Single-species population models with age structure and psychological effect in a polluted environment

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Abstract

This paper considers a single-population model with age structure and psychological effects in a polluted environment. We divide the single population into two stages of larval and adult structure. The model uses Logistic input, and the larvae are converted into adult bodies by constant ratio. We only consider adulthood. The role of psychological effects makes the contact between adult and environmental toxins a functional form, while the contact between larvae and environmental toxins is linear.

For the deterministic model embodied as a nonlinear time-varying system, we discuss the asymptotic stability of the system by Lyapunov one-time approximation theory, and give a sufficient condition for stability to be established.

Considering that the contact rate between biological and environmental toxins in nature is not always constant, we make the contact rate interfere with white noise, and then modify the contact rate into a stochastic process, thus establishing a corresponding random single-population model. According to Itô formula and Lyapunov in the function method, we first prove the existence of globally unique positive solutions for stochastic models under arbitrary initial conditions, and then give sufficient conditions for weak average long-term survival and random long-term survival for single populations in the expected sense.

Keywords: polluted environment, stochastic single-species population models, age structure, psychological effect, stability, persistence

1. Introduction

With the development of society, the problem of environmental pollution caused by human activities and industrial production has become increasingly serious, environmental pollution has always been the most threatening social and ecological problem. All kinds of pollutants discharged into the environment have a great impact on the normal survival of all kinds of organisms in the environment. All kinds of organisms, including human beings, are threatened by all kinds of poisons in the environment. At present, many scholars have established mathematical models to describe it.

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Since Hallam et al. [1–5], scholars have paid more and more attention to this problem. In 1983, Hallam et al. [1] first proposed a classical deterministic mathematical model for the survival of a single population in a polluted environment. Other scholars continue to study the environmental pollution model [6–11]. Srinivasu [12] considered the impact of biological death on the concentration of environmental toxins, studied the problem of population’s long-term survival and extinction, and considered to ensure the population’s long-term survival by increasing the amount of environmental toxins removed.

The growth of species is basically accompanied by a process of development, i.e. from adulthood to old age, from immature to mature, from young individual to adult individual. Each stage of the growth process often shows different characteristics, such as the young individual does not have the ability of reproduction and predation. Due to the limitation of intelligence development and survival experience, the young individual often does not have the ability of self avoiding danger. Compared with other stages of population, the ability of survival and resources competition of young individuals is weak. Young individuals are often easy to die, and unable to complete large-scale migration in space.

The adult population has the ability of reproduction and predation, a high level of intellectual development. It is also superior in survival experience and can actively avoid danger, thus has a strong survival ability and the ability to compete with other populations for limited survival resources. Some of them experience that one side is injured or even killed in order to compete for spouse. There are significant differences in the behavior of the same stage. In addition, there is always a relationship of mutual transformation between organisms in different stages, which will have an impact on the extinction and long-term survival of organisms. Therefore, it is of practical significance to study the population model with stage structure, which has also attracted the attention of many mathematicians and biologists.

The related work of stage structure population model can be traced back to the three-stage structure self feeding model proposed by Landahl and Hansen [13], in the 1970s, and the two-stage structure stochastic model proposed by Tognetti [14]. Later, many scholars also proposed different stage structure models [15]. But until 1990, Aiello and Freedman [16] proposed the single group delay model with two-stage structure. The related research only then truly ushers in the upsurge [17–23].

For vertebrates with a certain level of intelligence, when life safety is threatened by diseases, predators, environmental pollution and other external threats, there are often a series of psychological effects such as fear. Compared with other creatures with a low level of intelligence, psychological effects affect the daily behavior of the whole population to a certain extent.

In recent years, many scholars have tried to make the mathematical models to depict the influence of psychological effects. Many of them study human groups directly [24–30], and some of them study other creatures. In 2018, Kumar [31], studied the prey predator model of Allee effect induced by fear. In 2019, Lan et al. [32], considered the long-term behavior of a single population model with psychological effects under the pulse input of environmental toxins. Assuming that the internal growth rate of a single population is interfered by white noise, a corresponding stochastic model was established.

In this paper, we mainly consider vertebrates with sensory organs and highly differentiated nervous system. Compared with organisms without this structure in nature, sensory organs can interpret information in dynamic environment. For vertebrates, such as fish and birds, we can detect and process information in the physical world. When the environment is polluted, these organisms either bear the toxin concentration of surrounding environment or escapes from the living area when the psychological effect plays a role. Combined with the age structure, we naturally think that the intelligence level of young individuals is lower than that of adult individuals. Therefore, when considering the psychological effect, we should consider the two groups separately. Based on the above characteristics, we will study the single population with age structure and psychological effect in the polluted environment dynamic behavior.
In the first part of this paper, based on the previous studies, a single population model with age structure and psychological effects is established. The model is a nonlinear time-varying system. According to Lyapunov’s first approximation theory, we study the related properties of the original system by studying the stability of the zero solution of the approximate system.

In the second part of this paper, we consider that the contact rate between organisms and environmental poisons is disturbed by white noise, and then we establish a stochastic model. At the same time, we give the proof of the weak average persistence and stochastic persistence of the single population under certain conditions.

2. Stability analysis of deterministic model

2.1. Deterministic model

A single population model with psychological effects in polluted environment is proposed \[29\]

\[\dot{x}(t) = x(t)(b - d) - cx^2(t) - \alpha x(t)C_o(t) - \lambda x(t)g(C_e, \beta), \quad t \in [0, \infty) \]

The variables of the model are \(x = x(t)\), which means the number of population at time \(t\), \(C_o = C_o(t)\), the degree of individual toxins at time \(t\), \(C_e = C_e(t)\), the concentration of toxins in living environment at time \(t\), \(b, d, c\) represent the natural birth rate, mortality rate and biological intra-specific restriction factors when the toxin is zero. \(\alpha\) represent the response intensity of organism growth to the toxin. \(\lambda\) represent the contact rate between organism and environmental toxicant. \(\beta\) refers to the inhibition factors or psychological effects of the population in the polluted environment. To some extent, \(\beta\) describes the sensitivity of the population in the polluted environment. \(\alpha x(t)C_o(t)\) describes the removal amount of endotoxin in the organism at time \(t\). The toxic effect of the polluted environment is represented by \(g(C_e, \beta)\) \[26\]

\[g(C_e, \beta) = \frac{C_e}{1 + \beta C_e^p}\]

also known as contact rate or psychological effect. Age structure is introduced, and the specific model is as follows

\[\begin{align*}
\dot{x}(t) &= b y(t) - d_1 x(t) - \gamma x(t) - c_1 x^2(t) - \alpha_1 x(t)C_o(t) - \lambda_1 x(t)g_1(C_e(t), \beta), \\
\dot{y}(t) &= \gamma x(t) - d_2 y(t) - c_2 y^2(t) - \alpha_2 y(t)C_o(t) - \lambda_2 y(t)g_2(C_e(t), \beta),
\end{align*}\]

\(x, y\) refers to the number of young population and adult population, both of which are related to time \(t\). In this paper, we stipulate that the values of \(x\) and \(y\) are all in the range of \([1, \infty)\). \(\gamma\) represents the transformation rate from larva to adult, and the meaning of other symbols is consistent with the original. The toxin action in the polluted environment at time \(t\) is indicated by \(g_i(C_e, \beta), i = 1, 2\). Because there are differences in intelligence level and survival experience between larva and adult, we choose the following functions to describe the effect of environmental toxin respectively

\[g_1(C_e, \beta) = C_e, g_2(C_e, \beta) = \frac{C_e^p}{1 + \beta C_e^q}, 1 \leq p \leq q, p, q \in \mathbb{N} \]

It is assumed that the change rule of endotoxin concentration meets the following requirements

\[\dot{C}_o(t) = kC_e(t) - gC_o(t) - mC_o(t) - bC_o(t)\]

Among them, \(kC_e(t)\) is the ratio of biological absorption of environmental toxins at the time of \(t\). \(gC_o(t)\) is the ratio of biological elimination of toxins at the time of \(t\). \(mC_o(t)\) is the ratio of biological purification of
toxins at the time of $t$. $bC_\alpha(t)$ is the ratio of toxins lost by new individuals at the time of $t$. We always assume that the environmental capacity is large enough. We also assume that the toxins in new individuals should be very small, and the accumulation of toxins from birth to adulthood is very small, so the effect of toxin concentration on the density of a single population can be ignored. The change rate of toxin concentration in the environment is as follows

$$C_e(t) = u_e(t) - hC_e(t), \quad (6)$$

$hC_e(t)$ is the ratio of toxin reduction caused by environmental self-purification at time $t$. $hC_e(t)$ is the concentration of exogenous input toxin entering the environment at time $t$, and it is assumed to be a bounded non-negative differentiable function of time $t$.

