DOUBLING CONSTRUCTION FOR $O(m) \times O(n)$ INVARIANT SOLUTIONS TO THE ALLEN-CAHN EQUATION

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Abstract. We construct new families of two-ended $O(m) \times O(n)$-invariant solutions to the Allen-Cahn equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^{N+1}$, with $N \geq 7$, whose zero level sets diverge logarithmically from the Lawson cone at infinity. The construction is based on a careful study of the Jacobi-Toya system on a given $O(m) \times O(n)$-invariant manifold, which is asymptotic to the Lawson cone at infinity.

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1. Introduction

In this work we study existence and asymptotic behaviour of bounded, sign-changing solutions to the Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}^{N+1}. \quad (1.1)$$

Equation (1.1) was introduced in [3] to model the allocation of binary mixtures and it is the prototype equation for the continuous modelling of phase transition phenomena.

In the one dimensional case, the Allen-Cahn equation becomes the ordinary differential equation (ODE)

$$v''(t) + v(t) - v^3(t) = 0 \quad \text{in} \quad \mathbb{R}. \quad (1.2)$$

With the boundary conditions $v(\pm \infty) = \pm 1$, the equation (1.2) has an explicit solution given by

$$v_*(t) = \tanh \left( \frac{t}{\sqrt{2}} \right) \quad \text{for} \quad t \in \mathbb{R} \quad (1.3)$$

and up to translations, this solution is unique. Besides, $v_*$ is strictly monotone increasing, i.e. $v'_*(t) > 0$ for every $t \in \mathbb{R}$. In particular, $\{v_* = 0\} = \{0\}$.

Assume $N \geq 1$ and fix $\vec{a} \in \mathbb{R}^{N+1}$, a unit vector and $\xi_0 \in \mathbb{R}^{N+1}$. The function

$$u(\xi) := v_*(t), \quad t = \vec{a} \cdot (\xi - \xi_0) \quad \text{for} \quad \xi \in \mathbb{R}^{N+1} \quad (1.4)$$

is a bounded and sign-changing solution to (1.1) which is monotone in the direction of $\vec{a}$ and whose nodal set is the hyperplane with equation $\vec{a} \cdot (\xi - \xi_0) = 0$. We remark that, up to a translation and rotations of the axis, these solutions depend only on one variable and in this sense they are trivial.

In 1978 (see [17]), E. De Giorgi conjectured that if $1 \leq N \leq 7$, then for any solution to (1.1) which is monotone in one direction, the level sets $\{u = c\}$ must be parallel hyperplanes. This is equivalent to saying that for some unit vector $\vec{a}$ and some point $\xi_0$, the solution $u$ satisfies (1.4).

De Giorgi’s conjecture shows also evidence of the strong connection between the study of bounded solutions to the Allen-Cahn equation and the theory of minimal hypersurfaces, see for instance [37].

In [6, 24], De Giorgi’s conjecture was established in dimensions $N = 2, 3$. In [39], it was proven true in dimensions $4 \leq N \leq 8$, under the additional assumption

$$\lim_{\xi_{N+1} \rightarrow \pm \infty} u(\xi', \xi_{N+1}) = \pm 1.$$ 

Further evidence of the connection between solutions to (1.1) and the theory of minimal hypersurfaces is the Bernstein Conjecture, concerning rigidity of minimal hypersurfaces, see [5, 7, 8, 23, 41].
In [19] a counterexample, disproving the De Giorgi’s conjecture, was built for \( N = 8 \) using the non-trivial minimal graph \( \Gamma \) built in [8], as a counterexample to Bernstein’s conjecture. The solution found in [19] is bounded, monotone in one direction and its zero level set is close to the dilated surface \( \Gamma_\varepsilon = \varepsilon^{-1}\Gamma \), where \( \varepsilon > 0 \) is a small positive number. The main strategy is based upon the fact that \( \Gamma_\varepsilon \) is nearly flat around each of this points and hence the quantity \( v_\varepsilon(t) \) is a good approximation to a solution of (1.1), where \( t = t(\xi) \) is a choice of normal coordinate (signed distance) from \( \xi \) to \( \Gamma_\varepsilon \).

In [20], the same approach was used for the case \( N = 2 \) to construct a solution to (1.1) having zero level set close to a large dilation of an embedded minimal surface with finite total curvature, that satisfies certain non-degeneracy assumptions. One important example of such surfaces is the catenoid leading to an axially symmetric solution of (1.1).

The aforementioned construction was generalized to the case \( N \geq 3 \) in [2], where the authors built an axially symmetric solution to (1.1) having nodal set close to a large dilation of a logarithmic correction of the higher dimensional catenoid. This logarithmic correction, which is governed by the Liouville equation, is needed due to the fact that outside a large ball, the higher dimensional catenoid is asymptotic to two parallel planes.

In the works mentioned above, the solutions are constructed using the infinite dimensional Lyapunov-Schmidt reduction and based on the previous knowledge of what the nodal set should be, profiting also from the properties of those minimal hypersurfaces in each particular case.

To mention some further relevant works, let us introduce some notation. Throughout this work, for \( m \geq 2 \), we introduce the minimal hypersurface

\[
C_{m,n} := \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 = \frac{m-1}{n-1}|y|^2 \right\}
\]

known as the Lawson cone. In the case \( m = n \), \( C_{m,m} \) is known as the Simon’s cone. Also, observe that the Lawson cone is invariant under the action of the group of rotations \( O(m) \times O(n) \).

In [9, 10] existence and qualitative properties of saddle-shaped solutions to (1.1) are studied. The nodal set of this solutions is exactly the Simons cone \( C_{m,m} \).

We remark that for \( m, n \geq 2 \), the open set \( \mathbb{R}^{N+1} \setminus C_{m,n} \) has two connected components, namely

\[
E^{+}_{m,n} := \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 < \frac{m-1}{n-1}|y|^2 \right\},
\]

\[
E^{-}_{m,n} := \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 > \frac{m-1}{n-1}|y|^2 \right\},
\]

corresponding to the interior and the exterior of \( C_{m,n} \), respectively. The sets \( E^{\pm}_{m,n} \) help to describe an important feature of \( C_{m,n} \) that has been already studied in [26, 35, 42] and that is described in the next result.

**Theorem 1.1.** [35, 42] Let \( m, n \geq 3 \), \( n + m = N + 1 \geq 8 \). Then there exist two unique minimal hypersurfaces \( \Sigma_{m,n}^{\pm} \subset E^{\pm}_{m,n} \) which are asymptotic to \( C_{m,n} \) at infinity and \( d(\Sigma_{m,n}^{\pm}, \{0\}) = 1 \). Moreover, \( \Sigma_{m,n}^{\pm} \) are \( O(m) \times O(n) \)-invariant.
We are interested in solutions $u$ to (1.1) which are invariant under the action of the group of rotations $O(m) \times O(n)$. In this regard, the saddle-shaped solutions in [9, 10] enjoy this symmetry. We stress that functions with such symmetry are even in each of the variables.

Let $\Sigma$ be one of the minimal hypersurfaces $\Sigma^+_{m,n}$. In [38] the authors construct stable, $O(m) \times O(n)\text{-}\text{invariant}$ solutions to (1.1), changing sign once and having nodal set close to a large dilation of $\Sigma$. Their construction follows the approach from [19] using extensively the area-minimising character of the underlying cone. Also, from the results in [26], this construction can be generalized to more general minimal hypersurfaces asymptotic to an area-minimising cone.

In this work, we generalise this construction to built solutions to (1.1), but changing sign twice near a large dilation of $\Sigma$.

The first step in this generalisation is the following theorem, which is our first main result in this work.

**Theorem 1.2.** Let $m, n \geq 3$, $n + m = N + 1 \geq 8$ and let $a_* > 0$ be a constant. Let $\Sigma$ be one of the two minimal hypersurfaces constructed in Theorem 1.1, then there exists $\delta_* > 0$ small such that if $0 < \delta \leq \delta_*$, then the equation

$$\delta(\Delta_S w + |A_S|^2 w) = 2a_* e^{-\sqrt{2}\chi}$$

has a smooth solution which is $O(m) \times O(n)\text{-}\text{invariant}$.

Above $\Delta_S$ is the Laplace-Beltrami operator on $\Sigma$, $|A_S|$ is the norm of the second fundamental form and $\Delta_S + |A_S|^2$ is the Jacobi operator. The nonlinear, exponential term on the right hand side of (1.5) describes the Toda interaction between the two sheets of the zero level set of $u$, hence the name the Jacobi-Toda equation given to (1.5). Before we will describe the role it plays in the problem at hand in more details we will first explain why it is an important problem on its own.

Geometric analogs of the existence result for (1.1) proven in this paper are doubling construction for minimal surfaces (e.g. [29], [30], [31]) and connected sum construction for CMC (constant mean curvature) surfaces (e.g. [36]). Both are based on a similar idea of taking two copies of a given minimal or CMC surface and building a new, connected surface of the same type by inserting a catenoidal bridge between them. In general this requires also a deformation of the original surfaces. Likewise, in our case we want to "double" the zero level set $\Sigma$ of the solution $u$ of (1.1) and the Jacobi-Toda equation provides the "connection" between the two components. In this context a general principle would be: if for a given minimal surface one can solve the Jacobi-Toda equation (1.5) then a connected sum construction for (1.1) based on such surface should be possible (e.g. [18, 21]). As recent results in [12] and [43] show that the Jacobi-Toda equation also plays a crucial role in the problem of classification of finite Morse index solutions of the Allen-Cahn equation.

We state now the second result in this work, concerned directly with the existence of solutions to (1.1).

**Theorem 1.3.** Let $m, n \geq 3$, $n + m = N + 1 \geq 8$ and let $\Sigma$ be one of the two minimal hypersurfaces described in Theorem 1.1. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, there exists a solution $u_\varepsilon$ to (1.1) in $\mathbb{R}^{N+1}$ such that

(i) $u_\varepsilon$ is smooth and $O(m) \times O(n)\text{-}\text{invariant}$;

(ii) the zero level set of $u_\varepsilon$ is the disjoint union of 2 connected components, which are normal graphs over $\Sigma_\varepsilon := \varepsilon^{-1} \Sigma$ of $O(m) \times O(n)\text{-}\text{invariant}$ functions.
(iii) there exists a constant \( c > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and any \( R > 2\varepsilon^{-1} \),

\[
\int_{B_R} \frac{1}{2} \left| \nabla u_\varepsilon \right|^2 + \frac{1}{4} (1 - u_\varepsilon^2)^2 \leq cR^N.
\]

A few comments are now in order. First, from Theorem 1.3 and taking \( m, n \geq 3 \), there are two associated minimal hypersurfaces \( \Sigma^\pm_{m,n} \), each of which giving rise to a family of solutions. Since the nonlinearity is odd, if \( u \) is a solution then also \(-u\) is also a solution. Thus, we actually have 4 families of solutions.

Let \( \Sigma \) be as in Theorem 1.3. We remark also that the nodal set of the solution is governed by a Jacobi-Toda system associated to \( \Sigma \), namely:

\[
\varepsilon^2 (\Delta_{\Sigma} h_1 + |A_{\Sigma}|^2 h_1) - a_* e^{-\sqrt{\varepsilon} (h_2 - h_1)} = 0 \quad \text{in} \quad \Sigma, \tag{1.6}
\]

where \( \Delta_{\Sigma} \) and \( |A_{\Sigma}| \) are the Laplace-Beltrami operator and the norm of the second fundamental form on \( \Sigma \), respectively, and \( a_* > 0 \) is a constant depending only on the function \( v_* \) described in (1.3).

As we will see, the system (1.6) is decoupled into the system

\[
\Delta_{\Sigma} v_{0,1} + |A_{\Sigma}|^2 v_{0,1} = 0 \quad \text{in} \quad \Sigma, \tag{1.7}
\]

that can be solved using nondegeneracy of \( \Sigma \) (see Proposition 2.2 below) and Theorem 1.2 with \( \delta = \varepsilon^2 \).

The proof of Theorem 1.3 is based on the infinite dimensional Lyapunov-Schmidt reduction technique and as we will see, we can be more precise regarding the asymptotic behavior of the solution \( u_\varepsilon \). In particular the energy growth estimate follows from this.

To explain the previous paragraph, let \( \varepsilon \in (0, \varepsilon_0) \) and let \( \nu_{\Sigma} \) be the choice of the unit normal vector to \( \Sigma \) pointing towards the hyperplane \( \{0\} \times \mathbb{R}^n \). Let also \( \Sigma_\varepsilon := \varepsilon^{-1}\Sigma \) be a large dilation of \( \Sigma \). Consider a tubular neighbourhood of \( \Sigma_\varepsilon \) of the form

\[
\mathcal{N}_\varepsilon := \{ p + z \nu_{\Sigma}(e p) : p \in \Sigma_\varepsilon, \quad |z| < \varepsilon^{-1} \eta + c|p| \}
\]

for some \( \eta > 0 \) small.

Observe that for any \( \xi = p + z \nu_{\Sigma}(e p) \in \mathcal{N}_\varepsilon, \quad |z| = \text{dist}(\xi, \Sigma_\varepsilon) \) and the solution \( u_\varepsilon \) satisfies that

\[
u_\varepsilon(\xi) \sim v_\varepsilon(z - h_1(e p)) - v_\varepsilon(z - h_2(e p)) - 1
\]

while for \( \xi \) far from the set \( \mathcal{N} \), \( u_\varepsilon(\xi) \sim -1 \) at an exponential rate in \( |z| \).

Similar constructions have already been carried out for the equation (1.1) under different geometric settings. We mention for instance [22], where the authors build solutions to (1.1) in \( \mathbb{R}^2 \), having multiple ends governed at main order by a one-dimensional Toda System.

In [1] similar techniques as in [22] were used to construct solutions to (1.1) in \( \mathbb{R}^3 \) whose nodal set, outside a large ball, has multiple catenoidal like components. These components are governed by the Jacobi-Toda system associated to the catenoid.

Our developments are in the spirit of the construction of sign-changing solutions for the travelling wave problem for the Allen-Chan equation, carried out in [21], within the context of hypersurfaces.
with constant mean curvature, where the nodal set of the solutions is also governed by a Jacobi-Toda type system.

Finally, part \((iii)\) in Theorem 1.3 implies that for any \(\varepsilon \in (0, \varepsilon_0)\),

\[
\limsup_{R \to \infty} \frac{1}{R^N} \int_{B_R} \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4} (1 - u_\varepsilon^2)^2 < \infty
\]

and this suggests that these solutions should have finite Morse index, see [6, 24].

The rest of the paper is organized as follows: Section 2 discusses the topic of minimal hypersurfaces, minimal cones and area-minimising hypersurfaces. It also contains a detailed discussion on nondegeneracy properties of the hypersurfaces \(\Sigma_{m,n}\). Section 3 discusses the proof of Theorem 1.2 and the solution of system \((1.6)\). Section 4 presents the approximate solution to \((1.1)\) and gives the preliminary sketch of the proof of Theorem 1.3. In section 5 we present the Lyapunov-Schmidt reduction and we finish the proof of part \((i)\) and \((ii)\) in Theorem 1.3. Finally, in Section 6, we prove the energy estimate of the gradient in part \((iii)\) of Theorem 1.3.

2. \(O(m) \times O(n)\)-invariant minimal hypersurfaces

2.1. The Jacobi operator and stability. A hypersurface having zero mean curvature is not necessarily a minimiser of the area functional. One way to study the stability properties of a minimal hypersurface is through the study of the second variation of the area functional around the hypersurface, whenever the area functional is smooth enough.

To be more precise, let \(\Sigma \subset \mathbb{R}^N\) be a hypersurface with singular set \(\text{sing}(\Sigma)\) and normal vector \(\nu : \Sigma - \text{sing}(\Sigma) \to \mathbb{R}^N\). For any \(v \in C^\infty_c(\Sigma \setminus \text{sing}(\Sigma))\), consider the normal graph

\[
\Sigma_v := \{p + v(p)\nu(p) : p \in \Sigma\}.
\]

It is well known that \(\Sigma\) is a minimal hypersurface if it is a critical point (in some appropriate topology) of the functional

\[
C^\infty_c(\Sigma \setminus \text{sing}(\Sigma)) \ni v \mapsto \int_{\Sigma_v} 1 \, d\sigma,
\]

which is equivalent to saying that \(v = 0\) is a zero for the mean curvature operator,

\[
C^\infty_c(\Sigma \setminus \text{sing}(\Sigma)) \ni v \mapsto H_{\Sigma_v}.
\]

The second variation of \((2.1)\) and the first variation of \((2.2)\) at \(v = 0\) give rise to the quadratic form

\[
v \in C^\infty_c(\Sigma \setminus \text{sing}(\Sigma)) \mapsto \int_{\Sigma} \left(|\nabla v|^2 - |A_{\Sigma_v}|^2 v^2\right) d\sigma,
\]

which is characterised by the Jacobi operator of \(\Sigma\),

\[
J_\Sigma := \Delta_\Sigma + |A_{\Sigma_v}|^2,
\]

where \(\Delta_\Sigma\) is the Laplace-Beltrami operator and \(|A_{\Sigma_v}|\) is the norm of the second fundamental form of \(\Sigma\).

Stability properties of \(\Sigma\) can be studied with the help of the operator \(J_\Sigma\). We say that \(\Sigma\) is stable if for every \(v \in C^\infty_c(\Sigma \setminus \text{Sing}(\Sigma))\),

\[
\int_{\Sigma} (|\nabla v|^2 - |A_{\Sigma_v}|^2 v^2) d\sigma \geq 0.
\]

We also say that \(\Sigma\) is strictly stable if the inequality in \((2.4)\) is strict.
Observe that the minimal hypersurface $\Sigma$ is stable if and only if the eigenvalues of $-J_\Sigma$ are non-negative, while it is strictly stable if and only if all these eigenvalues are positive.

Next, we consider the case when $\Sigma$ is a minimal cone, which we denote by $C$ (saving $\Sigma$ for later purposes). We will also assume that $C$ is regular, i.e. sing$(C)$ consists only on one point which is assumed to be the origin.

Next, we analyse the stability properties of $C$. Let $B_\rho$ denote the ball in $\mathbb{R}^N$ and $S^{N-1}_\rho$ be the sphere in $\mathbb{R}^N$ both of them of radius $\rho > 0$ and centered at the origin.

Set $\Lambda := C \cap \partial B_1$ so that $\Lambda$ is a minimal submanifold of $\partial B_1$ and
\begin{equation}
\label{2.5}
C = \{ r \mathbf{p} : \mathbf{p} \in \Lambda, \ r > 0 \}.
\end{equation}

In the $(r, \mathbf{p})$-coordinates, the Jacobi operator $J_C = \Delta_C + |A_C|^2$ takes the form
\begin{equation}
\label{2.6}
J_C = \partial^2_r + \frac{N-1}{r} \partial_r + \frac{N^2}{r^2} J_\Lambda,
\end{equation}
where $J_\Lambda$ corresponds to the Jacobi operator of $\Lambda$.

Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$ be the sequences of eigenvalues of $-J_\Lambda$, counting multiplicities, and let $\{ \varphi_j \}_{j \geq 1}$ be an orthonormal basis for $L^2(\Lambda)$ with $\varphi_j$ being an eigenfunction associated to $\lambda_j$.

For any $\phi \in L^2(C)$, we perform the Fourier decomposition,
\begin{equation*}
\phi(r \mathbf{p}) = \sum_{j=1}^{\infty} \phi_j(r) \varphi_j(\mathbf{p}) \quad \text{with} \quad \phi_j(r) = \int_{\Lambda} \phi(r \mathbf{p}) \varphi_j(\mathbf{p}) d\sigma(\mathbf{p})
\end{equation*}
for all $r > 0$.

Therefore, $\phi$ is an eigenfunction of $-J_C$ with eigenvalue $\mu \in \mathbb{R}$ if and only if for every $j \in \mathbb{N}$,
\begin{equation}
\label{2.7}
- \Delta \varphi_j + \frac{\lambda_j}{|y|^2} \varphi_j = \mu \varphi_j \quad \text{in} \quad \mathbb{R}^N.
\end{equation}

Multiply (2.7) by $\varphi_j$ and integrate by parts, using that for $j > 1$, $\lambda_j > \lambda_1$ and the Hardy inequality (see [11], Proposition 1.20) to find that
\begin{equation*}
\mu \int_{\mathbb{R}^N} \phi_j^2 \, dy \geq \left( \frac{(N-2)^2}{2} + \lambda_1 \right) \int_{\mathbb{R}^N} \frac{\phi_j^2}{|y|^2} \, dy,
\end{equation*}
which yields that the cone $C$ is stable if
\begin{equation}
\label{2.8}
\lambda_1 \geq - \left( \frac{N-2}{2} \right)^2
\end{equation}
and strictly stable if
\begin{equation}
\label{2.9}
\lambda_1 > - \left( \frac{N-2}{2} \right)^2.
\end{equation}

We now focus in the particular instance when $C$ is a Lawson cone. To be more precise, let $m, n \geq 2$ and such that $N = m + n - 1$. Recall that the Lawson cone is defined as
\begin{equation}
\label{2.10}
C_{m,n} := \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x|^2 = \frac{m-1}{n-1} |y|^2 \right\}.
\end{equation}
In this case,

\[ \Lambda = \Lambda_{m,n} := S^{m-1}_{\rho_m} \times S^{n-1}_{\rho_n}, \]

where \( \rho_m := \sqrt{\frac{m-1}{N-1}} \) and \( \rho_n := \sqrt{\frac{n-1}{N-1}} \).

Since \( |A_{\Lambda_{m,n}}|^2 = N - 1 \), the Jacobi operator of \( \Lambda_{m,n} \) takes the form

\[ J_{\Lambda_{m,n}} = \Delta_{\Lambda_{m,n}} + (N - 1) \]

and the first eigenvalue of \(-J_{\Lambda_{m,n}}\) is \( \lambda_1 = -(N - 1) < 0 \). Thus, (2.9) translates into

\[ \frac{(N - 2)^2}{4} - (N - 1) > 0 \]

or equivalently \( N \geq 7 \).

Therefore, the Lawson cone \( C_{m,n} \) is stable whenever \( n + m \geq 8 \), \( n,m \geq 2 \) (see [38]) and its Jacobi operator \( J_{C_{m,n}} \) reads as

\[ J_{C_{m,n}} = \partial_{rr}^2 + \frac{N - 1}{r} \partial_r + \frac{1}{r^2} \left( \Delta_{\Lambda_{m,n}} + (N - 1) \right). \] (2.11)

2.2. Area-minimising hypersurfaces. Another property of the Lawson cone \( C_{m,n} \) that concerns with its stability is that \( C_{m,n} \) is not only a minimal hypersurface, but actually it is an area minimising hypersurface. Next, we discuss this topic in more detail.

First, define the perimeter of a subset \( E \subset \mathbb{R}^{N+1} \) in an open set \( \Omega \subset \mathbb{R}^{N+1} \) as

\[ \text{Per}(E, \Omega) := \sup \left\{ \int_E \text{div} X \, d\xi : X \in C_\infty^c(\Omega, \mathbb{R}^{N+1}) \right\}. \] (2.12)

If \( E \) has smooth boundary \( \partial E \), it follows from the Divergence Theorem that \( \text{Per}(E, \Omega) \) coincides with the \( N \)-dimensional Hausdorff measure of \( \partial E \cap \Omega \). However, the definition in (2.12) allows us to treat the case of sets \( E \) with non-smooth boundary \( \partial E \), like the case when \( \partial E = C_{m,n} \).

Up to a translation, there is no loss of generality in assuming that \( 0 \in \partial E \). Next, we define the concept of area minimising hypersurface.

**Definition 2.1.** We say that \( \Sigma := \partial E \) is an area-minimising hypersurface if for any \( \rho > 0 \) and for any smooth set \( F \subset \mathbb{R}^{N+1} \) such that \( F \setminus B_\rho = E \setminus B_\rho \),

\[ \text{Per}(E, B_{2\rho}) \leq \text{Per}(F, B_{2\rho}). \]

This definition is equivalent to say that \( \Sigma \) is a global minimiser of the area functional. Therefore, any area-minimising hypersurface is a minimal hypersurface.

We refer interested readers to [11, 13] and references therein, for a deeper understanding of the Definition 2.1 and related topics.

A relevant question concerns with the regularity of area-minimising hypersurfaces and, in general, of minimal hypersurfaces. In [41], Simons proved that if \( N \leq 6 \), \( N \)-dimensional area-minimising hypersurfaces are smooth. In higher dimension, area-minimising cones are known to exist (see [15]). This is the essence of the following theorem.

**Theorem 2.1.** [15]

(i) If \( N = m + n - 1 > 7 \) with \( n, m \geq 2 \), then the Lawson cone \( C_{m,n} \) is area-minimising.
(ii) If $N = m + n - 1 = 7$, then the Lawson cone $C_{m,n}$ has zero mean curvature everywhere except in the origin, which is singular. Moreover, it is area-minimising if and only if $|m - n| \leq 2$.

Theorem 2.1 provides an example of non smooth $N$-dimensional area-minimising hypersurfaces, at least of dimension $N \geq 7$. The first result in this direction was obtained in [8] for the case $n = m \geq 4$. The case $m + n > 8$ was treated in [33], while in [40] it is proven that $C_{3,5}$ is area minimising and that $C_{2,6}$ has zero mean curvature, but it is not a global minimiser of the area.

We finish this discussion with the notion of strictly area-minimising cones.

**Definition 2.2.** Let $C \subset \mathbb{R}^{N+1}$ be a cone and set $\Lambda := C \cap B_1$. Then $C$ is said to be strictly area-minimising if there exist constants $\theta > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any smooth hypersurface $\Gamma \subset \mathbb{R}^{N+1} \setminus B_{\varepsilon}$ such that $\Gamma \setminus B_1 = C \setminus B_1$

$$\mathcal{H}_N(\Lambda) \leq \mathcal{H}_N(\Gamma \cap B_1) - \theta \varepsilon^N,$$

where $\mathcal{H}_N$ stands for the $N$-dimensional Hausdorff measure in $\mathbb{R}^{N+1}$.

