The global existence and large time behavior of smooth compressible fluid in an infinitely expanding ball, I: 3D irrotational Euler equations

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Abstract
We are concerned with the global existence and large time behavior of compressible fluids (including the inviscid gases, viscid gases, and Boltzmann gases) in an infinitely expanding ball. Such a problem is one of the interesting models in studying the theory of global smooth solutions to multidimensional compressible gases with time-dependent boundaries and vacuum states at infinite time. Due to the conservation of mass, the fluid in the expanding ball becomes rarefied and eventually tends to a vacuum state meanwhile there are no appearances of vacuum domains in any part of the expansive ball, which is easily observed in finite time. In this paper, as the first part of our three papers, we will confirm this physical phenomenon for the compressible inviscid and irrotational gases by obtaining the exact lower and upper bound on the density function.

Keywords: compressible Euler equations, expanding ball, global existence, degenerate, weighted energy estimates, large time behavior

1. Introduction

In this paper, we consider the behavior of a compressible inviscid fluid in a 3D expanding ball given by \( \Omega_0 = \{ (t, x): t \geq 0, |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R_0(t) \} \), where \( R_0(t) \in C^0[0, \infty) \) satisfies \( R_0(0) = 1, R_0'(0) = 0, R_0''(0) = 0 \), and \( R_0(t) = 1 + Lt \) for \( t \geq 1 \) with some positive constant \( L \). From the expression of \( \Omega_0 \), we know that the ball \( S^3_t = \{ x: |x| \leq R_0(t) \} \) at the time \( t \) is artificially set by pulling out the initial unit ball \( S^3_0 = \{ x: |x| \leq 1 \} \) with a smooth speed and acceleration (see figure 1 below). Such a problem is one of the interesting models in studying the theory of global smooth solutions to multidimensional compressible gases with time-dependent boundaries and vacuum states at infinite time. From a physical point of view, this problem should be globally well-posed, however, this requires a strictly mathematical proof. There are three typical and different models to describe the movement of compressible fluids: Euler equations (viscosity of fluid is neglected), Navier–Stokes equations (viscosity of fluid is considered) and Boltzmann equations (microcosmic factors of fluid particles are considered), where Euler equations are the hyperbolic systems, Navier–Stokes equations are the hyperbolic-parabolic coupled systems, and Boltzmann equations are the scalar transport equations with a non-local collision operator. From a mathematical point of view, the properties and research methods on these three equations are rather different. A natural problem arises: with respect to the same physical phenomenon, are there the same global well-posedness results for the three different mathematical models? In this paper, we firstly establish the global well-posedness for the 3D irrotational Euler equations. The related global well-posedness on the Navier–Stokes model (part II) and the Boltzmann model (part III) can be found in [1] and [2] respectively.
impose the following initial-boundary conditions on $\Omega_0$ in (1.2) represents the solid wall condition. The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** If $\rho_0(x) \in H^4(S_0^0)$, $u_0(x) \in H^3(S_0^0)$, $\text{rot } u_0(x) \equiv 0$, and the compatibility conditions on $\{(t, x) : t = 0, x \in \partial S_0^0\}$ of $(\rho_0(x), u_0(x))$ hold, then there exists a constant $h_0 > 0$, and a small constant $\varepsilon_0 > 0$ depending only on $h_0$, such that when $\sup_{0 \leq \varepsilon \leq 1, \varepsilon \leq \varepsilon_0} |R_0(t)| + \|ho_0(x) - 1\|_{L^2(S_0^0)} + \|u_0(x)\|_{H^4(S_0^0)} < \varepsilon_0$, $R_0(t) = 1 +Lt$ for $t \geq 1$, and $0 < L < h_0$, problem (1.1)-(1.2) with $1 < \gamma < \frac{4}{3}$ admits a global solution $(\rho, u)$ in $\Omega_0$ satisfying

\[
(\rho(t, x), u(t, x)) \in C([0, \infty), H^4(S_0^0)) \cap C([0, \infty), H^3(S_0^0)),
\]

\[
\text{rot } u(t, x) \equiv 0, \quad (t, x) \in \Omega_0,
\]

\[
\frac{1}{2R^3(t)} \leq \rho(t, x) \leq \frac{3}{2R^3(t)} \quad \text{ for } \quad t \geq 1,
\]

and

\[
\sup_{x \in S_0^0} \left( \left| u(t, x) - \frac{Lx}{R(t)} \right| + \left| R(t) \nabla (u(t, x) - \frac{Lx}{R(t)}) \right| \right) \to 0 \text{ as } t \to +\infty.
\]

**Remark 1.1.** The pointwise estimate $\rho(t, x) \sim \frac{1}{R^4(t)}$ for large $t$ can be expected by the conservation of mass. In addition, the regularity assumptions of $\rho_0(x) \in H^4(S_0^0)$ and $u_0(x) \in H^3(S_0^0)$ in Theorem 1.1 are required since we will utilize the Sobolev imbedding lemma 6.1 and energy estimates (6.5) to derive the global time-decay estimates of $\nabla D \Phi$ and $\nabla^2 \Phi$ in section 6, where $\Phi$ is the solution to the linearized equation of the second-order quasilinear wave equation (1.10). On the other hand, if $\Omega_0 = (0, \infty) \times S_0$ (i.e., the compressible gases lie in a fixed domain), which corresponds to $L = 0$ in the expression of $R_0(t)$, then the perturbed solution $(\rho, u)$ will blow up in the general case (see [3] and so on).

**Remark 1.2.** When the fluid is governed by the compressible Navier–Stokes equations and the corresponding boundary condition is given by $u(t, x) = \frac{R'(t)x}{R(t)}$ for $(t, x) \in \partial \Omega_0$, the corresponding result was obtained in [1]. The case for a rarefied gas in $\Omega_0$ governed by the Boltzmann equation was also obtained in [2].
Remark 1.3. If the initial density contains vacuum, the local well-posedness of the compressible Euler system have been extensively studied, cf [4–10] and [11]. In addition, the authors in [12] and [13] establish global existence of solutions to the compressible Euler equations, in the case that a finite of ideal gas expands into vacuum, where the density \( \rho = 0 \) and the physical vacuum boundary condition on the vacuum surface play the essential role in obtaining the global solutions. However, the problem considered in this paper is not the case, and the vacuum is only the time asymptotic state, moreover we are required to overcome some huge difficulties having arisen from the fixed boundary value condition and non-vacuum state on the expansive boundary.

Remark 1.4. When the initial velocity \( u_0(x) \) is close to a linear field, the authors in [14] and [15] proved the global existence of smooth solution to the Cauchy problem of the compressible Euler system. And this is different from the case considered in this paper.

Remark 1.5. If the ball \( S_0 \) is pulled outwards rapidly, namely, when the number \( L \) is large, part of the region inside the ball may become vacuum in finite time (see [16] and so on).

Remark 1.6. Note that \( \hat{\rho}(t), \hat{u}(t, x) = \left( \frac{1}{R(t)} \right)^{\frac{2k}{c_0^2}}, \frac{L}{R(t)} \) with \( R(t) = 1 + L/t \) is a special solution to (1.1)–(1.2). In fact, theorem 1.2 gives the stability of this special solution. In addition, the smallness of \( L \) in theorem 1.1 is only used to prove the local existence of the solution to (1.1)–(1.2) and to obtain the smallness of \( (\rho(1, x) - 1, u(1, x)) \) in (6.11).

To prove theorem 1.1, we first solve an unsteady potential flow equation in the domain \( \Omega = \{ (t, x) : t \geq 0, |x| \leq R(t) \} \) with the initial-boundary conditions (1.2) (see the figure 2 below). Let \( \Phi(t, x) \) be the potential of velocity \( u = (u_1, u_2, u_3) \), i.e., \( u_i = \partial_i \Phi \) \((1 \leq i \leq 3)\). Then it follows from the Bernoulli law that

\[
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + h(\rho) = B_0, \tag{1.7}
\]

where \( h(\rho) = \frac{c_s^2(\rho)}{\gamma - 1} \) is the specific enthalpy, \( c_s(\rho) = \sqrt{P'(\rho)} \) is the local sound speed, \( \nabla = (\partial_1, \partial_2, \partial_3) \), \( B_0 = \frac{c_s^2(\rho_0)}{\gamma - 1} \) is the Bernoulli constant of a static state with the constant density \( \rho_0 \). By (1.7) and the implicit function theorem with \( h'(\rho) = \frac{c_s^2(\rho)}{\rho} > 0 \) for \( \rho > 0 \), the density function \( \rho(t, x) \) can be expressed as

\[
\rho = h^{-1} \left( B_0 - \partial_t \Phi - \frac{1}{2} |\nabla \Phi|^2 \right). \tag{1.8}
\]

where \( h^{-1} \) stands for the inverse function of \( h(\rho) \), and \( \nabla = (\partial_1, \partial_3) \). Substituting (1.8) into the first equation in (1.1) yields

\[
\partial_t (H(\nabla \Phi)) + \sum_{i=1}^{3} \partial_i (H(\nabla \Phi) \partial_i \Phi) = 0. \tag{1.9}
\]

In fact, for any \( C^2 \) solution \( \Phi \), (1.9) can be rewritten into the following second-order quasilinear equation

\[
\partial_t^2 \Phi + 2 \sum_{k=1}^{3} \partial_k \Phi \partial_k^2 \Phi + \sum_{i,j=1}^{3} \partial_i \Phi \partial_j \Phi \partial_i^2 \Phi - c_s^2(\rho) \Delta \Phi = 0. \tag{1.10}
\]

Denote the lateral boundary of \( \Omega \) by \( \partial \Omega = \{ (t, x) : t \geq 0, |x| = R(t) \} \). Then on \( \partial \Omega \),

\[
\sum_{i=1}^{3} \partial_i \Phi \cdot \frac{x_i}{|x|} = L. \tag{1.11}
\]

Due to the geometric property of \( \Omega \), it is convenient to work in the spherical coordinates \((r, \theta, \varphi)\):

\[
(x_1, x_2, x_3) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi), \tag{1.12}
\]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \), \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \varphi \leq \frac{\pi}{2} \).
Under the coordinate transformation (1.12), (1.10) becomes
\begin{align}
\frac{\partial^2_t \Phi}{r^2} + 2 \frac{\partial_t \Phi \partial_r \Phi}{r} + \frac{2}{r^2 \sin^2 \varphi} \partial_r \Phi \partial_r \Phi + \frac{2}{r^2} \partial_r \Phi \partial_r \Phi \\
+ \left( \frac{\partial_r \Phi}{r} - c^2(\rho) \right) \partial_r \Phi \\
+ \frac{1}{r^2} \left( \frac{\partial_r \Phi}{r} - c^2(\rho) \right) \partial_r^2 \Phi \\
+ \frac{2}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi + \frac{2 \partial_r \Phi \partial_r \Phi \partial_r \Phi}{r^2 \sin^2 \varphi} \partial_r \Phi \\
\frac{2}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi + \frac{2 \partial_r \Phi \partial_r \Phi \partial_r \Phi}{r^2 \sin^2 \varphi} \partial_r \Phi \\
- \frac{1}{r^2} \left[ 2c^2(\rho) + \frac{\partial_r \Phi}{r} \right] \partial_r \Phi + \frac{2}{r^4} \partial_r \Phi = 0.
\end{align}
(1.13)

Note that some coefficients in (1.13) have strong singularities near \( \varphi = 0 \). Consequently, as in [17] and [18], we rewrite (1.13) by introducing some smooth vector fields tangent to the sphere \( S^2 \).

Set
\begin{align}
Z_1 &= x_1 \partial_2 - x_2 \partial_1 = \partial_t, \\
Z_2 &= x_1 \partial_3 - x_3 \partial_1 = -\cos \varphi \cos \theta \partial_\theta - \sin \theta \partial_\varphi, \\
Z_3 &= x_3 \partial_2 - x_2 \partial_3 = -\cos \varphi \sin \theta \partial_\theta + \cos \theta \partial_\varphi.
\end{align}
(1.14)

Then it follows from direct computation that (1.13) can be written as
\begin{align}
\frac{\partial^2_t \Phi}{r^2} + 2 \frac{\partial_t \Phi \partial_r \Phi}{r} + \frac{3}{r^2} \partial_r \Phi \partial_r \Phi \\
+ \left( \frac{\partial_r \Phi}{r} - c^2(\rho) \right) \partial_r \Phi \\
+ \frac{2}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi + \frac{3}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi \\
+ \frac{1}{r^4} \partial_r \Phi \partial_r \Phi \partial_r \Phi \\
+ \frac{3}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi \\
+ \frac{3}{r^2} \partial_r \Phi \partial_r \Phi \partial_r \Phi \\
- \frac{2c^2(\rho)}{r} \partial_r \Phi = 0,
\end{align}
(1.15)

where \( \omega = \frac{1}{r} \), \( C_{ij}(\omega) = C_{ij}(\frac{1}{r}) \) and \( C_{ijk}(\omega) = C_{ijk}(\frac{1}{r}) \) are smooth functions of their arguments. Meanwhile, the boundary condition (1.11) becomes
\[ \partial_r \Phi = L, \quad \text{on} \quad \partial \Omega. \]
(1.16)

In addition, we impose the following initial perturbation:
\[ \Phi(0, x) = \frac{1}{2} L |x|^2 + \varepsilon \Phi_0(x), \quad \partial_t \Phi(0, x) = -\frac{1}{2} L^2 |x|^2 + \varepsilon \Phi_1(x), \]
(1.17)

where \( \varepsilon > 0 \) is a small parameter, \((\Phi_0(x), \Phi_1(x)) \in (H^3(S^0), H^4(S^0))\), and the initial-boundary value conditions (1.16)–(1.17) are compatible on \( S^0 \). Note that the initial data (1.17) can be replaced by \((\Phi(0, x), \partial_t \Phi(0, x)) = (\varepsilon \Phi_0(x), \varepsilon \Phi_1(x))\) when \( L > 0 \) is small. On the other hand, due to \( u_0 = \partial_t \Phi \) and (1.8), the initial conditions (1.17) can be realized by a small perturbation of the initial density and velocity of an irrotational flow.

**Theorem 1.2.** Under the above assumptions on the initial and boundary data, if \( \gamma \in \left[\frac{4}{3}, 1\right] \), then there exists a constant \( \varepsilon_0 \) depending on \( L, B_0 \) and \( \gamma \) such that problem (1.10) with (1.16)–(1.17) has a global solution \( \Phi(t, x) \in C([0, \infty), H^3(S^0) \cap C([0, \infty), H^4(S^0)) \) for \( \varepsilon < \varepsilon_0 \), where \( S_t = \{ x : r(t) \leq R(t) \} \). Moreover, \( \rho(t, x) > 0 \) and \( \lim_{t \to \infty} \rho(t, x) = 0 \) hold.

**Remark 1.7.** The linearized operator of the quasilinear wave equation (1.10) around the special expanding solution has the approximate form of
\begin{align}
\partial^2_t - \frac{\gamma}{(1 + L)^{3(\gamma - 1)}} \partial^2_t \partial^2_\varphi + 3 \frac{(\gamma - 1)}{1 + L} \partial_{\varphi},
\end{align}

On the other hand, if one considers the Cauchy problem of (1.10) for initial data as a small perturbation of a uniform constant density \( \rho_0 \) and velocity \((0, 0, q_0)\), that is,
\begin{align}
\partial^2_t \Phi + 2 \sum_{i=1}^3 \partial_i \Phi \partial^2_i \Phi + \partial_\varphi \Phi \partial^2_\varphi \Phi - c^2(\rho) \Delta \Phi = 0, \\
\Phi(t, x)|_{t=0} = \varepsilon \Phi_0(x), \quad \Phi(t, x)|_{t=0} = q_0 + \varepsilon \Phi_1(x), \quad x \in \mathbb{R}^3,
\end{align}
(1.18)

where \( \Phi(x) \in C_0^\infty(\mathbb{R}^3) \) \( (i = 0, 1) \), then (1.18) does not fulfill the ‘null-condition’ introduced in [19] and [20]. Therefore, according to the results obtained in [21–24] and [25], the classical solution to (1.18) blows up and then shock forms in finite time. Comparing this blowup result with theorem 1.1–1.2, the global existence of smooth solution to (1.10) together with a fixed wall condition comes from the rarefaction property of fluid.

We now give some remarks on the proof of theorem 1.2. Since the local solvability of problem (1.10) together with (1.16)–(1.17) is known as long as the vacuum does not appear, see, for example, [26], the proof of theorem 1.2 is based on the continuation argument. First of all, note that the linearized operator
\begin{align}
\mathcal{L} = \partial^2_t + \frac{2Lx}{R(t)} \partial_x^2 + \sum_{i,j=1}^3 \frac{L^2 x_i x_j}{R(t)^2} \partial^2_{x_i x_j} - \frac{\gamma}{R^{2+\gamma-1}(t)} \Delta
\end{align}

see (3.2), is different from the corresponding one in [18], which is
\begin{align}
\partial^2_t - \frac{1}{R^{2+\gamma-1}(t)} \partial^2_t + 2 \frac{\gamma \cdot \Delta}{R(t)} \partial_t,
\end{align}

The key ingredients in the analysis for the global existence and pointwise estimate are to obtain some weighted energy...
estimates by choosing some appropriate multiplier and anisotropic weights. For this, we need to
1. Show that the solution does not contain vacuum both on the boundary and inside the region;
2. Obtain the different time decay rates of the density and velocity of the solution when it tends to vacuum state at infinite time;
3. Fully use the Neumann-type boundary condition (1.16) on \( \Phi \) by applying the material derivative \( D_t = \partial_t + \frac{L_r}{R(t)} \partial_t \) on the solution, and estimate the radial derivatives and angular derivatives of \( \Phi \), together with some weighted Sobolev interpolation inequalities given in [27];
4. Introduce an anisotropic weighted Sobolev space for the solution \( \Phi \) to the linearized equation of (1.10) since \( D_t \Phi \) and \( \nabla \Phi \) have different decay rates. In this process, the chosen anisotropic Sobolev space should guarantee that the vacuum can not appear in any finite time but the vacuum state will certainly appear at infinity. This leads to a rather involved and complicated arguments for theorems 1.2 and 1.1.

The rest of the paper is organized as follows. In the next section, we will give some basic properties of the background solution and some preliminary weighted Sobolev interpolation inequalities. In section 3, we will reformulate problem (1.10) together with (1.16)–(1.17) by decomposing its solution as a sum of the background solution and a small perturbation so that its linearization can be studied clearly. In section 4, we will establish a uniform weighted energy estimate for the corresponding linear problem, where an appropriate multiplier is constructed. In section 5, the uniform higher order weighted estimates of \( \Phi \) are obtained by the analysis on the radial derivatives and angular derivatives of \( \Phi \), where the domain decomposition technique is applied.

Meanwhile, some delicate observations are given, and many complicated terms on the boundary will be treated very carefully. In section 6, we complete the proof of theorem 1.2 by applying the Sobolev embedding theorem and the continuation argument, and then theorem 1.1 follows from theorem 1.2 and the local existence result on the problem (1.1)–(1.2). In the last section, three future directions related to our problem are stated. In addition, some tedious but basic computations in lemma 5.1 are put in the appendix.

