A MODEL OF THREE-DIMENSIONAL LATTICE GRAVITY

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Abstract

A model is proposed which generates all oriented 3d simplicial complexes weighted with an invariant associated with a topological lattice gauge theory. When the gauge group is $SU_q(2)$, $q^n = 1$, it is the Turaev-Viro invariant and the model may be regarded as a non-perturbative definition of 3d simplicial quantum gravity. If one takes a finite abelian group $G$, the corresponding invariant gives the rank of the first cohomology group of a complex $C$: $I_G(C) = rank(H^1(C, G))$, which means a topological expansion in the Betti number $b^1$. In general, it is a theory of the Dijkgraaf-Witten type, i.e. determined completely by the fundamental group of a manifold.
1 Introduction

Lattice models have always been an useful tool in field theory. They often helped to look at a theory from another point of view, which led to better understanding and computational progress. The most recent example of such a kind was the matrix models of 2d gravity \[1\]. In that case lattice and continuum approaches have been developed in the close connection stimulating each other. The success of the matrix models made it desirable to extend this approach to higher dimensional euclidean gravity. The general idea is rather natural: the integral over all d-dimensional manifolds should be substituted by a sum over all d-dimensional simplicial complexes. If a topology is fixed, a lattice action may be chosen linear in the number of simplexes of every dimension. The partition function in the 3d case is of the form

\[Z_{\text{top}} = \sum_{C_{\text{top}}} e^{\alpha N_1 - \beta N_3} = \sum_{N_1, N_3} Z_{N_1 N_3} e^{\alpha N_1 - \beta N_3}\]  \hspace{1cm} (1)

where \textit{top} means a fixed topology; \(\sum_{C_{\text{top}}}\) is the sum over all 3d simplicial manifolds of the chosen topology. Let us remind that in odd dimensions manifolds have the zero Euler character, hence

\[\chi = N_0 - N_1 + N_2 - N_3 = 0 \]  \hspace{1cm} (2)

\(N_k\) is the number of simplexes of the \(k\)-th dimension in a complex \(C\), \textit{i.e.} points, links, triangles and tetrahedra, respectively. The other constraint is

\[N_2 = 2N_3 \]  \hspace{1cm} (3)

which means simply that every triangle is shared by exactly two tetrahedra. Owing to the constraints, if the volume is fixed, only one parameter remains in the 3d and 4d cases and one may hope that it should be related to a bare Newton coupling. Indeed, keeping all tetrahedra equilateral, one gets, from counting deficit angles associated with links, the lattice analog of the mean curvature \[2\]

\[\int d^3x \sqrt{g} R \sim a \left( 2\pi N_1 - 6N_3 \cos^{-1} \left( \frac{1}{3} \right) \right) \]  \hspace{1cm} (4)

where \(a\) is a lattice spacing.

For the fixed spherical topology, a 3-dimensional model of such a type was investigated numerically in refs. \[3, 4\] and a 4-dimensional one, in refs. \[5\]. It appears that the micro-canonical partition function \(Z_{N_1 N_3}\) is exponentially bounded at large \(N_3\)

\[Z_{N_1 N_3} \sim e^{\beta^* N_3}\]  \hspace{1cm} (5)

while with respect to \(N_1\) at \(N_3\) fixed its shape is, roughly speaking, gaussian \[4\]. Varying the lattice analog of the inverse Newton constant, \(\alpha\), one can only shift the position of the maximum changing continuously the mean curvature \[4\]. It appeared that the vacuum is not unique. In refs. \[4\] the first order phase
transition was found at some $\alpha_c > 0$ which separates phases of positive ($\alpha > \alpha_c$) and negative ($\alpha < \alpha_c$) mean curvatures (4). The remarkable feature of the first phase is that the mean curvature per unit volume, $2\pi N_1/N_3 - 6 \cos^{-1}(\frac{1}{2})$, does not depend on $N_3$ at all [4]. It means the existence of the continuum thermodynamical limit for the model (4). A similar transition also exists in the 4-d model and there is some hope that here it is of the second order [5]. If it is confirmed, one can find a non-trivial continuum limit in its vicinity. Anyhow, the lattice models of gravity are interesting in their own rights.

The aim of this paper is to construct a model which generates all 3-dimensional simplicial complexes within a perturbation expansion so that it might be regarded as a 3-d analog of the matrix models. The naive generalization, so-called tensor models [6], suffers from serious diseases. The main one is that they do not contain the sufficient number of parameters: it is impossible to perform any topological expansion within them. It makes these models uninteresting because of their non-universality. As we learnt from the matrix models, only a perturbation topological expansion might be universal [7]. So, one has somehow to control the topology.

It is well-known that, in the 3d space-time, integration over diffeomorphisms and local Lorentz rotations is equivalent to the ISO$(2, 1)$ Chern-Simons field theory [8]. Although that connection holds only on-shell, it is clear that, in general, every topology should be somehow weighted. So far the Turaev-Viro $SU(2)$ invariant has been considered as the lattice counterpart of the ISO$(2, 1)$ Chern-Simons partition function [9, 10]. Strictly speaking, the corresponding argumentation is heuristic[1] and it may be better to consider as general class of models as possible. As will be shown in this paper, the underlying structure of the Turaev-Viro [11] (or Ponzano-Regge in its original form [12]) partition function is 3-dimensional topological lattice gauge theory, and, simply taking different gauge groups, one is able to construct different invariants. This ”degree of freedom” appears to be rather useful and, as we hope, will lead to the better understanding of the problem of 3d gravity.

The paper is organized as follows. In Section 2, a model is formulated which generates all 3d simplicial complexes weighted with the Ponzano-Regge (i.e. non-regularized) partition function. From the point of view of the Regge calculus [8], this partition function corresponds to a discretization of 3d euclidean gravity [11]. A natural generalization leads to a whole class of models of such a type. In Section 3, the $Z_n$ gauge group is considered. It is shown that, in some scaling limit, an expansion in the Betti number $b_2$ can be performed. In Section 4, the case of $q$-deformed $SU(2)$ gauge group is considered, when the Ponzano-Regge construction leads to the Turaev-Viro invariant. Section 5 is devoted to a discussion.

