Synchronisation for scalar conservation laws via Dirichlet boundary

Ana Djurdjevac\(^1\) Tommaso Rosati\(^2\)

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Abstract

We provide an elementary proof of geometric synchronisation for scalar conservation laws on a domain with Dirichlet boundary conditions. Unlike previous results, our proof does not rely on a strict maximum principle, and builds instead on a quantitative estimate of the dissipation at the boundary. We identify a coercivity condition under which the estimates are uniform over all initial conditions, via the construction of suitable super- and sub-solutions. In lack of such coercivity our results build on \(L^p\) energy estimates and a Lyapunov structure.

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Introduction

This article concerns long-time properties of stochastic scalar conservation laws

\[
\partial_t u + \text{div} A(u) - \nu \Delta u = \xi , \quad u(0, x) = u_0(x) ,
\]

for \(t \geq 0\) and \(x \in \mathcal{D}\), where \(\mathcal{D}\) is a bounded domain with homogeneous Dirichlet boundary conditions

\[
u|_{(0, \infty) \times \partial \mathcal{D}} = 0 .
\]

In addition, \(A = (A_i)_{i=1}^d : \mathbf{R} \to \mathbf{R}^d\) is a vector-valued nonlinearity, \(u_0 \in C(\overline{\mathcal{D}})\) an initial value, \(\xi\) a noise and \(\nu > 0\) a viscosity parameter (later on, we will fix \(\nu = 1\)). Our aim is to prove exponential synchronisation:

\[
\|u_1^t - u_2^t\|_{L^1} \lesssim u_0^1, u_0^2 e^{-\mu t}, \quad \forall t \gg 1 ,
\]

\(^1\)Freie Universität Berlin, Germany; e-mail: adjurdjevac@zedat.fu-berlin.de

\(^2\)University of Warwick, UK; e-mail: t.rosati@warwick.ac.uk
for $u^i$ solution to (1) with initial condition $u_0^i$ and $\mu > 0$ some positive parameter.

Both ergodicity and synchronisation for scalar conservation laws have been much studied in past, motivated for instance by interest in mixing and turbulence for toy models in fluid dynamics. First results were obtained by Sinai [Sin91] in the periodic and viscous setting, then extended to the inviscid regime in a celebrated work [EKMS00]. Later works have concentrated on the analysis of the inviscid case ($\nu \ll 1$ or $\nu = 0$) or on infinite volume. In the latter case, we cite just some of the most recent works and refer the reader to the many references therein. Relevant to our discussion are for example results on the existence and uniqueness of invariant measures for Burgers’ on the line [DGR21] and a recent existence result for a general class of viscous scalar conservation laws [DDG22].

On finite volume much research has concentrated on the study of synchronisation either uniformly over the viscosity $\nu$ (in the regime $\nu \ll 1$) or for the inviscid case $\nu = 0$. We highlight in this direction a series of results by Boritchev [Bor18, Bor16, Bor14] (the latter being a review article) and by Debussche and Vauvelle [DV10, DV15], see also the many references therein. The work [Bor18] establishes an exponential convergence to the invariant measure in the inviscid case, building on the particular Lagrangian structure appearing if $\nu = 0$. This result does not extend to the case $0 < \nu \ll 1$. Instead, if $\nu > 0$ only polynomial rate of convergence is known [Bor16], uniformly over the viscosity parameter. Such result builds on Kruzhkov’s maximum principle in the periodic setting, which guarantees a bound on the derivative of the form $|\partial_x u| \leq C t^{-1}$, if no forcing is present and for strictly convex $A$. The result by Debussche and Vauvelle [DV15] proves instead uniqueness of the invariant measure in the inviscid case beyond the convexity assumption used by Boritchev, but their result does not provide uniform convergence rates over $\nu$. All mentioned results work in particular settings in which the random forcing $\xi$ in (1) allows for “small-noise zones” (in the language of [Bor16]), namely where the forcing vanishes, or is particularly small, for some time: in these zones one can make use of the $t^{-1}$ decay provided by the maximum principle to obtain synchronisation with a polynomial rate of convergence. Such argument does not allow, for instance, for $\xi$ to be a deterministic, time-independent and non-zero function.

By contrast, in the viscous and periodic case exponential convergence holds for just about any noise, as long as the equation is well defined, as a consequence of a strong maximum principle, following for example the original argument by Sinai. In fact, since scalar conservation laws contract over time in $L^1$, the aim in proving synchronisation is to show that the contraction is strict most of the time, providing quantitative estimates on its rate. For viscous equations with periodic boundary the contraction constant can be controlled by the (strictly positive) infimum of the fundamental solution to a linear equation.

This leads us to the present case of viscous equations with Dirichlet boundary conditions. We immediately remark that our results do not hold uniformly over the viscosity parameter, and indeed in the following we will assume that

$$\nu = 1.$$  

Yet even in this setting exponential convergence is not at first evident, since the original argument through the strict maximum principle does not work: the infimum of a fundamental solution to a linear equation with Dirichlet boundary is zero, attained at the boundary. A natural idea is to fix this issue by separating the boundary effect from the bulk behavior. Such is the approach taken by Shirikyan and coauthors (including one of the present authors) [Shi18b, Shi18a, DjS22], yet these results require the presence of a non-degenerate force and the solution of a control problem.

In the present work we take a different approach, and provide a bound on the contraction constant through an elementary estimate on the dissipation of mass at the boundary, which does not require any non-degeneracy condition. In particular, our approach solves for example the second open problem in [DjS22]. In our proof, such dissipation at the boundary
is an effect of the viscosity, rather than the nonlinearity, so there is no hope for estimates that are uniform over $\nu$. Yet we believe that if the forcing additionally satisfies conditions leading to “small-noise regions” as described above, then under the assumption that $A$ is coercive (a weaker assumption than the uniform convexity in the work by Boritchev) similar uniform estimates as those presented in [Bor16] should hold.

Coercivity appears because under such assumption in presence of Dirichlet boundary one can construct explicit super- and sub-solutions which guarantee (in absence of forcing) that $|u_t| \leq Ct^{-1}$ for some $C > 0$ depending on the domain and uniform over all initial conditions. These bounds take the rôle of Kruzhkov’s maximum principle in [Bor16]. They were employed already in [Shi18b, DjS22], and are related to $N$-waves and the appearance of shocks in scalar conservation laws. In lack of coercivity we still obtain our result, but without uniformity over all initial conditions, and we rely on $L^p$ energy estimates in the spirit of [GR00].

The construction of explicit super- and sub-solutions is particularly interesting as it suggests a possibility of proving coming down from infinity, a property understood for $\Phi^A_t$ equations [MW17], and global in time well-posedness for a wide class of scalar conservation laws driven by irregular noise (cf. [ZZZ22, PR19] for results for KPZ on the real line) without the use of the Cole-Hopf transformation, which is in general not available.

We leave this question to future investigations. Similarly, we did not present here the most general possible setting: we believe that our results apply for instance to inhomogeneous boundary or quasilinear strictly elliptic equations. In addition, synchronisation as in (3) is only one exemplary longtime property, and following similar arguments as in our proofs one can recover the existence a one-point attractor, often referred to as a one force, one solution principle.

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Notation and conventions

We define $\mathbb{N} = \{0, 1, 2, \ldots \}$ and $\mathbb{R}_+ = [0, \infty)$. For any bounded measurable set $\mathcal{D} \subseteq \mathbb{R}^d$ and $p \in [1, \infty]$ consider $L^p(\mathcal{D})$ the space of functions (modulo changes on null-sets) defined by the norm $\| \varphi \|_{L^p} = (\int_{\mathcal{D}} |\varphi(x)|^p \, dx)^{\frac{1}{p}}$, with the usual convention for $p = \infty$. For brevity we shall use the shorthand notation $\| \varphi \|_p = \| \varphi \|_{L^p}$. We write $(f, g) = \int_{\mathcal{D}} f(x) g(x) \, dx$ whenever the integral is defined. For any two topological spaces $X, Y$ we write $C(X; Y)$ for the space of continuous functions $f: X \to Y$, with the topology of uniform convergence on all compact sets. If $Y = \mathbb{R}$ we simply write $C(X) = C(X; \mathbb{R})$. Similarly, for $m, n, k \in \mathbb{N}$ we write $C^k(\mathbb{R}^m; \mathbb{R}^n) \subseteq C(\mathbb{R}^m; \mathbb{R}^n)$ for the space of $k$ times differentiable functions, with all derivatives continuous (and the topology of uniform convergence on compact sets for all derivatives). Function $F = (F_i)_{i=1}^d$ belongs to $C^1(\mathcal{D}; \mathbb{R}^d)$ if and only if for every $i = 1, \ldots, d$, $D^\alpha F_i$ continuously extends to $\overline{\mathcal{D}}$ for every multiindex $\alpha$, $|\alpha| \leq k$. For $F = (F_i)_{i=1}^d \in C^1(\mathcal{D}; \mathbb{R}^d)$ we define $\text{div}(F) = \sum_{i=1}^d \partial_{x_i} F_i$. For any set $\mathcal{X}$ and functions $f, g: \mathcal{X} \to \mathbb{R}$ we write $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq C g(x)$ for all $x \in \mathcal{X}$. If the proportionality constant depends on some parameter $\vartheta$, we may highlight this by writing $f \lesssim_{\vartheta} g$. For
any $\gamma \in (0,1]$ we denote by $C^\gamma(D)$ the Hölder space. We will denote by $P_t$ the Dirichlet heat semigroup on $D$, meaning that $P_t f$ solves $\partial_t(P_t f) = \Delta(P_t f)$ on $D$ with $P_0 f = f$ and $P_t f = 0$ on $\partial D$.