2. Background solution and some preliminaries

In this section, we analyze the background solution to (1.10) with (1.16)–(1.17) when the initial data (1.17) are given:

\[
\hat{\Phi}(0, x) = \frac{1}{2} L|x|^2, \quad \partial_t \hat{\Phi}(0, x) = -\frac{1}{2} L^2 |x|^2. \quad (2.1)
\]

In this case, the density \( \rho(x) \) and velocity \( u(t, x) = \nabla_t \Phi(t, x) \) in \( \Omega \) take the form: \( \rho(t, x) = \hat{\rho}(t, r), \ u(t, x) = \frac{\nabla \hat{U}}{r}(t, r) \). Consequently, problem (1.10) with (1.16) and (2.1) is equivalent to

\[
\begin{align*}
&\rho^2 \partial_t \hat{\rho} + \partial_t (\rho^2 \hat{U}) = 0, \\
&\partial_t (\rho^2 \hat{U}) + \partial_t (\rho^2 \hat{U}) + \rho^2 \hat{P} = 0, \\
&\hat{\rho}(0, r) = 1, \quad \hat{U}(0, r) = \frac{L_r}{R(t)}. 
\end{align*}
\]

One can easily check that (2.2) has a solution

\[
\hat{\rho}(t, r) = \frac{1}{R^3(t)}, \quad \hat{U}(t, r) = \frac{L_r}{R(t)}. \quad (2.3)
\]

Then for \( 1 < \gamma < \frac{4}{3} \), it follows from \( \alpha = \partial_t \Phi, \ (1.7) \) and (2.3) that (1.10) with (1.16) and (2.1) has a solution

\[
\begin{align*}
\frac{\alpha}{\gamma} (\gamma - 1)(4 - 3\gamma)L &+ B_1 t + \frac{L_r^2}{2(1 + L)} \\
- \frac{\gamma}{(\gamma - 1)(4 - 3\gamma)L} &+ \frac{l}{(1 + L)^{\beta/3}} \quad (2.4)
\end{align*}
\]

where \( B_0 = \frac{\gamma}{\gamma - 1} \). Next, we include the weighted Sobolev interpolation inequality from [27] that will be used in lemma 2.4.

Lemma 2.1. Suppose \( s, \tau, p, \alpha, \beta, q, a \) are real numbers, and \( j \geq 0, m > 0 \) are integers, satisfying

\[
\begin{cases}
p, q \geq 1, \frac{j}{m} \leq a \leq 1, s > 0, \\
\frac{1}{s} + \frac{1}{n} > 0, \frac{1}{p} + \frac{\alpha}{n} > 0, \frac{1}{q} + \frac{\beta}{n} > 0, \\
m - j - \frac{n}{p} \quad \text{is not a non-negative integer.}
\end{cases}
\]

There exists a positive constant \( C \) such that the following inequality holds for all \( v \in C_{0}^{\infty}(\mathbb{R}^n) \):

\[
||x|^s \nabla_x^\tau v||_{L^\infty} \leq C ||x|^p \nabla_x^\alpha \nabla_x^\beta v||_{L^p} ||x|^q ||_{L^q}^{\tau}, \quad (2.6)
\]

if and only if the following conditions hold:

\[
\frac{1}{s} + \frac{\tau - j}{n} = a \left( \frac{1}{p} + \frac{\alpha}{n} \right) + (1 - a) \left( \frac{1}{q} + \frac{\beta}{n} \right)
\]

with \( \tau \leq a \alpha + (1 - a) \beta; \quad (2.7) \)

if \( \frac{1}{q} + \frac{\beta}{n} = \frac{1}{p} + \frac{\alpha - m}{n} \), then

\[
\alpha (\alpha - m) + (1 - a) \beta + j \leq \tau; \quad (2.8)
\]

if \( a = \frac{j}{m} \), then

\[
\tau = a \alpha + (1 - a) \beta. \quad (2.9)
\]

Corollary 2.1. For the domain \( D \) (\( = \Omega_0 \) or \( \Omega \)) defined in section 1, if a function \( u(x) \in C^{\infty}(\mathbb{R}^n) \) and

\[
u_{1/0} \in H^\infty_n = 0, \quad (2.10)
\]
where \( H = \sqrt{R(T)^2 + \left( T + \frac{1}{L} \right)^2} > 0 \) is a constant, \( y = (t, x) \) and \( |y| = \sqrt{t^2 + |x|^2} \), then

(i) (2.6) still holds under the restriction (2.5) and (2.7)–(2.9), moreover, the constant \( C \) on the right-hand side of (2.6) is independent of \( T \).

(ii) When \( m = 2, 1 < \gamma < \frac{5}{3} \) and \( 0 < \delta < 3\gamma \), for \( \mu > 0 \),

\[
\begin{align*}
||y|^{2\mu+2-\delta} \nabla u|_{L^3(Q)} & \leq C ||y|^{\mu+1-\delta} \nabla^2 u|_{L^3(Q)} ||y|^{\mu+1-\delta} u |_{L^3(Q)}, \\
||y|^{3\gamma-3-\delta} \nabla u|_{L^3(Q)} & \leq C ||y|^{\gamma-3-\delta} \nabla^2 u|_{L^3(Q)} ||y|^{\gamma-3-\delta} u |_{L^3(Q)},
\end{align*}
\]

from (2.6) with \( \tau = \alpha = \beta = \frac{\gamma - \delta - 3}{2} \). And this completes the proof of the corollary. \( \square \)

As in [18], in order to apply lemma 2.1 or corollary 2.3 to derive some weighted Sobolev inequalities in \( Q \) without the restriction (2.10), we need to establish an extension result as stated in

**Lemma 2.2.** Set \( y = (t, x), \exists Q_T = \{y : 0 < t < T, |y| \leq R(t)\} \) for \( T > 1 \) and \( D_\delta = \{y : (t + \frac{1}{L})^2 + |x|^2 \leq S^2, t \geq 0, |x| \leq R(t)\} \) for \( S > 1 + \frac{1}{L} \).

If \( u(t, x) \in C^3(Q_T) \) and \( R(t)^3 \nabla_\alpha u \in L^p(Q_T) \) (\(|\alpha| \leq 3\)) with some \( \beta \in \mathbb{R} \), then there exists an extension \( Eu = u \) in \( Q_T \), \( Eu|_{D_\delta} \equiv 0 \) and

\[
|R(t)^3 \nabla_\alpha (u|_{D_\delta})| \leq C |R(t)^3 u|_{L^p(Q_T)}', \quad |R(t)^3 \nabla_\alpha (Eu|_{D_\delta})| \leq C \sum_{|\beta| \leq |\alpha|} |R(t)^{3-|\alpha|+|\beta|} \nabla_\beta^\alpha u|_{L^p(D_{\delta})},
\]

where \( H = \sqrt{R(T)^2 + \left( T + \frac{1}{L} \right)^2} \), and \( C > 0 \) is independent of \( T \).

**Proof.** Let \( \hat{E} \) be an extension operator defined by

\[
(\hat{E}u)(t, x) = \sum_{j=1}^4 \lambda_j u \left( \frac{4j(T-t)}{4L^2 + 4L + 1}, \frac{2}{L} x \right), \quad T < t \leq H = \frac{1}{L}, \quad |x| \leq \sqrt{H^2 - \left( \frac{1}{L} + \frac{1}{L} \right)^2}.
\]

where \( \nabla = \nabla_x \) and \( \nabla^2 = \nabla^2_x \).

**Proof.** (i) The proof is completely parallel to that of lemma 2.1 (one can check the details in [27]), then we omit it here.

(ii) In (2.5) and (2.7)–(2.9) of lemma 2.1, set \( s = 4, p = 2, q = \infty, a = \frac{1}{2} \) and \( j = 1, m = 2, n = 4 \), one can conclude that (2.11) and (2.12) come from (2.6) with \( \tau = \frac{2\mu + 2 - \delta}{4}, \alpha = \frac{\mu + 1 - \delta}{2}, \beta = \frac{\mu + 1}{2} \) and \( \tau = \frac{2\mu + 3 - \delta}{4}, \alpha = \frac{\mu + 3 - \delta}{2}, \beta = \frac{\mu + 3 - \delta}{2} \), respectively. (2.13) and (2.14) are from (2.6) with \( \tau = \frac{2\mu + 2 - 3|\alpha| - 1 - \delta}{4}, \alpha = \frac{2\mu + 3 - 3|\alpha| - 1}{2}, \beta = \frac{\mu + 1 - \delta}{2} \) and \( \tau = \alpha = \beta = \frac{\mu + 3 - \delta}{2} \), respectively. (2.15) is

\[
||y|^{6\gamma-6-\delta} \nabla u|_{L^3(Q)} \leq C ||y|^{\mu+1-\delta} \nabla^2 u|_{L^3(Q)} ||y|^{\mu+1-\delta} u |_{L^3(Q)}.
\]
Consequently, one has that for $(t, x) \in D_t$, 
\[ C_t R(t) \lesssim \left(1 + \frac{1}{L} \right)^2 + |x|^2 \lesssim C_t R(t). \]  
(2.17)

This yields for $|\alpha| \leq 3$
\[ |R(t)^{3} \hat{E} u|_{L^{\infty}(\mathbb{R}^3)} \lesssim |C R(t)^{3} | \hat{u}|_{L^{\infty}(\mathbb{R}^3)}, \]
\[ |R(t)^{3} \nabla_{x}^{\alpha} \hat{E} u|_{L^{2}(\mathbb{R}^3)} \lesssim |C R(t)^{3} \nabla_{x}^{\alpha} u|_{L^{2}(\mathbb{R}^3)}. \]
(2.18)

Next, we construct the extension operator $E$ starting from the operator $\hat{E}$. In terms of the geometric property of $D_t$, it is convenient to use the spherical coordinates. Let $x_0 = t + \frac{1}{L}$, and
\[ s^2 = \sum_{i=0}^{3} x_i^2, \quad \omega_i = \frac{x_i}{s}, \quad i = 0, 1, 2, 3. \]
Denoting by $\tilde{u}(\omega, \omega) = u(s \omega)$ with $\omega = (\omega_0, ..., \omega_3)$. Let $\tilde{E}$ be an extension operator defined by
\[ (\tilde{E}u)(\omega, \omega) = \sum_{j=1}^{2} \nu_j \tilde{E}u = \sum_{j=1}^{2} \nu_j \tilde{E}u + j(1 - H - j, \omega), \quad (t, x) \in D_{s} \setminus \partial D_{s}. \]
(2.19)

where $\sum_{j=1}^{2} (-1)^j \nu_j = 1$ for $k = 0, 1, 2, 3$. Note that
\[ 1 \leq \frac{s}{H + j(1 - H - j, \omega)} \leq \frac{9}{8} \quad \text{for} \quad H \leq s \leq \frac{9}{8} \quad \text{and} \quad 0 \leq j \leq 3. \]

This together with (2.18) yields
\[ |(R(t)^{3} \tilde{E} u)|_{L^{\infty}(\mathbb{R}^3)} \lesssim |C R(t)^{3} \tilde{u}|_{L^{\infty}(\mathbb{R}^3)}, \]
\[ |R(t)^{3} \nabla_{x}^{\alpha} \tilde{E} u|_{L^{2}(\mathbb{R}^3)} \lesssim |C R(t)^{3} \nabla_{x}^{\alpha} \tilde{u}|_{L^{2}(\mathbb{R}^3)}. \]
(2.20)

Choose a $C^\infty$ smooth function $\eta(s)$ with $\eta(s) = 1$ for $s \leq 1$ and $\eta(s) = 0$ for $s > \frac{9}{8}$ and set
\[ E u(t, x) = \eta \left( \frac{s}{H} \right) \tilde{E} u. \]
Then $E u$ satisfies (2.16), and thus lemma 2.2 is proved.

The $Z$-fields introduced in (1.14), as shown in [17], have the following properties.

Lemma 2.3.
(i) $[Z_1, Z_2] = -Z_1[Z_2, Z_3] = -Z_1[Z_1, Z_3] = -Z_2$,
(ii) $[Z_i, \partial_t] = 0, \quad Z_i = 0, \quad [Z_i, \partial_r] = 0$.
(iii) $\nabla f \cdot \nabla g = \partial_t f \cdot \partial_t g + \frac{1}{r^2} \sum_{i=1}^{3} Z_i f \cdot Z_i g$

for any $C^1$ smooth functions $f$ and $g$.
(iv) $|Z| \leq |\nabla|$, for any $C^1$ smooth function $v$, here and below $Z \in [Z_1, Z_2, Z_3]$. 
(v) $\partial_0 = \frac{\partial}{r} + \frac{\partial}{r^2} Z_1 - \frac{\partial}{r^2} Z_3, \quad \partial_2 = -\frac{\partial}{r^2} Z_1 + \frac{\partial}{r^2} Z_3 + \frac{\partial}{r} Z_2$.

Remark 2.1. If $u \in C^m(\mathbb{R}^3)$ with $m \in \mathbb{N}$, then by lemma 2.3 we have
\[ |\nabla^m u| \sim |\partial_r^m u| + \frac{|\partial_r^{m-1} Z u|}{r} + \frac{|\partial_r^{m-2} Z Z u|}{r^2} + \cdots \]
(2.18)

As a direct consequence of lemma 2.2, we have the following weighted inequalities, which will be used often in section 5.

Lemma 2.4. Define $D_t = \partial_t + \frac{Lr}{R(t)} \partial_r$ and $S_0 = R(t) D_t$. If

$1 < \gamma < \frac{5}{3}, \quad 0 < \delta \leq \frac{3}{5}(\gamma - 1), \quad u(y) \in C^4(Q_T) Q_T = \Omega_0 \cap \{t \leq T\}$, \quad $\Omega \cap \{t \leq T\}$ with $y = (t, x)$, then there exists a generic positive constant $C$ independent of $T$ such that

\[ \begin{align*}
&\text{(i)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(ii)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(iii)} \quad \left| \int R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(iv)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(v)} \quad \left| \int R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(vi)} \quad \left| \int R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(vii)} \quad \left| \int R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}. \\
\end{align*} \]

Proof. Let $E$ be the extension operator given in lemma 2.2. Then we have
\[ \begin{align*}
&\text{(i)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla E S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(ii)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla E S_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&\text{(iii)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&(\text{Applying (2.11) for } E S_0 u); \\
&\text{(iv)} \quad |R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)} \lesssim |C R(t)^{\frac{2}{9} + \delta - \gamma} \nabla Z_0 u|_{L^{\frac{9}{8}}(\mathbb{R}^3)}; \\
&(\text{By Lemma 2.2}).
\end{align*} \]
3. Reformulation of problem (1.10) with (1.16) and (1.17)

Firstly, we state a local solvability result on problem (1.10) with (1.11) and (1.17).

**Lemma 3.1.** There exists a $T_0 > 0$ such that the problem (1.10) with (1.16) and (1.17) has a local solution $\Phi(t,x) \in C([0,T_0],H^2(S_t)) \cap C^1([0,T_0],H^4(S_t))$ with $S_t = \Omega \cap \{x : |x| = R(t)\}$. Moreover,

$$
\|\Phi(t,x) - \hat{\Phi}(t,x)\|_{C([0,T_0],H^2(S_t))} + \|\Phi(t,x) - \hat{\Phi}(t,r)\|_{C^1([0,T_0],H^4(S_t))} \leq C\varepsilon,
$$

where $\hat{\Phi}(t,x)$ is given in (2.4).

**Proof.** The quasilinear equation (1.10) is strictly hyperbolic with respect to $t$. Thus, by the standard Picard iteration as in [26], one can derive that lemma 3.1 holds. □

Lemma 3.1 gives the local existence of solution $\Phi$ to (1.10) in the standard Sobolev space $C([0,T_0],H^2(S_t)) \cap C^1([0,T_0],H^4(S_t))$. Furthermore, by the usual energy estimate for hyperbolic equations, one has

$$
\sum_{k=0}^{3} \int_{S_{T_0}} (\nabla_{x}^{k} D_{t} \Phi)^{2} + (\nabla_{x}^{k} \nabla_{x} \Phi)^{2})dS
$$

$$
+ \sum_{k=0}^{3} \int_{\Omega_{T_0}} (\nabla_{x}^{k} D_{t} \Phi)^{2} + (\nabla_{x}^{k} \nabla_{x} \Phi)^{2})dt \leq C(T_0)\varepsilon^2,
$$

where $\Phi = \Phi - \hat{\Phi}$, $C(T_0)$ is a constant depending on $T_0$. In this case, one cannot use the continuation argument to directly get the global existence of solution $\Phi$.

Next, we reformulate (1.10) with (1.16)–(1.17). Let $\hat{\Phi} = \Phi - \hat{\Phi}$. Then (1.10) can be reduced to

$$
\mathcal{L}\hat{\Phi} = \hat{f} \quad \text{in } \Omega, \quad (3.1)
$$

where

$$
\mathcal{L}\Phi = \partial_{t}^{2}\hat{\Phi} + 2\sum_{i=1}^{3} \partial_{x}^{i}\partial_{t}^{i}\hat{\Phi} + \sum_{i,j=1}^{3} \partial_{x}^{i}\partial_{\Phi}^{j}\partial_{t}^{i}\hat{\Phi} + 2\sum_{i=1}^{3} \partial_{x}^{i}\partial_{\Phi}^{i}\Phi + 2\sum_{i=1}^{3} \partial_{x}^{i}\partial_{\Phi}^{i}\partial_{t}^{i}\hat{\Phi}
$$

$$
- \varepsilon^{2}\Delta\hat{\Phi} + \frac{3L(\gamma - 1)}{R(t)}(\partial_{t}\Phi + \sum_{i=1}^{3} \partial_{x}^{i}\partial_{\Phi}^{i}\Phi),
$$

and

$$
\hat{f} = \sum_{i=1}^{3} f_{0i} \partial_{x}^{i}\hat{\Phi} + \sum_{1 \leq i < j \leq 3} f_{ij} \partial_{x}^{i}\partial_{x}^{j}\hat{\Phi} + \sum_{i=1}^{3} f_{0i} \partial_{x}^{i}\hat{\Phi} + f_{0}
$$

$$
(3.2)
$$

(vi) and (vii) follow from the definition of $Z$-fields in (1.14) and lemma 2.3. □

with
Here we point out that \( f \) is actually an error function including the quadratic terms \( O(|\nabla \Phi|^2 + |\nabla \Phi||\nabla^2 \Phi|) \) and third-order terms \( O(|\nabla^3 \Phi|^2) \).

For later analysis, we use Z-fields to rewrite \( \mathcal{L} \Phi \) as follows:

\[
\mathcal{L} \Phi = \partial^2_t \Phi + 2\partial_t \Phi \partial^2_t \Phi + ((\partial_t \Phi)^2 - c^2) \partial^2_t \Phi \\
- \frac{c^2}{r^2} \sum_{i=1}^3 Z^2_i \Phi + \frac{\gamma - 1}{r} \left( r \partial^2_t \Phi + 2 \partial_t \Phi \partial_t \Phi + \right) \partial_t \Phi \\
+ \frac{1}{r} (2r \partial^2_t \Phi + (\gamma + 1) r \partial_t \Phi \partial^2_t \Phi) \\
+ 2(\gamma - 1)(\partial_t \Phi)^2 - 2 \varepsilon_3^2) \partial_t \Phi.
\]

(3.3)

On the lateral boundary \( \partial \Omega \) of \( \Omega \), \( \Phi \) satisfies

\[
\partial_t \Phi = 0.
\]

(3.4)

In addition, we have the following initial data of \( \Phi \) from (1.17)

\[
\Phi(0, x) = \varepsilon \Phi_0(x), \quad \partial_t \Phi(0, x) = \varepsilon \Phi_1(x).
\]

(3.5)

**Remark 3.1.** By (2.3) or (2.4), we know that \( \mathcal{L} \Phi \) in (3.2) has the following form at the \( t \)-axis:

\[
\partial^2_t \Phi - \frac{\gamma}{R^{3(\gamma - 1)}(t)} (\partial^2_t \Phi + \partial^2_\Phi \Phi + \frac{3L(\gamma - 1)}{R(t)} \partial_t \Phi),
\]

(3.6)

which is strictly hyperbolic but degenerate as \( t \to \infty \). On the other hand, the operator \( \mathcal{L}_t \Phi = \partial^2_t \Phi + \frac{2L}{R(t)} \partial^2_\Phi + \sum_{i=1}^3 L \partial^2_i \Phi - \frac{\gamma}{R^{3(\gamma - 1)}(t)} \Delta + \frac{2L}{R(t)} \partial_t \Phi \) in (3.2) is different from [18] where the corresponding linear operator is

\[
\partial^2_t \Phi - \frac{1}{R^{5(\gamma - 1)}} \Delta + \frac{L}{R(t)} \partial_t \Phi.
\]

Recently, with respect to the semilinear wave equations

\[
\partial^2_t u - \Delta u + \mu \frac{u}{(1 + u)^\alpha} \partial_t u = f(u),
\]

(3.7)

where \( \mu > 0 \) and \( \alpha > 0 \) are suitable constants, there have been extensive works on the global existence or blowup results for different nonlinear function \( f(u) \), cf [28–30] and the references therein. However, for the critical exponent \( \alpha = 1 \) and near the critical value \( \mu = 1 \), there are still some open questions on the blowup or global existence of solutions to (3.7). Here, (3.6) corresponds to the critical case of (3.7) when \( L = 1 \) and \( \gamma \) is close to \( \frac{4}{3} \).