For instance, if $n, m \geq 2$ and $N = m + n - 1 > 7$, then $C_{m,n}$ is strictly area minimising. The same is true if $n, m \geq 3$ and $N = m + n - 1 = 7$. This agrees with the fact that in this case $C_{m,n}$ is strictly stable.

### 2.3. Minimal hypersurfaces asymptotic to a cone

Next, we discuss the existence and asymptotic behaviour of smooth minimal hypersurfaces that are asymptotic to the cone $C_{m,n}$, $m,n \geq 3$, $m+n \geq 8$. In this work, these hypersurfaces are the core of the construction of sign changing solutions to (1.1).

In what follows we make extensive use of some of the symmetries of the cone $C_{m,n}$. To be more precise, consider the group $O(m) \times O(n)$. From (2.10) it is clear that $C_{m,n}$ is invariant under the action of this group.

A function $v : C_{m,n} \to \mathbb{R}$ is invariant under the action of $O(m) \times O(n)$ if and only if there exists $v : (0, \infty) \to \mathbb{R}$ such that

$$v(rp) = v(r) \quad \text{for all} \quad r > 0, \quad p \in \Lambda_{m,n}.$$

We are next interested in the solutions of the homogeneous equation

$$J_{C_{m,n}} v = 0 \quad \text{in} \quad C_{m,n},$$

also known as Jacobi fields of $C_{m,n}$. In particular, we are interested in the $O(m) \times O(n)$--invariant Jacobi fields.

It is straightforward to verify that the only two $O(m) \times O(n)$--invariant Jacobi fields of $C_{m,n}$ are

$$u_\pm(rp) = u_\pm(r) = r^{\gamma_\pm},$$

where $\gamma_\pm$ are the roots (usually referred to as indicial roots) of

$$\gamma^2 + \gamma(N-2) + (N-1) = 0,$$

that is

$$\gamma_\pm = -\frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - (N-1)}.$$

Observe that $\gamma_\pm \in (-\infty, 0)$ if and only if $N \geq 7$. 
Let $E^\pm$ denote the two connected components of $\mathbb{R}^{N+1} \setminus C_{m,n}$, where $E^-$ is the component containing the hyperplane $\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : y = 0\}$.

The following result summarises the discussion in this section.

**Theorem 2.2.** \cite{4,26,35,42}

Let $m,n \geq 3$, $m+n \geq 8$. Then there exist two unique minimal hypersurfaces $\Sigma^\pm_{m,n} \subset E^\pm$ satisfying that

(i) $\Sigma^\pm_{m,n}$ are smooth;

(ii) $\text{dist}(\Sigma^\pm_{m,n}, \{0\}) = 1$;

(iii) for any $\xi \in E^\pm$, the ray $\{\lambda \xi : \lambda > 0\}$ intersects $\Sigma^\pm_{m,n}$ exactly once;

(iv) $\Sigma^\pm_{m,n}$ are $O(m) \times O(n)$-invariant;

(v) there exist constants $R^\pm_x = R^\pm_x(C_{m,n}) > 0$, $c > 0$ and $O(m) \times O(n)$-invariant functions $w^\pm : C_{m,n} \setminus B_{R^\pm} \to \mathbb{R}$ such that

$$w^+ > 0, \quad w^- < 0 \quad \text{in} \quad C_{m,n} \setminus B_{R^\pm}$$

respectively, for any $p \in C_{m,n} \setminus B_{R^\pm}$

$$w^\pm(p) = c|p|^{\gamma \pm} (1 + o(1)) \quad \text{as} \quad |p| \to \infty$$

(2.13)

and

$$\Sigma^\pm_{m,n} = \{p + w^\pm(p)\nu_{C_{m,n}}(p) : p \in C_{m,n} \setminus B_{R^\pm}\},$$

where $\nu_{C_{m,n}} : C_{m,n} \setminus \{0\} \to \mathbb{R}^{N+1}$ is the choice of the normal vector to $C_{m,n}$ pointing towards $E^+$. 

**Proof.** In Theorem 2.1 from \cite{26}, given an area minimising cone $C$, the authors prove the existence of two unique smooth area minimising hypersurfaces $\Sigma^\pm_{m,n} \subset E^\pm$ with $d(\Sigma^\pm_{m,n}, \{0\}) = 1$ and asymptotic to $C_{m,n}$ in the sense that, outside a ball, they are normal graphs over $C_{m,n}$ of functions $w^+ > 0$ and $w^- < 0$ respectively. It is also proven that the scaling $\lambda \Sigma^\pm_{m,n}$, $\lambda > 0$, foliates $E^\pm$ respectively. The decay rate of these graphs is given by Theorem 3.2 in \cite{26}, provided $C$ is strictly area minimising, which is the case if $C = C_{n,m}$ for such $m+n = 8$ and $m,n \geq 3$ or $n+m \geq 9$ and $m,n \geq 2$.

As for (iv), in \cite{4} the authors prove the existence of two $O(m) \times O(n)$-invariant stable minimal hypersurfaces $\Gamma^\pm_{m,n} \subset E^\pm$ which are asymptotic to $C_{m,n}$ at infinity and that satisfy (i),(ii) and (iii), for $n+m \geq 8$, $m,n \geq 3$. By the uniqueness result in \cite{35}, we find that $\Sigma^\pm_{m,n} = \Gamma^\pm_{m,n}$, thus for such $m,n$, (iv) is satisfied too. \hfill \qed

**Remark 2.1.** The restrictions about $m$ and $n$ are crucial in the proof of Theorem 2.2, since Theorems 2.1 and 3.2 from \cite{26} relies on the strict minimality of the cones. In dimension $N = m+n-1 \leq 6$ there exist no area-minimising cones.
2.4. The Jacobi operator on an \(O(m) \times O(n)\)-invariant minimal hypersurface. In what follows we set \(\Sigma := \Sigma_{m,n}\). This represents no significant restriction in our developments since \(\Sigma_{m,n}^+ = \sigma^{-1}(\Sigma_{n,m}^-)\), where \(\sigma(x,y) = (y,x)\). In particular, if a family of solutions \(u_\varepsilon\) to the Allen-Cahn equation satisfying the properties of Theorem 1.3 with \(\Sigma = \Sigma_{n,m}^-\) exists, then the family \(v_\varepsilon := u_\varepsilon \circ \sigma\) will enjoy the same properties with \(\Sigma = \Sigma_{m,n}^+\).

Similar to \(C_{m,n}\), the set \(\mathbb{R}^{N+1} \setminus \Sigma\) has two connected components one, which we denote, abusing the notation, by \(E^\pm\). We make the convention that \(E^+\) is the connected component containing the hyperplane \(\{0\} \times \mathbb{R}^n\).

The \(O(m) \times O(n)\)-invariance of \(\Sigma\) implies that \(\Sigma\) is generated by a smooth, regular curve \(\Upsilon: \mathbb{R} \to \mathbb{R}^2\), \(\Upsilon(s) := (a(s), b(s))\) in the half-plane
\[
Q := \{(a,b) \in \mathbb{R}^2 : a > 0\}
\]
and such that \(\Upsilon(0) = (1,0)\) and \(\Upsilon'(0) = (0,1)\).

To be more precise, let \(\nu_\Sigma: \Sigma \to \mathbb{R}^{N+1}\) be the choice of the unit normal vector to \(\Sigma\), pointing towards \(E^+\). For any \(p \in \Sigma\), there exists a unique \((s,x,y) = (s(p), x(p), y(p)) \in \mathbb{R} \times S^{m-1} \times S^{n-1}\) such that
\[
p := (a(s)x, b(s)y).
\]

We stress that the function
\[
p \in \Sigma \mapsto s(p) \in \mathbb{R}
\]
is surjective, but not injective.

From (2.14), we compute for \(p := (a(s)x, b(s)y)\),
\[
\nu_\Sigma(p) = (-b'(s)x, a'(s)y)
\]
and the principal curvatures of \(\Sigma\) are computed as
\[
\lambda_0 = \frac{-a''b' + a'b''}{(a')^2 + (b')^2}^{3/2},
\]
\[
\lambda_i = \frac{b'}{a\sqrt{(a')^2 + (b')^2}}, \quad 1 \leq i \leq m - 1,
\]
\[
\lambda_j = \frac{-a'}{b\sqrt{(a')^2 + (b')^2}}, \quad m \leq j \leq m + n - 2.
\]

The principal curvatures of \(\Sigma\) allow us to compute \(H_\Sigma\) and \(|A_\Sigma|^2\) as follows:
\[
H_\Sigma = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_{m+n-2}
\]
\[
|A_\Sigma|^2 = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{m+n-2}^2.
\]

Without any loss of generality, assume that \(\Upsilon\) is parametrised by arch-length, i.e.
\[
(a')^2 + (b')^2 = 1,
\]
so the fact that \(\Sigma\) is a smooth \(O(m) \times O(n)\)-invariant minimal surface together with (2.16),(2.17) and (2.18), yield
\[
H_\Sigma = -a''b' + b''a' + (m - 1)\frac{b'}{a} - (n - 1)\frac{a'}{b} = 0
\]
and

$$|A_{\Sigma}|^2 = \left(-a''b' + a'b''\right)^2 + (m - 1) \left(\frac{b'}{a} \right)^2 + (n - 1) \left(\frac{a'}{b} \right)^2. \quad (2.20)$$

On the other hand, from parts (ii), (iii) and (v) in Theorem 2.2, we find that for some $R > 0$, $C_{m,n} \setminus B_R$ is diffeomorphic to $\Sigma$ via the $O(m) \times O(n)$--invariant mapping

$$C_{m,n} \setminus B_R \ni p \mapsto p + w^-(p) v_{C_{m,n}}(p) \in \Sigma.$$ 

Therefore, the definition of $O(m) \times O(n)$--invariant function extends naturally to functions defined over $\Sigma$.

To be more precise, a function $v : \Sigma \to \mathbb{R}$ is invariant under the action of $O(m) \times O(n)$ if and only if there exists $v : \mathbb{R} \to \mathbb{R}$ such that for every $s \in \mathbb{R}$, $x \in (0,2\pi)^{m-1}$ and $y \in (0,2\pi)^{n-1}$,

$$v(p) = v(s(p)).$$

Roughly speaking, a function defined on $\Sigma$ is $O(m) \times O(n)$--invariant if it depends only on the arch-length parameter $s$ of the profile curve $\gamma$ or equivalently it depends only $|p|$.

Observe that (2.20) implies that $|A_{\Sigma}|^2$ depends only of the arch-length variable $s$ and hence it is $O(m) \times O(n)$--invariant.

Next, study the invertibility theory for the linear equation

$$\Delta_{\Sigma} q + |A_{\Sigma}|^2 q = f \quad \text{in} \quad \Sigma \quad (2.21)$$

in the class of $O(m) \times O(n)$-invariant functions.

2.5. The Emden-Fowler change of variables in the Jacobi operator. In terms of the coordinates $p = (a(s)x, b(s)y)$ in (2.14), the Laplace-Beltrami operator acting on $O(m) \times O(n)$-invariant functions $q(p) = q(s(p))$ reads as

$$\Delta_{\Sigma} q = \partial^2_s q + \alpha(s) \partial_s q \quad \text{with} \quad \alpha(s) := (m - 1) \frac{a'}{a} + (n - 1) \frac{b'}{b}. \quad (2.22)$$

Setting $\beta(s(p)) := |A_{\Sigma}(p)|^2$, the equation (2.21) becomes

$$\partial^2_s q + \alpha(s) \partial_s q + \beta(s) q = f \quad \text{in} \quad \mathbb{R}, \quad (2.23)$$

where we have set $f(p) = f(s(p))$.

The $O(m) \times O(n)$--invariance allows us to restrict ourselves to the case when $q$ and $f$ are even and consequently, we study (2.23) for $s > 0$ with the boundary condition $\partial_s q(0) = 0$.

Consider the Emden-Fowler change of variables $s = e^t$ and for $t \in \mathbb{R}$, set

$$\tilde{\alpha}(t) := \alpha(e^t)e^t - 1 \quad \text{and} \quad \tilde{\beta}(t) := \beta(e^t)e^{2t}. \quad (2.24)$$

Consider also,

$$p(t) := \exp \left(- \int_0^t \frac{\tilde{\alpha}(\tau)}{2} d\tau \right) \quad \text{for} \quad t \in \mathbb{R}, \quad (2.25)$$

so that $p(t)$ solves

$$2\frac{\partial p}{p} + \tilde{\alpha}(t) = 0. \quad (2.26)$$

We look for a solution to (2.23) having the form $q(s) = p(t)u(t)$. From (2.24) and (3.37), we find that

$$\partial^2_s q + \alpha(s) \partial_s q + \beta(s) q = 0.$$
DOUBLING CONSTRUCTION FOR $O(M) \times O(N)$ INVARIANT SOLUTIONS TO THE ALLEN-CAHN EQUATION

\[\begin{align*}
&= \partial_t^2 u + \left(2 \frac{\partial p}{p} + \tilde{\alpha}(t)\right) \partial_t u + \left(\frac{\partial^2 p}{p^2} + \tilde{\alpha}(t) \frac{\partial p}{p} + \tilde{\beta}(t)\right) u \\
&= \partial_t^2 u + \left(\frac{\partial^2 p}{p} + \tilde{\alpha}(t) \frac{\partial p}{p} + \tilde{\beta}(t)\right) u.
\end{align*}\]

Set

\[\tilde{f}(t) := \frac{e^{2t}}{p(t)} f(e^t) \quad \text{and} \quad V(t) := \frac{\partial^2 p}{p} + \tilde{\alpha}(t) \frac{\partial p}{p} + \tilde{\beta}(t)\]

for $t \in \mathbb{R}$.

From (2.24) and (2.25),

\[V(t) = -\frac{1}{4} \left(\alpha(e^t)e^t - 1\right)^2 + \frac{1}{2} \left(\alpha'(e^t)e^{2t} + \alpha(e^t)e^t\right) + \beta(e^t)e^{2t}\]

and from the equation (2.23) for $q$, we find that $u$ must solve

\[\partial_t^2 u + V(t)u = \tilde{f} \quad \text{in} \quad \mathbb{R}.\]

In order to solve (2.28), we must analyze the asymptotic behavior of $V(t)$ as $t \to \pm \infty$. This analysis is done by studying the asymptotic behavior at $s = 0$ and at infinity of the functions $a$ and $b$ related to (2.14), as well as its derivatives.

First, notice that $a$ is an even function while $b$ is odd and since $\Upsilon(0) = (1, 0)$ and $\Upsilon'(0) = (0, 1)$, then

\[a(0) = 1 \quad \text{and} \quad b(0) = 0,\]

\[a'(0) = 0 \quad \text{and} \quad b'(0) = 1.\]  

Let $T, N : \mathbb{R} \to \mathbb{R}^2$ denote the tangent and a choice of the unit normal vector to $\Upsilon$ respectively, so that $\{T, N\}$ is a Frenet frame with positive orientation. Thus,

\[T(s) = (a'(s), b'(s)) \quad \text{and} \quad N(s) = (-b'(s), a'(s)).\]

Let $k : \mathbb{R} \to \mathbb{R}$ denote the curvature of $\Upsilon$. Thus, $k$ is an even function that does not change sign and from (2.19),

\[k := -T' \cdot N = -a''b + a'b'' = -(m - 1) \frac{b'}{a} + (n - 1) \frac{a'}{b}.\]  

Performing a Taylor expansion around $s = 0$, we fix $s_0 \in (0, 1)$ such that for any $s \in (0, s_0)$,

\[a(s) = 1 + \frac{a''(0)}{2} s^2 + O(s^4)\]

\[b(s) = s + \frac{b^{(3)}(0)}{6} s^3 + O(s^5).\]  

Also,

\[0 = \lim_{s \to 0} H_{\Sigma} = (m - 1) - a''(0)n\]

so that

\[a''(0) = \frac{m - 1}{n}.\]

On the other hand, from Taylor expansion, (2.30) and (2.31),

\[k(s) = -\frac{m - 1}{n} + O(s^2) \quad \text{for} \quad s \in (0, s_0).\]
Recall that \( \beta(s) = |A_{\Sigma}(s(p))|^2 \). From (2.19), (2.20) and (2.32), performing again the Taylor expansion for \( \beta(s) \) around zero we find that for any \( s \in (0, s_0) \),

\[
\beta(s) = \frac{N(m-1)}{n} + \mathcal{O}(s^2).
\]  

(2.33)

As for the asymptotic behavior of \( \beta(s) \) at infinity, we proceed as follows.

Since outside a ball \( \Sigma \) is the normal graph over the cone \( C_{m,n} \) of an \( O(m) \times O(n) \)-invariant function \( w^-(p) = w^-(r) \), \( r = |p| \), (see part v in Theorem 2.2), then

\[
(a(s), b(s)) = \frac{1}{\sqrt{N-1}} (\sqrt{m-1}, \sqrt{n-1}) r + \frac{1}{\sqrt{N-1}} (\sqrt{n-1}, \sqrt{m-1}) w^-(r)
\]

where \( s = s(p) \) is the arch length parameter along \( \Upsilon \).

Thus, for some \( r_0 > 0 \) fixed,

\[
s = \int_{r_0}^r \sqrt{1 + (\partial_r w^-(r'))^2} dr' = r + O(r^{\gamma + \alpha}) \quad \text{as} \quad r \to \infty
\]

with \( \alpha > 0 \).

On the other hand,

\[
\lim_{s \to \infty} \frac{a(s)}{s} = \lim_{s \to \infty} a'(s) = \sqrt{\frac{m-1}{N-1}}
\]

and so we can fix \( c_1 \) and \( s_1 \in (s_0, \infty) \) such that for any \( s \in (s_1, \infty) \),

\[
\begin{align*}
\alpha(s) &= \sqrt{\frac{m-1}{N-1}} s + \sqrt{\frac{n-1}{N-1}} c_1 s^{\gamma^+} + \mathcal{O}(s^{\gamma^+ - \alpha}), \\
b(s) &= \sqrt{\frac{n-1}{N-1}} s - \sqrt{\frac{m-1}{N-1}} c_1 s^{\gamma^+} + \mathcal{O}(s^{\gamma^+ - \alpha})
\end{align*}
\]  

(2.35)

and these expressions can be differentiated.

Putting together (2.20) and (2.35), for any \( s > s_1 \),

\[
\beta(s) = \frac{N-1}{s^2} + \mathcal{O}(s^{-3}).
\]  

(2.36)

We also remark that \( \partial_r \beta < 0 \) in \( \mathbb{R} \).

Next, we study the asymptotic behavior of the function

\[
\alpha = (m-1) \frac{a'}{a} + (n-1) \frac{b'}{b} \quad \text{in} \quad \mathbb{R}.
\]

By fixing \( s_0 > 0 \) smaller and \( s_1 > s_0 \) larger if necessary, we find from (2.31) that for any \( s \in (0, s_0) \),

\[
\alpha(s) = \frac{n-1}{s} + \mathcal{O}(s)
\]  

(2.37)

and from (2.35) that for any \( s > s_1 \),

\[
\alpha(s) = \frac{N-1}{s} + \mathcal{O}(s^{-2})
\]  

(2.38)
Summarising, the function $\beta : \mathbb{R} \to (0, \infty)$ is a positive, smooth, even and strictly decreasing function such that for some $c_0 > 0$,

$$
\beta(s) = \begin{cases} 
  c_0 + \mathcal{O}(s^2), & 0 < s < s_0 \\
  N - 1 + \mathcal{O}(s^{-3}), & s > s_1,
\end{cases} \tag{2.39}
$$

while the function $\alpha : (0, \infty) \to \mathbb{R}$ is a positive, smooth function such that

$$
\alpha(s) = \begin{cases} 
  \frac{n-1}{s} + \mathcal{O}(s), & 0 < s < s_0 \\
  \frac{N-1}{s} + \mathcal{O}(s^{-2}), & s > s_1.
\end{cases} \tag{2.40}
$$

Next, denote $T_0 := \ln(s_0)$ and $T_1 := \ln(s_1)$.

From (2.24), (2.39) and (2.40),

$$
\tilde{\alpha}(t) = \begin{cases} 
  (n-2) + \mathcal{O}(e^{2t}) & \text{for } t < T_0, \\
  (N-2) + \mathcal{O}(e^{-t}) & \text{for } t > T_1
\end{cases} \tag{2.41}
$$

and

$$
\tilde{\beta}(t) = \begin{cases} 
  c_0 e^{2t} + \mathcal{O}(e^{4t}) & \text{for } t < T_0, \\
  N - 1 + \mathcal{O}(e^{-t}) & \text{for } t > T_1.
\end{cases} \tag{2.42}
$$

On the other hand, from (2.25) and (2.41),

$$
p(t) = \begin{cases} 
  e^{-\frac{(n-2)^2}{4} t} \left(1 + \mathcal{O}(e^{2t})\right) & \text{for } t < T_0 \\
  e^{-\frac{(N-2)^2}{4} t} \left(1 + \mathcal{O}(e^{-t})\right) & \text{for } t > T_1.
\end{cases} \tag{2.43}
$$

We remark that the asymptotic behaviours described in (2.41), (2.42) and (2.43) can be differentiated in the variable $t$.

From this discussion and a straightforward computation we find that

$$
\frac{\partial_{tt} p(t)}{p(t)} + \tilde{\alpha}(t) \frac{\partial_{t} p(t)}{p(t)} = \begin{cases} 
  -\frac{(n-2)^2}{4} + \mathcal{O}(e^{2t}), & \text{for } t < T_0 \\
  -\frac{(N-2)^2}{4} + \mathcal{O}(e^{-t}), & \text{for } t > T_1
\end{cases} \tag{2.44}
$$

Putting together (2.27), (2.42) and (2.44), we find that

$$
V(t) = \begin{cases} 
  -\frac{(n-2)^2}{4} + \mathcal{O}(e^{2t}), & \text{for } t < T_0 \\
  -\frac{(N-2)^2}{4} + (N - 1) + \mathcal{O}(e^{-t}), & \text{for } t > T_1
\end{cases} \tag{2.45}
$$

and this relations can be differentiated. We also recall that $N \geq 7$ and so

$$
\left(\frac{N-2}{2}\right)^2 - (N-1) > 0.
$$
2.6. Jacobi fields of $\Sigma$. The minimality of $\Sigma$ is invariant under dilation and this allows us to find an explicit smooth and $O(m) \times O(n)$-invariant Jacobi field for $\Sigma$, namely the function

$$\Sigma \ni p \mapsto p \cdot \nu_\Sigma(p).$$

In addition, this Jacobi field does not change sign. This follows from the fact that the family $$\{\lambda \Sigma\}_{\lambda > 0}$$ is a foliation of the connected component of $\mathbb{R}^{N+1} \setminus C_{m,n}$ containing $\Sigma$.

We will use this information to prove the following Proposition. Recall that

$$\alpha \in (0, \infty)$$

and consider $c_1 > 0$ and $\alpha > 0$ the constants in (2.35).

**Proposition 2.1.** There exist exactly two $O(m) \times O(n)$-invariant linearly independent Jacobi fields $v_\pm(p) = v_\pm(s(p)) > 0$ of $\Sigma$ such that

(i) $v_+(s)$ is smooth, even in the variable $s$, $v_+(0) = 1$ and

$$v_+(s) = c_1 s^{\gamma_1} \left(1 + O(s^{-\alpha})\right) \quad \text{as } s \to \infty$$

(ii) $v_-(s)$ is smooth except at $s = 0$, where it is singular and for some $c_2 > 0$

$$v_-(s) = \begin{cases} s^{-(n-2)}(1 + O(s^2)) & \text{as } s \to 0 \\ c_2 s^{\gamma_2} (1 + O(s^{-\alpha})) & \text{as } s \to \infty, \end{cases}$$

where $\alpha \in (0, 1)$. Moreover, relation (2.46) and (2.47) can be differentiated.

**Proof.** Set $v_+(p) = -p \cdot \nu_\Sigma(p)$ for $p \in \Sigma$. We know that $v_+$ is a smooth Jacobi field which does not change sign. Directly from (2.14) and (2.15), we find for $p = (a(s)x, b(s)y)$ that

$$v_+(p) = -p \cdot \nu_\Sigma(p)$$

so that $v_+(s) := v(p) = a(s) b'(s) - a'(s) b(s)$ and hence it is even in the variable $s$. Since $v_+$ does not change sign and $v_+(0) = 1$, then $v_+ > 0$ in $\Sigma$.

From (2.35) and a direct computation,

$$v_+(s) = c_1 s^{\gamma_1} \left(1 + O(s^{-\alpha})\right) \quad \text{as } s \to \infty,$$

where we assume that $\alpha \in (0, 1)$.

This relation can be differentiated in the sense that

$$sv_+^{(s)}(s) = c_1 s^{\gamma_1} s^{\gamma_1} \left(1 + O(s^{-\alpha})\right) \quad \text{as } s \to \infty$$

and this proves (2.46) and completes the proof of (i).

