Fractional integral inequalities for different functions

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Abstract: In this paper, we establish several inequalities for different convex mappings that are connected with the Riemann-Liouville fractional integrals. Our results have some relationships with certain integral inequalities in the literature.

Keywords: Hadamard Inequality, Riemann-Liouville Fractional Integration, Minkowski Inequality.

1 Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and let \( a, b \in I \), with \( a < b \). The following inequality:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known in the literature as Hadamard’s inequality. Both inequalities hold in reversed direction if \( f \) is concave.

In [1], Godunova and Levin introduced the following class of functions.

Definition 1. A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is said to belong to the class of \( Q(I) \) if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in (0, 1) \) satisfies the inequality:

\[
f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.
\]

They also noted that all nonnegative monotonic and nonnegative convex functions belong to this class and also proved the following motivating result:

If \( f \in Q(I) \) and \( x, y, z \in I \), then

\[
f(x)(x - y)(x - z) + f(y)(y - x)(y - z) + f(z)(z - x)(z - y) \geq 0.
\]

In fact (3) is even equivalent to (2). So it can alternatively be used in the definition of the class \( Q(I) \).

In [9], Dragomir et al., defined the following new class of functions.
Definition 2. A function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is \( P \) function or that \( f \) belongs to the class of \( P(I) \), if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in [0, 1] \), satisfies the following inequality:

\[
f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y).
\] (4)

The power mean \( M_r(x, y; \lambda) \) of order \( r \) of positive numbers \( x, y \) is defined by

\[
M_r(x, y; \lambda) = \begin{cases} 
(\lambda x^r + (1 - \lambda)y^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\
x^{\lambda - \frac{1}{r}}y^{rac{1}{r}}, & \text{if } r = 0.
\end{cases}
\]

In [14], Pearce et al. generalized this inequality to \( r \)-convex positive function \( f \) which is defined on an interval \( [a, b] \), for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \);

\[
f(\lambda x + (1 - \lambda)y) \leq M_r(f(x), f(y); \lambda) = \begin{cases} 
(\lambda [f(x)]^r + (1 - \lambda) [f(y)]^r)^{\frac{1}{r}}, & \text{if } r \neq 0 \\
[f(x)]^{\lambda - \frac{1}{r}}[f(y)]^{rac{1}{r}}, & \text{if } r = 0.
\end{cases}
\]

We have that \( 0 \)-convex functions are simply log-convex functions and \( 1 \)-convex functions are ordinary convex functions.

In [19], Varošanec introduced the following class of functions.

Definition 3. Let \( h : J \subset \mathbb{R} \rightarrow \mathbb{R} \) be a positive function. We say that \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) is \( h \)-convex function or that \( f \) belongs to the class \( SX(h, I) \), if \( f \) is nonnegative and for all \( x, y \in I \) and \( \lambda \in (0, 1) \), we have

\[
f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y).
\] (5)

If the inequality in (5) is reversed, then \( f \) is said to be \( h \)-concave, i.e., \( f \in SV(h, I) \).

Obviously, if \( h(\lambda) = \lambda \), then all nonnegative convex functions belong to \( SX(h, I) \) and all nonnegative concave functions belong to \( SV(h, I) \); if \( h(\lambda) = \frac{1}{\lambda} \), then \( SX(h, I) = Q(I) \); if \( h(\lambda) = 1 \), then \( SX(h, I) \supseteq P(I) \) and if \( h(\lambda) = \lambda^s \), where \( s \in (0, 1) \), then \( SX(h, I) \supseteq K_2^s \). For some recent results for \( h \)-convex functions we refer to the interested reader to the papers [3], [4] and [15].

In [9], Dragomir et al. proved two inequalities of Hadamard type for class of Godunova-Levin functions and \( P \)-functions.

