RIGIDITY, RESIDUES AND DUALITY: OVERVIEW AND RECENT PROGRESS

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Abstract. In this article we explain the theory of rigid residue complexes in commutative algebra and algebraic geometry, summarizing the background, recent results and anticipated future results. Unlike all previous approaches to Grothendieck Duality, the rigid approach concentrates on the construction of rigid residue complexes over rings, and their intricate yet robust properties. The geometrization, i.e. the passage to rigid residue complexes on schemes and Deligne-Mumford (DM) stacks, by gluing, is fairly easy. In the geometric part of the theory, the main results are the Rigid Residue Theorem and the Rigid Duality Theorem for proper maps between schemes, and for tame proper maps between DM stacks.

Contents

0. Introduction \hspace{1cm} 1
1. The Squaring Operation and Rigid Complexes over Rings \hspace{1cm} 5
2. Rigid Dualizing Complexes over Rings \hspace{1cm} 6
3. Rigid Residue Complexes over Rings \hspace{1cm} 9
4. Rigidity, Residues and Duality for Schemes \hspace{1cm} 11
5. Rigidity, Residues and Duality for DM Stacks \hspace{1cm} 14
6. References \hspace{1cm} 19

0. Introduction

This article is about the rigid approach to Grothendieck Duality (GD). We survey the main features of this approach, discussing both results that were already obtained, and future expected results.

Throughout this article we work over a fixed base ring \( \mathbb{K} \), which is a finite dimensional regular noetherian commutative ring (e.g. a field or the ring of integers \( \mathbb{Z} \)). All rings are commutative \( \mathbb{K} \)-rings, and by default they are essentially finite type (EFT) over \( \mathbb{K} \).

In Section 1 we review the squaring operation. Given a ring \( A \), the squaring operation is a functor \( \text{Sq}_{A/\mathbb{K}} \) from the derived category \( D(A) \) to itself. This is a quadratic functor, and – unless \( A \) is flat over \( \mathbb{K} \) – its construction requires the use of commutative and noncommutative DG rings; see [Ye6]. The squaring operation has backward functoriality over any ring homomorphism \( u : A \to B \), and forward functoriality when \( u \) is essentially étale.

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A rigid complex over $A$ (relative to $K$) is a pair $(M, \rho)$, where $M \in D^b(A)$, and $\rho : M \rightarrow S\text{R}_A(K)(M)$ is an isomorphism in $D(A)$. There is a pretty obvious notion of rigid morphism between rigid complexes (see Definition 1.11). This notion extends to rigid backward morphisms over any ring homomorphism $u : A \rightarrow B$, and to rigid forward morphisms when $u$ is essentially étale.

A rigid dualizing complex over $A$ is a rigid complex $(R_A, \rho_A)$ as above, such that $R_A$ is a dualizing complex over $A$ in the sense of [RD]. This concept is discussed in Section 2. The ring $A$ has a rigid dualizing complex $(R_A, \rho_A)$, and it is unique up to a unique rigid isomorphism. If $u : A \rightarrow B$ is a finite ring homomorphism, then there is a unique nondegenerate rigid backward morphism $\text{tr}^\text{rig}_u : (R_B, \rho_B) \rightarrow (R_A, \rho_A)$ called the rigid trace morphism. If $v : A \rightarrow A'$ is an essentially étale ring homomorphism, then there is a unique nondegenerate rigid forward morphism $\text{q}^\text{rig}_v : (R_A, \rho_A) \rightarrow (R_{A'}, \rho_{A'})$ called the rigid étale-localization morphism. The morphisms $\text{tr}^\text{rig}_u$ and $\text{q}^\text{rig}_v$ commute with each other in the appropriate sense.

Having the rigid dualizing complex $(R_A, \rho_A)$ on every EFT $K$-ring $A$ allows to construct the twisted induction pseudofunctor, in a totally algebraic way (rings only, no geometry), equipped with forward and backward functorialities. All the material up to this point (the content of Sections 1-2) is in the papers [Ye6] and [OSY], the latter joint with M. Ornaigh and S. Singh.

Looking to the future, we plan to study a more refined object than the rigid dualizing complex: it is the rigid residue complex $(\mathcal{K}_A, \rho_A)$ of a ring $A$. Given an essentially étale ring homomorphism $v : A \rightarrow A'$, there is the nondegenerate rigid étale-localization homomorphism $\text{q}^\text{res}_v : \mathcal{K}_A \rightarrow \mathcal{K}_{A'}$. For a finite ring homomorphism $u : A \rightarrow B$ there is the nondegenerate rigid trace homomorphism $\text{tr}^\text{res}_u : \mathcal{K}_B \rightarrow \mathcal{K}_A$. The latter extends to a functorial backward homomorphism $\text{tr}^\text{res}_u : \mathcal{K}_B \rightarrow \mathcal{K}_A$ for an arbitrary ring homomorphism $u$, called the ind-rigid trace, but $\text{tr}^\text{res}_u$ is only a homomorphism of graded $A$-modules. We should stress that $\text{q}^\text{res}_v$ and $\text{tr}^\text{res}_u$ are actual homomorphisms of complexes (see Remark 3.12 regarding notation). This topic, summarized in Section 3, is planned for the paper [Ye8].

Rigid residue complexes can be easily glued on finite type (FT) $K$-schemes, and they still have the ind-rigid trace $\text{tr}^\text{res}_f : f_!(\mathcal{K}_Y) \rightarrow \mathcal{K}_X$ for an arbitrary scheme map $f : Y \rightarrow X$. The twisted induction pseudofunctor from Section 2 now becomes the geometric twisted inverse image pseudofunctor $f \mapsto f^!$. We expect to prove the Rigid Residue Theorem 4.10 and the Rigid Duality Theorem 4.11 for proper maps of EFT $K$-schemes, thus recovering almost all the results from the original book “Residues and Duality” [RD], yet in a very explicit way. This topic is covered in Section 4, and planned for the paper [Ye9].

For every scheme $X$ the assignment $X' \mapsto \mathcal{K}_{X'}$ is a quasi-coherent sheaf on the small étale site $X_{et}$. This implies that every finite type Deligne-Mumford (DM) $K$-stack $\mathfrak{X}$ admits a rigid residue complex $\mathcal{K}_X$. Here too we have the twisted inverse image pseudofunctor $f \mapsto f^!$. For a map $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ of FT DM $K$-stacks there is the ind-rigid trace $\text{tr}^\text{res}_f : f_!(\mathcal{K}_Y) \rightarrow \mathcal{K}_X$. Under a mild technical condition, we expect to prove the Rigid Residue Theorem 5.11 for proper maps of DM stacks, and the Rigid Duality Theorem 5.13 for such maps that are also tame. This is the content of Section 5 of the article, and will be written in the paper [Ye10].

To finish the introduction, a few words on the history of the rigid approach to GD. Rigid dualizing complexes were invented by M. Van den Bergh in the 1990’s, in the non-commutative setting. This concept was imported to commutative algebra by J.J. Zhang and the author around 2005, where it was made functorial, and it was also expanded to the arithmetic setting (namely allowing a base ring $K$ that is not a field). More details on
the history of the topic, with references, can be found in Remark 2.14. Many of the ideas in this article were already mentioned in our lectures from 2013, see Ye5.

Acknowledgments. Because this is a project spanning almost 15 years (and counting), it is impossible to thank all the people who had contributed to it. Therefore I shall only mention my collaborator James J. Zhang with whom the project began; my current collaborators Mattia Ornaghi and Saurabh Singh; Martin Olsson and Dan Edidin who assisted me in my attempts to understand DM stacks; and Jack Hall and Amnon Neeman who explained some of their work to me. The project is currently supported by the Israel Science Foundation grant no. 824/18.

1. The Squaring Operation and Rigid Complexes over Rings

We fix a base ring \( \mathbb{K} \), which is regular noetherian of finite Krull dimension (e.g. a field or \( \mathbb{Z} \)). Recall that a commutative \( \mathbb{K} \)-ring \( A \) is called essentially finite type (EFT) if it is a localization of a finite type \( \mathbb{K} \)-ring. Note that such a ring \( A \) is noetherian and of finite Krull dimension. The category of EFT commutative \( \mathbb{K} \)-rings is denoted by \( \text{Rng}_{/\text{et}} \mathbb{K} \). All rings in this paper belong to \( \text{Rng}_{/\text{et}} \mathbb{K} \).

For a ring \( A \) we denote by \( \text{M}(A) \) the abelian category of \( A \)-modules, and by \( \text{C}(A) \) the DG category of complexes of \( A \)-modules. The strict subcategory \( \text{C}_{\text{str}}(A) \subseteq \text{C}(A) \) has the same objects, but its morphisms are the degree 0 cocycles (i.e. the degree 0 homomorphisms \( \phi : M \rightarrow N \) that commute with the differentials). The derived category is \( \text{D}(A) \). The categorical localization functor \( Q : \text{C}_{\text{str}}(A) \rightarrow \text{D}(A) \) is the identity on objects, and it is universal for inverting quasi-isomorphisms in \( \text{C}_{\text{str}}(A) \). Note that \( \text{C}_{\text{str}}(A) \) is an abelian category, \( \text{D}(A) \) is a triangulated category, and the functor \( Q : A \)-linear. Inside \( \text{D}(A) \) there is the full triangulated subcategory \( \text{D}^+_{\text{str}}(A) \) of complexes with bounded finitely generated cohomology. All the necessary facts about derived categories can be found in our book Ye7. Furthermore, the contents of Sections 1-3 are treated in detail in this book, under the assumption that the rings in question are flat over \( \mathbb{K} \).

