Chiral de Rham complex. II

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To Dmitry Borisovich Fuchs, on his sixtieth birthday

This note is a sequel to [MSV]. It consists of three parts. The first part is an expanded version of the last section of op. cit. We give here certain construction of vertex algebras which includes in particular the ones appearing in the above note.

In the second part we show how the cohomology ring $H^*(X)$ of a smooth complex variety $X$ could be restored from the correlation functions of the vertex algebra $R\Gamma(X; \Omega^h_X)$.

In the third part, we prove first a useful general statement that the sheaf of loop algebras over the tangent sheaf $T_X$ acts naturally on $\Omega^h_X$ for every smooth $X$ (see §1). The $\mathbb{Z}$-graded vertex algebra $H^*(X; \Omega^h_X)$ seems to be a quite interesting object (especially for compact $X$). In §2, we compute $H^0(\mathbb{C}P^N; \Omega^h_{\mathbb{C}P^N})$ as a module over $\widehat{\mathfrak{sl}}(N+1)$.

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Part I: Chiral Weyl modules

§1. Recollections on vertex algebras

We will use the language of Kac’s book [K] and of the original Borcherds’ paper [B]. All omitted proofs may be found in [K].

1.1. Let \( V = V^{ev} \oplus V^{odd} \) be a super vector space. The parity of an element \( a \in V \) will be denoted by \( \tilde{a} \in \mathbb{Z}/2\mathbb{Z} \).

We will denote by \( V[[z, z^{-1}]] \) the space of all formal sums \( f(z) = \sum_{i \in \mathbb{Z}} a_i z^i; a_i \in V \), in the even variable \( z \). Similarly, we denote \( V[[z, z^{-1}], w, w^{-1}]] := V[[z, z^{-1}]][[w, w^{-1}]] \).

The subspace of Laurent power series, with \( a_i = 0 \) for \( i<<0 \) will be denoted by \( V((z)) \subset V[[z, z^{-1}]] \). We denote by \( \partial_z : V[[z, z^{-1}]] \to V[[z, z^{-1}]] \) the operator of differentiation by \( z \). We will also write \( f(z)' \) instead of \( \partial_z f(z) \). If \( A \) is a linear operator, \( A^{(k)} \) will denote the operator \( A^k/k! , k \in \mathbb{Z}_{\geq 0} \).

1.2. A vertex algebra is a super vector space \( V = V^{ev} \oplus V^{odd} \) together with the following data.

(a) An element \( |0\rangle \in V^{ev} \) called vacuum vector.

(b) An even linear map \( V \to \text{End}(V)[[z, z^{-1}]]; a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \) \hspace{1cm} (1.2.1)

For each \( a \in V \), the power series \( a(z) \) must be a field, which means that for each \( b \in V \), \( a_{(n)} b = 0 \) for \( n \gg 0 \). Here \( a_{(n)}(b) \in V \). The coefficients \( a_{(n)} \) are called Fourier modes of the field \( a(z) \).

(c) An even endomorphism \( \partial : V \to V \).

The following axioms must hold.

(Vacuum) \( |0\rangle(z) = \text{Id} \) (the constant power series). \( \partial |0\rangle = 0 \). For each \( a \in V \), \( a_{(n)} |0\rangle = 0 \) for \( n \geq 0 \), and \( a_{(-1)} |0\rangle = a \).

(Translation Invariance) For each \( a \in V \), \( [\partial, a(z)] = \partial_z a(z) \).

(Locality) For all \( a, b \in V \), there exists \( N \in \mathbb{Z}_{\geq 0} \) such that \( (z-w)^N [a(z), b(w)] = 0 \).

Here

\[ [a(z), b(w)] := \sum_{n,m} [a_{(n)}, b_{(m)}] z^{-n-1} w^{-m-1} \in \text{End}(V)[[z, z^{-1}, w, w^{-1}]] \]

1.3. The following basic Borcherds identity follows from and together with the Vacuum axiom is equivalent to the axioms of vertex algebra:

\[ \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)} = \]
\[ \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(n+m-j)} b_{(k+j)} - (-1)^{\tilde{a} \tilde{b}} \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)} a_{(m+j)} \quad (1.3.1) \]

for each \( a, b \in V, m, n, k \in \mathbb{Z} \). Here by definition,

\[ \binom{n}{j} = \frac{n(n-1) \cdots (n-j+1)}{j!} \quad (1.3.2) \]

for \( n \in \mathbb{C}, j \in \mathbb{Z}_{\geq 0} \). Note that

\[ \binom{n}{j} = 0 \text{ if } n \in \mathbb{Z}, j > n \geq 0 \quad (1.3.3) \]

We set

\[ \binom{n}{j} = 0 \text{ if } n, j \in \mathbb{Z}; j < 0 \]

Note the following useful particular case of (1.3.1) (corresponding to \( m = 0 \)) which is in fact equivalent to (1.3.1),

\[ (a_{(n)} b)_{(k)} = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(n-j)} b_{(k+j)} - (-1)^{\tilde{a} \tilde{b}} \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)} a_{(j)} \quad (1.3.4) \]

It is instructive to think of a vertex algebra as a super vector space equipped with an infinite number of (nonassociative, noncommutative) "multiplications"

\[ (\cdot) : a, b \mapsto a_{(n)} b \quad (n \in \mathbb{Z}) \quad (1.3.5) \]

satisfying the quadratic relations (1.3.1).

Another important commutativity formula is

\[ a_{(n)} b = (-1)^{\tilde{a} \tilde{b}} \sum_{j=0}^{\infty} (-1)^{j+n+1} \partial^{(j)}(b_{(n+j)} a) \quad (1.3.6) \]

for each \( n \in \mathbb{Z} \). Setting \( n = 0 \) in (1.3.1), we get

\[ [a_{(m)}, b_{(k)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)} b)_{(m+k-j)} \quad (1.3.7) \]

for each \( m, k \in \mathbb{Z} \) (Borcherds formula). In particular,

\[ [a_{(0)}, b_{(k)}] = (a_{(0)} b)_{(k)}, \quad (1.3.8) \]

in other words,

\[ a_{(0)} (b_{(k)} c) = (a_{(0)} b)_{(k)} c + (-1)^{\tilde{a} \tilde{b}} b_{(k)} (a_{(0)} c), \quad (1.3.9) \]

i.e. the operators \( a_{(0)} \) are derivations (of parity \( \tilde{a} \)) of all multiplications \( b_{(k)} c \).
It follows from the Vacuum and Translation invariance axioms that
\[ \partial a = a_{(-2)}|0 \]  
(1.3.10)

Substituting \( m = 0, n = -2, b = |0 \) in (1.3.1) and taking into account that
\[ \left( \begin{array}{c} -2 \\ j \end{array} \right) = (-1)^j(j + 1) \]  
(1.3.11)
we get
\[ (\partial a)_k = -ka_{(k-1)}, \]  
(1.3.12)
in other words,
\[ (\partial a)(z) = \partial_z a(z) \]  
(1.3.13)
Iterating (1.3.9) we get
\[ a_{(-1-j)} = (\partial^{(j)} a)_{(-1)} \]  
(1.3.14)
for all \( j \in \mathbb{Z}_{\geq 0} \). Applying (1.3.1) with \( m = 0, k = -2 \) to the vacuum vector, and taking into account (1.3.12), we deduce
\[ \partial(a_{(n)}b) = (\partial a)_{(n)}b + a_{(n)}(\partial b), \]  
(1.3.15)
i.e. \( \partial \) is an even derivation of each multiplication \((n)\).

1.4. Language of fields. Let \( V \) be a vertex algebra. Let \( \text{Fields}(V) \) denote the subspace of \( \text{End}(V)[[z,z^{-1}]] \) consisting of all power series \( f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n-1} \) such that for each \( a \in V \) there exists \( N \in \mathbb{Z} \) such that \( f_n(a) = 0 \) for each \( n \geq N \). This space inherits the obvious \( \mathbb{Z}/(2) \)-grading from \( \text{End}(V) \).

We define a map \( \pi : \text{Fields}(V) \rightarrow V \) by
\[ \pi(f(z)) = f_{(-1)}(|0\rangle) \]  
(1.4.1)
By the vacuum axiom
\[ \pi(a(z)) = a \ (a \in V) \]  
(1.4.2)
We set
\[ f(z) := \sum_{n \geq 0} f_n z^{-n-1}; \ f(z) = \sum_{n < 0} f_n z^{-n-1} \]  
(1.4.3)
For two fields \( f(z), g(z) \), we set
\[ f(z)g(w) := f(z) + g(w) + (-1)^j g(w) f(z)_{(-1)} \in \text{End}(V)[[z, z^{-1}, w, w^{-1}]] \]  
(1.4.4)
We can set \( z = w \) in this expression and get the well-defined element \( f(z)g(z) : \) of \( \text{Fields}(V) \). One defines the fields \( f(z)_{(n)} g(z) \ (n \in \mathbb{Z}) \) by the formulas
\[ f(z)_{(-1)} g(z) := f(z)g(z) : \]  
(1.4.5)
\[ f(z)_{(-1-j)} g(z) = \partial_{z^j}(f(z)_{(-1)} g(z)) \ (j \geq 0) \]  
(1.4.6)
(cf. (1.3.11)); and
\[ f(z)_j g(z) = \text{Res}_w [f(w), g(z)] (w - z)^j \quad (j \geq 0) \]  \hspace{1cm} (1.4.7)

where \( \text{Res}_w \) denotes the coefficient at \( w^{-1} \).

We have
\[ (a_{(n)} b)(z) = a(z)_{(n)} b(z) \]  \hspace{1cm} (1.4.8)

for each \( a, b \in V, \ n \in \mathbb{Z} \). Therefore,
\[ a_{(n)} b = \pi(a(z)_{(n)} b(z)) \]  \hspace{1cm} (1.4.9)

The Borcherds formula (1.3.4) may be rewritten as
\[ a(z) b(w) = \sum_{j=0}^{\infty} \frac{(a_{(j)} b)(w)}{(z - w)^{j+1}} + : a(z) b(w) : \]  \hspace{1cm} (1.4.10)

This is understood as an identity in \( V[[z, z^{-1}, w, w^{-1}]] \); one understands the fractions \( (z - w)^{-j-1} \) as the elements of this space using the binomial formula in the region \( |z| > |w| \):
\[ \frac{1}{(z - w)^{j+1}} = \sum_{m=0}^{\infty} (-1)^m \binom{-j-1}{m} w^m z^{-j-1-m} = \]
\[ = \sum_{n=0}^{\infty} \binom{n}{j} w^n z^{-j-n-1}, \]  \hspace{1cm} (1.4.11)

cf. (1.3.3). The identity (1.4.10) will be written as
\[ a(z) b(w) \sim \sum_{j=0}^{\infty} \frac{(a_{(j)} b)(w)}{(z - w)^{j+1}} \]  \hspace{1cm} (1.4.12)

(the \textit{operator product expansion} of the fields \( a(z), b(z) \)).

1.5. A \textit{morphism} of vertex algebras \( f : V \rightarrow V' \) is a linear operator sending vacuum vector to the vacuum vector, and such that \( f(a_{(n)} b) = f(a)_{(n)} f(b) \) for all \( a, b \in V, \ n \in \mathbb{Z} \).

A linear operator \( d : V \rightarrow V \) of parity \( \tilde{d} \) is called a \textit{derivation} of the vertex algebra \( V \) if
\[ d(a_{(n)} b) = d(a)_{(n)} b + (-1)^{\tilde{d} a} a_{(n)} d(b) \]  \hspace{1cm} (1.5.1)

Thus, for each \( a \in V, \ a_{(0)} \) is a derivation of \( V \), and \( \partial \) is an even derivation of \( V \).

The \textit{tensor product} \( V \otimes W \) of two vertex algebras is their tensor product as vector spaces with the vacuum vector \( |0\rangle_V \otimes |0\rangle_W \) and the following state-field correspondence
\[ (a \otimes b)(z) = a(z) \otimes b(z) \]  \hspace{1cm} (1.5.2)
i.e.

\[(a \otimes b)(n) = \sum_{k \in \mathbb{Z}} a_{(k)} \otimes b_{(n-k-1)}\]  \hfill (1.5.3)

It follows that

\[\partial_{V \otimes W} = \partial_V \otimes \text{Id}_W + \text{Id}_V \otimes \partial_W\]  \hfill (1.5.4)

1.6. A **graded** vertex algebra is a vertex algebra \(V\), together with an even diagonalizable linear operator \(H : V \rightarrow V\) (*Hamiltonian*) such that

\[[H, a(z)] = z\partial_z a(z) + (Ha)(z)\]  \hfill (1.6.1)

for each \(a \in V\). We will denote by \(V_\Delta\) the eigenspace of \(H\) corresponding to the eigenvalue \(\Delta\). The eigenvalues of \(H\) are called **conformal weights**. For \(a \in V_\Delta\), we will write the field \(a(z)\) in the form

\[a(z) = \sum_{i \in -\Delta + \mathbb{Z}} a_i z^{-i-\Delta}\]  \hfill (1.6.2)

Thus,

\[a_i = a_{(i+\Delta-1)}; \quad a_{(n)} = a_{n-\Delta+1}\]  \hfill (1.6.3)

The identity (1.6.1) means that \(a_n\) has conformal weight \(-n\), i.e.

\[a_n(V_\Delta) \subset V_{\Delta-n}\]  \hfill (1.6.4)

In other words,

\[\langle n \rangle : V_\Delta \otimes V_{\Delta'} \rightarrow V_{\Delta+\Delta'-n-1}\]  \hfill (1.6.5)

In particular,

\[\langle -1 \rangle : V_\Delta \otimes V_{\Delta'} \rightarrow V_{\Delta+\Delta'}\]  \hfill (1.6.6)

It follows that

\[|0\rangle \in V_0\]  \hfill (1.6.7)

since \(a_{(-1)}|0\rangle = a\), and

\[\partial(V_\Delta) \subset V_{\Delta+1}\]  \hfill (1.6.8)

since \(\partial a = a_{(-2)}|0\rangle\).

The OPE formula (1.4.12) will be rewritten as

\[a(z)b(w) \sim \sum_{j=0}^{\infty} \frac{(a_j \Delta + 1 b)(w)}{(z-w)^{j+1}} (a \in V_\Delta)\]  \hfill (1.6.9)

or

\[[a_m, b(z)] = \sum_{j=0}^{\infty} \binom{m + \Delta - 1}{j} (a_{j-\Delta+1} b)(z) \cdot z^{m-j+\Delta-1} (a \in V_\Delta)\]  \hfill (1.6.10)

1.7. A **conformal vertex algebra of central charge** \(c \in \mathbb{C}\) is a vertex algebra \(V\) equipped with an even vector \(L \in V\) satisfying the conditions (a) — (c) below.
(a) The field \( L(z) \) satisfies the OPE
\[
L(z)L(w) \sim \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w}
\] (1.7.1)

Equivalently, if we write \( L(z) = \sum L_n z^{-n-2} \) (so that \( L_{(n)} = L_{n-1} \)), then the components \( L_n \) satisfy the Virasoro commutation relations
\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12} \cdot c \cdot \delta_{m,-n}
\] (1.7.2)

(b) \( L_{-1} = \partial \)

(c) The operator \( L_0 \) is diagonalizable.

It follows that \( V \) is a graded vertex algebra with Hamiltonian \( L_0 \).

We have
\[
L(z)b(w) \sim \sum_{n=-1}^{\infty} \frac{(L_n b)(w)}{(z-w)^{n+2}} =
\]
\[
= \frac{b(w)'}{z-w} + \frac{\Delta b(w)}{(z-w)^2} + \ldots \quad (b \in V_\Delta)
\] (1.7.3)
or
\[
[L_m, b(z)] = \sum_{j=0}^{\infty} \binom{m+1}{j} (L_{j-1} b)(z) \cdot z^{m-j+1} =
\]
\[
= z^{m+1}b(z)' + (m+1)\Delta z^m b(z) + \ldots \quad (b \in V_\Delta)
\] (1.7.4)

1.8. A conformal algebra is a superspace \( R = R^{ev} \oplus R^{odd} \), together with an even linear operator \( \partial : R \to R \) and a collection of even operations \( (n) : R \otimes R \to R \), \( n \in \mathbb{Z}_{\geq 0} \) satisfying the axioms (C0) — (C3) below.

(C0) For each \( a, b \in R \), \( a_{(n)} b = 0 \) for \( n > 0 \).

(C1) \( (\partial a)_{(n)} b = -na_{(n-1)} b \)

(C2) \( a_{(n)} b = (-1)^{\hat{a}\hat{b}} \sum_{j=0}^{\infty} \partial^{(j)} (b_{(n+j)} a) \)

(C3) \( a_{(m)}(a_{(n)} c) = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)} b)_{(m+n-j)} c + (-1)^{\hat{a}\hat{b}} b_{(n)} (a_{(m)} c) \)

One has an obvious forgetful functor
\[
(\text{Vertex algebras}) \to (\text{Conformal algebras})
\] (1.8.1)

This functor admits a left adjoint
\[
U : (\text{Conformal algebras}) \to (\text{Vertex algebras})
\] (1.8.2)

called the vertex envelope, cf. [K], 4.7.

In the language of [BD], vertex algebras correspond to (unital) chiral algebras over the disk (we have the equivalence of categories). Conformal algebras correspond to \( \text{Lie}^* \)-algebras. The analogue of the functor \( U \) is called the chiral envelope.
1.9. A vertex algebra $V$ is called \textit{holomorphic} if $a_{(n)} = 0$ for each $a \in V$, $n \geq 0$.

