Striped superconductors: How the cuprates intertwine spin, charge and superconducting orders

Erez Berg,1 Eduardo Fradkin,2 Steven A. Kivelson,1 and John M. Tranquada3
1Department of Physics, Stanford University, Stanford, California 94305-4060, USA
2Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3080, USA
3Condensed Matter Physics & Materials Science Department, Brookhaven National Laboratory, Upton, New York 11973-5000, USA

Abstract. Recent transport experiments in the original cuprate high temperature superconductor, La2−xBaxCuO4, have revealed a remarkable sequence of transitions and crossovers which give rise to a form of dynamical dimensional reduction, in which a bulk crystal becomes essentially superconducting in two directions while it remains poorly metallic in the third. We identify these phenomena as arising from a distinct new superconducting state, the “striped superconductor,” in which the superconducting order is spatially modulated, so that its volume average value is zero. Here, in addition to outlining the salient experimental findings, we sketch the order parameter theory of the state, stressing some of the ways in which a striped superconductor differs fundamentally from an ordinary (uniform) superconductor, especially concerning its response to quenched randomness. We also present the results of DMRG calculations on a model of interacting electrons in which sign oscillations of the superconducting order are established. Finally, we speculate concerning the relevance of this state to experiments in other cuprates, including recent optical studies of La2−xSrxCuO4 in a magnetic field, neutron scattering experiments in underdoped YBa2Cu3O6+x, and a host of anomalies seen in STM and ARPES studies of Bi2Sr2CaCu2O8+δ.
1. Introduction

In this paper we carefully characterize in terms of its broken symmetries, a novel superconducting state of matter, the “pair-density-wave” (PDW), with special focus on the “striped superconductor”, a unidirectional PDW. We present a concrete microscopic model of interacting electrons which we show, using density matrix renormalization group (DMRG) methods, has a striped superconducting ground state. There is an intimate relation between PDW and charge density wave (CDW) order, as a consequence of which the striped superconductor exhibits an extreme sensitivity to quenched disorder which inevitably leads to glassy behavior. This is qualitatively different from the familiar effects of disorder in uniform superconductors.

On the experimental side, we first draw attention to a set of recently discovered transport anomalies in the high temperature superconductor, La$_{2-x}$Ba$_x$CuO$_4$, which are particularly prominent for $x = 1/8$. We will be particularly interested in the spectacular dynamical layer decoupling effects recently observed in this system [1] which indicate that the effective inter-layer Josephson coupling becomes vanishingly small with decreasing temperature. These experiments suggest that a special symmetry of the state is required to explain this previously unsuspected behavior. While no comprehensive theory of these observations currently exists, even at the phenomenological level, we show how the salient features of these observations can be straightforwardly rationalized under the assumption that La$_{2-x}$Ba$_x$CuO$_4$ is a striped superconductor. We outline some further experiments that could critically test this assumption. Finally, we speculate about the possible role of striped superconducting order as the source of a number of salient experimental anomalies in a much broader spectrum of high temperature superconductors, including recent experiments on magnetic field induced layer decoupling in La$_{2-x}$Sr$_x$CuO$_4$ [2], the notable evidence of a local gap with many characteristics of a superconducting gap in STM and ARPES experiments in Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$ [3, 4, 5], and, most speculatively of all, experiments indicative of time reversal symmetry breaking in the pseudo-gap regime of YBa$_2$Cu$_3$O$_{6+δ}$.

The striped superconductor is a novel state of strongly correlated electronic matter in which the superconducting, charge and spin order parameters are closely intertwined with each other, rather than merely coexisting or competing. As we show here (and discussed in [6, 7]) the striped superconductor arises from the competing tendencies existing in a strongly correlated system, resulting in an inhomogeneous state in which all three forms of order are simultaneously present. The striped superconductor is thus a new type of electronic liquid crystal state [8].† In particular, as we shall

† Electronic liquid crystals [8] are quantum states of matter that spontaneously break some of the translation and/or rotational symmetries of an electronic system. In practice, these symmetries are typically not the continuous symmetries of free space, but rather the various discrete symmetries of the host crystal. Although an electronic smectic (stripe ordered state) has the same order parameter as a CDW or an SDW, it is a more general state that does not necessarily derive from a nesting vector of an underlying Fermi surface. The liquid crystal picture offers a broader perspective on the individual phases and on their phase transitions [9]. In particular, the way in which the PDW phase intertwines
see, the symmetry breaking pattern of the striped superconductor naturally explains
the spectacular layer decoupling effects observed in experiments in La$_{2-x}$Ba$_x$CuO$_4$.
In contrast, in any state with uniform superconducting order, dynamical inter-layer
decoupling could only arise if a somewhat unnatural fine tuning of the inter-layer
couplings led to a sliding phase [10].

The striped superconductor has an order parameter describing a paired state with
non-vanishing wave vector, $Q$, the ordering wave vector of the unidirectional PDW.
As such, this state is closely related by symmetry to the Fulde-Ferrell [11] (FF) and
Larkin-Ovchinnikov [12] (LO) states. The order parameter structure of the PDW state,
involving several order parameters coupled to each other, also evokes the $SO(5)$ approach
of a unified description of antiferromagnetism and a uniform $d$-wave superconductor
[13, 14]. The relation of the present discussion to these other systems and to earlier
theoretical works on the same and closely related subjects is deferred to Sec. 8. The
physics of stripe phases in the cuprate superconductors has been reviewed in Ref.[15]
and more recently in Ref.[16].

It is important to stress that the macroscopically superconducting phase of the
cuprates reflects the existence of a spatially uniform $Q = 0$ component of the order
parameter, whether or not there is substantial finite range superconducting order at
non-zero $Q$. One might therefore reasonably ask whether striped superconductivity,
even if interesting in its own right, is anything but an exotic oddity, with little or no
relation to the essential physics of high temperature superconductivity. The answer to
this question is at present unclear, and will not be addressed to any great extent in the
present paper. However, we wish to briefly speculate on a way in which the striped
superconductor could play an essential role in the broader features of this problem. In
BCS theory, the superconducting state emerges from a Fermi liquid in which the strong
electronic interactions are already accounted for in the self-energy of the quasiparticles.
The cuprates are different, in that the superconductivity, especially in underdoped
materials, emerges from a pseudogap phase for which there is no commonly accepted
model. As we will show, striped superconductivity has features in common with the
pseudogap phase, such as a gapless nodal arc and antinodal gaps. We speculate that the
pseudogap phase might be associated with fluctuating striped superconductivity, a state
that we do not yet know how to treat. Nevertheless, analysis of the ordered PDW state
and comparison to observations of stripe-ordered cuprates is a starting point. Indeed,
comparison (see Subsection 9.1 of recent photoemission results on LBCO $x=1/8$ with
transport and optical properties suggests that a “uniform” $d$-wave state (i.e. one with
a non-zero uniform component of the order parameter) develops on top of a striped
superconductor, resulting in a fully superconducting Meissner state, albeit one with
substantial coexisting short-range correlated stripe order.

The rest of this paper is organized as follows: In Section 2 we give an order
parameter description of the pair-density wave state. In Secs. 3 and 4 we summarize
charge, spin, and superconducting orders is unnatural in terms of a Fermi surface instability, but not
so from the liquid crystalline perspective.
the experimental evidence for this state, with Sec. 3 focussing on the strongest case, La$_{2-x}$Ba$_x$CuO$_4$, and Sec. 4 on other cuprates. In Section 5 we discuss the microscopic mechanisms for the formation of a PDW state. In Subsection 5.1 we implement these microscopic considerations by constructing a specific model that exhibits a PDW phase. The central conceptual ingredient is a microscopic mechanism leading to the formation of $\pi$ junctions in an unidirectional PDW state, which is given in Subsection 5.1.1 using perturbative arguments and then checked numerically using the density matrix renormalization group (DMRG) (in Subsection 5.1.2). A solvable microscopic model is discussed in Subsection 5.2. The quasi-particle spectrum of the PDW state is discussed in Subsection 5.3. Next, the Landau-Ginzburg theory of the PDW phase is discussed in Section 6. In Section 7 we show that the PDW state, in three dimensional layered structures (orthorhombic and LTT) as well as at grain (twin) boundaries, leads to time-reversal symmetry breaking effects. In Section 8 we discuss the connections that exist between the PDW state and other states discussed in the literature, particularly the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states. Section 9 is devoted to our conclusions.

This paper is partly a review of our recent work on the theory of the PDW state[6, 7] and of other related work, with an updated discussion of the current experimental status. However in this paper we have also included many new results, particularly the DMRG analysis of PDW states in strongly correlated systems of Section 5.1.2 and the connection bewteen the PDW state and non-collinear order and time-reversal symmetry breaking of Section 7.

2. The Order Parameter of a Striped Superconductor

The order parameter whose non-zero expectation value defines a superconducting state is

$$\phi_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \equiv \langle \psi_{\sigma}^{\dagger}(\mathbf{r}) \psi_{\sigma'}^{\dagger}(\mathbf{r}') \rangle,$$

where $\psi_{\sigma}^{\dagger}(\mathbf{r})$ is the fermionic field operator which creates an electron with spin polarization $\sigma$ at position $\mathbf{r}$. Further distinctions between different superconducting states can be drawn on the basis of the spatial and spin symmetries of $\phi$. In crystalline solids, all familiar superconducting states respect the translational symmetry of the solid, $\phi(\mathbf{r} + \mathbf{R}, \mathbf{r}' + \mathbf{R}) = \phi(\mathbf{r}, \mathbf{r}')$, where $\mathbf{R}$ is any Bravais lattice vector. Consequently, the symmetries of the state can be classified by the irreducible representations of the point group - colloquially as s-wave, d-wave, p-wave, etc. In the absence of spin-orbit coupling, superconducting states can be classified, as well, by their transformation under spin rotations as singlet or triplet. Finally, the superconducting state can either preserve or break time reversal symmetry (as in $p_x + ip_y$).

In the presence of quenched disorder, the underlying Hamiltonian does not have any particular spatial symmetries, so the classification of distinct superconducting states by their symmetries (other than time reversal), at first seems difficult. However, there are
several ways that this can be accomplished \cite{17}, of which the most obvious is to consider
the symmetries of the configuration averaged order parameter
\[ \overline{\phi_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r'})} \equiv \langle \psi_{\sigma}^{\dagger}(\mathbf{r})\psi_{\sigma'}^{\dagger}(\mathbf{r'}) \rangle, \] (2)
where \( \langle ... \rangle \) signifies the thermal average, and \( (\ldots) \) signifies an average over realizations
of the disorder configuration. It is clear, for example, that under most circumstances,
a macroscopic "phase sensitive" measurement of the symmetry of the order parameter
will give \cite{17} a result consistent with a classification based on the symmetry of the
configuration averaged order parameter.

The striped superconductor is an example of a state, which has more generally
called \cite{18,19,20} a pair density wave (PDW), in which the translational symmetry
of the crystal is spontaneously broken as well, so that \( \phi(\mathbf{r} + \mathbf{R}, \mathbf{r'} + \mathbf{R}) \) exhibits non-trivial
dependence on \( \mathbf{R} \). However, this by itself, is insufficient to identify a new state of matter.
In a system with coexisting charge-density-wave (CDW) and superconducting order, the
CDW itself introduces a new periodicity into the problem, which must generically be
reflected in a spatial modulation of \( \phi \), as well.\footnote{The problem of coexisting stripe and superconducting order in strongly correlated systems has been
the focus of numerous studies in the literature. Sachdev, Vojta, and coworkers have investigated this
problem in detail in the context of generalized 2D \( t-J \) models in the large \( N \) approximation\cite{21,22,23}.
This problem has also been discussed in one-dimensional systems\cite{24}.}
As discussed in Sec. 6 an analysis of the implications of a generic theory of coupled order parameters implies \cite{22} that in a
state of coexisting order, a (possibly small) modulation of the superconducting order
with the same spatial period as that of the CDW will be induced. None-the-less, in
such a state, there still exists a "dominant" uniform component to the superconducting
order parameter, which we define as the spatial average of the SC order parameter:
\[ \phi_{\sigma,\sigma'}^{(0)}(\mathbf{r}, \mathbf{r'}) \equiv N^{-1} \sum_{\mathbf{R}} \langle \psi_{\sigma}^{\dagger}(\mathbf{r} + \mathbf{R})\psi_{\sigma'}^{\dagger}(\mathbf{r'} + \mathbf{R}) \rangle, \] (3)
where \( N \) is the number of unit cells in the system.

Instead, the pure PDW in a crystal is a state in which \( \phi \) is non-zero, but all uniform
components vanish, \( \phi_{\sigma,\sigma'}^{(0)}(\mathbf{r}, \mathbf{r'}) = 0 \) for any \( \mathbf{r} \) and \( \mathbf{r'} \). Just as a CDW is often defined
in terms of a fundamental harmonic, so a PDW state is characterized by the smallest
value of the crystal momentum, \( \mathbf{Q} \), for which
\[ \phi_{\sigma,\sigma'}^{(\mathbf{Q})}(\mathbf{r}, \mathbf{r'}) \equiv N^{-1} \sum_{\mathbf{R}} \exp[i\mathbf{Q} \cdot \mathbf{R}] \langle \psi_{\sigma}^{\dagger}(\mathbf{r} + \mathbf{R})\psi_{\sigma'}^{\dagger}(\mathbf{r'} + \mathbf{R}) \rangle, \] (4)
has a non-vanishing expectation value.\footnote{As with a uniform superconducting state, distinct PDW states with the same pattern of translation
symmetry breaking can also be distinguished by different patterns of point group symmetry breaking.
However, since the ordering vector (or vectors) already break the point group down to a smaller
subgroup, which is then all that is left of the original symmetry for this purpose. For instance, in a
tetragonal crystal, a striped superconductor with \( \mathbf{Q} \) along the x direction, can be classified as having
\( s \)-wave or \( d_{xy} \)-wave symmetry, based on whether or not the the order parameter changes sign under
reflection through a symmetry plane parallel to the x axis, but any distinction one would like to draw
between a striped version of an \( s \)-wave and a \( d_{x^2-y^2} \)-wave superconductor are in precise, not based on}
Note that the theory of coupled order parameters [7] implies that the existence of PDW order with ordering vector $\mathbf{Q}$ generically implies the existence of CDW order with ordering vector $2\mathbf{Q}$, but so long as $\phi^0 = 0$, no CDW ordering with wave vector $\mathbf{Q}$ is expected. A “striped superconductor” refers to the special case in which the independent ordering vectors are all parallel to each other (“unidirectional PDW”).

One of the prime new characteristics of a striped superconductor which is different from a uniform superconductor is its complex sensitivity to quenched disorder. As we shall see, for much the same reasons that disorder destroys long-range CDW order, under most relevant circumstances, even weak disorder causes the configuration averaged PDW order parameter to vanish:

$$\overline{\phi_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}')} = 0.$$  \hfill (5)

However, as in the case of an XY spin-glass, this is not the whole story: It is possible to define an analogue of the Edwards-Anderson order parameter,

$$Q_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \equiv \overline{|\langle \psi_{\sigma_1}^\dagger(\mathbf{r}_1)\psi_{\sigma_2}^\dagger(\mathbf{r}_2)\psi_{\sigma_3}^\dagger(\mathbf{r}_3)\psi_{\sigma_4}^\dagger(\mathbf{r}_4) \rangle|^2},$$  \hfill (6)

which vanishes in the normal high temperature phase, but which can be non-zero in a low temperature superconducting glass phase, where one exists. Moreover, in such a phase, as we will see, we generically expect time-reversal symmetry to be spontaneously broken. In analogy with the XY spin-glass, we expect that in two dimensions, the superconducting glass phase is stable only at $T = 0$ and for weak enough disorder, although in three dimensions it can exist below a non-zero superconducting glass transition temperature.\footnote{In [7], the possibility is discussed that in three dimensions there might also exist a superconducting version of a Bragg glass phase, in which $\phi$ exhibits quasi-long-range order. We have not further studied this potentially interesting state.}

There is one more extension that is useful—we define a charge $4e$ superconducting order parameter:

$$\phi^{(4)}(1, 2, 3, 4) \equiv \langle \psi_{\sigma_1}^\dagger(\mathbf{r}_1)\psi_{\sigma_2}^\dagger(\mathbf{r}_2)\psi_{\sigma_3}^\dagger(\mathbf{r}_3)\psi_{\sigma_4}^\dagger(\mathbf{r}_4) \rangle$$  \hfill (7)

where we have introduced a compact notation in which $1 \equiv (\sigma_1, \mathbf{r}_1)$, etc. Naturally, in any state with charge $2e$ superconducting order, $\phi_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \neq 0$, some components of the charge $4e$ order parameter will also be non-zero. This can be seen from the theory of coupled order parameters presented in Sec. \footnote{In [7], the possibility is discussed that in three dimensions there might also exist a superconducting version of a Bragg glass phase, in which $\phi$ exhibits quasi-long-range order. We have not further studied this potentially interesting state.}. At mean-field level, it can be seen by applying Wick’s theorem to the expression in Eq. (7) to express $\phi^{(4)}$ as a sum of pairwise products of $\phi$’s: $\phi^{(4)}(1, 2, 3, 4) \sim \phi(12)\phi(34) + \phi(14)\phi(23) - \phi(13)\phi(24)$.