1For example, in ISO$(2, 1)$ Chern-Simons theory there is no reason to quantize the coupling constant $k$ in contrast with the $SU(2)$ case.
2 General construction

The basic object is a set of real functions of 3 variables $\phi(x, y, z)$ (where $x, y, z \in G$ for some compact group $G$) invariant under simultaneous right shifts of all variables by $u \in G$.

$$\phi(x, y, z) = \phi(xu, yu, zu); \quad \bar{\phi}(x, y, z) = \phi(x, y, z)$$ (6)

We also demand the cyclic symmetry

$$\phi(x, y, z) = \phi(z, x, y) = \phi(y, z, x)$$ (7)

The general Fourier decomposition of such a function is of the form

$$\phi(x, y, z) = \sum_{j_1, j_2, j_3} \sum_{\{m, n, k\}} \Phi_{m_1, m_2, m_3; k_1 k_2 k_3} D_{m_1, n_1}(x) D_{m_2, n_2}(y) D_{m_3, n_3}(z)$$ (8)

$$\int d\omega D_{n_1 k_1}^j(\omega) D_{n_2 k_2}^{j'}(\omega) D_{n_3 k_3}^{j''}(\omega)$$ (8)

$$n_i, m_i, k_i = 1, \ldots, d_j; \quad (d_j \text{ is the dimension of an irrep } j); \quad D_{nm}^j(x) \text{ are matrix elements obeying the orthogonality condition}$$

$$\int dx D_{nm}^j(x) \overline{D}_{nm'}^{j'}(x) = \frac{1}{d_j} \delta^{jj'} \delta_{nn'} \delta_{mm'}$$ (9)

Throughout the paper all measures are assumed to be normalized to the unity:

$$\int_G dx \equiv \frac{1}{\text{rank}(G)} \sum_{g \in G} 1 = 1$$ (10)

The integral of three matrix elements is proportional to a product of two Clebsch-Gordan coefficients $(j_1 j_2 n_1 n_2 \mid j_1 j_2 j_3 n_3)$. We shall use the following notation:

$$\int dx D_{m_1 n_1}^{j_1}(x) D_{m_2 n_2}^{j_2}(x) D_{m_3 n_3}^{j_3}(x) = \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{array} \right)$$ (11)

In the $SU(2)$ case, $\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{array} \right)$ is called the Wigner 3$j$-symbol:

$$\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{array} \right) = \frac{(-1)^{j_1 - j_2 + n_3}}{\sqrt{2j_3 + 1}} (j_1 j_2 n_1 n_2 \mid j_1 j_2 j_3 - n_3)$$ (12)

The Fourier coefficients,

$$A_{m_1, m_2, m_3}^{j_1, j_2, j_3} = \frac{1}{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}$$

$$\sum_{k_1 k_2 k_3} \Phi_{m_1 m_2 m_3; k_1 k_2 k_3}^{j_1, j_2, j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{array} \right)$$ (13)
are complex numbers having symmetries of 3\(j\)-symbols except for the condition

\[ m_1 + m_2 + m_3 = 0 \]  

(14)

An action of interest can be constructed with those functions as follows

\[
S = \frac{1}{2} \int dxdydz \phi^2(x, y, z) - \frac{\lambda}{4L} \int dxdydz u dvw \phi(x, y, z) \phi(x, u, v) \phi(y, v, w) \phi(z, w, u)
\]  

(15)

If the variables are attached to edges, the first term can be regarded as two glued triangles and the second, as four triangles forming a tetrahedron. Integrating out all group variables, one gets in the \(SU(2)\) case

\[
S = \frac{1}{2} \sum_{\{j_1, j_2, j_3\}} \sum_{\{-j_k \leq m_k \leq j_k\}} |A_{j_1, j_2, j_3}^{m_1, m_2, m_3}|^2
- \frac{\lambda}{4L} \sum_{\{j_1, \ldots, j_6\}} \sum_{\{-j_k \leq m_k \leq j_k\}} (-1)^{j_1 j_2} A_{j_1, j_2, j_3, j_4, j_5, j_6}^{m_1, m_2, m_3, m_4, m_5}
\]

\[
A_{j_1, j_2, j_3, j_4, j_5, j_6}^{m_1, m_2, m_3, m_4, m_5} \text{ with } \{j_1, j_2, j_3\} \text{ and } \{j_4, j_5, j_6\}
\]

(16)

If the coefficients \(A_{j_1, j_2, j_3}^{m_1, m_2, m_3}\) obey the condition following from the reality of \(\phi(x, y, z)\) (eq. (15)),

\[
A_{j_1, j_2, j_3}^{m_1, m_2, m_3} = (-1)^{j_1 + j_2} A_{j_1, j_2, j_3}^{m_1, m_2, m_3}
\]

(17)

and the measure of integration is taken to be

\[
D\phi = \prod_{\{j, j', j''\}} \prod_{\{-j \leq m \leq j\}} dA_{j, j', j''}^{m, m', m''}
\]

(18)

where \(\prod_{\{j, j', j''\}}\) means the product over all triplets \((j, j', j'')\) obeying the triangle inequality: \(|j' - j''| \leq j \leq j' + j''\), then the partition function will generate all possible 3d simplicial complexes weighted with corresponding (non-regularized) Ponzano-Regge partition functions, \(i.e.,\)

\[
Z = \int D\phi \ e^{-S} = \sum_{\{C\}} \lambda^{N_3(C)} \prod_{\{j\}} \sum_{L \in C} \left(2j_L + 1\right) \prod_{T \in C} \left\{ j_{T_1}, j_{T_2}, j_{T_3}, j_{T_4}, j_{T_5}, j_{T_6} \right\}
\]

