Commutators of lepton mass matrices associated with seesaw and leptogenesis

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Abstract

The origin of tiny neutrino masses and the baryon number asymmetry of the Universe are naturally interpreted by the canonical seesaw and leptogenesis mechanisms, in which there are the heavy Majorana neutrino mass matrix $M_R$, the Dirac neutrino mass matrix $M_D$, the charged-lepton mass matrix $M_\ell$ and the effective (light) neutrino mass matrix $M_\nu$. We find that $\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_\rho M_\rho^\dagger \right] \right)$, $\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_\nu M_\nu^\dagger \right] \right)$ and $\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_D M_D^\dagger \right] \right)$ can serve for a basis-independent measure of CP violation associated with lepton-number-violating decays of heavy neutrinos, flavor oscillations of light neutrinos and lepton-flavor-violating decays of charged leptons, respectively. We first calculate these quantities with the help of a standard parametrization of the $6 \times 6$ flavor mixing matrix, and then discuss their implications on both leptogenesis and CP violation at low energy scales. A comparison with the weak-basis invariants of leptogenesis as proposed by Branco \textit{et al} is also made.

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The commutator of quark mass matrices $M_u$ and $M_d$, or equivalently the commutator of quark Yukawa coupling matrices $Y_u$ and $Y_d$, has proved to be a quite useful measure of weak CP violation in the standard model (SM) \cite{1}. Given the quark masses $m_u \simeq 1.38$ MeV, $m_d \simeq 2.82$ MeV, $m_s \simeq 57$ MeV, $m_c \simeq 0.638$ GeV, $m_b \simeq 2.86$ GeV and $m_t \simeq 172.1$ GeV at the electroweak energy scale $\mu = M_Z$ \cite{2}, one may easily arrive at

$$\text{Im} \left( \text{det} \left[ Y_u Y_d^\dagger, Y_d Y_u^\dagger \right] \right) = \frac{27}{v^{12}} \mathcal{J}_q \left( m_t^2 - m_u^2 \right) \left( m_c^2 - m_u^2 \right) \left( m_d^2 - m_u^2 \right) \left( m_b^2 - m_u^2 \right) \left( m_s^2 - m_u^2 \right) \left( m_s^2 - m_u^2 \right) \simeq 6.0 \times 10^{-20}, \quad (1)$$

where $v \simeq 246$ GeV denotes the vacuum expectation value of the SM Higgs field, $\mathcal{J}_q \simeq 2.96 \times 10^{-5}$ is the Jarlskog invariant of CP violation in the quark sector \cite{3}, and the relation $M_{u,d} = Y_{u,d} v / \sqrt{2}$ has been used. The very small number obtained in Eq. (1) is ten orders of magnitude smaller than $\eta \equiv n_b/n_\gamma \simeq 6.2 \times 10^{-10}$, the observed baryon number asymmetry of the Universe \cite{3}. This is one of the reasons why the SM itself cannot account for the cosmological matter-antimatter asymmetry.

A simple extension of the SM is to add three right-handed neutrinos and allow lepton number violation in the lepton sector,

$$- \mathcal{L}_{\text{lepton}} = \bar{\ell}_L Y H E_R + \bar{\ell}_L Y \tilde{H} N_R + \frac{1}{2} N_R^c M_R^T N_R + \text{h.c.}, \quad (2)$$

where the notations for the SM fields are self-explanatory, $N_R$ (or $N_R^c$) is the column vector of the right-handed neutrino fields (or its charge-conjugate counterpart), and $M_R$ stands for the symmetric Majorana mass matrix. The scale of $M_R$ is expected to be much larger than $v$, because the right-handed neutrinos are SU(2)$_L$ singlets. In this case the effective mass matrix of three light neutrinos is approximately given by the seesaw relation \cite{4}

$$M_\nu \simeq - \frac{1}{2} Y_\nu \frac{v^2}{M_R} Y_\nu^T = - M_D \frac{1}{M_R} M_D^T, \quad (3)$$

where $M_D = Y_L v / \sqrt{2}$ is usually referred to as the Dirac neutrino mass matrix. The smallness of the mass scale of $M_\nu$ is therefore attributed to the largeness of the mass scale of $M_R$. A special bonus of this canonical seesaw is the effectiveness of the leptogenesis mechanism \cite{5}, which provides an elegant interpretation of $\eta \simeq 6.2 \times 10^{-10}$ for the observable Universe thanks to the lepton-number-violating and CP-violating decays of heavy Majorana neutrinos.

