Algorithms for Computing Abelian Periods of Words

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Abstract

Constantinescu and Ilie (Bulletin EATCS 89, 167–170, 2006) introduced the notion of an Abelian period of a word. A word of length $n$ over an alphabet of size $\sigma$ can have $\Theta(n^2)$ distinct Abelian periods. The Brute-Force algorithm computes all the Abelian periods of a word in time $O(n^2 \times \sigma)$ using $O(n \times \sigma)$ space. We present an off-line algorithm based on a select function having the same worst-case theoretical complexity as the Brute-Force one, but outperforming it in practice. We then present on-line algorithms that also enable to compute all the Abelian periods of all the prefixes of $w$.

Keywords: Abelian period; Abelian repetition; weak repetition; design of algorithms; text algorithms; Combinatorics on Words

1. Introduction

An integer $p > 0$ is a (classical) period of a word $w$ of length $n$ if $w[i] = w[i + p]$ for every $1 \leq i \leq n - p$. Classical periods have been extensively studied in Combinatorics on Words \cite{13} due to their direct applications in data compression and pattern matching.

The Parikh vector of a word $w$ enumerates the cardinality of each letter of the alphabet in $w$. For example, given the alphabet $\Sigma = \{a,b,c\}$, the Parikh vector of the word $w = \text{aaba}$ is $(3, 1, 0)$. The reader can refer to \cite{3} for a list of applications of Parikh vectors.

An integer $p$ is an Abelian period of a word $w$ over a finite alphabet $\Sigma = \{a_1, a_2, \ldots, a_\sigma\}$ if $w$ can be written as $w = u_0u_1 \cdots u_ku_k$ where for $0 < i < k$ all the $u_i$’s have the same Parikh vector $P$ such that $\sum_{i=1}^{k-1} P[i] = p$ and the Parikh vectors of $u_0$ and $u_k$ are contained in $P$. For example, the word $w = \text{abbbabb}$ can be written as $w = u_0u_1u_2u_3$, with $u_0 = a$, $u_1 = \text{bab}$, $u_2 = \text{bba}$ and $u_3 = \text{bb}$, and 3 is an Abelian period of $w$.

This definition of Abelian period matches the one of weak repetition (also called Abelian power) when $u_0$ and $u_k$ are the empty word and $k > 2$ \cite{7}.

In recent years, several efficient algorithms have been designed for an Abelian version of the classical pattern matching problem, called the Jumbled Pattern Matching problem \cite{1,5,11,14,15}, defined as the problem of finding the occurrences of a substring in a text up to a permutation of the letters in the substring, i.e., the occurrences of any substring of the text having the same Parikh
vector as the pattern. However, apart from the greedy off-line algorithm given in [4], no efficient
ing algorithms are known for computing all the Abelian periods of a given word\footnote{\cite{4}}.

In this article, we present several off-line and on-line algorithms for computing all the Abelian
periods of a given word. In Section 2 we give some basic definitions and fix the notation. Section 3
presents off-line algorithms, while Section 4 presents on-line algorithms. In Section 5 we give some
experimental results on execution times. Finally, Section 6 contains conclusions and perspectives.

2. Definitions and notation

Let $\Sigma = \{a_1, a_2, \ldots, a_\sigma\}$ be a finite ordered alphabet of cardinality $\sigma$ and $\Sigma^*$ the set of words
over $\Sigma$. We set $\text{ind}(a_i) = i$ for $1 \leq i \leq \sigma$. We denote by $|w|$ the length of $w$. We write $w[i]$ the $i$-th
symbol of $w$ and $w[i..j]$ the factor of $w$ from the $i$-th symbol to the $j$-th symbol included, with
$1 \leq i \leq j \leq |w|$. We denote by $|w|_a$ the number of occurrences of the letter $a \in \Sigma$ in the word $w$.

The Parikh vector of a word $w$, denoted by $P_w$, counts the occurrences of each letter of $\Sigma$ in $w$,
i.e., $P_w = (|w|_{a_1}, \ldots, |w|_{a_\sigma})$. Notice that two words have the same Parikh vector if and only if one
is obtained from the other by permuting letters (in other words, one is an anagram of the other).
We denote by $P_w(i, m)$ the Parikh vector of the factor of length $m$ beginning at position $i$ in the
word $w$.

Given the Parikh vector $P_w$ of a word $w$, we denote by $P_w[i]$ its $i$-th component and by $|P_w|$ its
norm, that is the sum of its components. Thus, for $w \in \Sigma^*$ and $1 \leq i \leq \sigma$, we have $P_w[i] = |w|_{a_i}$
and $|P_w| = \sum_{i=1}^{\sigma} P_w[i] = |w|$. Finally, given two Parikh vectors $P, Q$, we write $P \subset Q$ if $P[i] \leq Q[i]$
for every $1 \leq i \leq \sigma$ and $|P| < |Q|$.

Definition 1 (\cite{6}). A word $w$ has an Abelian period $(h, p)$ if $w = u_0 u_1 \cdots u_{k-1} u_k$ such that:

- $P_{u_0} \subset P_{u_1} = \cdots = P_{u_{k-1}} \supset P_{u_k}$,
- $|P_{u_0}| = h, |P_{u_1}| = p$.

We call $u_0$ and $u_k$ resp. the head and the tail of the Abelian period. Notice that the length
t $t = |u_k|$ of the tail is uniquely determined by $h, p$ and $|w|$, namely $t = (|w| - h) \mod p$.

