Thouless bandwidth formula in the Hofstadter model

Stéphane Ouvry and Shuang Wu

LPTMS, CNRS-Faculté des Sciences d’Orsay, Université Paris Sud, 91405 Orsay Cedex, France

E-mail: stephane.ouvry@u-psud.fr

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Abstract
We generalize the Thouless bandwidth formula to its \(n\)th moment. We obtain a closed expression in terms of polygamma, zeta and Euler numbers.

Keywords: Thouless bandwidth, \(n\)th moment, Hofstadter spectrum

1. Introduction

In a series of stunning papers stretching over almost a decade [1], Thouless obtained a closed expression for the bandwidth of the Hofstadter spectrum [2] in the \(q \to \infty\) limit. Here the integer \(q\) stands for the denominator of the rational flux \(\gamma = 2\pi p/q\) of the magnetic field piercing a unit cell of the square lattice; the numerator \(p\) is taken to be 1 (or equivalently \(q-1\)). In the following, \(p\) will always be understood as equal to 1.

Let us recall that in the commensurate case where the lattice eigenstates \(\Psi_{m,n} = e^{i k_c k} \Phi_m\) are \(q\)-periodic \(\Phi_{m+q} = e^{i k_b} \Phi_m\), with \(k_c, k_b \in [-\pi, \pi]\), the Schrödinger equation

\[\Phi_{m+1} + \Phi_{m-1} + 2 \cos(k_y + \gamma m) \Phi_m = e \Phi_m\]

reduces to the \(q \times q\) secular matrix

\[m_{p/q}(e, k_x, k_y) = \begin{pmatrix}
2 \cos(k_x) - e & 0 & \cdots & 0 & c^{-ip_b} \\
1 & 2 \cos(k_y + \frac{2\pi p}{q}) - e & 1 & \cdots & 0 \\
0 & 1 & (\cdot) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

acting as

*Author to whom any correspondence should be addressed.*
on the $q$-dimensional eigenvector $\Phi = \{\Phi_0, \Phi_1, \ldots, \Phi_{q-1}\}$. Thanks to the identity
\[
\det(m_{p/q}(e, k_i, k_i)) = \det(m_{p/q}(0, 0)) - 2(1)^q \cos(qk_i) - 1 + \cos(qk_i) - 1,
\]
the eigenenergy equation $\det(m_{p/q}(e, k_i, k_i)) = 0$ rewrites [3] as
\[
\det(m_{p/q}(e, 0, 0)) = 2(-1)^q \cos(qk_i) - 1 + \cos(qk_i) - 1.
\]
(3)
The polynomial
\[
b_{p/q}(e) = -\sum_{j=0}^{[\frac{q}{2}]} a_{p/q}(2j)e^{2j}
\]
materializes in $\det(m_{p/q}(e, 0, 0))$
\[
\det(m_{p/q}(e, 0, 0)) + 4(-1)^q = (-1)^q e^q b_{p/q}(1/e)
\]
so that (3) becomes
\[
e^q b_{p/q}(1/e) = 2(\cos(qk_i) + \cos(qk_i))
\]
The coefficients $a_{p/q}(2j)$’s (with $a_{p/q}(0) = -1$) in the polynomial $b_{p/q}(e)$ above are related to the Kreft coefficients [4] so that
\[
a_{p/q}(2j) = (-1)^{j+1} \sum_{k_1=0}^{q-2j} \sum_{k_2=0}^{k_1} \ldots \sum_{k_{j-1}=0}^{k_{j-2}} 4 \sin^2 \left( \frac{\pi (k_1 + 2j - 1)}{q} \right)
\]
\[
\times 4 \sin^2 \left( \frac{\pi (k_2 + 2j - 3)}{q} \right) \ldots 4 \sin^2 \left( \frac{\pi (k_j + 1)}{q} \right)
\]
(4)
How to get this explicit expression is explained in Kreft’s paper.

