A general expression is discussed for pion probability distributions coming from relativistic heavy ion collisions. The general expression contains as limits: 1) The disoriented chiral condensate (DCC), 2) the negative binomial distribution and Pearson type III distribution, 3) a binomial or Gaussian result, 4) and a Poisson distribution. This general expression helps in dealing with other distributions such as the signal to noise laser distribution. The similarities and differences of the DCC distribution with these other distributions will be studied. A connection with the theory of disordered systems will be discussed which include spin-glasses, randomly broken objects, random and chaotic maps.

The purpose of this paper is to discuss a general expression for the pion probability distribution which may be used to analyze pions coming from relativistic heavy ion collisions. The study of pions is of current interest for several reasons. First pions are the main component of the produced particles coming from such collisions. Several thousand pions are now observed at CERN SPS experiments and this number may go up by a factor of 10 at RHIC energies. Secondly, the behavior of pions may signal the formation of the quark-gluon plasma as, for example, in the disoriented chiral condensate (DCC) picture [1–4]. Thirdly, the fluctuations in pions have been discussed in terms of intermittency and fractal structure [5] coming from non-Poissonian effects. A distribution which has been used to discuss these phenomena is the negative binomial (NB) distribution [6,7]. The NB distribution also has an important feature known as Koba-Nielsen-Olesen (KNO) scaling [8] which the Poisson distribution lacks. Fourthly, the Bose-Einstein correlations amongst pions is an important property used in Hanbury-Brown-Twiss (HBT) experiments, and such correlations have also been proposed for the formation of a pion laser [11,12]. Bose-Einstein condensation of atoms in a laser trap is a very recently observed phenomena in another area of physics. Distributions which extrapolate between Poisson and Bose-Einstein and negative binomial have been developed, such as the signal to noise model of Glauber-Lach. This S/N model was originally developed in quantum optics, but has also used for particle production and a review can be found in [13]. The expression to be developed extends the range of extrapolation by including not only these distribution but others. Specifically, a general expression will be developed which contains as special limiting cases: 1) The disoriented chiral condensate (DCC), the NB distribution, a binomial distribution [14] or Gaussian like distribution, and the Poisson distribution. The general distribution also approximates the S/N model. A connection with the theory of disordered systems [15,16] will also be discussed further. These disordered systems included spin-glasses [17,18], randomly broken objects [19,20], random permutations [21], random maps [16,19], and chaotic maps [22]. For example, the disoriented chiral condensate gives a probability distribution which comes from a random direction of the isospin vector and a connection of the DCC state with the theory of disordered systems [15,16] will also be discussed further. These disordered systems included spin-glasses [17,18], randomly broken objects [19,20], random permutations [21], random maps [16,19], and chaotic maps [22].

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\[ P_k(N, x, \gamma) = \frac{N!}{k!(N-k)!} \frac{\text{Beta}(N-k+x,k+\gamma)}{\text{Beta}(x,\gamma)} \]  

where \( k \) is a random variable which can take on values \( k = 0, 1, 2, ..., N \), and \( x, \gamma \) are parameters. The Beta function that appears in Eq. (1) is \( \text{Beta}(\omega, z) = \Gamma(\omega)\Gamma(z)/\Gamma(\omega + z) \) where \( \Gamma(z) \) is a gamma function. In probability theory, \( P_k(N, x, \gamma) \) is known as a Polya distribution [23]. This distribution was used in Ref. [25] to describe cluster yields coming from high energy heavy ion collisions and a connection of it with the theory of disordered systems was discussed briefly in Ref. [13]. Its usefulness as a model for pion probability distribution will be developed.