We next, prove (ii). With the Emden-Fowler change of variables $s = e^t$ and using the function defined in (2.25), we write

$$v_+(s) = p(t) u_+(t) \quad \text{for } t \in \mathbb{R}.$$

From (2.28) with $\tilde{f} = 0$, we see that $u_+$ must solve the ODE

$$\partial_t^2 u_+ + V(t) u_+ = 0 \quad \text{in } \mathbb{R}. \quad (2.50)$$

Setting,

$$\lambda := \frac{n-2}{2} \quad \text{and} \quad \Lambda := \sqrt{\left(\frac{N-2}{2}\right)^2 - (N-1)} \quad (2.51)$$

we find from (2.43) and (2.46) that for some \( c > 0 \),
\[
  u_+(t) = \begin{cases} 
    e^M (1 + \mathcal{O}(e^{2t})) & t < T_0 \\
    c_1 e^{\Lambda t} (1 + \mathcal{O}(e^{-\alpha t})) & t > T_1.
  \end{cases}
\] (2.52)

Proceeding in a similar fashion using (2.43), (2.46), (2.49), (2.52) and the fact that,
\[
  \partial_t u_+(t) = \frac{e^t}{p(t)} \partial_x v_+(e^t) - \frac{\partial_t p(t)}{p(t)} u_+(t),
\]
we find that
\[
  \partial_t u_+(t) = \begin{cases} 
    \lambda e^M (1 + \mathcal{O}(e^{2t})) & t < T_0 \\
    \Lambda c_1 e^{\Lambda t} (1 + \mathcal{O}(e^{-\alpha t})) & t > T_1.
  \end{cases}
\] (2.53)

Next, we find the second predicted Jacobi field of \( \Sigma \). Set,
\[
u_-(t) := u_+(t) \int_t^\infty \frac{1}{u_+(\tau)^2} d\tau \quad \text{for} \quad t \in \mathbb{R}.
\] (2.54)

Clearly, \( u_- \) is smooth and positive in \( \mathbb{R} \). From (2.50) and the variation of parameters formula, we conclude that \( u_- \) solves
\[
  \partial_t^2 u_- + V(t) u_- = 0 \quad \text{in} \quad \mathbb{R}
\] (2.55)
and \( u_+, u_- \) are linearly independent.

As for the asymptotic behavior of \( u_- \), we estimate directly from (2.52) and (2.54), to find that after normalisation of the solution, for some \( c_2 > 0 \),
\[
  u_-(t) = \begin{cases} 
    e^{-\Lambda t} (1 + \mathcal{O}(e^{2t})) & t < T_0 \\
    c_2 e^{-\Lambda t} (1 + \mathcal{O}(e^{-\alpha t})) & t > T_1.
  \end{cases}
\] (2.56)
where we have taken \( T_0 \) smaller and \( T_1 > T_0 \) larger if necessary, but still fix.

Similarly,
\[
  \partial_t u_-(t) = \begin{cases} 
    -\lambda e^{-\Lambda t} (1 + \mathcal{O}(e^{2t})) & t < T_0 \\
    -\Lambda c_2 e^{-\Lambda t} (1 + \mathcal{O}(e^{-\alpha t})) & t > T_1.
  \end{cases}
\] (2.57)

Going back to the original coordinate \( p = (a(s)x, b(s)y) \), we define
\[
  v_-(p) = v_-(s) := p(\log(s)) u_-(\log(s)).
\]

We conclude that \( v_- \) is smooth and positive in \((0, \infty)\). From (2.55), \( v_-(s) \) is another Jacobi field and the classical ODE theory yields that \( v_+(s), v_-(s) \) form a fundamental set for all the Jacobi fields of \( \Sigma \) that are \( O(m) \times O(n) \)-invariant.

Finally, from (2.43), (2.56) and (2.57),
\[
  v_-(s) = \begin{cases} 
    s^{-(n-2)}(1 + \mathcal{O}(s^2)) & s \to 0 \\
    s^{\gamma_1} (1 + \mathcal{O}(s^{-\alpha})) & s \to \infty
  \end{cases}
\]
and this relation can be differentiated, i.e.
\[
  sv_-(s) = \begin{cases} 
    s^{-(n-2)}(1 + \mathcal{O}(s^2)) & s \to 0 \\
    s^{\gamma_1} (1 + \mathcal{O}(s^{-\alpha})) & s \to \infty
  \end{cases}
\]

This proves (2.47) and completes the proof of the proposition.

\[\square\]
At this point, a few important remarks are in order.

**Remark 2.2.**

(i) We stress out that in the upcoming developments, the asymptotics described in expressions (2.52), (2.53), (2.56) and (2.57) will play a crucial role. Particularly, in the proof of Proposition 2.2 (see below).

(ii) Since minimal hypersurfaces in $\mathbb{R}^{N+1}$ are invariant under translations and rotations also, one expects the existence of other linearly independent Jacobi fields. However, these Jacobi fields are not $O(m) \times O(n)$–invariant and hence we do not study them here.

### 2.7. The Jacobi equation

Next, we introduce suitable function spaces to study invertibility theory for (2.21).

For $\beta \in (0, 1)$, $\mu > 0$ and a function $f : \Sigma \to \mathbb{R}$ we set

$$
\|f\|_{\infty, \mu} := \| (s(p)^2 + 2)^{\frac{\beta}{2}} f \|_{L_\infty(\Sigma)},
$$

$$
\|f\|_{C^{0,\beta}_\mu(\Sigma)} := \sup_{\xi \in \Sigma} (s(p)^2 + 2)^{\frac{\beta}{2}} \|f\|_{C^{0,\beta}(B_1(p))} \tag{2.58}
$$

and we consider the Banach space $C^{0,\beta}_\mu(\Sigma)$ defined as the space of $O(m) \times O(n)$–invariant functions $f \in C^{0,\beta}_\mu(\Sigma)$ for which the norm

$$
\|f\|_{C^{0,\beta}_\mu(\Sigma)} < \infty. \tag{2.59}
$$

We also consider the Banach space $C^{2,\beta}_\mu(\Sigma)$ defined as the space of $O(m) \times O(n)$–invariant functions $q \in C^{2,\beta}_\mu(\Sigma)$ for which the norm

$$
\|q\|_{C^{2,\beta}_\mu(\Sigma)} := \|D^2_q q\|_{C^{0,\beta}_\mu(\Sigma)} + \|\nabla_{\Sigma} q\|_{\infty, 1+\mu} + \|q\|_{\infty, \mu} < \infty. \tag{2.60}
$$

The following proposition shows that in this functional analytic setting, the linear operator in (2.21) has an inverse.

**Proposition 2.2.** Let $\beta \in (0, 1)$ and $\mu > 0$. Then, there exists a constant $c > 0$ such that for any $f \in C^{0,\beta}_\mu(\Sigma)$, there exists a solution $q \in C^{2,\beta}_\mu(\Sigma)$ to (2.21) such that

$$
\|q\|_{C^{2,\beta}_\mu(\Sigma)} \leq c \|f\|_{C^{0,\beta}_\mu(\Sigma)}. \tag{2.61}
$$

**Proof.** Given any $f \in C^{0,\beta}_\mu(\Sigma)$, we write $f(p) = f(s)$ and look for a solution of (2.21) having the form

$$
q(s) := p(t) u(t) \quad \text{for} \quad s = e^t > 0.
$$

Therefore, $u(t)$ must solve (2.28) with

$$
\dot{f}(t) := e^{2t} \|p\| f(e^t) \quad \text{for} \quad t \in \mathbb{R}
$$

and where the function $p(t)$ and the potential $V(t)$ are described in (2.43) and (2.45).

Observe that

$$
|\dot{f}(t)| \leq \begin{cases} 
  e^{(\frac{\beta}{2}+1)t} \|f\|_{C^{0,\beta}_\mu(\Sigma)} & \text{for} \quad t \leq T_0, \\
  e^{(\frac{\beta}{2}-\mu)t} \|f\|_{C^{0,\beta}_\mu(\Sigma)} & \text{for} \quad t \geq T_1.
\end{cases}
$$
A solution of (2.28) is given by the formula:

\[
u(t) = u_+(t) \int_{-\infty}^t u_-(\tau)W^{-1}(\tau)f(\tau)\,d\tau - u_-(t) \int_{-\infty}^t u_+(\tau)W^{-1}(\tau)f(\tau)\,d\tau \tag{2.62}\]

for \(t \in \mathbb{R}\).

Observe that the Wronskian \(W(t)\) of \(u_+(t)\) and \(u_-(t)\) is constant. Directly from (2.52), (2.53), (2.56), (2.57) and (2.62) we find that for some \(C > 0\) depending only on \(\mu\) and \(N\),

\[
|u(t)| + |\partial_t u(t)| \leq C \begin{cases} 
\exp((\frac{1}{2}+1)t)\|f\|_{C^{0,\beta}([\Sigma])} & \text{for } t \leq T_0, \\
\exp((\frac{1}{2}-\mu)t)\|f\|_{C^{0,\beta}([\Sigma])} & \text{for } t \geq T_1.
\end{cases} \tag{2.63}
\]

Pulling back the change of variables, we find from (2.63), that \(q(s) \sim s^2\) as \(s \to 0^+\). In particular, \(q(0) = \partial_s q(0) = 0\) so that \(q\) can be extended to an even function over \(\mathbb{R}\).

We also find from (2.23), (2.39), (2.40) and (2.63) that

\[
\|D^2_q q\|_{\infty,2+\mu} + \|\nabla_{\Sigma} q\|_{\infty,1+\mu} + \|q\|_{\infty,\mu} \leq c\|f\|_{C^{0,\beta}([\Sigma])}, \tag{2.64}
\]

We finish the proof of the estimate (2.61) by applying standard Hölder regularity. Since the coefficient \(\alpha(s)\) in (2.23) is singular at the origin, we rather use regularity theory to the corresponding partial differential equation (2.21). Thus, using (2.21) we notice that for \(s(p) \in (0, \frac{1}{2})\) we have

\[
\|q\|_{C^{2,\beta}([-\frac{1}{2}, \frac{1}{2}])} \leq c\left(\|q\|_{L^{\infty}(B_{\frac{1}{2}}(p))} + \|f\|_{C^{0,\beta}(B_{\frac{1}{2}}(p))}\right) \leq \|f\|_{C^{0,\beta}(B_{\frac{1}{2}}(p))}.
\]

If \(s(p) \geq \frac{1}{2}\), the same estimate follows from (2.23), (2.64) and the fact that \(\alpha\) is bounded and smooth outside a neighbourhood of the origin. This completes the proof of the proposition. \(\Box\)

3. The Jacobi-Toda equation

In this section we provide a detailed proof of the Theorem 1.2. We proceed by studying solvability theory for the equation

\[
\delta J_\Sigma h - 2a_+ e^{-\sqrt{2}h} = 0, \tag{3.1}
\]

where \(\delta > 0\) is a small parameter and \(a_+ > 0\) is a constant. We also recall that \(J_\Sigma\) is the Jacobi operator of \(\Sigma\) described in (2.21), (2.22) and (2.23).

In what follows we will use the following notation. For a function \(v\) defined on \(\Sigma\), set

\[
E_\delta(v) := \delta J_\Sigma v - 2a_+ e^{-\sqrt{2}v} \tag{3.2}
\]

and also denote

\[
Q(t) := e^{-\sqrt{2}t} - 1 + \sqrt{2}t \quad \text{for } t \in \mathbb{R}. \tag{3.3}
\]

To solve (3.1), we look for an \(O(m) \times O(n)\)-invariant solution \(h\) having the form

\[
h = v + q \quad \text{in } \Sigma,
\]

where \(v\) is an approximate solution and \(q\) is a small correction to get a genuine solution of (3.1).

A direct calculation shows that (3.1) becomes

\[
\delta J_\Sigma q + 2\sqrt{2}a_+ e^{-\sqrt{2}v}q = -E_\delta(v) + 2a_+ e^{-\sqrt{2}v}Q(q) \quad \text{in } \Sigma. \tag{3.4}
\]
The strategy consists in selecting \( v \) as accurately as possible so that the term \( E_\delta(v) \) is small for \( \delta > 0 \) small, in some suitable topology that allows us to study solvability theory for the linear operator
\[
L_\delta(q) := \delta J_\Sigma q + 2\sqrt{2}a_*e^{-\sqrt{2}\delta}q.
\]

3.1. The approximate solution. Let us now choose the approximate solution \( v \). In this part, we will make extensive use of the Lambert function \( W : [0, \infty) \rightarrow \mathbb{R} \) defined implicitly as the solution of the algebraic equation
\[
W(z)e^{W(z)} = z
\]
for any given \( z \geq 0 \). It is well known that
\[
W(z) = \begin{cases} 
  z - z^2 + O(z^4), & \text{as } z \to 0^+ \\
  \log(z) - \log \left( \log(z) \right) + O \left( \frac{\log(\log(z))}{\log(z)} \right), & \text{as } z \to \infty
\end{cases}
\]
and these relations can be differentiated. This is the essence of the following lemma.

**Lemma 3.1.** The function \( W \in C^\infty[0, \infty) \) and satisfies that for any \( i \in \mathbb{N} \), there exists a constant \( C_i > 0 \) such that for any \( z \geq 0 \),
\[
|W^{(i)}(z)| \leq \frac{C_i}{(1 + z)^i}.
\]

**Proof.** We briefly sketch the proof of (3.6). Set
\[
f_1(z) := \frac{z}{1 + z} \quad \text{for } z \geq 0
\]
and observe that for any \( l \in \mathbb{N} \),
\[
f_1^{(l)}(z) = \frac{(-1)^{l+1}}{(1 + z)^{l+1}}.
\]
Define recursively for \( i \geq 1 \),
\[
f_{i+1}(z) = if_i(z) - \frac{z}{z+1}f'_i(z) \quad \text{for } z \geq 0.
\]
We claim that for any \( i \geq 1 \) and any \( l \geq 0 \), \( f_i^{(l)} \in L^\infty(0, \infty) \), for some \( c_i > 0 \),
\[
f_i(z) = c_i z^i (1 + o(1)), \quad z \to 0^+
\]
and this relation can be differentiated.

This previous claim is obviously true for \( i = 1 \). As for the case \( i = 2 \), a direct computation yields that
\[
f_2(z) = \frac{z^2(z+2)}{(z+1)^3} \quad \text{for } z \geq 0
\]
and the claim holds true also in this case.

We proceed next by induction assuming that for some \( c_i > 0 \), \( f_i(z) = i(i-1)z^{i-1}(1 + o(1)) \) as \( z \to 0^+ \). Then, as \( z \to 0^+ \),
\[
f_{i+1}(z) = ic_i z^i (1 + o(1)) - \frac{z}{1 + z} \left( ic_i z^{i-1}(1 + o(1)) \right)
\]
\[
= c_{i+1} z^{i+1} (1 + o(1)),
\]
for some \( c_{i+1} > 0 \). This completes the proof of the claim.
Again, an argument by induction shows that
\[ W^{(i)}(z) = \frac{(-1)^{i+1}}{z^i} f_i(W(z)) \quad \text{for} \quad z \geq 0. \]  

Thus (3.6) follows from (3.8) and (3.9). This completes the proof of the lemma. \[ \square \]

Next lemma states that we can choose an approximate solution as accurate as needed.

**Lemma 3.2.** For any \( \delta > 0 \) and for any \( j \geq 0 \), there exist \( O(m) \times O(n) \)-invariant functions \( \omega_0, \ldots, \omega_j \) defined on \( \Sigma \) which are smooth and such that the function \( v_j \) defined by
\[ v_j := \omega_0 + \cdots + \omega_j \]  
and written in the coordinate \( s = s(p) \) as \( v_j(p) = v_j(s) \), satisfies the estimate
\[ |v_j - \frac{1}{\sqrt{2}} \left( \log(s^2 + 2) + \log(\delta) \right)| \leq \frac{C}{|\log(\delta)|^{j/2}}, \]

Moreover, the error defined in (3.2) for \( v_j \) is given by
\[ E_{\delta}(v_j) = \delta A_{\Sigma}^2 \omega_j \quad \text{in} \quad \Sigma \]
and in the coordinate \( s = s(p) \), satisfies the estimate
\[ |E_{\delta}(v_j)| \leq \frac{C_j \delta}{(1 + s^2)(\log(s^2 + 2))^{j/2}} |\log(\delta)|^{j/2} \]

for some constant \( C_j > 0 \) depending only on \( j \).

**Proof.** We proceed recursively to find \( \omega_0, \omega_1, \ldots, \omega_j \). For \( j = 0 \), choose \( \omega_0 \) solving the algebraic equation
\[ \delta |A_\Sigma|^2 \omega_0 - 2a \star e^{-\sqrt{2} \omega_0} = 0 \quad \text{in} \quad \Sigma. \]

A direct calculation yields that
\[ \omega_0 := \frac{1}{\sqrt{2}} W \left( \frac{2 \sqrt{2} a \star}{\delta |A_\Sigma|^2} \right), \]

where \( W \) is the Lambert function.

Write \( \omega_0(s) = \omega_0(p) \) for \( s = s(p) \). If \( \delta > 0 \) is small enough then, \( \delta^{-1} |A_\Sigma|^2 \) is large and using the function \( \beta(s) = |A_\Sigma(p)|^2 \), we find from (3.5) that
\[ \omega_0(s) = \frac{1}{\sqrt{2}} \log \left( \frac{2 \sqrt{2} a \star}{\delta \beta(s)} \right) - \frac{1}{\sqrt{2}} \log \left( \log \left( \frac{2 \sqrt{2} a \star}{\delta \beta(s)} \right) \right) \]
\[ + O \left( \frac{\log(\delta^{-1}(s^2 + 2))}{\log(\delta^{-1}(s^2 + 2))} \right) \]

for any \( s \in \mathbb{R} \).

From (3.6) and iterating the chain rule (see [27]), we find that for any \( i \in \mathbb{N} \), there exists \( C_i > 0 \), depending only on \( \Sigma \), such that
\[ |\partial_s^{(i)} \omega_0| \leq \frac{C_i}{(1 + s)^i}. \]
In particular, there exists \( C > 0 \), independent of \( \delta > 0 \), such that (abusing the notation)

\[
|\Delta_{\Sigma} w_0| \leq \frac{C}{(1 + s)^2},
\]

so that (3.12) holds true for \( j = 0 \).

Next, we choose \( w_1 \) solving the algebraic equation

\[
|A_{\Sigma}|^{-2} \Delta_{\Sigma} w_0 + w_1 - w_0 (e^{-\sqrt{2}w_1} - 1) = 0
\]

from where we find that

\[
w_1 := -w_0 - |A_{\Sigma}|^{-2} \Delta_{\Sigma} w_0 + \frac{1}{\sqrt{2}} W \left( \frac{2 \sqrt{|A_{\Sigma}|}}{|\delta| |A_{\Sigma}|^2} e^{\sqrt{2} |A_{\Sigma}|^{-2} \Delta_{\Sigma} w_0} \right).
\]

Using (3.18),

\[
E_\delta(v_1) = E_\delta(w_0) + \delta \Delta_{\Sigma} w_1 + \delta |A_{\Sigma}|^2 w_1 - \delta |A_{\Sigma}|^2 w_0 (e^{-\sqrt{2}w_1} - 1)
\]

so that (3.12) holds true for \( j = 1 \).

On the other hand, the asymptotic expansion

\[
-a - b + W(ae^{-a+b}) = -b \left( \frac{1}{a} \right) + O \left( \frac{1}{a^2} \right) \quad \text{as} \quad a \to \infty
\]

holds uniformly for \( b \) on any compact interval of \([0, \infty)\). Also, (3.17) yields that the term \( |A_{\Sigma}|^{-2} \Delta_{\Sigma} w_0 \) is uniformly bounded in the hypersurface \( \Sigma \) and in the parameter \( \delta \).

Therefore, from (3.15) and (3.19), there exists \( C > 0 \), independent of \( \delta > 0 \) such that for any \( s \in \mathbb{R} \),

\[
|w_1(s)| \leq \frac{C}{\log(s + 2) \log(s + 2)},
\]

where we have written \( w_1(s) = w_1(p) \) for \( s = s(p) \).

Next, we show how to proceed recursively to find \( w_j \). Write

\[
a_0 := \sqrt{2} w_0, \quad b_0 := \sqrt{2} |A_{\Sigma}|^{-2} \Delta_{\Sigma} w_0
\]

and for \( j \geq 1 \), define

\[
\begin{cases}
  a_j := e^{-\sqrt{2}w_j}, \\
  b_j := \sqrt{2} |A_{\Sigma}|^{-2} \Delta_{\Sigma} w_j \\
  \sqrt{2} w_{j+1} := -a_j - b_j + W(a_j e^{a_j + b_j}).
\end{cases}
\]

We remark that \( w_{j+1} \) solves the algebraic equation

\[
b_j + \sqrt{2} w_{j+1} - a_j (e^{-\sqrt{2}w_{j+1}} - 1) = 0
\]

or equivalently,

\[
\delta \Delta_{\Sigma} w_j + \delta |A_{\Sigma}|^2 w_{j+1} - \delta |A_{\Sigma}|^2 w_0 e^{-\sqrt{2}(w_1 + \cdots + w_j)} (e^{-\sqrt{2}w_{j+1}} - 1) = 0.
\]
We notice also that (3.20) yields
\[
\begin{align*}
u_{j+1} &= -w_0 e^{-\sqrt{2}w_{j+1} + u_j} - |A_\Sigma|^{-2} \Delta_\Sigma u_j \\
&\quad + \frac{1}{\sqrt{2}} W \left( \sqrt{2}w_0 \exp(- \sqrt{2}(w_1 + \cdots + w_j) - w_0 e^{-\sqrt{2}(w_1 + \cdots + w_j)} - |A_\Sigma|^{-2} \Delta_\Sigma u_j) \right).
\end{align*}
\]

Next, we prove that for any \( j \geq 0 \), (3.12) holds true. The cases \( j = 0 \) and \( j = 1 \) are already proven.

Proceeding inductively, for \( j \geq 1 \) we assume that \( E_\delta(w_j) = \delta \Delta_\Sigma u_j \). A direct calculation yields that
\[
E_\delta(w_{j+1}) = E_\delta(w_j) + \delta \Delta_\Sigma u_{j+1} + \delta |A_\Sigma|^2 w_{j+1} - 2a_1 e^{-\sqrt{2}w_j}(e^{-\sqrt{2}w_{j+1}} - 1)
\]
\[
= \delta \Delta_\Sigma u_{j+1} + \delta |A_\Sigma|^2 w_{j+1} - \delta |A_\Sigma|^2 w_0 e^{-\sqrt{2}(w_1 + \cdots + w_j)}(e^{-\sqrt{2}w_{j+1}} - 1).
\]

From (3.22), we conclude that \( E_\delta(w_{j+1}) = \delta \Delta_\Sigma u_{j+1} \). This proves the inductive steps and concludes the proof of (3.12).

Write \( w_j(s) = u_j(p) \) for \( s = s(p) \) and for \( j \in \mathbb{N} \). Next, we show that for any \( j \in \mathbb{N} \) and any \( i \in \mathbb{N} \cup \{0\} \),
\[
|\partial_s^{(i)} w_j(s)| \leq \frac{C}{(s+1)^i (\log(s+2))^j} |\log \delta|^j
\]
for any \( s \geq 0 \). (3.24)

From (3.16), estimate (3.24) holds true for \( j = 0 \) and for any \( i \geq 1 \). Next, assume that \( w_1, \ldots, w_j \) satisfy (3.24). We prove that (3.24) holds true also for \( j + 1 \).

Differentiating the third equation in (3.20) and using (3.9),
\[
\partial_s(\sqrt{2} w_{j+1}) = -\frac{a_j}{1 + W(a_j e^{a_j+b_j})} \partial_s \left( \frac{b_j}{a_j} \right) + \frac{\partial_s a_j}{a_j} \frac{\sqrt{2} w_{j+1}}{1 + W(a_j e^{a_j+b_j})}
\]
\[
= -\frac{a_j}{1 + W(a_j e^{a_j+b_j})} \left( \frac{\partial_s b_j}{a_j} - \frac{\partial_s a_j}{a_j^2} b_j \right) + \frac{\partial_s a_j}{a_j} \frac{\sqrt{2} w_{j+1}}{1 + W(a_j e^{a_j+b_j})}
\]
for \( j \geq 1 \).

Since (3.24) holds true for \( w_j \), for some constant \( C > 0 \), depending only on \( j \) and for any \( s \geq 0 \),
\[
|b_j| \leq \frac{C}{\log(s+2)^j |\log \delta|^j}.
\]
On the other hand, since we are assuming that \( w_1, \ldots, w_j \) satisfy (3.24) and noticing that
\[
a_j = a_{j-1} e^{-\sqrt{2}(w_{j-1} + w_j)}
\]
\[
a_0 = a_0 e^{-\sqrt{2}(w_1 + \cdots + w_j)},
\]
then for some constant \( c > 0 \), that is independent of \( \delta > 0 \) small,
\[
a_j \geq ca_0.
\]
(3.27)

Putting together (3.26) and (3.27), we find that for any \( s \geq 0 \),
\[
\frac{b_j}{a_j} \leq \frac{C}{\log(s+2)^j |\log \delta|^j}.
\]
(3.28)

where again \( C > 0 \) is a constant independent of \( \delta > 0 \).
Using (3.19), (3.20) and (3.28), for any \( s \geq 0 \),
\[
|w_{j+1}| \leq \frac{C}{\log (s + 2)^{\frac{s+1}{2}} |\log \delta|^\frac{s+1}{2}}.
\] (3.29)

As a by-product of the previous analysis, we also find from (3.10) and (3.15) that for any \( j \geq 0 \), (3.11) holds true.

Using an induction procedure over \( j \) and (3.20), it can be proven that there exists a constant \( C \) depending on \( j \), but not on \( \delta > 0 \) such that
\[
|\partial_s b_j| \leq C(s^2 |\partial_s^3 w_j| + s |\partial_s^2 w_j| + |\partial_s w_j|) \leq \frac{C}{(s + 1)(\log (s + 2))^{\frac{s+1}{2}} |\log \delta|^{\frac{s+1}{2}}}
\] (3.30)
and
\[
|\partial_s a_j| = \frac{|\partial_s a_0 - \sqrt{2} \partial_s (w_1 + \cdots + w_j)|}{C} \leq \frac{C}{(s + 1)(\log (s + 2))^{\frac{s+1}{2}} |\log \delta|^{\frac{s+1}{2}}}
\] (3.31)

Putting together (3.25), (3.28), (3.29), (3.30) and (3.31), we find that (3.24) holds true for \( i = 1 \).

We finish the proof of (3.24) for any \( i \geq 1 \) by differentiating (3.25), using the result in [27] and performing an inductive procedure over \( i \), with an arbitrary, but fixed \( j \).

Estimate (3.13) readily follows from (2.37), (2.38), (3.24) and the fact that the functions we are dealing with are even and smooth. This completes the proof of the lemma.

3.2. The linearised Jacobi-Toda operator. Let \( j \in \mathbb{N} \) be fixed, to be specified later. Consider the function \( v_j(p) = v_j(s(p)) \) for \( p \in \Sigma \), defined in (3.10).

