Estimation and inference for high-dimensional nonparametric additive instrumental-variables regression

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Abstract

The method of instrumental variables provides a fundamental and practical tool for causal inference in many empirical studies where unmeasured confounding between the treatments and the outcome is present. Modern data such as the genetical genomics data from these studies are often high-dimensional. The high-dimensional linear instrumental-variables regression has been considered in the literature due to its simplicity albeit a true nonlinear relationship may exist. We propose a more data-driven approach by considering the nonparametric additive models between the instruments and the treatments while keeping a linear model between the treatments and the outcome so that the coefficients therein can directly bear causal interpretation. We provide a two-stage framework for estimation and inference under this more general setup. The group lasso regularization is first employed to select optimal instruments from the high-dimensional additive models, and the outcome variable is then regressed on the fitted values from the additive models to identify and estimate important treatment effects. We provide non-asymptotic analysis of the estimation error of the proposed estimator. A debiasing procedure is further employed to yield valid inference. Extensive numerical experiments show that our method can rival or outperform existing approaches in the literature. We finally analyze the mouse obesity data and discuss new findings from our method.

Keywords: Causal inference; Group lasso; High-dimensional inference; Instrumental variables; Nonparametric additive models.

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1. Introduction

The method of instrumental variables has been extensively used in observational studies to control for unmeasured confounding. If measurements of the treatments and the outcome are confounded by unobserved variables, the casual effects of the endogenous treatments can be identified if instrumental variables are available. The instrumental variables need to be independent of the unmeasured confounders and can only affect the outcome indirectly through the treatment variables. The method originates from early research on structural equations in econometrics (Wright, 1928; Anderson and Rubin, 1949), and has become increasingly popular in biostatistics and epidemiology with Mendelian randomization as one of the most exciting recent applications (Davey Smith and Ebrahim, 2003; Didelez and Sheehan, 2007; Lin et al., 2015; Burgess et al., 2017). The general setting of this method involves modeling the instrument-treatment and treatment-outcome relationships. The classical two-stage least squares approach assumes linearity of both relationships and is widely used in the low-dimensional setting. However, in many concurrent studies, data are often high-dimensional. For example, gene expression data collected to identify genetic variants associated with complex traits in genome-wide association studies are usually high-dimensional. Many factors such as unmeasured environmental conditions may induce spurious associations and distort the true relationships between the gene expressions and the outcome of interest. Nevertheless, the random assortment of the genes transferred from parents to offspring resembles the use of randomization in experiments, and single nucleotide polymorphisms (SNPs) therefore serve as natural instrumental variables. The SNPs are also high-dimensional.

Recent developments of the instrumental-variables regression have introduced regularization as a means to address the high dimensionality issue (Gautier and Rose, 2011; Belloni et al., 2012; Fan and Liao, 2014; Cheng and Liao, 2015; Belloni et al., 2022). For example, Belloni et al. (2012) use the lasso to select optimal instruments from a large pool when the number of treatments remains fixed or low-dimensional. Various procedures using different types of regularization have been proposed thereafter. See Hansen and Kozbur (2014) and Fan and Zhong (2018), among others. Linear methods in which the instruments and treatments are both high-dimensional have also been considered (Lin et al., 2015; Zhu, 2018; Gold et al., 2020). Lin et al. (2015) demonstrate an application of the high-dimensional linear instrumental-variables regression to genetic genomics. However, nonlinear effects of the SNPs on the gene expressions are likely to exist as can be seen from some recent articles that employ different kernel-based procedures to capture possible nonlinear relationships (Wang et al., 2015; Zhang and Ghosh, 2017; Zhan et al., 2017; Zhao et al., 2019). While these methods keep fully nonparametric forms in linking the gene expressions and SNPs, they are not very effective when applied to the high-dimensional regime. Zhu (2018) also considers the high-dimensional linear instrumental-variables regression for peer effect estimation in econometrics. Specifically, to analyze the effects of peers’ output on a firm’s production output using panel data, the Research and Development expenditures of peer firms from a previous period are treated as potential instrumental variables for the endogenous treatments. Nevertheless, when the linear relationships are in question, which likely are, the approach by Zhu (2018) may lead to unignorable bias.
Specification of the outcome equation, either in a parametric or nonparametric form, is often based on expert knowledge or domain theory. The treatment model, however, can be more data-driven and should involve nonlinear relationships when possible to reduce bias (Newey, 1990; Fan and Zhong, 2018). To better approximate the treatments using optimal instruments, a general nonparametric model can be beneficial. A substantial body of the recent literature on high-dimensional nonparametric estimation focuses on the additive models (see, e.g., Huang et al., 2010, and references therein). To this end, we consider the high-dimensional additive models to capture the nonlinear effects of the large number of instrumental variables on the treatments. We keep the linearity assumption for the outcome model so that its coefficients directly bear causal interpretations. We allow the dimensions of both the instrumental variables and the treatments to be larger than the sample size. Similar to the regularized two-stage framework for the high-dimensional linear instrumental-variables regression, our proposed procedure consists of a first stage in which we use the group lasso to select important instruments to best predict the treatments, and a second stage in which we employ lasso to regress the outcome on the first-stage predictions to perform variable selection and estimation. We provide rigorous non-asymptotic analysis of the estimator and further employ a debiasing procedure to establish valid inference.

In contrast to existing methods in the literature, the present work has the following favorable features and makes several contributions to the high-dimensional instrumental-variables regression. Firstly, the proposed procedure is more data-adaptive which allows possible nonlinear instrument-treatment relationships under high dimensions. A few recent articles from the machine learning literature adopt deep learning to better estimate the instrumental-treatment relationships (Hartford et al., 2017; Xu et al., 2020). However, these methods typically require the dimensions of the instruments and the treatments both be smaller than the sample size, and are not directly applicable to the setting considered in the present article. Secondly, for the high-dimensional additive models in the first stage, we develop a probabilistic bound for the estimation error of the group lasso estimator. Compared with existing work in this area (e.g., Huang et al., 2010), we explicitly derive the non-asymptotic probabilistic bounds of the estimation errors, which may be of independent interest. Similar probabilistic bounds for the estimation error of the second stage are also provided. Lastly, we provide statistical inference for the causal parameters of interest by leveraging the debiasing procedures under high dimensionality. It is recognized that inference for high-dimensional models is typically difficult even when endogeneity is not present (Javanmard and Montanari, 2014; Zhang and Zhang, 2014; van de Geer et al., 2014). Gold et al. (2020) consider inference in the high-dimensional linear instrumental-variables regression to deal with endogeneity. The present work goes beyond that by establishing valid inference in the more flexible additive models. Hence, our work enriches the literature on high-dimensional inference that explicitly handles endogeneity.

2. The sparse additive instrumental-variables model

Suppose we have \( n \) independent and identically distributed observations from a population of interest. Let \( y_i, x_i, \) and \( z_i \) denote the \( i \)th observations of the outcome, the \( p \times 1 \) vector of treatment variables, and the \( q \times 1 \) vector of instrumental variables, respectively, where \( i = 1, \ldots, n \). Without loss of generality, assume the \( x_i \)'s and \( y_i \)'s are centered. Consider the
following joint modeling framework:

$$y_i = x_i^T \beta + \eta_i, \quad x_{i\ell} = \sum_{j=1}^{q} f_{j\ell}(z_{ij}) + \varepsilon_{i\ell} \quad (i = 1, \ldots, n; \ell = 1, \ldots, p),$$

where $\eta_i \sim N(0, \sigma^2_\eta)$ and $\varepsilon_{i\ell} \sim N(0, \sigma^2_{\varepsilon})$. Assume the treatment variables are endogeneous in the sense that $E(\eta_i \mid x_{i\ell}) \neq 0$, and the instrumental variables satisfy $E(\eta_i \mid z_{ij}) = E(\varepsilon_{i\ell} \mid z_{ij}) = 0$. The $f_{j\ell}(\cdot)$'s are unknown smooth functions with compact support $[a, b]$, where $a < b$. To ensure identifiability, assume $E\{f_{j\ell}(z_{ij})\} = 0$ for each $i, j$ and $\ell$. This is commonly assumed in the literature on additive models. We also impose some smoothness conditions on the $f_{j\ell}(\cdot)$'s and set the function class of consideration to a Hölder space $\mathcal{F}$.

**Assumption 1** For $j = 1, \ldots, q$ and $\ell = 1, \ldots, p$, the function $f_{j\ell}$ belongs to $\mathcal{F}$, where

$$\mathcal{F} = \left\{ f : |f^{(k_0)}(z') - f^{(k_0)}(z)| \leq C|z' - z|^{\alpha_0}, \quad z, z' \in [a, b]; \quad \sup_{z \in [a, b]} |f(z)| \leq C_0 \right\}$$

with $d = k_0 + \alpha_0 > 1.5$ and a universal constant $C_0 > 0$.

This assumption is common in nonparametric regression (see, e.g., Fan et al., 2015; Stone, 1985; Huang et al., 2010). Other similar assumptions such as existence of high-order continuous derivatives are also widely adopted (Assumption A3, Horowitz and Mammen, 2004).

In model (1), we are mainly interested in estimating the average treatment effects, $\beta$. The linear setting has been investigated by Lin et al. (2015) and Zhu (2018), where $\sum_{j=1}^{q} f_{j\ell}(z_{ij}) = z_i^T \gamma_\ell$. Here, we relax the linearity assumption and embrace the more general nonparametric additive form. Define $J_\ell = \{ j : f_{j\ell} \neq 0 \}$ for $\ell = 1, \ldots, p$, and $\mathcal{L} = \{ \ell : \beta_\ell \neq 0 \}$.

The sparsity assumption for the high-dimensional additive model entails that $|J_\ell| \leq r$ for all $\ell$ and some positive integer $r$, where $|J_\ell|$ denotes the cardinality of the set $J_\ell$. Similarly, we assume $s$-sparsity in the second stage with $|\mathcal{L}| \leq s$, where $s$ is a positive integer. To rewrite model (1) in matrix form, let $Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, $X = (x_1, \ldots, x_n)^T \in \mathbb{R}^{n \times p}$, $\eta = (\eta_1, \ldots, \eta_n)^T \in \mathbb{R}^n$, $F_j = (F_{j1}, \ldots, F_{jp}) \in \mathbb{R}^{n \times p}$ with $F_{j\ell} = \{ f_{j\ell}(z_{ij}), \ldots, f_{j\ell}(z_{in}) \}^T \in \mathbb{R}^n$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathbb{R}^{n \times p}$ with $\varepsilon_\ell = (\varepsilon_{1\ell}, \ldots, \varepsilon_{n\ell})^T \in \mathbb{R}^n$ for $j = 1, \ldots, q$ and $\ell = 1, \ldots, p$. Then model (1) can be rewritten as

$$Y = X\beta + \eta, \quad X = F + \varepsilon,$$

where $F = \sum_{j=1}^{q} F_j$. To handle high dimensionality and endogeneity, we allow $p, q \gg n$ and develop a two-stage penalized estimation framework.

In the first stage, we estimate each univariate function $f_{j\ell}$ via the B-spline approximation. Let $a = \xi_0 < \xi_1 < \cdots < \xi_K < \xi_{K+1} = b$ be an equal-distanced partition of $[a, b]$, where $K = [n^\nu]$ is a positive integer for some $0 < \nu < 0.5$. Let $I_{Kt} = [\xi_t, \xi_{t+1})$ for $t = 0, \ldots, K - 1$ and $I_{KK} = [\xi_K, \xi_{K+1}]$. Let $\{ \phi_k(\cdot) \}_{k=1}^{m}$ be the normalized B-splines such that each of them is (i) a polynomial function of degree $L$ on $I_{Kt}$ for $t = 0, \ldots, K$, and (ii) up to $(L - 1)$ times continuously differentiable on $[a, b]$, where $L > 1$ is an integer and $m = K + L$. A well-known property of such normalized basis functions is that $0 \leq \phi_k(z) \leq 1$ and $\sum_{k=1}^{m} \phi_k(z) = 1$ for any $z \in [a, b]$ (Schumaker, 2007, Theorem 4.20). Given the $z_{ij}$'s,
let \( \psi_{kj}(\cdot) = \phi_k(\cdot) - n^{-1} \sum_{i=1}^{n} \phi_k(z_{ij}) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, q \). We will denote \( \psi_k(\cdot) = \psi_{kj}(\cdot) \) when no confusion arises. Now approximate the additive functions using \( \{\psi_k(\cdot)\}_{k=1}^{m} \):

\[
x_{i\ell} \approx \sum_{j=1}^{q} \sum_{k=1}^{m} \gamma_{k\ell j} \psi_k(z_{ij}) + \varepsilon_{i\ell} \quad (i = 1, \ldots, n; \ell = 1, \ldots, p).
\]

(2)

Let \( U = (U_1, \ldots, U_q) \in \mathbb{R}^{n \times qm} \), where for each \( i \) and \( j \), \( U_j = (U_{1j}, \ldots, U_{nj})^T \in \mathbb{R}^{n \times m} \) and \( U_{ij} = \{\psi_1(z_{ij}), \ldots, \psi_m(z_{ij})\}^T \). Further define the parameter matrix \( \Gamma = (\gamma_1, \ldots, \gamma_p) \in \mathbb{R}^{qm \times p} \), where for each \( j \) and \( \ell \), \( \gamma_{\ell j} = (\gamma_{1\ell j}, \ldots, \gamma_{q\ell j})^T \in \mathbb{R}^{qm} \) and \( \gamma_{\ell j} = (\gamma_{1\ell j}, \ldots, \gamma_{m\ell j})^T \in \mathbb{R}^m \). The approximation in (2) becomes \( X \approx U \Gamma + \varepsilon \).

**Lemma 1** For each \( f_{j\ell} \in \mathcal{F} \), there exists \( \hat{r}_{j\ell} = (\hat{r}_{1j\ell}, \ldots, \hat{r}_{m\ell j})^T \) such that with probability at least \( 1 - 2(pqm)^{-2} \), the following holds

\[
\sup_{z \in [a,b]} |f_{j\ell}(z) - \hat{f}_{n j\ell}(z)| \leq 2C_L m^{-d} + 2C_0 \{\log(pqm)/n\}^{1/2},
\]

where \( \hat{f}_{n j\ell}(z) = \sum_{k=1}^{m} \hat{r}_{k j\ell} \phi_k(z) \) and \( C_L \) is a universal constant depending only on \( L \).

Lemma 1 characterizes the approximation error of the centered B-splines \( \{\phi_k(\cdot)\}_{k=1}^{m} \) to each \( f_{j\ell}(\cdot) \) with corresponding coefficients \( \hat{r}_{j\ell} \). Define \( \hat{r}_\ell = (\hat{r}_{1\ell}, \ldots, \hat{r}_{q\ell})^T \in \mathbb{R}^{qm} \). Lemma 1 implies that an intermediate step of recovering \( f_{j\ell} \) is to estimate \( \hat{r}_\ell \) by considering the following penalized problem:

\[
\hat{r}_\ell = \arg \min_{\gamma_\ell \in \mathbb{R}^{qm}} \left\{ \frac{1}{2n} \|X_\ell - U \gamma_\ell\|_2^2 + \lambda_\ell \sum_{j=1}^{q} \|\gamma_{j\ell}\|_2 \right\} \quad (\ell = 1, \ldots, p),
\]

(3)

where \( X_\ell \) is the \( \ell \)th column of \( X \) and \( \lambda_\ell \geq 0 \) is a tuning parameter. This is a group lasso problem (Yuan and Lin, 2006) and is motivated by the fact that when \( f_{j\ell} = 0 \), the vector \( \hat{r}_{j\ell} = 0 \). After obtaining the predicted treatments \( \hat{X} = (\hat{X}_1, \ldots, \hat{X}_p) \) with \( \hat{X}_\ell = U \hat{r}_\ell \), we plug \( \hat{X} \) into the following lasso problem to estimate \( \beta \) in the second stage:

\[
\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|Y - \hat{X} \beta\|_2^2 + \mu \|\beta\|_1 \right\}
\]

(4)

for some tuning parameter \( \mu \geq 0 \). Estimation with high-dimensional predictors has been a popular research topic in the past two decades. We note that the above formulation is slightly different from the original lasso problem due to the observed data being replaced by their estimations from the first stage. This turns out to be more involved when showing the estimation consistency.

3. Non-asymptotic analysis

We provide an estimation error bound for the first-stage group lasso problem. Compared with existing results in this area (Huang et al., 2010; Ravikumar et al., 2009), we make contributions by explicitly deriving the non-asymptotic probability bound. Based on this bound, we establish a similar error bound for the parameter of interest in the second-stage lasso problem. Define \( \Sigma_\ell = E(U^T U/n) \). We make the following assumptions.
Assumption 2 Each instrumental variable \( z_{ij} \) has a continuous density on \([a, b]\) and the density is bounded away from zero and infinity.

Assumption 3 There exists a universal constant \( \rho \in (0, 1) \) such that

\[
\min \left\{ \frac{\gamma^T \Sigma U \gamma}{\|\gamma\|_2^2} : |J| \leq r, \gamma \in \mathbb{R}^{pqm} \setminus \{0\}, \sum_{j \in J} \|\gamma_j\|_2 \leq 3 \sum_{j \in J} \|\gamma_j\|_2 \right\} \geq \frac{\rho}{m},
\]

where \( J \subseteq \{1, \ldots, q\} \) is an index set, \( J^c \) denotes its complement, and \( \gamma_J = (\gamma_{j\ell} : j \in J)^T \).

Assumption 2 is rather standard in the high-dimensional additive models (Huang et al., 2010; Fan and Zhong, 2018). Assumption 3 is often called the group restricted eigenvalue condition (Lounici et al., 2011; Lv et al., 2018; Lu et al., 2020). This is a natural extension of the restricted eigenvalue condition for the standard lasso and Dantzig selector problems (Bickel et al., 2009). When the instrumental variables are independent, \( \Sigma_U \) is a block diagonal matrix with diagonals \( \Sigma_j = E(U_j^T U_j/n) \). It is well known that \( \lambda_{\min}(\Sigma_j) \geq c_s/m \) (Lian, 2012; Huang et al., 2010) when each instrumental variable is uniformly distributed, where \( c_s > 0 \) is a constant depending on the smoothness of the B-splines. Denote \( a_n = o(b_n) \) if \( \lim_{n \to \infty} a_n/b_n = 0 \), \( a_n = O(b_n) \) if there exists a positive constant \( C_1 \) such that \( \lim \sup_{n \to \infty} a_n/b_n \leq C_1 \), and \( a_n = \Theta(b_n) \) if there exists positive constants \( C_2 \) and \( C_3 \) such that \( C_2 \leq \lim \inf_{n \to \infty} a_n/b_n \leq \lim \sup_{n \to \infty} a_n/b_n \leq C_3 \).

Theorem 2 Suppose Assumptions 1–3 hold. There exist positive constants \( c_1, c_2, \) and \( c_3 \) such that if

\[
\lambda_{\max} = \max_{\ell} \lambda_\ell = \max\left[ c_1 \sigma_{\max} \left( \frac{\log(pqm)}{n} \right)^{1/2} c_2 rmn^{-(2d+1)/2} + c_3 \left( \frac{\log(pqm)}{mn} \right)^{1/2} \right],
\]

then for sufficiently large \( n \), with probability at least \( 1 - 20(pqm)^{-1} \), the regularized estimator \( \hat{\gamma}_\ell \) in (3) satisfies

\[
\max_{\ell} \left\| \sum_{j=1}^q F_{j\ell} - U_{\ell} \hat{\gamma}_\ell \right\|_2 \leq \frac{50rm\lambda_{\max}^2}{\rho}, \quad \max_{\ell} \sum_{j=1}^q \left\| \hat{\gamma}_{j\ell} - \hat{\gamma}_{j\ell} \right\|_2 \leq \frac{32rm\lambda_{\max}}{\rho},
\]

where \( \sigma_{\max} = \max_{\ell} \sigma_\ell \), \( m = \Theta\{n^{1/(2d+1)}\} \), and \( r^2 = o\{n/m^4 \log(pqm)\} \).

The performance of the group lasso estimator depends crucially on the eigen behavior of the empirical covariance matrix \( U^T U/n \). While it can be shown that the group restricted eigenvalue condition for the empirical covariance matrix is satisfied under Assumption 3, this does come with a price on the rate of the sparsity level, that is, \( r^2 = o\{n/m^4 \log(pqm)\} \). Similar requirements can be found in Corollary 1 of Raskutti et al. (2010). In view of the conditions of Theorem 2, it is easy to verify that \( \lambda_{\max}^2 = O\{r^2 \log(pqm)/n\} \). Thus, to ensure the consistency of the average in-sample prediction, it is required that \( r^3 = o\{n^{2d/(2d+1)} / \log(pqm)\} \), while for the estimation consistency of the coefficients, \( r^4 = o\{n^{(2d-1)/(2d+1)} / \log(pqm)\} \) is required. This is a more restrictive requirement than that in the standard lasso, but it is expected due to the unspecified additive functional forms. In contrast to Theorem 1
of Huang et al. (2010) that only gives the convergence rates, our result is completely non-asymptotic. Moreover, our result allows the sparsity $r$ to grow with the sample size and dimension of the data while this is not allowed in Huang et al. (2010). Guaranteed by Theorem 1 part (i) of Huang et al. (2010), we can directly compare the estimation consistency result obtained here with part (ii) of their theorem, and when $r$ is a fixed number, it is easy to show they are the same. Other aligned results include Ravikumar et al. (2009) and Lu et al. (2020). Ravikumar et al. (2009) obtain the out-of-sample risk consistency while both the explicit rate and the in-sample error bound remain unclear. Lu et al. (2020) consider a kernel-sieve hybrid estimator and obtain a similar non-asymptotic bound.