A single population model with age structure and psychological effects in the polluted environment is established:

$$\begin{aligned}
\dot{x}(t) &= b y(t) - d_1 x(t) - \gamma x(t) - c_1 x(t)^2(t) - \alpha_1 x(t) C_\alpha(t) - \lambda_1 x(t) g_1(C_e(t), \beta), \\
\dot{y}(t) &= \gamma x(t) - d_2 y(t) - c_2 y(t)^2(t) - \alpha_2 y(t) C_\alpha(t) - \lambda_2 y(t) g_2(C_e(t), \beta), \\
\dot{C}_\alpha(t) &= k C_e(t) - g C_\alpha(t) - m C_\alpha(t) - b C_\alpha(t), \\
\dot{C}_e(t) &= u_e(t) - h C_e(t).
\end{aligned} \quad (7)$$

The study of Bronstein [33], Cantalupo [34], Castro [35], Dziewczynskia [36] showed that Betta splendens Regan was very aggressive in courtship, and the death of adult was mainly caused by intra-specific competition. Hagvar [37] and Uka [38] showed that the death of adult was mainly caused by intra-specific competition.

Therefore, according to these biological examples, When $d_2 y(t) \ll c_2 y^2(t)$, so $d_2 y(t) + c_2 y^2(t) \approx c_2 y^2(t)$, we modify the model [7] and rewrite it as

$$\begin{aligned}
\dot{x}(t) &= b y(t) - d x(t) - \gamma x(t) - c_1 x(t)^2(t) - \alpha_1 x(t) C_\alpha(t) - \lambda_1 x(t) C_e(t), \\
\dot{y}(t) &= \gamma x(t) - c_2 y(t)^2(t) - \alpha_2 y(t) C_\alpha(t) - \frac{\lambda_2 y(t) C_e(t)}{1 + \beta C_e(t)}, \\
\dot{C}_\alpha(t) &= k C_e(t) - g C_\alpha(t) - m C_\alpha(t) - b C_\alpha(t), \\
\dot{C}_e(t) &= u_e(t) - h C_e(t).
\end{aligned} \quad (8)$$

2.2. Stability analysis of nonlinear time-varying system

From (8), we can get

$$\begin{aligned}
C_\alpha(t) &= k \int_0^t C_e(s) e^{-(g+m+b)(t-s)} ds + C_\alpha(0) e^{-(g+m+b)t}, \\
C_e(t) &= \int_0^t u_e(s) e^{-b(t-s)} ds + C_e(0) e^{-bt}.
\end{aligned} \quad (9)$$

Denote $X(t) = (x(t), y(t))^T$, (9) could be described as follows

$$\dot{X}(t) = A(t) X(t) + G(X(t)), \quad (10)$$
Let By Routh-Hurwitz theorem, where
\[ A(t) = \begin{pmatrix} -d - \gamma \alpha_1 C_0(t) & -\alpha_2 C_0(t) \\ \gamma & -\frac{1}{1+\beta C_0(t)} \end{pmatrix}, \]
\[ G(X(t)) = \begin{pmatrix} -c_1 x_1^2(t) \\ -c_2 y_2(t) \end{pmatrix} \]
where \( A \) is the characteristic equation as follows
\[ \text{When } |A| > 0, \text{ the characteristic equation } A(t) \text{ gives the following roots } \lambda_i \text{ satisfying } \lambda_i |A| > 0. \]
\[ b \]
\[ -\alpha_2 C_0(t) \]
\[ \frac{1}{1+\beta C_0(t)} \]
\[ -\alpha_2 C_0(t) \]

Omitting the high-order term \( G(x(t)) \), obtaining the approximate linear system
\[ X(t) = A(t)X(t), \]
\[ (12) \]
where \( A(t) \) is called the first order approximate equation of \( (10) \). According to Lyapunov’s first approximation theory \([39]\), if \( G(X(t)) \) is a higher order infinitesimal of \( X(t) \) in the neighborhood of \( X = (0,0)^t \), the stability of \( (10) \) can often be studied by using the stability of linear system \( (12) \). In fact, we can easily get
\[ \lim_{t \to \infty} \frac{\|G(X(t))\|}{\|X(t)\|} = 0, \]
\[ (13) \]
So next we discuss the stability of linear system \( (12) \).

**Theorem 1.** If \( |A(t)| > 0 \), there exists an \( \varepsilon \) such that \( |a_{i,j}(t)| \leq \varepsilon \), then the trivial solution of model \( (12) \) is globally asymptotically stable.

**Proof.** Let
\[ A(t) = (a_{i,j}(t))_{2 \times 2} = \begin{pmatrix} -\Gamma_1(t) & b \\ \gamma & -\Gamma_2(t) \end{pmatrix}, \]
\[ (14) \]
where \( a_{i,j}(t) \) are differentiable with \( |a_{i,j}(t)| \leq b \vee (d + \gamma \alpha_1 + \lambda_1) \vee (\alpha_2 + \lambda_2) \vee \gamma \geq a \). The characteristic equation of \( A(t) \) gives the following roots
\[ \lambda(A(t)) = -(\Gamma_1(t) + \Gamma_2(t)) \pm \sqrt{(\Gamma_1(t) - \Gamma_2(t))^2 + 4by}. \]
\[ (15) \]
When \( |A(t)| > 0 \), each eigenvalue of \( A(t) \) has a negative real part, i.e. \( \text{Re}\lambda(A(t)) < 0 \). Rewrite the characteristic equation as follows
\[ f_A(\lambda) = \lambda^2 + p_1(t)\lambda + p_2(t) = 0, \]
\[ (16) \]
where
\[ p_1(t) = -\sum_{i=1}^{2} a_{i}(t) = \Gamma_1(t) + \Gamma_2(t), \quad p_2(t) = |A(t)| = \Gamma_1(t)\Gamma_2(t) - by. \]
\[ (17) \]
By Routh-Hurwitz theorem,
\[ \Delta_1(t) = p_1(t) > 0, \quad \Delta_2(t) = p_1(t)p_2(t) > 0. \]
\[ (18) \]
Let
\[ x = x_1, y = x_2, \]
\[ (19) \]
to find \( \nu(x_1, x_2) \) satisfying
\[ \sum_{i=1}^{2} \frac{\partial \nu}{\partial x_i} \sum_{j=1}^{2} a_{i,j} x_j = -2 \prod_{i=1}^{2} \Delta_i(t) \sum_{j=1}^{2} x_j^2. \]
\[ (20) \]
According to Barabashin formula,

\[ v(x_1, x_2) = \frac{\Delta_1(t)\Delta_2(t)}{\Delta(t)} \begin{vmatrix} 0 & x_1^2 & 2x_1x_2 & x_2^2 \\ 1 & a_{11} & a_{21} & 0 \\ 0 & a_{12} & a_{11} + a_{22} & a_{21} \\ 1 & 0 & a_{12} & a_{22} \end{vmatrix}, \]  

(21)

\[ \Delta(t) = \begin{vmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{11} + a_{22} & a_{21} \\ 0 & a_{12} & a_{22} \end{vmatrix}, \]  

