POSITIVE SOLUTIONS OF SOME NONLINEAR BVPS INVOLVING SINGULARITIES AND INTEGRAL BCS

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Abstract. In this paper we discuss the existence of positive solutions of some nonlocal boundary value problems subject to integral boundary conditions and where the involved nonlinearity might be singular.

1. Introduction. In this paper we establish new results on the existence of positive solutions of the following nonlocal boundary value problem (BVP)

\[- u''(t) = g(t)h(u(t)), \quad t \in (0, 1),\]

where \( h \) is a non-negative function, allowed to be singular, with the nonlocal boundary conditions (BCs)

\[u'(0) + \alpha[u] = 0, \quad \beta u'(1) + u(\eta) = 0, \quad \eta \in [0, 1].\]

We shall take \( \alpha[u] \) to be a positive functional given by

\[\alpha[u] = A_0 + \int_0^1 u(s) dA(s),\]

involving a Lebesgue-Stieltjes integral. This type of BC includes, as particular cases, multi-point problems when

\[\alpha[u] = \sum_{i=1}^m \alpha_i u(\xi_i)\]

and a continuously distributed case when

\[\alpha[u] = \int_0^1 \alpha(s) u(s) ds.\]

Multi-point and integral BCs are widely studied objects, see for example [4, 8, 10, 11, 12, 21, 22, 24, 27, 28, 29, 30, 31] and the reference therein.

One motivation for studying (1)-(2) is that this type of BVP arises in some heat flow problems. For example the special case \( \alpha[u] = 0 \) can be seen as a model for a heated bar of length 1 with a thermostat, where a controller at \( t = 1 \) adds or removes heat according to the temperature detected by a sensor at \( t = \eta \). This case has been extensively studied by Infante and Webb [9, 25, 26], who were motivated by previous work of Guidotti and Merino [3]. The case of a nonzero term \( \alpha[u] \),

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studied by Infante and Webb in [10], has also a physical interpretation, for example when $\alpha[u] = \alpha u(\xi)$, the four point BCs

$$u'(0) + \alpha u(\xi) = 0, \quad \beta u'(1) + u(\eta) = 0, \quad \xi, \eta \in [0, 1].$$

(3)
can be seen as a model for a heated bar, this time with two controllers, and two sensors at $t = \xi$ and $t = \eta$.

Here we prove the existence of multiple positive solutions of (1)-(2) under suitable conditions.

As an application we also establish new results for second order differential equations of the form

$$\lambda u''(t) + g(t)h(u(t)) = 0, \quad t \in (0, 1).$$

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subject to the four point BCs (3). Such type of equations have been widely studied, under different BCs, see for example [5, 6, 16, 18, 19, 20, 23] and the references therein.

Our approach is to rewrite the BVP (1)-(2) as a perturbed Hammerstein integral equation of the form

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s)g(s)h(u(s)) ds$$

(5)

where $\alpha[u]$ is a positive functional, $\gamma$ is a positive continuous function and $h$ is a positive function that is allowed to have singularities. This type of integral equation, with $h$ non singular, has been studied recently by Infante and Webb [10, 30], whereas the Hammerstein case, corresponding to $\gamma(t) \equiv 0$, but this time with a singular $h$, has been investigated by Lan [17].

The methods employed here rely on fixed point index theory, in particular we make use of results and ideas from the papers [10, 17].

We mention that, with the same technique, one may study the case of a more general $f(t, u)$ rather than $g(t)h(u)$. Here for brevity and clarity, we refrain from doing do so.

2. Positive solutions of perturbed Hammerstein integral equations with singularities. We are interested in finding positive solutions of the integral equation

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t, s)g(s)h(u(s)) ds := Tu(t).$$

(6)

We shall achieve this by seeking fixed points of an auxiliary operator $\tilde{T}$ in the cone of continuous function

$$K = \{ u \in C[0, 1] : \min\{u(t) : t \in [0, 1]\} \geq c\|u\| \},$$

(7)
a type of cone first used by Krasnosel’kii, see e.g. [13], and D. Guo, see e.g. [2].

Our main assumptions on $h, g, k, \alpha, \gamma$ satisfy are the following:

$(C_1)$ There exist constants $0 \leq r_1 < r_2$ such that $h : [r_1, r_2] \to [0, \infty)$ is continuous.

$(C_2)$ $k : [0, 1] \times [0, 1] \to [0, \infty)$ is continuous.