To provide an estimation error bound for $\hat{\beta}$ defined by (4), we make an extra assumption on the population covariance matrix $\Sigma_F = E(F^TF/n)$.

**Assumption 4** There exists a constant $\kappa > 0$ such that

$$\min \left\{ \frac{\beta^T \Sigma_F \beta}{\|\beta_L\|_2^2} : |L| \leq s, \beta \in \mathbb{R}^p \setminus \{0\}, \sum_{\ell \in L^c} |\beta_\ell| \leq 3 \sum_{\ell \in L} |\beta_\ell| \right\} \geq \kappa,$$

where $L \subset \{1, \ldots, p\}$ is an index set, $L^c$ denotes its complement, and $\beta_L = (\beta_\ell : \ell \in L)^T$.

Assumption 4 is the restricted eigenvalue condition on $\Sigma_F$ and is useful for deriving the error bounds in the second-stage lasso problem. This assumption imposes some requirements on the covariance structures of the treatment matrix $X$ and the noise variables $\varepsilon$. For example, when $\text{cov}(\varepsilon_t, \varepsilon_{t'}) = 0$ for $\ell \neq \ell'$ and the minimum eigenvalue of $\Sigma_X = E(X^TX/n)$ is larger than $\max_{\ell} \xi_\ell$, the above condition immediately holds. To provide an estimation error bound for $\hat{\beta}$, we restrict the parameter space of consideration to an $L_1$-ball $\|\beta\|_1 \leq B$ for some $B > 0$. Similar technique has been frequently used in the literature (see, e.g., Lin et al., 2015). This restriction can be further relaxed to the $L_\infty$-ball $\|\beta\|_\infty \leq B$, but it may lead to a sacrifice of the convergence rate.

**Theorem 3** Suppose Assumptions 1–4 hold. Let the regularization parameter $\lambda_{\max}$ be chosen as in Theorem 2. Further assume $\lambda_{\max}$ satisfies $560C_0 \lambda_{\max}(2rm/\rho)^{1/2} \leq \kappa^2/(4rs)$. If we choose the second-stage regularization parameter as

$$\mu = 2r \lambda_{\max}(7 \sigma_0 + 8\sqrt{5}B \sigma_{\max} + 30B)(2m/\rho)^{1/2},$$

then with probability at least $1 - 234(pqm)^{-1}$, the estimator $\hat{\beta}$ in (4) satisfies

$$\|\hat{\beta} - \beta\|_1 \leq \frac{64}{\kappa^2} s \mu, \quad \|X(\hat{\beta} - \beta)\|_2^2 \leq \frac{64}{\kappa^2} ns \mu^2.$$

As far as we know, Theorem 3 is the first to present a non-asymptotic estimation error bound for the two-stage additive model. Straightforward analysis shows that consistency is guaranteed if we take $\mu^2 = O\{r^4 \log(pqm)/n^{d/(2d+1)}\}$ and $s^2 r^5 = o\{n^{d/(2d+1)}/\log(pqm)\}$. When $r$ is fixed, we have $s^2 = o\{n/[m \log(pqm)]\}$. This almost recovers the sparsity in the classical lasso setting when $d$ is large enough. Since the two-stage linear model considered by Lin et al. (2015) and Zhu (2018) is a special case of our setting, the empirical results in Section 5 demonstrate similar performance between the two models when the true relationship in both stages is linear.
4. Inference

We develop a method to draw inference on each component of the outcome regression parameters $\beta$. When endogeneity is absent, various methods via debiasing the penalized estimator have been proposed to conduct valid inference for high-dimensional models (Zhang and Zhang, 2014; van de Geer et al., 2014; Javanmard and Montanari, 2014). A recent article by Gold et al. (2020) considers the endogeneity issue and adapts the parametric one-step update procedure to construct confidence intervals for parameters in the high-dimensional two-stage linear model. We extend the approach therein to draw inference under the more general setting entailed by model (1).

A key step in deriving the debiased estimator is to utilize the conditional moment restriction $E(\eta \mid Z) = 0$. This equation entails the orthogonality condition $E(\Gamma^T U^T \eta) = 0$. Here $\Gamma$ is any fixed coefficient matrix and will later be set to $d_i$ and $d_i$ be the $i$th row of $D$. By Lemma 1 and Theorem 2, the estimate $\hat{D} = U\hat{T}$ is a good estimate of $D$, where $\hat{T} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_p)$. The orthogonality condition then implies that the empirical counterpart based on the estimate $\hat{D}$ is approximately equal to zero, that is,

$$E_n \{ h(y_i, x_i, \hat{d}_i; \beta) \} := -\hat{D}^T (Y - X\beta)/n \approx 0,$$

where $E_n(w_i) = n^{-1} \sum_{i=1}^n w_i$ is the expectation with respect to the empirical measure. The one-step update to the second-stage estimator $\hat{\beta}$ can thus be written as

$$\tilde{\beta} = \hat{\beta} - \hat{\Omega} E_n \{ h(y_i, x_i, \hat{d}_i; \hat{\beta}) \} = \hat{\beta} + \hat{\Omega} \hat{D}^T (Y - X\hat{\beta})/n,$$

where $\hat{\Omega}$ is some estimate of $\Omega = \Sigma_p^{-1}$. Similar to Gold et al. (2020), we construct the estimator $\hat{\Omega}$ from a modification of the constrained $L_1$-minimization approach to sparse precision matrix estimation proposed by Cai et al. (2011). The rows $\hat{\theta}_\ell$ of the estimator $\hat{\Omega}$ are obtained as solutions to the following program:

$$\min_{\theta_{\ell} \in \mathbb{R}^p} \|\theta_{\ell}\|_1, \text{ subject to } \|\hat{\Sigma}_F \theta_{\ell} - e_{\ell}\|_\infty \leq v \quad (\ell = 1, \ldots, p), \quad (5)$$

where $e_{\ell}$ is the $\ell$th canonical basis vector in $\mathbb{R}^p$ and $v > 0$ is the tolerance parameter. The following lemma characterizes a decomposition of the one-step estimator $\tilde{\beta}$.

**Lemma 4** The one-step estimator $\tilde{\beta} = \hat{\beta} + \hat{\Omega} \hat{D}^T (Y - X\hat{\beta})/n$ satisfies $\sqrt{n}(\tilde{\beta} - \beta) = \Omega D^T \eta/\sqrt{n} + \sum_{k=1}^4 R_k$, where

$$R_1 = (\hat{\Omega} - \Omega) D^T \eta/\sqrt{n}, \quad R_2 = \Omega (\hat{D} - D)^T \eta/\sqrt{n},$$

$$R_3 = \hat{\Omega} \hat{D}^T (X - \hat{D})(\beta - \hat{\beta})/\sqrt{n}, \quad R_4 = \sqrt{n}(\hat{\Omega} \hat{\Sigma}_F - I)(\beta - \hat{\beta}).$$

Lemma 4 implies that to establish the asymptotic normality of each component $\tilde{\beta}_\ell$, it suffices to make sure each remainder term $\|R_k\|_\infty = o_p(1)$, $k = 1, 2, 3, 4$. The $L_1$-bound on $\hat{\theta}_\ell - \theta_\ell$, which is needed for controlling the remainder terms, becomes manageable when the following restriction on the population precision matrix is imposed.
Assumption 5 There exist some positive number $m_\Omega$, tolerance $b \in [0, 1)$, and generalized sparsity level $s_\Omega$ such that the population precision matrix $\Omega \in U(m_\Omega, b, s_\Omega)$, where

$$U(m_\Omega, b, s_\Omega) = \left\{ \Omega = (\theta_{\ell \ell'})_{\ell, \ell' = 1}^p \geq 0 : \|\Omega\|_1 \leq m_\Omega; \max_{\ell \in \{1, \ldots, p\}} \sum_{\ell' = 1}^p |\theta_{\ell \ell'}|^b \leq s_\Omega \right\}$$

and $\|\Omega\|_1 = \sup_\ell \|\theta_\ell\|_1$.

We assume the event that the rows $\theta_\ell$ of $\Omega$ are feasible for the minimization program (5) has probability approaching one, that is, $P(\|\Omega \hat{\Sigma}_F - I\|_\infty \leq \nu) \to 1$ as $n \to \infty$. The validity of such a requirement mainly depends on the choice of the tolerance $\nu$. We give a theoretical choice of $\nu$ in Theorem 5 and provide some rate conditions under which $\sqrt{n}(\hat{\beta}_\ell - \beta_\ell)/\omega_\ell$ is asymptotically normal, where $\omega^2_\ell = \sigma^2_0 \theta_\ell$ and $\theta_\ell$ is the $\ell$th diagonal entry of $\Omega$.

Theorem 5 Suppose Assumptions 1–5 and the conditions of Theorems 2–3 hold. Assume each element $\theta_{\ell \ell'} > \vartheta$ for some universal constant $\vartheta > 0$ and let $\nu = 36C_0 m_\Omega \lambda_{\max} r(2rm/p)^{1/2}$. If the following rate conditions hold

$$r^{(7-5b)/2} \left\{ \frac{\log(pqm)}{n} \right\}^{(1-b)/2} \left[ m^{1/2} + \left\{ \log(pqm) \right\} \right]^{1/2} = o(1),$$

$$r^2 (m^3/n)^{1/2} \log(pqm) = o(1), \quad r^{7/2} s(m^2/n)^{1/2} \log(pqm) = o(1),$$

then $\|R_k\|_\infty = o_p(1)$ as $n \to \infty$, where $k = 1, 2, 3, 4$. Moreover, $\sqrt{n}(\hat{\beta}_\ell - \beta_\ell)/\omega_\ell$ converges in distribution to the standard normal distribution.

The rates in Theorem 5 are required to make the remainder terms negligible. When the sparsity parameter $r$ in the first stage is fixed, the requirement for the sparsity level of the second stage is $s = o(n^{1/2}m/\log(pqm))$. This is almost the same as the requirement for the debiased lasso: $s = o(n^{1/2}/\log(p))$ (see, e.g., Javanmard and Montanari, 2014; van de Geer et al., 2014). By Theorem 5, we can construct a confidence interval for $\beta_\ell$ if a consistent estimator $\hat{\omega}_\ell$ is available. Given $\ell \in \{1, \ldots, p\}$ and $\alpha \in (0, 1)$, an asymptotic $100(1 - \alpha)$% confidence interval for $\beta_\ell$ is $[\hat{\beta}_\ell - z_\alpha \hat{\omega}_\ell/\sqrt{m}, \hat{\beta}_\ell + z_\alpha \hat{\omega}_\ell/\sqrt{m}]$, where $z_\alpha = \Phi^{-1}(1 - \alpha/2)$ and $\Phi$ is the cumulative distribution function of the standard normal distribution. The following theorem provides a way to construct a consistent estimator $\hat{\omega}_\ell$.

Theorem 6 Suppose the conditions of Theorems 2–5 hold. Define

$$\hat{\omega}_\ell = \tilde{\sigma}_0 (\tilde{\beta}_\ell \tilde{\Gamma}^U U^T \tilde{U} \tilde{\Gamma} \theta_\ell/n)^{1/2}, \quad \tilde{\sigma}_0 = n^{-1/2}\|Y - X\hat{\beta}\|_2.$$

Then $\hat{\omega}_\ell$ is a consistent estimator of $\omega_\ell$ for each $\ell \in \{1, \ldots, p\}$.

5. Simulation

We conduct simulation studies to evaluate the finite-sample performance of the proposed methods. Our objective is to test both the estimation and inferential procedures under a variety of experiments. For estimation purpose, we compare our procedure with the classical
We finally generate the outcome response according to $y = \beta_0 + \beta \gamma + \eta$, where $\gamma$ is the sampling strategy for the coefficients $\gamma$ vector ($\beta$ vector stage lasso estimator (Up-2SLS-L) proposed by Gold et al. (2020).

For inferential purpose, we compare our method with the updated two-stage penalized least squares (PLS) and the two-stage least-square method with lasso settings, we take $\gamma$ are generated from linear and nonlinear models, respectively. In both settings, we take $p = q = 600$ and vary $n$ from 100 to 2100. Experiments for other values of $(p, q, n)$ are provided in the Appendix. We generate the instrumental variables $z_i$ of the $i$th observation from the multivariate normal distribution with zero mean and covariance matrix $\Sigma_Z = \{(\Sigma_Z)_{jj'}\}_j$, where $(\Sigma_Z)_{jj'} = 0.2|j-j'|$ for $j, j' = 1, \ldots, q$. To generate the noise vector $(\eta_i, \varepsilon_{i1}, \ldots, \varepsilon_{ip})^T$, we sample from another normal distribution with zero mean and covariance matrix $\Sigma = (\Sigma_{\ell\ell'})$, where $\Sigma_{\ell\ell'} = 0.2|\ell-\ell'|$ for $\ell, \ell' = 2, \ldots, p + 1$ and $\Sigma_{11} = 1$. We also set $\Sigma_{1\ell}$ for $\ell = 2, \ldots, 6$ and five other random selected entries from the first column of $\Sigma$ to 0.3. All other entries are set to zero. We finally set $\Sigma_{1\ell} = \Sigma_{\ell1}$ for $\ell = 2, \ldots, p + 1$ to make $\Sigma$ symmetric. Note that the nonzero $\Sigma_{1\ell}$’s induce endogeneity in the data. In the linear setting, we generate the treatment variables $x_i$ according to $x_i = \Gamma^T z_i + \varepsilon_i$, where $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{q \times p}$ is a sparse coefficient matrix obtained by sampling $r = 5$ nonzero entries of each column from the uniform distribution $U(0.75, 1)$. In the nonlinear setting, we generate $x_i$ from the following equation:

$$x_{i\ell} = \gamma_{1\ell} z_{i1}^2 + \gamma_{2\ell} z_{i2} + \gamma_{3\ell} z_{i3}^2 + \gamma_{4\ell} \sin(\pi z_{i4}) + \gamma_{5\ell} z_{i5}^2 + \varepsilon_{i\ell}, \quad (\ell = 1, \ldots, p).$$

The sampling strategy for the coefficients $\gamma_{ij}$’s are the same as that in the linear setting. We finally generate the outcome response according to $y_i = x_i^T \beta + \eta_i$, where the coefficient vector $\beta$ is generated by sampling $s = 5$ nonzero components from the uniform distribution over two disjoint intervals $U\{-1, -0.75\} \cup \{0.75, 1\}$.

In all simulations, we use the Bayesian information criterion to select the first-stage tuning parameters $m$ and $\lambda_\ell$, and the five-fold cross validation to select the second-stage regularization parameter $\mu$. We report the $L_1$ error $\|\hat{\beta} - \beta\|_1$ of each method based on one hundred replications. The results are summarized in Table 1. It can be seen that when the instrument-treatment relationship is linear, our method is as good as 2SLS-L. As the sample size increases, the $L_1$ errors of both 2SLS-L and our method decrease, whereas that of PLS increases due to ignorance of endogeneity. When the instrument-treatment relationship is nonlinear, the performance of our method is similar to that in the linear setting. The $L_1$ error of our method is the smallest in almost all settings and exhibits a decreasing trend as the sample size increases. In contrast, neither PLS nor 2SLS-L shares such trend in their performance under these settings. We can clearly see that the 2SLS-L method induces a large bias since it suffers losses from model misspecification. It is also implied that model misspecification may have a heavier impact on 2SLS-L than on PLS. This may be related to the specific setting of our simulations. Overall, the results from Table 1 demonstrate the estimation consistency of our estimator.

Next, we evaluate the performance of the inferential procedure proposed in Section 4 and consider a more challenging nonlinear setting:

$$x_{i\ell} = -8\gamma_{1\ell} z_{i1}^2 + 2\gamma_{2\ell} \sin(\pi z_{i2}) + 2\gamma_{3\ell} \log(z_{i3}^2) + \gamma_{4\ell}(10z_{i4})^3 + \gamma_{5\ell} z_{i5}^2 + \varepsilon_{i\ell}, \quad (\ell = 1, \ldots, p).$$

We calculate the 95% confidence interval of each element of $\beta$ based on our method and the Up-2SLS-L based inferential procedure. The coverage probabilities and interval lengths
Table 1: $L_1$ estimation loss of each method averaged over 100 replications with standard deviation shown in parentheses for $p = 600$.

| Sample | Linear | Nonlinear |
|--------|--------|-----------|
|        | Our method | 2SLS-L | PLS | Our method | 2SLS-L | PLS |
| 100    | 1.26 (0.53) | 2.52 (1.19) | 0.86 (0.22) | 2.96 (1.41) | 1.19 (0.38) | 1.25 (0.38) |
| 300    | 0.50 (0.23) | 0.59 (0.30) | 0.51 (0.16) | 0.74 (0.28) | 1.27 (0.79) | 0.79 (0.27) |
| 600    | 0.34 (0.15) | 0.27 (0.11) | 0.46 (0.17) | 0.43 (0.17) | 1.73 (1.26) | 0.89 (0.27) |
| 900    | 0.25 (0.10) | 0.23 (0.08) | 0.52 (0.14) | 0.32 (0.13) | 2.90 (1.98) | 1.10 (0.23) |
| 1200   | 0.19 (0.09) | 0.19 (0.08) | 0.61 (0.18) | 0.27 (0.13) | 3.35 (2.36) | 1.18 (0.18) |
| 1500   | 0.17 (0.08) | 0.18 (0.08) | 0.70 (0.25) | 0.24 (0.10) | 4.17 (3.29) | 1.27 (0.16) |
| 1800   | 0.16 (0.07) | 0.17 (0.08) | 0.81 (0.29) | 0.21 (0.09) | 4.74 (4.03) | 1.34 (0.17) |
| 2100   | 0.14 (0.05) | 0.16 (0.07) | 1.09 (0.42) | 0.21 (0.11) | 5.61 (4.38) | 1.43 (0.16) |

averaged over all elements are shown in Table 2. Our confidence intervals have coverage probabilities close to the nominal level of 0.95. In contrast, the Up-2SLS-L based confidence intervals have coverage probabilities well below the nominal level. Moreover, their intervals are much wider than ours under all settings. These are expected as the Up-2SLS-L based inferential procedure is proposed under the linear setting and may not perform well under the nonlinear setting. The results from Table 2 validate our theory in Section 4.

Table 2: Coverage probabilities and lengths of the 95% confidence intervals by our method and the Up-2SLS-L based inferential procedure. Numbers shown are multiplied by one hundred.

| Dimension | Sample size | Our method | Up-2SLS-L |
|-----------|-------------|------------|-----------|
|            | Coverage    | Length     | Coverage  | Length     |
| 250        | 92.0        | 0.396      | 87.3      | 2.663      |
| 400        | 93.5        | 0.264      | 89.0      | 1.585      |
| 500        | 94.0        | 0.245      | 87.1      | 2.102      |
| 600        | 93.7        | 0.140      | 87.8      | 2.614      |

6. Real data analysis

We further illustrate our proposed method by analyzing the mouse obesity data described by Wang et al. (2006). This data set consists of genotype, gene expression and clinical information about the F2 intercross mice. The genotypes are characterized by the SNPs at an average density of 1.5 cM across the whole genome and the gene expressions in the liver tissues of the mice are profiled by microarrays. We are interested in the causal effect of the gene expressions on the body weights of the mice. We consider the data collected from $n = 287$ (144 female and 143 male) mice with $q = 1250$ SNPs and $p = 2816$
genes. Since there are only three genotypes, a sparse high-dimensional linear model between the SNPs and the gene expressions is often postulated. However, some recent articles suspect that nonlinear effects may exist in the data (Li et al., 2014; Zhang and Ghosh, 2017; Guha Majumdar et al., 2020). This has motivated us to consider the nonlinear setting.

Before applying our method to the data, we adjust for confounding induced by the sex of the mice. We first regress the body weight on the sex and subtract the estimated effect from the body weight. We then apply our proposed estimation method to the data with the adjusted body weight as the outcome. We use the five-fold cross validation to select the tuning parameters. The resultant model includes 28 genes. To increase the stability and interpretability of our analysis, we apply the stability selection approach proposed by Meinshausen and Bühlmann (2010) to compute the selection probability of each gene over one hundred subsamples of size \( \lfloor n/2 \rfloor \) for a sequence of values of the tuning parameter \( \mu \).