If $V$ is a holomorphic vertex algebra, then the operation $ab := a_{(-1)}b$ is super-commutative and associative, and the vacuum vector is a unity. The operator $\partial$ is an even derivation, $\partial(ab) = (\partial a)b + a\partial b$. The remaining operations $(-n-1)$, $n \geq 0$, can be recovered by the formula (1.3.14).

This gives an equivalence of the categories of holomorphic vertex algebras and the category of commutative associative unital superalgebras with an even derivation, cf. [K], 1.4.

In the language of [BD], holomorphic vertex algebras correspond to \textit{commutative} chiral algebras over the disk. The above mentioned equivalence is translated into the equivalence of the category of commutative chiral algebras and that of $\mathcal{D}$-algebras (commutative algebras in the category of $\mathcal{D}$-modules over the disk).

1.10. Let $V$ be a vertex algebra. A \textit{module} over $V$ is a super vector space $M$, together with an even linear map $V \to \text{End}(M)[[z, z^{-1}]]$, $a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)}z^{-n-1}$ (1.10.1) such that for all $a \in V$, $m \in M$, $a_{(n)}m = 0$ for $n >> 0$, $|0\rangle(z) = \text{Id}_M$, and the Borcherds identity (1.3.1) holds true.

If $V$ is graded, then $M$ is called graded if it is equipped with a direct sum decomposition $M = \bigoplus M_\Delta$ such that for all $a \in V_\Delta$, $m \in M_{\Delta'}$, $n \in \mathbb{Z}$, we have $a_{(n)}m \in M_{\Delta+\Delta'-n-1}$, cf (1.6.5).

§2. \textbf{Restricted vertex algebras}

2.1. Let $(V, H)$ be a graded vertex algebra. Consider the condition

(P) $V$ has no negative integer conformal weights.

Vertex algebras satisfying this condition will be called \textit{restricted}.

2.2. Let us fix a restricted vertex algebra $V$. To simplify the formulas, we will assume that $V$ is purely even. All the considerations below have the obvious "super" ($\mathbb{Z}/(2)$-graded) version. The operation $a_{(-1)}b$ will be denoted simply by $ab$ and referred to as a multiplication by $a$.

We are going to investigate, what kind of a structure on the subspace $V_{\leq 1} := V_0 \oplus V_1$ is induced by our vertex algebra.

(a) The space $V_0$ is a commutative associative unital $\mathbb{C}$-algebra with respect to the operation $ab$. The unity equals $|0\rangle$.

Note that $a_{(n)}b = 0$ for $a, b \in V_0$, $n \geq 0$ (2.2.1)
since the operation \((n)\) has conformal weight \(-n-1\), cf. (1.6.10). Now, the commutativity of \(V_0\) follows from the commutativity formula (1.3.6).

Applying (1.3.1) with \(m = 0\), \(n = k = -1\), to an element \(c\), and taking into account that
\[
\binom{-1}{j} = (-1)^j
\]
we get
\[
(ab)c = a(bc) + \sum_{j=0}^{\infty} a_{(-2-j)}b_{(j)}c + \sum_{j=0}^{\infty} b_{(-2-j)}a_{(j)}c
\]
for all \(a, b, c\). If \(a, b, c \in V_0\) then the sums disappear, and we get the associativity. The Vacuum axiom implies that \(|0\rangle\) is a unity.

This algebra will be denoted \(A\). The vacuum will be denoted \(1\).

There is a map \(A \otimes V \rightarrow V\), \(a \otimes b \mapsto ab\). However, it does not make \(V\) into an \(A\)-module: the multiplication by \(A\) is not associative in general.

We have the map \(\partial : A \rightarrow V_1\). Let \(\Omega \subset V_1\) denote the subspace spanned by the elements \(a\partial b\), \(a, b \in A\). Thus, \(\partial\) induces the map
\[
d : A \rightarrow \Omega
\]
(b) The left multiplication by \(A\) makes \(\Omega\) a left \(A\)-module. We have
\[
ad b = (db)a \quad (a, b \in A)
\]
Let us write down a particular case of (2.2.3):
\[
(ab)c - a(bc) = a_{(-2)}b_{(0)}c + b_{(-2)}a_{(0)}c
\]
for \(a, b \in V_0\), \(c \in V_1\)
\[
(2.2.6)
\]
For \(b, c \in V_0\), we have \(0 = \partial(b_{(0)}c) = (\partial b)_{(0)}c + b_{(0)}\partial c\), but \((\partial b)_{(0)} = 0\) by (1.3.12). It follows that
\[
b_{(0)}\partial c = 0 \text{ for } b, c \in V_0
\]
Therefore,
\[
(ab)\partial c = a(b\partial c) \text{ for } a, b, c \in V_0
\]
It follows that \(\Omega\) is a left \(A\)-module.
(c) The map \(d\) is a derivation, i.e.
\[
d(ab) = adb + bda
\]
This follows from (1.3.15).

Let us denote by \(T\) the quotient space \(V_1/\Omega\).
(d) The left multiplication by \(A\) makes \(T\) into a left \(A\)-module.
Note that by (1.3.12),
\[ a_{(-2)} b = \partial a \cdot b \]  
(2.2.10)

Thus, we can rewrite (2.2.3) as
\[ (ab)c - a(bc) = \partial a \cdot (b_{(0)} c) + \partial b \cdot (a_{(0)} c) \]  
(2.2.11)

Consider the operation
\[ (0) : V_1 \otimes V_1 \rightarrow V_1 \]  
(2.2.12)

(e) The operation (2.2.12) induces a Lie bracket on \( \mathcal{T} \), to be denoted [ , ].

By (1.3.6), we have
\[ a_{(0)} b = -b_{(0)} a + \partial(b_{(1)}) a \]  
(2.2.13)

Therefore, composition
\[ V_1 \otimes V_1 \xrightarrow{(0)} V_1 \rightarrow \mathcal{T} \]  
(2.2.14)

is skew-symmetric. On the other hand, by (1.3.1) with \( m = 0, n = -1, k = 0 \), we have
\[ (a \partial b)_{(0)} c = -\partial a \cdot (b_{(0)} c) + \partial b \cdot (a_{(0)} c) \text{ for } a, b \in V_0, \ c \in V_1 \]  
(2.2.15)

It follows that the composition (2.2.14) is zero on the subspace \( \Omega \otimes V_1 + V_1 \otimes \Omega \). The Jacobi identity holds since the operators \( a_{(0)} \) are derivations of all operations (k).

We will omit the proof of claims (f) — (n) below; they are proved in a similar manner by application of Borcherds identity and commutativity formula. The reader may want to perform these (easy) calculations on its own.

Consider the operation
\[ (0) : V_1 \otimes A \rightarrow A \]  
(2.2.16)

(f) The operation (2.2.16) vanishes on the subspace \( \Omega \otimes A \) and induces on \( A \) a structure of a module over the Lie algebra \( \mathcal{T} \).

This action will be denoted by \( \tau(a) \) (\( a \in A, \ \tau \in \mathcal{T} \)).

(g) The Lie algebra \( \mathcal{T} \) acts on \( A \) by derivations,
\[ \tau(ab) = \tau(a)b + a\tau(b) \]  
(2.2.17)

(h) We have
\[ (a\tau)(b) = a\tau(b) \]  
(2.2.18)

The properties (d) — (h) mean that \( \mathcal{T} \) is a Lie algebroid over \( A \), cf. [BFM], 3.2.1.

(i) The operation (2.2.12) induces on the space \( \Omega \) a structure of a module over the Lie algebra \( \mathcal{T} \).

This action will be denoted \( \tau(\omega) \) or \( \tau\omega \) (\( \tau \in \mathcal{T}, \ \omega \in \Omega \)).
(j) We have
\[ \tau(a\omega) = a\tau(\omega) + \tau(a)\omega \quad (a \in A, \; \tau \in \mathcal{T}, \; \omega \in \Omega) \] (2.2.19)

(k) The differential \( d : A \rightarrow \Omega \) is compatible with the \( \mathcal{T} \)-module structure.

It follows from (j) and (k) that

(l) we have
\[ \tau(ab) = \tau(a)db + ad(\tau(b)) \quad (\tau \in \mathcal{T}, \; a, b \in A) \] (2.2.20)

Consider the operation
\[ (1) : V_1 \otimes V_1 \rightarrow A \] (2.2.21)

(m) The map (2.2.21) vanishes on the subspace \( \Omega \otimes \Omega \). Therefore, it induces the pairing
\[ \langle \cdot, \cdot \rangle : \Omega \otimes \mathcal{T} \otimes \mathcal{T} \otimes \Omega \rightarrow A \] (2.2.22)

This pairing is \( A \)-bilinear and symmetric. We have
\[ \langle \tau, ab \rangle = a\tau(b) \quad (\tau \in \mathcal{T}, \; a, b \in A) \] (2.2.23)

(n) We have
\[ (a\tau)(\omega) = a\tau(\omega) + \langle \tau, \omega \rangle da \quad (a \in A, \; \tau \in \mathcal{T}, \; \omega \in \Omega) \] (2.2.24)

2.3. Let us denote by \( \hat{T} \) the space \( V_1/dA \). The operation (2.2.4) induces a Lie bracket on the space \( \hat{T} \). The subspace \( \Omega/dA \subset \hat{T} \) is an abelian Lie ideal. The adjoint action of \( \mathcal{T} = \hat{T}/(\Omega/dA) \) coincides with the action defined by (i) and (l).

Thus, we have an extension of Lie algebras
\[ 0 \rightarrow \Omega/dA \rightarrow \hat{T} \rightarrow \mathcal{T} \rightarrow 0 \] (2.3.1)

Note that this extension is not central in general.

Let us denote the space \( V_1 \) by \( \tilde{T} \). We can form a dg Lie algebra
\[ A \rightarrow \tilde{T} \rightarrow \hat{T} \] (2.3.2)

living in degrees \(-2, -1, 0\). The first arrow is \( d \), the second one is the projection. The \((-1, -1)\)-component of the bracket is given by the operation (2.2.21), the \((0, -1)\)-component is induced by the operation (0).

2.3.1. Remark. In fact, the dg Lie algebra (2.3.2) is a part of a bigger dg Lie algebra which one can associate with an arbitrary vertex algebra \( V \). It is defined as
\[ V \rightarrow V \rightarrow V/\partial V \] (2.3.1.1)
and lives in degrees \(-2, -1, 0\). The first arrow is \(\partial\). The operation \(_{(0)}\) induces the \(V/\partial V\)-module structure on \(V\): this will be the \((0, -1)\)-component of the bracket. The \((-1, -1)\)-component is given by the symmetric operation

\[
[a, b]^{-1, -1} = \sum_{j \geq 0} (-1)^j \frac{\partial^j}{(j + 1)!} \partial_{(j+1)} b
\]  

(2.3.1.2)

cf. [B], Section 9. The \((0, -2)\)-component is trivial. This definition was inspired by [BFM], 6.4.

2.4. We have an exact sequence of vector spaces

\[
0 \longrightarrow \Omega \longrightarrow \tilde{T} \longrightarrow T \longrightarrow 0 \quad (2.4.1)
\]

Both arrows are compatible with the left multiplication by \(A\). Let \(\pi\) denote the projection \(\pi : \tilde{T} \longrightarrow T\).

Let us define the "bracket" \([, ,] : \Lambda^2\tilde{T} \longrightarrow \tilde{T}\) by

\[
[x, y] = \frac{1}{2} (x_{(0)} y - y_{(0)} x) \quad (x, y \in \tilde{T}) \quad (2.4.2)
\]

This bracket does not make \(\tilde{T}\) into a Lie algebra: the Jacobi identity is in general violated. Set

\[
J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y] \quad (x, y, z \in \tilde{T}) \quad (2.4.3)
\]

Consider the operation (2.2.21).

(a) The operation (2.2.21) is symmetric. It will be denoted by \(\langle x, y \rangle\).

Let us define the map \(I : \Lambda^3\tilde{T} \longrightarrow A\) by

\[
I(x, y, z) = \langle x, [y, z] \rangle + \langle y, [z, x] \rangle + \langle z, [x, y] \rangle \quad (2.4.4)
\]

(b) We have

\[
J(x, y, z) = \frac{1}{6} dI(x, y, z) \quad (2.4.5)
\]

In fact, by (1.3.6),

\[
[x, y] = x_{(0)} y - \frac{1}{2} d\langle x, y \rangle \quad (2.4.6)
\]

It follows that

\[
[[x, y], z] = [x, y]_{(0)} z - \frac{1}{2} d\langle [x, y], z \rangle =
\]

\[
= (x_{(0)} y)_{(0)} z - \frac{1}{2} d\langle [x, y], z \rangle = x_{(0)} y_{(0)} z - y_{(0)} x_{(0)} z - \frac{1}{2} d\langle [x, y], z \rangle
\]

Similarly,

\[
[[y, z], x] = (y_{(0)} z)_{(0)} x - \frac{1}{2} d\langle [y, z], x \rangle,
\]

\[
[[z, x], y] = (z_{(0)} x)_{(0)} y - \frac{1}{2} d\langle [z, x], y \rangle.
\]
and a similar identity for \([z, x, y]\). Now we use the identities
\[
(y(0)z)(0)x = -x(0)y(0)z + d\langle x, y(0)z \rangle,
\]
and the similar one for \((z(0)x)(0)y\), to get
\[
J(x, y, z) = d\left(\langle x, y(0)z \rangle - \langle y, x(0)z \rangle\right) - \frac{1}{2}d\langle [x, y, z] \rangle + \text{(cyle)}
\]
Note that the left hand side of this equality, as well as the second summand in the right hand side (equal to \(-1/2 \cdot dI(x, y, z)\)), are completely skew symmetric with respect to all permutations of the letters \(x, y, z\). Therefore, we can skew symmetrize the first summand as well, which gives \(2/3 \cdot dI(x, y, z)\). The identity (2.4.5) follows.

(c) We have
\[
\langle ax, y \rangle = a\langle x, y \rangle - \pi(x)\pi(y)(a) \quad (a \in A, \ x, y \in \mathcal{T}) \tag{2.4.7}
\]
This follows from (1.3.1) with \(m = 0, n = -1, k = 1\).

(d) We have
\[
\langle [x, y], z \rangle + \langle y, [x, z] \rangle = \pi(x)(\langle y, z \rangle) - \frac{1}{2}\pi(y)(\langle x, z \rangle) - \frac{1}{2}\pi(z)(\langle x, y \rangle) \tag{2.4.8}
\]
To check this, we use the identity
\[
x(0)(y, z) = \langle x(0)y, z \rangle + \langle y, x(0)z \rangle \tag{2.4.9}
\]
and take into account that
\[
\langle da, z \rangle = \pi(z)(a) \quad (a \in A, \ z \in \mathcal{T}) \tag{2.4.10}
\]

2.5. Let us choose a splitting
\[
s : \mathcal{T} \longrightarrow \mathcal{T} \tag{2.5.1}
\]
of the projection \(\pi\). Let us define the map
\[
\langle \ , \ \rangle : S^2\mathcal{T} \longrightarrow A \tag{2.5.2}
\]
by
\[
\langle \tau_1, \tau_2 \rangle = \langle s(\tau_1), s(\tau_2) \rangle \tag{2.5.3}
\]
(we use the lower index \(s\) in the notation if we want to stress the dependence on the splitting \(s\)). Let us define the map
\[
c^2 = c^2_s : \Lambda^2\mathcal{T} \longrightarrow \Omega \tag{2.5.4}
\]
by
\[
c^2(\tau_1, \tau_2) = s([\tau_1, \tau_2]) - [s(\tau_1), s(\tau_2)] \tag{2.5.5}
\]
Let us define the map \( K : \Lambda^3 T \to A \) by
\[
K(\tau_1, \tau_2, \tau_3) = \langle s(\tau_1), s(\tau_2, \tau_3) \rangle + \langle s(\tau_2), s(\tau_3, \tau_1) \rangle + \langle s(\tau_3), s(\tau_1, \tau_2) \rangle
\] (2.5.6)
Let us define the map
\[
c^3 = c^3 : \Lambda^3 T \to A
\]
by
\[
c^3(\tau_1, \tau_2, \tau_3) = -\frac{1}{2}K(\tau_1, \tau_2, \tau_3) + \frac{1}{3}I(s(\tau_1), s(\tau_2), s(\tau_3))
\] (2.5.7)
cf. (2.4.4). Let us regard \( c^2 \) (resp. \( c^3 \)) as a Lie algebra cochains living in \( C^2(T; \Omega) \) (resp., in \( C^3(T; A) \)). Recall that the Lie differential \( d_{\text{Lie}} \) is defined by
\[
d_{\text{Lie}}(c_i)(\tau_1, \ldots , \hat{\tau}_i, \ldots ) = \sum_{p<q} (-1)^{p+q-1} c^i([\tau_p, \tau_q], \tau_1, \ldots , \hat{\tau}_p, \ldots , \hat{\tau}_q, \ldots , \tau_{i+1}) +
\sum_{p} (-1)^p \tau_p c^i(\tau_1, \ldots , \hat{\tau}_p, \ldots , \tau_{i+1})
\] (2.5.8)
(a) We have
\[
d_{\text{Lie}}(c^2) = dc^3
\] (2.5.9)
In fact, one sees from the definition (2.5.5) that
\[
d_{\text{Lie}}(c^2)(\tau_1, \tau_2, \tau_3) = -[s([\tau_1, \tau_2]), s(\tau_3)] + (\text{cycle}) - s(\tau_1(0)) s([\tau_2, \tau_3]) + (\text{cycle}) +
\]
\[+ s(\tau_1(0)) s(\tau_2), s(\tau_3)] + (\text{cycle}) - J(s(\tau_1), s(\tau_2), s(\tau_3))
\]
Now, (2.5.9) follows from (2.4.6) and (2.4.5).
(b) We have
\[
d_{\text{Lie}}(c^3) = 0
\] (2.5.10)
To check this, note that \( c^3 \) is a sum of two summands, as in (2.5.7). Also, let us split the map \( d_{\text{Lie}} \) into \( d' + d'' \) where
\[
d'(\gamma)(\tau_1, \ldots ) = \sum_{p<q} (-1)^{p+q-1} \gamma([\tau_p, \tau_q], \tau_1, \ldots , \hat{\tau}_p, \ldots , \hat{\tau}_q, \ldots )
\]
and
\[
d''(\gamma)(\tau_1, \ldots ) = \sum_{p} (-1)^p \tau_p \gamma(\tau_1, \ldots , \hat{\tau}_p, \ldots )
\]
Correspondingly, \( d_{\text{Lie}}(c^3) \) splits into four summands.

