SET-THEORETIC YANG-BAXTER SOLUTIONS
VIA FOX CALCULUS

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Dedicated to Professor Louis H. Kauffman for his 60th birthday

ABSTRACT
We construct solutions to the set-theoretic Yang-Baxter equation using braid group representations in free group automorphisms and their Fox differentials. The method resembles the extensions of groups and quandles.

Keywords: Set-theoretic Yang-Baxter equation, Fox free differential calculus, biracks, biquandles.

For a set $X$, a mapping $R : X \times X \to X \times X$ is called a solution to the set-theoretic Yang-Baxter equation (SYBE for short), if it satisfies the relation

$$(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R)$$

where 1 denotes the identity map. Often $R$ is required to be invertible, which we do not impose in this paper, and we concentrate on the above equation. The set theoretic Yang-Baxter equations are studied in detail in the papers, for example, [4, 7, 8, 15, 19]. They have been subjects of active investigations, not only from quantum algebra points of view, but also for applications to knot theory.
particular, algebraic systems called racks [10, 13], quandles [12, 15], biracks, bi-quandles [2, 9] have been actively studied in recent years in knot theory (see also [5, 6, 18], for example). It is of interest for applications to knots to have ample examples of such algebraic structures at hand to compute and use for knot invariants. The purpose of this note is to construct examples using representations of braid groups to the automorphisms of free groups [20] and the Fox free differential calculus [11]. The construction resembles constructions of group and quandle extensions by 2-cocycles.

Let $F_n = \langle x_1, \ldots, x_n \rangle$ be the free group of rank $n$. A (Fox) derivative [11] is a 1-cocycle, i.e., a homomorphism $d : \mathbb{Z}F_n \to \mathbb{Z}F_n$ satisfying the 1-cocycle condition (the Leibniz rule): $d(uv) = d(u) + u \cdot d(v)$ for any $u, v \in F_n$. Fox showed that any derivative is uniquely written as a linear combination of $\partial_{x_i} = \partial_{x_i} \delta$ defined (through linear extension) by $\partial_{x_i}(x_j) = \delta(x_i, x_j)$ for $i, j = 1, \ldots, n$, where $\delta$ denotes the Kronecker’s delta.

The chain rule for Fox calculus [11] is written as follows. Let $\lambda : Y = \langle y_1, \ldots, y_\ell \rangle \to X = \langle x_1, \ldots, x_n \rangle$ be a free group homomorphism. Then for any $f \in \mathbb{Z}Y$,

$$\frac{\partial f^\lambda}{\partial x_j} = \sum_k \left( \frac{\partial f}{\partial y_k} \right)^\lambda \frac{\partial y_k^\lambda}{\partial x_j},$$

where $f^\lambda$ is defined as $\hat{\lambda}(f)$ by the homomorphism $\hat{\lambda} : \mathbb{Z}Y \to \mathbb{Z}X$ induced from $\lambda$.

Wada [20] considered representations of braid groups to the automorphisms of free groups of the following type: For a standard generator $\sigma_i$ of the $n$-string braid group $B_n$, an isomorphism of $\langle x_1, \ldots, x_n \rangle$ written as

$$(x_i, x_{i+1}) \mapsto (u(x_i, x_{i+1}), v(x_i, x_{i+1})),$$

is assigned, where $u, v$ are finite words in $x_i, i = 1, \ldots, n$. Such representations are also studied independently by A. J. Kelly [14] as mentioned in the review article by Przytycki [17] of Wada’s paper.

The equalities for the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ are written as

\begin{align*}
(1) & \quad u(u(x, y), u(v(x, y), z)) = u(x, u(y, z)) \\
(2) & \quad v(u(x, y), u(v(x, y), z)) = u(v(x, u(y, z)), v(y, z)) \\
(3) & \quad v(v(x, y), z) = v(v(x, u(y, z)), v(y, z))
\end{align*}

where $(x, y, z) = (x_i, x_{i+1}, x_{i+2})$ and $\sigma_i$ denotes a standard braid generator. This computation is represented by diagrams as depicted in Fig. 1. If $u$ and $v \in F(x, y)$
and $v$ are elements of Baxter set with the operation $R$ satisfy the above equalities (1), (2) and (3), then any group $G$ becomes a Yang-Baxter set with the operation $R : G^2 \rightarrow G^2$ defined by $R(x, y) = (u(x, y), v(x, y))$ for $x, y \in G$.