There are two reasons to consider this order parameter. In the first place, it is clear from the above that even in the PDW state, although the uniform component of $\phi$ vanishes, the uniform component of $\phi^4 \sim \phi^Q\phi^{-Q} \neq 0$. More importantly, $\phi^{(4)}$ ordering can be more robust than the PDW ordering. Specifically, under some broken symmetries but on quantitative differences in local pairing correlations, and so do not define distinct phases of matter. However, a checkerboard PDW, with symmetry related ordering vectors $\mathbf{Q}$ and $\mathbf{Q}'$ along the $x$ and $y$ axes, respectively, can be classified as s-wave or $d_{x^2-y^2}$-wave, depending on how it transforms under rotation by $\pi/2$.\footnote{In [7], the possibility is discussed that in three dimensions there might also exist a superconducting version of a Bragg glass phase, in which $\phi$ exhibits quasi-long-range order. We have not further studied this potentially interesting state.}
circumstances [7,25], it is possible for thermal or quantum fluctuations to destroy the PDW order by restoring translational symmetry without restoring large-scale gauge symmetry; in this case, appropriate components of $\phi^{(4)}$ remain non-zero, even though $\phi$ vanishes identically. This is the only potentially realistic route we know of to charge 4e superconductivity.||

In the absence of spin-orbit coupling, distinct phases with translationally invariant charge 4e ordering can be classified according to the total spin of the order parameter, which in this case can be spin 2, 1, or 0. Manifestly, any charge 4e superconducting state which results from the partial melting of a singlet PDW will itself have spin 0. As with paired superconductors, the charge 4e order parameter can also be classified according to its transformation properties under action of the point-group of the host crystal. For instance, in a crystal with a $C_4$ symmetry, taking the points $r_i$ to lie on the vertices of a square, the transformation properties of $\phi^{(4)}$ under rotation by $\pi/2$ can be used to classify distinct spin-0 states as being d-wave or s-wave.

The definitions given here in terms of possible behaviors of the order parameter are natural from a taxonomic viewpoint. In particular, the striped superconductor seems at first to be a rather straightforward generalization of familiar uniform superconducting states. However, both at the microscopic level of the “mechanism” of formation of such a state, and at the phenomenological level of macroscopically observable implications of the state, the problem is full of subtleties and surprises, as discussed below.

3. Striped superconductivity in La$_{2-x}$Ba$_x$CuO$_4$ and the 214 family

We now summarize some of the observations that lead to the conclusion that La$_{2-x}$Ba$_x$CuO$_4$ with $x = 1/8$ is currently the most promising candidate experimental system as a realization of a striped superconductor [1,6].

Firstly, the existence of “stripe order” is unambiguous. It is well known (from neutron [27,28,29] and X-ray [30,31,28] scattering studies) that unidirectional CDW (charge-stripe) and SDW (spin-stripe) orders exist in La$_{2-x}$Ba$_x$CuO$_4$. Such spin and charge stripe orders were originally studied in La$_{1.48}$Nd$_{0.4}$Sr$_{0.12}$CuO$_4$ [32,33,34], and they have now been confirmed in La$_{1.8-x}$Eu$_{0.2}$Sr$_x$CuO$_4$ [35,36,37]. Furthermore, substantial spin stripe order has been observed in La$_2$CuO$_{4.11}$ [38], Zn-doped La$_{2-x}$Sr$_x$CuO$_4$ [39], and in the spin-glass regime of La$_{2-x}$Sr$_x$CuO$_4$ (0.02 < $x$ < 0.055) [40]. While the spin-stripe order in underdoped but superconducting La$_{2-x}$Sr$_x$CuO$_4$ (0.055 < $x$ < 0.14) is weak in the absence of an applied magnetic field, it has been observed [41,42,43] that readily accessible magnetic fields (which partially suppress the superconducting order) produce well-developed and reproducible spin-stripe order. For La$_{2-x}$Ba$_x$CuO$_4$ with $x = 1/8$, the charge ordering temperature is

|| It is possible to cook up models in which charge 4e superconductivity arises in systems in which electrons can form 4 particle bound-states, but do not form 2 particle or many particle bound states (phase separation) - see, for example, [26]. However, this involves unrealistically strong attractive interactions and an unpleasant amount of fine tuning of parameters.
53 K and the spin ordering temperature is 40 K [28].

In addition to spin and charge ordering, $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$ with $x = 1/8$ exhibits transport and thermodynamic behavior that is both striking and complex. We will not rehash all of the details here (see [28]); however, there are two qualitative features of the data on which we would like to focus: 1) With the onset of spin-stripe order at 40 K,† there is a large (in magnitude) and strongly temperature dependent enhancement of the anisotropy of the resistivity and other properties, such that below 40 K the in-plane charge dynamics resemble those of a superconductor, while in the $c$-direction the system remains poorly metallic. The most extreme illustration of this occurs in the temperature range $10 \text{ K} < T < 16 \text{ K}$, in which the in-plane resistivity is immeasurably small, while the $c$-axis resistivity is in the 1–10 mΩ-cm range, so that the resistivity anisotropy ratio is consistent with infinity. 2) Despite the fact that signatures of superconductivity onset at temperatures in excess of 40 K, and that angle resolved photoemission (ARPES) has inferred a “gap” [44, 45] of order 20 meV, the fully superconducting state (i.e., the Meissner effect and zero resistance in all directions) only occurs below a critical temperature of 4 K. It is very difficult to imagine a scenario in which a strong conventional superconducting order develops locally on such high scales, but fully orders only at such low temperatures in a system that is three dimensional, non-granular in structure, and not subjected to an external magnetic field.

Evidence that similar, although somewhat less extreme transport and thermodynamic anomalies accompany stripe ordering can be recognized, in retrospect, in other materials in the 214 family. For example, in $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ with $x = 0.08$ and 0.10, the anisotropy of the resistivity ($c$-axis vs. in-plane) rapidly grows towards $10^4$ as the superconducting $T_c$ is approached from above [46]. In the case of $\text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4$, evidence for dynamical layer decoupling is provided by measurements of the anisotropic onset of the Meissner effect [47]. In contrast, the resistivity ratio in this material [48] only reaches $10^3$; this may be limited by enhanced in-plane resistivity due to disorder [49]. Moreover, an unexpectedly strong layer decoupling in the charge dynamics produced by the application of a transverse magnetic field in $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ has been observed [2], but only in the underdoped range of $x$ where the magnetic field also induces spin stripe order [41, 43].

We shall see that the anomalous sensitivity of a striped superconductor to quenched disorder can account for the existence of a broad range of temperatures between the onset of strongly developed superconducting correlations on intermediate scales and the actual macroscopic transition temperature to a state of long-range coherence. Moreover, given the crystal structure of the Low Temperature Tetragonal (LTT) phase of $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$, there is a special symmetry of the striped superconducting state which produces interlayer decoupling. Specifically, because the stripes in alternate planes are oriented perpendicular to one another (as shown in Fig.5a), there is no first order Josephson coupling between neighboring planes. Indeed, analogous features of the spin-

† Static charge stripe order onsets at 54 K, at the LTT/LTO transition. [30]
striped superconductors, which have gone largely unnoticed in the past, are accounted for [6] by the same geometric features of the striped state. When spin-stripe order occurs, the in-plane correlation length can be very long, in the range of 100–600 Å [38, 28], but the interplanar correlation length is never more than a few Å [38, 50], a degree of anisotropy that cannot be reasonably explained simply on the basis of the anisotropy in the magnitude of the exchange couplings [51]. Furthermore, despite the presence of long correlation lengths, true long-range spin-stripe order has never been reported. It will be made clear that the superconducting stripe order and the spin stripe order share the same periodicity and the same geometry, so both the interlayer decoupling and the suppression by quenched disorder of the transition to a long-range ordered state can be understood as arising from the same considerations applied to both the unidirectional SDW and PDW orders.

4. Experiments in other cuprates

Data which clearly reveal the existence of spin-stripe order, or that provide compelling evidence of PDW order in other families of cuprates is less extensive. However, considerable evidence of a tendency to spin stripe order in YBa$_2$Cu$_3$O$_{6+x}$ has started to accumulate [15], and there are some persistent puzzles concerning the interpretation of various experiments in a number of cuprates that, we would like to speculate, may reflect the presence of PDW order. In this section, we will mention some of these puzzles, and will return to discuss why they may be indicative of PDW order in Sec. 7.

YBa$_2$Cu$_3$O$_{6+x}$ is often regarded as the most ideal cuprate, having minimal structural and chemical disorder, and less tendency to stripe or any other type of charge ordering than the 214 cuprates. (Sometimes the cuprates with the highest transition temperatures, such as HgBa$_2$Cu$_3$O$_{4+δ}$, are viewed as being similarly pristine.) However, in its underdoped regime it is well known that YBa$_2$Cu$_3$O$_{6+x}$ exhibits temperature-dependent in-plane anisotropic transport [52] as well as fluctuating spin stripe order [53, 15, 54]. Recent neutron scattering experiments have provided strong evidence that underdoped YBa$_2$Cu$_3$O$_{6+x}$ (with $x \sim 0.45$) has nematic order below a critical temperature $T_c \sim 150$ K [55]. Even more recent neutron scattering experiments by Hinkov et al. [56] on the same sample find that a modest $c$-axis magnetic field stabilizes an incommensurate static spin ordered state, detectable as a pair of peaks in the elastic scattering displaced by a distance in the crystallographic $a$ direction from the Neél ordering vector. Given the newfound evidence of spin-stripe related structures in YBa$_2$Cu$_3$O$_{6+x}$, it is plausible that here, too, striped superconductivity may occur. However, the differences in the 3D crystal structure, and especially the weak orthorhombicity, would make the macroscopic properties of a PDW distinctly different in YBa$_2$Cu$_3$O$_{6+x}$ than in the 214 cuprates.

A remarkable recent discovery is that underdoped YBa$_2$Cu$_3$O$_{6+x}$ appears to exhibit signatures of spontaneous time-reversal symmetry breaking (at zero magnetic field) below a critical temperature comparable to that for the nematic ordering. [57, 58]
Various theoretical scenarios for the existence of time-reversal symmetry breaking predated these experiments, and so in some sense predicted them. However, given that both nematic order and time-reversal symmetry breaking are seemingly present simultaneously in the same samples with comparable critical temperatures, it is reasonable to hope that both phenomena have an underlying common explanation. If we think of the superconducting order parameter as an XY pseudo-spin, then the PDW order is a form of collinear antiferromagnetism, and time-reversal symmetry breaking corresponds to non-collinear order of the pseudospins. As we will show in Sec. weak time reversal symmetry breaking can occur in a PDW state due to various patterns of geometric frustration in three dimensions or as a consequence of the existence of certain types of defects, such as twin boundaries. There is a large body of STM and ARPES data, especially on Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and Bi$_2$Sr$_2$CuO$_{6+\delta}$, which has revealed a surprisingly rich and difficult to interpret set of spectral features associated with the d-wave superconducting gap and a d-wave pseudo-gap whose origin is controversial. Indeed, there is a clear “nodal-anti-nodal dichotomy” in the behavior of the measured single-particle spectral functions. Some aspects of the data are suggestive that there is a single superconducting origin of all gap features, with anisotropic effects of superconducting fluctuations leading to the observed dichotomy. Other aspects suggest that there are at least two distinct origins of the near-nodal and the antinodal gaps. It is possible that PDW ordering tendencies can synthesize both aspects of the interpretation. In the presence of both uniform and PDW superconducting order, there are two distinct order parameters, both of which open gaps on portions of the Fermi surface, but they are both superconducting, and so they can smoothly evolve into one another. (Note that an early study of modulated structures seen in STM concluded that they could be understood in terms of just such a two-superconducting-gap state.)

More generally, one of the most remarkable features of the pseudo-gap phenomena is the existence of what appears to be superconducting fluctuations, detectable for instance in the Nernst and magnetization signal, over a surprisingly broad range of temperatures and doping concentrations. At a broad-brush level, these phenomena are a consequence of a phase stiffness scale that is small compared to the pairing scale. However, it is generally difficult to understand the existence of such a broad fluctuational regime on the basis of any sensible microscopic considerations. The glassy nature of the ordering phenomena in a PDW may hold the key to this central paradox of HTC phenomenology, as it gives rise to an intrinsically broad regime in which superconducting correlations extend over large, but not infinite distances.

5. Microscopic considerations

From a microscopic viewpoint, the notion that a PDW phase could be stable at first sounds absurd. Intuitively, the superconducting state can be thought of as the condensed state of charge $2e$ bosons. However, in the absence of magnetic fields, the ground-state
of a bosonic fluid is always node-less, independent of the strength of the interactions, and therefore cannot support a state in which the superconducting order parameter changes sign. Thus, for a PDW state to arise, microscopic physics at scales less than or of order the pair-size, $\xi_0$, must be essential. This physics reflects an essential difference between superfluids of paired fermions and preformed bosons [71].

Our goal in this section is to shed some light on the mechanism by which strongly interacting electrons can form a superconducting ground-state with alternating signs of the order parameter. We will consider the case of a unidirectional (striped) superconductor, but the same considerations apply to more general forms of PDW order. We will not discuss the origin of the pairing which leads to superconductivity. Likewise, we will not focus on the mechanism of translation symmetry breaking by the density wave, as that is similar to the physics of CDW and SDW formation. Our focus is on the sign alternation of $\phi$. Thus, in much of this discussion, we will adopt a model in which we have alternating stripes of superconductor and correlated insulator. The system looks like an array of extended superconductor-insulator-superconductor (SIS) junctions, and we will primarily be concerned with computing the Josephson coupling across the insulating barriers. If the effective Josephson coupling is positive, then a uniform phase (normal) superconducting state is favored, but if the coupling is negative (favoring a $\pi$ junction), then a striped superconducting phase is found.

So long as time reversal symmetry is neither spontaneously nor explicitly broken, the Josephson coupling, $J$ between two superconductors must be real. If it is positive, as is the usual case, the energy is minimized by the state in which the phase difference across the junction is 0; if it is negative, a phase difference of $\pi$ is preferred, leading to a “$\pi$ junction.” $\pi$ junctions have been shown, both theoretically and experimentally, to occur for two distinct reasons: they can be a consequence of strong correlation effects in the junction region between two superconductors [71, 72, 73] or due to the internal structure (e.g. d-wave symmetry) of the superconductors, themselves [74, 75].

In Ref. [7], we have provided examples of $\pi$ junctions which build on the first set of ideas. Unlike the previously studied cases, these $\pi$ junctions were extended (i.e., $J$ is proportional to the cross sectional “area” of the junction). However, since the problem was solved analytically (treating the tunneling between the superconducting and the insulating regions by perturbation theory), we were limited to somewhat artificial models. For example, tunneling between the sites in the insulating regime was neglected. In this Section, we first summarize the perturbative results of [7], and then present numerical (DMRG) results for an extended SIS junction. Under some circumstances, $J > 0$, but we also find a considerable region of parameter space where where $J < 0$. Finally, we discuss how this result can be generalized to an infinite array of junctions, forming a 2D unidirectional PDW.
5.1. A Solved model

Let us consider the following explicit model for a single SIS junction. The three decoupled subsystems are described by the Hamiltonian

$$H_0 = H_L + H_B + H_R,$$

(8)

The right ($R$) and left ($L$) superconducting regions and the barrier ($B$) region are one dimensional Hubbard models,

$$H_\alpha = \sum_{i\sigma} \left( -tc_{\alpha,i,\sigma}^\dagger c_{\alpha,i,\sigma} + \text{h.c.} - \mu_\alpha n_{\alpha,i} \right) + U_\alpha \sum_{i} n_{\alpha,i,\uparrow} n_{\alpha,i,\downarrow},$$

(9)

$c_{\alpha,i+1,\sigma}$ is a creation operator of an electron on chain $\alpha = L, R$ or $B$ at site $i$ with spin $\sigma$, and we have introduced the notation $n_{\alpha,i,\sigma} = c_{\alpha,i,\sigma}^\dagger c_{\alpha,i,\sigma}$ and $n_{\alpha,i} = \sum_\sigma n_{\alpha,i,\sigma}$. The left and right superconducting chains are characterized by a negative $U_R = U_L = -|U_{L,R}|$, while the insulating barrier has a positive $U_B > 0$. The chemical potentials of the left and right superconductors are the same, $\mu_R = \mu_L$, but different from $\mu_B$, which is tuned such that the barrier chain is half filled (and therefore insulating).