The summand \(-\frac{1}{2} d'K\) contains terms of the type
\[
\langle s([\tau_1, \tau_2]), s(\tau_3, \tau_4) \rangle
\] (2.5.11)
(six terms) and the terms of the type
\[
\langle s(\tau_1), s([\tau_2, \tau_3, \tau_4]) \rangle,
\]
these terms cancel out, due to the Jacobi identity in $\mathcal{T}$.

The summand $-\frac{1}{2}d''K$ has the terms of the type

$$-\tau_1((s(\tau_2), s(\tau_3, \tau_4))) + \tau_2((s(\tau_1), s(\tau_3, \tau_4)))$$

(six groups). The following identity is an easy consequence of (2.4.8):

$$\frac{3}{2}\left\{\pi(x)(y, z) - \pi(y)(x, z)\right\} = 2\langle x, y, z \rangle + \langle y, [x, z] \rangle - \langle x, [y, z] \rangle$$

Therefore, (2.5.12) is equal to

$$\frac{2}{3}\left\{-2\langle [s(\tau_1), s(\tau_2)], s(\tau_3, \tau_4)\rangle - \langle s(\tau_2), [s(\tau_1), s(\tau_3, \tau_4)]\rangle + \langle s(\tau_1), [s(\tau_2), s(\tau_3, \tau_4)]\rangle\right\}$$

(2.5.14)

The summand $\frac{1}{3}d'I$ contains six groups of the type

$$\langle [s(\tau_3, \tau_4)], [s(\tau_1), s(\tau_2)]\rangle + \langle s(\tau_1), [s(\tau_2), s(\tau_3, \tau_4)]\rangle + \langle s(\tau_2), [s(\tau_3, \tau_4), s(\tau_1)]\rangle$$

(2.5.15)

The second the third terms in (2.5.15) cancel out with the similar terms in (2.5.14).

Using (2.5.13) and (2.4.5), it is easy to deduce the identity

$$\tau_1([s(\tau_2), [s(\tau_3), s(\tau_4)]]) + \ldots = 3\{[s(\tau_1), s(\tau_2)], [s(\tau_3), s(\tau_4)]\} -$$

$$\{[s(\tau_1), s(\tau_3)], [s(\tau_2), s(\tau_4)]\} + \{[s(\tau_1), s(\tau_4)], [s(\tau_2), s(\tau_3)]\}$$

(2.5.16)

where $\ldots$ in the left hand side mean the complete skew-symmetrization (twelve summands altogether). It follows that

$$\frac{1}{3}d''I = -\{[s(\tau_1), s(\tau_2)], [s(\tau_3), s(\tau_4)]\} + \{[s(\tau_1), s(\tau_3)], [s(\tau_2), s(\tau_4)]\} -$$

$$\{[s(\tau_1), s(\tau_4)], [s(\tau_2), s(\tau_3)]\}$$

(2.5.17)

Finally, since

$$\langle [s(\tau_1), s(\tau_2)], c(\tau_3, \tau_4)\rangle = \langle s(\tau_1, \tau_2), c(\tau_3, \tau_4)\rangle,$$

we have

$$\langle s(\tau_1, \tau_2), [s(\tau_3, \tau_4)]\rangle = \langle [s(\tau_1), s(\tau_2)], s(\tau_3, \tau_4)\rangle + \langle s(\tau_1, \tau_2), [s(\tau_3), s(\tau_4)]\rangle -$$

$$\{[s(\tau_1), s(\tau_2)], [s(\tau_3), s(\tau_4)]\}$$

(2.5.18)

Using this relation, we see that the sum of the remaining summands equals zero. This proves (2.5.10).

The identities (a) and (b) mean that the pair $c = (c^2, c^3)$ form a 2-cocycle of the Lie algebra $\mathcal{T}$ with coefficients in the complex $A \rightarrow \Omega$. 
(c) We have
\[
\langle [\tau_1, \tau_2], \tau_3 \rangle + \langle \tau_2, [\tau_1, \tau_3] \rangle = \tau_1(\langle \tau_2, \tau_3 \rangle) - \frac{1}{2} \tau_2(\langle \tau_1, \tau_3 \rangle) - \frac{1}{2} \tau_3(\langle \tau_1, \tau_2 \rangle) - \\
- \langle \tau_2, c^2(\tau_1, \tau_3) \rangle - \langle \tau_3, c^2(\tau_1, \tau_2) \rangle
\]  
(2.5.19)

This follows from 2.4 (d).

Let us investigate the effect of the change of a splitting. Let \( s' : T \longrightarrow \tilde{T} \) be another splitting of \( \pi \). The difference \( s' - s \) lands in \( \Omega \); let us denote it
\[
\omega = \omega_{s,s'} : T \longrightarrow \Omega \tag{2.5.20}
\]
We regard \( \omega \) as a 1-cochain of \( T \) with coefficients in \( \Omega \). Let us define a 2-cochain
\[
\alpha = \alpha_{s,s'} \in C^2(T; A) \text{ by}
\]
\[
\alpha(\tau_1, \tau_2) = \frac{1}{2} \left( \langle \omega(\tau_1), \tau_2 \rangle - \langle \tau_1, \omega(\tau_2) \rangle \right)
\]  
(2.5.21)

(d) We have
\[
c^3_s - c^3_{s'} = d_{\text{Lie}}(\omega) - d\alpha \tag{2.5.22}
\]

Indeed, the left hand side of (2.5.22) is equal to
\[
\omega([\tau_1, \tau_2]) - [s(\tau_1), \omega(\tau_2)] - [\omega(\tau_1), s'(\tau_2)]
\]

By (2.4.6), we have
\[
[s(\tau_1), \omega(\tau_2)] = \tau_1 \omega(\tau_2) - \frac{1}{2} d(\langle \tau_1, \omega(\tau_2) \rangle),
\]
and the similar expression for \([\omega(\tau_1), s'(\tau_2)]\). The identity (2.5.22) follows.

(e) We have
\[
-d_{\text{Lie}}(\alpha) = c^3_s - c^3_{s'} \tag{2.5.23}
\]

Indeed we have
\[
d_{\text{Lie}} \alpha(\tau_1, \tau_2, \tau_3) = \frac{1}{2} \left\{ \langle \omega([\tau_1, \tau_2]), \tau_3 \rangle - \ldots - \langle [\tau_2, \tau_3], \omega(\tau_1) \rangle \right\} + \\
+ \frac{1}{2} \left\{ -\tau_1(\langle \omega(\tau_2), \tau_3 \rangle) + \ldots + \tau_3(\langle \tau_1, \omega(\tau_2) \rangle) \right\}
\]
and
\[
c^3_s(\tau_1, \tau_2, \tau_3) - c^3_{s'}(\tau_1, \tau_2, \tau_3) = -\frac{1}{2} \left\{ \langle \omega(\tau_1), [\tau_2, \tau_3] \rangle + \ldots + \langle \tau_3, \omega([\tau_1, \tau_2]) \rangle \right\} + \\
+ \frac{1}{3} \left\{ \langle \omega(\tau_1), [\tau_2, \tau_3] \rangle + \ldots + \langle \tau_3, [\tau_1, \omega(\tau_2)] \rangle \right\}
\]  
(2.5.24)

The summands of the type \( \langle \tau_1, \omega([\tau_2, \tau_3]) \rangle \) in (2.5.24) are equal to the corresponding summands in the expression for \( -d_{\text{Lie}} \alpha \). It remains to consider the summands
containing $\omega(\tau_i)$. Consider for example the summands in (2.5.24) containing $\omega(\tau_1)$. There is one,

$$-\frac{1}{2} \langle \omega(\tau_1), [\tau_2, \tau_3] \rangle$$

in the first bracket, and three,

$$\frac{1}{3} \left\{ \langle \omega(\tau_1), [\tau_2, \tau_3] \rangle + \langle \tau_2, [\omega(\tau_3), \tau_1] \rangle + \langle \tau_3, [\tau_1, \omega(\tau_2)] \rangle \right\}$$

in the second one. In the above sum, replace the second term using (2.4.8),

$$\langle \tau_3, [\omega(\tau_1), \tau_2] \rangle = \langle [\tau_2, \tau_3], \omega(\tau_1) \rangle - \tau_2(\langle \tau_3, \omega(\tau_1) \rangle) + \frac{1}{2} \tau_3(\langle \tau_2, \omega(\tau_1) \rangle),$$

and similarly replace the third term $\langle \tau_3, [\tau_1, \omega(\tau_2)] \rangle$. Summing up, we get the same as in $-d_{Lie}\alpha$.

The properties (d) and (e) mean that

$$c_s - c_s' = d_{Lie}(\beta) \quad (2.5.25)$$

where $\beta = \beta_{s,s'} := (\omega, \alpha) \in C^1(\mathcal{T}; A \rightarrow \Omega)$.

Therefore, we have assigned to our vertex algebra a canonically defined ”characteristic class”

$$c(V) = c(V_{\leq 1}) \in H^2(\mathcal{T}; A \rightarrow \Omega) \quad (2.5.26)$$

as the cohomology class of the cocycle $c_s$.

2.6. Note that we have

$$(ab)x - a(bx) = -\pi(x)(a)db - \pi(x)(b)da \quad (a, b \in A, \ x \in \tilde{T}) \quad (2.6.1)$$

This follows from (1.3.1) with $m = 0, n = -1, k = -1$.

Let us introduce the mapping

$$\gamma = \gamma_s : A \otimes \mathcal{T} \rightarrow \Omega \quad (2.6.2)$$

so that

$$\gamma(a, \tau) = s(a\tau) - as(\tau) \quad (2.6.3)$$

(a) We have

$$\gamma(ab, \tau) = \gamma(a, b\tau) + a\gamma(b, \tau) + \tau(a)db + \tau(b)da \quad (2.6.4)$$

This follows from (2.6.1).

We have

$$(ax)(0)y = a(x(0)y) - \pi(y)(a)x + \langle x, y \rangle da - d(\pi(x)\pi(y)(a)) \quad (2.6.5)$$
(a ∈ A, x, y ∈ ᵀ), which follows from (1.3.1) with m = 0, n = −1, k = 0. This implies (using (2.4.6) and (2.4.7))

$$[a x, y] = a [x, y] − π(y)(a)x + \frac{1}{2}⟨x, y⟩ da − \frac{1}{2}d(π(x)π(y)(a)) \quad (2.6.6)$$

(a ∈ A, x, y ∈ ᵀ).

(b) We have

$$⟨a τ₁, τ₂⟩ = a⟨τ₁, τ₂⟩ + ⟨γ(a, τ₁), τ₂⟩ − τ₁τ₂(a) \quad (2.6.7)$$

This follows from (2.4.7).

(c) We have

$$c^2(a τ₁, τ₂) = ac^2(τ₁, τ₂) + γ(a, [τ₁, τ₂]) − γ(τ₂(a), τ₁) −\frac{1}{2}⟨τ₁, τ₂⟩ da + \frac{1}{2}d(τ₁ τ₂(a)) − \frac{1}{2}d(⟨τ₂, γ(a, τ₁)⟩) \quad (2.6.8)$$

(a ∈ A, τ₁ ∈ ᵀ).

This follows from (2.6.6) and (2.6.7).

(d) We have

$$c^3(a τ₁, τ₂, τ₃) = ac^3(τ₁, τ₂, τ₃) + \frac{1}{2}τ₁[τ₂, τ₃](a) − \frac{1}{2}\{⟨τ₂, γ(a, [τ₃, τ₁])⟩ − ⟨τ₃, γ(a, [τ₂, τ₁])⟩ + ⟨τ₂, γ(τ₃(a), τ₁)⟩ − ⟨τ₃, γ(τ₂(a), τ₁)⟩\} + \frac{1}{2}\{⟨τ₂, τ₃γ(a, τ₁)⟩ − ⟨τ₃, τ₂γ(a, τ₁)⟩\} − \frac{1}{2}[τ₂, τ₃, γ(a, τ₁)] \quad (2.6.9)$$

(e) Exercise. Check that the formulas (c) and (d) are compatible with the identity $d_{L_α}(c^2) = dc^3$.

Hint: use 2.2 (n).

2.7. Filtered algebras. In fact, we do not really need the gradation on our vertex algebras. Let us call a vertex algebra $V$ filtered if it is equipped with an increasing exhaustive filtration $\ldots \subset V_{≤ i} \subset V_{≤ i+1} \subset \ldots \subset V$ such that $|0⟩ ∈ V_{≤ 0}$ and

$$V_{≤ i} (n) V_{≤ j} ⊂ V_{≤ i+j−n−1} \quad (2.7.1)$$

cf. (1.6.5). As a consequence, $∂V_{≤ i} ⊂ V_{≤ i+1}$. (The corresponding graded space $gr V$ is a graded vertex algebra.)

Let us call $V$ restricted if $V_{≤ −1} = 0$. For such $V$, again $A := V_{≤ 0}$ is a commutative algebra. Define $Ω ⊂ V_{≤ 1}$ as the $C$-subspace spanned by $a ∂ b$, $a, b ∈ A$. Set

$$A := V_{≤ 1}/Ω; \ T := V_{≤ 1}/(A + Ω) \quad (2.7.2)$$

Then $A$ is an $A$-Lie algebroid (Atiyah algebra of $V$) which is an $A$-extension of the Lie algebroid $T$, cf. [BFM]. The discussion 2.2 — 2.6 carries over to the filtered case, with $T$ replaced by $A$. 
§3. Prevertex algebras

3.1. Let us call the data (a) — (f) below a prevertex algebra.

(a) A commutative algebra $A$.

(b) An $A$-module $\Omega$, together with an $A$-derivation $d : A \to \Omega$. We assume that $\Omega$ is generated as a vector space by the elements $adb (a, b \in A)$, i.e. the canonical map $\Omega^1(A) := \Omega^1(A) \to \Omega$ is surjective.

(c) An $A$-Lie algebroid $T$. Define the action of $T$ on $\Omega$ by

$$\tau(adb) = \tau(a)db + ad(\tau(b)),$$

(cf. (2.2.9). We assume that this action is well defined. It follows that $d$ is compatible with the action of $T$.

We assume that the formula

$$\langle \tau, adb \rangle = a\tau(b)$$

gives a well defined $A$-bilinear pairing $T \times \Omega \to A$.

(d) A $\mathbb{C}$-bilinear mapping $\gamma : A \times T \to \Omega$ satisfying 2.6 (a).

(e) A $\mathbb{C}$-bilinear symmetric mapping $\langle , \rangle : T \times T \to A$ satisfying 2.6 (b).

(f) A skew symmetric map $c^2 : \Lambda^2 T \to \Omega$. This map should satisfy 2.5 (c), 2.6 (c), and the property (3.1.7) below. Let us define the map

$$[ , ]' := [ , ] - c^2 : \Lambda^2 T \to T \oplus \Omega$$

In the last formula, we extend the symmetric pairing $\langle , \rangle$ to the whole space $T \oplus \Omega$, by using (3.1.2), (e), and setting it equal to zero on $\Omega \times \Omega$.

Now, with $c^3$ defined above, we require that

$$d_{Lie}(c^2) = dc^3; \quad d_{Lie}(c^3) = 0$$

3.2. Given a prevertex algebra $P = (A, \Omega, T, \ldots)$, set $V_0 = A; \quad V_1 = T \oplus \Omega$. Define the operation $(-2) : V_0 \times V_0 \to V_1$ by

$$a(-2)b = bda$$
Define the operations \((-1) : V_0 \times V_1 \rightarrow V_1\) and \((-1) : V_1 \times V_0 \rightarrow V_1\) by

\[
a_{(-1)} \omega = a \omega; \quad a_{(-1)} \tau = a \tau + \gamma(a, \tau)
\]  

(3.2.2)

and

\[
\omega_{(-1)} a = a \omega; \quad \tau_{(-1)} a = a \tau + \gamma(a, \tau) + d(\tau(a))
\]  

(3.2.3)

\((a \in A, \omega \in \Omega, \tau \in T)\).

