Born-Infeld black hole in the isolated horizon framework

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In this work we probe the Born-Infeld (BI) black hole in the isolated horizon framework. It turns out that the BI black hole is consistent with the heuristic model for colored black holes proposed by Ashtekar et al [(2001) Class. Quant. Grav. 18 919-940]. The model points to the unstability of the BI black hole.

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I. INTRODUCTION

The Reissner-Nordström (RN) solution (characterized by its charge and mass) turns out to be the final fate of a charged star, having as uncharged limit the Schwarzschild black hole, then it is of interest to investigate in more detail its nonlinear electromagnetic generalization, in particular, the Born-Infeld field.

The Einstein-Born-Infeld (EBI) generalization of the RN black hole was obtained by García-Salazar-Plebański (GSP) in 1984 [2]. Two years after, Demianski [3] presented the static spherically symmetric solution that nowadays is known as the EBIon, it is the most known solution of the EBI equations; in the spirit of the concept of geon introduced by Wheeler in the sixties, this means electromagnetic radiation held together by its self-gravitational attraction. These two solutions, GSP and Demianski’s, are actually the same, differing only by a constant.

Remarkable properties of the EBI black hole arise in the context of the isolated horizon formalism, recently put forward by Ashtekar and co-workers in a series of papers in Phys. Rev. D and Class. and Q. Grav. [1]. In this approach it is pointed out the unsatisfactory (uncomplete) description of a black hole given by concepts such as ADM mass and event horizon, for instance, specially if one is dealing with hairy black holes. To remedy this uncompleteness, Ashtekar et al have proposed alternatively the isolated horizon formalism, that furnish a more complete description of what happens in the neighborhood of the horizon of a hairy black hole. Moreover, they conjecture about the relationship between the colored black holes and their solitonic analogs [3]: the ADM mass contains two contributions, one attributed to the black hole horizon and the other to the outside hair, captured by the solitonic residue. In the present communication we show that the EBI colored black hole and the corresponding solitonic solution have most of the properties of the model proposed in [1].

The EBI black hole corresponds to the solution for the field equations arising from the Einstein-Born-Infeld action

\[ S = \int d^4x \sqrt{-g} \{ R(16\pi)^{-1} + L \}, \]

where \( R \) denotes the scalar curvature, \( g := \det|g_{\mu\nu}| \) and \( L \), the electromagnetic part, is assumed to depend in nonlinear way on the invariants of \( F_{\mu\nu} \), the nonlinear generalization of the electromagnetic field, \( F_{\mu\nu} \),

\[ L = -\frac{1}{2} P^{\mu\nu} F_{\mu\nu} + K(P,Q), \]

where \( P \) and \( Q \) are the invariants of \( P_{\mu\nu} \), \( K(P,Q) \) is the so called structural function which for the Born-Infeld nonlinear electrodynamics is given by

\[ K = b^2(1 - \sqrt{1 - 2P/b^2 + Q^2/b^4}), \]

where \( b \) is the maximum field strentgh and the relevant parameter of the BI theory.

The EBI solution for a static spherically symmetric (SSS) spacetime is given by:

\[ ds^2 = -\psi dt^2 + \psi^{-1} dr^2 + r^2(\theta^2 + \sin^2 \theta d\phi^2), \]

\[ \psi = 1 - \frac{2m}{r} + \frac{\sqrt{3}}{3} b^2 r^2 (1 - \sqrt{1 + \frac{a^4}{r^4}} + \frac{4q^2}{3r} g(r)), \]

\[ g'(r) = -(r^4 + a^4)^{-\frac{1}{2}}, \]

where \( g'(r) = \frac{dg(r)}{dr} \), \( m \) is the mass parameter, \( q \) is the electric charge (both in lenght units), \( a^4 = q^2/b^2 \) and \( b \) is the Born-Infeld parameter given in units of [length]^{-1}. The nonvanishing components of the electromagnetic field are

\[ F_{rt} = q(r^4 + a^4)^{-\frac{1}{2}}, \quad P_{rt} = \frac{q}{r^2}. \]

