Mathematical quantum Yang-Mills theory revisited
II: Mass without mass

Alexander Dynin

Professor Emeritus of Mathematics, Ohio State University,
Columbus, OH 43210, USA, dynin@math.ohio-state.edu

Abstract

Massless Dirac equation for spinor multiplets is minimally coupled with a unitary representation of an arbitrary compact semisimple gauge group. The spectrum of the quantized interaction Hamiltonian has a positive mass gap running along the classical energy scale.

2010 AMS Subject Classification: Primary 81T13

1 Introduction

1.1 Physics

The concept of classically massless neutrinos is a cornerstone of the standard model of weak interactions. However, an experimental evidence by T. Kajita and A. McDonald groups (2015 Nobel prize) led to a notion of oscillation between electronic and muonic neutrinos presumably because of a difference in their masses. This is inconsistent with the standard model unless the masses have quantum origin.

F. Wilczek’s QCD Lite is the Dirac-Yang-Mills dynamics (DYM) of the light $u$ and $d$ quarks stripped of their Lagrangian mass terms. Lattice simulation of interaction between the massless quarks and massless gluons produces (with no Higgs field) 99% of the mass of visible universe.

Quotation from Wilczek (see [17, Subsection 1.2.1]):

My central points are most easily made with reference to two triplets and two anti-triplets of handed fermions, all with zero mass. Of course I have in mind that the gauge group represents color, and that one set of triplet and antitriplet will be identified with the quark fields $u$, $u_R$ and the other with $d$, $d_R$.

Upon demanding renormalizability, this theory appears to contain precisely one parameter, the coupling [constant]. It is, in units $\hbar = c = 1$,
a pure number. I am ignoring the $\theta$ parameter, which has no physical content here, since it can be absorbed into the definition of the quark fields. Mass terms for the gluons are forbidden by gauge invariance. MASS TERMS FOR THE QUARKS ARE FORBIDDEN BY $SU(2)_L \otimes SU(2)_R$ FLAVOR SYMMETRY.

Besides the coupling constants $\kappa$, the classical Lagrangians of quark flavors of the standard model differ only in their phenomenological quadratic mass terms. After these mass terms are discarded, as in the QCD Lite, the Lagrangians become indistinguishable.

By [16], physical constants of a quantum field theory, such as masses, are running along the classical energy scale.

### 1.2 Mathematics

The present paper shows that the spectra of second quantized massless Yang-Mills-Dirac (DYM) Hamiltonians (with arbitrary semi-simple compact gauge Lie groups) are discrete. In particular, the spectra have positive mass gaps. The sizes of the latter define the quantum masses.

This paper is a continuation of [4] where the Clay Mathematics Institute "Quantum Yang-Mills theory" problem (see [11]) has been solved in the case of Yang-Mills fields in the vacuum. The problem requires a mathematical proof that

for any compact semisimple global gauge group, a nontrivial quantum Yang-Mills theory exists on the four-dimensional Minkowski spacetime and has a positive mass gap.

Accordingly, the mass of a classically massless matter field is equal to the spectral mass gap of its quantum interaction with a gauge field associated with a global compact semisimple gauge Lie group.

As in [4] the analysis is based on study of the Cauchy problem for coupled Dirac-Yang-Mills (DYM) equations. Because of the local gauge symmetry, the Cauchy problem is underdetermined: the solution set is covariant relative the action of the gauge group. To reduce the gauge degree of freedom one uses a global gauge cross section, aka gauge (see [7, Section I.1]). Such is the Hamiltonian (aka temporal) gauge (see [15]).

In this gauge the Cauchy problem for coupled Dirac-Yang-Mills (DYM) equations has a unique global solution (Theorem 2.1 below).

As in [4], the proof of existence involves running cutoffs of initial data on the Euclidean balls $\|x\| < R$. This spatial renormalization converges as the radius $R$ increases to infinity.

In the first order formalism the DYM equation becomes an infinite-dimensional Hamilton equation on the manifold of solutions of partial differential constraint equation [7, Equation Chapter III, Equation (4.3)].
By R. Hamilton’s tame implicit function theorem (see [9]), the constraint manifold carries a global chart of a Frechet vector space (Theorem 2.2 below).

The latter facilitates Berezin-Fock quantization citeBerezin of the DYM Hamiltonian with a positive spectral mass gap (Theorem 3.1 below).

1.3 Nomenclature

To convert dimensional physical magnitudes into pure mathematical ones, the natural units of quantum field theory are used: Planck’s $\hbar$ (relevant for quantum effects), Einstein’s $c$ (relevant for relativistic effects), and a characteristic length unit, e.g. fm (relevant at nucleonic MeV energies).

