Solving Linear Constraints in Elementary Abelian $p$-Groups of Symmetries

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Abstract
Symmetries occur naturally in CSP or SAT problems and are not very difficult to discover, but using them to prune the search space tends to be very challenging. Indeed, this usually requires finding specific elements in a group of symmetries that can be huge, and the problem of their very existence is NP-hard. We formulate such an existence problem as a constraint problem on one variable (the symmetry to be used) ranging over a group, and try to find restrictions that may be solved in polynomial time. By considering a simple form of constraints (restricted by a cardinality $k$) and the class of groups that have the structure of $\mathbb{F}_p$-vector spaces, we propose a partial algorithm based on linear algebra. This polynomial algorithm always applies when $k = p = 2$, but may fail otherwise as we prove the problem to be NP-hard for all other values of $k$ and $p$. Experiments show that this approach though restricted should allow for an efficient use of at least some groups of symmetries. We conclude with a few directions to be explored to efficiently solve this problem on the general case.

keywords: symmetries, linear algebra, complexity

1 Introduction
Symmetries are permutations of input symbols that, when applied to an instance of a computational problem, leave its solution invariant. Since naïve algorithms reproduce these symmetries in their search space, it is tempting to use them as a pruning device. A typical example is the pigeon-hole problem in propositional logic: its symmetries show that any pigeon can be swapped with any other (and similarly for the holes), and therefore play equivalent rôles.

Since invariance is stable by composition, the set of symmetries of an instance forms a permutation group. This means that the information they provide has a mathematical structure that should yield nice computing properties (even if
they occur only at the meta-level). In fact, some techniques from computational group theory have been employed to discover symmetries and to use them.

However, it seems that algorithms can use symmetries in a straightforward way only if their search space preserves the structure of the group of symmetries in some sense (subtrees should somehow correspond to subgroups). This is usually not the case for the most efficient algorithms developed in the field of AI. Hence many different methods have been developed in order to prune the search space with symmetries, either by designing special algorithms or by modifying instances through symmetry-breaking. Much work has been devoted to this subject, see e.g. [12] and the references therein.

One feature common to these methods is that they ideally assume the ability to produce symmetries that have particular properties, suitable for the pruning scheme. But this problem also happens to be NP-hard, which explains why symmetries that do not result in the best pruning may be used. The time spent on searching and using symmetries does not always pay off. This suggests that the group structure may not be sufficient to induce enough computational properties to ensure efficient pruning.

Our aim in this paper is to investigate ways of finding suitable symmetries in polynomial time. To this purpose we first formulate this search problem by a language of constraints as simple as possible. This is the topic of Section 3. We then consider restrictions of this search problem that confer deeper mathematical structure to the groups. The idea is to transform the constraints into linear equations, which would then be easy to solve by means of basic computer algebra. To the best of our knowledge, although restrictions to vector spaces have already been considered in the literature, such a transformation represents a novel approach. This means that we need to assume that the groups are also vector spaces, and to develop ways of efficiently working with symmetries as vectors (i.e., essentially of computing their coordinates in a suitable basis). This is developed in Sections 4 and 5, leading to a polynomial algorithm that solves so-called linear constraints. We will then see in Section 6 that, even with the simple constraints and vector spaces, the search problem remains NP-hard in most cases. Experiments in Section 7 illustrate the efficiency of this polynomial algorithm on random samples of linear constraints, compared to a general purpose algorithm. We suggest in the conclusion a few directions for using this approach in a wider setting.

2 Definitions

We do not recall the most basic definitions and notations from group theory or linear algebra, such as cycles or bases, which can be found in standard textbooks, e.g. [3], except in order to settle notations.

Given a finite set \( A \), we denote by \( \text{Sym}(A) \) the group of permutations of \( A \). If \( g_1, \ldots, g_m \) are permutations of \( A \), we denote by \( \left[ g_1, \ldots, g_m \right] \) the subgroup of \( \text{Sym}(A) \) generated by these permutations. For \( a \in A \) and \( g, g' \in \text{Sym}(A) \), the image of \( a \) by \( g \) is denoted by \( a^g \), and the composition of permutations \( g' \circ g \),
by \( gg' \), so that \( a^{gg'} = (a^g)g' \). From a computational point of view, it is obvious that the product \( gg' \) can be performed in time linear in \(|A|\). The order of \( g \) is the smallest positive integer \( n \) such that \( g^n \) is the identity.

Let \( G \) be a permutation group on \( A \), the orbit of \( a \) in \( G \) (or \( G \)-orbit of \( a \)) is \( a^G = \{a^g \mid g \in G\} \). The set of \( G \)-orbits forms a partition of \( A \), denoted by \( \mathcal{OP}(G) \). The group \( G \) is transitive if it has only one orbit (i.e., \( \mathcal{OP}(G) = \{A\} \)). It is easy to see that \( \mathcal{OP}([g]) \) can be obtained from the cycles of \( g \), e.g. if \( A = \{1, \ldots, 6\} \) then \( \mathcal{OP}([(1 2)(3 4 5)]) = \{\{1, 2\}, \{3, 4, 5\}, \{6\}\} \).

The refinement order on partitions of \( A \) (\( P \subseteq P' \) iff \( \forall O \in P, \exists O' \in P' \) s.t. \( O \subseteq O' \)) is a complete lattice; the least upper bound \( P \sqcup P' \) is obtained by merging the non-disjoint elements of \( P \) and \( P' \). The smallest partition is \( \bot_A = \{\{a\} \mid a \in A\} \) and the greatest is \( \top_A = \{A\} \). Given two permutation groups \( G \) and \( G' \) on \( A \), \( \mathcal{OP}([G \sqcup G']) = \mathcal{OP}(G) \sqcup \mathcal{OP}(G') \) (see \([8]\) chap. 7). Hence, starting with \( m \) generators \( g_1, \ldots, g_m \) the orbit partition \( \mathcal{OP}([g_1, \ldots, g_m]) = \bigsqcup_{i=1}^m \mathcal{OP}([g_i]) \) can be computed in time polynomial in \( m \) and \(|A|\).

A group \( G \) is an elementary Abelian \( p \)-group if it is Abelian and its non-trivial elements have order \( p \), a prime number. It is simple to test this property on the generators of a group: \( G \) is an elementary Abelian \( p \)-group iff its generators commute and have order \( p \). If this is the case we adopt the additive notation, e.g. \( (a b) + (c d) = (a b)(c d) \), \( 2(a b c) = (a b c)^2 = (a c b) \) and \( 0 \) is the identity. By considering multiplication by an integer as an external product on the set of integers modulo \( p \), we confine to \( G \) the structure of an \( \mathbb{F}_p \)-vector space. Conversely, every \( \mathbb{F}_p \)-vector space is isomorphic to an elementary Abelian permutation \( p \)-group.

Since the class of elementary Abelian \( p \)-groups is closed under homomorphic images and for all \( O \in \mathcal{OP}(G) \) the restriction to \( O \) operator is a morphism from \( G \) to \( \text{Sym}(O) \), if \( G \) is an \( \mathbb{F}_p \)-vector space then so is its image \( G_{\mid O} = \{g_{\mid O} \mid g \in G\} \) (although it may not be a subgroup of \( G \)). The groups \( G_{\mid O} \) are the transitive constituents of \( G \).