(22)

which could be written as

\[ v(x_1, x_2) = C(t) \sum_{i,j=1}^{2} V_{ij}(t)x_ix_j, \]  

(23)

with \( V_{ij}(t) = v_{ij}(t), C(t) = \frac{\Delta_1(t)\Delta_2(t)}{\Delta(t)} \), where \( v_{ij} \) is the determinant obtained by exchanging the first column and the \( 2x_1x_2 \)-column (\( x_2^2 \)-column) and then removing the first row and the first column in the exchanged determinant. Therefore

\[ v(x_1, x_2) = \sum_{i,j=1}^{2} V_{ij}(t)x_ix_j, \]  

(24)

with \( V_{ij}(t) = C(t)v_{ij}(t) \) and \( V_{ij}(t) = V_{ji}(t) \). Next, we will prove that function \( v(x_1, x_2) \) is positive definite, given quadratic form

\[ W = -\prod_{i=1}^{2} \Delta(t) \sum_{j=1}^{2} x_j^2 \]  

(25)

From (20), we can find the uniquely determined \( v(x_1, x_2) \), so the \( v(x_1, x_2) \) expressed by (24) should be consistent with the Lyapunov function in reference [40].

\[
\begin{align*}
\Delta_1(t) & \Delta_2(t) = \Delta_1(t) \sum_{i,j=1}^{2} V_{ij}(t)x_ix_j \\
& = \Delta_2(t) \sum_{j=1}^{2} x_j^2 + \sum_{j=1}^{2} \left( \int_{s=1}^{s=p+1} \Delta_s(t) \Delta^2_{p,j}(x_1, x_2) \right) \\
& = \Delta_2(t) \sum_{j=1}^{2} x_j^2 + \sum_{j=1}^{2} \Delta_1(t) \Delta^2_{1,j}(x_1, x_2) \\
& = \Delta_2(t) \sum_{j=1}^{2} x_j^2 + \Delta_1(t) [(a_{22}x_1 - a_{12}x_2)^2 + (a_{21}x_1 - a_{11}x_2)^2] \\
& = \Delta_2(t) \sum_{j=1}^{2} x_j^2 + \Delta_1(t) [(\gamma_1(t)x_1 + bx_2)^2 + (\gamma_1 + |\gamma_2(t)|x_2)^2] \\
\end{align*}
\]  

(26)

Mark \( \Delta_{p,s}(x_1, x_2) \) can be found in reference [40], obviously

\[ v(x_1, x_2) \geq \Delta_2(t)(x_1^2 + x_2^2) = -\lambda_1\lambda_2(\lambda_1 + \lambda_2)(x_1^2 + x_2^2) \geq \delta^3(x_1^2 + x_2^2) \]  

(27)

Therefore, the above function \( v(x_1, x_2) \) is positive definite, so we can take the positive definite function \( v(x_1, x_2) \) as the Lyapunov function of the system (12),

\[
\begin{align*}
\frac{dv}{dt} & = -2\Delta_1(t)\Delta_2(t) \sum_{j=1}^{2} x_j^2 + \sum_{i,j=1}^{2} V_{ij}(t)x_ix_j \\
& \]  

(28)
where
\[ \sum_{i,j=1}^{2} V_{ij}(t)x_i x_j \leq \sum_{i,j=1}^{2} |V_{ij}(t)| x_i x_j \leq \frac{1}{2} \sum_{i,j=1}^{2} |\dot{V}_{ij}(t)(x_i^2 + x_j^2)| = \sum_{i=1}^{2} \sum_{i=1}^{2} |\dot{V}_{ij}(t)| x_i^2 \]  
(29)

Because \( V_{ij}(t) = C(t)v_{ij}(t) \), so
\[ \dot{V}_{ij}(t) = C(t)v_{ij}(t) + C(t)\dot{v}_{ij}(t) = \left( \sum_{i,j=1}^{2} \frac{\partial C}{\partial a_{ij}} \dot{a}_{ij} \right) v_{ij}(t) + C(t)\dot{v}_{ij}(t) \]  
(30)

Let \( |\dot{a}_{ij}(t)| \leq \varepsilon \), then
\[ |\dot{V}_{ij}(t)| \leq \varepsilon \left( \sum_{i,j=1}^{2} |\dot{a}_{ij}(t)| \right) |v_{ij}(t)| + \varepsilon |C(t)|P_{ij} \]  
(31)

where \( P_{ij} \) is the sum of the absolute values of the algebraic cofactors of the elements whose derivative is not zero in the determinant of \( V_{ij} \) (if an element appears in the form of \( a_{ij} + a_{jj} \), multiply by 2 before the absolute value of its algebraic cofactors), which is obtained from (31)
\[ |\dot{V}_{ij}(t)| \leq \varepsilon D(t)|v_{ij}(t)| + \varepsilon |C(t)|P_{ij}(t). \]  
(32)

Among them \( \sum_{i,j=1}^{2} \frac{\partial C}{\partial a_{ij}} = D(t) \), so
\[ \left( \frac{dv}{dt} \right) \leq -2\Delta_1(t)\Delta_2(t) \sum_{j=1}^{2} x_j^2 + \varepsilon \left[ D(t)Q_1(t) + |C(t)|P_1(t) \right] x_1^2 + D(t)Q_2(t) + |C(t)|P_2(t) \left( x_2^2 \right), \]  
(33)

where
\[ Q_1(t) = \sum_{j=1}^{2} |V_{1j}(t)|, Q_2(t) = \sum_{j=1}^{2} |V_{2j}(t)|, P_1(t) = \sum_{j=1}^{2} P_{1j}(t), P_2(t) = \sum_{j=1}^{2} P_{2j}(t), \]  
(34)

Take
\[ \varepsilon = \min \left\{ \frac{\Delta_1(t)\Delta_2(t)}{D(t)Q_1 + |C(t)|P_1}, \frac{\Delta_1(t)\Delta_2(t)}{D(t)Q_2 + |C(t)|P_2} \right\} > 0 \]  
(35)

so when \( |\dot{a}_{ij}(t)| \leq \varepsilon \), there are
\[ \left( \frac{dv}{dt} \right) \leq -2\Delta_1(t)\Delta_2(t) \sum_{j=1}^{2} x_j^2 \]  
(36)

By using the condition \( \text{Re}(\lambda(A(t))) \leq -\delta < 0 \)
\[ \left( \frac{dv}{dt} \right) \leq -K^* \sum_{j=1}^{2} x_j^2 \]  
(37)

where \( K^* \) is a constant independent of \( t \), so \( \left( \frac{dv}{dt} \right) \) is negative definite.

Since \( V(x_1, x_2) \) is positive definite and has an infinitesimal upper limit (obtained immediately from \( |a_{ij}(t)| \leq \delta \)), combined with the conclusion proved previously, the zero solution of the system (12) is asymptotically stable. Because the system (12) is linear, the stability has global properties. \( \square \)
3. Survival analysis of stochastic single population model

3.1. Stochastic Model

Because the contact rate between a single population and environmental toxins is inevitably affected by other conditions, such as weather conditions, temperature and other nearby noises, it is natural to think that constant contact rate \( \lambda_i \) is replaced by random variable \( \lambda_i + \sigma_i \xi_i(t) \), where \( \xi_i(t) \) is white noise, and \( \xi_i(t) = \frac{dB_i(t)}{dt} \), \( B_i(t) \) are two independent standard Brown motions defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Therefore, we get random single species with psychological effects group model

\[
\begin{align*}
    dx(t) &= \left[by(t) - dx(t) - \gamma x(t) - c_1x^2(t) - \alpha_1 x(t)C_\sigma(t) - \lambda_1 x(t)C_e(t)\right]dt - \sigma_1 x(t)C_e(t)dB_1(t), \\
    dy(t) &= \left[\gamma x(t) - c_2y^2(t) - \alpha_2 y(t)C_\sigma(t) - \frac{\lambda_2 y(t)C_e^p(t)}{1 + \beta C_e^q(t)}\right]dt - \frac{\sigma_2 y(t)C_e^p(t)}{1 + \beta C_e^q(t)}dB_2(t), \\
    dC_o(t) &= [kC_e(t) - gC_o(t) - mC_o(t) - bC_o(t)]dt, \\
    dC_e(t) &= [u_e(t) - hC_e(t)]dt.
\end{align*}
\]

(38)

For the convenience of subsequent derivation, we propose the following lemmas:

**Lemma 2.** Under the condition of non-pollution, the system (38) has a unique equilibrium point \((A, B)\) in the first quadrant.