The following lemma gives an upper bound on the number of Abelian periods of a word.

Lemma 2.1. A word $w$ of length $n$ over an alphabet $\Sigma$ of cardinality $\sigma$ can have $\Theta(n^2)$ different
Abelian periods.

Proof. The word $w = (a_1 a_2 \cdots a_\sigma)^{n/\sigma}$ has Abelian period $(h, p)$ for any $p \equiv 0 \mod \sigma$ and every $h$
such that $0 \leq h \leq \min(p - 1, n - p)$. Therefore, $w$ has $\Theta(n^2)$ different Abelian periods. \hfill $\square$

A natural order can be defined on the Abelian periods of a word.

Definition 2. Two distinct Abelian periods $(h, p)$ and $(h', p')$ of a word $w$ are ordered as follows:
$(h, p) < (h', p')$ if $p < p'$ or $(p = p'$ and $h < h')$.

We are interested in computing all the Abelian periods of a word. However, the algorithms we
present in this paper can be easily adapted to give the smallest Abelian period only.

\footnote{\cite{6}}
3. Off-line algorithms

3.1. Brute-Force algorithm

In Figure 1, we present a Brute-Force algorithm computing all the Abelian periods of an input word \( w \) of length \( n \). For each possible head of length \( h \) from 1 to \( \lfloor (n - 1)/2 \rfloor \) the algorithm tests all the possible values of \( p \) such that \( p > h \) and \( h + p \leq n \). It is a reformulation of the algorithm given in [7].

```plaintext
AbelianPeriod-BruteForce(w, n)
1   for h ← 0 to \( \lfloor (n - 1)/2 \rfloor \) do
2       p ← h + 1
3       while h + p ≤ n do
4         if (h, p) is an Abelian period of w then
5             Output(h, p)
6         p ← p + 1
```

Figure 1: Brute-Force algorithm for computing all the Abelian periods of a word \( w \) of length \( n \).

Example 1. For \( w = \text{abaababa} \) the algorithm outputs \((1, 2), (0, 3), (2, 3), (1, 4), (2, 4), (3, 4), (0, 5), (1, 5), (2, 5), (3, 5), (0, 6), (1, 6), (2, 6), (0, 7), (1, 7) \) and \((0, 8)\). Among these periods, \((1, 2)\) is the smallest.

Theorem 3.1. The algorithm AbelianPeriod-BruteForce computes all the Abelian periods of a given word of length \( n \) in time \( O(n^2 \times \sigma) \) with \( O(n \times \sigma) \) space.

Proof. The correctness of the algorithm comes directly from Definition 1. In a preprocessing phase, all the prefixes of the word are computed and stored in a table. This takes time \( O(n) \) and space \( O(n \times \sigma) \). In this way, the computation of the Parikh vector of a factor of the word can be done by computing the difference between two Parikh vectors in the table. Since the algorithm performs \( \sum_{h=0}^{\lfloor (n-1)/2 \rfloor} \sum_{p=h+1}^{n-h} n/p = O(\sum_{h=1}^{n} \sum_{p=h}^{n} n/p) = O(n^2) \) many comparisons between two Parikh vectors, and since each comparison takes \( O(\sigma) \) time, the overall time and space complexity are as claimed (output periods are not stored).

3.2. Select-based algorithm

Let us introduce the select function [16] defined as follows.

Definition 3. Let \( w \) be a word of length \( n \) over alphabet \( \Sigma \), then \( \forall a \in \Sigma \):

- \( \text{select}_a(w, 0) = 0 \);
- \( \forall 1 \leq i \leq |w|_a, \text{select}_a(w, i) = j \) if and only if \( j \) is the position of the \( i \)-th occurrence of letter \( a \) in \( w \);
- \( \forall i > |w|_a, \text{select}_a(w, i) \) is undefined.
In order to compute the select function of a word \( w \), we consider an array \( S_w \) of size \( |w| \) storing the (ordered) positions of the occurrences of the letter \( a_1 \) in \( w \), then the positions of the occurrences of the letter \( a_2 \) and so on, up to the positions of the occurrences of the letter \( a_\sigma \). In addition to \( S_w \), we also consider an array \( C_w \) of \( \sigma + 1 \) elements defined by: 

\[
C_w[1] = 1, \quad C_w[i] = \sum_{j=1}^{i-1} |w|_{a_j} + 1 \quad \text{for } 1 < i \leq \sigma \quad \text{and} \quad C_w[\sigma + 1] = |w| + 1.
\]

In fact, \( C_w[i] - 1 \) is the number of occurrences of letters strictly smaller than \( a_i \) in \( w \). Array \( C_w \) serves as an index to access \( S_w \). Hence, for a letter \( a \in \Sigma \) and \( i > 0 \), we have:

\[
select_a(w,i) = \begin{cases} 
S_w[C_w[ind(a)] + i - 1] & \text{if } i \leq C_w[ind(a)] + 1 - C_w[ind(a)], \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

**Example 2.** For \( w = abaababa \), the select function uses the following three arrays:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
a & b & a & a & b & a & b & a \\
\end{array}
\quad \begin{array}{c}
ind \\
1 & 2 \\
\end{array}
\quad \begin{array}{cccc}
1 & 2 & 3 \\
\hline
\end{array}
\quad \begin{array}{cccc}
S_w \\
1 & 3 & 4 & 6 & 8 & 2 & 5 & 7 \\
\end{array}
\quad \begin{array}{cccc}
C_w \\
1 & 6 & 9 \\
\end{array}
\]

Then, for instance, \( select_b(w,2) = S_w[C_w[ind(b)] + 2 - 1] = S_w[7] = 5 \), meaning that the second \( b \) in \( w \) appears in position 5.

