Exact Solutions to Special High Dimensional $O(n)$ Models, Dimensional Reductions, gauge redundancy, and special Frustrated Spin and Orbital models

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This work addresses models (e.g. potential models of directed orbital systems- the manganates) in which an effective reduction dimensionality occurs as a result of a new symmetry which is intermediate between that of global and local gauge symmetry. This path towards dimensional reduction is examined in simple $O(n)$ spin models and lattice gauge theories. A high temperature expansion is employed to map special anisotropic high dimensional models into lower dimensional variants. We show that it is possible to have an effective reduction in the dimension without the need of compactifying some dimensions. These models are frustrated and display a symmetry intermediate between local and global gauge symmetries. Some solutions are presented. Our dimensional reductions are a generalization of the trivial dimensional reduction that occur in pure two dimensional gauge theories. It will be further seen that the absence of a “phase interference” effect plays an important role in high dimensional problems. By identifying another (“permutational”) symmetry present in the large $n$ limit, we will further show how to generally map global high dimensional spin systems onto a one dimensional chain and discuss implications.

I. INTRODUCTION

In this article, a high temperature expansion is employed to examine mappings between problems of various dimensionality. By these reductions we will exactly solve an anisotropic two dimensional XY and some other models. It will be seen that the absence of a “phase interference” effect plays an important role in high dimensional problems; this observation will allow us to examine various high dimensional models by examining newly constructed one dimensional systems. We will show how an effective reduction dimensionality might occur as a result of an enhanced symmetry which lies midway between a full blown local symmetry and the much more restricted global symmetry. Such a symmetry might lead to an effective reduction in the dimension without the need of actually compactifying any dimensions. By further identifying a permutational symmetry present in the large $n$ limit, we will further show how to generally map global high dimensional spin systems onto a one dimensional chain and discuss implications.

In section [1] We are going to start off by discussing a momentum space high temperature expansion. Next, by analytically continuing the momenta and making them complex (section [11]), quaternion (section [14]), and of a general matrix form we will be able to exactly solve a variety of two, three, and four dimensional Ising and $O(n)$ models. Many of these models are artificial and correspond to non hermitian anisotropic long range interactions. Some of the resulting models will be short ranged and of real hermitian form. We will exactly solve some special simple two dimensional anisotropic XY models. These models may serve as simple caricatures for directed orbital models for transition metal oxides (e.g. the manganates). These models possess a symmetry which is intermediate between that of global and local theories. Similar models also occur naturally in the low energy regime of frustrated pyrochlore and checkerboard antiferromagnets.

We will then extend these techniques to the quantum case and discuss “2+1 dimensional Bethe Ansatz” solutions in section [14] where we will also solve a special two dimensional anisotropic XY model.

Next, we show in Section [15] that the trivial dimensional reduction of a 1+1 dimensional gauge theory to a one dimensional spin chain can be thought of as a trivial example of our general mode for dimensional reductions.

After solving these models, in subsection [15A] we will apply the high temperature expansion to reduce the three dimensional nearest neighbor Ising ferromagnet into an illusory formal solvable expression written in a determinant notation. In this method, the high temperature coefficient to each order is arrived at by looking a single linear relation originating from a matrix of (super exponentially) increasing order.

In subsection [15B], we will map the nearest neighbor $d$ dimensional Ising models to a one dimensional model whose spectrum is the sum of $d$ mutually incommensurate tight binding like spectra. This in turn will enable us to map the $d$ dimensional Ising model onto a a single spin 1/2 quantum particle in 0+1 dimensions.

In section [16], we will discuss a “permutational” symmetry present in the $O(n \rightarrow \infty)$ models but violated to $O(\beta^4)$ in finite $n$ problem. This will allow us to map any large $n$ problem onto a one dimensional lattice system.

One of the main motifs of this article is the absence or presence of “momentum interference” effects in high and low dimensional problems respectively. Another recur-
ring motif is that of a novel symmetry which interpolates between global and local symmetry and which effectively lowers the dimensionality of the problem.

II. A MOMENTUM SPACE HIGH TEMPERATURE EXPANSION

We will for the most part examine simple classical spin models of the type

\[ H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} \hat{V}(\vec{x}, \vec{y})[\hat{S}(\vec{x}) \cdot \hat{S}(\vec{y})]. \] (1)

Here, the sites \( \vec{x} \) and \( \vec{y} \) lie on a (generally hypercubic) lattice of size \( N \). The spins \( \{S(\vec{x})\} \) are normalized and have \( n \) components. The kernel \( \hat{V}(\vec{x}, \vec{y}) \) is translation invariant. The \( n = 1 \) case simply corresponds to the Ising model.

Later on, when we will discuss quantum models we will replace the scalar product by a slightly more complicated product.

The Hamiltonian in the (non-symmetrical) Fourier basis is diagonal \( f(\vec{k}) = \sum_{\vec{x}} F(\vec{x}) e^{-i\vec{k} \cdot \vec{x}} \); \( F(\vec{x}) = \frac{1}{N} \sum_{\vec{k}} f(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \) and reads

\[ H = \frac{1}{2N} \sum_{\vec{k}} \hat{v}(\vec{k}) |s(\vec{k})|^{2}, \] (2)

where \( \hat{v}(\vec{k}) \) and \( s(\vec{k}) \) are the Fourier transforms of \( \hat{V}(\vec{x}) \) and \( \hat{S}(\vec{x}) \) respectively.

For simplicity, we will set the lattice constant to unity-i.e., on a hypercubic lattice (of side \( L \)) with periodic boundary conditions (which we will assume throughout) the wave-vector components \( k_{i} = \frac{2\pi r_{i}}{L} \) where \( r_{i} \) is an integer (and the real space coordinates \( x_{i} \) are integers).

In the up and coming \( V \equiv \beta \hat{V} \). For an invertible \( \hat{V}(\vec{x}, \vec{y}) \), and an inverse temperature \( 0 < \beta < \infty \) (via a Hubbard-Stratonovich transformation [4], see also Appendix A), the partition function:

\[ Z = \text{Tr} \exp \left\{ -\frac{1}{2} \sum_{\vec{x}, \vec{y}} V(\vec{x}, \vec{y}) S(\vec{x}) S(\vec{y}) \right\} \]

\[ = \sqrt{\text{det}(-\frac{2V}{\pi})} \int \prod_{\vec{x}} d\eta(\vec{x}) \exp[-\hat{H}^{\text{dual}}\{\eta(\vec{x})\}]] \] (3)

where

\[ \hat{H}^{\text{dual}}\{\eta(\vec{x})\} = -\frac{1}{2} \sum_{\vec{x}, \vec{y}} \eta(\vec{x}) V^{-1}(\vec{x}, \vec{y}) \eta(\vec{y}) \]

\[ - \sum_{\vec{x}} \ln \cosh(\eta(\vec{x})) \] (4)

and analogously for the \( O(n > 1) \) model

\[ \hat{H}^{\text{dual}}\{\eta(\vec{x})\} = \frac{1}{2} \sum_{\vec{x}, \vec{y}} V^{-1}(\vec{x}, \vec{y}) \eta(\vec{x}) \cdot \eta(\vec{y}) \]

\[ - \sum_{\vec{x}} \ln(\sqrt{2/|\eta(\vec{x})|})^{n/2-1} I_{n/2-1}(|\eta(\vec{x})|) | \] (5)

with \( I_{n/2-1}(z) \) a Bessel function.

The kernel within the dual Hamiltonian \( \hat{H}^{\text{dual}} \) is the inverse of the kernel \( V \) appearing in the original Hamiltonian. It is Coulomb like \( \delta(\vec{x} - \vec{y}, l) \) for the nearest neighbor kernel \( V(\vec{x}, \vec{y}) = -\delta(\vec{x} - \vec{y}, l) \).

The \( \cosh \) and Bessel function terms appearing in Eqs.(4) and (5) may be viewed as terms of constraint securing the normalization of \( \hat{S}(\vec{x}) \) at every site \( \vec{x} \). The rotational invariance of the scalar product in Eqn.(3) is manifested by the rotational invariance \((|\eta| \text{ dependence}) \) of the Bessel function terms.

If a magnetic field \( h(\vec{x}) \) were applied, the argument of the \( \cosh \) would be replaced \( \cosh(\eta + h) \) with a similar occurrence \((|\eta| \rightarrow |\eta + h|) \) for the Bessel function appearing for the \( O(n > 1) \) models.

The \( q \)-state Potts model can be viewed as a spin model in which the \( q \) possible polarizations of the spins lie at the vertices of a \((q - 1)\) dimensional tetrahedron-in this manner the scalar product amongst any two non identical spins is \( \cos^{-1}(-(1/(q - 1))) \). Employing this representation, the sum \( \sum_{\vec{x}} \ln \cosh(\eta(\vec{x}) \) may be replaced by \( \sum_{\vec{x}} \ln \sum_{i=1}^{q} \exp[\hat{S}_i \cdot \eta] \) where the sum is over the \( q \) polarizations of the \((q - 1)\) dimensional spin \( \hat{S}_i \).

For a \( q \)-state clock model, the Bessel functions are similarly replaced by \( \sum_{\vec{x}} \ln \sum_{i=1}^{q} \exp[\hat{S}_i \cdot \eta] \). For the 4-state clock model which we will consider later on this is \( \sum_{\vec{x}} \ln[2(\cosh(\eta_1 + \cosh(\eta_2))] \). We will introduce, in section(V), in the context of the directed electronic orbitals appearing in the manganates, “directed clock-models”. These models share the same generating function \( \sum_{\vec{x}} \ln \sum_{i=1}^{q} \exp[\hat{S}_i \cdot \eta] \) albeit with a polarization \((\alpha)\) dependent kernel \( V_{\alpha,\beta}(\vec{x}, \vec{y}) \).

The partition function of the Ising spins reads

\[ Z = \sqrt{\text{det}(-\frac{2V}{\pi})} \int D\eta e^{-\beta H} \]

\[ = \sqrt{\text{det}(-\frac{2V}{\pi})} \int \prod_{\vec{x}} d\eta(\vec{x}) \]

\[ \exp \left[ -\frac{1}{2} \sum_{\vec{x}, \vec{y}} \eta(\vec{x}) V^{-1}(\vec{x}, \vec{y}) \eta(\vec{y}) \right] \]

\[ \prod_{\vec{x}} \cosh(\eta(\vec{x})). \] (6)

For a translationally invariant interaction \( V(\vec{x}, \vec{y}) = V(\vec{x} - \vec{y}) \), this may be written in momentum space
For asymptotically large shift in the on site interaction (or chemical potential) in real space such a constant shift amounts to a trivial property of the series become much more transparent. However, as we will see, in momentum space some place. Thus the resultant series is an expansion in momentum space formulation was rederived in the first inversion function is, trivially, where the sum is over all lattice sites \( \vec{x} \). Thus the partition function is, trivially,

\[
Z = N \sum_{\vec{x}, \ldots, \vec{y}} \prod_{i=1}^{N} \frac{1}{(2m_i)!} \eta^{2m_i}(\vec{x}_i)_{0}
\]

(9)

where \( \langle \ \rangle_0 \) denotes an average with respect to the unperturbed Gaussian weight

\[
\exp \left[ -\frac{1}{2N} \sum_{\vec{k}} v^{-1}(\vec{k}) \eta(\vec{k}) \eta(-\vec{k}) \right],
\]

(10)

and \( N \) is a normalization constant.

Each contraction \( \langle \eta(\vec{x}) \eta(\vec{y}) \rangle \) (or \( \langle \eta(\vec{k}) \eta(-\vec{k}) \rangle \)) leads to a factor of \( V(\vec{x}, \vec{y}) = \beta \tilde{V}(\vec{x}, \vec{y}) \) (or to \( \beta \tilde{v}(\vec{k}) / N \) in momentum space). Thus the resultant series is an expansion in the inverse temperature \( \beta \). In the up and coming we will focus attention on the momentum space formulation of this series. All the momentum space algebra presented above is only a slight modification to the well known high temperature series. All the momentum space algebra presented above is only a slight modification to the well known high temperature expansions usually generated by the Hubbard Stratonovich transformation directly applied to the real space representation of the fields \( \eta(\vec{x}) \). It is due to the naiveity of the author that such a redundant momentum space formulation was rederived in the first place. However, as we will see, in momentum space some properties of the series become much more transparent. To make \( V \) invertible and the expansion convergent, we will shift \( \hat{v}(\vec{k}) \) by a constant

\[
\hat{v}(\vec{k}) \to \hat{v}(\vec{k}) + A
\]

(11)

with \( A = \text{const} > 2d \) such that \( \hat{v}(\vec{k}) \) is strictly positive. In real space such a constant shift amounts to a trivial shift in the on site interaction (or chemical potential)

\[
\hat{V}(\vec{x}, \vec{y}) \to \hat{V}(\vec{x}, \vec{y}) + A. \delta_{\vec{x}, \vec{y}}
\]

(12)

For asymptotically large \( |\eta| \) the nontrivial

\[
\hat{H}_0^\text{dual} = -\sum_{\vec{x}} \ln |\cosh \eta(\vec{x})|
\]

(13)

is linear in \( |\eta| \). The Gaussian generating

\[
\tilde{H}_0^\text{dual} = -\frac{1}{2} \sum_{\vec{x}, \vec{y}} \eta(\vec{x}) V^{-1}(\vec{x}, \vec{y}) \eta(\vec{y})
\]

(14)

is a positive quadratic form and dominates at large \( \eta \). Note that the convergence of this series in Eqn.(8) is better than that appearing in the canonical field theories where the perturbing \( H_1 \) dominates over \( H_0 \) for large fields and, as a consequence, all canonical expansions are blatantly asymptotic.

