Is the phase transition in the Heisenberg model described by the 
\((2 + \epsilon)\)-expansion of the Nonlinear \(\sigma\)-Model?

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Abstract

Nonlinear \(\sigma\)-model is an ubiquitous model. In this paper, the \(O(N)\) model where the \(N\)-component spin is a unit vector, \(S^2 = 1\), is considered. The stability of this model with respect to gradient operators \((\partial_\mu S \cdot \partial_\nu S)^s\), where the degree \(s\) is arbitrary, is discussed. Explicit two-loop calculations within the scheme of \(\epsilon\)-expansion, where \(\epsilon = (d - 2)\), leads to the surprising result that these operators are relevant. In fact, the relevancy increases with the degree \(s\). We argue that this phenomenon in the \(O(N)\)-model actually reflects the failure of the perturbative analysis, that is, the \((2 + \epsilon)\)-expansion. It is likely that it is necessary to take into account non-perturbative effects if one wants to describe the phase transition of the Heisenberg model within the context of the non-linear \(\sigma\)-model. Thus, uncritical use of the \((2+\epsilon)\)-expansion may be misleading, especially for those cases for which there are not many independent checks.
I. INTRODUCTION

The nonlinear $\sigma$-model is an ubiquitous model that can describe systems ranging from quantum spins \[1\] to disordered electronic systems \[2\]. In the present paper we shall discuss this model with $O(N)$ symmetry. A particularly attractive method to solve this model is the $(2 + \epsilon)$-expansion \[3\], where the spatial dimensionality, $d$, defines $\epsilon = d - 2$. In this method, one takes advantage of the proximity of the non-trivial fixed point to the zero temperature fixed point in the limit $\epsilon \to 0$. One then expands around the zero temperature fixed point to obtain information about the non-trivial fixed point. Such arguments are clearly powerful, especially because there are not many explicit analytical techniques to solve this model. The purpose of the present paper is to examine this method more critically.

We begin by showing that, in the most natural definition of the problem, one must examine the role of gradient operators of degree higher than two, although, by power-counting, the higher order gradient terms are irrelevant. The task of this paper is to show that when fluctuations are taken into account, the higher order gradient operators become relevant, more so as the number of gradients increases. The simplest interpretation is that the fixed point is infinitely unstable, and the method fails. This surprising result was first discovered in a related model in the context of Anderson localization of an electron in a random potential \[4\]. However, we believe that this phenomenon in the $O(N)$-model actually reflects the failure of the perturbative analysis, that is of the $(2 + \epsilon)$-expansion. To appreciate this conclusion, consider the $O(3)$ Heisenberg model for which much is known from high temperature series expansion, accurate Monte-Carlo calculations, and from the $(4 - d)$-expansion of the $\phi^4$ field theory. None of these methods give any hint of any pathological behavior. Indeed, Kehrein, Wegner and Pismak have shown that in the $N$-component $\phi^4$ model the one-loop contributions always make the canonically irrelevant operators even more irrelevant \[5\]. Thus, it is very likely that it is necessary to take into account non-perturbative effects if one wants to describe this phase transition within the context of the non-linear $\sigma$-model. At the very least, the results obtained from this expansion should be taken with some caution, especially in those cases in which there are not many independent checks.

To one-loop order, the anomalous dimensions of the high gradient operators of the $O(N)$ model were first calculated by Wegner \[6\]. In an earlier paper \[7\], we briefly reported the corresponding two-loop results, and we showed that the relevance of the high-gradient operators persists to two-loop order. Since the method and the unexpected results may not be familiar, we have decided to give a full account in the present paper. We also analyze the renormalization group equations, and we discuss the physical consequences of the renormalization group flows.

The plan of the paper is as follows. In Sec. II, we define the high gradient operators, and, in Sec. III, we set up the background field method. In Sec. IV, we calculate the one-loop correction to the anomalous dimension to illustrate the efficacy of the background field method. In Sec. V, we present the two-loop calculation. In Sec. VI, we discuss the flows of the renormalization group equations. Sec. VII contains our conclusions, and there are four Appendices.
II. HIGH GRADIENT OPERATORS

It is useful to begin with a soft-spin model because, in many instances, the non-linear \( \sigma \)-model arises from a microscopic situation in which both the direction and the magnitude of the order parameter field is allowed to fluctuate. One then argues that as long as the \( O(N) \) symmetry is spontaneously broken, the interactions between the goldstone modes are precisely those given by the non-linear \( \sigma \)-model \[8\]. To be specific let us consider the \( O(N) \)-invariant \( \phi^4 \) field theory and explore how the high gradient operators arise. For this field theory, the action is given by

\[
S[\phi] = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} r \phi^2 + u(\phi^2)^2 \right].
\]  

(2.1)

Below the mean field transition, set

\[
\phi(x) = \rho(x) S(x), \quad S^2(x) = 1,
\]

(2.2)

and separate out the direction, \( S(x) \), of the order parameter field \( \phi(x) \) from its magnitude \( \rho(x) \). If we write

\[
\rho(x) = m + \Delta \rho(x),
\]

(2.3)

where \( m \) is the mean field value of the magnitude of the order parameter, not containing any loop corrections, \textit{i.e.}, fluctuations. Then

\[
S[\phi] = \int d^d x \left\{ \frac{1}{2} (m + \Delta \rho(x))^2 (\partial_\mu S)^2 + \frac{1}{2} (\partial_\mu \Delta \rho)^2 + \frac{1}{2} r (m + \Delta \rho(x))^2 + u(m + \Delta \rho(x))^4 \right\}
\]

(2.4)

In principle, we can integrate out the fluctuations of the magnitude of the order parameter, \( \Delta \rho(x) \). Because \( \rho(x) \) is a massive field, the effective action is local on scales larger than the inverse mass. It is now clear that, for scales larger than the inverse mass or the correlation length, the effective action will involve gradient operators to infinite order in a functional Taylor expansion. From symmetry, and from the locality of the action on scales larger than the correlation length associated with the fluctuations of the magnitude of the order parameter field, we find that the effective action can be cast in the form

\[
S_{\text{eff}} = \frac{1}{2T} \int d^d x \left\{ (\partial_\mu \vec{S})^2 + U_4 (\partial_\mu \vec{S})^4 + V_4 (\partial_\mu \vec{S} \cdot \partial_\nu \vec{S})^2 + \cdots \right\}
\]

(2.5)

where \( T, U_4, V_4, \ldots \), are the coupling constants; we are measuring the temperature \( T \) in units of the bare spin stiffness constant.

From power-counting, the gradient terms with powers larger than 2 are irrelevant and are usually dropped. If \( U_{2s} \), for \( s > 1 \), denotes the coupling constant associated with the operator \( (\partial_\mu S)^{2s} \), the full scaling dimension, \( y_{2s} \), of \( (U_{2s}/T) \) is defined by

\[
y_{2s} = 2(1 - s) + \epsilon + \text{loop corrections},
\]

(2.6)
where $\epsilon = (d - 2)$; the dimension $(1/T)$ is $\epsilon$, plus loop corrections. Hence, if we ignore the loop corrections, the contribution of the operator $(\partial_\mu S)^{2s}$ vanishes as $\Lambda^{-2s}$ as the momentum cutoff in the theory $\Lambda \to \infty$, for small $\epsilon$. In contrast, the second gradient term is relevant for $\epsilon > 0$ and marginal for $\epsilon = 0$.

The neglect of the higher-order derivative terms leads to the usual partition function of the nonlinear sigma model:

$$Z = \int D S \delta (S^2(x) - 1) \exp \left( -\frac{1}{2T} \int d^d x (\partial_\mu S)^2 \right).$$

This is also the na"ive continuum limit of the lattice Heisenberg model for which the Hamiltonian is

$$H/T = -\frac{1}{T} \sum_{<ij>} S_i \cdot S_j,$$

where the temperature is measured in units of the bare spin stiffness constant. There are well-known arguments that the lattice Heisenberg model is in the same universality class as the $N$-component $\phi^4$ field theory. Therefore, all three models, the nonlinear $\sigma$-model, the $N$-component $\phi^4$ field theory, and the lattice Heisenberg model of $N$-component unit vector spins should belong to the same universality class, sharing the same long-distance and critical properties.

However, power counting does not necessarily determine the effect of the operators. One must examine how loop corrections, or fluctuations, affect the picture. For the nonlinear $\sigma$ model, it may appear unlikely that the corrections to the dimensions of the high derivative terms coming from fluctuations can overcome the canonical dimension. However, we shall show that, for large $s$, not only the one-loop correction to the canonical dimension is sufficiently large to render the full dimension positive, but the two-loop correction is even larger. So, to two-loop order, the operators $(\partial_\mu S)^{2s}, s > 1$, are relevant, contrary to the common description of the nonlinear $\sigma$-model.

III. THE RIEMANNIAN MANIFOLD AND THE BACKGROUND FIELD METHOD

The present calculation is nearly impossible without an efficient method. In this section, we discuss a formalism that is efficient. The first step is to express the action as an invariant in the space of cosets $O(N)/O(N - 1)$ [2,10]. The length on this manifold is $ds^2 = d\sigma^2 + (d\pi^i)^2$, but because $S^2 = \sigma^2 + \pi^2 = 1$, we can eliminate $\sigma$ using $\sigma d\sigma + \pi \cdot d\pi = 0$. Therefore, the line element is

$$ds^2 = g_{ij}(\pi) d\pi^i d\pi^j,$$

where the metric, $g_{ij}(\pi)$, is

$$g_{ij}(\pi) = \delta_{ij} + \frac{\pi_i \pi_j}{1 - \pi^2}. \quad (3.2)$$
There is no unique way of choosing coordinates on this manifold. The set \( \{ \pi^i \} \) and the set \( \{ \tilde{\pi}^i \} \) will produce two different metrics, but they are related by the transformation equation

\[
g_{ij}(\pi) = \frac{\partial \tilde{\pi}^k}{\partial \pi^i} \frac{\partial \tilde{\pi}^l}{\partial \pi^j} g_{kl}(\tilde{\pi}).
\]

(3.3)

The action in Eq. (2.7) can now be written as

\[
S(\pi) = \frac{1}{2T} \int d^d x g_{ij}(\pi) \partial_\mu \pi^i \partial_\mu \pi^j.
\]

(3.4)

Similarly, the high derivative operators can be written as

\[
(\partial_\mu S \cdot \partial_\nu S) \cdots (\partial_\mu S \cdot \partial_\nu S) = (g_{ij} \partial_\mu \pi^i \partial_\nu \pi^j) \cdots (g_{pq} \partial_\mu \pi^p \partial_\nu \pi^q)
\]

(3.5)

Clearly, the high derivative operators and the action in Eq. (2.7) are invariant under the reparametrizations of the sphere, since \( \partial_\mu \pi^i \) transforms as a vector under reparametrization.

The invariant measure is

\[
[D\pi] = \prod_i \prod_x \sqrt{g(\pi)} d\pi^i(x),
\]

(3.6)

where \( g(\pi) = \det(g_{ij}) \).

We briefly recall the background field method [11]. The strength of this method is that covariant expressions can be handled easily and the explicit covariance can be maintained at each step of the calculation. This turns to be important for an efficient organization of the calculations to be described. There is another important reason for using this method. When the fluctuations around the background field are integrated out, operators arise that are not invariant under the reparametrizations of the sphere. One obtains operators that are proportional to the classical equation of motion

\[
\frac{\delta S}{\delta \pi_i} = \partial_\mu \partial^\mu \pi^i + \Gamma^i_{jk} \partial_\mu \pi^j \partial^\mu \pi^k;
\]

(3.7)

where

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} [\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}]
\]

(3.8)

are the Christoffel symbols. Since \( \Gamma^i_{jk} \) does not transform as a tensor, the equation of motion is not invariant, and the operators proportional to the equation of motion are not invariant quantities. These are the redundant operators [12] that do not affect the critical properties and can be removed by the reparametrizations of the sphere. Let us consider the shift

\[
\pi \to \pi + \theta f(\pi),
\]

(3.9)

in the partition function. Here \( f \) is a smooth function, and \( \theta \) is an infinitesimal parameter. Then, the redundant operator, \( O_{\text{red}} \), is

\[
O_{\text{red}} = \int d^d x \left\{ f \frac{\delta S}{\delta \pi} - \frac{\delta f}{\delta \pi} \right\}.
\]

(3.10)
The first term comes from the action and the second from the measure. As stated above, the redundant operators can be removed by a reparametrization. We adopt dimensional regularization and the minimal subtraction scheme for the calculation of the renormalization constants. In the dimensional regularization scheme, the contribution of the measure term can be set to zero, and the redundant operators disappear if the equation of motion is satisfied, which is an advantage of using the background field method.

