ON THE UNIQUENESS AND STABILITY OF
ENTROPY SOLUTIONS OF NONLINEAR DEGENERATE
PARABOLIC EQUATIONS WITH ROUGH COEFFICIENTS

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Abstract. We study nonlinear degenerate parabolic equations where the flux function \( f(x, t, u) \) does not depend Lipschitz continuously on the spatial location \( x \). By properly adapting the “doubling of variables” device due to Kružkov [25] and Carrillo [12], we prove a uniqueness result within the class of entropy solutions for the initial value problem. We also prove a result concerning the continuous dependence on the initial data and the flux function for degenerate parabolic equations with flux function of the form \( k(x)f(u) \), where \( k(x) \) is a vector-valued function and \( f(u) \) is a scalar function.

1. Introduction. The main subject of this paper is uniqueness and stability properties of entropy solutions of nonlinear degenerate parabolic equations where the flux function depends explicitly on the spatial location. In particular, this paper is concerned with the case where the flux function does not depend Lipschitz continuously on the spatial variable. Our study is motivated by applications where one frequently encounters flux functions possessing minimal smoothness in the spatial variable.

The problems that we study are initial value problems of the form

\[
\begin{align*}
    u_t + \text{div} f(x, t, u) &= \Delta A(u) + q(x, t, u), & (x, t) \in \Pi_T = \mathbb{R}^d \times (0, T), \\
    u(x, 0) &= u_0(x), & x \in \mathbb{R}^d,
\end{align*}
\]

where \( T > 0 \) is fixed, \( u(x, t) \) is the scalar unknown function that is sought, \( u_0(x) \) is the initial datum, \( f = f(x, t, u) \) is called the flux function, \( A = A(u) \) is the diffusion function, and \( q = (x, t, u) \) is the source term. The coefficients \( f, A, q \) of problem (1.1) are given functions satisfying certain regularity assumptions. The regularity assumptions on \( f, q \) will be given later.

For the initial value problem (1.1) to be well-posed, we must require that \( A(\cdot) \) satisfies

\[
A \in \text{Lip}_{\text{loc}}(\mathbb{R}) \quad \text{and} \quad A(\cdot) \text{ is nondecreasing with } A(0) = 0.
\]

The second part of (1.2) implies that the nonlinear operator \( u \mapsto \Delta A(u) \) is of degenerate elliptic type, and hence many well known nonlinear and linear partial differential equations are special cases of (1.1). In particular, the scalar conservation law \( (A' \equiv 0) \) is a “simple” special case. Included are also the heat equation, porous medium type equations characterized by one-point degeneracy, two-phase reservoir
flow equations characterized by the two-point degeneracy, as well as strongly degenerate convection-diffusion equations where \( A'(s) \equiv 0 \) for all \( s \) in some interval \([\alpha, \beta]\). Consequently, partial differential equations of the type (1.1) model a wide variety of phenomena, ranging from porous media flow [34], via flow of glaciers [20] and sedimentation processes [9], to traffic flow [37].

We recall that if the problem (1.1) is non-degenerate (uniformly parabolic), it is well known that it admits a unique classical solution. This contrasts with the case where (1.1) is allowed to degenerate at certain points, that is, \( A'(s) = 0 \) for some values of \( s \). Then solutions are not necessarily smooth (but typically continuous) and weak solutions must be sought. On the other hand, if \( A'(s) \) is zero on an interval \([\alpha, \beta]\), (weak) solutions may be discontinuous and are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution.

Roughly speaking, we call a function \( u \in L^1 \cap L^\infty \) an entropy solution of the initial value problem (1.1) if

\[
\begin{cases}
(i) & |u - c| + \text{div} \left[ \text{sign} \,(u - c) \left( f(x, t, u) - f(x, t, c) \right) \right] \\
& + \text{sign} \,(u - c) \left( \text{div} f(x, t, c) - q(x, t, u) \right) \\
& - \Delta |A(u) - A(c)| \leq 0 \text{ in } D' \forall c \in \mathbb{R}, \\
(ii) & \nabla A(u) \text{ belongs to } L^2.
\end{cases}
\]

In addition, we require that the initial function \( u_0 \) is assumed in the strong \( L^1 \) sense. We refer to §2 for a precise definition of an entropy solution.

The mathematical (\( L^1/BV \)) theory of parabolic equations was initiated by Oleı́nık [28]. She proved well-posedness of the initial value problem in the non-degenerate case with \( A(u) = u \), and showed that weak solutions are in this case classical.

In the hyperbolic case \( (A' \equiv 0) \) with the flux \( f = f(x, t, u) \) depending (smoothly) on \( x \) and \( t \), the notion of entropy solution was introduced independently by Kružkov [25] and Vol’pert [35] (the latter author considered the smaller \( BV \) class). These authors also proved general existence, uniqueness, and stability results for the entropy solution, see also Oleı́nık [28] for similar results in the convex case \( f_{uu} \geq 0 \).

In the mixed hyperbolic-parabolic case \( (A' \geq 0) \), the notion of entropy solution goes back to Vol’pert and Hudjaev [36], who were the first to study strongly degenerate parabolic equations. These authors showed existence of a \( BV \) entropy solution using the viscosity method and obtained some partial uniqueness results in the \( BV \) class (i.e., when the first order partial derivatives of \( u \) are finite measures). In the one-dimensional case, Wu and Yin [38] later provided a complete uniqueness proof in the \( BV \) class. Further results in the one-dimensional case were obtained by Bénilan and Touré [3, 4] using nonlinear semigroup theory.

As for the uniqueness issue in the multi-dimensional case, Brézis and Crandall [6] established uniqueness of weak solutions when \( f \equiv 0 \). Later, under the assumption that \( A(s) \) is strictly increasing, Yin [39] showed uniqueness of weak solutions in the \( BV \) class. Bénilan and Gariepy [2] showed that the \( BV \) weak solution studied in [39] is actually a strong solution. The assumption that \( u_t \) should be a finite measure was removed in [40, 41].

An important step forward in the general case of \( A(\cdot) \) being merely nondecreasing was made recently by Carrillo [12], who showed uniqueness of the entropy solution for a particular boundary value problem with the boundary condition "\( A(u) = 0 \)". His method of proof is an elegant extension of the by now famous “doubling of
variables” device introduced by Kružkov [25]. In [12], the author also showed existence of an entropy solution using the semigroup method. A related paper is that of Chen and DiBenedetto [13], see also Tassa [31] and uniqueness for piecewise smooth weak solutions.

In [7] (see also [30]), the uniqueness proof of Carrillo was adopted to several initial-boundary value problems arising the theory of sedimentation-consolidation processes [9], which in some cases call for the notion of an entropy boundary condition (see also [8] for the BV approach).

In the present paper we generalize Carrillo’s uniqueness result [12] by showing that it holds for the Cauchy problem with a flux function \( f = f(x, t, u) \) where the spatial dependence is non-smooth (non-Lipschitz). Only the case \( f = f(u) \) was studied in [12]. Moreover, we also establish continuous dependence on the flux function in the case \( f(x, t, u) = k(x)f(u) \).

With the assumptions on the diffusion function \( A \) already given (see (1.2)), we now present the (regularity) assumptions that are needed on the flux function \( f \) and the source term \( q \), with the those on \( f \) being the most important ones. Concerning the source term \( q : \mathbb{R}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R} \), we assume that \( q(x, t, 0) = 0 \) \( \forall x, t \) and

\[
q(\cdot, u) \in L^1(0, T; L^\infty(\mathbb{R}^d)) \forall u; \quad q(x, t, \cdot) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \text{ uniformly in } x, t. \tag{1.4}
\]

With the phrase “uniformly in \( x, t' \)” in (1.4), we mean

\[
|q(x, t, v) - q(x, t, u)| \leq C|v - u|, \quad \forall x, t, v, u,
\]

for some constant \( C > 0 \) (independently of \( x, t, v, u \)).

Concerning the flux function \( f : \mathbb{R}^d \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^d \), we assume that

\[
f(x, t, 0) = f_x(x, t, 0) = 0. \tag{1.8}
\]

Moreover, we assume that

\[
f(\cdot, u) \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^d)) \forall u; \quad f(x, t, \cdot) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \text{ uniformly in } x, t; \tag{1.5}
\]

\[
f_x(\cdot, u) \in L^1(0, T; L^\infty(\mathbb{R}^d)) \forall u; \quad f_x(x, t, \cdot) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \text{ uniformly in } x, t, \tag{1.5}
\]

where \( f_x = f_x(x, t, u) \) in (1.6) denotes the function obtained by taking the divergence of the flux \( f = f(x, t, u) \) with respect to the first variable. With the phrase “uniformly in \( x, t' \)” in (1.5) and (1.6), we mean

\[
|f(x, t, v) - f(x, t, u)|, \quad |f_x(x, t, v) - f_x(x, t, u)| \leq C|v - u|, \quad \forall x, t, v, u,
\]

for some constant \( C > 0 \) (independently of \( x, t, v, u \)).