So far all of this has been very general. Now let us consider the \( d \) dimensional nearest neighbor Ising model. To avoid carrying minus signs around, we will consider an Ising antiferromagnet whose partition function is, classically, identically equal to that of a ferromagnet. The exchange constant \( J \) will be set to unity. The corresponding momentum space kernel

\[
\hat{v}(\vec{k}) = A + \sum_{i=1}^{d} [e^{i \vec{k}_i} + e^{-i \vec{k}_i}].
\]

(15)

The resulting high temperature series (\( \ln Z = -N \sum_{p} f_p \beta^p \)) must be identical to the one derived by conventional real space methods which is convergent for all \( \beta < \beta_c \) (the inverse critical temperature): we reiterate that this convergence proves that the series we obtain is not asymptotic and free from many of the pathologies of common diagrammatic expansions.

As the reader might guess, the uniform shift \( A \) only leads to a trivial change in \( Z \) (i.e. to a multiplication by the constant exp\[\{-\beta A N\}] in the forthcoming we will set \( A = 0 \). Owing to the relation

\[
\int_{-\pi}^{\pi} dk \ e^{ikN'} = 2\pi \delta_{N',0}
\]

(16)

all loop integrals are trivial.

All stated here and below also holds exactly for finite size (of sides \( L > 1 \)) systems with periodic boundary conditions. Here

\[
\sum_{k_n} \exp[ik_n N'] = L \delta_{N',0}
\]

(17)

replaces Eqn.(13). In Eqn.(17) the sum is over all \( k_n = \frac{2\pi n}{L} \) with \( -\pi < k_n \leq \pi \). Note, however, that Eqn.(17) is far more general and holds for a large class of finite size systems.

In the thermodynamic limit, a simple integral is of the type

\[
\int_{\sum_{i=1}^{d} \tilde{k}_i = 0}^{} \prod_{i=1}^{M} \frac{d^d \tilde{k}_i}{(2\pi)^d} \prod_{i=1}^{N} \tilde{v}^n(\tilde{k}_i)
\]

\[
= \sum_{n=\sum_{i=1}^{d} (\gamma_i + \delta_i)} \left[ \frac{n!}{\prod_{l=1}^{d} \gamma_l! \delta_l!} \right]^M
\]

(18)
(when \( n = 1 \) this integral is \( 2d \) etc). A related integral reads
\[
\int \frac{d^dk}{(2\pi)^d} \prod_{i=1}^{\ell} \delta(\vec{k}_i) \prod_{i=1}^{M} \left[ [\hat{v}(\vec{k}_i)]^{n_i} \right] = \sum_{\gamma_i, \delta_i} \prod_{i=1}^{M} \frac{\gamma_i! \delta_i!}{\prod_{i=1}^{M} \gamma_i! \delta_i!} \tag{19}
\]
where in the last sum \( M \geq n \). Note that the high temperature expansion becomes simpler if instead of symmetrizing the interaction (each bond being counted twice-twice once by each of the two interacting spins) one considers
\[
\hat{v}(\vec{k}) = 2 \sum_{l=1}^{d} \exp[i\vec{k}l]. \tag{20}
\]
Here each spin interacts with only \( d \) (and not \( 2d \)) of its neighbors separated from it by one positive distance along the \( d \) axes \( \hat{e}^d \). The factor of 2 originates as each bond is now counted only once and therefore the corresponding bond strength is doubled. The partition function \( Z \) is trivially unchanged.

In the most general diagram the propagator momenta \( \{q_a\} \) (\( 1 \leq a \leq \) number of propagators) are linear combinations of the independent loop momenta \( \{\vec{k}_b\}_{b=1}^{\text{loops}} \)
\[
q_a = M_{ab} \vec{k}_b \tag{21}
\]
with integer coefficients \( M_{ab} \).

Symmetry factors aside, the value of a given diagram reads
\[
\int \prod_b \frac{d^dk_b}{(2\pi)^d} \int_a v(\sum M_{ab} \vec{k}_b) = \int \prod_b \frac{d^dk_b}{(2\pi)^d} \prod_a \left[ (2\beta)^d \exp[i \sum_b M_{ab} \vec{k}_b] \right] = (2\beta)^d \text{propagators}
\]
\[
\times \left( \int \prod_{b=1}^{\text{loops}} \frac{d^dk_b}{(2\pi)^d} \prod_{l=1}^{d} \exp[i \sum_{a,b} M_{ab} \vec{k}_b] \right) \tag{22}
\]
- i.e. upon expansion of the outer sum, just a product of individual loop integrals of the type encountered in Eqn.\( (16) \) for each \( \vec{k}_b \).

The latter integral in \( d = 1 \) reads
\[
I = \prod_{b=1}^{\text{loops}} \delta(\sum_{a=1}^{M} M_{ab}, 0) \tag{23}
\]
with a Kronecker delta. The situation in \( d > 1 \) becomes far richer. One may partition the integral into all possible subproducts of momenta corresponding to different spatial components \( l \) satisfying the delta function constraints. In a given diagram not all propagators need to correspond to the same spatial component \( \vec{k}^d \). The combinatorics of "in how many ways loops may be chosen to satisfy the latter delta function constraints?" boils down to the standard counting of closed loops in real space. Explicitly, in general \( d \),
\[
I = \sum_{\text{all partitions of } \{q_a\} \text{ into } d \text{ sets } C_l} \prod_{l=1}^{d} \prod_{b=1}^{\text{loops}} \delta(\sum_{a \in C_l} M_{ab}, 0). \tag{24}
\]

Note that entirely identical expressions would be reached for \( O(n > 1) \) models- the diagrams to be drawn and their integral expressions \( s \) are exactly the same as for the Ising system. The sole difference is that in the \( O(n > 1) \) case, the vertices of various orders (given by the Taylor expansion of the Bessel functions) will have different weights.

We may imagine coloring, for each individual term in Eqn.\( (24) \), the different sets \( C_l \) of propagator lines \( \{q_a\} \) corresponding to different dimensions by different colors to produce a graph colored with \( d \) colors. It is amusing to note that, by the four-color theorem, the most general topology of planar diagrams is present at \( d = 4 \) dimensions.

Some simple technical details of lattice perturbation theory vis a vis the standard continuum theories are presented in [14].

### III. An Exact Solution to a Special Three Dimensional Ising Model

If the spin spin interaction on a cubic lattice has a non hermitian kernel of the form
\[
\hat{V}(x_1, x_2, x_3) = -\delta_{x_2,x_0} \delta_{x_3,x_0} (\delta_{x_1,1} + \delta_{x_1,-1}) - \frac{1}{\pi} (-1)^x \sinh \frac{\pi \lambda}{\lambda^2 + x_3^2} \sin \delta x_1 \frac{\lambda (\delta_{x_2,1} + \delta_{x_2,-1})}{2} + i x_3 (\delta_{x_2,1} - \delta_{x_2,-1}) \tag{25}
\]
then, for all real \( \lambda \), the Helmholtz free energy per spin \( f(h = 0, \beta > \ln(1 + \sqrt{2})/2) \) is
\[
\beta f = - \ln(2 \cosh 2\beta) - \frac{1}{2\pi} \int_0^\pi d\phi \ln \frac{1}{2}(1 + \sqrt{1 - \kappa^2 \sin^2 \phi}) \tag{26}
\]
where
\[
\kappa = \frac{2 \sinh 2\beta}{\cosh^2 2\beta} \tag{27}
\]
and the internal energy per spin

\[ u = - \coth 2\beta \left[ 1 + \frac{2}{\pi} \kappa' K_1(\kappa) \right] \quad (28) \]

where \( K_1(\kappa) \) is a complete elliptic integral of the first kind

\[ K_1(\kappa) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}} \quad (29) \]

and

\[ \kappa' = 2 \tanh^2(2\beta) - 1. \quad (30) \]

The proof is quite simple. The expressions presented are the corresponding intensive quantities for the two dimensional nearest neighbor Ising model \([1]\).

Let us start by writing down the diagrammatic expansion (absolutely convergent for \( T > T_c \)) for the two dimensional Ising model (where \( T_c = 2/\ln(1 + \sqrt{2}) \)) and set

\[ \hat{v}(q_1, q_2) \rightarrow \hat{v}(q_1, q_2 + i\lambda q_3) \quad (31) \]

in all integrals. This effects

\[ \int_{-\pi}^{\pi} dk_2 e^{ik_2 N^{\text{loop}}} \rightarrow \int_{-\pi}^{\pi} dk_2 e^{ik_2 N^{\text{loop}}} \int_{-\pi}^{\pi} dk_3 e^{-\lambda k_3 L^{\text{loop}}} \quad (32) \]

for all individual loop integrals. The integrals are non-vanishing only if \( \{ N_b = 0 \} \) for all loops \( b \). When \( N^{\text{loop}} = 0 \) for a given loop then the second \( (k_3) \) integration leads to a trivial multiplicative constant (by one). In the original summation

\[ \sum_{k_b} \rightarrow \frac{L^d}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} d^dk_b \quad (33) \]

and thus an additional factor of \( L \) is introduced for each independent connected diagram. Thus the free energy per unit volume for a system with the momentum space kernel \( \hat{v}(k_1, k_2 + i\lambda k_3) \) is the same as for the two dimensional nearest neighbor ferromagnet with the kernel \( \hat{v}(k_1, k_2) \). Fourier transforming \( \hat{v}(k_1, k_2 + i\lambda k_3) \) and symmetrizing \( [V(\vec{x}) + V(-\vec{x})]/2 \rightarrow V(\vec{x}) \) one finds the complex real space kernel \( V(x_1, x_2, x_3) \) presented above \([16]\).

For the more general

\[ \hat{V}(x_1, x_2) = -\delta_{x_2, 0} \delta_{x_3, 0} (\delta_{x_1, 1} + \delta_{x_1, -1}) \]

\[ -\mathcal{J} \delta_{x_1, 0} \left[ \frac{(1)^{x_3}}{\lambda^2 + x_3^2} \right] \left\{ (\delta_{x_2, 1} + \delta_{x_2, -1}) + i(\delta_{x_2, -1} - \delta_{x_2, 1}) \frac{x_3}{\lambda} \right\} \quad (34) \]

we may define

\[ r \equiv \frac{\pi \mathcal{J}}{2\lambda \sinh \pi \lambda} \quad (35) \]

to write the free energy as the free energy of an anisotropic two dimensional nearest neighbor ferromagnet with the kernel

\[ \hat{V}(x_1, x_2) = -\delta_{x_2, 0} (\delta_{x_1, 1} + \delta_{x_1, -1}) \]

\[ -r \delta_{x_1, 0} (\delta_{x_2, 1} + \delta_{x_2, -1}) \quad (36) \]

i.e. a nearest neighbor Ising ferromagnet having the ratio of the exchange constants along the \( x_1 \) and \( x_2 \) axes equal to \( r = J_2/J_1 \) (throughout we set \( J_1 \) to unity). Introducing

\[ u = \frac{1}{\sinh 2\beta \sinh 2\beta r} \quad (37) \]

and defining

\[ F(\theta) \equiv \ln \{ 2[\cosh 2\beta \cosh 2\beta r + u^{-1} (1 - 2u \cos \theta + u^2)^{1/2}] \}, \quad (38) \]

we may write the free energy density as

\[ \beta f = \frac{1}{2\pi} \int_0^\pi F(\theta) \, d\theta. \quad (39) \]

Note that the same result applies for the “ferromagnetic”

\[ \hat{V}(x_1, x_2, x_3) = -\delta_{x_2, 0} \delta_{x_3, 0} (\delta_{x_1, 1} + \delta_{x_1, -1}) \]

\[ -\mathcal{J} \delta_{x_1, 0} \left[ \frac{1}{\lambda^2 + x_3^2} \right] \left\{ (\delta_{x_2, 1} + \delta_{x_2, -1}) + i(\delta_{x_2, -1} - \delta_{x_2, 1}) \frac{x_3}{\lambda} \right\} \quad (40) \]

The proof is simple- the partition function is identically the same if evaluated with the kernel in Eqn.\( 40 \) instead of that in Eqn.\( 36 \) if the spins are flipped

\[ S(\vec{x}) \rightarrow (-1)^{x_3} S(\vec{x}). \quad (41) \]

What about correlation functions? The equivalence of the partition functions \( Z[\beta, h(x_1, x_2)] \) implies that in the planar directions the correlations are the same as for two dimensional ferromagnet.

Changing the single momentum coordinate

\[ v(k_1, k_2) \rightarrow v(k_1, k_2 + i\lambda k_3) \]

effects

\[ G(k_1, k_2) \rightarrow G(k_1, k_2 + i\lambda k_3) \quad (42) \]

in the momentum space correlation function. The algebra is identically the same.
The connected spatial correlations (for $T \neq T_c$) are exponentially damped along the $x_2$ axis as they are in the usual two dimensional nearest neighbor ferromagnet (see (33) for the two dimensional correlation function).

Of course, all of this can also be extended to one-dimensional like spin systems.