Wada classified such representations for words of up to word lengths 10 for $u$ and $v$. His list of such functions consists of $(u, v) = (x, y), (y^{-1}, x), (y^{-1}, x^{-1}), (y, y^m xy^{-m}), (y, yx^{-1} y), (y^{-1}, y xy), (x^{-1} y^{-1} x, y^2 x)$.

Define

$$u_1(x, y) = \frac{\partial}{\partial x} u(x, y), \quad u_2(x, y) = \frac{\partial}{\partial y} u(x, y),$$

$$v_1(x, y) = \frac{\partial}{\partial x} v(x, y), \quad v_2(x, y) = \frac{\partial}{\partial y} v(x, y).$$

These are elements of $ZF_2$.

**Lemma 1.** Suppose $u, v \in F(x, y)$ satisfy the equations (1), (2), and (3). Then $u_i, v_i, i = 1, 2$, satisfy the following equalities in $ZF_3$.

$$u_1(u(x, y), u(v(x, y), z))u_1(x, y) + u_2(u(x, y), u(v(x, y), z))u_1(v(x, y), z)v_1(x, y) = u_1(x, u(y, z)),$$

$$u_1(u(x, y), u(v(x, y), z))u_2(x, y) + u_2(u(x, y), u(v(x, y), z))u_1(v(x, y), z)v_2(x, y) = u_2(x, u(y, z))u_1(y, z),$$

$$u_2(u(x, y), u(v(x, y), z))u_2(v(x, y), z) = u_2(x, u(y, z))u_2(y, z),$$

$$v_1(u(x, y), u(v(x, y), z))u_1(x, y) + v_2(u(x, y), u(v(x, y), z))u_1(v(x, y), z)v_1(x, y) = u_1(v(x, u(y, z)), v(y, z))v_1(x, u(y, z)),$$

$$v_1(u(x, y), u(v(x, y), z))u_2(x, y) + v_2(u(x, y), u(v(x, y), z))u_1(v(x, y), z)v_2(x, y) = u_1(v(x, u(y, z)), v(y, z))v_2(x, u(y, z))u_1(y, z) + u_2(v(x, u(y, z)), v(y, z))v_1(y, z),$$

$$v_2(u(x, y), u(v(x, y), z))u_2(v(x, y), z) = \ldots$$
where the second factors are as follows.

is a SYBE solution.

Then the map

By similar calculations for the equations (1), (2), (3) for derivative s with respect to

Proof. First consider the right-hand side $u(x, u(y, z))$ of the equation (1). Let

\[ \lambda : F(a, b) \to F(x, y, z) \]

be defined by $\lambda(a) = x$ and $\lambda(b) = u(y, z)$ that corresponds to the composition $u(x, u(y, z))$. Applying the chain rule, we obtain

\[
\frac{\partial}{\partial y} u(x, u(y, z)) = \left( \frac{\partial u(a, b)}{\partial a} \right)^x \left( \frac{\partial x}{\partial y} \right) + \left( \frac{\partial u(a, b)}{\partial b} \right)^y \left( \frac{\partial y}{\partial y} \right) = u_2(x, u(y, z))u_1(y, z).
\]

By similar calculations for the equations (1), (2), (3) for derivatives with respect to $x, y$ and $z$, respectively, we obtain the above nine equalities.

Let $u, v \in F(x, y)$ be solutions to the equations (1), (2) and (3). Define maps
\[ \tilde{u}, \tilde{v} : (G \times V)^2 \to G \times V \] by

\[
\tilde{u}((x, a), (y, b)) = (u(x, y), u_1(x, y) a + u_2(x, y) b),
\]

\[
\tilde{v}((x, a), (y, b)) = (v(x, y), v_1(x, y) a + v_2(x, y) b)
\]

for $x, y \in G$ and $a, b \in V$, where $V$ is a $G$-module.

Theorem 2. Let $u, v \in F(x, y)$ be solutions to the equations (1), (2) and (3). Then the map $R = (\tilde{u}, \tilde{v}) : (G \times V)^2 \to (G \times V)^2$ defined by

\[ R((x, a), (y, b)) = (\tilde{u}((x, a), (y, b)), \tilde{v}((x, a), (y, b))) \]

is a SYBE solution.