(19)

where \(\sum_{\{C\}}\) is the sum over all oriented 3d simplicial complexes; \(N_3(C)\) is the number of tetrahedra in a complex \(C\); \(\sum_{\{j\}}\) is the sum over all possible configurations of \(j\)'s (colorings of links); \(\prod_{L \in C}\) is the product over all links \(L\) in \(C\); \(\prod_{T \in C}\) is the product over all tetrahedra \(T\) \((j_{T_i}, i = 1, \ldots, 6)\); are six momenta attached to edges of a tetrahedron \(T\). \(\left\{ j_{T_1}, j_{T_2}, j_{T_3}, j_{T_4}, j_{T_5}, j_{T_6} \right\}\) is the Racah-Wigner
6j-symbol attached to a tetrahedron $T$. We use the normalization for which the 6j-symbol is symmetric with respect to permutations of columns:

$$\begin{align}
\{ j_1 & \ j_2 \ j_3 \\ j_4 & \ j_5 \ j_6 \} = \sum_{\{ j_i \leq m_i \leq j_i \} \leq j_i} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)
\left( \begin{array}{ccc} j_5 & j_6 & j_1 \\ m_5 & m_6 & m_1 \end{array} \right)
\left( \begin{array}{ccc} j_6 & j_4 & j_2 \\ m_6 & m_4 & m_2 \end{array} \right)
\left( \begin{array}{ccc} j_4 & j_5 & j_3 \\ m_4 & m_5 & m_3 \end{array} \right)
\tag{20}
\end{align}$$

Eq. (19) is formal and has to be somehow regularized. Let us postpone a discussion on that and firstly make several remarks. Eq. (8) means that we are considering functions of two independent variables. If we drop the cyclic symmetry condition (7), then we shall have the representation

$$\phi(x, y, z) = f(xz^+, yz^+) = \sum_{j_1 \geq m_1, n_1 < j} \sum_{j_2 \geq m_2, n_2 < j} \sum_{j_3 \geq m_3, n_3 < j} F_{j_1 j_2 j_3}^{m_1 n_1 m_2 n_2} D_{m_1 n_1}^{j_1} (x) D_{m_2 n_2}^{j_2} (y) D_{m_3 n_3}^{j_3} (z)$$

Hence,

$$\tilde{A}_{j_1, j_2, j_3}^{m_1 m_2 m_3} = \sum_{n_1, n_2} \sqrt{\frac{(2j_3 + 1)}{(2j_1 + 1)(2j_2 + 1)}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ n_1 & n_2 & m_3 \end{array} \right) F_{j_1 j_2 j_3}^{m_1 n_1 m_2 n_2}$$

From eq. (17) it follows that

$$F_{j_1 j_2}^{m_1 n_1 m_2 n_2} = (-1)^{m_1 + m_2 + n_1 + n_2} F_{j_1 j_2}^{m_1 n_1 m_2 n_2}$$

and, if the correlator of the Fourier coefficients is of the form

$$\langle F_{j_1 j_2}^{m_1 n_1 m_2 n_2} F_{j_1 j_2}^{m_1' n_1' m_2 n_2'} \rangle = (2j_1 + 1)(2j_2 + 1) \delta_{j_1, j_1'} \delta_{j_2, j_2'} \delta_{m_1 + m'_1, 0} \delta_{m_2 + m'_2, 0} \delta_{n_1 + n'_1, 0} \delta_{n_2 + n'_2, 0} \delta_{j_3, j_3'} \delta_{m_3, m'_3} \tag{24}$$

then

$$\langle \tilde{A}_{j_1, j_2, j_3}^{m_1, m_2, m_3} \tilde{A}_{j_1', j_2', j_3'}^{m_1', m_2', m_3'} \rangle = \sum_{n_1, n_2} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ n_1 & n_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j_3' \\ n_1 & n_2 & -m'_3 \end{array} \right) \delta_{j_1, j_1'} \delta_{j_2, j_2'} \delta_{m_1 + m'_1, 0} \delta_{m_2 + m'_2, 0} \delta_{m_3 + m'_3, 0} \delta_{j_3, j_3'} \delta_{m_3, m'_3}$$

In terms of the function $f(x, y)$ the action (13) takes the form

$$S = \frac{1}{2} \int dxdy f^2(x, y)$$

$$-\frac{\lambda}{4!} \int dxdydwuh(x, y)h(xw, uw)h(v, u)h(vw, yw) \tag{26}$$
where \( h(x, y) = \frac{1}{2}(f(x, y) + f(yx^+, x^+) + f(y^+, xy^+)). \)

For a general compact group, the action (26), as well as (15), together with the reality condition

\[
\mathcal{F}(x, y) = f(x, y) \tag{27}
\]

may be regarded as a definition of the model. The underlying mathematical structure here is topological lattice gauge theory. It can be seen as follows. The action (15) generates 3d complexes so that two 3j-symbols are attached to every triangle. Such a combination can be obtained integrating three matrix elements as in eq. (11). All lower indices of the matrix elements are summed up inside tetrahedra forming 6j-symbols. It is easy to notice that the partition function (19) can be written then in the form

\[
Z = \sum_{\{C\}} \lambda^{N_3} \sum \prod_{\{j;m,n\}} \prod_{t \in C} dx_t \prod_{L \in C} D_{m_1 n_1 t}^{j_1} (x_t) D_{m_2 n_2 t}^{j_2} (x_t) D_{m_3 n_3 t}^{j_3} (x_t) = \sum_{\{C\}} \lambda^{N_3} Z_{\text{gauge}}(C) \tag{28}
\]

where \( \prod_{\{L \in C\}} \) and \( \prod_{\{t \in C\}} \) are products over all links and triangles, respectively. The matrix elements being multiplied around links produce characters and then, summing over representations, one gets a \( \delta \)-function for every link. Its argument, \( \prod_{\{L \in C\}} x_{tL} \), are the product of group elements \( x_{tL} \) around a link \( L \). Triangles are oriented, the change of an orientation leading to the conjugaison: \( x \rightarrow x^+ = x^{-1} \). All products have to be performed taking the orientation into account. Although it is not the fact for a general compact group, in the \( SU(2) \) case our model generates only oriented complexes. It follows immediately from (17) (or (23)) and the form of the action (16).

