WEBS OF LAGRANGIAN TORI IN PROJECTIVE SYMPLECTIC MANIFOLDS

JUN-MUK HWANG, RICHARD M. WEISS

Abstract. For a Lagrangian torus $A$ in a simply-connected projective symplectic manifold $M$, we prove that $M$ has a hypersurface disjoint from a deformation of $A$. This implies that a Lagrangian torus in a compact hyperkähler manifold is a fiber of an almost holomorphic Lagrangian fibration, giving an affirmative answer to a question of Beauville’s. Our proof employs two different tools: the theory of action-angle variables for algebraically completely integrable Hamiltonian systems and Wielandt’s theory of subnormal subgroups.

Keywords. holomorphic symplectic geometry, compact hyperkähler manifold, holomorphic Lagrangian torus, action-angle variables, subnormal subgroups

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1. Introduction

The goal of this paper is to prove the following. See Definition 6.3 for the terminology.

Theorem 1.1. Let $M$ be a simply-connected projective manifold with a (holomorphic) symplectic form and let $A \subset M$ be a Lagrangian torus. Then $M$ has a hypersurface disjoint from a deformation of $A$.

Recall that a simply-connected compact Kähler manifold with a (holomorphic) symplectic form $\omega$ is called a compact hyperkähler manifold if $H^0(M, \Omega^2_M) = \mathbb{C}\omega$ (cf. [Hu]). One central problem in compact hyperkähler manifolds is to find a good condition for the existence of holomorphic or almost holomorphic fibrations on a compact hyperkähler manifold. In the survey [Be] of problems in hyperkähler geometry, Beauville asked whether the existence of a Lagrangian torus in $M$ gives rise to such a fibration (Question 6 in [Be]). As observed by Greb-Lehn-Rollenske (Corollary 5.6 of [GLR]), Theorem 1.1 implies the following, which gives an affirmative answer to Beauville’s question.

Theorem 1.2. Let $A \subset M$ be a Lagrangian torus in a compact hyperkähler manifold. Then there exists a meromorphic map $f : M \dasharrow B$ dominant over a projective variety $B$, such that on a nonempty Zariski open subset $M^o \subset M$ with $A \subset M^o$, $f|_{M^o}$ is a proper smooth morphism and $A$ is a fiber of $f$.

The deduction of Theorem 1.2 from Theorem 1.1 is a combination of a number of prominent results in hyperkähler geometry, in particular, [COP] and [Vo], as well as the standard hyperkähler machinery ([Hu]). We will not discuss this deduction, referring the reader to [GLR].

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Our proof of Theorem 1.1 uses completely different ideas and requires little knowledge of hyperkähler geometry. There are two crucial ingredients in our proof, one geometric and one algebraic. It is easy to see that deformations of a Lagrangian torus $A \subset M$ give rise to a multi-valued holomorphic foliation on a Zariski open subset in $M$. If this foliation is univalent, Theorem 1.1 is easily obtainable. Thus the key issue is how to deal with the multi-valuedness. To handle this difficulty, we are going to study the monodromy action of this multi-valued foliation (cf. Definition 3.6). The main geometric ingredient, Proposition 6.6, of our proof is the integrability of the local distribution given by a pair of sheets of the multi-valued foliation. This is established by means of the theory of action-angle variables for completely integrable Hamiltonian systems (see, e.g., [GS], Section 44). This ‘pairwise integrability’ gives some restrictions on the monodromy action, which is an action of a finite group on a finite set. However, these restrictions on the monodromy action do not immediately give us a solution of the problem. It turns out that a non-trivial result on the actions of finite groups on finite sets is required. This is our key algebraic ingredient, Theorem 2.4. Logically speaking, it belongs to abstract group theory, independent of geometry. Its proof uses Wielandt’s work on subnormal subgroups ([Wi]) and may be of independent interest.

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Conventions
1. Throughout the paper, a manifold is always connected, unless stated otherwise. A variety may have finitely many irreducible components. A projective manifold is a nonsingular irreducible projective variety.
2. When we say an open set, we mean it in the classical topology. An open set in Zariski topology will be called a Zariski open set.
3. For a projective subvariety $A$ in an algebraic variety $M$, $[A] \in \text{Hilb}(M)$ denotes the point of the Hilbert scheme determined by $A$. We abuse the term ‘deformation’ as follows. A deformation of $A$ means a subvariety of $M$ corresponding to a point in an irreducible Zariski open subset containing $[A]$ in $\text{Hilb}(M)$. A small deformation of $A$ means a subvariety of $M$ corresponding to a point in a classical neighborhood (or the germ) of $[A]$ in $\text{Hilb}(M)$.

2. A result on actions of finite groups on finite sets

This section is devoted to the study of certain actions of finite groups. The content of this section will be used only at the very end of Section 6, where we use Theorem 2.4 to complete the proof of Theorem 1.1.

Definition 2.1. A subgroup $H$ of a group $G$ is subnormal in $G$ if there exist a natural number $\ell$ and a chain of subgroups

$$H = F_\ell \subset F_{\ell-1} \subset \cdots \subset F_1 \subset F_0 = G$$

such that $F_i$ is normal in $F_{i-1}$ for all $i = 1, 2, \ldots, \ell$.

Our main tool is the following result of Wielandt proved in Satz 2 of [Wi]. See 6.7.4 of [KS] for a particularly beautiful proof.
Theorem 2.2. Let $G$ be a finite group. For a subgroup $H \subset G$ and $g \in G$, denote by $\langle H, gHg^{-1} \rangle$ the subgroup generated by $H$ and $gHg^{-1}$. If $H$ is subnormal in $\langle H, gHg^{-1} \rangle$ for all $g \in G$, then $H$ is subnormal in $G$.

Definition 2.3. We will consider triples $(X, G, H)$ consisting of a finite set $X$, a finite group $G$ acting on $X$ transitively, and a normal subgroup $H \triangleleft G_x$ of the stabilizer $G_x$ of a distinguished point $x \in X$. Given such a triple and an element $y \in X$, define $H_y := gHg^{-1}$, where $g \in G$ is an element such that $y = g \cdot x$. Since $H$ is a normal subgroup of $G_x$, $H_y$ is independent of the choice of $g$. Given a subset $Y \subset X$, say, $Y = \{y_1, \ldots, y_m\}$, denote by $|Y|$ the cardinality of $Y$ and denote by $\langle Y \rangle$, or $\langle y_1, \ldots, y_m \rangle$, the subgroup of $G$ generated by $\cup_{y \in Y} H_y$. For example, $H = \langle x \rangle$. A triple $(X, G, H)$ will be called trivial if $|X| = 1$. A triple $(X, G, H)$ will be called special if the following two conditions are satisfied.

1. $(X)$ acts transitively on $X$.
2. For any two distinct elements $y \neq z \in X$, $y$ and $z$ are not in the same $\langle y, z \rangle$-orbit.

Our result is the following.

Theorem 2.4. There are no non-trivial special triples.

Proof. Suppose that there exists a special triple $(X, G, H)$ with $|X| > 1$. Choose one such $(X, G, H)$ with minimal possible $|X| > 1$ and among those with minimal $|X|$, one with minimal $|G|$. If $(X, G, H)$ is a special triple, then so is $(X, \langle X \rangle, H)$. By the minimality of $|G|$, we have $G = \langle X \rangle$.

Lemma 2.5. There is no normal subgroup $N \triangleleft G$ such that $H \subset N \neq G$. In particular, $H$ is not a subnormal subgroup of $G$.

Proof. Assume the contrary and choose such a normal subgroup $N$. Then for any $g \in G$, $gHg^{-1} \subset gNg^{-1} = N$.

Thus $H_g \subset N \neq G$ for all $y \in X$. This contradicts $\langle X \rangle = G$. □

Lemma 2.6. For a subgroup $F \subset G$, let $F \cdot x$ denote the $F$-orbit containing $x$ and let $F^\circ = \langle F \cdot x \rangle$. Suppose that $F$ is a proper subgroup of $G$ containing $H$. Then $H \subset F^\circ \triangleleft F$ and either $F^\circ \neq F$ or $H = F^\circ = F$.

Proof. By definition, $F^\circ$ is generated by $\{fHf^{-1}, f \in F\}$. Since $H \subset F$, we have $H \subset F^\circ \triangleleft F$.

Suppose that $F^\circ = F$. Then $F^\circ$ acts transitively on $F \cdot x$ and $F = \langle F \cdot x \rangle$. This implies that $(F \cdot x, F, H)$ is a special triple. If $|F \cdot x| \neq 1$, then, by the minimality assumption, $F \cdot x = X$ implying $F = F^\circ = \langle F \cdot x \rangle = \langle X \rangle = G$, a contradiction. Thus $|F \cdot x| = 1$, which implies $F = F^\circ = \langle F \cdot x \rangle = \langle x \rangle = H$. □

Now we derive a contradiction as follows. Pick any $y \neq x \in X$. Define $F_1 := \langle x, y \rangle$. By Definition 2.3 (2), $x$ and $y$ are in two different $F_1$-orbits. This implies that
$H \subset F_1 \neq G$. If $H = F_1$, we stop here. If $H \neq F_1$, an application of Lemma 2.6 gives $F_2 := F_1^g$ satisfying 

$$H \subset F_2 \lhd F_1$$

with $F_2 \neq F_1$. If $H = F_2$, we stop. Otherwise, we can repeat the process to get $F_3 = F_2^g$ satisfying 

$$H \subset F_3 \lhd F_2 \lhd F_1$$

with $F_3 \neq F_2$. Repeating this, we get a natural number $\ell$ and a sequence of subgroups 

$$H = F_1 \lhd F_{\ell-1} \lhd \cdots \lhd F_2 \lhd F_1$$

such that $F_i$ is a proper normal subgroup of $F_{i-1}$ for each $i$. Thus $H$ is subnormal in $F_1 = \langle x, y \rangle$ for any choice of $y$. In other words, $H$ is subnormal in $\langle H, gHg^{-1} \rangle$ for any $g \in G$. By Wielandt’s Theorem, $H$ is subnormal in $G$. This is a contradiction to Lemma 2.5. \hfill \square

3. Webs of submanifolds

As explained in the introduction, our main object of study is a multi-valued foliation on a projective manifold arising from deformations of an algebraic submanifold. It is convenient to introduce the following to describe such a multi-valued foliation.

