Mathematical modeling of intrinsic Josephson junctions with capacitive and inductive couplings

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Abstract. We investigate the current voltage characteristics (CVC) of intrinsic Josephson junctions (IJJs) with two types of couplings between junctions: capacitive and inductive. The IJJ model is described by a system of coupled sine-Gordon equations which is solved numerically by the 4th order Runge-Kutta method. The method of numerical simulation and numerical results are presented. The magnetic field distribution is calculated as the function of coordinate and time at different values of the bias current. The influence of model parameters on the CVC is studied. The behavior of the IJJ in dependence on coupling parameters is discussed.

1. Introduction

Intrinsic Josephson effect arising in tunneling of Cooper pairs between superconducting layers CuO\textsubscript{2} inside of HTSC [1] gives a base to consider the HTSC as a system of coupled Josephson junctions (JJ). A stack of IJJ is one of the promising objects of superconducting electronics and studied intensively now [2, 3, 4, 5]. Recently discovered coherent electromagnetic radiation from the IJJ stack in the terahertz frequency range [6] opens a possibility for various applications. The coupled JJ system is described by the system of nonlinear and nonequilibrium equations [7, 8, 9, 10] which should be solved numerically. An important problem is related to the coupling between JJ [8, 9].

In this paper we study the CVCs and spatiotemporal features of long Josephson junctions with inductive and capacitive couplings within the framework of the model proposed by Machida and Sakai [9].

2. Model and method of simulation

We consider the IJJ stack of a length \( L \) where the width of the junction is much less than Josephson penetration depth \( \lambda_j \). The superconducting and insulating layers have thicknesses \( d_s \) and \( d_i \), respectively. Bias current \( I \) flows perpendicularly to the layers. The \( l \)-th Josephson junction is formed by \( l \)-th and \( l-1 \)-th superconducting layers. It is described by the gauge-invariant phase difference \( \varphi_l(t) \). The system of equations describing the phase dynamics of Josephson junctions with inductive and capacitive couplings has a form
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\[
\begin{aligned}
\frac{\partial \varphi_i}{\partial \tau} &= \sum_{j=1}^{N} C_{ij} V_j \\
\frac{\partial V_i}{\partial \tau} &= \sum_{j=1}^{N} \left( \mathcal{L}_{ij}^{-1} \frac{\partial^2 \varphi_j}{\partial x^2} \right) - \beta V_i - \sin \varphi_i + I
\end{aligned}
\]  

(1)

where \( V_i \) is the dimensionless voltage of \( i \)-th JJ normalized to \( V_0 = \hbar \omega_p/(2e) \), \( \hbar \) – Planck’s constant, \( e \) – charge of electron, \( \omega_p = \sqrt{8\pi d_i e c / \hbar e} \) – the plasma frequency, \( j_c \) – the critical current density, \( \varepsilon \) – the dielectric constant of insulating layer, \( \beta = \sigma V_0 / (\delta j c) \) – the dissipation parameter, \( \sigma \) – the electrical conductivity, \( I \) is the bias current normalized to \( j_c \).

The matrices \( C \) and \( \mathcal{L} \) describe the capacitive and inductive couplings, respectively and have the following form:

\[
C = \begin{pmatrix}
D^C & s^C & 0 & \ldots & s^C \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\ldots & 0 & s^C & D^C & s^C & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^C & \ldots & 0 & s^C & D^C
\end{pmatrix}
\quad \text{and} \quad
\mathcal{L} = \begin{pmatrix}
1 & S & 0 & \ldots & S \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\ldots & 0 & S & 1 & S & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
S & \ldots & 0 & S & 1
\end{pmatrix}
\]

where \( D^C = 1 + (2\lambda_e / d_i) \text{coth}(d_i / \lambda_e) \) and \( s^C = -\lambda_c / \text{sinh}(d_i / \lambda_c) \) are effective electrical length and capacitive coupling parameter, respectively normalized to \( d_i \), \( \lambda_c \) – Debye screening length, \( S = s^L / D^L \) – normalized inductive coupling parameter, \( D^L = d_i + 2\lambda_L \text{coth}(d_i / \lambda_L) \) – effective magnetic thickness, \( s^L = -\lambda_L / \text{sinh}(d_i / \lambda_L) \) and \( \lambda_L \) – London penetration depths.

Coordinate \( x \) is normalized to the Josephson penetration depth \( \lambda_j = \sqrt{\hbar c^2 / (8\pi e c \lambda L)} \) and time \( \tau \) is normalized to \( \omega_p^{-1} \).

For numerical simulation of the system (1) we use two independent numerical methods. First method is based on the standard explicit finite-difference algorithm. Within the framework of another approach we employ the 2nd order finite-difference approximation of (1) only in the coordinate \( x \) and then, we use the 4th order Runge-Kutta algorithm for numerical solution of the resulting system of time-dependent ordinary differential equations. Both algorithms provide the same numerical results. Below, we present results obtained in the framework of the second approach.

