Neural Control of Discrete Weak Formulations: Galerkin, Least-Squares & Minimal-Residual Methods with Quasi-Optimal Weights

Ignacio Brevis†, Ignacio Muga†, Kristoffer G. van der Zee∗

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Abstract

There is tremendous potential in using neural networks to optimize numerical methods. In this paper, we introduce and analyse a framework for the neural optimization of discrete weak formulations, suitable for finite element methods. The main idea of the framework is to include a neural-network function acting as a control variable in the weak form. Finding the neural control that (quasi-) minimizes a suitable cost (or loss) functional, then yields a numerical approximation with desirable attributes. In particular, the framework allows in a natural way the incorporation of known data of the exact solution, or the incorporation of stabilization mechanisms (e.g., to remove spurious oscillations).

The main result of our analysis pertains to the well-posedness and convergence of the associated constrained-optimization problem. In particular, we prove under certain conditions, that the discrete weak forms are stable, and that quasi-minimizing neural controls exist, which converge quasi-optimally. We specialize the analysis results to Galerkin, least-squares and minimal-residual formulations, where the neural-network dependence appears in the form of suitable weights. Elementary numerical experiments support our findings and demonstrate the potential of the framework.

†Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Chile
∗School of Mathematical Sciences, University of Nottingham, UK; Corresponding author, kg.vanderzee@nottingham.ac.uk
Dedicated to J. Tinsley Oden.
1 Introduction

In recent years there has been tremendous interest in the merging of neural networks and machine-learning algorithms with traditional methods in scientific computing and computational science [24, 17, 27, 39]. In this paper we demonstrate how neural networks can be utilized to optimize finite element methods.

In one of its most familiar mathematical forms, the finite element method is a discretization technique for partial differential equations (PDEs) based on a weak formulation using discrete subspaces, i.e., the exact solution \( u \in U \) is approximated by \( u_h \in U_h \), which is the unique solution of the discrete problem:

\[
\text{Find } u_h \in U_h : \quad b(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h, \tag{1}
\]

where \( U_h \) is a discrete subspace of the infinite-dimensional Hilbert or Banach space \( U \) (typically a Sobolev space on a domain \( \Omega \subset \mathbb{R}^d \)), \( V_h \) is a subspace of a Hilbert or Banach space \( V \) with \( \dim V_h = \dim U_h \), \( b : U \times V \to \mathbb{R} \) is a continuous bilinear form, \( f : V \to \mathbb{R} \) a continuous linear form, and the exact solution \( u \) satisfies \( b(u, v) = f(v) \) for all \( v \in V \).

It is well-known that the accuracy of \( u_h \) can be improved by enlarging \( U_h \) (e.g., by refining the underlying finite element mesh). However, for a fixed value of \( h \), the particular \( u_h \) defined by (1) may be very unsatisfactory. In fact, there is no reason why a certain quantity of interest of \( u_h \) is accurate at all, or why the approximation inherits certain qualitative features of the exact solution. Indeed, the discrete problem (1) is a rigid statement in the sense that it identifies a single element in \( U_h \), irrespective of desired attributes, whereas there could be many other elements in \( U_h \) that are far superior.

1.1 Neural optimization of discrete weak forms

The objective of this work is to propose and analyse a framework for the neural optimization of discrete weak formulations to significantly improve quantitative and qualitative attributes of discrete approximations. In particular, we consider Galerkin, least-squares, and minimal-residual formulations.

The main idea of the framework is that it incorporates a neural-network function \( \xi \) as a control variable in the discrete test space \( V_h(\xi) \). That is, the approximation \( u_h = u_h(\xi) \) now depends on \( \xi \) and solves the discrete problem:

\[
\text{Find } u_h(\xi) \in U_h : \quad b(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h(\xi). \tag{2}
\]

Then, in order to obtain a desired approximation \( u_h(\xi) \), we aim to find a neural-network function \( \xi \) that quasi-minimizes a desired cost (or loss) functional:

\[
J(u_h, \xi) \longrightarrow \text{quasi-min}. \tag{3}
\]

---

1 When \( U_h = V_h \), this is a Galerkin method; otherwise it is a Petrov–Galerkin method.
2 Indeed, a priori error analysis reveals that \( \|u - u_h\|_U \leq C \inf_{w_h \in U_h} \|u - w_h\|_U \), provided \( b(\cdot, \cdot) \) satisfies a discrete inf–sup condition on \( U_h \times V_h \); see e.g., [38, 19].
3 E.g., the value \( u_h(x_0) \) for some point \( x_0 \in \Omega \) is generally quite distinct from \( u(x_0) \).
4 E.g., \( u_h \) may exhibit spurious oscillations, while \( u \) is monotone.
5 We also allow for the inclusion of a regularization term in the cost functional; see Section 2.1.
The notion of quasi-minimization is critical when aiming to minimize over a set of neural-network functions (i.e., the set of functions implemented by neural networks of a fixed architecture); see Section 2.2 for further details (in particular, Definitions 2.1 and 2.2).

The quasi-minimization problem (3) is essentially a nonstandard PDE-constrained optimization, with the nonstandard part being the dependence of the state problem (2) on $\xi$ via the discrete test space $V_h(\xi)$. Importantly, $V_h(\xi)$ will be parameterized by $\xi$ in such a way so as to ensure stability of the discrete problem (2). Moreover, as will become clear in the following sections, the basis functions in $V_h(\xi)$ need not be computed explicitly, but equivalent formulations to (2) can be used, which instead incorporate $\xi$ by means of suitable weight functions. These formulations essentially lead to a PDE-constrained optimization with a nonlinear control-to-state map.

1.2 Potential of the methodology

There are two main benefits of having neural control of discrete weak forms:

- **Incorporation of data:** Knowledge of quantities of the exact solution can be taken into account in a natural way by setting, for example,

$$J(u_h, \xi) = \frac{1}{2} |q(u_h, \xi) - \bar{q}|^2,$$

where $q : U \to \mathbb{R}$ is a functional measuring the quantity of interest and $\bar{q} \in \mathbb{R}$ is known data.\(^6\) Minimizing such a $J(\cdot)$ ensures that the discrete solution $u_h$ to (2) is data-driven in the sense that $u_h$ becomes constrained by the data.\(^7\) We note that multiple quantities can be taken into account using, for example,

$$J(u_h, \xi) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} \frac{1}{2} |q_i(u_h, \xi) - \bar{q}_i|^2,$$

or, more generally, using some operator $Q : U \to Z$; see Section 2.

- **Incorporation of stabilization mechanisms:** Qualitative attributes of the discrete solution can be enhanced by minimizing a suitably-chosen $J(\cdot)$. In this way discrete solutions can be enforced to, e.g., satisfy an a priori known maximum principle, have monotone (or spurious oscillation free) behavior around discontinuities and layers, or have a certain discrete wave number (i.e., free from pollution). In the past decades, many different stabilized finite element methods have been proposed (and analyzed) that impose such attributes [21, 10, 26, 20, 15, 40]. Within our framework such a method is naturally obtained after (quasi-)minimization (i.e., method (2) with $\xi = \bar{\xi}$). As an example, Guermond [21] advocates the $L^1$-minimization of the residual; in other words, within our framework one would choose:

$$J(u_h, \xi) = \|f - Bu_h, \xi\|_{L^1(\Omega)},$$

where $f - Bu_h, \xi$ is the strong form of the residual.

The idea of using neural networks to parameterize the test space was initially proposed in our earlier work [8], where it was restricted to minimal-residual formulations within a parametric

\(^6\)The data $\bar{q}$ represents $q(u)$, and it could be obtained through experiments, high-fidelity computation, or otherwise.

\(^7\)This is somewhat similar in spirit to physics-informed neural networks (PINN) [43], where however a single neural-network function minimizes a combination of the residual and data misfit.
PDE setting. The current work presents significantly more general settings and formulations as well as analyses of their well-posedness and convergence.

While the above shows examples of $J(\cdot)$ corresponding to unsupervised learning (i.e., there is no need to know the exact solution $u$), when the original problem is parametric itself (e.g., a parametric PDE), supervised learning becomes meaningful. Indeed, in that case, the data may be the exact solution $u_\lambda$ for certain parameters $\lambda_i$, $i = 1, \ldots, N_{\text{data}}$. This then allows for the training of finite element discretizations with superior accuracy in quantities of interest even on very coarse meshes. We refer to our earlier work [8] for the methodology and illustrative examples in that case.

1.3 Main contributions: Well-posedness, convergent quasi-minimizers, weighted conforming formulations

Let us briefly outline the main contributions of this work. The first main contribution is the analysis of an abstract constrained-optimization problem associated to (3); see Section 2. In particular, we consider an abstract state problem equivalent to (2), but in the form of a mixed system with a $\xi$-dependent bilinear form. We prove, under suitable conditions, that the state problem is well-posed (uniformly with respect to $\xi$); see Proposition 2.9. Furthermore, we present differentiability conditions (on the $\xi$-dependence) that allow us to prove the existence of quasi-minimizers (within sets of neural-network functions, of some size $n$) to the associated constrained optimization (3), which converge quasi-optimally (upon $n \to \infty$); see Corollary 2.12 for details.

We note that our analysis is based on a fundamental result for the quasi-minimization of strongly-convex and differentiable functionals (see Theorem 2.A), which is of independent interest and applies, e.g., to the analysis of deep Ritz methods [54, 42, 37] and PINN methods [48, 35, 11].

The second main contribution of this work is the application of our framework to certain weak formulations used by conforming finite element methods; see Section 3. In these applications, the neural-network control variable $\xi$ will appear by means of suitable weights in the bilinear forms. In particular, we will analyse weighted least-squares, weighted Galerkin, and weighted minimal-residual formulations.

For weighted least-squares and weighted minimal-residual formulations, suitable conditions on the weights imply (via the abstract result of the first main contribution) stability of the discrete problem (uniformly in $\xi$). Furthermore, suitable differentiability conditions on the weights imply existence of (quasi-optimally) convergent quasi-minimizers of the associated constrained minimization.

On the other hand, for weighted Galerkin, it turns out that stability is not immediate, and may require constraints on $\xi$ depending on the problem at hand. Therefore, neural control is far more convenient for least-squares and minimal-residual formulations, the fundamental reason being the inherent stability that comes with their underlying minimization principle.

We support our findings with numerical experiments in Section 4. While our theoretical results directly apply to any linear operator, we choose the advection-reaction PDE to illustrate various numerical aspects, viz., the incorporation of data (Section 4.1), the quasi-optimal convergence of quasi-minimizers (Section 4.2), and the incorporation of $L^1$-type stabilization (Section 4.3).

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8The mixed system is motivated by residual-minimization theory [14, 36]: Minimal residual formulations are equivalent to mixed systems, which in turn are equivalent to Petrov–Galerkin formulations.

9In essence, the reason for instability relates to a discrete inf-sup condition of a weighted bilinear form.
1.4 Related work

There are a number of works related to ours.

**Optimizing numerical methods:** Traditionally, the incorporation of known data or other desired attributes in numerical PDE approximations is achieved via the method of Lagrange multipliers, see e.g., Evans, Hughes & Sangalli [20], Kergrene, Prudhomme, Chamoin & Laforêt [28], and references therein. More recently, neural networks have been proposed to learn the parameters that define a numerical method; see Ray & Hesthaven [45], Mishra [33] and others [2, 16, 53, 47]. Interestingly, a recent learning methodology for adaptive mesh refinement has been proposed that ensures optimal convergence; see Bohn & Feischl [6]. Within the context of optimizing finite-element formulations, a minimal-residual framework that ensures stability was proposed in our previous work [8]. Our current work contributes to these developments by providing the analysis of a general framework for neural optimization of finite element methods.

