New way to achieve synchronization in spatially extended systems

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We study the spatio-temporal behavior of simple coupled map lattices with periodic boundary conditions. The local dynamics is governed by two maps, namely, the sine circle map and the logistic map respectively. It is found that even though the spatial behavior is irregular for the regularly coupled (nearest neighbor coupling) system, the spatially synchronized (sometime chaotically synchronized) as well as periodic solution may be obtained by the introduction of three long range couplings at the cost of three nearest neighbor couplings.

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Recently there have been lot of interests in the small world network systems\textsuperscript{[1–5]}. The small world network systems lie in between the regular and the random networks where only a small fraction of long range links are introduced at the cost of the equal number of regular short range links keeping the total number of links conserved. The properties of such systems have been thoroughly studied very recently because of the fact that many biological, technological or social networks fall in this category. Intuitively it is expected that the information or source in the small world network will spread quickly through out the system because of the presence of some long range links.

On the other hand a number of phase models have been proposed over the recent years to describe the dynamical behavior of large population of nonlinear oscillators subject to a variety of coupling mechanisms\textsuperscript{[1–2]}. A major phenomenon that can be observed is the possibility of self synchronization among the members of the population. These can represent the fire flies, heart pace maker cells, pancreatic beta cells, neurons etc. as well as the circuit arrays\textsuperscript{[1]]. Different cells (units) in different systems may be locally governed by some specific rule. For example it may be governed by the sine circle map, logistic map or some differential equations depending on the physical or biological system. Furthermore different cells may be coupled to each other in various possible ways.

Even if the local dynamics is regular, the spatial as well as the temporal behavior of the regularly coupled system may be irregular. It might happen other way also, i.e., the dynamics of regularly coupled chaotic oscillators may turn out to be regular. We are interested to confine ourselves in the parameter regime where the spatial behavior of a regularly coupled lattice becomes irregular. Then the question we ask is whether it is possible to control the spatial irregularity by introducing some small fraction of long range couplings at the cost of the equal number of nearest neighbor couplings keeping the total number of couplings conserved. If so, then that would be a new way to control spatial irregularity in a spatially extended systems. Furthermore, if the solution turns out to be synchronized spatially but chaotic temporally, it will find good application to the secret communication network. We would like to investigate these aspects in one dimensional lattice where the local dynamics of individual lattice site is governed either by the sine circle map or the logistic map\textsuperscript{[1]}. The coupling we consider here is the simple unidirectional nearest neighbor coupling. First we discuss the unidirectionally coupled map lattice with the local dynamics defined by the sine circle map and then by the logistic map.

Here we consider a one dimensional periodic lattice of size $N$. The local dynamics at any site is governed by the following rule:

$$x(t + 1) = f(x(t)) = x(t) + \Omega - \frac{K}{2\pi} \sin(2\pi x(t)) \mod 1$$

(1)

where $x$ is the dynamical variable, $t$ is the time, $\Omega$ is the frequency and $K$ is the nonlinearity parameter. The map given by Eq. (1) is known as the sine circle map and very much similar to the dynamical equation in the circuits of Josephson Junction arrays. If all the sites in the system are independent of each other, the dynamics of the individual sites will be governed by Eq. (1). However, we are interested in the dynamics of the system when all the sites are directly or indirectly coupled to each other. The lattice sites may interact with each other in a various possible ways. But, here, we consider the situation where the dynamical variable at the $i^{th}$ site may be governed by the following rule:

$$x_i(t + 1) = (1 - \epsilon)f(x_i(t)) + \epsilon f(x_{i+1}(t)); \quad 1 \leq i < N,$$

and

$$x_N(t + 1) = (1 - \epsilon)f(x_N(t)) + \epsilon f(x_1(t))$$

(2)

where $\epsilon$ is the coupling parameter. The spatio-temporal behavior of the system has been studied extensively.\textsuperscript{[1]}

Even though the local dynamics is chaotic, the global dynamics of the system may turn out to be chaotic or synchronous.
regular depending on the parameters, initial condition, as well as the nature of the coupling. Our intention is to start with a random initial condition and fix the parameters such that the dynamics of the system becomes chaotic both spatially as well as temporally with the regular nearest neighbor coupling as described by Eq. (2).

