HALF-AUTOMORPHISM GROUP OF A CLASS OF BOL LOOPS

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Abstract. A Bol loop is a loop that satisfies the Bol identity \((xy)z = x(yz)\). If \(L\) is a loop and \(f : L \to L\) is a bijection such that \(f(xy) \in \{f(x)f(y), f(y)f(x)\}\), for every \(x, y \in L\), then \(f\) is called a half-automorphism of \(L\). In this paper, we describe the half-automorphism group of a class of Bol loops of order \(4m\).

1. Introduction

A loop is a set \(L\) with a binary operation \(\cdot\) and a neutral element 1 \(\in L\) such that for every \(a, b \in L\) the equations \(ax = b\) and \(ya = b\) have unique solutions \(x, y \in L\), respectively.

Let \((L, \ast)\) and \((L', \cdot)\) be loops. A bijection \(f : L \to L'\) is a half-isomorphism if \(f(xy) \in \{f(x) \cdot f(y), f(y) \cdot f(x)\}\), for every \(x, y \in L\). A half-automorphism is defined as expected. We say that a half-isomorphism (half-automorphism) is proper if it is neither an isomorphism (automorphism) nor an anti-isomorphism (anti-automorphism).

In the year 1957, W.R. Scott proved that there is no proper half-homomorphism between two groups. He also gave an example of a loop of order 8 that has a proper half-automorphism, so Scott’s result cannot be generalized to all loops. In the last decade, many facts about half-isomorphisms between nonassociative loops have been proved. Gagola and Giuliani extended Scott’s result to Moufang loops of odd order \([3]\). Grishkov et al showed that there is no proper half-automorphism of a finite automorphic Moufang loop \([6]\). Kinyon, Stuhl and Vojtěchovský generalized the previous results to a more general class of Moufang loops \([7]\). In \([8]\), Giuliani and dos Anjos showed that there is no proper half-isomorphism between automorphic loops of odd order.

Investigations on loops that have proper half-automorphisms were also made. Gagola and Giuliani established conditions for the existence of proper half-automorphisms for certain Moufang loops of even order, including Chein loops \([4]\). A similar work was done by Giuliani, Plaumann and Sabinina for certain diassociative loops \([11]\). In \([9]\), Giuliani and dos Anjos described the half-automorphism group for a class of automorphic loops of even order, and the same was done for Chein loops in \([10]\). In this paper, we describe the half-automorphism group of the Bol loops of the form \(L_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times M\), where \(M\) is an abelian group of exponent greater than 2.

2. Preliminaries

Let \(L\) be a loop. Denote by \(Z(L)\), \(N(L)\), \(N_x(L)\), \(N_u(L)\) and \(N_p(L)\) the center, nucleus, left nucleus, middle nucleus and right nucleus of \(L\), respectively. The commutant of \(L\) is defined by \(C(L) = \{a \in L \mid ax = xa \forall x \in L\}\). The nuclei of \(L\) are subgroups of \(L\) and the center of \(L\) is an abelian subgroup of \(L\). The commutant of \(L\) in general is not a subloop of \(L\).

2010 Mathematics Subject Classification. Primary: 20N05. Secondary: 20B25.

Key words and phrases. Bol loops, Half-automorphisms, Half-automorphism group.
A right Bol loop is a loop that satisfies the right Bol identity

\[(2.1) \quad x((yz)y) = ((xy)z)y,\]

and a left Bol loop is a loop that satisfies the left Bol identity

\[(2.2) \quad (x(yx))z = x(y(xz)).\]

Right (left) Bol loops are power-associative, right (left) alternative, and have the right (left) inverse property. Other basic results about these loops can be found in [13, 14].

A Moufang loop is a loop that is both a right and a left Bol loop.

Let \((L, \ast)\) be a loop. The \textit{opposite loop} of \(L\) is the loop \(L^{\text{op}} = (L, \cdot)\) with the operation \(\cdot\) defined by \(x \cdot y = y \ast x\). If \(L\) is a right Bol loop, then \(L^{\text{op}}\) is a left Bol loop (and vice versa).

**Proposition 2.1.** Let \(L\) be a right Bol loop. If \(L\) has an anti-automorphism, then \(L\) is a Moufang loop.

**Proof.** Let \(\varphi\) be an anti-automorphism of \(L\). Then \(\varphi\) is an isomorphism from \(L\) into \(L^{\text{op}}\). Thus \(L\) is also a left Bol loop, and hence \(L\) is a Moufang loop. \(\square\)

Consequently, if \(\varphi\) is a half-automorphism of a right Bol loop \(L\) that is not Moufang, then \(\varphi\) is either a proper half-automorphism or an automorphism of \(L\). The same can be shown for left Bol loops. In fact, every result about right Bol loops dualizes to left Bol loops. So, from now on, we will only work with right Bol loops, and we will call them simply Bol loops.

One can search for properties of half-isomorphisms of loops in [10, 8]. However, to make this text sort of self contained, some results follow:

**Proposition 2.2.** ([10, Proposition 2.2]) Let \(Q\) and \(Q'\) be loops and \(f : Q \to Q'\) be a half-isomorphism. If \(H\) is a subloop of \(Q'\), then \(f^{-1}(H)\) is a subloop of \(Q\).

**Definition 2.3.** Let \(L, L'\) be loops. A half-isomorphism \(f : L \to L'\) is called special if the inverse mapping \(f^{-1} : L' \to L\) is also a half-isomorphism.

**Proposition 2.4.** ([10, Corollary 2.7]) Every half-automorphism of a finite loop is special.

**Proposition 2.5.** ([8, Proposition 2.13]) Let \(L\) and \(L'\) be power-associative loops, and \(f : L \to L'\) be a half-isomorphism. Then \(f(x^n) = f(x)^n\), for all \(x \in L\) and \(n \in \mathbb{Z}\).

As a direct consequence of Proposition 2.5, we have that every half-isomorphism between power-associative loops preserves the order of the elements.

### 3. A construction of a class of Bol loops

We start this section pointing out a result from Foguel, Kinyon and Phillips [2] that establish when one can define a Bol loop over a right transversal of a group. Let \(G\) be a group. A subset \(B\) of \(G\) is called a twisted subgroup if \(1, x^{-1}, xyx \in B\), for all \(x, y \in B\).

