Quantum speedup in a memory environment

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Memory (non-Markovian) effect is found to be able to accelerate quantum evolution [S. Deffner and E. Lutz, Phys. Rev. Lett. 111, 010402 (2013)]. In this work, for an atom in a structured reservoir, we show that the mechanism for the speedup is not only related to non-Markovianity but also to the population of excited states under a given driving time. In other words, it is the competition between non-Markovianity and population of excited states that ultimately determines the acceleration of quantum evolution in memory environments. A potential experimental realization for verifying the above phenomena is discussed by using a nitrogen-vacancy center embedded in a planar photonic crystal cavity under current technologies.

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I. INTRODUCTION

When it comes to the ultimate physical limit of a computer, two questions arise immediately: how much information can it store, and how fast can it run [1]? The former is related to the famous Boltzmann’s entropy formula that restricts the number of (qu)bits available to a physical system [2]. The latter concerns the speed of quantum evolution, and is limited by the remarkable time-energy uncertainty relation: \( \tau \geq \frac{\pi \hbar}{2 \Delta E} \), where \( \Delta E \) is the energy spread of the quantum system. This fundamental relation determines that it takes at least \( \tau_{\text{QSL}} = \max \{ \frac{\pi \hbar}{2 \Delta E}, \frac{\pi \hbar}{2E} \} \), the so-called quantum speed limit (QSL) time, to evolve to an orthogonal state for a quantum system with energy spread \( \Delta E \) or average energy \( E \). This QSL time has been proved to be tight [5], can be extended to nonorthogonal state cases [6], and plays an important role in a great deal of quantum tasks such as quantum computation [7], quantum communication [8], quantum optimal control [9], and quantum metrology [10].

Due to the inevitable coupling to the surrounding environment, quantum systems are usually treated as open [11]. Significant experimental progress has been made in controlling the dynamics of open systems [12–14]. How to detect the limit to quantum non-unitary evolution of open systems now becomes paramountly important. In recent years, efforts have been made towards the description of QSL time in the evolution of open systems [15, 16]. A unified lower bound, including both Mandelstam-Tamm and Margolus-Levitin types, has been derived by Deffner and Lutz [16]. By von Neumann trace inequality and Cauchy-Schwarz inequality, the QSL time between an initial open system state \( \rho = |\psi_0\rangle \langle \psi_0| \) and its target state \( \rho_T \), governed by the time-dependent non-unitary equation \( \dot{\rho}_t = \mathcal{L}_t \rho_t \) with \( \mathcal{L}_t \) a super-operator, is given by

\[
\tau_{\text{QSL}} = \max \{ \tau_1, \tau_2, \tau_\infty \},
\]

where \( \tau_p \) is the Schatten p-norm, \( \alpha_1, \ldots, \alpha_n \) are the singular values of \( A \), and \( B(\rho, \rho_T) = \arccos \sqrt{\langle \psi_0 | \rho_T | \psi_0 \rangle} \) denotes the Bures angle between the initial pure state \( \rho = |\psi_0\rangle \langle \psi_0| \) and the target state \( \rho_T \).

To study the environmental effects on QSL time, one method is to evaluate the characteristic of the intrinsic speed of the quantum evolution, i.e., given a driving time, how fast can a quantum system evolve. Interestingly, it is discovered that non-Markovian effect can speed up the quantum evolution with a damped Jaynes-Cummings model for an atom resonantly coupled to a leaky single mode cavity: QSL decreases when the non-Markovian effect becomes stronger [16]. Several questions, however, naturally arise: (i) is the speedup phenomenon a rather general feature which also exists in other physical models? (ii) is non-Markovianity the only key factor for speeding up quantum evolution? (iii) what is the mechanism for quantum acceleration in memory environments?

To address these questions, in this paper, we consider a two-level atom embedded in a photonic crystal cavity (PCC) with the periodic dielectric structures forming photonic band gaps (PBG) [17, 18]. This structured reservoir has been widely used to form atom-photon bound states [19, 20], strong localization of light [21], as well as entanglement preservation [22]. In this setting, the transition from no-speedup to speedup of quantum evolution is observed when the atomic frequency is approaching and going inside the band edge. QSL time is found to be related to two quantities: non-Markovianity during the driving time and excited population at \( \tau \). We illustrate that it is the competition between the two quantities that ultimately determines the speedup of quantum evolution.
evolution. Finally, a possible experimental realization for our illustrated phenomena by a single nitrogen-vacancy (N-V) center embedded in a two-dimensional PCC is discussed.

The work is organized as follows. In Sec. II, we introduce the physical model with a two-level atom embedded in a PCC and derive the relevant QSL time in Sec. III. In Sec. IV, we explore the mechanism for the speedup of quantum evolution and suggest a PCC-N-V based experimental proposal in Sec. V. Finally, we discuss and summarize the results in Sec. VI.

