Some Functorial Properties of Nilpotent Multipliers

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Abstract

In this paper, we are going to look at the \( \mathcal{N}_cM(G) \), as a functor from the category of all groups, \( \text{Group} \), to the category of all abelian groups, \( \text{Ab} \), and focusing on some functional properties of it. In fact, by using some results of the first author and others and finding an explicit formula for the \( c \)-nilpotent multiplier of a finitely generated abelian group, we try to concentrate on the commutativity of the above functor with the two famous functors \( \text{Ext} \) and \( \text{Tor} \).

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1. Introduction

Let \( G \cong F/R \) be a group, presented as a quotient group of a free group \( F \) by a normal subgroup \( R \). Then the Baer-invariant of \( G \), after R. Baer [1], with respect to the variety \( \mathcal{V} \), denoted by \( \mathcal{V}M(G) \), is defined to be

\[
\mathcal{V}M(G) = \frac{R \cap V(F)}{[Rv^{*}F]},
\]

where \( V(F) \) is the verbal subgroup of \( F \) with respect to \( \mathcal{V} \) and

\[
[Rv^{*}F] = \langle v(f_1, \ldots, f_{i-1}, v^{r}, f_{i+1}, \ldots, f_n)v(f_1, \ldots, f_{i}, \ldots f_n)^{-1} \mid r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbb{N} >.
\]

It can be proved that the Baer-invariant of a group \( G \) is independent of the choice of the presentation of \( G \) and it is always an abelian group (See [8]).

In particular, if \( \mathcal{V} \) is the variety of abelian groups, \( \mathcal{A} \), then the Baer-invariant of \( G \) will be \( (R \cap F')/[R, F] \), which, following Hopf [6], is isomorphic to the second
cohomology group of $G$, $H_2(G, \mathbb{C}^*)$, in finite case, and also is isomorphic to the well-known notion the Schur multiplier of $G$, denoted by $M(G)$. The multiplier $M(G)$ arose in Schur’s work [15] of 1904 on projective representations of a group, and has subsequently found a variety of other applications. The survey article of Wiegold [19] and the books by Beyl and Tappe [2] and Karpilovsky [7] form a fairly comprehensive account of $M(G)$.

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer-invariant of the group $G$ will be

$$
\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]},
$$

where $\gamma_{c+1}(F)$ is the $(c + 1)$st term of the lower central series of $F$ and $[R, 1 F] = [R, F]$, $[R, c F] = [(R, c-1 F), F]$, inductively. The above notion is also called the $c$-nilpotent multiplier of $G$ and denoted by $M^{(c)}(G)$ (see [3]).

The following theorem permit us to look at the notion of the Baer-invariant as a functor.

**Theorem 1.1.**

Let $\mathcal{V}$ be an arbitrary variety of groups. Then, using the notion of the Baer-invariant, we can consider the following covariant functor from the category of all groups, $\mathcal{G}roup$, to the category of all abelian groups, $\mathcal{A}b$

$$
\mathcal{V}M(-) : \mathcal{G}roup \rightarrow \mathcal{A}b,
$$

which assigns to any group $G$ the abelian group $\mathcal{V}M(G)$.

**Proof.** Let $G$ be an arbitrary group. By the properties of the Baer-invariant, $\mathcal{V}M(G)$ is independent of the choice of a presentation of $G$ and it is always abelian. So $\mathcal{V}M(-)$ assigns an abelian group to each group $G$. Also, if $G_1$ and $G_2$ are two arbitrary groups with the following presentations:

$$
1 \rightarrow R_1 \rightarrow F_1 \xrightarrow{\pi_1} G_1 \rightarrow 1 \quad , \quad 1 \rightarrow R_2 \rightarrow F_2 \xrightarrow{\pi_2} G_2 \rightarrow 1 ,
$$

and if $\phi : G_1 \rightarrow G_2$ is a homomorphism, then, using the universal property of free groups, there exists a homomorphism $\overline{\phi} : F_1 \rightarrow F_2$. It is easy to see that $\overline{\phi}$ induces a homomorphism

$$
\tilde{\phi} : \frac{R_1 \cap V(F_1)}{[R_1 V^* F_1]} \rightarrow \frac{R_2 \cap V(F_2)}{[R_2 V^* F_2]},
$$

i.e. $\tilde{\phi} : \mathcal{V}M(G_1) \rightarrow \mathcal{V}M(G_2)$ is a homomorphism from the Baer-invariant of $G_1$ to the Baer-invariant of $G_2$. It is a routine verification to see that the above assignment is a functor from $\mathcal{G}roup$ to $\mathcal{A}b$ (see also [8]). $\Box$
§2. Elementary Results

