i'(\hat{\beta}_F - \beta_0) \right| \leq O_p(q^{4/r_x}n^{2/r_x})(O(1) + o_p(1))O_p\left(\frac{\ln q s_0^2}{n}\right)
\]
\[
= O_p(\frac{\sqrt{\ln q s_0}q^{4/r_x}n^{2/r_x}}{n^{1/2}})^2 = o_p(1),
\]
(A.102)
by Assumption 4(iv). Using (A.102) and (A.101) in (A.97) thus gives
\[
\|\hat{\Sigma}_{zu} - \Sigma_{zu}\| = O_p(q^{4/r_x}n^{2/r_x}\sqrt{\ln q}/n^{1/2}) = o_p(1).
\]
(A.103)
Insert (A.96) and (A.103) into (A.92) to get
\[
\|\hat{V}_d - \hat{V}_d\|_\infty = O_p\left(\frac{q^{4/r_s}s_0n^{2/r_s}\sqrt{\ln q}}{n^{1/2}}\right) = o_p(1). \tag{A.104}
\]

By the definition of the CLIME program one has for \( j = 1, ..., p \)
\[
\|\hat{\Gamma}_j\|_1 \leq \|\Gamma_j\|_1.
\]

By \( \Gamma \in U(m_\Gamma, f, s_\Gamma) \) being symmetric and Lemma A.8 one has with probability approaching one
\[
\max_{1 \leq j \leq p} \|\hat{\Gamma}_j\|_1 = \|\hat{\Gamma}\|_\infty \leq \|\Gamma\|_{l_\infty} = \|\hat{\Gamma}\|_{l_1} \leq m_\Gamma. \tag{A.105}
\]

Next, use (A.104) and (A.105) in (A.91) to bound (A.88).
\[
|e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j - e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j| \leq O_p(m_\Gamma^2)O_p\left(\frac{q^{4/r_s}s_0n^{2/r_s}\sqrt{\ln q}}{n^{1/2}}\right)
\]
\[
= O_p\left(\frac{m_\Gamma^2s_0q^{4/r_s}n^{2/r_s}\sqrt{\ln q}}{n^{1/2}}\right) = o_p(1). \tag{A.106}
\]

We now turn to (A.89) and note first that
\[
|e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j - e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j| 
\leq |e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j - e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j| + |e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j|
\leq \|e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j\|_2 \|\hat{V}_d - \hat{V}_d\|_\infty.
\tag{A.107}
\]

Then by Lemma S.1(iii)
\[
\|\hat{V}_d - \hat{V}_d\|_\infty \leq [\|\frac{X^\prime X}{n} \hat{W}_d - \hat{W}_d\|_\infty]2 \|\hat{\Sigma}_{zu} - \Sigma_{zu}\|_\infty.
\tag{A.108}
\]

By Lemmas A.1 and A.2
\[
P\left(\max_{1 \leq i \leq q} \max_{1 \leq t = 1 \leq m \leq q} \sum_{i=1}^{n} Z_{it}Z_{im}u_i^2 - EZ_{it}Z_{im}u_i^2 > t_9\right) \leq \exp(-C\kappa_n + \frac{KEM_7^2}{n\kappa_n}) = o(1),
\]
for a \( t_9 = O(\sqrt{\ln q/n^{1/2}}) \) via Assumption 4(iii). Thus,
\[
\|\hat{\Sigma}_{zu} - \Sigma_{zu}\|_\infty = O_p\left(\frac{\sqrt{\ln q}}{n^{1/2}}\right). \tag{A.111}
\]

Then insert (A.96) and (A.111) in (A.110) to get that
\[
\|\hat{V}_d - \hat{V}_d\|_\infty = O_p\left(\frac{\sqrt{\ln q}}{n^{1/2}}\right). \tag{A.112}
\]

Use (A.112) in (A.109) together with (A.105)
\[
|e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j - e_j^\prime \hat{\Gamma}_d \hat{\Gamma}^\prime e_j| = O_p(m_\Gamma^2)O_p\left(\frac{\sqrt{\ln q}}{n^{1/2}}\right) = o_p(1), \tag{A.113}
\]

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by Assumption 4(iv).

We now turn to (A.108) and begin by noting that

$$|e_j' \hat{V}_d \bar{Y}_d e_j - e_j' \hat{V}_d \bar{V}_d e_j| \leq \|e_j' \hat{V}_d\|_F^2 \|\bar{V}_d - V_d\|_\infty. \quad (A.114)$$

Next, by definition of $\bar{V}_d$ and $V_d$, and addition and subtraction

$$\bar{V}_d - V_d = \left( \frac{X'Z}{n} \right) W_{d/s} \sum_{zu} W_{d/s} \left( \frac{Z'X}{n} \right) - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \frac{W_{d/s}}{q} \sum_{xu}$$

$$= \left[ \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \left( \frac{X'Z}{n} \hat{W}_d \sum_{zu} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right)$$

We proceed by bounding each of the terms in the above display. Consider first (A.115).