4. The first-order weighted energy estimate and reformulation of (3.2)–(3.3)

In this section, we derive the weighted energy estimate of \( \nabla \Phi \) for the linear part (3.3) together with (3.4)–(3.5). Set \( \Omega_T = \Omega \cap \{0 < t < T\} \) and \( S_T = \{t = T\} \). On the other hand, the operator

\[
\frac{R(t)}{\varepsilon^2} \int_{S_T} (D_t \Phi)^2 dx + T^{n-3(\gamma - 1)} \int_{S_T} (\Delta \Phi)^2 dx \\
+ C \int_{S_T} (R(t)^{n-1-\frac{1}{2}} (D_t \Phi)^2 + R(t)^{n-1-3(\gamma - 1)} (\Delta \Phi)^2) \, dx \\
\leq \int_{S_T} \mathcal{L} \Phi \cdot \mathcal{M} \Phi \, dx + C\varepsilon^2.
\]

(4.1)

where \( D_t = \partial_t + \sum_i \delta_i \partial_i \partial_t \Phi = \partial_t + \frac{L}{R(t)} \partial_t \) is the material derivative, \( C > 0 \) is a generic positive constant depending only on the initial data, and \( \varepsilon > 0 \) is a small fixed constant.

**Remark 4.1.** The choice of \( \mu = 6\gamma - 9 \in (4.1) \) is necessary because of the following two reasons: first, to guarantee the positivity of \( III \) in (4.2), \( \mu \leq 6\gamma - 9 \); second, by the Bernoulli law (1.7), we have \( c^2(\mu) = c^2(\hat{\mu}) - (\gamma - 1) \). \( D_t \Phi = \frac{\gamma - 1}{2} |\nabla \Phi|^2 \). Notice that only the estimate of

\[
|D_t \Phi| \leq C \varepsilon R(t)^{\mu+3+2} \text{ and } |\Delta \Phi| \leq C \varepsilon R(t)^{\mu-3(\gamma - 1) + 3}
\]

can be obtained as shown in sections 5 and 6. On the other hand, \( c^2(\mu(t)) = \gamma R(t)^{3(\gamma - 1)} \) holds. Therefore, in order to guarantee the absence of vacuum for any finite time \( t \in \Omega \), we need to choose the constant \( \mu \) such that

\[
-\frac{\mu + 3}{2} \leq -3(\gamma - 1),
\]

that is, \( \mu \geq 6\gamma - 9 \). In combination, \( \mu = 6\gamma - 9 \).

**Proof.** Choosing \( \mathcal{M} \Phi = R(t)^\mu (a(t, r) \partial_t \Phi + b(t, r) \partial_t \Phi) \), where the non-negative functions \( a(t, r) \) and \( b(t, r) \) will be determined later, then

\[
\int_{S_T} \mathcal{L} \Phi \cdot \mathcal{M} \Phi \, dx = I + II + III.
\]

(4.2)
where
\[
I = \int_{\mathcal{S}_r} \frac{R(t)^\mu}{2\sqrt{1 + L^2}} (L(t, r) - b(t, r)) \\
\times \left( \partial_t \Phi \right)^2 - \frac{\partial_{r}^2}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \\
+ 2((L^2 - \ell^2)a(t, r) - L b(t, r)) \partial_t \Phi \partial_r \Phi \\
+ L(L^2 - \ell^2)(b(t, r)(\partial_t \Phi)^2) dS,
\]
\[II = H_1 - H_2,
\]
\[III = \int_{\mathcal{S}_r} \left( A(t, r)(\partial_t \Phi)^2 + B(t, r)\partial_t \Phi \partial_r \Phi \right) + C(t, r)(\partial_t \Phi)^2 + \frac{1}{r^2} D(t, r) \sum_{i=1}^{3} (Z_i \Phi)^2 \right) dtdx,
\]
with
\[
H_1 = \int_{\mathcal{S}_r} R(t)^\mu \frac{1}{2} a(t, r)(\partial_t \Phi)^2 + b(t, r)\partial_t \Phi \partial_r \Phi \\
+ (b(t, r)\partial_t \Phi - \frac{1}{2}(\partial_t \Phi)^2 - \ell^2)a(t, r))(\partial_t \Phi)^2 \\
+ \frac{\partial_{r}^2}{r^2} a(t, r) \sum_{i=1}^{3} (Z_i \Phi)^2 \right) dx,
\]
\[
H_2 = \int_{\mathcal{S}_r} \left[ \frac{1}{2} a(0, r)(\partial_t \Phi)^2 + b(0, r)\partial_t \Phi \partial_r \Phi \right. \\
\left. - \frac{1}{2}(\partial_t \Phi)^2 - \ell^2)a(0, r))(\partial_t \Phi)^2 \\
+ \frac{\partial_{r}^2}{r^2} a(0, r) \sum_{i=1}^{3} (Z_i \Phi)^2 \right] dx \bigg|_{r=0},
\]
\[
A(t, r) = \frac{1}{2} \partial_t (R(t)^\mu a(t, r)) - R(t)^\mu \partial_t (a(t, r) \partial_t \Phi) \\
- 2R(t)^\mu r^{-1} a(t, r) \partial_t \Phi \\
+ r^{-2} \partial_t \left( \frac{1}{2} r^2 R(t)^\mu b(t, r) \right) + (\gamma - 1) (\partial_t \Phi)^3 \\
+ \frac{1}{r} \partial_t \Phi R(t)^\mu a(t, r),
\]
\[
B(t, r) = -R(t)^\mu \partial_t ((\partial_t \Phi)^2 - \ell^2)a(t, r) \\
- \frac{2}{r} R(t)^\mu a(t, r)(\partial_t \Phi)^2 - \ell^2 - \partial_t (R(t)^\mu b(t, r)) \\
+ R(t)^\mu a(t, r)(2\partial_t \Phi + (\gamma + 1) \partial_t \Phi \partial_t \Phi \\
+ \frac{2(\gamma - 1)}{r} (\partial_t \Phi)^2 - \frac{2}{r} \partial_t \Phi \\
+ (\gamma - 1) (\partial_t \Phi)^3 + \frac{2}{2} \partial_t \Phi R(t)^\mu b(t, r),
\]
\[
C(t, r) = \frac{1}{2} \partial_t ((\partial_t \Phi)^2 - \ell^2) R(t)^\mu a(t, r) \\
- r^{-2} \partial_t \left( \frac{1}{2} R(t)^\mu b(t, r) r^{-2} (\partial_t \Phi)^2 - \ell^2 \right) \\
- \partial_t (R(t)^\mu b(t, r) \partial_t \Phi) + R(t)^\mu b(t, r) \\
\times (2(\partial_t \Phi + (\gamma + 1) \partial_t \Phi \partial_t \Phi \\
+ \frac{2(\gamma - 1)}{r} (\partial_t \Phi)^2 - \frac{2}{r} \partial_t \Phi \\
+ \partial_t \Phi R(t)^\mu b(t, r) r^{-2} (\partial_t \Phi)^2 - \ell^2 \right) \right),
\]
\[
D(t, r) = -\frac{1}{2} \partial_t (\ell^2 R(t)^\mu a(t, r)) - \partial_t \left( \frac{1}{2} R(t)^\mu b(t, r) r^{-2} \right).
\]
In view of the boundary condition (3.4), we have
\[
I = \int_{\mathcal{S}_r} \frac{1}{2\sqrt{1 + L^2}} (L(t, r) - b(t, r)) \\
\times \left( \partial_t \Phi \right)^2 - \frac{\partial_{r}^2}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) dS.
\]
To guarantee $I \geq 0$, it requires that on the boundary $r = R(t),$
\[
b(t, R(t)) = La(t, R(t)).
\]
In this case,
\[
I = 0.
\]
Next, we consider $II$. To fulfill $H_1 > 0$, it requires that on $t = T$
\[
\begin{cases}
(a(t, r) > 0, \\
b(t, r)^2 - a(t, r)(2b(t, r)\partial_t \Phi - ((\partial_t \Phi)^2 - \ell^2)a(t, r)) \leq 0.
\end{cases}
\]
This means that on $t = T$,
\[
\begin{cases}
a(t, r) > 0, \\
\partial_t \Phi - \ell \leq \frac{b(t, r)}{a(t, r)} \leq \partial_t \Phi + \ell.
\end{cases}
\]
Thus, combining (4.3) and (4.6) yields
\[
b(t, r) = \partial_t \Phi - a(t, r).
\]
On the other hand, by $\ell^2 = \gamma R(t)^{-2(\gamma - 1)}$, we have
\[
II_1 = \int_{\mathcal{S}_r} \frac{1}{2} R(t)^\mu a(t, r)((\partial_t \Phi + \partial_t \Phi \partial_t \Phi)^2 \\
+ \ell^2((\partial_t \Phi)^2 + \frac{1}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2) \right) dS \\
\geq C \int_{\mathcal{S}_r} a(T, r)(R(T)^\mu (D_t \Phi)^2 + R(T)^{\mu - 3(\gamma - 1)} \\
\times \left( (\partial_t \Phi)^2 + \frac{1}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) dS.
\]
It follows from initial data (3.5) that
\[
|II_2| \leq C \ell^2.
\]
Finally, we deal with $III$. In fact, we only need to choose $a(t, r) \equiv a(t)$. In this case, direct computation yields
If we choose \( a(t) > 0 \) and \( a'(t) < 0 \), then
\[
A(t, r) > 0, \quad B(t, r)^2 - 4A(t, r)C(t, r) < 0, \quad D(t, r) > 0.
\]

From this, we naturally set
\[
\mu = 6\gamma - 9, \quad a(t) > 0 \quad \text{and} \quad a'(t) < 0.
\]

If we choose
\[
a(t) = 1 + R(t)^{-\delta} \quad \text{with} \quad \delta > 0,
\]
then
\[
III = \int_{\Omega} \left\{ \frac{\tilde{F}}{2} R(t)^{\mu-1-\delta} (\partial_r \Phi + \partial_r \phi) (\partial_r \Phi + \partial_r \phi) + \frac{\gamma}{2} R(t)^{\mu-3\gamma-1} (L(5 - 3\gamma) a(t) - R(t)a'(t)) \times \left( (\partial_r \Phi)^2 + \frac{3}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) \right\} \, dt \, dx
\]
\[
\geq C \int_{\Omega} \left\{ R(t)^{\mu-1-\delta} (D_i \Phi)^2 + R(t)^{\mu-1-3\gamma-1} \times \left( (\partial_r \Phi)^2 + \frac{3}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) \right\} \, dt \, dx.
\]

Substituting (4.4), (4.8)–(4.9) and (4.13) into (4.2) yields
\[
R(T)^{\mu} \int_{S_r} (D_i \Phi)^2 \, dS + R(T)^{\mu-3\gamma-1} \times \int_{S_r} \left( (\partial_r \Phi)^2 + \frac{3}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) \, dS
\]
\[
+ C \int_{\Omega} \left( R(t)^{\mu-1-\delta} (D_i \Phi)^2 + R(t)^{\mu-1-3\gamma-1} \times \left( (\partial_r \Phi)^2 + \frac{3}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 \right) \right) \, dt \, dx
\]
\[
\leq \int_{\Omega} \mathcal{L} \Phi \cdot \mathcal{M} \Phi \, dt \, dx + C \varepsilon^2.
\]

This together with lemma 2.1 gives (4.1), and it completes the proof of the lemma.

Note that the material derivative \( D_t \) plays a crucial role in the energy estimate (4.1). Then it is necessary that we rewrite equation (3.2) and formula (3.3) by using the operator \( D_t \) as follows:
\[
\begin{aligned}
\mathcal{L} \Phi & = D_t^2 \Phi - \varepsilon^2 \Delta \Phi + \frac{3l(\gamma - 1)}{R(t)} D_t \Phi, \\
\dot{f} & = \sum_{i=1}^{3} \frac{\tilde{F}}{2} \partial_i \Phi + \sum_{1 \leq i < j \leq 3} \tilde{F}_{ij} \partial_i^2 \Phi + \sum_{i=1}^{3} \tilde{F}_i \partial_i^2 \Phi + \tilde{F}_0
\end{aligned}
\]
where
\[
\begin{aligned}
\tilde{F}_0 & = -2 \partial_i \Phi, \\
\tilde{F}_{ij} & = - \partial_i \Phi \partial_j \Phi, \\
\tilde{F}_{ii} & = - (\gamma - 1) (D_i \Phi + \frac{1}{2} \sum_{j=1}^{3} (\partial_j \Phi)^2) - \frac{(3\gamma - 5) L}{2 R(t)} \sum_{i=1}^{3} (\partial_i \Phi)^2.
\end{aligned}
\]
Especially, in the spherical coordinates, (4.15) has the form
\[
\mathcal{L} \Phi = D_t^2 \Phi - \varepsilon^2 \left( \partial_r \Phi + \frac{1}{r^2} \sum_{i=1}^{3} (Z_i \Phi)^2 + \frac{2}{r} \partial_r \Phi \right) + \frac{3l(\gamma - 1)}{R(t)} D_t \Phi = \dot{f},
\]
Here, we point out that the precise expression of \( \hat{f} \) is useful in the higher order energy estimates of \( \Phi \), cf section 5.

5. Higher order weighted energy estimates of \( \Phi \)

In this section, we will derive the higher order energy estimates of \( \Phi \) to (3.1) with (3.4)–(3.5). For this, we need to take care of the difficulties coming from the Neumann boundary condition (3.4), the asymptotic degeneracy of some coefficients in (4.16), and the different decay rates of \( D_r \Phi \) and \( \nabla_r \Phi \).

**Theorem 5.1.** Let \( \Phi \in C^4(\Omega) \) be the solution to (3.1) with (3.4)–(3.5), and assume with some \( \delta > 0 \) and \( 1 < \gamma < \frac{4}{3} \):

\[
\begin{align*}
\| \nabla \Phi \| & \leq M e^{-R(t)^{-3(\gamma-1)+\frac{\delta}{2}}}, \\
\| R(t) \nabla D_r \Phi \| & \leq M e^{-R(t)^{-3(\gamma-1)+\frac{\delta}{2}}}, \\
\| R(t) \nabla^2 \Phi \| & \leq M e^{-R(t)^{-3(\gamma-1)+\frac{\delta}{2}}},
\end{align*}
\]

Then for sufficiently small \( \varepsilon > 0 \) and \( 0 \leq k \leq 2 \), we have

\[
\int_{\Omega_r} (R(t))^{\mu+2k} (\nabla_{t,x} D_r \Phi)^2 + R(T)^{\mu-3(\gamma-1)+2k} (\nabla_{t,x} \nabla \Phi)^2 \right) dS + \int_{\Omega_0} (R(t))^{\mu-1-\delta+2k} (\nabla_{t,x} D_r \Phi)^2 \\
+ R(t)^{\mu-3(\gamma-1)+2k} (\nabla_{t,x} \nabla \Phi)^2 \right) dt dx \leq C \varepsilon^2,
\]

and

\[
\int_{\Omega_r} (R(t))^{\mu-\delta+6} (\nabla_{t,x} D_r \Phi)^2 \\
+ R(T)^{\mu-3(\gamma-1)+6} (\nabla_{t,x} \nabla \Phi)^2 \right) dS + \int_{\Omega_0} (R(t))^{\mu+5-\delta} (\nabla_{t,x} D_r \Phi)^2 \\
+ R(t)^{\mu-5-3(\gamma-1)} (\nabla_{t,x} \nabla \Phi)^2 \right) dt dx \leq C \varepsilon^2,
\]

where \( \mu = 6 \gamma - 9 \), \( 0 < \delta \leq \frac{3(\gamma - 1)}{5} \), \( C > 0 \) is independent of \( M \), and the domains \( \Omega_r, S_T \) are defined in section 4.

In order to prove theorem 5.1, we will apply the induction on \( k \) in (5.3)–(5.4) to establish the following estimates respectively:

(i) \( D_r S^k \Phi \) and \( \nabla_r S^k \Phi \) with \( S^k = S^k_0 Z^j \) and \( 1 \leq k = l_1 + l_2 \leq 3 \) (in this case, all the tangent derivatives of \( \nabla_r \Phi \) up to the third order are estimated, where the tangent derivative means the one of boundary \( \partial \Omega \));

(ii) \( D_r S^j \Phi \) and \( \nabla_r S^j \Phi \) with \( S_l = r \partial_r \) (in this case, together with the case \( k = 1 \) in (i), all the second-order derivatives \( \nabla^2_{t,x} \Phi \) are analyzed);

(iii) \( D_r S^j \Phi, \nabla_r S^j \Phi, D_r S^j \Phi \) and \( \nabla_r S^j \Phi \) (in this case, together with the case \( k = 2 \) in (i), all the estimates of third-order derivatives \( \nabla^3_{t,x} \Phi \) are given);

(iv) \( D_r S^j \Phi, \nabla_r S^j \Phi, D_r S^j \Phi, \nabla_r S^j \Phi, D_r S^j \Phi \) and \( \nabla_r S^j \Phi \) (in this case, together with the case \( k = 3 \) in (i), all the fourth-order derivatives \( \nabla^4_{t,x} \Phi \) are estimated).

These estimates will be given in lemmas 5.2–5.5 respectively. Firstly, we establish the tangent derivative estimates of \( \Phi \) under the suitable induction assumption. Set \( S_0 = R(t) D_t \) and \( S^m = S^m_0 Z^j \) with \( m = l_1 + l_2 \), which are tangent to the boundary \( \partial \Omega \). Then we have

**Lemma 5.1 (Tangent derivative estimates).** Under the assumptions of theorem 5.1, if (5.3) holds for \( 0 \leq k \leq m \) with \( 1 \leq m \leq 2 \), then

\[
R(T)^{\mu} \int_{\Omega_r} (D_r S^m \Phi)^2 dx + R(T)^{\mu-3(\gamma-1)} \int_{\Omega_r} (\nabla_r S^m \Phi)^2 dx \\
+ \int_{\Omega_0} (R(t))^{\mu-1-\delta} (D_r S^m \Phi)^2 + R(T)^{\mu-3(\gamma-1)} (\nabla_r S^m \Phi)^2 dx \\n\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_r} \sum_{i=0}^m (R(t))^{\mu-1-\delta+2i} (\nabla^i_{t,x} D_r \Phi)^2 \\
+ R(t)^{\mu-3(\gamma-1)+2i} (\nabla^i_{t,x} \nabla \Phi)^2 \right) dx,
\]

where \( 0 < \delta \leq \frac{3(\gamma - 1)}{5} \). Especially, for \( m = 0 \), the following estimate holds

\[
R(T)^{\mu} \int_{\Omega_r} (D_r \Phi)^2 dx + R(T)^{\mu-3(\gamma-1)} \int_{\Omega_r} (\nabla_r \Phi)^2 dx \\
+ \int_{\Omega_0} (R(t))^{\mu-1-\delta} (D_r \Phi)^2 + R(T)^{\mu-3(\gamma-1)} (\nabla_r \Phi)^2 \right) dx \\n\leq C \varepsilon^2.
\]
For $m = 3$, if (5.2)–(5.3) hold, then
\[
R(T)\mu \int_{S_T} (D_t S^m \Phi)^2 dx + R(T)\mu^{-3(\gamma - 1)} \int_{S_T} (\nabla_t S^m \Phi)^2 dx
\]
\[
+ \int_{\Omega_T} (R(t)\mu^{-1-\delta} (D_t S^m \Phi)^2)
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
\leq C\varepsilon^2 + C\varepsilon \int_{\Omega_T} \sum_{l=0}^{2} (R(t)\mu^{-1-\delta} (\nabla_t S^m\Phi)^2)
\]
\[
+ C\varepsilon \int_{\Omega_T} (R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)-\delta} (\nabla_t S^m \Phi)^2) dt dx.
\]
(5.6)

**Remark 5.1.** For the case when $m = 0$ in (5.5), we do not require any induction assumption.

**Remark 5.2.** Note that the normal derivatives of $\Phi$ are also included on the right-hand side of (5.5) and (5.6), which implies that we have not obtained the close estimates on the tangent derivative estimates of $\Phi$. However, since the coefficients of normal derivatives of $\Phi$ in (5.5)–(5.6) are small, then together with some subsequent normal derivative estimates, we can derive (5.3).