We focus on the Hofstadter spectrum bandwidth defined in terms of the $2q$ edge-band energies $e_r(4)$ and $e_r(-4)$, $r = 1, 2, \ldots, q$, solutions of
\[
e^q b_{p/q}(1/e) = 4 \text{ and } e^q b_{p/q}(1/e) = -4,
\]
respectively (see figures 1 and 2). If one specifies an ordering for the $e_r(4)$’s and the $e_r(-4)$’s
\[
e_1(4) \leq e_2(4) \leq \ldots \leq e_q(4) \quad \text{and} \quad e_1(-4) \leq e_2(-4) \leq \ldots \leq e_q(-4)
\]
the bandwidth is
\[
(-1)^{q+1} \sum_{r=1}^{q} (-1)^r (e_r(-4) - e_r(4)).
\]
(5)
The Thouless formula is obtained in the $q \to \infty$ limit as
\[
\lim_{q \to \infty} (-1)^{q+1} \sum_{r=1}^{q} (-1)^r (e_r(-4) - e_r(4)) = \frac{32}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k + 1)^2}
\]
(6)
(see also [5]). We aim to extend this result to the $n$th moment defined as
\[
(-1)^{q+1} \sum_{r=1}^{q} (-1)^r (e_r^n(-4) - e_r^n(4)).
\]
(7)
which is a natural generalization of (5); one can think of it as
\[
\int_{-4}^{4} \tilde{\rho}_{p/q}(e) \frac{e}{q} \left( e^{n-1} \right) \, de
\]
where \(\tilde{\rho}_{p/q}(e)\) is the indicator function with value 1 when \(|e^{p/q}(1/e)| \leq 4\) and 0 otherwise.

Let us stress that the bandwidth \(n\)th moment defined for \(n\) odd as
\[
(-1)^{q+1} \sum_{r=1}^{q} (-1)^r \left( e_r(-4) - e_r(4) \right)^n,
\]
is of particular interest. We will come back to this moment in the conclusion.

Trivially (7) vanishes when \(n\) is even—we will see later how to give a non-trivial meaning to the \(n\)th moment in this case. Therefore, we focus on (7) when \(n\) is odd and, additionally, when \(q\) is odd, in which case it simplifies further to
\[
-2 \sum_{r=1}^{q} (-1)^r e_r(4) = 2 \sum_{r=1}^{q} (-1)^r e_r(-4)
\]
thanks to the symmetry \(e_r(-4) = -e_{q+1-r}(4)\). As said above, the \(e_r(4)\)'s are the roots of \(e^{p/q}(1/e) = 4\) that is, by the virtue of (4), those of
\[
\det(m_{p/q}(e, 0, 0)) = 0.
\]

2. The first moment: Thouless formula

The key point in the observation of Thouless [1] is that if evaluating the first moment rewritten in (9) as
\[-2 \sum_{r=1}^{q} (-1)^r e_r(4)\]
when \(q\) is odd seems at first sight intractable, still, thanks to:
- \(\det(m_{p/q}(e, 0, 0))\) factorizing as
where

\[
m_{p/q}^{++}(e) = \begin{pmatrix}
  e - 2 & 2 & 0 & \cdots & 0 & 0 \\
  1 & e - 2 \cos\left(\frac{2\pi p}{q}\right) & 1 & \cdots & 0 & 0 \\
  0 & 1 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 1 & e - 2 \cos\left(2\pi \frac{1}{2} \frac{2\pi p}{q}\right) - 1 \\
\end{pmatrix}
\]

\[
m_{p/q}^{--}(e) = \begin{pmatrix}
  e - 2 \cos\left(\frac{2\pi p}{q}\right) & 1 & 0 & \cdots & 0 & 0 \\
  1 & e - 2 \cos\left(\frac{4\pi p}{q}\right) & 1 & \cdots & 0 & 0 \\
  0 & 1 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 1 & e - 2 \cos\left(\frac{2\pi p}{q} + \frac{1}{2}\right) + 1 \\
\end{pmatrix}
\]

are matrices of size \((q + 1)/2\) and \((q - 1)/2\), respectively, so that the \(e_r(4)\)'s split into two packets \(e_r^{++}, r = 1, 2, \ldots, (q + 1)/2\), the roots of \(\det(m_{p/q}^{++}(e)) = 0\) and \(e_r^{--}, r = 1, 2, \ldots, (q - 1)/2\), those of \(\det(m_{p/q}^{--}(e)) = 0\). And thanks to:

- \(\sum_{r=1}^{q} (-1)^r e_r(4)\) happening to rewrite as

\[-\sum_{r=1}^{q} (-1)^r e_r(4) = \sum_{r=1}^{q} |e_r^{++}| - \sum_{r=1}^{q} |e_r^{--}|\]
(9) becomes tractable since it reduces to the sum of the absolute values of the roots of two polynomial equations.

