Introducing a multi-particle collision method for the evolution of dense stellar systems code

Crash-test $N$-body simulations

Pierfrancesco Di Cintio$^{1,2}$, Mario Pasquato$^{3,4}$, Hyunwoo Kim$^5$, and Suk-Jin Yoon$^5$

1 Dipartimento di Fisica e Astronomia & CSDC, Università di Firenze, via G. Sansone 1, I–50019 Sesto Fiorentino, Italy
e-mail: pierfrancesco.dicintio@unifi.it
2 INFN - Sezione di Firenze, via G. Sansone 1, I–50019 Sesto Fiorentino, Italy
3 INAF, Osservatorio Astronomico di Padova, vicolo dell’Osservatorio 5, I–35122 Padova, Italy
e-mail: mario.pasquato@oapd.inaf.it
4 INFN- Sezione di Padova, Via Marzolo 8, I–35131 Padova, Italy
5 Department of Astronomy & Center for Galaxy Evolution Research, Yonsei University, Seoul 120-749, Republic of Korea
e-mail: sjyoon@yonsei.ac.kr

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ABSTRACT

Context. Stellar systems are broadly divided into collisional and non-collisional. The latter are large-$N$ systems with long relaxation timescales and can be simulated disregarding two-body interactions, while either computationally expensive direct $N$-body simulations or approximate schemes are required to properly model the former. Large globular clusters and nuclear star clusters, with relaxation timescales of the order of a Hubble time, are small enough to display some collisional behaviour and big enough to be impossible to simulate with direct $N$-body codes and current hardware.

Aims. We introduce a new method to simulate collisional stellar systems, and validate it by comparison with direct $N$-body codes on small-$N$ simulations.

Methods. The Multi-Particle collision for Dense stellar systems Code (mpcdss) is a new code for evolving stellar systems with the Multi-Particle Collision method. Such method amounts to a stochastic collision rule that allows to conserve exactly the energy and momentum over a cluster of particles experiencing the collision. The code complexity scales with $N \log N$ in the number of particles. Unlike Monte-Carlo codes, mpcdss can easily model asymmetric, non-homogeneous, unrelaxed and rotating systems, while allowing us to follow the orbits of individual stars.

Results. We evolve small ($N \approx 3 \times 10^4$) star clusters with mpcdss and with the direct-summation code nbody6, finding a similar evolution of key indicators. We then simulate different initial conditions in the $10^6 - 10^9$ star range.

Conclusions. mpcdss bridges the gap between small, collisional systems that can be simulated with direct $N$-body codes and large noncollisional systems. mpcdss in principle allows us to simulate globular clusters such as Omega Centauri and M54 and even the nuclear star cluster, beyond the limits of current direct $N$-body codes in terms of the number of particles.

Key words. Methods: numerical – globular clusters: general – Galaxy: bulge – Galaxies: dwarfs

1. Introduction

Our understanding of the formation and dynamical evolution of dense stellar systems such as globular clusters (hereafter GC) and nuclear stellar clusters (hereafter NSC) has a crucial impact on Galactic archaeology (see e.g. Chung et al. 2019), multi-messenger astronomy (where it allows us to better constrain compact object mergers; see e.g. Belczynski et al. 2002; Banerjee et al. 2010; Bae et al. 2014; Ziosi et al. 2014; Breivik et al. 2016; Rodriguez et al. 2016; Hurley et al. 2016; Askar et al. 2017; Chatterjee et al. 2017; Arca Sedda et al. 2018; Kremer et al. 2018; Di Carlo et al. 2019; Bouffanais et al. 2019; Rastello et al. 2019; Antonini & Gieles 2020), cosmology and supermassive black hole science (with star clusters acting both as nurseries of intermediate-mass black hole seeds and delivery mechanism to the galactic centers; see e.g. Capuzzo-Dolcetta 1993; Ebisuzaki et al. 2001; Portegies Zwart et al. 2004, 2006; Capuzzo-Dolcetta & Micocchi 2008; Antonini et al. 2012; Arca-Sedda & Capuzzo-Dolcetta 2014), and even stellar astrophysics (as clusters are key to the formation of stellar exoticia; see e.g. Fabian et al. 1975; Bailyn 1995; Portegies Zwart et al. 2001; Fregeau et al. 2004; Verbunt & Lewin 2006; Leigh et al. 2007; Portegies Zwart et al. 2010; Pasquato et al. 2014; van den Berg 2019; Wang et al. 2020).

