Anharmonic oscillator, negative dimensions and inverse factorial convergence of large orders to the asymptotic form

P.V. Pobylitsa
Petersburg Nuclear Physics Institute,
Gatchina, St. Petersburg, 188300, Russia

Abstract

The spectral problem for $O(D)$ symmetric polynomial potentials allows for a partial algebraic solution after analytical continuation to negative even dimensions $D$. This fact is closely related to the disappearance of the factorial growth of large orders of the perturbation theory at negative even $D$. As a consequence, certain quantities constructed from the perturbative coefficients exhibit fast inverse factorial convergence to the asymptotic values in the limit of large orders. This quantum mechanical construction can be generalized to the case of quantum field theory.

1 Introduction

The $O(D)$ symmetric anharmonic oscillator

$$H = \frac{1}{2} \sum_{i=1}^{D} \left( p_i^2 + x_i^2 \right) + V \left( \sum_{i=1}^{D} x_i^2 \right)$$

(1.1)

with a polynomial potential $V(r^2)$ is a good theoretical toy for testing various ideas used in quantum field theory. In particular, one should mention:

1) large orders of perturbation theory [1, 2, 3, 4, 5, 6, 7, 8, 9],
2) fermion representations for boson theories in negative dimensions [10, 11],
3) hidden symmetries, quasi-exactly solvable models [12, 13, 14], correspondence between integrable quantum field theoretical models and ordinary differential equations [15, 16, 17, 18, 19], correspondence between $O(D)$ symmetric and $D = 1$ anharmonic oscillator systems [5, 20, 21, 22, 23],
4) Regge theory of complex angular momenta in nonrelativistic quantum mechanics [24, 25, 26].

There is a vast literature where the anharmonic oscillator is studied from various points of view. The aim of this paper is to bring some of these ideas
together. The paper consists of two parts. The first part is devoted to the spectrum of the anharmonic oscillator at even negative $D$

$$D = 0, -2, -4, -6, \ldots$$  \hspace{1cm} (1.2)

understood in the sense of analytical continuation. A remarkable feature of these values of $D$ is that a part of the spectrum can be found by solving algebraic equations. In principle, this fact is known since long ago [10] but as usual with exactly solvable models, one can arrive at the same results using many different ways, and the alternative derivations discussed here may deserve attention.

The second part of the paper deals with the asymptotic behavior of large orders of the perturbation theory. As is well known, the perturbative series for the anharmonic oscillator is divergent because of the factorial growth of the coefficients. However, one can show that in even negative dimensions (1.2) the factorial growth disappears and the perturbative series becomes convergent. Taken alone, this fact is quite natural and not too exciting. But this observation gives an interesting solution to another old problem.

Let us return to the case of general $D$ where we have the factorial growth of perturbative coefficients. It is well known that the asymptotic behavior of large-$k$ orders of the perturbation theory is reached very slowly because of the $O(k^{-1})$ corrections that are usually rather large in practically interesting cases. Therefore it would be rather interesting to construct quantities for which the asymptotic regime is reached faster than $O(k^{-1})$. This problem is especially acute in quantum field theoretical models where only several first orders of the perturbation theory can be usually computed. In this paper we construct and study special quantities whose approach to the asymptotic regime is factorially fast instead of the traditional slow $O(k^{-1})$ convergence to the asymptotic form. It is interesting that the same construction can be used in quantum field theory. In fact, this paper appeared as an attempt to understand some superfast convergence effects that are seen in quantum field theory [27].

Before starting with a detailed derivation of the exact solution for the spectrum of the anharmonic oscillator at negative even dimensions $D$, it makes sense to describe the final result. For simplicity let us concentrate on levels with zero angular momentum $l = 0$.

At negative even $D$ a part of the discrete spectrum of energy $E$ is given by roots of the algebraic equation

$$R_M(E) = 0 , \hspace{1cm} (1.3)$$

$$M = -\frac{D}{2} = 0, 1, 2, 3, \ldots \hspace{1cm} (1.4)$$

where $R_M(E)$ are certain polynomials of $E$ depending on potential $V$. In this paper a rather elegant representation for these polynomials will be derived

$$R_M(E) = \det [J_+ + V (2J_-) - E] . \hspace{1cm} (1.5)$$

Here

$$J_\pm = J_1 \pm iJ_2 \hspace{1cm} (1.6)$$
are standard spin matrices for spin
\[ j = \frac{M}{2} = -\frac{D}{4} = 0, \frac{1}{2}, \frac{3}{2}, \ldots \] (1.7)

Several comments should be made about the history of the exact solution for the anharmonic oscillator. The limit \( D \to 0 \) was considered by Dolgov and Popov [6]. From the point of view of representation (1.5) this case corresponds to spin \( j = 0 \) with the polynomial \( R_0 = -E \) leading according to eq. (1.3) to only level controlled by the exact solution \( E = 0 \). Although this case is trivial from the spectral point of view, the work of Dolgov and Popov [6] contains several interesting results. In particular, the wave function corresponding to the level \( E = 0 \) was computed in Ref. [6].

The general case of arbitrary negative even \( D \) was studied by Dunne and Halliday in Ref. [10] where a method was suggested for a calculation of polynomials describing the spectrum. In principle, this solves the problem of the spectrum. However, some issues were not clarified. Testing some special cases, the authors of [10] found an unexpected factorization of their polynomials into “elementary polynomials”. In the framework of the current paper the “elementary polynomials” of Dunne and Halliday are nothing else but polynomials \( R_M(E, g) \) given by eq. (1.5). An important role in the disentanglement of the polynomial structure is played by a “hidden” \( sl(2) \) symmetry which stands behind the spin representation (1.5) and explains the miracles observed in Ref. [10].

In this paper we discuss several derivations of polynomials (1.5). One method described in section 2 uses an explicit expression for the Hamiltonian of the anharmonic oscillator in terms of differential operators obeying \( sl(2) \) algebra. Another way to polynomials (1.5) considered in Sec. 2.6 is based on the analysis of a recursion relation for the coefficients of the power series for the wave function. In the third method (Sec. 4.4) one performs analytical continuation of the \( l \) decomposition for the partition function. One more method (Sec. 5) uses a fermion representation for the partition function continued analytically to negative dimensions. This approach is very close to the method of original work [10] but we implement this method so that the \( sl(2) \) algebra remains explicit.

An important aspect of the problem of the anharmonic oscillator in negative dimensions is the precise definition of negative dimensions. It is interesting that the problem of analytical continuation to negative or complex dimensions is equivalent to the problem of analytical continuation to complex angular momenta which is a corner stone of Regge theory (Sec. 4.4).

As was already mentioned, at negative even \( D \) we meet interesting phenomena in large orders of the perturbation theory. The energy of the ground state of the anharmonic oscillator has a perturbative expansion
\[ E(g, D) = \sum_{k=0}^{\infty} E^{(k)}(D)g^k \] (1.8)

with coefficients \( E^{(k)}(D) \) which are polynomials of \( D \) so that there are no problems with analytical continuation of \( E^{(k)}(D) \) to negative dimensions. In the
case of the quartic anharmonic oscillator

\[ V(r^2) = \frac{1}{2} r^2 + g r^4 \tag{1.9} \]

the large-\( k \) asymptotic behavior of \( E^{(k)}(D) \) is described by formula \[4\]

\[ E^{(k)}(D) \xrightarrow{k \to \infty} (-1)^{k+1} \Gamma \left( k + \frac{D}{2} \right) 3^{k+\frac{D}{2}} \frac{2^{D/2}}{\pi \Gamma(D/2)} \left[ 1 + O(k^{-1}) \right]. \tag{1.10} \]

An interesting feature of this formula is that the RHS vanishes at even negative \( D \) because of the factor \( \Gamma \left( D/2 \right)^{-1} \) on the RHS \( \text{(1.10)} \). In section \( \text{6.1} \) we show that the factorial divergence of series \( \text{(1.8)} \) disappears at negative even values of \( D \) so that the perturbative series has a nonzero convergence radius.

Functions \( E^{(k)}(D) \) are polynomials of \( D \) of degree \( k + 1 \) and they have \( k + 1 \) roots:

\[ E^{(k)}(\nu_{k,r}) = 0 \quad (1 \leq r \leq k + 1). \tag{1.11} \]

The last part of the paper is devoted to asymptotic properties of these roots in the limit of large \( k \). One could wonder why we should care about the roots \( \nu_{k,r} \).

The answer is that these roots have a very interesting property: in the limit \( k \to \infty \) some subsets of the roots are convergent to points \( D = -4, -6, -8, \ldots \) and this convergence is factorially fast. In quantum mechanics this superfast factorial convergence may be an amazing but useless curiosity. However, in the context of quantum field theory analogous effects are important. Indeed, in quantum field theory one is usually limited to a rather small amount of perturbative terms. Four or five orders are usually considered as a great achievement. Therefore the existence of quantities with factorial convergence to the asymptotic form instead of the slow \( O(k^{-1}) \) convergence of \( \text{(1.10)} \) is extremely interesting for field theoretical applications. The case of the anharmonic oscillator allows us to study this phenomenon in detail.

## 2 \( O(D) \) symmetric Hamiltonians and \( sl(2) \) algebra

### 2.1 Schrödinger equation

The \( D \)-dimensional Hamiltonian with a spherically symmetric potential \( \text{(1.1)} \) becomes in the polar coordinates

\[ H_r = \frac{1}{2} \left( -\frac{d^2}{dr^2} - \frac{D-1}{r} \frac{d}{dr} \right) + \frac{l(l+D-2)}{2r^2} + V(r^2). \tag{2.1} \]

In \( D \geq 2 \) dimensions, parameter \( l \) runs over nonnegative integer values

\[ l = 0, 1, 2, 3, \ldots \quad (D \geq 2) \tag{2.2} \]
The degeneracy of the \( l \) levels is given by
\[
m(D, l) = \frac{(2l + D - 2)}{l!} \frac{\Gamma(D + l - 2)}{\Gamma(D - 1)}.
\tag{2.3}
\]
The cases \( D = 1, D = 2 \) are somewhat exceptional. But they still can be described by analytical continuation of (2.3) in \( D \) (at fixed integer nonnegative \( l \)):

For \( D = 2 \) and nonnegative integer \( l \), we obtain
\[
m(2, l) = 2 - \delta_{l0},
\tag{2.4}
\]
which can interpreted in terms of the \( \pm l \) degeneracy for \( l \not= 0 \).

For \( D = 1 \), we have
\[
m(1, l) = \delta_{l0} + \delta_{l1}
\tag{2.5}
\]
so that the only allowed values are \( l = 0, 1 \), and they correspond to the states with positive and negative parity respectively.

After the separation of the factor \( r^l \) from the wave function one arrives at
\[
H_D = r^{-l} H_r r^l = \frac{1}{2} \left( -\frac{d^2}{dr^2} - \frac{D - 1}{r} \frac{d}{dr} \right) + V(r^2)
\tag{2.6}
\]
where
\[
D = D + 2l.
\tag{2.7}
\]
Thus the spectrum depends on \( D \) and \( l \) only via parameter \( D \):
\[
E_{nl}(D) = E_{n0}(D + 2l) = E_{n0}(D).
\tag{2.8}
\]
In the case \( D = 1 \) is the values \( l = 0, 1 \) correspond the states with positive and negative parity respectively.

Introducing the variable
\[
\zeta = \frac{r^2}{2}
\tag{2.9}
\]
we bring \( H_D \) (2.6) to the form
\[
H_\zeta = \left( -\zeta \frac{d^2}{d\zeta^2} - \frac{D}{2} \frac{d}{d\zeta} \right) + V(2\zeta).
\tag{2.10}
\]
The corresponding Schrödinger equation is
\[
\left[ \left( -\zeta \frac{d^2}{d\zeta^2} - \frac{D}{2} \frac{d}{d\zeta} \right) + V(2\zeta) - E_n(D) \right] \psi_n(\zeta) = 0.
\tag{2.11}
\]
Below we assume for simplicity that the potential \( V(2\zeta) \) is polynomial in \( \zeta \).