Being additive is usually one of the important property that a functor may have. Unfortunately, the $c$-nilpotent multiplier functor $N_c M(\cdot)$ is not additive even if we restrict ourself to abelian groups. The following theorems can prove this claim.

**Theorem 2.1** (I. Schur [14], J. Wiegold [16]).

Let $G = A \times B$ be the direct product of two groups $A$ and $B$. Then

$$M(G) \cong M(A) \oplus M(B) \oplus (A_{ab} \otimes B_{ab}) .$$

**Theorem 2.2** (B. Mashayekhy and M.R.R. Moghaddam [11]).

Let $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, be a finite abelian group, where $n_i + 1 | n_i$ for all $1 \leq i \leq k - 1$ and $k \geq 2$. Then, for all $c \geq 1$, the $c$-nilpotent multiplier of $G$ is

$$N_c M(G) \cong \mathbb{Z}_{n_2}^{(b_2)} \oplus \mathbb{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(b_k-b_{k-1})} ,$$

where $\mathbb{Z}_{n}^{(n)}$ denotes the direct sum of $n$ copies of the cyclic group $\mathbb{Z}_n$, and $b_i$ is the number of basic commutators of weight $c + 1$ on $i$ letters (see [5]).

One of the interesting corollary of Theorem 2.2 is that the $c$-nilpotent multiplier functors can preserve every elementary abelian $p$-group.

**Corollary 2.3.**

Let $G = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$ ($k$-copies) be an elementary abelian $p$-group. Then, for all $c \geq 1$, $N_c M(G)$ is also an elementary abelian $p$-group.

**Proof.** By Theorem 2.2 we have

$$N_c M(G) \cong \mathbb{Z}_p^{(b_2)} \oplus \mathbb{Z}_p^{(b_3-b_2)} \oplus \ldots \oplus \mathbb{Z}_p^{(b_k-b_{k-1})} = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p \ (b_k \text{- copies}).$$

Hence the result holds. Note that $|G| = p^n$ and $|N_c M(G)| = p^{b_k}. \Box$

In 1952, C. Miller [12] proved that the Schur multiplier of a free product is isomorphic to the direct sum of the Schur multipliers of the free factors. In other words, he proved that the Schur multiplier functor $M(\cdot)$ is coproduct-preserving.

**Theorem 2.4** (C. Miller [12]).

For any group $G_1$ and $G_2$,

$$M(G_1 * G_2) \cong M(G_1) \oplus M(G_2) ,$$

where $G_1 * G_2$ is the free product of $G_1$ and $G_2$.

Now, with regards to the above theorem, it seems natural to ask whether the $c$-nilpotent multiplier functors, $N_c M(\cdot)$, $c \geq 2$, are coproduct-preserving or not. To answer the question, first we state an important theorem of J. Burns and G. Ellis [3, Proposition 2.13 and its Erratum] which is proved by a homological method.

**Theorem 2.5** (J. Burns and G. Ellis [3]).

Let $G$ and $H$ be two arbitrary groups, then there is an isomorphism

$$N_2 M(G * H) \cong$$
$\mathcal{N}_2 M(G) \oplus \mathcal{N}_2 M(H) \oplus (M(G) \otimes H_{ab}) \oplus (G_{ab} \otimes M(H)) \oplus \text{Tor}_1^Z(G_{ab}, H_{ab})$.

Now, using the above theorem and properties of tensor product and $\text{Tor}_1^Z$, we can prove that the second nilpotent multiplier functor $\mathcal{N}_2 M(-)$, preserves the coproduct of a finite family of cyclic groups of mutually coprime order.

