ON SERRE INTERSECTION MULTIPLICITY CONJECTURE

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ABSTRACT. We try to briefly explain the original ideas implying the proof of vanishing part of Serre multiplicity conjecture. A replacement for the proof based on intersection theory taught in W. Fulton is included. The author also discusses a connection between Serre positivity question and general positivity in Hodge theory.

INTRODUCTION

In 1950’s J. Serre made a definition of Intersection multiplicity of two finitely generated modules over a regular local ring $A$ by a type of Euler characteristic, namely Tor-formula. Specifically, for $M, N$ two finite $A$-modules he defines their intersection multiplicity

$$\chi^A(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i l(\text{Tor}_i^A(M, N))$$

subject to the condition that $\text{length}(M \otimes_A N) < \infty$. The formula

$$\dim(M) + \dim(N) \leq \dim(A)$$

will hold, [S]. This may also be proved using the Auslander-Buchsbaum relation for modules over a regular ring $(A, \mathfrak{m})$ ring,

$$\text{pd}(M) + \text{depth}(M) = \dim(A)$$

and the formula $\dim(N) \leq \text{pd}(M)$ when $l(M \otimes_A N) < \infty$ known as Intersection Theorem, [RO4]. Serre definition agrees with classical multiplicity of intersections of two varieties $f = 0$ and $g = 0$ in the plane as the length of the quotient ring $A/(f,g)$. In this case only the first term in the sum is non zero. The significance of Serre definition concerns the effect of higher Tor’s in multiplicity calculations. The necessity of higher Tor’s naturally is being understood when considering more complicated examples. In case of proper intersections of varieties Serre definition

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agrees with the Hilbert-Samuel definition for multiplicity, that if \( Y = \text{Spec}(A/p) \) and \( Z = \text{Spec}(A/q) \) be subvarieties in \( X = \text{Spec}(A) \) that intersect properly (the dimension condition reads as \( \text{ht}(p) + \text{ht}(q) = \dim(A) \)). Then it holds that \( \sqrt{p + q} = m \), and we have

\[
\chi^A(A/p, A/q) = e_{\dim_q}(q, A/p)
\]

where \( e_{\dim_q}(q, A/p) \) is Hilbert-Samuel multiplicity of the \( A \)-module \( A/q \) with respect to the ideal \( q \). He proved the positivity of the Euler characteristic in several special cases that were enough for most of purpose, for instance in the case where the ring \( A \) is essentially of finite type over a field (or a discrete valuation ring) \( k \). Serre conjectured the vanishing of multiplicity in non-proper intersections, and its positivity in the proper case, for general regular local rings, \([S]\).

**Conjecture:** \([S]\) Assume \( A \) is a regular local ring and \( M, N \) finite \( A \)-modules with \( l(M \otimes_A N) < \infty \), then

1. If \( \dim M + \dim N < \dim A \), then \( \chi^A(M, N) = 0 \)
2. In case \( \dim M + \dim N = \dim A \), called proper intersection, \( \chi^A(M, N) > 0 \).

Serre actually proves most of the fundamental properties of the intersection multiplicity over regular rings, especially when they are essentially of finite type over a field or a discrete valuation ring using the method of reduction to the diagonal

\[
A/p \otimes_A A/q \cong (A/p \otimes_k A/q) \otimes_{A \otimes_k A} A
\]

and the completed tensor product of A. Grothendieck, \([S]\). Serre’s proof essentially uses the flat structure over the field \( k \).

Two subvarieties \( Y \) and \( Z \) of codimensions \( p \) and \( q \) in \( X \) are said to intersect properly if \( Y \cap Z \) has codimension \( p + q \). Two key steps in building up intersection theory on smooth quasi-projective varieties is first the definition of intersection multiplicity and second the Moving lemma.

**Theorem 0.1.** (Chow’s Moving Lemma) Lets \( X \) be a smooth quasi projective variety over a field \( k \), and \( Y \) and \( Z \) closed integral subschemes of \( X \). Then the cycle \( Y \) is rationally equivalent to a cycle \( \eta \) which meets \( Z \) properly.

The vanishing part of the conjecture of Serre was proved by H. Gillet and C. Soule, \([GS1]\), and also independently by P. Roberts, \([RO1]\).

**Theorem 0.2.** (Vanishing, Roberts \([RO1]\), Gillet and Soule \([GS1]\)) Assume \( M, N \) are finitely generated modules over a complete intersection (hence regular) \( A \), such that \( M, N \) both have finite projective dimension (modules over regular rings always satisfy this) and \( M \otimes_A N \) has finite length. If \( \dim M + \dim N < \dim A \), then \( \chi^A(M, N) = 0 \).
The proof of P. Roberts uses a theory of local chern characters for commutative rings. He uses basic properties of local chern characters with an application of Auslander-Buchsbaum theorem mentioned above. He specifically uses a property of local chern character as

$$\chi(F_\bullet) = \text{ch}(F_\bullet), \tau(A)$$

for a regular local ring $A$, where ch is the local chern character on the $K$-theory of perfect $A$-complexes, and $\tau$ is the Todd genus (Riemann-Roch homomorphism on regular schemes). The work of Roberts concerns important developments of the theory of commutative rings toward geometric ideas on intersection theory. It fulfills several different formulations of the question and relates the Serre’s question to other conjectures in commutative algebra. Gillet and Soule systematically define intersection theory as a theory on $K_0$. They use the $\lambda$-ring structure on $K_0$ with support of a regular ring

$$\lambda^k : \bigoplus_Y K^Y_0(X) \to \bigoplus_Y K^Y_0(X), \quad k \geq 0$$

(not homomorphisms) and employ the Adams operations

$$\psi^k : K^Y_0(X) \to K^Y_0(X), \quad k \geq 0$$

as group homomorphisms. They closely analyze the eigen-values of Adams operations on the graded parts of $K_0$ with support with respect to the filtration by the codimension of support. Serre’s definition of intersection multiplicity, naturally lifts to a product on $K_0(A)$. That the $K$-theory of the regular ring $A$ concerns an intersection theory. In this way the intersection multiplicity can be read as a cup product in $K_0(A)$. The fact is intersection theory may be considered as a theory on $K$-groups of rings. The chern character $\text{ch}$ and the Riemann-Roch map transform this product into the Chow ring of $A$, where one may use Kodaira vanishing theorem in order to establish the vanishing of multiplicity. The theory of intersection multiplicity at some stage is the theory of Riemann-Roch algebra. The basic proofs on the Euler characteristic of Cartan-Serre defining the intersection multiplicity concerns various Riemann-Roch formulas. The Grothendieck Riemann-Roch theorem explains the existence of a functorial homomorphism

$$\text{ch} : K_0(X) \to CH^*(X)$$

for a regular scheme $X$, satisfying certain axiomatic properties which also characterize it. A well known method to study the Chow ring of a variety is to analyze various filtrations naturally defined on it. There are many well defined filtrations on Chow groups in the literature. We recall the coniveau (also called filtration by codimension of support) and $\gamma$-filtrations and study the multiplicativity of the filtrations.
With in the time a significant proof of non-negativity was given by O. Gabber \cite{RO2}. His proof involves both algebraic and geometric insights concerning multiplicity as an Euler characteristic. The proof does not recognize anything on vanishing or positivity separately. It only states the non-negativity. It closely studies the deformation of Euler characteristic in several coverings in a way showing that it is always changing by a non-negative multiple. Gabber’s work gives beautiful ideas on Euler characteristic of doubly graded modules (complexes). It also provides a way to produce interesting examples. This proof is not published in any paper by O. Gabber!. An exposition of that can be found in some papers by P. Roberts.

