Supermixed labyrinth fractals

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February 16, 2018

Keywords: fractal, dendrite, graph, tree, Sierpiński carpets, length of paths, arc length

AMS Classification: 28A80, 28A75, 51M25, 05C05, 05C38, 54D05, 54F50

Abstract

Labyrinth fractals are dendrites in the unit square. They were introduced and studied in the last decade first in the self-similar case [3, 4], then in the mixed case [5, 6]. Supermixed fractals constitute a significant generalisation of mixed labyrinth fractals: each step of

* L.L. Cristea is supported by the Austrian Science Fund (FWF), stand-alone project P27050-N26, and by the Austrian Science Fund (FWF) project F5508-N26, which is part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”
† G. Leobacher is supported by the Austrian Science Fund (FWF) project F5508-N26, which is part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”
the iterative construction is done according to not just one labyrinth pattern, but possibly to several different patterns. In this paper we introduce and study supermixed labyrinth fractals and the corresponding prefractals, called supermixed labyrinth sets, with focus on the aspects that were previously studied for the self-similar and mixed case: topological properties and properties of the arcs between points in the fractal. The facts and formulae found here extend results proven in the above mentioned cases. One of the main results is a sufficient condition for infinite length of arcs in mixed labyrinth fractals.

1 Introduction

Labyrinth fractals are a special family of Sierpiński carpets in the plane, and are dendrites in the unit square. They were defined and studied in the last decade, first in the self-similar case [3, 4], then in the more general case, as mixed labyrinth fractals [5, 6]. The self-similar labyrinth fractals are generated by one labyrinth pattern, while in the case of mixed labyrinth fractals a sequence of patterns may be used, i.e., one labyrinth pattern for each step of the iterative construction. In both cases, in order to study the arcs between points in the fractals, one uses the paths in the graphs (more precisely, trees) associated to the patterns and to the prefractals, called labyrinth sets of some level \( n \geq 1 \), respectively. An important role is played by the path matrix associated to a labyrinth pattern and, respectively, to a labyrinth set of level \( n \).

The results on labyrinth fractals have already found applications in physics, in different contexts like, e.g., the study of planar nanostructures [9], the fractal reconstruction of complicated images, signals and radar backgrounds [10], and recently in the construction of prototypes of ultra-wide band radar antennas [17]. Moreover, in very recent work [18] fractal labyrinths are used in combination with genetic algorithms for the synthesis of big robust antenna arrays and nano-antennas in telecommunication. Let us remark here that supermixed labyrinth fractals, as well as self-similar and mixed labyrinth fractals, are so-called finitely ramified carpets, in the terminology used in physicists’ work regarding the modelling of porous structures [21]. Moreover, physicists use so-called disordered fractals to study diffusion in disordered media [1], more precisely, they investigate the diffusion in fractals obtained by mixing different Sierpiński carpet generators.
This leads to the idea that supermixed labyrinth fractals are a class of such fractals that can be used as models for future research. Here we also remark that random Koch curves are related, e.g., to arcs between exits in supermixed labyrinth fractals, and are of interest to theoretical physicists in the context of diffusion processes, e.g., [20]. Since labyrinth fractals are dendrites, let us mention here that very recent research in materials engineering [11] shows that dendrite growth, a largely unsolved problem, plays an essential role when dealing with high power and energy lithium-ion batteries. In the context of fractal dendrites there is also recent work [10] in crystal growth which lead us to the conclusion that certain families of our labyrinth fractals are suitable as models for such phenomena.

While the self-similar and mixed labyrinth fractals are appealing because of their elementary setup and their elegant algebraic treatment, they do not always provide convincing models for applications. This is overcome by the introduction of supermixed labyrinth fractals: for those we have, at each step of the iterative construction, not just one labyrinth pattern, but in general, a finite collection of labyrinth patterns, according to which the construction of the prefractal is done. Again, the graph of each of the resulting prefractals is a tree, and the supermixed labyrinth fractal is a dendrite, thus the paths between given vertices, and, respectively, the arcs that connect points in the fractal are unique. Here, in order to obtain the length of paths in the prefractals, the path matrix does not work in the same way as in the mixed and self-similar case. Here we need to introduce counting matrices, specific to the supermixed case. In this more general case the relations that hold when passing from one iteration to the next one are not anymore “encoded” by the powers of a matrix or by products of matrices, as in the self-similar or mixed case, respectively. In the supermixed case the formulæ contain sums of products of path matrices of the patterns and counting matrices associated to the iterations.

Of course, one could also approach supermixed labyrinth fractals as $V$-variable fractals, see, e.g., [2, 8], or in the context of graph directed constructions, see, e.g., [13], or that of graph directed Markov systems [14], but here the idea was to remain in the same framework as in the case of self-similar and mixed labyrinth fractals, the objects that we generalise here.

Finally, we mention that there is very recent research on fractal dendrites [19], where self-similar dendrites are constructed by using polygonal systems in the plane, a method based on IFSs that is dif-
different from the construction method used for self-similar labyrinth fractals [3, 4].

Let us now give a short outline of the paper. First, we recall notions about labyrinth fractals and introduce the concepts of supermixed labyrinth set and fractal in Section 2. We also prove that supermixed labyrinth sets are labyrinth patterns.

In Section 3 we prove that every supermixed labyrinth fractal is a dendrite. Next we define the exits of supermixed labyrinth fractals and of squares of a given level in the short Section 4.

Next in Section 5 we describe how paths in the graphs of prefractals of supermixed labyrinth fractals are constructed iteratively. We recall the definition of the path matrix of a labyrinth pattern and define counting matrices, a concept specific to supermixed labyrinth fractals. The main result of this section is Theorem 2 which gives a recursive formula for the path matrices of supermixed labyrinth sets of different levels.

Section 6 is devoted to constructing, and exploring properties of arcs in supermixed labyrinth fractals.

The concept of blocked labyrinth pattern is recalled in Section 7. We review existing results about arcs in self-similar and mixed labyrinth fractals constructed with blocked patterns. Moreover, we recall properties of the path matrix of blocked labyrinth patterns.

Finally, we formulate and prove one of our main results in Section 8: a sufficient condition for infinite length of any arc between distinct points of a mixed labyrinth fractal is $\sum_{k=1}^{\infty} \frac{1}{m_k} = \infty$, where $(m_k)_{k \geq 1}$ is the sequence of widths of the patterns that define the fractal. We also remark on difficulties in adapting the proof method to the supermixed case.

2 Patterns, labyrinth patterns, super-mixed labyrinth sets and supermixed labyrinth fractals

In order to construct labyrinth fractals we use labyrinth patterns. Figures 1 and 2 show labyrinth patterns and illustrate the first two steps of the construction described below.

Let $x, y, q \in [0, 1]$ such that $Q = [x, x + q] \times [y, y + q] \subseteq [0, 1] \times [0, 1]$.  

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Then for any point \((z_x, z_y) \in [0, 1] \times [0, 1]\) we define the function

\[ P_{Q(z_x, z_y)} = (qz_x + x, qz_y + y). \]

Let \( m \geq 1 \). \( S_{i,j,m} = \{(x, y) \mid \frac{i}{m} < x \leq \frac{i+1}{m} \text{ and } \frac{j}{m} < y \leq \frac{j+1}{m}\} \) and \( S_m = \{S_{i,j,m} \mid 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq m-1\} \).

We call any nonempty \( A \subseteq S_m \) an \( m \)-pattern and \( m \) its width.

Let \( \{\tilde{A}_k\}_{k=1}^\infty \), with \( \tilde{A}_k = \{A_{k,h} \mid h = 1, \ldots, s_k\} \), where \( s_k \geq 1 \) (the number of patterns of the collection \( \tilde{A}_k \)), for all \( k \geq 1 \), be a sequence of nonempty collections of non-empty patterns and \( \{m_k\}_{k=1}^\infty \) be the corresponding width-sequence, i.e., for any \( k \geq 1 \) we have \( A \subseteq S_{m_k} \), for all \( A \in \tilde{A}_k \). Throughout this paper we assume \( s_1 = 1 \).

![Three labyrinth patterns](image)

Figure 1: Three labyrinth patterns (all of width 4), from left to right: the unique pattern \( A_{1,1} \in \tilde{A}_1 \), followed by the (two) patterns \( A_{2,1}, A_{2,2} \in \tilde{A}_2 \)

We denote \( m(n) = \prod_{k=1}^n m_k \), for all \( n \geq 1 \). We let \( W_1 := A_{1,1} \), and call it the set of white squares of level 1. Then we define \( B_1 = S_{m_1} \setminus W_1 \) as the set of black squares of level 1. For \( k \geq 1 \) let \( \phi_{k+1} : W_k \to \tilde{A}_{k+1} \), i.e., \( \phi_{k+1} \) assigns to every square \( W \in W_k \) a pattern in \( \tilde{A}_{k+1} \). For \( n \geq 2 \) we define the set \( W_n \) of white squares of level \( n \) as follows.

\[
W_n := \bigcup_{W \in \phi_n(W_{n-1}), W_{n-1} \in W_{n-1}} \{P_{W_{n-1}}(W)\}, \text{ for all } n \geq 2. \tag{1}
\]

We note that \( W_n \subseteq S_{m(n)} \), and we define the set of black squares of level \( n \) by \( B_n = S_{m(n)} \setminus W_n \). For \( n \geq 1 \), we define \( L_n = \bigcup_{W \in W_n} W \). One can immediately see that \( \{L_n\}_{n=1}^\infty \) is a monotonically decreasing sequence of compact sets. We call \( L_\infty := \bigcap_{n=1}^\infty L_n \) the limit set defined by the sequence of collections of patterns \( \{\tilde{A}_k\}_{k=1}^\infty \).
A graph $G$ is a pair $(V, E)$, where $V = V(G)$ is a finite set of vertices, and the set of edges $E = E(G)$ is a subset of $\{\{u, v\} \mid u, v \in V, u \neq v\}$. We write $u \sim v$ if $\{u, v\} \in E(G)$ and we say $u$ is a neighbour of $v$. The sequence of vertices $\{u_i\}_{i=0}^n$ is a path between $u_0$ and $u_n$ in a graph $G \equiv (V, E)$, if $u_0, u_1, \ldots, u_n \in V$, $u_{i-1} \sim u_i$ for $1 \leq i \leq n$, and $u_i \neq u_j$ for $0 \leq i < j \leq n$. The sequence of vertices $\{u_i\}_{i=0}^n$ is a cycle in $G \equiv (V, E)$, if $u_0, u_1, \ldots, u_n \in V$, $u_{i-1} \sim u_i$ for $1 \leq i \leq n$, $u_i \neq u_j$ for $1 \leq i < j \leq n$, and $u_0 = u_n$. A tree is a connected graph that contains no cycle. For any two distinct vertices in a tree there exists a unique cycle-free path in the tree that connects them.