The anomalous dimension of a composite operator $O(x)$ is computed by computing the divergences created by the insertion in the correlation function, defined by

$$\Gamma^{(n)}_O = \langle O(x) \pi(x_1) \cdots \pi(x_n) \rangle. \quad (3.11)$$

The divergence of the correlation function is not only proportional to the inserted operator, but it is usually a linear combination of a number of other operators that are said to mix with the inserted operator. The renormalized operator $O_i^R$ is defined as

$$O_i^R = \sum_j Z_{ij} O_j, \quad (3.12)$$

where $Z_{ij}$ are the renormalization constants. The renormalized correlation function $\Gamma^{(n)}_{O,R}$ in the momentum space is given by

$$\Gamma^{(n)}_{O_i,R}(q,p_i,g,\mu) = Z^{(n)} Z_{ij} \Gamma^{(n)}_{O_j}(q,p_i,g_0), \quad (3.13)$$

where $g_0$ is the bare coupling, the temperature $T$, and $Z$ is the wave function renormalization of the field. This leads to the renormalization group equation

$$\left[ \left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \gamma(g) \right) + \gamma_{ij}(g) \right] \Gamma^{(n)}_{O_j,R}(q,p_i,g,\mu) = 0. \quad (3.14)$$

The quantity $\gamma(g)$ is the anomalous dimension of the field, and $\gamma_{ij}$ is the anomalous dimension (matrix) corresponding to the operator $O$, and it is defined by

$$\gamma_{ij}^O = - \sum_k (Z^{-1})_{ik} \left( \mu \frac{d}{d\mu} \right) Z_{kj}, \quad (3.15)$$

where the derivative is computed with fixed bare couplings. One then diagonalizes the matrix $\gamma^O$ at the fixed point $g = g^*$ and obtains

$$\left[ \mu \frac{\partial}{\partial \mu} - \frac{n}{2} \eta + \gamma_\alpha(g^*) \right] \Gamma^{(n)}_{O_{\alpha,R}}(q,p_i,g^*,\mu) = 0, \quad (3.16)$$

where $O_\alpha$ is an eigenoperator. The quantities $\gamma_\alpha$ are the eigenvalues associated with the eigenoperators $O_\alpha$, and $\eta = \gamma(g^*)$ is the anomalous dimension of the field $\pi(x)$. The solution of this equation leads to the behavior

$$\lim_{\rho \to \infty} \Gamma^{(n)}_{O_{\alpha,R}}(q,p_i,g^*,\mu) = \rho^D \rho^{-y_\alpha} \Gamma^{(n)}_{O_{\alpha,R}}(q/\rho, p_i/\rho), \quad (3.17)$$

where $D$ is the usual dimension of the correlation function without the operator insertion, and $y_\alpha$ is given by
\[ y_\alpha = d - \left[ O_\alpha \right] - \gamma_\alpha (g^*), \] 

(3.18)

where \([O_\alpha]\) is the engineering dimension of the operator \(O_\alpha\). Of course, the largest eigenvalue of the matrix \(\gamma_{ij}(g^*)\) controls the critical behavior of the operators \(O_i\).

In the background field method [13], the initial step is to write the field \(\pi(x)\) in terms of new fields:

\[ \pi(x) = \psi(x) + \eta(x). \] 

(3.19)

The field \(\psi(x)\) is chosen to satisfy the classical equation of motion, and \(\eta(x)\) represents quantum fluctuations. If we substitute this decomposition in the action and in the definition of the composite operator and expand in powers of \(\eta\), the expansion is not manifestly covariant. To obtain a covariant expansion, the field \(\eta\) is written in terms of normal coordinates \(\xi^i\) defined by the connection \(\Gamma^k_{ij}\) in Eq. (3.8). The normal coordinate \(\xi^i\) is the tangent vector to the geodesic of the field manifold between \(\psi(x)\) and \(\pi(x)\) with a magnitude equal to the geodesic distance. In terms of the coordinates \(\psi\) and \(\pi\), the normal coordinates, \(\xi\), have the expansion [13]

\[ \xi = \pi - \psi + \frac{1}{2} \Gamma^i_{mn} (\pi - \psi)^m (\pi - \psi)^n + \cdots \] 

(3.20)

A systematic method to find the normal coordinate expansion of an arbitrary operator is given in Ref. [13]. First, we note that for the manifold \(O(N)/O(N - 1)\), \(\Gamma^k_{ij} = \pi^k g_{ij}\), and the curvature tensor \(R^m_{npq}\) is defined by the equation:

\[ R^m_{npq} = \partial_p \Gamma^m_{nq} + \Gamma^i_{nq} \Gamma^m_{ip} - [p \leftrightarrow q]. \] 

(3.21)

Therefore,

\[ R^m_{npq} = g_{mq} g_{np} - g_{pq} g_{mn}, \] 

(3.22)

\[ \nabla_m R_{pqrs} = 0, \] 

(3.23)

where \(\nabla_m\) is the usual covariant derivative on the manifold.

For simplicity, let us introduce the notation

\[ G_{\mu\nu}(\pi) \equiv g_{mn} \partial_\mu \pi^m \partial_\nu \pi^n, \] 

(3.24)

and we use \([G_{\mu\nu}(\pi)]_{\xi^n}\) to denote the \(n\)th term of the Taylor expansion of the operator \(G_{\mu\nu}(\pi)\) in terms of the normal coordinates \(\xi\). With this notation, the first four terms of the normal coordinate expansion are

\[ [G_{\mu\nu}(\pi)]_{\xi^1} = g_{mn} (\psi) D_\mu \psi^m \partial_\nu \xi^n + (\mu \leftrightarrow \nu), \] 

(3.25)

\[ [G_{\mu\nu}(\pi)]_{\xi^2} = g_{mn} (\psi) D_\mu \xi^m D_\nu \xi^n + R_{mnij}(\psi) \partial_\mu \psi^m \partial_\nu \psi^n \xi^i \xi^j, \] 

(3.26)

\[ [G_{\mu\nu}(\pi)]_{\xi^3} = \frac{1}{6} R_{mijk}(\psi) \partial_\mu \psi^m D_\nu \xi^i \xi^j + (\mu \leftrightarrow \nu), \] 

(3.27)

\[ [G_{\mu\nu}(\pi)]_{\xi^4} = \frac{1}{6} R^p_{ijkl}(\psi) R_{pklm}(\psi) \partial_\mu \psi^m \partial_\nu \psi^n + R_{mnij} D_\mu \xi^m D_\nu \xi^n \xi^i \xi^j; \] 

(3.28)
where the covariant derivative $D_\mu$ is defined by

$$D_\mu \xi^m \equiv \partial_\mu \xi^m + \Gamma^m_{st} \partial_\mu \psi^s \xi^t,$$

and we have used Eq. (3.23). The expansion of an invariant quantity in terms of the normal coordinates is manifestly reparametrization invariant: the coefficients that multiply the monomials $\xi^m(x)$ are tensors. Note that these coefficients, the curvature tensor and the metric, depend on the classical field $\psi(x)$ and are not affected by the integration over the quantum field, $\xi$.

The divergences that determine the renormalization constants $Z_{ij}$ are obtained by integrating over the quantum field $\xi$. More specifically, we calculate the one-particle irreducible diagrams of the expectation value

$$\langle O(e)(\psi, \xi) \rangle = \frac{\int [d\xi] O(e)(\psi, \xi) e^{-S(e)(\psi, \xi)}}{\int [d\xi] e^{-S(e)(\psi, \xi)}},$$

where $O(e)$ and $S(e)$ stand for all the terms of $O(\xi^n)$ with $n \geq 2$ of the expansion in terms of the normal coordinates. The expansion of $e^{-S(e)}$ in powers of $\xi$ and the integration $[d\xi]$ generates the diagramatic expansion.

In order to calculate the Feynmann diagrams generated from Eq. (3.30), it is necessary to compute the propagator of the field $\xi$. However, when we use Eq. (3.26) and Eq. (3.29) to obtain the noninteracting part of the action, we find that

$$[S]_\xi = \frac{1}{2} \int dx \left\{ g_{mn} D_\mu \xi^m D^\mu \xi^n + R_{iklj} \partial_\mu \psi^i \partial_\nu \psi^j \xi^k \xi^l \right\}$$

$$= \frac{1}{2} \int dx \left\{ g_{mn} \partial_\mu \xi^m \partial_\nu \xi^n + \ldots \right\}.$$  (3.31)

This leads to a complicated propagator that depends on the metric. Although it is possible to continue with the calculation, it is simpler to perform another transformation of coordinates to obtain the more common propagator. The transformation is between the curved system of coordinates to the tangent system of coordinates:

$$\xi^a = e^a_i(\psi) \xi^i,$$

$$\xi^i = e^a_i \xi^a,$$

$$e^a_i(\psi) g_{ij}(\psi) e^b_j(\psi) = \delta^{ab}.$$  (3.34)

where the local matrix $e^a_i$ is known as the Vielbein. Here, we follow the convention of using the earlier letters of the latin alphabet (a, b, etc.) for the local indices and the latter indices (i, j, etc.) for the covariant indices. Furthermore, $\delta^{ab}$ is the diagonal matrix with the diagonal elements (1, 1, ..., 1), so there is no distinction between covariant and contravariant indices.

In terms of these local coordinates, the covariant derivative becomes

$$D_\mu \xi^a = e^a_m D_\mu \xi^m,$$

$$= \partial_\mu \xi^a + A(\psi)_\mu^b \psi^b \xi^a.$$  (3.35)
where the quantity $A(\psi)_{\mu}^{ab}$ has dimension unity and transforms as a gauge field under the rotations of the tangent frames defined by $e_i^a$. So, the only way in which $A(\psi)_{\mu}^{ab}$ appears in the calculation is through the field strength $F_{\mu\nu}^{ab}$, which is

$$F_{\mu\nu}^{ab} = \partial_\mu A_{\nu}^{ab} - \partial_\nu A_{\mu}^{ab} + A_{\mu}^{ac}A_{\nu}^{cb} - A_{\nu}^{ac}A_{\mu}^{cb} = e_i^a e_j^b R_{i\mu j\nu} \partial_\mu \psi^m \partial_\nu \psi^n.$$  

(3.36)

The substitution of the local coordinates $\xi^a$ in Eq. (3.31) yields

$$\langle S \rangle^{\xi^2} = \frac{1}{2} \int dx \left\{ D_\mu \xi^a D_\mu \xi^a + e_i^a e_j^b R_{i\mu j\nu} \partial_\mu \psi^m \partial_\nu \psi^n \xi^a \xi^b \right\}$$

$$= \frac{1}{2} \int dx \left\{ (\partial_\mu \xi^a + A_{\mu}^{ab} \xi^b)(\partial_\mu \xi^a + A_{\mu}^{ab} \xi^b) + R_{i\mu j\nu} \partial_\mu \psi^m \partial_\nu \psi^n \xi^a \xi^b \right\}$$  

(3.37)

from which we obtain the usual propagator

$$\langle \xi^a(x) \xi^b(y) \rangle = \delta^{ab} \int dp \frac{e^{ip(x-y)}}{p^2 + m^2},$$  

(3.38)

In this equation $m^2$ is an infrared cutoff.