The conditions in (1.4)-(1.6) are sufficient to make sense to the notion of entropy solution (see §2). In the general case, however, we need one additional regularity assumption on the \( x \) dependency of \( f \) to get uniqueness of the entropy solution. Inspired by Capuzzo-Dolcetta and Perthame [10], we assume that

\[
(F(x, t, v, u) - F(y, s, v, u)) \cdot (x - y) \geq -\gamma |v - u| |x - y|^2, \quad \forall x, y, t, v, u, \tag{1.7}
\]

for some constant \( \gamma > 0 \) (independent of \( x, t, v, u \)), where

\[
F(x, t, v, u) := \text{sign } (v - u) \left[ f(x, t, v) - f(x, t, u) \right]. \tag{1.8}
\]

Note that condition (1.7) does not imply that \( f \) is Lipschitz continuous in the spatial variable \( x \). We remark that if \( f = f(x, u) \) is of the form

\[
f = k(x)h(u),
\]
for some vector valued function \( k : \mathbb{R}^d \to \mathbb{R}^d \), and a Lipschitz continuous function 
\( h \), then (1.7) reduces to
\[
(k(x) - k(y)) \cdot (x - y) \geq -\gamma |x - y|^2, \quad \forall x, y, t, v, u, \tag{1.9}
\]
for some constant \( \gamma > 0 \) (depending also on the Lipschitz constant of \( h \)). As pointed out in [10], this condition requires a bound only on the matrix \( \nabla_x k + (\nabla_x k)^T \) (the symmetric part of the Jacobian \( \nabla_x k \)) and \( k \) itself need not belong to any Sobolev space. To see this, let \( z = x - y \) and rewrite the left-hand side of (1.9) as follows
\[
(k(x) - k(y)) \cdot (x - y) = \int_0^1 \frac{d}{d\xi} [(k(y + \xi z) - k(y)) \cdot z] \, d\xi
\]
\[
= \int_0^1 \nabla_x k(y + \xi z) \cdot z \, d\xi
\]
\[
= \frac{1}{2} \int_0^1 (\nabla_x k + (\nabla_x k)^T) (y + \xi z) \cdot z \, d\xi,
\]
since \( \frac{1}{2} (\nabla_x k - (\nabla_x k)^T) (y + \xi z) \cdot z \equiv 0 \).

In [10], the authors showed the universality of (1.7) by proving that under this condition, uniqueness holds for the Kružkov-Vol’pert entropy solution of hyperbolic equations, the Crandall-Lions viscosity solution of Hamilton-Jacobi equations, and the DiPerna-Lions regularized solution of transport equations. With the present paper, we add to that list uniqueness of the entropy solution of degenerate parabolic equations. More precisely, we prove the following theorem:

**Theorem 1.1 (Uniqueness).** Assume that (1.2) and (1.4)-(1.7) hold. Let \( v, u \) be two entropy solutions of (1.1) with initial data \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \). Then \( v = u \) a.e. in \( \Pi_T = \mathbb{R}^d \times (0, T) \).

By combining the arguments used in the present paper by those used in [18], Theorem 1.1 can be proved even for a large class of weakly coupled systems of degenerate parabolic equations.

We next restrict our attention to problems of the form
\[
\begin{align*}
\partial_t u + \text{div} \{ k(x) f(u) \} &= \Delta A(u), \quad (x, t) \in \Pi_T, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^d, \tag{1.10}
\end{align*}
\]
where \( k : \mathbb{R}^d \to \mathbb{R}^d, f : \mathbb{R} \to \mathbb{R} \), and \( f(0) = 0 \). Problems of the form (1.10) occur in several important applications. Our first result for (1.10) states that in the \( L^\infty(0, T; BV(\mathbb{R}^d)) \) class of entropy solutions, an \( L^1 \) contraction principle actually holds provided
\[
f \in \text{Lip}_{\text{loc}}(\mathbb{R}); \quad k \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d); \quad k, \text{div} k \in L^\infty(\mathbb{R}^d). \tag{1.11}
\]
More precisely, we prove the following theorem:

**Theorem 1.2** (\( L^1 \) contraction). Assume that (1.2) and (1.11) hold. Let \( v, u \in L^\infty(0, T; BV(\mathbb{R}^d)) \) be entropy solutions of (1.10) with initial data \( v_0, u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d) \), respectively. Then for almost all \( t \in (0, T) \),
\[
\|v(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|v_0 - u_0\|_{L^1(\mathbb{R}^d)}.
\]
In particular, there exists at most one entropy solution of (1.10).
We remark that the existence of an $L^\infty(0, T; BV(\mathbb{R}^d))$ entropy solution of (1.10) is guaranteed if $\text{div} k \in BV(\mathbb{R}^d)$. This follows from the results obtained by Karlsen and Risebro [21], who prove convergence (within the entropy solution framework) of finite difference schemes for degenerate parabolic equations with rough coefficients. For an overview of the literature on numerical methods for approximating entropy solutions of degenerate parabolic equations, we refer to the first section of [21] and the lecture notes [15] (see also the references given therein).

Throughout this paper the coefficient $k(x)$ is not allowed to be discontinuous. In the one-dimensional hyperbolic case ($A' \equiv 0$) with $k(x)$ depending discontinuously on $x$, the equation (1.1) is often written as the following $2 \times 2$ system:

$$\begin{align*}
  u_t + f(k, u)_x &= 0, & k_t &= 0.
\end{align*}$$  \hspace{1cm} (1.12)

If $\partial f/\partial u$ changes sign, then this system is non-strictly hyperbolic. This complicates the analysis, and in order to prove compactness of approximated solutions a singular transformation $\Psi(k, u)$ has been used by several authors [32, 17, 24, 23]. In these works convergence of the Glimm scheme and of front tracking was established in the case where $k$ may be discontinuous. If $k \in C^2(\mathbb{R}^d)$, then convergence of the Lax-Friedrichs scheme and the upwind scheme was proved in [28]. Under weaker conditions on $k$ ($k' \in BV$) and for $f$ convex in $u$, convergence of the one-dimensional Godunov method for (1.12) (not for (1.1)) was shown by Isaacson and Temple in [19]. Recently, convergence of the one-dimensional Godunov method for (1.1) was shown by Towers [33] in the case where $k$ is piecewise continuous. In this case, the Kružkov entropy condition (1.3) no longer applies, and in [24] a wave entropy condition analogous to the Oleãälik entropy condition introduced in [28] was used to obtain uniqueness, see also [23]. Klausen and Risebro [22] analyzed the case of discontinuous $k$ by “smoothing out” the coefficient $k$ and then passing to the limit as the smoothing parameter tends to zero. In particular, they showed that the limit “entropy” solution satisfied the $L^3$ contraction property. We intend to study the degenerate parabolic problem (1.10) when $k(x)$ is discontinuous in future work.

Theorem 1.2 gives the desired continuous dependence on the initial data in degenerate parabolic problems of the type (1.10). Next we will establish continuous dependence also on the flux function. To this end, let us also introduce the problem

$$\begin{align*}
  v_t + \text{div}(l(x)g(v)) &= \Delta A(v), & (x, t) &\in \Pi_T,
  v(x, 0) &= v_0(x), & x &\in \mathbb{R}^d,
\end{align*}$$  \hspace{1cm} (1.13)

where $l: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $g: \mathbb{R} \rightarrow \mathbb{R}$, and $g(0) = 0$. We are interested in estimating the $L^1$ difference between the entropy solution $v$ of (1.13) and the entropy solution $u$ of (1.10). Now we assume that

$$\begin{align*}
  f, g &\in \text{Lip}_{\text{loc}}(\mathbb{R}); \quad k, l \in W^{1,1}(\mathbb{R}^d) \cap C(\mathbb{R}^d); \quad k, l, \text{div} k, \text{div} l \in L^\infty(\mathbb{R}^d). \hspace{1cm} (1.14)
\end{align*}$$

Under these assumptions, we prove the following continuous dependence result:

**Theorem 1.3** (Continuous dependence). Assume that the regularity conditions (1.2) and (1.14) hold. Let $v, u \in L^\infty(0, T; BV(\mathbb{R}^d))$ be entropy solutions of (1.13), (1.10) with initial data $v_0, u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, respectively. For definiteness, let us assume that $v, u$ take values in the closed interval $I \subset \mathbb{R}$ and that there are constants $V_v, V_u > 0$ such that

$$\begin{align*}
  |v(\cdot, t)|_{BV(\mathbb{R}^d)} &\leq V_v \quad \forall t \in (0, T), \quad |u(\cdot, t)|_{BV(\mathbb{R}^d)} \leq V_u \quad \forall t \in (0, T).
\end{align*}$$
Then for almost all $t \in (0, T)$,
\[
\|v(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|v_0 - u_0\|_{L^1(\mathbb{R}^d)} + t \left( C^{q,v}_1 \|l - k\|_{L^\infty(\mathbb{R}^d)} + C^{q,f}_2 \|l - k\|_{BV(\mathbb{R}^d)} + C^f_3 \|g - f\|_{L^\infty(I)} + C^{g,v}_4 \|g - f\|_{Lip(I)} \right)
\wedge \left( C^{q,u}_1 \|l - k\|_{L^\infty(\mathbb{R}^d)} + C^{q,f}_2 \|l - k\|_{BV(\mathbb{R}^d)} + C^f_3 \|g - f\|_{L^\infty(I)} + C^{g,u}_4 \|g - f\|_{Lip(I)} \right),
\]
where $C^{q,v}_1 = \|v\|_{Lip(I)} V_v$, $C^{q,u}_1 = \|u\|_{Lip(I)} V_u$, $C^{q,f}_2 = \|f\|_{L^\infty(I)}$, $C^f_3 = \|l\|_{BV(\mathbb{R}^d)}$, $C^{g,v}_4 = \|k\|_{L^\infty(\mathbb{R}^d)} V_v$, $C^{g,u}_4 = \|l\|_{L^\infty(\mathbb{R}^d)} V_u$, and $a \wedge b = \min(a, b)$.

Results on continuous dependence on the flux function in scalar conservation laws with $k(x) \equiv 1$ have been obtained by Lucier [27] and Bouchut and Perthame [5]. Finally, we mention that Cockburn and Gripenberg [14] have obtained a result regarding continuous dependence on both the flux function and the diffusion function in (1.10) when $k(x) = 1$. Their result does not, however, imply uniqueness of the entropy solution since their “doubling of variables” argument requires that one works with (smooth) approximate solutions. By properly combining the ideas in the present paper with those in [14], one can prove a version of Theorem 1.3 which also includes continuous dependence on the diffusion function $A$, see [16].

The remaining part of this paper is organized as follows: In the next section we introduce (precisely) the notion of entropy solution as well as stating and proving a version of an important lemma due to Carrillo [12]. Equipped with our version of Carrillo’s lemma, Theorems 1.1, 1.2, and 1.3 are proved respectively in §3, §4, and §5. Finally, in §6 (an appendix) we provide a proof of the weak chain rule needed in the proof of Carrillo’s lemma.