For $A \subseteq S_m$, we define $G(A) \equiv (V(G(A)), E(G(A)))$ to be the graph of $A$, i.e., the graph whose vertices $V(G(A))$ are the white squares in $A$, and whose edges $E(G(A))$ are the unordered pairs of white squares, that have a common side. The top row in $A$ is the set of all white squares in $\{S_{i,m-1,m} \mid 0 \leq i \leq m-1\}$. The bottom row, left column, and right column in $A$ are defined analogously. A top exit in $A$ is a white square in the top row, such that there is a white square in the same column in the bottom row. A bottom exit in $A$ is defined analogously. A left exit in $A$ is a white square in the left column, such that there is a white square in the same row in the right column. A right exit in $A$ is defined analogously. While a top exit together with the corresponding bottom exit constitute a vertical exit pair, a left exit and the corresponding right exit constitute a horizontal exit pair.

We recall the definition of a labyrinth pattern, see [3 page 3].
Definition 1. A non-empty \( m \times m \)-pattern \( A \subseteq S_m \), \( m \geq 3 \), is called an \( m \times m \)-labyrinth pattern (in short, labyrinth pattern) if \( A \) satisfies

- **The tree property.** \( G(A) \) is a tree.
- **The exits property.** \( A \) has exactly one vertical exit pair, and exactly one horizontal exit pair.
- **The corner property.** If there is a white square in \( A \) at a corner of \( A \), then there is no white square in \( A \) at the diagonally opposite corner of \( A \).

Definition 2. For any labyrinth pattern \( A \in S_m \) whose horizontal and vertical exit pair lies in row \( r \) and column \( c \), respectively, and \( r, c \in \{1, \ldots, m\} \), we call the ordered pair \((r,c)\) the exits positions pair of the pattern \( A \).

Assumptions 1. Let \( (\tilde{A}_k)_{k \geq 1} \) be a sequence of collections of labyrinth patterns, where \( \tilde{A}_k = \{A_{k,h}, h = 1, \ldots, s_k\} \), with \( s_1 = 1 \), and \( s_k \geq 1 \) for \( k \geq 2 \).

- **Pairwise tree consistency.** For all \( n \geq 1 \), \( \phi_{n+1} \) has the following property: if \( W, W' \in \mathcal{W}_n \) are neighbours in \( G(\mathcal{W}_n) \), then the restriction of the graph \( G(\mathcal{W}_{n+1}) \) to the subset of vertices that correspond to the (white) squares of level \( n \) that are contained in \( W' \) and \( W'' \) is a tree, for all neighbouring squares \( W, W' \in \mathcal{W}_n \).
- **Exits consistency.** All patterns in the collection \( \tilde{A}_k \) have the same exits positions pair \((r_k,c_k)\) and the same width \( m_k \), for all \( k \geq 1 \).
- **Corner consistency.** For all \( k \geq 1 \), if in a pattern \( A \in \tilde{A}_k \) there is a white square in a corner then there exists no pattern \( A' \in \tilde{A}_k \) with a white square at the diagonally opposite corner.

Throughout this article we assume that the Assumptions 1 hold.

Proposition 1. \( \mathcal{W}_n \) has the tree property, the exits property and the corner property, i.e., \( \mathcal{W}_n \) is a labyrinth pattern.

**Proof.** The proof works by induction. For \( \mathcal{W}_1 = \mathcal{A}_{1,1} \) all three properties are satisfied, since \( \mathcal{A}_{1,1} \) is a labyrinth pattern. Let \( n \geq 2 \). We assume that \( \mathcal{W}_{n-1} \) has the tree property, the exits property and the corner property, and show that then \( \mathcal{W}_n \) also has these three properties. The corner property follows immediately from the corner property of \( \mathcal{W}_{n-1} \) and from the corner property and the corner hypothesis satisfied by the labyrinth patterns in \( \tilde{A}_n \).
The exits property of \( W_n \) follows immediately from the exits property of \( W_{n-1} \) and from the exits property and the exits hypothesis for the labyrinth patterns in \( \tilde{A}_n \).

In order to prove that \( W_n \) has the tree property, we proceed analogously as in the case of mixed or self-similar labyrinth fractals. \( \mathcal{G}(W_n) \) is connected, by the connectedness of the tree \( \mathcal{G}(W_{n-1}) \), and by the exits property and the exits hypothesis for the patterns in \( \tilde{A}_n \). Now, we give an indirect proof for the fact that \( \mathcal{G}(W_n) \) has no cycles. Therefore, we assume that there is a cycle \( C = \{u_0, u_1, \ldots, u_r\} \) of minimal length in \( \mathcal{G}(W_n) \). For \( u \in \mathcal{V}(\mathcal{G}(W_n)) \) let \( w(u) \) be the white square in \( \mathcal{V}(\mathcal{G}(W_{n-1})) \) which contains \( u \) as a subset. Let \( j_0 = 0, v_0 = w(u_0), j_k = \min\{i : w(u_i) \neq v_{k-1}, j_{k-1} < i \leq r\} \), and \( v_k = w(u_{j_k}) \), for \( k \geq 1 \).

Let \( m \) be minimal such that the set \( \{i : w(u_i) \neq v_m, j_m < i \leq r\} \) is empty. Then we have, in \( \mathcal{G}(W_{n-1}), v_{i-1} \sim v_i \), for \( 1 \leq i \leq m \).

If \( \mathcal{G}(W_{n-1}) \) induced on the set \( \{v_0, v_1, \ldots, v_m\} \) contains a cycle in \( \mathcal{G}(W_{n-1}) \), this contradicts the induction hypothesis. Since by the tree property of labyrinth patterns \( \mathcal{G}(A_{n,j}) \) is a tree, for all \( A_{n,j} \in \tilde{A}_n \), it follows that not all white squares of the cycle \( C \) in \( \mathcal{G}(W_n) \) can be contained in \( v_0 \), wherefrom it follows that \( m \geq 1 \). Thus \( \mathcal{G}(W_n) \) induced on the set \( \{v_0, v_1, \ldots, v_m\} \) is a tree with more than one vertex. From the cycle-free mixing hypothesis we obtain the case \( m = 2 \) is not possible, thus \( m \geq 3 \). The fact that \( \mathcal{G}(W_n) \) induced on the set \( \{v_0, v_1, \ldots, v_m\} \) is a tree implies that, in order for the cycle \( C \) to exist in \( \mathcal{G}(W_n) \) there exists a square \( v_k \in W_{n-1} \), in the mentioned tree, such that in this square is “crossed” by two distinct paths in \( \mathcal{G}(W_n) \), i.e., there exist two disjoint paths \( p_1, p_2 \) in \( \mathcal{G}(W_n) \) that connect pairs of white squares of level \( n \) that lie on the same two (distinct) sides of \( v_k \), e.g., the top and bottom side, or the top and left side or, due to symmetry arguments, one can chose to any other such pair of sides. Since \( \tilde{A}_n \) consists only of labyrinth patterns, the graph \( \mathcal{G}(W_n) \) restricted to the squares that lie inside \( v_k \) is connected, and thus there has to exist a path in \( \mathcal{G}(W_n) \), inside \( v_k \), that connects a square of level \( n \) lying on \( p_1 \) with a square of level \( n \) that lies on \( p_2 \). This produces a new cycle, shorter than \( C \).

This contradicts the assumption that \( u_0, u_1, \ldots, u_r \) is a cycle of minimal length.

\( \square \)

\textbf{Remark 1.} The tree property implies the uniqueness of a cycle-
free path between any distinct (white) squares in $\mathcal{V}(G(A))$, for any labyrinth pattern $A$, and any distinct (white) squares in $\mathcal{V}(G(W_n))$, for any supermixed labyrinth set $W_n$ of level $n$.

**Definition 3.** For $n \geq 2$ we call $W_n$ an $m(n) \times m(n)$ supermixed labyrinth set (in short, supermixed labyrinth set) of level $n$, and the limit set

$$L_\infty = \bigcap_{n\geq 1} \bigcup_{W \in W_n} W$$

the supermixed labyrinth fractal generated by the sequence of collections of labyrinth patterns $(\tilde{A}_k)_{k \geq 1}$.

**Remark 2.** One can immediately see that supermixed labyrinth fractals generalise mixed labyrinth fractals and self-similar labyrinth fractals. If $s_k = 1$, for all $k \geq 1$, then $W_n$ is a mixed labyrinth set of level $n$, and $L_\infty$ a mixed labyrinth fractal, as defined in [5]. If we use only one pattern throughout the construction, we recover the self-similar case from [3, 4].

### 3 Topological properties of supermixed labyrinth fractals

Recall that for any sequence $\{\tilde{A}_k\}_{k=1}^\infty$ of collections of labyrinth patterns we assert Assumptions 1 from Section 2.