Regarding the positivity conjecture many persons have participated in the literature, such as S. Dutta, M. Hochster and Mc Laughlin, provided a counterexample to positivity when the projective dimensions of the modules are not finite (of course the ring $A$ is not regular in this case!). They give an example of a module $M$ with $\text{pd}(M) < \infty$ and $l(M) < \infty$ over the singular ring $A = k[[x, y, z, w]]/(xy - zw)$ such that for $N = A/(x, z)$ one obtains
\[
\chi(M, N) = -1
\]

Many other counter-examples found by other people later on, \cite{DHM}, \cite{D}, \cite{RO3}. The positivity conjecture is still an open problem.

Throughout the text we work with a local noetherian ring $A$ with maximal ideal $m$. We shall consider finitely generated $A$-modules, which we shortly call them finite $A$-modules. A regular local ring would be a local noetherian ring $(A, m)$ such that the minimal number of generators for $m$ is $\dim(A)$. It is the same as $\dim(m/m^2) = \dim(A)$. Quotients of regular local rings by regular sequences are called complete intersection rings. A Cohen-Macaulay module $M$ is one with $\text{depth}(M) = \dim(M)$ where $\text{depth}(M)$ is the lengths or any maximal $M$-sequence in $m$ and $\dim(M) = \dim(A/\text{ann}(M))$.

1. Serre’s Work

Let $A$ be a local ring and $M$ a finite $A$-module. Assume $a$ is an ideal such that $M/aM$ has finite length. The Hilbert-Samuel polynomial of $M$ with respect to the ideal $a$ is defined by
\[
P_{M}^{a}(n) = l(M/a^nM), \quad n >> 0
\]
when $A$ is noetherian and $a = \langle x_1, ..., x_k \rangle$, then
\[
\chi(K \otimes M) = e_k(a, M)
\]
where $K_\bullet$ is the Koszul complex on $<x_1, ..., x_k>$ and $e_k(a, M) = k! \times$ (coefficient of $n^k$ in $P^a_M$) is the Samuel multiplicity. By using a spectral sequence argument one can show that
\[
\chi(K_\bullet \otimes M) = \chi(\bar{K}_\bullet \otimes _{gr(A)} gr aM)
\]
where $\bar{K}_i = \oplus a^n K_i/a^{n+1}K_i$. This leads to the formula
\[
\chi(K_\bullet \otimes M) = \sum_p (-1)^p \binom{k}{p} P^a_M(i - p)
\]
where the right hand side is equal to $e_k(a, M)$, cf. [RO2].

Let $A$ be a regular local ring, and $M, N$ finitely generated $A$-modules such that $M \otimes_A N$ has finite length. J. P. Serre [S], defines the intersection multiplicity as
\[
\chi^A(M, N) := \sum (-1)^i l(Tor^A_i(M, N))
\]
He proves the basic fact that in this case:

(2) $\dim M + \dim N \leq \dim A$

will hold and makes the following question, known as Serre Multiplicity conjecture.

(1) If $\dim M + \dim N < \dim A$, then $\chi^A(M, N) = 0$

(2) In case $\dim M + \dim N = \dim A$, called proper intersection, $\chi^A(M, N) > 0$.

The condition, $M, N$ both have finite projective dimensions implies that the sums in (1) have finitely many terms. Also the condition, $M \otimes_A N$ has finite length, implies, all the $Tor^A_i(M, N)$ and hence all $Ext^i_A(M, N)$ have finite length. This makes the former criteria meaningful. In case of proper intersection of subvarieties $Y = \text{Spec}(A/p)$ and $Z = \text{Spec}(A/q)$ in $X = \text{Spec}(A)$ the dimension condition reads as $\text{ht}(p) + \text{ht}(q) = \dim(A)$. This implies $\sqrt{p + q} = m$, [S]. It is easy to prove that in proper intersections we have
\[
\chi^A(A/p, A/q) = e_{\dim q}(q, A/p)
\]
This shows Serre definition agrees with the original definition of Hilbert-Samuel multiplicity, [RO2].

**Remark 1.1.** The quantity
\[
\xi_A(M, N) := \sum (-1)^i l(Ext^i_A(M, N))
\]
called the Euler form carries equivalent data as $\chi$ by;
\[ \chi^A(M, N) = (-1)^{\dim N} \xi_A(M, N) \]

Both \( \chi^A, \xi_A \) are additive functions in each variable on exact sequences of \( A \)-modules, [CHA].

J. Serre proves the positivity in the equi-characteristic case. His proof is based on the Cohen structure theorem and the reduction to the diagonal. In this case the regular ring may be assumed to be of finite type over a field \( k \) or a discrete valuation ring \( k \). By reduction to the diagonal we mean \( Y \cap Z \cong (Y \times Z) \cap \Delta \) where \( \Delta \) is the diagonal in \( X \times X \). In terms of coordinate rings it can be written as

\[ A/p \otimes_A A/q \cong (A/p \otimes_k A/q) \otimes_{A \otimes_k A} A \]

where \( A = k[x_1, ..., x_n] \). One has \( Tor_i^B(M \otimes_k N, A) = Tor_i^A(M, N) \), where \( B = A \otimes_k A \). When \( A = k[[x_1, ..., x_n]] \), he regards \( N \) as a module over a similar ring but the variables are changed with \( y_1, ..., y_n \) formally and writes

\[ M \otimes N \cong (M \hat{\otimes}_k N) \otimes_{k[[x_1, ..., x_n, y_1, ..., y_n]]} k[[x_1, ..., y_n]]/I \]

where \( I = (x_1 - y_1, ..., x_n - y_n) \) and \( y_1, ..., y_n \) are some new variables. \( \hat{\otimes} \) is the completed tensor product, meaning that one also completes the modules \( M, N \) with respect to maximal ideals while tensoring. Specifically,

\[ M \hat{\otimes} N := \lim_{\leftarrow} (M/m^n M \otimes N/n^n N) \]

and then shows that,

\[ \dim(M \hat{\otimes} N) \leq \dim(M) + \dim(N) \]

Serre transforms the multiplicity question on \( M, N \) to that of \( (M \hat{\otimes}_k N) \) and the diagonal \( \delta \), by resolving the diagonal over \( A \otimes_k A \). Specifically he considers the convergent spectral sequence

\[ E^2_{pq} = Tor_p^{A \hat{\otimes}_k A}((A \hat{\otimes}_k A) / \delta, \widehat{Tor}_q^k(M, N)) \Rightarrow Tor_{p+q}^A(M, N) \]

implies

\[ Tor_p^{A \hat{\otimes}_k A}((A \hat{\otimes}_k A) / \delta, \widehat{Tor}_q^k(M, N)) \Rightarrow Tor_{p+q}^A(M, N) \]

and
\[ \chi^A(M,N) = \sum (-1)^p \text{Tor}_p^A((A \hat{\otimes}_k A) / \mathfrak{a}, \hat{\text{Tor}}_q^k(M,N)) = \sum_p (-1)^p l(H_p(\mathfrak{a}, M \hat{\otimes}_k N)) = e_0(M \hat{\otimes}_k N) \]

Here \( H_p(\mathfrak{a}, M \hat{\otimes}_k N) \) is the Koszul cohomology with respect to the parameter system \( \{x_i \hat{\otimes} 1 - 1 \hat{\otimes} x_i\} \), and \( e \) concerns Hilbert-Samuel multiplicity. The work of Serre in \[S\] were actually enough for the most purpose of geometers, specially in complex geometry, \[S\], \[SK\].