1 Main results

The precise setting in which we will work is described below.

**Assumption 1.** Here we collect the requirements on the domain, the boundary conditions, the nonlinear vector field $A$ and the noise.

1. *(Domain and boundary conditions)* Let $D \subseteq \mathbb{R}^d$ be an open domain with $C^2$ boundary $\partial D$.

2. *(Nonlinearity)* We assume that $A \in C^1(\mathbb{R};\mathbb{R}^d)$, and that there exists an $a \in [1,\infty]$ such that one of the following conditions holds, depending on the value of $a$.

   - *(Component-wise coercivity: if $a = \infty$)* There exist constants $\alpha,\beta \in \mathbb{R}^+$ such that
     \[
     A_i'(u) \text{sign}(u) \geq \alpha |u| - \beta, \quad \forall i \in \{1,\ldots,d\}, u \in \mathbb{R}.
     \]  

   - *(Polynomial growth: if $a < \infty$)* There exists a constant $C > 0$ such that
     \[
     |A(u)| \leq C (1 + |u|)^a, \quad |A'(u)| \leq C (1 + |u|)^a.
     \]

3. *(Noise)* Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ enhanced with a map $\vartheta: \mathbb{Z} \times \Omega \to \Omega$ such that $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is an ergodic, invertible metric dynamical system, supporting for some $n \in \mathbb{N}$ cocycles

   \[
   \psi_0: \Omega \to C(\mathbb{R}_+ \times D), \quad \text{and} \quad B^k: \Omega \to C(\mathbb{R}_+), \quad \forall k \in \{1,\ldots,n\},
   \]

   such that $\{B_k^k\}_{k=1}^n$ is a set of i.i.d. Brownian motions and $\psi_0$ satisfies for any $p \geq 0$ the moment estimate

   \[
   \sup_{t \geq 0} \mathbb{E}||\psi_0(t,\cdot)||_{\infty}^p < \infty. \tag{5}
   \]

   Then define

   \[
   \xi(\omega, t, x) = \psi_0(\omega, t, x) + \sum_{k=1}^{n} \psi_k(x) B_k^k(\omega),
   \]

   for some $\{\psi_k\}_{k=1}^n \subseteq C^{\alpha_0}(D)$, for some $\alpha_0 \in (0,1)$. If $a < \infty$ (for $a$ as in the previous point of the assumption), then we additionally assume that $\psi_0$ is not random, that is $\psi(\omega, t, x) = \psi_0(t, x)$, so that (5) reads

   \[
   \sup_{t \geq 0} \sup_{x \in D} |\psi_0(t, x)| < \infty.
   \]

   We observe that under Assumption 1 the noise $\xi$ can potentially be degenerate, indeed even $\xi(\omega, t, x) = h(x)$ for a Hölder-continuous, time-independent function $h$ is allowed. This reflects the fact that the main mechanism behind synchronisation is the structure of the equation, and in particular order preservation, cf. Lemma 8, which leads to an $L^1$ contraction. In particular, our choice of $\xi$ covers the setting of [DjS22] and [Shi18b], where the authors require respectively a one-dimension and a two-dimensional noise to obtain mixing. We remark the slight difference between the settings $a = \infty$ and $a < \infty$. In the first we have
extremely strong $L^\infty$ estimates, which allow us to treat also non-Markovian settings (which is the case if $\psi_0$ is allowed to be a generic cocycle): instead, if $a < \infty$, we need to use $L^p$ bounds that are less strong (in particular not uniform over all initial conditions). It is then convenient to have some kind of Markov structure on the solutions $u$: the requirement that $\psi_0$ be deterministic guarantees such a structure, but milder assumptions might be sufficient (for example requiring a finite range of dependence).

As for the regularity of the noise, it is of course of extreme interest to understand whether our methods apply to low regularity, for example due to the connection with the KPZ equation [Hai13]. Our assumption that $\psi_k \in C^\alpha$ for $k \in \{1, \ldots, n\}$ and $\alpha > 0$ reflects the fact that we need $z = (\partial_t - \Delta)^{-1} \xi \in C^\beta$ for some $\beta \geq 1$, cf. Lemma 5: we need $z$ differentiable to be able to construct the super-solution $\varphi^+$ to (7) that is required for the $L^\infty$ a-priori estimate in the case $a = \infty$, see the proof or Proposition 3.

As for the nonlinearity, the two requirements in Assumption 1 are substantially different. If $a = \infty$ we do not need any growth assumption on $A$, as we can use the coercivity to construct super- and sub-solutions that allow to control the norm $\|u_t\|_\infty$ at any positive time $t > 0$, uniformly over all initial conditions $u_0$: this control is somewhat in the spirit of the coming down from infinity property of the $\Phi^4$ model [MW17]. The coercivity assumption we impose holds component-wise, namely for each component $A'_i$ for $i \in \{1, \ldots, d\}$. In the case of smooth (in both space and time) driving noise this condition can be relaxed to the more natural

$$\sum_{i=1}^d A'_i(u)\text{sign}(u) \geq a|u| - \beta .$$

On the other hand, in the case $a < \infty$ we do not assume any particular structure on $A$, other than some polynomial growth. Here we use $L^p$ energy estimates in the spirit of [GR00], and our results are not uniform over the initial condition.

Finally, the requirement that the boundary $\partial D$ is of class $C^2$ is purely technical, so that we can apply the theory of analytic semigroups as presented in [Lan95, Chapters 2, 3].

In the setting we have introduced, let us now clarify the notion of solution that we will work with. It will be useful, also for later convenience, to decompose the solution $u$ to (1) as follows:

$$u = z + \varphi ,$$

where $z$ contains the stochastic forcing and $\varphi$ all the rest. Namely

$$\partial_t z = \Delta z + \psi_0 + \sum_{k=1}^n \psi_k \hat{B}_k^t , \quad z(0, \cdot) = 0 , \quad z|_{(0,\infty) \times \partial D}(t, \cdot) = 0 , \quad (6)$$

so that $\varphi$ should solve

$$\partial_t \varphi = \Delta \varphi - \text{div}(A(\varphi + z)) , \quad \varphi(0, \cdot) = u_0(\cdot) , \quad \varphi|_{(0,\infty) \times \partial D} = 0 . \quad (7)$$

**Definition 2.** Under Assumption 1, for any $\rho \in [a, \infty]$ and $u_0 \in L^\rho$ we say that $u$ is a mild solution to (1) if $u = z + \varphi$, with $z$ satisfying (6) and $\varphi \in C([0,\infty); L^\rho)$ given by

$$\varphi_t = P_t u_0 - \int_0^t P_{t-s} \text{div}(A(\varphi_s + z_s)) \, ds .$$

The next result guarantees that Equation (1) is well-posed for all times, together with suitable a-priori estimates, depending on the choice of $a$ in Assumption 1.
Proposition 3. Under Assumption 1 there exists a $g(a, d) \in [a, \infty]$ such that for any $u_0 \in L^p$ there exists a unique mild solution to (1) in the sense of Definition 2. In addition for some $C_1, C_2 \in (0, \infty)$

$$
\sup_{t \geq 1} E\left[ \sup_{u_0 \in L^\infty} \left\| u_s \right\|_{\infty} \right] \leq C_1 , \\
\sup_{t \geq 1} E\left[ \sup_{s \in [t, t+1]} \left\| u_s \right\|_{\infty} \right] \leq C_1 (e^{-C_2 t} \| u_0 \|_{L^p} + 1) ,
$$

if $a = \infty$ ,

if $1 \leq a < \infty$ .

The main result of this work is then the following theorem.

Theorem 4. For any $u_0, v_0 \in L^p$, with $g(a, d)$ as in Proposition 3, denote with $u_t, v_t$ the solution to (1) with initial datum $u_0, v_0$ respectively. Then there exists a constant $C > 0$ such that

$$
\lim_{t \to \infty} \sup_{u_0, v_0 \in L^\infty} \frac{1}{t} \log (\| u_t - v_t \|_{L^p}) \leq -C , \\
\lim_{t \to \infty} \sup_{u_0, v_0 \in L^\infty} \frac{1}{t} \log (\| u_t - v_t \|_{L^p}) \leq -C ,
$$

if $a = \infty$ ,

if $1 \leq a < \infty$ .

Although the statement of the theorem considers only synchronization in the $L^1$ norm, which is the simplest possible choice, our result can be lifted to higher regularity norms following similar arguments as in [Ros22] or [Ros21].

2 A priori estimates

This section is essentially devoted to a proof of Proposition 3, plus a corollary. Eventually we will require an estimate for $\varphi$, but let us start with estimates on $z$. Here we require parabolic regularity estimates for the Dirichlet heat semigroup $P_t$ on $\mathcal{D}$. Namely, we will use that $P_t$ is an analytic semigroup, cf. [Lun95, Chapters 2, 3], so that for any $0 < \alpha < \beta < \infty$ with $\alpha, \beta \not\in \mathbb{N}$ one can estimate, for some $\lambda > 0$

$$
\sup_{t \geq 0} \left\{ t^{\frac{\alpha}{\lambda}} e^{\lambda t} \| P_t f \|_{C^\beta(\mathcal{D})} \right\} \lesssim_{\alpha, \beta} \| f \|_\alpha . \tag{8}
$$

This estimate leads us to the following uniform bound on $z$.