Theorem 1. Let \( f \in Q(I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then the following inequality holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{4}{b - a} \int_a^b f(x)dx.
\] (6)

Theorem 2. Let \( f \in P(I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then the following inequality holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{2}{b - a} \int_a^b f(x)dx \leq 2[f(a) + f(b)].
\] (7)
In [11], Ngoc et al., established following theorem for \( r \)–convex functions:

**Theorem 3.** Let \( f : [a, b] \to (0, \infty) \) be \( r \)–convex function on \([a, b]\) with \( a < b \). Then the following inequality holds for \( 0 < r \leq 1 \):

\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \left( \frac{r}{r+1} \right)^{\frac{1}{r}} (f'(a) + f'(b))^\frac{1}{r}.
\]

(8)

For related results on \( r \)–convexity see the papers [10] and [20].

In [16], Sankaya et al. proved the following Hadamard type inequalities for \( h \)–convex functions.

**Theorem 4.** Let \( f \in SX(h, I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then

\[
\frac{1}{2h(\frac{1}{2})} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq [f(a) + f(b)] \int_0^1 h(\alpha)d\alpha.
\]

(9)

In [17], Sankaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.

**Theorem 5.** Let \( f : [a, b] \to \mathbb{R} \) be positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J_a^a (b) + J_b^a (a) \right] \leq \frac{f(a) + f(b)}{2}
\]

(10)

with \( \alpha > 0 \).

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 4.** Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_a^a f \) and \( J_b^a f \) of order \( \alpha > 0 \) with \( \alpha \geq 0 \) are defined by

\[
J_a^a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
J_b^a f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du \). Here is \( J_0^0 f(x) = J_b^0 f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see [2], [5]–[8], [12], [13], [17] and [18].

The main purpose of this paper is to present new Hadamard’s inequalities for fractional integrals via functions that belongs to the classes of \( Q(I) \), \( P(I) \), \( SX(h, I) \) and \( r \)–convex.
2 MAIN RESULTS

**Theorem 6.** Let \( f \in Q(I) \), \( a, b \in I \) with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). Then the following inequality for fractional integrals hold:
\[
f \left( \frac{a+b}{2} \right) \leq \frac{2\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_a^\alpha(b) + J_b^\alpha(a) \right]
\]
with \( \alpha > 0 \).

**Proof.** Since \( f \in Q(I) \), we have
\[
2(f(x) + f(y)) \geq f \left( \frac{x+y}{2} \right)
\]
for all \( x, y \in I \) (with \( \lambda = \frac{1}{2} \) in (1.2)).

If we choose \( x = ta + (1-t)b \) and \( y = (1-t)a + tb \) in above inequality, we get
\[
2 \left[ f \left( ta + (1-t)b \right) + f \left( (1-t)a + tb \right) \right] \geq f \left( \frac{a+b}{2} \right).
\]
(12)

Then multiplying both sides of (12) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1] \), we obtain
\[
2 \int_0^1 t^{\alpha-1} \left[ f \left( ta + (1-t)b \right) + f \left( (1-t)a + tb \right) \right] dt \geq f \left( \frac{a+b}{2} \right) \int_0^1 t^{\alpha-1} dt
\]
\[
2 \int_a^b \left( \frac{b-u}{b-a} \right)^{\alpha-1} f(u) \frac{du}{b-a} + 2 \int_a^b \left( \frac{v-a}{b-a} \right)^{\alpha-1} f(v) \frac{dv}{b-a} \geq \frac{1}{\alpha} f \left( \frac{a+b}{2} \right)
\]
\[
2\Gamma(\alpha+1) \left[ J_a^\alpha(b) + J_b^\alpha(a) \right] \geq f \left( \frac{a+b}{2} \right).
\]
The proof is complete.

**Remark.** If we choose \( \alpha = 1 \) in Theorem 6, then the inequalities (11) become the inequalities (6).

**Theorem 7.** Let \( f \in P(I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then one has inequality for fractional integrals:
\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_a^\alpha(b) + J_b^\alpha(a) \right] \leq 2(f(a) + f(b))
\]
(13)
with \( \alpha > 0 \).

