RANDOM WALK MODELS FOR NONTRIVIAL IDENTITIES OF BERNOULLI AND EULER POLYNOMIALS

LIN JIU, ITALO SIMONELLI, AND HENG YUE*

Abstract. We consider the 1-dimensional reflected Brownian motion and 3-dimensional Bessel process and the general models. By decomposing the hitting times of consecutive sites into loops, we obtain identities, called loop identities, for the generating functions of the hitting times. After proving this decomposition both combinatorially and inductively, we consider the case that sites are equally distributed. Then, from loop identities, we derive expressions of Bernoulli and Euler polynomials, in terms of Euler polynomials of higher-orders.

1. Introduction

Random walks, of various types, have been comprehensively studied and widely applied in physics, engineering, and especially many fields of mathematics. The connection between random walk models and special polynomials, especially Bernoulli and Euler polynomials, was not obvious, since those polynomials, though with numerous applications in different areas, mainly appear in number theory and combinatorics. However, the recent work by the first author and Vignat [3], on the 1-dimensional reflected Brownian motion and 3-dimensional Bessel process, discovered and proved that some non-trivial identities involving Bernoulli and Euler polynomials of order $p$, denoted by $B^{(p)}_n(x)$ and $E^{(p)}_n(x)$ and defined via their exponential generating functions

\begin{align}
\left(\frac{t}{e^t - 1}\right)^p e^{xt} &= \sum_{n=0}^\infty B^{(p)}_n(x) \frac{t^n}{n!} \\
\left(\frac{2}{e^t + 1}\right)^p e^{xt} &= \sum_{n=0}^\infty E^{(p)}_n(x) \frac{t^n}{n!}.
\end{align}

In particular, $B_n(x) = B^{(1)}_n(x)$ and $E_n(x) = E^{(1)}_n(x)$ are the ordinary Bernoulli and Euler polynomials; and Bernoulli numbers $B_n = B_n(1)$ and Euler numbers $E_n = 2^n E_n(1/2)$ are special evaluations. See, e.g., [5, Chap. 24] for details.

This study originally arises from early work [4, Eq. (3.8)], where the first author, Moll, and Vignat expressed the usual Euler polynomials as a linear combination of higher-order Euler polynomials: for any positive integer $N$,

\[ E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^\infty p^{(N)}_\ell E^{(\ell)}_n \left( \frac{\ell - N}{2} + Nx \right). \]

It is surprising that the positive coefficients $p^{(N)}_\ell$ also appear as transition probabilities in the context of a random walk over a finite number of sites [4].

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*corresponding author.

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which reveals the possibility to connect random walks and $E_n^{(p)}(x)$, as well as $B_n^{(p)}(x)$.

Results in [3] are obtained by decomposing the successive hitting times ONLY of two, three, and four fixed levels, i.e., walks with one or two loops. Therefore, it is the purpose of this paper to generalize this work into general $n$ loops,

- by both inclusion-exclusion principle and induction;
- in order to obtain the general hitting time decomposition for $n$ loops;
- and to derive the corresponding identities involving $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$.

Hence, this paper is organized as follows. In Section 2, we introduce basic notation for random walk, especially the generating function of the hitting time; the model of 1-loop and 2-loop cases are recalled as examples. In Section 3, we generate the model to general $n$-loops, with two proofs, both combinatorially, by inclusion-exclusion principle, and inductively. In Section 4, we recall the three umbral symbols: Bernoulli, Euler and uniform symbols, with their important properties and connection. These formulas are crucial to derive identities in Section 5. In this last section, we first consider the 1-dimensional reflected Brownian motion model; as an analogue, the loop decomposition and identities in 3-dimensional Bessel process model are also derived.

2. Preliminaries: loops

One can find similar summary in [3], except for a slight change in the notation: see Def. 2.1 below. We still include them to make this paper self-contained.

2.1. Notation for paths and loops. We begin with some notation on the moment generating functions of random walks among loops.

**Definition 2.1.** Consider sites $a < b$ and a third site $c$, different from $a$ and $b$.

- We let $\phi_{a \rightarrow b}$ be the moment generating function of the hitting time of site $b$ starting from site $a$; also let $\phi_{b \rightarrow a}$ be the counterpart from $b$ to $a$. Therefore, $L_{a,b} := \phi_{a \rightarrow b} \phi_{b \rightarrow a}$.

  is the moment generating function of the hitting time
  - starting from $a$;
  - hitting $b$ first;
  - and finally returning to $a$.

  Note the symmetry that $L_{b,a} = \phi_{b \rightarrow a} \phi_{a \rightarrow b} = L_{a,b}$. Thus, this is the loop between sites $a$ and $b$.

- Also, we let $\phi_{a \rightarrow b|f}$ be the moment generating function of the hitting time of site $b$ starting from site $a$ before hitting site $c$; and similarly for $\phi_{b \rightarrow a|f}$. It is easy to see, $\phi_{a \rightarrow b|f} = \phi_{a \rightarrow b}$ if $c > b$ and $\phi_{a \rightarrow b|f} = 0$ if $a < c < b$.

- If $a$ is the $m$th site, denoted by $a_m$; and $b$ is the $n$th site as $a_n$, we shall use $\phi_{m \rightarrow n}$, for simplicity, instead of $\phi_{a_m \rightarrow a_n}$.

- Finally, let $\phi_{m \rightarrow n|k}$ be the moment generating function of the hitting time from the $m$th site to the $n$th site before hitting the $k$th site.

- We can similarly use $t_{a \rightarrow b}$ etc. for the hitting times, rather than their moment generating functions. For instance, $t_{m \rightarrow n|k}$ is the hitting time from the $m$th site to the $n$th site before hitting the $k$th site.
Here, it is important and also convenient for us to let
\[ L_n = \phi_{(n-1)\rightarrow n[\mu-2\gamma]} \cdot \phi_{n\rightarrow(n-1)[\mu+1\gamma]} \]
denote the moment generating function of the hitting time of the loop between the (consecutive) \((n-1)\)th site and the \(n\)th site.

**Example 2.2.** We first recall the 1-loop and 2-loop cases, already studied in [3]. The 1-loop case can be viewed in Fig. 2.1, in which we assume the initial site is \(a_0\), namely there is no other site to the left of \(a_0\). It is not hard to see the hitting time decomposition as follows:
\[ t_{0\rightarrow2} = t_{0\rightarrow1} + t_{1\rightarrow0\phi} + t_{0\rightarrow1} + \cdots + t_{1\rightarrow0\phi} + t_{0\rightarrow1} + t_{1\rightarrow2\phi}, \]
\(k\) copies
namely, there can be \(k\) copies of \(L_1\) in the moment generating functions, for \(k = 0, 1, 2, \ldots\). When considering the moment generating functions of both sides, independence turns the summation into products. Therefore, we have
\[ \phi_{0\rightarrow2} = \phi_{0\rightarrow1}\phi_{1\rightarrow2\phi} \sum_{k=0}^{\infty} \left( \phi_{1\rightarrow0\phi}\phi_{0\rightarrow1} \right)^k = \frac{\phi_{0\rightarrow1}\phi_{1\rightarrow2\phi}}{1 - L_1}. \]  
(See also [3, Eq. (2.5)].) In addition, the 2-loop case is
\[ \phi_{0\rightarrow3} = \frac{\phi_{0\rightarrow1}\phi_{1\rightarrow2\phi}\phi_{2\rightarrow3\phi}}{1 - (L_1 + L_2)}, \]
by a combinatorial enumeration [3, Eq. (2.6)].