Lemma 5. Under Assumption 1, the process $z$ defined by (6) satisfies for some $\gamma > 1, p > 0$ and $T > 0$ the following moment bound

$$
\sup_{t \geq 0} E\left[ \sup_{t \leq s \leq t+T} \| z \|_{C^\gamma(\mathcal{D})}^p \right] \leq C(T, \gamma, p) , \tag{9}
$$

for some $C(T, \gamma, p) \in (0, \infty)$.

Proof. Let us rewrite (omitting the dependence on $\omega$)

$$
z(t, x) = \int_0^t [P_{t-s} \psi_0(s, \cdot)](x) \, ds + \sum_{k=1}^n \int_0^t [P_{t-s} \psi_k](x) \, dB^k_s .
$$

Since the first term is simpler than the latter terms, let us just treat one of the last addends. Define $z^k(t) := \int_0^t [P_{t-s} \psi_k](x) \, ds$ and for $a_0$ as in Assumption 1, fix any $1 < \gamma < 1 + a_0$. 

6
Then for \( p \geq 2 \) by BDG we have for any \( x, y \in \mathcal{D} \)
\[
\mathbb{E} \left[ |\nabla z^k_t(x) - \nabla z^k_t(y)|^p \right] \lesssim \left( \int_0^t |\nabla P_{t-s} \psi_k(x) - \nabla P_{t-s} \psi_k(y)|^2 \, ds \right)^{\frac{p}{2}} \\
\lesssim |x - y|^{p(\gamma - 1)} \int_0^t \|P_{t-s} z^k\|^{2\gamma} \, ds \\
\lesssim |x - y|^{p(\gamma - 1)} \int_0^t e^{-2\lambda(t-s)} t^{-2-\alpha/\gamma} \, ds \lesssim |x - y|^{p(\gamma - 1)},
\]
so that by (8), by our continuity assumption on \( \psi_k \) in Assumption 1, and by the Kolmogorov continuity test we obtain for any \( \gamma' < \gamma \)
\[
\sup_{t \geq 0} \mathbb{E} \|z(t, \cdot)\|_C^{p(\gamma'/\gamma)} \lesssim \infty.
\]

From here (9) follows along similar lines to those we have just sketched to obtain some temporal regularity, which eventually guarantees the uniform estimate in time. □

Now we are ready to prove the uniform bounds in Proposition 3.

Proof of Proposition 3. Case \( a = \infty \). As for the well-posedness of the equation for initial data \( u_0 \in L^\infty \), we will consider mild solutions in the sense of the Definition 2. Here through a classical Picard contraction argument one obtains that for any \( M > 0 \) and almost all \( \omega \in \Omega \) there exists a \( T^{\text{fin}}(M, \omega) > 0 \) such that for all \( u_0 \in L^\infty \) satisfying \( \|u_0\|_\infty \leq M \), there exists a unique mild solution \( \varphi \) to (7) on \([0, T^{\text{fin}}(M, \omega)]\).

To construct global solutions in time we need an a-priori \( L^\infty \) estimate. For this reason define
\[
\varphi^+(t, x) = \frac{a + b \sum_{i=1}^d x_i}{t}.
\]
Let us fix an arbitrary \( T > 0 \). We will then find \( a(T, \omega), b(T, \omega) > 0 \) such that \( \varphi^+ \) is a super-solution to (7) on \([0, T]\) for any initial condition \( u_0 \in L^\infty \), in the sense that if \( \varphi \) is the solution to (7) with initial condition \( u_0 \), then
\[
\varphi(t, x) \leq \varphi^+(t, x), \quad \forall t \in (0, T^{\text{fin}}(\|u_0\|_\infty, \omega) \wedge T), x \in \mathcal{D}.
\]
To prove that \( \varphi^+ \) is a super-solution we have to fix the parameters \( a, b \). We start by defining
\[
a = b(1 + D \|z\|_{\infty, T} + T\beta/\alpha),
\]
where \( \alpha, \beta \) are as in the coercivity requirement of Assumption 1 and where
\[
D = \max_{x \in \mathcal{D}} \sum_{i=1}^d |x_i|, \quad \|z\|_{\infty, T} = \sup_{0 \leq t \leq T} \|z\|_\infty.
\]
Such choice of \( a \) will be motivated by the upcoming calculations. Indeed, we find that
\[
\partial_t \varphi^+ + \text{div}(A(\varphi^+ + z)) - \Delta \varphi^+ = -\frac{\varphi^+}{t} + \sum_{i=1}^d A'_i(\varphi^+ + z) \partial_i(\varphi^+ + z)
\]
\[
= -\frac{\varphi^+}{t} + \sum_{i=1}^d A'_i(\varphi^+ + z) \left[ \frac{b}{t} + \partial_i z \right].
\]
Our aim is then to show that if $b > 1$ is chosen sufficiently large, then
\[ \sum_{i=1}^{d} A_i' (\varphi^+ + z) \left[ \frac{b}{t} + \partial_i z \right] \geq \frac{\varphi^+}{t}. \] (10)

Let us observe that by construction, since $b \geq 1$
\[ \frac{\beta}{\alpha} + bt^{-1} - z \leq \varphi^+ \leq bt^{-1} (1 + 2D \|z\|_{\infty,T} + T\beta/\alpha). \] (11)

In particular it suffices to prove
\[ \sum_{i=1}^{d} A_i' (\varphi^+ + z) \left[ \frac{b}{t} + \partial_i z \right] \geq \frac{1 + 2D + T\|z\|_{\infty,T} + T\beta/\alpha}{b} \left( \frac{b}{t} \right)^2, \]

Using $\varphi^+ + z \geq \beta/\alpha$, which in turn, by our component-wise coercivity requirement in Assumption 1, implies $A_i' (\varphi^+ + z) \geq 0$, we can bound
\[ \sum_{i=1}^{d} A_i' (\varphi^+ + z) \left[ \frac{b}{t} + \partial_i z \right] \geq \frac{1}{2} \sum_{i=1}^{d} A_i' (\varphi^+ + z), \]

if we assume that
\[ bt^{-1} \geq 2 \sum_{i=1}^{d} \|\partial_i z\|_{\infty,T}. \] (12)

Note that the lower bound in (12) is finite almost surely, since by Lemma 5 we have $z \in C^\gamma$ for some $\gamma > 1$. In particular, under this condition and using the coercivity assumption on $A$ in Assumption 1 we can further reduce the problem to proving
\[ \frac{1}{2} \sum_{i=1}^{d} (\alpha (\varphi^+ + z) - \beta) \geq \frac{1 + 2D + T\|z\|_{\infty,T} + T\beta/\alpha}{b} \left( \frac{b}{t} \right)^2, \]

and again via (11) we reduce this to
\[ \frac{1}{2} \sum_{i=1}^{d} \alpha \frac{b}{t} \geq \frac{1 + 2D + T\|z\|_{\infty,T} + T\beta/\alpha}{b} \left( \frac{b}{t} \right)^2, \]

which is satisfied for
\[ b \geq \frac{2}{\alpha d} \frac{1 + 2D + T\|z\|_{\infty,T} + T\beta/\alpha}{\alpha d}. \] (13)

Combining (12) and (13), let us define
\[ b := \max \left\{ 2 \frac{1 + 2D + T\|z\|_{\infty,T} + T\beta/\alpha}{\alpha d}, 2 \sum_{i=1}^{d} \|\partial_i z\|_{\infty,T} \right\}. \]

Then we have proven that $\varphi^+$ is a super-solution. Analogously one can construct sub-solutions.

Case $a \in [1, \infty)$. In this case we refrain from proving well-posedness of the equation, as this is already well understood, see for example [GR00, Theorem 2.1]. Instead we concentrate on the uniform bounds and start by establishing an $L^p$ energy estimate. We can compute, assuming for simplicity $p \in 2N \setminus \{0\}$:
\[ d\|u\|_{L^p}^p = p \langle u^{p-1}, \Delta u \rangle dt - \text{div}(A(u)) \text{ dt} + \psi_0 \text{ dt} + \sum_{k=1}^{n} \psi_k \text{ dB}_k^k \]
\[ + \sum_{k=1}^{n} \frac{p(p-1)}{2} \langle u^{p-2}, \psi_k^2 \rangle \text{ dt}. \] (14)
In the spirit of [GR00], the core of the estimate lies in the cancellation
\[
\langle u^{p-1}, \text{div}(A(u)) \rangle = 0 .
\] (15)

