MULTICIPILITY OF SOLUTIONS FOR A FRACTIONAL KIRCHHOFF TYPE PROBLEM

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ABSTRACT. In this paper, by using the (variant) Fountain Theorem, we obtain that there are infinitely many solutions for a Kirchhoff type equation that involves a nonlocal operator.

1. Introduction. In this article, we investigate multiplicity of solutions for the following problem

\[
\begin{align*}
- (a + b \|u\|_{X_0}^2) \mathcal{L}_K u(x) &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
\]

where \( \Omega \) is an open bounded set in \( \mathbb{R}^N \), \( N > 2s \) with \( s \in (0, 1) \), \( a, b > 0 \), \( f \) is a continuous function whose properties will be introduced later, \( \| \cdot \|_{X_0} \) is a functional norm which is defined in (9) and \( \mathcal{L}_K \) is a nonlocal operator defined as follows:

\[
\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^N} (u(x + y) + u(x - y) - 2u(x)) K(y) dy, \quad x \in \mathbb{R}^N.
\]

Here \( K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty) \) is a measurable function which satisfies

\[
\begin{align*}
\gamma K(x) &\in L^1(\mathbb{R}^N) \quad \text{with } \gamma(x) = \min\{|x|^2, 1\}; \\
there exists \theta > 0 \text{ such that } K(x) &\geq \theta |x|^{-(N+2s)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}; \\
K(x) = K(-x) &\text{ for any } x \in \mathbb{R}^N \setminus \{0\}.
\end{align*}
\]

A typical example is \( K(x) = |x|^{-(N+2s)} \). In this case \( \mathcal{L}_K u(x) = -(-\Delta)^s u(x) \) is the fractional Laplace operator which (up to normalization factors) can be defined as

\[
-(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{N+2s}} dy, \quad \text{for } x \in \mathbb{R}^N.
\]

In the classical Laplace operator case, problem (1) is related to the stationary analogue of the equation

\[
u_{tt} - (a + b \int_{\Omega} |\nabla u|^2) \Delta u = g(x, u),
\]

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proposed by Kirchhoff [14] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. Kirchhoff’s model takes into account the changes in length of the string produced by transverse vibrations. Some early studies of Kirchhoff equations were those of Bernstein [3] and Pohozaev [21]. Equation (5) received much attention only after Lions [16] proposed an abstract framework to the problem. More recently, by variational methods, Alves [1], Ma-Rivera [18] studied the existence of one positive solution, and He-Zou [12, 13] obtained the existence of infinitely many positive solutions for problem (5) respectively. Perera-Zhang [20] studied the existence of nontrivial solutions of problem (5) via the Yang index theory; Zhang-Perera [35] and Mao-Zhang [19] established the existence of sign-changing solutions of problem (5) via invariant sets of descent flow. We refer to [5, 6, 7, 8, 15, 17, 31, 33, 34] for more existence results of the Kirchhoff type equations.

Recently, much attention has been focused on studying the existence and multiplicity of solutions to problem (1) when \(a = 1, b = 0\). This type operator seems to have a prevalent role in physical situations such as combustion and dislocations in mechanical systems or in crystals. This problem has been studied by many authors, see [10, 22, 23, 24, 25, 26, 27, 28, 29] and the references therein.

The Kirchhoff type problem involving an integrodifferential operator was first studied by Fiscella and Valdinoci in a recent paper [11]. They considered the following Kirchhoff type problem:

\[
\begin{cases}
-M \left( \|u\|_{X_0}^2 \right) L_K u(x) = \lambda f(x, u) + |u|^{2^*-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is an open bounded set, \(2^* = \frac{2N}{N-2s}\) with \(s \in (0,1)\), \(N > 2s\), \(L_K\) is a nonlocal operator defined in (2), \(M\) and \(f\) are continuous functions. Based on the Mountain Pass Theorem, the authors in [11] obtained the existence of a nontrivial solution to (6).

Moreover, Sun and Teng [30] investigated problem (1) when \(K(x) = |x|^{-(N+2s)}\), which is a special case of problem (1). The authors proved the existence and multiplicity of solutions by using the Mountain Pass Theorem and the symmetric Mountain Pass Theorem together with truncation techniques.

Motivated by the above results, in the present paper, we establish that there are infinitely many solutions to problem (1) by the Fountain Theorem.

In order to state our main results, let us introduce the functional space that we will use in the following, which was introduced in [25]. For fixed \(s \in (0, 1)\), \(N > 2s\), \(\Omega \subset \mathbb{R}^N\) is an open bounded set with Lipschitz boundary, and \(X\) is the linear space of Lebesgue measurable functions from \(\mathbb{R}^N\) to \(\mathbb{R}\) such that the restriction to \(\Omega\) of function \(g\) in \(X\) belongs to \(L^2(\Omega)\) and

the map \((x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}\) is in \(L^2(\mathbb{R}^{2N} \setminus (C\Omega \times C\Omega), dx dy)\),

where \(C\Omega = \mathbb{R}^N \setminus \Omega\). Moreover, set

\[X_0 = \{g \in X : g = 0 \ a.e. \ \text{in } \mathbb{R}^N \setminus \Omega\}.
\]

