STOCHASTIC NON-AUTONOMOUS HOLLING TYPE-III PREY-PREDATOR MODEL WITH PREDATOR'S INTRA-SPECIFIC COMPETITION

SAMPURNA SENGUPTA AND PRITHA DAS*
Department of Mathematics
Indian Institute of Engineering Science and Technology, Shibpur
Howrah, West Bengal 711103, India

DEBASIS MUKHERJEE
Department of Mathematics
Vivekananda College, Thakurpukur, Kolkata-700063, India

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Abstract. The objective of this article is to study the significance of dynamical properties of non-autonomous deterministic as well as stochastic prey-predator model with Holling type-III functional response. Firstly, uniform persistence of the deterministic model has been demonstrated. Secondly, stochastic non-autonomous prey-predator system with Holling type-III functional response is proposed. The existence of a global positive solution has been derived. Sufficient conditions for non-persistence in mean, weakly persistence in mean, extinction have been derived. Moreover the sufficient conditions for permanence of the system have been established. The analytical results are verified by numerical simulation.

1. Introduction. The first major development in modern mathematical ecology was done when Lotka [26] and Volterra [36] published works for a predator-prey competing species. Dynamical behavior of the ecological model system depends on the functional form of interaction between prey and predator representing the biological interaction in natural world. Among the many aspects of the prey-predator relationship, the key factor is the functional response.

Functional response is defined as the intake rate of predator as a function of density of food. Nonlinear functional responses were originally proposed by Holling (1959) [13] on the basis of a general argument concerning the allocation of a predator’s time between two activities: prey searching and prey handling. Holling considered predation of small mammals on pine sawflies and observed that predation rates increased with increasing prey population density. Holling type I functional response is noticed in passive predators like spiders. The number of flies captured in the net is proportional to fly density. Death rate of prey due to predation is constant. Search rate of Holling type II is constant. Prey mortality reduces with...
prey density. For example, small mammals kill most of gypsy moth pupae in scattered populations of gypsy moth. However in high-density defoliating populations, small mammals kill a minor proportion of pupae. Plateau depicts predator saturation \[8\]. Mathematically, this model is equivalent to the model of enzyme kinetics developed in 1913 by Leonor Michaelis and Maud Menten. Holling type-III (sigmoid) functional responses may arise from a variety of mechanisms. Switching to alternative food sources is one of them. It is well-known that sigmoid functional responses may stabilize an otherwise unstable steady state of prey and predators in Lotka-Volterra models \[1\]. Holling type-III functional response happens when predators increase their search activity with increasing density of prey. Number of prey consumed per predator increases rapidly, then reaches a saturation amount with increasing prey density. When prey density is low, predator response to prey is depressed. If prey density exceeds the upper limit of interval of prey density, then the death rate of prey starts declining for which number of prey will get out of control until some other factors such as diseases or prey food shortage will stop their reproduction. This phenomenon is called “escape from natural enemies”, first discovered by Takahashi \[35\].

Justification of considering the type III functional response in the following situation are:

1. For large prey-density, number of safe hiding places for prey is limited.
2. If prey density is so small that predators find prey infrequently, then predators lacked experience to learn the best ways to search and attack that prey species and the prey encounter rate is extremely low in that case.
3. When prey density is low, predators switch to alternative food sources. There should be one predator and two or more prey species in a system for prey switching. When all prey densities are almost same, then predators will select any one of them randomly. Now if one of prey density reduces much, the predator will leave it and select other more common prey species with higher density. Switching to alternative food play a significant role in improving the persistence of prey-predator model systems. In 1975, Murdoch et al. \[32\] studied this behavior with guppy fish as predator and tubificids, fruit flies as preys. When number of fruit fly decreases, guppy fishes switch to alternative food tubificids.

The non-autonomous system where time appears explicitly is very important to take care field populations which in real world, live in a seasonal environment and changes with time. T. Caraballo, R. Colucci, X. Han \[2\] demonstrated existence of global pullback attractor, co-existence of species and chaotic behavior numerically for a non-autonomous semi-Kolmogorov population model with periodic forcing. Some authors have considered the non-autonomous systems to study the models with seasonally varying parameters. Wang and Li \[37\] discussed periodic solution and permanence for a delayed, non-autonomous, ratio dependent model with Holling type-III functional response. Chen and Shi \[5\] studied non-autonomous, delayed, ratio dependent model with Holling type-III functional response and obtained conditions for permanence and existence of the positive periodic solution. Intra-specific competition among predators has been introduced in our non-autonomous deterministic model and persistence of the deterministic system has been studied.

Inter-specific and intra-specific competitions both have major role in the direct year to year density dependence. Intra-specific competition may be straightforward, where predators fight among themselves for the limited resources. For example, white faced Capuchin monkeys compete between themselves directly for
the inadequate source of food. Some predators, like lions, compete among their pack for getting the good part of the killed prey. The dominant individuals take bigger share of food but weaker individuals do not get sufficient food. Thus their death rate increases. Sometimes, competition can be of passive way, when stronger individual consumes almost completely the available food or, if some population grows but the resources stays limited, then after some time that population can not increase. For example, in juvenile wolf spiders, all the siblings feed on the insufficient food resource so that after a certain period of time the resource become exhausted. This ultimately cause a decrease in the population size due to starvation [18]. Plants from the same species, like two oak trees growing beside each other or huge amount of grass plants in same place, receive sunlight, water and all nutrients from the same resource pool. They have to fight to get the sufficient resources for surviving [14]. Birds like sparrows, crows, hopping around some house or shop, have to compete each other to get the crumbs dropped by humans from their foods [38]. In the subarctic birch forest of Finland, the Clethrionomys rufocanus, generally called as gray-sided vole, typically coexist and they involve in intense competition among themselves for food and breeding space [11].

In real situation, parameters involved in the model always fluctuate around some average value due to continuous variation in the environment. May [30] has revealed that all the parameters involved in the model exhibit random variation as the factors controlling them are not constant. Hence equilibrium distributions obtained from the deterministic analysis are not so realistic, as they vary randomly around some average value. Large amplitude fluctuation in population density leads to extinction of some species, which may be observed in some ecological model systems. The oscillatory co-existence of two or more species with fluctuating periodicity can not be captured within the deterministic setup apart from quasi-periodic or chaotic oscillation for a range of parameter values. In order to study the dynamics of interacting population in more realistic environment, we need to analyze the associated stochastic model [9, 10, 7]. However, there is no unified process to formulate a stochastic model for the interacting populations under consideration. The available stochastic formalisms can be divided into two broad classes, one is discrete or continuous time Markov chain modeling and the other is the noise added systems. Again, these two classes can be partitioned further. T. Caraballo, R. Colucci1 and X. Han [3] studied semi-Kolmogorov system under random environments and has shown the existence of global random attractor for the random system. In another research article, using the classical theory of random dynamical systems, the existence of unique random attractor of the predation model of plankton community under fluctuation environment has been established by T. Caraballo et al. [4]. Over the last one decade, some significant investigations are carried out for prey-predator models within fluctuating environment to study the existence and uniqueness of solutions, stochastic persistence to understand the dynamical behavior in real situation [15, 22, 31]. Some important works have been done by the researchers on non-autonomous stochastic prey-predator model with different types of functional responses. Li and Zhang [24] used Beddington-DeAngelis functional response. Zhang et al. [40] used Crowley-Martin functional response. Wu, Zou and Wang [39] used Holling type-I functional response. But no work has been done on stochastic, non-autonomous prey-predator system with Holling type-III functional response till date as per authors’ knowledge.
The present paper is organized as follows. In Section 2, the non-autonomous deterministic model with Holling type-III functional response has been formulated by introducing intra-specific competition among predators and uniform persistence of the model system has been established. In Section 3, the stochastic model has been formulated by incorporating white noise terms into the growth rates of prey and predator populations and then the existence of unique positive global solution for the stochastic model has been established. In Section 4, sufficient conditions have been derived, under which extinction, weakly persistence in the mean, strongly persistence in the mean for both prey and predator species are observed. In Section 5, the parametric restrictions required for stochastic permanence have been derived. Finally results obtained for the stochastic model have been validated and explained with numerical experiments in section 6. Ecological interpretations of obtained results have been provided in the concluding Section 7.