In this paper, smooth means infinitely differentiable.

The symbol $*$ is used for a Hermitian conjugation. The bracketless notation, say $z^*w$, means $\langle z | w \rangle$.

The Lorentz metric has the signature $(+,-,-,-)$.

2 Classical DYM Hamiltonian

2.1 Yang-Mills bosons

The global gauge group $G$ of a Yang-Mills theory is a connected semisimple compact Lie group with the Lie algebra $g$ of skew-symmetric matrices $X = -X^T$.

The Lie algebra carries the adjoint representation $\text{Ad}(g)X = gxg^{-1}$, $g \in G$, $X \in g$, of the group $G$ and the corresponding representation $\text{ad}(X)y = [X,Y]$, $X,Y \in g$. The adjoint representation is orthogonal with respect to the positive definite $\text{ad}$-invariant scalar product

$$X \cdot Y := \text{trace}(\text{ad}X^T \text{ad}Y) = -\text{trace}(\text{adXadY}),$$

the negative of Killing form on $g$.

The local gauge group $\tilde{G}$ is the group of smooth $G$-valued functions $g(x)$ on $\mathbb{R}^{1+3}$ with the point-wise group multiplication. The local gauge Lie algebra $\tilde{g}$ of $g$-valued functions on $\mathbb{R}^{1+3}$ consists of infinitely differentiable $g$-valued functions on $\mathbb{R}^{1+3}$ with the point-wise Lie bracket.

$\tilde{G}$ acts via the pointwise adjoint action on the real vector space $\mathcal{A}$ of gauge fields $A = A_\mu(x) \in \tilde{g}$.

Gauge fields $A$ define the covariant partial derivatives of

$$\partial_{A,\mu}X := \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \mathcal{A}.$$  \hspace{1cm} (2.2)

Any $\tilde{g} \in \tilde{G}$ defines the affine gauge transformation

$$A_\mu \rightarrow A^\tilde{g}_\mu := \text{ad}(\tilde{g})A_\mu - (\partial_\mu \tilde{g})\tilde{g}^{-1}, \quad A \in \mathcal{A},$$ \hspace{1cm} (2.3)
so that $A^{\hat{g}_1\hat{g}_2} = A^{\hat{g}_1\hat{g}_2}$.

Relativistic Yang-Mills curvature $F(A)$ is the antisymmetric tensor

$$F(A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu].$$

(2.4)

The curvature is gauge invariant:

$$\text{Ad}(g)F(A) = F(A^g).$$

(2.5)

YM fields are solutions of the relativistic Yang-Mills equation in the vacuum of the $g$-valued partial differential equation

$$\partial_{A,\mu} F_{\mu,\nu} = 0.$$  

(2.6)

This is a relativistic gauge invariant semi-linear 2nd order partial differential $g$-valued equation for one unknown $g$-valued variable $A$.

Every $\mathbf{G}$-orbit contains a connection $A$ with $A_0 = 0$ (see [15]).

Henceforth to reduce the gauge arbitrariness we impose this Hamiltonian (aka temporal) gauge condition.

The splitting $\mathbb{R}^{1+3} = \mathbb{R} \times \mathbb{R}^3$ implies $A_\mu = (A_0,A_j)$, $j = 1,2,3$, where $A_0$ are scalar fields and $A_j$ are Euclidean vector fields. This yields the splitting

$$F^{\mu\nu} = (E^i := F^{0j}, \ B^i := (1/2)\epsilon^{ijk} \partial_{A,j} \partial_{A,k}).$$

(2.7)

The vector fields $E^i$ and $B^i$ are gauged versions of electric and magnetic fields.

### 2.2 Larks

Let the Minkowski space-time $\mathbb{R}^{1+3} = \mathbb{R} \times \mathbb{R}^3$ be proper, i.e. time and space oriented. Proper Lorentz transformations preserve both the space and time orientations; the orthochronous Lorentz transformations preserve the time orientation but may flip the space orientation. Actually the orthochronous Lorentz group is generated by proper Lorentz transformations and the improper Lorentz parity transformation of the spatial inversion.

$$P(x^0,x^1,x^2,x^3) := (x^0,-x^1,-x^2,-x^3).$$

(2.8)

The group $\text{SL}(2,\mathbb{C})$ of complex $2 \otimes 2$-matrices with the unit determinant is the double cover group of the group of proper Lorentz transformations. Two different $2 \otimes 2$-matrices cover the same proper Lorentz transformation if and only if they differ by sign.