In fact, one can rewrite by integration by parts
\[
\langle u^{p-1}, \text{div}(A(u)) \rangle = - \sum_{i=1}^{d} \int_{D} \partial_{x_i} H_i(u)(x) \, dx + c ,
\]
where \( n(x) \) is the outer unit normal to the boundary \( \partial D \) at \( x \in \partial D \), \( \Sigma \) is the \((d - 1)\)-dimensional Hausdorff measure on the boundary and \( H_i : \mathbb{R} \to \mathbb{R} \) is defined as the primitive
\[
H_i(a) = \int_0^a (p - 1)a^{p-2}A_i(r) \, dr .
\]
Then, by the divergence theorem
\[
\sum_{i=1}^{d} \int_{D} \partial_{x_i} H_i(u)(x) \, dx = \int_{\partial D} n(x) \cdot (u^{p-1}A(u))(x) \, d\Sigma(x) = 0 .
\]
Therefore (15) is proven. Moreover, we have that by the Poincaré’s inequality
\[
\frac{1}{p-1} \langle u^{p-1}, \Delta u \rangle = - \int_{D} (u^{p-2})^{\nabla u} \, dx = - \| u^{p/2-1} \nabla u \|_{L^2}^2 = - \frac{2}{p} \| \nabla (u^2) \|_{L^2}^2 .
\]
Finally, for any \( \varepsilon \in (0, 1) \) we can bound by Young’s inequality for products
\[
\langle u^{p-1}, \psi_0 \rangle \leq \| u \|_{L^{p-1}} \| \psi_0 \|_{\infty} \leq \frac{p-1}{p} \varepsilon \| u \|_{L^p}^{p-1} + \frac{1}{p} \varepsilon^{-p} \| \psi_0 \|_{\infty} .
\]
\[
\langle u^{p-2}, \sum_{k=1}^{n} \psi_k^2 \rangle \leq \| u \|_{L^{p-2}} \| \sum_{k=1}^{n} \psi_k^2 \|_{\infty} \leq \frac{p-2}{p} \varepsilon \| u \|_{L^p}^{p} + \frac{2}{p} \| \sum_{k=1}^{n} \psi_k^2 \|_{\infty}^{\frac{p}{p}} .
\]
In this way, denoting with \( M_t^{(p)} \) the martingale
\[
M_t^{(p)} = \sum_{k=1}^{n} \int_{t}^{t} p(u^{p-1}, \psi_k) \, dB_t^k ,
\] (16)
we have obtained the overall estimate by choosing \( \varepsilon \) sufficiently small and up to decreasing the value of the constant \( c > 0 \):
\[
d\| u \|_{L^p} \lesssim_{p, \psi} \{ -c \| u \|_{L^p} + 1 + \| \psi_0(t, \cdot) \|_{\infty} \} \, dt + \text{d}M_t^{(p)} .
\] (17)

Moreover, by the BDG inequality, we can control the martingale term \( M_t^{(p)} \) as follows, for any \( t, h \geq 0 \):
\[
\mathbb{E} \left[ \sup_{t \leq s \leq t+h} \left( M_s^{(p)} - M_t^{(p)} \right) \right] \lesssim \mathbb{E} \left[ \left( \int_{t}^{t+h} d\langle M^{(p)} \rangle_s \right)^{\frac{1}{2}} \right] = \mathbb{E} \left[ \left( \sum_{k=1}^{n} \int_{t}^{t+h} p^2 (u_s^{p-1}, \psi_k)^2 \, ds \right)^{\frac{1}{2}} \right] \lesssim_{p, \psi} \sqrt{n} \mathbb{E} \left[ \sup_{t \leq s \leq t+h} \| u_s \|_{L^{p-1}}^{p-1} \right] .
\]
Now we are ready to close our estimates. From (17), together with the moment bound on \( \|\psi_0(t,\cdot)\| \) in Assumption 1, we obtain for some \( c_1, c_2 > 0 \)
\[
\mathbb{E}[\|u_{t}\|_{L^p}^p] \leq e^{-c_1 t} \mathbb{E}[\|u_0\|_{L^p}^p] + c_2 .
\]  
(18)
Furthermore, again from (17), we have for any \( h \in (0,1) \)
\[
\mathbb{E} \left[ \sup_{t \leq s \leq t+h} \|u_s\|_{L^p}^p \right] \leq \mathbb{E}[\|u_t\|_{L^p}^p] + C_1 \mathbb{E} \left[ \int_t^{t+h} (1 + \|\psi_0(s,\cdot)\|_{L^p}) \, ds \right] + C_2 \sqrt{h} \mathbb{E} \left[ \sup_{t \leq s \leq t+h} \|u_s\|_{L^{p-1}}^p \right] + \mathbb{E}[\|u_t\|_{L^p}^p] \geq \mathbb{E}[\|u_t\|_{L^p}^p] + \sqrt{h} \left( 1 + \mathbb{E} \left[ \sup_{t \leq s \leq t+h} \|u_s\|_{L^{p-1}}^p \right] \right) ,
\]
so that by a Gronwall-type argument, choosing \( h \) sufficiently small and iterating the bound, we deduce that
\[
\mathbb{E} \left[ \sup_{t \leq s \leq t+1} \|u_s\|_{L^p}^p \right] \lesssim \mathbb{E}[\|u_t\|_{L^p}^p] \lesssim e^{-c_1 t} \mathbb{E}[\|u_0\|_{L^p}^p] + 1 ,
\]  
(19)
where in the last estimate we used (18). This concludes the proof of the a-priori \( L^p \) estimate: note that if we could have chosen \( p = \infty \) the proposition would be proven.

Instead, to obtain the \( L^\infty \) estimate we simply bootstrap our argument using the Schauder estimates as in (8). Indeed, for any \( t, h \geq 0 \) we represent the solution \( u_{t+h} \) by
\[
u_{t+h} = P_h u_t + \int_0^h P_{h-t} [\text{div}(A(u_{t+r})) + \psi_0] \, dr + \sum_{k=1}^n \int_t^{t+h} P_{t+h-k} \psi_k \, dB_k^p .
\]
Now recall the following Schauder estimates and Sobolev embeddings for \( \alpha \geq 0, \alpha \geq \beta > 0 \) and \( \kappa > 0 \):
\[
\|P_t \varphi\|_{W^{\alpha,p}} \lesssim t^{-\frac{\alpha}{p}} \|\varphi\|_{L^p} , \quad \|P_t \varphi\|_{W^{\beta,p}} \lesssim t^{-\frac{\beta}{p}} \|\varphi\|_{W^{\beta,p}} ,
\]
\[
\|\varphi\|_{L^\infty} \lesssim \|\varphi\|_{W^{\frac{\alpha}{\alpha-p},p}} .
\]  
(20)
Then for \( p \geq 1 \) and \( \kappa > 0 \) (to be chosen respectively sufficiently large and sufficiently small later on) we find, for \( h > 0 \)
\[
\|u_{t+h}\|_{L^\infty} \lesssim h^{-\frac{1}{p} \left( \frac{\alpha}{\alpha-p} + \kappa \right)} \|u_t\|_{L^p} + \int_0^h \|P_{h-t} [\text{div}(A(u_{t+r}))]\|_{W^{\frac{\alpha}{\alpha-p},p}} \, dr + \|z_{t,h}\|_{L^\infty} ,
\]
where \( z_{t,h} = \int_t^{t+h} P_{h-r} \psi_0(r,\cdot) \, dr + \int_t^{t+h} P_{h-r} \psi_k \, dB_k^p \). As for the second term, we estimate
\[
\int_0^h \|P_{h-t} [\text{div}(A(u_{t+r}))]\|_{W^{\frac{\alpha}{\alpha-p},p}} \, dr \lesssim \int_0^h (h-r)^{-\frac{1}{p} \left( \frac{\alpha}{\alpha-p} + 2\kappa \right)} \|\text{div}(A(u_{t+r}))\|_{W^{1-\alpha,p}} \, dr .
\]
Hence, assuming that \( p \) is chosen sufficiently large and \( \kappa \) sufficiently small, so that
\[
\gamma \overset{\text{def}}{=} 1 + \frac{d}{p} + 2\kappa < 2 ,
\]
and estimating
\[
\|\text{div}(A(u_{t+r}))\|_{W^{1-\alpha,p}} \lesssim \|A(u_{t+r})\|_{L^p} \lesssim \|u_{t+r}\|_{L^{sp}} ,
\]
we obtain
\[
\sup_{0 \leq s \leq h^\gamma/2} \|u_{t+s}\|_{L^\infty} \lesssim \|u_t\|_{L^p} + h \sup_{0 \leq s \leq h} \|u_{t+s}\|_{L^{sp}} + \sup_{0 \leq s \leq h} \|z_{t,s}\|_{L^\infty} .
\]  
(21)
Combining (19) with (21), together with our estimates on \( z_{t,s} \) from Lemma 5 we can conclude. \( \square \)
As a by-product of the proof of Proposition 3 we have actually proven the following statement.

**Corollary 6.** Assume that \( a = \infty \). For almost every \( \omega \in \Omega \) there exists a constant \( c(\omega) \in (0, \infty) \) such that

\[
\sup_{u_0 \in L^\infty} \sup_{1 \leq s \leq 2} \| u_s(\omega) \|_{L^\infty} < c(\omega).
\]

## 3 Synchronisation

Throughout this section we consider, for two fixed initial conditions \( u_0, v_0 \in L^6 \), where \( g(d, a) \in [a, \infty] \) is chosen as in Proposition 3, the difference

\[
w(t, x) = u(t, x) - v(t, x), \quad \forall (t, x) \in [0, \infty) \times \mathcal{D},
\]

where \( u_t, v_t \) are the solutions to (1) with respectively initial conditions \( u_0, v_0 \). Then \( w \) solves

\[
\partial_t w + \text{div}(B(u, v)w) - \Delta w = 0, \quad w|_{(0, \infty) \times \partial \mathcal{D}} = 0,
\]

with initial condition \( w_0 = 0 \) and where

\[
B(u, v) = \frac{A(u) - A(v)}{u - v}, \quad B \in C(\mathbb{R}^2; \mathbb{R}^d).
\]

Here the fact that \( B \in C(\mathbb{R}^2; \mathbb{R}^d) \) follows from the requirement that \( A \in C^1 \) in Assumption 1. It will be convenient to consider solutions to (22) for general initial conditions \( w_0 \). In this case we write

\[
(t, x) \mapsto \Phi(u, v, w_0; t, x)
\]

for the solution to (22) with initial condition \( w_0 \in L^1 \) and driven by paths \( u, v \in C([0, \infty); L^6) \), with \( g \geq g(a, d) \), the latter as in Proposition 3.