Define the operations \((0) : V_0 \times V_1 \rightarrow V_0\) and \((0) : V_1 \times V_0 \rightarrow V_0\) by

\[
\tau_{(0)} a = \tau(a), \quad a_{(0)} \tau = -\tau(a), \quad \omega_{(0)} a = a_{(0)} \omega = 0
\]  

(3.2.4)

Define the operation \((0) : V_1 \times V_1 \rightarrow V_1\) by

\[
\omega_{(0)} \omega_1 = 0; \quad \tau_{(0)} adb = \tau(a) db + ad(\tau(b)); \quad (adb)_{(0)} \tau = -\tau(a) db + \tau(b) da
\]  

(3.2.5)

and

\[
\tau_{(0)} \tau_2 = [\tau_1, \tau_2] - c^2(\tau_1, \tau_2) + \frac{1}{2} d(\tau_1, \tau_2)
\]  

(3.2.6)

Define the symmetric operation \((1) : V_1 \times V_1 \rightarrow V_0\) by

\[
\omega_{(1)} \omega_1 = 0; \quad \tau_{(1)} (adb) = a \tau(b); \quad \tau_{(1)} \tau_2 = \langle \tau_1, \tau_2 \rangle
\]  

(3.2.7)

This defines a structure of a one-truncated vertex algebra on the space \(V_{\leq 1} := V_0 \oplus V_1\). This means that we have the vacuum vector in \(V_0\), the map \(\partial := d : V_0 \rightarrow V_1\), together with the operations

\[
(n) : V_i \times V_j \rightarrow V_{i+j-n-1}
\]

for all \(i, j, n\) such that \(i, j, i + j - n - 1 \in \{0, 1\}\). These operations satisfy the Borcherds identities whenever the indices belong to the indicated range, the impossible operations being set to zero.

Let us give examples of prevertex algebras.

**3.3. Example.** Let \(T\) be a Lie algebra over \(\mathbb{C}\) equipped with a symmetric invariant bilinear form \(\langle , \rangle\). Set \(A = \mathbb{C}, \Omega = 0, c^2 = 0, \gamma = 0\). This defines a prevertex algebra.

Note that the component \(c^3\) defined by the rule (f) is equal to

\[
c^3(\tau_1, \tau_2, \tau_3) = -\frac{1}{2} \langle \tau_1, [\tau_2, \tau_3] \rangle
\]  

(3.3.1)

**3.4. Example.** Let \(A\) be a \(\mathbb{C}\)-algebra; set \(\Omega = \Omega^1_{\mathbb{C}}(A)\). Let \(T_0\) be an abelian Lie algebra over \(\mathbb{C}\) acting by derivations on \(A\).

(For example, let \(A\) be smooth, and assume that there exists a basis \(\{\tau_i\}\) of the left \(A\)-module \(T := \text{Der}_{\mathbb{C}}(A)\) consisting of commuting vector fields. Let \(T_0\) be the vector space spanned by this basis.)
Set $\mathcal{T} = A \otimes_{\mathbb{C}} \mathcal{T}_0$. There is a unique Lie bracket on $\mathcal{T}$ making it a Lie algebroid over $A$,
\[
[a_{\tau_1}, b_{\tau_2}] = a_{\tau_1}(b)\tau_2 - b_{\tau_2}(a)\tau_1 \quad (\tau_1, a, b \in A) \quad (3.4.1)
\]

We set $\langle \tau_1, \tau_2 \rangle = 0$; $\gamma(a, \tau) = 0$; $c^2(\tau_1, \tau_2) = 0$; $c^3(\tau_1, \tau_2, \tau_3) = 0$ for $a \in A$, $\tau_i \in \mathcal{T}_0$. Then the formulas 2.6 (a) — (d) define a unique extension of the operations $\gamma, (\ , \ ), c^2$ and $c^3$ to the whole space $\mathcal{T}$.

Namely,
\[
\gamma(a, b\tau) = -\tau(a)db - \tau(b)da \quad (3.4.2)
\]
\[
\langle a_{\tau_1}, b_{\tau_2} \rangle = -a_{\tau_2}\tau_1(b) - b_{\tau_1}\tau_2(a) - \tau_1(b)\tau_2(a) \quad (3.4.3)
\]

It is convenient to write down $c = (c^2, c^3)$ as a sum of a simpler cocycle and a coboundary,
\[
c^2(a_{\tau_1}, b_{\tau_2}) = 'c^2(a_{\tau_1}, b_{\tau_2}) + d\beta(a_{\tau_1}, b_{\tau_2}) \quad (3.4.4)
\]
where
\[
'c^2(a_{\tau_1}, b_{\tau_2}) = \frac{1}{2}\{\tau_1(b)d\tau_2(a) - \tau_2(a)d\tau_1(b)\} \quad (3.4.5)
\]
\[
\beta(a_{\tau_1}, b_{\tau_2}) = \frac{1}{2}\{b_{\tau_1}\tau_2(a) - a_{\tau_2}\tau_1(b)\} \quad (3.4.6)
\]

and
\[
c^3(a_{\tau_1}, b_{\tau_2}, c_{\tau_3}) = 'c^3(a_{\tau_1}, b_{\tau_2}, c_{\tau_3}) + d_{Lie}\beta(a_{\tau_1}, b_{\tau_2}, c_{\tau_3}) \quad (3.4.7)
\]
where
\[
'c^3(a_{\tau_1}, b_{\tau_2}, c_{\tau_3}) = \frac{1}{2}\{\tau_1(b)\tau_2(c)\tau_3(a) - \tau_3(c)\tau_2(a)\tau_1(b)\} \quad (3.4.8)
\]

A straightforward check shows that this gives a prevertex algebra.

Note that the cocycle $(', c^2, 'c^3)$ coincides with (minus one half of) the one from [MSV], (5.17-18) if $A$ is the polynomial ring.

### 3.5. Example.

Here is a common generalization of the above two examples. Namely, in the set up of 3.4, assume that $\mathcal{T}_0$ is an arbitrary Lie algebra over $\mathbb{C}$ acting by derivations on $A$, and equipped with a symmetric invariant bilinear form $\langle , \rangle$. There is a unique extension of the Lie structure on $\mathcal{T}_0$ to a Lie algebroid structure on $\mathcal{T} := A \otimes_{\mathbb{C}} \mathcal{T}_0$, given by
\[
[a_{\tau_1}, b_{\tau_2}] = ab[\tau_1, \tau_2] + a_{\tau_1}(b)\tau_2 - b_{\tau_2}(a)\tau_1 \quad (3.5.1)
\]
Set $\gamma(a, \tau) = 0$ for $\tau \in \mathcal{T}_0, a \in A$. There exists a unique extension of the function $\gamma$ to the whole $\mathcal{T}$, given by
\[
\gamma(a, b\tau) = -\tau(a)db - \tau(b)da \quad (3.5.2)
\]

There exists a unique extension of the pairing $\langle , \rangle$ to the whole $\mathcal{T}$ satisfying (2.6.7), namely
\[
\langle a_{\tau_1}, b_{\tau_2} \rangle = ab(\tau_1, \tau_2) - a_{\tau_2}\tau_1(b) - b_{\tau_1}\tau_2(a) - \tau_1(b)\tau_2(a) \quad (3.5.3)
\]
The cochain $c^2$ is defined as the unique extension of the zero cocycle on $\mathcal{T}_0$ satisfying (2.6.8). It is given by
\[
c^2 = 'c^2 + \nu c^3 + d\beta \quad (3.5.4)
\]
where $\beta$ is as in (3.3.6),

\[
\prime c^2(a_1, b_2) = \frac{1}{2} \left\{ \tau_1(b)d\tau_2(a) - \tau_2(a)d\tau_1(b) \right\} - [\tau_1, \tau_2](a)db - [\tau_1, \tau_2](b)da
\]

and

\[
\" c^2(a_1, b_2) = \frac{1}{2} \langle \tau_1, \tau_2 \rangle (adb - bda)
\]

(this is a "Kac-Moody" type summand). The component $c^3$ is given by

\[
c^3 = ' c^3 + " c^3
\]

where

\[
' c^3(a_1, b_2, c_3) = \frac{1}{2} \left\{ \tau_1(b)\tau_2(c)\tau_3(a) - \tau_1(c)\tau_2(a)\tau_3(b) \right\}
\]

and

\[
" c^3(a_1, b_2, c_3) = -\frac{1}{2} abc \langle \tau_1, [\tau_2, \tau_3] \rangle
\]

cf. (3.3.1). We have

\[
d_{Lie}(') c^2) = d(' c^3); d_{Lie}(" c^2) = d(\" c^3)
\]

and both \( ' c^3, " c^3 \) are Lie 3-cocycles.

3.6. Base change. The previous example is a particular case of the following construction. Let $P = (A, \Omega, T, \ldots)$ be a prevertex algebra, and let $A'$ be a commutative $A$-algebra.

Assume that we are given the data (a), (b) below.

(a) An action of the Lie algebra $T$ on the algebra $A'$ by derivations, extending its action on $A$ and such that

\[(a \tau)(b) = a \tau(b) \quad (a \in A, b \in A')\]

(b) A mapping $\gamma : A' \times T \rightarrow \Omega_{A'}$ extending the map 3.1 (d) such that 2.6 (a) is satisfied for all $a \in A', b \in A$. Here $\Omega_{A'} := A' \otimes_A \Omega$.

We call the data (a), (b) the base change data.

Set $T_{A'} := A' \otimes_A T$. Due to (a), there is a unique $A'$-Lie algebroid structure on $T_{A'}$, extending the given $A$-Lie algebroid structure on $T$. Formulas (2.6.4), (2.6.7) and (2.6.8) define the extension of the operations $\gamma$, $(\ , \ )$ and $c^2$ to the space $T_{A'}$.

This defines a new prevertex algebra $P_{A'} = (A', \Omega_{A'}, T_{A'}, \ldots)$.

3.7. The additive group of the space $Hom_C(T, \Omega) = C^1_{Lie}(T; \Omega)$ is acting on the set of data 2.7 (a) — (f) with fixed $A, \Omega, T, d$. Namely, for $w : T \rightarrow \Omega$, let us define the new

\[
\gamma_w(a, \tau) = \gamma(a, \tau) + w(\tau) - aw(\tau)
\]

\[(\tau_1, \tau_2)_w = \tau_1, \tau_2 + w(\tau_1), \tau_2 + \langle \tau_1, w(\tau_2) \rangle
\]
\[ c^2_w(\tau_1, \tau_2) = c^2(\tau_1, \tau_2) + d_{\text{Lie}} w(\tau_1, \tau_2) + d\alpha(\tau_1, \tau_2) \] (3.7.3)

\[ c^3_w(\tau_1, \tau_2, \tau_3) = d_{\text{Lie}} \alpha(\tau_1, \tau_2, \tau_3) \] (3.7.4)

where

\[ \alpha(\tau_1, \tau_2) = \frac{1}{2} \left\{ \langle \tau_1, w(\tau_2) \rangle - \langle \tau_2, w(\tau_1) \rangle \right\} \] (3.7.5)

The resulting truncated vertex algebras are isomorphic; if we denote the corresponding weight one spaces by \( V_1, V_{w1} \) where \( V_{w1} = V_1 \) as a vector space, then the isomorphism \( \phi_w : V_1 \rightarrow V_{w1} \) is defined by

\[ \phi_w(\tau, \omega) = (\tau, \omega + w(\tau)) \quad (\tau \in T, \ \omega \in \Omega) \] (3.7.6)

\( \phi_w \) being the identity on \( V_0 = V_{0w} \). We have obviously

\[ \phi_{w+w'} = \phi_w \circ \phi_{w'} \] (3.7.7)

This means that the truncated vertex algebra is defined uniquely, up to a unique isomorphism, by an element of the quotient of the set of the above data by the action of \( \text{Hom}_C(T, \Omega) \).

3.8. Let \( V \) be a restricted vertex algebra, and let \( \mathcal{M} \) be a graded \( V \)-module. We say that \( \mathcal{M} \) is restricted if it does not contain the negative integer conformal weights.

Suppose that \( \mathcal{M} \) is restricted. Again we assume to simplify the notations that \( V \) and \( M \) are even. We will adopt the notations of 2.2. We will denote the operation \( a_{(-1)}m \) simply by \( am \). Consider the weight zero component \( M = M_0 \). The operation \( am \) makes \( M \) a left \( A \)-module.

The operation \( (-1) : V_1 \otimes M \rightarrow M \) vanishes on the subspace \( \Omega \otimes M \), and thus induces the operation

\[ T \otimes M \rightarrow M, \ \tau \otimes m \mapsto \tau m \] (3.8.1)

making \( M \) a module over the Lie algebra \( T \). This operation satisfies

\[ a(\tau m) = (a\tau)m \] (3.8.2)

and

\[ \tau(am) = a(\tau m) + \tau(a)m \] (3.8.3)

\( (a \in A, \ \tau \in T, \ m \in M) \).

For example, if \( T \) coincides with the algebra of vector fields, this means that \( M \) is a \( D \)-module over \( A \).

§4. Vertex envelope

4.1. We have a functor

\[ (\text{Restricted vertex algebras}) \rightarrow (\text{Commutative algebras}) \] (4.1.1)
sending a vertex algebra $V$ to its conformal weight zero component. To shorten the notation, by a "commutative algebra" we mean a "commutative associative unital algebra".

We claim that this functor admits a left adjoint.

To construct it, we note that the forgetful functor from the category of the commutative algebras with a derivation to that of commutative algebras admits a left adjoint. It sends an algebra $R$ to the algebra $R_\partial$ which is the quotient of the free commutative algebra over $R$ generated by symbols $\partial^{(n)}r$, $r \in R$, $n \in \mathbb{Z}_{\geq 0}$, over the relations

\[
\partial^{(0)}r = r; \quad \partial^{(n)}(r_1 + r_2) = \partial^{(n)}r_1 + \partial^{(n)}r_2; \\
\partial^{(n)}(r_1r_2) = \sum_{p+q=n} \partial^{(p)}r_1 \cdot \partial^{(q)}r_2 \tag{4.1.2}
\]

The derivation acts as $\partial(\partial^{(n)}r) = (n+1)\partial^{(n+1)}r$.

Now, the left adjoint to (4.1.1) assigns to a commutative algebra $R$ the holomorphic vertex algebra corresponding to $R_\partial$, cf. 1.8.

4.2. We have a functor

\[
(Restricted vertex algebras) \longrightarrow (One – truncated vertex algebras) \tag{4.2.1}
\]

sending a vertex algebra $V$ to $V_{\leq 1} = V_0 \oplus V_1$. This functor admits a left adjoint, $U_1$. Namely, given a one-truncated algebra $W$, we first form the non-commutative nonassociative algebra $W'$ spanned by $\partial^{(n)}W$, $n \geq 0$ subject to relations (4.1.2). We introduce the operations $(n)$ on $W'$ using (1.3.12) and (1.3.4). By definition, $U_1W$ is the quotient of $W'$ by the Borcherds relations (1.3.1). We have $(U_1W)_{\leq 1} = W$.

The size of $U_1W$ is that of the symmetric algebra on the vector space

\[
\oplus_{n \geq 0} \partial^n A \oplus \oplus_{n \geq 0} \partial^n T
\]

(where $(A, T, \ldots)$ is the prevertex algebra corresponding to $W$). $U_1W$ contains as a subalgebra the holomorphic vertex algebra corresponding to $A$.

**Remark.** Essentially the same construction works in the filtered case, 2.7. Vertex algebras of this type appear as chiral algebras of twisted differential operators, cf. 4.4 below and [MSV], 5.15.

Let us call a restricted vertex algebra $V$ split if it is given together with a splitting (2.5.1). We have defined in 2.2 — 2.6 a functor

\[
P : (Split restricted vertex algebras) \longrightarrow (Prevertex algebras) \tag{4.2.2}
\]

This functor admits a left adjoint, to be denoted by $V$, and called vertex envelope. Namely, given a prevertex algebra, we first take the corresponding one-truncated vertex algebra, 3.2, and then apply $U_1$.

4.3. **Example.** If $P$ is as in 3.3 then $V(P)$ is the vacuum (level 1) representation of the affine Lie algebra corresponding to $(T, (\ , \ ))$, cf. [K], 4.7.
4.4. Example. If $P$ is as in 3.4, we get the algebras studied in [MSV], cf. op. cit., 6.9.

4.5. Chiral Weyl modules. Let us return again to the even situation. Let $V$ be a restricted vertex algebra, let $\mathcal{P}(V) = (A, \Omega, T, \ldots)$ be the corresponding prevertex algebra. Assume that the Lie algebra $T$ coincides with $\text{Der}(A)$.

In 3.8 we have defined a functor

$$(\text{Restricted } V \text{-modules}) \rightarrow (D_A \text{-modules}) \quad (4.5.1)$$

where $D_A$ is the algebra of differential operators on $A$. This functor admits a left adjoint

$W : (D_A \text{-modules}) \rightarrow (\text{Restricted } V \text{-modules}) \quad (4.5.2)$

Its construction is similar to 4.2, and even simpler. We leave it to the reader.

For a $D_A$-module $M$, the $V$-module $W(M)$ is called the chiral Weyl module corresponding to $M$. 
Part II: Vertex algebras and coinvariants

In §1 we focus on two important features of an arbitrary conformal vertex algebra \( V \). Firstly, with any such an algebra and any smooth curve we associate a sheaf of Lie algebras, see 1.1-1.3. We use this sheaf to define a space of (co)invariants (or “conformal blocks”) in the situation when there are several \( V \)-modules attached to several points on the curve; these spaces arrange in a vector bundle with flat connection, see 1.6. In 1.8-9 we explain that one can represent a horizontal section of the bundle associated with several copies of \( V \) attached to points on \( \mathbb{C}P^1 \) as a matrix element of a product of fields.

Secondly, we show in 1.4 that the Lie algebra of Fourier components of fields associated with any conformal vertex algebra affords an antiinvolution. This gives a duality functor on the category of \( V \)-modules, see 1.7.