Following Ye6 and Ye7 Chapter 3, by commutative DG \( \mathbb{K} \)-ring we mean a nonpositive strongly commutative DG ring \( \hat{A} = \bigoplus_{i \leq 0} \hat{A}^i \), equipped with a ring homomorphism \( \mathbb{K} \rightarrow \hat{A}^0 \). Strong commutativity means that \( b \cdot a = (-1)^{i \cdot j} \cdot a \cdot b \) for all \( a \in \hat{A}^i \) and \( b \in \hat{A}^j \), and that \( a \cdot a = 0 \) when \( i \) is odd. The DG ring \( \hat{A} \) is said to be \( \mathbb{K} \)-flat if it is \( \mathbb{K} \)-flat as a DG \( \mathbb{K} \)-module.

Let \( \hat{A} \) be a ring. A \( \mathbb{K} \)-flat commutative DG ring resolution of \( A \) is a quasi-isomorphism \( \hat{A} \rightarrow A \) from a \( \mathbb{K} \)-flat commutative DG \( \mathbb{K} \)-ring \( \hat{A} \). Such resolutions always exist.

Here is the main theorem of Ye6:

**Theorem 1.1 (YZ1, Ye6).** There is a functor

\[
\text{Sq}_{A/\mathbb{K}} : \text{D}(A) \rightarrow \text{D}(A)
\]

called the squaring operation, equipped with a canonical isomorphism

\[
\text{Sq}_{A/\mathbb{K}}(M) \cong \text{RHom}_{\hat{A} \otimes_{\mathbb{K}} \hat{A}}(A, M \otimes_{\mathbb{K}} M)
\]

for every \( \mathbb{K} \)-flat commutative DG ring resolution \( \hat{A} \rightarrow A \).

See Ye6 Theorem 0.3.4] for the precise statement. It is interesting to mention that the proof requires the use of noncommutative DG rings.

**Theorem 1.2 (YZ1, Ye6).** The functor \( \text{Sq}_{A/\mathbb{K}} \) is a quadratic functor, namely for a morphism \( \phi : M \rightarrow N \) in \( \text{D}(A) \) and an element \( a \in A \), there is equality

\[
\text{Sq}_{A/\mathbb{K}}(a \cdot \phi) = a^2 \cdot \text{Sq}_{A/\mathbb{K}}(\phi).
\]
Definition 1.3 ([YZ1], [Ye6]). A rigid complex over \( A \) relative to \( K \) is a pair \((M, \rho)\), consisting of a complex \( M \in \text{D}^b(A) \), and an isomorphism \( \rho : M \cong \text{Sq}_{A/K}(M) \) in \( \text{D}(A) \), called a rigidifying isomorphism for \( M \).

Definition 1.4 ([YZ1], [Ye6]). Suppose \((M, \rho)\) and \((N, \sigma)\) are rigid complexes. A rigid morphism \( \phi : (N, \sigma) \to (M, \rho) \) is a morphism \( \phi : N \to M \) in \( \text{D}(A) \), such that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{\sigma} & \text{Sq}_{A/K}(N) \\
\phi \downarrow & & \downarrow \text{Sq}_{A/K}(\phi) \\
M & \xrightarrow{\rho} & \text{Sq}_{A/K}(M)
\end{array}
\]

in \( \text{D}(A) \) is commutative.

We denote by \( \text{D}(A)_{\text{rig}/K} \) the category of rigid complexes and rigid morphisms between them.

Definition 1.5 ([Ye7]). A complex \( M \in \text{D}^b(A) \) is said to have the derived Morita property if the derived homothety morphism \( A \to \text{RHom}_A(M, M) \) in \( \text{D}(A) \) is an isomorphism.

Dualizing complexes (to be recalled later) and tilting complexes (see [Ye7, Chapter 14]) have the derived Morita property.

Theorem 1.6 ([YZ1], [Ye6]). If \((M, \rho)\) is a rigid complex, such that \( M \) has the derived Morita property, then the only automorphism of \((M, \rho)\) in \( \text{D}(A)_{\text{rig}/K} \) is the identity.

The key to the proof of this theorem is the quadratic property of the functor \( \text{Sq}_{A/K} \). See Theorem 1.1.

The squaring operation \( \text{Sq}_{A/K} \) has two kinds of functorialities, which we describe next. First there is backward functoriality. Given a ring homomorphism \( u : A \to B \), and complexes \( M \in \text{D}(A) \) and \( N \in \text{D}(B) \), a morphism \( \theta : N \to M \) in \( \text{D}(A) \) is called a backward morphism over \( u \). The backward morphism \( \theta \) is called nondegenerate if the morphism \( N \to \text{RHom}_A(B, M) \) in \( \text{D}(B) \), which corresponds to \( \theta \) by adjunction, is an isomorphism.

The next theorem extends Theorem 1.1.

Theorem 1.7 ([YZ1], [Ye6]). Let \( u : A \to B \) be a ring homomorphism, and let \( \theta : N \to M \) be a backward morphism in \( \text{D}(A) \) over \( u \). Then there is a backward morphism

\[
\text{Sq}_{u/K}(\theta) : \text{Sq}_{B/K}(N) \to \text{Sq}_{A/K}(M)
\]

in \( \text{D}(A) \) over \( u \), with an explicit formula given suitable DG ring resolutions \( \hat{A} \to A \) and \( \hat{B} \to B \). The operation \( \text{Sq}_{u/K}(\theta) \) is functorial in \( u \) and \( \theta \).

For a precise statement see [Ye6, Theorem 0.3.5].

Definition 1.8 ([YZ1], [OSY]). Let \( u : A \to B \) be a ring homomorphism, \( (M, \rho) \in \text{D}(A)_{\text{rig}/K} \), and \( (N, \sigma) \in \text{D}(B)_{\text{rig}/K} \). A rigid backward morphism \( \tilde{\theta} : (N, \sigma) \to (M, \rho) \) over \( u \) is a backward morphism \( \tilde{\theta} : N \to M \) in \( \text{D}(A) \) over \( u \), such that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{\sigma} & \text{Sq}_{B/K}(N) \\
\theta \downarrow & & \downarrow \text{Sq}_{u/K}(\theta) \\
M & \xrightarrow{\rho} & \text{Sq}_{A/K}(M)
\end{array}
\]

in \( \text{D}(A) \) is commutative.

Next is a generalization of Theorem 1.6.
**Theorem 1.9** ([YZ1], [OSY]). In the situation of Definition 1.8 if $N \in D(B)$ has the derived Morita property, then there is at most one nondegenerate rigid backward morphism $\theta : (N, \sigma) \to (M, \rho)$ over $u$.

The second functoriality of the squaring operation is forward functoriality. A homomorphism $\varphi : A \to A'$ is called essentially étale if it is EFT and formally étale. We know that an essentially étale homomorphism is flat. Given an essentially étale homomorphism $\varphi : A \to A'$, and complexes $M \in D(A)$ and $M' \in D(A')$, a morphism $\lambda : M \to M'$ in $D(A)$ is called a forward morphism over $\varphi$. The forward morphism $\lambda$ is called nondegenerate if the morphism $A' \otimes_A M \to M'$ in $D(A')$, which corresponds to $\lambda$ by adjunction, is an isomorphism.

**Theorem 1.10** ([YZ1], [OSY]). Let $\varphi : A \to A'$ be an essentially étale ring homomorphism, and let $\lambda : M \to M'$ be a forward morphism in $D(A)$ over $\varphi$. Then there is a forward morphism

$$Sq_{\varphi/K}(\lambda) : Sq_{A/K}(M) \to Sq_{A'/K}(M')$$

in $D(A)$ over $\varphi$, with an explicit formula given suitable DG ring resolutions $\tilde{A} \to A$ and $\tilde{B} \to B$. The operation $Sq_{\varphi/K}(\lambda)$ is functorial in $\varphi$ and $\lambda$.

In [YZ1] we only considered the case of a localization homomorphism $\varphi : A \to A'$. The essentially étale case, proved in [OSY], is much more difficult, and it relies on a detailed study of the diagonal embedding of $\text{Spec}(A')$ in $\text{Spec}(A' \otimes_A A')$.

**Definition 1.11** ([YZ1], [OSY]). Let $\varphi : A \to A'$ be an essentially étale ring homomorphism, let $(M, \rho) \in D(A)_{\text{rig}/K}$, and let $(M', \rho') \in D(A'_{\text{rig}/K})$. A rigid forward morphism $\lambda : (M, \rho) \to (M', \rho')$ over $\varphi$ is a forward morphism $\lambda : M \to M'$ in $D(A)$ over $\varphi$, such that the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\rho} & Sq_{B/K}(M) \\
\downarrow{\lambda} & & \downarrow{Sq_{\varphi/K}(\lambda)} \\
M' & \xrightarrow{\rho'} & Sq_{A'/K}(M')
\end{array}$$

in $D(A)$ is commutative.

Here is a second generalization of Theorem 1.6.

**Theorem 1.12** ([OSY]). In the situation of Definition 1.11 if $M' \in D(A')$ has the derived Morita property, then there is at most one nondegenerate rigid forward morphism $\lambda : (M, \rho) \to (M', \rho')$ over $\varphi$.

The backward and forward squaring operations commute with each other. This is the content of the next theorem.

**Theorem 1.13** ([OSY]). Suppose we are given ring homomorphisms $u : A \to B$ and $v : A \to A'$, such that $v$ is essentially étale, and a backward morphism $\theta : N \to M$ in $D(A)$ over $u$. Define the ring $B' := A' \otimes_A B$, and the complexes $M' := A' \otimes_A M \in D(A')$ and $N' := B' \otimes_B N \in D(B')$. Let $\theta' : N' \to M'$ be the induced backward morphism in $D(A')$, and let $q_M : M \to M'$ and $q_N : N \to N'$ be the canonical nondegenerate forward morphisms. We get the following commutative diagrams

$$\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{v} & & \downarrow{w} \\
A' & \xrightarrow{u'} & B'
\end{array} \quad \begin{array}{ccc}
M & \xleftarrow{\theta} & N \\
\downarrow{q_M} & & \downarrow{q_N} \\
M' & \xleftarrow{\theta'} & N'
\end{array}$$
in $\text{Rng} / \text{et} \mathbb{K}$ and in $\mathcal{D}(A)$. Then the diagram

\[
\array{\text{Sq}_{A/K}(M) & \text{Sq}_{B/K}(N) \\
\text{Sq}_{u/K}(q_m) & \text{Sq}_{u/K}(q_N) \\
\text{Sq}_{A/K}(M') & \text{Sq}_{B/K}(N')}
\]

in $\mathcal{D}(A)$ is commutative.