**Corollary 2.6.**

Let $\{Z_{n_i} | 1 \leq i \leq m\}$ be a family of cyclic groups of mutually coprime order. Then

$$\mathcal{N}_2 M(\prod_{i=1}^m \ast Z_{n_i}) \cong \oplus \sum_{i=1}^m \mathcal{N}_2 M(Z_{n_i}),$$

where $\prod_{i=1}^m \ast Z_{n_i}$ is the free product of $Z_{n_i}$'s, $1 \leq i \leq n$.

**Proof.** By using induction on $m$ and the following properties the result holds.

$$\mathcal{N}_2 M(Z_{n_i}) \cong 1, \text{Tor}_1^Z(Z_{n_i}, Z_{n_j}) \cong Z_{n_i} \otimes Z_{n_j} = 1, \text{ for all } i \neq j. \square$$

Note that the first author has generalized the above corollary to the variety of nilpotent groups of class at most $c$, $\mathcal{N}_c$, for all $c \geq 2$ as follows.

**Theorem 2.7** (B. Mashayekhy [10]).

Let $\{Z_{n_i} | 1 \leq i \leq m\}$ be a family of cyclic groups of mutually coprime order. Then

$$\mathcal{N}_c M(\prod_{i=1}^m \ast Z_{n_i}) \cong \oplus \sum_{i=1}^m \mathcal{N}_c M(Z_{n_i}), \text{ for all } c \geq 1.$$

In the following example, we are going to show that the condition of being mutually coprime order for the family of cyclic groups $\{Z_{n_i} | 1 \leq i \leq m\}$ is very essential in the above results. In other words, we show that the second nilpotent multiplier functor, $\mathcal{N}_2 M(-)$, is not coproduct preserving, in general.

**Example.**

Let $D_\infty = \langle a, b | a^2 = b^2 = 1 \rangle \cong C_2 \ast C_2$ be the infinite dihedral group. Then

$$\mathcal{N}_2 M(D_\infty) \neq \mathcal{N}_2 M(Z_2) \oplus \mathcal{N}_2 M(Z_2).$$

**Proof.** By Theorem 2.5 we have

$$\mathcal{N}_2 M(D_\infty) \cong \mathcal{N}_2 M(Z_2) \oplus \mathcal{N}_2 M(Z_2) \oplus Z_2 \otimes M(Z_2) \oplus M(Z_2) \oplus Z_2 \oplus \text{Tor}_1^Z(Z_2, Z_2) \cong \text{Tor}_1^Z(Z_3, Z_2) \cong Z_2 \otimes Z_2 \cong Z_2.$$

But $\mathcal{N}_2 M(Z_2) \oplus \mathcal{N}_2 M(Z_2) = 1$. Hence the result holds. $\square$

**Note.**

In 1980 M.R.R. Moghaddam [12] proved that in general, the Baer-invariant functor commutes with direct limit of a directed system of groups.

We know that every functor can preserve any split exact sequence as a split sequence. This property gives us the following interesting result.
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Theorem 2.8.
Let \( G = T \vartriangleleft N \) be the semidirect product (splitting extension) of \( N \) by \( T \) under \( \theta \). Then \( VM(T) \) is a direct summand of \( VM(G) \), for every variety of groups \( V \).

Note that K.I. Tahara [15] 1972, and W. Haebich [4] 1977, tried to obtain a result similar to the above theorem for the Schur multiplier of a semidirect product with an emphasis on finding the structure of the complementary factor \( M(T) \) of \( M(G) \), as much as possible. Also, a generalization of Haebich’s result [4] presented by the first author in [9].

Finally, the properties of right and left exactness are some of the most interesting properties that a functor may have. In the following, we show that the \( c \)-nilpotent multiplier functors are not right or left exact.

Theorem 2.9.
For every \( c \geq 1 \), the \( c \)-nilpotent multiplier functor, \( N_cM(\cdot) \), is not right exact.

Proof. Let \( G \) be a group such that \( N_cM(G) \neq 1 \) (note that by Theorem 2.2, we can always find such a group \( G \)). Let \( F \) be a free group and \( \pi : F \to G \) be an epimorphism (we can always consider a free presentation for a group \( G \)). Now by definition of the Baer-invariant we have \( N_cM(F) = 1 \) (consider the free presentation \( 1 \to 1 \to F \to F \to 1 \) for \( F \)). Therefore, it is easy to see that \( N_cM(F) \to N_cM(G) \) is not onto. \( \square \)

Theorem 2.10.
The \( c \)-nilpotent multiplier functor, \( N_cM(\cdot) \), is not left exact, in general.