The three subsystems are coupled together by a single-particle hopping term,

$$H' = -t' \sum_{i,\sigma} \left[ c_{L,i,\sigma}^\dagger c_{B,i,\sigma}^\dagger + c_{R,i,\sigma}^\dagger c_{B,i,\sigma} + \text{h.c.} \right].$$

(10)

The left and right chains are characterized by a spin gap and by dominant superconducting fluctuations, as a result of their negative $U$’s. The inter-chain hopping term $H'$ induces a finite Josephson coupling between the local superconducting order parameters of the two chains, via virtual hopping of a Cooper pair through the barrier chain.

5.1.1. Perturbative analysis  For completeness, let us briefly review the perturbative treatment of the inter-chain hopping term (10) given in [7]. The leading (fourth order) contribution to the Josephson coupling is given by

$$J = \frac{(t')^4}{\beta} \int d1 \, d2 \, d3 \, d4 \, F_L(1,2) F_R^*(4,3) \Gamma(1,2;3,4)$$

(11)

where $1 \equiv (\tau_1, i_1)$ etc.,

$$\int \, d1 \equiv \sum_{i_1} \int_0^\beta d\tau_1$$

(12)

(in the limit $\beta \to \infty$) and

$$F_\alpha(1,2) \equiv \left\langle T_\tau \left[ c_{\alpha,i_1,\uparrow}^\dagger(\tau_1) c_{\alpha,i_2,\downarrow}^\dagger(\tau_2) \right] \right\rangle$$

(13)

$$\Gamma(1,2;3,4) \equiv \left\langle T_\tau \left[ c_{i_1,\uparrow}^\dagger(\tau_1) c_{i_2,\downarrow}^\dagger(\tau_2) c_{i_3,\downarrow}(\tau_3) c_{i_4,\uparrow}(\tau_4) \right] \right\rangle$$

where we have made the identification $c_{i,\sigma}^\dagger \equiv c_{B,i,\sigma}^\dagger$. Our purpose is to determine the conditions under which $J < 0$. For the sake of simplicity, let us consider the case in which the gap to remove a particle from the barrier, $\Delta_h$, satisfies $\Delta_s \ll \Delta_h \ll \Delta_p,$
where $\Delta_s$ is the spin gap on the superconducting chains, and $\Delta_p$ is the gap to insert a particle in the barrier. These conditions can be met by tuning appropriately the chemical potentials on the three chains and setting $U_B$ to be sufficiently large.

In [7] it is shown that, quite generally, $J$ can be written as a sum of two terms

$$J = J_1 + J_2$$

where, in terms of the spin-spin correlation function, $\langle \vec{S}(1) \cdot \vec{S}(2) \rangle$ of the barrier chain,

$$J_1 = \frac{(t')^4}{4 \beta (\Delta_h)^2} \int d1 \, d2 \, |F_L(1, 2)|^2$$

$$J_2 = -\frac{3(t')^4}{4 \beta (\Delta_h)^2} \int d1 \, d2 \, |F_L(1, 2)|^2 \langle \vec{S}(1) \cdot \vec{S}(2) \rangle$$

Explicitly, $J_1 > 0$, while for generic circumstances one finds that $J_2 < 0$. The overall sign of $J$ is therefore non-universal, and determined by which term is bigger. We can, however, identify the conditions under which $J_2$ dominates. Upon a Fourier transform, $|F_L(1, 2)|^2$ is peaked around two values of the momentum $q$, at $q = 0$ and $2k_F$, in which $2k_F = \pi n$ where $n$ is the number of electrons per site in the left and right chains. Since, upon Fourier transforming, $\langle \vec{S}(1) \cdot \vec{S}(2) \rangle$ is peaked at momenta $q = 0$ and $\pi$, (as can be seen, e.g., from a bosonized treatment of the half filled chain) we expect that $J_2$ in Eq. (15) is maximized when $n = 1$, i.e. when the superconducting chains are half filled. The requirement of proximity to half filling becomes less and less stringent when $|U_{R,L}|$ is increased, since then the gap $\Delta_s$ in the superconducting chains increases and the peaks in $|F_L(q)|^2$ become more and more broad. These qualitative expectations are confirmed by numerical DMRG simulations, presented in the next subsection.

5.1.2. Numerical results  We have performed DMRG simulations of the model $H = H_0 + H'$ in Eq. (8,10), with the following parameters: $t = t' = 1$, $\mu_B = 6$, $U_B = 10$, and variable $U_L = U_R \equiv -|U_{L,R}|$ and $\mu_L = \mu_R \equiv \mu_{L,R}$. Most of the calculations were done with systems of size $3 \times 24$. In a small number of parameter sets, we have verified that the results do not change when we increase the system size to $3 \times 36$. Up to $m = 1600$ states where kept in these calculations. The results (both ground state energies and local measurements) where extrapolated linearly in the truncation error [76], which is in the range $10^{-5} - 10^{-6}$.

In order to measure the sign of the Josephson coupling from the calculations, we have applied pairing potentials on the left and right chains, adding the following term to Eq. (8):

$$H_{\text{pair}} = - \sum_{i,\alpha=L,R} \Delta_\alpha c_{\alpha,i,\uparrow}^{\dagger} c_{\alpha,i,\downarrow}^{\dagger} + \text{h.c.}$$

In the presence of this term, the number of particles in the calculation is conserved only modulo 2. The average particle number is fixed by the overall chemical potential. Two methods were employed to determine the sign of $J$. (a) Pairing potentials of either the same sign, $\Delta_R = \Delta_L$, and of opposite signs, $\Delta_R = -\Delta_L$, where applied to the two
chains. The ground state energies in the two cases are $E_+$ and $E_-$, respectively. Then $J = E_- - E_+ [\text{17}]$. (b) A pairing potential was applied to the left chain only, $\Delta_L > 0$, while $\Delta_R = 0$. The induced pair field

$$\phi_{R,i} \equiv \langle c_{R,i,\uparrow}^\dagger c_{R,i,\downarrow} \rangle $$  \hspace{1cm} (17)$$

on the right chain was measured. Its sign indicates the sign of $J$. This is the method we used in most calculations. Method (a) was applied to a small number of points in parameter space, and found to produce identical results to those of method (b) for the sign of $J$.

Fig. 1 shows the local expectation values of the particle number, spin and pair field operators along the three chains for $|U_{L,R}| = 2.5$ and various values of $\mu_{LR}$. The density of electrons on the left and right chains increases as $\mu_{LR}$ increases, while the density on the middle chain is kept close to one particle per site. A positive pair potential of strength $\Delta_L = 0.1$ was applied on the left chain, inducing a positive pair field $\phi_L = \langle c_{L,i,\uparrow}^\dagger c_{L,i,\downarrow} \rangle > 0$, while $\Delta_R = 0$. A negative induced pair field $\phi_R$ on the right chain indicates that the effective Josephson coupling $J$ between the left and right chains is negative. Note that $J$ is negative for the two upper rows (in which $\langle n_{L,R} \rangle = 0.9, 0.83$ respectively), while for the two lower rows (where $\langle n_{L,R} \rangle = 0.75, 0.66$) it becomes positive. This is in agreement with our expectation, based on the perturbative analysis of the previous subsection, that when the superconducting chains are close to half filling ($\langle n \rangle = 1$), the negative $J_2$ term dominates and the overall Josephson coupling is more likely to become negative.

The middle column in Fig. 1 shows the expectation value of the $z$ component of the spin along the three chains. In order to visualize the spin correlations, a Zeeman field of strength $h = 0.5$ was applied to the $i = 1$ site of the middle chain. The results clearly indicate that the two outer chains have a spin gap (and therefore have a very small induced moment), while in the half filled middle chain there are strong antiferromagnetic correlations. Interestingly, as the chemical potential on the outer chains is decreased, the spin correlations along the middle chain become incommensurate. This seems to occur at the same point where the Josephson coupling changes sign (between the second and third row in Fig. 1). This phenomenon was observed for other values of $U_{L,R}$ as well. The incommensurate correlations can be explained by the further-neighbor Ruderman-Kittel-Kasuya-Yosida (RKKY)-like interaction which are induced in the middle chain by the proximity of the outer chains. Upon decreasing the inter-chain hopping $t'$ to 0.7, the spin correlations in the middle chain become commensurate over the entire range of $\mu_{L,R}$ (and the region of negative $J$ increases). Why $J > 0$ seems to be favored by incommensurate correlations in the middle chain is not clear at present.

Fig. 2 shows the phase diagram of the three chain model as a function of the density $\langle n_{L,R} \rangle$ and the attractive interaction $|U_{L,R}|$ on the outer chains. In agreement with the perturbative considerations, proximity to $\langle n_{L,R} \rangle = 1$ and large $|U_{L,R}|$ (compared to the bandwidth $4t$) both favor a negative Josephson coupling between the outer chains.
Figure 1. (Color online.) The left, middle and right columns show the average density $\langle n_i \rangle$, $z$ component of the spin $\langle S^z_i \rangle$ and pair field $\langle c_{i\downarrow}c_{i\uparrow} \rangle$, respectively, as a function of position $i$ along the chains, calculated by DMRG for $3 \times 24$ systems. Circles, diamonds and dots refer to the left, middle and right chains, respectively. The attractive interaction on the superconducting (left and right) chains is $|U_{L,R}| = 2.5$ in all calculations. A pairing term [Eq. (16)] was applied with $\Delta_L = 0.1$ and $\Delta_R = 0$. The other model parameters are given by: $t = t' = 1$, $\mu_B = 6$, $U_B = 10$. Each row corresponds to a single calculation with a specific value of the chemical potential $\mu_{L,R}$ (and hence a particular particle density) on the superconducting chains.

5.2. Extension to an infinite array of coupled chains

The model presented in the previous subsection includes only a single extended $\pi$ junction. However, it is straightforward to extend this model to an infinite number of coupled chains with alternating $U$. So long as the Josephson coupling across a single junction is small, we expect that the extension to an infinite number of chains will not change it by much. Therefore, in the appropriate parameter regime in Fig. 2, the superconducting order parameter changes sign from one superconducting chain to the
next, forming a striped superconductor (or unidirectional PDW).

In order to demonstrate that there are no surprises in going from three chains to two dimensions, we have performed a simulation for a $5 \times 12$ system composed of 5 coupled chains with alternating $U = -3, 8, -3, 8, -3$. As before, the density of particles on the $U = 8$ chains was kept close to $\langle n \rangle = 1$, making them insulating, while the density of particles on the $U = -3$ (superconducting) chains was varied. As before, the hopping parameters are $t = t' = 1$. A pair field $\Delta = 0.1$ was applied on the bottom superconducting chain, and the induced superconducting order parameter was measured across the system. Up to $m = 2300$ states were kept. Fig. 3 shows the induced pair fields and the expectation value of $S^z$ throughout the system in two simulations, in which the average density of particles on the superconducting chains was $\langle n_{sc} \rangle = 0.7, 0.47$. As expected according to the phase diagram in Fig. 2, in the $\langle n_{sc} \rangle = 0.7$ case the order parameter changes sign from one superconducting chain to the next, while in the $\langle n_{sc} \rangle = 0.47$ case the sign is uniform. It therefore seems very likely that under the right conditions, the two dimensional alternating chain model forms a striped superconductor.
5.3. Quasiparticle spectrum of a striped superconductor

The quasiparticle spectrum of a uniform superconductor is typically either fully gapped, or gapless only on isolated nodal points (or nodal lines in 3D). This is a consequence of the fact that, due to time reversal symmetry, the points $\mathbf{k}$ and $-\mathbf{k}$ have the same energy. Since the order parameter carries zero momentum, any point on the Fermi surface is thus perfectly nested with its time reversed counterpart, and is gapped unless the gap function $\Delta_{\mathbf{k}}$ vanishes at that point.

For a striped superconductor, the situation is different. Since the order parameter has non-zero momentum $\mathbf{Q}$, only points that satisfy the nesting condition $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}+\mathbf{Q}}$, where $\varepsilon_{\mathbf{k}}$ is the single particle energy, are gapped for an infinitesimally weak order. Therefore, generically there are portions of the Fermi surface that remain gapless [78]. This is similar to the case of a CDW or SDW, which generically leave parts of the (reconstructed) Fermi surface gapless, until the magnitude of the order parameter reaches a certain critical value. The spectral properties of a striped superconductor where studied in detail in Refs. [79] [80].

As an illustration, we present in Fig. 4 the spectral function $A(\mathbf{k}, \omega = 0)$ of a superconductor with band parameters fitted to the ARPES spectrum of LSCO [45] and a striped superconducting order parameter with a single wavevector $\mathbf{Q} = (2\pi/8, 0)$ of
Figure 4. (Color online.) (a) The spectral function $A(k,\omega = 0)$ for a striped superconductor. The band parameters used in the calculation where fitted to the ARPES spectrum of LSCO \cite{15}: $t = 0.25$, $t' = -0.031863$, $t'' = 0.016487$, $t''' = 0.007612$, where $t$, $t'$, ..., are nearest neighbor hopping, second-nearest neighbor hopping and so on, chemical potential $\mu = -0.16235$. (All the parameters above are measured in eV.) The striped superconducting order parameter has a wavevector of $Q = (2\pi/8, 0)$, and its magnitude is $\Delta_Q = 60$meV. The order parameter is of "d-wave character", in the sense that it is of opposite sign on $x$ and $y$ oriented bonds. The thin solid line shows the underlying bare Fermi surface, and the dotted line shows the Fermi surface in the presence of the PDW. (b) $A(k,\omega)$ for the same model parameters along a cut in $k$-space.

magnitude $\Delta_Q = 60$meV. Note that a portion of the Fermi surface around the nodal (diagonal) direction remains ungapped (a “Fermi arc” \cite{81, 82}), while both antinodal directions [around $(\pi, 0)$ and $(0, \pi)$] are gapped. The Fermi arc is in fact the back side of a reconstructed Fermi pocket, but only the back side has a sizable spectral weight \cite{83}. Its length depends on the magnitude of the order parameter: the larger the magnitude of $\Delta_Q$, the smaller is the arc. Note that $A(k,\omega = 0)$ is not symmetric under rotation by $\pi/2$, because the striped superconducting order breaks rotational symmetry. However, in a system with an LTT symmetry (such as LBCO near $x = 1/8$ doping) both orientations of stripes are present, and an ARPES experiment would see the average of the picture in Fig. 4 and its rotation by $\pi/2$.

6. Order parameter theory of the PDW state

In this section, we explore the aspects of the theory of a PDW that can be analyzed without reference to microscopic mechanisms. We focus on the properties of ordered states at $T = 0$, far from the point of any quantum phase transition, where for the most part fluctuation effects can be neglected. (The one exception to the general rule is that, where we discuss effects of disorder, we will encounter various spin-glass related phases where fluctuation effects, even at $T = 0$, can qualitatively alter the phases.) For
simplicity, most of our discussion is couched in terms of a Landau theory, in which the effective free energy is expanded in powers of the order parameters; this is formally not justified deep in an ordered phase, but it is a convenient way to exhibit the consequences of the order parameter symmetries.

6.1. Order parameters and symmetries

We will now define the various order parameters introduced in this section and discuss their symmetry properties. The striped superconducting order parameter $\Delta_Q$ is a charge $2e$ complex scalar field, carrying momentum $Q$. To define it microscopically, we write

$$\phi(r, r') \equiv \left\langle \psi^\dagger(r) \psi^\dagger(r') \right\rangle = F(r - r') \left[ \Delta_0 + \Delta_Q e^{iQ \cdot R} + \Delta_{-Q} e^{-iQ \cdot R} \right],$$

(18)

where $R = (r + r')/2$, $F(r - r')$ is some short range function (for a “d-wave-like” striped superconductor, $F(r)$ changes sign under $90^\circ$ rotation), and $\Delta_0$ is the uniform $Q = 0$ component of the order parameter.† In the rest of this subsection, we set $\Delta_0 = 0$. The effect of $\Delta_0$ is discussed in subsection 6.3. To be concrete, we assume that the host crystal is tetragonal, and that there are therefore two potential symmetry related ordering wave vectors, $Q$ and $\bar{Q}$, which are mutually orthogonal, so $\Delta_{\bar{Q}}$ must be treated on an equal footing with $\Delta_Q$. (The discussion is easily generalized to crystals with other point-group symmetries.) Similarly, for simplicity, spin-orbit coupling is assumed to be negligible.