According to the conditions of \(K\), by Lemma 11 in [24], we have that \(C_0^2(\Omega) \subseteq X_0\), and so \(X\) and \(X_0\) are nonempty. The spaces \(X\) and \(X_0\) are endowed, respectively, with the norms defined by

\[\|g\|_X = \|g\|_{L^2(\Omega)} + \left( \int_{\Omega} |g(x) - g(y)|^2 K(x - y) dx \ dy \right)^{1/2},\]

where \(K(x) = |x|^{-(N+2s)}\) for \(x \in \mathbb{R}^N\).
and
\[ \|g\|_{X_0} = \left( \int_Q |g(x) - g(y)|^2 K(x-y) \, dx \, dy \right)^{1/2}, \tag{9} \]
where \( Q = \mathbb{R}^{2N} \setminus ((\mathcal{C} \Omega) \times (\mathcal{C} \Omega)) \subset \mathbb{R}^{2N} \). Since \( g \in X_0 \), then the integral in (9) can be extended to all \( \mathbb{R}^{2N} \). Moreover, the norm on \( X_0 \) given in (9) is equivalent to the usual one defined in (8), by Lemmas 6 and 7 in [25].

With the norm given in (9), \( X_0 \) is a Hilbert space with scalar product defined as
\[ \langle u, v \rangle_{X_0} = \int_Q (u(x) - u(y))(v(x) - v(y)) K(x-y) \, dx \, dy, \tag{10} \]
see Lemma 7 in [25]. For the framework of functional Sobolev spaces and fractional Laplace, we refer the reader to the survey of Di Nezza, Palatucci and Valdinoci [9].

**Definition 1.1.** We say that \( u \) is a weak solution of problem (1), if \( u \) satisfies
\[ (a + b\|u\|_{X_0}^2) \int_Q (u(x) - u(y))(\phi(x) - \phi(y)) K(x-y) \, dx \, dy = \int_{\Omega} f(x, u(x))\phi(x) \, dx, \tag{11} \]
for all \( \phi \in X_0 \).

Let \( J : X_0 \to \mathbb{R} \) be the energy functional associated with problem (1) defined by
\[ J(u) = \frac{a}{2} \|u\|_{X_0}^2 + \frac{b}{4} \|u\|_{X_0}^4 - \int_{\Omega} F(x,u) \, dx, \tag{12} \]
where \( F \) is the primitive function of \( f \) with respect to the second variable, that is, \( F(x,t) = \int_0^t f(x,r) \, dr \). We can see that \( J \in C^1(X_0, \mathbb{R}) \) and for any \( \phi \in X_0 \), there holds
\[ \langle J'(u), \phi \rangle_{X_0} = (a + b\|u\|_{X_0}^2) \int_Q (u(x) - u(y))(\phi(x) - \phi(y)) K(x-y) \, dx \, dy \]
\[ - \int_{\Omega} f(x,u)\phi(x) \, dx. \tag{13} \]

Now, we suppose that the right hand side of (1) is a Caratheodory function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) verifying the following conditions:
\[ (F_1) \lim_{|x| \to 0} \frac{f(x,s)}{|s|} = 0 \text{ uniformly for } x \in \Omega; \]
\[ (F_2) \ f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \text{ and for some } 2 < p < 2^*, \text{ with } 2^* = +\infty \text{ for } N \leq 2s \text{ and } 2^* = \frac{2N}{N-2s} \text{ for } N > 2s, \text{ there exists a constant } C > 0 \text{ such that} \]
\[ |f(x,t)| \leq C(1 + |t|^{p-1}) \text{ for all } x \in \Omega, t \in \mathbb{R}. \]

In the seminal work of Ambrosetti and Rabinowitz [2], a common feature is that the nonlinearity \( f(x,u) \) satisfies the Ambrosetti-Rabinowitz’s 2–superlinear condition:
\[ (AR) \text{ there exist } \mu > 2 \text{ and } R > 0 \text{ such that} \]
\[ 0 < \mu F(x,t) \leq tf(x,t) \text{ for all } x \in \Omega, |t| \geq R. \]

The role of (AR) is to ensure the boundedness of the Palais-Smale sequences of the Euler-Lagrange functional. This condition is crucial in the applications of critical point theory. Indeed, (AR) condition implies that for some \( c, d > 0 \),
\[ F(x,u) \geq c|u|^\mu - d \text{ for all } x \in \Omega, u \in \mathbb{R}. \]
However, the second term in the Euler-Lagrange functional $J$ of problem (1) is the 4-power of norm $\| \cdot \|_{X_0}$. Therefore, in our first result, we assume that $f(x,u)$ is the Ambrosetti-Rabinowitz’s 4–superlinear condition:

(F3) there exist $\alpha > 4$ and $R > 0$ such that

$$0 < \alpha F(x,t) \leq tf(x,t) \quad \text{for all } x \in \Omega, \; |t| \geq R.$$ 

**Theorem 1.2.** We assume $f(x,u)$ satisfies that (F1), (F2), (F3) and (F4) $f(x,t)$ is odd with respect to $t$, that is,

$$f(x,-t) = -f(x,t) \quad \text{for all } x \in \Omega, \; t \in \mathbb{R}.$$ 

Then problem (1) has a sequence of solutions $u_k$ such that $J(u_k) \to \infty$ as $k \to \infty$.

