M Times Photon Subtraction-Addition Coherent Superposition Operated Odd-Schrödinger-cat State: Nonclassicality and Decoherence

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Abstract We introduce a new non-Gaussian state (MCSO-OSCS), generated by \( m \) times coherent superposition operation \( a \cos \theta + a^\dagger \sin \theta \) (MCSO) on odd-Schrödinger-cat state \(|\alpha_0\rangle - |\alpha_0\rangle\) (OSCS), whose normalized constant is shown to be related to Hermite polynomials. We investigate the nonclassical properties of the MCSO-OSCS through Mandel’s Q-parameter, quadrature squeezing, the photocount distribution and Wigner function (WF), which is turned out to be influenced by parameters \( m \), \( \theta \) and \( \alpha_0 \). Especially the volume of negative region of WF could increase through controlling the parameters \( m \), \( \theta \) and \( \alpha_0 \). We also investigate the decoherence of the MCSO-OSCS in terms of the fadeaway of the negativity of WF in a thermal environment.

Keywords Non-Gaussian state · Wigner function · Coherent superposition operation · odd-Schrödinger-cat state · Decoherence

1 Introduction

Nonclassical states with non-Gaussian Wigner function (WF) have brought great interest in quantum optics and quantum information science [1]. For example, a single-photon state with non-Gaussian behavior in phase space has been applied to quantum information processing. In particular, any single-mode nonclassical state has become a sufficient resource to generate a two-mode entanglement via a beam-splitter [2]. Recently, non-Gaussian states have attracted more attention of both experimentalists and theoreticians [3–7]. It is possible to generate and manipulate various non-Gaussian states through subtracting or adding photon operation or photon subtraction-addition coherent superposition operation on usual
quintum states or Gaussian states [8]. For example, the photon subtraction transforms a Gaussian entangled state (two-mode squeezed state) to a non-Gaussian entangled state on a nonlocality test [9] and entanglement distillation [10]. The photon addition can also transform a classical state to a nonclassical state [11]. In laboratory the operation of photon subtraction or addition is now realized practically [12, 13]. In Ref. [14], Lee et al. have considered a coherent superposition of photon subtraction and addition, \( ta + r a^\dagger \), acting on a coherent state and a thermal state to form non-Gaussian states, and proposed the experimental scheme to implement this elementary coherent operation. Furthermore, a non-Gaussian state was obtained theoretically through \( m \) times coherent superposition of photon subtraction and addition, \( (ta + r a^\dagger)^m \), acting on thermal state [15] and coherent state [16], respectively.

On the other hand, as a kind of nonclassical state, the so-called Schrödinger cat states (SCS, quantum superposition of coherent states [17]), play an important role in fundamental tests of quantum theory [18, 19] and in various quantum information processing tasks, including quantum computation [20], quantum teleportation [21] and precision measurements [22, 23]. There have been a great deal of theoretical and experimental attempts to generate a Schrödinger-cat-like state and considerable experimental progresses have been achieved in recent years [24–28]. Such as in Ref. [28] a Schrödinger-cat-like state is generated via a coherent superposition of photonic operations.

Enlightened by these ideas, we shall operate the coherent superposition operator \( \Omega^m \) on odd-Schrödinger-cat state (OSCS) to construct a new non-Gaussian quantum state (MCSO-OSCS) which is supposed to be realized in experiment. In this paper, we focus on studying its nonclassical properties by deriving analytically several expressions, such as normalized constant, sub-Poissonian statistics, photocount distribution and Wigner function. In fact, systems are usually surrounded by a thermal reservoir, and decoherence becomes an important topic in the fields of quantum optics. Therefore, we shall also discuss the decoherence effect of the MCSO-OSCS caused by a thermal environment in this paper.

The paper is organized as follows. In Section 2, the MCSO-OSCS is constructed and its normalized constant turns out to be related with the Hermite polynomial. In Section 3, nonclassical properties of the MCSO-OSCS, such as sub-Poissonian statistics, quadrature squeezing properties and photocount distribution are calculated analytically and discussed in details. The fidelity between MCSO-OSCS and its original state (OSCS) is also discussed in this section. In Section 4, the explicit analytic expression of WF for the MCSO-OSCS is derived. Nonclassical properties of the MCSO-OSCS are also discussed in terms of the negativity of WF. In Section 5, the decoherence of the MCSO-OSCS in a thermal environment is investigated. In Section 6, we end our work with main conclusions.

### 2 Normalization of the MCSO-OSCS

Theoretically, the MCSO-OSCS can be introduced by repeated application of coherent superposition operator \( \Omega \) to the OSCS (\( |\alpha_0\rangle - |\alpha_0\rangle \)) for \( m \) times, i.e.,

\[
|\psi_m\rangle = \Omega^m (|\alpha_0\rangle - |\alpha_0\rangle),
\]

where \( \Omega = a \cos \theta + a^\dagger \sin \theta \) with \([a, a^\dagger] = 1\) and \( \theta \in (0, \pi/2) \), \( m \) is the order of coherent superposition operator (a non-negative integer), \( |\alpha_0\rangle \) is a coherent state of amplitude \( |\alpha_0| \). The density operator of the MCSO-OSCS is \( \rho_m = N_m^{-1} |\psi_m\rangle \langle \psi_m| \), where \( N_m \) is a normalized constant of the MCSO-OSCS determined by \( \text{Tr} \rho_m = 1 \). If \( \Omega^m \) operates on the
technique of integration within an ordered product (IWOP) of operators \([32]\), we have