**Lemma 7.** There exists a \( \Gamma(a, d) \geq g(a, d) \) (the latter as in Proposition 3), such that for every \( u, v \in C([0, \infty); L^6) \) and every \( w_0 \in L^1 \) there exists a unique solution \( w \) to (22), which can in addition be represented by

\[
w(t, x) = \int_{\mathcal{D}} \Gamma(u, v; t, x, y) w_0(y) \, dy, \quad \forall t > 0.
\]

where \( \Gamma(u, v; \cdot) \in C([0, \infty) \times \mathcal{D} \times \mathcal{D}) \).

**Proof.** Clearly it suffices to construct the fundamental solution \( (t, x) \mapsto \Gamma(u, v; t, x, y) \), which solves (22) with initial condition the Dirac delta centered at \( y \): \( x \mapsto \delta_y(x) \). Note that by Assumption 1, in particular via our growth requirement on \( A \) and \( A' \), we can find for any \( \zeta > 0 \) a \( \Gamma(a, d) > 0 \) such that \( B(u, v) \in C([0, \infty); L^6) \) if \( u, v \in C([0, \infty); L^6) \). Then the fundamental solution can be constructed via classical arguments, by smoothing the initial data from the Besov space \( B^{6}_{\infty, 1} \) to \( L^6 \), with \( \zeta^{-1} + (\zeta')^{-1} = 1 \), see for example [GP17, Chapter 6].

The property which eventually delivers synchronisation is an \( L^1 \) contraction principle for (1), whose proof is elementary and relies on the following computation:

\[
\int_{\mathcal{D}} w_t \, dx = \int_0^t \int_{\mathcal{D}} \Delta w_s + \text{div}(B(u_s, v_s)w_s) \, dx \, ds + \int_{\mathcal{D}} w_0 \, dx
\]

\[
= \int_0^t \int_{\partial \mathcal{D}} \mathbf{n} \cdot (\nabla w_s + B(u_s, v_s)w_s) \, d\Sigma \, ds + \int_{\mathcal{D}} w_0 \, dx
\]

\[
= \int_0^t \int_{\partial \mathcal{D}} \mathbf{n} \cdot \nabla w_s \, d\Sigma \, ds + \int_{\mathcal{D}} w_0 \, dx,
\]

(23)
by the divergence theorem. In the last line we have used that \( w|_{\partial D} = 0 \). Now we observe that if \( w_0 \geq 0 \), then by a maximum principle \( w_t \geq 0 \) for any \( t \geq 0 \) and in addition (see Lemma 9 below for a proof)

\[
\int_{\partial D} n \cdot \nabla w_t \, d\Sigma \leq 0 ,
\]

which from (23) implies that for \( w_0 \geq 0 \) we have

\[
\int_D w_t(x) \, dx \leq \int_D w_0(x) \, dx .
\]

From this we deduce the contraction principle as follows.

**Lemma 8.** Under Assumption 1, for any \( u_0, v_0 \in L^p \), with \( g(a,d) \) as in Proposition 3 and \( u_t, v_t \) solutions to (1) with respectively initial conditions \( u_0 \) and \( v_0 \), and \( w_t = u_t - v_t \), for any \( t \geq 0 \) we have that \( \|w_t\|_{L^1} \leq \|w_0\|_{L^1} \).

**Proof.** We decompose \( w_0 = (w_0)^+ - (w_0)^- \) in its positive and negative part, namely with \( x_+ = \max\{x,0\} \) and \( x_- = \max\{-x,0\} \). Then by linearity of (22) we have \( w_t = w_t^+ - w_t^- \), where \( w_t^\pm \) is the solution to (22) with initial condition \( (w_0)^\pm \). Then by a maximum principle we have \( w_t^\pm \geq 0 \). In addition, by (25) we conclude that \( \|w_t\|_{L^1} \leq \|w_0^+\|_{L^1} + \|w_0^-\|_{L^1} \leq \|w_0\|_{L^1} \).

In particular, the essence of our proof of synchronisation is to show that a strict contraction happens with positive probability, namely that \( \|w_t\|_{L^1} < \zeta \|w_0\|_{L^1} \) for some random and shift-invariant \( \zeta \in [0,1] \) such that \( P(\zeta < 1) > 0 \) (actually the argument in the case \( a < \infty \) will be slightly more involved, as the \( \zeta \) we obtain will not be shift-invariant). For this purpose, we will need a quantitative bound on the flux at the boundary (24).

To state the next result, consider the solution \([0, \infty) \ni s \mapsto X_s^t \in \mathbb{R}^d\) to the SDE

\[
dX_s^t = B(u_{t+s}, v_{t+s})(X_s) \, ds + \, dw_s ,
\]

where \( W \) is a \( d \)-dimensional Brownian motion, and \( \tau \) is the stopping time \( \tau = \inf\{t \geq 0 : X_s^t \in \partial D\} \): the solution to the SDE is defined only up to time \( \tau \), so for simplicity we define \( X_s^t = X_{\tau}^t \) for \( \tau \geq s \). Since both \( u \) and \( v \) lie in \( C([t, t+h] \times \overline{D}) \) for any \( t, h > 0 \) in both cases \( a < \infty \) and \( a = \infty \) by Proposition 3, it is straightforward to see that (26) admits a unique solution. In particular, the quantity

\[
p_{t,t+h} = \inf_{y \in D} \mathbb{P}_y(X_t \in \partial D) \in (0,1)
\]

is well defined for any \( t, h > 0 \). The next result tells us that \( p_{t,t+h} \) bounds the dissipation at the boundary of solutions to (22).

**Lemma 9.** Under Assumption 1 consider \( u_0, v_0 \in L^p \), for \( g(a,d) \in [a, \infty] \) as in Proposition 3 and \( u_t, v_t \) the solutions to (1) with respectively initial condition \( u_0 \) and \( v_0 \). Moreover, for any non-negative initial condition \( \Phi_0 = w_0 \geq 0, w_0 \in L^p \), let \( (t,x) \mapsto \Phi(u,v,w_0,t,x) \) be the solution to (22) with initial condition \( w_0 \). Then for any \( t, h > 0 \) and \( \omega \in \Omega \) there exists a \( p_{t,t+h}(\omega, u_0, v_0) \in (0,1) \) such that

\[
\int_0^{t+h} \int_{\partial D} n(x) \cdot \nabla \Phi(u,v,w_0; s, x) \, d\Sigma(x) \, ds \leq -p_{t,t+h} \|w_0\|_{L^1} .
\]

**Proof.** We can represent the solution \( w \) through the kernel \( \Gamma \) associated to (22). Namely for any \( y \in D \), let \( \Gamma(u,v,t,x,y) = \Phi(u,v,\delta_y; t,x) \) be the solution to (22) with initial condition \( \Gamma(u,v;0,x,y) = \delta_y(x) \). Then by linearity of (22) we have \( \Phi(u,v,w_0; t,x) = \int_0^t \int_D \Gamma(u,v,w_0; s,x) \, dx \, ds \).\]
there exist constants we have

\( \frac{8}{9} \) and the calculation in (\ref{eq:example})

consider

\( \text{the choice of the domain } \mathcal{D} \)

Then consider the ode

As in the proof of Lemma \ref{lem:example}.

Proof.

This result follows from Girsanov’s transformation. Fix the drift (\( s, x \)) defined by

\( \psi^t_s(x) = -B(u_{t+s}, v_{t+s})(x) + C h^{-1} c_1, \quad \forall s \in [0, h], \)

for \( x \in \mathcal{D} \), with \( c_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d \) and \( C > 0 \) a positive constant to be chosen later on. Then consider the ode

