A CONVERGING LAGRANGIAN CURVATURE FLOW IN THE SPACE OF ORIENTED LINES

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Abstract. Under mean radius of curvature flow, a closed convex surface in Euclidean space expands exponentially to infinity. In the 3-dimensional case the oriented normals to the flowing surface, considered as a surface in the space of all oriented lines, evolves by a curvature flow which preserves the Lagrangian condition.

We prove that this flow converges to a holomorphic Lagrangian section, which form the set of oriented lines through a point. Thus the oriented normal lines of the surface converge to the oriented normals of a round sphere whose centre is determined by the initial surface.

These Lagrangian holomorphic surfaces are $\alpha$-surfaces in the space of oriented lines, a conformally flat neutral Kaehler 4-manifold, and the flow is briefly discussed in the context of anti-selfdual neutral 4-manifolds.

Consider the evolution of a sphere $f : S^n \times [0, \infty) \to \mathbb{R}^{n+1}$ by mean radius of curvature flow (MRCF):

$$\frac{\partial f}{\partial t} = \sum_{j=1}^{n} r_j \mathbf{N},$$

where $\mathbf{N}$ is the unit normal vector and $r_1, r_2, \ldots, r_n$ are the radii of curvature of $S_t = f_t(S^n) \subset \mathbb{R}^{n+1}$.

As noted in [1], this flow, referred to there as the inverse harmonic mean curvature flow, is expanding and the support function $r$ of the surface evolves by the linear strictly parabolic equation

$$\left( \frac{\partial}{\partial t} - \Delta_{S^n} \right) r = nr.$$

As a result, the support function for a closed convex surface increases exponentially and the surface expands to infinity. Moreover, by rescaling the flow about the origin, the surface converges to a round sphere with centre at 0 and radius given by the initial surface [6].

In what follows we consider this flow for $n = 2$ and the induced flow on the oriented normal lines, considered as a surface in the space of all oriented lines. In particular, recall that, given a smooth oriented convex surface $S_t$ in $\mathbb{R}^3$, the set of oriented normal lines to $S_t$ forms a surface $\Sigma_t$ in the space $L(\mathbb{R}^3)$ of all oriented lines of $\mathbb{R}^3$. This surface is of necessity Lagrangian with respect to the canonical symplectic structure on $L(\mathbb{R}^3)$ and, since $S_t$ is convex, $\Sigma_t$ is a section of the bundle $\pi : L(\mathbb{R}^3) = TS^2 \to S^2$. 

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The flow of the Lagrangian section induced by MRCF in local holomorphic coordinates is the following parabolic system for the complex function $F : \mathbb{C} \rightarrow \mathbb{C}$:

$$ \left( \frac{\partial}{\partial t} - \Delta S^2 \right) F = -\frac{2\xi}{1 + \xi} \partial F. $$

In contrast to the flow in $\mathbb{R}^3$, here the flow converges:

**Main Theorem:**

*Under mean radius of curvature flow, any initial Lagrangian section converges smoothly to a holomorphic Lagrangian section.*

A holomorphic Lagrangian section corresponds to the oriented lines that pass through a fixed point in $\mathbb{R}^3$, and so, while the convex surfaces in $\mathbb{R}^3$ run out to infinity under the flow, their normal lines converge to the normals of a round sphere (without rescaling the flow).

To put this in context, mean curvature flow in $\mathbb{R}^3$ does not preserve the Lagrangian condition (the Kähler metric is not Einstein) and so cannot be interpreted in terms of flowing surfaces in $\mathbb{R}^3$. On the other hand, mean curvature flow in $\mathbb{R}^3$ does not lead to a linear parabolic system in $\mathbb{L}(\mathbb{R}^3)$, and so there is little obvious advantage to considering this flow in the space of oriented lines.

Here the canonical Kähler structure on $\mathbb{L}(\mathbb{R}^3)$ has neutral metric signature $(2, 2)$, and so 2-planes may be both holomorphic and Lagrangian (unlike the definite signature case). Moreover, since the metric is anti-selfdual (in fact, it is conformally flat) such planes are integrable. The leaves of this foliation are called $\alpha-$surfaces (the name deriving from the original Riemannian case [7]) and it is to these surfaces that the flow converges. It appears that, by suitable choice of initial surface with asymptotic boundary conditions, the flow can be made to converge to the other type of $\alpha$-surfaces, namely the oriented normal lines to a plane.