**IV. THE ONE LOOP CALCULATION**

We now compute the one-loop correction to demonstrate how the background field method works and to reproduce the results obtained by Wegner [5]. A simple rescaling of the field $\xi$ shows that, to one-loop order, we only need to expand the action and the operator up to order $O(\xi^2)$. Using the notation introduced in the previous section, we have to compute the one-particle irreducible diagrams corresponding to the expectation value

$$\langle \left[ G^s_{\mu\nu}(\pi) \right]^{\xi^2} \rangle = \frac{\int [d\xi] \left[ G^s_{\mu\nu}(\pi) \right]^{\xi^2} e^{[S(\pi)]^{\xi^2}}}{\int [d\xi] e^{[S(\pi)]^{\xi^2}}}$$  

(4.1)

Consider the simplest case $s = 2$. Then

$$[G^2_{\mu\nu}(\pi)]^{\xi^2} = G_{\mu\nu}(\psi) [G_{\alpha\alpha}(\pi)]^{\xi^2} + [G_{\mu\nu}(\pi)]^{\xi^1} [G_{\alpha\alpha}(\pi)]^{\xi^1} +$$

$$+ [G_{\mu\nu}(\pi)]^{\xi^2} G_{\alpha\alpha}(\psi),$$  

(4.2)

where the expression for $[G_{\mu\nu}]^{\xi^2}$ is given in Eq. (3.26) and, the term $[G_{\mu\nu}(\pi)]^{\xi^1} [G_{\alpha\alpha}(\pi)]^{\xi^1}$ can be found from Eq. (3.25):

$$[G_{\mu\nu}(\pi)]^{\xi^1} [G_{\alpha\beta}(\pi)]^{\xi^1} = e_m e_i^a \partial_\mu \psi^m \partial_\alpha \psi^b D_\nu \xi^c D_\alpha \xi^a$$

$$+ (\alpha \leftrightarrow \beta) + (\mu \leftrightarrow \nu)$$

$$+ (\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu).$$  

(4.3)

Note that the metric has been written in terms of the Vielbeins. Substituting in Eq. (4.1) and remembering that $\psi(x)$ is a classical field, we get
\[
\langle [G_{\mu\nu}^2(\pi)]_{\xi^2} \rangle = G_{\mu\mu}(\psi)\langle [G_{\alpha\alpha}(\pi)]_{\xi^2} \rangle + G_{\mu\mu}(\psi)\langle [G_{\alpha\alpha}(\pi)]_{\xi^2} \rangle \\
+ \langle [G_{\mu\mu}(\pi)]_{\xi^1} [G_{\alpha\alpha}(\pi)]_{\xi^1} \rangle.
\]

The extraction of the divergences of the one-loop diagrams is carried out with the help of the dimensional regularization scheme; the details are given in Appendix A. We find

\[
\langle [G_{\mu\mu}(\pi)]_{\xi^2} \rangle_W = -\frac{1}{2\pi\epsilon} R_{mn}[\partial_\mu \psi^m \partial_\nu \psi^n - \frac{\delta_{\mu\nu}}{2} \partial_\mu \psi^n \partial_\nu \psi^n],
\]

\[
\langle [G_{\mu\nu}(\pi)]_{\xi^1}[G_{\alpha\beta}(\pi)]_{\xi^1} \rangle_W = \frac{1}{4\pi\epsilon} R_{mpq}[\epsilon_{\mu\alpha\delta_\nu\psi^m \partial_\gamma \psi^n \partial_\rho \psi^\beta \partial_\sigma \psi^q + \partial_\mu \psi^m \partial_\nu \psi^n \partial_\rho \psi^p \partial_\sigma \psi^q] + \\
(\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta) + (\mu \leftrightarrow \nu, \alpha \leftrightarrow \beta).
\]

For the manifold \(O(N)/O(N - 1)\), we can substitute the expression for \(R_{mpq}\) given earlier and obtain

\[
\langle [G_{\mu\nu}(\pi)]_{\xi^2} \rangle_W = \frac{N - 2}{2\pi\epsilon} [G_{\mu\nu}(\psi) - \frac{\delta_{\mu\nu}}{2} G_{\alpha\alpha}(\psi)],
\]

\[
\langle [G_{\mu\nu}(\pi)]_{\xi^1}[G_{\alpha\beta}(\pi)]_{\xi^1} \rangle_W = 2G_{\mu\nu}(\psi)G_{\alpha\beta}(\psi) - [G_{\mu\alpha}(\psi)G_{\nu\beta}(\psi) + (\mu \leftrightarrow \nu)] \\
+ \left\{ \frac{\delta_{\mu\alpha}}{2} \left[ (G_{\gamma\gamma}(\psi)G_{\nu\beta} - G_{\nu\gamma}(\psi)G_{\beta\gamma}) \right] \right\} \\
+ \{\alpha \leftrightarrow \beta\} + \{\mu \leftrightarrow \nu\} + \{\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu\}.
\]

These equations are identical to those obtained by Wegner [5]. Note that the operator \(G_{\mu\nu}^2(\pi)\) mixes with the operator \(G_{\mu\beta}G_{\beta\nu}\). Furthermore, it also follows that the operators that mix with \(G_{\mu_1\nu_1}(\pi)\ldots G_{\mu_n\nu_n}(\pi)\) are the cyclic products of the form

\[
O^{\text{cyc}} \equiv (g_{ij} \partial_\alpha \pi^i \partial_\beta \pi^j)(g_{mn} \partial_\beta \pi^m \partial_\rho \pi^n)\ldots (g_{kt} \partial_\nu \pi^k \partial_\alpha \pi^l)
\]

This is expected since within the dimensional regularization scheme, the operators that mix have the same symmetry and the same dimension. Surprisingly, the operators

\[
(g_{ij}(\partial_{\mu_1})_{m_1} \pi^i (\partial_{\mu_2})_{m_2} \pi^j)(g_{kl}(\partial_{\mu_3})_{m_3} \pi^k (\partial_{\mu_4})_{m_4} \pi^l)\ldots,
\]

with \(m_1 + m_2 + m_3 + \cdots = 2s\) do not mix, even though they have the same dimension, and are also invariant.

We shall be studying the renormalization of the operators \(O^{\text{cyc}}\). Let us imagine that there are \(N\) such operators denoted by the set \(\{O^{\text{cyc}}_i\}\). We shall denote all other operators by \(P\), and let there be \(M\) such operators. Then, the renormalized operators can be expressed as

\[
O^R_i = \sum_{j=1}^{N} C_{ij} O_j + \sum_{j=1}^{M} E_{ij} P_j,
\]

\[
P^R_i = \sum_{j=1}^{N} F_{ij} O_j + \sum_{j=1}^{M} D_{ij} P_j.
\]
The one-loop matrix $Z_{ij}^{(1)}$ for the invariant operators with $2s$ gradients will have the form

$$Z^{(1)}(\epsilon, t) = \begin{bmatrix} C^{(1)} & 0 \\ 0 & D^{(1)} \end{bmatrix}. \tag{4.12}$$

Wegner \cite{5} has shown that the largest eigenvalue comes from the matrix $C^{(1)}$, and consequently we shall not concern ourselves with the matrix $D^{(1)}$.

Although we know the form of the operators that mix under renormalization, it is still a difficult task to diagonalize the matrix $C_{ij}^{(1)}$ since we do not have a simple way of classifying the operators. Near two dimensions the problem can be solved if we introduce conformal coordinates \cite{4}, so that

$$\partial_{+} = \partial_x + i \partial_y, \quad \partial_{-} = \partial_x - i \partial_y. \tag{4.13}$$

In terms of these new coordinates, the divergences in Eq. (4.7) and Eq. (4.8) take very simple forms:

$$\langle [G_{++}(\pi)]_{\xi^2} \rangle = \frac{(N-2)}{2\pi \epsilon} G_{++}(\psi), \tag{4.14}$$

$$\langle [G_{--}(\pi)]_{\xi^2} \rangle = \frac{(N-2)}{2\pi \epsilon} G_{--}(\psi), \tag{4.15}$$

$$\langle [G_{+-}(\pi)]_{\xi^1} [G_{+-}(\pi)]_{\xi^1} \rangle = -\frac{1}{\pi \epsilon}[G_{+-}(\psi)^2 - G_{++}(\psi)G_{--}(\psi)], \tag{4.16}$$

while all other possibilities vanish. A vanishing result simply means that the expectation value is free from divergences.

Let us now introduce the following notations:

$$H \equiv g_{kl}\partial_{+} \pi^k \partial_{-} \pi^l, \tag{4.17}$$

$$A \equiv g_{kl}\partial_{+} \pi^k \partial_{+} \pi^l, \tag{4.18}$$

$$B \equiv g_{kl}\partial_{-} \pi^k \partial_{-} \pi^l. \tag{4.19}$$

We now derive the important result that the operator $H^s$ mixes only with the set of operators

$$\left\{ H^s, H^{s-2}(AB), H^{s-4}(AB)^2, \ldots, H^2(AB)^{\frac{s-1}{2}}, (AB)^{\frac{s}{2}} \right\}. \tag{4.20}$$

Let us denote the normal coordinate expansion of an operator by

$$Q(\pi) = Q(\psi) + q_1(\psi, \xi) + q_2(\psi, \xi) + \cdots \tag{4.21}$$

and the one-loop contribution by

$$\langle Q(\pi) \rangle_{1L} = \langle q_2(\psi, \xi) \rangle. \tag{4.22}$$

The last equation follows because only the quadratic term contributes to one-loop order, and the average implies an integration over the normal coordinates. From Eqs. (4.14–4.16), it follows that
\[ \langle a_2(\psi, \xi) \rangle_{1L} = \nu I A(\psi), \]  \hspace{1cm} (4.22)  
\[ \langle b_2(\psi, \xi) \rangle_{1L} = \nu I B(\psi), \]  \hspace{1cm} (4.23)  
\[ \langle h_1(\psi, \xi) h_1(\psi, \xi) \rangle_{1L} = -2I \left[ H^2(\psi) - A(\psi) B(\psi) \right], \]  \hspace{1cm} (4.24)

where \( \nu = (N - 2) \) and \( I = 1/2\pi \epsilon \).

The one-loop computation, using Eqs. (4.22, 4.24), shows that
\[ \langle H^s \rangle_{1L} = -s(s - 1)I \left[ H^s(\psi) - H^{s-2}A(\psi) B(\psi) \right]. \]  \hspace{1cm} (4.25)  

Similarly,
\[ \langle H^{s-2}AB \rangle_{1L} = I H^{s-2}(\psi) A(\psi) B(\psi) \left[ (s - 1)(s - 2) - 2\nu \right] - I(s - 1)(s - 2) H^{s-4}(\psi) [A(\psi) B(\psi)]^2. \]  \hspace{1cm} (4.26)

Recursively, one can show that \( H^s \) mixes only with the operators listed above. It is now easy to show that
\[ \langle (AB)^l H^{s-2l} \rangle_{1L} = -I H^{s-2l}(\psi) [A(\psi) B(\psi)]^l \left[ (s - 2l)(s - 2l - 1) - 2\nu l \right] + I(s - 2l)(s - 2l - 1) H^{s-2l-2}(\psi) [A(\psi) B(\psi)]^{l+1}. \]  \hspace{1cm} (4.27)

From these equations we can now derive the renormalization matrix \( Z^{(1)}(t, \epsilon) \).

Using Eq. (4.27), we can read off the matrix \( Z_{ij}(t, \epsilon) \), which is a \([\frac{d}{2}] \times [\frac{d}{2}] \) matrix:
\[
Z^{(1)}(\epsilon, t) - 1 = \frac{t}{2\pi \epsilon} \begin{bmatrix}
  a_{11} & a_{12} & 0 & 0 & \cdots \\
  0 & a_{22} & a_{23} & 0 & \cdots \\
  0 & 0 & a_{33} & a_{34} & 0 \\
  0 & 0 & 0 & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix},
\]

where
\[ a_{j+1j+1} = (s - 2j)(s - 2j - 1) - 2\nu j, \]  \hspace{1cm} (4.28)  
\[ a_{j+1j+2} = -(s - 2j)(s - 2j - 1), \]  \hspace{1cm} (4.29)

\( j = 0, 1, 2, \ldots, [\frac{d}{2}] \). Note that there is a change in sign because to calculate \( Z \) we have to subtract a divergence. Because this matrix is upper triangular, the diagonal elements are the eigenvalues. The largest eigenvalue is the element \( a_{11} \). Using Eq. (3.13) and (3.18), we get
\[ y^{(1)}_{2s} = d - 2s - \gamma^{(1)}(\epsilon) \]  \hspace{1cm} (4.30)
\[ = 2 + \epsilon \left( 1 + \frac{s(s - 1)}{N - 2} \right), \]  \hspace{1cm} (4.31)

where we have substituted the well-known one-loop value of the fixed point \( t^* = 2\pi \epsilon / \nu \).

Therefore, for sufficiently large \( s \), regardless of how small \( \epsilon \) is, an infinite number of high gradient operators become relevant.
For the two-loop calculation we shall also need the eigenvectors of the matrix $C^{(1)}$. They are compactly contained in the matrix $S$ that diagonalizes $C^{(1)}$. The matrix $S$ is

$$S = \begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} & \cdots \\
0 & 1 & x_{23} & x_{24} & \cdots \\
0 & 0 & 1 & x_{34} & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \quad (4.32)$$

where

$$x_{ii+l} = \frac{a_{ii+1}a_{i+1i+2} \cdots a_{i+l-1i+l}}{(a_{ii} - a_{i+l+1})(a_{ii} - a_{i+2i+2}) \cdots (a_{ii} - a_{i+li+l})} \quad (4.33)$$

It is easily verified that $SC^{(1)}S^{-1}$ is diagonal and the diagonal elements are the diagonal elements of the matrix $C^{(1)}$.

**V. THE TWO LOOP CALCULATION**

The two-loop calculation is more involved. We need to subtract the subdivergences from the two-loop diagrams and to regulate the infrared divergences. The method for subtracting divergences that we follow is due to Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ). This method consists of subtracting directly the subdivergences from each Feynannnn diagram by using the forest formula of Zimmerman \[15\]. For a complete discussion of the forest formula and detailed examples, we refer the reader to Collins \[16\]. With respect to infrared divergences, their presence is due to the absence of a mass term in the action. Infrared divergences make the computation of the loop integrals ambiguous when dimensional regularization is used. The poles due to the ultraviolet and the infrared divergences tend to cancel each other, leading in some cases to a vanishing result. The easiest way to solve this problem is to introduce an infrared cutoff in every propagator, \textit{i.e.}, we let $p^2 \rightarrow p^2 + m^2$ in the internal lines of a diagram. Of course, other choices of cutoff should not affect the final result since the dependence on the cutoff appears only in terms corresponding to subdivergences. These terms are eventually subtracted with the forest formula.