However, modelling self-gravitating $N$-body systems with a realistic number of stellar particles (in some cases well above $10^6$) over several relaxation times is extremely challenging in terms of computational resources due to the super-quadratic scaling of complexity with the number of particles in direct summation codes (Makino & Hut 1988; Aarseth 2003). Current state-of-the-art direct $N$-body simulations with $10^6$ particles need several months of computer time on dedicated GPU clusters to follow the evolution of typical globular clusters (Wang et al. 2016).
This is an issue especially for simulating collisional systems, where the effects of relaxation driven by two-body interactions cannot be neglected.

In Fig. 1 we show that a large fraction of the Milky Way globular clusters is both in the collisional regime and contains a sufficiently high number of stars to make detailed modelling based on direct \(N\)-body simulations infeasible, especially when the need of obtaining a significant number of realizations of the same system is taken into account.

Several approximated alternatives to the direct \(N\)-body approach\(^1\) which do not share its prohibitive computational cost exist (see Heggie 2016, for an excellent review). The family of so-called Monte-Carlo codes, which essentially solve the Fokker-Planck equation (Hénon 1971a,b, 1975; Stodolkiewicz 1982, 1986; Joshi et al. 2000; Freitag & Benz 2001; Giersz 2001; Freitag & Benz 2002; Giersz 2006; Pattabiraman et al. 2013; Giersz et al. 2013; Hypki & Giersz 2013; Pijloo et al. 2015; Rodriguez et al. 2018; Sollima & Ferraro 2019) is perhaps the most successful among these. Monte-Carlo simulations however are generally limited to spherically symmetric systems, with the notable exception of the code developed by Vasiliev (2015), which is unfortunately not in widespread use. Among other issues, this limitation was shown to lead to discrepancies between direct \(N\)-body and Monte-Carlo (or any Fokker-Planck solver that assumes spherical symmetry) in the presence of an external tidal field such as the Galactic one (Takahashi & Portegies Zwart 2000).

In this work, (the first of a series of three) we introduce a new simulation scheme, the Multi-Particle collision for Dense stellar systems code (hereafter mpdss), which combines an essentially linear scaling of computational complexity in the number of particles with the ability to model configurations with arbitrary geometries. Here we introduce the structure of the code and present a first series of tests on the dynamical evolution of globular clusters, without including the stellar evolution modules, paving the way to the application to the two most massive star clusters in the Milky Way\(^2\) (M54 and Omega Centauri). Both clusters show a spread in metallicity (Sarajedini & Layden 1995; Lee et al. 1999), hinting at a non-trivial dynamical history which possibly includes one or more mergers (see e.g. Cole et al. 2017) which may still affect present-day observable properties (Amaro-Seoane et al. 2013; Pasquato & Chung 2016), and have large masses, thus being beyond the limit of what is currently modellable with "honest" \(N\)-body simulations.

This paper is structured as follows: in Section 2 we introduce the numerical methods used in mpdss to compute the gravitational field, treat the collisions and propagate the simulations particle trajectories, and we discuss the efficiency of our implementation. In Section 3 we compare a set of test simulations of collisional evaporation of smaller GCs using mpdss and the state-of-the-art direct \(N\)-body code \(\sigma\)ove\(\nu\) (Aarseth 2003; Ntator & Aarseth 2012). In Section 4 we present the results of numerical simulations of core collapse and mass segregation. In Section 5 we discuss our findings and, finally, Section 6 summarizes.

2. Overview of the numerical method

2.1. Stochastic collisions: the Multi-Particle Collision (MPC) method

In our numerical code we resolve the collisional interactions between stars using the so-called multi-particle collision method (hereafter MPC), originally, MPC was introduced by Malevanets & Kapel (1999, 2004) in the context of numerical hydrodynamics for the simulation of mesoscopic fluids (e.g. polymers in solution, colloidal fluids). It has been shown that the method yields Galilean-invariant dynamics, that the Navier-Stokes equations are recovered in the continuum limit, and that relaxation towards thermodynamical equilibrium is correctly modeled (see Gompper et al. 2009, for a detailed review). Recently, the MPC techniques have been also used in plasma physics to treat heat transport problems in reduced models in 1D (Di Cintio et al. 2015; Cirillo et al. 2018; Lepri et al. 2019) and 2D (Di Cintio et al. 2017b,a, 2018).

