Curlicues generated by circle homeomorphisms

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Abstract

We investigate the curves in the complex plane which are generated by sequences of real numbers being the lifts of the points on the orbit of an orientation preserving circle homeomorphism. Geometrical properties of these curves such as boundedness, superficiality, local discrete radius of curvature are linked with dynamical properties of the circle homeomorphism which generates them: rotation number and its continued fraction expansion, existence of a continuous solution of the corresponding cohomological equation and displacement sequence along the orbit.

Keywords: circle homeomorphism, curlicue, rotation number, cohomological equation, superficial curve

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Introduction

The term curlicue is probably mostly used in various visual arts, for example it can be a recurring decorative motif in architecture, calligraphy or fashion design. In this article we look at mathematical curlicues:

Definition 0.1 A curlicue $\Gamma = \Gamma(u)$, where $u = (u_n)_{n=0}^{\infty} \subset \mathbb{R}$, is a piece-wise linear curve in $\mathbb{C}$ passing consecutively through the points $z_0 = 0 \in \mathbb{C}$, and $z_1, z_2, \ldots$, where

$$z_n = \sum_{k=0}^{n-1} \exp(2\pi i u_k), \quad n = 1, 2, \ldots \quad (1)$$

In other words,

$$z_n = z_{n-1} + \exp(2\pi i u_{n-1}), \quad n = 1, 2, \ldots \quad (2)$$

A curlicue can be obtained from an arbitrary sequence $(u_n)_{n=0}^{\infty}$ of real numbers. However, in this paper we assume that $u_n := \Phi^n(x_0), \ n = 0, 1, \ldots, x_0 \in \mathbb{R}$, with $\Phi : \mathbb{R} \to \mathbb{R}$ being a lift of an orientation preserving circle homeomorphism $\varphi : S^1 \to S^1$, where $\mathbb{R}$ covers $S^1$ via the standard projection: $p : \mathbb{R} \to S^1, \ p(t) = \exp(2\pi it)$. Construction of such a curlicue is illustrated in Figure 1. Sometimes we will also denote $\Gamma$ as $\Gamma((u_n))$ and when the generating homeomorphism $\Phi$ is clear from the context, we will write $\Gamma(x_0)$ to distinguish between the curves generated by the same homeomorphism but along the orbits of different initial points $x_0$.

The name curlicue for such a curve is not accidentally connected with the artistic notion of a curlicue: indeed, these curves, obtained for various sequences $(u_n)_{n=0}^{\infty} \subset \mathbb{R}$, can form beautiful shapes as one can see, for example, in the papers of Dekking and Mendès-France (6), who studied geometrical properties of such curves (superficiality and dimension), Berry and Golberg (3).
Sinai (19) or Cellarosi (11) who studied and developed techniques of renormalisation and limiting distributions of classical curlicues, i.e., for $u_n = \alpha n^2$. Many fantastic pictures of curlicues can be found also in the work of Moore and van der Poorten (17), who gave nice description of the work [3]. However, we would like to draw attention to dynamically generated curlicues, i.e., the curves $\Gamma$, where $(u_n)$ is obtained from an orbit of a given map $f$ since reflecting the dynamics of $f$ in the structure of $\Gamma$ might be in general an intriguing question.

We also remark that in the existing literature the term curlicues (if used at all) often refers to spiral-like components of the curve $\Gamma$ (which usually has both straight-like and spiral-like parts). However, in the current paper by a “curlicue” we mean the whole curve $\Gamma$, defined as above. Perhaps it is also worth mentioning that the curlicue can be interpreted as a trajectory of a particle in the plane which starts in the origin at time $t = 0$ and moves with a constant velocity, changing its direction at instances $t = 0, 1, 2, 3, ..., \ldots$, where the new direction is given by a number $2\pi u_t \in [0, 2\pi)$ (compare [6]). Thus $\Gamma$ can be seen as a trajectory of a walk obtained through some dynamical system, similarly as, for example, in the work of Avila and collaborators (2) on returns to zero of the deterministic random walk (a stochastic process arising from irrational rotations).

In this study we are mainly interested not in “ergodic” but rather in geometric properties of curlicues such as boundedness and superficiality (defined below). Although dynamics of circle homeomorphisms is now well understood (see e.g. [13]), it turns out that it is not so trivial to give complete description of curlicues determined by them. In the first section we prove that geometric properties of such curves are inevitably connected with rationality of the rotation number $\rho$ of the circle homeomorphism $\varphi$. However, unless $\varphi$ is a rigid rotation, this relation is not so straightforward. In particular, there are no simple criteria for deciding whether $\Gamma$ is bounded or not (even for equidistributed sequences $(u_n)_{n=0}^{\infty}$, see [6]). In section 2 it is deduced that for $\rho \in \mathbb{R} \setminus \mathbb{Q}$ boundedness and shape of $\Gamma$ depend on the solution of the corresponding cohomological equation. Further, in section 3 we estimate growth rate and superficiality of an unbounded curve $\Gamma$ with $\rho \in \mathbb{R} \setminus \mathbb{Q}$ satisfying some further (generic) properties. The last section is devoted to a local discrete radius of curvature.

1 Dependance on the rotation number

In [6] (Example 4.1) the following result was stated for $\varphi$ being the rotation by $\rho$: 

![Figure 1: Construction of a curlicue](image-url)
\textbf{Fact 1.1} Let $u_n := n\varrho$. Then
\begin{equation}
|z_n| = \left| \frac{\sin n\pi \varrho}{\sin \pi \varrho} \right|
\end{equation}
and the points $z_n$ lie on a circle with radius
\begin{equation}
R = \frac{1}{2|\sin \pi \varrho|}
\end{equation}
and center
\begin{equation}
C = \left( \frac{1}{2}, \frac{1}{2} \cot \pi \varrho \right).
\end{equation}
Furthermore,
\begin{enumerate}
  \item[a)] if $\varrho \in \mathbb{Q}$ (and $\varrho \neq 0 \mod 1$), then $\Gamma(u)$ is a regular polygon (convex or star) with $q$ sides, where $\varrho = p/q$ ($p$ and $q$ relatively prime);
  \item[b)] if $\varrho \in \mathbb{R} \setminus \mathbb{Q}$, then $\Gamma(u)$ is dense in an annulus with radii
\begin{equation}
r_1 = \frac{1}{2} |\cot \pi \varrho| \quad \text{and} \quad r_2 = \frac{1}{2|\sin \pi \varrho|}
\end{equation}
and
\begin{equation}
dim \Gamma = 2.
\end{equation}
\end{enumerate}
For the precise definition of the dimension $\dim \Gamma$ see e.g. [6]. By regular star polygon we mean self-intersecting, equilateral equiangular polygon, which can be constructed by connecting every $p$-th point out of $q$ points regularly spaced on the circle. For example, regular star polygon in Figure 2a is obtained by joining every third vertex of a regular decagon until the starting vertex is reached. Regular polygons can be described by their Schlӓfli symbols $\{q, p\}$ where $p \geq 2$ and $q$ are relatively prime integers.

\textbf{Remark 1.2} If $\Gamma$ is a curve generated by rotation $\mathcal{R}_\varrho$ with $\varrho = \frac{p}{q}$, then it is a regular polygon with Schlӓfli symbol $\{q, p\}$.

It is easy to notice that rotation numbers of the form $1/q$ and $(q-1)/q$ correspond to $q$-sided regular convex polygons.

The Fact 1.1 deals with the simplest situation when the curve is generated by a circle rotation $\mathcal{R}_\varrho$. Clearly, the properties of $\Gamma$ are determined by rationality of $\varrho$. This simple observation is a starting point for our investigations: we ask what changes if one considers slightly more general case, i.e. when $\Gamma$ is generated by an orientation preserving circle homeomorphism (we remark that all homeomorphims of $S^1$ considered here are assumed to be orientation preserving, even if not stated directly).

\begin{figure}[h]
\centering
\begin{subfigure}{0.3\textwidth}
  \includegraphics[width=\textwidth]{figure2a}
  \caption{$\varrho = 3/10$, $k = 1000$}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
  \includegraphics[width=\textwidth]{figure2b}
  \caption{$\varrho = -\ln(0.5)$, $k = 100$}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
  \includegraphics[width=\textwidth]{figure2c}
  \caption{$\varrho = -\ln(0.5)$, $k = 1000$}
\end{subfigure}
\caption{$\Gamma_k((n\varrho))$, a curlicue generated by the rotation $\mathcal{R}_\varrho$, for different values of $k$ and $\varrho$.}
\end{figure}

Before we proceed, a few essential definitions and results obtained by other authors must be recalled.
Definition 1.3 A bounded sequence \( \{u_0, u_1, u_2, \ldots \} \) of real numbers is equidistributed in the interval \([a, b]\) if for any subinterval \([c, d]\) \(\subset [a, b]\) we have
\[
\lim_{n \to \infty} \frac{|\{u_0, u_1, u_2, \ldots u_{n-1}\} \cap [c, d]|}{n} = \frac{d-c}{b-a},
\]
where \(|\{u_0, u_1, u_2, \ldots u_{n-1}\} \cap [c, d]|\) denotes the number of elements of the sequence, out of the first \(n\)-elements, in the interval \([c, d]\).