2. Persistence of deterministic model. The deterministic non-autonomous prey-predator model with Holling type-III functional response is expressed by,

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t) \left[ r(t)(1-x(t)) - \frac{a_1(t)x(t)y(t)}{1+b(t)x^2(t)} \right] \\
\frac{dy(t)}{dt} &= y(t) \left[ -c(t) - d(t)y(t) + \frac{a_2(t)x^2(t)}{1+b(t)x^2(t)} \right]
\end{align*}
\]

(1)

where,
- \(x(t)\) = Prey population density
- \(y(t)\) = Predator population density
- \(r(t)\) = Logistic growth rate of prey population
- \(a_1(t)\) = Predation rate of predators on prey population
- \(a_2(t)\) = Corresponding conversion rate
- \(b(t)\) = Handling time of predators for each prey that is consumed
- \(c(t)\) = Natural death rate
- \(d(t)\) = Death rate of predators by intra-specific competition

We assume all the coefficients are continuous, bounded and non-negative functions on \(\mathbb{R}_+ = (0, +\infty)\). For convenience, we use the following notations throughout this paper. If \(f(t)\) is a continuous bounded function on \(\mathbb{R}_+\), we define \(f^M = \max_{t \in \mathbb{R}_+} f(t)\), \(f^L = \min_{t \in \mathbb{R}_+} f(t)\), \(f^u = \sup_{t \in \mathbb{R}_+} f(t)\), \(f^l = \inf_{t \in \mathbb{R}_+} f(t)\), \(f^* = \lim_{t \to \infty} \sup f(t)\), \(f_* = \lim_{t \to \infty} \inf f(t)\).

Now, to investigate the persistence of the system, the following lemma [37][17] is considered.

**Lemma 2.1.** Consider the following equation

\[ u'(t) = u(t)[d_1 - d_2 u(t)] \text{ where } d_2 > 0 \]

If \(d_1 > 0\) then \(\lim_{t \to \infty} u(t) = \frac{d_1}{d_2} \)

**Theorem 2.2.** If

\[
\begin{align*}
a_2^M &> c^* b^L \\
r^L &> \frac{a_1^M}{2 \sqrt{b^L}} \beta^1 \\
\beta_2^2 &> \frac{c^M}{a_2^L} \\
\end{align*}
\]

(2) (3) (4)
where,
\[ \beta_1 = \frac{a_2 M - c b L}{b L d L} \]
\[ \beta_2 = \frac{2 \sqrt{b L} r L - a_1^M \beta_1}{2 \sqrt{b L} r M} \]
then system (1) is uniformly persistent.

Proof. Since,
\[ \frac{dx(t)}{dt} \leq x(t)[r M - r L x(t)] \]
We consider the following equation,
\[ u'(t) = u(t)[r M - r L u(t)] \]
By Lemma 2.1 we have
\[ \lim_{t \to \infty} u(t) = \frac{r M}{r L} \]
By comparison theorem there exists \( T_1 > 0 \) such that,
\[ x(t) \leq \frac{r M}{r L} \quad \forall t \geq T_1 \]
\[ = \beta_0 \quad \text{(say)} \]
Also for predator species there exists \( T_2 > 0 \) such that,
\[ y(t) \leq \beta_1 \quad \forall t \geq T_2 \]
Now,
\[ \frac{dx(t)}{dt} \geq x(t) \left[ r L - r M x(t) - \frac{a_1^M \beta_1}{2 \sqrt{b L}} \right] \]
This together with (3) implies that, there exists \( T_3 > 0 \) such that \( x(t) \geq \beta_2 \quad \forall t \geq T_3 \)
Similarly for predator species there exists \( T_4 > 0 \) such that,
\[ y(t) \geq \frac{1}{d M} \left[ \frac{a_2^L \beta_2^2}{1 + b M \beta_2^2} - c M \right] = \beta_3 \quad \forall t \geq T_4 \]
We choose \( T = max\{T_i\} : i = 1,2,3,4 \). For any \( t > T \) we have \( \beta_2 \leq x(t) \leq \beta_0 \) and \( \beta_3 \leq y(t) \leq \beta_1 \)
Therefore, \( D = \{(x(t), y(t)) : \beta_2 \leq x(t) \leq \beta_0, \beta_3 \leq y(t) \leq \beta_1\} \) is the invariant set of (1). Hence the system is uniformly persistent.

3. Existence of global positive solution of the stochastic model. Environment fluctuations affect population systems in real world. Therefore, several authors have established stochastic perturbation into deterministic models to analyze the effect of environmental variability on the population dynamics. Our main focus of this paper is the dynamical behavior of non-autonomous prey-predator model with Holling type-III functional response in presence of environmental driving forces.

We get the following model by introducing multiplicative white noise terms into the growth equations of both the prey and predator populations in our deterministic system (1)
\[ dx(t) = x(t)[r(t)(1 - x(t)) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)}]dt + \sigma_1(t)x(t)dB_1(t) \]
\[ dy(t) = y(t)[-c(t) - d(t)y(t) + \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)}]dt + \sigma_2(t)y(t)dB_2(t) \]
where the functions \( r_1(t), a_1(t), b(t), \sigma(t) \) all are continuous bounded functions on \( \mathbb{R}_+ = [0, \infty) \).

\( B_1(t), B_2(t) \) are mutually independent Brownian motions.

\( \sigma_1(t), \sigma_2(t) \) represent the intensities of the white noise.

A stochastic differential equation has a unique global solution for any initial value, if coefficients of the equation satisfy the linear growth condition and the local Lipschitz condition \([28]\). The coefficients of our model do not satisfy linear growth condition but they are locally Lipschitz continuous. So the solution of our system (5) may explode at a finite time. Now to prove that our system (5) has a global positive solution, we will use the Lyapunov analysis method.