Equation (2.11) was derived for positive integer dimensions \( D \). However, we can use this equation for analytical continuation of eigenenergies \( E_n(D) \) to arbitrary \( D \). To this aim we solve equation (2.11) imposing the following boundary conditions on eigenfunctions \( \psi_n(\zeta) \):
1) $\psi_n(\zeta)$ must be analytical at $\zeta = 0$:

$$\phi(\zeta) = \sum_{k=0}^{\infty} p_k \zeta^k,$$  \hspace{1cm} (2.12)

2) $\psi_n(\zeta)$ must decay at $\zeta \to +\infty$.

### 2.2 $sl(2)$ representation for $O(D)$ symmetric Hamiltonians

Let us define parameter

$$j = \frac{D}{4} \hspace{1cm} (2.13)$$

and operators

$$T_+ = -\zeta \frac{d^2}{d\zeta^2} + 2j \frac{d}{d\zeta},$$  \hspace{1cm} (2.14)

$$T_0 = -\zeta \frac{d}{d\zeta} + j,$$  \hspace{1cm} (2.15)

$$T_- = \zeta.$$  \hspace{1cm} (2.16)

These operators provide a representation of $sl(2)$ algebra

$$[T_+, T_-] = 2T_0, \quad [T_+, T_0] = -T_+, \quad [T_-, T_0] = T_-$$  \hspace{1cm} (2.17)

with the Casimir operator

$$C_2 = \frac{1}{2} (T_+ T_- + T_- T_+) + T_0 T_0,$$  \hspace{1cm} (2.18)

$$[C_2, T_a] = 0.$$  \hspace{1cm} (2.19)

In representation (2.14) – (2.16) we have

$$C_2 = j(j + 1).$$  \hspace{1cm} (2.20)

In terms of operators (2.14) – (2.16) Hamiltonian (2.10) becomes

$$H_T = T_+ + V(2T_-).$$  \hspace{1cm} (2.21)

In the case of the quartic anharmonic oscillator (1.9) we have

$$H^{Q\alpha}_T = T_+ + T_- + 4gT_-^2.$$  \hspace{1cm} (2.22)

For integer or half-integer $j$ one could expect a simplification of the problem of the anharmonic oscillator due to the presence of finite dimensional irreducible representations of $sl(2)$. However, we meet a problem on this way: in the case of finite dimensional representations of $sl(2)$ there exists a vector $\psi$ annihilated by $T_-:

$$T_- \psi = 0, \quad \psi \neq 0,$$  \hspace{1cm} (2.23)
whereas for operator $T_-$ equation
\[ T_- \psi(\zeta) = \zeta \psi(z) = 0 \] (2.24)
leads to $\psi = 0$ [or to $\psi(z) = c\delta(z)$ if one allows for generalized functions]. In the next section we show how this problem can be circumvented in negative even dimensions.

2.3 Effective spin Hamiltonian for even negative dimensions $\mathcal{D}$

Let us consider the case of positive integer or half-integer $j$
\[ j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \] (2.25)
According to eq. (2.13) this corresponds to negative even values (1.2) of parameter $D$. These values are nonphysical. We assume analytical continuation in the sense of solutions of Schrödinger equation with boundary conditions described in Sec. 2.1.

Let us consider the subspace $\mathcal{H}_j$ of functions $f(\zeta)$ obeying conditions
\[ f^{(k)}(0) = 0 \quad (0 \leq k \leq 2j) \] (2.26)
This subspace is invariant under the action of operators $T_a$ (2.14) – (2.16). Therefore we can reduce the action of operators $T_a$ from the space $\mathcal{H}$ of differentiable functions to the factor space $\mathcal{H}/\mathcal{H}_j$. Expression (2.21) for Hamiltonian $H_T$ shows that $\mathcal{H}_j$ is invariant also under the action of $H_T$. Therefore we can also reduce the action of $H_T$ from $\mathcal{H}$ to $\mathcal{H}/\mathcal{H}_j$. The reduction to $\mathcal{H}/\mathcal{H}_j$ is also possible for the spectral problem
\[ (H_T - E) \psi = 0 \] (2.27)
Certainly after this reduction only some finite part of the infinite spectrum can survive. Note that the factor space $\mathcal{H}/\mathcal{H}_j$ is isomorphic to the space of polynomials $P_{2j}(\zeta)$ of degree $2j$. The action of operators $T_a$ (2.14) – (2.16) on these polynomials is described by the usual differentiation algebra extended by an additional formal rule
\[ \xi^{2j+k} = 0 \quad \text{for} \quad k \geq 1. \] (2.28)
The action of operators $T_a$ in the space of these polynomials is equivalent to the spin-$j$ irreducible $(2j+1)$-dimensional representation of $sl(2)$. This is obvious from the dimension $2j+1$ of this representation and from the eigenvalue $j(j+1)$ of the Casimir operator (2.20). Therefore after the reduction of the spectral problem (2.27) to the factor space $\mathcal{H}/\mathcal{H}_j$ we arrive at the matrix eigenvalue problem
\[ H_j = J_+ + V(2J_-), \] (2.29)
\[ (H_j - E) \phi = 0 \] (2.30)
where $J_a$ are $(2j + 1)$-dimensional matrices of the spin-$j$ representation of $sl(2)$. These $sl(2)$ matrices $J_a$ are connected by relations

$$J_0 = J_3,$$  \hspace{1cm} (2.31)

$$J_{\pm} = J_1 \pm iJ_2$$ \hspace{1cm} (2.32)

with standard spin matrices $J_{1,2,3}$ for the spin-$j$ representation of $su(2)$.

Thus the spectral problem reduces to solving the equation

$$\det (H_j - E) = 0.$$ \hspace{1cm} (2.33)

Obviously

$$R_{2j}(E) = \det (H_j - E) = \det [J_+ + V(2J_-) - E]$$ \hspace{1cm} (2.34)

is a polynomial of degree $2j + 1$ so that we deal with an algebraic equation

$$R_{2j}(E) = 0$$ \hspace{1cm} (2.35)

where $j = -D/4$ according to (2.13).

One should keep in mind that matrix Hamiltonian (2.29) describes only a part of the spectrum. Since $sl(2)$ matrices $J_{\pm}$ are real, polynomials $R_{2j}(E)$ are also real. Nevertheless the eigenvalues of $H_j$ may be complex.

In the case of quartic anharmonic oscillator (1.9) matrix Hamiltonian (2.29) reduces to

$$H_j = J_+ + J_- + 4gJ_3^2.$$ \hspace{1cm} (2.36)

2.4 Connection with quasi-exactly solvable problems

The Hamiltonians of type (2.36) appear in the context of quasi-exactly solvable (QES) potentials [12, 13, 14]. However, there is a certain difference between the $sl(2)$ representation discussed here and in traditional QES problems. Our differential operators $T_a$ (2.14) – (2.16) differ from the operators $T'_a$ used in QES problems:

$$T'_+ = 2j\xi - \xi^2 \frac{d}{d\xi},$$ \hspace{1cm} (2.37)

$$T'_0 = -j + \xi \frac{d}{d\xi},$$ \hspace{1cm} (2.38)

$$T'_- = \frac{d}{d\xi}. \hspace{1cm} (2.39)$$

Both sets of differential operators obey $sl(2)$ algebra (2.17). In fact, $T_a$ and $T'_a$ are connected (at least formally) by a Laplace transformation combined with the change of $j \rightarrow -1 - j$. Nevertheless the two classes of problems are different. In particular, the methods discussed here allow us to compute a part of the spectrum but not the corresponding wave functions $\psi_n(\zeta)$ obeying equation...
It should be emphasized that one has to distinguish two types of wave functions at negative $D$:

1) solutions of Schrödinger equation (2.11) at negative even $D$,
2) eigenvectors of the effective spin Hamiltonian (2.29).

It is interesting that in the case of the sextic QES potential, Dunne and Halliday [10] could trace the connection between the exact eigenfunctions in physical dimensions $D = 1, 2, 3, \ldots$ and the eigenvectors of the effective finite-dimensional matrix problem corresponding to $D = 0, -2, -4 \ldots$

### 2.5 Example: harmonic oscillator

In the case of the harmonic oscillator

$$V^{\text{HO}}(r^2) = \frac{1}{2} r^2,$$

$$V^{\text{HO}}(2\zeta) = \zeta,$$

the calculation of determinant (2.30) is trivial:

$$\det (J_+ + J_- - E) = \det (2J_1 - E) = \det (2J_3 - E) = 2^{2j+1} \prod_{n=-j}^{j} \left( n - \frac{E}{2} \right)$$

so that solutions of eq. (2.33) are

$$E = -2j, -2(j-1), \ldots, 2(j-1), 2j$$

$$= \frac{D}{2}, \frac{D}{2} + 2, \ldots, -\frac{D}{2} - 2, -\frac{D}{2} \quad (D \leq 0).$$

(2.43)

It is instructive to compare this algebraic solution with the full spectrum of the harmonic oscillator. Labeling levels with $n = 0, 1, 2, 3, \ldots$ in each $l$ sector, we can write

$$E_{nl}(0, D) = \frac{D}{2} + (2n + l).$$

(2.44)

Combining $D$ and $l$ into parameter $D$ (2.7) we arrive at the expression

$$E = \frac{D}{2} + 2n, \quad n = 0, 1, 2, \ldots$$

(2.45)

that has a trivial analytical continuation in $D$. For negative even $D = -4j$ the lowest $2j + 1$ levels (2.45) obviously coincide with the solution of the spin problem (2.43).

In Fig. 1 we show the “Regge trajectories” (the relation between the analytical continuation in $D$ and Regge theory is discussed in Sec. 4.1) connecting physical values $D = 1, 2, 3, \ldots$ and with the special points $D = 0, -2, -4, \ldots$ It is remarkable that any trajectory corresponding to a physical level after analytical continuation to negative $D$ sooner or later enters into the domain described by the roots of polynomial (2.42).
2.6 Power series and recursion relations

The analytical continuation of levels $E_n(D)$ to negative $D$ uses solutions $\phi(\zeta)$ (2.12) of Schrödinger equation (2.11) that are analytical at $\zeta = 0$. For polynomial potentials

\[
V(2\zeta) = \sum_k w_k \zeta^k,
\]

\[
w_k = 0 \quad \text{if} \quad k \leq 0 \quad \text{or} \quad k > L
\]

obeying condition

\[
V(0) = 0
\]

let us define

\[
u_k = \begin{cases} w_k & \text{if } 1 \leq k \leq L, \\ -E & \text{if } k = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Then

\[
V(2\zeta) - E = \sum_{k=-\infty}^{\infty} u_k \zeta^k.
\]
Inserting this series and expansion (2.12) into eq. (2.11) one obtains the recursion relation for the coefficients $p_k$ of series (2.12):

$$a_{k+1}(D)p_{k+1} = \sum_{m=0}^L u_m p_{k-m}, \quad (2.51)$$

$$a_k(D) = k \left( k - 1 + \frac{1}{2} D \right). \quad (2.52)$$

If $D \neq 0, -2, -4, \ldots$ then we can solve this equation iteratively starting from some $p_0 \neq 0$ and obtain

$$p_{k+1} = \frac{p_0 s_{k+1} \{ \{ u_m \}, D \}}{a_{k+1}(D)} \quad (2.53)$$

where $s_{k+1} \{ \{ u_m \}, D \}$ are polynomials of $u_m$. Once the series (2.12) is constructed, one has impose the boundary condition on $\phi(\zeta)$ at $\zeta \to +\infty$ and this will fix the spectrum.