**Remark 1.** From the assumption (F3), after integrating, we obtain that there exists $C > 0$ such that

$$C(|u|^\alpha - 1) \leq F(x,u) \quad \text{for all } x \in \Omega, \; u \in \mathbb{R}. \quad (14)$$

Moreover, by simple calculations, we can see that (F3) implies that

$$\lim_{|u| \to +\infty} \frac{F(x,u)}{|u|^4} = +\infty. \quad (15)$$

Hence $F(x,u)$ grows in a 4–superlinear rate as $|u| \to +\infty$.

It is well known that (F3) guarantees the boundedness of the $(PS)_c$ sequence of the corresponding functional $J$. Then we can apply the Fountain Theorem in [32] to get the desired result. However, there are many functions which are 4–superlinear but do not satisfy (F3) for any $\alpha > 4$. For example

$$f(x,u) = u^3(4 + \ln(1 + |u|)).$$

So without condition (F3), it is difficult to derive the boundedness of the $(PS)_c$ sequence of the corresponding functional. In this case, inspired by the variant Fountain Theorem in [36], we have the following result.

**Theorem 1.3.** In addition to conditions (F1) and (F2), suppose that the following conditions are satisfied:

1. $(S_1) \lim_{|u| \to +\infty} \frac{F(x,u)}{|u|^4} = \infty$ uniformly for $x \in \Omega$;

2. $(S_2) F(x,u) := \frac{1}{4} f(x,u) u - F(x,u)$ is increasing with $u$, for $x \in \Omega$.

Then problem (1) has a sequence of solutions $u_k$ such that $J(u_k) \to \infty$ as $k \to \infty$.

This paper is organized as follows. In Section 2, we give some preliminaries. We prove Theorems 1.2 and 1.3 in Section 3 and Section 4 respectively.

2. Preliminaries. In this paper we use the following notations: $X_0$ denotes Hilbert space given by (7) with the norm $\| \cdot \|_{X_0}$ given in (9), $X_0^*$ denotes the dual space for $X_0$, $L^r(\Omega)$ denotes Lebesgue space with the norm $| \cdot |_r$.

Taking into account Lemma 8 in [25], we have the following result.

**Lemma 2.1.** The embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1,2^*)$, while it is compact whenever $r \in [1,2^*)$. Moreover, there exists a positive constant $c(\theta)$ depending on $\theta$ (which is given in (3)), such that

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{H^s(\mathbb{R}^N)} \leq c(\theta)\|u\|_{X_0}.$$ 

Furthermore, there is a constant $c_r > 0$ such that for every $u \in X_0$,

$$|u|_r \leq c_r \|u\|_{X_0}. \quad (16)$$
Next, we introduce the definition of the $(PS)_c$ condition.

**Definition 2.2.** Let $J \in C^1(X_0, \mathbb{R})$ and $c \in \mathbb{R}$. The functional $J$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset X_0$ such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in} \quad X_0^*$$

as $n \to \infty$, has a convergent subsequence.

Since $X_0$ is a reflexive and separable Banach space, then there exist $e_j \in X_0$ and $e_j^* \in X_0^*$ ($j = 1, 2, \cdots$) such that

$$\langle e_i, e_j^* \rangle = \delta_{ij} \quad \text{where} \quad \delta_{ij} = 1 \quad \text{for} \quad i = j \quad \text{and} \quad \delta_{ij} = 0 \quad \text{for} \quad i \neq j,$$

and

$$X_0 = \text{span}\{e_j \mid j = 1, 2, \cdots\}, \quad X_0^* = \text{span}\{e_j^* \mid j = 1, 2, \cdots\}. $$

Set $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \bigoplus_{j=k}^{\infty} X_j$.

**Theorem 2.3.** [32, Theorem 3.6: the Fountain Theorem] Let $J \in C^1(X_0, \mathbb{R})$ be an even functional. Assume that for each $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that

(a1) $\alpha_k := \max_{u \in Y_k, \|u\|_{X_0} = \rho_k} J(u) \leq 0$;

(a2) $b_k := \inf_{u \in Z_k, \|u\|_{X_0} = \gamma_k} J(u) \to \infty$;

(a3) $J$ satisfies the $(PS)_c$ condition for every $c > 0$.

Then $J$ has an unbounded sequence of critical values.

Consider the following $C^1$ functional $\Phi_\lambda : X_0 \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \tag{17}$$

The following variant Fountain Theorem was established in [36].