**Theorem 3.1.** For any initial value \((x_0, y_0) \in \mathbb{R}_+^2\) there exists a unique solution \((x(t), y(t))\) of system on \( t \geq 0 \) and the solution will remain in \( \mathbb{R}_+^2 \) with probability 1.

**Proof.** Since, coefficients of model (5) are locally Lipschitz continuous, for any given initial value \((x_0, y_0) \in \mathbb{R}_+^2\) there exists a unique solution \((x(t), y(t))\) on \( t \in [0, \tau_e) \); where, \( \tau_e \) is explosion time.

We have to show, \( \tau_e = \infty \).

Let, \( n_0 > 0 \) be sufficiently large for \( x_0, y_0 \) lying in the interval \([\frac{1}{n_0}, n_0]\). For each integer \( n > n_0 \), we define the stopping times, \( \tau_n = \inf \{t \in [0, \tau_e) : x(t) \notin (\frac{1}{n}, n) \text{ or } y(t) \notin (\frac{1}{n}, n)\} \) where, we set \( \inf \phi = \infty \) where, \( \phi \) is the empty set.

Now obviously \( \tau_n \) is increasing as \( n \to \infty \)

Let

\[
\tau_\infty = \lim_{n \to \infty} \tau_n
\]

Hence, \( \tau_\infty \leq \tau_e \) almost surely.

Now left to show \( \tau_\infty = \infty \).

We assume that this is false. Then there exists a pair of constants \( T > 0 \) and \( \epsilon \in (0, 1) \) such that \( P(\tau_\infty \leq T) > \epsilon \).

Consequently there exists an integer \( n_1 \geq n_0 \) such that

\[
P\{\tau_n \leq T\} > \epsilon; \text{ for } n \geq n_1
\]

We now define a \( \mathbb{C}^2 \)-function \( V : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) by \( V(x, y) = (x - 1 - \log x) + (y - 1 - \log y) \), which is non-negative.

Applying Ito’s formula to our model (5), we obtain

\[
dV = (x - 1)(r(t)(1 - x(t)) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)})dt + \frac{\sigma_1^2(t)}{2} dt + (y - 1)(-c(t)
\]

\[
- d(t) y(t) + \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} dt + \frac{\sigma_2^2(t)}{2} dt + (x - 1) a_1(t) d B_1(t) + (y - 1) a_2(t) d B_2(t)
\]

Then,

\[
L V = r(t) x(t) - r(t) x^2(t) - \frac{a_1(t)x^2(t)y(t)}{1 + b(t)x^2(t)} - r(t) x(t) + \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)}
\]

\[
+ \frac{\sigma_1^2(t)}{2} + \frac{a_2(t)x^2(t) y(t)}{1 + b(t)x^2(t)} - c(t) y(t) - d(t) y^2(t) - \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} + c(t)
\]

\[
+ d(t) y(t) + \frac{\sigma_2^2(t)}{2}
\]
Taking supremum and infimum of the coefficients we get,

\[
LV \leq 2r^a(t)x(t) - r^l(t)x^2(t) - r^l(t) + \frac{a^u(t)}{b'(t)} + \frac{(\sigma^u)^2}{2}
\]

\[
+ y(t)(-c'(t) + \frac{a^u(t)}{b'(t)} + d''(t)) + e^u(t) - d'(t)y^2(t) + \frac{(\sigma^u)^2}{2}
\]

\[
\leq K
\]

where, \( K \) is a positive number.

LV is called the diffusion operator \([29]\) of Ito’s process associated with the \( \mathbb{C}^2 \)-function \( V \) and if \( x(t) \) is the Ito’s process with \( dx(t) = f(t)dt + g(t)dB(t) \), then

\[
dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(t)dB(t)
\]

almost surely.

Putting the previous result \( LV \leq K \) in (7) we have,

\[
dV(x(t), y(t)) \leq Kdt + (x - 1)\sigma_1(t)dB_1(t) + (y - 1)\sigma_2(t)dB_2(t)
\]

Integrating we get,

\[
\int_0^{\tau_n \wedge T} dV \leq \int_0^{\tau_n \wedge T} Kdt + \int_0^{\tau_n \wedge T} \sigma_1(t)x(s) - 1dB_1(s)
\]

\[
+ \int_0^{\tau_n \wedge T} \sigma_2(t)(y(s) - 1)dB_2(s)
\]

where \( \tau_n \wedge T = \min\{\tau_n, T\} \)

Now taking expectation both sides of the above inequality,

\[
EV(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + KE(\tau_n \wedge T)
\]

\[
\leq V(x_0, y_0) + KT
\]

(8)

Let, \( \Omega_n = \{\tau_n \leq T\} \) for \( n \geq n_1 \)

Then by(6), \( P(\Omega_n) \geq \epsilon \)

For every \( \omega \in \Omega_n \) there exists at least one of \( x(\tau_n, \omega), y(\tau_n, \omega) \) equaling either \( n \) or \( \frac{1}{n} \).

Therefore \( V(x(\tau_n, \omega), y(\tau_n, \omega)) \) is no less than \( \min\{(n - 1 - \log n), (\frac{1}{n} - 1 - \log \frac{1}{n})\} \)

From (8) we get,

\[
V(x_0, y_0) + KT \geq E[1_{\Omega_n(\omega)} V(x(\tau_n), y(\tau_n))]
\]

\[
\geq \epsilon \min\{(n - 1 - \log n), (\frac{1}{n} - 1 - \log \frac{1}{n})\}
\]

where \( 1_{\Omega_n(\omega)} \) = indicating function of \( \Omega_n \).

Now when \( n \to \infty \) we have \( \infty > V(x_0, y_0) + KT = \infty \) which is a contradiction.

We must have \( \tau_\infty = \infty \).

So the solution of our system (5) will not explode at a finite time with probability 1, which completes our proof.

4. Stochastic persistency and extinction scenario.

**Lemma 4.1.** The solution of system (5) with initial value \( (x_0, y_0) \in \mathbb{R}^+ \) have the following properties: \( \lim_{t \to \infty} \sup \frac{\ln x(t)}{t} \leq 0 \), \( \lim_{t \to \infty} \sup \frac{\ln y(t)}{t} \leq 0 \) almost surely.
Lemma 4.2. Considered. The proof of the following lemma 4.2 can be found in [23].