As with the one-loop calculation, we begin by considering the normal coordinate expansion of the operator $(G_{\mu\nu})^s$ and of the action. From a simple rescaling of the fields we learn that the expansion in terms of the normal coordinate needs to be carried out to order $O(\xi^4)$. This time, a calculation of the divergences for the case $s = 4$ is sufficient to determine the divergences for arbitrary $s$. The different possibilities that arise in the expansion of $(G_{\mu\nu})^4$ are:

$$O(\xi^2) : [G_{\mu\nu}(\pi)]_{\xi^2}, [G_{\mu\nu}(\pi)]_{\xi^1}[G_{\alpha\beta}(\pi)]_{\xi^1}; \quad (5.1)$$

$$O(\xi^3) : [G_{\mu\nu}(\pi)]_{\xi^3}, [G_{\mu\nu}(\pi)]_{\xi^2}[G_{\alpha\beta}(\pi)]_{\xi^1},$$

$$[G_{\mu\nu}(\pi)]_{\xi^1}[G_{\alpha\beta}(\pi)]_{\xi^1}[G_{\eta\rho}]_{\xi^1}; \quad (5.2)$$

$$O(\xi^4) : [G_{\mu\nu}(\pi)]_{\xi^4}, [G_{\mu\nu}(\pi)]_{\xi^3}[G_{\alpha\beta}(\pi)]_{\xi^2},$$

$$[G_{\mu\nu}(\pi)]_{\xi^2}[G_{\alpha\beta}(\pi)]_{\xi^1}[G_{\eta\rho}]_{\xi^2}, [G_{\mu\nu}(\pi)]_{\xi^1}[G_{\alpha\beta}(\pi)]_{\xi^1}[G_{\eta\rho}]_{\xi^1}[G_{\eta\rho}]_{\xi^1}. \quad (5.3)$$
Except for the possibility of an arbitrary number of two-point insertions in the internal lines, Fig. 1 shows schematically the two types of diagrams that arise in the two-loop calculation.

FIG. 1. The general form of the two-loop diagrams. The wiggly lines represent functions that depend on the background field $\psi$, which is a classical field. Diagrams of type (a) lead to contributions proportional to double poles in $\epsilon$ and are therefore not necessary for the computation of anomalous dimension. Only graphs of type (b) need to be considered.

In Appendix B it is shown that the graphs of Fig. 1(a) are either finite or lead to double poles in $\epsilon$ after the forest formula is applied to them. Hence, this type of graph plays no role in determining critical exponents. Moreover, the term of order $O(\xi^4)$ appearing in the expansion of $(G_{\mu\nu})^s$, and in the action, can be discarded because the two-loop diagrams generated by the 4-point vertices are of the type shown in Fig. 1(a). In view of this result, the two-loop calculation requires only to carry out the expansion of an operator $Q(\pi)$ in terms of normal coordinates to order $O(\xi^3)$,

$$Q(\pi) = Q(\psi) + q_1(\psi, \xi) + q_2(\psi, \xi) + q_3(\psi, \xi) + \cdots,$$

and the only expectation values needed are:

$$O(\xi^2) : \langle [G_{\mu\nu}]_{\xi^2} \rangle, \langle [G_{\mu\nu}]_{\xi^1} [G_{\alpha\beta}]_{\xi^1} \rangle$$

$$O(\xi^3) : \langle [G_{\mu\nu}]_{\xi^1} [G_{\alpha\beta}]_{\xi^1} \rangle, \langle [G_{\mu\nu}]_{\xi^1} [G_{\alpha\beta}]_{\xi^1} [G_{\kappa\lambda}]_{\xi^1} \rangle$$

In principle, it is also necessary to calculate the divergences of the operators in Eq. (4.10) as they may mix with the operators we are considering. However, we shall show that this mixing may be neglected without affecting our conclusions. A detailed calculation of the expectation value $\langle [G_{\mu_1\nu_1}]_{\xi^1} [G_{\mu_2\nu_2}]_{\xi^2} \rangle$ is given in Appendix B.

We emphasize again that we shall study only the renormalization of the set of operators

$$\left\{ H^s, H^{s-2}(AB), H^{s-4}(AB)^2, \cdots, H^2(AB)^{\frac{s}{2}-1}, (AB)^{\frac{s}{2}} \right\}.$$

The calculation is tedious, but the essential pieces are given in Appendix D. We find that

$$\langle H^s \rangle_{2L} = -\frac{H^s(\psi)}{\omega} \left[ 3\nu s + \frac{s(s-1)}{2}(\nu+6) + 2s(s-1)(s-2) \right]$$

$$+ \frac{H^{s-2}(\psi)A(\psi)B(\psi)}{\omega} \left[ \frac{s(s-1)}{2}(3-2 \nu) + 2s(s-1)(s-2) \right],$$

where $\omega = 24\pi^2\epsilon$. Similarly,
\[(H^{s-2}AB)_{2L} = \frac{H^s(\psi)}{\omega} [9 - \nu + 14(s - 2)]
+ \frac{H^{s-2}(\psi)A(\psi)B(\psi)}{\omega} \left[ (7\nu - 9) - (s - 2)(9\nu + 14) - \frac{1}{2}(s - 2)(s - 3)(\nu + 2) - 2(s - 2)(s - 3)(s - 4) \right]
+ \frac{H^{s-4}(\psi) (A(\psi) B(\psi))^2}{\omega} \left[ \frac{1}{2}(s - 2)(s - 3)(2 - \frac{13\nu}{2}) \right.
+ 2(s - 2)(s - 3)(s - 4) \right]. \tag{5.8}
\]

The general case is even more tedious, but the necessary ingredients are given in Appendix D. We find that
\[
\langle H^{s-2}AB \rangle_{2L} = r'_{l,l-1} H^{s-2l+2}(\psi)(A(\psi) B(\psi))^{l-1} + r'_{l,l} H^{s-2l}(\psi)(A(\psi) B(\psi))^l
+ r'_{l,l+1} H^{s-2l-2}(\psi)(A(\psi) B(\psi))^{l+1} + O\left(\frac{1}{\epsilon^2}\right) + \text{other operators}, \tag{5.9}
\]
where
\[
r'_{l,l} = -\frac{1}{\omega} \left[ 2(s - 2l)(s - 2l - 1)(s - 2l - 2) + 14l^2(s - 2l)
+ 2l(s - 2l)(s - 2l - 1) + 4l^2(l - 1) + \frac{1}{2}(s - 2l)(s - 2l - 1)(\nu + 6) + 6\nu l(s - 2l)
- l^2(\nu - 9) - 6\nu l - 3\nu l(l - 1) \right], \tag{5.10}
\]
\[
r'_{l,l+1} = \frac{1}{\omega} \left[ 2(s - 2l)(s - 2l - 1)(s - 2l - 2) - 2l(s - 2l)(s - 2l - 1),
+ \frac{1}{2}(s - 2l)(s - 2l - 1)(s - \frac{13}{2}\nu) \right] \tag{5.11}
\]
\[
r'_{l,l-1} = \frac{1}{\omega} \left[ l^2(9 - \nu) + 4l^2(s - 2l) + 4l^2(l - 1) \right]. \tag{5.12}
\]

The matrix \(\gamma\) in Eq. (3.13) can now be written in the form
\[
\gamma(t, \epsilon) = -at - 2bt^2 + O(t^3) \tag{5.13}
\]
where the matrices \(a\) and \(b\) are
\[
a = \begin{pmatrix} C^{(1)} & 0 \\ 0 & D^{(1)} \end{pmatrix}, \quad b = \begin{pmatrix} C^{(2)} \\ X \\ Y \\ D^{(2)} \end{pmatrix}. \tag{5.14}
\]

The matrix \(C^{(1)}\) was obtained in the previous section. From Eqs. (5.10, 5.12) we find that the matrix \(C^{(2)}\) is...
\[
C^{(2)} = \frac{1}{12\pi^2} \begin{bmatrix}
  r_{11} & r_{12} & 0 & 0 & 0 & \cdots \\
  r_{21} & r_{22} & 0 & 0 & 0 & \cdots \\
  0 & r_{32} & r_{33} & 0 & 0 & \cdots \\
  0 & 0 & r_{43} & r_{44} & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

(5.15)

where

\[
\begin{align*}
  r_{j+1,j+1} &= -\omega r_{j,j}, \\
  r_{j+1,j+2} &= -\omega r_{j,j+1}, \\
  r_{j+1,j} &= -\omega r_{j-1,j},
\end{align*}
\]

(5.16, 5.17, 5.18)

and \( j = 0, 1, \ldots, \left[ \frac{\nu}{\epsilon} \right] \), \( \nu \equiv N - 2 \).

To calculate the anomalous dimension correct to two-loop order, it is sufficient to use elementary first order nondegenerate perturbation theory since the eigenvalues of \( C^{(1)} \) are nondegenerate. Let \( U \) be the transformation matrix that diagonalizes the one loop matrix, where

\[
U = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}.
\]

(5.19)

The matrix \( S \) was given in the previous section. Therefore,

\[
U\gamma U^{-1} = -t \begin{pmatrix} SC^{(1)}S^{-1} & 0 \\ 0 & TD^{(1)}T^{-1} \end{pmatrix} - 2t^2 \begin{pmatrix} SC^{(2)}S^{-1} & SXT^{-1} \\ S^{-1}Y^{(1)}T & TD^{(2)}T^{-1} \end{pmatrix}.
\]

(5.20)

The two-loop correction to the anomalous dimension \( y_{2s} \) is contained in the diagonal entries of the second matrix. This is the reason why we could ignore calculating the mixing matrices \( X \) and \( Y \). We of course do not have a rigorous proof that a level crossing does not occur and that we do not need the block \( D^{(2)} \). However, this is highly unlikely within perturbation theory. In any case, we shall show that by ignoring this block we already obtain an eigenvalue corresponding to a positive full dimension \( y_{2s} \). A larger eigenvalue can only make things worse, while a smaller eigenvalue does not change our conclusions. It is easy to see that the we need only the \((11)\)-element, \( W_{11} \), of the matrix \( SC^{(2)}S^{-1} \), where

\[
W_{11} = -\frac{1}{\omega} [r_{11} + x_{12}r_{21}],
\]

(5.21)

with the matrix elements given above.

Using the fixed point \( t = t^* = \frac{2\pi^2}{\nu}(1 - \frac{\epsilon}{\nu}) \), and choosing \( s \) sufficiently large, we find that, to two-loop order, the full dimension \( y_{2s} \) is given by

\[
y_{2s} = d - 2s + \frac{\epsilon s^2}{N - 2} \left[ 1 + O\left(\frac{1}{s}\right) \right] + \left[ \frac{\epsilon^2 s^3}{(N - 2)^2} \right] \left[ \frac{2}{3} + O\left(\frac{1}{s}\right) \right] + O(\epsilon^3).
\]

(5.22)

This is the central result of our paper. It shows that for any \( \epsilon \), however small, we can always find an infinite number of high gradient operators that have positive scale dimension.
VI. RENORMALIZATION GROUP FLOWS

The main result of this paper is Eq. (5.22). For any $\epsilon$, however small, there are an infinite number of high gradient operators with positive anomalous dimension $y_{2s}$. In fact, the dimension is larger, larger the power of the gradient operator. The two-loop contribution has not changed the picture obtained from the one-loop calculation [3], but has compounded the problem because the two-loop contribution is even larger than the one-loop contribution for sufficiently large $s$.

The most curious phenomenon is the lack of feedback of the high gradient operators to the gradient operators of lower powers. It has been shown from a perturbative argument [3] that the gradient operators of power $2s$ contribute to the renormalization of the operators of powers $(4s - 2)$. Thus, the situation is very different from the $\phi^4$ theory around four dimensions. In that instance, the gaussian fixed point becomes unstable below 4 dimensions because the operator $u\phi^4$ becomes relevant. However, a new non-trivial fixed point can be found because of the feedback of this operator to the renormalization of the coupling associated with $\phi^2$ term. Thus, in contrast to $\phi^4$ theory, it is not possible to locate a new stable fixed point within the $(2 + \epsilon)$-expansion as described in the present paper.

