Equal opportunities lead to maximum wealth inequality

Ben-Hur Francisco Cardoso
Departamento de Economia e Relações Internacionais, Universidade Federal de Santa Catarina, Florianópolis, SC, Brazil

Sebastián Goñi-Funez
Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil

José Roberto Iglesias
Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil and Instituto Nacional de Ciência e Tecnologia de Sistemas Complexos, INCT-SC, CBPF, Rio de Janeiro, RJ, Brazil

(Dated: January 21, 2022)

If wealthier people have advantages in having higher returns than poor, inequality will unequivocally increase, but is equal opportunity enough to prevent it? According to several models in economics and econophysics, no. They all display wealth concentration as a peculiar feature of its dynamics, even though no individual can have repeating gains advantage. Here, generalizing these particular models, we present a rigorous analytical demonstration, using master equation formalism, that any fair market that gives each agent the same expected return conducts the system to maximum inequality.

The remarkable growth of wealth concentration during the XXTh Century in almost all countries is often attributed to some rich-biased advantage due to market failures, producing a self-feeding cascade. Although this type of mechanism promotes wealth concentration indeed, we could be tempted to think that it would not happen in a hypothetical efficient or fair market, where no individual has repeating advantages.

Let us start by reviewing some models which show us that such reasoning may be not correct. Consider a system of $N$ agents, where $x_{i,t} \geq 0$ is the wealth of agent $i$ at time $t$. Since agents face risk, the future market opportunities are not predictable and the realized growth rate of individual wealth of a given agent in the transition $t \rightarrow t+1$ can be greater or lower than the average. However, an efficient market prevents the arbitrage opportunities, i.e., the expected growth rate of individual wealth is the same for all agents. Therefore, there exists a positive constant $\alpha_t$ such that

$$E_t[x_{i,t+1}] = \alpha_t x_{i,t} \tag{1}$$

for all agent $i$. In other words, no agent is systemically favored. That is why we can call this market fair, with equal opportunities. However, all studied models satisfying Eq. (1)–random growth models, kinetic exchange models and unbiased kinetic exchange models – converge to the condensed state, where one agent concentrates all the disposable wealth.

Within the random growth model framework, the individual wealth dynamics follows

$$x_{i,t+1} = \alpha_{i,t} x_{i,t}, \quad E_t[\alpha_{i,t}] = \alpha_t \tag{2}$$

that is, at time $t$ each agent $i$ extracts a random return $\alpha_{i,t} > 0$ from a common probability density function $g_t(\alpha)$ with mean $\alpha_t$.

In the unbiased kinetic exchange models, two agents are sequential and randomly chosen to exchange wealth. So, let assume a wealth exchange between random selected agents $i$ and $j$

$$x_{i,t+1} = x_{i,t} + \Delta_{i,t} \text{ and } x_{j,t+1} = x_{j,t} + \Delta_{j,t}, \tag{3}$$

where $\Delta_{i,t}$ is the stochastic gain of agent $i$ such that $\Delta_{i,t} + \Delta_{j,t} = 0$. In mathematical terms, this zero-sum exchange process describe an efficient market if $E_t[\Delta_{i,t}] = E_t[\Delta_{j,t}] = 0$.

The best-known unbiased kinetic exchange model is the Yard-Sale:

$$\Delta_{i,t} = \eta_t \lambda_t \min(x_{i,t}, x_{j,t}), \quad \eta_t \in \{-1, 1\}, \quad E_t[\eta_t] = 0, \tag{4}$$

where $0 \leq \lambda_t \leq 1$ can be a constant or a random number. This rule was generalised in two ways. First, restricting exchanges between agents linked by a static or dynamic network. Second, introducing agent specific parameters $\lambda_i$ such that

$$\Delta_{i,t} = \eta_t \min\{\lambda_i x_{i,t}, \lambda_j x_{j,t}\}, \quad \eta_t \in \{-1, 1\}, \quad E_t[\eta_t] = 0, \tag{5}$$

where $\lambda_i$ is the fraction of the wealth that agent $i$ will risk in the exchange.