Left-handed (vs. right-handed) Weyl spinors space $\mathbb{C}^2$ (vs. $\overline{\mathbb{C}}^2$) carries the defining representations of the matrix group $\text{SL}(2,\mathbb{C})$ and its complex conjugate $\text{SL}(2,\overline{\mathbb{C}})$. The corresponding left and right two-signed representations of the proper Lorentz group are irreducible, antidual, and non-equivalent. They are intertwined by the representation of improper Lorentz parity transformation.
Remark 2.1 Occasionally the group $SL(2, \mathbb{C})$ is claimed to be isomorphic to the group $SU(2) \otimes SU(2)^*$. However the Lie algebras $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{su}(2) \times \mathfrak{su}^*(2)$ of these groups have different real dimensions!

Actually, $\mathfrak{sl}(2, \mathbb{C})$ is the complex envelope of the real compact semi-simple Lie algebra $\mathfrak{su}(2) \times \mathfrak{su}^*(2)$. Thus, by the principle of analytic continuation (see [18, Chapter VI, Section 41]), the tensor algebra of finite-dimensional representations of $\mathfrak{su}(2) \times \mathfrak{su}^*(2)$ is isomorphic via analytic continuation to the tensor algebra of their holomorphic and anti-holomorphic extensions to representations of $\mathfrak{sl}(2, \mathbb{C})$.

The representations algebra is generated by the fundamental representations of $\mathfrak{su}(2) \times \mathfrak{su}^*(2)$.

Massless Dirac fields $\Psi = \Psi(t, x) : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^2$ are solutions of the massless Dirac equation

$$\gamma^\mu \partial_\mu \Psi = 0,$$

where $\gamma_\mu$ are Dirac $(4 \times 4)$-matrices. The spinors $\Psi$ are subject to the Dirac conjugation $\Psi^\dagger := \Psi^* \gamma_0$.

Let the global gauge group $G$ be an irreducible unitary representation of a compact semi-simple Lie group on a finite multiplet of massless Dirac spinor fields $\Psi \in \mathfrak{g}^*(\mathbb{R}^{1+3}, \mathbb{C}^2)$.

Lark fields $\Lambda(t, x) := (\Psi, A)(t, x)$ are solutions of the DYM partial differential $\mathfrak{g}$-valued equation consisting of minimally coupled Dirac (2.9) and Yang-Mills (2.6) equations (see e.g. [5, Section 2.1]).

$$\gamma^\mu \partial_\mu \Lambda_{\mu} = 0, \quad \partial_\lambda F_{\mu \nu} = J_{\lambda, \mu},$$

where $J_{\lambda, \mu} := \overline{\Psi} \gamma_\mu A \Psi$ is the source DYM current. The equation implies the gauged conservation law

$$\partial_\lambda J_{\lambda, \mu} = 0.$$

Remar 2.2 Larks with $G = \mathfrak{su}(2) \times \mathfrak{su}^*(2)$ are gauge covariant massless leptons. Larks with $G = \text{the fundamental representation of } SU(3)$ are pairs of Wilczek lite quarks.

The DYM Equation (2.10) is equivalent to the system of the first order partial differential $\mathfrak{g}$-valued equations, the combination of the system of dynamical evolution equations

$$\partial_t \Psi = i\gamma_0 \gamma_k \partial_{A,k} \Psi, \quad \partial_t \overline{\Psi} = i\gamma_0 \gamma_k \partial_{A,k} \overline{\Psi}, \quad \partial_t A_k = E_k,$$

$$\partial_t E_k = -(1/2) \varepsilon_{klm} \partial_{A,l} B_m + J_k, \quad \partial_t B_k = (1/2) \varepsilon_{klm} \partial_{A,l} A_m,$$

1 Actually the massless Dirac equation is the direct product of Weyl equations for left and right 2-spinor fields. However Weyl equations are not considered in this paper.

2 “But O, Wreneagle Almighty, wouldn’t un be a sky of a lark” (see [10, Page 383]).
and the non-dynamical system of constraint equations

\[ B_k = (1/2)e_{kln}\partial_{A_l}A_m, \quad C(x) := \partial_k E_k - [A_k, E_k] + J_0 = 0. \]  

(2.14)

The constraints are conserved on solutions of the evolution equations (2.12) and (2.13) in view of Bianchi identity and Equation (2.11).