**Theorem 2.4.** [36, Theorem 2.1: the variant Fountain Theorem] Assume that $\Phi_\lambda$ defined above satisfies

(b1) $\Phi_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$, Furthermore,

$$\Phi_\lambda(-u) = \Phi_\lambda(u) \quad \text{for all} \quad (\lambda, u) \in [1, 2] \times X_0;$$

(b2) $B(u) \geq 0$ for all $u \in X_0$; Moreover, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_{X_0} \to \infty$;

(b3) there exists $\rho_k > \gamma_k > 0$ such that

$$\alpha_k(\lambda) := \max_{u \in Y_k, \|u\|_{X_0} = \rho_k} \Phi_\lambda(u) < \beta_k(\lambda) := \inf_{u \in Z_k, \|u\|_{X_0} = \gamma_k} \Phi_\lambda(u), \quad \forall \lambda \in [1, 2].$$

Then

$$\beta_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k, \|u\|_{X_0} = r_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2], \tag{18}$$

where

$$B_k = \{u \in Y_k : \|u\|_{X_0} \leq r_k\}, \quad \text{and} \quad \Gamma_k = \{\gamma \in C(B_k, X_0) : \gamma \text{ is odd, } \gamma|_{\partial B_k} = \text{id}\}. $$

Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n \|u_n^k(\lambda)\|_{X_0} < \infty, \quad \Phi_\lambda(u_n^k(\lambda)) \to 0, \quad \Phi(u_n^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as} \quad n \to \infty. \tag{19}$$
3. Proof of Theorem 1.2.

Proof. We first show that condition \((a_1)\) holds in Theorem 2.3. From (14),

\[
J(u) = \frac{a}{2} \|u\|^2_{X_0} + \frac{b}{4} \|u\|^4_{X_0} - \int_{\Omega} F(x, u) \, dx
\]

\[
\leq \frac{a}{2} \|u\|^2_{X_0} + \frac{b}{4} \|u\|^4_{X_0} - C_1 |u|_{\alpha}^\alpha + C_1 |\Omega|.
\]

Since all norms are equivalent on the finite dimensional space \(Y_k\) and \(\alpha > 4\), there exists large \(\rho_k > 0\) such that

\[
a_k := \max_{u \in Y_k, \|u\|_{X_0} = \rho_k} J(u) < 0.
\]

That is, condition \((a_1)\) holds.

From \((F_1)\) and \((F_2)\), we get that for any \(\varepsilon > 0\), there exists \(C_\varepsilon > 0\) such that

\[
F(x, u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^p \quad \text{for all } x \in \Omega, \ u \in \mathbb{R}.
\]  

(21)

Let us define

\[
\beta_k := \sup_{u \in Z_k, \|u\|_{X_0} = 1} |u|^p.
\]  

(22)

By [32, Lemma 3.8], we have that \(\beta_k \to 0\) as \(k \to \infty\). Thus, we find

\[
J(u) = \frac{a}{2} \|u\|^2_{X_0} + \frac{b}{4} \|u\|^4_{X_0} - \int_{\Omega} F(x, u) \, dx
\]

\[
\geq \frac{a}{2} \|u\|^2_{X_0} + \frac{b}{4} \|u\|^4_{X_0} - C_\varepsilon |u|^p - \varepsilon |u|^2
\]

\[
\geq \frac{a}{2} \|u\|^2_{X_0} - C_\varepsilon |u|^p - \varepsilon |u|^2
\]

\[
\geq \frac{a}{2} \|u\|^2_{X_0} - C_\varepsilon |u|^p - \varepsilon c_2^2 |u|^2_{X_0}
\]

\[
= \left( \frac{a}{2} - \varepsilon c_2^2 \right) \|u\|^2_{X_0} - C_\varepsilon |u|^p
\]  

(23)

where \(c_2^2\) is given by (16). Choosing

\[
\gamma_k = \left( \frac{4C_\varepsilon \beta_k^p}{a} \right)^{\frac{1}{2p}}.
\]

Then for \(u \in Z_k\) with \(\|u\|_{X_0} = \gamma_k\), we have

\[
b_k := \inf_{u \in Z_k, \|u\|_{X_0} = \gamma_k} J(u) \geq \left( \frac{a}{4} - \varepsilon c_2^2 \right) \left( \frac{4C_\varepsilon \beta_k^p}{a} \right)^{\frac{1}{2p}}.
\]  

(24)

By choosing \(\varepsilon > 0\) small enough such that \(\frac{a}{4} - \varepsilon c_2^2 > 0\). This together with \(\beta_k \to 0\) as \(k \to \infty\) implies

\[
b_k \to +\infty \quad \text{as } k \to \infty.
\]  

(25)

So \((a_2)\) holds.

Next we prove \(J\) satisfies \((PS)_c\) condition. Indeed, let \(\{u_n\}\) be the \((PS)_c\) sequence of \(J\), that is, for every \(c > 0\),

\[
J(u_n) \to c, \quad J'(u_n) \to 0 \quad \text{in } X_0^* \quad \text{as } n \to \infty.
\]  

(26)
Let $\eta \in (\frac{1}{\alpha}, \frac{1}{4})$, using (14) and (26), for $n$ sufficiently large, we obtain
\[
c + 1 + \|u_n\|_{X_0} \geq J(u_n) - \eta \langle J'(u_n), u_n \rangle
\]
\[
= a \left( \frac{1}{2} - \eta \right) \|u_n\|_{X_0}^2 + b \left( \frac{1}{4} - \eta \right) \|u_n\|_{X_0}^4
\]
\[
+ \int_{\Omega} (\eta f(x, u_n) - F(x, u_n)) \, dx
\]
\[
\geq a \left( \frac{1}{2} - \eta \right) \|u_n\|_{X_0}^2 + (\alpha \eta - 1) \int_{\Omega} F(x, u_n) \, dx
\]
\[
\geq a \left( \frac{1}{2} - \eta \right) \|u_n\|_{X_0}^2 + C_2 (\alpha \eta - 1) |u_n|_{\alpha}^\alpha - C_3
\]
\[
\geq a \left( \frac{1}{2} - \eta \right) \|u_n\|_{X_0}^2 - C_3.
\]
This yields that $\{u_n\}$ is bounded in $X_0$.