Definition 1.4 The sequence \( \{u_0, u_1, u_2, \ldots \} \) is said to be equidistributed modulo 1 (alternatively, uniformly distributed modulo 1) if the sequence of fractional parts of its elements, i.e. the sequence \( \{u_0 - \lfloor u_0 \rfloor, u_1 - \lfloor u_1 \rfloor, u_2 - \lfloor u_2 \rfloor, \ldots\} \), is equidistributed in the interval \([0, 1]\).

Let us also remind that an arbitrary curve \( \Gamma \) is rectifiable if its length is finite and is said to be locally rectifiable if all its closed subcurves are rectifiable (see e.g. [10]). For a locally rectifiable curve \( \Gamma := \gamma([0, \infty)) \) \((\gamma : [0, \infty) \to \mathbb{R}^2 \text{ a continuous function})\), we denote by \( \Gamma_t \) the beginning part of \( \Gamma \) which has length \( t \). \( \Gamma \) is called bounded if \( \text{Diam}\Gamma < \infty \) (otherwise, \( \Gamma \) is called unbounded). For \( \varepsilon > 0 \) we define
\[
\Gamma^\varepsilon := \{y : x \in \Gamma, d(x, y) < \varepsilon\}
\]

Definition 1.5 An unbounded curve \( \Gamma \) is superficial if
\[
\lim_{t \to \infty} \frac{t}{\text{Diam}\Gamma_t} = \infty.
\]
In turn, a bounded curve \( \Gamma \) is superficial if
\[
\lim_{\varepsilon \to 0} \frac{\text{Area}\Gamma^\varepsilon}{\varepsilon} = \infty,
\]
where by Area we mean a 2-dimensional Lebesgue measure.

The authors of [6] prove a very useful criterion for a sequence to be equidistributed modulo 1.

Theorem 1.6 (\([6]\)) Let \( \Gamma = \Gamma(u) \) be a curve generated by the sequence \( u = (u_n)_{n=0}^{\infty} \).

The sequence \( \{u_n\} \) is equidistributed modulo 1 if and only if for each positive integer \( q \) the curve \( \Gamma(qu) \) is superficial.

By \( \Gamma(qu) \) we denote a curve generated by the sequence \( (qu_n)_{n=0}^{\infty} \), i.e. a curve passing through the points \( z_0 = 0 \in \mathbb{C} \) and
\[
z_n(q) := \sum_{k=0}^{n-1} \exp(2\pi i ku_k), \quad n \in \mathbb{N}.
\]
It is well known (cf. Weyl Equidistribution Theorem) that the sequence \( (n\varrho)_{n=0}^{\infty} \) for \( \varrho \in \mathbb{R} \setminus \mathbb{Q} \) is equidistributed modulo 1. Hence,

Corollary 1.7 The curve generated by the irrational rotation \( \{u_n = \Phi^n(x_0), \Phi(x) = x + \varrho\} \) is superficial.

From the proof of Theorem 3.1 in [6] one concludes

Fact 1.8 If the sequence \( \{u_n\} \) determines a bounded curve \( \Gamma(u) \) and if infinitely many \( u_n \) are different modulo 1, then the curve \( \Gamma(u) \) is superficial.

Proposition 1.9 Let \( \Gamma \) be a curve generated by an orientation preserving circle homeomorphism \( \varphi \) with an irrational rotation number \( \varrho \in \mathbb{R} \setminus \mathbb{Q} \). It follows that:
1. If \( \varphi = R_\varrho \) is the rotation, then \( \Gamma \) is bounded and superficial.

2. If \( \Gamma \) is bounded, then it is also superficial.

**Proof.** Suppose that \( \varphi = R_\varrho \). Then \( \Gamma \) is superficial, which is covered by Corollary [1.7]. It is also bounded as it is contained in some annulus on the account of Theorem [1.1].

The second statement is justified by the Fact [1.8].

Note that if \( \varphi \) is not a rigid rotation, but its rotation number \( \varrho \) is irrational, then it is (semi-)conjugated to \( R_\varrho \) by a map \( \gamma : S^1 \to S^1 \) having a continuous surjective non-decreasing lift (in fact, if \( \varphi \) is transitive then it is conjugated to the rotation and \( \gamma \) is a homeomorphism, see [13]). Then the unique invariant ergodic measure for \( \varphi \) is given as \( \mu(A) = \Lambda(\gamma(A)) \), where \( \Lambda \) denotes the Lebesgue measure, and the sequence of fractional parts of \( u_n = \varphi^n(x_0) \) is equidistributed with respect to the measure \( \mu \). Therefore, if \( u_n \) is equidistributed modulo 1 (in the sense of Definition [1.4]), then the measure \( \mu \) coincides with the Lebesgue measure which implies that \( \gamma \) is also a rotation (the conjugating homeomorphism is always unique up to the composition with a rotation). Consequently, we obtain that \( \varphi \) is a rotation which is a contradiction. Therefore, the sequence \( u_n := \Phi^n(x_0), x_0 \in \mathbb{R} \), where \( \Phi \) is a lift of an orientation preserving circle homeomorphism \( \varphi \), is equidistributed modulo 1 if and only if \( \varphi \) is an irrational rotation.

In particular, when \( \varphi \) has irrational rotation number but is not transitive, then the measure \( \mu \) is concentrated on the minimal Cantor-like set \( \Delta \) and we immediately obtain that \( u_n \) is not equidistributed modulo 1.

We would like to analyse superficiality and boundedness of curves determined not by the rotations, but by general circle homeomorphisms. For this purpose we start with the case of rational rotation number.

**Proposition 1.10** Let \( \Gamma(u) \) be generated by \( u_n = \Phi^n(x_0) \), where \( \Phi \) is a lift of a circle homeomorphism \( \varphi \) with \( \varrho = p/q \) (\( p \) and \( q \) relatively prime). Suppose that \( \varphi \) is conjugated to the rational rotation \( R_\varrho \).

Then \( \Gamma(u) \) is not superficial, independently of the choice of \( x_0 \), and the following conditions are equivalent:

1. \( \frac{1}{q} \sum_{k=0}^{q-1} \exp(2\pi i \Phi^k(x_0)) = 0 \in \mathbb{C} \),
2. \( \Gamma(u) \) is bounded,
3. \( \Gamma(u) \) is an equilateral \( q \)-polygon.

Moreover, \( \Gamma(u) \) is a regular polygon for every \( x_0 \in \mathbb{R} \) if and only if \( \varphi = R_\varrho \). In this case and with \( x_0 = 0 \mod 1 \), the points \( z_0, z_1, z_q, z_{q+1}, z_{2q}, z_{2q+1}, \ldots, z_{aq}, z_{aq+1}, \ldots \) lie on the line \( \text{Im}(z) = 0 \).

Thus the curves generated by homeomorphisms conjugated to rational rotations, in contrast to those generated by pure rational rotations, can be unbounded and in case they are bounded, they might be equilateral but not regular polygons (i.e. not equiangular). Of course, they can be convex as well as not convex.

**Proof of Proposition 1.10.** We remark that \( \Gamma(\{\Phi^n(x_0)\}) \) is bounded if and only if

\[
\sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n-1} \exp(2\pi i \Phi^k(x_0)) \right| < \infty.
\]

Firstly, we will prove the equivalence of conditions 1.-3.

Suppose that 1. is satisfied, which in this case this is equivalent to

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i \Phi^k(x_0)) = 0.
\]
Assume that, on the contrary, \( \Gamma(u) \) is not bounded. In particular, this implies that \( z_q \neq z_0 \) because otherwise \( \Gamma(u) \) would be a closed curve. So let \( c = |z_q - z_0| \), where \( c > 0 \). Then by periodicity of the orbit \( \{ \varphi^n(\exp(2\pi ix_0)) \} \), we obtain \( |z_{q^n} - z_0| = 2c \) and inductively, \( |z_{q^n} - z_0| = nc \). But then \( \lim_{n \to \infty} \frac{|z_{q^n}|}{q} = \frac{c}{q} \) which contradicts 1. On the other hand, if \( \Gamma \) is bounded then its Birkhoff average must vanish which means that 1. holds. These arguments give equivalence of 1. and 2.

Now assume that 2. is satisfied. By periodicity of the orbit of \( \exp(2\pi ix_0) \) \( \in S^1 \) this means that \( z_q = z_0 \) since otherwise \( \Gamma \) would grow unbounded in the direction of \( v = z_q - z_0 \). But if \( z_q = z_0 \) then \( \Gamma \) must be an equilateral polygon with \( q \) sides (the fact that the sides of this polygon must be of equal length is simply due to the fact that they are vectors of length 1 by definition of a curlicue) and we obtain that 2. \( \implies \) 3. The case 3. \( \implies \) 2. is trivial.

We already now that if \( \varphi = \mathcal{R}_\theta \) with \( \theta = p/q \), then \( \Gamma(u) \) is a regular polygon with \( q \)-sides for every \( x_0 \). On the other hand, if \( \Gamma(u) \) is a regular polygon with \( q \)-sides for every \( x_0 \), then all the displacements \( \Phi^k(x_0) - \Phi^{k-1}(x_0) \mod 1 \) for every \( k \in \mathbb{N} \) and \( x_0 \in \mathbb{R} \) must be equal to \( p/q \) which means that \( \varphi \) is a rigid rotation. Since then for \( x_0 = 0 \mod 1 \) we have \( y_n = \sum_{k=0}^{n-1} \sin(2\pi \frac{k\theta}{q}) \), where \( z_n = (x_n, y_n) \), the last statement follows easily.