In §§2,3 we explain what all of this means in the case of the sheaf \( \Omega_{X}^{ch} \) introduced in [MSV]. In particular, we reproduce the well-known physics calculation that represents the structure coefficients of the cohomology ring of \( X \) as matrix elements of products of fields, see 3.2.

\( \S 1. \) The spaces of (co)invariants

1.1. We place ourselves in the situation of I.1.7, that is, assume that we are given a vertex algebra \( V \) along with a Virasoro field \( L(z) \in \text{Fields}(V) \) satisfying (I.1.7.1-2). Consider the space of Fourier modes of fields \( \text{Lie}(V) = \{ \int v(z)z^m, v \in V, m \in \mathbb{Z} \} \). By (I.1.3.7), \( \text{Lie}(V) \) is a Lie subalgebra of \( \text{End}(V) \). In particular, \( \text{Lie}(V) \) is a \( \mathcal{V}_{ir} \)-module and \( adL_0 \) is diagonalizable:

\[
\text{Lie}(V) = \bigoplus_{m \in \mathbb{Z}} \text{Lie}(V)_m, \quad \text{Lie}(V)_m = \{ A \in \text{Lie}(V) : [L_0, A] = mA \}. \tag{1.1.1}
\]

Observe that \( id \in \text{End}(V) \) belongs to \( \text{Lie}(V) \) as \( id = \int |0\rangle(z^{-1}) \). Below we shall sometimes quotient the Lie ideal \( \mathcal{C}id \) out.

Define:

\[
\text{Lie}(V)_\leq = \{ A \in \text{Lie}(V) : (adL_{-1})^m A = 0 \text{ if } m >> 0 \}/\mathcal{C}id, \tag{1.1.2}
\]

\[
\text{Lie}(V)_{\leq n} = \bigoplus_{m \leq n} \text{Lie}(V)_m \tag{1.1.3}
\]

\[
\tilde{\text{Lie}}(V) = \lim_{\rightarrow -\infty \leftarrow m} \text{Lie}(V)/\text{Lie}(V)_{\leq m} \tag{1.1.4}
\]

\[
\tilde{\text{Lie}}(V)_\leq = (\lim_{\rightarrow -\infty \leftarrow m} \text{Lie}(V)_\leq/(\text{Lie}(V)_{\leq m} \cap \text{Lie}(V)_\leq))/\mathcal{C}id. \tag{1.1.5}
\]
One easily checks that the bracket on Lie(\(V\)) extends by continuity to that on \(\tilde{\text{Lie}}(\{\})\) making the latter into a Lie algebra, and that \(\tilde{\text{Lie}}(\{\}) \leq \tilde{\text{Lie}}(\{\})\) is a Lie subalgebra.

Further, the Lie algebra of vector fields on the formal disk \(\mathbb{C}[[z]]d/dz \subset \text{Vir}\) operates on \(\tilde{\text{Lie}}(\{\}) \leq\), the action of the subalgebra \(\mathbb{C}[[z]]d/dz\) being integrable. The formal geometry of Gelfand and Kazhdan produces then a universal sheaf of Lie algebras. It means that for any smooth curve \(C\) we get a sheaf of Lie algebras, \(\text{Lie}_{C}(V)\) on \(C\), so that \(\Gamma(\text{Spec}\mathbb{C}[[z]], \text{Lie}_{\text{Spec}\mathbb{C}[[z]]}(V)) = \tilde{\text{Lie}}(\{\}) \leq\). The correspondence \(C \mapsto \text{Lie}_{C}(V)\) is functorial with respect to étale morphisms.

1.2. Here is a somewhat more explicit description of the sheaf \(\text{Lie}_{C}(V)\). Let \(v \in V_{\Delta}\) be a Vir-singular vector. This means that \(v\) does not belong to \(\mathbb{C}[0]\) and is annihilated by \(L_{i}, \ i \geq 1\). In the language of fields we get, due to (I.1.7.3), that:

\[
L(z)v(w) \sim \frac{v(w)'}{z-w} + \frac{\Delta v(w)}{(z-w)^2},
\]

that is to say, that \(v(z)\) is a primary field. A glance at (I.1.3.7) or (I.1.7.4) shows that

\[
[L_{m}, v_{(n)}] = (-n + (m + 1)(\Delta - 1))v_{(m+n)}.
\]

In other words, under the action of Vir elements \(v_{(n)} \in \text{Lie}(V)\) transform as if \(v_{(n)}\) were equal to \(z^{n}(dz)^{-\Delta+1}\) and therefore \(\sum_{n \geq 0} v_{(n)} \in \tilde{\text{Lie}}(\{\}) \leq\) can be regarded as a formal jet of the section of the bundle of \(-\Delta + 1\)-differentials. It follows that when “spread” over a curve a Vir-singular vector of conformal weight \(\Delta\) gives a subsheaf of \(\text{Lie}_{C}(V)\) isomorphic to the sheaf of \(-\Delta + 1\) differentials.

Of course not any element of \(V\) is singular. However \(V\) possesses the filtration by conformal weights: \(\ldots V_{\leq 0} \subset V_{\leq 1} \subset \ldots\), where \(V_{\leq m} = \oplus_{i \leq m} V_{i}\). By definition elements \(L_{m}, m > 0\) preserve this filtration and act trivially on \(\oplus_{m} V_{\leq m}/ \oplus_{m-1} V_{\leq m-1}\).

Using (I.1.7.4) once again we get that \(\tilde{\text{Lie}}(\{\}) \leq\) has a filtration such that the corresponding graded object is isomorphic, as a \(\mathbb{C}[[z]]d/dz\)-module, to a direct sum of modules of \(\Delta\)-differentials over \(\mathbb{C}^{*}\). There is one such module for each \(v \in V_{-\Delta+1}\) if \(\Delta \neq 1\) or for each \(v \in V_{0}/\mathbb{C}[0]\). If \(v = L_{-1}w, v, w \in V\), then the corresponding modules are equal (as subspaces of \(\tilde{\text{Lie}}(\{\})\)).

Putting all of this together we get that \(\text{Lie}_{C}(V)\) has a filtration such that the corresponding graded object is a direct sum of sheaves of \(\Delta\)-differentials with appropriate \(\Delta\). Further, locally in the presence of a coordinate, for example when \(C\) is either Spec\(\mathbb{C}[[z]]\) or Spec\(\mathbb{C}[z, (z - z_{1})^{-1}, \ldots, (z - z_{m})^{-1}]\), \(\text{Lie}_{C}(V)\) is a free sheaf of \(\mathcal{O}_{C}\)-modules whose fiber is isomorphic to \(V/(L_{-1}V + \mathbb{C}[0])\).

1.3. Here we collect several Lie algebras that arise in the case of a curve with marked points. By construction \(\text{Lie}(V)\) is a central extension of \(\Gamma(\mathbb{C}^{*}, \text{Lie}_{\mathbb{C}^{*}}(V))\). Likewise \(\text{Lie}(\{\})\) is a central extension of \(\Gamma(\text{Spec}\mathbb{C}([[z]]), \text{Lie}_{\text{Spec}\mathbb{C}([[z]])}(V))\).

Let \(P\) be a point of \(C\), \(U\) the formal neighborhood of \(P\), \(U' = U - P\). We have then the Lie algebras \(\Gamma(U', \text{Lie}_{U'}(V))\) and \(\Gamma(C - P, \text{Lie}_{C}(V))\) along with the “localization at \(P\)” map

\[
i_{P} : \Gamma(C - P, \text{Lie}_{C}(V)) \to \Gamma(U', \text{Lie}_{U'}(V)).
\]
Since any isomorphism $\mathbb{C}[[z]] \to O_U$ (a choice of a local coordinate) determines an isomorphism $\Gamma(\text{Spec}\mathbb{C}((z)), \text{Lie}_{\text{Spec}\mathbb{C}((z))}(V)) \to \Gamma(U', \text{Lie}_{U'}(V))$, there arises a Lie algebra, to be denoted by $\text{Lie}(V)$, a central extension of $\Gamma(U', \text{Lie}_{U'}(V))$ with a distinguished central element $K$. Again any isomorphism $\mathbb{C}[[z]] \to O_U$ determines an isomorphism $\text{Lie}(V) \to \text{Lie}(V)^P$ preserving $K$. The algebra $\text{Lie}(V)^P$ is sometimes referred to as “$\text{Lie}^i(V)$ sitting at $P$”. Given a collection of points $\{P_1, \ldots, P_m\} \subset \mathcal{C}$ consider the Lie algebra $\oplus_{i=1}^m \Gamma(U'_i, \text{Lie}_{U'_i}(V))$, where $U'_i$ is the formal punctured neighborhood of $P_i$, and its central extension $\text{Lie}(V)^{P_1, \ldots, P_m}$, the Baer sum of central extensions

$$0 \to \mathbb{C} \to \text{Lie}(V)^{P_1} \to \Gamma(U'_1, \text{Lie}_{U'_1}(V)) \to 0$$

(Take the Baer sum of central extensions by $\mathbb{C}$ means to take the direct sum of central extensions and then set $K$ in all summands equal each other.)

Another ingredient is the Lie algebra $\Gamma(\mathcal{C} - \{P_1, \ldots, P_m\}, \text{Lie}_{\mathcal{C}}(V))$; because of its importance we shall denote it simply as $\text{Lie}(V)$, suppressing some of the data on which it actually depends. There is a Lie algebra morphism (cf. (1.3.1))

$$i_{P_1, \ldots, P_m} = \bigoplus_j i_{P_j} : \text{Lie}(V)_{\text{out}} \to \bigoplus_{i=1}^m \Gamma(U'_i, \text{Lie}_{U'_i}(V)).$$

(1.3.2)

The key point is that the last map lifts to a Lie algebra morphism

$$\hat{i}_{P_1, \ldots, P_m} = \bigoplus_j i_{P_j} : \text{Lie}(V)_{\text{out}} \to \text{Lie}(V)^{P_1, \ldots, P_m}$$

(1.3.3)

This can be proved by methods of [BFM, 2.2.1-2.3.4], that is, by showing that the central extensions in question are Tate ones and then using the residue theorem. We shall skip this argument as below in the case of interest for us we shall exhibit a direct construction.

1.4. Let us now discuss in greater detail the case when $\mathcal{C} = \mathbb{C}P^1$. We assume fixed an embedding $\mathbb{C} \hookrightarrow \mathbb{C}P^1$ and a coordinate $z$ on $\mathbb{C}$. First of all we shall explain that to construct the splitting (1.3.3) in the case of $\mathbb{C}P^1$ means to construct a certain Lie algebra anti-involution $\eta : \text{Lie}(V) \to \text{Lie}(V)$ preserving the central element.

Begin with the case of 2 marked points and let the points be 0 and $\infty$. In this case $\text{Lie}(V)_{\text{out}} = \Gamma(\mathbb{C}^*, \text{Lie}_{\mathbb{C}^*}(V))$. Observe that the coordinate change $z \mapsto 1/z$ induces a Lie algebra involution:

$$\hat{\eta} : \Gamma(\mathbb{C}^*, \text{Lie}_{\mathbb{C}^*}(V)) \to \Gamma(\mathbb{C}^*, \text{Lie}_{\mathbb{C}^*}(V)).$$

(1.4.1)

Choose the local coordinates at 0 and $\infty$ to be $z$ and $1/z$ respectively. This allows to identify $\Gamma(U'_{0}, \text{Lie}_{U'_{0}}(V)) = \Gamma(U'_{\infty}, \text{Lie}_{U'_{\infty}}(V)) = \text{Lie}(V)/\mathbb{C}K$. Under this identification, both $i_0$ and $i_{\infty}$ are isomorphisms of $\text{Lie}(V)$ with $\text{Lie}(V)/\mathbb{C}K \subset \text{Lie}(V)/\mathbb{C}K$ such that $i_{\infty} = i_0 \circ \hat{\eta}$. We can and will assume that $i_0 = id$ (it is simply a matter of notation); then $i_{\infty} = \hat{\eta}$.

By definition both $i_0$ and $i_{\infty}$ lift to isomorphisms of central extensions

$$i_0, i_{\infty} : \text{Lie}(V) \to \text{Lie}(V).$$

(1.4.2)
Because of our conventions, we can set \( \hat{i}_0 = \text{id} \). Then existence of the splitting (1.3.3) in the situation in question is equivalent to existence of \( \hat{i}_\infty \) in (1.4.2) so that \( i_\infty(K) = -K \). It follows that the map \( \eta \) defined to be equal \(-i_\infty \) satisfies the following conditions:

\[
\eta : \text{Lie}(V) \to \text{Lie}(V) \text{ is an antiinvolution,}
\]

\[
\eta(K) = K,
\]

\[
\eta_{\text{Lie}(V)/CK} = -\bar{\eta}.
\]

As we explained, the data (1.4.3-5) is equivalent to the datum (1.3.3) in the case of 2 marked points \( 0, \infty \in \mathbb{C}P^1 \).

1.5. Let us now construct the splitting (1.3.3) in the case of several points on \( \mathbb{C}P^1 \). For the sake of definiteness assume that \( P_1 = \infty \), while \( P_i = z_i \in \mathbb{C}, i = 2, \ldots, m \). Choose \( z - z_i \) as a local coordinate at \( P_i \) and \( 1/z \) as that at \( \infty \). Trivializing the sheaf \( \text{Lie}_{\mathbb{C}P^1}(V) \) over \( \mathbb{C} \) using the coordinate \( z \) (see 1.2) we identify the Fourier component \( B_{(j)} \) of any field \( B(z) \) with a section of a trivial bundle over \( \mathbb{C} \) with fiber: image of \( \mathbb{C}B_{(-1)}(0) \subset V \) in \( V/(L_{-1}V + \mathbb{C}[0]) \), see the very end of 1.2. The translation invariance axiom shows then that under this identification \( B_{(j)} \) becomes \( z^j \). Similarly, denote by \( B_{(j)}^\prime \) the section of the same bundle identified with the function \((z - z_s)^j \); in particular, \( B_j^\prime = B_j \). In this notation the localization maps (1.3.1-2) are simply Laurent series expansions of rational functions. We have:

\[
i_{P_1, \ldots, P_m} : \text{Lie}(V)_{\text{out}} \to \oplus_{i=1}^m \Gamma(U_i^\prime, \text{Lie}_{U_i^\prime}(V)),
\]

\[
i_{P_1, \ldots, P_m} = \oplus_i i_{P_i},
\]

(1.5.1)

where

\[
i_{P_i} : \text{Lie}(V)_{\text{out}} \to \Gamma(U_i^\prime, \text{Lie}_{U_i^\prime}(V)),
\]

(1.5.2)

is defined as follows

\[
i_{z_i}(B_{(j)}^{z_s}) = -\partial_{z_s}(-j+1)(\sum_{i=0}^{\infty} \frac{B_{(i)}^{z_s}}{(z_s - z_i)^{j+1}}) \text{ if } j < 0,
\]

(1.5.3)

(note that the expression in the r.h.s. of the last equality is nothing but \( (B_{(j)}[0])(z_s - z_i) \), see (1.1.4.3) for the definition)

\[
i_{\infty}(B_{(j)}^{z_s}) = -\eta((B_{(j)}[0])(z_s))_+ \text{ if } j < 0,
\]

(1.5.4)

(see (1.1.4.3) for the definition of \( (B_{(j)}[0])(z_s)_+ \))

\[
i_{z_i}(B_{(j)}^{z_s}) = \sum_{i \geq 0} \binom{j}{i} (z_i - z_s)^{j-i} B_{(i)}^{z_s} \text{ if } j \geq 0,
\]

(1.5.5)

\[
i_{\infty}(B_{(j)}) = -\eta(B_{(j)}) \text{ if } j \in \mathbb{Z}.
\]

(1.5.6)
**Remark.** The formula (1.5.3) is obtained by expanding the function $1/z - z_s$ at $z_t$:

$$
\frac{1}{z - z_s} = -\sum_{i=0}^{\infty} \frac{(z - z_t)^i}{(z_s - z_t)^{i+1}}
$$

and identifying $(z - z_t)^i$ in the r.h.s. with $B_{(i)}^\ast$. A similar remark applies to (1.5.5) with $1/z - z_s$ replaced with $(z - z_s)^j$. Of course (1.5.3) follows from (1.5.5).

In the presence of a coordinate at any point $P$ any splitting $V = V' \oplus \mathbb{C}[0]$ gives a splitting $\text{Lie}(V)^P = \Gamma(U, \text{Lie}_U(V)) \oplus \mathbb{C}K$, $P \in U$. We have already fixed coordinates at our points and we now choose arbitrarily a splitting $V = V' \oplus \mathbb{C}[0]$ to get splittings

$$
\text{Lie}(V)^P_i = \Gamma(U'_i, \text{Lie}_{U'_i}(V)) \oplus \mathbb{C}K. \quad (1.5.7)
$$

Given that, there arise canonical embeddings $\Gamma(U'_i, \text{Lie}_{U'_i}(V)) \hookrightarrow \text{Lie}(V)^P_i$. We use these embeddings to define (cf. (1.3.2-3)):

$$
\hat{i}_{P_1,...,P_m} : \text{Lie}(V)_{\text{out}} \rightarrow \bigoplus_{i=1}^{\text{m}} \text{Lie}(V)^P_i
$$

to be

$$
\hat{i}_{P_1,...,P_m} = i_{P_1,...,P_m}. \quad (1.5.8)
$$

In the case of 2 points, $\infty$ and 0 the fact that this $\hat{i}_{P_1,...,P_m}$ is a Lie algebra homomorphism follows from 1.4. If the points in question are $\infty$ and $z_2$ then the desired claim follows from the following observation: operator $L_{-1}$ annihilates the cocycle determined by the splittings $(1.5.7)$. Finally, the case of more than 2 points is readily reduced to that of 2 points by representing a rational function as a linear combination of $(z - z_t)^j$.