Proof. It follows from system (5) that,
\[ \begin{align*}
    dx & \leq x(t) \left[ r(t)(1 - x(t)) \right] dt + \sigma_1(t)x(t)dB_1(t), \\
    dy & \leq y(t) \left[ \frac{a_2(t)}{b(t)} - d(t)y(t) \right] dt + \sigma_2(t)y(t)dB_2(t)
\end{align*} \]
(9)

Set,
\[ \begin{align*}
    d\bar{x} & = \bar{x}(t) \left[ r(t)(1 - \bar{x}(t)) \right] dt + \sigma_1(t)\bar{x}(t)dB_1(t), \\
    d\bar{y} & = \bar{y}(t) \left[ \frac{a_2(t)}{b(t)} - d(t)\bar{y}(t) \right] dt + \sigma_2(t)\bar{y}(t)dB_2(t)
\end{align*} \]
(10)

where \((\bar{x}(t), \bar{y}(t))\) is a solution of system (10) with initial value \(x_0 > 0\) and \(y_0 > 0\). By the comparison theorem for stochastic differential equations, it is easy to have,
\[ x(t) \leq \bar{x}(t) \quad y(t) \leq \bar{y}(t) \] (11)

almost surely (a.s.) where, \(t \in [0, +\infty)\)

We can write using lemma 3.4 in [25],
\[ dx(t) = x(t)\left[ r(t) - a_{11}(t)x(t) \right] dt + \alpha(t)dB(t) \] where \(a_{11}(t), \alpha(t)\) are all non-negative functions defined on \(\mathbb{R}_+\). If \(a_{11} > 0\) then,
\[ \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln |x(t)|}{\ln t} \leq 1 \]

almost surely. It follows from (10) and (11) that,
\[ \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln x(t)}{\ln t} \leq \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln \bar{x}(t)}{\ln t} \leq 1 \]
\[ \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln y(t)}{\ln t} \leq \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln \bar{y}(t)}{\ln t} \leq 1 \]

almost surely.

Again,
\[ \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln x(t)}{t} = \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln x(t)}{\ln t} \cdot \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln t}{t} \]
\[ \leq \lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln t}{t} \]
\[ = 0 \]

It leads to \(\lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln x(t)}{t} \leq 0\) a.s.

Similarly, \(\lim_{t \to \infty} \sup_{t \to \infty} \frac{\ln y(t)}{t} \leq 0\) a.s. \(\Box\)

Now to show stochastic persistence and extinction, another lemma 4.2 is also considered. The proof of the following lemma 4.2 can be found in [23].

Lemma 4.2. Suppose that, \(x(t) \in C[\Omega \times \mathbb{R}_+, \mathbb{R}^+_0]\), where \(\mathbb{R}^+_0 := \{a : a > 0, a \in \mathbb{R}\}\).

1. If there are positive constants \(\lambda_0, T, \lambda \geq 0\) such that,
\[ \ln x(t) \leq \lambda T - \lambda_0 t + x(s)ds + \sum_{i=1}^{n} \beta_i B_i(t) \]
for \(t \geq T\), where \(\beta_i\) is a constant, \(1 \leq i \leq n\). Then
\(\langle x \rangle^* \leq \frac{\lambda}{\lambda_0}\) almost surely.
where, \( \langle x(t) \rangle = \frac{1}{t} \int_0^t x(s) \, ds \) and \( \langle x(t) \rangle^* = \lim_{t \to \infty} \sup \frac{1}{t} \int_0^t x(s) \, ds \).

(2) If there are positive constants \( \lambda_0, T, \lambda \geq 0 \) such that,

\[
\ln x(t) \geq \lambda T - \lambda_0 \int_0^t x(s) \, ds + \sum_{i=1}^n \beta_i B_i(t)
\]

for \( t \geq T \), where \( \beta_i \) is a constant, \( 1 \leq i \leq n \). Then

\[
\langle x \rangle^* \geq \frac{\lambda}{\lambda_0}
\]

almost surely (a.s.).

Now, applying Itô’s formula to equation (5),

\[
dlnx = \left( r(t) - \frac{\sigma_1^2(t)}{2} - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1+b(t)x^2(t)} \right) dt + \sigma_1(t)dB_1(t) \tag{12}
\]

\[
dlny = \left( -c(t) - \frac{\sigma_2^2(t)}{2} - d(t)y(t) + \frac{a_2(t)x^2(t)}{1+b(t)x^2(t)} \right) dt + \sigma_2(t)dB_2(t) \tag{13}
\]

Let, \( h_1(t) = r(t) - \frac{\sigma_1^2(t)}{2} \) and \( h_2(t) = -c(t) - \frac{\sigma_2^2(t)}{2} \). Then \( \langle h_2 \rangle^* < 0 \).

For prey population \( x(t) \) of our system (5) we will prove the following theorem.

**Theorem 4.3.**

(1) If \( \langle h_1 \rangle^* < 0 \), then \( x(t) \) will be extinct almost surely.

(2) If \( \langle h_1 \rangle^* = 0 \), then \( x(t) \) will be non-persistent in the mean almost surely.

(3) If \( \langle h_1 \rangle^* > 0 \), then \( x(t) \) will be weakly persistent in the mean almost surely.

**Proof.** (1) We follow [21]. It follows from (12) that,

\[
\ln x(t) - \ln x_0 = \int_0^t \left[ h_1(s) - r(s)x(s) - \frac{a_1(s)x(s)y(s)}{1+b(s)x^2(s)} \right] ds + \int_0^t \sigma_1(s)dB_1(s) \tag{14}
\]

Therefore,

\[
\ln x(t) - \ln x_0 \leq \int_0^t h_1(s) ds + \int_0^t \sigma_1(s)dB_1(s)
\]

We set, \( M_1(t) = \int_0^t \sigma_1(s)dB_1(s) \) which is a martingale whose quadratic variation is, \( \langle M_1, M_1 \rangle_t = \int_0^t \sigma_1^2(s) ds \leq (\sigma_1^2)^2 t \).

Now using strong law of large numbers for martingale yields,

\[
\lim_{t \to \infty} \frac{M_1(t)}{t} = 0 \text{ a.s.} \tag{15}
\]

Then,

\[
\frac{\ln x(t) - \ln x_0}{t} \leq \frac{1}{t} \int_0^t h_1(s) ds + \frac{M_1(t)}{t} \tag{16}
\]

Taking superior limit on both sides of (16) we see,

\[
\lim_{t \to \infty} \sup \frac{\ln x(t)}{t} \leq \langle h_1 \rangle^* < 0
\]

So, \( \lim_{t \to \infty} x(t) = 0 \)

(2) From (14),

\[
\frac{\ln x(t) - \ln x_0}{t} \leq \langle h_1 \rangle - r(s)x(t) \leq \frac{M_1(t)}{t} \tag{17}
\]

Now from the property of superior limit and (15) we get that, For arbitrary \( \epsilon > 0 \) there exists \( T > 0 \) such that,

\[
\langle h_1 \rangle \leq \langle h_1 \rangle^* + \frac{\epsilon}{2} \text{ and } \frac{M_1(t)}{t} \leq \frac{\epsilon}{2} \quad \forall t \geq T
\]
Substituting these in (17),
\[ \ln x(t) - \ln x_0 \leq (\langle h_1 \rangle^* + \epsilon)t - r^t(s) \int_0^t x(s) \, ds \]
Now, \( \langle h_1 \rangle^* = 0 \) is our assumption, then,
\[ \ln x(t) \leq \epsilon t - r_1(s)x_0 \int_0^t \frac{x(s)}{x_0} \, ds \]
By \( r^t(s) > 0 \) and lemma 4.2, we have,
\[ \langle x(t) \rangle^* \leq \epsilon \]
By arbitrariness of \( \epsilon \), \( \langle x(t) \rangle^* \leq 0 \).