The solutions set of Equation (2.10) is invariant under the action of the local gauge group $\tilde{G}$. Every its orbit contains a smooth connection $A$ subject to Hamiltonian, aka temporal gauge (see [15]). Henceforth I consider only such connections.

**Theorem 2.1** There exists a unique smooth global solution $\Lambda(t,x)$ of the Cauchy problem on Minkowski space-time for DYM equations with arbitrary smooth initial data

\[ \lambda(x) := (\Psi := \Psi(0,x), a := A(0,x), e := E(0,x)). \]  

(2.15)

A solution at $(t,x)$ is uniquely defined by their initial data $\lambda(x)$ on the spatial balls $B_t := \{ x : |x| < t \} \subset \mathbb{R}^3$ (i.e. $B_t$ is the dependence domain of $\Lambda(t,x)$ at $(t,x)$).

In particular $\Lambda(t,x)$ are pointwise limits of solutions with initial data cutoffs in $B_t$ as $t \rightarrow \infty$.

**Proof** The Equations (2.12) and (2.13) is a first order hyperbolic semi-linear partial differential system for independent components $(\Psi, B, A)$ with the linear symmetric principal part and a polynomial field $f$ as a non-linear term with $f(0) = 0$. Let $\beta_r(t,x) \geq 0$ be a smooth cutoff function with compact support that is equal 1 on the solid cone $\{ (t,x) : |x| < r - t, 0 < t < r \}$.

By [12], Theorem 6.11 with Note 3 and Theorem 7.3], the semi-linear system with cutoff nonlinearity $\beta f$ has a unique solution in the frustum

\[ \mathbb{F}_r := \{ (t,x) : |x| < r - t, 0 < t \leq r/2 \}, \]  

(2.16)

where $\beta_r f = f$.

Furthermore, the one-to-one operator from smooth initial data on the base $t = 0$ of $\mathbb{F}_r$ to smooth data on its top $t = r/2$ is a continuous map in the topology of the Frechet vector spaces. Then, by the open map theorem for Frechet spaces, the inverse operator is continuous, i.e. the Cauchy problems on the frusta are well-posed and their solutions have the unit propagation speed.

Next, consider smooth Cauchy data $\lambda$ on the whole Euclidian space $\mathbb{R}^3$ with no restrictions at the spatial infinity.

Let $\alpha_{2j}(x) \geq 0$ be smooth functions with compact support in the balls $B_{2j} = \{|x| < 2j, j = 1, 2, \ldots \}$ such that $\alpha_j(x) = 1$ on the balls $B_j = \{|x| < j \}$. Then the solutions of the Cauchy problem for the evolution system with the initial data $\alpha_j(x)\Lambda(0,x)$ have unique solutions $\Lambda_j(t,x)$ on the strips $C_j := \{ (t,x) : 0 < t < j, x \in \mathbb{R}^3 \}$.

As $j$ converges to infinity, the balls $B_j$ increase to $\mathbb{R}^3$, the cylinders $C_j$ to the upper half $t > 0$ of the Minkowski space, the initial datum $\alpha_j(x)\lambda(0,x)$ converges pointwise
to the initial datum \( \lambda(0, x) \), and then \( \lambda_j(t, x) \) converges to the unique global solution of the Cauchy problem for Equations (2.12), (2.13) on \( \mathbb{R}^{1,3} \) constraint by Equation (2.14).

The system of DYM Equations (2.12), (2.13), and (2.14) for fields \( \Psi, A, E, B \) is equivalent to the shorter system for \( \Psi, \lambda, E \) (see [14] Equation 1.1), the combination of dynamical evolution equations

\[
\begin{align*}
\partial_t \Psi &= i \gamma_0 \gamma_k \partial_{A,k} \Psi, & \partial_t \overline{\Psi} &= i \gamma_0 \gamma_k \overline{\partial_{A,k} \Psi}, & \tag{2.17} \\
\partial_t A_k &= E_k, & \partial_t E_k &= -(1/2) \varepsilon_{klm} \partial_{A,l} B_m + J_k, & \tag{2.18}
\end{align*}
\]

and the non-dynamical system of constraint equations

\[
C(x) := \partial_k E_k - [A_k, E_k] + J_0 = 0. \tag{2.19}
\]

**Remark 2.3** In the special case of YM equations (when \( J = 0 \)) Theorem 2.7 is due to [8] under the weaker condition that the initial datum belongs to local Sobolev spaces \( H^2 \). For DYM equations a local version of Theorem 2.7 is due to [5] and [14]. Actually, by [12] Theorem 6.11 with Note 3 and Theorem 7.3, the global Theorem 2.7 holds under the same Sobolev condition.

