Magnetic fields and factored two-spheres

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Abstract

A magnetic monopole is placed at the centre of a ball whose surface $S^2$ is tiled by the symmetry group, $\Gamma$, of a regular solid. The quantum mechanics on the two-dimensional quotient $S^2/\Gamma$ is developed and the monopole charge is found to be quantised in an expected manner. The heat-kernel and $\zeta$–functions are evaluated and the Casimir energy is computed. Numerical approaches to the calculation of the derivative of the Barnes $\zeta$–function are presented.

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1. Introduction

In previous work, [1], we have discussed quantum field theory on orbifold factors of spheres $S^n/\Gamma$ where $\Gamma$ is a discrete subgroup of the orthogonal group, $O(n + 1)$. In this paper we wish to present an extension in the two-dimensional case ($n = 2$) to the situation where there is a uniform radial magnetic field passing through the surface of the sphere. This can be thought of as due to a magnetic monopole at the centre of a ball in an embedding $\mathbb{R}^3$. The motivation is partly to investigate the interplay between a magnetic field and a non-trivial topology/geometry induced by identification. Specifically we would be interested in what happens to the Dirac quantization in topologically interesting or singular manifolds (orbifolds). There is also continuing statistical mechanical interest in magnetic fields and two-dimensional domains.

The quotient group is the complete symmetry group of a regular solid and can be generated by reflections in the three (concurrent) planes of symmetry of the solid. The elements fall into two sets depending on whether they contain an even or an odd number of reflections. The even subset forms the rotational subgroup denoted now by $\Gamma \in SO(3)$. The odd subset is denoted by $\Gamma'$. If $\gamma'$ is any fixed element of $\Gamma'$ then as $\gamma$ runs over $\Gamma$, $\gamma\gamma'$ exhausts $\Gamma'$. In particular we can choose $\gamma' = \sigma$ where $\sigma$ is a reflection in a symmetry plane and so the complete group is

$$\Gamma' = \Gamma \cup \Gamma' = \Gamma \cup \Gamma \sigma. \quad (1)$$

The standard classification of finite subgroups (reflection groups) is given by Meyer, [2], and we use his notation. The general construction of the quotients, $S^2/\Gamma'$, which in this case are certain geodesic triangles on $S^2$ is sketched later. The vertices of these triangles are singular points. Joining them to the origin of the ambient $\mathbb{R}^3$ produces singular strings and our analysis can be extended to this three-dimensional setting [3]. The two-dimensional models that we study can be thought of as toy models for more general ‘textures’ in higher dimensions as laid down by Kibble, [see e.g. [4]].

Our main interest is in setting up the general framework and then applying it to some specific calculations such as the evaluation of vacuum energies. This will involve a certain amount of technical manipulation, especially with the properties of the Barnes $\zeta$–function. This function appears quite commonly, particularly in spherically symmetric situations and in the presence of magnetic fields or harmonic oscillators and we expect that our methods will have an applicability beyond the immediate one here.
2. Modes and group actions on the full sphere.

As modes on the full sphere, we can take the angular part of the Schrödinger equation solutions derived long ago by Tamm [5] and Fierz [6]. It would be more rigorous to use a fibre bundle formulation (Greub and Petry [7], Wu and Yang [8]) or geometric quantisation [9,10] but this is unnecessary for our purposes. For definiteness, we employ the modes denoted by \((Y_{qm})_a\) in Wu and Yang, corresponding to the string running down the negative \(z\)–axis. The modes are, up to normalisation, the SU(2) representation matrices \(D^{(l)*}_{m,-q}(\phi, \theta, -\phi)\) with \(l = |q|, |q + 1|, \ldots, -l \leq m \leq l\) [11]). \(2q\) is the monopole number with \(q \equiv e q/\hbar\), and \(2q \in \mathbb{Z}\). If the string runs up the positive \(z\)-axis, the modes are \(D^{(l)*}_{m,-q}(\phi, \theta, \phi)\).

The corresponding eigenvalues of \(H_{S^2}\), the angular part of \(- (\nabla - ieA)^2\), are

\[
\lambda_l = l(l+1) - q^2 = \left(l + \frac{1}{2}\right)^2 - \frac{1}{4} - q^2
\]

with degeneracy \(2l + 1\).

It is helpful to give the explicit form of the angular eigenfunctions in spherical polar coordinates,

\[
Y^{(l)}_{qm}(\theta, \phi) = N_{q_{lm}} \sin^{|q+m|}(\theta/2) \cos^{q-m}(\theta/2) P^{(l)}_{|q+m|+|q-m|}(\cos \theta) e^{i(q+m)\phi}
\]

where \(N_{q_{lm}}\) is a normalisation constant and the \(P^{\alpha, \beta}_n(x)\) are Jacobi polynomials. In the Wu-Yang formalism these are the solutions (sections) in the upper hemisphere. In the lower hemisphere the potential is taken to have a string along the positive \(z\)-axis. The two sets of solutions are related on the equator by the factor \(\exp(i2q\phi)\). Making this single valued gives the quantisation condition on \(q\), in this approach.

The basic means of finding the modes is to write \(H_{S^2}\psi = \lambda\psi\) explicitly as a differential equation in spherical polar coordinates,

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + i \frac{2q}{1 + \cos \theta} \frac{\partial \psi}{\partial \phi} - q^2 \frac{1 - \cos \theta}{1 + \cos \theta} \psi = \lambda \psi,
\]

and solve it assuming the separation \(\psi(\theta, \phi) = \exp(iu\phi)\text{P} (\cos \theta)\). Using this method we can suppose initially that \(q\) is arbitrary which leads to the same solutions as in (3) but which are characterised...
by two integers \( u \in \mathbb{Z}^+ \) and \( v \in \mathbb{Z} \), \([5]\). The relationship to the \( SU(2) \) labels \( l \) and \( m \) is given by

\[
   l = u + \frac{1}{2}(|q + m| + |q - m|), \quad v = m - q
\]

and thus it would appear that we could have any value of \( q \) as long as \( u \) and \( v \) are integers (with suitable \( l, m \in \mathbb{R} \)). However, in this case the solutions would not be single-valued and all the solutions would vanish on the string axis. Setting \( 2q \in \mathbb{Z} \) gives the same values of \( l \) and \( m \) as before.

As usual, when one tries to adapt wave-functions to some symmetry (here \( \Gamma' \)) there is the problem that the magnetic potential, \( \mathbf{A} \), might not possess the same symmetry so that compensating gauge transformations are necessary. Peierls, \([12]\), calls this process ‘umeichen’. It is a well known situation, with extensive discussion,[13], which we have encountered in an earlier calculation on the tetrahedron, \([14]\), an orbifold factor of the plane. A similar procedure has also been applied to factors of the Poincaré half-plane.

In the present case, under the action of \( \Gamma' \), the string is rotated and reflected and has to be brought back to its original position if we are to implement the identification in \( S^2/\Gamma' \) consistently. The equation that contains the relevant information is the behaviour of the modes under arbitrary rotations-reflections. For the moment we deal with the easier, pure rotational case, \( g \in SO(3) \). Then the behaviour is an elementary consequence of the \( SU(2) \) group combination law and one has explicitly (Wu and Yang \([8]\), Frenkel and Hraskó \([15]\))

\[
   Y^{(l)}_{qm}(g^{-1}\hat{r}) = e^{i\alpha(g)}Y^{(l)}_{qm}(\hat{r})D^{(l)}_{mm'}(g), \quad g \in SO(3).
\]

The exponential factor is the compensating gauge rotation with

\[
   \Lambda_g(\hat{r}) = \alpha(g) + \gamma(g) - \Omega_g
\]

where \( \alpha, \beta \) and \( \gamma \) are the Euler angles of the rotation \( g \) and \( \Omega_g \) is the solid angle subtended by the geodesic triangle on the (unit) sphere cut out by \( -\mathbf{r} \), the string axis, \( \mathbf{n} \), (here the negative \( z \)-axis) and the rotated string axis, \( g\mathbf{n} \),

\[
   \Omega_g(\hat{r}) = \Omega(g\mathbf{n}, \mathbf{n}; \mathbf{r}), \quad \mathbf{r} = (\hat{r}, r).
\]

(Our conventions regarding rotations and actions are generally those of Brink and Satchler \([16]\).) The gradient of \( \Omega \) gives the gauge transformation between the potentials of the two strings.
An alternative expression for $\Lambda$ is (Wu and Yang [8])

$$\Lambda_g(\hat{r}) = \phi_g - \phi - A_g$$

(8)

where $A_g$ is the angle at $-\hat{r}$ in the above mentioned triangle on $S^2$ and $g^{-1}\hat{r} = (\theta_g, \phi_g)$

In order that (6) be consistently iterated, it is necessary that the group combination (cocycle) condition,

$$\Lambda_{gh}(\hat{r}) = \Lambda_g(\hat{r}) + \Lambda_h(\hat{r})$$

(9)

should hold, and this can be checked directly from (7). The geometrical details are to be found in the useful article by Frenkel and Hraskó [15].

The magnetic rotation operator, $T_g$, is defined, on scalars, by the action

$$(T_g \psi)(\mathbf{r}) = e^{-iq\Lambda_g(\hat{r})} \psi(g^{-1}\mathbf{r}),$$

(10)

or, equivalently,

$$\langle \mathbf{r} | T_g = e^{-iq\Lambda_g(\hat{r})} \langle g^{-1}\mathbf{r} |.$$

(11)

In the present spherical case, because of (9), the magnetic rotations provide a true, as opposed to a ray, representation of the double of $\text{SO}(3)$ (denoted $\text{SO}^\ast(3)$) in the sense that

$$T_{gh} = T_g T_h$$

and

$$T_E = (-1)^{2q} 1,$$

(12)

where $E$ is a $2\pi$ rotation. This is not unexpected considering that for $q$ a half odd-integer all the monopole harmonics have half odd-integer angular momentum. $\text{SO}^\ast(3)$ is isomorphic to $\text{SU}(2)$.

Magnetic rotations on the plane [17] also provide true representations but not so translations, unless the flux through a fundamental domain is quantised [18].

We record the action of $T_g$ on the modes which is easily obtained from (10) and (6),

$$T_g Y_{qm'}^{(l)}(\mathbf{r}) = Y_{qm}^{(l)}(\hat{r}) D_{m'm}^{(l)}(g).$$

(13)
3. Parity and reflections.

The singularity (string) preserving magnetic parity operator on monopole wavefunctions is defined by the action, [15],

\[(\Pi \psi)(r) = e^{-iq\Omega(-n.n\cdot r)} \psi(-r)\]  

(14),

under the inversion \(\iota : r \rightarrow -r\).

On the modes we have, as an easy calculation shows,

\[\Pi Y_{qm}^{(l)} = (-1)^l Y_{-qm}^{(l)}.\]  

(15)

The parity operator can be used to extend the magnetic rotation operator \(T\) to include reflections. The reflection, \(\sigma\), in the plane with normal \(t\) can be written as a rotation through \(\pi\) about the axis \(t\) combined with the parity inversion \(\iota\). This can be written as \(\sigma = g_\pi \iota = \iota g_\pi\). The extension of \(T\) to reflections is thus defined by

\[T_\sigma \psi = T_{ig_\pi} \psi = T_\iota T_{g_\pi} \psi = \Pi T_{g_\pi} \psi.\]

From (10) and (14)

\[(T_\sigma \psi)(r) = e^{-iq\Omega_\sigma(\hat{r})+iq\pi} (R_{g_\pi} \psi)(-r),\]  

(16)

and on the modes,

\[(T_\sigma Y_{qm'}^{(l)})(\hat{r}) = (-1)^l Y_{-qm}^{(l)}(\hat{r}) D_{mm'}^{(l)}(g_\pi).\]  

(17)

4. Construction of the domains \(S^2/\Gamma'\)

It is convenient now to formalize briefly what we mean by the space \(S^2/\Gamma'\) where \(\Gamma'\) is a finite subgroup of \(O(3)\). A region of the sphere \(F \in S^2\) is called a fundamental domain for \(\Gamma\) if it satisfies the following criteria,

(i) \(F\) is open in \(S^2\),

(ii) \(F \cap \gamma F = \emptyset, \quad \forall \gamma \in \Gamma' - \{id\},\)

(iii) \(S^2 = \bigcup_{\gamma \in \Gamma'} \overline{\gamma F}\).

We also assume that \(F\) is connected.

Physically, \(F \cup \partial F\) represents the space on which our theory is defined, and all the copies of \(F\) on \(S^2\) must be physically equivalent.
The space $S^2/\Gamma'$ is the sphere $S^2$ with the points $r$ and $\gamma r$ identified for all $\gamma \in \Gamma'$. If $\Gamma'$ acts freely on $S^2$, i.e. there are no fixed points, then $S^2/\Gamma'$ is closed and can be taken to be $\mathcal{F}$ with identified boundary. If there are fixed points, these will be contained in $\partial \mathcal{F}$ and can be considered as boundary singular points of $S^2/\Gamma'$.

When $\Gamma$ is purely rotational the fixed points form a discrete set of which there are two, or three, in $\partial \mathcal{F}$. For the extended, reflection groups, as already indicated, the boundary $\partial \mathcal{F}$ is constructed from the intersections of two or three reflection planes with the sphere. Thus there are transformations that map $\mathcal{F}$ into adjacent domains and which leave part of the boundary of $\mathcal{F}$ fixed. The conclusion here is that, unlike the rotational case, the boundary $\partial \mathcal{F}$ is a real boundary and the physical manifold is termed a Möbius triangle.