We set the threshold probability to 0.5 so that the number of genes selected is reasonable. The results are shown in Table 3.

| Gene Name         | Selection Probability | Gene Name         | Selection Probability |
|-------------------|-----------------------|-------------------|-----------------------|
| Vwf               | 0.77                  | Krtap19-2         | 0.59                  |
| Akap12            | 0.63                  | Tmem184c          | 0.74                  |
| 2010002N04Rik     | 0.84                  | Igfbp2           | 0.51                  |
| Slc43a1           | 0.76                  | Gstm2            | 0.91                  |
| Ccnl2             | 0.54                  | D14Abb1e         | 0.52                  |
| B4galnt4          | 0.71                  |                   |                       |

We observe that there are five genes: Igfbp2, Gstm2, Vwf, 2010002N04Rik, and Ccnl2 that are also selected in Lin et al. (2015). These overlapping genes are highly likely to be connected to obesity from a biological point of view. This indicates the effectiveness and stability of our method in finding the risky genes. Table 3 also shows our method identifies some other genes that were not previously found, possibly due to the nonparametric form we consider in the first stage. In fact, some genes selected with high probability by our method have been verified by many biological studies. In particular, Solute Carrier Family 43 Member 1 (Slc43a1) is a Protein Coding gene which can encode the amino acid transporters that are known to regulate the transmembrane transport of phenylalanine. Gill et al. (2010) found that the expression of Slc43a1 in the fat mice group is quite different from that in the lean mice group so Slc43a1 is a potential factor leading to obesity.

We also apply our inferential procedure to the data to quantify the uncertainty associated with our estimation. We use the R package Flare to obtain the optimal precision matrix from solving (5). Then we construct the confidence intervals based on the debiased estimator. Table 4 presents the causal effects of the genes on the body weight whose corresponding confidence intervals do not contain zero. Note that the confidence intervals are wide due to possibly low signal-to-noise ratio and small sample size of the data. The result is generally consistent with that obtained by the stable selection. The confidence intervals that are far away from zero include Vwf, 2010002N04Rik, Gstm2, Gp1d1, Slc43a1, Igfbp2, which are also shown to have high selection probability in both Table 3. Moreover, several other genes
from Table 4 that are shown to have significant causal effects on the body weight are newly found and have been confirmed to have close biological relation with obesity. For example, a recent study in Wang et al. (2019) shows that by silencing Anxa2, the obesity-induced insulin resistance is attenuated and our result confirms such a positive relation. Cyp4f15 genes are known to control the omega-hydroxylated fatty acids in the liver tissue and such acids can be used for energy production (Hardwick et al., 2009). Another independent study shows that the downregulation of Cyp4f15 happens in the liver tissue among the group of mice fed with high-fat diet (Gai et al., 2020). The negative causal effect obtained in our result coincides with these findings.

Table 4: 95% confidence intervals for the causal effects of the genes on the body weights of the mice. Shown are only the genes whose corresponding intervals do not contain zero.

| Gene Name     | Confidence Interval | Gene Name     | Confidence Interval |
|---------------|---------------------|---------------|---------------------|
| Anxa5         | (0.010, 7.269)      | Kif22         | (0.615, 7.930)      |
| Vwf           | (0.500, 7.841)      | Gstm2         | (0.537, 8.231)      |
| Aqp8          | (0.066, 6.855)      | Gpld1         | (−7.448, −0.447)    |
| Lamc1         | (0.094, 5.877)      | Slep43a1      | (−6.641, −1.412)    |
| Acot9         | (0.056, 8.298)      | Abca8a        | (−7.152, −0.072)    |
| Anxa2         | (1.086, 9.331)      | Cyp4f15       | (−7.468, −0.250)    |
| 2010002N04Rik | (1.343, 8.240)      | Igfbp2        | (−6.451, −0.666)    |
| Msr1          | (0.004, 6.783)      |               |                     |

7. Discussion

Motivated by the data-driven modeling spirit, we develop a high-dimensional additive instrumental-variables regression method with a sound non-asymptotic analysis and valid inference procedure when both instruments and treatments are allowed to be high-dimensional. There are a lot of directions that are worth further exploration. Firstly, while we estimate the nonparametric functions separately in the first-stage, it would be helpful to take into account the correlations among the treatments and borrow information across those regressions. Secondly, although it is convenient for causal interpretation when considering the linear outcome model, many applications in econometrics and biostatistics have considered nonparametric method to model the relationships between treatments and outcome. One may further relax our second-stage model setting by considering a high-dimensional single-index or nonparametric additive outcome model (Radchenko, 2015; Huang et al., 2010). Thirdly, it is of great interest to extend our results to handling other types of outcome data, for instance, binary or survival outcome. We plan to pursue these and other related issues in future research.

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Appendix

We first define the notation that will be used throughout the Appendix. Suppose \( A \in \mathbb{R}^{m \times n} \). Let \( \|A\|_2 \) be the largest singular value of \( A \) and \( \|A\|_\infty \) the largest absolute value of the entries of \( A \). Also, let \( \|A\|_1 = \max_{j \in [n]} \|A_j\|_1 \), where \( A_j \) is the \( j \)th column of \( A \) and \( [n] = \{1, \ldots, n\} \). Denote by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) the minimum and maximum eigenvalues of \( A \), respectively, when \( A \) is symmetric.

Appendix A: some useful Lemmas

In this appendix present some useful lemmas that will be used for proving the main results.

**Lemma 7** (Hoeffding Bound, Proposition 2.5 of Wainwright, 2019) Suppose random variables \( X_i, i = 1, \ldots, n \) are independent, and \( X_i \) has mean \( \mu_i \) and sub-Gaussian parameter \( \sigma_i \). Then for all \( t \geq 0 \), we have

\[
\Pr\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_i) \right| \geq t \right\} \leq 2 \exp\left( -\frac{n^2 t^2}{2 \sum_{i=1}^{n} \sigma_i^2} \right).
\]

Specifically, if \( X_i \) is bounded in \([a, b]\), we have

\[
\Pr\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_i) \right| \geq t \right\} \leq 2 \exp\left( -\frac{2 nt^2}{(b-a)^2} \right).
\]

Lemma 8 follows from Lemma 5 of Stone (1985), which is a well-known result on B-spline approximation and has been frequently used in the additive models.

**Lemma 8** For each \( f_{j\ell} \in \mathcal{F} \), there exists \( \tilde{f}_{nj\ell} = \sum_{k=1}^{m} \tilde{\gamma}_{kj\ell} \phi_k \) such that

\[
\sup_{z \in [a, b]} |f_{j\ell}(z) - \tilde{f}_{nj\ell}(z)| \leq C_L m^{-d},
\]

where \( C_L > 0 \) is a constant depending only on the degree \( L \) of the B-splines.

The next lemma shows the minimum and maximum eigenvalues of the spline matrix have the same order. Denote the non-centered B-spline matrix by \( \tilde{U} = (\tilde{U}_1, \ldots, \tilde{U}_q) \in \mathbb{R}^{n \times qm} \), where \( \tilde{U}_j = (\tilde{U}_{1j}, \ldots, \tilde{U}_{nj})^T \in \mathbb{R}^{n \times m} \) and \( \tilde{U}_{ij} = \{\phi_1(Z_{ij}), \ldots, \phi_m(Z_{ij})\}^T \in \mathbb{R}^m \).

**Lemma 9** (Lemma 6.2 in Shen et al., 1998) For \( j = 1, \ldots, q \), we have

\[
\frac{3c_*}{m} - 2\|P_j - P_j^n\|_\infty \leq \lambda_{\min}\left( \frac{\tilde{U}_j^T \tilde{U}_j}{n} \right) \leq \lambda_{\max}\left( \frac{\tilde{U}_j^T \tilde{U}_j}{n} \right) \leq \frac{c_*}{2m} + 2\|P_j - P_j^n\|_\infty,
\]

where \( 0 < c_* < 1 < c^* \) and \( \|P_j - P_j^n\|_\infty = \sup_z \{\Pr(Z_j \leq z) - n^{-1} \sum_{i=1}^{n} 1\{Z_{ij} \leq z\}\} \) with \( \Pr_j \) and \( \Pr_j^n \) being the population and empirical distributions of \( Z_j \), respectively.
The following result gives a bound on $\|P_j - P_n^j\|_\infty$.

**Lemma 10** (Glivenko–Cantelli Theorem, Corollary 4.15 of Wainwright, 2019) Let $P$ be the distribution of a random variable $X$ and $P_n$ the empirical distribution based on $n$ i.i.d. copies, $X_1, \ldots, X_n$, of $X$. Then, we have

$$\|P_n - P\|_\infty \leq 8\{(\log(n + 1)/n)^{1/2} + \delta\}$$

with probability at least $1 - \exp(-n\delta^2/2)$ for any $\delta \geq 0$.

Combining Lemmas 9 and 10, we obtain the following result on the B-spline matrix.

**Lemma 11** If $8\{(\log(n + 1)/n)^{1/2} + \{2/m^3\}^{1/2}\} \leq \min\{c^*/m, c^*/4m\}$, then for each $j = 1, \ldots, q$, with probability at least $1 - \exp(-nm^{-3})$, we have

$$\frac{c^*}{m} \leq \lambda_{\min}\left(\frac{\tilde{U}_j^T \tilde{U}_j}{n}\right) \leq \lambda_{\max}\left(\frac{\tilde{U}_j^T \tilde{U}_j}{n}\right) \leq \frac{c^*}{m}.$$  

**Proof** Apply the Glivenko–Cantelli Theorem (Lemma 10) with $\delta = \sqrt{2}/m^{3}$. \hfill \qed

**Lemma 12** (Bernstein’s Inequality, see Proposition 2.14 of Wainwright, 2019) Let $X_1, \ldots, X_n$ be i.i.d. random variables such that $|X_i| \leq b$ almost surely for some constant $b > 0$. Then for any $\delta > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} \{X_i - \mathbb{E}(X_i)\}\right| \geq \delta\right) \geq 1 - 2\exp\left[-n\delta^2/(2(b^2 + \delta/3))\right].$$

**Lemma 13** (Concentration Inequality for $\chi^2$ Variable) Suppose $X_1, \ldots, X_n$ are i.i.d. standard normal random variables. Then for $t \geq 0$,

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^{n} X_i^2 - 1\right| \geq t\right) \leq 2\exp\left(-nt^2/8\right).$$

**Proof** See Example 2.11 of Wainwright (2019). \hfill \qed

We present a key concentration result which will be used in the following proofs.

**Lemma 14** (Lemma B.1 in Lounici et al., 2011) Let $\xi_1, \ldots, \xi_N$ be i.i.d. standard normal random variables. Moreover, let $v = (v_1, \ldots, v_N) \neq 0$, $\eta_v = \sum_{i=1}^{N} (\xi_i^2 - 1)v_i/(\sqrt{2}\|v\|_2)$ and $m(v) = \|v\|_\infty/\|v\|_2$. Then for all $x > 0$, we have

$$\mathbb{P}(|\eta_v| > x) \leq 2\exp\left(-\frac{x^2}{2\{1 + \sqrt{2}m(v)\}}\right).$$

The following lemma validates the restricted eigenvalue condition given Assumption 3.
Lemma 15 Suppose Assumption 3 holds. If $160r\{m^4 \log(pqm)/n\}^{1/2} \leq \rho/2$, with probability at least $1 - 6(qm)(pqm)^{-2}$,

$$
\min \left\{ \frac{\gamma^T U U^T \gamma}{n \| \gamma \|_2^2} : |J| \leq r, \gamma \in \mathbb{R}^{qm}\backslash\{0\}, \sum_{j \in J^c} \| \gamma_j \|_2 \leq 3 \sum_{j \in J} \| \gamma_j \|_2 \right\} \geq \frac{\rho}{2m}.
$$

Proof Note that

$$
\Sigma - \frac{U^T U}{n} = \frac{n - 1}{n} \mathbb{E} \left[ \left( \frac{\tilde{U} - \mathbb{E}(\tilde{U})}{n} \right)^T \left( \frac{\tilde{U} - \mathbb{E}(\tilde{U})}{n} \right) \right] - \frac{\mathbb{E}(\tilde{U} - \mathbb{E}(\tilde{U}))^T (\tilde{U} - \mathbb{E}(\tilde{U}))}{n} + \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n^2}.
$$

Let $\tilde{\Sigma} = \mathbb{E}[(\tilde{U} - \mathbb{E}(\tilde{U}))^T (\tilde{U} - \mathbb{E}(\tilde{U}))]/n$. Then, we have

$$
\left\| \Sigma - \frac{U^T U}{n} \right\|_\infty \leq \left\| \frac{n - 1}{n} \tilde{\Sigma} - \frac{\tilde{U}^T (\tilde{U} - \mathbb{E}(\tilde{U}))}{n} \right\|_\infty + \left\| \frac{\mathbb{E}(\tilde{U})^T \mathbb{E}(\tilde{U})}{n} - \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n} \right\|_\infty + \left\| \frac{\tilde{U}^T 11^T \tilde{U}}{n^2} - \frac{\mathbb{E}(\tilde{U})^T \tilde{U}}{n} \right\|_\infty := T_1 + T_2 + T_3.
$$

For $T_2$, we can write

$$
T_2 = \left\| \frac{\mathbb{E}(\tilde{U})^T \mathbb{E}(\tilde{U})}{n} - \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n} \right\|_\infty = \left\| \frac{\mathbb{E}(\tilde{U})^T \mathbb{E}(\tilde{U})}{n} - \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n} \right\|_\infty = \max_{j, j', k, k'} \left\| \mathbb{E}\{\phi_k(Z_{ij})\} \mathbb{E}\{\phi_{k'}(Z_{ij'})\} - \sum_{i=1}^n \frac{\phi_k(Z_{ij})}{n} \mathbb{E}\{\phi_{k'}(Z_{ij'})\} \right\|_\infty
$$

where $\tilde{U}_i^T$ is the $i$th row of $\tilde{U}$. By the property of the B-spline matrix, we have $0 \leq \phi_k(z) \leq 1$ and $0 \leq \mathbb{E}\{\phi_k(Z_{ij})\} \leq 1$. Therefore,

$$
T_2 = \left\| \frac{\mathbb{E}(\tilde{U})^T \mathbb{E}(\tilde{U})}{n} - \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n} \right\|_\infty \leq \max_{j, k} \left\| \mathbb{E}\{\phi_k(Z_{ij})\} - \sum_{i=1}^n \frac{\phi_k(Z_{ij})}{n} \right\|_\infty.
$$

Now apply Lemma 7 with $t = \{4 \log(pqm)/n\}^{1/2}$ and take the union bound to obtain

$$
T_2 = \left\| \frac{\mathbb{E}(\tilde{U})^T \mathbb{E}(\tilde{U})}{n} - \frac{\tilde{U}^T \mathbb{E}(\tilde{U})}{n} \right\|_\infty \leq \left\{ \frac{4 \log(pqm)}{n} \right\}^{1/2},
$$
which holds with probability at least $1 - 2(qm)(pqm)^{-2}$. For $T_3$, the same bound can be obtained with the same probability since

$$T_3 = \left\lVert \frac{\bar{U}^T 11^T \bar{U} - \frac{\mathbb{E}(\bar{U})^T \bar{U}}{n}}{n^2} \right\rVert_\infty$$

$$= \max_{j, j', k, k'} \left\lVert \frac{\sum_{i=1}^n \phi_k(Z_{ij}) \{ \sum_{i=1}^n \phi_{k'}(Z_{ij'}) \} - \{ \sum_{i=1}^n \phi_k(Z_{ij}) \} \mathbb{E}\{\phi_{k'}(Z_{ij'})\}}{n} \right\rvert$$

$$\leq \max_{j, j', k, k'} \left\lVert \frac{\sum_{i=1}^n \phi_k(Z_{ij}) \{ \sum_{i=1}^n \phi_{k'}(Z_{ij'}) \} - \mathbb{E}\{\phi_{k'}(Z_{ij'})\}}{n} \right\rvert.$$

Now we bound $T_1$

$$T_1 = \left\lVert \frac{n-1}{n} \bar{\Sigma} - \frac{\{ \bar{U} - \mathbb{E}(\bar{U}) \}^T \{ \bar{U} - \mathbb{E}(\bar{U}) \}}{n} \right\rVert_\infty$$

$$\leq \left\lVert \bar{\Sigma} - \frac{\{ \bar{U} - \mathbb{E}(\bar{U}) \}^T \{ \bar{U} - \mathbb{E}(\bar{U}) \}}{n} \right\rVert_\infty + \frac{1}{n} \lVert \bar{\Sigma} \rVert_\infty$$

$$\leq \max_{j, j', k, k'} \left[ \mathbb{E}\{\phi_k(Z_{ij})\phi_{k'}(Z_{ij'}) - \mathbb{E}\{\phi_k(Z_{ij})\} \mathbb{E}\{\phi_{k'}(Z_{ij'})\}]$$

$$\quad - \frac{1}{n} \sum_{i=1}^n [\phi_k(Z_{ij}) - \mathbb{E}\{\phi_k(Z_{ij})\}] [\phi_{k'}(Z_{ij'}) - \mathbb{E}\{\phi_{k'}(Z_{ij'})\}] \right\rvert + \frac{2}{n},$$

where the last inequality holds because each entry of $\bar{\Sigma}$ can be bounded

$$|\mathbb{E}\{\phi_k(Z_{ij})\phi_{k'}(Z_{ij'}) - \mathbb{E}\{\phi_k(Z_{ij})\} \mathbb{E}\{\phi_{k'}(Z_{ij'})\}]| \leq 2.$$

Note $|\phi_k(Z_{ij}) - \mathbb{E}\{\phi_k(Z_{ij})\}| \leq 1, \forall j \in \{1, \ldots, q\}, k \in \{1, \ldots, m\}$, so we have

$$\left\lvert [\phi_k(Z_{ij}) - \mathbb{E}\{\phi_k(Z_{ij})\}] [\phi_{k'}(Z_{ij'}) - \mathbb{E}\{\phi_{k'}(Z_{ij'})\}] \right\rvert \leq 1.$$

Applying Lemma 7 again with $t = \{8 \log(pqm)/n\}^{1/2}$, we obtain by the union bound argument over $j, j' \in \{1, \ldots, q\}, k, k' \in \{1, \ldots, m\}$

$$T_1 \leq \{8 \log(pqm)/n\}^{1/2} + 2/n \leq 6\{\log(pqm)/n\}^{1/2}, \quad (6)$$

which holds with probability at least $1 - 2(qm)^2(pqm)^{-4}$. Thus, we have

$$\left\lVert \bar{\Sigma} - \frac{U^T U}{n} \right\rVert_\infty \leq T_1 + T_2 + T_3 \leq 10 \left\lvert\frac{\log(pqm)}{n}\right\rvert^{1/2},$$
which holds with probability at least \(1 - 6(qm)(pqm)^{-2}\). Consider \(\gamma\) such that \(\sum_{j \in J} \|\gamma_j\|_2 \leq 3 \sum_{j \in J} \|\gamma_j\|_2\), then by Assumption 3, we have

\[
\frac{\gamma^T U^T U \gamma}{n} = \frac{\gamma^T U^T U \gamma}{n} - \gamma^T \Sigma \gamma + \gamma^T \Sigma \gamma \geq \frac{\rho \|\gamma_j\|_2^2}{m} - \Bigg| \frac{\gamma^T U^T U \gamma}{n} - \gamma^T \Sigma \gamma \Bigg|
\]

\[
\geq \frac{\rho \|\gamma_j\|_2^2}{m} - \|\gamma\|_2 \|\Sigma - \frac{U^T U}{n}\|_\infty = \frac{\rho \|\gamma_j\|_2^2}{m} - \left( \sum_{j \in J} \|\gamma_j\|_1 + \sum_{j \in J} \|\gamma_j\|_1 \right)^2 \|\Sigma - \frac{U^T U}{n}\|_\infty
\]

\[
\geq \frac{\rho \|\gamma_j\|_2^2}{m} - \left( \sum_{j \in J} \|\gamma_j\|_2 + 3 \sum_{j \in J} \|\gamma_j\|_2 \right)^2 \|\Sigma - \frac{U^T U}{n}\|_\infty
\]

\[
= \frac{\rho \|\gamma_j\|_2^2}{m} - 16m \left( \sum_{j \in J} \|\gamma_j\|_2 \right)^2 \|\Sigma - \frac{U^T U}{n}\|_\infty
\]

\[
\geq \frac{\rho \|\gamma_j\|_2^2}{m} - 16m \sum_{j \in J} \|\gamma_j\|_2^2 \left( \frac{\log(pqm)}{n} \right) \|\Sigma - \frac{U^T U}{n}\|_\infty = \frac{\rho \|\gamma_j\|_2^2}{m} - 16m \|\gamma_j\|_2^2 \|\Sigma - \frac{U^T U}{n}\|_\infty.
\]

It follows from (6) that

\[
\frac{\gamma^T U^T U \gamma}{n} \geq \frac{\rho \|\gamma_j\|_2^2}{m} - 160rm \|\gamma_j\|_2 \left( \frac{\log(pqm)}{n} \right) \|\Sigma - \frac{U^T U}{n}\|_\infty \geq \left( \frac{\|\gamma_j\|_2^2 \rho}{2m} \right)^{1/2}
\]

holds with probability at least \(1 - 6qm(pqm)^{-2}\), as long as \(160r\{m^4 \log(pqm)/n\}^{1/2} \leq \rho/2\). This concludes the lemma.