Let \( A \) be a complete discrete valuation ring, with residue field \( k \). Suppose that \( A \) and \( k \) have the same characteristic, and \( k \) is perfect. Then \( A \) is isomorphic to \( k[[T]] \). This fact can be proved by showing that \( A \) contains a system of representatives of the residue field, which is a field. If \( S \) is such a system of representatives, then any \( a \in A \) can be written as a convergent series \( a = \sum s_n \pi^n \), \( s_n \in S \). Now suppose that \( A \) and \( k \) have different characteristic. This is possible only when \( \text{char}(A) = 0 \) and \( \text{char}(k) = p > 0 \). Then \( v(p) = e \geq 0 \) is called the absolute ramification index of \( A \). The injection \( \mathbb{Z} \hookrightarrow A \) extends by continuity to an injection of the ring \( \mathbb{Z}_p \) of p-adic integers into \( A \). When the residue field is a finite field with \( q = p^f \) element, then \( A \) is a free \( \mathbb{Z}_p \) module of rank \( n = ef \). For any perfect field \( k \) of characteristic \( p \), there exists a complete discrete valuation ring and only one up to a unique isomorphism which is absolutely un-ramified and has \( k \) as its residue field. It is denoted \( W(k) \). It satisfies the universal property that for any \( A \) a complete discrete valuation ring of characteristic unequal to that of its residue field \( k \). Let \( e \) be its absolute ramification index. Then there exists a unique homomorphism of \( W(k) \) into \( A \) which makes commutative the diagram:

\[
\begin{array}{ccc}
W(k) & \rightarrow & A \\
\downarrow & & \downarrow \\
& k & 
\end{array}
\]

This homomorphism is injective and \( A \) is a free \( W(k) \)-module of rank equal to \( e \), \[S\].

As stated Serre proved the positivity in the unramified non-equicharacteristic case. His method can be applied to show that if we work over a regular ring with non-equicharacteristic the completion and the reduction to the diagonal will prove the positivity. We are going to investigate the analogue of the his strategy in the equicharacteristic situation. The goal is to obtain the corresponding equation to (8). Suppose we are given the regular local ring \((\hat{R}, \mathfrak{m})\) in the equi-characteristic and the finitely generated modules \( M, N \) satisfying the Serre condition \( l(M \otimes_A N) < \infty \). Denote by \( \hat{R}, \hat{M}, \hat{N} \) the completions. By the structure theorem the equation (5) would be replaced by

\[ M \otimes_A N = (M \hat{\otimes}_k N) \otimes_{W(X,Y)} W(X,Y) / I \]
where $W(X, Y)$ is a Witt ring. Then the question is how one can relate $\text{Tor}_*(M, N)$ and $\overline{\text{Tor}}_*(M, N)$.

We end up with some remarks related to derived algebraic geometry. Recall that if $A$ is a $k$-algebra and $k$ a field, for any two $A$-modules $M, N$ we have

$$M \otimes_A N = A \otimes_{A \otimes_k A} (M \otimes_k N)$$

This was the formula used for reduction to the diagonal in $\text{Spec}(A \otimes_k A)$. Notice that here the point was essentially based on the fact that every thing being flat over $k$. If $M$ and $N$ be simplicial $A$-modules and $A$ $R$-algebra, then this leads to the module

$$HH^R(A, M \otimes^L_R N) := A \otimes^L_{A \otimes^L_R A} (M \otimes^L_R N)$$

The conclusion is that one can reduce the question of intersection over $A$ to that over $R$ which is normally assumed to be simpler, or then iterate this.

2. The work of P. Robert

The Cartan-Eilenberg Euler characteristic defining intersection multiplicity can be written in terms of projective resolutions. If $E_\bullet$ and $F_\bullet$ be free resolutions of the $A$-modules $M, N$ (which may be taken to be finite, by the regularity of $A$), then

$$(9) \quad \chi^A(M, N) = \chi(E_\bullet \otimes F_\bullet) = (-1)^{\text{codim } M} \chi(E_\bullet^\vee \otimes F_\bullet)$$

where the right hand side is the usual Euler characteristic of the complex $E_\bullet \otimes F_\bullet$. The latter makes sense for the complex is supported on the maximal ideal of $A$. By a perfect $A$-complex we mean a bounded complex of finitely generated free (Projective) $A$-modules. The support of such complex would be the closed subspace $\text{Supp}(G_\bullet)$ where the localization $(G_\bullet)_p$ has non-trivial homology. Then the dimension of the complex is defined to be $\dim \text{Supp}(G_\bullet)$. The local chern character is an analogue of the usual chern character, on the $K$-theory of perfect $A$-complexes. We only need the formal properties of this theory to explain the vanishing part of the conjecture.

The local Chern character of a perfect complex may be written as

$$\text{ch}(G_\bullet) = \text{ch}^0(G_\bullet) + \text{ch}^1(G_\bullet) + \ldots , \quad \text{ch}_i(G_\bullet) : CH_k(\text{Spec}A)_Q \to A_{k-i}(\text{Supp}(G_\bullet))_Q$$

The Euler characteristic and local Chern character are related by

$$(10) \quad \chi(G_\bullet) = \text{ch}(G_\bullet).\text{td}(A)$$
which is called local Riemann-Roch theorem. When \( G_\bullet \) is supported at the maximal ideal (7) becomes the simple one

\[ \chi(G_\bullet) = \text{ch}(G_\bullet)[\text{Spec}(A/m)] \]

The correct language to work with characteristic classes is to consider them as maps on the \( K_0 \) groups, or rings. Then the chern character becomes a ring homomorphisms between \( K_0 \) and \( \text{CH}^*(A) = \text{CH}(\text{Spec}(A)) \).

**Proof by P. Roberts:** \([\text{RO1}], [\text{RO2}], [\text{RO3}], [\text{RO4}]\) Let \( M, N \) be as in the conjecture and \( F_\bullet, G_\bullet \) be their free resolutions respectively. By the local Riemann-Roch

\[ \chi(F_\bullet \otimes G_\bullet) = \text{ch}(F_\bullet \otimes G_\bullet)[A] = \sum_{i+j=d} \text{ch}_i(F_\bullet).\text{ch}_j(G_\bullet)[A] \]

Let \( X \) and \( Y \) denote the support of \( M, N \) respectively, these are also the support of \( F_\bullet, G_\bullet \). If \( d - j > \text{dim}(Y) \) then \( \text{ch}_j(G_\bullet)[A] = 0 \), and similarly when \( d - j > \text{dim}(X) \) then \( \text{ch}_i(F_\bullet)[A] = 0 \). These are the only cases when \( \text{dim}(X) + \text{dim}(Y) < d \).

P. Roberts systematically studies the inter-relation of Serre’s conjecture with other theorems and conjectures in commutative algebra, see \([\text{RO4}]\).

### 3. Proof of vanishing conjecture by H. Gillet and C. Soule

In geometry we replace \( A \) with a noetherian scheme \( X \). For \( Y \subset X \) a closed subset, \( K_0^Y(X) \) may be defined similar to the usual \( K_0(X) \) for bounded complexes having support in \( Y \). Then we will have the natural product;

\[ \cup : K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y \cap Z}(X) \]

given by \([E_\bullet] \otimes_{\mathcal{O}_X} [F_\bullet] \). So if we set:

\[ K_0^\sigma(X) = \bigoplus_{Y \subset X} K_0^Y(X) \]

then we obtain a ring structure with unit the complex \([\mathcal{O}_X] \). The \( K \)-theory with support satisfies all the natural functorial properties with respect to flat pull-backs or proper push-forwards. Let \( K_0^f \) be the \( K \)-theory of finitely generated modules (probably over a singular space or ring-sometimes denoted by \( G_0 \)). If we define,
\[ \cap : K_0^Y(X) \times K'_0(Z) \to K'_0(Y \cap Z), \quad [E_\bullet] \cap [M] = \sum_{i \geq 0} (-1)^i[H_i(E_\bullet \otimes_{O_X} M)] \]

then for \( f \) a flat proper morphism, we have the familiar projection formula

\[ f_*(f^*(\beta) \cap \alpha) = \beta \cap f_*(\alpha) \]

with appropriate \( \beta, \alpha \), where \( f_*[M] = \sum (-1)^i[R^i f_* M] \). If \( X \) is a regular scheme, then the map \( K_0^Y(X) \to K^Y_0(Y) \) defined by \([E_\bullet] \mapsto \sum (-1)^i[H^i(E_\bullet)]\) would be an isomorphism. In this case if \( Y, Z \) be closed subsets then the product structure on \( K^Y_0(X) \) induces the following pairing

\[ (14) \quad K'_0(Y) \otimes K'_0(Z) \to K'_0(Y \cap Z) \]

\[ [E_\bullet] \otimes [F_\bullet] \mapsto \sum_i (-1)^i[\text{Tor}_i^O X(E_\bullet, F_\bullet)] \]

**Theorem 3.1. [GS1]** The exterior powers endow \( K^\sigma_0(X) = \bigoplus_{Y \subset X} K^Y_0(X) \) with a \( \lambda \)-ring structure.

This means that there exists a collection of maps \( \{\lambda^k : K^\sigma_0(X) \to K^\sigma_0(X)\}_{k \geq 0} \) given by exterior powers, satisfying some combinatorial conditions, reflecting the simplicial structure of these rings, [GS1]. Then we have a collection of ring homomorphisms

\[ (15) \quad \{\psi_k : K^\sigma_0(X) \to K^\sigma_0(X)\}, \]

called Adams operations. They are defined by certain axiomatic properties similar to chern classes. The restrictions, \( \psi_k : K^Y_0(X) \to K^Y_0(X) \) are group homomorphisms. If \( X = \text{Spec}(A) \), with \( A \) noetherian they are defined by

\[ \psi_k[K(a)] = k[K(a)] \]

where \( K(a) \) is the single Koszul complex \( A \xrightarrow{\times a} A, \ a \in A \). The single Koszul complexes are considered as the building blocks of the \( \lambda \)-ring \( K^\sigma_0(X) \), [GS1].

**The proof by H. Gillet and C. Soule:** [GS1] If \( F^m \) is the filtration by co-dimension of support, i.e.

\[ (16) \quad F^m K^Y_0(X) := \lim_{\substack{z \subset X \\
\text{codim}_{X}(Z) \geq m}} \text{Im}(K^\sigma_0(X) \to K^Y_0(X)) \]
The vector space $K^Y_0(X)$ decomposes as
\[ \bigoplus_{i \geq \text{codim}(Y)} \text{Gr}_i K^Y_0(X)_\mathbb{Q}, \]
and
\[ \text{Gr}_i F \psi_k : \text{Gr}_i K^Y_0(X)_\mathbb{Q} \to \text{Gr}_i K^Y_0(X)_\mathbb{Q} \]
is just multiplication by $k^i$. We have the product
\[ F^m K^Y_0(X)_\mathbb{Q} \otimes F^n K^Y_0(X)_\mathbb{Q} \to F^{m+n} K^Y_0(X)_\mathbb{Q} \]
If $\alpha = \sum \alpha_i$, $\beta = \sum \beta_i$, then $\alpha \cup \beta = \sum \alpha_i \cup \beta_j$. One checks that $\psi_k(\alpha_i \cup \beta_j) = k^i \alpha_i \cup \beta_j$ and thus
\[ \alpha_i \cup \beta_j \in \text{Gr}_{i+j} K^Y \cap Z(X) \]
It follows that when
\[ \text{codim}(\text{Supp}(M)) + \text{codim}(\text{Supp}(N)) > d = \text{dim } A \]
then
\[ \chi(M, N) = ([M] \cup [N]) \cap [\mathcal{O}_X] = \sum (-1)^i l(Tor^A_i(M, N)) = 0 \]
In this way the intersection multiplicity can be written as a cup product on the $K$-theory of perfect $A$-complexes which is the same as usual $K_0(A)$ when $A$ is regular.

In [GS1] Gillet and Soulé conclude with a similar theory of Chow groups with supports of regular rings (more general complete intersection rings). That is on a regular scheme $X$ with closed subvarieties $Y, Z$ there exists a pairing
\[ CH^p_Y(X) \otimes CH^q_Z(X) \to CH^{p+q}_{Y \cap Z}(X)_\mathbb{Q} \]
Similar to $K_0$ with supports $\bigoplus_Y CH^*_Y(X)_\mathbb{Q}$ would be a ring with unit $[X]$. Their strategy is use a $K$-theory with support of $\gamma$-filtration. However the intersection theory with support obliges to extend the coefficients to $\mathbb{Q}$. In fact, they establish an isomorphism
\[ CH^p_Y(X) \otimes [1/(d-1)!] \cong Gr^p K^Y_0(X) \otimes [1/(d-1)!] \]
\[ \Rightarrow CH^p_Y(X)_\mathbb{Q} \cong Gr^p K^Y_0(X)_\mathbb{Q} \]
Using reduction to the diagonal one shows that the intersection product agrees with the previously defined product, [GS1]. Intersection theory on singular varieties can be explained using supports and deformation to the normal cone, see section 5 in this text.
We shall work with families of closed subsets of a scheme which are closed under taking subsets and finite unions. Let \( \Phi \) be such a family on the scheme \( X \). Similar to section 3, define

\[
K^\Phi_0(X) := \lim_{Y \in \Phi} K^Y_0(X)
\]

One has a product structure as \( K^\Phi_0(X) \otimes K^\Psi_0(X) \to K^{\Phi \wedge \Psi}_0(X) \), for two families \( \Phi \) and \( \Psi \), where \( \Phi \wedge \Psi \) is the natural join of the two families (taking intersections).

