MULTIPARAMETER SCHUR Q-FUNCTIONS ARE SOLUTIONS OF
BKP HIERARCHY

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Abstract. We prove that multiparameter Schur Q-functions, which include as
specializations factorial Schur Q-functions and classical Schur Q-functions, provide
solutions of BKP hierarchy.

1. Introduction

Integrable systems of KP (Kadomtsev-Petviashvilli) type hierarchy of partial dif-
ferential equations corresponding to infinite-dimensional Lie algebra of type A, and
its type B variant, BKP hierarchy, have as solutions renowned families of symmetric
functions: Schur polynomials in KP case and Schur Q-polynomials in BKP case [2],
[3], [4], [11], [12], [16], [17], [18], etc. In this note we show that multiparameter Schur
Q-functions also provide solutions of BKP hierarchy.

Multiparameter Schur Q-functions $Q^{(a)}_{\lambda}$ were introduced and studied combinato-
rially in [9]. These symmetric functions are interpolation analogues of the classi-
cal Schur Q-functions depending on a sequence of complex valued parameters
$a = (a_0, a_1, \ldots)$. Definition of multiparameter Schur Q-functions is reproduced in (7.1).
Classical Schur Q-functions correspond to $a = (0, 0, 0, \ldots)$, and with evaluation
$a = (0, 1, 2, 3, \ldots)$ multiparameter Schur Q-functions are called factorial Schur Q-
functions. These families of symmetric functions proved to be useful in study of a
number of questions of representation theory and algebraic geometry. Here are few
examples.

The authors of [1], [15] described Capelli polynomials of queer Lie superalgebra
which form a distinguished family of super-polynomial differential operators indexed
by strict partitions acting on an associative superalgebra. The eigenvalues of these
Capelli polynomials are expressed through factorial Schur Q-functions.

In [6], [8] the equivariant cohomology of a Lagrangian Grassmannian of a symplectic
or orthogonal types is studied. The restrictions of Schubert classes to the set of points
fixed under the action of a maximal torus of the symplectic group are calculated in
terms of factorial symmetric functions. Further in [7] factorial Schur Q-functions are

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parameter Schur Q-functions, vertex operators.
used to write generators and relations for the equivariant quantum cohomology rings of the maximal isotropic Grassmannians of types B, C and D.

In [5] the center of the twisted version of Khovanov’s Heisenberg category is identified with the algebra generated by classical Schur Q-functions (denoted as $B_{\text{odd}}$ in the exposition below). Factorial Schur Q-functions are described as closed diagrams of this category.

The goal of this note is to show that multiparameter Schur Q-functions $Q_{\lambda}^{(a)}$ are solutions of BKP hierarchy. The origin for this phenomena lies in the fact proved in [13] that generating functions of multiparameter Schur Q-functions and of classical Schur Q-functions coincide.

While BKP hierarchy is described in a wide range of literature on integrable systems and solitons, for the completeness of exposition and for the convenience of the reader we formulate the whole setting of BKP hierarchy in terms of generating functions of symmetric functions with neutral fermions bilinear identity (5.1) as a starting point. We avoid to use any other facts than the well-known properties of symmetric functions that can be found in the classical monograph [14], and through the text we provide the references to the corresponding chapters and examples of that monograph.

It is worth to mention that formulation of KP and BKP integrable systems solely in terms of symmetric functions can be found e.g. in [10]. The authors of [10] start with the bilinear identities in integral form, then, using the Cauchy type orthogonality properties of symmetric functions (c.f. [14] Chapter III (8.13)), they arrive at Plucker type relations, and the later ones are transformed into the collection of partial differential equations of Hirota derivatives that constitute the hierarchy. As it is mentioned above, our route is traced differently employing the properties of generating functions of complete, elementary symmetric functions and power sums. We obtain differential equations of the hierarchy in Hirota form as coefficients of Taylor expansions. One of the advantages of this approach is that it directly addresses the corresponding vertex operators actions, since the later ones are also ‘generating functions’ (formal distributions).

The paper is organized as follows. In Section 2 we recall some facts about complete, elementary symmetric functions, power sums and classical Schur Q-functions. In Section 3 we describe the action of neutral fermions on the space generated by classical Schur Q-functions. In Section 4 we review properties of generating functions for multiplication operators and corresponding adjoint operators and deduce vertex operator form of the formal distribution of neutral fermions. In Section 5 we review all the steps of recovering BKP hierarchy of partial differential equations in Hirota form from the neutral fermions bilinear identity. In Section 6 we make simple observation that immediately shows that classical Schur Q-functions are solutions of BKP hierarchy (which recovers the result of [17]). In Section 7 we introduce multiparameter Schur Q-functions, and using the observation of Section 6, we show that $Q_{\lambda}^{(a)}$ are also solutions of BKP hierarchy.
MULTIPARAMETER SCHUR Q-FUNCTIONS ARE SOLUTIONS OF BKP HIERARCHY

2. Schur Q-functions

Let $\mathcal{B}$ be the ring of symmetric functions in variables $(x_1, x_2, \ldots)$. Consider the families of the following symmetric functions:

- Elementary symmetric functions: $\{e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \mid k = 0, 1, \ldots\}$
- Complete symmetric functions: $\{h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} \mid k = 0, 1, \ldots\}$
- Symmetric power sums: $\{p_k = \sum x_i^k \mid k = 0, 1, \ldots\}$