In this part, we follow the conventions and notations from subsections 2.5, 2.6 and 2.7. We study solvability theory for the linear problem
\[
\delta J_k q + 2\sqrt{2} a_s e^{-\sqrt{2} q} q = \delta f \quad \text{in} \quad \Sigma,
\] (3.32)
where \( f : \Sigma \to \mathbb{R} \) is continuous and \( O(m) \times O(n) \)-invariant, i.e. \( f(p) = f(s) \) with \( f : \mathbb{R} \to \mathbb{R} \) continuous and even.

3.3. The Emden-Fowler change of variables. As in subsection 2.5, we first describe (3.32) in suitable coordinates. Using the symmetries, we write \( f(p) = f(s) \) and \( q(p) = q(s) \) so that (2.21) reduces to the ODE
\[
\partial_s^2 q + \alpha(s) \partial_s q + \beta(s)(1 + \sqrt{2} w_0 e^{-\sqrt{2} q}) q = f \quad \text{in} \quad \mathbb{R},
\] (3.33)
where we recall that \( v_j \) is defined and estimated in Lemma 3.2 and
\[
\beta(s) := |A_k|^2, \quad \alpha(s) := (m - 1) \frac{a'}{a} + (n - 1) \frac{b'}{b}.
\]
Even more, from (3.5), (3.11) and (3.15) in Lemma 3.2 and setting \( \sigma := \log \left( \frac{2\sqrt{2} a_s}{\delta} \right) \),
\[
v_j(s) = \frac{1}{\sqrt{2}} \sigma + \frac{1}{\sqrt{2}} \log (\beta^{-1}(s)) - \frac{1}{\sqrt{2}} \log \left( \sigma + \log (\beta^{-1}(s)) \right) + o(1)
\] (3.34)
for any \( s \in \mathbb{R} \) and as a corollary of the proof of Lemma 3.2, this relation can be differentiated.

Since we are looking for an even solution, we study the equation for \( s \geq 0 \) and consider the Emden-Fowler change of variables \( s = e^t \).

We recall from (2.24)
\[
\tilde{\alpha}(t) := \alpha(e^t) e^t - 1 \quad \text{and} \quad \tilde{\beta}(t) := \beta(e^t) e^{2t}
\] (3.35)
and
\[
p(t) := \exp \left( - \int_0^t \tilde{\alpha}(\tau) \, d\tau \right) \quad \text{for} \quad t \in \mathbb{R}
\] (3.36)
and that \( p(t) \) is chosen solving the ODE
\[
2 \frac{\partial p}{p} + \tilde{\alpha}(t) = 0.
\] (3.37)

Next, we denote
\[
\tilde{w}(t) := 1 + \sqrt{2} w_0(e^t) e^{-\sqrt{2}(v_j(e^t) - w_0(e^t))} \quad \text{for} \quad t \in \mathbb{R}
\] (3.38)
and
\[
Q := \frac{\partial^2 p}{p} + \tilde{\alpha} \frac{\partial p}{p} + \tilde{w} \tilde{\beta} \quad \text{in} \quad \mathbb{R}
\] (3.39)
Equation (3.33) reduces to the ODE
\[
\partial_t^2 v + Q(t) v = \tilde{f} \quad \text{in} \quad \mathbb{R},
\] (3.40)
where as in (2.27),
\[
\tilde{f}(t) := \frac{e^{2t}}{p(t)} f(e^t).
\]

3.4. The homogeneous problem. Next we study the solutions of the homogeneous equation
\[
\partial_t^2 v + Q(t) v = 0 \quad \text{in} \quad \mathbb{R}.
\] (3.41)

Since the coefficient \( Q(t) \) is smooth, from the standard theory of linear ODE’s, we can select two smooth solutions \( v(t), \tilde{v}(t) \) of (3.41) that satisfy some additional conditions making them linearly independent.

The first step in studying these solutions consists in analysing the asymptotic behavior of the potential \( Q(t) \) as \( t \to \pm \infty \).

We recall from (2.41), (2.42) and (2.43) that for some \( T_0, T_1 \in \mathbb{R} \) with \( T_0 < 0 < T_1 \) and for some \( c_0 > 0 \),
\[
\tilde{\alpha}(t) = \begin{cases} 
(n - 2) + \mathcal{O}(e^{2t}) & \text{for} \quad t < T_0, \\
(N - 2) + \mathcal{O}(e^{-t}) & \text{for} \quad t > T_1,
\end{cases}
\] (3.42)
\[
\tilde{\beta}(t) = \begin{cases} 
c_0 e^{2t} + \mathcal{O}(e^{4t}) & \text{for} \quad t < T_0, \\
N - 1 + \mathcal{O}(e^{-t}) & \text{for} \quad t > T_1
\end{cases}
\] (3.43)
and
\[
p(t) = \begin{cases} 
e^{-\frac{n-2}{2} t} (1 + \mathcal{O}(e^{2t})) , & \text{for} \quad t < T_0, \\
e^{-\frac{n-2}{2} t} (1 + \mathcal{O}(e^{-t})) , & \text{for} \quad t > T_1.
\end{cases}
\] (3.44)
We recall also that the asymptotic behaviors described in (3.42), (3.43) and (3.44) can be differentiated in the variable $t$.

From (3.34),

$$\tilde{w}(t) = \begin{cases} \sigma + \mathcal{O}(1), & \text{for } t < T_0 \\ \sigma + 2t + \mathcal{O}(\ln(t)), & \text{for } t > T_1 \end{cases}$$

and this relations can be differentiated.

Putting together (2.25), (3.42), (3.43) and (3.45),

$$Q(t) = \begin{cases} -\frac{(n-2)^2}{4} + c_0 \sigma e^{2t} + \mathcal{O}(e^{2t}) + \mathcal{O}(\sigma e^{4t}), & \text{for } t < T_0 \\ (N-1)\sigma + 2t + \mathcal{O}(\ln(t)) + \mathcal{O}(\sigma e^{-t}) + \mathcal{O}(te^{-t}), & \text{for } t > T_1 \end{cases}$$

and this relations can be differentiated.

From (3.46), notice that the potential $Q(t)$ has three different qualitative regimes regarding its sign. We describe next the solutions $v(t), \tilde{v}(t)$ of (3.41) in each of these regions.

Recall from (2.51) that we have set

$$\lambda := \frac{n-2}{2}$$

and notice from (3.46) that for some constants $C_0, C_1, c_1 > 0$,

$$\begin{cases} -\lambda^2 + c_0 \sigma e^{2t} \leq Q(t) \leq -\lambda^2 + C_0 \sigma e^{2t}, & \text{for } t < T_0 \\ 0 < c_1 \sigma \leq Q(t) \leq C_1 \sigma, & T_0 < t \leq T_1 \\ c_2 (\sigma + t) \leq Q(t) \leq C_2 (\sigma + t), & t > T_1. \end{cases}$$

### 3.5. Estimates for negative potential.

Write

$$Q(t) := -(\lambda^2 + q(t)) \quad \text{for } t \in \mathbb{R}. \quad (3.48)$$

Next, fix $\eta > \frac{1}{2}$ and assume that

$$\sigma > \left(\frac{2C_0}{\lambda^2}\right)^{\frac{1}{2\eta-1}}, \quad (3.49)$$

where $C_0 > 0$ is given in (3.47).

Define $t_\sigma := -\eta \log(\sigma)$ and notice from (3.47) and (3.49) that

$$-\lambda^2 + c_0 \sigma e^{2t} \leq Q(t) \leq -\lambda^2 \quad \text{in } (-\infty, t_\sigma). \quad (3.50)$$

Write the solution $v(t)$ of (3.41) as $v(t) = e^{-\lambda t}x(t)$. Setting $y(t) := e^{-2\lambda t}x'(t)$, we find that $x(t)$ and $y(t)$ must solve the system

$$\begin{cases} x' = e^{2\lambda t}y \\ y' = q(t)e^{-2\lambda t}. \end{cases} \quad (3.51)$$

We select the solution $v(t)$ by fixing the initial conditions for $(x, y)$,

$$x(t_\sigma) = 1 \quad \text{and} \quad y(t_\sigma) = 0. \quad (3.52)$$
Integrating (3.51), we have for \( t \leq t_\sigma \),

\[
x(t) = 1 - \int_t^{t_\sigma} e^{2\lambda \tau} y(\tau) d\tau, \quad y(t) = - \int_t^{t_\sigma} q(\tau)e^{-2\lambda \tau} x(\tau) d\tau
\]  

(3.53)

and from Fubini's Theorem and since \( n > 2 \),

\[
x(t) = 1 + \int_t^{t_\sigma} x(\zeta) q(\zeta) \frac{1 - e^{-2\lambda(\zeta-t)}}{2\lambda} d\zeta.
\]

We estimate for \( t \leq t_\sigma \),

\[
|x(t)| \leq 1 + \int_t^{t_\sigma} |x(\zeta)||q(\zeta)| \frac{1 - e^{-2\lambda(\zeta-t)}}{2\lambda} d\zeta,
\]

and using the Gronwall inequality and (3.47),

\[
|x(t)| \leq \exp\left( \int_t^{t_\sigma} \frac{|q(\zeta)|}{2\lambda} d\zeta \right)
\]

\[
\leq \exp\left( c\sigma \int_t^{t_\sigma} e^{2\zeta} \right)
\]

\[
\leq e^{\frac{c\sigma}{2} e^{1-2\eta}}
\]

for \( t \leq t_\sigma \). Since \( \eta > \frac{1}{2} \), we conclude that \( x \in L^\infty(-\infty, t_\sigma) \) and by choosing \( \sigma \) larger if necessary, we have that

\[
\|x\|_{L^\infty(-\infty, t_\sigma)} \leq e^{\frac{c\sigma}{2} e^{1-2\eta}} \leq 2,
\]

which is uniformly bounded in \( \sigma \).

Consequently, from (3.47) and the equation for \( y(t) \) in (3.53),

\[
|y(t)| \leq c\sigma \int_t^{t_\sigma} e^{2(1-\lambda)\tau} d\tau \leq \begin{cases} 
\frac{c\sigma e^{2(1-\lambda)(t_\sigma-t)} - e^{2(1-\lambda)1}}{2(1-\lambda)} & \lambda \neq 1 \\
\frac{c\sigma (t_\sigma-t)}{2(1-\lambda)} & \lambda = 1.
\end{cases}
\]

Using again (3.47) and the first equation in system (3.51), we get the estimate

\[
|x'(t)| \leq \begin{cases} 
\frac{c\sigma}{2(1-\lambda)}(\sigma^{-2\eta} e^{2\lambda(t_\sigma-t)} - e^{2t}) & \lambda \neq 1 \\
\frac{c\sigma e^{2t}(t_\sigma-t)}{2(1-\lambda)} & \lambda = 1,
\end{cases}
\]

(3.55)

for \( t \leq t_\sigma \). By an elementary maximisation argument, it follows that

\[
|x'(t)| \leq c\sigma^{1-2\eta}.
\]

(3.56)

Since,

\[
|x(t) - 1| = |x(t) - x(t_\sigma)| \leq \int_t^{t_\sigma} |x'(\tau)| d\tau \leq \int_t^{t_\sigma} |x'(\tau)| d\tau
\]

by integrating (3.55) and using integration by parts for the case \( \lambda = 1 \), we get that for any \( t < t_\sigma \),

\[
|x(t) - 1| \leq c\sigma^{1-2\eta}.
\]

(3.57)
Choosing again $\sigma$ larger if necessary, we obtain that $x \geq \frac{1}{2}$ in $(-\infty, t_\sigma)$. Using (3.51) and the fact that $q(t) < 0$, we conclude that $y' < 0$.

Proceeding in a similar fashion and since $y(t_\sigma) = 0$, we have $y > 0$ in $(-\infty, t_\sigma)$ and therefore $x' > 0$.

Recall that $v(t) = e^{-\lambda t} x(t)$, so that $\partial_t v = (-\lambda x + x')e^{-\lambda t}$. From this remark and (3.52),

$$v(t_\sigma) = \sigma^{\eta \lambda} \quad \text{and} \quad \partial_t v(t_\sigma) = -\lambda \sigma^{\eta \lambda}.$$

On the other hand, from (3.55), (3.56) and (3.57),

$$(1 - c\sigma^{1 - 2\eta})e^{-\lambda t} \leq v(t) \leq (1 + c\sigma^{1 - 2\eta})e^{-\lambda t} \quad (3.58)$$

and

$$-\lambda(1 + c\sigma^{1 - 2\eta})e^{-\lambda t} \leq \partial_t v(t) \leq -\lambda(1 - c\sigma^{1 - 2\eta})e^{-\lambda t} \quad (3.59)$$

for any $t \in (-\infty, t_\sigma)$.

We select the second linearly independent solution $\tilde{v}(t)$ of (3.41) by setting

$$\tilde{v}(t) := v(t) \int_{-\infty}^{t} v^{-2}(\tau)d\tau \quad \text{for} \quad t \in (-\infty, t_\infty) \quad (3.60)$$

and directly from (3.60) we find that

$$\frac{1}{2\lambda}(1 - c\sigma^{1 - 2\eta})e^{\lambda t} \leq \tilde{v}(t) \leq \frac{1}{2\lambda}(1 + c\sigma^{1 - 2\eta})e^{\lambda t},$$

$$\frac{1}{2}(1 - c\sigma^{1 - 2\eta})e^{\lambda t} \leq \partial_t \tilde{v}(t) \leq \frac{1}{2}(1 + c\sigma^{1 - 2\eta})e^{\lambda t} \quad (3.61)$$

for $t \in (-\infty, t_\sigma)$.

Observe that

$$\tilde{v}(t_\sigma) = \frac{1}{2\lambda} \sigma^{-\eta \lambda}(1 + O(\sigma^{1 - 2\eta}))$$

$$\partial_t \tilde{v}(t_\sigma) = \frac{1}{2\lambda} \sigma^{-\eta \lambda}(1 + O(\sigma^{1 - 2\eta}))$$

so that the Wronskian of $v(t)$ and $\tilde{v}(t)$ is given by

$$W(t) = v(t_\sigma)\partial_t \tilde{v}(t_\sigma) - \tilde{v}(t_\sigma)\partial_t v(t_\sigma)$$

$$= \sigma^{\eta \lambda} \frac{\sigma^{-\eta \lambda}}{2\lambda} (-\lambda \sigma^{\eta \lambda}) + O(\sigma^{1 - 2\eta})$$

$$= 1 + O(\sigma^{1 - 2\eta}). \quad (3.62)$$

3.6. The transition region for the potential. Recall that we are assuming that $\sigma$ is large enough. Set

$$T_\sigma := \frac{1}{2} \log(\sigma) + \frac{1}{2} \log \left( \frac{\lambda^2 + 1}{c_0} \right),$$

where $\lambda = \frac{n-2}{2}$ and $c_0 > 0$ is given in (3.46).

Since $t_\sigma = -\eta \log(\sigma)$,

$$T_\sigma - t_\sigma = \left( \eta - \frac{1}{2} \right) \log(\sigma) + \frac{1}{2} \log \left( \frac{c_0}{\lambda^2 + 1} \right).$$
Next, we estimate \( v \) and \( \tilde{v} \) in the intermediate region \((t_\sigma, T_\sigma)\), where the potential \( Q(t) \) makes its only transition from negative to positive. Observe also that from our choice of \( T_\sigma \) and from (3.47), \( Q(t) = -\lambda^2 + q(t) \geq 1 \) for \( t \geq T_\sigma \). Observe also that \( q(t) > 0 \) for \( t \in [t_\sigma, T_\sigma] \).

We begin by analysing \( v(t) \) via the system (3.51).

Set

\[
A(t) := \begin{pmatrix} 0 & e^{2\lambda t} \\ q(t)e^{-2\lambda t} & 0 \end{pmatrix} \quad \text{for} \quad t \in [t_\sigma, T_\sigma]
\]

and observe that

\[
\begin{pmatrix} \partial_t x \\ \partial_t y \end{pmatrix} = A(t) \begin{pmatrix} x \\ y \end{pmatrix}
\]

and also that

\[
\|A(t)\| := \sup_{\|p\|=1} |A(t)p| = \max\{ |\mu| : \mu \text{ is an eigenvalue of } A(t) \}.
\]

Since the eigenvalues of \( A(t) \) are given by \( \pm \sqrt{q(t)} \), we find from (3.47) that \( \|A(t)\| \leq C\sigma^{\frac{1}{2}}e^t \) for \( t \in [t_\sigma, T_\sigma] \). Consequently,

\[
|v(t)| = |(x(t), y(t))| = |(x(t), y(t_\sigma))| + \int_{t_\sigma}^{t} |A(\tau)||v(\tau, y(\tau))|d\tau
\leq |(x(t), y(t_\sigma))| + C\sigma^{\frac{1}{2}}\int_{t_\sigma}^{t} e^\tau |(x(\tau), y(\tau))|d\tau.
\]

By the Gronwall inequality and the choices of \( t_\sigma \) and \( T_\sigma \),

\[
|x(t)| \leq |x(t_\sigma)| \exp\left( C\sigma^{\frac{1}{2}}\int_{t_\sigma}^{t} e^\tau d\tau \right)
\leq |x(t_\sigma)| \exp\left( C\sigma^{\frac{1}{2}}(e^{T_\sigma} - e^{t_\sigma}) \right)
\leq C|x(t_\sigma)|.
\]

In conclusion, the above discussion and (3.51) yield that for any \( t \in [t_\sigma, T_\sigma] \),

\[
|x(t)| + |x'(t)| \leq C. \tag{3.63}
\]

Going back to \( v \), we find that for any \( t \in [t_\sigma, T_\sigma] \),

\[
|v(t)| + |v'(t)| \leq C e^{-\lambda t}. \tag{3.64}
\]

It remains to estimate \( \tilde{v} \) and \( \partial_t \tilde{v} \) in \([t_\sigma, T_\sigma]\). Writing \( \tilde{v} = e^{\lambda t}\tilde{x} \) and \( \tilde{y} = e^{2\lambda t}\partial_t \tilde{x} \), (3.41) for \( \tilde{v} \) becomes the system

\[
\begin{cases}
\partial_t \tilde{x} = e^{-2\lambda t}\tilde{y} \\
\partial_t \tilde{y} = q(t)e^{2\lambda t}\tilde{x}(t)
\end{cases}
\]

for \( t \in [t_\sigma, T_\sigma] \).

Using the estimates (3.61) for \( \tilde{v} \) and \( \partial_t \tilde{v} \),

\[
\tilde{x}(t_\sigma) = \frac{1}{2\lambda} + O(\sigma^{1-2\eta}), \quad \tilde{y}(t_\sigma) = O(\sigma^{1-2\eta}).
\]
Using this initial data and (3.65), we proceed as we did above to prove (3.63) to find that
\[ |\tilde{x}(t)| + |\tilde{x}'(t)| \leq C. \tag{3.66} \]

Consequently, for some \( C > 0 \) independent of \( \sigma \),
\[ |\tilde{v}(t)| + |\tilde{v}'(t)| \leq C e^{\lambda t} \tag{3.67} \]
for \( t \in [t_\sigma, T_\sigma] \).

### 3.7. Strictly positive potential.

Now we will study the behaviour of the solutions \( v, \tilde{v} \) in the interval \((T_\sigma, \infty)\). In this interval, we have \( Q \geq 1 \) and hence introduce the change of variables
\[ \xi(t) = \int_{T_\sigma}^t Q(\tau) \frac{1}{2} d\tau, \quad \forall t \geq T_\sigma. \]

Write
\[ v(t) = Q(t)^{-\frac{1}{4}} w(\xi(t)) \quad \text{for} \quad t \geq T_\sigma \tag{3.68} \]
and observe from (3.41) that \( w \) must solve
\[ \partial^2_\xi w + (1 + V(\xi)) w = 0 \quad \text{in} \quad (0, \infty), \tag{3.69} \]
where
\[ V(\xi) := -\frac{\partial^2_\xi Q(t(\xi))}{4Q^2(t(\xi))} + \frac{5(\partial_t Q)^2(t(\xi))}{Q^3(t(\xi))}. \]

Since \( \partial_t Q \) and \( \partial^2_t Q \) are bounded in \([T_\sigma, \infty)\), we find from (3.44) that \( V \in L^1(0, \infty) \) and
\[ \int_0^\infty |V(\xi)| d\xi = \int_0^{\infty} |V(\xi(t))| Q^{\frac{1}{2}}(t) dt \leq \int_{T_\sigma}^{T_1} c dt + \int_{T_1}^{\infty} \frac{c}{(\sigma + t)^{\frac{1}{2}}} dt \leq K \log \sigma. \tag{3.70} \]

Moreover, differentiating (3.68) and evaluating at \( t = T_\sigma \), we get
\[ w(0) = Q(T_\sigma)^{\frac{1}{4}} v(T_\sigma) \quad \text{and} \quad \partial_\xi w(0) = \frac{\partial_t Q(T_\sigma)}{4Q^2(T_\sigma)} v(T_\sigma) - \frac{\partial_t v(T_\sigma)}{Q(T_\sigma)^{\frac{1}{2}}}, \]
which together with (3.64) yield that
\[ |w(0)| + |\partial_\xi w(0)| \leq c |v(T_\sigma)| + |\partial_t v(T_\sigma)| \leq C_\sigma^\frac{1}{2}. \tag{3.71} \]

The aim now is to estimate \( w \) and \( \partial_\xi w \) in the interval \((T_\sigma, \infty)\). In order to do so, we multiply (3.69) by \( \partial_\xi w \) to find that
\[ \partial_\xi (\partial_\xi w) + (w)^2 = -2V(\xi) w \partial_\xi w \]
and a direct integration yields that
\[ (\partial_\xi w(\xi))^2 + (w(\xi))^2 = \int_0^\xi -2V(z) w \partial_\xi w dz \leq \int_0^\xi |V(z)| \left((\partial_\xi w)^2 + (w)^2\right) dz. \]
From the Gronwall inequality, (3.70) and (3.71),
\[ |\partial_\xi w(\xi)| + |w(\xi)| \leq c\sigma^K(|w(0)| + |\partial_\xi w(0)|) \leq C\sigma^{\frac{1}{2}+K} \]
(3.72)
for any \( \xi > 0 \).

Going back to \( v \) and \( \partial_tv \),
\[ |v(t)| + |\partial_tv(t)| \leq c\sigma^{\frac{1}{2}+K}Q(t)^{-\frac{1}{4}} \]
(3.73)
for any \( t > T_\sigma \).

Using (3.67), the same argument applied to \( \tilde{v} \) yields for \( t > T_\sigma \) that
\[ |\tilde{v}(t)| + |\partial_t\tilde{v}(t)| \leq c\sigma^K(|\tilde{v}(T_\sigma)| + |\partial_t\tilde{v}(T_\sigma)|)Q(t)^{-\frac{1}{4}} \leq C\sigma^{-\frac{1}{4}+K}Q(t)^{-\frac{1}{4}}. \]
(3.74)

Putting together (3.58), (3.59), (3.64), (3.67), (3.73), (3.74) and (3.62), we have proven the following Proposition.

**Proposition 3.1.** There exist two linearly independent solutions \( v \) and \( \tilde{v} \) to the homogeneous equation (3.41) such that
\[ |v(t)| + |\partial_tv(t)| \leq ce^{-\lambda t}, \]
\[ |\tilde{v}(t)| + |\partial_t\tilde{v}(t)| \leq ce^\lambda t, \quad \forall t \leq T_\sigma \]
(3.75)
and
\[ |v(t)| + |\partial_tv(t)| \leq c\sigma^{\frac{1}{2}+K}Q(t)^{-\frac{1}{4}}, \quad \forall t > T_\sigma. \]
(3.76)

Moreover,
\[ W(t) \equiv W(t_\sigma) = 1 + o(1) \quad \text{as} \quad \sigma \to \infty. \]

3.8. **The linearised Jacobi-Toda equation.** Here we consider equation (3.33) with a non trivial even continuous right-hand side \( f : \Sigma \to \mathbb{R} \).

The topology of \( f \) is motivated by the behavior of the error \( E_\delta(v_j) \) described in (3.13). Thus, we introduce the following functional analytic setting.

For \( \beta \in (0, 1) \), \( \mu > 0 \) and \( \varrho \in \mathbb{R} \), we introduce the norms
\[ \|f\|_{*,\mu,\varrho} := \sup_{p \in \Sigma} (s(p)^2 + 2)^{\frac{\varrho}{2}} \log(s(p) + 2)\|f\|_{L^\infty(B_1(p))}. \]
(3.77)
\[ \|f\|_{D^{0,\beta}_\mu(\Sigma)} := \sup_{p \in \Sigma} (s(p)^2 + 2)^{\frac{\beta}{2}} \log(s(p) + 2)\|f\|_{C^{0,\beta}(B_1(p))}. \]
(3.78)
and we consider the Banach space \( D^{0,\beta}_\mu(\Sigma) \) defined as the space of \( O(m) \times O(n) \)-invariant functions \( f \in C^{0,\beta}_{loc}(\Sigma) \) for which the norm
\[ \|f\|_{D^{0,\beta}_\mu(\Sigma)} < \infty. \]
(3.79)

We also consider the Banach space \( D^{2,\beta}_\mu(\Sigma) \) defined as the space of \( O(m) \times O(n) \)-invariant functions \( q \in C^{2,\beta}_{loc}(\Sigma) \) for which the norm
\[ \|q\|_{D^{2,\beta}_\mu(\Sigma)} := \sigma^{-1}||D_\Sigma^2q||_{D^{0,\varrho+2,\beta-1}_\mu(\Sigma)} + ||\nabla_\Sigma q||_{*,\mu,\varrho+1,\varrho} + ||q||_{*,\mu,\varrho} < \infty. \]
(3.80)
The following proposition shows that (3.32) has an inverse in this functional analytic setting and also it allows us to estimate its size.