**Definition 3.2.** (co-niveau and niveau filtrations) The coniveau filtration is the decreasing filtration

\[
F^i_{\text{cod}}(K_0(X)) := \text{Image}(K^{X_{\geq i}}_0(X) \to K_0(X)), \quad i \geq 0
\]

where \( X_{\geq i} \) is the closed subsets of codimension at least \( i \) (\( X_{\leq j} \) may be considered as subsets of dimension at most \( j \) for niveau filtration). The coniveau filtration on \( G^0_0(X) \) is defined similarly by the codimension of the support of modules.

Some conjectural phenomenon happens for co-niveau filtration similar to Serre multiplicity conjecture. It is well known theorem proved by A. Grothendieck, that if the base scheme is essentially of finite type over a field then the product structure on \( K_0 \) is compatible with the coniveau filtration. However for general regular noetherian scheme we have the following conjecture.

**Conjecture:** On a regular noetherian scheme \( X \) we have

\[
F^i_{\text{cod}}(K^Y_0(X)) \ast F^j_{\text{cod}}(K^Z_0(X)) \subset F^{i+j}_{\text{cod}}(K^{Y \cap Z}_0(X))
\]

One can easily show that

**Theorem 3.3.** The above conjecture implies Serre vanishing conjecture.

**Proof.** We may suppose the supports \( Y, Z \) of the two modules \( M, N \) intersect in the close point \( x \in X = \text{Spec}(A) \). If \([M] \in F^{p}_{\text{cod}}K^{Y}_0(X)\) and \([N] \in F^{q}_{\text{cod}}K^{Z}_0(X)\), with \( p+q > n = \text{dim}(A) \), then

\[
\chi(M, N) = [M] \cup [N] \in F^{p+q}_{\text{cod}}K^x_0(X) \subset F^{n+1} = 0
\]

□

**Definition 3.4.** (\( \gamma \)-filtration) The \( \gamma \) operations are defined by

\[
\gamma^n : K_0(X) \to K_0(X), \quad \gamma^n(x) = \lambda^n(x + (n-1)[O_X])
\]

where \( \lambda^n(E) = \Lambda^nE \) are the exterior powers which define the \( \lambda \)-structure on \( K_0(X) \). The \( \gamma \)-filtration is the multiplicative filtration on \( K_0(X) \) such that \( F^1_\gamma \) is the class of vector bundles that are locally of rank 0.
It is a theorem by Grothendieck that the natural transformations
\[ c_k(E) := \gamma^k(E - \text{rank}(E)) \]
satisfy the axioms for the chern classes. The corresponding chern character is a
natural transformation
\[ ch : K_0(X) \to Gr^\ast_\gamma(K_0(X)) \]

**Theorem 3.5.** [G] If \( X \) is a regular scheme, the chern character induces
\[ ch : Gr_{\text{cod}}(K_0(X))_Q \xrightarrow{\cong} Gr^\ast_\gamma(K_0(X))_Q \]
Furthermore, there are isomorphisms
\[ ch_k : CH^k(X)_Q \xrightarrow{\cong} Gr^k_\gamma(K_0(X))_Q \]
The multiplicativity of the coniveau filtration is also a consequence of the theorem, [G].

4. de Jang alterations and O. Gabber reduction for \(( \geq 0 )\)

The proof of O. Gabber, [RO2] is based on a theorem of de Jang. He reduces the
question of intersection over a general regular local ring to corresponding questions
of intersections on projective schemes. The theorem of de Jang states that for any
regular (affine) scheme \( V \) there exists a projective \( \phi : X \to V \) which is of finite type
over \( V \). Assume we are concerned with the intersection of two regular schemes \( Y, Z \)
of finite type over the regular local ring \(( A, m, k )\). Consider \( Z' \) to be the projective
scheme over \( Z \) which is finite of degree \( n \), obtained from the de Jang theorem. Set \( A' = A[X_1, \ldots, X_n] \) with quotient field \( K \). Then \( Z' \) can be considered as a
closed subscheme of \( \text{Proj}(K[X_1, \ldots, X_n]) \), say with homogeneous ideal \( I \). Set \( Y' = \text{Proj}(k(Y)[X_1, \ldots, X_n]) \) and \( P = \text{Proj}(A[X_1, \ldots, X_n]) \).

The strategy is to compare \( \chi(O_{Y'}, O_{Z'}) \) with \( \chi(O_Y, O_Z) \). Let \( F \) and \( G \) be the
\( A' \)-projective resolutions of \( Y', Z' \), respectively. Consider the Cech complex
\[ C^\bullet : 0 \to \prod A'_{X_i} \to \prod A_{X_i X_j} \to \cdots \to A_{X_1 \ldots X_n} \to 0 \]
Using the relation
\[ (C^\bullet \otimes_{A'} F) \otimes_R G_c \cong C^\bullet \otimes_{A'} (F \otimes_R G_c) \cong C^\bullet \otimes_{A'} (F \otimes_{A'} G_c) \]
and the projection formula \( \phi_{A'} F_c \otimes G_c = \phi_{A'} (F_c \otimes \phi^c(G_c)) \) for the projective resolutions
of \( O_{Y'} \) and \( O_{Z'} \). Gabber shows that
\[ \chi(Y', Z') = \chi(Y, \phi_{A'} F) = \chi(Y, (O_Z)^n) = n \chi(Y, Z) \]
since $\phi_n(\mathcal{O}_Y) \cong (\mathcal{O}_Y)^n$. In another step he replaces the rings by $Gr_1A'$ and $Gr_1B$ where $B = A' \otimes_A k[Y]$. Set $E = \text{Proj}(Gr_1A'), \ M = \text{Proj}(gr_1B)$. He shows

$$\chi_E(M, Z') = \chi(Y', Z')$$

One may consider a filtration of $M$ whose successive quotients $M_i$ are annihilated by a power of $m$ the maximal ideal of $A$. Then,

$$\chi_E(M, Z') = \sum_i \chi(M_i, Z') = \sum_i \chi_{E_s}(M_i, Z'_s)$$

where $E_s = \text{Proj}(Gr_1A' \otimes_A k)$ and $Z'_s = \text{Proj}(A'/I \otimes_A k)$.

**Proposition 4.1.** (Gabber) \cite{RO2} The positivity conjecture holds for the $Y, Z$ if and only if

$$\chi_{E_s}(\text{Proj}(gr_1B \otimes k), Z'_s) \geq 0.$$  

The theorem completes the reduction step and the proof. de Jang theorem can be regarded as a weak version of Hironaka resolution of singularity theorem. It can be stated in a more general statement as, given a variety $X$, there exists $\phi : Y \to X$ with $Y$ non-singular and $\phi$ proper surjective. de Jang method of alterations can be applied with the method of deformation to the normal cone to define an intersection theory with support on singular varieties, but with coefficient in $\mathbb{Q}$, see sec. 5 below.