**Definition 3.1.** Let $M$ be a projective manifold. A *web of submanifolds* on $M$ is the following data, to be denoted by $\mathcal{W} = [\mu : \mathcal{U} \to M, \rho : \mathcal{U} \to \mathcal{K}]$.

1. A generically finite surjective morphism $\mu : \mathcal{U} \to M$ from a projective manifold $\mathcal{U}$.
2. A projective morphism $\rho : \mathcal{U} \to \mathcal{K}$ with connected fibers onto a projective manifold $\mathcal{K}$ with a Zariski open subset $\mathcal{K}^{\text{bihol}} \subset \mathcal{K}$ such that for each $a \in \mathcal{K}^{\text{bihol}}$,
   1. $\rho^{-1}(a)$ is smooth;
   2. $\mu_{|\rho^{-1}(O_a)} : \rho^{-1}(O_a) \to \mu(\rho^{-1}(O_a))$ is biholomorphic for some open neighborhood $O_a$ of $a$ in $\mathcal{K}^{\text{bihol}}$;
   3. $\mu(\rho^{-1}(b)) \neq \mu(\rho^{-1}(a))$ if $b \in \mathcal{K}, b \neq a$.

For a point $a \in \mathcal{K}^{\text{bihol}}$, the submanifold $\mu(\rho^{-1}(a))$ in $Z$ is called a *member of the web* $\mathcal{W}$.

We skip the proof of the following easy proposition.

**Proposition 3.2.** Given a web $\mathcal{W} = [\mu : \mathcal{U} \to M, \rho : \mathcal{U} \to \mathcal{K}]$ of submanifolds, there exists a nonempty Zariski open subset $M^{\text{et}} \subset M$ such that 

$$\mathcal{U}^{\text{et}} := \mu^{-1}(M^{\text{et}}) \subset \rho^{-1}(\mathcal{K}^{\text{bihol}})$$

and $\mu|_{\mathcal{U}^{\text{et}}} : \mathcal{U}^{\text{et}} \to M^{\text{et}}$ is étale.

**Notation 3.3.** In the setting of Proposition 3.2, let $d$ be the degree of $\mu$. For $y \in M^{\text{et}}$, write $\mu^{-1}(y) = \{y_1, \ldots, y_d\}$ and $I = \{1, \ldots, d\}$. For each $i \in I$, we set 

$$A_i^y := \mu(\rho^{-1}(\mu(y_i))).$$

For each pair $(i, j) \in I \times I$, let $A_{ij}^y$ be the irreducible component of $\mu^{-1}(A_i^y)$ containing $y_j$. This notation is not very precise, because it involves an ordering of $\mu^{-1}(y)$. But this should not cause confusion, because it will be applied to a given point $x \in M^{\text{et}}$ and points $y$ in a sufficiently small neighborhood of $x$ where we can always fix an ordering of $\mu^{-1}(y)$ in a uniform manner.

We recall the following standard topological fact.
Lemma 3.4. Let \( \rho : U \to K \) be a proper morphism between projective manifolds with connected fibers. Let \( E \subset U \) be a proper subvariety. Then there exists a proper subvariety \( B \subset K \) such that the restriction \( \rho' \) of \( \rho \) to the complement of \( B \) and \( E \), i.e.,

\[
\rho' : \rho^{-1}(K \setminus B) \setminus E \to K \setminus B,
\]

is locally differentiably trivial over the base in the sense of [CMP] Theorem 4.1.2.

Proof. By Ehresman’s Theorem ([CMP] Theorem 4.1.2) it suffices to prove that \( \rho' \) is evenly submersive in the sense of [CMP], p. 133. Replacing \( U \) by a log-resolution of \( (U, E) \), we may choose a Zariski open \( K_1 \subset K \) such that \( E_1 := \rho^{-1}(K_1) \cap E \) is the union of smooth hypersurfaces with simple normal crossing and the restriction of \( \rho \) on each component of \( E_1 \) and each intersection stratum of the components is a smooth morphism. In other words, \( E_1 \subset \rho^{-1}(K_1) \) is relatively simple normal crossing with respect to \( \rho \). In particular, for a given point \( x \in K_1 \) and \( y \in \rho^{-1}(x) \), there exists a neighborhood \( O_x \) of \( x \) in \( K_1 \) and a neighborhood \( U_y \) of \( y \) in \( \rho^{-1}(O_x) \) such that

1. there exists a biholomorphic map \( h : U_y \to W \times O_x \), where \( W \) is a domain in \( \mathbb{C}^n, n = \dim U - \dim K_1 \),
2. \( \rho'|_{U_y} = p_2 \circ h \), where \( p_2 : W \times O_x \to O_x \) is the projection to the second factor, and
3. \( h(E \cap U_y) \) is the product of a simple normal crossing hypersurface in \( W \) and \( O_x \).

From the compactness of \( E \cap \rho^{-1}(x) \), finitely many of such neighborhoods \( U_y \) cover \( E \). Thus we can fix the neighborhood \( O_x \) for all \( y \in \rho^{-1}(x) \). Now if we choose \( B \) to be the complement \( K \setminus K_1 \), then the corresponding \( \rho' \) is evenly submersive.

Proposition 3.5. In the setting of Proposition 3.2, there exists a proper subvariety \( C \subset K \) containing \( K \setminus K_{\text{bihol}} \) such that the restriction of \( \rho \) to \( U^\text{et} \setminus \rho^{-1}(C) \) is locally differentiably trivial over \( K \setminus C \) and the morphism \( \mu \) gives an embedding of \( \rho^{-1}(a) \cap U^\text{et} \) into \( M^\text{et} \) for each \( a \in K \setminus C \).

Proof. Let \( E \subset U \) be \( \mu^{-1}(M \setminus M^\text{et}) \) and apply Lemma 3.4 to obtain \( B \subset K \). Setting \( C = B \cup (K \setminus K_{\text{bihol}}) \), we have the result.

Definition 3.6. In the setting of Proposition 3.5, define \( M_o := M^\text{et} \setminus \mu(\rho^{-1}(C)) \). Fix a point \( x \in M_o \). Let \( X \) be the finite set \( \mu^{-1}(x) = \{ x_1, \ldots, x_d \} \) using Notation 3.3 and let \( \mathfrak{S}_X \) be the symmetry group on \( X \). The étale cover \( U^\text{et} \to M^\text{et} \) induces a natural homomorphism

\[
\alpha : \pi_1(M^\text{et}, x) \to \mathfrak{S}_X
\]

whose image will be denoted by \( G \). By the connectedness of \( U^\text{et} \), \( G \) acts transitively on \( X \). For each \( 1 \leq i \leq d \), let \( H_i \subset G \) be the image of the homomorphism

\[
\alpha \circ \lambda_i : \pi_1(A^i \cap M^\text{et}, x) \to \mathfrak{S}_X,
\]

where

\[
\lambda_i : \pi_1(A^i_x \cap M^\text{et}, x) \to \pi_1(M^\text{et}, x)
\]

is induced by the inclusion \( A^i_x \subset M \).

Proposition 3.7. In Definition 3.6, for all \( 1 \leq i \leq d \), denote by \( G_i \subset G \) the isotropy subgroup of \( x_i \in X \). We have \( G_i = g_iG_1g_i^{-1} \) and \( H_i = g_iH_1g_i^{-1} \) for any \( g_i \in G \) with \( g_i \cdot x_1 = x_i \) and \( H_i \) is a normal subgroup of \( G_i \).
Proof. Choose \( g'_i \in \pi_1(M^{\text{et}}, x) \) with \( \alpha(g'_i) = g_i \). Let
\[
\gamma_i : [0, 1] \to U^{\text{et}}
\]
be a path representing \( g'_i \), i.e.,
\[
\gamma_i(0) = x_1, \quad \gamma_i(1) = x_i \quad \text{and} \quad \text{the class of } \mu \circ \gamma_i \text{ belongs to } g'_i.
\]
Since \( X \cap \rho^{-1}(C) = \emptyset \), we can assume that \( \gamma_i([0, 1]) \) is disjoint from \( \rho^{-1}(C) \). Set \( c_i := \rho(\gamma_i(1)) \in K \setminus \mathcal{C} \). Then the family
\[
\{\mu(\rho^{-1}(c_t)) \cap M^{\text{et}}, t \in [0, 1]\}
\]
is locally differentiably trivial with \( \mu(\rho^{-1}(c_0)) \cap M^{\text{et}} = A^1_x \cap M^{\text{et}} \) and \( \mu(\rho^{-1}(c_1)) \cap M^{\text{et}} = A^i_x \cap M^{\text{et}} \).