The short evolution system of (2.17) and (2.18) is rewritten as the functional Hamiltonian equation [7] Chapter III, Equations (4.4)-(4.6)] for fields

\[
\psi(t) := \Psi(t, x), \quad a(t) := a_k(t, x), \quad e_k(t) := F_{0k}(t, x),
\]

that are solutions of

\[
\begin{align*}
\partial_t \psi^* &= i \partial_x \psi^*, & \partial_t \psi &= -i \partial_x \psi^* H, & \partial_t a_k &= \partial_x H, & \partial_t e &= -\partial_x H, & \tag{2.21} \\
H &= \frac{1}{2} \int_{\mathbb{R}^3} d^3x \left( i \psi^* \gamma_0 \gamma_k \partial_{A,k} \psi + \kappa^{-2}(\| \text{curl}^{(a)} a \|^2 + \| e \|^2) \right), & \tag{2.22}
\end{align*}
\]

where

\[
\text{div}^{(a)} e = \text{div} e - [a_k, e_k], \quad (\text{curl}^{(a)} a)_k := (\text{curl} a)_k - (1/2) \varepsilon_{ijk}[a_j, a_k],
\]

and \( \kappa \) is a dimensionless coupling constant.

The time-independent Hamiltonian functional \( H \) is well defined by the initial values of (2.20) with compact supports in the initial Euclidean space \( \mathbb{R}^3 \).

Equation (2.21) shows that \( (\psi^* = \psi^*(0, x), \psi = \psi(0, x)) \) and \( (a = a(0, x), e = e(0, x)) \) are pairs of canonically conjugate variables with respect to the Poisson brackets on the Cauchy data defined via the Jacobi identity by the Poisson brackets

\[
\begin{align*}
\{ H, \psi^* \} &= \delta H / \delta \psi, & \{ H, \psi \} &= -\delta H / \delta \psi^*, & \tag{2.24} \\
\{ H, a \} &= \delta H / \delta e, & \{ H, e \} &= -\delta H / \delta a. & \tag{2.25}
\end{align*}
\]
2.3 DYM phase space

By equation (2.2), the dimensionality homogeneity implies \( a \propto L^{-1} \) in the natural units \( c = 1, \hbar = 1 \), where \( L \) is a characteristic length. Consequently,

\[
a \propto L^{-1}, \quad e \propto L^{-2}, \quad \psi^* \propto L^{-3/2}, \quad \psi \propto L^{-3/2}, \quad H \propto L^{-1}, \quad C \propto L^{-2}.
\]

Thus the \( R \)-scaling \( x \mapsto Rx, \ 0 < R < \infty \), transforms the Hamiltonian functional \( H \) into \( R^{-1}H \).

By Theorem 2.1, larks \( \Lambda(t,x) \) are uniquely defined by their Cauchy data on the spatial balls \( B(R) := \{ x : |x| < R \} \subset \mathbb{R}^3 \), and then, via the scaling covariance, by the restricted Hamiltonian functional on \( B := B(1) \):

\[
H_B := \int_B d^3x \left( i\psi^* \kappa^{-2} (\text{curl}^{(0)} a)^2 + ||e||^2 \right),
\]

(2.27)

Sobolev-Hilbert spaces \( \mathcal{A}^s \) of connections \( a(x) \) are the completions of the spaces of smooth connections with compact supports in the open unit ball \( B \) with respect to the norm squares

\[
||a||^2_s := \int_B dx \left( a(1 - \Delta)^s a \right) < \infty.
\]

(2.28)

The topological intersection \( \mathcal{A} := \cap \mathcal{A}^s \) is a real Frechet space, a subspace of the Hilbert space \( \mathcal{A}^0 \).

Similarly we define the real Frechet spaces \( \mathcal{E} := \cap \mathcal{E}^s \) of connections \( e(x) \), \( \mathcal{I} \) of real scalar fields \( u(x) \) on \( B \) with values in \( \text{ad} \mathbb{G} \), the complex Frechet vector space \( \mathcal{D} \) of spinor fields on \( B \), and their Hilbert space completions \( \mathcal{E}^0(B), \mathcal{I}^0(B), \mathcal{D}^0(B) \).

By Sobolev imbedding theorem, the elements of \( \mathcal{A}, \mathcal{E}, \mathcal{I}, \mathcal{D} \) are smooth fields.

By [9, Corollaries 1.3.7, and 1.3.8.], \( \mathcal{A}, \mathcal{E}, \mathcal{I}, \mathcal{D} \) are tame Frechet spaces.