$$\| \left( \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \right) \left( \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \right) \|_\infty$$

$$\leq \| \left( \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \right) \|_\infty \| \sum_{xu} \|_\infty, \quad (A.118)$$

where we used Lemma S.1 (iii) for the estimate. Consider the first term on the right side of (A.118):

$$\| \left( \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \right) \|_\infty = \| \left( \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \right) \|_\infty \| \sum_{xu} \|_\infty$$

$$\leq \| \frac{X'Z}{n} \hat{W}_d - \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \|_\infty + \| \sum_{xu} \frac{W_{d/s}}{q} \sum_{zu} \|_\infty$$

$$\leq q \| \frac{X'Z}{n} \sum_{xu} \|_\infty \| \hat{W}_d - W_d \|_1$$

$$= \| \frac{1}{n} \sum_{i=1}^n (X_i Z_i' - EX_i Z_i') \|_\infty \| \hat{W}_d - W_d \|_1$$

$$+ \| \frac{1}{n} \sum_{i=1}^n EX_i Z_i' \|_\infty \| \hat{W}_d - W_d \|_1. \quad (A.119)$$

where we used triangle inequality for the first inequality and Lemma S.1(iv) for the second inequality. Consider the terms on the right-side of (A.119). By Lemma A.3(iv), Lemma A.5 and $\hat{W}_d$ being diagonal

$$\| \frac{1}{n} \sum_{i=1}^n (X_i Z_i' - EX_i Z_i') \|_\infty \| \hat{W}_d - W_d \|_1 = O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right), \quad (A.120)$$

Next, arguing as in (A.86) we obtain

$$\| EX_i Z_i' \|_\infty \| \hat{W}_d - W_d \|_1 = O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right) \quad (A.121)$$

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Using (A.120) and (A.121) results in
\[
\left\| \left( \frac{X'Z\hat{W}_d}{n} - \frac{\Sigma_{xz}}{q} \right)\right\|_{t_\infty} = O_p \left( \frac{\sqrt{\ln q}}{n^{1/2}} \right) + O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right) = O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right).
\]
(A.122)

By using the generalized version of Hölder’s inequality we can bound \(\|\Sigma_{zu}\|_{\infty}\) in (A.118).
\[
\|\Sigma_{zu}\|_{\infty} = \max_{1 \leq i \leq q} \max_{1 \leq m \leq q} E Z_d Z_m u_i^2 \leq [E|Z_d|^3]^{1/3}[E|Z_m|^3]^{1/3}[Eu_i^6]^{1/3} \leq C
\]
where we used that \(r_z > 12, r_u > 8\). Thus,
\[
\|\Sigma_{zu}\|_{\infty} = O(1).
\]
(A.123)

Insert (A.122) and (A.123) into (A.118) to obtain
\[
\left\| \left( \frac{X'Z\hat{W}_d}{n} - \frac{\Sigma_{xz}}{q} \right)\right\|_{t_\infty} = O_p \left( \frac{\ln q s_0^2}{n} \right),
\]
(A.124)

which establishes the rate for (A.115). Now consider (A.116)
\[
\left\| \left( \frac{X'Z\hat{W}_d}{n} - \frac{\Sigma_{xz}}{q} \right)\right\|_{t_\infty} \leq \|X'Z\hat{W}_d\|_{t_\infty} \|\Sigma_{zu}\|_{\infty} \|W_d\|_{t_\infty} = O(1),
\]
(A.125)

where we use (S.3) for the first inequality and the dual norm inequality on p.44 in van de Geer (2016) for the second one. Next, by Lemma S.1(vi),
\[
\|W_d\|_{t_\infty} \leq q\|EX_i Z_i\|_{t_\infty} \|W_d\|_{t_\infty} = O(1),
\]
(A.126)

where we use (A.10), and Assumption 1, with \(W_d\) being diagonal, and \(\min_{1 \leq i \leq q} \sigma_i^2\) being bounded away from zero. Now use (A.122), (A.123) and (A.126) in (A.125) to get
\[
\left\| \left( \frac{X'Z\hat{W}_d}{n} - \frac{\Sigma_{xz}}{q} \right)\right\|_{t_\infty} = O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right).
\]
(A.127)

Next, (A.117) obeys the same bound as (A.116) by the two matrices being each others transposes. Thus,
\[
\left\| \left( \frac{X'Z\hat{W}_d}{n} - \frac{\Sigma_{xz}}{q} \right)\right\|_{t_\infty} = O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right).
\]
(A.128)

Now use (A.124) in (A.115), (A.127) in (A.116) and (A.128) in (A.117) to get
\[
\|\hat{V}_d - V_d\|_{t_\infty} = O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right) + O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right) + O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right) = O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right).
\]
(A.129)

Using (A.129) in (A.114) together with (A.105) yields
\[
|c'_j \hat{V}_d \hat{W}_e j - c'_j \hat{V}_d \hat{W}_e j| = O_p(m^2)O_p \left( \frac{\sqrt{\ln q} s_0}{n^{1/2}} \right) = o_p(1),
\]
(A.130)

since by Assumption 4(iv) \(m^2 \sqrt{\ln q} = o(1)\). Next, by (A.113) and (A.130) we have that the following holds for (A.89):
\[
|c'_j \hat{V}_d \hat{W}_e j - c'_j \hat{V}_d \hat{W}_e j| = o_p(1).
\]
(A.131)
Finally, we bound (A.90). By Lemma 3.1 in the supplement of van de Geer et al. (2014), we have
\[
|\epsilon_j' \tilde{\Gamma} \epsilon e_j - \epsilon_j' \Gamma \epsilon e_j| \leq Eigmax(V_d)^2 ||\tilde{\Gamma} e_j - \Gamma e_j||_2^2 + 2||V_d \Gamma e_j||_2 ||\tilde{\Gamma} e_j - \Gamma e_j||_2.
\]
(A.132)