II. PHYSICAL MODEL

A two-level atom (open system) embedded in a PCC is coupled to the radiation field (environment) with the Hamiltonian ($\hbar = 1$) [11]

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k^\dagger a_k + i \sum_k g_k \left( a_k^\dagger \sigma_- - a_k \sigma_+ \right),$$

where $\omega_0$ is the resonant transition frequency of the atom between the excited and the ground states, $\sigma_{ \pm }$ are the Pauli raising and lowering operators, $\omega_k$ and $a_k (a_k^\dagger)$ are, respectively, the frequency, the annihilation and creation operators of the $k \text{th}$ mode of the reservoir with the real coupling constant $g_k = \omega_0 (2 \pi \omega_k V)^{-1/2} e_k \cdot d$. Here $\epsilon_0$ is the free space permittivity, $V$ and $e_k$ refer to the normalized volume and the unit polarization vector of the radiation field, respectively, and $d$ is the dipole moment of the atom.

The master equation for the reduced density matrix of the atom is $\dot{\rho}_t = \mathcal{L}_t \rho_t$ with

$$\mathcal{L}_t \rho_t = i \epsilon_t [\sigma_+ \sigma_-, \rho_t] + \gamma_t (\sigma_+ \rho_t + \rho_t \sigma_- - 2 \sigma_- \rho_t \sigma_+),$$

where $\epsilon_t =$Im$(\dot{b}_t b_t^\dagger)$ and $\gamma_t =$Re$(\dot{b}_t b_t^\dagger)$ are time-dependent Lamb shift and decay rate respectively, and $b_t$ is the decoherence function depending on certain reservoir structures [11]. The reduced density matrix of the atom with an initial state $\rho = (\rho_{mn})$ (in matrix form) can be evaluated as [11]

$$\rho_t = \Lambda_t \rho = \begin{pmatrix} \rho_{11} |b_t|^2 & \rho_{10} b_t \\ \rho_{01} b_t^\dagger & 1 - \rho_{11} \end{pmatrix},$$

with $\Lambda_t$ the quantum map.

In this work, we consider an ideal photonic crystal with isotropic photon dispersion relation approximated by $\omega_k = \omega_0 + A(k - k_0)^2$ near the band edge [20], where $\omega_0$ is the upper band-edge frequency, and $A = \omega_0 / k_0^2$ with $k_0 \simeq \omega_0 / c$ being a specific wave vector with respect to the point-group symmetry of the PCC. The Laplace transform of $b_t$ is $b_s = [s - (i \delta t)^{3/2} / \sqrt{s - i \delta}]^{-1}$ with $\delta^{3/2} = \omega_0^{7/2} / (6 \pi \epsilon_0 \hbar c^3)$ and $\delta = \omega_0 - \omega_c$ [20]. The decoherence function $b_t$ can then be calculated by the standard inverse Laplace transform as

$$b_t = \mathcal{L}^{-1} \left( \hat{b}_s \right) = \sum_{j<k} \sum_{k \neq l} e^{-\pi t_j} \frac{x_j^2 + i s x_j + (i \delta t_j)^{3/2} / \sqrt{x_j + i \delta}}{(x_j - x_k)(x_k - x_l)} \text{Erf} \left( \sqrt{-(x_j + i \delta)} \right),$$

where $x_{j,k,l}$ ($j, k, l = 1, 2, 3$) are the three parameters of the equation $s^3 - i \delta s^2 + j \beta^3 = (s + x_1)(s + x_2)(s + x_3)$, and Erf$(\cdot)$ is the error function.

III. QUANTUM SPEED LIMIT TIME

In this section, we evaluate the intrinsic speed for the evolution between the initial state $\rho$ and the final state $\rho_\tau$, with $\tau$ the actual driving time. For convenience and without loss of generality, the initial state is set to be the excited state $\rho = |1\rangle \langle 1| [16]$. It is readily checked that the maximum in Eq. (1) is $\tau_\infty$, for $E_{1\infty}^2 = E_1^2 / 2 = E_1^2 / \sqrt{2}$. In the light of Eqs. (1) and (4), the QSL time of the above model can be derived as

$$\tau_{\text{QSL}} = \frac{1 - P_t}{\tau_\infty \int_0^\infty |\partial_t P_t| dt},$$

with $P_t = |b_t|^2$ denoting the population of excited states at time $t$.