The following lemma is crucial for proving Theorem 2.

**Lemma 16** For every \(\ell \in \{1, \ldots, p\}\), consider the random event \(A_\ell = \cap_{j=1}^q A_{j,\ell}\), where

\[A_{j,\ell} = \left\{ \frac{1}{n} \|U_j^T \varepsilon_\ell\|_2^2 \leq \frac{\lambda_\ell}{4} \right\}.
\]

If \(\lambda_\ell \geq 4e^s \sigma_\ell \{14 \log(pqm)/n\}^{1/2}\), then \(P(A_\ell^c) \leq 3(pqm)^{-2}q\).

**Proof** Note that

\[
P(A_{j,\ell}) = P \left( \frac{1}{n^2} \xi_j^T U_j U_j^T \xi_j \leq \frac{\lambda_\ell^2}{16} \right) = P \left( \sum_{i=1}^n \nu_{j,i} (\xi_i^2 - 1) \sqrt{2} \|\nu_j\|_2 \leq x_{j,\ell} \right),
\]

where \(\xi_1, \ldots, \xi_n\) are i.i.d standard normal random variables, \(\nu_{j,1}, \ldots, \nu_{j,n}\) denote the eigenvalues of the matrix \(U_j U_j^T / n\) among which the positive ones are the same as those of \(\Psi_j = U_j^T U_j / n\), and

\[
x_{j,\ell} = \frac{\lambda_\ell^2 n/(16 \sigma_\ell^2) - \text{tr}(\Psi_j)}{\sqrt{2} \|\Psi_j\|_F}.
\]
We bound from the above the probability of \( \mathcal{A}_j^c \) using Lemma 14. Specifically, choose \( \nu = \nu_j = (\nu_{j,1}, \ldots, \nu_{j,n}) \), \( x = x_j \) and \( m(\nu) = \|\Psi_j\|_2/\|\Psi_j\|_F = \|v_j\|_\infty/\|v_j\|_2 \) to obtain
\[
P(\mathcal{A}_j^c) \leq 2 \exp \left\{ - \frac{x_{j\ell}^2}{2(1 + \sqrt{2}x_{j\ell}\|\Psi_j\|_2/\|\Psi_j\|_F)} \right\}.
\]
Now we find the appropriate \( \lambda_{j\ell} \) such that the above probability approaches one as \( p \) and \( q \) increase with \( n \). Let
\[
\exp \left\{ - \frac{x_{j\ell}^2}{2(1 + \sqrt{2}x_{j\ell}\|\Psi_j\|_2/\|\Psi_j\|_F)} \right\} \leq (pqm)^{-2},
\]
which implies \( x_{j\ell}^2 \geq 4 \log(pqm)(1 + \sqrt{2}x_{j\ell}\|\Psi_j\|_2/\|\Psi_j\|_F) \). Equivalently, we need to have
\[
x_{j\ell} = \frac{\lambda_{j\ell}^2 n/(16\sigma_{j\ell}^2) - \text{tr}(\Psi_j)}{\sqrt{2}\|\Psi_j\|_F} \\
\geq 2\sqrt{2}\|\Psi_j\|_2/\|\Psi_j\|_F \log(pqm) + \left[ 2\{2\|\Psi_j\|_2/\|\Psi_j\|_F \log(pqm)\}^2 + 4 \log(pqm) \right]^{1/2},
\]
or
\[
\lambda_{j\ell}^2 n/16\sigma_{j\ell}^2 \geq \text{tr}(\Psi_j) + 4\|\Psi_j\|_2 \log(pqm) + \left[ 4\{2\|\Psi_j\|_2 \log(pqm)\}^2 + \|\Psi_j\|_F \log(pqm) \right]^{1/2}.
\]
Therefore, a sufficient condition for the above is (by noting \( (a+b)^{1/2} \leq a^{1/2} + b^{1/2}, \forall a, b \geq 0 \))
\[
\lambda_{j\ell}^2 n/16\sigma_{j\ell}^2 \geq \text{tr}(\Psi_j) + 8\|\Psi_j\|_2 \log(pqm) + \left\{ 8\|\Psi_j\|_F \log(pqm) \right\}^{1/2}.
\]
(7)
Therefore, when (7) holds, \( P(\mathcal{A}_j^c) \leq 2(pqm)^{-2} \). It suffices to find the probability of (7) when \( \lambda_{j\ell} \geq 4c^* \sigma_{j\ell} \{14 \log(pqm)/n\}^{1/2} \). By Lemma 11, when \( \{2 \log(pqm)m^3/n\}^{1/2} \leq 1 \), with probability at least \( 1 - \exp(-nm^{-3}) \geq 1 - (pqm)^{-2} \), the following inequality
\[
\text{tr}(\Psi_j) = \text{tr}\left( \frac{U_j^T U_j}{n} \right) = \text{tr}\left( \frac{\tilde{U}_j^T \tilde{U}_j}{n} - \frac{\tilde{U}_j^T 11^T \tilde{U}_j}{n^2} \right) \leq \text{tr}\left( \frac{\tilde{U}_j^T \tilde{U}_j}{n} \right) \leq m \frac{c^*}{m} = c^*
\]
holds, where the first inequality follows from the positive semidefiniteness of \( \tilde{U}_j^T 11^T \tilde{U}_j/n^2 \). Similarly, \( \|\Psi_j\|_2 \leq c^*/m \) and \( \|\Psi_j\|_F \leq \{\|\Psi_j\|_2 \text{tr}(\Psi_j)\}^{1/2} \leq c^*/\sqrt{m} \). Therefore, when \( \lambda_{j\ell} \geq 4c^* \sigma_{j\ell} \{14 \log(pqm)/n\}^{1/2} \),
\[
\lambda_{j\ell}^2 n/(16\sigma_{j\ell}^2) \geq c^* + \frac{8c^* \log(pqm)}{m} + \left\{ \frac{8(c^*)^2 \log(pqm)}{m} \right\}^{1/2}
\]
\[
\geq \text{tr}(\Psi_j) + 8\|\Psi_j\|_2 \log(pqm) + \left\{ 8\|\Psi_j\|_F \log(pqm) \right\}^{1/2}
\]
holds with probability at least \( 1 - (pqm)^{-2} \). As a result, \( \mathcal{A}_j^c \) happens with probability at least \( 1 - 3(pqm)^{-2} \) when \( \lambda_{j\ell} \geq 4\sigma_{j\ell} c^* \{14 \log(pqm)/n\}^{1/2} \). The proof is complete by taking the union bound argument.
Lemma 17 (Error Bound for Approximation) Under the assumption of Lemma 8, the event
\[ D_t := \left\{ \frac{1}{\sqrt{n}} \left\| U \gamma_t - \sum_{j=1}^{q} F_{j} \right\|_2 \leq 4rC_Lm^{-d} + 4r\{C_0^2 \log(pqm)/n\}^{1/2} \right\} \]
happens with probability at least \( 1 - 2r(pqm)^{-2} \).

Proof First, note that
\[ \frac{1}{n} \left\| U \gamma_t - \sum_{j=1}^{q} F_{j} \right\|_2^2 = \frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{j} \tilde{f}_{j}(Z_{ij}) - \sum_{j} \tilde{\gamma}_{jk} \psi_k(Z_{ij}) \right\}^2 \]
\[ = \frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{j \in I_t} \tilde{f}_{j}(Z_{ij}) - \sum_{j \in I_t} \tilde{\gamma}_{jk} \psi_k(Z_{ij}) \right\}^2 \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in I_t} \left\{ \tilde{f}_{j}(Z_{ij}) - \tilde{\gamma}_{jk} \psi_k(Z_{ij}) \right\}^2 \leq r \sum_{j \in I_t} \sup_{z \in [a,b]} |f_j(z) - \tilde{f}_{j}(z)|^2. \]

Applying Lemma 1 with a union bound argument, with probability at least \( 1 - 2r(pqm)^{-2} \), we have
\[ \frac{1}{\sqrt{n}} \left\| U \gamma_t - \sum_{j=1}^{q} F_{j} \right\|_2 \leq 4rC_Lm^{-d} + 4r\{C_0^2 \log(pqm)/n\}^{1/2}. \]

This completes the proof. \[ \square \]

The next lemma resembles Lemma 15 but focuses on the classical restricted eigenvalue condition.

Lemma 18 Suppose Assumption 4 holds. Then, with probability at least \( 1 - 2(pqm)^{-2} \),
\[ \min \left\{ \beta^T F^T \beta : |\beta| \leq s, \beta \in \mathbb{R}^p \setminus \{0\}, \sum_{\ell \in \mathcal{L}} |\beta_\ell| \leq 3 \sum_{\ell \in \mathcal{L}} |\beta_\ell| \right\} \geq \frac{\kappa}{2} \]
holds when \( 64sr^2C_0\{\log(pqm)/n\}^{1/2} \leq \kappa/2 \) and \( \{\log(pqm)/n\} \leq 1/2 \).

Proof Consider \( \beta \) satisfying \( \sum_{\ell \in \mathcal{L}} |\beta_\ell| \leq 3 \sum_{\ell \in \mathcal{L}} |\beta_\ell| \). Then,
\[ \frac{\beta^T F^T \beta}{n} = \beta^T \Sigma_f \beta + \frac{\beta^T F^T F \beta}{n} \]
\[ \geq \|\beta\|_2^2 \kappa - \beta^T (\Sigma_f - \frac{F^T F}{n}) \beta \geq \|\beta\|_2^2 \kappa - \|\beta\|_2^2 \Sigma_f - \frac{F^T F}{n} \|_\infty \]
\[ \geq \|\beta\|_2^2 \kappa - \left( \sum_{\ell \in \mathcal{L}} |\beta_\ell| + \sum_{\ell \in \mathcal{L}} |\beta_\ell| \right)^2 \|\Sigma_f - \frac{F^T F}{n} \|_\infty \]
\[ \geq \|\beta\|_2^2 \kappa - \left( 3 \sum_{\ell \in \mathcal{L}} |\beta_\ell| \right)^2 \|\Sigma_f - \frac{F^T F}{n} \|_\infty \]
\[ \geq \|\beta\|_2^2 \kappa - \left( 3 \sum_{\ell \in \mathcal{L}} |\beta_\ell| \right)^2 \|\Sigma_f - \frac{F^T F}{n} \|_\infty \]
\[ \geq \|\beta\|_2^2 \kappa - 16 \|\beta\|_1 \|\Sigma_f - \frac{F^T F}{n} \|_\infty \geq \|\beta\|_2^2 \kappa - 16s \|\beta\|_2^2 \|\Sigma_f - \frac{F^T F}{n} \|_\infty \].
We first bound $\|\Sigma_f - F^TF/n\|_\infty$,
\[\|\Sigma_f - \frac{F^TF}{n}\|_\infty = \max_{\ell,\ell'} \mathbb{E} \left[ \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_j) \right\} \left\{ \sum_{j=1}^{q} f_{j\ell'}(Z_j) \right\} \right] - \sum_{i=1}^{n} \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_{ij}) \right\} \left\{ \sum_{j=1}^{q} f_{j\ell'}(Z_{ij}) \right\} / n.\]

As there are at most $r$ nonzero summands in $\sum_{j=1}^{q} f_{j\ell}(z)$ for every $\ell \in \{1, \ldots, p\}$, we have
\[\left| \sum_{j=1}^{q} f_{j\ell}(z) \right| \leq rC_0, \forall \ell \in \{1, \ldots, p\},\]
which implies
\[\left| \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_j) \right\} \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_j) \right\} \right| \leq r^2C_0^2.\]

Now applying Lemma 12 with $b = r^2C_0^2$ and $\delta = 4r^2C_0^2 \{\log(pqm)/n\}^{1/2}$, with probability at least $1 - 2\exp \left[ -8r^4C_0^4 \log(pqm)/(r^4C_0^4 + \frac{4}{3}r^4C_0^4(\log(pqm)/n)^{1/2}) \right]$, we have
\[
\mathbb{E} \left[ \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_j) \right\} \left\{ \sum_{j=1}^{q} f_{j\ell'}(Z_j) \right\} \right] - \sum_{i=1}^{n} \left\{ \sum_{j=1}^{q} f_{j\ell}(Z_{ij}) \right\} \left\{ \sum_{j=1}^{q} f_{j\ell'}(Z_{ij}) \right\} / n \\
\leq 4r^2C_0^2 \left\{ \frac{\log(pqm)}{n} \right\}^{1/2}.\]

When $\{\log(pqm)/n\}^{1/2} \leq 1/2$, we have
\[1 - 2\exp \left[ -\frac{8r^4C_0^4 \log(pqm)}{r^4C_0^4 + \frac{4}{3}r^4C_0^4(\log(pqm)/n)^{1/2}} \right] \geq 1 - 2(pqm)^{-4}\]
and by taking the union bound we get
\[\|\Sigma_f - \frac{F^TF}{n}\|_{\infty} \leq 4r^2C_0^2 \left\{ \frac{\log(pqm)}{n} \right\}^{1/2},\]
which holds with probability at least $1 - 2p^2(pqm)^{-4}$. It follows that
\[\frac{\beta^TF^TF\beta}{n} \geq \|\beta_L\|^22\kappa - 64s\|\beta_L\|^2r^2C_0^2 \left\{ \frac{\log(pqm)}{n} \right\}^{1/2}\]
holds with probability at least $1 - 2p^2(pqm)^{-4}$. When $64sr^2C_0^2 \{\log(pqm)/n\}^{1/2} \leq \kappa/2$, we have $\beta^TF^TF\beta/n \geq \kappa\|\beta_L\|^2/2$ holds with probability at least $1 - 2(pqm)^{-2}$. This concludes the lemma. \[\Box\]
Section B: Proof of Results in Section 2

Proof of Lemma 1 in the main text

Proof By the assumption $\mathbb{E}\{f_{j\ell}(z_{ij})\} = 0$ and Lemma 8, we know

$$
\sup_{z \in [0, 1]} |f_{j\ell}(z) - \tilde{f}_{nj\ell}(z)|
\leq \sup_{z \in [0, 1]} |f_{j\ell}(z) - \tilde{f}_{nj\ell}(z)| + \left| \frac{1}{n} \sum_{k=1}^{m_n} \sum_{i=1}^{n} \tilde{\gamma}_{jkt} \phi_k(Z_{ij}) - \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) \right|
+ \left| \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) - \mathbb{E}(f_{j\ell}(Z_{ij})) \right|
\leq C_L m_n^{-d} + \frac{1}{n} \sum_{i=1}^{n} |\tilde{f}_{nj\ell}(Z_{ij}) - f_{j\ell}(Z_{ij})| + \left| \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) - \mathbb{E}(f_{j\ell}(Z_{ij})) \right|
\leq 2C_L m_n^{-d} + \left| \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) - \mathbb{E}(f_{j\ell}) \right|,
$$

where the first inequality follows from the triangle inequality and the second to the last inequalities follow from Lemma 8. Finally, noting $|f(Z_{ij})| \leq C_0$ and applying Lemma 7 with $t = \{4C_0^2 \log(pqm)/n\}^{1/2}$, we have

$$
\left| \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) - \mathbb{E}(f_{j\ell}) \right| \leq \{4C_0^2 \log(pqm)/n\}^{1/2},
$$

which holds with probability at least $1 - 2(pqm)^{-2}$. This completes the proof. \hfill \blacksquare

Section C: Proof of Results in Section 3

Proof of Theorem 2 in the main text

Consider the $\ell$th optimization problem in (3). By the optimality of $\tilde{\gamma}_\ell$ in (3), we have $\forall \gamma_\ell \in \mathbb{R}^m$,

$$
\frac{1}{n} \left\| U\tilde{\gamma}_\ell - X_\ell \right\|^2_2 + 2\lambda_\ell \sum_{j=1}^{q} \left\| \tilde{\gamma}_{j\ell} \right\|^2 \leq \frac{1}{n} \left\| U\gamma_\ell - X_\ell \right\|^2_2 + 2\lambda_\ell \sum_{j=1}^{q} \left\| \gamma_{j\ell} \right\|^2.
$$

Recall that $\sum_{j=1}^{q} F_{j\ell} + \varepsilon_\ell = X_\ell$. Therefore,

$$
\frac{1}{n} \left\| U\tilde{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell} + U\tilde{\gamma}_\ell - U\gamma_\ell + \varepsilon_\ell \right\|^2_2 + 2\lambda_\ell \sum_{j=1}^{q} \left\| \tilde{\gamma}_{j\ell} \right\|^2 \leq \frac{1}{n} \left\| U\gamma_\ell - \sum_{j=1}^{q} F_{j\ell} + U\gamma_\ell - U\gamma_\ell + \varepsilon_\ell \right\|^2_2 + 2\lambda_\ell \sum_{j=1}^{q} \left\| \gamma_{j\ell} \right\|^2.
$$
Thus, we obtain

\[
\begin{align*}
&\frac{1}{n} \|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|_2^2 + \frac{1}{n} \|\varepsilon_\ell\|_2^2 + \frac{1}{n} \left\| \sum_{j=1}^{q} F_{j\ell} - U\bar{\gamma}_\ell \right\|^2_2 \\
&+ \frac{2}{n} (U\hat{\gamma} - U\bar{\gamma})^T (\varepsilon_\ell + U\bar{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell}) + \frac{2}{n} \varepsilon_\ell^T \left( U\bar{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell} \right) + 2\lambda\ell \sum_{j=1}^{q} \|\bar{\gamma}_{j\ell}\|_2 \\
&\leq \frac{1}{n} \|U\gamma_\ell - U\bar{\gamma}_\ell\|_2^2 + \frac{1}{n} \|\varepsilon_\ell\|_2^2 + \frac{1}{n} \left\| \sum_{j=1}^{q} F_{j\ell} - U\bar{\gamma}_\ell \right\|^2_2 \\
&+ \frac{2}{n} (U\gamma_\ell - U\bar{\gamma}_\ell)^T (\varepsilon_\ell + U\bar{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell}) + \frac{2}{n} \varepsilon_\ell^T \left( U\bar{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell} \right) + 2\lambda\ell \sum_{j=1}^{q} \|\bar{\gamma}_{j\ell}\|_2,
\end{align*}
\]

which implies that

\[
\begin{align*}
&\frac{1}{n} \|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|_2^2 + 2\lambda\ell \sum_{j=1}^{q} \|\bar{\gamma}_{j\ell}\|_2 \\
&\leq \frac{1}{n} \|U\gamma_\ell - U\bar{\gamma}_\ell\|_2^2 + \frac{2}{n} \left( \sum_{j=1}^{q} (U\gamma_{j\ell} - U\bar{\gamma}_{j\ell}) \right)^T (\varepsilon_\ell + U\bar{\gamma}_\ell - \sum_{j=1}^{q} F_{j\ell}) + 2\lambda\ell \sum_{j=1}^{q} \|\bar{\gamma}_{j\ell}\|_2 \\
&\leq \frac{1}{n} \|U\gamma_\ell - U\bar{\gamma}_\ell\|_2^2 + \sum_{j=1}^{q} \frac{2}{n} (\gamma_{j\ell}^T - \hat{\gamma}_{j\ell}) U_{j\ell}^T \varepsilon_\ell \\
&+ \sum_{j=1}^{q} \frac{2}{n} (\hat{\gamma}_{j\ell}^T - \bar{\gamma}_{j\ell}) U_{j\ell}^T \left( U\gamma_\ell - \sum_{j=1}^{q} F_{j\ell} \right) + 2\lambda\ell \sum_{j=1}^{q} \|\gamma_{j\ell}\|_2 \\
&\leq \frac{1}{n} \|U\gamma_\ell - U\bar{\gamma}_\ell\|_2^2 + \sum_{j=1}^{q} \frac{2}{n} \|U_{j\ell}^T \varepsilon_\ell\|_2 \|\hat{\gamma}_{j\ell} - \gamma_{j\ell}\|_2 \\
&+ \sum_{j=1}^{q} \frac{2}{n} \left\| U_{j\ell}^T \left( U\gamma_\ell - \sum_{j=1}^{q} F_{j\ell} \right) \right\|_2 \|\bar{\gamma}_{j\ell} - \gamma_{j\ell}\|_2 + 2\lambda\ell \sum_{j=1}^{q} \|\gamma_{j\ell}\|_2,
\end{align*}
\]