The order parameters that may couple to $\Delta_Q$ and their symmetry properties are as follows: The nematic order parameter $N$ is a real pseudo-scalar field; the CDW $\rho_K$ with $K = 2Q$ is a scalar field; $\vec{S}_Q$ is a neutral spin-vector field. All these order parameters are electrically neutral. Under spatial rotation by $\pi/2$, $N \rightarrow -N$, $\rho_K \rightarrow \rho_K$, $\vec{S}_Q \rightarrow \vec{S}_Q$, and $\Delta_Q \rightarrow \pm \Delta_Q$, where $\pm$ refers to a d-wave or s-wave version of the striped superconductor. Under spatial translation by $r$, $N \rightarrow N$, $\rho_K \rightarrow e^{iK \cdot r} \rho_K$, $\vec{S}_Q \rightarrow e^{iQ \cdot r} \vec{S}_Q$, and $\Delta_Q \rightarrow e^{iQ \cdot r} \Delta_Q$. Note that since the SDW and CDW orders are real, $\vec{S}_Q \ast = \vec{S}_{-Q}$ and $\rho_K \ast = \rho_{-K}$. Generally, $\Delta_Q$ and $\Delta_{-Q}$ are independent.

† A state in which both components of the SC order parameter coexist, $\Delta_0 \neq 0$ and $\Delta_Q \neq 0$ is certainly not “uniform”. Even a weak $\Delta_Q \neq 0$ implies the existence of a modulation of the local amplitude of the SC order parameter, and a SC state is “truly uniform” only if $\Delta_Q = 0$. Nevertheless, as we will see in Section 6.3, the properties of a SC state in which both order parameters coexist are largely dominated by the “uniform” component $\Delta_0$, and the striking features of the PDW state are not directly observable. In this sense, the uniform-PDW coexisting SC state is effectively “uniform.”
Specifically, the emphasis in this section is on the interrelation between striped superconducting order and other orders. There is a necessary relation between this order and CDW and nematic (or orthorhombic) order, since the striped superconductor breaks both translational and rotational symmetries of the crystal. From the microscopic considerations, above, and from the phenomenology of the cuprates, we also are interested in the relation of superconducting and SDW order. The Landau effective free energy density can then be expanded in powers of these fields:

\[ \mathcal{F} = \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \ldots \]  

(19)

where \( \mathcal{F}_2 \), the quadratic term, is simply a sum of decoupled terms for each order parameter,

\[ \mathcal{F}_3 = \gamma_s [\rho_{-\mathbf{K}} \vec{S}_Q \cdot \vec{S}_Q + \rho_{-\mathbf{K}} \vec{S}_Q \cdot \vec{S}_Q + \text{c.c.}] \]

(20)

\[ + \gamma_\Delta [\rho_{-\mathbf{K}} \Delta^*_Q \Delta_Q + \rho_{-\mathbf{K}} \Delta^*_Q \Delta_Q + \text{c.c.}] \]

\[ + g_\Delta N [\Delta^*_Q \Delta_Q + \Delta^*_Q \Delta_Q - \Delta^*_Q \Delta_Q - \Delta^*_Q \Delta_Q] \]

\[ + g_s N [\vec{S}^*_Q \cdot \vec{S}_Q - \vec{S}^*_Q \cdot \vec{S}_Q] \]

\[ + g_c N [\rho_{-\mathbf{K} \rho \mathbf{K}} - \rho_{-\mathbf{K} \rho \mathbf{K}}] \]

and the fourth order term, which is more or less standard, is shown explicitly below.

The effect of the cubic term proportional to \( \gamma_s \) on the interplay between the spin and charge components of stripe order has been analyzed in depth in [84]. Similar analysis can be applied to the other terms. In particular, the \( \gamma_\Delta \) and \( g_\Delta \) terms imply\(^\dagger\) that the existence of superconducting stripe order (\( \Delta_Q \neq 0 \), and \( \Delta_Q = 0 \)), implies the existence of nematic order (\( N \neq 0 \)) and charge stripe order with half the period (\( \rho_{2Q} \neq 0 \)). However, the converse statement is not true: while CDW order with ordering wave vector \( 2Q \) or nematic order tend to promote PDW order, depending on the magnitude of the quadratic term in \( \mathcal{F}_2 \), PDW order may or may not occur.

One new feature of the coupling between the PDW and CDW order is that it produces a sensitivity to disorder which is not normally a feature of the superconducting state. In the presence of quenched disorder, there is always some amount of spatial variation of the charge density, \( \rho(r) \), of which the important portion for our purposes can be thought of as being a pinned CDW, that is, a CDW with a phase which is a pinned, slowly varying function of position, \( \rho(r) = |\rho_{\mathbf{K}}| \cos[\mathbf{K} \cdot \mathbf{r} + \phi(r)] \). Below the nominal striped superconducting ordering temperature, we can similarly express the PDW order in terms of a slowly varying superconducting phase, \( \Delta(r) = |\Delta_Q| \exp[i \mathbf{Q} \cdot \mathbf{r} + i \theta_Q(r)] + |\Delta_{-Q}| \exp[-i \mathbf{Q} \cdot \mathbf{r} + i \theta_{-Q}(r)] \). The resulting contribution to \( \mathcal{F}_3 \) is

\[ \mathcal{F}_{3,\gamma} = 2\gamma_\Delta |\rho_{\mathbf{K}} \Delta_Q \Delta_{-Q}| \cos[2\theta_{-}(\mathbf{r}) - \phi(\mathbf{r})]. \]

(21)

\(^\dagger\) Note that the \( \gamma_\Delta \) term is odd under a particle-hole transformation, which takes \( \rho_{\mathbf{K}} \rightarrow -\rho_{\mathbf{K}} \). Therefore, if the system has exact particle-hole symmetry, this term vanishes, and there is no necessary relation between \( \Delta_Q \) and \( \rho_{\mathbf{K}} \). Microscopic systems are generically not symmetric under charge conjugation. However, some real systems (e.g. the cuprates) are not too far from being particle-hole symmetric, and therefore in these systems \( \gamma_\Delta \) is expected to be relatively small.
where
\[ \theta_{\pm}(r) \equiv [\theta_Q(r) \pm \theta_{-Q}(r)]/2; \tag{22} \]
\[ \theta_{\pm Q}(r) = [\theta_{+}(r) \pm \theta_{-}(r)]. \]

The aspect of this equation that is notable is that the disorder couples directly to a piece of the superconducting phase, \( \theta_{-} \). No such coupling occurs in usual 0 momentum superconductors.

It is important to note that the condition that \( \Delta(r) \) be single valued implies that \( \theta_Q(r) \) and \( \theta_{-Q}(r) \) are defined modulo \( 2\pi \). Correspondingly, \( \theta_{\pm} \) are defined modulo \( \pi \), subject to the constraint that if \( \theta_{\pm} \to \theta_{\pm} + \pi m_{\pm} \) then \( m_{+} + m_{-} \) must be an even integer. Since \( \phi \) and \( \theta_{-} \) are locked to each other at long distances, the possible topological excitations of the coupled PDW-CDW system are thus point defects in 2D and line defects in 3D classified by the circulation of \( \theta_{+} \) and \( \phi \) on any enclosing contour. The elementary topological defects thus are: a) An ordinary superconducting vortex, about which \( \Delta \theta_{+} = 2\pi \) and \( \Delta \phi = 0 \). b) A bound-state of a half vortex and a dislocation,§ about which \( \Delta \theta_{+} = \pi \) and \( \Delta \phi = 2\pi \). c) A double dislocation (or dislocation bound state) about which \( \Delta \theta_{+} = 0 \) and \( \Delta \phi = 4\pi \). All these defects have a logarithmically divergent energy in 2D, or energy per unit length in 3D; the prefactor of the logarithm is determined by the superfluid stiffness for the vortex, the elastic modulus of the CDW for the double vortex, and an appropriate sum of these two stiffnesses for the half vortex. Consequences of this rich variety of topological defects are discussed in [6, 25, 86].

An important consequence of the coupling between the superconducting and CDW phase is that the effect of quenched disorder, as in the case of the CDW itself, destroys long-range superconducting stripe order. (This statement is true [87], even for weak disorder, in dimensions \( d < 4 \).) Naturally, the way in which this plays out depends on the way in which the CDW state is disordered.

In the most straightforward case, the CDW order is punctuated by random, pinned dislocations, i.e. \( 2\pi \) vortices of the \( \phi \) field. The existence of the coupling in Eq. [21] implies that there must be an accompanying \( \pi \) vortex in \( \theta_{-} \). The condition of single-valued-ness implies that there must also be an associated half-vortex or anti-vortex in \( \theta_{+} \). If these latter vortices are fluctuating, they destroy the superconducting state entirely, leading to a resistive state with short-ranged striped superconducting correlations. If they are frozen, the resulting state is analogous to the ordered phase of an \( XY \) spin-glass: such a state has a non-vanishing Edwards-Anderson order parameter, spontaneously breaks time-reversal symmetry, and, presumably, has vanishing resistance but no Meissner effect and a vanishing critical current. In 2D, according to conventional wisdom, a spin-glass phase can only occur at \( T = 0 \), but in 3D there can be a finite temperature glass transition [88].

In 3D there is also the exotic possibility that, for weak enough quenched disorder, the CDW forms a Bragg-glass phase, in which long-range order is destroyed, but no free

§ The possibility of half vortices in a striped superconductor and their effect on the phase diagram in the clean case was discussed by D. F. Agterberg and H. Tsunetsugu [85].
dislocations occur \[89, 90, 91\]. In this case, \(\phi\) can be treated as a random, but single-valued function - correspondingly, so is \(\theta\). The result is a superconducting Bragg-glass phase which preserves time reversal symmetry and, presumably, acts more or less the same as a usual superconducting phase. It is believed that a Bragg-glass phase is not possible in 2D \[90\].

Another perspective on the nature of the superconducting state can be obtained by considering a composite order parameter which is proportional to \(\Delta_Q\). There is a cubic term which couples a uniform, charge \(4e\) superconducting order parameter, \(\Delta_4\), to the PDW order:

\[
\mathcal{F}_3' = g_4\{\Delta_4^* [\Delta_Q \Delta_{-Q} + \Delta_Q \Delta_{-Q}] + \text{c.c.}\}
\]

This term implies that whenever there is PDW order, there is also necessarily charge \(4e\) uniform superconducting order. However, since this term is independent of \(\theta_\pm\), it would be totally unaffected by Bragg-glass formation of the CDW. The half-vortices in \(\theta_\pm\) discussed above can simply be viewed as the fundamental (hc/4e) vortices of a charge \(4e\) superconductor.

Some additional physical insight can be gained by examining the quartic terms \((\mathcal{F}_4\) in Eq. [19]). Let us write all the possible fourth order terms consistent with symmetry:

\[
\mathcal{F}_4 = u \left( \tilde{S}_Q \cdot \tilde{S}_Q \Delta_Q^* \Delta_{-Q} + \tilde{S}_Q \cdot \tilde{S}_Q \Delta_Q^* \Delta_{-Q} + \text{c.c.} \right) \\
+ \left( v_+ [\tilde{S}_Q \cdot \tilde{S}_Q + \tilde{S}_Q \cdot \tilde{S}_Q] + \tilde{v}_+ [||\rho_K||^2 + ||\rho_K||^2] \right) \\
\times (||\Delta_Q||^2 + ||\Delta_{-Q}||^2 + ||\Delta_Q||^2 + ||\Delta_{-Q}||^2) \\
+ \left( v_- [\tilde{S}_Q \cdot \tilde{S}_Q - \tilde{S}_Q \cdot \tilde{S}_Q] + \tilde{v}_- [||\rho_K||^2 - ||\rho_K||^2] \right) \\
\times (||\Delta_Q||^2 + ||\Delta_{-Q}||^2 - ||\Delta_Q||^2 - ||\Delta_{-Q}||^2) \\
+ vN^2 \left\{ (||\Delta_Q||^2 + ||\Delta_{-Q}||^2) + (||\Delta_Q||^2 + ||\Delta_{-Q}||^2) \right\} \\
+ \lambda_+ \left\{ (||\Delta_Q||^2 + ||\Delta_{-Q}||^2)^2 + (||\Delta_Q||^2 + ||\Delta_{-Q}||^2)^2 \right\} \\
+ \lambda_- \left\{ (||\Delta_Q||^2 - ||\Delta_{-Q}||^2)^2 + (||\Delta_Q||^2 - ||\Delta_{-Q}||^2)^2 \right\} \\
+ \lambda (||\Delta_Q||^2 + ||\Delta_{-Q}||^2)(||\Delta_Q||^2 + ||\Delta_{-Q}||^2) \\
+ \ldots
\]

where we have explicitly shown all the terms involving \(\Delta_Q\), while the terms \ldots represent the remaining quartic terms all of which, with the exception of those involving \(N\), are exhibited explicitly in [84].

There are a number of features of the ordered phases which depend qualitatively on the sign of various couplings. Again, this is very similar to what happens in the case of CDW order - see, for example, [92, 93]. For instance, depending on the sign of \(\lambda\), either unidirectional (superconducting stripe) or bidirectional (superconducting checkerboard) order is favored.

On physical grounds, we have some information concerning the sign of various terms in \(\mathcal{F}_4\). The term proportional to \(u\) determines the relative phase of the spin
and superconducting stripe order—we believe \( u > 0 \) which thus favors a \( \pi/2 \) phase shift between the SDW and the striped superconducting order, i.e. the peak of the superconducting order occurs where the spin order passes through zero. The other interesting thing about this term is that it implies an effective cooperativity between spin and striped superconducting order. The net effect, i.e. whether spin and striped superconducting order cooperate or fight, is determined by the sign of \( |u| - v_+ - v_- \), such that they cooperate if \( |u| - v_+ - v_- > 0 \) and oppose each other if \( |u| - v_+ - v_- < 0 \). It is an interesting possibility that spin order and superconducting stripe order can actually favor each other even with all “repulsive” interactions. The term proportional to \( \lambda_- \) determines whether the superconducting stripe order tends to be real (\( \lambda_- > 0 \)), with a superconducting order that simply changes sign as a function of position, or a complex spiral, which supports ground-state currents (\( \lambda_- < 0 \)).

### 6.3. Coexisting uniform and striped order parameters

Finally, we comment on the case of coexisting uniform and striped superconducting order parameters. Such a state is not thermodynamically distinct from a regular (uniform) superconductor coexisting with a charge density wave, even if the uniform superconducting component is in fact weaker than the striped component. Therefore, we expect many of the special features of the striped superconductor (such as its sensitivity to potential disorder) to be lost. Here, we extend the Landau free energy to include a uniform superconducting component, and show that this is indeed the case.

We will now analyze the coupling of a striped superconducting order parameter \( \Delta_Q \) to a uniform order parameter, \( \Delta_0 \). In this case, we have to consider in addition to the order parameters introduced in Sec. 6 a CDW order parameter with wavevector \( Q \), denoted by \( \rho_Q \). The additional cubic terms in the Ginzburg-Landau free energy are

\[
F_{3,u} = \gamma_Q \Delta_0 \left[ \rho_Q \Delta_{-Q} + \rho_{-Q} \Delta_Q + \rho_Q \Delta_{-Q} + \rho_{-Q} \Delta_Q \right] + \text{c.c.} \\
+ g_\rho \left[ \rho_{-2Q} \rho_Q^2 + \rho_{-2Q} \rho_Q^2 + \text{c.c.} \right].
\]  

(25)

Eq. (25) shows that if both \( \Delta_0 \) and \( \Delta_Q \) are non-zero, there is necessarily a coexisting non-zero \( \rho_Q \), through the \( \gamma_Q \) term. The additional quartic terms involving \( \Delta_0 \) are

\[
F_{4,u} = u_\Delta \left( \Delta_0^2 \Delta_{-Q} + \Delta_0^2 \Delta_Q \Delta_{-Q} + \text{c.c.} \right) + \delta |\Delta_0|^2 |\Delta_Q|^2 + |\Delta_Q|^2 \\
+ |\Delta_0|^2 \left[ u_\rho \left( |\rho_Q|^2 + |\rho_Q|^2 \right) + u'_\rho \left( |\rho_{2Q}|^2 + |\rho_{2Q}|^2 \right) \right] \\
+ v' |\Delta_0|^2 |\vec{S}_{-Q} \cdot \vec{S}_Q + \vec{S}_{-Q} \cdot \vec{S}_Q|.
\]  

(26)

Let us now consider the effect of quenched disorder. Following the discussion preceding Eq. (22) we write the order parameters in real space as

\[
\Delta (r) = |\Delta_0| e^{i\delta_0} + |\Delta_Q| e^{i(\theta_Q + Q \cdot r)} + |\Delta_{-Q}| e^{i(\theta_{-Q} - Q \cdot r)}
\]  

(27)

and

\[
\rho (r) = |\rho_Q| \cos (Q \cdot r + \phi_Q) + |\rho_{2Q}| \cos (2Q \cdot r + \phi).
\]  

(28)
Let us assume that the disorder nucleates a point defect in the CDW, which in this case corresponds to a $2\pi$ vortex in the phase $\phi_Q$. By the $g_\rho$ term in Eq. 25, this induces a $4\pi$ vortex in $\phi$. (Note that in the presence of $\rho_Q$, a $2\pi$ vortex in $\phi$ is not possible.) The $\gamma_\Delta$ term in Eq. 21 then dictates a $2\pi$ vortex in the phase $\theta_- = (\theta_Q - \theta_{-Q})/2$. However, unlike before, this vortex does not couple to the global superconducting phase $\theta_+ = (\theta_Q + \theta_{-Q})/2$. Therefore, an arbitrarily small uniform superconducting component is sufficient to remove the sensitivity of a striped superconductor to disorder, and the system is expected to behave more or less like a regular (uniform) superconductor, albeit with a modulated amplitude of the order parameter.