In Fig. 1, we depict the QSL time $\tau_{\text{QSL}}$ in unit of $1/\beta$ (blue solid curves) together with different actual driving time $\tau$ (red dashed lines) as functions of $\delta/\beta$, where the transition from no-speedup to speedup is clearly shown. When the atom transition frequency goes far outside the

![FIG. 1: (Color online) Quantum speed limit time $\tau_{\text{QSL}}$ in unit of $1/\beta$ (blue solid curves) for an atom embedded in an ideal isotropic photonic crystal cavity as a function of $\delta/\beta$, the detuning between the atom transition frequency and the upper band-edge frequency, for different driving time $\tau = 1, 3, 5, 10$ (in unit of $1/\beta$) (red dashed lines) in (a) $\sim$ (d), respectively.](image-url)
band gap (e.g., $\delta/\beta = 10$), the QSL time is actually the driving time. However, a remarkable phenomenon appears when the atom transition frequency is approaching and going inside the band edge ($\delta/\beta \rightarrow 0$), the QSL time $\tau_{\text{QSL}}$ will begin to decrease, implying the intrinsic speedup of quantum evolution. Obviously, the deeper the atomic transition frequency lies in the band edge, the smaller the $\tau_{\text{QSL}}$ time will be in a general trend. On the other hand, if the driving time is not very long, e.g., in Fig. 1(a), the speedup phenomenon only occurs inside the band edge ($\delta/\beta < 0$), which is quite different from Fig. 1(b)~(d), where the speedup region lies even outside the band edge.

How can we explain the above phenomena? Can we claim that the reason for the speedup in the above model is solely due to the memory (non-Markovian) effect of the reservoir? To answer these questions, in the following we first describe the non-Markovian behavior of the evolution of this model.

IV. MECHANISM FOR THE SPEEDUP OF QUANTUM EVOLUTION

The non-Markovianity measure we employ here is based on the total amount of information, quantified by trace distance $D(\Lambda_t \rho_1, \Lambda_t \rho_2) = \| \Lambda_t \rho_1 - \Lambda_t \rho_2 \| / 2$ of a pair of evolved quantum states ($\rho_1, \rho_2$), flowing back from the environment. The gradient of trace distance $G_t = \partial_t D(\Lambda_t \rho_1, \Lambda_t \rho_2)$ represents the information flow, with positive value indicating information flowing back to the system. Here $\Lambda = \{ \Lambda_t \}_{t \in [0, \tau]}$ denotes the dynamical map. The non-Markovianity is defined as the total backflow of information [23]

$$\mathcal{N}(\Lambda) = \max_{\rho_1, \rho_2} \int_{G_t > 0} G_t dt,$$

with the maximization over all initial state pairs $(\rho_1, \rho_2)$. There exists no general analytical method to find the optimal initial state pair $(\rho_1, \rho_2)$ [24]. Therefore, we will use numerical calculations instead.

In Fig. 2, the integrals $\int_{G_t > 0} G_t dt$ in the definition of $\mathcal{N}(\Lambda)$ of 2000 random initial state pairs $(\rho_1, \rho_2)$ [green (light gray) curves] are generated with $\delta/\beta = -10$ as an example. Clearly all these pairs yield smaller values than that of the state pair $(|0\rangle, |1\rangle)$ [red (dark gray) curve]. It is easy to check that for this optimal state pairs, the trace distance of the evolved states can be written as $D_t = |b_t|^2 = P_t$ ($D_t$ denotes the optimal trace distance in the following). Based on this, we take the non-Markovianity as

$$\mathcal{N}(\Lambda) = \int_{\partial_t P_t > 0} \partial_t P_t dt. \quad (8)$$

To associate the non-Markovianity with the QSL time, we rewrite Eq. (8) as

$$\mathcal{N}(\Lambda) = \frac{1}{2} \left[ \int_0^\tau |\partial_t P_t| dt + P_\tau - 1 \right], \quad (9)$$

Consequently, the QSL time is reduced to

$$\tau_{\text{QSL}} = \frac{\tau}{2\mathcal{N}(\Lambda)} + 1. \quad (10)$$

Clearly, the QSL time is related to two quantities: the non-Markovianity $\mathcal{N}(\Lambda)$ within the driving time and the
atomic excited population $P_r$. As an illustration, the two quantities $\mathcal{N}(\Lambda)$ [red (dark gray) curves] and $P_r$ [light blue (light gray) curves], as functions of $\delta/\beta$, are drawn in Fig. 3.

Equation (10) implies that the transition point from no-speedup to speedup of quantum evolution is just the point when the Markovian environment becomes non-Markovian, for $\tau_{\text{QSL}} = \tau$ when $\mathcal{N}(\Lambda) = 0$. It is therefore easy to account for the phenomenon that the speedup only takes place within the bandgap edge when $\tau = 1$ [Fig. 1(a)], for the transition point from Markovian to non-Markovian environment just occurs inside the bandgap edge when the driving time is short [Fig. 3(a)].