If a complex is fixed, the model is equivalent to 3d gauge theory with fields defined on links of the dual \( \phi^4 \) graphs and the pure gauge condition on dual faces. If \( \delta \)-functions were substituted by, for example, the heat-kernel weights

\[
\delta(x) = \sum_j d_j \chi_j(x) \leftarrow \mathcal{G}(x) = \sum_j d_j \chi_j(x) e^{-\epsilon C_j} \tag{29}
\]

where \( C_j \) is the quadratic Casimir, one would have just the ordinary lattice gauge theory. The former is the weak coupling limit of the latter.

Now, to prove the topological invariance, we need only formal properties of the group measure and \( \delta \)-functions. We have to investigate transformations of \( Z_{\text{gauge}}(C) \) under topology preserving deformations of a complex \( C \). Two complexes are of the same topological type (homeomorphic), if they can be connected by a sequence of elementary ”continuous” deformations (moves). These moves can be defined as follows [3, 4]: if some subcomplex of a \( d \)-dimensional complex can be identified with a part of the boundary of the \( (d+1) \)-dimensional simplex, it is substituted by the rest of the boundary. In the 3-dimensional case
there are two pairs of mutually inverse moves shown in Fig. 1 (a) and (b). The first pair is called the triangle-link exchange (the dual diagrams are shown in Fig. 2(a)). A pair of tetrahedra glued together by faces is substituted by three tetrahedra sharing the new link. For the first configuration we have the integral of the type

\[
\int dx_1 dx_2 dx_3 dy_1 dy_2 dy_3 dw D_{m_1 n_1}^{j_1} (x_1 x_2^+) D_{m_2 n_2}^{j_2} (x_2 x_3^+) D_{m_3 n_3}^{j_3} (x_3 x_1^+) \\
D_{m_1' n_1'}^{j_1'} (y_1 y_2^+) D_{m_2' n_2'}^{j_2'} (y_2 y_3^+) D_{m_3' n_3'}^{j_3'} (y_3 y_1^+) \\
D_{a_1 b_1}^{l_1} (x_1 y_1^+) D_{a_2 b_2}^{l_2} (x_2 y_2^+) D_{a_3 b_3}^{l_3} (x_3 y_3^+) \tag{30}
\]

\(x\)'s and \(y\)'s stay for faces of the upper and lower tetrahedra, respectively, and \(w\), for the common face. It is clear that the dependence on \(w\) can be removed by the shift \(x_1 \to x_1 w^+; x_2 \to x_2 w^+; x_3 \to x_3 w^+\). In the second case the situation is quite analogous: there are three triangles (\(w\)'s) and one link (\(\delta\)-function) inside the subcomplex. The counterpart of eq. (30) is

\[
\int dx_1 dx_2 dx_3 dy_1 dy_2 dy_3 dw_1 dw_2 dw_3 D_{m_1 n_1}^{j_1} (x_1 w_1 x_2^+) D_{m_2 n_2}^{j_2} (x_2 w_2 x_3^+) \\
D_{m_3 n_3}^{j_3} (x_3 w_3 x_1^+) D_{m_1' n_1'}^{j_1'} (y_1 w_1 y_2^+) D_{m_2' n_2'}^{j_2'} (y_2 w_2 y_3^+) D_{m_3' n_3'}^{j_3'} (y_3 w_3 y_1^+) \\
D_{a_1 b_1}^{l_1} (x_1 y_1^+) D_{a_2 b_2}^{l_2} (x_2 y_2^+) D_{a_3 b_3}^{l_3} (x_3 y_3^+) \delta (w_1 w_2 w_3, 1) \tag{31}
\]

where all \(w\)-integrations are trivial due to the \(\delta\)-function.

Instead of proving the invariance under the moves in Fig.1(b), we can consider the case of two tetrahedra glued along three faces (Fig.2(b)). This configuration can be obtained removing one of the links in Fig.1(b) by the triangle-link exchange. The appearing integral is of the form

\[
\int dw_1 dw_2 dw_3 \delta (w_1 w_2^+) \delta (w_2 w_3^+) \delta (w_3 w_1^+) D_{m_1 n_1}^{j_1} (w_1) D_{m_2 n_2}^{j_2} (w_2) D_{m_3 n_3}^{j_3} (w_3) \\
= \delta (1, 1) \int dw D_{m_1 n_1}^{j_1} (w) D_{m_2 n_2}^{j_2} (w) D_{m_3 n_3}^{j_3} (w) \tag{32}
\]

which means that, up to \(\delta (1, 1) = \text{rank}(G)\), those two glued tetrahedra are equivalent to a single triangle. We see that the partition function (19) can be finally written down as

\[
Z = \sum_{\{C\}} \lambda^{N_3} (\text{rank}(G))^{N_0 - 1} I_C (C) \tag{33}
\]

where \(I_C (C)\) is a topological invariant associated with a group \(G\).

For finite groups, our model is well defined, as in this case the rank is equal to the number of group elements. For continuous compact groups, the \(q\)-deformation provides us with a regularization of the model (notice that the substitution (23) destroys the topological properties of the gauge theory). For example, in the \(SU_q(2)\) case, \(q^n = 1\),

\[
\text{rank}(SU_q(2)) = \frac{n}{2 \sin^2 \left( \frac{\pi}{n} \right)} \tag{34}
\]
Indeed, the representations of the $q$-anals of compact groups resemble the classical representations. And, while one is working with $3j$ and $6j$-symbols not permuting momenta, as we did above, the $R$-matrix does not appear and all formal manipulations coincide in both cases. We see that quantum groups are here on equal footing with finite groups. That is why in the next section we shall concentrate ourselves on the simpler latter case.