Algorithm **ComputeSelect** (see Figure 2) computes the two arrays \( C_w \) and \( S_w \) used by the select function.

**Proposition 3.2.** Algorithm **ComputeSelect** runs in \( O(n + \sigma) \) time and space.

**Proof.** The time complexity comes from the fact that the for loops in lines 2–3 and 4–5 are executed \( O(\sigma) \) times, the for loop in lines 6–8 is executed \( n \) times, and all the other instructions take constant time. \qed

Once the arrays \( C_w \) and \( S_w \) have been computed, each call to the select function is answered in constant time.

The Brute-Force algorithm tests all possible pairs \((h,p)\), but it is clear that, for a given value of \( h \), some pairs \((h,p)\) cannot be Abelian periods of \( w \). For example, let \( w = abaaaaabaa \) and \( h = 2 \). Since \( P_w(1,h) \) has to be contained in \( P_w(h+1,p) \), the pairs \((2,3), (2,4)\) and \((2,5)\) cannot be Abelian periods of \( w \). Indeed, the minimal value of \( p \) such that \((2,p)\) can be an Abelian period of \( w \) is 6, in order to contain the second \( b \) of \( w \).

This remark leads us to introduce two arrays, \( M_w \) and \( G_w \), which allow one to skip, for each value \( h \) of the head, a number of values of \( p \) that are not compatible with \( h \).

The array \( M_w \) is defined as follows:

**Definition 4.** Let \( w \) be a word of length \( n \) over the alphabet \( \Sigma \). Then \( \forall 0 \leq h \leq \lfloor (n - 1)/2 \rfloor \), \( M_w[h] \) is defined by

\[
M_w[h] = \begin{cases} 
\min\{p \mid P_w(1,h) \subset P_w(h+1,p)\} & \text{if } \forall a \in \Sigma, 2 \times |w[1..h]|_a \leq |w|_a, \\
-1 & \text{otherwise.}
\end{cases}
\]

4
In other words, if \( \text{select}_a(w, 2 \times |w[1..h]|) \) is defined for all the letters \( a \in \Sigma \), then
\[
M_w[h] = \max \{h + 1, \max \{\text{select}_a(w, 2 \times |w[1..h]|) \mid a \in \Sigma \} - h\};
\]
otherwise, \( M_w[h] = -1 \).

The algorithm \( \text{COMPUTE}_M(w, n, C_w, S_w) \) (see Figure 3) builds the array \( M_w[h] \) processing the positions of \( w \) from left to right.

**Proposition 3.3.** Algorithm \( \text{COMPUTE}_M(w, n, C_w, S_w) \) computes the array \( M_w \) in time and space \( O(n + \sigma) \).

**Proof.** The correctness of the algorithm comes directly from Definition 4. The time complexity comes from the fact that the \( \text{for} \) loop in lines 2–3 is executed \( \sigma \) times, the \( \text{for} \) loops in lines 4–9 and 10–12 are executed \( O(n) \) times, and all the other instructions take constant time. \( \square \)
Proposition 3.4. Let \( w \) be a word of length \( n \) over the alphabet \( \Sigma \), and \( h \) such that \( 0 \leq h \leq \lfloor (n - 1)/2 \rfloor \). If \( \mathcal{M}_w[h] = -1 \), then \( \forall h' \geq h \) one has \( \mathcal{M}_w[h'] = -1 \), and \( h' \) cannot be the length of the head of an Abelian period of \( w \).

**Proof.** If \( \mathcal{M}_w[h] = -1 \), then by definition there exists a letter \( a \in \Sigma \) such that \( 2 \times |w[1..h]|_a > |w|_a \). Therefore, one cannot find a value \( p \) such that \( |w[1..h]|_a \leq |w[(h + 1)..(h + p)]|_a \). It is clear that this is also true for any value \( h' > h \).

The array \( G_w \) is defined as follows:

**Definition 5.** Let \( w \) be a word of length \( n \) over the alphabet \( \Sigma \). Then, for every \( h \) such that \( 0 \leq h \leq \lfloor (n - 1)/2 \rfloor \), \( G_w[h] \) is defined by

\[
G_w[h] = \max \{ \text{select}_a(w, i + 1) - \text{select}_a(w, i) \mid a \in \Sigma, \ h < \text{select}_a(w, i) < \text{select}_a(w, i + 1) \leq n \}.
\]

In fact, \( G_w[h] \) is the maximal value \( j' - j \) such that \( h < j < j' \) and \( w[j] = w[j'] \), for some \( j \) and \( j' \).

The array \( G_w \) can be computed by the algorithm \( \text{ComputeG}(w, n) \) (see Figure 4) processing the positions of \( w \) from right to left.

**Figure 4:** Algorithm computing the array \( G_w \).

Proposition 3.5. Algorithm \( \text{ComputeG}(w, n) \) computes the array \( G_w \) in time and space \( O(n + \sigma) \).

**Proof.** The correctness of the algorithm comes directly from Definition 5. The time complexity comes from the fact that the \( \text{for} \) loop in lines 2–3 is executed \( \sigma \) times, the \( \text{for} \) loop in lines 4–10 is executed \( n \) times, and all the other instructions take constant time.

