Self-Intersecting Periodic Curves in the Plane

J. Howie & J. F. Toland

Abstract

Suppose a smooth planar curve $\gamma$ is $2\pi$-periodic in the $x$ direction and the length of one period is $\ell$. It is shown that if $\gamma$ self-intersects, then it has a segment of length $\ell - 2\pi$ on which it self-intersects and somewhere its curvature is at least $2\pi/(\ell - 2\pi)$. The proof involves the projection $\Gamma$ of $\gamma$ onto a cylinder. (The complex relation between $\gamma$ and $\Gamma$ was recently observed analytically in [1], see also [5, Ch. 10]). When $\gamma$ is in general position there is a bijection between self-intersection points of $\gamma$ modulo the periodicity, and self-intersection points of $\Gamma$ with winding number 0 around the cylinder. However, our proof depends on the observation that a loop in $\Gamma$ with winding number 1 leads to a self-intersection point of $\gamma$.

Mathematics Subject Classification: Primary 53A04, Secondary 55M25

Let a smooth $2\pi$-periodic curve $\gamma$ in the $(x, y)$-plane be parametrized by arc-length as follows:

$$\begin{align*}
\gamma &= \{p(s) : s \in \mathbb{R}\}, \\
p(s) &= (u(s), v(s)), \\
\{u(s + \ell) = 2\pi + u(s), \\
v(s + \ell) = v(s), \\
u'(s)^2 + v'(s)^2 = 1,
\end{align*}$$

The length of one period of $\gamma$ is $\ell$ and $q \in \gamma$ is called a crossing if $q = p(s_1) = p(s_2)$ and $s_1 \neq s_2$. Note that crossings exist if and only if $p$ is not injective. A crossing $q$ is called simple if there are exactly two real numbers $s_1 \neq s_2$ with $p(s_1) = p(s_2) = q$ and if $p'(s_1) \neq p'(s_2)$ when $p(s_1) = p(s_2)$ and $s_1 \neq s_2$. Note that the smooth curve $\gamma$ can be approximated arbitrarily closely by smooth curves in general position, that is with all crossings simple. If $\gamma$ is in general position, then it follows from the smoothness that the set of crossings is discrete, and hence finite by compactness. Let $p'(s) = (\cos \vartheta(s), \sin \vartheta(s)), \ s \in \mathbb{R}$, where $\vartheta$ is smooth [3 Prop. 2.2.1]. The goal is to establish the following which is intuitively obvious. (A periodic segment of $\gamma$ is a segment of the form $\{p(t) : t \in [a, a + \ell]\}$.)

**Proposition.** Suppose that all crossings of $\gamma$ are simple.

(a) If $p$ is injective on every interval of length $\ell - 2\pi$, $p$ is injective.

(b) If $p$ is not injective its curvature is somewhere at least $2\pi/(\ell - 2\pi)$. 

1
(c) If \( p \) is not injective and \( \vartheta \) is periodic, then \( \gamma \) has a periodic segment which contains two crossings.

The global problem of bounding from below the maximum curvature of a self-intersecting periodic planar curve arose in a study of water waves beneath an elastic sheet. In the model \([6]\), the sheet energy increases with the curvature and, roughly speaking, the conclusion needed was that sheets of certain energies could not self-intersect.

**Remark.** Periodicity of \( \vartheta \) in the Proposition does not follow from that of \( p \), as the first diagram below shows. Part (c) of the Proposition is illustrated in the second diagram, where \( \vartheta \) is periodic.

\[
x = -\pi \quad x = \pi \quad x = -\pi \quad x = \pi
\]