The MPC scheme alternates a streaming step (corresponding to non-collisional evolution) and a collision step. In three spatial dimension, the collision step amounts to a rotation of the parti-

\(^1\) Note that direct \(N\)-body solutions are themselves not exact, due to the chaotic nature of the problem (see e.g. Di Cintio & Casetti 2019, 2020 and references therein) and the finite precision of the numerics involved (e.g. see Breen et al. 2019, which also proposes a creative alternative simulation scheme.).

\(^2\) With the exception of the nuclear star cluster (Walcher et al. 2005; Misgeld & Hilker 2011; Neumayer 2017) which is in principle also amenable to simulation with MPDSS and has been also studied with direct \(N\)-body simulations (Agarwal & Miotasavlejović 2011; Perets & Mastrobuono-Battisti 2014) in the context of the so-called repeated accretion scenario (Antonini et al. 2012; Arca-Sedda & Capuzzo-Dolcetta 2014).
Fig. 2. Evolution of the virial ratio $-2K/U$ (top panel), the total energy $E$ in units of the initial energy $E_0$ (middle panel), and the norm of the total angular momentum $L$ (bottom panel) for the same two initial conditions evolved with the exact angular momentum preserving scheme (light blue) and the faster MPC rotation with random axis (purple). The initial conditions have a slight angular momentum due to the randomized initialisation procedure for stellar velocities.

Fig. 3. Typical orbits extracted from a $N$–body simulation (upper left-hand panels) and a MPC simulation (upper right-hand panels) projected on the $x–y$ plane, and their phase-space sections in the $r–v_r$ subspace (lower panels).
cle’s velocity vectors in the centre of mass frame of each cell\(^3\) onto which the simulation domain has been partitioned.

At the beginning of the collision step the code evaluates in every cell the centre of mass (c.o.m.) velocity

\[
u_{\text{com},i} = \frac{1}{m_{\text{tot},i}} \sum_{j=1}^{n_i} m_j \mathbf{v}_j; \quad m_{\text{tot},i} = \sum_{j=1}^{n_i} m_j
\]

and the relative velocities \(\delta \mathbf{v}_j = \mathbf{v}_j - \mathbf{u}_i\). For each cell then, a random axis \(\mathbf{R}_i\) and rotation angle \(\alpha_i\) are sampled from uniform distributions. At this point, the vectors \(\delta \mathbf{v}_j\) are rotated around \(\mathbf{R}_i\) of \(\alpha_i\) and then converted back to the simulation frame, so that for the \(j\)-th particle in cell \(i\) the new velocity reads

\[
\mathbf{v}'_j = \mathbf{u}_i + \delta \mathbf{v}_{j,\perp} \cos(\alpha_i) + (\delta \mathbf{v}_{j,\perp} \times \mathbf{R}_i)\sin(\alpha_i) + \delta \mathbf{v}_{j,\parallel},
\]

where \(\delta \mathbf{v}_{j,\perp}\) and \(\delta \mathbf{v}_{j,\parallel}\) are the relative velocity components perpendicular and parallel to \(\mathbf{R}_i\), respectively.

Such operation conserves exactly the total kinetic energy \(K_i\) and the three components of the momentum \(P_i\) in cell \(i\) (e.g. see Ryder 2005; Di Cintio et al. 2017a). For the rigorous proof), By introducing a constraint on the rotation angles \(\alpha_i\), we conserve a component of the angular momentum vector of the cell \(\mathbf{L}_i\) by defining \(\alpha_i\) such that

\[
\sin(\alpha_i) = -\frac{2a_i b_i}{a_i^2 + b_i^2}; \quad \cos(\alpha_i) = \frac{a_i^2 - b_i^2}{a_i^2 + b_i^2},
\]

where

\[
a_i = \sum_{j=1}^{N_i} \left| \mathbf{r}_j \times (\mathbf{v}_j - \mathbf{u}_i) \right|; \quad b_i = \sum_{j=1}^{N_i} \mathbf{r}_j \cdot (\mathbf{v}_j - \mathbf{u}_i).
\]

In the formulae above, \(\mathbf{r}_j\) are the particle positions vectors, and the notation \([x]\) means that one is taking (without loss of generality) the component of the vector \(x\) parallel to the \(z\) axis of the cell’s coordinate system, so that the \(z\) component of the cell angular momentum is conserved.