5. Projection to $S^2/\Gamma$

Still restricting to purely rotational $\Gamma$, we will now formally project everything down to $S^2/\Gamma$ in a more or less standard fashion.

In order that a function $\tilde{\psi}(r)$ on $S^2$ project down consistently to $S^2/\Gamma$, it is necessary that it satisfy the magnetic periodicity condition

$$T_g \tilde{\psi}(r) = a(g^*) \psi(r), \quad g \in \Gamma^*,$$

(18)

where $a(g^*)$ forms a one-dimensional representation of $\Gamma^*$, the double of $\Gamma$, and labels the projection. For ease we have put $r$ in place of $\tilde{r}$. We show later that there is a minimal choice for $a(g^*)$.

If $q$ is integral, there is no need to introduce the group doubles. However, if one does, there is a simple duplication, which is easily dealt with in practice as we can show with the following, somewhat superfluous, constructions.

Since the action of $\Gamma^*$ covers the sphere $S^2$ twice, we introduce the trivial double covering, $S^2^*$, of two identical copies of $S^2$, with $S^2^*/\Gamma^* = S^2/\Gamma$.

The modes themselves do not change sign on a $2\pi$ rotation, even when $l$ is half odd-integral, and neither does the wavefunction. So $\psi^*(r) = \psi^*(E r)$, where $E$ is a $2\pi$ rotation, and therefore, $\tilde{\psi}(r) = \psi^*(r)$.

We then have, somewhat non-rigorously, for the function, $\psi(r)$, on $S^2^*/\Gamma^*$ the numerical equality

$$\psi(r) = \psi^*(r)$$

(19)

where we do not distinguish between the projected and original coordinates, both being denoted by $r$. The $r$ on the right belongs to the fundamental domain of $\Gamma^*$.
on $S^2\bullet$, which is, of course, isomorphic to that of $\Gamma$ on $S^2$. Then

$$T_{g\bullet}\psi(r) = a(g\bullet)\psi(r). \quad (20)$$

The heat-kernel on $S^2$, written in mode form,

$$K_{S^2}(r, r'; \tau) = \sum_{l=|q|}^{\infty} e^{-\lambda_l \tau} \text{Tr} \left[ Y_q^{(l)}(r) Y_q^{(l)}(r') \right], \quad (21)$$

propagates arbitrary wavefunctions on $S^2$ and satisfies the important switching relation

$$K_S(r, g^{-1}r'; \tau) = e^{-iq \Lambda_g(r') - iq \Lambda_{g^{-1}}(r)} K_S(gr, r'; \tau)$$

which realised in coordinate representation the operator rotational symmetry

$$T_g K_{S^2} = K_{S^2} T_g$$

(see (11)).

The heat-kernel that propagates wavefunctions on $S^2\bullet/\Gamma\bullet$ obeying (20) is, by general theory,

$$K_{S^2\bullet/\Gamma\bullet}(r, r') = \sum_{g \in \Gamma\bullet} a(g\bullet) T_{g\bullet^{-1}} K_{S^2\bullet}(r, r') \quad (22)$$

which is referred to as the pre-image form of this propagator in terms of that on $S^2\bullet$.

The double group $\Gamma\bullet$ can be decomposed

$$\Gamma\bullet = \Gamma \cup E\Gamma$$

and the sum over $\Gamma\bullet$ reduced to a sum over the subset $\Gamma$

$$K_{S^2\bullet/\Gamma\bullet}(r, r') = \sum_{g \in \Gamma} (a(g) T_{g^{-1}} + a(Eg) T_{(Eg)^{-1}}) K_{S^2\bullet}(r, r'). \quad (23)$$

We know, (12), that $T_E = (-1)^{2q} \mathbf{1}$ and so from (20), $a(E) = (-1)^{2q}$. Hence $a(E) T_E = \mathbf{1}$ and so

$$K_{S^2\bullet/\Gamma\bullet}(r, r') = 2 \sum_{g \in \Gamma} a(g) T_{g^{-1}} K_{S^2\bullet}(r, r'), \quad (24)$$

which means we can finally write what was almost obvious from the start, the preimage sum,

$$K_{S^2/\Gamma}(r, r') = \sum_{g \in \Gamma} a(g) T_{g^{-1}} K_{S^2}(r, r'). \quad (25)$$

7
The factor of 2 is a normalization (volume) factor between $S^2_\bullet$ and $S^2$, $K_{S^2_\bullet} = 2K_{S^2}$.

This result shows that we can effectively ignore the complication of the double group and just proceed with $\Gamma$ as usual.

Given an arbitrary field $\tilde{\phi}(\mathbf{r})$ on $S^2$, a quasi-periodic field on $S^2$ and hence, via (19), a field on $S^2/\Gamma$ is constructed by the projection

$$\psi(\mathbf{r}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} a(g) T_{g^{-1}} \tilde{\phi}(\mathbf{r}). \quad (26)$$

In particular the projected (i.e. adapted) modes are

$$Y_{QM}(l) (\mathbf{r}) = \frac{1}{\sqrt{|\Gamma|}} \sum_{g \in \Gamma} a(g) (T_{g^{-1}} Y_{QM}(l)) (\mathbf{r}) = \sqrt{|\Gamma|} Y_{QM}(l) P_{m'm'} \quad (27)$$

where, using (13),

$$P_{m'm'} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} a(g) D_{m'm'}(g^{-1}) \quad (28)$$

is a hermitian projection matrix, $P^2 = P$. Since the eigenvalues of $P$ are 0 and 1, this shows that in a suitable, i.e. diagonal, basis the projected eigenfunctions are a subset of the unprojected ones, a general result and independent of the magnetic field.

Depending on whether $q$ and $l$ are integral or half odd-integral together, the representations $a$ and $D$ of the doubled elements, $Eg$, have the same or opposite sign as those of $\gamma$ and we can see explicitly that it is adequate to restrict the sum in (28) to $\Gamma$, as we have done.

Let us pursue this a little further to make sure everything is satisfactory. It is much more elegant to express everything in abstract form but we will retain the coordinate representation. The discussion is a textbook on in applied group theory.

From (27, using periodicity, one easily derives the integral

$$\int_{S^2/\Gamma} Y_{QM}(l)^* (\mathbf{r}) Y_{QM'}(l) (\mathbf{r}) d\mathbf{r} = P_{m'm'}.$$ 

Diagonalising $A$, $A = UDU^{-1}$, introduces the linear combinations $Y^{(\Gamma)}U$, which, from (27) equals $\sqrt{|\Gamma|} YU D$ showing that certain linear combinations of the modes on $S^2$ vanish and others do not, after adaptation which takes on the nature of a filtering process. We write for the nonzero combinations

$$Y_{QM}(l) = \sqrt{|\Gamma|} Y_{QM}(l) U_{m\alpha} \quad (29)$$
which form a complete, orthogonal set on $S^2/\Gamma$.

The factors of $\sqrt{|\Gamma|}$ have been chosen to give suitably normalised eigenfunctions on $S^2/\Gamma$ so that

$$K_{S^2/\Gamma}(r, r'; \tau) = \sum_{l=|q|}^{\infty} e^{-\lambda_l \tau} \text{Tr} \left[ Y^{(l)}_q(r) Y^{(l)}_q(r') \right].$$  \hfill (30)

The matrix $U$ reduces the $l$-representation of $\Gamma^\bullet$ into irreducible ones and, from (28), the range of $\alpha$, i.e. the degeneracy, is the number of times the irreducible $a$-representation occurs in this decomposition.

5. Effect of fixed points

The standard theory of coverings applies when the covering group acts freely. This is not so in the present case and the result is a singular space – an orbifold. We can still maintain the standard covering terminology, as we already have done when referring to fundamental domains, in an obvious way by firstly removing the fixed points, so that the standard theory applies, and then extending the action of the covering group, $\Gamma$, to these points.

We now investigate the implications of the periodicity condition (18) when extended to the fixed point set of $\Gamma$. For any $\gamma \in \Gamma$ there are two fixed points on the sphere, $\pm r_\gamma$. Going back to the mathematical result (6), we see that the monopole modes on the full sphere satisfy the identity

$$Y^{(l)}_{qm'}(\pm r_\gamma) = e^{iq\Lambda_\gamma(\pm r_\gamma)} Y^{(l)}_{qm}(\pm r_\gamma) \mathcal{D}^{(l)}_{m'm'}(\gamma),$$  \hfill (31)

where $\gamma$ is the rotation about $\pm r_\gamma$, obviously.

Using the relation (29) we can determine the action of the magnetic rotation operator $T_\gamma$ on the $\mathcal{Y}$,

$$T_\gamma \mathcal{Y}(r) = \sqrt{|\Gamma|} (T_\gamma Y)(r) U = e^{-iq\Lambda_\gamma(r)} \sqrt{|\Gamma|} Y(\gamma^{-1}r) U = e^{-iq\Lambda_\gamma(r)} \mathcal{Y}(\gamma^{-1}r)$$

as expected. However we also have, from (27),

$$T_\gamma \mathcal{Y}(r) = a(\gamma) \mathcal{Y}(r)$$

and therefore

$$e^{-iq\Lambda_\gamma(r)} \mathcal{Y}(\gamma^{-1}r) = a(\gamma) \mathcal{Y}(r)$$
and of course
\[ e^{-iq \Lambda_{\gamma}(r)} \psi(\gamma^{-1} r) = a(\gamma) \psi(r). \]

If there were no fixed points there would be no problem to discuss. However if \( r = r_{\gamma} \) is fixed by \( \gamma \) we have the consistency condition
\[
\begin{align*}
 e^{-iq \Lambda_{\gamma}(r_{\gamma})} \psi(r_{\gamma}) &= a(\gamma) \psi(r_{\gamma}) \\
 e^{-iq \Lambda_{\gamma}(r_{\gamma})} \mathcal{Y}(r_{\gamma}) &= a(\gamma) \mathcal{Y}(r_{\gamma})
\end{align*}
\]
(32)

This says that the phase is undetermined at the fixed points. If the fixed point is removed there is no problem. We are then effectively working arbitrarily close to the fixed point and all the points \( \gamma r, \gamma \in \Gamma \), belong to different images of the fundamental domain. When the fixed point is put back and the limit \( r \to r_{\gamma} \) taken, all these images coalesce and the phase becomes indeterminate, unless something special happens such as \( e^{-iq \Lambda_{\gamma}(r_{\gamma})} \) equalling \( a(\gamma) \) or if the wave function has a node at \( r_{\gamma} \). We analyse this situation further.

Since \( \Gamma \) can be generated by a set of cyclic rotations, it is helpful firstly to take the case when \( \Gamma \) is cyclic, \( C_k \), about the \( z \)-axis. Everything in (28) is then explicit. Let the generator of \( C_k \) be \( \gamma \), \( \tilde{\gamma}^{\nu} = 1 \). Also let the generator of the double group \( \mathbb{Z}^* \) be \( \gamma^* \) with \( (\gamma^*)^\nu = E \) and \( (\gamma^*)^{2\nu} = 1 \). \( E \) is the \( 2\pi \) doubling rotation.

The representations \( a(\gamma^*) \) are
\[
a((\gamma^*)^p) = e^{2\pi i pr/\nu}, \quad p = 0, 1, \ldots, 2\nu - 1,
\]
(33)
where, for integral \( q \), the label \( r \), is in the range 0 to \( \nu - 1 \), while for half-odd integral \( q, r = (2s + 1)/2 \) with \( 0 \leq s \leq \nu - 1 \).

\[ D_{mm'}^{(l)}(\gamma^*) \]
is diagonal
\[
D_{mm'}^{(l)}((\gamma^*)^p) = e^{-2\pi ipm/\nu} \delta_{mm'}, \quad -l \leq m, m' \leq l,
\]
(34)
and \( P_{mm'} \) is easily found,
\[
P_{mm'} = \frac{1}{\nu} \frac{1 - e^{2\pi i (m+r)/\nu} \delta_{mm'}}{1 - e^{2\pi i (m+r)/\nu} \delta_{mm'}}.
\]
(35)
This expression vanishes unless \( m + r \) is 0 mod \( \nu \), in which case it equals unity making the filtering obvious.

It is comforting to check things by looking at the explicit forms of some modes. For \( q = 0 \), the modes \( Y_{m}^{(l)} \) are ordinary spherical harmonics. At the north pole only
the \( m = 0 \) component survives which then forces \( r \) to be zero and (32) is trivially satisfied since \( a(\gamma) = 1 \).

One can proceed generally but let us use the modes exhibited in Wu and Yang section 7. For \( q = 1/2 \) we see that only the \( m = -1/2 \) survives at the north pole making \( r = 1/2 \) and now to check (32) we need the expression for \( \Lambda \). For a rotation through \( \omega \) about the z-axis, from (7) or (8), \( \Lambda = -\omega \), and again the check works.

For \( q = 1 \), the nonzero mode corresponds to \( r = 1 \).

In general one finds that \( r = q \) and looking at the \( \phi \) dependence of the modes, \( Y_{m q}^{(l)} \), (3), \( i.e. \exp \left( i(m+q)\phi \right) \), we see that the modes \( Y^{\Gamma(l)} \) are periodic as \( \phi \) increases by \( 2\pi/\nu \) and so, therefore, is the wave function.