**Lemma 1.** Let $\{\tilde{A}_k\}_{k=1}^\infty$ be a sequence of collections of labyrinth patterns and let $n \geq 1$. Then, from every black square in $G(B_n)$ there is a path in $G(B_n)$ to a black square of level $n$ in $G(B_n)$.

**Proof.** The proof uses Proposition 1 and works in the same way as in the case of labyrinth sets occurring in the construction of self-similar labyrinth fractals [3, Lemma 2].

We state the following two results without the proof, since the proofs given in the self-similar case [3] work also in the more general case of the supermixed labyrinth sets. For more details and definitions we refer to the papers [3, 4].

**Lemma 2.** Let $\{\tilde{A}_k\}_{k=1}^\infty$ be a sequence of collections of labyrinth patterns. If $x$ is a point in $([0, 1] \times [0, 1]) \setminus L_n$, then there is an arc $a \subseteq ([0, 1] \times [0, 1]) \setminus L_{n+1}$ between $x$ and a point in the boundary $\text{fr}([0, 1] \times [0, 1])$. 

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Corollary 1. Let \( \{ \tilde{A}_k \}_{k=1}^\infty \) be a sequence of collections of labyrinth patterns and let \( n \geq 1 \). If \( x \) is a point in \( ([0,1] \times [0,1]) \setminus L_\infty \), then there is an arc \( a \subseteq ([0,1] \times [0,1]) \setminus L_\infty \) between \( x \) and a point in \( \text{fr}([0,1] \times [0,1]) \).

Now, let us recall that a continuum is a compact connected Hausdorff space, and a dendrite is a locally connected continuum that contains no simple closed curve.

Theorem 1. Any supermixed labyrinth fractal \( L_\infty \) generated by a sequence of collections of labyrinth patterns is a dendrite.

Proof. The proof, which uses the Hahn-Mazurkiewicz-Sierpiński theorem [12, Theorem 2, p.256] as well as the Jordan Curve theorem, is almost identical to the the proof of Theorem 1 in [5] but uses Corollary 1 in place of the corresponding result there. \( \square \)

As a consequence of the fact that \( L_\infty \) is a dendrite, for any two points \( x \neq y \) in \( L_\infty \) there exists a unique arc in \( L_\infty \) that connects them [12, Theorem 3, par. 47, V, p. 181]. Throughout this paper we denote by \( a(x,y) \) the unique arc in \( L_\infty \) with endpoints \( x \) and \( y \).

4 Exits

Let \( W_n^{\text{top}} \in W_n \) be the top exit of \( W_n \), for \( n \geq 1 \). We call \( \bigcap_{n=1}^\infty W_n^{\text{top}} \) the top exit of \( L_\infty \). The other exits of \( L_\infty \), \( W_n^{\text{bottom}}, W_n^{\text{left}}, W_n^{\text{right}} \), are defined analogously. We note that the exits property for \( W_n \) yields that \( (x,1), (x,0) \in L_\infty \) if and only if \( (x,1) \) is the top exit of \( L_\infty \) and \( (x,0) \) is the bottom exit of \( L_\infty \). The analogous holds also for the left and the right exit of the fractal.

Let \( n \geq 1, W \in W_n \), and \( t \) be the intersection of \( L_\infty \) with the top edge of \( W \). Then we call \( t(W) \) the top exit of \( W \). Analogously we define the bottom exit \( b(W) \), the left exit \( l(W) \) and the right exit \( r(W) \) of \( W \). Each of these four exits is unique, by the uniqueness of the four exits of a supermixed mixed labyrinth fractal and by the fact that each set of the form \( L_\infty \cap W \), with \( W \in W_n \), is a supermixed labyrinth fractal scaled by the factor \( m(n) \). Thus, we have defined the notion of exit for three different types of objects: for supermixed labyrinth sets of level \( n \), for \( L_\infty \), and for squares in \( W_n \), for \( n \geq 1 \).

The following result immediately follows from the construction of supermixed labyrinth fractals.
Proposition 2. Let \( \{ \tilde{A}_k \}_{k=1}^{\infty} \) be a sequence of collections of labyrinth patterns, as above.

(a) If \( (x_1^t, x_2^t) \) and \( (x_1^b, x_2^b) \) are the Cartesian coordinates of the top exit and bottom exit, respectively, in \( L_\infty \), then
\[
x_1^t = x_1^b = \sum_{k=1}^{\infty} \frac{c_k - 1}{m(k)}, \quad x_2^t = 1, \quad x_2^b = 0.
\]

(b) If \( x_1^l, x_2^l \) and \( x_1^r, x_2^r \) are the Cartesian coordinates of the left exit and right exit, respectively, in \( L_\infty \), then
\[
x_2^l = x_2^r = \sum_{k=1}^{\infty} \frac{r_k m_k - m_k}{m(k)}, \quad x_1^l = 0, \quad x_1^r = 1.
\]

5 Paths in supermixed labyrinth sets.
Path matrices and counting matrices

As in the setting of self-similar labyrinth fractals or mixed labyrinth fractals, the first step is to study paths in prefractals, i.e., in this case in (the graphs of) supermixed labyrinth sets. More precisely, we look at the construction method for paths between distinct exits of supermixed labyrinth sets. Sometimes we will just skip “supermixed” when it is understood from the context that we deal with supermixed objects.

Therefor, let us first introduce some notation. We call a path in \( G(W_n) \) a \( \mathbb{I} \)-path if it leads from the top to the bottom exit of \( W_n \). The \( \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{A}, \mathbb{U}, \mathbb{E} \)-paths lead from left to right, top to right, right to bottom, bottom to left, and left to top exits, respectively. Formally, we denote such a path by \( \text{path}_i(W_n) \), with \( i \in \{ \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{A}, \mathbb{U}, \mathbb{E} \} \). Let \( \mathbb{I}(n), \mathbb{M}(n), \mathbb{L}(n), \mathbb{A}(n), \mathbb{U}(n) \), and \( \mathbb{E}(n) \) be the length of the respective path in \( G(W_n) \), for \( n \geq 1 \), and \( \mathbb{I}_{k,h}, \mathbb{M}_{k,h}, \mathbb{L}_{k,h}, \mathbb{A}_{k,h}, \mathbb{U}_{k,h}, \mathbb{E}_{k,h} \) the length of the respective path in \( G(A_{k,h}) \), for \( k \geq 1 \) and \( 1 \leq h \leq s_k \). Formally, we denote such a path in (the graph of) this pattern by \( \text{path}_i(A_{k,h}) \), with \( i \in \{ \mathbb{I}, \mathbb{M}, \mathbb{L}, \mathbb{A}, \mathbb{U}, \mathbb{E} \} \). By the length of a path in a labyrinth set (of level \( n \)) or labyrinth pattern we mean the number of squares (of level \( n \)) in the path. For \( n = k = 1 \) the two path lengths coincide, i.e., \( \mathbb{I}(1) = \mathbb{I}_{1,1}, \ldots, \mathbb{E}(1) = \mathbb{E}_{4,1} \).

Both in the case of self-similar fractals \( [3, 4] \) and in the case of mixed labyrinth fractals \( [5] \) we used the same method for the construction of paths in prefractals. Before describing this construction
method for the case of supermixed labyrinth sets, let us remark some facts that play an essential role in the reasoning about paths in supermixed labyrinth sets and arcs in supermixed labyrinth fractals.

**Remark 3.** One can prove (e.g., by using an indirect proof) that if a square \( W \in \mathcal{V}(\mathcal{G}(W_n)) \) is one of the four exits of \( W_n \), then it has as a subset the square \( U \in \mathcal{V}(\mathcal{G}(W_{n+1})) \) which is the exit in \( W_{n+1} \) of the same type (top, bottom, left, or right) as \( W \) in \( W_n \). Moreover, from the tree property of \( \mathcal{G}(W_n) \), for all \( n \geq 1 \), it also follows that for any distinct exits \( W', W'' \) of \( W_n \), the corridor \( \Gamma(p_n(W', W'')) = \bigcup_{W \in p_n(W', W'')} W \) of the path \( p_n(W', W'') = \{W' = W_1, W_2, \ldots, W_r = W''\} \) in \( \mathcal{G}(W_n) \) contains as a subset the corridor \( \Gamma(p_{n+1}(U', U'')) = \bigcup_{U \in p_{n+1}(U', U'')} U \) of the path \( p_{n+1}(U', U'') = \{U' = U_1, U_2, \ldots, U_r = U''\} \) in \( W_{n+1} \), where \( U' \) and \( U'' \) is, respectively, the same type of exit in \( W_{n+1} \), as \( W' \) and \( W'' \) in \( W_n \). We remark that the purpose of the index \( n \) in the notation \( p_n(W', W'') \) for the path is to emphasise that the path is in the graph \( \mathcal{G}(W_n) \).

![Figure 3: Paths from top to bottom and from left to right exit of \( A_1 \)](image)

In the following, let us describe the construction of paths that connect exits of supermixed labyrinth sets of level \( n \geq 1 \). Therefor, let us consider, e.g., the path between the bottom and the right exit in \( W_n \), for some fixed \( n \geq 1 \). For any other pair of distinct exits in \( W_n \), the construction follows the same steps. Let us assume \( n \geq 2 \), since for \( n = 1 \) the mentioned path concides with the path from the bottom to the right exit of the pattern \( A_{1,1} \) and we are done.