The two dimensional spin-spin kernel

\[
\hat{V}(x_1, x_2) = -\frac{1}{\pi} (-1)^{x_2} \frac{\sin \pi \lambda}{\lambda^2 + x_2^2} \delta_{x_1,0} \left[ \lambda (\delta_{x_2,1} + \delta_{x_2,-1}) + i x_2 (\delta_{x_2,1} - \delta_{x_2,-1}) \right]
\]

leads to one dimensional behavior with oscillations along the $x_2$ axis. More precisely, the momentum space correlator for any $O(n)$ system in one dimension reads

\[
G(k) = \frac{e^{r} e^{i r \vec{k} \cdot \vec{x}}}{e^{r} - e^{i r \vec{k} \cdot \vec{x}}} + \frac{1}{e^{i k r} - 1}.
\]

For the Ising ($n = 1$) system $r \equiv -[\ln \tanh \beta]$. One may set $k = k_1 + i \lambda k_2$ and compute the inverse Fourier transform. The correlations along $x_2$ are oscillatory with temperature dependent wavevectors. In other words, the effective correlation length along $x_2$ is infinite.

If the two dimensional spin-spin interactions in Eqn.(33) are augmented by an additional on-site magnetic field $h$ then the two dimensional partition function reads

\[
Z = \lambda_+^N \lambda^-^N
\]

where, as usual,

\[
\lambda_{\pm} = e^{\beta} \cosh \beta h \pm \sqrt{e^{2\beta} \sinh^2 \beta h + e^{-2\beta}}.
\]

That “complexifying” the coordinates should not change the physics is intuitively obvious: if one shifts $k_1 \rightarrow k_1 + \lambda k_2 \equiv k'_1$ with real $\lambda$ in the one dimensional $\hat{v}(k_1)$ then the resulting kernel $\hat{v}(k'_1)$ plainly describes a stack on chains parallel to $(1, \lambda)$; the spins interact along the chain direction yet the chains do not interact amongst themselves. The free energy density should be identically equal to that of a one dimensional system- no possible dependence on $\lambda$ can occur. All that we have done in the above is allow $\lambda$ to become complex.

In

\[
\hat{v}(k_1, k_2) \rightarrow \hat{v}(k_1, k_2 + \lambda k_3)
\]

the correlation functions along a direction orthogonal to the direction $(1, \lambda)$ in the $(x_2, x_3)$ plane vanish. If $\lambda$ is complex then there are no correlations along an orthogonal direction in the “complex” space.

Formally, all this stems from the trivial fact that the measure $dk_1dk_2 = const(dkdk^*)$, whereas $\hat{v}(k)$ depends only on the single complex coordinate $k$.

Other trivial generalizations are possible. For example, we may make both $k_1$ and $k_2$ complex in $\hat{v}(k_1, k_2)$ to generate a four dimensional spin kernel

\[
\hat{V}(x_1, x_2, x_3, x_4) = -\frac{1}{\pi} (-1)^{x_2} \frac{\sin \pi \lambda_1}{\lambda_1^2 + x_2^2} \delta_{x_1,0} \left[ \lambda_2 (\delta_{x_2,1} + \delta_{x_2,-1}) + ix_2 (\delta_{x_2,1} - \delta_{x_2,-1}) \right]
\]

which leads to the canonical (nearest neighbor) two dimensional behavior for arbitrary $\lambda_1$ and $\lambda_2$.

All of this need not be restricted to cubic lattice models. We may also analytically continue $\hat{v}(k)$ of the triangular lattice etc.

For the triangular antiferromagnet/ferromagnet

\[
\hat{v}(k) = \pm 2 [\cos k_1 + \cos \left( \frac{k_1}{2} + \frac{\sqrt{3}}{2} k_2 \right)
\]

\[
+ \cos \left( -\frac{k_1}{2} + \frac{\sqrt{3}}{2} k_2 \right)].
\]

Making $k_2$ complex leads to interactions on a layered triangular lattice etc.

**IV. HIGHER $O(N > 1)$ MODELS AND SYMMETRIES INTERMEDIATE BETWEEN GLOBAL AND LOCAL**

The multi-component $O(n > 1)$ spin models display a far richer variety of possible higher dimensional extensions.

The constant $\lambda$ need not be only a complex number, it may also be quaternion or correspond to more general set of matrices.

Let us examine the extension to “quaternion” $\vec{k}$ (or with the matrices $i\sigma_1, i\sigma_2$, and $i\sigma_3$ (with $\{\sigma_n\}$ the Pauli matrices) taking on the role of $i, j$ and $k$). If we were dealing with an $O(2)$ (or XY) model, the same high temperature expansion could be reproduced.

We may envisage a trivial extension to the usual scalar product Hamiltonian

\[
H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} \hat{V}(\vec{x} - \vec{y})\vec{S}(\vec{x}) \cdot \vec{S}(\vec{y})
\]

to one in which the kernel is no longer diagonal in the internal spin indices $\alpha, \beta = 1, 2$.

\[
H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} \hat{V}_{\alpha, \beta}(\vec{x} - \vec{y})\vec{S}_\alpha(\vec{x}) \vec{S}_\beta(\vec{y}).
\]
The Hubbard–Stratonovich transformation for $Z = Tr\{\exp[-\beta H]\}$ with the bilinear $H = \frac{1}{2} \sum_{\kappa} S_\kappa V_{\kappa\rho} S_\rho$ proceeds as before. The super-index $\kappa = (\vec{x}, \alpha)$ now labels both the physical coordinate $\vec{x}$ and the polarization $\alpha$. And, just as before, the trace of $\exp[i S_\kappa \eta_b]$ over all $\{S_\kappa\}$ is $\prod_{\vec{x}} \int_{\eta_b = -1}^{\eta_b = 1} (|\eta_b(\vec{x})|)$. The high temperature expansion of section (II) may be reproduced word for word with the kernel $V_{\alpha\beta}(\vec{x} - \vec{y})$ now generated by each contraction of $\eta_\alpha(\vec{x})$ and $\eta_\beta(\vec{y})$, or in momentum space $v_{\alpha\beta}(\vec{k}) \delta_{\vec{E} + \vec{E}' = 0}$ is generated by the contraction of $\eta_\alpha(\vec{k})$ and $\eta_\beta(\vec{k}')$. In the evaluation of the free energy or the partition function, a given bubble diagram reads

$$\left( \int \prod_{b=1}^{\text{loops}} \frac{d^d k_b}{(2\pi)^d} \sum_{l=1}^{d} Tr\{\exp[i \sum_{a,b} M_{ab} k_l]\} \right)$$

(52)

where, as before, $a$ spans the number of propagators, $b$ spans the number of loops, and $k_l$ with $1 \leq l \leq d$ denotes the $l$-th component of the $d$-dimensional wavevector within the $b$-th loop $k_b$.

Let us start off with a nearest neighbor one dimensional XY chain and consider the transformation

$$\exp[i k_1] \rightarrow \exp[i (k_1 + i\lambda_1 k_2 \sigma_1 + i\lambda_2 k_3 \sigma_2 + i\lambda_3 k_4 \sigma_3)]$$

$$\equiv \exp[i k_1 - \vec{w} \cdot \vec{\sigma}].$$

(53)

In the argument of the exponential the identity matrix 1 commutes with $\vec{w} \cdot \vec{\sigma}$ and the exponential may be trivially expanded as

$$\exp[i k_1 - \vec{w} \cdot \vec{\sigma}] = \exp[i k_1] \exp[-\vec{w} \cdot \vec{\sigma}].$$

(54)

Once again in higher dimensions (this time $d = 4$) the $k_1$ integration will reproduce the familiar

$$\int_{-\pi}^{\pi} dk_1 \exp[i N' k_1] = 2\pi \delta_{N',0}$$

(55)

for each independent loop momentum $\vec{k}$ and the integration over the remaining $k_2, k_3,$ and $k_4$ components will simply yield multiplicative constants.

Thus the kernel

$$v(\vec{k}) = \exp[i k_1 - \vec{w} \cdot \vec{\sigma}] + h.c.$$ (56)

will generate a four dimensional XY model which has the partition function of a nearest neighbor one dimensional XY chain. The corresponding real space kernel

$$\hat{V}_{\alpha\beta}(x_1, x_2, x_3, x_4) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \exp[i \vec{k} \cdot \vec{x}] \left( \exp[i (k_1 + i\lambda_1 k_2 \sigma_1 + i\lambda_2 k_3 \sigma_2 + i\lambda_3 k_4 \sigma_3)]_{\alpha\beta} + h.c. \right).$$

In general, this kernel is no longer diagonal in the internal spin coordinates (unless $\vec{w}$ happens to be oriented along $\sigma_3$).

$$|\vec{w}| = \sum_{i=2}^{4} \lambda_i^2 - k_i^2$$

$$\hat{w} = \left( \frac{\lambda_1 k_2}{|\vec{w}|}, \frac{\lambda_2 k_3}{|\vec{w}|}, \frac{\lambda_3 k_4}{|\vec{w}|} \right)$$

(57)

and

$$\exp[-\vec{\sigma} \cdot \vec{w}] = \cosh(|\vec{w}|) - \sinh(|\vec{w}|)(\vec{\sigma} \cdot \hat{w})$$

(58)

For general $\{\lambda_i\}_{i=1}^{3}$ the inverse Fourier transform is non-trivial. The scalar product is replaced by a more complicated product amongst the components. The real space interaction kernel is once again algebraically long ranged along the $x_2, x_2$ and $x_4$ axes.

Just as quaternions (or Pauli matrices $\{\sigma_i\}$) may be employed we may also consider the transformation

$$k_1 \rightarrow k_1 + k_{\mu+2} \gamma^\mu$$

(59)

with the Dirac matrices $\{\gamma^\mu\}$, with its associated propagator $\exp[i (k_1 + k_{\mu+2} \gamma^\mu)]$ to produce solvable models in $2 \leq d \leq 5$ dimensions (depending on how many momentum components are contracted with the gamma matrices). The partition function of these models is identically that of a nearest neighbor one dimensional system. An infinite number of variations along this theme are possible.

In this manner we may generate a multitude of solvable, slightly more realistic, models with a real hermitian kernel.

The hermitian kernel

$$\hat{v}(\vec{k}) = -2 \cos(k_1 + k_2 \sigma_3)$$

(60)

leads to the partition function of a nearest neighbor one dimensional $O(2)$ chain.

The proof amounts to a reproduction of the calculations given above for the four dimensional $O(2)$ system.

In real space the latter kernel (Eqn.(52)) leads to the $O(2)$ Hamiltonian

$$H = \sum_{(ij) \text{ along } (1,1)} S_i^{(1)} S_j^{(1)}$$

$$- \sum_{(ij) \text{ along } (1,-1)} S_i^{(2)} S_j^{(2)}$$

(61)

where the $n = 1$ (or $x$ component) of the spins interact in the first term and only the $y$-components of the $O(2)$ spins appear in the second term; the indices $i$ and $j$ are the two dimensional square lattice coordinates. The first term corresponds to interactions along the $(1,1)$ direction in the plane and the second term corresponds to interactions
along the $(1,-1)$ diagonal. The two spin components satisfy $[S_i^{(1)}]^2 + [S_i^{(2)}]^2 = 1$ i.e. are normalized at every site $i$. By our mapping, this model trivially has the partition function of a one dimensional nearest neighbor $O(2)$ spin chain. Thus its free energy per site $f = -\beta^{-1} \ln[I_0(\beta)]$ where $I_0$ denotes the associated Bessel function.

It amusing to trivially note by inspection that this explicitly works for works the periodic $2 \times 2$ system on the torus where the two diagonals decouple and along each diagonal one has the interactions of the standard $O(2)$ symmetric one dimensional XY model.

This system is frustrated-glancing at the Hamiltonian of Eqn.(24) we note, that it is impossible to saturate the bonds associated with both the $x$ and $y$ polarizations of the spins simultaneously. The state in which the $x$ component interactions are minimized is geometrically orthogonal to one in which the $y$-component interactions are minimized. In fact, we just saw that the frustration is so large that it thwarts the generic low temperature two dimensional quasi (algebraic) long range order and leads to a canonical one dimensional like behavior instead.

We may also generate interactions along arbitrary tilted rays by considering
\[
\hat{v}(\vec{k}) = -2 \cos(k_1 + \lambda k_2 \sigma_3)
\]
(62) to show that the Hamiltonian
\[
H = - \sum_{\langle ij \rangle} S_i^{(1)} S_j^{(1)} + \sum_{\langle \langle ij \rangle \rangle} S_i^{(2)} S_j^{(2)}
\]
(63)
is equivalent to
\[
H = - \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j
\]
(64)
By tuning $\lambda \to 0$, the interactions along the two “clapping” directions $(1, \pm \lambda)$ fold back and degenerate into the original one dimensional, $O(2)$ symmetric, chain.

Similarly, by choosing the diagonal
\[
\hat{v}(\vec{k}) = -2 \cos(k_1 P_+ + k_2 P_-),
\]
(65)
with the projection operators $P_{\pm} \equiv \frac{1}{2}(1 \pm \sigma_3)$, we may prove that
\[
H = - \sum_{\langle \langle ij \rangle \rangle} S_i^{(1)} S_j^{(1)} + \sum_{\langle \langle ij \rangle \rangle} S_i^{(2)} S_j^{(2)}
\]
(66)
is equivalent to the one dimensional
\[
H = - \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j
\]
(67)
Here as all $\hat{v}$ matrices are diagonal in the internal spin indices, the traces of the two dimensional system (Eqn.(24)) are identically the same as those of the one dimensional system of Eqn.(27). Schematically, in the two dimensional system, for each given loop momenta $\vec{k}^b$ having the two spatial components $(k_1^b, k_2^b)$, the integrals are exactly the same as they are for the one dimensional case:
\[
Tr \int \frac{dk_1^b}{2\pi} \int \frac{dk_2^b}{2\pi} \left[ \begin{array}{cc} e^{ik_1^b N_1} & 0 \\ 0 & e^{ik_2^b N_1} \end{array} \right] ...
\]
\[
= Tr \int \frac{dk_1^b}{2\pi} \int \frac{dk_2^b}{2\pi} \left[ \begin{array}{cc} e^{ik_1^b N_1} & 0 \\ 0 & e^{ik_1^b N_1} \end{array} \right] ...
\]
After integrations, each one of the identical diagonal entries will be given by Eqns.(24). These are the exactly the same integrals that we have evaluated for the Ising case. As told, it is merely the weights of the vertices which change in the general $O(n)$ model. The dimensional reduction of Eqn.(64) is ambivalent to the special nearest neighbor short range (or even lattice character) of the problem. If the $x$-components of the spins interact with a kernel which is $x_1$ direction dependent and the $y$-components would interact with the same kernel along the $x_2$ direction then a dimensional reduction would be possible. In general, and in the continuum in particular, one may, of course, perform rotations within the external momentum space and internal spin space in order to examine dimensional reductions for all interaction directions.