Equations (6) and (7) are very useful in the following calculations.

\[
\int \frac{d^2 z}{\pi} \exp \left( \frac{\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^4}{\zeta^2 - 4fg} \right) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp \left(-\frac{-\xi \eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg} \right),
\]

In order to obtain the normalized constant \(N_m\), and we note that the operator \(\Omega\) is not always Hermitian due to \(\Omega \neq \Omega^\dagger\) when \(\cos \theta \neq \sin \theta\), we shall derive the normal ordering form of \(\Omega^m\) firstly. Recalling the generating function of the Hermite polynomial \(H_m(x)\) \([30]\), i.e.

\[
\sum_{m=0}^\infty \frac{t^m}{m!} H_m(x) = \exp \left( 2xt - t^2 \right)
\]

using the Baker-Hausdorff formula

\[
e^{A}e^{B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]} = e^{B}e^{A}e^{-\frac{1}{2}[B,A]}\]

and the technique of integration within an ordered product (IWOP) of operators \([32]\), we have

\[
e^{\lambda \Omega} = e^{\lambda \Omega + \frac{1}{2} \lambda^2 \sin 2\theta} = \sum_{m=0}^\infty \frac{\lambda^m}{m!} \Omega^m : H_m \left( \frac{i\Omega}{\sin 2\theta} \right) : ,
\]

where the symbol \(:\) stands for the normally ordering. Comparing (3) with the expansion of \(e^{\lambda \Omega}\), i.e.

\[
e^{\lambda \Omega} = \sum_{m=0}^\infty \frac{\lambda^m}{m!} \Omega^m ,
\]

we can easily obtain the normal ordering form of \(\Omega^m\):

\[
\Omega^m = \left(-\frac{i}{2} \sqrt{\sin 2\theta}\right)^m : H_m \left( \frac{i\Omega}{\sin 2\theta} \right) : .
\]

Similarly,

\[
\Omega^m = \left(-\frac{i}{2} \sqrt{\sin 2\theta}\right)^m : H_m \left( \frac{i\Omega^\dagger}{\sin 2\theta} \right) : .
\]

From (4) and (5), we also give the following relations

\[
\langle \beta | \Omega^m | \alpha \rangle = \left(-\frac{i}{2} \sqrt{\sin 2\theta}\right)^m H_m \left[ \frac{i(\alpha \cos \theta + \beta^* \sin \theta)}{\sin 2\theta} \right] \langle \beta | \alpha \rangle ,
\]

and

\[
\langle \beta | \Omega^m | \alpha \rangle = \left(-\frac{i}{2} \sqrt{\sin 2\theta}\right)^m H_m \left[ \frac{i(\beta^* \cos \theta + \alpha \sin \theta)}{\sin 2\theta} \right] \langle \beta | \alpha \rangle ,
\]

where \(|\alpha\rangle\) and \(|\beta\rangle\) are coherent states and \(\langle \beta | \alpha \rangle = \exp \left[-\frac{1}{2} (|\alpha|^2 + |\beta|^2) + \beta^* \alpha \right]\) \([33, 34]\).

Equations (6) and (7) are very useful in the following calculations.

Next, according to \(\text{Tr} \rho_m = 1\), we obtain

\[
N_m = \text{Tr} \left[ |\psi_m\rangle \langle \psi_m | \right] = \langle \psi_m | \psi_m \rangle .
\]

Substituting (1), (4)–(5) into (8), and inserting the completeness relation of the coherent state \(\int \frac{d^2 z}{\pi} |z\rangle \langle z | = 1\), with the help of (2) and the following integral formula

\[
\int \frac{d^2 z}{\pi} \exp \left( \xi z \eta + \xi^2 \eta + f z^2 + g z^4 \right)
\]

\[
= \frac{1}{\sqrt{\xi^2 - 4fg}} \exp \left(-\frac{-\xi \eta + \xi^2 g + \eta^2 f}{\xi^2 - 4fg} \right),
\]

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whose convergent condition is \( \text{Re}(\zeta \pm f \pm g) < 0, \) \( \text{Re}[(\zeta^2 - 4fg)/(\zeta \pm f \pm g)] < 0, \) we obtain

\[
N_m = 2\chi^m \left[ \sum_{k=0}^{m} (-1)^m A |H_{m-k} (B)|^2 - \sum_{k=0}^{m} (-1)^k A |H_{m-k} (C)|^2 \exp \left(-2|\alpha_0|^2\right) \right],
\]

where we have set

\[
A = \tan^k \theta \frac{2^k (m!)^2}{k! [(m - k)!]^2}, \quad \chi = -\frac{1}{4} \sin 2\theta,
\]

\[
B = i \frac{\sqrt{2}}{2} \left[ (\tan \theta)^{\frac{1}{2}} \alpha_0 + (\tan \theta)^{-\frac{1}{2}} \alpha_0^* \right],
\]

\[
C = i \frac{\sqrt{2}}{2} \left[ (\tan \theta)^{\frac{1}{2}} \alpha_0 - (\tan \theta)^{-\frac{1}{2}} \alpha_0^* \right],
\]

and used the recurrence relation of \( H_m (x) \):

\[
\frac{\partial}{\partial x^l} H_m (x) = \frac{2^l m!}{(m - l)!} H_{m-l} (x).
\]

Equation (10) indicates that the normalization factor \( N_m \) is just related to a Hermite polynomial. Obviously, when \( m = 0 \), the MCSO-OSCS reduces to the odd SCS. The analytical expression of \( N_m \) is important for further investigating the properties of MCSO-OSCS. For MCSO-ESCS, we can change the negative sign “−” before the second sign of sum in (10) to the positive sign “+” and obtain its normalized constant.