The first step is to find the path between the bottom and the right exit of \( W_1 \), which is equivalent to constructing the path between the
Figure 4: Paths from top to right and from bottom to right exit of $A_1$

Figure 5: Paths from left to bottom and top to left exit of $A_1$

Figure 6: Paths from bottom to top and from bottom to right exit of $A_{2,1}$

bottom and the right exit of $A_{1,1}$. Now we denote each square in this path according to its neighbours within this path, i.e., we establish
Figure 7: Paths from left to right and from left to top exit of $A_{2,2}$

Figure 8: The path (in lighter grey) from the bottom to the right exit of the labyrinth set $W_2$ shown in Figure 2

its type: if it has a top and a bottom neighbour it is called a $\square$-square (with respect to the path), and it is called a $\mathcal{T}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{D}$-square if its neighbours are at left-right, top-right, bottom-right, left-bottom, and left-top, respectively. If a square in the mentioned path is an exit of the pattern (the labyrinth set), it is supposed to have a neighbour outside the side of the exit. A bottom exit, e.g., is supposed to have a neighbour below, outside the bottom, additionally to its inside neighbour. We repeat this procedure for all possible paths between two exits in $G(W_1)$, as shown in Figure 3, 4, and 5.

We introduce the notation $\mathcal{J} = \{\mathcal{T}, \mathcal{E}, \mathcal{L}, \mathcal{R}, \mathcal{B}, \mathcal{D}\}$.

Now, as a next step, in order to construct the $\mathcal{T}$-path in $G(W_2)$, which is shown in Figure 8, we replace each $j$-square $W$ of the $\mathcal{T}$-path
in $\mathcal{G}(W_1)$ with the $j$-path in $\mathcal{G}(\phi_2(W))$, where $j \in \mathcal{J}$. Some of the paths in the patterns in $\tilde{A}_2$ are shown in Figures 6 and 7. In general, for any pair of exits and $n \geq 1$, we replace each marked square $W$ in the path in $\mathcal{G}(W_n)$ with its corresponding path in the graph of the pattern $\phi_{n+1}(W)$ and thus obtain the path of the supermixed labyrinth set $\mathcal{G}(W_{n+1})$.

**Notation.** For any square $W$ in a given path between two exits of a labyrinth pattern or labyrinth set, if $W$ is a square of type $j \in \mathcal{J}$ within this path, we write $\text{type}(W) = j$.

We recall the definition of path matrices of labyrinth patterns or labyrinth sets of some level.

**Definition 4.** The path matrix of a labyrinth pattern $A$ is a non-negative $6 \times 6$ matrix whose rows and columns are both indexed according to the set $\mathcal{J} = \{\mathbb{1}, \mathbb{2}, \mathbb{3}, \mathbb{4}, \mathbb{5}, \mathbb{6}\}$ and whose entry on the position $(i,j) \in \mathcal{J} \times \mathcal{J}$ is the number of squares of type $j$ in the path of type $i$ in $\mathcal{G}(A)$. If $A = W_n$, then we obtain the path matrix of the labyrinth set $W_n$ of level $n \geq 1$.

For a sequence of collections of labyrinth patterns $\{\tilde{A}_k\}_{k \geq 1}$, with $\tilde{A}_k = \{A_{k,1}, \ldots, A_{k,s_k}\}$, for $k \geq 1$ and the corresponding sequence $\{W_n\}_{n \geq 1}$ of supermixed labyrinth sets of level $n$, we introduce in the following some more notation for matrices that are useful in the study of paths in supermixed labyrinth sets.

**Notation.** For all $n \geq 1$ we denote the path matrix of the labyrinth set $W_n$ by

$$M(W_n) := M(n) = (m^{(n)}_{i,j})_{i,j \in \mathcal{J}}.$$ 

For all $k \geq 1$ and $h = 1, \ldots, s_k$ we denote the path matrix of the labyrinth pattern $A_{k,h}$ by

$$M(A_{k,h}) := M_{k,h} = (m^{k,h}_{i,j})_{i,j \in \mathcal{J}}.$$ 

**Definition 5.** For $n \geq 1$ and $h \in \{1, \ldots, s_{n+1}\}$ the $h$-th counting matrix for step $n$, $Q_{n,h} = (q_{i,j}^{n,h})_{i,j \in \mathcal{J}}$, is defined by

$$q_{i,j}^{n,h} := \sum_{W \in \text{path}_i(W_n)} \mathbf{1}_{\phi_{n+1}(W) = A_{n+1,h}} \cdot \mathbf{1}_{\text{type}(W) = j}.$$ 

In the above formula, $\mathbf{1}$ denotes in each case the corresponding indicator function. In other words, the entry $q_{i,j}^{n,h}$ is the number of $j$-squares
in the path of type $i$ in $\mathcal{G}(W_n)$ which at the next step are “substituted” according to the pattern $A_{n+1,h}$, with $i, j, n, h$ as above.

The following proposition is an immediate consequence of the definitions of the path matrices and the construction method given at the beginning of this section.

**Proposition 3.** With the above notation we have

\[
\begin{pmatrix}
1
1
1
1
1
1
\end{pmatrix}
= M_{k,h} \cdot
\begin{pmatrix}
1
1
1
1
1
1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1
1
1
1
1
1
\end{pmatrix}
= M(n) \cdot
\begin{pmatrix}
1
1
1
1
1
1
\end{pmatrix}.
\] (3)

The next result establishes identities that describe the relations between path matrices of labyrinth patterns, supermixed labyrinth sets of some level and the counting matrices introduced before. In this theorem we extend results obtained for path matrices in the case of self-similar [3, 4] and mixed labyrinth fractals [5].

**Theorem 2.** With the above notation we have, for all $n \geq 1$,

\[
M(n) = Q_{n,1} + \cdots + Q_{n,s_{n+1}}
\] (4)

and

\[
M(n+1) = \sum_{h=1}^{s_{n+1}} Q_{n,h} \cdot M_{n+1,h}.
\] (5)

**Proof.** First, we prove the equality (4).

By the definition of the matrices $Q_{n,h}$, $h = 1, \ldots, s_{n+1}$, we have

\[
\sum_{h=1}^{s_{n+1}} q_{i,j}^{n,h} = \sum_{h=1}^{s_{n+1}} \sum_{W \in \text{path}_i(W_n)} 1_{\phi_{n+1}(W) = A_{n+1,h}} \cdot 1_{\text{type}(W) = j}
\]

\[
= \sum_{W \in \text{path}_i(W_n)} 1_{\text{type}(W) = j} \sum_{h=1}^{s_{n+1}} 1_{\phi_{n+1}(W) = A_{n+1,h}}
\]

\[
= \sum_{W \in \text{path}_i(W_n)} 1_{\text{type}(W) = j} = m_{i,j}^{(n)}.
\]
Now, in order to prove the formula \((5)\), let us start by computing, for 
\((i, j) \in J \times J\), the entry in the row \(i\) and column \(j\) of the \(6 \times 6\) matrix 
\[
\sum_{h=1}^{s_{n+1}} Q_{n,h} M_{n+1,h},
\]
i.e. the number
\[
\sum_{h=1}^{s_{n+1}} \sum_{\nu \in J} q_{i,\nu} \cdot m_{\nu,j}^{n+1,h}.
\]
By the definition of the counting matrix given in formula \((2)\) and the definition of \(m_{\nu,j}^{n+1,h}\), that can be expressed by the equation
\[
m_{\nu,j}^{n+1,h} = \sum_{W \in \text{path}_{\nu}(A_{n+1,h})} 1_{\text{type}(W)=j},
\]
for \(\nu, j \in J, n \geq 1, 1 \leq h \leq s_{n+1}\),
the above double sum equals
\[
\sum_{h=1}^{s_{n+1}} \sum_{\nu \in J} \sum_{W \in \text{path}_{\nu}(A_{n+1,h})} 1_{\phi_{n+1}(W)=A_{n+1,h}} \cdot 1_{\text{type}(W)=\nu} \sum_{W \in \text{path}_{\nu}(A_{n+1,h})} 1_{\text{type}(W)=j},
\]
Since, for all \(W \in \text{path}_{i}(W_{n})\),
\[
\sum_{\nu \in J} 1_{\text{type}(W)=\nu} \sum_{W \in \text{path}_{\nu}(A_{n+1,h})} 1_{\text{type}(W)=j} = m_{\nu,j}^{n+1,h},
\]
it follows that the above quadruple sum equals
\[
\sum_{h=1}^{s_{n+1}} \sum_{W \in \text{path}_{i}(W_{n})} 1_{\phi_{n+1}(W)=A_{n+1,h}} \cdot m_{\nu,j}^{n+1,h} = m_{i,j}^{(n+1)},
\]
by the definition of \(M(n+1)\) and the construction methods of paths in labyrinth sets of level \(n \geq 2\).

**Remark 4.** For \(s_{n+1} = 1\), (for some \(n \geq 1\), we have \(M(n) = Q_{n,1}\)
and thus in this case we recover the formula \(M(n+1) = M(n) \cdot M_{n+1}\) that holds for mixed labyrinth fractals \([5]\). Moreover, let us recall here, that throughout our consideration we have \(s_{1} = 1\), i.e., the first step of the construction is defined by exactly one pattern.

From the above theorem we immediately obtain the following recursion equation for the counting matrices.

**Corollary 2.** With the above notation, we define \(Q_{0,1} := I\), the identity \(6 \times 6\)-matrix, and for all \(n \geq 1\) we have
\[
Q_{n,1} + \cdots + Q_{n,s_{n+1}} = Q_{n-1,1} \cdot M_{n,1} + \cdots + Q_{n-1,s_{n}} \cdot M_{n,s_{n}}.
\]
Lemma 3. Let \( A \subseteq S_m \) be a labyrinth pattern (or \( A = W_n \) is a labyrinth set of level \( n \)) with exits \( W_{\text{top}}, W_{\text{bottom}}, W_{\text{left}}, W_{\text{right}} \). Then \( p(W_{\text{top}}, W_{\text{bottom}}) \cap p(W_{\text{left}}, W_{\text{right}}) \neq \emptyset \).

More precisely, one of the following cases can occur:

(a) \( p(W_{\text{top}}, W_{\text{bottom}}) \cap p(W_{\text{left}}, W_{\text{right}}) = \{V_1\} \), for some \( V_1 \in \mathcal{V}(G(A)) \).