We shall give the loop decomposition for \(n\)-loops in the following section.

3. **General \(n\)-loop Decomposition**

In this section, we shall give the expression of the general \(n\)-loop formula (see Fig. 3.1, the black paths), as the generalization of (2.1) and (2.2). Again, we assume the walk begins at the site \(a_0\) and only walk to its right on consecutive sites.
between consecutive sites, which generalizes the one given in Example 2.2, can be
natorial interpretation to the terms appearing in Theorem 3.1.
Proof of Thm. 3.1, by inclusion-exclusion principle.

So we can recover the case of two loops, i.e., (2.2).

Apparently, if there are only two loops, 

\[ \phi_{a_1 < a_2 < \cdots < a_{n+1}}. \] Namely the walk will consider \( a_0 \) as its starting point and there are no sites to the left of \( a_0 \).

**Theorem 3.1.**

\[
\phi_{0 \to (n+1)} = \phi_{0 \to 1} \prod_{j=1}^{n} \phi_{j \to (j+1)} \left( \sum_{k \geq 0} \sum_{l=1}^{k} \prod_{i=1}^{l} L_{i} \right) 
\]

\[
= \phi_{0 \to 1} \prod_{j=1}^{n} \phi_{j \to (j+1)} \sum_{k \geq 0} \sum_{l=1}^{k} \left( L_{1} + L_{2} + \cdots + L_{n} \right) + \sum_{l=1}^{n} (-1)^{l+1} \left( L_{j_1}, L_{j_2}, \ldots, L_{j_l} \right)^{k}
\]

\[
= \phi_{0 \to 1} \prod_{j=1}^{n} \phi_{j \to (j+1)} \frac{1}{1 - \left( L_{1} + L_{2} + \cdots + L_{n} \right) + \sum_{l=1}^{n} (-1)^{l+1} \left( L_{j_1}, L_{j_2}, \ldots, L_{j_l} \right) ^{k}},
\]

where

\[ ** = \{(i_1, i_2, \cdots, i_k) : 1 \leq i_t \leq n, \text{and } i_t = i_{t+1}, i_t = i_{t+1} + 1, \text{or } i_t < i_{t+1} \}; \]

and

\[ *= \{ n \geq j_1 > \cdots > j_l \geq 1, \ l \geq 2, \ j_m - j_{m+1} \geq 2 \}. \]

**Remark.** As one can tell, terms in \(*\) are loops without consecutive ones; and they are listed in a descending order, for its combinatorial interpretation later in the proof. Before the proof of Thm. 3.1 we would like to present several example for small number of loops.

**Example 3.2.** The formulas for \( n = 2, 3, 4, 5 \), given by Theorem 3.1 are listed as follows.

\[
\phi_{0 \to 3} = \frac{\phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3} \phi_{3 \to 4} \phi_{4 \to 5}}{1 - \left( L_1 + L_2 \right)},
\]

\[
\phi_{0 \to 4} = \frac{\phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3} \phi_{3 \to 4} \phi_{4 \to 5} \phi_{5 \to 6}}{1 - \left( L_1 + L_2 + L_3 + L_4 + L_5 L_1 \right)},
\]

\[
\phi_{0 \to 5} = \frac{\phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3} \phi_{3 \to 4} \phi_{4 \to 5} \phi_{5 \to 6}}{1 - \left( L_1 + L_2 + L_3 + L_4 + L_5 L_2 - L_4 L_1 L_3 L_1 \right)},
\]

\[
\phi_{0 \to 6} = \frac{\phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3} \phi_{3 \to 4} \phi_{4 \to 5} \phi_{5 \to 6}}{1 - \left( L_1 + \cdots + L_5 L_3 L_2 + L_5 L_2 - L_3 L_1 - L_4 L_1 L_3 L_1 \right)}.
\]

Apparently, if there are only two loops, \( L_1 \) and \( L_2 \), then \(*\), as well as \(*\) is empty. So we can recover the case of two loops, i.e., (2.2).

**Proof of Thm. 3.1.** by inclusion-exclusion principle. The proof will give a combinatorial interpretation to the terms appearing in Theorem 3.1.

Let \( n \geq 1 \). An arbitrary decomposition of \( t_{0 \to (n+1)} \) as a sum of hitting times between consecutive sites, which generalizes the one given in Example 2.2, can be written as

\[
(A) \quad t_{0 \to (n+1)} = \sum_{j=0}^{2k + n + 1} t_{i_j \to i_{j+1}}.
\]
where

(i) $i_0 = 0$, $i_1 = 1$, $i_{2k+n+1} = n+1$;

(ii) if $0 \leq j < 2k + n$, $0 \leq i_j, i_{j+1} \leq n$, and $|i_j - i_{j+1}| = 1$;

(iii) if $i_j \neq 0$, $|i_j^* - i_j| = 1$ and $|i_j^* - i_{j+1}| = 2$;

(iv) if $i_j = 0$, $t_{i_j \rightarrow i_{j+1}} = t_{i_j \rightarrow i_{j+1} = t_{0 \rightarrow 1}}$.

It is not hard to see that as we move through the sum in $(A)$, in the direction of increasing $j$, every time we encounter a stopping time which corresponds to a step to the left, i.e., from a site $t_j$ to site $t_{j+1} = t_j - 1$, a loop is formed. These loops can only occur between consecutive sites, and the order in which they appear is important: different walks with the same collection of loops will have different orderings. The ordering of the loops has an additional constraint, given by the assumption the walk only moves between consecutive sites. That is, if $t_{j+1} < t_j$, then $t_{j+1} = t_j - 1$. No such constraint is needed if $t_{j+1} > t_j$. Hence every decomposition of $t_{0 \rightarrow (n+1)}$ in $(A)$ can be uniquely written as

$$(B) \quad t_{0 \rightarrow (n+1)} = t_{0 \rightarrow 1} + \sum_{j=1}^{n} t_{j \rightarrow (j+1)}j^* + \sum_{t=1}^{k} L_i,$$

where

(a) the order in which we sum the $l_i$’s is important;

(b) $1 \leq i_t \leq n$, and $i_t = i_{t+1}$, or $i_t = i_{t+1} + 1$, or $i_t < i_{t+1}$.

Clearly this correspondence can be reversed, and $(B)$ can be used to express the moment generating function of $t_{0 \rightarrow (n+1)}$ in terms of the moment generating functions of loops. That is,

$$(3.1) \quad \phi_{0 \rightarrow n+1} = \phi_{0 \rightarrow 1} \prod_{j=1}^{n} \phi_{j \rightarrow (j+1)}j^* \left( \sum_{k=0}^{\infty} \sum_{**}^{L_i} \prod_{t=1}^{k} L_i \right),$$

where

$$** = \{(i_1, i_2, \cdots, i_k) : \text{the } i_t \text{’s satisfy (b).}\}$$

Let $k$ be arbitrary but fixed. We are going to apply the method of inclusion exclusion to recover $\sum_{**}^{L_i}$ from $(L_1 + L_2 + \cdots + L_n)^k$.