It remains to show non-superficality of \( \Gamma(u) \). For bounded case there is nothing to prove. Similarly, if \( \Gamma \) is unbounded then we check the condition \( \lim_{t \to \infty} \text{Diam}\Gamma_t \leq q/c < \infty \). By choosing the subsequence \( t_n = nq \) we obtain that \( \text{Diam}\Gamma_{t_n} \geq |z_{nq} - z_0| = nc \), where \( c = |z_q - z_0| \) and consequently \( \lim_{n \to \infty} \text{Diam}\Gamma_{t_n} \leq q/c < \infty \). 

**Example 1** Let \( \varphi \) be conjugated to a rational rotation, i.e. \( \Phi = h^{-1} \circ \mathcal{R}_\theta \circ h \), where \( \theta \in \mathbb{Q} \) and \( h \) is a lift of some other orientation preserving circle homeomorphisms. For example, define:

\[
h(x) = \begin{cases} 
\frac{3}{4} x, & 0 \leq x \leq \frac{3}{4}; \\
\frac{1}{4} (x - 1), & \frac{3}{4} \leq x \leq \frac{5}{4}; \\
\frac{1}{4} (x + 1), & \frac{5}{4} \leq x \leq 1.
\end{cases}
\]

Then

\[
h^{-1}(x) = \begin{cases} 
\frac{3}{4} x, & 0 \leq x \leq \frac{1}{4}; \\
\frac{1}{4} (x + 1), & \frac{1}{4} \leq x \leq \frac{3}{4}; \\
\frac{1}{4} (x - 1), & \frac{3}{4} \leq x \leq 1.
\end{cases}
\]

By applying the rule \( h(x + 1) = h(x) + 1 \) (similarly, for \( h^{-1}(x) \)) we extend \( h \) onto an orientation preserving homeomorphism of \( \mathbb{R} \).

Let then \( \theta = 1/4 \). For an arbitrary choice of \( x_0 \) the orbit \( \{ \varphi^n(\hat{x}_0) \}_{n=0}^{\infty} \), \( \hat{x}_0 = \exp(2\pi ix_0) \), is periodic with period 4. In particular, we compute that \( \Phi(0) = 3/8, \Phi^2(0) = 1/2, \Phi^3(0) = 7/8, \Phi^4(0) = 1 \mod 1 \) and \( \Phi^5(0) \mod 1 = 0(0) \) etc. Thus the displacements \( \Phi^k(0) - \Phi^{k-1}(0) \) along the trajectory are not equal but, as we easily verify, their average vanishes:

\[
\frac{1}{4} \sum_{k=0}^{3} \exp(2\pi i \Phi^k(0)) = (0, 0)
\]

According to the above proposition the curve \( \Gamma \), evaluated over \( \{ \Phi^n(0) \} \), is an equilateral polygon, but not regular: it is closed as the average is 0 but it is not regular since the displacements are not all equal. Indeed, the displacements are alternatingly equal to 3/8 and 1/8 and, as we see in Figure 65, \( \Gamma \) is a rhombus but not a square.

**Example 2** Let \( x_0 = 0 \) and \( h \) be as in Example 1 but take \( \theta = 1/5 \). In this case the orbit of \( x_0 \) is obviously periodic (modulo 1) with period 5 but the exponential average along the orbit does not vanish: \( \frac{1}{5} \sum_{k=0}^{4} \exp(2\pi i \Phi^k(0)) \approx (-0.0078, -0.0273) \). Thus \( \Gamma \) is unbounded, as reflected in Figure 66.

Finally, let us also remark that the boundedness of the curve in Proposition 1.10 might depend on \( x_0 \). Indeed, consider for example the lift \( \Phi = h^{-1} \circ \mathcal{R}_{1/2} \circ h \), where \( h(x) = -2(x - 1/2)^2 + 1/2 \).
Figure 3: $\Gamma_{100}(0)$ generated by $\varphi$ conjugated with rational rotation as in Example 1 and 2.

for $0 \leq x \leq 1/2$ and $h(x) = 2(x - 1/2)^2 + 1/2$ for $1/2 \leq x \leq 1$ and $R_{1/2}(x) = x + 1/2$ is a lift of a rotation by $\pi$. Let $\Gamma(x)$ denote the curve generated by the orbit $\{\Phi^n(x)\}$. Compare $\Gamma(x_0)$ and $\Gamma(y_0)$ where $x_0 = 0$, $y_0 \in (0, 1/2)$. Then $\Gamma(x_0)$ is bounded whereas $\Gamma(y_0)$ is not.

Now we move to the case of a so-called semi-periodic homeomorphism, which is a homeomorphism with rational rotation number but not conjugated to the rotation, i.e. when apart from periodic orbits we also have some non-periodic ones.

**Proposition 1.11** Suppose that $\Gamma = \Gamma(x_0)$ is a curve generated by the lift of the orbit of $x_0$ of a semi-periodic circle homeomorphism $\varphi$ with $\varrho = \frac{p}{q}$. Then:

- If $x_0$ is a periodic point, then the equivalence of conditions 1.-3. in Proposition 1.10 also applies to $\Gamma$. Similarly, $\Gamma$ (bounded or not) is not superficial and it is a regular polygon if and only if the displacements along the periodic orbit $\{\varphi^n(\exp(2\pi i x_0))\}$ are all equal.

- If $x_0$ is not a periodic point and $\Gamma$ is bounded, then $\Gamma$ is also superficial.

**Proof.** The first statement can be proved exactly as Proposition 1.10. As for the second statement, when $x_0$ is not a periodic point, then its orbit is attracted by some periodic orbit of $\varphi$ (see e.g. [13]) but infinitely many (all) $u_n$’s are different modulo 1 and thus, again on the account of Fact 1.8, bounded $\Gamma$ is also superficial.

In particular we realize that periodic orbits of circle homeomorphisms may give rise to unbounded curves.

## 2 Connection with the cohomological equation

In this part we are going to show how the shape of a bounded curlicue generated by a transitive circle homeomorphism $\varphi$ is related to the solution of the induced cohomological equation. Till the end of this part we assume that $\varphi \in \mathbb{R} \setminus \mathbb{Q}$.

It is clear that when the Birkhoff sums are bounded, i.e. $\sup_{n \in \mathbb{N}} |\sum_{k=0}^{n-1} \exp(2\pi i u_k)| < \infty$, then the curve $\Gamma$ is bounded as well. On the other hand, when the Birkhoff sums are unbounded and the Birkhoff average does not vanish, i.e. $\lim_{n \to \infty} \frac{1}{n} |\sum_{k=0}^{n-1} \exp(2\pi i u_k)| \neq 0$, then the curlicue $\Gamma(u)$ is unbounded and grows in the direction of the nonzero vector $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi i u_k) = (c_1, c_2) \in \mathbb{C}$.

In order to verify how consecutive points $z_n$ are located in the plane $\mathbb{C}$ let us recall the classical Denjoy-Koksma inequality:
Theorem 2.1 (cf. [11]) Let \( \varphi : S^1 \to S^1 \) be a homeomorphism with irrational rotation number \( q \) and \( g : S^1 \to \mathbb{R} \) a real function (not necessary continuous) with bounded variation \( \text{Var}(g) \). Then

\[
| \sum_{i=0}^{q_n-1} g(\varphi^i(x)) - q_n \int_{S^1} g \, d\mu | \leq \text{Var}(g), \quad \forall x \in S^1,
\]

where \( q_n \) is a denominator of a rational approximation of \( q \) by the continued fraction expansion and \( \mu \) is the only invariant Borel probability measure of \( \varphi \).

If we consider

\[
z_n(x_0) := \sum_{k=0}^{n-1} \exp(2\pi i \Phi^k(x_0)) = \sum_{k=0}^{n-1} (\cos(2\pi \Phi^k(x_0)) + i \sin(2\pi \Phi^k(x_0)))
\]

then denoting \( g^{(1)}(x) := \cos(2\pi x) \) and \( g^{(2)}(x) := \sin(2\pi x) \) we can apply the inequality (8), respectively to \( g^{(1)} \) and \( g^{(2)} \). Assume further that the Birkhoff average disappears:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i \Phi^k(x_0)) = \int_{[0,1]} \exp(2\pi i x) \, d\hat{\mu}(x) = 0 \in \mathbb{C},
\]

where \( \hat{\mu} \) is the measure \( \mu \) lifted to \([0,1]\). Then

\[
\int_{[0,1]} g^{(1)}(x) \, d\hat{\mu}(x) = 0 \quad \text{and} \quad \int_{[0,1]} g^{(2)}(x) \, d\hat{\mu}(x) = 0.
\]

Now, as \( \text{Var}(g^{(1)}) = \text{Var}(g^{(2)}) = 4 \) for \( z_n(x_0) = a_n + ib_n \) we obtain that \( |a_{q_n} - a_0| \leq 4 \) and \( |b_{q_n} - b_0| \leq 4 \) where \( a_0 = b_0 = 0 \) by the construction of the curlicue. This justifies the first part of the following claim:

Proposition 2.2 Let \( \varphi \) be a circle homeomorphism with irrational rotation number \( q \) and a lift \( \Phi : \mathbb{R} \to \mathbb{R} \) and suppose that the Birkhoff average vanishes \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i \Phi^k(x)) = 0 \).

