Logarithmically Complete Monotonicity of a Function Involving the Gamma Functions

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Abstract: The monotonic functions were first introduced by S. Bernstein as functions which are non-negative with non-negative derivatives of all orders. He proved that such functions are necessarily analytic and he showed later that if a function is absolutely monotonic on the negative real axis then it can be represented there by a Laplace-Stieltjes integral with non-decreasing determining function and converse. Somewhat earlier F. Hausdorff had proved a similar result for completely monotonic sequences which essentially contained the Bernstein result. Bernstein was evidently unaware of Hausdorff’s result, and his proof followed entirely independent lines. Since then many studies have been written on monotonic functions. In this work, we mainly have proved that a certain function involving ratio of the Euler gamma functions and some parameters is completely and logarithmically completely monotonic. Also, we have given the sufficient conditions for this function to be respectively completely and logarithmically completely monotonic. As applications, some inequalities involving the volume of the unite ball in the Euclidean space $\mathbb{R}^n$ are obtained. The established results not only unify and improve certain known inequalities including, but also can generate some new inequalities and the given results could trigger a new research direction in the theory of inequalities and special functions.

Keywords: Completely Monotonic, Inequality, Logarithmically Completely Monotonic Function, Gamma Function

1. Introduction

A function $f$ is said to be completely monotonic [1] on interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on $I$ and satisfies for all $u > 0$ and $k \in \mathbb{N}$

$$0 \leq (-1)^k f^{(k)}(u) < \infty.$$ 

A positive function $f$ is said to be logarithmically completely monotonic (see for example [2]) on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on $I$ and its logarithm $\ln f$ satisfies for $k \in \mathbb{N}$

$$0 \leq (-1)^k [\ln f(u)]^k < \infty.$$ 

The class of completely monotonic functions on $(0, \infty)$ may be characterized by [1] as:

$f(u)$ is completely monotonic for $0 < u < 1$ if and only if

$$f(u) = \int_0^\infty e^{-su} \, d\mu(s),$$

where $\mu(s)$ is non-decreasing and the integral converges for $0 < u < \infty$.

It is proved that $f$ is logarithmically completely monotonic if and only if $f^\alpha$ is completely monotonic for all $\alpha > 0$ [3]. It is known that any logarithmically completely monotonic function must be completely monotonic, but not conversely [4].

The logarithmically completely monotonic function was characterized as the infinitely divisible completely monotonic functions [5]. Recently, the completely monotonic or logarithmically completely monotonic functions have been the subject of intensive research. For more details we refer the reader to [6]–[12].

The gamma function $\Gamma(u)$ is defined for $u > 0$ by the integral

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} \, dt,$$

where $u > 0$. The gamma function $\Gamma(u)$ is an analytic function on the complex plane $\mathbb{C}$ excluding the non-positive integers and it satisfies the functional equation

$$\Gamma(u) = \frac{\Gamma(u+1)}{u}$$

for $u > 0$. The gamma function has a simple pole at each non-positive integer $u = -n$ with residue $1/n!$. The gamma function is closely related to the factorial function $n!$ for non-negative integers $n$.

The digamma function $\psi(u) = \Gamma'(u)/\Gamma(u)$ is the logarithmic derivative of the gamma function. The digamma function has a simple pole at each non-positive integer $u = -n$ with residue $-1/n$. The digamma function is closely related to the harmonic numbers $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for positive integers $n$.

The polygamma function $\psi^{(n)}(u)$ is the $n$th derivative of the digamma function. The polygamma function has a simple pole at each non-positive integer $u = -n$ with residue $n/((n-1)!)$.

The gamma function is used extensively in statistics, probability, and combinatorics. It is defined for all complex numbers except the non-positive integers and its values can be extended to other domains using analytic continuation.

The gamma function is a fundamental tool in the study of special functions, and it plays a crucial role in the theory of asymptotic expansions, integral representations, and series expansions.

The gamma function has a wide range of applications in various fields such as physics, engineering, economics, and computer science. It is used in the study of quantum field theory, statistical mechanics, and other areas of theoretical physics. It is also used in the analysis of probabilistic models, financial mathematics, and other areas of applied mathematics.

The gamma function is an important special function that has a rich history and a wide range of applications. It is a fundamental tool in the study of asymptotic expansions, integral representations, and series expansions. It has a wide range of applications in various fields such as physics, engineering, economics, and computer science.
\[ \Gamma(u) = \int_0^\infty t^{u-1}e^{-t} \, dt. \]

The logarithmic derivative of \( \Gamma(u) \), denoted by \( \psi(u) = \frac{\Gamma'(u)}{\Gamma(u)} \), is called the psi or digamma function and the \( \psi^{(k)}(u) \) for \( k \in \mathbb{N} \) are called the polygamma functions.

