Shape *versus* timing: linear responses of a limit cycle with hard boundaries under instantaneous and static perturbation

Y. Wang¹, J.P. Gill², H.J. Chiel²,³, P.J. Thomas²,⁴

¹ Mathematical Biosciences Institute, The Ohio State University, Columbus, OH
² Department of Biology, Case Western Reserve University, Cleveland, OH
³ Depts of Biomedical Engineering, Neurosciences, Case Western Reserve University, Cleveland, OH
⁴ Dept. Mathematics, Applied Mathematics, and Statistics, Case Western Reserve University, Cleveland, OH

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Abstract

When dynamical systems that produce rhythmic behavior operate within hard limits, they may exhibit limit cycles with sliding components, that is, closed isolated periodic orbits that make and break contact with a constraint surface. Examples include heel-ground interaction in locomotion or firing rate rectification in neural networks. In many rhythmic systems, robustness against external perturbations involves response of both the shape and the timing of the limit cycle trajectory. The existing methods of infinitesimal phase response curve (iPRC) and variational analysis are well established for quantifying changes in timing and shape, respectively, for smooth systems. These tools have recently been extended to nonsmooth dynamics with transversal crossing boundaries. In this work, we further extend both iPRC and variational methods to nonsmooth systems with sliding components, for both instantaneous and parametric perturbations. We observe a new feature of the isochrons in a planar limit cycle with hard sliding boundaries: a nonsmooth kink in the asymptotic phase function, originating from the point at which the limit cycle smoothly departs the constraint surface, and propagating away from the hard boundary into the interior of the domain. Moreover, the classical variational analysis neglects timing information and is restricted to instantaneous perturbations. By defining the “infinitesimal shape response curve” (iSRC), we incorporate timing sensitivity of an oscillator to describe the shape response of this oscillator to parametric perturbations. In order to extract timing information, we develop a “local timing response curve” (lTRC) that measures the timing sensitivity of a limit cycle within any given region. We demonstrate in a specific example that taking into account local timing sensitivity in a nonsmooth system greatly improves the accuracy of the iSRC over global timing analysis given by the iPRC.
1 Introduction

A runner adjusts her stride and posture as she leans into a hill; a climber adjusts the timing and tension of his grasp as he works along a steepening incline; a frog adjusts the location and intensity of wiping as it tries to remove an irritant from its skin. In each of these scenarios, a neuromechanical motor control system exhibits periodic motions that make and break contact with a physical substrate, and adjust the shape and timing of the motion in response to parametric changes in the environmental conditions. As these examples illustrate, limit cycles with sliding components are commonplace in biological motor control systems. In this paper we develop the variational and phase response curve analysis needed to understand the stability and robustness of such systems under parametric variation.

A limit cycle with sliding component (LCSC) is a closed, isolated, periodic orbit of an $n$-dimensional, autonomous, deterministic nonsmooth dynamical system, in which the trajectory is constrained to move along a surface of dimension $k < n$ during some portion of the orbit. As an example, consider a person running. The dynamics of neural activity, muscle activation, joint position, and center-of-mass together form a system of nonlinear ordinary differential equations in $n \gg 1$ dimensions. Models of such systems are often constructed to exhibit asymptotically stable limit cycle trajectories (Holmes et al., 2006; Revzen and Guckenheimer, 2011; Guckenheimer and Javeed, 2018). Each of the runner’s legs passes alternately through a swing phase and a stance phase, in which the foot is respectively free or in contact with the ground. At the points of making contact (the heel strike) and breaking contact (liftoff of the toe) the dynamics makes a nonsmooth transition into and out of a lower dimensional submanifold of the state space. While the foot is in contact with the ground, the dynamics has fewer degrees of freedom than when the foot moves unconstrained.

Transitions onto and off of constraint surfaces occur in many motor control systems. For example, in the transition from biting (attempting to grasp food) to swallowing, an organism makes an initial physical contact with the food; any forces exerted by the food on the organism appear abruptly upon first contact. When a mollusk such as *Aplysia californica* swallows a long stipe of seaweed, the grasper organ repeatedly makes and breaks contact as it pulls seaweed into the gut, through movements orchestrated by a central pattern generator (CPG) circuit (Sutton et al., 2004; Chiel, 2007). In hindlimb scratching, another CPG driven behavior, the foot repeatedly makes and breaks contact with the underbelly (Barajon et al., 1992; Gelfand et al., 2004; Mortin and Stein, 1989).

The neural dynamics internal to central pattern generators may also exhibit nonsmooth sliding components. Consider a firing rate model for a half-center oscillator CPG with inhibition-mediated synaptic connections. Inhibition tends to drive the firing rate of the postsynaptic cell towards zero. Once an inhibited cell shuts off, its firing rate cannot be further reduced: there is a hard boundary at zero firing rate. Recordings of CPG activity in both vertebrate and invertebrate preparations show cells with bursts of activity switching on and switching off during different phases of activity patterns. While one cell’s firing rate is held at zero, other system components continue to evolve.

The study of limit cycle motions in such piecewise smooth systems requires analytical tools beyond the existing arsenal of phase response curves and variational analysis,
developed for systems with smooth (differentiable) right-hand sides (Spardy et al., 2011a,b). The resilience of neuromotor activity under perturbations of modest size may be studied via variational analysis. This analysis is particularly tractable in two extremes: when perturbations cause small instantaneous dislocations of the trajectory, and when perturbations are sustained over long times (parametric perturbation). For small instantaneous displacements, analysis in terms of infinitesimal phase response curves (iPRC) is well established when the underlying dynamics is smooth, i.e. differentiable (Park et al., 2017). The iPRC analysis has been extended to nonsmooth dynamics, provided the flow is always transverse to any switching surfaces at which nonsmooth transitions occur (Shirasaka et al., 2017; Park et al., 2018; Chartrand et al., 2018; Wilson, 2019). Limit cycles with sliding components violate the transverse flow condition; to our knowledge, we are the first to extend iPRC analysis to this case. Variational analysis has likewise been extended to nonsmooth dynamics for studying the linearized effect on the shape of a trajectory following a small instantaneous perturbation (Bernardo et al., 2008; Leine and Nijmeijer, 2013). Again, this analysis requires the transverse flow condition, excluding limit cycles with sliding components. In the present paper we extend both iPRC and variational methods to the LCSC in nonsmooth systems, for instantaneous as well as for parametric perturbations.

Parametric perturbations arise in motor control systems in several circumstances. As a runner advances along a steeper and steeper path, the general pattern of neural activity and biomechanics remains similar, but the timing and intensity of muscle activations changes in detail to accommodate the changing slope of the hill. As the sea slug progressively ingests a stipe of seaweed, its thickness may increase and its pliability may decrease; the motor activation pattern must adjust accordingly. The CPG regulating respiration adjusts the breathing rate as the runner’s metabolic demand increases. In each of these examples, the dynamical system exhibiting the LCSC becomes a family of dynamical systems indexed by one or more parameters. In general, a small fixed change in a parameter gives rise to a new limit cycle, with different shape and timing than the original. This paper develops the mathematical framework required to analyze the changes in shape and timing wrought by parametric changes, such as changing mechanical or metabolic load, on motor control systems with CPG-driven limit cycles with sliding components.

We review the classical variational and phase response curve analysis for the response of smooth dynamical systems to instantaneous perturbations in §2.1. In order to account for the response of an oscillator to parametric perturbations, we derive an infinitesimal shape response curve (iSRC) in §2.2. In contrast to standard variational analysis, which neglects timing changes, the iSRC takes into account both timing and shape changes arising due to a parametric perturbation.

The infinitesimal phase response curve (iPRC) captures the change in timing of an oscillator due to an instantaneous perturbation, as well as the global change in period in response to a parametric perturbation. However, in many applications, the impact of a perturbation on local timing can be as important as the global effects. For instance, in the feeding of the marine mollusk Aplysia californica, a static perturbation such as a force applied to the animal’s food can only be felt when its grasper or jaws are closed on the food. Similarly, any motor control system that operates by making
and breaking physical contact (walking, scratching or grasping) would experience perturbations limited to a discrete component of the limit cycle. In these cases, one would need to compute the local timing changes of the trajectory during the phase when the perturbation exists (e.g., the grasper is closed) to understand the robustness of this system. Such local change is often different from the global timing change and hence cannot be obtained using the iPRC. To this end, in §2.3 and Appendix B, we develop a local timing response curve (lTRC) that is analogous to the iPRC but measures the local timing sensitivity of a limit cycle within any given local region. Development of the lTRC leads to a piecewise-specified version of the iSRC which often exhibits greater accuracy.

While the applicability of the classical perturbative methods from §2 is generally limited by the constraint that the dynamics of the system is smooth, some elements of the methods have already been generalized to nonsmooth systems with only transversal crossing boundaries, which will be reviewed in §3.1. Moreover, we extend these methods to the LCSC case in nonsmooth systems, both for instantaneous and parametric perturbations, in §3.2. Throughout, we consider nonsmooth systems with degree of smoothness one or higher; that is, systems with continuous trajectories, also known as Filippov systems (Bernardo et al., 2008). Our main result is Theorem 3.7, which describes the behavior of the variational and infinitesimal phase response curve dynamics when the LCSC of a Filippov system enters and exits a hard boundary. Appendix C gives a proof of the theorem. Numerical algorithms for implementing these methods are presented in Appendix D. In §4, we illustrate both the theory and algorithms using a planar model, comprising a limit cycle with a linear vector field in the interior of a simply connected convex domain with a hard boundary condition. In this example, we show that under certain circumstances (e.g. non-uniform perturbation), the iSRC together with the ITRC provides a more accurate representation of the combined timing and shape responses to static perturbations than using the global iPRC alone. Surprisingly, we discover nondifferentiable “kinks” in the isochron function that propagate backwards in time along an osculating trajectory that encounters the hard boundary exactly at the liftoff point (the point where the limit cycle trajectory smoothly departs the boundary). Lastly, we discuss limitations of our methods and possible future directions in §5.

Appendix A provides a table of symbols used in the paper.

## 2 Linear responses of smooth systems

In this section we review the classical theory for linear approximation of the effects of instantaneous and sustained perturbation on the timing and shape of a limit cycle trajectory in the smooth case.

Consider an $n$-dimensional $C^1$ dynamical system (a system of ordinary differential equations in $n$ variables with velocity $F(x)$ having continuous first derivatives with respect to the components of $x$),

$$\frac{dx}{dt} = F(x),$$

(2.1)
with a period $T$ stable limit cycle (LC) solution $\gamma(t) = \gamma(t+T)$, that is, a closed, isolated periodic orbit attracting all trajectories originating within some open set containing the LC. The effects of small, instantaneous perturbations on the shape and timing of trajectories near the LC are captured by the variational and infinitesimal phase response curve analysis, which we review in §2.1. In §2.2, we also derive the infinitesimal shape response curve (iSRC) to account for the combined shape and timing response of $\gamma(t)$ under static perturbations. To obtain a more accurate iSRC when the limit cycle experiences different timing sensitivities in different regions within the domain, in §2.3, we introduce the local timing response curve (lTRC). In contrast to the iPRC, which measures the global shift in the period $T$, the ITRC lets us compute the timing change of $\gamma(t)$ within regions bounded between specified Poincaré sections.

### 2.1 Shape and timing response to instantaneous perturbations

Suppose a small, brief perturbation is applied at time $t_0$ such that there is a small abrupt perturbation in the state space. We have

$$\tilde{\gamma}(t_0) = \gamma(t_0) + \varepsilon P,$$  \hspace{1cm} (2.2)

where $\tilde{\gamma}$ indicates the trajectory subsequent to the instantaneous perturbation, $\varepsilon$ is the magnitude of the perturbation, and $P$ is the unit vector in the direction of the perturbation in the state space. As we show below, the effects of the small brief perturbation $\varepsilon P$ on the shape and timing of the limit cycle trajectory are given, respectively, by the solution of the variational equation (2.3), and the iPRC which solves the adjoint equation (2.5).

The evolution of a trajectory $\tilde{\gamma}(t)$ close to the limit cycle $\gamma(t)$ may be approximated as

$$\tilde{\gamma}(t) = \gamma(t) + u(t) + O(\varepsilon^2),$$

where $u(t)$ satisfies the variational equation

$$\frac{du}{dt} = DF(\gamma(t))u$$  \hspace{1cm} (2.3)

with initial displacement $u(t_0) = \varepsilon P$ given by (2.2), for small $\varepsilon$. Here $DF(\gamma(t))$ is the Jacobian matrix evaluated along $\gamma(t)$.

On the other hand, an iPRC of an oscillator measures the timing sensitivity of the limit cycle to infinitesimally small perturbations at every point along its cycle. It is defined as the shift in the oscillator phase $\theta \in [0, T)$ per size of the perturbation, in the limit of small perturbation size. The limit cycle solution takes each phase to a unique point on the limit cycle, $x = \gamma(\theta)$, and its inverse maps each point on the cycle to a unique phase, $\theta = \phi(x)$. We extend the domain of $\phi(x)$ to points in the basin of attraction $\mathcal{B}$ of the limit cycle by defining the asymptotic phase: $\phi(x) : \mathcal{B} \rightarrow [0, T)$ with

$$\frac{d\phi(x(t))}{dt} = 1, \quad \phi(x(t)) = \phi(x(t + T)).$$

If $x_0 \in \gamma(t)$ and $y_0 \in \mathcal{B}$, then we say that $y_0$ has the same asymptotic phase as $x_0$ if $\|x(t; x_0) - y(t; y_0)\| \rightarrow 0$, as $t \rightarrow \infty$. This means that $\phi(x_0) = \phi(y_0)$. The set of all
points off the limit cycle that have the same asymptotic phase as the point \( x_0 \) on the limit cycle is the *isochron* with phase \( \phi(x_0) \).

Suppose \( \varepsilon P \) applied at phase \( \theta \) results in a new state \( \gamma(\theta) + \varepsilon P \in \mathcal{B} \), which corresponds to a new phase \( \tilde{\theta} = \phi(\gamma(\theta) + \varepsilon P) \). The phase difference \( \tilde{\theta} - \theta \) defines the phase response curve (PRC) of the oscillator. One defines the iPRC as the vector function \( z : [0, T) \to \mathbb{R}^n \) satisfying
\[
 z(\theta) \cdot P = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\phi(\gamma(\theta) + \varepsilon P) - \theta) = \nabla_x \phi(\gamma(\theta)) \cdot P
\] for arbitrary unit perturbation \( P \). The first equality serves as a definition, while the second follows from routine arguments (Brown et al., 2004; Ermentrout and Terman, 2010; Schwemmer and Lewis, 2012; Park et al., 2017). It follows directly that the vector iPRC is the gradient of the asymptotic phase and it captures the phase (or timing) response to perturbations in any direction \( P \) in state space. Since the vector field \( F \) is assumed to be \( C^1 \), the iPRC is a continuous \( T \)-periodic solution satisfying the *adjoint equation* (Schwemmer and Lewis, 2012),
\[
 \frac{d}{dt} z = -DF(\gamma(t))^\top z, \tag{2.5}
\]
with the normalization condition
\[
 F(\gamma(\theta)) \cdot z(\theta) = 1. \tag{2.6}
\]

**Remark 2.1.** By direct calculation, one can show that the solutions to the variational equation and the adjoint equation satisfy \( u^\top z = \text{constant} \):
\[
 \frac{d(u^\top z)}{dt} = \frac{d}{dt} u^\top z + u^\top \frac{dz}{dt} = u^\top DF^\top z + u^\top (-DF^\top z) = 0. \tag{2.7}
\]

This relation holds for both smooth and nonsmooth systems with transverse crossings (Park et al., 2018).

For completeness, we note that differences between phase variables, as in (2.4), will be interpreted as the *periodic difference*, \( d_T(\phi(x), \phi(y)) \). That is, if two angular variables \( \theta \) and \( \psi \) are defined on the circle \( S \equiv [0, T) \), then we set
\[
 d_T(\theta, \psi) = \begin{cases} 
 \theta - \psi + T, & \theta - \psi < -\frac{T}{2} \\
 \theta - \psi, & -\frac{T}{2} \leq \theta - \psi \leq \frac{T}{2} \\
 \theta - \psi - T, & \theta - \psi > \frac{T}{2}
\end{cases} \tag{2.8}
\]
which maps \( d_T(\theta, \psi) \) to the range \([-T/2, T/2]\). In what follows we will simply write \( \theta - \psi \) for clarity rather than \( d_T(\theta, \psi) \).

### 2.2 Shape and timing response to sustained perturbations

In this section we study the effects of sustained perturbations on the shape and timing of the LC solution. In contrast to the instantaneous perturbation considered in the
previous section, changes in each aspect of shape and timing can now influence the  
other, and hence a variational analysis of the combined shape and timing response of  
limit cycles under constant perturbation is needed (see (2.17)).

Suppose a sustained perturbation on (2.1) leads to the perturbed system

$$\frac{dx}{dt} = F_\varepsilon(x),$$  

(2.9)

with a stable limit cycle solution \( \gamma_\varepsilon(t) \) with period \( T_\varepsilon \) depending smoothly on \( \varepsilon \) over some range. In particular, when \( \varepsilon = 0 \) we use the notation \( F_0, \gamma_0, \) and \( T_0 \) to denote these  
quantities, each of which are equivalent to \( F, \gamma, \) and \( T, \) respectively, introduced earlier  
for the unperturbed system (2.1). To simplify notation, we will drop the subscript 0  
except where required to avoid confusion. The perturbed periodic solution \( \gamma_\varepsilon(t) \) can  
be represented, to leading order, by the single variable system

$$\frac{d\theta}{dt} = 1 + z(\theta)^\top G(x, t),$$  

(2.10)

where \( G(x, t) = \varepsilon \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0} \) represents the \( O(\varepsilon) \) perturbation of the vector field, \( \theta \in [0, T_0) \) is the asymptotic phase as defined above, and \( z : \theta \in [0, T_0] \rightarrow \mathbb{R}^n \) is the  
iPRC. Suppose that for \( 0 \leq \varepsilon \ll 1 \) we can represent \( T_\varepsilon \) with a uniformly convergent  
series

$$T_\varepsilon = T_0 + \varepsilon T_1 + O(\varepsilon^2)$$  

(2.11)

where \( T_1 \) is the linear shift in the limit cycle period in response to the static pertur-  
bation. \( T_1 > 0 \) if increasing \( \varepsilon \) increases the period. From (2.10), \( T_1 \) can be calculated  
using the iPRC as

$$T_1 = -\int_0^{T_0} z(\theta)^\top \frac{\partial F_\varepsilon(\gamma(\theta))}{\partial \varepsilon} \bigg|_{\varepsilon=0} d\theta.$$  

(2.12)

To understand how the static perturbation changes the shape of the limit cycle  
\( \gamma(t) \), we need to rescale the time coordinate of the perturbed solution so that \( \gamma(t) \) and  
\( \gamma_\varepsilon(t) \) may be compared at corresponding time points. That is, for \( \varepsilon > 0 \) we wish to  
introduce a perturbed time \( \tau(t) \) so that we can write the perturbed limit cycle solution  
uniformly in \( t \) as

$$\gamma_\varepsilon(\tau(t)) = \gamma_0(t) + \varepsilon \gamma_1(t) + O(\varepsilon^2).$$  

(2.13)

We define the \( T_0 \)-periodic function \( \gamma_1(t) \) to be the \textit{infinitesimal shape response curve}  
\textit{iSRC).}

We show next that \( \gamma_1(t) \) obeys an inhomogeneous variational equation (2.17). This  
equation resembles (2.3), but has two additional non-homogeneous terms arising, re-  
spectively, from time rescaling \( t \rightarrow \tau(t) \), and directly from the constant perturbation  
acting on the vector field.

We require that the perturbed time satisfy the consistency and smoothness conditions

$$\frac{d\tau}{dt} > 0, \quad \text{and} \quad \frac{1}{\varepsilon} \left( \int_{t=0}^{T_0} \left( \frac{d\tau}{dt} \right) dt - T_0 \right) = T_1 + O(\varepsilon), \quad \text{as} \ \varepsilon \rightarrow 0.$$  

(2.14)
These constraints do not determine the value of the derivative of \( \tau \), which we write as \( d\tau/dt = 1/\nu_\varepsilon(t) \). In general, the iSRC will depend on the choice of \( \nu_\varepsilon(t) \). However, some natural choices are particularly well adapted to specific problems, as we will see. Initially, we will make the simple ansatz \( \nu_\varepsilon(t) = \text{const} \), i.e. we will assume uniform local timing sensitivity. Later we will introduce local timing response curves to exploit alternative time rescalings for greater accuracy.

To derive (2.17), we assume \( \nu_\varepsilon(t) = \text{const} \) and set the scaling factor to be \( \nu_\varepsilon = T_\varepsilon/T_0 \). Moreover, we assume that \( \nu_\varepsilon \) can be written as a uniformly convergent series

\[
\nu_\varepsilon = 1 - \varepsilon \nu_1 + O(\varepsilon^2),
\]

where \( \nu_1 = T_1/T_0 \) represents the relative change in frequency. In terms of \( \nu_\varepsilon \), the rescaled time for \( \gamma_\varepsilon \) can be written as \( \tau(t) = t/\nu_\varepsilon \in [0,T_\varepsilon] \) for \( t \in [0,T_0] \). Differentiating (2.13) with respect to \( t \) \( \frac{d\gamma_\varepsilon}{dt} = \frac{d\gamma_\varepsilon}{d\tau} \frac{d\tau}{dt} \), substituting the ansatz \( \frac{d\tau}{dt} = \frac{1}{\nu_\varepsilon} \), and rearranging leads to

\[
\frac{d\gamma_\varepsilon}{dt} = \nu_\varepsilon \left( \gamma'(t) + \varepsilon \gamma'_1(t) + O(\varepsilon^2) \right)
= (1 - \varepsilon \nu_1 + O(\varepsilon^2)) \left( \gamma'(t) + \varepsilon \gamma'_1(t) + O(\varepsilon^2) \right)
= \gamma'(t) - \varepsilon \nu_1 \gamma'(t) + \varepsilon \gamma'_1(t) + O(\varepsilon^2)
= F_0(\gamma(t)) + \varepsilon (-\nu_1 F_0(\gamma(t)) + \gamma'_1(0)) + O(\varepsilon^2).
\] (2.15)

where \( ' \) denotes the derivative with respect to \( t \). On the other hand, expanding the right hand side of (2.9) gives

\[
\frac{d\nu_\varepsilon}{d\tau} = F_\varepsilon(\gamma_\varepsilon(\tau)) = F_0(\gamma(t)) + \varepsilon \left( DF_0(\gamma(t)) \gamma_1(t) + \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0} \right) + O(\varepsilon^2).
\] (2.16)

Equating (2.15) and (2.16) to first order, we find that the linear shift in shape produced by a static perturbation, i.e. the iSRC, satisfies

\[
\frac{d\gamma_1(t)}{dt} = DF_0(\gamma(t)) \gamma_1(t) + \nu_1 F_0(\gamma(t)) + \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0},
\] (2.17)

with period \( T_0 \), as claimed before.

It remains to establish the initial condition \( \gamma_1(0) \). Let \( p_0 = \gamma(0) \) denote an arbitrary base point chosen along the limit cycle. If \( \Sigma \) is any Poincaré section transverse to the unperturbed limit cycle at \( p_0 \), then let \( p_\varepsilon \) denote the intersection point where the LC under static perturbation \( |\varepsilon| > 0 \) crosses \( \Sigma \). We may choose \( p_\varepsilon \) as a reference point anchoring the comparison of the perturbed and unperturbed limit cycles by setting \( \gamma_\varepsilon(0) = p_\varepsilon \). Then the initial condition for the shape response curve \( \gamma_1(0) \) is given by \( (\gamma_\varepsilon(0) - \gamma_0(0))/\varepsilon \) and lies in the direction of \( (dp_\varepsilon/d\varepsilon)|_{\varepsilon=0} \). If \( \Sigma_a \) and \( \Sigma_b \) are two Poincaré sections through \( p_0 \), with initial shape displacements \( \gamma_1^{a,b}(0) \), it can be readily shown that \( \gamma_1^{b}(0) \in \text{Span} \{ \gamma_1^{a}(0), F_0(p) \} \). Thus the collection of initial infinitesimal displacement vectors \( \gamma_1(0) \) lies within a two-dimensional plane, which is a projection of the tangent space to the manifold defined by the family of limit cycles under variations in \( \varepsilon \). For nonsmooth systems discussed in the balance of the paper, the ambiguity in the choice of the reference section \( \Sigma \) is largely removed by setting \( \Sigma \) equal to one of the switching or contact boundaries.
To our knowledge, we are the first to derive the variational equation of the combined shape and timing response of a limit cycle to sustained perturbation, in either the smooth case (here) or the nonsmooth case (below).

The accuracy of the iSRC in approximating the linear change in the limit cycle shape evidently depends on its timing sensitivity, that is, the choice of the relative change in frequency $\nu_1$. In the preceding derivation, we chose $\nu_1$ to be the relative change in the full period by assuming the limit cycle has constant timing sensitivity. It is natural to expect that different choices of $\nu_1$ will be needed for systems with varying timing sensitivities along the limit cycle. This possibility motivates us to consider local timing surfaces which divide the limit cycle into a number of segments, each distinguished by its own timing sensitivity properties. For each segment, we show that the linear shift in the time that $\gamma(t)$ spends in that segment can be estimated using the lTRC derived in §2.3. The lTRC is analogous to the iPRC in the sense that they obey the same adjoint equation, but with different boundary conditions.

### 2.3 Local timing surfaces

The iPRC captures the net effect on timing of an oscillation – the phase shift – due to a transient perturbation (2.4), as well as the net change in period due to a sustained perturbation (2.12). In order to study the impact of a perturbation on local timing as opposed to the global timing, we introduce the notion of local timing surfaces that separate the limit cycle trajectory into segments with different timing sensitivities. Examples of local timing surfaces in smooth systems include the passage of neuronal voltage through its local maximum or through a predefined threshold voltage, and the point of maximal extension of reach by a limb. In nonsmooth systems, switching surfaces at which dynamics changes can also serve as local timing surfaces. For instance, in the feeding system of *Aplysia californica* as discussed in §1, the open-closed switching boundary of the grasper defines a local timing surface.

Whatever the origin of the local timing surface or surfaces of interest, it is natural to consider the phase space of a limit cycle as divided into multiple regions. Hence we may consider a smooth system $d\mathbf{x}/dt = F_0(\mathbf{x})$ with a limit cycle solution $\gamma(t)$ passing through multiple regions in succession (see Figure 1). In each region, we assume that $\gamma(t)$ has constant timing sensitivity. To compute the relative change in time in any given region, we define a local timing response curve (lTRC) to measure the timing shift of $\gamma(t)$ in response to perturbations delivered at different times in that region. Below, we illustrate the derivation of the lTRC in region I and show how it can be used to compute the relative change in time in this region, denoted by $\nu^I_1$.

Suppose that at time $t^{\text{in}}$, $\gamma(t)$ enters region I upon crossing the surface $\Sigma^{\text{in}}$ at the point $\mathbf{x}^{\text{in}}$; at time $t^{\text{out}}$, $\gamma(t)$ exits region I upon crossing the surface $\Sigma^{\text{out}}$ at the point $\mathbf{x}^{\text{out}}$ (see Figure 1). Denote the vector field under a constant perturbation by $F_\varepsilon(\mathbf{x})$ and let $\mathbf{x}_\varepsilon$ denote the coordinate of the perturbed trajectory. Let $T^0_1 = t^{\text{out}} - t^{\text{in}}$ denote the time $\gamma(t)$ spent in region I and let $T^1_\varepsilon$ denote the time the perturbed trajectory spent in region I. Assume we can write $T^1_\varepsilon = T^0_1 + \varepsilon T^1_1 + O(\varepsilon^2)$ as we did before. It
follows that the relative change in time of $\gamma(t)$ in region I is given by

$$\nu^I_1 = \frac{T^I_1}{T^I_0} = \frac{T^I_1}{t^{out} - t^{in}}.$$ 

The goal is to compute $\nu^I_1$, which requires an estimate of $T^I_1$. To this end, we define the local timing response curve $\eta^I(t)$ associated with region I. We show that $\eta^I(t)$ satisfies the adjoint equation (2.21) and the boundary condition (2.22).

Let $T^I(x)$ for $x$ in region I be the time remaining until exiting region I through $\Sigma^{out}$, under the unperturbed vector field. This function is at least defined in some open neighborhood around the reference limit cycle trajectory $\gamma(t)$ if not throughout region I. For the unperturbed system, $T^I$ satisfies

$$\frac{dT^I(x(t))}{dt} = -1$$

(2.18)

along the limit cycle orbit $\gamma(t)$. Hence

$$F(x) \cdot \nabla T^I(x) = -1$$

(2.19)

for all $x$ for which $T^I$ is defined. We define $\eta^I(t) := \nabla T^I(x(t))$ to be the local timing response curve (LTRC) for region I. It is defined for $t \in [t^{in}, t^{out}]$. We show in Appendix B that $T^I_1$ can be estimated as

$$T^I_1 = \eta^I(x^{in}) \cdot \frac{\partial x^{in}}{\partial \varepsilon} \bigg|_{\varepsilon = 0} + \int_{t^{in}}^{t^{out}} \eta^I(\gamma(t)) \cdot \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon = 0} dt,$$

(2.20)

where $x^{in}$ denotes the coordinate of the perturbed entry point into region I. We may naturally view $\eta^I$ as either a function of space, as in (2.20), or as a function of time,
evaluated e.g. along the limit cycle trajectory. Comparing (2.20) with (2.12), the integral terms have the same form, albeit with opposite signs. In addition, (2.20) has an additional term arising from the impact of the perturbation on the point of entry to region I. On the other hand, the impact of the perturbation on the exit point, denoted by $\eta^I(x^{\text{out}}) \cdot \frac{\partial x^{\text{out}}}{\partial \varepsilon}|_{\varepsilon=0}$, is always zero because the exit boundary $\Sigma^{\text{out}}$ is a level curve of $T^I$; in other words, $T^I \equiv 0$ at $\Sigma^{\text{out}}$. This indicates that the ITRC vector $\eta^I$ associated with a given region is always perpendicular to the exit boundary of that region.

Similar to the iPRC, it follows from (2.19) that $\eta^I$ satisfies the adjoint equation

$$\frac{d\eta^I}{dt} = -DF(\gamma(t))^T \eta^I$$

(2.21)

together with the boundary (normalization) condition at the exit point

$$\eta^I(x^{\text{out}}) = \frac{-n^{\text{out}}}{n^{\text{out}} F(x^{\text{out}})}$$

(2.22)

where $n^{\text{out}}$ is a normal vector of $\Sigma^{\text{out}}$ at the unperturbed exit point $x^{\text{out}}$. The reason $\eta^I$ at the exit point has the direction $n^{\text{out}}$ is because $\eta^I$ is normal to the exit boundary as discussed above.

To summarize, in order to compute $\nu^I = T^I / (t^{\text{out}} - t^{\text{in}})$, we need numerically to find $t^{\text{in}}, t^{\text{out}}$ and evaluate (2.20) to estimate $T^I$, for which we need to solve the boundary problem of the adjoint equation (2.21)-(2.22) for the ITRC $\eta^I$. The procedures to obtain the relative change in time in other regions $\nu^I, \nu^{I1}, \nu^{I2}, \cdots$ are similar to computing $\nu^I$ in region I and hence are omitted. The existence of different timing sensitivities of $\gamma(t)$ in different regions therefore leads to a piecewise-specified version of the iSRC (2.17) with period $T_0$,

$$\frac{d\gamma^I(t)}{dt} = DF^I_0(\gamma(t))\gamma^I(t) + \nu^I F^I_0(\gamma(t)) + \frac{\partial F^I_0(\gamma(t))}{\partial \varepsilon}|_{\varepsilon=0},$$

(2.23)

where $\gamma^I, F^I_0, F^I$ and $\nu^I$ denote the iSRC, the unperturbed vector field, the perturbed vector field, and the relative change in time in region $j$, respectively, with $j \in \{I, II, III, \cdots\}$. Note that in a smooth system as concerned in this section, $F^I_0 \equiv F_0$ for all $j$.

In §4 we will show in a specific example that the iSRC with piecewise-specified timing rescaling has much greater accuracy in approximating the linear shape response of the limit cycle to static perturbations than the iSRC using a global uniform rescaling.

Remark 2.2. The derivation of the ITRC in a given region still holds as long as the system is smooth in that region. Hence the assumption that $F(x)$ is smooth everywhere can be relaxed to $F(x)$ being piecewise smooth.

3 Linear responses of nonsmooth systems with continuous solutions

Nonsmooth dynamics arises in many areas of biology and engineering. However, methods developed for smooth systems (discussed in §2) do not extend directly to under-
standing the changes of periodic limit cycle orbits in nonsmooth systems, because their Jacobian matrices are not well defined (Chartrand et al., 2018; Wilson, 2019). Specifically, nonsmooth systems exhibit discontinuities in the time evolution of the solutions to the variational equations, \( u (2.3) \) and \( \gamma_1 (2.17) \), and the solutions to the adjoint equations, \( z (2.5) \) and \( \eta (2.21) \). Following the terminology of Park et al. (2018) and Leine and Nijmeijer (2013), we call the discontinuities in \( z \) and \( \eta \) “jumps” and call the discontinuities in \( u \) and \( \gamma_1 \) “saltations”. Qualitatively, we use “jumps” to refer to discontinuities in the timing response of a trajectory, and “saltations” in the shape response. Since \( z \) and \( \eta \) satisfy the same adjoint equation, they have the same discontinuities. Similarly, \( u \) and \( \gamma_1 \) obey versions of the variational equation with the same homogeneous term and different nonhomogeneous terms; since the jump conditions arise from the homogeneous terms (involving the Jacobian matrix in (2.3), (2.17)) we will presume that \( u \) and \( \gamma_1 \) satisfy the same saltation conditions at the transition boundaries. In the rest of this section, we characterize the discontinuities in the solutions to the adjoint equation in terms of \( z \), and discuss nonsmoothness of the variational dynamics in terms of \( u \).

As mentioned in the introduction, we consider nonsmooth systems with degree of smoothness one or higher (Filippov systems); that is, systems with continuous solutions. In such systems, two types of discontinuities can occur in the trajectory: transversal crossing boundaries and hard boundaries. We review the existing methods for computing the saltations in \( u \) (and \( \gamma_1 \)) and the jumps in \( z \) (and \( \eta \)) associated with transversal crossing boundaries in §3.1, following Bernardo et al. (2008); Leine and Nijmeijer (2013) and Park (2013); Park et al. (2018). In §3.2 we characterize the discontinuous behavior of the iPRC and the variational dynamics for nonsmooth systems with hard boundaries, stated here as Theorem 3.7.

### 3.1 Transversal crossing boundary

It is sufficient to consider a Filippov system with a single boundary to illustrate the discontinuities of \( z \) and \( u \) at any boundary transversal crossing point.

**Definition 3.1.** A two-zone system with uniform degree of smoothness one (or higher) is described by

\[
\frac{d\mathbf{x}}{dt} = F(\mathbf{x}) := \begin{cases} F^I(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^I \\ F^II(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^II \end{cases}
\]

(3.24)

where \( \mathcal{R}^I := \{ \mathbf{x} | H(\mathbf{x}) < 0 \} \) and \( \mathcal{R}^II := \{ \mathbf{x} | H(\mathbf{x}) > 0 \} \) for a smooth function \( H \), and the vector fields \( F^I, F^II : \mathcal{R} \rightarrow \mathbb{R}^n \) are smooth (at least \( C^1 \)). We assume each region \( \mathcal{R}^I, \mathcal{R}^II \) has non-empty interior, and we write \( \overline{\mathcal{R}} \) for the closure of \( \mathcal{R} \) in \( \mathbb{R}^n \). The **switching boundary** is the \( \mathbb{R}^{n-1} \)-dimensional manifold \( \Sigma := \overline{\mathcal{R}}^I \cap \overline{\mathcal{R}}^II = \{ \mathbf{x} | H(\mathbf{x}) = 0 \} \).

Suppose at time \( t = t_p \) a limit cycle solution \( \gamma(t) \) of (3.24) crosses the boundary \( \Sigma \) from \( \mathcal{R}^I \) to \( \mathcal{R}^II \) transversely (see Figure 2A):

\[
n_p \cdot F^I(\mathbf{x}_p) > 0 \quad n_p \cdot F^II(\mathbf{x}_p) > 0,
\]

(3.25)
Figure 2: Examples of trajectories of nonsmooth systems with a boundary $\Sigma$. (A) The trajectory of a two-zone nonsmooth system (3.24) intersects the boundary $\Sigma$ transversely at $x_p$ (black dot). $F^I$ and $F^{II}$ denote the vector fields in the two regions $\mathcal{R}^I$ and $\mathcal{R}^{II}$. Components of $F^I$ and $F^{II}$ normal to $\Sigma$ at the crossing point have the same sign, allowing for the transversal crossing. (B) The trajectory of a nonsmooth system (3.33) hits the hard boundary $\Sigma$ at the landing point (red dot) and begins sliding along $\Sigma$ under the vector field $F^{\text{slide}}$. At the liftoff point (blue dot), the trajectory naturally reenters the interior. $F^{\text{interior}}$ denotes the vector field in the interior domain.

where the boundary crossing point is denoted as $x_p := \lim_{t \to t_p^-} \gamma(t) = \lim_{t \to t_p^+} \gamma(t)$ and $n_p = \nabla H(x_p)$ refers to the vector normal to $\Sigma$ at $x_p$.

Below we show how to characterize discontinuities of $u$ and $z$ for (3.24) at $x_p$. As discussed before, the iSRC $\gamma_1$ and the lTRC $\eta$ experience the same discontinuities as $u$ and $z$, respectively.

For a sufficiently small instantaneous perturbation, the displacement $u(t)$ evolves continuously over the domain in which (3.24) is smooth and can be obtained to first order in the initial displacement by solving the variational equation (2.3). As $\gamma$ crosses $\Sigma$ at time $t_p$, $u(t)$ exhibits discontinuities (or “saltations”) since the Jacobian evaluated at $x_p$ is not uniquely defined. The discontinuity in $u$ at $x_p$ can be expressed with the saltation matrix $S_p$ as

$$u_p^+ = S_p u_p^-$$

where $u_p^- = \lim_{t \to t_p^-} u(t)$ and $u_p^+ = \lim_{t \to t_p^+} u(t)$ represent the displacements between perturbed and unperturbed solutions just before and just after the crossing, respectively. It is straightforward to show (cf. Leine and Nijmeijer (2013) §7.2 or Bernardo et al. (2008) §2.5) that $S_p$ can be constructed using the vector fields in the neighborhood of the crossing point and the vector $n_p$ normal to the switching boundary at $x_p$ as

$$S_p = I + \left(\frac{F_p^+ - F_p^-}{n_p^\top F_p^-}\right)n_p^\top$$

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where \( F_p^- = \lim_{x \to x_p^-} F(x) \), \( F_p^+ = \lim_{x \to x_p^+} F(x) \) are the vector fields of (3.24) just before and just after the crossing at \( x_p \). Throughout this paper, \( I \) denotes the identity matrix with size \( n \times n \).

**Remark 3.2.** If the vector field \( F \) evaluated along the limit cycle is continuous when crossing the boundary \( \Sigma \) transversely, so that \( F_p^- = F_p^+ \) and \( n_p^T F_p^- \neq 0 \), then the saltation matrix \( S_p \) at such a boundary crossing point is the identity matrix, and there is no discontinuity in \( u \) or \( \gamma_1 \) at time \( t_p \).

Now we consider discontinuous jumps in the iPRC \( z \) for (3.24). This curve obeys the adjoint equation (2.5) and is continuous within the interior of each subdomain in which (3.24) is smooth. When the limit cycle path crosses the switching boundary at the point \( x_p \), \( z \) exhibits a discontinuous jump which can be characterized by the jump matrix \( J_p \)

\[
z_p^+ = J_p z_p^-
\]

(3.28)

where \( z_p^- = \lim_{t \to t_p^-} z(t) \) and \( z_p^+ = \lim_{t \to t_p^+} z(t) \) are the iPRC just before and just after crossing the switching boundary at time \( t_p \). As discussed in Park et al. (2018), the relation (2.7) between \( u \) and \( z \) for smooth systems remains valid at any transversal boundary crossing point. In other words, \( u_p^T z_p^- = u_p^T z_p^+ \) holds at the transversal crossing point \( x_p \). This leads to a relation between the saltation and jump matrices at \( x_p \)

\[
J_p^T S_p = I.
\]

(3.29)

The saltation matrix \( S_p \) given by (3.27) has full rank at any transverse crossing point. It follows that \( J_p \) can be written as

\[
J_p = (S_p^{-1})^T.
\]

### 3.2 Sliding motion on a hard boundary

The existence of the saltation and jump matrices discussed in §3.1 is guaranteed by the transversal flow condition (3.25), which, however, will no longer hold when part of the limit cycle slides along a boundary (e.g., Figure 2B). As an example of a hard boundary at which the transverse flow condition would break down, consider the requirement that firing rates in a neural network model be nonnegative. When a nerve cell ceases firing because of inhibition, its firing rate will be held at zero until the balance of inhibition and excitation allow spiking to resume. At the point at which the firing rate first resumes positive values, the vector field describing the system lies tangent to the constraint surface rather than transverse to it.

In this section, we establish the conditions relating \( u \) and \( z \) at non-transversal crossings, including the landing point at which a sliding motion begins, and the liftoff point at which the sliding terminates (see Fig. 2B). We gather our main results in Theorem 3.7. To this end, we begin with precise definitions of hard boundary, sliding region, sliding vector field and the liftoff condition.
**Definition 3.3.** Consider a system with domain $\mathcal{R}$. We call a surface $\Sigma$ a hard boundary if it is part of the boundary of the closure of $\mathcal{R}$.

In order to describe motions that are confined to slide along a hard boundary during a component of the trajectory, we will consider two vector fields: $F_{\text{interior}}$, defined on the closure of the domain $\mathcal{R}$ (that is, the whole of the domain, including its bounding surface $\Sigma$) and a vector field $F_{\text{slide}}$ that is tangent to the hard boundary (see Definition 3.4). The motion along a trajectory is specified differently depending on the location of a point. For a point in the interior, denoted $\mathcal{R}_{\text{interior}}$, the dynamics is determined by $F_{\text{interior}}$. For a point on the boundary, the velocity obeys either $F_{\text{interior}}$ or else $F_{\text{slide}}$ that is tangent to $\Sigma$, depending on whether $F_{\text{interior}}$ is directed inwardly or outwardly at a given boundary point. This dual definition of the vector field has the effect that points driven into the boundary do not exit through the hard boundary, but rather slide along the boundary until the interior vector field allows them to reenter the domain.

**Definition 3.4.** The sliding region $(\mathcal{R}_{\text{slide}})$ is defined as the portion of a hard boundary $\Sigma$ for which

$$\mathcal{R}_{\text{slide}} = \{x \in \Sigma | n_x \cdot F_{\text{interior}}(x) > 0\},$$

where $n_x$ is a unit normal vector of $\Sigma$ at $x$ that points away from the interior domain and $F_{\text{interior}}(x)$ denotes the smooth vector field in $\mathcal{R}_{\text{interior}}$.

By construction, the sliding vector field $F_{\text{slide}}$ for flows confined to $\mathcal{R}_{\text{slide}}$ must have a vanishing normal component. While any vector field with vanishing normal component could be considered for $F_{\text{slide}}$, in this paper we adopt the natural choice of setting $F_{\text{slide}}$ to be the continuation of the interior vector field in the component tangential to $\mathcal{R}_{\text{slide}}$,

$$F_{\text{slide}}(x) = F_{\text{interior}}(x) - (n_x \cdot F_{\text{interior}}(x))n_x.\tag{3.31}$$

The vector field $F_{\text{slide}}$ chosen in this way lies tangent to the hard boundary $\Sigma$, and the flow exits the sliding region $\mathcal{R}_{\text{slide}}$ as the trajectory crosses the liftoff boundary $L$, defined as

$$L = \{x \in \Sigma | n_x \cdot F_{\text{interior}}(x) = 0\}.\tag{3.32}$$

Thus the liftoff boundary constitutes the edge of the sliding region of the hard boundary.

**Remark 3.5.** Our definition of the sliding region and sliding vector field is consistent with that in Bernardo et al. (2008) §5.2.2, except that the system of interest in this paper is only defined on one side of the sliding region. However, our main Theorem 3.7, below, holds in either case. Hence our results also apply to Filippov systems with sliding regions bordered by vector fields on either side, as in the example the stick-slip oscillator (Leine and Nijmeijer (2013) §6.5).

Using the preceding notation, in the neighborhood of a sliding region in the hard boundary $\Sigma$, a system with a limit cycle component confined to the sliding region takes the following form

$$\frac{dx}{dt} = F(x) := \begin{cases} F_{\text{interior}}(x), & x \in \mathcal{R}_{\text{interior}} \\ F_{\text{slide}}(x), & x \in \mathcal{R}_{\text{slide}} \subset \Sigma \end{cases}. \tag{3.33}$$

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Figure 3: Trajectory from the interior with vector field $F^{\text{interior}}$ making a transverse entry to a hard boundary $\Sigma$, followed by motion confined to the sliding region $R^{\text{slide}}$, then a smooth liftoff at $L$ back into the interior of the domain. Red dot: landing point (point at which the trajectory exits the interior and enters the hard boundary surface). Blue dot: liftoff point (point at which the trajectory crosses the liftoff boundary $L$ and reenters the interior). After a suitable change of coordinates, the geometry may be arranged as shown, with the hard boundary $\Sigma$ coinciding with one coordinate plane. Downward vertical arrow: $n$, the outward normal vector for $\Sigma$. The region $n \cdot F^{\text{interior}} > 0$ defines the sliding region within $\Sigma$; the condition $n \cdot F^{\text{interior}} = 0$ defines the liftoff boundary $L$.

where $F^{\text{slide}}$ and $R^{\text{slide}}$ are given by (3.31) and (3.30). Recall that $F^{\text{interior}}$ is smooth (cf. Definition 3.4).

**Definition 3.6.** In a general Filippov system which locally at a hard boundary $\Sigma$ has the form (3.33), we call a closed, isolated periodic orbit that passes through a sliding region a limit cycle with sliding component, denoted as LCSC.

To identify the liftoff point at which the LCSC reenters the interior of the domain, we require the nondegeneracy condition that the trajectory crosses the liftoff boundary $L$ at a finite velocity. Specifically, the outward normal component of the interior velocity should switch from positive (outward) to negative (inward) at the liftoff boundary, as one moves in the direction of the flow (see Fig. 3). That is,

$$\left[\nabla(n_x \cdot F^{\text{interior}}(x)) \cdot F^{\text{slide}}(x)\right]|_{x \in L} < 0.$$  \hfill (3.34)\]

Note that the liftoff condition (3.32) together with the nondegeneracy condition (3.34) uniquely defines a liftoff point for the LCSC. Note that at the liftoff point, we have $F^{\text{slide}} = F^{\text{interior}}$.

We assume that under an appropriate smooth change of coordinates, we may transform the hard boundary $\Sigma$ into a plane, so that $\Sigma$ has a constant normal vector $n$. 




























throughout $R_{\text{slide}}$ (cf. Fig. 3). Theorem 3.7 gathers together several conclusions about the variational and infinitesimal phase response curve dynamics of a LCSC local to a sliding boundary that follow from these definitions (see Appendix C for the proof):

**Theorem 3.7.** Consider a general LCSC described locally by (3.33) in the neighborhood of a hard boundary $\Sigma$ with a constant normal vector $n$, and with a liftoff point defined by (3.32). Assume that within the stable manifold of the limit cycle there is a well defined asymptotic phase function $\phi(x)$ satisfying $d\phi/dt = 1$ along trajectories. Assume that $\phi$ is Lipschitz continuous, and assume that on the constraint surface $\Sigma$, the directional derivatives of $\phi$ with respect to directions tangential to the surface are Lipschitz continuous, except (possibly) at the liftoff and landing points. Finally, assume the nondegeneracy condition (3.34) holds at the liftoff point. Then the following properties hold for the saltation matrix for $u$, and the jump matrix for $z$:

(a) At the landing point, the saltation matrix is $S = I - nn^\top$, where $I$ is the identity matrix.

(b) At the liftoff point, the saltation matrix is $S = I$.

(c) Along the sliding region, the component of $z$ normal to $\Sigma$ is zero.

(d) The normal component of $z$ is continuous at the landing point.

(e) The tangential components of $z$ are continuous at both landing and liftoff points.

We make the following additional observations about Theorem 3.7:

**Remark 3.8.**

- It follows from (a) in Theorem 3.7 that the component of $u$ normal to $\Sigma$ vanishes when the LCSC hits $\Sigma$. Once on the sliding region, the Jacobian used in the variational equation switches from $DF_{\text{interior}}$ to $DF_{\text{slide}}$ where $F_{\text{slide}}$ has zero normal component by construction (3.31). Hence, the normal component of $u$ is stationary over time, and remains zero on the sliding region.

- It follows from parts (d) and (e) in Theorem 3.7 that the jump matrix of a LCSC at the landing point is trivial (identity matrix).

- The assumption that the asymptotic phase function $\phi(x)$ is differentiable with respect to the directions forming a basis of the constraint surface is necessary for the proof of part (e). A stable limit cycle arising in a $C^r$-smooth vector field, for $r \geq 1$, will have $C^r$ isochrons (Wiggins, 1994; Josic et al., 2006). In (Park et al., 2018) and (Wilson, 2019) the authors assume differentiability of the phase function with respect to a basis of vectors spanning a switching surface. The assumption we require here is similarly plausible; it appears to hold at least for the model systems we have considered.

**Remark 3.9.** Theorem 3.7 excludes discontinuities in $z$ except at the liftoff point, and then only in its normal component. Since the normal component of $z$ along each sliding component of a LCSC is zero by Theorem 3.7, a discontinuous jump occurring at a liftoff point must be a nonzero instantaneous jump, which cannot be specified directly in terms of the value of $z$ prior to the jump. However, a time-reversed version of the jump matrix at the liftoff point, denoted as $J$, is well defined as follows:

$$z_{\text{lift}}^- = Jz_{\text{lift}}^+$$

(3.35)
where \( \mathbf{z}_{\text{lift}}^- \) and \( \mathbf{z}_{\text{lift}}^+ \) are the iPRC just before and just after the trajectory crosses the liftoff point in forwards time, and \( \mathcal{J} \) at the liftoff point has the same form as the saltation matrix \( S \) at the corresponding landing point

\[
\mathcal{J} = I - nn^\top. 
\]  

That is, the component of \( \mathbf{z} \) normal to \( \Sigma \) becomes \( 0 \) as the trajectory enters \( \Sigma \) in backwards time.

**Remark 3.10.** Combining Theorem 3.7, Remarks 3.8 and 3.9, we summarize the behavior of the solutions of the variational and adjoint equations \( \mathbf{u} \) and \( \mathbf{z} \) in limit cycles with sliding components:

|   | Landing | Sliding | Liftoff |
|---|---------|---------|---------|
| \( \mathbf{u} \) | \( S = I - nn^\top \) | \( \mathbf{u}^\perp = 0 \) | \( S = I \) |
| \( \mathbf{z} \) | \( \mathcal{J} = I \) | \( \mathbf{z}^\perp = 0 \) | \( \mathcal{J} = I - nn^\top \) |

where \( S \) is the regular saltation matrix and \( \mathcal{J} \) is the time-reversed jump matrix.

**Remark 3.11.** It follows directly from Remark 3.10 that the relation between the saltation and jump matrices at a transversal boundary crossing point \( J^TS = I \) (see (3.29)) is no longer true at a landing or a liftoff point. Instead, the following condition holds

\[
J_p^TS_p = I - nn^\top
\]

where \( J_p \) and \( S_p \) denote the time-reversed jump matrix and the regular saltation matrix at a landing or a liftoff point.

We illustrate the behavior of a limit cycle with sliding component via an analytically tractable planar model in §4. In this example, we will see that a nonzero instantaneous jump discussed in Remark 3.9 can occur in the normal component of \( \mathbf{z} \) at the liftoff point, reflecting a “kink” or nonsmooth feature in the isochrons (cf. Figure 5). In that example system, the discontinuity in the iPRC reflects a curve of nondifferentiability in the asymptotic phase function propagating backwards along a trajectory from the liftoff point to the interior of the domain (Figure 5A). The presence of a discontinuous jump from zero to a nonzero normal component in \( \mathbf{z} \) in forward time implies that numerical evaluation of the iPRC (presented in Appendix D) should be accomplished via backward integration along the limit cycle.

In addition to the iPRC, we also provide numerical algorithms for calculating the lTRC, the variational dynamics and the iSRC for a LCSC in a general nonsmooth system with hard boundaries, in Appendix D.

### 4 Shape and timing response in a planar limit cycle model with sliding components

In this section, we apply our methods to a two dimensional, analytically tractable model that has a single interior domain with purely linear flow and hard boundary
constraints that create a limit cycle with sliding components (LCSC). We find the surprising result that the isochrons exhibit a nonsmooth “kink” propagating into the interior of the domain from the locations of the liftoff points, i.e. the points where the limit cycle smoothly departs the boundary. In addition, we show that using local timing response curve analysis gives significantly greater accuracy of the shape response than using a single, global, phase response curve.

MATLAB source code for simulating the model and reproducing the figures is available: https://github.com/yangyang-wang/LC_in_square.

In the interior of the domain $[-1,1] \times [-1,1]$, we take the vector field of a simple spiral source to define the interior dynamics of the planar model

$$\frac{dx}{dt} = F(x) = \begin{bmatrix} \alpha x - y \\ x + \alpha y \end{bmatrix}$$

(4.37)

where $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\alpha$ is the expansion rate of the source at the origin. The rotation rate is fixed at a constant value 1. The Jacobian matrix $DF$ evaluated along the limit cycle solution in the interior of the domain is

$$DF = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}$$

(4.38)

In what follows we will require $0 < \alpha < 1$, so we have a weakly expanding source. For illustration, $\alpha = 0.2$ provides a convenient value. Every trajectory starting from the interior, except the origin, will eventually collide with one of the walls at $x = \pm 1$ or $y = \pm 1$ (in time not exceeding $\frac{1}{2\alpha} \ln(2/(x(0)^2 + y(0)^2))$). As in §3.2, we set the sliding vector field when the trajectory is traveling along the wall to be equal to the continuation of the interior vector field in the component parallel to the wall, while the normal component is set to zero (except where it is oriented into the domain interior).

The resulting vector fields of the planar LCSC model $F(x)$ on the interior and along the walls are given in Table 1, and illustrated in Fig. 4B.

| $x$ range | $y$ range | $dx/dt$ | $dy/dt$ |
|-----------|-----------|---------|---------|
| $|x| < 1$  | $|y| < 1$  | $\alpha x - y$ | $x + \alpha y$ |
| $x = 1$  | $-1 \leq y < \alpha$ | $0$ | $1 + \alpha y$ |
| $x = 1$  | $\alpha \leq y < 1$ | $\alpha - y$ | $1 + \alpha y$ |
| $y = 1$  | $1 \geq x > -\alpha$ | $\alpha x - 1$ | $0$ |
| $y = 1$  | $-\alpha \geq x > -1$ | $\alpha x - 1$ | $x + \alpha$ |
| $x = -1$ | $1 \geq y > -\alpha$ | $0$ | $-1 + \alpha y$ |
| $x = -1$ | $-\alpha \geq y > -1$ | $-\alpha - y$ | $-1 + \alpha y$ |
| $y = -1$ | $-1 \leq x < \alpha$ | $\alpha x + 1$ | $0$ |
| $y = -1$ | $\alpha \leq x < 1$ | $\alpha x + 1$ | $x - \alpha$ |

Table 1: Vector field of the planar LCSC model on the interior and along the boundaries.
Figure 4: Simulation result of the planar LCSC model with parameter $\alpha = 0.2$. (A): Time series of the limit cycle $\gamma(t)$ generated by the planar LCSC model over one cycle with initial condition $\gamma(0) = [1, \alpha]^T$. (B): Projection of $\gamma(t)$ onto $(x, y)$ phase space (solid black) and the osculating trajectory (dashed black), starting near the center that ends up running into the wall at the liftoff point $[1, \alpha]^T$ (black star). Red arrows represent the vector field.

i.e., $(F_{\text{interior}} \cdot n_{\text{wall}})|_{\text{wall}} = 0$ (see (3.32)). For instance, on the wall $x = 1$ with a normal vector $n = [1, 0]^T$, we compute

$$F_{\text{interior}}|_{\text{wall}} \cdot n_{\text{wall}} = (\alpha x - y)|_{\text{wall}} = \alpha - y = 0.$$ 

It follows that $y = \alpha$ defines the liftoff condition on the wall $x = 1$. For this planar model, there are four lift-off points with coordinates $(1, \alpha), (-\alpha, 1), (-1, -\alpha), (\alpha, -1)$ on the walls $x = 1, y = 1, x = -1, y = -1$, respectively.

Denote the LCSC produced by the planar model by $\gamma(t)$, whose time series over $[0, T_0]$ is shown in Figure 4A, where $T_0$ is the period. The projection of $\gamma(t)$ onto the $(x, y)$-plane is shown in the right panel, together with an osculating trajectory that starts near the center and ends up running into the wall $x = 1$ at the lift-off point $(1, \alpha)$ (black star).

Next we implement algorithms given in Appendix D to find the timing and shape responses of the LCSC to both instantaneous perturbations and sustained perturbation. We start with finding the iPRC for the LCSC to understand the timing response, and then solve the variational equation to find the linear shape response of the planar LCSC model to an instantaneous perturbation. Lastly, we compute the iSRC when the applied sustained perturbations are both uniform and nonuniform, to understand the shape response of the planar LCSC model to sustained perturbations.

4.1 Infinitesimal phase response analysis

In the case of weak coupling or small perturbations of a strongly stable limit cycle, a linearized analysis of the phase response curve – the iPRC – suffices to predict the
behavior of the perturbed system. When trajectories slide along a hard boundary, however, the linearized analysis breaks down. For nonsmooth systems such as the LCSCs, the asymptotic phase function $\phi(x)$ may itself be nonsmooth at certain locations, even when it remains well defined; its gradient (i.e., the iPRC) may therefore be discontinuous at those locations. Nevertheless, one may be able to derive a consistent first order approximation to the phase response curve notwithstanding that the directional derivative (2.4) may not be well defined, as discussed in §3.

The dynamics of the planar LCSC model are smooth except for the discontinuities when crossing the switching boundaries, that is, entering or exiting the walls. The iPRC, $z(t)$, will be continuous in the interior domain as well as in the interior of the four boundaries. As discussed in §3.2, the discontinuity of iPRCs only occurs at the liftoff point. By Remark 3.10, the time-reversed jump matrix at a liftoff point, which takes the iPRC just after crossing the liftoff point to the iPRC just before crossing the liftoff point in backwards time, is given by

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

when the trajectory leaves the walls $x = \pm 1$, and is given by

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

when the trajectory leaves the walls $y = \pm 1$.

Figure 5A shows the limit cycle (solid black curve), the osculating trajectory (dashed black curve) corresponding to the liftoff point (black star, $\theta = 0$) and the isochrons computed from a direct method, starting from a grid of initial conditions and tracking the phase of final locations (colored scalloped curves). There appears to be a “kink” in the isochron function, propagating backwards in time along the trajectories that encounter the boundaries exactly at the liftoff points, such as the dashed curve. This apparent discontinuity in the gradient of the isochron function in the interior of the domain exactly corresponds, at the boundary, with the point of discontinuity occurring in the iPRC along the limit cycle (cf. Remark 3.9). According to Figure 5, the isochron curves are perpendicular to the sliding region of the wall at which the interior vector field is pointing outward. That is, the normal component of the iPRC when the trajectory slides along a wall is equal to 0. There is no jump in $z$ when the trajectory enters the wall, but instead a discontinuous jump from zero to nonzero occurs in the normal component of $z$ at the liftoff point. All of these observations are consistent with iPRCs $z$ (Figure 5, right) that are computed using Algorithm for $z$ in §D.1 based on Theorem 3.7.

