Diversity enhanced synchronization in a small-world network of phase oscillators

Tayeb Nikfard,1 Farhad Shahbazi,1,a) and Reihan Kouhi Esfahani1
Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

(Dated: 28 August 2019)

In this work, we study the synchronization of a group of phase oscillators (rotors) in the small-world (SW) networks. The distribution of intrinsic angular frequency of the rotors are given by a Lorenz probability density function with zero mean and the width \( \gamma \), and their dynamics are governed by the Kuramoto model. We find that the partially synchronized states of identical oscillators (with \( \gamma = 0 \)) in the SW network, become more synchronized when \( \gamma \) increases up to an optimum value, where the synchrony in the system reaches a maximum and then start to fall. We discuss that the reason for this "diversity enhanced synchronization" is the weakening and destruction of topological defects presented in the partially synchronized attractors of the Kuramoto model in SW network of identical oscillators. We also show that introducing the diversity in the intrinsic frequency of the rotary agents makes the fully synchronized state in the SW networks, more fragile than the one in the random networks.

It is an expectation that the synchronization in a population of coupled phase oscillators with Kuramoto interaction, decreases by increasing the diversity in the intrinsic frequency of the oscillators. However, we found that the synchronization of the Kuramoto model in a small-world network is enhanced by considering a Lorentz distribution for the intrinsic frequency of oscillators. We discuss that the reason for such a counterintuitive result is the destruction of defect patterns, formed in the small-world network of identical oscillators1, as the result of diversity.

I. INTRODUCTION

Exploring the collective behaviors in a group of interacting dynamic agents is a profound goal in the field of complex systems. One of such collective behaviors is the synchronization of self-sustained oscillators or rotors with nonlinear interaction. Strengthening the mutual coupling between the oscillators tends to the convergence of the frequency and phase of individuals and as a result, synchronization emerges in the system. Some examples of synchronization in natural populations are synchronous flashing of fireflies, flocking of birds and fishes, the simultaneous firing of neurons in a neuronal network2–6.

A simple model which elegantly displays the emergence of synchrony in a network of phase oscillators was introduced by Y. Kuramoto7,8. In the Kuramoto model, the coupling between any pair of rotors is given by the sine of their phase difference. The exact solution of this model in a complete network indicates the continuous growth of synchrony at a critical coupling7. The Kuramoto model and its variants have been used to model the opinion formation dynamics in a society9–14. In this context, synchronization is equivalent to the formation of consensus in society.

Recently, there has been much interest in studying synchronization in complex networks15. In particular, the effect of network topology on the synchronization of the Kuramoto model in complex networks has widely been investigated16. For instance, it has been shown that the fully synchronized state of an assembly of identical phase oscillators in a scale-free network is more robust against the noise than the random or small-world (SW) networks17.

Of interests, are the multiple stationary states of the Kuramoto model of identical oscillators in the regular and SW networks. It has been shown that in a regular network consisting of \( N \) nodes in a ring, in which each node is connected to its \( k \) nearest neighbors, the stationary state is not unique for \( k/N < 0.34 \)18. The phase profiles in these attractors are in the form of some helical patterns, which are topologically distinct and characterized by integer winding numbers17.

The SW networks, owning both the high clustering coefficient of the regular and small mean path length of random networks, are constructed by rewiring of the links of a regular network by a small probability \( p \sim 0.01 \)19,20. Many real networks, such as some brain neuronal connectivities or human friendship networks are shown to have SW characteristics21. The interesting properties of SW networks motivated some studies on the synchronization of the Kuramoto model in these networks1,20,22–25.

Interestingly, the multiplicity in the attractors of the Kuramoto model in an SW network of phase oscillators with equal intrinsic frequencies has also been observed11. Rewiring the links of the regular lattice tends to the transformation of the helical patterns to the partially synchronized state with isolated or quasi-periodic extended defects. Indeed, the helical states with a small winding number are converted to the phase texture with a few numbers of isolated point defects, where the phase difference of oscillators vary continuously from 0 to \( \pi \) upon moving toward the defect center. However, the states with a higher winding number are more robust against link rewiring and are transformed to agitated helical patterns.