It is well-known ([14] Chapter I.2), that each of these families generate $\mathcal{B}$ as a polynomial ring:

$$\mathcal{B} = \mathbb{C}[p_1, p_2, p_3, \ldots] = \mathbb{C}[e_1, e_2, e_3, \ldots] = \mathbb{C}[h_1, h_2, h_3, \ldots].$$

Combine the families $h_k, e_k, p_k$ into generating functions

$$H(u) = \frac{\prod_i x_i + u}{\prod_i x_i - u}, \quad E(u) = \frac{\sum_k e_k u^k}{1 + x_i/u}, \quad P(u) = \frac{\sum_k p_k u^k}{1 + x_i/u}.$$

The following facts are well-known ([14] Chapter I.2).

**Lemma 2.1.**

$$H(u) = \prod_i \frac{1}{1 - x_i/u}, \quad E(u) = \prod_i 1 + x_i/u, \quad E(-u)H(u) = 1,$$

$$H(u) = \exp \left( \sum_{n \geq 1} \frac{1}{n} p_n u^n \right), \quad E(u) = \exp \left( \sum_{n \geq 1} \frac{(-1)^{n-1} p_n}{n} u^n \right).$$

We introduce one more family of symmetric functions $\{Q_k = Q_k(x_1, x_2, \ldots)\}$ with $(k = 0, 1, \ldots)$ as the coefficients of generating function

$$Q(u) = \sum_{k \geq 0} Q_k u^k = \prod_i \frac{u + x_i}{u - x_i}. \quad (2.1)$$

Then from Lemma 2.1 and (2.1) we get immediately following relations.

**Lemma 2.2.**

$$Q(u) = E(u)H(u) = R(u)^2, \quad R(u) = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \frac{p_n}{nu^n} \right),$$

where $\mathbb{N}_{odd} = \{1, 3, 5, \ldots\}.$
Note that $Q(u)Q(-u) = 1$, which implies that $Q_r$ with even $r$ can be expressed algebraically through $Q_r$ with odd $r$:

$$Q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1}Q_r Q_{2m-r} + \frac{1}{2} (-1)^{m-1}Q_m^2.$$ 

More generally, Schur $Q$-functions $Q_\lambda$ labeled by strict partitions are defined as a specialization of Hall-Littlewood polynomials ([14] Chapter III.2).

**Definition 2.1.** Let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l)$ be a strict partition. Let $l \leq N$. Schur $Q$-polynomial $Q_\lambda(x_1, \ldots x_N)$ is the symmetric polynomial in variables $x_i$’s defined by the formula

$$Q_\lambda(x_1, \ldots, x_N) = \frac{2^l}{(N-l)!} \sum_{\sigma \in S_N} \prod_{i=1}^{l} x_{\lambda_i}^{x_{\sigma(i)}} \prod_{1 \leq i < j \leq N} \frac{x_{\sigma(i)} + x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}. \quad (2.2)$$

Alternatively, Schur $Q$-polynomial $Q_\lambda = Q_\lambda(x_1, \ldots x_N)$ for $N > l$ is the coefficient of $u^{-\lambda_1} \cdots u^{-\lambda_l}$ in the formal generating function

$$Q(u_1, \ldots, u_l) = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}} \frac{Q_\lambda}{u^{\lambda_1} \cdots u^{\lambda_l}} = \prod_{1 \leq i < j \leq l} \frac{u_j - u_i}{u_j + u_i} \prod_{i=1}^{l} Q(u_i), \quad (2.3)$$

where it is understood that

$$\frac{u_j - u_i}{u_j + u_i} = 1 + 2 \sum_{r \geq 1} (-1)^r u_i^r u_j^{-r},$$

and $Q(u)$ is given by (2.1) ([14] Chapter III, (8.8)). Schur $Q$-polynomials have a stabilization property, hence, one can omit the number $N$ of variables $x_i$’s as long as it is not less than the length of the partition $\lambda$ and consider $Q_\lambda$ as functions of infinitely many variables $(x_1, x_2 \ldots)$.

### 3. Action of neutral fermions on bosonic space $\mathcal{B}_{odd}$

Consider the subalgebra $\mathcal{B}_{odd}$ of $\mathcal{B}$ generated by odd ordinary Schur $Q$-functions: $\mathcal{B}_{odd} = \mathbb{C}[Q_1, Q_3, \ldots]$. It is known that $\mathcal{B}_{odd}$ is also a polynomial algebra in odd power sums $\mathcal{B}_{odd} = \mathbb{C}[p_1, p_3, \ldots]$ and that Schur $Q$-functions $Q_\lambda$ labeled by strict partitions constitute a linear basis of $\mathcal{B}_{odd}$ ([14] Chapter III.8, (8.9)).