After the trajectory lifts off the east wall ($x = 1$) at the point marked $\theta = 0$ in Figure 5A (black star), a perturbation along the positive $x$-direction (resp., positive $y$-direction) causes a phase delay (resp., advance). While the timing sensitivity of the LCSC to small perturbations in the $x$-direction reaches a local maximum before reaching the next wall $y = 1$, the phase advance caused by the $y$-direction perturbation decreases continuously to 0 as the trajectory approaches $y = 1$. As the trajectory is
Figure 5: iPRCs for the planar LCSC model with parameter $\alpha = 0.2$. (A): Trajectories and isochrons for the LCSC model. The solid black and dashed black curves are the same as in Figure 4B. The colored scalloped curves are isochrons of the LCSC $\gamma(t)$ (black solid) corresponding to 50 evenly distributed phases $nT_0/50$, $n = 1, \ldots, 50$. We define the phase at the liftoff point (black star) to be zero. (B): iPRCs for the planar LCSC model. The blue and red curves represent the iPRC for perturbations in the positive $x$ and $y$ directions, respectively. The intervals during which $\gamma(t)$ slides along a wall are indicated by the shaded regions. While $y = +1$, the iPRC vector is parallel to the wall ($z_y \equiv 0$) and oriented opposite to the direction of flow ($z_x < 0$). Similarly, on the remaining walls, the iPRC vector has zero normal component relative to the active constraint wall, and parallel component opposite the direction of motion.

sliding along the wall ($y = 1$), the positive $y$-direction perturbation that is normal to the wall has no effect on the LCSC and hence will not affect its phase. Moreover, we showed in Theorem 3.7 that a perturbation in the negative $y$-direction also has no effect on the phase, since the perturbed trajectory returns to the wall within time $O(\varepsilon)$, with a net phase offset that is at most $O(\varepsilon^2)$, where $\varepsilon$ is the size of the perturbation. As the trajectory lifts off the wall, there is a discontinuous jump in $z_y$, so that a negative $y$-direction perturbation applied immediately after the liftoff point leads to a phase advance. On the other hand, on the sliding region of the wall $y = 1$, a perturbation along the positive $x$-direction, against the direction of the flow, results in a phase delay, which decreases in size as the phase increases, and becomes 0 upon reaching the next wall, $x = -1$. The timing sensitivity of the LCSC to perturbations applied afterwards are similar to what are observed in the first quarter of the period due to the $\mathbb{Z}_4$-symmetry $\sigma(x, y) = (-y, x)$.

The linear change in the oscillation period of the LCSC in response to a static perturbation can then be estimated by taking the integral of the iPRC multiplying the given perturbation, as shown in the last step in Algorithm for $z$ (§ D.1). As noted before, the change in period will be needed to solve (2.17) for the iSRC to understand how this perturbation affects the shape of the LCSC.
In this example, the interior vector field (4.37) is linear. Therefore its Jacobian is constant, and the iPRC may be obtained analytically (Park et al., 2018). The resulting curves are indistinguishable from the numerically calculated curves shown in Fig. 5B.

4.2 Variational analysis

Suppose a small instantaneous perturbation, applied at time $t = 0$, leads to an initial displacement $u(0) = \gamma(0) - \gamma(0)$, where $\gamma(0) = [1, \alpha]$ is the liftoff point (black star in Figs. 4B and 5A) as in the previous section. We use the variational analysis to study how this perturbation evolves over time.

Similar to the iPRC, $u(t)$ will be continuous everywhere in the domain except when entering or exiting the walls. In contrast to the iPRC, $u$ is continuous at all liftoff points, but exhibits discontinuous saltations when the trajectory enters a wall. According to Theorem 3.7, the saltation matrix $S$, which takes $u$ just before entering a wall to $u$ just after entering the wall in forwards time, for the planar LCSC model is given by

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

when the trajectory enters the walls $x = \pm 1$, and is given by

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

when the trajectory enters the walls $y = \pm 1$.

Solutions to the variational equation of the planar LCSC model with the given initial condition $u(0)$ can be computed using Algorithm for $u$ in §D.3. As discussed in Remark D.3, an alternative way to find the displacement $u(t)$ is to compute the fundamental solution matrix $\Phi(t, 0)$ by running Algorithm for $u$ twice and then to evaluate $u(t) = \Phi(t, 0)u(0)$. The advantage of the latter approach is that once $\Phi(t, 0)$ is obtained, it can be used to compute $u(t)$ with any given initial value by evaluating a matrix multiplication instead of solving the variational equation.

Here, by taking $[1, 0]$ and $[0, 1]$ as the initial conditions for $u$ at the liftoff point A, we apply Algorithm for $u$ to compute the time evolution of the two columns for the fundamental matrix $\Phi(t, 0)$. A simple calculation shows that the monodromy matrix $\Phi(T_0, 0)$ has an eigenvalue +1, whose eigenvector $[0, 1]$ is tangent to the limit cycle at the liftoff point, as expected (Remark D.4). It follows that if the initial displacement at the liftoff point is along the limit cycle direction, then the displacement after a full period becomes the same as the initial one. To see this, we take the initial displacement $u(0) = [0, \varepsilon]$ where $\varepsilon = 0.1$ to be the tangent vector of the limit cycle at the liftoff point, and compute $u(t)$, $x$ and $y$ components of which are shown in black solid curves in Figure 6B,D. The saltations in $u$ at time when the trajectory hits the walls can be clearly distinguished in the plot. Moreover, $u(T_0) = u(0)$ as we expect.

To further validate the accuracy of $u$, we solve and plot $\gamma(t)$ with $\gamma(0) = [1, \alpha]$ (black, Figure 6A,C) and the perturbed trajectory $\tilde{\gamma}(t)$ with $\tilde{\gamma}(0) = [1, \alpha] + u(0)$ (red dotted, Figure 6A,C). The differences between the two trajectories along the $x$-direction
Figure 6: Linear shape response $u(t)$ of the LCSC trajectory $\gamma(t)$ to an instantaneous perturbation applied at $\gamma(0) = [1, \alpha]$ where $\alpha = 0.2$. The initial displacement is $u(0) = [0, \varepsilon]$ where $\varepsilon = 0.1$. (A, C) Time series of $\gamma(t)$ (black solid) and $\tilde{\gamma}(t)$ (red dotted) with a perturbed initial condition $\tilde{\gamma}(0) = [1, \alpha + \varepsilon]$. (B, D) The difference between $\tilde{\gamma}(t)$ and $\gamma(t)$ obtained by direct calculation from the left panels (black) and the displacement solution $u(t)$ obtained using the Algorithm for $u$ (red dotted). (A) and (B) show trajectories and the displacement along the $x$-direction, while (C) and (D) show trajectories and displacements along the $y$-direction. Shaded regions have the same meanings as in Figure 5.

and the $y$-direction are indicated by the black lines in Figure 6B and D, both showing good agreements with the approximated displacements computed from the variational equation, indicated by the red dotted lines in Figure 6B and D. Such an approximation becomes better as the perturbation size $\varepsilon$ gets smaller (simulation result not shown).

Next, we study the effects of static perturbations on the timing and shape using the iPRC and iSRC.

### 4.3 Shape response analysis

In this section, we illustrate how to compute the iSRC $\gamma_1$, the linear shape responses of the LCSC to small static perturbations. Recall that we use $\gamma_0(t)$ with period $T_0$ and $\gamma_\varepsilon(t)$ with period $T_\varepsilon$ to denote the original and the perturbed LCSC solutions. We write $\gamma_1$ for the linear shift in the limit cycle shape in response to the static perturbation as
indicated by (2.13), which we also repeat here:

\[ \gamma_\varepsilon(\tau(t)) = \gamma_0(t) + \varepsilon \gamma_1(t) + O(\varepsilon^2), \]

where the time for the perturbed LCSC is rescaled to be \( \tau(t) \) to match the unper- turbed time points. The iSRC \( \gamma_1 \) satisfies the nonhomogeneous variational equation (2.17). To solve this equation, an estimation of the timing scaling factor \( \nu_1 \), determined by the choice of time rescaling \( \tau(t) \), is needed. Here we consider two kinds of static perturbations on the planar LCSC model: global perturbation and piecewise perturbation.

**Global perturbation.** We apply a small static perturbation to the planar LCSC model by increasing the model parameter \( \alpha \) by \( \varepsilon \) globally: \( \alpha \to \alpha + \varepsilon \). To compare the LCSCs before and after perturbation at corresponding time points, we rescale the perturbed trajectory uniformly in time so that \( \tau(t) = T_1 t/T_0 \). It follows that \( \nu_1 = T_1 / T_0 \), where the linear shift \( T_1 := \lim_{\varepsilon \to 0} (T_\varepsilon - T_0)/\varepsilon \) can be estimated using the iPRC as discussed in §2.2 (see (2.12)).

Using **Algorithm for \( \gamma_1 \) with uniform rescaling**, we numerically compute the iSRC \( \gamma_1(t) \) for \( \varepsilon = 0.01 \). The \( x \) and \( y \) components of \( \varepsilon \gamma_1(t) \) are shown by the red curves in Figure 7A, both of which show good agreement with the numerical displacement \( \gamma_\varepsilon(\tau(t)) - \gamma_0(t) \) (black solid), as expected from our theory.

For \( \varepsilon \) over a range \([0, 0.01]\), we repeat the above procedure and compute the Euclidean norms of both the numerical displacement vector \( \gamma_\varepsilon(\tau(t)) - \gamma(t) \) (Figure 7B, black solid) and the approximated displacement vector \( \varepsilon \gamma_1(t) \) (Figure 7B, red dotted) over one cycle. From the plot, we can see that the iSRC with uniform rescaling of time gives a good first-order \( \varepsilon \) approximation to the shape response of the planar LCSC model to a global static perturbation.

**Piecewise perturbation.** Uniform rescaling of time as used above is the simplest choice among many possible rescalings, and is shown to be adequate in the global perturbation case for computing an accurate iSRC. As discussed in §2, in certain cases we may instead need the technique of local timing response curves (lTRCs) to obtain nonuniform choices of rescaling for greater accuracy.

As an illustration, we add two local timing surfaces \( \Sigma^\text{in} \) and \( \Sigma^\text{out} \) to the planar LCSC model (see Figure 8A). We denote the subdomain above \( \Sigma^\text{in} \) and \( \Sigma^\text{out} \) by region I \((R^I)\) and denote the remaining subdomain by region II \((R^II)\). Moreover, we introduce a new parameter \( \omega \), the rotation rate of the source at the origin, that has previously been fixed at 1, and rewrite the interior dynamics of the planar LCSC model as

\[ \frac{dx}{dt} = F(x) = \begin{bmatrix} \alpha x - \omega y \\ \omega x + \alpha y \end{bmatrix}. \]  

(4.39)

The vector fields on a given wall are obtained by replacing the coefficient of \( y \) in \( dx/dt \) (in Table 1) by \( -\omega \) and replacing the coefficient of \( x \) in \( dy/dt \) (in Table 1) by \( \omega \) on that wall.
Figure 7: iSRC of the LCSC model to a small perturbation $\alpha \rightarrow \alpha + \varepsilon$ with unperturbed parameter $\alpha = 0.2$. (A) Time series of the difference between the perturbed and unperturbed solutions along the $x$-direction (top panel) and the $y$-direction (bottom panel) with $\varepsilon = 0.01$. The black curve denotes the numerical displacement computed by subtracting the unperturbed solution trajectory from the perturbed trajectory, after globally rescaling time. The red dashed curve denotes the product of $\varepsilon$ and the shape response curve solution. Shaded regions have the same meanings as in Figure 5. (B) The norm of the numerical difference (black) and the product of $\varepsilon$ and the iSRC (red dashed) grow linearly with respect to $\varepsilon$ with nearly identical slope, indicating that the iSRC is very good for approximating the numerical difference over a range of $\varepsilon$ and improves with smaller $\varepsilon$.

We apply a static piecewise perturbation to the system by letting $(\alpha, \omega) \rightarrow (\alpha + \varepsilon, \omega - \varepsilon)$ over region I but not region II. Such a piecewise constant perturbation affects both the expansion and rotation rates of the source in region I, and hence will lead to different timing sensitivities of $\gamma(t)$ in the two regions. It is therefore natural to use piecewise uniform rescaling when computing the shape response curve as opposed to using a uniform rescaling. In the following, we first compute the lTRC (see Figure 8) and use it to estimate the two time rescaling factors for $R^I$ and $R^{II}$, which are denoted by $\nu^I_1$ and $\nu^{II}_1$, respectively. We then show the iSRC computed using the piecewise uniform rescaling factors provides a more accurate representation of the shape response to the piecewise static perturbation than using a uniform rescaling (see Figure 9 and 10).

Although the lTRC $\eta$ is defined throughout the domain, estimating the effect of the perturbation localized to region I only requires evaluating the lTRC in this region. Figure 8B shows the time series of $\eta^I$ for the planar LCSC model in region I, obtained by numerically integrating the adjoint equation (2.21) backward in time with the initial condition of $\eta^I$ given by its value at the exit point of region I denoted by $x^{out}$ (see Algorithm for $\eta^I$).

Similar to the iPRC, the $y$ component of the lTRC $\eta^I$ shown by the red curve in
Figure 8: ITRC of the planar LCSC model under perturbation \((\alpha, \omega) \rightarrow (\alpha + \varepsilon, \omega - \varepsilon)\) over region I with unperturbed parameters \(\alpha = 0.2\) and \(\omega = 1\) held fixed in region II. (A) Projection of the limit cycle solution to the planar model with two new added switching surfaces \(\Sigma^{in}\) (green dashed line) and \(\Sigma^{out}\) (blue dashed line) onto its phase plane. (B) Time series of the ITRC \(\eta^I\) from \(t^{in}\) (the time of entry into region I at \(x^{in}\)) to \(t^{out}\) (the time of exiting region I at \(x^{out}\)). A discontinuous jump occurs when the trajectory exits the wall \(y = 1\) indicated by the right boundary of the shaded region, which has the same meaning as in Figure 5.

Figure 8B is zero along the wall \(y = 1\), and the only discontinuous jump of \(\eta^I\) occurs at the liftoff point. Note that \(\eta^I\) is defined as the gradient of the time remaining in \(R^I\) until exiting through \(\Sigma^{out}\). If the \(x\) or \(y\) component of \(\eta^I\) is positive then the perturbation along the positive \(x\)-direction or \(y\)-direction increases the time remaining in \(R^I\), and the exit from \(R^I\) will occur later. On the other hand, if the \(x\) or \(y\) component \(\eta^I\) is negative then the perturbation along the positive \(x\)-direction or \(y\)-direction decreases the time remaining in \(R^I\), and the exit from \(R^I\) will occur sooner. The relative shift in time spent in \(R^I\) caused by a static perturbation can therefore be estimated using the ITRC (see (2.20)) as illustrated in the last step of Algorithm for \(\eta^I\). Note that the first term in (2.20) implies that the timing change in a region generically depends on the shape change at the corresponding entry point, leading to the possibility of bidirectional coupling between timing and shape changes. However, in this planar system, a perturbed trajectory with \(\varepsilon \ll 1\) will converge back to the original trajectory, within region II (where the perturbation is absent), in finite time. Under these circumstances, there is no shift between the perturbed and unperturbed trajectories in the entry location to region I. Hence, in this case, the local timing shift does not depend on the shape change.

Let \(T^I_0\) denote the time spent in region I, and let \(T^{II}_0 = T_0 - T^I_0\) denote the time spent in region II (recall \(T_0\) is the total period). The linear shift in \(T^I_0\), denoted by \(T^I_1\), can be estimated using the ITRC \(\eta^I\) as discussed above. By definition the two time rescaling
factors required to compute the iSRC are given by \( \nu^I_1 = \frac{T^I_1}{T^I_0} \) and \( \nu^{II}_1 = \frac{T_1 - T^I_1}{T^{II}_0} \), where the global relative change in period, \( T_1 \), can be estimated using the iPRC as discussed before. With \( \nu^I_1 \) and \( \nu^{II}_1 \) known, we take \( x^m \), the coordinate of the entry point into \( R^I \), as the initial condition for \( \gamma(t) \) and apply Algorithm for \( \gamma_1 \) with piecewise uniform rescaling to compute the iSRC \( \gamma_1 \) for \( \epsilon = 0.1 \). The \( x \) and \( y \) components of \( \epsilon \gamma_1 \) are shown by the red dashed curves in Figure 9B, both of which show good agreement with the numerical displacement \( \gamma_\epsilon(\tau(t)) - \gamma(t) \) (black solid curves). Here the rescaling \( \tau(t) \) is piecewise uniform:

\[
\tau(t) = \begin{cases} 
   t^I + T^I_\epsilon (t - t^I)/T^I_0, & \gamma(t) \in R^I \\
   t^I + T^I_\epsilon + T^{II}_\epsilon (t - t^{out})/T^{II}_0, & \gamma(t) \in R^{II}
\end{cases} \tag{4.40}
\]

where \( T^I_\epsilon \) denotes the time \( \gamma_\epsilon \) spends in \( R^i \) with \( i \in \{I, II\} \). It follows that the exit time of the trajectory from region I before (Figure 9B, vertical blue line) and after (Figure 9B, vertical magenta line) perturbation are the same.

As a comparison, for \( \epsilon = 0.1 \), we also compute the iSRC and the numerical displacement using the uniform rescaling of time as we did in the global perturbation case (see Figure 9A). The difference between the vertical blue and magenta lines (the time when the unperturbed and perturbed trajectories leave region I) indicates region I and region II have different timing sensitivities. As expected, the resulting \( \epsilon \gamma_1 \) no longer shows good agreement with the numerical displacement obtained from subtracting the unperturbed solution from the rescaled perturbed solution.

Piecewise uniform rescaling, on the other hand, leads to a more accurate iSRC for the LCSC model (4.39) than uniform rescaling, when the LCSC \( \gamma(t) \) experiences distinct timing sensitivities for \( \epsilon = 0.1 \). Fig. 9 contrasts the accuracy of the linearized shape response using global (A) versus local (B) timing response curves, for \( \epsilon = 0.1 \). We also show the same conclusion holds for other \( \epsilon \) values. To this end, for \( \epsilon \) over a range of \([0, 0.1]\) we repeat the above procedure and compute the Euclidean norms of both the numerical displacement vector and the displacement vector approximated by the iSRC, as illustrated in Figure 10A. The numerical and approximated norm curves using the uniform rescaling are shown in red solid and red dotted lines, while the numerical and approximated norm curves using the piecewise uniform rescaling are shown in blue solid and blue dotted lines. Unsurprisingly, the norms of the displacements between the perturbed and original trajectory grow approximately linearly with respect to \( \epsilon \), and the displacement norms with piecewise uniform rescaling are smaller than that with uniform rescaling. The fact that the difference between the lines in red is much bigger than the difference between the lines in blue suggests that the piecewise uniform rescaling gives a more accurate iSRC than using the uniform rescaling for \( \epsilon \in [0, 0.1] \), as we expect. This heightened accuracy is further demonstrated in Figure 10B, where the relative difference between the numerical and approximated norms with uniform rescaling (red curve) is significantly larger than the relative difference when using piecewise uniform rescaling (blue curve).
Figure 9: A small perturbation is applied to the planar model over region I in which \((α, ω) \rightarrow (α + ε, ω - ε)\) with unperturbed parameters \(α = 0.2, ω = 1\) and perturbation \(ε = 0.1\). Time series of the difference between the perturbed and unperturbed solutions along the \(x\)-direction (top panel) and the \(y\)-direction (lower panel) using (A) the global rescaling factor and (B) two different rescaling factors within regions I and II. The vertical blue dashed line denotes the exit time of the unperturbed trajectory from Region I, while the vertical magenta solid line denotes the exit time of the rescaled perturbed trajectory from Region I. Other color codings of lines are the same as in Figure 7A. Shaded regions have the same meanings as in Figure 5.