**Proposition 3.1.** ([2, Proposition 5.2]) Let \(G\) be a group, \(H \leq G\), and \(B \subset G\) a right transversal of \(H\) in \(G\). If \(B\) is a twisted subgroup of \(G\), then \(B\) with the operation

\[(3.1) \quad x \cdot y = z, \text{ if } xy = hz, \text{ for some } h \in H,\]
is a Bol loop. Conversely, if $H$ is core-free and $(B, \cdot)$ is a Bol loop, then $B$ is a twisted subgroup of $G$.

Let $M$ be an abelian group. Define $L_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times M$ and consider the following operation on $L_M$:

$$
(3.2)\quad (l, s, x) \cdot (u, v, y) = \begin{cases} 
(l, s, xy), & \text{if } u = v = 0, \\
(l + u, s + v, x^{-1}y), & \text{otherwise.}
\end{cases}
$$

In the following, we will show that $(L_M, \cdot)$ is a Bol loop.

The generalized dihedral group of $M$ can be defined by $D(M) = M \cup Mr$, where $r \not\in M$, $r^2 = 1$ and $rxx = x^{-1}$, for every $x \in M$. Consider the direct product $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times D(M)$. We have that $H = 0 \times 0 \times \{1, r\}$ is a subgroup of $G$ of order 2. Let $B = \{(0, 0, x), (l, s, rx) | x \in M, l, s \in \mathbb{Z}_2, (l, s) \neq (0, 0)\}$.

The set $B$ is a right transversal of $H$ in $G$. Furthermore, $B$ contains $(0, 0, 1)$, the identity of $G$.

**Proposition 3.2.** $B$ is a twisted subgroup of $G$.

*Proof.* Since $(l, s, rx)^{-1} = (l, s, rx)$ and $(0, 0, x)^{-1} = (0, 0, x^{-1})$, for every $l, s \in \mathbb{Z}_2$ and $x \in M$, we get that $B$ is closed under inverses. Let $x, y \in M$ and $l, s, u, v \in \mathbb{Z}_2$ be such that $(l, s) \neq (0, 0)$ and $(u, v) \neq (0, 0)$. Then

$$
(0, 0, x)(0, 0, y)(0, 0, x) = (0, 0, xy), \\
(0, 0, x)(l, s, ry)(0, 0, x) = (l, s, xryx) = (l, s, ry), \\
(l, s, rx)(0, 0, y)(l, s, rx) = (2l, 2s, rxyrx) = (0, 0, y^{-1}), \\
(l, s, rx)(u, v, ry)(l, s, rx) = (2l + u, 2s + v, rxyrx) = (u, v, ry^{-1}x^2).
$$

Thus $XYX \in B$, for every $X, Y \in B$, and hence $B$ is a twisted subgroup of $G$. \hfill \Box

As a consequence of Propositions 3.1 and 3.2 we have that $(B, \cdot)$ is a Bol loop, where $\cdot$ is the operation given by (3.1). For $x, y \in M$ and $l, s, u, v \in \mathbb{Z}_2$ such that $(l, s) \neq (0, 0)$, $(u, v) \neq (0, 0)$ and $(l + u, s + v) \neq (0, 0)$, we have:

$$
(0, 0, x)(0, 0, y) = (0, 0, 1)(0, 0, xy), \\
(0, 0, x)(l, s, ry) = (0, 0, 1)(l, s, rx^{-1}y), \\
(l, s, rx)(0, 0, y) = (0, 0, 1)(l, s, rxy), \\
(1, 1, rx)(1, 1, ry) = (0, 0, 1)(0, 0, x^{-1}y), \\
(l, s, rx)(u, v, ry) = (0, 0, r)(l + u, s + v, rx^{-1}y),
$$

and then the operation $\cdot$ is given by the following rules:

$$
(3.3)\quad (0, 0, x) \cdot (0, 0, y) = (0, 0, xy), \\
(0, 0, x) \cdot (l, s, ry) = (l, s, rx^{-1}y), \\
(l, s, rx) \cdot (0, 0, y) = (l, s, rxy), \\
(1, 1, rx) \cdot (1, 1, ry) = (0, 0, x^{-1}y), \\
(l, s, rx) \cdot (u, v, ry) = (l + u, s + v, rx^{-1}y).
$$

Define $\psi : (L_n, \cdot) \to (B, \cdot)$; $\psi((0, 0, x)) \mapsto (0, 0, x)$ and $\psi((l, s, x)) \mapsto (l, s, rx)$, where $(l, s) \neq (0, 0)$. Comparing (3.2) and (3.3), we get that $\psi$ is an isomorphism, and hence $(L_M, \cdot)$ is a Bol loop.
Proposition 3.3. If \( M \) is an elementary abelian 2-group, then \((L_M, \cdot)\) is also an elementary abelian 2-group. If \( M \) is not an elementary abelian 2-group, then \((L_M, \cdot)\) is a nonassociative, noncommutative Bol loop.

Proof. If \( M \) is an elementary abelian 2-group, then \( x^{-1} = x \), for every \( x \in M \), and so we can see that \((B, \cdot)\) is commutative and has exponent 2 by \((3.3)\). Thus \((B, \cdot)\) is a commutative Moufang loop of exponent 2, and hence it is an elementary abelian 2-group. Now consider that \( M \) is not an elementary abelian 2-group. Then there exists \( x \in M \) such that \( x \neq x^{-1} \). Using \((3.2)\), we get:

\[
\begin{align*}
((0,1,x) \cdot (1,0,x)) \cdot (0,1,x) &= (1,1,0) \cdot (0,1,x) = (1,0,x), \\
(0,1,x) \cdot ((1,0,x) \cdot (0,1,x)) &= (0,1,x) \cdot (1,1,0) = (1,0,x^{-1}).
\end{align*}
\]

Then \((L_M, \cdot)\) is nonassociative and noncommutative.

Corollary 3.4. If \( M \) is an abelian group with exponent greater than 2, then \((L_M, \cdot)\) is not a Moufang loop. In particular, \( L_M \) has no anti-automorphisms.

Proof. In the proof of Proposition \(3.3\), we saw that \((L_M, \cdot)\) does not satisfy the flexible identity, and then \((L_M, \cdot)\) is not a Moufang loop. The rest of the claim follows from Proposition \(2.1\).

Let \( L \) be a power associative loop and let \( x \) be an element of \( L \). Then the order of \( x \) will be denoted by \( o(x) \). From \((3.2)\), we can easily obtain the order of the elements of \( L_M \) and a straightforward calculation gives us the nuclei, commutant and the center of \( L_M \).