There is a bounded neighbourhood \( U \) of \( 0 \in \mathbb{C} \) such that for every \( x_0 \in \mathbb{R} \) the points \( z_{q_n}(x_0) \in \Gamma(x_0) \), corresponding to the closest-return times of \( \varphi \), return to this neighbourhood:

\[
z_{q_n}(x_0) \in U \quad \text{for} \quad n = 0, 1, \ldots
\]

If, additionally, \( \varphi \) is a \( C^{1+\text{bv}} \) circle diffeomorphism, then

\[
\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \ \forall x \in \mathbb{R} \ |z_{q_n}(x)| < \varepsilon.
\]

The second statement in the above Proposition asserts that for sufficiently large \( n \) the points \( z_{q_n}(x_0) \) of the curlicue \( \Gamma(x_0) \) fall into arbitrarily small neighbourhood of \( 0 \), and this convergence is uniform with respect to \( x_0 \), provided that \( \varphi \) is enough smooth. This follows from the improved version of the Denjoy-Koksma inequality:

Theorem 2.3 (Corollary in [18]) If \( \varphi \) is a \( C^{1+\text{bv}} \) circle diffeomorphism and \( g \) is \( C^1 \), it holds that

\[
\| \sum_{i=0}^{q_n-1} g(\varphi^i) - q_n \int_{S^1} g \, d\mu \|_{C^0} \to 0, \quad \text{as} \quad n \to \infty.
\]

If we do not assume that the Birkhoff average vanishes but that it equals \( c = (c_1, c_2) \), where \( c \) is some nonzero vector in the complex plane, then we obtain the analog of Proposition 2.2 saying that the curlicue visits neighborhoods of some points on the straight line in the direction of the vector \( c \).
Corollary 2.4 Let \( \varphi \) be a circle homeomorphism with irrational rotation number \( g \) and a lift \( \Phi : \mathbb{R} \to \mathbb{R} \) and suppose that the Birkhoff average \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i g^k(x)) = c \), where \( c = (c_1, c_2) \in \mathbb{C} \).

Then there exists a constant \( M \) such that for every \( x \in \mathbb{R} \) we have
\[
|z_{q_n}(x) - q_n c| \leq M.
\]

Moreover, if \( \varphi \) is \( C^{1+\varepsilon} \) diffeomorphism then
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall x \in \mathbb{R} \ |z_{q_n}(x) - q_n c| < \varepsilon.
\]

However, in case of the vanishing Birkhoff average, the Denjoy-Koksma inequality does not explain in fact whether the curlicue is bounded or not. Nonetheless, this can be achieved by considering the so-called cohomological equation. Let us start from recalling the famous Gottschalk-Hedlund Theorem:

Theorem 2.5 (cf. [8]) Let \( X \) be a compact metric space and \( T : X \to X \) a minimal homeomorphism. Given a continuous function \( g : X \to \mathbb{R} \) there exists a continuous function \( u : X \to \mathbb{R} \) such that
\[
g = u - u \circ T
\]
if and only if there exists \( K < \infty \) such that
\[
\sup_{x \in X} \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n-1} g(T^k(x)) \right| < K. \tag{10}
\]

Note that every two continuous solutions of the cohomological equation for a minimal homeomorphism \( \varphi \) of the compact metric space \( X \) differ by a constant (i.e. if \( u \) is such a solution, then \( \tilde{u} := u + c \), where \( c \) is arbitrary constant, is also a solution). Moreover, from minimality of \( \varphi \) we have that if for some \( x_0 \) \( \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n-1} g(T^k(x_0)) \right| < K \), then \( \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n-1} g(T^k(x)) \right| < 2K \) for arbitrary \( x \in X \), i.e. it suffices to verify (10) along the orbit of an arbitrary point.

In our setting, \( X = S^1 \). We assume from now till the end of this section that \( \varphi : S^1 \to S^1 \) is minimal (i.e. its rotation number \( \rho \in \mathbb{R} \setminus \mathbb{Q} \) and, moreover, \( \varphi \) is transitive). In this case \( \varphi \) is conjugated to the irrational rotation \( R_{\theta} \). Therefore \( \varphi \) satisfies the assumptions of Theorem 2.5. We consider \( S^1 \) as the quotient space \( S^1 = \mathbb{R}/\mathbb{Z} \) (equivalently, as the interval \( [0,1] \) with endpoints identified). Denote by \( g : S^1 \to \mathbb{C} \) an exponential function \( g(x) = \exp(2\pi i x) \) and identify \( \varphi : S^1 \to S^1 \) with its lift \( \Phi \) by \( \varphi = \Phi \mod 1 \). The following functional equation:
\[
\exp(2\pi i x) + u(x) = u(\varphi(x)), \quad \forall x \in \mathbb{R}, \tag{11}
\]
where \( u : S^1 \to \mathbb{C} \) is the unknown of the problem, will be referred to as the cohomological equation in our further considerations. Traditionally, the cohomological equation is given with respect to the functions \( g \) and \( u \) taking real values but in our case one can equivalently consider \( u = (u_1, u_2) \) and \( g = (g_1, g_2) \) with \( u_i, g_i : S^1 \to \mathbb{R}, \ i = 1,2 \). The condition on bounded Birkhoff sums now takes the form:
\[
\sup_{x \in X} \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n-1} \exp(2\pi i \varphi^k(x)) \right| < K. \tag{12}
\]

Notice that if there exists a continuous solution \( u \) of the equation (11), then by integrating both sides with respect to the invariant measure \( \mu \) of \( \varphi \) we obtain that \( \int_{S^1} \exp(2\pi i x) \, d\mu = 0 \). In other words, vanishing of the Birkhoff averages \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i \varphi^k(x)) = 0 \) is a necessary condition for the existence of a continuous solution of (11). Obviously bounded Birkhoff sums already imply bounded Birkhoff averages.
Let us for a while consider cylinder maps (see e.g. [11]): If \( \varphi : S^1 \to S^1 \) is a minimal homeomorphism, \( g : S^1 \to \mathbb{C} \) is a continuous function and \( Y \) denotes the product space \( Y = S^1 \times \mathbb{C} \), then the transformation \( F : Y \to Y \) given as
\[
F(x, \xi) := (\varphi(x), \xi + g(x))
\]
is called a cylinder transformation. In the current work we consider cylinder transformations of the following form: \( F(x, \xi) = (\varphi(x), \xi + \exp(2\pi i x)) \). Assume that \( u(x) \) is the solution of the cohomological equation \((11)\). In this case
\[
F(x, u(x)) = (\varphi(x), u(x) + \exp 2\pi i x) = (\varphi(x), u(\varphi(x)))
\]
i.e. \( S := \{(x, y) \in Y : x \in S^1, y = u(x)\} \) is an invariant section of \( F (F(S) \subseteq S) \). We are ready to state

**Proposition 2.6** Let \( \varphi : S^1 \to S^1 \) be a transitive homeomorphism with a lift \( \Phi : \mathbb{R} \to \mathbb{R} \). Then the curve \( \Gamma \) generated by an arbitrary orbit \( \{\Phi^n(x)\} \) is bounded and superficial if and only if the cohomological equation \((11)\) has a continuous solution \( u : S^1 \to \mathbb{C} \).

Moreover, if \( \Gamma \), evaluated over the orbit of some point \( x_0 \), is bounded then the points \( z_n \) of \( \Gamma \) lie on the curve \( u(S^1) \subseteq \mathbb{C} \), where \( u \) is a continuous solution of \((11)\) satisfying \( u(x_0) = 0 \). In this case also \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \exp(2\pi i \Phi^k(x_0)) = 0 \).

**Proof.** The first statement follows from Theorem 2.5 since the assumption on the bounded exponential sums is equivalent to the assumption that the curlicue is bounded. Moreover, on the account of Fact 2.5 we know that if \( \Gamma \) is bounded, then it must also be superficial.

The second part of the proposition can be concluded e.g. from the Main Theorem and examples in the work [11] of Coronel, Navas, Ponce, which studies cocycles of isometries over minimal dynamics and existence of the induced continuous sections. However for the convenience of the Reader we present its direct short proof below. To start with, notice that \( F^n(x, \xi) = (\varphi^n(x), \xi + \sum_{k=0}^{n-1} \exp(2\pi i \varphi^k(x))) \), as \( g(x) = \exp(2\pi i x) \) and \( \varphi(x) = \Phi(x) \mod 1 \). Thus if we choose a point \( x_0 \) and let \( u \) denote the continuous solution of \((11)\) such that \( u(x_0) = 0 \), then by substituting \( \xi = u(x_0) = 0 \) we get \( F^n(x_0, 0) = F^n(x_0, u(x_0)) = (\varphi^n(x_0), \sum_{k=0}^{n-1} \exp(2\pi i \varphi^k(x_0))) \). Consequently, \( \pi_2(F^n((x_0), u(x_0))) = \sum_{k=0}^{n-1} \exp(2\pi i \varphi^k(x_0)) \) \( = z_n \), where \( z_n \) is the \( n \)-th vertex of the curlicue evaluated over the orbit \( \{\varphi^i(x_0)\} \) and \( \pi_2 \) is the projection onto the second coordinate (onto the complex plane). But \( F^n(x_0, u(x_0)) = (\varphi^n(x_0), u(\varphi^n(x_0))) \) since we are on the invariant section \( S \subseteq S^1 \times \mathbb{C} \). Thus \( z_n \in \pi_2(S) \). Precisely, \( z_n = u(\varphi^n(x_0)) \) and as \( u : S^1 \to \mathbb{C} \) is continuous, \( u(S^1) = \pi_2(S) \) is a bounded closed curve in the complex plane \( \mathbb{C} \) with vertices \( z_n \) lying on it.