Indeed, by using [1]

\[ \frac{2i}{\pi} \int_{-ix}^{ix} \left( \frac{z}{z-a} - 1 \right) \, dz = \frac{4a}{\pi} \arctan \left( \frac{x}{a} \right), \]

\[ \lim_{x \to \infty} \frac{4a}{\pi} \arctan \left( \frac{x}{a} \right) = 2|a| \]

and

\[ \lim_{x \to \infty} \frac{2i}{\pi} \int_{-ix}^{ix} \left( \frac{z}{z-a} - 1 \right) \, dz = \lim_{x \to \infty} \frac{2i}{\pi} \int_{-ix}^{ix} (-\log \frac{z-a}{z}) \, dz \]

one gets

\[ 2 \left( \sum_{r=1}^{q+1} |e_{r}^{+}| - \sum_{r=1}^{q} |e_{r}^{-}| \right) = \frac{2i}{\pi} \lim_{x \to \infty} \int_{-ix}^{ix} \log \left( \frac{\det(m_{p/q}^{-1}(z))}{\det(m_{p/q}(z))} \right) \, dz. \quad (10) \]

Making [1] further algebraic manipulations on the ratio of determinants in (10), in particular in terms of particular solutions \( \{\Phi_0, \Phi_1, \ldots, \Phi_{q-1}\} \) of (2)—on the one hand \( \Phi_0 = 0 \) and on the other hand \( \Phi_{(q-1)/2} = \Phi_{(q+1)/2} \)—and then for large \( q \) taking in (1) the continuous limit lead to, via the change of variable \( y = qx/(8\pi) \),

\[ \lim_{q \to \infty} 2q \left( \sum_{r=1}^{q+1} |e_{r}^{+}| - \sum_{r=1}^{q} |e_{r}^{-}| \right) = 32 \int_{0}^{\infty} \log \left( \frac{\Gamma(3/4 + y)^{2}}{\Gamma(1/4 + y)^{2}} \right) \, dy. \]

This last integral gives the first moment

\[ \lim_{q \to \infty} q \left( \sum_{r=1}^{q} (-1)^{r} e_{r}(4) = \frac{4}{\pi} \left( \psi^{(1)} \left( \frac{1}{4} \right) - \pi^{2} \right) \quad (11) \right. \]

which is a rewriting of (6) (\( \psi^{(1)} \) is the polygamma function of order 1).

3. The \( n \)th moment

3.1. \( n \) odd

To evaluate the \( n \)th moment one follows the steps above by first noticing in (9) that

\[ - \sum_{r=1}^{q} (-1)^{r} e_{r}(4) = \sum_{r=1}^{q+1} |e_{r}^{+}| - \sum_{r=1}^{q} |e_{r}^{-}| \]

holds. Then using

\[ \frac{2i}{\pi} \int_{-ix}^{ix} \left( \frac{z^{n}}{z-a} - \sum_{k=0}^{n-1} a^{k} z^{n-1-k} \right) \, dz = \frac{4a^{n}}{\pi} \arctan \left( \frac{x}{a} \right), \]

\[ \lim_{x \to \infty} \frac{4a^{n}}{\pi} \arctan \left( \frac{x}{a} \right) = 2|a^{n}| \]

and
\[
\lim_{x \to \infty} \frac{2i}{\pi} \int_{-i}^{i} \left( \frac{e^x}{x - a} - \sum_{k=0}^{n-1} a^k x^{-k-1} \right) d\zeta = \lim_{x \to \infty} \frac{2i}{\pi} \int_{-i}^{i} -n e^{z-1} \left( \log \frac{z - a}{z} + \sum_{k=1}^{n-1} \frac{a^k}{k^2} \right) d\zeta
\]

one gets
\[
2 \left( \sum_{r=1}^{\infty} |e_r^{+}|^n - \sum_{r=1}^{\infty} |e_r^{-}|^n \right) = 2i \lim_{x \to \infty} \int_{-i}^{i} n e^{z-1} \left( \log \frac{z \det(\Delta_{r}^{+}(z))}{\det(\Delta_{r}^{-}(z))} \right)
- \sum_{k=1}^{n-1} \sum_{r=1}^{\infty} (e_r^{+})^k - \sum_{r=1}^{\infty} (e_r^{-})^k \right) d\zeta.
\]