Also, it was proposed an exchange rule such that

$$\Delta_{i,t} = \epsilon_t \lambda_i x_{j,t} - (1 - \epsilon_t)\lambda_t x_{i,t}, \quad \epsilon_t \in \{0, 1\}, \quad E_t[\epsilon_t] = \frac{x_{i,t}}{x_{i,t} + x_{j,t}}, \tag{6}$$

In these models, it is generally assumed that $\alpha_t > 1$, that is, the average wealth grows with time.
where $0 \leq \lambda_t \leq 1$ can be a random or constant number. Finally, a similar model was proposed by one of the authors\[11\], where

$$\Delta_{i,t} = \eta_{i,t} \frac{x_{i,t} x_{j,t}}{x_{i,t} + x_{j,t}}, \quad \eta_{i,t} \in \{-1, 1\}, \quad \mathbb{E}[\eta_{i,t}] = 0. \quad (7)$$

As stated above, all these models (Eqs. 2, 4, 5, 6 and 7) converge to the condensed state. In a previous paper\[10\], working with a master equation in the thermodynamic limit ($N \to \infty$), we proved that condensation is the asymptotic state of any model with agents only characterized by their wealth, whose dynamics are given by binary exchanges and without wealth growth (the unbiased kinetic exchange model). Here, we extend the result by relaxing these three assumption, proving that any efficient market satisfying Eq. 1 converges to maximum inequality.

**The master equation**

Let be a system of agents where each one is in a state from the set of all possible ones represented by $\Omega$. For notation simplicity, if an agent is in the state $\Gamma = (\Gamma^0, \Gamma^1, \cdots) \in \Omega$, $\Gamma^0 = x$ is the agent’s wealth. The system can be represented by the probability density function $p_t(\Gamma)$, where $p_t(\Gamma) d\Gamma$ is the fraction of agents within $d\Gamma$ of $\Gamma$ at time $t$. This probability density function must obey the following properties:

(i) Negative wealth is not allowed, then, for all $t$

$$x < 0 \Rightarrow p_t(\Gamma) = 0. \quad (8)$$

(ii) For all $t$, the probability density function is normalized

$$\int_\Omega d\Gamma p_t(\Gamma) = 1. \quad (9)$$

(iii) For all $t$, we define the average wealth as

$$\mu_t = \int_\Omega d\Gamma x p_t(\Gamma). \quad (10)$$

The dynamics of $p_t(\Gamma)$ can be represented by the Master Equation\[13\][19]

$$p_{t+1}(\Gamma) = \int_\Omega d\Gamma' p_t(\Gamma') \omega_t(\Gamma' \to \Gamma, p_t) \quad (11)$$

where $\omega_t(\Gamma' \to \Gamma, p_t) d\Gamma$ is the probability of an agent to be within $d\Gamma'$ of $\Gamma$ at time $t + 1$, conditioned on being in the state $\Gamma'$ at time $t$\[9\]. For simplicity, we use the notation $\omega_t(\Gamma' \to \Gamma) \equiv \omega_t(\Gamma' \to \Gamma, p_t)$.

The transition rate $\omega_t(\Gamma' \to \Gamma)$ must obey the following properties:

(i) Negative wealth is not allowed, then, for all $t$

$$x < 0 \text{ or } x' < 0 \Rightarrow \omega_t(\Gamma' \to \Gamma) = 0. \quad (12)$$

(ii) Agents live forever; that is, one agent in a given state $\Gamma' \in \Omega$ at $t$ must be in another state in $\Omega$ at $t + 1$. This requires that the integral over all the transitions must be 1, so we have for all $t$ that

$$\int_\Omega d\Gamma \omega_t(\Gamma' \to \Gamma) = 1, \quad \forall \Gamma' \in \Omega. \quad (13)$$

Also, this condition guarantees the normality of $p_t$ for all $t$ (Eq. 9).

(iii) As the market is fair, the expected individual growth rate of wealth is equal for all agents, that is, there is a positive constant $\alpha_t$ for all $t$ such that

$$\int_\Omega d\Gamma x \omega_t(\Gamma' \to \Gamma) = \alpha_t x', \quad \forall \Gamma' \in \Omega. \quad (14)$$

(iv) All agents face risk, that is, even if the random growth rate has the same expected value $\alpha_t$ for all, this randomness can make the individual growth in a given instance less or greater than that average. So, for all $t$ and $\Gamma' \in \Omega$ such that $x' > 0$

$$\int_\Omega d\Gamma |x - \alpha_t x'| \omega_t(\Gamma' \to \Gamma) > 0. \quad (15)$$

This rule cannot be valid for agents with zero wealth ($x' = 0$), since it will contradict Eq. 13

Using Eqs. 11, 14 and 15, it is ease to verify that

$$\mu_{t+1} = \alpha_t \mu_t. \quad (16)$$

That state transition rate can be driven by an autonomous change (like random growth models) or by interactions among agents (like unbiased kinetic exchange models or any kind of exchange model among $m > 2$ agents). Below we describe two examples.