$(C_3)$ There exist a measurable function $\Phi : [0, 1] \to [0, \infty)$ and a constant $c_1 \in (0, 1]$ such that

$$c_1 \Phi(s) \leq k(t, s) \leq \Phi(s)$$

for $t \in [0, 1]$ and almost every $s \in [0, 1]$.

$(C_4)$ $g \Phi \in L^1[0, 1]$, $g \geq 0$ a.e., and $\int_0^1 \Phi(s)g(s) ds > 0$. 
(C₅) \( \gamma : [0, 1] \to [0, \infty) \) is continuous and there exists a constant \( c_2 \in (0, 1] \) such that
\[ \gamma(t) \geq c_2 \| \gamma \| \text{ for } t \in [0, 1]. \]

(C₆) \( \alpha : K \to [0, \infty) \) is a continuous functional with
\[ \alpha[u] = A_0 + \int_0^1 u(s) \, dA(s), \]
where \( dA \) is a Lebesgue-Stieltjes measure with \( A_1 := \int_0^1 dA(s) < \infty \).

(C₇) The function \( t \mapsto k(t, s) \) is integrable with respect to the measure \( dA \), that is
\[ K(s) := \int_0^1 k(t, s) \, dA(t) \]
is well defined.

The above assumptions will enable us to use the well-known fixed point index for compact maps (see for example [1] or [2]) on the cone \((C)\), with \( c = \min\{c_1, c_2\} \).

In order to use the results of [10], we extend \( h \) to all of \([0, \infty)\) in a similar way to that of Lan [17]. We define \( \tilde{h}(u) : [0, \infty) \to [0, \infty) \) as
\[
\tilde{h}(u) :=
\begin{cases}
  h(r_1) & \text{if } 0 \leq u \leq r_1, \\
  h(u) & \text{if } r_1 \leq u \leq r_2, \\
  h(r_2) & \text{if } r_2 \leq u < \infty.
\end{cases}
\]
and consider the operator
\[
\tilde{T}u(t) := \gamma(t) \alpha[u] + \int_0^1 k(t, s)g(t)\tilde{h}(u(s)) \, ds.
\]
(8)

By construction
\[ Tu = \tilde{T}u \text{ for } u \in C(r_1, r_2), \]
where
\[ C(r_1, r_2) = \{ u \in K : r_1 \leq u(t) \leq r_2 \text{ for } t \in [0, 1]\}. \]

We now look for fixed points of \( \tilde{T} \) in \( C(r_1, r_2) \) to find solutions of (6). First of all we note that \( \tilde{T} : K \to K \) is compact, that is, \( \tilde{T} \) is continuous and maps bounded sets in precompact sets.

**Theorem 1.** [10] Assume that \((C_1)-(C_6)\) hold. Then \( \tilde{T} \) maps \( K \) into \( K \) and is compact.

**Definition 1.** We write \( K_r = \{ u \in K : ||u|| < r \}, \) \( \overline{K}_r = \{ u \in K : ||u|| \leq r \} \), and define
\[ \Gamma = \int_0^1 \gamma(t) \, dA(t). \]

Let \( q : [0, 1] \to [0, \infty) \) denote the continuous function \( q(u) = \min\{u(t) : t \in [0, 1]\} \).

We make use of the open set
\[ V_\rho = \{ u \in K : q(u) < \rho \}. \]
The set \( V_\rho \) was introduced in [10] and is equal to the set called \( \Omega_{\rho/c} \) in [14]. One advantage of using \( V_\rho \) is that it makes clearer that choosing \( c \) as large as possible yields a weaker condition to be satisfied by the nonlinearity \( h \) in Lemma 2.
Lemma 1. [10] $V_\rho$ defined above has the following properties.
(a) $V_\rho$ is open relative to $K$.
(b) $K_\rho \subset V_\rho \subset K_{\rho/c}$.
(c) $u \in \partial V_\rho$ if and only if $q(u) = \rho$.
(d) If $u \in \partial V_\rho$, then $\rho \leq u(t) \leq \rho/c$ for $t \in [0,1]$.

The following Lemmas are special cases of Lemmas 2.5 and 2.7 of [10]. The first gives conditions which imply that the fixed point index is 0.

Lemma 2. Assume that there exist $\rho > 0$ such that $u \neq \tilde{T}u$ for $u \in \partial V_\rho$ and a positive constant $\alpha_0$ such that

$$\alpha[u] \geq \alpha_0 \rho \quad \text{for} \quad u \in \partial V_\rho,$$

and

$$c_2 \|\gamma\| \alpha_0 + \tilde{h}_{\rho,\rho/c} \cdot \frac{1}{M} \geq 1,$$

where

$$\tilde{h}_{\rho,\rho/c} = \inf \{ \tilde{h}(u)/\rho : \rho \leq u \leq \rho/c \} \quad \text{and} \quad \frac{1}{M} = \inf \{ \int_0^1 k(t,s)g(s)\,ds \}.$$ 

Then the fixed point index, $i_K(\tilde{T},V_\rho)$, is 0.