In the sequel, unless stated otherwise, \( a, b, c \) denote members of \( A \), \( u, v, w \) denote vectors (i.e., permutations on \( A \)), \( \mathbf{h} \) and \( \mathbf{f} \) denote bases of vector spaces, and the \( x_i \)'s coordinates in vector spaces (i.e., integers modulo \( p \)).

3 Constraints on symmetries

As mentioned above, techniques designed to compute generators for a group of symmetries are well-known. Basically, they consist in building a graph with the elements to be permuted as vertices, and with edges and labels that depend on the instance, so that its automorphism group is the expected group of symmetries. Building this graph is usually straightforward and its automorphism group can then be computed with e.g. the well-known program nauty \([8]\). Though not polynomial, this algorithm has a low average complexity and is very efficient.

We thus assume given a group of symmetries that is specified by a generating set of \( m \) permutations \( g_1, \ldots, g_m \) of a set \( A \). The group \( G = \langle g_1, \ldots, g_m \rangle \) is the set in which a permutation \( g \) satisfying a given constraint is searched for. In
this section we define a language for expressing constraints on $g$ that is both useful and simple. We begin with an example.

We consider the problem MMC from [2]. Given a model $M$ of a formula built on propositional variables which are linearly ordered (say, $x < y < z$), $\gamma M$ being the interpretation defined by $\gamma M(a) = M(a^\circ)$, and given a group $G$ by its generators, the problem MMC consists in deciding whether there exists a $g \in G$ such that $\gamma M < M$ (interpretations are ordered lexicographically). This problem is required for computing symmetry-breaking predicates, and is shown in [2] to be NP-complete. We can translate $\gamma M < M$ as: $M(x^\circ)M(y^\circ)M(z^\circ) < M(x)M(y)M(z)$, and then as

$$M(x^\circ) < M(x) \lor \quad M(x^\circ) = M(x) \land M(y^\circ) < M(y) \lor \quad M(x^\circ) = M(x) \land M(y^\circ) = M(y) \land M(z^\circ) < M(z).$$

A translation into disjunctive normal form yields:

- $M(x^\circ) < M(x)$
- $M(x^\circ) = M(x) \lor M(y^\circ) < M(y)$
- $M(x^\circ) = M(x) \lor M(y^\circ) = M(y) \land M(z^\circ) < M(z)$.

If $X$ denotes $\{a \mid M(a) < M(x)\}$ then the first disjunct translates into $x^\circ \in X$. Similarly, let $X' = \{a \mid M(a) = M(x)\}$ and $Y = \{a \mid M(a) < M(y)\}$, the second disjunct is $x^\circ \in X' \lor y^\circ \in Y$, etc. Thus the problem MMC can be expressed by means of boolean combinations of atomic constraints of the form $x^\circ \in X$, where $\sigma$ is the (unique) variable ranging in $G$. Note that the negation of any atomic constraint can be expressed as an atomic constraint\(^1\)(with the complement set), hence negation can be ruled out from the language. Since the set of solutions of a disjunction is the union of the solutions of each disjunct, we focus on solving conjunctions of atomic constraints.

**Definition 1** We address the computational problem GC, for Group Constraint, that takes as input a set $A$ of cardinality $n$, permutations $g_1, \ldots, g_m$ of $A$, and a constraint $\varphi$ which is a conjunction of atomic formulas of the form $x^\circ \in X$, where $\sigma$ is a unique variable and $x \in A, X \subseteq A$. The decision problem GC consist in checking for the existence of a permutation $g \in [g_1, \ldots, g_m]$ that satisfies all the conjuncts in $\varphi$ (a conjunct $x^\circ \in X$ is satisfied by $g$ if $x^\circ \in X$).

The associated search problem is the computation of such a $g$ if it exists.

A number of transformations on the constraint that preserve the set of solutions can be applied. These are:

$$x^\circ \in X \land x^\circ \in Y \quad \rightarrow \quad x^\circ \in X \cap Y,$$

$$\varphi \quad \rightarrow \quad \varphi \land x^\circ \in x^G.$$

The correctness of the first transformation is obvious and that of the second trivial since $x^\circ \in x^G$ holds for all $g \in G$. By applying these transformations

\(^1\)This is no longer true if we break down $x^\circ \in \{x_1, \ldots, x_n\}$ into $x^\circ = x_1 \lor \ldots \lor x^\circ = x_n$, hence atomic constraints of the form $x^\circ = y$ would not necessarily be simpler to handle.
systematically (together with associativity-commutativity of conjunction), assuming that \( A = \{a_1, \ldots, a_n\} \) it is always possible to transform any given constraint into an equivalent constraint of the form \( a_i^1 \in A_1 \land \ldots \land a_i^n \in A_n \), where \( A_i \subseteq a_i^G \) for all \( 1 \leq i \leq n \). Computing this normal form is linear in the length of \( \varphi \) and \(|A|\).

This normal form may be represented as a function \( C \) from \( A \) to \( 2^A \), where \( C(a_i) = A_i \). The problem \( gc \) then consists in finding a \( g \in G \) such that \( \forall a \in A, a^g \in C(a) \), and we may consider \( C \) directly as an input of the problem, so that an instance of \( gc \) is a tuple \( \langle A, g_1, \ldots, g_m, C \rangle \) (or \( \langle A, G, C \rangle \) in short). We say that \( C \) is a \( k \)-constraint if \( \forall a \in A, |C(a)| \leq k \).

As mentioned above, in the sequel we only consider, for every prime \( p \), the restriction of \( gc \) to the instances where \( G = [g_1, \ldots, g_m] \) is an elementary Abelian \( p \)-group, i.e., an \( \mathbb{F}_p \)-vector space.

4 From permutations to linear algebra

Before giving the technical details of the transformation from a problem on permutations to a problem in linear algebra, we briefly summarize the way we proceed.

Since our approach is based on the transitive constituents \( G|_O \) of \( G \) (for \( O \in \mathcal{OP}(G) \)), we first define a vector space \( F \) that contains as subspaces both the group \( G \) and the \( G|_O \)'s. We then prove a fundamental property of the \( G|_O \)'s and use it to realize a polynomial test of linear dependence restricted to these subspaces. This is used to compute a basis of each \( G|_O \) and to obtain the coordinates of any \( u \in G|_O \) in this basis, in polynomial time. This directly yields a basis of \( F \) and the coordinates of any \( u \in F \) in this basis. It is then standard to transform any linear variety of \( F \), such as \( G \), as a set of linear equations using basic linear algebra. Section 5 will be devoted to applying these techniques to the constraint \( C \). In the sequel we write \( \mathcal{P} \) for the orbit partition \( \mathcal{OP}(G) \).

4.1 The super-space \( F \)

The groups \( G \) and the \( G|_O \)'s are of course all included in \( \text{Sym}(A) \), but \( \text{Sym}(A) \) is not elementary Abelian hence not a vector space. Let \( F \) be the permutation group on \( A \) generated by the transitive constituents of \( G \), i.e., \( F = [\bigcup_{O \in \mathcal{P}} G|_O] \).

Lemma 1 \( F \) is an \( \mathbb{F}_p \)-vector space that contains \( G \) and \( F = \bigoplus_{O \in \mathcal{P}} G|_O \).