where the last inequality holds due to the Cauchy–Schwarz inequality. By Lemma 16 and the choice of $\lambda_\ell$, we have

\[
\begin{align*}
&\frac{1}{n} \|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|_2^2 + 2\lambda\ell \sum_{j=1}^{q} \|\bar{\gamma}_{j\ell}\|_2 \\
&\leq \frac{1}{n} \|U\gamma_\ell - U\bar{\gamma}_\ell\|_2^2 + \frac{\lambda\ell}{2} \sum_{j=1}^{q} \|\gamma_{j\ell} - \bar{\gamma}_{j\ell}\|_2 \\
&+ \sum_{j=1}^{q} \frac{2}{n^{1/2}} \|U_{j\ell}\|_2 \frac{2}{n^{1/2}} \|U\gamma_\ell - \sum_{j=1}^{q} F_{j\ell}\|_2 \|\bar{\gamma}_{j\ell} - \gamma_{j\ell}\|_2 + 2\lambda\ell \sum_{j=1}^{q} \|\gamma_{j\ell}\|_2
\end{align*}
\]
with probability at least $1 - 3q(pqm)^{-2}$. Then by Lemma 11, we have
\[
\frac{1}{n} \|U\hat{\gamma}_t - U\bar{\gamma}_t\|_2^2 + 2\lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt}\|_2^2 \\
\leq \frac{1}{n} \|U\gamma_t - U\bar{\gamma}_t\|_2^2 + \frac{\lambda t}{2} \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 \\
+ \sum_{j=1}^q \left(\frac{c^*}{m} \frac{4}{n^2} \|U\bar{\gamma}_t - \gamma_{jt}\|_2 \right) \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + 2\lambda t \sum_{j=1}^q \|\gamma_{jt}\|_2,
\]
which holds with probability at least $1 - 4q(pqm)^{-2}$ (taking the union bound here) when $1 - \exp(-nm^{-3}) \geq 1 - (pqm)^{-2}$. Then by Lemma 17, we have
\[
\frac{1}{n} \|U\hat{\gamma}_t - U\bar{\gamma}_t\|_2^2 + 2\lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt}\|_2 \\
\leq \frac{1}{n} \|U\gamma_t - U\bar{\gamma}_t\|_2^2 + \frac{\lambda t}{2} \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + 2\lambda t \sum_{j=1}^q \|\gamma_{jt}\|_2 \\
+ \left[8rC_L(c^*/2NM^{-1/2}) + 8r \left\{\frac{c^*C_0^2 \log(pqm)}{nm}\right\}^{1/2}\right] \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 \\
\leq \frac{1}{n} \|U\gamma_t - U\bar{\gamma}_t\|_2^2 + \frac{\lambda t}{2} \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + \frac{\lambda t}{2} \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + 2\lambda t \sum_{j=1}^q \|\gamma_{jt}\|_2 \\
\leq \frac{1}{n} \|U\gamma_t - U\bar{\gamma}_t\|_2^2 + \lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + 2\lambda t \sum_{j=1}^q \|\gamma_{jt}\|_2,
\]
which holds with probability at least $1 - 6q(pqm)^{-2}$ since $r \leq q$. Setting $\gamma_t = \hat{\gamma}_t$, we have
\[
\frac{1}{n} \|U\hat{\gamma}_t - U\bar{\gamma}_t\|_2^2 + \lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 \\
\leq 2\lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 + 2\lambda t \sum_{j=1}^q \|\bar{\gamma}_{jt}\|_2 - 2\lambda t \sum_{j=1}^q \|\gamma_{jt}\|_2 \\
\leq 4\lambda t \sum_{j \in J_t} \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2.
\]
Therefore, we know $\lambda t \sum_{j=1}^q \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 \leq 4\lambda t \sum_{j \in J_t} \|\gamma_{jt}\|_2$, which implies that
\[
\sum_{j \in J_t} \|\hat{\gamma}_{jt} - \gamma_{jt}\|_2 \leq 3 \sum_{j \in J_t} \|\gamma_{jt}\|_2
\]
holds with probability at least $1 - 6q(pqm)^{-2}$. Then, combine with Lemma 15 to obtain
\[
\|\gamma_{J_t} - \bar{\gamma}_{J_t}\|_2 \leq \frac{\|U\hat{\gamma}_t - U\bar{\gamma}_t\|_2}{n^{1/2}} \left(\frac{2m}{\rho}\right)^{1/2},
\]
which holds with probability at least \(1 - 12(qm)(pqm)^{-2}\). We also know from inequality (8) that
\[
\frac{1}{n} \|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|_2^2 \leq 4\lambda_\ell \sum_{j \in J_\ell} \|\hat{\gamma}_{j\ell} - \bar{\gamma}_{j\ell}\|_2 \leq 4\lambda_\ell \sqrt{r} \|\hat{\gamma}_{J_\ell} - \bar{\gamma}_{J_\ell}\|_2,
\]
which holds with probability at least \(1 - 6q(pqm)^{-2}\), where the second inequality follows from the Cauchy–Schwarz inequality. Then, it follows that
\[
\frac{\|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|}{n^{1/2}} \leq 4\lambda_\ell r^{1/2} \left(\frac{2m}{\rho}\right)^{1/2}
\]
holds with probability at least \(1 - 18(qm)(pqm)^{-2}\). It holds with the same probability that
\[
\sum_{j=1}^{q} \|\hat{\gamma}_{j\ell} - \bar{\gamma}_{j\ell}\|_2 \leq 4\lambda_\ell \sum_{j \in J_\ell} \|\hat{\gamma}_{j\ell} - \bar{\gamma}_{j\ell}\|_2 \leq 4n^{1/2} \|\hat{\gamma}_{J_\ell} - \bar{\gamma}_{J_\ell}\|_2
\]
\[
\leq 4r^{1/2} \frac{\|U\hat{\gamma}_\ell - U\bar{\gamma}_\ell\|_2}{\sqrt{n}} \left(\frac{2m}{\rho}\right)^{1/2} \leq 32\lambda_\ell \frac{m}{\rho},
\]
Now we apply Lemma 17 to obtain
\[
\left\| \sum_{j=1}^{q} F_{j\ell} - U\hat{\gamma}_\ell \right\|_2 = \left\| \sum_{j=1}^{q} F_{j\ell} - U\bar{\gamma}_\ell + U\bar{\gamma}_\ell - U\hat{\gamma}_\ell \right\|_2
\]
\[
\leq \left\| \sum_{j=1}^{q} F_{j\ell} - U\bar{\gamma}_\ell \right\|_2 + \|U\bar{\gamma}_\ell - U\hat{\gamma}_\ell\|_2
\]
\[
\leq 4rn^{1/2} C_L m^{-d} + 4rn^{1/2} C_0 \{\log (pqm)/n\}^{1/2} + 4\lambda_\ell (rn)^{1/2} \left(\frac{2m}{\rho}\right)^{1/2}
\]
\[
\leq 5\lambda_\ell (rn)^{1/2} \left(\frac{2m}{\rho}\right)^{1/2},
\]
which holds with probability at least \(1 - 20(qm)(pqm)^{-2}\), where the last inequality follows from the definition of \(\lambda_\ell\). The proof is complete by taking the union bound over all \(\ell\).

**Proof of Theorem 3**

**Lemma 19** Suppose Assumptions 1–4 hold. If the regularization parameters satisfy
\[
560C_0 \lambda_{\text{max}} \left(\frac{2rm}{\rho}\right)^{1/2} \leq \frac{\kappa^2}{4rs}, \tag{9}
\]
then with probability at least \(1 - 62(pqm)^{-1}\), the matrix \(\hat{X} = U\hat{\Gamma}\) satisfies
\[
\min \left\{ \frac{\|\hat{X}\beta\|}{n^{1/2} \|\beta_{\mathcal{L}}\|} : |\mathcal{L}| \leq s, \beta \in \mathbb{R}_p \setminus \{0\}, \|\beta_{\mathcal{L}}\| \leq 3\|\beta_{\mathcal{L}}\|, \|\beta_{\mathcal{L}}\| \leq 3\|\beta_{\mathcal{L}}\| \right\} \geq \frac{\kappa}{2}
\]
when \(n\) is sufficiently large.
Proof For any subset $\mathcal{L} \subset \{1, \ldots, p\}$ with $|\mathcal{L}| \leq s$ and any $\delta \in \mathbb{R}^p$ such that $\delta \neq 0$ and $\|\delta_{\mathcal{L}^c}\|_1 \leq 3\|\delta_{\mathcal{L}}\|_1$, we have

$$\frac{\delta^T (F^T F - \hat{X}^T \hat{X}) \delta}{n\|\delta\|_2^2} \leq \frac{\|\delta\|_2^2 \max_{1 \leq \ell, \ell' \leq p} \left| \left( \sum_{j=1}^q F_{j\ell} \right)^T \left( \sum_{j=1}^q F_{j\ell'} \right) - \hat{X}^T_{\ell} \hat{X}_{\ell} \right|}{n\|\delta\|_2^2}. $$

Since $\|\delta_{\mathcal{L}^c}\|_1 \leq 3\|\delta_{\mathcal{L}}\|_1$, we have $\|\delta\|_2^2 = (\|\delta_{\mathcal{L}}\|_1 + \|\delta_{\mathcal{L}^c}\|_1)^2 \leq 16\|\delta_{\mathcal{L}}\|_1^2 \leq 16s\|\delta_{\mathcal{L}}\|_2^2$, which implies that

$$\frac{\delta^T (F^T F - \hat{X}^T \hat{X}) \delta}{n\|\delta\|_2^2} \leq \frac{16s \max_{1 \leq \ell, \ell' \leq p} \left| \left( \sum_{j=1}^q F_{j\ell} \right)^T \left( \sum_{j=1}^q F_{j\ell'} \right) - \hat{X}^T_{\ell} \hat{X}_{\ell} \right|}{n}. $$

To bound the entrywise maximum, we write

$$\left| \left( \sum_{j=1}^q F_{j\ell} \right)^T \left( \sum_{j=1}^q F_{j\ell'} \right) - \hat{X}^T_{\ell} \hat{X}_{\ell} \right|$$

$$= \left| \left( \hat{X}_{\ell} - \sum_{j=1}^q F_{j\ell} \right)^T \left( \hat{X}_{\ell'} - \sum_{j=1}^q F_{j\ell'} \right) + \left( \hat{X}_{\ell} - \sum_{j=1}^q F_{j\ell} \right)^T \left( \sum_{j=1}^q F_{j\ell} \right) \right|$$

$$\leq \left| \left( \hat{X}_{\ell} - \sum_{j=1}^q F_{j\ell} \right)^T \left( \hat{X}_{\ell'} - \sum_{j=1}^q F_{j\ell'} \right) \right| + \left| \left( \hat{X}_{\ell} - \sum_{j=1}^q F_{j\ell} \right)^T \left( \sum_{j=1}^q F_{j\ell} \right) \right|$$

$$= T_1 + T_2 + T_3.$$ 

For $T_1$, by the Cauchy–Schwarz inequality and Theorem 2, we have

$$T_1 \leq \left\| \hat{X}_{\ell} - \sum_{j=1}^q F_{j\ell} \right\|_2 \cdot \left\| \hat{X}_{\ell'} - \sum_{j=1}^q F_{j\ell'} \right\|_2 \leq 25n\lambda_{\max}^2 \frac{2m}{r},$$

which holds with probability at least $1 - 20(pqm)^{-1}$. For $T_2$, note that $\left\| \sum_{j=1}^q F_{j\ell} \right\|_2 \leq C_0 n^{1/2}r$, $\forall \ell \in \{1, \ldots, p\}$ due to the sparsity assumption. Then, we have

$$T_2 \leq \left\| U \hat{\gamma}_{\ell} - \sum_{j=1}^q F_{j\ell} \right\|_2 \cdot \left\| \sum_{j=1}^q F_{j\ell'} \right\|_2 \leq 5n\lambda_{\max} \left( \frac{2rm}{\rho} \right)^{1/2} r C_0$$

which holds with probability at least $1 - 20(pqm)^{-1}$. Similarly, it holds with the same probability that

$$T_3 \leq 5n\lambda_{\max} \left( \frac{2rm}{\rho} \right)^{1/2} r C_0.$$ 

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Therefore, we have

\[ \max_{1 \leq \ell, \ell' \leq p} \left\| \left( \sum_{j=1}^{q} F_{j \ell'} \right) \left( \sum_{j=1}^{q} F_{j \ell} \right) - \hat{X}_\ell \hat{X}_\ell \right\| \leq 35n \lambda_{\max} r C_F \left( \frac{2rm}{\rho} \right)^{1/2}, \]

which holds with probability at least \( 1 - 60(pqm)^{-1} \) when \( \lambda_{\max}(2m/\rho)^{1/2} \leq r^{1/2} C_0 \). We know from Lemma 18 that

\[ \min \left\{ \frac{\|f\beta\|}{\sqrt{n} \|\beta_L\|} : \|L\| \leq s, \beta \in \mathbb{R}^p \backslash \{0\}, \|\beta_L\| \leq 3\|\beta_L\|_1 \right\} \geq \frac{\sqrt{2}\kappa}{2} \]

holds with probability at least \( 1 - 2(2qm)^{-2} \). It follows from (9) and (10) that

\[ \frac{\delta^T (F^T F - \hat{X}^T \hat{X}) \delta}{n \|\delta_L\|^2} \leq 560 \lambda_{\max} C_0 rs \left( \frac{2rm}{\rho} \right)^{1/2} \leq \frac{\kappa^2}{4} \]

holds with probability at least \( 1 - 60(pqm)^{-1} \). Therefore, with probability at least \( 1 - 62(pqm)^{-1} \), we get

\[ \frac{\beta^T (\hat{X}^T \hat{X}) \beta}{n \|\beta_L\|^2} \geq \frac{\kappa^2}{4} \]

This completes the proof.

**Lemma 20** Under Assumptions 1-4, if the regularization parameters \( \lambda_\ell \)'s are chosen as in Theorem 2 and

\[ \mu = 2 \lambda_{\max} r \left( \frac{2m}{\rho} \right)^{1/2} \left( 7\sigma_0 + 8 \sqrt{5} B \max_{\ell} \sigma_\ell + 30B \right), \]

then with probability at least \( 1 - 86(pqm)^{-1} \), the regularized estimator \( \hat{\beta} \) of (4) satisfies

\[ \frac{1}{2n} \| \hat{X} (\hat{\beta} - \beta) \|_2^2 + \frac{\mu}{2} \| \hat{\beta} - \beta \|_1 \leq 2\mu \| \hat{\beta}_L - \beta_L \|_1. \]

**Proof** By the optimality of \( \hat{\beta} \), we have

\[ \frac{1}{2n} \| Y - \hat{X} \hat{\beta} \|_2^2 + \mu \| \hat{\beta} \|_1 \leq \frac{1}{2n} \| Y - \hat{X} \beta \|_2^2 + \mu \| \beta \|_1. \]

Substituting \( Y = X\beta + \eta \), we have

\[ \| Y - \hat{X} \hat{\beta} \|_2^2 = \| \eta - (\hat{X} \hat{\beta} - X\beta) \|_2^2 \]

\[ = \| \eta \|_2^2 + \| \hat{X} \hat{\beta} - X\beta \|_2^2 - 2\eta^T (\hat{X} \hat{\beta} - X\beta) \]

\[ = \| \eta \|_2^2 + \| \hat{X} \hat{\beta} - X\beta \|_2^2 + 2\eta^T (\hat{X} \hat{\beta} - X\beta) \]

\[ = \| \eta \|_2^2 + \| \hat{X} (\hat{\beta} - \beta) \|_2^2 + \| (\hat{X} - X)\beta \|_2^2 - 2\eta^T (\hat{X} \hat{\beta} - X\beta) \]

\[ + 2\beta^T (\hat{X} - X)^T \hat{X} (\hat{\beta} - \beta) \]
and
\[ \|Y - \hat{X} \beta\|_2^2 = \|\eta - (\hat{X} - X)\beta\|_2^2 = \|\eta\|_2^2 + \|\hat{X} - X\|_2^2 - 2\eta^T (\hat{X} - X)\beta. \]

It then follows that
\[ \frac{1}{2n} \|\hat{X}(\hat{\beta} - \beta)\|_2^2 \leq \mu \|\beta\|_1 - \mu \|\hat{\beta}\|_1 + \frac{1}{n} \eta^T \hat{X}(\hat{\beta} - \beta) - \frac{1}{n} \beta^T (\hat{X} - X)^T \hat{X}(\hat{\beta} - \beta) \]
\[ \leq \mu \|\beta\|_1 - \mu \|\hat{\beta}\|_1 + \frac{1}{n} \hat{X}^T \eta - \frac{1}{n} \hat{X}^T (\hat{X} - X)\beta \leq \tau \|\hat{\beta}\|_1. \] (11)

Next, we find a probability bound for the following event
\[ \left\| \frac{1}{n} \hat{X}^T \eta - \frac{1}{n} \hat{X}^T (\hat{X} - X)\beta \right\|_{\infty} \leq \frac{\mu}{2}. \]

Substituting \( \hat{X} = U\hat{\Gamma} \) and \( X = F + \varepsilon \), we write
\[ \frac{1}{n} \hat{X}^T \eta - \frac{1}{n} \hat{X}^T (\hat{X} - X)\beta = \frac{1}{n} (U\hat{\Gamma})^T \eta - \frac{1}{n} (U\hat{\Gamma})^T (U\hat{\Gamma} - X)\beta \]
\[ = \frac{1}{n} \hat{\Gamma}^T U^T \eta - \frac{1}{n} \hat{\Gamma}^T U^T (U\hat{\Gamma} - F - \varepsilon)\beta \]
\[ = \frac{1}{n} (\hat{\Gamma}^T U^T - F^T)\eta + \frac{1}{n} F^T \eta + \frac{1}{n} (\hat{\Gamma}^T U^T - F^T)\varepsilon\beta + \frac{1}{n} F^T \varepsilon\beta \]
\[ = \frac{1}{n} (\hat{\Gamma}^T U^T - F^T) (U\hat{\Gamma} - F)\beta - \frac{1}{n} F^T (U\hat{\Gamma} - F)\beta \]
\[ =: T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \]

For \( T_1 \), it follows from Theorem 2 that
\[ \|T_1\|_\infty \leq \frac{1}{n} \max_{\ell} \left\| \sum_{j=1}^q F_{j\ell} - U\hat{\gamma}_\ell \right\|_2 \leq 5\lambda_{\max} \tau^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} \left( \frac{\|\eta\|_2}{n^{1/2}} \right) \]
holds with probability at least \( 1 - 20(pqm)^{-1} \). For a Gaussian variable \( \eta_i \), \( \eta_i^2 \) is sub-exponential and follows the \( \sigma_0^2 \chi^2 \) distribution. Apply Lemma 13 with \( t = \{16 \log (pqm)/n\}^{1/2} \) to obtain
\[ \left\| \frac{1}{\sigma_0^2 n} \|\eta\|_2^2 - 1 \right\| \leq 4 \left\{ \frac{\log (pqm)}{n} \right\}^{1/2}, \]
which holds with probability at least \( 1 - 2(pqm)^{-2} \). Therefore, we have
\[ \|T_1\|_\infty \leq 5\lambda_{\max} \tau^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} \left( \frac{\|\eta\|_2}{n^{1/2}} \right)^{1/2} \]
\[ \leq 5\lambda_{\max} \tau^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} \left[ 4\sigma_0^2 \{\log (pqm)/n\}^{1/2} + \sigma_0^2 \right] \]
\[ \leq 5\lambda_{\max} \tau^{1/2} \left( \frac{10m}{\rho} \right)^{1/2} \sigma_0, \]
which holds with probability at least $1 - 22(pqm)^{-1}$ when $\log(pqm)/n \leq 1$. For $T_2$, we have

$$\|T_2\|_{\infty} = \frac{1}{n} \|f^T \eta\|_{\infty} \leq \frac{1}{n} \max_{\ell} \left| \left( \sum_{j=1}^{q} \mathcal{F}_{j\ell} \right)^T \right| \eta.$$  