Proof. Suppose \( G = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \). Then by Theorem 2.1 we have
\[
M(G) \cong M(\mathbb{Z}_4) \oplus M(\mathbb{Z}_4) \oplus (\mathbb{Z}_4 \otimes \mathbb{Z}_4) \cong \mathbb{Z}_4 .
\]

By a famous result on the Schur multiplier we know that every finite \( p \)-group can be embedded in a finite \( p \)-group whose Schur multiplier is elementary abelian \( p \)-group (see [7,17]). So there exists an exact sequence \( G \xrightarrow{\phi} H \to 1 \), where \( H \) is a finite \( 2 \)-group and \( M(H) \) is an elementary abelian 2-group. Hence \( M(\theta) : M(G) \to M(H) \) can not be a monomorphism. \( \square \)

3. Main Results

In this section, we will see the behaviour of the functor \( N_cM(\cdot) \) with the functors \( Ext^n_A(\mathbb{Z}_m, \cdot) \) and \( Tor^n_A(\mathbb{Z}_m, \cdot) \). First, by using notations and similar method of paper [11], we can present an explicit formula for the \( c \)-nilpotent multiplier of a finitely generated abelian groups as follows.

Theorem 3.1.
Let \( G \cong \mathbb{Z}^{(n)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k} \), be a finitely generated abelian group, where \( n \geq 0, n_{i+1}|n_i \) for all \( 1 \leq i \leq k-1 \) and \( k \geq 2 \). Then, for all \( c \geq 1 \), the
c-nilpotent multiplier of \( G \) is

\[
\mathcal{N}_cM(G) \cong \mathbb{Z}^{(b_n)} \oplus \mathbb{Z}^{(b_{n+1}-b_n)} \oplus \cdots \oplus \mathbb{Z}^{(b_{n+k}-b_{n+k-1})},
\]

where \( b_1 = b_0 = 0 \)

**Proof.** Clearly \( Z \otimes Z \cong Z, Z \otimes Z_{n_i} \cong Z_{n_i} \) and \( Z_{n_i} \otimes Z_{n_i+1} \cong Z_{n_i+1} \). Hence we have

\[
Z^{(t)} \otimes Z_{n_1} \otimes Z_{n_2} \otimes \cdots \otimes Z_{n_r} \cong Z_{n_r} \) and \( Z \otimes \cdots \otimes Z \cong Z.
\]

for all \( t \geq 0 \) and \( r \geq 1 \). Thus by theorem 2.3 of [11] we have

\[
\mathcal{N}_cM(Z^{(n)}) \cong T(Z, \ldots, Z)_{c+1} \cong Z^{(b_n)}.
\]

We remind that \( T(H_1, \ldots, H_n)_{c+1} \) is the summation of all the tensor products corresponding to the subgroup generated by all the basic commutators of weight \( c + 1 \) on \( n \) letters \( x_1, \ldots, x_n \), where \( x_i \in H_i \) for all \( 1 \leq i \leq n \). Now, by induction hypothesis assume

\[
\mathcal{N}_cM(Z^{(n)} \oplus Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k-1}) \cong Z^{(b_n)} \oplus Z^{(b_{n+1}-b_n)} \oplus \cdots \oplus Z^{(b_{n+k-1}-b_{n+k-2})}.
\]

Then we have

\[
\mathcal{N}_cM(Z^{(n)} \oplus Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}) \cong T(Z, \ldots, Z, Z_{n_1}, \ldots, Z_{n_k})_{c+1} \cong T(Z, \ldots, Z, Z_{n_1}, \ldots, Z_{n_k-1})_{c+1} \oplus L
\]

where \( L \) is the summation of all the tensor products of \( Z, Z_{n_1}, \ldots, Z_{n_k} \) corresponding to the subgroup generated by all the basic commutators of weight \( c + 1 \) on \( n + k \) letters which involve \( Z_{nk} \). Using (*), all those tensor product are isomorphic to \( Z_{nk} \). So \( L \) is the direct summand of \((b_{n+k} - b_{n+k-1})\)-copies of \( Z_{nk} \). Hence the result follows by induction. \( \square \)

For the rest of the paper we need the following lemmas.