Proof. \( F \) is generated by the union of generating sets of the groups \( G|_O \), i.e., by the set \( \{g_i|_O \mid 1 \leq i \leq m, O \in \mathcal{P}\} \). Since orbits are mutually disjoint and

\footnote{Note that solving such constraints is much simpler than computing any lex-leader formula as in [2], where restrictions to vector spaces are already considered. Our constraints are therefore not relevant to the problem of building lex-leader formulas, and are not meant to be used in the symmetry-breaking scheme of [2].}
permutations on disjoint sets commute, \( g_i O g_j O' = g_j O' g_i O \) if \( O \neq O' \). Furthermore \( g_i O g_j O = (g_j g_i) O = g_j O g_i O \) since \( G | O \) is Abelian. Hence \( F \) is Abelian, and similarly its non-trivial elements have order \( p \).

We therefore use the additive notation in \( F \), which yields \( F = \sum_{O \in \mathcal{P}} G | O \).

For any \( O \in \mathcal{P} \), let \( A' = A \setminus O \) and \( F_O = \sum_{O' \in O} G | O' \), since \( F_O \) is a subgroup of \( \text{Sym}(A') \), then \( G | O \cap F_O \subseteq \text{Sym}(O) \cap \text{Sym}(A') = \{0\} \). This proves that \( F \) is the (internal) direct sum of the \( G | O \)'s, which is written \( F = \bigoplus_{O \in \mathcal{P}} G | O \).

It is clear that \( G \) is a subspace of \( F \) since any \( u \in G \) can be written as \( u = \sum_{O \in \mathcal{P}} u_o \) and hence belongs to \( F \).

We call \( F \) the super-space of \( G \). Since the sum in Lemma \( 1 \) is direct, the decomposition of any vector \( u \) in \( F \) as a sum of elements of the \( G | O \)'s is unique, the dimension \( d \) of \( F \) is the sum of the dimensions \( d_O \) of \( G | O \) for \( O \in \mathcal{P} \), and a basis for \( F \) can be obtained by concatenating bases for the subspaces \( G | O \).

### 4.2 Orbits as affine spaces

For all \( O \in \mathcal{P} \), the transitive constituent \( G | O \) is of course transitive in \( O \). This trivial fact yields a fundamental property:

**Lemma 2** \( (O, G | O) \) is an affine space.

**Proof.** We define the external sum \( + : O \times G | O \to O \), for all \( a \in O \) and \( u \in G | O \), by \( a + u = a^u \) (the image of \( a \) by \( u \)) and prove that the two axioms of affine spaces hold. For all \( v \in G | O \) it is clear that

\[
(a + u) + v = (a^u)^v = a^{(u+v)} = a + (u + v).
\]

Consider \( \phi_a : G | O \to O \) such that \( \phi_a(u) = a + u \), we prove that \( \phi_a \) is bijective. It is obviously onto since \( G | O \) is transitive on \( O \): \( \forall b \in O \), \( \exists u \in G | O \) such that \( b = a^u = \phi_a(u) \). Assume now that \( \phi_a(u) = \phi_a(v) \), i.e., \( a^u = a^v \), then for any \( b \in O \), if \( w \in G | O \) is such that \( a^w = b \), then

\[
b^u = a^{(w+u)} = a^{(u+w)} = (a^v)^w = (a^v)^w = b^v,
\]

hence \( u = v \), and \( \phi_a \) is injective.

This is nothing more than a geometric interpretation of a known result: that transitive Abelian groups are “regular” (see [10 theorem 10.3.4]). This entails that \( |O| = |G | O| \), and that the external sum is regular, i.e., an equality \( a + u = b \) determines every term from the two others. In particular the unique vector \( u \in G | O \) such that \( a + u = b \) is usually written \( u = b - a \) (or \( ab \)), and we also write \( b = a - u \) for \( b = a + (-u) \), which is equivalent to \( b + u = a \). Note that there is one external sum (and difference) per orbit, but since they are disjoint it is unambiguous to denote them with the same symbol.

In the sequel we use the additive notation on sets \( S, S' \) of vectors or elements of an orbit, i.e., if \( \varepsilon \in \{+,-\} \) then \( S \varepsilon S' = \{ s \varepsilon s' | s \in S, s' \in S' \} \). If one set is a singleton, say \( S = \{s\} \), we write \( s \varepsilon S' \) for \( \{s \varepsilon s' | s \in S, s' \in S' \} \). This is of course compatible with the notations already used when \( S \) and \( S' \) are subspaces of \( F \).
4.3 Computing a basis for $F$

Since $G_{\mid O}$ is a finite $\mathbb{F}_p$-vector space, its cardinality must be $p^{d_O}$. But this cardinality is also that of $O$ which is known, hence the dimension $d_O$ can be computed: $d_O = \log_p |O|$. Furthermore, since $g_1\mid O, \ldots, g_m\mid O$ is a generating set for $G_{\mid O}$, it is possible to extract a basis $f_O$ of $G_{\mid O}$ from this set by discarding $m - d_O$ linearly dependent vectors. For this a test of linear dependence is required.

**Lemma 3** For any subspace $H$ of $G_{\mid O}$, any $u \in G_{\mid O}$ and any $a \in O$,

$$u \in H \text{ iff } a^u \in a^H.$$  

**Proof.** By regularity of the external sum, $u \in H$ iff $a + u \in a + H$. By definition $a + u = a^u$ and $a + H = \{a + v \mid v \in H\} = \{a^v \mid v \in H\} = a^H$. $\diamond$

Linear dependence is therefore reduced to computing the orbit of an arbitrary point $a \in O$. A basis of $G_{\mid O}$ can be built step by step by computing the corresponding orbit partition $Q$ of $O$, with the following function $B$:

$$B([\ ] \ h, Q) = h,$$

$$B([u \oplus l, h, Q) = B(l, h, Q) \quad \text{if } a^u \in a[Q],$$

$$B(l, [u \oplus h, Q) \quad \text{otherwise.}$$

Here $l$ is a list of vectors, $\oplus$ is the concatenation of lists and $[\ ]$ is the empty list.

**Lemma 4** $B([g_1\mid O, \ldots, g_m\mid O], [\ ], \perp)$ is a basis of $G_{\mid O}$.

**Proof.** Let $f_O = B([g_1\mid O, \ldots, g_m\mid O], [\ ], \perp)$. Since $\perp$ is the orbit partition of the trivial group $\{0\}$ on $O$, the invariant $Q = QP([h])$ is maintained throughout the computation. This means that the class $a[Q]$ of $a$ modulo $Q$ is the orbit $a^{[h]}$, and that $a^u \in a[Q]$ is equivalent to $u \in [h]$ by Lemma 3. Hence $u$ is added to $h$ if and only if it is linearly independent from the latter, which proves that $h$ remains free. This also proves that $[l@h]$ is invariant, hence $[f_O] = [g_1\mid O, \ldots, g_m\mid O] = G_{\mid O}$ and $f_O$ is free, it is thus a basis of $G_{\mid O}$. $\diamond$