**Proposition 3.2.** Let $\beta \in (0, 1)$ and $\varrho > 0$. There exist $\sigma_0 > 0$ and a constant $C > 0$ such that for any $\sigma \in (0, \sigma_0)$ and any $f \in D^{0, \beta}_2(\Sigma)$, equation (3.32) has a solution $q := F_1(f) \in D^{2, \beta}_{2, e^{-t}}(\Sigma)$ satisfying the estimate

$$\|q\|_{D^{2, \beta}_{2, e^{-t}}(\Sigma)} \leq C\sigma^{2K}\|f\|_{D^{0, \beta}_2(\Sigma)},$$

where $K > 0$ is the constant in the estimate (3.76).

**Proof.** Since $f(p) = f(s)$ with $f$ even, we can solve the equation just for $s \geq 0$, and then extend the solution to the whole $\mathbb{R}$ by reflection.

After the Emden-Fowler change of variables $s = e^t$, we are lead to consider equation (3.40) with $\tilde{f}$ satisfying the estimate

$$|\tilde{f}(t)| \leq C \begin{cases} e^{(\frac{\lambda}{2} + 1)t}\|f\|_{D^{0, \beta}_2(\Sigma)} & \text{for } t \leq T_0, \\ e^{\frac{N-2}{2}t}\|f\|_{D^{0, \beta}_2(\Sigma)} & \text{for } t \geq T_1. \end{cases}$$

Let $v(t), \tilde{v}(t)$ be the solutions of (3.41) described in Proposition 3.1. A solution to (3.40) is given by the variation of parameters formula

$$u(t) := v(t) \int_{-\infty}^{t} \tilde{v}(\tau)W^{-1}(\tau)\tilde{f}(\tau)d\tau - \tilde{v}(t) \int_{-\infty}^{t} v(\tau)W^{-1}(\tau)\tilde{f}(\tau)d\tau. \quad (3.83)$$

Recall that $\lambda = \frac{N-2}{2}$. From (3.75) and (3.76) in Proposition 3.1 and using (3.82) and (3.83), we find that

$$|u(t)| + |\partial_t u(t)| \leq c\sigma^{2K}\|f\|_{D^{0, \beta}_2(\Sigma)} \begin{cases} e^{\frac{\lambda}{2}t}, & t \leq T_0, \\ 1, & T_0 < t < T_1, \\ t^{\frac{1}{2}}e^{\frac{N-2}{2}t}, & t > T_1. \end{cases}$$

Setting $q(s) := p(t(s))u(t(s))$ for $s \geq 0$, we find that $q(s) \sim s^2$ as $s \to 0$, so that we can extend $q$ smoothly to $\mathbb{R}$.

From the previous analysis and setting $q(p) = q(s)$ for $s = s(p)$, we find that

$$\|q\|_{\ast, 0, e^{-\frac{1}{2}}} \leq c\sigma^{2K}\|f\|_{D^{0, \beta}_2(\Sigma)}, \quad (3.84)$$

On the other hand, since

$$\partial_s q(s) = \frac{1}{s} \left( \partial_s p(t(s))u(t(s)) + p(t(s))\partial_t u(t(s)) \right)$$

with $t(s) = \log(s)$, we find that $\partial_s q(s) \sim s$ as $s \to 0$ and the gradient of the function $q(p) = q(s)$ satisfies the estimate

$$\|\nabla \Sigma q\|_{\ast, 1, e^{-\frac{1}{2}}} \leq c\sigma^{2K}\|f\|_{D^{0, \beta}_2(\Sigma)}, \quad (3.85)$$

Since $\partial_s q(0) = 0$, the even extension of $q$ yields a $C^2(\mathbb{R})$ solution of (3.33) so that $q(p) = q(s)$ is a solution of (3.32) in $\Sigma$.

On the other hand, from (3.33), (3.84) and (3.85)

$$\sigma^{-1}\|D^2_{\Sigma}q\|_{\ast, 2, e^{-\frac{1}{2}}} + \|\nabla \Sigma q\|_{1, e^{-\frac{1}{2}}} + \|q\|_{\ast, 0, e^{-\frac{1}{2}}} \leq c\sigma^{2K}\|f\|_{D^{0, \beta}_2(\Sigma)}. \quad (3.86)$$
We conclude the proof of the proposition as follows. Since the coefficient $\alpha(s)$ in ODE (3.33) is singular at $s = 0$, we need to pass through the elliptic PDE on $\Sigma$ in order to estimate the Hölder norm of the second derivative near $s = 0$.

From the standard local Hölder estimates applied to the equation (3.32),
\[
\Delta_{\Sigma} q + |A_{\Sigma}|^2 q = f - \sqrt{2|A_{\Sigma}|^2 w_0} e^{-\sqrt{2(v_j - w_0)}} q \quad \text{in} \quad \Sigma,
\]
we find that
\[
\|q\|_{C^{2,\beta}(B_{\frac{1}{2}}(\zeta))} \leq C(\|q\|_{L^\infty(B_2(\zeta))} + \|f\|_{C^{0,\beta}(B_{\frac{1}{2}}(\zeta))} + \|q\|_{C^{0,\beta}(B_{\frac{1}{2}}(\zeta))}) \\
\leq C(\|q\|_{L^\infty(B_2(\zeta))} + \|f\|_{C^{0,\beta}(B_{\frac{1}{2}}(\zeta))} + \|\nabla q\|_{L^\infty(B_2(\zeta))}) \\
\leq c\sigma^{1+2\beta}\|f\|_{P^{0,\beta}_{2,\alpha}(\Sigma)}.
\]

This yields that,
\[
\|q\|_{P^{2,\beta}_{0,\alpha-\frac{1}{2}}(\Sigma)} \leq c\sigma^{2\beta}\|f\|_{P^{0,\beta}_{2,\alpha}(\Sigma)}
\]
and this completes the proof. \qed

3.9. A fixed point argument and the proof of Theorem 1.2. In this part, we use the linear theory studied in the previous subsections to prove Theorem 1.2. The ideas here will also be used in the the section 5.

Let $\sigma_0 > 0$ be as in Proposition 3.2 and let $\sigma := \log \left(\frac{2\sqrt{2a}}{\delta}\right)$ with $\sigma > \sigma_0$.

Fix $j \in \mathbb{N}$, to be specified later and let $v_j(p) = v_j(\kappa(p))$ be the approximate solution of (3.1) described in Lemma 3.2.

From (3.13),
\[
\|E_\delta(v_j)\|_{P^{0,\beta}_{2,\alpha-\frac{1}{2}}(\Sigma)} \leq C\delta^{\sigma-\frac{1}{2}}. \quad (3.87)
\]

We look for a solution of (3.1) having the form $v = v_j + q$ so that we solve equation (3.4) for $q$. Using the operator
\[
L_\delta(q) := \delta J_{\Sigma} q + 2\sqrt{2a} e^{-\sqrt{2\delta} v_j} q,
\]
this equation reads as
\[
L_\delta(q) = -E_\delta(v_j) + \sqrt{2\delta}|A_{\Sigma}|^2 v_j Q(q), \quad (3.88)
\]
where $Q$ is the quadratic term defined in (3.3).

Set
\[
\mathcal{B} := \left\{ q \in P^{2,\beta}_{2,\alpha-\frac{1}{2}}(\Sigma) : \|q\|_{P^{2,\beta}_{2,\alpha-\frac{1}{2}}(\Sigma)} \leq A\sigma^{2\beta-\frac{1}{2}} \right\},
\]
where $A > 0$ is a constant independent of $\sigma$ and to be specified later.

Next, observe there exists a constant $c > 0$ such that for any $q \in \mathcal{B}$,
\[
|Q(q)| \leq c q^2
\]
and consequently,
\[
\|\delta|A_{\Sigma}|^2 v_j Q(q)\|_{P^{0,\beta}_{2,\alpha-\frac{1}{2}}(\Sigma)} \leq c A^2 \delta^{4\beta-\frac{1}{2}}. \quad (3.89)
\]
On the other hand, for some \( c > 0 \) independent of \( j \) and \( \sigma \) and for any \( q, \tilde{q} \in B \),
\[
|Q(q) - Q(\tilde{q})| \leq c|q + \tilde{q}||q - \tilde{q}| \quad \text{in} \quad \Sigma
\]
so that
\[
\|\delta|A|^{2}v_{j}(Q(q) - Q(\tilde{q}))\|_{E^{0, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma)} \leq 2Ac\delta\sigma^{2K-\frac{d}{4}}||q - \tilde{q}||_{E^{0, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma)}.
\] (3.90)

Next, let \( R : D^{2, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma) \rightarrow D^{2, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma) \) be defined by
\[
R(q) := \delta^{-1}L_{\delta}^{-1}\left(-E_{\delta}(v_{j}) + \sqrt{2}\delta|A|^{2}v_{j}Q(q)\right).
\]
Observe that \( R \) is the resolvent operator of the nonlinear equation (3.88).

We fix \( j > 8K, \sigma > 0 \) large and \( A > 2C \) large, but independent of \( \sigma \) and \( j \), so that from (3.87) and (3.89), we find that
\[
\|R(q)\|_{D^{2, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma)} \leq C\sigma^{2K}\left(\sigma^{-\frac{d}{4}} + cA^{2}\sigma^{4K-j}\right)
\leq C\sigma^{2K-\frac{d}{4}}\left(1 + O(\sigma^{2K-\frac{d}{4}})\right)
\leq 2C\sigma^{2K-\frac{d}{4}}
\leq A\sigma^{2K-\frac{d}{4}}
\]
for any \( q \in B \). Thus, \( R : B \rightarrow B \) is well defined.

Next, we verify the contractive character of \( R \) restricted to \( B \). Using (3.90), we find that for some constant \( M > 0 \), independent of \( \sigma \) and \( j \),
\[
\|R(q) - R(\tilde{q})\| \leq \|\delta^{-1}L_{\delta}^{-1}\|\|A_{\Sigma}^{2}v_{j}(Q(q) - Q(\tilde{q}))\|_{E^{0, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma)}
\leq M\sigma^{2K-\frac{d}{4}}||q - \tilde{q}||_{D^{2, \beta}_{\epsilon, \frac{\delta}{4}}(\Sigma)}
\]
for any \( q, \tilde{q} \in B \). Since \( \sigma > 0 \) is large enough and \( j > 8K \), \( R : B \rightarrow B \) is a contraction.

A direct application of the contraction mapping principle yields the existence of a unique \( q \in B \) solving (3.88). This completes the proof of Theorem 1.2.

4. THE APPROXIMATE SOLUTION TO THE ALLEN-CAHN EQUATION

In this part we find an appropriate approximate solution to (1.1) and compute its error in a suitable coordinate system.

4.1. Fermi coordinates. Recall from Section 3 that in our developments we are \( \Sigma = \Sigma_{-m,n} \). First, we introduce the system of coordinates that we will use to describe the Laplacian near a dilated and translated version of the hypersurface \( \Sigma \).

Using (2.10) and the fact that \( \Sigma \) is asymptotic to the cone \( C_{m,n} \) stressed out in (2.35), we find \( \delta_{0} > 0 \) small and \( \eta_{0} \) with \( 0 < \eta_{0} < \frac{1}{4\sqrt{m+n}}\min(\sqrt{m+1}, \sqrt{n-1}) \), such that in the tubular neighbourhood
\[
\mathcal{N} := \{p + z\nu_{\Sigma}(p) : p \in \Sigma, \quad |z| < \delta_{0} + \eta_{0}|s(p)|\}
\]
of \( \Sigma \), the mapping
\[
X(p, z) := p + z\nu_\Sigma(p)
\] (4.1)
defines a system of local coordinates, known as the Fermi coordinates, where \( \nu_\Sigma \) is computed in (2.15).

Let \( \varepsilon > 0 \) be small, but fixed and consider the dilated hypersurface \( \Sigma_\varepsilon := \varepsilon^{-1}\Sigma \). Observe first that for any \( p \in \Sigma_\varepsilon \), there exists a unique
\[
(s, x, y) = (s(p), x(p), y(p)) \in \mathbb{R} \times S^{m-1} \times S^{n-1}
\]
such that
\[
p = \varepsilon^{-1}(a(s)x, b(s)y)
\]
and
\[
s(\varepsilon p) = \varepsilon s(p), \quad x(p) = x(\varepsilon p), \quad y(p) = y(\varepsilon p).
\]

The Fermi coordinates of \( \Sigma_\varepsilon \) are defined by
\[
X_\varepsilon(p, z) := p + z\nu_\Sigma(\varepsilon p)
\]
in the dilated neighbourhood
\[
\mathcal{N}_\varepsilon := \left\{ p + z\nu_\Sigma(\varepsilon p) : p \in \Sigma_\varepsilon, \ |z| < \frac{\delta_0}{\varepsilon} + \eta_0|s(p)| \right\}.
\]

Fix \( \alpha \in (0, \frac{1}{2}) \) and consider two smooth \( O(m) \times O(n) \)-invariant functions \( h_1, h_2 : \Sigma \to \mathbb{R} \) with \( h_l(p) = h_l(s(p)) \) for \( p \in \Sigma \) and \( l = 1, 2 \) and such that
\[
\frac{1}{\sqrt{2}} \left( l - \frac{3}{2} - \alpha \right) (\log(s^2 + 2) + 2|\log \varepsilon|) < h_l(s) < \frac{1}{\sqrt{2}} \left( l - \frac{3}{2} + \alpha \right) (\log(s^2 + 2) + 2|\log \varepsilon|) \tag{4.2}
\]
for \( s \in \mathbb{R} \). Assume also that \( h_l(p) = h_l(s) \) is even in the variable \( s \in \mathbb{R} \).

For \( l = 1, 2 \), the mapping
\[
X_{\varepsilon, h_l}(p, t) := p + (t + h_l(\varepsilon p))\nu_\Sigma(\varepsilon p)
\]
defines a diffeomorphism onto the tubular neighbourhood
\[
\mathcal{N}_{l, \varepsilon} = \left\{ X_{\varepsilon, h_l}(p, t) : |t| < \frac{1}{4\sqrt{2}} (\log(s(\varepsilon p)^2 + 2) + 2|\log \varepsilon|) \right\} \tag{4.3}
\]

Next, let \( l \in \{1, 2\} \). We compute the euclidean Laplacian in \( \mathcal{N}_{l, \varepsilon} \) for \( O(m) \times O(n) \)-invariant functions.

Let \( g = (g_{ij})_{N \times N} \) be a Riemannian metric on \( \Sigma \) with inverse \( g^{-1} = (g^{ij})_{N \times N} \). Using the Fermi coordinates in (4.1), the metric \( g \) induces a metric \( G = (G_{ij})_{(N+1) \times (N+1)} \) on \( \mathcal{N} \) whose entries are determined by the formulae
\[
G_{ij} = g_{ij} - 2A_{ij} z + z^2 \partial_i \nu_\Sigma \partial_j \nu_\Sigma \quad \text{for} \quad i, j = 1, \ldots, N,
\]
\[
G_{iz} = G_{zi} = 0 \quad \text{for} \quad i = 1, \ldots, N, \quad G_{zz} = 1.
\]
Writing $G^{-1} = (G^{ij})_{(N+1) \times (N+1)}$, the Laplace operator in the set $\mathcal{N}$ takes the form

$$
\Delta = \frac{1}{\sqrt{\det G}} \partial_i \left( \sqrt{\det G} G^{ij} \partial_j \right),
$$

where summation over repeated indexes is understood.

A direct computation yields that

$$
\Delta = \partial_z^2 + \partial_z (\log \sqrt{\det G}) \partial_z + G^{ij} \partial_{ij} + (\partial_i G^{ij} + \partial_j (\log \sqrt{\det G}) G^{ij}) \partial_i
$$

$$
= \partial_z^2 - H_{\Sigma_\epsilon} \partial_z + \Delta_{\Sigma_\epsilon},
$$

where $H_{\Sigma_\epsilon}$ and $\Delta_{\Sigma_\epsilon}$ are the mean curvature and the Laplace-Beltrami operator of the normally translated hypersurface

$$
\Sigma_\epsilon := \{ p + z \nu_{\Sigma_\epsilon}(p) : p \in \Sigma \}
$$

for $z$ fixed and such that for every $p \in \Sigma$, $(p, z) \in X^{-1}(\mathcal{N})$. We observe that for $|z| > \delta_0$, $\Sigma_\epsilon$ is only defined outside a compact set and has two connected components.

Let $z$ be arbitrary and as in the previous paragraph. Define $Q$ by

$$
Q(p, z) := H_{\Sigma_\epsilon}(p) - H_{\Sigma}(p) - z |A_{\Sigma}(p)|^2
$$

so that

$$
|(1 + s(p)) D_\Sigma Q| + |\partial_2 Q| + |Q| \leq C |z|^2 (1 + s(p))^{-3}.
$$

For $i, j = 1, \ldots, N$, define also $a^{ij}$ and $b^j$ as

$$
a^{ij}(p, z) := G^{ij}(p, z) - g^{ij}(p),
$$

$$
b^j(p, z) := \partial_i G^{ij}(p, z) + \partial_i (\log \sqrt{\det G(p, z)}) G^{ij}(p, z) - \partial_i g^{ij}(p) - \partial_i (\log \sqrt{\det g}) g^{ij}(p).
$$

The formal expansion for the euclidean Laplacian in the coordinates $X(p, z)$ in the set $\mathcal{N}$ reads as

$$
\Delta = \partial_z^2 - z |A_{\Sigma}|^2 \partial_z + \Delta_{\Sigma} - Q(p, z) \partial_z + a^{ij}(p, z) \partial_{ij} + b^j(p, z) \partial_j.
$$

Introducing the change of variables

$$
z = \varepsilon (t + h_t(\varepsilon p)) \quad \text{for} \quad (p, t) \in X_{\varepsilon, h_t}^{-1}(\mathcal{N}_{\ell, \epsilon}), \quad (4.4)
$$

the euclidean Laplacian can be computed in the set $\mathcal{N}_{\ell, \epsilon}$ in the coordinates $X_{\varepsilon, h_t}(p, t)$. These computations are collected in the following Lemma.

**Lemma 4.1.** *In the neighbourhood $\mathcal{N}_{\ell, \epsilon}$, the Laplacian in the $(p, t)$ coordinates is given by

$$
\Delta = \Delta_{\Sigma_\epsilon} + \partial_z^2 - \varepsilon^2 (\Delta_{\Sigma_\epsilon} |A_{\Sigma}|^2 h_t) \partial_t - \varepsilon^2 t |A_{\Sigma}|^2 \partial_t - 2 \varepsilon \nabla_{\Sigma_\epsilon} h_t \partial_t \nabla_{\Sigma_\epsilon} + \varepsilon^2 |\nabla_{\Sigma_\epsilon} h_t|^2 \partial_t^2
$$

$$
- \varepsilon \nabla \partial_t + (a^{ij} \partial_j^2 + \varepsilon b^j \partial_j) - \varepsilon^2 (a^{ij} \partial_i h_t + b^j \partial_j h_t) \partial_t - 2 \varepsilon a^{ij} \partial_i h_t \partial_j - \varepsilon^2 a^{ij} \partial_h \partial_j h_t, \quad (4.5)
$$

where $h_t$ and its derivatives are evaluated at $\varepsilon p$, while $Q$, $a^{ij}$ and $b^j$ are evaluated at $(\varepsilon p, \varepsilon(t + h_t(\varepsilon p)))$.*

We remark that the computations in Lemma 4.1 started off from and are valid for an arbitrary metric on $\Sigma$.

Next, we consider the Riemannian metric on $\Sigma$ that is induced by the parametrisation in (2.14). After rescaling and translating the neighbourhood $\mathcal{N}$ to $\mathcal{N}_{\ell, \epsilon}$ in this particular coordinates, it follows
from Lemma 4.1 that for smooth functions defined in \( \mathcal{N}_{\varepsilon,t} \), expressed in the coordinates \((p,t)\) with \( s = s(p) \in \mathbb{R} \) and being even in \( s \), we have

\[
\Delta = \Delta_{\varepsilon,s} + \varepsilon^2 - \varepsilon^2(\Delta_{\varepsilon,s}h_1)\partial_t - \varepsilon^2t|\Delta_{\varepsilon,s}h_1|^2\partial_t \\
- \varepsilon^2(h''_1 + \alpha(s)h'_1)\partial_t - 2\varepsilon h'_1\partial_{s \varepsilon} + \varepsilon^2(h'_1)^2 \partial_t^2 \\
- \varepsilon Q\partial_t + a\partial_s^2 + \varepsilon \beta(s) - \varepsilon^2(\alpha(s) + \beta(s))\partial_t - 2\varepsilon h'_1\partial_{s \varepsilon} + \varepsilon^2a(h'_1)^2 \partial_t^2,
\]

(4.6)

where

\[
\Delta_{\varepsilon,s} = \partial_s^2 + \varepsilon \alpha(s)\partial_s
\]

and

\[
\Delta_{\varepsilon,s}h_1 + |\Delta_{\varepsilon,s}h_1|^2 = \partial_s s h_1(s) + \alpha(s)\partial_s h_1(s) + \beta(s)h_1(s)
\]

with \( \alpha(s) \) and \( \beta(s) \) described in (2.22), (2.23), (2.40) and (2.39) and where \( h_1 \) and its derivatives are evaluated at \( \varepsilon s \).

Also, for \((p,z) \in \Sigma \times \mathbb{R} \), we have denoted

\[
Q(s(p), z) := Q(p, z), \quad a(s(p), z) := a^{ss}(p, z), \quad b(s(p), z) := b^s(p, z)
\]

and observe that \( Q, a \) and \( b \) are evaluated at \((\varepsilon s, \varepsilon(t + h_1(\varepsilon s)))\).

Proceeding as in the Appendix in [20] and as in Section 3 in [2], using the variables \((s,t)\) and setting \( h_1(p) = h_1(s(p)) \) for \( p \in \Sigma \), we find that

\[
|(1 + s(p))D_{\varepsilon,s}a^{ss}| + |\partial_s a^{ss}| + |a^{ss}| \leq C|z|(1 + s(p))^{-1}
\]

and

\[
|(1 + s(p))D_{\varepsilon,s}b^s| + |\partial_s b^s| + |b^s| \leq C|z|(1 + s(p))^{-2}.
\]

In particular, for every \((s,t) = (s(p), t)\) with \((p,t) \in X_{\varepsilon b_i}(\mathcal{N}_{\varepsilon,t})\),

\[
|Q(\varepsilon s, \varepsilon(t + h_1(\varepsilon s)))| \leq C_\varepsilon \frac{\varepsilon^2(t + h_1(\varepsilon s))^2}{(1 + |\varepsilon s|)^3} \\
|a(\varepsilon s, \varepsilon(t + h_1(\varepsilon s)))| \leq C \frac{\varepsilon(t + h_1(\varepsilon s))}{1 + |\varepsilon s|} \\
|b(\varepsilon s, \varepsilon(t + h_1(\varepsilon s)))| \leq C \frac{\varepsilon(t + h_1(\varepsilon s))}{(1 + |\varepsilon s|)^2}
\]

(4.7)

4.2. First approximation. In this part we choose the first approximation of the solution of equation (1.1). We focus to the region near the approximate zero level set using the Fermi coordinates \((p,z) \) of \( \Sigma \times \mathbb{R} \).

Recall that we have fixed two \( O(m) \times O(n) \)-invariant functions \( h_1, h_2 : \Sigma \to \mathbb{R} \), \( h_i(p) = h_i(s) \), with \( h_1, h_2 \) even in \( s \), \( h_1, h_2 \in C^2(\Sigma) \) and satisfying (4.2). Assume further that

\[
-\infty \equiv h_0 < h_1 < h_2 < h_3 \equiv +\infty
\]

(4.8)

and that for any \( s \in \mathbb{R} \),

\[
|h'_1(s)| \leq \frac{c}{|s| + 1}, \quad |h''_1(s)| \leq \frac{c}{(|s| + 1)^2}, \quad l = 1, 2.
\]

(4.9)

First, for \( l = 1, 2 \) and \( t \in \mathbb{R} \) set \( w_l(t) := (-1)^{l-1}v_*(t) \), where \( v_*(t) \) is the heteroclinic solution to (1.2) described in (1.3). For \((p,z) \in \Sigma \times \mathbb{R} \) define

\[
U_0(p, z) := w_1(z - h_1(\varepsilon p)) + w_2(z - h_2(\varepsilon p)) - 1.
\]

(4.10)
Set also
\[ S(u) = \Delta u + F(u), \quad F(u) = u(1 - u^2) \]
and let us now compute the error \( S(U_0) \) near the normal graphs of the functions \( p \mapsto h_i(\varepsilon p) \) over \( \Sigma_\varepsilon \).

**Lemma 4.2.** For \( l \in \{1, 2\} \) and for any \( (p, t) \in X_{\varepsilon, h_i}^{-1}(N_{l, \varepsilon}) \),
\[
(-1)^{l-1}S(U_0) = -\varepsilon^2(\Delta \Phi h_i + |A\Sigma|^2 h_i)v'_* - \varepsilon^2|A\Sigma|^2 t v''_* + \varepsilon^2|\nabla \Phi h_i|^2 v''_* \\
+ 6(1 - v_*^2)(e^{-\sqrt{t}e^{-\sqrt{t}(h_i - h_{l-1})}} - e^{\sqrt{t}e^{-\sqrt{t}(h_i - h_{l-1})}}) + R_\varepsilon(h_1, h_2),
\]
where \( R_\varepsilon(h_1, h_2) \) is such that
\[
|R_\varepsilon(h_1, h_2)| \leq C\varepsilon^{2+\gamma}(s(\varepsilon p)^2 + 2)\frac{2\varepsilon^4}{\rho^2} e^{-\rho |t|} \quad \text{in} \quad X_{\varepsilon, h_i}^{-1}(N_{l, \varepsilon})
\]
for some \( \gamma \in (0, \frac{1}{2}) \) and \( \rho \in (0, \sqrt{2}) \).

**Proof.** Our calculations are done for \( O(m) \times O(n) \)-invariant functions and hence we use the coordinates \((s, t)\), where \( s = s(p) \) and \( t = z - h_i(\varepsilon p) \). Recall also that \( h_i(\varepsilon p) = h_i(\varepsilon s) \).