**Corollary 2.7** Given an orientation preserving transitive circle homeomorphisms \( \varphi \) and denoting by \( \Gamma(x_0) \) a curve generated by the orbit of \( x_0 \), either \( \Gamma(x_0) \) is bounded for all \( x_0 \in S^1 \) or for every \( x_0 \) \( \Gamma(x_0) \) is unbounded. Moreover, in case \( \Gamma(x_0) \) is bounded, its vertices lie on a closed curve \( \tau \) \((\tau = u(S^1))\), which shape does not depend on the choice of the generating point \( x_0 \).

**Example 3.** We have numerically investigated the family of Arnold circle maps:
\[
\varphi(x) = x + \omega \frac{K}{2\pi} \sin(2\pi x) \mod 1
\]
with the results presented in Figure 11. Taking \( x_0 = 0 \) and \( K = 1 \), for \( \omega = 1/2 \) we obtained the horizontal segment of length 1 since in this case \( x_0 = 0 \) is a 2-periodic point of \( \varphi \). However, for \( \omega = 1/3 \) and \( \omega = 1/4 \) the curlicue \( \Gamma(0) \) accumulated along the straight line with some regular patterns visible after zooming in.

Further we consider the following example of the transitive circle homeomorphism \( \varphi \):
Figure 4: $\Gamma_n(x_0)$ generated by the Arnold map at $x_0 = 0$ with $K = 1$ and different values of $\omega$.

**Example 4.** Choose an irrational rotation number $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and let $h$ be as in Example 1. Then $h$ induces a transitive circle homeomorphism $\varphi$ with $\varphi = \Phi \mod 1$ where

$$\Phi = h^{-1} \circ R_\rho \circ h$$

and $R(x) = x + \rho$. One checks that

$$\int_0^1 \cos(2\pi h^{-1}(x)) \, dx = \int_0^1 \sin(2\pi h^{-1}(x)) \, dx = 0$$

that is, the Birkhoff average (for $\rho \in \mathbb{R} \setminus \mathbb{Q}$) vanishes:

$$\frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi i \Phi^k(x_0)) = \frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi i h^{-1}(x_0 + k\rho)) \to 0 \quad \text{as } n \to \infty$$

It is obvious that one can construct the whole family of such examples.

We numerically simulated two cases: $\rho = -\ln(0.5)$ and $\rho = \pi$ and for each of them obtained a bounded curve (suggesting that the corresponding Birkhoff sums are bounded and the corresponding cohomological equations have continuous solutions). The results are presented in Figure 5.

3 Growth rate and superficiality for unbounded curlicues

In this part we deal with curves $\Gamma$ generated by circle homeomorphisms $\varphi = \Phi \mod 1$ with irrational rotation number $\rho \in \mathbb{R} \setminus \mathbb{Q}$. We already know that if $\Gamma$ is bounded then its shape can be described by a solution of the certain cohomological equation. Notwithstanding, in case $\Gamma$ is unbounded we might always ask how fast $\text{Diam}(\Gamma_t)$ increases to $\infty$.

We recall that when the curlicue generated by $\varphi$ with irrational rotation number is bounded, then it is also superficial. However, not so much is known for the unbounded case. Below we provide some sufficient conditions under which $\Gamma$ is superficial, even when unbounded, and some estimates of the rate of its growth. The important tool is the Denjoy-Koksma inequality in different formulations.

**Definition 3.1** A real number $\rho$ for which there exists $C > 0$ and $r > 1$ satisfying

$$|\rho - \frac{p}{q}| \geq \frac{C}{q^{1+r}}$$

for all $p/q \in \mathbb{Q}$ is called Diophantine of type $r$. 
We remark that for every fixed $r > 1$ the set of real number of Diophantine type $r$ has full Lebesgue measure. Moreover (cf. e.g. [14]), the intersection of sets of Diophantine number of type $r$ over all $r > 1$ has full measure too.

In next definition we use the continued fraction expansion of $\varrho$ ($\varrho \in [0, 1) \cap \mathbb{R} \setminus \mathbb{Q}$):

$$
\varrho = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}} =: [a_0, a_1, a_2, \ldots],
$$

where $a_0, a_1, \ldots \in \mathbb{Z}$. By terminating the continued fraction expansion at some point

$$
\frac{p_n}{q_n} := \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}} =: [a_0, a_1, \ldots, a_n]
$$

we obtain rational approximation of $\varrho$:

$$
\varrho = \lim_{n \to \infty} \frac{p_n}{q_n},
$$

Figure 5: $\Gamma_n(x_0)$ generated by $\Phi = h^{-1} \circ R_{\varrho} \circ h$ mod 1 (see Example 4) with $x_0 = 0$ and different values of $\varrho$. 

(a) $\varrho = -\ln(0.5)$, $n \in \{100, 1000, 100000\}$

(b) $\varrho = \pi$, $n \in \{100, 1000, 100000\}$
where the integers $p_n$ and $q_n$ satisfy

\begin{align}
p_{n+1} &= a_n p_n + p_{n-1} \\
qu_{n+1} &= a_n q_n + q_{n-1}
\end{align}

for $n \geq 1$ and $p_0 = 0$, $q_0 = 1$, $p_1 = a_0$, $q_1 = a_0$. This rational approximation of $\varrho$ is the best possible in a sense that $\frac{p_n}{q_n}$ is the closest rational to $\varrho$ among all rational numbers $\frac{p}{q}$ such that $q < q_n$.

**Definition 3.2** A real number is of *constant type* if the continued fraction approximation $\frac{p_n}{q_n}$ has the property that $\frac{q_{n+1}}{q_n}$ is bounded.

**Remark 3.3** Equivalently, a real number $\varrho$ is of constant type when $\sup_{a_n} < \infty$. Alternatively, we say that $\varrho$ is of *bounded* type.

Let us firstly consider the situation when $\lim_{n \to \infty} \frac{q_{n+1}}{q_n}$ is the unique invariant Borel probability measure of $\varphi = \Phi \mod 1$. Then there exists a constant $M$ such that for every $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\left| z_n(x_0) \right| = \left| \sum_{k=0}^{n-1} \exp (2\pi i \Phi^k(x_0)) \right| \leq Mn^{1-1/r} \log(n).$$

**Proof.** Let $\frac{p_n}{q_n}$ be the convergent in the continued fraction expansion of $\varrho$. Then by Denjoy-Koksma inequality [13] we have

$$\left| z_{q_n}(x_0) \right| = \left| \prod_{k=0}^{q_{n-1}} \exp (2\pi i \Phi^k(x_0)) \right| \leq M',$$

where by $M'$ we can take $M' = \text{Var}(\exp(2\pi i x)) = 2\pi$. Now, by Euclidean division algorithm, for arbitrary $n$ there exists $m(n)$ (for short we will denote $m$) and the sequence of integers $c_0, c_1, \ldots, c_m$ such that $q_m \leq n < q_{m+1}$ and $n = c_m q_m + c_{m-1} q_{m-1} + \ldots + c_1 q_1 + c_0$ with $c_k \leq q_{k+1} / q_k$. Then by reapplying inequality [13] several times we obtain

$$\left| z_n(x_0) \right| = \left| \sum_{k=0}^{n} \exp (2\pi i \Phi^k(x_0)) \right| \leq (c_0 + \ldots + c_m) M' \leq \left( \sum_{k=0}^{m-1} \frac{q_{k+1}}{q_k} + c_m \right) M'. \quad (15)$$

Since $\varrho$ is Diophantine of type $r$, $\frac{C}{q_n^r} \leq \left| \varrho q_n - p_n \right| \leq \frac{1}{q_{n+1}^r}$ as for the continued fraction expansion it holds that $\left| \varrho - \frac{p_n}{q_n} \right| \leq \frac{1}{q_{n+1}^r}$. Thus $q_{k+1} \leq \frac{q_k^r}{C}$ and $\frac{q_{k+1}}{q_k} \leq C^{-1/r} q_{k+1}^{1/r}$. Consequently, we can further estimate

$$\left| z_n(x_0) \right| \leq \left( C^{-1/r} \sum_{k=1}^{m} q_k^{1-1/r} + \frac{n}{q_m} \right) M' \leq \left( C^{-1/r} m q_m^{1-1/r} + \frac{n}{q_m} \right) M'. \quad (15)$$