The first version of this paper appeared as a preprint in April 2000 on the Conservation Laws Preprint Server (http://www.math.ntnu.no/conservation/). Some parts of the proofs in that version are different from what can be found herein.

2. Preliminaries. We use the following definition of an entropy solution of (1.1):

Definition 2.1. An entropy solution of (1.1) is a measurable function $u = u(x, t)$ satisfying:

D.1 $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0, T; L^1(\mathbb{R}^d))$.

D.2 For all $c \in \mathbb{R}$ and all non-negative test functions $\phi$ in $C^\infty_c(\Pi_T)$, the following entropy inequality holds:

\[
\iint_{\Pi_T} \left[ (u - c)\phi_t + \text{sign}(u - c) \left( f(x, t, u) - f(x, t, c) \right) \cdot \nabla \phi + |A(u) - A(c)| \Delta \phi 
\right. \\
\left. - \text{sign}(u - c) \left( \text{div} f(x, t, u) - q(x, t, u) \right) \phi \right) \, dt \, dx \geq 0.
\]

(2.15)

D.3 $A(u) \in L^2(0, T; H^1(\mathbb{R}^d))$.

D.4 Essentially as $t \downarrow 0$,

\[
\int_{\mathbb{R}^d} |u(x, t) - u_0(x)| \, dx \to 0.
\]
Remark 2.2. (i) Observe that when $A' \equiv 0$, (2.15) reduces to the well-known entropy inequality for scalar conservation laws introduced by Kružkov [25] and Vol’pert [35].

(ii) Condition (D.4), i.e., that the initial datum $u_0$ should be taken by continuity, motivates the requirement of continuity with respect to $t$ in condition (D.1).

Let $u$ be an entropy solution. Then, since $A(u) \in H^1(\mathbb{R}^d)$ for a.e. $t \in (0,T)$, it follows from general theory of Sobolev spaces that

$$\nabla[A(u) - A(c)] = \text{sign}(A(u) - A(c)) \nabla A(u) \quad \text{a.e. in } \Pi_T,$$

also, $\text{sign}(A(u) - A(c)) = \text{sign}(u - c)$ provided $A(u) \neq A(c)$. Again since $A(u) \in H^1(\mathbb{R}^d)$ for a.e. $t \in (0,T)$, it follows that $\nabla A(u) = 0$ a.e. (w.r.t. $dt dx$) in $\{(x,t) \in \Pi_T : A(u(x,t)) = A(c)\}$. We therefore conclude that

$$\nabla[A(u) - A(c)] = \text{sign}(u - c) \nabla A(u) \quad \text{a.e. in } \Pi_T$$

and the entropy inequality (2.15) can be written equivalently as

$$\iint_{\Pi_T} \left( (u - c) \phi_t + \text{sign}(u - c) \left[ f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi 
- \text{sign}(u - c) (\text{div}(f(x,c) - q(x,t,u))\phi) \right) dt dx \geq 0,$$

for all $\phi \in C^0_c(\Pi_T)$. If we take $c > \text{ess sup } u(x,t)$ and $c < \text{ess inf } u(x,t)$ in (2.15), then we deduce that $u$ satisfies

$$\iint_{\Pi_T} \left( u\phi_t + f(x,t,u) \cdot \nabla \phi + A(u) \Delta \phi + q(x,t,u)\phi \right) dt dx = 0,$$

for all $\phi \in C^0_c(\Pi_T)$. Note that (1.5) implies

$$\|f(x,t,u)\|_{L^2(\Pi_T)}^2 \leq \text{Const } \|u\|_{L^\infty(\Pi_T)} \|u\|_{L^1(\Pi_T)} < \infty,$$

so that $f(x,t,u) - \nabla A(u) \in L^2(\Pi_T; \mathbb{R}^d)$. Similarly, (1.4) implies $q(x,t,u)$ belongs to $L^2(\Pi_T)$. An integration by parts in (2.17) followed by an approximation argument will then show that the equality

$$\iint_{\Pi_T} \left( u\phi_t + [f(x,t,u) - \nabla A(u)] \cdot \nabla \phi + q(x,t,u)\phi \right) dt dx = 0$$

holds for all functions $\phi \in L^2(0,T; H^{-1}(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^\infty(\mathbb{R}^d))$ that vanish for $t = 0$ and $t = T$.

We can even go one step further. To this end, let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H^1_0(\mathbb{R}^d)$. From (2.19), we conclude that

$$\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d)),$$

so that the equality

$$- \int_0^T \langle \partial_t u, \phi \rangle dt + \iint_{\Pi_T} \left( [f(x,t,u) - \nabla A(u)] \cdot \nabla \phi + q(x,t,u)\phi \right) dt dx = 0$$

holds for all $\phi \in L^2(0,T; H^1_0(\mathbb{R}^d)) \cap W^{1,1}(0,T; L^\infty(\mathbb{R}^d))$ that vanish for $t = 0$ and $t = T$. The fact that an entropy solution $u$ satisfies (2.20) will be important for the uniqueness proof.

We now set

$$A_\phi(z) = \int_{z_0}^z \psi(A(r)) \, dr,$$
where \( \psi : \mathbb{R} \to \mathbb{R} \) is a nondecreasing and Lipschitz continuous function and \( z_0 \in \mathbb{R} \).

Concerning the function \( A_\psi \), we shall need the following associated “weak chain rule”:

**Lemma 2.3.** Let \( u : \Pi_T \to \mathbb{R} \) be a measurable function satisfying the following four conditions:

- (a): \( u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0, T; L^1(\mathbb{R}^d)) \).
- (b): \( u(0, \cdot) = u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \).
- (c): \( \partial_t u \in L^2(0, T; H^{-1}(\mathbb{R}^d)) \).
- (d): \( A(u) \in L^2(0, T; H^1(\mathbb{R}^d)) \).

Then, for a.e. \( s \in (0, T) \) and every nonnegative \( \phi \in C_0^\infty(\mathbb{R}^d \times [0, T]) \), we have

\[
- \int_0^s \langle \partial_t u, \psi(A(u)) \phi \rangle dt = \int_0^s \int_{\mathbb{R}^d} A_\psi(u) \phi_t dt \, dx + \int_{\mathbb{R}^d} A_\psi(u(s)) \phi(x, s) \, dx - \int_{\mathbb{R}^d} A_\psi(u(s)) \phi(x, s) \, dx.
\]

Lemma 2.3 can proved more or less in the same way as the “weak chain rule” in Carrillo [12], see also Alt and Luckhaus [1] and Otto [29]. For the sake of completeness, a proof of Lemma 2.3 is given in §6 (the appendix).

In what follows, we shall frequently need a continuous approximation of \( \text{sign} \cdot \). For \( \varepsilon > 0 \), set

\[
\text{sign}_\varepsilon(\tau) = \begin{cases} 
-1, & \tau < \varepsilon, \\
\tau/\varepsilon, & \varepsilon \leq \tau \leq \varepsilon, \\
1, & \tau > \varepsilon.
\end{cases}
\]

Note that \( \text{sign}_\varepsilon(-r) = -\text{sign}_\varepsilon(r) \) and \( \text{sign}'_\varepsilon(-r) = \text{sign}'_\varepsilon(r) \) a.e.

We let \( A^{-1} : \mathbb{R} \to \mathbb{R} \) denote the unique left-continuous function satisfying \( A^{-1}(A(u)) = u \) for all \( u \in \mathbb{R} \), and by \( \bar{E} \) we denote the set

\[
E = \left\{ r : A^{-1}(r) \text{ discontinuous at } r \right\}.
\]

Note that \( E \) is associated with the set of points \( \{ u : A'(u) = 0 \} \) at which the operator \( u \mapsto \Delta A(u) \) is degenerate elliptic.

We are now ready to state and prove the following version of an important observation made by Carrillo [12]:

**Lemma 2.4** (Entropy dissipation term). Let \( u \) be an entropy solution of (1.1). Then, for any nonnegative \( \phi \in C_0^\infty(\Pi_T) \) and \( c \in \mathbb{R} \) such that \( A(c) \notin E \), we have

\[
\iint_{\Pi_T} \left[ |u - c| \phi_t + \text{sign} (u - c) \left( f(x, t, u) - f(x, t, c) - \nabla A(u) \right) \cdot \nabla \phi \\
- \text{sign} (u - c) \left( \text{div} f(x, t, c) - q(x, t, u) \right) \phi \right] dt \, dx 
\]

\[
= \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} |\nabla A(u)|^2 \text{sign}'_\varepsilon (A(u) - A(c)) \phi \, dt \, dx.
\]  

(2.22)

**Proof.** The proof is similar to the proof of the corresponding result in [12]. In (2.21), introduce the function \( \psi_\varepsilon(z) = \text{sign}_\varepsilon (z - A(c)) \) and set \( z_0 = c \). Notice that the conditions of Lemma 2.3 are satisfied and hence

\[
- \int_0^T \left\langle \partial_t u, \text{sign}_\varepsilon (A(u) - A(c)) \phi \right\rangle dt = \iint_{\Pi_T} A_{\psi_\varepsilon}(u) \phi_t \, dt \, dx.
\]
Since $u$ satisfies (2.20) and since $[\text{sign}_c (A(u) - A(c)) \phi] \in L^2(0,T;H^1_0(R))$ is an admissible test function vanishing for $t = 0$ and $t = T$, we have

$$- \int_0^T \left( \partial_t u, \text{sign}_c (A(u) - A(c)) \phi \right) dt$$

$$+ \iint_{\Pi_T} \left( [f(x,t,u) - f(x,t,c) - \nabla A(u)] \cdot \nabla (\text{sign}_c (A(u) - A(c)) \phi) \right.$$  

$$- \text{div} f(x,t,c) - q(x,t,u) (\text{sign}_c (A(u) - A(c)) \phi) \bigg) dt dx = 0,$$

which implies that

$$\iint_{\Pi_T} A_{\phi_i}(u) \phi_t dt dx$$

$$+ \iint_{\Pi_T} \left( [f(x,t,u) - f(x,t,c) - \nabla A(u)] \cdot \nabla (\text{sign}_c (A(u) - A(c)) \phi) \right.$$  

$$- \text{sign}_c (A(u) - A(c)) \text{div} f(x,t,c) - q(x,t,u) \phi \bigg) dt dx = 0.$$  