Some interesting limits of Eq. (1) are as follows. Setting \( x = 1, \gamma = 1/2 \) in Eq. (1), the \( P_k(N, x, \gamma) \) is \( P_k(N, 1, 1/2) \) is given by

\[ P_k(N, 1, 1/2) = \frac{(N!)^2}{(2N+1)!(k!)^2} \sim \frac{1}{2\sqrt{Nk}} \]  

Eq. (2) appears in pionic yields from the disoriented chiral condensate model of a QCD phase transition. Specifically \( 2k = n_0 \) is the number of neutral pions and \( 2N = n_0 + n_+ + n_- \) is the total number of pions with \( n_+ = n_- \) being the number of positive or negative pions. In obtaining \( 1/2\sqrt{Nk} \), Stirling’s approximation was used. The probability distribution for the neutral to total pion yield, \( R_3 = n_0/(n_0 + n_+ + n_-) \), is then \( P(R_3) \sim 1/2\sqrt{R_3} \). A general expression for the fluctuation of Eq. (1) is [15]
\[ \langle k^2 \rangle - \langle k \rangle^2 = \frac{x}{x + \gamma + 1} \langle k \rangle \left( 1 + \frac{\langle k \rangle}{\gamma} \right) \] (3)

where \( \langle k \rangle = (\gamma/(x + \gamma)) N \). For \( x = 1, \gamma = 1/2, \langle k \rangle = (1/3)N \) or \( \langle n_0 \rangle = 2N/3 = \langle n_+ \rangle = \langle n_- \rangle \). Thus \( \langle n_0^2 \rangle - \langle n_0 \rangle^2 = (4/5) \langle n_0 \rangle (1 + \langle n_0 \rangle) \) and \( \langle n_+^2 \rangle - \langle n_+ \rangle^2 = (1/5) \langle n_+ \rangle (1 + \langle n_+ \rangle) = (1/4)(\langle n_0^2 \rangle - \langle n_0 \rangle^2) \). The probability distribution of \( j \) \( n_+ \)’s or \( n_- \)’s is

\[ P_j(N, 1, 1/2) = \frac{(N!)^22^{2j}(2(N - j))!}{(2N + 1)!((N - j)!)^2} \sim \frac{1}{2\sqrt{N(N - j)}} \] (4)

The \( P_j(N, 1, 1/2) \) increases with \( j \) and behaves asymptotically like the arcsine distribution \( 1/\pi \sqrt{j(N - j)} \) for \( j \sim N \) when \( N \) is very large. This behavior is totally different than the \( \pi_0 \) behavior simply obtained from Eq.(3). As will be shown the \( \pi_0 \) behavior can be approximated by other distributions, but not this \( \pi_+ \) or \( \pi_- \) behavior which is coupled to the \( \pi_0 \) distribution. The arcsine distribution is also a special case of a distribution related to Eq.(1) to be discussed below. The constraint \( 2N = n_0 + n_+ + n_- \) represents a major difference between the DCC model and a NB description and other probability distributions usually applied to pion yields.

A binomial limit of Eq.(4) is obtained in the limit \( x \to \infty, \gamma \to \infty \). If \( x = 2\gamma \) so that \( \langle n_0 \rangle = \langle n_+ \rangle = \langle n_- \rangle = \frac{1}{3}(2N) \), then \( P_k(N) = \left( \frac{N}{k} \right) \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^{N-k} \). Also \( \langle k^2 \rangle - \langle k \rangle^2 = (2/3) \langle k \rangle \) from Eq.(4) and since \( n_0 = 2k: \langle n_0^2 \rangle - \langle n_0 \rangle^2 = (4/3) \langle n_0 \rangle \). For large \( N \), the binomial distribution is approximately a Gaussian. The DCC distribution for a large number of domains is also a binomial. In the limit in which each domain has \( k = 0 \) or 1 or \( n_0 = 0, 2 \) \( \pi_0 \)’s coming from each domain. The fluctuation in one domain is \( \langle k^2 \rangle - \langle k \rangle^2 = \sigma_k^2 \) and is given by Eq.(2) with \( x = 1, \gamma = 1/2 \), or using \( \langle n_1 \rangle = N_1/3: \langle k^2 \rangle - \langle k \rangle^2 = (2/15)N_1(1 + 2N_1/3) \). For \( m \) DCC cells, the total variance is \( \sigma_m^2 = m\sigma_k^2 = (2/15)N(1 + 2N/3m) \). When \( m = N, \sigma_m^2 = (2/9)N \) which is the binomial result.