Note that, for strictly two dimensional systems, Equation (2) becomes

\[
\mathbf{v}'_j = \mathbf{u}_i + \mathbf{g}_{\alpha_i,j} \cdot \delta \mathbf{v}_j,
\]

where now \(\mathbf{g}_{\alpha_i,j}\) is the 2D rotation matrix of an angle \(\alpha_i\) that, if chosen according to Equations (3,4), ensures the conservation of the scalar angular momentum, (see Di Cintio et al. 2017b) in addition to \(K_i\) and \(P_i\). Note also that, the conservation of the total angular momentum can be achieved even in three dimensional systems by choosing \(\mathbf{R}_i\) to be parallel to the direction of the cell’s angular momentum vector \(\mathbf{L}_i\), and taking in the definition of \(\alpha_i\) the component of the vector \(\mathbf{r}_j \times (\mathbf{v}_j - \mathbf{u}_i)\) parallel to the latter. For the simulations presented here, we limit ourselves to the standard rotation scheme with only one component of the total angular momentum conserved, as it is much less time consuming not having to determine cell by cell the direction of the angular momentum (pseudo)vector.

As in globular clusters the collision frequency strongly depends on the local values of the stellar density and velocity dispersion, we condition the MPC step to a cell-dependent probability accounting for the local degree of collisionality. We define first the cell-dependent MPC probability as

\[
p_i = \text{Erf} \left( \frac{\Delta t \pi G^2 \rho^3 \log \Lambda_i}{\sigma_i^3} \right),
\]

where \(\Delta t\) is the timestep, \(\rho_i\) the mean stellar density, \(\tilde{m}_i\) and \(\sigma_i\), the average particle mass and the velocity dispersion in the cell, respectively and \(\text{Erf}(\chi)\) is the standard error function. The cell-dependent Coulomb logarithm is defined as

\[
\log \Lambda_i = \log(\pi G^3 r_s^3 / 2 m_i^2),
\]

with \(r_s\) the typical scale length of the system. In the expression above \(\beta\) is a dimensionless constant fixed to \(2N_i\).

Once Equation (6) is evaluated in each cell, a random number \(p_{\text{i},\text{rand}}\) is sampled from a uniform distribution in the interval [0,1] and the multi-particle collision is applied for all cells for which \(p_{\text{i},\text{rand}} \leq p_i\). By doing so, particles in cells with higher collision frequency are more likely to be prone to a MPC step.

### 2.2. Deterministic dynamics

The collective dynamics of the systems is simulated by using a rather standard particle-mesh scheme (see e.g. Hockney & Eastwood 1988) that solves the Poisson equation

\[
\Delta \Phi_r = -4\pi G \rho_r
\]

on a spherical grid in polar coordinates \(N_r \times N_\theta \times N_r\) and interpolates \(\nabla \Phi\) at each particle position \(\mathbf{r}_r\).

The equations of motion between two MPC steps are solved with a standard second order leapfrog scheme with fixed timestep (see e.g. McLachlan & Atela 1992) of the order of \(10^{-2}\) in units of the system’s initial crossing time \(t_{\text{dyn}}\). In the preliminary simulations presented in this work, in order to further speed up the calculations, instead of solving Equation (8), we evaluate only the radial component of the gravitational field so that, in practice, the equations of motion become

\[
\mathbf{r}_r = -\frac{G M(r_i)}{r_i^3} \mathbf{r}_r,
\]
where $M(r_i)$ is the mass within the particles radial coordinate $r_i$. By doing so, when needed, the potential $\Phi(r_i)$ on particle $i$ can be obtained as

$$\Phi(r_i) = -G \left( \frac{M(r_i)}{r_i} + \sum_{j=i+1}^{N} \frac{m_j}{r_j} \right),$$

(10)

after having sorted the radial coordinates of all particles (see e.g. Pattabiraman et al. 2013; Rodriguez et al. 2018) as in standard Monte Carlo codes.

With such assumptions the initial spherical symmetry of the model is preserved, as no radial orbit instability can take place (Ciotti et al. 2007) and, if the collision step is deactivated, each particle orbit retains its original plane and all angular momentum vectors $J_i = m_i \mathbf{r}_i \times \mathbf{v}_i$ are conserved individually.