Proposition 3.5. Let \((u,v,x)\) be an element of \( L_M \). Then

\[
o((u,v,x)) = \begin{cases} o(x), & \text{if } u = v = 0, \\
2, & \text{otherwise.}
\end{cases}
\]

Proposition 3.6. For the Bol loop \( L_M \), the following holds

\[
N_{\lambda}(L_M) = \{(u,v,x) \mid o(x) = 2\}, N_{\mu}(L_M) = N_{\rho}(L_M) = \{(0,0,x) \mid x \in M\},
\]

and \( Z(L_M) = N(L_M) = C(L_M) = \{(0,0,x) \mid o(x) = 2\} \).

4. Half-automorphisms of the Bol loop \( L_M \)

In this section, suppose that \( M \) is a finite abelian group with exponent greater than 2. Let \( L_M \) be the Bol loop constructed in section \(3\). From now on, let us denote \( L_M \) as \( L_M = K \times M \), where \( K = \{1, a, b, c\} \) is the Klein group and the multiplication in \( L_M \) will be given by

\[
\begin{align*}
(1, x) \ast (1, y) &= (1, xy) \\
(A, x) \ast (1, y) &= (A, xy) \\
(1, x) \ast (B, y) &= (B, x^{-1}y) \\
(A, x) \ast (B, y) &= (AB, x^{-1}y),
\end{align*}
\]

where \( A, B \neq 1 \).

Recall that \( f : L_M \to L_M \) is a half-automorphism of \( L_M \) if \( f \) is a bijection of \( L_M \) and \( f(xy) \in \{f(x)f(y), f(Y)f(X)\} \), for every \( X \) and \( Y \) in \( L_M \).

Proposition 4.1. Let \( (1, M) = \{(1, x) \in L_M ; x \in M\} \) and let \( f \) be a half-automorphism of \( L_M \). Then \( f(1, M) = (1, M) \).
Proposition 4.3. Let \( A, B, C \in K \setminus \{1\} \) and \( x, y, z \in M \). If \( A = B \), then \( (C, z) = (A, x)(B, y) = (A, x)(A, y) = (1, x^{-1}y) \) and \( C = 1 \), which is a contradiction. Then \( A \neq B \) and similarly, \( A \neq C \) and \( B \neq C \). So, \( H = \{(1, 1), (A, x), (B, y), (C, z)\} \), where \( \{A, B, C\} = \{a, b, c\} \). Since \( (C, z) = (A, x)(B, y) = (B, y)(A, x) \), we get \( z = x^{-1}y = xy^{-1} \), and then the order of \( xy^{-1} \) divides 2. If \( xy^{-1} = 1 \), then \( x = y \) and \( H = \{(1, 1), (A, x), (B, x), (C, 1)\} \). By \( (A, x)(C, 1) = (C, 1)(A, x) \), we get \( o(x) = 2 \). Then either \( H = (K, 1) \) or \( H = \{(1, 1), (A, x), (B, x), (C, 1)\} \) with \( o(x) = 2 \). Suppose that the order of \( xy^{-1} \) is equal to 2. Then \( x \neq y \) and \( 1 = xy^{-1}x^{-1} = x^2y^2 \), and so \( x^2 = y^2 \). If \( x = 1 \), then \( H = \{(1, 1), (A, 1), (B, y), (C, y)\} \), where \( o(y) = 2 \). If \( y = 1 \), then \( H = \{(1, 1), (A, x), (B, 1), (C, x)\} \), where \( o(x) = 2 \). Consider \( x, y \neq 1 \). By \( (A, x)(C, x^{-1}y) = (C, x^{-1}y)(A, x) \), we get \( x^2y = x^2y^{-1} \), and so \( y^2 = x^4 \). Since \( x^2 = y^2 \), we obtain \( x^2 = x^4 \) and then \( x^2 = 1 \). Hence, \( H = \{(1, 1), (A, x), (B, y), (C, xy)\} \), where \( x, y \in M \) are distinct elements of order 2. We have established the following result.

Proposition 4.3. Let \( H \in \mathcal{H}_M \). There are three possibilities:

(i) \( H = (K, 1) \),

(ii) \( H = \{(1, 1), (A, x), (B, x), (C, 1)\} \), with \( \{A, B, C\} = \{a, b, c\} \) and \( o(x) = 2 \),

(iii) \( H = \{(1, 1), (A, x), (B, y), (C, xy)\} \), with \( \{A, B, C\} = \{a, b, c\} \) and \( x \neq y \) are elements of order equal to 2.

In particular, if \( M \) has odd order, then \( \mathcal{H}_M = \{(K, 1)\} \).

Corollary 4.4. If \( H \in \mathcal{H}_M \), then \( L_M = H(1, M) = (1, M)H \).

Corollary 4.5. If \( |M| \) is odd, then \( (K, 1) \) is the only subloop of order 4 of \( L_M \).
Proof. Let $H$ be a subloop of $L_M$ of order 4. Then $|H \cap (1,M)| = 1$. By Proposition 3.5, $o(X) \leq 2$, for every $X \in H$. Then $H \cong K$, and the result follows by Proposition 4.3. \qed

**Corollary 4.6.** Consider $M$ as a finite abelian group of even order and exponent greater than 2 and write $M = C_{2i_1} \times C_{2i_2} \times \ldots \times C_{2i_s} \times M_1$, where $|M_1|$ is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$. Then $|H_M| = 4^s$.

**Proof.** Each $C_{2i_j}$ has only one element of order 2. Denote such element by $x_j$. Consider $n_1, n_2$ and $n_3$ as the numbers of elements in $H_M$ of the types (i), (ii) and (iii) according to Proposition 4.2 respectively. Then $|H_M| = n_1 + n_2 + n_3$ and $n_1 = 1$.

We have that $n_2 = 3 \cdot n'_2$, where $n'_2$ is the number of elements of order 2 in $M$. Each such element has the form $x_{i_1}x_{i_2}\ldots x_{i_r}$, where $r \leq s$ and $1 \leq l_1 < l_2 < \ldots < l_r \leq s$. Then $n'_2 = 2^s - 1$ and we get $n_2 = 3 \cdot (2^s - 1)$.