**Neural networks for PDEs:** The use of neural networks for approximating directly the solution to PDEs has received widespread interest since the works by E & Yu [18], Sirignano & Spiliopoulos [49], Berg & Nyström [3] and Raissi, Perdikaris & Karniadakis [43], amongst others. Recently, there have been a number of ideas that propose an adaptive construction of neural-network approximations; see Ainsworth & Dong [1], Liu, Cai & Chen [31] and Uriarte, Pardo & Omella [52]. Neural networks can also be used to obtain the coefficients of the basis expansion used by a standard (linear) approximation [23, 29].

**Neural networks for inverse PDEs:** In the context of inverse problems involving PDEs, the use of neural networks to represent unknown PDE coefficients (fields) and constitutive models has been explored by, e.g., Teichert, Natarajan, Van der Ven & Garikipati [50], Berg & Nyström [4] and Xu & Darve [55]. These works are similar to the current work in the sense that standard (finite element) methods are used to solve the PDE, while a neural network is embedded within the discrete formulation. We note that the analysis provided by our current work can be extended to those inverse problems.

**Error analysis for neural-network approximations:** There are a number of works containing a priori error analysis for neural-network based PDE approximations. For those related to the deep Ritz method; see Xu [54, Section 5], Pousin [42, Section 3], and Müller & Zeinhofer [37]. For those related to physics-informed neural networks (PINN) and least-squares methods; see Sirignano & Spiliopoulos [49, Section 7], Mishra & Molinaro [35, 34], Pousin [42, Section 4] and Cai, Chen & Liu [11]. Recently, a posteriori error analysis has also been studied, in particular goal-oriented analysis using the dual-weighted residual (DWR) methodology; see, e.g., Roth, Schröder and Wick [46], Minakowski & Richter [32] and Chakraborty, Wick, Zhuang & Rabcewicz [12]. We note that in our current work, while we have in mind the error analysis for neural-control approximations, the abstract analysis presented in Section 2 is essentially an extension of the above-mentioned a priori analysis to a certain class of problems involving a convex and differentiable cost functional.

2 Abstract framework

In this section we present the analysis of the abstract state equation (in the form of a mixed system) and the associated optimization problem. We essentially follow the classical theory of optimal control (PDE-constrained optimization) by Lions [30]; see also, [25, 51, 7]. Our resulting optimization problem bears similarity to that of parameter identification of PDE coefficients; see Rannacher & Vexler [44] and references therein for its error analysis. While we present our abstract framework within Hilbert spaces (and using a quadratic cost), we note that extensions
to Banach spaces are feasible, but not within the scope of the current work.

2.1 Discrete state problem and associated cost functional

Let $X$ be a Hilbert space for the control variable, $U$ and $V$ be Hilbert spaces for trial and test functions, respectively, $U_h \subset U$ be a discrete (finite element) subspace, and $\hat{V} \subseteq V$. In all that follows, we think of $h$ (hence $U_h$) as being fixed. Given $\xi \in X$ and $f \in V^*$ (the dual of $V$), we consider the discrete state problem given by:

\[
\begin{aligned}
\text{Find } (r, u_h) \in \hat{V} \times U_h :
\quad & a(\xi; r, v) + b(u_h, v) = f(v), \quad \forall v \in \hat{V}, \\
\quad & b(w_h, r) = 0, \quad \forall w_h \in U_h ,
\end{aligned}
\]

where $b(\cdot, \cdot)$ is a continuous bilinear form on $U \times V$, i.e., $b(\cdot, \cdot) \in \mathcal{L}(U \times V; \mathbb{R})$, and for each $\xi \in X$, $a(\xi; \cdot, \cdot)$ is a continuous bilinear form on $V \times V$, i.e., $a(\xi; \cdot, \cdot) \in \mathcal{L}(V \times V; \mathbb{R})$. To explicitly indicate the dependence of $r$ and $u_h$ on $\xi$, we use the notation:

$(r_\xi, u_{h, \xi}) = \text{solution of (4a)--(4b) for a given } \xi$.

In Section 2.4, we demonstrate that (4a)--(4b) is equivalent to (2) for a particular choice of $V_h(\xi)$; see Proposition 2.10. The discrete problem in (4a)--(4b) is essentially a general formulation, which for a specific choice of $a(\cdot, \cdot, \cdot)$ and $\hat{V}$ reduces to a (weighted) Galerkin, least-squares or minimal residual method; see Section 3.

Next, let $Z$ be a Hilbert space, and let $Q : U \to Z$ be a linear continuous (observation) operator. Then, given an observation $z_o \in Z$ and regularization parameter $\alpha \geq 0$, we consider the cost (or loss) functional $J : U_h \times X \to \mathbb{R}$ defined by:

\[
J(w_h, \xi) := J_1(w_h) + \alpha j_2(\xi),
\]

where

\[
\begin{aligned}
J_1(w_h) & := \frac{1}{2} \|Q(w_h) - z_o\|_Z^2, \\
\alpha j_2(\xi) & := \frac{1}{2} \|\xi\|_X^2.
\end{aligned}
\]

The associated reduced cost functional $j : X \to \mathbb{R}$ is then given by:

\[
j(\xi) := j_1(\xi) + \alpha j_2(\xi),
\]

where $j_1 : X \to \mathbb{R}$ is defined by:

\[
j_1(\xi) := J_1(u_{h, \xi}) = \frac{1}{2} \|Q(u_{h, \xi}) - z_o\|_Z^2,
\]

While ideally we would like to minimize $j(\cdot)$ over (the infinite-dimensional) $X$, we proceed by considering neural-network approximations.

\footnote{Later on, when considering minimal residual formulations, $\hat{V}$ will be a discrete (finite element) subspace of $V$, but for the other formulations $\hat{V} = V$.}
2.2 Neural quasi-minimization

To accommodate neural optimization, we consider the subset $\mathcal{M}_n \subset X$ consisting of all functions implemented by neural networks of a fixed architecture parameterized by $n$.\(^{11}\) We shall simply refer to $\mathcal{M}_n$ as a set of neural-network functions, and we think of $n$ as a measure of the size of the architecture (e.g., the total number of neurons, or total number of parameters).

When aiming to minimize $j(\cdot)$, a significant complication is that the set $\mathcal{M}_n$ may not be closed (topologically) in $X$.\(^{12}\) Hence, even though $j(\cdot)$ may have an infimum on $\mathcal{M}_n$, there may not be a minimizer in $\mathcal{M}_n$. Therefore, one should not aim to completely minimize $j(\cdot)$, but instead use a relaxed notion of quasi-minimization as used by Shin, Zhang & Karniadakis [48]\(^{13}\) (for which the existence of an infimum implies the existence of a quasi-minimizer):

**Definition 2.1 (Quasi-minimizers and quasi-minimizing sequences)**

Let $j : X \to \mathbb{R}$ be a cost functional.

(i) Let $\delta_n > 0$ and $\mathcal{M}_n \subset X$ be a subset of $X$ (not necessarily closed in $X$). A function $\bar{\xi}_n \in \mathcal{M}_n$ is said to be a **quasi-minimizer** of $j(\cdot)$ if the following holds true:\(^{14}\)

$$j(\bar{\xi}_n) \leq \inf_{\xi_n \in \mathcal{M}_n} j(\xi_n) + \frac{\delta_n}{2}. \quad (8)$$

(ii) Consider a sequence of subsets $(\mathcal{M}_n)_{n \in \mathcal{N}}$ of $X$, with $\mathcal{N}$ being a strictly-increasing sequence of natural numbers. A sequence $(\xi_n)_n$, with $\bar{\xi}_n \in \mathcal{M}_n$, is said to be a **quasi-minimizing sequence** if (9) holds true for all $n \in \mathcal{N}$ with $\delta_n > 0$ such that:

$$\delta_n \to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

In summary, the neural optimization problem that we consider is the following:

**Definition 2.2 (The quasi-minimizing control problem)**

The following statements are equivalent.

**Reduced quasi-minimizing control problem**: For $j(\cdot)$ given by (7), we aim to quasi-minimize $j(\cdot)$, i.e., given $\delta_n > 0$,

$$\begin{cases}
\text{Find } \bar{\xi}_n \in \mathcal{M}_n : \\
\quad j(\bar{\xi}_n) \leq \inf_{\xi_n \in \mathcal{M}_n} j(\xi_n) + \frac{\delta_n}{2}.
\end{cases} \quad (9)$$

**Constrained quasi-minimizing control problem**: For $J(\cdot, \cdot)$ given by (5), we aim to quasi-minimize $J(u_h, \xi)$ subject to (4a)–(4b), i.e., given $\delta_n > 0$,

$$\begin{cases}
\text{Find } \bar{\xi}_n \in \mathcal{M}_n : \\
\quad J(u_h, \bar{\xi}_n, \xi_n) \leq \inf_{\xi_n \in \mathcal{M}_n} J(u_h, \xi_n, \eta_n) + \frac{\delta_n}{2}.
\end{cases} \quad (10)$$

\(^{11}\)In the terminology of Petersen, Raslan and Voigt [41], the set $\mathcal{M}_n$ consists of the *realisations* of all possible neural networks of some fixed architecture (and some given activation function). While a neural network is identified with the set of weight and bias parameters, its realisation is the function implemented by the network.

\(^{12}\)For example, [41, Theorem 3.1] shows that, under mild conditions on the architecture and activation function, $\mathcal{M}_n$ is not a closed subset of $L^2(\Omega)$ (or, more generally, $L^p(\Omega)$, with $0 < p < \infty$), unless, e.g., an upper bound is imposed on the weight parameters [41, Proposition 3.7].

\(^{13}\)Quasi-minimization can also be thought of as solving the minimization problem up to some optimization accuracy, cf. [37].

\(^{14}\)Observe that if $j(\cdot)$ has an infimum on $\mathcal{M}_n$, then immediately a quasi-minimizer exists (in $\mathcal{M}_n$). This is true simply by the definition of the infimum.
Example 2.3 (Need for quasi-minimizers) Let us discuss a simple example illustrating the non-existence of minimizers, hence the need for quasi-minimizers.\footnote{This is essentially an example of a PINN problem, i.e., minimizing a strong residual and boundary condition in least-squares sense. It is not difficult to construct a similar example for a neural control problem.}

Let \( x = (x_1, x_2) \in \Omega = (0, 1)^2 \subset \mathbb{R}^2 \). Given \( z \in (0, 1) \), let \( \chi_{[z,1]} \) denote the characteristic function of the subset \([z,1]\).\footnote{That is, \( \chi_{[z,1]}(x_1) = 1 \) if \( x_1 \in [z,1] \) and \( = 0 \) otherwise.} Consider the following cost functional:

\[
j(\xi) = \frac{1}{2} \int_0^1 \int_0^1 \left( \frac{\partial \xi}{\partial x_2} \right)^2 dx_1 dx_2 + \int_0^1 (\xi - \chi_{[z,1]})^2 dx_1
\]

for \( \xi \in \mathbb{X} = \left\{ \eta \in L^2(\Omega) \mid \frac{\partial \eta}{\partial x_2} \in L^2(\Omega) \right\} \). Minimizing \( j(\cdot) \) over \( \mathbb{X} \) solves a first-order PDE (constant advection in the direction of the \( x_2 \)-axis) with discontinuous data given by \( \chi_{[z,1]} \), which is a well-posed problem \cite{5}.