**Proof.** Under the condition of non-pollution, the system (38) is simplified as

\[
\begin{align*}
    \dot{x}(t) &= by(t) - dx(t) - \gamma x(t) - c_1x^2(t), \\
    \dot{y}(t) &= \gamma x(t) - c_2y^2(t),
\end{align*}
\]

(39)

The equilibrium point of the system is required, that is, to find the solution of the following equations

\[
\begin{align*}
    0 &= by(t) - dx(t) - \gamma x(t) - c_1x^2(t), \\
    0 &= \gamma x(t) - c_2y^2(t),
\end{align*}
\]

(40)

Since the parameters of the system are all positive, there is only one solution in the first quadrant of the equation group, which is set as \((A, B)\). For the convenience of writing, we write the equation group (40) in the following form:

\[
\begin{align*}
    x &= \tilde{a}y^2, \\
    y &= \tilde{b}x + \tilde{c}x^2,
\end{align*}
\]

(41)

of which

\[
\tilde{a} = \frac{c_2}{\gamma}, \quad \tilde{b} = \frac{d + \gamma}{b}, \quad \tilde{c} = \frac{c_1}{b}.
\]

(42)

The solution of equations (41) in the first quadrant is \((A, B)\), where

\[
A = \tilde{a}B^2, \quad B = \Lambda - \frac{\tilde{b}}{3\tilde{a}\tilde{c}}, \quad \Lambda = \left(\frac{\sqrt{27\tilde{c}} + 4\tilde{a}\tilde{b}^2}{2\sqrt{27\tilde{a}^2\tilde{c}^2}} + \frac{1}{2\tilde{a}\tilde{c}}\right)^{\frac{1}{2}}
\]

(43)

\[
\square
\]

**Lemma 3.** For models (8) and (38), if \( k \leq g + m + b, \; u_e^* \leq h \), then, \( 0 \leq C_o(t) < 1, \; 0 \leq C_e(t) < 1 \) for \( t \in \mathbb{R}_+ \).
Proof. The explicit solutions $C_o(t)$, $C_e(t)$ are as follows:

$$
C_o(t) = k \int_0^t C_e(s)e^{-(g+m+b)(t-s)}ds + C_o(0)e^{-(g+m+b)t},
$$

$$
C_e(t) = \int_0^t u_p(s)e^{-ht(s)}ds + C_e(0)e^{-ht}.
$$

for the given initial conditions $0 \leq C_o(0) < 1$, $0 \leq C_e(0) < 1$, the explicit solutions can be estimated

$$
C_o(t) \leq 1 - e^{-(g+m+b)t} + C_o(0)e^{-(g+m+b)t} < 1,
$$

$$
C_e(t) \leq 1 - e^{-ht} + C_e(0)e^{-ht} < 1.
$$

So $0 \leq C_o(t) < 1$ and $0 \leq C_e(t) < 1$ hold. \hfill \Box

3.2. Existence and uniqueness of a global positive solution

**Theorem 4.** For any initial value $(x(0), y(0), C_o(0), C_e(0)) \in \mathbb{R}_+^4$, there is a unique solution of model (38), and the solution does not leave $\mathbb{R}_+^4$ according to probability 1.

**Proof.** Our proof refers to the work of Mao [41]. By Lyapunov method mentioned in Lemma 2 of [42], and the references [43, 44], the coefficients of model (38) are locally Lipschitz continuous. On $[0, \tau_e)$, for any initial value $(x(0), y(0), C_o(0), C_e(0)) \in \mathbb{R}_+^4$, there is a local solution $(x(t), y(t), C_o(t), C_e(t))$, where $\tau_e$ is the blow up time. In order to prove that the solution is global, we need to prove that $\tau_e = \infty$ holds almost surely. Let $m_0 > 1$ be large enough, for $m > m_0$, each component of the initial value $(x(0), y(0), C_o(0), C_e(0)) \in \mathbb{R}_+^4$ is in the interval $[\frac{1}{m}, m_0]$. Define the stop time

$$
\tau_m = \inf \left\{ t \in [0, \tau_e) : \min \{x(t), y(t), C_o(t), C_e(t)\} \leq \frac{1}{m} \text{ or } \max \{x(t), y(t), C_o(t), C_e(t)\} \geq m \right\}.
$$

In this paper, we set $\inf \phi = \infty$. Obviously, when $m \to \infty$, $\tau_m$ increases and $\tau_m < \tau_e$. We denote $\tau_\infty = \lim_{m \to \infty} \tau_m$. We claim that for all $t \geq 0$, $(x(t), y(t), C_o(t), C_e(t)) \in \mathbb{R}_+^4$, and $\tau_\infty = \infty$. Use the counter evidence method below. Otherwise, for any $m > m_0$, there is a pair of constants $T > 0$, $\varepsilon \in (0, 1)$ such that $\forall \mathbb{P}(\tau_m \leq T) \geq \varepsilon$. We define a $C^2$-function as follows: $V_1 : \mathbb{R}_+^4 \to \mathbb{R}_+$, and let $x = x(t), y = y(t), C_o = C_o(t), C_e = C_e(t)$.

$$
V_1(t) = (x - 1 - \ln x) + (y - 1 - \ln y) + (C_o - 1 - \ln C_o) + (C_e - 1 - \ln C_e).
$$

For all $t \in [0, \tau_e)$, according to Itô’s formula, together with $V_1(t) := V_1(x(t), y(t), C_o(t), C_e(t))$

$$
dV_1(t) = \left(1 - \frac{1}{x}\right)dx + \left(1 - \frac{1}{y}\right)dy + \left(1 - \frac{1}{C_o}\right)dC_o + \left(1 - \frac{1}{C_e}\right)dC_e
$$

$$
+ \frac{1}{2x^2}(dx)^2 + \frac{1}{2y^2}(dy)^2
$$

$$
= \mathcal{L}V_1 dt - (x - 1)\sigma_1C_e dB_1(t) - (y - 1)\sigma_2C_e dB_2(t)
$$

$$
+ \frac{\sigma_2^2C_e^p}{1 + \beta C_e^q} dB_2(t)
$$

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where

\[
\mathcal{L} V_1(t) = (x - 1) \left( \frac{b^2}{y} - d - \frac{c_1}{y} - \alpha_1 C_0(t) - \lambda_1 t \right) + \frac{1}{2} \sigma_1^2 \int_0^t C_2 e^{\lambda_1 (t-s)} ds \\
+ (y - 1) \left( \frac{x}{y} - c_2 y - \alpha_2 C_0 - \frac{\lambda_2 C_0'}{1 + \beta C_0'} \right) + \frac{\sigma_2^2 C_2'}{2(1 + \beta C_0')^2} \\
+ kC_e - (g + m + b)C_0 - k \frac{C_e}{C_0} + g + m + b + u_e - C_e - \frac{u_e}{C_e} + h \\
\]

\[
< (c_1 - d) x - c_1 x^2 + (b + c_2) y - c_2 y^2 + (\alpha_1 + \alpha_2) + (\lambda_1 + \lambda_2) + \frac{\sigma_1^2 + \sigma_2^2}{2} \\
+ y + k + g + m + b + d + (u_e)^* + h \\
\leq \frac{(c_1 - d)^2}{4c_1} + \frac{(b + c_2)^2}{4c_2} + (\alpha_1 + \alpha_2) + (\lambda_1 + \lambda_2) + \frac{\sigma_1^2 + \sigma_2^2}{2} \\
+ y + k + g + m + b + d + (u_e)^* + h \\
:= M > 0.
\]