(2.23)

Since $A(r) > A(c)$ if and only if $r > c$, $\text{sign}_c (A(r) - A(c)) \rightarrow 1$ as $\varepsilon \downarrow 0$ for any $r > c$. Similarly, $\text{sign}_c (A(r) - A(c)) \rightarrow -1$ as $\varepsilon \downarrow 0$ for any $r < c$. Consequently, whenever $A(c) \notin E$,

$$A_{\psi_i}(u) \rightarrow |u - c| \text{ a.e. in } \Pi_T \text{ as } \varepsilon \downarrow 0.$$

Moreover, we have $|A_{\psi_i}(u)| \leq |u - c| \in L^1_{\text{loc}}(\Pi_T)$, so by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} A_{\psi_i}(u) \phi_t dt dx = \iint_{\Pi_T} |u - c| \phi_t dt dx.$$

For $c$ such that $A(c) \notin E$, we have

$$\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \left[ f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla [\text{sign}_c (A(u) - A(c)) \phi] dt dx$$

$$= \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \left[ f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \text{sign}_c (A(u) - A(c)) \phi dt dx$$

$$+ \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \text{sign}_c (A(u) - A(c)) \left[ f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi dt dx$$

$$\equiv \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \text{sign}_c (A(u) - A(c)) \left( f(x,t,u) - f(x,t,c) \right) \cdot \nabla A(u) \phi dt dx$$

$$- \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \left| \nabla A(u) \right|^2 \text{sign}_c (A(u) - A(c)) \phi dt dx$$

$$+ \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \text{sign}_c (A(u) - A(c)) \left[ f(x,t,u) - f(x,t,c) - \nabla A(u) \right] \cdot \nabla \phi dt dx.$$
One can check that

\[
I_1 = \lim_{\varepsilon \downarrow 0} \int_{\Pi_T} \left( \text{div} \mathcal{Q}_\varepsilon(x, t, A(u)) - \mathcal{Q}_{z\varepsilon}(x, t, A(u)) \right) \phi \, dt \, dx,
\]

where \(\mathcal{Q}_\varepsilon\) is defined as

\[
\mathcal{Q}_\varepsilon(x, t, z) = \frac{1}{\varepsilon} \int_{\min(z, A(c)+\varepsilon)}^{\min(z, A(c)-\varepsilon)} \left( f(x, t, A^{-1}(r)) - f(x, t, A(c)) \right) \, dr,
\]

and

\[
\mathcal{Q}_{z\varepsilon}(x, t, z) = \frac{1}{\varepsilon} \int_{\min(z, A(c)+\varepsilon)}^{\min(z, A(c)-\varepsilon)} \left( f_x(x, t, A^{-1}(r)) - f_x(x, t, A(c)) \right) \, dr.
\]

Since \(f = f(x, t, u)\) and \(f_x = f_x(x, t, u)\) are locally Lipschitz continuous with respect to \(u\) (uniformly in \(x, t\)), \(\mathcal{Q}_\varepsilon(x, t, z)\) and \(\mathcal{Q}_{z\varepsilon}(x, t, z)\) tend to zero as \(\varepsilon \downarrow 0\) for all \(z\) in the image of \(A\) (uniformly in \(x, t\)). Consequently, by the Lebesgue dominated convergence theorem,

\[
I_1 = \lim_{\varepsilon \downarrow 0} \int_{\Pi_T} \left( -\mathcal{Q}_\varepsilon(x, t, A(u)) \cdot \nabla \phi - \mathcal{Q}_{z\varepsilon}(x, t, A(u)) \phi \right) \, dt \, dx = 0.
\]

Observe that for each \(c \in \mathbb{R}\) such that \(A(c) \notin E\),

\[
\text{sign} \ (u - c) = \text{sign} \ (A(u) - A(c)) \quad \text{a.e. in } \Pi_T.
\]

Therefore, from the Lebesgue bounded convergence theorem, it follows that

\[
I_2 = \iint_{\Pi_T} \left( \text{div} f(x, t, u) - q(x, t, u) \right) \phi \, dt \, dx
\]

and

\[
\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \text{sign}_\varepsilon (A(u) - A(c)) \left( \text{div} f(x, t, c) - q(x, t, u) \right) \phi \, dt \, dx
\]

\[
= \iint_{\Pi_T} \text{sign} (u - c) \left( \text{div} f(x, t, c) - q(x, t, u) \right) \phi \, dt \, dx.
\]

Therefore, letting \(\varepsilon \downarrow 0\) in (2.23), we obtain the desired equality (2.22). \(\square\)

3. Proof of Theorem 1.1. Equipped with the results derived in §2 (in particular Lemma 2.4), we now set out to prove Theorem 1.1 using the “doubling of variables” device, which was introduced by Kružkov [25] as a tool for proving the uniqueness \((L^1\) contraction property) of the entropy solution of first order hyperbolic equations. We refer to Carrillo [11, 12], Otto [29], and Cockburn and Gripenberg [14] for applications of the “doubling” device in the context of second order parabolic equations. The presentation that follows below is inspired by Carrillo [12].
Let $\phi \in C^0_c(\Pi_T \times \Pi_T)$, $\phi \geq 0$, $\phi = \phi(x, t, y, s)$, $v = v(x, t)$, and $u = u(y, s)$. We shall also need to introduce the "hyperbolic" sets

$$\mathcal{E}_v = \{(x, t) \in \Pi_T : A(v(x, t)) \in E\}, \quad \mathcal{E}_u = \{(y, s) \in \Pi_T : A(u(y, s)) \in E\}.$$

Observe that we have

$$\text{sign} (v - u) = \text{sign} (A(v) - A(u)) \quad (3.24)$$

a.e. (w.r.t. $dt \, dx \, ds \, dy$) in $[\Pi_T \times (\Pi_T \backslash \mathcal{E}_v)] \cup [\Pi_T \backslash \mathcal{E}_u] \times \Pi_T$ and

$$\nabla_x A(v) = 0 \text{ a.e. (w.r.t. } dt \, dx \text{) in } \mathcal{E}_v,$$

$$\nabla_y A(u) = 0 \text{ a.e. (w.r.t. } ds \, dy \text{) in } \mathcal{E}_u. \quad (3.25)$$

From the definition of entropy solution, Lemma 2.4, and the first part of (3.25), we have

$$- \iint_{\Pi_T \times \Pi_T} \left( |v - u| \phi_t + \text{sign} (v - u) \left[ f(x, t, v) - f(x, t, u) - \nabla_x A(v) \right] \cdot \nabla_x \phi ight. \left. - \text{sign} (v - u) \left( \text{div}_x f(x, t, u) - q(x, t, v) \right) \phi \right) dt \, dx \, ds \, dy \leq - \lim_{\varepsilon \downarrow 0} \iint_{(\Pi_T \backslash \mathcal{E}_v) \times \Pi_T} \left| \nabla_x A(v) \right|^2 \text{sign}'_x (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy \quad (3.26)$$

$$= - \lim_{\varepsilon \downarrow 0} \iint_{(\Pi_T \backslash \mathcal{E}_u) \times (\Pi_T \backslash \mathcal{E}_v) \times \Pi_T} \left| \nabla_x A(v) \right|^2 \text{sign}'_x (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy. \quad (3.27)$$

The inequality (3.26) is obtained by using Lemma 2.4 with $v(x, t)$ where $(x, t)$ is not in the hyperbolic set $\mathcal{E}_u$, noting that the integral over $\Pi_T \backslash \mathcal{E}_v$ is less than the integral over $\Pi_T$. Finally, (3.27) follows from (3.25).

Similarly, using Lemma 2.4 for $u = u(y, s)$, and the second part of (3.25), we find the inequality

$$- \iint_{\Pi_T \times \Pi_T} \left( |u - v| \phi_t + \text{sign} (u - v) \left[ f(y, s, u) - f(y, s, v) - \nabla_y A(u) \right] \cdot \nabla_y \phi ight. \left. - \text{sign} (u - v) \left( \text{div}_y f(y, s, u) - q(y, s, v) \right) \phi \right) dt \, dx \, ds \, dy \leq - \lim_{\varepsilon \downarrow 0} \iint_{(\Pi_T \backslash \mathcal{E}_u) \times (\Pi_T \backslash \mathcal{E}_v) \times \Pi_T} \left| \nabla_y A(u) \right|^2 \text{sign}'_y (A(u) - A(v)) \phi \, dt \, dx \, ds \, dy \quad (3.28)$$

Observe that whenever $\nabla_x A(v)$ is defined,

$$\iint_{\Pi_T} \nabla_x A(v) \cdot \nabla_y (\text{sign}_x (A(v) - A(u)) \phi) \, ds \, dy = \nabla_x A(v) \cdot \iint_{\Pi_T} \nabla_y (\text{sign}_x (A(v) - A(u)) \phi) \, ds \, dy = 0,$$

or more conveniently

$$- \iint_{\Pi_T} \text{sign}_x (A(v) - A(u)) \nabla_x A(v) \cdot \nabla_y \phi \, ds \, dy = \iint_{\Pi_T} \nabla_y \text{sign}_x (A(v) - A(u)) \cdot \nabla_x A(v) \phi \, ds \, dy. \quad (3.29)$$
Similarly, for a.e. \((y, s) \in \Pi_T\),
\[
- \iint_{\Pi_T} \text{sign}_c (A(u) - A(v)) \nabla_y A(u) \cdot \nabla_x \phi \, dt \, dx = \iint_{\Pi_T} \nabla_x \text{sign}_c (A(u) - A(v)) \cdot \nabla_y A(u) \phi \, dt \, dx. \tag{3.30}
\]