For a proof, we project \( \gamma \) onto the cylinder \( C = S^1 \times \mathbb{R} \), where \( S^1 = \{ e^{i\phi} : \phi \in \mathbb{R} \} \). Let \( P : \mathbb{R} \to C \) be given by \( P(s) = (e^{i\vartheta(s)}, v(s)) \) and let \( \Gamma = \{ P(s) : s \in [0, \ell] \} \). Thus the projection of the periodic, non-compact curve \( \gamma \) in \( \mathbb{R}^2 \) onto \( C \) is the compact curve \( \Gamma \). Now \( \Gamma \) has a crossing \( Q \) if \( P(s_0) = P(t_0) = Q \) for some \( 0 \leq t_0 < s_0 < \ell \) and we note that \( P(s_0) = P(t_0) \) if and only if \( p(s_0) = p(t_0) + k(2\pi, 0) = p(t_0 + k\ell), \quad k \in \mathbb{Z} \),

where \( k = \#(\Gamma_Q) \), the winding number around \( C \) of

\[
\Gamma_Q = \{ P(s) : s \in [t_0, s_0] \},
\]

a loop at \( Q \). Crossings of \( \Gamma \) with winding number \( k \) correspond to the existence of horizontal chords with length \( 2|k|\pi \) connecting points of \( \gamma \). Significantly for the Proposition, there is a one-to-one correspondence between crossings of \( \gamma \) and crossings of \( \Gamma \) with winding number zero. Note that \( \#(\Gamma) = 1 \), since \( P(\ell) = P(0) \) and \( p(\ell) = p(0) + (2\pi, 0) \).

**Lemma 1.** Suppose that \( \#(\Gamma_Q) \in \{0, 1\} \) for a crossing \( Q \) of \( \Gamma \). Then \( p \) is not injective on some interval of length \( \ell \).

**Proof.** By hypothesis \( \Gamma_Q := \{ P(s) : s \in [t_0, s_0] \}, \ [t_0, s_0] \subset [0, \ell] \) and

\[
u(s_0) = u(t_0) + 2k\pi \text{ for } k \in \{0, 1\}, \quad v(s_0) = v(t_0).
\]
If \( k = 0 \), \( p(s_0) = p(t_0) \) and the conclusion holds. If \( k = 1 \),
\[
p(s_0) = p(t_0 + \ell), \quad 0 < t_0 + \ell - s_0 < \ell,
\]
and again the conclusion holds.

**Remark.** Note that if \( \#(\Gamma_Q) = -1 \), the proof of Lemma 1 leads only to the conclusion that there is an interval of length \( 2\ell \) on which \( p \) is not injective, as illustrated in the example below.

![Diagram](image)

The segment \( 1 \to 2 \to 3 \to 4 \), in which arrows denote increasing arc-length, represents one period of \( \gamma \) in \( \mathbb{R}^2 \). The dashed curve \( 5 \to 6 \to 7 \to 8 \) represents the next period. The segment numbered 1 contains a sub-loop of \( \Gamma \) on \( C \) with winding number \(-1\) and the construction just described leads to the crossing \( O \) on \( \gamma \). However, the length of the corresponding closed sub-arc of \( 1 \to 2 \to 3 \to 4 \to 5 \) in \( \mathbb{R}^2 \) lies between \( \ell \) and \( 2\ell \) which does not vindicate the Proposition. However, there is another crossing \( * \) on \( \gamma \), and the closed loop \( 4 \to 5 \to 6 \) satisfies the conclusion of the Proposition.

The following is the key.

**Lemma 2.** Suppose the crossings of \( \Gamma \) are all simple. For any loop at \( \bar{Q} \) of the form \( \Gamma_{\bar{Q}} = \{ P(s) : s \in [a, b] \} \), \( P(a) = P(b) = \bar{Q} \), with \( \#(\Gamma) > 1 \), there exists a sub-loop at \( \bar{Q}_1 \) of the form \( \Gamma_{\bar{Q}_1} := \{ P(s) : s \in [a_1, b_1] \} \), \( P(a_1) = P(b_1) = \bar{Q}_1 \), \( a \leq a_1 < b_1 < b \), with \( \#(\Gamma_{\bar{Q}_1}) = 1 \).

**Proof.** Since \( \#(\Gamma_{\bar{Q}}) > 1 \) it follows from the topology of the cylinder that \( \Gamma_{\bar{Q}} \) has a crossing. The proof is by induction on the number of crossings.

If \( \Gamma_{\bar{Q}} \) has only one crossing, \( \Gamma_{\bar{Q}} \) is the union of two loops, \( \Gamma_1 \) and \( \Gamma_2 \), based at a point of \( \Gamma_{\bar{Q}} \). Since they have no crossings, each has winding number \( \pm 1 \) or 0. Since the sum of their winding numbers is \( \#(\Gamma_{\bar{Q}}) > 1 \), each has winding number 1 and \( \#(\Gamma_{\bar{Q}}) = 2 \). If \( \bar{Q} \in \Gamma_2 \), then the sub-path \( \Gamma_1 \) satisfies the conclusion of the lemma, and vice versa.