By the tail bound of a standard normal random variable $X$

$$\mathbb{P}(|X| \geq t) \leq 2 \left( \frac{2}{\pi} \right)^{1/2} \frac{\exp(-t^2/2)}{t},$$

we have

$$\mathbb{P}(\|T_2\|_{\infty} \geq t) \leq \mathbb{P}\left\{ \frac{1}{n} \max_{\ell} \left| \left( \sum_{j=1}^{q} \mathcal{F}_{j\ell} \right)^T \eta \right| \geq t \right\}$$

$$\leq p \mathbb{P}\left( \frac{1}{n} \sum_{j=1}^{q} \mathcal{F}_{j\ell} \eta \geq t \right) = p \mathbb{P}\left( \frac{1}{n} \sum_{j=1}^{q} \mathcal{F}_{j\ell} \eta \geq t \right)$$

$$= p \mathbb{P}\left[ \frac{\left| \sum_{\ell=1}^{n} \mathcal{F}_{j\ell} \eta \right|}{\left( \sum_{\ell=1}^{n} \left( \sum_{j'=1}^{q} \mathcal{F}_{j'\ell}(Z_{ij'}) \right)^2 \right)^{1/2} \sigma_0} \geq \frac{nt}{\left( \sum_{\ell=1}^{n} \left( \sum_{j'=1}^{q} \mathcal{F}_{j'\ell}(Z_{ij'}) \right)^2 \right)^{1/2} \sigma_0} \right]$$

$$\leq 2p \left( \frac{2}{\pi} \right)^{1/2} \exp \left[ \frac{-n^2t^2}{2\sigma_0^2 \sum_{\ell=1}^{n} \left( \sum_{j'=1}^{q} \mathcal{F}_{j'\ell}(Z_{ij'}) \right)^2} \right] \frac{\sigma_0}{nt} \left( \sum_{\ell=1}^{n} \left( \sum_{j'=1}^{q} \mathcal{F}_{j'\ell}(Z_{ij'}) \right)^2 \right).$$

Note that $\sum_{i=1}^{n}\left( \sum_{j'=1}^{q} \mathcal{F}_{j'\ell}(Z_{ij'}) \right)^2 \leq r^2C_0^2n$. By setting $t = 2C_0\{r^2 \log(pqm)/n\}^{1/2} \sigma_0$, we obtain

$$\mathbb{P}\left[ \|T_2\|_{\infty} \geq 2C_0\{r^2 \log(pqm)/n\}^{1/2} \sigma_0 \right] \leq p(pqm)^{-2}.$$

Therefore, we can bound $T_2$ as

$$\|T_2\|_{\infty} \leq 2C_0\sigma_0 \left\{ \frac{r^2 \log(pqm)}{n} \right\}^{1/2},$$

which holds with probability at least $1 - p(pqm)^{-2}$. Now for $T_3$, we have

$$\|T_3\|_{\infty} \leq \|\beta]\| \frac{1}{n} \left\| (U^T - F)^T \varepsilon \right\|_{\infty} \leq B \frac{1}{n} \max_{\ell, \ell'} \left\| U^T \varepsilon - \sum_{j=1}^{q} \mathcal{F}_{j\ell} \right\|_2 \|\varepsilon\|_2.$$

It follows from Theorem 2 that

$$\max_{\ell} \left\| U^T \varepsilon - \sum_{j=1}^{q} \mathcal{F}_{j\ell} \right\|_2 \leq 5 \lambda_{\max} r^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} \sqrt{n},$$

which implies that

$$\|T_3\|_{\infty} \leq 5B \lambda_{\max} r^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} \max_{\ell'} \frac{\|\varepsilon_{\ell'}\|_2}{\sqrt{n}}.$$
holds with probability at least \(1 - 20(pqm)^{-1}\). Now applying Lemma 13 again with \(t = \{16\log(pqm)/n\}^{1/2}\), we get

\[
\left| \frac{\|\varepsilon_t\|_2^2}{n\sigma_t^2} - 1 \right| \leq 4 \left( \frac{\log(pqm)}{n} \right)^{1/2}
\]

with probability at least \(1 - 2(pqm)^{-2}\), which implies

\[
\frac{\|\varepsilon_t\|_2^2}{n^{1/2}} \leq 4\sigma_t^2 \left( \frac{\log(pqm)/n}{1/2} + \sigma_t^2 \right) \leq \sqrt{5} \sigma_t
\]

with probability at least \(1 - 2(pqm)^{-2}\) when \(\log(pqm)/n \leq 1\). Therefore, we have

\[
\|T_3\|_\infty \leq 5\sqrt{5} \max_{t'} \sigma_t B_\lambda \max_{t'} r^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} = 5B \max_{t'} \sigma_t \lambda \left( \frac{10m}{\rho} \right)^{1/2},
\]

which holds with probability at least \(1 - 22(pqm)^{-1}\). Similar to \(T_2\), we can bound \(T_4\) as

\[
\|T_4\|_\infty = \frac{1}{n} \|F^t \varepsilon \|_\infty \leq \frac{\|\beta\|_1}{n} \|F^t \varepsilon \|_\infty \leq \frac{B}{n} \|F^t \varepsilon \|_\infty \leq \frac{B}{n} \max_{t, t'} \left( \sum_{j=1}^q F_{jt} \right)^T \varepsilon_{t'}.
\]

Therefore, we have

\[
\mathbb{P}(\|T_4\|_\infty \geq t) \leq p^2 \mathbb{P} \left\{ \frac{B}{n} \left( \sum_{j=1}^q F_{jt} \right)^T \varepsilon_{t'} \geq t \right\} = p^2 \mathbb{P} \left\{ \frac{B}{n} \left( \sum_{j\in J_t} F_{jt} \right)^T \varepsilon_{t'} \geq t \right\}
\]

\[
= p^2 \mathbb{P} \left[ \sqrt{\frac{\left( \sum_{j=1}^q F_{jt} \right)^T \varepsilon_{t'}}{B} \left( \sum_{j\in J_t} (f_j t(Z_{ij}))^2 \right)} \right] \geq \frac{nt}{B^2 \sum_{j\in J_t} (f_j t(Z_{ij}))^2} \leq 2p^2 \left( \frac{2}{n} \right)^{1/2} \exp \left[ -n^2 t^2 \frac{\|F^t \varepsilon_{t'}\|_\infty^2}{B^2 \sum_{j\in J_t} (f_j t(Z_{ij}))^2} \right] \frac{\|F^t \varepsilon_{t'}\|_\infty^2}{B^2 \sum_{j\in J_t} (f_j t(Z_{ij}))^2}.
\]

Since \(\sum_{j=1}^n (\sum_{j\in J_t} f_j t(Z_{ij}))^2 \leq r^2 C_0^2 n\), by setting \(t = \max_{t'} \sigma_t B C_0 \left( 8n^2 \log(pqm)/n \right)^{1/2}\), we obtain

\[
\|T_4\|_\infty \leq \max_{t'} \sigma_t B C_0 \left( 8n^2 \log(pqm)/n \right)^{1/2},
\]

which holds with probability at least \(1 - p^2(pqm)^{-4}\). Now we bound \(T_5\) as

\[
\|T_5\|_\infty \leq \max_{t, t'} \left\| (\hat{F}^T U^t - F^t) (U \hat{\Gamma} - F) \right\|_\infty \leq \frac{B}{n} \max_{t, t'} \left\| \sum_{j=1}^q F_{jt} - U \hat{\gamma}_t \right\|_2 \left\| \sum_{j=1}^q F_{jt} - U \hat{\gamma}_{t'} \right\|_2 \leq \frac{25 B \lambda_{\max}^2 (2m)^2}{\rho},
\]

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which holds with probability at least $1 - 20(pqm)^{-1}$. Finally, we bound $T_6$ as

$$
\|T_6\|_\infty \leq \frac{\|\beta\|_1}{n} \|F^\top (U \hat{T} - F)\|_\infty \leq \frac{B}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^q F_{j\ell} \right\|_2 \left\| U \hat{\gamma}_{\ell'} - \sum_{j=1}^q F_{j\ell'} \right\|_2 \leq \frac{B}{n} (rn)^{1/2} C_0 \max_{\ell'} \left\| U \hat{\gamma}_{\ell'} - \sum_{j=1}^q F_{j\ell'} \right\|_2 \leq 5B \max_{\ell} \lambda_{\max} \left( \frac{2m}{\rho} \right)^{1/2},
$$

which holds with probability at least $1 - 20(pqm)^{-1}$. Therefore, we have

$$
\left\| \frac{1}{n} \hat{X}^\top \eta - \frac{1}{n} \hat{X}^\top (\hat{X} - X) \beta \right\|_\infty \leq \|T_1\|_\infty + \|T_2\|_\infty + \|T_3\|_\infty + \|T_4\|_\infty + \|T_5\|_\infty + \|T_6\|_\infty \leq 5\sigma_0 \lambda_{\max} \left( \frac{2rm}{\rho} \right)^{1/2} + 2C_0 \sigma r \left\{ \log \frac{(pqm)}{n} \right\}^{1/2} + 5B \max_{\ell} \sigma_\ell (r)^{1/2} \lambda_{\max} \left( \frac{10rn}{\rho} \right)^{1/2} + \max_{\ell} \sigma_\ell BC_0 r \left\{ \frac{8 \log (pqm)}{n} \right\}^{1/2} + 25B \lambda_{\max}^2 \rho \frac{2m}{\rho} + 5B \max_{\ell} \lambda_{\max} \left( \frac{2m}{\rho} \right)^{1/2} \leq 7\sigma_0 \lambda_{\max} \left( \frac{2rm}{\rho} \right)^{1/2} + 8B \max_{\ell} \sigma_\ell \lambda_{\max} \left( \frac{10m}{n} \right)^{1/2} + 30rB \lambda_{\max} \left( \frac{2m}{\rho} \right)^{1/2} \leq \lambda_{\max} r \left( \frac{2m}{\rho} \right)^{1/2} \left( 7\sigma_0 + 8\sqrt{5} B \max_{\ell} \sigma_\ell + 30B \right),
$$

which holds with probability at least $1 - 86(pqm)^{-1}$ when $\lambda_{\max} (2m/\rho)^{1/2} \leq 1$, where we use the definition of $\lambda_{\max}$ in the third inequality. This, together with (11), implies

$$
\frac{1}{2n} \| \hat{X} (\hat{\beta} - \beta) \|_2^2 \leq \mu \| \beta \|_1 - \mu \| \hat{\beta} \|_1 + \frac{\mu}{2} \| \hat{\beta} - \beta \|_1.
$$

Adding $\mu \| \hat{\beta} - \beta \|_2^2 / 2$ to both sides yields

$$
\frac{1}{2n} \| \hat{X} (\hat{\beta} - \beta) \|_2^2 + \frac{\mu}{2} \| \hat{\beta} - \beta \|_1 \leq \mu (\| \beta \|_1 - \| \hat{\beta} \|_1 + \| \hat{\beta} - \beta \|_1) = \mu (\| \beta \|_1 - \| \hat{\beta} \|_1 - \| \hat{\beta} \|_1 + \| \hat{\beta} - \beta \|_1) = \mu (\| \beta \|_1 - \| \hat{\beta} \|_1 + \| \hat{\beta} - \beta \|_1),
$$

where the last inequality follows from the triangle inequality. This completes the proof. ■

Now we are ready to prove our main results. We note from Lemma 20 that with probability at least $1 - 86(pqm)^{-1}$, we have

$$
\frac{1}{2n} \| \hat{X} (\hat{\beta} - \beta) \|_2^2 \leq 2\mu \| \hat{\beta} - \beta \|_1 \leq 2\mu s^{1/2} \| \hat{\beta} - \beta \|_2
$$

and

$$
\frac{\mu}{2} \| \hat{\beta} - \beta \|_1 \leq 2\mu \| \hat{\beta} - \beta \|_1.
$$

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By Lemma 19, with probability at least $1 - 148(pqm)^{-1}$, we have
\[
\|\hat{\beta}_L - \beta_L\|_2 \leq \frac{2\|\hat{X}(\hat{\beta} - \beta)\|_2}{n^{1/2} \kappa}.
\]
Combining (12) and (13), we obtain
\[
\|\hat{X}(\hat{\beta} - \beta)\|_2 \leq \frac{64}{\kappa^2} \eta \mu
\]
and
\[
\|\hat{\beta} - \beta\|_1 \leq 4\|\hat{\beta}_L - \beta_L\|_1 \leq 4s^{1/2}\|\hat{\beta}_L - \beta_L\|_2 \leq \frac{64}{\kappa^2} s \mu,
\]
which hold with probability at least $1 - 234(pqm)^{-1}$.

Section D: Proof of Results in Section 4

Proof of Lemma 4

Proof Note that
\[
\begin{align*}
\tilde{\beta} &= \bar{\beta} + \hat{\Omega}\hat{D}^T(Y - X\hat{\beta})/n \\
&= \bar{\beta} + \hat{\Omega}\hat{D}^T(X(\beta - \bar{\beta}) + \eta)/n \\
&= \bar{\beta} + \hat{\Omega}\hat{D}^T(\hat{D}(\beta - \bar{\beta}) + (X - \hat{D})(\beta - \bar{\beta}) + \eta)/n \\
&= \beta + \hat{\Omega}\hat{D}^T\eta/n + \hat{\Omega}\hat{D}^T(X - \hat{D})(\beta - \bar{\beta})/n + (\hat{\Omega}\hat{\Sigma}_d - I)(\beta - \bar{\beta}).
\end{align*}
\]
Now decompose the second term on the rightmost-hand side of the above display as
\[
\hat{\Omega}\hat{D}^T\eta/n = \hat{\Omega}D^T\eta/n + \hat{\Omega}(\hat{D} - D)^T\eta/n
\]
\[
= \Omega D^T\eta/n + (\hat{\Omega} - \Omega)D^T\eta/n + \hat{\Omega}(\hat{D} - D)^T\eta/n,
\]
which completes the proof.

Proof of Theorem 5

Proof In this proof, we will temporarily assume $\|R_k\|_\infty = o_p(1), k = 1, 2, 3, 4$, which will be elaborated in the next section. First, we show $\omega_\ell$ is bounded away from zero for large $n$, that is, $\omega_\ell \geq c$ for some constant $c > 0$. This follows immediately since $\Omega_\ell\ell$ is lower bounded by a constant. Define
\[
T_{\ell,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\theta_i^T \Gamma^T \tilde{U}_i^T) \frac{\eta_i}{\omega_\ell},
\]
where $\tilde{U}_i$ is the $i$th row of $\tilde{U}$. Then, we have
\[
T_{\ell,n} - n^{1/2} \frac{\hat{\beta}_\ell - \beta_\ell}{\omega_\ell} = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} - \sum_{i=1}^{4} R_{\ell i} \right| \\
\leq \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{\sqrt{n} \omega_\ell} \right| + \sum_{i=1}^{4} \| R_{\ell i} \|_\infty \min_\ell \omega_\ell.
\]

As noted above, we will control the remainder terms $\| R_{k} \|_\infty$ in the next section and for now we assume they are all $o_p(1)$. Now we have
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| = \left| \frac{1}{n^{1/2} \omega_\ell} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right| = \left| \frac{1}{n^{1/2} \omega_\ell} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right| \\
= \left| \frac{1}{n^{1/2} \omega_\ell} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right|.
\]

where the first equality follows from the fact
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| \\
= \left| \frac{1}{n^{1/2} \omega_\ell} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i - \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i + \frac{1}{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right| \\
= \left| \frac{1}{n^{1/2} \omega_\ell} \left( \frac{1}{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right) \right|.
\]

Apply the Hölder’s inequality to get
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| \\
= \left| \frac{1}{n^{1/2} \omega_\ell} \left( \max_{1 \leq \ell} \sum_{j=1}^{q} F_{j \ell} \tilde{\gamma}_j - \frac{1}{n} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T 11^T \eta \right) \right| \\
\leq \left\{ m_{\Omega} \max_{1 \leq \ell} \sum_{j=1}^{q} \left| F_{j \ell} \right| \right\} \left\{ \sum_{j=1}^{q} \left| \tilde{\gamma}_j \right| \right\} \left\{ \sum_{i=1}^{n} \left| \theta_i^\ell \right| \right\} \left\{ \sum_{i=1}^{n} \left| \eta_i \right| \right\}
\]

where $\tilde{\gamma}_j$ is the $j$th row of $\tilde{\gamma}$. Then, we have
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| \\
= \left| \frac{1}{n^{1/2} \omega_\ell} \left( \max_{1 \leq \ell} \sum_{j=1}^{q} \left| F_{j \ell} \right| \right) \left\{ \sum_{j=1}^{q} \left| \tilde{\gamma}_j \right| \right\} \left\{ \sum_{i=1}^{n} \left| \theta_i^\ell \right| \right\} \left\{ \sum_{i=1}^{n} \left| \eta_i \right| \right\}
\]

Then, we can write
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| \\
\leq \left\{ m_{\Omega} \max_{1 \leq \ell} \sum_{j=1}^{q} \left| F_{j \ell} \right| \right\} \left\{ \sum_{j=1}^{q} \left| \tilde{\gamma}_j \right| \right\} \left\{ \sum_{i=1}^{n} \left| \theta_i^\ell \right| \right\} \left\{ \sum_{i=1}^{n} \left| \eta_i \right| \right\}
\]

Finally, we have
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^\ell \Gamma^\ell \tilde{U}_i^T \eta_i}{\omega_\ell} - \frac{\theta_i^\ell D^\ell \eta}{n^{1/2} \omega_\ell} \right| \\
\leq \left\{ m_{\Omega} \right\} \left\{ \sum_{i=1}^{n} \left| \eta_i \right| \right\} \left\{ \sum_{i=1}^{n} \left| \theta_i^\ell \right| \right\} \left\{ \sum_{i=1}^{n} \left| \eta_i \right| \right\}
\]

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It then follows from the Gaussian tail probability that
\[
P\left(\left| \frac{1}{n^{1/2}\sigma_0} \sum_{i=1}^{n} \eta_i \right| \geq t \right) \leq 2 \left( \frac{2}{\pi} \right)^{1/2} \exp\left(-\frac{t^2}{2}\right).
\]

Setting \( t = 2\{\log(n)\}^{1/2} \), we obtain with probability at least \( 1 - n^{-1} \)
\[
\left| \frac{1}{n} \sum_{i=1}^{n} \eta_i \right| \leq \sigma_0 \left\{ \frac{\log(n)}{n} \right\}^{1/2}.
\]

Similarly, by Lemma 7, we have
\[
\left| \frac{1}{n} \sum_{i=1}^{n} f_{j\ell}(Z_{ij}) \right| \leq \left\{ 4C_0^2 \log\left(pqm\right)/n \right\}^{1/2}
\]
with probability at least \( 1 - 2(pqm)^{-2} \). By the union bound argument, we have
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^T \Gamma \tilde{U}_i \eta_i \omega_{\ell} - \frac{\theta_0^T D^T \eta}{n^{1/2}\omega_{\ell}} \right| \leq \sigma_0 \{\log(n)\}^{1/2} \frac{m_{\theta} r \omega_{\ell}}{\omega_{\ell}} \left[ C_L m^{-d} + 2C_0 \{\log(pqm)/n\}^{1/2} \right]
\]
with probability at least \( 1 - 2(pqm)^{-1} - n^{-1} \). Since \( \omega_{\ell} \) is lower bounded by a constant when \( n \) is large, we conclude
\[
\left| \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^T \Gamma \tilde{U}_i \eta_i \omega_{\ell} - \theta_0^T D^T \eta \right| = o_p(1).
\]

It follows that
\[
\left| T_{\ell,n} - n^{1/2} \frac{\tilde{\beta}_\ell - \beta_\ell}{\omega_{\ell}} \right| = o_p(1).
\]

Therefore, \( T_{\ell,n} \) and \( \sqrt{n}(\tilde{\beta}_\ell - \beta_\ell)/\omega_{\ell} \) share the same weak limit. Now we show \( T_{\ell,n} \) converges in distribution to the standard normal. Note that
\[
T_{\ell,n} = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \theta_i^T \Gamma \tilde{U}_i \eta_i \omega_{\ell}
= \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{\theta_i^T \Gamma \tilde{U}_i \eta_i}{\omega_{\ell}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \theta_i^T \Gamma \mathbb{E}(\tilde{U}_i \tilde{U}_i^T) \Gamma \theta_i \right\}^{1/2} \sigma_0 \frac{\theta_0^T \mathbb{E}(\Gamma \tilde{U}_i \tilde{U}_i^T/n) \theta_0}{\sigma_0 \left\{ \theta_0^T \mathbb{E}(\Gamma \tilde{U}_i \tilde{U}_i^T/n) \theta_0 \right\}^{1/2}}
\]
\[
\times \sigma_0 \left\{ \theta_0^T \mathbb{E}(\Gamma \tilde{U}_i \tilde{U}_i^T/n) \theta_0 \right\}^{1/2}.
\]

It follows that
\[
P\left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\theta_i^T \Gamma \tilde{U}_i \eta_i}{\omega_{\ell}} \leq t \right] = \mathbb{E} \left[ \mathbb{P} \left\{ \frac{1}{n^{1/2}} \sum_{i=1}^{n} \frac{\theta_i^T \Gamma \tilde{U}_i \eta_i}{\omega_{\ell}} \leq t \right\} \right] = \Phi(t).
\]