\( \partial_s X^t_s \psi^t_s = B(u_{t+s}, v_{t+s})(X^t_s \psi^t_s) + \psi^t_s(X^t_s \psi^t_s), \forall s \in [0, h], \)

so that \( X^t_s \psi^t = X^t_0 \psi^t + C e_1 \not\in \mathcal{D} \) for all \( X^0_n \psi^0_n \in \mathcal{D} \), provided \( C > 0 \) is chosen sufficiently large, depending on the domain \( \mathcal{D} \).
Now, since \( \psi^t \in \mathcal{CM} \), the Cameron–Martin space of \( W \), we find that under the probability measure \( \mathbb{P}_y^{\psi^t} \) given by the Radon–Nikodym derivative
\[
\frac{d\mathbb{P}_y^{\psi^t}}{d\mathbb{P}_y} = \exp \left\{ \int_0^h \psi^t_s \, dW_s - \frac{1}{2} \int_0^h |\psi^t_s|^2 \, ds \right\},
\]
the solution \( X^t_s \) to the SDE (26) solves the SDE
\[
dX^t_s = Cc \, ds + d\tilde{W}_s, \quad X^t_0 = y, \quad \forall s \in [0, h \wedge \tau],
\]
where \( \tilde{W} \) is a Brownian motion under \( \mathbb{P}_y^{\psi^t} \). In particular, under \( \mathbb{P}_y^{\psi^t} \) we have \( X^t_h = y + \tilde{W}_h + Cc \). Hence if \( C > 0 \) is sufficiently large so that \( (D + B_1(0) + Cc_1) \cap D = \emptyset \) (here \( B_1(0) \) is the ball of unit Euclidean radius centered about zero and we consider the sum of sets \( A + B = \{a + b : a \in A, b \in B\} \)), we can conclude that
\[
\inf_{y \in \mathcal{D}} \mathbb{P}_y^{\psi^t}(X^t_h \not\in D) \geq \mathbb{P}_y^{\psi^t}(|\tilde{W}_h| \leq 1) \geq \mathbb{P}_y^{\psi^t}(|\tilde{W}_1| \leq 1) = c,
\]
for some constant \( c > 0 \), where we used that \( h \in [0, 1] \). Now we would like to pass from the probability measure \( \mathbb{P}_y^{\psi^t} \) to the original probability measure \( \mathbb{P}_y \):
\[
\mathbb{P}_y^{\psi^t}(A) = \mathbb{E}_y \left[ 1_A \exp \left\{ \int_0^h \psi^t_s(X^t_s) \, dW_s - \frac{1}{2} \int_0^h |\psi^t_s(X^t_s)|^2 \, ds \right\} \right] \\
\leq \mathbb{P}_y(A)^{\frac{1}{2}} \mathbb{E}_y \left[ \exp \left\{ 2 \int_0^h \psi^t_s(X^t_s) \, dW_s - 2 \int_0^h |\psi^t_s(X^t_s)|^2 \, ds \right\} \right]^{\frac{1}{2}} \\
\leq \mathbb{P}_y(A)^{\frac{1}{2}} \mathbb{E}_y \left[ \exp \left\{ \int_0^h |\psi^t_s(X^t_s)|^2 \, ds \right\} \right]^{\frac{1}{2}}.
\]
In addition, we have that
\[
\sup_{x \in \mathcal{D}} |\psi^t_s(x)| \leq \sup_{0 \leq s \leq h} \|B(u_{t+s}, v_{t+s})\|_\infty + Ch^{-1}.
\]
We therefore conclude
\[
\inf_{y \in \mathcal{D}} \mathbb{P}_y(X^t_h \not\in D) \geq \left( \inf_{y \in \mathcal{D}} \mathbb{P}_y^{\psi^t}(X^t_h \not\in D) \right)^{\frac{1}{2}} \mathbb{E}_y \left[ \exp \left\{ \int_0^h |\psi^t_s(X^t_s)|^2 \, ds \right\} \right]^{-\frac{1}{2}} \\
\geq \mathbb{E}_y \left[ \exp \left\{ \int_0^h |\psi^t_s(X^t_s)|^2 \, ds \right\} \right] \\
\geq \mathbb{E}_y \left[ \exp \left\{ -h^{-1} \left( C + \sup_{0 \leq s \leq h} \|B(u_{t+s}, v_{t+s})\|_\infty \right) \right\} \right],
\]
with \( c \) as in (28) and by Jensen’s inequality. This concludes the proof of the lemma, up to choosing a smaller \( c > 0 \).

We want to make use of the previous result in combination with the ergodic theorem. Therefore, let us write \( \mathcal{S}_t(\omega) \) for the (random) solution map to (1) at time \( t \geq 0 \): namely \( \psi(t, \omega) = \mathcal{S}_t(\omega)u_0 \). Then we can bound, for every \( n \in \mathbb{N} \setminus \{0\} \):
\[
\sup_{0 \leq s \leq 1} \|B(u_{n+s}, v_{n+s})\|_\infty \leq \sup_{u_0, v_0 \in L^\infty} \|B(\mathcal{S}_{1+s}(\varphi^{n-1} \omega)u_0, \mathcal{S}_{1+s}(\varphi^{n-1} \omega)v_0)\|_\infty \\
\leq \sup_{a, b \in B(\varphi^{n-1} \omega)} B(a, b) \overset{\text{def}}{=} c_B(\varphi^{n-1} \omega) < \infty,
\]
(29)
with $c(\omega)$ as in Corollary 6 and where $B_0(0)$ is the ball of radius $\rho > 0$ about zero.

**Proof of Theorem 4.** As usual we distinguish the case $a = \infty$ from the case $a < \infty$.

**Case $a = \infty$.** Here we can use Lemma 9 together with (23) (and following the notation in the proof of Lemma 9) to bound, for every $n \in \mathbb{N} \setminus \{0\}$

$$
\|w_{n+1}\|_{L^1} \leq \|w_{n+1}^+\|_{L^1} + \|w_{n+1}^-\|_{L^1} \\
\leq (1 - p_{n,n+1})(\|w_{n}^+\|_{L^1} + \|w_{n}^-\|_{L^1}) \leq (1 - p_{n,n+1})\|w_{n}\|_{L^1},
$$

Therefore we obtain

$$
\lim_{n \to \infty} \frac{1}{n} \log \|w_n\|_{L^1} \leq \lim_{n \to \infty} \frac{1}{n} \left( \sum_{i=0}^{n-1} \log (1 - p_{i,i+1}) + \log \|w_0\|_{L^1} \right).
$$

Now we use the bound in Lemma 11 together with (29) to obtain by the ergodic theorem, almost surely

$$
\lim_{n \to \infty} \frac{1}{n} \log \|w_n\|_{L^1} \leq \lim_{n \to \infty} \frac{1}{n} \left( \sum_{i=1}^{n-1} \log \left( 1 - \exp \left\{ -c_2^2 B(\vartheta_i - 1) \right\} \right) \right) \\
= \mathbb{E} \left[ \log \left( 1 - \exp \left\{ -c_2^2 \right\} \right) \right] < 0.
$$

Since almost surely

$$
\lim_{n \to \infty} \frac{1}{n} \left\{ \log (1 - p_{0,1}) + \log \|w_0\|_{L^1} \right\} = 0,
$$

which concludes the proof.

**Case $a < \infty$.** Here we use the results of Section 3.1 below. In that section we introduce a constant $R_2 \in (0, \infty)$ and a sequence of stopping times

$$
0 \leq \tau_0^{\text{in}} < \cdots < \tau_i^{\text{in}} < \tau_i^{\text{out}},
$$

such that $\max\{\|u_i\|_{\infty}, \|v_i\|_{\infty}\} \leq R_2$ for $t \in [\tau_i^{\text{in}}, \tau_i^{\text{out}}]$ and $i \in \mathbb{N}$. Then let us define

$$
B = B(R_2) = \max \{B(u, v) : \max\{|u|, |v|\} \leq R_2 \}.
$$

Moreover, by construction $T_i^\infty = \tau_i^{\text{out}} - \tau_i^{\text{in}} \leq 1$, so that we are in the setting of Lemma 11 and can bound

$$
\log \|w_t\|_{L^1} \leq \sum_{i : \tau_i^{\text{out}} \leq t} \log p_{\tau_i^{\text{in}}, \tau_i^{\text{out}}} + \log \|w_{\tau_i^{\text{in}}}\|_{L^1} \leq - \sum_{i : \tau_i^{\text{out}} \leq t} c(T_i^\infty, B) + \log \|w_{\tau_i^{\text{in}}}\|_{L^1},
$$

where the constants $c(T_i^\infty, B)$ are given by

$$
c(T_i^\infty, B) = - \log \left\{ 1 - \exp \left( (T_i^\infty)^{-1} (C + B) \right) \right\}, \quad (30)
$$

with $c, C > 0$ defined in Lemma 11. Now by Proposition 12 there exits an $\eta = \eta(B) > 0$ such that

$$
\limsup_{t \to \infty} \frac{1}{t} \log \|w_t\|_{L^1} \leq -\eta,
$$

so that the result is proven.
3.1 Lyapunov estimates

In this section we consider the setting of Assumption 1 in the case \( a < \infty \). We then choose two initial condition \( u_0, v_0 \in L^p \), for \( \varphi(a, d) \) as in Proposition 3, and associated respectively to two solutions \( u_t, v_t \) of (1) driven by the same noise. The aim of this section is then to derive estimates on the time that the couple \( u_t = (u_t, v_t) \) spends in the “center” of the state space, i.e. a ball of finite volume about the origin in \( L^p \). Here we will make use of the last requirement in Assumption 1, that \( \psi_0 \) is not random, so that the process \( u_t \) is Markov.