The second result implies that the index is 1.

Lemma 3. Suppose $\Gamma < 1$ and assume that there exists $\rho > 0$ such that $u \neq \tilde{T}u$ for $u \in \partial K_\rho$ and

$$\frac{A_0 \|\gamma\|}{\rho(1-\Gamma)} + \left( \frac{\|\gamma\|}{1-\Gamma} \int_0^1 K(s)g(s)\,ds + \frac{1}{M} \right) \tilde{h}^{0,\rho} \leq 1,$$

where

$$\tilde{h}^{0,\rho} = \sup \{ \tilde{h}(u)/\rho : 0 \leq u \leq \rho \} \quad \text{and} \quad \frac{1}{m} = \sup \{ \int_0^1 k(t,s)g(s)\,ds \}.$$ 

Then we have $i_K(\tilde{T},K_\rho) = 1$.

Note that in Lemma 2.7 of [10] one needs to control the growth of $\tilde{h}$ on a larger set, here this is not needed, due to the fact that $K$ contains only positive functions.

The above results valid for $\tilde{T}$ allow us to give the following new result on existence of multiple positive solutions for Eq. (6).

Theorem 2. Eq. (6) has one positive solution in $C(r_1,r_2)$ if either of the following conditions hold.

(H1) There exist $\rho_1, \rho_2$ with $r_1 \leq c\rho_1 < \rho_1 < \rho_2 \leq c\rho_2$ such that

$$\left( I^0_{\rho_1} \right), \quad \left( I^0_{\rho_2} \right), \, u \neq \tilde{T}u \quad \text{for} \quad u \in \partial V_{\rho_2}.$$

(H2) There exist $\rho_1, \rho_2$ with $r_1 \leq \rho_1 < c\rho_2 < \rho_2 \leq r_2$ such that

$$\left( I^0_{\rho_1} \right), \quad \left( I^0_{\rho_2} \right), \, u \neq \tilde{T}u \quad \text{for} \quad u \in \partial K_{\rho_2}.$$

Eq. (6) has two positive solutions in $C(r_1,r_2)$ if one of the following conditions hold.

(S1) There exist $\rho_1, \rho_2, \rho_3$ with $r_1 \leq c\rho_1 < \rho_1 < \rho_2 < c\rho_3 < \rho_3 \leq r_2$ such that

$$\left( I^0_{\rho_1} \right), \quad \left( I^0_{\rho_2} \right), \, u \neq \tilde{T}u \quad \text{for} \quad u \in \partial V_{\rho_2} \quad \text{and} \quad \left( I^0_{\rho_3} \right) \text{ hold}.$$

(S2) There exist $\rho_1, \rho_2, \rho_3$ with $r_1 \leq \rho_1 < c\rho_2 < c\rho_3 < \rho_3 \leq c\rho_2$ such that

$$\left( I^0_{\rho_1} \right), \quad \left( I^0_{\rho_2} \right), \, u \neq \tilde{T}u \quad \text{for} \quad u \in \partial K_{\rho_2} \quad \text{and} \quad \left( I^0_{\rho_3} \right) \text{ hold}.$$
Proof. Suppose \((H_1)\) holds. Then it follows from properties of fixed point index (see for example [7, 15]), that the map \(\tilde{T}\) has a fixed point \(u\) in \(V_{\rho_2} \setminus K_{\rho_1}\). Then \(\rho_1 \leq ||u|| \leq \rho_2/c\) and \(c\rho_1 \leq u(t) \leq \rho_2/c\) for \(t \in [0,1]\), hence \(u \in C(r_1, r_2)\). Since \(Tu = T\tilde{u}\) for \(u \in C(r_1, r_2)\), Eq. (6) has one positive solution in \(C(r_1, r_2)\). The other assertions are proved similarly.

Remark 1. It is possible to state results for three or more positive solutions by similar arguments, we refer the reader to [14] for the type of results that may be stated.