The solution given by García-Salazar-Plebański [2] corresponds to

\[ g(r) = \int_r^{\infty} \frac{ds}{\sqrt{s^2 + a^2}} = \frac{1}{2a} F(\arccos \left\{ \frac{r^2 - a^2}{r^2 + a^2} \right\}, \frac{1}{\sqrt{2}}). \]
while the one given by Demianski corresponds to
\[ g(r) = -\int_0^\infty \frac{ds}{\sqrt{s^4 + a^4}} = -\frac{1}{2a} F(\arccos \left( \frac{a^2 - r^2}{a^2 + r^2} \right), \frac{1}{2}). \]

Choosing \( g(r) \) as in Eq. (8) or Eq. (9) has as a consequence a different behavior of the solution at the origin. The metric function \( \psi \) with \( g(r) \) given in Eq. (8) (GSP solution) diverges at \( r \to 0 \) (even when \( m = 0 \)), it corresponds to the black hole solution. The other one, meaning \( \psi \) with \( g(r) \) given in Eq. (9) (Demianski), is the so called EBlon, a particlelike solution that is finite at the origin (for \( m = 0 \)). The integrals of Eqs. (8) and (9) are related by
\[
\int_r^\infty \frac{ds}{\sqrt{s^4 + a^4}} = -\int_0^r \frac{ds}{\sqrt{s^4 + a^4}} + \text{Const},
\]
where \( K(\frac{1}{2}) \) is the complete elliptic integral of the first kind. In the limit of large distances, \( r \to \infty \), asymptotically the solution corresponds to the RN solution. Also when the BI parameter goes to infinity, \( b \to \infty \), we recover the linear electromagnetic (Einstein-Maxwell) RN solution. In the uncharged limit, \( b = 0 \) (or \( q = 0 \)), it is recovered the Schwarzschild black hole.

### A. Hairy EBI Black hole

When the black hole is not completely determined by global charges defined at spatial infinity such as ADM mass, angular momentum or electric charge, but rather it possesses short range charges (hair) that vanish at infinity, then it is a hairy black hole. This is the case for the EBI black hole, since the location and size of the horizon depends on the parameter \( bq \) (the corresponding metric function was studied in [3]), however, at infinity it is undistinguishable from a RN black hole characterized only by its charge \( q \) and mass \( m \).

In a series of papers Ashtekar et al have proposed a more complete description to characterize a hairy black hole, based on quantities defined at the horizon. This formalism is intended to deal with situations more general than SSS hairy black holes and it involves the canonical formalism of gravity. Furthermore, they proposed a formula relating the horizon mass and the ADM mass of the colored black hole solution with the ADM mass of the soliton solution of the corresponding theory,

\[
M^{(n)}_{\text{hor}} = M^{(n)}_{\text{ADM}} - M^{(n)}_{\text{sol}},
\]

where the superscript \( n \) indicates the colored version of the hole; in the papers of Ashtekar et al this \( n \) refers to the Yang-Mills hair, labeled by this parameter, corresponding to \( n = 0 \) the Schwarzschild limit (absence of YM charge). This relation has been proved numerically to work for the Einstein-Yang-Mills (EYM) black hole. For the case studied here the \( n = 0 \) version shall correspond to the distinct hairy black holes labeled by distinct (continuous) BI parameter, \( b \); to the limit \( b \to \infty \) corresponds the linearly charged case, that is precisely the RN black hole; in the limit \( b \to 0 \) (or \( q = 0 \)) we arrive to the Schwarzschild black hole (our \( n = 0 \) also).

It turns out that the EBI black hole and the corresponding EBlon solution fulfill the relation between the masses as well as most of the properties of the model for the colored black hole, as will be shown in what follows. The EBI black hole case has the additional advantage of being an exact solution.