Introducing the uniform grid in the interval \( 0 < x < L \) with the stepsize \( \Delta x \) we approximate the 2nd derivative of the phase difference with respect to the coordinate \( x \) in (1) as follows:

\[
\frac{\partial^2 \varphi_i}{\partial x^2} = \frac{2(\varphi_i^2 - \varphi_i^1)}{\Delta x^2} - \frac{2H_{\text{ext}}}{\Delta x}, \quad \frac{\partial^2 \varphi_i^m}{\partial x^2} = \frac{\varphi_i^{m+1} - 2\varphi_i^m + \varphi_i^{m-1}}{\Delta x^2}; \quad \frac{\partial^2 \varphi_i^M}{\partial x^2} = \frac{2(\varphi_i^{M+1} - \varphi_i^M)}{\Delta x^2} + \frac{2H_{\text{ext}}}{\Delta x}
\]

where \( H_{\text{ext}} \) is the external magnetic field and \( \varphi_i^m = \varphi_i(x_m) \) \( (m \) is the \( x \)-node number, \( 1 \leq m \leq M \), \( x_1 = 0, x_M = L \)).

We fix quantity \( I \) and put \( V_i = \varphi_i = 0 \) as the initial conditions of our numerical simulation. At each step of time of the Runge-Kutta process we average the phase difference and voltage in coordinate as \( \bar{V}(t) = 1/L \int_0^L V_i(x,t) dx \) using the Simpson quadrature formula for numerical integration. Then the voltage is averaged in time as \( V = 1/(T_{\text{max}} - T_{\text{min}}) \int_{T_{\text{min}}}^{T_{\text{max}}} \bar{V}(t) dt \) using the rectangle quadrature method.

The presented procedure allows us to obtain the value of \( V \) for the fixed value \( I \), i.e. the point of CVC. Increasing the current \( I \) by the stepsize \( \Delta I \), we repeat the simulation in accordance to the above algorithm. The current \( I \) increases up to \( I_{\text{max}} \), then \( I \) is decreased back to zero.
Our simulations have been done with \( \beta = 0.2 \) and \( H_{\text{ext}} = 0 \). The system (1) is solved in the time interval \([0, 300]\) with the time stepsize \( \Delta \tau = 0.025 \) and coordinate stepsize \( \Delta x = 4 \Delta \tau \). The bias current is increased from the value \( I = 0.01 \) till \( I_{\text{max}} = 1.1 \) with the stepsize \( \Delta I = 0.005 \) and then decreased back to \( I_{\text{min}} = 0 \). In the interval \( I = 0.2 \div 1 \) the current step is taken \( \Delta I = 0.0001 \). A noise with amplitude \( \delta I = \pm 10^{-8} \) is added to the bias current.

3. Results and discussions

Let us start with the consideration of the single JJ. Fig.1 (a) shows the one-loop CVC of the single JJ with length \( L = 3 \). We see branches related to the fluxon states in JJ shown by hole arrows.

![Figure 1](image1)

**Figure 1.** (a) CVC of the single JJ at \( L = 3, \beta = 0.2, H_{\text{ext}} = 0 \); (b) Spatiotemporal dependence of internal magnetic field \( H(t, x) \) in JJ at \( I = 0.9 \); (c) The same at \( I = 0.85 \).

The fluxon branches are in the agreement with the similar results of the Refs [11, 12]. Position of branches is related to the number of fluxons in JJ. The spatiotemporal dynamics of the system is demonstrated in Fig.1 (b,c) where the distribution of the internal magnetic field \( H(t, x) \) is presented for \( I = 0.9 \) and \( I = 0.85 \). At \( I = 0.9 \) the magnetic field strength is about \( 10^{-7} \). The distribution of \( H(t, x) \) at \( I = 0.85 \) is shown in Fig.1 (c) which demonstrates two fluxon states.

Let us now consider the system of JJ with the inductive coupling.

![Figure 2](image2)

**Figure 2.** (a) CVC of the stack with \( N = 3 \) JJ with inductive coupling at \( L = 3, \beta = 0.2, H_{\text{ext}} = 0, S = -0.1 \); (b) Spatiotemporal dependence of magnetic field \( H(t, x) \) in first JJ at \( I = 0.9 \); (c) The same at \( I = 0.81 \).

Fig.2 (a) demonstrates the CVC of the stack of 3 JJ with \( L = 3 \) taking into account the inductive coupling with coupling parameter \( S = -0.1 \). The value \( I = 0.94 \) corresponds to the
additional branch in the CVC which is not observed in the case of the single junction. Fig.2 (b) shows the spatiotemporal dependence of magnetic field at $I = 0.94$. The two fluxon state is realized at this $I$ value. In the interval of current $0.77 < I < 0.822$ the CVC demonstrates a chaotic behavior, while in the case of single junction chaotic feature is not observed. We assume that the chaotic behavior here is related to the appearing of unstable fluxons (see Fig.2 (c)).

Fig.3 shows the CVC of the stack of 3 JJ with $L = 3$ taking into account the capacitive coupling between junctions with parameters $s^C = -0.02$, $D^C = 1.04$. CVC illustrates a chaotic behavior like in the inductive coupling case. We note that some branches in CVC are not related to the fluxons. In particular, branches in the CVC at $V = 1.87235$ and $V = 0.908239$, which shown with hollow arrows correspond to the states with one and two JJ in rotating state, respectively.

4. Conclusions
We developed the simulation method of CVC for the system of JJ with inductive and capacitive couplings and demonstrate the fluxon branch and chaotic behavior. We showed that chaotic behavior is related to the appearance of unstable fluxons.

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6. References
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