One-dimensional continuum finite-difference model of the dynamics of a dusty medium in aerodynamic, electric and gravitational fields

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Abstract. This paper presents a mathematical model of the dynamics of a gas suspension under the action of an aerodynamic field, an electric field, and a gravitational field. The continuum model is used to describe the dynamics of the disperse component. The intercomponent momentum exchange included the aerodynamic drag force, the dynamic Archimedes force, and the added mass force. The Coulomb force acting on dispersed particles was taken into account. The model assumes the solution of the equations of conservation of “average density”, momentum and energy for the dispersed phase. The electric field was described by the Poisson equation. The equations of the mathematical model were supplemented with boundary conditions. The equations of mechanics were integrated by explicit finite-difference method of McCormack.

1. Introduction

Studies of the dynamics of inhomogeneous media are applicable in many areas [1-10]. Including in plasma physics [3]. The heterogeneity of the studied media is one of the elements of scientific novelty in modern studies of hydrodynamics. As a branch of physics, hydrodynamics has a significant reliance on mathematical mods - methods of mathematical physics. In [3], the dynamics of monodisperse quartz particles in plasma is numerically modeled. In [5], electrohydrodynamic processes in the atmosphere are mathematically described taking into account the inhomogeneous structure of clouds. To study processes involving heterogeneous effects, it is necessary to use various branches of science [11]. The results presented in this study are theoretical in nature and allow bringing together such physical and mechanical concepts as viscosity, compressibility of the medium, intercomponent interaction and electric field.

2. Mathematical model

The dynamics of the carrier medium is described by the one-dimensional Navier-Stokes equation. The right-hand sides of the equations contain the terms responsible for the intercomponent momentum exchange \(-F(u,u_i)\) and intercomponent heat transfer \(-Q(T,T_i)\) [7,9,10]:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial \left(\rho u^2 + p - \tau\right)}{\partial x} = -F + \alpha \frac{\partial \rho}{\partial x}
\]
\[
\frac{\partial(e)}{\partial t} + \frac{\partial}{\partial x}\left[(e + p - \tau)u - \lambda \frac{\partial T}{\partial x}\right] = -Q - |F|(u - u_i) + \alpha \frac{\partial(\rho u)}{\partial x}
\]

\[\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x} = 0\]

\[\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i^2)}{\partial x} = F - F_c - g \rho - \alpha \frac{\partial p}{\partial x}\]

\[\frac{\partial (e_i)}{\partial t} + \frac{\partial (e_i u_i)}{\partial x} = Q\]

\[\frac{\partial^2 \phi}{\partial x^2} = \rho \phi_{0}\]

Equation (1) is the gas mass continuity equation, equation (2) is the Navier-Stokes equation describing the continuity of the viscous gas momentum, equation (3) is the gas energy conservation equation. Equation (4) is the equation of continuity of the "average density" - \(\rho_1\) of the carrier medium. The average density is the product of the physical density of the material, a constant \(\rho_{0}=\text{const}\) and the volumetric content of the dispersed component \(\alpha = \alpha(x,t)\), which is a function of space and time variables. Equation (5) is the equation of continuity of the momentum of the dispersed phase, equation (6) is the equation of conservation of the energy of the dispersed component. Equation (5) takes into account both the intercomponent momentum exchange \(-F\), and the Coulomb force acting on particles \(-F_C\), as well as the gravitational settling of particles, the term: \(-\rho_1g\). Equation (7) is the Poisson equation describing the electric field created both by the electric charge of dispersed particles and by external electric potentials. Intercomponent momentum exchange \(F\) involves several forces: \(F = F_{AR} + F_A + F_{AM}\); where \(F_{AR}\) is the force of aerodynamic drag (8), \(F_A\) is the dynamic force of Archimedes (9), \(F_{AM}\) is the force of added masses (10) [1]:

\[F_{AR} = \frac{3\alpha}{4d} C_d \rho |u - u_i|(u - u_i)\]

\[F_A = \alpha \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right)\]

\[F_{AM} = 0.5 \alpha \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} - u_i \frac{\partial u}{\partial x} \right)\]

\[Q = 6\alpha \eta \lambda (T - T_1) / d^2\]

\(Nu_1 = 2 \exp\left(-M_1\right) + 0.459 Re_1^{0.55} Pr^{0.33}, C_d = C_d^0 \eta(\alpha) \psi(M_1), C_d = \frac{24}{Re_1} + \frac{4}{Re_1^{0.5}} + 0.4, \psi(M_1) = 1 + \exp\left(-\frac{0.427}{M_1^{0.63}}\right), \eta(\alpha) = (1 - \alpha)^{-25}, M_1 = |u - u_i| / c_0, Re_1 = \rho |u - u_i| d / \mu, \)

\[Pr = \gamma C_p \mu / \lambda, \tau = \frac{4}{3} \alpha \frac{\partial u}{\partial x}\]