The number $n_3$ is equal to $6 \cdot n'_3$, where $n'_3$ is the number of subsets of $M$ isomorphic to $K$. It is not difficult to see that $n'_3 = \frac{1}{3} \cdot \frac{n'_2^3}{2(2^s - 2)!} = \frac{(2^s - 1)(2^s - 2)}{6}$. Then $n_3 = (2^s - 1)(2^s - 2)$ and we have

$$|H_M| = n_1 + n_2 + n_3 = 1 + 3 \cdot (2^s - 1) + (2^s - 1)(2^s - 2) = 4^s.$$ \qed

**Corollary 4.7.** Let $f$ be a half-automorphism of $L_M$. Then there exist a unique $f' \in Aut(K)$ and $x, y \in M$ such that $o(x), o(y) \leq 2$ and

$$f(A,1) = (f'(A), \alpha_{(x,y)}(A)),$$

where $\alpha_{(x,y)}(1) = 1, \alpha_{(x,y)}(a) = x, \alpha_{(x,y)}(b) = y$ and $\alpha_{(x,y)}(c) = xy$.

**Proof.** The existence of $f' \in Aut(K)$ and $x, y \in M$ satisfying the conditions of the claim is a consequence of Propositions 4.2 and 4.3 and the fact that $Aut(K)$ is the symmetric group $S_3$. The unicity is trivial. \qed

**Remark 4.8.** In the conditions of Corollary 4.7 the mapping $\alpha_{(x,y)} : K \to M$ is a homomorphism. Furthermore, from Proposition 3.6 it follows that $(1, \alpha_{(x,y)}(A)) \in Z(L_M)$, for every $A \in K$.

**Proposition 4.9.** Let $f$ be a half-automorphism of $L_M$. For every $x \in M$, consider $f''(x) \in M$ as $(1, f''(x)) = f(1,x)$. Then $f'' : M \to M$ is an automorphism of $M$.

**Proof.** First, observe that By Proposition 4.1 $f''$ is well-defined. Now, for every $x$ and $y$ in $M$, we have

$$(1, f''(xy)) = f(1, xy) = f((1, x)(1, y)) \in \{ f(1, x)f(1, y), f(1, y)f(1, x) \}.$$

Note that $\{ f(1, x)f(1, y), f(1, y)f(1, x) \} = \{ (1, f''(x)f''(y)) \}$. Then $f''(xy) = f''(x)f''(y)$. \qed

Every element $(A, x)$ in $L_M$ can be written as $(A, 1)(1, x)$. By Corollary 4.7 there are $f' \in Aut(K)$ and $u, v \in M$ such that $f(A, 1) = (f'(A), \alpha_{(u,v)}(A))$, where $o(u), o(v) \leq 2$. For $A \neq 1$,

$$f(A, x) = f(((A, 1)(1, x)) \in \{ (f'(A), 1)(1, f''(x)), (1, f''(x))(f'(A), 1) \},$$

and so

$$f(A, x) \in \{ (f'(A), f''(x)\alpha_{(u,v)}(A)), (f'(A), f''(x^{-1})\alpha_{(u,v)}(A)) \}.$$
Then if \( f' \in \text{Aut}(K), \ f'' \in \text{Aut}(M) \) and \( u, v \in M \) are elements such that \( o(u), o(v) \leq 2 \), we define \( F_{(f', f'', u, v)}^+, F_{(f', f'', u, v)}^- : L_M \rightarrow L_M \) by

\[
F_{(f', f'', u, v)}^+(A, x) = (f'(A), f''(x)o(u,v)(A)) \quad \text{and} \quad F_{(f', f'', u, v)}^-(A, x) = \begin{cases} (f'(A), f''(x)o(u,v)(A)) & \text{if } A = 1, \\ (f'(A), f''(x^{-1})o(u,v)(A)) & \text{otherwise}. \end{cases}
\]

For making the notation easier, we will write \( f_{(u,v)}^+ \) and \( f_{(u,v)}^- \) instead of \( F_{(f', f'', u, v)}^+ \) and \( F_{(f', f'', u, v)}^- \), respectively. By Corollary 4.11, \( f_{(u,v)}^+ \) and \( f_{(u,v)}^- \) are bijections. Furthermore, for every \( A \in K \) and \( x \in M \),

\[
(4.1) \ f_{(u,v)}^+(A, x) = (1, o(u,v)(A))\ast f_{(1,1)}^+(A, x) \quad \text{and} \quad f_{(u,v)}^-(A, x) = (1, o(u,v)(A))\ast f_{(1,1)}^-(A, x).
\]

**Proposition 4.10.** In the conditions above, \( f_{(u,v)}^+ \) is an automorphism and \( f_{(u,v)}^- \) is a proper half-automorphism of \( L_M \).

**Proof.** Since \( o(u,v) \) is a homomorphism and \( (1, o(u,v)(A)) \in \mathcal{Z}(L_M) \), for every \( A \in K \) (Remark 4.3), we only have to prove that \( f_{(1,1)}^+(A) \) is an automorphism and \( f_{(1,1)}^- \) is a proper half-automorphism by (4.1). Denote \( f_{(1,1)}^+ \) and \( f_{(1,1)}^- \) by \( f^+ \) and \( f^- \), respectively. First, let us prove that \( f^+ \) is an automorphism. Let \( (A, x), (B, y) \in L_M \). When \( B = 1 \) we have

\[
f^+((A, x)(1, y)) = f^+(A, xy) = (f'(A), f''(xy)) = (f'(A), f''(x))(1, f''(y)) = f^+(A, x)f^+(1, y).
\]

If \( B \neq 1 \), then

\[
f^+((A, x)(B, y)) = f^+(AB, x^{-1}y) = (f'(AB), f''(x^{-1}y)) = (f'(A)f'(B), f''(x^{-1})f''(y)) = f^+(A, x)f^+(B, y).
\]

Now, let us prove that \( f^- \) is a proper half-automorphism. Clearly, \( f^-((1, x)(1, y)) = f^-((1,1)) \) for every \( x, y \in M \). A straightforward calculation shows us that, for \( B \neq 1 \), we have \( f^-((1, x)(B, y)) = f^-(B, y)f^-(1, x) \). Also, for \( A \neq 1 \), we get \( f^-((A, x)(1, y)) = f^-((1,1))f^-((A, x)) \) and \( f^-((A, x)(A, y)) = f^-((A,1))f^-((A, x)) \). Finally, for \( A \neq B \), and both not equal to 1, we have

\[
f^-((A, x)(B, y)) = f^-((AB, x^{-1}y)) = (f'(AB), f''(xy^{-1})) = f^-((A, x)f^-((B, y)).
\]

