THE RIEMANN-STIELTJES DIAMOND-ALPHA INTEGRAL ON TIME SCALES

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Abstract. In this paper, we define and study the Riemann–Stieltjes diamond-alpha integral on time scales. Many properties of this integral will be obtained. The Riemann–Stieltjes diamond-alpha integral contains the Riemann–Stieltjes integral and diamond-alpha integral as special cases.

1. Introduction

The calculus on time scales was introduced for the first time in 1988 by Hilger [1] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [2-8].

Two versions of the calculus on time scales, the delta and nabla calculus, are now standard in the theory of time scales [3, 4]. In 2006, the diamond-alpha integral on time scales was introduced by Sheng, Fadag, Henderson, and Davis [10], as a linear combination of the delta and nabla integrals. The diamond-alpha integral reduces to the standard delta integral for $\alpha = 1$ and to the standard nabla integral for $\alpha = 0$. We refer the reader to [9, 10, 11] for a complete account of the recent diamond-alpha integral on time scales. In 2009, the Riemann diamond-alpha integral on time scales, as a more basic notion of diamond-alpha integral, was introduced by A.B. Malinowska and D.F.M. Torres [12]. In this paper we define the Riemann–Stieltjes diamond-alpha integral on time scales.
scales, which give a common generalization of the Riemann diamond-alpha integral and the Riemann–Stieltjes integral [8]. We also prove the corresponding main theorems of the Riemann–Stieltjes diamond-alpha integral.

2. Preliminaries

A time scale \( T \) is a nonempty closed subset of real numbers \( \mathbb{R} \) with the subspace topology inherited from the standard topology of \( \mathbb{R} \). For \( a, b \in T \) we define the closed interval \([a, b]_T\) by \([a, b]_T = \{ t \in T : a \leq t \leq b \}\). For \( t \in T \) we define the forward jump operator \( \sigma(t) \) by \( \sigma(t) = \inf\{ s > t : s \in T \} \) where \( \inf\emptyset = \sup\{ T \} \), while the backward jump operator \( \rho(t) \) is defined by \( \rho(t) = \sup\{ s < t : s \in T \} \) where \( \sup\emptyset = \inf\{ T \} \).

If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \), we say that \( t \) is left-scattered. If \( \sigma(t) = t \), we say that \( t \) is right-dense, while if \( \rho(t) = t \), we say that \( t \) is left-dense. A point \( t \in T \) is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function \( \mu(t) \) and the backward graininess function \( \eta(t) \) are defined by \( \mu(t) = \sigma(t) - t \), \( \eta(t) = t - \rho(t) \) for all \( t \in T \) respectively. If \( \sup T \) is finite and left-scattered, then we define \( T^k := T \setminus \sup T \), otherwise \( T^k := T \); if \( \inf T \) is finite and right-scattered, then \( T_k := T \setminus \inf T \), otherwise \( T_k := T \). We set \( T^k_k := T^k \cap T_k \).

A function \( f : [a, b]_T \to \mathbb{R} \) is called regulated provided its right-sided limits exist at all right-dense point of \([a, b]_T \) and its left-sided limits exist at all left-dense point of \((a, b)_T \).

A function \( f : T \to \mathbb{R} \) is delta differentiable at \( t \in T_k \) if there exists a number \( f^{\Delta}(t) \) such that, for each \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that

\[
|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|
\]

for all \( s \in U \). We call \( f^{\Delta}(t) \) the delta derivative of \( f \) at \( t \) and we say that \( f \) is delta differentiable if \( f \) is delta differentiable for all \( t \in T^k \).