Since the solution of our system (5) is non-negative, we get \( \langle x(t) \rangle^* = 0 \). i.e, prey population \( x(t) \) is non-persistent in the mean almost surely.

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Since the solution of our system (5) is non-negative, we get \( \langle x(t) \rangle^* = 0 \). i.e, prey population \( x(t) \) is non-persistent in the mean almost surely.

Now, for the predator population we are going to prove the following theorem.
Theorem 4.4. (1) If \((r^2)_* (h_2)_* + (a_2(t))_* (h_1)_* < 0\) then \(y(t)\) will go to extinction almost surely.
(2) If \((r^2)_* (h_2)_* + (a_2(t))_* (h_1)_* = 0\) then predator population \(y(t)\) will be non-persistent in the mean almost surely.
(3) If \((h_2)_* + \left\{ \frac{a_2(t)x^2(t)}{1+x^2(t)} \right\}_* > 0\) then \(y(t)\) will be weakly persistent in the mean almost surely, where \((\bar{x}(t), \bar{y}(t))\) is the solution of equation (10) with initial value \((x_0, y_0) \in \mathbb{R}^2\).

Proof. (1) If \((h_1)_* \leq 0\) then from previous theorem, \(\langle x(t) \rangle_* = 0\).

By using (13),
\[
\frac{\ln y(t) - \ln y_0}{t} \leq (h_2)_* + a_2(t) x^2(t) + \frac{M_2(t)}{t}
\]
therefore, \([t^{-1} \ln y(t)]_* \leq (h_2)_* < 0\) then, \(\lim_{t \to \infty} y(t) = 0\)

Now, if \((h_1)_* > 0\), it follows from the property of superior limit, interior limit and (15) that for sufficiently small \(\epsilon\) there exists a \(T > 0\) such that,
\[
\frac{\ln x(t) - \ln x_0}{t} \leq (h_1)_* + \epsilon - r_* x(t) + \epsilon \forall t > T
\]
Applying lemma 4.2 and arbitrarness of \(\epsilon\) gives,
\[
\langle x(t) \rangle_* \leq \frac{(h_1)_*}{r_*} (22)
\]
Putting in (13),
\[
[t^{-1} \ln y(t)]_* \leq (h_2)_* + (a_2(t))_* \frac{(h_1)_*}{(r^2)_*} (23)
\]
Then, \([t^{-1} \ln y(t)]_* \leq \frac{(r^2)_* (h_2)_* + (a_2(t))_* (h_1)_*}{(r^2)_*} < 0\) that is, \(\lim_{t \to \infty} y(t) = 0\) almost surely.

(2) In case (1) we have shown that if \((h_1)_* \leq 0\) then \(\lim_{t \to \infty} y(t) = 0\). i.e, \(\langle y(t) \rangle_* = 0\).

Now, we have to show that, \(\langle y(t) \rangle_* = 0\) when \((h_1)_* > 0\).

Otherwise, if \((y(t))_* > 0\) then from lemma 4.1 and using (23) we get,
\[
0 = [t^{-1} \ln y(t)]_* \leq (h_2)_* + (a_2(t))_* \langle x^2(t) \rangle_* (24)
\]
for \(\epsilon > 0\) arbitrary there exists a \(\bar{T} > 0\) such that,
\[
\langle h_2 \rangle < \langle h_2 \rangle_* + \frac{\epsilon}{3},
\]
\[
\langle a_2(t)x^2(t) \rangle < (a_2(t))_* \langle x^2(t) \rangle_* + \frac{\epsilon}{3} \forall t > \bar{T}
\]
\[
\frac{M_2(t)}{t} < \frac{\epsilon}{3}
\]
Putting in (13),
\[
\frac{\ln y(t) - \ln y_0}{t} \leq (h_2)_* + \epsilon + (a_2(t))_* \langle x^2(t) \rangle_* - (d(t))_* \langle y(t) \rangle
\]
By lemma 4.2 and (24),
\[
\langle y(t) \rangle_* \leq \frac{(h_2)_* + \epsilon + (a_2(t))_* \langle x^2(t) \rangle_*}{(d(t))_*}
\]
By (22) and arbitrariness of \( \epsilon \) we get,
\[
(y(t))^* \leq \frac{(h_2)^*(r^2)_* + (a_2(t))^*(h_2^2)^*}{(d(t))_*(r^2)_*} = 0
\]
which is a contradiction.

So \( (y(t))^* = 0 \) a.s.

(3) We need to show that \( (y(t))^* > 0 \) a.s.

Else for arbitrary \( \epsilon_2 > 0 \) there exists a solution \( (\hat{x}(t), \hat{y}(t)) \) of system (5) with positive initial value \( (x_0, y_0) \in \mathbb{R}^2_+ \) such that, \( P\{ (\hat{y}(t))^* < \epsilon_2 > 0 \} \).

Let \( \epsilon_2 \) be sufficiently small so that,
\[
(h_2)^* + \left\langle \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} \right\rangle^* > d^* + \frac{2a_2^2 \sigma_1^2}{(b')^2 x^4} \tag{25}
\]

From (13),
\[
\frac{\ln \hat{y}(t) - \ln y_0}{t} = (h_2) + \left\langle \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} \right\rangle - \langle d(t) \hat{y}(t) \rangle + \frac{M_2(t)}{t} + \left\langle \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} - \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} \right\rangle
\]

Here \( (\hat{x}(t), \hat{y}(t)) \) is the solution of model (10) with initial value \( (x_0, y_0) \in \mathbb{R}^2_+ \).

\( \hat{x}(t) \leq \hat{x}(t) \leq \hat{y}(t) \) a.s. for \( t \in [0, +\infty) \), because of,
\[
\frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} - \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} = \frac{a_2(t) (\hat{x}^2(t) - \hat{x}^2(t))}{(1 + b(t) \hat{x}^2(t))(1 + b(t) \hat{x}^2(t))}
\]
\[
\geq -\frac{a_2(t) (\hat{x}(t) - \hat{x}(t))}{b(t)} - \frac{a_2(t) (\hat{x}(t) - \hat{x}(t))}{b(t)} = -\frac{2a_2(t)}{b(t)} (\hat{x}(t) - \hat{x}(t))
\]

Then calculating we get,
\[
\frac{\ln \hat{y}(t) - \ln y_0}{t} \geq (h_2) + \left\langle \frac{a_2(t) \hat{x}^2(t)}{1 + b(t) \hat{x}^2(t)} \right\rangle - \langle d(t) \hat{y}(t) \rangle + \frac{M_2(t)}{t} - \left\langle \frac{2a_2^2 \sigma_1^2}{(b')^2 (\hat{x}(t) - \hat{x}(t))} \right\rangle
\]
\[
\tag{27}
\]

Now we consider the Lyapunov function, \( V_2(t) = | \ln \hat{x}(t) - \ln \hat{y}(t) | \) Then \( V_2(t) \) is a positive function on \( \mathbb{R}_+ \).