Let \( \mathcal{M}_n \) be the set of two-layer neural-network functions \( \Omega \mapsto \mathbb{R}^2 \mapsto \mathbb{R} \) using two neurons and ReLU activation in the hidden layer, i.e.,

\[
\mathcal{M}_n = \left\{ \xi_n : \Omega \to \mathbb{R} \mid \xi_n(x) = \sum_{i=1}^{2} a_i \text{ReLU}(w_i \cdot x - b_i), a_i, b_i \in \mathbb{R}, w_i \in \mathbb{R}^2 \right\}.
\]

Note that an infimizing sequence of \( j(\cdot) \) in \( \mathcal{M}_n \) is given by:

\[
\xi_m(x) = \begin{cases} 
0 & 0 \leq x_1 < z_m := (1 - \frac{1}{m})z, \\
\frac{x_1 - z_m}{z - z_m} & z_m \leq x_1 < z, \\
1 & z \leq x_1 \leq 1,
\end{cases}
\]

for \( m = 1, 2, 3, \ldots \), but whose limit \( \xi_m \to \bar{\xi} \) in \( \mathbb{X} \) as \( m \to \infty \) is a discontinuous function (with \( j(\bar{\xi}) = 0 \)). Therefore the infimizer \( \xi \) does not exist in \( \mathcal{M}_n \subset C(\Omega) \).

On the other hand, quasi-minimizers \( \bar{\xi}_n \) do exist in \( \mathcal{M}_n \), in particular, \( \xi_m \) as defined above is a quasi-minimizer for \( m \) large enough.\footnote{Indeed, one can verify by direct calculation that \( m \) must be such that \( \frac{1}{2}(z - z_m) \leq \frac{\delta_2}{2} \), i.e., \( m \geq \frac{2}{z \delta_2} \).}

\( \Box \)

2.3 Analysis of reduced control problem

We first proceed with the analysis of the reduced control problem (9). Let the state operators \( R_h : \mathbb{X} \to \mathbb{V} \) and \( S_h : \mathbb{X} \to U_h \) be defined by:

\[
R_h(\xi) := r_{h,\xi}, \quad \forall \xi \in \mathbb{X}, \quad (11a)
\]

\[
S_h(\xi) := u_{h,\xi}, \quad \forall \xi \in \mathbb{X}, \quad (11b)
\]

where \( r_{h,\xi} \) and \( u_{h,\xi} \) are the first and second component, respectively, of the solution to the mixed system (4). Then the reduced cost \( j(\cdot) \) given in (7) can be written as follows:

\[
j(\xi) = j_1(\xi) + \alpha j_2(\xi) = J_1(S_h(\xi), \xi) + \alpha J_2(\xi) = \frac{1}{2} \|Q \circ S_h(\xi) - z_0\|_2^2 + \frac{\alpha}{2} \|\xi\|_\mathbb{X}^2. \quad (12)
\]

Our main result depends on the following fundamental theorem, which is of independent interest:
Theorem 2.A (Differentiable, strongly-convex quasi-minimization)

Let \( j : X \to \mathbb{R} \) be a cost functional. Assume that \( j(\cdot) \) is Gâteaux differentiable with derivative \( j' : X \to X^* \) being Lipschitz continuous, i.e., there is a constant \( L > 0 \) such that
\[
\|j'(\xi) - j'(\eta)\|_{X^*} \leq L\|\xi - \eta\|_X, \quad \forall \xi, \eta \in X,
\]
Furthermore, assume that \( j(\cdot) \) is strongly convex, i.e., there is a constant \( \gamma > 0 \) such that
\[
\langle j'(\xi) - j'(\eta), \xi - \eta \rangle_{X^* \times X} \geq \gamma\|\xi - \eta\|_X^2, \quad \forall \xi, \eta \in X. \tag{13}
\]
Then the following hold true:

(i) \( j(\cdot) \) has a unique minimizer \( \bar{\xi} \in X \), which satisfies:
\[
j'\big(\bar{\xi}\big) = 0 \quad \text{in } X^*.
\]

(ii) For any subset \( M_n \subset X \), \( j(\cdot) \) has a quasi-minimizer \( \bar{\xi}_n \in M_n \) that satisfies (8).

(iii) Any quasi-minimizer \( \bar{\xi}_n \) in \( M_n \) satisfies the following quasi-optimal error estimate:
\[
\|\bar{\xi} - \bar{\xi}_n\|_X \leq \left( \frac{L}{\gamma} \inf_{\xi_n \in M_n} \|\bar{\xi} - \xi_n\|_X^2 + \frac{\delta_n}{\gamma} \right)^{1/2} \tag{14}. 
\]

\textbf{Proof} See Appendix A.1. □

We now analyse when our \( j(\cdot) \) satisfies the assumptions of Theorem 2.A.

Theorem 2.B (Reduced control problem: Differentiability & strong convexity)

Let \( \alpha > 0 \) and \( j(\cdot) = j_1(\cdot) + \alpha j_2(\cdot) \) be as in (12). Let \( Q \in L(U, Z) \). Assume \( S_h : X \to U_h \) is differentiable, \( S_h(\cdot) \) and \( S'_h(\cdot) \) are uniformly bounded on \( X \), and \( S'_h(\cdot) \) is Lipschitz continuous. Then:

(i) \( j_1, j_2, j : X \to \mathbb{R} \) are Gâteaux differentiable with \( j'_1, j'_2, j' : X \to X^* \) Lipschitz continuous.

Additionally, assume \( \alpha \) is sufficiently large. Then:

(ii) \( j : X \to \mathbb{R} \) is strongly convex, i.e., there is a constant \( \gamma > 0 \) such that (13) holds true.\(^\mathbf{18}\) □

\textbf{Proof} See Appendix A.2. □

Corollary 2.4 (Reduced control problem: (Quasi-)minimizers & quasi-optimality)

Under the conditions of Theorem 2.B, the statements (i), (ii) and (iii) of Theorem 2.A hold true. □

\textbf{Proof} The results of Theorem 2.B are the assumptions of Theorem 2.A. □

Remark 2.5 (Quasi-optimal rates) The first part on the right-hand side of the quasi-optimality result (14) can be estimated in terms of \( n \) using results from neural-network approximation theory; see, e.g., Yarotsky [56], Gühring, Kutyniok and Petersen [22], and references therein. Such a result may be useful in finding a proper balance of \( \delta_n \) as \( n \to \infty \). Alternatively, the choice of \( \delta_n \) may be found through a proper \( \text{a posteriori} \) estimator, which seems to be an open problem. □

\(^{18}\)In particular, when \( \alpha > L_1 \), where \( L_1 \) is the Lipschitz constant of \( j'_1(\cdot) \), then \( \gamma = \alpha - L_1 \).
Remark 2.6 (Condition on $\alpha$) The proof of Theorem 2.B reveals that the condition that $\alpha$ is sufficiently large may be weakened if $j_1$ has additional structure (e.g., convexity). Indeed, convexity of $j_1$ guarantees that $j$ will be strongly convex, with strongly convexity constant equal to $\alpha > 0$. If the case, there is no need of Lipschitzness of $j'_1$ in order to prove statement (iii) of Theorem 2.B, only $\alpha > 0$ will be enough. Furthermore, statement (v) of Theorem 2.B becomes:

$$\|\hat{\xi} - \hat{\xi}_n\|_X < \left( \frac{\alpha + L_1}{\alpha} \inf_{\eta_n \in M_n} \|\hat{\xi} - \eta_n\|_X^2 + \frac{\delta_0}{\alpha} \right)^{1/2}.$$  

Remark 2.7 (Physics-informed neural networks (PINN))
Theorem 2.A can be applied to PINN [43] (for neural-network approximations to PDEs). Indeed, consider

$$j(\xi) = \frac{1}{2} \|f - B\xi\|_L^2,$$

where $f - B\xi$ is an abstract residual in some abstract Hilbert space $L$ (which may include the PDE residual, initial condition and boundary conditions, as in [35], as well as a data residual, as in [34]). If $B : X \to L$ is a linear operator, then the assumptions of Theorem 2.A (Lipschitz continuity and strong convexity) hold true.

Remark 2.8 (Deep Ritz method)
Theorem 2.A can also be applied to the Deep Ritz method [18]. Indeed, consider

$$j(\xi) = \frac{1}{2} b(\xi, \xi) - f(\xi),$$

where $b \in \mathcal{L}(X \times X; \mathbb{R})$ is a coercive bilinear form and $f \in X^*$. For such a $j(\cdot)$, the assumptions of Theorem 2.A (Lipschitz continuity and strong convexity) hold true.

2.4 Analysis of constrained control problem
We now proceed with the analysis of the constrained control problem (10). We begin by providing conditions that guarantee the well-posedness of the state problem.

Proposition 2.9 (Stability of the state problem) Let $a(\xi; \cdot, \cdot) \in \mathcal{L}(V \times V; \mathbb{R})$ for each $\xi \in X$, and let $b(\cdot, \cdot) \in \mathcal{L}(U \times V; \mathbb{R})$. For $U_h \subset U$ and $\hat{V} \subseteq V$, let the kernel subspace $\hat{K} := \{v \in \hat{V}: b(w_h, v) = 0, \forall w_h \in U_h\}$. Then, the following statements hold true:

(i) For each $\xi \in X$, problem (4) is well-posed (for any $f \in V^*$) if and only if there exist constants $\alpha_h \equiv \alpha_h(\xi) > 0$ and $\beta_h > 0$ such that:

$$\inf_{v_1 \in \hat{K}} \sup_{v_2 \in \hat{K}} \frac{a(\xi; v_1, v_2)}{\|v_1\|_V \|v_2\|_V} \geq \alpha_h,$$  \hspace{1cm} (15a)\]

$$\{v_2 \in \hat{K} : a(\xi; v_1, v_2) = 0, \forall v_1 \in \hat{K}\} = \{0\},$$

$$\inf_{w_h \in U_h} \sup_{v \in \hat{V}} \frac{b(w_h, v)}{\|w_h\|_U \|v\|_V} \geq \beta_h.$$  \hspace{1cm} (15b)\]

(ii) If (15) is satisfied, then the following a priori bound holds true for the solution $u_h \in U_h$ of problem (4):

$$\|u_h\|_U \leq \frac{1}{\beta_h} \left( 1 + \frac{\|a(\xi; \cdot, \cdot)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\alpha_h} \right) \|f\|_{V^*}.$$\hspace{1cm} (19)\]

\[19\] Only when $\hat{V}$ is infinite-dimensional, one needs the extra hypothesis in (15a), and whenever $a(\xi; \cdot, \cdot)$ is an equivalent inner product on $V$, then this condition is automatically satisfied. Indeed, zero is the only element in $V$ which is orthogonal to itself.