A negative binomial limit can also be obtained from Eq.(4) in the limit \( N \to \infty x \to \infty, \rho = x/N \) and \( p = \rho/(1+\rho) \). Then

\[ P_k(p, \gamma) = \left( \frac{\gamma + k - 1}{k} \right) p^\gamma (1 - p)^k \]

\[ = \left( \frac{\gamma + k - 1}{k} \right) \frac{1}{(1 + \langle k \rangle/\gamma)^k} \left( \frac{\langle k \rangle/\gamma}{1 + \langle k \rangle/\gamma} \right) \] (5)

where \( \langle k \rangle = (\gamma/(x + \gamma)) N = \gamma/\rho \) and \( \langle k^2 \rangle - \langle k \rangle^2 = \langle k \rangle (1 + \langle k \rangle/\gamma) \) from Eq.(3). The \( P_k \) of Eq.(4) is the NB distribution with \( \gamma \) now the NB parameter. The NB distribution is of current interest since this distribution fits many experimental observations as, for example, in the multiplicity distribution of hadrons produced in e+e− collisions at LEP and PEP-PETRA and in hadron and heavy ion collisions. Van Hove and Giovanni have discussed a clan model which gives a NB distribution. Carruthers and Shih summarized various mechanisms that lead to NB distributions and show that this distribution can be put into a correspondence with self-similar Cantor sets or fractal structures. The NB distribution has been used to discuss issues related to intermittency. Taking \( \gamma = 1/2 \), the same value of \( \gamma \) as in the DCC distribution, then

\[ P_k(1/2) = \frac{(2k)!}{2^k(k!)^2} \left( \frac{1}{1 + 2 \langle k \rangle} \right)^{1/2} \left( \frac{2 \langle k \rangle}{1 + 2 \langle k \rangle} \right)^k \] (6)

The factor \( (2k)!/2^k(k!)^2 \) also appears in the DCC distribution and its Stirling limit is \( 1/\sqrt{\pi k} \) which gives the \( 1/\sqrt{\pi \gamma} \) behavior associated with the DCC state. The factor \( (2 \langle k \rangle/(1 + 2 \langle k \rangle))^k \) gives a geometric series decrease or exponential decrease in \( P_k \). Writing \( (2 \langle k \rangle/(1 + 2 \langle k \rangle)) = e^{-ak} \) with \( a = \ln(1 + 1/2 \langle k \rangle) \approx 1/2 \langle k \rangle \) results in \( P_k(1/2) \approx \left( 1/\sqrt{2\pi k \langle k \rangle} \right) e^{-k/2(\langle k \rangle)} \). For general \( \gamma \),

\[ P_k(\gamma) \approx \left( k^{-1}/\Gamma(\gamma) \right) (\gamma/\langle k \rangle)^\gamma e^{-k/\langle k \rangle} \] (7)

which is a Pearson type III distribution.
The Poisson limit of the NB is realized when $\gamma \gg \langle k \rangle$. Both the NB and DCC have larger than Poisson fluctuations. Fluctuations of pions larger than Poisson are to be expected just on the basis of Bose-Einstein statistics, with the Poisson limit obtained from Maxwell-Boltzmann statistics. A 20% enhancement in fluctuations above those of Poisson statistics occurs in both hydrodynamic and thermodynamic models from Bose-Einstein correlations \(^2\). An important feature of the NB and Poisson type distribution is that $(k) P_k(\gamma)$ is a universal function of the scaled variable $k/\langle k \rangle$ in the limit $k \to \infty$, $\langle k \rangle \to \infty$, which is known as KNO scaling \(^3\). For example the Poisson Type III is simply $k P_k(\gamma) = (\gamma^2/\Gamma(\gamma))(k/\langle k \rangle)^{\gamma-1} e^{-k/\langle k \rangle}$ and the DCC distribution is $\langle k \rangle P_k \equiv (1/2\sqrt{3}) \left( \frac{N_L}{\alpha} \right)^{1/2}$. However, the Poisson distribution does not obey KNO scaling.