As told, on the lattice, all of this need not apply to special short range interactions. More generally, we may replace $k_1 \to k_1 + k_2 \sigma_3$ within an arbitrary kernel
\[
\hat{v}(k_1) = -2 \sum_{r=1}^{R} v_r \cos(rk_1) 
\]
of spatial range $R \geq 1$. Employing Eqn.(24) once again we note that any XY spin system with a Hamiltonian
\[
H = - \sum_{\langle \langle ij \rangle \rangle} \tilde{V}_{ij} S_i^{(1)} S_j^{(1)} + \sum_{\langle \langle ij \rangle \rangle} \tilde{V}_{ij} S_i^{(2)} S_j^{(2)} 
\]
(69)
with a general translationally invariant $\tilde{V}_{ij} = f(|\vec{i} - \vec{j}|)$ of an arbitrary range $R$ (i.e. one which does not necessarily link only nearest neighbor diagonal sites $i$ and $j$) has exactly the same partition function of a one dimensional system whose Hamiltonian
\[
H = - \sum_{\langle ij \rangle} \tilde{V}_{ij} \vec{S}_i \cdot \vec{S}_j
\]
(70)
with the arbitrary range interaction kernel \( \hat{V}_{ij} \).

Similarly, the partition function of an XY system having the three dimensional kernel \( \hat{v}(\vec{k}) = -2 \cos k_1 - 2 \cos (k_2 + \gamma \sigma_3) \) is equal to that of the two dimensional nearest neighbor XY ferromagnet which exhibits Kosterlitz-Thouless like behavior at sufficiently large inverse temperature \( \beta \) (or temperatures \( T < T_{KT} \)). Here each of two spin components interacts within a different subplane. The coupling between the two spin components due to the normalization constraint apparently plays no role in leaving the system two dimensional.

In a similar manner, any \( d' \) dimensional spin system (at temperatures \( T > T_c \)) can be made have an effective dimensionality \( 1 \leq d \leq d' \) without the physical need of actual geometrical compactification.

The models above display special symmetries, intermediate between those of the standard two spin \( O(n) \) models which display a global group symmetry and lattice gauge theories which a local group symmetry at every lattice site. Note that the transformation

\[
\begin{align*}
S_{\alpha=1}^{(x_1,x_2)} &\rightarrow \eta_{x_1} S_{\alpha=1}^{(x_1,x_2)} \\
S_{\alpha=2}^{(x_1,x_2)} &\rightarrow \eta_{x_2} S_{\alpha=2}^{(x_1,x_2)}
\end{align*}
\]

with the \( (2L) \) arbitrary, “gauge”, degrees of freedom \( \eta_{x_1} = \pm 1 \) leaves Eqn. (70) invariant. Here \( (x_1, x_2) \) denote the two spatial components of the spin coordinate \( \vec{x} \). Note that by this symmetry, any particular state (including the ground state) is, at least, \( 2^{2L} \) degenerate—yet another manifestation of “frustration”. The effective gauge redundancy of Eqn. (71) is midway between a full blown gauge theory (where redundant gauge degrees of freedom live on every site) and a globally symmetric theory where only one global symmetry exists. The number of gauge degrees of freedom in Eqn. (70) is the same as that in a one dimensional \( Z_2 \) gauge system. In general, the manifold of the gauge degrees of freedom can assume any effective dimensionality in between the two standard extremes- \( d \) (gauge theories) and zero (globally symmetric spin models). The \( O(4) \) spin model

\[
H = - \sum_{(ij) \mathrm{along \ (1,1)}} \left( S_i^{(1)} S_j^{(1)} + S_i^{(3)} S_j^{(3)} \right) - \sum_{(ij) \mathrm{along \ (1,-1)}} \left( S_i^{(2)} S_j^{(2)} + S_i^{(4)} S_j^{(4)} \right)
\]

displays \( (2L) \) independent \( SO(2) \) (or \( U(1) \)) gauge degrees of freedom \( \eta \) residing on each individual diagonal. Explicitly

\[
\begin{align*}
S_{\alpha=1}^{(x_1,x_2)} + i S_{\alpha=3}^{(x_1,x_2)} &\rightarrow \eta_{d_1} S_{\alpha=1}^{(x_1,x_2)} + i \eta_{d_3} S_{\alpha=3}^{(x_1,x_2)} \\
S_{\alpha=2}^{(x_1,x_2)} + i S_{\alpha=4}^{(x_1,x_2)} &\rightarrow \eta_{d_2} S_{\alpha=2}^{(x_1,x_2)} + i \eta_{d_4} S_{\alpha=4}^{(x_1,x_2)}
\end{align*}
\]

with arbitrary phases \( \eta_{d_1} \) along the \( 2L \) diagonals \( d_{1,2} \), is a symmetry. By our mapping, this two dimensional system with \( \sim L \) redundant \( O(2) \) gauge degrees of freedom is equivalent to an isotropic, nearest neighbor, \( O(4) \) spin chain having only a single zero dimensional global symmetry. The variations are endless.

Although we have focused attention on equivalence to simple models with rotationally invariant scalar products (e.g. Eqn. (70)), much of what we said is also generally valid for non-isotropic actions \([17]\).

Finally, in order to set the stage for the discussion of a cartoon of directed electronic orbitals (e.g. \( p \) or \( d \) orbitals appearing in some systems (e.g. the manganates), we will now introduce “directed clock models”. These models are just the standard \( N \)-state clock models with the additional twist that an interaction (“hopping”) can occur only along an axis parallel to the spin direction (a cartoon for the direction along which the electronic orbital is extended). The high temperature diagrammatic expansion for these models may proceed as before, with the provision that the generating function (which merely sets the weights of all vertices) is no longer a rotationally invariant Bessel function but rather \( \sum_N \exp[\vec{S} \cdot \vec{n}] \). For the directed 4-state clock model, the four possible spin orientations are \((\pm 1,0)\) and \((0,\pm 1)\) and the Hamiltonian is none other than that of Eqn. (70). Our mapping will show that this is equivalent to the standard (undirected) one dimensional nearest neighbor 4-state clock model whose partition function is trivially twice of that the nearest neighbor one dimensional Ising model. The generating function is

\[
\prod_{\vec{x}} [2(\cosh \eta_1 + \cosh \eta_2)] \\
\prod_{\vec{x}} [2\cosh(\eta_{+})] \prod_{\vec{x}} [2\cosh(\eta_{-})],
\]

with \( \eta_{\pm} \equiv (\eta_1 \pm \eta_2)/2 \). The partition function is a direct product of the partition function terms for two independent Ising systems. This has a much clearer intuitive meaning (see Appendix B)—the direct product space of two Ising spins sitting at every site gives rise to two dimensional spins whose components are \((\pm 1, \pm 1)\). Upon rotation and normalization, this becomes the four state clock model just discussed.

In general, it is easy to see that the partition function of the \( (2^d) \) state directed clock model (in which each spin has \( d \) components) embedded in an external space of \( d \) dimensions is equivalent to that of the one dimensional Ising model (more precisely, \( d \) replicas of Ising systems).

We may similarly address [17] the “squared directed 4-state clock model”

\[
H = - \sum_{(ij) \mathrm{along \ (1,0)}} \left[ S_{i}^{(1)} S_{j}^{(1)} \right]^2.
\]
to show that its partition function is equivalent to that of a one-dimensional model with
\[ H = - \sum_{i} S_{i}^2 S_{i+1}^2 = \text{const} \ (= -N), \] (76)
which is trivial.

The reader is referred to Appendix B for a perspective on the conventional, non-directed, and directed clock models in \( n \geq 2 \) internal spin dimensions.

V. POTENTIAL PHYSICAL REALIZATIONS

Some frustrated systems might have realizations close to the special symmetry points at which we proved the occurrence of exact dimensional reductions.

A. The Manganates

Before giving a superficial flavor of one potential system where models and symmetries such as those introduced in the previous section might be of real practical relevance, let us quickly outline the logic. As seen in the central figure of Fig. (1), orbital wave functions can be cigar like and highly direction. In such a case, overlap integrals will be much larger for hopping or interactions along one axis than along another. Hopping and/or spin-orbit couplings might depend on the state of the orbital direction (“polarization”).

Polarization dependent interactions akin to those in Eqn. (61) may be of relevance in discussing directed orbital problems in two dimensions (e.g. those associated with the electronic orbitals in transition metal oxides of perovskite structure). The non-isotropic electronic orbitals lead to overlap terms which are strongly direction dependent. In these materials (e.g. \( \text{La}_{2-x}\text{Sr}_x\text{MnO}_3 \)), the surrounding oxygen atoms introduce crystal field splitting of the d orbitals of the caged transition metal (e.g. Mn). Wave functions pointing towards the oxygen ions may attain higher energy than those that point in between. The two “bad” higher energy d orbitals are \( |x^2 - y^2 \rangle \) and \( |3z^2 - r^2 \rangle \) (customarily denoted as \( e_g \) orbitals) are depicted in Fig. (1). The ground state is determined by Hund’s rule. A Mn\(^{3+} \) ion, for example, has a d\(^4 \) configuration and consequently, one electron populates on the states shown in Fig. (1). These states dominate the orbital physics.

The directed four or six state clock models may be regarded as oversimplified cartoons for the orbital lobes in systems such as these. The overlap terms allow hopping only in the direction lying along the axis of symmetry of the orbital. They are a very crude model for

\[ - \sum_{\langle ij \rangle \text{ along } (0,1)} [S_{i}^{(2)} S_{j}^{(2)}]^{2} \] (75)

where the hopping amplitudes \( \{ t_{ij} \} \) depend on internal indices indicating the relative orientation between the orbitals at sites \( i \) and \( j \). The hopping amplitudes between two neighboring Mn atoms depend on the overlap of the d-orbitals with the oxygen p-orbitals which lie in between.

Another origin of such a behavior may result from spin-orbit couplings.

B. Frustrations and Transitions within the checkerboard and Pyrochlore lattices

As demonstrated by E. Berg et al. [10], the S = 1/2 Heisenberg antiferromagnet on the frustrated checkerboard and pyrochlore lattices, can be related to models of the form similar to those that we have looked at till now.

For instance [10], in the low energy regime, the Heisenberg model on the checkerboard lattice shown in Fig. (1), is related to the Ising like model

\[ H = -J \sum_{\langle ij \rangle} (\hat{S}_{i} \cdot \hat{e}_{ij})(\hat{S}_{j} \cdot \hat{e}_{ij}) - h \sum_{i} S_{i}^z. \] (78)

Here the directors \( \hat{e}_{ij} \) may point along directions at angles \( \pi/3(-\pi/3) \) away from the x axis for horizontal (vertical) bonds respectively. The spins correspond to vertical or horizontal dimers [10]. Had the directors \( \hat{e}_{ij} \) been along orthogonal axis, we would just obtain a trivial counterpart of the Hamiltonian of Eqn. (11). At the orthogonal point (if the directors were orthogonal), the system is exactly one dimensional. Sans the applied magnetic field, the free energy density at this point is \( -\beta^{-1} \ln \cosh \beta \) times a combinatorial factor equal to the logarithm of the number of ways in which the plane may be tiled with horizontal and vertical lines such that each point belongs to exactly one line.

A deviation of \( \{ \hat{e}_{ij} \} \) from orthogonality, leads to the removal of the degeneracy associated with the symmetries of
Eqn. (70) and its likes, and the system is indeed expected to display canonical two dimensional behavior and have a transition at a finite temperature.

A similar model may be written for the pyrochlore lattice [13].

Thus, some geometrically frustrated models are close to yet removed from the special symmetry points where dimensional reduction would occur.

VI. SPECIAL TWO DIMENSIONAL BETHE ANSATZ SOLUTIONS

We may consider quantum (spin, fermionic, bosonic) extensions in which the unperturbed action $S_0$ links two nearest neighbor sites. By the Trotter formula, the partition function may be written as a lattice sum over a $(d+1)$ dimensional system of thickness $\beta$ (the inverse temperature) along the imaginary time axis (the $(d+1)$-th direction). The $(d+1)$-th momentum component $k_{d+1}$ is Matsubara frequency $\omega$ whose allowed values $\{\omega_n\}$ are dictated by the periodic/antiperiodic boundary conditions along the imaginary time axis that must be imposed for a bosonic/fermionic system respectively. The Green’s function conjugate to $S_0$ in the undualized (non Hubbard-Stratonovich transformed) system reads

$$G_0 = \left[ \beta \left( -2 \sum_{l=1}^{d+1} \cos k_l + A \right) \right]^{-1}. \quad (79)$$

If the magnitude of the squared mass $|m_0|^2 = |A| = const > 2(d+1)$ then a geometric series expansion in $A^{-1}$ may be performed to generate a sum of harmonics. Within each diagrammatic term, we may take $k_1 \rightarrow k_1 + i\lambda k_2$ or other similar extensions to create high dimensional problems that have the same partition function as that of a solvable canonical one dimensional system.