**Proof.** Note that on $B_T$
\[
S^m \left( \sum_{j=1}^{3} \varepsilon_j \partial_j \Phi \right) = S^m (\partial_r \Phi) = 0.
\]
(5.7)

This, together with theorem 4.1 and (3.5), yields
\[
R(T)\mu \int_{S_T} (D_t S^m \Phi)^2 dx + R(T)\mu^{-3(\gamma - 1)} \int_{S_T} (\nabla_t S^m \Phi)^2 dx
\]
\[
+ \int_{\Omega_T} (R(t)\mu^{-1-\delta} (D_t S^m \Phi)^2)
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
\leq C\varepsilon^2 + C\varepsilon \int_{\Omega_T} \sum_{l=0}^{2} (R(t)\mu^{-1-\delta} (\nabla_t S^m\Phi)^2)
\]
\[
+ C\varepsilon \int_{\Omega_T} (R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)-\delta} (\nabla_t S^m \Phi)^2) dt dx.
\]
(5.8)

We divide the estimates (5.5) and (5.6) into three cases:
1. $S^m = S^m_0$
2. $S^m = Z^m$
3. $S^m = S^m_0 Z^m_1$ \((1 \leq l_1, l_2 \leq m - 1, l_1 + l_2 = m)\)

**Case (1)** $S^m = S^m_0$

It follows from lemma A.1 in appendix that (5.5) and (5.6) hold for $S^m = S^m_0$.

**Case (2)** $S^m = Z^m$

By lemma A.2 in appendix, (5.5) and (5.6) hold for $S^m = Z^m$.

**Case (3)** $S^m = S^m_0 Z^m_1$ \((1 \leq l_1, l_2 \leq m - 1, l_1 + l_2 = m)\)

Since the estimations for this case are similar to the Case (1) and Case (2), we omit the details. Combining Cases (1)–(3), we complete the proof of lemma 5.1.

Although the weighted estimates of $D_t \Phi$ and $D_t^2 \Phi$ have been obtained from lemma 5.1, we cannot use equation (4.16) to derive the estimate of $\Delta \Phi$ directly since only more precise estimate of $\Delta \Phi$ is required in theorem 5.1. More concretely, it follows from (4.16) and lemma 5.1 that
\[
\varepsilon^2 \Delta \Phi = D_t^2 \Phi + \frac{3L(\gamma - 1)}{R(t)} D_t \Phi - f
\]
and
\[
R(T)\mu^{-2-6(\gamma - 1)} \int_{S_T} (\Delta \Phi)^2 dx \leq C\varepsilon^2.
\]
(5.9)

However, the required estimate of $\Delta \Phi$ in theorem 5.1 is
\[
R(T)\mu^{-2-3(\gamma - 1)} \int_{S_T} (\Delta \Phi)^2 dx \leq C\varepsilon^2,
\]
which is a more precise estimate than (5.9). Therefore, we should take other ingredients to derive the estimates on the higher order radial derivatives of $\Phi$. This will be realized in subsequent lemma 5.2–5.5.

**Lemma 5.2 (The second order radial derivative estimates).**

Under the assumptions of theorem 5.1, we have
\[
R(T)\mu \int_{S_T} (D_t S^m \Phi)^2 dx + R(T)\mu^{-3(\gamma - 1)} \int_{S_T} (\nabla_t S^m \Phi)^2 dx
\]
\[
+ \int_{\Omega_T} (R(t)\mu^{-1-\delta} (D_t S^m \Phi)^2)
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
\leq C\varepsilon^2 + C\varepsilon \int_{\Omega_T} \sum_{l=0}^{1} (R(t)\mu^{-1-\delta} (\nabla_t S^m\Phi)^2)
\]
\[
+ C\varepsilon \int_{\Omega_T} (R(t)\mu^{-1-3(\gamma - 1)} (\nabla_t S^m \Phi)^2) dt dx
\]
\[
+ R(t)\mu^{-1-3(\gamma - 1)-\delta} (\nabla_t S^m \Phi)^2) dt dx.
\]
(5.10)

**Corollary 5.1.** Lemma 5.2 together with lemma 5.1 for $m = 1$ yields
\[ \sum_{0 \leq k + l \leq 1} \left( R(T)^\mu \int_{\Omega r} (D_i S_k^l S_l^i \Phi)^2 dx + R(T)^{\mu - 3(\gamma - 1)} \int_{\Omega r} (\nabla_i S_k^l S_l^i \Phi)^2 dx \right) \\
+ \sum_{0 \leq k + l \leq 1} \int_{\Omega r} (R(t)^{\mu - 1 - \delta} (D_i S_k^l S_l^i \Phi)^2 + R(t)^{\mu - 3(\gamma - 1)} (\nabla_i S_k^l S_l^i \Phi)^2) dtdx \]
\[ \leq C\varepsilon^2 + C \varepsilon \int_{\Omega r} (S_i f - 2c^2 \Delta \Phi) \cdot M\Sigma_i \Phi dtdx. \]
\[ (5.11) \]

Since \( S_i \) and \( Z \) are expressed in terms of \( s_i \partial_i (1 \leq i, j \leq 3) \), similar to the proofs of (A.27)–(A.38) and (A.40) in appendix, we have
\[ \left| \int_{\Omega r} S_i f \cdot M\Sigma_i \Phi dtdx \right| \]
\[ \leq C\varepsilon^2 + C\varepsilon \left( R(T)^\mu \int_{\Omega r} (D_i S_i \Phi)^2 dx + R(T)^{\mu - 3(\gamma - 1)} \right) \\
\times \int_{\Omega r} (\nabla_i S_i \Phi)^2 dx \]
\[ + C \varepsilon \int_{\Omega r} \sum_{i=0}^{1} (R(t)^{\mu - 1 - \delta} (D_i S_i \Phi)^2 + R(t)^{-3(\gamma - 1)} (\nabla_i S_i \Phi)^2) \]
\[ + R(t)^{\mu - 3(\gamma - 1) + 2\delta (\nabla_i S_i \Phi)^2} dtdx. \]
\[ (5.12) \]

Finally, substituting (5.12)–(5.13) into (5.11) yields (5.10), and this completes the proof of the lemma.

**Lemma 5.3 (The third-order radial derivative estimates).** Under the assumptions of theorem 5.1, we have
\[ R(T)^\mu \int_{\Omega r} (D_i S_i \Phi)^2 dx + R(T)^{\mu - 3(\gamma - 1)} \int_{\Omega r} (\nabla_i S_i \Phi)^2 dx \\
+ \sum_{i=0}^{1} (R(t)^{\mu - 1 - \delta} (D_i S_i \Phi)^2 + R(t)^{-3(\gamma - 1)} (\nabla_i S_i \Phi)^2) \]
\[ \leq C\varepsilon^2 + C \varepsilon \int_{\Omega r} \sum_{i=0}^{1} R(t)^{\mu - 1 - \delta + 2\delta (\nabla_i S_i \Phi)^2} \]
\[ + R(T)^{\mu - 3(\gamma - 1) + 2\delta (\nabla_i S_i \Phi)^2} dtdx. \]
\[ (5.14) \]

**Remark 5.3.** The biggest difference between the proofs of lemmas 5.1 and 5.3 comes from the treatments on the boundary condition. In order to prove lemma 5.3, we need to derive a new boundary condition for the higher order radial derivative of \( \Phi \) in terms of the equation (4.16) (see (5.17) below), other than take the tangential derivatives on the boundary condition (5.3) directly to obtain (5.7) as in the proof of lemma 5.1. Obviously, the boundary condition (5.17) is much more complicated than (5.7). This will bring more extra terms to be treated in the higher order energy estimates of \( \Phi \). Same efforts are also required in subsequent lemmas 5.4 and 5.5.

**Corollary 5.2 (Third-order mixed tangent-radial derivative estimates).** Under the assumptions of theorem 5.1, as in the
proof of lemma 5.1, we have

\[ R(T)^{\mu} \int_{\Omega_T} (D_sS_t^2)\Phi^2 dx + R(T)^{\mu-3(\gamma-1)} \int_{\Omega_T} (\nabla_sS_t^2\Phi)^2 dx \\
+ \int_{\Omega_T} (R(t)^{\mu-1-\delta}(D_sS_t^2\Phi)^2 dx \\
+ R(T)^{\mu-1-3(\gamma-1)}(\nabla_sS_t^2\Phi)^2 dx) dtdx \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \frac{1}{2} (R(t)^{\mu-1-\delta}(D_sS_t^2\Phi)^2 dx \\
+ \sum_{0 \leq i \leq 2} \int_{\Omega_T} (R(t)^{\mu-1-\delta}(D_sS_t^2\Phi)^2 dx \\
+ R(T)^{\mu-1-3(\gamma-1)}(\nabla_sS_t^2\Phi)^2 dx) dtdx \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \frac{1}{2} (R(t)^{\mu-1-\delta}(D_sS_t^2\Phi)^2 dx \\
+ R(T)^{\mu-1-3(\gamma-1)}(\nabla_sS_t^2\Phi)^2 dx) dtdx \\
+ \int_{\Omega_T} (R(t)^{\mu-5-\delta}(\nabla_sS_t^2\Phi)^2 dx \\
+ \sum_{0 \leq i \leq 2} \int_{\Omega_T} (R(t)^{\mu-1-\delta}(D_sS_t^2\Phi)^2 dx \\
+ R(T)^{\mu-1-3(\gamma-1)}(\nabla_sS_t^2\Phi)^2 dx) dtdx) \frac{1}{2}. \\
(5.15) \]

where

\[ A = \varepsilon^2 - (\gamma - 1)D_t\Phi - \frac{\gamma - 1}{2R(t)} \sum_{i=1}^{3} (Z_i\Phi)^2, \]

\[ G = -\frac{\gamma - 1}{R(t)} \sum_{i=1}^{3} (Z_i\Phi)^2 - \frac{2L}{R(t)} \sum_{i=1}^{3} C_{ij}Z_i\Phi Z_j\Phi - \frac{4}{R(t)^2} \sum_{i,j=1}^{3} C_{ijkl}Z_i\Phi Z_j\Phi Z_k\Phi. \]

For convenience, we rewrite (5.17) as follows

\[ S_t^3\Phi - S_t^2\Phi - 2\sum_{i=1}^{3} Z_i^2\Phi - \chi \left( \frac{r}{R(t)} \right) \frac{G}{A} = 0 \quad \text{on } B_T, \]

(5.18)

where \( \chi(s) \) is a smooth cut-off function

\[ \chi(s) = \begin{cases} 
1, & \text{for } \frac{2}{3} \leq s \leq 1, \\
0, & \text{for } 0 \leq s \leq \frac{1}{3}, \\
\text{smooth connection,} & \text{for } \frac{1}{3} \leq s \leq \frac{2}{3},
\end{cases} \]

(5.19)

On the other hand, direct computation yields

\[ \mathcal{L}(S_t^2 - S_t^1)\Phi = (S_t^2 - S_t^1)\mathcal{L}\Phi + 4\varepsilon^2 \Delta S_t\Phi - 2\varepsilon^2 \Delta \Phi. \]

(5.20)

As in (4.2), we have

\[ \int_{\Omega_T} \mathcal{L}(S_t^2 - S_t^1)\Phi \cdot \mathcal{M}(S_t^2 - S_t^1)\Phi dx \\
= -\int_{\Omega_T} R(t)^{\mu} \varepsilon^2 a(t)D_t(S_t^2 - S_t^1)\Phi \\
\cdot \partial_t(S_t^2 - S_t^1)\Phi dx \\
+ \left( \int_{\Omega_T} R(T)^{\mu}a(t)D_t(S_t^2 - S_t^1)\Phi \right) ds \\
+ \frac{\delta}{2} R(t)^{\mu-1-\delta}(D_t(S_t^2 - S_t^1)\Phi)^2 \\
+ \frac{\gamma}{2} R(T)^{\mu-1-3(\gamma-1)}(L(5 - 3\gamma)a(t) \\
- R(t)a'(t))|\nabla_t(S_t^2 - S_t^1)\Phi|^2 |dx. \]

(5.21)

Substituting (5.20) into (5.21), it follows

\[ S_t^3\Phi - S_t^2\Phi - 2\sum_{i=1}^{3} Z_i^2\Phi - \frac{G}{A} = 0, \]

(5.17)

Proof of Lemma 5.3. In order to derive the third-order radial derivative estimate, we need to study a higher order boundary condition on \( B_T \). Differentiating (4.16) with respect to \( r \) and applying (5.7) yield on \( B_T \)

\[ S_t^3\Phi - S_t^2\Phi - 2\sum_{i=1}^{3} Z_i^2\Phi - \frac{G}{A} = 0, \]

(5.17)
\[
R(T)^\mu \int_{S_T} (D_t(S_t^2 - S_t)\Phi)^2 dx + R(T)^{\mu - 3(\gamma - 1)}
\times \int_{S_T} (\nabla_t(S_t^2 - S_t)\Phi)^2 dx
+ \int_{S_T} (R(t)^{\mu - 1 - \ell}(D_t(S_t^2 - S_t)\Phi)^2
+ R(T)^{\mu - 1 - 3(\gamma - 1)}(\nabla_t(S_t^2 - S_t)\Phi)^2) dtdx
\leq C_\varepsilon^2 + C_\varepsilon \int_{S_T} \int_0^\infty \left( \frac{R(t)}{1 + L^2} \right)^{\mu - 3(\gamma - 1) - 2(\nabla_t D_t\Phi)^2}
+ \left( \frac{R(t)}{1 + L^2} \right)^{\mu - 3(\gamma - 1) - 2(\nabla_t \Phi)^2} dtdx.
\]

In view of (5.18) and divergence theorem, we have
\[
\int_{S_T} \frac{R(t)^\mu}{1 + L^2} \varepsilon^2 (a(t)D_t(S_t^2 - S_t)\Phi \cdot \partial_t(S_t^2 - S_t)\Phi) dtdx
= \int_{S_T} R(t)^\mu \varepsilon^2 (a(t)\partial_t D_t(S_t^2 - S_t)\Phi - D_t\Phi) dx
\times \left( \chi \left( \frac{r}{R(t)} \frac{G}{A} + 2 \sum_{i=1}^3 Z_i^2 \Phi \right) \right)
\times \int_{S_T} \left( \frac{r}{R(t)} \frac{G}{A} + 2 \sum_{i=1}^3 Z_i^2 \Phi \right) dx
\times \int_{S_T} \left( \frac{r}{R(t)} \frac{G}{A} + 2 \sum_{i=1}^3 Z_i^2 \Phi \right) dx.
\]

And it follows from (5.2) that
\[
\left( \frac{R(t)}{R(t)} \frac{G}{A} \right) \leq C_\varepsilon(R(t)^{-3(\gamma - 1)}(|S_t^2\Phi| + |Z^2\Phi|) + |Z\Phi|).
\]

Next we deal with \( \int_{S_T} (S_t^2 - S_t)^\lambda \cdot M(S_t^2 - S_t)\Phi dtdx \). Similar to (A.4), we have
\[
(S_t^2 - S_t)^\lambda = I_1 + I_2 + I_3.
\]

On the other hand, by the expressions of \( \tilde{f}_{ij}, \tilde{f}_0 \) and (5.7),

\[
\int_{S_T} \int_0^\infty \left( \frac{\varepsilon^2 \Delta S_t\Phi}{} + \frac{|\varepsilon^2 \Delta \Phi|}{} \right) \cdot |M(S_t^2 - S_t)\Phi| dtdx
\leq C_\varepsilon^2 + C_\varepsilon \int_{S_T} \int_0^\infty \left( \frac{R(t)^{\mu + 3(\gamma - 1)}(\nabla_{t, l} D_t\Phi)^2}{1 + L^2} + R(T)^{\mu + 3(\gamma - 1)}(\nabla_{t, l} \nabla_t \Phi)^2 \right) dtdx.
\]

Then as in (A.5), thanks to (A.7) and (5.8) in the case of \( S^2 = ZS_0 \), it follows
Similar to (5.23)–(5.26), we obtain
\[ \left| \int_{B_r} \sum_{i=1}^{3} x_i \cdot \left( \frac{1}{2} a(t) R(t)^\mu \int_{\Omega} \left( D_i (S_i^3 - S_i) \Phi \right)^2 \right) + a(t) R(t)^\mu D_i (S_i^2 - S_i) \Phi \sum_{j=1}^{3} \int_{\Omega} \partial_j (S_j^3 - S_j) \Phi \right) dS \right| \]
\[ \leq C \varepsilon^2 + C \varepsilon \int_{B_r} \left( \sum_{i=1}^{3} R(t)^{\mu - 1 - \frac{\varepsilon}{3}} (\nabla_{li} D_i \Phi)^2 \right) + \sum_{l=0}^{2} R(t)^{\mu - 1 - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx. \]

Thus, as in (A.36), it follows
\[ \left| \int_{\Omega} \left( \partial_l \cdot \mathcal{L} (S_l^3 - S_l) \Phi \right) dtdx \right| \]
\[ \leq C \varepsilon^2 + C \varepsilon \int_{\Omega} \left( R(T) \int_{B_T} (D_i (S_i^3 - S_i) \Phi)^2 dx \right) + R(T)^{\mu - 3(\gamma - 1)} \int_{B_T} (\nabla_{li} (S_i^3 - S_i) \Phi)^2 dx \]
\[ + \sum_{l=0}^{2} R(t)^{\mu - 1 - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx. \]

Then, similar to (A.37) and (A.43), it follows
\[ \int_{\Omega} \left| [I_3] + [I_3] \cdot |\mathcal{L} (S_l^3 - S_l) \Phi| \right| dtdx \]
\[ \leq C \varepsilon^2 + C \varepsilon \sum_{l=0}^{2} \int_{B_T} \left( R(t)^{\mu - 1 - \frac{\varepsilon}{3}} (\nabla_{li} D_i \Phi)^2 \right) + R(t)^{\mu - 1 - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx. \]

Substituting (5.26)–(5.28) and (5.32)–(5.33) into (5.22), we complete the proof of lemma 5.3. \( \square \)

**Corollary 5.3 (Estimate on \( D_l S_l^2 \Phi \) and \( \nabla_l S_l^2 \Phi \)).** Under the assumptions of theorem 5.1, as the estimation in the proof of lemma 5.2, we have
\[ R(T)^{\mu} \int_{B_T} (D_l (S_l^3 - S_l) \Phi)^2 dx + R(T)^{\mu - 3(\gamma - 1)} \int_{B_T} (\nabla_l (S_l^3 - S_l) \Phi)^2 dx \]
\[ + \int_{B_T} \left( R(T)^{\mu - \frac{\varepsilon}{3}} (\nabla_{li} D_i \Phi)^2 \right) + R(t)^{\mu - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx \]
\[ \leq C \varepsilon^2 + C \varepsilon \sum_{l=0}^{2} \int_{B_T} \left( R(T)^{\mu - 1 - \frac{\varepsilon}{3}} (\nabla_{li} D_i \Phi)^2 \right) + R(t)^{\mu - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx \]
\[ + C \varepsilon \int_{B_T} \left( R(T)^{\mu - 1 - \frac{\varepsilon}{3}} (\nabla_{li} D_i \Phi)^2 \right) + R(t)^{\mu - 3(\gamma - 1) + 2l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx \]
\[ + C \varepsilon \int_{B_T} \left( R(T)^{\mu - 3(\gamma - 1) - l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx \]
\[ + R(t)^{\mu - 3(\gamma - 1) - l} (\nabla_{li} \nabla_i \Phi)^2 \right) dtdx. \]

Next, we will give the estimates on \( \nabla_{li} \Phi \) in lemmas 5.4–5.6.