Define operators $\{\varphi_k\}_{k \in \mathbb{Z}}$ acting on the coefficients of generating functions $Q(u_1, \ldots, u_l)$ by the rule

$$\Phi(v)Q(u_1, \ldots, u_l) = Q(v, u_1, \ldots, u_l) \quad (3.1)$$

with $\Phi(v) = \sum_{m \in \mathbb{Z}} \varphi_m v^{-m}$. Then in the expansion (2.3)

$$\varphi_m : Q_\lambda \mapsto Q_{(m, \lambda)}.$$
Observe that from (2.3)

\[(\Phi(u)\Phi(v) + \Phi(v)\Phi(u))Q(u_1, \ldots, u_l)\]

\[= 2 \left( 1 + \sum_{r \geq 1} (-1)^r u^r v^{-r} + \sum_{r \geq 1} (-1)^r v^r u^{-r} \right) A(u, v, u_1, \ldots, u_l)\]

\[= 2 \sum_{r \in \mathbb{Z}} \left( \frac{-u}{v} \right)^r A(u, v, u_1, \ldots, u_l),\]

where

\[A(u, v, u_1, \ldots, u_l) = \prod_{1 \leq j \leq l} \frac{(u_j - u)(u_j - v)}{(u_j + u)(u_j + v)} Q(u)Q(v)Q(u_1, \ldots, u_l).\]

Using that \(\delta(u, v) = \sum_{r \in \mathbb{Z}} u^r v^{-(r+1)}\) is a formal delta distribution with the property \(\delta(u, v) a(u) = \delta(u, v) a(v)\) and that \(Q(u)Q(-u) = 1\), we get

\[(\Phi(u)\Phi(v) + \Phi(v)\Phi(u)) Q(u_1, \ldots, u_l) = 2v \delta(-u, v) Q(u_1, \ldots, u_l). \quad (3.2)\]

Since coefficients of the expansion of \(Q(u_1, \ldots, u_l)\) in powers of \(u_1, \ldots, u_l\) include Schur Q-functions \(Q_{\lambda}\), and the later form a linear basis of \(B_{\text{odd}}\), one can show that (3.1) provides the action of well-defined operators \(\{\varphi_k\}_{k \in \mathbb{Z}}\) on \(B_{\text{odd}}\):

\[\varphi_k(Q_{\lambda}) = Q_{(k, \lambda)}.\]

Relation (3.2) on generating functions is equivalent to the commutation relations

\[[\varphi_m, \varphi_n]_+ = 2(-1)^m \delta_{m+n,0} \quad \text{for} \quad m, n \in \mathbb{Z}. \quad (3.3)\]

Thus, operators \(\{\varphi_i\}_{i \in \mathbb{Z}}\) and 1 provide the action for Clifford algebra \(Cl_\varphi\) of neutral fermions on the Fock space \(B_{\text{odd}}\). Note that for any strict partition \(\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l)\)

\[Q(\lambda_1, \ldots, \lambda_l) = \varphi_{\lambda_1} \cdots \varphi_{\lambda_l}(1), \quad (3.4)\]

or in terms of generating functions,

\[Q(u_1, \ldots, u_l) = \Phi(u_1) \cdots \Phi(u_l)(1). \quad (3.5)\]

Formulae (3.4), (3.5) sometimes are called vertex operator realization of Schur Q-functions.

4. VERTEX OPERATOR FORM OF FORMAL DISTRIBUTION OF NEUTRAL FERMIONS

It will be convenient for us to consider \(B_{\text{odd}}\) as the subring of the ring of symmetric functions \(\mathcal{B}\). This allows us to recover the well-known vertex operator form of the formal distribution of neutral fermions \(\Phi(u)\) from no-less celebrated properties of generating functions of complete and elementary symmetric functions. All of these properties are discussed in [14] Chapter I.
The ring of symmetric functions $B$ possesses a bilinear form $(\cdot, \cdot)$ ([14] Chapter I (4.5)) defined on the linear basis of monomials of power sums labeled by partitions $\lambda$ and $\mu$ as

$$(p_{\lambda_1} \ldots p_{\lambda_l}, p_{\mu_1} \ldots p_{\mu_l}) = z_\lambda \delta_{\lambda, \mu},$$

where $z_\lambda = \prod i^m m_i!$ and $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$.

We will use this form and its restriction to $B_{\text{odd}}$ to define adjoint operators of the multiplication operators. By definition, given an element $f \in B$, the operator $f^\perp$ adjoint to the operator of multiplication by $f$ is given by the rule

$$(f^\perp g, h) = (g, fh)$$

for any $g, h \in B$.

Chapter I. 5 Example 3 of [14] contains the following statement. Consider a symmetric function $f = f(p_1, p_2, \ldots)$ expressed as a polynomial in power sums $p_i$. Then adjoint operator on $B$ to the multiplication operator by $f$ is given by

$$f^\perp = f \left( \frac{\partial}{\partial p_1}, \frac{2\partial}{\partial p_2}, \frac{3\partial}{\partial p_3}, \ldots \right).$$

In particular $p_n^\perp = n\partial/\partial p_n$.

Combine the corresponding adjoint operators of the families $h_k, e_k, p_k$ and $Q_k$ into generating functions

$$H^\perp(u) = \sum_{k \geq 0} h_k^\perp u^k, \quad E^\perp(u) = \sum_{k \geq 0} e_k^\perp u^k, \quad P^\perp(u) = \sum_{k \geq 1} p_k^\perp u_k, \quad Q^\perp(u) = \sum_{k \geq 0} Q_k^\perp u_k.$$

Then (4.1) immediately implies the following relations.