Hence it may be concluded that

\[
dV_1(t) < Mdt - (x - 1) \sigma_1 C_2 dB_1(t) - (y - 1) \frac{\sigma_2 C_2'}{1 + \beta C_0'} dB_2(t).
\]

Integrate the two sides of (47) from 0 to \( \tau_m \land T \), and then take the expectation to obtain the following inequality

\[
\mathbb{E}(V_1(\tau_m \land T)) < V_1(0) + \mathbb{E} \int_0^{\tau_m \land T} Md s \\
- \mathbb{E} \int_0^{\tau_m \land T} (x(s) - 1) \sigma_1 C_2(s) dB_1(s) \\
- \mathbb{E} \int_0^{\tau_m \land T} (y(s) - 1) \frac{\sigma_2 C_2'(s)}{1 + \beta C_0'(s)} dB_2(s) \\
\leq V_1(0) + MT.
\]

Note that \( \Omega_m = \{ \tau_m \land T \} \), then \( \mathbb{P}(\Omega_m) \geq \epsilon \). For any \( w \in \Omega_m \), by the definition of stop time, there is

\[
x(\tau_m, \omega) \land y(\tau_m, \omega) \land C_0(\tau_m, \omega) \land C_e(\tau_m, \omega) = \frac{1}{m},
\]

or

\[
x(\tau_m, \omega) \lor y(\tau_m, \omega) \lor C_0(\tau_m, \omega) \lor C_e(\tau_m, \omega) = m.
\]

Therefore, there are

\[
V_1(\tau_m, \omega) \geq \min \left\{ m - 1 - \ln m, \frac{1}{m} - 1 + \ln m \right\}
\]

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Then inequality (48) is equivalent to
\[
V_1(0) + MT \geq \mathbb{E}[1_{\Omega_m}(\omega) \cdot V_1(\tau_m \wedge T)]
\]
\[
\geq \mathbb{P}\{\Omega_m\} \cdot \min\left\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\right\}
\]
\[
\geq \varepsilon \min\left\{m - 1 - \ln m, \frac{1}{m} - 1 + \ln m\right\}.
\]
where \(1_{\Omega_m}(\omega)\) is the demonstrative function of \(\Omega_m\). Let \(m \to \infty\), we can get
\[
+\infty > V_1(0) + MT \geq +\infty.
\]
The contradiction arises, therefore, \(\mathbb{P}\{\tau_m = \infty\} = 1\).

3.3. Weakly persistent in the mean

**Theorem 5.** Let \(X(0) = (x(0), y(0), C_o(0), C_e(0))\) is the initial value, \(X(t)\) be a solution of model (38). If the coefficients of (38) satisfy
\[
\sigma_1^2 < \frac{bB}{A} + c_1A - \frac{b}{2} - \frac{\gamma}{2}, \quad \sigma_2^2 < \frac{\gamma A}{B} + c_2B - \frac{b}{2} - \frac{\gamma}{2},
\]
\[
k < 2(g + m + b), \quad \sigma_1^2A^2 + \sigma_2^2B^2 < 2h - k - 1.
\]
Then the solution \(X(t)\) of (38) has the properties:
\[
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \int_{0}^{t} \left(\eta_1(x(s) - A)^2 + \eta_2(y(s) - B)^2 + \eta_3C_o^2(s) + \eta_4C_e^2(s)\right) ds \leq K.
\]
In other words, in the long run, the single population near the pollution-free equilibrium point \((A, B, 0, 0)\) is weak persistent in the mean.

**Proof.** We set new variables
\[
u_1(t) = x(t) - A, \quad \nu_2(t) = y(t) - B, \quad v(t) = C_o(t), \quad w(t) = C_e(t),
\]
where the expressions of \(A\) and \(B\) are given in Lemma 2. Therefore, the model (38) is rewritten as follows:

\[
\begin{align*}
\frac{du_1}{dt} &= (u_1 + A) \left(\frac{b(u_2 + B)}{u_1 + A} - \frac{bB}{A} - c_1u_1 - \alpha_1v - \lambda_1w\right) dt - (u_1 + A) \sigma_1 dB(t), \\
\frac{du_2}{dt} &= (u_2 + B) \left(\frac{\gamma (u_1 + A)}{u_2 + B} - \frac{\gamma A}{B} - c_2u_2 - \alpha_2v - \frac{\lambda_2 w^p}{1 + \beta w^q}\right) dt - (u_2 + B) \frac{\sigma_2 w^p}{1 + \beta w^q} dB(t), \\
\frac{dv}{dt} &= [k u - (g + m + b)v] dt, \\
\frac{dw}{dt} &= [u_e - hw] dt.
\end{align*}
\]
We define a \(C^2\)-function \(V_2 = u_1^2 + u_2^2 + v^2 + w^2\). From Itô’s formula
\[
\frac{dV_2}{dt} = \mathcal{L}V_2 dt - 2(u_1 + A) \sigma_1 u_1 dB(t) - 2(u_2 + B) \frac{\sigma_2 u_2 w^p}{1 + \beta w^q} dB(t),
\]
\[
\text{in which, } \mathcal{L} = \frac{bB}{A} + c_1A - \frac{b}{2} - \frac{\gamma}{2}, \quad \mathcal{L}^* = \frac{\gamma A}{B} + c_2B - \frac{b}{2} - \frac{\gamma}{2},
\]
\[
\text{and } \mathcal{L} < \mathcal{L}^*.
\]
by the elementary equality $2xy \leq x^2 + y^2$ for $x > 0$ and $y > 0$, we get

$$
\mathcal{L}V_2 = 2u_1 (u_1 + A)\left(\frac{b(u_2 + B)}{u_1 + A} - \frac{bB}{A} - c_1 u_1 - \alpha_1 v - \lambda_1 w\right) + (u_1 + A)^2 \sigma_1^2 w^2
+ 2u_2 (u_2 + B)\left(\frac{\gamma(u_1 + A)}{u_2 + B} - \frac{\gamma A}{B} - c_2 u_2 - \alpha_2 v - \frac{\lambda_2 w^2}{1 + \beta w}\right)
+ \sigma_2^2 \left(\frac{w^2}{(1 + \beta w)^2}\right)
+ 2v\left[kw - (g + m + b)v\right] + 2w (u_e - hw)

< 2b\eta u_1 u_2 - \frac{2bB}{A} \eta u_1^2 + \frac{2\gamma A}{B} \eta u_1^2 - 2c_1 \eta u_1^2 + \sigma_1^2 u_1^2
+ 2\sigma_1^2 Au_1 w^2 + \sigma_1^2 A^2 w^2 - 2c_2 Bu_2 + \sigma_2^2 u_2^2 + 2\sigma_2^2 Bu_2 w^2 + \sigma_2^2 B^2 w^2
+kw^2 + kv^2 - 2\eta u_1 u_2 - 2\eta u_1 \eta u_2 - \eta u_1 \eta u_2 - \eta u_2^2 - 2hw^2

\leq -\eta u_1^2 - \eta u_2^2 - \eta u_3^2 - \eta u_4 w^2 + K,

where

$$
\eta_1 = \frac{2bB}{A} + 2c_1 A - 2\sigma_1^2 - b - \gamma,
\eta_2 = \frac{2\gamma A}{B} + 2c_2 B - 2\sigma_2^2 - b - \gamma,
\eta_3 = 2\eta u_1 \eta u_2 - 2\eta u_1 \eta u_2 - \eta u_1 \eta u_2 - \eta u_2^2 - 2hw^2

\leq -\eta u_1^2 - \eta u_2^2 - \eta u_3^2 - \eta u_4 w^2 + K,

\text{where}

$$
\eta_1 = \frac{2bB}{A} + 2c_1 A - 2\sigma_1^2 - b - \gamma,
\eta_2 = \frac{2\gamma A}{B} + 2c_2 B - 2\sigma_2^2 - b - \gamma,
$$

Integrating from 0 to $t$ on both sides of (58) and taking the expectation

$$
\mathbb{E}\int_0^t \left(\eta_1 u_1^2(s) + \eta_2 u_2^2(s) + \eta_3 v^2(s) + \eta_4 w^2(s)\right) ds < \mathbb{E}(V_2(0) - V_2(t)) + K t \leq K t.