On other hand, we can note that \( f^-(B, y)f^-(A, x) = (f'(A), f''(y)f''(x^{-1})) \) and, since there exists \( w \in M \) such that \( w \neq w^{-1} \), we conclude that \( f^- \) is a proper half-automorphism of \( L_M \). \( \square \)

**Proposition 4.11.** Let \( g \) be an automorphism of \( L_M \). Then \( g = g_{(u,v)}^+ \).

**Proof.** First, observe that \( g_{(u,v)}^+(1, y) = g_{(u,v)}^-(1, y) \) for every \( y \in M \) and, if \( o(y) \leq 2 \), \( g_{(u,v)}^+(A, y) = g_{(u,v)}^-(A, y) \) for every \( A \in K \). Then suppose, towards a contradiction, that \( g(A, y) = g_{(u,v)}^-(A, y) \) for some \( A \in K \setminus \{1\} \) and \( y \in M \) with \( o(y) > 2 \). For every \( x \in M \), \( g((A, y)(A, x)) = g(A, y)g(A, x) \) and \( g((A, y)(A, x)) = g(1, y^{-1}x) = (1, g''(y^{-1}x)) \). If \( g(A, x) = g_{(u,v)}^+(A, x) \), then \( g(A, y)g(A, x) = (1, g''(y)g''(x)) \) which implies \( g''(y) = g''(y^{-1}) \), which is a contradiction since \( o(y) > 2 \). On other hand, if \( g(A, x) = g_{(u,v)}^-(A, x) \), then \( g(A, y)g(A, x) = (1, g''(y)g''(x^{-1})) \), and so \( o(y^{-1}x) \leq 2 \) for every \( x \in M \), which is a contradiction because \( M \) has exponent greater than 2. Therefore, \( g = g_{(u,v)}^+ \). \( \square \)
Proposition 4.12. Let \( g \) be a proper half-automorphism of \( L_M \). Then \( g = g_{(u,v)} \).

Proof. For every \((A,x) \in L_M\), \( g(A,x) \in \{g_{(u,v)}^+(A,x), g_{(u,v)}^-(A,x)\} \) and, by Proposition 4.11, \( g \neq g_{(u,v)}^+ \). Also, \( g_{(u,v)}^+(A,x) = g_{(u,v)}^-(A,x) \) whenever either \( A = 1 \) or \( o(x) \leq 2 \). Thus suppose, towards a contradiction, that \( g(A,x) = g_{(u,v)}^+(A,x) \) for some \( A \in K - \{1\} \) and \( x \in M \) with \( o(x) > 2 \). Since \( g \) is not an automorphism of \( L_M \), there is \((\epsilon, \theta) \in L_M \) such that \( g(\epsilon, \theta) = g_{(u,v)}^+(\epsilon, \theta) \neq g_{(u,v)}^+(\epsilon, \theta) \) and then \( \epsilon \neq 1 \) and \( o(\theta) > 2 \). First, suppose \( \epsilon = A \) and set \( \theta = z \), that is, \( g(A,z) = g'((A), g''((z^{-1}) \alpha(u,v)(A)) \).

Thus suppose, towards a contradiction, that \( g(A,x)(A,z) = g(1, x^{-1}z) = (1, g''((x^{-1}z)) \in \{g(A,x)g(A,z), g(A,z)g(A,x)\} \).

First, if \( g(B, x^{-1}z) = g_{(u,v)}^+(B, x^{-1}z) \), then we have either \( g(B, x^{-1}z) = (g'(B), g''((x^{-1}z^{-1}) \alpha(u,v)(B)) \) and \( z = z^{-1} \) or \( g(B, z)g(A,x) = (g'(B), g''((z^{-1} \alpha((u,v)(B))) and \( x = x^{-1} \) and, in both cases, we get a contradiction. Finally, if \( g(B, x^{-1}z) = g_{(u,v)}^-(B, x^{-1}z) \), in the same way as before, we get either \( x = x^{-1} \) or \( z = z^{-1} \) and so, a contradiction. Hence, \( g = g_{(u,v)}^+ \) as desired. \( \square \)

Theorem 4.13. Let \( L_M \) be the Bol loop constructed in the Section 3 and let \( \text{Half}(L_M) \) be the group of half-automorphisms of \( L_M \). Then \( \text{Half}(L_M) \) is the set

\[ \{F^+_{(f',f'',u,v)}, F^-_{(f',f'',u,v)} \mid f' \in \text{Aut}(K), f'' \in \text{Aut}(M), u, v \in M, o(u), o(v) \leq 2\} \]

Furthermore:

(a) If \(|M|\) is odd, then \( |\text{Half}(L_M)| = 2.|\text{Aut}(K)|.|\text{Aut}(M)| \).

(b) If \(|M|\) is even, then \( |\text{Half}(L_M)| = 2^{2s+1}.|\text{Aut}(K)|.|\text{Aut}(M)| \), where the abelian group \( M \) is the direct product \( C_{2^{i_1}} \times C_{2^{i_2}} \times ... \times C_{2^{i_s}} \times M_1 \), \(|M_1|\) has odd order, \( s \geq 1 \) and \( i_j \geq 1 \), for all \( j \), and \( C_n \) denotes the cyclic group of order \( n \).

Proof. By Corollary 3.4 and Propositions 4.11 and 4.12, we have that \( \text{Half}(L_M) \) is the set of all mappings \( F^+_{(f',f'',u,v)} \) and \( F^-_{(f',f'',u,v)} \). If \(|M|\) is odd, then \( M \) has no elements of order 2 and we get (a). Now consider that \(|M|\) is even and \( M = C_{2^{i_1}} \times C_{2^{i_2}} \times ... \times C_{2^{i_s}} \times M_1 \). In the proof of Corollary 4.6 we saw that \( M \) has \( 2^s - 1 \) elements of order 2. Then \( |\text{Half}(L_M)| = 2.(2^s)^2.|\text{Aut}(K)|.|\text{Aut}(M)| = 2^{2s+1}.|\text{Aut}(K)|.|\text{Aut}(M)| \). \( \square \)

5. Half-automorphisms group of \( L_M \)

In this section we present the structure of the half-automorphisms group of the loop \( L_M \). Recall that if \( f \) is a half-automorphism of \( L_M \), then by Corollary 4.7 and Proposition 4.9, the mappings \( f' \) and \( f'' \) automorphisms of \( K \) and \( M \), respectively.

Lemma 5.1. If \( f, g \) are half-automorphisms of \( L_M \), then

(a) \((gf)' = gf' \) and

(b) \((gf)'' = g''f'' \).