Now we make the inductive hypothesis that the lemma holds for any loop \( \Gamma_{\bar{Q}} \) of the form in the lemma with no more than \( N - 1 \) crossings, \( N \geq 2 \).
Suppose a loop $\Gamma_{\hat{Q}} = \{P(s) : s \in [\hat{a}, \hat{b}]\}$, $P(\hat{a}) = P(\hat{b}) = \hat{Q}$, has $N$ crossings. Choose one of them, $P(s_1) = P(t_1) =: \hat{Q}$, say. This splits $\Gamma_{\hat{Q}}$ into two loops, $\Gamma_1$ and $\Gamma_2$, based at $\hat{Q}$. If they both have winding number 1, then the result follows, exactly as in the case $N = 1$ above. Otherwise one of them, $\Gamma_1$ say, has winding number at least 2 and no more than $N - 1$ crossings.

Now, momentarily, let $\hat{Q}$ be the origin of arc length so that $\Gamma_1 = \{P(s) : s \in [0, \hat{t}]\}$ where $s$ is arc length measured from $\hat{Q}$ along $\Gamma_1$. Then, by induction, there is a loop $\Gamma_{11}$ in $\Gamma_1$, satisfying the conclusion of the lemma with $[0, \hat{t}]$ instead of $[a, b]$, and winding number 1.

If $\Gamma_{11}$ does not contain $\hat{Q}$, then $\Gamma_{11}$ with the original parametrization satisfies the conclusion of the lemma.

If $\Gamma_{11}$ does contain $\hat{Q}$, then its complement in $\hat{\Gamma}$ is a sub-path $\Gamma_{12} = \{P(s) : s \in [a', b'] \subset [a, b]\}$ of $\hat{\Gamma}$, with winding number not smaller than 1 and no more than $N - 1$ crossings.

If the winding number of $\Gamma_{12}$ is 1, then we are done. If it exceeds 1, then the required conclusion follows from the inductive hypothesis.

**Lemma 3.** If $(\Gamma_Q) > 1$ for a crossing $Q$ of $\Gamma$, then $p$ is not injective on some closed interval of length $\ell$.

**Proof.** Assume first that all the crossings of the original curve $\Gamma$ are simple. Putting $\Gamma = \Gamma_Q$ in Lemma 2 gives the existence of a crossing of $\Gamma$ with winding number 1. The required result follows by Lemma 1 when all the crossings of $\Gamma$ are simple. If the crossings of $\Gamma$ are not all simple, apply the conclusion of Lemma 2 to a uniform periodic approximation $\gamma_1$ of $\gamma$ parametrized by a smooth periodic function $p_1$ with the property that each crossing of $\Gamma_1$ is simple and close to a crossing of $\Gamma$. The required result in the general case will follow by a simple limiting argument.

**Proof of the Proposition.** (a) If $p$ is not injective, $\Gamma$ has a crossing, $Q$. Suppose $P(t_0) = P(s_0), 0 \leq t_0 < s_0 < \ell$. Then, in the notation of (11), $\Gamma_Q = \{P(s) : s \in [t_0, s_0]\}$ and there is a minimal sub loop $\Gamma_{Q_1} = \{P(s) : s \in [t_1, s_1]\}$ of $\Gamma_Q$ (a loop in $\Gamma_Q$ which has no proper sub loop) $[t_1, s_1] \subset [t_0, s_0], P(s_1) = P(t_1) =: Q_1$. Since $\Gamma_{Q_1}$ has no crossings, $|\#(\Gamma_{Q_1})| \leq 1$.

Now we observe that if $p$ is not injective, then it is not injective on some interval of length $\ell$. If $(\Gamma_{Q_1}) \in \{0, 1\}$, the observation holds by Lemma 1. If $(\Gamma_{Q_1}) = -1$, since $(\Gamma) = 1$, the complement of $\Gamma_{Q_1}$ in $\Gamma$ has winding number 2 and the observation holds, by Lemma 3.