5. Intersection on singular schemes

For all schemes (quasi-projective or singular) there exists a functorial homomorphism

$$\tau := \tau_X : K_0(X) \to CH_*(X)_\mathbb{Q}$$

called Riemann-Roch homomorphism or (Todd genus), where $K_0(X)$ is the Grothendieck group of finite modules up to short exact sequences (sometimes denoted $G_0$ or $K'_0$). In the smooth category this homomorphism is an isomorphism and is defined by

$$\tau(E) = ch(E).Td(X),$$

where

$$ch(E = \oplus L_i) = \sum \exp(c_1(L_i)), \quad Td(E) = \prod \frac{c_1(L_i)}{1 - e^{c_1(L_i)}}$$

for line bundles $L_i$. The strategy to extend the definition of $\tau$ to singular category, proceeds by embedding $X \hookrightarrow M$ in a smooth scheme $M$ and then define

$$\tau(E) = ch^X_M(E).Td(M), \quad ch^X_M(E) \in H^*(M, M \setminus X)$$
The definition of \( ch_X^\tau(E) \) uses McPherson graph construction in a way that extension of \( \tau \) to the singular category is unique. Therefore, it is an isomorphism for all schemes when tensored with \( \mathbb{Q} \). This homomorphism satisfies

\[
\tau(\mathcal{O}_V) = [V] + \text{lower dimension terms}
\]

Let \( E_* \) be any complex of vector bundles on a scheme \( X \) which is exact off a closed subscheme \( Y \). Then for any coherent sheaf \( F \) on \( X \), one has the following Riemann-Roch formula

\[
\sum (-1)^i \tau_Y[H^i(E_* \otimes F)] = Ch_X^\tau(E_*) \cap \tau_X(F)
\]

In case of a regular embedding \( f : Y \hookrightarrow X \) of codimension \( d \) and normal bundle \( N \), the above formula provides

\[
\sum (-1)^i \tau_Y[\text{Tor}^i(\mathcal{O}_X, F)] = td(N)^{-1} \cap f^*\tau_X(F)
\]

It follows that when \( \dim(\text{Supp}F) = n \) one has

\[
\sum (-1)^i Z_{n-d}[\text{Tor}^i(\mathcal{O}_X, F)] = td(N)^{-1} \cap f^*Z_n(F)
\]

When \( X \) is regular and \( V, W \) closed subsets, applying the above formula to the diagonal embedding \( X \hookrightarrow X \times_X X \) and \( F = \mathcal{O}_{V \times W} \) gives the formula

\[
[V],[W] = \sum (-1)^i Z_m[\text{Tor}^i(\mathcal{O}_V, \mathcal{O}_W)], \quad m = \dim(V) + \dim(W) - \dim(V \cap W)
\]

This formula proves the vanishing conjecture!, [E] page 364. The above formula re-suggest this idea that many of the numerical equalities that we have in intersection theory are influenced from more general ones on divisors, and are actually divisorial identities. Riemann-Roch theorems are of this type. The formula (21) suggest that the expression \( \sum (-1)^i \tau_Y[H^i(E_* \otimes F)] \) is a candidate to generalize the Tor formula. Using the identification \( \tau \) it follows from (21) that the intersection theory on a general scheme can be described using cap product with local chern characters.

As said before one standard method to extend the intersection theory is via the Chow group with support. There are several ways to do this. One due to H. Gillet and Soulé uses \( \gamma \)-filtration on \( K \)-theory. The second due to Robert, Kleiman, Thorup, Fulton is via the definition of operational Chow groups \( CH_{op}^p(X) \), where an element \( CH_{op}^p(X) \) consists of giving for every map \( f : Y \to X \) of varieties, homomorphisms \( \cap \alpha : CH_q(Y) \to CH_{q-p}(Y), \forall q \geq 0 \) which satisfy various compatibilities. In this way he introduces a theory of chern classes which is a generalization of the local chern characters we already mentioned. If \( X \) is a nodal elliptic curve over a field
one can show that $CH^1_{op}(X) = \mathbb{Z}$. These groups in many cases miss some information on algebraic cycles. One analogously defines operational Grothendieck groups $K^0_{op}(X)$ and the corresponding Todd genus $\tau$ where similar formulas as (20) and (21) holds. The third is via the definition of deformation to the normal cone (due to A. Weber) and uses the alteration technique of de Yang as mentioned before. in all cases the extension of the coefficients to $\mathbb{Q}$ is necessary. Let $Y, Z \hookrightarrow X$ be closed and $p_Y : \tilde{Y} \to Y$ and $p_Z : \tilde{Z} \to Z$ be the varieties obtained from $Y, Z$ via the alteration theorem of de Jang with finite degrees $m_Y, m_Z$ respectively.

$$
\begin{array}{ccccc}
\tilde{Y} & \leftarrow & \tilde{Y} \times \tilde{Z} & \longrightarrow & \tilde{Z} \\
\downarrow \pi_Y & & \downarrow \pi_Z & & \text{de Jang alterations} \\
Y & \overset{i_Y}{\longrightarrow} & X & \leftarrow & Z
\end{array}
$$

Set $f := \pi_Y \circ i_Y$ and $g := \pi_Z \circ i_Z$. Then define the intersection with support as

$$
Y, [Z] := \frac{1}{m_Y} f_* f^! [Z] = \frac{1}{m_Y m_Z} f_* f^! [X] = \frac{1}{m_Y} g_* g^! [Y] := Z, [Y]
$$

where the middle equality uses W. Fulton intersection theory. This definition makes sense as soon as we extend the coefficients to $\mathbb{Q}$.

As already mentioned the strict positivity part of Serre multiplicity conjecture is still an open challenging problem. We open this section in order to mention some special cases that the conjecture can be proved, and also give a counter example in the singular case. The positivity of Cartan-Eilenberg Euler characteristic can be proved in some special cases. For instance if the $A$-modules $M, N$ are Cohen-Macaulay. Assume $M, N$ are so and $\dim(M) + \dim(N) = \dim(A)$. By the Auslander-Buchsbaum theorem

$$
Pd(M) + \dim(M) = \dim(A)
$$

for $A$ regular or complete intersection, the length of the minimal free resolution of $M$ is $\dim(A) - \dim(M) = \dim(N)$. Since $N$ is also Cohen-Macaulay, the condition on the length of the resolution implies that $Tor_i(M, N) = 0, \ i > 0$. Thus $\chi(M, N) = l(M \otimes N) > 0$.

Serre also stated the conjecture on the higher Euler characteristics, that is the sums

$$
\chi_i(M, N) = \sum_{j=i}^d (-1)^{j-i} l(Tor_i(M, N))
$$

He proves that in the equi-characteristic case $\chi_i(M, N) \geq 0$ and if $Tor_i(M, N) \neq 0$ then $\chi_i(M, N) > 0$. The conjecture on the higher Tor’s motivates the conjecture on
rigidity of Tor. It says that over a regular local ring $A$ the Tor functor is rigid (if $\text{Tor}_i(M,N) = 0$ then $\text{Tor}_j(M,N) = 0$, $j > i$ for all finite $M,N$.