Note that, if $n = 1, 2$, $(b)$ necessarily holds, and from $(3.1)$ we immediately obtain

$$\phi_{0 \rightarrow 2} = \phi_{0 \rightarrow 1} \phi_{1 \rightarrow 2} \sum_{k=0}^{\infty} L_1^k = \frac{\phi_{0 \rightarrow 1} \phi_{1 \rightarrow 2} \phi_{1 \rightarrow 2}}{1 - L_1},$$
and
\[ \phi_{0 \to 3} = \phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3} \sum_{k=0}^{\infty} (L_1 + L_2)^k = \frac{\phi_{0 \to 1} \phi_{1 \to 2} \phi_{2 \to 3}}{1 - (L_1 + L_2)}, \]
which coincide with (2.1) and (2.2), respectively (see also Example (3.2)).

Let \( n \geq 3 \). Before applying the method of inclusion exclusion it is convenient to introduce an additional notation. Since property (b) can be expressed in terms of the order in which multiplication of the \( L_i \)'s is performed, we write

\[ (L_1 + L_2 + \cdots + L_n)^k = \prod_{i=1}^{k} (L_1 + L_2 + \cdots + L_n)_{(i)} \]
and use these additional subscripts to label the forbidden products of pair of moment generating functions of loops (m.g.f. of loops) in the expansion of \( (L_1 + L_2 + \cdots + L_n)^k \). We write \( (L_i L_j)_{(s)} \) when \( i - j > 1 \), \( L_i \) comes from the factor \( (L_1 + L_2 + \cdots + L_n)_{(s)} \), and \( L_j \) comes from \( (L_1 + L_2 + \cdots + L_n)_{(s+1)} \). Moreover when we write \( (L_i L_j)_{(s)} (L_r L_u)_{(v)} \), we assume that

\[ v \geq s + 1, \quad \text{and} \quad v = s + 1 \quad \text{if and only if} \quad L_j = L_r. \]

In this latter case,

(3.2) \[ (L_i L_j)_{(s)} (L_r L_u)_{(s+1)} \]

(3.2) naturally extends to products of several forbidden pairs of m.g.f. of loops, thus producing forbidden tuples \( (L_{i_1} L_{i_2} \cdots L_{i_l}) \), where \( i_j \geq i_{j+1} + 2, 1 \leq j \leq t - 1 \).

The method of inclusion exclusion now gives that \( \sum_{k \geq 0} \sum_{t=1}^{k} \prod_{i=1}^{k} L_{i_t} \) can be written as

(3.3) \[ \sum_{k \geq 0} \left[ (L_1 + \cdots + L_n)^k + \sum_{l=1}^{k} (-1)^{l-1} \sum_{s_j} \prod_{j=1}^{l} (L_{i_j} L_{s_j}) \right] \sum_{(l)} (L_1 + \cdots + L_n)^{(n-\#l)}, \]
where \( \#l \) denotes the number of distinct \( L_r \) in \( \prod_{j=1}^{l} (L_{i_j} L_{s_j}) \), and \( \sum \) runs over of all possible \( \{ (i_j, t_j)_{s_j}, 1 \leq j \leq l, s_j < s_{j+1}, i_j > t_{j+1} \} \). Note that this set could be empty.

Next we rewrite (3.3) in a very simple form. First, whenever possible we apply (3.2) and its extensions, then we drop the subscripts (now they do not provide any additional information), but we keep parentheses around each forbidden quantity. Finally, we combine like terms. We view each forbidden tuple, i.e, \( (L_{i_1} L_{i_2}), (L_{i_3} L_{i_4} L_{i_5}), \cdots \) as distinct variables, and, for every \( k \), we collect all monomials of degree \( k \) in these variables. We claim that this procedure reduces (3.3) to

(3.4) \[ \sum_{k \geq 0} \left( (L_1 + L_2 + \cdots + L_n) + \sum_{s'} (-1)^{l+1} (L_{j_1} L_{j_2} \cdots L_{j_l}) \right)^k, \]
where \( s' = \{ n \geq j_1 > \cdots > j_l \geq 1, l \geq 2, j_m - j_{m+1} \geq 2 \} \).
The number of these terms can be obtained by viewing each pair of parentheses is easy to see that this number is counting the number of ways they can be arranged on a line, order is important. It shows that all the terms of degree forbidden pairs of m.g.f. of loops from forbidden tuples for arbitrary positive integers (j_1^{(1)},\ldots,j_s^{(1)}),\ldots,(j_1^{(r)},\ldots,j_s^{(r)}), and s = (s_1 + 1)t_1 + \cdots + (s_r + 1)t_r. In (3.3), the terms are introduced when the method of inclusion exclusion is applied to remove the forbidden pairs of m.g.f. of loops from (L_1 + \cdots + L_n)^{t_1s_1 + t_2s_2 + \cdots + t_rs_r + t_{r+1}}.

The number of these terms can be obtained by viewing each pair of parentheses (\cdots) in (3.5) as distinct objects, (\cdots)^{t_i} implies (\cdots) is repeated \(t_i\) times, and counting the number of ways they can be arranged on a line, order is important. It is easy to see that this number is

\[
\frac{(t_1 + t_2 + \cdots + t_r + t_{r+1})!}{t_1!t_2!\cdots t_r!t_{r+1}!} = \binom{m}{t_1,t_2,\ldots,t_r,t_{r+1}}.
\]

This shows that all the terms of degree \(m\) in (3.4) can be found in (3.3). Moreover it is easy to see that these are the only terms of degree \(m\) in (3.3), thus proving the equivalence between (3.3) and (3.4).

Hence,

\[
\phi_{0\to(n+1)} = \phi_{0\to1} \prod_{j=1}^{n} \phi_{j\to(j+1)}(\sum_{k=0}^{\infty} \prod_{l=1}^{k} L_{j_l}^{(l)} + 1)
\]

\[
= \phi_{0\to1} \prod_{j=1}^{n} \phi_{j\to(j+1)}(\sum_{k=0}^{\infty} (L_1 + L_2 + \cdots + L_n) + \sum_{s}(-1)^{t+1}(L_{j_1}, L_{j_2}, \ldots, L_{j_l}))
\]

\[
= \phi_{0\to1} \prod_{j=1}^{n} \phi_{j\to(j+1)}(\sum_{k=0}^{\infty} (L_1 + L_2 + \cdots + L_n) + \sum_{s}(-1)^{t}(L_{j_1}, L_{j_2}, \ldots, L_{j_l}))
\]

The proof is now complete. \(\square\)
Next we provide an alternative proof of Theorem 3.1 by using induction on \( n \).

To simplify the notation in the proof, we shall use

\[
\sum (k,l,n) = \sum^* (-1)^{l+1} L_{j_1} \cdots L_{j_l},
\]

where \( * = \{ k = j_1 < \cdots < j_l \leq n, 1 \leq l \leq n - k + 1, j_{m+1} - j_m \geq 2 \} \), in which we reordered the loops in the ascending order.