This theorem implies that there is no conflict between Definitions 1.8 and 1.11 when $A = B = A'$ and $u = v = \text{id}_A$, in which case the morphisms $\theta$ and $\lambda$ in these definitions are both forward and backward morphisms.

2. Rigid Dualizing Complexes over Rings

Again $\mathbb{K}$ is a regular base ring, and $A$ is an EFT $\mathbb{K}$-ring. The next definition is taken from [RD], except that the derived Morita property (Definition 1.5) had not yet been introduced when [RD] was written.

**Definition 2.1.** A complex $R \in \mathcal{D}^b(A)$ is called dualizing if it has finite injective dimension and the derived Morita property.

**Definition 2.2 ([VdB], [YZ2], [Ye7], [OSY]).** A rigid dualizing complex over $A$ relative to $\mathbb{K}$ is a rigid complex $(R, \rho)$, as in Definition 1.5 such that $R$ is a dualizing complex.

**Theorem 2.3 ([VdB], [YZ2], [OSY]).** Let $A$ be an EFT $\mathbb{K}$-ring. The ring $A$ has a rigid dualizing complex $(R_A, \rho_A)$, and it is unique, up to a unique isomorphism in $\mathcal{D}(A)_{\text{rig}/\mathbb{K}}$.

The uniqueness part of this theorem is a combination of a result of Grothendieck (see [RD] Theorem V.3.1) or [Ye7] Theorem 13.1.35), with a variation of Theorem 1.6 (see [Ye7] Theorem 13.5.4). Existence is much harder. The strategy of proof in [OSY] is this: we factor the structure homomorphism $u : \mathbb{K} \to A$ into $u = u_{\text{loc}} \circ u_{\text{fin}} \circ u_{\text{pl}}$, where $u_{\text{pl}} : \mathbb{K} \to A_{\text{pl}}$ is a homomorphism to a polynomial ring over $\mathbb{K}$; $u_{\text{fin}} : A_{\text{pl}} \to A_{\text{fl}}$ is a surjection; and $u_{\text{loc}} : A_{\text{fl}} \to A$ is a localization homomorphism. Then we show that the dualizing complexes $R_{\text{fl}}, R_{\text{ft}}$ and $R$, over the rings $A_{\text{pl}}, A_{\text{fl}}$ and $A$ respectively, that are constructed in the course of the proof of [Ye7] Theorem 13.1.34, are all rigid. This is achieved using induced rigidity for essentially smooth ring homomorphisms, and coinduced rigidity for finite ring homomorphisms.

**Theorem 2.4 ([YZ2], [OSY]).** Let $u : A \to B$ be a finite ring homomorphism. There exists a unique nondegenerate rigid backward morphism

$$\text{tr}^{\text{rig}}_u : (R_B, \rho_B) \to (R_A, \rho_A)$$

in $\mathcal{D}(A)$ over $u$, called the rigid trace morphism.

**Corollary 2.5 ([YZ2], [OSY]).** Let $A \xrightarrow{u} B \xrightarrow{v} C$ be finite ring homomorphisms. Then $\text{tr}^{\text{rig}}_u \circ \text{tr}^{\text{rig}}_v = \text{tr}^{\text{rig}}_{v \circ u}$, as backward morphisms $R_C \to R_A$ in $\mathcal{D}(A)$ over $v \circ u$.

**Theorem 2.6 ([YZ2], [OSY]).** Let $v : A \to A'$ be an essentially étale ring homomorphism. There exists a unique nondegenerate rigid forward morphism

$$\text{q}^{\text{rig}}_v : (R_A, \rho_A) \to (R_{A'}, \rho_{A'})$$

in $\mathcal{D}(A)$ over $v$, called the rigid étale-localization morphism.
Corollary 2.7 ([YZ2], [OSY]). Let $A \overset{\alpha}{\to} A' \overset{\delta}{\to} A''$ be essentially étale ring homomorphisms. Then $q_{A'}^{\text{rig}} \circ q_{A''}^{\text{rig}} = q_{A'}^{\text{rig}}$, as forward morphisms $R_A \to R_{A''}$ in $D(A)$ over $\alpha \circ \delta$.

Theorem 2.8 ([YZ2], [OSY]). Suppose $u : A \to B$ and $v : A \to A'$ are ring homomorphisms, with $u$ finite and $v$ essentially étale. Define the ring $B' := A' \otimes_A B$. We get the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{v} & & \downarrow{w} \\
A' & \xrightarrow{u'} & B'
\end{array}
\]

in $\text{Rng}_{/\text{et}} \mathbb{K}$, in which $u'$ is finite and $w$ is essentially étale. Then the diagram

\[
\begin{array}{ccc}
R_A & \xleftarrow{\text{tr}^{\text{rig}}_{u'}} & R_B \\
\downarrow{q_v^{\text{rig}}} & & \downarrow{q_w^{\text{rig}}} \\
R_{A'} & \xleftarrow{\text{tr}^{\text{rig}}_{u'}} & R_{B'}
\end{array}
\]

in $D(A)$ is commutative.

For a ring $A$ with rigid dualizing complex $(R_A, \rho_A)$, let us define the rigid auto-duality functor

\[
(2.9) \quad D_A^{\text{rig}} := \text{RHom}_A(-, R_A) : D(A) \to D(A).
\]

According to [RD], or [Ye7, Chapter 13], the functor $D_A^{\text{rig}}$ induces an equivalence $D_A^{\text{rig}} : D_A^\star(A)^{\text{op}} \to D_A^{-\star}(A)$, where $\star$ and $-\star$ are reverse boundedness conditions. In particular there is the evaluation isomorphism

\[
(2.10) \quad \text{ev} : \text{Id} \xrightarrow{\sim} D_A^{\text{rig}} \circ D_A^{\text{rig}}
\]

of triangulated functors from $D_I^\star(A)$ to itself.

The next discussion involves 2-categories and pseudofunctors. The only textbook reference for 2-categories seems to be in [MC, Chapter XII]. There is a brief discussion of 2-categories and pseudofunctors in [SP, Section \text{tag=003G}] and a summary of 2-categorical notation in Section 8.1 of our book [Ye7]. A detailed exposition of 2-categories and pseudofunctors can be found in Sections 1-2 of our paper [Ye4]. We should stress that our 2-categories are strict, and our pseudofunctors are normalized.

Recall that to a ring homomorphism $u : A \to B$ we can associate the derived induction functor

\[
(2.11) \quad \text{LInd}_u := B \otimes_A^L (-) : D_I^-(A) \to D_I^-(B).
\]

It is the left derived functor of the induction functor $\text{Ind}_u := B \otimes_A (-) : \text{M}(A) \to \text{M}(B)$.

Consider the 2-category of $\mathbb{K}$-linear triangulated categories, which we denote by $\text{TrCat/} \mathbb{K}$. Derived induction is actually a pseudofunctor

\[
(2.12) \quad \text{LInd} : \text{Rng}_{/\text{et}} \mathbb{K} \to \text{TrCat/} \mathbb{K},
\]

sending a ring $A$ to the category $D_I^-(A)$, and a ring homomorphism $u : A \to B$ to the functor $\text{LInd}_u$. The composition isomorphisms of $\text{LInd}$ arise from those of the plain induction pseudofunctor $\text{Ind}$; and these are just the associativity isomorphisms for tensor products.

The next result is our ring-theoretic version of the geometric twisted inverse pseudofunctor $f \mapsto f^I$ from [RD]. See Theorem 4.9 for our geometric result.
Theorem 2.13 ([YZ2], [OSY]). There is a unique pseudofunctor
\[ \text{TwInd} : \text{Rng} / e K \to \text{TrCat} / K \]
called twisted induction, with these properties:

1. To an object \( A \in \text{Rng} / e K \) it assigns the triangulated category \( D^+_f (A) \).
2. To a morphism \( u : A \to B \) in \( \text{Rng} / e K \) it assigns the triangulated functor
   \[ \text{TwInd}_u := D^+_B \circ \text{LInd}_u \circ D^+_A : D^+_f (A) \to D^+_f (B). \]
3. The composition isomorphisms of \( \text{TwInd} \) come from the evaluation isomorphisms of the rigid auto-duality functors (2.10), combined with the composition isomorphisms of the pseudofunctor \( \text{LInd} \).

In the next section we shall see how \( \text{TwInd} \) interacts with rigid traces and rigid étale-localizations.

Remark 2.14. Observe that the pseudofunctor \( \text{TwInd} \) is constructed purely by algebraic methods, namely without geometry, and especially without global duality, which is a key ingredient in all earlier approaches to GD. The only other such purely algebraic construction we are aware of is the very recent one by P. Scholze and D. Clausen [Sc]. They use condensed mathematics, and so far it works only for FT \( \mathbb{Z} \)-rings.

Remark 2.15. Here are a few historical details on the squaring operation and rigid dualizing complexes.

Noncommutative dualizing complexes were introduced in our 1992 paper [Ye3]. These are complexes of bimodules over a NC ring \( A \), central over a base field \( K \), satisfying conditions similar to those of commutative dualizing complexes.