**Lemma 5.4 (Estimate on \( D_l S_l^2 \Phi \) and \( \nabla_l S_l^2 \Phi \)).** Under the assumptions of theorem 5.1, we have
\[ R(T)^{\mu} \int_{S_T} (D_i SS_i^2)^2 dx + R(T)^{\mu-3(\gamma-1)} \int_{S_T} (\nabla_i SS_i^2)^2 dx \]
\[ + \int_{S_T} (R(t)^{\mu-1-\epsilon} (D_i SS_i^2)^2) dt dx \]
\[ + R(T)^{\mu-1-3(\gamma-1)} (\nabla_i SS_i^2)^2 dt dx \]
\[ \leq C \varepsilon^2 + C \varepsilon^2 \sum_{l=0}^{2} \int_{S_T} (R(t)^{\mu-1-\epsilon+2l} (\nabla_i D_i)^2)^2 dt dx \]
\[ + R(T)^{\mu-1-3(\gamma-1)+2} (\nabla_i \nabla_i)^2 dt dx \]
\[ \sum_{i=1}^{3} S_{Z_i}^2 \Phi \]
\[ = \nabla_i (S_{Z_i}^2 \Phi - SS_i^2 \Phi) - 3 \sum_{i=1}^{3} S_{Z_i}^2 \Phi \]
\[ \nabla_i \left( \frac{r}{R(t)} \right) \left( \frac{G}{A} \right) = 0 \text{ on } B_T. \]  

**Proof.** Applying \( S \) to (5.18) yields \( S_1(SS_i^2 \Phi - SS_i^2 \Phi) = 3 \sum_{i=1}^{3} S_{Z_i}^2 \Phi \) \( \nabla_i \left( \frac{r}{R(t)} \right) \left( \frac{G}{A} \right) = 0 \) on \( B_T \). (5.36)

In addition, direct computation yields \( \mathcal{L}Z(S_1^2 - S_i) \Phi = Z(S_1^2 - S_i) \mathcal{L} \Phi + 4 \varepsilon^2 \Delta ZS_1 \Phi - 2 \varepsilon^2 \Delta Z \Phi \). (5.37)

Similarly to (5.21), we obtain \( \int_{S_T} \lambda S(S_1^2 - S_i) \Phi - MS(S_1^2 - S_i) \Phi dt dx \)
\[ = - \int_{S_T} \frac{R(t)^{\mu}}{\sqrt{1 + L^2}} \varepsilon^2 a(t) D_i S(S_1^2 - S_i) \Phi \]
\[ \cdot \partial \partial S(S_1^2 - S_i) \Phi ds \]
\[ + \left( \int_{S_T} \mathcal{L}S(S_1^2 - S_i) \Phi \right) \]
\[ + \left( \int_{S_T} \left( \frac{1}{2} R(t)^{\mu} a(t) D_i S(S_1^2 - S_i) \Phi \right)^2 \right) ds \]
\[ + \int_{S_T} \left( \frac{1}{2} R(t)^{\mu-1-\epsilon} (D_i S(S_1^2 - S_i) \Phi)^2 \right) ds \]
\[ + \frac{\gamma}{2} R(t)^{\mu-1-3(\gamma-1)} (L(5 - 3 \gamma) a(t)) \]
\[ - R(t)^{a'(t)}(\nabla_i S(S_1^2 - S_i) \Phi)^2 \]  

As in (5.23), we have \( \int_{S_T} \frac{R(t)^{\mu}}{\sqrt{1 + L^2}} \varepsilon^2 a(t) D_i S(S_1^2 - S_i) \Phi \cdot \partial \partial S(S_1^2 - S_i) \Phi dt \) \[ = \int_{S_T} R(t)^{\mu} \varepsilon^2 a(t) \left( \Delta (D_i S(S_1^2 - S_i) \Phi - D_i S \Phi) \right) \]
\[ \times \left( \frac{r}{R(t)} \right) \left( \frac{G}{A} \right) + 2 \sum_{i=1}^{3} S_{Z_i}^2 \Phi \right) dt dx \]  

(5.40)

Note that, for \( C^1 \)-smooth functions \( f \) and \( g \), \( \int_{S_T} (x_i \partial_i f - x_i \partial_i g) g dx = \int_{S_T} (\partial_i(x_i f g) - \partial_i(x_i g)) \]
\[ = \int_{S_T} f(x_i \partial_i g - x_i \partial_i g) dx. \]  

Then \( \int_{S_T} Z f \cdot g dx = - \int_{S_T} f \cdot Z g dx. \) (5.41)

Hence, if we choose \( S = Z \), then it follows from (5.40) that \( \int_{S_T} R(t)^{\mu} \varepsilon^2 a(t) \left( \Delta (D_i S(S_1^2 - S_i) \Phi - D_i S \Phi) \right) \]
\[ \times \left( \frac{r}{R(t)} \right) \left( \frac{G}{A} \right) + 2 \sum_{i=1}^{3} S_{Z_i}^2 \Phi \right) dt dx \]  

(5.42)

In addition, it follows from (5.2) that
\[ \left| \frac{r}{R(t)} \right| \left( \frac{G}{A} \right) \leq C \varepsilon(R(t)^{3(\gamma-1)} - L^2 S_1^2 \Phi \]
\[ + R(t)^{-3(\gamma-1)}(S_1^2 \Phi + |Z_1^3 \Phi| + |Z_2^2 \Phi| + |Z_3^2 \Phi|), \]  

(5.43)

(5.44)
Combining (5.40) and (5.42)–(5.44), we have that from (5.39)

\[
\left| \int_{\Omega_T} \frac{R(t)^\mu}{\sqrt{1 + L^2}} e^{2a(t)D_tZ(S_t^2 - S_t)} \cdot \partial_t Z(S_t^2 - S_t) \psi dS \right| \\
\leq C e^{2} \sum_{i=0}^{2} \int_{\Omega_T} (R(t)^\mu)^{-1 - \delta + 2i} (\nabla_{i,t}^2 D_t \Phi)^2 \\
+ R(t)^{\mu - 1 - 3(\gamma - 1) + 2i} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx \\
+ C e^{2} \int_{\Omega_T} (R(t)^{\mu + 5 - \delta}) (\nabla_{i,t}^2 D_t \Phi)^2 \\
+ R(t)^{\mu + 5 - 3(\gamma - 1) - \delta} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx \\
+ \int_{\Omega_T} R(t)^{\mu - 1} |Z^2 \Phi|^2 \cdot |\Delta(D_t S_t \Phi - D_t \Phi)| dtdx.
\]
(5.45)

By Hölder inequality and lemma 2.4, we have

\[
|R(t)^{\mu - 1} (Z^2 \Phi)^2 \Delta(D_t S_t \Phi - D_t \Phi)| L^2(\Omega_T) \leq C |R(t)^{\mu - 1} (\nabla Z \Phi)^2 \sum_{i=0}^{3} ((R(t) \nabla_i D_t \Phi) |L^2(\Omega_T)) \\
\leq C |R(t)^{2i + 5 - 3(\gamma - 1) + 2i} (\nabla Z \Phi)^2 |L^2(\Omega_T) [R(t)^{\mu - 1 - 3\delta} \sum_{i=0}^{3} ((R(t) \nabla_i D_t \Phi) |L^2(\Omega_T)) \\
\leq C e^{2} \sum_{i=0}^{2} \int_{\Omega_T} (R(t)^{\mu - 1 - \delta + 2i} (\nabla_{i,t}^0 D_t \Phi)^2 \\
+ R(t)^{\mu - 1 - 3(\gamma - 1) + 2i} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx \\
+ C e^{2} \int_{\Omega_T} (R(t)^{\mu + 5 - \delta}) (\nabla_{i,t}^0 \nabla_t \Phi)^2 \\
+ R(t)^{\mu + 5 - 3(\gamma - 1) - \delta} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx.
\]
(5.46)

Substituting (5.46) into (5.45) yields

\[
\left| \int_{\Omega_T} \frac{R(t)^\mu}{\sqrt{1 + L^2}} e^{2a(t)D_tZ(S_t^2 - S_t)} \cdot \partial_t Z(S_t^2 - S_t) \Phi dS \right| \\
\leq C e^{2} \sum_{i=0}^{2} \int_{\Omega_T} (R(t)^{\mu - 1 - \delta + 2i} (\nabla_{i,t}^0 D_t \Phi)^2 \\
+ R(t)^{\mu - 1 - 3(\gamma - 1) + 2i} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx \\
+ C e^{2} \int_{\Omega_T} (R(t)^{\mu + 5 - \delta}) (\nabla_{i,t}^0 D_t \Phi)^2 \\
+ R(t)^{\mu + 5 - 3(\gamma - 1) - \delta} (\nabla_{i,t}^0 \nabla_t \Phi)^2) dtdx.
\]
(5.47)
Thus, together with (5.48)–(5.50), this yields lemma 5.4 for the case $S = Z$.

Next, we deal with the case of $S = S_0$. From (A.5), we have

$$
\int_{\Omega_T} D_t f \cdot \text{g} \, dt dx = \int_{\Omega_T} D_t f g + f D_t g \, dt dx.
$$

(5.51)

If we choose $S = S_0$, then it follows from (5.40) and $[S_0, R(t)^2 \Delta] = 0$ that

$$
\int_{\Omega_T} R(t)^m \partial_\alpha \partial_\beta (\Delta \nabla (S_0 S_1 \Phi) - D_0 S_0 \Phi)
\times \left( \chi \left( \frac{r}{R(t)} \right) S_0 \frac{G}{A} + 2 \sum_{i=1}^{3} S_i Z_i^2 \Phi \right) \, dt dx
\leq C \sum_{i=1}^{3} \left| S_i \right|^2 \Phi \, dt dx
$$

(5.52)

And it follows from (5.2) that

$$
\left| \chi \left( \frac{r}{R(t)} \right) S_0 \frac{G}{A} \right| \leq C \left( R(t)^{-3(\gamma - 1)} \sum_{i=0}^{1} |S_i| \right) \frac{G}{A} + |S_i|^2 \Phi
$$

(5.53)

Thus, combining (5.40) and (5.52)–(5.54), we have

$$
\left| \int_{\Omega_T} \frac{R(t)^m}{\sqrt{1 + L^2}} \partial_\alpha \partial_\beta (\Delta \nabla (S_0 S_1 \Phi) - D_0 S_0 \Phi) \cdot \partial_\gamma \Phi \, dt dx \right|
\leq C \sum_{i=0}^{1} \left| S_i \right|^2 \Phi + |S_i|^2 \Phi + R(t) |S_i|^2 D_1 \Phi \right|.
$$

(5.54)

Thus, combining (5.40) and (5.52)–(5.54), we have

$$
\left| \int_{\Omega_T} \frac{R(t)^m}{\sqrt{1 + L^2}} \partial_\alpha \partial_\beta (\Delta \nabla (S_0 S_1 \Phi) - D_0 S_0 \Phi) \cdot \partial_\gamma \Phi \, dt dx \right|
\leq C \sum_{i=0}^{1} \left| S_i \right|^2 \Phi + |S_i|^2 \Phi + R(t) |S_i|^2 D_1 \Phi \right|.
$$

(5.55)

Substituting (5.38) and (5.55) into (5.39) yields

$$
R(T)^m \int_{\Omega_T} (D_t S_0 S_1 \Phi) \, dt dx
+ R(T)^m \int_{\Omega_T} (\nabla_\alpha S_0 S_1 \Phi) \, dt dx
\leq C \sum_{i=0}^{1} \left| S_i \right|^2 \Phi + |S_i|^2 \Phi + R(t) |S_i|^2 D_1 \Phi \right|.
$$

(5.56)

Similar to (5.27), it follows

$$
\left| \int_{\Omega_T} \left( |S_0| \partial_\alpha \partial_\beta (\Delta \nabla (S_1 \Phi) + |S_0| \partial_\alpha \partial_\beta \Phi \right) \right|
\leq C \sum_{i=0}^{1} \left| S_i \right|^2 \Phi + |S_i|^2 \Phi \cdot |S_0| \partial_\alpha \partial_\beta \Phi \right| dt dx
$$

(5.57)
As (5.32)–(5.33) and (5.47), we have
\[
\left| \int_{\Omega_T} (|S_0(S_i^2 - S_i)\mathcal{L} \Phi| + |(S_i^2 - S_i)\mathcal{L} \Phi|) \, dx \right|
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} (S_i^2 - S_i) \mathcal{D} \Phi^2 \, dx
\]
\[+ \int_{\Omega_T} \left( \sum_{i=1}^{n} (S_i^2 - S_i) \mathcal{D} \Phi^2 \right) \, dx \]
\[= \left( \int_{\Omega_T} (S_i^2 - S_i) \mathcal{D} \Phi^2 \, dx \right)^2 \frac{1}{2}. \quad (5.58)\]

Together with (5.56)–(5.58), lemma 5.4 is proved for the case of \( S = S_0 \) and it then completes the proof of the lemma. \( \square \)

**Lemma 5.5 (The fourth-order radial derivative estimates).**
Under the assumptions of theorem 5.1, we have
\[
R(T)\mu^{-\delta} \int_{\Omega_T} (D_i^3 S_i^2 \Phi) \, dx
\]
\[+ R(T)\mu^{-3(\gamma - 1) - \delta} \int_{\Omega_T} (S_i^5 S_i^2 \Phi) \, dx
\]
\[+ \int_{\Omega_T} (R(T)\mu^{-1 - \delta} (D_i^3 S_i^2 \Phi)) \, dx
\]
\[= \left( \int_{\Omega_T} (D_i^3 S_i^2 \Phi) \, dx \right)^2 \frac{1}{2}. \quad (5.60)\]

**Proof of lemma 5.5.** Set
\[
N_i = S_i^3 - S_i^2 - 2 \sum_{j=1}^{3} Z_j^i \Phi. \quad (5.61)\]
Then it follows from (5.18) that on $B_T$

$$N\Phi \equiv (N_1 - N_2)\Phi = 0.$$  

(5.63)

On the other hand,

$$\mathcal{L}N\Phi = N\mathcal{L}\Phi + [\mathcal{L}, N_1]\Phi - [\mathcal{L}, N_2]\Phi,$$  

(5.64)

where

$$[\mathcal{L}, N_1]\Phi = -6\varepsilon^2 \Delta S^2_1\Phi + 8\varepsilon^2 \Delta S_1\Phi - 4\varepsilon^2 \Delta \Phi,$$  

(5.65)

and

$$[\mathcal{L}, N_2]\Phi \leq C\varepsilon \left(\frac{r}{R(t)}\right)^{3(\gamma - 1) - 1 - 2} \right) \sum_{0 \leq l \leq 1} \left( \frac{1}{2} |S_1^2| |S_1| \right) + R(t)^{-3(\gamma - 1) + \delta - 2} \sum_{0 \leq l \leq 2} |S_1^3| Z\Phi | + R(t)^{-1} S_1^3 D_1\Phi$$

$$+ R(t)^{-3(\gamma - 1) + \delta - 1} \sum_{1 \leq l \leq 2} R(t)|\nabla_1 Z\Phi|$$

$$+ R(t)^{-3(\gamma - 1) + \delta - 1} \nabla_1^2 D_1\Phi),$$  

(5.66)

Note that the term $R(t)^{-1} S_1^3 D_1\Phi$ is on the right-hand side of (5.66). Thus, we can not choose the multiplier $\mathcal{M}N\Phi$ as in theorem 4.1 to derive the energy estimate of $N\Phi$ from (5.64). Otherwise, we cannot control the term $\int_{B_T} R(t)^{\mu - 1} S_1^3 D_1\Phi \cdot D_1 N_1\Phi |dtdx$. Instead, we will use a new multiplier $\mathcal{M}N\Phi = R(t)^{\mu - 1} D_1 N_1\Phi$. It follows that

$$\int_{B_T} \mathcal{L}N\Phi \cdot \mathcal{M}N\Phi |dtdx$$

$$= \left( \int_{B_T} - \int_{\mathbb{R}^3} \frac{1}{2} R(t)^{\mu - \delta}(D_1 N_1\Phi)^2 + \varepsilon^2 |\nabla_1 N\Phi|^2)dsight.$$  

$$+ \int_{B_T} \left\{ \frac{\delta}{2} R(t)^{\mu - 1 - \delta} \nabla_1^2 D_1 N_1\Phi \right\} |dtdx.$$  

(5.67)

This yields

$$\int_{B_T} \mathcal{L}N\Phi \cdot \mathcal{M}N\Phi |dtdx$$

$$= \int_{B_T} \left[ \frac{\delta}{2} R(t)^{\mu - 1 - \delta} \nabla_1^2 D_1 N_1\Phi \right] |dtdx.$$  

(5.68)
Then similar to (A.8), we have
\[
\left| \int_{\Omega_T} M_1 \cdot \vec{\mathcal{M}} N \Phi dt dx \right| \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \sum_{i=0}^2 (R(t)^{\mu_1-3\gamma_1+1} D_i^2 \Phi^2) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx \\
+ C \varepsilon \int_{\Omega_T} (R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2) dt dx \\
+ R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx. \tag{5.73}
\]
Moreover, by the expression of $N$ and lemmas 5.1–5.4, corollary 1.5, we have
\[
\left| \int_{\Omega_T} M_2 \cdot \vec{\mathcal{M}} N \Phi dt dx \right| \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \sum_{i=0}^2 (R(t)^{\mu_1-3\gamma_1+1} D_i^2 \Phi^2) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx \\
+ C \varepsilon \int_{\Omega_T} (R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2) dt dx \\
+ R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx. \tag{5.74}
\]
Then as for (A.10)–(A.11), (A.22), (A.37)–(A.39), (A.43)–(A.44) and (5.10), we obtain
\[
\left| \int_{\Omega_T} M_2 \cdot \vec{\mathcal{M}} N \Phi dt dx \right| \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \sum_{i=0}^2 (R(t)^{\mu_1-3\gamma_1+1} D_i^2 \Phi^2) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx \\
+ C \varepsilon \int_{\Omega_T} (R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2) dt dx \\
+ R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx. \tag{5.75}
\]
Finally, substituting (5.69), (5.71) and (5.74)–(5.75) into (5.68) completes the proof of the lemma. \qed

As shown in corollary 5.4, the operators $S_1$ and $Z$ are used in the energy estimates. However, $R(t)\delta_i$ ($1 \leq i \leq 3$) are equivalent to $S_1$ and $Z$ only when $r > \frac{1}{3} R(t)$. Hence, we still need to obtain estimates for $r \leq \frac{1}{3} R(t)$. For this purpose, set $\nu(s) = 1 - \chi(s)$ supported in $[0, \frac{2}{3}]$ with $\chi(s)$ defined in (5.19).

**Lemma 5.6 (Estimates near $r = 0$).** Under the assumptions of theorem 5.1, we have
\[
\int_{S_T} R(T)^{\mu-\delta} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^3 \Phi)^2 \\
+ R(T)^{\mu-3\gamma_1-1} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^4 \Phi)^2 \right) dS \\
+ \int_{\Omega_T} \left( R(t)^{\mu-1-\delta} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^3 \Phi)^2 \right) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^4 \Phi)^2 \right) dt dx \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \sum_{i=0}^2 (R(t)^{\mu_1-3\gamma_1+1} D_i^2 \Phi^2) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx \\
+ C \varepsilon \int_{\Omega_T} (R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2) dt dx \\
+ R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx. \tag{5.76}
\]
\[
\int_{S_T} \left( R(T)^{\mu-\delta} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^3 \Phi)^2 \\
+ R(T)^{\mu-3\gamma_1-1} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^4 \Phi)^2 \right) dS \\
+ \int_{\Omega_T} \left( R(t)^{\mu-1-\delta} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^3 \Phi)^2 \right) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \frac{r}{R(t)} \right) (D_i (R(t) \nabla_i)^4 \Phi)^2 \right) dt dx \\
\leq C \varepsilon^2 + C \varepsilon \int_{\Omega_T} \sum_{i=0}^2 (R(t)^{\mu_1-3\gamma_1+1} D_i^2 \Phi^2) dt dx \\
+ R(t)^{\mu-1-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx \\
+ C \varepsilon \int_{\Omega_T} (R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2) dt dx \\
+ R(t)^{\mu+5-3\gamma_1+2} \left( \nabla_{1,t}^l, \nabla_{1,x}^l \Phi \right)^2 dt dx. \tag{5.77}
\]

**Proof.** By direct computation, we have
\[
(R(t) \delta_i) \mathcal{L} = \mathcal{L}(R(t) \delta_i). \tag{5.78}
\]
This, together with theorem 4.1, yields for $0 \leq k \leq 2$,
\[
\int_{S_{tr}} \left[ R(T)^\mu \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2 \right] dS + R(T)^{\mu - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( \nabla_i (R(t) \nabla_i) \psi \right)^2 dS + \int_{\Omega_T} \left[R(t)^{\mu - 1 - \beta} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2 \right] dtdx + \frac{R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( \nabla_i (R(t) \nabla_i) \psi \right)^2}{dS} + \frac{\int_{\Omega_T} R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2}{dtdx} \leq C \int_{\Omega_T} R(t)^{\mu - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2 dtdx + \nu \left( \frac{r}{R(t)} \right) \lambda \left( R(t) \nabla_b \psi \right)^2 dtdx.
\]
Note that the function $\nu \left( \frac{r}{R(t)} \right)$ has a compact support away from $r = 0$ so that the first term on the right-hand side of (5.79) can be estimated as in corollary 5.4.

