Other Families of Rational Solutions to the KPI Equation

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Author’s contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

Aims / Objectives: We present rational solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of polynomials in \( x \), \( y \) and \( t \) depending on several real parameters. We get an infinite hierarchy of rational solutions written as a quotient of a polynomial of degree \( 2N(N+1) - 2 \) in \( x \), \( y \) and \( t \) by a polynomial of degree \( 2N(N+1) \) in \( x \), \( y \) and \( t \), depending on \( 2N - 2 \) real parameters for each positive integer \( N \).

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Conclusion: We construct explicit expressions of the solutions in the simplest cases \( N = 1 \) and \( N = 2 \) and we study the patterns of their modulus in the \((x, y)\) plane for different values of time \( t \) and parameters. In particular, in the study of these solutions, we see the appearance not yet observed of three pairs of two peaks in the case of order 2.

Keywords: Kadomtsev-Petviashvili equation; rational solutions; patterns of configurations.

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## 1 Introduction

We consider the Kadomtsev-Petviashvili (KPI) equation in the following normalization

\[(4u_t - 6uu_x + u_{xxx})_x = 3u_{yy},\]  \hspace{1cm} (1.1)

where subscripts \(x\), \(y\) and \(t\) denote partial derivatives.

This equation first appears in 1970, in a paper written by Kadomtsev and Petviashvili [1]. The equation (1.1) was derived as a model for surface and internal water waves [2], and in nonlinear optics [3].

A lot of methods have been used to solve this equation; these methods are reviewed in the book by Ablowitz and Clarkson published in 1991 [4]. Matveev used the Darboux transformation to construct solutions [5] in 1979. Dubrovin gave for the first time solutions in terms of Riemann theta functions in the framework of algebraic geometry in 1981 [6]. Freeman and Nimmo constructed in 1983 solutions in terms of wronskians [7, 8] by means of tau functions. The inverse scattering method has been applied to obtain solutions of the KPI equations in 1999 [9].

The first rational solutions were constructed in 1977 by Manakov, Zakharov, Bordag, Its and Matveev [10]. Among the various researches concerning rational solutions of the KPI equation, we can mention the works of the following authors: Krichever in 1978 [11, 12], Satsuma and Ablowitz in 1979 [13], Pelinovsky and Stepanyants in 1993 [14], Pelinovsky in 1994 [15].

We present multi-parametric families of rational solutions to KPI equation as a quotient of two polynomials in \(x\), \(y\) and \(t\) depending on several real parameters. These solutions presented here belong to an infinite hierarchy of solutions of rational solutions written as a quotient of a polynomial of degrees \(2N(N+1) - 2\) in \(x\), \(y\), and \(t\) by a polynomial of degree \(N(N+1)\) in \(x\), \(y\), and \(t\), depending on \(2N-2\) real parameters for each positive integer \(N\). We limit here the study to the case of the simplest \(N = 1\) and \(N = 2\).

We study the patterns of the modulus of the solutions in the \((x, y)\) plane for different values of time \(t\) and parameters.

This work is part of a project to find rational solutions to different classical partial differential equations. New types of rational solutions are proposed in this article.

We will give the ideas of the proof of the results in the case of order \(N\) in a future article. The proofs of the following results in the simple cases of the first orders can be carried out by hand or with the help of a formal calculation software.

## 2 Rational Solutions of Order 1

**Theorem 2.1.** The function \(v\) defined by

\[v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)^2}\]  \hspace{1cm} (2.1)

is a rational solution to the KPI equation (1.1)
with
\[ n(x, y, t) = -512 x^2 + (768 t + 1024)x + 512 y^2 - 768 t - 288 t^2, \]
and
\[ d(x, y, t) = 16 x^2 + (-24 t - 32)x + 9 t^2 + 16 y^2 + 24 t + 32. \]

Proof. We will give the ideas of the proof of the result in the case of order \( N \) in a future article. It can be carried out by hand or with the help of a formal calculation software: it is sufficient to replace the expression of the solution given by (2.1) and check that (1.1) is verified. \( \square \)

We represent the modulus of the solution for different values of \( t \) in the plane \((x, y)\). For \( t = 0 \), the maximum of the modulus is equal to 4. When the values of \( t \) increase, the maximum of the modulus of the solution decreases rapidly. For example for \( t = 10^3 \), this value is equal to 0.16.