**1.6.** So far we have been dealing with “algebras attached to points on a curve”. One can as well attach to points modules over these algebras. More precisely, let $\mathcal{C}$ be a curve with $m$ marked points $P_1,...,P_m$ and coordinates $z_i$ on the formal neighborhoods $U_i \ni P_i$, $i = 1,...,m$. This gives canonical identifications $\text{Lie}(V) = \text{Lie}(V)^{P_i}$, $i = 1,...,m$. Therefore, if $M_1,...,M_m$ is a collection of $\text{Lie}(V)$-modules of the same central charge, then $M_1 \otimes M_2 \otimes \cdots \otimes M_m$ is a $\text{Lie}(V)^{P_1} \otimes \cdots \otimes M_m$-module, meaning that elements of $\text{Lie}(V)^{P_i}$ act on the $i$-th factor of the tensor product $M_1 \otimes M_2 \otimes \cdots \otimes M_m$.

Due to (1.3.3), $M_1 \otimes M_2 \otimes \cdots \otimes M_m$ is also a $\text{Lie}(V)_{\text{out}}$-module. Hence there arises the space of coinvariants

$$
(M_1 \otimes M_2 \otimes \cdots \otimes M_m)_{\text{Lie}(V)_{\text{out}}} := \frac{M_1 \otimes M_2 \otimes \cdots \otimes M_m}{\text{Lie}(V)_{\text{out}}(M_1 \otimes M_2 \otimes \cdots \otimes M_m)}. \quad (1.6.1)
$$

We shall often make use of the dual space

$$
\langle M_1, M_2, ..., M_m \rangle (P_1,...,P_m) := ((M_1 \otimes M_2 \otimes \cdots \otimes M_m)_{\text{Lie}(V)_{\text{out}}})^\ast. \quad (1.6.2)
$$

Explicitly, $\langle M_1, M_2, ..., M_m \rangle (P_1,...,P_m)$ consists of linear functionals

$$
\langle \cdot, \cdot, \cdot, \rangle : M_1 \otimes M_2 \otimes \cdots \otimes M_m \rightarrow \mathbb{C}, \quad v_1 \otimes \cdots \otimes v_m \mapsto \langle v_1, ..., v_m \rangle
$$
satisfying the following \( \text{Lie}(V)_{\text{out}} \)-invariance condition:

\[
\sum_{t=1}^{m} (v_1, \ldots, v_{t-1}, i_{P_t}(X)v_t, v_{t+1}, \ldots, v_m) = 0
\]

for any \( X \in \text{Lie}(V)_{\text{out}} \).

By definition the space \( \langle M_1, M_2, \ldots, M_m \rangle(P_1, \ldots, P_m) \) depends on the choice of local coordinates. However, if the modules \( M_i \) are all integrable with respect to the algebra \( \mathbb{C}[\![z]\!]d/dz \), then the two such spaces associated with different coordinates are canonically identified. This means that there arises a vector bundle \( \langle M_1, M_2, \ldots, M_m \rangle_{\mathbb{C}} \) over the configuration space \( \mathbb{C}^m - \text{diagonals} \), the fiber over the point \( (P_1, \ldots, P_m) \) being \( \langle M_1, M_2, \ldots, M_m \rangle(P_1, \ldots, P_m) \).

Denote by \( L_{-1}^{(i)} \) the linear transformation of \( \langle M_1, M_2, \ldots, M_m \rangle(P_1, \ldots, P_m) \) acting as \( L_{-1} \) on the \( i \)-th factor. It is well known that the operators

\[
\Delta_t = \partial/\partial z_t - L_{-1}^t, \quad t = 1, \ldots, m
\]

define a flat connection on the bundle \( \langle M_1, M_2, \ldots, M_m \rangle_{\mathbb{C}} \).

We now try to explain that in many cases this construction admits a transparent representation-theoretic or “vertex-theoretic” interpretation.

1.7. Denote by \( V-\text{Mod} \) the category of restricted graded \( V \)-modules. Existence of the antiinvolution \( \eta \) (see (1.4.3)) allows us to define the following duality functor

\[
D : V-\text{Mod} \to V-\text{Mod}, \quad M \mapsto D(M) := \oplus_{n \in \mathbb{Z}} M_n^*.
\]

The \( V \)-module structure on \( D(M) \) is defined in the following manner:

\[
(b(z)f)(.) := f(\eta(b(z))), = \sum_{i \in \mathbb{Z}} z^{-i-1} f(\eta(b(i))).
\]

for all \( b \in V, f(.) \in D(M) \).

Now consider the case of the 2 marked points, \( \infty \) and 0, on \( \mathbb{CP}^1 \) with the standard choice of local coordinates \( z \) and \( z^{-1} \). Comparing the definition of \( \eta \) in 1.3, or the formulas in 1.5, with (1.6.3) we immediately get the canonical embedding:

\[
\text{Hom}_V(M_1, M_2) \hookrightarrow \langle D(M_2), M_1 \rangle(\infty, 0),
\]

\[
F \mapsto \{(x, y) \mapsto y(F(x))\}.
\]

Further, if homogeneous subspaces \( M_2 \) are finite dimensional, then (1.7.3) is an isomorphism. This finiteness condition sometimes fails as it does in some of our concrete examples to be considered below.

1.8. Consider now the case of 3 points on \( \mathbb{CP}^1 \). Attach modules \( D(V), V, V \) to the points \( \infty, z \) and 0 respectively.

**Lemma.** The functional \( \Phi_z(., ., .) \in (D(V) \otimes V \otimes V)^* \) defined by the formula

\[
\Phi_z(a, b, c) = a(b(z)c)
\]
actually belongs to \(\langle D(V), V, V \rangle (\infty, z, 0)\).

Consider \(\langle D(V), V, V \rangle_{\mathbb{C}^*}\), a vector bundle over \(\mathbb{C}^*\) obtained by restricting \(\langle D(V), V, V \rangle_\mathbb{C}\), a trivial bundle over \(\mathbb{C}^*\) - diagonals defined in 1.6, to \(\{ (\infty, z, 0), z \in \mathbb{C}^* \} \subset (\mathbb{C}^1)^3 \) - diagonals. Of course, the connection (1.6.4) restricts to the \(\langle D(V), V, V \rangle_{\mathbb{C}^*}\). The translation invariance axiom (see I.1.2) along with (I.1.7.4) implies that the section \(z \mapsto \Phi_z\) is horizontal. Indeed we have:

\[
\Delta(\Phi_z)(a, b, c) = a((b(z))' - (L_{-1}b)(z))c = a((b(z))' - [L_{-1}, b(z)])c = 0.
\]

1.9. More generally, we have the following result. Observe that although the formal product of fields

\[
a_1(z_1)a_2(z_2)a_3(z_3) \cdots a_{m-1}(z_{m-1})a_m(z_m), \ a_i \in V
\]
does not make sense as an operator on \(V\), the matrix element

\[
y(a_1(z_1)a_2(z_2)a_3(z_3) \cdots a_{m-1}(z_{m-1})a_m(z_m)x), \ x \in V, y \in D(V)
\]
is a well-defined formal Laurent series. Further, it easily follows from the OPE formula (1.1.6.9) that this series converges to a certain rational function in the region \(|z_1| > |z_2| > \cdots > |z_m|\).

**Lemma.** The functional \(\Phi_{z_2, \ldots, z_{m-1}}(\ldots, \ldots) \in (D(V) \otimes V \otimes \cdots \otimes V)^*\) defined by the formula

\[
\Phi_{z_2, \ldots, z_{m-1}}(a_1, a_2, \ldots, a_{m-1}, a_m) = a_1(a_2(z_2)a_3(z_3) \cdots a_{m-1}(z_{m-1})a_m)
\]

actually belongs to \(\langle D(V), V, \ldots, V \rangle (\infty, z_2, \ldots z_{m-1}, 0)\).

**Proof.** By way of preparation let us remark that due to the locality axiom the OPE (I.1.4.12) formula can be rewritten as the following 2 identities (see also [K], Theorem 2.3 (ii)):

\[
[a_s(z_s), B(z_t)_+] = \sum_{i=0}^{\infty} \frac{(B_{(i)}a_s)(z_s)}{(z_t - z_s)^{i+1}}, \tag{1.9.1}
\]

\[
[B(z_t)_-, a_s(z_s)] = \sum_{i=0}^{\infty} \frac{(B_{(i)}a_s)(z_s)}{(z_t - z_s)^{i+1}}. \tag{1.9.2}
\]

When compared to (1.5.3) the last equalities rewrite as follows:

\[
[a_s(z_s), B(z_t)_+] = -(i_{z_s}(B_{(i)}^*)(a_s))(z_s), \tag{1.9.3}
\]

\[
[B(z_t)_-, a_s(z_s)] = -(i_{z_s}(B_{(i)}^*)(a_s))(z_s). \tag{1.9.4}
\]

Now turn to the proof proper. We have to show that

\[
(X \Phi_{z_2, \ldots, z_{m-1}})(a_1, a_2, \ldots, a_{m-1}, a_m) = 0
\]
for any \(X \in \text{Lie}(V)_{out}\). When restricted to \(\mathbb{C}\), \(X\) may only have poles at \(z_t\), \(2 \leq t \leq m-1\). It means that it is enough to consider the following cases: \(X\) is either \(B_{(j)}^2\), \(j < 0\) or \(B_{(j)}\), \(j \geq 0\). Let for simplicity \(X\) be equal \(B_{(-1)}^2\), \(1 < t < m\). We have:

\[
a_1(a_2(z_2) \cdots a_{t-1}(z_{t-1})(B_{(1)}-a_t) a_t a_{t+1}(z_{t+1}), \ldots, a_{m-1}(z_{m-1})a_m)
\]

\[
= a_1(a_2(z_2), \ldots, B(z_t), a_t(z_t), \ldots, a_{m-1}(z_{m-1})a_m)
\]

\[
+ a_1(a_2(z_2), \ldots, a_t(z_t) B(z_t) \ldots, a_{m-1}, a_m)
\]

\[
= \sum_{s=2}^{t-1} a_1(a_2(z_2), \ldots, [a_s(z_s), B(z_t)], \ldots, a_t(z_t), \ldots, a_{m-1}(z_{m-1})a_m)
\]

\[
+ \sum_{s=t+1}^{m-1} a_1(a_2(z_2), \ldots, a_t(z_t), \ldots, [B(z_t), a_s(z_s)], \ldots, a_{m-1}(z_{m-1})a_m)
\]

\[
+ a_1(B(z_t), a_2(z_2), \ldots, a_t(z_t), \ldots, a_{m-1}(z_{m-1})a_m)
\]

\[
+ a_1(a_2(z_2), \ldots, a_t(z_t), \ldots, a_{m-1}(z_{m-1})B(z_t) \ldots a_m).
\]

The terms in the r.h.s of this equality are interpreted as follows:

by (1.3.4) each summand in the summations \(\Sigma_s\) equals:

\[-a_1(a_2(z_2), \ldots, (i_z, (B_{(-1)}^m(a_s)(z_s), \ldots, a_{m-1}(z_{m-1})a_m));\]

the last summand equals, directly by (1.5.3),

\[-a_1(a_2(z_2), \ldots, a_{m-1}(z_{m-1})(i_{z_{m-1}=0}(B_{(-1)}^m(a_m))));\]

while the one before the last one, by (1.5.4), equals

\[-i_{z_{m-1}}(B_{(-1)}^m(a_1)(a_2(z_2) \ldots a_{m-1}(z_{m-1})a_m).\]

Collecting all the terms in the l.h.s. we obtain the desired equality \(B_{(-1)}^m \Phi_{z_2, \ldots, z_{m-1}} = 0\). The case of \(B_{-1}^m, j < -1\) is obtained from the one considered by taking derivatives. The case \(\check{X} = B_{(j)}, j \geq 0\) is treated similarly.

As in 1.8 one can consider the embedding

\[
\pi: (\mathbb{C}^*)^{(m-2)} \backslash \{\text{diagonals}\} \rightarrow \mathbb{C}^m \backslash \{\text{diagonals}\} \quad (1.9.5)
\]

obtained by keeping two points equal 0 and \(\infty\). There arises the pull-back \(\pi^* \langle D(V), V, ..., V \rangle\) to \((\mathbb{C}^*)^{(m-2)}\) and one proves, exactly as in the end of 1.8, that \((z_2, ..., z_{m-1}) \mapsto \Phi_{z_2, ..., z_{m-1}}(\cdot, ..., \cdot)\) is its horizontal section.

\[\S 2. \text{An application to the chiral \textit{De Rham complex}}\]

Here we shall explain what the abstract constructions of \(\S 1\) mean in several concrete examples. In the end we shall explain how to recover the multiplication in the cohomology ring of a smooth manifold by looking at correlation functions.
2.1. Consider the vertex algebra $O_N$. It is generated by the fields

$$a^i(z) = \Sigma_{s\in\mathbb{Z}} a^i_s z^{-s-1}$$  \hspace{1cm} (2.1.1)  
$$b^i(z) = \Sigma_{s\in\mathbb{Z}} b^i_s z^{-s}$$  \hspace{1cm} (2.1.2)  

so that the Fourier components $a^i_s, b^i_s \in \text{Lie}(O_N)$ satisfy the relations

$$[a^i_s, b^j_t] = \delta_{ij}\delta_{s,-t}\mathbb{K}.$$  \hspace{1cm} (2.1.3)  

These relations mean that we are dealing with the Heisenberg algebra, $H_N$, linearly spanned by $a$'s and $b$'s. $O_N$ is a level 1 ($\mathbb{K} \mapsto 1$) representation of this algebra; it is generated by the “vacuum” vector, $|0\rangle$, satisfying the relations:

$$a^i_s|0\rangle = b^i_s|0\rangle = 0 \text{ if } s \geq 0, t > 0.$$  

The Virasoro field is as follows:

$$L(z) = \Sigma_{i=1}^N :b^i(z) a^i(z):$$  \hspace{1cm} (2.1.4)  

One easily checks that $b^i_0|0\rangle$ is a $Vir$-singular vector of weight 0. It follows that (cf. 1.2):

$$[L_s, a^i_j] = -ja^{i,s+j},$$  \hspace{1cm} (2.1.5)  
$$[L_s, b^i_j] = -(s+j)b^{i,s+j}.$$  \hspace{1cm} (2.1.6)  

It follows that as elements of $\Gamma(\mathbb{C}^*, \text{Lie}_{\mathbb{C}^*}(O_N))$, $a^i_j$ and $b^i_j$ are identified with $z^j$ and $z^{-j}dz$ respectively. Hence the automorphism $\bar{\eta}$ of $\Gamma(\mathbb{C}^*, \text{Lie}_{\mathbb{C}^*}(O_N))$ induced by the coordinate change $z \mapsto z^{-1}$ (see (1.4.1)) operates on these elements as follows:

$$\bar{\eta}(a^i_j) = a^i_{-j}, \hspace{0.5cm} \bar{\eta}(b^i_j) = -b^i_{-j}.$$  \hspace{1cm} (2.1.7)  

These formulas suggest how to lift $\bar{\eta}$ to an automorphism $\hat{\eta}$ (cf. (1.4.2)) at least when restricted to $H_N$:

$$\hat{\eta}(a^i_j) = a^i_{-j}, \hspace{0.5cm} \hat{\eta}(b^i_j) = -b^i_{-j}, \hspace{0.5cm} \hat{\eta}(\mathbb{K}) = -\mathbb{K}$$  \hspace{1cm} (2.1.8)  

It is now easy to find the anti-automorphism $\eta$; here is how it operates on $H_N$:

$$\eta(a^i_j) = a^i_{-j}, \hspace{0.5cm} \eta(b^i_j) = b^i_{-j}, \hspace{0.5cm} \eta(\mathbb{K}) = \mathbb{K}.$$  \hspace{1cm} (2.1.9)  

It is also easy to extend the anti-automorphism $\eta$ from $H_N$ to the entire $\text{Lie}(O_N)$ – easier than the automorphism $\hat{\eta}$. Each element of $\text{Lie}(V)$ is an infinite sum of monomials in $a$'s and $b$'s; to evaluate $\eta$ on such a series one has to apply $\eta$ to each summand regarded as an element of the universal enveloping algebra of $H_N$. A similar procedure does not apply to $\hat{\eta}$.

2.2. As we argued in [MSV], the vertex algebra $O_N$ is associated with $\mathbb{C}^N$ with a fixed coordinate system; this is reflected, in particular, in the fact that the conformal weight 0 component of $O_N$ is $\mathbb{C}[b^1_0, ..., b^N_0]$. Passing to various completions we get
the vertex algebras $O_N$, $O_N^{an}(U)$, $O_N^{sm}(U)$ associated with a formal disk, an open set in analytic category and an open set in smooth category resp.; as the conformal weight 0 component they have respectively: the algebra of formal power series in $b_1^0,...,b_N^0$, the algebra of analytic functions, or the algebra of smooth functions on the given open set $U$. What was said in 2.1 carries over to these completions without serious changes.

Let $f(b)$ be a non-constant function in any of the mentioned categories identified with an element of the conformal weight 0 component of the corresponding vertex algebra $O_N$. Denote also by $f(b)$ the corresponding element $O_N$ and by $f(b)_j$, $j \in \mathbb{Z}$ the corresponding elements of $\Gamma(C^*, \text{Lie}_C^*(O_N))$.