It should be noted that if (8) is used, the azimuthal angle of the north pole changes by \( \omega \) even though it is a fixed point. The string’s location may be unchanged, but a nonzero compensating gauge transformation is still required.

It is clear geometrically that the same results will hold if \( C_k \) is cyclic about any axis so that it is consistent to set

\[
a(\gamma) = e^{-iq\Lambda(\gamma)} r(\gamma),
\]

(36)

for any \( \Gamma \). The consequence of replacing \( r(\gamma) \) by \( -r(\gamma) \) is discussed in the next section.

The minimal choice (36) clearly corresponds to untwisted fields in the sense that the more general form

\[
a(\gamma) = e^{-iq\Lambda(\gamma)} b(\gamma),
\]

(37)

where \( b(\gamma) \) is some nontrivial representation of \( \Gamma^* \), encodes physics over and above the magnetic monopole, such as Aharonov-Bohm fluxes through the fixed points. In this latter case, (18) implies that the wavefunction vanishes at the fixed points.

6. Charge quantisation on rotational domains.

We now have expressions for the modes and heat-kernel on \( S^2/\Gamma \) in the rotational case and have tacitly assumed that all integer values of \( 2q \) are allowed. However \( 2q \) need only form a subset of the integers. A geometric, Dirac type argument will firstly be used to calculate this subset.

Choose a fundamental domain \( \mathcal{F} \) which does not contain the string. Let \( L \subset \mathcal{F} \) be a piecewise continuous loop in \( \mathcal{F} \) and construct the parallel propagator,

\[
\mathcal{I}[L] = e^{-ie\int_L A} = e^{-iq\Omega},
\]

(38)
where $A$ is the potential one-form, $\Omega$ is the area within $L$. This follows from the explicit form of the monopole potential. As $L$ shrinks to a point in $\mathcal{F}$, $\mathcal{I}[L]$ clearly tends to unity as the area vanishes. Now $L$ also describes a loop with (oriented) area $-(|\mathcal{F}| - \Omega)$ in $S^2/\Gamma$, since $S^2/\Gamma$ is closed for rotational $\Gamma$ (except for the fixed points which we can ignore for these purposes as a set of measure zero). Thus if we let $L$ expand to fill the boundary $\partial \mathcal{F} \subset S^2$, the propagator will also take the value 1. This requires that $q|\mathcal{F}|$ be a multiple of $2\pi$ which gives the possible values of the magnetic charge as

$$q = \frac{\mid \Gamma \mid \cdot n}{2}, \quad n \in \mathbb{Z},$$

the expected value since the magnetic charge per closed domain, $\overline{q} \equiv q/|\Gamma|$, equals the Dirac value, $n/2$.

If the string passes through $\mathcal{F}$ there is an extra phase $-4\pi q$ in (38) which makes no difference to the final result.

For the cyclic group, $C_k$, according to (33), and (36),

$$a(\gamma^p) = e^{2\pi ipq/\nu} = (-1)^np \quad (40)$$

so all the $a(\gamma)$ are $\pm 1$.

A further restriction occurs if $C_k$ is a subgroup of a point group $\Gamma$, for then

$$a(\gamma^\bullet) = e^{i\pi|\Gamma|n/k},$$

but, by observation, for a noncyclic point group, all $|\Gamma|/k$ are even and the $a(\gamma)$ are unity. Furthermore the monopole charge $q$ is integral and there are no spinor modes.

The quantisation (39) or (36), will now be derived in another way. The consistency condition (32) must hold for both fixed points of $\gamma$. Geometry shows, if orientation effects are taken correctly into account, that $\Lambda(\gamma r_\gamma) = -\Lambda(\gamma r_\gamma)$. This also follows from the mode transformation (31) and the behaviour under parity (14), which involves $q \to -q$.

The two definitions of $a(\gamma)$ require

$$q = \frac{k_\gamma}{2} n_\gamma$$

for some integer $n_\gamma$, where $k_\gamma$ is the order of the cyclic subgroup generated by $\gamma$. Since $\Lambda_\gamma(r_\gamma) = 2\pi/n_\gamma$, all the $a(\gamma)$ are $\pm 1$.

If $\Gamma$ is not cyclic, the minimum requirement of (41) is $q = (K/2)p$ where $p \in \mathbb{Z}$ and $K$ is the LCM of all the $n_\gamma$. Looking at the character tables of the double
point groups it is easily checked that it is not possible to reconcile the tabulated \( \pm 1 \) representations \( a(\gamma^\bullet) \) with (41) unless all the \( a(\gamma^\bullet) \) are unity, implying that \( p \) is even or \( q = (2K/2)p' \) for \( p' \in \mathbb{Z} \). But \( 2K \) is nothing but the group order \( \Gamma \) leading to (39) for all \( n \). QED.

7. Integrated kernels and zeta functions

Special interest attaches itself to the integrated kernel,

\[
K_\Gamma(\tau) \equiv \int_{S^2/\Gamma} K_{S^2/\Gamma}(r, r; \tau) \, dr = \sum_{g \in \Gamma} a(g) \int_{S^2/\Gamma} (T_{g^{-1}}K_{S^2})(r, r; \tau) \, dr
\]

\[
= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} a(g) \int_{S^2} (T_{g^{-1}}K_{S^2})(r, r; \tau) \, dr. \tag{42}
\]

It is convenient to write everything in terms of the covering \( S^2 \) quantities because we can use the mode form (21) in (42) to obtain

\[
K_\Gamma(\tau) = \sum_{l=|q|}^\infty d_\Gamma(l) e^{-\lambda_l \tau} \tag{43}
\]

with

\[
d_\Gamma(l) = \frac{1}{|\Gamma|} \sum_{g} a(g) \chi^{(l)}(g^{-1}), \tag{44}
\]

where we have used the orthogonality of the \( S^2 \) monopole modes and \( \chi^{(l)}(g) \) is the usual \( \text{SO}^\bullet(3) \sim \text{SU}(2) \) character of \( g \). In particular

\[
\int_{S^2} \text{Tr} [(T_g Y_{ql})(r)Y_{ql}^+(r)] \, dr = \chi_l(g). \tag{45}
\]

Algebraically we can see that the degeneracy, \( d_\Gamma(l) \), is the number of times the irreducible representation ‘\( a \)’ occurs in the \( l \)-representation. Since degeneracies are real, \( a(g) \) in (44) and in the following equations, can be replaced by its real part, \( \text{Re} a(g) \), although in the present case the minimal \( a(g) = \pm 1 \).

(44) shows that the summand in \( d_\Gamma(l) \), is a class function so that we can make a convenient geometrical decomposition of the traced kernel, as in our earlier work [1]. The preimage sum, \( i.e. \) the sum over \( \Gamma \), is firstly replaced by a sum over conjugacy classes, \( \{ g \} \),

\[
d_\Gamma(l) = \frac{2l + 1}{|\Gamma|} + \frac{1}{|\Gamma|} \sum_{\{ g \}} a(g) |\{ g \}| \chi^{(l)}(g)
\]

13
with \( g \) being a typical element in \( \{ g \} \).

We now recall that the elements of a class correspond to rotations through one fixed angle about a set of conjugate axes. For a given set of such axes, one corresponding class can be considered to be the primitive class, all others associated with these axes then being generated by this one. Thus the sum over all classes can be rewritten as a sum over primitive classes and powers of these. Let \( k \) be the generic order of the rotation associated with the generic primitive class \( \{ \hat{g} \} \) so that \( \hat{g}^k = \text{id} \). Then \( |\{ \hat{g} \}| \) is just the number, \( n_k \), of conjugate \( k \)-fold axes and we can write

\[
d_\Gamma(l) = \frac{2l+1}{|\Gamma|} + \frac{1}{|\Gamma|} \sum_{g} n_k \sum_{p=1}^{k-1} a^p(\hat{g}) \chi^{(l)}(\hat{g}^p)
\]

which is the same as when there is no magnetic field, apart from the restriction \( l \geq |q| \) and one sees again, cf., that the cyclic groups, \( C_k \), form the basic building blocks. This can be made explicit as follows. In the case that \( \Gamma \) is just \( C_k \), one has from (46)

\[
d_k(l) = \frac{1}{k} \sum_{p=0}^{k-1} a^p(\hat{g}) \chi^{(l)}(\hat{g}^p) = \frac{2l+1}{k} + \frac{1}{k} \sum_{p=1}^{k-1} a^p(\hat{g}) \chi^{(l)}(\hat{g}^p)
\]

so that, substituting back,

\[
d_\Gamma(l) = \frac{d_1(l)}{|\Gamma|} (1 - \sum_{g} n_k) + \frac{1}{|\Gamma|} \sum_{g} k n_k d_k(l),
\]

where \( d_1(l) = 2l + 1 \) is the full sphere degeneracy.

Having now the degeneracies and the eigenvalues, we can turn to the explicit construction of the integrated heat-kernel (43) and its Mellin transform, the \( \zeta \)-function,

\[
\zeta_\Gamma(s) = \sum_l \frac{d_\Gamma(l)}{\lambda_l^s}.
\]

The eigenvalues are given by (2) and we face the old problem of computing spherical spectral quantities. We will approach this by firstly looking at a ‘linearised’ system, one whose eigenvalues are \( l + 1/2 \). The corresponding heat-kernel, denoted \( \tilde{K}_\Gamma(\tau) \), can be considered as that for the pseudo-operator \( (H_{g^2} + 1/4 + q^2)^{1/2} \) and will allow us to find the \( \zeta \)-function for the eigenvalues \( (l + 1/2)^2 \) quickly and that for the \( \lambda_l \) of (2), more elaborately. The expressions are also of some statistical mechanical interest if \( \tau \) is interpreted as an inverse temperature.
From (43), we have the linearised, or square root, kernel

$$K_\Gamma(\tau) = \sum_{l=|q|}^{\infty} d_\Gamma(l) e^{-(l+1/2)\tau}$$  \hspace{1cm} (50)

with degeneracies given by (44).

In accordance with (48), it is sufficient to consider the cyclic group case, $K_k(\tau)$ and $\zeta_k(s)$.

Since the cyclic axis is immaterial, we choose it to lie along the $z$-axis and could use the expressions in the previous section since $d_k(l)$ is $\text{Tr} P$, where $P$ is given by (35) with $r = q$. Therefore

$$\tilde{K}_k(\tau) = \frac{1}{k} \sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} e^{-(l+1/2)\tau} \frac{1 - e^{2\pi i (m+q)}}{1 - e^{2\pi i (m+q)/k}}.$$  \hspace{1cm} (51)

The summations can be relabelled using (5) and performed, but it is perhaps more elegant to substitute (44) into (43) and do the $l$-summation first, as in [1].

Remembering the charge quantisation condition, (40), the degeneracies are

$$d_k(l, q) = \frac{1}{k} \sum_{p=0}^{k-1} \cos(\pi np) \frac{\sin((2l + 1)\pi p/k)}{\sin(\pi p/k)}, \quad q = nk/2,$$  \hspace{1cm} (52)

where $n$ is even or odd. If $n$ is even, the degeneracies are identical to the case when $q = 0, d_k(l, q) = d_k(l, 0)$.

We now introduce the generating function

$$h_k(\sigma, q) = \sum_{l=|q|}^{\infty} d_k(l, q) \sigma^l$$  \hspace{1cm} (53)

closely connected with the traced heat-kernel, (43), if $\sigma = e^{-\tau}$. For $n$ even the only effect of the monopole is to make the series start at $l = |q|$. However it is better to continue with the summations. We have, for both integral and half odd-integral $q$,

$$h_k(\sigma, q) = \frac{1}{k} \sum_{l=|q|}^{\infty} \sum_{p=0}^{k-1} \cos(\pi np) \frac{\sin(2l + 1)\pi p/k}{\sin(\pi p/k)} \sigma^l$$

$$= \frac{1}{k} \sum_{p=0}^{k-1} \cos(\pi np) \sum_{l=|q|}^{\infty} \frac{\sin(2l + 1)\pi p/k}{\sin(\pi p/k)} \sigma^l$$

$$= \sigma^q \left( \frac{1 + \sigma + 2q(1 - \sigma)}{k(1 - \sigma)^2} \right) + \frac{1}{k} \sum_{p=1}^{k-1} \cos(\pi np) \sum_{l=|q|}^{\infty} \frac{\sin(2l + 1)\pi p/k}{\sin(\pi p/k)} \sigma^l$$

$$= \sigma^q \left( \frac{1}{1 - \sigma} \frac{1 + \sigma_k}{1 - \sigma} + \frac{2q}{k(1 - \sigma)} \right)$$  \hspace{1cm} (54)
and so the corresponding heat-kernel is related in the simple way

\[ \tilde{K}_k^q(\tau) = e^{-q\tau} \left( \tilde{K}_k(\tau) + \frac{2q}{k} \tilde{K}_\infty(\tau) \right) \]  

(55)
to the monopole-less \((q = 0)\) expression, [1],

\[ \tilde{K}_k(\tau) = \frac{\coth(k\tau/2)}{2\sinh(\tau/2)}. \]

The decomposition into conjugacy classes, (48), shows that this relation will follow through for all groups \(\Gamma\),

\[ \tilde{K}_k^q(\tau) = e^{-q\tau} \left( \tilde{K}_\Gamma(\tau) + 2q\tilde{K}_\infty(\tau) \right), \]  

(56)

where \(\tilde{K}_\Gamma(\tau)\) has been determined in [1],

\[ \tilde{K}_\Gamma(\tau) = \frac{\cosh(d_0\tau/2)}{2\sinh(d_1\tau/2)\sinh(d_2\tau/2)}. \]  

(57)

Here \(d_0, d_1\) and \(d_2\) are integer invariants associated with the reflection group having \(\Gamma\) as its rotation subgroup.