**Example.** Let $O = \{1, \ldots, 8\}$ and

$$g_1 = (1 2)(3 4)(5 6)(7 8),$$

$$g_2 = (1 5)(2 6)(3 7)(4 8),$$

$$g_3 = (1 3)(2 4)(5 7)(6 8),$$

We choose $a = 1$, then

$$f_O = B([g_1, g_2, g_3], [\ ], \perp)$$

$$= B([g_2, g_3], [g_1], QP(g_1)) \text{ since } 1^{g_1} = 2 \not\in 1[\perp] = \{1\}$$

$$= B([g_3], [g_2, g_1], QP(g_2) \cup QP(g_1)) \text{ since } 1^{g_2} = 5 \not\in 1[QP(g_1)] = \{1, 2\}$$

$$= B([\ ], [g_3, g_2, g_1], \perp) \text{ since } 1^{g_3} = 3 \not\in 1[QP(g_2) \cup QP(g_1)] = \{1, 2, 5, 6\}$$

$$= [g_3, g_2, g_1].$$
Of course the algorithm can be interrupted once \( h \) has \( d_O \) elements (or equivalently when \( Q = T_O \)). Building \( f_O \) requires at most \( m \) recursive calls and the computation of exactly \( d_O \leq m \) orbit partitions of \( O \), each being polynomial in \( |O| \leq n \), hence computing \( f_O \) is computed in time polynomial in \( n \) and \( m \).

The bases \( f_O \) can be concatenated (in an arbitrary order) to form a basis \( f \) of \( F \). The length \( d \) of this basis may be greater than \( m \), but since

\[
    d_O = \log_p |O| \leq \frac{|O|}{p},
\]

necessarily

\[
    d = \sum_{O \in P} d_O \leq \sum_{O \in P} \frac{|O|}{p} = \frac{n}{p},
\]

hence computing \( f \) is again polynomial in \( n \) and \( m \).

### 4.4 Computing coordinates in the basis for \( F \)

The coordinates of any vector \( u \in F \) in basis \( f \) can be obtained by computing, for all \( O \in P \), the coordinates of \( u_{i_O} \in G_{i_O} \) in the basis \( f_O \), and by concatenating these coordinates in the same order as the one used to build \( f \). We show how to compute the coordinates in \( f_O \) of the permutations in \( G_{i_O} \).

Since \( f_O \) is a basis of \( G_{i_O} \), there is a 1-1 correspondence from the tuples \( \langle x_1, \ldots, x_{d_O} \rangle \in (\mathbb{F}_p)^{d_O} \) to the elements of \( G_{i_O} \), given by \( \sum_{i=1}^{d_O} x_i h_i \), which can be computed in polynomial time: each \( x_i h_i \) requires composing \( x_i - 1 < p \) times the permutation \( h_i \) of \( O \) with itself, hence computing a permutation from its coordinates in \( f_O \) can be computed in time \( O(d_O p |O|) \), which is bounded by \( O(n \log n) \) (since \( p \) is a constant).

Computing the coordinates in \( f_O \) of a given \( u \in G_{i_O} \) means computing the inverse of the previous correspondence. This can be performed by browsing through all possible values \( \langle x_1, \ldots, x_{d_O} \rangle \in (\mathbb{F}_p)^{d_O} \) until \( \sum_{i=1}^{d_O} x_i h_i \) equals \( u \). Since \( |O| = |G_{i_O}| = p^{d_O} \), this requires at most \( |O| \) iterations, hence can be computed in time \( O(d_O p |O|^2) \), which is bounded by \( O(n^2 \log n) \).

The same technique may be applied if \( u \) is given as \( b - a \), where \( a, b \in O \). In this case it is necessary to check whether \( b = a + \sum_{i=1}^{d_O} x_i h_i \), i.e., whether \( b \) is the image of \( a \) by the permutation \( \sum_{i=1}^{d_O} x_i h_i \). The complexity is thus the same as above. Note that this also allows to compute \( b - a \) explicitly as a permutation.

From a practical point of view we should avoid repeated computations of permutations from coordinates: this could be done by storing values in a suitable array. Using our geometric interpretation, we choose arbitrarily an origin \( a \in O \) and define the coordinates of any point \( b \in O \) as those of the vector \( b - a \) relative to \( f_O \). We can therefore fill an array that associates its coordinates to each entry \( b \in O \), by browsing through all possible coordinates as explained above. Since this array has \( |O| \) entries, filling it takes polynomial time. Then, given a permutation \( u \), we need only compute the image \( b = u^a \) and pick the coordinates of \( b \) in the array; these are the coordinates of \( u \).
In the sequel we write \( f = f_1, \ldots, f_d \), and for any \( u \in F \), if \( u = \sum_{i=1}^{d} x_i f_i \) where the \( x_i \in \mathbb{F}_p \) are the coordinates of \( u \) in \( f \), we write \( u_f = f(x_1 \cdots x_d) \) the column matrix of these coordinates.

### 4.5 A characterization of linear varieties in \( F \)

Since the super-space \( F \) is isomorphic to the vector space \( (\mathbb{F}_p)^d \), and this isomorphism can be computed (through the coordinates in \( f \)) in both directions, computations with matrices can be substituted for computations with permutations. This means that standard algorithms from linear algebra apply, in particular Gaussian elimination. Note that exact computations can be performed in \( \mathbb{F}_p \), including division, in time at most quadratic in the number of bits (see, e.g. [5, p.117]), hence in constant time in the present context.

In particular, it is now straightforward to test whether a family of \( l \leq d \) vectors \( u_1, \ldots, u_l \in F \) is linearly dependent, by first computing the coordinates \( u_i = \sum_{j=1}^{d} x_{ij} f_j \) and then by performing Gaussian elimination on the \( l \times d \)-matrix \( (x_{ij}) \) (which requires a number of operations at most cubic in \( d \)). The family is linearly dependent iff Gaussian elimination yields a zero row in the resulting matrix (the number of non-zero lines after Gaussian elimination is the rank of the matrix, i.e., the dimension of the space \( \langle u_1, \ldots, u_l \rangle \)).

Assume we are given a linear variety \( v + H \) of \( F \) by the permutation \( v \) and a generating set for the subspace \( H \). We can compute the coordinates in \( f \) of these permutations and, using the previous procedure, extract a basis \( h_1, \ldots, h_{d'} \) of \( H \) from the generators of \( H \), where \( d' \) is the dimension of \( H \). The vectors \( h_1, \ldots, h_{d'} \) together with the vectors in \( f \) form a generating family of \( F \); the free family \( h_1, \ldots, h_{d'} \) can therefore be completed into a basis \( h = h_1, \ldots, h_d \) of \( F \) by adding \( d - d' \) vectors taken from \( f \). If \( P \) denotes the matrix whose \( i \)th column is \( (h_i)_f \) then \( P \) is the change of basis matrix from \( h \) to \( f \): \( u_f = Pu_h \) for all \( u \in F \). This matrix is invertible and its inverse can be computed using the Gauss-Jordan algorithm in time cubic in \( d \).