Proposition 3.6. Let \( w \) be a word of length \( n \) over the alphabet \( \Sigma \). For every \( h \) such that \( 0 \leq h \leq \lfloor (n - 1)/2 \rfloor \), if \( p \) is such that \( h < p < \max\{\mathcal{M}_w[h], \lfloor (G_w[h] + 1)/2 \rfloor \} \), then \( (h, p) \) is not an Abelian period of \( w \).
Proof. From the definition of $\mathcal{M}_w[h]$, it directly follows that if $p < \mathcal{M}_w[h]$, then $(h, p)$ cannot be an Abelian period of $w$.

Given $h$, let $a \in \Sigma$ be such that there exists $1 \leq i < n$ and $\text{select}_a(w, i + 1) − \text{select}_a(w, i) = \mathcal{G}_w[h]$. Let $j = \text{select}_a(w, i)$ and $j′ = \text{select}_a(w, i + 1)$. If $p < [(\mathcal{G}_w[h] + 1)/2]$, setting $k = \min\{k′ \mid h + k′p \geq j\}$, then $h + (k + 1)p < j′$ and $|w[h + kp + 1..h + (k + 1)p]|_a = 0$. Therefore, $(h, p)$ cannot be an Abelian period of $w$ (see Figure 5).

Arrays $\mathcal{M}_w$ and $\mathcal{G}_w$ give, for every head length $h$, a minimal value for a possible $p$ such that $(h, p)$ can be an Abelian period of $w$. This allows us to skip a number of values for $p$ that cannot give an Abelian period. Our next off-line algorithm based on the $\text{select}$ function will make use of these arrays.

The following lemma shows how to check if $(h, p)$ is an Abelian period of $w$ (except for the tail) using the $\text{select}$ function.

**Lemma 3.7.** Let $w$ be a word of length $n$ over the alphabet $\Sigma$. Let $H = \mathcal{P}_w(1, h)$ and $\mathcal{P} = \mathcal{P}_w(h + 1, p)$. Let $i = h + kp$ such that $0 < k, p \leq n - i$, and $(h, p)$ is an Abelian period of $w[1..i]$ (with an empty tail). Then the following two conditions are equivalent:

1. $(h, p)$ is an Abelian period of $w[1..i]$;
2. for all $a \in \Sigma$

$$\text{select}_a(w, H[\text{ind}(a)]) + \left(1 + \lfloor \frac{i}{p} \rfloor \right) \times \mathcal{P}[\text{ind}(a)] \leq i + p.$$  

Proof. Since $(h, p)$ is an Abelian period of $w[1..i]$ with $i = h + kp$ for some $k > 0$, then $|w[1..i]|_a = H[\text{ind}(a)] + k \times \mathcal{P}[\text{ind}(a)]$ for each letter $a \in \Sigma$. Notice that since $h < p$ we have $k = \lfloor i/p \rfloor$.

1. $(1 \Rightarrow 2)$. The fact that $(h, p)$ is an Abelian period of $w[1..i + p]$ implies that, for all $a \in \Sigma$, $|w[1..i + p]|_a = H[\text{ind}(a)] + (k + 1) \times \mathcal{P}[\text{ind}(a)]$. Thus, by definition of $\text{select}$, we have $\text{select}_a(w, H[\text{ind}(a)] + (1 + \lfloor i/p \rfloor) \times \mathcal{P}[\text{ind}(a)]) \leq i + p$.

2. $(2 \Rightarrow 1)$. The fact that $\text{select}_a(w, H[\text{ind}(a)] + (1 + \lfloor i/p \rfloor) \times \mathcal{P}[\text{ind}(a)]) \leq i + p$ implies that $|w[1..i + p]|_a = H[\text{ind}(a)] + (k + 1) \times \mathcal{P}[\text{ind}(a)]$. We know that $|w[1..i]|_a = H[\text{ind}(a)] + k \times \mathcal{P}[\text{ind}(a)]$. By difference, $|w[i + 1..i + p]|_a = \mathcal{P}[\text{ind}(a)]$. Since this is true for all $a \in \Sigma$, we have $\mathcal{P}_w(i + 1, p) = \mathcal{P}$, and therefore $(h, p)$ is an Abelian period of $w[1..i + p]$.

Figure 6 presents the algorithm **ABELIANPERIOD-SHIFT** based on the previous lemma.
Algorithm 6: Algorithm checking whether \((h, p)\) is an Abelian period of the prefix of \(w\) of length \(n - ((n - h) \mod p)\).

**Proposition 3.8.** Algorithm \textsc{AbelianPeriod-Shift}(\(h, p, w, n, C_w, S_w\)) returns true if and only if \((h, p)\) is an Abelian period of the prefix of \(w\) of length \(n - ((n - h) \mod p)\) in time \(O(n/p \times \sigma)\) and space \(O(\sigma)\).

**Proof.** The correctness comes directly from Lemma 3.7. The while loop in line 3 is executed \(n/p\) times and the for loop in line 4 is executed \(\sigma\) times, thus the time complexity is \(O(n/p \times \sigma)\). The algorithm only needs to access two Parikh vectors, so the space used is \(O(\sigma)\). \qed

Using Propositions 3.6 and 3.8, algorithm \textsc{AbelianPeriod-Select}, given in Figure 7, computes all the Abelian periods of a word \(w\) of length \(n\).