Proof. (a) By Corollary 4.7 \( f(A,1) = (f'(A), \alpha(x,y)(A)), g(A,1) = (g'(A), \alpha(x',y')(A)) \) and \((gf)(A,1) = ((gf)'(A), \alpha(x'',y'')(A)) \), for every \( A \in K \). Then for \( A \in K \), we have
Since $g$ is a half-automorphism and $g(1, \alpha_{(x,y)}(A)) = (1, g''(\alpha_{(x,y)}(A)))$, there exists $\epsilon \in \{-1, 1\}$ such that $g((f'(A), \alpha_{(x,y)}(A))) = (g'f'(A), g''(\alpha_{(x,y)}(A))'')\alpha_{(x',y')}(A))$. Then $(gf)'(A) = g'f'(A)$.

(b) For $x \in M$, we have that
\[
(1, (gf)'(x)) = (gf)(1, x) = g(1, f'(x)) = (1, g'' f''(x)),
\]
and then $(gf)'' = g'' f''$.

Setting $K$ as $\{a, b, c\}$, the elements of $Aut(K)$ are the permutations $I$, $(a b)$, $(a c)$, $(b c)$, $(a b c)$ and $(a c b)$, where $I$ is the identity mapping. Let $u, v$ be elements of $M$ such that $o(u), o(v) \leq 2$. It is easy to see that, for all $A \in K$:
\[
\alpha_{(u,v)}(I(A)) = \alpha_{(u,v)}(A), \quad \alpha_{(u,v)}((a b)(A)) = \alpha_{(u,v)}(A),
\]
(5.1)
\[
\alpha_{(u,v)}((a c)(A)) = \alpha_{(u,v)}(A), \quad \alpha_{(u,v)}((b c)(A)) = \alpha_{(u,v)}(A),
\]
\[
\alpha_{(u,v)}((a b c)(A)) = \alpha_{(v,u)}(A), \quad \alpha_{(u,v)}((a c b)(A)) = \alpha_{(uv,u)}(A).
\]

Then for $f' \in Aut(K)$ and $A \in K$, we have
\[
\alpha_{(u,v)}(f'(A)) = \alpha_{(\overline{\mathcal{F}}_1(u,v), \overline{\mathcal{F}}_2(u,v))}(A),
\]
for some mappings $\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2 : M \times M \rightarrow M$. When $u \neq v$, we get $\overline{\mathcal{F}}_1(u, v) = \overline{\mathcal{F}}(u)$ and $\overline{\mathcal{F}}_2(u, v) = \overline{\mathcal{F}}(v)$, where:
\[
\overline{\mathcal{F}}(u) = u, \overline{\mathcal{F}}(v) = v, (a b)(u) = v, (a b)(v) = u,
\]
(5.3)
\[
(ac)(u) = uv, (ac)(v) = v, (bc)(u) = u, (bc)(v) = uv,
\]
\[
(ab c)(u) = v, (abc)(v) = uv, (acb)(u) = uv, (acb)(v) = u.
\]

It is known that if $g'' \in Aut(M)$, then $o(g''(u)) = o(u)$ and $o(g''(v)) = o(v)$, and, therefore, $\alpha_{(g''(u), g''(v))}$ is well-defined. Moreover, a straightforward calculation shows us the following

Lemma 5.2. Let $g'' \in Aut(M)$ and $u, v, u', v' \in M$ be such that their orders are smaller or equal to 2. Then for all $A \in K$:
(a) $g''(\alpha_{(u,v)}(A)) = \alpha_{(g''(u), g''(v))}(A)$.
(b) $\alpha_{(u,v)}(A)\alpha_{(u',v')}(A) = \alpha_{(uv,u,v')}(A)$.

Theorem 1.1.3 gives us a label of every element of $Half(L_M)$. In the following, we describe the behavior of these elements in such label.

Proposition 5.3. Let $F^+_{(f', f'', u,v)} F^-_{(f', f'', u,v)} F^+_{(g', g'', u',v')}, F^-_{(g', g'', u',v')}$ be half-automorphisms of $L_M$, where $f', f''$, $g'$ are automorphisms of the Klein group $K$, $f'', g''$ are automorphisms of the abelian group $M$ and $u, v$ are elements of $M$ of order at most 2. Then
(a) $F^+_{(g', g'', u',v')} F^+_{(f', f'', u,v)} = F^+_{(g' f', g'', g''(u) \overline{\mathcal{F}}_1(u',v') \overline{\mathcal{F}}_2(u',v'))}$,
(b) $F^-_{(g', g'', u',v')} F^-_{(f', f'', u,v)} = F^-_{(g' f', g'', g''(u) \overline{\mathcal{F}}_1(u',v') \overline{\mathcal{F}}_2(u',v'))}$,
(c) $F^+_{(g', g'', u',v')} F^-_{(f', f'', u,v)} = F^-_{(g' f', g'', g''(u) \overline{\mathcal{F}}_1(u',v') \overline{\mathcal{F}}_2(u',v'))}$ and
(d) $F^-_{(g', g'', u',v')} F^-_{(f', f'', u,v)} = F^+_{(g' f', g'', g''(u) \overline{\mathcal{F}}_1(u',v') \overline{\mathcal{F}}_2(u',v'))}$.\]
Proof. We will just prove item (b). The others can be, similarly, proved. Denote $F_{(g',g'',u',v')}^+$ and $F_{(f',f'',u',v')}^-$ by $g$ and $f$, respectively. Then for every $A, B$ in $K$ and $x, y$ in $M$, we have $g(B, y) = (g'(B), g''(y)\alpha_{(u',v')}(B))$ and

$$f(A, x) = \begin{cases} (1, f''(x)), & \text{if } A = 1, \\ (f'(A), f''(x^{-1})\alpha_{(u',v')}(A)), & \text{otherwise.} \end{cases}$$

If $A = 1$, then

$$g(f(A, x)) = g((1, f''(x))) = (1, g'' f''(x)),$$

and if $A \neq 1$,

$$g(f(A, x)) = g((f'(A), f''(x^{-1})\alpha_{(u',v')}(A))) = (g'(f'(A)), g''(f''(x^{-1}))g''(\alpha_{(u',v')}(A))\alpha_{(u',v')} f'(A))) = (g' f'(A), g'' f''(x^{-1})\alpha_{(g''(u) f'(A), g''(v) f'(A))}(A)).$$

\[ \square \]

**Corollary 5.4.** $Half(L_M) \cong C_2 \times Aut(L_M)$.

**Proof.** It is a consequence of Propositions 4.10, 4.11 and 5.3. \[ \square \]

In order to reach a full description of the group $Half(L_M)$, we aim to describe the automorphisms group $Aut(L_M)$.