Going if necessary to a subsequence, still denote it $u_n$, there exists $u \in X_0$ such that $u_n \rightarrow u$ weakly in $X_0$, that is
\[
\int_{\Omega} (u_n(x) - u_n(y)) \langle \phi(x) - \phi(y) \rangle K(x - y) \, dxdy
\]
\[
\rightarrow \int_{\Omega} (u(x) - u(y)) \langle \phi(x) - \phi(y) \rangle K(x - y) \, dxdy \quad \text{for } \forall \phi \in X_0, \quad (27)
\]
as $n \rightarrow \infty$. Moreover, by [27, Lemma 9], we have
\[
u_n \rightarrow u \quad \text{weakly in } L^2(\mathbb{R}^N);
\]
u_n \rightarrow u \quad \text{in } L^r(\mathbb{R}^N) \text{ for } r \in [1, 2^*);
u_n \rightarrow u \quad \text{a.e. in } \mathbb{R}^N,

as $n \rightarrow \infty$. On the other hand, from [4, Theorem IV-9], there exists $\ell \in L^r(\mathbb{R}^N)$ with $1 \leq r < 2^*$, such that
\[
|u_n(x)| \leq \ell(x) \quad \text{a.e. in } \mathbb{R}^N.
\]

Observe that
\[
\langle J'(u_n) - J'(u), u_n - u \rangle_{X_0}
\]
\[
= a \langle u_n - u, u_n - u \rangle_{X_0} + b \|u_n\|_{X_0}^2 \langle u_n, u_n - u \rangle_{X_0} - b \|u\|_{X_0}^2 \langle u, u_n - u \rangle_{X_0}
\]
\[
- \int_{\Omega} (f(x, u_n) - f(x, u)) \langle u_n - u \rangle \, dx
\]
\[
= a \langle u_n - u, u_n - u \rangle_{X_0} + b \|u_n\|_{X_0}^2 \langle u_n, u_n - u \rangle_{X_0} - b \|u\|_{X_0}^2 \langle u, u_n - u \rangle_{X_0}
\]
\[
+ b \|u_n\|_{X_0}^2 \langle u, u_n - u \rangle_{X_0} - b \|u\|_{X_0}^2 \langle u, u_n - u \rangle_{X_0}
\]
\[
- \int_{\Omega} (f(x, u_n) - f(x, u)) \langle u_n - u \rangle \, dx
\]
\[
= (a + b \|u_n\|_{X_0}^2) \langle u_n - u \rangle_{X_0} - b \|u\|_{X_0}^2 \langle u, u_n - u \rangle_{X_0}
\]
\[
- \int_{\Omega} (f(x, u_n) - f(x, u)) \langle u_n - u \rangle \, dx.
\]

It is clear that the left hand side of (28) tends to zero as $n \rightarrow \infty$. From (27), we have
\[
b(\|u_n\|_{X_0}^2 - \|u\|_{X_0}^2) \langle u, u_n - u \rangle_{X_0} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
Moreover, by Hölder inequality, and using condition $(F_2)$, we can get
\[
\int_{\Omega} |(f(x, u_n) - f(x, u))(u_n - u)|\,dx \leq |f(x, u_n) - f(x, u)|_{r'}|u_n - u|_r \to 0 \quad \text{as } n \to \infty,
\]
where \( r \in (1, 2^*) \) and \( r' = \frac{r}{r-1} \). Using these facts and (28), we deduce that
\[
\|u_n - u\|_{X_0} \to 0 \quad \text{as } n \to \infty. \tag{29}
\]
Hence we get \((a_3)\).

Applying Theorem 2.3, problem (1) has a sequence of solutions \( u_k \) such that \( J(u_k) \to \infty \) as \( k \to \infty \).

4. Proof of Theorem 1.3. In order to use Theorem 2.4 to prove Theorem 1.3, we define
\[
\Phi_{\lambda}(u) = \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|_{X_0}^4 - \lambda \int_{\Omega} F(x, u)\,dx := A(u) - \lambda B(u),
\]
for \( \lambda \in [1, 2] \), where
\[
A(u) = \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|_{X_0}^4, \quad \text{and} \quad B(u) = \int_{\Omega} F(x, u)\,dx.
\]
We have \( \Phi_{\lambda}(-u) = \Phi_{\lambda}(u) \) for all \( (\lambda, u) \in [1, 2] \times X_0 \). Moreover, \( A(u) \to \infty \) as \( \|u\|_{X_0} \to \infty \), and \( B(u) \geq 0 \).

Next, we show that \( \Phi_{\lambda} \) satisfies the assumptions in Theorem 2.4.