A function \( f : T \to \mathbb{R} \) is nabla differentiable at \( t \in T_k \) if there exists a number \( f^{\nabla}(t) \) such that, for each \( \varepsilon > 0 \), there exists a neighborhood \( V \) of \( t \) such that

\[
|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|
\]

for all \( s \in V \). We call \( f^{\nabla}(t) \) the nabla derivative of \( f \) at \( t \) and we say that \( f \) is nabla differentiable if \( f \) is nabla differentiable for all \( t \in T_k \).
Let \( t, s \in \mathbb{T} \) and define \( \mu_{t,s} := \sigma(t) - s \) and \( \eta_{t,s} := \rho(t) - s \). A function \( f : \mathbb{T} \to \mathbb{R} \) is diamond-\( \alpha \) differentiable at \( t \in \mathbb{T}_k \) if there exists a number \( f^{\diamond \alpha}(t) \) such that, for each \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that, for all \( s \in U \),
\[
|\alpha(f(\sigma(t)) - f(s))\eta_{t,s} + (1 - \alpha)(f(\rho(t)) - f(s))\mu_{t,s} - f^{\diamond \alpha}(t)\mu_{t,s}\eta_{t,s}| \leq \varepsilon|\mu_{t,s}\eta_{t,s}|.
\]

3. The Riemann-Stieltjes diamond-\( \alpha \) integral

A partition of \([a, b]_\mathbb{T} \) is any finite ordered subset
\[
P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_\mathbb{T}, \text{ where } a = t_0 < t_1 < \ldots < t_n = b.
\]

Each partition \( P = \{t_0, t_1, \ldots, t_n\} \) of \([a, b]_\mathbb{T} \) decomposes it into subintervals \([t_{i-1}, t_i]_\mathbb{T}, \ i = 1, 2, \ldots, n \), such that for \( i \neq j \) one has \([t_{i-1}, t_i]_\mathbb{T} \cap [t_{j-1}, t_j]_\mathbb{T} = \emptyset\).

By \( \mathcal{P}([a, b]_\mathbb{T}) \) we denote the set of all partitions of \([a, b]_\mathbb{T} \). Let \( P_n, P_m \in \mathcal{P}([a, b]_\mathbb{T}) \). If \( P_m \subset P_n \) we call \( P_n \) a refinement of \( P_m \). If \( P_n, P_m \) are independently chosen, then the partition \( P_n \cup P_m \) is a common refinement of \( P_n \) and \( P_m \). Let \( g : [a, b]_\mathbb{T} \to \mathbb{R} \) be a real-valued non-decreasing function on \([a, b]_\mathbb{T} \). For the partition \( P \) we define the set
\[
g(P) = \{g(a) = g(t_0), g(t_1), \ldots, g(t_n) = g(b)\} \subset g([a, b]_\mathbb{T}).
\]

The image \( g([a, b]_\mathbb{T}) \) is not necessarily an interval in the classical sense, because our interval \([a, b]_\mathbb{T} \) may contain scattered points. From now on let \( g : [a, b]_\mathbb{T} \to \mathbb{R} \) be always a non-decreasing real function on the considered interval \([a, b]_\mathbb{T} \).

Let \( f : [a, b]_\mathbb{T} \to \mathbb{R} \) be a real-valued bounded function on \([a, b]_\mathbb{T} \). We denote
\[
\underline{M} = \sup\{f(t) : t \in [a, b]_\mathbb{T}\}, \quad \underline{m} = \inf\{f(t) : t \in [a, b]_\mathbb{T}\},
\]
\[
\overline{M} = \sup\{f(t) : t \in (a, b]_\mathbb{T}\}, \quad \overline{m} = \inf\{f(t) : t \in (a, b]_\mathbb{T}\},
\]
and for \( 1 \leq i \leq n \),
\[
\underline{M}_i = \sup\{f(t) : t \in [t_{i-1}, t_i]_\mathbb{T}\}, \quad \underline{m}_i = \inf\{f(t) : t \in [t_{i-1}, t_i]_\mathbb{T}\},
\]
\[
\overline{M}_i = \sup\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\}, \quad \overline{m}_i = \inf\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\},
\]