3 Nonclassical Properties of MCSO-OSCS

In this section, we shall discuss the nonclassical properties of the MCSO-OSCS in terms of sub-Poissonian statistics, quadrature squeezing properties and photocount distribution.

3.1 Mandel’s Q-Parameter

The Mandel’s Q-parameter measures the deviation of the variance of the photon number distribution of the field state under consideration from the Poissonian distribution of the coherent state, which has been defined as [35]

\[
Q = \frac{\langle a^+ a^2 \rangle}{\langle a^+ a \rangle} - \langle a^+ a \rangle.
\]

The quantum states have the Poissonian, sub-Poissonian and super-Poissonian statistics for \( Q = 0, Q < 0 \) and \( Q > 0 \), respectively. It is well known that the negativity of Q-parameter refers to the nonclassical character of the state, but a state may be nonclassical even though Q-parameter is positive as pointed out in [36].
Using (4), (5), \( \rho_m = N^{-1}_m |\psi_m\rangle \langle \psi_m| \) and IWOP technique of operators, one can calculate \( \{a^\dagger a\} \) as

\[
\{a^\dagger a\} = \text{Tr} \left( \rho_m a^\dagger a \right) \\
= N^{-1}_m \chi^m \sum_{k=0}^m I \left\{ (-1)^{m-k} \left( k + 1 + |\alpha_0|^2 \right) |H_{m-k} (B)|^2 \right. \\
- \left( k + 1 - |\alpha_0|^2 \right) |H_{m-k} (C)|^2 e^{-2|\alpha_0|^2} \\
+2\text{Re} \left[ R^* \alpha_0^* (m-k) H_{m-k} (-B) H_{m-k-1} (B^*) \right] \\
-2\text{Re} \left[ R^* \alpha_0^* (m-k) H_{m-k} (C^*) H_{m-k-1} (C) e^{-2|\alpha_0|^2} \right] \} - 1,
\]

where

\[
R = i \sqrt{2} \tan \theta, \quad I = \frac{2 (1)^k (m!)^2}{k! ([m-k]!)^2} |R|^{2k},
\]

and get the value of \( \langle a^2 a^\dagger \rangle \) as

\[
\langle a^2 a^\dagger \rangle = f_1 (\alpha_0) + f_1 (-\alpha_0) - f_2 (\alpha_0) - f_2 (-\alpha_0),
\]

where

\[
f_1 (\alpha_0) = N^{-1}_m \chi^m \frac{\partial^{2m+2}}{\partial t^m \partial s^m \partial \lambda \partial \eta} e^{-|\alpha_0|^2 - K^* t + K s - s^2 - t^2} \frac{e^{(H+H_0+|\alpha_0|^2)/(1-4\lambda \eta)}}{\sqrt{1-4\lambda \eta}},
\]

\[
f_2 (\alpha_0) = N^{-1}_m \chi^m \frac{\partial^{2m+2}}{\partial t^m \partial s^m \partial \lambda \partial \eta} e^{-|\alpha_0|^2 - K^* t + K s - s^2 - t^2} \frac{e^{(H-L_0-|\alpha_0|^2)/(1-4\lambda \eta)}}{\sqrt{1-4\lambda \eta}},
\]

and

\[
K = i \sqrt{2} \sqrt{\cot \theta} \alpha_0, \\
H = -|R|^2 t s + R^2 t^2 \eta + \alpha_0^2 \eta + \alpha_0^2 \lambda + R^* s^2 \lambda, \\
H_0 = R \alpha_0 t - R^* \alpha_0^* s + 2R \alpha_0^* t \eta - 2R^* \alpha_0 \lambda s, \\
L_0 = R \alpha_0 t + R^* \alpha_0^* s - 2R \alpha_0^* t \eta - 2R^* \alpha_0 \lambda s.
\]

Here \( s, t, \lambda, \eta \) are parameters introduced into the calculation process and will be eliminated after finishing the calculation by setting them to zero. Furthermore, one can use the relation \( [a, a^\dagger] = 1 \) to obtain

\[
\langle a^\dagger a^2 \rangle = \langle a^2 a^\dagger \rangle - 4 \langle a^\dagger a \rangle - 2.
\]

Substituting (14), and (19) into (13), and using the method of numerical calculation we can study the property of Mandel’s Q-parameter for the MCSO-OSCS. The Q-parameters of MCSO-OSCS as a function of \( \alpha_0 \) are depicted in Fig. 1 for several different values of \( m, \theta \). Here \( \alpha_0 \) is set as a real number, the same as in Figs. 2, 3 and 8. It is interesting to note that the values of Q-parameter are always smaller than zero for different \( m \) values under the given \( \theta = \frac{\pi}{4} \) (see Fig. 1a), which indicates sub-Poissonian statistics. In addition, the absolute value of Q-parameter decreases with the increment of \( \alpha_0 \), and tends to zero eventually, which indicates that all states with different \( m \) values will tend to the Poissonian statistics (the distribution of a coherent state) when the value of \( \alpha_0 \) is large enough. However, we can see that the range of the Q-parameter is \([-1, 0.5]\) in Fig. 1b with different \( m \) values.
Mandel’s Q-parameter ($Q$) of MCSO-OSCS as a function of $\alpha_0$ (here $\alpha_0$ is set as a real number) a $\theta = \frac{\pi}{4}$, with $m = 0, 1, 2, 3$; b $\theta = \frac{\pi}{8}$, with $m = 0, 1, 2, 3$; c $m = 1$, with $\theta = \frac{\pi}{10}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$.  