On the other hand, by a similar argument for $\int_{\Omega_T} L \nabla^2 \psi \cdot \nabla \nabla^2 \psi dtdx$ in (A.35), we can obtain
\[
\int_{\Omega_T} L \nabla \nabla \psi \cdot \nabla \nabla^2 \psi dtdx \leq C \epsilon^2 + C \epsilon \int_{\Omega_T} \frac{\int_{\Omega_T} \left|R(t)^{\mu - 1 - \beta} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2 \right| dtdx}{dS} + R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( \nabla_i (R(t) \nabla_i) \psi \right)^2 dtdx + \frac{\int_{\Omega_T} R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2}{dtdx} \leq C \epsilon^2 + C \epsilon \int_{\Omega_T} R(t)^{\mu - 1 - \beta} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2 dtdx + R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( \nabla_i (R(t) \nabla_i) \psi \right)^2 dtdx + \frac{\int_{\Omega_T} R(t)^{\mu - 1 - 3(\gamma - 1)} \nu \left( \frac{r}{R(t)} \right) \left( D_i (R(t) \nabla_i) \psi \right)^2}{dtdx} \right] dtdx.
\]
Then (5.76) follows from (5.79)–(5.80) and corollary 5.4. In addition, similar to (5.76), we obtain (5.77). And this completes the proof of the lemma. \[\square\]

Based on lemmas 5.1–5.6 and remark 5.2–Corollary 5.4, we are ready to prove theorem 5.1.

\section*{Proof of theorem 5.1.} By corollary 5.4 and lemma 5.6, for sufficiently small $\epsilon > 0$, we have
\[
\sum_{k=0}^2 \int_{S_{tr}} \left( R(T)^{\mu + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 \right) dS + R(T)^{\mu - 3(\gamma - 1) + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dS + \int_{S_{tr}} \left( R(T)^{\mu + 6 - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 \right) dS + R(T)^{\mu - 3(\gamma - 1) + 6 - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dS + \sum_{k=0}^2 \int_{\Omega_T} \left( R(t)^{\mu - 1 - \beta + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 \right) dtdx + R(t)^{\mu - 1 - 3(\gamma - 1) + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx + \int_{\Omega_T} \left( R(t)^{\mu + 5 - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 \right) dtdx + R(t)^{\mu + 5 - 3(\gamma - 1) - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx \leq C \epsilon^2 + C \epsilon \int_{\Omega_T} R(t)^{\mu - 1 - \beta + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx + R(t)^{\mu - 1 - 3(\gamma - 1) + 2k} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx + \int_{\Omega_T} R(t)^{\mu + 5 - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx + R(t)^{\mu + 5 - 3(\gamma - 1) - \epsilon} \left( \nabla_{i, t}^k \nabla_t \psi \right)^2 dtdx \right] dtdx.
\]
then it follows from (5.81) that
\[
\begin{align*}
\int_{\Omega_T} \sum_{l=0}^{2} (R(t))^{\mu-1-\delta/2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx &+ R(t)^{\mu-3(\gamma - 1) + 2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&+ \int_{\Omega_T} (R(t))^{\mu+5-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&+ \int_{\Omega_T} (R(t))^{\mu+5-3(\gamma - 1)-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&\leq C \left( \sum_{l=0}^{2} (R(t))^{\mu-1-\delta/2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx + R(t)^{\mu-3(\gamma - 1) + 2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx + \int_{\Omega_T} (R(t))^{\mu+5-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx + \int_{\Omega_T} (R(t))^{\mu+5-3(\gamma - 1)-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \right)^{\frac{1}{2}},
\end{align*}
\]
which implies
\[
\begin{align*}
\int_{\Omega_T} \sum_{l=0}^{2} (R(t))^{\mu-1-\delta/2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx &+ R(t)^{\mu-3(\gamma - 1) + 2} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&+ \int_{\Omega_T} (R(t))^{\mu+5-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&+ \int_{\Omega_T} (R(t))^{\mu+5-3(\gamma - 1)-\delta} (\nabla_{t,x}^l D, \Phi)^2 dt \, dx \\
&\leq C \xi^2.
\end{align*}
\]
Substituting this into (5.81) derives (5.2) and (5.3), and then it completes the proof of theorem 5.1. \(\square\)

**Remark 5.4.** For the spherically symmetric solution, one can relax the restriction on \(1 < \gamma < \frac{4}{3}\) in theorem 1.1 and theorem 5.1 to \(1 < \gamma < \frac{5}{3}\). The main reasons come from the simplified boundary conditions (5.18) and (5.64), and the assumption (5.2) in the domain \(\{(t,x): t > 0, r > \frac{1}{3} R(t)\}\) can be replaced by
\[
\begin{align*}
|Z\Phi| &\leq M_{\sigma} R(t)^{-\sigma} \quad \text{if} \quad \gamma \in \left(1, \frac{4}{3}\right] \cup \left[\frac{4}{3}, \frac{5}{3}\right), \quad |Z\Phi| \leq M_{z} \ln R(t) \quad \text{if} \quad \gamma = \frac{4}{3};
\end{align*}
\]
where \(\sigma = \min \{0, 3(\gamma - 1) - 1\}\). We omit the details of the estimation for this special case.

6. Proofs of theorem 1.1 and 1.2

To complete the proof of theorem 1.2, as in [17], the following estimate is needed.

**Lemma 6.1.** For \(1 \leq l \leq T_0\) and \(k_0 \geq 4\), we have that for a smooth function \(\varphi(t, x) \in H^{k_0}(\Omega_T)\)
\[
\sum_{0 \leq l \leq k_0-4} |t^l \nabla_t^{l+1} \varphi(t, x)|^2 \leq C t^{-3} \sum_{0 \leq l \leq k_0} |t^l \nabla_t^{l+1} \varphi(t, x)|^2 dx.
\]

**Proof.** For any \(t_1 \in [1, T_0]\), set \((t', x') = \frac{1}{t_1} (t, x)\).

Then
\[
\nabla_{t,x}^{l+1} \varphi = \frac{1}{t_1^l} \nabla_{t',x'}^{l+1} \varphi, \quad \forall \; k \in \mathbb{N}.
\]

Define \(D_{k} = \{(t', x'): l' = 1, |x'| \leq L\}\). Then by the Sobolev imbedding theorem and by noting that \(D_{k}\) satisfies the uniform interior cone condition,
\[
|\nabla_{t,x}^{l+1} \varphi|^2 (1, x') \leq C \int_{D_{k}} |\nabla_{t',x'}^{l+1} \varphi|^2 (1, x') dx'.
\]

In view of (6.2), one has
\[
|\nabla_{t,x}^{l+1} \varphi|^2 (t_1, x) \leq \frac{1}{t_1^l} |\nabla_{t,x}^{l+1} \varphi|^2 (1, x') \leq C \int_{D_{k}} |\nabla_{t',x'}^{l+1} \varphi|^2 (1, x') dx' = C \frac{1}{t_1^l} \sum_{0 \leq l \leq k_0} |t^l \nabla_t^{l+1} \varphi|^2 (t_1, x) \frac{1}{t_1^3} dx
\]
\[
= C \frac{1}{t_1^l} \int_{\Omega_T} \sum_{0 \leq l \leq k_0} |t^l \nabla_t^{l+1} \varphi|^2 (t_1, x) dx.
\]

This yields (6.1) for \(l = 0\). The cases of \(1 \leq l \leq k_0 - 4\) can be estimated similarly and this completes the proof of the lemma. \(\square\)

We now first prove theorem 1.2 as follows.

**Proof of theorem 1.2.** It follows from lemma 6.1 that, for \(0 \leq t \leq T\),
\[
\begin{align*}
\sum_{0 \leq l \leq k_0-4} |R(t)^{l+1} \nabla_t^{l+1} \varphi|^2 &\leq CR(t)^{-3} \int_{\Omega_T} \sum_{0 \leq l \leq k_0} |R(t)^{l+1} \nabla_t^{l+1} \varphi|^2 dx, \\
|R(t)^{k+1} \varphi|^2 &\leq CR(t)^{-3} \int_{\Omega_T} \sum_{0 \leq l \leq k_0} |R(t)^{l+1} \nabla_t^{l+1} \varphi|^2 dx, \\
\sum_{0 \leq l \leq k_0} |R(t)^{l+1} \nabla_t^{l+1} D, \Phi|^2 &\leq CR(t)^{-3} \int_{\Omega_T} \sum_{0 \leq l \leq k_0} |R(t)^{l+1} \nabla_t^{l+1} D, \Phi|^2 dx.
\end{align*}
\]
Thus, the (5.2) gives
\[
\begin{align*}
\int_{S_t} \sum_{0 \leq l \leq 2} |R(t)^{l} \nabla_{s}^l \Phi|^2 dx & \leq \varepsilon^2 R(t)^{-\mu + 3(\gamma - 1)}, \\
\int_{S_t} \sum_{0 \leq l \leq 2} |R(t)^{l} \nabla_{s}^l D_{i} \Phi|^2 dx & \leq \varepsilon^2 R(t)^{-\mu},
\end{align*}
\]
and
\[
\begin{align*}
\int_{S_t} \sum_{0 \leq l \leq 2} |R(t)^{l} \nabla_{s}^l \Phi|^2 dx & \leq \varepsilon^2 R(t)^{-\mu + 3(\gamma - 1) + \delta}, \\
\int_{S_t} \sum_{0 \leq l \leq 2} |R(t)^{l} \nabla_{s}^l D_{i} \Phi|^2 dx & \leq \varepsilon^2 R(t)^{-\mu + \delta}, \\
\int_{S_t} \sum_{0 \leq l \leq 3} |R(t)^{l} \nabla_{s}^l D_{i} \Phi|^2 dx & \leq \varepsilon^2 R(t)^{-\mu + 3(\gamma - 1) - \delta}.
\end{align*}
\]
Hence, we obtain
\[
\begin{align*}
\left| \nabla_{s} \Phi \right| & \leq C \varepsilon R(t)^{-\frac{3(\gamma - 1)}{2}}, \\
\left| R(t)^{l} \nabla_{s}^l D_{i} \Phi \right| & \leq C \varepsilon R(t)^{-3(\gamma - 1) - \delta}, \\
\left| R(t)^{l} \nabla_{s}^l D_{i} \Phi \right| & \leq C \varepsilon R(t)^{-\frac{3(\gamma - 1) - \delta}{2}}.
\end{align*}
\]

On the other hand, by the stream line equation
\[
\frac{dx(t)}{dt} = \frac{L_{i}}{R(t)}, \quad x_{i}(0) = x_{i}^{0}, \quad (i = 1, 2, 3),
\]
we have
\[
x_{i}(t) = x_{i}^{0} R(t) \quad (i = 1, 2, 3),
\]
where \((x_{1}^{0}, x_{2}^{0}, x_{3}^{0})\) is the initial point. Then integrating along the stream line, we have for \(1 < \gamma < \frac{4}{3}\) that
\[
\begin{align*}
|R(t) \nabla_{s} \Phi(t, x(t))| & \leq \left| \nabla_{s} \Phi(0, x(0)) \right| + \int_{0}^{t} \left| D_{i} R(t) \nabla_{s} \Phi \right| dt \\
& \leq C \varepsilon \left( 1 + R(t)^{-3(\gamma - 1) + \frac{\delta}{2}} \right),
\end{align*}
\]
which implies
\[
|\nabla_{s} \Phi| \leq C \varepsilon R(t)^{-1} + R(t)^{-3(\gamma - 1) + \frac{\delta}{2}} \leq C \varepsilon R(t)^{-3(\gamma - 1) + \frac{\delta}{2}}.
\]
Thus, the first inequality in (5.1) is proved. On the other hand, it follows from corollary 5.4 and theorem 5.1 that
\[
\begin{align*}
\sum_{0 \leq b + c \leq 2} \left( R(t)^{b} \int_{S_{t}} (D_{i} S^{0} S_{s}^{c} \Phi)^{2} dx \right) & + R(t)^{\mu - 3(\gamma - 1)} \int_{S_{t}} (\nabla_{s} S^{0} S_{s}^{c} \Phi)^{2} dx \\
& + \sum_{b + c + d \geq 3} \left( R(t)^{b} \int_{S_{t}} (D_{i} S^{0} S_{s}^{c} S_{s}^{d} \Phi)^{2} dx \right) \\
& + R(t)^{\mu - 3(\gamma - 1)} \int_{S_{t}} (\nabla_{s} S^{0} S_{s}^{c} S_{s}^{d} \Phi)^{2} dx \\
& \leq C \varepsilon^{2}.
\end{align*}
\]
Subsequently, one has
\[
|Z D_{i} \Phi| \leq C \varepsilon R(t)^{-3(\gamma - 1)}, \quad |\nabla_{s} D_{i} \Phi| \leq C \varepsilon R(t)^{-\frac{3(\gamma - 1)}{2}}.
\]
In addition, for \(1 < \gamma < \frac{4}{3}\) and \(r > \frac{1}{3} R(t)\), integrating along the stream yields
\[
\begin{align*}
|Z \Phi(t, x(t))| & \leq |Z \Phi(0, x(0))| + \int_{0}^{t} |D_{i} (Z \Phi)| dt \\
& \leq C \varepsilon \left( 1 + R(t)^{1 - 3(\gamma - 1) - \delta} \right) \leq C \varepsilon R(t)^{1 - 3(\gamma - 1) - \delta}.
\end{align*}
\]
Note that the generic constant \(C > 0\) appeared in this section depends only on the initial data. Then we can choose the constant \(M = 2C\) in (5.1)–(5.2) for small \(\varepsilon > 0\) so that (5.1) and (5.2) hold. In this case, by the Bernoulli law (1.7), we have \(c^{2}(\rho) = c^{2}(\hat{\rho}) - (\gamma - 1) D_{i} \hat{\Phi} - \frac{\gamma - 1}{2} \left| \nabla_{s} \hat{\Phi} \right|^{2}\), which gives \(C (\gamma - 1) - C \varepsilon R(t)^{3(\gamma - 1)} < c^{2}(\rho) < CR(t)^{3(\gamma - 1)} + C \varepsilon R(t)^{3(\gamma - 1)}\). Thus, one obtains \(c^{2}(\rho) > 0\) for any \(t > 0\) and small \(\varepsilon > 0\). Therefore, the proof of theorem 1.2 is completed by the local existence result in lemma 3.1 and continuation argument.

Finally, theorem 1.1 follows.

**Proof of theorem 1.1.** Under the assumptions of theorem 1.1, it follows from lemma 3.1 and the smallness of \(L\) that (1.1)–(1.2) has a local solution \((\rho(\cdot, x), u(\cdot, x))\) that satisfies
\[
\|\rho_{0}(1, x) - 1\|_{HS^{3(\gamma - 1)}} + \|u(1, x)\|_{HS^{3(\gamma - 1)}} < \delta_{0},
\]
and
\[
\text{rot } u(1, x) \equiv 0,
\]
where \( S^l = \{ x : |x| = 1 + L \}, \delta_0 > 0 \) is a small number depending only on \( L \) and \( \epsilon_0 \). This, together with theorem 1.2 for the time \( t \geq 1 \), yields theorem 1.1.

\[ \square \]

7. Future directions

In this paper, together with [1, 2], we have shown the global existence of smooth solutions to the compressible irrotational Euler equations, compressible Navier–Stokes equations and Boltzmann equations when polytropic gases lie in a 3D expanding ball with the approximate constant speed. In fact, there are still many related problems that can be discussed or studied:

- If the compressible full Euler equations are rotational, can we get the global stability of smooth solutions as in theorem 1.1 and theorem 1.2?
- If the expanding speed of the initial ball is not an approximate constant, i.e., the 3D expanding ball is given by \( \Omega = \{(t, x) : t \geq 0, |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R(t)\} \), where \( R(t) = (1 + L) \) for \( t \geq 1 \), the constants \( L > 0 \) and \( \alpha > 0 \) with \( \alpha = 1 \) hold, can we get the global existence of smooth solutions for the compressible Euler equations, compressible Navier–Stokes equations and Boltzmann equations respectively?
- By our knowledge, combining the results in this paper with [1, 2] yields the first global existence result of non-trivial smooth solutions simultaneously for the compressible irrotational Euler equations, compressible Navier–Stokes equations and Boltzmann equations. A natural and interesting problem arises: in this case, can we get the global stable boundary layer when the viscosity coefficients of gases vanish?

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Appendix

In the appendix, we will prove that lemma 5.1 holds for two special cases of \( S^m = S_0^m \) and \( S^m = Z^m \).

Lemma A.1. (5.5) and (5.6) hold for \( S^m = S_0^m \).

Proof. At first we derive an explicit representation of \( LS_0^m \Phi \) for later use. By direct computation, one has
\[
S_0 L = LS_0 - 2LL + 3(\gamma - 1)\epsilon^2 \Delta. \tag{A.1}
\]
By induction, for \( 1 \leq m \leq 3 \), we obtain
\[
LS_0^m = S_0^m L + B_{1m} + B_{2m} \tag{A.2}
\]
with
\[
B_{1m} = \sum_{0 \leq i \leq m-1} C_{im} S_0^i L + \sum_{0 \leq i \leq m-2} C'_{im} S_0^i (\epsilon^2 \Delta),
\]
\[
B_{2m} = -3m(\gamma - 1)S_0^{m-1} (\epsilon^2 \Delta), \tag{A.3}
\]
where \( B_{2m} \Phi \) contains the \((m + 1)-\)th order (the highest order) derivatives of \( \Phi \), but \( B_{1m} \Phi \) only includes the terms \( \nabla_\alpha^m \Phi \) with \( |\alpha| \leq m \) (the lower order derivatives of \( \Phi \)) and the term \( \nabla_\alpha^{m+1} \Phi \) with small coefficients. In addition, from equation (4.15), we have that for \( 0 \leq m \leq 3 \)
\[
S_0^m L \Phi = I_1^m + I_2^m + I_3^m, \tag{A.4}
\]
where
\[
I_1^m = \sum_{i=1}^3 \tilde{f}_0 \partial_i D_i S_0^m \Phi + \sum_{1 \leq i < j \leq 3} \tilde{f}_0^2 \partial_i \partial_j S_0^m \Phi + \sum_{1 \leq i \leq 3} \tilde{f}_0^2 \partial_i^2 S_0^m \Phi,
\]
\[
I_2^m = \sum_{i=1}^3 \tilde{f}_0 [S_0^m, \partial_i D_i] \Phi + \sum_{0 \leq \alpha \leq m-3} \tilde{f}_0^0 [S_0^m, \partial_i] \Phi + \sum_{1 \leq i \leq 3} \tilde{f}_0 [S_0^m, \partial_i^2] \Phi,
\]
\[
I_3^m = \sum_{1 \leq i < j \leq m} C_{lm} \sum_{l_i + l_j = l, l_i \geq 1} \hat{C}_{l_{i,l_j}} \left\{ \sum_{i=1}^3 \tilde{f}_0^l \partial_i S_0^l \Phi \left( \partial_i D_i \Phi \right) + \sum_{1 \leq k \leq 3} (S_0^l \tilde{f}_0^l) S_0^k \left( \partial_i D_i \Phi \right) \right\} + S_0^m \tilde{f}_0^0.
\]
Based on the above preparation, we now treat \( \int_{\Omega} LS_0^m \Phi \cdot MS_0^m \Phi dtdx \) on the right-hand side of (5.8) as follows. This procedure is divided into five parts.