This means that the coordinates of any vector \( u \) in \( h \) (or in any basis) can be computed in polynomial time through \( u_h = P^{-1} u_f \). Membership of \( u \) to the subspace \( H \) may be checked simply by making sure the last \( d - d' \) coordinates of \( u_h \) are equal to zero. This can be expressed by building the diagonal \( d \times d \)-matrix \( D \) with ones on the last \( d - d' \) positions of the diagonal and zeroes elsewhere:

\[
D = \begin{pmatrix}
0 & 0 \\
0 & I
\end{pmatrix},
\]

where \( I \) is the \( (d - d') \times (d - d') \) identity matrix. Thus vector \( u \in F \) belongs to \( H \) iff \( Du_h = 0 \). Let \( M_H = DP^{-1} \), we have shown that:

**Lemma 5** From any linear variety \( v + H \) of \( F \) can be computed in polynomial time a \( d \times d \)-matrix \( M_H \) such that \( \forall u \in F \),

\[
u \in v + H \text{ iff } M_H u_f = M_H v_f.
\]
Conversely, it is well known that the set of solutions $u$ of any system of linear equations on $d$ unknowns (the coordinates of $u$ in $f$) is either empty or a linear variety of $F$.

5 Solving linear constraints

The vector space $G$ being a linear variety of $F$ by Lemma 1, its elements are characterized as the solutions $u$ of a system of linear equations $M_G u f = 0$, as shown in Lemma 5 (by taking $v = 0$). We now investigate how to characterize the constraint $C$ by another system of linear equations.

5.1 Constraints as sets of vectors

The first step towards this characterization is to explicitly represent the set of vectors $u \in E$ that satisfy a constraint $C$. Let $V_O = \bigcap_{a \in O} C(a) - a$ for all $O \in \mathcal{P}$.

**Lemma 6** A vector $u \in F$ satisfies $C$ iff $u \in \sum_{O \in \mathcal{P}} V_O$.

*Proof.* Assume a vector $u \in F$ satisfies the constraint $C$, i.e., for all $a \in A$, $a^n \in C(a)$. Then for all $O$ in the orbit partition $\mathcal{P}$ and for all $a \in O$, since $a^n = a^{u_O} \in O$ and $(O, G|_O)$ is an affine space, we may write $a + u_O \in C(a)$, and since $C(a) \subseteq O$, this is equivalent to $u_O \in C(a) - a$. But this is true for all $a \in O$, hence $u_O \in V_O$. Since $u = \sum_{O \in \mathcal{P}} u_O$, vector $u$ must belong to $\sum_{O \in \mathcal{P}} V_O$.

Conversely, let $u \in \sum_{O \in \mathcal{P}} V_O$ and let $a \in A$. If $O = a^O$, then $u_O \in V_O$, hence in particular $u_O \in C(a) - a$, i.e., $a^n = a + u_O \in C(a)$. Hence $u$ satisfies the constraint $C$. \hfill \Diamond

The sets $V_O$ can be computed for each $O \in \mathcal{P}$ by selecting an arbitrary $a \in O$ and computing the coordinates of $c - a$ in $f_O$ for all $c \in C(a)$, and this operation is polynomial in $n$ for each $c$ as mentioned in section 1.4. Provided $C$ is a $k$-constraint this yields at most $k$ elements in $C(a) - a$. Then, $V_O$ is the set of those vectors $u$ from $C(a) - a$ that also belong to $C(b) - b$ for all $b \in O \setminus \{a\}$, i.e., such that $b^n \in C(b)$, which requires that $u$ be also computed as an explicit permutation. Computing the coordinates of all the elements of $V_O$ is therefore polynomial in $n$ and $k$.

5.2 Linear constraints

The size of the set $\sum_{O \in \mathcal{P}} V_O$ is $\prod_{O \in \mathcal{P}} |V_O|$. If one of the sets $V_O$ is empty, then obviously constraint $C$ is unsatisfiable. But in general this set is exponential in the number of $G$-orbits (and therefore in $n$, the worst case being $2^2$ with $2^2$ orbits of size 2). This motivates the following definition.
Definition 2 \( C \) is a linear constraint if \( \sum_{O \in P} V_O \) is either empty or a linear variety of \( F \).

In order to check whether a constraint \( C \) is linear or not, we provide a simple characterization of this property. For \( O \in P \), let \( w_O \) be an arbitrary element of \( V_O \) and \( E_O = V_O - w_O \).

Lemma 7 \( \sum_{O \in P} V_O \) is a linear variety of \( F \) iff \( \forall O \in P, \dim[E_O] = \log_p |V_O| \).

Proof. Since \( F = \bigoplus_{O \in P} G_{|O|} \) and \( V_O \subseteq G_{|O|} \), it is clear that \( \sum_{O \in P} V_O \) is a linear variety of \( F \) iff \( V_O \) is a linear variety of \( G_{|O|} \) for all \( O \in P \). If this is so then \( E_O \) is a subspace of \( G_{|O|} \) that does not depend on \( w_O \). Hence this is equivalent to all the \( E_O \)'s being subspaces, hence to \( [E_O] = E_O \). Again this is equivalent to \( p^{\dim[E_O]} = |[E_O]| = |E_O| = |V_O| \). \( \diamond \)

The dimension of \( [E_O] \) is the rank of the matrix formed by the coordinates of the vectors in \( E_O \), which can be computed in polynomial time and compared to \( \log_p |V_O| \). Computing this rank is of course useless if \( \log_p |V_O| \) is not an integer. Hence the linearity of \( C \) can be tested in time polynomial in \( n \) (since \( |P| \leq n \)). Note that the case where \( V_O \) is a singleton corresponds to \( E_O \) being reduced to the trivial subspace \( \{0\} \) of dimension 0.

Let \( w = \sum_{O \in P} w_O \) and \( E = \bigoplus_{O \in P} E_O \) for every \( O \in P \), so that \( \sum_{O \in P} V_O \) is the linear variety \( w + E \) (assuming it is not empty).

Theorem 8 If \( C \) is linear then the problem gc is equivalent to a system of linear equations on coordinates of a solution in \( f \), and this system can be computed and solved in polynomial time.

Proof. If \( \sum_{O \in P} V_O = \emptyset \) then the instance \( \langle A, G, C \rangle \) of gc has no solution, which is equivalent to the linear equation \( 0 = 1 \). Otherwise \( \sum_{O \in P} V_O = w + E \) and by Lemma 5 any \( u \in F \) is a solution of the instance \( \langle A, G, C \rangle \) iff \( u \in G \) and \( u \in w + H \), which is equivalent by Lemma 5 to

\[
\begin{align*}
\mathcal{M}_G u_f &= 0 \\
\mathcal{M}_E u_f &= \mathcal{M}_E w_f.
\end{align*}
\]

This is a system of \( 2d \) linear equations on \( d \) unknowns, it can be solved by Gaussian elimination in time cubic in \( d \). \( \diamond \)

Corollary 9 The problem gc restricted to \( p = k = 2 \) is polynomial.

Proof. Every non-empty \( V_O \) has at most \( k = 2 \) elements. If \( V_O = \{u\} = u + \{0\} \) then \( \dim[E_O] = \dim(0) = 0 = \log_2 |V_O| \); if \( V_O = \{u, v\} = u + \{0, v - u\} \) then \( \dim[E_O] = \dim[v - u] = 1 = \log_2 |V_O| \), hence according to Lemma 7 the constraint \( C \) is linear. \( \diamond \)

This result holds both for the decision and the search problem.
6 NP-Completeness results

6.1 Constraints with more than 2 elements

If one $V_O$ has more than 2 elements then $C$ may not be a linear constraint, and therefore the previous polynomial algorithm may not apply. In fact, we now prove that allowing constraints of a cardinality greater than 2 makes the decision problem $gc$ NP-hard, whatever the value of $p$. We proceed by reduction from the problem of 1-satisfiability of positive $k$-clauses.