In the RHS of (12) the polynomial \( n^{-1} \sum_{k=1}^{n-1} \frac{d \theta}{d\theta} \) cancels the positive or null exponents in the expansion around \( z = \infty \) of the logarithm term \( \frac{d \theta}{d\theta} \). Likewise, in (13), the same mechanism takes place for \( -z^n-1 \left( \sum_{k=1}^{n-1} \sum_{r=1}^{\infty} (e_r^{+})^k - \sum_{r=1}^{\infty} (e_r^{-})^k \right) \) with respect to \( \frac{d \theta}{d\theta} \). Additionally, the polynomials can be reduced to their \( k \) even components. Further algebraic manipulations in (13) and, when \( q \) is large, taking the continuous limit, lead to, via the change of variable \( y = qz/(8\pi i) \),
\[
\lim_{q \to \infty} 2q^n \left( \sum_{r=1}^{\infty} |e_r^{+}|^n - \sum_{r=1}^{\infty} |e_r^{-}|^n \right) = (8\pi i)^{n-1} 32
\]
\[
\int_{0}^{\infty} ny^{n-1} \left( \log \left( \frac{\Gamma(3/4 + y)^2}{\sqrt{\Gamma(1/4 + y)^2}} \right) + \sum_{k=2,k \text{ even}}^{n-1} \frac{E_k}{k 4^k y^k} \right) dy.
\]

To go from (13) to (14) one has used that for \( k \) even, necessarily \( 2 \)
\[
\lim_{q \to \infty} q^n \left( \sum_{r=1}^{\infty} (e_r^{+})^k - \sum_{r=1}^{\infty} (e_r^{-})^k \right) = (2\pi i)^{k} |E_k|
\]
where the \( E_k \)’s are the Euler numbers, a result which is also strongly supported by numerical simulations. Indeed in (14), as it was the case in (12,13), the polynomial \( \sum_{k=2,k \text{ even}}^{n-1} \frac{E_k}{k 4^k y^k} \) cancels the positive or null exponents in the expansion around \( y = \infty \) of the logarithm term \( \frac{d \theta}{d\theta} \)—see the appendix for a proof. It amounts to a fine tuning at the infinite upper integration limit so that after integration the end result is finite. Performing this last integral gives the \( n \)th moment—see the appendix for details
\[
\lim_{q \to \infty} q^n \left( \sum_{r=1}^{\infty} (-1)^r (e_r^{+}(4) - e_r^{-}(4)) \right) = \frac{4}{\pi} \left( (-1)^{n-1} q^{2n} \left( \frac{1}{4} \right) - 2^n \left( 2^{n+1} - 1 \right) \zeta(n+1) n! \right)
\]
\[\text{More generally the } k \text{th moments } \sum_{r=1}^{n} (e_r^{+})^k \text{ and } \sum_{r=1}^{n} (e_r^{-})^k \text{ can be directly retrieved from the coefficients of } \det(\Delta_{r}^{+}(e)) \text{ and } \det(\Delta_{r}^{-}(e)), \text{ respectively. In particular, one finds } \sum_{r=1}^{n} e_r^{+} = 2 \text{ and } \sum_{r=1}^{n} e_r^{-} = -2; \text{ for } k \text{ odd } \lim_{q \to \infty} \sum_{r=1}^{n} (e_r^{+})^k = 4^k/2 \text{ and } \lim_{q \to \infty} \sum_{r=1}^{n} (e_r^{-})^k = -4^k/2; \text{ for } k \text{ even } \lim_{q \to \infty} 1/q \left( \sum_{r=1}^{n} (e_r^{+})^k \right) = \lim_{q \to \infty} 1/q \left( \sum_{r=1}^{n} (e_r^{-})^k \right) = \left( \frac{1}{4} \right)^2 / 2. \text{ This last result can easily be understood in terms of the number } \left( \frac{1}{4} \right)^2 \text{ of closed lattice walks with } k \text{ steps [6].} \]
which generalizes the Thouless formula (11) to \( n \) odd \( (\psi^{(n)}) \) is the polygamma function of order \( n \).