These defect patterns have nontrivial topologies and hence are protected against small perturbations. Nevertheless, introducing an additive uncorrelated white noise to the Kuramoto dynamics with large enough strength, gives rise to the destruction of these defects, hence producing a more uniform phase pattern with higher amount of synchrony, a phe-

---

a)Electronic mail: shahbazi@cc.iut.ac.ir
nomenon which is called stochastic synchronization\(^1\).

Motivated by the striking results of the Kuramoto model in an SW network of similar oscillators, in this work we aim to study the effect of diversity on the defect patterns discussed above. The enhancement in the collective firing of individuals induced by diversity been reported in excitable media\(^26,27\). More recently, the idea of the enhancement of synchronization by breaking the symmetry in a population has been put forwarded, entitled “asymmetry-induced synchronization” (AISync)\(^28-31\). To include the diversity, in this work we consider a Lorentz distribution for the intrinsic angular frequencies of the phase oscillators.

The paper is organized as the following. In section II, we define the model and the numerical methods of quantifying the synchronization. Section III represents the results and discussion and section IV is devoted to the concluding remarks.

II. MODEL AND METHOD

Consider a system of \(N\) self-sustained oscillators, residing on the vertices of a network in such a way that the interaction between any pair of oscillators is represented by a link between the two. In this work, we consider the symmetric mutual coupling that is the Kuramoto coupling given by the sine of their phase difference. Then the model is given by the following set of coupled differential equations\(^3\):

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{\lambda}{k_i} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, ..., N
\]  

(1)

where \(\theta_i\) and \(\omega_i\) denote the phase and intrinsic frequency of the oscillator at node \(i\), respectively, \(a_{ij}\) denotes the elements of the adjacency matrix (i.e. \(a_{ij} = 1\) if \(i\) and \(j\) are connected and \(a_{ij} = 0\) otherwise), \(\lambda > 0\) is the coupling strength between any two connected oscillators \(i\) and \(j\), and \(k_i\) is the degree of the node \(i\).

The intrinsic frequencies \(\omega_i\) are randomly chosen from a Lorentz probability density function with zero mean and width \(\gamma\)

\[
g(\omega; \gamma) = \frac{\gamma}{\pi(\omega^2 + \gamma^2)}.
\]  

(2)

In the case that the mean frequency is not zero (\(\omega \neq 0\)), one can always sets it to zero by moving to a rotating frame with the angular velocity \(\bar{\omega}\), i.e. \(\theta'(t) = \theta(t) - \bar{\omega}t\).

The coupling \(\lambda\) in equation (1) can be absorbed to the time variable, \(t\), to give a dimensionless parameter \(\tau = \lambda t\), henceforth eq. (1) turns to

\[
\frac{d\theta_i}{d\tau} = \omega_i' + \frac{1}{k_i} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, ..., N
\]  

(3)

in which \(\omega_i' = \omega_i / \lambda\) whose distribution is given by \(g(\omega'; \gamma') = \lambda g(\omega; \gamma)\) and \(\gamma' = \gamma / \lambda\). Therefore, the only effect of \(\lambda\) is the rescaling of the width of frequency distribution, hence in the rest of this work, we set it to unity.

In order to characterize the synchrony among the oscillators at any time in the model, an order parameter can be defined as:

\[
r(\tau) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j(\tau))].
\]  

(4)

Depending on the degree of synchrony, \(r\) takes a value between 0 and 1. \(r = 0\) indicates a totally random phase distribution or a phase locked state, whereas \(0 < r < 1\) indicates a partially synchronized state. As the system becomes more coherent, \(r\) will approach to 1.

While the order parameter \(r\) gives a global sense of synchrony among the oscillators, the local dynamics in the stationary state can be represented by the pairwise correlation matrix \(D\), defined as\(^32\):

\[
D_{ij} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{\tau_i}^{\tau_i + \Delta t} \cos(\theta_i(\tau) - \theta_j(\tau))d\tau.
\]  

(5)

The matrix elements \(D_{ij}\) take a value in the interval \([-1, 1]\). \(D_{ij} = 1\) denotes full synchrony between oscillators \(i\) and \(j\), while \(D_{ij} = -1\), represents an anti-phase state (i.e. \(|\theta_i - \theta_j| = \pi\)). \(\tau_i\) is the time needed to reach a stationary state.