Remark. Notice that we have reversed the order of subscript in the newly defined product notation. Indeed, \( \sum^* \) and \( \sum^* \) are mathematically equivalent. Although we believe \( \sum^* \) does a better job in conveying the combinatorics idea behind the loop identity, it is for the simplicity of expression that we choose to use the reversed order in the following proof.

It is easy to see that

\[
\sum (k,l,n) = L_k - L_k \sum_{j=k+2}^n \sum_{l,j} = L_k \left( 1 - \sum_{j=k+2}^n \sum_{l,j} \right).
\]

Hence, we can further rewrite Thm. 3.1 as

\[
\phi_{0 \rightarrow n+1} = \phi_{0 \rightarrow 1} \prod_{j=1}^n \phi_{j \rightarrow (j+1) | j \geq 1} \frac{1}{1 - \sum_{k=1}^n \sum_{l,j,k,l,n}}.
\]

Proof of Thm. 3.1 by induction. For the case \( n = 1 \), (3.8) reduces to

\[
\phi_{0 \rightarrow 2} = \phi_{0 \rightarrow 1} \phi_{1 \rightarrow 2} | \phi_{1 \rightarrow 2} = \frac{1}{1 - L_1},
\]

the same as (2.1). Suppose (3.8) holds for \( n = m - 1 \). Then, we need to show

\[
\phi_{0 \rightarrow (m+1)} = \phi_{0 \rightarrow 1} \prod_{j=1}^m \phi_{j \rightarrow (j+1) | j \geq 1} \frac{1}{1 - \sum_{k=1}^m \sum_{l,j,k,l,m}}.
\]
We combine the first two loops together, which reduces to $m - 1$ new loops, labeled as $L'_2$, $L'_3$, $L'_4$, etc. See Fig. 3.1. Note that $\phi_{0 \to 2}$ is given by (3.9); and similarly

$$\phi_{2 \to 0} = \frac{\phi_{1 \to 2} \phi_{2 \to 1}}{1 - \phi_{1 \to 2} \phi_{2 \to 1}}.$$

Hence,

$$L'_2 = \phi_{0 \to 2} \phi_{2 \to 0} = \frac{L_1 L_2}{1 - L_1 (1 - L_2)}.$$

In addition,

$$L'_3 = \phi_{2 \to 3} \phi_{3 \to 2}$$

$$= \phi_{2 \to 3} \sum_{k=0}^{\infty} (\phi_{1 \to 2} \phi_{2 \to 1})^k \phi_{3 \to 2}$$

$$= (\phi_{2 \to 3} \phi_{3 \to 2}) \frac{1}{1 - \phi_{1 \to 2} \phi_{2 \to 1}}$$

$$= \frac{L_3}{1 - L_2}$$

and $L'_k = L_k$, for all $4 \leq k \leq m$. To further simplify expressions, we need a new summation symbol:

$$\sum_{(k,l,m)} = \sum_{*} (-1)^{j_1 + \cdots + j_l} L'_{j_1} \cdots L'_{j_l},$$

where $* = \{k = j_1 < \cdots < j_l \leq n, 1 \leq l \leq n - k + 1, j_{m+1} - j_m \geq 2\}$, which is exactly the same as (3.6), with all $L$'s replaced by $L'$. Now apply (3.8) for sites $0, 2, 3, \ldots, m + 1$, (i.e., with $m - 1$ loops,) to get

$$\phi_{0 \to (m+1)} = \phi_{0 \to 2} \phi_{2 \to 3} \prod_{j=3}^{m} \phi_{j \to (j+1)} \frac{1}{1 - \sum_{k=2}^{m} \sum_{(k,l,m)}}$$

$$= \phi_{0 \to 2} \phi_{2 \to 1} \phi_{1 \to 2} \frac{1}{1 - L_1} \prod_{j=3}^{m} \phi_{j \to (j+1)} \frac{1}{1 - L_2} \frac{1}{1 - \sum_{k=2}^{m} \sum_{(k,l,m)}}$$

$$= \phi_{0 \to 1} \prod_{j=1}^{m} \phi_{j \to (j+1)} \frac{1}{1 - \sum_{k=2}^{m} \sum_{(k, l, m)}} (1 - L_1) (1 - L_2) \left( 1 - \sum_{k=1}^{m} \sum_{(k,l,m)} \right).$$

Therefore, (3.10) is equivalent to

$$(3.11) \quad (1 - L_1) (1 - L_2) \left( 1 - \sum_{k=2}^{m} \sum_{(k,l,m)} \right) = 1 - \sum_{k=1}^{m} \sum_{(k,l,m)} \frac{1}{1 - L_1} (1 - L_2).$$

By applying (3.7), we have the left-hand side

$$(1 - L_1) (1 - L_2) \left( 1 - \sum_{(2,l,m)} - \sum_{(3,l,m)} - \sum_{k=4}^{m} \sum_{(k,l,m)} \right)$$
\[ = 1 - L_1 - L_2 + L_1 L_2 - (1 - L_1) (1 - L_2) \left[ L_2 \left( 1 - \sum_{k=4}^{m} \sum_{(k,l,m)} \right) \right] \\
\quad + L_3' \left( 1 - \sum_{k=5}^{m} \sum_{(k,l,m)} \right) \right] - \sum_{k=4}^{m} \sum_{(k,l,m)} \sum_{(k,l,m)} + L_1 \sum_{k=4}^{m} \sum_{(k,l,m)} + L_2 \sum_{k=4}^{m} \sum_{(k,l,m)} \\
- L_1 L_2 \sum_{k=4}^{m} \sum_{(k,l,m)} \right] \\
= 1 - L_1 - L_2 + L_1 L_2 - L_1 L_2 \left( 1 - \sum_{k=4}^{m} \sum_{(k,l,m)} \right) - L_3 \left( 1 - \sum_{k=5}^{m} \sum_{(k,l,m)} \right) \\
+ L_1 L_3 \left( 1 - \sum_{k=5}^{m} \sum_{(k,l,m)} \right) - \sum_{k=4}^{m} \sum_{(k,l,m)} \sum_{(k,l,m)} + L_1 \sum_{k=4}^{m} \sum_{(k,l,m)} + L_2 \sum_{k=4}^{m} \sum_{(k,l,m)} \\
- L_1 L_2 \sum_{k=4}^{m} \sum_{(k,l,m)} \right]. \\
\]

Note that the two term underlined cancel; and since \( L_k' = L_k \) for \( k \geq 4 \), all the \( \sum \) terms above are actually \( \sum \). Therefore, the left-hand side of (3.11) is

\[ = 1 - L_1 \left[ 1 - L_3 \left( 1 - \sum_{k=5}^{m} \sum_{(k,l,m)} \right) \right] - \sum_{k=4}^{m} \sum_{(k,l,m)} \sum_{(k,l,m)} - L_2 \left( 1 - \sum_{k=4}^{m} \sum_{(k,l,m)} \right) \]

\[ - L_3 \left( 1 - \sum_{k=5}^{m} \sum_{(k,l,m)} \right) \right] - \sum_{k=4}^{m} \sum_{(k,l,m)} \sum_{(k,l,m)} \]

\[ = 1 - \sum_{k=1}^{m} \sum_{(k,l,m)} \sum_{(k,l,m)} , \]

which is the right-hand side of (3.11). \( \square \)

4. Preliminaries: umbral random symbols

Again, one can find similar summary of this section in [3]; but we are restating them here, for self-containedness. We will let \( \mathcal{B}, \mathcal{E}, \) and \( \mathcal{U} \) be the Bernoulli, Euler, and uniform (umbral) symbols, respectively. They are defined as follows.