As an example, in Figure 2 we show the evolution of the energy, virial ratio $-2K/U$ and angular momentum for an equilibrium isotropic Plummer (1911) model with $10^3$ particles (for the details of the simulations see the Section below) evolved with angular momentum conserving and non-angular momentum conserving MPC rotation schemes. Remarkably, in both cases, the virial equilibrium is preserved and the total energy $E$ shows only little fluctuations, as small as $10^{-3}$, at variance with the direct $N$–body simulations where a systematic drift in $E$ is already visible at earlier times.

3. Comparison with direct $N$-body

3.1. Orbital structure

Introducing a somewhat stochastic mechanism of scattering between individual particle orbits via the MPC scheme, might at first seem to be inducing dramatic differences in the dynamical behaviour of orbit evolved with MPCDSS with respect to standard direct $N$–body integrators.

As a first test of reliability of MPC simulations we have evolved a small systems of $10000 \leq N \leq 32000$ particles with the direct $N$–body code NBODY6 and with MPCDSS and compared the structure of the orbits as resolved by the two schemes for identical initial conditions.

The models used for the two simulations sets have been set up as follows. We considered an isolated spherical isotropic model with Plummer (1911) density distribution

$$\rho(r) = \frac{3}{4\pi} \frac{M_0^2}{(r^2 + r_0^2)^{\frac{3}{2}}},$$

(11)

with total mass $M$ and scale radius $r_0$, related to the half mass radius by $r_{\text{half}} \approx 1.3 r_0$. In both the $N$-body and MPC case the systems have been evolved up to 1000 dynamical times defined by

$$t_{\text{dyn}} = \sqrt{\frac{r_0^3}{GM}},$$

(12)

with fixed timestep $\Delta t = t_{\text{dyn}}/100$ and neglecting stellar evolution and formation of binaries (such that each simulation particle represents an individual star retaining its mass throughout the whole run).

Surprisingly, the usage of the MPC step to resolve collisional processes not alter qualitatively the behaviour of the orbits themselves with respect to standard $N$–body simulation. In the upper panels of Figure 3 we show the projections on the $x$–$y$ plane of two orbits in a Plummer model with $N = 2 \times 10^4$ integrated with a direct $N$–body code (black lines) and with MPCDSS (red/light gray lines). Being always confined within less than two half-mass radii, the particles experience in both cases several ”close encounters” thus being subject to dramatic changes in orbital inclination and precession frequency. Direct $N$–body and MPC dynamics results in a large degree of phase-space ”diffusion” with particles exploring the whole energetically accessible region as shown in the lower panels of Fig. 3.

Moreover, following Di Cintio & Casetti (2019) we have also studied for a broad range of initial conditions the Fourier spectra of the radial coordinate $r$ for individual particle orbits. In general, the stochastic collision rule does not alter significantly the structure of the spectrum of a given orbit obtained from the same initial condition, with respect to a direct $N$–body evolution. In Figure 4 we show the modulus squared $|r(\omega)|^2$ of the radial coordinate $r$ for a particle propagated in the same Plummer model with $N = 20000$ with the two simulation approaches. Remarkably the structure of the fundamental frequency (and a large fraction of higher harmonics at larger values of $\omega$) are preserved, thus leading to speculate that MPC dynamics can be sufficiently trusted even for larger systems.

3.2. Core collapse with mass spectrum

The effect of multiple mass populations on the dynamical evolution of globular clusters is of prime importance. The present implementation of our MPC code offers the interesting chance to study multi-mass systems without adding extra computational complexity. We performed a set of additional tests with NBODY6 and with MPCDSS simulating the evolution up to and after core collapse of Plummer models with mass spectrum.

For the sake of simplicity, instead of using the multi-slope Kroupa (2002) mass function, in this work the particle masses $m_i$ have been extracted from a pure power-law mass function of the form

$$f(m) = \frac{C}{m^\alpha}; \quad m_{\text{min}} \leq m \leq m_{\text{max}},$$

(13)

where $\alpha > 0$, and the normalization constant $C$ depends on the minimum-to-maximum-mass ratio $m_{\text{min}}/m_{\text{max}}$ so that $\int_{m_{\text{min}}}^{m_{\text{max}}} f(m) dm = M$.

We run simulations for a range of $\alpha$ spanning from 0.6 to 3.0 by intervals of 0.1, and in the case of the direct $N$-body simulations we run 10 different realizations (with a different seed for the initial conditions) for each value of $\alpha$.

In both the $N$-body and MPC case the systems have been evolved for $10^4$ dynamical times, corresponding to roughly 20 two body (collisional) relaxation times of a model with the same total mass and number of equal mass particles, given by

$$t_{2b} \approx \frac{0.138N}{\log N} t_{\text{dyn}},$$

(14)

Again, as a rule, in all sets of MPCDSS simulations we have fixed the timestep $\Delta t = t_{\text{dyn}}/100$, neglected stellar evolution and the contribution of binaries$^4$.

With such simplifications, in terms of computational time the evolution of the cluster up to $4.2t_{2b}$ takes a few days on a dedicated GPU workstation for the direct $N$–body simulation, and roughly one hour on a single core of a 64 bit INTEL® machine, when using a $N_r = 32 \times 16 \times 16$ grid in polar coordinates to perform the collisions and a $N_{\text{grid}} = 2000$ radial grid to evaluate the

$^4$ Note that in direct $N$-body simulations, even if started without binaries, may form a few new binaries dynamically.
gravitational field in monopole approximation (see Eq. 9) for the MPC simulation.

For the two simulations sets we have extracted and compared as a function of time the number of escapers $N_{\text{esc}}$ (defined as the number of particles being at $r > 17r_s$ with positive energy, see e.g. Fukushige & Heggie 2000), the central density and velocity dispersion $\rho_0$ and $\sigma_0$, as well as the mass function in the core.

In Figure 5 we present the time evolution of the number of escapers $N_{\text{esc}}$ for choices of $\alpha$ in Equation (13). We find that in all cases the MPC evolution (solid line) recovers the quasi-linear trend of $N_{\text{esc}}$ with time. However, some discrepancies between MPC and $N$–body simulations are observed in particular at low values of $\alpha$ (i.e. "flatter" mass spectrum). For the best agreement case, represented by the simulations with $\alpha = 2.3$ (and corresponding to a Salpeter 1955 mass function), we show in Figure 6 the evolution of the central density $\rho_0$ and central velocity dispersion $\sigma_0$ defined within the Lagrangian radius enclosing 8% of the total mass $M$. In this case, the evolution of both quantities with the MPC code (squares) matches remarkably well that obtained using the nbody6 (circles).

In addition, for each simulation of the two sets we take the time of core-collapse $t_{\text{cc}}$ as the time at which the minimum value reached by $r^{2\%}$ (i.e. the 3D Lagrangian radius enclosing the 2% of the total mass). Fig. 7 shows the time of core collapse as a function of the initial mass-function power law $\alpha$. As expected, simulations starting with a shallower mass-function have more heavy particles, slowing global core collapse. Thus low-$\alpha$ runs take longer to reach core-collapse. We fit the relation between $\alpha$ and $t_{\text{cc}}$ using a second order polynomial.
isotropic Plummer density profiles in a broad range of systems. We have performed a broad spectrum of numerical experiments, except for a few values of $\alpha$. The result is generally within the range spanned by the direct N-body realizations of the initial condition with choices of timestep and N-body simulations (green, thick line) is superimposed. The MPC result is, however, well below $10^3 t_{\text{dyn}}$ for $\alpha = 1.5$ and $2$. In both cases the core collapse time $t_{cc}$ is well below $10^3 t_{\text{dyn}}$, as shown in the right panel of Figure 8. Remarkably, for low values of $\alpha$ there seems to be a somewhat non-monotonic trend of $t_{cc}$ with $N$. When expressing $t_{cc}$ in units of the collisional relaxation time $t_{\text{dyn}}$, the picture is inverted and large $\alpha$, even smaller values of $t_{cc}$, are associated with large $N$. We have also performed additional simulations with MPC models with different values of $\alpha$ and mass spectrum finding that, surprisingly, for the models with $\alpha$ in the range between 1.5 and 3 the asymptotic slope of the density profile in the inner regions has a better matching to the predicted $r^{-2.23}$ trend, as shown in Figure 11 for the $\alpha = 1.5$ and $\alpha = 2$ cases with $N = 10^5$.