The case $D = -4j = 0, -2, -4, \ldots$ is exceptional because in this case at $k = -D/2 = 2j$ we have a zero in the denominator on the RHS of (2.53). Therefore instead of the determination of $p_{2j}$ we will arrive at the condition

$$s_{2j+1} \{ \{ u_m \}, -4j \} = 0. \quad (2.54)$$

Since $s_{2j+1} \{ \{ u_m \}, D \}$ is a nonzero polynomial in $E$ (via $u_0 = -E$), this condition does not hold for the most of energies. In fact, in the case $D = 0, -2, -4, \ldots$ the majority of eigenfunctions have power expansion (2.12) with

$$p_0 = p_1 = \ldots = p_{2j+1} = 0, \quad p_{2j+2} \neq 0. \quad (2.55)$$

However, if condition (2.54) holds then we can proceed with iterations and construct the series (2.12). Thus in the special case (2.54) we have two solutions (2.12) of Schrödinger equation (2.11) analytical at $\zeta = 0$: one solution with $p_0 \neq 1$ and the second solution (2.55). Thus all solutions of Schrödinger equation (2.12) are regular in the case (2.54). This means that taking an appropriate linear combination we can satisfy boundary conditions at infinity so that values of $E$ obeying equation (2.54) automatically belong to the spectrum.

It is easy to see that this construction is equivalent to the spin problem (2.33). Indeed, recursion relation (2.51) is nothing else but vector equation (2.33) written in the basis diagonalizing matrix $J_0$. Now we understand that polynomials $s_{2j+1} \{ \{ w_m \}, -4j \}$ appearing in eq. (2.54) must coincide up to a constant with polynomials $R_{2j}(E)$ defined by eq. (2.34)

$$s_{2j+1} \{ \{ u_m \}, -4j \} = c_j \det \left[ J_+ + V (2J_0) - E \right] \quad (2.56)$$

with some coefficients $c_j$.

Let us derive equation (2.56) more carefully and compute coefficients $c_j$. First we define a $(N + 1) \times (N + 1)$ matrix

$$C_{kn}^{(N)} = u_{k-n} - a_k \delta_{k,n-1} \quad 1 \leq k, n \leq N + 1. \quad (2.57)$$
According to eq. (2.49) we assume that \( u_k = 0 \) if \( k < 0 \).

Then recursion relation (2.51) takes the form

\[
\sum_{n=1}^{N+1} C_{kn}^{(N)} p_{n-1} = \delta_{k,N+1} a_{N+1} p_{N+1} \quad (1 \leq k \leq N + 1).
\]  

(2.58)

Matrix (2.57) is quasitriangular. Its determinant can be computed by making with its rows the same linear manipulations as in the recursive calculation of \( s_{k+1} (\{u_m\}, \mathcal{D}) \) in eq. (2.53). As a result, one obtains

\[
\det \left[ C^{(N)} (\{u_m\}, \mathcal{D}) \right] = s_{N+1} (\{u_m\}, \mathcal{D}) \prod_{k=1}^{N} a_k (\mathcal{D}).
\]  

(2.59)

Now we set \( \mathcal{D} = -4j \), \( N = 2j \). Using eq. (2.52), we find

\[
\prod_{k=1}^{2j} a_k (-2j) = (-1)^{2j} [(2j)!]^2 .
\]  

(2.60)

Thus

\[
\det \left[ C^{(2j)} (\{u_m\}, -4j) \right] = (-1)^{2j} [(2j)!] s_{2j+1} (\{u_m\}, -4j).
\]  

(2.61)

Therefore algebraic equation (2.54) takes the form

\[
\det \left[ C^{(2j)} (\{u_m\}, -4j) \right] = 0.
\]  

(2.62)

In practical calculations one can compute \( \det C^{(2j)} \) either directly using definition (2.57)

\[
\det C^{(2j)} = \det_{1 \leq k,n \leq 2j+1} \left[ w_{k-n} - k(k-1-2j) \delta_{k,n-1} - E \delta_{kn} \right]
\]  

(2.63)

or solving recursion relations iteratively and finding in this way \( s_{2j+1} (\{u_m\}, -4j) \). Both methods lead to the same results:

\[
\det C^{(0)} = -E,
\]  

(2.64)

\[
\det C^{(1)} = E^2 - w_1,
\]  

(2.65)

\[
\det C^{(2)} = -E^3 + 4w_2 + 4Ew_1,
\]  

(2.66)

\[
\det C^{(3)} = E^4 - 10E^2w_1 + 9w_1^2 - 24w_2E - 36w_3 ,
\]  

(2.67)

\[
\det C^{(4)} = -E^5 + 20E^3w_1 - 64Ew_1^2 + 84w_2E^2 - 192w_1w_2 + 288Ew_3 + 576w_4 .
\]  

(2.68)

Note that recursion relation (2.58) for \( N = 2j \) is nothing else but the equation

\[
[T_+ + V (2T_-) - E] \phi(\zeta) = 0
\]  

(2.69)
written in the basis $\zeta^k$, $k = 0, 1, 2, \ldots 2j$. This basis coincides with the standard basis for spin-$j$ representation up to the subtleties of the normalization and enumeration. Obviously the determinant of matrix $C^{(2j)}$ is invariant with the respect to this trivial change of the basis. Therefore

$$\det C^{(2j)} = \det [J_+ + V(2J_-) - E] \quad (2.70)$$

where the spin $j$ representation is assumed on the RHS. Combining this result with eq. (2.42) we see that we have obtained old equation (2.29).

Certainly this “new derivation” of eq. (2.29) is nothing else but an explicit detailed version of the compact arguments of Sec. 2.3 presented there in terms of the $sl(2)$ algebra acting in the factor space $\mathcal{H}/\mathcal{H}_j$.

### 2.7 Properties of polynomials $R_{2j}(E)$

Comparing eqs. (2.34), (2.63), (2.70) we see that we have three equivalent representations for polynomials $R_{2j}(E)$

$$R_{2j}(E) = \det [J_+ + V(2J_-) - E] = \det \frac{1}{1 \leq k,n \leq 2j+1} \left[ w_{k-n} - k (k-1-2j) \delta_{k,n-1} - E \delta_{kn} \right]$$

$$= (-1)^{2j} [2j]! s_{2j+1} \left( \{ u_m \} , -4j \right) . \quad (2.72)$$

Representation (2.71) is written in terms of spin-$j$ matrices $J_\pm = J_1 \pm iJ_2$ and explicitly expresses the $sl(2)$ symmetry standing behind this construction. Representation (2.72) written in terms of coefficients $w_k$ of the polynomial potential $V (2.40)$ is nothing else but eq. (2.71) transformed to a basis with a nonstandard normalization which allows us to get rid of cumbersome square roots appearing in the traditional expressions for the matrix elements of $J_\pm$. Representation (2.73) is based on the iterative solution (2.53) of recursion relations (2.51) and is useful for computer calculations of polynomials $R_{2j}(E)$.

The higher coefficients of polynomials $R_{2j}(E)$ can be computed using the large-$E$ expansion:

$$\det [E - J_+ - V(2J_-)] = E^{2j+1} \exp \left\{ \frac{\text{Tr} \ln \left[ 1 - \frac{J_+ + V(2J_-)}{E} \right]}{E^2j+1} \right\}$$

$$= E^{2j+1} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} E^{-n} \text{Tr} [J_+ + V(2J_-)]^n \right\} . \quad (2.74)$$

We have

$$\text{Tr} [J_+ + V(2J_-)] = 0 ,$$

$$\text{Tr} \left\{ [J_+ + V(2J_-)]^2 \right\} = 2 \text{Tr} [J_+ V(2J_-)] = 2w_1 \text{Tr} (J_+ J_-) ,$$

$$\text{Tr} [J_+ + V(2J_-)]^3 = 3 \text{Tr} \left[ (J_+)^2 V(2J_-) \right] = 3w_2 \text{Tr} \left[ (J_+)^2 (J_-)^2 \right] \quad (2.75)$$
where \(w_k\) are coefficients of the polynomial potential (2.46). A straightforward spin algebra gives

\[
\text{Tr}(J_+J_-) = \frac{2}{3} j(j + 1)(2j + 1),
\]

\[
\text{Tr}\left[(J_+^2)(J_-^2)\right] = \frac{2}{15} j(j + 1)(2j + 1)(2j - 1)(2j + 3).
\]

Finally we obtain

\[
(-1)^{2j+1} R_{2j}(E) = E^{2j+1} - \frac{2}{3} j(j + 1)(2j + 1) w_1 E^{2j-1} - \frac{2}{15} j(j + 1)(2j + 1)(2j - 1)(2j + 3) w_2 E^{2j-2} + O(E^{2j-3}),
\]

(2.78)

In the case of harmonic oscillator (2.40) we have according to (2.42)

\[
R_{2j}^{\text{HO}}(E) = \det(J_+ + J_- - E) = 2^{2j+1} \prod_{n=-j}^{j} \left(n - \frac{E}{2}\right).
\]

(2.79)

### 2.8 Symmetry \(D \to 4 - D\)

Relation

\[
\zeta^{-1+(D/2)} \left(-\frac{d^2}{d\zeta^2} - \frac{D}{2} \frac{d}{d\zeta}\right) \zeta^{1-(D/2)} = -\frac{d^2}{d\zeta^2} - \frac{4 - D}{2} \frac{d}{d\zeta}
\]

(2.80)

shows that equation (2.11) has a symmetry

\[
\phi_{4-D}(\zeta) = \zeta^{-1+(D/2)} \phi_D(\zeta),
\]

(2.81)

\[
D \to 4 - D.
\]

(2.82)

However, one should be careful about the boundary conditions. Indeed, for \(D < 2\) the factor \(\zeta^{-1+(D/2)}\) is singular so that regular solutions \(\phi_D(\zeta)\) may correspond to singular solutions of \(\phi_{4-D}(\zeta)\). In other words, some part of the \(D < 2\) spectrum may be lost when one turns from \(D < 2\) to \(4 - D > 2\).

According to results of Sec. 2.6 at \(D = 2M = 0, -2, -4, \ldots\) there are two types of regular solutions (2.12)

1) solutions with \(p_0 \neq 0\),
2) solutions with \(p_0 = \ldots = p_M = 0, p_{M+1} \neq 0\).

Obviously solutions \(\phi_D(\zeta)\) of the first type with \(p_0 \neq 0\) generate singular functions \(\phi_{4-D}(\zeta)\) (2.81). Therefore the spectrum of the \(4 - D\) problem does not contain that part of the \(D \to 4 - D\) which corresponds to wave functions (2.12) with \(p_0 = 0\). We see that under the change \(D \to 4 - D\) we lose exactly those states which are described by the roots of polynomials \(R_M(E)\).