The Virasoro field is given by (2.1.4). As $f(b)|0\rangle$ is a $\text{Vir}$-singular vector of weight 0 we get (analogously to (2.1.6):

$$[L_s, f(b)_j] = -(s + j)f(b)_{s+j}. \quad (2.2.1)$$

Therefore we have the following analogue of (2.1.9)

$$\eta(a_j) = -a_{-j}, \quad \eta(f(b)_j) = f(b)_{-j}, \quad \eta(K) = K. \quad (2.2.2)$$

An extension of (2.2.2) to the entire $\text{Lie}(O_N^{an})$ or $\text{Lie}(O_N^{sm})$ is exactly as explained in the end of 2.1.

Observe by the way that in any restricted vertex algebra $V$ Fourier components of a field $v(z)$ associated with $v \in V_0$ transform as differential forms.

2.3. Our exposition in §1 was suited for the pure even case. It is obvious however that the same could have been done in the supercase by changing signs in certain places in the standard way. Consider, for example, the vertex algebra $\Lambda_N$ based on the Clifford algebra $Cl_N$ in the same way as $O_N$ is based on $H_N$. The basis of $Cl_N$ is as follows: $\phi_i^j, \psi_i^j$ (odd), $K$ (even); the relations are:

$$[\phi_i^j, \psi_i^j] = \delta_{ij} \delta_{s,-t} K \quad (2.3.1)$$

The vacuum $|0\rangle \in \Lambda_N$ satisfies

$$\phi_s^i |0\rangle = \psi_s^i |0\rangle = 0, \quad s > 0, t \geq 0, \quad (2.3.2)$$

$$K|0\rangle = |0\rangle. \quad (2.3.3)$$

The Virasoro field is given by

$$L(z) = \Sigma_{i=1}^N :\phi^i(z)\psi^i(z): \quad (2.3.4)$$

Similarly to (2.1.5-6) we get

$$[L_s, \psi_j^i] = -j \psi_{s+j}^i, \quad (2.3.5)$$

$$[L_s, \phi_j^i] = -(s + j)\phi_{s+j}^i. \quad (2.3.6)$$
It follows that $\phi^i_j$ is identified with $z^i - 1dz$, while $\psi^i_j$ with $z^j$. Going through the same steps is in 2.1, we obtain the following formulas for the antiinvolution $\eta$ (cf. (2.1.9):

$$
\eta(\phi^i_j) = -\phi^i_{-j}, \eta(\psi^i_j) = \phi^i_j, \eta(\mathbb{K}) = \mathbb{K}.
$$

(2.3.7)

(To check that the relations (2.3.1) are indeed preserved one has to really use the sign rule.) The recipe to extend $\eta$ to the entire Lie($\mathcal{O}_N$), which is explained in the sentences following (2.1.9), carries over to the present situation word for word.

2.4. What was done in 2.1-2.2 and 2.3. can be combined and applied in the obvious manner to the vertex algebras $\Omega \cdot N := \mathcal{O}_N \otimes \Lambda_N$, $\hat{\Omega} \cdot N := \mathcal{O}_N \otimes \Lambda_N$, $\Omega^{an}_N := \mathcal{O}^{an}_N \otimes \Lambda_N$, $\Omega^{sm}_N := \mathcal{O}^{sm}_N \otimes \Lambda_N$. An $\eta$ is defined to act on $a$’s and $b$’s by (2.1.9) and on $\phi$’s and $\psi$’s by (2.3.7).

All the algebras we just listed possess the four remarkable fields

$$
L(z) = \Sigma_{i=1}^N : b^i(z) : a^i(z) : + : \phi^i(z) : \psi^i(z) :,
$$

(2.4.1)

(the Virasoro field),

$$
G(z) = \Sigma_{i=1}^N : b^i(z) : \psi^i(z) : , \quad J(z) = \Sigma_{i=1}^N : \phi^i(z) : \psi^i(z) : , \quad Q(z) = \Sigma_{i=1}^N : \phi^i(z) : a^i(z) :.
$$

(2.4.2)

Fourier components of these fields satisfy the $N = 2$-superconformal algebra relations. We shall not list all of these here restricting ourselves to the following ones:

$$
[L_i, Q_j] = -jQ_{i+j},
$$

(2.4.3a)

$$
[Q_0, G(z)] = -L(z).
$$

(2.4.3b)

According to (1.2), (2.4.3a) means that $Q_{-1}|0\rangle$ generates a subsheaf of $\text{Lie}_C(\Omega_N)$ isomorphic to the structure sheaf for any smooth curve $C$. In particular,

$$
Q_0 = \int Q(z) \in \Gamma(C, \text{Lie}_C(\Omega_N)),
$$

(2.4.4)

since $Q_0$ is identified with a constant function.

2.5. We now extend the antiinvolution $\eta$ on $\Omega_N$ (see (2.1.9, 2.3.7)) to the sheaf of vertex algebras $\Omega^\mathbb{C}_X$, which was associated with a smooth manifold $X$ in [MSV]. This can be easily and uniformly explained for either an analytic or $C^\infty$-manifold $X$.

Recall that $\Omega^\mathbb{C}_X$ was defined in each of the 2 settings by considering an atlas of $X$, associating with each chart a copy of a vertex algebra of the type considered in 2.2 and then gluing the algebras over intersections by lifting the usual transition functions to operators acting on the algebras. Let us write down the relevant formulas.
Let $U$ and $\tilde{U}$ be two open subsets of $X$, $b^1, ..., b^N$ and $\tilde{b}^1, ..., \tilde{b}^N$ being the respective coordinate functions. On $U \cap \tilde{U}$ they are related by

$$\tilde{b}^i = g^i(b^1, ..., b^N); \ b^i = f^i(\tilde{b}^1, ..., \tilde{b}^N). \quad (2.5.1)$$

Associated to $U$ and $\tilde{U}$ there are $\mathcal{O}_N(U)$ and $\mathcal{O}_N(\tilde{U})$; the first one is generated by the fields $a^i(z), b^i(z), \phi^i(z), \psi^i(z)$, the second one by the fields $\tilde{a}^i(z), \tilde{b}^i(z), \tilde{\phi}^i(z), \tilde{\psi}^i(z)$. We also have 2 vertex algebras associated to $U \cap \tilde{U}$: one is an extension of $\mathcal{O}_N(U)$ another is that of $\mathcal{O}_N(\tilde{U})$. We identify these two algebras by assuming that the fields are related by the following transformation $G$ lifting (2.5.1):

$$G(\tilde{b}^i(z)) = g^i(b^1, ..., b^N)(z), \quad (2.5.2)$$

$$G(\tilde{\phi}^i(z)) = \left(\frac{\partial f^i}{\partial \tilde{b}^j}\phi^j\right)(z), \quad (2.5.3)$$

$$G(\tilde{\psi}^i(z)) = \left(\frac{\partial f^i}{\partial \tilde{b}^j}\phi^j\right)(z), \quad (2.5.4)$$

$$G(\tilde{\phi}^i(z)) = a^i\left(\frac{\partial f^i}{\partial \tilde{b}^j}\phi^j + \frac{\partial^2 f^i}{\partial \psi^j \partial \tilde{\phi}^j} \phi^j \psi^k\right)(z). \quad (2.5.5)$$

Denote by $\eta_U$ ($\eta_{\tilde{U}}$ resp.) the antiautomorphism $\eta$ specialized to $\mathcal{O}_N(U)$ ($\mathcal{O}_N(\tilde{U})$ resp.). It is an easy exercise on definitions in [MSV] to show that

$$\eta_U \circ G = G \circ \eta_{\tilde{U}}. \quad (2.5.6)$$

This equality of course means that the collection of antiinvolutions $\{\eta_U\}$ glues to give an antiinvolution of the sheaf:

$$\eta : \text{Lie}(\Omega^c_X) \to \text{Lie}(\Omega^c_X) \quad (2.5.7)$$

2.6. Of the four fields in (2.4.1-2) only two, $G(z)$ and $L(z)$, are in general invariant under the transformation (2.5.2-5); it follows, in particular, that $\Gamma(X, \Omega^c_X)$ is a restricted conformal vertex algebra. The two other fields, $Q(z)$ and $J(z)$, are only invariant up to the addition of a total derivative; it follows that the Fourier modes $J_0$ and $Q_0$ are operators acting on the sheaf $\Omega^c_X$. The action of $J_0$ on $\Gamma(X, \Omega^c_X)$ arising in this way equips $\Gamma(X, \Omega^c_X)$ with a gradation (by eigenspaces of $J_0$, or, as they say, by fermionic charge), while $Q_0$ satisfies $Q_0^2 = 0$. It follows that $\Gamma(X, \Omega^c_X)$ is a complex (infinite in both directions) and we proved in [MSV] that its cohomology $H_Q(\Gamma(X, \Omega^c_X))$ is canonically isomorphic to the de Rham cohomology of $X$: $H_{DR}(X)$. The de Rham complex $C_{DR}(X)$ is, in fact, canonically identified with the conformal weight 0 component of $\Gamma(X, \Omega^c_X)$ and this embedding is a quasiisomorphism.

Given any $\Gamma(X, \Omega^c_X)$-module $M$ one can similarly consider the $Q$-cohomology group

$$H_Q(M) = \frac{\text{Ker}(Q_0 : M \to M)}{\text{Im}(Q_0 : M \to M)}.$$
For example, when applied to $D(\Gamma(X, \Omega^c_X))$, this gives

$$H_Q(D(\Gamma(X, \Omega^c_X))) = H_Q(\Gamma(X, \Omega^c_X))^* = H_{DR}(X)^*.$$ (2.6.1)

### 2.7. One remark concerning the “size” of $D(\Gamma(X, \Omega^c_X))$ is in order. This module also has a gradation by conformal weight

$$D(\Gamma(X, \Omega^c_X)) = \oplus_{n \geq 0} D(\Gamma(X, \Omega^c_X))_n, \quad D(\Gamma(X, \Omega^c_X))_n = (\Gamma(X, \Omega^c_X)_n)^*.$$  

One can argue that the space $D(\Gamma(X, \Omega^c_X))_n$ is often much “bigger” than $\Gamma(X, \Omega^c_X)_n$. For example, since $\Gamma(X, \Omega^c_X)_0$ is the de Rham complex of $X$, the dual $D(\Gamma(X, \Omega^c_X))_0$ is the space of distributions. If $X$ is a compact manifold and we work in the $C^\infty$-setting, then there is the following embedding

$$\Gamma(X, \Omega^c_X)_0 \hookrightarrow D(\Gamma(X, \Omega^c_X))_0, \quad \omega(\nu) = \int_X \omega \wedge \nu.$$ (2.7.1)

If $\Omega^c_X$ were a coherent locally trivial sheaf, then the same procedure applied to all conformal weights spaces would give an embedding $\Gamma(X, \Omega^c_X) \hookrightarrow D(\Gamma(X, \Omega^c_X))$. It is not, but $\Omega^c_X$ does possess a filtration so that the associated graded sheaf is indeed coherent and locally trivial. Therefore we can define a smaller space $\tilde{D}(\Gamma(X, \Omega^c_X)) \subset D(\Gamma(X, \Omega^c_X))$ so that the graded object $Gr(\tilde{D}(\Gamma(X, \Omega^c_X)))$ with respect to the corresponding dual filtration is the space of distributions of the type (2.7.1). It is easy to see that $\tilde{D}(\Gamma(X, \Omega^c_X))$ is also a $\Gamma(X, \Omega^c_X)$-module.
§3 Correlation functions

In this section we shall keep to the vertex algebra $\Gamma(X, \Omega^2_X)$ in the $C^\infty$-setting.

3.1. Let $M_1, \ldots, M_m$ be restricted $\Gamma(X, \Omega^2_X)$-modules. Attaching $M_i$ to $P_i \in \mathcal{C}$ we get a bundle over the configuration space with infinite dimensional fiber $\otimes_i M_i$. Since $\tilde{z} \mathbb{C}[z] dz$ acts trivially on the conformal weight zero component of each $M_i$, this bundle has a trivial subbundle with fiber $\otimes_i (M_i)_0$. Passing to $\langle M_1, \ldots, M_m \rangle_{\mathcal{C}}$, as explained in 1.6, we get a trivial quotient bundle, to be denoted by $(M_1, \ldots, M_m)_{\mathcal{C},0}$, with fiber “the space of $\text{Lie}(\Gamma(X, \Omega^2_X))$-invariant functionals restricted to $\otimes_i (M_i)_0$.

Define $Z(M) = \ker(Q_0 : M \to M)$. We can further quotient the bundle $\langle M_1, \ldots, M_m \rangle(0)_{\mathcal{C}}$ to get the bundle $\langle Z(M_1), \ldots, Z(M_m) \rangle_{\mathcal{C},0}$ whose fiber is a result of restricting that of $(M_1, \ldots, M_m)_{\mathcal{C},0}$ to $\otimes_i Z(M_i)_0$

We are now in a position to reproduce a well-known physics calculation that shows, in our terminology, that the connection on $(M_1, \ldots, M_m)_{\mathcal{C}}$ (see (1.6.4)) descends on the bundle $\langle Z(M_1), \ldots, Z(M_m) \rangle_{\mathcal{C},0}$ and the result is the trivial connection. Let $\langle \cdot, \cdot, \cdot \rangle$ be an element of the fiber of $(M_1, \ldots, M_m)$; apply the vertical component of the connection $\Delta_1$ and restrict the result to $\otimes_i (Z(M_i)_0$.

We have

\[ \langle \omega_1, \ldots, L_{-1} \omega_1, \omega_{t+1}, \ldots, \omega_m \rangle = -\langle \omega_1, \ldots, [Q_0, G_{-1}] \omega_t, \omega_{t+1}, \ldots, \omega_m \rangle = 0 \]

(3.1.1)

Here we have used that, firstly, $L_{-1} = -[Q_0, G_{-1}]$ by (2.4.3b), secondly, that $Q_0 \in \text{Lie}(\Gamma(\Omega^2_X))_{\text{out}}$ by (2.4.4) and, finally, that $\omega_i \in Z(M_i)$.

Similar calculation shows that

\[ \langle \omega_1, \ldots, \omega_m \rangle = 0 \text{ if for some } i, \omega_i \in \text{Im} Q_0. \]

(3.1.2)

3.2. Let us see what this construction gives us when $\mathcal{C} = \mathbb{C} \mathbb{P}^1$ and the modules are either $\Gamma(X, \Omega^2_X)$ or $D(\Gamma(X, \Omega^2_X))$. We shall keep to the case when $P_1 = \infty$, $P_i = z_i, i = 2, \ldots, m - 1$, $P_m = 0$, $M_1 = D(\Gamma(X, \Omega^2_X))$, $M_i = \Gamma(X, \Omega^2_X)$.

In this case the bundle $\pi^*(D(\Gamma(X, \Omega^2_X)), \Gamma(X, \Omega^2_X), \ldots, \Gamma(X, \Omega^2_X))_{\mathcal{C}}$ is trivial (see (1.9.5) for the definition of $\pi$) because our standard choice of the local coordinate $z - z_i$ at the point $z = z_i$ depends on the point “smoothly”. Hence any section of this bundle can be regarded as a functional, depending on $z_2, \ldots, z_{m-1}$, on $D(\Gamma(X, \Omega^2_X)) \otimes (\Gamma(X, \Omega^2_X))^{\otimes (m - 1)}$. However the connection is not trivial and the functional $\Phi_{z_2, \ldots, z_{m-1}}(\ldots)$ defined by (1.9.1) is a horizontal section of this bundle non-trivially depending on $z_2, \ldots, z_{m-1}$. What (3.1.1) tells us is that when restricted to the conformal weight 0 components, the functional $\Phi_{z_2, \ldots, z_{m-1}}(\ldots)$ is actually independent of $z_2, \ldots, z_{m-1}$. In fact one can avoid using (3.1.1): given $\omega_i^* \in D(\Gamma(X, \Omega^2_X))_0, \omega_i \in \Gamma(X, \Omega^2_X), i = 2, \ldots, m$, it is easy to calculate $\Phi_{z_2, \ldots, z_{m-1}}(\omega_1^*, \omega_2, \ldots, \omega_m)$ by directly using (1.9.1) and obtain:

\[ \Phi_{z_2, \ldots, z_{m-1}}(\omega_1^*, \omega_2, \ldots, \omega_m) = \omega_1^*(\omega_2 \wedge \ldots \wedge \omega_m). \]

(3.2.1)

In r.h.s. of (3.2.1) the wedge product of forms is used; it makes sense for, as we reminded the reader in 2.6, the conformal weight 0 component of $\Gamma(X, \Omega^2_X)$ is exactly the space of global differential forms.
Assuming further that $\omega_1^*$ actually belongs to $\tilde{D}(\Gamma(X, \Omega^c_X))_0$ (see 2.7), that is, equals a differential form, one rewrites (3.2.1) in the following nicer form:

$$\Phi_{z_2, \ldots, z_{m-1}}(\omega_1^*, \omega_2, \ldots, \omega_m) = \int_X \omega_1^* \wedge \omega_2 \wedge \ldots \wedge \omega_m.$$  \hspace{1cm} (3.2.2)

The last formula suggests to regard the map

$$\Phi_\emptyset : \tilde{D}(\Gamma(X, \Omega^c_X)) \otimes \Gamma(X, \Omega^c_X) \to \mathbb{C},$$ \hspace{1cm} (3.2.3)

arising in the case of 2 points, as a “chiralization” of the standard pairing of forms.