Proof. We compute

\[
(R \times 1)(1 \times R)(R \times 1)((x, a), (y, b), (z, c)) = \]

\[
( (u(u(x, y), u(v(x, z), A)), (v(u(x, y), u(v(x, z), B)), (v(v(x, y), z), C)) ,
\]

\[
(1 \times R)(R \times 1)(1 \times R)((x, a), (y, b), (z, c)) =
\]

\[
( (u(x, u(y, z)), A'), (u(v(x, u(y, z)), v(y, z)), B'), (v(v(x, u(y, z)), v(y, z)), C')) )
\]

where the second factors are as follows.

\[
A = u_1(u(x, y), u(v(x, y), z))[u_1(x, y) a + u_2(x, y) b]
\]

\[
+ u_2(u(x, y), u(v(x, y), z)) \times
\]
we obtain $[u_1(v(x, y), z)(v_1(x, y) a + v_2(x, y) b ) + u_2(v(x, y), z) c ]$,

$$B = v_1(u(x, y), u(v(x, y), z))[u_1(x, y) a + u_2(x, y) b ] + v_2(u(x, y), u(v(x, y), z)) \times [u_1(v(x, y), z)(v_1(x, y) a + v_2(x, y) b ) + u_2(v(x, y), z) c ],$$

$$C = v_1(v(x, y), z)[v_1(x, y) a + v_2(x, y) b ] + v_2(v(x, y), z) c,$$

$$A' = u_1(x, u(y, z)) a + u_2(x, u(y, z))[u_1(y, z) b + u_2(y, z) c ],$$

$$B' = u_1(v(x, u(y, z)), v(y, z)) \times [v_1(x, u(y, z)) a + v_2(x, u(y, z))[u_1(y, z) b + u_2(y, z) c ],$$

$$C' = v_1(v(x, u(y, z)), v(y, z)) \times [v_1(x, u(y, z)) a + v_2(x, u(y, z))[u_1(y, z) b + u_2(y, z) c ] + v_2(v(x, u(y, z)), v(y, z))[v_1(y, z) b + v_2(y, z) c ].$$

By grouping coefficients for $a, b, c$ in the equations $A = A'$, $B = B'$, and $C = C'$, we obtain the result by Lemma 1. \hfill \Box

The above calculations can be visualized directly by diagrams of the Reidemeister type III move. For a given coloring by group elements at a crossing as in Fig. 2 assign small beads at the top arcs as depicted, representing elements ($a, b$ in the figure) of a $G$-module $V$. Below the crossing, elements of $V$ denoted by $\alpha$ and $\beta$ in the figure, that are the images of linear maps $u_1(x, y) a + u_2(x, y) b$ and $v_1(x, y) a + v_2(x, y) b$, are represented by another pair of beads as depicted. Let elements $a, b, c \in V$ be assigned to the three arcs at the top of Fig. 1 from left to right, respectively. At the bottom, arcs receive three beads representing the images of the composition of linear maps, corresponding to three crossings. The outcomes are $A, B, C$ in the above calculation for the left-hand side, and $A', B', C'$ for the right-hand side.

![Figure 2: vectors assigned to colored arcs](image)

**Example 3.** For the second last example in Wada’s list $u = y^{-1}$ and $v = xy$, we obtain

\[ u_1 = 0, \quad u_2 = -y^{-1}, \quad v_1 = y, \quad v_2 = 1 + yx, \]
so that for any group $G$ and any $G$-module $V$, $G \times V$ is a Yang-Baxter set by

$$R((x, a), (y, b)) = ( (y^{-1}, (-y^{-1})b ), (xy, ya + (1 + yx)b ) ).$$

Similarly, for the last example in Wada’s list $u = x^{-1}y^{-1}x$ and $v = y^{2}x$, we obtain

$$R((x, a), (y, b)) = ( (x^{-1}y^{-1}x, (-x^{-1} + x^{-1}y^{-1})a + (-x^{-1}y^{-1})b ), (y^{2}x, (y^{2})a + (1 + y)b ) ).$$

**Remark 4.** For the 4th and 5th examples in the Wada’s list, the operation $x \ast y = v(x, y)$ (where $v(x, y) = y$ does not play an essential role) defines a rack, and the equalities in Lemma 1 give rise to the condition of rack algebras and modules defined in [1]; in their notation, $\eta_{x,y} = v_{1}(x, y)$ and $\tau_{x,y} = v_{2}(x, y)$. Hence in these cases, Theorem 2 can be used to produce examples of rack modules.

On the other hand, the equalities in Lemma 1 can be regarded as providing a natural definition of birack algebras and modules, from point of view of [1]. Specifically, a birack algebra can be defined with generators

$$\{ u_{i}(x, y), v_{j}(x, y) \mid i, j = 1, 2, \quad x, y \in X \}$$

with relations stated in Lemma 1, where $X$ is a birack defined in [2]. Furthermore, again from [1], it is expected that these could be used as twisted coefficients for generalizations of the homology theory of the SYBE [4]. Applications to knot invariants are also expected.

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