We recall that $m(n) \leq \frac{2 \log q_m(n)}{\log 2}$ since $q_n = a_{n-1} q_{n-1} + q_{n-2} \geq 2q_{n-2}$, $n \geq 2$ and by induction we immediately obtain $q_n \geq 2^n / 2$. Thus also $m(n) \leq \frac{2 \log q_m(n)}{\log 2}$ as $q_m \leq n$. This gives the estimation for the first term: $C^{-1/r} m q_m^{1-1/r} \leq C^{-1/r} \frac{2 \log n}{2 \log 2} n^{1-1/r}$. From $n \leq q_{m+1} \leq \frac{q_{m+1}}{q_m}$ we obtain that $\frac{n}{q_m} \leq C^{-1/r} n^{1-1/r}$ which is the estimation for the second term. Finally, we arrive at

$$\left| z_n(x_0) \right| \leq M n^{1-1/r} \log n,$$

where $M$ is the constant connected only with $\text{Var}(\exp 2\pi i x)$ and $C$. \qed
Remark 3.5 In [12] and [14] a similar fact is shown for the irrational rotation $\varphi(x) = x + \varphi \mod 1$ and for arbitrary function $g(x)$ with bounded variation $\text{Var}(g)$ (here $g(x) = \exp(2\pi ix)$) and such that $\int_0^1 g(x)dx = 0$ (cf. the proof of Lemma 12 in [12] and the proof of Lemma 4.1 in [14]). However, the proof remains valid for any homeomorphism with irrational rotation number (of Diophantine type $r$) provided that the condition $\int_0^1 g(x)dx = 0$ is replaced with $\int_0^1 g(x)d\mu(x) = 0$ since the Denjoy-Koksma inequality is valid for an arbitrary such homeomorphism.

Theorem 3.6 Assume that $\varphi$ is of constant type and that $\int_0^1 \exp(2\pi ix)\ d\mu(x) = 0$. Then there exists a constant $M$ such that for every $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$|z_n(x_0)| = |\sum_{k=0}^{n-1} \exp(2\pi i\Phi^k(x_0))| \leq M \log(n)$$

Proof. Similarly, as in [14] we can estimate

$$|z_n(x_0)| = |\sum_{k=0}^{n} \exp(2\pi i\Phi^k(x_0))| \leq (c_0 + \ldots c_m)M' \leq \left(\sum_{k=0}^{m} \frac{q_{k+1}}{q_k}\right) M'$$

where $M' = \text{Var}(\exp(2\pi ix))$. Now, since $\varphi$ is of bounded type, there exists a constant $C$ such that $q_{i+1}/q_i < C$, $i = 0, 1, \ldots, m$ and thus we have $|z_n(x_0)| < mM' C$ but again $m \leq \frac{2\log n}{\log 2}$ which gives the desired estimate.

The following theorem is a counterpart of the corresponding Proposition 2.3 in [9] on irrational rotations but it is clear from the proof therein that this proposition extends to homeomorphisms with irrational rotation number and such that $\int_0^1 g(x)\ d\mu(x) = 0$. Therefore we omit the proof.

Theorem 3.7 Suppose that $\varphi \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $a_m < (m+1)^{1+\varepsilon}$, with $\varepsilon > 0$, for any $m$ large enough, where $a_m$’s are the integers appearing in continued fraction expansion of $\varphi$ ($\varphi = [0; a_0, a_1, a_2, \ldots]$). If $\int_0^1 \exp(2\pi i x)\ d\mu(x) = 0$, then for every $x_0 \in \mathbb{R}$

$$|z_n(x_0)| = |\sum_{k=0}^{n-1} \exp(2\pi i x)| = O(\log^{2+\varepsilon} n).$$

We remark that the set of irrationals whose partial quotients satisfy the condition $a_m < (m + 1)^{1+\varepsilon}$ with some fixed $\varepsilon > 0$, for any $m$ large enough, is of full measure (cf. [9] and references therein).

Now we easily formulate:

Theorem 3.8 Let $\varphi = \Phi \mod 1$ be an orientation preserving circle homeomorphism with irrational rotation number $\varphi$ and the invariant measure $\mu$. Let $\Gamma$ be a curve generated by $\varphi$. Suppose that the Birkhoff average $\int_{[0,1]} \exp(2\pi i x)\ d\mu(x) = 0$ vanishes and that the rotation number $\varphi$ is either of constant type or Diophantine of type $r > 1$ or satisfies hypothesis of Theorem 3.7. In any of the above cases, the curve $\Gamma$ is superficial.

Proof. Note that if $\Gamma$ is bounded then we immediately obtain that it is superficial, since infinitely many $u_n$’s are different modulo 1.

Suppose now that $\Gamma$ is unbounded. Then from Theorems 3.3, 3.6 or 3.7 it follows that there exist a constant $M$ and a function $l(\cdot)$ such that $l(n) = o(n)$ (i.e. $\lim_{n \to \infty} \frac{l(n)}{n} = 0$) and that for every $x_0 \in \mathbb{R}$ we have:

$$|z_n(x_0)| \leq M l(n).$$

Thus if $\Gamma$ is generated by the orbit of $x_0$ then

$$\lim_{n \to \infty} \frac{n}{\text{Diam}\Gamma_n} \geq \lim_{n \to \infty} \frac{n}{2|z_n(x_0)|} \geq \lim_{n \to \infty} \frac{n}{Ml(n)} = \infty.$$
and it follows that $\Gamma$ is superficial.

Next we consider the case when the Birkhoff average does not vanish. It is clear that similarly we can obtain the following

**Proposition 3.9** Let $\varphi = \Phi \mod 1$ be an orientation preserving circle homeomorphism with irrational rotation number $\varrho$ and the invariant measure $\mu$. Let $\Gamma$ be a curve generated by $\varphi$. Suppose that the Birkhoff average $\int_{[0, 1]} \exp(2\pi ix) \, d\mu(x) = v$ does not vanish and that the rotation number $\varrho$ is either of constant type or Diophantine type $r > 1$ or satisfies hypotheses of Theorem 3.7. Then there exists a constant $M$ and a function $l(\cdot)$ satisfying $l(n) = o(n)$ and such that for any $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$|z_n(x_0) - nv| \leq Ml(n).$$

Below we show that irrational rotation number and non-vanishing Birkhoff average always leads to non-superficial curlicues, even without the additional assumptions on $\varrho$:

**Theorem 3.10** Let $\varphi = \Phi \mod 1$ be an orientation preserving circle homeomorphism with irrational rotation number $\varrho$ and the invariant measure $\mu$. Let $\Gamma$ be a curve generated by $\varphi$. Suppose that the Birkhoff average $\int_{[0, 1]} \exp(2\pi ix) \, d\mu(x) = v$ does not vanish. Then the curve $\Gamma$ is not superficial.

**Proof.** Let us take the increasing sequence $q_n \to \infty$ of the closest returns. By Denjoy-Koksma for some constant $M$ we have

$$q_n |v| - M \leq |z_{q_n}(x_0)| \leq M + q_n |v|$$

and

$$\lim_{n \to \infty} \frac{q_n}{\text{Diam} \Gamma_{q_n}} \leq \lim_{n \to \infty} \frac{q_n}{|z_{q_n}(x_0)|} \leq \lim_{n \to \infty} \frac{q_n}{q_n |v| - M} = \frac{1}{|v|},$$

which violates the condition that $\lim_{t \to \infty} \frac{1}{\text{Diam} \Gamma_t} = \infty$. □

**Corollary 3.11** Let $\varphi = \Phi \mod 1$ be an orientation preserving circle homeomorphism with irrational rotation number $\varrho$ and $\Gamma$ be a curve generated by $\varphi$. Then

1. If $\Gamma$ is bounded, then it is always superficial.

2. If $\Gamma$ is unbounded and the Birkhoff average $\int_{[0, 1]} \exp(2\pi ix) \, d\mu(x) = v \neq 0$ does not vanish, then $\Gamma$ is never superficial.

3. If $\Gamma$ is unbounded but the Birkhoff average $\int_{[0, 1]} \exp(2\pi ix) \, d\mu(x) = 0$ vanishes and $\varrho$ satisfies assumptions of Theorem 3.4 or Theorem 3.6 or Theorem 3.7, then $\Gamma$ is superficial.

### 4 Local discrete radius of curvature

The last geometric feature we are going to study is the local discrete radius of curvature, which after Sinai ([19]) we define as:

**Definition 4.1** Let $z_n := \sum_{k=0}^{n-1} \exp(2\pi i u_k)$. The radius of the circle which goes through the consecutive points $z_{n-1}$, $z_n$ and $z_{n+1}$ on the curve $\Gamma(u)$ is called local discrete radius of curvature and denoted $r_n$.

Direct calculations justify the following (compare [19]):

**Fact 4.2** If $\Gamma$ is generated by the sequence $u = (u_n)$, then

$$r_n = \frac{1}{2} \csc \left( \frac{\eta_n}{2} \right),$$

where $\eta_n$ is the displacement between the elements $u_{n-1}$ and $u_n$:

$$\eta_n := 2\pi (u_n - u_{n-1}).$$
In particular, when \( u_n = \Phi^n(x_0) \) for some \( \Phi : \mathbb{R} \to \mathbb{R} \), then
\[
\eta_n := 2\pi \Psi(\Phi^{n-1}(x_0)),
\]
where \( \Psi := \Phi - \text{Id} \) is the displacement function of \( \Phi \). Thus when \( \Gamma \) is generated by an orientation preserving homeomorphism \( \varphi : S^1 \to S^1 \), the sequence \( \{r_n\} \) has exactly the same properties as the displacement sequence of an orientation preserving circle homeomorphism, studied in [15] and [16].