Lastly, by [9, Example 1.2.2.(3)], the semi-linear partial differential constraint self-transformation of \( \mathcal{D} \times \mathcal{A} \times \mathcal{E} \)

\[
(\psi^*, \psi, a, e) \mapsto i\psi^* \gamma_0 a \psi + \text{div} e - [a_k, e_k]
\]

(2.29)
is tame. Then the next lemma is a corollary of the R. Hamilton implicit function theorem [9, Theorems 3.3.1 and 3.3.4.].

**Lemma 2.1** If \( (\psi^*, a^\circ, e^\circ) \) is a solution of the constraint equation \( c = 0 \) then there is a unique smooth tame map \( e(\psi, a) \) of a neighborhood \( \mathcal{N}^\circ \) of \( (\psi^*, a^\circ) \) in \( \mathcal{D} \times \mathcal{A} \) to \( \mathcal{E} \) such that

\[
C(\psi, a, e(\psi, a)) = 0.
\]

(2.30)

**PROOF** It suffices to check that the partial derivative with respect to \( e \) of the constraint functional \( C(\psi^*, \psi, a, e) \), (which is linear in \( e \))

\[
\partial_e C = \text{div}^{[a]} e
\]

(2.31)
is continuous with respect to \((\psi, a)\), surjective, and has right inverse. Since Laplacian \(\triangle\) commutes with \((1 - \triangle)^s\), it is tame mapping from \(\mathcal{S}\) to itself. Since the inverse \(\triangle^{-1}: \mathcal{S}^2 \to \mathcal{S}\) exists, the operator \(\triangle\) is invertible in \(\mathcal{S}\).

The gauge Laplacian \(\triangle[a] := \|\text{div}^{[a]}\|^2\) differs from the usual Laplacian \(\triangle\) by a first order differential operator. Therefore it is a tame Fredholm operator of zero index in \(\mathcal{S}\).

If \(\triangle[a]u = 0\) then
\[
(\triangle[a]u)u = (\text{grad}^{[a]}u)(\text{grad}^{[a]}u) = 0,
\] (2.32)
so that \(\text{grad}u - [a, u] = 0\), or \(\partial_k u = [a_k, u]\).

The computation
\[
(1/2)\partial_k(uu) = (\partial_k uu) = [a_k, u]u = -\text{trace}(a_k uu - ua_k u) = 0
\] (2.33)
implies that the solutions \(u \in \mathcal{S}\) are constant. Because they vanish on the ball boundary, they vanish on the whole ball. Since the index of the Fredholm operator \(\triangle[a]\) is zero, its range is a closed subspace with the codimension equal to the dimension of its null space. Thus the operators
\[
\text{div}^{[a]}\text{grad}^{[a]}: \mathcal{S} \to \mathcal{S}, \quad a \in \mathcal{A},
\] (2.34)
are surjective and have the right inverse \(\text{grad}^{[a]}(\triangle[a])^{-1}\). QED

By their existence and uniqueness, the local maps \(e = e(\psi, a)\) convert the solutions set of the constraint equation (2.14) into a trivial bundle with the base \(\mathcal{D} \times \mathcal{A}\) and fibers \(\mathcal{E}(\psi, a)\) that consist of solutions for the equation
\[
\text{div} e = -i\psi \gamma_0 a \psi + [a_k, e_k].
\] (2.35)
The fiber \(\mathcal{E}_\perp := \mathcal{E}(\psi, 0)\) consists of solutions for the transversality equation \(\text{div} e = 0\).

The term of (2.27)
\[
\|\text{curl}^{[a]}\|^2 + \|e\|^2 = -F_{jk}F_{jk}
\] (2.36)
is the curvature of the time-independent gauge fields \(a(x)\). Thus \(H\) is invariant under smooth local time-independent gauge group.

By \[3, Proposition 1\], the closure of the local gauge Lie group \(\overline{\mathcal{G}}^1\) in the Sobolev space \(\mathcal{W}^{1}(\mathbb{B})\) is an infinite-dimensional compact group with a continuous action in the Hilbert space \(\mathcal{A}^0\). The action orbits are compact so that the squared continuous Hilbert norm \(\|a\|^2\) has an absolute minimal value on every orbit which is attained at a weakly transversal \(a\), \(\text{div} a = 0\).

By Sobolev embedding theorem, \(\mathcal{W}^{1}(\mathbb{B}) \subset \mathcal{W}^6(\mathbb{B})\). Therefore the functional \(H\) has a unique continuation to \(\overline{\mathcal{G}}^1\)-orbits in \(\mathcal{D}^0 \times \mathcal{A}^0 \times \mathcal{E}^0\) and is constant on each of them.