This symmetry \(D \to 4 - D\) is illustrated in Fig. 2. It should be stressed that the symmetry \(D \to 4 - D\) is relevant only for integer even values of \(D\).
3 Quartic anharmonic oscillator

3.1 Polynomials $R_{2j}(E, g)$

In the case of the quartic anharmonic oscillator (1.9) we have

$$V(2\zeta) = \zeta + 4g\zeta^2 = w_1 + w_2\zeta^2,$$

$$w_1 = 1, \quad w_2 = 4g.$$  \hfill (3.1)

Now one can use one of representations (2.71) – (2.73) and compute $R_{2j}(E)$. In order to avoid large coefficients it is convenient to define

$$\tilde{R}_{2j}(E, g) = (-2)^{-2j} R_{2j}(E) = (-2)^{-2j} \det(J_+ + J_- + 4gJ^2 - E).$$  \hfill (3.2)

At $g = 0$ we have according to (2.42)

$$\tilde{R}_{2j}(E, 0) = -2 \prod_{k=0}^{2j} \left( \frac{E - 2j + k}{2} \right).$$  \hfill (3.3)

Let us define

$$\tilde{R}'_{2j}(E, g) = \tilde{R}_{2j}(E, g) - \tilde{R}_{2j}(E, 0).$$  \hfill (3.4)

The first polynomials $\tilde{R}'_{2j}(E, g)$ are

$$\tilde{R}'_1(E, g) = 0,$$  \hfill (3.5)

$$\tilde{R}'_2(E, g) = 4g,$$  \hfill (3.6)

$$\tilde{R}'_3(E, g) = 12Eg,$$  \hfill (3.7)

$$\tilde{R}'_4(E, g) = 3(-16 + 7E^2)g,$$  \hfill (3.8)

$$\tilde{R}'_5(E, g) = 4(7E^3 - 55E - 200g)g,$$  \hfill (3.9)

$$\tilde{R}'_6(E, g) = \frac{9}{2}(7E^4 - 124E^2 + 192 - 1000Eg)g,$$  \hfill (3.10)

$$\tilde{R}'_7(E, g) = \frac{9}{2}(7E^5 - 230E^3 + 1183E - 3056E^2g + 10976g)g.$$  \hfill (3.11)

Now we find using eqs. (3.1) and (3.3)

$$\tilde{R}_0(E, g) = -E,$$  \hfill (3.12)

$$\tilde{R}_1(E, g) = -\frac{1}{2}(E^2 - 1),$$  \hfill (3.13)

$$\tilde{R}_2(E, g) = -\frac{1}{4}E(E^2 - 4) + 4g,$$  \hfill (3.14)

$$\tilde{R}_3(E, g) = -\frac{1}{8}(E^2 - 9)(E^2 - 1) + 12gE.$$  \hfill (3.15)
3.2 Ground state at $D = -4$

The ground state has $l = 0$ so that the case of the ground state in $D = -4$ dimensions corresponds to $D = D = -4$. The spectrum is described by the roots of polynomial $\tilde{R}_2(E, g)$ (3.15)

$$\tilde{R}_2(E, g) = 0$$

i.e.

$$E^3 - 4E - 16g = 0.$$ 

Let us define

$$E = \frac{2}{\sqrt{3} \varepsilon},$$

$$h = 3^{3/2} g.$$ (3.19)

Then

$$\varepsilon^3 - 3\varepsilon - 2h = 0.$$ (3.20)

The solutions can be found using Cardano formula:

$$\varepsilon = \sqrt[3]{h + \sqrt{h^2 - 1}} + \sqrt[3]{h - \sqrt{h^2 - 1}}.$$ (3.21)

Keeping in mind applications to the case of the perturbation theory in small $g$ it is convenient to transform this expression to the form

$$\varepsilon = e^{5\pi i/6} \sqrt[3]{\sqrt{1 - h^2 + i h} + e^{-5\pi i/6} \sqrt[3]{\sqrt{1 - h^2 - i h}}}$$

$$= 2\text{Re} \left[ \frac{-\sqrt{3} + i}{2} \sqrt[3]{\sqrt{1 - h^2 - i h}} \right]$$ (3.22)

which allows for the perturbative expansion

$$\varepsilon = -\sqrt{3} + \frac{h}{3} + \ldots$$ (3.23)

This leads to the perturbative expansion for the energy of the ground state in $D = -4$ dimensions

$$E(g, -4) = -2 + 2g + 3g^2 + 8g^3 + \frac{105}{4} g^4 + 96g^5 + \frac{3003}{16} g^6 + 1536g^7 + \ldots$$ (3.24)

In Sec. 4 we will compute the asymptotic behavior (3.23) of this series.

4 Analytical continuation to negative $D$

4.1 Negative $D$ and Regge theory

We have already made some comments about the role of Schrödinger equation (2.11) for the analytical continuation in $D$. Remember that parameter $D$ (2.7)
is built of $D$ and $l$ so that the problem of analytical continuation in $D$ has two equivalent formulations:

1) analytical continuation in $D$ at fixed $l$,
2) analytical continuation in $l$ at fixed $D$.

The second approach allows us to use some results from Regge theory of complex angular momenta [24, 25, 26]. Although the traditional Regge theory deals with the scattering problem whereas we are interested in polynomial potentials, the analysis of the behavior of the solutions of Schrödinger equation (2.11) at $\zeta \to 0$ is the same in Regge theory of the potential scattering and in our case. In particular, in Regge theory for potentials regular at $r = 0$, the Regge poles are at points

$$l = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$$

(4.1)

Combining this with the general formula (2.7) applied to the case $D = 3$

$$D = 3 + 2l \quad (D = 3)$$

(4.2)

we see that $l$-points (4.1) correspond to negative even values $D = 0, -2, -4, -6, \ldots$.

From the point of view of the historical perspective it is also interesting that analytical continuation in $l$ for potentials growing at infinity, e.g. for the oscillator energies (2.44), was discussed in the context of naive models for quark confinement [25, 26].

Although the discussion of analytical continuation in terms of $l$ is preferable from the point of view of explicit connections with Regge theory, we choose the $D$ representation. The $D$ language is natural for the analysis of analytical properties of the partition function (Sec. 4.4) and for the fermion representation at negative $D$ (5).

4.2 $D$ dependence of levels

We have already discussed the dependence of energy $E_n(D)$ on $D$ for the harmonic oscillator (see Fig. 1). If we switch on a small anharmonicity then the trajectories will be deformed but qualitative features will be preserved. However, in the case of a strong deviation from the harmonic regime (or at larger values of $|D|$) some new phenomena may occur. In Fig. 2 we show the $D$ dependence of levels for the quartic anharmonic oscillator (1.9) with $g = 1$. We see that the trajectories of the ground state and of the first excited state meet together at $D \approx -2.6$. Polynomials $R_{2j}$ have $2j + 1$ roots but some of these roots may be complex. In Fig. 2 we see that this happens for polynomials $R_2, R_3, R_4$.

4.3 Various aspects of the problem of analytical continuation in $D$

The problem of analytical continuation in $D$ can be studied for various quantities:
Figure 2: Trajectories continuing the levels of the quartic anharmonic oscillator with \( g = 1 \) analytically in \( D \) from physical values \( D = 1, 2, 3, \ldots \) (marked with crosses) to arbitrary values of \( D \). The levels at \( D = 0, -2, -4, \ldots \) that are described by roots of polynomials \( R_{2j} \) \( (j = -D/2) \) are marked with circles. Note that the trajectories of the ground state and of the first excited state meet together at \( D \approx -2.6 \). Some polynomials \( R_{2j} \) have complex roots so that the number of real levels associated with these polynomials is smaller than \( 2j + 1 \). The dashed horizontal lines connect levels related by the symmetry \( D \to 4 - D \) (for even \( D \)).
1) analytical continuation of separate energy levels

\[ E_{nl}(D) = E_{n0}(D + 2l) = E_{n0}(D), \]  

(4.3)

2) analytical continuation of coefficients \( E_n^{(k)}(D) \) of the perturbative expansion

\[ E_{n0}(D, g) = \sum_{k=1}^{\infty} E_n^{(k)}(D)g^k \]  

(4.4)

for anharmonic oscillator,

3) analytical continuation of the partition function

\[ Z(\beta, D) = \text{Tr} \exp(-\beta H). \]  

(4.5)

Let us comment briefly on the analytical properties of these quantities.

The situation with analytical continuation of the perturbative coefficients \( E_n^{(k)}(D) \) in expansion (4.3) is simple because \( E_n^{(k)}(D) \) are polynomials of \( D \). However, the asymptotic nature of the perturbative series (4.4) does not allow us to draw simple conclusions about the analyticity in \( D \) of exact levels \( E_{n0}(D, g) \) from the trivial analyticity of \( E_n^{(k)}(D) \).

The simplest example of analytical continuation in \( D \) provides the harmonic oscillator. Its partition function trivially factorizes into \( D \) components

\[ Z^{(0)}(\beta, D, 0) = \left[ Z^{(0)}(\beta, 1, 0) \right]^D \]  

(4.6)

with an obvious analyticity in \( D \). The case of anharmonic potentials will be considered in the next section.

### 4.4 Analytical continuation of the partition function

Now let us study analytical continuation of the partition function

\[ Z(\beta, D) = \text{Tr} \exp(-\beta H) \quad (D = 1, 2, 3, \ldots) \]  

(4.7)

in \( D \). At physical values of \( D = 1, 2, 3, 4, \ldots \) one can use the \( l \) decomposition

\[ Z(\beta, D) = \sum_{n, l=0}^{\infty} m(D, l) \exp[-\beta E_{nl}(D)]. \]  

(4.8)

Here \( m(D, l) \) is the degeneracy of the \( l \)-levels in the \( D \)-dimensional space given by eq. (2.3). Using eq. (2.8), we find from eq. (4.8)

\[ Z(\beta, D) = \sum_{l=0}^{\infty} m(D, l) \left\{ \sum_n \exp[-\beta E_{n0}(D + 2l)] \right\}. \]  

(4.9)
Let us define
\[ z(\beta, D) = \sum_{n=0}^{\infty} \exp[-\beta E_n(D)]. \quad (4.10) \]

Then
\[ Z(\beta, D) = \sum_{l=0}^{\infty} m(D, l) z(\beta, D + 2l). \quad (4.11) \]

We want to use this series for analytical continuation of \( Z(\beta, D) \) to negative \( D \). It is easy to check that for positive integer \( l = 0, 1, 2, 3 \ldots \), function \( m(D, l) \) is analytical in \( D \) in the whole complex plane. Analytical continuation of \( m(D, l) \) to \( D = 1 \) and \( D = 2 \) was already discussed in Sec. 2.1. Now let us consider the general case. Using the relation
\[ \Gamma(z)\Gamma(1-z) = \pi \sin \pi z, \quad (4.12) \]
we can transform eq. (2.3) to the form
\[ m(D, l) = (-1)^l \frac{(2l + D - 2)}{l!} \frac{\Gamma(2 - D)}{\Gamma(3 - D - l)}. \quad (4.13) \]

Combining representations (2.3) and (4.13), we see that \( m(D, l) \) is regular for all \( D \). The regularity of \( m(D, l) \) allows us to use representation (4.11) for analytical continuation of \( Z(D) \) to negative \( D \).

At negative even values \( D = -2M = 0, -2, -4, \ldots \) we have an important simplification. Indeed, according to eq. (4.13)
\[ m(-2M, l) = (-1)^{l+1} \frac{(2l - 2M - 2)}{l!} \frac{\Gamma(2 + 2M)}{\Gamma(3 + 2M - l)}. \quad (4.14) \]

We see that
\[ m(-2M, l) = 0 \quad \text{if} \quad l \geq 2M + 3. \quad (4.15) \]

Therefore series (4.11) reduces to a finite sum:
\[ Z(\beta, -2M) = \sum_{l=0}^{2M+1} (-1)^l \frac{2(-l + M + 1)}{l!} \frac{(1 + 2M)!}{(2 + 2M - l)!} z(\beta, -2M + 2l). \quad (4.16) \]

Now we change the summation variable from \( l \) to \( k = l - M + 1 \).
\[ (4.17) \]

Then
\[ Z(\beta, -2M) = \sum_{k=-M}^{M+1} \frac{(-1)^{k-M} 2k [(1 + 2M)!]}{(M + 1 - k)!(1 + M + k)!} z(\beta, 2 + 2k). \quad (4.18) \]
Note that $k = 0$ does not contribute to this sum. Combining the contributions of $k$ and $-k$ we arrive at

$$Z(\beta, -2M) = \sum_{k=1}^{M+1} \frac{(-1)^{k-M}2k[(1 + 2M)!]}{(M + 1 - k)(1 + M + k)!} \times [z(\beta, 2 + 2k) - z(\beta, 2 - 2k)]. \quad (4.19)$$

Now we use the $D \rightarrow 4 - D$ symmetry (2.81) connecting the levels of $z(\beta, 2 + 2k)$ and $z(\beta, 2 - 2k)$. As was explained in Sec. 2.8, this symmetry is partial: when one passes from $D = 2 - 2k$ to $D = 2 + 2k$ one loses the states described by the solutions of the algebraic equation $R_k(E) = 0$. Therefore

$$z(\beta, 2 - 2k) = z(\beta, 2 + 2k) + \left[ \text{Tr} \exp \left(-\beta H_{j = -(k-1)/2}\right) \right] \quad (4.20)$$

where $H_j$ is the matrix Hamiltonian (2.29) corresponding to the spin

$$j = \frac{k - 1}{2}. \quad (4.21)$$

Now eq. (4.19) takes the form

$$Z(\beta, -2M) = \sum_{k=1}^{M+1} \frac{(-1)^{k-M+1}2k[(1 + 2M)!]}{(M + 1 - k)(1 + M + k)!} \times \text{Tr} \exp \left(-\beta H_{j = (k-1)/2}\right). \quad (4.22)$$

Let us replace the summation variable $k$ to $j$ (4.21)

$$Z(\beta, -2M) = \sum_{j=0}^{M/2} (-1)^{2j-M} \frac{2(2j + 1) [(1 + 2M)!]}{(M - 2j)!(2 + M + 2j)!} \times \text{Tr} \exp \left(-\beta H_{j}\right). \quad (4.23)$$

Here the summation runs over integer and half-integer $j$.