In theorems and propositions below we list these properties. We provide only reduced proofs since some parts of the proofs follow along the same lines as the corresponding statements in [15] and [16].

**Theorem 4.3** Let \( r_n \) be the local discrete radius of curvature of the curve \( \Gamma \) generated by the orbit of \( x_0 \) under the lift \( \Phi \) of an orientation preserving circle homeomorphism \( \varphi \) with the rotation number \( \varphi \). Then

1. If \( \varphi \) is a rotation \( \mathcal{R}_\varphi \) by \( 2\pi \varphi \), then the sequence \( r_n \) is constant: \( r_n = \frac{1}{2}|\text{cosec}(\pi \varphi)| \).

2. If \( \varphi \) is conjugated to the rational rotation by \( 2\pi \varphi \), where \( \varphi = \frac{p}{q} \), then the sequence \( r_n \) is \( q \)-periodic.

3. For a semi-periodic circle homeomorphism \( \varphi \), the sequence \( r_n \) is asymptotically periodic. Precisely, if \( g(\varphi) = p/q \) then:
\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \; \forall n > N \; \forall k \in \mathbb{N} \; \exists \kappa \in \{0, 1, \ldots, N\} \; |r_{n+k\kappa} - r_n| < \varepsilon \tag{17}
\]

4. If \( \varphi : S^1 \to S^1 \) is a homeomorphisms with an irrational rotation number \( \varphi \), then the following cases occur:
   - if \( \varphi \) is transitive, then the sequence \( \{r_n\} \) is almost strongly recurrent, i.e.
   \[
   \forall \varepsilon > 0 \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \; \forall k \in \mathbb{N} \cup \{0\} \; \exists \kappa \in \{0, 1, \ldots, N\} \; |r_{n+k+i} - r_n| < \varepsilon
   \]
   and \( \{r_n\} \) is dense in a set
   \[
   \left[ \min_{x \in [0,1]} g(\Psi(x)), \max_{x \in [0,1]} g(\Psi(x)) \right] = \left[ \min_{x \in [0,1]} g(\Omega(x)), \max_{x \in [0,1]} g(\Omega(x)) \right],
   \]
   where \( g(x) := \frac{1}{2}|\text{cosec}(\pi x)| \) and \( \Omega(x) := G^{-1}(x + \varphi) - G^{-1}(x) \), with \( G \) being the lift of a homeomorphism \( \gamma \) conjugating \( \varphi \) with the corresponding rotation;
   - if \( \varphi \) is not transitive, then the sequence \( \{r_n(x_0)\} \) is almost strongly recurrent, provided that \( x_0 \in \Delta \), where \( \Delta \) denotes the lift of the minimal invariant set for \( \varphi \);
   - if \( \varphi \) is not transitive and \( x_0 \notin \Delta \), then for an arbitrary \( y \in \Delta \) there exist increasing sequences \( \{n_k\} \) and \( \{\tilde{n}_k\} \) such that for every \( l \in \mathbb{Z} \)
   \[
   \lim_{k \to \infty} r_l(\Phi^{n_k}(x_0)) = r_l(y) \quad \text{and} \quad \lim_{k \to \infty} r_l(\Phi^{-\tilde{n}_k}(x_0)) = r_l(y),
   \]
   where \( r_n(y) \) denotes the sequence of local discrete radii of curvature for a curve generated by the orbit of \( y \in \mathbb{R} \) (i.e. for \( \tilde{u}_n = \Phi^n(y) \), \( n = 0, 1, \ldots \));
   - in any case, for not transitive \( \varphi \) the sequence \( r_n \) asymptotically accumulates on the set \( g(D) \), where \( D := \{\Psi(x) \mod 1 : x \in \Delta\} \) (if \( \varphi \) transitive, then \( \Delta = S^1 \)).
Proof. Points 1.-3. are obvious consequences of the dynamics of the homeomorphism $\varphi$ (compare also with Remark 1.4, Remark 1.5 and Proposition 2.5 in [15], the corresponding statements for the displacement sequence). The statement 4. is a consequence of the fact that the displacement sequence for a transitive circle homeomorphism is almost strongly recurrent (cf. Proposition 3.3 and its proof in [13]) and the fact that the function $g(x) = \frac{1}{2}\csc(\pi x)$ is uniformly continuous on the interval $[\delta, 1 - \delta]$, where $\Psi([0, 1]) \subset [\delta, 1 - \delta] \subset (0, 1)$. The claims concerning the accumulation set for $\{r_n\}$ follow from Proposition 2.1 in [16].

We shall also ask about the distribution of the elements of the sequence $\{r_n(x_0)\}$ if $\varphi$ has irrational rotation number. Formally, we define:

**Definition 4.4** Let $A \subset \mathbb{R}^+ = [0, +\infty)$ be a Borel subset. We define the distribution $\omega$ of the elements of $\{r_n(x_0)\}$ as

$$\omega(A) := \lim_{n \to \infty} \left| \left\{ k \in \{1, 2, ..., n\} : r_k(x_0) \in A \right\} \right|$$

The following assures that $\omega$ does not depend on the choice of the generating point $x_0$:

**Proposition 4.5** If $\varphi$ has irrational rotation number $\varrho$, then for every Borel set $A \subset \mathbb{R}$ we have

$$\omega(A) = \int_{[0,1]} \chi_A \circ F \, d\mu = \mu(\{F^{-1}(A)\})$$

where $F(x) := \frac{1}{2}|\csc(\pi \Psi(x))|$ and $\mu$ is the unique invariant ergodic measure for the homeomorphism $\varphi = \Phi \mod 1$. In particular, the average local discrete radius of curvature equals:

$$\hat{r} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} r_k = \frac{1}{2}|\csc(\pi \varrho)|$$

**Proof.** The above statement follows from the fact that $\varphi$ is uniquely ergodic. We recall that due to the unique ergodicity of $\varphi = \Phi \mod 1$, the convergence in (18) to $\int_{[0,1]} \chi_A \circ F \, d\mu$ is uniform with respect to $x_0$.

The next two theorems are immediate consequences of, correspondingly, Proposition 2.8 and Theorem 2.17 in [16].

**Proposition 4.6** Suppose that $\varphi_n$ is a sequence of homeomorphisms with irrational rotation numbers $\varrho_n$ which converges in the metric of $C^0(S^1)$ to the homeomorphism $\varphi$ with irrational rotation number $\varrho$. Let $\omega^{(n)}$ and $\omega$ be the corresponding distributions of local radii of curvature, each of which is evaluated over an arbitrary orbit of the corresponding homeomorphism. Then $\omega^{(n)} \rightharpoonup \omega$, where $\rightharpoonup$ denotes the weak convergence of measures.

The intermediate distributions $\omega^{(n)}$ are well-defined (i.e. do not depend on the choice of the orbit) if the intermediate homeomorphisms $\varphi_n$ all have irrational rotation numbers (implying that the invariant ergodic measures $\mu^{(n)}$ are unique). However, in general for every homeomorphism $\varphi$ and every choice of the generating point $x$ we can define *sample distributions* of local radii of curvature for the first $n$ points on the curve $\Gamma$ (i.e. for the curve $\Gamma_n$ of length $n$):

$$\omega_{n, x} := \frac{1}{n} \sum_{k=1}^{n} \delta_{r_k(x)}$$

where $\delta_{r_k(x)}$ is the Dirac delta centered at $r_k(x)$.

Suppose that $\varphi : S^1 \to S^1$ is a homeomorphism with irrational rotation number and that $\omega$ is the distribution of local radii of curvature for the curlicues it generates. Let $\tilde{\varphi}$ be another homeomorphism, with rational or irrational rotation number, and $\tilde{r}_n(x_0)$ denote the local discrete radius of curvature at the point $\tilde{z}_n(x)$ on the curlicue $\tilde{\gamma}$ generated over the orbit $\{\tilde{\varphi}^n(x)\}$. Denote a sample distribution of radius of curvature associated with $\tilde{\varphi}$ as: $\tilde{\omega}_{n, x} := \frac{1}{n} \sum_{k=1}^{n} \delta_{\tilde{r}_k(x)}$. 


Proposition 4.7 Let \( \varphi \) be a homeomorphism with irrational rotation number and the distribution \( \omega \) of local radii of curvature.

For every \( \varepsilon > 0 \) there exists a neighborhood \( U \subset C^0(S^1) \) of \( \varphi \) such that for every homeomorphism \( \tilde{\varphi} \in U \) and every \( x_0 \in [0,1] \) we have

\[
d_F(\lim_{n \to \infty} \tilde{\omega}_{n,x}, \omega) < \varepsilon
\]

where \( d_F \) denotes the Fortet-Mourier metric.

Note that the convergence in the \( d_F \) metric implies also weak convergence. Therefore Propositions 4.6 and 4.7 provide some kind of stability for distributions of the elements of the sequence \( r_n \).