**Lemma 4.1.**

$$H^\perp(u) = \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right), \quad E^\perp(u) = \exp \left( \sum_{n \geq 1} (-1)^{n-1} \frac{\partial}{\partial p_n} u^n \right).$$

$$Q^\perp(u) = E^\perp(u) H^\perp(u) = R^\perp(u)^2, \quad R^\perp(u) = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{\partial}{\partial p_n} u^n \right).$$

where $\mathbb{N}_{\text{odd}} = \{1, 3, 5, \ldots\}$.

The proof of next lemma is outlined in [14] Chapter I.5 Example 29.

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1Traditionally, one uses rescaled form on $B_{\text{odd}}$ defined as $(p_\lambda, p_\mu) = 2^{-l(\lambda)} z_\lambda \delta_{\lambda, \mu}$, where $l(\lambda)$ is the number of parts of $\lambda$, but rescaling is not necessary for our purposes, since in the rescaled form $p_n^\perp = n/2 \cdot \partial/\partial p_n$ (see [14] Chapter III.8. Example 11)
Lemma 4.2. The following commutation relations on generating functions of multiplication and adjoint operators acting on $\mathcal{B}$ hold.

\[
H^\perp(u) \circ H(v) = (1 - u/v)^{-1} H(u) \circ H^\perp(v),
\]
\[
H^\perp(u) \circ E(v) = (1 + u/v) E(u) \circ H^\perp(v),
\]
\[
E^\perp(u) \circ H(v) = (1 + u/v) H(u) \circ E^\perp(v),
\]
\[
E^\perp(u) \circ E(v) = (1 - u/v)^{-1} E(u) \circ E^\perp(v).
\]

Corollary 4.1.

\[
H^\perp(u) \circ Q(v) = \frac{v + u}{v - u} Q(u) \circ H^\perp(v),
\]
\[
E^\perp(u) \circ Q(v) = \frac{v + u}{v - u} Q(u) \circ E^\perp(v),
\]
\[
R^\perp(u) \circ Q(v) = \frac{v + u}{v - u} Q(u) \circ R^\perp(v).
\]

Proof. For the first and second one we use that $Q(u) = E(u)H(u)$. Observe that

\[H^\perp(u)|_{\mathcal{B}_{odd}} = E^\perp(u)|_{\mathcal{B}_{odd}} = R^\perp(u)|_{\mathcal{B}_{odd}}.\]

In other words, since $Q_k$ does not depend on even power sums $p_{2r}$, we can add terms $\partial/\partial p_{2r}$ in the sum under the exponent when applying to elements of $\mathcal{B}_{odd}$:

\[
R(u)^\perp(Q(v)) = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \frac{\partial}{\partial p_n} u^n \right) Q(v) = \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_n} u^n \right) Q(v) = H^\perp(u)Q(v).
\]

We arrive at the vertex operator form of formal distribution of neutral fermions.

Proposition 4.1.

\[
\Phi(v) = Q(v) R(-v)^\perp = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \frac{2p_n}{n} \frac{1}{v^n} \right) \exp \left( - \sum_{n \in \mathbb{N}_{odd}} \frac{\partial}{\partial p_n} v^n \right). \tag{4.2}
\]

Proof. From Corollary 4.1, the action of the operator $Q(v) R(-v)^\perp$ on the coefficients of generating function $Q(u_1, \ldots, u_l)$ coincides with the action of $\Phi(v)$:

\[
Q(v) R(-v)^\perp(Q(u_1, \ldots, u_l)) = Q(v) \prod_{1 \leq i < j \leq l} \frac{u_j - u_i}{u_j + u_i} R(-v)^\perp \left( \prod_{i=1}^l Q(u_i) \right)
\]
\[
= Q(v) \prod_{1 \leq j < i \leq l} \frac{u_j - u_i}{u_j + u_i} \prod_{i=1}^l \frac{v - u_i}{v + u_i} \prod_{i=1}^l Q(u_i) = Q(v, u_1, \ldots, u_l).
\]

Since coefficients of $Q(u_1, \ldots, u_l)$ contain linear basis of $\mathcal{B}_{odd}$, the equality (4.2) follows. \qed
5. Neutral fermions bilinear identity

In this section we use the vertex operator form \((4.2)\) of \(\Phi(u)\) to convert neutral fermions bilinear identity into BKP hierarchy of partial differential equations. This is a well-known procedure that we provide here in details for the convenience of the reader. Then a simple observation why Schur Q-functions constitute solutions of neutral fermions bilinear identity, and hence of BKP hierarchy, follows.

Let
\[
\Omega = \sum_n \varphi_n \otimes (-1)^n \varphi_{-n}.
\]
One looks for the solutions in \(B_{\text{odd}} \otimes B_{\text{odd}}\) of \textit{neutral fermions bilinear identity}
\[
\Omega(\tau \otimes \tau) = \tau \otimes \tau, \tag{5.1}
\]
where \(\tau = \tau(\tilde{p}) = \tau(2p_1, 2p_3/3, 2p_5/5, \ldots)\) and \(\tilde{r} = (2r_1, 2r_3/3, 2r_5/5, \ldots)\). It is known \cite{2,3,4,11} that \((5.1)\) is equivalent to an infinite integrable system of partial differential equations called BKP hierarchy.