(b) if there exist \( V_1, V_2 \in \mathcal{V}(G(A)) \), \( V_1 \neq V_2 \), with \( p(W_{\text{top}}, W_{\text{bottom}}) \cap p(W_{\text{left}}, W_{\text{right}}) \supseteq \{V_1, V_2\} \), then the path \( p(V_1, V_2) \) in \( G(A) \) satisfies \( p(V_1, V_2) \subseteq p(W_{\text{top}}, W_{\text{bottom}}) \) and \( p(V_1, V_2) \subseteq p(W_{\text{left}}, W_{\text{right}}) \).

Proof (Sketch). Let \( m \) be the width of \( A \) and let us consider the lattice \( L_m := \{1, \ldots, m\}^2 \). Then \( G(A) \) induces on \( L_m \) a subgraph \( L(A) \) in a natural way. In order to prove the first assertion of the lemma it is enough to show that a (cycle-free) path in \( L(A) \) that connects \((x, 1)\) with \((x, m)\) intersects any (cycle-free) path \( L_m \) that connects \((1, y)\) with \((m, y)\), where \( x, y \in \{1, \ldots, m\} \). We use the natural embedding of \( L_m \) and \( L(A) \) in the plane, scaled by factor \( 1/m \).

From this embedding we obtain two curves in the unit square, one leading from top to bottom and one leading from left to right, corresponding to the two mentioned paths in the induced graph. By \cite{15}[Lemma 2], these two curves intersect.

Since the intersection of the two corresponding paths in \( G(A) \) is non-empty, it can be either a single vertex, which yields case (a), or contain more than one vertex, which leads to case (b), where the assertion can be easily obtained by indirect proof, based on the fact that the graph of a labyrinth pattern (or set) is a tree. \( \square \)

The above result easily yields the following.

Lemma 4. In every labyrinth pattern \( A \subseteq S_m \) (or \( A = W_n \) labyrinth set of level \( n \)) with exits \( W_{\text{top}}, W_{\text{bottom}}, W_{\text{left}}, W_{\text{right}} \) the lengths of the paths in \( G(A) \) between exits satisfy

\[
\ell(p(W_{\text{top}}, W_{\text{bottom}})) + \ell(p(W_{\text{left}}, W_{\text{right}})) = \\
\max\{\ell(p(W_{\text{left}}, W_{\text{bottom}})) + \ell(p(W_{\text{top}}, W_{\text{right}})) , \ell(p(W_{\text{left}}, W_{\text{top}})) + \ell(p(W_{\text{bottom}}, W_{\text{right}}))\}
\]  

(7)
6 Arcs in supermixed labyrinth fractals

The following lemma establishes a connection between paths in supermixed labyrinth sets and arcs in labyrinth fractals. It provides tools for the construction and study of arcs in the fractal. Its proof works analogously as in the case of self-similar labyrinth fractals (see, e.g., [3, Lemma 6]) by using a theorem from Kuratowski’s book [12, Theorem 3, par. 47, V, p181].

Lemma 5 (Arc construction). Let \(a, b \in L_\infty\), with \(a \neq b\). For all \(n \geq 1\), there are \(W_n(a), W_n(b) \in \mathcal{V}(\mathcal{G}(W_n))\) such that

(a) \(W_1(a) \supseteq W_2(a) \supseteq \ldots\),
(b) \(W_1(b) \supseteq W_2(b) \supseteq \ldots\),
(c) \(\{a\} = \bigcap_{n=1}^\infty W_n(a)\),
(d) \(\{b\} = \bigcap_{n=1}^\infty W_n(b)\).

(e) The set \(\bigcap_{n=1}^\infty \left( \bigcup_{W \in p_n(W_n(a), W_n(b))} W \right)\) is an arc between \(a\) and \(b\) in \(L_\infty\).

The proof of the following two propositions works analogously to the case of mixed labyrinth fractals [5].

Proposition 4. Let \(n, k \geq 1\), \(\{W_1, \ldots, W_k\}\) be a (shortest) path between the exits \(W_1\) and \(W_k\) in \(\mathcal{G}(W_n)\), \(K_0 = W_1 \cap \text{fr}([0, 1] \times [0, 1])\), \(K_k = W_k \cap \text{fr}([0, 1] \times [0, 1])\), and \(c\) be a curve in \(L_n\) from a point of \(K_0\) to a point of \(K_k\). The length of any parametrisation of \(c\) is at least \((k - 1)/(2 \cdot m(n))\).

Proposition 5. Let \(e_1, e_2\) be two exits in \(L_\infty\), and \(W_n(e_1), W_n(e_2)\) be the exits in \(\mathcal{G}(W_n)\) of the same type as \(e_1\) and \(e_2\), respectively, for some \(n \geq 1\). If \(a\) is the arc that connects \(e_1\) and \(e_2\) in \(L_\infty\), \(p\) is the path in \(\mathcal{G}(W_n)\) from \(W_n(e_1)\) to \(W_n(e_2)\), and \(W \in W_n\) is a \(\mathbb{I}\)-square with respect to \(p\), then \(W \cap a\) is an arc in \(L_\infty\) between the top and the bottom exit of \(W\). If \(W\) is an other type of square, the corresponding analogous statement holds.

Based on an alternative definition of the box counting dimension \([7,\text{Definition 1.3.}]\) and making use of the property that one can use, instead of \(\delta \to 0\), appropriate sequences \((\delta_k)_{k \geq 0}\) for the computation of the box counting dimension (see the above reference), one can
show that the following result also holds in the context of supermixed labyrinth fractals.

**Proposition 6.** If $a$ is an arc between the top and the bottom exit in the supermixed labyrinth fractal $L_\infty$ then
\[
\liminf_{n \to \infty} \frac{\log(|I(n)|)}{\sum_{k=1}^{n} \log(m_k)} = \dim_B(a) \leq \dim_B(a) = \limsup_{n \to \infty} \frac{\log(|I(n)|)}{\sum_{k=1}^{n} \log(m_k)}.
\]

For the other pairs of exits, the analogous statement holds.

**Lemma 6.** Let $L_\infty$ be a supermixed labyrinth fractal with the top, bottom, left and right exit denoted by $t_\infty, b_\infty, l_\infty, r_\infty$, respectively. Then the arcs $a(t_\infty, b_\infty)$ and $a(l_\infty, r_\infty)$ in $L_\infty$ have non-empty intersection.

**Proof (idea).** One easy way to prove this is to apply [Lemma 2] to the arcs connecting exits of the supermixed labyrinth fractal that lie on opposite sides of the unit square.

With the above notations, we have:

**Lemma 7.** $a(t_\infty, b_\infty) \cap a(l_\infty, r_\infty)$ is either a point or a common subarc of $a(t_\infty, b_\infty)$ and $a(l_\infty, r_\infty)$ in the supermixed labyrinth fractal $L_\infty$.

**Proof.** We sketch the proof of a more general version of the lemma. We show that if $a_1, a_2$ are arcs in a dendrite $D$ and $a_1 \cap a_2 \neq \emptyset$ then $a_1 \cap a_2$ is a point or a (proper) arc such that $a \subseteq a_1$ and $a \subseteq a_2$.

Since $a_1 \cap a_2 \neq \emptyset$, there exists an $x \in D$ such that $x \in a_1 \cap a_2$. Assume now, there exists $y \in D$, $y \neq x$, such that $y \in a_1 \cap a_2$. This means that there exist arcs, $a_1(x, y) \subseteq a_1$ and $a_2(x, y) \subseteq a_2$, that connect $x$ and $y$ in the dendrite. Since the arc between any two distinct points in a dendrite is unique [12] Theorem 3, par. 47, V, p. 181], we must have $a_1(x, y) = a_2(x, y) = a(x, y)$, $a(x, y) \subseteq a_1$, $a(x, y) \subseteq a_2$.

The above lemma, and simple combinatorial arguments lead to the following result (see also Figure 9).

**Lemma 8.** Let $t_\infty, b_\infty, l_\infty, r_\infty$ be the exits of a supermixed labyrinth fractal $L_\infty$. If there exist $g_1, g_2$ in $L_\infty$ such that $g_1 \neq g_2$ and $g_1, g_2 \in a(t_\infty, b_\infty) \cap a(l_\infty, r_\infty)$, then, the following positions of the points $g_1$ and $g_2$ with respect to the four exits are possible (up to symmetry):
Figure 9: Relative positions of the exits in $L_\infty$ and the points $g_1, g_2$: the two cases described in Lemma 8.

(a) $g_1$ separates the points $t_\infty$ and $g_2$ on the arc $a(t_\infty, b_\infty)$ and $g_1$ separates the points $l_\infty$ and $g_2$ on the arc $a(l_\infty, r_\infty)$.

(b) $g_2$ separates the points $t_\infty$ and $g_1$ on the arc $a(t_\infty, b_\infty)$ and $g_1$ separates the points $l_\infty$ and $g_2$ on the arc $a(l_\infty, r_\infty)$.

The following proposition is an immediate consequence of the facts in the Lemmas 7 and 8.

**Proposition 7.** Let $L_\infty$ be a supermixed labyrinth fractal with the top, bottom, left and right exit denoted by $t_\infty, b_\infty, l_\infty, r_\infty$, respectively. Then

$$\ell(a(t_\infty, b_\infty)) + \ell(a(l_\infty, r_\infty)) = \max\{\ell(a(t_\infty, l_\infty)) + \ell(a(b_\infty, r_\infty)), \ell(a(t_\infty, r_\infty)) + \ell(a(l_\infty, b_\infty))\}. \quad (8)$$

### 7 Blocked labyrinth patterns

An $m \times m$-labyrinth pattern $A$ is called **horizontally blocked** if the row (of squares) from the left to the right exit contains at least one black square. It is called **vertically blocked** if the column (of squares) from the top to the bottom exit contains at least one black square. Analogously we define for any $n \geq 1$ a horizontally or vertically blocked labyrinth set of level $n$. One can easily check that horizontally or vertically blocked $m \times m$-labyrinth patterns only exist for $m \geq 4$. For
example, the labyrinth patterns shown in Figure 1 are horizontally and vertically blocked, while those in Figure 10 are not blocked.