**Lemma 3.2.**

For any abelian groups \( A \) and \( B \), we have

(i) \( \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, B) \cong B/mB \).

Also, \( \text{Ext}^1_{\mathbb{Z}}(A, B) = 0 \), for all \( n \geq 2 \).

(ii) If \( A \) and \( B \) are finite abelian groups, then

\[
\text{Ext}^1_{\mathbb{Z}}(A, B) \cong \text{Ext}^1_{\mathbb{Z}}(B, A).
\]

(iii) \( \text{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, B) \cong B/m \), where \( B/m = \{ b \in B : mb = 0 \} \). Also, \( \text{Tor}^1_{\mathbb{Z}}(A, B) = 0 \), for all \( n \geq 2 \), and \( \text{Tor}^1_{\mathbb{Z}}(A, B) \cong \text{Tor}^1_{\mathbb{Z}}(B, A) \).

**Proof.** See [14, Chapters 7, 8]. \( \square \)
Lemma 3.3.

Let $A$ and $\{B_k\}_{k \in I}$ be abelian groups. Then for all $n \geq 0$ the following isomorphism hold.

\[
(i) \quad \text{Ext}_n^1(A, \bigoplus_{k \in I} B_k) \cong \bigoplus_{k \in I} \text{Ext}_n^1(A, B_k), \quad \text{Ext}_n^1(\bigoplus_{k \in I} B_k, A) \cong \bigoplus_{k \in I} \text{Ext}_n^1(B_k, A).
\]

\[
(ii) \quad \text{Tor}_n^1(A, \bigoplus_{k \in I} B_k) \cong \bigoplus_{k \in I} \text{Tor}_n^1(A, B_k), \quad \text{Tor}_n^1(\bigoplus_{k \in I} B_k, A) \cong \bigoplus_{k \in I} \text{Tor}_n^1(B_k, A).
\]

Proof. See [14]. □

It is obvious that the functor $N_cM(-)$ commutes with the functors $\text{Ext}_n^1(Z_m, -)$, and $\text{Tor}_n^1(Z_m, -)$ for all $n \geq 2$, by lemma 3.2. Now we are going to pay our attention to the functors $\text{Ext}_n^1(Z_m, -)$, $\text{Ext}_n^1(-, Z_m)$, and $\text{Tor}_n^1(Z_m, -)$.

Theorem 3.4.

Let $D \cong Z^{(n)} \oplus Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k}$, be a finitely generated abelian group, where $n \geq 0$, $n_{i+1}|n_i$ for all $1 \leq i \leq k - 1$. Then, for all $c \geq 1$, the following isomorphisms hold.

(i) $N_cM(\text{Ext}_n^1(Z_m, D)) \cong Z_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

(ii) $\text{Ext}_n^1(Z_m, N_cM(D)) \cong Z_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

(iii) $N_cM(\text{Ext}_n^1(D, Z_m)) \cong (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

(iv) $\text{Ext}_n^1(N_cM(D), Z_m) \cong (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

(v) $N_cM(\text{Tor}_n^1(D, Z_m)) \cong (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

(vi) $\text{Tor}_n^1(N_cM(D), Z_m) \cong (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}^{(b_{n_{i+1}} - b_{n_i} - 1)}).

Proof. (i) By Lemma 3.3(i), $\text{Ext}_n^1(Z/mZ, Z) \cong Z/mZ \cong Z_m$. Now by using Lemmas 3.3(i) and 3.2(i), we have

\[
\text{Ext}_n^1(Z_m, D) \cong (\text{Ext}_n^1(Z_m, Z))^{(n)} \oplus (\oplus \sum_{i=1}^{k} \text{Ext}_n^1(Z_m, Z_{n_i}))
\]

\[
\cong Z_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} Z_{n_i}/mZ_{n_i}).
\]