**Example 1.** The agent’s state in the class of random growths models (Eq. 2) is restricted to wealth $\Gamma = \{x\}$. So, in terms of the master equation formulation, their dynamics can be written as

$$\omega_t(\Gamma' \to \Gamma) = \int_0^\infty d\alpha \ g_t(\alpha) \delta(\alpha x' - x), \quad (17)$$

where $g_t(\alpha)$ is the probability density function of random returns at time $t$.

**Example 2.** The agent’s state in the almost all unbiased kinetic exchange models (Eq. 3) is restricted to wealth
and a fixed risk propensity $\Gamma = \{x, \lambda\}$. For example, the
dynamics of Eq. 4 can be written as
\[
\omega_t(\Gamma' \rightarrow \Gamma) = \frac{1}{2} \delta(x' - \lambda) \int \Omega d\Gamma'' p_t(\Gamma'') \times \\
\left[ \frac{\delta(x - x' + \lambda' x'') + \delta(x - x' - \lambda' x'')}{\mu_t} \right]. \quad (18)
\]

In addition to the other models reviewed in the Introduction, the procedure made in Examples 1 and 2 can be
easily extended to include higher order interactions, i.e.,
with exchanges among $m > 2$ agents.

The ever-increasing inequality

To measure inequality, we use the Gini index \[20\]:
\[
G_t = \frac{1}{2} \int \Omega^2 d\Gamma_1 \frac{|x - x_1|}{\mu_t} p_t(\Gamma) p_t(\Gamma_1). \quad (19)
\]

The Gini index varies between 0, which corresponds to perfect equality (everyone has the same wealth) and 1,
that corresponds to maximum inequality, associated with the marginal probability density function of wealth \[16, 21\]
\[
\delta(x) + \lim_{N \rightarrow \infty} \frac{\delta(x - \mu_t N)}{N}. \quad (20)
\]

This means that all wealth is in hands of an infinitesimal part of the system, with measure zero. Then, this
probability density function is such that $p_t(\Gamma) = 0$ for all $x > 0$ \[16, 21, 23\], also satisfying Eqs. 9 and 10.

Now, we can enunciate and proof the following proposition: In an efficient market, $G_{t+1} \geq G_t$ for all $t$ and
\[
\lim_{t \rightarrow \infty} G_t = 1.
\]

Proof: Given the definition of Gini index (Eq. 19), and the master equation (Eq. 11), we have that
\[
G_{t+1} = \frac{1}{2} \int \Omega^2 d\Gamma_1 \frac{|x - x_1|}{\mu_{t+1}} p_{t+1}(\Gamma) p_{t+1}(\Gamma_1) = \frac{1}{2} \int \Omega^2 d\Gamma' d\Gamma'_{t+1} p_t(\Gamma) p_{t+1}(\Gamma_1) \times \int \Omega^2 d\Gamma_1 \frac{|x - x_1|}{\mu_{t+1}} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1). \quad (21)
\]

Now, since the integral of absolute value is always equal or greater than the absolute value of the integral \[24\], the
Eqs. 13, 14, and 16 implies
\[
\int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x - x_1|}{\mu_{t+1}} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1) \geq \frac{\alpha_t |x' - x'_1|}{\mu_{t+1}} = \frac{|x' - x'_1|}{\mu_t}. \quad (22)
\]

So, given the Eqs. 21, 22, and 19 we find that
\[
G_{t+1} = \frac{1}{2} \int \Omega^2 d\Gamma' d\Gamma'_{t+1} p_t(\Gamma) p_{t+1}(\Gamma_1) \times \int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x - x_1|}{\mu_{t+1}} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1) \geq \frac{1}{2} \int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x' - x'_1|}{\mu_t} p_t(\Gamma) p_{t+1}(\Gamma_1) \rightarrow G_{t+1} \geq G_t \quad (23)
\]