3 Topological gauge theory for finite groups and the $Z_p$ model.

The topological lattice model appeared in the previous section is a particular example of the Dijkgraaf-Witten theory [12]. Actually, it is the simplest model of such a type. Dijkgraaf and Witten introduced a topological action, which exists, however, not for all groups. In our case there is no action and, therefore, there are no corresponding restrictions. Two other peculiarities are (i) the gauge fields are defined on dual edges rather than on links of a triangulation; (ii) since \( \sum_{\{C\}} \) runs over all possible complexes, we should take into consideration non-manifolds as well. Nevertheless, the model bears general properties of the Dijkgraaf-Witten one. The main is that its partition function is determined completely by the fundamental group.

Among lattices generated perturbatively there are such that, strictly speaking, do not obey the definition of the simplicial complex (for example, shown in Fig. 2 (b)). It forces us to work with more general cell complexes.

From now on, we shall consider simultaneously triangulations and dual $\phi^4$ lattices denoting quantities defined for the latter by the tilde. So, at the beginning we have a cell complex dual to a triangulation: 0-cells are counterparts of tetrahedra, 1-cells of triangles, 2-cells of links and 3-cells of vertices of the triangulation. Since analogs of eqs. (30,31,32) are valid for general polyhedra as well, we can shrink a 1-cell identifying two 0-cells forming its boundary (Fig. 3(a)); delete a 2-cell joining two 3-cells a common boundary of which it was (Fig. 3(b)) and drop a subcomplex homotopic to a 3d spherical ball (Fig. 3(c)). Of course, all these manipulations are possible only when a final complex is homotopic to an initial one. So, we have in hands the powerful apparatus of the cell homology theory.

Given a cell complex $\tilde{C}$, one can easily calculate the corresponding invariant as follows:

1) all complexes under consideration should be put in the form where there are only one 3-cell $\sigma^3$ and only one 0-cell $\sigma^0$. It is always possible for oriented connected manifolds.
2) a gauge variable $g_i \in G$ is put into correspondence to every 1-cell $\sigma^1_i$; $i = 1, \ldots, n_1$.
3) each 2-cell $\sigma^2_j$; $j = 1, \ldots, n_2$; gives a $\delta$-function with the argument equal to the ordered product of the gauge variables along its boundary, $\partial \sigma^2_j$, taking an orientation into account (the inversion $g \to g^{-1}$ corresponding to the moving

\footnote{A more complete discussion will be given in Section 4.}
in the opposite direction). If the boundary is empty, one has to substitute
\( \delta(1, 1) = \text{rank}(G) \).
Finally, one gets
\[
\tilde{I}_G(\tilde{C}) = \int_G \prod_{i=1}^{n_1} dg_i \prod_{j=1}^{n_2} \delta( \prod_{\ell \in \partial \sigma_j^2} g_\ell, 1)
\]  \hspace{1cm} (35)

\( n_1 \) and \( n_2 \) are the numbers of 1-cells and 2-cells, respectively.

Let us point out the simple fact that the \( n_2 \) conditions
\[
\prod_{\ell \in \partial \sigma_j^2} g_\ell = 1
\]  \hspace{1cm} (36)
can be regarded as the defining relations of the fundamental group \( \pi_1(\tilde{C}) \), if one considers \( G \) as the free group on \( n_1 \) generators.

From eq. (35) it follows that
\[
\tilde{I}_G(\tilde{C}) = \text{rank}(\pi_1(\tilde{C}) \hookrightarrow G)
\]  \hspace{1cm} (37)
which is reminiscent of theories of the Dijkgraaf-Witten type. \( \pi_1(\tilde{C}) \hookrightarrow G \) means the homomorphism of \( \pi_1(\tilde{C}) \) into a finite group \( G \) defined by the above construction.

From eq. (35) it follows that \( \tilde{I}_G(\tilde{C}) \) is multiplicative with respect to the connected sum of two 3d complexes, \( \tilde{C} = \tilde{C}_1 \# \tilde{C}_2 \),
\[
\tilde{I}_G(\tilde{C}) = \tilde{I}_G(\tilde{C}_1) \tilde{I}_G(\tilde{C}_2)
\]  \hspace{1cm} (38)
The operation \( \# \) is commutative, hence, eq. (37) can be regarded as a representation of this semi-group.

An interesting case is abelian groups. Since
\[
H_1(\tilde{C}, G) = \pi_1(\tilde{C})/[\pi_1(\tilde{C}), \pi_1(\tilde{C})]
\]  \hspace{1cm} (39)
\( (i.e. \) the first homology group is a commutated fundamental group), we have in this case
\[
\tilde{I}_G(\tilde{C}) = \text{rank}(H_2(\tilde{C}, G))
\]  \hspace{1cm} (40)
where \( H_2(\tilde{C}, G) \) is the second homology group of a complex \( \tilde{C} \) with coefficients in \( G \).

To prove eq. (40) let us note that there are only one 0-cell \( \sigma^0 \) and only one 3-cell \( \sigma^3 \) and for all 1-cells \( \sigma_i^1; i = 1, \ldots, n_1 \)
\[
\partial \sigma_i^1 = 0
\]  \hspace{1cm} (41)
where \( \partial \) is the standard homologic boundary operator \( (\partial : \sigma^k \rightarrow \sigma^{k-1}) \). Because of the orientability,
\[
\partial \sigma^3 = 0
\]  \hspace{1cm} (42)
as well (i.e. there are no exact 2-cells) and, hence, every 2-cell having zero boundary gives a generator of \( H_2(\tilde{C}, G) \). But it is exactly the condition that is coded in the arguments of the \( \delta \)-functions in eq.(35): \( \text{rank}(H_2(\tilde{C}, G)) \) is equal to the number of times the \( \delta \)-functions "have worked".

The group \( H_2(\tilde{C}, G) \) is isomorphic to \( H^1(C, G) \) by the Poincaré duality generated by the transformation from \( \phi^1 \) graphs to triangulations and vice versa.