One can see that for every $n, m \in Z$, we have $Z_m/nZ_m \cong Z_{(n,m)}$. Therefore

\[
\text{Ext}_n^1(Z_m, D) \cong Z_m^{(n)} \oplus (\oplus \sum_{i=1}^{k} Z_{(n_i, m)}).
\]

Now, by Theorem 2.2 and by noting that $(m, n_{i+1}))(m, n_i)|m$ we have

\[
N_cM(\text{Ext}_n^1(Z_m, D))
\]
\[
\cong Z_{m}^{(b_{2}-b_{1})} \oplus Z_{m}^{(b_{1}-b_{2})} \oplus \ldots \oplus Z_{m}^{(b_{n}-b_{n-1})} \oplus Z_{(n_{1}, m)}^{(b_{n+1}-b_{n})} \oplus \ldots \oplus Z_{(n_{k}, m)}^{(b_{n+k}-b_{n+k-1})}
\cong Z_{m}^{(b_{n}_{k})} \oplus (\oplus_{i=1}^{k} Z_{(n_{i}, m)}^{(b_{n+i}-b_{n+i-1})}).
\]

(ii) By Theorem 3.1 and Lemmas 3.3(i) and 3.2(i), we have

\[
\text{Ext}^{1}_{Z}(Z_{m}, N_{c}M(D)) \cong \text{Ext}^{1}_{Z}(Z_{m}, Z)^{(b_{n})} \oplus (\oplus_{i=1}^{k} (\text{Ext}^{1}_{Z}(Z_{m}, Z_{n_{i}}))^{(b_{n+i}-b_{n+i-1})})
\cong Z_{m}^{(b_{n})} \oplus (\oplus_{i=1}^{k} Z_{(n_{i}, m)}^{(b_{n+i}-b_{n+i-1})}).
\]

(iii) By Lemmas 3.3(ii) and 3.2(ii) we have

\[
\text{Tor}^{1}_{Z}(Z_{m}, D) \cong (\text{Tor}^{1}_{Z}(Z_{m}, Z))^{(n)} \oplus (\oplus_{i=1}^{k} \text{Tor}^{1}_{Z}(Z_{m}, Z_{n_{i}})) \cong \oplus_{i=1}^{k} Z_{n_{i}}[m].
\]

Note that \(\text{Tor}^{1}_{Z}(Z_{m}, Z) \cong 1\) and \(Z_{n}[m] \cong Z_{(n, n)}\). So we have \(\text{Tor}^{1}_{Z}(Z_{m}, D) \cong \oplus_{i=1}^{k} Z_{(n_{i}, m)}\). Now by Theorem 2.2 the result holds.

(iv) Again by using Theorem 3.1 and Lemmas 3.3(ii) and 3.2(ii), we have

\[
\text{Tor}^{1}_{Z}(Z_{m}, N_{c}M(D)) \cong (\text{Tor}^{1}_{Z}(Z_{m}, Z)^{(b_{n})} \oplus (\oplus_{i=1}^{k} \text{Tor}^{1}_{Z}(Z_{m}, Z_{n_{i}})^{(b_{n+i}-b_{n+i-1})})
\cong \oplus_{i=1}^{k} \text{Tor}^{1}_{Z}(Z_{m}, Z_{n_{i}})^{(b_{n+i}-b_{n+i-1})} \cong \oplus_{i=1}^{k} Z_{(n_{i}, m)}^{(b_{n+i}-b_{n+i-1})}. \quad \square
\]

In the following corollary you can find some of main results of the paper.

**Corollary 3.5.**

Let \(D\) be an arbitrary finitely generated abelian group. Then

(i) \(N_{c}M(\text{Ext}^{1}_{Z}(Z_{m}, D)) \cong \text{Ext}^{1}_{Z}(Z_{m}, N_{c}M(D)).\)

(ii) If \(D\) is also finite, then

\[
N_{c}M(\text{Tor}^{1}_{Z}(Z_{m}, D)) \cong \text{Tor}^{1}_{Z}(Z_{m}, N_{c}M(D)),
\]

\[
N_{c}M(\text{Ext}^{1}_{Z}(D, Z_{m})) \cong \text{Ext}^{1}_{Z}(N_{c}M(D), Z_{m}).
\]

(iii) If \(D\) is infinite, then

\[
N_{c}M(\text{Tor}^{1}_{Z}(Z_{m}, D)) \not\cong \text{Tor}^{1}_{Z}(Z_{m}, N_{c}M(D)).
\]

\[
N_{c}M(\text{Ext}^{1}_{Z}(D, Z_{m})) \not\cong \text{Ext}^{1}_{Z}(N_{c}M(D), Z_{m}).
\]
This means that the $c$-nilpotent multiplier functors, $N_cM(-)$ do not commute with $\text{Tor}^Z_A(Z_m, -)$ and $\text{Ext}^1_Z(-, Z_m)$, in infinite case.