Since $G_t \leq 1$ for all $t$, we immediately find that $G_t = 1 \Rightarrow G_{t+1} = G_t$. Now, let us proof that reciprocate:
$G_{t+1} = G_t \Rightarrow G_t = 1$. From Eqs. 19 and 21 $G_{t+1} - G_t = 0$ means that
\[
0 = \frac{1}{2} \int \Omega^2 d\Gamma' d\Gamma'_{t+1} p_t(\Gamma) p_{t+1}(\Gamma_1) \left[ - \frac{|x' - x'_1|}{\mu_t} + \int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x' - x'_1|}{\mu_t} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1) \right]. \quad (24)
\]

By the Eq. 22 the term in square brackets of Eq. 23 is always non-negative, so the equality only holds if
\[
\int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x' - x'_1|}{\mu_t} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1) = \frac{|x' - x'_1|}{\mu_t} \quad (25)
\]

for all $\Gamma', \Gamma_1' \in \Omega$ such that $p_t(\Gamma') > 0$ and $p_{t+1}(\Gamma_1') > 0$. By Eq. 14 the particular case of $x = 0$ trivially satisfies the Eq. 25. Furthermore, we must have $p_t(\Gamma') = 0$ for all $x' > 0$. Otherwise, there would be at least one $x' > 0$ such that $p_t(\Gamma') > 0$; in that case, for $x_1 = x'$, the left side of Eq. 25 becomes
\[
\int \Omega^2 d\Gamma' d\Gamma'_{t+1} \frac{|x - x_1|}{\mu_t} \omega_t(\Gamma' \rightarrow \Gamma) \omega_t(\Gamma_1' \rightarrow \Gamma_1) \geq \int \Omega d\Gamma |x - \alpha_t x'| \omega_t(\Gamma' \rightarrow \Gamma) > 0,
\]

by Eq. 16 contradicting the right side of Eq. 25 that is zero when $x' = x'_1$. So, we conclude that
\[
G_{t+1} - G_t = 0 \Rightarrow \left[ p_t(\Gamma) = 0 \forall x > 0 \right] \Rightarrow G_t = 1.
\]

Summarizing, $G_t = 1 \Leftrightarrow G_{t+1} = G_t$ and, complementary, $G_t < 1 \Leftrightarrow G_{t+1} > G_t$. Then, $\sup_{t \in \mathbb{N}} \{G_t\} = 1$. By the well known monotone convergence theorem, we find
\[
\lim_{t \rightarrow \infty} G_t = \sup_{t \in \mathbb{N}} \{G_t\} = 1, \quad (26)
\]

that is, we asymptotically obtain condensation, Q.E.D.

Final comments

Previous numerical \[4, 13\] and analytical \[6, 11, 16, 23\] results show that efficient markets lead to maximum
inequality, or condensation, in specific model cases. However, our contribution is the first analytical demonstration that such fate occurs for all efficient markets, regardless of their particularities.

We might think that a system without regulatory policies is fair since no individual has systematic or a priori advantages [1][8]. However, we have demonstrated here in full generality that such apparent “fairness” does not prevent inequality from increasing up to a maximum.

Thus, what are the prepositions’ implications here demonstrated in the real world? The worldwide data about economies show a clear tendency toward increasing inequalities after relaxing taxation policies [20]. We can cite the US case, where the fraction of national wealth in the hands of the top 1% risen from 22% in 1980 to 37% in 2015 [22]. Given that a “fair” market cannot impede inequality from increasing, some mechanism to favor the poorest is necessary, such as taxation and redistribution of wealth, to guarantee less unequal societies.