Let \( \alpha \in [0, 1] \). The upper Darboux-Stieltjes \( \diamond \alpha \)-sum of \( f \) with respect to the partition \( P \), denoted by \( U(f, g, P) \), is defined by
\[
U(f, g, P) = \sum_{i=1}^{n} (\alpha\overline{M}_i + (1 - \alpha)\underline{M}_i)(g(t_i) - g(t_{i-1})),
\]
while the lower Darboux-Stieltjes $\diamond'$-sum of $f$ with respect to the partition $P$, denoted by $L(f, g, P)$, is defined by

$$L(f, g, P) = \sum_{i=1}^{n} (\alpha m_i + (1 - \alpha) m_i)(g(t_i) - g(t_{i-1})).$$

Note that

$$U(f, g, P) \leq \sum_{i=1}^{n} (\alpha M + (1 - \alpha) M)(g(t_i) - g(t_{i-1})) = (\alpha M + (1 - \alpha) M)(g(b) - g(a))$$

and

$$L(f, g, P) \geq \sum_{i=1}^{n} (\alpha m + (1 - \alpha) m)(g(t_i) - g(t_{i-1})) = (\alpha m + (1 - \alpha) m)(g(b) - g(a)).$$

Thus, we have:

$$(\alpha m + (1 - \alpha) m)(g(b) - g(a)) \leq L(f, g, P) \leq U(f, g, P) \leq (\alpha M + (1 - \alpha) M)(g(b) - g(a)).$$

**Definition 3.1.** Let $I = [a, b]_T$, where $a, b \in T$. The upper Darboux-Stieltjes $\diamond'$-integral from $a$ to $b$ with respect to function $g$ is defined by

$$\int_{a}^{b} f(t) \diamond' g(t) = \inf_{P \in \mathcal{P}([a, b]_T)} U(f, g, P);$$

The lower Darboux-Stieltjes $\diamond'$-integral from $a$ to $b$ with respect to function $g$ is defined by

$$\int_{a}^{b} f(t) \diamond' g(t) = \sup_{P \in \mathcal{P}([a, b]_T)} L(f, g, P).$$

If $\int_{a}^{b} f(t) \diamond' g(t) = \int_{a}^{b} f(t) \diamond' g(t)$, then we say that $f$ is Riemann-Stieltjes $\diamond'$-integrable with respect to $g$ on $[a, b]_T$, and the common value of the integrals, denoted by $\int_{a}^{b} f(t) \diamond' g(t)$, is called the Riemann-Stieltjes $\diamond'$-integral.

**Definition 3.2.** Let $I = [a, b]_T$, where $a, b \in T$. The upper Darboux-Stieltjes $\Delta$-integral from $a$ to $b$ with respect to function $g$ is defined by

$$\int_{a}^{b} f(t) \Delta g(t) = \inf_{P \in \mathcal{P}([a, b]_T)} U(f, g, P)$$
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where \( U(f, g, P) \) denote the upper Darboux-Stieltjes sum of \( f \) with respect to the partition \( P \) and

\[
U(f, g, P) = \sum_{i=1}^{n} M_i (g(t_i) - g(t_{i-1})) \text{, } M_i = \sup \{ f(t) : t \in [t_{i-1}, t_i) \}
\]

The lower Darboux-Stieltjes \( \Delta \)-integral from \( a \) to \( b \) with respect to function \( g \) is defined by

\[
\int_{a}^{b} f(t) \Delta g(t) = \sup_{P \in \mathcal{P}([a, b])} L(f, g, P).
\]

where \( L(f, g, P) \) denote the lower Darboux-Stieltjes sum of \( f \) with respect to the partition \( P \) and

\[
L(f, g, P) = \sum_{i=1}^{n} m_i (g(t_i) - g(t_{i-1})) \text{, } m_i = \inf \{ f(t) : t \in [t_{i-1}, t_i) \}
\]

If \( \int_{a}^{b} f(t) \Delta g(t) = \int_{a}^{b} f(t) \nabla g(t) \), then we say that \( f \) is \( \Delta \)-integrable with respect to \( g \) on \( [a, b] \), and the common value of the integrals, denoted by \( \int_{a}^{b} f(t) \Delta g(t) \), is called the Riemann-Stieltjes \( \Delta \)-integral. Similarly, we can give the definition of the Riemann-Stieltjes \( \nabla \)-integral.