Note that \(\Omega\) is the constant coefficient of the formal distribution \(\Phi(u) \otimes \Phi(-u)\), or, in terms of residue,
\[
\Omega = \text{Res}_{u=0} \frac{1}{u} \Phi(u) \otimes \Phi(-u). \tag{5.2}
\]
We identify \(B_{\text{odd}} \otimes B_{\text{odd}}\) with \(\mathbb{C}[p_1, p_3, \ldots] \otimes \mathbb{C}[r_1, r_3, \ldots]\) – two copies of polynomial rings, where variables in each of them play role of power sum symmetric functions.

Set \(\tilde{p} = (2p_1, 2p_3/3, 2p_5/5, \ldots)\) and \(\tilde{r} = (2r_1, 2r_3/3, 2r_5/5, \ldots)\). Then \(\partial_{\tilde{p}_n} = 2\partial_{\tilde{p}_n}/n\) and
\[
\Phi(u) \tau \otimes \Phi(-u) \tau = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{(\tilde{p}_n - \tilde{r}_n)}{u^n} \right) \exp \left( - \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{2}{n} \left( \frac{\partial}{\partial \tilde{p}_n} - \frac{\partial}{\partial \tilde{r}_n} \right) u^n \right) \tau(\tilde{p}) \tau(\tilde{r}).
\]
Introduce the change of variables
\[
p_n = x_n - y_n, \quad \tilde{r}_n = x_n + y_n.
\]
Then
\[
\Phi(u) \tau \otimes \Phi(-u) \tau = \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} -2y_n \frac{1}{u^n} \right) \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{2}{n} \frac{\partial}{\partial y_n} u^n \right) \tau(x-y) \tau(x+y)
\]
with \((x \pm y) = (x_1 \pm y_1, x_3 \pm y_3, x_5 \pm y_5, \ldots)\).

\textbf{Definition 5.1.} Let \(P(D)\) be a multivariable polynomial in the collection of variables \(D = (D_1, D_2, \ldots)\), let \(f(x), g(x)\) be differentiable functions in \(x = (x_1, x_2, \ldots)\).

\textit{Hirota derivative} \(P(D)f \cdot g\) is a function in variables \((x_1, x_2, \ldots)\) given by expression
\[
P(D)f \cdot g = P(\partial_{z_1}, \partial_{z_2}, \ldots) f(x + z) g(x - z)|_{z=0},
\]
where \(x \pm z = (x_1 \pm z_1, x_2 \pm z_2, \ldots)\).
For example,
\[ D^n_i f \cdot g = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\partial^k f}{\partial x_i^k} \frac{\partial^{n-k} g}{\partial x_i^{n-k}}, \]
which implies in particular that odd Hirota derivatives are tautologically zero when \( f = g \):
\[ D^{2n+1}_i f \cdot f = 0 \quad \text{for } n = 0, 1, 2, \ldots. \]
The following lemma allows one to rewrite bilinear identity (5.1) in terms of Hirota derivatives.

**Lemma 5.1.**
\[ \exp \left( \sum_{n \in \mathbb{N}_{odd}} 2 \frac{\partial}{\partial y_n} u^n \right) \tau(x - y) \tau(x + y) = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \left( y_n + \frac{2}{n} u^n \right) D_n \right) \tau \cdot \tau. \]

**Proof.** By Taylor series expansion,
\[ e^{a \partial/\partial_y} g(y) = \sum_{n=0}^{\infty} \frac{a^n g^{(n)}(y)}{n!} = g(y + a). \quad (5.3) \]
Applying (5.3) twice with \( t = (t_1, t_3, t_5, \ldots), \tilde{u} = (2u, 2u^3/3, 2u^5/5, \ldots), \)
\[ \exp \left( \sum_{n \in \mathbb{N}_{odd}} 2 \frac{\partial}{\partial y_n} u^n \right) \tau(x - y) \tau(x + y) = \tau(x + y + \tilde{u}) \tau(x - y - \tilde{u}) \]
\[ = \tau(x + y + \tilde{u} + t) \tau(x - (y + \tilde{u} + t)) |_{t=0} \]
\[ = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \left( y_n + \frac{2}{n} u^n \right) \frac{\partial}{\partial t_n} \right) \tau(x + t) \tau(x - t) |_{t=0}. \]
Thus, we can write in terms of Hirota derivatives
\[ \Phi(u) \tau \otimes \Phi(-u) \tau = \exp \left( \sum_{n \in \mathbb{N}_{odd}} \frac{-2y_n}{u^n} \right) \exp \left( \sum_{n \in \mathbb{N}_{odd}} \frac{2}{n} D_n u^n \right) \exp \left( \sum_{n \in \mathbb{N}_{odd}} y_n D_n \right) \tau \cdot \tau. \quad (5.4) \]

In order to compute \( \text{Res}_{u=0} \frac{1}{u} \Phi(u) \tau \otimes \Phi(-u) \tau \), which is just the coefficient of \( u^0 \) of \( \Phi(u) \tau \otimes \Phi(-u) \tau \), we recall the following well-known facts on the composition of exponential series with generating series. Their proofs can be done e.g. by induction, or again found in [14] Chapter I.
Proposition 5.1. Let \( S(u) = \sum_{k=0}^{\infty} S_k \frac{u^k}{k!} \) and \( X(u) = \sum_{k=1}^{\infty} X_k \frac{u^k}{k!} \) be related by \( \exp(X(u)) = S(u) \).