\(^3\overline{\mathcal{G}}^1\) is not a Lie group.
Therefore the space of smooth constraint initial data has the total Frechet vector space chart of the direct product $E \perp \times (D \times A \perp)$ of weakly transversal fields with the flat parallel transport preserving the norm $\|e\|$.

All in all we have proved

**Theorem 2.2** DYM Hamiltonian functional $H(\psi^*, \psi, a, e)$, constrained by Equation (2.14), is uniquely determined by its restriction to the Frechet space $A \perp \times E \perp \times D$.

Now modify the DYM Hamiltonian as the scaling invariant integral

$$H_R(\psi^*, \psi, a, e) := (R/2) \int_{B(R)} d^3x (i\psi^* \gamma_0 \partial_{a,h} \psi + \kappa^{-2}(\|\text{curl}^{(a)} a\|^2 + \|e\|^2)) \quad (2.37)$$

In view of the scaling covariance let us set $R = 1$.

Then the space of *dimensionless* complex combinations of transversal Yang-Mills fields

$$z := (1/\sqrt{2})(a + ie), \quad z^* := (1/\sqrt{2})(a - ie). \quad (2.38)$$

admits the global Frechet vector space chart $Z := A \perp \times E \perp \times D^* \perp \times D \perp$ on the phase space of the Hamiltonian system (2.21).

### 3 Quantum DYM Hamiltonian

Fix a Yang-Mills field $a' \in A \perp$, $e' \in E \perp$ and consider the partial Hamiltonian

$$H = (1/2) \int_B d^3x (i\psi^* \gamma_0 \partial_{a,h} \psi + \kappa^{-2}(\|\text{curl}^{(a')} a'\|^2 + \|e'\|^2) =: H'(\overline{\psi}, \psi) + K,$$

where $K$ is a constant.

Replace $H$ with $H'$ since the Hamiltonian dynamics on the complex phase space $D^* \perp \times D \perp$ is the same if Hamiltonians differ by an additive constant.

The Hamiltonian $H'$ is a Hermitian real-valued quadratic form of the essentially selfadjoint operator $i\gamma_0 \partial_{a,h}$ on the domain $D \perp$.

Let $D$ denote its selfadjoint closure in $D^0$. Since $D^2$ is a strongly elliptic second order partial differential operator on $B$, the spectrum of $D^2$, and therefore the spectrum of $D$, are sequences of finitely multiple eigenvalues that converge to $\infty$ (see [11, Section 1.3]).

Denote $\mathcal{F}$ the Fock functor (see [13, Epigraph to Section X.7]). Then the quantum Hamiltonian $\hat{D}$ is an unbounded operator in the fermionic Fock space $\mathcal{F} D^0$ over the Hilbert space $D^0$ by [2, Section 6, Theorem 1], the operator $\hat{D}$ is essentially self-adjoint in $\mathcal{F} D^0$ on the domain $\mathcal{F} D \perp$.

**Theorem 3.1** The spectrum of $\hat{H}'$ is sequence of finitely multiple eigenvalues converging to $\infty$, and therefore has a mass gap, i.e. the zero spectral point is isolated.
PROOF  $D$ is an orthogonal sum of the operators $D_+, D_-, D_0$ such that

$$i\bar{\psi}D_+\psi \geq 0, \quad i\bar{\psi}D_-\psi \leq 0, \quad i\bar{\psi}D_0\psi = 0, \quad \psi \in \mathcal{D}. \quad (3.2)$$

Then the spectrum of $D_+$ is a sequence of positive eigenvalues of $D$ converging to $+\infty$, the spectrum of $D_-$ is a sequence of negative eigenvalues of $D$ converging to $-\infty$, and the spectrum of $D_0$ is the zero point (all eigenvalues are counted according to their multiplicity).

Let

$$\mathcal{D}_0 = \mathcal{D}_+^0 \oplus \mathcal{D}_-^0 \oplus \mathcal{D}_0^0 \quad (3.3)$$

be the corresponding orthogonal expansion of the Hilbert space $\mathcal{D}_0$.