Noncommutative rigid dualizing complexes were invented by M. Van den Bergh in 1997, in his seminal paper [VdB]. Observe the similarity between the squaring operation (Theorem 1.1) and Hochschild cohomology. This is not a coincidence. An idea similar to that of a NC rigid dualizing complex was independently discovered by M. Kontsevich in the late 1990’s, and it is one of the cornerstones of the theory of Calabi-Yau categories that emerged from the work of Kontsevich and others. These NC theories are explained in [Ye7, Chapter 18].

J.J. Zhang and the author imported the concept of rigid dualizing complexes back to commutative algebra and algebraic geometry. We added two features to the original concept of NC rigidity of Van den Bergh: the functoriality of rigid complexes, and the passage from base field to base ring, which is sometimes referred to as the arithmetic setting. This was done in our papers [YZ1] and [YZ2] from around 2008. As already mentioned in Section 1, the generalization from base field to base ring required the use of DG ring resolutions.

Unfortunately there were several serious errors regarding the manipulation of DG rings in the paper [YZ1], and these errors affected some constructions and proofs in [YZ1] and [YZ2]. Interestingly, our main results in these two paper were correct, as we now know.

The errors in [YZ1] and [YZ2] were discovered by the authors of [AILN] in 2010, and they also fixed one of these errors. (However, contrary to the norms of ethical scientific conduct, the authors of [AILN] failed to inform us of the errors in our work before publishing their own paper.)

This accident – the errors regarding DG rings – took a lot of time and effort to correct. Some of the errors were fixed in our paper [Ye6] from 2016, and the rest are corrected.
in the upcoming paper [OSY], which is joint work with M. Ornaghi and S. Singh. This is one of the main reasons the project discussed in this article was delayed for so many years.

3. Rigid Residue Complexes over Rings

Now we enter the zone of “anticipated results”: many of the results still only have sketchy proofs, and should therefore be regarded with caution. Such a theorem will be labeled Theorem (*).

Recall that we are still working in $\text{Rng}/\mathfrak{e}$, where $K$ is a regular noetherian base ring. If $L \in \text{Rng}/\mathfrak{e}$, $K$ is a field, then its rigid dualizing complex $R_L$ must be isomorphic to $L[\bar{d}]$ for some integer $\bar{d}$. We define the rigid dimension of $L$ to be $\text{rig.dim}_K(L) := \bar{d}$.

Example 3.1. If the base ring $K$ is a field, then $\text{rig.dim}_K(L) = \text{tr.deg}_K(L)$, the transcendence degree. On the other hand, for $K = \mathbb{Z}$ and $L = \mathbb{F}_p$ we have $\text{rig.dim}_\mathbb{Z}(\mathbb{F}_p) = -1$.

Now take an arbitrary $A \in \text{Rng}/\mathfrak{e}$. For a prime ideal $\mathfrak{p} \subseteq A$, with residue field $k(\mathfrak{p})$, we define $\text{rig.dim}_A(\mathfrak{p}) := \text{rig.dim}_K(k(\mathfrak{p}))$. The resulting function $\text{rig.dim}_A : \text{Spec}(A) \rightarrow \mathbb{Z}$ has the expected property: it drops by 1 if $\mathfrak{p} \subseteq \mathfrak{q}$ is an immediate specialization of primes. The function $\text{rig.dim}_A$ is bounded, since $\text{Spec}(A)$ is finite dimensional. See [Ye7] Theorem 13.3.3.

For any $\mathfrak{p} \in \text{Spec}(A)$ we denote by $J(\mathfrak{p})$ the injective hull of the $A_{\mathfrak{p}}$-module $k(\mathfrak{p})$. This is an indecomposable injective $A$-module. According to the Matlis classification, every injective $A$-module is a direct sum, often infinite, of such modules $J(\mathfrak{p})$. See [Ye7] Section 13.2] for a detailed discussion, and Remark 3.13 on the possible functorial properties of $J(\mathfrak{p})$.

The following definition is a refinement of the concept of residual complex from [RD].

Definition 3.2. A rigid residue complex over $A$ relative to $K$ is a rigid dualizing complex $(\mathcal{K}_A, \rho_A)$ of this special sort: for every $d \in \mathbb{Z}$ there is an isomorphism of $A$-modules

$$\mathcal{K}_A^{-d} \cong \bigoplus_{\substack{\mathfrak{p} \in \text{Spec}(A) \\ \text{rig.dim}_A(\mathfrak{p}) = d}} J(\mathfrak{p}).$$

Recall the category of rigid complexes $D(A)_{\text{rig/K}}$ from Definition 1.4.

Definition 3.3. A morphism of rigid residue complexes $\phi : (\mathcal{K}_A, \rho_A) \rightarrow (\mathcal{K}'_A, \rho'_A)$ is a homomorphism $\phi : \mathcal{K}_A \rightarrow \mathcal{K}'_A$ in $\text{Cstr}(A)$, such that $Q(\phi) : (\mathcal{K}_A, \rho_A) \rightarrow (\mathcal{K}'_A, \rho'_A)$ is a morphism in $D(A)_{\text{rig/K}}$. We denote by $\text{C}(A)_{\text{res/K}}$ the category of rigid residue complexes.

Note the unusual mixed nature of this definition: a morphism $\phi$ in $\text{C}(A)_{\text{res/K}}$ is literally isomorphism of complexes, but the condition it must satisfy is in the derived category.

Theorem 3.4 ([Ye7], [OSY]). Let $A$ be an EFT $K$-ring. The ring $A$ has a rigid residue complex $(\mathcal{K}_A, \rho_A)$ and it is unique, up to a unique isomorphism in $\text{C}(A)_{\text{res/K}}$.

This theorem is not actually stated in either of these references, yet it is easily proved by combining several results from them (Theorem 2.3 above, proved in [OSY], and [Ye7] Theorems 13.3.15 and 13.3.17)).

Let us now mention several important functorial properties of rigid residue complexes.

Theorem (*). 3.5 ([Ye8]). Suppose $\psi : A \rightarrow A'$ is an essentially étale homomorphism of EFT $K$-rings. There is a unique nondegenerate forward homomorphism $\psi^\text{res} : \mathcal{K}_A \rightarrow \mathcal{K}_{A'}$ in
for every integer $i$, homomorphism. Let

Theorem 3.6 (Ye8). The assignment $A' \mapsto K_{A'}$ extends uniquely to a quasi-coherent sheaf on the small étale site of $\text{Spec}(A)$.

This corollary will be crucial when we get to DM stacks.

Theorem 3.7 (Ye8). Suppose $u : A \rightarrow B$ is a finite homomorphism of EFT $K$-rings. There is a unique nondegenerate backward homomorphism $\text{tr}^\text{res}_u : K_B \rightarrow K_A$ in $\text{C}_{\text{str}}(A)$, called the rigid trace homomorphism, such that $Q(\text{tr}^\text{res}_u) : (K_B, \rho_B) \rightarrow (K_A, \rho_A)$ is the rigid trace morphism $\text{tr}^\text{rig}_u$ from Theorem 2.7.

The uniqueness implies that the rigid trace is functorial on finite homomorphisms.

Theorem 3.8 (Ye8). Suppose we are given a finite ring homomorphism $u : A \rightarrow B$ and an essentially étale ring homomorphism $v : A \rightarrow A'$. Define the ring $B' := A' \otimes_A B$, and the homomorphisms $u' : A' \rightarrow B'$ and $w : B \rightarrow B'$ as in Theorem 2.8 which are finite and essentially étale, respectively. Then the diagram

in $\text{C}_{\text{str}}(A)$ is commutative.

Theorem 3.9 (Ye8). For every homomorphism $u : A \rightarrow B$ in $\text{Rng}_{/A} K$ there is a unique backward homomorphism of graded $A$-modules $\text{tr}^\text{res}_u : K_B \rightarrow K_A$, called the ind-rigid trace homomorphism, satisfying these conditions:

- If $u$ is a finite homomorphism, then $\text{tr}^\text{res}_u = \text{tr}^\text{res}_u$, the rigid trace from Theorem 3.7.
- The ind-rigid trace $\text{tr}^\text{res}_u$ commutes with the rigid étale-localizations $q^\text{res}_w$ from Theorem 3.5.
- Functoriality: $\text{tr}^\text{res}_{u' \circ w} = \text{tr}^\text{res}_u \circ \text{tr}^\text{res}_w$.

Remark 3.10. If $u : A \rightarrow B$ is not finite, then usually the ind-rigid trace $\text{tr}^\text{res}_u$ is not a morphism in $\text{C}_{\text{str}}(A)$, i.e. it does not commute with the differentials. This same behavior already occurred for the trace in [RD, Section VI.4]. See the Rigid Residue Theorem 4.10 for the case of a proper map of schemes.

Observe that the rigid auto-duality functor $D^\text{rig}_A$ from 2.9 now becomes

$$D^\text{rig}_A = \text{Hom}_A(-, K_A) : D_1(A)^{\text{op}} \rightarrow D_1(A).$$

This means that the twisted induction pseudofunctor $\text{TwInd}$ gets a lot more structure from the results of this section.

A ring homomorphism $v : A \rightarrow A'$ is called faithfully étale if it is étale and faithfully-flat. In other words, if the map of affine schemes $\text{Spec}(A') \rightarrow \text{Spec}(A)$ is étale and surjective.

Theorem 3.11 (Étale Codescent, Ye8). Suppose $v : A \rightarrow A'$ is a faithfully étale ring homomorphism. Let $A'' := A' \otimes_A A'$, and let $w_1, w_2 : A' \rightarrow A''$ be the two inclusions. Then for every integer $i$ the sequence of $A$-module homomorphisms

$$K_A^i \rightarrow K_{A'}^i \rightarrow K_{A''}^i \rightarrow 0$$
is exact.

We have not encountered any similar result in the literature. For us this theorem will be required for producing ind-rigid traces for maps of DM stacks.