**Proof.** According to (4) with \( x = ta + (1-t)b \), \( y = (1-t)a + tb \) and \( \lambda = \frac{1}{2} \), we find that
\[
f \left( \frac{a+b}{2} \right) \leq f \left( ta + (1-t)b \right) + f \left( (1-t)a + tb \right)
\]
(14)
for all $t \in [0, 1]$. Thus multiplying both sides of (14) by $t^{\alpha - 1}$ and integrating the resulting inequality with respect to $t$ over $[0, 1]$, we have

$$f \left( \frac{a+b}{2} \right) \int_0^1 t^{\alpha - 1} dt \leq \int_0^1 t^{\alpha - 1} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] dt$$

and

$$\frac{1}{\alpha} f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ J_{a^+}^\alpha (b) + J_{b^-}^\alpha (a) \right]$$

and

$$f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha}} \left[ J_{a^+}^\alpha (b) + J_{b^-}^\alpha (a) \right]$$

and the first inequality is proved.

Since $f \in P(f)$, we have

$$f(ta + (1-t)b) \leq f(a) + f(b)$$

and

$$f((1-t)a + tb) \leq f(a) + f(b).$$

By adding these inequalities, we get

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq 2 \left[ f(a) + f(b) \right].$$

(15)

Then multiplying both sides of (15) by $t^{\alpha - 1}$ and integrating the resulting inequality with respect to $t$ over $[0, 1]$, we have

$$\int_0^1 t^{\alpha - 1} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] dt \leq 2 \int_0^1 t^{\alpha - 1} dt \left[ f(a) + f(b) \right]$$

and

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{a^+}^\alpha (b) + J_{b^-}^\alpha (a) \right] \leq 2 \left( f(a) + f(b) \right)$$

and thus the second inequality is proved.

**Remark.** If we choose $\alpha = 1$ in Theorem 7, then the inequalities (13) become the inequalities (7).

**Theorem 8.** Let $f : [a, b] \to (0, \infty)$ be $r$–convex function on $[a, b]$ with $a < b$ and $0 < r \leq 1$. Then the following inequality for fractional integral inequalities holds:

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{a^+}^\alpha (b) + J_{b^-}^\alpha (a) \right] \leq \left[ \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r \left[ f(a) \right]^r + \left( \frac{\beta(\alpha, \frac{r+1}{r})}{\alpha + \frac{1}{r}} \right)^r \left[ f(b) \right]^r \right] ^\frac{1}{r}$$

and

$$\left[ \left( \frac{\beta(\alpha, \frac{r+1}{r})}{\alpha + \frac{1}{r}} \right)^r \left[ f(a) \right]^r + \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r \left[ f(b) \right]^r \right] ^\frac{1}{r}.$$

**Proof.** Since $f$ is $r$–convex function and $r > 0$, we have

$$f(ta + (1-t)b) \leq (t \left[ f(a) \right]^r + (1-t) \left[ f(b) \right]^r) ^\frac{1}{r}$$

and

$$f((1-t)a + tb) \leq ((1-t) \left[ f(a) \right]^r + t \left[ f(b) \right]^r) ^\frac{1}{r}.$$
for all $t \in [0, 1]$.

By adding these inequalities we have

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq (t [f(a)]^r + (1-t) [f(b)]^r)^{\frac{1}{r}} + ((1-t) [f(a)]^r + t [f(b)]^r)^{\frac{1}{r}}.$$  

Then multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating the resulting inequality with respect to $t$ over $[0, 1]$, we obtain

$$\int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt$$

$$\leq \int_0^1 t^{\alpha-1} (t [f(a)]^r + (1-t) [f(b)]^r)^{\frac{1}{r}} dt + \int_0^1 t^{\alpha-1} ((1-t) [f(a)]^r + t [f(b)]^r)^{\frac{1}{r}} dt.$$  

It is easy to observe that

$$\int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt = \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ J_a^\alpha (b) + J_b^\alpha (a) \right]$$  