## 5 Fermion representation

### 5.1 Derivation

Let us start from the case of the quartic anharmonic oscillator (1.9). We can write the path integral for the partition function

$$Z^{QO}(\beta, D, g) = \int_{q(0) = q(\beta)} Dq \exp \left\{- \int_0^\beta dt \left[ \frac{1}{2} \sum_{i=1}^D (\dot{x}_i^2 + x_i^2) + g \left( \sum_{i=1}^D x_i^2 \right)^2 \right] \right\}. \quad (5.1)$$
One can use the standard trick with the auxiliary field $\sigma$

$$Z^{QO}(\beta, D, g) = \int_{q(0)=q(T)} Dq \int D\sigma$$

$$\times \exp \left\{ - \int_0^\beta dt \left[ \frac{1}{2} \sum_{i=1}^D x_i \left( -\partial_t^2 + 1 + i\sqrt{8g\sigma} \right) x_i + \frac{\sigma^2}{2} \right] \right\}$$

$$= \int D\sigma \left[ \text{Det} \left( -\partial_t^2 + 1 + i\sqrt{8g\sigma} \right) \right]^{-D/2} \exp \left( - \int_0^\beta dt \frac{\sigma^2}{2} \right). \quad (5.2)$$

For negative even $D$

$$D = -2M \quad (5.3)$$

we can use the fermion representation

$$\left[ \text{Det} \left( -\partial_t^2 + 1 + i\sqrt{8g\sigma} \right) \right]^{-D/2}$$

$$= \int D\psi^+ D\psi \exp \left\{ - \int_0^\beta dt \left[ \sum_{i=1}^M \left( \partial_t \psi_i^+ \right) \left( \partial_t \psi_i \right) + 4g \left( \sum_{i=1}^M \psi_i^+ \psi_i \right)^2 \right] \right\}. \quad (5.4)$$

Integrating out the field $\sigma$, we obtain

$$Z^{QO}(\beta, D, g)_{D=-2M}^{D} \int D\psi^+ D\psi$$

$$\times \exp \left\{ - \int_0^\beta dt \left[ \sum_{i=1}^M \left( \partial_t \psi_i^+ \right) \left( \partial_t \psi_i \right) + V \left( \sum_{i=1}^M \psi_i^+ \psi_i \right) \right] \right\}. \quad (5.5)$$

The same trick can be repeated (at least formally) for an arbitrary polynomial central potential $V(r^2)$ if one uses more intricate integral representations involving several auxiliary boson fields $\sigma$. In this way one arrives at the fermion representation for partition function (4.5) with an arbitrary polynomial $V$:

$$Z(\beta, D)^{D=-2M} \int D\psi^+ D\psi$$

$$\times \exp \left\{ - \int_0^\beta dt \left[ \sum_{i=1}^M \left( \partial_t \psi_i^+ \right) \left( \partial_t \psi_i \right) \right] \right\}. \quad (5.6)$$

Now we introduce the auxiliary fermion field $\chi_i$:

$$\int D\chi^+ D\chi \exp \left\{ \int_0^\beta dt \sum_{i=1}^M \left[ \left( \partial_t \psi_i^+ \right) \chi_i - \chi_i^+ \left( \partial_t \psi_i \right) - \chi_i^+ \chi_i \right] \right\}$$

$$= \exp \left[ - \int_0^\beta dt \sum_{i=1}^M \left( \partial_t \psi_i^+ \right) \left( \partial_t \psi_i \right) \right]. \quad (5.7)$$
Then

\[
Z(\beta, D) \overset{D=-2M}{=} \int D\psi^+ D\chi^+ D\chi
\times \exp \left\{ \int_0^\beta dt \left[ \sum_{i=1}^M \left[ (\partial_t \psi_i^+) \chi_i - \chi_i^+ (\partial_t \psi_i) - \chi_i^+ \chi_i \right] \right.ight.
\left. - V \left( 2 \sum_{i=1}^M \psi_i^+ \psi_i \right) \right\} .
\]

(5.8)

Next we change the notation for the integration variables:

\[
\psi_i \rightarrow a_{i1},
\]

(5.9)

\[
\psi_i^+ \rightarrow a_{i2},
\]

(5.10)

\[
\chi_i \rightarrow a_{i2},
\]

(5.11)

\[
\chi_i^+ \rightarrow a_{i1}
\]

(5.12)

so that

\[
Z(\beta, D) \overset{D=-2M}{=} \int Da^+ Da
\times \exp \left\{ - \int_0^\beta dt \left[ \sum_{i=1}^M \sum_{\alpha,\beta=1}^2 a_{i\alpha}^+ \left( \delta_{\alpha\beta} \partial_t + \frac{1}{2} (\sigma_1 + i\sigma_2)_{\alpha\beta} \right) a_{i\beta} + V \left( \sum_{i=1}^M \sum_{\alpha,\beta=1}^2 a_{i\alpha}^+ (\sigma_1 - i\sigma_2)_{\alpha\beta} a_{i\beta} \right) \right] \right\} .
\]

(5.13)

where \(\sigma_a\) are standard Pauli matrices.

The integral on the RHS corresponds to the effective fermion Hamiltonian

\[
H_F = \sum_{i=1}^M \sum_{\alpha,\beta=1}^2 \frac{1}{2} a_{i\alpha}^+ (\sigma_1 + i\sigma_2)_{\alpha\beta} a_{i\beta} + V \left( \sum_{i=1}^M \sum_{\alpha,\beta=1}^2 a_{i\alpha}^+ (\sigma_1 - i\sigma_2)_{\alpha\beta} a_{i\beta} \right)
\]

(5.14)

with anticommutation relations

\[
\{a_{m\alpha}, a_{n\beta}^+\} = \delta_{mn} \delta_{\alpha\beta},
\]

(5.15)

\[
\{a_{m\alpha}, a_{n\beta}\} = \{a_{m\alpha}^+, a_{n\beta}^+\} = 0 .
\]

(5.16)

Thus we have derived the formula

\[
Z(\beta, D) \overset{D=-2M}{=} \text{Tr} \left[ e^{-\beta H_F} P_F \right]
\]

(5.17)

which should be understood in the sense of analytical continuation of the bosonic partition function \(\text{Tr} e^{-\beta H}\) from positive integer dimensions to negative even
dimensions. On the RHS we have a trace in a $2^{2M}$ dimensional vector space. $P_F$ is the operator counting the fermion parity of the state:

$$P_F = (-1)^{N_F + M},$$  \hspace{1cm} (5.18)

$$N_F = \sum_{i,\alpha} a_{i\alpha}^+ a_{i\alpha}.$$  \hspace{1cm} (5.19)

$P_F$ appears in eq. (5.17) because the integral over Grassmann variables (5.13) inherits periodic boundary conditions of the original boson integral (5.1).

Now we define

$$\tilde{T}_a = \frac{1}{2} \sum_{i=1}^{M} \sum_{\alpha,\beta=1}^{2} a_{i\alpha}^+ (\sigma_a)_{\alpha\beta} a_{i\beta}.$$  \hspace{1cm} (5.20)

Obviously

$$[\tilde{T}_a, \tilde{T}_b] = i\varepsilon_{abc} \tilde{T}_c.$$  \hspace{1cm} (5.21)

Introducing $sl(2)$ generators

$$\tilde{T}_\pm = \tilde{T}_1 \pm i\tilde{T}_2;$$  \hspace{1cm} (5.22)

$$\tilde{T}_0 = \tilde{T}_3.$$  \hspace{1cm} (5.23)

we can write Hamiltonian (5.14) as

$$H_F = \tilde{T}_+ + V(2\tilde{T}_-).$$  \hspace{1cm} (5.24)

This effective Hamiltonian has a form similar to form (2.21).

### 5.2 Degeneracy factors

Note that in the first derivation of the spin Hamiltonian $H_j$ (2.29) via the $sl(2)$ expression $H_T$ (2.21) we worked with the “effective dimension” $D$ given by (2.27) keeping in mind that the spectrum depends on $D$ and $l$ only via $D$. On the contrary, in the derivation based on the fermion Hamiltonian $H_F$ (5.24) we can trace the contributions of levels with different $l$. Strictly speaking, $H_F$ (5.24) is not completely equivalent to $H_j$ (2.29) because $H_F$ acts in the $2^{2M}$ dimensional space corresponding to the reducible representation of $sl(2)$ (or $su(2)$ if one prefers a more familiar physical interpretation)

$$\bigotimes_{i=1}^{M} \left[0 \oplus \left[0\bigoplus \left[\frac{1}{2}\right]\right]\right].$$  \hspace{1cm} (5.25)

Here we use notation $[j]$ for irreducible representations corresponding to spin $j$. Each factor $[0 \oplus \left[0\bigoplus \left[\frac{1}{2}\right]\right]$ in (5.25) is associated with the states in the $i$-th sector:

$$a_{i\beta}^+ |0\rangle \rightarrow \sqrt{1/2} \left[\frac{1}{2}\right],$$  \hspace{1cm} (5.26)

$$|0\rangle \rightarrow |0\rangle,$$  \hspace{1cm} (5.27)

$$a_{i1}^+ a_{i2}^+ |0\rangle \rightarrow |0\rangle.$$  \hspace{1cm} (5.28)
Tensor product (5.25) can be decomposed in irreducible representations \([j]\)

\[ \bigotimes_{i=1}^{M} \left( [0] \oplus [0] \oplus \left[ \frac{1}{2} \right] \right) = \bigoplus_{j=0}^{M/2} n(j, M) [j] \]  

(5.29)

with degeneracies

\[ n(j, M) = \frac{2 (2j + 1) \Gamma(2M + 2)}{\Gamma (M - 2j + 1) \Gamma(3 + M + 2j)} \]  

(5.30)

which can be computed using the recursion relation

\[ n(j, M + 1) = 2n(j, M) + n \left( j - \frac{1}{2}, M \right) + n \left( j + \frac{1}{2}, M \right) . \]  

(5.31)

Now we apply decomposition (5.29) to the calculation of the partition function (5.17):

\[ Z(\beta, D)_{D \rightarrow M/2} = \sum_{j=0}^{M/2} (-1)^{2j - M} n(j, M) \left[ \text{Tr} \exp \left( -\beta H^j \right) \right] . \]  

(5.32)

Inserting \(n(j, M)\) from eq. (5.30), we arrive at the result coinciding with (4.24). Thus fermion representation (5.6) for the partition function leads to the same result as the direct calculation of Sec. 4.4.