Neural control of discrete weak formulations
Note that the state equations (4a)–(4b) can then be written as follows:

(iii) Furthermore, if \( a(\xi, \cdot, \cdot) \) is an equivalent inner-product on \( V \), with associated norm \( \| \cdot \|_V^\xi := \sqrt{a(\xi; \cdot, \cdot)} \), i.e., for some \( C_{1,\xi}, C_{2,\xi} > 0 \):

\[
C_{1,\xi} \| v \|_V \leq \| v \|_{V^\xi} \leq C_{2,\xi} \| v \|_V, \quad \forall v \in V,
\]

then \( \alpha_h = (C_{1,\xi})^2 \) in (15a), and additionally, the following improved a prior bound holds true:

\[
\| u_h \|_U \leq \frac{C_{2,\xi}}{C_{1,\xi}^2} \beta_h \| f \|_{V^\xi}. \tag{17}
\]

**Proof** See Appendix A.3. □

To establish the equivalence between the mixed system (4) and the Petrov–Galerkin statement (2), let us define the operators \( A : X \to \mathcal{L}(\hat{V}, \hat{V}^*) \) and \( B \equiv B_h \in \mathcal{L}(U_h; \hat{V}^*) \) by:

\[
A(\xi)\hat{v} := a(\xi; \hat{v}, \cdot) \in \hat{V}^*, \quad \forall \xi \in X, \forall \hat{v} \in \hat{V};
\]

\[
Bw_h := b(w_h, \cdot) \in \hat{V}^*, \quad \forall w_h \in U_h. \tag{18a}
\]

Note that the state equations (4a)–(4b) can then be written as follows:

\[
A(\xi)r + Bw_h = f \quad \text{in } \hat{V}^*, \tag{19a}
\]

\[
B^*r = 0 \quad \text{in } (U_h)^*. \tag{19b}
\]

**Proposition 2.10 (Equivalent Petrov–Galerkin problem)**

Assume the conditions of Proposition 2.9, including the well-posedness condition (15b). Instead of (15a), assume the stronger hypothesis (full inf-sup, instead of just on the kernel):

\[
\inf_{v_1 \in \hat{V}} \sup_{v_2 \in \hat{V}} \frac{a(\xi; v_1, v_2)}{\| v_1 \| \| v_2 \|_V} \geq \alpha_h, \tag{20a}
\]

\[
\left\{ v_2 \in \hat{V} : a(\xi; v_1, v_2) = 0, \forall v_1 \in \hat{V} \right\} = \{0\}. \tag{20b}
\]

Let the test space \( \mathcal{V}_h(\xi) \) be given by:

\[
\mathcal{V}_h(\xi) = \left\{ v \in V \left| A(\xi)^* v = B w_h \text{ for some } w_h \in U_h \right. \right\}. \tag{21}
\]

Then the state problem (4) is equivalent to the Petrov–Galerkin problem (2) with \( \mathcal{V}_h(\xi) \) given by (21). □

**Proof** See Appendix A.4. □

Finally, we now present (differentiability) conditions on \( \xi \mapsto A(\xi) \) that guarantee the (differentiability) requirements on \( \xi \mapsto S_h(\xi) \) in Theorem 2.B and Corollary 2.4. Once in place, existence of (quasi)-minimizers and quasi-optimal convergence follow immediately for the constrained control problem.

To anticipate the connection between derivatives \( A' \) and \( S'_h \) (as well as \( R'_h \)),\(^{20}\) note that a formal differentiation of (19) (with \( r = R_h(\xi) \) and \( u_h = S_h(\xi) \)) with respect to \( \xi \) in the

\[\text{Recall that the Gâteaux derivative of, e.g., } A \text{ at } \xi \in X \text{ in the direction } \eta \in X \text{ is given by } A'(\xi)\eta = \lim_{t \to 0} \frac{A(\xi + t\eta) - A(\xi)}{t}, \text{ provided the limit exists in } \mathcal{L}(\hat{V}, \hat{V}^*). \text{ If the map } \eta \mapsto A'(\xi)\eta \text{ is linear and continuous from } X \to \mathcal{L}(\hat{V}, \hat{V}^*), \text{ then } A \text{ is Gâteaux differentiable at } \xi \in X.\]

---

\(^{20}\)Recall that the Gâteaux derivative of, e.g., \( A \) at \( \xi \in X \) in the direction \( \eta \in X \) is given by \( A'(\xi)\eta = \lim_{t \to 0} \frac{A(\xi + t\eta) - A(\xi)}{t} \), provided the limit exists in \( \mathcal{L}(\hat{V}, \hat{V}^*) \). If the map \( \eta \mapsto A'(\xi)\eta \) is linear and continuous from \( X \to \mathcal{L}(\hat{V}, \hat{V}^*) \), then \( A \) is Gâteaux differentiable at \( \xi \in X \).
direction \( \eta \in \mathbb{X} \) yields:

\[
A(\xi) R_h(\xi) \eta + BS'_h(\xi) \eta = -A'(\xi) R_h(\xi) \eta \quad \text{in } \hat{V}^*,
\]

\[
B^* R'_h(\xi) \eta = 0 \quad \text{in } (\mathbb{U}_h)^*.
\]

One may therefore expect that suitable conditions on \( A(\cdot) \) will imply desired conditions on \( S_h(\cdot) \) (and \( R_h(\cdot) \)):

**Proposition 2.11 (State differentiability)**

Let \( R_h(\cdot) \) and \( S_h(\cdot) \) be the state operators as defined in (11), and let \( A(\cdot) \) be as defined in (18a). Assume the conditions of Proposition 2.9, including the well-posedness conditions (15). Then, the following statements hold true:

(i) If \( A(\cdot) \) has a Gâteaux derivative at \( \xi \in \mathbb{X} \) in the direction \( \eta \in \mathbb{X} \), then \( R_h(\cdot) \) and \( S_h(\cdot) \) have a Gâteaux derivative at \( \xi \) in the direction \( \eta \).

(ii) If \( A(\cdot) \) is Gâteaux-differentiable at \( \xi \), then so are \( R_h(\cdot) \) and \( S_h(\cdot) \).

(iii) If \( A(\cdot), A'(\cdot) \) and \( \sigma_h^{-1}(\cdot) \) are uniformly bounded on \( \mathbb{X} \), then \( R'_h(\cdot) \) and \( S'_h(\cdot) \) are also uniformly bounded on \( \mathbb{X} \).

(iv) Additionally, if \( A'(\cdot) \) is Lipschitz continuous, then \( R'_h(\cdot) \) and \( S'_h(\cdot) \) are Lipschitz continuous as well. \( \square \)

**Proof** See Appendix A.5.

**Corollary 2.12 (Constrained problem: (Quasi-)minimizers & quasi-optimality)**

Let \( J(\omega, \xi) = J_1(\omega) + \alpha j_2(\xi) \) as in (5) with \( Q \in \mathcal{L}(\mathbb{U}; \mathbb{Z}) \). Let the associated \( j(\cdot) \) be as in (12). Under the conditions of Propositions 2.9 and 2.11, and assuming \( \alpha \) is sufficiently large, the statements (i), (ii) and (iii) of Theorem 2.A hold true. In other words, the constrained control problem (10) has a quasi-minimizer in \( \mathcal{M}_n \) that converges quasi-optimally to the unique minimizer in \( \mathbb{X} \). \( \square \)

**Proof** The results of Propositions 2.9 and 2.11, together with \( \alpha \) sufficiently large, are the assumptions of Theorem 2.B, whose results are the assumptions of Theorem 2.A. \( \square \)

### 3 Conforming weak formulations with suitable control

In this section, we study various weighted versions of conforming weak formulations, viz., least-squares, Galerkin and minimal-residual formulations. The aim is to propose suitable \( \xi \)-dependent weighting within the weak forms, in order to be able to prove the assumptions of Propositions 2.9 and 2.11. By Corollary 2.12, we can then conclude that the corresponding constrained neural-control problem has desired properties (existence of quasi-minimizers and quasi-optimal convergence).

In what follows, we often consider a positive weight function \( \omega \). We shall use the notation \( \varpi := 1/\omega \) to indicate the (multiplicative) inverse of \( \omega \).
3.1 Weighted least-squares formulations

Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Let $B : \mathbb{H}_B \to L^2(\Omega)$ be a linear differential operator in strong form, where $\mathbb{U} = \mathbb{H}_B$ denotes the graph space

$$\mathbb{H}_B := \left\{ w \in L^2(\Omega) \left| Bw \in L^2(\Omega) + \text{boundary conditions} \right. \right\}.$$ 

We further assume that $\mathbb{H}_B$ is a Hilbert space when endowed with the inner product

$$(w_1, w_2)_{\mathbb{H}_B} := (w_1, w_2)_{L^2(\Omega)} + (Bw_1, Bw_2)_{L^2(\Omega)}, \hspace{1cm} \forall w_1, w_2 \in \mathbb{H}_B,$$

and that $B$ is boundedly invertible from $\mathbb{H}_B$ onto $\mathbb{V} := L^2(\Omega) =: \mathbb{V} = \hat{\mathbb{V}}$.

Given $f \in L^2(\Omega)$, a positive weight function $\omega : L^2(\Omega) \to L^\infty(\Omega)$, a control $\xi \in \mathbb{X} = L^2(\Omega)$, and a conforming discrete finite element space $\mathbb{U}_h \subseteq \mathbb{H}_B$, we aim to find $u_h \equiv S_h(\xi) \in \mathbb{U}_h$, which is the solution of the weighted least-squares problem:

$$u_h = \arg \min_{w_h \in \mathbb{U}_h} \frac{1}{2} \left\| \sqrt{\omega(\xi)}(f - Bw_h) \right\|^2_{L^2(\Omega)}.$$

The optimality condition of such a minimizer is:

$$\left( \omega(\xi)(f - Bw_h), Bw_h \right)_{L^2(\Omega)} = 0, \hspace{1cm} \forall w_h \in \mathbb{U}_h. \hspace{1cm} (22)$$

In particular, notice that we can directly identify the test space in (2) as $\mathbb{V}_h(\xi) = \omega(\xi)B\mathbb{U}_h = \{ v \in L^2(\Omega) \mid v = \omega(\xi)Bw_h \text{ for some } w_h \in \mathbb{U}_h \}$.

To establish the connection with the general mixed system (4), we set $r = \omega(\xi)(f - Bw_h)$ so that (22) is equivalent to:

$$\begin{align*}
\left( \omega(\xi)r, v \right)_{L^2(\Omega)} + (Bw_h, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}, \hspace{1cm} \forall v \in \mathbb{V}, \hspace{1cm} (23a) \\
(Bw_h, r)_{L^2(\Omega)} &= 0, \hspace{1cm} \forall w_h \in \mathbb{U}_h. \hspace{1cm} (23b)
\end{align*}$$

Thus, in this case the bilinear forms $a(\xi; \cdot, \cdot) \in \mathcal{L}(\mathbb{V} \times \mathbb{V}; \mathbb{R})$ and $b(\cdot, \cdot) \in \mathcal{L}(\mathbb{H}_B \times \mathbb{V}; \mathbb{R})$ in (4) are given by

$$\begin{align*}
a(\xi; v_1, v_2) &:= \left( \omega(\xi)v_1, v_2 \right)_{L^2(\Omega)}, \hspace{1cm} \forall v_1, v_2 \in \mathbb{V} = L^2(\Omega), \hspace{1cm} (24a) \\
b(w, v) &:= (Bw, v)_{L^2(\Omega)}, \hspace{1cm} \forall w \in \mathbb{H}_B, \forall v \in \mathbb{V}. \hspace{1cm} (24b)
\end{align*}$$

Proposition 3.1 (Weighted least squares) Let $\varpi : L^2(\Omega) \to L^\infty(\Omega)$ be a differentiable map, such that for some positive constants $\varpi_{\text{min}}, \varpi_{\text{max}}, \varpi'_{\text{min}}$, and $\varpi_L$, the application $\varpi(\cdot)$ satisfies

- $\varpi_{\text{min}} \leq \varpi(\xi) \leq \varpi_{\text{max}}$, for all $\xi \in L^2(\Omega)$;
- $\|\varpi'(\xi)\|_{L^2(\Omega)} \leq \varpi'_{\text{min}}$, for all $\xi \in L^2(\Omega)$;
- $\|\varpi'(\xi_1) - \varpi'(\xi_2)\|_{L^2(\Omega)} \leq \varpi_L\|\xi_1 - \xi_2\|_{L^2(\Omega)}$, for all $\xi_1, \xi_2 \in \mathbb{R}$.