We can easily get the following two theorems.

**Theorem 3.3.** If \( f : [a, b] \to \mathbb{R} \) is Riemann–Stieltjes \( \Delta \)-integrable and Riemann–Stieltjes \( \nabla \)-integrable with respect to \( g : [a, b] \to \mathbb{R} \) on the interval \( [a, b] \), then it is Riemann–Stieltjes \( \diamond \alpha \)-integral with respect to \( g \) on \( [a, b] \) and

\[
\int_{a}^{b} f(t) \diamond g(t) = \alpha \int_{a}^{b} f(t) \Delta g(t) + (1 - \alpha) \int_{a}^{b} f(t) \nabla g(t).
\]

**Theorem 3.4.** Let \( f : [a, b] \to \mathbb{R} \) is Riemann–Stieltjes \( \diamond \alpha \)-integrable with respect to \( g : [a, b] \to \mathbb{R} \) on the interval \( [a, b] \).

(i) If \( \alpha = 1 \), then \( f \) is Riemann–Stieltjes \( \Delta \)-integrable with respect to \( g \) on \( [a, b] \).

(ii) If \( \alpha = 0 \), then \( f \) is Riemann–Stieltjes \( \nabla \)-integrable with respect to \( g \) on \( [a, b] \).

(iii) If \( 0 < \alpha < 1 \), then \( f \) is Riemann–Stieltjes \( \Delta \)-integrable and Riemann–Stieltjes \( \nabla \)-integrable with respect to \( g \) on \( [a, b] \).

(iv) If \( g \equiv t \), then the Riemann–Stieltjes \( \diamond \alpha \)-integral reduces to the standard diamond-alpha integral.
The following theorems may be showed in the same way as Theorem 5.5 and Theorem 5.6 in [4] or Theorem 3.5 and Theorem 3.6 in [8].

**Theorem 3.5.** Let  $L(f, g, P) = U(f, g, P)$ for some $P \in \mathcal{P}([a, b]_{\mathbb{T}})$, then the function $f$ is Riemann–Stieltjes $\diamond \alpha$-integrable on the interval $[a, b]_{\mathbb{T}}$ with respect to $g$ and

$$
\int_{a}^{b} f(t) \circ \alpha g(t) = \int_{a}^{b} f(t) \circ g(t) = L(f, g, P) = U(f, g, P).
$$

**Theorem 3.6.** Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a bounded function on the interval $[a, b]_{\mathbb{T}}$. Then the function $f$ is Riemann–Stieltjes $\circ \alpha$-integrable on the interval $[a, b]_{\mathbb{T}}$ with respect to $g$ if and only if for every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}([a, b]_{\mathbb{T}})$ such that

$$
U(f, g, P) - L(f, g, P) < \epsilon.
$$

The following Lemma can be found in [8].

**Lemma 3.7.** Let $I = [a, b]_{\mathbb{T}}$ be a closed (bounded) interval in $\mathbb{T}$ and let $g$ be continuous on $[a, b]_{\mathbb{T}}$. For every $\delta > 0$ there is a partition $P_{\delta} = \{ t_{0}, t_{1}, \ldots, t_{n} \} \in \mathcal{P}([a, b]_{\mathbb{T}})$ such that for each $i$ one has:

$$
g(t_{i}) - g(t_{i-1}) \leq \delta \quad \text{or} \quad g(t_{i}) - g(t_{i-1}) > \delta \land \rho(t_{i}) = t_{i-1}.
$$

**Theorem 3.8.** A bounded function $f$ on $[a, b]_{\mathbb{T}}$ is Riemann-Stieltjes $\circ \alpha$-integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_{\delta} \in \mathcal{P}([a, b]_{\mathbb{T}})$ implies

$$
U(f, g, P_{\delta}) - L(f, g, P_{\delta}) < \epsilon.
$$

Proof. If for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_{\delta} \in \mathcal{P}([a, b]_{\mathbb{T}})$ implies

$$
U(f, g, P_{\delta}) - L(f, g, P_{\delta}) < \epsilon,
$$

then we have that $f$ on $[a, b]_{\mathbb{T}}$ is integrable by Theorem 3.6.