Now,
\[
||V_d \Gamma e_j||_2 = \sqrt{\epsilon_j' \Gamma V_d^2 \Gamma e_j} \\
\leq \sqrt{Eigmax(V_d)^2 \epsilon_j' \Gamma e_j} \\
\leq \sqrt{Eigmax(V_d)^2 (Eigmax(\Gamma))^2} = \frac{Eigmax(V_d)}{Eigmin(\Sigma)} \\
= O(1),
\]
(A.133)
where we used that \(\Gamma\) and \(V_d\) are symmetric, \(\epsilon_j' e_j = 1\) and \(\Gamma = \Sigma^{-1}\). Finally Assumption 4(ii) was used. Next,
\[
||\hat{\Gamma} e_j - \Gamma e_j||_2 \leq \max_{1 \leq j \leq p} ||\hat{\Gamma}_j - \Gamma_j||_2 = O_p \left( s_\Gamma(m_\Gamma \mu)^{1-f} \right),
\]
(A.134)
by \(\epsilon_j' e_j = 1\) and Lemma A.9. Using (A.133) and (A.134) in (A.132) yields, along with the rate of \(\mu\) in (A.71),
\[
|\epsilon_j' \tilde{\Gamma} \epsilon e_j - \epsilon_j' \Gamma \epsilon e_j| \leq O(1)O_p \left( s_{\Gamma}^2 (m_\Gamma \mu)^{2(1-f)} \right) + O(1)O_p \left( s_{\Gamma} (m_\Gamma \mu)^{1-f} \right) \\
= O_p \left( s_{\Gamma} m_\Gamma^2 - 2f s_{0} - f \ln q^{(1-f)/2} \right) = o_p(1),
\]
(A.135)
by Assumption 4(i). Thus, (A.90) is asymptotically negligible. As (A.88) and (A.89) have also been shown to be asymptotically negligible in (A.106) and (A.131), (A.87) is established.

This concludes Step 2 since by Steps 2a-b we have shown that \(t_{W_{d1}} - t_{W_{d2}} = o_p(1)\). Inspection of the above arguments also shows that (A.87) is valid uniformly over \(B_{s_0}(s_0) = \{ ||\beta||_{l_0} \leq s_0 \}\).

**Step 3.** Here we show that \(t_{W_{d2}} = o_p(1)\). Note that the denominator of \(t_{W_{d2}}\) is the same as that of \(t_{W_{d1}}\), which is bounded away from 0 with probability converging to one. It thus suffices to show that the numerator of \(t_{W_{d2}}\) vanishes in probability. To this end, note that this numerator is upper bounded by \(|\epsilon_j' \Delta|\) where
\[
\Delta = \left[ \tilde{\Gamma} \hat{\Sigma} - I_p \right] \sqrt{n}(\hat{\beta} - \beta_0) = \left[ \Gamma \left( \frac{X'Z \hat{W}_d}{n} \frac{Z'X}{n} \right) - I_p \right] \sqrt{n}(\hat{\beta} - \beta_0).
\]

Next, note that
\[
|\epsilon_j' \Delta| = \left| \left( \epsilon_j' (\tilde{\Gamma} \hat{\Sigma} - I_p) \right) \sqrt{n}(\hat{\beta} - \beta_0) \right| \\
\leq ||\epsilon_j' (\tilde{\Gamma} \hat{\Sigma} - I_p)||_\infty ||\sqrt{n}(\hat{\beta} - \beta_0)||_1 \\
\leq ||\epsilon_j||_1 ||\tilde{\Gamma} \hat{\Sigma} - I_p||_\infty \sqrt{n} ||\hat{\beta} - \beta_0||_1 \\
= O_p(\mu) \sqrt{n} ||(\hat{\beta} - \beta_0)||_1,
\]
where we used Hölder’s inequality for the estimates. Furthermore, we used that by definition of the CLIME program \(||\tilde{\Gamma} \hat{\Sigma} - I_p||_\infty \leq \mu\). Then, by (A.71)
\[
\mu = O \left( m_\Gamma s_0 \sqrt{\ln q \over \sqrt{n}} \right).
\]
which together with with Theorem 1(ii) gives that
\[
|\epsilon_j' \Delta| = O \left( m_\Gamma s_0 \sqrt{\ln q \over \sqrt{n}} \right) O_p \left( \sqrt{\ln q s_0 \over \sqrt{n}} \right) n^{1/2} = O_p \left( m_\Gamma s_0 \sqrt{\ln q \over \sqrt{n}} \right) = o_p(1),
\]
(A.136)
by Assumption 4(i). This concludes Step 3 upon noting that the above estimates are valid uniformly over $B_{\ell_0}(s_0) = \{\|\beta_0\|_{\ell_0} \leq s_0\}$ by Theorem 1(ii).
9 Supplementary Appendix

In this part of the paper we present auxiliary technical lemmas and their proofs. We start with some matrix norm inequalities. Let $A$ be a generic $q \times p$ matrix and $x$ a $p \times 1$ vector. Define $a'_l$, $l = 1, \ldots, q$ as the $l$th row of the matrix $A$. Finally, let $B$ be a $p \times q$ matrix, and $F$ a square $q \times q$ matrix while $x$ is a vector of conformable dimension.