The present work aims to construct the commutators of lepton mass matrices and explore their relations with CP violation in both lepton-flavor-violating and lepton-number-violating processes. We find that Im $\left( \text{det} \left[ M_D^T M_D, M_R^T M_R \right] \right)$, Im $\left( \text{det} \left[ M_t M_l^T, M_{\nu} M_{\nu}^T \right] \right)$ and Im $\left( \text{det} \left[ M_{\ell} M_{\ell}^T, M_{\nu} M_{\nu}^T \right] \right)$ can serve for a basis-independent measure of CP violation associated with the lepton-number-violating decays of heavy neutrinos, the flavor oscillations of light neutrinos and the lepton-flavor-violating decays of charged leptons, respectively. We calculate these rephasing-invariant quantities in terms of a standard parametrization of the $6 \times 6$ flavor mixing matrix, and discuss their implications on both leptogenesis and CP violation at low energy scales. For the sake of comparison, we also calculate the weak-basis invariants of leptogenesis as defined by Branco et al \cite{6} and point out their similarity with and difference from our invariant Im $\left( \text{det} \left[ M_D^T M_D, M_R^T M_R \right] \right)$. 

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After the SU(2)$_L$ × U(1)$_Y$ symmetry is spontaneously broken to U(1)$_{em}$, Eq. (2) becomes

\[ -\mathcal{L}'_{\text{lepton}} = \bar{E}_LM_\ell E_R + \frac{1}{2} \left( \nu_L N_R^c \right) \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} + \text{h.c.}, \tag{4} \]

where $M_\ell = Y_\ell v/\sqrt{2}$. Without loss of generality, we choose a convenient lepton flavor basis in which $M_\ell = \hat{M}_\ell \equiv \text{Diag}\{m_e, m_\mu, m_\tau\}$ holds. The overall $6 \times 6$ neutrino mass matrix is symmetric, and it can be diagonalized by a unitary matrix containing 15 angles and 15 phases [7]:

\[ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) A \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{array}{c} V_0 \\ 0 \end{array} \right) \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{M}_\nu \\ 0 \end{pmatrix}, \tag{5} \]

where $\hat{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$ and $\hat{M}_N \equiv \text{Diag}\{M_1, M_2, M_3\}$ with $m_i$ or $M_i$ (for $i = 1, 2, 3$) being the physical masses of light or heavy Majorana neutrinos, and $V_0$ or $U_0$ is a $3 \times 3$ unitary matrix which consists of three mixing angles and three CP-violating phases. This basis transformation allows us to express the weak charged-current interactions of six neutrinos in terms of their mass states:

\[ -\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (e \mu \tau)_L \gamma^\mu \begin{pmatrix} V & R \end{pmatrix} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} W_\mu^\dagger + \text{h.c.}, \tag{6} \]

where $V \equiv AV_0$ is the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix [8] responsible for flavor oscillations of the light neutrinos $\nu_i$ (for $i = 1, 2, 3$), and $R$ measures the strength of charged-current interactions of the heavy neutrinos $N_i$ (for $i = 1, 2, 3$). The relationship between $V$ and $R$ is $VV^\dagger = AA^\dagger = 1 - RR^\dagger$ [9]. Since $R$ and $S$ describe the mixing between light and heavy neutrinos, the magnitudes of their elements are constrained to be at most of $\mathcal{O}(0.1)$ [10]. The exact expressions of $A$, $B$, $R$, $S$, $U_0$ and $V_0$ can be found in Ref. [7]. Here we only quote

\[ V_0 = \begin{pmatrix} c_{12}c_{13} & \hat{s}_{12}c_{13} & s_{13} \\ -\hat{s}_{12}c_{23} - c_{12}s_{13}s_{23} & c_{12}c_{23} - \hat{s}_{12}\hat{s}_{13}s_{23} & c_{13}s_{23} \\ s_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23} & -c_{12}\hat{s}_{23} - \hat{s}_{12}\hat{s}_{13}c_{23} & c_{13}c_{23} \end{pmatrix}, \tag{7} \]