Fig. 1. Solution of order 1 to KPI, on the left for \( t = 0 \); in the center for \( t = 10 \); on the right for \( t = 10^3 \)

3 Rational Solutions of Order 2 Depending on 2 Parameters

We always consider the KPI equation defined by (1.1).

**Theorem 3.1.** The function \( v \) defined by
\[ v(x, y, t) = -2 \frac{n(x, y, t)}{d(x, y, t)}, \tag{3.1} \]
is a rational solution to the KPI equation (1.1), quotient of a polynomial \( n(x, y, t) \) of degree \( 2N(N + 1) - 2 = 10 \) in \( x, y, t \) by a polynomial \( d(x, y, t)^2 \) of degree \( 2N(N + 1) = 12 \) in \( x, y, t \) depending on 2 real parameters \( a_1, b_1 \)

with \( n \) and \( d \) defined by
\[ n(x, y, t) = -73383542784 x^{10} + (550376570880 t + 1712282664960) x^9 + (-18590497505280 t - 1857520926720 t^2 - 11557907988480 t - 220150628352 y^2) x^8 + (1320903770112 t y^2 + 34673723965440 t^2 + 111542985031680 t + \]

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As already said, we will give the ideas of the proof of the result in the case of order 1.1. It can be carried with the help of a formal calculation software. We have to replace the expression

$$\text{expression}$$

in the plane ($x, y$).

**Proof.** As already said, we will give the ideas of the proof of the result in the case of order $N$ in a future article. It can be carried with the help of a formal calculation software. We have to replace the expression of the solution given by (3.1) in (1.1) and check that the relation is verified.

We represent the modulus of the solution for different values of $t$ in the plane ($x, y$).

**Fig. 2.** Solution of order 2 to KPI for $t = 0$, on the left $a_1 = 0, b_1 = 10^3$; in the center $a_1 = 0, b_1 = 0$; on the right $a_1 = 10^3, b_1 = 0$

In this case of order 2, two types of configurations were highlighted.

For $a_1 \neq 0$ and $b_1 = 0$, we get a figure with 3 peaks in which height decreases when $a_1$ grows.

Conversely, for $b_1 \neq 0$ and $a_1 = 0$, we see the appearance of 6 peaks on a ring in which height decreases as $b_1$ grows.

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Fig. 3. Solution of order 2 to KPI for $t = 1$, on the left $a_1 = 0, b_1 = 0$; in the center $a_1 = 0, b_1 = 10^5$; on the right for $a_1 = 10^5, b_1 = 0$

Fig. 4. Solution of order 2 to KPI, on the left for $t = 10, a_1 = 0, b_1 = 10^5$; in the center for $a_1 = 0, b_1 = 0$; on the right for $a_1 = 10^5, b_1 = 0$

Fig. 5. Solution of order 2 to KPI, on the left for $t = 0, a_1 = 10^4, b_1 = 10^4$; in the center for $t = 1, a_1 = 10^5, b_1 = 10^5$; on the right for $t = 10, a_1 = 10^5, b_1 = 10^5$
For $a_1 \neq 0$ and $b_1 \neq 0$, we see the appearance of 3 couples of peaks on a ring in which height decreases as $a_1$ and $b_1$ grow.

4 Conclusion

a We have constructed rational solutions to KPI equation. To the best of our knowledge, all these solutions and configurations have never been presented before.

b We propose ourselves in a future research to give the general method to construct rational solutions to KPI equation at order $N$. We will show that these solutions can be written as a quotient of a polynomial of degree $2N(N+1)-2$ in $x$, $y$ and $t$ by a polynomial of degree $2N(N+1)$ in $x$, $y$ and $t$, depending on $2N-2$ parameters.

c The method described in the present paper provides a powerful tool to get explicit solutions to the KPI equation and to understand the behavior of solutions.

d These solutions are different from those derived from the NLS equation [16, 17, 18, 19].

e In both cases, peak modules are observed to decrease very rapidly with time $t$. What is new in the study of these solutions is the appearance not yet observed of 3 pairs of two peaks in the case of order 2. One can hope to discover other new configurations in the study of solutions for higher orders.

Competing Interests

Author has declared that no competing interests exist.

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