Since the usual (uniform) superconducting order and the PDW break distinct symmetries, nothing can be said, in general, about the conditions in which they will coexist. However, microscopic considerations can, in some cases, yield generic statements, too. For example, in a striped SC, a uniform component of the order parameter can be generated by dimerizing the stripe order, such that the positive and negative strips of superconducting order are made alternately broader and narrower. In any structure (such as the LTT structure of LBCO), in which there is zero Josephson coupling between neighboring layers, a coupling is generated, thus lowering the energy of the system, in proportion to the square of the dimerization. Presumably, so long as the PDW period is incommensurate with the underlying lattice, there is also a quadratic energy cost to dimerization which is related to an appropriate generalized elastic constant of the PDW. However, if the PDW has a long period, this elastic constant will be vanishingly small. Thus, any long period, incommensurate PDW may generically be expected to be unstable toward the generation of a small amount of uniform SC order.

**7. Non-collinear order and time reversal symmetry breaking**

In a layered system, PDW order in the planes can lead to frustration of the inter-plane Josephson coupling, which naturally explains the layer decoupling seen in 1/8 doped LBCO. In analogy with frustrated magnetic systems (in which the superconducting order is thought of as an $XY$ pseudo-spin), this frustration can also lead to various forms of non-collinear order. In the PDW case, such non-collinear orders break time-reversal symmetry and are accompanied by spontaneous equilibrium currents.

In this section, we give detailed predictions for the patterns of bulk time-reversal symmetry breaking and spontaneous currents in various lattice geometries. We will discuss this problem at zero temperature and at a classical level. It is worth noting that the non-collinear order, where it occurs, results in a partial lifting of the frustration. In the case of a PDW in the LTT structure (relevant to $\text{La}_{2-x}\text{Ba}_x\text{CuO}_4$), we shall show that it results in a non-vanishing effective Josephson coupling between planes.

---

† The problem of the 3D phase transition in a system with an effective layer decoupling is largely unsolved. See, however, the recent work of Raman, Oganesyan and Sondhi.
Figure 5. (Color online.) (a) Model for a striped superconductor with an LTT structure. Solid (dashed) lines represent positive (negative) Josephson couplings. The arrow on the center of each link indicates the direction of the equilibrium current across that link. The red arrows on the vertices represent the superconducting phases. (b) Same as (a) for an orthorhombic striped superconductor, where the charge stripes are shifted by half a period from one layer to the next. (c) An in-plane domain wall.

and hence, in a sense, spoils the strict layer decoupling we have touted. However, this effective Josephson coupling is equivalent to a higher order coupling [6] (due to coherent tunneling of two Cooper pairs), both in terms of its small magnitude, and its dependence on the cosine of twice the difference of the superconducting phases on neighboring planes. (See Eqs. 30 and 32.) Note also that defects (such as point defects, domain walls or twin boundaries) can lead to additional intra-plane time reversal symmetry breaking, that can drive the system into a glassy superconducting state (as discussed in Sec. 6).‡

Let us start with the case of the LBCO LTT structure, in which the stripe direction rotates by $90^\circ$ between adjacent planes. We model the system by a 3D discrete lattice of Josephson junctions, shown in Fig. § § The lattice spacing in the plane is the inter-stripe distance $\lambda$, and $c$ is the inter-plane distance. Each lattice point has a single degree of freedom $\theta_r$, which is the local value of the superconducting phase at that point.

‡ An in-plane magnetic field can also change the inter-layer frustration, leading to small violations of the layer decoupling effect. If large enough such effects can be used to detect a PDW state. A similar effect can also take place in junctions between an FFLO state and a uniform superconductor [95].

§ Note that we are actually considering a simplified version of the LBCO LTT structure. The structure in Fig. § has two planes per unit cell, while the LBCO LTT structure has four. The difference is that in LBCO, the charge stripes in second neighboring planes (which are parallel to each other) are shifted by half a period relative to one another, while in Fig. § they are not. However, the considerations we discuss here are the same for two structures, and the resulting non-collinear ground states are similar.
$J, -J', J''$ are the intra-stripe, the inter-stripe and the inter-plane Josephson couplings, respectively. We assume that $J > J' \gg J'' > 0$, corresponding to a unidirectional striped superconductor in the planes. For any collinear configuration, the Josephson coupling between the planes vanishes. However, if the staggered order parameter in each plane is rotated by $90^\circ$ relative to its neighbors, then the energy can be lowered by distorting the phases periodically with respect to the collinear configuration in each plane. We use a variational ansatz for the phases $\theta_a$ of the form

$$\theta_a = \frac{1 + (-1)^z}{2} y \pi + \frac{1 - (-1)^z}{2} \left( x + \frac{1}{2} \right) \pi + (-1)^{x+y+z} \theta$$

where $\theta_a = (x, y, z)$ is the integer valued position vector ($x$ and $y$ are measured in units of $\lambda$, and $z$ is measured in units of $c$), and the distortion angle $\theta$ is a variational parameter.

The Josephson energy per site as a function of $\theta$ is

$$E_{\text{LT T}}(\theta) = -(J + J') \cos 2\theta - J'' \sin 2\theta.$$  

(30)

The inter-plane coupling energy gain is linear in $\theta$, whereas the cost in intra-plane coupling energy is quadratic in $\theta$. Thus the distortion occurs for any non-zero value of the inter-plane coupling $J''$. Minimizing Eq. (30), we get

$$\tan 2\theta = \frac{J''}{J + J'}.$$  

(31)

The equilibrium currents across the three types of links are $J = J \sin 2\theta$, $J' = J' \sin 2\theta$ and $J'' = J'' \cos 2\theta = J + J'$, where Eq. (31) was used in the last relation. The directions of the currents are as indicated in Fig. 5a. Associated with these currents is a magnetic field with non-zero components in all three directions. The wavevector associated with this pattern is $Q = \left( \frac{x}{\lambda}, \frac{x}{\lambda}, \frac{1}{c} \right)$, where $\lambda$ is the inter-stripe distance (for LBCO at $x = 1/8$, $\lambda \approx 4a$ where $a$ is the Cu-Cu distance) and $c$ is the inter-plane distance.

The non-collinear distortion in the above pattern induces an effective non-zero inter-plane coupling. However, the effective inter-layer coupling is (taking the limit $J'' \ll J, J'$)

$$J_{\text{eff}} \simeq \frac{(J'')^2}{4(J + J')}.$$  

(32)

and is therefore much smaller than the bare inter-plane coupling $J''$. Note, moreover, that the induced Josephson coupling between two neighboring planes with PDW superconducting phases $\theta_i$ and $\theta_j$ has the form $J_{\text{eff}} \cos[2(\theta_i - \theta_j)]$, i.e. its period in the relative phase is $\pi$.

Next, we consider the case of an orthorhombic structure (such as the LTO phase of LBCO). In this case, rotational symmetry in the plane is broken in the same way in every plane, and the stripes are all in the same direction. However, we assume that due to Coulomb interactions, the charge stripes are shifted by half a period between adjacent planes. (Such a shift is indeed observed between second neighbor planes in the LTT phase of LBCO, in which the stripe direction is parallel.) Therefore, the inter-plane coupling is frustrated due to the resulting “zigzag” geometry. We shall show below that
the ground state has spiral order which partially relieves this frustration. Introducing a spiral twist angle $\theta$, such that $\theta_r = 2x\theta$ (as shown in Fig. 5b), costs an energy $E_{\text{ORT}}(\theta)$ per stripe, given by

$$\frac{E_{\text{ORT}}(\theta)}{L} = J' \cos 2\theta - 2J'' \cos \theta$$

(33)

where $L$ is the length of each stripe. The minimum is for $\cos \theta = \frac{J'}{2J''}$. Therefore a spiral distortion occurs for any $J'' < 2J'$. The currents along this links are $J'' = -J' = J'' \sin \theta$, and their directions are indicated in Fig. 5b. Each plane carries a uniform current which flows perpendicular to the stripes, and an equal and opposite current flows between the planes. The magnetic field associated with these currents is pointing parallel to the stripe direction, and its lowest Fourier component is at wavevector $Q = (0, 0, \frac{2\pi}{c})$.

Finally, we turn to the case of a domain wall in the PDW order, depicted in Fig. 5c. (Such a defect is very costly energetically, but it is favored by a twin boundary in the crystal structure.) The Josephson coupling across the domain wall vanishes for any collinear configuration. The energy can be lowered by distorting the phases in the pattern shown in Fig. 5c, which is closely analogous to the minimum energy configuration in the LTT case (Fig. 5a). The superconducting phases $\theta_r$ are given by

$$\theta_r = \begin{cases} (x + \frac{1}{2})\pi - (-1)^y\theta_x & (x < 1) \\ y\pi + (-1)^y\theta_x & (x \geq 1) \end{cases}$$

(34)

where the distortion angle $\theta_x$ depends on the distance from the domain wall $x$. (In our notation, $x = 0$ and $1$ are the two columns on either side of the domain wall.) The energy is

$$\frac{E_{\text{DIS}}(\{\theta_x\})}{L} = -J \sum_{x=1}^{\infty} \cos (\theta_{x+1} - \theta_x) - J' \sum_{x=1}^{\infty} \cos (2\theta_x)$$

$$- J' \sum_{x=-\infty}^{0} \cos (\theta_{x-1} - \theta_x) - J \sum_{x=-\infty}^{0} \cos (2\theta_x) - \tilde{J} \sin (\theta_{x=1} + \theta_{x=0})$$

(35)

Here, $\tilde{J}$ is the Josephson coupling across the domain wall, and $L$ is the number of sites along the domain wall. For simplicity, we assume that $\tilde{J} \ll J, J'$, in which case $\theta_x \ll 1$ and we may expand Eq. (35) to second order in $\theta_x$. Minimizing $E_{\text{DIS}}(\{\theta_x\})$, we obtain the following solution:

$$\theta_x = \begin{cases} \theta_<> e^{\alpha x} & (x < 1) \\ \theta_> e^{-\beta x} & (x \geq 1) \end{cases}$$

(36)

where $\alpha = 2 \sinh^{-1} \left( \sqrt{\frac{J'}{J}} \right)$, $\beta = 2 \sinh^{-1} \left( \sqrt{\frac{J'}{J}} \right)$, $\theta_<$ = $\frac{\tilde{J}}{J'(1-e^{-\alpha})+4J}$ and $\theta_>$ = $\frac{\tilde{J}}{J'(1-e^{-\beta})+4J}$. Associated with the distortion of the superconducting phases is a periodic pattern of spontaneous currents, shown in Fig. 5c, with periodicity of two inter-stripe distances.
Similar considerations apply to an in-plane Josephson junction between a striped superconductor and a uniform superconductor, if the boundary is perpendicular to the stripe direction. Therefore, in such a junction time reversal symmetry is also broken. The critical current is of order \( \tilde{J}^2_{\min}\{J,J'\} \). [This follows from the same considerations as the effective inter-plane coupling in the LTT case, Eq. 32.] It is thus suppressed relative to the critical current of a Josephson junction between uniform superconductors, which is of order \( \tilde{J} \), as a result of the frustration of the Josephson coupling across the junction. Similarly to inter-plane coupling in the LTT case, the period of the coupling between a uniform and a striped superconductor in the relative phase is \( \pi \), i.e. half of the period of the coupling between two uniform superconductors.

8. Connections and History

The notion of a superconducting state with spontaneously generated oscillations in the sign of the order parameter has cropped up, under various guises, a number of times in the past. It is worthwhile to recount some of these circumstances, not only in the interest of scholarship, but also to broaden the range of phenomena which can be addressed within the same conceptual framework.

8.1. Josephson \( \pi \) junctions

Since the superconducting order parameter is a charge \( 2e \) scalar field, it is often assumed that it is possible to think of the superconducting state as a Bose condensed state of charge \( 2e \) bosons. In contrast, most classic treatments of the subject [96] emphasize that many features of BCS theory, especially those associated with quasiparticle coherence factors, cannot be understood in this way. At the very least, a bosonic theory is inadequate to capture basic features of the groundstate of any superconductor which has gapless quasiparticles, either because of the order parameter symmetry (e.g. d-wave) or because of scattering from magnetic impurities (gapless superconductor).

Even ignoring the possibility of gapless quasiparticles, there are qualitative possibilities in a fermionic system that cannot occur in a bosonic system. A feature of a time reversal invariant bosonic system is that the ground-state can be chosen to be real and nodeless. Thus, the order parameter in a Bose-condensed system must have a phase which is independent of position. The \( \pi \) junctions, which we have been discussing, are possible only because of the composite character of the superconducting order parameter [71].

There have been several previous theoretical studies which have found circumstances under which \( \pi \) junctions might occur [71, 72, 97]. More recently, the existence of such \( \pi \) junctions in the predicted circumstances have been confirmed by experiment. The first such experiments [74, 75] were significant as the “phase sensitive” measurements which definitively established the d-wave symmetry of the superconducting order in the cuprates. More recently, however, mesoscopic \( \pi \)
junctions between two s-wave superconductors have been constructed and characterized \[73\]. In our opinion, these latter experiments are also landmarks in the study of superconductivity. They establish that \(\pi\) junctions, the essential ingredient for the existence of striped superconductors, are physically possible.

8.2. FFLO states

In a superconductor with negligible spin-orbit coupling, it is possible to generate an imbalance in the population of up and down spin quasiparticles, either by applying a magnetic field in a geometry in which it predominantly couples to the electron spins, or by injecting a non-equilibrium population of quasiparticles from a neighboring ferromagnet \[98\]. In the related systems of cold fermionic atomic gases, it is possible to vary the population of up and down spin atoms independently, and to study the effect of this population imbalance on the superfluid state \[99\] \[100\] \[25\]. While a first order quenching of the superconducting state is possible under these circumstances, there has also been considerable discussion of the possibility of spatially modulated superconducting states, so called FFLO states \[11\] \[12\]. Two distinct states of this sort have been considered: (1) The FF state \[11\], in which the order parameter has constant magnitude but a phase which twists as a function of position according to \(\theta = \Delta k_F \cdot r\), where \(\Delta k_F\) is the difference between the up spin and down spin Fermi momentum. (2) The LO state \[12\], in which the order parameter remains real, but oscillates in sign with a period \(L = 2\pi/|\Delta k_F|\).

The LO state is similar in structure to the striped superconductor considered here. In the order parameter theory presented in Sec. \(6.2\), it corresponds to \(\lambda_- > 0\) in Eq. \(24\). The parallel with the FF state (which is realized in the order parameter theory for \(\lambda_- < 0\)) is less crisp, but when superconducting striped spirals which spontaneously break time reversal symmetry arise due to the appropriate type of geometric frustration of the Josephson couplings (as discussed in Sec. \(7\)), states that are in many ways analogous to the FF state also occur in striped superconductors. Thus, many of the physical phenomena we have discussed in this paper are pertinent to the FFLO phases in more weakly correlated systems, with the added richness \[25\] in the case of cold atomic gases that there are conserved quantities associated with the continuous rotational invariance of the underlying Hamiltonian.