The networks that we study in this work are created by Watts-Strogatz (WS) algorithm\(^19,20\). In this algorithm, starting from a regular lattice with a given degree of \(k\) for each node, one rewires the links with the probability \(p\). It has been shown that for \(0.005 \lesssim p \lesssim 0.05\), this process gives rise to an SW network with small mean path length and large clustering coefficient. Increasing the rewiring probability tends to lower the clustering coefficient while keeping the mean path length small which are the peculiarities of a random network given by \(p = 1\).

III. RESULTS AND DISCUSSION

To obtain the time evolution of the phase of the oscillators and also the order parameter (4), we use the fourth order Runge-Kutta method for integrating the set of equations (3). The integration time step is set to \(d\tau = 0.01\) and the initial phase distribution is taken from a box distribution in the interval \([-\pi, \pi]\).

Figures 1-(a) and 1-(b) show the time dependence (in logarithmic scale) of the order parameter for three frequency distribution widths \(\gamma = 0.02, 0.06, 0.25\) and similar initial phase distributions for a random and SW network with the rewiring probability \(p = 0.03\), respectively. As it is clear from this figure, the order parameter reaches the stationary state for the random network in \(~100\) simulation time steps, however, reaching to a stationary state takes much longer time (~\(10^4\) simulation steps) for the SW network. More interestingly, while in the random network the stationary order parameter \((r_m)\) decreases by increasing the frequency width \(\gamma\), the variation of the stationary order parameter in the SW network is not monotonic versus \(\gamma\). Indeed, \(r_m\) moves up when \(\gamma\) rises from 0.02 to 0.06 and then falls down by increasing to \(\gamma = 0.25\).
FIG. 1. (Color online) Order parameters $r$ versus simulation time (in log scale) for $\gamma = 0.02, 0.06, 0.25$ in (a) a random ($p = 1$) and (b) an SW ($p = 0.03$) network of $N = 1000$ oscillators and mean degree $< k > = 10$.

We proceed to find the dependence of long-time order parameter $r_{\infty}$ to the width of frequency distribution $\gamma$. Since it has been shown that the stationary state of the Kuramoto model in an SW network is not unique for delta distributed ($\gamma = 0$) intrinsic frequencies\(^1\), we investigate the full and partial synchronized states, separately.

In the random network, the full synchrony ($r = 1$) is the only attractor when all the oscillators have the same intrinsic frequency, so it is easy to find the variation of the stationary order parameter $r_{\infty}$ in terms of $\gamma$ for this network. However, since the stationary state is dependent on the initial condition in SW network, for a comparison with the random network we consider an initial phase distribution ending up to the fully synchronized state in SW network when $\gamma = 0$, and obtain the variation of stationary order parameter ($r_{\infty}$) versus $\gamma$. Such a comparison is illustrated in figure 2. These results are obtained by averaging over 10 different realizations of frequency distributions for each $\gamma$ and a fixed WS network of $p = 1$ for the random and $p = 0.03$ for the SW network. This figure shows a faster falling of $r_{\infty}$ versus $\gamma$ for the SW network than

FIG. 2. (Color online) Stationary order parameters $r_{\infty}$ versus the width of intrinsic frequency distribution $\gamma$, for a random ($p = 1$) and an SW ($p = 0.03$) network of $N = 1000$ and $< k > = 10$. These results are obtained by averaging over 10 different realizations of frequency distribution and the error bars are less than the size of the symbols. The initial phase distributions of the SW network are chosen in the ways that end up to the full synchronized state at $\gamma = 0$.\(^1\)

FIG. 3. (Color online) Stationary order parameters $r_{\infty}$ versus the width of intrinsic frequency distribution $\gamma$, for an SW network (starting from a partially synchronized state at $\gamma = 0$) of $N = 1000$ and $< k > = 10$. Three different initial conditions $a, b, c$ are chosen and the results are obtained by averaging over 10 different realizations of the frequency distribution. The error bars show the standard error of the mean.
the random network, indicating the more resistance of synchrony in the random networks, against the diversity in intrinsic frequency.