Let $\Sigma$ be a finite set of propositional variables (which will be denoted by Greek letters), a positive $k$-clause is a subset $C \subseteq \Sigma$ of cardinality $k$. Let $S$ be a finite set of such clauses, then $S$ is 1-satisfiable if there is an interpretation $I \subseteq \Sigma$ such that every clause $C \in S$ contains exactly one element in $I$.

Given $\Sigma$ and $S$, we build an instance of the decision problem $gc$ (restricted to $F_p$-vector spaces) whose satisfiability is equivalent to the 1-satisfiability of $S$. Furthermore, the construction is polynomial in the size of the input clause set, i.e., it is polynomial in $|\Sigma|$ and $|S|$ (not necessarily in $k$, which is a constant).

This transformation consists in interpreting propositional variables and clauses in $F_p$.

We consider the space of functions from $\Sigma$ to $F_p$, written $F^\Sigma_p$, with the standard sum $(u+v)(\alpha) = u(\alpha) + v(\alpha)$ and the external product $(xu)(\alpha) = xu(\alpha)$ for all $\alpha \in \Sigma$, $u, v \in F^\Sigma_p$ and $x \in F_p$. This is an $F_p$-vector space of dimension $|\Sigma|$, with generating set $\{\delta_\beta \mid \beta \in \Sigma\}$, where $\delta_\beta(\alpha) = 1$ if $\alpha = \beta$ and 0 otherwise.

We construct a permutation group as a homomorphic image of the elementary Abelian $p$-group $F^\Sigma_p$. The elements to be permuted are those of the set $A_S = \bigcup_{C \in S} F^C_p$ of functions from $C$ to $F_p$, where $C$ is a positive $k$-clause belonging to $S$. The cardinality of $A_S$ is $\sum_{C \in S} p^{|C|} = p^k|S|$. Each element of $A_S$ is a function that associates integers modulo $p$ to $k$ propositional variables, hence it can be encoded in constant size.

Example. We consider the set $\Sigma = \{\alpha, \beta, \gamma\}$ and assume that $p = 2$. There are exactly 8 functions from $\Sigma$ to $F_2$, which we denote according to the following scheme.

| $\Sigma$ | $g_0$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ |
|----------|------|------|------|------|------|------|------|------|
| $\alpha$ | 0    | 0    | 0    | 0    | 1    | 1    | 1    | 1    |
| $\beta$  | 0    | 0    | 1    | 1    | 0    | 0    | 1    | 1    |
| $\gamma$ | 0    | 1    | 0    | 1    | 0    | 1    | 0    | 1    |

Let $C = \{\alpha, \beta, \gamma\}$, which is a positive 3-clause on $\Sigma$, and $S = \{C\}$. Since $C = \Sigma$, we have $A_S = \bigcup_{C' \in S} F^C_p = F^C_p = \{g_0, \ldots, g_7\}$.

$\diamondsuit$
Let \( f \) denote the function from \( \mathbb{F}^S_p \) to \( \text{Sym}(A_S) \) defined for all \( u \in \mathbb{F}^S_p \), \( C \in S \) and \( w \in \mathbb{F}^C_p \), by \( w^{f(u)} = w + u_C \). It is straightforward to verify that \( f(u) \) is indeed a permutation of \( A_S \), and that \( f(u)f(v) = f(u + v) \); hence \( f \) is a group morphism and the group \( G_S \) generated by the permutations \( \{ f(\delta_\alpha) \mid \alpha \in \Sigma \} \) is an elementary Abelian \( p \)-group. This generating set can obviously be computed in time polynomial in \( |\Sigma| \) and \( |A_S| \).

Example. We compute \( f(g_3) \). For all \( w \in \mathbb{F}^S_p \), \( w^{f(g_3)} = w + g_3 \). We have \( g_0 + g_3 = g_3, g_1 + g_3 = g_2, \) etc. and we easily obtain, in cycle notation

\[
\begin{align*}
f(g_3) &= (g_0 \ g_3)(g_1 \ g_2)(g_4 \ g_7)(g_5 \ g_6).
\end{align*}
\]

\( \diamond \)

Finally, let \( C_S \) be the \( k \)-constraint on \( G_S \) defined for all \( C \in S \) and \( w \in \mathbb{F}^C_p \), by

\[
C_S(w) = \{ w + (\delta_\alpha)_C \mid \alpha \in C \}.
\]

Example. Since \( \delta_C(\alpha) = g_4, \delta_C(\beta) = g_2 \) and \( \delta_C(\gamma) = g_1 \), we have for all \( w \in \mathbb{F}^S_p \),

\[
C_S(w) = \{ w + g_4, w + g_2, w + g_1 \}.
\]

This yields for instance \( C_S(g_3) = \{ g_7, g_1, g_2 \} \).

\( \diamond \)

Lemma 10 \( S \) is 1-satisfiable iff \( C_S \) is satisfiable in \( G_S \).

Proof. First assume that \( S \) is 1-satisfiable, i.e., there is an interpretation \( I \subseteq \Sigma \) such that every clause \( C \in S \) has exactly one element in \( I \). Let \( u \in \mathbb{F}^C_p \) be defined by \( u(\alpha) = 1 \) if \( \alpha \in I \) and 0 otherwise. For \( C \in S \), if \( \{ \alpha \} = C \cap I \) then it is clear that \( u_C = (\delta_\alpha)_C \), so that for all \( w \in \mathbb{F}^C_p \), \( w^{f(u)} = w + u_C = w + (\delta_\alpha)_C \in C_S(w) \). Thus \( f(u) \in G_S \) satisfies \( C_S \).

Conversely, suppose there is an element \( f(u) \) of \( G_S \) that satisfies \( C_S \), let \( I = \{ \alpha \in \Sigma \mid u(\alpha) = 1 \} \) and let \( C \) be any clause in \( S \). For all \( w \in \mathbb{F}^C_p \), since \( w^{f(u)} \in C_S(w) \) there is an \( \alpha \in C \) such that \( w^{f(u)} = w + (\delta_\alpha)_C \), hence such that \( u_C = (\delta_\alpha)_C \). Necessarily \( u(\alpha) = 1 \) and therefore \( \alpha \in C \cap I \). If \( \beta \in C \cap I \) we similarly obtain \( u_C = (\delta_\beta)_C \), hence \( \beta = \alpha \). This proves that \( C \cap I \) is a singleton for all \( C \in S \), and that \( I \) 1-satisfies \( S \). \( \diamond \)

Theorem 11 For any prime \( p \), the problem of solving \( k \)-constraints in \( \mathbb{F}_p \)-vector spaces is NP-complete if \( k \geq 3 \).

This follows from the NP-completeness of the problem of determining 1-satisfiability of a set of positive 3-clauses (see [4] problem L04, p. 259).