We write
\[
(-1)^{l-1}S(U_0) = E_1 + E_2, \quad E_1 := (-1)^{l-1}F(U_0), \quad E_2 := (-1)^{l-1}\Delta U_0.
\]
First we consider the term \( E_1 \). Observe that for
\[
F(U_0) = \sum_{j=1}^{2} F(w_j(t + h_l - h_j)) + F(U_0) - \sum_{j=1}^{2} F(w_j(t + h_l - h_j)).
\]
Since \( F \) is odd for \( l \in \{1, 2\} \),
\[
(-1)^{l-1}F((-1)^{l-1}v) = F(v), \quad (-1)^{l-1}F((-1)^{l-2}v) = (-1)^{l-1}F((-1)^{l}v) = -F(v).
\]
Thus,
\[
(-1)^{l-1} \left( F(U_0) - \sum_{j=1}^{2} F(w_j(t + h_l - h_j)) \right)
= (-1)^{l-1}F(U_0) - F(v_*) + F(v_*(t + h_l - h_{l-1})) + F(v_*(t + h_l - h_{l+1}))
\]
By the Mean Value Theorem, there exists \( \xi_1 \) between \( w_l(t) \) and \( U_0(p, t) \) such that
\[
F(U_0) - F(w_l) = F'(w_l)(U_0 - w_l) + \frac{1}{2} F''(w_l + \xi_1(U_0 - w_l))(U_0 - w_l)^2.
\]
Using that \( F \) is odd, we get
\[
(-1)^{l-1}F(U_0) - F(v_*) = F'(v_*)((-1)^{l-1}U_0 - v_*)
+ \frac{1}{2} F''(v_*) + \xi_1((-1)^{l-1}U_0 - v_*)((-1)^{l-1}U_0 - v_*)^2.
\]
Since,
\[
(-1)^{l-1}U_0 - v_* = \text{sign}(l - j) - v_*(t + h_l - h_j), \quad j \in \{1, 2\}, \quad j \neq l,
\]
performing a Taylor expansion we find that
\[
F(v_*(t + h_l - h_{l-1})) = F(1) + F'(1)(v_*(t + h_l - h_{l-1}) - 1)
+ \frac{1}{2} F''(1 + \xi_2(v_*(t + h_l - h_{l-1}) - 1))(v_*(t + h_l - h_{l-1}) - 1)^2
\]
and
\[
F(v_*(t + h_l - h_{l+1})) = F(-1) + F'(1)(v_*(t + h_l - h_{l+1}) + 1) + \frac{1}{2} F''(-1 + \xi_3(v_*(t + h_l - h_{l+1}) + 1))(v_*(t + h_l - h_{l+1}) + 1)^2. \tag{4.16}
\]

Putting together (4.14), (4.15) and (4.16) we find that for \( j \in \{1, 2\}, j \neq l, \)
\[
(-1)^{l-1} \left( F(U_0) - \sum_{j=1}^{k} F(w_j(t + h_l - h_j)) \right) =
\begin{align*}
&- (2 + F'(v_*))(v_*(t + h_l - h_{l+1}) + v_*(t + h_l - h_{l-1})) \\
&+ \frac{1}{2} F''(v_* + \xi_1((-1)^{l-1}U_0 - v_*))(v_*(t + h_l - h_j) - \text{sign}(l - j))^2 \\
&+ \frac{1}{2} F''(1 + \xi_2(v_*(t + h_l - h_{l-1}) - 1))(v_*(t + h_l - h_{l-1}) - 1)^2 \\
&+ \frac{1}{2} F''(-1 + \xi_3(v_*(t + h_l - h_{l+1}) + 1))(v_*(t + h_l - h_{l+1}) + 1)^2,
\end{align*}
\tag{4.17}
\]

Finally, using the asymptotic behaviour of \( v_* \), we have
\[
E_1 = (-1)^{l-1} \sum_{j=1}^{k} F(w_j(t+h_l-h_j)) + 6(1-v_*^2)(e^{-\sqrt{2}l}e^{-\sqrt{2}(h_l-h_{l-1})} - e^{\sqrt{2}l}e^{-\sqrt{2}(h_{l+1}-h_l)}) + R_1, \tag{4.18}
\]
where
\[
R_{1,\varepsilon} := -3(1 + v_*^2)(v_*(t + h_l - h_{l-1}) - 1 + 2e^{-\sqrt{2}l}e^{-\sqrt{2}(h_l-h_{l-1})}) \\
- 3(1 + v_*^2)(v_*(t + h_l - h_{l+1}) + 1 - 2e^{\sqrt{2}l}e^{-\sqrt{2}(h_{l+1}-h_l)}) \\
+ \frac{1}{2} F''(v_* + \xi_1((-1)^{l-1}U_0 - v_*))(v_*(t + h_l - h_j) - \text{sign}(l - j))^2 \\
+ \frac{1}{2} F''(1 + \xi_2(v_*(t + h_l - h_{l-1}) - 1))(v_*(t + h_l - h_{l-1}) - 1)^2 \\
+ \frac{1}{2} F''(-1 + \xi_3(v_*(t + h_l - h_{l+1}) + 1))(v_*(t + h_l - h_{l+1}) + 1)^2 \tag{4.19}
\]
for \( j \in \{1, 2\}, j \neq l. \)

Next, we compute \( E_2. \) Observe that
\[
(-1)^{l-1} \Delta U_0 = \Delta v_* - \Delta v_*(t + h_l - h_j) \quad \text{for} \quad j \in \{1, 2\}, j \neq l.
\]
Moreover, from (4.6) we find that
\[
\Delta v_* = v''_* - \varepsilon^2 (h''_* + \alpha h'_* + \beta h_*) v'_* - \varepsilon^2 \beta v_* - \varepsilon^2 (h'_*)^2 v''_* \\
- \varepsilon \xi v''_* - \varepsilon^2 (\alpha h''_* + \beta h'_*) v'_* + \varepsilon^2 (h_*)^2 v''_* \\
= v''_* - \varepsilon^2 (h''_* + \alpha h'_* + \beta h_*) v'_* - \varepsilon^2 \beta v_* - \varepsilon^2 (h'_*)^2 v''_* + R_{2,\varepsilon},
\]
where $\alpha$, $\beta$ and $h_1$ and its derivatives are all evaluated at $\varepsilon s$, while $Q$, $a$ and $b$ are evaluated at $(\varepsilon s, \varepsilon(t + h_1(\varepsilon s)))$.

Similarly, for $j \neq l$,
\[
(-1)^j \Delta v_j(t + h_l - h_j) = \Delta w_j(t + h_l - h_j) = w_j''(t + h_l - h_j) - \varepsilon^2 (h_j'' + \alpha h_j' + \beta(t + h_l)) w_j'(t + h_l - h_j)
\]
\[
+ \varepsilon^2 (h_j'')^2 w_j'(t + h_l - h_j) + \varepsilon^2 (aw_j''(t + h_l - h_j)(h_j')^2
\]
\[
- \varepsilon^2 (ah_j'' + bh_j') w_j'(t + h_l - h_j) - \varepsilon Q w_j'(t + h_l - h_j),
\]
\[
w_j''(t + h_l - h_j) - \varepsilon^2 (h_j'' + \alpha h_j' + \beta(t + h_l)) w_j'(t + h_l - h_j)
\]
\[
+ \varepsilon^2 (h_j')^2 w_j''(t + h_l - h_j) + R_{3, \varepsilon}
\]
for $j \in \{1, 2\}$ with $j \neq l$. Therefore,
\[
(-1)^j \Delta v_j(t + h_l - h_j) = w_j''(t + h_l - h_j) - \varepsilon^2 (h_j'' + \alpha h_j' + \beta(t + h_l)) w_j'(t + h_l - h_j)
\]
\[
+ \varepsilon^2 (h_j')^2 w_j''(t + h_l - h_j) + R_{3, \varepsilon}
\]
for $j \in \{1, 2\}$ with $j \neq l$ and consequently,
\[
E_2 = v_*'' - \varepsilon^2 (h_1'' + \alpha h_1' + \beta h_1)v_*' - \varepsilon^2 \beta v_*'\]
\[
+ \varepsilon^2 (h_1')^2 v_*''
\]
\[
+ w_j''(t + h_l - h_j) - \varepsilon^2 (h_j'' + \alpha h_j' + \beta(t + h_l)) w_j'(t + h_l - h_j)
\]
\[
+ \varepsilon^2 (h_j')^2 w_j''(t + h_l - h_j) + R_{2, \varepsilon} + R_{3, \varepsilon}.
\]

Setting $R_{\varepsilon} := R_{1, \varepsilon} + R_{2, \varepsilon} + R_{3, \varepsilon}$, we find that (4.11) holds true.

It remains to prove the estimate (4.12) for the remainder term $R_{\varepsilon}$. To do so, we proceed as follows. Using the inequalities in (4.2), we find the lower estimate
\[
|t + h_j - h_l| \geq |h_j - h_l| - |t|
\]
\[
= |h_j - h_l| - (1 + \alpha)|t| + \alpha|t|
\]
\[
\geq \frac{1}{\sqrt{2}} \left( 1 - 2\alpha - \frac{1 + \alpha}{4} \left( |\varepsilon s|^2 + 2 \right) \|\log(|\varepsilon s|)^2 + 2\| \log \varepsilon \| \right) \alpha|t|
\]
and hence we conclude that
\[
e^{-2\sqrt{2}|t + h_j - h_l|} \leq e^{-2\sqrt{2}(1 - 2\alpha - \frac{1 + \alpha}{4} \|\log(|\varepsilon s|^2 + 2 \| \log \varepsilon \|) \| e^{-2\sqrt{2} \alpha|t|}
\]
\[
\leq e^{3(1 - 3\alpha)} (|\varepsilon s|^2 + 2)^{-\frac{3}{2}(1 - 3\alpha)} e^{-2\alpha|t|}.
\]

We note that
\[
3(1 - 3\alpha) = 2 + \gamma > 2
\]
provided $\alpha > 0$ is small enough.

Putting together the previous estimate and the estimates in (4.7), the estimate in (4.12) follows. This completes the proof of the Lemma.

4.3. Improvement of the approximation. In this subsection we improve the approximation $U_0$ defined in (4.10). More precisely, we cancel the terms of order $\varepsilon^2$ or smaller.

We begin by writing
\[
6(1 - v_*^2) e^{-\sqrt{t}} = a_* v_*' + g_0(t) \quad \text{with} \quad \int_\mathbb{R} g_0(t) v_*'(t) dt = 0
\]
and by noticing that
\[
\int_\mathbb{R} v''(t)v'(t)dt = \int_\mathbb{R} t(v'(t))^2 dt = 0. \tag{4.23}
\]
In order to improve the approximation \(U_0\), we solve the ODE’s
\[
\psi''_0 + (1 - 3v^2_0)\psi_0 = g_0, \quad \psi'_1 + (1 - 3v^2_1)\psi_1 = -v'_1, \quad \psi'_2 + (1 - 3v^2_2)\psi_2 = tv'_1 \quad \text{in} \quad \mathbb{R}. \tag{4.24}
\]
Since \(v'_1\) is a positive solution of the equation
\[
\psi''(t) + (1 - 3v^2)\psi(t) = 0 \quad \text{in} \quad \mathbb{R},
\]
directly from the variation of parameters formula and the orthogonality conditions in (4.22) and (4.23), we find that
\[
\psi_0(t) = v'_1(t) \int_0^t (v'_1(r))^{-2} \left( \int_\tau^\infty v'_1(\xi)g_0(\xi)d\xi \right) d\tau,
\]
\[
\psi_1(t) = \frac{1}{2}tv'_1(t) \quad \text{and}
\]
\[
\psi_2(t) = v'_1(t) \int_0^t (v'_1(r))^{-2} \left( \int_\tau^\infty \xi v'_1(\xi)^2d\xi \right) d\tau.
\]
We also find that for any \(i \in \mathbb{N} \cup \{0\}\), there exists \(C_i > 0\) such that
\[
\left\| (1 + e^{2\sqrt{7}t}|\chi_{t>0}|)\partial_t\psi_i \right\|_{L^\infty(\mathbb{R})} \leq C_i
\]
and for any \(g \in (0,\sqrt{2})\) and for any \(j = 1, 2\) and any \(i \in \mathbb{N} \cup \{0\}\), there exists \(\tilde{C}_i > 0\)
\[
\left\| e^{e^{42}t}\partial_t\psi_j \right\|_{L^\infty(\mathbb{R})} \leq \tilde{C}_i.
\]
Furthermore, since \(\psi_1\) and \(\psi_2\) are odd functions in the variable \(t\), then
\[
\int_\mathbb{R} \psi_1(t)v'_1(t)dt = \int_\mathbb{R} \psi_2(t)v'_1(t)dt = 0.
\]
Next, we proceed as in subsection 5.2 in [1]. First, define \(\eta_j : \Sigma \times \mathbb{R} \to \mathbb{R}\) by the formula
\[
(-1)^{j-1}\eta_j(p, z) := -e^{-\sqrt{7}(\epsilon_1 p - \epsilon_0 z)}\psi_0(h_0(\epsilon_0) - z) + e^{-\sqrt{7}(\epsilon_1 p - \epsilon_0 z)}\psi_0(z - h_0(\epsilon_0))
\]
\[
+ e^2\psi_1(z - h_0(\epsilon_0)) + e^2|A_{\Sigma}(\epsilon_0)|^2\psi_2(z - h_0(\epsilon_0))
\]
and set \(\tilde{\eta}_j(p, t) := (-1)^{j-1}\eta_j(p, t + h_0(\epsilon_0))\). We consider the approximation
\[
U_1(p, z) := U_0(p, z) + \eta(p, z), \quad \eta(p, z) := \sum_{j=1}^2 \eta_j(p, z).
\]
Using the variable \(t := z - h(t)\) in \(X_{\epsilon, h}(N_{t, z})\), we have for \(j \in \{1, 2\}\) with \(j \neq l\), that
\[
(-1)^{l-1}S(U_1) = (-1)^{l-1}S(U_0) + \Delta N_{t, z}\tilde{\eta}_l + F'(w_l)\tilde{\eta}_l + (F'(U_0) - F'(w_l))\tilde{\eta}_l
\]
\[
+ \Delta N_{t, z}\tilde{\eta}_j(t + h_l - h_j) + F'(U_0)\tilde{\eta}_j(t + h_l - h_j) + (-1)^{l-1}Q_{U_0}(\eta),
\]
where we have denoted
\[
Q_{U_0}(\eta) = F(U_0 + \eta) - F(U_0) - F'(U_0)\eta.
\]
Using (4.6) and (4.7) in \(N_{t, z}\), we find that
\[
(\Delta N_{t, z} + F'(w_0)(\tilde{\eta}_l(\epsilon s))^2\psi_1(t) + \tilde{\eta}_l(\epsilon s)(\tilde{\eta}_l(\epsilon s))^2\psi_2(t)) = \tilde{\eta}_l(\epsilon s)tv'_1 + \tilde{\eta}_l(\epsilon s)^2v''_1 + R_{4, l}(h_1, h_2)
\]
\[
(\Delta N_{t, z} + F'(w_0)(\tilde{\eta}_l(\epsilon s))^2\psi_1(t) + \tilde{\eta}_l(\epsilon s)(\tilde{\eta}_l(\epsilon s))^2\psi_2(t)) = \tilde{\eta}_l(\epsilon s)tv'_1 + \tilde{\eta}_l(\epsilon s)^2v''_1 + R_{4, l}(h_1, h_2)
\]
Lemma 4.3. Assume the hypothesis in Lemma 4.1. We look for a solution to the Allen-Cahn equation of the form

\[ (\Delta_{N_1} + F'(w)) \left( -e^{-\sqrt{2}(h_{1-1})} \psi_0(-t) + e^{-\sqrt{2}(h_{1+1}-h_1)} \psi_0(t) \right) = \]

\[ -e^{-\sqrt{2}(h_{1-1})} g_0(-t) + e^{-\sqrt{2}(h_{1+1}-h_1)} g_0(t) + R_{i,\varepsilon}(h_1, h_2), \]

with

\[ |R_{i,\varepsilon}(h_1, h_2)| \leq C\varepsilon^3(s(\varepsilon p)^2 + 2)^{-1/2} e^{-\rho|t|}, \quad i = 4, 5. \]

The following lemma summarises the computations of the error \( S(U_1) \).

**Lemma 4.3.** Assume the hypothesis in Lemma (4.11). The error \( S(U_1) \) in \( X_{\varepsilon,h_1}(N_{\varepsilon,l}) \) is given by

\[ (-1)^{l+1} S(U_1) = \varepsilon^2 (\Delta_{\Sigma} h_l + |A_{\Sigma}|^2 h_l) v_*' + a_* (e^{-\sqrt{2}(h_{l-1})} - e^{-\sqrt{2}(h_{l+1}-h_1)}) v_*' + R_{6,\varepsilon}(h_1, h_2), \]

with

\[ |R_{6,\varepsilon}(h_1, h_2)| \leq C\varepsilon^{2+\gamma}(s(\varepsilon p)^2 + 2)^{-1/2} e^{-\rho|t|}. \] (4.26)

5. The Lyapunov-Schmidt Reduction

In this section, we perform an infinite dimensional Lyapunov-Schmidt reduction procedure and finish the construction of the solution of (1.1) predicted in Theorem 1.3. Again many of the developments are in the lines of those in [1, 20].

5.1. A gluing procedure. We begin the developments in this part by construction a global approximation of (1.1). We introduce a smooth cutoff function \( \chi \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1 \) and

\[ \chi(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2. \end{cases} \]

For \( (p, z) \in \Sigma_\varepsilon \times \mathbb{R} \), we set

\[ \zeta(p, z) := \chi \left( |z| - \frac{4}{\sqrt 2} (\log(s(\varepsilon p)^2 + 2) + \log \varepsilon) + 2 \right) \]

and

\[ \tilde{\zeta}(\xi) := \begin{cases} \zeta \circ X_{\varepsilon}^{-1}(\xi), & \text{for } \xi \in N_\varepsilon \\ 0, & \text{otherwise}. \end{cases} \] (5.1)

Similarly, we set

\[ \tilde{U}_1(\xi) := \begin{cases} U_1 \circ X_{\varepsilon}^{-1}(\xi), & \text{for } \xi \in N_\varepsilon \\ 0, & \text{otherwise}. \end{cases} \] (5.2)

We define our global approximation as

\[ w(\xi) := \tilde{\zeta}(\xi) \tilde{U}_1(\xi) - (1 - \tilde{\zeta}(\xi)), \] (5.3)

and we look for a solution to the Allen-Cahn equation of the form

\[ u = w + \varphi, \]

where \( \varphi : \mathbb{R}^{N+1} \to \mathbb{R} \) is a small correction in an appropriate topology to be determined later.

We introduce next some useful notation. For \( l = 1, 2 \) and a function \( u : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R} \), we set

\[ u^l_\varepsilon(\xi) := \begin{cases} u \circ X_{\varepsilon,h_l}(\xi), & \text{for } \xi \in N_{\varepsilon,l} \subset \mathbb{R}^{N+1} \\ 0, & \text{otherwise}. \end{cases} \] (5.4)
On the other hand, given any $O(m) \times O(n)$-invariant function $v : \mathbb{R}^{N+1} \to \mathbb{R}$, we define for $1 \leq l \leq 2$ and $(p, t) \in \Sigma_{c} \times \mathbb{R}$,
\[
v_{t}^{l}(p, t) := \begin{cases} v \circ X_{\varepsilon,t_{l}}(p, t) & \text{if } (p, t) \in X_{\varepsilon,K}^{-1}(N_{\varepsilon,l}) \\ 0, & \text{otherwise.} \end{cases}
\] (5.5)

The notation just introduced can be explained as follows: $u_{t}^{l}$ refers to an expression of $u$ in natural or euclidean coordinates while $u^{l}$ refers to a function expressed in natural coordinates pushed forward to Fermi coordinates.

For any integer $i \geq 1$ and $(p, t) \in \Sigma_{c} \times \mathbb{R}$, we set
\[
\chi_{i}(p, t) := \chi \left( |t| - \frac{1}{4\sqrt{2}} (\log(s\varepsilon p)^{2} + 2|\log \varepsilon|) + i \right)
\]
and we look for a correction of the form
\[
\varphi = \sum_{i=1}^{2} \chi_{3, i}^{\sharp} \varphi_{l} + \psi.
\] (5.6)

Using the fact that $\chi_{4, l}^{\sharp} \chi_{3, l}^{\sharp} = \chi_{4, l}^{\sharp}$, the Allen-Cahn equation can be written as
\[
0 = S(w + \varphi) = S(w) + \Delta \varphi + F'(w)\varphi + Q_{w}(\varphi) = \sum_{l=1}^{2} \chi_{3, l}^{\sharp}(\Delta \varphi_{l} + F'(w)\varphi_{l}) + \chi_{4, l}^{\sharp} S(w) + \chi_{4, l}^{\sharp} Q_{w}(\varphi_{l} + \psi) + \chi_{4, l}^{\sharp}(F'(w) + 2)\psi
\]
\[
+ \Delta \psi + (2 - (1 - \sum_{l=1}^{2} \chi_{4, l}^{\sharp})(F'(w) + 2))\psi + (1 - \sum_{l=1}^{2} \chi_{4, l}^{\sharp}) S(w)
\]
\[
+ \sum_{l=1}^{k} 2\nabla \chi_{3, l}^{\sharp} \cdot \varphi_{l} + \Delta \chi_{3, l}^{\sharp} \varphi_{l} + (1 - \chi_{4, l}^{\sharp}) Q_{w}(\psi + \sum_{i=1}^{2} \chi_{3, i}^{\sharp} \varphi_{i}),
\]
where we have denoted
\[
Q_{w}(\varphi) = F(w + \varphi) - F(w) - F'(w)\varphi.
\]

Since we look for an $O(m) \times O(n)$-invariant solution, also $\psi$ and $\varphi_{1}, \varphi_{2}$ have to satisfy these symmetries. In other words, $\varphi_{l} = \varphi_{l}^{\sharp}$, for some functions $\varphi_{l} : \Sigma_{c} \times \mathbb{R} \to \mathbb{R}$ of the $(p, t)$-variables, which are $O(m) \times O(n)$-invariant. Therefore we have to solve the system given by the equations
\[
\Delta \psi + \left( 2 - (1 - \sum_{l=1}^{2} \chi_{4, l}^{\sharp})(F'(w) + 2) \right) \psi + \left( 1 - \sum_{l=1}^{2} \chi_{4, l}^{\sharp} \right) S(w)
\]
\[
+ \sum_{l=1}^{2} 2\nabla \chi_{3, l}^{\sharp} \cdot \nabla \varphi_{l} + \Delta \chi_{3, l}^{\sharp} \varphi_{l} + (1 - \chi_{4, l}^{\sharp}) Q_{w}(\psi + \sum_{i=1}^{2} \chi_{3, i}^{\sharp} \varphi_{i}) = 0 \quad \text{in } \mathbb{R}^{N+1}, \quad l = 1, 2
\]
and
\[
\Delta_{\Sigma_{c}} \varphi_{l} + \partial_{t}^{2} \varphi_{l} + F'(v_{s})\varphi_{l} + \chi_{4} S(w_{l}^{\sharp}) + \chi_{4} Q_{w}^{l}(\varphi_{l} + \psi_{l}^{\sharp}) + \chi_{4}(F'(w_{l}^{\sharp}) + 2)\psi_{l}^{\sharp}
\]
\[
+ \chi_{4} (\Delta_{N_{c,s}} - \partial_{l}^{2} - \Delta_{\Sigma_{c}}) \varphi_{l} \left( F'(w_{l}^{\sharp}) - F'(v_{s}) \right) \chi_{2} \varphi_{l} = 0 \quad \text{in } \Sigma_{c} \times \mathbb{R}, \quad l = 1, 2,
\]
where $\Delta_{N_{c,s}}$ represents the Laplacian in the $(p, t)$-coordinates, given by Lemma 4.1.
Using the decay of the error both along the surface and in the orthogonal direction, it is possible to prove the following result about the behaviour of the error far from the interfaces.

**Lemma 5.1.** In the previous notations, for some $\gamma > 0$, 
\[
\left| \left( 1 - \sum_{i=1}^{2} \chi_{i}^{\alpha} \right) S(w) \right| \leq C\varepsilon^{2+\gamma}(\varepsilon|\xi|^2 + 2)^{-\frac{2+\gamma}{2}}, \quad \forall \xi \in \mathbb{R}^{N+1}. \tag{5.9}
\]

**Proof.** Here we estimate the error far from the interfaces. To be more precise, let $\xi = X_{\varepsilon,h_{\gamma}}(p,t)$ with $p \in \Sigma_{\varepsilon}$ and
\[
\frac{1}{4\sqrt{2}} \left( \log(s(\varepsilon p)^2 + 2) + 2|\log \varepsilon| \right) - 3 \leq |t| \leq \frac{1}{4\sqrt{2}} \left( \log(s(\varepsilon p)^2 + 2) + 2|\log \varepsilon| \right) - 2.
\]
Then, for $s = s(p)$
\[
e^{-\sqrt{2}|t|} \left( e^{-\sqrt{2}|h_{1} - h_{2}|} + e^{-\sqrt{2}|h_{2} - h_{1}|} \right) \leq e^{-(1-\alpha)\sqrt{2}|t|} e^{-(1-2\alpha)} \left( \log(s(\varepsilon p)^2 + 2) + 2|\log \varepsilon| \right) e^{-\alpha \sqrt{2}|t|}
\leq e^{-\sqrt{2}(1-2\alpha)} \left( \log(s(\varepsilon p)^2 + 2) + 2|\log \varepsilon| \right) e^{-\alpha \sqrt{2}|t|}
= e^{\frac{\alpha}{2} - \frac{\alpha}{2} \left( s(\varepsilon p)^2 + 2 \right)^{-1} + \frac{\alpha}{2} e^{-\alpha \sqrt{2}|t|}}
\leq C e^{\frac{\alpha}{2} - 4\alpha} \left( |s(\varepsilon p)|^2 + 2 \right)^{-1} + \frac{\alpha}{2} e^{-\alpha \sqrt{2}|t|}
\leq C e^{\frac{\alpha}{2} - 4\alpha} \left( |\varepsilon s|^2 + 2 \right)^{-1} + \frac{\alpha}{2} e^{-\alpha \sqrt{2}|t|}.
\]
Thus the conclusion follows by setting \( \min \left\{ \frac{\alpha}{2} - 4\alpha, \frac{\alpha}{2} - \frac{\alpha}{2} \right\} =: 2 + \gamma > 2 \) with any $\alpha \in (0, \frac{1}{2})$. \qed

In view of Lemma 5.1, we can find a solution $\psi = \psi(\phi_{1}, \phi_{2}, h_{1}, h_{2})$ to equation (5.7), for any fixed $\phi_{1}, \phi_{2}$ and $h_{1}, h_{2}$. This will be done in subsection 5.3, thanks to coercitivity of the bilinear form associated with the linear operator. After that, we will plug this solution into system (5.8), which will be solved with respect to $(\phi_{1}, \phi_{2}, h_{1}, h_{2})$.