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv e^{i\delta_{ij}}s_{ij}$, and $s_{ij} \equiv \sin \theta_{ij}$ with $\theta_{ij}$ and $\delta_{ij}$ being angles and phases (for $1 \leq i < j \leq 6$), respectively. In view of the smallness of the nine active-sterile mixing angles $\theta_{ij}$ (for $i = 1, 2, 3$ and $j = 4, 5, 6$), we arrive at the following excellent approximations:

\[ A \simeq B \simeq 1 - \mathcal{O}(s_{ij}^2) \text{, } R \simeq -S^\dagger \simeq \begin{pmatrix} \hat{s}_{14} & \hat{s}_{15} & \hat{s}_{16} \\ \hat{s}_{24} & \hat{s}_{25} & \hat{s}_{26} \\ \hat{s}_{34} & \hat{s}_{35} & \hat{s}_{36} \end{pmatrix} + \mathcal{O}(s_{ij}^3). \tag{8} \]

In this case we have $M_D \simeq R\hat{M}_N U_0^T$, $M_R \simeq U_0\hat{M}_N U_0^T$ and $V_0\hat{M}_\nu V_0^T + R\hat{M}_N R^T \simeq 0$. The seesaw relation in Eq. (3) can therefore be reexpressed as $M_\nu \equiv V_0\hat{M}_\nu V_0^T \simeq V_0\hat{M}_\nu V_0^T \simeq -R\hat{M}_N R^T$.

Let us first calculate the commutator $\left[ M_D^\dagger M_D, M_R^\dagger M_R \right]$, which is essentially associated with CP violation in the lepton-number-violating decays of $N_i$ at very high energy scales. Given the good
approximations made in Eq. (8), the result is
\[
\left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \simeq U_0^* \tilde{M}_N \left[ R^1 R, \tilde{M}_N^2 \right] \tilde{M}_N U_0^T \\
\simeq U_0^* \tilde{M}_N \left( \begin{array}{ccc}
0 & -M_1 M_2 \Delta_{12} \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i5}^* & -M_1 M_3 \Delta_{13} \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i6}^* \\
M_1 M_2 \Delta_{12} \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i5}^* & 0 & -M_2 M_3 \Delta_{13} \sum_{i=1}^3 \hat{s}_{i5} \hat{s}_{i6}^* \\
M_1 M_3 \Delta_{13} \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i6}^* & M_2 M_3 \Delta_{13} \sum_{i=1}^3 \hat{s}_{i5} \hat{s}_{i6}^* & 0
\end{array} \right) \tilde{M}_N U_0^T ,
\] (9)
where \( \Delta_{ij} \equiv M_i^2 - M_j^2 \) (for \( i, j = 1, 2, 3 \)).
The determinant of this commutator is purely imaginary, and thus we obtain
\[
\text{Im} \left( \det \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right) \simeq \text{det} \left( \tilde{M}_N^2 \right) \text{Im} \left( \det \left[ R^1 R, \tilde{M}_N^2 \right] \right) \simeq 2 M_1^2 M_2^2 M_3^2 \Delta_{12} \Delta_{13} \Delta_{23} \chi_N ,
\] (10)
in which
\[
\chi_N \equiv \text{Im} \left[ \left( \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i5}^* \right) \left( \sum_{i=1}^3 \hat{s}_{i4} \hat{s}_{i6}^* \right) \left( \sum_{i=1}^3 \hat{s}_{i5} \hat{s}_{i6}^* \right) \right]
\] (11)
is a Jarlskog-like quantity associated with the effects of CP violation in the decays of heavy Majorana neutrinos. So \( \text{Im} \left( \det \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right) \) will vanish if \( \Delta_{ij} = 0 \) holds or if \( \chi_N = 0 \) holds. We see that \( \chi_N \) depends on six independent phase differences, such as \( \delta_{i4} - \delta_{i5} \) and \( \delta_{i5} - \delta_{i6} \) (for \( i = 1, 2, 3 \)). Of course, the CP-violating asymmetries \( \varepsilon_{i\alpha} \) (or \( \varepsilon_i = \varepsilon_{i\alpha} + \varepsilon_{i\mu} + \varepsilon_{i\tau} \) in the unflavored case) between the lepton-number-violating decay modes \( N_i \rightarrow \ell_\alpha + H \) and \( N_i \rightarrow \ell_\alpha + \ell_\beta \) must depend on the same phase differences [7]. However, \( \chi_N \neq 0 \) is in general a necessary but not sufficient condition for \( \varepsilon_{i\alpha} \neq 0 \) or \( \varepsilon_i \neq 0 \), and hence \( \chi_N = 0 \) cannot guarantee \( \varepsilon_{i\alpha} = 0 \) or \( \varepsilon_i = 0 \) (or vice versa) either.