Equation (10) also provides us with a route to explore the mechanism for the intrinsic speedup of quantum evolution. For illustration, we consider the driving time $\tau = 10$ as an example [Fig. 3(d)]. When the atom transition frequency is far outside the bandgap edge frequency (e.g., $\delta/\beta = 10$), the population will not be trapped, i.e., $P_r = 10 \approx 0$, and the non-Markovianity also approaches zero, therefore, $\tau_{\text{QSL}} = \tau$. When the atomic transition frequency is around the bandgap edge (e.g., $\delta/\beta \in [-1,0]$), $\tau_{\text{QSL}}$ is dependent on both the population $P_r$ and non-Markovianity $\mathcal{N}(\Lambda)$, for the atomic excited population is trapped, i.e., $P_r \neq 0$, and the non-Markovianity is large. Finally, if the atomic transition frequency is deeply inside the bandgap edge (e.g., $\delta/\beta = -10$), the non-Markovianity becomes small, and $\tau_{\text{QSL}}$ is mainly dependent on $P_r$. Summarizing, it is the competition between $P_r$ and $\mathcal{N}(\Lambda)$ that takes responsibility for the intrinsic speedup of quantum evolution.

We note that our conclusion is not only restricted to the above physical model, it can also be used to explain the phenomena illustrated in Ref. [16], where the optimal trace distance for non-Markovianity of a two-level atom resonantly coupled to a leaky single mode cavity is found to be $D_t = |b| = \sqrt{t_r}$ [25]. Since the monotonicity of $\sqrt{t_r}$ and $P_r$ is the same, for simplicity, we still use $P_r$ as the trace distance (not optimal) for non-Markovianity $\mathcal{N}(\Lambda)$. Therefore Eq. (10) still holds with $\mathcal{N}(\Lambda)$ replaced by $\mathcal{N}(\Lambda)$. It is easy to check that no population will be trapped, i.e., $P_r = 0$ when $\tau \to \infty$. Thus

$$\tau_{\text{QSL}} = \frac{\tau}{2\sqrt{\mathcal{N}(\Lambda) + 1}},$$

which is inversely proportional to non-Markovianity $\mathcal{N}(\Lambda)$ if $\tau$ is longer enough. In this case, when $\mathcal{N}(\Lambda) = 0$ (Markovian), $\tau_{\text{QSL}} = \tau$. Consequently, the non-Markovian effect becomes the unique reason for speeding up quantum evolution in this model.

V. POSSIBLE EXPERIMENTAL REALIZATION

In this section, we briefly discuss a possible experimental realization for testing the above phenomena using a composite isotropic planar PCC-N-V system [shown in Fig. 4(a)] [26], where an external magnetic field $B$ is applied along the [111] axis of the N-V center to lift the degeneracy between the states $m_s = \pm 1$. In our case, the ground state $|0\rangle = |E_0, m_s = -1\rangle$ and the excited state $|1\rangle = (|E_-, m_s = +1\rangle + |E_+, m_s = -1\rangle)/\sqrt{2}$ are employed as the two-level system coupled to the modes of the PCC with coupling constant $g_s$.

![FIG. 4: (Color online) (a). Schematic diagram for the planar PCC-N-V system. (b). Energy levels of the electronic spin where $|0\rangle = |E_0, m_s = -1\rangle$ and $|1\rangle = (|E_-, m_s = +1\rangle + |E_+, m_s = -1\rangle)/\sqrt{2}$ are employed as the two-level system coupled to the modes of the PCC with coupling constant $g_s$.](Image 317x625 to 562x740)

Though our present work mainly focuses on a specific open system model, further study on more complicated non-Markovian systems characterized by other non-Markovian master equations [11], e.g., post-Markovian master equation [38], will be of great interest and importance.

On the other hand, as the non-Markovian effect plays a significant role in the speedup of quantum evolution, one
interesting question then arises: will non-Markovianity also be associated with some other speed limits? In quantum many-body systems, for instance, there exists a maximum speed of information propagation in discrete quantum systems with local interactions. This speed limit, known as Lieb-Robinson bound [39], has been extensively studied [40] and was observed in experiment with a one-dimensional ultracold gas of bosonic atoms [41]. Since quantum systems are always subject to decoherence and dissipation in practice, recent studies of Lieb-Robinson bound have been extended to classical Markovian dynamics [42] as well as quantum Markovian [43] and non-Markovian [44] conditions. It is desirable to further investigate the effects of noise and memory environments on the Lieb-Robinson velocity bound, and the interplay between QSL and the Lieb-Robinson bound.

In summary, for a model of two-level atom embedded in a PBG structured reservoir, the transition from no-speedup to speedup of quantum evolution has been observed. In particular, we have established a linkage between the QSL time and non-Markovianity as well as the population of excited states for a given driving time. We have found that it is the competition between the two quantities that finally determines the speedup of quantum evolution under noise. The phenomenon we illustrated in this work is finally analyzed by real PCC-N-V based experimental data. Our approach may be of both theoretical and experimental interests in exploring the ultimate limits to quantum computers in memory environments.

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