In fact, the result can be strengthened by dropping the Lagrangian assumption on the initial surfaces, and the limit will still be holomorphic Lagrangian (see Proposition 5). Moreover, $\alpha$-surfaces are a peculiarity of ASD $(2, 2)$ metrics, and these surfaces are holomorphic curves in the case where the metric is also Kähler. They therefore offer a generalization of divisors on a complex surface, and a suitably generalized evolution can be used to flow to these special divisors in neutral ASD 4-manifolds. This generalization will be explored more fully in future work.

The proof involves showing that under MRCF the Lagrangian sections flow by a linear strictly parabolic equation system. Then, utilizing spherical harmonics to solve the equation in terms of the initial spectral decomposition we study the asymptotic behaviour. The fundamental result for the parabolic equation that we use is contained in Proposition 4.

In the next section we describe the geometric relationship between the $\mathbb{R}^3$ and $\mathbb{L}(\mathbb{R}^3)$. Comparison of MRCF in the two spaces is done in section 2, while in the final section we prove the Main Theorem.

1. **The Space of Oriented Lines**

The space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines of Euclidean $\mathbb{R}^3$ can be identified with $TS^2$, the total space of the tangent bundle to the 2-sphere. $TS^2$ carries a neutral Kähler structure $\mathcal{G}, \mathbb{J}, \Omega$ which is invariant under the Euclidean group acting on oriented lines. In what follows, the terms holomorphic and Lagrangian refer to the complex
structure $J$ and symplectic structure $\Omega$, respectively. The metric $G$ is of neutral signature, hence planes can be both holomorphic and Lagrangian. Further details on the neutral Kähler structure can be found in [2] [3].

For local computations, let $\xi$ be the standard complex coordinate on $S^2$ coming from stereographic projection from the south pole. Extend this to complex coordinates on an open set of $T S^2$ by identifying $X \in T \xi S^2$ with $(\xi, \eta) \in \mathbb{C}^2$ when

$$X = \eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}}.$$  

Consider the set of oriented normal lines to a surface $S$. These form a Lagrangian surface $\Sigma \subset T S^2$. As there are no flat points, the Gauss map of $S^2$ is invertible and hence, $\Sigma$ is a Lagrangian section of the canonical bundle $\pi : T S^2 \to S^2$. In terms of local coordinates the surface $\Sigma$ is given by $\xi \mapsto \xi, \eta = F(\xi, \bar{\xi})$ for some complex valued function $F$.

The link between these holomorphic coordinates and flat coordinates $(x^1, x^2, x^3)$ in $\mathbb{R}^3$ is provided by the map $\Phi : T S^2 \times \mathbb{R} \to \mathbb{R}^3$:

$$x^1 + ix^2 = \frac{2(\eta - \bar{\eta}) + 2\xi(1 + \xi \bar{\xi})r}{(1 + \xi \bar{\xi})^2} \quad x^3 = \frac{-2(\eta \bar{\xi} + \bar{\eta} \xi) + (1 - \xi^2 \bar{\xi})r}{(1 + \xi \bar{\xi})^2},$$

which sends an oriented line $(\xi, \eta)$ and a real number $r$ to the point on the line in $\mathbb{R}^3$ that is an oriented distance $r$ from the closest point on the line to the origin.

These equations can be recast as

$$\eta = \frac{1}{2}(x^1 + ix^2 - 2x^3\xi - (x^1 - ix^2)\xi^2) \quad r = \frac{(x^1 + ix^2)\bar{\xi} + (x^1 - ix^2)\xi + x^3(1 - \xi \bar{\xi})}{1 + \xi \bar{\xi}}.$$  

The perpendicular distance $\chi$ of an oriented line $(\xi, \eta)$ to the origin is found to be

$$\chi^2 = \frac{4\eta \bar{\eta}}{(1 + \xi \bar{\xi})^2}. $$

**Definition 1.** The support function of a convex surface is the map $r : S \to \mathbb{R}$ which takes a point $p$ to the signed distance between $p$ and the point on the oriented normal line to $S$ at $p$ which lies closest to the origin. Alternatively it is the signed perpendicular distance between the oriented tangent plane to $S$ at $p$ and the origin.