![Labyrinth patterns example](image)

Figure 10: Examples of labyrinth patterns, that are neither horizontally nor vertically blocked

Let us recall some important results obtained in the case of self-similar and mixed labyrinth fractals constructed based on blocked labyrinth patterns.

In the self-similar case the following result was proven [4, Theorem 3.18]:

**Theorem 3.** Let $L_\infty$ be the (self-similar) labyrinth fractal generated by a horizontally and vertically blocked labyrinth pattern of width $m$.

with path matrix $M$ and $r$ be the spectral radius of $M$.

(a) Between any two points in $L_\infty$ there is a unique arc $a$.

(b) The length of $a$ is infinite.

(c) The set of all points, at which no tangent to $a$ exists, is dense in $a$.

(d) $\dim_B(a) = \frac{\log(r)}{\log(m)}$.

For the case of mixed labyrinth fractals, the following two results were proven in [6].

**Theorem 4.** There exist sequences $\{A_k\}_{k=1}^\infty$ of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set $L_\infty$ has the property that for any two points in $L_\infty$ the length of the arc $a \subset L_\infty$ that connects them is finite. For almost all points $x_0 \in a$ (with respect to the length) there exists the tangent at $x_0$ to the arc $a$.

**Proposition 8.** There exist sequences $\{A_k\}_{k=1}^\infty$ of (both horizontally and vertically) blocked labyrinth patterns, such that the limit set $L_\infty$ has the property that for any two points in $L_\infty$ the length of the arc $a \subset L_\infty$ that connects them is infinite.
At this point, we state a conjecture for the supermixed labyrinth fractals.

**Conjecture 1.** Let \( \{ \tilde{A}_k \}_{k \geq 1} \) be a sequence of collections of labyrinth patterns, such that for all \( k \geq 1 \) all patterns in \( \tilde{A}_k \) are horizontally and vertically blocked, and the width sequence \( \{ m_k \}_{k \geq 1} \) satisfies \( \sum_{k=1}^{\infty} \frac{1}{m_k} = \infty \). Then the supermixed labyrinth fractal \( L_\infty \) generated by the sequence \( \{ \tilde{A}_k \}_{k \geq 1} \) has the property that between any two distinct points in the fractal, the arc that connects them in the fractal has infinite length.

In the next section we give a proof of the result for the mixed case and we will highlight some of the difficulties that occur when attempting to generalise the method to the supermixed case.

In the remainder of this section we recall some facts from [4] about path matrices of blocked labyrinth patterns.

**Lemma 9** (Lemma 3.3 in [4]). Let \( M = (m_{i,j})_{i,j \in J} \) be the path matrix of a blocked labyrinth pattern. Then

(a) \( m_{1,1} = m_{1,1} \) and \( m_{1,1} = m_{1,1} \).
(b) \( m_{-,-} = m_{-,-} \) and \( m_{-,-} = m_{-,-} \).
(c) \( m_{-L} + m_{L} = m_{-L} + m_{L} \) and \( m_{-R} + m_{R} = m_{-R} + m_{R} \).
(d) \( m_{-L} + m_{L} = m_{-L} + m_{L} \) and \( m_{-R} + m_{R} = m_{-R} + m_{R} \).

**Lemma 10** (Lemma 3.4 in [4]). Let \( M = (m_{i,j})_{i,j \in J} \) be the path matrix of a blocked labyrinth pattern. Then

\( m_{i,j} \geq 1 \) for \( j \in \{L,L,R,R\} \) and \( m_{i,j} \geq 1 \) for \( j \in \{-L,-L,-R,-R\} \).

**Lemma 11** (Lemma 3.5 in [4]). Let \( M = (m_{i,j})_{i,j \in J} \) be the path matrix of a blocked labyrinth pattern. Then

\[
\begin{align*}
m_{L,L} & \geq 1 \text{ or } m_{L,L} \geq 1, \\
m_{R,R} & \geq 1 \text{ or } m_{R,R} \geq 1, \\
m_{L,-L} & \geq 1 \text{ or } m_{L,-L} \geq 1, \\
m_{R,-R} & \geq 1 \text{ or } m_{R,-R} \geq 1.
\end{align*}
\]

Since any labyrinth set of level \( n \geq 1 \) can be viewed as a labyrinth pattern, the above lemma also holds for any (blocked) labyrinth set of some level \( n \geq 1 \).
8 Arcs of infinite length: the mixed case

In this section, we prove the following theorem, which is a special case of Conjecture 1 and which solves the conjecture on mixed labyrinth fractals formulated in recent work [6].

**Theorem 5.** Let \( \{A_k\}_{k \geq 1} \) be a sequence of horizontally and vertically blocked labyrinth patterns, such that the corresponding sequence of widths \( \{m_k\}_{k \geq 1} \) satisfies the condition \( \sum_{k \geq 1} \frac{1}{m_k} = \infty \). Then, for all \( x, y \in L_\infty \) with \( x \neq y \) the arc in \( L_\infty \) that connects \( x \) and \( y \) has infinite length.

For this we recall the concept of a reduced path matrix of a labyrinth pattern (or set) as defined in [4].

Let \( A \subset S_m \) be a labyrinth pattern and \( M \) its path matrix. Then the reduced path matrix \( \overline{M} \) of \( A \) is a \( 4 \times 4 \) matrix, obtained from \( M \) as follows: first, we add the fifth row to the third row of \( M \), and then we add the sixth row to the fourth row of \( M \) und thus obtain a \( 4 \times 6 \) matrix, from which we delete the last two columns to obtain \( \overline{M} \). Since any labyrinth set of some level \( n \) (mixed or supermixed or generated by only one labyrinth pattern) can be viewed as a labyrinth pattern, the definition of a reduced path matrix works for any labyrinth set of some level \( n \geq 1 \). According to the way it is constructed, the rows of a reduced path matrix are indexed by \( \{\bl, \bs, \bw, \br\} \) (in this order), and its columns are indexed by \( \{\bl, \bs, \bw, \br\} \) (in this order).

**Lemma 12.** Let \( M_1, M_2 \) be two path matrices of labyrinth patterns. Then

\[
\overline{M}_1 \cdot \overline{M}_2 = \overline{M}_1 \cdot \overline{M}_2.
\]

**Proof.** In order to prove this, we use the definitions of the path matrix and the reduced path matrix of a labyrinth pattern. Therefor, let \( M_1 = (m^1_{i,j})_{i,j \in J} \) and \( M_2 = (m^2_{i,j})_{i,j \in J} \) be path matrices of labyrinth patterns, \( \overline{M}_1 = (\overline{m}^1_{i,j})_{i,j \in J \times J'} \), \( \overline{M}_2 = (\overline{m}^2_{i,j})_{i,j \in J \times J'} \), and \( \overline{M}_1 \cdot \overline{M}_2 = (\overline{m}_{i,j})_{i,j \in J \times J'} \), where \( J = \{\bl, \bs, \bw, \br\} \) and \( J' = \{\bl, \bs, \bw, \br\} \) are sets of indices. The idea is to prove that for all \( (i, j) \in J \times J' \)

\[
\overline{m}_{i,j} = \sum_{(r,r') \in \{(\bl,\bl), (\bs,\bs), (\bw,\bw), (\br,\br)\}} \overline{m}^1_{i,r} \cdot \overline{m}^2_{r',j}.
\]
Due to symmetry, it is enough to consider four cases corresponding, e.g., to \((i,j) \in \{((1,1),(1,5)),((5,1),(5,5))\}\). For all these cases \((9)\) is obtained from the definition of products of matrices, by applying the results from Lemma 9.

**Remark 5.** The above result can also be proven by using the following arguments. From Lemma 9 we know that any path matrix \(M\) is of the form

\[
M = \begin{pmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} & m_{34} + a & m_{34} + b \\
m_{41} & m_{42} & m_{43} & m_{44} & m_{43} + c & m_{44} + d \\
m_{51} & m_{52} & m_{53} & m_{54} & m_{53} - a & m_{54} - b \\
m_{61} & m_{62} & m_{63} & m_{64} & m_{63} - c & m_{64} - d
\end{pmatrix},
\]

with integers \(m_{i,j} i, j \in \{1, \ldots, 6\}\) and \(a, b, c, d\). Further note from the definition of the reduced path matrix that \(M = P_LMP_R\) with

\[
P_L := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}, \quad P_R := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Now it is straightforward to verify (for example using a computer algebra system) that \(P_LM_1M_2P_R = P_LM_1P_RP_LM_2P_R\) for any two matrices \(M_1, M_2\) having this form.

Lemma 12 yields a simple formula for the reduced path matrix of mixed labyrinth sets:

**Corollary 3.** The reduced path matrix of a mixed labyrinth set of level \(n\) is the product of the reduced path matrices of the patterns that define it, i.e.,

\[
\overline{M(n)} = \prod_{k=1}^{n} M_k,
\]

where \(M(n) = \prod_{k=1}^{n} M_k\), and \(M_k\) is the path matrix of the pattern \(A_k\) that defines the \(k\)-th step of the construction, \(k \geq 1\).
For any integer $m \geq 4$, we define the virtual reduced path matrix

$$L(m) := \begin{pmatrix}
m - 2 & 0 & 1 & 1 \\
0 & m - 2 & 1 & 1 \\
1 & 1 & m - 1 & 0 \\
1 & 1 & 0 & m - 1
\end{pmatrix}.$$  

We use the attribute “virtual” because there need not exist a pattern to which a given virtual reduced path matrix $L(m)$ corresponds. The significance of these matrices is that they provide us with lower bounds on “real” reduced path matrices.