The intersection multiplicity on singular varieties is probably encoded by higher Tor’s. A simple example is the intersection multiplicity on a hypersurface $f = 0$ with isolated singularity. Then the intersection multiplicity of two subvarieties $M,N$ intersecting at the isolated singular point is given by

$$\Theta(M,N) = l(\text{Tor}_{2i}(M,N)) - l(\text{Tor}_{2i+1}(M,N)), \quad i >> 0$$

It is a general fact discovered by D. Eisenbud that a minimal free resolution of a finite $R = A/(f)$-module is eventually periodic. $\Theta(M,N)$ was defined by Hochster and is called Hochster Theta function.

The Serre intersection multiplicity can be negative on non-regular rings. Hochster-Dutta-McLaughlin [DHM], give an example of two modules $M,N$ over $A = k[x,y,u,v]/(xy-wv)$ with $\chi(M,N) = -1$. We illustrate the following similar example generalized by Levine, [L]. Set $A = k[x,y,z,u,v,w]/(ux+vy+wz)$ localized at the maximal ideal $m = (x,y,z,u,v,w)$. Let $p = (u,v,w)$. We wish to construct an $A$-module $N$ such that $\chi(N,A/p) = -2$.

The module $A/p$ has a minimal free resolution

(25) \[ ... \xrightarrow{\phi_3} A^4 \xrightarrow{\phi_4} A^4 \xrightarrow{\phi_3} A^4 \xrightarrow{\phi_4} A^3 \xrightarrow{\phi_3} A \rightarrow 0 \]

where

$$\phi_1 = (u \ v \ w) \quad \phi_2 = \begin{pmatrix} x & 0 & -w & v \\ y & w & 0 & -u \\ z & -v & u & 0 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} 0 & u & v & w \\ u & 0 & v & w \\ v & -z & -y & x \\ w & y & -x & 0 \end{pmatrix} \quad \phi_4 = \begin{pmatrix} 0 & x & y & z \\ x & 0 & -w & v \\ y & w & 0 & -u \\ z & -v & u & 0 \end{pmatrix}$$

If we assume $l(N) = 55$ after tensoring this resolution with $N$, we get

(26) \[ \chi(N,A/p) = 55 - 165 + 220 - \text{rank}(\phi_4 \otimes N) = 110 - \text{rank}(\phi_4 \otimes N) \]

In the construction in [D], $\text{rank}(\phi_4 \otimes N) = 112$. See [DHM], [L] and [RO3] for other examples.

Serre multiplicity conjecture is known for graded regular rings.
Theorem 5.1. [RO3] If $A$ is a graded regular ring and $M, N$ graded finite modules with $l(M \otimes N) < \infty$, then the Serre multiplicity conjecture holds for $\chi(M, N)$. That is $\chi(M, N) \geq 0$ and is $\chi(M, N) > 0$ if and only if $\dim(M) + \dim(N) = \dim(A)$.

6. Intersection theory and Higher K-groups

We briefly screen some generalization of the previous ideas into higher $K$-theory, following [SU]. The group $K_{n+1}$ of an exact category can be understood as an obstruction of equivalences in $K_n$. This is a generalization of trivial facts in the homotopy groups of topological spaces or even homology groups of complexes. Because similar argument says, the difference of two representatives of a null class in $\pi_n$ can be measured by a class in $\pi_{n+1}$, etc. A similar basic fact holds for homologies of complexes. Let $S$ be a subcategory of an exact category $E$. The localization sequence of the $K$-groups, is an exact sequence

$$\ldots \to K_{m+1}(E/S) \to K_m(S) \to K_m(E) \to K_m(E/S) \to \ldots$$

For instance if $A$ be a Dedekind domain with quotient field $F$, $E$ be the exact category of finite $A$-modules, and $S$ to the subcategory of torsion $A$-modules, Then $E/S$ is the category of coherent locally free sheaves on the point $\text{Spec}(F)$ and $S = \bigcup_{\varphi \neq 0} \mathcal{P}(\text{Spec}(k(\varphi)))$, where $\mathcal{P}(\text{Spec}(k(\varphi)))$ is the category of coherent sheaves with support at $k(\varphi)$. The localization sequence is the Gersten exact sequence in this case,

$$\ldots \to K_{m+1}(F) \to \bigoplus_{\varphi \neq 0} K_m(k(\varphi)) \to K_m(A) \to K_m(F) \to \ldots$$

Let $\mathcal{K}$ be the Zariski simplicial sheaf

$$U \to \mathbb{Z} \times BGL(\Gamma(U, \mathcal{O}_X))^+$$

Then one can define a cohomology

$$K^Y_m(X) := H^{-m}_Y(X, \mathcal{K})$$

Similarly one can define a $A$-ring structure on the groups $\bigoplus_{m>0, Y \subset X} K^Y_m(X)$ and obtain Adams operations
Gersten conjectured that for any regular noetherian finite dimensional scheme $X$ and $p \geq 0$, the group $CH^p(X)$ is isomorphic to the Zariski cohomology group $H^p(X, K_p)$, where $K_p : U \to K_p(U)$ is the sheaf of $K$-groups. In case, the intersection pairing

\[
(32) \quad CH^p(X) \otimes CH^q(X) \to CH^{p+q}(X)
\]

can be defined as

\[
(33) \quad H^p(X, K_p) \times H^q(X, K_q) \to H^{p+q}(X, K_{p+q})
\]

M. Rost generalizes this structure to also include higher Chow groups. He considers any covariant functor $M$ from the category of fields to the category of $(\mathbb{Z}$ or $\mathbb{Z}/2)$ graded-Abelian groups together with

- Transfers $tr_{E/F} : M(E) \to M(F)$ of degree 0 for all $E/F$ finite field extension.
- For every discrete valuation $v$ of $F$ a boundary map
  \[
  \partial_v : M(F) \to M(k(v))
  \]
  of degree -1.
- A pairing $F^* \times M(F) \to M(F)$ of degree 1, which extends to a pairing $K^*_M(F) \times M(F) \to M(F)$, which makes $M(F)$ a graded module over Milnor $K$-theory ring.

If $X$ is an algebraic variety over $F$ define the complex $C^*(X, M, q)$ by

\[
(C^p(X, M, q) = \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x)), \quad \{\partial_v : M_{q-p}(k(x)) \to M_{q-p-1}(k(v))\}_v
\]

Then one defines the higher Chow groups as

\[
A^p(X, M, q) = H^p(C^*(X, M, q))
\]

In case $M$ is the Milnor $K$-theory or Quillen $K$-theory one has $A_p(X, K^M, p) \cong CH^p(X)$. There exists a well defined theory of chern classes with coefficients in the Milnor $K$-theory

\[
c_n : K_p(X) \to H^{n-p}(X, K^M_n)
\]

satisfying the properties op. cit, [G].
Intersection multiplicity may be defined on Deligne-Mumford stacks in a similar way. One needs first to fix the definition of dimension for these stacks, which is well defined. The dimension of a Deligne-Mumford stack $X$, is defined to be the dimension of $X$ for any atlas $x \to X$. The dimension of a stack can be negative. For instance $\dim(\ast/G) = -\dim(G)$. If $X$ is a Deligne-Mumford stack over a field then one can define the Chow groups $CH^p(X)$ as the quotient of reduced irreducible substacks of codimension $p$, and the rational equivalence defined by rational functions on stacks. Then many of the aforementioned constructions and identities remains true, however one is forced to extend the coefficients to $\mathbb{Q}$. One still can show that $CH^p(X)_{\mathbb{Q}} = H^p_{et}(X, K_p(O_X))_{\mathbb{Q}}$ which leads to a $K$-theoretic definition of intersection product on $X$, $[G]$, $[J]$.