However, the FFLO states arise from the explicit breaking of time reversal symmetry. Absent a magnetic field, Kramer’s theorem implies perfect nesting between time-reversed pairs of states on opposite sides of the Fermi surface, so BCS pairing always occurs preferentially at \(k = 0\). This constraint is removed when time reversal symmetry is explicitly broken. One can think of the FFLO states as taking advantage of the “best” remaining approximate nesting vector, \(\Delta k_F\), in the two-particle channel. Alternatively, one can think of the LO state as consisting of a set of discommensurations \[98\] \[25\] such that the excess spin-up quasiparticles are incorporated in mid-gap states localized near the core of the discommensuration.
The energetic considerations that lead to the FFLO states are thus very different than the strong-coupling physics that gives rise to the striped superconductor.† The fact that the FFLO states explicitly break time reversal symmetry implies that they are macroscopically distinct (as phases of matter) from the striped superconductors that preserve this symmetry. Even in comparison with striped states which spontaneously break time reversal symmetry, the distinction remains that the FFLO states have a net magnetization, while the striped superconductor does not. Conversely, the FFLO states generally have no particular relation to other flavors of electronic ordering, while striped superconductors, as is characteristic of all electronic liquid crystals, embody a subtle interplay between multiple ordering tendencies. Specifically, since the striped superconductor seems to be generally associated with the strong coupling physics of doped antiferromagnets, there is a natural sense in which antiferromagnetism, charge density wave formation, and striped superconductivity are intertwined.

8.3. Intertwined orders and emergent symmetries

One explicit way in which the relation between several order parameters can be more intimate than in a generic theory of “competing orders” is if there is an emergent symmetry at low energies which unifies them. In particular, the order parameter structure of the PDW state, involving several order parameters coupled to each other, evokes the $SO(5)$ approach of a unified description of antiferromagnetism and uniform $d$-wave superconductivity [13, 14]. Indeed, by tuning the parameters of the effective Landau-Ginzburg theory that we presented in other sections it is possible to achieve an effective enlarged symmetry which makes it possible “rotate” the striped superconducting order and charge stripe order parameters into each other. Even if the enlarged symmetry is not exact, a rotation of the order parameters is possible but with a finite energy cost (similar to a “spin flop”.) It is also worth noting that a symmetry which allows a similar form of unification of $d$-wave superconductivity, electron nematicity, and $d$-density wave order [61] (dDW) has recently been found to exist under special circumstances by Kee et al. [103]. It is therefore possible that there could exist additional forms of striped superconducting states which interleave these orders.

Thus, it is possible to view the PDW state as a “liquid crystalline” analog of the $SO(5)$ scenario. Indeed, the possibility of an $SO(5)$ “spiral” was discussed previously by Zhang [104]. However, it should be noted that in the context of any conventional Landau-Ginzburg treatment of a system of competing orders, a general theorem [105] precludes a sign change of any component of the order parameter, and hence precludes the existence of spirals. In order to get a PDW state from an interplay between $d$-wave superconductivity and antiferromagnetism, unconventional gradient dependent interactions between the different order parameters, such as those discussed in [105],

† FFLO states in the absence of magnetic fields have been shown to exist for special band structures in 1D [101] and 2D [102].
must play a significant role in the physics.

In other words, in addition to the standard couplings allowed by a theory with several order parameters, the existence of a stripe order (for instance) in the charge order parameter must be able to induce a texture in the superconducting order as well. A useful analogy to keep in mind is the McMillan-deGennes theory of the nematic-smectic transition in classical liquid crystals in which the nematic order parameter acts as a component of a gauge field thus coupling to the phase of the smectic order, or in blue phases of liquid crystals. (For a detailed discussion of these topics in liquid crystals see, e.g. [106, 107].) In fact, Ref [25] presents a theory of FFLO states in ultra-cold atoms with gauge-like couplings (i.e. covariant derivative couplings) that relate the stripe (and spiral) order to the superconducting order.

In addition to the conceptual advantages, noted above, the liquid-crystal picture of the PDW state offers a direct way to classify the phase transitions (both quantum and thermal) out of this state. Thus, in addition to a direct transition to a normal state, intermediate phases characterized with composite order parameters, are also possible leading to an interesting phase diagram. We will explore these issues in a separate publication [86].

8.4. PDW states in Hubbard and t-J models

In the context of the cuprates, there have been several studies looking for a striped superconducting state in the $t-J$ or Hubbard models. On the one hand, extensive, but not exhaustive DMRG calculations by White and Scalapino [77, 108] have consistently failed to find evidence in support of any sort of spontaneously occurring $\pi$ junctions. On the other hand, a number of variational Monte Carlo and renormalized mean field calculations have concluded that the striped superconductor is either the ground-state of such a model [109] under appropriate circumstances, or at least close in energy to the true ground state [110, 111, 80]. These latter calculations are certainly encouraging, in the sense that they suggest that there is no obvious energetic reason to rule out the existence of spontaneously occurring PDW order in strongly correlated electronic systems. However, the fact remains that no spontaneous $\pi$-junction formation has yet been observed in DMRG or other “unbiased” studies of the $t-J$ or the purely repulsive Hubbard models, indicating that there remain basic unsettled issues concerning the microscopic origins of $\pi$ junctions.

9. Final thoughts

In this paper, we have introduced the PDW phase and studied its properties theoretically. In terms of symmetry, the PDW is distinct from the standard uniform superconductor. While some of its properties are similar to those of a uniform superconductor (e.g., zero resistance), others are markedly different: most importantly, the existence of a Fermi surface (and hence a finite density of states) in the ordered phase.
the possibility of frustration of the inter-layer coupling (depending on the lattice geometry), and the strong sensitivity to (non-magnetic) disorder. Generically, the PDW state in the presence of weak disorder is expected to give way to a “superconducting glass” phase, in which the configuration average of the local superconducting order parameter vanishes, but the Edwards-Anderson order parameter is non-zero (and hence gauge symmetry is broken).

Even though the ordered PDW state itself is time reversal invariant, time reversal symmetry breaking is a very natural consequence of PDW order, either in the superconducting glass phase, or as a way of relieving the frustration of the Josephson couplings in some crystal structures. Specifically, frustration can lead to non-collinear ground state configurations of the superconducting pseudo-spins (representing the local phase of the superconducting order), which are analogous to the non-collinear ground states which are often found in frustrated spin systems. An even more exotic state that can naturally emerge from a “parent” PDW state is a superconductor with a charge 4\(e\) order parameter [6, 7, 25], which can result when the CDW part of the PDW order is melted by either quantum or thermal fluctuations.

The occurrence of PDW states in microscopic models is an intrinsically strong coupling effect, since PDW order (much like CDW or SDW) is not an instability of a generic Fermi surface. In this paper, we have provided a “proof of principle” of a not-too-contrived, strongly correlated, microscopic model with a PDW ground state. This model mimics some features of the striped state found in the cuprates (e.g., it has charge stripes separated by \(\pi\)-phase-shifted spin stripes). Whether a PDW state can be found in more realistic models, which include such features as uniformly repulsive interactions and a d-wave-like order parameter, remains to be settled.

Doped Mott insulators are strongly correlated systems whose ground states have a strong tendency to form liquid-crystalline-like inhomogeneous phases [8, 112, 113, 114, 115, 116]. In this regard, the PDW state is an electronic liquid crystal phase in which the superconducting and charge/spin orders do not compete with each other but rather are intertwined. As some of us have noted earlier [117, 118] the observation of a high pairing scale in such an electronically inhomogeneous state is suggestive of the existence of an optimal degree of inhomogeneity for superconductivity. Indeed, recent ARPES data suggest that the stripe order that develops in La\(_{2-x}\)Ba\(_x\)CuO\(_4\) does not suppress the pairing scale.† The fact that the pairing scale is large in this material suggests that the development of charge stripe order suppresses the development of superconducting coherence but not pairing. In fact, it gives credence to the argument that there is a connection between the emergence of charge order and the mechanism of superconducting pairing [117, 118].

However, at present, it is unclear to what extent PDW order should be expected to be common where stripe order occurs. On the purely theoretical side, PDW order has proven elusive in DMRG studies of models [77, 108] with entirely repulsive

† ARPES data in La\(_{2-x}\)Ba\(_x\)CuO\(_4\) shows a substantial and weakly doping dependent anti-nodal gap accross \(x = 1/8\) [44, 45], where the signatures of the PDW state are strongest.
interactions. Indeed, in a previous publication[7], we showed that in any weakly interacting superconductor, $\pi$ junctions can only occur under exceedingly fine-tuned circumstances. It is clear from variational calculations[109, 110, 111, 80] that for strong interactions, the differences in energy between the PDW and uniform sign superconducting states in striped systems is relatively small; what particular features of the microscopic physics tip the balance one way or another is still not clear. Accordingly, it is not clear, in the absence of unambiguous experimental evidence, whether in the context of the cuprates, we should expect the PDW state to be a rare occurrence, perhaps stabilized by some particular detail of the electronic structure of $La_{2-x}Ba_xCuO_4$, or if instead we should infer that some degree of local PDW order exists in any cuprate in which evidence of local stripe correlations can be adduced.

To close this Section, we turn to discuss the evidence for PDW states in the cuprate high temperature superconductors. The analysis of the PDW state was motivated by the experimental observations on $La_{2-x}Ba_xCuO_4$. Having studied the nature of this phase, we will now discuss to what extent the signatures of the PDW state are consistent with experiment. Finally, we speculate on the possible relevance of these ideas to other members of the cuprate family.

9.1. Striped SC phases in $La_{2-x}Ba_xCuO_4$ and 214 cuprates

As already discussed in Sec. 3, the onset of clearly identifiable 2D superconducting correlations in $La_{2-x}Ba_xCuO_4$ with $x = \frac{1}{8}$ occurs at $\sim 40$ K, together with the onset of static spin-stripe order. It would be natural to associate this behavior with the simultaneous onset of local PDW order; however, an attempt to reach a consistent interpretation of a broad range of results leads to a more nuanced story.

The original motivation for applying the PDW concept to $La_{1.875}Ba_{0.125}CuO_4$ was to explain the dynamical layer decoupling through the frustration of the interlayer Josephson coupling in the LTT phase [6, 109], as discussed in Sec. 7. It provides a compelling account‡ for the induced dynamical layer decoupling produced in underdoped $La_{2-x}Sr_xCuO_4$ by a modest c-axis magnetic field [2]. Moreover, the sensitivity of the PDW to disorder which limits the growth of the superconducting correlation length within the planes, provides a natural explanation for the existence of an enormously enhanced “superconducting fluctuation” regime, characterized by enhanced contributions of local superconductivity to the electrical conductivity and to (strongly anisotropic) diamagnetism, but with no global phase coherence. Thus, it naturally accounts for the most dramatic aspects of the experimental data [1] below the spin ordering temperature $T_{SDW}$. We consider this strong evidence that the basic ingredients of the theory are applicable to the stripe ordered state of $La_{2-x}Ba_xCuO_4$ and closely

‡ Since $La_{2-x}Sr_xCuO_4$ retains the LTO structure to low temperatures, and the spin correlations in the c-direction measured at zero field are extremely short-ranged [119], it is unclear whether the charge stripes in neighboring planes tend to be perpendicular to each other, as in the LTT materials, or parallel but offset by half a period from each other, as in the YBa$_2$Cu$_3$O$_{6+z}$ bilayers. In either case, the interlayer Josephson coupling for a PDW would be highly frustrated.
related materials. In addition, the observed transition at temperature $T_{3D}$ into a state with zero resistance in all direction has a natural interpretation in terms of an assumed PDW state as the superconducting glass transition [7]. Besides having zero resistance, the glass phase presumably shows no Meissner effect and zero critical current. If this latter identification is correct, it leads to the further prediction that this phase should be characterized by various phenomena associated with slow dynamics, characteristic of spin glasses, as well as with breaking of time reversal symmetry. The experimental detection of such phenomena below $T_{3D}$ supercurrents) would serve as further confirmation of the existence of a PDW in this material. (For example, the glass phase would likely exhibit a metastable zero-field Kerr effect [58].)

One can also look for evidence for the PDW in single-particle properties. One of the key features of the PDW stripes is the gapping of single-particle excitations in the antinodal region, as illustrated in Fig. 4; in contrast, the nodal states remain ungapped. From the underlying band-structure, one sees that the largest contribution to the density of states with energies near $E_F$ comes from the antinodal regions (where the dispersion is relatively flat); thus, the onset of local PDW order should have a major impact on properties sensitive to the total density of states. Conversely, properties that are largely determined by near nodal quasiparticle dynamics, which presumably includes the quasiparticle contribution to the in-plane conductivity, may be less strongly affected.

Observed striking changes in various transport properties of several stripe order cuprates can be interpreted in this light as being suggestive of the appearance of local PDW order at the onset of charge-stripe order at $T_{CO}$ (which is generally somewhat higher than $T_{SDW}$). In La$_{2-x}$Ba$_x$CuO$_4$ and Nd- and Eu-doped La$_{2-x}$Sr$_x$CuO$_4$, it is observed that the in-plane thermopower drops dramatically below $T_{CO}$ as does the Hall resistivity as does the Hall resistivity [120, 121, 122] as does the Hall resistivity [120, 121, 122] as does the Hall resistivity [120, 121, 122]. Furthermore, the opening of a superconducting-like gap as the temperature drops below $T_{CO}$ results in an observed suppression of the in-plane optical conductivity at frequencies below 40 meV. In contrast, the in-plane DC-resistivity changes relatively little [120, 121, 122] upon cooling through $T_{CO}$.

Putting aside the issue of the onset-temperature, the notion that stripe ordered cuprates exhibit local PDW order is also supported by ARPES studies. For example, measurements on stripe-ordered La$_{1.48}$Nd$_{0.48}$Sr$_{0.12}$CuO$_4$ at $T = 15$ K ($>2T_c$) reveal a gapless nodal arc of states covering roughly a third of the nominal Fermi surface, as well as a gap reaching 30 meV in the antinodal region [128]. Temperature-dependent ARPES measurements on La$_{2-x}$Ba$_x$CuO$_4$ with $x = \frac{1}{8}$ indicate that, for temperatures above the spin-ordering transition, there is a gapless nodal arc of states, together with a substantial antinodal gap [45].

However, there are several aspects of this story which require further analysis. Firstly, there is the issue that different aspects of the crossovers we would like to identify with the onset of local PDW order appear to onset at different temperatures. This is not necessarily inconsistent, as a crossover (as opposed to a phase transition) can appear to occur at somewhat different temperatures depending on what quantity is measured and
how the data is analyzed. Nonetheless, the drop in the thermopower and Hall number appears to have a very sharp onset at $T_{CO}$, while the superconducting like drop in the in-plane resistivity at $T_{SDW}$ is also very sharp, at least in 1/8 doped La$_{2-x}$Ba$_x$CuO$_4$. [In this sense, it is reminiscent of the situation $^{129}$ in O doped La$_x$CuO$_4$, where the sharply defined spin ordering and superconducting ordering transitions occur at the same temperature (in zero field) with very small uncertainty.]

A still more perplexing issue arises in correlating the onset of the signatures of 2D superconductivity in La$_{1.875}$Ba$_{0.125}$CuO$_4$ with the thermal evolution of the ARPES $^{44}$ $^{45}$ spectrum. Below $T_{SDW}$, there is clear evidence of the appearance of a $d$-wave-like gap in the nodal region, with the scale of this second gap being smaller than the pre-existing antinodal gap $^{15}$. This behavior suggests that uniform $d$-wave superconductivity develops and coexists with the PDW superconductor below $T_{SDW}$. However, this is somewhat problematic, as the proposed explanation of the dynamical interlayer decoupling and the bounded growth of superconducting correlations that occurs below $T_{SDW}$ rests on the assumed (near) absence of a uniform component of the order parameter in each plane. Reconciling the uniform $d$-wave component of the order parameter inferred spectroscopically from ARPES studies of La$_{2-x}$Ba$_x$CuO$_4$ with the apparently almost complete absence of such a component inferred from bulk transport measurements on the same material is a challenge for future work. It may be significant, however, that ARPES studies of La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ $^{128}$ and La$_{1.8-x}$Eu$_{0.2}$Sr$_x$CuO$_4$ $^{130}$ appear consistent with pure PDW order, (i.e., there is no $d$-wave gap in the nodal region), although the PDW appears to set in at around $T_{CO}$, which can be substantially greater than $T_{SDW}$ in these materials.