Algorithm 7: Algorithm computing all the Abelian periods of word \(w\) of length \(n\), based on the \textit{select} function.

**Theorem 3.9.** Algorithm \textsc{AbelianPeriod-Select} computes all the Abelian periods of a word of length \(n\) in time \(O(n^2 \times \sigma)\) and space \(O(n \times \sigma)\).
4.1. Two-dimensional array

to store efficiently the Abelian periods of the prefixes of \( w \).

On-line algorithms

The test whether a pair \((h,p)\) is an Abelian period of the word is done by calling the function \( \text{ABELIAN\_PERIOD\_SHIFT} \) in line 8 and, if this returns true, by verifying the compatibility of the tail in line 10 (output periods are not stored). By Proposition 3.8, each call to the function \( \text{ABELIAN\_PERIOD\_SHIFT} \) takes \( O(n/p \times \sigma) \) time, while the test on the tail is performed in \( O(\sigma) \) time. Thus, the overall time complexity of the algorithm \( \text{ABELIAN\_PERIOD\_SELECT} \) is \( O(\sum_{h=0}^{(n-1)/2} \sum_{p=h+1}^{n-h} (n/p \times \sigma)) = O(\sum_{h=1}^{n} \sum_{p=h}^{n} (n/p \times \sigma)) = O(n^2 \times \sigma) \).

The space needed by the preprocessing phase is \( O(n \times \sigma) \), needed for the computation of the table of the Parikh vectors of the prefixes of \( w \), whereas the while loop in lines 5–13 only requires \( O(\sigma) \) additional space.

4. On-line algorithms

We now propose three on-line algorithms to compute all the Abelian periods of a word \( w \) using dynamic programming. The idea is to find combinatorial constraints to determine which Abelian periods of the prefix \( w[1..i-1] \) are still Abelian periods of the prefix \( w[1..i] \). Moreover, one has to store efficiently the Abelian periods of the prefixes of \( w \). The three algorithms we describe below use for this purpose three different data structure: a two-dimensional table, lists and heaps, respectively.

The following proposition states that if \((h,p)\) is not an Abelian period of a word \( w \), with \( h+p \leq n = |w| \), then it cannot be an Abelian period of any word having \( w \) as a prefix.

**Proposition 4.1.** Let \( w \) be a word of length \( n \) and let \( h,p \) such that \( h+p \leq n \). If \((h,p)\) is not an Abelian period of \( w \), then \((h,p)\) is not an Abelian period of \( wa \) for any letter \( a \in \Sigma \).

**Proof.** If \((h,p)\) is not an Abelian period of \( w \), at least one of the following three cases holds:

1. \( P_w(1,h) \not\subset P_w(h+1,p) \);
2. there exist two distinct indices \( h \leq i, i’ \leq |w|-p+1 \) such that \( i = kp+h+1 \) and \( i’ = k’p+h+1 \) with \( k \) and \( k’ \) two integers and \( P_w(i,p) \neq P_w(i’,p) \);
3. \( t = (|w| - h) \text{ mod } p \) and \( P_w(|w|-t+1,t) \not\subset P_w(|w|-p-t+1,p) \).

If case 1 holds, then \( P_{wa}(1,h) \not\subset P_{wa}(h+1,p) \); if case 2 holds, then \( P_{wa}(i,p) \neq P_{wa}(i’,p) \); finally, if case 3 holds, then \( P_{wa}(|w|-t+1,t+1) \not\subset P_{wa}(|w|-p-t+1,p) \). In all cases, \((h,p)\) is not an Abelian period of \( wa \).

4.1. Two-dimensional array

The first algorithm (given in Figure 8) uses a two-dimensional array \( T \) and the property stated in Proposition 4.1 to compute all the Abelian periods of an input word \( w \) on-line. The algorithm processes the positions of \( w \) in increasing order from left to right. When processing position \( i \),
the value of $T[h, p]$ is set to $j$ if and only if $w[1 \ldots j]$ is the longest prefix of $w[1 \ldots i]$ having Abelian period $(h, p)$. Hence, if $j = i - 1$, the algorithm checks whether $w[1 \ldots i]$ has Abelian period $(h, p)$, and updates $T[h, p]$ accordingly.

**AbelianPeriod-array**($w, n$)

1. $T[0, 1] \leftarrow 1$
2. for $i \leftarrow 2$ to $n$ do
3.     for $p \leftarrow 1$ to $i - 1$ do
4.         for $h \leftarrow 0$ to $\min\{p - 1, i - p - 1\}$ do
5.             if $T[h, p] = i - 1$ then
6.                 $d \leftarrow (i - h) \mod \ell$
7.                 if $d \neq 0$ then
8.                     if $\mathcal{P}_w(i - d + 1, d) \subset \mathcal{P}_w(i - d - p + 1, p)$ then
9.                         $T[h, p] \leftarrow i$
10.                    else if $\mathcal{P}_w(i - p + 1, p) = \mathcal{P}_w(i - 2 \times p + 1, p)$ then
11.                        $T[h, p] \leftarrow i$
12.                     for $h \leftarrow 0$ to $\lfloor i/2 \rfloor - 1$ do
13.                         if $\mathcal{P}_w(1, h) \subset \mathcal{P}_w(h + 1, i - h)$ then
14.                             $T[h, i - h] \leftarrow i$
15.                     else $T[h, i - h] \leftarrow -1$
16.     return $T$

Figure 8: On-line dynamic programming algorithm for computing all the Abelian periods of a word $w$ of length $n$ using a two-dimensional array.