Fig. 1  

and for given $\theta$ ($\theta = \frac{\pi}{8}$). That indicates the MCSO-OSCS with small value of $\theta$ may not exhibit sub-Poissonian statistics but exhibit super-Poissonian statistics.

In particular, when $m = 1$, the MCSO-OSCS deduces to the COSCS ($\Omega (|\alpha_0\rangle - |{-\alpha_0}\rangle)$) [29]. From Fig. 1c, We can see that the range of the Q-parameter is $[-1, 0.7]$. From the criteria of Q-parameter, one finds that the MCSO-OSCS exhibits sub-Poissonian statistics for $\theta = \frac{\pi}{10}, \frac{\pi}{6}$ and $\frac{\pi}{2}$. In the area of $\theta \in (0, \frac{\pi}{2})$, the larger the value of $\theta$ is, the more the sub-Poissonian statistics exhibits. A similar conclusion can be seen in Ref. [29].

3.2 Quadrature Squeezing Properties of MCSO-OSCS

Other than sub-Poissonian statistics, nonclassicality can be also characterized through squeezing effects, which do not allow classical interpretation of photoelectric counting events. Here, we consider an appropriate quadrature operator $X_\theta = a e^{-i\theta} + a^\dagger e^{i\theta}$, and the squeezing can be characterized by $\langle (\Delta X_\theta)^2 \rangle_{\text{min}} < 1$ with respect to angle $\theta$, or by the normal ordering form $\langle : (\Delta X_\theta)^2 : \rangle_{\text{min}} < 0$ [37]. Upon expanding the terms in $\langle : (\Delta X_\theta)^2 : \rangle_{\text{min}}$, one can minimize its value over the whole angle $\theta$, which is given by [38]

$$S = \langle : (\Delta X_\theta)^2 : \rangle_{\text{min}} = -2 \left|\langle a^\dagger a \rangle^2 - \langle a^{\dagger^2} \rangle \right| + 2 \left|\langle a^\dagger a \rangle - 2 \langle a^\dagger \rangle \right|^2. \quad (20)$$

Then its negative value in the range $[-1, 0)$ indicates squeezing (or nonclassicality). Similarly, using the integration formula (9), we obtain

$$\langle a^\dagger \rangle = 0. \quad (21)$$
Fig. 2 Quadrature squeezing ($S$) of MCSO-OSCS as a function of $\theta$ a $\alpha_0 = 0.1$, with $m = 1, 3, 5, 7$; b $\alpha_0 = 0.1$, with $m = 0, 2, 4, 10$; c $m = 1$, with $\alpha_0 = 0.1, 0.5, 1.0, 1.5$

and

$$\langle a \dagger a \rangle = f_3 (\alpha_0) + f_3 (-\alpha_0) - f_4 (\alpha_0) - f_4 (-\alpha_0),$$  

(22)

where

$$f_3 (\alpha_0) = N_m^{-1} \chi^m \frac{\partial^{2m}}{\partial t^m \partial s^m} \left( R t + \alpha_0^* \right)^2 e^{-t^2-s^2-|R|^2 s t+2B t-2B^* s} \bigg|_{s=t=0},$$

$$f_4 (\alpha_0) = N_m^{-1} \chi^m \frac{\partial^{2m}}{\partial t^m \partial s^m} \left( R t + \alpha_0^* \right)^2 e^{-t^2-s^2-|R|^2 s t-2C^* t-2C s-2|\alpha|^2} \bigg|_{s=t=0}.$$  

(23)

Using (14), (21), (22) and (20), one can obtain the expression of the quadrature squeezing $S$ of MCSO-OSCS. We plot the quadrature squeezing as a function of $\theta$ for different $m$ values and for given $\alpha_0$ value, (say $\alpha_0 = 0.1$), see Fig. 2a and b. It is interesting to find that the MCSO-OSCS can exhibit squeezing when the parameter $m$ is odd ($m = 1, 3, 5, 7$) and the angle $\theta$ is smaller than a threshold (different $m$ has different $\theta$ threshold), while can not exhibit squeezing when the parameter $m$ is even ($m = 0, 2, 4, 10$) for any angle $\theta$. Furthermore, we find that the original state ($m = 0$) can not exhibit squeezing, which implies that the odd times coherent superposition operation ($\Omega^m$, $m$ is odd.) can achieve squeezing. A small angle $\theta$ corresponds to the case that the subtracting photon operation is in the ascending, which indicates that subtracting photon operation benefits to squeezing in the case of odd $m$.

In Fig. 2c, we plot $S$ as a function of $\theta$ for different $\alpha_0$ values ($\alpha_0 = 0.1, 0.5, 1.0, 1.5$) and for given $m$ values, (say $m = 1$). We find that small value of $\alpha_0$ is helpful to squeezing under the condition that the angle $\theta$ is smaller than the threshold (around $\theta = 0.47$).