5 Discussion

Rhythmic motions making and breaking contact with a constraining boundary, and subject to external perturbations, arise in motor control systems such as walking, running, scratching, biting and swallowing, as well as other natural and engineered hybrid systems (Branicky, 1998). Dynamical systems describing such rhythmic motions are therefore nonsmooth and often exhibit limit cycle trajectories with sliding components. In smooth dynamical systems, classical analysis for understanding the change in periodic limit cycle orbits under weak perturbation relies on the Jacobian linearization of the flow near the limit cycle. These methods do not apply to nonsmooth systems, for which the Jacobian matrices are not well defined. In this work, we describe for the first time the variational analysis and the infinitesimal phase response curves (iPRC) for limit cycles with sliding components (LCSC). Moreover, we give a rigorous derivation of the saltation matrix associated with the variational dynamics and the closely related jump matrix for the iPRC at the hard boundary crossing point. We also report, for the first time, how the presence of a liftoff point, where a limit cycle leaves a constraint surface, can create a nondifferentiable “kink” in the asymptotic phase function, propagating backwards in time along an osculating trajectory (see Figure 5A). Most significantly, we have defined the infinitesimal shape response curve (iSRC) to analyze the joint variation of both shape and timing of limit cycles with sliding compo-
Figure 10: A small perturbation is applied to the planar model over region I in which \((\alpha, \omega) \rightarrow (\alpha + \varepsilon, \omega - \varepsilon)\) with unperturbed parameters \(\alpha = 0.2, \omega = 1\). (A): Values of the Euclidean norm of \((\gamma_{\varepsilon}(\tau(t)) - \gamma(t))\) computed numerically (solid curve) versus those computed from the iSRC (dashed curve), as \(\varepsilon\) varies. The norms grow approximately linearly with respect to \(\varepsilon\). The approximation obtained by the iSRC when using piecewise uniform rescaling (blue) is closer to the actual simulation than using the uniform rescaling (red). (B): The relative difference between the actual and approximated norms with a uniform rescaling (red) is larger than that when piecewise uniform rescaling is used (blue). The difference between the two curves expands as \(\varepsilon\) increases.

We show that taking into account local timing sensitivity within a switching region improves the accuracy of the iSRC over global timing analysis alone. This improvement in accuracy is facilitated by our introduction of a novel local timing response curve (lTRC) measuring the timing sensitivity of an oscillator within a given local region.

Our results clarify an important distinction between the effects of the boundary encounter on the timing and shape changes in limit cycles with sliding components. We have extended both iPRC and variational analysis developed for smooth limit cycle systems to the LCSC case, presented here as Theorem 3.7. In addition, our analysis yields an explicit expression for the iPRC jump matrix that characterizes the behavior of the iPRC at the landing and liftoff points. Surprisingly, we find that the iPRC experiences no discontinuity when the trajectory first contacts a hard boundary, while the variational equation suffers a discontinuity, captured by the saltation matrix. Even more interesting, at the liftoff point – where the saltation matrix for the variational problem is trivial – the iPRC does show a discontinuous change, captured by a non-trivial jump matrix. Specifically, there is a discontinuous jump from zero to a nonzero normal component in the iPRC. Consequently, numerical evaluation of the iPRC must be obtained by backward integration along the limit cycle, as discussed in §D.1. Finally, we find that both the iPRC and the variational dynamics have zero normal components
during the sliding component of the limit cycle, due to dimensional compression at the hard boundary.

Standard variational and phase response curve analysis typically neglect changes in timing or shape, focusing instead on only one of the two aspects (Kuramoto, 1975). However, in many applications such as motor control systems, both the shape and timing of the trajectory are often affected under slow or parametric perturbations. In this paper, we consider both timing and shape aspects using the iSRC. We have discussed two ways of incorporating timing changes into the iSRC: uniform timing rescaling based on the global timing analysis (iPRC) and piecewise uniform timing rescaling based on the local timing analysis (lTRC). As demonstrated in the planar system example in §4, when the trajectory exhibits approximately constant timing sensitivities, the iSRC with global timing rescaling is good enough for approximating the shape change (see Figure 7); otherwise, we need take into account local timing changes to increase the accuracy of the iSRC (see Figure 9). LCSC with piecewise timing sensitivities naturally arise in many motor control systems due to nonuniform perturbations. For instance, the friction of the ground acts as a perturbation during the stance phase of locomotion, when a leg generates ground reaction forces, and is absent during the swing phase; a force applied to the food can only be felt when an animal is biting on the food. Local timing analysis (lTRC) will then provide a better understanding of such systems compared with the global timing analysis (iPRC).

Other investigators have also considered variational (Bernardo et al., 2008; Leine and Nijmeijer, 2013) and phase response analysis in nonsmooth systems (Shirasaka et al., 2017; Park et al., 2018; Chartrand et al., 2018; Wilson, 2019), but these studies were subject to transverse flow conditions. Our work extends both variational and iPRC analysis to the LCSC case in which the transversal crossing condition fails. Combined timing and shape responses of limit cycles to perturbations have also been explored in other works. Monga and Moehlis (2018) examined energy-optimal control of the timing of limit cycle systems including spiking neuron models and models of cardiac arrhythmia. They showed that when one of the nontrivial Floquet multipliers of an unperturbed limit cycle system has magnitude close to unity, control inputs based solely on standard phase reduction, which neglects the effect on the shape of the controlled trajectory, can dramatically fail to achieve control objectives. They and other authors have introduced augmented phase reduction techniques that use a system of coordinates (related to the Floquet coordinates) transverse to the limit cycle to improve the accuracy of phase reduction and control (Castejon et al., 2013; Wilson and Moehlis, 2015, 2016; Wilson and Ermentrout, 2018; Monga et al., 2018; Wilson, 2019). These methods require the underlying dynamics be smoothly differentiable, and rely on calculation of the Jacobian (first derivative) and in some cases the Hessian (second derivative) matrices (Wilson and Ermentrout, 2018). For nonsmooth limit cycle systems with sliding components, our analysis is the first to address the combined effects of shape and timing, an essential element of improved control in biomedical applications as well as for understanding mechanisms of control in naturally occurring motor control systems.

For trajectories with different timing sensitivities in different regions, we rely on the local timing response curve (lTRC) to estimate the relative shift in time in each
sub-region, in order to compute the full infinitesimal shape response curve (iSRC). Conversely, solving for the ITRC in a given region may also require an understanding of the impact of the perturbation on the entry point associated with that region (see (2.20)). Thus, in general, the iSRC and the ITRC are interdependent. While we have not derived a closed-form expression for the shape and timing response in the most general case, we have provided effective algorithms for solving each of them separately, which requires preliminary numerical work to find the trajectory shape shift at the entry point. In the future, it may be possible to derive general closed-form expressions for the iSRC and ITRC in systems with distinct timing sensitivities.

While our methods are illustrated using a planar limit cycle system with hard boundaries, they apply to higher dimensional systems as well.

For instance, preliminary investigations suggest that the methods developed in this paper are applicable to analyzing the nonsmooth dynamics arising in the control system of feeding movements in the sea slug *Aplysia* (Shaw et al., 2012, 2015; Lyttle et al., 2017). More generally, limit cycles with discontinuous trajectories arise in neuroscience (e.g., integrate and fire neurons) and mechanics (e.g., ricochet dynamics). If such systems manifest limit cycles with sliding components, our methods could be combined with variational methods adapted for piecewise continuous trajectories (Coombes et al., 2012; Shirasaka et al., 2017).

It was observed heuristically by Lyttle et al. (2017) that sensory feedback could in some circumstances lead to significant robustness against an increase in applied load, in the sense that although modest relative increases in external load (c. 20%) led to comparable changes in both the timing and shape of trajectories, the net effect on the performance (rate of intake of food) was an order of magnitude smaller (c. 1%). Similarly, Diekman et al. (2017) showed that in a model for control of a central pattern generator regulating the breathing rhythm, mean arterial partial pressure of oxygen (PPO$_2$) remained approximately constant under changing metabolic loads when chemosensory feedback from the arterial PPO$_2$ to the central pattern generator was present, but varied widely otherwise (Diekman et al., 2017). Understanding how rhythmic biological control systems respond to such perturbations and maintain robust, adaptive performance is one of the fundamental problems within theoretical biology. Solving these problems will then require variational analysis along the lines we develop here. Nonsmooth dynamics arise naturally in many biological systems (Aihara and Suzuki, 2010; Coombes et al., 2012), and thus, the approach in this paper is likely to have broad applicability to many other problems in biology.

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### A Table of Common Symbols

| Symbol | Meaning |
|--------|---------|
| \( x \) | state variables |
| \( t \) | time |
| \( \theta(t) \) | phase of a limit cycle |
| \( \phi(x) \) | asymptotic phase of a stable limit cycle |
| \( F(x) \) | unperturbed velocity vector field |
| \( \gamma(t) \) | unperturbed limit cycle solution |
| \( T \) | period of the unperturbed limit cycle |
| \( \varepsilon P \) | small instantaneous perturbation vector |
| \( \tilde{\gamma}(t) \) | trajectory near limit cycle after instantaneous perturbation |
| \( \mathbf{u}(t) \approx \tilde{\gamma}(t) - \gamma(t) \) | displacement from limit cycle after instantaneous perturbation |
| \( \varepsilon \) | sustained (parametric) perturbation |
| \( F_\varepsilon(x) \) | perturbed velocity vector field |
| \( \gamma_\varepsilon(t) \) | perturbed limit cycle solution |
| \( T_\varepsilon \) | period of the perturbed limit cycle |
| \( F_0 = F \) | zeroth-order term of Taylor expansion of \( F_\varepsilon \) around \( \varepsilon = 0 \) |
| \( \gamma_0 = \gamma \) | zeroth-order term of Taylor expansion of \( \gamma_\varepsilon \) around \( \varepsilon = 0 \) |
| \( T_0 = T \) | zeroth-order term of Taylor expansion of \( T_\varepsilon \) around \( \varepsilon = 0 \) |
| \( F_1 = \partial F_\varepsilon / \partial \varepsilon \bigg|_{\varepsilon=0} \) | first-order term of Taylor expansion of \( F_\varepsilon \) around \( \varepsilon = 0 \) |
| \( \gamma_1 = \partial \gamma_\varepsilon / \partial \varepsilon \bigg|_{\varepsilon=0} \) | first-order term of Taylor expansion of \( \gamma_\varepsilon \) around \( \varepsilon = 0 \), also called the infinitesimal shape response curve (iSRC) |
| \( T_1 = \partial T_\varepsilon / \partial \varepsilon \bigg|_{\varepsilon=0} \) | first-order term of Taylor expansion of \( T_\varepsilon \) around \( \varepsilon = 0 \) |
| \( DF; D_w \phi \) | Jacobian matrix; directional derivative of \( \phi \) in \( w \) direction |
| \( I \) | identity matrix |
| \( S \) | saltation matrix (for variation equation) |
| \( J \) | jump matrix (for adjoint equation) |
| \( \mathcal{J} \) | time-reversed jump matrix (for adjoint equation) |
| \( \Sigma^i \) | boundary \( i \) |
| \( \mathcal{R}^j \) | region \( j \) |
| \( F^j(x) \) | velocity vector field in region \( j \) |
| \( \nu_\varepsilon = T_0 / T_\varepsilon \) | relative frequency of perturbed limit cycle |
| \( \nu_1 = T_1 / T_0 \) | first-order term of Taylor expansion of \( \nu_\varepsilon \) around \( \varepsilon = 0 \), also called the relative change in frequency |
| \( \mathcal{T}^j \) | time remaining in region \( j \) along a trajectory |
| \( \mathbf{u}(t) \) | variational dynamics governed by (2.3) |
| \( \mathbf{z}(t) = \nabla_x \phi(\gamma(t)) \) | infinitesimal phase response curve (iPRC) governed by (2.5) |
| \( \gamma_1(t) \) | infinitesimal shape response curve (iSRC) governed by (2.17) |
| \( \eta^j(t) = \nabla_x \mathcal{T}^j(\gamma(t)) \) | local timing response curve (ITRC) governed by (2.21) |
B Derivation of Equation 2.20

This section establishes equation (2.20), which specifies the first-order change in the transit time through region I, or $T^I_1$:

$$T^I_1 = \eta^I(x^{in}) \cdot \frac{\partial x^{in}}{\partial \varepsilon} \bigg|_{\varepsilon=0} + \int_{t^{in}}^{t^{out}} \eta^I(\gamma(t)) \cdot \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0} dt,$$

Recall $T^I(x)$ is the time remaining until exiting region I through $\Sigma^{out}$, under the unperturbed vector field, starting from location $x$; $\eta^I := \nabla T^I(x)$ is the local timing response curve (ITRC) for region I, defined for the component of the trajectory lying within region I, i.e. for times $t \in [t^{in}, t^{out}]$; and $x^{in}_\varepsilon$ is the coordinate of the perturbed entry point into region I.

We consider a single region $\mathcal{R}$ with entry surface $\Sigma^{in}$ and exist surface $\Sigma^{out}$. We assume that these two surfaces are fixed, independent of static perturbation with size $\varepsilon$. The limit cycle solution $x = \gamma_\varepsilon(\tau)$ satisfies

$$\frac{dx}{d\tau} = F_\varepsilon(x)$$

where $\tau$ is the time coordinate of the perturbed trajectory. Moreover, $\gamma_\varepsilon(\tau)$ enters $\mathcal{R}$ at $x^{in}_\varepsilon \in \Sigma^{in}$ when $\tau = t^{in}_\varepsilon$ and exits at $x^{out}_\varepsilon \in \Sigma^{out}$ when $\tau = t^{out}_\varepsilon$. Since the system is autonomous, we are free to choose the reference time along the limit cycle orbit. For convenience of calculation, we set $t^{out}_\varepsilon \equiv 0$ for all $\varepsilon$.

Denote the transit time that $\gamma_\varepsilon$ spends in $\mathcal{R}$ by $T^\mathcal{R}_\varepsilon$. It follows that $t^{in}_\varepsilon = -T^\mathcal{R}_\varepsilon$, where $\varepsilon$ can be 0. Assuming that the transit time has a well behaved expansion in $\varepsilon$, we write

$$T^\mathcal{R}_\varepsilon = T^\mathcal{R}_0 + \varepsilon T^\mathcal{R}_1 + O(\varepsilon^2) \quad (B.41)$$

where $T^\mathcal{R}_0$ is the transit time for the unperturbed trajectory and $T^\mathcal{R}_1$ is the linear shift in the transit time. In the rest of this section, we drop the superscript $\mathcal{R}$ on $T^\mathcal{R}_\varepsilon$, $T^\mathcal{R}_0$ and $T^\mathcal{R}_1$ for simplicity.

Our goal is to prove that $T_1$ is given by (2.20). We do this in two steps. First, we show that the transit time $T_\varepsilon$ can be expressed in terms of the perturbed vector field and perturbed local timing response curve (see (B.43)). Second, we expand the expression for $T_\varepsilon$ to first order in $\varepsilon$ to obtain the expression for $T_1$.

Since the time remaining to exit, denoted as $T_\varepsilon$, decreases at a constant rate along trajectories, for arbitrary $\varepsilon$ we have

$$-1 = \frac{dT_\varepsilon}{d\tau} = F_\varepsilon(\gamma_\varepsilon(\tau)) \cdot \eta_\varepsilon(\gamma_\varepsilon(\tau)), \quad (B.42)$$

where $\eta_\varepsilon(x) = \nabla T_\varepsilon(x)$ is defined as the local timing response curve under perturbation. By (B.42), the transit time $T_\varepsilon$ is therefore given by

$$T_\varepsilon = \int_{\tau=t^{in}_\varepsilon}^{t^{out}_\varepsilon} F_\varepsilon(\gamma_\varepsilon(\tau)) \cdot \eta_\varepsilon(\gamma_\varepsilon(\tau)) d\tau. \quad (B.43)$$
In this expression, we integrate *backwards in time* along the limit cycle trajectory, from the egress point \( x^\text{out}_\varepsilon \) at time \( t^\text{out}_\varepsilon \), to the ingress point \( x^\text{in}_\varepsilon \) at time \( t^\text{in}_\varepsilon \):

For \( \varepsilon = 0 \), and taking into account \((B.42)\), this integral reduces to

\[
T_0 = \int_{\tau = t^\text{out}_\varepsilon}^{t^\text{in}_\varepsilon} F_0(\gamma_0(\tau)) \cdot \eta_0(\gamma_0(\tau)) \, d\tau = \int_{\tau = t^\text{out}_\varepsilon}^{t^\text{in}_\varepsilon} (-1) \, d\tau = t^\text{out}_\varepsilon - t^\text{in}_\varepsilon = 0 - (-T_0),
\]

since \( t^\text{in}_\varepsilon = -T_0 \) and \( t^\text{out}_\varepsilon \equiv 0 \).

In order to derive an expression for \( T_1 \), the first order shift in the transit time, we need to expand \((B.43)\) to first order in \( \varepsilon \). To this end, we need to know the Taylor expansions for all terms in \((B.43)\).

Suppose we can expand \( F_\varepsilon, T_\varepsilon, \) and \( \eta_\varepsilon \) as follows:

\[
\begin{align*}
F_\varepsilon(x) &= F_0(x) + \varepsilon F_1(x) + O(\varepsilon^2), & \text{as } \varepsilon \to 0, \\
T_\varepsilon(x) &= T_0(x) + \varepsilon T_1(x) + O(\varepsilon^2), & \text{as } \varepsilon \to 0, \\
\eta_\varepsilon(x) &= \eta_0(x) + \varepsilon \eta_1(x) + O(\varepsilon^2), & \text{as } \varepsilon \to 0,
\end{align*}
\]

where \( \eta_0(x) = \nabla T_0(x) \) is the unperturbed local timing response curve.

Following the idea of deriving the infinitesimal shape response curve in §2.2, we write the portion of the perturbed limit cycle trajectory within region \( \mathcal{R} \) in terms of the unperturbed limit cycle, plus a small correction,

\[
\gamma_\varepsilon(\tau) = \gamma(\nu_\varepsilon \tau) + \varepsilon \gamma_1(\nu_\varepsilon \tau) + O(\varepsilon^2)
\]

where \(-T_\varepsilon \leq \tau \leq 0\) and \( \nu_\varepsilon = \frac{T_0}{T_\varepsilon} \).

Now we expand \((B.43)\) to first order

\[
T_\varepsilon = \int_{\tau = 0}^{-T_\varepsilon} \left[ F_0(\gamma_0(\nu_\varepsilon \tau)) + \varepsilon D F_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \gamma_1(\nu_\varepsilon \tau) + \varepsilon F_1(\gamma_0(\nu_\varepsilon \tau)) \right] \cdot \left[ \eta_0(\gamma_0(\nu_\varepsilon \tau)) + \varepsilon D \eta_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \gamma_1(\nu_\varepsilon \tau) + \varepsilon \eta_1(\gamma_0(\nu_\varepsilon \tau)) \right] \, d\tau + O(\varepsilon^2)
\]

\[
= \int_{\tau = 0}^{-T_\varepsilon} F_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \eta_0(\gamma_0(\nu_\varepsilon \tau)) d\tau + \varepsilon \left[ F_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \eta_1(\gamma_0(\nu_\varepsilon \tau)) + F_1(\gamma_0(\nu_\varepsilon \tau)) \cdot \eta_0(\gamma_0(\nu_\varepsilon \tau)) \right] d\tau + \\
\varepsilon \left[ F_0(\gamma_0(\nu_\varepsilon \tau)) \cdot D \eta_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \gamma_1(\nu_\varepsilon \tau) + D F_0(\gamma_0(\nu_\varepsilon \tau)) \cdot \gamma_1(\nu_\varepsilon \tau) \cdot \eta_0(\gamma_0(\nu_\varepsilon \tau)) \right] \, d\tau + O(\varepsilon^2)
\]

\[
= \frac{1}{\nu_\varepsilon} \int_{t = 0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) dt + \varepsilon \left[ F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \right] dt + \\
\varepsilon \left[ F_0(\gamma_0(t)) \cdot D \eta_0(\gamma_0(t)) \cdot \gamma_1(t) + D F_0(\gamma_0(t)) \cdot \gamma_1(t) \cdot \eta_0(\gamma_0(t)) \right] dt + O(\varepsilon^2)
\]

To order \( O(1) \), we recover

\[
T_0 = \int_{t = 0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \, dt.
\]
This leads to $T_\varepsilon = \frac{1}{\nu_\varepsilon} T_0$, as required for consistency. We are therefore left with

\[
0 = \int_{t=0}^{-T_0} \left[ F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \right] dt \quad (B.49)
\]

\[
+ \int_{t=0}^{-T_0} \left[ F_0(\gamma_0(t)) \cdot D\eta_0(\gamma_0(t)) \cdot \gamma_1(t) + DF_0(\gamma_0(t)) \cdot \gamma_1(t) \cdot \eta_0(\gamma_0(t)) \right] dt + O(\varepsilon)
\]

\[
= \int_{t=0}^{-T_0} \left[ F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \right] dt
\]

\[
+ \int_{t=0}^{-T_0} \left[ F_0(\gamma_0(t))^T D\eta_0(\gamma_0(t)) + \eta_0(\gamma_0(t))^T DF_0(\gamma_0(t)) \right] \cdot \gamma_1(t) dt + O(\varepsilon)
\]

where the second equality follows from rearranging orders of factors in the second integral.