If \( M_1 \ll M_2 \ll M_3 \) holds and the leptogenesis mechanism works at temperature \( T \simeq M_1 \) [5], then it should not be difficult for the quantity
\[
\frac{\text{Im} \left( \det \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right)}{T^{12}} \simeq -2 \left( \frac{M_2}{M_1} \right)^4 \left( \frac{M_3}{M_1} \right)^6 \chi_N
\] (12)
to be comparable with or larger than \( \eta \simeq 6.2 \times 10^{-10} \) in magnitude. Taking \( M_3 \sim 10^2 M_2 \sim 10^4 M_1 \) and \( \theta_{i4} \sim \theta_{i5} \sim \theta_{i6} \lesssim \mathcal{O}(10^{-7}) \) for example, we expect that \( |\chi_N| \lesssim \mathcal{O}(10^{-42}) \) holds and thus the magnitude of \( \text{Im} \left( \det \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right) / T^{12} \) is in general possible to reach the \( \mathcal{O}(10^{-10}) \) level. In fact, a number of specific seesaw-plus-leptogenesis models have so far been proposed to successfully account for the observed baryon number asymmetry of the Universe [11].

We proceed to calculate the commutator \( \left[ M_\ell^\dagger M_\ell, M_\nu^\dagger M_\nu^\dagger \right] \), which is directly relevant to leptonic CP violation in neutrino oscillations [12]. Given the flavor basis \( M_\ell = \tilde{M}_\ell \) and the good approximation \( M_\nu \simeq V_0 \tilde{M}_\nu V_0^T \), it is straightforward to arrive at
\[
\text{Im} \left( \det \left[ M_\ell^\dagger M_\ell, M_\nu^\dagger M_\nu^\dagger \right] \right) \simeq 2 \Delta_{e\mu} \Delta_{e\tau} \Delta_{12} \Delta_{13} \Delta_{23} \delta_J \tau ,
\] (13)
in which $\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2$ (for $\alpha, \beta = e, \mu, \tau$) and $\Delta_{ij}' \equiv m_i^2 - m_j^2$ (for $i, j = 1, 2, 3$) are defined, and $\mathcal{J}_\nu = c_{12} s_{12} c_{13} s_{13} c_{23} s_{23} \sin \delta$ with $\delta \equiv \delta_{13} - \delta_{12} - \delta_{23}$ is just the Jarlskog invariant of $V_0$. Note that the mixing angles and CP-violating phases of $V_V$ and $J_X$ are actually correlated with those of $R$ due to the seesaw relation $V_0 M_\nu V_0^T + R M_N R^T \simeq 0$. The latter allows us to express Eq. (13) in terms of the parameters of $\hat{M}_N$ and $R$ besides $\Delta_{\alpha\beta}$, but the result is rather lengthy and thus less instructive. A key point is that $\chi' = 0$ does not necessarily lead to $\mathcal{J}_\nu = 0$, or vice versa, as one can see from the seesaw formula. Similarly, it is possible for either $\varepsilon_{1\alpha} = 0$ or $\varepsilon_{\alpha} = 0$, but $\mathcal{J}_\nu \neq 0$ or $\varepsilon_{1\alpha} \neq 0$ (or $\varepsilon_i \neq 0$) but $\mathcal{J}_\nu = 0$ to hold. Hence there is in general no direction connection between leptogenesis and CP violation at low energy scales.

Let us estimate the magnitude of $\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_\mu M_\mu^\dagger \right] \right)$ in order to give one a ball-park feeling of how small it is. The values of three charged-lepton masses are $m_e \simeq 0.48657$ MeV, $m_\mu \simeq 105.718$ MeV and $m_\tau \simeq 1746.17$ MeV at the electroweak scale $\mu = M_Z$. Furthermore, a global analysis of current neutrino oscillation data yields $\Delta'_{12} \simeq -7.5 \times 10^{-5}$ eV$^2$, $\Delta'_{13} \simeq \Delta'_{23} \simeq \mp 2.4 \times 10^{-3}$ eV$^2$, $\theta_{12} \simeq 34^\circ$, $\theta_{13} \simeq 9^\circ$, $\theta_{23} \simeq 41^\circ$ and $\delta \sim 250^\circ$. We therefore obtain $\mathcal{J}_\nu \sim -3.3 \times 10^{-2}$ and $\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_\mu M_\mu^\dagger \right] \right) \sim -2.3 \times 10^{-64}$ GeV$^{12}$. Because $\mathcal{J}_\nu$ measures the strength of leptonic CP violation in flavor oscillations of the light neutrinos, we anticipate some appreciable CP-violating effects to occur in the forthcoming long-baseline experiments.