Since $T[h, p] = i$ if and only if $w[1 \ldots i]$ is the longest prefix of $w$ having Abelian period $(h, p)$, one has that $(h, p)$ is an Abelian period of $w$ if and only if $T[h, p] = n$.

**Example 3.** For $w = \text{abaababa}$, the algorithm computes the following array $T$:

| $h \backslash p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|
| 0               | 1 | 3 | 8 | 6 | 8 | 8 | 8 | 8 |
| 1               |   | 8 | 6 | 8 | 8 | 8 | 8 | 8 |
| 2               |   |   | 8 | 8 | 8 | 8 |
| 3               |   |   |   | 8 | 8 |

Cells in which $T[h, p] = |w| = 8$ correspond to pairs $(h, p)$ output by algorithm AbelianPeriod-BruteForce of Example 1. Empty cells on the left part of the array correspond to cases in which $h \geq p$, while empty cells on the right part correspond to cases in which $h + p > |w|$.

**Theorem 4.2.** The algorithm AbelianPeriod-array computes all the Abelian periods of a given word of length $n$ in time $\Theta(n^3 \times \sigma)$ and space $\Theta(n^2)$.

**Proof.** The correctness of the algorithm comes from Proposition 4.1. The time complexity of the algorithm is due to the three for loops of lines 2 to 4. The space complexity is due to the array $T$. \qed
4.2. Lists

Also the algorithm AbelianPeriod-list, given in Figure 9, processes the position of $w$ in increasing order from left to right. When processing position $i$, it only stores the pairs $(h, \ell)$ such that $w[1..i-1]$ has Abelian period $\ell$ with head $h$.

AbelianPeriod-list($w, n$)

1. $L \leftarrow \{(0,1)\}$
2. for $i \leftarrow 2$ to $n$ do
3. $L' \leftarrow \emptyset$
4. for all $(h, \ell) \in L$ do
5. $d \leftarrow (i-h) \mod \ell$
6. if $d \neq 0$ then
7. if $P_w(i-d+1,d) \subset P_w(i-d-\ell+1,\ell)$ then
8. $L' \leftarrow L' \cup \{(h, \ell)\}$
9. else if $P_w(i-\ell+1,\ell) = P_w(i-2 \cdot \ell+1,\ell)$ then
10. $L' \leftarrow L' \cup \{(h, \ell)\}$
11. for $h \leftarrow 1$ to $\lfloor i/2 \rfloor - 1$ do
12. if $P_w(1,h) \subset P_w(h+1,i-h)$ then
13. $L' \leftarrow L' \cup \{(h, i-h)\}$
14. $L \leftarrow L'$

Figure 9: On-line dynamic programming algorithm for computing all the Abelian periods of a word $w$ of length $n$ using lists.

Theorem 4.3. The algorithm AbelianPeriod-list computes all the Abelian periods of a given word of length $n$ in time $O(n^3 \times \sigma)$ and space $O(n^2)$.

Proof. The correctness of the algorithm comes from Proposition 4.1. The space complexity for the list $L$ is given by Lemma 2.1. The time complexity of the algorithm is due to the two for loops of lines 2 and 3 and the maximal number of elements in the list $L$. \hfill \qed

4.3. Heaps

The following proposition shows that the set of Abelian periods of a prefix of a word can be partitioned into subsets depending on the length of the tail. In some cases, all the periods of a subset can be processed at once by inspecting only the smallest period of the subset.

Proposition 4.4. Let $w$ have $s$ Abelian periods $(h_1, p_1) < (h_2, p_2) < \cdots < (h_s, p_s)$ such that $(|w| - h_i) \mod p_i = t > 0$ for $1 \leq i \leq s$. For any letter $a \in \Sigma$, if $(h_1, p_1)$ is an Abelian period of $w_a$, then $(h_2, p_2), \ldots, (h_s, p_s)$ are also Abelian periods of $w_a$.

Proof. Since $(h_1, p_1) < (h_2, p_2) < \cdots < (h_s, p_s)$ are Abelian periods of $w$, we can write $w = u_{i,0} u_{i,1} \cdots u_{i,k_i-1} u_{i,k_i}$ with $|u_{i,0}| = h_i$, $|u_{i,j}| = p_i$ and $|u_{i,k_i}| = t$ for $1 \leq i \leq s$ and $1 \leq j \leq k_i$. If $(h_1, p_1)$ is an Abelian period of $w_a$, then $P_{u_{i,k_i-1}} \subseteq P_{u_{i,k_i}}$. Since $|u_{i,k_i}| = |u_{i,k_i}|$ and $|u_{i,k_i-1}| \leq |u_{i,k_i-1}|$, we have that $P_{u_{i,k_i-1}} \subseteq P_{u_{i,k_i}}$ for $2 \leq i \leq s$. Hence, $(h_2, p_2), \ldots, (h_s, p_s)$ are Abelian periods of $w_a$ (see Figure 10). \hfill \qed
The algorithm \textsc{AbelianPeriod-heap}, given in Figure 11, uses the property stated in Proposition 4.4 for computing all the Abelian periods of an input word \( w \) by gathering all the ongoing periods \( (h, p) \) with the same tail length in a heap where the element in the root is the smallest period.