Conversely, suppose that $f$ is Riemann–Stieltjes $\circ \alpha$-integrable with respect to $g$ on $[a, b]_{\mathbb{T}}$. If $\alpha = 1$ or $\alpha = 0$ then, $f$ is Riemann–Stieltjes $\Delta$-integrable or $\nabla$-integrable with respect to function $g$ on $[a, b]_{\mathbb{T}}$. Therefore condition holds from [8,Theorem 2.6]. Now, let $0 < \alpha < 1$, $f$ is Riemann–Stieltjes $\circ \alpha$-integrable with respect to function $g$, then $f$ is Riemann–Stieltjes $\Delta$-integrable or $\nabla$-integrable. According to [8,Theorem 2.6], for each $\epsilon > 0$ there exists $\delta' > 0$ and $\delta'' > 0$ such that $P_{\delta'} \in \mathcal{P}([a, b]_{\mathbb{T}})$, $P_{\delta''} \in \mathcal{P}([a, b]_{\mathbb{T}})$ we have

$$
U(f, g, P_{\delta'}) < \int_{a}^{b} f(t) \circ \alpha g(t) + \frac{\epsilon}{2}, \quad \int_{a}^{b} f(t) \circ \alpha g(t) - \frac{\epsilon}{2} < L(f, g, P_{\delta'}).$$
Let functions be Riemann-Stieltjes integrable with respect to arbitrary real numbers. Then, 
\[ \int_a^b f(t) \circ \alpha g(t) - \frac{\epsilon}{2} < L(f, g, P_\delta) \leq U(f, g, P_\delta) < \int_a^b f(t) \circ \alpha g(t) + \frac{\epsilon}{2}. \]
Because \( \int_a^b f(t) \circ \alpha g(t) = \int_a^b f(t) \circ \alpha g(t), \) then 
\[ U(f, g, P_\delta) - L(f, g, P_\delta) < \epsilon. \]

The proofs of the following three results are very similar to the proofs of Theorems 3.5, 3.6 and 3.7 in [8] respectively and hence the proofs are omitted.

**Theorem 3.9.** Let functions \( f_1, f_2 : T \to \mathbb{R} \) be Riemann-Stieltjes integrable with respect to \( g : T \to \mathbb{R} \) on the interval \([a, b]_T\), and \( \alpha, \beta \) be arbitrary real numbers. Then, \( \alpha f_1 \pm \beta f_2 \) is Riemann-Stieltjes integrable with respect to \( g : T \to \mathbb{R} \) on \([a, b]_T\) and 
\[ \int_a^b (\alpha f_1(t) \pm \beta f_2(t)) \circ \alpha g(t) = \alpha \int_a^b f_1(t) \circ \alpha g(t) \pm \beta \int_a^b f_2(t) \circ \alpha g(t). \]

**Theorem 3.10.** Let \( f : T \to \mathbb{R} \) be Riemann-Stieltjes integrable with respect to \( g_1, g_2 : T \to \mathbb{R} \) on the interval \([a, b]_T\), and \( \alpha, \beta \) be arbitrary real numbers. Then, \( f \) is Riemann-Stieltjes integrable with respect to \( \alpha g_1 \pm \beta g_2 \) on \([a, b]_T\) and 
\[ \int_a^b f(t) \circ (\alpha g_1(t) \pm \beta g_2(t)) = \alpha \int_a^b f(t) \circ g_1(t) \pm \beta \int_a^b f(t) \circ g_2(t). \]