4.1. Bernoulli \( \mathcal{B} \). The Bernoulli symbol \( \mathcal{B} \) satisfies the evaluation rule

\[ (x + \mathcal{B})^n = B_n(x). \]

In fact, \( \mathcal{B} \) can be viewed as a random variable [2] Thm. 2.3], i.e., \( \mathcal{B} = iL_B - 1/2 \), for \( i^2 = -1 \) and \( L_B \) is the random variable on \( \mathbb{R} \), with density \( p_B(t) = \pi \text{sech}^2(\pi t)/2 \). Hence, the evaluation rule is equivalent to the expectation operator. Moreover, for
any suitable function \( f \), i.e., one making integrals absolutely convergent,)

\[
f(x + B) = \mathbb{E}\left[ f\left( x + iL_B - \frac{1}{2} \right) \right] = \frac{\pi}{2} \int_{\mathbb{R}} f\left( x + it - \frac{1}{2} \right) \text{sech}^2(\pi t) dt.
\]

In particular, \( f(x) = x^n \) yields (4.1). In addition, we have

\[
B_n^{(p)}(x) = \left( x + B^{(p)} \right)^n = (x + B_1 + \cdots + B_p)^n
\]

for a set of \( p \) independent umbral symbols (or random variables) \( (B_i)_{i=1}^p \), satisfying:

- if \( i \neq j \), so that \( B_i \) and \( B_j \) are independent, then we evaluate

\[
B_i^n B_j^m = B_{n+m}.
\]

- and if \( i = j \), then

\[
B_i^n B_j^m = B_{n+m}.
\]

Now, with (4.1), we deduce that

\[
e^{B_t} = \frac{t}{e^t - 1}, \quad e^{(2B+1)} = \frac{t}{\sinh t}, \quad \text{and} \quad e^{(2B^{(p)}+p)} = \left( \frac{t}{\sinh t} \right)^p.
\]

4.2. Euler \( \mathcal{E} \). The Euler symbol \( \mathcal{E} \) can be similarly defined via the random variable interpretation that \( \mathcal{E} = iL_E - 1/2 \), where \( L_E \)'s density is given by \( p_E(t) = \text{sech}(\pi t) \). Then, \( (\mathcal{E} + x)^n = E_n(x) \); and in particular, for sum of independent symbols \( \mathcal{E}^{(p)} = \mathcal{E}_1 + \cdots + \mathcal{E}_p \),

\[
E_n^{(p)}(x) = \left( x + \mathcal{E}^{(p)} \right)^n.
\]

Therefore, (4.1) yields

\[
e^{\mathcal{E}} = \frac{2}{e^t + 1}, \quad e^{(2\mathcal{E}+1)} = \tanh t, \quad \text{and} \quad e^{(2\mathcal{E}^{(p)}+p)} = \tanh^p t.
\]

Moreover, from the generating functions, (4.2) and (4.3),

\[
e^{2B_t} = \frac{2t}{e^{2t} - 1} = \frac{t}{e^t - 1} \cdot \frac{2}{e^t + 1} = e^{(B+\mathcal{E})},
\]

we can have \( 2B = B + \mathcal{E} \), namely for any suitable function \( f \), \( f(x+2B) = f(x+B+\mathcal{E}) \).

4.3. Uniform \( \mathcal{U} \). The uniform symbol \( \mathcal{U} \) is the uniform random variable on \([0,1]\), i.e., \( \mathcal{U} \sim U[0,1] \), so that the evaluation is

\[
\mathcal{U}^n = \int_0^1 t^n dt = \frac{1}{n+1}.
\]

Easily, we have

\[
e^{t\mathcal{U}} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = e^t - 1 \cdot \frac{t}{t}, \quad e^{t(2\mathcal{U} - 1)} = \frac{\sinh t}{t}, \quad \text{and} \quad e^{t(2\mathcal{U}^{(p)} - p)} = \left( \frac{\sinh t}{t} \right)^p,
\]

for the sum of independent symbols \( \mathcal{U}^{(p)} = \mathcal{U}_1 + \cdots + \mathcal{U}_p \). An important link between \( B \) and \( \mathcal{U} \) is the cancellation rule. Note that

\[
e^{t(B+\mathcal{U})} = e^{t\mathcal{U}} e^{tB} = \frac{e^t - 1}{e^t - 1} \cdot \frac{t}{e^t - 1} = 1.
\]

So for a suitable function \( f \),

\[
f(x + B + \mathcal{U}) = f(x).
\]
In what follows, we will use independent copies of the three symbols. In order to distinguish them, we shall denote independent uniform symbols by $U, U', \ldots$ and $U^{(p)}, U'^{(p)}, \ldots$; and similarly for the other two symbols.

5. Identities of Bernoulli and Euler polynomials

Now, with the loop decomposition (3.8) and evaluation of symbols (4.2), (4.3), and (4.5), we can derive certain identities. Note that, even in the case of two loops, it does not seem possible to further simplify the expressions completely in terms of Bernoulli and Euler polynomials, but in terms of the three symbols; see e.g., [3, Thms. 3.4 and 4.2]. We would like to set all the sites equally distributed, namely $a_j = j$, for $j = 0, 1, 2, \ldots, n$, throughout this section.

5.1. 1-dim reflected Brownian motion on $\mathbb{R}^+$. In this case, for three consecutive sites $a < b < c$, the generating functions of the corresponding hitting times can be found in [1, p. 198 and p. 355] with variable $w$:

\[
\phi_{a \to b} = \frac{\cosh(aw)}{\cosh(bw)},
\]

\[
\phi_{b \to a|c} = \frac{\sinh((c-b)w)}{\sinh((c-a)w)},
\]

\[
\phi_{b \to c|a} = \frac{\sinh((b-a)w)}{\sinh((c-a)w)}.
\]

In this case, we begin with $a_0 = 0$ as the initial site and then apply the formulas above to have, for $n \geq 1$.

(5.1) \[ \phi_{0 \to n} = \frac{1}{\cosh(nw)} \quad \text{and} \quad \phi_{n \to n+1|a \to r} = \phi_{n \to n-1|a \to r} = \frac{1}{2 \cosh(w)}. \]

Before we state and prove the general formula, we first compute an example of 3-loop case, which is not included in [3].