**Lemma 4.1.** Suppose that \((F_1), (F_2)\) and \((S_1)\) hold. Then there exists \( r_k > \rho_k > 0 \) such that
\[
(i) \quad \alpha_k(\lambda) := \max_{u \in Y_k, \|u\|_{X_0} = \rho_k} \Phi_{\lambda}(u) \leq 0;
\]
\[
(ii) \quad \beta_k(\lambda) := \inf_{u \in Z_k, \|u\|_{X_0} = r_k} \Phi_{\lambda}(u) > 0.
\]

**Proof.** We first prove \((i)\). From \((S_1)\), we get for any \( L > 0 \), there exists a constant \( C_L \) such that \( F(x, u) \geq L\|u\|_4^4 - C_L \) for all \( u \in \mathbb{R} \). Since \( Y_k \) is a finite dimensional space, then there exists \( c_{k, r} > 0 \) such that \( \|u\|_r \geq c_{k, r}\|u\|_{X_0} \) for all \( u \in Y_k \). Therefore, for \( u \in Y_k \),
\[
\Phi_{\lambda}(u) = \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|_{X_0}^4 - \lambda \int_{\Omega} F(x, u)\,dx
\]
\[
\leq \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|_{X_0}^4 - Lc_{k, r}^4\|u\|_{X_0}^4 + C_0
\]
\[
= \frac{a}{2}\|u\|_{X_0}^2 + \left[ \frac{b}{4} - Lc_{k, r}^4 \right]\|u\|_{X_0}^4 + C_0,
\] \tag{30}
where \( C_0 \) is a positive constant. Choosing \( L \) such that \( Lc_{k, r}^4 > \frac{b}{4} \), then for \( \|u\|_{X_0} = \rho_k \) with \( \rho_k > 0 \) large enough, we get that \((i)\) holds.

Next we show \((ii)\). By conditions \((F_1)\) and \((F_2)\), as the same proof of \((25)\), we have that \( \beta_k(\lambda) \to \infty \) as \( k \to \infty \). Thus \((ii)\) holds.

We see that all conditions of Theorem 2.4 hold, then for all \( \lambda \in [1, 2] \), there exists a sequence \( \{u_n^k(\lambda)\}_{n=1}^{\infty} \) such that
\[
\sup_n \|u_n^k(\lambda)\|_{X_0} < \infty, \quad \Phi_{\lambda}(u_n^k(\lambda)) \to 0, \quad \text{as } n \to \infty
\]
\[
\Phi(u_n^k(\lambda)) \to \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k, \|u\|_{X_0} = r_k} \Phi_{\lambda}(\gamma(u)) \geq \beta_k(\lambda) \geq 0 \quad \text{as } n \to \infty.
\]
Moreover, since $\zeta_k(\lambda) \leq \|u\|_{L^r(\Omega)}$, we define $A_k(\lambda) := \zeta_k(\lambda)$, and $X_0 \hookrightarrow L^r(\Omega)$ is compact for $2 \leq r < 2^*$, by standard argument. Let $\{u_k^n(\lambda)\}_{n=1}^{\infty}$ have a convergent subsequence. Suppose $u_k^n(\lambda) \rightarrow u^k(\lambda)$ as $n \rightarrow \infty$, then we have

$$\Phi'_\lambda(u_k^n(\lambda)) = 0, \quad \Phi(\lambda)(u_k^n(\lambda)) \in [\beta_k(\lambda), a_k^\infty]$$

for almost every $\lambda \in [1, 2]$. Now we take a sequence $\{\lambda_n\} \subset [1, 2]$ with $\lambda_n \rightarrow 1$, and for simplicity, write $u_n := u^k(\lambda_n)$, then

$$\Phi'_\lambda(u_n) = 0, \quad \Phi(\lambda_n)(u_n) \in [\beta_k(\lambda_n), a_k^\infty]. \quad (31)$$

**Lemma 4.2.** Under the assumptions of Theorem 1.3, the sequence $\{u_n\}$ is bounded in $X_0$.

**Proof.** Assume by contradiction that $\|u_n\|_{X_0} \rightarrow \infty$ as $n \rightarrow \infty$. Define $w_n := \frac{u_n}{\|u_n\|_{X_0}}$, then $\|w_n\|_{X_0} = 1$. Thus, up to a subsequence, still denoted by $w_n$, there exists $w \in X_0$ such that $w_n \rightarrow w$ weakly in $X_0$. By [27, Lemma 9], we have that

$$w_n \rightarrow w \quad \text{in} \quad L^r(\Omega) \quad \text{for} \ r \in [1, 2^*]; \quad w_n \rightarrow w \quad \text{a.e. in} \ \Omega,$$

as $n \rightarrow \infty$, and by [4, Theorem IV-9] there exists $\ell \in L^r(\Omega)$ such that $\|w_n(x)\| \leq \ell(x)$ a.e. in $\mathbb{R}^N$ for any $1 \leq r < 2^*$. Next we distinguish the following two cases:

(i) $w \neq 0$ and (ii) $w = 0$.