Now, we consider the case of a partially synchronized state in the SW network. Most of the initial phase distributions tend to reach the inhomogeneous stationary phase patterns with \( r < 1 \) for \( \gamma = 0 \). Figure 3 represents that how \( r_\infty \) varies in terms of \( \gamma \) when three of such phase distributions are chosen as the initial conditions of equation (3). The results are averaged over 10 different frequency distributions \( (g(\omega)) \) on a WS network with \( p = 0.03 \) and the error bars indicate the standard error of the mean (SEM) over the stationary state of these 10 simulations. All the plots in this figure indicate the enhancement of synchrony by increasing the width of frequency distribution up to \( \gamma \approx 0.06 \) and then declining the synchrony for \( \gamma > 0.06 \).

To gain a detailed insight into this interesting \( r_\infty \) versus \( \gamma \) behavior, in figure 4, we illustrate the density plots of correlation matrix elements given by Eq. (5), for one of such initial conditions, i.e. SW-c, for \( \gamma = 0.0, 0.02, 0.04, 0.06 \). This figure represents the evolution of the defect pattern in terms of \( \gamma \), which gives rise to its weakening by increasing the frequency PDF width \( \gamma \) and formation of a more homogeneous state, and consequently the growth of synchrony at \( \gamma = 0.06 \). After the defects being disappeared, increasing \( \gamma \) naturally tends to monotonically decrease of synchrony. This is similar to the stochastic synchronization in an SW network of identical phase oscillators, where a maximum enhancement in the synchrony takes place at a specific applied noise strength on the oscillators. In this work, we name this phenomenon "diversity enhanced synchronization".

To make sure that these results are not the artifact of using only one network realization, we performed the above simulations on 10 SW networks with the same number of nodes, degree and rewiring probability, but with different rewiring realizations and observed the diversity enhanced synchroniza-
tion in all of them. However, the value of $\gamma$ at which the maximum synchrony occurs could differ a little from sample to sample. Moreover, we found that increasing the rewiring probability $p$ gives rise to declining of the optimum value of $\gamma$ at which the maximum synchrony takes place (Fig. 5). For $p > 0.15$ there is no diversity enhancement of synchronizion and the system behaves like a random network.

IV. CONCLUSION

A small-world network of rotating agents with the Kuramoto dynamics is shown to have multiple stationary attractors, characterized by isolated or extended defects. The agents in such defects are in an anti-phase state with the rest of rotors. It has already been shown that applying an external spatially uncorrelated white noise on this system, up to optimal strength, increases the synchrony in the system by destroying the defects. Here, we show that such an enhancement in the synchrony of a Kuramoto model in an SW network occurs by increasing the diversity in the intrinsic frequencies of the oscillators. This work may provide a simple model, indicating the important role of diversity in rising the coherency in human societies which are known to have small-world connectivity.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the Sheikh Bahaei National High-Performance Computing Center (SBNHPCC) for providing computing facilities and time. SBNHPCC is supported by the scientific and technological department of presidential office and Isfahan University of Technology (IUT).