**Proposition 9.** Let $M$ be the matrix of a horizontally and vertically blocked labyrinth pattern of width $m \geq 4$ and $L(m)$ the virtual reduced path matrix with parameter $m$. Furthermore, let $0 < c < 1$.

Then we have, elementwise,

$$M(1, 1, 1 + c, 1 + c)_{\text{top}} \geq L(m)(1, 1, 1 + c, 1 + c)_{\text{top}}. \quad (11)$$

**Proof.** Due to symmetry it is enough to prove the inequalities for the first and third entry of the vectors resulting from this multiplication.

We introduce some temporary notation: for squares along a path from one exit to another one we discriminate between the directions in which the square may be passed. Thus we introduce the corresponding “oriented” types of squares $\mathbf{I}, \mathbf{R}, \mathbf{A}, \mathbf{S}$, and we write $m[I]$ for the number of $\mathbf{I}$-squares in the path from top to bottom and similar for the other symbols.

Given a path from the top exit to the bottom exit, i.e., a path of type $\mathbf{I}$, we count the directed square types along the path. Starting at the top exit, each square of type $\mathbf{I}$ displaces us by one unit down, zero units to the left or right. Each square of type $\mathbf{R}$ displaces us by $1/2$ unit down, $1/2$ units to the right. Whatever the shape of the path, finally we must have a total displacement by $m$ units down and zero units to the left or right. In other words, we may assign a 2-dimensional vector to each of the square types, where $v(\mathbf{I}) = (0, -1)$, $v(\mathbf{R}) = (1/2, -1/2)$ and so on. Of course, this assignment is not injective.

Let $p = 1/\log_2(1 + c)$, and define $||(x, y)||_p := (|x|^p + |y|^p)^{1/p}$, for every vector with real entries $(x, y)$. Note that for this choice of $p$ we have $||(1, 1)||_p = 1 + c$.

Consider a path from top to bottom, consisting of the squares $s_1, \ldots, s_n$. Since the total displacement from top to bottom is $(0, -m)$, we need to have $\sum_{k=1}^n v(s_k) = (0, -m)$. 

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The first entry of $\mathbf{M}(1, 1, 1 + c, 1 + c)^\top$ equals
\[
m_{1,1} + m_{1,2} + (1 + c)(m_{1,3} + m_{1,4})
= m_{1,1} + m_{1,2} + \frac{1}{2}(1 + c)(m_{1,3} + m_{1,4} + m_{1,5} + m_{1,6})
= m_{1,1} + m_{1,2} + \frac{1}{2}(1 + c)(m_{1,3} + m_{1,4} + m_{1,5} + m_{1,6})
+ m_{1,1} + m_{1,2} + \frac{1}{2}(1 + c)(m_{1,3} + m_{1,4} + m_{1,5} + m_{1,6})
= \sum_{k=1}^{n} \|v(s_k)\|_p.
\]
(Here we have used that $m_{1,3} = m_{1,4}$ and $m_{1,5} = m_{1,6}$, from Lemma [9].)

From Lemma [10] we know that the path of type $1$ in a blocked labyrinth pattern (or set) has to contain at least one square of type $3$.

Now, since the path is “oriented”, it induces an orientation on those squares, so in the path we have at least one occurrence of $3$ or of $4$.

That is, there exists $k_1 \in \{1, \ldots, n\}$ such that $\text{type}(s_{k_1}) \in \{3, 4\}$. In the same way, there exist $k_2, k_3, k_3 \in \{1, \ldots, n\}$ such that $\text{type}(s_{k_2}) \in \{3, 4\}, \text{type}(s_{k_3}) \in \{3, 4\}, \text{type}(s_{k_4}) \in \{3, 4\}$.

Note that $\sum_{k=1}^{n} v(s_k) = (0, -m)$ and that $\sum_{k \in \{k_1, k_2, k_3, k_4\}} \|v(s_k)\|_p = 2(1 + c)$. Furthermore, it is straightforward to check that
\[
\sum_{k \in \{k_1, k_2, k_3, k_4\}} v(s_k) \in \{(0, 0), \pm(0, 2), \pm(2, 0), \pm(1, 1), \pm(1, -1)\}
\]
and thus $\left\| \sum_{k \in \{k_1, k_2, k_3, k_4\}} v(s_k) \right\|_p \leq 2$. By the triangle inequality for the $p$-norm,
\[
\sum_{k=1}^{n} \|v(s_k)\|_p \geq \sum_{k \in \{k_1, k_2, k_3, k_4\}} \|v(s_k)\|_p + \sum_{k \in \{k_1, k_2, k_3, k_4\}} v(s_k) \|_p
\geq 2(1 + c) + \left\| (0, -m) - \sum_{k \in \{k_1, k_2, k_3, k_4\}} v(s_k) \right\|_p
\geq 2(1 + c) + \| (0, -m) \|_p - \sum_{k \in \{k_1, k_2, k_3, k_4\}} v(s_k) \|_p
\geq 2(1 + c) + m - 2 = m + 2c.
\]
But the last expression is just the first entry of $L(m)(1, 1, 1 + c, 1 + c)^\top$.

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By symmetry, we get the inequality for the second entry.

Next we consider the concatenation of a path from top to right with that from left to bottom. An example of such a concatenation is illustrated in Figure 11.

![Figure 11: The concatenation of paths in two adjacent copies of the pattern (labyrinth set of level 1): from top to right and from left to bottom](image)

Let the combined path again consist of “directed squares” $s_{k_1}, \ldots, s_{k_n}$. The third entry of $\overline{M}(1, 1 + c, 1 + c)^\top$ equals

\[
m_{s_{k_1}} + m_{s_{k_2}} + (1 + c)(m_{s_{k_1}} + m_{s_{k_2}}) = m_{s_{k_1}} + m_{s_{k_2}} + \frac{1}{2}(1 + c)(m_{s_{k_1}} + m_{s_{k_2}} + m_{s_{k_1}} + m_{s_{k_2}}) + m_{s_{k_1}} + m_{s_{k_2}} + \frac{1}{2}(1 + c)(m_{s_{k_1}} + m_{s_{k_2}} + m_{s_{k_1}} + m_{s_{k_2}})
\]

\[= \sum_{k=1}^n \|v(s_k)\|_p.
\]

When we pass from the first to the second line in the above formula, we apply the assertion 3 in Lemma 9c. By the facts in Lemma 11 adapted to the case of “oriented” paths and squares, it follows that at least one of the squares in a “concatenated” path is of type $\text{I}$ or $\text{II}$. Thus there exists $k_1 \in \{1, \ldots, n\}$ such that $\text{type}(s_{k_1}) \in \{\text{I}, \text{II}\}$. In the same way there exists $k_1 \in \{1, \ldots, n\}$ such that $\text{type}(s_{k_2}) \in \{\text{I}, \text{II}\}$.
Note that $\sum_{k=1}^{n} v(s_k) = (m, -m)$, such that $\| \sum_{k=1}^{n} v(s_k) \|_p = (1 + c)m$, and that $\|v(s_{k_1})\|_p + \|v(s_{k_2})\|_p = 2$ and $\|v(s_{k_1}) + v(s_{k_2})\|_p = 1 + c$.

Thus, by the triangle inequality for the $p$-norm,

$$\sum_{k=1}^{n} \|v(s_k)\|_p \geq \|s_{k_1}\|_p + \|s_{k_2}\|_p + \left\| \sum_{k \notin \{k_1, k_2\}} v(s_k) \right\|_p$$

$$= 2 + \left\| \sum_{k=1}^{n} v(s_k) - v(s_{k_1}) - v(s_{k_2}) \right\|_p$$

$$\geq 2 + \left\| \sum_{k=1}^{n} v(s_k) \right\|_p - \|v(s_{k_1}) + v(s_{k_2})\|_p$$

$$= 2 + (1 + c)(m - 1).$$

But the last expression equals the third entry of $L(m)(1, 1, 1+c, 1+c)^\top$.

By symmetry, we get the inequality for the fourth entry. This completes the proof. \qed

**Lemma 13.** For every $c \in (0, 1)$ there exists $\kappa > 0$ such that

$$L(m)(1, 1, 1+c, 1+c)^\top \geq m(1 + \kappa/m)(1, 1, 1+c, 1+c)^\top,$$

elementwise, for all $m \geq 4$.

**Proof.** The assertion holds true for $\kappa$ if

$$\begin{pmatrix}
m + 2c \\
m + 2c \\
(1+c)m + (1 - c) \\
(1+c)m + (1 - c)
\end{pmatrix} \geq m(1 + \kappa/m) \begin{pmatrix} 1 \\ 1 \\ 1 + c \\ 1 + c \end{pmatrix},$$

elementwise, i.e., if $m + 2c \geq m(1 + \kappa/m)$ and $(1 + c)m + (1 - c) \geq m(1 + \kappa/m)(1 + c)$. This is the case if $\kappa \leq \min(2c, \frac{1-c}{1+c})$. Thus, setting $\kappa = \min(2c, \frac{1-c}{1+c}) > 0$ gives the result. \qed

Since $\overline{M}$ has positive entries we have, elementwise,

$$\frac{1}{m} \overline{M} \lambda \geq \frac{\min_i \lambda_i}{1 + c} \frac{1}{m} \overline{M}(1, 1, 1+c, 1+c)^\top,$$

for all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^\top$ with positive real entries and all $0 < c < 1$.