In order to comprehend the different characters in the case of odd $m$ and even $m$ in above figures, we calculate the fidelity between the new state (MCSO-OSCS) with the original state (OSCS). Here the fidelity measures how close the new state (density matrix is $\rho_m$) is
Fig. 3  Fidelity ($F$) between MCSO-OSCS and OSCS as a function of $\alpha_0$ a $\theta = \pi/4$, with $m = 0, 2, 4, 1, 3$; b $m = 2$, with $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

to the original state (density matrix is $\rho_o$), which is defined as $F_{\theta,m} = \text{Tr}(\rho_m \rho_o) / \text{Tr}(\rho_o^2)$ [8]. In general, $0 \leq F \leq 1$. $F = 1$ shows that the two states are same, while $F = 0$ shows that the two states are anamorphic absolutely. As the similar procedure of deriving the normalization constant, the fidelity can be calculated out as

$$F_{\theta,m} = \frac{N_m^{-1} \chi^m}{2 \left(1 - e^{-2|\alpha_0|^2}\right)} \left[(-1)^m + 1 \left[H_m(B^*) - e^{-2|\alpha_0|^2} H_m(C)\right]\right]^2.$$  (24)

In Fig. 3a, we plot the fidelity $F_{\theta,m}$ as a function of $\alpha_0$ for different $m$ values with given $\theta$ values. In particular, when $m = 0$, (24) reduces to $F_{\theta,m} = 1$ (as shown in Fig. 3a), which indicates that the MCSO-OSCS is reduced to the OSCS, as expected. It is obvious that $F_{\theta,m} = 0$ when $m = 1, 3$, which means the two states are anamorphic absolutely, and $F_{\theta,m} \neq 0$ when $m = 2, 4$, which means the two states have certain similarity. So we can comprehend why the new state can show squeezing character as $m$ is odd while the original state can not show. When $m$ is an even number and $\theta = \pi/3$, the fidelity increases monotonously with the increment of $\alpha_0$ and tends to 1 eventually, which indicates that the coherent superposition operation has no influence on the field when the field is strong enough. Comparing with the curves of $m = 0, 2, 4$, we find that the smaller the value of $m$ is, the larger the value of $F_{\theta,m}$ will be. In order to see the effect of different $\theta$ values on the fidelity, we plot the fidelity as the function of $\alpha_0$ for different $\theta$ values and given $m$ values ($m = 2$), see Fig. 3b. It is shown that the fidelity decreases as $\theta$ increases.

3.3 Photocount Distribution of MCSO-OSCS

For the case of a single radiation mode of registering $n$ photoelectrons in the time interval $T$, the photon counting distribution $P (n)$ is given by [39],

$$P (n) = \text{Tr} \left[ \xi \left(\xi \xi^\dagger a \right)^n \frac{e^{-\xi \xi^\dagger a}}{n!} \right],$$  (25)

where $\xi \propto T$ is called the quantum efficiency (a measure) of the detector, $\rho$ is a single-mode density operator of the light field concerned. When $\xi = 1$, $P (n)$ becomes the photon number distribution (PND) for a given state. By virtue of the technique of IWOP of operators, Fan and Hu deduced a reformed formula as showed in reference [40],

$$P (n) = \xi^n \left(\xi - 1\right)^n \int \frac{d^2z}{\pi} e^{-\xi |z|^2} L_n \left(\xi |z|^2\right) \mathcal{Q} \left(\sqrt{1 - \xi z}\right).$$  (26)
where \( Q(\beta) = \langle \beta | \rho | \beta \rangle \) is the Q-function, \(|\beta\rangle\) is the coherent state, and \( L_n(x) \) is the Laguerre polynomials. Once the Q-function of \( \rho \) is known, it is easy to calculate the photocount distribution of MCSO-OSCS from (26).

The Q-function of MCSO-OSCS is given by

\[
Q(\beta) = \langle \beta | \rho_m | \beta \rangle = N_m^{-1} \chi^m \left( \langle \beta : H_m(a, a^\dagger) : (\langle a_0 \rangle - |\alpha_0\rangle) \left(\langle a_0 \rangle - (-\alpha_0)\right) : H_m^*(a, a^\dagger) : |\beta\rangle\right),
\]

(27)

where \( H_m(a, a^\dagger) = H_m(i\Omega/\sqrt{\sin 2\theta}) \). Then substituting (27) into (26) and using (9) and the two-variable Hermite polynomials expression of Laguerre polynomials [30]

\[
L_n(z,z^*) = \frac{(-1)^n}{n!} H_{n,n}(z,z^*) = \left(\frac{-1}{n!}\right) \frac{\partial^{2n}}{\partial \mu^n \partial \nu^n} e^{-\mu z + \nu z^*} \bigg|_{\mu=\nu=0},
\]

we obtain the final result of \( P(n) \)

\[
P(n) = 2T_m \sum_{j=0}^m \sum_{l,k=0}^n A_{j,k} \left[ e^{-\xi |\alpha_0|^2} H_{m-l-j} \left(\frac{K-J^*}{2}\right) H_{m-j-k} \left(\frac{J-K^*}{2}\right) - (-1)^{n-k} e^{(\xi-2)|\alpha_0|^2} H_{m-l-j} \left(\frac{K+J}{2}\right) H_{m-j-k} \left(\frac{K+J^*}{2}\right) \right],
\]