(61)
$$

Further, we obtain

$$
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E}\int_0^t \left(\eta_1 u_1^2(s) + \eta_2 u_2^2(s) + \eta_3 v^2(s) + \eta_4 w^2(s)\right) ds \leq K.

(62)
$$

3.4. Stochastic permanence

**Theorem 6.** If the coefficients of the model satisfy

$$
\limsup_{t \to \infty} \left(\alpha_1 + \alpha_2\right) C_0(t) + \lambda_1 C_e(t) + \frac{\lambda_2 C_p(t)}{1 + \beta C_e(t)} + 3\sigma^2 C^2_e(t)\right) \leq 4 \sqrt{by} - \eta - \frac{c^2_1 + c^2_2}{2} - d - \gamma,

(63)
$$

where $\sigma = \max(\sigma_1, \sigma_2)$, then the total number of the juveniles and the adults is stochastic permanence.

**Proof.** We choose positive constants $\theta$ and $\eta$ such that $\theta \geq 1, \eta \geq 1$, and define $C^2$-function $V_3 : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$
V_3 = e^{\eta \left(1 + \frac{1}{x^2} + \frac{1}{y^2}\right)}.

(64)
$$
By Itô’s formula, we obtain that
\[
dV_3 = \eta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \left( \frac{\sigma_1 C_{\varepsilon}^p}{x^2} dB_1(t) + \frac{\sigma_1 C_{\varepsilon}^p}{y^2 (1 + \beta C_{\varepsilon}^p)} dB_2(t) \right)
\]
\[
+ \theta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \frac{3}{x^4} + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-2} \frac{2}{x^6} (dx)^2
\]
\[
+ \theta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \frac{3}{y^4} + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-2} \frac{2}{y^6} (dy)^2
\]
\[
= \mathcal{L} V_3 dt + M(t),
\]
where
\[
M(t) = 2 \theta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \left( \sigma_1 C_{\varepsilon}^p \frac{3}{x^2} + \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \frac{2(\theta-1)}{x^4} \right)
\]
\[
+ \frac{\sigma_1 C_{\varepsilon}^p}{(1 + \beta C_{\varepsilon}^p)^2} \left( \frac{3}{y^2} + \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \frac{2(\theta-1)}{y^4} \right)
\]
\[
\mathcal{L} V_3 = \eta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \left( b \frac{\partial}{\partial x} - d - \gamma - c_1 x - \alpha_1 C_o - \lambda_1 C_e \right)
\]
\[
+ \frac{1}{x^2} \left( \frac{\gamma}{y} - c_2 y - c_2 x - \alpha_2 C_o - \lambda_2 C_e \right)
\]
\[
+ \theta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \left( \sigma_1 C_{\varepsilon}^p \frac{3}{x^2} + \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \frac{2(\theta-1)}{x^4} \right)
\]
\[
+ \frac{\sigma_1 C_{\varepsilon}^p}{(1 + \beta C_{\varepsilon}^p)^2} \left( \frac{3}{y^2} + \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-1} \frac{2(\theta-1)}{y^4} \right)
\]
\[
\Rightarrow \theta e^{\eta t} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta-2} (I_{31} + I_{32} + I_{33}).
\]
After simplification, we derive
\[
I_{31} = \eta \left[ 1 + 2 \left( \frac{1}{x^2} + \frac{1}{y^2} \right) + \left( \frac{1}{x^2} + \frac{1}{y^2} \right)^2 \right],
\]
\[
I_{32} < -2 \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \frac{\sqrt{2} \gamma y}{xy} - \frac{1}{x^2} \right) \left( d + \gamma + \alpha_1 C_o + \lambda_1 C_e \right)
\]
\[
- \frac{1}{y^2} \left( \sigma_2 C_o + \frac{\lambda_2 C_{\varepsilon}^p}{1 + \beta C_{\varepsilon}^p} \right) \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \frac{c_1}{x} + \frac{c_2}{y} \right),
\]
\[
I_{33} = 3 \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \sigma_1 C_{\varepsilon}^p \frac{3}{x^2} + \frac{\sigma_2 C_{\varepsilon}^p}{y^2 (1 + \beta C_{\varepsilon}^p)^2} \right) + 2(\theta-1) \left( \frac{\sigma_1 C_{\varepsilon}^p}{x^4} + \frac{\sigma_2 C_{\varepsilon}^p}{y^2 (1 + \beta C_{\varepsilon}^p)^2} \right)
\]
\[
< 3 \sigma^2 C_{\varepsilon}^p \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \left( \frac{1}{x^2} + \frac{1}{y^2} \right) + 2(\theta-1) \sigma^2 C_{\varepsilon}^p \left( \frac{1}{x^4} + \frac{1}{y^4} \right),
\]
Integrating (65), together with (63), of Theorem 6, we get
\[ I_{31} + I_{32} + I_{33} < \frac{\eta}{\theta} + \frac{2c_1}{x^2} + \frac{2c_1}{x^3} + \frac{2c_2}{y} + \frac{2c_2}{y^3} \]
\[ + \left( \frac{2\eta}{\theta} + 2(d + \gamma + \alpha_1 C_\omega + \lambda_1 C_e) + 3\sigma^2 C_e^2 + 1 \right) \frac{1}{x^2} \]
\[ + \left( \frac{2\eta}{\theta} + 2 \left( \alpha_2 C_\omega + \frac{\lambda_2 C_e}{1 + \beta C_e} \right) + 3\sigma^2 C_e^2 + 1 \right) \frac{1}{y^2} \]
\[ + \left( \frac{\eta}{\theta} + 2 (d + \gamma + \alpha_1 C_\omega + \lambda_1 C_e) + (3 + 2(\theta - 1))\sigma^2 C_e^2 \right) \frac{1}{x^3} \]
\[ + \frac{\eta}{\theta} + 2 \left( \alpha_2 C_\omega + \frac{\lambda_2 C_e}{1 + \beta C_e} \right) + (3 + 2(\theta - 1))\sigma^2 C_e^2 \]
\[ + 2 \left( \frac{c_1 + c_2}{2} - 4\sqrt{by} \right) \left[ \frac{1}{x^3} - \frac{4\sqrt{by}}{xy} \right]^2. \]

By condition (63) of Theorem 6, we get
\[ I_{31} + I_{32} + I_{33} < \frac{\eta}{\theta} + \frac{2c_1}{x^2} + \frac{2c_1}{x^3} + \frac{2c_2}{y} + \frac{2c_2}{y^3} \]
\[ + \left( \frac{2\eta}{\theta} + 2(d + \gamma + \alpha_1 + \lambda_1) + 3\sigma^2 + 1 \right) \frac{1}{x^2} \]
\[ + \left( \frac{2\eta}{\theta} + 2 \left( \alpha_2 + \alpha_3 \right) + 3\sigma^2 + 1 \right) \frac{1}{y^2} \]
\[ + \left( \frac{\eta}{\theta} + 2 (d + \gamma + \alpha_1 + \lambda_1) + (3 + 2(\theta - 1))\sigma^2 \right) \frac{1}{x^3} \]
\[ + \left( \frac{\eta}{\theta} + 2 (\alpha_2 + \lambda_3) + (3 + 2(\theta - 1))\sigma^2 \right) \frac{1}{y^3} \]
\[ := \quad F(x, y). \]

It is obvious that
\[ H(x, y) := F(x, y)G(x, y), \quad G(x, y) = \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^{\theta - 2} \]

admits an upper boundary for all \(x\) and \(y\) in the domain \(\mathbb{R}_+\). Let us denote
\[ H_1 = \sup_{(x, y) \in \mathbb{R}_+^2} H(x, y) < \infty. \]