Lemma S.1. (i).
\[ \|Ax\|_1 \leq \left[ \sum_{l=1}^{q} \|a'_l\|_\infty \right] \|x\|_1. \]

(ii).
\[ \|Bx\|_\infty \leq q \|B\|_\infty \|x\|_\infty. \]

(iii).
\[ \|BF\|_\infty \leq \|B\|_\infty \|F\|_1 = \|B\|_2. \]

(iv).
\[ \|FB\|_\infty \leq p \|B\|_\infty \|F\|_1. \]

(v).
\[ \|F\|_\infty \leq q \|F\|_\infty \|B\|_\infty. \]

(vi).
\[ \|B\|_\infty \leq q \|B\|_\infty \|A\|_\infty. \]

(vii).
\[ \|B\|_\infty \leq q \|B\|_\infty \|A\|_\infty. \]

Proof. (i) Using Hölder’s inequality
\[ \|Ax\|_1 = \left\| \begin{bmatrix} a'_1x \\ \vdots \\ a'_q x \end{bmatrix} \right\|_1 = \sum_{l=1}^{q} |a'_lx| \leq \left[ \sum_{l=1}^{q} \|a'_l\|_\infty \right] \|x\|_1. \] (S.1)

(ii) Letting $b'_j$ be the $j$th row of $B$ and $B_{jl}$ the $(j,l)$th entry of $B$, it follows by Hölder’s inequality that
\[ \|Bx\|_\infty = \left\| \begin{bmatrix} b'_1x \\ \vdots \\ b'_p x \end{bmatrix} \right\|_\infty = \max_{1 \leq j \leq p} \|b'_jx\|_\infty \leq \max_{1 \leq j \leq p} \left[ \sum_{l=1}^{q} \|B_{jl}\| \right] \|x\|_1 \leq q \left[ \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} |B_{jl}| \right] \|x\|_\infty. \] (S.2)

(iii) Let $a_j$ be the $j$th column of $A$, $j = 1, \ldots, p$. Then,
\[ BA = \begin{bmatrix} b'_1 \\ \vdots \\ b'_j \\ \vdots \\ b'_p \end{bmatrix} \begin{bmatrix} a_1, \ldots, a_j, \ldots, a_p \end{bmatrix} = \begin{bmatrix} b'_1a_1 & \cdots & b'_1a_p \\ \vdots & \vdots & \vdots \\ b'_j a_1 & \cdots & b'_j a_p \\ \vdots & \vdots & \vdots \\ b'_p a_1 & \cdots & b'_p a_p \end{bmatrix}. \]
Thus, by Hölder’s inequality
\[
\|BA\|_\infty = \max_{1 \leq j \leq p} \max_{1 \leq k \leq p} |b_j^t a_k| \\
\leq \max_{1 \leq j \leq p} \max_{1 \leq k \leq p} \|b_j\|_1 \|a_k\|_\infty \\
= \left[ \max_{1 \leq j \leq p} \|b_j\|_1 \right] \left[ \max_{1 \leq k \leq p} \|a_k\|_\infty \right] \\
= \|B\|_{1\infty} \|A\|_{1\infty},
\] (S.3)

Next, using \(A = FB'\), where \(F\) is a generic \(q \times q\) matrix, it follows from the dual norm inequality in section 4.3 of van de Geer (2016) that
\[
\|FB'\|_\infty \leq \|F\|_1 \|B'\|_{1\infty},
\] (S.4)
Combine (S.3) and (S.4) to get
\[
\|BB'\|_{1\infty} \leq \|B\|_{1\infty} \|B'\|_{1\infty} \|F\|_1.
\] (S.5)

(iv) Denoting the \(j\)th row of \(B\) by \(b_j^t\), for \(j = 1, \ldots, p\) and the columns of \(F\) by \(f_l\) for \(l = 1, \ldots, q\):
\[
BF = \begin{bmatrix}
    b_1^t \\
    \vdots \\
    b_j^t \\
    \vdots \\
    b_p^t
\end{bmatrix}
\begin{bmatrix}
    f_1, \ldots, f_l, \ldots, f_q
\end{bmatrix}
= \begin{bmatrix}
    b_1^t f_1 & \cdots & b_1^t f_l & \cdots & b_1^t f_q \\
    \vdots & \cdots & \vdots & \cdots & \vdots \\
    b_j^t f_1 & \cdots & b_j^t f_l & \cdots & b_j^t f_q \\
    \vdots & \cdots & \vdots & \cdots & \vdots \\
    b_p^t f_1 & \cdots & b_p^t f_l & \cdots & b_p^t f_q
\end{bmatrix}.
\]

Then
\[
\|BF\|_{1\infty} = \max_{1 \leq j \leq p} \sum_{l=1}^q |b_j^t f_l| \leq \max_{1 \leq j \leq p} \sum_{l=1}^q \|b_j\|_\infty \|f_l\|_1 \\
\leq \|B\|_1 \|f\|_{1\infty} = q \|B\|_1 \|F\|_1,
\] (S.6)
where we used the definition \(\|\cdot\|_{1\infty}\) in the last equality.

(v) Denoting the \(m\)th row of \(F\) by \(f_m^t\), \(m = 1, \ldots, q\) and the \(j\)th column of \(B'\) by \(b_j\), \(j = 1, \ldots, p\) we first observe
\[
F'B' = \begin{bmatrix}
    f_1^t \\
    \vdots \\
    f_m^t \\
\end{bmatrix}
\begin{bmatrix}
    b_1, \ldots, b_j, \ldots, b_p
\end{bmatrix}
= \begin{bmatrix}
    f_1^t b_1 & \cdots & f_1^t b_j & \cdots & f_1^t b_p \\
    \vdots & \cdots & \vdots & \cdots & \vdots \\
    f_m^t b_1 & \cdots & f_m^t b_j & \cdots & f_m^t b_p
\end{bmatrix}.
\]