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for all \( t \in \mathbb{R} \), where \( \Phi(t) \) denotes the CDF of the standard normal distribution. Finally, we note that

\[
\frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\theta_i^T \Gamma^T \mathbb{E}(\tilde{U}_i^T \tilde{U}_i) \Gamma \theta_i}{\sigma_0} \right\}^{1/2} = \left( \frac{n}{n-1} \right)^{1/2},
\]

where we use the fact that \( \mathbb{E}(U^T U/n) = \frac{n-1}{n} \mathbb{E}(\tilde{U}^T \tilde{U}/n) \). Then, by the triangular inequality, we have

\[
\left\| \mathbb{E} \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty} \leq \left\| \mathbb{E} \left( \frac{n-1}{n} \frac{F^T F}{n} - \frac{n-1}{n} \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty} + \left\| \frac{1}{n} \mathbb{E} \left( \frac{F^T F}{n} \right) \right\|_{\infty}
\]

\[
= \left\| \mathbb{E} \left( \frac{n-1}{n} \frac{F^T F}{n} - \frac{n-1}{n} \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty} + \left\| \frac{1}{n} \mathbb{E} \left( \frac{F^T F}{n} \right) \right\|_{\infty}
\]

\[
\leq \frac{n-1}{n} \left\| \mathbb{E} \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty} + \frac{n-1}{n} \left\| \mathbb{E} \left( \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty}
\]

It then follows from the triangular and Jensen’s inequalities that

\[
\left\| \mathbb{E} \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U \Gamma}{n} \right) \right\|_{\infty} \leq \frac{n-1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \left( \sum_{j=1}^{q} F_{j\ell} \right) \left( \sum_{j=1}^{q} F_{j\ell'} - \tilde{U}_\gamma e' \right) \right) \right\|_{2} + \frac{n-1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \left( \tilde{U}_\gamma e' \right)^T \left( \sum_{j=1}^{q} F_{j\ell} \right) \right) \right\|_{2} + \frac{1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \sum_{j=1}^{q} F_{j\ell} \right) \right\|_{2} \sqrt{n}
\]

\[
+ \frac{n-1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \left( \tilde{U}_\gamma e' \right)^T \left( \sum_{j=1}^{q} F_{j\ell} \right) \right) \right\|_{2} + \frac{1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \sum_{j=1}^{q} F_{j\ell} \right) \right\|_{2} \sqrt{n}
\]

\[
+ \frac{n-1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \left( \tilde{U}_\gamma e' \right)^T \left( \sum_{j=1}^{q} F_{j\ell} \right) \right) \right\|_{2} + \frac{1}{n} \max_{\ell, \ell'} \left\| \mathbb{E} \left( \sum_{j=1}^{q} F_{j\ell} \right) \right\|_{2} \sqrt{n}.
\]
Now we apply the Cauchy–Schwarz inequality to get

\[
\left\| E \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U}{n} \right) \right\|_\infty \leq \frac{n - 1}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^{q} F_{j\ell'} \right\|_{n^{1/2}} \left\| \sum_{j=1}^{q} F_{j\ell} - \tilde{U} \gamma_{\ell} \right\|_{n^{1/2}} + \frac{n - 1}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^{q} F_{j\ell} - \tilde{U} \gamma_{\ell} \right\|_{n^{1/2}} + \frac{1}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^{q} F_{j\ell} \right\|_{n^{1/2}} \left\| \sum_{j=1}^{q} F_{j\ell'} \right\|_{n^{1/2}} + \frac{1}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^{q} F_{j\ell} \right\|_{n^{1/2}} \left\| \sum_{j=1}^{q} F_{j\ell'} \right\|_{n^{1/2}} \leq \frac{n - 1}{n} rC_0 E \max_{\ell} \left\| \sum_{j=1}^{q} F_{j\ell} - \tilde{U} \gamma_{\ell} \right\|_{n^{1/2}} + \frac{n - 1}{n} \max_{\ell, \ell'} \left\{ \left( \left\| \sum_{j=1}^{q} F_{j\ell} \right\|_{n^{1/2}} + \left\| \sum_{j=1}^{q} F_{j\ell'} \right\|_{n^{1/2}} \right) \right\} + \frac{1}{n} \max_{\ell, \ell'} \left\| \sum_{j=1}^{q} F_{j\ell} \right\|_{n^{1/2}} \left\| \sum_{j=1}^{q} F_{j\ell'} \right\|_{n^{1/2}} \leq \frac{n - 1}{n} rC_0 E \max_{\ell} \sum_{j \in J_{\ell}} \left\| \sum_{j \in J_{\ell}} F_{j\ell} - \tilde{U} \gamma_{j\ell} \right\|_{n^{1/2}} + \frac{n - 1}{n} \max_{\ell} \left( \sum_{j \in J_{\ell}} \left\| \tilde{U} \gamma_{j\ell} - \sum_{j=1}^{q} F_{j\ell} \right\|_{n^{1/2}} + rC_0 \right) \max_{\ell} \left\| \sum_{j=1}^{q} F_{j\ell} - \tilde{U} \gamma_{\ell} \right\|_{n^{1/2}} + \frac{r^2 C_0^2}{n}.
\]

Finally, we obtain

\[
\left\| E \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U}{n} \right) \right\|_\infty \leq \frac{n - 1}{n} r^2 C_0 \sup_{z \in [a, b]} \left| f_{j\ell}(z) - \sum_{k=1}^{m_n} \tilde{\gamma}_{j\ell k} \phi_k(z) \right| + \frac{r^2 C_0^2}{n} + \frac{n - 1}{n} \left\{ r \sup_{z \in [a, b]} \left| f_{j\ell}(z) - \sum_{k=1}^{m_n} \tilde{\gamma}_{j\ell k} \phi_k(z) \right| + rC_0 \right\} \left\{ r \sup_{z \in [a, b]} \left| f_{j\ell}(z) - \sum_{k=1}^{m_n} \tilde{\gamma}_{j\ell k} \phi_k(z) \right| + rC_0 \right\},
\]

where we apply the Cauchy–Schwarz and triangular inequalities. Now by Lemma 8, we have

\[
\left\| E \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U}{n} \right) \right\|_\infty \leq \frac{n - 1}{n} r^2 C_0 C_L m_n - d + \frac{n - 1}{n} \left( rC_L m_n - d + rC_0 \right) rC_L m_n - d + \frac{r^2 C_0^2}{n} \leq 3 \frac{n - 1}{n} r^2 C_0 C_L m_n - d + \frac{r^2 C_0^2}{n}.
\]
The above inequality holds when \( C_L m_n^{-d} \leq C_0 \). It then follows that
\[
\left| \theta_T^T \mathbb{E} \left( \frac{F^T F}{n} \right) \theta_T - \theta_T^T \mathbb{E} \left( \frac{\Gamma^T U^T U \Gamma}{n} \right) \theta_T \right| \leq \| \theta_T \|_1^2 \left\| \mathbb{E} \left( \frac{F^T F}{n} - \frac{\Gamma^T U^T U}{n} \right) \right\|_{\infty}
\]
\[
\leq 3m_n^2 \left( \frac{n-1}{n} \right)^r C_0 C_L m_n^{-d} + m_n^2 \frac{r^2 C_0^2}{n},
\]
from which we know
\[
\sigma_0 \left\{ \theta_T^T \mathbb{E} \left( \frac{\Gamma^T U^T U \Gamma}{n} \right) \theta_T \right\}^{1/2} \rightarrow \omega_\ell.
\]
Next, we show, for a consistent estimator \( \hat{\omega}_\ell \),
\[
n^{1/2} (\tilde{\beta}_\ell - \beta_\ell) / \hat{\omega}_\ell \sim \mathcal{N}(0, 1).
\]
Since \( \omega_\ell \) is lower bounded by a constant when \( n \) is large, for a consistent estimator \( \hat{\omega}_\ell \), we know \( \hat{\omega}_\ell = \Omega_p(1) \). Therefore, we have
\[
\left| \frac{\omega_\ell}{\hat{\omega}_\ell} - 1 \right| = o_p(1).
\]
By Slutsky’s theorem, we have
\[
n^{1/2} (\tilde{\beta}_\ell - \beta_\ell) / \hat{\omega}_\ell = \frac{n^{1/2} (\tilde{\beta}_\ell - \beta_\ell) \omega_\ell}{\hat{\omega}_\ell} \sim \mathcal{N}(0, 1).
\]

**Proof of Lemma 6**

**Proof** Note that
\[
\hat{\omega}_\ell^2 = \hat{\sigma}_0^2 \frac{\Gamma^T U^T U \Gamma}{n} \hat{\theta}_T.
\]
We first prove \( \hat{\sigma}_0 \) is a consistent estimator of \( \sigma_0 \) by showing \( \hat{\sigma}_0^2 \rightarrow \sigma_0^2 \) in probability. Note that
\[
\left| \hat{\sigma}_0^2 - \sigma_0^2 \right| = \left| \frac{1}{n} \| Y - X \hat{\beta} \|_2^2 - \sigma_0^2 \right| = \left| \frac{1}{n} \| X \beta + \eta - X \hat{\beta} \|_2^2 - \sigma_0^2 \right|
\]
\[
= \left| \frac{1}{n} \| \eta \|_2^2 + \frac{1}{n} \| X \beta - X \hat{\beta} \|_2^2 + \frac{2}{n} \eta^T (X \beta - X \hat{\beta}) - \sigma_0^2 \right|
\]
\[
\leq \left| \frac{1}{n} \| \eta \|_2^2 - \sigma_0^2 \right| + \frac{\| X \beta - X \hat{\beta} \|_2^2}{n} + \frac{2 \| \eta \|_2 \| X \beta - X \hat{\beta} \|_2}{n^{1/2}}
\]
\[
\leq \left| \frac{1}{n} \| \eta \|_2^2 - \sigma_0^2 \right| + \frac{64 \kappa^2 \mu^2}{n^{1/2}} + \frac{8 \| \eta \|_2 \| X \beta - X \hat{\beta} \|_2}{n^{1/2}}
\]
\[
\leq \left| \frac{1}{n} \| \eta \|_2^2 - \sigma_0^2 \right| + \frac{64 \kappa^2 \mu^2}{n^{1/2}} + \frac{8 \| \eta \|_2 \| X \beta - X \hat{\beta} \|_2}{n^{1/2}}
\]
\[
\leq \left| \frac{1}{n} \| \eta \|_2^2 - \sigma_0^2 \right| + \frac{64 \kappa^2 \mu^2}{n^{1/2}} + \frac{8 \| \eta \|_2 \| X \beta - X \hat{\beta} \|_2}{n^{1/2}}.
\]
which holds with probability at least $1 - 234 (pqm)^{-1}$ by applying Theorem 3. From Theorem 3, we have
\[
\left| \frac{1}{n} \| \eta \|_2^2 - \sigma_0^2 \right| \leq 4 \sigma_0^2 \left( \frac{\log(pqm)}{n} \right)^{1/2}
\]
with probability at least $1 - 2(pqm)^{-2}$, which implies
\[
| \tilde{\sigma}_0^2 - \sigma_0^2 | \leq 4 \sigma_0^2 \left( \frac{\log(pqm)}{n} \right)^{1/2} + \frac{64}{\kappa^2} \mu \| \theta \|_2^2 + 2 \sqrt{\frac{5}{2}} \sigma_0 \kappa s^{1/2} \mu
\]
holds with probability at least $1 - 236 (pqm)^{-1}$ when $\log(pqm_n)/n \leq 1$. Then we consider
\[
\left| \frac{\hat{\theta}^T U^{T} U \Gamma \hat{\theta} - \hat{\theta}^T F \Gamma \hat{\theta} + \hat{\theta}^T U \Gamma \hat{\theta}}{n} \right|
\]
\[
\leq \left| \left( \frac{\hat{\theta}^T U^{T} U \Gamma \hat{\theta} - \hat{\theta}^T F \Gamma \hat{\theta}}{n} \right)_{1} \right| + \left| \left( \frac{\hat{\theta}^T U \Gamma \hat{\theta}}{n} - \frac{\hat{\theta}^T F \Gamma \hat{\theta}}{n} \right)_{\infty} \right|
\]
\[
=: T_1 + T_2 + T_3
\]
For $T_1$, we know that
\[
T_1 \leq \| \theta \|_2^2 \left( \frac{\hat{\theta}^T U^{T} U \Gamma \hat{\theta} - \frac{\hat{\theta}^T F \Gamma \hat{\theta}}{n}}{n} \right)_{\infty} \leq 35 m_{\Omega}^2 \lambda_{\text{max}} r C_0 \left( \frac{2 \lambda_{\max}}{\lambda} \right)^{1/2}
\]
with probability at least $1 - 60 (pqm)^{-1}$ when $\lambda_{\text{max}} \{2m/\rho\}^{1/2} \leq \sqrt{r} C_0$, where we use the definition of $\theta$ in the first inequality and Lemma 19 in the second inequality. For $T_2$, we have
\[
T_2 \leq \left| \frac{\hat{\theta}^T F \Gamma \hat{\theta} - \hat{\theta}^T F \Gamma \hat{\theta}}{n} \right| \leq \| \theta \|_1 \| \theta - \hat{\theta} \|_1 \left( \frac{\hat{\theta}^T F \Gamma \hat{\theta}}{n} \right)_{\infty} \leq m_{\Omega} r^2 C_0^2 \| \theta - \hat{\theta} \|_1,
\]
where we use the fact that
\[
\left| \left( \sum_{j=1}^q f_{j\ell}(Z_j) \right) \left( \sum_{j=1}^q f_{j\ell}(Z_j) \right) \right| \leq r^2 C_0^2.
\]
Applying Lemma 24, we have
\[
T_2 \leq 2 \lambda_{\max} m_{\Omega} s_{\Omega} r^2 C_0^2 (2m_{\Omega} v)^{1-b}
\]
with probability at least $1 - 62(pqm)^{-1}$, where $v = 36 m \lambda_{\max} r C_0 \{2 rm / \rho\}^{1/2}, 0 \leq b < 1$. For $T_3$, we have

$$T_3 = \left| \theta_1^T \left( \frac{F^T F}{n} - \frac{\Gamma^T \mathbf{E}(U^T U) \Gamma}{n} \right) \theta_1 \right|$$

$$\leq \| \theta_1 \|_1^2 \left\| \frac{F^T F}{n} - \frac{\Gamma^T \mathbf{E}(U^T U) \Gamma}{n} \right\|_{\infty}$$

$$\leq m_\Omega^2 \left( 3 \frac{n-1}{n} r^2 C_L C_0 m_n^{-d} + \frac{r^2 C_0^2}{n} \right),$$

which follows from Theorem 5. Therefore, we have

$$\left| \frac{\hat{\theta}_1^T U^T U \hat{\theta}_1 - \theta_1^T \Gamma^T \mathbf{E}(U^T U) \Gamma \theta_1}{n} \right|$$

$$\leq 35 m_\Omega \lambda_{\max} r C_0 \left\{ \frac{2 rm}{\rho} \right\}^{1/2} + 2 c_b m_\Omega s \Omega r^2 C_0^2 (2 m_\Omega v)^{1-b} + m_\Omega^2 \left( 3 \frac{n-1}{n} r^2 C_L C_0 m_n^{-d} + \frac{r^2 C_0^2}{n} \right).$$

Now we have

$$|\hat{\omega}_1^2 - \omega_1^2| = \left| \frac{\sigma_0^2 \hat{\theta}_1^T U^T U \hat{\theta}_1 - \sigma_0^2 \theta_1^T \mathbf{E}(U^T U \Gamma) \theta_1}{n} \right|$$

$$\leq \left| \frac{\sigma_0^2 \hat{\theta}_1^T U^T U \hat{\theta}_1 - \sigma_0^2 \theta_1^T \mathbf{E}(U^T U \Gamma) \theta_1}{n} \right| + \left| \frac{\sigma_0^2 \hat{\theta}_1^T U^T U \hat{\theta}_1 - \sigma_0^2 \theta_1^T \mathbf{E}(U^T U \Gamma) \theta_1}{n} \right|$$

$$\leq \left( \frac{4 \sigma_0^2 \left( \log(pqm) \right)}{n} \right)^{1/2} + \frac{64}{\kappa^2} s \mu^2 + \frac{2 \sqrt{5} \sigma_0}{\kappa^2} s^{1/2} \left| \frac{\hat{\theta}_1^T U^T U \hat{\theta}_1 - \theta_1^T \mathbf{E}(U^T U \Gamma) \theta_1}{n} \right|$$

$$\leq 35 m_\Omega \lambda_{\max} r C_0 \left\{ \frac{2 rm}{\rho} \right\}^{1/2} + 2 c_b m_\Omega s \Omega r^2 C_0^2 (2 m_\Omega v)^{1-b} + m_\Omega^2 \left( 3 \frac{n-1}{n} r^2 C_L C_0 m_n^{-d} + \frac{r^2 C_0^2}{n} \right).$$

We also have

$$\left| \frac{\hat{\theta}_1^T U^T U \hat{\theta}_1}{n} \right| \leq \frac{\hat{\theta}_1^T}{n} \left\| \frac{F^T U}{n} - \frac{\Gamma^T \mathbf{E}(U^T U) \Gamma}{n} \right\|_{\infty}$$

$$\leq m_\Omega^2 \left( \left\| \frac{F^T U}{n} - \frac{\Gamma^T \mathbf{E}(U^T U) \Gamma}{n} \right\|_{\infty} + \left\| \frac{F^T U}{n} \right\|_{\infty} \right)$$

$$\leq 35 m_\Omega^2 \lambda_{\max} r C_0 \left\{ \frac{2 rm}{\rho} \right\}^{1/2} + m_\Omega^2 r^2 C_0^2.$$
Therefore, we get

\[
\left| \hat{\omega}_i^2 - \omega_i^2 \right| \leq \left[ 4\sigma_0^2 \left\{ \frac{\log(pqm)}{n} \right\} \right]^{1/2} + 64\sigma_0^2 s\mu^2 + 2\sqrt{5}\sigma_0^2 s^{1/2}\mu
\]

\[
\cdot \left[ 35m_0^2 \lambda_{\text{max}} rC_0 \left( \frac{2rm}{\rho} \right) \right]^{1/2} + m_0^2 r^2 C_0^2
\]

\[
+ \sigma_0^2 \left[ 35m_0^2 \lambda_{\text{max}} rC_0 \left( \frac{2rm}{\rho} \right) \right]^{1/2} + 2\sigma_0^2 m_0^2 r^2 C_0^2 (2m_0^2) \right]^{1-b}
\]

\[
+ m_0^2 \left( \frac{3n-1}{n} r^2 C_0 m_n - d + r^2 C_0^2 \right)
\].

\[\blacksquare\]

**Section E: Control of Remainder Terms**

We derive specific conditions to make sure the remainder terms satisfy \( \|R_k\|_\infty = o_p(1), k = 1, 2, 3, 4 \). We first find the probability of the event \( \|\Omega \hat{\Sigma}_d - I\|_\infty = o_p(1) \).

**Lemma 21** Suppose Assumption 5 holds, then

\[
\|\Omega \hat{\Sigma}_d - I\|_\infty \leq m_0^2 \lambda_{\text{max}} rC_0 \left( \frac{2rm}{\rho} \right)^{1/2}
\]

holds with probability at least \( 1 - 62(pqm)^{-1} \), when \( \lambda_{\text{max}} \{2m/\rho\}^{1/2} \leq r^{1/2}C_0 \).

**Proof** Note that

\[
\|\Omega \hat{\Sigma}_d - I\|_\infty = \left\| \Omega \left\{ \frac{\hat{\Gamma}^T U^T U \hat{\Gamma}}{n} - \mathbb{E} \left( \frac{F^T F}{n} \right) \right\} \right\|_\infty
\]

\[
\leq \|\Omega\|_1 \left\| \frac{\hat{\Gamma}^T U^T U \hat{\Gamma}}{n} - \mathbb{E} \left( \frac{F^T F}{n} \right) \right\|_\infty
\]

\[
\leq m_0 \left\| \Sigma_f - \frac{F^T F}{n} + \frac{F^T F}{n} - \hat{X}^T \hat{X} \right\|_\infty
\]

\[
\leq m_0 \left( \left\| \Sigma_f - \frac{F^T F}{n} \right\|_\infty + \left\| \frac{F^T F}{n} - \hat{X}^T \hat{X} \right\|_\infty \right).
\]

From Lemma 18, we have

\[
\left\| \Sigma_f - \frac{F^T F}{n} \right\|_\infty \leq 4r^2 C_0^2 \left\{ \frac{\log(pqm)}{n} \right\}^{1/2}
\]

with probability at least \( 1 - 2p^2(pqm)^{-4} \). Also, from Lemma 19, we have

\[
\left\| \frac{F^T F}{n} - \frac{\hat{X}^T \hat{X}}{n} \right\|_\infty \leq 35\lambda_{\text{max}} rC_0 \left( \frac{2rm}{\rho} \right)
\]
with probability at least $1 - 60(pqm)^{-1}$, when $\lambda_{\max}(2m/\rho)^{1/2} \leq r^{1/2}C_0$. Therefore, we have

$$\|\Omega \hat{\Sigma}d - I\|_\infty \leq m_\Omega 36\lambda_{\max}rC_0 \left(\frac{2rm}{\rho}\right)^{1/2}$$

with probability at least $1 - 62(pqm)^{-1}$ when $\lambda_{\max}(2m/\rho)^{1/2} \leq r^{1/2}C_0$. Therefore, we have

By Lemma 21, $v$ can be chosen as $36m_\Omega \lambda_{\max}rC_0 \frac{\sqrt{2rm}}{\rho}$. The next result provides a probabilistic bound for $\|U^T \eta / n\|_\infty$.