The drag coefficient of a single particle \(C_d\) is calculated taking into account the compressibility of the medium (factor \(\psi\)) corrected for multiplicity of particles (factor \(\eta\)) (\(C_d\) is the drag coefficient of a single spherical particle). \(Nu_1\) is the relative Nuselt number, \(Re_1\) and \(M_1\) are the relative Reynolds and Mach numbers, \(\tau(u)\) – function of viscous gas stresses. The electric field is described in the following way [12]:
\[
div E = \frac{\rho}{\varepsilon \varepsilon_0}, \quad E = -\nabla \varphi, \quad \Delta^2 \varphi = -\frac{\rho}{\varepsilon \varepsilon_0}, \quad \rho_e = \alpha \rho_0, \quad q_0 = \rho_1, \quad \varepsilon_0 = \frac{10^{-9}}{36\pi} F / m
\]  

(13)

where \(q_0\) is the specific charge per unit mass of the solid fraction, \(\varphi\) is the electric field potential, \(E\) is the electric field strength, \(\varepsilon\) is the relative dielectric constant of air, \(\varepsilon_0\) is the absolute dielectric constant of air. The Coulomb force acting on the particles of the dispersed component of the gas suspension is determined by formula (14):

\[
F_e = -\rho_e \frac{\partial \varphi}{\partial x}
\]

(14)

The equations of the mathematical model (1)-(14) were supplemented with boundary conditions. The velocities of the mixture components at the boundaries of the computational domain were set equal to zero. For the rest of the dynamic functions, the homogeneous Neumann boundary conditions were set. For the electric field potential, values were set at opposite ends of the computational domain [13,14]:

\[
\begin{align*}
    u(t,1) &= 0, \quad u(t,N) = 0, \\
    u(t,1) &= u(t,N) = 0, \\
    \rho(t,1) &= \rho(t,2), \\
    \rho(t,N-1) &= \rho(t,N), \\
    \rho(t,1) &= \rho(t,N-1), \\
    e(t,1) &= e(t,2), \\
    e(t,N) &= e(t,1), \\
    e(t,1) &= e(t,N-1), \\
    p(t,1) &= p(t,2), \\
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\]

3. Solution methods

Let us consider the application of the algorithm of the explicit McCormack method using the example of a scalar nonlinear partial differential equation (15) [13]:

\[
\frac{\partial f}{\partial t} + \frac{\partial a(f)}{\partial x} = c(f)
\]

(15)

The algorithm of McCormack’s explicit finite-difference method for nonlinear equation (15) has the form (16) - (17):

\[
\begin{align*}
    f_i^* &= f_i^n - \frac{\Delta t}{\Delta x} \left( a_{i+1}^n - a_i^n \right) + \Delta t a_i^n \\
    f_i^{n+1} &= 0.5(f_i^n + f_i^*) - 0.5 \frac{\Delta t}{\Delta x} \left( a_i^n - a_{i-1}^n \right) + 0.5 \Delta t a_i^n
\end{align*}
\]

(16) (17)

Here \(\Delta x\) is the step in the spatial direction, \(\Delta t\) is the step in time. The Poisson equation for the potential of the electric field was solved by the finite difference method - the method of establishment applied on the gas-dynamic computational grid. The scheme of the establishment method is as follows [15]. For a one-dimensional inhomogeneous second-order differential equation (18):

\[
\frac{df}{dx^2} = h(x)
\]

(18)

To implement the method for equation (18), a non-stationary expressions is written (19), (20):

\[
\begin{align*}
    \frac{\partial m}{\partial t} &= -\frac{\partial^2 m}{\partial x^2} - h(x) \\
    \lim_{\tau \to \infty} m(t,x) &= f(x)
\end{align*}
\]

(19) (20)

The function \(m(t,x)\) is found from the solution of a finite-difference equation:

\[
\frac{m_{i+1}^{n+1} - m_{i}^{n+1}}{\Delta t} = \frac{m_{i+2}^{n} - 2m_{i+1}^{n} + m_{i}^{n}}{\Delta x^2} + h(\Delta x)
\]

(21)