To solve (5.8) we will rely on the infinite dimensional Lyapunov-Schmidt reduction. This allows to overcome the fact that the operator $\Delta_{\Sigma} + \partial_{\ell}^{2} + F^{\langle v_{*} \rangle}$ has a one-dimensional kernel spanned by $v_{*}$. In order to set up the reduction scheme we set
\[
N_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2}) := \chi_{i} Q_{w_{i}}(\phi_{1} + \psi_{i}) + \chi_{i} (\partial_{\ell}^{2}(w_{i}) + 2) \psi_{i} + \chi_{2}(\Delta X_{i} - \partial_{\ell}^{2} - \Delta_{\Sigma}) \phi_{i}
+ (F^{\langle w_{i} \rangle} - F^{\langle v_{*} \rangle}) \chi_{2} \phi_{i},
\]
\[
P_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2})(p) := \int_{\mathbb{R}} \left( \chi_{i} S(w_{i}) + N_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2}) \right) v_{i}(t) dt \quad \text{for} \quad p \in \Sigma_{\varepsilon}. \tag{5.10}
\]

First we fix $h_{1}, h_{2}$ and we find a solution $(\phi_{1}, \phi_{2}) = (\phi_{1}(h_{1}, h_{2}), \phi_{2}(h_{1}, h_{2}))$ to the system
\[
\Delta_{\Sigma} \phi_{i} + \partial_{\ell}^{2} \phi_{i} + F^{\langle v_{*} \rangle} \phi_{i} = -\chi_{i} S(w_{i}) - N_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2}) + P_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2})(p) v_{i}
\int_{\mathbb{R}} \phi_{i}(p,t) v_{i}(t) dt = 0, \quad \forall p \in \Sigma_{\varepsilon}, \quad l = 1, 2
\]
where $\psi = \psi(\varphi_{1}, \varphi_{2}, h_{1}, h_{2})$ is the solution found above. In other words, for any $h_{1}, h_{2}$ fixed, equation (5.8) can be solved up to Lagrange multipliers $P_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2})$, which depend on $h_{1}$ and $h_{2}$ (we recall that also $\psi, \phi_{1}$ and $\phi_{2}$ do depend on $h_{1}$ and $h_{2}$). This system, known as the auxiliary equation will be treated in Subsection 5.4. Finally we will determine $h_{1}, h_{2}$ by solving the system
\[
P_{i}(\psi, \phi_{1}, \phi_{2}, h_{1}, h_{2}) = 0, \quad l = 1, 2,
\]
known as the bifurcation equation. Equivalently, we have to choose $h_1$ and $h_2$ in order for the Lagrange multipliers to vanish. Integrating over $\mathbb{R}$ and using Lemma 4.3, it is possible to see that this is equivalent to solve a non-linear system of the form

$$
\varepsilon^2 J_{\Sigma} h_1 + a_* e^{-\sqrt{\gamma}(h_2 - h_1)} = \varepsilon^2 f_1(p, h_1, h_2) \quad \text{in } \Sigma
$$

for $h_1$ and $h_2$ satisfying (4.2) and (4.9) with

$$
|f_l(p, h_1, h_2)| \leq \varepsilon^2 \gamma (s(p)^2 + 2)^{2+\mu} \quad \text{for } p \in \Sigma, \quad l = 1, 2,
$$

where $\mu := \min\{\gamma, \bar{\gamma}\} > 0$ with $\gamma$ and $\bar{\gamma}$ begin the constants in Lemmas 4.2 and 5.1 and where from

$$
a_* = \|v_*\|_{L^2[\mathbb{R}]}^{-2} \int_{\mathbb{R}} 6(1 - v_*^2)e^{-\sqrt{\gamma} v_*'(t)}dt > 0
$$

is the constant in (4.22).

Setting

$$
\begin{aligned}
  v_1 &:= h_1 + h_2, \\
v_2 &:= h_2 - h_1,
\end{aligned}
\quad
\begin{aligned}
  g_1 &:= f_1 + f_2, \\
g_2 &:= f_2 - f_1,
\end{aligned}
$$

we reduce the bifurcation system (5.12) reduces to

$$
J_{\Sigma} v_1 = g_1 \left( p, \frac{v_1 - v_2}{2}, \frac{v_1 + v_2}{2} \right),
$$

$$
\varepsilon^2 J_{\Sigma} v_2 - 2a_* e^{-\sqrt{\gamma} v_2} = \varepsilon^2 g_2 \left( p, \frac{v_1 - v_2}{2}, \frac{v_1 + v_2}{2} \right).
$$

We look for a solution of the form

$$
v_l = v_{0,l} + q_l,
$$

where the pair $(v_{0,1}, v_{0,2})$ solves the homogeneous Jacobi-Toda system

$$
J_{\Sigma} v_{0,1} = 0
$$

$$
\varepsilon^2 J_{\Sigma} v_{0,2} - 2a_* e^{-\sqrt{\gamma} v_{0,2}} = 0.
$$

From Proposition 2.2, $v_{0,1} = 0$, while Theorem 1.2 with $\delta = \varepsilon^2 > 0$, provides the existence of $v_{0,2}$. Consequently, the pair $(q_1, q_2)$ is determined by solving the nonlinear system

$$
J_{\Sigma} q_1 = \tilde{g}_1(p, q_1, q_2)
$$

$$
\varepsilon^2 J_{\Sigma} q_2 + 2\sqrt{2a_*} e^{-\sqrt{\gamma} q_2} = \varepsilon^2 \tilde{g}_2(p, q_1, q_2)
$$

in $\Sigma$, (5.15)

where $v = v_{0,2}$ has the asymptotic behavior of the approximate solution to the Jacobi-Toda equation described in Lemma 3.2, i.e.

$$
v_{0,2} \sim \frac{1}{\sqrt{2}} W \left( \frac{2\sqrt{2a_*}}{\varepsilon^2 |A_{\Sigma}|^2} \right).
$$
The right-hand sides in (5.15) are given by
\[
\begin{align*}
\tilde{g}_1(p, q_1, q_2) &= g_1 \left( p, -\frac{v_{0,2}}{2} + \frac{q_1 - q_2}{2}, \frac{v_{0,2}}{2} + \frac{q_1 + q_2}{2} \right) \\
\tilde{g}_2(p, q_1, q_2) &= g_2 \left( p, -\frac{v_{0,2}}{2} + \frac{q_1 - q_2}{2}, \frac{v_{0,2}}{2} + \frac{q_1 + q_2}{2} \right) \\
&\quad + 2a_* e^{-2\sqrt{2}v_{0,2}Q(q_2)} - 2\sqrt{2}a_* e^{-\sqrt{2}v} (e^{-\sqrt{2}v_{0,2} - v} - 1)q_2.
\end{align*}
\]

For further details about system (5.15), we refer to subsection 5.5.

**Remark 5.1.** The solution \((h_1, h_2)\) of the system (5.12) will have the form
\[
h_1 = -\frac{v_{0,2}}{2} + \frac{q_1 - q_2}{2}, \quad h_2 = \frac{v_{0,2}}{2} + \frac{q_1 + q_2}{2}.
\]

From (3.11) in Lemma 3.2 and the developments in subsection 3.9, we are able to conclude that \((h_1, h_2)\) satisfies (4.2) and (4.9), since \((q_1, q_2)\) are determined as small perturbations of solutions to the Jacobi-Toda system.

Roughly speaking, the two connected components of the zero level set of \(u_\varepsilon\) diverge logarithmically from the cone \(C_{m,n}\) at infinity.

### 5.2. Function spaces

In order to treat equation (5.7) and system (5.11) we introduce some function spaces.

For \(\beta \in (0, 1), \mu > 0\) and functions \(g \in C^0_{\text{loc}}(\mathbb{R}^{N+1})\), we introduce the norm (c.f. (2.58)):
\[
\|g\|_{\infty, \mu} := \|(|\varepsilon|^2 + 2)^{\mu/2} g\|_{L^\infty(\mathbb{R}^{N+1})}.
\]

Moreover, we say that \(g \in Y^\beta_{\mu, \rho}\) if it is \(O(m) \times O(n)\)-invariant and the norm
\[
\|g\|_{Y^\beta_{\mu, \rho}} := \sup_{\xi \in \mathbb{R}^{N+1}} (2 + |\varepsilon|^2)^{\mu/2} \|D^2 g\|_{C^0(\Sigma_{\varepsilon}^\beta(B_1(\xi)))} + \|D\psi\|_{\infty, 2+\mu} + \|\psi\|_{\infty, 2+\mu}
\]
is finite.

We also say that a function \(\psi \in C^2_{\text{loc}}(\mathbb{R}^{N+1})\) is in \(X_{\mu}^{\beta}\) if it is \(O(m) \times O(n)\)-invariant and the norm
\[
\|\psi\|_{X_{\mu}^{\beta}} := \sup_{\xi \in \mathbb{R}^{N+1}} (2 + |\varepsilon|^2)^{\mu/2} \|D^2 \psi\|_{C^0(\Sigma_{\varepsilon}^\beta(B_1(\xi)))} + \|D\psi\|_{\infty, 2+\mu} + \|\psi\|_{\infty, 2+\mu}
\]
is finite.

Given \(\beta \in (0, 1), \rho \in (0, \sqrt{2}), \mu > 0\) and a function \(f \in C^0_{\text{loc}}(\Sigma_{\varepsilon} \times \mathbb{R})\), we define the norm
\[
\|f\|_{\infty, \mu, \rho} := ||(s(\varepsilon^2 + 2)^{\mu/2} \cosh(t)^\rho f\|_{L^\infty(\Sigma_{\varepsilon} \times \mathbb{R})},
\]

Furthermore, we say that \(f \in Y_{\mu, \rho}^{\beta, \rho}\) if it is \(O(m) \times O(n)\)-invariant and
\[
\|f\|_{Y_{\mu, \rho}^{\beta, \rho}} := \sup_{p \in \Sigma_{\varepsilon}, t \in \mathbb{R}} (s(\varepsilon^2 + 2)^{\mu/2} \cosh(t)^\rho f\|_{C^0(\Sigma_{\varepsilon} \times \mathbb{R})}, \quad I_{\rho, t} := B_1(p) \times (t, t + 1)
\]
is finite. Moreover, for \(O(m) \times O(n)\)-invariant functions \(\phi \in C^2_{\text{loc}}(\Sigma_{\varepsilon} \times \mathbb{R})\), we say that \(\phi \in X_{\mu, \rho}^{\beta, \rho}\) if
\[
\|\phi\|_{X_{\mu, \rho}^{\beta, \rho}} := \sup_{p \in \Sigma_{\varepsilon}, t \in \mathbb{R}} (s(\varepsilon^2 + 2)^{\mu/2} \cosh(t)^\rho \|D^2 \phi\|_{C^0(\Sigma_{\varepsilon} \times \mathbb{R})} + \|D\phi\|_{\infty, 2+\mu, \rho} + \|\phi\|_{\infty, 2+\mu, \rho}
\]
is finite.
5.3. The equation far from the nodal set. The aim of this subsection is to solve equation (5.7), with $h_1$, $h_2$, $\phi_1$ and $\phi_2$ fixed. Recall that we have set $\mu := \min\{\gamma, \bar{\gamma}\} > 0$, with $\gamma$ and $\bar{\gamma}$ defined as in Lemmas 4.2 and 5.1. We also refer the reader back to subsections 2.7 and 3.8 for the respective definition of the spaces $C^{2,\beta}_{\infty,\mu}(\Sigma)$ and $D^{2,\beta}_{\mu,\frac{1}{2}}(\Sigma)$.

**Proposition 5.1.** Let $\beta \in (0, \frac{1}{2})$, $\rho > 0$ be given and let $\Lambda_0, \Lambda_1 > 0$ be fixed constants. Let also $h_1, h_2$ be of the form (5.16), with $q_1 \in C^{2,\beta}_{\infty,\mu}(\Sigma)$, $q_2 \in D^{2,\beta}_{\mu,\frac{1}{2}}(\Sigma)$ such that

$$\|q_1\|_{C^{2,\beta}_{\infty,\mu}(\Sigma)} < \Lambda_0 \varepsilon^\mu, \quad \|q_2\|_{D^{2,\beta}_{\mu,\frac{1}{2}}(\Sigma)} < \Lambda_0 \varepsilon^\mu.$$ 

Let $\phi_1, \phi_2 \in X^{\beta}_{\mu,\rho}$ be such that

$$\|\phi_1\|_{X^{\beta}_{\mu,\rho}} < \Lambda_1 \varepsilon^{2+\mu},$$

Set $\varphi_1 = \phi_1^\varepsilon$. Then there exists a unique solution $\psi := \psi(\varphi_1, \varphi_2, h_1, h_2) \in X^{\beta}_{\mu,\rho}$ to equation (5.7) satisfying

$$\|\psi(\varphi_1, \varphi_2, h_1, h_2)\|_{X^{\beta}_{\mu,\rho}} \leq \Lambda_2 \varepsilon^{2+\mu},$$

$$\|\psi(\varphi_1, \varphi_2, h_1, h_2) - \psi(\varphi_1, \varphi_2, h_1, h_2)\|_{X^{\beta}_{\mu,\rho}} \leq c \varepsilon^{2+\mu} \left( \|\varphi_1^\varepsilon - \varphi_1^\varepsilon\|_{X^{\beta}_{\mu,\rho}} + \|\varphi_2^\varepsilon - \varphi_2^\varepsilon\|_{X^{\beta}_{\mu,\rho}} \right)$$

$$\|\psi(\varphi_1, \varphi_2, h_1, h_2) - \psi(\varphi_1, \varphi_2, h_1, h_2)\|_{X^{\beta}_{\mu,\rho}} \leq c \varepsilon^{2+\mu} \left( \|q_1^\varepsilon - q_1^\varepsilon\|_{C^{2,\beta}_{\infty,\mu}(\Sigma)} + \|q_2^\varepsilon - q_2^\varepsilon\|_{D^{2,\beta}_{\mu,\frac{1}{2}}(\Sigma)} \right),$$

for some $c, \Lambda_2 > 0$.

The proof basically consists of two steps. First we construct a right inverse of the operator

$$-\Delta + V_\varepsilon, \quad V_\varepsilon := 2 - (1 - \sum_{l=1}^2 \chi_{\varepsilon,l}^\varepsilon)(F'(w) + 2)$$

using the fact that the potential $V_\varepsilon$ is positive and bounded away from 0, uniformly in $\varepsilon$ (see Proposition 5.2). After that, we will use a perturbation argument, based on the contraction mapping theorem.

To carry out the first step we consider the equation:

$$-\Delta \psi + V_\varepsilon(\xi)\psi = g.$$ (5.22)

**Proposition 5.2.** Let $\beta \in (0, 1)$, $\mu > 0$, and $g \in Y^\varepsilon$. Then, for $\varepsilon > 0$ small enough, there exists a solution $\psi := F_3(g) \in X^{\beta}_{\mu,\rho}$ to (5.22). Moreover, it satisfies

$$\|\Psi(g)\|_{X^{\beta}_{\mu,\rho}} \leq c \|g\|_{Y^{\beta}_{\mu,\rho}},$$ (5.23)

for some constant $c > 0$.

First we state the following a priori estimate, which will be used to treat the linear problem.

**Lemma 5.2.** Let $\beta \in (0, 1)$, $\mu > 0$ and $\psi$ be a bounded solution to (5.22) with $g \in Y^{\beta}_{\mu,\rho}$. Then

$$\|\psi\|_{\infty, 2+\mu} \leq c \|g\|_{\infty, 2+\mu}.$$
Proof. The idea is to take a point $\xi_0 \in \mathbb{R}^{N+1}$ and to prove that
\[
|\psi(\xi_0)| \leq \lambda((\varepsilon \xi_0)^2 + 2)^{-\frac{2+\mu}{2}}.
\]
In order to do so, we take $\nu > 0$ (to be fixed) and we compare $\psi$ with the barrier
\[
v_{\lambda, \nu}(\xi) := \lambda((\varepsilon \xi)^2 + 2)^{-\frac{2+\mu}{2}} + \nu((\varepsilon\xi)^2 + 2)^{-\frac{2+\mu}{2}}
\]
in a ball of radius $R_0 > |\xi_0|$ so large that
\[
|\psi(\xi)| \leq \|\psi\|_{L^\infty(\mathbb{R}^{N+1})} \leq \nu((\varepsilon R_0)^2 + 2)^{\frac{2+\mu}{2}}, \; \forall \xi \in \partial B_{R_0}.
\]
In fact, in such a ball, we have
\[
(-\Delta + V_\varepsilon)(\psi - v_{\lambda, \nu}) \leq ((g)_{\infty, 2+\mu} - \lambda(2 - \delta)(2\varepsilon^2 + 2)^{\frac{2+\mu}{2}} = 0
\]
with $0 < 2 - \delta < 2$, provided $\varepsilon$ is small enough and
\[
\lambda = \frac{\|g\|_{\infty, 2+\mu}}{2 - \delta}.
\]
Therefore, applying the maximum principle and letting $\nu \to 0$, we have the statement. \hfill \Box

Now we can prove Proposition 5.2.

Proof of Proposition 5.2. We take a smooth cutoff function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > 2$. For $k \geq 0$, we set $\chi_R(x) = \chi(|\varepsilon x| - R)$ and $g_R := \chi_R g$, so that $g_R \to g$ uniformly as $R \to \infty$ and we consider the unique solution $\psi_R$ to the problem
\[
\begin{cases}
-\Delta \psi_R + V_\varepsilon(x)\psi_R = g_R, & \text{in } \mathbb{R}^{N+1} \\
\psi_R \in H^1(\mathbb{R}^{N+1}).
\end{cases}
\]
Since $g$ is $O(m) \times O(n)$-invariant, then, by uniqueness, so is $\psi_R$. Moreover, by a bootstrap argument, it is possible to prove that $\psi_R \in L^\infty(\mathbb{R}^{N+1})$. Thus Lemma 5.2, gives the bound
\[
\|\psi_R\|_{L^\infty(\mathbb{R}^{N+1})} \leq \|\psi_R\|_{\infty, 2+\mu} \leq c\|g_R\|_{2+\mu} \leq \|g\|_{\infty, 2+\mu}.
\]

By the standard elliptic regularity and and the Ascoli-Arzelà theorem, there exists a sequence $R_k \to \infty$ such that $\psi_{R_k}$ converges uniformly on compact subsets to a bounded solution $\psi$ to (5.22), which itself satisfies
\[
\|\psi\|_{2+\mu} \leq c\|g\|_{2+\mu}.
\]
To conclude, the estimate in the norm $\|\cdot\|_{L^\infty}^\mu$ follows from the elliptic estimates. \hfill \Box

Now we can prove Proposition 5.1.

Proof of Proposition 5.1. Equation (5.7) can be reduce to the fixed point problem
\[
\psi = -F_3 \left( (1 - \sum_{l=1}^2 \lambda_{4,i,l}^\beta) S(w) + \sum_{l=1}^2 2\nabla \chi_{3,l}^\beta \cdot \nabla \varphi_l + \Delta \chi_{3,l}^\beta \varphi_l + (1 - \lambda_{3,i}^\beta) Q_w(\psi + \sum_{l=1}^2 \chi_{3,i}^\beta \varphi_l) \right),
\]
which can be uniquely solved in the ball
\[
B_{\Lambda_1} := \{ \psi \in X_{\mu, \rho}^{\beta} : \|\psi\|_{X_{\mu, \rho}^{\beta}} < \Lambda_1 \varepsilon^{2+\mu} \},
\]
using the contraction mapping theorem and thanks to Lemmas 4.3 and 5.1. We leave the details to the reader and refer to [19], section 5 for details on similar developments. \hfill \Box
5.4. The auxiliary equation. In this subsection we will deal with the auxiliary system (5.11) with $h_1, h_2$ fixed.

First we introduce the spaces
\[ X_{\mu,\rho}^{\beta} := \left\{ \phi \in X_{\mu,\rho}^{\beta} : \int_{\mathbb{R}} \phi(p, t) v'_t(t) dt = 0, \forall p \in \Sigma_{\epsilon} \right\}, \]
\[ Y_{\mu,\rho}^{\beta} := \left\{ f \in Y_{\mu,\rho}^{\beta} : \int_{\mathbb{R}} f(p, t) v'_t(t) dt = 0, \forall p \in \Sigma_{\epsilon} \right\}. \]

**Proposition 5.3.** Let $\beta \in (0, 1/2)$, $\rho > 0$ be given and let $h_1, h_2$ be of the form (5.16), with $q_1 \in C_{\infty, \mu}^{2, \beta}(\Sigma)$, $q_2 \in D_{\mu, \rho}^{2, \beta}(\Sigma)$ such that
\[
\|q_1\|_{C_{\infty, \mu}^{2, \beta}(\Sigma)} < \Lambda_0 \epsilon^\mu, \quad \|q_2\|_{D_{\mu, \rho}^{2, \beta}(\Sigma)} < \Lambda_0 \epsilon^\mu,
\]
with $\Lambda_0 > 0$ fixed. Then there exists a unique solution $(\phi_1, \phi_2) = (\phi_1(h_1, h_2), \phi_2(h_1, h_2)) \in X_{\mu,\rho}^{\beta} \times X_{\mu,\rho}^{\beta}$ to system (5.11) satisfying
\[
\|\phi_1(h_1, h_2)\|_{X_{\mu,\rho}^{1, \beta}} + \|\phi_2(h_1, h_2)\|_{X_{\mu,\rho}^{1, \beta}} \leq \Lambda_1 \epsilon^{2+\mu},
\]
\[
\|\phi_1(h_1, h_2) - \phi_1(h_1, h_2)\|_{X_{\mu,\rho}^{0, \beta}} \leq c \epsilon^{2+\mu}(\|q_1\|_{C_{\infty, \mu}^{2, \beta}(\Sigma)} + \|q_2\|_{D_{\mu, \rho}^{2, \beta}(\Sigma)})
\]
for some $c, \Lambda_1 > 0$.

Once again, first we will construct a right inverse of the operator $-\Delta_{\Sigma_{\epsilon}} - \partial_t^2 + (3v^2 - 1)$ under the suitable orthogonality condition, then we will apply a fixed point argument. For this purpose, we consider the linear problem
\[
- \Delta_{\Sigma_{\epsilon}} \phi - \partial_t^2 \phi + (3v^2 - 1)\phi = f \quad \text{in } \Sigma_{\epsilon} \times \mathbb{R}.
\]
In order to be able to solve equation (5.26), $f$ has to satisfy the orthogonality condition
\[
\int_{\mathbb{R}} f(p, t) v'_t(t) dt = 0 \quad \text{for } p \in \Sigma_{\epsilon},
\]
and we look for a solution $\phi$ which satisfies (5.27). The aim of this subsection is to prove the following result.

**Proposition 5.4.** Let $\beta \in (0, 1)$, $\rho \in (0, \sqrt{2})$ and $\mu \in (0, 1)$. Then, for any $f \in Y_{\mu,\rho}^{\beta}$, there exists a unique solution $\phi := \Pi(f) \in X_{\mu,\rho}^{\beta}$ to (5.26). Moreover
\[
\|\phi\|_{X_{\mu,\rho}^{\beta}} \leq c\|f\|_{Y_{\mu,\rho}^{\beta}}.
\]

The proof of Proposition 5.4 involves several steps, which will be dealt with the aid of some Lemmas and Remarks.

Since the decay of $f$ along the surface is slow, $f$ is not necessarily in $L^2(\Sigma_{\epsilon} \times \mathbb{R})$, thus we start with a truncated problem with right-hand side $f_R := f 1_{\chi R}$, where $\chi_R(p) := \chi(s(\epsilon p) - R)$ and $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth cutoff function such that
\[
\chi(\tau) = \begin{cases} 1, & \tau > 2 \\ 0, & \tau < 1 \end{cases}, \quad 0 \leq \chi \leq 1.
\]
Lemma 5.3 (Existence for a truncated problem). Let $\beta \in (0, 1)$, $\rho \in (0, \sqrt{2})$, $\gamma \in (0, 1)$ and $f \in \mathcal{Y}_{\rho, \rho}^{\beta}$. Then, for any $R > 0$, there exists a unique solution $\phi_R \in H^1(\Sigma \times \mathbb{R})$ to (5.26) such that

$$
\int_{\mathbb{R}} \phi_R(p, t)v'_s(t)dt = 0, \quad \forall p \in \Sigma.
$$

Moreover, $\phi \in C^2(\Sigma \times \mathbb{R})$, it is $O(m) \times O(n)$-invariant and bounded.