Similarly, in the case of 3 points, $\Phi_2(\omega_1^*, \omega_2, \omega_3)$ gives the structure constants of the algebra of global differential forms.

Finally, by (3.1.2) and (2.6.1), $\Phi_{z_2, \ldots, z_{m-1}}(\ldots)$ descends to a functional

$$\Psi_m : H_{DR}(X)^* \otimes H_{DR}(X)^{\otimes (m-1)} \to \mathbb{C}.$$ \hspace{1cm} (3.2.4)

The latter is something very well known: $\Psi_2(\ldots)$ is the Poincare duality, $\Psi_3(\ldots, \ldots)$ gives the structure constants of the cohomology ring of $X$ etc.
Part III: Calculation of $\Gamma(\mathbb{CP}^N, \Omega^\varphi_{\mathbb{CP}^N})$

§1. An embedding $L(Vect_X) \hookrightarrow End(\Omega^\varphi_X)$

1.1. Let $X$ be a smooth manifold, $TX$ the total space of the tangent bundle and $\Pi TX$ the supermanifold obtained by changing the parity of all the fibers of the projection $TX \rightarrow X$; in other words, $\Pi TX$ is that supermanifold whose structure sheaf is the sheaf of differential forms on $X$. The action of vector fields on forms by the Lie derivative gives an embedding of Lie algebras:

$$\pi : Vect_X \hookrightarrow Vect_{\Pi TX},$$

which in coordinates reads as follows:

$$\pi(f^i(b)a^j) = f^i(b_a^j + \frac{\partial f^j(b)}{\partial b^s} b^s \phi^s \psi^j).$$

In the last formula we keep to the following conventions: $\{b^i\}$ is a coordinate system, $\phi^j = db^i$, so that $\{b^i, \phi^j\}$ is a coordinate system on $\Pi TX$; $a_i$ stands for the even vector field $\partial/\partial b^i$, while $\psi^i$ signifies the odd one $\partial/\partial \phi^i$.

1.2. Let us do a chiral version of 1.1.

Let $Vect_{\mathbb{C}^N}$ be the Lie algebra of polynomial vector fields on $\mathbb{C}^n$ and let

$$L(Vect_{\mathbb{C}^N}) = Vect_{\mathbb{C}^N} \otimes \mathbb{C}[t, t^{-1}]$$

be the corresponding loop algebra. For $\tau \in Vect_{\mathbb{C}^N}$ set $\tau_n = \tau \otimes t^n \in L(Vect_{\mathbb{C}^N})$.

Consider the vertex algebra $\Omega_N$, which was reviewed in II.2.4, and the space of fields $Fields(\Omega_N)$ associated with $\Omega_N$, which was defined in I.1.4. Attach to $\tau = f^i(b)a^j \in Vect_{\mathbb{C}^N}$ a field $\tau(z) \in Fields(\Omega_N)$:

$$\tau(z) = :a^i(z)f^j(b(z)) : + : \frac{\partial f^j(b(z))}{\partial b^s}(b(z)) \phi^s(z) \psi^j(z) :.$$

Let $\{\tau(z)_n, n \in \mathbb{Z}\}$ be Fourier coefficients of $\tau(z)$ so that (in accordance with (I.1.6.2))

$$\tau(z) = \sum_{n \in \mathbb{Z}} \tau(z)_n z^{-n-1}.$$

**Lemma.** The map

$$\hat{\pi} : L(Vect_{\mathbb{C}^N}) \rightarrow End(\Omega_N), \tau_n \mapsto \tau(z)_n$$

is a Lie algebra homomorphism.

**Proof.** We have to check that

$$[\hat{\pi}(\tau_n), \hat{\pi}(\xi_m)] = \hat{\pi}([\tau, \xi]_{n+m}).$$

(1.2.3)
In order to do so we compute the OPE of the corresponding elements of $\text{Fields}(\Omega_N)$,

$$\tau(z) =: a^i(z) f^i(b(z)) : + \frac{\partial f^i}{\partial b^s}(b(z)) \phi^s(z) \psi^j(z) :$$

and

$$\xi(z) =: a^i(z) g^i(b(w)) : + \frac{\partial g^i}{\partial b^s}(b(w)) \phi^s(z) \psi^j(z) :,$$

by making use of Wick’s theorem. In this way we get 2 singular terms: one having pole of degree 2 and another one having pole of degree 1. The degree 2 pole arises because of double pairings between $a$ and $b$ in the first summands of each of the expressions above and because of double pairings between $\phi$ and $\psi$ in the second summands; as a result we have:

$$\frac{1}{(z-w)^2} \left\{ \frac{\partial f^i}{\partial b^i}(b(z)) \frac{\partial f^l}{\partial b^l}(b(w)) + \frac{\partial f^i}{\partial b^i}(b(z)) \frac{\partial f^s}{\partial b^s}(b(w)) \right\} = 0.$$

The degree one pole is caused by single pairings. The corresponding calculation is parallel to the classical calculation of the bracket $[\tau, \xi]$; here is the result:

$$\frac{[\tau, \xi](w)}{(z-w)}.$$

Therefore

$$\tau(z) \xi(w) \sim \frac{[\tau, \xi](w)}{(z-w)}.$$

Using (I.1.6.10) we get the desired equality (1.2.3).

1.3. Corollary. On a smooth manifold $X$, the homomorphisms $\hat{\pi}$ glue to the homomorphism:

$$\hat{\pi}_X : L(\Gamma(X, TX)) \to \text{End}(\Omega^c_{ch}X)$$

(1.3.1)

(See [MSV] or II.2.5 for the definition of the sheaf $\Omega^c_{ch}X$.)

1.4. An attempt to replace $\Omega_N$ with a purely even vertex algebra $O_N$ (see II.2.1) in Lemma 1.2 fails: as we showed in [MSV], a mapping similar to (1.2.2) may only give an embedding of a certain extension of $L(Vect_{\mathbb{C}}N)$.

§2 Calculation of $\Gamma(\mathbb{CP}^N, \Omega^c_{ch}\mathbb{CP}^N)$

2.1. Let $G$ be a simple complex Lie group and $\mathfrak{g} = \text{Lie}G$. It immediately follows from 1.3 that if $X$ is a $G$-space or, at least, if there is a Lie algebra homomorphism:

$$\rho : \mathfrak{g} \to \Gamma(X, TX),$$

then $\Omega^c_{ch}X$ is a sheaf of $\mathfrak{g}$-modules of level 0 due to the map

$$\hat{\pi}_X \circ \hat{\rho} : \hat{\mathfrak{g}} \to \text{End}(\Omega^c_{ch}X),$$

(2.1.2a)
where $\hat{\rho}$ is the obvious continuation of $\rho$ to the homomorphism of loop algebras:

$$\hat{\rho} : L(\mathfrak{g}) \to L(\Gamma(X, TX)), \quad \hat{\rho}(g \otimes t^n) = \rho(g) \otimes t^n. \quad (2.1.2b)$$

Focus on the case of $X = G/P$, where $P \subset G$ is a parabolic subgroup. Let $X_0 \in G/P$ be a big cell. Then there is a $\mathfrak{g}$-module embedding:

$$\Gamma(X, \Omega^h_X) \hookrightarrow \Gamma(X_0, \Omega^h_X).$$

One can think of $\Gamma(X, \Omega^h_X)$ as those sections of $\Omega^h_X$ over $X_0$ that are “regular outside $X_0$. A little thought shows that in fact

$$\Gamma(X, \Omega^h_X) = \Gamma(X_0, \Omega^h_X)^{g\text{-int}}, \quad (2.1.3)$$

where $\Gamma(X_0, \Omega^h_X)^{g\text{-int}}$ stands for the maximal $\mathfrak{g}$-integrable submodule or, equivalently, for the maximal submodule on which $\mathfrak{g}$ operates locally finitely.

**2.2. Example.** Let $X$ be $\mathbb{CP}^1$, $X_0 = \mathbb{C}$. Let $\{E_{ij} \in \mathfrak{gl}_2, 1 \leq i, j \leq 2\}$ be the standard basis of $\mathfrak{gl}_2$. In the homogeneous coordinates $(y_0 : y_1)$ on $\mathbb{CP}^1$ one has the following specialization of (2.1.1):

$$\rho : \mathfrak{gl}_2 \to \Gamma(\mathbb{CP}^1, T\mathbb{CP}^1),$$

$$\rho(E_{ij}) = y_{i+1} \partial / \partial y_{i+1}. \quad (2.2.1)$$

Passing to the coordinate $b = y_1/y_0$ on the big cell $\mathbb{C} \subset \mathbb{CP}^1$ and keeping to the conventions of 1.1, one has:

$$\rho(E_{12}) = a, \quad \rho(E_{21}) = -b^2 a. \quad (2.2.2)$$

Then (1.1.2) takes the following form

$$\pi \circ \rho(E_{12}) = a, \quad \pi \circ \rho(E_{21}) = -b^2 a - 2b \phi \psi. \quad (2.2.3)$$

Passing further to the loop algebras one obtains the following version of (2.1.2):

$$\hat{\pi} \circ \hat{\rho}(E_{12} \otimes t^n) = a(z), \quad \hat{\pi} \circ \hat{\rho}(E_{21} \otimes t^n) = -b(z)^2 a(z) - 2b(z) \phi(z) \psi(z) \phi(z) \partial \phi(z)^n. \quad (2.2.3)$$

Observe that $J(z) = \phi(z) \psi(z)$ is a “free boson” meaning that the following OPE is valid: $J(z)J(w) \sim J(w)/(z - w)^2$. The space $\Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1})$ is graded by fermionic charge, that is to say, by eigenvalues of $J_0$: $\Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1}) = \oplus_{n \in \mathbb{Z}} \Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1})^{(i)}$. Hence, the boson-fermion correspondence (see e.g. [K] 5.1) tells us that $\Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1})^{(i)}$ is an irreducible representation of the Lie algebra of Fourier components of the fields $a(z), b(z), J(z)$ generated by the vector $\phi_{-i+1} \phi_{-i+2} \cdots \phi_0 |0\rangle$ if $i \geq 0$, or $\psi_i \phi_{i-1} \cdots \phi_{-1} |0\rangle$ if $i < 0$.

With all of this in mind one checks (2.2.3) against (2.10) in [FF2] and concludes that each $\Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1})^{(i)}$ is a Wakimoto module over $\hat{sl}_2$ and the complex $(\Gamma(\mathbb{C}, \Omega^h_{\mathbb{CP}^1}), Q_0)$ is the two-sided resolution of the trivial representation constructed in [FF1]. Finally a glance at the diagram $III_-$ in [FF2]4.2 allows one to use
(2.1.3) in order to obtain a rather explicit description of the \(\hat{sl}_2\)-module structure of \(\Gamma(\mathbb{CP}^1, \Omega^{ch}_{\mathbb{CP}^1})\). Let us formulate the result leaving the details of this calculation out.

Let \(V_m\) denote the simple \(m + 1\)-dimensional \(\hat{sl}_2\)-module. Represent \(\hat{sl}_2\) as a direct sum of the loop algebra \(L(sl_2)\) and \(\mathbb{C}K\), where \(K\) is the standard central element. The subalgebra \(L(sl_2)_{+} \subset \hat{sl}_2\) of loops regular at 0 maps onto \(sl_2\) by means of the evaluation at 0 map. Hence \(V_m\) becomes an \(L(sl_2)_{+}\)-module and one defines the Weyl module \(V_m\) of zero central charge as follows:

\[
V_m = \text{Ind}_{L(sl_2)_{+} \otimes \mathbb{C}K}^{\hat{sl}_2} V_m;
\]

it is assumed that \(K\) operates on \(V_m\) as 0.

\(V_m\) has a unique irreducible quotient to be denoted \(L_m\). In fact, \(V_m\) has a unique proper submodule isomorphic to \(L_{m+2}\).

**Lemma.** \(\Gamma(\mathbb{CP}^1, \Omega^{ch}_{\mathbb{CP}^1})^{(i)}\) has a filtration by \(\hat{sl}_2\)-submodules

\[
F_0 \subset F_1 \subset \cdots \subset \bigcup_{m=0}^{\infty} F_m = \Gamma(\mathbb{CP}^1, \Omega^{ch}_{\mathbb{CP}^1})^{(i)}
\]

so that:

if \(i \leq 0\), then

\[
F_0 = V_{-2i}, \quad \frac{F_m}{F_{m-1}} = V_{-2i+4m}, m \geq 1;
\]

if \(i > 0\), then

\[
F_0 = L_{2i+2}, \quad \frac{F_m}{F_{m-1}} = V_{2i+4m}, m \geq 1.
\]

To further elaborate on the link between our approach and that of [FF2] let us mention that the collection of fields

\[
\{\psi(z), b(z)\psi(z), b(z)^2\psi(z)\} \subset \text{End}(\Gamma(X_0, \Omega^{ch}_{\mathbb{CP}^1})[[z, z^{-1}]]
\]

(2.2.4)

is the vertex operator associated with the adjoint representation of \(sl_2\) and that the “chiral de Rham differential” \(Q_0 = \int a(z)\phi(z)\) coincides with the “screening charge”.

In fact, the states \(\psi_{-1}|0\rangle, b_0\psi_{-1}|0\rangle, b_0^2\psi_{-1}|0\rangle \in \Gamma(X_0, \Omega^{ch}_{\mathbb{CP}^1})\), to which the fields \(\psi(z), b(z)\psi(z), b(z)^2\psi(z)\) correspond belong to \(\Gamma(\mathbb{CP}^1, \Omega^{ch}_{\mathbb{CP}^1})\). Therefore (2.2.4) can be sharpened as follows

\[
\{\psi(z), b(z)\psi(z), b(z)^2\psi(z)\} \subset \text{End}(\Gamma(\mathbb{CP}^1, \Omega^{ch}_{\mathbb{CP}^1})[[z, z^{-1}]]
\]

(2.2.5)

There is a well-known superaffine Lie algebra \((\hat{sl}_2)_{\text{super}}\) (of zero central charge), see e.g. [K] 2.5, obtained by taking the semi-direct product of the usual loop algebra with the module of loops in the adjoint representation and declaring the
latter subspace odd. The formula (2.2.5) simply means that the sheaf $\Omega_{\mathbb{CP}^1}^{ch}$ is actually a sheaf of $(\hat{sl}_2)_{\text{super}}$-modules.

2.3. More generally one can consider $X$ equal $\mathbb{C}P^N$, $X_0 = \mathbb{C}^N$. By (2.1.2) $\Gamma(\mathbb{C}P^N, \Omega_{\mathbb{CP}^N}^{\text{ch}})$ is a $\hat{g}_{N+1}$-module. Let $(y_0 : ... : y_N)$ be the homogeneous coordinates on $\mathbb{C}P^N$; $b^i = y_i/y_0$, $i = 1, ..., N$ is a coordinate system on $X_0$. The formula (2.2.3) generalizes as follows:

$$\hat{\pi} \circ \hat{\rho}(E_{ij} \otimes t^n) = b^{i-1}(z)a^{j-1}(z) \cdot_n + \phi^{i-1}(z)\psi^{j-1}(z) \cdot_n, \quad i, j \neq 1, \quad (2.3.1a)$$

$$\hat{\pi} \circ \hat{\rho}(E_{1j} \otimes t^n) = a^{j-1}(z)\cdot_n, \quad j \neq 1, \quad (2.3.1b)$$

$$\hat{\pi} \circ \hat{\rho}(E_{11} \otimes t^n) = -\sum_{i=1}^N b^{i-1}(z)b^i(\cdot_n) \cdot_n$$

$$-\sum_{i=1}^N : b^{i-1}(z)\phi^i(z)\psi^i(z) :_n - \sum_{i=1}^N : b^i(z)\phi^{i-1}(z)\psi^i(z) :_n, \quad i \neq 1. \quad (2.3.1c)$$

We again have $\Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}}) = \oplus_{i \in \mathbb{Z}} \Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}})^{(i)}$. By construction, each $\Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}})^{(i)}$ is a generalized Wakimoto module attached in [FF1] to the maximal parabolic subgroup of $SL_{N+1}$, or rather, (2.3.1a-c) provide an explicit description of this module. By [MSV] Theorem 2.4 the complex $(\Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}}), Q_0)$ is a two-sided resolution of the trivial representation composed of such modules.

By (2.1.3), $\Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}})$ is the maximal $sl_{N+1}$-integrable submodule of the described generalized Wakimoto module. To find an analogue of Lemma 2.2 one needs more information about generalized Wakimoto modules than is available now.

What was said in the end of 2.2 about vertex operators and the structure of a superaffine algebra module on our sheaf carries over to the present situation easily. For example, the vertex operator associated with the adjoint representaion of $sl_{N+1}$

$$\{e_{ij}(z), 1 \leq i, j \leq N + 1\} \subset \text{End}(\Gamma(\mathbb{C}^N, \Omega_{\mathbb{CP}^N}^{\text{ch}})[[z, z^{-1}]] \quad (2.3.2)$$

is given (over $\mathbb{C}^N$) by the following variation of (2.3.1):

$$e_{ij}(z) = : b^{i-1}(z)\psi^{j-1}(z) :_n, \quad i, j \neq 1, \quad (2.3.3a)$$

$$e_{1j}(z) = \psi^{j-1}(z), \quad j \neq 1, \quad (2.3.3b)$$

$$e_{11}(z) = -\sum_{i=1}^N b^{i-1}(z)b^i\cdot_n \quad (2.3.3c)$$

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