When processing \( w[i] \), the algorithm processes every heap \( H \) for the different tail lengths:

- if the period \( (h, p) \) at the root of \( H \) is a period of \( w[1..i] \), then by Proposition 4.4 all the elements of \( H \) are Abelian periods of \( w[1..i] \). If the tail length becomes equal to \( p \), then \( (h, p) \) is removed from the current heap and is put into a new heap corresponding to the empty tail.

- if the period \( (h, p) \) at the root of \( H \) is not a period of \( w[1..i] \), then it is removed from \( H \) and the same process is applied until a pair \( (h', p') \) is an Abelian period of \( w[1..i] \) or the heap becomes empty. This procedure is realized by function \textsc{ExtractUntilOK} in line 8.

Finally, all the degenerate cases \( (h, p) \) such that \( h < p \) and \( h + p = i \) have to be inserted in the heap corresponding to the empty tail (lines 12 to 15).

The function \textsc{Root}(\( H \)) returns the smallest element of the heap \( H \), the function \textsc{Insert}(\( H, e \)) inserts element \( e \) in the heap \( H \), while the function \textsc{Remove}(\( H \)) removes the smallest element of the heap \( H \).

\textbf{Theorem 4.5.} The algorithm \textsc{AbelianPeriod-heap} computes all the Abelian periods of a given word of length \( n \) in time \( O(n^3 \log n \times \sigma) \) and space \( O(n^2) \).

\textbf{Proof.} The space memory depends on the total number of nodes of the heaps. Since one node corresponds exactly to one Abelian period, the maximum number of nodes is then bounded by \( n^2 \).

For the same reason, during each execution of the \textbf{for} loop starting in line 4, the maximum number of nodes removed or inserted by \textsc{ExtractUntilOK}, \textsc{Remove} and \textsc{Insert} functions is bounded by \( n^2 \). Each of these functions takes time at most \( \log n \). Comparing two Parikh vectors in line 7 takes time at most \( \sigma \). The time complexity of this loop is then \( O(n^2 \log n \times \sigma) \).

The \textsc{Insert} function in the \textbf{while} loop starting in line 13 is called at most \( n \) times. The time complexity of this loop is then \( O(n \log n) \).

Since these two loops are executed \( n \) times (loop \textbf{for} starting in line 2) the time complexity of this algorithm is \( O(n^3 \log n \times \sigma) \). \qed

5. Experimental results

Practical performances of the two off-line algorithms have been compared. They both have been implemented in C in a homogeneous way using the table of the Parikh vectors of the prefixes...
```
ABELIAN_PERIOD-heap(w, n)
1  L ← list with one heap containing (0, 1)
2  for i ← 2 to n do
3    NewHeap ← ∅
4    for all H ∈ L do
5      (h, p) ← ROOT(H)
6      t ← p - ((i - h) mod p)
7      if P_w(i - t + 1, t) ⊆ P_w(i - t - p + 1, p) then
8         EXTRACTUNTILOK(H)
9      else if t = p then
10         REMOVE(H)
11         INSERT(NewHeap, (h, p))
12    h ← 0
13    while h < ⌊(i + 1)/2⌋ and \(P_w(1, h) \subset P_w(h + 1, i - h)\) do
14       INSERT(NewHeap, (h, i - h))
15    h ← h + 1
16  L ← L ∪ NewHeap
17 return L
```

Figure 11: On-line algorithm for computing all the Abelian periods of a word \(w\) of length \(n\) using heaps.

of the word, and run on test sets of random words (3 000 words each) of different lengths (from 10 to 10 000) on different alphabet sizes (2, 3, 4, 8 and 16). Tests were performed on a MacBook Pro laptop running Mac OS X with a 2.2 GHz processor and 2 GB RAM.

A first remark is that most of the Abelian periods of a word have only one occurrence of the factor of length \(p\), i.e., are such that \(h + 2p \geq |w|\). We call these latter trivial Abelian periods. To give an idea, the prefix of length 4 181 of the Fibonacci word \(F = \text{abaababaabab} \cdots\) has 3 453 511 Abelian periods, but only 538 739 (i.e., about 15.6%) are non-trivial. The same proportion holds for longer prefixes of the Fibonacci word. But the Fibonacci word is probably one of the words with the highest proportion of non-trivial Abelian periods. Note that the word \(a^{2090}\text{ba}^{2090}\) of the same length has 2 914 854 Abelian periods, and all of them are trivial.

If one considers all the Abelian periods (that is, both trivial and non-trivial) running times of the two algorithms are very close, and seem to depend on the machine architecture more than on the algorithm itself (results not shown). If instead one computes non-trivial Abelian periods only, the select-based algorithm significantly improves on the Brute-Force one, and the gap increases when the alphabet size increases. In fact, even if the worst-case complexity of the two algorithms depend on \(\sigma\), the select-based algorithm seems to have an average behavior independent from the alphabet size. In Figure 12 we show results for alphabet sizes 2 and 16. These tests also suggest that the select-based algorithm becomes much faster than the brute-force algorithm when the word length increases.
Figure 12: Average running times (in ms), over 3000 random words, of the Brute-Force and select-based algorithms on alphabet size 2 (top) and 16 (bottom), in the case where $h + 2p \leq |w|$, i.e., for at least two repetitions of the Abelian period.
6. Conclusion and perspectives

This paper is the first attempt to give algorithms for computing all the Abelian periods of a word. As shown in Lemma 2.1, the total number of Abelian periods of a word can be quadratic in its length. We gave an $O(n^2 \times \sigma)$ time off-line algorithm based on the select function that in practice appears to be significantly faster than the Brute-Force one, as discussed in the experimental part section. We also presented three on-line algorithms that compute the Abelian periods of all the prefixes of the word.