**Proof.** (i) It is clear by parts (i), (ii) of the previous theorem.
(ii) By putting $n = 0$ in parts (iii) to (vi) of the previous theorem, the result holds.
(iii) Since $D$ is finite, so $n \geq 1$. Hence the result holds by the previous theorem
parts (iii) to (vi). □

We know that $\text{Hom}(Z_m, Z) \cong 0$ and $\text{Hom}(Z, Z_m) \cong Z_m$. So by similar methods of Theorem 3.4 we are going to indicate the behaviour of functor $N_cM(-)$ with $\text{Ext}^1_Z(Z_m, -) = \text{Hom}(Z-m, -), \text{Ext}^2_Z(-, Z_m) = \text{Hom}(-, Z_m)$, and $\text{Tor}^0_Z(Z_m, -) = Z_m \otimes -$ as the following theorem.

**Theorem 3.6.**

For any finitely generated abelian group $D \cong \mathbb{Z}^{(n)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, we have

(i) $N_cM(\text{Hom}(Z_m, D)) \cong \mathbb{Z}^{(b_2)}_{(m,n_2)} \oplus \ldots \oplus \mathbb{Z}^{(b_k-b_{k-1})}_{(m,n_k)}$.

(ii) $\text{Hom}(Z_m, N_cM(D)) \cong \mathbb{Z}^{(b_{n+1}-b_n)}_{(m,n_1)} \oplus \ldots \oplus \mathbb{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.

(iii) If $D$ is finite, then $N_cM(\text{Hom}(Z_m, D)) \equiv \text{Hom}(Z_m, N_cM(D))$.

If $D$ is infinite, then $N_cM(\text{Hom}(Z_m, D)) \ncong \text{Hom}(Z_m, N_cM(D))$.

(iv) $N_cM(\text{Hom}(D, Z_m)) \equiv \text{Hom}(N_cM(D), Z_m) \cong \mathbb{Z}^{(b_n)}_{(m,n_1)} \oplus \mathbb{Z}^{(b_{n+1}-b_n)}_{(m,n_1)} \oplus \ldots \oplus \mathbb{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.

(v) $N_cM(Z_m \otimes D) \cong Z_m \otimes N_cM(D) \equiv \mathbb{Z}^{(b_n)}_{(m,n_1)} \oplus \mathbb{Z}^{(b_{n+1}-b_n)}_{(m,n_1)} \oplus \ldots \oplus \mathbb{Z}^{(b_{n+k}-b_{n+k-1})}_{(m,n_k)}$.

Now, in the following we are going to show that our conditions in the previous results are essential. In general case $\text{Ext}^1_Z(A, -)$ and $\text{Tor}^0_Z(A, -)$, where $A$ is not cyclic, do not commute with $N_cM(-)$, for $i = 0, 1$.

**Some Examples.**

(a) $N_cM(\text{Ext}^1_Z(Z_n \oplus Z_n, Z_n)) \cong \mathbb{Z}^{(b_2)}_{(n)} \ncong 1 \cong \text{Ext}^1_Z(Z_n \oplus Z_n, N_cM(Z_n))$, i.e

$N_cM(\text{Ext}^1_Z(-, A)) \ncong \text{Ext}^1_Z(N_cM(-), A)$.

(b) $N_cM(\text{Ext}^1_Z(Z_n, Z_n \oplus Z_n)) \cong \mathbb{Z}^{(b_2)}_{(n)} \ncong 1 \cong \text{Ext}^1_Z(N_cM(Z_n), Z_n \oplus Z_n)$, i.e

$N_cM(\text{Ext}^1_Z(A, -)) \ncong \text{Ext}^1_Z(A, N_cM(-))$.