Integrating (65), together with \(x_0 = x(0), \ y_0 = y(0)\), which gives that
\[ e^{\theta} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right) \theta \left( 1 + \frac{1}{x_0^2} + \frac{1}{y_0^2} \right) \theta \frac{H_1 \theta}{\eta} (e^{\eta} - 1) + \int_{x_0}^{x} M(s)ds, \]
taking the expectation on both sides, then
\[
E \left[ \frac{1}{2^\theta} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^\theta \right] < E \left[ \frac{e^{-\eta t}}{2^{\theta}} \left( 1 + \frac{1}{x_0^2} + \frac{1}{y_0^2} \right)^\theta + \frac{H_1 \theta}{2^\theta} (1 - e^{-\eta t}) + \frac{e^{-\eta t}}{2^\theta} \int_0^t M(s) ds \right]. \tag{75}
\]
Therefore, together with (75), taking supremum limit, we derive that
\[
\limsup_{t \to \infty} \mathbb{E} \left[ \frac{1}{2^\theta} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^\theta \right] \leq \limsup_{t \to \infty} \mathbb{E} \left[ \frac{1}{2^\theta} \left( 1 + \frac{1}{x^2} + \frac{1}{y^2} \right)^\theta \right] < \frac{H_1 \theta}{2^\theta} : = H_2. \tag{76}
\]
For any given \( \varepsilon > 0 \), we let
\[
\xi = \left( \frac{\varepsilon}{H_2} \right)^{\frac{1}{\theta}},
\]
the Chebyshev inequality gives that
\[
\mathbb{P}\{x + y < \xi\} = \mathbb{P}\{(x + y)^{-\theta} > \xi^{-\theta}\} \leq \frac{\mathbb{E}\{(x + y)^{-2\theta}\}}{\xi^{-2\theta}} = \xi^{2\theta} E \{(x + y)^{-2\theta}\}.
\]
Combined with (76), we have
\[
\lim_{t \to \infty} \mathbb{P}\{x + y < \xi\} < \varepsilon, \tag{77}
\]
which implies that
\[
\liminf_{t \to \infty} \mathbb{P}\{x + y \geq \xi\} \geq 1 - \varepsilon. \tag{78}
\]
We define a \( C^2 \)-function \( V_4 = e^l (x^m + y^n)^l, 0 < l < 1, m, n \in \mathbb{N}_+ \). By Itô’s formula again
\[
dV_4 = e^l (x^m + y^n)^l dt + le^l (x^m + y^n)^l - 1 (m x^{m-1} dx + n y^{n-1} dy) + \frac{1}{2} l(l - 1) e^l (x^m + y^n)^{l-2} \left( m^2 x^{2(m-1)} (dx)^2 + n^2 y^{2(n-1)} (dy)^2 \right) + \frac{1}{2} le^l (x^m + y^n)^{l-1} \left( m(m - 1) x^{m-2} (dx)^2 + n(n - 1) y^{n-2} (dy)^2 \right) = \mathcal{L}V_4 dt - le^l (x^m + y^n)^{l-1} \left( \frac{m \sigma_1 x^m C_n dB_1(t)}{1 + \beta C_n^2} + \frac{m \sigma_2 y^n C_n^\theta}{1 + \beta C_n^2} dB_2(t) \right).
\]
\[ L V_4 = e'(x^m + y^n)l + le'(x^m + y^n)^{l-1}\left[ (b^Y_x - d - \gamma - c_1 x - \alpha_1 C_o - \lambda_1 C_e) m x^m + \right. \\
+ \left. (y^x - c_2 y - \alpha_2 C_o - \frac{\lambda_1 C_e^p}{1 + \beta C_e}) n y^n \right] \\
+ \frac{1}{2} l (l - 1) e'(x^m + y^n)^{l-2} \left( m^2 \sigma_1^2 x^{2m} C_e^p + \frac{n^2 \sigma_2^2 y^{2n} C_e^{2p}}{(1 + \beta C_e)^2} \right) \\
+ \frac{1}{2} l e'(x^m + y^n)^{l-1} \left( m(m - 1) \sigma_1^2 x^{m^2} C_e^p + \frac{n(n - 1) \sigma_2^2 y^{n^2} C_e^{2p}}{(1 + \beta C_e)^2} \right) \]

\[ < e'(x^m + y^n)^{l-1} \left\{ x^m + y^n + l \left[ (b^Y_x - d - \gamma - c_1 x - \alpha_1 C_o - \lambda_1 C_e) m x^m + \right. \\
+ \left. (y^x - c_2 y - \alpha_2 C_o - \frac{\lambda_1 C_e^p}{1 + \beta C_e}) n y^n \right] \\
+ \frac{1}{2} l (m(m - 1) \sigma_1^2 x^{m^2} C_e^p + \frac{n(n - 1) \sigma_2^2 y^{n^2} C_e^{2p}}{(1 + \beta C_e)^2} \right) \right\} \]

\[ \leq e'(x^m + y^n)^{l-1} \left\{ x^m + y^n \right\} + \frac{b m l}{2} (x^{2(m-1)} + y^2) + \frac{m l}{2} (x^2 + y^{2(n-1)}) - c_1 m l x^{m+1} - c_2 n l y^{n+1} \]

\[ + \left[ l \left( -d - \alpha_1 C_o - \lambda_1 C_e \right) m x^m + \left( -\alpha_2 C_o - \frac{\lambda_1 C_e^p}{1 + \beta C_e} \right) n y^n \right] \\
+ \frac{1}{2} l (m(m - 1) \sigma_1^2 x^{m^2} C_e^p + \frac{n(n - 1) \sigma_2^2 y^{n^2} C_e^{2p}}{(1 + \beta C_e)^2} \right) \]}

\[ = e'(x^m + y^n)^{l-1} \left[ x^m \right. \\
+ \left[ 1 + \frac{1}{2} m(m - 1) \sigma_1^2 C_e^p - m l (d + \alpha_1 C_o + \lambda_1 C_e) \right] x^m \\
+ \left. \left[ 1 + \frac{n(n - 1) \sigma_2^2 C_e^{2p}}{2(1 + \beta C_e)^2} - n l (\alpha_2 C_o + \frac{\lambda_1 C_e^p}{1 + \beta C_e}) \right] y^n \right) \]

\[ + \frac{b m l}{2} (x^{2(m-1)} + y^2) + \frac{m l}{2} (x^2 + y^{2(n-1)}) \]

We take \( m = n = 2 \), then

\[ L V_4 \leq e'(x^2 + y^2)^{l-1} \left\{ -2c_1 l x^3 - 2c_2 y^3 \\
+ \left[ 1 + l \sigma_1^2 C_e^p - 2 l (d + \alpha_1 C_o + \lambda_1 C_e) + b l + \gamma \right] x^2 \\
+ \left[ 1 + \frac{l \sigma_2^2 C_e^{2p}}{2(1 + \beta C_e)^2} - 2 l \left( \alpha_2 C_o + \frac{\lambda_1 C_e^p}{1 + \beta C_e} \right) + b l + \gamma \right] y^2 \right\} \]

\[ := e' J(x, y) \leq C e' \]

taking the expectation on both sides of (80), then

\[ \mathbb{E}(e'(x^2 + y^2)^l) - (x^2(0) + y^2(0))^l < C(e' - 1), \]
where $C$ is a constant. Due to $x + y \leq x^2 + y^2$, we have
\[
\liminf_{t \to \infty} \mathbb{E}(x^2 + y^2) < C.
\] (82)

For any given $\varepsilon > 0$, we let
\[
N = \left(\frac{C}{\varepsilon}\right)^rac{1}{2},
\] (83)
by Chebyshev inequality
\[
\mathbb{P}\{x + y > N\} = \mathbb{P}\{(x + y)\frac{1}{2} > N\frac{1}{2}\} \leq \frac{\mathbb{E}(x + y)^\frac{1}{2}}{N^\frac{1}{2}}.
\] (84)

Combined with (82), we have
\[
\liminf_{t \to \infty} \mathbb{P}\{x + y > N\} < \varepsilon,
\] (85)
which implies that
\[
\limsup_{t \to \infty} \mathbb{P}\{x + y \leq N\} \geq 1 - \varepsilon.
\] (86)

In other words, the single population is random and persistent.