First of all let’s check the relation between the masses Eq. (12). The horizon and ADM masses as functions of the horizon radius \( r_h \) for the EBI solution are given, respectively, by

\[
M^{(b)}_{\text{hor}}(r_h) = \frac{r_h}{2} + \frac{b^2 r_h}{3} - \frac{\sqrt{r_h^2 + a^4}}{3} - \frac{2q^2}{3} \int_0^{r_h} \frac{ds}{\sqrt{a^4 + s^4}},
\]

\[
M^{(b)}_{\text{ADM}}(r_h) = \frac{r_h}{2} + \frac{b^2 r_h}{3} - \frac{\sqrt{r_h^4 + a^4}}{3} + \frac{2q^2}{3} \int_{r_h}^{\infty} \frac{ds}{\sqrt{a^4 + s^4}},
\]

In Fig. it is shown \( M^{(b)}_{\text{ADM}}(r_h) \) for different values of the BI parameter \( b \) compared with \( M^{(b)}_{\text{ADM}}(r_h) \) of Reissner-Nordström and the bare black hole (Schwarzschild).

The mass of the soliton can be obtained by letting \( r \to 0 \) in the ADM mass, Eq. (12), we obtain \( M^{(b)}_{\text{sol}} = \)

\[
M^{(b)}_{\text{sol}(r \to 0)} = \frac{2q^2}{3} \int_0^{\infty} \frac{ds}{\sqrt{a^4 + s^4}}.
\]
greater than the ‘dressed’ mass, $M^{(0)}_{\text{ADM}}(r_h)$, shown in this figure for different values of the BI parameter $b$ (in parenthesis) compared with the corresponding to Reisner-Nords"orm, $M^{(0)}_{\text{ADM}}(r_h)$, and the bare black hole mass $M^{(0)}_{\text{hor}}(r_h)$ (Schwarzschild).

FIG. 1: It is shown the ADM mass as function of the horizon radius $r_h$, $M^{(0)}_{\text{ADM}}(r_h)$ for different values of the BI parameter $b$ (in parenthesis) compared with the corresponding to Reisner-Nords"orm, $M^{(0)}_{\text{ADM}}(r_h)$, and the bare black hole mass $M^{(0)}_{\text{hor}}(r_h)$ (Schwarzschild).

$M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h)$. From these expressions one can trivially check that they satisfy Eq. (12). Moreover, in the heuristic model for the colored black hole there are predictions that the EBI solution satisfies. We enumerate them as stated in [3].

(i) The bare black hole horizon mass, $M^{(0)}_{\text{hor}}(r_h)$, is greater than the ‘dressed’ mass, $M^{(n)}_{\text{hor}}(r_h)$, for all $n$ and all values of the horizon radius, $r_h$. The corresponding for the EBI case is $M^{(0)}_{\text{hor}}(r_h) < M^{(0)}_{\text{hor}}(r_h)$, which amounts to

$$\frac{r_h}{2} > \frac{r_h}{2} - \frac{b^2}{3} \left( \sqrt{1 + \frac{a^2}{r^4}} - 1 \right) + \frac{2q^2}{3} \int_0^{r_h} \frac{ds}{\sqrt{a^4 + s^4}},$$

(15)

since the term in square brackets is positive, the inequality holds for all $b$, for all $r_h$. Moreover, the horizon masses satisfy the inequality $M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h)$. This is shown in Fig 2.

(ii) For all $b$ and all $r_h$, the surface gravity of the coloured black hole is less that the one for the bare black hole, i.e. $\kappa_{(b)}(r_h) < \kappa_{(0)}(r_h)$,

$$\frac{1}{2r_h} \left( 1 + 2b^2(r_h^2 - \sqrt{r_h^2 + a^4}) \right) < \frac{1}{2r_h},$$

(16)

the inequality reduces to

$$r_h^2(1 - \sqrt{1 + \frac{a^4}{r^4}}) < 0,$$

(17)

that in fact is satisfied since $\sqrt{1 + \frac{a^4}{r^4}} > 1$ for $q, b \neq 0$. It is shown in Fig. 3 for some values of $b$.

(iii) From Figs. 2 and 3 it is manifest that both $M^{(0)}_{\text{hor}}(r_h)$ and $\kappa_{(b)}(r_h)$, for fixed $r_h$, are monotonically decreasing functions of $b$. In other words, as $b \to 0$, the coloured black hole tends to the bared one.