(c) $N_cM(\text{Tor}^0_Z(Z_n \oplus Z_n, Z_n)) \cong \mathbb{Z}^{(b_2)}_{(n)} \cong 1 \cong \text{Tor}^0_Z(Z_n \oplus Z_n, N_cM(Z_n))$, i.e

$N_cM(\text{Tor}^0_Z(-, A)) \cong \text{Tor}^0_Z(N_cM(-), A)$.

(d) $N_cM(\text{(Z_n \oplus Z_n \otimes Z_n)}) \equiv \mathbb{Z}^{(b_2)}_{n} \ncong \mathbb{Z}_n \oplus Z_n \otimes N_cM(Z_n))$, i.e

$N_cM(A \otimes -) \ncong (A \otimes N_cM(-))$.

(e) $N_cM(\text{Hom}(Z_n \oplus Z_n, Z_n) \cong \mathbb{Z}^{(b_2)}_{(n)} \cong 1 \cong \text{Hom}(Z_n \oplus Z_n, N_cM(Z_n))$, i.e

$N_cM(\text{Hom}(A, -)) \cong \text{Hom}(A, N_cM(-))$. 


(f) $N_c M(\text{Hom}(Z_{14} \oplus Z_2, Z_6 \oplus Z_3)) \cong Z_2^{(b_2)} \not\cong 1 \cong \text{Hom}(Z_{14} \oplus Z_2, Z_3^{(b_2)}) \cong \text{Hom}(Z_{14} \oplus Z_2, N_c M(Z_6 \oplus Z_3))$, i.e.

$N_c M(\text{Hom}(A, -)) \not\cong \text{Hom}(A, N_c M(-)).$

(g) $N_c M(\text{Hom}(Z_6 \oplus Z_2, Z_9 \oplus Z_3)) \cong Z_3^{(b_2)} \not\cong 1 \cong \text{Hom}(Z_2^{(b_2)}, Z_9 \oplus Z_3)) \cong \text{Hom}(N_c M(Z_6 \oplus Z_2), Z_9 \oplus Z_3)$, i.e.

$N_c M(\text{Hom}(-, A)) \not\cong \text{Hom}(N_c M(-), A).$

(h) $M(\text{Hom}(D, Z_m)) \not\cong \text{Hom}(M(D), Z_m)$, and $M(D \otimes Z_m) \not\cong M(D) \otimes Z_m,$

when $D$ is not abelian: Because one can see that $\text{Hom}(S_n, Z_2) \cong Z_2$, for each $n \geq 2$. Also we know that $M(S_n) \cong Z_2$, for each $n \geq 4$, see [7, theorem 2.12.3]. Now

$1 \cong M(\text{Hom}(S_n, Z_2)) \not\cong \text{Hom}(M(S_n), Z_2) \cong Z_2,$

Moreover $S_n \otimes Z_2 \cong S_n / S_n' \otimes Z_2 \cong Z_2 \otimes Z_2 \cong Z_2$. Then

$1 \cong M(S_n \otimes Z_2)) \not\cong M(S_n) \otimes Z_2 \cong Z_2.$

The functor $S = A \otimes -$, where $A$ is a non-cyclic group does not commute with the functor $N_c M(-)$. Put $A = Z_{m_1} \oplus Z_{m_2}, G = Z_n$, where $n \mid m_i$. Then $A \otimes G \cong Z_{m_1} \oplus Z_{m_2}$, where $m_i = (n, n_i)$, for $i = 1, 2$. Clearly $m_2 \mid m_1$, so by Theorem 3.1 we have $N_c M(A \otimes G) \cong Z_{m_2}^{(b_2)}$. On the other hand, we have $A \otimes N_c M(G) \cong A \otimes 1 = 1$. Hence $N_c M(A \otimes G) \not\cong A \otimes N_c M(G)$.

We should also point out that the Theorem 3.1 shows that the $c$-nilpotent multiplier functor, $N_c M(-)$, does not preserve the tensor product, for $N_c M(Z_m \otimes G_{ab}) \not\cong N_c M(Z_m) \otimes N_c M(G_{ab}) = 1$. □

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