The following results were investigated in [13]:

The function
\[
\frac{\Gamma(u+v+1)/\Gamma(v+1)\Gamma(u+v+2)/\Gamma(u+1)}{(u+v+1)^{\frac{1}{\alpha}}(u+1)^{\frac{1}{\beta}}} \tag{1}
\]
is decreasing with respect to \( u \geq 1 \) for fixed \( v \geq 0 \). Consequently, for positive real numbers \( u \geq 1 \) and \( v \geq 0 \), we have
\[
\frac{u+v+1}{u+v+2} \leq \frac{\Gamma(u+1)/\Gamma(u+1)\Gamma(u+v+1)/\Gamma(u+1)}{(u+v+1)^{\frac{1}{\alpha}}(u+1)^{\frac{1}{\beta}}} \tag{2}
\]

The function (1) was proved to be logarithmically completely monotonic with respect to \( u \in (0, \infty) \) for \( v \geq 0 \) and so is its reciprocal for \(-1 < v \leq -\frac{1}{2} \) [14].

Consequently, the inequality (2) is true for \((u, v) \in (0, \infty) \times [0, \infty) \) and reversed for \((u, v) \in (0, \infty) \times (-1, -\frac{1}{2}) \). For \((u, v) \in (0, \infty) \times (0, \infty) \) and \( \alpha \in [0, \infty) \), the function
\[
\frac{\Gamma(u+1)/\Gamma(u+1)}{(u+1)^{\frac{1}{\alpha}}} \tag{3}
\]
was proved in [15] to be strictly increasing (or decreasing, respectively) with respect to the single variable \( u \in (0, \infty) \) if and only if \( 0 \leq \alpha \leq \frac{1}{2} \) (or \( \alpha \geq -1 \) respectively), to be strictly increasing with respect to \( \nu \) on \([0, \infty) \) if and only if \( 0 \leq \alpha \leq 1 \) and to be logarithmically concave with respect to the tow

\[
(u, v) \in (0, \infty) \times (0, \infty) \text{ if } 0 \leq \alpha \leq \frac{1}{2}.
\]

For given \( \nu \in (-1, \infty) \) and \( \alpha \in (-\infty, \infty) \), we define the function
\[
h_{\alpha, \nu}(u) = \begin{cases} 
\frac{\Gamma(u+1)/\Gamma(v+1)}{(u+1)^{\frac{1}{\alpha}}} e^{\psi(\nu+1)}, & u \in (-\nu, 1, \infty) \setminus \{0\} \\
\frac{1}{(v+1)^{\alpha}} \psi(v+1), & u = 0
\end{cases} \tag{4}
\]

It is clear that the ranges of \( u, v \) and \( \alpha \) in the function \( h_{\alpha, \nu}(u) \) extend the corresponding ones in the functions (1) and (3) which were ever discussed in [15].

In this work, we prove that the function \( h_{\alpha, \nu}(u) \) is logarithmically completely monotonic function in some cases.

### 2. Results

Our main results are the following:

**Theorem 1.** Let \( \nu > -1 \). Then

a) The function (4) is logarithmically completely

monotonic with respect to \((-\nu - 1, \infty)\) if and only if

\[ \alpha \geq \max\{\frac{1}{\nu}, \frac{1}{\nu+1}\}. \]

b) If \( \alpha \leq \min\{\frac{1}{\nu}, \frac{1}{\nu+1}\} \), the reciprocal of the function (4) is logarithmically completely monotonic with respect to \( u \in (-\nu - 1, \infty) \).

c) The necessary condition for the reciprocal of the function (4) to be logarithmically completely monotonic with respect to \( u \in (-\nu - 1, \infty) \) is \( \alpha \leq 1 \).

Theorem 1 extends and generalizes the logarithmically complete monotonicity of the function (1) established in [14] and a part of the results in [15].

From the theorems we get the following corollary:

**Corollary 1.** For \( t > 0 \), \( v + 1 > 0 \) and \( u + v + 1 > 0 \), the double inequality
\[
\left( \frac{u+v+1}{u+v+2} \right)^{\alpha} \leq \frac{\Gamma(u+1)/\Gamma(v+1)\Gamma(u+v+1)/\Gamma(v+1)}{(u+v+1)^{\frac{1}{\alpha}}(u+1)^{\frac{1}{\beta}}} \leq \left( \frac{u+v+1}{u+v+2} \right)^{\beta} \tag{5}
\]
holds if \( \alpha \geq \max\{\frac{1}{\nu}, \frac{1}{\nu+1}\} \) and \( \beta \leq \min\{\frac{1}{\nu}, \frac{1}{\nu+1}\} \).