The zeta function corresponding to the linear heat kernel is easily determined as the Mellin transform of (57). In fact we shall find it more useful to consider the slightly more general zeta function defined by,

\[ \zeta_s^q(s,a | d_1,d_2) = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} e^{\tau/2 - a\tau} \tilde{K}_k^q(\tau) d\tau \]

(58)

where \(\zeta_H\) is the Hurwitz zeta function, and \(\zeta_2\) is the two-dimensional Barnes zeta function defined for \(s > 2\) by, [19],

\[ \zeta_2(s,a | d_1,d_2) = \sum_{n_1,n_2=0}^{\infty} \frac{1}{(a + n_1d_1 + n_2d_2)^s}. \]  

(59)

\(\zeta_H\) is actually a one-dimensional Barnes \(\zeta\)-function.

To conclude this section we shall present some properties of the Barnes zeta function needed later.

The function \(\zeta_2(s,a | d_1,d_2)\) has simple poles at \(s = 1,2\) whose residues can be written in terms of generalised Bernoulli polynomials

\[ \text{Res}_{s \to r} \zeta_2(s,a | d_1,d_2) = \frac{(-1)^r}{d_1d_2} B_{2-r}^{(2)}(a | d_1,d_2) \]  

(60)
for $r = 1, 2$. Here we have used the more standard notation as in Erdelyi [20]. The values of the Barnes $\zeta$–function at negative integers are also given in terms of generalised Bernoulli polynomials. For $n \in \mathbb{Z}^+$ we have,

$$
\zeta_2(-n, a|d_1, d_2) = \frac{1}{(n+1)(n+2)d_1d_2}B_{2+n}^{(2)}(a|d_1, d_2).
$$

(61)

The explicit generalised Bernoulli polynomials required in this paper are,

$$
B_0^{(2)}(a|d_1, d_2) = 1
$$

$$
B_1^{(2)}(a|d_1, d_2) = a - \frac{1}{2}(d_1 + d_2)
$$

$$
B_2^{(2)}(a|d_1, d_2) = a^2 - (d_1 + d_2)a + \frac{1}{6}(d_1^2 + 3d_1d_2 + d_2^2)
$$

$$
B_3^{(2)}(a|d_1, d_2) = a^3 - \frac{3}{2}(d_1 + d_2)a^2 + \frac{1}{2}(d_1^2 + 3d_1d_2 + d_2^2)a - \frac{1}{4}(d_1^2d_2 + d_1d_2^2).
$$

(62)

Finally we present a useful Bernoulli identity which will be used to simplify some expressions later on,

$$
B_n^{(2)}(d_1 + d_2 - a|d_1, d_2) = (-1)^n B_n^{(2)}(a|d_1, d_2).
$$

(63)

8. Extension to reflection groups

Before showing how to deal with the eigenvalues (2), we extend the analysis to orbifolds $S^2/\Gamma'$ where $\Gamma'$ is a finite reflection group – the complete symmetry group of a regular solid as outlined in section 1. The domain of interest is a Möbius triangle on $S^2$.

As shown in section 3, under reflection, the magnetic charge, $q$, changes sign and the projection has to take this into account. The rotational projection is given by (22) and (27) and all that is necessary is to combine this with the group decomposition (1). We start by writing down the projected modes

$$
W_{qm}^{(l)}(r) = Y_{qm}^{\Gamma(l)}(r) + a(\sigma)T_\sigma Y_{-qm}^{\Gamma(l)}(r),
$$

(64)

where $\sigma$ is a reflection, say in one of the symmetry planes, so that $a(\sigma) = \pm 1$. We can choose either sign.

Note the extended periodicity condition

$$
T_\gamma W_{qm}^{(l)}(r) = a(\gamma)a(\sigma)W_{-qm}^{(l)}(r), \quad \gamma' = \gamma\sigma.
$$
which shows that the wavefunction on the domain $\gamma'F$ has monopole charge $-q$ if that on $F$ has charge $q$.

The group decomposition of $\Gamma'$ gives

$$S^2 = \left( \bigcup_{\gamma \in \Gamma} \gamma F \right) \cup \left( \bigcup_{\gamma' \in \Gamma'_1} \gamma' F \right)$$

$$= \bigcup_{\gamma \in \Gamma} \gamma (1 + \sigma) F,$$

and we see that the only possible theory using images requires that adjacent domains on the sphere have opposite numerical monopole charge. The charge is $q$ on $\gamma F$ and $-q$ on $\gamma' F$ for all $\gamma \in \Gamma, \gamma' \in \Gamma'_1$.

For the theory to be consistent, we must at least show that the fundamental domains with charge $q$ and $-q$ define equivalent physical theories. There are two points. The first is that the sign of the charge is arbitrary, being essentially a matter of definition for the observer. The second point is that any physically significant quantities will depend only on $F_{\mu\nu} F^{\mu\nu} \sim q^2$. Our theory therefore has the possibility of being consistent and we now derive the values of $q$ for which it is consistent.

Across the boundaries of the fundamental domains we have $B \rightarrow -B$ and $A \rightarrow -A$. For consistency we should define these vector quantities to vanish on the reflecting boundaries. Consider now the parallel propagator, (38), where the loop $L \in F$. Since we have defined $A$ to vanish on $\partial F$ we have $\mathcal{I}[\partial F] = 1$. Thus for an arbitrary loop approaching the boundary we require trivial parallel transport just as in the pure rotational case. This gives the possible values of $q$ as

$$q = |\Gamma'| \frac{n'}{2} = |\Gamma| n', \quad n' \in \mathbb{Z},$$

so $q$ is integral and there are no spinor modes.

The existence of fixed points imposes certain conditions. The situation is more restricting than in the pure rotational case because the fixed points form a continuous set, the boundary of our domain, $F$. Let $\sigma$ be a reflection ($\sigma^2 = 1$). The fixed point set, $\mathcal{P}$, is the intersection of the reflecting plane with the $S^2$ i.e. a great circle. Extending the periodicity condition (18) to $\Gamma'$ we see that $\psi$ would have to satisfy

$$e^{-iq\Omega_\sigma(r)+iq\pi} \psi(r) = a(\sigma) \psi(r), \quad \forall r \in \mathcal{P}.$$  

(66)\]

Since $\Omega_\sigma(r)$ is not constant on $\mathcal{P}$, this equation cannot be satisfied on all of $\mathcal{P}$ unless one of $q = 0, \psi|_\mathcal{P} = 0, n \perp \mathcal{P}$ or $n \in \mathcal{P}$ is true. Applying this argument to the two,
or three, reflection generators removes the last two possibilities and we are left with either \( q = 0 \) or \( \psi|_P = 0 \) (or both).

The fact that the magnetic charge, \( q \), has to be of opposite sign on adjacent domains under the action of \( \Gamma' \) for the image method to work indicates, crudely, that \( q \) is zero on the boundary and suggests that our construction satisfies the restrictions following from (66).

Developing this nonrigorous analysis, since the magnetic field \( B \) vanishes on the reflecting boundaries we must insist that the monopole modes take the value \( W^{(l)}_{0m} \) on the boundaries. Thus the consistency equation (66) is satisfied. Let \( \gamma \) be an arbitrary rotation in \( \Gamma \). The rotational consistency (32) is satisfied for all \( \gamma \) since \( q \) is an integer multiple of \( |\Gamma| \) and, as already stated, \( a(\gamma) = 1, \forall \gamma \). It would thus appear that all the values of \( q \) in (65) produce a consistent theory.

For clarity we restate that the mode labelled by \( qlm \) takes the values

\[
W^{(l)}_{qm} \quad \text{on} \quad \Gamma \mathcal{F} \\
W^{(l)}_{-qm} \quad \text{on} \quad \Gamma'_1 \mathcal{F} \\
W^{(l)}_{0m} \quad \text{on} \quad \partial(\Gamma \mathcal{F}) \sim -\partial(\Gamma'_1 \mathcal{F}).
\]

The modes (64) can be used to define a heat-kernel analogous to the rotational case. We skip straight to the linear heat-kernel which may be written

\[
\tilde{K}^q_{\Gamma'}(\tau) = \int_{S^2/\Gamma'} \sum_{l=q}^{\infty} e^{-(l+1/2)\tau} \text{Tr} \left[ W^{(l)}_q(r) W^{(l)}_q(r)^\dagger \right] dr.
\]

Here \( W^{(l)}_q \) is the vector of solutions \( W^{(l)}_{qm} \). We can extend this integral to all \( S^2 \) just as we did in equation (42), in the rotational case (this is not entirely trivial but it is possible using the invariance of the theory under \( q \rightarrow -q \)),

\[
\tilde{K}^q_{\Gamma'}(\tau) = \frac{1}{|\Gamma'|} \int_{S^2} \sum_{l=q}^{\infty} e^{-(l+1/2)\tau} \text{Tr} \left[ W^{(l)}_q(r) W^{(l)}_q(r)^\dagger \right] dr.
\]

The next step is to use the explicit rotational modes (27) or (29) (with \( a(g) = 1 \), equations (13) and (17), and the invariance of the heat kernel under charge reversal. The result is

\[
\tilde{K}^q_{\Gamma'}(\tau) = \frac{1}{2} \tilde{K}^q_{\Gamma}(\tau) + \frac{a(\sigma)}{|\Gamma'|} \sum_{\gamma' \in \Gamma_1} \int_{S^2} \sum_{l=q}^{\infty} e^{-(l+1/2)\tau} \text{Tr} \left[ (T_{\gamma'} Y^{(l)}_{-q})(r) Y^{(l)}_{-q}(r)^\dagger \right] dr. \quad (68)
\]
This equation actually represents two heat kernels for the cases \( a(\sigma) = \pm 1 \).
For \( a(\sigma) = -1 \) we shall use the term ‘Dirichlet’ and write \( K^q_D(\tau) \). For \( a(\sigma) = +1 \) we use the term ‘Neumann’ and write \( K^q_N(\tau) \). These names are used in analogy to the case with no monopole field, \( q = 0 \). In this case \( W_{0m}^{(l)} \) is, by construction, a solution on the whole sphere (see (67)) and must satisfy \( T_\gamma W_{0m}^{(l)} = a(\sigma)W_{0m}^{(l)} \) everywhere. On the reflecting boundaries, \( a(\sigma) = -1 \) requires \( W_{0m}^{(l)} \) to vanish i.e. Dirichlet boundary conditions, and \( a(\sigma) = +1 \) requires the normal derivative of \( W_{0m}^{(l)} \) to vanish, i.e. Neumann boundary conditions.

We now turn to some specific calculations of the heat kernels. Using (45) we can write the second term in (68) as,

\[
\frac{a(\sigma)}{|\Gamma'|} \int_{S^2} \sum_{l=q}^{\infty} \chi_l(\gamma')e^{-(l+1/2)\tau}.
\] (69)

Our first calculation is for the reflection group \( \Gamma' \) with rotational subgroup \( C_k \). In this case \( \Gamma'_1 \) consists of \( k \) reflection planes with a common invariant axis, and with angle \( 2\pi/k \) between adjacent planes. If we take one plane to be the \( z-x \) plane then we can write \( \Gamma'_1 = \{ \Pi g_\pi \hat{\gamma}^p | p = 0, 1, 2, \ldots, k-1 \} \) where \( \Pi \) is parity, \( g_\pi \) is a rotation by angle \( \pi \) about the \( y \) axis, and \( \hat{\gamma} \) is a rotation by angle \( 2\pi/k \) about the \( z \)-axis. Using the explicit result

\[
D^{(l)}_{mm'}(g_\pi) = (-1)^{l-m} \delta_{m,m'}
\] (70)

and equations (15), (34) we find \( \chi_l(\gamma') = 1 \) for elements \( \gamma' \in \Gamma'_1 \) and all \( l \). Thus (69) is trivial to calculate in this case, and from (68), (55) we find

\[
\tilde{K}^q_D(\tau) = e^{-q\tau} \frac{e^{-k\tau/2}}{4 \sinh(\tau/2) \sinh(k\tau/2)} + \frac{q}{2k} \frac{e^{-q\tau}}{2 \sinh(\tau/2)}
\] (71)

\[
\tilde{K}^q_N(\tau) = e^{-q\tau} \frac{e^{k\tau/2}}{4 \sinh(\tau/2) \sinh(k\tau/2)} + \frac{q}{2k} \frac{e^{-q\tau}}{2 \sinh(\tau/2)}.
\] (72)

The first term in each of these expressions is simply \( \exp(-q\tau) \) times the monopole-less heat kernel [1]. Notice that the extra monopole contribution is the same in both heat kernels.