Since the functor $\mathcal{F}$ is abelian we have

$$\hat{D} = \mathcal{F}D_+ \oplus \mathcal{F}D_0 \oplus \mathcal{F}D_- \quad (3.4)$$

Let $\{e_k\}$ be an orthonormal eigenbasis for $D_+$ with eigenvalues $\lambda_k > 0$; $\{e_l\}$ be an orthonormal eigenbasis for $D_-$ with negative eigenvalues $\lambda_l < 0$; and $\{e_m\}$ be an orthonormal basis for $D_0$ with the eigenvalues $\lambda_m = 0$. Note that $D_0$ has a finite rank.

Then

• $\mathcal{F}D_+$ has an orthonormal basis of $n!^{-1/2} \wedge_{k=1}^n e_k$, $n \geq 0$, with the eigenvalues $\sum_{k=1}^n e\lambda_k$;

• $\mathcal{F}D_-$ has an orthonormal eigenbasis of $n!^{-1/2} \wedge_{l=1}^n e_l$, $n \geq 0$, with the eigenvalues $\sum_{l=1}^n \lambda_l$;

• $\mathcal{F}D_0$ has a finite orthonormal eigenbasis of $n!^{-1/2} \wedge_{m=1}^n e_m$, $n \geq 0$, with zero eigenvalues.

Thus spectrum of the quantum Hamiltonian $\hat{H}' = \mathcal{F}D$ is a sequence of eigenvalues converging to $\infty$ and the vacuum eigenvalue 0 is an isolated point of the spectrum of $\hat{H}_\mathbb{B}$, so that the spectrum has a positive mass gap. QED

Remark 3.1 By the dilation covariance, the restricted Hamiltonian over the Euclidean ball $\mathbb{B}(R)$,

$$H_R(\bar{\psi}, \psi) := \int_{\mathbb{B}(R)} d^3x i\bar{\psi}i\gamma_j \partial_{\kappa A, j} \psi = R^{-1}H_\mathbb{B}. \quad (3.5)$$

Thus the mass gap of $\hat{H}_R$ is running in the inverse proportion to $R$, i.e. in the direct proportion to the effective classical energy. Incidentally it depends on the selected $\alpha'$. 

11
References

[1] Quantum Yang-Mills theory, [http://www.claymath.org/prizeproblems/index.html](http://www.claymath.org/prizeproblems/index.html)

[2] F. A. Berezin, The method of second quantization (Academic Press, 1966).

[3] Dell’Antonio, G. and Zwanziger, D., Every gauge orbit passes inside the Gribov horizon, Comm. Math. Phys., 138 (1991), 259-299.

[4] A. Dynin, Mathematical quantum Yang-Mills theory revisited, Russian Journal of Mathematical Physics, 24, (1), 26-43 (2017).

[5] Y. Choquet-Bruhat, D. Christodoulou, Existence of global solutions of the Yang-Mills, Higgs and spinor Weyl equations in 3+1 dimensions, Annales scientifiques de l’E.N.S., 4e serie, 14, 481-506, 1981.

[6] D.M. Eardly, V. Moncrief, The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. II, Commun. Math. Phys., 83(1982),194-212.

[7] L. Faddeev, A. Slavnov, Gauge fields, introduction to quantum theory, addison-Wesley, 1991.

[8] Goganov, M. V., Kapitanskii, L. V., Global Solvability of the Initial Problem for Yang-Mills-Higgs Equations, Zapiski LOMI 147, (1985); J. Sov. Math. 37(1987), 802-822.

[9] R. Hamilton, The inverse function theorem of Nash-Moser, Bulletin of the AMS, 7 (1982), 65-222.

[10] Joyce J., Finnegans Wake, Wordsworth Classics, Various printings.

[11] O. A. Ladyzhenskaya, The boundary value problems of mathematical physics (Springer-Verlag, 1985).

[12] S. Mizohata, The theory of partial differential equations, Cambridge, 1973.

[13] M. Reed, B. Simon, II. Methods of modern mathematical physics, Academic Press, 1972.

[14] G. Schwartz, J. Sniaticky, Yang-Mills and Dirac fields in a bag, existence and uniqueness theorems. Comm. Math. Phys., 168(1995), 441-453.

[15] I. Segal, The Cauchy problem for Yang-Mills equations, J. Func. Anal 33 (1979), 175-194.

[16] E. C. G. Stueckelberg, A. Petermann, La normalisation des constantes dans la theorie des quanta, Helv. Phys. Acta 26, 499-520 (1953).
[17] F. Wilczek, *Four Big Questions with Pretty Good Answers*, Werner Heisenberg Centennial Symposium ”Developments in Modern Physics” (Buschhorn, G., Wess, J., eds), 79-98, Springer, 2004.

[18] D. Zhelobenko, *Compact Lie groups and their representations*, AMS 1973.