Proof. Since we have multiplied by a cutoff function and we have exponential decay in $t$, $f_R \in L^2(\Sigma \times \mathbb{R})$, thus equation (5.26) can be attached with variational techniques. In fact, due to the spectral decomposition of the ordinary differential operator $-\partial_t^2 + (3v_s^2 - 1)$, the corresponding functional

$$
\int_{\Sigma \times \mathbb{R}} |\nabla_{\Sigma} \phi|^2 + \int_{\Sigma \times \mathbb{R}} (\partial_t \phi)^2 + \int_{\Sigma \times \mathbb{R}} (3v_s^2 - 1)\phi^2d\sigma(p)dt - \int_{\Sigma \times \mathbb{R}} f_R\phi d\sigma(p)dt
$$

is coercive and lower semicontinuous on the closed subspace

$$
Z := \left\{ \phi \in H^1(\Sigma \times \mathbb{R}) : \int_{\mathbb{R}} \phi(p, t)v'_s(t)dt = 0, \forall p \in \Sigma \right\},
$$

thus it has a unique minimiser $\phi_R \in Z$, which fulfils

$$
\int_{\Sigma \times \mathbb{R}} \left( (\nabla_{\Sigma} \phi_R, \nabla_{\Sigma} \psi)_{\Sigma} + \partial_t \phi_R \partial_t \psi + (3v_s^2 - 1)\phi_R \psi \right) d\sigma(p)dt = \int_{\Sigma \times \mathbb{R}} f_R\psi d\sigma(p)dt, \quad \forall \psi \in Z.
$$

(5.28)

In order to prove that $\phi$ is a true weak solution we have to show that (5.28) is satisfied for any $\psi \in H^1(\Sigma \times \mathbb{R})$. This follows from a direct computation after writing any $\psi \in H^1(\Sigma \times \mathbb{R})$ as

$$
\psi = \tilde{\psi} + a(p)v'_s(t), \quad \tilde{\psi} \in Z, \quad a(p) := \frac{\int_{\mathbb{R}} \psi(p, t)v'_s(t)dt}{\int_{\mathbb{R}} v'_s(t)^2dt}
$$

and we use the fact that $v'_s$ is in the kernel of $-\Delta_{\Sigma} - \partial_t^2 + (3v_s^2 - 1)$. Symmetry and regularity for $\phi_R$ follow from symmetry and regularity of $f$ and uniqueness. The fact that $\phi \in L^\infty(\Sigma \times \mathbb{R})$ follows from a bootstrap argument. \hfill \Box

In order to prove the decay of the solution and the estimates in the required weighted norms we need an a priori estimate, which relies on the following well-known result, which is proved, for instance in [19].

Lemma 5.4. Let $\phi$ be a bounded solution to

$$
-\Delta_{RN} \phi + \partial_t^2 \phi + (3v_s^2 - 1)\phi = 0, \quad \forall (y, t) \in \mathbb{R}^N \times \mathbb{R}.
$$

Then $\phi(y, t) = ce^t(t)$, for some constant $c \in \mathbb{R}$.

Lemma 5.5 (A priori estimate). Let $\beta \in (0, 1)$, $\rho \in (0, \sqrt{2})$, $\gamma \in (0, 1)$ and $f \in \mathcal{Y}_{\rho, \rho}^{\beta}$. Let $\phi$ be a bounded solution to (5.26). Then

$$
\|\phi\|_{L^\infty(\Sigma \times \mathbb{R})} \leq c\|f\|_{L^\infty(\Sigma \times \mathbb{R})}.
$$
Proof. Assume, by contradiction, that there exist a sequence $\varepsilon_n \to 0$ and sequences $\phi_n, f_n \in L^\infty(\Sigma_c \times \mathbb{R})$ such that

$$-\Delta_{\Sigma_c} \phi_n - \partial_t^2 \phi_n + (3v^2 - 1)\phi_n = f_n$$

$$\int_{\mathbb{R}} \phi_n(p, t) dt = 0 \quad \forall \ p \in \Sigma_c$$

$$\|\phi_n\|_{L^\infty(\Sigma_c \times \mathbb{R})} = 1, \quad \|f_n\|_{L^\infty(\Sigma_c \times \mathbb{R})} \to 0.$$ Let us consider a sequence $(p_n, t_n) \in \Sigma_c \times \mathbb{R}$ such that $|\phi_n(p_n, t_n)| \geq \frac{1}{2}$ and a parametrisation $Y : B_\theta \subset \mathbb{R}^N \to \Sigma$ of $\Sigma$ such that $Y(0) = \varepsilon_n p_n$ and the metric at 0 is the identity. In these coordinates, the laplacian reads

$$\Delta_{\Sigma_c} := g^{ij}(\varepsilon_n y) \partial_{ij} + \varepsilon_n b^j(\varepsilon_n y) \partial_i$$

so that the equation reads

$$-g^{ij}(\varepsilon_n y) \partial_{ij} \tilde{\phi}_n - \varepsilon_n b^j(\varepsilon_n y) \partial_i \tilde{\phi}_n - \partial_t^2 \tilde{\phi}_n + (3v^2 - 1)\tilde{\phi}_n = \tilde{f}_n,$$ \quad $\forall (y, t) \in B_{\theta \varepsilon^{-1}} \times \mathbb{R},$

where $\tilde{\phi}_n(y, t) := \phi_n(p, t)$ and $\tilde{f}_n(y, t) := f_n(p, t)$. Note that $|\tilde{\phi}_n(0, t_n)| = |\phi_n(p_n, t_n)| \geq \frac{1}{2}$.

Let us assume first that $t_n$ is bounded. Then, up to a subsequence, $t_n \to t_\infty \in \mathbb{R}$ and, by the Ascoli-Arzelà theorem, $\tilde{\phi}_n$ converges uniformly on compact subsets to a bounded solution $\phi_\infty$ to

$$-\Delta_{\mathbb{R}^N} \phi_\infty - \partial_t^2 \phi_\infty + (3v^2 - 1)\phi_\infty = 0$$

in $\mathbb{R}^{N+1}$ such that $|\phi_\infty(0, t_\infty)| \geq \frac{1}{2}$, which yields that $\phi_\infty = C v_\ast(t)$, with $C \neq 0$. However, the orthogonality condition

$$\int_{\mathbb{R}} \phi_\infty v_\ast dt = 0, \quad \forall y \in \mathbb{R}^N$$

yields that $C = 0$, a contradiction.

If $t_n$ is unbounded, say $t_n \to \infty$, the situation is similar. In this case we set

$$\phi_n^\varepsilon(y, t) := \tilde{\phi}_n(y, t + t_n)$$

and we get, in the limit, a bounded solution $\phi_\infty$ to

$$-\Delta_{\mathbb{R}^N} \phi_\infty - \partial_t^2 \phi_\infty + 2\phi_\infty = 0$$

in $\mathbb{R}^{N+1}$ such that $|\phi_\infty(0, \infty)| \geq \frac{1}{2}$, which contradicts the maximum principle and the boundedness of $\phi_\infty$. \hfill $\square$

**Lemma 5.6 (Decay in $t$).** Let $\beta \in (0, 1)$, $\rho \in (0, \sqrt{2})$, $\mu \in (0, 1)$ and $f \in Y_{\perp, \frac{1}{2}, \mu, \rho}^{\varepsilon, \beta}$. Let $\phi$ be a bounded solution to (5.26). Then

$$|\phi(p, t)| \cosh(t)^\rho \leq c(\|\phi\|_{L^\infty(\Sigma_c \times \mathbb{R})} + \|f \cosh(t)^\rho\|_{L^\infty(\Sigma_c \times \mathbb{R})}), \quad \forall (p, t) \in \Sigma_c \times \mathbb{R}.$$\hfill $\square$

**Proof.** The proof relies on comparing the solution with a barrier. For any $\nu > 0$ and $\lambda > 0$, we set

$$v_{\lambda, \nu}(t) := (\lambda \cosh^{-\rho}(t) + \nu \cosh^\rho(t))e^{\nu R_0}.$$ The idea is to apply the maximum principle to bounded domain

$$\Omega := \Sigma_{c, \varepsilon^{-1} R_0} \times (t_1, t_2), \quad \Sigma_{c, \varepsilon^{-1} R_0} := \Sigma_c \cap B_{\varepsilon^{-1} R_0},$$
with \( R_0 > 0, t_1 > 0, t_2 > 0 \) to be determined. First we note that, in order to apply the maximum principle, \( f'(v_\ast(t)) \) must be uniformly negative, thus we choose \( t_1 > 0 \) such that \( 1 - \frac{3 \nu^2}{2} < -\frac{\sqrt{2}}{2} \) for \( |t| > t_1 \). Then we observe that, since \( e^{\nu R_0} \geq 1 \), for \( p \in \Sigma_\ast \) and \( t = t_1 \) we have

\[
\phi(p, t_1) \leq \|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \leq \lambda \cosh^{-\rho}(t_1) \leq v_{\lambda, \nu}(t_1)
\]

provided \( \lambda \geq \|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \cosh^\rho(t_1) \). For \( |t| = t_2 \), we have

\[
\phi(p, t_2) \leq \|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \leq \nu \cosh^\rho(t_2) \leq v_{\lambda, \nu}(t_2)
\]

provided \( t_2 \) is large enough. Moreover, on \( \partial\Sigma_{\varepsilon, -1} R_0 \times (t_1, t_2) \), we have

\[
v_{\lambda, \nu}(t) \geq (\lambda \cosh^{-\rho}(t_2) + \nu \cosh^\rho(t_1)) e^{\nu R_0} \geq \|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \geq \phi(p, t)
\]

provided \( R_0 > 0 \) is large enough. Differentiation shows that in \( \Omega \) the following differential inequality holds

\[
(-\Delta_{\Sigma_\ast} - \partial_t^2 + (3 \nu^2 - 1))(\phi - v_{\lambda, \nu}) \leq \left(\|f\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} - \lambda e^{\nu R_0} \left(1 - \frac{\rho^2}{2}\right)\right) \cosh(t)^{-\rho} \leq 0
\]

provided

\[
\lambda \geq \frac{2\|f\|_{L^\infty(\Sigma_\ast \times \mathbb{R})}}{e^{\nu R_0}(2 - \rho^2)}.
\]

In conclusion, applying the maximum principle and letting \( \nu \to 0 \), we have

\[
\phi \leq \lambda \cosh^{-\rho}(t) \text{ in } \Sigma_\ast \times (t_1, \infty), \quad \lambda = \|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \cosh^\rho(t_1) + \frac{2\|f\|_{L^\infty(\Sigma_\ast \times \mathbb{R})}}{2 - \rho^2}.
\]

If \( \phi \) satisfies the hypothesis of the Lemma, then also \(-\phi\) and \(\phi(p, -t)\) do, hence the proof is concluded. \( \square \)

**Lemma 5.7** (Decay in \( p \)). Let \( \beta \in (0, 1) \), \( \rho \in (0, \sqrt{2}) \), \( \rho \in (0, 1) \) and \( f \in Y_{\beta, \mu, \rho}^{\varepsilon, 2} \). Let \( \phi \) be a bounded solution to (5.26). Then

\[
\|\phi\|_{2+\mu, \rho, \infty} \leq c(\|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} + \|f\|_{2+\mu, \rho, \infty}).
\]

**Proof.** For \( p \in \Sigma_\ast \), we define

\[
\varphi(p) := \int_{\mathbb{R}} \phi(p, t)^2 dt.
\]

Due to the exponential decay in \( t \) provided by Lemma 5.6, \( \psi \) is well defined and bounded with

\[
\|\varphi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} \leq c(\|\phi\|_{L^\infty(\Sigma_\ast \times \mathbb{R})} + \|f\|_{L^\infty(\Sigma_\ast \times \mathbb{R})})
\]

A computation shows that

\[
\Delta_{\Sigma_\ast} \varphi = \int_{\mathbb{R}} (2\phi \Delta_{\Sigma_\ast} \varphi - 2|\nabla_{\Sigma_\ast} \phi|^2) dt,
\]

thus, multiplying (5.26) by \( \phi \) and integrating over \( \mathbb{R} \),

\[
\Delta_{\Sigma_\ast} \varphi = 2 \int_{\mathbb{R}} |\nabla_{\Sigma_\ast} \phi|^2 dt + 2 \int_{\mathbb{R}} (\partial_t \phi)^2 dt + 2 \int_{\mathbb{R}} (3 \nu^2 - 1) \phi^2 dt - 2 \int_{\mathbb{R}} \phi f dt,
\]

therefore, by the spectral properties of the ordinary differential operator \(-\partial_t^2 + (3 \nu^2 - 1)\), we have

\[
\Delta_{\Sigma_\ast} \varphi \geq 3 \int_{\mathbb{R}} \phi^2 dt - 2 \int_{\mathbb{R}} f \phi dt,
\]
and hence, by the Young inequality, \( \psi \geq 0 \) satisfies the differential inequality

\[
-\Delta_{\Sigma_c} \varphi(p) + 2\varphi(p) \leq \int_{\mathbb{R}} f(p, t)^2 dt, \quad \forall p \in \Sigma_c.
\]

Therefore, using the barrier

\[
w_{\lambda, \nu}(p) := \lambda(s(\varepsilon p)^2 + 2)^{-(2+\mu)} + \nu(s(\varepsilon p)^2 + 2)^{2+\mu},
\]

with

\[
\lambda := \frac{2\|f\|_{2+\mu, p, \infty}^2}{2-\delta} + c(\mu)\|\varphi\|_{L^\infty(\Sigma_c \times \mathbb{R})}
\]

and \( \nu > 0 \) arbitrarily small, the maximum principle gives

\[
0 \leq \varphi(p) \leq c\|f\|_{2+\mu, p, \infty}^2(s(\varepsilon p)^2 + 2)^{\frac{-2\mu}{2+\mu}}
\]
or equivalently

\[
(s(\varepsilon p)^2 + 2)^{2+\mu} \int_{\mathbb{R}} \phi(p, t)^2 dt \leq c\|f\|_{2+\mu, p, \infty}^2.
\]

By the elliptic estimates, we have

\[
(s(\varepsilon p)^2 + 2)^{\frac{2+\mu}{2}} \|\phi\|_{L^\infty(t_0, t)} \leq c\|f\|_{2+\mu, p, \infty}, \quad \forall p \in \Sigma_c.
\]

In order to prove that the decay along the surface is uniform in \( t \), we use the barrier

\[
\tilde{w}_{\lambda, \nu}(p, t) := \tilde{c}(\mu, \rho)\|f\|_{2+\mu, p, \infty}(s(\varepsilon p)^2 + 2)^{-\frac{2+\mu}{2+\nu}} \cosh(t) - \nu(s(\varepsilon p)^2 + 2)^{\frac{2+\mu}{2+\nu}} \cosh(t)^\rho
\]

where \( \tilde{c}(\mu, \rho) > 0 \) is a suitable constant, in a region of the form

\[
\{(p, t) \in \Sigma_c \times \mathbb{R} : t_0 < t < t_1, |s(\varepsilon p)| \leq s_0\},
\]

with \( t_0 \) so large that \( 3\varepsilon^2 - 1 > 1 + \frac{2}{2\nu} \) for \( t \in (t_0, \infty) \). This concludes the proof. \( \square \)

Now we can conclude the proof of Proposition 5.4.

**Proof.** Given \( f \in Y^1_{\lambda, \mu, \rho} \), for any \( R > 0 \), by Lemma 5.3 it is possible to find a bounded \( O(m) \times O(n) \)-invariant solution \( \phi_R \in H^1(\Sigma_c \times \mathbb{R}) \) to the truncated equation

\[
-\Delta_{\Sigma_c} \phi_R - \partial_t^2 \phi_R + (3\mu^2 - 1)\phi_R = f_R
\]

which satisfies the orthogonality condition (5.27) and, by the a priori estimate provided in Lemma 5.5,

\[
\|\phi_R\|_{L^\infty(\Sigma_c \times \mathbb{R})} \leq c\|f_R\|_{L^\infty(\Sigma_c \times \mathbb{R})} \leq c\|f\|_{L^\infty(\Sigma_c \times \mathbb{R})}.
\]

Therefore, by the Ascoli-Arzelà theorem, there exists a sequence \( R_k \to \infty \) such that \( \phi_{R_k} \) converges uniformly on compact subsets to a bounded solution \( \phi \) to (5.26).

Since \( \phi_R \) satisfies the orthogonality condition (5.27) and is \( O(m) \times O(n) \)-invariant, then also \( \phi \) does. Moreover, by Lemmas 5.6 and 5.7, \( \phi \) has the suitable decay both in \( t \) and in \( p \)

\[
\|\phi\|_{2+\mu, p, \infty} \leq c(\|\phi\|_{L^\infty(\Sigma_c \times \mathbb{R})} + \|f\|_{2+\mu, p, \infty}).
\]

Once again by the a priori estimate, namely by Lemma 5.5,

\[
\|\phi\|_{2+\mu, p, \infty} \leq \|f\|_{2+\mu, p, \infty}.
\]

By the elliptic estimates, we can see that \( \phi \in X_0 \) and

\[
\|\phi\|_{X^1_{\lambda, \mu, \rho}} \leq c\|f\|_{Y^1_{\lambda, \mu, \rho}}.
\]
Now we can prove Proposition 5.3.

**Proof.** System (5.11) can be formulated as a fixed point problem in the form
\[
\phi_l = F_l\left(-\chi_4 S(w), l - N_l(\psi, \phi_1, \phi_2, h_1, h_2) + P_l(\psi, \phi_1, \phi_2, h_1, h_2)(p)\psi', h_1, h_2\right), \quad l = 1, 2,
\]
where \(\psi = \psi(\phi_1, \phi_2, h_1, h_2)\) is the correction found in Proposition 5.1.

Thanks to Lemma 4.3 and to the size of \(\psi\) in \(\varepsilon\), the above problem has a unique solution in the ball
\[
B_{\Lambda_1} = \{(\phi_1, \phi_2) \in X^{4, \beta}_{\mu, \rho} \times X^{4, \beta}_{\mu, \rho} : \|\phi_1\|_{X^{4, \beta}_{\mu, \rho}} + \|\phi_2\|_{X^{4, \beta}_{\mu, \rho}} < \Lambda_1 \varepsilon^{2+\mu}\},
\]
provided \(\Lambda_1 > 0\) is large enough. The details of the nonlinear argument are similar to the proof of Proposition 5.1. A similar proof can be found in Section 6 from [19]. This concludes the proof. □

5.5. The bifurcation equation. We recall that the bifurcation equation is actually a system, given by
\[
P_l^\beta(\psi, \phi_1, \phi_2, h_1, h_2) = 0, \quad l = 1, 2,
\]
where \(\psi, \phi_1\) and \(\phi_2\) are constructed in subsections 5.3 and 5.4.

As we shown in subsection 5.1, this turns out to be equivalent to a system of the form (5.15), which can be solve using a fixed point argument in the ball
\[
B_{\Lambda_0} := \{(q_1, q_2) \in C^{2, \beta}_\infty(\Sigma) \times D^{2, \beta}_{\mu, \rho}(\Sigma) : \|q_1\|_{C^{2, \beta}_\infty(\Sigma)} < \Lambda_0 \varepsilon^\mu, \|q_2\|_{D^{2, \beta}_{\mu, \rho}(\Sigma)} < \Lambda_0 \varepsilon^\mu\},
\]
provided \(\Lambda_1 > 0\) is large enough, but independent of \(\varepsilon > 0\).

This is due mainly to the fact that, thanks to Proposition 3.2, the right inverse of \(\Delta_\Sigma + |A_\Sigma|^2 + 2\sqrt{2}u_\varepsilon e^{-2}e^{-\sqrt{\varepsilon}r}\) is of order \(|\log \varepsilon|^2\) and Lemma 4.3, providing the size of the error. Once again, the details are left to the reader. Similar proofs can be found in [19].

6. Estimate of the energy on a ball

In this section we will prove point (1.3) and hence This concluding the proof of Theorem 1.3. Recall that the developments in Section 5 have yielded a solution \(u_\varepsilon = w + \varphi\) to (1.1), where \(w\) is described in (5.2) and \(\varphi\) is described in (5.6).

To show
\[
\int_{B_R} \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4} (1 - u_\varepsilon)^2 \leq c R^N,
\]
we first claim that
\[
\int_{B_R} |\nabla u_\varepsilon|^2 \leq c R^N.
\]

Our developments make extensive use of the coordinates \((p, t)\)-coordinates. First we observe that the volume element in the Fermi coordinates \((p, z)\) of \(\Sigma\) is given by
\[
\sqrt{|\det G|} = \sqrt{|\det g|} + P(p, z),
\]
being \(P\) a polynomial in \(z\) such that \(P(p, 0) = 0\) (see (4.3) for the definition of \(G\) and \(g\)).
With the change of variables in (4.4), the volume element in the \((p, t)\)-coordinates in \(X_{\epsilon_R}(\mathcal{N}_{\epsilon_R,b_R})\) is given by
\[
\sqrt{\det g_{\epsilon}} + P(\epsilon p, \epsilon(t + h_1(\epsilon p))) = \epsilon^{-(m-1)}a(s(\epsilon p))^{m-1} \epsilon^{-(n-1)}b(s(\epsilon p))^{n-1} + P(\epsilon p, \epsilon(t + h_1(\epsilon p))),
\]
being \(g_{\epsilon}\) the metric of \(\Sigma_{\epsilon} = \epsilon^{-1}\Sigma\).

Let \(p\) is any point in \(\Sigma_{\epsilon} \cap \partial B_R\) and set \(\mathfrak{s}(R) := \mathfrak{s}(p)\) and
\[
t_\epsilon := \frac{1}{4\sqrt{2}} \left(\log(\epsilon p) + 2|\log \epsilon|\right).
\]

Assume that \(R > 2\epsilon^{-1}\). Then
\[
\int_{B_R} |\nabla u|^2 \, d\xi \leq \int_0^{\mathfrak{s}(R)} \epsilon^{-(m-1)}a(\epsilon s)^{m-1} \epsilon^{-(n-1)}b(\epsilon s)^{n-1} \, ds \int_0^{t_\epsilon} v_\epsilon'(t)^2 \, dt
\]
\[
\leq \epsilon^{-(N-1)} \int_0^{\mathfrak{s}(R)} (1 + \epsilon s)^{m-1} \epsilon^{n-1} \, ds
\]
\[
= \epsilon^{-(N-1)} \int_0^{\mathfrak{s}(R)} (1 + s)^{m-1} s^{n-1} \, ds
\]
\[
= \epsilon^{-(N-1)} \left( \int_0^1 (1 + s)^{m-1} s^{n-1} \, ds + \int_1^{\mathfrak{s}(R)} (1 + s)^{m-1} s^{n-1} \, ds \right)
\]
\[
\leq \epsilon^{-(N-1)} \left(1 + (\epsilon \mathfrak{s}(R))^N\right) \leq \epsilon^{-(N-1)} (1 + (\epsilon R)^N) \leq cR^N.
\]

Setting \(\tilde{\varphi} = \sum_{i=1}^k \lambda_{3,i} \tilde{\varphi}_i\),
\[
\int_{B_R} |\nabla \tilde{\varphi}|^2 \, d\xi \leq \epsilon^{4+2\mu} \int_0^{\mathfrak{s}(R)} \frac{\epsilon^{-(m-1)}a(\epsilon s)^{m-1} \epsilon^{-(n-1)}b(\epsilon s)^{n-1}}{(\epsilon s)^2 + 2 + \mu} \, ds \int_0^{t_\epsilon} e^{-2\mu t} \, dt
\]
\[
\leq \epsilon^{4+2\mu-(N-1)} \int_0^{\mathfrak{s}(R)} \frac{(1 + \epsilon s)^{m-1} \epsilon^{n-1}}{(\epsilon s)^2 + 2 + \mu} \, ds
\]
\[
= \epsilon^{4+2\mu-N} \int_0^{\mathfrak{s}(R)} \frac{(1 + s)^{m-1} s^{n-1}}{(s^2 + 2 + \mu)} \, ds
\]
\[
= \epsilon^{4+2\mu-N} \left( \int_0^1 \frac{(1 + s)^{m-1} s^{n-1}}{(s^2 + 2 + \mu)} \, ds + \int_1^{\mathfrak{s}(R)} \frac{(1 + s)^{m-1} s^{n-1}}{(s^2 + 2 + \mu)} \, ds \right)
\]
\[
\leq \epsilon^{4+2\mu-N} \left(1 + (\epsilon \mathfrak{s}(R))^{N-4-2\mu}\right)
\]
\[
\leq \epsilon^{4+2\mu-N} \left(1 + (\epsilon R)^{N-4-2\mu}\right)
\]
\[
\leq cR^{N-4-2\mu}.
\]
Finally, we estimate the gradient of $\psi$, to find that

$$
\int_{B_R} |\nabla \psi|^2 d\xi \leq c \varepsilon^{4+2\mu} \int_{B_R} \frac{1}{(|\xi| + 2)^{2+\mu}} d\xi \\
= c \varepsilon^{4+2\mu} \int_0^{\varepsilon(R)} \frac{\rho^N}{((\rho^2 + 2)^{2+\mu} \varepsilon) d\rho} \\
= c \varepsilon^{4+2\mu} \int_0^{\varepsilon(R)} \frac{(\varepsilon^{-1} \rho)^N}{(\rho^2 + 2)^{2+\mu} \varepsilon} d\rho \\
= c \varepsilon^{4+2\mu - N-1} \left( \int_0^{\varepsilon(R)} \frac{\rho^N}{(\rho^2 + 2)^{2+\mu}} d\rho + \int_{\varepsilon(R)}^{1} \frac{\rho^N}{(\rho^2 + 2)^{2+\mu}} d\rho \right) \\
\leq c \varepsilon^{4+2\mu - N-1} \left( 1 + \int_{1}^{\varepsilon(R)} \rho^{N-4-2\mu} d\rho \right) \\
\leq c \varepsilon^{4+2\mu - N-1} \left( 1 + \varepsilon(R)^{N-3-2\mu} \right) \\
\leq c R^{N-3-2\mu}.
$$

Similar estimates hold for the mixed terms and this completes the proof of the claim. To show that

$$
\int_{B_R} (1 - u_\varepsilon^2)^2 \leq CR^N
$$

a similar argument as in the proof of the above claim can be used. We leave the details to the reader. The proof of the theorem is now complete.

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