Moreover, we have also performed additional simulations for Plummer models with different values of $\alpha$ and mass spectra finding that, surprisingly, for the models with $\alpha$ in the range between 1.5 and 3 the asymptotic slope of the density profile in the inner regions has a better matching to the predicted $r^{-2.23}$ trend, as shown in Figure 11 for the $\alpha = 1.5$ and $\alpha = 2$ cases with $N = 10^5$.

Fig. 7. Time of core collapse $t_{cc}$ in units of the dynamical time $t_{\text{dyn}}$ as function of the mass spectrum exponent $\alpha$ for MPC (red empty squares) and N–body (small black circles) simulations. A parabolic fit to the N–body simulations (green, thick line) is superimposed. The MPC result is generally within the range spanned by the direct N-body realizations, except for a few values of $\alpha$ on the high end, where core collapse happens earlier in direct N-body simulations.

4. Numerical experiments and results

We have performed a broad spectrum of numerical experiments to determine the range of applicability of our newly introduced simulation method. First of all we have investigated the process of core collapse of single component models starting with isotropic Plummer density profiles in a broad range of systems sizes spanning from $10^3$ to $10^6$. For this set of numerical experiments we follow the evolution of the three dimensional and the associated Lagrangian radii containing different fractions of the total number of particles (or mass) between 2% and 90%.

In line with expectations, we find from the MPC simulations that for large values $N$ the time of core collapse becomes asymptotically larger (in units of $t_{\text{dyn}}$), as shown in the left panel of Figure 8 for the equal masses case with $N$ ranging from $10^4$ to $10^6$. Remarkably, for low values of $N$ there seems to be a somewhat non-monotonic trend of $t_{cc}$ with $N$. We performed additional MPC simulations with $N$ as small as $10^4$ using different realizations of the initial condition with choices of timestep and grid size, and such trend persists.

When expressing $t_{cc}$ in units of the collisional relaxation time $t_{\text{dyn}}$ (cfr. Eq. 14), the picture is inverted and large $N$ systems reach core collapse earlier in units of their intrinsic $t_{cc}$, see right panel of same Figure.

In Figure 9 we show for the $N = 10^5$ case the radial density profile at different times between $t_{25}$ and $18 t_{25}$. It is evident as at already around $2 t_{25}$ (corresponding roughly to $23400 t_{\text{dyn}}$ for this value of $N$) the density has significantly departed from the initial Plummer profile (marked in figure by the thin black line). Remarkably, at later times the inner part of the density slope approaches the $r^{-2.23}$ trend (heavy dashed line) as predicted by Cohn (1980) (see also Heggie & Stevenson 1988), and in nice agreement with the Monte-Carlo simulations by Joshi et al. (2000) and Pattabiraman et al. (2013). For times larger than roughly $8 t_{25}$ the re-expansion of the outer regions becomes evident, as can be seen also from the evolution of the Lagrangian radii in Figure 10. We find a core-collapse time $t_{cc}$ (i.e. the time at which the central part of the cluster reaches the highest concentration that we measure here as the minimum attained by the Lagrangian radius containing the 2% of the simulation particles) of about $10 t_{25}$ (indicated by the vertical line in Fig. 10), in rather good agreement with the Monte-Carlo simulations of Joshi et al. (2000); Hurley & Shara (2012) and the N–body simulations of Küpper et al. (2008), finding values between 10 and 15 $t_{25}$ for models with initial conditions analogous to the ones used in our simulations (i.e. Plummer profile, $N = 10^5$ and no mass spectrum).

We observe that, in general, for fixed number of particles and total mass $M$, models with a mass spectra reach the core collapse faster in units of $t_{\text{dyn}}$ than the associated equal mass case. This can be seen from Figure 12 where we show the Lagrangian radii enclosing the same fractions of the total number of particles as in Fig. 10, but for two models with mass spectrum with exponents $\alpha = 1.5$ and 2. In both cases the core collapse time $t_{cc}$ is well below $10^3 t_{\text{dyn}}$, as shown in the right panel of Figure 8. Remarkably, for low values of $\alpha$ there seems to be a somewhat non-monotonic trend of $t_{cc}$ with $N$. When expressing $t_{cc}$ in units of the collisional relaxation time $t_{\text{dyn}}$, the picture is inverted and large $\alpha$, even smaller values of $t_{cc}$, are associated with large $N$. We have also performed additional simulations with MPC models with different values of $\alpha$ and mass spectra finding that, surprisingly, for the models with $\alpha$ in the range between 1.5 and 3 the asymptotic slope of the density profile in the inner regions has a better matching to the predicted $r^{-2.23}$ trend, as shown in Figure 11 for the $\alpha = 1.5$ and $\alpha = 2$ cases with $N = 10^5$.