However, some Abelian periods exist just as a consequence of the existence of smaller ones. For instance, in the word $w = \text{abaababa}$ of Example 1, the fact that $(1, 4), (1, 6), (3, 4)$ are Abelian periods for $w$ is just a consequence of the fact that $(1, 2)$ is. So, let us define the cutting positions of an Abelian period $(h, p)$ as follows:

$$\text{Cut}_w(h, p) = \{ k = h + jp \mid 1 \leq k \leq |w| \text{ and } 0 \leq j \}.$$ 

We say that an Abelian period $(h, p)$ of $w$ is non-deducible if there does not exist another Abelian period $(h', p')$ of $w$ such that $\text{Cut}_w(h, p) \subseteq \text{Cut}_w(h', p')$. Anyway, even the number of non-deducible Abelian periods can still be quadratic.

It seems quite clear that balanced words (words such that for any letter $a \in \Sigma$ the difference of the number of $a$’s in any two factors of the same length is bounded by 1) are the words with the maximum number of Abelian periods. In a recent paper, together with Alessio Langiu and Filippo Mignosi [8], we studied the Abelian repetitions in Sturmian words and gave a formula for computing the smallest Abelian period of the Fibonacci finite words. Preliminary experiments toward this results were done using the algorithms presented in this paper.

On the opposite side, it remains to obtain a bound on the minimal Abelian period given a word length and an alphabet size. Simple modifications of the presented algorithms would allow one to compute the minimal Abelian period of each factor of a word.

References

[1] G. Badkobeh, G. Fici, S. Kroon, and Zs. Lipták. Binary Jumbled String Matching for Highly Run-Length Compressible Texts. Information Processing Letters, 113:604–608, 2013.
[2] P. Burcsi, F. Cicalese, G. Fici, and Zs. Lipták. On Table Arrangements, Scrabble Freaks, and Jumbled Pattern Matching. In P. Boldi and L. Gargano, editors, Proceedings of the 5th International Conference on Fun with Algorithms, FUN 2010, volume 6099 of Lecture Notes in Computer Science, pages 89–101. Springer, 2010.
[3] P. Burcsi, F. Cicalese, G. Fici, and Zs. Lipták. Algorithms for Jumbled Pattern Matching in Strings. International Journal of Foundations of Computer Science, 23(2):357–374, 2012.
[4] P. Burcsi, F. Cicalese, G. Fici, and Zs. Lipták. On Approximate Jumbled Pattern Matching in Strings. Theory of Computing Systems, 50(1):35–51, 2012.
[5] F. Cicalese, G. Fici, and Zs Lipták. Searching for Jumbled Patterns in Strings. In J. Holub and J. Žd’árek, editors, Proceedings of the Prague Stringology Conference, PSC 2009., pages 105–117. Czech Technical University in Prague, 2009.
[6] S. Constantinescu and L. Ilie. Fine and Wilf’s theorem for abelian periods. Bulletin of the European Association for Theoretical Computer Science, 89:167–170, 2006.
[7] L. J. Cummings and W. F. Smyth. Weak repetitions in strings. Journal of Combinatorial Mathematics and Combinatorial Computing, 24:33–48, 1997.
[8] G. Fici, A. Langiu, T. Lecroq, A. Lefèvre, F. Mignosi, and É. Prieur-Gaston. Abelian Repetitions in Sturmian Words. In O. Carton and M.-P. Béal, editors, Proceedings of the 17th International Conference on Developments in Language Theory, DLT 2013, volume 7907 of Lecture Notes in Computer Science, pages 227–238. Springer, 2013.
[9] G. Fici, T. Lecroq, A. Lefebvre, and É. Prieur-Gaston. Computing Abelian Periods in Words. In J. Holub and J. Zdárek, editors, *Proceedings of the Prague Stringology Conference, PSC 2011*, pages 184–196. Czech Technical University in Prague, 2011.

[10] G. Fici, T. Lecroq, A. Lefebvre, É. Prieur-Gaston, and W. F. Smyth. Quasi-Linear Time Computation of the Abelian Periods of a Word. In J. Holub and M. Bálik, editors, *Proceedings of the Prague Stringology Conference, PSC 2012*, pages 103–110. Czech Technical University in Prague, 2012.

[11] E. Giaquinta and S. Grabowski. New algorithms for binary jumbled pattern matching. *Information Processing Letters*, 113(14-16):538–542, 2013.

[12] T. Kociumaka, J. Radoszewski, and W. Rytter. Fast algorithms for abelian periods in words and greatest common divisor queries. In N. Portier and T. Wilke, editors, *Proceedings of the 30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013*, volume 20 of LIPIcs, pages 245–256. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.

[13] M. Lothaire. *Algebraic Combinatorics on Words*. Cambridge University Press, 2002.

[14] T. M. Moosa and M. Sohel Rahman. Indexing permutations for binary strings. *Information Processing Letters*, 110(18-19):795–798, 2010.

[15] T. M. Moosa and M. Sohel Rahman. Sub-quadratic time and linear space data structures for permutation matching in binary strings. *Journal of Discrete Algorithms*, 10:5–9, 2012.

[16] G. Navarro and V. Mäkinen. Compressed full-text indexes. *ACM Computing Surveys*, 39(1):2, 2007.