5.3 Connection with the results of Dunne and Halliday

If one uses representation

\[ a_{i2}^+ = \theta_{2i-1} , \]  

(5.33)

\[ a_{i2} = \frac{\partial}{\partial \theta_{2i-1}} , \]  

(5.34)

\[ a_{i1} = \theta_{2i} , \]  

(5.35)

\[ a_{i1}^+ = \frac{\partial}{\partial \theta_{2i}} \]  

(5.36)

then Hamiltonian \(H_F\) takes the form

\[ H_F = \sum_{i=1}^{M} \frac{\partial}{\partial \theta_{2i-1}} \frac{\partial}{\partial \theta_{2i-1}} + V \left( \sum_{i=1}^{M} \theta_{2i-1} \theta_{2i} \right) \]  

(5.37)

\[ H_F = \sum_{i=1}^{M} \sum_{\alpha, \beta=1}^{2} \frac{1}{2} a_{i\alpha}^+ \left( \sigma_1 + i \sigma_2 \right)_{\alpha \beta} a_{i\beta} + V \left( \sum_{i=1}^{M} \sum_{\alpha, \beta=1}^{2} a_{i\alpha}^+ \left( \sigma_1 - i \sigma_2 \right)_{\alpha \beta} a_{i\beta} \right) \]  

(5.38)
used by Dunne and Halliday [10]. The advantage of this form is the possibility to work with the pseudo-boson variables

\[ \xi_i = \theta_{2i-1} \theta_{2i}, \]  
\[ \xi_i \xi_k = -\xi_k \xi_i, \]  
\[ (\xi_i)^2 = 0 \]

(5.39) (5.40) (5.41)

and to look for the eigenstates of \( H_F \) in the form of polynomials \( \Psi(\{\xi_i\}) \) in \( \xi_i \).

In our representation these polynomial wave functions correspond to the states of the form

\[ P(\{a_{i1}^+ a_{i1}\})|\Omega\rangle \]

(5.42)

where \( P \) are polynomials of \( a_{i1}^+ a_{i1} \), and \( |\Omega\rangle \) is fixed by the conditions

\[ a_{i1}^+ |\Omega\rangle = a_{i2} |\Omega\rangle = 0 \quad (1 \leq i \leq M). \]

(5.43)

This corresponds to working in the subspace of states obeying the constraint

\[ \sum_{\beta=1}^{2} a_{i\beta}^+ a_{i\beta} \psi = \psi \quad (1 \leq i \leq M). \]

(5.44)

In terms of decomposition (5.29) this constraint selects states (5.26) in each \( i \)-sector. Therefore states (5.42) belong to the subspace:

\[ \bigotimes_{i=1}^{M} \left[ \frac{1}{2} \right] = \bigoplus_{j=\{M/2\}}^{M/2} \tilde{n}_j(j,M)[j], \]

(5.45)

where \( \{x\} \) stands for the fractional part of \( x \), and the degeneracy factors

\[ \tilde{n}(j,M) = \frac{(-1)^{M+2j} + 1}{2} \frac{(2j+1)M!}{((\frac{M}{2} + j + 1)!(\frac{M}{2} - j)!} \]

(5.46)

follow from the recursion relation

\[ \tilde{n}(j,M + 1) = \tilde{n}(j - \frac{1}{2},M) + \tilde{n}(j + \frac{1}{2},M). \]

(5.47)

Since Dunne and Halliday [10] (working with coupling constant \( \lambda = 8g \)) compute their characteristic polynomials \( P_M(E,\lambda) \) in the subspace associated with decomposition (5.45), their polynomials \( P_M(E,\lambda) \) factorize into our polynomials \( R_M(E,\lambda) \) (2.34):

\[ [P_M(E,\lambda)]_{\lambda=8g} = \text{const} \prod_{j=\{M/2\}}^{M/2} [R_{2j}(E,g)]^{\tilde{n}_j}. \]

(5.48)
For example,

\[ P_1 = \text{const } R_1, \]
\[ P_2 = \text{const } R_0 R_2, \]
\[ P_3 = \text{const } R_2^2 R_3, \]
\[ P_4 = \text{const } R_0^2 R_3 R_4. \]

In this way the \( sl(2) \) algebra explains the “miracles” observed in [10].

6 Large orders of the perturbation theory

6.1 Disappearance of the factorial divergence at negative even \( D \)

In a general case the perturbative series for the energy of the ground state of the \( D \)-dimensional quartic anharmonic oscillator \[ E(g,D) = \sum_{k=0}^{\infty} E^{(k)}(D) g^k \] (6.1) has factorially growing coefficients [7]:

\[ E^{(k)}(D) \overset{k \to \infty}{=} (-1)^{k+1} \Gamma \left( k + \frac{D}{2} \right) 3^{k+\frac{D}{2}} \frac{2^{D/2}}{\pi^{D/2}} \Gamma \left( \frac{D}{2} \right) \left[ 1 - \frac{1}{6k} \left( \frac{5}{3} + \frac{9}{2}D + \frac{7}{4}D^2 \right) + O(k^{-2}) \right]. \] (6.2)

However, at negative even values of \( D \) function \( \Gamma \left( \frac{D}{2} \right) \) vanishes. Note that \( \Gamma \left( \frac{D}{2} \right) \) appears on the RHS as a common factor so that at negative even values of \( D \) all terms of the systematic expansion in \( 1/k \) become zero. This hints that at \( D = 0, -2, -4 \) the factorial growth of perturbative coefficients disappears. The same happens in for the potential

\[ V(r^2) = \frac{1}{2} r^2 + gr^{2N}. \] (6.3)

In this case the factorial structure is different [4]

\[ E^{(k)}(D) \overset{k \to \infty}{=} [k(N-1)]! \left[ -\frac{1}{\pi} \frac{(N-1)^{D/2}}{\Gamma(D/2)} \right] \left[ \frac{\Gamma \left( \frac{2N}{N-1} \right)}{\Gamma \left( \frac{N}{N-1} \right)} \right]^{D/2} \]

\[ \times k^{(D/2)-1} \left\{ \left[ \frac{1}{2} \frac{\Gamma \left( \frac{2N}{N-1} \right)}{\Gamma \left( \frac{N}{N-1} \right)} \right]^{N-1} \right\}^k \left[ 1 + O(k) \right]. \] (6.4)
but we still have the same factor of \( \Gamma \left( \frac{D}{2} \right)^{-1} \).

This factor of \( \Gamma \left( \frac{D}{2} \right)^{-1} \) also appears in the asymptotic expression for \( E^{(k)}(D) \) in the \( O(D) \) symmetric model studied in Ref. [28]. This model describes a particle on a \( D \)-dimensional sphere with a potential breaking the \( O(D + 1) \) symmetry of the sphere to \( O(D) \).

One can easily trace the origin of this universal factor \( \Gamma \left( \frac{D}{2} \right)^{-1} \). In the semiclassical path integral derivation of large-order asymptotic formulas, the common factor of \( \Gamma \left( \frac{D}{2} \right)^{-1} \) comes from the integration over rotational collective coordinates of the classical solution breaking the \( O(D) \) symmetry of the problem to \( O(D - 1) \). As a result of this integration, the final result is proportional to the surface area of \( (D - 1) \)-dimensional sphere in the \( D \)-dimensional space:

\[
S_D = \frac{2 \pi^{D/2}}{\Gamma(D/2)}. \quad (6.5)
\]

Let us study the asymptotic behavior of \( E^{(k)}(D) \) at negative even \( D = -2M \). We will consider the case when the potential has the form

\[
V(r) = \frac{1}{2} r^2 + gU(r^2) \quad (6.6)
\]

where \( U(r^2) \) is some polynomial. We know that at \( D = -2M \) the energy of the ground state (understood as usual in the sense of analytical continuation) is given (at least for small \( g \)) by one of the roots of the algebraic equation \( (1.3) \). Since \( R_M(E, g) \) is a polynomial in both \( g \) and \( E \), the resulting dependence of \( E(g, -2M) \) is regular for most values of \( g \). The radius of convergence of the perturbative expansion of \( E(g, -2M) \) is controlled by the singularity of \( E(g, -2M) \) closest to \( g = 0 \). These singularities appear at values of \( g \) corresponding to degenerate roots of polynomials \( R_M(E, g) \) which are described by equations

\[
R_M(E_0, g_0) = \left. \frac{\partial}{\partial E} R_M(E, g_0) \right|_{E = E_0} = 0. \quad (6.7)
\]

Note that at \( g = 0 \) polynomial \( R_M(E, g) \) is given by expression \( (2.42) \) which has no degenerate roots so that the point \( g = 0 \) is regular and we have a nonzero convergence radius. Equations \( (6.7) \) may have several solutions. One has to choose the solution corresponding to the singularity closest to the point \( g = 0 \) (on the main Riemann sheet). In the vicinity of the degenerate point \( g_0, E_0 \) we can use the Taylor expansion

\[
R_M(E, g) = \frac{1}{2} (E - E_0)^2 \left[ \frac{\partial^2}{\partial E^2} R_M(E, g_0) \right]_{E = E_0} + (g - g_0) \left[ \frac{\partial}{\partial g} R_M(E_0, g) \right]_{g = g_0} + \ldots \quad (6.8)
\]

Now equation \( (1.3) \) leads to the following expression valid in the vicinity of the singularity of \( E(g) \) at \( g = g_0 \) is described by equation

\[
E(g) \overset{g=g_0}{\longrightarrow} E_0 - \sqrt{c \left( 1 - \frac{g}{g_0} \right)} \quad (6.9)
\]
where

\[
c = -2g_0 \left[ \frac{\partial}{\partial g} R_M(E_0, g) \right]_{g=g_0} \left\{ \left[ \frac{\partial^2}{\partial E^2} R_M(E, g_0) \right]_{E=E_0} \right\}^{-1}.
\]

(6.10)

Expanding

\[
\sqrt{1 - \frac{g}{g_0}} = -\frac{1}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma \left( k - \frac{1}{2} \right)}{\Gamma \left( k + 1 \right)} \left( \frac{g}{g_0} \right)^k
\]

(6.11)

and using the limit \( k \to \infty \)

\[
\frac{\Gamma \left( k - \frac{1}{2} \right)}{\Gamma \left( k + 1 \right)} \to k^{-3/2},
\]

(6.12)

we find from eq. (6.9)

\[
E^{(k)}(D) \xrightarrow{k \to \infty} \frac{1}{2} \sqrt{\frac{c(D)}{\pi}} k^{-3/2} \left[ g_0(D) \right]^{-k} \quad (D = -2M).
\]

(6.13)

### 6.2 Cases \( D = 0, -2, -4, -6 \)

Now we want to concentrate on the case of the quartic anharmonic oscillator (1.9) and to consider cases \( D = 0, -2, -4, -6. \)

The values \( D = 0 \) and \( D = -2 \) are trivial because the corresponding polynomials \( \tilde{R}_0 \) (3.13) and \( \tilde{R}_1 \) (3.14) do not depend on \( g \) so that the energy is given by

\[
E(0) = 0,
\]

(6.14)

\[
E(-2) = -1
\]

(6.15)

and almost all perturbative coefficients \( E^{(k)} \) vanish:

\[
E^{(k)}(0) = 0
\]

(6.16)

\[
E^{(k)}(-2) = -\delta_{k0}
\]

(6.17)

In the case \( D = -4 \) we insert polynomial \( \tilde{R}_2(E, g) \) (3.15) into eq. (6.7) and find two solutions

\[
E_0(-4) = \pm \frac{2}{\sqrt{3}}, \quad g_0(-4) = \pm 3^{-3/2}.
\]

(6.18)

Taking into account that we are interested in the energy of the ground state that corresponds to

\[
[E(g, D)]_{g=0} = \frac{D}{2},
\]

(6.19)

\[
E(0, -4) = -2,
\]

(6.20)
one can check that the relevant singularity corresponds to the upper signs in eq. \(6.13\)

\[
E_0(-4) = -\frac{2}{\sqrt{3}}, \quad g_0(-4) = 3^{-3/2}.
\]

Now we find from eq. \(6.10\)

\[
c(-4) = \frac{8}{9}
\]

and insert this into eq. \(6.13\)

\[
E^{(k)}(-4) \overset{k \to \infty}{\longrightarrow} \frac{1}{3} \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} k^{-3/2} 3^{3k/2} g^k
\]

for the coefficients \(E^{(k)}(D)\) of expansion \(6.1\) at \(D = -4\).