Part 1. Estimate on \( \int_{\Omega_x} I_1^m \cdot MS_0^m \Phi dtdx \)

Note that for \( C^k \)-smooth functions \( g \), one has
\[
D_t g = \partial_t g + \sum_{i=1}^3 \partial_i \left( \frac{L}{R(t)} x_i g \right) - \frac{3L}{R(t)} g.
\]
Then
\[
\int_{\Omega_t} D_t g dx = \int_{S_t} g dS - \int_{\Omega_t} 3L_0 R(t) g dx.
\] (A.5)

In addition, for \( m \leq 3 \), direct computation yields
\[
I_m^m = \mathcal{L} S_m^m \Phi
\]
\[
= D_t \left( \frac{1}{2} a(t) R(t) \mu \sum_{i,j=1}^{3} (\tilde{f}_{ij} \partial_i S^m_0 \Phi \partial_j S^m_0 \Phi) \right)
+ \sum_{i=1}^{3} \partial_i \left( \frac{1}{2} a(t) R(t) \mu \tilde{f}_{0i} (D_t S^m_0 \Phi)^2 \right)
+ a(t) R(t) \mu D_t S^m_0 \Phi \sum_{i=1}^{3} \tilde{f}_{ij} \partial_j S^m_0 \Phi
- \frac{1}{2} a(t) R(t) \mu (D_t S^m_0 \Phi)^2 \sum_{i=1}^{3} \partial_i \tilde{f}_{0i}
- a(t) \mu L R(t)^{\mu-1} \sum_{i,j=1}^{3} (\tilde{f}_{ij} \partial_i S^m_0 \Phi \partial_j S^m_0 \Phi)
- a(t) R(t) \mu D_t S^m_0 \Phi \sum_{i=1}^{3} (\partial_i \tilde{f}_{ij} \partial_j S^m_0 \Phi)
+ \frac{1}{2} \sum_{i,j=1}^{3} (D_t (a(t) R(t) \mu \tilde{f}_{ij}) \partial_i S^m_0 \Phi \partial_j S^m_0 \Phi).
\] (A.6)

On the other hand, by the expressions of \( \tilde{f}_{ij} \) and \( \tilde{f}_{0ij} \), and (5.7) on \( B_t \), we observe that
\[
\sum_{i,j=1}^{3} x_i \left[ \left( \frac{1}{2} a(t) R(t) \mu \tilde{f}_{0ij} (D_t S^m_0 \Phi) \right)^2
+ a(t) R(t) \mu D_t S^m_0 \Phi \sum_{j=1}^{3} \tilde{f}_{ij} \partial_j S^m_0 \Phi \right]
= -a(t) R(t) \mu (D_t S^m_0 \Phi)^2 \sum_{i=1}^{3} (x_i \partial_i \tilde{f}_{ij})
- a(t) R(t) \mu D_t S^m_0 \Phi \sum_{i=1}^{3} (x_i \partial_i \tilde{f}_{ij})
- (\gamma - 1) a(t) R(t) \mu D_t S^m_0 \Phi
\times (D_t \Phi + \frac{3}{2} \sum_{k=1}^{3} (\partial_k \Phi)^2) \sum_{i=1}^{3} x_i \partial_i S^m_0 \Phi
= 0.
\] (A.7)

Thus, by (A.6) together with (A.5) and (A.7), it follows from the expressions of \( \tilde{f}_{ij}, \tilde{f}_{0ij}, \tilde{f}_{00} \) and assumption (5.1) that
\[
\int_{\Omega_t} I_m^m \cdot \mathbf{S}_m^m \Phi dx dt \leq C \varepsilon^2 + C \varepsilon \left( R(T)^{\mu} \int_{S_t} (D_t S^m_0 \Phi)^2 dx + R(T)^{\mu-3(\gamma-1)} \int_{S_t} (\nabla S^m_0 \Phi)^2 dx \right)
+ \int_{\Omega_t} (R(t)^{\mu-1-\delta} (D_t S^m_0 \Phi)^2 dx)
+ R(t)^{\mu-3(\gamma-1)} \int_{\Omega_t} (\nabla S^m_0 \Phi)^2 dx dt
+ R(t)^{\mu-3(\gamma-1)} \int_{\Omega_t} (\nabla S^m_0 \Phi)^2 dx dt.
\] (A.8)

where we have used
\[
|a(t) R(t)^{\mu} (D_t S^m_0 \Phi)^3 \sum_{i=1}^{3} \partial_i \tilde{f}_{00}|
\leq C \varepsilon \varepsilon (R(t)^{\mu-1-3(\gamma-1)-\delta} (D_t S^m_0 \Phi)^2)
\leq C \varepsilon \varepsilon (R(t)^{\mu-1-\delta} (D_t S^m_0 \Phi)^2) \quad \text{for} \quad 0 < \delta \leq \gamma - 1.

Part 2. Estimate on \( \int_{\Omega_t} I_m^m \cdot \mathcal{L} S_m^m \Phi dx dt \)

It follows from the expressions of \( \tilde{f}_{ij}, \tilde{f}_{0ij}, \tilde{f}_{00} \), and (5.1) that
\[
|I_m^m| \leq C \varepsilon \left( R(t)^{-3(\gamma-1)-1+\delta} \sum_{0 \leq |\ell| \leq m-1} |\nabla \mathcal{S}^{1+1}_0 \Phi| \right)
+ R(t)^{-3(\gamma-1)} \sum_{0 \leq |\ell| \leq m-1} |\nabla^2 \mathcal{S}^{1}_0 \Phi|.
\]

This derives that for \( m \leq 2 \)
\[
\int_{\Omega_t} |I_m^m| \cdot \mathcal{L} S_m^m \Phi dx dt \leq C \varepsilon \int_{\Omega_t} \sum_{i=0}^{m} (R(t)^{-1-\delta+2i} (\nabla^i \mathcal{L} D_s \Phi)^2 dx)
+ R(t)^{-3(\gamma-1)+1} \sum_{0 \leq |\ell| \leq m-1} |\nabla^2 \mathcal{S}^{1}_0 \Phi| dx dt.
\] (A.9)

and for \( m = 3 \)
\[
\int_{\Omega_t} |I_m^m| \cdot \mathcal{L} S_m^m \Phi dx dt \leq C \varepsilon \int_{\Omega_t} \sum_{i=0}^{2} (R(t)^{-1-\delta+2i} (\nabla^i \mathcal{L} D_s \Phi)^2 dx)
+ R(t)^{-5(\gamma-1)+1} \sum_{0 \leq |\ell| \leq m-1} |\nabla^2 \mathcal{S}^{1}_0 \Phi| dx dt.
\] (A.10)

Part 3. Estimate on \( \int_{\Omega_t} I_3^m \cdot \mathcal{L} S_3^m \Phi dx dt \)

First, we treat the case of \( \int_{\Omega_t} I_3^m \cdot \mathcal{L} S_3^m \Phi dx dt \) with \( m \leq 2 \). For \( m \leq 2 \), as in Part 2, it follows from the expression of \( f_3 \) and assumption (5.1) that
\[
|I_3^m| \leq C \varepsilon \left( R(t)^{-3(\gamma-1)+1+\delta} \sum_{0 \leq |\ell| \leq m} |\nabla \mathcal{S}^{1}_0 \Phi| \right)
+ R(t)^{-3(\gamma-1)+1} \sum_{0 \leq |\ell| \leq m} |\nabla^2 \mathcal{S}^{1}_0 \Phi|.
\]

which yields for \( m \leq 2 \)
\[
\int_{\Omega_t} |I_3^m| \cdot \mathcal{L} S_3^m \Phi dx dt \leq C \varepsilon \int_{\Omega_t} \sum_{i=0}^{m} (R(t)^{-1-\delta+2i} (\nabla^i \mathcal{L} D_s \Phi)^2 dx)
+ R(t)^{-3(\gamma-1)+1} \sum_{0 \leq |\ell| \leq m} |\nabla^2 \mathcal{S}^{1}_0 \Phi| dx dt.
\] (A.11)

Next we deal with \( \int_{\Omega_t} I_3^m \cdot \mathcal{L} S_3^m \Phi dx dt \). Note that the difficult terms in \( I_3^m \) are those that include the products of third-order derivatives of \( \Phi \) because there are no related weighted \( L^\infty \) estimates in (5.1). For the convenience, we decompose \( I_3^m \) into \( J_1 \) and \( J_2 \) by using \( S_3^m = LR(t)D_s + R(t)^2 D_s^2 \), where only \( J_2 \) contains the products of third-order derivatives of \( \Phi \).
Namely,
\[ I_2^3 = J_1 + J_2, \]
where
\[
J_1 = \sum_{1 \leq i \leq 3} C_{3m} \left\{ \sum_{l \neq i} \sum_{l \neq j \neq 0, l \neq 1} \tilde{C}_{ikl} \left( \sum_{i=1}^{3} (S_{ij}^l \tilde{f}_0^i) S_{ij}^l (\partial_i D_i \Phi) \\
+ \sum_{1 \leq i \leq 3} (S_{ij}^l \tilde{f}_0^i) S_{ij}^l (\tilde{\partial}_0^2 \tilde{\Phi}) \right) \right\}
+ \sum_{1 \leq i \leq 3} \tilde{C}_{3i} \left( \sum_{i=1}^{3} (S_{ij}^l \tilde{f}_0^i) S_{ij}^l (\partial_i D_i \Phi) \right)
+ \sum_{1 \leq i \leq 3} \tilde{C}_{31} \left( \sum_{i=1}^{3} (S_{ij}^l \tilde{f}_0^i) S_{ij}^l (\tilde{\partial}_0^2 \tilde{\Phi}) \right)
+ \sum_{1 \leq i \leq 3} (LS_{ij}^0 \tilde{f}_0^i) S_{ij}^l (\partial_i D_i \Phi) + S_{ij}^0 \tilde{f}_0^i,
\]
and
\[
J_2 = C_{33} \tilde{C}_{321} \left[ \sum_{i=1}^{3} (R(t)^2 D_i \tilde{f}_0^i) S_{ij}^l (\partial_i D_i \Phi) \right]
+ \sum_{1 \leq i \leq 3} \left( R(t)^2 D_i \tilde{f}_0^i \right) S_{ij}^l (\tilde{\partial}_0^2 \tilde{\Phi}) \right].
\]
By assumption (5.1) and the expressions of \( \tilde{f}_0^i, \tilde{f}_0^j, \tilde{f}_0^k \), direct computation yields
\[
|J_1| \leq C \left[ R(t)^{3\gamma -1} - 1 + \frac{2}{\gamma + 1} \sum_{i,j \leq 2} |\nabla_S^l \phi| \right] + R(t)^{3\gamma -1} \sum_{\tilde{0} \leq i \leq 2} |\nabla^2 \phi| \right].
\]
On the other hand, by the expression of \( f_0 \), we further decompose \( J_2 \) into \( J_2 = J_{21} + J_{22} \) so that only \( J_{22} \) contains the product terms of third-order derivatives of \( \phi \). More precisely,
\[
J_2 = C_{33} \tilde{C}_{321} \left[ \sum_{i=1}^{3} (R(t)^2 D_i \tilde{f}_0^i) S_{ij}^l (\partial_i D_i \Phi) \right]
+ \sum_{1 \leq i \leq 3} \left( R(t)^2 D_i \tilde{f}_0^i \right) S_{ij}^l (\tilde{\partial}_0^2 \tilde{\Phi}) \right].
\]
and
\[
|J_{21}| \leq C \left[ R(t)^{3\gamma -1} - 1 + \frac{2}{\gamma + 1} \sum_{i,j \leq 2} |\nabla_S^l \phi| \right] + R(t)^{3\gamma -1} \sum_{\tilde{0} \leq i \leq 2} |\nabla^2 \phi| \right].
\]
Combining (A.14) and (A.15) yields
\[
\int_{\Omega} |(J_1 + J_2) \cdot \mathcal{M} \dot{\phi}]dtdx
\leq C \sum_{l=0}^{1} \int_{\Omega} \left[ R(t)^{\gamma - \frac{1}{2} + 2} \left( \nabla_{i,x}^l D_i \dot{\phi} \right)^2 \right]
+ R(t)^{\mu - 3\gamma -1 + 2} \left( \nabla_{i,x}^l \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2 dtdx
+ \int_{\Omega} (R(t)^{\mu + 5\gamma - 1}) \left( \nabla_{i,x}^l D_i \dot{\phi} \right)^2
dx + R(t)^{\mu + 5\gamma - 1} \left( \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2 dtdx.
\]
Finally, we estimate \( J_{22} \cdot \mathcal{M} \dot{\phi} ]dtdx \). To overcome the difficulty induced by the lack of weighted \( L^\infty \) estimates of \( |\nabla_{i,x}^l \dot{\phi} | \) in \( J_{22} \), we will use the interpolation inequalities in corollary 2.1 and lemma 2.4. In fact, by (5.1) and the expression of \( J_{22} \), it is sufficient to estimate some typical terms in \( \int_{\Omega} |J_{22} \cdot \mathcal{M} \dot{\phi} |dtdx \) as follows:
(A) Estimate on \( |R(t)^{\mu - 1}(\partial \dot{\phi}) D_i \dot{\phi} \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} |_{L^2(x,t)} \)
\[
|R(t)^{\mu - 1}(\partial \dot{\phi}) D_i \dot{\phi} \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} |_{L^2(x,t)} \]
\leq C \int_{\Omega} \sum_{l=0}^{1} \left[ R(t)^{\mu + \delta} \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right] \left( \nabla_{i,x}^l D_i \dot{\phi} \right)
+ C \int_{\Omega} (R(t)^{\mu + 5\gamma - 1}) \left( \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2
dx + R(t)^{\mu + 5\gamma - 1} \left( \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2 dtdx.
\]
(B) Estimate on \( |R(t)^{\mu}(\partial \dot{\phi}) \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} |_{L^2(x,t)} \)
\[
|R(t)^{\mu}(\partial \dot{\phi}) \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} \partial \dot{\phi} \partial \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} D_i \dot{\phi} |_{L^2(x,t)} \]
\leq C \int_{\Omega} \sum_{l=0}^{1} \left[ R(t)^{\mu + \delta} \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right] \left( \nabla_{i,x}^l D_i \dot{\phi} \right)
+ C \int_{\Omega} (R(t)^{\mu + 5\gamma - 1}) \left( \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2
dx + R(t)^{\mu + 5\gamma - 1} \left( \nabla_{i,x}^l \nabla_{i,x}^l \dot{\phi} \right)^2 dtdx.
\]
First, from the expressions of $B_{1m}$ and $\hat{f}$, and by (5.1), we have
\begin{align*}
|B_{11}\Phi| & \leq C\varepsilon (R(t))^{-3(\gamma - 1) - \frac{2}{3}} |\nabla \Phi(D_{1})| + R(t)^{-3(\gamma - 1) - 1} |\nabla^{2}\Phi| \\
& \quad + R(t)^{-3(\gamma - 1) - 1 - \frac{2}{3}} |\nabla\Phi|,
\end{align*}
and
\begin{align*}
|B_{1m}\Phi| & \leq C\varepsilon \sum_{0 \leq \ell \leq m - 1} (R(t))^{-3(\gamma - 1) - \frac{2}{3}} |\nabla_{\ell} S_{0}\Phi| \\
& \quad + R(t)^{-3(\gamma - 1) - 1} |\nabla^{2} S_{0}\Phi| + R(t)^{-3(\gamma - 1) - 1 - \frac{2}{3}} |\nabla_{\ell} S_{0}\Phi| \\
& \quad \times \sum_{0 \leq \ell \leq m - 2} |\Delta S_{0}\Phi|, \quad m = 2, 3.
\end{align*}

Since (5.3) holds for $t \leq m - 1$, we have from (A.21)–(A.22) that for $m \leq 2$,
\begin{align*}
\int_{\Omega_{t}} |B_{1m}\Phi \cdot \mathcal{M}_{S_{0}}^{m}\Phi| dx \\
& \leq C\varepsilon \sum_{j = 0}^{m} \int_{\Omega_{t}} (R(t))^{-1 - \frac{2}{3}} |\nabla_{j} S_{0}\Phi|^{2} dx \\
& \quad + C\varepsilon \int_{\Omega_{t}} (R(t))^{-1 - 3(\gamma - 1) - 2} |\nabla_{j}^{2} S_{0}\Phi|^{2} dx \\
& \quad + C\varepsilon \int_{\Omega_{t}} (R(t))^{-1 - 3(\gamma - 1) - \frac{2}{3}} |\nabla_{j}^{3} S_{0}\Phi|^{2} dx,
\end{align*}
and for $m = 3$,
\begin{align*}
\int_{\Omega_{t}} |B_{1m}\Phi \cdot \mathcal{M}_{S_{0}}^{m}\Phi| dx \\
& \leq C\varepsilon \sum_{j = 0}^{m} \int_{\Omega_{t}} (R(t))^{-1 - \frac{2}{3}} |\nabla_{j} S_{0}\Phi|^{2} dx \\
& \quad + C\varepsilon \int_{\Omega_{t}} (R(t))^{-1 - 3(\gamma - 1) - 2} |\nabla_{j}^{2} S_{0}\Phi|^{2} dx \\
& \quad + C\varepsilon \int_{\Omega_{t}} (R(t))^{-1 - 3(\gamma - 1) - \frac{2}{3}} |\nabla_{j}^{3} S_{0}\Phi|^{2} dx.
\end{align*}

Part 5. Estimate on $\int_{\Omega_{t}} B_{2m}\Phi \cdot \mathcal{M}_{S_{0}}^{m}\Phi dx$

Note that $B_{2m}\Phi$ contains the $(m + 1)$-th (the highest order) derivatives of $\Phi$ and $B_{2m}\Phi \cdot \mathcal{M}_{S_{0}}^{m}\Phi$ contains the term $\nabla_{t, x}^{m} \Phi \nabla^{3}_{x} \Phi$ ($|\alpha| = |\beta| = m + 1$). Thanks to the definite sign of the coefficient $-3m(\gamma - 1) < 0$ in $B_{2m}$ and (3.1), $\nabla_{t, x}^{m} \Phi \nabla^{3}_{x} \Phi$ with $|\alpha| = |\beta| = m + 1$ can be controlled. Since
\begin{equation}
D_{i} S_{0}^{m}\Phi = \sum_{0 \leq \ell \leq m - 1} C_{i m} D_{i} S_{0}^{\ell} \Phi + R(t) S_{0}^{m - 1} D_{i}^{2} \Phi,
\end{equation}
it follows from (3.1) that
\begin{align*}
D_{i} S_{0}^{m}\Phi = \sum_{0 \leq \ell \leq m - 1} C_{i m} D_{i} S_{0}^{\ell} \Phi + R(t) S_{0}^{m - 1} \\
\times \left( \hat{\varphi} \Delta \Phi - \frac{3L(\gamma - 1)}{R(t)} D_{i} \Phi + \mathcal{L} \Phi \right).
\end{align*}
Direct computation yields
\[
\int_{\Omega_T} B_{2m} \Phi \cdot \mathcal{M} S_0^m \phi dtdx
= \int_{\Omega_T} -3m(\gamma - 1) R(t)^{\mu} a(t) \{S_0^{m-1}(\xi^2 \Delta \phi)^2
+ \left( \sum_{0 \leq i \leq m-1} C_{ln D_i, D_i} S_0^i \phi + R(t) S_0^{m-1} \right)
\times \left( \mathcal{L} \Phi - \frac{3L(\gamma - 1)}{R(t)} D_i \phi \right) \} S_0^{m-1}(\xi^2 \Delta \phi) \} dtdx
\leq -3m(\gamma - 1) \int_{\Omega_T} R(t)^{\mu} a(t) \{ \sum_{0 \leq i \leq m-1} C_{ln D_i, D_i} S_0^i \phi
+ R(t) S_0^{m-1} \left( \mathcal{L} \Phi - \frac{3L(\gamma - 1)}{R(t)} D_i \phi \right) \} S_0^{m-1}(\xi^2 \Delta \phi) \} dtdx.
\]
Note that \( \sum_{0 \leq i \leq m-1} C_{ln D_i, D_i} S_0^i \phi \) only contains at most \( m \) - th order derivatives of \( \Phi \), then we have by (5.1) that, for \( 1 \leq m \leq 1 \),
\[
\int_{\Omega_T} R(t)^{\mu - 1 - \delta} \left( \sum_{0 \leq i \leq m-1} C_{ln D_i, D_i} S_0^i \phi \right) dtdx \leq C \epsilon^2.
\]
(5.29) On the other hand, we have
\[
|S_0^{m-1} \mathcal{L} \phi| \leq C \epsilon \left( \sum_{0 \leq i \leq m} R(t)^{3(\gamma - 1) - 1 - \frac{\gamma}{2}} |\nabla \phi| S_0^i \phi \right)
+ \sum_{0 \leq i \leq m-1} R(t)^{3(\gamma - 1)} |\nabla ^2 S_0^i \phi| \right).
\]
(5.30) Therefore, putting (5.29)–(5.30) into (5.28) yields for \( m \leq 2 \)
\[
\int_{\Omega_T} |B_{2m} \Phi \cdot \mathcal{M} S_0^m \phi| dtdx
\leq C \epsilon^2 + C \epsilon \left( \sum_{i=0}^{m} \int_{\Omega_T} R(t)^{\mu - 1 - \delta} |(\nabla \phi)^2| dtdx
+ R(t)^{\mu - 1 - 3(\gamma - 1)} |(\nabla \phi)^2| dtdx \right),
\]
(5.31) and for \( m = 3 \)
\[
\int_{\Omega_T} |B_{2m} \Phi \cdot \mathcal{M} S_0^m \phi| dtdx
\leq C \epsilon^2 + C \epsilon \left( \sum_{i=0}^{2} \int_{\Omega_T} R(t)^{\mu - 1 - \delta} |(\nabla \phi)^2| dtdx
+ R(t)^{\mu - 3(\gamma - 1)} |(\nabla \phi)^2| dtdx \right)
+ R(t)^{\mu + 5 - 3(\gamma - 1)} |(\nabla \phi)^2| dtdx.
\]
(5.32)

Consequently, putting (A.9)–(A.10), (A.21), (A.24), (A.29)–(A.30) into (5.8), we complete the proofs of (5.5) and (5.6) for \( S^m = S_0^m \). For the case of \( m = 0 \), (5.5) comes directly from theorem 4.1, (A.9)–(A.10) and (A.21).