We can also calculate the heat kernels for the group \( \Gamma' \) with dihedral rotational subgroup \( D_k \). We take \( \Gamma'_1 = \sigma D_k \) where \( \sigma \) is a reflection in the \( x-y \) plane, and \( D_k \) is the dihedral group with the \( k \)-fold cyclic group about the \( z \) axis, and a rotation by \( \pi \) about the \( y \) axis. Using (34) and (70), we find \( \chi_l(\gamma') = 1 \) for elements \( \gamma' \in \Gamma'_1 \)
involving the rotation by angle $\pi$ about $y$. These elements are $|\Gamma|$ in number and contribute to (69) the expression
\[ \frac{a(\sigma)}{8} \frac{e^{-q\tau}}{\sinh(\tau/2)}. \] (73)

The remaining elements can be written $\Pi g_\pi \hat{\gamma}^p$ where $g_\pi$ is a rotation by $\pi$ about the $z$ axis, and $\Pi$, $\hat{\gamma}$ are as defined above. Using (15) and (34) we find the transformation matrix $(-1)^{l+m}\delta_{m'}^{m} \exp(-i2\pi mp/k)$. This transformation is non-trivial and we have to construct a sum similar to (51).

For technical variety we use the relations (5) so that the sum can be written,
\[ \frac{a(\sigma)}{4} e^{-\tau/2} \sum_{u=0}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{kt} (-e)^{-\tau(u+(|tk+q|+|tk-q|)/2)} \]
\[ = a(\sigma) e^{-q\tau} \frac{\cosh(k\tau/2)}{8 \cosh(\tau/2) \sinh(k\tau/2)}. \] (74)

Adding together (73) and (74), and using (55) gives the results from (68),
\[ \tilde{K}_D^q(\tau) = e^{-q\tau} \frac{e^{-(k+1)\tau/2}}{4 \sinh(2\tau/2) \sinh(k\tau/2)} + \frac{q}{4k} \frac{e^{-q\tau}}{\sinh(\tau/2)} \] (75)
\[ \tilde{K}_N^q(\tau) = e^{-q\tau} \frac{e^{(k+1)\tau/2}}{4 \sinh(2\tau/2) \sinh(k\tau/2)} + \frac{q}{4k} \frac{e^{-q\tau}}{\sinh(\tau/2)}. \] (76)

Again the first terms are simply $\exp(-q\tau)$ times the $q = 0$ case.

For all the heat kernels derived above the extra monopole contribution is simply given by
\[ \frac{q}{|\Gamma'|} \frac{e^{-q\tau}}{\sinh(\tau/2)}. \] (77)

Since we may construct the heat kernel for an arbitrary reflection group from the heat kernels calculated above, [2], the simplicity of (77) leads us to conclude that in the general case,
\[ \tilde{K}_D^q(\tau) = e^{-q\tau} \frac{e^{-d_0\tau/2}}{4 \sinh(d_1\tau/2) \sinh(d_2\tau/2)} + \tilde{q}' \frac{e^{-q\tau}}{\sinh(\tau/2)} \] (78)
\[ \tilde{K}_N^q(\tau) = e^{-q\tau} \frac{e^{d_0\tau/2}}{4 \sinh(d_1\tau/2) \sinh(d_2\tau/2)} + \tilde{q}' \frac{e^{-q\tau}}{\sinh(\tau/2)}. \] (79)

These two equations are the culmination of this section.
The first terms in these heat kernels are just $\exp(-q\tau)$ times the $q = 0$ expressions [1], and we have defined the monopole charge per reflection domain

$$\bar{q}' = \frac{q}{|\Gamma'|}.$$ 

Equation (68) predicts that the linear Dirichlet and Neumann heat kernels should add up to give the rotational linear heat kernel (56), and this is seen to be true (in performing this sum we must use the relation $\bar{q} = 2\bar{q}'$ for the same value of $q$).

The linear zeta functions for $\tilde{K}_D^q$ and $\tilde{K}_N^q$ are calculated using equation (58) with $\tilde{K}_I^q(\tau)$ suitably replaced. In terms of the Barnes zeta function we find,

$$\zeta_D^q(s,a) = \zeta_2(s,a + q + d_0|d_1,d_2) + \bar{q}'\zeta_H(s,a + q).$$

(80)

$$\zeta_N^q(s,a) = \zeta_2(s,a + q|d_1,d_2) + \bar{q}'\zeta_H(s,a + q)$$

(81)

It should be remembered that although we may add these two zeta functions to produce the rotational zeta function, the physical theories are completely different due to the changing sign of the magnetic field. Thus the procedure can only be regarded as a formal trick.

9. General zeta function and its derivative

Equations (58), (80) and (81) give zeta functions for the eigenvalues $(l+a)^2$ if $s$ is replaced with $2s$. In general we need the zeta functions for the more general eigenvalues $\lambda_l = (l+a)^2 - \alpha^2$, with suitable constants $a$ and $\alpha$. For example we may add curvature coupling and mass terms to the Hamiltonian $H_{S^2}$. In this case the eigenvalue equation is,

$$(H_{S^2} + \xi R + m^2)Y_{qm}^{(l)} = \lambda_l Y_{qm}^{(l)}$$

and we find $a = 1/2$, $\alpha^2 = q^2 + (1/4 - 2\xi) - m^2$ ($R = 2$ on the unit two-sphere). It is assumed that $\alpha^2$ is positive. To analyse these general zeta functions we use similar methods to those found in [21]. For brevity we shall just write $\zeta(s)$ to represent a general zeta function. For $s > 1$, we have the explicit mode sum

$$\zeta(s) = \sum_{l=|q|}^{\infty} \frac{d(l)}{[(l+a)^2 - \alpha^2]^s}.$$ 

(82)
The function $d(l)$ is the degeneracy of the eigenvalue $\lambda_l$ for some linear zeta function (i.e. rotational, Dirichlet or Neumann). If we assume that $|\alpha| < |q| + a$ we can perform a binomial expansion on the summand which leads to a continuation of $\zeta(s)$ given by

$$
\zeta(s) = \sum_{r=0}^{\infty} \alpha^{2r} \frac{\Gamma(s + r)}{r! \Gamma(s)} \zeta^q(2s + 2r, a).
$$

(83)

In the above equation $\zeta^q(s, a)$ is intended to be any one of the rotational, Dirichlet, or Neumann zeta functions defined by equations (58), (80) and (81) respectively.

In order to tie in with [21] we generalise to the case where $\zeta^q(s)$ represents an arbitrary zeta function on a $d$-dimensional space with simple poles (only) at $s = 1, 2, \ldots, d$. This is the situation encountered in [21] where $\zeta^q(s)$ would just be a $d$-dimensional Barnes function $\zeta_d(s, a|d)$ (of course the label $q$ is defunct in the general case). Near the poles we define,

$$
\zeta^q(s + r, a) = \frac{N_r}{s} + R_r + O(s), \quad s \to 0,
$$

for $r = 1, 2, \ldots, d$. For our three cases (with $d = 2$) the residues $N_r$ for $r = 1, 2$ can be calculated from the specific forms of the zeta functions and (60).

The important fact is that the series (83) reduces to a finite sum when $s$ is a negative integer. Thus we concentrate on these values of $s$ and find for $n \in \mathbb{Z}^+$,

$$
(-\alpha^2)^{-n} \zeta(-n) = 2 \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-\alpha^2)^{-r} \zeta^q(-2r, a) - \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-\alpha^2)^{-r} (\psi(n + 1) - \psi(r + 1)) \zeta^q(-2r, a)
$$

$$
\sum_{r=1}^{u} \alpha^{2r} \frac{n!(r - 1)!}{(r + n)!} \left\{ R_{2r} + \frac{1}{2} N_{2r} (\psi(r) - \psi(n + 1)) \right\} + \sum_{r=u+1}^{\infty} \alpha^{2r} \frac{n!(r - 1)!}{(r + n)!} \zeta^q(2r, a).
$$

(84)

The number $u$ in the above is defined as $[d/2]$ where $d$ is the dimension of the space under consideration. The derivative of the zeta function at $s = -n$ can also be calculated from (83),

$$
(-\alpha^2)^{-n} \zeta'(-n) = 2 \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-\alpha^2)^{-r} \zeta'^q(-2r, a) - \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-\alpha^2)^{-r} (\psi(n + 1) - \psi(r + 1)) \zeta'^q(-2r, a) -
$$

$$
\sum_{r=1}^{u} \alpha^{2r} \frac{n!(r - 1)!}{(r + n)!} \left\{ R_{2r} + \frac{1}{2} N_{2r} (\psi(r) - \psi(n + 1)) \right\} + \sum_{r=u+1}^{\infty} \alpha^{2r} \frac{n!(r - 1)!}{(r + n)!} \zeta'^q(2r, a).
$$

(85)
where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function.

The problem now rests on the evaluation of the infinite sum on the last line of this expression. We show that this sum can be written in finite terms. It is expected that the sum will be finite since we expect $\zeta'(-n)$ to be finite, and all other terms on the right hand side are finite. A word of caution is that in singular situations, there is the possibility that logarithmic terms, $\log \tau$, may appear in the asymptotic expansion of the heat-kernel. If this were so, more care would have to be taken over the evaluation of the determinants. However no such terms occur here.

Using the integral representation (58) extended to the arbitrary zeta function $\zeta^q(s, a)$, we may write the last line in (85) as,

$$2n! \sum_{n=-u+1}^{\infty} \alpha^{2r} \frac{r!}{(r + n)!} \int_0^\infty \tau^{2r-1} e^{\tau/2 - a \tau} \tilde{K}^q(\tau) d\tau,$$

where $\tilde{K}^q(\tau)$ is the linear heat kernel associated with $\zeta^q(s, a)$. Since (86) is assumed finite we may take the sum inside the integral. Thus our problem can be reduced to evaluating the sequence of sums,

$$T_n(\tau) = n! \sum_{r=1}^{\infty} \frac{r! \tau^{2r}}{(2r)!(r + n)!}.$$

Using the simple result $\sqrt{\pi}(2r)! = 2^{2r}r! \Gamma(r + 1/2)$ and changing the summation variable to $r' = r + n$ gives the result

$$T_n(\tau) = n! \sqrt{\pi} \left(\frac{1}{2}\right)^{1/2 - n} I_{-n-1/2}(\tau) - \sum_{r=0}^{n} \binom{n}{r} (2r)! (-\tau^2)^{-r},$$

where $I_{\nu}(x)$ is the modified Bessel function.

To find a closed form for $T_n(\tau)$ we employ a useful integral representation given in reference [22]. For $\nu > 0$,

$$\Gamma\left(\frac{1}{2} + \nu\right) I_{-\nu}(x) =$$

$$\frac{2}{\sqrt{\pi}} \left(\frac{1}{2}\right)^{\nu} \left[ \int_{-1}^{1} e^{-xt}(1 - t^2)^{\nu-1/2} dt + \sin(\pi \nu) \int_{1}^{\infty} e^{-xt}(t^2 - 1)^{\nu-1/2} dt \right].$$

Setting $\nu = n + 1/2$ in this expression we may expand the integrand factors $(1-t^2)^n$ using the binomial theorem leaving simple exponential integrals. After a little work we find,

$$T_n(\tau) = \sum_{r=0}^{n} \binom{n}{r} (-1)^r \left\{ (-\tau^2)^{-n} (2\tau)^r (2n - r)! \frac{1}{2} (e^\tau + (-1)^r e^{-\tau}) - (2r)! \tau^{-2r} \right\}.$$
Having found a suitable expression for $T_n$ we can now go back to (86) and write it in the new form,

$$
\int_0^\infty \left\{ 2T_n(\alpha \tau) - \sum_{r=1}^{u} (\alpha \tau)^{2r} \frac{n!(r-1)!}{(r+n)!(2r-1)!} \right\} \tau^{s-1} e^{-\tau/2 - a \tau} \tilde{K}^q(\tau) d\tau. \tag{88}
$$

We have introduced into the integral a regulator $\tau^s$ (the expression that we want is given for the value $s = 0$). The continuation variable $s$ has been introduced so that we may evaluate the integrals of the individual terms in the sum definition (87) of $T_n$, and of the sum subtracted from it, before we perform the sums. We assume that $s$ is large enough so that all the individual integrals are well defined. In fact this requires $s > 2n$. Performing the integrations leaves,

$$
(-\alpha^2)^{-n} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (2\alpha)^r (2n-r)! \Gamma(s+r-2n) \times \nabla\{\zeta^q(s+r-2n,a+\alpha) + (-1)^r \zeta^q(s+r-2n,a-\alpha)\} - \nabla 2 \sum_{r=0}^{n} (-\alpha^2)^{-r} \left( \begin{array}{c} n \\ r \end{array} \right) (2r)! \Gamma(s-2r) \zeta^q(s-2r,a) - \sum_{r=1}^{u} \alpha^{2r} n!(r-1)!(2r)! \Gamma(s+2r) \zeta^q(s+2r,a) \tag{89}
$$

As $s \to 0$ all of these terms diverge, although taking all terms together we must get a finite result i.e. all poles must cancel as $s \to 0$. This cancellation of the poles leads to the equation,

$$
(-\alpha^2)^{-n} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (2\alpha)^r \{\zeta^q(r-2n,a-\alpha) + (-1)^r \zeta^q(r-2n,a+\alpha)\} - \nabla 2 \sum_{r=0}^{n} (-\alpha^2)^{-r} \left( \begin{array}{c} n \\ r \end{array} \right) \zeta^q(-2r,a) - \sum_{r=1}^{u} \alpha^{2r} n!(r-1)! \Gamma(s+r-2n) \zeta^q(r-2n,a+\alpha) \tag{90}
$$