For a relativistic spin-1/2 system, we may employ the naive fermionic propagator

$$G_0 = [\beta(\gamma_{\mu} \sin k_\mu + m_0)]^{-1} \quad (80)$$

and once again expand the resulting diagram in powers of $m_0^{-1}$ to obtain a sum of harmonics. This holds for all canonical lattice theories in which the kernel are periodic functions of the momenta and thus, when well behaved, may be expressed as sums of harmonics.

Eqn. (65) forces all diagrams with substitutions such as $k_1 \rightarrow k_1 + i\lambda k_2$ to attain the same value as they would attain in the original system.

If an exact Bethe Ansatz solution is known for a fermionic one dimensional system with nearest neighbor hopping then in some cases it may be extended to two dimensions where the hopping matrix element $K(x_1,x_2)$ for separation $\vec{x} = (x_1,x_2)$ attains exactly the same form as the two-dimensional kernel $\hat{V}(x_1,x_2)$ just given previously. Loosely speaking, if in a fictitious electronic system the hopping matrix element would be the non hermitian

$$K(x_1,x_2) = -\frac{1}{\pi} \left( \frac{-1}{\lambda_1} \sin \frac{\pi \lambda_1}{\lambda_2^2 + \delta_{x_1,0}^2} \right) \left[ \lambda_1 (\delta_{x_2,1} + \delta_{x_2,-1}) + ix_3 (\delta_{x_2,1} - \delta_{x_2,-1}) \right] \quad (81)$$

then the system would essentially be a one dimensional Luttinger liquid along $x_1$. If the kernel would instead decay algebraically along the imaginary time axis the problem would become that of dissipating system.

A hermitian hopping matrix element could also do the trick if we were to consider polarization dependent hopping (or “interactions”) analogous to those in Eqn. (61) and slightly more complicated variants. As noted earlier, such models and the more general equivalence of the two dimensional system of Eqn. (54) with the one dimensional one of Eqn. (73) may be of relevance in discussing directed orbital problems in two dimensions in systems like the manganates [13].

VII. DIMENSIONAL REDUCTION IN LATTICE GAUGE THEORIES

Similarly, the standard lattice gauge theories may be mapped to higher dimensional lattice gauge theories. Let us further show also the opposite- the dimensional reduction that trivially occurs in going from a two dimensional gauge theory to a one dimensional spin chain is one special case in our general way of constructing models of different dimensions that are equivalent.

FIG. 2. The checkerboard lattice. The two singlet ground states of the crossed plaquettes may be represented by an Ising pseudospin lying along the horizontal axis. The effective interaction amongst these pseudo-spins is encapsulated by Eqn. (78).
A pure lattice gauge theories (for any group) may be mapped onto a Coulomb model with nontrivial onsite interactions. The advantage is that in the Coulomb gas all interactions are between pairs and our machinery for dimensional reduction can be brought to bear. To avoid carrying too many indices around we will now consider the Abelian \( Z_2 \) (Ising) group.

Let us consider the even sublattice sites
\[
\sum_{a=1}^{d} x_a \equiv 0 \pmod{2} \tag{82}
\]
in a hypercubic lattice, where \( x_a \) labels the \( a \)-th component of the position vector of a site on the lattice. On this sublattice dual fields \( \zeta_i \) are introduced.

Each bond variable \( U_{ij} \) is connected to one and exactly one lattice site (\( i \) or \( j \)) which belongs to the even sublattice (where the fields \( \zeta \) exist). The partition function may then be schematically written as
\[
Z = \int DU \exp[K \sum \sum UUUU] = \int DU \prod_{\alpha \beta \gamma \delta} \exp \left( \frac{1}{2} V_{\alpha \beta \gamma \delta} U_{\alpha \beta} U_{\gamma \delta} \right) \tag{83}
\]
where the kernel \( V_{\alpha \beta \gamma \delta} = K \delta_{|\alpha \beta - \gamma \delta|} \) where the Greek indices denote the bonds and \( \alpha \beta \) denotes the site common to bonds \( \alpha \) and \( \beta \). The distance between two sites was marked by \( |\alpha \beta - \gamma \delta| \). At each sublattice site \( i \) there are \( 2d(d-1) \) fields \( \{\zeta_i\} \). As each link \( U_{ij} \) has only a single end point \( i \) which lies in the even sublattice, only a dependence on one point \( \zeta_i \) will remain when integrating (or tracing) over the link variables that have one end point at site \( i \),
\[
\int DU \exp[-iU\zeta] = \prod_i f(\bar{\zeta}) \tag{84}
\]
where the \( 2d(d-1) \) fields have been lumped into the vector \( \bar{\zeta} \). The effective dual action
\[
S_{eff} = -\frac{1}{2} V_{\alpha \beta \gamma \delta} \delta_{\alpha \beta} \zeta_{\gamma \delta} - \sum_i \ln f(\bar{\zeta}_i). \tag{85}
\]
This is a \( 2d(d-1) \) component spin system with pair interactions and a nontrivial onsite interaction. As \( V \) corresponds to nearest neighbor interactions on the even sublattice (\( \sim k^2 \) for small wave numbers), its inverse corresponds to Coulombic interactions (\( \sim k^{-2} \)) in the long wavelength limit. More precisely, the dual model corresponds to coupled Coulomb gases having a net of \( 2d(d+1) \) components (in the Abelian case) instead of one.

The function \( f(\bar{\zeta}) \) is always even and is further always invariant under the group symmetry of \( U \). In the argument of the exponent of Eqn.(82)
\[
Tr \left[ (\gamma U g^{-1} g U g^{-1} \zeta)^p \right] = Tr \left[ (\gamma U g^{-1} \zeta)^p \right]
\]
for any integer power \( p \geq 1 \) in the expansion, where the cyclic property of the trace and the group invariance of the measure \( DU \) were employed. A pure gauge theory is equivalent to gauge invariant charges interacting by pair Coulomb potentials.

As the two dimensional \( Z_2 \) lattice gauge theory is equivalent to the one dimensional Ising chain, its dual, a special isotropic two dimensional Coulomb gas with spin-spin interactions, has a canonical one dimensional behavior. (Its partition function \( Z \equiv [(cosh K)^L + (sinh K)^L]^L \) (with \( L \) the linear extent of the system) if the two dimensional system has periodic boundary conditions along both axis (each of length \( L \)). This is a simple example of a dimensional reduction. The same applies to a \( U(1) \) system where in the two dimensional dual model- a special, eight component XY model with Coulomb like interactions- we will find that the free energy density \( -N^{-1} \ln Z \) tends to \( (\ln |I_0(K)|) \) with \( I_0 \) a Bessel function.

In fact, by selecting the temporal gauge, in which the gauge links \( U_0(n) = 1 \) along all links along the time (0) direction, any two (or 1+1) dimensional gauge theory with plaquette terms can be recast as a classical one dimensional model having nearest neighbor interactions. If we dualize any two dimensional gauge theory we obtain a special, spatially symmetric, two dimensional Coulomb gas that is a nearest neighbor one dimensional system in disguise.

To get an appreciation for what is happening, let us examine matters in detail. For the \( Z_2 \) lattice gauge theory in \( d = 2 \) dimensions, the vector \( \zeta \) is four-dimensional and at each site,
\[
f(\bar{\zeta}) = 16 (\cos \zeta_1 \cos \zeta_2 \cos \zeta_3 \cos \zeta_4 + \sin \zeta_1 \sin \zeta_2 \sin \zeta_3 \sin \zeta_4), \tag{87}
\]
where we have marked in a counterclockwise fashion the bonds \( \{U\} \) about a given site \( i \) by \( U_1,U_2,U_3 \) and \( U_4 \). With these bonds we identify 4 dual fields in the following way- \( \zeta_1 \) arises from a coupling to the two bond product \( U_4 U_1, \zeta_2 \) arises from a coupling to the product \( U_3 U_2 \), and so on cyclically.

The interaction kernel in the 4 dimensional internal space of \( \bar{\zeta}(\bar{k}) \) (the components of \( \zeta \) in internal and momentum space), reads
\[
v = -K \begin{pmatrix} 0 & A_2 \\ A_2 & 0 \end{pmatrix}, \tag{88}
\]
with the two by two submatrix
\[
A_2 = \begin{pmatrix} \cos(k_1 + k_2) & 0 \\ 0 & \cos(k_1 - k_2) \end{pmatrix}, \tag{89}
\]
where $k_1$ and $k_2$ denote the $x$ and $y$ components of the momentum in the plane.

As detailed earlier, for all $K < K_c$, a symmetry operation may be performed to map any spin model to onto a higher ($D > 1$) dimensional one, by merely replacing a one dimensional momentum coordinate on a spin, $k$, by a linear combination $\sum_{j=1}^{D} M_j k_j$ where each of the elements $\{M_j\}$ is a constant matrix, in the argument of the interaction kernel. This is exactly what is done here. The kernel that appears in our two dimensional problem, $\cos k$, may be obtained from a one dimensional kernel with the substitution $\cos k \rightarrow \cos(k_1 + k_2 \sigma_3)$ with $\sigma_3$ the Pauli matrix to the standard nearest neighbor kernel on a spin chain. This transforms the argument of the cosine, $k$, into a two dimensional matrix and makes the interaction look two dimensional (involving both $k_1$ and $k_2$) while in reality, the system is one dimensional spin chain in disguise.

The interesting question is, of course, whether we may see similar dimensional reductions (exact or approximate) in higher dimensional gauge theories of higher groups.

VIII. PHASE INTERFERENCE AND A MAPPING ONTO A SINGLE SPIN PROBLEM

We will now map classical high dimensional ($d > 1$) spin problems onto translationally invariant systems in one dimension.

For the nearest neighbor ferromagnet, the basic idea is to make a comparison between the the kernels

$$\hat{v}_d = -2 \sum_{l=1}^{d} \cos k_l$$
$$\hat{v}_1 = -2 \sum_{l=1}^{d} \cos c_l k_1$$

in $d$ and in one spatial dimensions respectively. Note that the one dimensional problem has, in general, $d$ different harmonics of the single momentum coordinate $k_1$. If the coefficients $c_l$ are integers equal to $R_l$ then the one dimensional problem amounts to a ferromagnetic chain in which each spin interacts with $d$ other spins at distances $\{R_l\}_{l=1}^{d}$ away. When evaluating the loop integrals, we will find that the partition function/ free energy deviate from their $d$ dimensional values due to “interference” between the various harmonics in $\hat{v}_1(k_1)$. If all harmonics acted independently in the effective one dimensional problem then the $d$ dimensional result would be reproduced. The advantages of the form $\hat{v}_d$ are obvious. Perhaps the most promising alloy is that of “incommensurate dimensional reduction”. If this method may be also extended to fermionic systems that it would suggest that we might examine d-dimensional problem described by $\hat{v}_d$ by a bosonization of the one-dimensional problem of $\hat{v}_1$. A solution to the one dimensional problem posed by $\hat{v}_1$ with mutually incommensurate coefficients $\{c_l\}_{l=1}^{d}$ will immediately lead to the corresponding $d-$ dimensional quantities.

A. Commensurate dimensional reduction

Consider a one dimensional ferromagnetic lattice model with a nearest neighbor (or Range = 1) interaction and a Range = $n$ interaction: $\hat{v}(k_1) = -2(\cos k_1 + \cos nk_1)$. The loop integrals containing terms of the cos $k_1$ origin and terms stemming from cos $nk_1$ are independent to low orders. Mixed terms can survive only to orders $(1/T)^{n+1}$ and higher. The delta function constraints $\sum q_1 = 0$ would generate terms identical to those of the two dimensional nearest neighbor model with $\hat{v}(k_1, k_2) = -2(\cos k_1 + \cos k_2)$ with the two delta function constraints for the two separate components of the momentum, $\sum q_1 = 0$ and $\sum q_2 = 0$, at every vertex. Thus, in a sense, this system is two dimensional up to $T^{-n}$. Similarly, if a one dimensional chain has interactions of length one lattice constant, $R = n$ and $R = m$ i.e.

$$\hat{v}(k_1) = -2(\cos k_1 + \cos nk_1 + \cos mk_1 + \cos sk_1)$$

then this system, as fleshed out by a $1/T$ expansion for the partition function or for the free energy, is three dimensional up to order $n$ if $m = n^2$. (If $m = n + 1$ then the triangular ferromagnet is generated). Similar, higher dimensional extensions similarly follow from the lack of commensurability of the cosine arguments (for four dimensions

$$\hat{v}(k_1) = -2(\cos k_1 + \cos nk_1 + \cos mk_1 + \cos sk_1)$$

with $s = n^3$ etc.)

The corresponding transfer matrices are trivial to write down. For the three dimensional case:

$$\langle s_1...s_{n^2}|T|s'_1...s'_{n^2}\rangle = \exp[\beta(\frac{1}{2} (s_1 s_2 + ... + s_{n^2-1} s_{n^2})$$
$$+ \frac{1}{2} (s'_1 s'_2 + ... + s'_{n^2-1} s'_{n^2})$$
$$+ \frac{1}{2} (s_1 s_{n+1} + s_2 s_{n+2} + ... + s_{n^2-n} s_{n^2})$$
$$+ \frac{1}{2} (s'_1 s'_{n+1} + s'_2 s'_{n+2} + ... + s'_{n^2-n} s'_{n^2})$$
$$+ s_{n^2} s'_1 + (s_1 s_{n+1} + s_{n^2-n} + s_{n^2+n} + ... + s_{n^2} s'_n)$$
$$+(s'_1 s'_2 + ... + s'_{n^2} s'_{n^2})$$
$$= \exp[\beta T_{31}].)$$

In higher dimensional generalizations, $T_{ji}$ will be slightly more nested with the span $n^2$ replaced by $n^{d-1}$. In
Eqn. (93), the matrix indices \(i\) and \(j\) are written in binary numerals in terms of the \(n^2\) spins

\[
j = \sum_{\alpha=1}^{n^2} (s_{\alpha} + 1)2^{n-2}
\]

(94)

where \(\alpha\) is the one dimensional coordinate along the bra spin indices. By Eqn. (93), \(T_{ji}\) are expressed in terms of a sum over two digit products where the digits are those that appear in the binary (length \(n \) spin) representation of the coordinates \(j\) and \(i\).