Then the following statements hold:

\[
S_k = \sum_{s=1}^{k} \sum_{l_1 + 2l_2 + \ldots + sl_s = k, l_i \geq 1} \frac{1}{l_1! \ldots l_s!} X_1^{l_1} \ldots X_s^{l_s},
\]

\[
S_k = \det \frac{1}{n!} \begin{pmatrix} X_1 & -1 & 0 & 0 & \ldots & 0 \\ 2X_2 & X_1 & -2 & 0 & \ldots & 0 \\ 3X_3 & 2X_2 & X_1 & -3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ kX_k & (k-1)X_{k-1} & (k-2)X_{k-2} & (k-3)X_{k-3} & \ldots & X_1 \end{pmatrix},
\]

\[
X_k = (-1)^{k-1} \frac{k}{k} \det \begin{pmatrix} S_1 & 1 & 0 & 0 & \ldots & 0 \\ 2S_2 & S_1 & 1 & 0 & \ldots & 0 \\ 3S_3 & 2S_2 & S_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ kS_k & S_{k-1} & S_{k-2} & S_{k-3} & \ldots & S_1 \end{pmatrix}.
\]

Example 5.1.

\[
S_0 = 1, \quad S_1 = X_1, \quad S_2 = \frac{1}{2} X_1^2 + X_2, \quad S_3 = S_3 + X_2X_1 + \frac{1}{6} X^3,
\]

\[
S_4 = X_4 + X_3X_1 + \frac{1}{2} X_2^2 + \frac{1}{2} X_2X_1^2 + \frac{1}{24} X_1^4.
\]

By Lemma 2.1, when \( X \) variables in these formulas are interpreted as normalized power sums \( X_k = p_k/k \), \( S_k \)’s are identified with complete symmetric functions \( h_k \)’s.

Example 5.2. Let \( X_{2k} = 0 \) for \( k = 1, 2, \ldots \). Then the first few \( S_n = S_n(X_1, 0, X_3 \ldots) \) are given by

\[
S_0 = 1, \quad S_1 = X_1, \quad S_2 = \frac{1}{2} X_1^2, \quad S_3 = X_3 + \frac{1}{6} X^3,
\]

\[
S_4 = X_3X_1 + \frac{1}{24} X_1^4, \quad S_5 = \frac{1}{120} X_1^5 + \frac{1}{2} X_1^2X_3 + X_5,
\]

\[
S_6 = \frac{1}{720} X_1^6 + \frac{1}{6} X_1^3X_3 + \frac{1}{2} X_3^2 + X_1X_5,
\]

\[
S_7 = \frac{1}{5040} X_1^7 + \frac{1}{24} X_1^4X_3 + \frac{1}{2} X_1X_3^2 + \frac{1}{2} X_1X_5 + X_7.
\]

Note that by Lemma 2.2 when \( X \) variables in these formulas are interpreted as odd normalized power sums \( X_{2k+1} = \frac{2p_{2k+1}}{2k+1} \), \( S_k \)’s are identified with Schur \( Q \)-functions \( Q_k \)’s.
Using the statement of Proposition 5.1, we can write the coefficient of $u^0$ of (5.4) as

$$\sum_{m=0}^{\infty} S_m(\tilde{y}) S_m(\tilde{D}) \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} y_n D_n \right) \tau \cdot \tau = \tau(x - y) \cdot \tau(x + y), \tag{5.5}$$

where $\tilde{y} = (-2y_1, 0, -2y_3, \ldots), \tilde{D} = (2D_1, 0, 2D_3/3, 0, \ldots)$.

Note that $S_0 = 1$ and $\exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} y_n D_n \right) \tau \cdot \tau = \tau(x - y) \cdot \tau(x + y)$, hence we can rewrite (5.5) as

$$\sum_{m=1}^{\infty} S_m(\tilde{y}) S_m(\tilde{D}) \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} y_n D_n \right) \tau \cdot \tau = 0. \tag{5.6}$$

To obtain the equations of BKP hierarchy, one expands the left hand side of (5.6) in monomials $y_1^{m_1} y_2^{m_2} \ldots y_N^{m_N}$ to obtain as coefficients Hirota operators in terms of $D_k$'s.

For example, let us compute the coefficient of $y_2^3$. In the expansion of

$$(S_1(\tilde{y}) S_1(\tilde{D}) + S_2(\tilde{y}) S_2(\tilde{D}) + S_3(\tilde{y}) S_3(\tilde{D}) + \ldots)(1 + \sum y_i D_i + \frac{1}{2} (\sum y_i D_i)^2 + \ldots)$$

the term $y_3^2$ appears in $S_3(\tilde{y}) S_3(\tilde{D}) \times y_3 D_3$ and in $S_6(\tilde{y}) S_6(\tilde{D}) \times 1$. Using the expansions of Example 5.2, the coefficient of $y_3^2$ is

$$-2S_3(\tilde{D})D_3 + 2S_6(\tilde{D}) = \frac{8}{45}(D_1^6 - 5D_1 D_3 - 5D_3^2 + 9D_1 D_5),$$