Eq. (32) allows us to determine the Betti number \( \text{mod } G \):

\[
b_1 = \left\lfloor \frac{\log \tilde{I}_G(\tilde{C})}{\log \text{rank}(G)} \right\rfloor
\]

where \( \lfloor x \rfloor \) means the integer part of \( x \).

Now, let us give several simple examples for the cyclic group \( \mathbb{Z}_p \).

1) Sphere \( S^3 \). There are no 1- and 2-cells at all.

\[
\tilde{I}_G(S^3) = 1
\]  

2) Lenses \( L^q = S^3/\mathbb{Z}_q \). There is one 1-cell and one 2-cell: \( \partial \sigma^2 = q \sigma^1 \).

\[
\tilde{I}_G(L^q) = \int_G dg \delta(g^q, 1)
\]

\[
\tilde{I}_{\mathbb{Z}_p}(L^q) = \begin{cases} 
  p & , p = q \\
  1 & , p \neq q 
\end{cases}
\]  

3) \( S^1 \times S^2 \) There is one 1-cell and one 2-cell: \( \partial \sigma^2 = 0 \).

\[
\tilde{I}_G(S^1 \times S^2) = \int_G dg \delta(1, 1) = \text{rank}(G)
\]

\[
\tilde{I}_{\mathbb{Z}_p}(S^1 \times S^2) = p
\]  

4) \( S^1 \times M_r^2 \) where \( M_r^2 \) is a 2d oriented surface with \( r \) handles \( r \geq 1 \):

\[
\tilde{I}_G(S^1 \times M_r^2) = \int_G dg \prod_{i=1}^r df_i dh_i \delta(\prod_{j=1}^r h_j f_j h_j^{-1} f_j^{-1}, 1)
\]

\[
\prod_{j=1}^r \delta(gh_j g^{-1} h_j^{-1}, 1) \delta(g f_j g^{-1} f_j^{-1}, 1)
\]

\[
\tilde{I}_{\mathbb{Z}_p}(S^1 \times M_r^2) = p^{2r+1}
\]  

The consideration so far involved more or less standard things and now let us discuss peculiarities. First, we should extend our construction to non-manifolds. In three dimensions there is no general restriction on the Euler character but in our case \( \chi \) defined by eq. (32) appears to be non-negative. It can be seen as follows. For each vertex in a complex, tetrahedra touching it form a 3-ball with a non-trivial, in general, 2d boundary. Let us denote \( \chi_i^{(2)} = 2(1 - p_i) \) the 2d
Euler character of the boundary of the ball for the $i$-th vertex. Summing over vertices one gets

$$\sum_{i=1}^{N_0} \chi_i^{(2)} = 2N_0 - 2 \sum_{i=1}^{N_0} p_i$$

(51)

On the other hand, this quantity can be obtained counting the numbers of simplexes of different dimensions. A simple algebra gives

$$\chi = \sum_{i=1}^{N_0} p_i \geq 0$$

(52)

By definition a complex is a manifold, iff $\forall i : p_i = 0$.

The Euler character can be as well expressed through the Betti numbers:

$$\chi = b_0 - b_1 + b_2 - b_3$$

(53)

and, since, for oriented connected complexes, always $b_0 = b_3$, we have the inequality

$$b_2 \geq b_1$$

(54)

The dual quantities, $\tilde{b}_i = b_{3-i}$ and $\tilde{\chi} = -\chi$ by the Poincaré duality, which reads $H^k(C, Z_p) = H_{3-k}(C, Z_p)$. Hence, our invariant is sensitive to $b_1$.

For manifolds, in eq. (35), the number of integrations is always equal to the number of $\delta$-functions ($n_1 = n_2$). In general, there can be an excess of variables. It means that, at least for abelian groups, the invariant does not distinguish between manifolds and non-manifolds. For every manifold there are infinitely many non-manifolds (having different $\chi$’s) giving the same answer. Therefore, the choice $G = Z_p$ looks rather reasonable. The invariant gives essentially $p^{b_1}$ (up to subtleties clearly seen in the case of lenses). And, if we weigh links with $\mu/p$, triangles with $p$ and tetrahedra with $\lambda/p$, the partition function will take the form

$$\log Z = \sum_{\{C_c\}} Q(C) \lambda^{N_3} \mu^{N_1} p^{b_2-1}$$

(55)

where the factor $Q(C) = I_{Z_p}(C)/p^{b_1} < p; \sum_{\{C_c\}}$ is the sum over connected oriented complexes.

So, we arrive at the following generalization of the 2d matrix models

$$Z = \int \prod_{a,b,c=1}^{\mu/p} \prod_{i,j,k} d\phi_{i,j,k}^{abc} \exp \left\{ -\frac{1}{2} \sum_{a,b,c=1}^{\mu/p} \sum_{i,j,k} |\phi_{i,j,k}^{abc}|^2 \right\}$$

$$+ \frac{\lambda p}{4!} \sum_{a,b,c}^{\mu/p} \sum_{i,j,k}^{\mu/p} \phi_{i,j,k}^{abc} \phi_{i-l,j-m,k-n}^{ade} \phi_{-i,j-m,k-n}^{beg} \phi_{-j,m,-n}^{cde}$$

(56)
Lower indices, $i, j, k, l, m, n$, are taken $\text{mod } p$; $\phi_{i,j,k}^{abc} = 0$, unless $i + j + k = 0 \pmod{p}$, and all sums and products run over this set of indices. The field $\phi_{i,j,k}^{abc}$ has to obey the following additional conditions