As usual, the partition function

\[
Z = Tr[T^N] = \sum_i \lambda_i^N
\]

(95)

where \(\{\lambda_i\}\) are the eigenvalues of the transfer matrix. In the following \(\lambda_{\text{max}}\) will denote the largest eigenvalue.

Note that the transfer matrix eigenvalues correspond to periodic boundary conditions as strictly required: we employed translational invariance to write the partition function expansion in Fourier space. If the boundary conditions are not periodic then all of this is void.

By the lack of interference effects, the connected diagrams for the free energy are correct to \(O(\beta^n)\) for interactions with \(m = n^2\) (Eqn. [1]). This along with

\[
\beta f = \sum_p f_p \beta^p = -\ln Z_N = -\ln \lambda_{\text{max}}
\]

(96)

(where the last equality holds in the limit of large system size \(N\)) and a canonical power expansion having an infinite radius of convergence

\[
\lambda_{\text{max}} = \exp[-\sum_p f_p \beta^p] = \sum_{m=0}^{\infty} \left( \frac{1}{m!} (-\sum_p f_p \beta^p)^m \right) = \sum_p b_p \beta^p
\]

(97)

imply that \(\lambda_{\text{max}}\) is correct to \(O(\beta^n)\).

If we already know the lower order coefficients corresponding to \(O(\beta^p)\) with \(p = 0, 1, 2, 3, \ldots, (n - 1)\) when we consider the \(\text{Range} = n\) problem. then we may set out to compute is the coefficient of \(\beta^n\). This will amount to extracting only a single unknown (i.e. the term corresponding to the coefficient of \(\beta^n\)) from a transfer matrix eigenvalue equation, This can be done to the next order etc. recursively. A simple yet very long single linear relation gives the largest transfer matrix eigenvalue to each higher order in the inverse temperature \(\beta\).

Explicitly, we can write longhand

\[
\det(T - \lambda) = \epsilon_{i_1 \ldots i_{n^2}} \prod_{j=1}^{n^2} [T_{ji} (\beta) - \lambda (\beta) \delta_{ji}] = 0
\]

(98)

All one has to do is to yank out the \(O(\beta^n)\) term from the sum over products of \(2^n\) terms. If we pull out a power \(p_{ji}\), from each of the elements \([T_{ji}(\beta) - \lambda(\beta) \delta_{ji}]\) then those will need to satisfy

\[
\sum_{j=1}^{n^2} p_{ji} = n.
\]

(99)

In each element of the transfer matrix, \(\exp[\beta T_{ji}]\), the coefficient of \(\beta^n\) is trivially \(T_{ji}/n!\). For each permutation (or “path” of coordinates \((i, j)\) within the matrix to be summed for the evaluation of the determinant

\[
\left[ \sum_{\text{path}} T_{ji} \right] = \sum_{p_1, p_2, \ldots, p_{n^2}} \frac{n!}{p_1! p_2! \ldots p_{n^2}!} \prod p_{p} = n
\]

(100)

is exactly the product of the coefficients of \(\beta^n\) such that those lead a net power \(\beta^n\) (i.e. satisfying Eqn. (99)) from that particular path.

It follows that net contributions stemming from \(T_{ji} (\beta)\) in Eqn. (101) are

\[
F(n) = \frac{1}{n!} \sum_{\text{all paths}} \epsilon_{\text{path}} \left[ T_{ji} \right] (j_1 \rightarrow j_2) \]

(101)

where \(\epsilon_{\text{path}}\) simply denotes the sign of the given permutation \(\{j\}_1^n \rightarrow \{i\}_n\).

To take into account the products including \(\lambda(\beta)\) let us define

\[
D_{2m}^{(n; p)} = \frac{1}{(n - p)!} \sum_{\text{all \(2m\) paths}} \epsilon_{\text{path}} \sum_{(j_1)} \left[ T_{ji} \right]^{n - p}
\]

(102)

where the summation is over paths going through all possible \((2m)\) given points on the diagonal and \(\sum\) denotes a summation over \(T_{ji}\) in the given paths sans the contributions from the \((2m)\) diagonal points. The determinant path in \(\sum_{\text{all \(2m\) paths}}\) can pass through more than \((2m)\) diagonal points— it is just that we need to sum over all those that in their pass also traverse all possible sets of \((2m)\) given points on the diagonal and where those points are excluded from the second \(T_{ji}\) summation.

The \(\beta^n\) component of Eqn. (101) is

\[
F(n) + \sum_{p=0}^{n-2} 2^{n^2 - 1} D_m^{(n; p)} \sum_{p_{1} \ldots p_{n^2}} b_{p_1} \ldots b_{p_{n^2}} = 0
\]

(103)

where \(\{b_p\}_{p=0}^{n-1}\) appear in the expansion of the largest eigenvalue (Eqn. (77)).

If the coefficients \(\{b_p\}_{p=0}^{n-1}\) then Eqn. (103) where \(b_n\) appears there only linearly gives \(b_n\) and one may then proceed to look at the eigenvalue equation for the next value of \(n\).
In the absence of an external field, for \( n = 1 \) the largest eigenvalue for the \( d \) dimensional problem is \( \lambda_+ = 2 \cosh d\beta \) and other eigenvalue \( \lambda_- = 2 \sinh d\beta \). For the three dimensional case setting \( m = n^2 = n = 1 \) leads to the one dimensional momentum space kernel \( \hat{v}(k_1) = -6 \cos k_1 \) which in real space corresponds to

\[
H = -3 \sum_i s_i s_{i+1}.
\] (104)

Of the two eigenvalues only \( \lambda_+ \) has a nonzero \( b_0 \). Knowing the values \( b_0 = 2 \) and \( b_1 = 0 \) of the largest eigenvalue we may proceed to find \( b_2 \) from the Eqn.(103) when \( n = 2 \) etc. This, of course, may be extended to systems with magnetic fields.

Setting \( b_0 = 2 \) in Eqn.(103) we find

\[
b_n = \sum_{m=1}^{2^{n^2-1}} m^{4m} D_m^{(n;n)} = \sum_{m=2}^{2^{n^2-1}} b_n \sum_{p=0}^{n-1} b_{p_1} b_{p_2} D_{m}^{(n;p)} \sum_{i=1}^{n} \epsilon_{path}.
\] (105)

This is the rather explicit recursive relation giving \( b_n \) once \( \{b_p\}_{p=0}^{n-1} \) are known. To find out the coefficient of \( b_n \) let us look at

\[
\mathcal{T}_s^{(n;n)} = \sum_{exactly \ 2s \ diagonal \ paths} \epsilon_{path},
\] (106)

where the summation admits only those permutations that lead to exactly 2s diagonal elements. For the 2s diagonal path in the matrix \( T \): \( \epsilon_{path} = 1 \). For all paths threading 2s-2 diagonal sites \( \epsilon_{path} = -1 \) and the number of such paths is \( \binom{2s}{2} \). For the general case the last quantity amounts to

\[
\mathcal{T}_s^{(n;n)} = (-1)^s \binom{2n^2}{2s}.
\] (107)

Evaluating, we find

\[
D_m^{(n;n)} = \sum_{s=m}^{2^{n^2-1}} \mathcal{T}_s^{(n;n)} = \frac{(2n^2)!}{(4m)!(2n^2 - 4m)!} \times F_{pq} \{ 1, \frac{m}{2} - 2n^2 - 1, \frac{1}{2} + \frac{m}{2} - 2n^2 - 1 \}, \frac{1}{2} + \frac{m}{2}, 1 + 2m, 1 \} \] (108)

where \( F_{pq} \) is a generalized hypergeometric function.

Insofar as simple geometric visualization is concerned, it is amusing to note, as shown in Fig.(3) for the two dimensional case, that employing the usual high temperature expansion in powers of \( \tanh^2 \beta \) one would reach the same conclusion regarding the correctness (up to order \( n \)) of the partition function evaluated for a width \( n \) slab vis a vis the partition function of the two dimensional system. We may look at the a finite thickness \( n \) slab of the two dimensional lattice along which we apply periodic boundary conditions. Let us now draw a string going along one row of length \( n \), after which it would jump to the next row, scan it for \( n \) sites, jump to the next one, and so on. On the one dimensional laced string, the system is translationally invariant and the interactions are of ranges \( R = 1, n \). Or, explicitly, by counting the number of closed loops in real space (employing the standard, slightly different, diagrammatic expansion in powers of \( \tanh^2 \beta \), we see that the terms in this expansion are also identical up to order \( \tanh^2 \beta \) [21].

A lacing of a two dimensional slab by a one dimensional string, as shown in the figure, is one of the backbones of Density Matrix Renormalization Group Theory when applied to two dimensional problems.

**B. Incommensurate dimensional reduction**

The one dimensional continuum limit \( (-\Lambda < k_1 < \Lambda) \) momentum space kernel

FIG. 3. Lacing of an \((n \times L)\) slab (with \( L \to \infty \)). All real space high temperature diagrams up to order \((\tanh^n \beta)\) such as the closed rectangle shown above are of equal value in both this one dimensional system with interaction ranges \( R = 1, n \) and the full \((L \times L)\) two dimensional model with nearest neighbor interactions. The width of the slab \( "n" \) serves as an inverse temperature axis in the following sense- the wider it is, the higher order in \( 1/T \) that we may advance towards the full two dimensional model. We may get an analogous “three-dimensional” slab with “two imaginary time axis” if we set \( m = n^2 \). The path will then thread \( n^2 \) sites before continuing upward.
\[ \hat{v}(k_1) = -2 \sum_{l=1}^{d} \cos c_l k_1 \]  

(109)

with mutually incommensurate \( \{c_l\} \) will reproduce the expansion with the kernel in Eqn.(15) for the \( d \) dimensional hypercubic lattice system \( [24] \).

The proof is quite easy. The most general integral is of the form

\[ \prod_{a=1}^{d} \sum_{l=1}^{d} \exp[i \sum_{b=1}^{d} M_{ab} k_b c_l] \]  

(110)

where the index \( b \) runs over the various independent loop momenta (\( k \) is still merely a scalar for this one dimensional problem). Unless, for a given \( k_b \) integration, the argument of the exponent is identically zero (i.e. corresponding to a term that would be generated in the \( d \)-dimensional nearest neighbor problem) an “interference term” results. However, such a term is down by \( O(\Lambda^{-1}) \) by comparison to the “good” noninterference terms that occur in the \( d \)-dimensional problem. The canonical loop integral reads

\[ \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} dk_b \exp[i c k_b] \rightarrow \frac{\pi \delta(c)}{\Lambda} \]  

(111)

where \( c \) is a linear combination of the \( d \) coefficients \( \{c_l\} \). Unless \( c \) vanishes identically (i.e. unless a “good” noninterference term occurs), then \( c \neq 0 \) if \( \{c_l\} \) are chosen to be mutually incommensurate. Whenever \( c = 0 \), the integral in Eqn.(111) transforms into the Kronecker delta \( \delta(c,0) = 1 \) in the limit \( \Lambda \rightarrow \infty \) and the integral attains exactly the same value that it would take on for the \( d \)-dimensional hypercubic lattice system. Similarly, in this limit all “bad” interference terms will evaporate in all diagrams.

As this holds for any diagram, this conclusion is also valid for any set of diagrams (not only those encountered in computation of the partition function or the free energy), e.g. \( C_{Y, \chi} \) etc.

As a consequence we may state that for a single spin 1/2 particle with an action

\[ S = \int d\omega \overline{\psi}(-\omega) \hat{v}(\omega) \psi(\omega) \]  

(112)

where \( \overline{\psi}(\tau) \) is the two component spinor, the partition function is identically the same as that of the \( d \) dimensional nearest neighbor ferromagnet.

The proof of this statement trivially follows from breaking up the \([0, \beta] \) segment along the imaginary time axis into \( L \) pieces and allowing \( L \rightarrow \infty \).

We have mapped the entire three dimensional Ising model onto a single spin 1/2 quantum particle \( [23] \).

The Fourier transform of Eqn.\([109] \) on a lattice reads

\[ \hat{V}(x_1) = -2 \sum_{l=1}^{d} [(c_l + x_1) \sin \pi(c_l - x_1) \]

\[ + (c_l - x_1) \sin \pi(x_1 + c_l)] \]

\[ \times [(c_l - x_1)(c_l + x_1)]^{-1}, \]  

(113)

a sum of shifted Coulomb like interactions. If an average \( (2\Lambda)^{-1} \int_{-\Lambda}^{\Lambda} dc_l \) is performed over each of the \( d \) coefficients \( \{c_l\} \) then the resulting quantities are the exact \( d \)-dimensional ones. If “bad” interference terms result then their value is \( O(1/e) \). In the averages, it is the large coefficients that dominate and these will lead to a vanishing average value of all interference terms.