Therefore, given a closed path
\[
\beta_0 : [0, 1] \to A^1_x \cap M^{\text{et}}, \quad \beta_0(0) = \beta_0(1) = x
\]
representing an element of
\[
\pi_1(\mu(\rho^{-1}(c_0)) \cap M^{\text{et}}, x) = \pi_1(A^1_x \cap M^{\text{et}}, x),
\]
we can find a continuous family of closed paths
\[
\{\beta_t : [0, 1] \to \mu(\rho^{-1}(c_t)) \cap M^{\text{et}}, \quad \beta_t(0) = \beta_t(1) = \mu(\gamma_i(t)), \quad t \in [0, 1]\}.
\]
Thus in \( M^{\text{et}} \), \( \beta_0 \) is homotopic to
\[
(\mu \circ \gamma_i)^{-1} \cdot \beta_1 \cdot (\mu \circ \gamma_i)
\]
for some closed path \( \beta_1 \) based at \( x \) representing an element of \( \pi_1(A^1_x \cap M^{\text{et}}, x) \). This implies that
\[
[\beta_0] = (g'_i)^{-1} \cdot [\beta_1] \cdot g'_i \quad \text{in } \pi_1(\mathcal{M}^{\text{et}}, x),
\]
proving \( H_i = g_iH_1g_i^{-1} \). Setting \( i = 1 \), we conclude that \( H_1 \) is normal in \( G_1 \). Therefore \( H_i = g_iH_1g_i^{-1} \) is normal in \( g_iG_1g_i^{-1} = G_i \) for all \( i \).

\[\square\]

**Definition 3.8.** Let \( f : M' \to M \) be a generically finite surjective morphism between two irreducible nonsingular varieties. Given an irreducible subvariety \( A \subset M \), we say that \( f \) splits over \( A \) if for each irreducible component \( A' \) of \( f^{-1}(A) \) satisfying \( f(A') = A \), the restriction \( f|_{A'} : A' \to A \) is birational.

**Proposition 3.9.** In Definition \[\ref{definition:splitting}\], denote by \( \mathcal{H} \subset G \) the subgroup generated by \( H_1, \ldots, H_d \). Assume that \( \mathcal{H} \) does not act transitively on \( X \). Then there exists a projective manifold \( M' \) and a generically finite surjective morphism \( f : M' \to M \) which is not birational and splits over \( \mu(\rho^{-1}(a)) \) for a general \( a \in \mathcal{K} \).

**Proof.** Put \( U_o := \mu^{-1}(M_o) \), where \( M_o \) is as in Definition \[\ref{definition:splitting}\]. Given two point \( u, v \in U_o \) with \( \mu(u) = \mu(v) \), write \( u \sim v \) if the following holds: there exist a point \( w \in U_o \) with \( \mu(w) = \mu(u) = \mu(v) \) and an irreducible component of \( \mu^{-1}(\mu(\rho^{-1}(\rho(w))) \cap M^{\text{et}}) \) containing both \( u \) and \( v \). Now let \( \sim \) be the equivalence relation on \( U_o \) generated by \( \sim \). In other words, two points \( u \) and \( v \) are equivalent, \( u \approx v \), if there exists an integer \( \ell \geq 1 \) and a sequence of points in \( U_o \)
\[
u = u_1, u_2, \ldots, u_{\ell-1}, u_\ell = v \]
such that \( u_i \sim u_{i+1} \) for each \( 1 \leq i \leq \ell - 1 \). Over a Zariski open subset \( M_1 \subset M_o \), this gives an \( \text{étale} \) equivalence relation, i.e., the equivalence classes on \( U_1 := \mu^{-1}(M_1) \) determine an \( \text{étale} \) factorization \( U_1 \to M'_1 \to M_1 \) of \( \mu|_{U_1} \). From the definition of the
equivalence relation, the étale morphism \( M'_1 \to M_1 \) splits over \( \mu(\rho^{-1}(a)) \cap M_a \) for a general \( a \in \mathcal{K} \).

We claim that \( M'_1 \to M_1 \) is not bijective if \( \mathcal{H} \) does not act transitively on \( X = \mu^{-1}(x), x \in M_1 \). To see the claim, note that for two points \( x_i, x_j \in X, x_i \sim x_j \) if and only if \( A^k_{x_i} = A^k_{x_j} \) for some \( k \in I = \{1, \ldots, d\} \), using Notation 3.3. The latter is equivalent to \( x_j = H_k \cdot x_i \) for some \( k \). Thus the equivalence classes of \( \approx \) in \( X \) are just the orbits of the group \( \mathcal{H} \). This proves the claim.

Since the equivalence relation \( \approx \) on \( \mathcal{U} \) is an algebraic equivalence relation, we have a generically finite morphism \( f : M' \to M \) compactifying \( M'_1 \to M_1 \) and a dominant rational map \( q : \mathcal{U} \dashrightarrow M' \) satisfying \( \mu = f \circ q \). By the assumption that the group \( \mathcal{H} \) does not act transitively on \( X \) and the previous claim, we see that \( f \) is not birational.

By the definition of \( M' \), we know that \( f \) splits over \( \mu(\rho^{-1}(a)) \) for a general \( a \in \mathcal{K} \). □

**Proposition 3.10.** Let \( \mathcal{W} = (\mu : \mathcal{U} \to M, \rho : \mathcal{U} \to \mathcal{K}) \) be a web of submanifolds on a projective manifold \( M \). Let \( f : M' \to M \) be a generically finite surjective morphism. Assume that \( f \) splits over \( \mu(\rho^{-1}(a)) \) for a general \( a \in \mathcal{K} \). Then \( \mu(\rho^{-1}(a)) \) is disjoint from the reduced branch divisor \( D \subset M \) of the morphism \( f \).

**Proof.** Suppose that \( \mu(\rho^{-1}(a)) \) has non-empty intersection with \( D \). By the generality of \( a \in \mathcal{K} \), we can assume that \( \mu(\rho^{-1}(a)) \) passes through a general point \( y \) of an irreducible component of \( D \) and the divisor \( D' := D \cap \mu(\rho^{-1}(a)) \) on \( \mu(\rho^{-1}(a)) \) is smooth at \( y \). Pick a ramification point \( z \in M' \) with \( f(z) = y \). Then \( z \), the morphism is locally analytically equivalent to a cyclic branched covering of degree \( \geq 2 \).

Pick an irreducible curve \( C \subset \mu(\rho^{-1}(a)) \) such that \( C \) intersects \( D' \) transversally at \( y \). Then \( f^{-1}(C) \) has an irreducible component \( C' \) through \( z \) such that \( C' \to C \) is locally a cyclic branched covering near \( z \) of degree \( \geq 2 \). The irreducible component of \( f^{-1}(\mu(\rho^{-1}(a))) \) containing \( z \) cannot be birational over \( \mu(\rho^{-1}(a)) \) because it must contain \( C' \) and we can choose \( C' \) to pass through any general point of \( \mu(\rho^{-1}(a)) \). This contradicts the assumption that \( f \) splits over \( \mu(\rho^{-1}(a)) \). □

### 4. Pairwise integrable webs of submanifolds

The term ‘web’ in the previous section has its origin in ‘web geometry’ in differential geometry. In this section, we need to view a web from this original viewpoint of local differential geometry. To be precise, we introduce the following definition.

**Definition 4.1.** Let \( U \) be a complex manifold. A regular web on \( U \) is a finite number of integrable subbundles

\[
W^i \subset T(U), \ i \in I := \{1, 2, \ldots, d\}
\]

for some integer \( d \geq 1 \) such that for any pair \( (i, j) \in I \times I \), the intersection \( W^i \cap W^j \subset T(U) \) is also a subbundle. This implies that the sum \( W^{ij} = W^i + W^j \subset T(U) \) with fiber at \( x \in U \)

\[
W^{ij}_x := W^i_x + W^j_x \subset T_x(U)
\]

is a subbundle of rank \( \text{rk}(W^i) + \text{rk}(W^j) - \text{rk}(W^i \cap W^j) \). A regular web is pairwise integrable if for any pair \( (i, j) \), \( W^{ij} = W^{ji} \) is integrable.

**Remark 4.2.** We are interested in local differential geometry of a regular web. So we will assume that all leaves of integrable distributions are closed in the complex manifold \( U \).

The following three lemmata are immediate.
Lemma 4.3. Let $U$ be a complex manifold and let $\{W^i \subset T(U), i \in I\}$ be an arbitrary finite collection of integrable subbundles of $T(U)$. Then there exists a nonempty Zariski open subset $U' \subset U$ such that the restriction $\{W^i|_{U'}, i \in I\}$ defines a regular web on $U$.

Lemma 4.4. Let $\{W^i, i \in I\}$ be a regular web on a complex manifold $U$. Given a connected open subset $U' \subset U$, the restriction $\{W^i|_{U'}, i \in I\}$ is a regular web on $U'$. If $\{W^i|_{U'}, i \in I\}$ is pairwise integrable for some nonempty $U' \subset U$, then $\{W^i, i \in I\}$ is also pairwise integrable.