After using Ito’s formula, by (10) and (12) we get,
\[
d^* V_2(t)
\]
\[
= \left[ r(t) - \frac{\sigma_1^2(t)}{2} - r(t) \hat{x}(t) \right] dt + \sigma_1(t) dB_1(t)
\]
\[
- \frac{a_1(t) \hat{x}(t) \hat{y}(t)}{1 + b(t) \hat{x}^2(t)} dt + \sigma_1(t) dB_1(t)
\]
\[
= \left[ r(t) - \frac{\sigma_1^2(t)}{2} - r(t) \hat{x}(t) \right] dt - \left[ r(t) - \frac{\sigma_1^2(t)}{2} - r(t) \hat{x}(t) - \frac{a_1(t) \hat{x}(t) \hat{y}(t)}{1 + b(t) \hat{x}^2(t)} \right] dt
\]
\[
\tag{28}
\]
taking superior limit, and using (25),

\[ \text{Stochastic permanence.} \]

5. If

\[ \text{Proof.} \]

Firstly, we have to show that, for \( p > 0 \),

\[ \begin{align*}
    \ln \hat{y}(t) - \ln y_0 & \geq \langle h_2 \rangle + \left\langle \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} \right\rangle - \langle d(t) \rangle \hat{y}(t) + \frac{M_2(t)}{t} - \frac{2a_2^u a_1^u}{(b')^2 r^2} \langle \hat{y}(t) \rangle \\
    \text{taking superior limit, and using (25),} \\
    \left\langle t^{-1} \ln \hat{y}(t) \right\rangle & \geq \langle h_2 \rangle^* + \left\langle \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} \right\rangle^* - \left( d^u + \frac{2a_2^u a_1^u}{(b')^2 r^2} \right) \epsilon_2 > 0 \\
    \text{which contradicts lemma 4.1. So, } \langle y(t) \rangle^* > 0 \text{ almost surely.} \\
    \text{That is } y(t) \text{ is weakly persistent in the mean a.s.} \quad \Box
\end{align*} \]

5. Stochastic permanence.

**Theorem 5.1.** If

\[ 2(\max\{\sigma_1^u, \sigma_2^u\})^2 < \min\{r^l - \frac{a_1^u}{b^r}, \frac{a_2^l}{b^u} - c^u\} \quad (31) \]

Then system (5) is stochastically permanent.

**Proof.** Firstly, we have to show that, for \( \epsilon > 0 \) there exists a constant \( \delta > 0 \) such that,

\[ P_*\{|x(t)| \geq \delta\} \geq 1 - \epsilon \]

let, \( \theta > 0 \) is an arbitrary constant which satisfy the following equation,

\[ 2(\theta + 1)(\max\{\sigma_1^u, \sigma_2^u\})^2 < \min\{r^l - \frac{a_1^u}{b^r}, \frac{a_2^l}{b^u} - c^u\} \quad (32) \]

By (31) there exists a constant \( p > 0 \) satisfying,

\[ \min\{r^l - \frac{a_1^u}{b^r}, \frac{a_2^l}{b^u} - c^u\} - 2(\theta + 1)(\max\{\sigma_1^u, \sigma_2^u\})^2 - p \] > 0 \quad (33)

Define \( V(x, y) = x + y \) then,

\[ \begin{align*}
    dV(x, y) = & \left\{ \begin{array}{lr}
        x(t) \left( r(t) - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)} \right) + y(t) \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} dt \\
        y(t)(-c(t) - d(t)y(t)) dt + \sigma_1(t)x(t)dB_1(t) + \sigma_2(t)y(t)dB_2(t)
        \end{array} \right. \\
\end{align*} \]
Again let, $U(x, y) = \frac{1}{V(x, y)}$. Using Ito’s formula we get,

$$dU(X) = \left[-U^2(X) \left( r(t) - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)} + y(t) \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} \right) \right] dt$$

$$- U^2(X)y(t)(-c(t) - d(t)y(t)) dt + U^3(X)x^2(t)\sigma_1^2(t) dt$$

$$+ U^3(X)y^2(t)\sigma_2^2(t) dt - U^2(X)(\sigma_1(t)x(t)dB_1(t) + \sigma_2(t)y(t)dB_2(t))$$

$$= LU(X) dt - U^2(X)(\sigma_1(t)x(t)dB_1(t)) - U^2(X)(\sigma_2(t)y(t)dB_2(t))$$

We choose a positive constant $\theta$ such that it obeys (31). Then,

$$L(1 + U(X))^\theta = \theta(1 + U(X))^{\theta-1}LU(X) + \frac{1}{2} \theta(\theta - 1)(1 + U(X))^{\theta-2}U^4(X)(x^2\sigma_1^2(t) + y^2\sigma_2^2(t))$$

So we may choose some sufficiently small positive constant $p$ such that it satisfies (32).

Next we take $W(X) = e^{pt}(1 + U(X))^\theta$.  

$$LW(X) = pe^{pt}(1 + U(X))^\theta + e^{pt}L(1 + U(X))^\theta$$

$$= e^{pt}(1 + U(X))^{\theta-2}(p(1 + U(X))^2 - \theta U^2(X)x(t)(r(t) - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)}))$$

$$- \theta U^2(X)y(t)(-c(t) - d(t)y(t)) + \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} - \theta U^3(X)y(t)(-c(t) - d(t)y(t))$$

$$- \theta U^3(X)(x(t)(r(t) - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)})) + \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)}$$

$$+ \theta U^3(X)(x^2(t)\sigma_1^2(t) + y^2(t)\sigma_2^2(t)) + \frac{\theta(\theta + 1)}{2}U^4(X)(x^2(t)\sigma_1^2(t) + y^2(t)\sigma_2^2(t))$$

(34)

Obviously,

$$\theta U^3(X)(x^2(t)\sigma_1^2(t) + y^2(t)\sigma_2^2(t)) \leq \theta U(X)(2 \max\{\sigma_1^2, \sigma_2^2\})^2$$

$$\frac{\theta(\theta + 1)}{2}U^4(X)(x^2(t)\sigma_1^2(t) + y^2(t)\sigma_2^2(t)) \leq \frac{\theta(\theta + 1)}{2}U^2(X)(2 \max\{\sigma_1^2, \sigma_2^2\})^2$$

Hence,

$$LW(X) \leq e^{pt}(1 + U(X))^{\theta-2}(p + \theta \max\{r^u, d^u\}) + (2p - \theta \min\{r^l - \frac{a_1^u}{b^l}, \frac{a_2^u}{b^l} - c^u\})$$

$$+ \theta \max\{r^u, d^u\} + 4\theta(\max\{\sigma_1^2, \sigma_2^2\})^2U(X) + (p - \theta \min\{r^l - \frac{a_1^u}{b^l}, \frac{a_2^u}{b^l} - c^u\})$$

$$+ \frac{\theta(\theta + 1)}{2}(2 \max\{\sigma_1^2, \sigma_2^2\})^2U^2(X)$$

Using (32) there exists a positive constant $s$ such that, $LW(X) \leq se^{pt}$.