Note that since $F_0 \cdot \eta_0 \equiv -1$ everywhere, we have the identity

\[
0 = \frac{\partial}{\partial x_j} \left( \sum_i \eta^i F^i \right) = \sum_i \frac{\partial \eta^i}{\partial x_j} F^i + \sum_i \eta^i \frac{\partial F^i}{\partial x_j} \quad (B.50)
\]

where $F^i$ and $\eta^i$ are the $i$-th components for $F_0$ and $\eta_0$; $x_j$ denotes the $j$th component of $x$ for $j \in \{1, \cdots, n\}$. It follows that $F_0^j (D\eta_0) + \eta_0^j (DF_0) = 0$ in (B.49), leaving only

\[
0 = \int_{t=0}^{-T_0} \left[ F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \right] dt. \quad (B.51)
\]

Since $F_0(\gamma_0(t)) = d\gamma_0/dt$ and $\eta_1(x) = \partial \eta_\varepsilon (x)/\partial \varepsilon|_{\varepsilon=0} = \partial \nabla T_\varepsilon (x)/\partial \varepsilon|_{\varepsilon=0}$,

\[
\int_{t=0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) dt = \int_{t=0}^{-T_0} \left( \frac{d\gamma_0}{dt} \right) \cdot \left( \frac{\partial}{\partial \varepsilon} \left[ \nabla T_\varepsilon (\gamma_0(t)) \right] \right)|_{\varepsilon=0} dt
\]

\[
= \int_{t=0}^{-T_0} \left( \frac{d\gamma_0}{dt} \right) \cdot \nabla \left( \frac{\partial}{\partial \varepsilon} \left[ T_\varepsilon (\gamma_0(t)) \right] \right)|_{\varepsilon=0} dt
\]

\[
= \int_{t=0}^{-T_0} \frac{d}{dt} \left( \frac{\partial}{\partial \varepsilon} \left[ T_\varepsilon (\gamma_0(t)) \right] \right)|_{\varepsilon=0} dt
\]

\[
= \left( \frac{\partial}{\partial \varepsilon} \left[ T_\varepsilon (x_0^{in}) \right] \right)|_{\varepsilon=0} - \left( \frac{\partial}{\partial \varepsilon} \left[ T_\varepsilon (x_0^{out}) \right] \right)|_{\varepsilon=0}
\]

\[
= \left( \frac{\partial}{\partial \varepsilon} \left[ T_\varepsilon (x_0^{in}) \right] \right)|_{\varepsilon=0} - 0
\]

\[
= T_1(x_0^{in}).
\]

Therefore

\[
T_1(x_0^{in}) = \int_{t=-T_0}^{0} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) dt = \int_{t=-T_0}^{0} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) dt. \quad (B.52)
\]

The second equality follows from our convention that $t_0^{in} = -T_0$ and $t_0^{out} \equiv 0$. 

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We notice that
\[
T_\varepsilon = T_\varepsilon(x^{in}_\varepsilon) = T_0 + \varepsilon \left( T_1(x^{in}_0) + \nabla T_0(x^{in}_0) \cdot x^{in}_1 \right),
\]
where we have made use of the Taylor expansion
\[
x^{in}_\varepsilon = x^{in}_0 + \varepsilon x^{in}_1 + O(\varepsilon^2), \quad \text{as} \ \varepsilon \to 0.
\]
Equating the first order terms in (B.41) and (B.53) leads to
\[
T_1 = T_1(x^{in}_0) + \eta_0(x^{in}_0) \cdot x^{in}_1.
\]
Substituting (B.52) into (B.54), we finally obtain
\[
T_1 = \eta_0(x^{in}_0) \cdot x^{in}_1 + \int_{t=t_0^\text{in}}^{t_0^\text{out}} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) \, dt
\]
which is (2.20), as desired.

C Proof of Theorem 3.7

In this section we present a proof of Theorem 3.7, which we restate for the reader’s convenience.

**Theorem.** Consider a general LCSC described locally by (3.33) in the neighborhood of a hard boundary \( \Sigma \) with a constant normal vector \( n \), and with a liftoff point defined by (3.32). Assume that within the stable manifold of the limit cycle there is a well defined asymptotic phase function \( \phi(x) \) satisfying \( d\phi/dt = 1 \) along trajectories. Assume that \( \phi \) is Lipschitz continuous, and assume that on the constraint surface \( \Sigma \), the directional derivatives of \( \phi \) with respect to directions tangential to the surface are Lipschitz continuous, except (possibly) at the liftoff and landing points. Finally, assume the nondegeneracy condition (3.34) holds at the liftoff point. Then the following properties hold for the saltation matrix for \( u \), and the jump matrix for \( z \):

(a) At the landing point, the saltation matrix is \( S = I - nn^T \), where \( I \) is the identity matrix.

(b) At the liftoff point, the saltation matrix is \( S = I \).

(c) Along the sliding region, the component of \( z \) normal to \( \Sigma \) is zero.

(d) The normal component of \( z \) is continuous at the landing point.

(e) The tangential components of \( z \) are continuous at both landing and liftoff points.

**Proof.** We choose coordinates \( x = (w,v) = (w_1,w_2,\ldots,w_{n-1},v) \) so that within a neighborhood containing both the landing and liftoff points, the hard boundary corresponds to \( v = 0 \), the interior of the domain coincides with \( v > 0 \), and the unit normal vector for the hard boundary is \( n = (0,\ldots,0,1) \). Writing the velocity vector \( F = (f_1,f_2,\ldots,f_{n-1},g) \) in these coordinates. In addition, we use \( F^\text{slide} \) to denote the vector field for points on the sliding region, whereas the dynamics of other points is governed by \( F^\text{int} \). The transversal intersection condition for the trajectory entering the hard boundary is \( g^\text{int}(x_{\text{land}},0) < 0 \) (cf. eq. (3.30)); note that \( n \) defined here points in
the opposite direction from the outward normal vector in (3.30)). At points \( x \in \mathcal{L} \) on the liftoff boundary, \( F_{\text{slide}} \) and \( F_{\text{int}} \) coincide and we will use whichever notation seems clearer in a given instance. Under the nondegeneracy condition at the liftoff point (3.34), we can further arrange the coordinates \((w_1, \ldots, w_{n-1})\) so that the unit vector normal to the liftoff boundary \( \mathcal{L} \) at the liftoff point is \( \ell = (0, \ldots, 0, 1, 0) \), and \( g_{\text{int}} \geq 0 \iff w_{n-1} \geq 0 \). With these coordinates, the nondegeneracy condition (3.34) is \( F_{\text{slide}}(x_{\text{lift}}) \cdot \ell = f_{n-1}^{\text{slide}}(x_{\text{lift}}) > 0 \).

(a) At the landing point, the saltation matrix is \( S = I - \mathbf{n}\mathbf{n}^\intercal \), where \( I \) is the identity matrix. The saltation matrix at a transition from the interior to a sliding motion along a hard boundary is given in (Bernardo et al. (2008), Example 2.14, p. 111) as

\[
S = I + \frac{(F_{\text{slide}} - F_{\text{int}})\mathbf{n}^\intercal}{\mathbf{n}^\intercal F_{\text{int}}},
\]

provided the trajectory approaches the hard boundary transversally.

It follows from the definition of the sliding vector field \( F_{\text{slide}} \) given by (3.31) that

\[
S = I - \mathbf{n}\mathbf{n}^\intercal,
\]

as claimed.

(b) At the liftoff point, the saltation matrix is \( S = I \). We adapt the argument in (Bernardo et al. (2008), §2.5) to our hard boundary/liftoff construction. The essential difference is that the trajectory is not transverse to the hard boundary at the liftoff point, indeed \( \mathbf{n}^\intercal F = 0 \) at \( x_{\text{lift}} \), so eq. (C.56) does not give a well defined saltation matrix. However, by replacing the vector \( \mathbf{n} \) normal to the hard boundary with the vector \( \ell \) normal to the liftoff boundary, we recover an equation analogous to (C.56), as we will show. Since \( F_{\text{slide}} = F_{\text{int}} \) at the liftoff point, we conclude that the saltation matrix at the liftoff point reduces to the identity matrix.

Let \( \Phi_I \) and \( \Phi_{II} \) denote the flow operators on the sliding region and in the domain complementary to the sliding region, respectively. That is, \( \Phi_I(x, t) \) takes initial point \( x \in \mathcal{R}_{\text{slide}} \) at time zero to \( \Phi_I(x, t) \) at time \( 0 \leq t \leq \mathcal{T}(x) \). So \( \Phi_I \) is restricted to act for times up to the time \( \mathcal{T}(x) \) at which the trajectory starting at \( x \) reaches the liftoff point, \( \Phi_I(x, \mathcal{T}(x)) \in \mathcal{L} \). Such a trajectory necessarily has initial condition \( x = (w_1, \ldots, w_{n-1}, 0) \) satisfying \( w_{n-1} < 0 \), by our coordinatization. Let \( x_a \in \mathcal{R}_{\text{slide}} \) be a point on the periodic limit cycle solution, so that \( \Phi_I(x_a, \mathcal{T}(x_a)) = x_{\text{lift}} \). Write \( \tau = \mathcal{T}(x_a) \) for the time it takes for the trajectory to reach the liftoff point after passing location \( x_a \). We require a first-order accurate estimate of the effect of the boundary on the displacement between the unperturbed trajectory and a nearby trajectory. If we make a small (size \( \varepsilon \)) perturbation into the domain interior, away from the constraint surface, the normal component of the perturbed trajectory will return to zero within a time interval of \( O(\varepsilon) \) duration, before the two trajectories reach the liftoff boundary. Therefore we need only consider perturbations tangent to the constraint surface.

Let \( x'_a \in \mathcal{R}_{\text{slide}} \) denote a point near \( x_a \), and suppose it takes time \( \mathcal{T}(x'_a) = \tau + \delta \) for the trajectory through \( x'_a \) to liftoff, at some point \( x'_{\text{lift}} \in \mathcal{L} \). There are two cases
to consider: either $\delta \geq 0$ or else $\delta \leq 0$. The two cases are handled similarly; we focus on the first for brevity. In case $\delta > 0$, the original trajectory arrives at the liftoff boundary before the perturbed trajectory, and the point $x'_b = \Phi I(x'_a, \tau) \in \mathcal{R}^{\text{slide}}$. We write $x'_b = x_{\text{lift}} + \Delta x_b$ (see Fig. 11B) and expand the flow operator as follows:

$$\Phi I(x'_a, \delta) = x'_b + \delta F^{\text{slide}}(x'_a) + \frac{\delta^2}{2} \left( \nabla^{\text{slide}} F^{\text{slide}}(x'_a) \right) \cdot F^{\text{slide}}(x'_a) + O(\delta^3)$$

$$= x_{\text{lift}} + \Delta x_b + \delta F^{\text{slide}}(x_{\text{lift}}) + \delta \left( \nabla^{\text{slide}} F^{\text{slide}}(x_{\text{lift}}) \right) \cdot \Delta x_b$$

$$+ \frac{\delta^2}{2} \left( \nabla^{\text{slide}} F^{\text{slide}}(x'_b) \right) \cdot F^{\text{slide}}(x'_b) + O(\delta),$$

(C.57)

where $\nabla^{\text{slide}}$ is the gradient operator restricted to $x = (x_1, \ldots, x_{n-1})$. The Taylor expansion in (C.57) is justified in a neighborhood of $x'_b$ contained in the sliding region of the hard boundary. The transversality of the intersection of the reference trajectory with $L$ (that is, $F_{n-1}(x_{\text{lift}}) > 0$) means that $\delta$ and $|\Delta x_b|$ will be of the same order. We write $O(n)$ to denote terms of order ($|\Delta x_b|^p \delta^{n-p}$) for $0 \leq p \leq n$.

Next we estimate $\delta$ and the location $x'_{\text{lift}}$ at which the perturbed trajectory crosses $L$. To first order,

$$\ell^T x'_b = \ell^T F^{\text{slide}}(x'_a) \delta$$

(C.58)

$$\ell^T (x_{\text{lift}} + \Delta x_b) = \ell^T \left( F^{\text{slide}}(x_{\text{lift}} + \Delta x_b) \right) \delta$$

(C.59)

$$\ell^T \Delta x_b = \ell^T \left( F^{\text{slide}}(x_{\text{lift}}) + \left( \nabla^{\text{slide}} F^{\text{slide}}(x_{\text{lift}}) \right) \cdot \Delta x_b \right) \delta$$

$$= \ell^T F^{\text{slide}}(x_{\text{lift}}) \delta + O(2)$$

$$\delta = \frac{\ell^T \Delta x_b}{\ell^T F^{\text{slide}}(x_{\text{lift}})} + O(2).$$

(C.60)

(C.61)

Combining this result with (C.57), the perturbed trajectory’s liftoff location is

$$x'_{\text{lift}} = x_{\text{lift}} + \Delta x_b + F^{\text{slide}}(x_{\text{lift}}) \delta + O(2).$$

(C.62)

Meanwhile, as the perturbed trajectory proceeds to $L$, during a time interval of duration $\delta$, the unperturbed trajectory has reentered the interior and evolves according to $\Phi II$, the flow defined for all initial conditions not within the sliding region. At a time $\delta$ after reaching $L$, the unperturbed trajectory is located, to first order, at a point $x_c = x_{\text{lift}} + F^{\text{int}}(x_{\text{lift}}) \delta + O(2)$. Thus, combining (C.62) and (C.63) the displacement between the two trajectories immediately following liftoff of the perturbed trajectory, $\Delta x_c = x'_c - x_c$, is given (to
first order) by
\[
\Delta x_c = x_{\text{lift}} - x_c \\
= x_{\text{lift}} + \Delta x_b + \mathbf{F}^{\text{slide}}(x_{\text{lift}})\delta - (x_{\text{lift}} + \mathbf{F}^{\text{int}}(x_{\text{lift}})\delta) \\
= \Delta x_b + \left(\mathbf{F}^{\text{slide}}(x_{\text{lift}}) - \mathbf{F}^{\text{int}}(x_{\text{lift}})\right)\delta \\
= \Delta x_b + \frac{\left(\mathbf{F}^{\text{slide}}(x_{\text{lift}}) - \mathbf{F}^{\text{int}}(x_{\text{lift}})\right)\ell^T \Delta x_b}{\ell^T \mathbf{F}^{\text{slide}}(x_{\text{lift}})} \\
= S_{\text{lift}} \Delta x_b + O(2).
\]

Therefore, the saltation matrix at the liftoff point is
\[
S_{\text{lift}} = I + \frac{\left(\mathbf{F}^{\text{slide}}(x_{\text{lift}}) - \mathbf{F}^{\text{int}}(x_{\text{lift}})\right)\ell^T}{\ell^T \mathbf{F}^{\text{slide}}(x_{\text{lift}})}. \tag{C.64}
\]

We take the vector field on the sliding region to be the projection of the vector field defined for the interior onto the boundary surface (cf. (3.31)). Therefore for our construction \(\mathbf{F}^{\text{slide}}(x_{\text{lift}}) = \mathbf{F}^{\text{int}}(x_{\text{lift}})\), and hence \(S_{\text{lift}} = I\), as claimed. We note that equation (C.64) will hold for more general constructions as well. This concludes the proof of part (b).

In parts (c) and (d) of the proof, our goal is to show the normal component of the iPRC is zero along the sliding region on \(\Sigma\) and is continuous at the landing point. To this end, we compute the normal component of the iPRC using its definition (2.4), which in \((w, v)\) coordinates takes the form
\[
z_v := z \cdot n = \lim_{\varepsilon \to 0} \frac{\phi(x + \varepsilon n) - \phi(x)}{\varepsilon}, \tag{C.65}
\]
where \(\phi(x)\) denotes the asymptotic phase at point \(x\) on the limit cycle. That is, we apply a small instantaneous perturbation to the limit cycle, either while it is sliding along \(\Sigma\) (part c) or else just before landing (part d), in the \(n\) direction, and estimate the phase difference between the perturbed and unperturbed trajectories (cf. Fig. 11).

(c) Along the sliding region, the component of \(z\) normal to \(\Sigma\) is zero. By (C.65) the normal component of the iPRC for a point on the sliding component of the trajectory, denoted by \(x_a = (w_a, 0)\) is given by
\[
z_v(x_a) = \lim_{\varepsilon \to 0} \frac{\phi(w_a, \varepsilon) - \phi(w_a, 0)}{\varepsilon}. \tag{C.66}
\]
By \(x'_a = (w_a, \varepsilon)\) we denote a point that is located at a distance of \(\varepsilon\) above \(x_a\). Our goal is to show \(z_v(x_a) = 0\).

The perturbed trajectory from \(x'_a\) is governed by the interior flow \(\Phi_{\Pi}\) until it reaches the sliding region at a point \(x'_c \in \Sigma\), after some time \(\tau\). Meanwhile the unperturbed trajectory from \(x_a\) is governed by the sliding flow \(\Phi_{\Pi}\) until it crosses the liftoff point at \(L\) (Fig. 11, dotted line).
Figure 11: Unperturbed trajectory (black curve) and a perturbed trajectory (red curve) near the hard boundary \( \Sigma \) (horizontal plane) in the \((w, v)\) phase space. Dashed line: intersection of liftoff boundary \( \mathcal{L} \) and \( \Sigma \). (A) Trajectory moves downward towards the sliding region (the area in \( \Sigma \) where \( g < 0 \)), hits \( \Sigma \) at the landing point \( x_{\text{land}} \), and exits \( \Sigma \) at the liftoff point \( x_{\text{lift}} \). (B) Construction for the proof of part (b). An instantaneous perturbation tangent to \( \Sigma \) is made to the point \( x_a \) at \( t = 0 \), pushing it to a point \( x'_a \in \Sigma \). The trajectory starting at \( x_a \) (resp., \( x'_a \)) reaches the liftoff point \( x_{\text{lift}} \) (resp., \( x'_{\text{lift}} \)) after time \( \tau \), and reaches \( x_c \) (resp., \( x'_{\text{lift}} \)) after additional time \( \delta \). The displacements \( \Delta x_b = x'_b - x_{\text{lift}} \) and \( \Delta x_c = x'_{\text{lift}} - x_c \) differ by an amount captured, to linear order, by the saltation matrix. (C) Construction for the proof of part (c). An instantaneous perturbation with size \( \varepsilon \) in the positive \( v \)-direction (green arrow) is made to the point \( x_a \in \Sigma \), pushing it off the boundary to an interior point \( x'_a \). After time \( \tau \), the trajectory starting at \( x'_a \) (resp., \( x_a \)) reaches a landing point \( x'_{\text{land}} \) (resp., \( x_{\text{land}} \)). (D) The same perturbation (green arrow) as in panel (C) is applied to the point \( x_a \) located at a distance of \( h \) above \( \Sigma \), pushing it to a point \( x'_a \). The trajectory starting at \( x_a \) lands on \( \Sigma \) at \( x_{\text{land}} \). After the same amount of time, the perturbed trajectory starting at \( x'_a \) reaches \( x'_b \). After additional time \( \tau \), the two trajectories reach \( x_c \) and \( x'_{\text{land}} \), respectively.

To first order in \( \varepsilon \), the time for the perturbed trajectory \( x'(t) \) to return to the constraint surface is

\[
\tau(\varepsilon) = -\frac{\varepsilon}{g_{\text{int}}(w_a, \varepsilon)} + O(\varepsilon^2) = -\frac{\varepsilon}{g_{\text{int}}(w_a, 0)} + \varepsilon D_v g_{\text{int}}(w_a, 0) + O(\varepsilon^2) + O(\varepsilon^4)
\]

\[
= -\frac{\varepsilon}{g_{\text{int}}(w_a, 0)} + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.
\]

(C.67)
Because \( x_a = (w_a, 0) \) is in the sliding region, \( g^{\text{int}}(w_a, 0) < 0 \); we conclude that \( \tau \) and \( \varepsilon \) are of the same order. We use \((p)\) to denote terms of order \( p \) in \( \varepsilon \) or \( \tau \).