Analogous to the commutator $\left[ H_\ell H_\ell^\dagger, H_\mu H_\mu^\dagger \right]$, the leptonic commutator $\left[ M_\ell M_\ell^\dagger, M_\mu M_\mu^\dagger \right]$ can be used to describe CP violation within the heavy Majorana neutrino sector. Given the approximation $M_R \simeq U_0 \hat{M}_N U_0^T$ and the parametrization $U_0$ where $\Delta_{\alpha\beta}$ and $\Delta_{ij}$ have already been defined, and $\mathcal{J}_N = c_{45} s_{45} c_{46} s_{46} c_{56} s_{56} \sin \delta'$ with $\delta' \equiv \delta_{46} - \delta_{45} - \delta_{56}$, which in turn lead to $\mathcal{J}_N = \mathcal{J}_\nu$. Whether such a heavy-light neutrino symmetry is phenomenologically useful remains an open question.

Another interesting commutator is $\left[ M_\ell M_\ell^\dagger, M_\mu M_\mu^\dagger \right]$, which should more or less be associated with the lepton-flavor-violating decays of charged leptons [14]. Taking $M_D \simeq R \hat{M}_N U_0^T$, we find

$$\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_D M_D^\dagger \right] \right) \sim 2 \Delta_{e\mu} \Delta_{e\tau} \Delta_{\mu\tau},$$

$$\text{Im} \left( \det \left[ M_\ell M_\ell^\dagger, M_D M_D^\dagger \right] \right) \sim 2 \Delta_{e\mu} \Delta_{e\tau} \Delta_{\mu\tau} \text{Im} \left[ \frac{c_{45} c_{46}}{-s_{45} s_{56} - c_{45} s_{46} s_{56}} \right].$$

\[\text{(16)}\]
At this point it makes sense to comment on the weak-basis invariants \( X \) on possible new physics behind them. It is therefore desirable to search for \( \mu \rightarrow e + \gamma \) and other possible lepton-flavor-violating channels, so as to fully probe the seesaw mechanism and its parameter space. In the minimal supersymmetric standard model extended with three heavy Majorana neutrinos, for example, there is some parameter space for the branching ratios of \( \mu \rightarrow e + \gamma \) and \( \tau \rightarrow \mu + \gamma \) decay modes to be close to their respective upper bounds as set by the present experiments [16]. The next-generation experiments of this kind are expected to impose more stringent constraints on such rare or forbidden processes in the SM and on possible new physics behind them.

At this point it makes sense to comment on the weak-basis invariants \( I_i \) as proposed by Branco et al [9] in the canonical seesaw mechanism. Now that the several leptonic commutators discussed above are independent of the flavor basis of weak interactions taken for the lepton mass matrices, one of them should be more or less equivalent to \( I_i \), which are defined as

\[
I_1 \equiv \text{Im} \text{Tr} \left[ \left( M_D^\dagger M_D \right) \left( M_R^\dagger M_R \right) M^*_R \left( M_D^\dagger M_D \right)^* M_R \right], \\
I_2 \equiv \text{Im} \text{Tr} \left[ \left( M_D^\dagger M_D \right) \left( M_R^\dagger M_R \right)^2 M^*_R \left( M_D^\dagger M_D \right)^* M_R \right], \\
I_3 \equiv \text{Im} \text{Tr} \left[ \left( M_D^\dagger M_D \right) \left( M_R^\dagger M_R \right)^2 M^*_R \left( M_D^\dagger M_D \right)^* M_R \right],
\]

(17)