Case (i), by $\Phi'_\lambda(u_n) = 0$, we get

$$0 = \langle \Phi'_\lambda(u_n), u_n \rangle_{X_0} = a\|u_n\|_{X_0}^2 + b\|u_n\|_{X_0}^4 - \lambda_n \int_\Omega f(x, u_n)u_n dx.$$

Dividing both sides of the above equality by $\lambda_n\|u_n\|_{X_0}^4$, we obtain

$$\int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|_{X_0}^4} dx = \frac{a}{\lambda_n\|u_n\|_{X_0}^2} + \frac{b}{\lambda_n} \leq c < \infty.$$

On the other hand, by Fatou’s Lemma and condition (S1), we have

$$\int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|_{X_0}^4} dx = \int_{\{w_n(x) \neq 0\}} \frac{|w_n|^4 f(x, u_n)u_n}{\|u_n\|_{X_0}^4} dx \rightarrow \infty,$$

which is a contradiction.

Case (ii), set

$$\Phi_{\lambda_n}(t_n u_n) = \max_{t \in [0, 1]} \Phi_{\lambda_n}(t u_n).$$

For any given $M > 0$, define $w_n^* = \sqrt{4c_*}w_n$ with $c_* > 0$. By (21), we get

$$\int_\Omega F(x, w_n^*) dx \leq \epsilon \int_\Omega |w_n^*|^2 + C_\epsilon \int_\Omega |w_n^*|^p dx \rightarrow 0, \quad (32)$$

as $n \rightarrow \infty$. Then for $n$ large enough, we have

$$\Phi_{\lambda_n}(t_n u_n) \geq \Phi_{\lambda_n}(w_n^*) = \frac{a}{2}\|w_n^*\|_{X_0}^2 + \frac{b}{4}\|w_n^*\|_{X_0}^4 - \lambda_n \int_\Omega F(x, w_n^*) dx$$

$$= 2ac_* + 4bc_*^2 - \lambda_n \int_\Omega F(x, w_n^*) dx \geq 2ac_*,$$

since $c_* > 0$ can be large arbitrarily, we then get that $\lim_{n \rightarrow \infty} \Phi_{\lambda_n}(t_n u_n) \rightarrow \infty$. 


Moreover, by the definition of \( t_n \), we have that \( \langle \Phi_{\lambda_n}'(t_n u_n), t_n u_n \rangle_{X_0} = 0 \). Thus, by (S\( \lambda_n \)), we have
\[
\infty \leftarrow \Phi_{\lambda_n}(t_n u_n) = \Phi_{\lambda_n}(t_n u_n) - \frac{1}{4} \langle \Phi_{\lambda_n}'(t_n u_n), t_n u_n \rangle_{X_0} \\
= \frac{a t_n^2}{4} \| u_n \|_{X_0}^2 + \lambda_n \int_\Omega \left[ \frac{1}{4} f(x, t_n u_n)t_n u_n - F(x, t_n u_n) \right] dx \\
\leq \frac{a t_n^2}{4} \| u_n \|_{X_0}^2 + \lambda_n \int_\Omega \left[ \frac{1}{4} f(x, u_n)u_n - F(x, u_n) \right] dx \\
= \Phi_{\lambda_n}(u_n) - \frac{1}{4} \langle \Phi_{\lambda_n}'(u_n), u_n \rangle_{X_0} \in [\beta_k(\lambda_n), a_k^\ast]
\]
as \( n \to \infty \). This is a contradiction and this completes the proof of Lemma 4.2. \( \square \)

**Lemma 4.3.** The sequence \( \{ u_n \} \) has a convergent subsequence with the limit \( u^k \in X_0 \) for \( k \) large.

**Proof.** By Lemma 4.2, \( \{ u_n \} \) is bounded in \( X_0 \), then there is a subsequence of \( \{ u_n \} \), denoted by itself, such that \( u_n \to u^k \) weakly in \( X_0 \) as \( k \to \infty \). By the same argument given in Section 3, we get \( u_n \to u^k \) strongly in \( X_0 \) as \( k \to \infty \). \( \square \)

**Proof of Theorem 1.3.** Recall that \( u_n := u^k(\lambda_n) \). From Lemmas 4.1-4.3 and Lemma 2.1, by the standard argument, we obtain that there is a convergent subsequence of \( \{ u_n \} \) when \( \lambda_n \to 1 \), such that \( u_n \to u^k \) for some \( u^k \in X_0 \). On the other hand, we have
\[
J(u_n) = \Phi_1(u_n) = \Phi_{\lambda_n}(u_n) + (\lambda_n - 1) \int_\Omega F(x, u_n) dx.
\]
Moreover, \( \{ u_n \} \) is bounded in \( X_0 \), this together with \( (F_2) \) and (16) implies
\[
\left| \int_\Omega F(x, u_n) dx \right| < \infty \quad \text{as} \quad n \to \infty.
\]
Using these facts and (31), we obtain
\[
J(u^k) = \lim_{n \to \infty} \Phi_{\lambda_n}(u_n) \in [\beta_k(\lambda_n), a_k^\ast], \tag{33}
\]
and
\[
\lim_{n \to \infty} \langle J'(u_n), v \rangle_{X_0} = 0 \quad \text{for all} \quad v \in X_0. \tag{34}
\]
In view of \( J \in C^1(X_0, \mathbb{R}) \), we have \( J'(u_n) \to J'(u^k) \) in \( X_0^* \). Therefore, for every \( v \in X_0 \),
\[
\left| \langle J'(u_n) - J'(u^k), v \rangle_{X_0} \right| \leq \| J'(u_n) - J'(u^k) \|_{X_0^*} \| v \|_{X_0} \to 0
\]
as \( n \to \infty \). This means that \( \langle J'(u^k), v \rangle_{X_0} = 0 \) for all \( v \in X_0 \), that is, \( J'(u^k) = 0 \) in \( X_0^* \). By (33) and \( \beta_k(\lambda_n) \to +\infty \) as \( k \to \infty \), we then get that \( \{ u^k \}_{k=1}^\infty \) is an unbounded sequence of critical points of \( J(u) \). This completes the proof of Theorem 1.3.