For a more complete understanding, consider the renormalization group equations. From Eq. (5.13), it is simple to see that

\[
\frac{dt}{dl} = -(d - 2)t + (N - 2)\frac{t^2}{2\pi} + (N - 2)\frac{t^3}{(2\pi)^2} + \cdots \tag{6.1}
\]

\[
\frac{1}{U_{2s}} \frac{dU_{2s}}{dt} = d - 2s + \frac{ts(s - 1)}{2\pi} + \frac{t^2[s^3 + O(s^2)]}{12\pi^2} + \cdots \tag{6.2}
\]

where $e^{l}$ is the rescaling factor, and $U_{2s}$ is the coupling constant associated with an eigenoperator. The equation for $t$ is the well-known [3] equation. To recover the previous result for $y_{2s}$, it is only necessary to substitute the fixed point value of $t$ in the equation for $U_{2s}$. Written in this form, it is clear that the difficulties persist for $d = 2$; substitution of $\epsilon = 0$ in Eq. (5.22) misleadingly leads one to believe that there are no difficulties in $d = 2$. The renormalization group flows associated with these equations for $d = 2$ and $d = 2 + \epsilon$ are shown in Figs. 2 and 3.

Let us first consider the case $d = 2 + \epsilon$. In principle, for $t > t^*$, there are two possibilities: (1) $U_{2s} \to 0$, and (2) $U_{2s} \to \infty$. The first possibility does not directly follow from these equations. This is the hypothetical case in which the higher order terms, not calculated here, bend the flows back. Because the flows are vertical as they approach $t^*$ from below, the magnetization must drop discontinuously at the transition [17]. Note that due to the growth of the couplings $U_{2s}$ the spins cannot fluctuate from the preferred direction because $U_{2s}$ multiplies the corresponding high gradient operator. Therefore, it is energetically infinitely costly to allow the gradient of the spins to be non-vanishing. In the second possibility, in which $U_{2s} \to \infty$, an infinite number of $U_{2s}$ grows simultaneously. This makes it exceedingly costly for the spins in the system to point in different directions, regardless of $t$. In other words, the spins cannot disorder. Non-perturbative effects, not contained, in the $(2 + \epsilon)$-expansion are necessary to achieve an order-disorder transition.

The case $d = 2$ is not much different. The growth of $U_{2s}$ as $t$ increases leads us, once again, to one of the previous conclusions, that is, the system cannot disorder. However,
there appears to be a region in which the $U_{2s}$ initially decreases with increasing length scale. Therefore, for shorter length scales, the perturbative renormalization group equations may be approximately valid. In this region, it may be possible to match the solution of these low temperature renormalization group equations to strong coupling calculations. Effects not contained within the scheme of the perturbative renormalization group, such as the instanton effects described by Belavin and Polyakov [18] may be essential to reach a satisfactory description of this model.

VII. CONCLUSIONS

The results derived in this paper are perplexing. While $(2 + \epsilon)$-expansion has never been successful [19] in deriving the critical properties of the Heisenberg model, it has been useful in providing a conceptual framework. In contrast, the expansion in $(4 - d)$ of the $\phi^4$-field theory has been both quantitatively and conceptually useful. To us, the results derived raise serious doubts about the usefulness of the $(2 + \epsilon)$-expansion, because it is virtually certain that the phase transition in the Heisenberg model for $d > 2$ is described by two relevant operators, the temperature and the magnetic field and not by infinitely many relevant operators. This conviction is supported by precise finite size scaling analysis as well as theoretical work based on the expansion around $d = 4$.

It might be argued that because the high gradient operators do not feed into the equation for the temperature, the disordering transition is well-described by the conventional analysis [20]. This argument has little force as an infinite number of parameters must be fine tuned to be zero. On the other hand, for small $\epsilon$, the pathological effect of the high gradient operators will be felt at very long length scales. Thus, the conventional analysis may be approximately valid for shorter length scales. More precise statements are difficult to make.
FIG. 3. The renormalization flows for $d = 2$. $U_{2s}$ is the charge associated with the high gradient operator for sufficiently large $s$

It is also possible that higher loop corrections may make the anomalous dimension of the high gradient operators negative. While this cannot be ruled out, such a situation will still imply rather unusual properties of the $(2 + \epsilon)$-expansion, if it is necessary to go to very high orders to eliminate the pathological behavior of this expansion.

Based on our present understanding of the $O(N)$ model for which there are many precise theoretical checks, we are forced to end on a negative note. In the theory of Anderson localization, $(2 + \epsilon)$-expansion has been used to derive a number of interesting conclusions concerning the distribution of the fluctuation of the moments of the conductance [21]. In fact, it is precisely the context in which the anomalous behavior of the high gradient operators [4] was first discovered. While these effects may exist, it is difficult to accept them on the basis of the $(2 + \epsilon)$-expansion. The most recent work of Dupré [22], in which a numerical simulation of a hyperbolic superplane model is carried out, indicates that there is only one relevant operator at the Anderson localization transition, as was conceived originally by Abrahams et al. [23].

In conclusion, we note that the $(2 + \epsilon)$-expansion does not seem to reproduce what we believe to be the correct behavior of the lattice Heisenberg model. It is possible that non-perturbative effects involving topological excitations are important. Such considerations have been discussed in the past [24,19] and have been emphasized recently in careful numerical simulations [25].

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APPENDIX A:

In this appendix we provide an account of the calculation of Eqs. (4.5-4.6). For convenience, we define the dimension of the fields as follows

\[ [\partial \xi] = [\xi] = 0, \quad [A_\mu] = [\partial \psi] = 1. \]  

(A1)

Thus, any average of the field \( \xi(x) \) will have dimension zero.

The one-loop divergences of the operator \( G_{\mu\nu}(\pi) \) are obtained from the normal coordinate expansion through second order in the field \( \xi(x) \) \([13]\). Making use of Eqs. (3.25-3.26) the normal coordinate expansion for the action and the operator are

\[ [S(\pi)]_{\xi^2} = \frac{1}{2} \int dx \left[ (\partial_\mu \xi^a + A^a_{\mu} \xi^b)(\partial_{\nu} \xi^a + A^a_{\nu} \xi^b) + R_{mn}\partial_\mu \psi^m \partial_\nu \psi^n \xi^a \xi^b \right], \]  

(A2)

\[ [G_{\mu\nu}(\pi)]_{\xi^2} = \left[(\partial_\mu \xi^a + A^a_{\mu} \xi^b)(\partial_{\nu} \xi^a + A^a_{\nu} \xi^b) + R_{mn}\partial_\mu \psi^m \partial_\nu \psi^n \xi^a \xi^b \right]. \]  

(A3)

Since the operator \( G_{\mu\nu}(\pi) \) has dimension \( d_G = 2 \), it can only lead to divergent terms of the same dimension. Then, by inspection, we find that the one-loop divergence of the operator \( G_{\mu\nu}(\pi) \) is given by

\[ \langle [G_{\mu\nu}(\pi)]_{\xi^2} \rangle_{1L} = D_1 + D_2 + D_3 + D_4, \]  

(A4)

where the divergent contractions are:

\[ D_1 = [R_{mn}\partial_\mu \psi^m \partial_\nu \psi^n + A^a_{\mu} A^a_{\nu} \langle [\xi^a_1(x) \xi^a_2(y)] \rangle], \]  

(A5)

\[ D_2 = -\frac{1}{2}[R_{mn}\partial_\alpha \psi^m \partial_\alpha \psi^n + A^a_{\mu} A^a_{\nu} \langle [\xi^a_1(x) \xi^a_2(y) \partial_\mu \xi^c(y) \partial_\nu \xi^c(y)] \rangle], \]  

(A6)

\[ D_3 = -A^a_\mu A^a_\alpha \langle [\xi^a_1(x) \partial_\alpha \xi^a_2(y) \partial_\nu \xi^c(y)] \rangle + (\mu \leftrightarrow \nu), \]  

(A7)

\[ D_4 = \frac{1}{2} A^a_{\beta_1} A^a_{\beta_2} \langle [\xi^a_{\beta_1}(x) \partial_\alpha \xi^a_{\beta_2}(y) \partial_\nu \xi^a_1(y) \partial_\delta \xi^a_2(y) \partial_\mu \xi^a_3(z) \partial_\nu \xi^a_4(z)] \rangle. \]  

(A8)

The diagrams associated with these contractions are shown in Fig. 4.

Note that the divergences of these diagrams are logarithmic, so we can safely set the external momenta to zero. Direct and straightforward calculations give the following results:

\[ D_1 = I[R_{mn}\partial_\mu \psi^m \partial_\nu \psi^n + A^a_{\mu} A^a_{\nu} \langle [\xi^a_1(x) \xi^a_2(y)] \rangle], \]  

(A9)

\[ D_2 = -\frac{1}{2} I \delta_{\mu\nu} [R_{mn}\partial_\alpha \psi^m \partial_\alpha \psi^n + A^a_{\mu} A^a_{\nu} \langle [\xi^a_1(x) \xi^a_2(y) \partial_\mu \xi^c(y) \partial_\nu \xi^c(y)] \rangle], \]  

(A10)

\[ D_3 = -2 I A^a_\mu A^a_\nu, \]  

(A11)

\[ D_4 = \frac{1}{2} I [2 A^a_\mu A^a_\nu + \delta_{\mu\nu} A^a_\mu A^a_\nu] \]  

(A12)

where \( I = -\frac{1}{2\pi e} \). Thus,

\[ \langle [G_{\mu\nu}]_{\xi^2} \rangle_{1L} = [R_{mn}\partial_\mu \psi^m \partial_\nu \psi^n - \frac{1}{2} \delta_{\mu\nu} R_{mn} \partial_\alpha \psi^m \partial_\alpha \psi^n]. \]  

(A13)
It should be noted that the terms that depend on $A^{ab}_\mu$ cancel out. This is a consequence that the only possible covariant term, $F^{ab}_\mu e_\alpha e_\beta = R_{\alpha\beta\gamma\delta} \partial_\mu \psi^\gamma \partial_\nu \psi^\delta$, is antisymmetric under the interchange between $\mu$ and $\nu$, and then, it cannot appear in the right hand side of the foregoing equation.

Turning next to the derivation of Eq. (4.6), we use Eq. (3.25) to obtain

$$[G_{\mu\nu}(\pi)]_{\xi} = e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} D_{\mu} \xi^{a} D_{\kappa} \xi^{b} + (\kappa \leftrightarrow \lambda, \mu \leftrightarrow \nu),$$

where $D_{\mu} \xi^{a}$ is given in Eq. (3.35). Using the fact that the dimension of the operator $G_{\mu\nu} G_{\kappa\lambda}$ is $d_{|\phi|} = 4$, we can find that the average of the first term of Eq. (A14) is given by

$$\langle e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} D_{\mu} \xi^{a} D_{\kappa} \xi^{b} \rangle_{1L} = E_{1} + E_{2} + E_{3} + E_{4} + E_{5},$$

where

$$E_{1} = e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} A^{ad}_{\mu} A^{be}_{\kappa} \langle \xi^{d} \xi^{e} \rangle,$$

$$E_{2} = -\frac{1}{2} e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} \left[ R_{\rho a \lambda \kappa} \partial_{\alpha} \psi^{\rho} \partial_{\nu} \psi^{\beta} + A^{a1}_{\alpha} A^{a2}_{\alpha} \right] \langle \xi^{a1} (x) \xi^{a2} (y) \partial_{\mu} \xi^{a} (x) \partial_{\nu} \xi^{b} (y) \rangle,$$

$$E_{3} = -e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} A^{ad}_{\mu} A^{bf}_{\kappa} \langle \xi^{d} (x) \partial_{\alpha} \xi^{f} (y) \partial_{\mu} \xi^{a} (x) \partial_{\nu} \xi^{b} (y) \rangle + (\mu \leftrightarrow \kappa, a \leftrightarrow b),$$

$$E_{4} = -e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} A^{ab1}_{\gamma} \langle \xi^{d} (x) \partial_{\alpha} \xi^{b1} (y) \partial_{\mu} \xi^{a} (x) \partial_{\nu} \xi^{b} (y) \rangle,$$

$$E_{5} = \frac{1}{2} e_{\nu} e_{\alpha} \partial_{\lambda} \psi^{\beta} \partial_{\nu} \psi^{\alpha} A^{b1}_{\gamma} A^{a1}_{\nu} \langle \xi^{a1} (x) \partial_{\alpha} \xi^{a2} (x) \xi^{b1} (y) \partial_{\mu} \xi^{b2} (y) \partial_{\nu} \xi^{c1} (z) \partial_{\mu} \xi^{c2} (z) \rangle.$$

The diagrams corresponding to these contractions are shown in Fig. 4. With the exception of $E_{4}$, all the contractions have dimension 4 and are logarithmically divergent, so we can
FIG. 5. One-loop diagrams contributing to the expectation value \( \langle [G_{\mu\nu}]_1^1 [G_{\alpha\beta}]_1^1 \rangle \).