3.2. \( n \) even

As said above, the \( n \)th moment trivially vanishes when \( n \) is even. In this case, we should rather consider a \( n \)th moment restricted to the positive—or equivalently by symmetry negative—half of the spectrum\(^3\). In the \( q \) odd case it is

\[
\begin{align*}
&= \sum_{r=1}^{(q-1)/2} (-1)^r (e_r^+(4) - e_r^+(4)) + e_r^{(q+1)/2} \left( (-1)^{\frac{q+1}{2} - 4} \right) \\
&= \sum_{(q+3)/2}^q (-1)^r (e_r^+(4) - e_r^+(4)) + e_r^{(q+1)/2} \left( (-1)^{\frac{q+1}{2} - 4} \right).
\end{align*}
\]

(17)

It is still true that

\[
\begin{align*}
\sum_{r=1}^{q+1} (e_r^+)^n - \sum_{r=1}^q (e_r^-)^n &= \sum_{(q+3)/2}^q (-1)^r (e_r^+(4) - e_r^+(4)) + e_r^{(q+1)/2} \left( (-1)^{\frac{q+1}{2} - 4} \right) \\
\end{align*}
\]

where, since \( n \) is even, absolute values are not needed anymore, a simpler situation. It follows that the RHS of (16) also gives, when \( n \) is even, twice the \( q \to \infty \) limit of the half-spectrum \( n \)th moment as defined in (17), up to a factor \( q^n \).

3.3. Any \( n \)

One reaches the conclusion that

\[
\begin{align*}
\frac{4}{\pi} \left( (-1)^{n-1} \psi^{(n)} \left( \frac{1}{4} \right) - 2^n \left( 2^n - 1 \right) \zeta(n+1)n! \right) &= \frac{2}{\pi} 4^{n+1} n! \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{2k+1} \right)^{n+1} \\
&= \frac{2}{\pi} n! \left( \zeta(n+1, 1/4) - \zeta(n+1, 3/4) \right) \\
&= \frac{2}{\pi} \left( -1 \right)^{n+1} \left( \psi^{(n)} \left( \frac{1}{4} \right) - \psi^{(n)} \left( \frac{3}{4} \right) \right)
\end{align*}
\]

(18)

yields \( q^n \) times the \( n \)th moment when \( n \) is odd\(^4\)

\(^3\) Instead of \( n \int_{-4}^{4} \hat{\rho}_{\gamma/4}(e)e^{e-1} \, de \) one considers

\[
n \int_{-4}^{4} \hat{\rho}_{\gamma/4}(e)e^{e-1} \, de = n \int_{0}^{4} \hat{\rho}_{\gamma/4}(e)e^{e-1} \, de.
\]

\(^4\) When \( n \) is odd it is also twice the half-spectrum \( n \)th moment

\[
n \int_{-4}^{4} \hat{\rho}_{\gamma/4}(e)e^{e-1} \, de = 2n \int_{0}^{4} \hat{\rho}_{\gamma/4}(e)e^{e-1} \, de.
\]
and twice the half-spectrum \( n \)th moment when \( n \) is even. Numerical simulations confirm this result convincingly (even though the convergence is slow when \( n \) becomes large). In the \( n \) even case one already knows from (15) that (18) simplifies further to

\[ 2|E_n|(2\pi)^n \]

from which one gets for the \( n \rightarrow \infty \)-moment scaling

\[ \frac{8}{\pi} n! 2^{2n}. \]

4. Conclusion and opened issues

(18) is certainly a simple and convincing \( n \)th moment generalization of the Thouless band-width formula (6). It remains to be proven on more solid grounds for example in the spirit of [5].

In the definition of the \( n \)th moment (7), one can view the exponent \( n \) as a magnifying loupe of the Thouless first moment. (18) was obtained for \( p = 1 \) (or \( q - 1 \)); it would certainly be interesting to understand what happens for \( p \neq 1 \) where numerical simulations indicate a strong \( p \) dependence when \( n \) increases, an effect of the \( n \)-zooming inherent to the \( n \)th moment definition (7).

In the \( n \) even case, twice the half-spectrum \( n \)th moment ends up being equal to

\[ 2|E_n|\left(\frac{2\pi}{q}\right)^n \]

a result that can be interpreted as if, at the \( n \)-zooming level, they were \( 2|E_n| \) bands each of length \( \frac{2\pi}{q} \). It would be interesting to see if this Euler counting has a meaning in the context of lattice walks [6] (twice the Euler number \( 2|E_n| \) counts the number of alternating permutations in \( S_n \)).