By Hölder’s inequality in the first inequality, and definition of norms afterwards
\[
\|F'B'\|_{1\infty} = \max_{1 \leq m \leq q} \sum_{j=1}^p |f_m^t b_j| \leq \max_{1 \leq m \leq q} \sum_{j=1}^p \|f_m\|_1 \|b_j\|_\infty \\
\leq p \max_{1 \leq j \leq p} \|b_j\|_\infty \max_{1 \leq m \leq q} \|f_m\|_1 = p \|B\|_1 \|F\|_{1\infty}.
\] (S.7)
(vi) By Hölder’s inequality

\[ \|FB'\|_1 = \max_{1 \leq j \leq p} \sum_{m=1}^{q} |f'_m b_j| \]

\[ \leq \max_{1 \leq j \leq p} \sum_{m=1}^{q} \|f_m\|_1 \|b_j\|_\infty \]

\[ \leq \left[ q \max_{1 \leq m \leq q} \|f_m\|_1 \right] \left[ \max_{1 \leq j \leq p} \|b_j\|_\infty \right] \]

\[ = q\|F\|_{1\infty} \|B\|_\infty, \]

(vii) By (S.3), and letting \( b'_j \) be the \( j \)th row of \( B \) we obtain:

\[ \|BA\|_\infty \leq \|B\|_{1\infty} \|A\|_\infty = \left[ \max_{1 \leq j \leq p} \|b'_j\|_1 \right] \|A\|_\infty \]

\[ \leq \left[ q \max_{1 \leq j \leq p} \max_{1 \leq l \leq q} |B_{jl}| \right] \|A\|_\infty \]

\[ = q\|B\|_{1\infty} \|A\|_\infty \quad \text{(S.8)} \]

(viii) Observe that

\[ |x'BFAx| \leq \|x\|_1 \|BFAx\|_\infty \]

\[ \leq \|x\|_1^2 \|BFA\|_\infty \]

\[ \leq \|x\|_1^2 \|B\|_\infty \|FA\|_1 \]

\[ \leq q\|x\|_1^2 \|B\|_{1\infty} \|FA\|_1 \]

\[ \leq q\|x\|_1^2 \|B\|_\infty \|FA\|_1 \]

\[ \leq q\|x\|_1^2 \|B\|_{1\infty} \|FA\|_1 \quad \text{(S.9)} \]

where we use Hölder’s inequality for the first and second estimates. The third estimate uses the dual norm inequality of Section 4.3 in van de Geer (2016) while (vi) was used for the last estimate.

The following lemma shows that the adaptive restricted eigenvalue, as defined prior to Lemma A.4, is bounded away from zero with high probability for the first step GMM estimator.

**Lemma S.2.** Let Assumptions 1 and 2 be satisfied. Then, for \( n \) sufficiently large, the set

\[ A_3 = \{ \phi_{\Sigma_{xz}}^2 (s_0) \geq \phi_{\Sigma_{xz}}^2 (s_0) / 2 \}, \]

has probability at least \( 1 - \exp(-C_{\kappa_n}) - \frac{KEM^2}{n\kappa_n} \), for universal positive constants \( C, K \). Furthermore, the probability of \( A_3 \) tends to one as \( n \to \infty \).

**Proof of Lemma S.2.** We begin by noting that for any \( p \times 1 \) vector \( \delta \)

\[ \frac{1}{q} \left| \delta' \frac{X'Z X}{n} \delta \right| = \frac{1}{q} \left| \delta' \left( \frac{X'Z}{n} - \Sigma_{xz} \right) \left( \frac{Z'X}{n} - \Sigma_{zx} \right) \delta + 2 \delta' \left( \frac{X'Z}{n} - \Sigma_{xz} \right) \Sigma'_{xz} \delta \right| \]

\[ \geq \frac{1}{q} \left| \delta' \Sigma_{xz} \Sigma'_{xz} \delta \right| - \frac{2}{q} \left| \delta' \left( \frac{X'Z}{n} - \Sigma_{xz} \right) \Sigma'_{xz} \delta \right| \]

\[ - \frac{1}{q} \left| \delta' \left( \frac{X'Z}{n} - \Sigma_{xz} \right) \left( \frac{Z'X}{n} - \Sigma_{zx} \right) \delta \right| \quad \text{(S.10)} \]

The second term on the right side of (S.10) can be bounded as follows:

\[ \frac{2}{q} \left| \delta' \left( \frac{X'Z}{n} - \Sigma_{xz} \right) \Sigma'_{xz} \delta \right| \leq \frac{2}{q} \|\delta\|_{1\infty}^2 \|\left( \frac{X'Z}{n} - \Sigma_{xz} \right) \Sigma'_{xz} \|_\infty \]

\[ \leq 2 \|\delta\|_{1\infty}^2 \|\left( \frac{X'Z}{n} - \Sigma_{xz} \right) \|_\infty \|\Sigma'_{xz} \|_\infty \]

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where for the second inequality we used (S.8). For the third term on the right side of (S.10) we get in the same way as above
\[
\frac{1}{q} \left| \delta'(\frac{X'Z}{n} - \Sigma_{xx}) \left(\frac{Z'X}{n} - \Sigma'_{xz}\right) \right| \leq \|\delta\|_1^2 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2.
\] (S.11)

Inserting the estimates of the above two equation displays in (S.10) yields
\[
\frac{1}{q} \left| \frac{Z'X}{n} \delta \right|^2 \geq \frac{1}{q} \left| \Sigma_{xx} \delta \right|^2 - 2\|\delta\|_1^2 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 - \|\delta\|_1^2 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2.
\] (S.12)

Note that we have the restriction \(|\delta_{S0}|_1 \leq 3\sqrt{s_0}||\delta_{S0}\||_2\). Add \(|\delta_{S0}|_1\) to both sides of this to get \(|\delta|_1 \leq 4\sqrt{s_0}||\delta_{S0}\||_2\) where we also used the Cauchy-Schwarz inequality. Thus,
\[
\frac{||\delta||_1^2}{||\delta_{S0}||_2^2} \leq 16s_0.
\] (S.13)