Other cases of negative even \(D\) can be analyzed in the same way. For example, solving equation \(6.7\) for the polynomial \(R_3(E, g)\) given by eq. \(3.16\) we find for \(D = -6\)

\[
g_0(-6) = \frac{5\sqrt{13} - 1}{36 \sqrt{3} \left(5 + 2\sqrt{13}\right)},
\]

\[
E_0(-6) = -\sqrt{\frac{1}{3} \left(5 + 2\sqrt{13}\right)},
\]

\[
c(-6) = \frac{2}{9} \left(5 - \frac{1}{\sqrt{13}}\right)
\]

which results in the asymptotic behavior

\[
E^{(k)}(-6) \overset{k \to \infty}{\longrightarrow} \frac{1}{6} \sqrt{\frac{2}{\pi}} \left(5 - \frac{1}{\sqrt{13}}\right) k^{-3/2} [g_0(-6)]^{-k}.
\]

### 6.3 Roots of polynomials \(E^{(k)}(D)\)

Coefficients \(E^{(k)}(D)\) of the perturbative expansion \(6.1\) are polynomials of degree \(k + 1\). Let us denote the \(k + 1\) roots of these polynomials \(\nu_{k,r}\):

\[
E^{(k)}(\nu_{k,r}) = 0 \quad (1 \leq r \leq k + 1)
\]

According to eqs. \(6.16\), \(6.17\), at \(k \geq 1\) polynomials \(E^{(k)}(D)\) have a common factor of \(D(D + 2)\), i.e.

\[
E^{(k)}(D) = D(D + 2)P_k(D)
\]

where \(P_k(D)\) is a polynomial of degree \(k - 1\) with \(k - 1\) roots \(\nu_{k,r}\). This means that the roots \(\nu_{k,r}\) include values 0 and -2:

\[
\{\nu_{k,r}\} = \{0, -2, \ldots\}
\]
Note that the function \([\Gamma \left( \frac{D}{2} \right)]^{-1}\) on the RHS of eq. (6.2) has zeros at
\[ D = 0, -2, -4, -6, \ldots \] (6.31)
The roots of the asymptotic expression (6.2) at \(D = 0, -2\) appear explicitly in the factorized expression on the RHS of eq. (6.29). The other roots should appear asymptotically at large \(k\). In other words, in the full set of roots \(\nu_k, r\) there must be subsets converging to values \(D = -4, -6, \ldots\).

Let us study the first subset converging to
\[ D = -2M \quad (M = -2, -3, \ldots) \]
We label this subset of roots with the index \(r = r(k, -2M)\):
\[
E^{(k)}(\nu_{k, r(k, -2M)}) = 0, \quad \lim_{k \to \infty} \nu_{k, r(k, -2M)} = -2M.
\] (6.32)

We want to derive a large-\(k\) asymptotic formula for \(2M + \nu_{k, r(k, -2M)}\). To this aim we need an asymptotic formula for \(E^{(k)}(D)\) in the double limit \(D \to -2M\) and \(k \to \infty\). This asymptotic formula has the form
\[
E^{(k)}(D) \xrightarrow{k \to \infty, D \to -2M} (-1)^{k+1} \left( k + \frac{D}{2} \right)^{3k+D/2} \frac{2^{D/2}}{\pi \Gamma \left( \frac{D}{2} \right)} + E^{(k)}(-2M). \quad (6.33)
\]

Note that we dropped the \(1/k\) correction present in eq. (6.2) but added the extra term \(E^{(k)}(-2M)\) which becomes essential at \(D \to -2M\) because of the suppressing factor \([\Gamma \left( \frac{D}{2} \right)]^{-1}\) in the leading term. At \(D \to -2M\) we can simplify eq. (6.33)
\[
E^{(k)}(D) \xrightarrow{k \to \infty, D \to -2M} (-1)^{k+1} k! k^{-M-1} 3^{-M} 2^{-M} \frac{2^{M}}{\pi \Gamma \left( \frac{D}{2} \right)} + E^{(k)}(-2M). \quad (6.34)
\]
Here
\[
\Gamma \left( \frac{D}{2} \right) \xrightarrow{D \to -2M} \frac{(-1)^M}{M!} \frac{1}{\frac{D}{2} + M}. \quad (6.35)
\]

Inserting this and eq. (6.13) into eq. (6.34), we obtain
\[
E^{(k)}(D) \xrightarrow{k \to \infty, D \to -2M} (-1)^{k+1} k! k^{-M-1} 3^{-M} \frac{2^{M}}{\pi \Gamma \left( \frac{D}{2} \right)} \left( \frac{D}{2} + M \right) + \frac{1}{2} \sqrt{\frac{\pi (-2M)}{\pi}} k^{-3/2} [g_0(-2M)]^{-k}. \quad (6.36)
\]

Using this asymptotic formula, we solve equation
\[
E^{(k)}(\nu_{k, r(k, -2M)}) \nu_{k, r(k, -2M)} \xrightarrow{D \to -2M} 0
\] (6.37)
and find

\[ \nu_{k,r}(k,-2M) + 2M \xrightarrow{k \to \infty} \frac{(-6)^M}{M!} \sqrt{\pi c(-2M)} \frac{(-1)^k}{k!} k^{-M-1/2} [3g_0(-2M)]^{-k} . \]  

Using above expressions (6.18), (6.22), (6.24), (6.26), for \( c(-2M) \) and \( g_0(-2M) \) at \( M = 2, 3 \), we obtain

\[ \nu_{k,r}(k,-4) + 4 \xrightarrow{k \to \infty} 12 \sqrt{2\pi} \frac{(-1)^k}{k!} k^{-3/2} 3^{k/2} , \]  

\[ \nu_{k,r}(k,-6) + 6 \xrightarrow{k \to \infty} 12 \sqrt{2\pi} \left( \frac{5}{\sqrt{13}} \right) \frac{(-1)^{k+1}}{k!} k^{5/2} \left[ \frac{5\sqrt{13} - 1}{12 \sqrt{3 (5 + 2\sqrt{13})}} \right]^{-k} . \]  

Asymptotic formulas (6.39), (6.40) agree with results of the direct numerical calculation using methods of Refs. [1, 2, 5, 7]. The roots approaching \(-4\) appear rather early. They are listed in Table 1. For small odd values \( k = 5, 7, 9 \) these roots have small imaginary parts. Starting from \( k = 10 \), the roots \( \nu_{k,r}(k,4) \) close to \(-4\) become real and exhibit a very fast factorial convergence to \(-4\) which is described by the asymptotic formula (6.39). Already at \( k = 11 \) this formula works with accuracy better than 7%.

At large values of \( k \) there appear roots converging to other integer values \( D = -6, -8, \ldots \) The roots approaching the value \(-6\) are listed in Table 2. At large \( k \) they are well described by asymptotic formula (6.40).

### 6.4 Distribution of roots in the complex plane

One should keep in mind that apart from the stable roots at \( D = 0, -2 \) and the roots asymptotically approaching points \( D = -4, -6, \ldots \), there are many other roots \( \nu_{k,r} \). Most of them are complex. In order to get an impression about the general distribution of roots in the complex plane, we show them for the cases \( k = 7 \) (Fig. 3), \( k = 8 \) (Fig. 4), and \( k = 30 \) (Fig. 5) and \( k = 60 \) (Fig. 6). From the plots for \( k = 30 \) and \( k = 60 \), it is clearly seen that at large \( k \) there appears a certain regular structure in the distribution of complex roots. It is an interesting problem to give a complete asymptotic description of roots in the complex plane. Here we make only preliminary comments about the distribution of roots.

It is convenient to work with polynomials \( P_k(D) \) (6.29). Using eq. (1.10), we find:

\[ P_k(D) \xrightarrow{k \to \infty} (-1)^{k+1} \Gamma \left( k + \frac{D}{2} \right) \frac{2^{(D/2)-2}}{\pi \Gamma (D/2 + 2)} . \]  

The polynomial \( P_k(D) \) has roots \( \tilde{\nu}_{k,r} \) which coincide with the roots of \( E^{(k)}(D) \) up to the two stable roots 0, \(-2\):

\[ \{\nu_{k,1}, \nu_{k,2}, \ldots \nu_{k,k+1}\} = \{\tilde{\nu}_{k,1}, \tilde{\nu}_{k,2}, \ldots \tilde{\nu}_{k,k+1}, 0, -2\} . \]
| $k$ | $\nu_{k,r(k,-4)} + 4$ | $k$ | $\nu_{k,r(k,-4)} + 4$ |
|-----|----------------------|-----|----------------------|
| 5   | $-3.22834 \pm 0.426293$ | 33  | $-4.79523448 \times 10^{-26}$ |
| 6   | $-3.44545$            | 34  | $+2.55611474 \times 10^{-27}$ |
| 7   | $-3.63083 \pm 0.34226$ | 35  | $-1.32186040 \times 10^{-28}$ |
| 8   | $-3.76443$            | 36  | $+6.63762841 \times 10^{-30}$ |
| 9   | $-3.9583 \pm 0.226557$ | 37  | $-3.23912410 \times 10^{-31}$ |
| 10  | $+0.04231592827$      | 38  | $+1.53735607 \times 10^{-32}$ |
| 11  | $-0.01231265412$      | 39  | $-7.10198670 \times 10^{-34}$ |
| 12  | $+0.00178433080$      | 40  | $+3.19560318 \times 10^{-35}$ |
| 13  | $-0.00027422590$      | 41  | $-1.40147930 \times 10^{-36}$ |
| 14  | $+0.00003787462$      | 42  | $+5.99459657 \times 10^{-38}$ |
| 15  | $-4.86252995 \times 10^{-6}$ | 43  | $-2.50229008 \times 10^{-39}$ |
| 16  | $+5.80950053 \times 10^{-1}$ | 44  | $+1.01993415 \times 10^{-41}$ |
| 17  | $-6.49387664 \times 10^{-8}$ | 45  | $-4.06166432 \times 10^{-42}$ |
| 18  | $+6.81906230 \times 10^{-9}$ | 46  | $+1.58113430 \times 10^{-43}$ |
| 19  | $-6.75145346 \times 10^{-10}$ | 47  | $-6.01961315 \times 10^{-45}$ |
| 20  | $+6.32321355 \times 10^{-11}$ | 48  | $+2.24249068 \times 10^{-46}$ |
| 21  | $-5.61842521 \times 10^{-12}$ | 49  | $-8.17805838 \times 10^{-48}$ |
| 22  | $+4.74864227 \times 10^{-13}$ | 50  | $+2.92092144 \times 10^{-49}$ |
| 23  | $-3.82680164 \times 10^{-14}$ | 51  | $-1.02217301 \times 10^{-50}$ |
| 24  | $+2.94682238 \times 10^{-15}$ | 52  | $+3.50629616 \times 10^{-52}$ |
| 25  | $-2.17261017 \times 10^{-16}$ | 53  | $-1.17934411 \times 10^{-54}$ |
| 26  | $+1.53640755 \times 10^{-17}$ | 54  | $+3.89122545 \times 10^{-55}$ |
| 27  | $-1.04388321 \times 10^{-18}$ | 55  | $-1.25990807 \times 10^{-56}$ |
| 28  | $+0.82474923 \times 10^{-20}$ | 56  | $+4.00444478 \times 10^{-58}$ |
| 29  | $-4.2996193 \times 10^{-21}$ | 57  | $-1.24982870 \times 10^{-59}$ |
| 30  | $+2.61370107 \times 10^{-22}$ | 58  | $+3.83179265 \times 10^{-61}$ |
| 31  | $-1.53497157 \times 10^{-23}$ | 59  | $-1.5437888 \times 10^{-62}$ |
| 32  | $+8.71891707 \times 10^{-25}$ | 60  | $+3.41802128 \times 10^{-64}$ |