**Proposition 5.5.** Let $A = \{F_{(f',f'',1,1)}^+ \mid f' \in Aut(K), f'' \in Aut(M)\}$. Then

$$A \cong Aut(K) \times Aut(M).$$

**Proof.** Let $F_{(f',f'',1,1)}^+, F_{(g',g'',1,1)}^+ \in A$ and denote $f = F_{(f',f'',1,1)}^+$ and $g = F_{(g',g'',1,1)}^+$. Then $f(A, x) = (f'(A), f''(x))$ and $g(A, x) = (g'(A), g''(x))$, for every $A \in K$ and $x \in M$. Also

$$g f(A, x) = (g' f'(A), g'' f''(x)) = ((g f)'(A), (g f)''(x)).$$

Define the mapping $\Psi : A \rightarrow Aut(K) \times Aut(M)$ by $\Psi(F_{(f',f'',1,1)}^+) = (\phi, \varphi)$. Clearly, $\Psi$ is a bijection and, furthermore,

$$\Psi(g f) = ((g f)'(A), (g f)''(A)) = (g' f'(A), g'' f''(A)) = (g', g'') f' f'' = \psi(g) \psi(f).$$

Therefore, $\Psi$ is an isomorphism from $A$ to $Aut(K) \times Aut(M)$. \[ \square \]

The following result allows us to describe the group $Half(L_M)$ in terms of the automorphism group of $M$, for a special case of $M$.

**Theorem 5.6.** Let $K$ be the Klein group, let $M$ be an abelian group of odd order and let $L_M = K \times M$ be the Bol loop constructed in Section 3. Then

$$Aut(L_M) \cong S_3 \times Aut(M) \text{ and } Half(L_M) \cong C_2 \times S_3 \times Aut(M).$$

**Proof.** First observe that $M$ has no elements of order 2. By Proposition 4.11 and Theorem 4.13, $Aut(L_M) = \{F_{(f',f'',1,1)}^+ \mid f' \in Aut(K), f'' \in Aut(M)\}$. Since $Aut(K) \cong S_3$, the result follows from Corollary 5.4, Proposition 5.5. \[ \square \]
From now on we consider $|M|$ even. Then $M = C_{2i_1} \times C_{2i_2} \times \ldots \times C_{2i_s} \times M_1$, where $M_1$ is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all $j$. Set $H = \{x \in M \mid o(x) \leq 2\}$. It is known that $H$ is a subgroup of $M$ of exponent 2 and, by the proof of Corollary 4.6, we have that $|H| = 2^s$. Denote by $I_K$ and $I_M$ the identity mappings of $K$ and $M$, respectively. Define

$$\mathcal{B} = \{F^+_{(I_K,I_M,x,y)} \mid x, y \in H\}.$$  

By Proposition 5.3 (a), $\mathcal{B}$ is closed under compositions, and, since $M$ is finite, $\mathcal{B}$ is a subgroup of $Aut(L_M)$. Considering $\mathcal{A}$ as in Proposition 5.5 we have that $|\mathcal{A} \cap \mathcal{B}| = 1$, and so

$$\mathcal{A} \mathcal{B} = |\mathcal{A}|.|\mathcal{B}| = 2^{2s}.|Aut(K)|.|Aut(M)|$$

Proposition 5.7. Let $K$ be the Klein group, let $M$ be an abelian group of even order and let $L_M = K \times M$ be the Bol loop constructed in Section 3. $Aut(L_M) = \mathcal{A} \mathcal{B} = \mathcal{B} \mathcal{A}$. 

Proof. By Theorem 4.13 and Proposition 4.11 $|Aut(L_M)| = 2^{2s}.|Aut(K)|.|Aut(M)|$. Then the result follows from Equation 5.5.

Proposition 5.8. Using the notation above,

(a) $\mathcal{B} \cong H \times H \cong C_2^{2s}$,
(b) $\mathcal{B} \triangleleft Aut(L_M)$ and
(c) $Aut(L_M)/\mathcal{B} \cong \mathcal{A}$.

Proof. (a) We only have to prove that $\mathcal{B}$ has exponent 2. Let $f = F^+_{(I_K,I_M,u,v)} \in \mathcal{B}$. Since $f(A, x) = (A, x\alpha_{(u,v)}(A))$, we have that

$$f^2(A, x) = f(A, x\alpha_{(u,v)}(A)) = (A, x\alpha_{(u,v)}(A)\alpha_{(u,v)}(A)) = (A, x).$$

Hence, $f^2$ is the identity mapping of $L_M$ and we have that $\mathcal{B}$ has exponent 2.

(b) and (c) Define the mapping $\psi : Aut(L_M) \rightarrow \mathcal{A}$, $\psi(F^+_{(f',f'',u,v)}) = F^+_{(f',f'',1,1)}$. It is clear that $\psi$ is surjective. Let $F^+_{(j,f',f'',u,v)}, F^+_{(g',g'',u',v')} \in Aut(L_M)$. By Proposition 5.3 we have that $\psi(F^+_{(g',g'',u',v')}F^+_{(j,f',f'',u,v)}) = F^+_{(g',g'',f',f'',1,1)}$. and using (5.4) we get

$$\psi(F^+_{(g',g'',u',v')}F^+_{(j,f',f'',u,v)}) = F^+_{(g',g'',f',f'',1,1)} = \psi(F^+_{(g',g'',u',v')})\psi(F^+_{(j,f',f'',u,v)}).$$

Then $\psi$ is a homomorphism. Note that $Ker(\psi) = \{F^+_{(I_K,I_M,u,v)} \mid u, v \in H\} = \mathcal{B}$.

Hence, $\mathcal{B} \triangleleft Aut(L_M)$ and $Aut(L_M)/\mathcal{B} \cong \mathcal{A}$. 