Lemma 4.5. Given a regular web $\{W^i, i \in I\}$ on a complex manifold $U$ and a point $x \in U$, denote by $L^i_x$ the leaf of $W^i$ through $x$. Then $L^i_y = L^i_x$ for any $y \in L^i_x$. When $W^ij$ is integrable, denote by $L^ij_x$ the leaf of $W^ij$ through $x$. Then $L^ij_x, L^ij \subset L^ij_y$ and for any $y \in L^ij_x$,

$$L^ij_x = L^ij_y = L^ij_y.$$

Proposition 4.6. When $W^ij$ is integrable in Lemma 4.5, the germ of $L^ij_y$ at $y \in U$ is equal to that of $\bigcup_{z \in L^ij_y} L^ij_z$ and also that of $\bigcup_{z \in L^ij_y} L^ij_z$.

Proof. Set $\text{rk}(W^i) = r$ and $\text{dim} U = n$. Let $\Delta^r \times \Delta^{n-r}$ be the product of polydiscs of dimension $r$ and $n-r$, respectively. Let $p : \Delta^r \times \Delta^{n-r} \to \Delta^{n-r}$ be the projection. There exists a neighborhood $U_y$ of $y$ biholomorphic to $\Delta^r \times \Delta^{n-r}$ such that $L^i_x = p^{-1}(p(x))$ when we identify $U_y$ with the product of polydiscs. By the requirement that $W^ij$ is a vector subbundle of $T(U)$, $p^1|_{L^i_y \cap U_y}$ is a smooth morphism and by shrinking $U_y$ if necessary, we can assume that $p(L^ij_y \cap U_y)$ is a submanifold of dimension $\text{rk}(W^j) - \text{rk}(W^i \cap W^j)$ in $\Delta^{n-r}$. The germ of $\bigcup_{z \in L^ij_y} L^ij_z$ at $y$ is equal to that of the submanifold $p^{-1}(p(L^ij_y \cap U_y))$ and has dimension

$$r + \text{dim}(p(L^ij_y \cap U_y)) = r + \text{rk}(W^j) - \text{rk}(W^i \cap W^j) = \text{dim}(L^ij_y).$$

Because $L^i_y \subset L^ij_y$ and for each $z \in L^ij_y$, $L^ij_z \subset L^ij_y = L^ij_y$ by Lemma 4.5, we have

$$\bigcup_{z \in L^ij_y} L^ij_z \subset L^ij_y.$$

Since the germs of both sides at $y$ are smooth and both sides have the same dimension, their germs at $y$ coincide. \hfill \Box

Definition 4.7. Given a web $W = [\mu : U \to M, \rho : U \to K]$ of submanifolds on a projective manifold, let $M_o \subset M$ be the Zariski open subset in Definition 3.0 and let $U_o := \mu^{-1}(M_o)$. Denote by $d \geq 1$ the degree of $\mu$. For each point $x \in M_o$, there exists a neighborhood $U \subset M_o$ of $x$ such that

$$\rho^{-1}(U) = U^1 \cup U^2 \cup \cdots \cup U^d$$

is a disjoint union of connected open subsets $U^i \subset U$ and each $\mu^i := \mu|_{U^i}$ is a biholomorphic morphism from $U^i$ onto $U$. Let $T^i(U^i) \subset T(U^i)$ be the vector subbundle given by the relative tangent bundle of the smooth morphism $\rho|_{U^i}$. Let $W^i = d\mu^i(T^i(U^i))$ be the corresponding vector subbundle of $T(U)$. We say that $x \in M_o$ is a good point with respect to the web $W$ if $\{W^1, \ldots, W^d\}$ defines a regular web (in the sense of Definition 4.1) in a neighborhood of $x$. Let $M^{\text{good}} \subset M_o$ be the subset consisting of good points. By Lemma 4.3, $M^{\text{good}}$ is a nonempty Zariski open subset in $M$. For each point $x \in M^{\text{good}}$, the regular web $\{W^1, \ldots, W^d\}$ in a neighborhood of $x$ is called the induced regular web (at $x$). The web $W$ of submanifolds
is pairwise integrable if the induced regular web at some (hence any by Lemma \[4.4\]) point \(x \in M^{\text{good}}\) is pairwise integrable in the sense of Definition \[4.1\].

**Notation 4.8.** In the setting of Definition \[4.7\] for each pair \((i, j) \in I \times I\), recall from Notation \[3.3\] that \(A_{ij}^y \subset U\) is the irreducible component of \(\mu_y^{-1}(A_y^i)\) passing through \(y\). In particular, \(A_{ij}^y = \rho^{-1}(\rho(y))\). We set

\[
K_y^i := \rho(A_y^i), \quad A_{ij}^{y} := \text{closure of } \rho^{-1}(K_y^i \cap K^{\text{bihol}}) \quad \text{and } A_y^{ij} := \mu(A_{ij}^y).
\]

All of these are irreducible subvarieties.

**Proposition 4.9.** In the setting of Definition \[4.7\], choose a neighborhood \(x \in U \subset M^{\text{good}}\) and let \(\{W^1, \ldots, W^d\}\) be the regular web on \(U\) obtained from \(T^0(U^i)\). Using notation of Lemma \[4.5\] and Notation \[4.8\], we have the following for any \(y \in U\).

1. \(L_y^i\) is the connected component of \(A_y^i \cap U\) through \(y\).
2. If \(W\) is pairwise integrable, then \(L_y^{ij}\) is an irreducible component of \(A_y^{ij} \cap U\).

**Proof.** (i) is immediate from the definition of \(W^i\). (ii) is a consequence of (i) and Proposition \[4.6\]. \(\square\)

**Proposition 4.10.** Let \(W = [\mu : U \to M, \rho : U \to K]\) be a pairwise integrable web of submanifolds on a projective manifold \(M\). For a point \(x \in M^{\text{good}}\), choose a neighborhood \(x \in U \subset M^{\text{good}}\) equipped with the induced regular web \(\{W^1, \ldots, W^d\}\) on \(U\). Using Notation \[4.8\], we have the following.

1. \(A_y^i, A_y^j \subset A_y^{ij}, \rho(A_y^{ij}) = \rho(A_y^i)\) and \(A_{ij}^y\) is an irreducible component of \(\mu^{-1}(A_y^{ij})\) through \(y\).
2. \(A_y^{ij} = A_y^{ij}\).
3. If \(y \in L_x^i\), then \(A_y^{ij} = A_y^{ij}\).
4. If \(y \in L_x^{ij}\), \(A_y^{ij} = A_y^{ij}\) and \(A_y^{ij} = A_y^{ij}\).

**Proof.** (i) is immediate from the definition in Notation \[4.8\]. (ii) and (iii) follow from Lemma \[4.9\] and Proposition \[4.9\]. (iv) follows from (i) and (iii). \(\square\)

**Proposition 4.11.** In the setting of Proposition \[4.10\], let \(\tilde{A}_{ij}^x\) be the normalization of \(A_x^{ij}\) and let

\[
\sigma : \tilde{A}_{ij}^x \to \tilde{A}_{ij}^y \to A_y^{ij} \subset M
\]

be a desingularization of \(A_x^{ij}\) which leaves the smooth locus of \(\tilde{A}_{ij}^x\) intact. For any \(y \in L_x^i, \tilde{A}_{ij}^y \subset \tilde{A}_{ij}^x\) be the proper transforms of \(A_y^i\) and \(A_y^j\), respectively. Then there exist neighborhoods \(\mathcal{V}_y^i\) of \(\tilde{A}_{ij}^y\) and \(\mathcal{V}_y^{ij}\) of \(\tilde{A}_{ij}^y\) in \(\tilde{A}_{ij}^x\) equipped with morphisms

\[
\nu_y^i : \mathcal{V}_y^i \to A_y^i \cap \rho^{-1}(K^{\text{bihol}}) \quad \text{and } \nu_y^{ij} : \mathcal{V}_y^{ij} \to A_y^{ij} \cap \rho^{-1}(K^{\text{bihol}})
\]

such that

1. \(\nu_y^i(\tilde{A}_{ij}^y) = A_y^i\) and \(\nu_y^{ij}(\tilde{A}_{ij}^y) = A_y^{ij}\);
2. the four morphisms

\[
\sigma|_{\mathcal{V}_y^i} : \mathcal{V}_y^i \to M, \quad \sigma|_{\mathcal{V}_y^{ij}} : \mathcal{V}_y^{ij} \to M, \quad \nu_y^i : \mathcal{V}_y^i \to U \quad \text{and } \nu_y^{ij} : \mathcal{V}_y^{ij} \to U
\]

are embeddings;
(3) the four morphisms
\[ \sigma|_{A^i_y} : \tilde{A}^i_y \to A^i_y, \sigma|_{A^i_y} : \tilde{A}^i_y \to A^i_y, \nu^j_y|_{A^i_y} : \tilde{A}^i_y \to A^i_y \text{ and } \nu^j_y|_{A^i_y} : \tilde{A}^i_y \to A^i_y, \]
are birational;
(4) \( \mu \circ \nu^j_y = \sigma|_{V^j_y} \) and \( \mu \circ \nu^j_y = \sigma|_{V^j_y} \), i.e., the following diagrams, and those with \( i \) and \( j \) switched, commute.

\[
\begin{array}{ll}
V^i_y & \subset \tilde{A}^i_y \\
\downarrow \nu^i_y & \downarrow \sigma \\
U & \supset A^i_y
\end{array}
\begin{array}{ll}
A^i_y & \subset M \\
\downarrow \nu^j_y & \downarrow \sigma \\
A^i_y & \supset A^i_y
\end{array}
\]