Now consider an interval $[a, a + \ell]$ on which $p$ is not injective. Since $p(a + \ell) = p(a) + (2\pi, 0)$, it follows easily (from the diagram below!) that the length of any loop in this periodic segment of $\gamma$ does not exceed $\ell - 2\pi$. Hence there is an interval of length $\ell - 2\pi$ on which $p$ is not injective.
(b) A classical result [4] in the case of plane curves is the following [2]. Remark on p. 38.

Axel Schur (1921). Suppose that $\mathcal{U}_i = \{v_i(s) : s \in [0, S]\}$, $i = 1, 2$, are two plane curves parametrized by arc length, with the same length $S$ and with curvatures $\kappa_i(s)$ at $v_i(s)$. Suppose that $\mathcal{U}_1$ has no self-intersections and, along with the chord from $v_1(0)$ to $v_1(S)$, bounds a convex region. Furthermore, suppose that $|\kappa_2| \leq \kappa_1$ on $[0, S]$. Then $|v_2(s) - v_2(0)| \geq |v_1(s) - v_1(0)|$, $s \in [0, S]$.

Let $\mathcal{U}_2$ be a closed loop in $\gamma$ with length $S$ no greater than $\ell - 2\pi$ and suppose that at every point its curvature $|\kappa_2| \leq 2\pi(1 - \epsilon)/(\ell - 2\pi)$ for some $\epsilon > 0$. Let $\mathcal{U}_1$ be the segment of length $S$ of a circle of radius $(\ell - 2\pi)/(2\pi(1 - \epsilon))$. Now $|\kappa_2| \leq \kappa_1$, $\mathcal{U}_1$ is not closed but $\mathcal{U}_2$ is closed, which contradicts Schur’s result. Hence no such $\epsilon$ exists, which proves (b).

(c) Consider a periodic segment of $\gamma$ with only one crossing at an angle $\alpha$, as illustrated by the solid line in the diagram. Now extend this segment as a smooth closed curve of length $\ell + L$ with no further crossings (the extension is the dashed curve $\tilde{\gamma}$).

By the hypothesis of part (c),

$$\int_0^\ell \vartheta'(s) \, ds = 0,$$

and by construction, $\int_{\ell}^{\ell+L} \vartheta'(s) \, ds = -2\pi$.

So, from the hypothesis, the integral of $\vartheta'$ around the oriented loop $\gamma \cup \tilde{\gamma}$ is $-2\pi$. On the other hand, by the Hopf’s Umlaufsatz for curvilinear polygons [3].
§13.2,\[
\int_{\ell_1}^{\ell_2} \varphi'(s) ds = \pi + \alpha = \int_{\ell_2}^{\ell+L} \varphi'(s) ds + \int_{0}^{\ell_1} \varphi'(s) ds.
\]
This is impossible since $\alpha \notin \{0, \pi\}$, because all crossings are simple. This contradiction completes the proof.

References

[1] T. M. Apostol and M.A. Mnatsakanian, Unwrapping curves from cylinders and cones. *Amer. Math. Monthly.* 114 (2007), 388-416.

[2] S. S. Chern, Curves and surfaces in Euclidean space, in: *Studies in Global Geometry and Analysis*, Studies in Mathematics Volume 4, Math. Asoc. Amer, 1967, pp. 16–56.

[3] A. Pressley, *Elementary Differential Geometry*. Second Edition. Springer Undergraduate Mathematics Series. Springer, London, 2010.

[4] A. Schur. Über die Schwarzsche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung. *Mathematische Annalen* 83 (1921), 143-148. [http://www.digizeitschriften.de/main/dms/img/?PPN=PPN235181884](http://www.digizeitschriften.de/main/dms/img/?PPN=PPN235181884)

[5] H. Steinhaus, *Mathematical Snapshots*. Dover, Mineola NY, 1999.

[6] J. F. Toland, Heavy hydroelastic travelling waves. *Proc. R. Soc. Lond. A* 463 (2007), 2371-2397 (DOI : 10.1098/rspa.2007.1883)

J. Howie
Department of Mathematics and Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Edinburgh EH14 4AS

J. F. Toland
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY