Nonlinear dynamics of non-neutral Maxwellian plasma in external trapping field

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Abstract. Nonlinear dynamics of collisionless non-neutral plasma in an external electrical trapping field is considered. For such system, the time-dependent solution of nonlinear Vlasov-Poisson equations are obtained. Proceeding from this solution, the influence of initial conditions on the dynamics of charged layer is discussed. Conditions leading to the capture of two-dimensional charged collisionless layer in the external parabolic potential are presented.

1. Introduction

There has been a great deal of interest in examining dynamic states of a non-neutral collisionless plasma in an electrostatic or magnetic trap because they are important for understanding the fundamental physical properties of both non-neutral plasma and hydrodynamic systems as a whole \cite{1-6}. In this case, the formation of various dynamic states (so-called collisionless relaxation) is determined by the collective interaction of waves and particles. In particular, in \cite{2,5,7} it has been shown that non-neutral plasma does not reach the state of thermodynamic equilibrium during collisionless relaxation. Unlike conventional thermodynamic equilibrium, the states, in which plasma evolves, explicitly depend on the initial distribution of velocities, particle coordinates, and the external focusing field.

How may it occur? Which is the role of initial conditions and external governing factors? We can indicate a few points. As is known, the dynamics of longitudinal electrostatic waves in collisionless plasma tends to a nontrivial behavior even near the initial equilibrium state (see, for example, \cite{6-10} and references therein). As was shown in \cite{7-11}, depending on the initial conditions, one can observe both the O’Neil and the Landau scripta in collisionless plasma and in both cases the initial amplitude of the wave plays the role of a bifurcation parameter which determines the system evolution. Also, in plasma media near the Maxwellian initial equilibrium state, there may exist nonlinear, stable localized distribution functions which realize the energy maximum similar to the steady two-dimensional fluid vortices (see, for example, \cite{5,8,9}). These equilibrium electrostatic structures, so called phase-space holes, may be treated as Bernstein-Greene-Kruskal modes in which a population of charged particles is trapped in a self-created electrostatic potential.

Thus, nonlinearity of Vlasov-Maxwell system and the initial conditions may play the role of the driving parameter which determines the possibility of emergence of quasi-stationary states. This system possesses such a property because it precludes direct particle–particle collisions that tend to
randomize the particle velocities, and the interaction via the self-consistent electrostatic field cannot change the full system entropy. It means that there are many ways of evolution of the system from the initial state far from equilibrium to a state, where the distribution function practically ceases to change with time. However, one can compare the macroscopic quantities (velocity–space moments of the distribution function) with that obtained from the corresponding fluid models which always are workable near local equilibrium states described by the distribution function of the Maxwellian type [8,9,11,12].

So, it would be of interest to study the dynamics of non-neutral plasma slab via the formation of local equilibrium state formed in the self-generated and external holding fields (see, for example, [8,9,11,12] and references therein). In the present paper, following our previous works [9–12] we are going to study this case in two-dimensional geometry for the cloud of the single charged particles trapping by symmetric quadratic electrostatic field. In order to investigate the evolution of such system, we shall construct exact nonstationary solution of the two-dimensional Vlasov-Poisson equations, which has a local equilibrium form. Using this solution, we will discuss the physical conditions for the realization of local equilibrium states in charged media of the type under consideration.

2. Formulation of the kinetic problem

We consider the dynamics of two-dimensional slab of non-neutral, non-relativistic, collisionless medium constituted by particles with the same mass $m$ and charge $q$, which is confined in the quadratic potential

$$\Phi_{\text{ext}}(x,y) = k_x x^2 + k_y y^2,$$

where $k_x$ and $k_y$ are the constant components of the external focusing parameter in the $Oxy$ plane. Such dependence is typical for Paul, Kingdon and Penning traps. The dimensionless Vlasov-Poisson equations for the one-particle distribution function $f = f(t,x,y,v_x,v_y)$, which describe the electron plasma in the electrostatic approximation, are

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + \left( E_x - \frac{\partial \Phi_{\text{ext}}}{\partial x} \right) \frac{\partial f}{\partial v_x} + \left( E_y - \frac{\partial \Phi_{\text{ext}}}{\partial y} \right) \frac{\partial f}{\partial v_y} = 0, \quad \nabla \cdot \mathbf{E} = 4\pi n,$$

where $n$ is the density of particles forming a charged medium, $\mathbf{E} = -\nabla \Phi$ and $\Phi(t,x,y)$ are the self-consistent electric field and potential, respectively. The density, time and space have been normalized by the initial electron density $n_0$, the inverse plasma frequency $\omega_p = (4\pi n_0 e^2 / m_e)^{1/2}$ and the inverse Debye length $\lambda_D = (T_e / 4\pi n_0 e^2)^{1/2}$, where $m_e$ the mass, $e$ the charge, $T_e$ the temperature of electrons. The velocity and self-consistent potential are normalized by $\omega_p \lambda_D$ and $T_e / e$, respectively.