set the external momenta to zero. In the case of \( E_4 \), the divergence is linear and, thus, proportional to the external momenta. In the Fourier space, the calculation of \( E_4 \) yields

\[
\langle e^{b_1}_1 \partial_\gamma e^{b_2}_2 \partial_\mu e^{a}_a \partial_\xi e^{c}_c \rangle = \frac{i \delta^{b_1}_a \delta^{b_2}_c}{q^2 (q + p)^2} \int dq (p + q)_{\mu} q_{\nu} q_{\kappa} + \frac{i \delta^{b_1}_a \delta^{b_2}_c}{q^2 (q + p)^2} \int dq (p + q)_{\kappa} q_{\nu} q_{\mu} = \frac{1}{8\pi \epsilon} \delta^{b_1}_a \delta^{b_2}_c [p_\mu \delta_{\kappa\gamma} \gamma_{\kappa \mu} - p_\gamma \delta_{\mu\kappa} + \delta^{b_1}_a \delta^{b_2}_c [\mu \leftrightarrow \kappa \}}.
\]  

(A21)

Direct calculations give the following results:

\[
\begin{align*}
E_1 &= e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q \partial_\beta \psi^r \partial_\gamma \psi^s (A_\mu^a A_\delta^b I) , \\
E_2 &= -\frac{1}{2} \mu I e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q + A_\alpha^a A_\beta^b \right) , \\
E_3 &= 2 I e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q A_\beta^b \partial_\delta \psi^e \partial_\gamma \psi^s , \\
E_4 &= -\frac{1}{2} I e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q (A_\mu^a A_\delta^b - \partial_\delta \gamma_{\kappa \mu} ) , \\
E_5 &= -\frac{1}{2} I e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q [A_\alpha^a A_\beta^b \gamma_{\kappa \mu} + A_\alpha^a A_\beta^b + A_\alpha^a A_\beta^b] .
\end{align*}
\]

(A22)

Hence,

\[
\langle e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p D_\mu^a D_\alpha^b \rangle_{II} = -\frac{1}{2} \delta_{\mu \kappa} I R_{pqrs} \partial_\lambda \psi^t \partial_\nu \psi^p \partial_\alpha \psi^q + \frac{1}{2} I e^{b_1}_b e^{a}_a \partial_\lambda \psi^t \partial_\nu \psi^p [(\partial_\mu A_\kappa^a + \partial_\kappa A_\mu^a) + (A_\alpha^c A_\beta^b + A_\beta^c A_\mu^a)] .
\]  

(A24)

Then, we find that the terms that depend on the gauge field \( A_\mu^a \) lead to the tensor \( F_\mu^a \). In fact, it is not necessary to calculate \( E_5 \) since from gauge invariance and the result for \( E_4 \),
the coefficient that multiplies $F_{\mu\nu}^{ab}$ can be known. Terms that are not gauge invariant either cancel or are not divergent. Making use of the definition of $F_{\mu\nu}^{ab}$ in Eq. (3.36), we obtain the one-loop divergence

$$\langle [G_{\mu\nu}(\pi)]_{\xi_1} [G_{\kappa\lambda}(\pi)]_{\xi_1} \rangle = \frac{1}{2} I_\partial \lambda \partial^\nu \psi^\rho \left[ -\delta_{\mu\kappa} R_{\rho\pi\kappa\sigma} \partial_\sigma \psi^p \partial_\eta \psi^q + R_{\kappa\lambda\sigma\mu} \partial_\mu \psi^m \partial_\kappa \psi^n \right] + (\mu \leftrightarrow \nu) + (\kappa \leftrightarrow \lambda) + (\mu \leftrightarrow \nu, \kappa \leftrightarrow \lambda).$$

(A25)

It may be noted that the results of this appendix are valid for arbitrary Riemannian manifolds.

**APPENDIX B:**

In this appendix we discuss the calculation of the two-loop diagrams. First we study the contribution of the diagrams of the type (a) shown in Fig. 1. Let $G$ denote the graph in question, and $g_1$ and $g_2$ the corresponding subgraphs. Then, if $I_G$ denotes the Feynmann integral associated with $G$, we have that

$$I_G = I_{g_1} I_{g_2}. \quad \text{(B1)}$$

Since $g_i$ are simply one-loop graphs, dimensional regularization tell us that

$$I_{g_i} = \frac{a_i}{\epsilon} + b_i, \quad \text{(B2)}$$

where $a_i$ and $b_i$ are just constants that may depend on the external momenta of the graphs $g_i$. Note that, if the one-loop integral is finite, $a_i = 0$.

If $T$ denotes the operation that selects the poles in $\epsilon$, the application of the forest formula to $G$ yields

$$R(G) = I_G - T(I_{g_1}) I_{g_2} - T(I_{g_2}) I_{g_1}, \quad \text{(B3)}$$

where $R(G)$ denotes the overall divergence of the graph $G$. When $I_{g_1}$ and $I_{g_2}$ are both divergent, the above formula yields

$$R(G) = \left( \frac{a_1}{\epsilon} + b_1 \right) \left( \frac{a_2}{\epsilon} + b_2 \right) - \frac{a_1}{\epsilon} (\frac{a_2}{\epsilon} + b_2) - \frac{a_2}{\epsilon} (\frac{a_1}{\epsilon} + b_1)$$

$$= -\frac{a_1 a_2}{\epsilon^2} + \text{finite} \quad \text{(B4)}$$

If $I_{g_2}$ is finite, we obtain instead the result

$$R(G) = \left( \frac{a_1}{\epsilon} + b_1 \right) b_2 - \frac{a_1}{\epsilon} b_2$$

$$= \text{finite} \quad \text{(B5)}$$

From these results, we conclude that the overall divergence of the two-loop graph in Fig 1(a) is proportional to double poles in $\epsilon$. Thus, this type of graphs do not lead to corrections to the anomalous dimension.
Now we turn to the calculation of the averages in Eqs. (3.5-3.6). Since the calculation of the averages is quite similar, we only show the calculation of the average

\[ [G_{\mu_1\nu_1} G_{\mu_2\nu_2}]_{i2} = \langle [G_{\mu_1\nu_1}]_{i1} [G_{\mu_2\nu_2}]_{i2} \rangle + \langle [G_{\mu_2\nu_2}]_{i1} [G_{\mu_1\nu_1}]_{i2} \rangle \]  

(B6)

Making use of Eqs. (3.23-3.26)

\[ [G_{\mu_1\nu_1}]_{i1} [G_{\mu_2\nu_2}]_{i2} = [e_{\alpha a} \partial_{\alpha a} D_{\mu_1} \xi^{\alpha} + \mu_1 \leftrightarrow \nu_1] \times \\
[\epsilon_{kcd} D_{\mu_2} \xi^{k} D_{\nu_2} \xi^{l} + R_{kcd} \partial_{\mu_2} \psi^{k} \partial_{\nu_2} \psi^{l} \xi^{c} \xi^{d}] \]  

(B7)

Substituting \( D_{\mu} \xi^{a} \) (see Eq. (3.33)), we obtain

\[ [G_{\mu_1\nu_1}]_{i1} [G_{\mu_2\nu_2}]_{i2} = \sum_{i=1}^{6} Q_i(\psi, \xi), \]  

(B8)

where

\[ Q_1(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} \partial_{\mu_1} \xi^{\alpha} \partial_{\mu_2} \xi^{d} \partial_{\nu_2} \xi^{d} + (\mu_1 \leftrightarrow \nu_1), \]  

(B9)

\[ Q_2(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} [A_{\mu_1}^{ac_1} \xi^{c_1} \partial_{\mu_2} \xi^{d} \partial_{\nu_2} \xi^{d} + A_{\mu_2}^{dc_2} \xi^{c_2} \partial_{\mu_1} \xi^{d} + (\mu_1 \leftrightarrow \nu_1)] , \]  

(B10)

\[ Q_3(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} [A_{\mu_1}^{ac_1} A_{\mu_2}^{dc_2} \xi^{c_1} \xi^{c_2} \partial_{\nu_2} \xi^{d} + A_{\mu_1}^{ac_1} A_{\nu_2}^{dc_2} \xi^{c_1} \xi^{c_2} \partial_{\mu_2} \xi^{d} + A_{\mu_2}^{dc_2} \xi^{c_2} \partial_{\mu_1} \xi^{a} + (\mu_1 \leftrightarrow \nu_1)] , \]  

(B11)

\[ Q_4(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} [A_{\mu_1}^{ac_1} A_{\mu_2}^{dc_2} A_{\nu_2}^{dc_3} \xi^{c_1} \xi^{c_2} \xi^{c_3} + (\mu_1 \leftrightarrow \nu_1) , \]  

(B12)

\[ Q_5(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} \partial_{\mu_2} \psi^{m_2} \partial_{\nu_2} \psi^{n_2} R_{m_2b_2n_2} \xi^{b_1} \xi^{b_2} \partial_{\mu_1} \xi^{a} + (\mu_1 \leftrightarrow \nu_1) , \]  

(B13)

\[ Q_6(\psi, \xi) \equiv e_{\alpha a} \partial_{\alpha a} \psi^{n_1} \partial_{\mu_2} \psi^{m_2} \partial_{\nu_2} \psi^{n_2} R_{m_2b_2n_2} A_{\mu_1}^{ac_1} \xi^{b_1} \xi^{b_2} \xi^{c_1} + (\mu_1 \leftrightarrow \nu_1) . \]  

(B14)

In a similar fashion, the action \( S \) through third order in the field \( \xi \) is given by

\[ [S]_{i2} + [S]_{i3} = \int dx \left[ S_1 + S_2 + S_3 + S_4 + S_5 \right], \]  

(B15)

where

\[ S_1(\psi, \xi) \equiv A_{\alpha}^{ab} \xi^{b} \partial^{\alpha} \xi^{a} , \]  

(B16)

\[ S_2(\psi, \xi) \equiv \frac{1}{2} A_{\alpha}^{ab} A_{\alpha}^{ac} \xi^{b} \xi^{c} , \]  

(B17)

\[ S_3(\psi, \xi) \equiv \frac{1}{2} R_{ia_1a_2j} \partial_{\alpha} \psi^{i} \partial^{\alpha} \psi^{j} \xi^{a_1} \xi^{a_2} , \]  

(B18)

\[ S_4(\psi, \xi) \equiv \frac{2}{3} R_{ma_1a_2a_3} \partial_{\alpha} \psi^{m} \xi^{a_1} \xi^{a_2} \partial_{\alpha} \xi^{a_3} \]  

(B19)

\[ S_5(\psi, \xi) \equiv \frac{2}{3} R_{ma_1a_2a_3} A_{\alpha}^{ac} \partial_{\alpha} \psi^{m} \xi^{a_1} \xi^{a_2} \xi^{a_3} . \]  

(B20)

In principle, one has to consider all the divergent two-loop diagrams that arise from the contraction between one of the vertices \( Q_n \) and the vertices \( S_m \). However, the number of diagrams can be reduced by using the dimension of the operator \( G_{\mu_1\nu_1} G_{\mu_2\nu_2} \), which is 4, and
by calculating only those terms proportional to the derivative of the gauge field $\partial A$. Also, let us recall that we are only interested in the operators defined in Eq. 4.9. In this manner the only terms necessary in the calculation of $[G_{\mu_1 \nu_1} G_{\mu_2 \nu_2}]_{12}$ are

\[
W_1(\psi) = (-1)\langle Q_5(\psi, \xi) S_4(\psi, \xi) \rangle ,
\]

\[
W_2(\psi) = (-1)\langle Q_5(\psi, \xi) S_4(\psi, \xi) \rangle ,
\]

\[
W_3(\psi) = (-1)\langle Q_5(\psi, \xi) S_4(\psi, \xi) \rangle ,
\]

\[
W_4(\psi) = \langle Q_5(\psi, \xi) S_4(\psi, \xi) S_4(\psi, \xi) \rangle ,
\]

\[
W_5(\psi) = \langle Q_5(\psi, \xi) S_4(\psi, \xi) S_4(\psi, \xi) \rangle .
\]

From Eqs. (B9-B20), we see that the contractions of interest are

\[
\Gamma^{b_1b_2b_3;\alpha_1\alpha_2\alpha_3}_{\mu\nu}(x,y) \equiv \langle \xi^{b_1} \xi^{b_2} \partial_\mu \xi^{b_3}(x) \xi^{\alpha_1} \xi^{\alpha_2} \partial_\nu \xi^{\alpha_3}(y) \rangle ,
\]

\[
\Theta^{b_1b_2b_3;\alpha_1\alpha_2\alpha_3}_{\mu\alpha}(x,y) \equiv \langle \xi^{b_1} \partial_\mu \xi^{b_2} \partial_\nu \xi^{b_3}(x) \xi^{\alpha_1} \xi^{\alpha_2} \partial_\alpha \xi^{\alpha_3}(y) \rangle ,
\]

\[
\Lambda^{b_1b_2b_3;\alpha_1\alpha_2\alpha_3}_{\mu\nu\rho}(x,y) \equiv \langle \partial_\mu \xi^{b_1} \partial_\nu \xi^{b_2} \partial_\rho \xi^{b_3}(x) \xi^{\alpha_1} \xi^{\alpha_2} \partial_\alpha \xi^{\alpha_3}(y) \rangle ,
\]

\[
\Delta^{c_1c_2c_3;\alpha_1\alpha_2\alpha_3;\beta_1\beta_2}_{\mu_1\mu_2\mu_3\gamma}(x,y,z) \equiv \langle \partial_\mu_1 \xi^{c_1} \partial_\mu_2 \xi^{c_2} \partial_\mu_3 \xi^{c_3}(z) \xi^{\alpha_1} \xi^{\alpha_2} \partial_\alpha \xi^{\alpha_3}(x) \xi^{b_1} \partial_\gamma \xi^{b_2}(y) \rangle ,
\]

\[
\Xi^{c_1c_2c_3;\alpha_1\alpha_2\alpha_3;\beta_1\beta_2}_{\mu_1\mu_2\mu_3\alpha}(x,y,z) \equiv \langle \partial_\mu_1 \xi^{c_1} \partial_\mu_2 \xi^{c_2} \partial_\mu_3 \xi^{c_3}(z) \xi^{\alpha_1} \xi^{\alpha_2} \partial_\alpha \xi^{\alpha_3}(x) \xi^{b_1} \xi^{b_2}(y\rangle .
\]