which provides the Hirota bilinear form of the BKP equation that gives the name to the hierarchy:

$$(D_1^6 - 5D_1 D_3 - 5D_3^2 + 9D_1 D_5) \tau \cdot \tau = 0.$$

**Remark 5.1.** Writing the residue (5.2) as a contour integral, one gets BKP in its integral form:

$$\oint \frac{1}{2\pi i u} \exp \left( \sum_{n \in \mathbb{N}_{\text{odd}}} (\tilde{p}_n - \tilde{r}_n) \frac{1}{u^n} \right) \exp \left( - \sum_{n \in \mathbb{N}_{\text{odd}}} \frac{2}{n} \left( \frac{\partial}{\partial \tilde{p}_n} - \frac{\partial}{\partial \tilde{r}_n} \right) u^n \right) \tau(\tilde{p}) \tau(\tilde{r}) = \tau(\tilde{p}) \tau(\tilde{r}).$$

6. **Commutation relation for the bilinear identity.**

Our goal is to show that multiparameter Schur Q-functions are solutions of neutral fermions bilinear identity (5.1), thus they provide solutions of BKP hierarchy.

Let $X = \sum_{n>0} A_n \varphi_n$ for some $A_n \in \mathbb{C}$. From (3.3) one gets $X^2 = 0$.

**Proposition 6.1.**

$$\Omega(X \otimes X) = (X \otimes X)\Omega.$$
Proof.

\[ \Omega(X \otimes X) = \sum_{k \in \mathbb{Z}} \varphi_k X \otimes (-1)^k \varphi_{-k} X = \sum_{k \in \mathbb{Z}} (-X \varphi_k + [\varphi_k, X]_+) \otimes (-1)^k (-X \varphi_{-k} + [\varphi_{-k}, X]_+) \]

\[ = (X \otimes X)\Omega - \sum_{n>0} 2(-1)^n A_n \otimes X(-1)^n \varphi_n - \sum_{n>0} X\varphi_n \otimes 2(-1)^n A_n \]

\[ + 4 \sum_{i \in \mathbb{Z}} \sum_{m,n>0} (-1)^n A_n \delta_{n+i,0} \otimes (-1)^m A_m \delta_{m-i,0} \]

\[ = (X \otimes X)\Omega - 2 \otimes X^2 - X^2 \otimes 2 + 4 \sum_{m,n>0} (-1)^n A_n \otimes (-1)^m A_m \delta_{m+n,0}. \]

We use that \( X^2 = 0 \), and since both \( m \) and \( n \) in the last sum are always positive, the last term is also zero. \( \square \)

**Corollary 6.1.** Let \( \tau \in B_{\text{odd}} \) be a solution of (5.1), and let \( X = \sum_{n>0} A_n \varphi_n \) with \( A_n \in \mathbb{C} \). Then \( \tau' = X \tau \) is also a solution of (5.1).

Vertex operator presentation (3.4) of Schur Q-functions and Corollary 6.1 immediately imply that Schur Q-functions are solutions of (5.1), since constant function 1 is a solution of (5.1). This argument reproves the result of [17] and easily extends to more general case of multiparameter Schur Q-functions defined in the next section.

### 7. Multiparameter Schur Q-functions are solutions of BKP hierarchy

Multiparameter Schur Q-functions were introduced in [9] as generalizing of definition (2.2). Fix an infinite sequence of complex numbers \( a = (a_0, a_1, a_2, \ldots) \). Consider the analogue of a power of a variable \( x \)

\[ (x|a)^k = (x - a_0)(x - a_2)\ldots(x - a_{k-1}). \]

We also define a shift operation \( \tau : a_k \mapsto a_{k+1} \), so that

\[ (x|\tau a)^k = (x - a_1)(x - a_2)\ldots(x - a_k). \]

**Definition 7.1.** Let \( \alpha = (\alpha_1, \ldots, \alpha_l) \) be a vector with positive integer coefficients \( \alpha_i \in \mathbb{Z}_{>0} \). Multiparameter Schur Q-function in variables \( (x_1, \ldots, x_N) \) with \( l \leq N \) is defined by

\[ Q^{(a)}_{\alpha}(x_1, \ldots, x_N) = \frac{2^l}{(N-l)!} \sum_{\sigma \in \mathcal{S}_N} \prod_{i=1}^l (x_{\sigma(i)}|a)^{\alpha_i} \prod_{i\leq l, i<j \leq N} \frac{x_{\sigma(i)} + x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}. \tag{7.1} \]

When \( a = (0, 0, 0, \ldots) \) and \( \alpha \) is a strict partition, one gets back (2.2), the classical Schur Q-functions \( Q^{(a)}_{\alpha}(x_1, \ldots, x_N) \). The evaluation \( a = (0, 1, 2, \ldots) \) gives shifted
Schur Q-functions denoted as $Q_{\alpha}^*(x)$, those applications are outlined in the Introduction. The multiparameter Schur Q-functions enjoy a stability property, hence one can consider $Q_{\alpha}^{(a)}(x_1, x_2, \ldots)$ to be a function of infinitely many variables.