6.2 Constraints with at most 2 elements

If a \( V_O \) has exactly two elements but \( p \geq 3 \), then \( V_O \) cannot be a linear variety of \( F \) and the algorithm of Section 5.2 necessarily fails. We prove that allowing \( p \geq 3 \) makes the restriction of the decision problem \( GC \) to \( \mathbb{F}_p \)-vector spaces
NP-complete, even with constraints of cardinality at most 2. We proceed by
reduction from 1-satisfiability of positive \( p \)-clauses.

Given a set \( S \) of positive \( p \)-clauses on \( \Sigma \), we again consider the \( \mathbb{F}_p \)-vector
space \( \mathbb{F}_p^\Sigma \) and define the set

\[
A'_S = \Sigma \times \mathbb{F}_p \cup S \times \mathbb{F}_p,
\]
whose cardinality is \( p|\Sigma| + p|S| \). Let \( f' \) be the function from \( \mathbb{F}_p^\Sigma \) to \( \text{Sym}(A'_S) \)
defined for all \( u \in \mathbb{F}_p^\Sigma \), \( \langle \alpha, x \rangle \in \Sigma \times \mathbb{F}_p \) and \( \langle C, y \rangle \in S \times \mathbb{F}_p \), by

\[
\langle \alpha, x \rangle^{f'(u)} = \langle \alpha, x + u(\alpha) \rangle,
\]
\[
\langle C, y \rangle^{f'(u)} = \langle C, y + \sum_{\beta \in C} u(\beta) \rangle.
\]

It is straightforward to verify that \( f'(u) \) is a permutation of \( A'_S \): if \( \langle \alpha, x \rangle^{f'(u)} =
\langle \alpha', x' \rangle^{f'(u)} \) then \( \alpha = \alpha' \) and then \( x = x' \); if \( \langle C, y \rangle^{f'(u)} = \langle C', y' \rangle^{f'(u)} \) then \( C =
C' \) and then \( y = y' \). It is obvious that \( f'(u) f'(v) = f'(u + v) \), hence \( f' \) is a group
morphism and the group \( G'_S \) generated by the permutations \( \{ f'(\delta_\alpha) \mid \alpha \in \Sigma \} \) is
therefore an elementary Abelian \( p \)-group. This generating set can be computed
in time polynomial in \( |\Sigma| \) and \( |A'_S| \).

**Example.** We assume the same \( \Sigma, p, C \) and \( S \) as in the running example of
Section 6.1. Then

\[
A'_S = \{ \langle \alpha, 0 \rangle, \langle \alpha, 1 \rangle, \langle \beta, 0 \rangle, \langle \beta, 1 \rangle, \langle \gamma, 0 \rangle, \langle \gamma, 1 \rangle, \langle C, 0 \rangle, \langle C, 1 \rangle \}.
\]

The permutations \( f'(u) \) for \( u \in \mathbb{F}_2^\Sigma \) can again be expressed in cycle notation,
for instance:

\[
f'(g_2) = (\langle \beta, 0 \rangle \langle \beta, 1 \rangle)(\langle C, 0 \rangle \langle C, 1 \rangle),
\]
\[
f'(g_3) = (\langle \beta, 0 \rangle \langle \beta, 1 \rangle)(\langle \gamma, 0 \rangle \langle \gamma, 1 \rangle).
\]

\( \langle C, 0 \rangle \) is a fix-point of \( f'(g_3) \) since \( g_3(\alpha) + g_3(\beta) + g_3(\gamma) = 0 + 1 + 1 = 0 \).

Let \( C'_S \) be the 2-constraint on \( G'_S \) defined, for all \( \langle \alpha, x \rangle \in \Sigma \times \mathbb{F}_p \) and
\( \langle C, y \rangle \in S \times \mathbb{F}_p \), by

\[
C'_S(\langle \alpha, x \rangle) = \{ \langle \alpha, x \rangle, \langle \alpha, x + 1 \rangle \},
\]
\[
C'_S(\langle C, y \rangle) = \{ \langle C, y + 1 \rangle \}.
\]

**Example.** Obviously \( C'_S(\langle \alpha, 0 \rangle) = C'_S(\langle \alpha, 1 \rangle) = \{ \langle \alpha, 0 \rangle, \langle \alpha, 1 \rangle \} \), and similarly
for \( \beta \) and \( \gamma \). For the other two elements of \( A'_S \) we have:

\[
C'_S(\langle C, 0 \rangle) = \{ \langle C, 1 \rangle \} \text{ and } C'_S(\langle C, 1 \rangle) = \{ \langle C, 0 \rangle \}.
\]

\( \diamond \)

**Lemma 12** \( S \) is 1-satisfiable iff \( C'_S \) is satisfiable in \( G'_S \).
Proof. Assume $S$ is 1-satisfiable, let $I$ be an interpretation of $S$ and consider $u \in \mathbb{F}_p^n$ defined by $u(\alpha) = 1$ if $\alpha \in I$ and 0 otherwise. The constraint on any $\langle \alpha, x \rangle \in \Sigma \times \mathbb{F}_p$ is satisfied since $\langle \alpha, x \rangle f'(u) = \langle \alpha, x + u(\alpha) \rangle \in \mathcal{C}'_S((\alpha, x))$. For all $\langle C, y \rangle \in S \times \mathbb{F}_p$ there is a unique $\beta \in C$ such that $u(\beta) = 1$ and $u_{|C}$ is zero elsewhere, hence $\langle C, y \rangle f'(u) = \langle C, y + \sum_{\beta \in C} u(\beta) \rangle = \langle C, y + 1 \rangle \in \mathcal{C}'_S((C, y))$. This shows that $f'(u) \in G'_S$ satisfies $\mathcal{C}'_S$.

Conversely, suppose that an element $f'(u)$ of $G'_S$ satisfies $\mathcal{C}'_S$ and let $I = \{ \alpha \in \Sigma \mid u(\alpha) = 1 \}$. Then $u(\alpha) \neq 1$ for all $\alpha \in \Sigma \setminus I$, and $u(\alpha) \in \{0, 1\}$ since $\langle \alpha, x \rangle f'(u) \in \mathcal{C}'_S((\alpha, x))$, thus $u(\alpha) = 0$ (modulo $p$). Let $C$ be a clause in $S$, the constraint yields $\langle C, 0 \rangle f'(u) = \langle C, 1 \rangle$, hence $\sum_{\beta \in C} u(\beta) = 1$. The terms of this sum are either 0 or 1, hence at least one must be a 1. Furthermore, there are at most $p$ terms, hence only one can be a 1, say $u(\alpha) = 1$, then by definition of $u$, $\alpha$ is the only member of $C$ that belongs to $I$. Hence $I$ 1-satisfies $S$. ⊗

**Theorem 13** For any prime $p \geq 3$, the problem of solving 2-constraints in $\mathbb{F}_p$-vector spaces is NP-complete.

7 Experimental results

The polynomial algorithm for solving linear constraints has been implemented in the GAP system, using its facilities on permutations, matrix algebra and finite fields. The implementation, nicknamed **Solvect** (see “downloads” page on [capp.imag.fr](http://capp.imag.fr)), is straightforward except for the fact that coordinates in the transitive constituents are kept in memory and hence computed only once (this is performed while computing the orbit partition of $G$). Its performance has been measured against a general purpose group search algorithm provided in GAP, described in [6] and refined in [11]. The call to this algorithm is

```
ElementProperty([g_1, \ldots, g_m], g \mapsto \forall a \in A, a^g \in C(a));
```

which returns an element of $G = [g_1, \ldots, g_m]$ satisfying the specified property if there is one, and fail otherwise.