The function \(f\) is found from the finite-difference equations (21). The solution of the stationary linear differential equation (7) of type (18) was carried out each time after the transition from the \(n\)th to the \(n+1\)th time layer, so the function \(\varphi(x,t)\) is a non-stationary function – depends on time. To overcome some of the shortcomings of the numerical method used, a nonlinear grid correction algorithm was used, which will reduce the local maxima and increase the local minina of the grid function [14]. The correction algorithm was performed sequentially along all nodes. Let us consider
an algorithm for correcting the solution using the example of the function $f$. If the conditions are met \( \left( \delta f_{i+1/2} \cdot \delta f_{i+3/2} \right) < 0 \) or \( \left( \delta f_{i+1/2} \cdot \delta f_{i+3/2} \right) < 0 \) then the algorithm of the correction scheme is applied to the function $f$ in the $i$-th node: $\tilde{f}_i = f_i + \kappa \left( \delta f_{i+3/2} - \delta f_{i+1/2} \right)$. The subscript indicates the mesh node number. The notation is used here: $\delta f_{i-1/2} = f_i - f_{i-1}$, $\delta f_{i+1/2} = f_{i+1} - f_i$, $\delta f_{i+3/2} = f_{i+2} - f_{i+1}$. Otherwise: $\tilde{f}_i = f_i$, $\tilde{f}_i$ is the value of the function at the $i$-th node after the transition to the $(n+1)$-th time layer according to the McCormack scheme, $\kappa$ is the correction factor. The numerical solution of the equations of the mathematical model described here is implemented as a computer program in the Fortran programming language.

4. Calculation results

The channel length was $L=1$ m. The specific charge of the disperse component $q_0=0.001$ C/kg, $\varphi_1=15$ kV $\varphi_2=0$. Computational domain nodes $N=200$, Courant number $C=0.5$, correction factor $\kappa=0.15$. Figure 1 schematically shows the computational domain, positively charged dispersed particles move under the action of gravity and an electric field: since a positive potential is applied to the upper surface of the container, and the lower surface of the container has a potential equal to zero. Physical density of particle material $\rho_{10}=2700$ kg/m$^3$, volume content of the dispersed component $\alpha=0.0004$, the size of particles $d=200$ µm. The carrier medium has the physical properties of air.

![Figure 1. Schematic representation of the simulated process.](image)

Figure 2 shows the distribution of the potential of the electric field, it can be observed that between the two areas of potential reduction there is an area with a uniform distribution of the electric potential.

![Figure 2. Spatial distribution of the electric field potential. The specific charge of the disperse component $q_0=0.001$ C/kg, $\varphi_1=15$ kV $\varphi_2=0$.](image)
The distribution of the potential of the electric field is consistent with the Coulomb force, which reaches the highest value in the areas of the maximum gradient of the potential of the electric field - Figure 3.

![Figure 3. Spatial distribution of the Coulomb force. The specific charge of the disperse component $q_0=0.001$ C/kg, $\phi_1=15$ kV $\phi_2=0$.](image)

The velocity of the dispersed component reaches the highest value in the areas of the highest value of the Coulomb force – Figure 4.

![Figure 4. Spatial distribution of particle velocity. Physical density of particle material $\rho_1=2700$ kg/m$^3$, volume content of the dispersed component $\alpha=0.0004$, size of particles $d=200$ µm](image)

Due to the intercomponent exchange of momentum, the movement of dispersed particles results in the movement of gas - Figure 5.

![Figure 5. Spatial distribution of gas velocity. Initial gas temperature $T_0=293$ K. Initial gas density $\rho_0=1.21$ kg/m$^3$.](image)
Due to the movement of gas in the channel, an uneven distribution of gas pressure will form Figure 6.

Figure 6. Spatial distribution of gas pressure. Initial gas temperature $T_0=293$ K. Initial gas density $\rho_0=1.21$ kg/m$^3$.

5. Conclusion
The article describes the results of scientific research - the development of a theoretical model based on the theory of fluid, gas and plasma dynamics. It can be more definitely said that the methods of the dynamics of multiphase media and electrodynamics are used. The results of formalization of theoretical calculations are presented in the form of a nonlinear system of partial differential equations of parabolic type. As well as one linear ordinary differential equation implicitly related to the general system of equations for the dynamics of an inhomogeneous medium. The entire mathematical model has the form of a boundary value problem for an ordinary differential equation and a Cauchy boundary value problem for the rest of the model. To obtain the solution, methods of computational mathematics were used. The numerical algorithm is implemented as a computer program. Thus, the results obtained can be used in modeling the dynamics of dusty plasma and in calculating flows of electrohydrodynamics of inhomogeneous media. In the above calculations, a relationship was found between the settling rate of dispersed particles and the magnitude of the gradient of the electric field potential. Also, the use of the continuum modeling technique made it possible to reveal the reciprocal effects of the interaction of the mixture components.

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