Assume that the system is in the state of local equilibrium. Then we look for the partial solution of time-dependent system of Vlasov-Poisson equations (2)-(3) in the local-equilibrium form

$$f(t,x,y,v_x,v_y) = \frac{n}{\pi T^2} \exp \left( - \frac{(v-x)^2}{T} \right),$$

where $(v-x)^2 = (v_x-V_x)^2 + (v_y-V_y)^2$ and $n = n(t)$, $T = T(t,x,y)$, $V_x = V_x(t,x,y), V_y = V_y(t,x,y)$ are unknown functions to be determined.

In view of the symmetry of the slab, we set the additional conditions

$$\left. \frac{\partial \Phi}{\partial x} \right|_{x=0} = \left. \frac{\partial \Phi}{\partial y} \right|_{y=0} = 0,$$

$$k_x = k_y = k.$$

The relations (3) and (5) are satisfied by the function

$$\Phi(t,x,y) = \frac{n(t)}{2} (x^2 + y^2),$$

which corresponds to the self-generated field

$$\mathbf{E}(t,x,y) = -\nabla \Phi = -n(t) x \mathbf{e}_x - n(t) y \mathbf{e}_y.$$
3. Evolution equations
Comparing (1) with (8), taking into account (6), we conclude that for $k > 0$ external and self-consistent fields are oppositely directed: this makes it possible to confine the non-neutral plasma. For this condition, the substitution of (4) into (2) leads to

$$\nu^{(0)} \left[ \frac{\hat{n}}{n} - \frac{\hat{t}}{T} + \frac{\hat{V}}{T^2} V^2 - \frac{2}{T} (V_x V_x + V_y V_y) + \frac{2}{T} (n - k) (x V_x + y V_y) \right] +$$

$$\nu^{(1)} \left[ -2 \frac{\hat{T}}{T^2} V_x + \frac{2}{T} \frac{\partial V_x}{\partial x} - \frac{2 V_y}{T} \frac{\partial V_x}{\partial y} - \frac{2(n - k)x}{T} \right] +$$

$$\nu^{(1)} \left[ -2 \frac{\hat{T}}{T^2} V_y + \frac{2}{T} \frac{\partial V_y}{\partial y} - \frac{2 V_x}{T} \frac{\partial V_y}{\partial x} - \frac{2(n - k)y}{T} \right] +$$

$$\nu^{(2)} \left[ \frac{2}{T^2} \frac{\partial^2 V_x}{\partial x^2} + \frac{2}{T} \frac{\partial V_x}{\partial y} \right] + \nu^{(3)} \left[ \frac{\hat{T}}{T^2} + \frac{2}{T} \frac{\partial V_x}{\partial y} \right] = 0. \tag{9}$$

This relation is valid for any $v_x$ and $v_y$. Thus, the coefficients of each power of $v_x^2$, $v_y^2$ should be equal to zero for any $l$ and $s$. Nullifying the coefficients of $v_x v_y$, $v_x^{(2)}$ and $v_y^{(2)}$ in equation (9), we obtain

$$\frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y} \tag{10}$$

$$\frac{\partial v_x}{\partial x} = -\frac{1}{2} \frac{\partial v_x}{\partial y} = -\frac{1}{2} \frac{\partial v_y}{\partial x} \tag{11}$$

In the simplest case, if $T$ does not depend on the coordinate, from the relations (11) it follows that

$$v_x(t, x) = \frac{-1}{2} \frac{\partial v_x}{\partial y} = g(y, t), \tag{12}$$

$$v_y(t, y) = \frac{-1}{2} \frac{\partial v_y}{\partial x} = h(x, t). \tag{13}$$

The equations (12), (13) must satisfy the condition (10). Now we show that this relation will be satisfied when

$$h(x, t) = C x + D_1(t), \tag{14}$$

$$g(y, t) = -C y + D_2(t), \tag{15}$$

where $C$ is arbitrary constant, $D_1(t), D_2(t)$ are some functions. Indeed, since

$$\frac{\partial v_x}{\partial y} = \frac{\partial g}{\partial y}, \quad \frac{\partial v_y}{\partial x} = \frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}, \tag{16}$$

then relation (10) is satisfied.