Thus,

$$E[e^{pt}(1 + U(X))^\theta] \leq (1 + U(0))^\theta + \frac{s(e^{pt} - 1)}{p}$$
Therefore we get,

\[ P = \text{...} \]

Then,

\[ \lim_{t \to \infty} \sup_{t \to \infty} E \left[ \frac{1}{|x(t)|^\theta} \right] \leq 2^\theta \lim_{t \to \infty} E U^\theta(X) \]

\[ \leq 2^\theta \frac{S}{p} = M \]

Thus for any \( \epsilon > 0 \), letting \( \delta = \left( \frac{1}{M^\theta} \right)^n \) and by the Chebyshev’s inequality we obtain,

\[ P\{ |X(t)| < \delta \} = P\{ |X(t)|^{-\theta} > \delta^{-\theta} \} \]

\[ \leq \frac{E[|X(t)|^{-\theta}]}{\delta^{-\theta}} \]

\[ = \delta^\theta E[|X(t)|^{-\theta}] \]

Therefore we get, \( P_*\{ |X(t)| \geq \delta \} \geq 1 - \epsilon \).

Secondly, we are going to prove that for any, \( \epsilon > 0 \) there exists a constant \( \xi > 0 \) such that, \( P_*\{ |X(t)| \leq \xi \} \geq 1 - \epsilon \).

Define, \( V_0(X) = x^q + y^q \), where \( 0 < q < 1 \), \( X = (x, y) \in \mathbb{R}_+^2 \).

By Itô’s formula,

\[ dV(X(t)) = q x^q \left( r(t) - r(t)x(t) - \frac{a_1(t)x(t)y(t)}{1 + b(t)x^2(t)} + \frac{q - 1}{2} \sigma_1^2(t) \right) dt + q y^q (-c(t) - d(t)y(t)) \]

\[ + \frac{a_2(t)x^2(t)}{1 + b(t)x^2(t)} + \frac{q - 1}{2} \sigma_2^2(t) \right) dt + q x^q \sigma_1(t) dB_1(t) + q y^q \sigma_2(t) dB_2(t) \]

(35)

Let \( k_0 \) be so large that \( X_0 \) lies within the interval \( \left[ \frac{1}{k_0}, k_0 \right] \). For each integer \( k \geq k_0 \) we define the stopping time, \( \tau_k = \inf\{ t \geq 0 : X(t) \notin \left( \frac{1}{k}, k \right) \} \).

Obviously \( \tau_k \) increases as \( k \to \infty \).

So,

\[ E[\exp\{t \wedge \tau_k\} X^q(t \wedge \tau_k)] - X^q(0) \]

\[ \leq q E \int_0^{t \wedge \tau_k} \exp s \cdot x^q(s) \left( 1 + q(r(s) - r(s)x(s) \right. \]

\[ - \frac{1 - q}{2} \sigma_1^2(s) \right) ds + q E \int_0^{t \wedge \tau_k} \exp s y^q(s)(1 + q(-c(s)) \]

\[ - d(s)y(s) + \frac{a_2(s)}{b(s)} \frac{1 - q}{2} \sigma_2^2(s) \right) ds \]

\[ \leq E \int_0^{t \wedge \tau_k} (k_1 + k_2) \exp s ds \]

\[ \leq (k_1 + k_2)(\exp t - 1) \]

where \( k_1, k_2 \) are positive constants.

Letting \( k \to \infty \) we have,

\[ \exp t E[X^q(t)] \leq X^q(0) + (k_1 + k_2)(\exp t - 1) \]

In other words, we have shown that,

\[ \lim_{t \to \infty} \sup E[X^q(t)] \leq k_1 + k_2 \].
Thus for any given $\epsilon > 0$, choosing $\xi = \frac{(k_1 + k_2)\frac{1}{2}}{\epsilon}$ and by chebyshev inequality we get,

$$P\{|X(t)| > \xi\} = P\{|X(t)|^q > \xi^q\} \leq \frac{E[|X(t)|^q]}{\xi^q}$$

that is , $P\{|X(t)| > \xi\} \leq \frac{E[|X(t)|^q]}{\xi^q} \leq \epsilon$.

Consequently,

$$P\{|X(t)| \leq \xi\} \geq 1 - \epsilon.$$  \hspace{1cm} (36)

This completes the proof.

6. Numerical simulation and results.

6.1. Deterministic model. We have not discussed analytically the stability analysis of non-autonomous deterministic ecological system because in this case, stability analysis will probably not prove to be very useful. The concept of stability has been first defined in the mathematical form by Lyapunov(1892) [27], where the stability considers the behavior of a system solution if its initial state is in the neighborhood of an equilibrium point [16]. But it is not acceptable qualitative definition of ecosystem stability. As in the figure 1, it is shown that predator population goes to extinction while prey population exists for long time. Both prey and predator population co-exist after a long time. Lyapunov’s direct method has been extended to such systems by Lassalle and Rath (1963) [19]. It shows that if a system behaves properly for a sufficiently long time, it is expected to behave properly in future.

For numerical simulation of deterministic model, we consider, $r(t) = 0.2 + 0.05\sin t$, $b(t) = 0.22 + 0.02\sin t$, $c(t) = 0.1 + 0.05\sin t$, $d(t) = 0.2 + 0.01\sin t$.

In Figure 1 we choose, $a_1(t) = 0.1 + 0.01\sin t$, $a_2(t) = 0.02 + 0.01\sin t$ (since $a_2(t) < a_1(t)$). Then we see in Figure 1 that our system (1) is stable.

In Figure 2 we choose, $a_1(t) = 2 + 0.1\sin t$, $a_2(t) = 1 + 0.1\sin t$. Then we see in Figure 2 that our system (1) is unstable.

In Figure 3 we choose, $r(t) = 5 + 2.5\sin t$, $b(t) = 0.22 + 0.02\sin t$, $c(t) = 0.01 + 0.005\sin t$, $d(t) = 0.2 + 0.01\sin t$, $a_1(t) = 0.1 + 0.1\sin t$, $a_2(t) = 1 + 0.1\sin t$. Then $a_2^L = 1.1 > 0.001 = c^L, b^L$.

$$r^L = 4.95 > 3.55 = \frac{a_1^L}{2\sqrt{b^L}\beta_1} \text{ and } \frac{\beta_2}{1 + b^L \beta_2} = 0.0785 > 0.017 = \frac{c^M}{a_2^L}.$$  

This shows that the conditions of uniform persistence of the deterministic system in Theorem 2.2 are satisfied. The corresponding Figure 3 also validates this result depicting the persistence of the system.