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REFERENCES

[1] C. O. Alves, F. J. S. A. Corrêa and T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.*, 49 (2005), 85–93.

[2] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, 14 (1973), 349–381.

[3] S. Bernstein, Sur une classe d’équations fonctionnelles aux dérivées partielles, *Izv. Akad. Nauk SSSR Ser.*, 4 (1940), 17–26.

[4] H. Brézis, *Analyse fonctionelle. Théorie et applications*, Masson, Paris, 1983.

[5] C. Chen, J. Huang and L. Liu, Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave-convex nonlinearities, *Applied Mathematics Letters*, 26 (2013), 754–759.

[6] S. Chen and L. Li, Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$, *Nonlinear Anal. RWA*, 14 (2013), 1477–1486.

[7] B. Cheng and X. Wu, Existence results of positive solutions of Kirchhoff type problems, *Nonlinear Anal.*, 71 (2009), 4883–4892.

[8] F. J. S. A. Corrêa and G. M. Figueiredo, On a p-Kirchhoff equation via Krasnoselskii’s genus, *Appl. Math. Lett.*, 22 (2009), 819–822.

[9] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchiker’s guide to the fractional Sobolev spaces, *Bull. Sci. math.*, 136 (2012), 521–573.

[10] A. Fiscella, Saddle point solutions for non-local elliptic operators, preprint, (2012), available at http://arxiv.org/abs/1210.8401.

[11] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Analysis*, 94 (2014), 156–170.

[12] X. He and W. Zou, Infinitely many positive solutions of Kirchhoff type problems, *Nonlinear Anal.*, 70 (2009), 1407–1414.

[13] X. He and W. Zou, Multiplicity solutions of for a class of Kirchhoff type problems, *Acta Mathematicae Applicatae Sinica, English Series*, 26 (2010), 387–394.

[14] Kirchhoff and G. Mechanik, *Teubner*, Leipzig, 1883.

[15] Y. Li, F. Li and J. Shi, Existence of positive solutions to Kirchhoff type problems with zero mass, *J. Math. Anal. Appl.*, 410 (2014), 361–374.

[16] J. L. Lions, On some quations in boundary value problems of mathematical physics, in *Contemporary Developments in Continuum Mechanics and Partial differential Equations*, Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977, in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, 1978, 284–346.

[17] D. Liu and P. Zhao, Multiple nontrivial solutions to a p-Kirchhoff equation, *Nonlinear Anal.*, 75 (2012), 5032–5038.

[18] T. F. Ma and J. E. Muñoz Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, *Appl. Math. Lett.*, 16 (2003), 243–248.

[19] A. Mao and Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P. S. condition, *Nonlinear Anal.*, 70 (2009), 1275–1287.

[20] K. Perera and Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations*, 221 (2006), 246–255.

[21] S. I. Pohožaev, A certain class of quasilinear hyperbolic equations, *Mat. Sb. (N.S.)*, 168 (1975), 152–166.

[22] R. Servadei, The Yamabe equation in a non-local setting, *Advances in Nonlinear Analysis*, 2 (2013), 235–270.

[23] R. Servadei, Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity, *Contemp. Math.*, 595 (2013), 317–340.

[24] R. Servadei and E. Valdinoci, Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators, *Rev. Mat. Iberoam.*, 29 (2013), 1091–1126.

[25] R. Servadei and E. Valdinoci, Mountain Pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, 389 (2012), 887–898.

[26] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete and Continuous Dynamical Systems*, 5 (2013), 2105–2137.

[27] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, 367 (2015), 67–102.

[28] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.*, 12 (2013), 2445–2464.
[29] R. Servadei and E. Valdinoci, Fractional Laplacian equations with critical Sobolev exponent, preprint (2012), available at http://www.math.utexas.edu/mp_arc-bin/mpa?yn=12-58.

[30] G. F. Sun and K. M. Teng, Existence and multiplicity of solutions for a class of fractional Kirchhoff-type problem, Math. Commun., 19 (2014), 183–194.

[31] J. J. Sun and C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type equations, Nonlinear Analysis, 74 (2011), 1212–1222.

[32] M. Willem, Minimax Theorems, Birkhauser, Boston, Basel, Berlin, 1996.

[33] Q. L. Xie, X. P. Wu and C. L. Tang, Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent, Commun. Pure Appl. Anal., 12 (2013), 2773–2786.

[34] Y. W. Ye, Infinitely many solutions for Kirchhoff type problems, Differential Equations & Applications, 5 (2013), 83–92.

[35] Z. T. Zhang and K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl., 317 (2006), 456–463.

[36] W. Zou, Variant fountain theorem and their applications, Manuscripta Math., 104 (2001), 343–358.

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