We can choose $c$ to maximise $\min(2c, \frac{1-c}{1+c})$. The maximiser satisfies $2c = \frac{1-c}{1+c}$, such that $c = (\sqrt{17} - 3)/4 \approx 0.14$, and thus $\kappa = \frac{29}{31}$.
sup_{c \in (0,1)} \min(2c, \frac{1-c}{1+c}) = (\sqrt{17} - 3)/2 \approx 0.28.

Together with (12), we therefore get, for these values of $c$ and $\kappa$,

$$\frac{1}{m} \overline{M} \lambda \geq \frac{\min_i \lambda_i}{1+c} (1+\kappa/m)(1,1,1+c,1+c)^\top,$$ elementwise.

**Lemma 14.** Let $A_1, A_2, \ldots$ be a sequence of patterns with corresponding path matrices $M_1, M_2, \ldots$ and widths $m_1, m_2, \ldots$, respectively.

If $\sum_{k=1}^{\infty} \frac{1}{m_k} = \infty$ then we have, elementwise:

$$\sup_n \prod_{k=1}^{n} \left( \frac{1}{m_k} M_k \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \\ 1+c \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \\ \infty \\ \infty \end{pmatrix}.$$

**Proof.** Let again $c = (\sqrt{17} - 3)/4$ and $\kappa = 2c$, and let $K$ be an integer, $K > 1$. Then, for every vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^\top$ with positive entries,

$$\frac{1}{m_K} \overline{M} \lambda \geq \frac{\min_i \lambda_i}{1+c} (1+\kappa/m_K)(1,1,1+c,1+c)^\top.$$

Now, for every $1 \leq k \leq K - 1$ we have, by Proposition 9 and Lemma 13 elementwise,

$$\left( \frac{1}{m_k} M_k \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \end{pmatrix} \geq \left( \frac{1}{m_k} L(m_k) \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \end{pmatrix} \geq \left( 1 + \frac{\kappa}{m_k} \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \end{pmatrix},$$

and thus we obtain, for all vectors $\lambda$ with positive entries,

$$\prod_{k=1}^{K} \left( \frac{1}{m_k} M_k \right) \lambda \geq \left( \frac{\min_i \lambda_i}{1+c} \prod_{k=1}^{K} \left( 1 + \frac{\kappa}{m_k} \right) \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \end{pmatrix},$$ elementwise.

Using the above inequality with $\lambda = (1,1,1,1)^\top$, we have $\min_i \lambda_i = 1$ and therefore (elementwise) for any $K \geq 1$

$$\sup_n \prod_{k=1}^{n} \left( \frac{1}{m_k} M_k \right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1+c \\ 1+c \end{pmatrix} \geq \frac{1}{1+c} \prod_{k=1}^{K} \left( 1 + \frac{\kappa}{m_k} \right) \begin{pmatrix} 1 \\ 1+c \\ 1+c \end{pmatrix}.$$
From the lemma’s hypothesis \( \sum_{k \geq 1} \frac{1}{m_k} = \infty \) and thus, by known facts from calculus, \( \sup_K \prod_{k=1}^{K} (1 + \frac{\kappa}{m_k}) = \infty \), which, together with taking the limit \( K \to \infty \), yields, elementwise,

\[
\sup_n \prod_{k=1}^{n} \left( \frac{1}{m_k M_k} \right) \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} = \begin{pmatrix}
\infty \\
\infty \\
\infty \\
\infty
\end{pmatrix}.
\]

\[\Box\]

Proof of Theorem 5. By Lemma 14 and Proposition 4 it follows that in \( L_\infty \) we have

\[
\ell(a(t_\infty, b_\infty)) = \ell(a(l_\infty, r_\infty)) = \ell(a(t_\infty, r_\infty)) + \ell(a(l_\infty, b_\infty)) = \ell(a(l_\infty, t_\infty)) + \ell(a(b_\infty, r_\infty)) = \infty.
\]

The next step is to show that, under the theorem’s assumptions, all arcs between exits of the labyrinth fractal have infinite length, i.e.,

\[
\ell(a(t_\infty, r_\infty)) = \ell(a(l_\infty, b_\infty)) = \ell(a(l_\infty, t_\infty)) = \ell(a(b_\infty, r_\infty)) = \infty.
\]

Since \( \ell(a(t_\infty, r_\infty)) + \ell(a(l_\infty, b_\infty)) = \infty \), at least one of the arc lengths in the sum in infinite, therefore let us assume w.l.o.g that \( \ell(a(t_\infty, r_\infty)) = \infty \). Now, let us consider the arc \( a(l_\infty, b_\infty) \). Due to the corner property of labyrinth patterns, this arc cannot be totally contained in only one square of level 1, which in this case would be a square \( W \in V(G(W_1)) \) of type \( 3 \) in the path in \( G(W_1) \) from the left to the bottom exit of \( W_1 \). It follows that the path from the left exit to the bottom exit of \( W_1 \), contains, in addition to a square of type \( 3 \), at least one square of type \( 1 \) or \( 2 \). Let us assume, w.l.o.g., it contains a square of type \( 1 \).

Now, let us consider \( L'_\infty \), the mixed labyrinth fractal defined by the sequence of patterns \( A'_1, A'_2, \ldots \), where \( A'_k = A_{k+1} \), for all \( k \geq 1 \), with width sequence \( \{m'_k\}_{k \geq 1} \). We get \( \sum_{k \geq 1} \frac{1}{m'_k} = \infty \) and \( L'_\infty \) satisfies the assumptions of the theorem. On the other hand, \( L_\infty \) is a finite union of copies (scaled by factor \( 1/m_1 \)) of \( L'_\infty \). Moreover, the arc \( a(l_\infty, b_\infty) \) in \( L_\infty \) is a union of arcs between exits in some of the mentioned copies of \( L'_\infty \). Since one of the squares in the path of type \( 3 \)
in $G(W_1)$ is of type $\mathbf{I}$, it follows by [5] Proposition 3] or by Proposition 5 applied to mixed labyrinth fractals, that the arc $a(l_\infty, b_\infty)$ in $L_\infty$ has as subarc a copy (scaled by factor $1/m_1$) of the arc between the left and bottom exits in $L'_\infty$, which has infinite length. Therefore, it follows that $\ell(a(l_\infty, b_\infty)) = \infty$. Analogously one can prove that $\ell(a(l_\infty, t_\infty)) = \ell(a(b_\infty, r_\infty)) = \infty$.

Finally, by arguments analogous to those used in the self-similar case [4, Theorem 3.18], it follows that the arc between any two distinct points in the fractal $L_\infty$ has infinite length. 

\textbf{Remark 6.} The condition $\sum_{k \geq 1} \frac{1}{m_k} = \infty$ in Theorem 5 is not a necessary condition for a sequence of blocked labyrinth patterns to generate a fractal $L_\infty$ with the property that the length of any arc that connects distinct points in $L_\infty$ is infinite. This is shown, e.g., by the following example. Let $L_\infty$ be a self-similar labyrinth fractal generated by a horizontally and vertically blocked $m \times m$-pattern $A$. The fractal can also be viewed as being a mixed labyrinth fractal generated by the sequence of patterns $\{A_k\}_{k \geq 1}$, with $A_1 = A$, and $A_k = W_k(A)$, for $k \geq 2$, where $W_k(A)$ denotes the labyrinth set of level $k$ obtained in the construction of the self-similar fractal $L_\infty$. Thus, $m_k = m^k$, for all $k \geq 1$ and thus $\sum_{k \geq 1} \frac{1}{m_k} < \infty$. On the other hand, Theorem 3 applied for the self-similar labyrinth fractal $L_\infty$ yields that for every two distinct points in the fractal the length of the arc that connects them has infinite length.

\textbf{Remark 7.} For supermixed labyrinth sets we have no result corresponding to Corollary 3. That is, we have no “simple” formula for the reduced path matrix of supermixed labyrinth sets. This is one of the reasons why the method used in the proof of Theorem 5 does not work in the supermixed case.

\textbf{Remark 8.} Note that from Lemma 11 it follows trivially that $m_{\mathbf{E}1} + m_{\mathbf{F}1} \geq 1$. We discuss a geometric consequence of this:

Let $M$ be the path matrix of the pattern $A_2$ and let $M_1$ be the path matrix of the pattern $A_1$. Let us consider a path between exits in $W_2$ and also the corresponding path in $W_1$ (so that the path in $W_2$ lies entirely in the one in $W_1$).

For every pair of squares in this path in $W_1$ such that one is of type $\mathbf{E}$ and the other is of type $\mathbf{F}$, we know that there exist at least one square of type $\mathbf{E}$ and one square of type $\mathbf{F}$ in the path in $W_2$. This consideration plays an essential role in the proof of Proposition 9 which in turn is used in the proof of Theorem 5.
However, this need not hold in the supermixed case. In the following example, the path segment that connects the top exit of the left pattern to the bottom exit of the right pattern, substituted in adjacent squares of $W_1$ (see figure 12) contains neither a square of type $\square$ nor a square of type $\square$.

![Figure 12: The concatenation of paths in two adjacent copies of distinct labyrinth patterns (labyrinth set of level 1): from top to right and from left to bottom](image)

9 Conclusions

We have presented a new class of planar dendrites, termed supermixed labyrinth fractals, which generalises the established concept of (mixed) labyrinth fractal. We proved several useful results about these objects, in particular we were able to find a recursive formula for their path matrix, Theorem 2, which makes them amenable to concrete computation of discrete path lengths and related quantities. In addition to generalising earlier results on the topology of (mixed) labyrinth fractals, some of our results provide new insight also into the more special cases of self-similar and mixed labyrinth fractals. E.g., we establish important relations between the length of arcs between exits of the fractal and special subarcs of the fractal.

One of the main results is Theorem 5, were we give a sufficient condition for the infinite length of any arc in a mixed labyrinth fractal. Even though the method of proof is not strong enough to provide the same result for the supermixed case, we believe that it also constitutes
a valuable step towards the more general result, which therefore still remains open.

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