Note from (7.1) that for any permutation $\sigma \in S_l$,

$$Q_{\alpha}^{(a)}(x_1, \ldots, x_N) = (-1)^{\sigma}Q_{\sigma(a)}^{(a)}(x_1, \ldots, x_N),$$  \quad(7.2)$$

where $(-1)^{\sigma}$ is the sign of permutation $\sigma$ ([13], Proposition 3). Hence, $Q_{\alpha}^{(a)} = 0$ if $\alpha_i = \alpha_j$ for some $i, j$, and for a vector $\alpha = (\alpha_1, \ldots, \alpha_l)$ with positive distinct integer coefficients $\alpha_i \in \mathbb{Z}_{>0}$, function $Q_{\alpha}^{(a)}$ coincides up to a sign with another $Q_{\alpha'}^{(a)}$ labeled by strict partition $\alpha'$.

Consider a part of the generating function (2.3) of ordinary Schur Q-functions that corresponds only to positive values of $\lambda_i$:

$$Q^+(u_1, \ldots, u_l) = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}_{>0}} \frac{Q_{\lambda}^{(a)}}{u_1^{\lambda_1} \cdots u_l^{\lambda_l}}.$$  \quad(7.3)$$

By (7.2), every non-zero coefficient of $Q^+(u_1, \ldots, u_l)$ up to a sign coincides with a classical Schur Q-function labeled by an appropriate strict partition. In [13] the following remarkable observation is made.

**Theorem 7.1.** [13] For any sequence $a = (0, a_1, a_2, \ldots)$

$$Q^+(u_1, \ldots, u_l) = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}_{>0}} \frac{Q_{\lambda}^{(a)}}{(u_1 | a)^{\lambda_1} \cdots (u_l | a)^{\lambda_l}}.$$  \quad(7.4)$$

The theorem is proved by induction on the length of the vector $\lambda$ with the use of the fact that for any sequence $a = (0, a_1, a_2, \ldots)$

$$\sum_{i=1}^{\infty} \frac{(x | a)^i}{(u | a)^i} = \frac{x}{u - x}.$$  \quad(7.5)$$

Thus, Theorem 7.1 suggests that multiparameter Schur Q-functions are obtained from classical Schur Q-functions by the change of the basis of expansion $\{1/u^k\} \rightarrow \{1/(u | a)^k\}$ in the generating function $Q^+(u_1, \ldots, u_l)$. One can check the following transitions between expansions.
Theorem 7.2. For \( n = 0, 1, 2, \ldots \)
\[
(u - a_1) \ldots (u - a_n) = \sum_{k=0}^{\infty} (-1)^{n-k} h_{n-k}(a_1, \ldots, a_n) u^k,
\]
\[
\frac{1}{(u - a_1) \ldots (u - a_n)} = \sum_{k=0}^{\infty} h_{k-n}(a_1, \ldots, a_n) u^{-k},
\]
\[
u^n = \sum_{k=0}^{\infty} h_{n-k}(a_1, \ldots, a_{k+1}) (u - a_1) \ldots (u - a_k),
\]
\[
\frac{1}{u^n} = \sum_{k=0}^{\infty} (-1)^{n-k} e_{k-n}(a_1, \ldots, a_{k-1}) \frac{1}{(u - a_1) \ldots (u - a_k)}.
\]

Theorem 7.2 immediately implies the following relations between classical and multiparameter Schur Q-functions (see also [9] Theorem 10.2)

Corollary 7.1. For any integer vector \( \alpha = (\alpha_1, \ldots, \alpha_l) \) with \( \alpha_i \in \mathbb{Z}_{>0} \)
\[
Q^{(\alpha)}_{\lambda} = \sum_{\lambda_1, \ldots, \lambda_l \in \mathbb{Z}_{>0}} (-1)^{\sum \lambda_i - \sum \alpha_i} e_{\alpha_1 - \lambda_1}(a_1, \ldots, a_{\alpha_1 - 1}) \ldots e_{\alpha_l - \lambda_l}(a_1, \ldots, a_{\alpha_l - 1}) Q_{\lambda}.
\]

Proposition 7.1. Multiparameter Schur Q-functions \( Q^{(\alpha)}_{\lambda} \) are solutions of \((5.1)\).

Proof. Constant polynomial 1 is obviously a solution of \((5.1)\) in \( B_{\text{odd}} \). By vertex operator presentation \((3.4)\) and Corollary 7.1,

\[
Q^{(\alpha)}_{\lambda} = \sum_{\lambda_1, \ldots, \lambda_l \geq 0} A_{\lambda_1, \alpha_1} \ldots A_{\lambda_l, \alpha_l} \varphi_{\lambda_1} \ldots \varphi_{\lambda_l} \cdot 1
\]

with \( A_{n,k} = (-1)^{n-k} e_{k-n}(a_1, \ldots, a_{k-1}) \). Hence,

\[
Q^{(\alpha)}_{\lambda} = X_{\alpha_1} \cdot X_{\alpha_1} \cdot 1,
\]

where \( X_m = \sum_{s=0}^{\infty} (-1)^{m-s} e_{s-m}(a_1, \ldots, a_{s-1}) \varphi_s \), and Corollary 6.1 proves the statement. \(\square\)

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MULTIPARAMETER SCHUR Q-FUNCTIONS ARE SOLUTIONS OF BKP HIERARCHY

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