Now using integrating (3.29), (3.24), and (3.25), we find that
\[
- \iiint_{\Pi_T \times \Pi_T} \text{sign} (v - u) \nabla_x A(v) \cdot \nabla_y \phi \, dt \, ds \, dy \tag{3.31}
\]
\[
= - \iiint_{\Pi_T \times (\Pi_T \setminus \xi)} \text{sign} (A(v) - A(u)) \nabla_x A(v) \cdot \nabla_y \phi \, dt \, ds \, dy
\]
\[
= - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \xi) \times (\Pi_T \setminus \xi)} \nabla_y A(u) \cdot \nabla_x A(v) \text{sign}_{\varepsilon} (A(v) - A(u)) \phi \, dt \, ds \, dy.
\]

Similarly, using (3.30), (3.24), and (3.25), we find that
\[
- \iiint_{\Pi_T \times \Pi_T} \text{sign} (A(u) - A(v)) \nabla_y A(u) \cdot \nabla_x \phi \, dt \, ds \, dy \tag{3.32}
\]
\[
= - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \xi) \times (\Pi_T \setminus \xi)} \nabla_x A(u) \cdot \nabla_y A(v) \text{sign}_{\varepsilon} (A(v) - A(u)) \phi \, dt \, ds \, dy.
\]

Adding (3.27) and (3.31) yields
\[
- \iiint_{\Pi_T \times \Pi_T} \left( |v - u| \phi_t + \text{sign} (v - u) \left[ (f(y, s, u) - f(x, t, u)) \cdot \nabla_x \phi - \nabla_x A(u) \cdot \nabla_x \phi + \nabla_y \phi \right] 
\right. \\
\left. \quad - \text{sign} (v - u) \left( \text{div}_y f(x, t, u) - q(x, t, v) \right) \phi \right) \, dt \, ds \, dy
\]
\[
\leq - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \xi) \times (\Pi_T \setminus \xi)} \left( \nabla_x A(v)^2 - \nabla_y A(u) \cdot \nabla_x A(v) \right) \text{sign}_{\varepsilon} (A(v) - A(u)) \phi \, dt \, ds \, dy.
\]

Similarly, adding (3.28) and (3.32) yields
\[
- \iiint_{\Pi_T \times \Pi_T} \left( |u - v| \phi_s + \text{sign} (u - v) \left[ (f(y, s, u) - f(y, s, v)) \cdot \nabla_y \phi - \nabla_y A(u) \cdot \nabla_y \phi + \nabla_x \phi \right] 
\right. \\
\left. \quad - \text{sign} (u - v) \left( \text{div}_y f(y, s, v) - q(y, s, u) \right) \phi \right) \, dt \, ds \, dy
\]
\[
\leq - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \xi) \times (\Pi_T \setminus \xi)} \left( \nabla_y A(u)^2 - \nabla_x A(u) \cdot \nabla_y A(u) \right) \text{sign}_{\varepsilon} (A(u) - A(v)) \phi \, dt \, ds \, dy.
\]

Note that we can write
\[
\text{sign} (v - u) \left( f(x, t, v) - f(x, t, u) \right) \cdot \nabla_x \phi - \text{sign} (v - u) \text{div}_y f(x, t, u) \phi
\]
\[
= \text{sign} (v - u) \left( f(x, t, v) - f(y, s, u) \right) \cdot \nabla_x \phi
\]
and
\[
\text{sign} (v - u) \left( f(y, s, u) - f(y, s, v) \right) \cdot \nabla_y \phi - \text{sign} (u - v) \text{div}_y f(y, s, v) \phi
\]
\[
= \text{sign} (v - u) \left( f(x, t, v) - f(y, s, u) \right) \cdot \nabla_x \phi
\]
\[
- \text{sign} (v - u) \text{div}_y \left[ \left( f(x, t, v) - f(y, s, v) \right) \phi \right].
\]
Taking these identities into account when adding (3.33) and (3.34), we get
\[
- \iiint_{\Pi_T \times \Pi_T} \left[ |v - u| (\phi_t + \phi_s) + I_1 + I_2 + I_3 + I_4 \right] dt dx ds dy
\]
\[
\leq - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \varepsilon) \times (\Pi_T \setminus \varepsilon)} \left| \nabla_x A(v) - \nabla_y A(u) \right|^2 \text{sign}_\varepsilon^\prime (A(v) - A(u)) \phi \ dt \ dx \ ds \ dy
\]
\[
\leq 0,
\]
where
\[I_1 = \text{sign} (v - u) \left[ f(x, t, v) - f(y, s, u) \right] \nabla_{x+y} \phi,\]
\[I_2 = |A(v) - A(u)| \Delta_{xy} \phi,\]
\[I_3 = \text{sign} (v - u) \left| \nabla_x \left( f(y, s, u) - f(x, t, u) \right) \phi \right| - \text{div}_y \left( f(x, t, v) - f(y, s, v) \right) \phi,\]
\[I_4 = \text{sign} (v - u) \left( q(x, t, v) - q(y, s, u) \right) \phi,\]
and we have introduced the notation
\nabla_{x+y} = \nabla_x + \nabla_y, \quad \Delta_{xy} = \Delta_x + 2 \nabla_x \cdot \nabla_y + \Delta_y, \quad \partial_{t+s} = \partial_x + \partial_s.
\]
To obtain \(I_2\) we use the relation
\[
\text{sign} (v - u) \left( \nabla_x A(v) - \nabla_y A(u) \right)
\]
\[
= \text{sign} (v - u) \left( \nabla_x (A(v) - A(u)) + \nabla_y (A(v) - A(u)) \right)
\]
\[
= \nabla_{x+y} \left| A(v) - A(u) \right|.
\]
Next we introduce a non-negative function \(\delta \in C_0^\infty\), satisfying
\[
\delta(\sigma) = \delta(-\sigma), \quad \delta(\sigma) = 0, \quad \text{for} \quad |\sigma| \geq 1, \quad \text{and} \quad \int_R \delta(\sigma) \, d\sigma = 1,
\]
and set
\[
\delta_\rho(t) = \frac{1}{\rho} \delta \left( \frac{t}{\rho} \right), \quad \text{for} \quad \rho > 0.
\]
Let
\[
\omega_\rho(x) = \frac{1}{2\rho^d} \delta \left( \frac{|x|^2}{\rho^2} \right).
\]
Observe that
\[
\nabla_x \omega_\rho(x - y) = \frac{1}{\rho^{d+2}} (x - y) \delta' \left( \frac{|x - y|^2}{\rho^2} \right) = -\nabla_y \omega_\rho(x - y).
\]
We define \(\phi = \phi(x, t, y, s) \in C_0^\infty(\Pi_T \times \Pi_T)\) by
\[
\phi(x, t, y, s) = \psi \left( \frac{x + y}{2}, \frac{t + s}{2} \right) \omega_\rho \left( \frac{x - y}{2} \right) \delta_\rho \left( \frac{t - s}{2} \right),
\]
(3.36)
where $\psi = \psi(x, t) \in C_0^\infty(\Pi_T)$ is another non-negative test function. Note that
\[
\partial_t \delta_{\rho_0} \left( \frac{t-s}{2} \right) = 0, \quad \nabla_{x+y} \omega_\rho \left( \frac{x-y}{2} \right) = 0, \quad \Delta_{xy} \omega_\rho \left( \frac{x-y}{2} \right) = 0.
\]

After a few straightforward computations, we find that
\[
\partial_t \phi(x, t, y, s) = \partial_t \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \omega_\rho \left( \frac{x-y}{2} \right) \delta_{\rho_0} \left( \frac{t-s}{2} \right),
\]
\[
\nabla_{x+y} \phi(x, t, y, s) = \nabla_{x+y} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \omega_\rho \left( \frac{x-y}{2} \right) \delta_{\rho} \left( \frac{t-s}{2} \right),
\]
\[
\Delta_{xy} \phi(x, t, y, s) = \Delta_{xy} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \omega_\rho \left( \frac{x-y}{2} \right) \delta_{\rho} \left( \frac{t-s}{2} \right).
\]

After some manipulation (3.35) reads
\[
- \iiint_{\Pi_T \times \Pi_T} \left( \bar{I}_0 + \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4 \right) \omega_\rho \left( \frac{x-y}{2} \right) \delta_{\rho} \left( \frac{t-s}{2} \right) \]
\[
\quad + \bar{I}_5 \nabla_{x} \omega_\rho \left( \frac{x-y}{2} \right) dt dx ds dy \leq 0,
\]
where
\[
\bar{I}_0 = |v-u| \partial_{t+s} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right),
\]
\[
\bar{I}_1 = \text{sign} (v-u) \left( f(x, t, v) - f(y, s, u) \right) \nabla_{x+y} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right),
\]
\[
\bar{I}_2 = |A(v) - A(u)| \Delta_{xy} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right),
\]
\[
\bar{I}_3 = \text{sign} (v-u) \left\{ \left[ \text{div}_{x} \left( f(y, s, u) - f(x, t, u) \right) \right] \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) - \left[ \text{div}_{y} \left( f(x, t, v) - f(y, s, v) \right) \right] \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \right. \]
\[
\quad + \left( f(y, s, u) - f(x, t, u) \right) \cdot \nabla_{x} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \]
\[
\quad - \left( f(x, t, v) - f(y, s, v) \right) \cdot \nabla_{y} \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \right\},
\]
\[
\bar{I}_4 = \text{sign} (v-u) \left( q(x, t, v) - q(y, s, u) \right) \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right),
\]
and
\[
\bar{I}_5 = \left[ F(x, t, v, u) - F(y, s, v, u) \right] \psi \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \delta_{\rho} \left( \frac{t-s}{2} \right).
\]
Taking (1.7) and $\delta'(\sigma) \leq 0$ for $\sigma \geq 0$ into account, we have

\[
\begin{align*}
(F(x, t, v, u) - F(y, s, v, u)) \cdot \nabla_x \omega_\rho \left(\frac{x - y}{2}\right) & = \frac{1}{2} (F(x, t, v) - F(y, s, u)) \cdot (x - y) \frac{1}{\rho^{d+2}} \delta' \left(\frac{|x - y|^2}{4\rho^2}\right) \\
& \leq \frac{\gamma}{2} |v - u| \frac{|x - y|^2}{\rho^2} \frac{1}{\rho^d} \max |\delta'| \mathbf{1}_{|x - y| < 2\rho}.
\end{align*}
\] (3.39)