Since the time dependent radii enclosing a given fraction of the total mass $M$ or number of particles $N$ are not the same quantity for a model with different species, we have evaluated both types of “Lagrangian radii” for some of the models with mass spectrum. As expected, since as result of the more efficient dynamical friction, more massive stars tend to accumulate to the centre of the system, the Lagrangian radii computed for a given percentage of the mass of the model attain systematically smaller values than those evaluated for the same percentage of the total number of particles instead. However, the estimated core collapse times do not differ significantly with the two choices, independently on the number of particles in the simulation.

5. Discussion, conclusions and future prospects

We have introduced a new code for simulating the collisional evolution of dense stellar systems using the Multi Particle Collision (MPC) approach. Our code is characterized by an $N \log N$ scaling in the number of stars, which makes it suitable for simulating massive globular clusters and the Milky Way nuclear star cluster. These systems are for the foreseeable future well beyond the reach of direct-summation N-body codes because the latter scale quadratically with the number of stars. The MPC method is based on alternating streaming steps (where stars evolve in the smooth gravitational potential of the whole star system) and collision steps which are meant to model the relaxation effects induced by stellar encounters. Collision steps are cell-based rotations of the stellar velocity vectors which by construction conserve mass, momentum, and energy. The MPC approach abstracts away from the intricacies of two- and multiple-body encounters that require techniques such as softening or Kustaanheimo-Stiefel regularization in direct-summation grav-

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3 In our implementation of the method, angular momentum conservation is also insured, but this feature can be switched off to speed up calculations if necessary.
Fig. 8. Relations of the number of particles $N$ versus the time of core collapse $t_{cc}$ in units of the dynamical time (left panel) and relaxation time (right panel) for single component Plummer models.

Fig. 9. Evolution of the 3D density profile for a model with $N = 10^5$ and initial Plummer density distribution (thin solid line) with equal masses. The heavy dashed line marks the theoretical $r^{-2.23}$ profile.

Fig. 10. Evolution of the 3D Lagrangian radii enclosing 2%, 5%, 10%, 50% and 90% of the total number of particles $N$ for the same model in Fig. 9. The vertical dashed line marks the system’s core collapse time $t_{cc} = 10t_{2b} \approx 12000t_{dyn}$.

In this paper presented our MPC code and run a few test simulations, showing that the total energy and angular momentum of an isolated simulation are conserved. We also calculated the virial ratio (kinetic over potential energy), which is also conserved with remarkable accuracy. Finally, we validated our code by comparison to direct N-body simulations of star clusters. We find that the evolution of the central density, central velocity dispersion, and number of escapers as a function of time in MPC simulations closely follows that in direct N-body simulations over a wide range of stellar mass spectra. Additionally, the time at which core collapse is reached is also in good agreement with both direct N-body and theoretical analytical calculations.

In the future we plan to add a stellar evolution module and to introduce one or more schemes to simulate binary stars, in order to study the dynamics of compact objects (Mapelli 2016; Rastello et al. 2020). We will then run a large set of simulations which we plan to release publicly, including simulations meant to model specific objects, such as Omega Centauri and M 54. Finally, we will address the black-hole retention problem in star
clusters, focusing in particular on the fate of intermediate-mass black holes in the Galactic nuclear cluster.

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Fig. 11. Evolution of the 3D density profile for two models with $N = 10^5$ and initial Plummer density distribution (thin solid line) and mass spectra with $\alpha = 1.5$ (left panel) and 2.0 (right panel). As in Fig. 9, the heavy dashed line marks the theoretical $r^{-2.25}$ profile.

Fig. 12. Evolution of the 3D Lagrangian radii enclosing 2%, 5%, 10%, 50% and 90% of the total number of particles $N = 10^5$ for the two models shown in Fig. 11. The solid and dotted-dashed lines refer the $\alpha = 1.5$ and 2 cases, respectively. The two vertical dashed lines mark the core collapse times for the models $t_{cc} = 2000 t_{\text{dyn}}$ for $\alpha = 1.5$ and $t_{cc} \approx 773 t_{\text{dyn}}$ for $\alpha = 2.0$.

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