**Proof.** Let \( L^i_{y_i} \) be the connected component of \( \mu^{-1}(L^i_{y_i}) \) through \( y_i \). It is an open neighborhood of \( y_i \) in \( A^i_y \) and shrinking \( U \) if necessary, \( \rho|_{L^i_{y_i}} \) is a smooth morphism from \( y \in M^{\text{good}} \) and the germ of the submanifold \( \rho(L^i_{y_i}) \subset \mathcal{K}^{\text{bihol}} \) at \( \rho(y_i) \) is an irreducible component of the germ of \( K^i_{y_i} \) at \( \rho(y_i) \). Choose a neighborhood \( O_{\rho(y_i)} \) as in Definition 3.1 such that \( \rho(L^i_{y_i}) \cap O_{\rho(y_i)} \) is smooth and denote by \( [\rho(L^i_{y_i}) \cap O_{\rho(y_i)}] \) its connected component through \( \rho(y_i) \). Define
\[
\forall^i_{y_i} := \rho^{-1}([\rho(L^i_{y_i}) \cap O_{\rho(y_i)}]).
\]
Then \( \mu|_{\forall^i_{y_i}} \) is an embedding into \( M \). Its image lies in \( A^i_y \) and contains an open subset of \( A^i_y \). Thus we have a lifting
\[
\tilde{\mu}^i_{y_i} : \forall^i_{y_i} \to A^i_y
\]
which is an embedding satisfying
\[
\mu|_{\forall^i_{y_i}} = \sigma \circ \tilde{\mu}^i_{y_i} \text{ and } \tilde{\mu}^i_{y_i}(A^i_y) = \tilde{A}^i_y.
\]
Define
\[
\forall^i_{y} := \tilde{\mu}(\forall^i_{y_i}) \text{ and } \nu^i_y := (\tilde{\mu}^i_{y_i})^{-1}.
\]
Define \( \forall^j_y \) and \( \nu^j_y \) in the same way by switching \( i \) and \( j \). Then all of (1)-(4) are immediate from the construction. \( \square \)

**Proposition 4.12.** In the setting of Proposition 4.11, the submanifolds \( \tilde{A}^i_x \) and \( \tilde{A}^j_x \) of \( \tilde{A}^{ij}_x \) belong to the same irreducible component of the Hilbert scheme \( \text{Hilb}(A^{ij}_x) \) if and only if \( A^{ij}_x = A^{ij}_x \).

**Proof.** Note that from Proposition 4.11 (2), \( \tilde{A}^i_x \) and \( \tilde{A}^j_x \) correspond to smooth points of the Hilbert scheme \( \text{Hilb}(\tilde{A}^{ij}_x) \). So each of \( \tilde{A}^i_x \) and \( \tilde{A}^j_x \) belongs to a unique irreducible component of the Hilbert scheme. At these smooth points, the Hilbert scheme has dimension
\[
\dim A^{ij}_x - \dim A^i_x = \dim A^{ij}_x - \dim A^i_x = \text{rk}(W^{ij}) - \text{rk}(W^i \cap W^j).
\]
Thus, from Proposition 4.11 (2) and (3), all general deformations of \( \tilde{A}^i_x \) and \( \tilde{A}^j_x \) in \( \tilde{A}^{ij}_x \) are proper transforms of the \( \mu \)-images of the fibers of the families
\[
A^{ij}_{x_i} \cap \rho^{-1}(\mathcal{K}^{\text{bihol}}) \to K^{ij}_{x_i} \cap \mathcal{K}^{\text{bihol}} \text{ and } A^{ij}_{x_j} \cap \rho^{-1}(\mathcal{K}^{\text{bihol}}) \to K^{ij}_{x_j} \cap \mathcal{K}^{\text{bihol}},
\]
respectively. If \( A^{ij}_{x_i} = A^{ij}_{x_j} \), then the two families coincide. Thus \( \tilde{A}^i_x \) and \( \tilde{A}^j_x \) belong to the same irreducible component of the Hilbert scheme \( \text{Hilb}(A^{ij}_x) \).
On the other hand, if $A^ij_{x_i} \neq A^ij_{x_j}$, from the effectiveness assumption in Definition 3.1 (iii), deformations of $A^ij_y$ and $A^ij_y$ in $A^ij_x$ do not belong to the same irreducible component of $\text{Hilb}(A^ij_x)$. By Proposition 4.11 (2) and (3), this implies that $\tilde{A}^i_x$ and $\tilde{A}^j_x$ do not belong to the same irreducible component of $\text{Hilb}(A^ij_x)$. □

**Proposition 4.13.** In the setting of Proposition 4.9, for a general point $x \in M^\text{good}$, the variety $A^ij_x$ is nonsingular at $x$ for any pair $(i, j) \in I \times I$.

**Proof.** From Proposition 4.9 (ii), it is clear that the subvarieties in $M$ that can be realized as $A^ij_x$ form a family of dimension equal to $\dim M - \dim A^ij_x$. Thus their singular loci are contained in a proper subvariety in $M$. □

**Proposition 4.14.** In the notation of Definition 3.6 and Notation 4.8, choose $x \in M^\text{good}$ general in the sense of Proposition 4.13. Let $\text{Sm}(A^ij_x)$ be the smooth locus of $A^ij_x$ and let $H_{ij} \subset G$ be the image of

$$\alpha \circ \lambda_{ij} : \pi_1(\text{Sm}(A^ij_x) \cap M^{\text{et}}, x) \to \mathcal{G}_X,$$

where the homomorphism

$$\lambda_{ij} : \pi_1(\text{Sm}(A^ij_x) \cap M^{\text{et}}, x) \to \pi_1(M^{\text{et}}, x)$$

is induced by the inclusion $A^ij_x \subset M$. Then $H_{ij}$ contains the subgroups $H_i$ and $H_j$ in Definition 3.6.

**Proof.** The homomorphism

$$\lambda^o_i : \pi_1(A^i_x \cap \text{Sm}(A^ij_x) \cap M^{\text{et}}, x) \to \pi_1(M^{\text{et}}, x)$$

induced by the inclusion $A^i_x \subset M$ factors through $\lambda_{ij}$. Thus $H_{ij}$ contains the images of $\alpha \circ \lambda^o_i$.

Denote by $\theta_i$ the homomorphism

$$\theta_i : \pi_1(A^i_x \cap \text{Sm}(A^ij_x) \cap M^{\text{et}}, x) \to \pi_1(A^i_x \cap M^{\text{et}}, x)$$

induced by the obvious inclusion. Then $\lambda^o_i = \lambda_i \circ \theta_i$. The complement

$$(A^i_x \cap M^{\text{et}}) \setminus (A^i_x \cap \text{Sm}(A^ij_x) \cap M^{\text{et}})$$

is a proper subvariety in the nonsingular irreducible variety $A^i_x \cap M^{\text{et}}$ from $A^i_x \subset A^ij_x$ of Proposition 4.10. Thus $\theta_i$ must be surjective.

It follows that $H_{ij}$ contains the image of $\alpha \circ \lambda_i$ i.e., $H_i$. By the same reasoning, $H_{ij}$ contains $H_j$, too. □

**Proposition 4.15.** In the setting of Proposition 4.14, if $A^ij_{x_i} \neq A^ij_{x_j}$, then $x_i$ and $x_j$ are not in the same $H_{ij}$-orbit in $X = \mu^{-1}(x)$.

**Proof.** Since $A^ij_x$ is smooth at $x$, for each $x_k \in X$, there exists a unique irreducible component of $\mu^{-1}(A^ij_x)$ containing $x_k$. Thus the assumption $A^ij_{x_i} \neq A^ij_{x_j}$ implies $X \cap A^ij_{x_i} \neq X \cap A^ij_{x_j}$. To prove the proposition, it suffices to show that $X \cap A^ij_{x_i}$ (resp. $X \cap A^ij_{x_j}$) is the $H_{ij}$-orbit of $x_i$ (resp. $x_j$).

From Proposition 4.10 (i), $A^ij_{x_i}$ is the irreducible component of $\mu^{-1}(A^ij_x)$ through $x_i$. Thus

$$A^ij_{x_i} \cap \mu^{-1}(\text{Sm}(A^ij_x) \cap M^{\text{et}})$$

is precisely the connected component of $\mu^{-1}(\text{Sm}(A^ij_x) \cap M^{\text{et}})$ containing $x_i$. Then the $H_{ij}$-orbit of $x_i$ is

$$X \cap A^ij_{x_i} \cap \mu^{-1}(\text{Sm}(A^ij_x) \cap M^{\text{et}}) = X \cap A^ij_{x_i},$$
because $X = \mu^{-1}(x) \subset \mu^{-1}(\text{Sm}(A^y_j) \cap M^{et})$ by our choice of $x$. In the same way, the $H_{ij}$-orbit of $x_j$ is $X \cap A^y_{x_j}$. 