The relationship between the support function $r$ of $S$ and the Lagrangian section $F$ is

$$F = \frac{1}{2}(1 + \xi \bar{\xi})^2 \partial \bar{\partial} r.$$  

Define the complex slopes by

$$\partial F = -\bar{\sigma} \quad (1 + \xi \bar{\xi})^2 \partial \left( \frac{F}{(1 + \xi \bar{\xi})^2} \right) = \rho + i\lambda.$$  

A section is Lagrangian iff $\lambda = 0$ and this implies the existence of the real function $r$ satisfying equation (1.3). In addition, the radii of curvature of the surface $S$ are determined by

$$|\sigma|^2 = \frac{1}{4}(r_1 - r_2)^2 \quad (r + \rho)^2 = \frac{1}{4}(r_1 + r_2)^2.$$
Finally, translations in \( \mathbb{R}^3 \) act on our functions as follows. Suppose we consider the translation that takes the origin to \((x^1 + ix^2, x^3) = (\alpha, b)\). Then we have

\[
\eta \mapsto \eta + \frac{1}{2}(\alpha - 2b\xi - \bar{\alpha}\xi^2) \quad r \mapsto r + \frac{\alpha\xi + \bar{\alpha}\xi + b(1 - \xi\bar{\xi})}{1 + \xi\bar{\xi}},
\]

while \( \sigma \) and \( r + \rho \) are invariant under translations.

## 2. Mean Radius of Curvature Flow

Let us now consider the flow (0.1) for a strictly convex surface \( S_t \) in \( \mathbb{R}^3 \). Using coordinates \((x^1 + ix^2, x^3)\) on \( \mathbb{R}^3 \) and Gauss coordinates \( \xi \) on \( S_t \), let \( r_t : S^2 \to \mathbb{R} \) be the support function of \( S_t \). Then, differentiating equations (1.1) in time

\[
\frac{\partial}{\partial t}(x^1 + ix^2) = 2(1 + \xi\bar{\xi})\frac{\partial}{\partial t}\eta + \frac{2\xi}{1 + \xi\bar{\xi}}\frac{\partial}{\partial t}\bar{\eta} + \frac{2\xi}{1 + \xi\bar{\xi}}r,
\]

\[
\frac{\partial}{\partial t}x^3 = -\frac{2\bar{\xi}}{(1 + \xi\bar{\xi})^2}\frac{\partial}{\partial t}\eta - \frac{2\xi}{(1 + \xi\bar{\xi})^2}\frac{\partial}{\partial t}\bar{\eta} + \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}}\frac{\partial}{\partial t}r,
\]

and projecting

\[
\frac{\partial f}{\partial t} = \frac{\partial r}{\partial t}N = (r_1 + r_2)N.
\]

Finally, from the relationship between section and support (1.3) we have

\[
r_1 + r_2 = 2(r + \rho) = 2r + 2(1 + \xi\bar{\xi})\partial \left( \frac{F}{(1 + \xi\bar{\xi})^2} \right)
\]

\[
= 2r + (1 + \xi\bar{\xi})^2\partial r = 2r + \triangle S^2 r.
\]

We have therefore proven the first part of

**Proposition 1.** Under MRCF the support function evolves by

\[
\left( \frac{\partial}{\partial t} - \triangle S^2 \right) r = 2r,
\]

while the perpendicular distance function of the normal lines evolves by

\[
\left( \frac{\partial}{\partial t} - \triangle S^2 \right) \chi^2 = 2\chi^2 - 4(\rho^2 + |\sigma|^2).
\]

**Proof.** The first we have proven, the second follows from a similar calculation. \( \square \)

In the space of oriented lines, the set of oriented normal lines to \( S_t \) form a Lagrangian section given locally by a complex function \( F : \mathbb{C} \to \mathbb{C} \). To lift the flow to the space of oriented lines:

**Proposition 2.** Under MRCF the Lagrangian section \( F \) evolves in \( TS^2 \) by the linear parabolic system

\[
\left( \frac{\partial}{\partial t} - \triangle S^2 \right) F = -\frac{2\bar{\xi}}{1 + \xi\bar{\xi}}\partial F.
\]
Proof. Differentiate the relationship (1.3) in time to get
\[ \frac{\partial}{\partial t} F = \frac{1}{2} (1 + \xi \bar{\xi})^{2} \frac{\partial}{\partial t} \rho = \triangle_{S^2} F - \frac{2\xi}{1 + \xi \bar{\xi}} \partial F. \]

Finally, computing the flow of the derived quantities:

**Proposition 3.** Under MRCF the slopes evolve by

\[ \left( \frac{\partial}{\partial t} - \triangle_{S^2} \right) \rho = 2\rho \quad \left( \frac{\partial}{\partial t} - \triangle_{S^2} \right) \lambda = -2\lambda, \]
\[ \left( \frac{\partial}{\partial t} - \triangle_{S^2} \right) \sigma = -2(1 + 2\xi \bar{\xi})\sigma + 2(1 + \xi \bar{\xi})(\xi \bar{\partial} \sigma - \xi \partial \sigma). \]

Proof. Differentiate the defining relationships (1.4) in time and use the previous Proposition. \( \square \)

Note that the flow equation for \( \lambda \) is such that, if \( \lambda = 0 \) initially, it remains so for all time. Since \( \Omega|_{\Sigma} = \lambda d\xi \cdot A \), we say that the flow in \( TS^2 \) is Lagrangian, since it preserves the Lagrangian condition.