Comparing equations (90) and (84) we see that the pole cancellation is precisely the statement,

$$
\zeta(-n) = \frac{1}{2} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (2\alpha)^r \{\zeta^q(r-2n,a-\alpha) + (-1)^r \zeta^q(r-2n,a+\alpha)\}. \tag{91}
$$

This expression contains equally terms with arguments $a+\alpha$ and $a-\alpha$. We shall say that $\zeta(-n)$ is ‘symmetric’. It is a generalisation to general $n$ of the symmetric expression for $n = 0$ found in [21]. The methods used in this reference do not produce a suitable pole cancellation to give a symmetric result for $\zeta(-n)$.
The finite remainder part of (89) as \( s \to 0 \) is given by the expression,

\[
(-\alpha^2)^{-n} \sum_{r=0}^{n} \binom{n}{r} (2\alpha)^r \{ \zeta'(r - 2n, a - \alpha) + (-1)^r \zeta'(r - 2n, a + \alpha) \} - \\
2 \sum_{r=0}^{n} (-\alpha^2)^{-r} \binom{n}{r} \zeta'(-2r, a) - \sum_{r=1}^{u} \alpha^{2r} \frac{n!(r-1)!}{(r+n)!} R_{2r} + \\
(-\alpha^2)^{-n} \sum_{r=0}^{n} \binom{n}{r} (2\alpha)^r \psi(2n - r + 1) \{ \zeta(q(r - 2n, a - \alpha)) + (-1)^r \zeta(q(r - 2n, a + \alpha)) \} - \\
2 \sum_{r=0}^{n} (-\alpha^2)^{-r} \binom{n}{r} \psi(2r + 1) \zeta(q(-2r, a)) - \sum_{r=1}^{u} \alpha^{2r} \frac{n!(r-1)!}{(r+n)!} \psi(2r) N_{2r}.
\]

Inserting this expression into (85), and adding zero in the form of \( \psi(2n + 1) \) times (90) gives

\[
\zeta'(-n) = \sum_{r=0}^{n} \binom{n}{r} (2\alpha)^r \{ \zeta'(r - 2n, a - \alpha) + (-1)^r \zeta'(r - 2n, a + \alpha) \} - \\
\sum_{r=1}^{n} \binom{n}{r} (2\alpha)^r \sigma_r \{ \zeta(q(r - 2n, a - \alpha)) + (-1)^r \zeta(q(r - 2n, a + \alpha)) \} - \\
\sum_{r=0}^{n-1} (-\alpha^2)^{n-r} \binom{n}{r} (2\psi(2r + 1) - \psi(r + 1) + \psi(n + 1) - 2\psi(2n + 1)) \zeta(q(-2r, a)) - \\
(-1)^n \sum_{r=1}^{u} \alpha^{2(r+n)} \frac{n!(r-1)!}{2(r+n)!} (2\psi(2r) - \psi(r) + \psi(n + 1) - 2\psi(2n + 1)) N_{2r}.
\]

The quantities \( \sigma_r \) in the above equation are defined as the sums,

\[
\sigma_r = \sum_{k=0}^{r-1} \frac{1}{2n - k}.
\]

Equation (92) is not symmetric in the sense that, unlike (91), it does not depend only on the quantities \( a \pm \alpha \). If we assume that the sum over the residues is a true feature of \( \zeta'(-n) \), as it is for \( \zeta'(0) \) in [21], then we are still left with a sum over \( \zeta(q(-2r, a)) \). We will now re-write this sum in a more natural, \( i.e. \) symmetric, form.

To this end we introduce the intermediate zeta function, \( \zeta(s) \), on the space \( \overline{M} = \mathbb{R}^{2n} \times S^2/\Gamma \), which is given by

\[
\overline{\zeta}(s) = \frac{\Gamma(s-n)}{(4\pi)^n \Gamma(s)} \zeta(s-n).
\]

(93)
Combining this with (83) gives an expansion for $\zeta(s)$,

$$\zeta(s) = \sum_{r=0}^{\infty} \alpha^{2r} \frac{\Gamma(s + r)}{r! \Gamma(s)} \zeta^q(2s + 2r, a)$$  \(94\)

where we have defined the new linear zeta function via

$$\zeta^q(s, a) = \frac{\Gamma(\frac{1}{2}s - n)}{(4\pi)^n \Gamma(\frac{1}{2}s)} \zeta^q(s - 2n, a)$$  \(95\)

The dimension of $\mathcal{M}$ is $d = 2n + d$ (remember, for our monopole case $d = 2$). We see from (95) that $\zeta^q(s)$ has poles at $s = 2, 4, \ldots, 2n$ and $s = 2n + 1, 2n + 2, \ldots, \bar{d}$.

The dimension of the residues are given by the formulae,

$$\overline{N}_{2r} = \frac{2(-1)^r}{(4\pi)^n (r - 1)!(n - r)!} \zeta^q(2r - 2n, a), \quad r = 1, 2, \ldots, n$$  \(96\)

$$\overline{N}_{2n+r} = \frac{\Gamma(\frac{1}{2}r)}{(4\pi)^n \Gamma(\frac{1}{2}r + n)} N_r, \quad r = 1, 2, \ldots, d.$$  \(97\)

The purpose of making these new definitions is the functional similarity between (83) and (94). This implies that results like (91) and (92) should exist for $\zeta(s)$ in terms of $\zeta^q(s)$. The important point is that we know that $\zeta'(0)$ can be written as a symmetric part and a sum over the residues of $\zeta^q(s, a)$, either from [21], or setting $n = 0$ in (92). Thus we might expect $\zeta'(0)$ to consist of a symmetric part and a sum over the residues of $\zeta^q(s, a)$. Differentiating (93) and setting $s = 0$ gives,

$$\zeta'(0) = \frac{(-1)^n}{(4\pi)^n n!} (\zeta'(-n) + (\psi(n + 1) - \psi(1))\zeta(-n)).$$  \(98\)

Now $\zeta(-n)$ is symmetric in terms of $\zeta^q(s, a \pm \alpha)$ and extends easily to a symmetric form for $\zeta^q(s, a)$ using (95). Thus by our reasoning we expect $\zeta'(−n)$ to contain a sum over the residues $\overline{N}_r$. This is exactly what we find, and the final result, in this section, is the symmetrical expression, [23],

$$\zeta'(-n) = \sum_{r=0}^{n} \binom{n}{r} (2\alpha)^r \{\zeta'^q(r - 2n, a - \alpha) + (-1)^r \zeta'^q(r - 2n, a + \alpha)\} -$$

$$\sum_{r=1}^{n} \binom{n}{r} (2\alpha)^r \sigma_r \{\zeta^q(r - 2n, a - \alpha) + (-1)^r \zeta^q(r - 2n, a + \alpha)\} -$$

$$(-1)^n (4\pi)^n n! \sum_{r=1}^{\pi} \frac{\alpha^{2r}}{r} \rho_r \overline{N}_{2r},$$  \(99\)
with the definitions \( \bar{u} = \lfloor d/2 \rfloor \) and
\[
\rho_r = \psi(2r - 2n + 1) - \frac{1}{2} \psi(r - n + 1) - (\psi(2n + 1) - \frac{1}{2} \psi(n + 1))
\]
\[
= \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{k=0}^{r-1} \frac{1}{2k+1}.
\]

The conclusion of these manipulations is that, despite the apparent awkwardness of the binomial expansion, (83), to obtain the required \( \zeta \)-function, the quantities that we want are given in finite terms, (91), (92), (99), and involve only relatively standard functions such as generalised Bernoulli polynomials introduced via the Barnes \( \zeta \)-function.

10. Vacuum energy calculations

Simply as an example of the use of the preceding expressions, we evaluate some vacuum (Casimir) energies on \( S^2/\Gamma' \).

Let \( \zeta^q(s, a) \) represent one of the rotational, Dirichlet or Neumann linear zeta functions as in the previous section. Then \( \zeta^q(s, a) \) can be extended to the odd-dimensional space-time \( \mathbb{R} \times S^2/\Gamma \) (or \( \mathbb{R} \times S^2/\Gamma' \)) by defining a new zeta function \( \zeta(s) \) given by (with \( a = 1/2 \)),
\[
\zeta(s) = \frac{\Gamma(s - 1/2)}{(4\pi)^{1/2}\Gamma(s)} (4\pi)^{1/2} \zeta^q(2s, 1/2).
\]

(100)

This zeta function corresponds to the rather artificial case of a conformally coupled field in three dimensions with mass \( q^2 \). The vacuum energy associated with this physical situation is defined by the simple formula [24],
\[
E = -\frac{1}{2} \mu_r \frac{d}{ds} \left( \frac{\mu}{\mu_r} \right)^{2s} \zeta(s) \bigg|_{s=0}.
\]

(101)

In this equation \( \mu \) is an arbitrary mass scale and \( \mu_r = 1/r \) is the mass scale associated with the sphere radius \( r \) (which has the value \( r = 1 \)). Equation (101) is in fact just half the logarithmic determinant. Inserting equation (100) into (101) gives the simpler expression
\[
E = \frac{1}{2} \zeta^q(-1, 1/2),
\]

(102)

†For a scalar field conformally coupled in \( N \) dimensions we have \( 4\xi = (N - 2)/(N - 1) \). In fact we only require that \( \alpha^2 = 0 \) or equivalently \( m^2 = q^2 + (1/4 - 2\xi) \).
which is finite and independent of \( \mu \) (we have set \( r = 1 \) again).

The vacuum energy on the space \( \mathbb{R} \times \mathbb{R}^{2n} \times S^2/\Gamma(\epsilon) \) can also be found and gives \( E \) proportional to \( \zeta^q(-1 - 2n, \frac{1}{2}) \). The calculation for general \( n \) is entirely equivalent to the \( n = 0 \) case, which we now calculate.

Using the definitions (80), (80) and equation (61), we find from (102) the Dirichlet and Neumann vacuum energies,

\[
E_{\{ D \}} = \pm \frac{d_0}{48|\Gamma'|}(d_0^2 - d_1^2 - d_2^2) + \frac{\varphi'}{24}(3d_0^2 - d_1^2 - d_2^2 + \frac{1}{2} \pm 6d_0q - 2q^2).
\]

The constant \((q \text{ independent})\) terms are exactly the same as those calculated in [1] for \( q = 0 \), as required. Adding together the Dirichlet and Neumann vacuum energies gives the rotational vacuum energy

\[
E_\Gamma = \frac{\varphi}{24}(3d_0^2 - d_1^2 - d_2^2 - 2q^2),
\]

where we have used the relation \( \varphi = 2\varphi' \) for fixed \( q \). This vacuum energy necessarily vanishes for \( q = 0 \), as proved in [1]. We now list the vacuum energies \( E_{\{ D \}} \) for all possible reflection groups \( \Gamma' \),

\[
O^* : \pm \frac{29}{256} + \frac{\varphi'}{48}(383 \pm 108q - 4q^2)
\]

\[
Y^* : \pm \frac{89}{384} + \frac{\varphi'}{48}(1079 \pm 90q - 4q^2)
\]

\[
O|T : \pm \frac{11}{192} + \frac{\varphi'}{48}(167 \pm 36q - 4q^2)
\]

\[
D_n|C_n : \pm \frac{1}{96} + \frac{\varphi'}{48}(4n^2 - 1 \pm 6nq - 4q^2)
\]

\[
D^*_n(n \text{ even}), \quad D_{2n}|D_n(n \text{ odd}) : \pm \frac{(n + 1)(2n - 3)}{192n} \frac{\varphi'}{48}(4(n + 1)(n + 2) - 9 \pm 6(n + 1)q - 4q^2).
\]

Those for the corresponding rotational subgroups are obtained by adding the D and N values.

We now go on to calculate the vacuum energy on the even-dimensional space \( \mathbb{R}^{2n} \times S^2/\Gamma(\epsilon) \). Using equations (93), (98) and (101) we find the expression,

\[
E = \frac{(-1)^n+1}{2(4\pi)^n n!} \left( \zeta'(-n) + \left[ \ln \left( \frac{\mu}{\mu_r} \right)^2 + \psi(n + 1) - \psi(1) \right] \zeta(-n) \right).
\]

Here \( \zeta(s) \) is the general zeta function defined via equations (82) and (83).
An infinite contribution has been (arbitrarily) dropped to arrive at (103), the logarithmic term being a relic of this divergence. Since $\zeta(0)$ is not zero for any of the three monopole theories, we conclude from (103) that the vacuum energy is explicitly dependent on the arbitrary scale $\mu$. However for simplicity we shall assume $\mu = \mu_r = 1$ for the rest of this section. Our concern in this paper is not with realistic quantum field theory considerations.

Equations (91), (92) imply that the calculation for increasing $n$ merely requires the evaluation of more and more zeta functions and their derivatives. Thus we shall concentrate on the simplest case $n = 0$ corresponding to $S^2/\Gamma'$ itself. This will also allow us to compare with the results for $q = 0$ studied in [21].