Using Minkowski inequality, we have

$$\int_0^1 t^{\alpha-1} (t [f(a)]^r + (1-t) [f(b)]^r)^{\frac{1}{r}} dt \leq \left[ \left( \int_0^1 t^{\alpha-1} f(a) dt \right)^r + \left( \int_0^1 t^{\alpha-1} f(b) dt \right)^r \right]^{\frac{1}{r}}$$

$$= \left[ \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r [f(a)]^r + \left( \beta(\alpha, \frac{r+1}{r}) \right)^r [f(b)]^r \right]^{\frac{1}{r}}$$

and similarly

$$\int_0^1 t^{\alpha-1} ((1-t) [f(a)]^r + t [f(b)]^r)^{\frac{1}{r}} \leq \left[ \left( \int_0^1 t^{\alpha-1} (1-t) f(a) dt \right)^r + \left( \int_0^1 t^{\alpha-1} f(b) dt \right)^r \right]^{\frac{1}{r}}$$

$$= \left[ \beta(\alpha, \frac{r+1}{r}) \right]^r [f(a)]^r + \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r [f(b)]^r \right]^{\frac{1}{r}}.$$  

Thus

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[ J_a^\alpha (b) + J_b^\alpha (a) \right] \leq \left[ \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r [f(a)]^r + \left( \beta(\alpha, \frac{r+1}{r}) \right)^r [f(b)]^r \right]^{\frac{1}{r}}$$

$$+ \left[ \beta(\alpha, \frac{r+1}{r}) \right]^r [f(a)]^r + \left( \frac{1}{\alpha + \frac{1}{r}} \right)^r [f(b)]^r \right]^{\frac{1}{r}}.$$  

This proof is complete.

**Remark.** In Theorem 8, if we choose $\alpha = 1$, then we obtain the inequalities (8).
Theorem 9. Let \( f \in SX(h,I) \), \( a,b \in I \) with \( a < b \) and \( f \in L_1[a,b] \). Then one has inequality for \( h \)-convex functions via fractional integrals

\[
\frac{1}{ah(\frac{1}{2})} f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ J_{a^+}^{\alpha} (b) + J_{b^-}^{\alpha} (a) \right] \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] dt. \tag{16}
\]

Proof. According to (5) with \( x = ta + (1-t)b, y = (1-t)a + tb \) and \( \alpha = \frac{1}{2} \) we find that

\[
f \left( \frac{a+b}{2} \right) \leq h \left( \frac{1}{2} \right) f(ta + (1-t)b) + h \left( \frac{1}{2} \right) f((1-t)a + tb) \leq h \left( \frac{1}{2} \right) [f(ta + (1-t)b) + f((1-t)a + tb)]. \tag{17}
\]

Then multiplying the first inequality in (17) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain

\[
f \left( \frac{a+b}{2} \right) \int_0^1 t^{\alpha-1} dt \leq h \left( \frac{1}{2} \right) \int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt \leq \frac{1}{ah(\frac{1}{2})} f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ J_{a^+}^{\alpha} (b) + J_{b^-}^{\alpha} (a) \right] \tag{18}
\]

and the first inequality in (16) is proved.

Since \( f \in SX(h,I) \), we have

\[
f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)
\]

and

\[
f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y).
\]

By adding these inequalities we get

\[
f(tx + (1-t)y) + f((1-t)x + ty) \leq [h(t) + h(1-t)] [f(x) + f(y)]. \tag{19}
\]

By using (19) with \( x = a \) and \( y = b \) we have

\[
f(ta + (1-t)b) + f((1-t)a + tb) \leq [h(t) + h(1-t)] [f(a) + f(b)]. \tag{20}
\]

Then multiplying both sides of (20) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0,1]\), we get

\[
\int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt \leq \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] [f(a) + f(b)] dt,
\]

\[
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ J_{a^+}^{\alpha} (b) + J_{b^-}^{\alpha} (a) \right] \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] dt \tag{21}
\]

and thus the second inequality is proved. We obtain inequalities (16) from (18) and (21). The proof is complete.

Remark. In Theorem 9:
–if we choose \( h(t) = t \), then the inequalities (16) become the inequalities (10) of Theorem 5.
–if we take \( \alpha = 1 \), then we obtain the inequalities (9).
–Let \( \alpha = 1 \). If we choose \( h(t) = t \) and \( h(t) = 1 \), then (16) reduce to (1) and (7), respectively.

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