The diagrams associated with these contractions are shown in Fig. 6. To illustrate the
method of calculation, we now study the contraction $\Delta$. In Fourier space, we find

$$\Delta_{\mu_1\mu_2\mu_3\gamma}^{c_1c_2:a_1a_2a_3:b_1b_2}(l, p) = \left\{ [I_{E_1}(\mu_1, \mu_2, \mu_3)\delta^{c_1a_3}\delta^{c_2a_2}\delta^{c_1b_1}\delta^{b_2a_1} + I_{E_2}(\mu_1, \mu_2, \mu_3)\delta^{(b_1+b_2)}] \\
+ [I_{E_3}(\mu_2, \mu_1, \mu_3)\delta^{c_1a_3}\delta^{c_2a_2}\delta^{c_1a_1} + I_{E_4}(\mu_2, \mu_1, \mu_3)\delta^{(b_1+b_2)}] \\
+ [I_{E_5}(\mu_2, \mu_1, \mu_3)\delta^{c_2a_2}\delta^{c_1a_1} + I_{E_6}(\mu_2, \mu_1, \mu_3)\delta^{(b_1+b_2)}] \\
+ a_2 \leftrightarrow a_3 + \{ a_3 \rightarrow a_2; a_3 \rightarrow a_1 \} \right\} + \{ a_1 \leftrightarrow a_2 \},$$

(B32)

where $I_{E_i}$ are just the Feynman integrals of the diagrams of Fig. For instance the graph

![Diagrams](image)

FIG. 7. The diagrams which arise from the calculation of the contraction $\Delta$

$I_{E_1}$ yields

$$I_{E_1} = -i \int dqdk \frac{q_\alpha q_\mu_3 (k-q)_\mu_2 (k-l-p)_\mu_5 (k-l)_\gamma}{[(k-l)^2 + m^2][(k-l-p)^2 + m^2][(k-q)^2 + m^2][q^2 + m^2]}$$

where we have included the infrared cutoff in the propagators. By power counting, the integral is linearly divergent (the superficial degree of divergence is $\delta = 1$). Hence, by the Weinberg theorem [16], the overall divergence must be also linear, i.e., the subtraction of divergences cancels all the non-linear divergences. Thus, we can expand the propagators of the previous expression in terms of the external momenta $l$ and $p$, and neglect all the terms except for the linear term. This yields

$$iI'_{E_1} = (2p + 4l)_\delta \int dqdk \frac{k_\delta k_\mu_1 k_\gamma (k-q)_\mu_2 q_\mu_3 q_\alpha}{(k^2 + m^2)^{3}(q^2 + m^2)}$$

$$-l_\gamma \int dqdk \frac{k_\mu_1 (k-q)_\mu_2 q_\mu_3 q_\alpha}{(k^2 + m^2)^{2}(q^2 + m^2)}$$

$$-(l + p)_\mu_1 \int dqdk \frac{k_\gamma (k-q)_\mu_2 q_\mu_3 q_\alpha}{(k^2 + m^2)^{2}(q^2 + m^2)}.$$  

(B33)
Now the calculation is reduced to computing the logarithmic divergences of the integrals of the right hand side of the foregoing equation. One of the ways to calculate the divergence of this type integral is to write the most general tensor form compatible with the symmetries of the integral, and then to make all the possible contractions to obtain relations between the coefficients. In this way we obtain

\[ iE'_1 = (2p + 4l)u_1 \delta_{\mu_1\gamma} \delta_{\mu_2\mu_3} + u_2 (\delta_{\mu_1\mu_2} \delta_{\gamma\mu_3} + \delta_{\mu_2\gamma} \delta_{\mu_1\mu_3}) \]
\[ + (2p + 4l)u_1 \delta_{\mu_1\gamma} \delta_{\mu_2\alpha} + u_2 (\delta_{\mu_1\mu_2} \delta_{\gamma\alpha} + \delta_{\mu_2\gamma} \delta_{\mu_1\alpha}) \]
\[ + (2p + 4l)u_1 \delta_{\mu_1\gamma} \delta_{\mu_3\alpha} + u_2 (\delta_{\mu_1\mu_3} \delta_{\gamma\alpha} + \delta_{\mu_3\gamma} \delta_{\mu_1\alpha}) \]
\[ + (2p)\gamma [u_3 \delta_{\mu_1\mu_2} \delta_{\mu_3\alpha} + u_1 (\delta_{\mu_2\mu_3} \delta_{\mu_1\alpha} + \delta_{\mu_1\mu_3} \delta_{\mu_2\alpha})] \]
\[ + (l)_1 [u_4 \delta_{\mu_1\mu_2} \delta_{\mu_3\alpha} + u_5 (\delta_{\mu_2\mu_3} \delta_{\mu_1\alpha} + \delta_{\mu_2\mu_3} \delta_{\mu_1\alpha})] \]
\[ + (p)\mu_1 [u_6 \delta_{\mu_1\mu_2} \delta_{\mu_3\alpha} + u_7 (\delta_{\mu_2\mu_3} \delta_{\mu_1\alpha} + \delta_{\mu_2\mu_3} \delta_{\mu_1\alpha})] \]
\[ + (l)\mu_1 [u_4 \delta_{\mu_1\mu_2} \delta_{\mu_3\alpha} + u_5 (\delta_{\mu_2\mu_3} \delta_{\mu_1\alpha} + \delta_{\mu_2\mu_3} \delta_{\mu_1\alpha})] , \] (B34)

where the coefficients \( u_i \) are

\[ u_1 = \frac{1}{\Omega} [-6 + \epsilon/2], \quad u_2 = -\epsilon/\Omega, \quad u_3 = \frac{1}{\Omega} [6 - 5\epsilon/2], \] (B35)
\[ u_4 = 8\epsilon/\Omega, \quad u_5 = 4\epsilon/\Omega, \quad u_6 = \frac{1}{\Omega} [-12 + 13\epsilon], \] (B36)
\[ u_7 = \frac{1}{\Omega} [12 - 5\epsilon], \] (B37)

and \( \Omega = 1536\pi^2\epsilon^2 \).

Let us now turn to the subtraction of subdivergences of the graph \( IE_1 \). The three subgraphs of \( IE_1 \) are shown in Fig. [8]. Since the subgraph \( e_{11} \) is finite, the application of the Forest formula gives

\[ Re_1 = IE_1 - T(IE_{12})IE_{13} - T(IE_{13})IE_{12} \] (B38)

After the subtractions are carried out, we find that the overall divergence \( Re_1 \) is identical to the integral \( IE_1 \) except for the sign of the \( \frac{1}{\epsilon^2} \) term. The final result for \( [G_{\mu_1\nu_1} G_{\mu_2\nu_2}]_{12} \) is then

\[ [G_{\mu_1\nu_1} G_{\mu_2\nu_2}]_{12} \equiv \langle \{G_{\mu_1}\} \xi^{i} \{G_{\mu_2}\} \xi^{j} \rangle W + \langle \{G_{\mu_1}\} \xi^{j} \{G_{\mu_2}\} \xi^{i} \rangle W \]
\[ = [E_{m_1n_1m_2n_2} \Psi_{1\mu_1\nu_1\mu_2\nu_2}^{m_1n_1m_2n_2} + F_{pq,n_1n_2} \delta_{\mu_1\mu_2} \Psi_{\nu_1\nu_2}^{n_1n_2} \Psi^{pq}\alpha\beta] \]
\[ + [\mu_1 \leftrightarrow \nu_1] + [\mu_2 \leftrightarrow \nu_2] + [\mu_1 \leftrightarrow \nu_1, \mu_2 \leftrightarrow \nu_2] + \]
\[ G_{pq,\alpha\beta} [\delta_{\mu_2\nu_2} \Psi_{\mu_1\nu_1}^{m_1n_1} + \delta_{\mu_1\nu_1} \Psi_{\mu_2\nu_2}^{m_1n_1}] , \] (B39)

where
\[ E_{m_1 n_1, m_2 n_2} = -\frac{\epsilon}{\Omega} \left[ 48 R_{m_1} R^s_{n_2 m_1 m_2} + 40 R_{n_1 (n_2 p) q} R^{pq}_{m_1 m_2} \right], \quad (B40) \]
\[ F_{pq, n_1 n_2} = \frac{\epsilon}{\Omega} \left[ 16 R_{ps} R^s_{n_1 n_2 q} - 4 R_{p(n_1 s) t} R^{st}_{n_2 q} \right] + [n_1 \leftrightarrow n_2], \quad (B41) \]
\[ G_{pq, mn} = -\frac{\epsilon}{\Omega} \left[ 8 R_{ps} R^s_{m n q} + 16 R_{p(m s) t} R^{st}_{n q} \right] + [m \leftrightarrow n]. \quad (B42) \]

In Eqs. (B40)-(B42), we only show the single pole contribution, and \((ab)\) denotes symmetrization in the indices \(a\) and \(b\). Note that this result is only valid for manifolds which satisfy \(\nabla_t R_{mn pq} = 0\).

**APPENDIX C:**

In this appendix we write down our results. To simplify the expressions, in this appendix we adopt the notation
\[ \Psi^{m_1 m_2 \ldots}_{\mu_1 \mu_2 \ldots} \equiv \partial_{\mu_1} \psi^{m_1} \partial_{\mu_2} \psi^{m_2} \ldots. \quad (C1) \]

The expectation values of the terms of order \(O(\xi^2)\) give the following results:
\[ \langle [G_{\mu \nu}] \xi^2 \rangle = -\frac{64 \epsilon}{\Omega^2} R_{m a_1 a_2 a_3} R_m (a_1 a_2 a_3) [\Psi^{m n}_{\mu \nu} + \Psi^{m \alpha n}_{\alpha \mu \nu}], \quad (C2) \]
\[ \langle [G_{\mu_1 \nu_1}]_{\xi^1} [G_{\mu_2 \nu_2}]_{\xi^1} \rangle = [M_{m_1 n_1, m_2 n_2} \Psi^{m_1 n_1 m_2 n_2}_{\mu_1 \nu_1 \mu_2 \nu_2} + N_{p q, n_1 n_2} \delta_{\mu_1 \mu_2} \Psi^{n_1 n_2}_{\nu_1 \nu_2} \Psi^{p q}_{\alpha \alpha}] + [\mu_1 \leftrightarrow \nu_1] + [\mu_2 \leftrightarrow \nu_2] + [\mu_1 \leftrightarrow \nu_1, \mu_2 \leftrightarrow \nu_2], \quad (C3) \]

where
\[ M_{n_1 n_2 n_2} = -\frac{\epsilon}{\Omega} \left[ 24 R_{m_1 (st) n_1} R_{m_2 (st) n_2} + \frac{40}{3} R_{m_1 (nt) s} R_{m_2 (nt)} s t \right] + \frac{32}{3} R_{m_1 (nt) s} R_{m_2 (nt)} s t + [n_1 \leftrightarrow n_2] , \quad (C4) \]
\[ N_{pq, n_2} = -\frac{\epsilon}{\Omega} \left[ 12 R_{m_1 (st) n_1} R_{m_2 (st) n_2} + \frac{4}{3} R_{m_1 (nt) s} R_{m_2 (nt)} t s \right] + \frac{32}{3} R_{m_1 (nt) s} R_{m_2 (nt)} s t + [n_1 \leftrightarrow n_2] . \quad (C5) \]