**Example 5.1.** As stated in Ex. 3.2

\[
\phi_{0 \to 4} = \frac{\phi_{0 \to 1|0 \to 2|1 \to 3|2 \to 4|3} \phi_{0 \to 1|2 \to 3|1 \to 2}}{1 - (L_1 + L_2 + L_3 - L_1L_3)).
\]

Now, apply (5.1) to have

\[
\frac{1}{\cosh(4w)} = \frac{1}{8 \cosh^4 w} \sum_{k=0}^{\infty} \left( \frac{\sinh w}{\sinh(2w) \cosh w} + \frac{2 \sinh^2 w}{\sinh^2(2w)} \right)^k.
\]

- The left-hand side is simply $\text{sech}(4w) = \exp \{4w(2e + 1)\}.$
- For the right-hand side, we first simplify that

\[
\frac{\sinh w}{\sinh(2w) \cosh w} + \frac{2 \sinh^2 w}{\sinh^2(2w)} = \frac{1}{\cosh^2 w},
\]

\[
\frac{\sinh w}{\sinh(2w) \cosh w}, \frac{\sinh^2 w}{\sinh^2(2w)} = \frac{1}{8 \cosh^3 w}.
\]
Hence, we have
\[
\frac{1}{8 \cosh^2 w} \sum_{k=0}^{\infty} \left( \frac{1}{\cosh^2 w} - \frac{1}{8 \cosh^4 w} \right)^k
= \frac{1}{8 \cosh^4 w} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left( \frac{1}{\cosh^2 w} \right)^{k-\ell} \left( \frac{1}{8 \cosh^4 w} \right)^\ell
= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{1}{8^{\ell+1}} \exp \left\{ w(2E^{(2k+2\ell+4)} + 2k + 2\ell + 4) \right\}.
\]
Namely,
\[
\exp \left\{ 8E w + 4w \right\} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{1}{8^{\ell+1}} \exp \left\{ w(2E^{(2k+2\ell+4)} + 2k + 2\ell + 4) \right\},
\]
i.e.,
\[
\exp \left\{ 8E w \right\} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{1}{8^{\ell+1}} \exp \left\{ w(2E^{(2k+2\ell+4)} + 2k + 2\ell) \right\}.
\]
Multiplying both sides by \(\exp \{xw\}\) and comparing the coefficients of \(w^n\), we see
- the left-hand side yields
  \[\exp \left\{ (8E + x)w \right\} \Rightarrow (8E + x)^n = 8^n E_n \left( \frac{x}{8} \right);\]
- while the right-hand side gives
  \[
  \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{1}{8^{\ell+1}} (2E^{(2k+2\ell+4)} + 2k + 2\ell + x)^n
  = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{2^n}{8^{\ell+1}} E_n^{(2k+2\ell+4)} \left( \frac{x}{2} + k + \ell \right).
  \]
Therefore, we have
\[
E_n(x) = \frac{1}{4^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{2^n}{8^{\ell+1}} E_n^{(2k+2\ell)} (4x + k + \ell).
\]

Remark. It is easy to see that \(\phi_{0 \rightarrow 1}\) is twice of other paths: \(\phi_{n-n+1|a \rightarrow b} = \phi_{n-n-1|a \rightarrow b}\), which also causes the difference between \(L_1\) and \(L_j\)’s \(j = 2, 3, \ldots\). The following combinatorial enumeration is the key to find the general formulas.

Definition 5.2. Suppose \(S = \{a_1, a_2, \ldots, a_n\}\) is a set of a sequence of \(n\) mathematical objects.
(1) If the subindices of \(a_{j_1}, a_{j_2}, \ldots, a_{j_m}\) satisfy \(1 \leq j_1 \leq \cdots \leq j_m \leq n, j_k - j_{k-1} \geq 2\), we call it a nonadjacent product of order \(n\), length \(m\) with initial state \(j_1\).
(2) We define \(N(\ell, n)\) as the number of all nonadjacent products of order \(n\) and length \(\ell\) (without specific initial state).
(3) And finally we let \( n(a, \ell, m) \) be the number of different nonadjacent products of order \( n \), length \( \ell \) with initial state \( a \).

**Remark.** By convention, \( N(\ell, n) \) and \( n(a, \ell, m) \) can both be zero, if no such product exist.

**Proof.** Let \( \ell \) be an integer such that \( 3 \leq \ell \leq n \), then

\[
N(\ell, n) = N(\ell, n - 1) + N(\ell - 1, n - 2).
\]

All the nonadjacent products of order \( n \) and length \( \ell \) can be divide into two part: the first part consists of the existed nonadjacent products before adding \( a_n \), which are all nonadjacent products of order \( n \) and length \( \ell \); and the second part consists of the new nonadjacent products after adding \( a_n \), which are all nonadjacent products of order \( n - 2 \) and length \( \ell - 1 \).

**Theorem 5.3.** Suppose \( 2 \leq \ell \leq n \), then

\[
n(1, \ell, n) = \sum_{k=3}^{n} n(k, \ell - 1, n) = N(\ell - 1, n - 2).
\]

**Proof.** A nonadjacent product of order \( n \), length \( \ell \) and initial state 1 starts with \( a_1 \) and follows with \( a_j, j \geq 3 \). So the number of all nonadjacent products of order \( n \), length \( \ell \) and initial state 1 starts with \( a_1 \) equals to the number of all nonadjacent products of order \( n \), length \( \ell - 1 \) and initial state 1 starts with \( a_j, j \geq 3 \), which also equals to the number of all nonadjacent products of order \( n - 2 \) and length \( \ell - 1 \).

Note that \( \binom{n}{k} = 0 \) if \( k > n \), or \( k < 0 \). So we can identify \( N(\ell, n) \) as the binomial coefficients.

**Theorem 5.4.** Suppose \( 1 \leq \ell \leq n \), then

\[
N(\ell, n) = \binom{n - \ell + 1}{\ell}.
\]

**Proof.** It suffices to show that \( \binom{n-\ell+1}{\ell} \) satisfies the same initial conditions and recurrence relation with \( N(\ell, n) \), which is easy to see, since

\[
\binom{n-\ell+1}{\ell} = \binom{n-\ell}{\ell} + \binom{n-\ell}{\ell-1},
\]

coincides with (5.2). For the initial conditions, notice that when \( \ell = 1 \),

\[
N(1, n) = n = \binom{n}{1},
\]

which are exact the loops \( L_1, \ldots, L_n \).

Recall the multinomial coefficients: for \( k_1, \ldots, k_m \in \mathbb{N}, k_1 + \cdots + k_m \leq n \),

\[
\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! \cdots k_m!(n-k_1-\cdots-k_m)!}.
\]

In particular, for \( m = 1 \), we have the binomial coefficients \( \binom{n}{k} \).
Theorem 5.5. Let \( M = \lfloor m \rfloor - 1 \), and \( M' \) be the largest odd number less or equal to \( M \), then

\[
E_n\left(\frac{x}{m+1}\right) = \frac{1}{(m+1)^n} \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1, \ldots, n_M = 0}^{k} \begin{pmatrix} k \\ n_1, \ldots, n_M \end{pmatrix} \\
\times (-1)^{n_1+n_3+\cdots+n_{M'}} 4^{n_1+\cdots+n_M} \\
\times \left(\frac{(m-2)}{2^3} + \frac{(m-2)}{2^4}\right)^{n_1} \cdots \left(\frac{(m-M-1)}{2^{M+1}} + \frac{(m-M-1)}{2^{M+2}}\right)^{n_M} \\
\times E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)} (k + n_1 + 2n_2 + \cdots + Mn_M + x).
\]