\section{Pairwise Integrable Webs of Tori}

The goal of this section is to prove Proposition \ref{prop:5.10} about pairwise integrable webs of tori on projective manifolds. Its proof requires some standard results on deformations of submanifolds with trivial normal bundles. We start by recalling them.

\begin{definition}
A submanifold $A$ of a projective manifold $Z$ is \textit{unobstructed} if the Hilbert scheme $\text{Hilb}(Z)$ of $Z$ is smooth at $[A] \in \text{Hilb}(Z)$, the point corresponding to $A$. In this case, denote by $\text{Hilb}(Z)_A$ the (unique) irreducible component of $\text{Hilb}(Z)$ containing $[A]$, by $\xi_A : \text{Univ}(Z)_A \to \text{Hilb}(Z)_A$ the universal family and by $\eta_A : \text{Univ}(Z)_A \to Z$ the evaluation morphism.
\end{definition}

\begin{proposition}
For an unobstructed submanifold $A$ with trivial normal bundle in a projective manifold $Z$, denote by $\text{Hilb}(Z)^o_A \subset \text{Hilb}(Z)_A$ the open subset consisting of points parametrizing unobstructed submanifolds with trivial normal bundles. Denote by $\xi^o_A : \text{Univ}(Z)^o_A \to \text{Hilb}(Z)^o_A$ and $\eta^o_A : \text{Univ}(Z)^o_A \to Z$, the restrictions of $\xi_A$ and $\eta_A$, respectively. Then $\xi^o_A$ is a smooth projective morphism and $\eta^o_A$ is unramified. If furthermore $\eta_A$ is birational, then $\eta^o_A$ is biregular to its image which is a Zariski open subset in $Z$.
\end{proposition}

\begin{proof}
Since all members of $\text{Hilb}(Z)^o_A$ are smooth, $\xi^o_A$ is a smooth projective morphism. The evaluation morphism $\eta^o_A$ is unramified by the triviality of the normal bundles of members of $\text{Hilb}(Z)^o_A$. If $\eta_A$ is birational, the unramified morphism $\eta^o_A$ must be biregular over its image.
\end{proof}

\begin{remark}
Proposition \ref{prop:5.2} implies that an unobstructed submanifold with trivial normal bundle is a member of a web of submanifolds. Conversely, a member of a web of submanifolds is an unobstructed submanifold with trivial normal bundle. Thus one can replace Definition \ref{def:5.1} by Proposition \ref{prop:5.2} and develop all the theory starting from there. But we prefer Definition \ref{def:5.1} because it is more geometrically appealing (to us) and also the approach via Hilbert scheme plays a rather restricted role in this paper; it is used essentially only in this section.
\end{remark}

\begin{proposition}
Assume that $A$ in Proposition \ref{prop:5.2} is biregular to a complex torus. Suppose there exists a subtorus $S \subset A$ such that deformations of $S$ in $Z$ cover an open subset in $Z$. Then we have the following.

(i) There exists an open neighborhood $U \subset Z$ of $A$ equipped with a smooth projective morphism $f : U \to \Delta^n$ over a polydisc $\Delta^n$ of dimension $n = \dim Z - \dim A$, such that fibers of $f$ give deformations of $A$ in $U$.

(ii) There exists a smooth projective morphism $\zeta : U' \to U'$ over a complex manifold $U'$ whose fibers are deformations of $S$. Thus $S$ is unobstructed with trivial normal bundle in $Z$ and $\zeta$ induces a natural embedding of $U'$ into $\text{Hilb}(Z)^o_S$, realizing $U'$ as an open neighborhood of $[S] \in \text{Hilb}(Z)^o_S$.

(iii) There exists a smooth projective morphism $f' : U' \to \Delta^n$ with $f = f' \circ \zeta$ such that the fibers of $f'$ are the quotient tori of deformations of $A$ by deformations of $S$.
\end{proposition}

\begin{proof}
By Proposition \ref{prop:5.2} we can choose a polydisc neighborhood $\Delta^n \subset \text{Hilb}(Z)^o_A$ of $[A]$ such that setting $U^1 := \xi^{-1}_A(\Delta^n)$, the morphism $\eta_A|_{U^1}$ is biholomorphic to its
image $U := \eta_A(U^1) \subset Z$. Then we have $f : U \to \Delta^n$ satisfying

$$\xi_A|_{U^1} = f \circ \eta_A|_{U^1}.$$  

Since $\Delta^n$ contains no positive-dimensional compact subvariety, all deformations of $S$ in $U$ are contained in fibers of $f$. By shrinking $U$, we can assume that there exists a section $\Sigma \subset S$ in $S$ deformations of torus groups (analytic abelian scheme over $\Delta^n$) equipped with a family of subtori. Let $f' : U' \to \Delta^n$ be the family of quotient groups with $\xi : U \to U'$ the quotient morphism. Then $U'$, $f'$ and $\xi$ have the required properties. \qed

**Proposition 5.5.** In the setting of Proposition 5.4, assume that $\eta_S : \text{Univ}(Z)_S \to Z$ is birational. Then we have the following.

(a) There exists a Zariski open subset $Z^o \subset Z$ with a smooth projective morphism $\xi : Z^o \to M^o$ to a smooth variety $M^o$ whose fibers are deformations of $S$.

(b) For a general member $[A^i] \in \text{Hilb}(Z)_{S^i}$, there exists $[S^i] \in \text{Hilb}(Z)_{S^i}$ such that $S^i \subset A^i$ is a subtorus.

(c) $A^i$ in (b) lies in $Z^o$ and $\xi(A^i) \subset M^o$ is an unobstructed torus with trivial normal bundle, biregular to the quotient torus $A^i/S^i$.

**Proof.** By Proposition 5.2 applied to $S$ in place of $A$, we obtain a Zariski open subset $Z^o \subset Z$ as the image of $\eta_S^o$ and a morphism $\xi : Z^o \to M^o$ over a smooth variety $M^o \cong \text{Hilb}(Z)_{S^o}$, proving (a).

Since deformations of $S$ cover an open subset in $Z$, Proposition 5.4 shows that a general deformation of $A$ contains a deformation of $S$ as a subtorus. This proves (b).

To see (c), apply Proposition 5.4 in a neighborhood of $A^i$ containing $S^i$. All translates of $S^i$ inside $A^i$ are elements of $\text{Hilb}(Z)_{S^i}$, implying $A^i \subset Z^o$. That $\xi(A^i)$ is unobstructed and has trivial normal bundle is immediate from Proposition 5.4 (iii). \qed

**Proposition 5.6.** Let $A$ be an unobstructed submanifold in a projective manifold $Z$ with trivial normal bundle such that $\dim Z = 2 \dim A$. For any $[A'] \subset \text{Hilb}(Z)_A$, the intersection $A \cap A'$ has no isolated point.

**Proof.** From $\dim Z = 2 \dim A$, the intersection number $A \cdot A'$ is well-defined and equal to $A \cdot A$. Since a small deformation of $A$ is disjoint from $A$ by the triviality of the normal bundle, we have $A \cdot A' = 0$.

Suppose that $A \cap A'$ has an isolated point $z$. Regard $A \cdot A'$ as an intersection cycle in the sense of [Fu]. The isolated intersection point $z$ gives a positive contribution to $A \cdot A'$. The contribution from the other components of $A \cap A'$ is non-negative by Theorem 12.2 of [Fu] because the normal bundle of $A$ is trivial. This gives $A \cdot A' > 0$, a contradiction. \qed

**Proposition 5.7.** Let $A^1, A^2$ be two unobstructed tori with trivial normal bundles in a projective manifold $Z$. Assume that a connected component $S$ of $A^1 \cap A^2$ is a subtorus both in $A^1$ and in $A^2$, with $\dim Z = \dim A^1 + \dim A^2 - \dim S$. Assume furthermore that $S$ is unobstructed with trivial normal bundle in $Z$ and $\eta_S : \text{Univ}(Z)_S \to Z$ is birational. Then $\text{Hilb}(Z)_{A^1} \neq \text{Hilb}(Z)_{A^2}$.

**Proof.** Applying Proposition 5.5 to $S \subset Z$ with $A = A^1$ (resp. $A = A^2$), we see that $\xi(A^1)$ (resp. $\xi(A^2)$) is an unobstructed torus with trivial normal bundle in a projective manifold $M$ compactifying $M^o$. Since $S$ is a component of $A^1 \cap A^2$, we
see that $\zeta(A^1) \cap \zeta(A^2)$ has an isolated point $\zeta(S)$. If $\text{Hilb}(Z)_{A^1} = \text{Hilb}(Z)_{A^2}$, then $\dim A^1 = \dim A^2$ and $\zeta(A^2)$ is a member of $\text{Hilb}(\mathcal{M})_{\zeta(A^1)}$. From the assumption $\dim Z = \dim A^1 + \dim A^2 - \dim S$, we have

$$\dim \mathcal{M} = \dim Z - \dim S = 2(\dim A^1 - \dim S) = 2 \dim \zeta(A^1) = 2 \dim \zeta(A^2).$$

Applying Proposition 5.8 with $A := \zeta(A^1)$ and $A' := \zeta(A^2)$, we have a contradiction. \hfill $\square$

We will skip the proof of the following elementary lemma.