For a quotient stacks we define $CH^*(X/G/k) := CH^*_G(X/k)$ where $CH^*_G(X/k)$ is the equivariant Chow group of $X$. This definition is well defined, meaning that if $[X/G] = [Y/H]$ then $A^*_G(X) = A^*_H(Y)$. If $\bar{X}$ is the coarse moduli space of a quotient stack $X$, then one shows that the natural map $\pi : X \to \bar{X}$ induces isomorphism $CH^*(X)_{\mathbb{Q}} := CH^*(\bar{X})_{\mathbb{Q}}$

Let $G$ be a finite group, viewed as a group scheme over a field $k$ such that $(|G|, char(k)) = 1$. The Grothendieck group of vector bundles on the stack $[Spec(k)/G]$ can be identified by the representation ring of the finite group $G$, namely $Rep(G) = K^0_G(Spec(k))$. One has $H^*_{et}(Spec(k)/G, \mathbb{Q}) = H^*_{et}(BG, \mathbb{Q}) = \mathbb{Q}$. The diagram

$$
K^0_G(Spec(k)) \xrightarrow{ch_G} H^*_{et}(BG, \mathbb{Q})\xrightarrow{\pi_*} \mathbb{Q}, \text{ not commutes}
$$

where $\pi : [Spec(k)/G] \to Spec(k)$ is the obvious (non-representable) map of algebraic stacks, fails to be commutative. The top row is the $G$-equivariant Chern character, whereas the bottom row is the usual Chern character which is the rank map. The left column sends a representation to its $G$-invariant part. The difficulty with Riemann-Roch for algebraic stacks is seen by the lack of the commutativity of the diagram.
where $K : U \to K(U)$, and $U$ on the étale site. The idea is if one relaces the étale topology with another topology namely isoinvariant étale topology and the sheaf $K$ with the equivariant version $K^G$, the diagram

$$
\begin{array}{ccc}
K^0_G(\text{Spec}(k)) & \xrightarrow{ch^G} & H^0_{et}(BG, K^0) \\
\pi_* \downarrow & & \downarrow \pi_* \\
K^0(\text{Spec}(k)) & \xrightarrow{ch} & H^0_{et}(\text{Spec}(k), K^0)
\end{array}
$$

not commutes

where $K^0(\text{Spec}(k)) \xrightarrow{ch} H^0_{et}(\text{Spec}(k), K^0)$ does commute, see [J] for details.

8. Appendix: Local chern classes

For a topologixal space $X$ let $H^*(X, \mathbb{Z})$ be the integral cohomology in the sense of sheaf theory, and $\hat{H}^*(X, \mathbb{Z}) = \prod_i H^i(X, \mathbb{Z})$. Let also $H^*_Z(X, \mathbb{Z}) = H^*(X, X - Z, \mathbb{Z})$ for $Z$ closed. A theory of local chern classes consists in assigning to a complex $K^\bullet$ on $X$ with support in $Z$ a cohomology class

$$c^Z_\bullet(K^\bullet) \in \hat{H}^*_Z(X, \mathbb{Z})$$

satisfying:

- Functoriality. $c^Z(f^*L^\bullet) = f^*c^V(L^\bullet)$ for continuous $f$ such that $f(X - Z) \subset Y - V$.

- $r(c^Z_\bullet(K^\bullet)) + 1 = \prod_i c_\bullet(K^{2i})c_\bullet(K^{2i-1})^{-1}$, where $r : H^*_Z(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is the natural map.

One slightly proves the existence of a theory of chern characters

$$ch^Z(K^\bullet) \in \hat{H}^*_Z(X, \mathbb{Q})$$

with equivalent properties as follows;
• Functoriality. - $f^*ch^V(L^\bullet) = ch^Z(f^*L^\bullet)$

• $r(ch^Z(K^\bullet)) = \sum_i (-1)^ich(K^i)$.  

• Decalage. - $ch^Z(K^\bullet[1]) = -ch^Z(K^\bullet)$

• Additivity. - $ch^Z(K^\bullet \oplus L^\bullet) = ch^Z(K^\bullet) + ch^Z(L^\bullet)$ for $K^\bullet, L^\bullet$ having support in $Z$.

• Multiplicativity. - $ch^Z \cap V(K^\bullet \otimes L^\bullet) = ch^Z(K^\bullet) \cdot ch^V(L^\bullet)$ for $K^\bullet$ and $L^\bullet$ having support in $Z$ and $V$ respectively.

**Theorem 8.1.** (B. Iversen) A theory of local Chern classes exists and is unique.

The proof is based on the well-known McPherson graph construction. It concerns certain constructions on the cohomology of flag manifolds.

Set $1 + \hat{H}_Z^{ev}(X, \mathbb{Z})^+ = 1 + \prod_{i \geq 1} H^Z_i(X, \mathbb{Z})$, and define a product $*$ by

$$\left(1 + x_m + \ldots\right) * \left(1 + y_n + \ldots\right) = 1 - \frac{(n + m - 1)!}{(m - 1)! (n - 1)!} x_m y_n + \ldots$$

If we set $\tilde{c}^Z_\bullet(K^\bullet) = 1 + c^Z_\bullet(K^\bullet) \in 1 + \hat{H}_Z^{ev}(X, \mathbb{Z})^+$. Then one has the following relations:

$$f^* \tilde{c}^V(L^\bullet) = \tilde{c}^Z(f^*L^\bullet)$$

$$r(\tilde{c}^Z(K^\bullet)) = \prod_i \tilde{c}(K^{2i}) \tilde{c}(K^i)^{-1}. $$

$$\tilde{c}^Z(K^\bullet[1]) = \tilde{c}^Z(K^\bullet)^{-1}$$

$$\tilde{c}^Z(K^\bullet \oplus L^\bullet) = \tilde{c}^Z(K^\bullet) \cdot \tilde{c}^Z(L^\bullet).$$

$$\tilde{c}^Z \cap V(K^\bullet \otimes L^\bullet) = \tilde{c}^Z(K^\bullet) \ast \tilde{c}^V(L^\bullet)$$

In order to relate these to multiplicity in algebraic geometry one may define a local cycle class map

$$cl^Z \in H^{2d}_Z(X, \mathbb{Z}), \quad d = \dim(Z)$$

Its image in $H^{2d}(Z, \mathbb{Z})$ is the usual cycle map.

**Theorem 8.2.** Let $E^\bullet$ be complex of locally free coherent sheaves on $X$ with support on $Z$. Then

$$ch^X(E^\bullet) = \sum (-1)^i l(H^i E^\bullet).cl^X + \ldots$$
Local chern characters and Grothendieck-Riemann-Roch map generalized as in Section 5, motivates an intersection theory on quasi-projective and may be singular varieties. This theory has been basically developed by Fulton-McPherson, however it is so adhoc to provide sensitive examples.

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