$$\phi_{i,j,k}^{abc} = \phi_{j,i}^{bca} = \phi_{k,i,j}^{cab}$$

and

$$\phi_{i,j,k}^{abc} = \phi_{-i,-j,-k}^{abc}$$

If $p$ is odd, all complexes generated by the model are oriented and the above analysis is valid. In the formal limit $p \to 0$ only homologic spheres survive. But one should be very careful here. There are infinitely many topologies at $b_2 = 0$ (all lenses among them). In our model their number is cut by the volume, $N_3$. Hence, one should keep $p$ sufficiently large (at least larger than the biggest $q$ among appearing lenses $L^q$). It means that, for a given $\lambda$ away from a critical point $\lambda_c$ ($N_3$ is finite), one should take the limit $p \to \infty$ first. After that one may tend $\lambda \to \lambda_c$ performing simultaneously an analytical continuation to $p = 0$. It means a non-trivial scaling. In any case, one has somehow to remove a singularity at $\lambda = 0$ like it was done for the matrix models in refs. [7]. The problem, however, is whether the number of complexes with $b_2$ fixed is exponentially bounded. If it is so, the critical value $\lambda_c$ exists and the above program is self-consistent. If not, then a further topological classification is needed. For a fixed topology, the answer to that question is ”yes”. At least, numerical experiments clearly showed that the number of spheres homeomorphic to $S^3$ grows exponentially with the volume. This growth should be determined locally, as in the $2d$ case, i.e. independently of a topology. So, the question is ”How many topologies can one fill a given volume with?”’. But, even if the above program does make sense, hard technical problems still remain to be overcome.

4 The $SU_q(2)$, $q^n = 1$, model.

In the case of $q$-deformed $SU(2)$ group, some conceptual problems still remain. The main tool of our analysis in Section 2 was the Peter-Weyl theorem stating that the algebra of regular functions on a compact group is isomorphic to the algebra of matrix elements of finite dimensional representations. The $q$-analog of this theorem was proved in refs. [13] for $|q| < 1$. In this case there is one-to-one correspondence between representations of $SU_q(2)$ and $SU(2)$, and the notion of the matrix elements is naturally generalized. The main difference in the quantum case is that the tensor product is not commutative (for example, $\delta(x,y) \neq \delta(y,x)$). Although in this case there exists a definition of a rank which appears to be finite [14], the lattice topological gauge theory built with this group does not exist because of divergencies. $q^n = 1$ changes the situation drastically. The analysis of refs. [13] is not valid in this case and the whole subject has to be revised. On the other hand, the theory of representations of the quantized universal enveloping algebra $U_q(SL(2))$, when $q^n = 1$, was given in refs. [15] and in the most complete form in ref. [16].
As was established in [16], all highest weight irreps $\rho_j$ of $U_q(\mathfrak{sl}(2))$, when $q^n = 1$, fall into two classes:

a) dimension of $\rho_j$, $\dim(\rho_j) < M$, where $M = \begin{cases} \frac{n}{2} & n \text{ even} \\ n & n \text{ odd} \end{cases}$

These irreps are numbered by two integers $d$ and $z$: $(d, z)$, where $d = \dim(\rho_j)$, and the highest weights are

$$j = \frac{1}{2}(d - 1) + \frac{n}{4}z$$  \hspace{1cm} (59)

b) $\dim(\rho_j) = M$. In this case irreps $I^1_z$ are labeled by a complex number $z \in \mathbb{C} \setminus \{Z + \frac{2}{n} r \mid 1 \leq r \leq M - 1\}$ and have the highest weights

$$j = \frac{1}{2}(M - 1) + \frac{n}{4}z$$  \hspace{1cm} (60)

There are also indecomposable representations which are not irreducible but nevertheless cannot be expanded in a direct sum of invariant subspaces. They are labeled by an integer $2 \leq p \leq M$ and the complex number $z$: $I^p_z$. Their dimension $\dim(I^p_z) = 2M$.

In ref. [16] the following facts important for us were established:

1) If $n \geq 4$, irreps $(d, 0)$ are unitary only for even $n$.
2) Representations of the type $I^p_z$, $1 \leq p \leq M$ form a two sided ideal in the ring of representations (i.e., if at least one of them appears in a tensor product, then all representations in the decomposition will be of this type). Their quantum dimension vanishes: $\dim_q(I^p_z) = \begin{cases} [M] & , p = 1 \\ 2M & , p \geq 2 \end{cases} = 0$, where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$.
3) For the tensor product of two irreps the following formula takes place:

$$\langle i, z \rangle \otimes \langle j, w \rangle = \bigoplus_{k = |i - j| + 1; +3; +5,...}^{\min(i+j-1, 2M - i - j - 1)} \langle k, z + w \rangle \bigoplus_{\ell = r, r+2, r+4,...}^{i+j-M} I^\ell_{z+w}$$  \hspace{1cm} (61)

where $r = \begin{cases} 1 & , i + j - M \text{ odd} \\ 2 & , \text{otherwise} \end{cases}$

Eq. (61) means that the class of representations $(d, z)$ and $I^p_z$ with $z = 0$ form a ring with respect to the tensor product. The highest weights (59) are in the one-to-one correspondence with the ones at $|q| < 1$. Let us suppose that, for even $n \geq 4$, matrix elements of the first $n/2 - 1$ irreps of $SU_q(2)$, $|q| < 1$, allows a limit $q \to e^{\frac{2\pi i}{n}}$, and form (together with their descendants) the above mentioned ring. On the other hand, we can ignore $I^p_z$ representations, while we calculate integrals of products of matrix elements. Hence, we have to truncate the space of functions to integrate over to a subspace spanned by the matrix elements of irreps of the type $(d, 0)$, $1 \leq d \leq n/2 - 1$, for even $n \geq 4$. Then we have a guarantee that appearing invariants coincide with the Turaev-Viro
ones. This construction reminds very much the finite-groups one considered in the previous section.