Then, the following statements hold true:

(i) The bilinear forms in (24) satisfy the inf-sup conditions (15), and thus the mixed problem (23) is well-posed.

(ii) The state operator $S_h(\cdot) (\equiv u_h)$ of the mixed problem (23) is uniformly bounded on $\mathbb{X} = L^2(\Omega)$ and differentiable.
(iii) The derivative $S'_h(\cdot)$ is uniformly bounded on $X = L^2(\Omega)$ and Lipschitz continuous. □

Proof See Appendix A.6 ■

Remark 3.2 (Neural control of weighted least squares) Proposition 3.1 guarantees that the conditions of Propositions 2.9 and 2.11 are satisfied, hence Corollary 2.12 applies to the neural optimization of the above weighted least-squares formulation. □

3.2 Weighted Galerkin formulations

Consider a Hilbert space $U = V$ on $\Omega \subset \mathbb{R}^d$ and a bilinear form $b \in L^2(V \times V; \mathbb{R})$ satisfying (for some constant $\beta > 0$) the following conditions

\[
\sup_{v \in V} \frac{b(w, v)}{\|v\|_V} \geq \beta \|w\|_V, \quad \forall w \in V, \quad (25a)
\]

\[
\left\{ v \in V : b(w, v) = 0, \forall w \in V \right\} = \{0\}. \quad (25b)
\]

Given $f \in V^*$, the well-known Babuška–Brezzi theory (see, e.g., [19]) ensures the existence of an unique $u \in V$ such that

\[
b(u, v) = f(v), \quad \forall v \in V. \quad (26)
\]

Now, given a weight function $\omega : L^2(\Omega) \to W_+$ (the space $W_+$ will be clarified later), a control $\xi \in X = L^2(\Omega)$, and a conforming discrete subspace $U_h \subset V$, we consider the following weighted-Galerkin discretization of problem (26):

\[
\begin{cases}
\text{Find } u_h \equiv S_h(\xi) \in U_h : \\
b(u_h, \omega(\xi)v_h) = f(\omega(\xi)v_h), \quad \forall v_h \in U_h.
\end{cases} \quad (27)
\]

Notice that one can directly identify the test space in (2) as $V_h(\xi) = \omega(\xi)\mathbb{U}_h = \{v \in V | v = \omega(\xi)w_h \text{ for some } w_h \in \mathbb{U}_h\}$. We will show next that problem (27) admits also an equivalent mixed formulation of the type (4), and therefore it fits the abstract setting of Section 2.

First, we need to provide sense to the weighted object $\omega(\xi)v_h \in V$. Thus, we further consider an abstract Banach space $W = W(\Omega)$ of measurable functions on $\Omega$, such that for any $w \in W$, the multiplication operator $M_w : V \to V$ given by

\[M_w v := wv, \quad \forall v \in V,\]

is a well-defined linear and continuous map.

Example 3.3 (Multiplication in $H^1$) Let $V = H^1(\Omega)$. Then it is easy to see that the Sobolev space $W = W^{1,\infty}(\Omega)$ is a space of functions for which the multiplicative operator $M_w : H^1(\Omega) \to H^1(\Omega)$ is a well-defined linear and continuous map, for all $w \in W^{1,\infty}(\Omega)$. The latter is also true for Hilbert spaces $V \subset L^2(\Omega)$ containing at most first-order (weak) derivatives in $L^2(\Omega)$ (e.g., first-order graph spaces).

A particular subset of interest for us will be

\[W_+ := \left\{ w \in W | \exists w_{\text{min}} > 0 \text{ for which } w_{\text{min}} \leq w(x) \leq \frac{1}{w_{\text{min}}}, \forall x \in \Omega \right\}.\]

Notice that $\frac{1}{w} \in W_+$ iff $w \in W_+$. We can then define $M_w^{-1} := M_{\frac{1}{w}}$, which is justified by the fact that

\[M_w^{-1}(M_w v) = v = M_w(M_w^{-1} v), \quad \forall v \in V. \quad (28)\]
The adjoint operators of $M_w$ and $M_w^{-1}$ will be denoted by $M_w^*$ and $M_w^{-*}$ respectively. Using the relations (28) it is straightforward to see that the adjoint operators satisfy
\[
M_w^{-*}(M_w^* \ell) = \ell = M_w^*(M_w^{-*} \ell), \quad \forall \ell \in V^*.
\] (29)

We translate problem (27) into operator notation by means of the operator $B \in \mathcal{L}(V; V^*)$ such that $\forall \varpi \mapsto Bu := b(w, \cdot) \in V^*$. Notice that such an operator is invertible thanks to conditions (25). Problem (27) translates into finding $u_h \equiv S_h(\varpi) \in V_h$ such that
\[
\langle Bu_h, M_w^{-1} v_h \rangle = \langle f, M_w^{-1} v_h \rangle, \quad \forall v_h \in V_h.
\]

Hence, by means of the adjoint relation we get
\[
\langle M_w^{-*}(f - Bu_h), v_h \rangle = 0, \quad \forall v_h \in V_h.
\] (30)

Since $B$ is invertible, so is $B^* : V \rightarrow V^*$ defined by $\forall \varpi \mapsto b(\cdot, v) \in V^*$. Therefore, there exists a unique $r \in V$ such that $B^* r = M_w^{-*}(f - Bu_h)$ in $V^*$. Thus, multiplying this last equation by $M_{w(\varpi)}$, using (29), (30), and the definition of $r \in V$, we arrive to the mixed form
\[
\begin{aligned}
\begin{cases}
\langle B^* r, M_{w(\varpi)} v \rangle + b(u_h, v) = f(v), & \forall v \in V, \\
b(v_h, r) = 0, & \forall v_h \in V_h.
\end{cases}
\end{aligned}
\]
(31a)\hspace{1cm} (31b)

Observe that (31) has the structure of (4) for $\bar{V} := V = \mathbb{U}; \mathbb{U}_h := V_h$; and
\[
a(\varpi; r, v) := \langle B^* r, M_{w(\varpi)} v \rangle = b(\varpi(\xi), v, r).
\]

The next proposition establishes a sufficient condition for the well-posedness of (31), or equivalently (27).

**Proposition 3.4 (Weighted Galerkin)**

Let $b \in \mathcal{L}(V \times V; \mathbb{R})$ be a bilinear form satisfying the (inf-sup) conditions (25). Consider a conforming discrete subspace $V_h \subset V$ and let
\[
K := \left\{ v \in V : b(v_h, v) = 0, \forall v_h \in V_h \right\}.
\] (32)

Let $\varpi : L^2(\Omega) \rightarrow \mathbb{W}_+$ be a weight function such that
\[
|b(\varpi(\xi), v) v| \geq c_h(\xi)\|v\|^2_c, \quad \forall v \in K,
\]
(33)

for some positive function $c_h(\cdot) > 0$. Then, the following statements hold true:

(i) For any $f \in V^*$ and $\xi \in L^2(\Omega)$, problems (27) and (31) are well-posed.

(ii) If there exist uniform constants $\alpha > 0$ and $\varpi_\infty > 0$ such that $c_h(\xi) \geq \alpha$ and $\|\varpi(\xi)\|_W \leq \varpi_\infty$ for all $\xi \in L^2(\Omega)$, then the solution $u_h \equiv S_h(\cdot)$ to problems (27) and (31) is uniformly bounded on $\mathbb{H} = L^2(\Omega)$.

(iii) Additionally, if $\varpi(\cdot)$ is differentiable, then $S_h(\cdot)$ is also differentiable. Moreover, if $\varpi(\cdot)$ is uniformly bounded and Lipschitz-continuous, then also $S_h(\cdot)$ is uniformly bounded and Lipschitz-continuous.

\[\blacksquare\]

**Proof** See Appendix A.7.
Remark 3.5 (Neural control of weighted Galerkin) Proposition 3.4 guarantees that the conditions of Propositions 2.9 and 2.11 are satisfied, hence Corollary 2.12 applies to the neural optimization of the above weighted least-squares formulation.

□

Remark 3.6 (Inconvenient condition for weighted Galerkin) While for the weighted least-squares method the conditions on the weight are explicit (recall Proposition 3.1), for weighted Galerkin the condition (33) is problem dependent. Furthermore, Example 3.7 shows it may require inconvenient constraints on $\xi$. It seems therefore much more convenient to have neural control of least-squares formulations, or of dual minimal-residual formulations, as we will see in Section 3.3.

□

Example 3.7 (Weighted Galerkin for Laplacian) Let us illustrate the difficulty of condition (33) using the elementary Laplacian. Let $V = H^1_0(\Omega)$, $b(u, v) = \int_\Omega \nabla u \cdot \nabla v$ for all $u, v \in V$. Let $\varpi \in W^{1,\infty}(\Omega)$ such that $\varpi(x) \geq \varpi_{\min} > 0$ for all $x \in \Omega$.

In particular, let $\varpi(x) = \varpi_{\min} + cy \cdot (x - x_0)$ for some $c \in \mathbb{R}$ and $y, x_0 \in \mathbb{R}^d$ such that $y \cdot (x - x_0) \geq 0$ for all $x \in \Omega$. Then

$$b(\varpi v, v) = \int_\Omega (\varpi |\nabla v|^2 + c \varpi v \cdot \nabla v) \quad (34)$$

Consider any $v \in V$ so that $y \cdot \int_\Omega [(x - x_0) |\nabla v|^2 + v \nabla v] < 0$. Then there is a $c > 0$ such that $b(\varpi v, v) = 0$. This shows that (33) can not be satisfied in general without additional conditions on $\varpi$.

Indeed, from (34) a sufficient condition can be obtained. First notice that, for any $\varpi \in W^{1,\infty}(\Omega)$ such that $\varpi(x) \geq \varpi_{\min}$ for all $x \in \Omega$,

$$b(\varpi v, v) \geq \varpi_{\min} \|\nabla v\|_{L^2(\Omega)}^2 - \|\nabla \varpi\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

$$\geq (\varpi_{\min} - C_{\Omega} \|\nabla \varpi\|_{L^2(\Omega)}) \|\nabla v\|_{L^2(\Omega)}^2,$$

where a Poincaré inequality was used. Therefore, the constraint $C_{\Omega} \|\nabla \varpi\|_{L^2(\Omega)} < \varpi_{\min}$ is sufficient to guarantee (33). Unfortunately, since $\varpi = \varpi(\xi)$, such a condition translates into a constraint on $\nabla \xi$, which may be very inconvenient to impose in practice.

□

3.3 Weighted discrete-dual minimal residual formulations

Let $U_h \subset U$ and $V_h \subset V$ be discrete subspaces, and assume:

$$\begin{align}
\dim(V_h) &> \dim(U_h), \\
\exists \beta_h > 0 : \inf_{w_h \in U_h} \sup_{v_h \in V_h} \frac{b(w_h, v_h)}{\|w_h\|_U \|v_h\|_V} &\geq \beta_h. \quad \text{(35a)}
\end{align}$$

For each $\xi \in X$, we consider an equivalent (weighted) inner product $(\cdot, \cdot)_{V, \xi}$ on $V$, i.e., such that its induced norm

$$\forall v \mapsto \|v\|_{V, \xi} := \sqrt{(v, v)_{V, \xi}} \quad \text{satisfies (16)}.$$ 

The minimal-residual method that we consider is then: Given $\xi \in X$, find $r_h \in V_h$ and $u_h \equiv S_h(\xi) \in U_h$ such that

$$\begin{align}
(r_h, v_h)_{V, \xi} + b(u_h, v_h) &= f(v_h), \quad \forall v_h \in V_h, \quad \text{(36a)} \\
b(w_h, r_h) &= 0, \quad \forall w_h \in U_h. \quad \text{(36b)}
\end{align}$$
This has the structure of (4) for \( \tilde{V} := V_h \) and \( a(\xi; r, v) := (r, v)_{V, \xi} \).