**Remark 3.12.** A few words on notation. In Section 2 we used the notation $tr^{rig}_u$ and $q^{rig}_u$ for the rigid trace and rigid étale-localization morphisms, respectively, which are morphisms between dualizing complexes in $D(A)$. We use the modified notation $tr^{res}_u$ and $q^{res}_u$ for the corresponding homomorphisms between rigid residue complexes in $C_{str}(A)$. They are related as follows: $tr^{rig}_u = Q(tr^{res}_u)$ and $q^{rig}_u = Q(q^{res}_u)$, where $Q : C_{str}(A) \to D(A)$ is the categorical localization functor.

**Remark 3.13.** It is important to mention that the indecomposable injective $A$-module $J(p)$ is unique only up to isomorphism, and without added structure such an isomorphism cannot be made unique or "canonical". To see this, note that the automorphism group of $J(p)$ is the group of invertible elements $\tilde{A}^\times_p$ of the complete local ring $\tilde{A}_p$, and the subgroup of $\tilde{A}^\times_p$ consisting of the elements congruent to 1 modulo $p$ (usually a huge subgroup) acts trivially on the submodule $k(p) \subseteq J(p)$. This implies that there is nothing canonical about the embedding $k(p) \hookrightarrow J(p)$.

There are ways to enhance $J(p)$ with some extra structure. In our context, the rigid residue complex $(K_{A_p}, \rho_A)$ of $A_p$ (relative to $K$) has the property that the $A_p$-module $K^{d-1}_A$, where $d := \text{rig.dim}_K(p)$, is isomorphic to $J(p)$, by Definition 3.2. A more global way to "rigidify" $J(p)$ is to consider the $A$-module $\Gamma_p(K^{d-1}_A)$, the $p$-torsion submodule of $K^{d-1}_A$. The forward functoriality $q^{res}_p : K_A \to K_{A_p}$, which is associated to the localization ring homomorphism $\varpi : A \to A_p$, induces an isomorphism $\Gamma_p(K^{d-1}_A) \cong K^{d-1}_A$. According to Theorems 3.5, 3.7 and 3.8, the modules $K^{d-1}_A$ and $\Gamma_p(K^{d-1}_A)$ are functorial, in two directions, in the rings $A_p$ and $A$ respectively. Thus these are "canonical incarnations" of $J(p)$.

Other ways to obtain "canonical incarnations" of $J(p)$ are using *local duality* – this is implicit throughout [RD]; or using coefficient fields and higher completions (see e.g. [Ye1] and [Ye2]).

4. Rigidity, Residues and Duality for Schemes

Recall that $K$ is a regular finite dimensional noetherian base ring. We use the notation $\text{Sch}_K$ for the category of finite type (FT) $K$-schemes.

Consider a FT $K$-scheme $X$. If $U \subseteq X$ is an affine open set, then $A := \Gamma(U, O_X)$ is a FT $K$-ring. Given a quasi-coherent $O_X$-module, for every affine open set $U$ we get an $A$-module $M := \Gamma(U, M)$. The resulting functor $\Gamma(U, -) : \text{QCoh}(X) \to \text{Mod}(A)$ is exact.

**Remark 4.1.** The theory discussed in this section extends easily to *essentially finite type* $K$-schemes. But the geometry of EFT $K$-schemes is a bit messy (or perhaps the foundations have not been studied well enough yet), and therefore we will not talk about them here. The paper [Ye2] does deal with EFT $K$-schemes, by making special adjustments to overcome a few annoying technicalities.

If $V \subseteq U$ is an inclusion of affine open sets in $X$, then geometric restriction $\text{rest}_{V/U} : \Gamma(U, O_X) \to \Gamma(V, O_X)$ is an *étale ring homomorphism*. For $M \in \text{QCoh}(X)$ we get a nondegenerate forward homomorphism $\text{rest}_{V/U} : \Gamma(U, M) \to \Gamma(V, M)$ of $\Gamma(U, O_X)$-modules.

According to the Grothendieck-Matlis classification of injective $O_X$-modules from [RD] Section II.7], every quasi-coherent $O_X$-module $M$ admits a monomorphism $M \hookrightarrow I$,
where \( \mathcal{I} \) is an \( \mathcal{O}_X \)-module that is both quasi-coherent and injective in the category \( \text{Mod}(X) \) of all \( \mathcal{O}_X \)-modules. In particular, the object \( \mathcal{I} \) is injective in the full subcategory \( \text{QCoh}(X) \) of \( \text{Mod}(X) \). We call such an object \( \mathcal{I} \) a quasi-coherent injective \( \mathcal{O}_X \)-module.

Every quasi-coherent injective \( \mathcal{O}_X \)-module \( \mathcal{I} \) is a direct sum (often infinite) of indecomposable quasi-coherent injective \( \mathcal{O}_X \)-modules. These indecomposable quasi-coherent injective \( \mathcal{O}_X \)-modules are classified by points \( x \in X \). For every point \( x \) the corresponding indecomposable injective \( \mathcal{J}(x) \) is a skyscraper quasi-coherent \( \mathcal{O}_X \)-module with support the closed set \( \{ x \} \), and whose stalk at \( x \) is the injective hull of the residue field \( k(x) \) over the local ring \( \mathcal{O}_{X,x} \). To match this with the ring theoretic Matlis classification from Section 3, when \( X = \text{Spec}(A) \) is affine, and \( x = \mathfrak{p} \), then \( \mathcal{J}(x) \cong J(\mathfrak{p}) \) as modules over \( \mathcal{O}_{X,x} \cong \mathfrak{p} \).

**Definition 4.2** (Ye9). Let \( X \) be a FT \( \mathbb{K} \)-scheme. A rigid residue complex on \( X \) (relative to \( \mathbb{K} \)) is a pair \( (\mathcal{K}_X, \rho_X) \) consisting of these data:

(a) A bounded complex \( \mathcal{K}_X \) of quasi-coherent injective \( \mathcal{O}_X \)-modules,

(b) A collection \( \rho_X := \{ \rho_U \} \) indexed by the affine open sets \( U \subseteq X \), where each \( \rho_U \) is a rigidifying isomorphism for the complex of \( \Gamma(U, \mathcal{O}_X) \)-modules \( \Gamma(U, \mathcal{K}_X) \), in the sense of Definition 4.3.

There are two conditions:

(i) For every \( U \), the pair \( (\Gamma(U, \mathcal{K}_X), \rho_U) \) is a rigid residue complex over the ring \( \Gamma(U, \mathcal{O}_X) \), in the sense of Definition 3.2.

(ii) For every inclusion \( V \subseteq U \) of affine open sets, the nondegenerate forward homomorphism \( res_{V/U} : \Gamma(U, \mathcal{K}_X) \to \Gamma(V, \mathcal{K}_X) \) is the unique rigid étale-localization homomorphism between these rigid residue complexes (see Theorem 4.5).

The collection \( \rho_X \) is called a rigid structure on the complex \( \mathcal{K}_X \).

**Definition 4.3** (Ye9). Suppose \( (\mathcal{K}_X, \rho_X) \) and \( (\mathcal{K}_X', \rho_X') \) are two rigid residue complexes on the FT \( \mathbb{K} \)-scheme \( X \). A morphism of rigid residue complexes \( \phi : (\mathcal{K}_X, \rho_X) \to (\mathcal{K}_X', \rho_X') \) is a homomorphism \( \phi : \mathcal{K}_X \to \mathcal{K}_X' \) of complexes of \( \mathcal{O}_X \)-modules, such that for every affine open set \( U \), with \( A := \Gamma(U, \mathcal{O}_X) \), the homomorphism \( \Gamma(U, \phi) \) is a morphism in \( \mathcal{C}(A)_{\text{res/}} \).

The next four theorems are easy consequences of the corresponding theorems for rings in Section 3, combined with descent for QC sheaves on schemes.

**Theorem 4.4** (Ye9). Every FT \( \mathbb{K} \)-scheme \( X \) has a rigid residue complex \( (\mathcal{K}_X, \rho_X) \), and it is unique up to a unique isomorphism of rigid residue complexes.

**Theorem 4.5** (Ye9). For every map \( f : Y \to X \) between FT \( \mathbb{K} \)-schemes there is a unique homomorphism of graded quasi-coherent \( \mathcal{O}_X \)-modules \( \text{tr}^{\text{res}}_f : f_* (\mathcal{K}_Y) \to \mathcal{K}_X \), called the ind-rigid trace homomorphism, which extends the ind-rigid trace homomorphism \( \text{tr}^{\text{res}}_\phi \) on \( \mathbb{K} \)-rings from Theorem 3.9. The homomorphism \( \text{tr}^{\text{res}}_f \) is functorial in \( f \).

If \( f \) is a finite map of schemes, then \( \text{tr}^{\text{res}}_f \) is a homomorphism of complexes – this is immediate from the result for rings. The case of a proper map will be mentioned soon.

**Theorem 4.6** (Ye9). For every étale map \( g : X' \to X \) between FT \( \mathbb{K} \)-schemes there is a unique homomorphism of complexes of quasi-coherent \( \mathcal{O}_X \)-modules \( q^{\text{res}}_g : \mathcal{K}_X \to g_* (\mathcal{K}_{X'}) \), called the rigid étale-localization homomorphism, which extends the rigid étale-localizations homomorphism \( q^{\text{res}}_\phi \) on \( \mathbb{K} \)-rings from Theorem 3.5. The homomorphism \( q^{\text{res}}_g \) is functorial in such maps \( g \).
Theorem 4.7 ([Ye9]). The ind-rigid trace homomorphism $\text{tr}^{\text{res}}$ and the rigid étale-localization homomorphism $q^{\text{res}}_{(-)}$ commute with each other, as in Theorem 3.8 but geometrized.