Divide (S.12) by \(|\delta_{S0}|_2 > 0\) and use (S.13)
\[
\frac{1}{q} \frac{||\frac{Z'X}{n} \delta||_2^2}{||\delta_{S0}||_2^2} \geq \frac{1}{q} \frac{||\Sigma_{xx} \delta||_2^2}{||\delta_{S0}||_2^2} - 32s_0 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 - 16s_0 \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right)^2.
\] (S.14)

Using that \(\frac{1}{q} \frac{||\Sigma_{xx} \delta||_2^2}{||\delta_{S0}||_2^2} \geq \phi^2_{\Sigma_{xx}}(s_0)\) for all \(\delta\) satisfying \(|\delta_{S0}|_1 \leq 3\sqrt{s_0}||\delta_{S0}\||_2\) and minimizing the left hand side over these \(\delta\) yields
\[
\phi^2_{\Sigma_{xx}}(s_0) \geq \phi^2_{\Sigma_{xx}}(s_0) - 32s_0 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 - 16s_0 \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right)^2.
\]

Note that if
\[
32s_0 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 + 16s_0 \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right)^2 \leq \phi^2_{\Sigma_{xx}}(s_0)/2.
\] (S.15)

then \(\phi^2_{\Sigma_{xx}}(s_0) \geq \phi^2_{\Sigma_{xx}}(s_0)/2\). Thus,
\[
P \left( \phi^2_{\Sigma_{xx}}(s_0) < \phi^2_{\Sigma_{xx}}(s_0)/2 \right) \leq P \left( 32s_0 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 + 16s_0 \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right)^2 \right) \right) \leq \phi^2_{\Sigma_{xx}}(s_0)/2.
\]

Letting \(t_3\) be as in (A.58) define
\[
\epsilon_n = 32s_0t_3 ||\Sigma'_{xz}||_\infty + 16s_0(t_3)^2
\] (S.16)

and note that
\[
P \left( 32s_0 \left[ \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right]^2 \right) + 16s_0 \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty \right)^2 > \epsilon_n \right) \leq P \left( \left\| \frac{X'Z}{n} - \Sigma_{xx} \right\|_\infty > t_3 \right) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n}
\]

by (A.8). Since \(\epsilon_n \to 0\) by Assumption 2, for \(n\) sufficiently large, the above displays can be combined to
\[
P \left( \phi^2_{\Sigma_{xx}}(s_0) < \phi^2_{\Sigma_{xx}}(s_0)/2 \right) \leq \exp(-C\kappa_n) + \frac{KEM^2}{n\kappa_n} \to 0
\]

by Assumption 2.

The following lemma verifies the adaptive restricted eigenvalue condition for the two-step GMM estimator.
Lemma S.3. Let Assumptions 1 and 2 be satisfied. Then, for $n$ sufficiently large, the set

$$
A_4 = \{ \phi_{\Sigma_{xx}}^2 (s_0) \geq \phi_{\Sigma_{xx}}^2 (s_0)/2 \},
$$

has probability at least $1 - 10 \exp(-C\kappa_n) - \frac{K(2EM_1^2 + EM_0^2 + EM_1^2 + EM_0^2 + EM_1^2)}{n\kappa_n}$ for universal constants $C, K$. Furthermore, the probability of $A_4$ tends to one as $n \to \infty$.

Proof. (i). By adding and subtracting $\Sigma_{xx}, W_d$,

$$
\frac{1}{q} |\delta'(X'Z_n / n \hat{W}_d) - Z'X_n / n - \delta| = \frac{1}{q} |\delta'(X'Z_n / n \hat{W}_d - W_d + W_d) - (Z'X_n / n - \Sigma_{xx})\delta| \\
\geq \frac{1}{q} |\delta'(X'Z_n / n \hat{W}_d - W_d + W_d) - (Z'X_n / n - \Sigma_{xx})\delta|
- \frac{1}{q} |\delta'(X'Z_n / n - \Sigma_{xx})(W_d - W_d) - (Z'X_n / n - \Sigma_{xx})\delta|
- \frac{1}{q} |\delta'(\Sigma_{xx})(W_d - W_d) - (\Sigma_{xx})\delta|
- \frac{2}{q} |\delta'(X'Z_n / n - \Sigma_{xx})(\hat{W}_d - W_d)(\Sigma_{xx})\delta|
- \frac{2}{q} |\delta'(X'Z_n / n - \Sigma_{xx})(W_d)(\Sigma_{xx})\delta|,
$$

(S.17)

Now we consider the second term on the right side of the inequality in (S.17).

$$
\frac{1}{q} |\delta'(X'Z_n / n - \Sigma_{xx})(W_d - W_d) - (Z'X_n / n - \Sigma_{xx})\delta| \leq ||\delta||_2^2 \left[ ||X'Z_n / n - \Sigma_{xx}||_\infty^2 \right] ||W_d - W_d||_\infty,
$$

by Lemma S.1 (viii). By the same reasoning,

$$
\frac{1}{q} |\delta'(X'Z_n / n - \Sigma_{xx})(W_d - W_d)(\Sigma_{xx})\delta| \leq ||\delta||_2^2 ||\Sigma_{xx}||_\infty^2 ||W_d - W_d||_\infty.
$$