(29)

where we have set

\[
A_{j,k} = \frac{(-1)^{j+k} G^{j+l} G^{j+k} F^{n-l} F^{n-k}}{\bar{b}^{j+l} \bar{b}^{n-l} \bar{b}^{n-k}},
\]

\[
F = \sqrt{1 - \xi} \alpha_0, \quad G = -\sqrt{1 - \xi} R^*,
\]

\[
J = (1 - \xi) R\alpha_0, \quad T_m,n = N_m^{-1} \chi^m \frac{n! (m!)^2 \xi^n}{(1 - \xi)^n}.
\]

(30)

In order to discuss the photocount distribution of MCSO-OSCS, we plot the graph of \( P(n) \) for several given parameters \( \theta, \alpha_0, m, \) or \( \xi \) in Figs. 4 and 5. Comparing Fig. 4a with b, we find that for some given values of \( m (m = 4) \), \( \alpha_0 (\alpha_0 = 0.5 + i0.5) \) and \( \theta (\theta = \pi/4) \), the corresponding probability-peak of photocount distribution moves from \( n = 1 \) to \( n = 4 \) as \( \xi \) increases to 0.9, which means that the probability of registering big photon-numbers increases gradually while the probability of registering small photon-numbers decreases when we increase the time interval \( T \). Meanwhile, the larger the \( \xi \) is, the wider the tail of photocount distribution of MCSO-OSCS is. Comparing Fig. 4c with a, we find that for given values of \( \alpha_0 (\alpha_0 = 0.5 + i0.5), \theta (\theta = \pi/4) \) and \( \xi (\xi = 0.2) \), the probability of finding big photon-numbers increases with the increment of the parameter \( m (m = 1 \rightarrow m = 4) \). Comparing Fig. 5a with b, we find that for given values of \( \alpha_0 (\alpha_0 = 0.5+ i0.5), m (m = 4) \) and \( \xi (\xi = 0.9) \), the probability of finding big photon-numbers increases with the increment of the parameter \( \theta (\theta = \pi/8 \rightarrow \pi/3) \).

4 Wigner Function of the MCSO-OSCS

The WF is a quasi-probability distribution, which fully describes the state of a quantum system in phase space. The partial negativity of the WF is indeed a good indication of the highly nonclassical character of the state [41]. Therefore it is worth obtaining the WF.
Fig. 4 Photocount distribution $P(n)$ of MCSO-OSCS as a function of $n$ (photon number, a non-negative integer) for $\alpha_0 = 0.5 + i 0.5$ a $m = 4$, $\theta = \frac{\pi}{4}$, $\xi = 0.2$; b $m = 4$, $\theta = \frac{\pi}{4}$, $\xi = 0.9$; c $m = 1$, $\theta = \frac{\pi}{4}$, $\xi = 0.2$

for the states discussed above and using the negative region to check whether a state has nonclassicality. For a single-mode system, the WF $W(\alpha, \alpha^*)$ associated with a quantum state density matrix $\rho$ can be expressed as [42]:

$$W(\alpha) = \frac{1}{\pi} e^{2|\alpha|^2} \int d^2z \frac{1}{\pi} \langle -z | \rho | z \rangle e^{-2(\alpha^* z - \alpha z^*)}, \quad (31)$$

where $|z\rangle$ is the coherent state. Substituting $\rho_m = N_m^{-1} |\psi_m\rangle \langle \psi_m|$ into (31), we can finally obtain the WF of MCSO-OSCS:

$$W(\alpha) = W_{\alpha_0}(\alpha) + W_{-\alpha_0}(\alpha) - W'_{\alpha_0}(\alpha) - W'_{-\alpha_0}(\alpha), \quad (32)$$

Fig. 5 Photocount distribution $P(n)$ of MCSO-OSCS as a function of $n$ (photon number, a non-negative integer) for $\alpha_0 = 0.5 + i 0.5$ a $m = 4$, $\theta = \frac{\pi}{8}$, $\xi = 0.9$; b $m = 4$, $\theta = \frac{\pi}{8}$, $\xi = 0.9$
Fig. 6 Wigner function distributions $WF(\alpha = \text{Re }\alpha + i \text{Im }\alpha)$ of MCSO-OSCS as the function of Re$\alpha$ and Im$\alpha$ with $\theta = \frac{\pi}{2}, \alpha_0 = 1 + i$ a $m = 0$; b $m = 1$; c $m = 2$; d $m = 3$.

where we have set

$$W_{\alpha_0} (\alpha) = \sum_{k=0}^{m} D_m (-1)^k e^{-2|\alpha_0 - \alpha|^2} |H_{m-k} (-C^* + R\alpha)|^2,$$

$$W'_{\alpha_0} (\alpha) = \sum_{k=0}^{m} D_m (-1)^m e^{2\alpha_0^* \alpha - 2\alpha^* \alpha_0 - 2|\alpha|^2} H_{m-k} \left((B^* - R^* \alpha^*) H_{m-k} (B + R\alpha)\right),$$

(33)

and

$$D_m = \frac{2^{k-2m} N_m^{-1} (m!)^2}{\pi k! [(m-k)!]^2} (\sin 2\theta)^m \left(\tan^k \theta\right).$$

(34)

It is found that the sum of $W'_{\alpha_0} (\alpha)$ and $W'_{-\alpha_0} (\alpha)$ is a real function due to $W'_{-\alpha_0} (\alpha) = W'_{\alpha_0} (\alpha)^*$. By using (32), the WFs as a function of real and imaginary parts of $\alpha$ for several different values of $m$, $\alpha_0$ and $\theta$ are depicted in Figs. 6 and 7.

We can see clearly that WF distributions are non-Gaussian. In addition, as the evidence of the nonclassicality of the state, it is easy to see that there is a negative region of the WF.