For a 2D vector \( u = (u, v) \in L^p(D; \mathbb{R}^2) \) we will make use of the norms \( \|u\|_{L^p} = (\|u\|_{\ell^p}^p + \|v\|_{\ell^p}^p)^{1/2} \) for \( p \in [1, \infty] \), with the usual convention for \( p = \infty \). We fix any \( 0 < R_1 < R_2 < R_3 < \infty \) (we shall fix these values in Lemma 13) and define the “center” subsets of \( L^p \)

\[
C^- = \{ u \in L^p : \|u\|_{L^p} \leq R_1 \}, \quad C^m = \{ u \in L^p : \|u\|_{L^p} \leq R_2 \},
\]

\[
C^+ = \{ u \in L^p : \|u\|_{L^p} \leq R_3 \}.
\]

Our aim will then be to prove that the process \( u_t \) passes a substantial amount of time in the set \( C^+ \): this will guarantee an \( L^1 \) contraction with rate roughly dependent on \( R_1 \). Actually, since the contraction depends on the \( L^\infty \) norm of \( u_t \), we will have to additionally make use of parabolic regularisation to show that once in \( C^+ \), the process is likely to stay bounded also in \( L^\infty \). The reason why we consider the \( L^p \) norm in the definition of \( C^\pm \) is that in the energy estimates of Proposition 3 we have shown that if \( a < \infty \), the \( L^p \) norms are Lyapunov functionals for the process \( u_t \), so our estimates on the time passed in \( C^+ \) will follow from classical bounds on return times for Markov processes. In any case, for some \( 0 < R_1 < R_2 < \infty \) to be chosen later on, define also the following subsets of \( L^\infty \)

\[
C^-_\infty = \{ u \in L^\infty : \|u\|_\infty \leq R_1 \}, \quad C^+_\infty = \{ u \in L^\infty : \|u\|_\infty \leq R_2 \}.
\]

Then we consider two types of excursions: excursions in the center and from the center. Suppose that \( u_t \not\in C^- \) (otherwise we would have to define the stopping times starting from \( \sigma_0 = 0 \), set \( \tau_0 = 0 \) and then define for \( i \in \mathbb{N} \)

\[
\sigma_i = \begin{cases} 
\inf\{ t \geq \tau_i : u_t \in C^- \} & \text{if } u_{\tau_i} \notin C^m, \\
\tau_i & \text{if } u_{\tau_i} \in C^m,
\end{cases}
\]

\[
\tau_{i+1} = \inf\{ t \geq \sigma_i : u_t \notin C^+ \} \wedge (\sigma_i + 1),
\]

\[
\tau_{i+1}^{in} = \inf\{ t \geq \sigma_i : u_t \in C^\infty \} \wedge T_{\tau_{i+1}^{in}},
\]

\[
\tau_{i+1}^{out} = \inf\{ t \geq \tau_{i+1}^{in} : u_t \notin C^+ \} \wedge \tau_{i+1}.
\]

Then \( \{ \sigma_i - \tau_{i-1} \}_{i \in \mathbb{N}} \) is the succession of outer excursions lengths (return times from \( C^\infty \) to \( C^- \)) and \( \{ \tau_i - \sigma_{i-1} \}_{i \in \mathbb{N}\setminus\{0\}} \) is the succession of internal excursion lengths, roughly from \( C^- \) to \( C^\infty \): we have introduced the middle shell \( C^m \) so that for later convenience we have the additional property \( \tau_i \leq 1 \) (otherwise we could have simply defined \( \tau_{i+1} = \inf\{ t \geq \sigma_i : u_t \notin C^+ \} \) and the second case in the definition of \( \sigma_i \) would not be necessary, as well as the distinction between \( C^m \) and \( C^+ \)). Similarly we consider \( \{ T_{i}^{\infty} = \tau_{i+1}^{out} - \tau_{i}^{in} \}_{i \in \mathbb{N}\setminus\{0\}} \). For later convenience, let us also define the following stopping times, for a given \( u \in L^p \) (here \( u_t \) is the evolution of the Markov process \( (u_t, v_t) \) with initial condition \( u_0 = u) \):

\[
S(u) = \begin{cases} 
\inf\{ t \geq 0 : u_t \in C^- \} & \text{if } u \notin C^m, \\
0 & \text{if } u \in C^m,
\end{cases}
\]

\[
T(u) = \inf\{ t \geq 0 : u_t \notin C^+ \} \wedge 1,
\]

\[
T^{in}(u) = \inf\{ t \geq 0 : u_t \in C^\infty \} \wedge T(u),
\]

\[
T^{out}(u) = \inf\{ t \geq T^{in}(u) : u_t \notin C^+ \} \wedge T(u),
\]

for \( u \in C^m \)
Finally define for any $B > 0$

$$X_t = \sup \left\{ n \in \mathbb{N} : \sum_{i=0}^{n} S_i \leq t \right\}, \quad L_t(B) = \sum_{i=1}^{X_t} c(T_i^\infty, B),$$  \hspace{1cm} (33)

which are respectively the total number of excursions by time $t$ and a functional of the time spent in the $L^\infty$–norm center: the constant $c$ is defined in (30). We want to prove the following result.

**Proposition 12.** Under Assumption 1, in the case $a < \infty$, consider the center sets defined in (31) and (32) with constants $p(a,d) \in [a,\infty)$, $0 < R_1 < R_2 < R_3 < \infty$ and $0 < R_1 < R_2 < \infty$ as in Lemma 13. Then for any $B > 0$ there exists a positive (deterministic) $\eta(B) > 0$ and $p(a,d) \in [a,\infty)$ such that for any initial condition $u \in L^p$

$$\liminf_{t \to \infty} \frac{L_t(B)}{t} \geq \eta, \quad \mathbb{P}–\text{almost surely},$$

with $L_t(B)$ as in (33).

**Proof.** By the martingale law of large numbers (from here on MLLN) we find

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} c(T_i^\infty, B) \geq \inf_{u \in \mathbb{C}^m} \mathbb{E}[c(T^{\text{out}}(u) - T^{\text{in}}(u), B)] \overset{\text{def}}{=} \gamma_1 \in [0, \infty).$$  \hspace{1cm} (34)

In particular, if we can in addition prove the following statement:

$$\liminf_{t \to \infty} \frac{X_t}{t} \geq \inf_{u \in (\mathbb{C}^\infty)^\times} \frac{1}{\mathbb{E}[S(u)]} \overset{\text{def}}{=} \gamma_2 \in [0, \infty), \quad \mathbb{P}–\text{almost surely},$$  \hspace{1cm} (35)

we would be able to conclude

$$\liminf_{t \to \infty} \frac{L_t(B)}{t} = \liminf_{t \to \infty} \frac{X_t}{t} \frac{1}{X_t} \sum_{i=0}^{X_t} T_i^\infty \geq \frac{\inf_{u \in \mathbb{C}^m} \mathbb{E}[c(T^{\text{out}}(u) - T^{\text{in}}(u), B)]}{\sup_{u \in (\mathbb{C}^\infty)^\times} \mathbb{E}[S(u)]} = \gamma_1 \cdot \gamma_2 \overset{\text{def}}{=} \eta \in [0, \infty).$$

Finally, to show that $\eta > 0$ we would need

$$\inf_{u \in \mathbb{C}^m} \mathbb{E}[c(T^{\text{out}}(u) - T^{\text{in}}(u), B)] > 0, \quad \sup_{u \in (\mathbb{C}^\infty)^\times} \mathbb{E}[S(u)] < \infty.$$  \hspace{1cm} (36)

Hence, to conclude, we have to check (34), (35) and (36): the latter follows from Lemma 13, so we restrict to proving the first two.

**Proof of (34).** We make use of the MLLN applied to the following sum, with $\mathcal{G}_i = \mathcal{F}_{\sigma_i}$ for $i \in \mathbb{N}$:

$$P_n = \sum_{i=0}^{n} T_i^\infty - \nu_i, \quad \nu_i = \mathbb{E}[T_i^\infty | \mathcal{G}_i].$$

In particular, since $T_i^\infty \leq T_i \leq 1$, we have

$$\sum_{i=0}^{\infty} \frac{\mathbb{E}[T_i^\infty - \nu_i]^2}{i^2} < \infty,$$
so that P–almost surely and in $L^1$ we obtain the convergence $n^{-1}P_n \to 0$. Hence, since by the Markov property of $u_i$ we have $\nu_i \geq \gamma_1$ for each $i \in \mathbb{N}$, $\mathbb{P}$–almost surely, the claim follows.

Proof of (35). Suppose that (35) does not hold and that on the contrary, for some $\alpha \in (0, 1)$ and along a subsequence $\{t_k\}_{k \in \mathbb{N}}$ such that $t_k \uparrow \infty$, we have $X_{t_k} < \gamma_2 \alpha t_k$. From the definition of $X_t$ we deduce that

$$
\frac{1}{\gamma_2 \alpha t_k} \sum_{i=1}^{\gamma_2 \alpha t_k} S_i \geq \frac{t_k}{\gamma_2 \alpha t_k} \sim \frac{1}{\gamma_2 \alpha}, \quad \forall k \in \mathbb{N}.
$$

(37)

Now let $\mu_i = \mathbb{E}[S_i | \tilde{G}_i]$ where $\tilde{G}_i = \mathcal{F}_{\sigma_i}$ is the filtration generated by the excursion start times $\{\sigma_{i+1}\} \in \mathbb{N}$. Then

$$
M_n \overset{\text{def}}{=} \sum_{i=1}^{n} (S_i - \mu_i)
$$

is a martingale with respect to the discrete filtration $(\tilde{G}_i)_{i \in \mathbb{N}}$. Hence, as above, by the moment bound Lemma 13, we can conclude by the MLLN that almost surely and in $L^1$

$$
\frac{1}{n} M_n \to 0.
$$

In particular, by our assumption on $\alpha$ and $t_k$

$$
\frac{1}{\alpha} \frac{1}{\gamma_2} \leq \liminf_{k \to \infty} \frac{1}{\gamma_2 \alpha t_k} \sum_{i=1}^{\gamma_2 \alpha t_k} S_i \leq \limsup_{k \to \infty} \frac{1}{\gamma_2 \alpha t_k} \sum_{i=1}^{\gamma_2 \alpha t_k} \mu_i \leq \frac{1}{\alpha},
$$

which is a contradiction, since $\alpha \in (0, 1)$. This concludes the proof of our result. $\square$

We conclude with a moment estimate on our excursion times.