6.2. Stochastic model. We use the Milstein method mentioned in Higham [12] to substantiate our main results.

We consider the following discretized equations:

$$x_{k+1} = x_k + x_k[r(k\Delta t) - r(k\Delta t)x_k - \frac{a_1(k\Delta t)x_ky_k}{1 + b(k\Delta t)x_k^2}\Delta t]$$

$$+ \sigma_1(k\Delta t)x_k\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2(k\Delta t)}{2}x_k(\xi_k^2 - 1)\Delta t,$$
Figure 1. Numerical simulation for the deterministic system (1) with initial condition (0.2,0.3) by $a_1(t) = 0.1 + 0.01 \sin t$, $a_2(t) = 0.02 + 0.01 \sin t$ shows the stable behavior of prey and predator.

Figure 2. Numerical simulation for the deterministic system (1) with initial condition (0.2,0.3) by $a_1(t) = 2 + 0.1 \sin t$, $a_2(t) = 1 + 0.1 \sin t$ shows the unstable behavior of prey and predator.

Figure 3. Numerical simulation for the deterministic system (1) with (0.2,0.3) by $r(t) = 5 + 2.5 \sin t$, $b(t) = 0.22 + 0.02 \sin t$, $c(t) = 0.01 + 0.005 \sin t$, $d(t) = 0.2 + 0.01 \sin t$, $a_1(t) = 0.1 + 0.1 \sin t$, $a_2(t) = 1 + 0.1 \sin t$ shows that system is persistent.

\[
y_{k+1} = y_k + y_k[-c(k\Delta t) - d(k\Delta t)y_k + \frac{a_2(k\Delta t)x_k^2}{1 + b(k\Delta t)x_k^2}]\Delta t + \sigma_2(k\Delta t)y_k \sqrt{\Delta t}\eta_k + \frac{\sigma_2^2(k\Delta t)}{2} y_k(\eta_k^2 - 1)\Delta t
\]

where $\xi_k, \eta_k$ (k=1,2,...,n) are the Gaussian random variables $N(0,1)$. 
We choose, \( r(t) = 0.2 + 0.01 \sin t \), \( a_1(t) = 5 + 0.01 \sin t \), \( a_2(t) = 4 + 0.01 \sin t \), \( b(t) = 5 + 0.02 \sin t \), \( c(t) = 0.1 + 0.05 \sin t \), \( d(t) = 0.2 + 0.01 \sin t \).

In Figure 4 we choose, \( \sigma_1^2 = \sigma_2^2 = 0.21 + 0.02 \sin t \).

Then \( \langle h_1 \rangle^* = -0.01 < 0 \) and \( (r^2) \langle h_2 \rangle^* + (a_2(t))^* \langle h_1^2 \rangle^* = (0.19)^2 \times (-0.31) + 4.01 \times (-0.01)^2 < 0 \).

This shows that the conditions of extinction of both prey population \( x(t) \) and predator population \( y(t) \) of stochastic system in Theorem 4.3 and Theorem 4.4 are satisfied. The corresponding Figure-4(a) and 4(b) also validates this result.

In Figure 5 we choose \( \sigma_1^2 = 0.19 + 0.02 \sin t \), \( \sigma_2^2 = 0.09 + 0.02 \sin t \).

Then \( \langle h_1 \rangle^* = 0.01 > 0 \) and \( (r^2) \langle h_2 \rangle^* + (a_2(t))^* \langle h_1^2 \rangle^* = (0.19)^2 \times (-0.19) + 4.01 \times (0.01)^2 < 0 \).

This shows that the conditions of weakly persistent in the mean of prey population \( x(t) \) and extinction of predator population \( y(t) \) of stochastic system in Theorem 4.3 and Theorem 4.4 are satisfied. The corresponding Figure-5(a) and 5(b) also validates this result.

In Figure 6 we change the value of \( r(t) \) into \( r(t) = 2.2 + 0.01 \sin t \). And choose \( \sigma_1 = \sigma_2 = 0.02 + 0.01 \sin t \).

Then \( 2(\max \{\sigma_1^2, \sigma_2^2\})^2 = 0.0018 \) and \( \min \{r^l - \frac{a_1^u}{b^l}, \sigma_1^2 \} - c^u \} = 0.645 \). So we get, \( 2(\max \{\sigma_1^2, \sigma_2^2\})^2 < \min \{r^l - \frac{a_1^u}{b^l}, \sigma_1^2 \} - c^u \} \).

This shows that the conditions of stochastic permanence of stochastic system in Theorem 5.1 are satisfied. The corresponding Figure-6(a) and 6(b) also validates this result.

7. Conclusion. In this paper, we have studied firstly the uniform persistence of the non-autonomous deterministic model of prey-predator population with Holling type-III functional response. Then we have thoroughly investigated the existence of global positive solution, persistence, permanence and extinction of the corresponding stochastic system. We have derived the conditions for uniform persistence in the deterministic system. In real world we get examples of this phenomenon, as [6], in northeast Minnesota, wolves play a crucial role in the decline of white tailed deer population. They eat on an average of 11.5% of the white tailed deer population.
prey is weakly persistence in the mean

**Figure 5.** Numerical simulation for the system (5) with $\sigma_1^2 = 0.19 + 0.02 \sin t$, $\sigma_2^2 = 0.09 + 0.02 \sin t$ shows weakly persistence in the mean of prey and extinction of predator.

permanence of prey

permanence of predator

**Figure 6.** Numerical simulation for the system (5) with $r(t) = 2.2 + 0.01 \sin t$, $\sigma_1 = \sigma_2 = 0.02 + 0.01 \sin t$ shows permanence of both prey and predator.

In stochastic counterpart, existence of global positive solution is discussed. Persistence, permanence and extinction criteria are established for different values of system parameters. The threshold between persistence and extinction has been obtained for prey and predator species in stochastic environment which is very important for evaluating the risk of extinction of species in systems. The forcing intensity of fluctuating environment plays a vital role in the survival of prey and predator species. From biological point of view permanence of a system ensures the survival of both prey and predator species in the long run. Theorem 5.1 reveals that very small white noise will force the population to become stochastically permanent. Similarly, Theorem 4.3 and 4.4 tell us that large white noise can cause extinction of both prey and predator species. For example [34], bats and owls get affected by environmental noise and can’t find and hunt for prey. Gleaning bats such as the Bechstein’s bat don’t hunt in noisy areas generally. Some birds with lower frequencies can not tweet louder than the environmental noise which affect them
from communicating with and finding each other for mating and even propel them to fly away to less noisy environments. Thus large white noise put these animals and birds at risk of extinction by making once fulfilling environments unlivable. In absence of this large noise, these animals and birds get adapted to small noise by tweeting louder and remain bounded.

The common definition of stochastic permanence has some limitations as in this case, if there is only one species having positive lower bound and all other species go to extinction, the system is still permanent. So it can not guarantee that all the species have positive lower bound. But some researchers considered a more reasonable definition of permanence for stochastic population model, which is stochastically persistence in probability [20], [23] and [33]. In future we want to study the stochastic persistence in probability of the present model. Other than white noise, Levy noise is inescapable in real world. We want to consider stochastic population model systems with Levy noise in future.

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E-mail address: sampurna2492@gmail.com
E-mail address: prithadas02@math.iiests.ac.in
E-mail address: mukherjee1961@gmail.com