As shown in [36, Theorem 4.1], the mixed formulation (36) is equivalent to minimizing the residual as measured by a discrete-dual norm:

\[
\| \cdot \|_{V, \xi} = \sup_{v_h \in V_h} \frac{|f(v_h) - b(w_h, v_h)|}{\|v_h\|_{V, \xi}},
\]

Because \((\cdot, \cdot)_{V, \xi}\) and \(\| \cdot \|_{V, \xi}\) depend on \(\xi\), we refer to the above as a weighted discrete-dual minimal residual formulations.

**Proposition 3.8** Let the bilinear form \(b(\cdot, \cdot) \in \mathcal{L}(U \times V; \mathbb{R})\) and \((U_h, V_h)\) satisfy (35). Consider a parametrized set of equivalent inner-products

\[
\{ (\cdot, \cdot)_{V, \xi} \in \mathcal{L}(V \times V; \mathbb{R}) : \xi \in X \},
\]

whose induced norms \(\| \cdot \|_{V, \xi}\) satisfy (16) for some equivalence constants \(C_{1, \xi} > 0\) and \(C_{2, \xi} > 0\). Let \(A : X \rightarrow \mathcal{L}(V, V^*)\) be defined by \(A(\xi)v := (v, \cdot)_{V, \xi} \in V^*\), for all \(\xi \in X\) and \(v \in V\). Then, the following statements hold true:

(i) The mixed discrete formulation (36) is well-posed.

(ii) If there exist uniform constants \(\tilde{C}_1 > 0\) and \(\tilde{C}_2 > 0\) such that \(C_{1, \xi} \geq \tilde{C}_1\) and \(C_{2, \xi} \leq \tilde{C}_2\) for all \(\xi \in X\), then the solution \(u_h \equiv S_h(\cdot)\) to problems (36) and (37) is uniformly bounded on \(X\).

(iii) Additionally, if \(A(\cdot)\) is differentiable, then \(S_h(\cdot)\) is also differentiable. Moreover, if \(A'(\cdot)\) is uniformly bounded and Lipschitz-continuous, then also \(S'_h(\cdot)\) is uniformly bounded and Lipschitz continuous. \(\square\)

**Proof** See Appendix A.8.

**Remark 3.9 (Neural control of weighted residual minimization)** Proposition 3.8 guarantees that the conditions of Propositions 2.9 and 2.11 are satisfied, hence Corollary 2.12 applies to the neural optimization of the above weighted minimal-residual formulation. \(\square\)

**Example 3.10 (Weighted \(H^1(\Omega)\) inner-product)** Consider a differentiable weight function \(\omega : L^2(\Omega) \rightarrow L^\infty(\Omega)\), such that for some given constants \(\omega_{\text{max}} > \omega_{\text{min}} > 0\), and for all \(\xi \in L^2(\Omega)\), we have \(\omega_{\text{min}} \leq \omega(\xi) \leq \omega_{\text{max}}\). We further assume that \(\omega'(\cdot)\) is uniformly bounded and Lipschitz-continuous.

Given \(\xi \in L^2(\Omega)\), define the following weighted \(H^1(\Omega)\) inner-product:

\[
(v_1, v_2)_{H^1, \xi} := \int_\Omega \omega(\xi) \nabla v_1 \cdot \nabla v_2 + \int_\Omega v_1 v_2.
\]

Observe that

\[
\min\{1, \omega_{\text{min}}\}\|v\|_{H^1}^2 \leq (v, v)_{H^1, \xi} \leq \max\{1, \omega_{\text{max}}\}\|v\|_{H^1}^2, \quad \forall v \in H^1(\Omega).
\]

Hence, statement (ii) of Proposition 3.8 is satisfied with \(\tilde{C}_1 = \sqrt{\min\{1, \omega_{\text{min}}\}}\) and \(\tilde{C}_2 = \sqrt{\max\{1, \omega_{\text{max}}\}}\).

On the other hand, given \(\xi \in L^2(\Omega)\), the operator \(A(\xi)\) is defined by the following action:

\[
A(\xi)v = (\omega(\xi) \nabla v, \nabla(\cdot))_{L^2(\Omega)} + (v, v)_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega).
\]
Therefore, is easy to see that $A(\cdot)$ satisfies the statement (iii) of Proposition 3.8. Indeed, observe that $A'(\xi)\eta = (|\omega'(\xi)\eta|\nabla v, \nabla (\cdot))_{L^2(\Omega)}$ for any direction $v \in L^2(\Omega)$. Moreover, $A'(\cdot)$ is uniformly bounded and Lipschitz-continuous, since $\omega'(\cdot)$ is uniformly bounded and Lipschitz continuous.

Of course, for any $v_1, v_2 \in H^1(\Omega)$, we may have chosen the following equivalent inner-products where we can prove similar results:

$$(v_1, v_2)_{H^1, \xi} := (\nabla v_1, \nabla v_2)_{L^2(\Omega)} + (\omega(\xi)v_1, v_2)_{L^2(\Omega)},$$

$$(v_1, v_2)_{H^1, \xi} := (\omega(\xi)\nabla v_1, \nabla v_2)_{L^2(\Omega)} + (\omega(\xi)v_1, v_2)_{L^2(\Omega)}.$$ 

Also, for $H^0_0(\Omega)$, we could consider just $(\omega(\xi)\nabla v_1 \cdot \nabla v_2)_{L^2(\Omega)}$. \hfill $\square$

# 4 Numerical results

In this section, we consider numerical examples for the advection–reaction PDE in 1-D and 2-D. We consider both weighted least squares and weighted residual minimization.

We construct weight functions $\omega : L^2(\Omega) \to L^\infty(\Omega)$ that are based on algebraic expressions, i.e., for which $\omega(\xi)(x) = \omega(\xi(x))$ for $x \in \Omega$. These are convenient expressions, but the price to pay is that $\omega' : L^2(\Omega) \to L(\omega(L^2(\Omega); L^\infty(\Omega)))$ can not be Lipschitz. We do not believe this to have a major impact, and we leave the construction of more complicated weight functions for future investigation. While using algebraic weight functions, we have not observed any undesirable numerical effects. In fact, our results in Section 4.2 do demonstrate quasi-optimal convergence, as expected in our current theory.

## 4.1 Quantities of interest (point values)

### 4.1.1 Weighted least-squares approach

Let $\Omega = (0, 1) \subset \mathbb{R}$ and $r > 0$. Consider the advection-reaction problem

$$
\begin{cases}
  u' + ru &= r \text{ in } \Omega, \\
  u(0) &= 0.
\end{cases}
\tag{38}
$$

Since the exact solution to (38) is $u(x) = 1 - \exp(-rx)$, we observe that $u(x) \to 1$ when $r \to +\infty$, for all $x > 0$. Hence, for $r > 0$ sufficiently large, the exact solution has a boundary layer in the neighborhood of $x = 0$.

Let $\mathbb{U}_h \subset H^1_0(\Omega) := \{ w \in H^1(\Omega) : w(0) = 0 \}$ be the conforming subspace of continuous piecewise linear functions on the uniform mesh of $N$ elements of size $h = 1/N$. We use the weighted least squares method from (22), with weight function:

$$
\omega(\xi(x)) := 1 + \frac{M}{1 + \exp(-\xi(x))}, \quad M > 0.
\tag{39}
$$

It is well known that the standard least-squares solution (i.e., the one with $\omega(\xi) \equiv 1$) will exhibit overshoots around the boundary layer. Aiming to remedy this situation, we choose a cost functional that measures the distance to the exact solution at the point value $x = h$. In fact, we consider

$$
j(\xi) := \frac{1}{2} \left( u(h) - u_{h,\xi}(h) \right)^2 + \frac{\alpha}{2} \| \xi \|_{L^2}^2, \quad \alpha \geq 0.
$$

\[\text{Weighted Galerkin is not considered in view of Remark 3.6.}\]
Let $M_8$ be the set of neural network functions with one hidden layer, 8-neurons, and ReLU activation, i.e.,

$$M_8 := \left\{ \eta_8(x) = \sum_{j=1}^{8} c_j \text{ReLU}(W_j x + b_j) \mid c_j, W_j, b_j \in \mathbb{R} \right\}.$$ (40)

We then consider the neural optimization of $j(\cdot)$; see Definition (2.2).

For our first experiment, we choose a finite element space $U_h$ consisting of $N = 16$ elements of size $h = 1/16$. We set $r = 160$ and $\alpha = 0$. We compute least-squares approximations for several configurations of the weight function (39), varying the $M$ constant. Figure 1 (left) shows that the weight needs to have enough room for variability ($M = 100$) in order to pull down the cost functional to zero. Figure 1 (right) shows that our strategy is effective in reducing the overshoots of the finite element solution.

For the second experiment of this section, we fix $M = 100$ and we investigate variations of the $\alpha$-parameter. Figure 2 (left) suggest that the $L^2$-norm of $\xi$ has to be able to reach high values (case when $\alpha = 0$) in order to pull down to zero the cost functional. This is also related to allowing the weight to have more variability. Figure 2 (right) shows the impact of $\alpha$ reducing the overshoots of the finite element solution (the smaller $\alpha$, the better).

### 4.1.2 Weighted discrete-dual residual minimization approach

This experiment has exactly the same configuration of the previous experiment in Section 4.1.1, except that $S_h(\xi)$ is computed with the discrete-dual minimal residual methodology. First, the approximation (trial) space $U_h \subset L^2(\Omega)$ corresponds to the space of piecewise constants functions over the mesh. Additionally, we make use of a discrete test space $V_h \subset H^1_0(\Omega) := \{ v \in H^1(\Omega) : v(1) = 0 \}$ consisting in conforming piecewise linear functions over the refined uniform mesh of $2N = 32$ elements. The weighted discrete-dual residual minimization formulation that computes $S_h(\xi)$ is as follows: Find $r_h \in V_h$ and $u_h \equiv S_h(\xi) \in \bar{U}_h$ such that

$$\begin{aligned}
\int_0^1 \omega(\xi) r_h' v_h' - \int_0^1 u_h (v_h' - r v_h) &= r \int_0^1 v_h , \quad \forall v_h \in V_h , \\
- \int_0^1 w_h (r_h' - r r_h) &= 0 , \quad \forall w_h \in U_h .
\end{aligned}$$ (41)
As in the previous Section 4.1.1, the computation of $S_h$ is carried out for several configurations of the weight function $\omega(\xi)$ (see (39)), varying its $M$ constant. Figure 3 (left) shows that larger values of $M$ allow to pull down faster the cost functional in the training procedure. Figure 3 (right) shows how the overshoots of the finite element solutions are controlled.

The second experiment investigates variations of the $\alpha$-parameter. Figure 4 (left) suggest that the smaller $\alpha$, the better for faster minimization of $j(\cdot)$. Figure 4 (right) shows the impact of $\alpha$ reducing the overshoots of the finite element solution.

### 4.2 Convergence of artificial neural networks

Let $\Omega := (0,1) \subset \mathbb{R}$ be a one-dimensional domain and consider the simple advection problem

$$
\begin{cases}
u' = f & \text{in } \Omega, \\
u(0) = 0,
\end{cases}
$$

with $f(x) := \pi \sin(\pi x)$. Notice the exact solution to (42) is $u(x) = 1 - \cos(\pi x)$.