---

1. Lucas Chancel. Ten facts about inequality in advanced economies. WID. World Working Paper. (2019/15), 2019.
2. Thomas A DiPrete and Gregory M Eirich. Cumulative advantage as a mechanism for inequality: A review of theoretical and empirical developments. Annu. Rev. Sociol., 32:271–297, 2006.
3. Cristian F Moukarzel. Multiplicative asset exchange with arbitrary return distributions. Journal of Statistical Mechanics: Theory and Experiment, 2011(08):P08023, 2011.
4. Joseph E Fargione, Clarence Lehman, and Stephen Polasky. Entrepreneurs, chance, and the deterministic concentration of wealth. PloS one. 6(7):e20728, 2011.
5. Yuri Biondi and Simone Rigoli. Inequality, mobility and the financial accumulation process: a computational economic analysis. Journal of Economic Interaction and Coordination, 14(1):93–119, 2019.
6. Yuri Biondi and Stefano Olla. Financial accumulation implies ever-increasing wealth inequality. Journal of Economic Interaction and Coordination, pages 1–9, 2020.
7. Oren S Klass, Ofer Bilham, Moshe Levy, Ofer Malcai, and Sorin Solomon. The forbes 400, the pareto power-law and efficient markets. The European Physical Journal B, 55(2):143–147, 2007.
8. Moshe Levy and Haum Levy. Investment talent and the pareto wealth distribution: Theoretical and experimental analysis. Review of Economics and Statistics, 85(3):709–725, 2003.
9. Brian Hayes. Computing science: Follow the money. American Scientist, 90(5):400–405, 2002.
10. J R Iglesias and R M C De Almeida. Entropy and equilibrium state of free market models. The European Physical Journal B. 85(3):85, 2012.
11. Nicolas Bouleau and Christophe Chorro. The impact of randomness on the distribution of wealth: Some economic aspects of the wright–fisher diffusion process. Physica A: Statistical Mechanics and its Applications, 479:379–395, 2017.
12. Ben-Hur Francisco Cardoso, Sebastián Gonçalves, and José Roberto Iglesias. Wealth distribution models with regulations: Dynamics and equilibria. Physica A: Statistical Mechanics and its Applications, 551:124201, 2020.
13. S Risau Gusman, M F Laguna, and J R Iglesias. Wealth distribution in a network with correlations between links and success. In Econophysics of Wealth Distributions, pages 149–158. Springer, 2005.
14. R Bustos-Guajardo and Cristian F Moukarzel. Yard-sale exchange on networks: wealth sharing and wealth appropriation. Journal of Statistical Mechanics: Theory and Experiment, 2012(12):P12009, 2012.
15. M F Laguna, S Risau Gusman, and J R Iglesias. Economic exchanges in a stratified society: End of the middle class? Physica A: Statistical Mechanics and its Applications, 356(1):107–113, 2005.
16. Ben-Hur Francisco Cardoso, José Roberto Iglesias, and Sebastián Gonçalves. Wealth concentration in systems with unbiased binary exchanges. Physica A: Statistical Mechanics and its Applications, page 126123, 2021.
17. GM Caon, S Gonçalves, and J.R. Iglesias. The unfair consequences of equal opportunities: Comparing exchange models of wealth distribution. The European Physical Journal Special Topics, 143(1):69–74, 2007.
18. Guy Katriel. Directed random market: the equilibrium distribution. Acta Applicandae Mathematicae, 139(1):95–103, 2015.
19. LP Pitaevskii and EM Lifshitz. Physical kinetics, volume 10. Butterworth-Heinemann, 2012.
20. Amartya Sen, Master Amartya Sen, Sen Amartya, James E Foster, James E Foster, et al. On economic inequality. Oxford University Press, 1997.
21. Bruce M Boghosian, Adrian Devitt-Lee, Merek Johnson, Jie Li, Jeremy A Marcq, and Hongyan Wang. Oligarchy as a phase transition: The effect of wealth-attained advantage in a fokker–planck description of asset exchange. Physica A: Statistical Mechanics and its Applications, 476:15–37, 2017.
22. Bruce M Boghosian. Kinetics of wealth and the pareto law. Physical Review E, 89(4):042804, 2014.
23. Bruce M Boghosian, Merek Johnson, and Jeremy A Marcq. An h theorem for boltzmann’s equation for the yard-sale model of asset exchange. Journal of Statistical Physics, 161(6):1339–1350, 2015.
24. Frank Hansen and Gert K Pedersen. Jensen’s operator inequality. Bulletin of the London Mathematical Society, 35(4):553–564, 2003.
25. Cristian Fernando Moukarzel, Sebastián Gonçalves, José Roberto Iglesias, Manuel Rodriguez-Achach, and Rodrigo Huerta-Quintanilla. Wealth condensation in a multiplicative random asset exchange model. The European Physical Journal Special Topics, 143(1):75–79, 2007.
26. Thomas Piketty, Emmanuel Saez, and Gabriel Zucman. Distributional national accounts: methods and estimates for the united states. The Quarterly Journal of Economics, 133(2):553–609, 2018.