**Lemma 5.8.** A closed submanifold of a complex torus with trivial normal bundle is a subtorus.

**Proposition 5.9.** Let $Z$ be a projective manifold and let $A_1, A_2 \subset Z$ be two distinct tori with $A_1 \cap A_2 \neq \emptyset$. Assume that there exist open neighborhoods $A_1 \subset V_1$ and $A_2 \subset V_2$ equipped with smooth projective morphisms

$$\rho_1 : V_1 \to \Delta^{\dim Z - \dim A^1} \quad \text{and} \quad \rho_2 : V_2 \to \Delta^{\dim Z - \dim A^2}$$

such that $A_1 = \rho_1^{-1}(0)$ and $A_2 = \rho_2^{-1}(0)$. Then for a general point $u \in V_1 \cap V_2$, the connected component $S_u$ of

$$\rho_1^{-1}(\rho_1(u)) \cap \rho_2^{-1}(\rho_2(u))$$

with $u \in S_u$ is a subtorus both in $\rho_1^{-1}(\rho_1(u))$ and in $\rho_2^{-1}(\rho_2(u))$. Furthermore, this $S_u$ is unobstructed with trivial normal bundle in $Z$.

**Proof.** For each $t \in \Delta^{\dim Z - \dim A^2}$, $\rho_2^{-1}(t) \cap V_1$ is a closed subvariety of $V_1$. General fibers of the proper morphism $\rho_1|_{\rho_2^{-1}(t) \cap V_1}$ are unions of submanifolds with trivial normal bundles in the torus $\rho_2^{-1}(t)$. Thus they must be subtori of $\rho_2^{-1}(t)$ by Lemma 5.8. It follows that for general $u \in V_1 \cap V_2$, $S_u$ must be a subtorus of $\rho_2^{-1}(\rho_2(u))$, and by the same reasoning, also a subtorus in $\rho_1^{-1}(\rho_1(u))$. By the generality of $u$, deformations of $S_u$ cover an open subset in $Z$. Applying Proposition 5.8 with $S = S_u$ and $A = \rho_1^{-1}(\rho_1(u))$, we see that $S$ is unobstructed with trivial normal bundle in $Z$. \hfill $\square$

The next proposition is the main result of this section.

**Proposition 5.10.** Let $W$ be a pairwise integrable web on a projective manifold $M$ whose members are tori. Fix a general point $x \in M^{\text{good}}$ and choose a neighborhood $U \subset M^{\text{good}}$ as in Definition 4.7. Since $A^i_x$ is smooth at $x$ by Proposition 4.13, we may assume by shrinking $U$ that

$$\sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \to U \cap A^i_x$$

is biholomorphic. Using the notation of Proposition 4.11 and shrinking $U$ further if necessary, we have the following.

(i) For any $y \in U \cap A^i_x$, the component $S^i_y$ of $\tilde{A}^i_x \cap A^i_y$ through $\sigma^{-1}(y)$ is unobstructed with trivial normal bundle in $A^i_x$ and is a subtorus both in $\tilde{A}^i_x$ and $A^i_x$.

(ii) The germ of $S^i_y$ at $y$ is sent by $\sigma$ to that of the leaf of $W^i \cap W^j$ through $y$.

(iii) Small deformations of $S^i_y$ in $\tilde{A}^i_x$ are realized by $S^i_z$ for $z \in U \cap A^i_x$.

(iv) If $\eta_{S^i_y} : \text{Univ}(A^i_x)|_{S^i_y} \to A^i_x$ is birational, then $A^i_{x_1} \neq A^i_{x_2}$. 

Proof. (i) is a direct consequence of Proposition 4.11 and Proposition 5.9. (ii) is immediate from Proposition 4.9 (i).

For (iii), note that for $z \in U \cap A_x^{ij}$ with sufficiently small $U$, the submanifold $S_z^{ij}$ is a deformation of $S_y^{ij}$ and the family $\{S_z^{ij}, z \in U \cap A_x^{ij}\}$ covers an open set in $\sigma^{-1}(U)$. Since $S_y^{ij}$ has trivial normal bundle in $A_x^{ij}$, this implies that the family $\{S_z^{ij}, z \in U \cap A_x^{ij}\}$ corresponds to an open neighborhood of $[S_y^{ij}]$ in $\text{Hilb}(A_x^{ij})$, proving (iii).

To prove (iv), set

$$Z = A_x^{ij}, A^1 := \tilde{A}_x^{ij} \text{ and } A^2 := \tilde{A}_y^{ij}.$$ 

From (ii) we have $\dim Z = \dim A^1 + \dim A^2 - \dim S$. Applying Proposition 5.7, we have $\text{Hilb}(Z)_{A^1} \neq \text{Hilb}(Z)_{A^2}$. Thus $A_x^{ij} \neq A_y^{ij}$ by Proposition 4.12. □

6. WEBS OF LAGRANGIAN TORI

Definition 6.1. An even-dimensional vector space $V$ equipped with a non-degenerate form $\omega \in \wedge^2 V^*$ is a symplectic vector space. Given a subspace $W \subset V$, define

$$W^\perp := \{v \in V, \omega(v, w) = 0 \text{ for all } w \in W\}.$$ 

A subspace $W \subset V$ is Lagrangian if $W = W^\perp$. If $W$ is Lagrangian, $2 \dim W = \dim V$.

We will skip the proof of the following elementary lemma.

Lemma 6.2. Given a symplectic vector space $(V, \omega)$ and two Lagrangian subspaces $W^1, W^2 \subset V$,

$$W^1 \cap W^2 = (W^1 + W^2)^\perp \subset W^1 + W^2.$$ 

Definition 6.3. Let $M$ be a complex manifold. A symplectic form on $M$ is a closed holomorphic 2-form $\omega$ on $M$ such that for each $x \in M$, the tangent space $(T_x(M), \omega_x)$ is a symplectic vector space. The pair $(M, \omega)$ is called a symplectic manifold. A submanifold $A$ of a symplectic manifold $(M, \omega)$ is Lagrangian if for each $x \in A$, $T_x(A)$ is a Lagrangian subspace of $T_x(M)$. A Lagrangian submanifold is a Lagrangian torus if it is biholomorphic to a complex torus. A flat morphism with connected fibers $f : M \to B$ from a symplectic manifold $M$ onto a complex manifold $B$ is a Lagrangian fibration if each general fiber is a Lagrangian submanifold.

The next lemma is standard: see, e.g., p.220 of [GS].

Lemma 6.4. The cotangent bundle $T^*(B)$ of a complex manifold $B$ has a canonical holomorphic symplectic form $\omega_{st}$ on it. When $n = \dim B$, a holomorphic coordinate system $(q^1, \ldots, q^n)$ on a coordinate neighborhood $U \subset B$ and the associated functions $p^i$ on $T^*(U)$ given by $\frac{\partial}{\partial q^i}$ define a holomorphic coordinate system $(p^1, \ldots, p^n, q^1, \ldots, q^n)$ on $T^*(U)$ such that

$$\omega_{st}|_{T^*(U)} = dp^1 \wedge dq^1 + \cdots + dp^n \wedge dq^n.$$ 

With respect to $\omega_{st}$, the natural projection $\pi : T^*(B) \to B$ is a Lagrangian fibration and a section $\Sigma \subset T^*(B)$ of $\pi$ is a Lagrangian submanifold if and only if it is $d$-closed when regarded as a 1-form on $B$.

The following is a holomorphic version of the action-angle variables for completely integrable Hamiltonian systems, Theorem 44.2 in [GS]. It is a reformulation of Proposition 3.5 in [Hw], and its proof will be skipped.
Proposition 6.5. Let \((N, \omega)\) be a symplectic manifold and \(f : (N, \omega) \to B\) be a proper Lagrangian fibration such that each fiber is a complex torus and there exists a Lagrangian section \(\Sigma \subset N\). Then there exists an unramified surjective holomorphic map \(\chi : T^*(B) \to N\) with \(\pi = f \circ \chi\) such that

(i) for each \(b \in B\), \(\chi_b : T^*_b(B) \to f^{-1}(b)\) is the universal covering of the complex torus with \(\chi_b(0) = \Sigma \cap f^{-1}(b)\),

(ii) in the notation of Lemma 6.4, \(\omega_{\text{st}} = \chi^* \omega\), and, consequently,

(iii) each component of \(\chi^{-1}(\Sigma)\) is a Lagrangian submanifold in \(T^*(B)\), locally defining a closed 1-form on \(B\).

Proposition 6.6. In the setting of Proposition 6.5, let \(O \subset N\) be a connected open subset equipped with a smooth Lagrangian fibration \(\psi : O \to Q\), different from \(f|_U\). For each \(x \in U\), consider the subspace \(D_x \subset T(x)(U)\) defined by

\[
D_x := T_x\left(f^{-1}(f(x))\right) + T_x(\psi^{-1}(\psi(x))).
\]

By shrinking \(O\) if necessary, we may assume that \(\{D_x, x \in O\}\) defines a vector subbundle \(D \subset T(O)\). If for each \(x \in O\), \(f^{-1}(f(x)) \cap \psi^{-1}(\psi(x))\) is an open subset in a subtorus of \(f^{-1}(f(x))\), then \(D\) is integrable.

Proof. At each point \(x \in O\), define \(F_x \subset T_x(O)\) by

\[
F_x := T_x\left(f^{-1}(f(x))\right) \cap T_x(\psi^{-1}(\psi(x))).
\]

By shrinking \(O\) if necessary, \(F \subset T(O)\) defines a foliation on \(O\), whose leaves are open subsets of subtori in the fibers of \(f\). After shrinking \(B\), if necessary, we can extend \(F\) via translations in the fibers of \(f\) to a foliation on \(N\) whose leaves are subtori of the fibers of \(f\). Let \(S \subset N\) be the collection of these subtori intersecting \(\Sigma\) such that \(f|_S : S \to B\) is a family of subtori with a section \(\Sigma \subset S\) and leaves of \(F\) are just translates of fibers of \(f|_S\) (as in Proposition 5.4).