In the quantum case we have to correct a number of formulas of Section 2. For a unitary representation we still have

\[ D^j_{nm}(x^{-1}) = \overline{D}^j_{nm}(x) \]  

(62)

but the orthogonality condition (14) need to be modified as follows

\[ \int dx D^j_{nm}(x) \overline{D}^{j'}_{n'm'}(x) = \frac{q^{2m}}{[2j+1]} \delta^{j,j'} \delta_{n,m} \delta_{n',m'} \]

\[ \int dx \overline{D}^{j'}_{n'm'}(x) D^j_{nm}(x) = \frac{q^{-2n}}{[2j+1]} \delta^{j,j'} \delta_{n',n} \delta_{m',m} \]  

(63)

To integrate over \( SU_q(2) \) variables in eq. (62), we can use the following useful formula

\[ \overline{D}^j_{nm}(x) = (-q)^{m-n} D^j_{-n,-m}(x) \]  

(64)

which gives

\[ \int dx D^j_{n_1 m_1}(x) D^{j'}_{n_2 m_2}(x) = \frac{(-q)^{m_1+n_1}}{[2j+1]} \delta^{j,j'} \delta_{n_1,-n_2} \delta_{m_1,-m_2} \]  

(65)

Hence, we get, instead of eq. (62), the following ”hermiticity” condition

\[ A^{m_1,m_2,m_3}_{j_1,j_2,j_3} = (-1)^{j_1+j_2+j_3} (-q)^{m_1+m_2+m_3} A^{-m_1,-m_2,-m_3}_{j_1,j_2,j_3} \]  

(66)

Quantum 3j and 6j-symbols were investigated in ref. [17], which contains many useful formulas. The 3j symbol is connected to the Clebsch-Gordan coefficient as follows

\[ \binom{j_1 j_2 j_3}{n_1 n_2 n_3} = (-1)^{j_1-j_2} \frac{(-q)^{-n_3}}{\sqrt{[2j+1]}} (j_1 j_2 n_1 n_2 | j_1 j_2 j_3 - n_3) \]  

(67)

and the eq. (64) is still valid.

It is easy to see that

\[ \binom{j_1 j_2 j_3}{n_1 n_2 n_3} = q^{-2n_3} \binom{j_3 j_1 j_2}{n_3 n_1 n_2} \]  

(68)

And the cyclic symmetry condition in the form (6) cannot be imposed in the quantum case.

The Racah-Wigner 6j-symbol can be defined for example as follows

\[ \binom{j_1 j_2 j_3}{j_4 j_5 j_6} = \sum_{\{j_i \leq m_i \leq j_i\}} (-1)^{j_4+j_5+j_6} (-q)^{m_4+m_5+m_6} \]  

(69)
From which the analog of eq. (16) immediately follows

\[ S = \frac{1}{2} \sum_{j_1, j_2, j_3} \sum_{-m_k \leq j_k \leq m_k} |A_{j_1, j_2, j_3}|^2 - \]

\[ -\frac{\lambda}{4!} \sum_{j_1, ..., j_6} \sum_{-m_k \leq j_k \leq m_k} (-q) \sum_{k}^6 m_k A_{j_1, j_2, j_3} A_{j_4, j_5, j_6} A_{j_3, j_1, j_2} A_{j_3, j_4, j_5} A_{j_3, j_4, j_5} A_{j_3, j_4, j_5} \]

\[ \delta_q(x, y) = \frac{M - 1}{2j + 1} \sum_{j=0}^{M-1} q^{-2j} D_{ab}(x) D_{ab}(y) \]

With this definition

\[ \int dx f(x) \delta(x, y) = \int dx \delta(y, x) f(x) = f(y) \]

One can imagine every \( \delta \)-function in eq. (28) as an index loop going around a link of a triangulation. Matrices forming the argument of the \( \delta \)-function can be identified with intersections between the loop and triangles sharing the link. In the \( SU_q(2) \) case, such loops can form non-trivial knots and links [17]. If the corresponding links are trivial, equations (30), (31) and (32) are valid in the quantum case as well and we have the same proof of the topological invariance as for classical groups.

A thorough investigation of the model formulated in this section is beyond the scope of the present paper and will be given elsewhere. A discussion on calculations of the Turaev-Viro invariant for the lenses can be found in ref. [18]. In order to conclude, let us notice that this invariant is more sensitive than the one considered in Section 3. In principle, it can distinguish between manifolds having the same fundamental group, which makes it, potentially, to be a powerful tool in the theory of 3d manifolds.

5 Discussion

The models considered in this paper may be regarded as generalizing the well-known 2d matrix models to the 3d case. They are adequate to the problem...
of 3d euclidean quantum gravity, since they contain the sufficient number of parameters and allow a topological expansion to be performed.

In $d$-dimensional space a metric has $d(d - 1)/2$ angular degrees of freedom which can be simulated by summing over equilateral simplicial complexes. The other $d$ degrees of freedom are the gauge ones and one can simply ignore them while working with a fixed topology (as in numerical simulations in refs. [3, 4, 5]). However, a complete theory has to take into account both types of degrees of freedom. The aim of this paper was to formulate such a model. Different choices of the gauge group may be interpreted as different space structures. It would be interesting to solve the ”inverse” problem, i.e., to recover the geometry of a ”space” (if any) corresponding to a particular gauge group. The cyclic group $\mathbb{Z}_n$, from this point of view, corresponds to a space in which all lengths are quantized to be integers mod $n$ but, instead of the triangle inequality, one has the one-dimensional ”triangle equality”. It is, in a sense, actually a model of lattice quantum gravity but with a one-dimensional ”target space”.

The lattice gauge theory with a quantum gauge group also may be of interest. It is easy to introduce an action in it (e.g., by eq. (29)). In this case, the theory exists for general $q$ as well and can be generalized to an arbitrary quantum group à la Woronowicz. It is a theory with dynamical degrees of freedom and might be useful in a search for new physics.

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Figures

Fig. 1 — (a) The triangle-link exchange: the common triangle of two tetrahedra on the left is removed and three new triangles sharing the new link appear on the right. (b) The subdivision: 4 new tetrahedra fill an old one.

Fig. 2 — Dual graphs: (a) the triangle-link exchange; (b) two tetrahedra glued along three common faces (a self-energy insertion) are equivalent to a triangle.
Fig. 3 — A subcomplex can be substituted by a homotopic one: (a) $\sigma_1^0 = \sigma_2^0 = \sigma^0$; (b) $\sigma_1^2 \cup \sigma_2^2 = \sigma^2$; (c) a 3d ball is homotopic to a point.
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