Next, we are interested in how the radius \( r_n^a \) depends on the parameter \( a \) when \( \{ \varphi_n \} \) is a continuously (smoothly) parameterized family of circle homeomorphisms \( \varphi_n \). The following theorem says that the local discrete radius \( r_n(x_0) \) depends continuously on \( \varphi \) in \( C^0(S^1) \) topology (this dependence is also uniform with respect to \( n \in \mathbb{N} \)), at least for transitive homeomorphisms:

Proposition 4.8 Let \( \varphi : S^1 \to S^1 \) be a transitive homeomorphism with an irrational rotation number \( \rho \). Fix \( \varepsilon > 0 \). Then there exists a neighborhood \( U \subset C^0(S^1) \) of \( \varphi \) such that for every other transitive homeomorphism \( \tilde{\varphi} \in U \) with the same rotation number \( \rho(\tilde{\varphi}) = \rho \) we have

\[
\sup_{x_0 \in \mathbb{R}} \sup_{n \in \mathbb{N}} |r_n(x_0) - \tilde{r}_n(x_0)| < \varepsilon,
\]

where \( r_n(x_0) \) and \( \tilde{r}_n(x_0) \) denote local radii of curvature evaluated at \( x_0 \), respectively, for \( \varphi \) and \( \tilde{\varphi} \).

Proof. We will make use of Theorem 2.3 in [10], saying that the mapping \( \varphi \mapsto \gamma \) assigning to a homeomorphism with irrational rotation number a map \( \gamma : S^1 \to S^1 \) semi-conjugating it (or conjugating, if it is transitive) with the corresponding rotation is a continuous mapping from \( C^0(S^1) \) into \( C^0(S^1) \)-topology (up to some normalization, since every two (semi-)conjugacies of \( \varphi \) differ by an additive constant in the lift).

We recall that

\[
\begin{align*}
\gamma \in C^0(S^1) & \quad\quad \iff \quad\quad |(\Phi(\gamma, \Phi^{-1}(x_0)))| \leq 1, \\
\gamma \in C^0(S^1) & \quad\quad \iff \quad\quad |(\Phi(\gamma, \Phi^{-1}(x_0)))| \leq 1,
\end{align*}
\]

where \( \Phi \) and \( \tilde{\Phi} \) are corresponding lifts. Let \( \Psi = \Phi - \text{Id} \) and \( \tilde{\Psi} = \tilde{\Phi} - \text{Id} \) denote corresponding displacement functions. Then \( \Psi, \tilde{\Psi} : \mathbb{R} \to \mathbb{R} \) are continuous and periodic with period 1. Moreover, \( \Psi([0,1]) = \Psi(((k, k + 1)) \subset (k, k + 1) \) for some \( k \in \mathbb{Z} \), as there are no periodic points of \( \varphi \) (similarly for \( \tilde{\Psi} \)). However, the lift \( \Phi \) of \( \varphi \) can be chosen so that \( \Psi([0,1]) \subset (0, 1) \) (the shape of the curlicue and the radius of curvature do not depend on the choice of the lift). But as \( \Psi \) attains its lower and upper bounds there exists \( \delta \) such that \( \Psi([\delta, 1 - \delta]) \subset (0, 1) \). By considering sufficiently small neighborhood \( U \subset C^0(S^1) \) of \( \varphi \) we can assume that \( \tilde{\Psi}([\delta, 1 - \delta]) \subset \Psi(0, 1) \), \( \tilde{\Psi} \) being the displacement of an arbitrary \( \varphi \in U \). Now consider the function \( h(x) = \csc(\pi x) \) on the interval \( [\delta, 1 - \delta] \). Then \( h \) is a continuously differentiable function. There exists \( M \) such that \( |h'(x)| < M \) for every \( x \in [\delta, 1 - \delta] \). Fix \( \gamma \), which conjugates \( \varphi \) with the rotation by 2\( \pi \theta \) and let \( G \) be its lift. Let us also fix \( \omega \) and \( \tau < \omega \) which can be arbitrary small numbers such that \( |G^{-1}(x) - G^{-1}(y)| < \omega \) whenever \( |x - y| < \tau \). After possibly further decreasing the neighborhood \( U \), we can assume that for every transitive \( \tilde{\varphi} \in U \) there exists \( \tilde{\gamma} \) (semi-)conjugating \( \tilde{\varphi} \) with its corresponding rotation such that \( d_{C^0}(\gamma, \tilde{\gamma}) < \tau \) and \( d_{C^0}(\gamma^{-1}, \tilde{\gamma}^{-1}) < \tau \). Thus let us choose \( \tilde{\varphi} \in U \), which is a transitive homeomorphism with the same rotation number \( \rho \). Let \( x_0 \in \mathbb{R} \) be arbitrary. Then \( d_{C^0}(\gamma, \tilde{\gamma}), d_{C^0}(\gamma^{-1}, \tilde{\gamma}^{-1}) < \tau \) for some \( \gamma \) conjugating \( \varphi \) with the rotation. We can assume that \( |G(x_0) - \tilde{G}(x_0)| < \tau \) where \( \tilde{G} \) is a lift of \( \gamma \).

Notice that \( \Phi^n(x_0) = G^{-1}(G(x_0) + n \theta) \) and \( \tilde{\Phi}^n(x_0) = \tilde{G}^{-1}(\tilde{G}(x_0) + n \theta) \) for \( n = 0, 1, 2, \ldots \). Thus the
corresponding points on the orbits \( \{ \varphi^n(x) \} \) and \( \{ \tilde{\varphi}^n(x) \} \) remain \( 2\omega \)-close (independently of \( x_0 \) and \( n \)). Consequently we estimate:

\[
|r_n(x_0) - \tilde{r}_n(x_0)| < \frac{1}{2} M(|G^{-1}(G(x_0) + n\varrho) - G^{-1}(\tilde{G}(x_0) + n\varrho)| + \\
+|G^{-1}(\tilde{G}(x_0) + n\varrho) - G^{-1}(\tilde{G}(x_0) + n\varrho)| + \\
+|G^{-1}(G(x_0) + (n-1)\varrho) - G^{-1}(\tilde{G}(x_0) + (n-1)\varrho)| + \\
|G^{-1}(\tilde{G}(x_0) + (n-1)\varrho) - G^{-1}(\tilde{G}(x_0) + (n-1)\varrho)|) < M(\tau + \omega) < 2M\omega,
\]

which ends the proof.

If we do not require that the rotation numbers of \( \varphi \) and \( \tilde{\varphi} \) are the same then the above follows for fixed \( n \) (which is a simple observation). Namely,

**Remark 4.9** Let \( \varphi : S^1 \to S^1 \) be a transitive homeomorphism. Fix \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). There exists a neighborhood \( U \subset C^0(S^1) \) of \( \varphi \) such that for every other transitive homeomorphism \( \tilde{\varphi} \in U \) we have

\[
\sup_{x_0 \in \mathbb{R}} |r_n(x_0) - \tilde{r}_n(x_0)| < \varepsilon,
\]

where \( r_n(x_0) \) and \( \tilde{r}_n(x_0) \) denote local radii of curvature of curves generated by \( \varphi \) and \( \tilde{\varphi} \), respectively.

**Discussion**

We have established a number of properties of curlicues generated by orientation preserving circle homeomorphisms. In particular, we have studied the behaviour of the local discrete radius of curvature and perhaps it is worth noting that the spiral-like components of curves \( \Gamma = \Gamma((u_n)) \) (Definition 0.1) occur for those \( n \) where \( r_n \) is close to the minimum of \( G(x) = \frac{1}{2} \lvert \text{cosec}(\pi x) \rvert \) = 1/2 (see [19]). Moreover, the repetitive-like structure of the \( (r_n)_{n=1}^{\infty} \) sequence induced by circle homeomorphisms, as captured e.g. by Theorem 1.3, explains visible recurrence of “similar” parts of the curlicue as illustrated e.g. in Figure 4.

The first natural conclusion that we have drawn is that the geometrical properties of curlicues depend on the rationality of the rotation number of the generating circle homeomorphisms. Nevertheless, even for rational rotation number basic properties such as being bounded or not, might rather depend on the homeomorphism \( h \) conjugating \( \varphi \) with the corresponding rotation (if \( \varphi \) is conjugated to the rotation), as follows from Examples 1. and 2. On the other hand, for the irrational rotation number the relationship between the shape of the generated curve and the continuous solution of the corresponding cohomological equation seems to be an interesting observation. However, there are rather not explicit and easy to verify criteria assuring that such a solution exists, except for the case when the homeomorphism is the rotation (see e.g. [17]). Similarly, one cannot apriori determine whether the curve is superficial or not. Indeed, for bounded case or unbounded with non-zero Birkhoff average the situation is clear but in the remaining case it depends on more refined properties of the rotation number (Corollary 3.11). We know that the necessary condition for the curlicue to be bounded (and thus for the existence of a continuous solution of the cohomological equation) is the vanishing of the Birkhoff average. On the other hand, if the Birkhoff average does not vanish, then the curlicue is unbounded. Therefore, it would be interesting to determine what happens when the Birkhoff average equals zero but the induced curve is unbounded. Partially we answered this question in Theorems 5.1, 5.6 and Theorem 6.4, which allowed to establish superficiality and estimate the grow rate of such an unbounded curlicue. However, even providing a specific example of a transitive homeomorphism with vanishing Birkhoff average and unbounded curlicue (thus unbounded Birkhoff sums) seems a non-trivial task and further characterization of such curves may be a subject of further research.

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