Finally, returning to the bandwidth \( n \)th moment defined in (8) for \( n \) odd, and focusing again on \( q \) odd, one can expand

\[
\sum_{r=1}^{q} (-1)^r (e_r(-4) - e_r(4))^n = -2 \sum_{k=0}^{(n-1)/2} \binom{n}{k} (-1)^k \sum_{r=1}^{q} (-1)^r e_r(-4)^k e_r(4)^{n-k}
\]

where the symmetry \( e_r(-4) = -e_{q+1-r}(4) \) has again been used. The \( k = 0 \) term \(-2 \sum_{r=1}^{q} (-1)^r e_r(4)^{n} \) is the \( n \)th moment discussed above and one knows that multiplying it by \( q^n \) ensures in the \( q \rightarrow \infty \) limit a finite scaling. Let us also multiply in (19) the \( k = 1, \ldots, (n-1)/2 \) terms by \( q^n \); one checks numerically that

\[
\lim_{q \rightarrow \infty} -2q^n \sum_{r=1}^{q} (-1)^r e_r(-4)^k e_r(4)^{n-k} = -\frac{2k}{n} \sum_{r=1}^{q} (-1)^r e_r(4)^{n}
\]

\[ = \frac{n - 2k}{n} \left( (-1)^{n-1} \psi(n) \left(\frac{1}{4}\right) - 2^n (2^{n+1} - 1) \zeta(n+1)n! \right). \]

Using

\[
\sum_{k=0}^{(n-1)/2} \binom{n}{k} (-1)^k \frac{n - 2k}{n} = 0
\]

one concludes that in the \( q \rightarrow \infty \) limit the bandwidth \( n \)th moment is such that
\[
\lim_{q \to \infty} q^n \sum_{r=1}^{q} (-1)^r (e_r(-4) - e_r(4))^n = 0
\]
when \( n \) is odd and larger than 1, a fact which is also supported by numerical simulations. Similarly, when \( n \) is even, the bandwidth \( n \)th moment, now defined as
\[
\lim_{q \to \infty} q^n \sum_{r=1}^{q} (-1)^r (e_r(-4) - e_r(4))^n = 0.
\]
Clearly, multiplying the sum in (19) by \( q^n \) is insufficient, a possible manifestation of the fractal structure [5] of the band spectrum. We leave to further studies the question of finding the right scaling for the bandwidth \( n \)th moment.

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Appendix

We want to evaluate the integral (14)
\[
I_n = (8\pi i)^{n-1} \cdot 32 \int_{0}^{\infty} y^{n-1} \left( \log \left( \frac{\Gamma(3/4 + y)^2}{\Gamma(1/4 + y)^2} \right) + \sum_{k=2, k \text{ even}}^{n-1} \frac{E_k}{k^2 y^k} \right) dy
\]
for odd integers \( n \). We will see below that \(-\sum_{k=2, k \text{ even}}^{\infty} \frac{E_k}{k^2} \) is indeed the series expansion of \( \log \left( \frac{\Gamma(3/4 + y)^2}{\Gamma(1/4 + y)^2} \right) \) as \( y \to \infty \). So for \( y \to 0 \), the expression in parentheses is of order \( y^{1-n} \) (log \( y \) for \( n = 1 \)), and it is of order \( y^{-1-n} \) as \( y \to \infty \). We can now apply integration by parts to obtain
\[
I_n = -(8\pi i)^{n-1} \cdot 32 \int_{0}^{\infty} y^n \left( \frac{2\Gamma''(3/4 + y)}{\Gamma(3/4 + y)} - \frac{2\Gamma''(1/4 + y)}{\Gamma(1/4 + y)} \right) = 1 - \sum_{k=2, k \text{ even}}^{n-1} \frac{E_k}{k^2 y^{k+1}} dy.
\]
In the following, \( \Psi(x) = \Gamma'(x)/\Gamma(x) \) denotes the digamma function. We first show that
\[
2\Psi(3/4 + y) - 2\Psi(1/4 + y) = \frac{2\Gamma''(3/4 + y)}{\Gamma(3/4 + y)} - \frac{2\Gamma''(1/4 + y)}{\Gamma(1/4 + y)} = 2 \sum_{k=0, k \text{ even}}^{\infty} \frac{E_k}{4^k y^{k+1}}.
\]