Let $H_0^1(\Omega) := \{w \in H^1(\Omega) : w(0) = 0\}$ and let $U_h \subset H_0^1(\Omega)$ be the finite element subspace of continuous piecewise linear functions on a uniform mesh consisting of $N$ elements of size
Neural control of discrete weak formulations

Figure 4: Point value control for weighted discrete-dual residual minimization. Optimization of the cost functional for several values of $\alpha$ (left). Overshoot control of the discrete solutions (right).

$h = 1/N$. We consider the weighted least-squares formulation

$$
\begin{cases}
\text{Find } u_h \equiv S_h(\xi) \in U_h : \\
\int_0^1 \omega(\xi) (f - u'_h) w'_h = 0, \quad \forall w_h \in U_h ,
\end{cases}
$$

where the weight function is such that

$$
\omega(\xi(x)) := \frac{1}{2} + \frac{2}{1 + \exp(-\xi(x))} .
$$

(43)

Let $\mathcal{M}_n$ be the set of neural network functions with one hidden layer, $n$-neurons, and ReLU activation, i.e.,

$$
\mathcal{M}_n := \left\{ \eta_n(x) = \sum_{j=1}^n c_j \text{ReLU}(W_j x + b_j) \left| c_j, W_j, b_j \in \mathbb{R} \right. \right\}.
$$

Consider the cost functional

$$
J(\xi) := \frac{1}{2} \int_0^1 \left( f(x) - u'_h,\xi(x) \right)^2 \mathrm{d}x ,
$$

(45)

with $\tilde{\omega}(x) = 1 + \sin(\pi x/2)$.

Since the minimization of the cost functional and the discrete problem (43) are both weighted least-squares formulations of the same problem (42), we expect that $\omega(\xi_n) \to \tilde{\omega}$ as $n \to +\infty$, which is confirmed in Figure 5 (left). Additionally, solving for $\xi_n$ we get (see Figure 5 (right))

$$
\xi_n(x) \longrightarrow \xi(x) = - \ln \left( \frac{2}{\sin(\pi x/2) + 1/2} - 1 \right), \quad \text{as } n \to +\infty .
$$

To initialize the minimization algorithm, we have chosen $\xi_n(0) \in \mathcal{M}_n$ as the neural network function that (linearly) interpolates $\tilde{\xi}$ on a uniform mesh of $n - 1$ subintervals of $\Omega$ (i.e., having $n$ uniformly distributed nodal points). The space $U_h$ has been fixed to $N = 16$ uniform elements.

In Figure 6, we plot the error $\|\xi - \xi_n\|_{L^2}$, which confirms quasi-optimal convergence behaviour; indeed the asymptotic rate is $O(n^{-1/2})$, which is expected for our single-hidden-layer ReLU neural network approximations (continuous piecewise-linear polynomials).
Neural control of discrete weak formulations

Figure 5: Convergence of $\omega(\xi_n) \to \bar{\omega}$ (left) and $\xi_n \to \bar{\xi}$ (right), as $n \to +\infty$.

Figure 6: $L^2$ error of $\xi_n$ as $n \to +\infty$.

4.3 $L^1$-based controls

We now consider numerical experiments that incorporate a stabilization mechanism. We note that the employed cost functionals use an $L^1$-type norm, and hence do not fit within the currently presented theory. However our numerics show that desirable quasi-minimizers have been computed.

4.3.1 Minimizing the total variation

In this section we work exactly with the same problem of the previous Section 4.1.1, but we introduce a modification in the cost functional. Instead of minimizing the distance to the exact solution of a particular point value (supervised training), we take an unsupervised approach by minimizing the total variation of $u_h$ (i.e., the $L^1$-norm of $u'_h$). Hence, we consider the cost functional:

$$j(\xi) := \|u'_h, \xi\|_{L^1} + \frac{\alpha}{2} \|\xi\|^2_{L^2}, \quad \alpha \geq 0.$$  

For a fixed value of $M = 100$, Figure 7 (left) shows the behavior of the cost functional for different values of $\alpha$, indicating that this value has to be chosen small enough to speed up the...
minimization process. Figure 7 (right) shows the quality of overshoot reduction for several values of $\alpha$.

4.3.2 Minimizing the $L^1$ residual (1D domain)

This experiment is inspired by the example of Guermond [21, Section 4.6.2]. As usual $\Omega = (0, 1) \subset \mathbb{R}$. The idea is to interpret the following overconstrained problem:

\[
\begin{aligned}
&u' + u = 1 \quad \text{in } \Omega, \\
u(0) = u(1) = 0,
\end{aligned}
\]  

(46)

as the limiting case of a vanishing viscosity regime (i.e., an equivalent problem having an extra $-\varepsilon u''$ term that vanishes as $\varepsilon \to 0^+$). Of course, the exact solution that we want to approach ($u(x) = 1 - e^{-x}$) only satisfies one of the boundary conditions. However, any discrete solution in a $H^1_0(\Omega)$-conforming space must satisfy both constrains. In this case, it is well-known that the standard least-squares solution to this problem does not deliver satisfactory results. To remedy this drawback, we propose a cost functional that mimics the $L^1$ residual minimization as proposed in [21]. Thus, our (unsupervised) cost functional will be

\[
j(\xi) := \left\| 1 - u_{h,\xi} - u_{h,\xi}^{\text{residual}} \right\|_{L^1} + \frac{\alpha}{2} \left\| \xi \right\|_{L^2}^2, \quad \alpha \geq 0.
\]

We consider the weighted least-squares formulation for $u_{h,\xi}$, solved on a uniform mesh of $N = 8$ elements. For a fixed $M = 1000$ constant in the weight function (39), we compute the discrete solution for several values of the $\alpha$-parameter. Large values of $\alpha$ allow for small values of $\left\| \xi \right\|_{L^2}$, and thus the weight becomes almost constant (close to the standard least-squares approach). On the other hand, small values of $\alpha$ allow for more variability of the weight, and thus, we observe that we can recover a discrete solution mimicking the vanishing viscosity case (see Fig. 8).
4.3.3 Minimizing $L^1$ residual (2D domain)

This is the two-dimensional version of the previous example in Section 4.3.2. Let $\Omega = (0, 1)^2 \subset \mathbb{R}^2$. For an advection field $\vec{\beta} = (1, 0)$, we consider the over-constrained problem:

$$\begin{align*}
\vec{\beta} \cdot \nabla u + u &= 1 \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \{(x_1, x_2) \in \partial \Omega : x_1 = 0 \text{ or } x_1 = 1\}. 
\end{align*}$$

(47)

We approach (47) using a coarse (and over-constrained) finite element space of piecewise linears functions of the form

$$U_h \subset \{ w \in H^1_0(\Omega) : w(0, x_2) = w(1, x_2) = 0, \forall x_2 \in [0, 1] \}.$$

We use the weighted least-squares method:

$$\begin{align*}
\int_{\Omega} [\omega(\xi)(1 - u_h - \beta \cdot \nabla u_h) (\beta \cdot \nabla w_h + w_h)] &= 0, \quad \forall w_h \in U_h, 
\end{align*}$$

(48)

using the same weight (39) with $M = 1000$. On the other hand, the cost functional $j(\cdot)$ for this case is defined as

$$j(\xi) := \|1 - u_h,\xi - \beta \cdot \nabla u_h,\xi\|_{L^1} + \frac{\alpha}{2} \|\xi\|_{L^2}, \quad \alpha \geq 0.$$

The discrete neural network space where we minimize $j(\cdot)$ will be $\mathcal{M}_8$ (see (40)). Results for the $\alpha = 0$ case are depicted in Figure 9. We observe a strong correlation with the results in [21, Figure 9].

A Proofs

A.1 Proof of Theorem 2.A

(i) Strong convexity of $j$ implies coercivity, i.e., $j(\xi) \to +\infty$ when $\|\xi\|_X \to +\infty$. Moreover, $j$ is continuous in the strong topology since it is differentiable. Additionally, we know
that convexity plus continuity implies that $j$ is weakly lower semicontinuous (see, e.g. [9, Corollary 3.9]). We thus satisfy all the hypothesis of the theorem of existence of minimizers for coercive and sequentially weakly lower semicontinuous functionals [13, Theorem 9.3-1]. Moreover, strong convexity ensures that such a (global) minimizer $\bar{\xi} \in X$ is unique. Besides, global differentiability of $j$ implies the first-order necessary optimality condition $j'(\bar{\xi}) = 0$.

(ii) We now that $j$ has a global lower bound. Thus, by the infimum property, for any $\delta_n > 0$ there must exist $\bar{\xi}_n \in M_n$ such that

$$j(\bar{\xi}_n) < \inf_{\eta_n \in M_n} j(\eta_n) + \frac{\delta_n}{2}. \quad (49)$$

(iii) Let $\bar{\xi} \in X$ be the global minimizer and let $\bar{\xi}_n \in M_n$ satisfy (8). By characterization of strong convexity we have for all $t \in (0,1)$

$$j(\bar{\xi}_n) \leq j(t\bar{\xi} + (1-t)\bar{\xi}_n) \leq tj(\bar{\xi}) + (1-t)j(\bar{\xi}_n) - \frac{\gamma}{2}(1-t)\|\bar{\xi} - \bar{\xi}_n\|_X^2.$$ 

Thus, for all $t \in (0,1)$ and $\eta_n \in M_n$ we get

$$\frac{\gamma}{2}t\|\bar{\xi} - \bar{\xi}_n\|_X^2 \leq j(\bar{\xi}_n) - j(\bar{\xi}) < j(\eta_n) - j(\bar{\xi}) + \frac{\delta_n}{2}. \quad (50)$$

On the other hand, using the facts that $j'$ is $L$-Lipschitz and $j'(\bar{\xi}) = 0$, we deduce [13, cf. proof of Thm. 7.7-3, page 488]

$$j(\eta_n) - j(\bar{\xi}) = \int_0^1 \langle j'(s\eta_n + (1-s)\bar{\xi}), \eta_n - \bar{\xi} \rangle \, ds$$

$$= \int_0^1 \langle j'(s\eta_n + (1-s)\bar{\xi}) - j'(\bar{\xi}), \eta_n - \bar{\xi} \rangle \, ds$$

$$\leq L\|\eta_n - \bar{\xi}\|_X \int_0^1 s = \frac{L}{2}\|\eta_n - \bar{\xi}\|_X^2. \quad (51)$$

Hence, combining (50) with (51), taking the limit when $t \to 1$ and the infimum over all $\eta_n \in M_n$, we get the estimate

$$\gamma \|\bar{\xi} - \bar{\xi}_n\|_X^2 < L \inf_{\eta_n \in M_n} \|\bar{\xi} - \eta_n\|_X^2 + \delta_n,$$

from which (14) is deducted.
A.2 Proof of Theorem 2.9

We proceed to prove each one of the statements.

(i) Since $Z$ and $X$ are a Hilbert spaces, the quadratic maps $Z \ni z \mapsto \frac{1}{2}\|z\|_Z^2$ and $X \ni \xi \mapsto \frac{1}{2}\|\xi\|_X^2$ are differentiable. On the other hand, $S_h$ and $Q$ are also differentiable ($Q$ is linear), and thus $j_1$ is differentiable by means of the chain rule (see, e.g. [21, Theorem 2.20]). Moreover,

$$j_1'(\eta)(\cdot) = (QS_h(\eta), QS_h'(\eta)(\cdot))_Z = (S_h'(\eta)^*QS_h(\eta), \cdot)_X.$$ 

Thus, we conclude that $j_1$ is Lipschitz since

$$\|j_1'(\eta) - j_1'(\zeta)\|_X = \|S_h'(\eta)^*Q S_h(\eta) - S_h'(\zeta)^*Q S_h(\zeta)\|_X,$$

and

$$= \|S_h'(\eta)^*Q (S_h(\eta) - S_h(\zeta))\|_X + \|(S_h'(\eta) - S_h'(\zeta))^*Q S_h(\zeta)\|_X,$$

$$\leq \|Q\|^2_{(V,Z)} (M_S^2 + L_S^2 M_S) \|\eta - \zeta\|_X,$$

where we have used the mean value theorem together with

- the boundedness of $S_h'$, with bounding constant $M_S$;
- the Lipschitzness of $S_h'$, with Lipschitz constant $L_S$;
- the boundedness of $S_h$, with bounding constant $M_S$.