Other distribution have been developed which extrapolate between Poisson and NB and have been used to describe the multiplicity distribution of produced particles. These include the Glauber-Lach signal $S$ to noise $N_L$ model \(^4\) with $S$ the coherent signal level (Poisson emitter) and $N_L$ the thermal Bose-Einstein noise level. In Biyajima’s generalization of this model \(^5\), the probability of $k$ particles is given by a distribution which is called a laser distribution:

$$\begin{equation}
P(k) = \frac{(NL/\alpha)^k}{(1 + NL/\alpha)^{k+\alpha}} e^{-S/(1 + NL)} L_{k-1}^{\alpha-1} \left( \frac{-S}{NL} \right) \end{equation}$$

with $\alpha = 1$ the Glauber-Lach model and $L_k^{\alpha-1}$ is a generalized Laguerre function. The $\langle k \rangle = NL + S$ and $\langle k^2 \rangle - \langle k \rangle^2 = (NL + S)^2 + NL(NL + S) + 1/NL$. The Poisson limit is achieved when $NL \to 0$, while a NB limit follows for $S \to 0$. By comparison, the distribution of Eq.\(^3\) also contains these two limits as well as the DCC distribution and other distributions already mentioned. Fig.1 illustrates the similarity of Eq.\(^3\) for the DCC choice of $x = 1$, $\gamma = 1/2$ and for $N = 300$ with the distribution of Eq.\(^8\) with $\alpha = 1/2$, $S = 62$, $NL = 38$. Both distributions have $\langle k \rangle = 100$. In the DCC choice $k$ is restricted to $0 \leq k \leq N$ while for signal to noise models $0 \leq k \leq \infty$. The figure shows the interval $0 \leq k \leq 300$ only. As noted, a major difference between the DCC results and these other models is the constraint $2N = \pi_0 + \pi_+ + \pi_-$. This leads to totally different behaviors in the $\pi_0$ channel and the $\pi_+$, $\pi_-$ channel. This result suggests that event-by-event data should be investigated not only for the fall off in $\pi_0$ to total yield as $1/2\sqrt{N_L}$ but also the rise in the $\pi_+$ or $\pi_-$ yields as given by Eq.\(^3\). Also shown in this figure is a NB distribution with $\gamma = 1/2$ and $\langle k \rangle = 100$. If the fluctuations of the two distributions, Eq.\(^3\) and Eq.\(^8\) are equated then $\alpha$, $\gamma$ are connected by $\gamma = \alpha/(1 - b^2)$ where $b = S/(NL + S)$ is the fractional signal level. Very large Bose-Einstein enhancements to Poisson results can appear in other models such as the pion laser model of Ref. \(^1\).

![FIG. 1.](image)