**Theorem 3.11.** Let \( a, b, c \in T \) and \( a < b < c \). If \( f : T \to \mathbb{R} \) is bounded on \([a, c]_T\) and \( g : T \to \mathbb{R} \) is non-decreasing on \([a, c]_T\), then 
\[ \int_a^c f(t) \circ \alpha g(t) = \int_a^b f(t) \circ \alpha g(t) + \int_b^c f(t) \circ \alpha g(t). \]

**Theorem 3.12.** Let \( I = [a, b]_T \), where \( a, b \in T \). Every constant function \( f : T \to \mathbb{R}, f(t) \equiv c, \) is Riemann-Stieltjes integrable with respect to \( g \) on \([a, b]_T\) and 
\[ \int_a^b f(t) \circ \alpha g(t) = c(g(b) - g(a)). \]

Proof. Let \( P \in \mathcal{P}([a, b]_T) \) and \( P = \{t_0, \cdots, t_n\} \). Then we have 
\[ U(f, g, P) = L(f, g, P) = c \sum_{i=1}^n (g(t_i) - g(t_{i-1})) = c(g(b) - g(a)). \]
Hence, \( \int_a^b f(t) \circ_\alpha g(t) = \int_a^b f(t) \circ_{\alpha} g(t) = c(g(b) - g(a)). \)

The following theorem may be proved in much the same way as [4, Theorem 5.18, 5.19, 5.20, 5.21].

**Theorem 3.13.** Let \( I = [a, b]_T \), where \( a, b \in T \).

(i) Every monotone function \( f \) is Riemann–Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([a, b]_T \).

(ii) Every continuous function \( f \) is Riemann–Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([a, b]_T \).

(iii) Every bounded function \( f \) with only finitely many discontinuity points is Riemann–Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([a, b]_T \).

(iv) Every regulated function \( f \) is Riemann–Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([a, b]_T \).

**Theorem 3.14.** Let \( f : T \to \mathbb{R} \) and \( t \in T \). Then, \( f \) is Riemann-Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([t, \sigma(t)]_T \) and

\[
\int_t^{\sigma(t)} f(s) \circ_\alpha g(s) = (\alpha f(t) + (1 - \alpha)f(\sigma(t)))(g(\sigma(t)) - g(t)).
\]

Moreover, if \( 0 < \alpha \leq 1 \) and \( g \) is \( \circ_\alpha \)-differentiable at \( t \), then

\[
\int_t^{\sigma(t)} f(s) \circ_\alpha g(s) = \mu(t) g^\Delta(t) (\alpha f(t) + (1 - \alpha)f(\sigma(t))).
\]

**Proof.** If \( t = \sigma(t) \), then the equality is obvious. If \( t < \sigma(t) \), then \( P([t, \sigma(t)]_T) \) contains only one element given by

\[
t = s_0 < s_1 = \sigma(t).
\]

Since \([s_0, s_1]_T = \{t\} \) and \((s_0, s_1]_T = \{\sigma(t)\} \), we have

\[
U(f, g, P) = L(f, g, P) = \alpha f(t)(g(\sigma(t)) - g(t)) + (1 - \alpha)f(\sigma(t))(g(\sigma(t)) - g(t)).
\]

By Theorem 3.5, \( f \) is Riemann-Stieltjes \( \circ_\alpha \)-integrable with respect to \( g \) on \([t, \sigma(t)]_T \) and

\[
\int_t^{\sigma(t)} f(s) \circ_\alpha g(s) = (\alpha f(t) + (1 - \alpha)f(\sigma(t)))(g(\sigma(t)) - g(t)).
\]

By [9, Corollary 3.5., Theorem 3.9.], if \( 0 < \alpha \leq 1 \) and \( g \) is \( \circ_\alpha \)-differentiable at \( t \), then \( g \) is \( \Delta \) differentiable at \( t \) and \( g(\sigma(t)) - g(t) = \mu(t) g^\Delta(t) \).