Table 1: Roots $\nu_{k,r(k,-4)}$ of polynomials $P_D(N)$ approaching the value $-4$. 
| $k$ | $\nu_{k,r(k,-6)} + 6$ | $k$ | $\nu_{k,r(k,-6)} + 6$ |
|-----|-----------------|-----|-----------------|
| 21  | +0.03432898     | 41  | +1.44684 $\times 10^{-18}$ |
| 22  | -0.011016       | 42  | -1.55969 $\times 10^{-19}$ |
| 23  | +0.0020276      | 43  | +1.63996 $\times 10^{-20}$ |
| 24  | -0.00040899     | 44  | -1.68294 $\times 10^{-21}$ |
| 25  | +0.00007659     | 45  | +1.68654 $\times 10^{-22}$ |
| 26  | -0.000013876    | 46  | -1.65141 $\times 10^{-23}$ |
| 27  | +2.40442 $\times 10^{-6}$ | 47  | +1.58077 $\times 10^{-24}$ |
| 28  | -4.00519 $\times 10^{-7}$ | 48  | -1.47999 $\times 10^{-25}$ |
| 29  | +6.42173 $\times 10^{-8}$ | 49  | +1.35591 $\times 10^{-26}$ |
| 30  | -9.92472 $\times 10^{-9}$ | 50  | -1.21614 $\times 10^{-27}$ |
| 31  | +1.48040 $\times 10^{-9}$ | 51  | +1.06834 $\times 10^{-28}$ |
| 32  | -2.13385 $\times 10^{-10}$ | 52  | -9.19592 $\times 10^{-30}$ |
| 33  | +2.97550 $\times 10^{-11}$ | 53  | +7.75910 $\times 10^{-31}$ |
| 34  | -4.01814 $\times 10^{-12}$ | 54  | -6.41992 $\times 10^{-32}$ |
| 35  | +5.26006 $\times 10^{-13}$ | 55  | +5.21090 $\times 10^{-33}$ |
| 36  | -6.68131 $\times 10^{-14}$ | 56  | -4.15065 $\times 10^{-34}$ |
| 37  | +8.24173 $\times 10^{-15}$ | 57  | +3.24558 $\times 10^{-35}$ |
| 38  | -9.88147 $\times 10^{-16}$ | 58  | -2.49222 $\times 10^{-36}$ |
| 39  | +1.15242 $\times 10^{-16}$ | 59  | +1.87991 $\times 10^{-37}$ |
| 40  | -1.30830 $\times 10^{-17}$ | 60  | -1.39342 $\times 10^{-38}$ |

Table 2: Roots $\nu_{k,r(k,-6)}$ of polynomials $P_k(D)$ approaching the value $-6$. 

34
We can represent $P_k(D)$ in the form

$$P_k(D) = \beta_k \prod_{r=1}^{k-1} (D - \tilde{\nu}_{k,r}) .$$  \hfill (6.42)

The coefficient $\beta_k$ was computed in Ref. [6]:

$$\beta_k = (-1)^{k+1} \frac{2^{k-2}}{k+1} \frac{\Gamma(\frac{2k-1}{2})}{\Gamma(\frac{k+1}{2})} .$$  \hfill (6.43)

We find from eqs. (6.41), (6.42)

$$\frac{P_k(D)}{P_k(0)} = \prod_{r=1}^{k-1} \left( 1 - \frac{D}{\tilde{\nu}_{k,r}} \right) \xrightarrow{k \to \infty} \frac{(6k)^{D/2}}{\Gamma(D/2 + 2)} .$$  \hfill (6.44)

Differentiating the logarithm of this expression with respect to $D$, we obtain

$$\sum_{r=1}^{k-1} \frac{1}{D - \tilde{\nu}_{k,r}} \xrightarrow{k \to \infty} \frac{1}{2} \left[ \ln (6k) - \psi \left( \frac{D}{2} + 2 \right) \right]$$  \hfill (6.45)

where $\psi(z)$ is the logarithmic derivative of the $\Gamma$ function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \sum_{k=0}^{\infty} \left( \frac{1}{z+k} - \frac{1}{k+1} \right)$$  \hfill (6.46)

and $\gamma$ is Euler’s constant. Thus

$$\prod_{r=1}^{k-1} \frac{1}{D - \tilde{\nu}_{k,r}} \xrightarrow{k \to \infty} \frac{1}{2} \ln (6k) + \frac{1}{2} \left[ \gamma + \sum_{k=0}^{\infty} \left( \frac{2}{D + 4 + 2k} - \frac{1}{k+1} \right) \right] .$$  \hfill (6.47)

This relation requires the appearance of roots $\nu_{k,r}$ converging to $-4, -6, \ldots$ but does not forbid the existence of additional complex roots going to infinity or making a quasicontinuous distribution in the limit $k \to \infty$. Numerical calculations show a large amount of complex roots at large $k$ (see Figs. 5 and 6).

In the limit of large $k$, one can compute the product of all roots $\nu_{k,r}$ of the polynomials $P_k(D)$ setting $D = 0$ in eqs. (6.41), (6.42) and using expression (6.43) for $\beta_k$:

$$\prod_{r=1}^{k-1} \nu_{k,r} \xrightarrow{k \to \infty} \frac{1}{\beta_k 4\pi} \frac{3^k \Gamma(k)}{4k} \xrightarrow{k \to \infty} (-1)^{k+1} 3k^2 \left( \frac{k}{e\sqrt{3}} \right)^k .$$  \hfill (6.48)

This gives us an asymptotic estimate for the geometric average of all nonzero $k$ roots $\nu_{k,r}$ of $E_k(D)$ (including the root $\nu_{k,r} = -2$)

$$\langle |\nu_{k,r}| \rangle_{\nu_{k,r} \neq 0} = \left[ 2 \left( \prod_{r=1}^{k-1} |\tilde{\nu}_{k,r}| \right)^{1/k} \right]^{1/k} \xrightarrow{k \to \infty} \frac{k}{e\sqrt{3}} (6k^2)^{1/k} .$$  \hfill (6.49)

The growth of this quantity with $k$ shows that most of the roots go to infinity.
Figure 3: 8 roots $\nu_{8,r}$ of the polynomial $E^{(7)}(D)$. Two complex conjugate roots $-3.63 \pm i0.34$ are in the vicinity of the negative even value $D = -4$. 
Figure 4: 9 roots $\nu_{9,r}$ of the polynomial $E^{(8)}(D)$. One root at $-3.76443$ is close to the negative even value $-4$. 
Figure 5: 31 roots $\nu_{30,r}$ of the polynomial $E^{(30)}(D)$. One can see a discrete set of roots close to even negative values up to $D = -8$. Most of the roots belong to the quasicontinuous set formed in the complex plane.
Figure 6: 61 roots $\nu_{60,r}$ of the polynomial $E^{(60)}(D)$. One can see a discrete set of roots close to even negative values up to $D = -12$. There is also a quasicontinuous set formed in the complex plane.
7 Conclusions

Thus we have established a close connection between
1) exactly solvable features of the anharmonic oscillator in negative even dimensions,
2) disappearance of the factorial growth of perturbative coefficients in negative even dimensions,
3) fast inverse factorial convergence of roots of polynomials $E^{(k)}(D)$ to the negative even points.

Although the content of this paper was restricted to quantum mechanics, the main motivation comes from quantum field theory. In contrast to quantum mechanics where the large-order behavior of the perturbation theory can be easily tested using direct numerical calculations in cases when the analytical methods fail or raise doubts, in quantum field theory the power of both analytical and numerical tools is rather limited. The slow $O(k^{-1})$ convergence of perturbative coefficients to the asymptotic form [see e.g. eq. (6.2)] is rather disturbing in quantum mechanics but one still can reach the asymptotic regime in large orders. In quantum field theory calculations rarely go beyond four or five loops. In this situation the construction of quantities whose perturbative expansion has a fast factorial convergence to the asymptotic form is very important.

Polynomials $E^{(k)}(D)$ and their roots have analogs in quantum field theory. In many field theoretical models we have dependence on various parameters (e.g. number of colors, flavors etc.) and Feynman diagrams in any order lead to a polynomial (or fractional polynomial) dependence on these parameters. The control of this polynomial dependence is usually trivial compared to the hard work needed for the calculation of loop integrals. Therefore available multiloop results in quantum field theory provide many opportunities for testing the roots of these polynomials. There is some evidence that the inverse factorial convergence of roots found in the case of the anharmonic oscillator has analogs in some field theoretical models, e.g. in the $N$-component $O(N)$ symmetric model [27]. An indirect theoretical argument comes from the presence of an inverse $\Gamma$ function in field theoretical asymptotic expressions. This inverse $\Gamma$ function may generate zeros similar to the zeros at $D = 0, -2, -4 \ldots$ coming from the factor $[\Gamma(D)]^{-1}$ in eq. (1.11) in the case of the anharmonic oscillator. Since the convergence is very fast, it may happen that one can detect it (or its signals) in available multi-loop perturbative expressions.

Acknowledgments. I appreciate discussions with G.V. Dunne, L.N. Lipatov, V.Yu. Petrov and A. Turbiner.

References

[1] C.M. Bender, T.T. Wu, Phys. Rev. 184 (1969) 1231.
[2] C.M. Bender, T.T. Wu, Phys. Rev. D7 (1973) 1620.
[3] L.N. Lipatov, Sov. Phys. JETP 45 (1977) 216.
[4] E. Brezin, J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. D15 (1977) 1544.
[5] R. Seznec, J. Zinn-Justin, J. Math. Phys. 20 (1979) 1398.
[6] A. D. Dolgov and V. S. Popov, Phys. Lett. B86 (1979) 185.
[7] J. Zinn-Justin, J. Math. Phys. 22 (1981) 511.
[8] J. Zinn-Justin, Phys. Rep. 70 (1981) 109.
[9] J. Zinn-Justin, U.D. Jentschura, Ann. of Phys. 313 (2004) 197, 269.
[10] G.V. Dunne and I.G. Halliday, Nucl. Phys. B308 (1988) 589.
[11] G.V. Dunne, J. Phys. A22 (1989) 1719.
[12] A. Turbiner, Quasi-exactly Solvable Differential Equations, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3: New Trends in Theoretical Developments and Computational Methods, ed. N. H. Ibragimov, CRC Press, 1995 [hep-th/9409068].
[13] M.A. Shifman, ITEP Lectures on Particle Physics and Field Theory, vol. II, chapter VII, World Scientific (1999).
[14] A. Ushveridze, Quasi-Exactly Solvable Models in Quantum Mechanics, IOP Publishing, Bristol (1994).
[15] P. Dorey and R. Tateo, J. Phys. A32 (1999) L419.
[16] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, J. Statist. Phys 102 (2001) 567.
[17] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Adv. Theor. Math. Phys. 7 (2004) 711.
[18] P. Dorey, C. Dunning, R. Tateo, The ODE/IM Correspondence, arXiv: hep-th/0703066.
[19] P. Dorey, C. Dunning, F. Gliozzi and R. Tateo, On the ODE/IM correspondence for minimal models, arXiv: 0712.2010 [hep-th].
[20] A.A. Andrianov, Ann. of Phys. 140 (1982) 82.
[21] R. Damburg, R. Propin and V. Martyshchenko, J. Phys. A17 (1984) 3493.
[22] V. Buslaev and V. Grecchi, J. Phys. A26 (1993) 5541.
[23] A.A. Andrianov, Phys. Rev. D76 (2007) 025003.
[24] V. de Alfaro and T. Regge, Potential Scattering, North-Holland, Amsterdam (1965).
[25] P.D.B. Collins. An Introduction to Regge Theory and High Energy Physics, Cambridge University Press, Cambridge (1977).
[26] R.G. Newton, The Complex $J$-Plane, Benjamin, New York (1964).

[27] P.V. Pobylitsa, arXiv:0807.5136 [hep-th].

[28] V.A. Rubakov, O.Yu. Shvedov, Nucl. Phys. B434 (1995) 245.