When Eq.\(^3\) is rewritten as

$$P_k(N, x, \gamma) = \int \binom{N}{k} w^k (1 - w)^{N-k} u(w, x, \gamma) dw$$

important functions as $u(w, x, \gamma) = (\text{Beta}(x, \gamma))^{-1} w^{x-1}(1 - w)^{x-1}$ and $f(w, x, \gamma) = w^{-1} u(w)$ emerge. The $u(w, x, \gamma)$ is also the limit of $P_k(N, x, \gamma)$ as $N \to \infty$, $k \to \infty$, $k/N \to w$. The Stirling limit of Eq.\(^8\) corresponds to $u(w, 1, 1/2)$. 3
The distribution of Eq.(9) and \( u(w, x, \gamma) \) and its associated \( f(w, x, \gamma) \) play a prominent role in the theory of disordered systems such as spin-glasses, randomly broken objects, random and chaotic maps, etc. Table 1 summarise some of these connections. The choice of \( x \) and \( \gamma \) varies for the different areas as listed in the table. The last column gives the quantity described by Eq.(9). In spin-glass models, random hamiltonians based on an Ising interaction \( J\vec{\sigma}_i \cdot \vec{\sigma}_j \) (where \( J = \pm|J| \) is chosen randomly) are used to calculate rugged free-energy landscapes. The function \( f(w, x, \gamma) \) and its related probability function \( u(w, x, \gamma) \) give the distribution of well depths in the free energy landscape. For clusters and breaking processes, with initial size \( A \) into fragments of size \( k = 1, 2, ..., A \) with \( n_k \) the number of fragments of size \( k \), \( f(w, x, \gamma) \) is related to \( n_k \) by \( n_k \rightarrow f(w, x, \gamma) \) in the limit \( k, A \rightarrow \infty \) with \( w = k/A \). At \( x = 1 \), \( n_k \sim 1/k^\tau \), a power law distribution with Fisher exponent \( \tau = 2 - \gamma \). In the case of random permutations in Table 1, the \( n_k \) of the cluster example just given now becomes the number of cycles of length \( k \), when the symmetric group of permutations is represented by its cycle class structure. A cycle class description for pionic distributions was also described in Ref. [28]. The random map example of Table 1 is also a prototype model for disordered systems [20]. Here \( n_k \) is the number of attractors with \( k \) elements. These attractors are limit cycles produced in a random mapping of \( A \) points into itself (allowing multiple points to go into one point). The \( f(w, x, \gamma) \) and \( u(w, x, \gamma) = wf(w, x, \gamma) \) follow as in the cluster case above. Chaotic maps, such as the tent and quadratic map [22,23], have invariant distributions which are given by arcsine laws. The \( u(w, x, \gamma) \) for \( x = \gamma = 1/2 \) is the arcsine distribution \( u(w, x, \gamma) = \pi/(w(1-w))^{1/2} \).

| Area                                      | \( x \)   | \( \gamma \) | Quantity Described                  |
|-------------------------------------------|-----------|--------------|-------------------------------------|
| Spin Glass [17,18]                        |           |             | Well depths from random hamiltonians |
| Disoriented Chiral Condensate - DCC [1–5] | 1         | 1/2          | Pion probability distribution       |
| Randomly Broken Objects [16,19,20]         |           |             | Fragment sizes                      |
| Random Permutations [21]                   | 1         | 1            | Cycle length                        |
| Random Maps [16,19]                        | 1/2       | 1            | Limit cycle distribution            |
| Chaotic Maps; Tent and Quadratic [22,23]   | 1/2       | 1/2          | Invariant distribution              |
| Power Laws with Fisher Exponent [15]       | 1         | \( \gamma \) | Cluster or droplet distributions    |

\( \tau < 2 \) and \( \tau = 2 - \gamma \)
To summarize and conclude, this paper contained a general expression for pions coming from relativistic heavy ion collisions. Various limits of this general distribution contain frequently used pion distribution. One important example is the disordered chiral condensate distribution which is of current interest because of its use as a possible signal of the quark-gluon phase transition. Another important limit gives rise to the negative binomial distribution which appears in discussions of intermittency, KNO scaling, non-Poissonian fluctuations. Other limits include a binomial limit, or Gaussian like limit, and a Poisson limit. Distributions which extrapolate between Poisson and negative binomial such as the Glauber-Lach and laser distribution have been developed previously and this paper extends the range of limiting cases to include many other distributions. Comparisons are made between the DCC, NB distribution, and laser distribution, with each having large fluctuations. Similarities and differences which may be important in understanding the quark-gluon phase transition are discussed. For example, the DCC distribution has a probability distribution of $\pi^0$'s which falls a $1/\sqrt{k}$, Eq.(2), coupled with a $\pi^+$ or $\pi^-$ distribution which rises as $1/\sqrt{N-j}$, Eq.(4), as $j \to N$ for $j\pi^+$'s or $\pi^-$'s. The $\pi^0$ distribution can be approximated by a standard negative binomial model or a signal to noise laser model as shown in Fig.1. Thus, this unique coupling of the $\pi^0$ and $\pi^+$ or $\pi^-$ channel characterizes the DCC state more than just the often quoted $1/\sqrt{R_3}$ or $1/\sqrt{k}$ behavior in the probability distribution of $\pi^0$'s. Heavy ion pion production data may also be analyzed in terms of the general distribution. The connection of this general distribution with the theory of disordered systems is discussed. These connections included spin glasses, randomly broken objects, random permutations, random maps and chaotic maps.

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