The next result is really just a geometric version of Theorem 2.13 without any significant difficulties. For a scheme $X$ with rigid residue complex $(K_A, \rho_A)$, we define
\begin{equation}
D_X^{\text{rig}} := \mathcal{H}om_X(-, K_X) : D(X) \to D(X).
\end{equation}

Here $D(X)$ is the derived category of $\mathcal{O}_X$-modules. As in the affine case, there are equivalences $D_X^{\text{rig}} : D_c^*(X) \to D_c^{*,*}(X)$ for reversed boundedness conditions $\star$ and $-\star$.

Theorem 4.9 ([YZ2], [Ye9]). There is a unique pseudofunctor
\begin{equation}
\text{TwInvIm} : (\text{Sch}/\mathbb{K})^{\text{op}} \to \text{TrCat}/\mathbb{K}
\end{equation}
called twisted inverse image, with these properties:

1. To an object $X \in \text{Sch}/\mathbb{K}$ it assigns the triangulated category $D_c^+(X)$.
2. To a morphism $f : Y \to X$ in $\text{Sch}/\mathbb{K}$ it assigns the triangulated functor
\begin{equation}
\text{TwInvIm}_f = f^! := D_Y^{\text{rig}} \circ Lf^* \circ D_X^{\text{rig}} : D_c^+(X) \to D_c^+(Y).
\end{equation}
3. The composition isomorphisms of $\text{TwInvIm}$ come from the evaluation isomorphisms of the rigid auto-duality functors (4.8), combined with the composition isomorphisms of the pseudofunctor $f \mapsto Lf^*$.

The functorial properties of the rigid residue complexes (stated in the theorems above and below) provide the twisted inverse image pseudofunctor with more structure.

Here is the rigid version of [RD] Theorem VII.2.1.

Theorem 4.10. (Rigid Residue Theorem, [Ye9]) Let $f : Y \to X$ be a proper map between $FT \mathbb{K}$-schemes. Then the ind-rigid trace $\text{tr}^{\text{res}}_f : f_*(K_Y) \to K_X$ is a homomorphism of complexes.

The idea of the proof (imitating [RD]) is to reduce to the case when $X = \text{Spec}(A)$ for a local artinian ring $A \in \text{Rng}/\mathbb{K}$, and $Y = \mathbb{P}_A^1$, the projective line. We do it using Theorems 3.5, 4.6 and 4.7. Then we do a calculation of residues, which similar to the one in the proof of the classical Residue Theorem for $\mathbb{P}_A^1(\mathbb{C})_{\text{an}}$. The only minor complication is that a rational differential form on $Y$ relative to $A$, i.e. a form $\omega \in \Omega^1_{A(t)/A}$, where $A(t)$ is the total ring of fractions of $Y$, might have poles at closed points of $Y$ that do not belong to $Y(A)$. But the poles of $\omega$ always belong to $Y(A')$ for a suitable finite faithfully flat $A$-ring $A'$, and by performing the base change $A \to A'$ the calculation can proceed.

The next theorem is the rigid version of [RD] Theorem VII.3.3.

Theorem 4.11. (Rigid Duality Theorem, [Ye9]) Let $f : Y \to X$ be a proper map between $FT \mathbb{K}$-schemes. For every $N \in D_{\text{qc}}(Y)$ the morphism
\begin{equation}
Rf_!(\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_Y(N, K_Y)) \to \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_X(Rf_!(N), K_X)
\end{equation}
in $D(X)$, which is induced by the ind-rigid trace $\text{tr}^{\text{res}}_f : f_*(K_Y) \to K_X$, is an isomorphism.

The proof of imitates the proof of the corresponding theorem in [RD], once we have the Theorem 4.10 at hand.

Theorem 4.10 implies that for every complex $\mathcal{M} \in D_c^+(X)$ there is an ind-rigid trace morphism
\begin{equation}
\text{tr}^{\text{rig}}_{f, \mathcal{M}} : Rf_!(f^!(\mathcal{M})) \to \mathcal{M}
\end{equation}
in $D(X)$, which is functorial in $\mathcal{M}$. Here is our version of [RD] Corollary 3.4.
Corollary 4.13.  (Ye9) Let \( f : Y \to X \) be a proper map between FT \( K \)-schemes. For every \( \mathcal{M} \in D^+_c(X) \) and \( \mathcal{N} \in D_{qc}^-(Y) \) the morphism

\[
Rf_!(R\text{Hom}_Y(N, f^!(\mathcal{M}))) \to R\text{Hom}_X(Rf_!(\mathcal{N}), \mathcal{M})
\]

in \( D(X) \), which is induced by the ind-rigid trace \( \text{tr}_{f,!*}^{\text{rig}} \), is an isomorphism.

Remark 4.14. At this stage we have recovered almost all the results of the original book [RD]. One advantage of our rigidity approach is that it is much cleaner and shorter than the original approach in [RD]. We can also treat EFT maps of schemes. Another advantage of the rigidity approach, as we shall see next, is that it gives rise to a useful duality theory for DM stacks. On the down side, we must work with a fixed regular base ring \( K \).

For complements to the book [RD], and for alternative approaches, due to J. Lipman, B. Conrad, A. Neeman, the author of this article, and others, please consult [Co], [LH], [Ne1], [Ne2], [Ye7] and their references.

5. Rigidity, Residues and Duality for DM Stacks

Here is a brief recollection of facts about Deligne-Mumford (DM) stacks and quasi-coherent sheaves on them, extracted from [SP] and [Ol], with some additional elaboration. This will help us state our results.

As a first approximation, it useful to think of a DM stack \( \mathcal{X} \) as a scheme equipped with an extra structure: the points of \( \mathcal{X} \) are bunched together as objects of a groupoid, in an intricate geometric way. (This can be understood, for instance, as the way \( \mathcal{X} \) sits above its coarse moduli space \( X \); we will return to this idea later.)

To give a more precise description of DM stacks, we need to place ourselves in the realm of sheaves and stacks on \( (\text{Sch}/K)_{et} \), the big étale site of FT \( K \)-schemes.

For us a prestack of groupoids on \( \text{Sch}/K \) is a pseudofunctor \( \mathfrak{X} : (\text{Sch}/K)^{op} \to \text{Grpd} \), where \( \text{Grpd} \) is the 2-category of groupoids. The prestack \( \mathfrak{X} \) is a stack if it satisfies descent for morphisms and descent for objects with respect to coverings in \( (\text{Sch}/K)_{et} \). See [Ye4] for details on stacks, as viewed from the pseudofunctor approach.

Most algebraic geometry texts (including [Ol] and [SP]) prefer viewing a prestack of groupoids on \( \text{Sch}/K \) as a category fibered in groupoids. The passage from this approach to ours is this: a prestack of groupoids \( \mathfrak{X} \) in our sense is the same as a category fibered in groupoids equipped with a cleavage. See [Ye4] Remark 2.5.

A benefit of our approach to stacks of groupoids is that it lets us refer to a sheaf of sets as a very trivial kind of stack. Indeed, a sheaf of sets \( X \) on \( (\text{Sch}/K)_{et} \) can be seen as a stack of groupoids, in which the only local isomorphisms between local objects are the identity automorphisms.

We know how a scheme \( X \) becomes a sheaf of sets on \( (\text{Sch}/K)_{et} \); it is the Yoneda incarnation of \( X \) as the sheaf \( \text{Hom}(\_ , X) \). An algebraic space \( X \) is a more complicated sheaf of sets on \( (\text{Sch}/K)_{et} \); it is not representable, but only "locally representable". One way to express this condition is that the sheaf \( X \) is isomorphic to the quotient sheaf \( U_0/U_1 \) of an étale equivalence relation in schemes \( (U_1 \rightrightarrows U_0) \). Let’s refer to the isomorphism of sheaves \( p : U_0/U_1 \rightrightarrows X \) as a presentation of \( X \). See [Ol] Proposition 5.2.5.

There is an analogous way to say when an abstract stack \( \mathfrak{X} \) on \( (\text{Sch}/K)_{et} \) is a DM stack. The condition is that \( \mathfrak{X} \) is equivalent, in the 2-category of stacks of groupoids on \( (\text{Sch}/K)_{et} \), to the quotient stack \( [X_0/X_1] \) of an étale groupoid in algebraic spaces \( (X_1 \rightrightarrows X_0) \). Note that the groupoid \( (X_1 \rightrightarrows X_0) \) involves a third algebraic space \( X_2 \), with a map \( X_2 \to X_1 \times_{X_0} X_1 \) that encodes composition of morphisms. Again we call the equivalence \( p : [X_0/X_1] \rightrightarrows \mathfrak{X} \) a presentation of \( \mathfrak{X} \). See [SP] Theorem [tag=04TK] and Lemma [tag=05TK].
We shall use the term map of stacks for a 1-morphism \( f : \mathcal{X} \to \mathcal{Y} \) between stacks of groupoids on \((\text{Sch}/\mathcal{K})_{\text{et}}\).

Thus there are four stages, or levels of complexity, in the geometry of FT DM \( \mathcal{K} \)-stacks:

\[
\text{(5.1)} \quad \text{AfSch}/\mathcal{K} \subseteq \text{Sch}/\mathcal{K} \subseteq \text{AlgSp}/\mathcal{K} \subseteq \text{DMStk}/\mathcal{K}.
\]

These are the affine schemes, schemes, algebraic spaces and DM stacks, all of FT over \( \mathcal{K} \).

If we ignore the fact that DMStk/\( \mathcal{K} \) is a 2-category (i.e. if we forget the 2-morphisms between 1-morphisms of DM stacks), formula (5.1) describes full embeddings of categories. Morphisms in these categories will be called maps. In what follows all our stacks (and algebraic spaces and schemes) are FT over the base ring \( \mathcal{K} \); and the rings are EFT over \( \mathcal{K} \).