### IX. INCOMMENSURATE DIMENSIONAL REDUCTIONS OF QUANTUM MODELS

The incommensurate dimensional reduction averaging theorem which we just proved by interchanging the order of integrations over \( \{c_l\} \) and the loop momenta \( \{k_b\} \) holds for any translationally invariant system (spin, bosonic, fermionic) in which the unperturbed action \( S_0 \) links two nearest neighbor sites. For a \((d+1)\) dimensional quantum lattice system, the Green’s function conjugate to \( S_0 \) in the undualized (non Hubbard-Stratonovich transformed) system

\[ G_0 = \left[ \beta \left( -2 \sum_{l=1}^{d+1} \cos k_l + A \right) \right]^{-1}. \]  

(114)

As before, if the magnitude of the squared mass \( |A| = \text{const} > 2(d+1) \) then a geometric series expansion in \( \Lambda^{-1} \) may be performed to generate a sum of harmonics. For each given single diagrammatic term, our previous proof may be applied.

We note that fermionic electron (or other) fields might, perhaps, be formally bosonized on each individual chain. The one dimensional band,

\[ \hat{v}_1 = -2 \sum_{l=1}^{d} \cos c_l k_l, \]  

(115)

is now a simple sum of \( d \) cosines (au lieu of the standard single tight binding cosine). For each given set \( \{c_l\}_{l=1}^{d} \), the band dispersion of Eqn.(113) may be easily linearized about its respective Fermi points. Consequently the standard bosonization methodology may be applied. In the large \( \{c_l\} \) limit, the number of Fermi points becomes infinite. A one dimensional Fermi system with \((2n)\) Fermi points is expected to be equivalent to a \( d \)-dimensional system up to \( O(\beta^n) \).
The astute reader will note that we may just as well use Eqn. (114) to study a classical spin system (with the upper bound \( d + 1 \) replaced by \( d \)) and prove everything that we have previously stated without the need to consider the dual model. Such an approach is not as rigorous as that followed hitherto. First, the geometric series expansion in the harmonics \( \exp[i\vec{k}] \) converges only for a sufficiently absolute value of the mass- or, equivalently for temperatures far enough from the mean field temperature. Second, the expansion in \( G_0 \) is in general explicitly asymptotic as the perturbing piece is of higher order in the fields than \( H_0 \). The dual formulation was free of these pathologies.

X. PERMUTATIONAL SYMMETRY

The classical spin spherical model (or \( O(n \to \infty) \)) partition function

\[
Z = \text{const} \left( \prod_{\vec{k}} \frac{1}{\sqrt{\beta[\hat{V}(\vec{k}) + \mu]}} \right),
\]

where the chemical potential \( \mu \) satisfies

\[
\beta = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{\beta[\hat{V}(\vec{k}) + \mu]},}
\]

is invariant under permutations of \( \{\hat{V}(\vec{k})\} \to \{\hat{V}(P\vec{k})\} \). In the above, the permutations

\[
\{\vec{k}_i\}_{i=1}^N \to \{P\vec{k}\}
\]

correspond to all possible shufflings of the \( N \) wavevectors \( \vec{k}_i \). Although quite simple, this is not universally realized. Several authors have attempted to compute the critical exponents in the spherical limit (via an RG calculation) for systems having different minimizing manifolds yet all sharing the same relevant density of states. This quest was not very economical. As unrealized by these authors, by permutational symmetry these models are identical.

This simple invariance allows all \( d \)-dimensional translationally invariant systems to be mapped onto a one-dimensional one. Let us design an effective one dimensional kernel \( V_{eff}(k) \) by

\[
\int \delta[\hat{V}(\vec{k}) - \hat{V}]d^d k = |\frac{dV_{eff}}{dk}|^{-1} dV_{eff}(k) = |\hat{V}|^{-1} d\hat{V},
\]

The last relation secures that the density of states and consequently the partition function is preserved. For the two-dimensional nearest-neighbor ferromagnet:

\[
|\frac{dV_{eff}}{dk}|^{-1} = \rho(V_{eff})
\]

and consequently

\[
k(V_{eff}) = c_1 \int_0^{V_{eff}} F(\sin^{-1} \frac{2}{(3-u)u}) - \sqrt{4u - u^2}) du,
\]

where \( F(t, s) \) is an incomplete elliptic integral of the first kind. Eqn. (121) may be inverted and Fourier transformed to find the effective one dimensional real space kernel \( V_{eff}(x) \). We have just mapped the two dimensional nearest neighbor ferromagnet onto a one dimensional system. In a similar fashion, within the spherical (or equivalently the \( O(n \to \infty) \)) limit all high dimensional problems may be mapped onto a translationally invariant one dimensional problem. It follows that the, large \( n \), critical exponents of the \( d \) dimensional nearest neighbor ferromagnet are the same as those of translationally invariant one dimensional system with longer range interactions. We have just shown that a two dimensional \( O(n \gg 1) \) system may have the same thermodynamics as a one dimensional system. By permutational symmetry, such a mapping may be performed for all systems irrespective of the dimensionality of the lattice or of the nature of the interaction (so long as it translationally invariant). This demonstrates once again that the notion of universality (with dependence only on the order parameter symmetry, dimensionality etc.) may apply only to the canonical interactions.

The lowest order term breaking permutational symmetry in our high temperature expansion is \( \eta^4(\vec{x})\eta^4(\vec{y}) \). Thus permutational symmetry is broken to \( O(\vec{x}) \) for finite \( n \). For a constraining term (e.g. \( \sum_{\vec{x}} \ln[\cosh[\eta(\vec{x})]] \) for \( O(1) \) spins) symmetric in \( \{\eta(\vec{k})\} \) to a given order, one may re-arrange the non-constraining term \( \sum_{\vec{x}} \eta^{-1}(\vec{k})|\eta(\vec{k})|^2 = \sum_{\vec{x}} \eta^{-1}(P\vec{k})|\eta(P\vec{k})|^2 \) and relabel the dummy integration variables \( H[\{\eta(\vec{k})\}] \to H[\{\eta(P^{-1}\vec{k})\}] \) to effect the constraining term augmented to a shuffled spectra \( \hat{V}(P\vec{k}) \).

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XI. APPENDIX A: THE HUBBARD STRATONOVICH TRANSFORMATION FOR \( O(N) \) SPIN SYSTEMS

For the benefit of our uninitiated readers we present the standard Hubbard Stratonovich transformation. This “transformation” merely relies on the elementary Gaussian identity
\[
\int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}K\eta^2 + \phi\eta\right] = K^{-1/2} \exp\left[\frac{\phi^2}{2K}\right].
\] (122)

As well known, this easily generalizes (upon diagonalization) to
\[
\int (D\eta) \exp\left[-\frac{1}{2}(\eta, K\eta) + (\eta, \phi)\right] = (\det K)^{-1/2} \exp\left[\frac{1}{2}(\phi, K^{-1}\phi)\right].
\] (123)

For our particular purposes, note that
\[
\sqrt{\det(2\pi(-V^{-1}))} = \int D\eta \exp\left[-\frac{1}{2}(sV - \eta)(-V^{-1})(Vs - \eta)\right].
\] (124)

Thus
\[
Z = \sum_s \exp\left(-\frac{1}{2}sV\right) = \sqrt{\det\left(-\frac{2V}{\pi}\right)} \int D\eta \exp\left[\frac{1}{2}V^{-1}\eta\right] \sum_s \exp[-sV\eta]
\] (125)

where
\[
\bar{H} = -\sum_i \ln(\cosh \eta_i) - \frac{1}{2}(\eta, V^{-1}\eta).
\] (126)

In the above \(s = (S_1, S_2, \ldots, S_N)\) where \(i\) label the sites \(\vec{x}\). For an Ising system \(S(\vec{x}) = \pm 1\). In Eqn. (126), the relation
\[
\sum_s \exp(-s, \eta) = \sum_s \exp(-s_1\eta_1 - \ldots - s_N\eta_N) = \prod_{i=1}^{N}(2\cosh \eta_i),
\] (127)

was employed.

In an \(O(n)\) spin system, tracing out over the spins leads to
\[
\sum_s \exp(-s, \eta) = \prod_{i=1}^{N} \left[ \int d\Omega S_i \exp(-S_i \cdot \vec{\eta}) \right] = \prod_{i=1}^{N} \left[ \Gamma(n/2) \left( \frac{2}{\bar{\eta}_i} \right)^{n/2-1} I_{n/2-1}(\bar{\eta}_i) \right].
\] (128)

XII. APPENDIX B: THE EIGHT-FOLD “PORCUPINE” SPIN MODEL AND OTHER “PLATONIC” SPIN MODELS

For “\(Z_2 \otimes Z_2 \otimes Z_2\)” models, the three-dimensional spins explicitly read
\[
s = (\sigma_1, \sigma_2, \sigma_3)
\] (129)

with
\[
\sigma_i = \pm 1
\] (130)

(i.e. spin \(\frac{1}{2}\) like). This corresponds to the 8 possible states (the vertices of the unit cube) \((\pm 1, \pm 1, \pm 1)\), or in a more pompous notation- the eight direct product states of three spin-1/2 particles \(|\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle\) states. For most canonical rotationally and translationally invariant models \(\langle \sigma_1 \sigma_2 \sigma_3 | H | \sigma_1 \sigma_2 \sigma_3 \rangle\) decomposes into a sum of three decoupled Hamiltonians each living within its own individual two dimensional \((|\sigma_i\rangle)\) subspace. The three dimensional “single particle (spin)” problem will reduce to a sum of the three individual non-interacting Hamiltonians each corresponding to a single free (spin-1/2) “particle” having only two states. A Hamiltonian containing only \([S_i \cdot S_j]\) terms (with \(i\) and \(j\) lattice sites), amounts to
\[
H(|\{S_i\}\rangle) = \sum_{x_i, x_j} V(x_i, x_j) |S_i \cdot S_j\rangle = \sum_{\alpha=1}^{3} \mathcal{H}(\{\sigma_i^\alpha\})
\] (131)

where \(\mathcal{H}\) are Ising Hamiltonians, i.e.
\[
\mathcal{H}(\{\sigma_i^\alpha\}) = \sum_{x_1, x_2} \sigma_1^\alpha V(x_1, x_2) \sigma_2^\alpha.
\] (132)

The partition function
\[
Z = \prod_{\alpha=1}^{3} Z_\alpha = (Z_{Ising})^3.
\] (133)

The \(Z_\alpha^3\) model behaves exactly like an Ising model (or a \(q = 2\) Potts model). This may be extended to other \(\tilde{N}\) state clock models. When \(\tilde{N} = 2\) the “clock model” is trivially the \(q = 2\) Potts (Ising) model. When \(\tilde{N} = n + 1\) in the \(n\)-dimensional “clock model”, a mapping to an \((n + 1)\) state Potts model is possible (by constructing a regular \(n\) dimensional tetrahedra). For \(\tilde{N} = 2^n\) (the \(n\) dimensional hypercube), the model may be mapped (once again) onto \(n\) decoupled \(q = 2\) Potts models. In two dimensions, \(q_c = 4\) is the critical number of states required for the nature of the transition to vary in the unfrustrated ferromagnetic Potts model. Thus, in the unfrustrated case, as \(\tilde{N}\) is monotonically increased (for \(n > 3\)), the nature of the transition changes, at least
twice, from that with \((\tilde{N} = 2; \ 2 = q < q_c)\) to \((\tilde{N} = q = n + 1 > q_c)\) to the \(q \leq q_c\) case \((\tilde{N} = 2^n; \ 2 = q \leq q_c)\) once again. For all \(\tilde{N} < n + 1\) uniformly spaced polarizations, with overall isotropic orientation, the model is exactly equivalent to a \(q = \tilde{N}\) state Potts model; in this case, the discrete polarizations span an \(N\)–dimensional subspace embedded in the full \(n\)–dimensional spin space.

The three-dimensional “clock model” is restricted only to very special \(N\) values corresponding to the perfect solids (the six perfect polyhedra – the “Platonic solids” – when \(n = 3\)) and when \(n > 3\), the uniformly spaced spin polarizations are restricted to the vertices of the (special small set of allowed) regular polytopes possible for that value of \(n\). The \(\tilde{N} \to \infty\) limit may not be taken- non-infinite \(\tilde{N}\)’s are bounded by a finite number (e.g. \(\tilde{N}_{\text{max}} = 60\) for \(n = 3\)).

When \(\tilde{N} = \infty\) (not the limit) for a given \(n\) the continuous \(O(n)\) model is attained -this model has second order phase transitions in \(d = 3\) for its unfrustrated ferromagnetic variant.

When \(n = 3\), the only possible finite \(\tilde{N}\) variants are \(\tilde{N} = 4\) (the tetrahedron spin model that may be mapped onto the \(q = 4\) state Potts model as just discussed), \(\tilde{N} = 8\) (the cube or hexahedron - \(q_{\text{eff}} = 2\) as discussed in on page 1), \(\tilde{N} = 6\) (the octahedron), \(\tilde{N} = 20\) (the dodecahedron), \(\tilde{N} = 12\) (the icosahedron), and \(\tilde{N} = 60\) (the soccer-ball or truncated icosahedron).

For the \(\tilde{N} = 6, 12, 20, \) and 60 variants a mapping into a model similar to the Potts model is possible. Here the number of allowed individual two-spin energy states is no longer effectively two (as in the Potts models for equal and unequal spin polarizations of neighboring spins) but is slightly larger - a trivially generalized Potts model (e.g. there are three different possible values for a spin-spin interaction in the \(\tilde{N} = 6\) model).