By (S.13) and $\hat{W}_d$ and $W_d$ being positive definite matrices, (S.17) thus yields

$$
\frac{||\hat{W}_d^{1/2} Z'X_n / n \delta||_2^2}{q ||\delta_S_0||_2^2} \geq \frac{||W_d^{1/2} \Sigma_{xx}^2 \delta||_2^2}{q ||\delta_S_0||_2^2}
- 16s_0 \left[ ||X'Z_n / n - \Sigma_{xx}||_\infty \right]^2 \left( ||\hat{W}_d - W_d||_\infty + ||W_d||_\infty \right)
- 16s_0 ||\Sigma_{xx}||_\infty^2 ||\hat{W}_d - W_d||_\infty
- 32s_0 \left( ||X'Z_n / n - \Sigma_{xx}||_\infty ||\Sigma_{xx}||_\infty \right) \left( ||\hat{W}_d - W_d||_\infty + ||W_d||_\infty \right).
$$

(S.18)

Since $\phi_{\Sigma_{xx}}^2 (s_0) / 2$ for all $\delta \in \mathbb{R}^p$ such that $||\delta_S_0||_1 \leq 3\sqrt{s_0}||\delta||_2$ minimizing the left hand side of the above display over such $\delta$ yields

$$
\phi_{\Sigma_{xx}}^2 (s_0) \geq \phi_{\Sigma_{xx}}^2 (s_0) - a_n
$$
Note that if \(a_n \leq \phi_{2,\varepsilon_{zw}}^2(s_0)/2\), then \(\hat{\phi}_{2,\varepsilon_{zw}}^2(s_0) \geq \phi_{2,\varepsilon_{zw}}^2(s_0)/2\). Thus,

\[
P(\hat{\phi}_{2,\varepsilon_{zw}}^2(s_0) < \phi_{2,\varepsilon_{zw}}^2(s_0)/2) \leq P(a_n > \phi_{2,\varepsilon_{zw}}^2(s_0)/2)
\]

As argued just after (A.65) one has that \(||\hat{\sigma}_j^2 - \sigma_j^2|| \leq c_n\) where \(c_n\) is defined in (A.57). Define

\[
\epsilon_n := 16s_0(t_3)^2[c_1n + ||W_d||_{l_\infty}] + 16s_0(||\Sigma_{xz}||_{\infty})c_1n + 32s_0t_3||\Sigma_{xz}||_{\infty}(c_1n + ||W_d||_{l_\infty}).
\]

Then

\[
P(a_n > \epsilon_n) \leq P(a_n > \epsilon_n, \max_{1 \leq j \leq p} ||\hat{\sigma}_j^2 - \sigma_j^2|| \leq c_n) + P(\max_{1 \leq j \leq p} ||\hat{\sigma}_j^2 - \sigma_j^2|| > c_n)
\]

\[
\leq P\left(||X/Z - \Sigma_{xz}||_{l_{\infty}} > t_3\right) + P(\max_{1 \leq j \leq p} ||\hat{\sigma}_j^2 - \sigma_j^2|| > c_n)
\]

\[
\leq 10\exp(-C\kappa_n) + \frac{K[2EM^2_k + 5EM^3 + EM^3_k + EM^2_k]}{n\kappa_n}
\]

by (A.8) and Remark 3. By Assumption 2, the the right hand side of the above display converges to zero. Furthermore, by Lemma A.3(iv), \(t_3 = O(\sqrt{\ln q/n})\) and inspecting the proof of Lemma A.5 yields have \(c_{1n} = O(s_0\sqrt{\ln q/n})\) (upon noting that \(c_{1n} = O(c_n)\)). By (A.10) we have \(||\Sigma_{xz}||_{\infty} \leq C < \infty\), and Assumption 1 gives \(||W_d||_{l_\infty} = O(1)\). Thus

\[
\epsilon_n = O(s_0^2\sqrt{\ln q/n}) \rightarrow 0.
\]

by Assumption 2. Therefore, for \(n\) sufficiently large,

\[
P(\hat{\phi}_{2,\varepsilon_{zw}}^2(s_0) < \phi_{2,\varepsilon_{zw}}^2(s_0)/2) \leq P(a_n > \phi_{2,\varepsilon_{zw}}^2(s_0)/2) \leq P(a_n > \epsilon_n)
\]

\[
\leq 10\exp(-C\kappa_n) + \frac{K[2EM^2_k + 5EM^3 + EM^3_k + EM^2_k]}{n\kappa_n} \rightarrow 0.
\]

**Proof of Theorem 3.** We begin with part (i). For \(\epsilon > 0\) define the events

\[
A_{1n} = \{ \sup_{\beta_0 \in B_{t_0}} |\epsilon'_j\Delta| < \epsilon \},
\]

\[
A_{2n} = \left\{ \sup_{\beta_0 \in B_{t_0}} \frac{\sqrt{e_j'\Gamma\tilde{Y}_d\Gamma'_{\epsilon_j}}}{\sqrt{e_j'\Gamma\tilde{Y}_d\Gamma'_{\epsilon_j}}} - 1 < \epsilon \right\},
\]

\[
A_{3n} = \{ \epsilon_j'\Gamma X/Z \rightarrow_{nq} W_d Z'/n^{1/2} - e_j'\Gamma\Sigma_{xz}W_d Z'/n^{1/2} < \epsilon \}.
\]