![Graph](image1)

**FIG. 1.** All $x_i$’s at time $t=5001$ are plotted as a function of site index $i$ for a lattice of size 90 where all the sites are coupled to its nearest neighbor unidirectionally. The parameter values are $K = \sqrt{2}$, $\Omega = 0.3$ and $\epsilon = 0.5$ respectively.

For example, we chose the parameter values as $\epsilon=0.5$, $\Omega=0.3$ and $K = \sqrt{2}$. We consider the lattice size, $N=90$. We allow the system to evolve for 5000 steps before determining it’s state. The spatial behavior of the system described by Eq. (2) is shown in Fig. (1) where $x_i$’s are plotted as a function of $i$ for $t=5001$. We see that there is no regular spatial behavior. The temporal evolution for all the sites are also found to be irregular. The temporal evolution at the first site is shown in the Fig. (2) as an example.

We introduce the concept of small world network systems in the spatially extended system, namely, coupled map lattices. Suppose that we have a periodic one dimensional lattice of $N$ sites. If the lattice sites are coupled with its nearest neighbor site unidirectionally, there will be $N$ couplings or bonds. We will use the word bond in place of coupling for convenience. A fraction of these $N$ bonds are replaced by long range couplings. The replacement of nearest neighbor bond (coupling) by the long range bond is made randomly. In the process of replacement, we make sure that duplicates are not allowed and no part of the system becomes isolated. Under such rearrangements, the total number of bonds remain conserved. Since the replacement of bonds are done randomly, there will be many possible configurations for fixed number of bonds being replaced. Therefore, it is necessary to check if any of those configurations produces the synchronized solutions.

![Graph](image2)

**FIG. 2.** This shows the temporal evolution of the dynamical variable $x$ at site 1 for the same system with same parameters as used in Fig. 1.

So, we consider the system where $p$ fraction of nearest neighbor bonds are replaced by randomly chosen long range bonds. The parameter values are are chosen to be same as in Fig. (1). Initially all the sites are populated randomly and then they evolve according to Eq. (2) with the replacement of $p$ fraction of bonds. The solution appears to be stable after 5000 iterations. Since there are large number of possible configurations, the numerical experiment is carried out for 1000 configurations for a fixed value of $p$. The simulation is also carried out for different values of $p$ between 0 and 1. It is found that only two to ten percentage of configurations lead to synchronized solutions for each value of $p$. For $p = 0.5$ more number of configurations produce synchronized solution.

Thus we experience that there exist some configurations which may produce the synchronized solutions. Therefore, the control of the spatially chaotic behavior is possible by the above mentioned mechanism but the percentage value of configurations producing the synchronized solutions are very low. So, one has to go through various sets of re-wiring to achieve the synchronization and is therefore difficult to implement in the physical systems. One needs to know a re-wiring procedure where least number of bonds will be disturbed or replaced. Furthermore, one should know a systematic way of replacement of bonds to achieve the synchronization. Intuitively we suspect that the replacing of the nearest neighbor bonds may be made in a systematic manner and think that the bonds should be broken at regular interval in the lattice and the disturbed vertices should be connected in a regular fashion. More explicitly, let us divide the lattice into $n$ number of segments, each segment has length (L) equal to $\frac{N}{n}$. So, the bond between sites $i$ and $i+1$ should be broken and the $i^{th}$ site should be connected to $(i+L)^{th}$ site. Next, the bond between sites $(i+L)$ and $(i+L+1)$ should be broken and $(i+L)^{th}$ site should be connected
to \((i + 2L)^{th}\) site and so on. This procedure should be continued until the full lattice is covered. We note that if \(n = N\) or \(L = 1\), then the modified lattice remains same as the regular one. Therefore, we expect that \(n\) should be between 1 to \(N - 1\). We therefore do the numerical experiment for \(n=1, 2, 3, 4\) and so on. In fact we find that \(n=1\) and 3 cases work very well. Although, ideally, \(n=1\), would be the most suitable solution, we also study \(n=3\), since it is the most ideal case for studying multiple rewirings. We find that \(n=2\) and \(n=4\) have lesser probability to synchronize than \(n=3\), over wide regions of the \(\Omega - \epsilon\) phase space. We will discuss these concepts in details in a forthcoming article. For \(n=3\), the re-wiring rule is given explicitly for a lattice of size 90:

\[
x_i(t + 1) = f(x_i(t)) = \mu x_i(t)(1 - x_i(t))
\]

over a wide range of parameters. The phase diagram in the \(\epsilon - \Omega\) plane for \(k = \sqrt{2}\) is shown in Fig 4. The shaded region (in fig. 4) in the \(\epsilon - \Omega\) plane are the allowed parameter regime where one achieves the synchronized solution starting from a random initial conditions. We see that there is a large region in the \(\epsilon - \Omega\) plane where the synchronization is obtained through the re-wiring mechanism. The synchronized region for only one broken bond is larger than that of the 3 bond case by about 20-30 \%. However, for any other number of rewirings, the allowed part of the phase space is much smaller.

![Phase diagram in the $\Omega - \epsilon$ plane for the same system as described in Fig. 3. Here $K = \sqrt{2}$. The shaded region corresponds to the allowed parameter space to obtain the chaotic synchronization with the re-wiring mechanism.](image)

Here we consider the same system where the local dynamics is governed by the Logistic map. The logistic map has been extensively used to model a number of physical as well as biological phenomenon [11]. The map is given by

\[
x(t + 1) = f(x(t)) = \mu x(t)(1 - x(t))
\]

where \(\mu\) is the parameter controlling the local dynamics. The the dynamics of the variable \(x_i\) at the \(i^{th}\) site is governed by the Eq. (2) with \(f(x(t))\) as given in Eq. (3) where \(t\) represents the time, and \(\epsilon\) is the coupling strength. In this system also we find that one may obtain the spatially periodic solution and sometimes synchronized solution through the re-wiring rule as stated earlier.

In this system, we find that there is a critical value of the coupling strength \(\epsilon\) for each value of \(\mu\). For example, all \(x_i\)'s are plotted as a function of \(\epsilon\) for \(\mu=4\) in Figs. (5) and (6). Fig. 5 is for the system where all the sites are regularly coupled to its nearest neighbor. But Fig. 6 is for the rewired lattice. We clearly see from Fig. 5 that there is no spatial regularity. On the other hand,
for the rewired lattice (Fig. 6) there is a critical value of \( \epsilon \), say \( \epsilon_{cr} = 0.35 \) above which the solution is spatially periodic of period four. The spatial nature of the solution for the rewired lattice is verified in the entire \( \epsilon - \mu \) plane as shown in Fig 7. There are various regions corresponding to various kinds of solutions. The solution is spatially periodic with period one or four in the region I, the solution is temporally as well as spatially irregular in region II, the solution is spatially periodic of period four in region III and the solution is chaotically synchronized in region IV.

Thus we see that the re-wiring mechanism helps us to control the spatial irregularity in a large region of the parameter space. Furthermore we note the parameter region for both the spatial as well as temporal periodic solution also large. Therefore, this mechanism with the logistic map works well to control spatial and temporal irregularity in a large region of parameter space.

In summary, we have studied spatially extended system where the local dynamics are governed by the sine circle map and the logistic map. We have implemented the idea of small world network and investigated that spatial irregularity in the system can be controlled. We have shown that for a regularly coupled system showing irregular behavior spatially, synchronization can be achieved by disturbing a few regular links (nearest neighbor coupling) and establishing an equal number of long ranged links in a regular way. We notice that the region in the \( \Omega - \epsilon \) space where synchronization occurs depends on how many of these rewirings are done. For one and three rewirings, synchronization occurs over large regions of the \( \Omega - \epsilon \) space. We will establish the reason for this in a forthcoming article. Chaotic synchronization is achieved for the sine circle map with the re-wiring mechanism. On the other hand, spatially as well as temporally periodic solution is achieved for the logistic map. Thus we experience that the re-wiring mechanism helps us to achieve the chaotic synchronization for some local dynamics. The chaotic synchronization is an important solution in Josephson junction arrays. Furthermore it is very useful as far as the secret communication network is concerned. This mechanism may be tested with other local dynamics existing in physical or biological systems.
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