Define $\sigma : \mathcal{A} \rightarrow \mathcal{B}$, where, for $\alpha \in \mathcal{A}$, $\sigma(\alpha) = \sigma_\alpha$ and $\sigma_\alpha(\beta) = \alpha \beta \alpha^{-1}$, for all $\beta \in \mathcal{B}$. Since $\mathcal{B} \triangleleft Aut(L_M)$, then $\sigma$ is a homomorphism and we can construct the inner semidirect product $\mathcal{A} \ltimes \mathcal{B}$, where the operation is given by

$$(\alpha, \beta) \cdot (\alpha', \beta') = (\alpha \alpha', \beta \alpha \beta' \alpha^{-1}) \ (\alpha, \alpha' \in \mathcal{A}, \beta, \beta' \in \mathcal{B}).$$

Define $\psi : Aut(L_M) \rightarrow \mathcal{A} \ltimes \mathcal{B}$ by $\psi(\beta \alpha) = (\alpha, \beta)$. Then $\psi$ is a bijection. Furthermore, 

$$\psi((\beta \alpha)(\beta' \alpha')) = \psi(\beta(\beta \alpha^{-1}) \alpha \alpha') = (\alpha \alpha', \beta(\beta \alpha^{-1})) = (\alpha, \beta) \cdot (\alpha', \beta').$$

Thus $Aut(L_M) \cong \mathcal{A} \ltimes \mathcal{B}$. Hence, we established the following result.
**Theorem 5.9.** Let $M$ be a finite abelian group such that its exponent is greater than 2. Write $M = C_{2i_1} \times C_{2i_2} \times \ldots \times C_{2i_s} \times M_1$, where $M_1$ is an abelian group of odd order, $s \geq 1$ and $i_j \geq 1$, for all $j$. Then

$$\text{Aut}(L_M) \cong \mathcal{A} \ltimes \mathcal{B} \text{ and } \text{Half}(L_M) \cong C_2 \times (\mathcal{A} \ltimes \mathcal{B}),$$

where $\mathcal{A} \cong S_3 \times \text{Aut}(M)$ and $\mathcal{B} \cong C_2^s$.

We finish this paper with two examples.

**Example 5.10.** Let $M = C_3$, the cyclic group of order 3. Then $L_M$ is a nonassociative Bol loop of order 12, which is recognized by the command “RightBolLoop(12, 3)” in the library of loops of the LOOPS package [12] for GAP [5]. The Cayley table of this loop is presented below.

| * | 1 2 3 4 5 6 7 8 9 10 11 12 |
|---|-----------------------------|
| 1 | 1 2 3 4 5 6 7 8 9 10 11 12 |
| 2 | 2 1 4 3 6 5 8 10 11 9 12 7 |
| 3 | 3 5 6 2 4 1 10 9 12 11 7 8 |
| 4 | 4 6 5 1 3 2 9 11 7 12 8 10 |
| 5 | 5 3 2 6 1 4 12 7 10 8 9 11 |
| 6 | 6 4 1 5 2 3 11 12 8 7 10 9 |
| 7 | 7 9 11 8 12 10 1 5 4 6 3 2 |
| 8 | 8 10 12 7 11 9 2 1 6 5 4 3 |
| 9 | 9 7 8 11 10 12 4 3 1 2 5 6 |
| 10 | 10 8 7 12 9 11 3 2 5 1 6 4 |
| 11 | 11 12 10 9 8 7 6 4 2 3 1 5 |
| 12 | 12 11 9 10 7 8 5 6 3 4 2 1 |

Since $\text{Aut}(M) = C_2$, we have that

$$\text{Aut}(L_M) \cong C_2 \times S_3 \text{ and } \text{Half}(L_M) \cong C_2^2 \times S_3$$

by Theorem 5.6. Then $L_M$ has 24 half-automorphisms, from which 12 are proper. By using GAP computing, we can obtain expressions for these mappings in terms of permutations. The automorphisms of $L_M$ are the permutations:

$$I_d, \quad (7, 9)(8, 11)(10, 12), \quad (2, 8, 11)(4, 12, 10)(5, 9, 7),$$

$$(2, 9, 11, 5, 8, 7)(3, 6)(4, 12, 10), \quad (2, 11)(4, 10)(5, 7), \quad (2, 11, 8)(4, 10, 12)(5, 7, 9),$$

$$(2, 5)(3, 6)(7, 8)(9, 11)(10, 12), \quad (2, 5)(3, 6)(7, 11)(8, 9), \quad (2, 7)(3, 6)(4, 10)(5, 11)(8, 9),$$

$$(2, 7, 8, 5, 11, 9)(3, 6)(4, 10, 12), \quad (2, 8)(4, 12)(5, 9), \quad (2, 9)(3, 6)(4, 12)(5, 8)(7, 11)$$

and the proper half-automorphisms of $L_M$ are the permutations:

$$(2, 5)(7, 8)(9, 11)(10, 12), \quad (2, 5)(7, 11)(8, 9), \quad (2, 7)(4, 10)(5, 11)(8, 9),$$

$$(2, 7, 8, 5, 11, 9)(4, 10, 12), \quad (2, 9, 11, 5, 8, 7)(4, 12, 10), \quad (2, 9)(4, 12)(5, 8)(7, 11),$$

$$(3, 6), \quad (3, 6)(7, 9)(8, 11)(10, 12), \quad (2, 8, 11)(3, 6)(4, 12, 10)(5, 9, 7),$$

$$(2, 8)(3, 6)(4, 12)(5, 9), \quad (2, 11)(3, 6)(4, 10)(5, 7), \quad (2, 11, 8)(3, 6)(4, 10, 12)(5, 7, 9).$$

**Example 5.11.** Let $M$ be the group $C_4 \times C_2$. This group is recognized by the command “SmallGroup(8, 2)” in GAP and its Cayley table is presented below.
The Bol loop $L_M$ is nonassociative and has order 32. Furthermore, since $\text{Aut}(M) = D_8$, the dihedral group of order 8, we have that $\text{Aut}(L_M) \cong (S_3 \times D_8) \rtimes C_2$ and $\text{Half}(L_M) \cong C_2 \times ((S_3 \times D_8) \rtimes C_2)$ by Theorem 5.9. Then $L_M$ has 1536 half-automorphisms, from which 768 are proper.

Considering $K = \{1, a, b, c\}$ as the Klein group, the permutation $$((a, 1), (a, 4), (a, 3), (a, 7), (b, 1), (b, 4), (b, 3), (b, 7), (c, 2), (c, 6), (c, 5), (c, 8))$$ is an example of a proper half-automorphism of $L_M$ which does not fix the subgroup $(K, 1)$ of $L_M$. This permutation was obtained by using GAP computing with the LOOPS package.

Acknowledgments

Some calculations in this work have been made by using the LOOPS package [12] for GAP [5].

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