At time \( \tau \) following the perturbation, the location of the perturbed trajectory is

\[
x'_a = \Phi_{\Pi}(x'_a, \tau) = x'_a + \tau F_{\text{int}}(x'_a) + O(2)
\]

\[
= x_a + \varepsilon n + \tau (F_{\text{int}}(x_a) + \varepsilon n^T D F_{\text{int}}(x_a)) + O(2)
\]

\[
= x_a + (0, \ldots, 0, \varepsilon) - \frac{\varepsilon}{g_{\text{int}}(x_a)} (f^1_{\text{int}}(x_a), \ldots, f^1_{n-1}(x_a), g_{\text{int}}(x_a)) + O(2)
\]

\[
= x_a - \frac{\varepsilon}{g_{\text{int}}(x_a)} (f^1_{\text{int}}(x_a), \ldots, f^1_{n-1}(x_a), 0) + O(2).
\]

Simultaneously, the location of the unperturbed trajectory is

\[
x_b = \Phi_{\Pi}(x_a, \tau) = x_a + \tau F_{\text{slide}}(x_a) + O(2)
\]

\[
= x_a - \frac{\varepsilon}{g_{\text{int}}(x_a)} (f^1_{\text{int}}(x_a), \ldots, f^1_{n-1}(x_a), 0) + O(2),
\]

since for \( x \in \Sigma \), we have \( f_{\text{slide}}(x) = f_{\text{int}}(x) \) by construction. Comparing the difference in location of the two trajectories at time \( \tau \) after the perturbation, we see that

\[
||x'_a - x_b|| = O(\varepsilon^2).
\]

By assumption, the asymptotic phase function \( \phi(x) \) is \( C^1 \) with respect to displacements tangent to the constraint surface. Since both \( x_b \) and \( x'_b \) are on this surface, \( n^T (x'_b - x_b) = 0 \), and \( \phi(x'_b) = \phi(x_b) + O(\varepsilon^2) \). Therefore \( z_v(x_a) = 0 \) for points \( x_a \) on the sliding component of the limit cycle. This completes the proof of part (c).

(d) The normal component of \( z \) is continuous at the landing point. In order to show that the normal component of the iPRC \( (z_v) \) is continuous at the landing point, we prove that \( z_v \) has a well-defined limit at the landing point and moreover, this limit equals 0 which is the value of \( z_v \) at the landing point as proved in (c). To this end, consider a point on the limit cycle shortly ahead of the landing point, \( x_a = (w_a, h) \) with \( 0 < h \ll 1 \) fixed, (cf. Fig. 11D). By (C.65)

\[
z_v(x_a) = \lim_{\varepsilon \to 0} \frac{\phi(w_a, h + \varepsilon) - \phi(w_a, h)}{\varepsilon}.
\]

Our goal is to show \( \lim_{h \to 0} z_v(x_a) = z_v(x_{\text{land}}) = 0 \).

We consider the case \( \varepsilon > 0 \); the treatment for \( \varepsilon < 0 \) is similar. For \( \varepsilon > 0 \), when the unperturbed trajectory arrives at the constraint surface (at landing point \( x_{\text{land}} \)), the perturbed trajectory is at a point \( x'_b \) that is still in the interior of the domain. Denote the unperturbed landing time \( t = 0 \); denote the time of flight from initial point \( x_a \) to \( x_{\text{land}} \) by \( s \). Through an estimate similar to that in part (c), to first order in \( h \), we have

\[
s(h) = -\frac{h}{g_{\text{int}}(x_{\text{land}})} + O(h^2).
\]
Between \( t = -s \) and \( t = 0 \), the displacement between the perturbed trajectory \((x'(t))\) and the unperturbed trajectory \((x(t))\) satisfies
\[
\frac{d(x' - x)}{dt} = DF^\text{int}(x(t)) \cdot (x' - x) + O(\varepsilon^2),
\] (C.73)
with initial condition \(x'(s) - x(s) = \varepsilon n\). Because the interior vector field is presumed \(C^1\), for \(h, s \ll 1\) we have
\[
x'_b - x_{\text{land}} = x'_a - x_a + s (\varepsilon D_v F^\text{int}(x_{\text{land}}) + O(\varepsilon^2)) + O(s^2)
= \varepsilon n - h \left( \varepsilon \frac{D_v F^\text{int}(x_{\text{land}})}{g^\text{int}(x_{\text{land}})} + O(\varepsilon^2) \right) + O(h^2)
= (0, \ldots, 0, \varepsilon) - \frac{h \varepsilon}{g^\text{int}(x_{\text{land}})} (f^\text{int}_1(x_{\text{land}}), \ldots, f^\text{int}_{n-1,v}(x_{\text{land}}), g^\text{int}_v(x_{\text{land}})) + O(2)
= \left( -h \varepsilon \frac{f^\text{int}_v(x_{\text{land}})}{g^\text{int}(x_{\text{land}})}, \varepsilon - h \varepsilon \frac{g^\text{int}_v(x_{\text{land}})}{g^\text{int}(x_{\text{land}})} \right) + O(2).
\]
Here \( f^\text{int}_v = (f^\text{int}_{1,v}, \ldots, f^\text{int}_{n-1,v}) \), where \( f^\text{int}_{k,v} \) denotes \( \partial f^\text{int}_k / \partial v \), and \( O(2) \) denotes terms of order 2 in \( \varepsilon \) or \( h \) as in (c). In the rest of this proof, we drop the dependence of the functions on \( x_{\text{land}} \) for simplicity.

Since \( x_{\text{land}} \) is in the sliding region, it follows that \( x'_b \) is \( \varepsilon - h \varepsilon \frac{g^\text{int}_v}{g^\text{int}} + O(2) \) above the sliding region. Through a similar estimation as in part (c), to first order in \( \varepsilon \) and \( h \), the time for the perturbed trajectory to arrive at the sliding region is
\[
\tau(h, \varepsilon) = \frac{\varepsilon - h \varepsilon \frac{g^\text{int}_v}{g^\text{int}}}{-g^\text{int}} + O(2) = -\frac{\varepsilon}{g^\text{int}} + h \varepsilon \frac{g^\text{int}_v}{(g^\text{int})^2} + O(2).
\]

At time \( \tau \), the location of the perturbed trajectory is
\[
x'_{\text{land}} = \Phi_H(x'_b, \tau)
= x'_b + \tau F^\text{int}(x'_b) + O(2)
= x_{\text{land}} + (\frac{-h \varepsilon}{g^\text{int}}, \varepsilon - h \varepsilon \frac{g^\text{int}_v}{g^\text{int}}) + \tau F^\text{int}(x_{\text{land}}) + O(2)
= x_{\text{land}} + \left( -h \varepsilon \frac{f^\text{int}_v}{g^\text{int}}, \varepsilon - h \varepsilon \frac{g^\text{int}_v}{g^\text{int}} \right) + \left( -\frac{\varepsilon}{g^\text{int}} + h \varepsilon \frac{g^\text{int}_v}{(g^\text{int})^2} \right) (f^\text{int}_v, g^\text{int}) + O(2)
= x_{\text{land}} + \left( -h \varepsilon \frac{f^\text{int}_v}{g^\text{int}}, 0 \right) + \left( -\frac{\varepsilon}{g^\text{int}} + h \varepsilon \frac{g^\text{int}_v}{(g^\text{int})^2} \right) (f^\text{int}_v, 0) + O(2).
\] (C.75)

Simultaneously, the location of the unperturbed trajectory is
\[
x_c = \Phi_I(x_{\text{land}}, \tau)
= x_{\text{land}} + \tau F^\text{slide}(x_{\text{land}}) + O(2)
= x_{\text{land}} + \left( -\frac{\varepsilon}{g^\text{int}} + h \varepsilon \frac{g^\text{int}_v}{(g^\text{int})^2} \right) (f^\text{int}_v, 0) + O(2).
\] (C.76)

Comparing the difference between (C.75) and (C.76), we see that
\[
\|x'_{\text{land}} - x_c\| = O(h \varepsilon).
\] (C.77)
Recall that the asymptotic phase is assumed to be $C^1$, with respect to displacements tangent to $\Sigma$. Since $x'_{\text{land}}$ and $x_c$ are on $\Sigma$, it follows that

$$\phi(x'_{\text{land}}) - \phi(x_c) = O(h\varepsilon).$$

Therefore, by (C.71),

$$z_v(x_a) = \lim_{\varepsilon \to 0} \frac{\phi(x'_a) - \phi(x_a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\phi(x'_{\text{land}}) - \phi(x_c)}{\varepsilon} = O(h)$$

Consequently,

$$\lim_{h \to 0} z_v(x_a) = 0$$

as required. This completes the proof of part (d).

(e) The tangential components of $z$ are continuous at both landing and liftoff points. We denote the tangential components of the iPRC by $z_w$, where $w$ represents vectors in the $n-1$ dimensional tangent space of the hard boundary. The $n-1$ dimensional iPRC vector $z_w$ obeys a restricted (i.e. reduced dimension) adjoint equation given in terms of $f_w$, the $(n-1) \times (n-1)$ Jacobian derivative of $f$ with respect to the $n-1$ tangential coordinates ($w$), and $g_w$, the $1 \times (n-1)$ Jacobian derivative of $g$ with respect to the tangential coordinates, and $z_v$, the (scalar) component of $z$ in the normal direction

$$\frac{dz_w}{dt} = -f_w(w,v)^\top z_w - g_w(w,v)^\top z_v$$

along the limit cycle in the interior domain. On the other hand, along the sliding component of the limit cycle that is restricted to $\{\Sigma : v = 0\}$, $z_u$ satisfies

$$\frac{dz_w}{dt} = -f_w(w,0)^\top z_w.$$ 

By part (c), $z_v$ goes continuously to zero as the trajectory from the interior approaches the landing point. Therefore $z_w$ is continuous at the landing point.

Next we prove the continuity of $z_w$ at the liftoff point $x_{\text{lift}} = (w_{\text{lift}}, 0)$. Recall that in the coordinates employed, the unit vector tangent to $\Sigma$ and normal to $L$ at $x_{\text{lift}}$ is $\ell = (0, \ldots, 0, 1, 0)$ (cf. Fig. 12). Fix an arbitrary tangential unit vector $\hat{w}$ oriented away from the sliding region (such that $\ell^T \hat{w} > 0$). The left and right limits of $z_w$ at $x_{\text{lift}}$ are given by

$$z_w^-(x_{\text{lift}}) = \lim_{\varepsilon \to 0^-} \frac{\phi(w_{\text{lift}} - \varepsilon \hat{w}, 0) - \phi(w_{\text{lift}}, 0)}{-\varepsilon}$$

and

$$z_w^+(x_{\text{lift}}) = \lim_{\varepsilon \to 0^+} \frac{\phi(w_{\text{lift}} + \varepsilon \hat{w}, 0) - \phi(w_{\text{lift}}, 0^+)}{\varepsilon}.$$ 

By $x_a = (w_{\text{lift}} - \varepsilon \hat{w}, 0)$ and $x_b = (w_{\text{lift}} + \varepsilon \hat{w}, 0)$ we denote the two points that are located at a distance of $\varepsilon$ away from $x_{\text{lift}}$ along the $-\hat{w}$ and $\hat{w}$ directions, respectively (cf. Fig. 12). We will show that

$$z_w^-(x_{\text{lift}}) = z_w^+(x_{\text{lift}}).$$ 

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Figure 12: Unperturbed trajectory (black) leaves the hard boundary at the liftoff point $x_{\text{lift}}$, in the $(w, v)$ phase space. An instantaneous perturbation tangent to $\Sigma$ is made to the liftoff point at $t = \tau$, pushing it to $x_a$ on the sliding region or to $x_b$ that is outside the sliding region. The points $x_c$ and $x'_{\text{lift}}$ denote the positions of the unperturbed trajectory and the perturbed trajectory at $t = 0$.

The equality of these limits will establish that $z_w$ is continuous at the liftoff point.

First, we consider $z_w^+(x_{\text{lift}})$. Given $\hat{w}$, there exists a unique point $x'_{\text{lift}}$ at the liftoff boundary $\mathcal{L} \cap \Sigma$, and a time $\tau > 0$, such that the trajectory beginning from $x'_{\text{lift}}$ at time 0 passes directly over $x_b'$ at time $\tau$, in the sense that $\Phi_{\text{II}}(x'_{\text{lift}}, \tau) = (w_b, h)$, where $\Phi_{\text{II}}$ is the flow operator in the complement of the sliding region, $h > 0$ is the “height” of $x_b'$ above $x_b$, and $w_b$ is the coordinate vector along the tangent space of the hard boundary. Let $x_{\text{lift}} = (w_{\text{lift}}, 0)$ and $x'_{\text{lift}} = (w'_{\text{lift}}, 0)$. By our construction, $w_b = w_{\text{lift}} + \varepsilon \hat{w}$. Hence, the location of the perturbed trajectory at time $\tau$ is

$$
(w_b, h) = (w_{\text{lift}} + \varepsilon \hat{w}, h) = \Phi_{\text{II}}(x'_{\text{lift}}, \tau) = (w'_{\text{lift}}, 0) + (f^\text{int}(x'_{\text{lift}}), 0)\tau + O(\tau^2)
$$

$$
= (w_{\text{lift}}' + f^\text{int}(x'_{\text{lift}})\tau + O(\tau^2), O(\tau^2)).
$$

Hence

$$
w_{\text{lift}} - w'_{\text{lift}} = f^\text{int}(x'_{\text{lift}})\tau - \varepsilon \hat{w} + O(\tau^2), \quad (C.83)
$$

$$
h = O(\tau^2), \quad (C.84)
$$

and

$$
w_b - w'_{\text{lift}} = f^\text{int}(x'_{\text{lift}})\tau + O(\tau^2) \quad (C.85)
$$

On the other hand,

$$
\varepsilon \hat{w} + (w'_{\text{lift}} - w_b) = w_{\text{lift}} - w'_{\text{lift}}.
$$

By (C.83) and (C.85), the above equation becomes

$$
\varepsilon \hat{w} - f^\text{int}(x'_{\text{lift}})\tau = f^\text{int}(x'_{\text{lift}})\tau - \varepsilon \hat{w} + O(\tau^2).
$$

That is,

$$
\varepsilon \hat{w} = f^\text{int}(x'_{\text{lift}})\tau + O(\tau^2).
$$
Taking the inner product of both sides with the unit vector $\ell$ (normal to $L$), and noting that for sufficiently small $\varepsilon$, $\ell \cdot f^{\text{int}}(x_{\text{lift}}') > 0$ (our nondegeneracy condition), we have
\[
\tau = \varepsilon \frac{\ell^T \dot{w}}{\ell \cdot f^{\text{int}}(x_{\text{lift}}')} + O(\varepsilon^2),
\]
and hence $\tau = O(\varepsilon)$. Therefore, (C.84) becomes
\[
h = O(\varepsilon^2)
\]
and hence the phase difference between $x_b'$ and $x_b$ is
\[
\phi(x_b) - \phi(x_b') = O(\varepsilon^2)
\]
due to the assumption that $\phi$ is Lipschitz continuous.

Next we show (C.82) holds using (C.80) and (C.81). Let the unperturbed trajectory pass through $x_{\text{lift}}$ at time $\tau$, and let $x_c$ be the location of the unperturbed trajectory at time $t = 0$ (see Fig. 12). Let $\Delta x_c = x_{\text{lift}}' - x_c$ and $\Delta x_b = x_b' - x_{\text{lift}}$. Then by part (b),
\[
\Delta x_b - \Delta x_c = O(|\Delta x_b|^2);
\]
since the saltation matrix is equal to the identity matrix at the liftoff boundary. Since $x_{\text{lift}}, x_b, x_b'$ form a right triangle,
\[
|\Delta x_b|^2 = \varepsilon^2 + h^2 = \varepsilon^2 + O(\varepsilon^4),
\]
which implies that
\[
\Delta x_b - \Delta x_c = O(\varepsilon^2).
\]

Direct computation shows
\[
\begin{align*}
\phi(x_b) - \phi(x_{\text{lift}}) &= (\phi(x_b) - \phi(x_b')) + (\phi(x_b') - \phi(x_{\text{lift}})) \\
&= (\phi(x_{\text{lift}}') - \phi(x_c)) + O(\varepsilon^2) \\
&= D_w \phi(x_c) \cdot \Delta x_c + O(\varepsilon^2) \\
&= D_w \phi(x_c) \cdot \Delta x_b + O(\varepsilon^2) \\
&= D_w \phi(x_c) \cdot (x_b' - x_{\text{lift}}) + O(\varepsilon^2) \\
&= D_w \phi(x_c) \cdot (x_b - x_{\text{lift}}) + O(\varepsilon^2) \\
&= D_w \phi(x_c) \cdot \varepsilon \hat{w} + O(\varepsilon^2)
\end{align*}
\]
To obtain the second equality, we translate the trajectories backward in time by $\tau$ beginning from $x_b'$ and $x_{\text{lift}}$, respectively; shifting both trajectories by an equal time interval does not change their phase relationship. The $O(\varepsilon^2)$ difference arises from (C.87). The third equality follows from the assumption that $\phi$ is differentiable with respect to displacements tangent to the sliding region. The fourth equality uses (C.88); the fifth and seventh follow from the definitions; the sixth uses (C.86).

Recall the we assume $\phi$ to have Lipschitz continuous derivatives in the tangential directions at the boundary surface (except possibly at the landing and liftoff points). Under this assumption, taking the limit $\varepsilon \to 0^+$ leads to $x_c \to x_{\text{lift}}^-$ and hence
\[
z_w^+(x_{\text{lift}}) = D_w \phi(x_{\text{lift}}^-) \cdot \dot{w}
\]
by (C.81). On the other hand,

$$\phi(x_a) - \phi(x_{\text{lift}}) = D_w \phi(x_a) \cdot (x_a - x_{\text{lift}}) + O(\varepsilon^2)$$

$$= -D_w \phi(x_a) \cdot \hat{\varepsilon}w + O(\varepsilon^2).$$

(C.90)

Taking the limit $\varepsilon \to 0^+$ results in $x_a \to x_{\text{lift}}$ and hence (C.80) together with (C.90), implies

$$z_{\hat{w}}(x_{\text{lift}}) = D_w \phi(x_{\text{lift}}) \cdot \hat{w}.$$

Hence, (C.82) holds.

$\square$
D Numerical Algorithms

We will now describe how the results presented in §2 and §3 can be implemented as numerical algorithms. MATLAB code that implements these algorithms for the example system described in §4 is available: https://github.com/yangyang-wang/LC_in_square.

Consider a multiple-zone Filippov system generalized from (3.31),

$$\frac{dx}{dt} = F(x),$$

(D.91)

that produces a $T_0$-periodic limit cycle solution $\gamma(t) \subset \mathbb{R}^n$. Suppose $\gamma(t)$ includes $k$ sliding components confined to boundary surfaces denoted as $\Sigma^i \subset \mathbb{R}^{n-1}$, $i \in \{1, \ldots, k\}$. $\gamma(t)$ exits the $i$-th boundary $\Sigma^i$ at a unique liftoff point $x^i_{\text{lift}}$ given that the nondegeneracy condition (3.34) at $x^i_{\text{lift}}$ is satisfied. We denote the normal vector to $\Sigma^i$ at liftoff, landing, or boundary crossing points by $n^i$. We denote the interior domain by $\mathcal{R}^{\text{interior}}$, which can now consist of multiple subdomains separated by transversal crossing boundaries, and denote the piecewise smooth vector field in $\mathcal{R}^{\text{interior}}$ by $F^{\text{interior}}$. By (3.31), the sliding vector field on the sliding region $\mathcal{R}^{\text{slide}} \subset \Sigma^i$ is therefore

$$F^{\text{slide}}_i(x) = F^{\text{interior}}(x) - (n^i \cdot F^{\text{interior}}(x))n^i$$

(D.92)

Using this notation, the vector field (D.91) can be written as

$$F(x) := \begin{cases} 
F^{\text{interior}}(x), & x \in \mathcal{R}^{\text{interior}} \\
F^{\text{slide}}_i(x), & x \in \mathcal{R}^{\text{slide}}_i 
\end{cases}$$

(D.93)

and we denote the vector field after a static perturbation by

$$F_\varepsilon(x) := \begin{cases} 
F^{\text{interior}}_\varepsilon(x), & x \in \mathcal{R}^{\text{interior}} \\
F^{\text{slide}}_i(x), & x \in \mathcal{R}^{\text{slide}}_i 
\end{cases}$$

(D.94)

where $i \in \{1, \ldots, k\}$. Here we assume that the regions are independent of static perturbation with size $\varepsilon$.