Fig. 7 Wigner function distributions $WF(\alpha = \text{Re }\alpha + i \text{Im }\alpha)$ of MCSO-OSCS as the function of Re$\alpha$ and Im$\alpha$ with $m = 2$ a $\alpha_0 = 1 + i, \theta = \frac{\pi}{8}$; b $\alpha_0 = 2(1 + i), \theta = \frac{\pi}{4}$.
in each plot. From Fig. 6, we can see that WFs exist odd (even) negative peaks when the values of \( m \) are even (odd) for given \( \alpha_0 \) and \( \theta \), and exhibit more vibration character as the value of \( m \) increases. Meanwhile, we can find that the minimum value of the WF occurs at the center of the figure when \( m \) is an even number (see Fig. 6a and c). But the case is not true when \( m \) is an odd number (see Fig. 6b and d). Comparing Fig. 6a (\( \theta = \frac{\pi}{8}, m = 2, \alpha_0 = 1 + i \)) with Fig. 6c (\( \theta = \frac{\pi}{4}, m = 2, \alpha_0 = 1 + i \)), we can see that the width of the figure of WF in one direction increases as increasing the value of \( \theta \). Comparing Fig. 6c (\( \alpha_0 = 1 + i, m = 2, \theta = \frac{\pi}{4} \)) with Fig. 6b (\( \alpha_0 = 2 + 2i, m = 2, \theta = \frac{\pi}{4} \)), we can also see that WFs exhibit more vibration character as increasing the value of amplitude \(|\alpha_0|\).

The volume of the negative part of the WF was used in [43, 44] to describe the interference effects which determine the departure from classical behavior. In order to further evaluate how these parameters \( m, \alpha_0 \), and \( \theta \) affect the negative part of WF distribution for MCSO-OSCS, we shall consider the negative part volume of WF which may be written as

\[
\delta = \frac{1}{2} \int d^2 \alpha |W(\alpha)| - 1
\]  

(35)

By definition, the quantity \( \delta \) is equal to zero for coherent and squeezed vacuum states, as their WFs are non-negative. Given the Wigner function of a quantum state, we can obtain the negative part volume of WF through numerical integration.

In Fig. 8, we plot the negative part volume \( \delta \) of WF for MCSO-OSCS as a function of \( \theta \). It is shown that the negative part volume \( \delta \) gradually increases as \( \theta \) increases when \( m \neq 0 \). In addition, it is interesting to note that \( \delta \) is sensitive to parameter \( m \), and the value of \( \delta \) increases as \( m \) increases when parameter \( \theta \) is bigger than a threshold (see Fig. 8a). In other words, the MCSO-OSCS may exhibit more nonclassicality by increasing the value of \( m \). Meanwhile, the value of \( \delta \) increases as the value of \( \alpha_0 \) increases when parameter \( \theta \) is smaller than a threshold (see Fig. 8b).

5 The Decoherence of the MCSO-OSCS in a Thermal Environment

When the MCSO-OSCS evolves in a thermal channel, the evolution of the density matrix in the Born-Markov approximation and the interaction picture can be described by the master equation [45]

\[
\frac{d\rho}{dt} = \kappa (\bar{n} + 1) \left( 2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a \right) + \kappa \bar{n} \left( 2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger \right),
\]  

(36)

\[\begin{align*}
\text{Fig. 8} & \quad \text{The volume (}\delta\text{) of the negative part of the WF for MCSO-OSCS as the function of } \theta \quad \text{a } \alpha_0 = 0.1, \quad m = 0, 1, 2, 3; \quad \text{b } m = 1, \alpha_0 = 0.1, 1, 1.5, 2
\end{align*}\]
where $\kappa$ represents the dissipative coefficient and $\bar{n}$ ($\bar{n} = \frac{1}{\epsilon} \frac{\hbar \omega}{k_B T}$, $T$ is temperature.) denotes the average thermal photon number of the environment [46]. Using the thermal entangled state representation [47], the time evolution of distribution functions in the dissipative channels are derived [48, 49]. The evolutions of the WF is governed by the following integration equation

$$W (\gamma, \gamma^*, t) = \frac{2}{(2\bar{n} + 1) \Gamma} \int \frac{d^2\alpha}{\pi} W (\alpha, \alpha^*, 0) \exp \left[ -\frac{2 \left| \gamma - \alpha e^{-\kappa t} \right|^2}{(2\bar{n} + 1) \Gamma} \right], \quad (37)$$

where $\Gamma = 1 - e^{-2\kappa t}$ and $W (\alpha, \alpha^*, 0)$ is the WF of the initial state. Thus the WF at any time can be obtained by performing the integration when the initial WF is known.