Substituting the relations (12) and (13) into equation (9), we can equate to zero the coefficients before $x^2 y^4$ for any $l$ and $s$. Thus, we ensure that this equation is satisfied for arbitrary $x$ and $y$. As a result, we obtain the relations

$$\frac{a}{T} \left( \frac{T}{T} \right)^2 + \frac{2n - 2k}{T} \frac{\hat{T}}{T} = 0, \tag{17}$$

$$\frac{b}{T} + \frac{\hat{T}}{T} (D_1^2 + D_2^2) + \frac{2}{T} D_1 D_1 + \frac{2}{T} D_2 D_2 = 0, \tag{18}$$

$$D_1 = \frac{1}{2} T_1 \hat{T}, \tag{19}$$

$$D_2 = \frac{1}{2} T_2 \hat{T}. \tag{20}$$

Equations (19), (20) have the following solutions

$$D_1 = D_a T_1^2, \tag{21}$$

$$D_2 = D_b T_2^2, \tag{22}$$

where $D_a, D_b$ are constants. For simplicity we put $D_a = D_b = D_0$. In this case we have

$$D_1 = D_2 = D_0 T_2^2. \tag{23}$$

By substituting (23) into (12) and (13), for initial moment of time, we obtain
\[
D_0 = V_x(t = 0)T(t = 0)^{-\frac{1}{2}} = V_y(t = 0)T(t = 0)^{-\frac{1}{2}}.
\]

Thus, \(D_0\) determines the initial distribution of hydrodynamic velocities.

Taking into account (23), from the relation (18) we get
\[
\frac{\dot{n}}{n} + \frac{T}{4D_0^2} = 0.
\]
The solution of this equation is
\[
n = n_0 T^{1-4D_0^2}.
\]

For simplicity, we put the constant of integration equal to \(n_0\). This solution allows us to exclude the density \(n(t)\) from the equation (17), which as a result becomes
\[
\frac{d^2u}{dt^2} = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + 2n_0 e^{\alpha u} - 2k = 0,
\]

here we use the notation
\[
u = \ln T, \quad \alpha = 1 - 4D_0^2.
\]

By using (12), (13), (26) and (28), we express the parameters in the distribution (4) in explicit form through the function \(u\) and parameter \(\alpha\).

\[
n = n_0 e^{\alpha u}, T = e^u, V_x = -\frac{1}{2} \frac{du}{dt} x + \frac{1-\alpha}{4} e^{\frac{u}{2}}, \quad V_y = -\frac{1}{2} \frac{du}{dt} y + \frac{1-\alpha}{4} e^{\frac{u}{2}}.
\]

Thus, proceeding from these relations and (27) we can determine the evolution of the system worked out.

4. Dynamics of a charged slab

The physical properties of a non-neutral plasma are determined by the dynamic properties of the solutions of the equation (27), depending on the initial conditions. In the absence of heating or cooling sources in the system we have to put
\[
\left.\frac{du}{dt}\right|_{t=0} = 0,
\]
which corresponds to immobile non-neutral medium in initial moment of time, and to consider changing of relevant parameters in the local equilibrium distribution (4) for different initial temperatures \(u(t = 0) = u_0\), the values of the external focusing parameter \(k\) and the parameter \(\alpha\).

As is seen from relation (29), the parameter \(\alpha\), associated with the functions \(n\) and \(V_x, V_y\), defines the initial distribution of the hydrodynamic velocities. In the case of \(\alpha = 1\), the hydrodynamic velocities at the initial moment of time are equal to zero. Therefore, it is convenient to use this value in further calculations.

Figure 1 shows the evolution of the function \(u(t)\) associated with density, temperature and velocity by the relations (29) for different values of the focusing parameter \(0 < k < 4\) and the initial conditions \(u(t = 0) = u_0\), which corresponds to a immobile non-neutral plasma with nonzero initial temperature \(T(t = 0) = 1\).

In the case of small values \(k \leq 1\) (figure 1 (a)), when \(u < 0\), the charged layer is expanded accompanied by nonlinear oscillations of the function \(u(t)\). As is seen from the above dependences, increase in the parameter \(k\) leads to increase in the oscillation frequency. However, according to (27) and (29), these oscillations are most strongly manifested in the velocity field, while the other parameters show insignificantly sensitivity to oscillations of \(u\). The interval \(1 \leq k \leq 4\) corresponds to confinement of the plasma. For \(k = 1\) the function \(u\), and hence the temperature \(T\) and the density \(n\), do not change with time, the velocities \(V_x\) and \(V_y\) become equal to zero. A further increase in \(k\) leads to the emergence of strong nonlinear oscillations in all parameters. For \(k > 4\) the distribution \(u(t)\) exhibits a singular behavior, which is due to the dynamics of the velocity field.
Thus, from the point of view of plasma confinement, the value $k = 1$ is of the greatest interest, since here the parameters in the distribution (4) remain constant.