The inequality (5) generalizes and extends the inequality (2) and the main results in [16, 17].

For \( u + v > 0 \) and \( v + 1 > 0 \) the inequality
\[
\frac{\Gamma(u+1)/\Gamma(v+1)}{(u+1)^{\frac{1}{\alpha}}} \leq \frac{\Gamma(u+1)/\Gamma(v+1)}{(u+1)^{\frac{1}{\beta}}} \tag{6}
\]
is true if \( u > 1 \) and reversed if \( x < 1 \) and that the power \( \frac{1}{2} \) is the best possible.

**Theorem 2.** For any \( n \in \mathbb{N} \), let \( \Omega_n = \frac{\pi^n}{\Gamma(1+\frac{1}{n})} \). The following inequalities are true
\[
\sqrt[n+4]{\frac{\pi^{n+2}}{\Omega_n}} < \frac{1}{\pi^{n+2}} < \sqrt[n+4]{\frac{\pi^{n+2}}{n^{n+2}}} \tag{7}
\]

In order to prove our main results, the following lemma is needed.

**Lemma 1 [17, 18].** For \( \nu \in (0, \infty) \) and \( k \in \mathbb{N} \), we have
\[
\ln u - \frac{1}{u} < \psi(u) < \ln u - \frac{1}{2u} \tag{6}
\]
and
\[
\frac{(k-1)!}{u^k} + \frac{k!}{2u^{k+1}} \leq (-1)^k \psi^{(k)}(u) \leq \frac{(k-1)!}{u^k} + \frac{k!}{u^{k+1}} \tag{7}
\]

**Proof of Theorem 1.** For \( u \neq 0 \), taking the logarithm of \( h_{\alpha, \nu}(u) \) gives
\[
\ln h_{\alpha, \nu}(u) = \frac{\ln \Gamma(u+1) - \ln \Gamma(v+1)}{u} - \alpha \ln(u + v + 1).
\]
A direct differentiation yields
\[
\frac{\partial}{\partial u} \ln h_{\alpha,v}(u)^{(k)} = \frac{k!}{u^{v+1}} \Psi_{k=0}^{(1)} \left( \frac{-1}{u \psi((v+1)(u+v+1)) - \frac{(v+1)\ln(u+1)}{u^{v+1}}} - \frac{(-1)^{k-1}(k-1)\alpha}{(u+v+1)^{k+1}} \right)
\]
for \( k \in \mathbb{N} \), where \( \psi^{(1)}(u+v+1) \) and \( \psi^{(0)}(u+v+1) \) stand for \( \ln \Gamma(u+v+1) \) and \( \psi(u+v+1) \) respectively. From the relation (8) we get
\[
\left\{ u^{k+1} \ln h_{\alpha,v}(u) \right\}^{(k)} = (-1)^{k-1}u^k \left[ (-1)^{k-1} \psi^{(k)}(u+v+1) - \frac{(k-1)\alpha}{(u+v+1)^{k+1}} \right]
\]
Using the relations (7) and (8) we obtain
\[
\frac{(-1)^{k-1}(1-v)}{(u+v+1)^k} + \frac{k!}{u^{v+1}} \leq \frac{(-1)^{k-1}}{u^k} \left\{ u^{k+1} \ln h_{\alpha,v}(u) \right\}^{(k)} \leq \frac{(-1)^{k-1}(1-v)}{(u+v+1)^k} + \frac{k!}{u^{v+1}}
\]
for \( k \in \mathbb{N}, u \neq 0, v \in (-\infty, \infty) \) and \( \alpha \in (-\infty, \infty) \). Therefore,
\[
\frac{(-1)^{k-1}}{u^k} \left\{ u^{k+1} \ln h_{\alpha,v}(u) \right\}^{(k)} \begin{cases} \leq 0 & \text{if } \alpha \geq 1 \text{ and } \alpha \geq \frac{1}{v+1} \\ \geq 0 & \text{if } \alpha \leq 1 \text{ and } \alpha \leq \frac{1}{v+1} \end{cases}
\]
for \( k \in \mathbb{N}, v > 1 \text{ and } u \neq 0 \).