We consider the case of a massless field with minimal coupling, that is $m^2 = 0$ and $\xi = 0$. Thus we have $a = 1/2$ and $2\alpha = \sqrt{4q^2 + 1}$ (we shall still write $\alpha$ when convenient). From the definitions (80), (81), and equations (91), (61), we find for Dirichlet and Neumann boundary conditions

$$\zeta\{D\}_{N}(0) = \frac{1}{12} + \frac{1}{6|\Gamma'|}(d_0(d_0 - 1) + 1 \pm 6d_0q + 18q^2).$$

These expressions reduce to those found earlier for $q = 0$ [21]. For the rotational case we simply add the Dirichlet and Neumann results to give

$$\zeta(0) = \frac{1}{6} + \frac{1}{6|\Gamma'|}(d_0(d_0 - 1) + 1 + 18q^2).$$

To calculate the vacuum energy we have still to calculate $\zeta'(0)$. Considering equation (92) with $n = 0$ requires the evaluation of the residue $N_2$. For both Dirichlet and Neumann zeta functions we find from (60) and (62) the value $N_2 = 2/|\Gamma'|$. Using the derivative of the Hurwitz zeta function in [20] then gives the zeta function derivatives

$$\zeta'_D(0) = \zeta'_2(0, \frac{1}{2} + q - \alpha + d_0) + \zeta'_2(0, \frac{1}{2} + q + \alpha + d_0) +$$

$$\frac{\gamma'}{2|\Gamma'|} \ln \left\{ \Gamma(\frac{1}{2} + q - \alpha)\Gamma(\frac{1}{2} + q + \alpha) \right\} - \frac{\gamma'}{2|\Gamma'|} \ln(2\pi) - \frac{1 + 4q^2}{2|\Gamma'|}, \quad (104)$$

$$\zeta'_N(0) = \zeta'_2(0, \frac{1}{2} + q - \alpha) + \zeta'_2(0, \frac{1}{2} + q + \alpha) +$$

$$\frac{\gamma'}{2|\Gamma'|} \ln \left\{ \Gamma(\frac{1}{2} + q - \alpha)\Gamma(\frac{1}{2} + q + \alpha) \right\} - \frac{\gamma'}{2|\Gamma'|} \ln(2\pi) - \frac{1 + 4q^2}{2|\Gamma'|}. \quad (105)$$

(The rotational zeta function derivative is just the sum of these two.) The triangle inequality $|x| + |y| \geq \sqrt{x^2 + y^2}$ ($x, y \in \mathbb{R}$) implies that $1/2 + q \geq \alpha$. The equality
is only met for $q = 0$, and in this case we have to remove the singularity in the first term $\zeta'_2(0, 1/2 + q - \alpha|d_1, d_2)$ of the Dirichlet zeta function as in 21 (we shall do this later). For $q > 0$ all terms in (105) and (104) are well defined.

There is no known analytic form for the derivatives of the Barnes zeta functions appearing in (105), (104) and so we have to calculate them numerically. To do this we obviously need a continuation of the Barnes zeta function which is open to easy numerical computation. In [21] several efficient continuations are presented which are valid for positive integer values of $d_1$ and $d_2$. However as we shall show in the next section, it is useful to have an expression which is valid for all $d_1, d_2 \in \mathbb{R}^+$. We shall now derive such an expression.

Our starting point is the Plana sum formula which we display here [25],

$$\sum_{n=a}^{b} f(n) = \frac{1}{2} (f(a) + f(b)) + \int_{a}^{b} f(t) dt + i \int_{0}^{\infty} \frac{f(a + it) - f(a - it) - f(b + it) + f(b - it)}{e^{2\pi t} - 1} dt.$$  \hspace{1cm} (106)

To be valid $f(t)$ must be an analytic function in the region of the complex $t$ plane $a \leq \text{Re}(t) \leq b$, and the integrals must exist. Applying (106) twice to the sum definition (59) of the Barnes zeta function gives immediately, for $s > 2$,

$$\zeta_2(s, a|d_1, d_2) = \frac{1}{2} d_1^{-s} \zeta_H(s, \frac{a}{d_1}) + \frac{1}{d_2(s-1)} d_1^{-s} \zeta_H(s-1, \frac{a}{d_1}) + i \int_{0}^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ \frac{1}{2} \left( (a + id_2t)^{-s} - (a - id_2t)^{-s} \right) \right\} + \frac{1}{d_1(s-1)} \left\{ (a + id_2t)^{1-s} - (a - id_2t)^{1-s} \right\} + \int_{0}^{\infty} \frac{du}{e^{2\pi u} - 1} \int_{0}^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ (a + id_1 u - id_2t)^{-s} - (a - id_1 u + id_2t)^{-s} - (a - id_1 u - id_2t)^{-s} \right\}.$$  \hspace{1cm} (107)

It is simple to verify that we are meeting the conditions required for the validity of the sum formula. In order to get rid of the single integrals in (107) we use the Plana sum definition of the Hurwitz zeta function which is, from (106)

$$\zeta_H(s, a) = \frac{1}{2} a^{-s} + \frac{a^{1-s}}{s-1} + i \int_{0}^{\infty} \frac{dt}{e^{2\pi t} - 1} \left\{ (a + it)^{-s} - (a - it)^{-s} \right\}.$$  \hspace{1cm} (108)

The integral part of this expression is equivalent to the integrals appearing in (107).
To simplify the double integral in (107) we first perform a change of variables from \( t, u \) to \( d_1 u \pm d_2 t \). Following this, we use the easily proved formula,
\[
(x + iy)^s + (x - iy)^s = 2 \cos \left( \frac{s \tan^{-1} y}{x} \right),
\]
which is valid for Re \( x \geq 0 \). After a little work we find the more convenient form for the double integral,
\[
\frac{2}{d_1 d_2} \int_0^\infty \frac{dw}{e^{2\pi w}} \frac{G(w)}{e^{2\pi w} - 1} \frac{\cos \left( \frac{s \tan^{-1} (w/a)}{a^2 + w^2} \right)}{(a^2 + w^2)^{s/2}}. \tag{109}
\]
All the non-trivial \( d_1, d_2 \) dependence has been absorbed into the function \( G(w) \) which is independent of \( s \) and has the explicit form
\[
G(w) = (e^{2\pi w} - 1) \left\{ \int_0^w \frac{dy}{(e^{\delta_1 y} - 1)(e^{\delta_2 (w-y)} - 1)} - \int_0^\infty \frac{dy}{(e^{\delta_1 y} - 1)(e^{\delta_2 (w+y)} - 1)} - \int_0^\infty \frac{dy}{(e^{\delta_1 (w+y)} - 1)(e^{\delta_2 y} - 1)} \right\} \tag{110}
\]
where we have defined \( \delta_i = 2\pi / d_i, i = 1, 2 \). This function is symmetric under the interchange of \( d_1, d_2 \) as one would expect. The factor \( (e^{2\pi w} - 1) \) has been included into the definition to ensure that \( G(w) \) is finite as \( w \to 0 \).

All the integrals in the definition of \( G(w) \) are divergent at their lower limits, and the first integral is also divergent at its upper limit. However one can check that the combination is well defined. In fact by expanding the integrands at their limits of integration we find that for small \( \epsilon > 0 \),
\[
G(w) = (e^{2\pi w} - 1) \left\{ \int_\epsilon^w \frac{dy}{(e^{\delta_1 y} - 1)(e^{\delta_2 (w-y)} - 1)} - \int_\epsilon^\infty \frac{dy}{(e^{\delta_1 y} - 1)(e^{\delta_2 (w+y)} - 1)} - \int_\epsilon^\infty \frac{dy}{(e^{\delta_1 (w+y)} - 1)(e^{\delta_2 y} - 1)} \right\} + O(\epsilon).
\]
Thus \( G(w) \) is easy to calculate numerically, with the error being of order \( \epsilon \). We may also make \( \epsilon \) the lower limit of the integration over \( w \) in (109). Since the integrand with respect to \( w \) is finite at the lower limit, the error incurred will still be \( O(\epsilon) \).

The full expression for our continuation of the Barnes zeta function, after dealing with both the single and double integrals in (107), is
\[
\zeta_2(s, a|d_1, d_2) = -\frac{1}{4} a^{-s} - \frac{a^{2-s}}{d_1 d_2 (s-1)(s-2)} - \frac{(d_1 + d_2)a^{1-s}}{2d_1 d_2 (s-1)} + \frac{1}{2} d_1^{-s} \zeta_H(s, a|d_1) + \frac{1}{2} d_2^{-s} \zeta_H(s, a|d_2) + \frac{1}{d_1 d_2 (s-1)} \left\{ d_1^{2-s} \zeta_H(s-1, a|d_1) + d_2^{2-s} \zeta_H(s-1, a|d_2) \right\} + \frac{2}{d_1 d_2} \int_0^\infty \frac{G(w)dw}{e^{2\pi w} - 1} \frac{\cos \left( \frac{s \tan^{-1} (w/a)}{a^2 + w^2} \right)}{(a^2 + w^2)^{s/2}} \tag{111}
\]
Although this formula was derived for \( s > 2 \), it is actually a continuation to all values of (complex) \( s \) except at the points \( s = 1, 2 \) where there are simple poles. It is easy to check that these poles are correct in that their residues match those given in (60). From (111) we can calculate the derivative of the Barnes zeta function at \( s = 0 \),

\[
\zeta'(0, a|d_1, d_2) = -\frac{1}{4} \ln a - \frac{1}{2} \left( 1 - \frac{a}{d_1} - \frac{a}{d_2} \right) \ln(2\pi) + \frac{1}{2} a \ln a \left( \frac{1}{d_1} + \frac{1}{d_2} \right) - \\
\frac{a^2}{2d_1d_2} \left( \frac{5}{2} + \ln a \right) + \left( \frac{1}{2} - \frac{a}{d_1} \right) \left( \frac{1}{2} - \frac{a}{d_2} \right) \ln \frac{a^2}{d_1d_2} + \\
\left( \frac{1}{2} - \frac{a}{d_1} \right) \ln \Gamma\left( \frac{a}{d_2} \right) + \left( \frac{1}{2} - \frac{a}{d_2} \right) \ln \Gamma\left( \frac{a}{d_1} \right) + \frac{d_1^2 + d_2^2}{12d_1d_2} + \\
\frac{1}{d_1d_2} \int_0^\infty \frac{dw}{e^{2\pi w} - 1} \left( 2G(w) - d_1^2 w \ln(a^2 + d_1^2 w^2) - d_2^2 w \ln(a^2 + d_2^2 w^2) \right)
\]

(112)

Here we have used (108) again to convert derivatives of the Hurwitz zeta function \( \zeta_H(s, a) \) at \( s = -1 \) into integrals suitable for numerical evaluation.

The expression (112) can be used directly in equation (105) and (104). Figures 1 to 4 show plots of \( E = -\zeta'(0)/2 \) for small values of \( \theta \). The lines on the graphs for \( T, O, \) and \( Y \) are for labelling only. The graphs for \( C_k \) and \( D_k \) are plotted for all \( k \), although at the moment we are only concerned with the integral values \( k = 1, 2, \ldots \) marked with crosses. On each graph the value \( \theta \) of the charge per rotational domain is always twice the reflection value \( \theta' \) so as to give the same value of \( q \).
Fig. 1 Vacuum energies for $C_k$ and $D_k$ with $\overline{q'} = 0$. 
Fig. 2 Vacuum energies for $C_k$ and $D_k$ with $\overline{q}' = 1/2$. 
Fig. 3 Vacuum energies for $C_k$ and $D_k$ with $\tau' = 1$. 

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Fig. 4 Vacuum energies for groups $T$, $O$ and $Y$ as functions of $\overline{r}$.
The graphs with $\mathbf{q} = q = 0$ are exactly the same as those calculated in [21]. To plot the graphs in this case we have had to remove the divergence in the first Barnes zeta function derivative in (105). This is due to a zero mode appearing in the spectrum of the operator $H_{S^2}$. The divergence is logarithmic and can be removed by defining the so called $\Gamma$-modular form $\rho_2$ defined by [19,21],

$$\lim_{a \to 0} \zeta'_2(0, a|d_1, d_2) = -\ln a \ln \rho_2.$$ 

The leading term $-\ln a$ on the right hand side is the divergent zero mode contribution which must be removed. The form $\rho_2$ has been calculated in terms of the multiple gamma function by Barnes [19,26]. The $\Gamma$-modular form can be more easily calculated by using the simply proved relation

$$\zeta_2(s, a|d_1, d_2) = a^{-s} + d_1^{-s} \zeta_H(s, 1 + a d_1) + d_2^{-s} \zeta_H(s, 1 + a d_2) + \zeta_2(s, a + d_1 + d_2|d_1, d_2).$$

Differentiating this with respect to $s$ at $s = 0$, and taking the limit $a \to 0$ leaves the expression

$$-\ln \rho_2 = \frac{1}{2} \ln(d_1 d_2) - \ln(2\pi) + \zeta'_2(0, d_0 + 1|d_1, d_2).$$

For $q > 0$, the zero mode that we have just removed for $q = 0$ is still almost zero. This is the reason why the vacuum energy for the Dirichlet zeta function increases more rapidly with $\mathbf{q}'$ than in the Neumann case.