Now we write the results for the expectation values of the terms of \( O(\xi^3) \):
\[ \langle [G_{\mu \nu}]_{\xi^3} \rangle = \frac{128 \epsilon}{\Omega} R_{na_1 a_2 a_3} R_{n (a_1 a_2 a_3) \psi_{\mu \nu}} , \quad (C6) \]
\[ [G_{\mu 1 \nu} G_{\mu 2 \nu}]_{12} = \left[ E_{m_1 n_1, m_2 n_2} \Psi_{m_1 n_1, m_2 n_2} \Psi_{m_1 n_1, m_2 n_2} + F_{pq, n_1, n_2} \delta_{\mu \nu} \Psi_{m_1 n_1, n_2} \Psi_{m_2 n_2} \right] + \left[ \mu_1 \leftrightarrow \nu_1 \right] + \left[ \mu_2 \leftrightarrow \nu_2 \right] + \left[ \mu_1 \leftrightarrow \nu_1, \mu_2 \leftrightarrow \nu_2 \right] + G_{pq, n_1} \Psi_{m_1 n_1, n_2} \Psi_{m_2 n_2} \right] , \quad (C7) \]

where
\[ E_{m_1 n_1, m_2 n_2} = -\frac{\epsilon}{\Omega} \left[ 48 R_{m_1 s} R_{n_2 m_1 m_2} + 40 R_{m_1 n_2 p} R_{m_1 m_2} \right] , \quad (C8) \]
\[ F_{pq, n_1, n_2} = \frac{\epsilon}{\Omega} \left[ 16 R_{p_1 s} R_{n_1 n_2 q} - 4 R_{p_1 n_2} R_{n_1 n_2} \right] + [n_1 \leftrightarrow n_2] , \quad (C9) \]
\[ G_{pq, n_1} = -\frac{\epsilon}{\Omega} \left[ 8 R_{p_1 s} R_{n_1 n_2 q} + 16 R_{p_1 n_2} R_{n_1 n_2} \right] + [m_1 \leftrightarrow n_1] . \quad (C10) \]

Also,
\[ \langle [G_{\mu 1 \nu}]_{\xi^3} [G_{\mu 2 \nu}]_{\xi^3} [G_{\mu 3 \nu}]_{\xi^3} \rangle = \left\{ \Psi_{m_1 n_1, m_2 n_2, m_3 n_3} U_{m_1 n_1, m_2 n_2, m_3 n_3} + \left[ \mu_1 \leftrightarrow m_2; n_1 \leftrightarrow n_2 \right] + \left[ \mu_1 \leftrightarrow n_3; \mu_2 \leftrightarrow m_3 \right] \right\} + \left[ \mu_1 \leftrightarrow \nu_1 \right] , \quad (C11) \]

with
\[ U_{m_1 n_1, m_2 n_2, m_3 n_3} = \frac{4 \epsilon}{\Omega} \left\{ \left( R_{m_1 (nt) n_2} + 5 R_{m_1 (nt) n_1} \right) R_{m_2 (nt) m_3} + \left[ n_2 \leftrightarrow n_3; m_2 \leftrightarrow m_3 \right] \right\} , \quad (C12) \]
\[ V_{pq, m_1 n_1, n_2 n_2} = \frac{2 \epsilon}{\Omega} \left\{ 2 R_{p (n_1 n_2) s} R_{n_1 n_2} R_{m_1 (nt) s} + 3 R_{m_1 n_2} R_{p m_1 n_2} \right\} + \left[ n_1 \leftrightarrow n_2 \right] \]
\[ -4 \left[ R_{p (n_1 n_2) s} R_{n_1 n_2} R_{m_1 (nt) s} + \left[ n_1 \leftrightarrow n_2 \right] \right] - \left[ R_{m_1 n_2} n_1 + 5 R_{m_1 n_2} n_3 \right] R_{p (n_2) s} q \right] - \left[ n_1 \leftrightarrow n_2 \right] . \quad (C13) \]

To find the results for the manifold \( O(N)/O(N - 1) \), we substitute the curvature tensor, Eq. (3.22), in the previous results. This yields,
\[ \langle [G_{\mu \nu}]_\xi^2 \rangle = -\frac{192\epsilon}{\Omega} g_{mn}[\delta_{\mu \nu} \Psi_{\alpha \alpha}^{mn} + \Psi_{\mu \nu}^{mn}], \]  
(14)

\[ \langle [G_{\mu \nu \rho}]_\xi^2 [G_{\mu \nu \rho}]_\xi^2 \rangle = -\frac{16\epsilon}{\Omega} P_{m1n1,m2n2} \Psi_{\mu \nu \rho 1}^{m1n1,m2n2} \]
\[ + \frac{12\epsilon}{\Omega} Q_{pqkl} \Psi_{\alpha \alpha}^{pq} (\delta_{\mu \nu \rho} \Psi_{\nu \nu \rho}^{kl} + \delta_{\nu \rho \mu} \Psi_{\rho \rho \mu}^{kl} + (\mu_2 \leftrightarrow \nu_2)), \]  
(15)

where

\[ P_{m1n1,m2n2} = (18 - 4N)g_{m1n1}g_{m2n2} + (9N - 15)(g_{m1m2}g_{n1n2} + g_{m1n2}g_{n1m2}), \]  
(16)

\[ Q_{pqkl} = -10g_{klpq} + (11 - 3N)(g_{kpq} + g_{kqpl}). \]  
(17)

Similarly, the \(O(\xi^3)\) expectation values are

\[ \langle [G_{\mu \nu}]_\xi \rangle = \frac{384\epsilon(N - 2)}{\Omega} g_{mn} \Psi_{\mu \nu}^{mn}, \]  
(18)

\[ [G_{\mu \nu 1}, G_{\mu \nu 2}]_{12} = \frac{48\epsilon}{\Omega} X_{m1n1,m2n2} \Psi_{\mu \nu 1 \nu 2}^{m1n1,m2n2} + \]
\[ \frac{4\epsilon}{\Omega} Y_{pqkl} \Psi_{\alpha \alpha}^{pq} (\delta_{\mu \nu 1} \Psi_{\nu \nu 2}^{kl} + \delta_{\nu \rho 2} \Psi_{\rho \rho 1}^{kl} + (\mu_2 \leftrightarrow \nu_2)) + \]
\[ \frac{8\epsilon}{\Omega} Z_{pqkl} \Psi_{\alpha \alpha}^{pq} (\delta_{\mu \nu 1} \Psi_{\nu \nu 2}^{kl} + \delta_{\mu \nu 2} \Psi_{\mu \nu 1}^{kl}), \]  
(19)

where

\[ X_{m1n1,m2n2} = (20N - 66)g_{m1n1}g_{m2n2} + (17 - 2N)(g_{m2n1}g_{n1m1} + g_{m1m2}g_{n1n1}), \]  
(C20)

\[ Y_{pqkl} = (32N - 70)g_{klpq} + (23 - 10N)(g_{kpq} + g_{kql}), \]  
(C21)

\[ Z_{pqkl} = (26N - 82)g_{klpq} + (23 - 4N)(g_{kpq} + g_{kql}), \]  
(C22)

Finally,

\[ \langle [G_{\mu \nu 1}]_\xi^2 [G_{\mu \nu 2}]_\xi^2 [G_{\mu \nu 3}]_\xi^2 \rangle = \frac{4\epsilon}{\Omega} T_{m1n1,m2n2,m3n3} \Psi_{\mu \nu 1 \nu 2 \nu 3}^{m1n1,m2n2,m3n3} + \]
\[ \frac{2\epsilon}{\Omega} S_{pqmnkl} \Psi_{\alpha \alpha}^{pq} \times \]
\[ \{(\Psi_{\rho \rho 1}^{mn} [\delta_{\mu \nu 1} \Psi_{\nu \nu 2}^{kl} + \delta_{\nu \rho 2} \Psi_{\rho \rho 1}^{kl} + (\mu_2 \leftrightarrow \nu_2)] + \]
\[ \Psi_{\mu \nu 1}^{mn} [\delta_{\mu \nu 1} \Psi_{\nu \nu 2}^{kl} + \delta_{\nu \rho 2} \Psi_{\rho \rho 1}^{kl} + (\mu_2 \leftrightarrow \nu_2)] + \]
\[ \Psi_{\mu \nu 2}^{mn} [\delta_{\mu \nu 1} \Psi_{\nu \nu 2}^{kl} + \delta_{\nu \rho 2} \Psi_{\rho \rho 1}^{kl} + (\mu_2 \leftrightarrow \nu_2)]\}, \]  
(23)

with

\[ T_{m1n1,m2n2,m3n3} = -432g_{m1n1}g_{m2n2}g_{m3n3} + 144\{g_{m3n3} (g_{m1n1}g_{m2n2} + g_{m1n2}g_{m3n2}) + \]
\[ + [m_2 \leftrightarrow m_3; n_2 \leftrightarrow n_3] + [m_1 \leftrightarrow m_3; n_1 \leftrightarrow n_3]\} - 54\{g_{m1n2} (g_{m1m2}g_{m3n3} + g_{m2m3}g_{m1n3}) + \]
\[ + [m_2 \leftrightarrow m_3; n_2 \leftrightarrow n_3] + [m_1 \leftrightarrow m_3; n_1 \leftrightarrow n_3]\} - 54\{g_{m1n3} (g_{m1m3}g_{m2n2} + g_{m2m3}g_{m1n2}) + \]
\[ - 19\{g_{m1n1} (g_{m1m1}g_{m2n3} + g_{m1m2}g_{m2n3}) + g_{m1n2} (g_{m1n2} + g_{m1n3}) + [m \leftrightarrow n]\}. \]  
(C24)
APPENDIX D:

In this appendix we use conformal coordinates to simplify the two-loop results obtained in the previous appendix. Here, we set \( \omega = 24\pi^2\epsilon \) and use the following notation:

\[
A(\pi) = \partial_+ \pi^m \partial_+ \pi^n g_{mn},
B(\pi) = \partial_- \pi^m \partial_- \pi^n g_{mn},
H(\pi) = \partial_+ \pi^m \partial_- \pi^n g_{mn}.
\]

We denote the \( O(\xi^n) \) term of the normal expansion of \( A(\pi), B(\pi) \) and \( H(\pi) \) by \( a_n(\psi, \xi) \), \( b_n(\psi, \xi) \) and \( h_n(\psi, \xi) \) respectively. Straightforward manipulations yield

\[
\langle a_2 \rangle_{2L} = -\frac{3\nu}{\omega} A(\psi) , \quad \langle h_2 \rangle_{2L} = -\frac{9\nu}{\omega} H(\psi), \quad (D1)
\]

\[
\langle a_1 a_1 \rangle_{2L} = -\frac{9\nu}{\omega} A(\psi)^2 , \quad \langle a_1 h_1 \rangle_{2L} = -\frac{27\nu}{2\omega} A(\psi)H(\psi), \quad (D2)
\]

\[
\langle a_1 b_1 \rangle_{2L} = -\frac{9}{\omega} [H^2(\psi)(\nu + 1) + A(\psi)B(\psi)(\nu - 1)] , \quad (D3)
\]

\[
\langle h_1 h_1 \rangle_{2L} = -\frac{3}{2\omega} [H^2(\psi)(6 \nu + 2) + A(\psi)B(\psi)(3 \nu - 2)] . \quad (D4)
\]

We now turn to the expectation values of the terms of order \( O(\xi^3) \). We obtain

\[
\langle a_3 \rangle_{2L} = \frac{6\nu}{\omega} A(\psi) , \quad \langle h_3 \rangle = \frac{6\nu}{\omega} H(\psi), \quad (D5)
\]

\[
\langle [a a]_{12} \rangle_{2L} = \frac{12\nu}{\omega} A(\psi)^2 , \quad \langle [a h]_{12} \rangle_{2L} = \frac{21\nu}{2\omega} A(\psi)H(\psi) , \quad (D6)
\]

\[
\langle [a b]_{12} \rangle_{2L} = \frac{2}{\omega} [H^2(\psi)(4 \nu + 9) + A(\psi)B(\psi)(5 \nu - 9)], \quad (D7)
\]

\[
\langle [h h]_{12} \rangle_{2L} = \frac{3}{\omega} [H^2(\psi)(8 \nu - 3) + A(\psi)B(\psi)(3 - 2 \nu)] ; \quad (D8)
\]

where \( [ab]_{12} \equiv a_1 b_2 + a_2 b_1 \). Finally,

\[
\langle a_1 b_1 h_1 \rangle_{2L} = \frac{14}{\omega} [H^2(\psi) - A(\psi)B(\psi)H(\psi)] , \quad (D9)
\]

\[
\langle a_1 h_1 h_1 \rangle_{2L} = \frac{2}{\omega} [H^2(\psi)A(\psi) - A(\psi)^2 B(\psi)] , \quad (D10)
\]

\[
\langle a_1 a_1 b_1 \rangle_{2L} = \frac{4}{\omega} [H^2(\psi)A(\psi) - A(\psi)^2 B(\psi)] , \quad (D11)
\]

\[
\langle h_1 h_1 h_1 \rangle_{2L} = \frac{12}{\omega} [A(\psi)B(\psi)H(\psi) - H^3(\psi)] . \quad (D12)
\]

All other possibilities vanish.
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