For this case figure 2 shows the influence of the initial temperature for $u(t = 0) = 0$ on the dynamics of the resulting oscillations in the interval $-2 \leq u_0 \leq 4$

The values of the initial temperature greater than $u_0 = 4$ lead to the emergence of a singularity. This is due to rollover of the velocity profile that is shown in figure 3.

These graphs show that for $u_0 \geq 1$ we can observe plasma confinement, accompanied by the emergence of persistent nonlinear oscillations.

Thus, the dependencies presented in figures 1-3, point out to the existence of a range of $k$ and $u_0$, in which plasma is confined. For example, we specify a neighborhood near the point $k = 1, u_0 = 0$. Apparently, such states can be identified with stable, long-lived states that can be obtained experimentally if the distribution of the form (4) has already been established in the system.

Now we consider the dynamics of the nonlinear oscillations in the temperature dependence $u(t)$ under the variation of the parameter $\alpha$. 

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**Figure 1** Evolution of the temperature dependence $u(t)$ for $\alpha = 1$, $u'(t = 0) = u(t = 0) = 0$ under different values of the parameter $k$.

**Figure 2** Evolution of the temperature dependence $u(t)$ for $\alpha = 1$, $u'(t = 0) = u(t = 0) = 0$ under different values of the parameter $u_0$. 

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(a) 

(b) 

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(a) 

(b)
Figure 3 Dynamics of the first derivative of the function \( u(t) \) for \( \alpha = 1, k = 1, \) \( u'(t = 0) = 0 \) under different values of the initial temperature \( u_0. \)

Figure 4 (a) shows the plots of the function \( u(t) \) for various values of the parameter \( 0.4 \leq \alpha \leq 1 \) and fixed initial conditions \( k = 1.5, u(t = 0) = u(t = 0) = 0. \) As is seen from these dependencies, the amplitude of the oscillations of the function \( u(t) \) increases and shows a tendency to a singular behavior when the parameter \( \alpha \to 0. \) This leads to the fact that the range of values of the parameter \( k, \) where the charge layer can be confined, becomes smaller. For example, if \( \alpha = 0.4, \) this range decreases to \( 1 \leq k \leq 1.5. \) In the case of \( \alpha > 1, \) the values of the components of hydrodynamic velocities have an imaginary part.

By setting the value of initial temperature \( u_0 = 1 \) for \( 0.4 \leq \alpha \leq 1, k = 1.5, \) we obtained the set of graphs presented in figure 4 (b). The comparison of this dependences with dependences in figure 4 (a), makes it possible to investigate the influence of the initial temperature on the dynamics of the oscillations of the function \( u(t) \) for various \( \alpha: \) we see that the character of the amplitude variation for different \( u_0 \) and \( \alpha \) can differ. Since for the graphs shown in figure 4 (b), the decrease in the parameter \( \alpha \) leads to a decrease in the amplitude of the oscillations of the function \( u(t), \) then it becomes possible to select the conditions under which the function \( u(t) \) does not change with time. In our case, this is achieved for \( u_0 = 1, \alpha = 0.4, k = 1.5. \) Thus, by varying the parameters \( \alpha \) and \( k \) with initial temperature \( u_0, \) for which a decrease in parameter \( \alpha \) leads to a decrease in the amplitude of the oscillations of the function \( u(t), \) we can find a conditions for the constancy of temperature when the charged layer is confined.

Figure 4 Influence of the parameter \( \alpha = 1 \) in the dynamics of the oscillations of the function \( u(t) \) for \( k = 1.5, u'(t = 0) = 0 \) under different values of the initial temperature \( u_0. \)
5. Conclusion

In this paper we have considered the dynamics of non-neutral single-component plasma in the external electrostatic field with a quadratic potential for a two-dimensional geometry. As a particular solution of the Vlasov-Poisson equations, we used the distribution of the local-equilibrium form (4), whose macroscopic parameters are described by the dependencies (28) and (29) through the function $u(t)$, which is the logarithm of temperature, and the parameter $\alpha$. The function $u(t)$ satisfies the equation (27), which describes a series of non-stationary bounded solutions, depending on the initial temperatures, the external focusing parameter $k$ and the parameter $\alpha$. Analysis of this equation points out to existence of a range of initial conditions, in which nonstationary confinement of a non-neutral plasma is possible.

The two-dimensional model considered here describes a particular case of the evolution of a charged medium, since real systems have a higher spatial dimension, even in the absence of plasma confinement and dissipation processes. On the other hand, our exact solutions, that describe possible strongly nonlinear final states, can give a basic idea of the formation of time-dependent structures in a real system.

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