4. Numerical simulations

To support the main conclusions of this paper, we selected the appropriate parameters and initial values for each conclusion. The main method of numerical simulation is to discretize the original system and use matlab to observe the trend of the graph. Then verify the conclusions of this paper. At the same time of verifying random and persistent survival, under the premise that the remaining parameters are fixed, we take 4 sets of different psychological effect intensities for numerical simulation.

The parameters and their reference value are shown on this table:

| Parameter | Value | Reference value | Source |
|-----------|-------|-----------------|--------|
| $b$       | 0.3   | [0.1,0.3]       | 29,32,45 |
| $d$       | 0.01  | 0.01            | 29     |
| $\gamma$  | 0.5   | [0.2,0.4]       | 32,45  |
| $c_1$     | 0.02  |                | Assumed |
| $c_2$     | 0.05  | [0.1,0.4]       | 29,32,45 |
| $\alpha_1$| 0.2   |                | Assumed |
| $\alpha_2$| 0.1   | [0.1,0.5]       | 29,32,45 |
| $\lambda_1$| 0.07  |                | Assumed |
| $\lambda_2$| 0.05  | 0.05            | 29,32   |
| $k$       | 0.1   | [0.1,0.6]       | 29,32,45,46 |
| $g$       | 0.08  | [0.08,0.3]      | 29,32,45,46 |
| $m$       | 0.04  | [0.04,0.3]      | 29,32,45,46 |
| $h$       | 0.8   | [0.1,0.5]       | 29,32,45,47 |
| $u_e$     | [0,0.6]| [0,0.8]         | 29,45,46 |
| $\beta$   | [0.1,10]| [0.1,10]        | 29,32,45,46 |
| $\sigma_1, \sigma_2$ | [0.1,0.3] | [0.1,0.3] | 29 |
4.1. Simulation of globally unique positive solution

The solution of equations (41) in the first quadrant is the globally unique positive solution, noted as $(A, B)$. Take the value in Table 1 then we can get $A = 2.8078$ and $B = 5.2989$. We draw and calibrate $(A, B)$ as shown in the Figure 1.

![Figure 1. The globally unique positive solution of equations (41)](image)

4.2. Simulations of weakly persistent in the mean

Here we use numerical simulation results to support the conclusion of Theorem 3. We determine the parameters and initial values that meet the conditions of the theorem. Respectively, we select two toxin input levels for numerical simulation, through the image branch of numerical simulation to support the conclusion of subsection 3.3.

Considering that theorem 3 requires non-pollution, according to the actual situation, we take the smaller toxin input level $u_e(t) = 0.1$ and $u_e(t) = 0.1 + 0.1 \sin(t)$, and the psychological effect intensity $\beta = 0.1$, and then take the remaining parameters the same as Table 1 while the variable parameters are shown on this Table 2.
Table 2. Variable parameters and their value

| Group   | $u_e$      | $\beta$ | $\sigma_1$ | $\sigma_2$ |
|---------|------------|---------|-------------|-------------|
| Group 1 | 0.1        | 0.1     | 0.15        | 0.1         |
| Group 2 | $0.1 + 0.1 \sin(t)$ | 0.1     | 0.15        | 0.1         |

The diagrams below track the density curves of $x(t)$ and $y(t)$ in Figure 2a and Figure 2b and toxin concentration $C_o(t)$ and $C_e(t)$ over time in Figure 2c and Figure 2d when this population is weakly persistent in the mean.

![Graphs](image_url)

Figure 2. The density curves of $x(t)$ and $y(t)$, Toxin concentration $C_o(t)$ and $C_e(t)$ with different $u_e$.

4.3. Simulations of stochastic permanence

We take the parameters and initial values that meet the conditions of the Theorem 6, select the input levels of two toxins respectively, and group them according to the different noise intensity. At the same time, in order to verify the influence of the intensity of psychological effects on the dynamics of a single
population, we divide them under the premise that the other parameters are fixed. We determine the variable parameters as in Table 3, while others are consistent with Table 1.

The variable parameters are shown on this table:

| Figure | $u_e$     | $\beta$ | $\sigma_1, \sigma_2$ |
|--------|-----------|---------|----------------------|
| a      | 0.3       | 0.1     | 0.1 or 0.3           |
| b      | 0.3       | 1       | 0.1 or 0.3           |
| c      | 0.3       | 10      | 0.1 or 0.3           |
| d      | 0.3 + 0.3 sin$(t)$ | 0.1 | 0.1 or 0.3 |
| e      | 0.3 + 0.3 sin$(t)$ | 1 | 0.1 or 0.3 |
| f      | 0.3 + 0.3 sin$(t)$ | 10 | 0.1 or 0.3 |

The three diagrams in Figure 3, Figure 4 and Figure 5 respectively track the density curves of $x(t)$ and $y(t)$, toxin concentration $C_o(t)$ and $C_e(t)$ over time, when this population is stochastic permanence.

We first take $u_e(t) = 0.3$, and on the premise of fixing the noise intensity of the two groups, respectively take four groups of psychological effect intensity $\beta = 0.1, 1, 10$ for comparison. The results are shown in Figure 3a-3c and Figure 4a-4c.

Then we take $u_e(t) = 0.3 + 0.3 \sin(t)$, on the premise of fixing the noise intensity of the two groups, respectively take four groups of psychological effect intensity $\beta = 0.1, 1, 10$ for comparison, the results are shown in Figure 5a-5c and Figure 6a-6c.

The results show that when the psychological effect intensity increases, the adult population size will tend to be stable, which means that adults will actively evade the living area with high toxin concentration.
Figure 3. The density curves of $x(t)$ with different $u_e$ and $\beta$.

Figure 4. The density curves of $y(t)$ with different $u_e$ and $\beta$.
5. Discussion

The results of this paper can be further extended, especially for the single population model of such stage structure. There are many long-term behaviors that can be studied, and different methods can be used for theoretical practice. In reality, it can guide production and life in some fields, such as fishery production. When the production environment is polluted, it can be guided by relevant theories to take measures in time. At the same time, for the biological aquaculture which needs to separate the larva and adult, the subject can be developed into the theoretical basis for guiding the capture strategy [10], and also can be introduced into the control theory.

There is still space for further development for the two-stage structure model established in this paper. We assume that young individuals are transformed into adult individuals at a certain rate. In reality, the maturity process of any organism needs time accumulation, that is, the generation of time delay. In this regard, the model established in this paper also has certain limitations. Many scholars consider the existence of this factor, which leads to the time-delay term is used to describe the mature process [15–17].

6. Conclusion

In general, based on the previous studies, we introduce two-stage age structure, develop and expand some results. We develop the psychological effect function, select different psychological effect functions for different age structure, and extract some existing function types.

In subsection 2.1, a single population model with age structure and psychological effects in polluted environment is established, which is a nonlinear time-varying system. Then in subsection 2.2, we discuss the asymptotic stability of the system by Lyapunov first approximation theory, and give a sufficient condition for the stability.

In subsection 3.1 based on subsection 2.1, the exposure rate of organisms to environmental toxins is affected by white noise, and then the exposure rate is modified into a random process, and the corresponding random single population model is established. The subsequent contents are mostly proved by Lyapunov function method. In subsection 3.2, we verify the existence of the globally unique positive solution of the stochastic model. In subsection 3.3 near the non-pollution equilibrium point, we give the sufficient
conditions for the weak mean persistence of a single population in the expected sense. In subsection 3.4, we give the sufficient conditions for the random persistence of a single population.

In section 4, we make some numerical simulation to support our conclusions in Theorem 5 and Theorem 6.

Conflict of Interest

All authors stated that they have no conflict of interests in the paper.

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