Proof. Let \( n = m \) in (3.8) and apply (5.1) to have

\[
\text{sech}((m+1)w) = \text{sech} w \sum_{k=0}^{\infty} \left(n(1,1,m) \frac{\text{sech}^2 w}{2} + \right. \\
N(1,m-1) \frac{\text{sech}^2 w}{2^2} - n(1,2,m) \frac{\text{sech}^4 w}{2^3} - \\
N(2,m-1) \frac{\text{sech}^4 w}{2^4} + \cdots + (-1)^M n(1,M+1,m) \times \\
\left. \frac{\text{sech}(2^{M+2}) w}{2^{2M+1}} + (-1)^M N(M+1,m-1) \frac{\text{sech}(2^{M+2}) w}{2^{2M+2}} \right)^k
\]

\[
= \frac{\text{sech}^{(m+1)} w}{2m} \sum_{k=0}^{\infty} \left(\frac{\text{sech}^2 w}{2} + \frac{(m-1)}{2^2} \text{sech}^2 w \right. \\
\left. - \frac{(m-2)}{2^3} \text{sech}^4 w - \frac{(m-2)}{2^4} \text{sech}^4 w + \cdots + (-1)^M \times \\
\frac{(m-M-1)}{2^{M+1}} \text{sech}(2^{M+2}) w + (-1)^M \frac{(m-M-1)}{2^{2M+2}} \text{sech}(2^{M+2}) w \right)^k
\]

Applying (4.3) to get

\[
e^{(m+1)w(2\xi+1)} = \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1, \ldots, n_M = 0}^{k} \begin{pmatrix} k \\ n_1, \ldots, n_M \end{pmatrix} (-1)^{n_1+n_3+\cdots+n_{M'}} 4^{n_1+\cdots+n_M} \\
\times \left(\frac{(m-2)}{2^3} + \frac{(m-2)}{2^4}\right)^{n_1} \cdots \left(\frac{(m-M-1)}{2^{M+1}} + \frac{(m-M-1)}{2^{M+2}}\right)^{n_M} \\
\times e^{w(2\xi(2k+2n_1+4n_2+\cdots+2Mn_M+m)+2k+2n_1+4n_2+\cdots+2Mn_M+m+1)}.
\]
Figure 5.1. 3-dim Bessel Process

Multiplying by \( e^{wx} \) produces

\[
e^{w(2mE + 2E + x)} = \sum_{k=0}^{\infty} \frac{(m+1)^k}{2^{2k+m}} \sum_{n_1, \ldots, n_M=0}^{k} \binom{k}{n_1, \ldots, n_M} \frac{(-1)^{n_1+n_3+\cdots+n_M} 4^{n_1+\cdots+n_M}}{(m+1)^{n_1+\cdots+n_M}}
\]

\[\times \left( \frac{(m-2)}{2^3} + \frac{(m-2)}{2^4} \right)^n_1 \cdots \left( \frac{(m-M-1)}{2^{2M+1}} + \frac{(m-M-1)}{2^{2M+2}} \right)^n_M\]

\[\times e^{w(2E(2k+2n_1+4n_2+\cdots+2Mn_M+m+1)+2k+2n_1+4n_2+\cdots+2Mn_M+x))}.
\]

Now, we identify the coefficients of \( w^n \) on both sides:

- The left-hand side is simply

\[(2mE + 2E + x)^n = (2m + 2)^n \left( E + \frac{x}{2m + 2} \right)^n = (2m + 2)^n E_n \left( \frac{x}{2m + 2} \right);\]

- while the right-hand side, we only need to focus on the term

\[\left( 2E(2k+2n_1+4n_2+\cdots+2Mn_M+m) + 2k + 2n_1 + 4n_2 + \cdots + 2Mn_M + x \right)^n\]

\[= 2^n E_n^{(2k+2n_1+4n_2+\cdots+2Mn_M+m)} \left( k + n_1 + 2n_2 + \cdots + Mn_M + \frac{x}{2} \right).
\]

Therefore, simplification, with \( x \mapsto 2x \), gives the desired identity. \( \square \)

Example 5.6. The formulas derived from 4- and 5-loop cases are as follows.

\[E_n \left( \frac{x}{5} \right) = \frac{1}{5^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{5^k(-1)^\ell}{2^{2k+2\ell+4}} \binom{k}{\ell} E_n^{(2\ell+2k+5)} \left( x + \ell + k \right),\]

\[E_n \left( \frac{x}{6} \right) = \frac{1}{6^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{(-1)^{n_1} 3^{k-n_1-n_2}}{2^{k+3n_1+4n_2+5}} \binom{k}{n_1, n_2}
\]

\[\times E_n^{(2k+2n_1+4n_2+6)} \left( x + k + n_1 + 2n_2 \right).
\]

5.2. 3-dim Bessel process on \( \mathbb{R}^3 \). We can also find the generating functions for three consecutive sites \( a < b < c \).\footnote{pp. 463–464} with variable \( w \):

\[\phi_{a \rightarrow b} = \frac{b \sinh (aw)}{a \sinh (bw)},\]

\[\phi_{b \rightarrow a} = \frac{a \sinh ((c - b)w)}{b \sinh ((c - a)w)},\]

\[\phi_{c \rightarrow a} = \frac{c \sinh ((b - a)w)}{b \sinh ((c - a)w)}.
\]

As stated in \footnote{Rem. 4.1}, \( \phi_{a \rightarrow 0} \equiv 0 \) for \( 0 < a < b \). In this case, the first loop occurs between site 1 and 2 (instead of 0 and 1). We can still have \( (3.8) \) by shifting all the indices as \( a_m \mapsto m + 1 \), ONLY for the loops:
Theorem 5.7. For the 3-dimensional Bessel process on sites $0, 1, \ldots, m + 2$, we have

$$\phi_{0 \to (m+2)} = \phi_{0 \to 1} \prod_{j=1}^{m+1} \phi_{j \to (j+1)|j=1}^{1 - \sum_{k=1}^{m} \sum_{(k, l, n)}}.$$  

Let $M = \lceil m \rceil - 1$, and $M'$ be the largest odd number less or equal to $M$, then

$$B_{n+1} \left( \frac{2 + x}{m + 2} \right) - B_{n+1} \left( \frac{x}{m + 2} \right) = \frac{n + 1}{(m + 2)} \sum_{k=0}^{\infty} \frac{m^k}{2^{2k+m+1}} \sum_{n_1, \ldots, n_M=0}^{k} \left( \begin{array}{c} k \\ n_1, \ldots, n_M \end{array} \right) (-1)^{n_1+n_3+\cdots+n_M} \times \frac{4^{n_1+2n_3+\cdots+Mn_M M_1+\cdots+n_M}}{4^{n_1+2n_3+\cdots+Mn_M M_1+\cdots+n_M}} \times e^{2\phi(2k+2n_3+\cdots+2Mn_M + m)} \times e^{2k+2n_3+\cdots+2Mn_M + x}.$$  

Proof: The steps will be similar to that of (5.3), so we shall skip some direct but tedious calculation steps. By (5.4) and the well-known formula $\sinh(2w) = 2\sinh w \cosh w$, we have

$$\frac{(m+2)w}{\sinh((m+2)w)} = \frac{(m+2)w}{2} \sum_{k=0}^{\infty} \frac{\binom{m}{1} \text{sech}^2 w}{4^k} \cdot \sum_{k=0}^{\infty} \frac{(m-1) \text{sech}^4 w}{4^{k+1}} \cdot \sum_{k=0}^{\infty} \frac{(m-M) \text{sech}^{2M+2} w}{4^{M+1}}.$$  