Notice that the computation of the iSRC requires estimating the rescaling factors, for which we need to compute the iPRC or the ITRC depending on whether a global uniform rescaling (2.17) or a piecewise uniform rescaling (2.23) is needed. We hence first present the numerical algorithms for obtaining the iPRC in §D.1 and the ITRC in §D.2; the algorithm for solving the homogeneous variational equation for the linear shape responses of $\gamma(t)$ to instantaneous perturbations (the variational dynamics $u$) is presented in §D.3; lastly, in §D.4 we illustrate the algorithms for computing the linear shape responses of $\gamma(t)$ to sustained perturbations (the iSRC $\gamma_1$) with a uniform rescaling factor computed from the iPRC as well as with piecewise uniform rescaling factors computed from the ITRC.

For simplicity, we assume the initial time is $t_0 = 0$. 

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D.1 Algorithm for Calculating the iPRC $z$ for LCSCs

It follows from Remark 3.9 that the iPRC $z$ for the LCSCs need to be solved backward in time. While there is no discontinuity of $z$ at a landing point, a time-reversed version of the jump matrix at the liftoff point on the hard boundary $\Sigma^i$, denoted as $J^i_{\text{lift}}$, is given by

$$J^i_{\text{lift}} = I - n^i n^i\top,$$  \hspace{1cm} (D.95)

where $I$ is the identity matrix. $J^i_{\text{lift}}$ updates $z$ local to the liftoff point as

$$z^i_{\text{lift}}^- = J^i_{\text{lift}} z^i_{\text{lift}}^+$$  \hspace{1cm} (D.96)

where $z^i_{\text{lift}}^-$ and $z^i_{\text{lift}}^+$ are the iPRC just before and just after the trajectory crosses the liftoff point $x^i_{\text{lift},i}$ in forwards time.

We now describe an algorithm for numerically obtaining the complete iPRC $z$ for $\gamma(t)$, a stable limit cycle with sliding components along hard boundaries and transversal crossing boundaries as described before.

Algorithm for $z$

1) Fix an initial condition $x_0 = \gamma(0)$ on the limit cycle, and integrate (D.93) to compute $\gamma(t)$ over $[0, T_0]$.

2) Integrate the adjoint equation backward in time by defining $s = T_0 - t$ and numerically solve for the fundamental matrix $\Psi(s)$ over one period $0 \leq s \leq T_0$, where $\Psi$ satisfies

(a) $\Psi(0) = I$, the identity matrix.

(b) For $s$ such that $\gamma(T_0 - s)$ lies in the interior of the domain,

$$\frac{d\Psi}{ds} = A^\text{interior}(T_0 - s)\Psi$$

where $A^\text{interior}(t) = (DF^\text{interior}(\gamma(t)))\top$ is the transpose of the Jacobian of the interior vector field $F^\text{interior}$.

(c) For $s$ such that $\gamma(T_0 - s)$ lies within a sliding component along boundary $\Sigma^i$,

$$\frac{d\Psi}{ds} = A^i(T_0 - s)\Psi$$

where $A^i(t) = (DF^\text{slidei}(\gamma(t)))\top$ is the transpose of the Jacobian of the sliding vector field $F^\text{slidei}$, given in (D.92).

(d) At any time $t_p$ when $\gamma$ transversely crosses a switching surface with a normal vector $n_p$,

$$\Psi^- = J \Psi^+$$

where $\Psi^- = \lim_{s \to (T_0 - t_p)^-} \Psi(s)$ and $\Psi^+ = \lim_{s \to (T_0 - t_p)^+} \Psi(s)$ are the fundamental matrices just before and just after crossing the surface in forwards
\[ \mathcal{J} = S^T \text{ since } \mathcal{J}^T S = I \text{ as discussed in §3.1, where the saltation matrix at any transversal crossing point is} \]

\[ S = I + \frac{(F_p^+ - F_p^-)n_p^T}{n_p^T F_p^-} \]

where \( F_p^-, F_p^+ \) are the vector fields just before and just after the crossing in forwards time (see (3.27)).

(e) At a liftoff point on the \( i \)-th hard boundary \( \Sigma^i \) (in backwards time, a transition from the interior to \( \Sigma^i \)), update \( \Psi \) as

\[ \Psi^- = \mathcal{J}^i \Psi^+ \]

where \( \mathcal{J}^i = I - n^i n^{i\top} \) as defined in (D.95), and then switch the integration from the full Jacobian \( A^{\text{interior}} \) to the restricted Jacobian \( A^i \).

(f) At a landing point on the \( i \)-th hard boundary \( \Sigma^i \) (in backwards time, a transition from \( \Sigma^i \) to the interior) switch integration from the restricted Jacobian \( A^i \) to the full Jacobian \( A^{\text{interior}} \); no other change in \( \Psi \) is needed.

3) Diagonalize the fundamental matrix at one period \( \Psi(T_0) \); it should have a single eigenvector \( v \) with unit eigenvalue. The initial value for \( z_{\text{BW}} \) (represented in \emph{backwards time}) at the point \( \gamma(T_0) = \gamma(0) = x_0 \) is given by

\[ z_{\text{BW}}(0) = \frac{v}{F(x_0)} \cdot v \]

4) The iPRC in backward time over \( s \in [0, T_0] \) is given by \( z_{\text{BW}}(s) = \Psi(s)z_{\text{BW}}(0) \) and is \( T_0 \)-periodic. Equivalently, one may repeat step (2) by replacing \( \Psi(s) \) with \( z_{\text{BW}}(s) \) and replacing the initial condition \( \Psi(0) = I \) with \( z_{\text{BW}}(0) \) to solve for the complete iPRC.

5) The iPRC in forward time is then given by \( z(t) = z_{\text{BW}}(T_0 - t) \) where \( t \in [0, T_0] \).

6) The linear shift in period in response to the static perturbation can be calculated by evaluating the integral (see (2.12))

\[ T_1 = -\int_0^{T_0} z^\top(t) \frac{\partial F_{\varepsilon}(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0} dt \]

\textbf{Remark D.1.} An alternative way (in MATLAB) to do backward integration is reversing the time span in the numerical solver; that is, integrate the adjoint equation over \([T_0, 0]\) to compute \( z(t) \).

\section*{D.2 Algorithm for Calculating the ITRC for LCSCs}

The ITRC satisfies the same adjoint equation, (2.5), as the iPRC, and hence exhibits the same jump matrix at each liftoff, landing and boundary crossing point. It follows that the algorithm for the iPRC from §D.1 can mostly carry over to computing the ITRC.
Suppose the domain of $\gamma(t)$ can be divided into $m$ regions $R^1, \ldots, R^m$, each distinguished by its own timing sensitivity properties. We denote the ITRC in $R^j$ by $\eta^j$.

Below we describe the algorithm to compute $\eta^j$ in region $R^j$ bounded by the two local timing surfaces $\Sigma^\text{in}$ and $\Sigma^\text{out}$. Following the notations in §2, $t^\text{in}$ and $t^\text{out}$ denote the time of entry into and exit out of $R^j$, at locations $x^\text{in}$ and $x^\text{out}$, respectively. The algorithm for computing $\eta^j$ is described as follows.

**Algorithm for $\eta^j$**

1) Compute $\gamma$, the unperturbed limit cycle, and $T_0$, its period, by integrating (D.93).
2) Compute $t^\text{in}, t^\text{out}$ for region $j$. Evaluate $x^\text{in} = \gamma(t^\text{in})$, $x^\text{out} = \gamma(t^\text{out})$ and $T_0^j = t^\text{out} - t^\text{in}$.
3) Compute the boundary value for $\eta^j$ at the exit point $x^\text{out}$ (see (2.22))
   $$\eta^j(x^\text{out}) = -n^\text{out} \cdot F(x^\text{out})$$
   where $n^\text{out}$ is a normal vector to $\Sigma^\text{out}$.
4) Integrate the adjoint equation backward in time by defining $s = T_0 - t$ and numerically solve for $\eta^j_{\text{BW}}(s)$ (represented in backwards time) over $[T_0 - t^\text{out}, T_0 - t^\text{in}]$. $\eta^j_{\text{BW}}(s)$ satisfies the initial condition $\eta^j_{\text{BW}}(T_0 - t^\text{out}) = \eta^j(t^\text{out})$ computed from step (3) as well as conditions (b) through (f) from step (2) of Algorithm for $z$ in §D.1.
5) The ITRC in forward time is then given by $\eta^j(t) = \eta^j_{\text{BW}}(T_0 - t)$ where $t \in [t^\text{in}, t^\text{out}]$.
6) Compute $\gamma_\varepsilon$, the limit cycle under some small static perturbation $\varepsilon \ll 1$, and find $x^\text{in}_\varepsilon$, the coordinate of the intersection point where $\gamma_\varepsilon(t)$ crosses $\Sigma^\text{in}$. The linear shift in time in region $j$ in response to the static perturbation can be calculated by evaluating the integral (see (2.20))
   $$T_1^j = \eta^j(x^\text{in}) \cdot \frac{x^\text{in}_\varepsilon - x^\text{in}}{\varepsilon} + \int_{t^\text{in}}^{t^\text{out}} \eta^j(\gamma(t)) \cdot \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon} \bigg|_{\varepsilon=0} dt.$$

**Remark D.2.** All the local linear shifts in time sum up to the global linear shift in period, that is, $T_1 = \sum_{j=1}^m T_1^j$.

**D.3 Algorithm for Solving the Homogeneous Variational Equation for LCSCs**

Here we describe the algorithm for solving the homogeneous variational equation for linear displacement $u$, the shape response to an instantaneous perturbation. This makes use of Theorem 3.7, which describes different jumping behaviors of $u$ at liftoff, landing, and boundary crossing points. Unlike the iPRC and ITRC which require integration backwards in time, the variational dynamics can be solved with forward integration. This makes the algorithm comparatively simpler by allowing $\gamma(t)$ and $u(t)$ to be solved simultaneously.
Algorithm for $u$:

1) Fix an initial condition $x_0 = \gamma(0)$ on the limit cycle and an initial condition $u_0 = u(0)$ for the displacement at $\gamma(0)$ of the limit cycle.

2) Integrate the original differential equation \((D.93)\) and the homogeneous variational equation \((2.3)\) simultaneously forward in time and numerically solve for $u(t)$ over one period $0 \leq t \leq T_0$, where $u$ satisfies

   (a) $u(0) = u_0$.

   (b) For $t$ such that $\gamma(t)$ lies in the interior of the domain,

   $$\frac{du}{dt} = DF_{\text{interior}}(\gamma(t))u$$

   (c) For $t$ such that $\gamma(t)$ lies within a sliding component along boundary $\Sigma_i$,

   $$\frac{du}{dt} = DF_{\text{slide}_i}(\gamma(t))u$$

   where $DF_{\text{slide}_i}$ is the Jacobian of the sliding vector field $F_{\text{slide}_i}$ given in \((D.92)\).

   (d) At any time $t_p$ when $\gamma$ transversely crosses a switching surface with a normal vector $n_p$ separating vector field $F^-_p$ on the incoming side from vector field $F^+_p$ on the outgoing side,

   $$u^+ = S u^-$$

   where $u^- = \lim_{t \to t_p^-} u(t)$ and $u^+ = \lim_{t \to t_p^+} u(t)$ are the displacements just before and just after crossing the surface. By the definition for the saltation matrix at transversal crossing point \((3.27)\), we have

   $$S = I + \frac{(F^+_p - F^-_p)n_p^T}{n_p F^-_p}.$$ 

   (e) At a landing point on the $i$-th hard boundary $\Sigma^i$, update $u$ as

   $$u^+ = S^i u^-$$

   where $S^i = I - n^i n^{iT}$ (recall $n^i$ is the normal vector to $\Sigma^i$) and switch integration from the full Jacobian $DF_{\text{interior}}$ to the restricted Jacobian $DF_{\text{slide}_i}$.

   (f) At a liftoff point on the $i$-th hard boundary $\Sigma^i$, switch integration from the restricted Jacobian $DF_{\text{slide}_i}$ to the full Jacobian $DF_{\text{interior}}$; no other change in $u$ is needed.

Remark D.3. The fundamental solution matrix satisfies

$$\frac{d\Phi(t, 0)}{dt} = DF\Phi(t, 0), \text{ with } \Phi(0, 0) = I$$

and takes the initial perturbation $u(0)$ to the perturbation $u(t)$ at time $t$, that is,

$$u(t) = \Phi(t, 0)u(0).$$
Computing $\Phi$ therefore requires applying Algorithm for $u$ $n$ times, once for each dimension of the state space. Specifically, let $\Phi(t, 0) = [\phi_1(t, 0), ..., \phi_n(t, 0)]$. The $i$-th column $\phi_i(t, 0)$ is the solution of the variational equation (2.3) with the initial condition $\phi_i(0, 0) = e_i$, a unit column vector with zeros everywhere except at the $i$-th row where the entry equals 1.

**Remark D.4.** Once $\Phi$ is obtained, we can obtain the monodromy matrix, $M = \Phi(T_0, 0)$. It follows from the periodicity of $\gamma(t)$ that $M$ has $+1$ as an eigenvalue with eigenvector $v$ tangent to the limit cycle at $x_0$; this condition provides a partial consistency check for the algorithm.

### D.4 Algorithms for computing $\text{iSRC}$, the response to sustained perturbation

Now we discuss the calculation of $\text{iSRC}$ $\gamma_1$, the linear shape response to a sustained perturbation. While $\gamma_1$ shares the same saltation as $u$ at each liftoff, landing and boundary crossing point, $\gamma_1$ satisfies the nonhomogeneous version of the variational equation, (2.17) or (2.23), where one of the nonhomogeneous terms depends on the time scaling factor, $\nu_1$ or $\nu_1'$. Moreover, the initial condition for $\gamma_1$ depends on the given perturbation and hence needs to be computed in the algorithm whereas the initial value for $u$ is arbitrarily preassigned.

In the following, we first describe the algorithm for computing $\gamma_1$ using the global uniform rescaling and then consider using piecewise uniform rescaling.

**Algorithm for $\gamma_1$ with uniform rescaling**

1) Fix an initial condition $x_0 = \gamma(0)$ on the limit cycle.

2) Compute the linear shift in period $T_1$ using Algorithm for $z$, then evaluate $\nu_1 = T_1/T_0$.

3) Choose an arbitrary Poincaré section $\Sigma$ (this can be one of the switching boundaries for appropriate $x_0$) that is transverse to $\gamma$ at $x_0$. Compute $\gamma_\varepsilon$, the limit cycle under some fixed small static perturbation, and find $x_{0\varepsilon}$, the coordinate of the intersection point where $\gamma_\varepsilon(t)$ crosses $\Sigma$. The initial value for $\gamma_1$ at the initial point $x_0$ is then given by

$$\gamma_1(0) = \frac{x_{0\varepsilon} - x_0}{\varepsilon}$$

4) Integrate the original differential equation (D.93) with the initial condition $x_0$ and the nonhomogeneous variational equation (2.17) simultaneously forward in time and numerically solve for $\gamma_1$ over one period $0 \leq t \leq T_0$, where $\gamma_1$ satisfies

(i) $\gamma_1(0) = (x_{0\varepsilon} - x_0)/\varepsilon$.

(ii) For $t$ such that $\gamma(t)$ lies in the interior of the domain,

$$\frac{d\gamma_1}{dt} = DF_{\text{interior}}(\gamma(t))\gamma_1 + \nu_1 F_{\text{interior}}(\gamma(t)) + \left. \frac{\partial F_{\text{interior}}(\gamma(t))}{\partial \varepsilon} \right|_{\varepsilon=0}$$
(iii) For \( t \) such that \( \gamma(t) \) lies within a sliding component along boundary \( \Sigma_i \),

\[
\frac{d\gamma_1}{dt} = D F^{\text{slide}_i}(\gamma(t)) \gamma_1 + \nu_i F^{\text{slide}_i}(\gamma(t)) + \left. \frac{\partial F^{\text{slide}_i}(\gamma(t))}{\partial \varepsilon} \right|_{\varepsilon=0}
\]

where \( D F^{\text{slide}_i} \) is the Jacobian of the sliding vector field \( F^{\text{slide}_i} \) given in (D.92).

(iv) For transversal crossings, landing points, and liftoff points, apply (d), (e) and (f), respectively, from step 2) in Algorithm for \( u \) in §D.3, by replacing \( u \) with \( \gamma_1 \).

Next we consider the case when \( \gamma(t) \) exhibits \( m \) different uniform timing sensitivities at regions \( R_1, \ldots, R_m \), each bounded by two local timing surfaces, as discussed in §D.2. Piecewise uniform rescaling is therefore needed to compute the shape response curve. The procedure for obtaining \( \gamma_1 \) in this case is nearly the same as described in Algorithm for \( \gamma_1 \) with uniform rescaling, except we now need to compute various rescaling factors using the ITRC. This hence leads to different variational equations that need to be solved. On the other hand, the local timing surfaces naturally serve as the Poincaré sections that are required to compute the initial values for \( \gamma_1 \) in the uniform rescaling case.

**Algorithm for \( \gamma_1 \) with piecewise uniform rescaling**

1) Take the initial condition for \( \gamma(t) \) to be \( \gamma(0) = x_0 \in \Sigma \), where \( \Sigma \) is one of the local timing surfaces. Compute \( \gamma(t) \), the unperturbed trajectory, and \( \gamma_\varepsilon(t) \), the trajectory under some static perturbation \( 0 < \varepsilon \ll 1 \), by integrating (D.93).

2) For \( j \in \{1, \ldots, m\} \), compute \( T_0^j \), the time that \( \gamma(t) \) spends in region \( j \) and \( T_1^j \), the linear shift in time in region \( j \) using Algorithm for \( \eta_j^1 \), and then evaluate \( \nu_j^1 = T_1^j / T_0^j \).

3) Compute \( x_{0\varepsilon} \), the coordinate of the intersection point where \( \gamma_\varepsilon(t) \) crosses \( \Sigma \). The initial value for \( \gamma_1 \) at the initial point \( x_0 \) is given by

\[
\gamma_1(0) = \frac{x_{0\varepsilon} - x_0}{\varepsilon}
\]

4) Integrate the original differential equation (D.93) with the initial condition \( x_0 \) and the piecewise nonhomogeneous variational equation (2.23) simultaneously forward in time and numerically solve for \( \gamma_1 \) over one period \( 0 \leq t \leq T_0 \), where \( \gamma_1 \) satisfies

(i) \( \gamma_1(0) = (x_{0\varepsilon} - x_0)/\varepsilon \).

(ii) For \( t \) such that \( \gamma(t) \) lies in the intersection of the interior of the domain and region \( R_j \),

\[
\frac{d\gamma_1}{dt} = D F^{\text{interior}_j}(\gamma(t)) \gamma_1 + \nu_j^1 F^{\text{interior}_j}(\gamma(t)) + \left. \frac{\partial F^{\text{interior}_j}(\gamma(t))}{\partial \varepsilon} \right|_{\varepsilon=0}
\]

where \( D F^{\text{interior}_j} \) is the Jacobian of the interior vector field \( F^{\text{interior}_j} \) in \( R_j \).
(iii) For $t$ such that $\gamma(t)$ lies within the intersection of a hard boundary $\Sigma^i$ and region $\mathcal{R}^j$,

$$\frac{d\gamma_1}{dt} = DF_{\text{slide}}^i(\gamma(t))\gamma_1 + \nu^j F_{\text{slide}}^i(\gamma(t)) + \left. \frac{\partial F_{\text{slide}}^i(\gamma(t))}{\partial \varepsilon} \right|_{\varepsilon = 0} \varepsilon$$

where $DF_{\text{slide}}^i$ is the Jacobian of the sliding vector field $F_{\text{slide}}^i(x) = F_{\text{interior}}^j(x) - (n^i \cdot F_{\text{interior}}^j(x))n^i$ given in (D.92).

(iv) For transversal crossings, landing points, and liftoff points, apply (d), (e) and (f), respectively, from step 2) in Algorithm for $u$ in §D.3, replacing $u$ with $\gamma_1$.

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