By construction, these three invariants are only sensitive to the CP-violating phases which appear in leptogenesis, because \( M_D \) always appears in the form of \( M_D^\dagger M_D \). So CP invariance requires \( I_1 = I_2 = I_3 = 0 \). Given \( M_D \simeq R M_N^\dagger U_0^T \) and \( M_R \simeq U_0^\dagger M_N U_0^T \) in the seesaw approximation, we find that the structures of \( I_i \) (for \( i = 1, 2, 3 \)) are almost the same:

\[
I_1 \simeq \text{Im} \text{Tr} \left[ \left( R^\dagger R \right) \tilde{M}_N^5 \left( R^\dagger R \right)^* \tilde{M}_N^3 \right] = \sum_{i<j} \left( M_i M_j \right)^3 \left( M_j^2 - M_i^2 \right) \text{Im} \left[ \left( R^\dagger R \right)_{ij} \right]^2,
\]

\[
I_2 \simeq \text{Im} \text{Tr} \left[ \left( R^\dagger R \right) \tilde{M}_N^5 \left( R^\dagger R \right)^* \tilde{M}_N^3 \right] = \sum_{i<j} \left( M_i M_j \right)^3 \left( M_j^2 - M_i^2 \right) \text{Im} \left[ \left( R^\dagger R \right)_{ij} \right]^2,
\]

\[
I_3 \simeq \text{Im} \text{Tr} \left[ \left( R^\dagger R \right) \tilde{M}_N^5 \left( R^\dagger R \right)^* \tilde{M}_N^3 \right] = \sum_{i<j} \left( M_i M_j \right)^5 \left( M_j^2 - M_i^2 \right) \text{Im} \left[ \left( R^\dagger R \right)_{ij} \right]^2,
\]

(18)

where \( i, j = 1, 2, 3 \). It becomes obvious that \( I_1, I_2 \) and \( I_3 \) contain the same information about CP-violating phases. Taking \( I_1 \) for example and taking account of the the explicit parametrization of \( R \) in Eq. (8), we immediately arrive at

\[
I_1 \simeq \left( M_1 M_2 \right)^3 \Delta_{21} \text{Im} \left( \sum_{i=1}^3 \hat{s}_{14} \hat{s}_{i5}^* \right)^2 \left( M_1 M_3 \right)^3 \Delta_{31} \text{Im} \left( \sum_{i=1}^3 \hat{s}_{14} \hat{s}_{i6}^* \right)^2 \left( M_2 M_3 \right)^3 \Delta_{32} \text{Im} \left( \sum_{i=1}^3 \hat{s}_{15} \hat{s}_{i6}^* \right)^2.
\]

(19)

We see that \( I_1 \) depends on the same (six independent) CP-violating phases as \( X \) does, but they are not equivalent to each other. In fact, the CP-violating asymmetries \( \varepsilon_{i6} \) or \( \varepsilon_i \) vanish if all the three phase terms in Eq. (19) vanish, or equivalently the three invariants \( I_1, I_2 \) and \( I_3 \) are all vanishing [9].
In summary, we have constructed the commutators of lepton mass matrices and explored their relations with CP violation in the canonical seesaw and leptogenesis mechanisms. It is demonstrated that $\text{Im} \left( \text{det} \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right)$, $\text{Im} \left( \text{det} \left[ M_\ell M_\ell^\dagger, M_\nu M_\nu^\dagger \right] \right)$ and $\text{Im} \left( \text{det} \left[ M_\nu M_\nu^\dagger, M_D^\dagger M_D \right] \right)$ can serve for a basis-independent measure of CP violation associated with the lepton-number-violating decays of heavy Majorana neutrinos, the flavor oscillations of light Majorana neutrinos and the lepton-flavor-violating decays of charged leptons, respectively. We have calculated these rephasing-invariant quantities with the help of a standard parametrization of the $6 \times 6$ flavor mixing matrix, and discussed their implications on both leptogenesis and CP violation at low energy scales. We have also calculated the weak-basis invariants of leptogenesis as defined by Branco et al. and pointed out their similarity with and difference from $\text{Im} \left( \text{det} \left[ M_D^\dagger M_D, M_R^\dagger M_R \right] \right)$ in our case.

Finally, let us emphasize that the commutator language has played a very important role in the developments of Quantum Mechanics and Quantum Field Theories, and its applications in flavor physics have proved to be interesting and instructive for a basis-independent description of flavor mixing and CP violation. For example, the leptonic commutator $\left[ M_\ell M_\ell^\dagger, M_\nu M_\nu^\dagger \right]$ in vacuum and its counterpart in matter can help establish some direct relations between the effects of CP and T violation in vacuum and those in matter. This kind of study is therefore useful for the upcoming long-baseline neutrino oscillation experiments. The present work, which has offered a novel application of the commutator language in the canonical seesaw and leptogenesis mechanisms, is also meaningful and helpful to enrich the phenomenology of neutrino physics.

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