We now use the change of variables

\[
\tilde{x} = \frac{x + y}{2}, \quad \tilde{t} = \frac{t + s}{2}, \quad z = \frac{x - y}{2}, \quad \tau = \frac{t - s}{2},
\]

which maps $\Pi_T \times \Pi_T$ into

\[
\mathbb{R}^d \times \mathbb{R}^d \times \left\{ (\tilde{t}, \tau) : 0 \leq \tilde{t} + \tau \leq T, 0 \leq \tilde{t} - \tau \leq T \right\}.
\]

As usual with this change of variables [25],

\[
\partial_{\tilde{t} + s} \psi \left(\frac{x + y}{2}, \frac{t + s}{2}\right) = \psi_t(\tilde{x}, \tilde{t}), \quad \nabla_{x+y} \phi(x, t, y, s) = \nabla_{\tilde{x}} \psi(\tilde{x}, \tilde{t}).
\]

In addition it has the wonderful property of diagonalizing the operator $\Delta_{xy}$:

\[
\Delta_{xy} \psi \left(\frac{x + y}{2}, \frac{t + s}{2}\right) = \Delta_{\tilde{x}} \psi(\tilde{x}, \tilde{t}).
\]

We now employ this change of variables and Lebesgue’s differentiation theorem to obtain the following limits

\[
\begin{align*}
\lim_{\rho \downarrow 0} \iint_{\Pi_T \times \Pi_T} \int_0^1 \omega_\rho \left(\frac{x - y}{2}\right) \delta_\rho \left(\frac{t - s}{2}\right) \ dt \ dx \ dy \ ds
& = \iint_{\Pi_T} |v(x, t) - u(x, t)| \psi_t(x, t) \ dt \ dx, \\
(3.40)
\lim_{\rho \downarrow 0} \iint_{\Pi_T \times \Pi_T} \int_0^1 \omega_\rho \left(\frac{x - y}{2}\right) \delta_\rho \left(\frac{t - s}{2}\right) \ dt \ dx \ dy \ ds
& = \iint_{\Pi_T} F(x, t, v, u) \cdot \nabla_x \psi(x, t) \ dt \ dx, \\
(3.41)
\lim_{\rho \downarrow 0} \iint_{\Pi_T \times \Pi_T} \int_0^1 \omega_\rho \left(\frac{x - y}{2}\right) \delta_\rho \left(\frac{t - s}{2}\right) \ dt \ dx \ dy \ ds
& = \iint_{\Pi_T} |A(v(x, t)) - A(u(x, t))| \Delta \psi(x, t) \ dt \ dx, \\
(3.42)
\lim_{\rho \downarrow 0} \iint_{\Pi_T \times \Pi_T} \int_0^1 \omega_\rho \left(\frac{x - y}{2}\right) \delta_\rho \left(\frac{t - s}{2}\right) \ dt \ dx \ dy \ ds
& = \iint_{\Pi_T} F_x(x, t, v, u) \psi(x, t) \ dt \ dx.
(3.43)
\]
\[ \lim_{\rho \downarrow 0} \iiint_{\Pi_T \times \Pi_T} I_5 \omega_{\rho} \left( \frac{x - y}{2} \right) \delta_{\rho} \left( \frac{t - s}{2} \right) dt \, dx \, dy \, ds = \iint_{\Pi_T} Q(x, t, v, u) \psi(x, t) \, dt \, dx, \]

(3.44)

where

\[ F_x(x, t, v, u) := \text{sign} \, (v - u) \left( f_x(x, t, v) - f_x(x, t, u) \right) \]

and

\[ Q(x, t, v, u) = \text{sign} \, (v - u) \left( q(x, t, v) - q(x, t, u) \right). \]

To estimate the integral involving \( \bar{I}_5 \) we use (3.39), and find that

\[ \lim_{\rho \downarrow 0} \iiint_{\Pi_T \times \Pi_T} \bar{I}_5 \nabla_x \omega_{\rho} \left( \frac{x - y}{2} \right) dt \, dx \, dy \, ds \leq \text{Const} \iint_{\Pi_T} |v(x, t) - u(x, t)| \psi(x, t) \, dt \, dx, \]

(3.45)

for some nonnegative constant depending on \( \delta' \). Summing up, we have shown that for any two entropy solutions \( u \) and \( v \), and for any nonnegative test function \( \psi(x, t) \) that

\[ - \iint_{\Pi_T} \left( |v - u| \psi_t + F(x, t, v, u) \cdot \nabla \psi + |A(v) - A(u)| \Delta \psi + (F_x(x, t, v, u) + Q(v, u)) \psi \right) dt \, dx \]

(3.46)

\[ \leq \text{Const} \iint_{\Pi_T} |v - u| \psi \, dt \, dx. \]

(3.47)

Since \( F_x + Q \) is Lipschitz continuous in \( v - u \), (3.46) implies that

\[ - \iint_{\Pi_T} \left( |v - u| \psi_t + F(x, t, v, u) \cdot \nabla \psi + |A(v) - A(u)| \Delta \psi \right) dt \, dx \]

\[ \leq \text{Const} \iint_{\Pi_T} |v - u| \psi \, dt \, dx, \]

(3.48)

for some constant depending on \( f, f_x \) and \( q \). Fix \( 0 < t_1 < t_2 < T \) and set

\[ \chi_\rho(t) = \int_{-\infty}^{t} \left( \delta_{\rho}(\tau - t_1) - \delta_{\rho}(\tau - t_2) \right) d\tau, \]

and for \( r > 1 \) set

\[ \varphi_r(x) = \int_{\mathbb{R}} \delta(|x - y|) 1_{|y| < r} \, dy. \]

We have that

\[ \nabla \varphi_r(x) = 0, \quad \text{for } |x| < r - 1 \text{ or } |x| > r + 1. \]

Set \( \psi(x, t) = \varphi_r(x) \chi_\rho(t) \), then

\[ \lim_{r \rightarrow \infty} \iint_{\Pi_T} \left( F(x, t, v, u) \nabla \psi + |A(v) - A(u)| \Delta \psi \right) dt \, dx \]

(3.49)
\[
\leq C \lim_{r \to \infty} \int_0^T \int_{|x| - r \leq 1} \left( |v| + |u| \right) dx \, dt = 0,
\]

since \(u(\cdot, t)\) and \(v(\cdot, t)\) are in \(L^1(\mathbb{R}^d)\) for almost all \(t\). Therefore, after sending \(r \to \infty\), we are left with

\[
- \int_0^T \int_{\mathbb{R}^d} |v - u| \chi_q'(t) \, dx \, dt \leq \text{Const} \int_0^T \int_{\mathbb{R}^d} |v - u| \chi_p(t) \, dx \, dt.
\]

Next, letting \(\rho \downarrow 0\), yields

\[
\|u(\cdot, t_2) - v(\cdot, t_2)\|_{L^1(\mathbb{R}^d)} = \|u(\cdot, t_1) - v(\cdot, t_1)\|_{L^1(\mathbb{R}^d)}
\]

\[
\leq \text{Const} \int_{t_1}^{t_2} \|u(\cdot, \tau) - v(\cdot, \tau)\|_{L^1(\mathbb{R}^d)} \, d\tau.
\]

Grönwall’s inequality then states

\[
\|u(\cdot, t_2) - v(\cdot, t_2)\|_{L^1(\mathbb{R}^d)} \leq \|u(\cdot, t_1) - v(\cdot, t_1)\|_{L^1(\mathbb{R}^d)} \left( 1 + \text{Const} (t_2 - t_1) e^{\text{Const} (t_2 - t_1)} \right).
\]

Sending \(t_1 \downarrow 0\) and setting \(t_2 = t\) concludes the proof of the theorem.

4. **Proof of Theorem 1.2.** In this section, we restrict ourselves to problems of the form (1.10), i.e., \(f(x, t, u) = k(x)f(u)\) and \(q(x, t, u) \equiv 0\). Let

\[
u, v \in L^\infty(0, T; BV(\mathbb{R}^d))
\]

be two entropy solutions of (1.10) with initial data

\[
u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d),
\]

respectively. As before, we are interested in estimating the \(L^1\) distance between \(v\) and \(u\). In this case \(q \equiv 0\), and the term \(I_4\) disappear and (3.44) is zero. We now divert our attention to \(I_5\), which in this case reads

\[
I_5 = (k(x) - k(y)) \tilde{F}(v, u) \psi \delta_p,
\]

where we have used the notation \(\tilde{F}(v, u) = \text{sign} (v - u) (f(v) - f(u))\) and \(\psi, \delta_p\) are respectively (here and in the next section) short-hand notations for

\[
\psi \left( \frac{x + y}{2} \right)
\]

and

\[
\delta_p \left( \frac{t - s}{2} \right).
\]

Similarly, in what follows we let \(\omega_p\) be short-hand notation for

\[
\frac{x - y}{2}.
\]

To continue, we need the following lemma (a proof can be found in, e.g., [5]):