(iv) For fixed $b$, the function $\beta_{(b)}(r_h) = 2r_h\kappa_{(b)}(r_h) < 1$. This condition reduces to $\left\{ 1 + 2b^2(r_h^2 - \sqrt{r_h^2 + a^4}) \right\} < 1$ that is satisfied since $(r_h^2 - \sqrt{r_h^2 + a^4}) < 0$. The condition is illustrated in Fig 4 where $\kappa_{(b)}(r_h) = \frac{1}{2r_h} \left( 1 + 2b^2(r_h^2 - \sqrt{r_h^2 + a^4}) \right)$.

(v) This prediction is about the behavior of $M^{(0)}_{\text{hor}}(r_h)$ and is the only one that the EBI solution does not fulfil. Contrary to the EYM case, the $M^{(0)}_{\text{hor}}(r_h)$, as a function of $r_h$ (fixed $b$), does not increase monotonically, as can be seen in Fig. 2 that for $b = 1.5$ and larger, $M^{(0)}_{\text{hor}}(r_h)$ has a minimum and then increases monotonically. Asymptotically its slope tends to the one of the bare black hole. We note that for the EYM solution this prediction was

FIG. 2: The inequality satisfied by the horizon masses is shown in this figure for $b = 0.5$, $b = 1.5$ and $b = 2.5$, $M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h) > M^{(0)}_{\text{hor}}(r_h)$. The corresponding for the coloured black hole horizon mass, $M^{(0)}_{\text{hor}}(r_h)$, is greater than the ‘dressed’ mass, $M^{(0)}_{\text{hor}}(r_h)$, for all $n$ and all values of the horizon radius, $r_h$. The corresponding for the EBI case is $M^{(0)}_{\text{hor}}(r_h) < M^{(0)}_{\text{hor}}(r_h)$, which amounts to

$$\frac{r_h}{2} > \frac{r_h}{2} - \frac{b^2}{3} \left( \sqrt{1 + \frac{a^2}{r^4}} - 1 \right) + \frac{2q^2}{3} \int_0^{r_h} \frac{ds}{\sqrt{a^4 + s^4}},$$

(15)
FIG. 4: The function $\beta(b)(r_h) = 2r_h\kappa(b)(r_h) < 1$ is displayed. For the Schwarzschild black hole, $\beta_0(r_h) = 1$.

FIG. 5: It can be observed how the EBI ADM mass, $M_{ADM}^{(b)}(r_h)$, show the crossing of families corresponding to distinct BI parameter $b$. The plot shows the cases $b = 1$ and $b = 6$ (in parenthesis).

We have shown that the the static sector of the EBI theory is described by the heuristic model for the colored black holes proposed by Ashtekar et al. It is carried out provided that in the EBI theory there exist both exact solutions: the colored black hole and the soliton like solution. In such a manner that the predictions of the model can be demonstrated analytically, and they are shown in plots as a better illustration. There is only one prediction of the model that the EBI solution does not fulfil: $M_{hor}^{(b)}(r_h)$ does not increase monotonically, except for low values of $b$.

Additionally the EBI ADM masses, $M_{ADM}^{(b)}(r_h)$ show the crossing of families corresponding to distinct BI parameter $b$, as can be seen in the Fig. 5. This is also a feature in the EYM case.

B. Final Remarks

We have shown that the the static sector of the EBI theory is described by the heuristic model for the colored black holes proposed by Ashtekar et al. It is carried out provided that in the EBI theory there exist both exact solutions: the colored black hole and the soliton like solution. In such a manner that the predictions of the model can be demonstrated analytically, and they are shown in plots as a better illustration. There is only one prediction of the model that the EBI solution does not fulfil: $M_{hor}^{(b)}(r_h)$ does not increase monotonically, except for low values of $b$.

We also point out some differences in comparison with the EYM black hole whose study lead to the model [7]: the Abelian character of BI theory vs. non-Abelian EYM and that BI theory is described with a continuos parameter $b$ in contrast with the discreteness of the EYM parameter $n$.

In general it occurs that $M_{ADM}^{(b)} > M_{hor}^{(b)}$. Using the corresponding expressions Eqs. (14) and (13), this inequality reduces to inequality (18). Since the diference between the hamiltonian horizon mass and the ADM mass can be seen as the energy that is available for radiation to fall both into the black hole and to infinity, then the nonzero value of the Hamiltonian could be an indication of instability of the EBI solution. On this basis, one can conjecture that the EBI black hole is unstable.

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