For \( u > 0 \), the equation (9) means
\[
\left\{ u^{2k} \ln h_{\alpha,v}(u) \right\}^{(2k-1)} \begin{cases} \leq 0 & \text{if } \alpha \geq 1 \text{ and } \alpha \geq \frac{1}{v+1} \\ \geq 0 & \text{if } \alpha \leq 1 \text{ and } \alpha \leq \frac{1}{2(v+1)} \end{cases}
\]
and
\[
\left\{ u^{2k+1} \ln h_{\alpha,v}(u) \right\}^{(2k)} \begin{cases} \leq 0 & \text{if } \alpha \geq 1 \text{ and } \alpha \geq \frac{1}{v+1} \\ \geq 0 & \text{if } \alpha \leq 1 \text{ and } \alpha \leq \frac{1}{2(v+1)} \end{cases}
\]
for \( k \in \mathbb{N} \). From (8), we get
\[
\lim_{u \to 0} \left\{ u^{k+1} \ln h_{\alpha,v}(u) \right\}^{(k)} = 0 
\]
for \( k \in \mathbb{N} \) and \( v > 1 \). As a result,
\[
\left\{ \ln h_{\alpha,v}(u) \right\}^{(2k-1)} \begin{cases} < 0 & \text{if } \alpha \geq 1 \text{ and } \alpha \geq \frac{1}{v+1} \\ > 0 & \text{if } \alpha \leq 1 \text{ and } \alpha \leq \frac{1}{2(v+1)} \end{cases}
\]
and
\[
\left\{ \ln h_{\alpha,v}(u) \right\}^{(2k)} \begin{cases} > 0 & \text{if } \alpha \geq 1 \text{ and } \alpha \geq \frac{1}{v+1} \\ < 0 & \text{if } \alpha \leq 1 \text{ and } \alpha \leq \frac{1}{2(v+1)} \end{cases}
\]
for \( k \in \mathbb{N} \) and \( u \in (0, \infty) \), that is,
\[
\alpha \geq (u+v+1) \left[ \frac{1}{u^v} \sum_{i=0}^{(1-v)} \frac{(-1)^{i-1}u^{(1)}(u+v+1)}{i!} + \frac{\ln(\Gamma(u+1))}{u^2} \right]
\]
and
\[
\alpha \leq (u+v+1) \left[ \frac{1}{u^v} \sum_{i=0}^{(1-v)} \frac{(-1)^{i-1}u^{(1)}(u+v+1)}{i!} + \frac{\ln(\Gamma(u+1))}{u^2} \right]
\]
for \( k \in \mathbb{N} \), which is equivalent to the fact that the equations (11) and (12) hold for \( u \in (-v-1, 0) \). As a result, the equation (13) is valid for \( k \in \mathbb{N} \) and \( u \in (-v-1, 0) \). Therefore, the function \( h_{\alpha,v}(u) \) has the same logarithmically complete monotonicity properties on \((-v-1, 0)\) as on \((0, \infty)\).

Conversely, if \( h_{\alpha,v}(u) \) is logarithmically completely monotonic on \((-v-1, \infty)\), then \[\ln h_{\alpha,v}(u)\] is strictly positive on \((-v-1, \infty)\), which can be simplified as
\[
\ln h_{\alpha,v}(u) = \frac{1}{u^v} \sum_{i=0}^{(1-v)} \frac{(-1)^{i-1}u^{(1)}(u+v+1)}{i!} + \frac{\ln(\Gamma(u+1))}{u^2}
\]
and
\[
\ln h_{\alpha,v}(u) = \frac{1}{u^v} \sum_{i=0}^{(1-v)} \frac{(-1)^{i-1}u^{(1)}(u+v+1)}{i!} + \frac{\ln(\Gamma(u+1))}{u^2}
\]
From (6), it is easy to see that
\[ \lim_{u \to 0^+} [u^2 \psi(u)] = 0 \quad (17) \]

It is known that
\[ \Gamma(u + 1) = u \Gamma(u) \quad (18) \]

For \( u > 0 \). Taking the logarithm on both sides of (18), rearranging and taking limit lead to
\[ \lim_{u \to 0^+} [u \ln \Gamma(u)] = \lim_{u \to 0^+} [u \ln \Gamma(u + 1)] - \lim_{u \to 0^+} [u \ln u] = 0 \quad (19) \]

**3. Conclusion**

We have established the necessary and sufficient conditions for a certain function involving ratio of the gamma functions to be logarithmically complete monotonic properties. As a consequence, we derived some inequalities involving the gamma functions. The established results could trigger a new research direction in the theory of inequalities and special functions.

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