**Lemma 4.1.** Consider a function \(z = z(x)\) belonging to \(L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)\) and let \(h \in \text{Lip}(I_z)\). Then \(h(z)\) belongs to \(L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)\) and

\[
\left| \frac{\partial}{\partial x_j} h(z) \right| \leq \|h\|_{\text{Lip}(I_z)} \left| \frac{\partial}{\partial x_j} z \right| \text{ in the sense of measures, for } j = 1, \ldots, d,
\]

where \(I_z\) denotes the interval \([-\|z\|_{L^\infty(\mathbb{R}^d)}, \|z\|_{L^\infty(\mathbb{R}^d)}]\).
Note that the function \( \hat{F}(v, u) \) is locally Lipschitz continuous in \( v \) and \( u \) with Lipschitz constant that of \( f \). Now since \( v(\cdot, t) \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d) \) for each \( t \), by Lemma 4.1 \( \nabla_x \left( \hat{F}(v, u) \psi \right) \) is a finite measure. Then we have

\[
\int_{\Pi_T \times \Pi_T} \tilde{I}_5 \nabla_x \omega_\rho \, dt \, dx \, ds \, dy = - \int_{\Pi_T \times \Pi_T} \nabla_x \tilde{I}_5 \omega_\rho \, dt \, dx \, ds \, dy = - \int_{\Pi_T \times \Pi_T} \left[ \text{div}_x k(x) \hat{F}(u, v) \psi + (k(x) - k(y)) \nabla_x \left( \hat{F}(u, v) \psi \right) \right] \omega_\rho \delta_\rho \, dt \, dx \, ds \, dy = E_1 + E_2,
\]

where

\[
E_1 = - \int_{\Pi_T \times \Pi_T} \text{div}_x k(x) \hat{F}(u, v) \psi \omega_\rho \delta_\rho \, dt \, dx \, ds \, dy
\]

and

\[
E_2 = - \int_{\Pi_T \times \Pi_T} (k(x) - k(y)) \nabla_x \left( \hat{F}(u, v) \psi \right) \omega_\rho \delta_\rho \, dt \, dx \, ds \, dy.
\]

Since \( k \in C(\mathbb{R}^d) \) and \( \nabla_x \left( \hat{F}(u, v) \psi \right) \) is a finite measure, it follows that

\[
\lim_{\rho \downarrow 0} E_2 = 0,
\]

and

\[
\lim_{\rho \downarrow 0} E_1 = - \int_{\Pi_T} \nabla_x k(x) \hat{F}(u, v) \psi(x, t) \, dx
\]

\[
= - \lim_{\rho \downarrow 0} \int_{\Pi_T} \int_{\Pi_T} I_3 \omega_\rho \left( \frac{x - y}{2} \right) \delta_\rho \left( \frac{t - s}{2} \right) \, dx \, dy \, ds.
\]

Thus in this case (3.46) reads

\[
- \int_{\Pi_T} \left( |v - u| \psi(x, t) + k(x) \hat{F}(v, u) \cdot \nabla \psi(x, t) + |A(v) - A(u)| \Delta \psi(x, t) \right) \, dx \leq 0,
\]

which essentially is (3.48) with \( \text{Const} = 0 \). This implies the conclusion of the theorem.

5. **Proof of Theorem 1.3.** We are going to estimate the \( L^1 \) difference between the entropy solution \( v \) of (1.13) and the entropy solution \( u \) of (1.10). To do this, we proceed exactly as in the proof of Theorem 1.1. In what follows, let \( \phi = \phi(x, t, y, s) \) be a test function on \( \Pi_T \times \Pi_T \).

Let \( \tilde{G}(v, u) \) be defined similarly to \( \hat{F} \), but with \( f \) replaced by \( g \). Similarly to (3.33) and (3.34), we can derive the following integral inequalities for the entropy.
solutions $v = v(x, t)$ and $u = u(y, s)$ of (1.13) and (1.10):

$$
- \iiint_{\Pi_T \times \Pi_T} \left| v - u \right| \phi_t + \text{sign} (v - u) \left[ l(x) (g(v) - g(u)) \cdot \nabla_x \phi - \nabla_x A(v) \cdot \nabla_{x+y} \phi - \text{div}_x l(x) g(u) \phi \right] \, dt \, dx \, ds \, dy \\
\leq - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \mathcal{E}_\varepsilon) \times (\Pi_T \setminus \mathcal{E}_\varepsilon)} \left( \left| \nabla_x A(v) \right|^2 - \left| \nabla_y A(u) \cdot \nabla_x A(v) \right| \right) \times \text{sign}^\varepsilon (A(v) - A(u)) \phi \, dt \, dx \, ds \, dy,
$$

and

$$
- \iiint_{\Pi_T \times \Pi_T} \left| u - v \right| \phi_x + \text{sign} (u - v) \left[ k(y) (f(u) - f(v)) \cdot \nabla_y \phi - \nabla_y A(u) \cdot \nabla_{x+y} \phi - \text{div}_y k(y) f(v) \phi \right] \, dt \, dx \, ds \, dy \\
\leq - \lim_{\varepsilon \to 0} \iiint_{(\Pi_T \setminus \mathcal{E}_\varepsilon) \times (\Pi_T \setminus \mathcal{E}_\varepsilon)} \left( \left| \nabla_y A(u) \right|^2 - \left| \nabla_x A(v) \cdot \nabla_y A(u) \right| \right) \times \text{sign}^\varepsilon (A(u) - A(v)) \phi \, dt \, dx \, ds \, dy.
$$

Next we write

$$
l(x) (g(v) - g(u)) \cdot \nabla_x \phi - \text{sign} (v - u) \text{div}_x l(x) g(u) \phi = \text{sign} (v - u) \left[ l(x) g(v) - k(y) f(u) \right] \cdot \nabla_x \phi + \text{sign} (v - u) \text{div}_x \left[ (k(y) f(u) - l(x) g(u)) \phi \right]
$$

and

$$
\text{sign} (u - v) k(y) (f(u) - f(v)) \cdot \nabla_y \phi - \text{sign} (u - v) \text{div}_y k(y) g(v) \phi = \text{sign} (v - u) \left[ l(x) g(v) - k(y) f(u) \right] \cdot \nabla_y \phi - \text{sign} (v - u) \text{div}_y \left[ (l(x) g(v) - k(y) f(v)) \phi \right].
$$

Similarly to (3.35), by adding (3.33) and (3.34) we obtain

$$
- \iiint_{\Pi_T \times \Pi_T} \left| v - u \right| \partial_{t+s} \phi + I_1 + I_2 \right) \, dt \, dx \, ds \, dy \leq 0,
$$

where

$$
I_1 = \text{sign} (v - u) \left[ l(x) g(v) - k(y) f(u) \right] \nabla_{x+y} \phi + \left| A(v) - A(u) \right| \Delta_{x+y} \phi,
$$

$$
I_2 = \text{sign} (v - u) \left[ \text{div}_x \left[ (k(y) f(u) - l(x) g(u)) \phi \right] - \text{div}_y \left[ (l(x) g(v) - k(y) f(v)) \phi \right] \right].
$$

We now specify the test function $\phi$ as in (3.36). Before we continue, let us write $I_2 = I_{2,1} + I_{2,2}$ where

$$
I_{2,1} = \text{sign} (v - u) \left[ (k(y) f(u) - l(x) g(u)) \cdot \nabla_x \phi - \left( l(x) g(v) - k(y) f(v) \right) \cdot \nabla_y \phi \right],
$$

$$
I_{2,2} = \text{sign} (v - u) \left( \text{div}_y k(y) f(v) - \text{div}_x l(x) g(u) \right) \phi.
$$

With the test function $\phi$ defined in (3.36), we find that

$$
I_{2,1} = \text{sign} (v - u) \left[ k(y) (f(u) + f(v)) - l(x) (g(u) + g(v)) \right] \cdot \nabla_{x+y} \psi \omega_\rho \delta_\rho \\
+ \left[ l(x) \tilde{G}(v, u) - k(y) \tilde{F}(v, u) \right] \psi \nabla_x \omega_\rho \delta_\rho.
$$
Therefore, we have that

\[ I_{2,1,1} + I_{2,1,2}. \]

We now use integration by parts to show that

\[
\begin{aligned}
\iint_{\Pi_T \times \Pi_T} I_{2,1,2} \; dt \; dx \; ds \; dy \\
= - \iint_{\Pi_T \times \Pi_T} \left[ \frac{\partial}{\partial s} l(x) \tilde{G}(v, u) + l(x) \cdot \nabla_x \tilde{G}(v, u) \\
- k(y) \cdot \nabla_x \tilde{F}(v, u) \right] \omega_\rho \delta_\rho \; dt \; dx \; ds \; dy \\
- \iint_{\Pi_T \times \Pi_T} \left[ l(x) \tilde{G}(v, u) - k(y) \tilde{F}(v, u) \right] \omega_\rho \delta_\rho \frac{1}{2} \nabla \psi \; dt \; dx \; ds \; dy.
\end{aligned}
\]

Next we calculate

\[
- \frac{\partial}{\partial s} l(x) \tilde{G}(v, u) \phi + I_{2,2} \\
= \text{sign} \left( v - u \right) \frac{\partial}{\partial s} k(y) f(v) - \text{sign} \left( v - u \right) \frac{\partial}{\partial s} l(x) g(v) \phi \\
= \text{sign} \left( v - u \right) \left[ \frac{\partial}{\partial s} k(y) \left( f(v) - g(v) \right) - \left( \frac{\partial}{\partial s} k(y) - \frac{\partial}{\partial s} l(x) \right) g(v) \right] \phi.
\]

Adding \( I_{2,2} \) and \( I_{2,1,2} \) and integrating we thus find

\[
\begin{aligned}
\iint_{\Pi_T \times \Pi_T} \left( I_{2,2} + I_{2,1,2} \right) \; dt \; dx \; ds \; dy \\
= \iint_{\Pi_T \times \Pi_T} \left( \text{sign} \left( v - u \right) \left[ \frac{\partial}{\partial s} k(y) \left( f(v) - g(v) \right) - \left( \frac{\partial}{\partial s} k(y) - \frac{\partial}{\partial s} l(x) \right) g(v) \right] \\
+ \left( k(y) - l(x) \right) \cdot \nabla_x \tilde{G}(v, u) + k(y) \cdot \nabla_x \left( \tilde{F}(v, u) - G(v, u) \right) \right) \phi \; dt \; dx \; ds \; dy \\
+ \iint_{\Pi_T \times \Pi_T} \left[ l(x) \tilde{G}(v, u) - k(y) \tilde{F}(v, u) \right] \omega_\rho \delta_\rho \frac{1}{2} \nabla \psi \; dt \; dx \; ds \; dy.
\end{aligned}
\]