**Lemma 22**

$$\mathbb{P}\left[\frac{\|U^T \eta\|_\infty}{n} \leq 4\left\{\log(pqm/n)\right\}^{1/2}\right] \geq 1 - (pqm)^{-1}.$$  

**Proof** Write $\|U^T \eta / n\|_\infty$ as $\|U^T \eta\|_\infty = \max_{j,k} |U^T_{jk}\eta|$, where $U_{jk}$ is the $k$th column of spline matrix $U_j$. Recall the Gaussian tail bound

$$\mathbb{P}(|X| \geq t) \leq 2\left(\frac{2}{\pi}\right)^{1/2} \exp\left(-\frac{t^2}{2}\right).$$

Now we have

$$\mathbb{P}\left(\frac{\|U^T \eta\|_\infty}{n} \geq t\right) \leq qm_n \mathbb{P}\left(\frac{|U^T_{jk}\eta|}{n} \geq t\right)$$

$$= qm_n \mathbb{P}\left[\frac{|U^T_{jk}\eta|}{\sigma_0 \left\{\sum_{i=1}^n \psi_k^2(Z_{ij})\right\}^{1/2}} \geq \frac{nt}{\sigma_0 \left\{\sum_{i=1}^n \psi_k^2(Z_{ij})\right\}^{1/2}}\right]$$

$$\leq 2qm_n \left(\frac{2}{\pi}\right)^{1/2} \exp\left\{-\frac{n^2t^2}{2\sigma_0^2 \sum_{i=1}^n \psi_k^2(Z_{ij})\{\sum_{i=1}^n \psi_k^2(Z_{ij})\}^{1/2}/nt\right\}.$$  

Since $|\psi_k(Z_{ij})| \leq 2$, setting $t = 4\sigma_0 \{\log(pqm/n)\}^{1/2}$, we obtain

$$\mathbb{P}\left[\frac{\|U^T \eta\|_\infty}{n} \geq 4\sigma_0 \{\log(pqm/n)\}^{1/2}\right] \leq qm(pqm)^{-2}.$$  

**Lemma 23 (Lemma A.1 in Gold et al., 2020)** Suppose (i) $\|\Omega\|_1$ is bounded above by a constant $m_\Omega < \infty$, and (ii) $\hat{\Omega}$ is an estimate of $\Omega$ with rows $\theta_\ell$ obtained as solutions to (5). Then

$$\|\hat{\Omega} - \Omega\|_\infty \leq 2m_\Omega v$$

holds with probability at least $1 - 62(pqm)^{-1}$.  

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Proof Under the conditions of this Lemma, we know that

\[ \| \Omega \hat{\Sigma}_d - I \|_\infty \leq v, \quad \| \hat{\Omega} \hat{\Sigma}_d - I \|_\infty \leq v. \]

Then, we have

\[
\Omega - \hat{\Omega} = \left\{ I - \hat{\Omega} \mathbb{E} \left( \frac{F^T F}{n} \right) \right\} \Omega = \left[ I + \hat{\Omega} \left\{ \hat{\Sigma}_d - \mathbb{E} \left( \frac{F^T F}{n} \right) \right\} - \hat{\Omega} \hat{\Sigma}_d \right] \Omega \\
= (I - \hat{\Omega} \hat{\Sigma}_d) \Omega - \hat{\Omega} \left\{ \mathbb{E} \left( \frac{F^T F}{n} \right) - \hat{\Sigma}_d \right\} \Omega = (I - \hat{\Omega} \hat{\Sigma}_d) \Omega - \hat{\Omega} (I - \hat{\Sigma}_d \Omega).
\]

By Hölder’s inequality, we have

\[
\| (I - \hat{\Omega} \hat{\Sigma}_d) \Omega \|_\infty \leq \| I - \hat{\Sigma}_d \Omega \|_\infty \| \hat{\Omega}^T \|_1 \leq \| I - \hat{\Sigma}_d \Omega \|_\infty \| \hat{\Omega}^T \|_1 \leq m_\Omega v
\]

and

\[
\| (I - \hat{\Omega} \hat{\Sigma}_d) \Omega \|_\infty \leq \| \Omega \|_1 \| I - \hat{\Omega} \hat{\Sigma}_d \|_\infty \leq m_\Omega v.
\]

Therefore, we have

\[ \| \Omega - \hat{\Omega} \|_\infty \leq 2m_\Omega v. \]

We have shown in Lemma 21 that

\[ \| \Omega \hat{\Sigma}_d - I \|_\infty \leq 36m_\Omega \lambda_{\max} r C_0 \left( \frac{2rm}{\rho} \right)^{1/2} \]

with probability at least \(1 - 62 (pqm)^{-1}\). Therefore, the proof is complete by choosing \(v = 36m_\Omega \lambda_{\max} r C_0 (2rm/\rho)^{1/2}\).

Lemma 24 (Lemma A.2 in Gold et al., 2020) Suppose, in addition to the conditions of Lemma 23, \(\Omega \in \mathcal{U}(m_\Omega, b, s_\Omega)\). Then,

\[ \| \hat{\theta}_\ell - \theta_\ell \|_1 \leq 2c_b (2m_\Omega v)^{1-b} s_\Omega \]

for each \(\ell \in \{1, \ldots, p\}\), where \(c_b = 1 + 2^{1-b} + 3^{1-b}\).

Proof See the proof of Theorem 6 in Cai et al. (2011).
Control of $R_1$

**Lemma 25** Suppose the conditions of Theorem 3 and Lemma 24 hold. If

$$n^{1/2}c_b(2m_Ωv)^{1-b}s_Ω \left[ 4\sqrt{5}\sigma_0rC_Lm_n^{-d} + 14C_0\sigma_0 \left( \frac{r^2 \log(pqm)}{n} \right)^{1/2} \right] = o(1),$$

then $\|R_1\|_\infty = o(1)$ with probability at least $1 - 5(pqm)^{-1}$.

**Proof** Note that

$$R_1 = (\hat{Ω} - Ω)D^Tη/n^{1/2} = (\hat{Ω} - Ω)(Γ^T\Gamma - F^t + F^t)η/n^{1/2}$$

$$\leq \|\hat{Ω}^T - Ω^T\|_1 \|Γ^T\Gamma - F^t\|_\infty \|η/n^{1/2}\|_\infty + \|F^t\|_\infty \|η/n^{1/2}\|_\infty$$

$$= n^{1/2} \|\hat{Ω}^T - Ω^T\|_1 \|Γ^T\Gamma - F^t\|_\infty \|η/n\|_\infty + \|F^t\|_\infty \|η\|_\infty.$$

As is shown in the proof of Theorem 3, $\|F^t\|_\infty \leq 2C_0\sigma_0 \{r^2 \log(pqm) + n\}^{1/2}$ holds with probability at least $1 - p(pqm)^{-2}$. Applying Lemma 17 and the $\chi^2$ concentration bound, we have

$$\|Γ^T\Gamma - F^t\|_\infty \leq \frac{1}{n^{1/2}} \max_\ell \left\| \sum_{j=1}^q F_{\ell j} \right\|_2 \max_\ell \frac{\|η\|_2}{n^{1/2}}$$

$$\leq 4\sqrt{5}\sigma_0rC_Lm_n^{-d} + 4r\sigma_0 \{5C_0^2 \log(pqm)/n\}^{1/2}$$

with probability at least $1 - 4r(pqm)^{-2}$. It follows that

$$R_1 \leq n^{1/2} \|\hat{Ω}^T - Ω^T\|_1 \|F^t\|_\infty$$

$$\leq n^{1/2}c_b(2m_Ωv)^{1-b}s_Ω \left[ 4\sqrt{5}\sigma_0rC_Lm_n^{-d} + 14C_0\sigma_0 \left( \frac{r^2 \log(pqm)}{n} \right)^{1/2} \right]$$

holds with probability at least $1 - 5(pqm)^{-1}$. \hfill \Box

Control of $R_2$

**Lemma 26** Under the same conditions of Lemma 25, if

$$m_Ω128r\lambda_{\max} \{\log(pqm)\}^{1/2}m_n^{3/2} \rho = o(1),$$

then $\|R_2\|_\infty = o(1)$ with probability at least $1 - 19(pqm)^{-1}$.

**Proof** Note that

$$\|R_2\|_\infty = \hat{Ω}(\hat{Γ}^T\hat{Γ} - Γ^TΓ)\eta/n^{1/2}$$

$$\leq n^{1/2} \|\hat{Ω}^T\|_1 \|\hat{Γ} - Γ\|_1 \|U^t\eta\|_\infty$$

$$\leq n^{1/2}m_Ω \max_\ell \|\hat{γ}_\ell - γ_\ell\|_1 \|U^t\eta\|_\infty.$$
Recall from the proof of Theorem 2
\[
\max_{\ell=1,\ldots,p} \| \hat{\gamma}_\ell - \gamma_\ell \|_1 \leq \max_{\ell} m^{1/2} \sum_{j=1}^q \| \hat{\gamma}_{j\ell} - \gamma_{j\ell} \|_2 \leq m^{1/2} \frac{32r \lambda_{\max}}{\rho} \frac{m}{\rho}
\]
holds with probability at least \( 1 - 18(pqm)^{-1} \). Then, by Lemma 22, we obtain
\[
\| U^T \eta/n \|_\infty \leq 4 \left\{ \frac{\log(pqm)}{n} \right\}^{1/2}
\]
with probability at least \( 1 - (pqm)^{-1} \). It follows that
\[
\| R_2 \|_\infty \leq m \Omega \frac{128r \lambda_{\max} \{\log(pqm)\}^{1/2} \frac{m_3/2}{\rho}}
\]
holds with with probability at least \( 1 - 19(pqm)^{-1} \).

\[ \square \]

**Control of \( R_3 \)**

**Lemma 27** Under the same conditions of Lemma 25, if
\[
n^{1/2} m \Omega \lambda_{\max} \left( \frac{2 rm}{\rho} \right)^{1/2} \left( 30rC_0 + 16 \max_{\ell} \sigma_\ell \right) \frac{64}{\kappa^2} \mu = o(1),
\]
then \( \| R_3 \|_\infty = o(1) \) with probability at least \( 1 - 277(pqm)^{-1} \).

**Proof** We first apply H"older’s inequality to get
\[
\| R_3 \|_\infty \leq n^{1/2} \| \hat{\Omega}^T \|_1 \| \hat{D}^T (X - \hat{D})/n \|_\infty \| \hat{\beta} - \beta \|_1.
\]
Note that
\[
\hat{D}^T (X - \hat{D})/n = \hat{D}^T (F + E - \hat{D})/n = \hat{D}^T (F - \hat{D})/n + \hat{D}^T E/n.
\]
For the first term on the right-hand side above, we have
\[
\| \hat{\Gamma}^T U^T (F - \hat{D})/n \|_\infty \leq \| F^T (F - \hat{D})/n \|_\infty + \| (\hat{\Gamma}^T U^T - F^T) (F - \hat{D})/n \|_\infty
\]
\[
\leq \max_{\ell, \ell'} \left\{ \frac{\sum_{j=1}^q F_{j\ell'} \hat{\gamma}_{j\ell}}{n^{1/2}} \right\}_2 \left\{ \frac{\sum_{j=1}^q F_{j\ell'} - U \hat{\gamma}_{j\ell}}{n^{1/2}} \right\}_2 \| \hat{\gamma}_\ell \|_2 + \| \hat{\gamma}_\ell \|_2 \leq \frac{r}{\rho} \frac{2m}{\rho} \leq \frac{25 \lambda_{\max}^2 \rho}{\rho} \leq 30 \lambda_{\max} \frac{r^{3/2}}{\rho} C_0 \left( \frac{2m}{\rho} \right)^{1/2},
\]

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which holds with probability at least $1 - 20(pqm)^{-1}$ when $\lambda_{\text{max}} r^{1/2} (2m/\rho)^{1/2} \leq C_0$. For the second term, we have

\[
\frac{1}{n} \left\| \hat{D}^T E \right\| = \left\| \hat{\Gamma}^T U^T F / n \right\| \leq \left\| \frac{1}{n} \sum_{j=1}^q F_j \right\| + \left\| \hat{\Gamma}^T U^T F / n \right\|
\]

\[
\leq 5 \sqrt{5} \lambda_{\text{max}} \max_{\ell} \sigma_{\ell} r^{1/2} \left( \frac{2m}{\rho} \right)^{1/2} + \left\| \hat{F}^T E / n \right\|
\]

\[
\leq 16 \max_{\ell} \sigma_{\ell} \lambda_{\text{max}} r^{1/2} \left( \frac{2m}{\rho} \right)^{1/2}
\]

with probability at least $1 - 23(pqm)^{-1}$, where we use the arguments in the proof of Theorem 3 when bounding $\| T_3 \|_\infty$ and $\| T_4 \|_\infty$. It follows that

\[
\left\| \hat{D}^T (X - \hat{D}) / n \right\| \leq \lambda_{\text{max}} \left( \frac{2rm}{\rho} \right)^{1/2} \left( 30C_0 + 16 \max_{\ell} \sigma_{\ell} \right)
\]

holds with probability at least $1 - 43(pqm)^{-1}$. Finally, we have

\[
\frac{1}{n} \left\| R_3 \right\|_\infty \leq n^{1/2} \| \hat{\Omega}^T \|_1 \left\| \hat{D}^T (D - \hat{D}) / n \right\|_\infty \| \hat{\beta} - \beta \|_1
\]

\[
= n^{1/2} m_\Omega \lambda_{\text{max}} \left( \frac{2rm}{\rho} \right)^{1/2} \left( 30C_0 + 16 \max_{\ell} \sigma_{\ell} \right) \frac{64}{\kappa^2} s \mu
\]

with probability at least $1 - 277(pqm)^{-1}$.

\section*{Control of $R_4$}

\textbf{Lemma 28} Under the same conditions of Lemma 25, if

\[
n^{1/2} 36 m_\Omega \lambda_{\text{max}} r C_0 \left( \frac{2rm}{\rho} \right)^{1/2} \frac{64}{\kappa^2} s \mu = o(1),
\]

then $\| R_4 \|_\infty = o(1)$ with probability at least $1 - 296(pqm)^{-1}$.

\textbf{Proof} By Hölder’s inequality, we have

\[
\| R_4 \|_\infty \leq n^{1/2} \| \hat{\Omega} \hat{\Sigma}_d - I \|_\infty \| \hat{\beta} - \beta \|_1.
\]

Applying Lemma 21, we obtain

\[
\| R_4 \|_\infty \leq n^{1/2} 36 m_\Omega \lambda_{\text{max}} r C_0 \left( \frac{2rm}{\rho} \right)^{1/2} \| \hat{\beta} - \beta \|_1.
\]
which holds with probability at least $1 - 62(pqm)^{-1}$ when \( \lambda_{\text{max}}(2m/\rho)^{1/2} \leq r^{1/2}C_0 \). Apply Theorem 3 to get

\[
\|R_4\|_\infty \leq n^{1/2}36m\lambda_{\text{max}}rC_0 \left( \frac{2rm}{\rho} \right)^{1/2} \frac{64}{\kappa^2}s\mu,
\]

which holds with probability at least $1 - 296(pqm)^{-1}$.

Section F: Additional Experiments

In this section, we present more experiments to demonstrate the finite sample performance of our proposed estimator when the dimensions of the treatment and instrumental variables are large.

Table 5: \( L_1 \) error of each method averaged over one hundred replications with standard deviation shown in parentheses for \( p = 100 \).

| Sample size | Linear | Nonlinear |
|-------------|--------|-----------|
|              | Our method | 2SLS-L | PLS | Our method | 2SLS-L | PLS |
| 100          | 1.15 (0.42) | 1.48 (0.62) | 0.75 (0.27) | 1.94 (0.98) | 1.55 (0.74) | 0.94 (0.31) |
| 200          | 0.62 (0.33) | 0.62 (0.32) | 0.69 (0.28) | 1.06 (0.53) | 1.87 (0.81) | 0.84 (0.32) |
| 400          | 0.47 (0.26) | 0.41 (0.21) | 0.82 (0.34) | 0.65 (0.29) | 2.55 (0.95) | 1.06 (0.28) |
| 600          | 0.38 (0.18) | 0.35 (0.16) | 1.20 (0.51) | 0.45 (0.26) | 2.95 (1.11) | 1.27 (0.26) |
| 800          | 0.29 (0.14) | 0.28 (0.13) | 1.36 (0.42) | 0.38 (0.17) | 3.22 (1.52) | 1.38 (0.29) |
| 1000         | 0.23 (0.10) | 0.23 (0.10) | 1.44 (0.38) | 0.30 (0.13) | 3.49 (1.52) | 1.43 (0.26) |
| 1200         | 0.23 (0.10) | 0.24 (0.10) | 1.68 (0.37) | 0.27 (0.15) | 3.44 (1.28) | 1.46 (0.21) |
| 1400         | 0.19 (0.08) | 0.20 (0.09) | 1.75 (0.38) | 0.24 (0.13) | 3.83 (1.75) | 1.55 (0.21) |

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Table 6: $L_1$ error of each method averaged over one hundred replications with standard deviation shown in parentheses for $p = 200$.

| Sample size | Linear | | | Nonlinear | | |
|------------|--------|--------|--------|--------|--------|--------|
|            | Our method | 2SLS-L | PLS | Our method | 2SLS-IL | PLS |
| 100        | 1.14 (0.52) | 0.52 (0.76) | 0.80 (0.23) | 2.53 (1.47) | 1.21 (0.50) | 1.02 (0.31) |
| 200        | 0.65 (0.28) | 0.74 (0.38) | 0.74 (0.38) | 0.99 (0.41) | 1.35 (0.80) | 0.84 (0.29) |
| 400        | 0.42 (0.20) | 0.42 (0.20) | 0.64 (0.23) | 0.62 (0.24) | 2.03 (1.02) | 0.93 (0.29) |
| 600        | 0.30 (0.16) | 0.28 (0.14) | 0.73 (0.31) | 0.43 (0.19) | 2.49 (1.16) | 1.09 (0.27) |
| 800        | 0.26 (0.11) | 0.25 (0.10) | 0.92 (0.35) | 0.35 (0.15) | 3.42 (1.75) | 1.25 (0.24) |
| 1000       | 0.21 (0.09) | 0.20 (0.08) | 1.14 (0.43) | 0.30 (0.15) | 3.42 (1.65) | 1.35 (0.22) |
| 1200       | 0.21 (0.10) | 0.21 (0.10) | 1.36 (0.47) | 0.29 (0.15) | 3.92 (2.32) | 1.39 (0.24) |
| 1400       | 0.18 (0.07) | 0.19 (0.08) | 1.58 (0.47) | 0.24 (0.12) | 4.12 (2.43) | 1.47 (0.21) |

Table 7: $L_1$ error of each method averaged over one hundred replications with standard deviation shown in the parentheses when $p = 400$.

| Sample size | Linear | | | Nonlinear | | |
|------------|--------|--------|--------|--------|--------|--------|
|            | Our method | 2SLS-L | PLS | Our method | 2SLS-IL | PLS |
| 100        | 1.23 (0.68) | 2.05 (0.86) | 0.91 (0.34) | 2.69 (1.24) | 1.08 (0.45) | 1.14 (0.36) |
| 200        | 0.74 (0.60) | 1.00 (0.50) | 0.57 (0.21) | 1.11 (0.64) | 1.11 (0.63) | 0.83 (0.29) |
| 400        | 0.39 (0.16) | 0.36 (0.15) | 0.51 (0.18) | 0.57 (0.26) | 1.62 (0.98) | 0.86 (0.28) |
| 600        | 0.35 (0.15) | 0.29 (0.11) | 0.55 (0.21) | 0.43 (0.17) | 1.75 (1.04) | 1.00 (0.22) |
| 800        | 0.27 (0.13) | 0.23 (0.11) | 0.56 (0.19) | 0.37 (0.18) | 2.81 (1.50) | 1.14 (0.24) |
| 1000       | 0.24 (0.11) | 0.23 (0.09) | 0.56 (0.19) | 0.31 (0.12) | 3.20 (2.11) | 1.19 (0.22) |
| 1200       | 0.20 (0.08) | 0.20 (0.08) | 0.84 (0.37) | 0.23 (0.09) | 3.47 (1.90) | 1.29 (0.22) |
| 1400       | 0.19 (0.07) | 0.18 (0.08) | 0.91 (0.41) | 0.22 (0.10) | 4.12 (2.98) | 1.36 (0.20) |

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