Lemma 13. Under Assumption 1, in the case $a < \infty$, there exist $p(a, d) \in [a, \infty)$, $0 < R_1 < R_2 < \infty$ and $0 < \overline{R}_1 < \overline{R}_2 < R_3 < \infty$ such that for some $\kappa > 0$ and any $B > 0$:

$$
\inf_{u \in \mathcal{C}^a} \mathbb{E}[c(T^{\text{out}}(u) - T^{\text{in}}(u), B)] \geq 0, \quad \mathbb{E}[\exp(\kappa S(u))] \leq C(1 + \|u\|_{L^p}), \quad \forall u \in L^p,
$$

where the center sets are defined in terms of $R_i, \overline{R}_i$ as in (31) and (32).

Proof. Bound on $S(u)$. For $u \in L^p$, following the calculations in the proof of Proposition 3, we have that for some $c_1, c_2 > 0$:

$$
d\|u_t\|_{L^p}^p \leq (-c_1\|u\|_{L^p}^p + c_2)dt + dM_t^{(p)},
$$

(38)

for a continuous square integrable martingale $M_t^{(p)}$. Since for $u \in \mathcal{C}^a$ the bound is trivial, let us assume that $u \not\in \mathcal{C}^m$. Hence, for any $\kappa > 0$

$$
d(e^{\kappa t}\|u_t\|_{L^p}^p) \leq e^{\kappa t}(-c_1\|u\|_{L^p}^p + c_2 + \kappa\|u\|_{L^p}^p) dt + e^{\kappa t} dM_t^{(p)}.
$$

Assuming that $R_1^p \geq 2c_2c_1^{-1} + 1$ and $\kappa \leq c_1/2$, we deduce that for any $n \in \mathbb{N}$

$$
e^{\kappa S\wedge n}\|u_{S\wedge n}\|_{L^p}^p + \frac{c_1}{2} \int_0^{S\wedge n} e^{\kappa r} dr \leq \int_0^{S\wedge n} e^{\kappa r} dM_r^{(p)} + \|u_0\|_{L^p}^p.
$$

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In particular we conclude that
\[
\kappa^{-1} \mathbb{E} [e^{\kappa S^a - \kappa n}] = \mathbb{E} \left[ \int_0^{S^a \wedge n} e^{\kappa r} \, dr \right] \leq 2c_1^{-1} \|u\|^p.
\]
Sending \(n \to \infty\) allows us to deduce that \(\mathbb{E}[\exp\{\kappa S\}] < C(\kappa)(1 + \|u\|_{L^p}^p)\), which is the required bound.

**Bound on \(T(u)\).** It suffices to prove that
\[
\inf_{u \in \mathbb{C}^n} \mathbb{P}(T^{\text{in}}(u) < T^{\text{out}}(u)) > 0. \tag{39}
\]
As a first step towards deducing this bound, we will prove that for any \(\varepsilon, \delta \in (0, 1)\), provided \(0 < R_1(\varepsilon, \delta) < R_2(\varepsilon, \delta) < R_3(\varepsilon, \delta)\) and \(0 < \overline{\mathbb{P}}_1(\delta, \varepsilon) < \overline{\mathbb{P}}_2(\delta, \varepsilon) < \infty\) are sufficiently large, we have that
\[
\inf_{u \in \mathbb{C}^n} \mathbb{P}(T(u) > 2\delta) > 1 - \varepsilon, \quad \inf_{u \in \mathbb{C}^n} \mathbb{P}(T^{\text{in}}(u) < \delta) > 1 - \varepsilon. \tag{40}
\]
For the first bound we use (38) to find that for \(u_0 \in \mathbb{C}^n\) and a square integrable martingale \(M_t^{(p)}\) we have
\[
\|u_0\|_{L^p}^p \leq R_2^p + c_2t + M_t^{(p)}.
\]
Hence
\[
\mathbb{P}(T(u) \leq 2\delta) \leq \mathbb{P} \left( R_2^p + c_22\delta + \sup_{0 \leq s \leq 2\delta} M_s^{(p)} \geq R_3^p \right) = \mathbb{P} \left( \sup_{0 \leq s \leq 2\delta} M_s^{(p)} \geq R_3^p - R_2^p - c_2\delta \right).
\]
Now if we define \(\lambda = R_3^p - R_2^p - 2c_2\) (since \(\delta \in (0, 1)\)), and assuming \(\lambda > 0\) by choosing \(R_3\) sufficiently large, we obtain, by using the definition of the martingales in (16)
\[
\mathbb{P}(T(u) \leq 2\delta) \leq \mathbb{P}( \sup_{0 \leq s \leq 2\delta} M_s^{(p)} \geq \lambda)
\]
\[
\lesssim \lambda^{-2} \mathbb{E}[M_{2\delta T(u)}^{(p)}] \simeq \mathbb{E}(M^{(p)})_{2\delta T(u)} \lesssim \lambda^{-2} \mathbb{E} \int_0^{2\delta T(u)} \|u_s\|_{L^p}^{2(p-1)} \, ds
\]
\[
\lesssim \lambda^{-2} R_3^{2(p-1)} \delta \sim R_3^{-2} \delta,
\]
from the definition of \(\lambda\). Therefore choosing \(R_3(\varepsilon, \delta) \gg 1\) so that \(R_3^{-2} \delta \lesssim \varepsilon\) we obtain the desired estimate. Now we pass to a bound on \(T^{\text{in}}(u)\). Following (21), for \(p \geq 1\) sufficiently large so that
\[
\gamma \defeq 1 + \frac{\alpha}{p} + 2\kappa < 2,
\]
we find that
\[
\|u_\delta\|_{\infty} \lesssim \delta^{-\frac{\gamma}{1-\gamma}} \left\{ \|u\|_{L^p(A)} + \delta \sup_{0 \leq s \leq \delta} \|u_s\|_{L^p} + \sup_{0 \leq s \leq \delta} \|z_s\|_{\infty} \right\}.
\]
Therefore we obtain
\[
\mathbb{P}(T^{\text{in}}(u) \geq \delta) \leq \mathbb{P}( \sup_{0 \leq s \leq \delta} \|u_s\|_{\infty} \geq \overline{R}_1, T(u) \geq \delta) \leq \mathbb{P}(\|u_\delta\|_{\infty} \geq \overline{R}_1, T(u) \geq \delta)
\]
\[
\leq \overline{R}_1^{-1} \mathbb{E}[\|u_\delta\|_{\infty} \mathbb{1}_{T(u) \geq \delta}] \lesssim \overline{R}_1^{-1} \delta^{-\frac{\gamma}{1-\gamma}} \left\{ R_3^a + \mathbb{E} \left( \sup_{0 \leq s \leq \delta} \|z_s\|_{\infty} \right) \right\}.
\]
Now, choosing $\mathcal{R}_1(\delta, \varepsilon) \gg 1$ sufficiently large we obtain the desired estimate
\[
\sup_{u \in \mathbb{C}} \mathbb{P}(T^{in}(u) \geq \delta) < \varepsilon,
\]
which concludes the proof of (40). We now pass to (39), where we have
\[
\mathbb{P}(T^{out}(u) - T^{in}(u) \leq \delta) \leq \mathbb{P}(T^{out}(u) - T^{in}(u) \leq \delta, T(u) \geq 2\delta, T^{in}(u) < \delta) + 2\varepsilon
\leq \mathcal{R}_2^{-1} \mathbb{E} \left[ \sup_{T^{in}(u) \leq s \leq 2\delta} \|u_s\|_{\infty}^{1} \mathbb{1}_{\{T(u) \geq 2\delta, T^{in}(u) < \delta\}} \right] + 2\varepsilon.
\]
Now, to estimate the average above we follow again the bound (21), this time with initial condition in $L^\infty$. We obtain, for $p(a, d) \geq a$ sufficiently large
\[
\sup_{T^{in}(u) \leq s \leq \delta} \|u_s\|_{\infty} \lesssim \|u_{T^{in}}\|_{\infty} + 2\delta \sup_{T^{in}(u) \leq s \leq \delta} \|u_s\|_{p, T^{in}} + \sup_{T^{in}(u) \leq s \leq 2\delta} \|z_{T^{in}(u), t}\|_{\infty}.
\]
(41)
We conclude by Proposition 3 that
\[
\mathbb{E} \left[ \sup_{T^{in}(u) \leq s \leq 2\delta} \|u_s\|_{\infty}^{1} \mathbb{1}_{\{T(u) \geq 2\delta, T^{in}(u) < \delta\}} \right] \lesssim \mathcal{R}_1 + C\delta(1 + R^p_3) + \mathbb{E} \left[ \sup_{T^{in}(u) \leq s \leq 2\delta} \|z_{T^{in}(u), t}\|_{\infty} \right],
\]
so that by the bounds on $z$ in Lemma 5 and by choosing $\mathcal{R}_2 \gg 1$ sufficiently large, the bound (39) follows from (41). Hence the proof of our result is concluded.

\[\square\]

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