The probability of \(A_{1n}\) converges to one by (A.136) while the probability of \(A_{2n}\) tends to one by (A.87) and \(e_j'\Gamma \tilde{Y}_d\Gamma'_{\epsilon_j}\) being bounded away from zero. Finally, \(A_{3n}\) converges to one in probability by step 2a in the
proof of Theorem 2. Thus, every \( t \in \mathbb{R} \),
\[
\begin{align*}
|P \left( \frac{n^{1/2}(\hat{\beta}_j - \beta_j)}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t \right) - \Phi(t)| &= \left| P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t \right) - \Phi(t) \right| \\
&\leq \left| P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) - \Phi(t) \right| + P \left( \cup_{i=1}^{3} A_{in}^c \right).
\end{align*}
\]
Using that \( e_j^2 \Gamma_d \Gamma' e_j \) is bounded away from zero and does not depend on \( \beta_0 \) it follows that there exists a universal \( D > 0 \) such that
\[
\begin{align*}
P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) &= P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) \\
&= P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) \\
&\leq P \left( \frac{\epsilon_j^2 \Gamma_{\Sigma x z} W_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t(1 + \epsilon) + 2D \epsilon \right),
\end{align*}
\]
Thus, as the right hand side of the above display does not depend on \( \beta_0 \) it follows from the asymptotic normality of \( \frac{\epsilon_j^2 \Gamma_{\Sigma x z} W_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \) that for \( n \) sufficiently large
\[
\begin{align*}
\sup_{\beta_0 \in B_{\beta_0}} P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) &\leq P \left( \frac{\epsilon_j^2 \Gamma_{\Sigma x z} W_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t(1 + \epsilon) + 2D \epsilon \right) \\
&\leq \Phi(t(1 + \epsilon) + 2D \epsilon) + \epsilon,
\end{align*}
\]
Using the continuity of \( q \mapsto \Phi(q) \) it follows that for any \( \delta > 0 \) there exists a sufficiently small \( \epsilon \) such that
\[
\sup_{\beta_0 \in B_{\beta_0}} P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) \leq \Phi(t) + \delta + \epsilon.
\]
Following a similar reasoning as above one can also show that for any \( \delta > 0 \) and \( \epsilon > 0 \) sufficiently small
\[
\inf_{\beta_0 \in B_{\beta_0}} P \left( \frac{\epsilon_j^2 \hat{\Gamma}_j X' \hat{Z} \hat{W}_d \frac{Z'u}{n^{1/2}}}{\sqrt{e_j^2 \Gamma_d \Gamma' e_j}} \leq t, A_{1n}, A_{2n}, A_{3n} \right) \geq \Phi(t) - 2\epsilon - \delta.
\]
From (S.21) and (S.22) it can be concluded that

\[
\sup_{\beta_0 \in B_0} \left| P \left( \frac{n^{1/2}e_j'(\hat{b} - \beta_0)}{\sqrt{\Gamma_j'\Gamma_k'\Gamma_j}} \leq t \right) - \Phi(t) \right| \to 0.
\]

Part (ii) can be established in a similar fashion as the proof of Theorem 3ii in Caner and Kock (2018). We now turn to part (iii).

\[
n^{1/2} \sup_{\beta_0 \in B_0} \text{diam}(\hat{b}_j - z_{1-\alpha/2}n^{1/2} \hat{\sigma}_j^{1/2}, \hat{b}_j + z_{1-\alpha/2}n^{1/2})
\]

\[
= \sup_{\beta_0 \in B_0} 2\hat{\sigma}_j z_{1-\alpha/2}
\]

\[
= 2 \left[ \sup_{\beta_0 \in B_0} \sqrt{\Gamma_j'\Gamma_k'\Gamma_j} + o_p(1) \right] z_{1-\alpha/2}
\]

\[
= O_p(1),
\]

by Theorem 2(ii) for the second equality, and Assumption 4 (ii) for the last equality.

\[\square\]

### 9.1 Linear dynamic panel data models as a special case

In this section we show how the classic dynamic linear panel data model as studied in, eg., Arellano and Bond (1991) is covered by our framework as a special case upon taking first differences. To be precise, we consider the model

\[
y_{it} = \rho_0 y_{it-1} + x_{it}'\delta_0 + \mu_i + u_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T
\]

where \(|\rho_0| < 1\). \(y_{it}\) is a scalar, \(x_{it}\) is a \(K \times 1\) vector of strictly exogenous variables and \(\mu_i\) is the unobserved effect for individual \(i\) which can be correlated with \(y_{it-1}\) and \(x_{it}\). Assume for concreteness that \(y_{i0} = 0\) for \(i = 1, \ldots, n\). Since \((\rho_0, \delta_0)\) is the parameter of interest we have \(p = K + 1\) in the terminology of our paper. The \(\mu_i\) can be removed by taking first differences, arriving at

\[
\Delta y_{it} = \rho_0 \Delta y_{it-1} + \Delta x_{it}'\delta_0 + \Delta u_{it}, \quad i = 1, \ldots, n, \quad t = 2, \ldots, T, \quad (S.23)
\]

Upon stacking the observations across individuals and time, (S.23) is of the form (1). Next, imposing

\[
E[u_{it}|\mu_i, y_{i,1}^{t-1}, x_i^T] = 0, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T,
\]

where \(y_i^t = (y_{i1}, \ldots, y_{it})'\) and \(x_i^T = (x_{i1}', \ldots, x_{iT}')'\) implies that for each \(i \in \{1, \ldots, n\}\)

\[
E[y_{i,t-2}^t \Delta u_{it}] = 0, \quad t = 3, \ldots, T, \quad (S.25)
\]

\[
E[x_{it}\Delta u_{is}] = 0, \quad t = 1, \ldots, T, \quad s = 2, \ldots, T. \quad (S.26)
\]

This results in \(q = (T - 2)(T - 1)/2 + T(T - 1)K\) moment inequalities for each \(i = 1, \ldots, n\) thus fitting into (2). In particular we note that the number of instruments \(q\) can be larger than the sample size \(n(T - 2)\) even for moderate values of \(T\) and \(K\) thus resulting in a setting with many moments/instruments compared to the number of observations as studied in this paper.
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