A map \( f : \mathcal{Y} \to \mathcal{X} \) of DM stacks is surjective if it is locally essentially surjective on objects. (This is a property of a map between abstract stacks of groupoids, not just of algebraic stacks.)

A map \( f : \mathcal{Y} \to \mathcal{X} \) between DM stacks is called étale if there exist presentations \( p : [X_0/X_1] \xrightarrow{\sim} \mathcal{X} \) and \( q : [Y_0/Y_1] \xrightarrow{\sim} \mathcal{Y} \), and an étale map of algebraic spaces \( g : Y_0 \to X_0 \), such that the diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{q} & \mathcal{Y} \\
\downarrow{g} & & \downarrow{f} \\
X_0 & \xrightarrow{p} & \mathcal{X}
\end{array}
\]

of stacks is commutative up to isomorphism. For this condition we need to know what is an étale map \( g : Y_0 \to X_0 \) between algebraic spaces, and this is done by going down the staircase of embeddings (5.1). (This same yoga is how other local properties of a map \( f : \mathcal{Y} \to \mathcal{X} \) to be étale is in [Ol, Definition 8.2.6].)

Presentations of algebraic spaces and DM stacks allow us to describe quasi-coherent sheaves on these geometric objects. Thus, a presentation \( p : U_0/U_1 \xrightarrow{\sim} X \) of an algebraic space \( X \) induces an equivalence of abelian categories

\[
\text{(5.3)} \quad p^* : \text{QCoh}(X) \to \text{QCoh}(U_1 \supseteq U_0),
\]

where the latter refers to the category of equivariant quasi-coherent sheaves on the equivalence relation \((U_1 \supseteq U_0)\). Analogously, a presentation \( p : [X_0/X_1] \xrightarrow{\sim} \mathcal{X} \) of a DM stack \( \mathcal{X} \) induces an equivalence of abelian categories

\[
\text{(5.4)} \quad p^* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X_1 \supseteq X_0),
\]

where now \((X_1 \supseteq X_0)\) is an étale groupoid in algebraic spaces. See [Ol] Sections 7.1 and 9.2 and [SP] Proposition [tag=03M3] and Lemma [tag=05AV].

The discussion above suggests a mechanism for producing a quasi-coherent sheaf \( \mathcal{M}_X \) on a FT DM \( \mathcal{K} \)-stack \( \mathcal{X} \). Consider a presentation \([X_0/X_1] \xrightarrow{\sim} \mathcal{X}\), the finitely many schemes \( U \) involved in presenting the algebraic spaces \( X_i \), the finitely many affine schemes \( \text{Spec}(A) \) involved in suitable hypercoverings of these schemes, and the finitely many étale maps between these geometric objects, going down the staircase in the hierarchy (5.1).

The first step in this mechanism is to provide an \( A \)-module \( M_A \) for every affine scheme \( \text{Spec}(A) \) involved, and a consistent collection of nondegenerate forward homomorphisms \( M_A \to M_{A'} \) for all the étale ring homomorphisms \( A \to A' \) involved. The second step is to glue these modules into QC sheaves \( \mathcal{M}_U \) on all the schemes \( U \) involved. There will be an induced consistent collection of isomorphisms \( g^*(\mathcal{M}_U) \xrightarrow{\sim} \mathcal{M}_{U'} \) for the étale maps.
of groupoids in algebraic spaces (see [SP, Definition tag=043W].)

To specify a homomorphism \( \phi : f_!(\mathcal{N}) \to \mathcal{M} \) in \( \text{QCoh}(\mathcal{X}) \) we can use presentations. After having arranged for presentations \( p : [X_0/X_1] \to \mathcal{X} \) and \( q : [Y_0/Y_1] \to \mathcal{Y} \), and a map of algebraic spaces \( \tilde{f}_0 : Y_0 \to X_0 \), such that the diagram (5.2),

Finally, formula (5.4) will give us the desired quasi-coherent sheaf \( \mathcal{M} \).

Let \( \tilde{f} = (\tilde{f}_0, \tilde{f}_1) : (Y_1 \Rightarrow Y_0) \to (X_1 \Rightarrow X_0) \)

of groupoids in algebraic spaces (see [SP, Definition tag=043X and Section tag=047J]).

By pullback along \( p \) and \( q \) we get quasi-coherent sheaves \( \mathcal{M}_i \in \text{QCoh}(X_i) \) and \( \mathcal{N}_i \in \text{QCoh}(Y_i) \) satisfying the appropriate relations. Then we need to arrange for homomorphisms \( \tilde{f}_i : \tilde{f}_i^!(\mathcal{N}_i) \to \mathcal{M}_i \) that respect the relations; and these will determine \( \phi \) by the equivalence (5.4).

As before, this construction requires going down the staircase of the hierarchy (5.1).

After recalling and elucidating the mechanism of gluing quasi-coherent sheaves on DM stacks, we can finally talk about rigid residue complexes. As we shall see, the definition of a rigid residue complex on a DM stack \( \mathcal{X} \) is very similar to the definition for a scheme, with only a minor adjustment. (We will skip the algebraic space case, since it is identical to the DM stack case.)

**Definition 5.6** ([Ye10]). Let \( \mathcal{X} \) be a FT DM \( K \)-stack. A rigid residue complex on \( \mathcal{X} \) (relative to \( K \)) is a pair \((K_X, \rho_X)\) consisting of these data:

(a) A bounded complex \( K_X \) of quasi-coherent injective \( \mathcal{O}_X \)-modules.

(b) A collection \( \rho_X := \{\rho_{(U,g)}\} \) indexed by étale maps \( g : U \to \mathcal{X} \) from affine \( K \)

schemes \( U \), where each \( \rho_{(U,g)} \) is a rigidifying isomorphism for the complex of \( \Gamma(U, \mathcal{O}_U) \)-modules \( \Gamma(U, g^*(K_X)) \), in the sense of Definition 1.3.

There are two conditions:

(i) For every étale map \( g : U \to \mathcal{X} \) from an affine scheme \( U = \text{Spec}(A) \), letting \( K_A := \Gamma(U, g^*(K_X)) \), the pair \((K_A, \rho_{(U,g)})\) is a rigid residue complex over the ring \( A \), in the sense of Definition 3.2.

(ii) For every commutative diagram

\[
\begin{array}{ccc}
U_2 & \xrightarrow{h} & U_1 \\
\downarrow{g_1} & & \downarrow{g_1} \\
\mathcal{X} & \xrightarrow{g_2} & \\
\end{array}
\]

of étale maps, where \( U_1 \) and \( U_2 \) are affine schemes, the homomorphism \( h^* \) arising from the equality \( g_2 = g_1 \circ h \) is the unique nondegenerate rigid étale-localization homomorphism

\[
h^* : (\Gamma(U_1, g_1^*(K_X)), \rho_{(U_1,g_1)}) \to (\Gamma(U_2, g_2^*(K_X)), \rho_{(U_2,g_2)}),
\]

in the sense of Theorem (5.5).

The collection \( \rho_X \) is called a rigid structure on the complex \( K_X \).

**Theorem** (5.7). ([Ye10]) Let \( \mathcal{X} \) be a FT DM \( K \)-stack. There exists a rigid residue complex \((K_X, \rho_X)\), and it is unique up to a unique rigid isomorphism.
The proof is by applying the mechanism of gluing quasi-coherent sheaves that was explained above, where in the first step of affine schemes we use Corollary \(^{\circledast} 3.6\).

**Theorem\(^{\circledast} 5.8.\) (Ye10)** Let \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) be a map between FT DM \(\mathbb{K}\)-stacks. There is a homomorphism of graded quasi-coherent \(\mathcal{O}_X\)-modules \(\text{tr}^{\text{ind-res}}_f : f^!(\mathcal{K}_\mathfrak{f}) \rightarrow \mathcal{K}_X\) called the ind-rigid trace, extending the ind-rigid trace on FT \(\mathbb{K}\)-rings.

To prove this theorem we apply the construction explained above for homomorphisms between quasi-coherent sheaves, starting with Theorem \(^{\circledast} 3.11\) for affine schemes.

**Theorem\(^{\circledast} 5.9\) (Ye10).** There is a pseudofunctor

\[
\text{TwInvIm} : (\text{DMStk}/\mathbb{K})^{\text{op}} \rightarrow \text{TrCat}/\mathbb{K}, \quad \mathfrak{x} \mapsto \mathcal{D}_\mathcal{C}^+(\mathfrak{x}), \quad f \mapsto f^!
\]

called the twisted inverse image, with properties like in Theorem \(^{\circledast} 4.9\).

In this theorem we consider DMStk/\(\mathbb{K}\) as a 1-category, ignoring the 2-isomorphisms between 1-morphisms, as we had done in the hierarchy \(^{\circledast} 5.1\). However, the 2-category structure of DMStk/\(\mathbb{K}\) is reflected in this property: if \(f_0, f_1 : \mathfrak{f} \rightarrow \mathfrak{x}\) are maps in DMStk/\(\mathbb{K}\) for which a 2-isomorphism \(\gamma : f_0 \Rightarrow f_1\) exists, then the triangulated functors \(f_0^!, f_1^! : \mathcal{D}_\mathcal{C}^+(\mathfrak{x}) \rightarrow \mathcal{D}_\mathcal{C}^+(\mathfrak{f})\) are isomorphic. This implies that when \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) is an equivalence of stacks, the functor \(f^!\) will also be an equivalence.

The obvious question now is this: Do the Rigid Residue Theorem and the Rigid Duality Theorem hold for a proper map \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) between DM stacks? We only know a partial answer.

By the Keel-Mori Theorem [Ol, Theorem 11.1.2], a separated FT \(\mathbb{K}\)-stack \(\mathfrak{f}\) has a coarse moduli space \(\pi : \mathfrak{f} \rightarrow \mathfrak{y}\). The map \(\pi\) is proper and quasi-finite, and \(\mathfrak{y}\) is, in general, an algebraic space.