It also follows easily from the flow equations that \( |\sigma| \) and \( \rho \) remain bounded for all time and hence the surface in \( \mathbb{R}^3 \) remains strictly convex.

With a view to future generalizations, note that the symplectic form of \( TS^2 \) is exact \( \Omega = d\Theta \), with associated 1-form given by:
\[ \Theta = \eta (1 + \xi \bar{\xi})^{2} d\xi + \bar{\eta} (1 + \xi \bar{\xi})^{2} d\xi. \]

**Corollary 1.** Under MRCF the 1-form evolves by
\[ \frac{\partial}{\partial t} \Theta = \sqrt{|G|} \Gamma_{(12)}, \]
where \( G \) is the neutral Kähler metric and \( \Gamma_{(12)} \) is the induced Levi-Civita connection 1-form on the Lagrangian section. The parabolic nature of the flow is apparent from the equivalent expression
\[ \left( \frac{\partial}{\partial t} - \triangle_{HL} \right) \Theta = 0, \]
where \( \triangle_{HL} \) is the spherical Hodge-Laplacian acting on 1-forms.

3. **Proof of the Main Theorem**

Consider the flow
\[ \left( \frac{\partial}{\partial t} - \triangle_{S^2} \right) f = 2f, \quad (3.1) \]
for \( f : S^2 \times [0, \infty) \to \mathbb{R} \) with \( f(\cdot, 0) = f_0(\cdot) \).
Definition 2. Define the spherical area $A_{S^2}(f)$ of $f$ by:

$$A_{S^2}(f) = \iint_{S^2} f \, dA.$$ 

Proposition 4. The above flow converges if and only if the $A_{S^2}(f_0) = 0$.

For $A_{S^2}(f_0) = 0$, it converges smoothly to an eigenfunction for the spherical Laplacian with eigenvalue 2.

For $A_{S^2}(f_0) \neq 0$, there exists a constant $C$ depending only on $f_0$ such that

$$|f| \geq Ce^t.$$ 

Proof. The flow (3.1) is linear and strictly parabolic, and therefore by standard theory [4], given any initial function, there exists a smooth solution for all time. Let $f_t$ be the solution of the flow for some initial $f_0$.

Integrating the flow equation over the 2-sphere

$$\frac{\partial}{\partial t} A_{S^2}(f) = A_{S^2}(f).$$

Thus if $A_{S^2}(f_0) = 0$, then $A_{S^2}(f) = 0$ for all time, while $A_{S^2}(f_0) \neq 0$ implies exponential growth in time for the spherical area.

For fixed time $t$, decompose $f_t : S^2 \to \mathbb{R}$ in terms of spherical harmonics $Y_l^m : S^2 \to \mathbb{R}$ [5]:

$$f_t = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} Y_l^m,$$

where $B_{lm}$ are complex and satisfy

$$B_{lm} = (−1)^l B_{l,-m} \text{ for } m \neq 0 \text{ and } B_{l0} = B_{00}.$$

Since the flow is linear, we obtain a flow on the projection of $f$ onto the spectrum of the Laplacian:

$$\frac{\partial B_{lm}}{\partial t} = [2 - l(l+1)]B_{lm},$$

which integrate to yield

$$f_t = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{B}_{lm} e^{[2-l(l+1)]t} Y_l^m.$$ 

For convenience we have denoted $B_{lm}$ at $t = 0$ by $\hat{B}_{lm}$.

Splitting off the first few terms

$$f_t = \hat{B}_{00} e^{2t} Y_0^0 + \hat{B}_{1-1} Y_1^{-1} + \hat{B}_{10} Y_1^0 + \hat{B}_{11} Y_1^1 + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \hat{B}_{lm} e^{(2-l(l+1))t} Y_l^m$$

$$= \frac{1}{2} \sqrt{\frac{1}{\pi}} \hat{B}_{00} e^{2t} + \sqrt{\frac{3}{2\pi}} \hat{B}_{1-1} \frac{\xi}{1 + \xi} + \frac{1}{2} \sqrt{\frac{3}{2\pi}} \hat{B}_{10} \frac{1 - \xi \bar{\xi}}{1 + \xi} - \sqrt{\frac{3}{2\pi}} \hat{B}_{11} \frac{\xi}{1 + \xi} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \hat{B}_{lm} e^{(2-l(l+1))t} Y_l^m$$