11. Theory for a spherical slice

By a spherical slice of width $\beta$ we mean the space $S_\beta = \{ (\theta, \phi) | \theta \in [0, \pi], \phi \in [0, \beta] \}$ with the points $(\theta, 0)$ and $(\theta, \beta)$ identified (here $(\theta, \phi)$ are the spherical polar coordinates on the sphere which shall be used throughout this section). One might also term this space a `periodic lune'.

The starting point for the monopole theory on $S_\beta$ is the solution to the explicit differential equation (4) with arbitrary $q$. As mentioned in the discussion surrounding the differential equation the solutions are characterised by integers $u, v$, and are given by equations (3), (5). We shall now show that it is possible to define consistent monopole theories on $S_\beta$ corresponding to the rotational, Dirichlet and Neumann cases already given. The rotational case is considered first.

Define $k$ by $k = 2\pi/\beta$ and let $\hat{\gamma}$ denote a rotation by angle $\beta$ about the $z$ axis. Due to the identification of points in $S_\beta$ at $\phi = 0, \beta$, we will proceed conventionally and take the wave function to be single valued i.e. to have period $\beta$. Also,
for rapidity, the monopole charge quantisation will be obtained by requiring the connecting function \( \exp(i2q\phi) \) to have period \( \beta \). This yields

\[
q = \frac{k}{2}n, \quad n \in \mathbb{Z}.
\] (113)

Corresponding to these values of \( q \) we find \( a(\hat{\gamma}) = (-1)^n \). If \( k \) is an integer then \( S_\beta \) is a fundamental domain for the group \( C_k \) (generated by \( \hat{\gamma} \)), and the relation \(|C_k| = k\) gives an equivalence between the quantisation conditions (39) and (113). Thus we see that the quantisation condition for arbitrary \( k \) is a generalisation of the previous theory for \( C_k \) (with integer \( k \)). From single valuedness, \( \varphi(\theta, \beta) = \varphi(\theta, 0) \), for the monopole harmonics (5), we see from the explicit \( \phi \) dependence that,

\[
m = jk - q, \quad j \in \mathbb{Z}.
\] (114)

In this more general setting it would appear that we can always choose \( n \) to be odd, whereas before this was only possible for \( k \) odd also. From (5) we find the possible values of \( l \),

\[
l = u + \frac{k}{2}(|j| + |n - j|), \quad u \in \mathbb{Z}^+, \quad j \in \mathbb{Z}.
\]

The eigenvalues \( \lambda \) appearing in the mode equation (4) are still given by \( l(l + 1) - q^2 \). As in section 7 we shall calculate the linear heat kernel for the eigenvalues \((l + 1/2)\).

Using the values of \( q, l, \) and \( m \) above gives the sum form for the linear heat kernel analogous to (51), with (5),

\[
\tilde{K}_k^q(\tau) = e^{-\tau/2} \sum_{u=0}^{\infty} \sum_{j=-\infty}^{\infty} e^{-\tau(u+k(|j|+|j+n|))/2}.
\] (115)

Evaluating this sum gives exactly the result found before, equation (56), except that now \( k \) is not an integer, and \( q \) need not be an integer multiple of \( k \). A corollary of this result is that (56) is also valid for the case when \( \Gamma = C_k \) with \( k \) and \( 2q \) odd integers. The zeta function defined by equation (58) is also correct for \( d_1 = 1, \quad d_2 = k \) and hence we conclude that the vacuum energies calculated in the previous section are valid for arbitrary \( k \).

For the Dirichlet and Neumann theories defined in section 8 we consider the slice \( S_{\beta/2} \) and two reflection planes \( P_0 \) and \( P_1 \) which leave \( \phi = 0 \) and \( \phi = \beta/2 \) invariant respectively. Following the same arguments as in section 8 gives the values of the monopole charge

\[
q = kn',
\] (116)
which is equivalent to equation (65). Taking $\gamma'$ in (64) to be a reflection in $P_0$ and using (17) gives the modes $W_{qm}^{(l)}$ as the combinations,

$$W_{qm}^{(l)}(\theta, \phi) = Y_{qm}^{(l)}(\theta, \phi) + a(\sigma)Y_{q-m}^{(l)}(\theta, \phi)$$

Here $Y_{qm}^{(l)}$ are those defined in (3) with the string along the negative $z$ axis, and $a(\sigma) = \pm 1$ as before (the reflection in $P_0$ is expressed simply as $\phi \rightarrow -\phi$).

The $\phi$ dependent part of the modes $W_{qm}^{(l)}$ is given by

$$W_{qm}^{(l)} \sim e^{iq\phi} \times \begin{cases} \cos(m\phi), & a(\sigma) = +1 \\ \sin(m\phi), & a(\sigma) = -1 \end{cases}.$$  

Reflection in $P_0$ is equivalent to $\phi \rightarrow -\phi$ and we see explicitly that this transforms $W_{qm}^{(l)}(\theta, 0)$ into $a(\sigma)W_{q-m}^{(l)}(\theta, 0)$ as required. Reflection in $P_1$ is equivalent to $\phi \rightarrow \beta - \phi$ and results in the condition on $m$,

$$m = kj, \quad j \in \mathbb{Z}^+.$$  

(117)

For the Dirichlet case $a(\sigma) = -1$ the $m = 0$ mode is in fact zero and has to be removed. Comparing (117) with (114) (with $n = 2n'$ even) we see that the only difference between the rotation and reflection cases is that in the reflection case the values of $m$ are restricted to positive integers.

The linear heat kernel for the Neumann case $a(\sigma) = +1$ is given by

$$\tilde{K}_N^q(\tau) = e^{-\tau/2} \sum_{u=0}^{\infty} \sum_{j=0}^{\infty} e^{-\tau(u+k(|j+n'|+|j-n'|)/2)}.$$  

(118)

Explicit evaluation of the sum gives exactly the heat kernel as before, equation (55). The Dirichlet case involves subtracting the $j = 0$ term from (71) and yields the previous expression (75). Thus the zeta functions are given exactly as before and we conclude that the vacuum energy calculated for $C_k$ is in fact valid for all $k$.

To extend the results of $D_k$ to arbitrary $k$ we consider the slice of the upper hemisphere $S_\beta' = \{ (\theta, \phi) | \theta \in [0, \pi/2], \phi \in [0, \beta] \}$ with again $k = 2\pi/\beta$. This is a fundamental domain for $D_k$ when $k$ is an integer. The theory then follows as for the $C_k$ extension above, but we must include in this case a rotation about the $x$ axis by angle $\pi$. This rotation can be thought of as a reflection in the plane $P_0$ followed by a reflection in the $x - y$ plane, which we call $P_2$. The theory for $S_\beta'$ above is adapted to the reflection $P_0$ and thus we see that $P_2$ is the essential extra detail here.
The reflection $P_2$ is equivalent to the transformation $\theta \to \pi - \theta$ which does not affect the $\phi$ dependence of the modes $Y_{qm}^{(l)}(\theta, \phi)$. Since the extension to arbitrary $k$ is entirely linked with the $\phi$ dependence, we conclude that all the heat kernels, zeta functions and vacuum energies for $D_k$ can be extended to arbitrary $k$. The values of the monopole charge in the reflection case are calculated using the theory of section 8 as

$$q = 2kn', \quad n' \in \mathbb{Z},$$

which is the generalisation of (65) with $|\Gamma'| = 4k$.

12. Summary and discussion

We have thoroughly adapted Dirac’s monopole theory to the orbifold, $S^2/\Gamma$, for the cases that $\Gamma$ contains only rotations and when $\Gamma$ is generated by reflections. In the former case we imposed rotational (periodic) boundary conditions on the monopole solutions. In the latter we had a choice of boundary conditions defined so as to reproduce Dirichlet and Neumann conditions for no monopole charge, $q = 0$. We found that it was the monopole charge $\bar{q} = q/|\Gamma|$ through $S^2/\Gamma$ that was Dirac quantised with $2\bar{q} \in \mathbb{Z}$.

After all the formalities of the theory had been tidied we explicitly calculated the vacuum energies on the orbifolds $S^2/\Gamma$ and $\mathbb{R} \times S^2/\Gamma$. Formal expressions are given for the generalisation to the spaces $\mathbb{R}^{2n} \times S^2/\Gamma$ and $\mathbb{R} \times \mathbb{R}^{2n} \times S^2/\Gamma$. Finally we provided an extension of the monopole theory to arbitrary slices of the sphere and hemisphere. In this case the flux through the spherical region is still quantised, although now the overall monopole charge $q$ is not, in general, an integer or half odd-integer.

We feel that the scalar theory has been developed essentially to its analytical limit on the factored sphere. The next step would be the extension to $\mathbb{R}^3/\Gamma$ for $\Gamma$ a reflection or rotation group. This requires modes of the full Hamiltonian which are given by

$$Y_{qm}^{(l)}(\theta, \phi)J_{\nu}(kr)\sqrt{k/r}, \quad \nu = \sqrt{(l + 1/2)^2 - q^2}$$

with eigenvalues $k^2$. Since the radial dependence does not involve $m$, the underlying facts of the theory (modes on factored space, charge quantisation etc.) are the same as in the spherical case. However the heat-kernel calculation, and hence the $\zeta$–function, is completely different. Due to the complicated index, $\sqrt{(l + 1/2)^2 - q^2}$, closed forms do not seem possible and asymptotic methods are needed. One could
always arbitrarily add a term \( q^2 / r \) to the (total) Hamiltonian and then a closed form would exist. This fact suggests that there is some significance to this modification.

The spinor theory on the factored sphere \( S^2 / \Gamma \) has been considered by Chang [27]. He found a consistent theory only for \( \Gamma = C_k \) with \( k \) odd. For \( q \neq 0 \) we claim that the same restriction still holds. This follows from the lack of half odd-integral solutions to the scalar monopole problem for \( \Gamma \neq C_k \).

A possible extension of the scalar calculation would be to consider if (high temperature) Bose-Einstein condensation occurs. The general theory has been laid down by Toms [28] See also Kirsten and Toms [29]. Basically, all that is required is to ensure that the \( \zeta \)–function for the theory, and its derivative, are finite at zero as the chemical potential approaches a critical value. On the two-sphere we can use the calculations of \( \zeta(-n) \) and \( \zeta'(-n) \) given in section 9 to study the theory on \( \mathbb{R}^{2n} \times S^2 \). In fact we could also discuss \( \mathbb{R}^{2n} \times S^d \).

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References

1. Chang, Peter and Dowker, J.S. *Nucl. Phys.* **B395** (1993) 407.
2. Meyer, B. *Can. J. Math.* **6** (1954) 135.
3. Chang, Peter and Dowker, J.S. *Phys. Rev. D* (19).
4. *The Formation and Evolution of Cosmic strings* edited by G.W.Gibbons, S.W.Hawking and T.Vachaspati (Cambridge University Press, Cambridge, 1990).
5. Tamm, I. *Z. f. Phys.* **71** (1931) 141.
6. Fierz, M. *Helv.Phys.Acta* **17** (1944) 27.
7. Greub, W. and Petry, H.-R. *J. Math. Phys. Phys. Rev.* **D16** (1977) 1018.
8. Wu, T.T. and Yang, C.N. *Nucl. Phys.* **B107** (1976) 365; *Phys. Rev. D16* (1977) 1018.
9. Sniatycki, J.J. *Math. Phys. 15* (1974) 619.
10. Horvathy, P. *Int. J. Theor. Phys.* **20** (1981) 697.
11. Biedenharn,L.C. and Louck,J.D. *The Racah-Wigner Algebra in Quantum Theory*, (Addison-Wesley,Reading,Mass,1981).

12. Peierls,R. *Z. f. Phys.* 80 (1933) 763.

13. Janner,A. and Janssen,T. *Physica* 53 (1971) 1.

14. Dowker,J.S. *J. Math. Phys.* 28 (1987) 33.

15. Frenkel,A. and Hraskó,P. *Ann. Phys.* 105 (1977) 288.

16. Brink,D.M. and Satchler,G.R. *Angular Momentum* 2nd ed. Clarendon Press, Oxford, 1968.

17. Tam,W.G. *Physica* (19).

18. Opechowski,W. and Tam, W.G. *Physica* 42 (1969) 529.

19. Barnes,E.W. *Trans. Camb. Phil. Soc.* 19 (1903) 374.

20. A.Erdelyi,W.Magnus,F.Oberhettinger and F.G.Tricomi *Higher Transcendental Functions* Vol.I McGraw-Hill, New York, 1953.

21. Dowker,J.S. *Comm. Math. Phys.* 162 (1994) 633.

22. Magnus,W., Oberhettinger,F. and Soni, R.P., *Formulas and theorems for the special functions of mathematical physics*, 3rd ed. (Springer-Verlag, Berlin, 1966.

23. Cook, A.W. PhD Thesis, University of Manchester, 1996.

24. Dowker,J.S. and Kennedy,G. *J. Phys.* A11 (1978) 895.

25. Lindelöf,E. *Le Calcul des Résidus*, Gauthier-Villars, Paris, 1905.

26. Barnes,E.W. *Trans. Camb. Phil. Soc.* 19 (1903) 426.

27. Chang,Peter, PhD thesis, University of Manchester, 1993.

28. Toms,D.J. *Phys. Rev.* D50 (1994) 6457.

29. Kirsten,K. and Toms,D.J. *Bose-Einstein condensation in arbitrarily shaped cavities*, cond-mat/9810098.