3. Positive solutions of the BVP (1)-(2). We now consider the BVP

\[-u''(t) = g(t)h(u(t)), \text{ a.e. on } [0,1],\]

with boundary conditions

\[u'(0) + \alpha[u] = 0, \beta u'(1) + u(\eta) = 0, \eta \in [0,1],\]

The solution of \(-u'' = y\) under these BCs can be written

\[u(t) = (\beta + \eta - t)\alpha[u] + \int_0^1 y(s)ds + \int_0^\eta (\eta - s)g(s)ds - \int_0^t (t - s)g(s)ds.\]

By a solution of the BVP (13)-(14) we mean a solution \(u \in C[0,1]\) of the corresponding integral equation

\[u(t) = (\beta + \eta - t)\alpha[u] + \int_0^1 k(t, s)g(s)h(u(s))ds,\]

where

\[k(t, s) = \beta + \begin{cases} \eta - s, & s \leq \eta \\ 0, & s > \eta \end{cases}, \quad \begin{cases} t - s, & s \leq t \\ 0, & s > t \end{cases}.\]

Note that \(k(t, s)\) in (15) is the kernel for the special case \(u'(0) = 0\), studied by Infante and Webb in [9]. Here we discuss the case \(\beta + \eta > 1\), that leads to the existence of positive solutions.

Upper bounds

Note that

\[\|\gamma\| = \beta + \eta,\]

In [9] it was shown that when \(\beta + \eta > 1\) one may take

\[\Phi(s) = \begin{cases} \beta, & s > \eta \\ \beta + \eta - s, & s \leq \eta \end{cases}.\]

Lower bounds

Note \(\gamma(t)\) is a decreasing function of \(t\) in \([0,1]\) and \(\min_{t \in [0,1]} \gamma(t) = \beta + \eta - 1\). A simple calculation shows that \(k(t, s) \geq \beta + \eta - 1\) for \(t \in [0,1]\). This leads to

\[c = 1 - \frac{1}{\beta + \eta}.\]

Hence we work on the cone

\[K = \{u \in C[0,1], \min_{t \in [0,1]} u(t) \geq c||u||\},\]

with \(c\) as in (17).

We state a result for the existence of one positive solution, of course there are more general results, analogous to Theorem 2 and Remark (1).
Theorem 3. Let \( c \) be as in (17), \( m \) be as in (12) and \( M \) as in (10) and \( \Gamma < 1 \). Then the BVP (13), (14) has at least one positive solution, if either \((H_1)\) or \((H_2)\) of Theorem 2 hold.

Remark 2. Consider the BVP

\[- u''(t) = h(u(t)), \quad \text{a.e. on } [0,1],\]

with BCs

\[u'(0) + \alpha u(\xi) = 0, \quad \beta u'(1) + u(\eta) = 0, \quad 0 \leq \eta \leq 1, \quad 0 \leq \xi \leq 1.\]  

In this case \( g \equiv 1 \) and we may take \( A_0 = 0 \) and \( dA(s) \) the Dirac measure of weight \( \alpha > 0 \) at \( \xi \). Optimal values of the constants \( m \) and \( M \), for the special case \( u'(0) = 0 \), were given by Webb in [25] as follows:

\[
\frac{1}{m} = \beta + \eta^2/2 \quad \text{and} \quad \frac{1}{M} = \begin{cases} 
\frac{(\beta + \eta)^2/4}{\beta^2/4}, & \text{if } \beta < \eta, \\
\frac{(\beta^2 + \eta^2)/2}{\beta^2/4}, & \text{if } \eta \geq \beta \text{ and } \beta \leq 1, \\
\frac{(2\beta - 1 + \eta^2)/2}{\beta^2/4}, & \text{if } \beta > 1.
\end{cases}
\]  

Then \( i_K(\hat{T}, V_\rho) = 0 \), if \( \alpha |u| \geq \alpha_0 \rho \) for \( u \in \partial V_\rho \), and

\[c_2 \|\| \alpha + h_{\rho,\rho/c} \cdot \frac{1}{M} \| \geq 1.\]  

Here we may take \( \alpha_0 = \alpha \) since, for \( u \in \partial V_\rho \), we have \( \alpha |u| = \alpha u(\xi) \geq \alpha \rho \). With these BCs (21) reads

\[\alpha(\beta + \eta - 1) + \tilde{h}_{\rho,\rho/c} \cdot \frac{1}{M} \geq 1.\]  

We have that \( i_K(\hat{T}, K_\rho) = 1 \) if \( \Gamma < 1 \) and

\[
\left( \frac{(\beta + \eta)}{1 - \Gamma} \int_0^1 K(s) \, ds + \frac{1}{m} \right) \hat{h}^{0,\rho} \leq 1.
\]  