By (4.2) and (4.3), we deduce that

$$e^{(m+2)w(2B'+1)} = e^{2w(2B'+1)} \sum_{k=0}^{\infty} \frac{(m+2)\phi^k}{2^{2k+m+1}} \sum_{n_1, \ldots, n_M=0}^{k} \left( \begin{array}{c} k \\ n_1, \ldots, n_M \end{array} \right) \times (-1)^{n_1+n_3+\cdots+n_M} \times e^{2\phi(2k+2n_3+\cdots+2Mn_M + m)} \times e^{e^{(2k+2n_3+\cdots+2Mn_M + m) + 2k+2n_3+\cdots+2Mn_M + x}}.$$  

In order to cancel the $e^{4wB'}$ on the right-hand side, we multiplying by $e^{4wB' + wx}$ to have

$$e^{w(2m+4)B + 4L + x)} = \sum_{k=0}^{\infty} \frac{(m+2)\phi^k}{2^{2k+m+1}} \sum_{n_1, \ldots, n_M=0}^{k} \left( \begin{array}{c} k \\ n_1, \ldots, n_M \end{array} \right) \times (-1)^{n_1+n_3+\cdots+n_M} \times e^{2\phi(2k+2n_3+\cdots+2Mn_M + m)} \times e^{2k+2n_3+\cdots+2Mn_M + x}.$$  

Random Walks for Bernoulli and Euler Polynomials
When identifying the coefficient of $w^n$, the right-hand side directly gives the Euler polynomial of higher-orders; while the left-hand side is, by (4.4),

\[
((2m + 4)B + 4u + x)^n = \int_0^1 ((2m + 4)B + 4u + x)^{n+1} \bigg|_{u=1}^{u=0} \\
= \frac{(2m + 4)^{n+1}}{4(n + 1)} \left( B_{n+1} \left( \frac{x + 4}{2m + 4} \right) - B_{n+1} \left( \frac{x}{2m + 4} \right) \right).
\]

Simplification and substitution $x \mapsto 2x$ complete the proof. □

**Example 5.8.** The identities from three and four loops are given by

\[
B_{n+1} \left( \frac{x + 2}{5} \right) - B_{n+1} \left( \frac{x}{5} \right) \\
= \frac{n + 1}{5^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} \frac{1}{12^\ell} E_n^{(2k+2\ell+3)} (k + \ell + x),
\]

and

\[
B_{n+1} \left( \frac{x + 2}{6} \right) - B_{n+1} \left( \frac{x}{6} \right) \\
= \frac{n + 1}{6^n} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} \frac{3^\ell}{4^{2\ell}} E_n^{(2k+2\ell+4)} (k + \ell + x).
\]

We shall briefly verify the identities in the example for the readers by direct calculation of the generating functions of Bernoulli and Euler polynomials.

**Proof.** It suffices to show

\[
\sum_{n=0}^{\infty} \left( B_{n+1} \left( \frac{x + 2}{5} \right) - B_{n+1} \left( \frac{x}{5} \right) \right) \frac{t^{n+1}}{(n + 1)!} \\
= t \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} \frac{1}{12^\ell} E_n^{(2k+2\ell+3)} (k + \ell + x) \frac{t^n}{n!},
\]

where the left-hand side is

\[
\sum_{n=0}^{\infty} \left( B_{n+1} \left( \frac{x + 2}{5} \right) - B_{n+1} \left( \frac{x}{5} \right) \right) \frac{t^{n+1}}{(n + 1)!} \\
= \sum_{n+1=0}^{\infty} \left( B_{n+1} \left( \frac{x + 2}{5} \right) - B_{n+1} \left( \frac{x}{5} \right) \right) \frac{t^{n+1}}{(n + 1)!} - B_0 \left( \frac{x + 2}{5} \right) + B_0 \left( \frac{x}{5} \right) \\
= \left( \frac{t}{e^t - 1} \right) e^{\left( \frac{2}{5} \right)x + 2 \cdot \frac{x}{5} - t} - \left( \frac{t}{e^t - 1} \right) e^{\frac{1}{5}x - t} \\
= \frac{t(e^t - 1)}{e^t - 1} e^{\frac{12}{5}x}.
\]
while the right-hand side is

\[
\begin{align*}
&= t \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( -1 \right)^{\ell} \frac{1}{12^\ell} \sum_{n=0}^{\infty} \frac{E_n^{(2k+2\ell+3)}(k+\ell+x)}{n!} \frac{(t/5)^n}{n!} \\
&= t \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( -1 \right)^{\ell} \frac{1}{12^\ell} \sum_{n=0}^{\infty} \frac{E_n^{(2k+2\ell+3)}(k+\ell+x)}{n!} \frac{(t/5)^n}{n!} \\
&= t \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( -1 \right)^{\ell} \frac{1}{12^\ell} \left( \frac{2}{e^{\theta} + 1} \right) e^{(k+\ell+x)/5} \\
&= te^{\frac{x}{5}} \left( \frac{1}{e^{\theta} + 1} \right)^3 \sum_{k=0}^{\infty} 3^k e^{\frac{x}{5}} \left( \frac{1}{e^{\theta} + 1} \right)^{2k} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( -1 \right)^{\ell} \frac{1}{3^\ell} \left( \frac{1}{e^{\theta} + 1} \right)^{2\ell} e^{\frac{x}{5}} \\
&= te^{\frac{x}{5}} \left( \frac{1}{e^{\theta} + 1} \right)^3 \sum_{k=0}^{\infty} 3^k e^{\frac{x}{5}} \left( \frac{1}{e^{\theta} + 1} \right)^{2k} \left( 1 - \frac{1}{3} \left( \frac{1}{e^{\theta} + 1} \right)^2 \right) e^{\frac{x}{5}} \\
&= te^{\frac{x}{5}} \left( \frac{1}{e^{\theta} + 1} \right)^3 \left( 1 - \frac{1}{3} \left( \frac{1}{e^{\theta} + 1} \right)^2 \right) e^{\frac{x}{5}} \\
&= t(e^{\frac{x}{5}} - 1) e^{\frac{x}{5}} \\
\end{align*}
\]

\[\square\]

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**ZU CHONGZHI CENTER FOR MATHEMATICS AND COMPUTATIONAL SCIENCES, DUKE KUNSHAN UNIVERSITY, KUNSHAN, SUZHOU, JIANGSU PROVINCE, PR CHINA, 215316.**

**Email address:** lin_jiu@dukekunshan.edu.cn

**ZU CHONGZHI CENTER FOR MATHEMATICS AND COMPUTATIONAL SCIENCES, DUKE KUNSHAN UNIVERSITY, KUNSHAN, SUZHOU, JIANGSU PROVINCE, PR CHINA, 215316.**

**Email address:** italo.simonelli@dukekunshan.edu.cn

**CLASS OF 2023, DUKE KUNSHAN UNIVERSITY, KUNSHAN, SUZHOU, JIANGSU PROVINCE, PR CHINA, 215316.**

**Email address:** heng.yue@dukekunshan.edu.cn