By Proposition 6.3 and shrinking \(B\) further, we have \(\chi : T^*(B) \to N\) such that \(\chi^{-1}(\Sigma)\) is a union of Lagrangian sections of \(T^*(B) \to B\). The component of \(\chi^{-1}(\Sigma)\) containing the zero-section of \(T^*(B)\) is a vector subbundle \(\mathcal{F} \subset T^*(B)\) such that

\[
\chi|_{F_b} : F_b \to S_b := f^{-1}(b) \cap S
\]

is the universal cover of the subtorus \(S_b\) for each \(b \in B\).

Set \(n = \dim B\) and \(r = \text{rk}(F)\). We can find a set \(\{\Sigma^1, \ldots, \Sigma^n\}\) of components of \(\chi^{-1}(\Sigma)\) forming a frame for the vector bundle \(T^*(B)\) such that the subset \(\{\Sigma^1, \ldots, \Sigma^r\}\) forms a frame for the subbundle \(\mathcal{F} \subset T^*(B)\). Using Lemma 6.4 and shrinking \(B\) further, we have holomorphic functions \(q^1, \ldots, q^n\) on \(B\) such that the closed 1-forms \(dq^1, \ldots, dq^n\) represent \(\Sigma^1, \ldots, \Sigma^n \subset T^*(B)\). Then in terms of the coordinates

\[
(p^1 = \frac{\partial}{\partial q^1}, \ldots, p^n = \frac{\partial}{\partial q^n}, q^1, \ldots, q^n)
\]

on \(T^*(B)\) of Lemma 6.4, the foliation \(\chi^{-1}F\) induced by \(F\) on \(T^*(B)\) is given by the translates of the span of \(\frac{\partial}{\partial p^1}, \ldots, \frac{\partial}{\partial p^n}\). Since the distribution \(D\) coincides with \(F\) by Lemma 6.2, the pull-back distribution \(\chi^{-1}D\) on \(T^*(B)\) is given by the translates of the span of

\[
\frac{\partial}{\partial p^1}, \ldots, \frac{\partial}{\partial p^n}, \frac{\partial}{\partial q^{n-r+1}}, \ldots, \frac{\partial}{\partial q^n}.
\]

Thus \(\chi^{-1}D\) is integrable and so is \(D\). \(\square\)
**Definition 6.7.** Let $M$ be a projective symplectic manifold, i.e., a projective manifold equipped with a holomorphic symplectic form. A web of submanifold $W$ on $M$ is called a **web of Lagrangian tori** if its members are Lagrangian tori in $M$.

**Proposition 6.8.** A web of Lagrangian tori is pairwise integrable.

*Proof.* Let $W = [\mu : U \to M, \rho : U \to K]$ be a web of Lagrangian tori on a projective symplectic manifold $(M, \omega)$. The open set $N := \rho^{-1}(K^{bihol})$ is a symplectic manifold equipped with the symplectic form $(\mu|_N)^*\omega$ and $\rho|_N : N \to K^{bihol}$ is a Lagrangian fibration. We choose $U \subset M^{good}$ and $U^i \subset N$ as in Definition 4.7. Let $\rho^i := \rho|_{U^i}$. By the natural biholomorphic map $(\mu^1)^{-1} \circ \mu^i : U^i \to U^1$, we can regard $\rho^i$ as defined on $U^1$. Then

$$f = \rho|_N, \psi := \rho^i, B := K^{bihol}, D := W^{1i} \text{ and } O := U^1$$

satisfy the assumption of Proposition 6.6 by Proposition 5.10 (i). Thus Proposition 6.2 gives, for a point $y \in (\mu|_N)^{-1}(\rho^i)$, we see that deformations of $W^{1i}$ are tangent to $S$ by Proposition 5.10 (ii) and (iii), we see that deformations of $W^{1i}$ are integrable. By the same reasoning, we get the integrability of $W^{ij}$ for all pairs $(i, j)$. Thus $W$ is pairwise integrable. \qed

**Proposition 6.9.** In the setting of Proposition 6.8, use the notation of Proposition 5.10. For a general point $x \in M^{good}$, we have $Z := \overset{\sim}{A}^j_x$ and $S := S^i_x \subset Z$, an unobstructed torus with trivial normal bundle in $Z$ which is the connected component of $\overset{\sim}{A}^j_x \cap A^j_x$ containing $\sigma^{-1}(x)$. Then $\eta_S : \text{Univ}(Z)_S \to Z$ is birational.

*Proof.* We claim that there exists a non-empty Zariski open subset $Z \subset Z$ and a vector subbundle $\mathcal{N} \subset T(Z)$ such that small deformations of $S$ in $Z$ intersecting $Z$ are tangent to $\mathcal{N}$ with $\text{rk}(\mathcal{N}) = \dim S$. This implies that $\eta_S$ is birational.

To prove the claim, define the null subspace at a smooth point $y \in A^j_x$ by

$$\text{Null}^{ij}_y(\omega)_y := \{v \in T_y(A^j_x), \omega(v, T_y(A^j_y)) = 0\} = T_y(A^j_x)^\perp \cap T_y(A^j_y).$$

On a Zariski open subset $A \subset \text{Sm}(A^j_x)$, this defines a vector subbundle $\text{Null}^{ij}(\omega)|_A \subset T(A)$. Using the desingularization $\sigma : Z \to A^j_x$, define

$$Z := \sigma^{-1}(A) \cong A \text{ and } \mathcal{N} := d\sigma^{-1}(\text{Null}^{ij}(\omega)|_A) \subset T(Z).$$

When $\{W^i, i \in I\}$ is the regular web induced by $W$ in a neighborhood $U$ of $x$, Lemma 6.2 gives, for a point $y \in A \cap U$,

$$W^i_y \cap W^j_y = (W^i_y)^\perp \cap W^j_y = T_y(A^j_y)^\perp \cap T_y(A^j_y) = \text{Null}^{ij}(\omega)_y$$

because the germ of $A$ is that of a leaf of $W^{ij}$ by Proposition 4.9 (ii). Since the germs of deformations of $S$ in $Z$ correspond to those of leaves of $W^i \cap W^j|_A$ by Proposition 5.10 (ii) and (iii), we see that deformations of $S$ are tangent to $\mathcal{N}$ with $\text{rk}(\mathcal{N}) = \dim S$, proving the claim. \qed

Now we are ready to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** It is well-known that a Lagrangian torus $A \subset M$ is unobstructed with trivial normal bundle (e.g., by Theorem 8.7 in [DM]). Thus we have $\xi_A : \text{Univ}(M)^A_\alpha \to \text{Hilb}(M)^A_\alpha$ and $\eta_A : \text{Univ}(M)^A_\alpha \to M$ in Proposition 6.2. By choosing suitable projective manifolds compactifying $\text{Univ}(M)^A_\alpha$ and $\text{Hilb}(M)^A_\alpha$, we obtain a web of Lagrangian tori $W = [\mu : U \to M, \rho : U \to K]$ which has $A$ as a member.

Using the notation of Definition 3.6, suppose that the group $H \subset G$ generated by $H_1, \ldots, H_d$ acts intransitively on $X$. Then by Proposition 3.3, we have a factorization of $\mu : U \to M$ via a generically finite morphism $\mu' : U' \to M$ which is not birational and splits over a general member of $W$. Since $M$ is simply connected, the branch
divisor $D \subset M$ of $\mu'$ is a non-empty hypersurface. By Proposition 3.10, $D$ is disjoint from a general member of $W$ and we are done.

Thus we may assume that $H$ acts transitively on $X$. We claim that $(X, G, H)$ with $H = H_1$ is a special triple in the sense of Definition 2.3. From Proposition 3.10, $D$ is disjoint from a general member of $W$ and we are done.

Thus we may assume that $H$ acts transitively on $X$. We claim that $(X, G, H)$ with $H = H_1$ is a special triple in the sense of Definition 2.3. From Proposition 3.7, $H$ is a normal subgroup of the isotropy subgroup $G_1$ of $x_1 \in X$ and $H_i = H g_i H g_i^{-1}$ when $x_i = g_i \cdot x_1$. In terms of Definition 2.3, $\langle x \rangle = H$. Thus our assumption that $H$ acts transitively on $X$ is exactly Definition 2.3. The web $W$ is pairwise integrable by Proposition 6.8. For a general point $x$ and any pair $x_i \neq x_j$ of points on $X = \mu^{-1}(x)$, we have the torus $S$ in $Z = A^{ij}_x$ with $\eta_S$ birational by Proposition 6.9. This implies $A^{ij}_{x_i} \neq A^{ij}_{x_j}$ by Proposition 5.10. Applying Proposition 4.11, we see that $x_i \neq x_j$ do not belong to the same $H_{ij}$-orbit in $X$. Since $H_i, H_j \subset H_{ij}$ by Proposition 4.14, $x_i$ and $x_j$ do not belong to the same $\langle x_i, x_j \rangle$-orbit. This is the condition (2) of Definition 2.3. Thus the triple $(X, G, H_1)$ is a special triple.

By Theorem 2.4, we see that $d = 1$, i.e., $\mu$ is birational. By taking a general ample hypersurface $D' \subset K$ and letting $D = \mu(\rho^{-1}(D'))$, we see that $A$ is disjoint from $D$.  

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