The octahedron (\(\tilde{N} = 6\)) model is equivalent to a “three-color dilute Ising model” where each site is colored red, white, or blue. On each individual color cluster independent Ising spins interact amongst themselves. Consequently, a sum over all possible colorings (dilute inter penetrating color sites) of the lattice sites is to be performed.

References

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[2] T. T. Wu et al., Phys. Rev. B 13, 316 (1976)
[3] B. M. McCoy and T. T. Wu, “The Two-Dimensional Ising Model”, Harvard University, Cambridge, Mass. 1973.
[4] R. L. Stratonovich, Dokl. Akad. Nauk S.S.S.R. 115, 1907 (1957) (English translation- Sov. Phys. Dokl. 2, 416 (1959)). J. Hubbard, Phys. Rev. Lett. 3, 77 (1959)
[5] Explicitly, the inverse kernel of the nearest neighbor ferromagnet reads

\[ V^{-1}(\vec{x}, \vec{y}) = \beta^{-1} \int \frac{d^d k}{(2\pi)^d} \frac{\exp[i \vec{K} \cdot \vec{x}] \exp[i \vec{k} \cdot \vec{z}]}{A - 2 \sum_{i=1}^{d} \cos k_i} \]

(134)

where \(A\) denotes the strength of a possible additional onsite \(A[S(\vec{x})]^2\) interaction term which is merely a constant (\(A\)) for the normalized \(O(n)\) spins. For \(A = 2d\), the denominator is none other than the lattice Laplacian which scales as \(k^{-2}\) for small momenta and whose inverse Fourier transform is a real space Coulomb kernel. For \(A > 2d\), a Yukawa like screened Coulomb interaction kernel (with a screening length \(A^{-1/2}\)) appears.

[6] M. Wortis in Phase Transitions and Critical Phenomena, Volume 3, Edited by C. Domb and M. S. Green, Academic Press (1974)
[7] A. M. Polyakov “Gauge fields and Strings”, Harwood Academic Publishers (1987).
[8] The asymptotically convergent character of generic \(e\) expansions and \(\phi^4\) field theories was frequently overlooked and has led to quite a few incorrect conclusions in the past. For instance, the exact equivalence of the diagrammatic expansions for the uniform and random magnets in \((d-2)\) and \(d\) dimensions respectively, has led to the speculation that the two problems may be identical. If correct, this would have implied that the lower critical dimension of the random field Ising model would be three and not two as follows from Imry-Ma arguments. The flaw in such conclusions resting on the equivalence of two asymptotic diagrammatic series was pointed by D. S. Fisher [9]. Our expansion is convergent for \(\beta < \beta_c\) (the inverse critical temperature) and is explicitly free from many of the spurious pathologies present in canonical diagrammatic expansions.

[9] D. S. Fisher Phys. Rev. B 31, 7233-7251 (1985)
[10] If a twist \(\alpha\) is applied across the sample (e.g. \(\alpha = 1\) corresponds to periodic boundary conditions and \(\alpha = -1\) to antiperiodic boundary conditions) then \(\{z_n\}\), with \(z_n \equiv \exp[i k_n]\), are roots of the equation \(z^d - \alpha = 0\); the sum of these polynomial roots is zero. More generally, Eqn. (17) holds for any system in which the sum of the all allowed values, \(\sum_n z_n\), is equal to zero.
[11] V. V. Bazhanov, R. J. Baxter, J. Stat. Phys., 69 453 (1992), hep-ph/9212051.
[12] Y. Tokura and N. Nagaosa, Science 288, 462 (2000), and references therein
[13] J. W. Negele and H. Orland, “Quantum Many-Particle Systems” Frontiers in Physics, Addison-Wesley (1988)
[14] R. J. Baxter “Exactly solved models in statistical physics”, Academic Press (1982)
[15] As a technical aside, note that on the lattice (unlike the standard continuum theory) each vertex conserves momentum only up to a reciprocal lattice vector \(\vec{K} = 2\pi(n_1, n_2, ..., n_d)\) with \(n_i\) integers. Formally, this is as \(\vec{x}\) only take on discrete values on a lattice,

\[ \sum_{\vec{x}} \exp[i \vec{k} \cdot \vec{x}] = N \sum_{\vec{R}} \delta_{\vec{x}, \vec{R}} \]

(135)
where $\vec{K}$ are reciprocal lattice vectors. Thus at each vertex of order $2m$ we have
\[ \frac{1}{(2m)!} \sum_{\vec{K}} \delta(\vec{q}_{in} - \vec{K}, 0) \] (136)
where $\vec{q}_{in}$ denotes the net momenta flowing into the vertex. As for all lattice models the propagators are periodic $\hat{\psi}(\vec{q}) = \hat{\psi}(\vec{q} + \vec{K})$ for each of the $N$ reciprocal lattice vectors $\vec{K}$. As such the momentum conservation constraint may be replaced by
\[ \frac{1}{(2m)!} \sum_{\vec{K}} \delta(\vec{q}_{in} - \vec{K}, 0) \rightarrow N \frac{1}{(2m)!} \delta(\vec{q}_{in}, 0). \] (137)

In the aftermath, each vertex carries a factor of the volume ($N$) multiplying the discrete Kronecker delta function. After the momentum conservation constraints are taken care of at each vertex, we will be left with a summation over all remaining independent loop momenta $\{\vec{k}_0\}$. Each one of these summations is to be replaced by the standard
\[ \sum_{\vec{k}} \rightarrow N \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \frac{d^d \vec{k}}{(2\pi)^d}. \] (138)
Thus, at the end we will be left with a factor of
\[ N^{\text{loops}} + \text{vertices} - \text{propagator lines} \] (139)
multiplying an integral over the independent loop momenta $\{\vec{k}_0\}$ in Eqn.(23). Compounding all together, as in each closed diagram the number of independent loops + number of delta function momentum conserving vertices $= 1$ + number of propagator lines, a net factor of $N$ should be attributed to each independent closed bubble just as it does in the conventional continuum theories.

[16] As noted earlier, for any lattice model
\[ \hat{\psi}(\vec{k}) = \hat{\psi}(\vec{k} + \vec{K}), \] (140)
as $\exp[i \vec{K} \cdot \vec{x}] = 1$ in
\[ \hat{\psi}(\vec{k}) = \sum_{\vec{x}} \hat{V}(\vec{x}) \exp[i \vec{k} \cdot \vec{x}] \]. (141)
The sum over $\vec{x}$ spans all discrete lattice sites.
The way to analytically continue $\psi(\vec{k})$ when $k_3$ becomes complex is to make the $k_2$ dependence also invariant under shifts by $(2\pi)$. Within each Brillouin Zone the $k_2$ dependence stems from the analytic continuation of the functional dependence on the real part. The only complication is the Gibbs’ phenomenon in $k_3$ which occurs when taking the Fourier transform of the lattice $\hat{V}(\vec{x})$. This is slightly avoided for the real part of $V(x)$ as follows: Under parity and all individual reflections: $\hat{V}(x_1, x_2, x_3) = \hat{V}(x_1, x_2, -x_3)$ etc. This implies that in the Fourier transform $\hat{v}(\vec{k})$: $\hat{v}(k_1, k_2, k_3) = \hat{v}(k_1, k_2, -k_3)$. Thus, taking only the even part in $k_3$ of $\cos(k_2 + i\lambda k_3)$ we will only be left with the terms $\cos k_2 \cosh k_3$. Now, $\cosh k_3$ is continuous and periodic when the Brillouin Zone is chosen as $-\pi < k_3 \leq \pi$. No Gibbs phenomenon occurs. The left and right derivatives are still discontinuous at the zone edges.

[17] By replacing the scalar product of Eqn.(19) by a non-rotationally invariant product of the form
\[ \hat{S}(\vec{x}) \circ \hat{S}(\vec{y}) = \sum_{\alpha=1}^{n} [S_{\alpha}(\vec{x})]^\alpha [S_{\alpha}(\vec{y})]^\alpha \] (142)
with a power $z \neq 1$, the rotationally invariant Bessel functions (with prefactors) of Eqn.(19) are replaced by
\[ F(\vec{q}) = \sum_{\alpha=1}^{n} \int d\Omega \eta_\alpha \exp[i\eta_\alpha \hat{\Omega}_\alpha] \] (143)
with $\Omega_\alpha$ denoting the $\alpha$ component of the $n$-dimensional unit vector $\Omega$. The rotational invariance of the Bessel function terms in Eqn.(19) is replaced by the point group symmetry of $F(\vec{q})$ (that of the $n$-dimensional hypercube). By repeating the steps leading to Eqn.(20), we find that the partition function of
\[ H = - \sum_{\langle ij \rangle \text{ along } (1,1)} \hat{V}_{ij} [S_{ij}^{(1)}]^\alpha [S_{ij}^{(1)}]^\alpha \]
\[ - \sum_{\langle ij \rangle \text{ along } (1,-1)} \hat{V}_{ij} [S_{ij}^{(2)}]^\alpha [S_{ij}^{(2)}]^\alpha \] (144)
is identical to that of a one dimensional system with
\[ H = - \sum_{\langle ij \rangle} \hat{V}_{ij} [\hat{S}_i] \circ [\hat{S}_j]. \] (145)

[18] P. Fazekas, cond-mat/0011354
[19] E. Berg, E. Altman, and A. Auerbach, cond-mat/0206384
[20] For the $U(1)$ case, the real action
\[ S_R = -K \text{Re} \{ \sum_{\square} (UUUU) \}. \] (146)
As usual, in the non-Abelian case, the real part of trace of the plaquette product must be taken.
[21] It might be easier to compute the terms in the $\beta$ expansion for a three dimensional Ising nearest neighbor ferromagnet by going to a triangular lattice and deleting all bad “constructive interference” loop integrals. A line threading the triangular lattice will connect a spin with spins of distance $\text{Range} = 1$, $n$, and $n + 1$ along the line. “Bad interference terms” can result from loop integrals of the form ($\exp[ik] \exp[ink] \exp[-i(n+1)k]$)$^p$. This is the only way in which loop integrals can conspire to give nonzero results when $n \rightarrow \infty$. Thus, trivially, the high temperature expansion for the $3$ cubic ferromagnet is equivalent to that of the triangular ferromagnet up to order $\beta^{n+1}$. Similarly, if we set $s = n + 1$ we obtain a $2$-Dimensional lattice model in which each spin interacts with its four nearest neighbors and four diagonal neighbors.
[22] In the continuum limit, this corresponds to the kernel
\[ \hat{V}(x) = -\frac{\pi}{\Lambda} \sum_{l=1}^{d} \delta(|x - c_l|) \] (147)
or, in a lattice approximation ($-\pi < k_1 \leq \pi$), to
\[ V(x_1) = -2 \sum_{l=1}^{d} [(c_l + x_1) \sin \pi (c_l - x_1) + (c_l - x_1) \sin \pi (x_1 + c_l)] \times [(c_l - x_1)(c_l + x_1)]^{-1}, \]

(i.e. a scenario in which each spin is effectively composed of d shifted “Coulombic sources” (more precisely, each spin is composed of d pairs of “charges” generating shifted sinc potentials)).

[23] The astute reader will note that the partition function of the d−dimensional nearest neighbor Ising model on a lattice (with \(-\pi < k_i \leq \pi\)) is equivalent to that of the d−dimensional nearest neighbor Ising model in the continuum (with the same kernel \(\hat{v}(\vec{k})\) but with unrestricted \((-\Lambda < k \leq \Lambda)\))

[24] That permutational symmetry cannot, for finite \(n\), hold to all orders in \(\beta\) immediately follows from the fact that the normalization constraints of the \(O(n)\) spins are not permutationally symmetric. If

\[ \hat{S}(\vec{x}) = \frac{1}{N} \sum_{\vec{k}} \hat{s}(\vec{k}) \exp[i\vec{k} \cdot \vec{x}] \]

then the normalization constraint at each site \(\vec{x}\) reads:

\[ 1 = S^2(\vec{x}) = \frac{1}{N^2} \sum_{\vec{k},\vec{k}'} \hat{s}(\vec{k}) \cdot \hat{s}(\vec{k}') \exp[i(\vec{k} + \vec{k}') \cdot \vec{x}]. \]

A permutation of \(\{\hat{s}(\vec{k})\}\) (which is equivalent to a permutation of the spectra \(\{v(\vec{k})\}\) within the quadratic interaction kernel) would transform the \(N\) constrains of Eqn.(149) into another set of equations.

The sole normalization constraint which is permutationally symmetric is the global normalization constraint (appropriate to the \(O(n \to \infty)\) spherical model)

\[ 1 = \frac{1}{N^2} \sum_{\vec{k},\vec{k}'} \hat{s}(\vec{k}) \cdot \hat{s}(\vec{k}'). \]

This illustrates why permutational symmetry must trivially hold, to all orders, in the \(O(n \to \infty)\) limit (and only in this limit).

For the quantum spin system,

\[ \hat{s}(\vec{k}) \times \hat{s}(\vec{k}') = iN \hat{\pi}_{\vec{k},\vec{k}'}. \]

Just as with the normalization conditions for the classical spin models, these commutation relations are not invariant under permutations of \(\{\hat{s}(\vec{k})\}\).

We mention, in passing, that permutational symmetry trivially holds in all quadratic (non-interacting) translationally invariant Bose and Fermi systems. The respective commutation and anticommutation relations

\[ [a^\dagger(\vec{k}'), a(\vec{k})] = \{ c^\dagger(\vec{k}), c(\vec{k}') \} = \delta_{k,k'} \]

are invariant under permutations \(\hat{O}(k) \to \hat{O}(Pk)\) for all bosonic or fermionic \(\hat{O}\).