Substituting (32) into (37), we have

$$W (\gamma, \gamma^*, t) = W_{\alpha_0} (\gamma, \gamma^*, t) + W_{-\alpha_0} (\gamma, \gamma^*, t) - (W'_{\alpha_0} (\gamma, \gamma^*, t) + c.c.), \quad (38)$$

where

$$W_{\alpha_0} (\gamma, \gamma^*, t) = \sum_{k=0}^{m} \sum_{l=0}^{m-k} MVU^l e^{-2V |\gamma - \alpha_0 e^{-\kappa t}|^2} \left| H_{m-k-l} \left( -C^* + R\alpha_0 U + R\gamma e^{-\kappa t} V \right) \right|^2,$$

$$W'_{\alpha_0} (\gamma, \gamma^*, t) = \sum_{k=0}^{m} \sum_{l=0}^{m-k} MVU^l e^{-2V |\gamma|^2 V - 2|\alpha_0|^2 U + 2\gamma e^{-\kappa t} \alpha_0^* V - 2\gamma^* e^{-\kappa t} \alpha_0 V} \times H_{m-k-l} \left( -B^* + R^* \alpha_0^* U + R^* \gamma^* e^{-\kappa t} V \right)$$

$$\times H_{m-k-l} \left( B - R\alpha_0 U + R\gamma e^{-\kappa t} V \right), \quad (40)$$

![Fig. 9](image-url) The time evolution of Wigner function WF($\gamma = \text{Re}\gamma + i \text{Im}\gamma$) for MCSO-OSCS in the thermal environment as the function of Re$\gamma$ and Im$\gamma$ with $\theta = \frac{\pi}{3}, \alpha_0 = 1 + i, m = 1, \bar{n} = 0.2$ a $\kappa t = 0.001$; b $\kappa t = 0.05$; c $\kappa t = 0.1$; d $\kappa t = 3$
Fig. 10 Wigner function distributions $WF(\gamma = \text{Re}\gamma + i\text{Im}\gamma)$ of MCSO-OSCS in the thermal environment as the function of $\text{Re}\gamma$ and $\text{Im}\gamma$ for $\theta = \pi / 3$, $\alpha_0 = 1 + i$, $m = 1$, $\kappa t = 0.05$ with different parameter $\bar{n}$: a $\bar{n} = 0$; b $\bar{n} = 0.5$; c $\bar{n} = 2$; d $\bar{n} = 8$

and

\begin{align}
V &= \frac{1}{2\bar{n}! + 1},
U &= 1 - e^{-2\kappa t} V,
M &= \frac{N_m^{-1} (-1)^k 2^{2k+m} (m!)^2}{\pi k! l! ((m - k - l)!)^2} \frac{\sin^{k+l+m} \theta}{\cos^{k+l-m} \theta}.
\end{align}

Furthermore, when $t = 0$, $\Gamma = 0$, (38) reduces to (32), as expected.

In order to see the decoherence of the MCSO-OSCS in the thermal environment, we plot the time evolution of $WF(\gamma = \text{Re}\gamma + i\text{Im}\gamma)$ as a function of real and imaginary parts of $\gamma$ for different $t$ values and for a given $m$ value (say, $m = 1$) in Fig. 9. Time in each plot is fixed, and goes forward from Fig. 9a to d. It is shown that as time proceeds the negative part of WF and multi-peaks vibration structure disappear gradually, and finally the distribution evolves to a wave packet structure (see Fig. 9d), which means that the MCSO-OSCS has reduced to a thermal state. In Fig. 10, we plot the picture of $WF(\gamma = \text{Re}\gamma + i\text{Im}\gamma)$ as a function of $\text{Re}\gamma$ and $\text{Im}\gamma$ for different $\bar{n}$ values and for a given $m$ value (say, $m = 1$) at the given time (say, $\kappa t = 0.05$). It is interesting to note that the negative part of WF decreases as the average photon number $\bar{n}$ increases, i.e., the larger the $\bar{n}$ is, the more rapidly the nonclassicality vanishes, which means that the higher the temperature of thermal field is, the more rapidly the nonclassicality of the MCSO-OSCS vanishes. This result is consistent with that in Ref. [50].

6 Conclusions

In summary, we have investigated the nonclassicality of MCSO-OSCS which was obtained through $m$ times coherent superposition operator $a \cos \theta + a^\dagger \sin \theta$ operating on an odd-Schrödinger-cat state. For arbitrary $m$ value, through IWOP technique we have obtained an analytical expression of the normalization constant, which turns out to be related to the Hermite polynomial. The nonclassical properties of the state, such as sub-Poissonian statistics,
quadrature squeezing properties, and photocount distribution were also discussed in details. We found that MCSO-OSCS has more chances to exhibit sub-Poissonian statistics with bigger value of $\theta$ in the area of $\theta \in (0, \frac{\pi}{2})$. We also found that MCSO-OSCS can exhibit squeezing when the parameter $m$ is odd and the angle $\theta$ is smaller than a threshold, which indicates that the subtracting photon operation benefits to squeezing for odd $m$. Furthermore, the nonclassicality of MCSO-OSCS was investigated in terms of WF and the negative part volume of WF after deriving the analytical expression of WF. It was shown that the WF of the MCSO-OSCS always has negative values which implies the highly nonclassical properties of quantum states. The negative part volume of WF increases as $m$ increases when $m \neq 0$ and parameter $\theta$ is bigger than a threshold, and also increases as the value of $\alpha_0$ increases when parameter $\theta$ is smaller than a threshold. Especially, the negative part volume of WF increases with the increment of parameter $\theta$ except the case of $m = 0$. We have also investigated the decoherence of the MCSO-OSCS in terms of the fade-away of the negativity of WF in a thermal environment. It was shown that nonclassicality of the MCSO-OSCS decreases as time proceeds and the MCSO-OSCS reduces to the thermal state finally. It was also shown that nonclassicality is influenced by the temperature of environment, the higher the temperature is, the more rapidly the nonclassicality of the MCSO-OSCS vanishes. We expect that our results will be of benefit to instructing experiments, for example, the new state would become a useful resource to generate a two-mode entanglement via a beam-splitter.

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