To this end, we use the well-known integral representation [7, 5.9.12]
\[
\Psi(z) = \int_{0}^{\infty} \left( \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-z t}}{1 - e^{-t}} \right) dt,
\]
which yields
\[
2\Psi(3/4 + y) - 2\Psi(1/4 + y) = 2 \int_0^\infty \left( \frac{e^{-(-1/4-y)t}}{1-e^{-t}} - \frac{e^{-(3/4-y)t}}{1-e^{-t}} \right) dt
\]
\[
= 2 \int_0^\infty \frac{e^{(1/4-y)t}}{e^{t/2} + 1} dt
\]
\[
= \int_0^\infty \frac{e^{-yt}}{\cosh(t/4)} dt
\]
\[
= \int_0^\infty e^{-yt} \sum_{k=0, k \text{ even}}^\infty \frac{E_k}{4^k k!} \ dt
\]
\[
= \sum_{k=0, k \text{ even}}^\infty \frac{E_k}{4^k k!} \int_0^\infty t^{k} e^{-yt} \ dt
\]
\[
= \sum_{k=0, k \text{ even}}^\infty \frac{E_k}{4^k k! y^{k+1}}
\]
as desired. Integration also yields
\[
\log \left( \frac{\Gamma(3/4 + y)^2}{y \Gamma(1/4 + y)^2} \right) = - \sum_{k=2, k \text{ even}}^\infty \frac{E_k}{k4^k y^k}
\]

Now we consider the Mellin transform of \(2(\Psi(3/4 + y) - \Psi(1/4 + y))\), i.e.
\[
F(s) = \int_0^\infty 2(\Psi(3/4 + y) - \Psi(1/4 + y))y^{s-1} \ dy.
\]
The integral converges for \(0 < \text{Re} s < 1\). Using the same integral representation as before, we find
\[
F(s) = \int_0^\infty y^{s-1} \int_0^\infty \frac{e^{-yt}}{\cosh(t/4)} \ dt \ dy
\]
\[
= \int_0^\infty \frac{1}{\cosh(t/4)} \int_0^\infty y^{s-1} e^{-yt} \ dy \ dt
\]
\[
= \Gamma(s) \int_0^\infty \frac{t^{s-1}}{\cosh(t/4)} \ dt
\]
\[
= 2\Gamma(s) \int_0^\infty t^{s-1/4} \frac{e^{-t/4} - e^{-3t/4}}{1-e^{-t}} \ dt.
\]
Now we use the following integral representation of the Hurwitz zeta function [7, 25.11.25]:
\[
\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-at}}{1-e^{-t}} \ dt
\]
for \(a > 0\) and \(\text{Re} s > 1\). This gives us
\[
\zeta(s,1/4) - \zeta(s,3/4) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1/4}(e^{-t/4} - e^{-3t/4})}{1-e^{-t}} \ dt.
\]
a priori only for $\Re x > 1$. However, the integral also converges for $\Re x > 0$, so the identity remains true by analytic continuation. Hence we have

$$F(s) = 2 \Gamma(s) \Gamma(1 - s) (\zeta(1 - s, 1/4) - \zeta(1 - s, 3/4)).$$

The functional equations of the Gamma function and the Hurwitz zeta function yield

$$\Gamma(s) \Gamma(1 - s) = \frac{\pi}{\sin(\pi s)}$$

and

$$\zeta(1 - s, 1/4) - \zeta(1 - s, 3/4) = \frac{4 \Gamma(s) \sin(\pi s/2)}{(8\pi)^s} (\zeta(s, 1/4) - \zeta(s, 3/4)), $$

so

$$F(s) = \frac{4\pi \Gamma(s)}{(8\pi)^s \cos(\pi s/2)} (\zeta(s, 1/4) - \zeta(s, 3/4)).$$

Now we use the general property of the Mellin transform (see [8, p 19]) that subtracting off terms of the asymptotic expansion at either 0 or $\infty$ only changes the fundamental strip of the Mellin transform, but not the transform itself. Thus we have

$$I_n = - (8\pi i)^{n-1} \cdot 32 \cdot \frac{4\pi \Gamma(n+1)}{(8\pi)^{n+1} \cos(\pi (n+1)/2)} \left( \zeta(n+1, 1/4) - \zeta(n+1, 3/4) \right)$$

$$= \frac{2}{\pi} n! \left( \zeta(n+1, 1/4) - \zeta(n+1, 3/4) \right)$$

which is (16) and (18).

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