So we need

\[\Gamma = \int_0^1 \gamma(t) \, dA(t) = \alpha \gamma(\xi) = \alpha(\beta + \eta - \xi) < 1.\]  

Since \( K(s) = \alpha k(\xi, s) \), by direct calculation one gets

\[\int_0^1 K(s) \, ds = \alpha \int_0^1 k(\xi, s) \, ds = \alpha(\beta + \frac{1}{2} \eta^2 - \frac{1}{2} \xi^2).\]  

So (23) reads

\[
\left( \frac{\alpha(\beta + \eta)(\beta + \frac{1}{2} \eta^2 - \frac{1}{2} \xi^2)}{1 - \alpha(\beta + \eta - \xi)} + \beta + \frac{1}{2} \eta^2 \right) \hat{h}^{0,\rho} \leq 1.
\]  

Note that all the numbers in (22) and (25) can be calculated.

With the same technique we can allow nonlinearities with more than one singularity, as the next example shows.

Example 1. Take \( \alpha = \beta = 1/2 \), \( \xi = 1/4 \) and \( \eta = 3/4 \), and take

\[h(u) := \begin{cases} 
\frac{5}{u^2}, & \text{if } 0 < u \leq 1, \\
\frac{5}{2} + \frac{10}{u^2}(u - \frac{11}{16})^2, & \text{if } 1 < u < 10, \\
\frac{5}{(u-1)^2}, & \text{if } 10 \leq u < 11, \\
30 + \frac{1}{(u-1)^2}, & \text{if } u > 11.
\end{cases}\]
We need a suitable interval \([r_1, r_2]\) for our truncation. This time \(c = 1/5\). The \(i_K(T, V, \rho) = 0\) condition needs \(
olinebreak[3]\hat{h}_{\rho_1/c} \geq 28/13\) and \(i_K(T, V, \rho) = 1\) requires \(
olinebreak[3]\hat{h}_{\rho_1/c} \leq 32/55\).

The choice of \(r_1 = \rho_1 = 1\) and \(r_2 = \rho_2 = 10\) gives \(\hat{h}_{1,5} = 5/2 > 28/13\) and \(\hat{h}_{0,10} = 5/10 < 32/55\). This provides the existence of one positive solution with \(1 \leq u(t) \leq 10\) for every \(t \in [0, 1]\). On the other hand a choice of \(r_1 = \rho_1 = 12\) and \(r_2 = \rho_2 = 65\) gives \(\hat{h}_{12,60} > 5/2 > 28/13\) and \(\hat{h}_{0,65} = 31/65 < 32/55\). Thus we achieve the existence of a positive solution with \(12 \leq u(t) \leq 65\) for every \(t \in [0, 1]\).

As an application of Theorem 3 we consider the following eigenvalue problem:

\[
\lambda u''(t) + g(t)\hat{h}(u(t)) = 0, \quad t \in (0, 1) \tag{26}
\]

subject to BC (19).

**Theorem 4.** Let \(c\) be as in (17), \(m\) and \(M\) as in (20) and \(\Gamma\) as in (24). Then \(\lambda\) is an eigenvalue of the boundary value problem (26)-(19), with a corresponding positive eigenvector, if either

\((P_1)\) there exist \(\rho_1, \rho_2\) with \(r_1 \leq \rho_1 < \rho_2 \leq r_2\) such that

\[
\left(\frac{(\beta + \eta)}{1 - \Gamma} \int_0^1 K(s) \, ds + \frac{1}{m}\right)\hat{h}_{0, \rho_1} < \lambda < \frac{\hat{h}_{\rho_2, \rho_1/c}}{M(1 - \alpha(\beta + \eta - 1))}
\]

or

\((P_2)\) there exist \(\rho_1, \rho_2\) with \(r_1 \leq \rho_1 < \rho_2 \leq r_2\) such that

\[
\left(\frac{(\beta + \eta)}{1 - \Gamma} \int_0^1 K(s) \, ds + \frac{1}{m}\right)\hat{h}_{0, \rho_2} < \lambda < \frac{\hat{h}_{\rho_1, \rho_1/c}}{M(1 - \alpha(\beta + \eta - 1))}
\]

**Proof.** Suppose \(\lambda\) is fixed and satisfies \((P_1)\) and consider the equation

\[
u''(t) + g(t)\hat{h}(u(t)) = 0 \tag{27}
\]

with BC (19), where \(\hat{h}(u) = \lambda^{-1}\hat{h}(u)\). We can then apply Theorem 3 to the BVP (27)-(19), in order to obtain a positive solution of the BVP (26)-(19). The case \((P_2)\) is treated in a similar manner.

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