Theorem 3.15. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$. Then, $f$ is Riemann-Stieltjes $\diamondsuit_\alpha-$integrable with respect to $g$ on $[\rho(t), t]_\mathbb{T}$ and

$$\int_{\rho(t)}^{t} f(s) \diamondsuit_\alpha g(s) = (\alpha f(\rho(t)) + (1 - \alpha)f(t))(g(t) - g(\rho(t))).$$

Moreover, if $0 \leq \alpha < 1$ and $g$ is $\diamondsuit_\alpha-$differentiable at $t$, then

$$\int_{\rho(t)}^{t} f(s) \diamondsuit_\alpha g(s) = \eta(t)g^{\nabla}(t)(\alpha f(\rho(t)) + (1 - \alpha)f(t)).$$

Proof. If $t = \rho(t)$, then the equality is obvious. If $t > \rho(t)$, then $[\rho(t), t]_\mathbb{T}$ contains only one element given by

$$\rho(t) = s_0 < s_1 = t.$$ 

Since $[s_0, s_1]_\mathbb{T} = \{\rho(t)\}$ and $(s_0, s_1]_\mathbb{T} = \{t\}$, we have

$$U(f, g, P) = L(f, g, P) = \alpha f(\rho(t))(g(t) - g(\rho(t))) + (1 - \alpha)f(t)(g(t) - g(\rho(t))).$$

By Theorem 3.5, $f$ is Riemann-Stieltjes $\diamondsuit_\alpha-$integrable with respect to $g$ on $[\rho(t), t]_\mathbb{T}$ and

$$\int_{\rho(t)}^{t} f(s) \diamondsuit_\alpha g(s) = (\alpha f(\rho(t)) + (1 - \alpha)f(t))(g(t) - g(\rho(t))).$$

By [9, Corollary 3.5., Theorem 3.9.], if $0 \leq \alpha < 1$ and $g$ is $\diamondsuit_\alpha-$differentiable at $t$, then $g$ is $\nabla$ differentiable at $t$ and $g(t) - g(\rho(t)) = \eta(t)g^{\nabla}(t)$. \qed

By the definition of the Riemann-Stieltjes $\diamondsuit_\alpha-$integral, we have the following Corollary:

Corollary 3.16. Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:

(i) If $\mathbb{T} = \mathbb{R}$, then a bounded function $f$ is Riemann-Stieltjes $\diamondsuit_\alpha-$integrable with respect to $g$ on the interval $[a, b]_\mathbb{T}$ if and only if $f$ is Riemann-Stieltjes integrable with respect to $g$ on $[a, b]_\mathbb{T}$ in the classical sense. Moreover, then

$$\int_{a}^{b} f(t) \diamondsuit_\alpha g(t) = \int_{a}^{b} f(t)dg(t).$$
(ii) If $T = \mathbb{Z}$, then each function $f : \mathbb{Z} \to \mathbb{R}$ is Riemann-Stieltjes $\circ_\alpha$-integrable with respect to function $g : \mathbb{Z} \to \mathbb{R}$ on the interval $[a, b]_T$. Moreover
\[
\int_a^b f(t) \circ_\alpha g(t) = \sum_{t=a}^{b-1} (\alpha f(t) + (1 - \alpha) f(t + 1))(g(t + 1) - g(t)).
\]

(iii) If $T = h\mathbb{Z}$, then each function $f : h\mathbb{Z} \to \mathbb{R}$ is Riemann-Stieltjes $\circ_\alpha$-integrable with respect to function $g : h\mathbb{Z} \to \mathbb{R}$ on the interval $[a, b]_T$. Moreover
\[
\int_a^b f(t) \circ_\alpha g(t) = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} [\alpha f(kh - h) + (1 - \alpha) f(kh)](g(kh) - g(kh - h)).
\]

Acknowledgement

The authors are grateful to the referee for his or her careful reading of the manuscript and for valuable and helpful suggestions.

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