**Lemma A.2.** (5.5) and (5.6) hold for \( S^m = Z^m \).

**Proof.** By direct computation, we have
\[
Z \mathcal{L} = \mathcal{L} Z,
\]
that implies
\[
\mathcal{L} Z^m = Z^m \mathcal{L}.
\]
In addition, from equation (4.15), we have that for \( 0 \leq m \leq 3 \),
\[
Z^m \mathcal{L} \phi = K_1^m + K_2^m + K_3^m,
\]
where
\[
K_1^m = \sum_{i=1}^{3} \tilde{f}_{ij} \partial_i \partial_i Z^m \phi
+ \sum_{i=1}^{3} \tilde{f}_{ij} \partial_i \partial_i Z^m \phi
+ \sum_{i=1}^{3} \tilde{f}_{ij} \partial_i \partial_i Z^m \phi,
\]
\[
K_2^m = \sum_{i=1}^{3} \tilde{f}_{ij} \partial_i \partial_i Z^m \phi
+ \sum_{i=1}^{3} \tilde{f}_{ij} \partial_i \partial_i Z^m \phi.
\]

Based on the above preparation, we now estimate \( \int_{\Omega_T} \mathcal{L} Z^m \phi \cdot \mathcal{M} Z^m \phi dtdx \) on the right-hand side of (5.8) in the following three steps.

**Part 1.** Estimate on \( \int_{\Omega_T} K_1^m \cdot \mathcal{M} Z^m \phi dtdx \) replacing \( S_0^m \) in (A.8) by \( Z \), we have as in (A.8),
\[
\int_{\Omega_T} K_{m}^m \cdot \mathcal{M} Z^m \phi dtdx
\leq C \epsilon^2 + C \epsilon \left( R(T)^{\mu - 3(\gamma - 1)} |(\nabla \phi)^2| dtdx
+ \int_{\Omega_T} (\nabla Z^m \phi)^2 dtdx
+ \int_{\Omega_T} R(t)^{\mu + 5 - 3(\gamma - 1)} |(\nabla \phi)^2| dtdx \right).
\]
(5.36)

**Part 2.** Estimate on \( \int_{\Omega_T} K_2^m \cdot \mathcal{M} Z^m \phi dtdx \)

direct computation yields
\[
|K_2^m| \leq C \epsilon \left( \sum_{i=1}^{3} |\tilde{f}_{ij}| |\sum_{0 \leq i \leq m-1} \nabla \phi| D_i \phi| \right)
+ \sum_{0 \leq i \leq m} \left( \sum_{0 \leq i \leq m-1} \nabla \phi \right).
\]
Then it follows from the expressions of \( \tilde{f}_{ij}, \tilde{f}_{ij} \) and (5.1) that
\[
|K_2^m| \leq C \epsilon \left( R(t)^{3(\gamma - 1) - \frac{\gamma}{2}} \sum_{0 \leq i \leq m-1} |\nabla \phi| D_i \phi| \right)
+ R(t)^{3(\gamma - 1)} \sum_{0 \leq i \leq m} |\nabla ^2 \phi| D_i \phi| \right).
\]
which gives that for $m \leq 2$,

$$\int_{\Omega_T} |K^m_2 \cdot \mathcal{M}^m \Phi| \, dt \, dx \leq C \varepsilon \sum_{l=0}^{m} \int_{\Omega_T} (R(t))^{\mu_1-1-\delta} (\nabla_{l,x}^l \partial_x \Phi)^2 + R(t)^{\mu_1-1-3(\gamma-1)+2l} (\nabla_{l,x}^l \nabla_x \Phi^2) \, dt \, dx,$$

and for $m = 3$,

$$\int_{\Omega_T} |K^m_3 \cdot \mathcal{M}^m \Phi| \, dt \, dx \leq C \varepsilon \sum_{l=0}^{m} \int_{\Omega_T} (R(t))^{\mu_1-1-\delta} (\nabla_{l,x}^l \partial_x \Phi)^2 + R(t)^{\mu_1-1-3(\gamma-1)+2l} (\nabla_{l,x}^l \nabla_x \Phi^2) \, dt \, dx + C \varepsilon \int_{\Omega_T} (R(t))^{\mu_1-1-5(\gamma-1)-\delta} \cdot (\nabla_{l,x}^l \nabla_x \Phi)^2 \, dt \, dx. \quad (A.38)$$

### Part 3. Estimate on $\int_{\Omega_T} K^m_3 \cdot \mathcal{M}^m \Phi \, dt \, dx$

For $m = 1$, it follows from the expressions of $f_0, f_0, f_0$ and (5.1) that

$$|K^1_3| \leq C (R(t)^{3(\gamma-1)-1} \cdot 2 \cdot |\nabla Z\Phi| + |\nabla Z\Phi| + R(t)^{3(\gamma-1)-1} |\nabla Z\Phi|).$$

This yields

$$\int_{\Omega_T} |K^1_3 \cdot \Phi| \, dt \, dx \leq C \varepsilon \sum_{l=0}^{1} \int_{\Omega_T} (R(t))^{\mu_1-1-\delta} (\nabla_{l,x}^l \partial_x \Phi)^2 + R(t)^{\mu_1-1-3(\gamma-1)+2l} (\nabla_{l,x}^l \nabla_x \Phi^2) \, dt \, dx. \quad (A.39)$$

For $m = 2$,

$$K^2_3 = K^3_1 + K^3_2,$$

where

$$K^3_1 = \sum_{1 \leq l \leq 2} C_{2,l} \left\{ \sum_{1 \leq j \leq 2} C_{l,j} \left( \sum_{i=1}^{3} (Z^1_i f_0^1) Z^j \partial_x (\partial_x D_l \Phi) + \sum_{1 \leq j \leq 3} (Z^3_i f_0^3) Z^j (\partial_x \Phi) \right) \right\} - \frac{3(\gamma - 5)}{2R(t)} \sum_{i=1}^{3} \partial_x \Phi Z^i \partial_x \Phi - \frac{3(\gamma - 5)}{R(t)} \sum_{i=1}^{3} (Z_i \partial_x \Phi)^2,$$

$$K^3_2 = -\frac{3(\gamma - 5)}{R(t)} \sum_{i=1}^{3} (Z_i \partial_x \Phi)^2.$$

Therefore, it follows from the expressions of $f_0, f_0, f_0$ and (5.1) that

$$|K^3_3| \leq C \varepsilon \left\{ (R(t)^{-3(\gamma-1)-1} \cdot 2 \cdot |\nabla Z\Phi| + R(t)^{-3(\gamma-1)-1} \cdot 2 \cdot |\nabla Z\Phi|) + R(t)^{-3(\gamma-1)+2} (\nabla_{l,x}^l \Phi)^2 \right\}.$$
which gives

\[ \int_{\Omega_T} |K_{31}^3 \cdot MZ^3 \Phi| dt dx \]

\[ \leq C \int_{\Omega_T} \left( R(t) \mu^{-1-\delta+2(\nabla_{l,x}^3 D_\Phi)^2} + R(t)^{\mu-1-3(\gamma-1)+2(\nabla_{l,x}^2 \nabla_{l,x} \Phi)^2} dt dx \right. \]

\[ + C \int_{\Omega_T} \left( R(t)^{\mu+5-\delta} (\nabla_{l,x}^3 D_\Phi)^2 + R(t)^{\mu+5-3(\gamma-1)-\delta} (\nabla_{l,x}^2 \nabla_{l,x} \Phi)^2 \right) dt dx. \] (A.44)

We now turn to estimate \( \int_{\Omega_T} |K_{32}^3 \cdot MZ^3 \Phi| dt dx \). Again, to overcome the lack of weighted \( L^\infty \) estimates of \( |\nabla_{l,x}^3 \Phi| \) in \( K_{32}^3 \), we use the interpolation inequalities in corollary 2.1 and lemma 2.4. In fact, by (5.1) and the expression of \( K_{32}^3 \), it suffices to estimate the following typical terms in \( \int_{\Omega_T} |K_{32}^3 \cdot MZ^3 \Phi| dt dx \): (A) Estimate on \( |R(t)^{\mu} \partial_t Z^2 \Phi \partial_t Z^2 D_\Phi D_3 \Phi |_{L^2(\Omega_T)} \)

\[ |R(t)^{\mu} \partial_t Z^2 \Phi \partial_t Z^2 D_\Phi D_3 \Phi |_{L^2(\Omega_T)} \leq C \int_{\Omega_T} \left( R(t)^{3\gamma-1-\delta} (\nabla_{l,x}^2 \nabla_{l,x} \Phi)^2 + R(t) \mu^{-1-\delta} D_3 \Phi |_{L^2} \right) \]

\[ \times \left( \sum_{k=0}^{\gamma-3-\delta} \frac{1}{2} \frac{(\nabla_{l,x}^{3-k} \nabla_{l,x} \Phi)^2}{L^2} \right) \]

\[ \sum_{k=0}^{\gamma-3-\delta} \frac{1}{2} \frac{(\nabla_{l,x}^{3-k} \nabla_{l,x} \Phi)^2}{L^2}. \] (A.45)

(B) Estimate on \( |R(t)^{\mu} \partial_t Z^2 \Phi \partial_t Z^2 D_3 \Phi |_{L^2(\Omega_T)} \)

\[ |R(t)^{\mu} \partial_t Z^2 \Phi \partial_t Z^2 D_3 \Phi |_{L^2(\Omega_T)} \leq C \int_{\Omega_T} \left( R(t)^{3\gamma-3-\delta} (\nabla_{l,x}^2 \nabla_{l,x} \Phi)^2 + R(t)^{\mu-1-3(\gamma-1)+2(\nabla_{l,x}^3 \nabla_{l,x} \Phi)^2} |_{L^2} \right) \]

\[ \times \left( \sum_{k=0}^{\gamma-3-\delta} \frac{1}{2} \frac{(\nabla_{l,x}^{3-k} \nabla_{l,x} \Phi)^2}{L^2} \right) \]

\[ \sum_{k=0}^{\gamma-3-\delta} \frac{1}{2} \frac{(\nabla_{l,x}^{3-k} \nabla_{l,x} \Phi)^2}{L^2}. \] (A.46)
(C) Estimate on $|R(t)^{\mu}D_{\dot{Z}}\Phi_{t}D_{\dot{Z}}\Phi_{t}|_{L_{2}(\Omega_{T})}$

\[
|R(t)^{\mu}D_{\dot{Z}}\Phi_{t}D_{\dot{Z}}\Phi_{t}|_{L_{2}(\Omega_{T})} \\
\leq C \left| R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \right| |R(t)^{\frac{3}{2} - \frac{3}{2}}\partial_{\dot{t}}^{2}Z\Phi_{t}|_{L_{2}} \\
\times R(t)^{\frac{3}{2} - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \\
\leq C \left( \sum_{k=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \right)^{\frac{1}{2}} \\
+ \sum_{k=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \\
\times R(t)^{\frac{3}{2} - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}.
\]

(Appllying Lemma 2.4 (iv) and (vii) for $\phi$)

\[
\leq C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}^{2} \\
+ R(t)^{\mu - \frac{3}{2}(\gamma - 1) + 2}|(\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}|dtdx \\
+ C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}(\gamma - 1) + 2}|(\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}|dtdx.
\]

(D) Estimate on $|R(t)^{\mu - 1}\partial_{\dot{t}}^{2}Z\Phi_{t}D_{\dot{Z}}\Phi_{t}|_{L_{2}(\Omega_{T})}$

\[
|R(t)^{\mu - 1}\partial_{\dot{t}}^{2}Z\Phi_{t}D_{\dot{Z}}\Phi_{t}|_{L_{2}(\Omega_{T})} \\
\leq C |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} |R(t)^{\frac{3}{2} - \frac{3}{2}}\partial_{\dot{t}}^{2}Z\Phi_{t}|_{L_{2}} \\
\times R(t)^{\frac{3}{2} - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \\
\leq C \left( \sum_{k=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \right)^{\frac{1}{2}} \\
+ \sum_{k=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}} \\
\times R(t)^{\frac{3}{2} - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}.
\]

(Applying Lemma 2.4 (iii) and (vii) for $\phi$)

\[
\leq C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}^{2} \\
\times (\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2} + R(t)^{\mu - \frac{3}{2}(\gamma - 1) + 2}|(\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}|dtdx \\
+ C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}^{2} \\
\times (\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}dtdx.
\]

Substituting estimates of (A)–(D) into $\int_{\Omega_{T}} |K_{32}^{K_{3}} \cdot M Z \Phi_{t}|dtdx$ yields

\[
\int_{\Omega_{T}} |K_{32}^{K_{3}} \cdot M Z \Phi_{t}|dtdx \\
\leq C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}^{2} \\
+ R(t)^{\mu - \frac{3}{2}(\gamma - 1) + 2}|(\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}|dtdx \\
+ C \int_{\Omega_{T}} \sum_{i=0}^{2} |R(t)^{\mu - \frac{3}{2}}D_{\dot{Z}}\Phi_{t}|_{L_{2}}^{2} \\
\times (\nabla_{t, \dot{t}}^{\gamma} \Phi_{t})^{2}dtdx.
\]

Then putting (A.36)–(A.38), (A.43)–(A.44) and (A.48) into (5.8) yields (5.5) and (5.6) for the case $S^{m} = Z^{m}$.

References

[1] Yin H and Zhang L 2018 The global existence and large time behavior of smooth compressible fluids in infinitely expanding balls, II: 3D Navier-Stokes equations Discrete Cont. Dyn. Syst. 38 1063–102
[2] Yin H and Zhao W 2018 The global existence and large time behavior of smooth compressible fluids in infinitely expanding balls, III: 3D Boltzmann equation J. Differential equations 264 30–81
[3] Sideris T C 1985 Formation of singularities in three-dimensional compressible fluids Commun. Math. Phys. 101 475–85
[4] Chemin J Y 1990 Dynamique des gaz à masse totale finie Asymptotic Anal. 3 215–20
[5] Coutand D and Shkoller S 2011 Well-posedness in smooth function spaces for moving-boundary 1-D compressible Euler equations in physical vacuum Comm. Pure Appl. Math. 64 328–66
[6] Coutand D and Shkoller S 2012 Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum Arch. Ration. Mech. Anal. 206 515–616
[7] Fang J and Masmoudi N 2009 Well-posedness for compressible Euler equations with physical vacuum singularity Commun. Pure Appl. Math. 62 1327–85
[8] Fang J and Masmoudi N 2015 Well-posedness of compressible Euler equations in a physical vacuum Comm. Pure Appl. Math. 68 61–111
[9] Liu T-P, Xin Z and Yang T 1998 Vacuum states for compressible flow Discrete Contin. Dynam. Syst. 4 1–32
[10] Makino T, Ukai S and Kawashima S 1986 Sur la solution à support compact de laéquations d’Euler compressible Japan J. Appl. Math. 3 249–57
[11] Xin Z 1998 Blowup of smooth solutions to the compressible Navier–Stokes equation with compact density Comm. Pure Appl. Math. 51 229–40
[12] Hadžić M and Jiang J 2016 Expanding large global solutions of the equations of compressible fluid mechanics arXiv:1610.01666
[13] Sideris T C 2017 Global existence and asymptotic behavior of affine motion of 3D ideal fluids surrounded by vacuum Arch. Ration. Mech. Anal. 225 141–76
[14] Grassin M 1998 Global smooth solutions to Euler equations for a perfect gas Indiana Univ. Math. J. 47 1397–432
[15] Serre D 1997 Solutions classiques globales des équations d’Euler pour un fluide parfait comprressible Ann. Inst. Fourier (Grenoble) 47 139–53
[16] Courant R and Friedrichs K O 1948 Supersonic Flow and Shock Waves (New York: Interscience Publishers Inc.)
[17] Li J, Witt I and Yin H 2018 Global multidimensional shock waves for 2-D and 3-D unstable potential flow equations SIAM J. Math. Anal. 50 933–1009
[18] Xu G and Yin H 2015 On global multidimensional supersonic flows with vacuum states at infinity Arch. Ration. Mech. Anal. Vol.218 1189–238
[19] Christodoulou D 1986 Global solutions of nonlinear hyperbolic equations for small initial data Comm. Pure Appl. Math. 39 267–82
[20] Klainerman S 1986 The null condition and global existence to nonlinear wave equations, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa
21 Alinhac S 1999 Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions Ann. of Math. 149 97–127
22 Alinhac S 1999 Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II Acta Math. 182 1–23
23 Christodoulou D and Miao S 2014 Compressible flow and Euler’s equations Surveys of Modern Mathematics, 9 (Beijing: International Press, Somerville, MA; Higher Education Press)
24 Speck J 2016 Shock formation in small-data solutions to 3D quasilinear wave equations Mathematical Surveys and Monographs 214 (Providence, RI: American Mathematical Society) xxiii+515 pp.
25 Yin H 2004 Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data Nagoya Math. J. 175 125–64
26 Majda A 1984 Compressible fluid flow and systems of conservation laws in several space variables Applied Mathematical Sciences 53 (New York: Springer)
27 Lin C 1986 Interpolation inequalities with weights Comm. Partial Differential equations 11 1515–38
28 D’Abbicco M, Lucente S and Reissig M 2013 Semi-linear wave equations with effective damping Chin. Ann. Math. Ser. B 34 345–80
29 Wirth J 2006 Wave equations with time-dependent dissipation. I. Non-effective dissipation J. Differential equations 222 487–514
30 Wirth J 2007 Wave equations with time-dependent dissipation. II. Effective dissipation J. Differential equations 232 74–103