$$= \frac{1}{2} \sqrt{\frac{1}{\pi}} A_{S^2}(f_0) e^{2t} - \frac{\alpha \xi + \alpha \bar{\xi} + b(1 - \xi \bar{\xi})}{1 + \xi} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \hat{B}_{lm} e^{(2-l(l+1))t} Y_l^m,$$
where we note that, by the orthogonality properties of the spherical harmonics

\[ A_{S^2}(f_0) = \int \int_{S^2} f_0 \ dA = B_{00}, \]

and we have introduced \( \alpha \in \mathbb{C}, b \in \mathbb{R} : \)

\[ \alpha = -\sqrt{\frac{3}{2\pi}} B_{11}, \quad b = \frac{1}{2} \sqrt{\frac{1}{\pi}} B_{10}. \]

If \( A_{S^2}(f_0) \neq 0 \), then \( |f_t| \) exponentially blows up as \( t \to \infty \), while for \( A_{S^2}(f_0) = 0 \), \( f_t \) converges to an eigenfunction of the spherical Laplacian with eigenvalue equal to 2:

\[ f_t \to -\alpha \bar{\xi} + \bar{\alpha} \xi + b(1 - \xi \bar{\xi}) \frac{1}{1 + \xi \bar{\xi}} \quad \text{as} \quad t \to \infty, \]

as claimed. \( \square \)

**Proof of Main Theorem.**

Under MRCF we have seen that the support flows by equation (2.1). By Proposition 4 to determine the behaviour we need to compute the area \( A_{S^2}(r) \). Note that this integral is invariant under translation:

\[ A_{S^2}(r) = A_{S^2} \left( r - \frac{\alpha \bar{\xi} + \bar{\alpha} \xi + b(1 - \xi \bar{\xi})}{1 + \xi \bar{\xi}} \right). \]

If we move the origin to the interior of \( S \) we can make \( r > 0 \), and conclude that \( A_{S^2}(r) > 0 \). Thus, by Proposition 4, the support function blows up exponentially. More particularly, the spectral decomposition is

\[ r = r_{00} e^{2t} + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \rho_{lm} e^{(2-l(l+1))t} Y_{l}^{m}, \]

where \( r_{00} > 0 \). Clearly, \( r_{00} \) is the radius of the limit of the rescaled flow for \( re^{-2t} \) [6].

Now consider the flow in \( TS^2 \). By the evolution equation for \( \rho \) in Proposition 3 we require \( A_{S^2}(\rho) \). In this case, however we have

\[ A_{S^2}(\rho) = \frac{1}{4} \int \int_{S^2} \Delta_{S^2} \rho \ dA = 0, \]

and by Proposition 4

\[ \rho = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \rho_{lm} e^{(2-l(l+1))t} Y_{l}^{m}, \]

where the constants \( \rho_{lm} \) are determined by the initial surface. Let

\[ \alpha = -\sqrt{\frac{3}{2\pi}} \rho_{11}, \quad b = \frac{1}{2} \sqrt{\frac{1}{\pi}} \rho_{10}. \]

We claim that the oriented normal lines to the flowing surface converge to the set of oriented lines passing through \((x^1 + ix^2, x^3) = (\alpha, b)\).

First translate so that \((\alpha, b) = (0, 0)\) and note that

\[ A_{S^2}(\chi^2) = \int \int_{S^2} \frac{4\eta \bar{\eta}}{(1 + \xi \bar{\xi})^2} \ dA = \int \int_{S^2} 2\eta \bar{\eta} \ dA = -\int \int_{S^2} 2\rho \ dA \]
= \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \rho_{lm} r_{lm} e^{2 \left[2 - l(l+1)\right] t}, \]

where in the last line we have used the spectral decomposition of $r$ and $\rho$ and the orthogonality properties of the spherical harmonics.

Thus we have that $A_{S^2}(\chi^2) \to 0$ and, since the integrand is a smooth positive function, $\chi^2 \to 0$ as claimed. This completes the proof of the Main Theorem. \hfill \Box

**Proposition 5.** The flow converges to a Lagrangian holomorphic section even if the initial section is not Lagrangian.

**Proof.** Recall that the section is Lagrangian iff $\lambda = 0$. The flow equation for $\lambda$ given in Proposition 3 can be written

$$\left( \frac{\partial}{\partial t} - \triangle_{S^2} \right) \lambda e^{2t} = 0,$$

which by the maximum principle implies that there exists a constant $C$ depending only on $\lambda_0$ such that

$$|\lambda| \leq Ce^{-2t},$$

so that $\lambda \to 0$ as $t \to \infty$. \hfill \Box

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