9.2. Dynamical layer decoupling and quasi-two-dimensional behavior in the cuprates

The cuprate superconductors are layered materials with varying degrees of quasi-two-dimensional behavior. Evidence for quasi-2D behavior (and for dimensional crossover) in the cuprates has existed for a long time and it is well documented. It is thus useful to compare and contrast this well known behavior with the unexpected layer decoupling effect observed in La$_{2-x}$Ba$_x$CuO$_4$.

In a quasi-2D system, as a continuous thermodynamic superconducting phase transition is approached, the in-plane correlation length grows very rapidly. While at first the fluctuations have a markedly 2D character, very close to the phase transition they rapidly cross over to their ultimate 3D behavior. Dimensional crossover is observed, for instance, in dynamical probes of some cuprates. High frequency ($\sim$100 GHz) conductivity measurements in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (the most quasi-2D material among the cuprates) by Corson et al $^{131}$ showed that (at those frequencies) the fluctuation conductivity is 2D-like and exhibits Kosterlitz-Thouless behavior, as if the CuO$_2$ planes were effectively decoupled. Similarly, quasi-2D behavior in the dynamic conductivity (with frequencies in the range 1-10 GHz) has been observed in underdoped La$_{2-x}$Sr$_x$CuO$_4$ near $T_c$ (but not in overdoped La$_{2-x}$Sr$_x$CuO$_4$) by Kitano et al $^{132}$. By
probing the system at finite frequency, these experiments explore the correlations at a frequency dependent mesoscopic length scale, where sufficiently weak 3D couplings have negligible effect on the physics. By contrast, the resistive transition both in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and in La$_2-x$Sr$_x$CuO$_4$, measured at zero frequency in macroscopic samples, is not of the 2D XY (Kosterlitz-Thouless) type, but rather reflects the three-dimensional nature of these materials.

In contrast, the unusual layer decoupling effect observed in stripe-ordered La$_{2-x}$Ba$_x$CuO$_4$ takes place in a temperature range where the CuO$_2$ planes appear to become superconducting (well above the three-dimensional critical temperature). The layer decoupling effect is observed in the resistive transition and is thus not a dimensional crossover effect. As we noted above, in this regime La$_{2-x}$Ba$_x$CuO$_4$ behaves as if for some reason the effective inter-layer Josephson coupling is either turned off (which is unphysical) or is somehow frustrated.

Support for this idea is provided by recent Josephson resonance experiments in La$_{2-x}$Sr$_x$CuO$_4$ by Schafgans et al\cite{2}, which essentially measure the c-axis superfluid stiffness, $\rho_c$. In the absence of an external magnetic field, $\rho_c$ has the expected\cite{133,134,135} magnitude, i.e. $\rho_c$ is proportional to the normal state conductivity at $T_c$. However, for underdoped materials, $\rho_c$ becomes unmeasurably small in the presence of moderate magnetic fields ($B \leq 8T$). Magnetic fields are known to induce static spin-stripe order (as detected by neutron scattering experiments\cite{41}) in precisely the same range of field strengths and hole concentration. These experiments thus suggest\cite{2} that the “fluctuating stripe order” seen in La$_{2-x}$Sr$_x$CuO$_4$ at zero field may actually be of the PDW type and that dynamical layer decoupling occurs as static stripe order is stabilized in a magnetic field.\footnote{While it is tempting to reinterpret in hindsight the results of Corson et al\cite{131} as being indicative of “fluctuating PDW order” in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ we should note that the STM data on this material\cite{5} show a glassy pattern of short range stripe order at high bias. However, as we explained elsewhere in this paper, a glassy version of the PDW state would not exhibit a sharp layer decoupling effect.}

Indeed, in materials, including La$_{2-x}$Ba$_x$CuO$_4$ and La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$, which exhibit stripe order in zero field, $\rho_c$ is found\cite{136} to be orders of magnitude smaller than its “expected” value on the basis of the normal state conductivity.

9.3. Possible relevance to other cuprates

Although there are still open issues, the PDW state (or its glassy version) seems to offer a rather compelling explanation for what is otherwise an extremely surprising set of phenomena observed in stripe ordered cuprates. Could these ideas also be relevant to a broader range of phenomena in the cuprates? The direct empirical information available\cite{15} concerning the structure of any sort of static or fluctuating stripe order present in cuprates outside the 214 family is much less clear.\footnote{The results of recent neutron scattering studies of underdoped YBa$_2$Cu$_3$O$_{6+x}$ by Hinkov et al.\cite{55} have confirmed\cite{52} the existence of a nematic phase, derived from the weak melting of a stripe ordered state, onseting below a temperature comparable to the pseudogap onset-temperature, $T^*$. Still more} Consequently, any attempt to
achieve a theoretical understanding based on the assumed existence of a PDW state is necessarily speculative. We therefore present the discussion of this final section in the spirit of provocative conjectures, which we believe are deserving of further investigation.

ARPES studies of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and La$_{2-x}$Sr$_x$CuO$_4$ have revealed “Fermi arcs” of gapless states between antinodal pseudogaps. There has been a great deal of controversy over the nature of the antinodal pseudogap. Two recent studies of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ have reported signatures of Bogoliubov quasiparticles in the antinodal gap region, which was interpreted as being suggestive that the pseudogap is, at least in part, produced by superconducting fluctuations. On cooling through $T_c$, a $d$-wave gap appears along the nodal arc. In near optimally doped samples, as $T \to 0$, this $d$-wave gap and the pseudo-gap merge to form a single gap with a simple $[\cos(k_x) - \cos(k_y)]$ form. However, in underdoped samples, even as $T \to 0$, the nodal gap appears to have a different energy scale than the antinodal gap (i.e., they do not merge to form a simple $d$-wave gap). Thus, in some ways it is clear that there are two distinct gaps - an antinodal pseudo-gap that might be associated with some sort of “competing” order, and a nodal gap, which is clearly superconducting in the sense that it onsets quite sharply at $T_c$. However, in other ways it seems that all the gaps have some unifying superconducting character.

We propose that this puzzle may be resolved by postulating that there are two distinct gaps, both with superconducting character in the sense that one is associated with uniform the other with modulated superconducting order. Indeed, the measured quasiparticle spectral function in the pseudogap looks somewhat like that of the PDW state. Moreover, just such a combination of modulated and uniform superconducting orders has been previously proposed on phenomenological grounds to explain STM spectra in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ and other cuprates.

Seemingly more direct evidence of superconducting fluctuations in the normal state of La$_{2-x}$Sr$_x$CuO$_4$, Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$, and Bi$_2$Sr$_2$-$_y$La$_y$CuO$_6$ has been reported by Ong and coworkers based on measurements of the Nernst effect and diamagnetism. Nernst measurements on YBa$_2$Cu$_3$O$_{6+x}$ and STM studies of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ suggest that disorder may be important to the existence of the fluctuation effects over a substantial temperature range. It is intriguing that the onset temperatures of the enhanced Nernst response in La$_{2-x}$Sr$_x$CuO$_4$ has a maximum at $x \sim 0.1$, close to the optimum doping for stripe order. Moreover, Taillefer recently, the same authors have demonstrated that modest magnetic fields stabilize static spin-stripe order where primarily fluctuating (nematic) order existed at zero field.
and coworkers [151] have found close correlations between an enhanced Nernst signal and stripe order. Neither the observed sensitivity to disorder nor the association with stripe order, by themselves, necessarily negate the interpretation of these effects in terms of superconducting fluctuations; however, both would be unusual in the case of a simple, homogeneous d-wave superconductor. While we are far from having an explicit theory, it seems to us that these general trends are consistent with the existence of a disordered PDW state over at least a portion of the pseudogap phase. Specifically, Ong and coworkers [152] have reported the observation of a sublinear dependence of the magnetization on magnetic field ($M \sim -B^\alpha$ with $\alpha < 1$) in a relatively narrow but non-vanishing range of temperatures above $T_c$ in crystals of Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$. This behavior, if it truly persists in the limit $B \to 0$, must signify the existence of a distinct phase of matter in this range of temperatures, which we very tentatively propose could be a superconducting glass formed from a disordered PDW.+

One of the most intriguing recent discoveries in the cuprates involve several distinct observations of a rather subtle, and not fully understood, form of time reversal symmetry breaking in the pseudogap phase of YBa$_2$Cu$_3$O$_{6+x}$ [57, 58] and HgBa$_2$CuO$_{4+\delta}$ [59]. As we have seen, various forms of subtle time-reversal symmetry breaking can occur when frustration is added into the PDW mix. It is our hope that, with further work, a relation can be established between these two rather vague statements.

**Acknowledgments**

We thank Peter Abbamonte, Dimitri Basov, Hong Yao, Ruihua He, Srinivas Raghu, Aharon Kapitulnik, Eun-Ah Kim, Vadim Oganesyan, Gil Refael, Doug Scalapino, Dale Van Harlingen, Kun Yang, and Shoucheng Zhang for great discussions. This work was supported in part by the National Science Foundation, under grants DMR 0758462 (E.F.) and DMR 0531196 (S.A.K.), and by the Office of Science, U.S. Department of Energy under Contracts DE-FG02-91ER45439 through the Frederick Seitz Materials Research Laboratory at the University of Illinois (E.F.), DE-FG02-06ER46287 through the Geballe Laboratory of Advanced Materials at Stanford University (S.A.K. and E.B.), and DE-AC02-98CH10886 at Brookhaven (J.M.T.).

**References**

[1] Li Q, H"ucker M, Gu G D, Tsvelik A M and Tranquada J M 2007 Phys. Rev. Lett. 99 067001
[2] Schafgans A A, LaForge A D, Dordevic S V, Qazilbash M M, Komiyama S, Ando Y and Basov D N 2008 Towards two-dimensional superconductivity in La$_2$-$_x$Sr$_x$CuO$_4$ in a moderate magnetic field (unpublished)

+ Some of us have argued elsewhere [153, 118] that the pair correlations in hole-rich stripes correspond to spin singlet correlations, so that the pairing energy is reflected in the singlet-triplet excitation energy. The description of pairing and spin correlations within the charge stripes has much in common with the RVB perspective [154]; however, electronic self-organization into stripes certainly enhances, and may be necessary to realize this behavior in the CuO$_2$ planes [153, 118].
Striped superconductors

[3] Howald C, Eisaki H, Kaneko N and Kapitulnik A 2003 Proc. Natl. Acad. Sci. U.S.A. 100 9705
[4] Lang K M, Madhavan V, Hoffman J E, Hudson E W, Eisaki H, Uchida S and Davis J C 2002 Nature 415 412
[5] Kolsaka Y, Taylor C, Fujita K, Schmidt A, Lupien C, Hanaguri T, Azuma M, Takano M, Eisaki H, Takagi H, Uchida S and Davis J C 2007 Science 315 1380–1385
[6] Berg E, Fradkin E, Kim E A, Kivelson S, Oganesyan V, Tranquada J M and Zhang S 2007 Phys. Rev. Lett. 99 127003
[7] Berg E, Fradkin E and Kivelson S A 2009 Phys. Rev. B 79 064515
[8] Kivelson S A, Fradkin E and Emery V J 1998 Nature 393 550
[9] Sun K, Fregoso B M, Lawler M J and Fradkin E 2002 Phys. Rev. B 66 104517
[10] O’Hern C S, Lubensky T C and Toner J 1999 Phys. Rev. Lett. 83 2746
[11] Fulde P and Ferrell R A 1964 Phys. Rev. 135 A550
[12] Larkin A I and Ovchinnikov Y N 1964 Zh. Eksp. Teor. Fiz. 47 113 (Sov. Phys. JETP. 20, 762 (1965))
[13] Zhang S C 1997 Science 275 1089
[14] Demler E, Hanke W and Zhang S C 2004 Rev. Mod. Phys. 76 909
[15] Kivelson S A, Fradkin E, Oganesyan V, Bindloss I, Tranquada J, Kapitulnik A and Howald C 2003 Rev. Mod. Phys. 75 1201
[16] Vojta M 2009 Lattice symmetry breaking in cuprate superconductors: Stripes, nematics, and superconductivity (unpublished) (Preprint arXiv:0901.3145)
[17] Spivak B I, Oreto P and Kivelson S A 2008 Phys. Rev. B. 77 214523
[18] Chen H D, Vafek O, Yazdani A and Zhang S C 2004 Phys. Rev. Lett. 93 187002
[19] Melikyan A and Tešanović Z 2005 Phys. Rev. B 71 214511
[20] Balents L, Bartosch L, Burkov A, Sachdev S and Sengupta K 2005 Phys. Rev. B 71 144508
[21] Vojta M and Sachdev S 1999 Phys. Rev. Lett. 83 3916
[22] Vojta M, Zhang Y and Sachdev S 2000 Phys. Rev. B 62 6721
[23] Vojta M and Rösch O 2008 Phys. Rev. B 77 094504
[24] Aligia A A, Anfossi A, Arrachea L, Boschi C D E, Dobry A O, Gazza C, Montorsi A, Ortolani F and Torio M E 2007 Phys. Rev. Lett. 99 206401
[25] Radzihovsky L and Vishwanath A 2008 Quantum liquid crystals in imbalanced Fermi gas: fluctuations and fractional vortices in Larkin-Ovchinnikov states (unpublished) (Preprint arXiv:0812.3945)
[26] Kivelson S A, Emery V J and Lin H Q 1990 Phys. Rev. B 42 6523–6530
[27] Fujita M, Goka H, Yamada K, Tranquada J M and Regnault L P 2004 Phys. Rev. B 70 104517
[28] Tranquada J M, Gu G D, Hücke M, Kang H J, Klingerer R, Li Q, Wen J S, Xu G Y and v Zimmermann M 2008 Phys. Rev. B 78 174529
[29] Dunsiger S R, Zhao Y, Yamani Z, Buyers W J L, Dabkowski H and Gaulin B D 2008 Phys. Rev. B 77 224410
[30] Abbamonte P, Rusydi A, Smadici S, Gu G D, Sawatzky G A and Feng D L 2005 Nature Phys. 1 155
[31] Kim Y J, Gu G D, Gog T and Casa D 2008 Phys. Rev. B 77 064520
[32] Tranquada J M, Sternlieb B J, Axe J D, Nakamura Y and Uchida S 1995 Nature 375 561
[33] v Zimmermann M, Vigliante A, Niemöller T, Ichikawa N, Frello T, Uchida S, Andersen N H, Madsen J, Wochner P, Tranquada J M, Gibbs D and Schneider J R 1998 Europhys. Lett. 41 629
[34] Christensen N B, Rønnow H M, Mesot J, Ewings R A, Momono N, Oda M, Ido M, Enderle M, McMorrow D F and Boothroyd A T 2007 Phys. Rev. Lett. 98 197003
[35] Fink J, Schierle E, Weschke E, Geck J, Hawthorn D, Wadati H, Hu H H, Durr H A, Wizent N, Böchner B and Sawatzky G A 2008 Charge order in La_{1.8-x}Eu_{0.2}Sr_xCuO_4 studied by resonant soft x-ray diffraction (unpublished) (Preprint arXiv:0805.3452v1)
[36] Hücke M, Gu G D, Tranquada J M, v Zimmermann M, Klauss H H, Curro N J, Braden M and
Striped superconductors