The performance of **ElementProperty** depends essentially on the size of $G$, while **Solvect** depends mostly on $n = |A|$ and to a lesser extent on $d = \dim F$.

We thus perform two sets of experiments: the first in Table [1] is parametrized by the size of $G$ (which is $2^{\dim G}$) and the second in Table [2] is parametrized by $n$. In each case we measure the mean values of $n$, dim $G$ and $d$ as well as the times in milliseconds taken by the two solvers ($t_1$ for **Solvect** and $t_2$ for **ElementProperty**).

The samples are generated by choosing randomly the number and dimensions of transitive constituents, i.e., a sequence $d_1, \ldots, d_q$ of strictly positive integers, then computing generators for the transitive constituents and composing them randomly to produce generators for $G$. In the first experiment we guarantee that $G$ has the correct dimension, bounded by $\max_{i=1}^q d_i \leq \dim G \leq \sum_{i=1}^q d_i = d$. 

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Table 1: Experiment 1

| dim $G$ | $n$         | $d$          | $t_1$         | $t_2$         |
|---------|-------------|--------------|---------------|---------------|
| 5       | $26.4 \pm 56\%$ | $6.9 \pm 29\%$ | $0.384 \pm 310\%$ | $0.82 \pm 214\%$ |
| 10      | $270 \pm 150\%$  | $14.9 \pm 26\%$ | $1.46 \pm 137\%$  | $20.3 \pm 56\%$  |
| 15      | $785 \pm 277\%$  | $27.2 \pm 28\%$ | $5.4 \pm 143\%$  | $631 \pm 106\%$  |
| 20      | $1060 \pm 229\%$ | $35.4 \pm 28\%$ | $10.1 \pm 100\%$ | $19900 \pm 90\%$ |
| 25      | $2230 \pm 175\%$ | $52.2 \pm 31\%$ | $25.8 \pm 73\%$ | $-$            |
| 30      | $2730 \pm 148\%$ | $67.4 \pm 34\%$ | $45.3 \pm 66\%$ | $-$            |
| 35      | $2870 \pm 107\%$ | $79 \pm 35\%$ | $65.3 \pm 67\%$ | $-$            |
| 40      | $8510 \pm 94\%$  | $94.2 \pm 38\%$ | $147 \pm 69\%$  | $-$            |
| 45      | $7680 \pm 77\%$  | $125 \pm 37\%$ | $241 \pm 64\%$  | $-$            |
| 50      | $12800 \pm 60\%$ | $147 \pm 39\%$ | $436 \pm 75\%$ | $-$            |

In the second experiment we guarantee that $\sum_{i=1}^{q} 2^{d_i} = n$. 2-constraints are also generated randomly, with the following bias: half of them are guaranteed to be satisfiable (a solution is chosen randomly in $G$). Another bias has been introduced: we choose the $d_i$ between 1 and 13, because computing generators for an orbit of a size greater than $2^{13}$ takes too much time.

Since our random samples are by no means supposed to be representative, we also measure the standard deviation expressed as a percentage of the mean value. We test 1000 instances on the low values and 100 on the higher ones. Values are rounded to 3 digits. We see that ElementProperty can hardly be used on groups of size much bigger than $2^{20}$, while Solvect works well up to the limits of the memory used by GAP (the limit is reached with $n = 2^{17}$).

8 Conclusion and perspectives

We can therefore solve $k$-constraints in the class of $\mathbb{F}_p$-vector spaces in guaranteed polynomial time only when $k = p = 2$, and we have provided an algorithm to do so. For greater values of $k$ and $p$ the problem is NP-complete, which is quite surprising considering the rich structure of vector spaces and the relative simplicity of the constraints that were considered. These results confirm how difficult it can be to develop efficient algorithms for finding useful symmetries.

However, our algorithm may be used on linear constraints regardless of $k$ and $p$, and other experiments with Solvect suggests that many constraints are linear. Furthermore, checking the linearity of the constraint is fast.

But this still requires the group to be an elementary Abelian $p$-group, which seems unlikely unless the problem under consideration is of a geometric nature, for instance if a hypercube is involved (its group of symmetries is an elementary Abelian $p$-group).

---

3 When a process takes less than 4 ms, GAP measures its duration as either 0 or 4 ms, hopefully with a probability depending on this duration. In that case the mean value should be accurate, but standard deviation is obviously exaggerated.
| $n$ | dim $G$ | $d$ | $t_1$              | $t_2$              |
|-----|---------|-----|-------------------|-------------------|
| $2^1$ | 1       | 1   | 0.108 ± 600%      | 0.28 ± 375%      |
| $2^2$ | 1.81 ± 21% | 2   | 0.144 ± 517%      | 0.308 ± 351%     |
| $2^3$ | 2.96 ± 20% | 3.68 ± 13% | 0.212 ± 431% | 0.444 ± 294%     |
| $2^4$ | 4.44 ± 19% | 6.39 ± 22% | 0.344 ± 342% | 0.756 ± 211%     |
| $2^5$ | 6.12 ± 19% | 10.4 ± 30% | 0.716 ± 224% | 1.88 ± 139%      |
| $2^6$ | 8.05 ± 19% | 16.2 ± 35% | 1.4 ± 141% | 7.05 ± 145%      |
| $2^7$ | 10 ± 19% | 24.4 ± 36% | 3.08 ± 92% | 35.8 ± 188%      |
| $2^8$ | 12 ± 19% | 34.1 ± 40% | 6.96 ± 79% | 184 ± 277%       |
| $2^9$ | 14.5 ± 18% | 51.1 ± 33% | 17.5 ± 74% | 1740 ± 322%     |
| $2^{10}$ | 16.6 ± 17% | 67.6 ± 34% | 34.9 ± 81% | 15700 ± 426%    |
| $2^{11}$ | 18.5 ± 17% | 85.2 ± 37% | 64.4 ± 69% | 37600 ± 166%    |
| $2^{12}$ | 20.3 ± 15% | 113 ± 37% | 144 ± 88% | 187000 ± 315%   |
| $2^{13}$ | 23.3 ± 12% | 131 ± 32% | 209 ± 66% | -                |
| $2^{14}$ | 25.8 ± 9% | 189 ± 26% | 593 ± 80% | -                |
| $2^{15}$ | 28.8 ± 8% | 268 ± 23% | 1520 ± 60% | -                |
| $2^{16}$ | 30.9 ± 7% | 446 ± 17% | 6910 ± 55% | -                |

Table 2: Experiment 2

Abelian 2-group). In general it would be necessary to enforce this property by approximating the group of symmetries by one or several elementary Abelian $p$-subgroups. This is reasonable since symmetry pruning is meant to be fast, not complete with respect to just any group of symmetries. It seems possible to generalize these results to the class of Abelian groups in the line of [7], at the expense of our elegant geometric interpretation of linear constraints. We are currently investigating this generalization.

Identifying tractable restrictions of the generally intractable problem of finding selected symmetries is therefore a natural approach to efficient symmetry pruning. Methods using only special symmetries have already been tried with some success, as in [9] where only transpositions are considered. We therefore believe the present results open interesting perspectives.

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