**Definition 5.10.** (1) Let \(\mathfrak{f}\) be a separated FT DM \(\mathbb{K}\)-stack, with coarse moduli space \(\pi : \mathfrak{f} \rightarrow \mathfrak{y}\). We call \(\mathfrak{f}\) a coarsely schematic stack if its coarse moduli space \(\mathfrak{y}\) is a scheme.

(2) Let \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) be a separated map between FT DM \(\mathbb{K}\)-stacks. We say that \(f\) is a coarsely schematic map if for some surjective étale map \(U \rightarrow \mathfrak{y}\) from an affine scheme \(U\), the stack \(\mathfrak{f}' := \mathfrak{f} \times_{\mathfrak{x}} U\) is coarsely schematic.

This appears to be a rather mild restriction: most DM stacks that come up in examples are of this kind. See Remark 5.17 about the necessity of this condition.

**Theorem\(^{\circledast} 5.11\) (Rigid Residue Theorem, Ye10).** Suppose \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) is a proper coarsely schematic map of FT DM \(\mathbb{K}\)-stacks. Then the ind-rigid trace \(\text{tr}^{\text{ind-res}}_f : f^!(\mathcal{K}_\mathfrak{f}) \rightarrow \mathcal{K}_\mathfrak{x}\) is a homomorphism of complexes of \(\mathcal{O}_\mathfrak{x}\)-modules.

It is not expected that duality will hold in this generality. In fact, there are easy counter examples. The problem is not geometric, but rather a basic difficulty with representations of finite groups in positive characteristics.

Following [AOV] (see also [Ol, Definition 11.3.2]), a separated FT DM stack \(\mathfrak{f}\) is called tame if for every field \(L \in \text{Rng}_{/\mathbb{K}}\), the automorphism groups in the groupoid \(\mathfrak{f}(L)\), which are always finite, have orders prime to the characteristic of \(L\).

**Definition 5.12.** A separated map of FT DM \(\mathbb{K}\)-stacks \(f : \mathfrak{f} \rightarrow \mathfrak{x}\) is called a tame map if for some surjective étale map \(U \rightarrow \mathfrak{x}\) from an affine scheme \(U\), the stack \(\mathfrak{f}' := \mathfrak{f} \times_{\mathfrak{x}} U\) is tame.
Theorem (Rigid Duality Theorem, [Ye10]). Suppose \( f : \mathcal{Y} \to \mathcal{X} \) is a proper tame coarsely schematic map between FT DM \( \mathbb{K} \)-stacks. For every \( \mathcal{N} \in \mathcal{D}^c_\mathbb{Q}(\mathcal{X}) \) the morphism
\[
Rf_!(R\mathcal{H}om_\mathcal{Y}(\mathcal{N}, K_\mathcal{Y})) \to R\mathcal{H}om_\mathcal{X}(Rf_!(\mathcal{N}), K_\mathcal{X})
\]
in \( \mathcal{D}(\mathcal{X}) \), which is induced by the ind-rigid trace \( tr_{f,*} : f_!(K_\mathcal{Y}) \to K_\mathcal{X} \), is an isomorphism.

Theorem (5.11) implies that for every complex \( \mathcal{M} \in \mathcal{D}^c_\mathbb{A}(\mathcal{X}) \) there is an ind-rigid trace morphism \( tr_{f,*} : f_!(\mathcal{M}) \to \mathcal{M} \) in \( \mathcal{D}(\mathcal{X}) \), which is functorial in \( \mathcal{M} \).

Corollary (5.14). (Ye10) Suppose \( f : \mathcal{Y} \to \mathcal{X} \) is a proper tame coarsely schematic map between FT DM \( \mathbb{K} \)-stacks. For every \( \mathcal{M} \in \mathcal{D}^c_\mathbb{A}(\mathcal{X}) \) and \( \mathcal{N} \in \mathcal{D}^c_\mathbb{Q}(\mathcal{Y}) \) the morphism
\[
Rf_!(R\mathcal{H}om_\mathcal{Y}(\mathcal{N}, f^*(\mathcal{M}))) \to R\mathcal{H}om_\mathcal{X}(Rf_!(\mathcal{N}), \mathcal{M})
\]
in \( \mathcal{D}(\mathcal{X}) \), which is induced by the ind-rigid trace \( tr_{f,*} \), is an isomorphism.

Here is a sketch of the proofs of Theorems (5.11) and (5.13), which we are going to refer to as "Residue" and "Duality", respectively. Take a surjective étale map \( U \to \mathcal{X} \) from an affine scheme \( U \) such that the stack \( \mathcal{Y}' := \mathcal{Y} \times_{\mathcal{X}} U \) is coarsely schematic. Consider the commutative up to isomorphisms diagram of maps of stacks
\[
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{\pi'} & \mathcal{Y} \\
& \downarrow f' \swarrow f \\
U & \rightarrow & \mathcal{X}
\end{array}
\]
where \( f' \) is gotten from \( f \) by base change, and \( Y' \) is the coarse moduli space of \( \mathcal{Y}' \). The maps \( \pi' \) and \( g' \) are both proper.

Because the functor \( \mathcal{Q}Coh(\mathcal{X}) \to \mathcal{Q}Coh(U) \) is faithful, it suffices to prove "Residue" and "Duality" for the map \( f' \). Now \( Y' \) is a scheme, so the proper map \( g' : Y' \to U \) satisfies both "Residue" and "Duality", by Theorems (4.10) and (4.11).

It remains to verify "Residue" and "Duality" for the proper map \( \pi' : \mathcal{Y}' \to Y' \). These properties are étale local on \( Y' \). Namely let \( V'_1, \ldots, V'_r \) be affine schemes, and let
\[
\bigsqcup_i V'_i \to Y'
\]
be a surjective étale map. For any \( i \) let \( \mathcal{Y}'_i := \mathcal{Y}' \times_{Y'} V'_i \). We get this commutative (up to isomorphism) diagram of maps of stacks
\[
\begin{array}{ccc}
\bigsqcup_i \mathcal{Y}'_i & \xrightarrow{\pi'_i} & \mathcal{Y}' \\
\downarrow \bigsqcup_i \pi'_i & & \downarrow \pi' \\
\bigsqcup_i V'_i & \rightarrow & Y'
\end{array}
\]
It is enough to check "Residue" and "Duality" for the maps \( \pi'_i : \mathcal{Y}'_i \to V'_i \). Note that the affine scheme \( V'_i \) is the coarse moduli space of the stack \( \mathcal{Y}'_i \).

According to [Ol] Theorem 11.3.1] it is possible to choose a covering (5.15) such that
\[
\mathcal{Y}'_i \cong [W_i/G_i] \quad \text{and} \quad V'_i \cong W_i/G_i.
\]
Here \( W_i \) is an affine scheme, \( G_i \) is a finite group acting on \( W_i \), \([W_i/G_i]\) is the quotient stack, and \( W_i/G_i \) is the quotient scheme. Moreover, in the tame case we can assume that the order of the group \( G_i \) is invertible in the ring \( \Gamma(W_i, \mathcal{O}_{W_i}) \).
We have now reduced the problem to proving “Residue” and “Duality” for a map of stacks
\[ \pi : \mathcal{Y} \to \mathcal{V} = W/G, \]
where \( W = \text{Spec}(A) \) for some ring \( A \), \( G \) is a finite group acting on \( A \), and in the tame case the order of \( G \) is invertible in \( A \).

Let \( A^G \) be the ring of \( G \)-invariants inside \( A \). So \( A^G \) is a \( \mathbb{K} \)-ring, and the quotient scheme is \( W/G = \text{Spec}(A^G) \). Consider the skew group ring \( G \rtimes A \); this is the noncommutative (NC) ring appearing in [Ol, Exercise 9.E]. The NC ring \( G \rtimes A \) is finite over its center \( A^G \). It is known that \( \text{QCoh}(\mathcal{Y}) \approx \text{Mod}(G \rtimes A) \) as abelian categories, and the functor \( \pi_* \), on the geometric side goes to the \( G \)-invariants functor \( (-)^G \) on the NC algebraic side.

We finish the proof by using GD for NC rings, see [Ye7, Chapters 14-18].

**Remark 5.17.** It is very likely that the “coarsely schematic” condition can be removed from Theorems 5.11 and 5.13. This will probably require a better understanding of the geometry of algebraic spaces.

**Remark 5.18.** We do not know how to handle Artin stacks. This is because we do not understand the smooth functoriality of the squaring operation.

**Remark 5.19.** Let us end the article with a brief discussion of related work.

1. There is a preprint [Ni] by F. Nironi from 2008 on GD for DM stacks. It has neither been updated nor published. The details in that paper are not clear to us.

2. J. Hall and D. Rydh wrote several papers dealing with derived categories on stacks. In their paper [HR] from 2017 they prove a version of GD for a finite faithfully flat representable map \( f : \mathcal{Y} \to \mathcal{X} \) of Artin stacks.

3. A. Neeman wrote the paper [Ne2], last updated in 2017. It deals with GD in great generality, and also in a very abstract manner. The geometric objects considered in it range from schemes to Artin stacks; and the maps between them are required to be concentrated and compactifiable. For any such map \( f : \mathcal{Y} \to \mathcal{X} \) the functor \( \mathcal{R}f_* \) is shown to have a right adjoint \( f^* \); and under certain further conditions one has \( f^* \cong f^! \). According to Neeman (in a private communication), his work implies that our Corollary 5.14 holds, when \( f : \mathcal{Y} \to \mathcal{X} \) is a concentrated proper map between DM stacks.

There might be other work on global GD, perhaps also from the direction of derived algebraic geometry, but we are not aware of any at present.

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