Büchner B 2007 Physica C 460–462 170–173

[37] Teitel’baum G B, Büchner B and de Gronckel H 2000 Phys. Rev. Lett. 84 2949
[38] Lee Y S, Birgeneau R J, Kastner M A, Endoh Y, Wakimoto S, Yamada K, Erwin R W, Lee S H and Shirane G 1999 Phys. Rev. B 60 3643
[39] Hirota K 2001 Physica C 357–360 61–68
[40] Fujita M, Yamada K, Hiraka H, Gehring P M, Lee S H, Wakimoto S and Shirane G 2002 Phys. Rev. B 65 064505
[41] B Lake, H M Rønnow, N B Christensen, G Aeppli, K Lefmann, D F McMorrow, P Vorderwisch, P Smeibidl, N Mangkorntong, T Sasagawa, M Nohara, H Takagi and T E Mason 2002 Nature 415 299
[42] Khaykovich B, Wakimoto S, Birgeneau R J, Kastner M A, Lee Y S, Smeibidl P, Vorderwisch P and Yamada K 2005 Phys. Rev. B 71 220508(R)
[43] Chang J, Niedermayer C, Gilardi R, Christensen N, Ronnow H, McMorrow D, Ay M, Stahn J, Sobolev O, Hiess A, Pailhes S, Baines C, Momono N, Oda M, Ido M and Mesot J 2008 Phys. Rev. B 78 104525
[44] Valla T, Fedorov A V, Lee J, Davis J C and Gu G D 2006 Science 314 1914–1916
[45] He R H, Tanaka K, Mo S K, Sasagawa T, Fujita M, Adachi T, Mannella N, Yamada K, Koike Y, Hussain Z and Shen Z X 2008 Nat. Phys. 5 119
[46] Komiyama S, Ando Y, Sun X F and Lavrov A N 2002 Phys. Rev. B 65 214535
[47] Ding J F, Xiang X Q, Zhang Y Q, Liu H and Li X G 2008 Phys. Rev. B 77 214524
[48] Xiang X Q, Zhang Y Q, Ding J F and Li X G 2008 Physica C 468 2336–2340
[49] Fujiwara K, Noda T, Kojima K M, Eisaki H and Uchida S 2005 Phys. Rev. Lett. 95 07006
[50] Tranquada J M, Axe J D, Ichikawa N, Nakamura Y, Uchida S and Nachumi B 1996 Phys. Rev. B 54 7489
[51] Chakravarty S private communication
[52] Ando Y, Segawa K, Komiyama S and Lavrov A N 2002 Phys. Rev. Lett. 88 137005
[53] Mook H A, Dai P, Hayden S M, Aeppli G, Perring T G and Doan F 1998 Nature 395 580
[54] Stock C, Buyers W J L, Liang R, Peets D, Tun Z, Bonn D, Hardy W N and Birgeneau R J 2004 Phys. Rev. B 69 014502
[55] Hinkov V, Haug D, Fauqué B, Bourges P, Sidis Y, Ivanov A, Bernhard C, Lin C T and Keimer B 2008 Science 319 597
[56] Haug D, Hinkov V, Suchaneck A, Inosov D, Christensen N B, Niedermayer C, Bourges P, Sidis Y, Park J T, Ivanov A, Lin C T, Mesot J and Keimer B 2008 Magnetic field enhanced incommensurate magnetism in the underdoped high-temperature superconductor YBa2Cu3O6.45 (unpublished) (Preprint arXiv:0902.3335)
[57] Fauqué B, Sidis Y, Hinkov V, Pailhé N, Lin C T, Chaud X and Bourges P 2006 Phys. Rev. Lett. 96 197001
[58] Xia J, Schemm E, Deutscher G, Kivelson S A, Bonn D A, Hardy W N, Liang R, Siemons W, Koster G, Fejer M M and Kapitulnik A 2008 Phys. Rev. Lett. 100 127002
[59] Li Y, Balédent V, Barisic N, Cho Y, Fauqué B, Sidis Y, Yu G, Zhao X, Bourges P and Greven M 2008 Nature 455 372
[60] Varma C M 2005 Philos. Mag. 85 1657
[61] Chakravarty S, Laughlin R B, Morr D K and Nayak C 2001 Phys. Rev. B 63 094503
[62] Zhou X J, Yoshida T, Lee D H, Yang W L, Brouet V, Zhou F, Ti W X, Xiong J W, Zhao Z X, Sasagawa T, Kakeshita T, Eisaki H, Uchida S, Fujimori A, Hussain Z and Shen Z X 2004 Phys. Rev. Lett. 92 187001
[63] Yang H B, Rameau J D, Johnson P D, Valla T, Tsvelik A and Gu G D 2008 Nature 456 77–80
[64] Podolsky D, Demler E, Damle K and Halperin B I 2003 Phys. Rev. B. 67 094514
[65] Hoffman J E, McElroy K, Lee D H, Lang K M, Eisaki H, Uchida S and Davis J 2002 Science 297 1148–1151
[66] Vershinin M, Misra S, Ono S, Abe Y, Ando Y and Yazdani A 2004 Science 303 1005
Striped superconductors

[67] Wang Y, Ong N P, Xu Z A, Kakeshita T, Uchida S, Bonn D A, Liang R and Hardy W N 2002 Phys. Rev. Lett. 88 257003
[68] Podolsky D, Raghu S and Vishwanath A 2007 Phys. Rev. Lett. 99 117004
[69] Oganesyan V and Ussishkin I 2004 Phys. Rev. B 70 054503
[70] Emery V J and Kivelson S A 1995 Nature 374 434
[71] Spivak B I and Kivelson S A 1999 Phys. Rev. Lett. 82 2788
[72] van Dam J A, Nazarov Y V, E P A M Bakkers, Franceschi S D and Kouwenhoven L P 2006 Nature (London) 442 667
[73] Wollman D A, Harlingen D J V, Lee W C, Ginsberg D M and Leggett A J 1993 Phys. Rev. Lett. 71 2134
[74] Tseu C C, Kirtley J R, Chi C C, Yu-Jahnes L S, Gupta A, Shaw T, Sun J Z and Ketchen M B 1994 Phys. Rev. Lett. 73 593
[75] White S R and Chernyshev A L 2007 Phys. Rev. Lett. 99 127004
[76] White S R and Scalapino D J 1998 Phys. Rev. Lett. 80 1272
[77] Berg E, Chen C C and Kivelson S A 2008 Phys. Rev. Lett. 100 027003
[78] Baruch S and Orgad D 2008 Phys. Rev. B. 77 174502
[79] Yang K Y, Chen W Q, Rice T M, Sigrist M and Zhang F C 2008 Nature of Stripes in the Generalized t-J Model Applied to the Cuprate Superconductors (Preprint arXiv:0807.3789)
[80] Norman M R, Ding H, Randeria M, Campuzano J C, Yokoya T, Takeuchi T, Takahashi T, Mochiku T, Kadowaki K, Guptasarma P and Hinks D G 1998 Nature (London) 392 157
[81] Kanigel A, Norman M R, Randeria M, Chatterjee U, Suoma S, Kaninski A, Fretwell H M, Rosenkranz S, Shi M, Sato T, Takahashi T, Li Z Z, Raffy H, Kadowaki K, Hinks D, Ozyuzer L and Campuzano J C 2006 Nature Phys. 2 447
[82] Chakravarty S, Nayak C and Tewari S 2003 Phys. Rev. B 68 100504(R)
[83] Zachar O, Kivelson S A and Emery V J 1998 Phys. Rev. B. 57 1422
[84] Berg E, Fradkin E and Kivelson S A 2009 Charge 4e superconductivity from pair density wave order in certain high temperature superconductors (unpublished) (Preprint arXiv:0904.1230)
[85] Larkin A 1970 Z. E. Fiz. 35 1466 (Sov. Phys. JETP 31, 784 (1970))
[86] Binder K and Young A P 1986 Rev. Mod. Phys. 70 1545
[87] Giamarchi T and LeDoussal P 1995 Phys. Rev. B. 52 1242
[88] Fisher D S 1997 Phys. Rev. Lett. 78 1964 Chen Zeng, P. L. Leath and Daniel S. Fisher, Phys. Rev. Lett. 82, 1935 (1999)
[89] Partridge G B, Li W, Kamar R I, and Liao Y and Hulet R G 2006 Science 311 503–505
[90] Zwierlein M W, Schirotzek A, Schunck C H and Ketterle W 2006 Science 311 492–496
[91] Datta T 2009 Eur. Phys. J. B 67 197
[92] Kubo K 2008 J. Phys. Soc. Jpn. 77 043702
[93] Robertson J A, Kivelson S A, Fradkin E, Fang A C and Kapitulnik A 2006 Phys. Rev. B 74 134507
[94] Del Maestro A, Rosenow B and Sachdev S 2006 Phys. Rev. B 74 024520
[95] Raman K S, Oganesyan V and Sondhi S L 2009 Biot-Savart correlations in layered superconductors (unpublished) (Preprint arXiv:0902.1547)
[96] Yang K and Agterberg D F 2000 Phys. Rev. Lett. 84 4970
[97] Schrieffer J R 1964 Theory of Superconductivity (Redwood City, CA: Addison-Wesley)
[98] Bulaevskii L N, Kuzii V V, Sobyanin A A and Lebedev P N 1978 Solid State Comm. 25 1053
[99] Salkola M I and Schrieffer J R 1998 Phys. Rev. B. 57 14433
[100] Partridge G B, Li W, Kamar R I, and Liao Y and Hulet R G 2006 Science 311 503–505
[101] Zwierlein M W, Schirotzek A, Schunck C H and Ketterle W 2006 Science 311 492–496
[102] Datta T 2009 Eur. Phys. J. B 67 197
[103] Kubo K 2008 J. Phys. Soc. Jpn. 77 043702
[104] Kee H Y, Doh H and Grzesiak T 2008 J. Phys. Condens. Matter 76 255248
[105] Zhang S C 1998 J. Phys. Chem. Solids 59 1774
[106] Pryadko L P, Kivelson S A, Emery V J, Bazaliy Y B and Demler E A 1999 Phys. Rev. B 60 7541
Striped superconductors

[106] de Gennes P G and Prost J 1993 The Physics of Liquid Crystals (Oxford, UK: Oxford Science Publications/ Clarendon Press)

[107] Chaikin P M and Lubensky T C 1995 Principles of Condensed Matter Physics (Cambridge, UK: Cambridge University Press)

[108] DJ Scalapino and White S 2008 Private communication

[109] Himeda A, Kato T and Ogata M 2002 Phys. Rev. Lett. 88 117001

[110] Chaikin P M and Lubensky T C 1995 Principles of Condensed Matter Physics (Cambridge, UK: Cambridge University Press)

[111] Himeda A, Kato T and Ogata M 2002 Phys. Rev. Lett. 88 117001

[112] Raczkowski M, Capello M, Poilblanc D, Fréssard R and Oleš A M 2007 Phys. Rev. B 76 140505(R)

[113] Emery V J and Kivelson S A 1993 Physica C 209 597

[114] White S R and Scalapino D J 2000 Phys. Rev. B 61 6320

[115] Carlson E W, Emery V J, Kivelson S A and Orgad D 2004 The Physics of Conventional and Unconventional Superconductors ed Bennemann K H and Kettersson J B (Berlin: Springer-Verlag) [arXiv:cond-mat/0206217]

[116] Arrigoni E, Fradkin E and Kivelson S A 2004 Phys. Rev. B 69 214519

[117] Kivelson S A and Fradkin E 2007 Handbook of High Temperature Superconductivity ed Schrieffer J R and Brooks J (New York: Springer-Verlag) pp 569–595 (Preprint [arXiv:cond-mat/0507459])

[118] Lake B, Lefmann K, Christensen N B, Aeppli G, McMorrow D F, Rønnow H M, Vorwerk P, Smeibidl P, Mangkorntong N, Sasagawa T, Nohara M and Takagi H 2005 Nature Mat. 4 658–662

[119] Sera M, Ando Y, Kondoh S, Fukuda K, Sato M, Watanabe I, Nakashima S and Kumagai K 1989 Solid State Commun. 69 851

[120] Zhou J S and Goodenough J B 1997 Phys. Rev. B 56 6288–6294

[121] Huang M, Kataev V, Pommer J, Baberski O, Schablitz W and Büchner B 1998 J. Phys. Chem. Solids 59 1821

[122] Noda T, Eisaki H and Uchida S 1999 Science 286 265

[123] Adachi T, Noji T and Koike Y 2001 Phys. Rev. B 64 144524

[124] Takahata N, Sasagawa T, Sugioke T, Takagi H 2003 J. Phys. Soc. Jpn. 73 1123–1126

[125] Homec C C, Dordevic S V, Gu G D, Li Q, Valla T and Tranquada J M 2006 Phys. Rev. Lett. 96 257002

[126] Ichikawa N, Tranquada J M, Niemöller T, Gehring P M, Lee S H and Schneider J R 2000 Phys. Rev. Lett. 85 1738

[127] Chang J, Sassa Y, Guererro S, Månsson M, Shi M, Pailhés S, Bendounan A, Mottl R, Claesson T, Tjernberg O, Patthey L, Ido M, Oda M, Momono N, Mudry C and Mesot J 2008 New J. Phys. 10 103016

[128] Khaykovich B, Lee Y S, Erwin R W, Lee S H, Wakimoto S, Thomas K J, Kastner M A and Birgeneau R J 2002 Phys. Rev. B 66 014528

[129] Zabolotnyy V B, Kordyuk A A, Inosov D S, Evtsushinsky D V, Schuster R, Büchner B, Wizent N, Behr G, Fyon S, Takagi H, Follath R and Borisenko S V 2008 Evidence for Fermi surface reconstruction in the static stripe phase of La_{1.8-x}Eu_{0.2}Sr_{x}CuO_{4}, x = 1/8 (unpublished) (Preprint [arXiv:0809.2237])

[130] Corson J, Mallozzi R, Orenstein J, Eckstein J N and Bozovic I 1999 Nature 398 221

[131] Kitano H, Ohashi T, Maeda A and Tsukada I 2006 Phys. Rev. B 73 092504

[132] Basov D N, Timusk T, Dabrowski B and Jorgensen J D 1994 Phys. Rev. B 50 3511

[133] Dordevic S V, Singley E J, Basov D N, Komiya S, Ando Y, Bucher E, Homes C C and Strongin M 2002 Phys. Rev. B 65 134511
Stripped superconductors

[135] Homes C C, Dordevic S V, Strongin M, Bonn D A, Liang R, Hardy W N, Komiyama S, Ando Y, Yu G, Kaneko N, Zhao X, Greven M, Basov D N and Timusk T 2004 Nature 430 539

[136] A. Shafgans, APS March Meeting Invited Talk, Pittsburgh (2009), and D. N. Basov, private communication

[137] Shi M, Chang J, Pailhes S, Norman M R, Campuzano J C, Mansson M, Claesson T, Tjernberg O, Bendouman A, Patthey L, Momono N, Oda M, Ido M, Mudry C and Mesot J 2008 Phys. Rev. Lett. 101 047002

[138] Yoshida T, Hashimoto M, Ideta S, Fujimori A, Tanaka K, Mannella N, Hussain Z, Shen Z X, Kubota M, Ono K, Komiyama S, Ando Y, Eisaki H and Uchida S 2008 Universal versus Material-Dependent Two-Gap Behaviors in the High-Tc Cuprates: Angle-Resolved Photoemission Study of La$_{2-x}$Sr$_x$CuO$_4$ (unpublished) (Preprint arXiv:0812.0155)

[139] Norman M R, Pines D and Kallin C 2005 Adv. Phys. 54 715

[140] Hüfner S, Hossain M A, Damascelli A and Sawatzky G A 2008 Rep. Prog. Phys. 71 062501

[141] Kanigel A, Chatterjee U, Randeria M, Norman M R, Koren G, Kadowaki K and Campuzano J C 2008 Phys. Rev. Lett. 101 137002

[142] Lee W S, Vishik I M, Tanaka K, Lu D H, Sasagawa T, Nagaosa N, Devereaux T P, Hussain Z and Shen Z X 2007 Nature 450 81–84

[143] Howald C, Eisaki H, Kaneko N, Greven M and Kapitulnik A 2003 Phys. Rev. B 67 014533

[144] Gomes K K, Pasupathy A N, Pushp A, Ono S, Ando Y and Yazdani A 2007 Nature 447 569

[145] Slezak J A, Lee L, Wang M, McElroy K M, Fujita K, Andersen B M, Hirschfeld P J, Eisaki H, Uchida S and Davis J C 2008 PNAS 105 3203–3208

[146] Wise W D, Boyer M C, Chatterjee K, Kondo T, Takeuchi T, Ikuta H, Wang Y and Hudson E W 2008 Nat. Phys. 4 696–699

[147] Wang Y, Xu Z A, Kakeshita T, Uchida S, Ono S, Ando Y and Ong N P 2001 Phys. Rev. B 64 224519

[148] Li L, Wang Y, Naughton M J, Komiyama S, Ono S, Ando Y and Ong N P 2007 J. Magn. Magn. Mater. 310 460–466

[149] Rullier-Albenque F, Tourbot R, Alloul H, Lejay P, Colson D and Forget A 2006 Phys. Rev. Lett. 96 067002

[150] Pasupathy A N, Pushp A, Gomes K K, Parker C V, Wen J, Xu Z, Gu G, Ono S, Ando Y and Yazdani A 2008 Science 320 196–201

[151] Cyr-Choinière O, Daou R, Laliberté F, LeBoeuf D, Doiron-Leyraud N, Chang J, Yan Y Q, Zhou J S, Goodenough J B, Pyon S, Takagi H and Taillefer L 2009 Nature 458 743

[152] Li L, Wang Y, Naughton M J, Ono S, Ando Y and Ong N P 2005 Europhys. Lett. 72 451–457

[153] Emery V J, Kivelson S A and Zachar O 1997 Phys. Rev. B 56 6120

[154] Anderson P W, Lee P A, Randeria M, Rice T M, Trivedi N and Zhang F C 2004 J. Phys. Condens. Matter 16 R755