1R. K. Esfahani, F. Shabhazi, and K. A. Samani, “Noise-induced synchronization in small world networks of phase oscillators,” Physical Review E 86, 036204 (2012).
2A. T. Winfree, The geometry of biological time, Vol. 12 (Springer Science & Business Media, 2001).
3A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: a universal concept in nonlinear sciences, Vol. 12 (Cambridge university press, 2003).
4C. Manrubia, A. S. Mikhailov, et al., Emergence of dynamical order: synchronization phenomena in complex systems, Vol. 2 (World Scientific, 2004).
5A. Balanov, N. Janson, D. Postnov, and O. Sosnovtseva, Synchronization: from simple to complex (Springer Science & Business Media, 2008).
6S. Strogatz, Sync: How Order Emerges From Chaos In the Universe, Nature, and Daily Life (2012).
7Y. Kuramoto, “Self-entrainment of a population of coupled non-linear oscillators,” in International symposium on mathematical problems in theoretical physics (Springer, 1975) pp. 420–422.
8Y. Kuramoto, Chemical oscillations, waves, and turbulence, Vol. 19 (Springer Science & Business Media, 2012).
9J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, “The kuramoto model: A simple paradigm for synchronization phenomena,” Reviews of modern physics 77, 137 (2005).
10A. Pluchino, V. Latora, and A. Rapisarda, “Changing opinions in a changing world: A new perspective in sociophysics,” International Journal of Modern Physics C 16, 515–531 (2005).
11A. Pluchino, V. Latora, and A. Rapisarda, “Compromise and synchronization in opinion dynamics,” The European Physical Journal B-Condensed Matter and Complex Systems 50, 169–176 (2006).
12A. Pluchino, S. Boccaletti, V. Latora, and A. Rapisarda, “Opinion dynamics and synchronization in a network of scientific collaborations,” Physica A: Statistical Mechanics and its Applications 372, 316–325 (2006).
13H. Hong and S. H. Strogatz, “Conformists and contrarians in a kuramoto model with identical natural frequencies,” Physical Review E 84, 046202 (2011).
14H. Hong and S. H. Strogatz, “Kuramoto model of coupled oscillators with positive and negative coupling parameters: an example of conformist and contrarian oscillators,” Physical Review Letters 106, 054102 (2011).
15A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, “Synchronization in complex networks,” Physics reports 469, 93–153 (2008).
16F. A. Rodrigues, T. K. D. Peron, P. Ji, and J. Kurths, “The kuramoto model in complex networks,” Physics Reports 610, 1–98 (2016).
17H. Khoshbakhht, F. Shabhazi, and K. A. Samani, “Phase synchronization on scale-free and random networks in the presence of noise,” Journal of Statistical Mechanics: Theory and Experiment 2008, P10020 (2008).
18D. A. Wiley, S. H. Strogatz, and M. Girvan, “The size of the sync basin,” Chaos: An Interdisciplinary Journal of Nonlinear Science 16, 015103 (2006).
19D. J. Watts and S. H. Strogatz, “Collective dynamics of small-world?networks,” nature 393, 440 (1998).
20D. J. Watts, Small worlds: The dynamics of networks between order and randomness (Princeton University Press, 2001).
21S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, “Complex networks: Structure and dynamics,” Physics reports 424, 175–308 (2006).
22P. M. Gade and C.-K. Hu, “Synchronous chaos in coupled map lattices with small-world interactions,” Physical Review E 62, 6409 (2000).
23H. Hong, M.-Y. Choi, and B. J. Kim, “Synchronization on small-world networks,” Physical Review E 65, 026139 (2002).
24M. Barahona and L. M. Pecora, “Synchronization in small-world systems,” Physical review letters 89, 054101 (2002).
25F. A. Rodrigues, T. K. D. Peron, P. Ji, and J. Kurths, “The kuramoto model in complex networks,” Physics Reports 610, 1–98 (2016).
26J. H. Cartwright, “Emergent global oscillations in heterogeneous excitable media: The example of pancreatic $\beta$ cells,” Physical review E 62, 1149 (2000).
27C. J. Tessone, A. Scire, R. Toral, and P. Colet, “Theory of collective firing induced by noise or diversity in excitable media,” Physical Review E 75, 016203 (2007).
28Y. Nishikawa and A. E. Motter, “Symmetric states requiring system asymmetry,” Physical review letters 117, 114101 (2016).
28 Y. Zhang, T. Nishikawa, and A. E. Motter, “Asymmetry-induced synchronization in oscillator networks,” Physical Review E 95, 062215 (2017).
29 Y. Zhang and A. E. Motter, “Identical synchronization of nonidentical oscillators: when only birds of different feathers flock together,” Nonlinearity 31, R1 (2018).
30 A. E. Motter and M. Timme, “Antagonistic phenomena in network dynamics,” (2018).
31 J. Gómez-Gardenes, Y. Moreno, and A. Arenas, “Paths to synchronization on complex networks,” Physical review letters 98, 034101 (2007).