Now that we have removed all derivatives on \( \omega_\rho \), we can send \( \rho \rightarrow 0 \), in effect setting \( y = x, s = t \) and removing two integrals. Next, we choose \( \psi \approx 1_{t_1 < t < t_2} \), i.e.,

\[
\psi(x, t) = \psi_{r, \tilde{\rho}}(x, t) = \varphi_r(x) \chi_{\tilde{\rho}}(t), \quad r, \tilde{\rho} > 0, 
\]

where \( \varphi_r, \chi_{\tilde{\rho}} \) are as in the previous section. With this choice of \( \psi \), all integrals with integrands of the form

\[
\{ \text{Terms in } L^1(\mathbb{R}^d) \} \cdot \nabla \psi_{r, \tilde{\rho}} \quad \text{and} \quad \{ \text{Terms in } L^1(\mathbb{R}^d) \} \Delta \psi_{r, \tilde{\rho}}
\]

will vanish when \( r \rightarrow \infty \). Thus

\[
\lim_{\rho \rightarrow 0} \int_{\Pi_T} \left( I_1 + I_{2,1,1} + J \right) \; dt \; dx \; ds \; dy = 0. \quad (5.53)
\]

Therefore, we have that

\[
- \lim_{\rho \rightarrow 0} \int_{\Pi_T} \left( |v - u| \partial_{r+s} \phi + I_1 + I_2 \right) \; dt \; dx \; ds \; dy \quad (5.54)
\]

\[
- \lim_{\rho \rightarrow 0} \int_{\Pi_T} \left( |v - u| \partial_{r+s} \phi + I_2 + I_{2,1,2} \right) \; dt \; dx \; ds \; dy.
\]
Observe that by Lemma 4.1 we have
\[
\left| \frac{\partial}{\partial x_j} G(v, u) \right| \leq \|g\|_{\text{Lip}(I)} \left| \frac{\partial}{\partial x_j} v(x, t) \right|,
\]
(5.55)
for \( j = 1, \ldots, d \). Equipped with (5.55) and (5.53), we send \( \rho, \tilde{\rho} \downarrow 0 \) and \( r \uparrow \infty \) to obtain
\[
\|v(\cdot, t) - u(\cdot, t)\|_{L^2(I; \mathbb{R}^d)} \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left( \|\text{div}(x)\| f - g \|_{L^\infty(I)} + \|\text{div}(k(x) - l(x))\| g \|_{L^\infty(I)} \right.
\]
\[
+ \|k - l\|_{L^\infty(\mathbb{R}^d)} \|g\|_{\text{Lip}(I)} \sum_{j=1}^d \left| \frac{\partial}{\partial x_j} v(x, t) \right|
\]
\[
+ \|k\|_{L^\infty(\mathbb{R}^d)} \|f - g\|_{L^2(I)} \sum_{j=1}^d \left| \frac{\partial}{\partial x_j} v(x, t) \right| \right) dx \, dt.
\]
We now end up with the following continuous dependence estimate:
\[
\|v(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq (t_2 - t_1) \left( \|g\|_{\text{Lip}(I)} \sup_{t \in (0, T)} \|v(\cdot, t)\|_{\text{BV}(\mathbb{R}^d)} \|k - l\|_{L^\infty(\mathbb{R}^d)} \right.
\]
\[
+ \|g\|_{L^\infty(I)} \|k - l\|_{\text{BV}(\mathbb{R}^d)} + \|k\|_{L^\infty(\mathbb{R}^d)} \sup_{t \in (0, T)} \|v(\cdot, t)\|_{\text{BV}(\mathbb{R}^d)} \|f - g\|_{L^\infty(I)}
\]
\[
+ \|k\|_{L^\infty(\mathbb{R}^d)} \sup_{t \in (0, T)} \|v(\cdot, t)\|_{\text{BV}(\mathbb{R}^d)} \|f - g\|_{L^2(I)} \right).
\]
Sending \( t_1 \downarrow 0 \), setting \( t = t_2 \) and using symmetry, we finally conclude that Theorem 1.3 holds.

6. Appendix (proof of Lemma 2.3). In this appendix, we give a proof of Lemma 2.3. The proof follows Carrillo [12], but see also Alt and Luckhaus [1] and Otto [29]. Note that \( A_\psi \) is a nonnegative and convex function. Convexity implies that for a.e. \((x, t) \in \Pi_T\), we have
\[
A_\psi(u(x, t)) - A_\psi(u(x, t - \tau)) \leq (u(x, t) - u(x, t - \tau)) \psi(A(u(x, t))),(6.56)
\]
where we define \( u(t) = u_0 \) for \( t \in (-\tau, 0) \). In the sequel let \( \phi \in C^\infty_0(\mathbb{R}^d \times [0, T]) \). Multiplying the above inequality by \( \phi(x, t) \) yields
\[
A_\psi(u(x, t)) \phi(x, t) - A_\psi(u(x, t - \tau)) \phi(x, t - \tau)
\]
\[
+ A_\psi(u(x, t - \tau)) \phi(x, t - \tau) - A_\psi(u(x, t - \tau)) \phi(x, t)
\]
\[
= A_\psi(u(x, t)) \phi(x, t) - A_\psi(u(x, t - \tau)) \phi(x, t)
\]
\[
\leq (u(x, t) - u(x, t - \tau)) \psi(A(u(x, t)) \phi(x, t),\]
where we define \( \phi(x, t) = \phi(x, 0) \) for \( t < 0 \). We exploit the fact that
\[
A_\psi(u_0) \in L^1(\mathbb{R}^d), \quad A_\psi(u) \in L^\infty(0, T; L^1(\mathbb{R}^d)).
\]
Dividing (6.56) by $\tau$ and integrating over $\mathbb{R}^d \times (0, s)$, we get
\[
\frac{1}{\tau} \int_{s-\tau}^{s} \int_{\mathbb{R}^d} A_\psi(u(x,t)) \phi(x,t) \, dx \, dt - \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u_0(x)) \phi(x,0) \, dx \, dt \\
+ \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u(x,t-\tau)) (\phi(x,t-\tau) - \phi(x,t)) \, dx \, dt \\
\leq \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} (u(x,t) - u(x,t-\tau)) \psi(A(u(x,t))) \phi(x,t) \, dx \, dt. \quad (6.57)
\]
Since $\phi \in C_0^{\infty}(\mathbb{R}^d \times [0,T])$ and $A(u) \in L^2(0,T; H^1(\mathbb{R}^d))$, we have
\[
\psi(A(u)) \phi \in L^2(0,T; H_0^1(\mathbb{R}^d)).
\]
Therefore, exploiting that $u \in C(0,T; L^1(\mathbb{R}^d))$ and $\partial_t u \in L^2(0,T; H^{-1}(\mathbb{R}^d))$, we can let $\tau \downarrow 0$ in (6.57) and obtain
\[
\int_{\mathbb{R}^d} A_\psi(u(x,s)) \phi(x,s) \, dx - \int_{\mathbb{R}^d} A_\psi(u_0) \phi(x,0) \, dx - \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u) \phi_t \, dx \, dt \\
\leq \int_{0}^{s} (\partial_t u, \psi(A(u)) \phi) \, dt,
\]
for a.e. $s \in (0,T)$. Convexity implies also that for a.e. $(x,t) \in \Pi_T$ and $t > \tau$, we have
\[
A_\psi(u(x,t)) - A_\psi(u(x,t-\tau)) \geq (u(x,t) - u(x,t-\tau)) \psi(A(u(x,t-\tau))).
\]
Multiplying this inequality by $\phi(x,t-\tau)$ yields
\[
A_\psi(u(x,t)) \phi(x,t) - A_\psi(u(x,t-\tau)) \phi(x,t-\tau) \\
+ A_\psi(u(x,t)) (\phi(x,t-\tau) - \phi(x,t)) \\
= A_\psi(u(x,t)) \phi(x,t-\tau) - A_\psi(u(x,t-\tau)) \phi(x,t-\tau) \\
\geq (u(x,t) - u(x,t-\tau)) \psi(A(u(x,t-\tau))) \phi(x,t-\tau). \quad (6.58)
\]
After dividing (6.58) by $\tau$ and integrating over $\mathbb{R}^d \times (\tau, s)$, we obtain
\[
\frac{1}{\tau} \int_{s-\tau}^{s} \int_{\mathbb{R}^d} A_\psi(u(x,t)) \phi(x,t) \, dx \, dt - \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u(x,t)) \phi(x,t) \, dx \, dt \\
+ \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u(x,t)) (\phi(x,t-\tau) - \phi(x,t)) \, dx \, dt \\
\geq \frac{1}{\tau} \int_{0}^{s} \int_{\mathbb{R}^d} (u(x,t) - u(x,t-\tau)) \psi(A(u(x,t-\tau))) \phi(x,t-\tau) \, dx \, dt. \quad (6.59)
\]
Finally, similarly to (6.57), letting $\tau \downarrow 0$ in (6.59), we get, for a.e. $s \in (0,T)$,
\[
\int_{\mathbb{R}^d} A_\psi(u(x,s)) \phi(x,s) \, dx - \int_{\mathbb{R}^d} A_\psi(u_0) \phi(x,0) \, dx - \int_{0}^{s} \int_{\mathbb{R}^d} A_\psi(u) \phi_t \, dx \, dt \\
\geq \int_{0}^{s} (\partial_t u, \psi(A(u)) \phi) \, dt.
\]
This concludes the proof of the Lemma 2.3.
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