Finally, by making $L_1 := \|Q\|^2_{(V,Z)} (M_S^2 + L_S^2 M_S)$, it is straightforward to see that $L_1 + \alpha$ will be a Lipschitz constant for $j$.

(ii) Just observe that

$$\langle j_1'(\eta) - j_1'(\zeta), \eta - \zeta\rangle_{X^*, X} = \langle j_1'(\eta) - j_1'(\zeta), \eta - \zeta\rangle_{X^*, X} + \alpha\|\eta - \zeta\|_X^2,$$

$$\geq (-L_1 + \alpha)\|\eta - \zeta\|_X^2.$$

Thus, $j$ is strongly convex whenever $\alpha > 0$ is sufficiently large.

A.3 Proof of Proposition 2.9

The statements (i) and (ii) are classical from Babuška–Brezzi theory (see, e.g., Ern & Guermond [20, Theorem 49.13]). To prove statement (iii) first observe that

$$\sup_{\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}} \frac{a(\xi; \mathbf{v}_1, \mathbf{v}_2)}{\|\mathbf{v}_1\|_V \|\mathbf{v}_2\|_V} \geq \frac{a(\xi; \mathbf{v}_1, \mathbf{v}_1)}{\|\mathbf{v}_1\|_V \|\mathbf{v}_1\|_V} = \frac{a(\xi; \mathbf{v}_1, \mathbf{v}_1)}{\|\mathbf{v}_1\|_V \|\mathbf{v}_1\|_V} (C_{1,\xi})^2 = (C_{1,\xi})^2,$$

which confirms $\alpha_h = (C_{1,\xi})^2$ in (15a). For the a priori bound, since $a(\xi; \cdot, \cdot)$ is an equivalent inner-product on $\mathbb{V}$, consider $\mathbf{\hat{z}} \in \mathbb{V}$ such that

$$a(\xi; \mathbf{\hat{z}}, \mathbf{\hat{v}}) = b(u_h, \mathbf{\hat{v}}), \quad \forall \mathbf{\hat{v}} \in \mathbb{V}.$$

Hence,

$$\sup_{\mathbf{\hat{v}} \in \mathbb{V}} \frac{b(u_h, \mathbf{\hat{v}})}{\|\mathbf{\hat{v}}\|_{V,\xi}} = \sup_{\mathbf{\hat{v}} \in \mathbb{V}} \frac{a(\xi; \mathbf{\hat{z}}, \mathbf{\hat{v}})}{\|\mathbf{\hat{v}}\|_{V,\xi}} = \frac{a(\xi; \mathbf{\hat{z}}, \mathbf{\hat{z}})}{\|\mathbf{\hat{z}}\|_{V,\xi}} = \frac{b(u_h, \mathbf{\hat{z}})}{\|\mathbf{\hat{z}}\|_{V,\xi}}$$

(52)
Moreover,

\[ a(\xi, \hat{r}, \hat{z}) = a(\xi, \hat{z}, \hat{r}) = b(u_h, \hat{r}) = 0. \] (53)

Next, observe that

\[ \|u_h\|_U \leq \frac{1}{\beta_h} \sup_{v \in V} \frac{b(u_h, \hat{v})}{\|\hat{v}\|_V} \leq \frac{C_{2, \xi}}{\beta_h} \sup_{v \in V} \frac{b(u_h, \hat{v})}{\|\hat{v}\|_{V, \xi}} \] (by (15) and (16))

\[ = \frac{C_{2, \xi}}{\beta_h} \frac{b(u_h, \hat{\xi})}{\|\hat{\xi}\|_{V, \xi}} = \frac{C_{2, \xi}}{\beta_h} \left( f(\hat{z}) - a(\xi, \hat{r}, \hat{z}) \right) \] (by (52) and (4))

\[ \leq \frac{C_{2, \xi}}{C_{1, \xi} \beta_h} \frac{1}{\|\hat{\xi}\|_{V, \xi}}, \] (by (16) and (53))

from which (17) can be easily deduced.

### A.4 Proof of Proposition 2.10

Let \((r, u_h) \in \hat{V} \times \mathbb{U}_h\) solves the state problem (4), or equivalently (19) in operator form. Testing with elements in \(v_h \in V_h(\xi)\) we get

\[ \langle f, v_h \rangle = \langle A(\xi)r, v_h \rangle + \langle Bu_h, v_h \rangle = \langle r, A(\xi)^*v_h \rangle + \langle Bu_h, v_h \rangle \] (by (19a))

\[ = \langle r, Bu_h \rangle + \langle Bu_h, v_h \rangle = \langle Bu_h, v_h \rangle. \] (by definition of \(V_h(\xi)\))

Thus, (2) is satisfied.

Conversely, assume \(u_h \in \mathbb{U}_h\) satisfies the Petrov–Galerkin problem (2) with \(A(\xi)^*V_h(\xi) = B\hat{U}_h\). In particular,

\[ \langle f - Bu_h, v_0 \rangle = 0 \quad \forall v_0 \in \ker A(\xi)^* = \{ v_0 \in \hat{V} \mid A(\xi)^*v_0 = 0 \}. \]

Hence, by orthogonality,\(^{22}\) there exists an \(r \in \hat{V}\) such that \(A(\xi)r = f - Bu_h\), which is (19a).

Next, let \(v_h \in \mathbb{U}_h\). Since \(A(\xi)^*\) is surjective (by (20)), there exists a \(v_{w_h} \in \hat{V}\) such that \(A(\xi)^*v_{w_h} = Bu_h\), in other words, \(v_{w_h} \in V_h(\xi)\). Therefore,

\[ \langle Bu_h, r \rangle = \langle A(\xi)^*v_{w_h}, r \rangle = \langle A(\xi)r, v_{w_h} \rangle = \langle f - Bu_h, v_{w_h} \rangle = 0, \]

which verifies (19b).

### A.5 Proof of Proposition 2.11

Let us start proving statements (i), (ii) and (iii) at the same time.

Recall the definition of the kernel space \(\mathring{K} := \ker B^* \subset \hat{V}\). For any \(\xi \in X\), consider the restricted operator \(A(\xi)|_{\mathring{K}} : \mathring{K} \rightarrow \mathring{K}^*\), as well as the restriction \(f|_{\mathring{K}} \in \mathring{K}^*\). Observe that the inf-sup condition (15) ensures that \(A(\xi)|_{\mathring{K}}\) is a boundedly invertible linear operator. Thus, given a direction \(\eta \in X\) and \(t \in \mathbb{R}\), from the first equation of the mixed system (19) (restricted to \(\mathring{K}\)) we obtain that

\[ A(\xi + t\eta)|_{\mathring{K}} R_h(\xi + t\eta) = f|_{\mathring{K}} \]

\[ A(\xi)|_{\mathring{K}} R_h(\xi) = f|_{\mathring{K}}. \] (54a)

\[ ^{22}\text{That is, } (\ker(A^*))^{-1} = \operatorname{ran} A; \text{ see, e.g., } [19, \text{Lemma C.34}]. \]
In particular, continuity of \( A(\cdot) \) implies continuity of \( R_h(\cdot) \). Moreover, using the inf-sup condition (15), it is clear that
\[
\|R_h(\cdot)\|_V \leq \frac{\|f\|_{\tilde{V}^*}}{\alpha_h(\cdot)}.
\] (55)
Next, adding the term \( A(\xi)|_\tilde{K}R_h(\xi + \eta t) \) on both sides of equation (54a), rearrange it, and subtracting equation (54b) we get
\[
R_h(\xi + \eta t) - R_h(\xi) = \left[ A(\xi)|_\tilde{K} \right]^{-1} (A(\xi)|_\tilde{K} - A(\xi + \eta t)|_\tilde{K}) R_h(\xi + \eta t),
\]
from which, if \( A'(\xi)\eta \) exists, we imply that \( R_h(\cdot) \) has a Gâteaux derivative and
\[
R'_h(\xi)\eta = - \left[ A(\xi)|_\tilde{K} \right]^{-1} A'(\xi)\eta|_\tilde{K} R_h(\xi).
\] (56)
Finally, if \( A(\cdot) \) is Gâteaux-differentiable at \( \xi \), then using the inf-sup condition (15), the boundedness of the linear operator \( A'(\xi) \), and the estimate (55), we imply
\[
\|R'_h(\xi)\|_V \leq \frac{\|A'(\xi)\|_{\tilde{V}(\tilde{\tilde{Y}}, \tilde{\tilde{V}}^*)} \|R_h(\xi)\|_V}{\alpha_h(\cdot)} \leq \frac{\|A'(\xi)\| \|f\|_{\tilde{V}^*}}{\alpha_h(\cdot)} \|\eta\|_X,
\] (57)
which proves that \( R_h(\cdot) \) is Gâteaux-differentiable at \( \xi \). Besides, if \( A'(\cdot) \) and \( \alpha_h^{-1}(\cdot) \) are uniformly bounded on \( X \), then \( R'_h(\cdot) \) is uniformly bounded on \( X \).

Now is the turn of \( S_h \). From the mixed system (19) we deduce
\[
BS_h(\xi + \eta t) = f - A(\xi + \eta t) R_h(\xi + \eta t) \\
BS_h(\xi) = f - A(\xi) R_h(\xi).
\]
Since \( B \) is boundedly invertible onto its closed range we get
\[
S_h(\xi + \eta t) - S_h(\xi) = B^{-1} \left( [A(\xi) - A(\xi + \eta t)] R_h(\xi + \eta t) + A(\xi) [R_h(\xi) - R_h(\xi + \eta t)] \right).
\]
Therefore, if \( A'(\xi)\eta \) exists, then we already know that \( R'_h(\xi)\eta \) exists, and thus
\[
S'_h(\xi)\eta = B^{-1} \left( - [A'(\xi)\eta] R_h(\xi) - A(\xi) R'_h(\xi)\eta \right).
\] (58)
Moreover, if \( A(\cdot) \) is Gâteaux-differentiable, then using the inf-sup condition (15) and the estimate (57) we get
\[
\|S'_h(\xi)\|_V \leq \frac{1}{\beta_h} \|B[S'_h(\xi)\eta]\|_{\tilde{V}^*} \leq \frac{\|A'(\xi)\| \|R_h(\xi)\|_V + \|A(\xi)\|_{\tilde{L}(\tilde{\tilde{Y}}, \tilde{\tilde{V}}^*)} \|R'_h(\xi)\|_{\tilde{L}(\tilde{Y}, \tilde{V})} \|\eta\|_X}{\beta_h} \leq \frac{\|A'(\xi)\| \|f\|_{\tilde{V}^*}}{\alpha_h \beta_h(\cdot)} \left( 1 + \frac{\|A(\xi)\|_{\tilde{L}(\tilde{\tilde{Y}}, \tilde{\tilde{V}}^*)}}{\alpha_h(\cdot)} \right) \|\eta\|_X,
\] (59)
which proves that \( S_h(\cdot) \) is Gâteaux-differentiable. Besides, it is clear from (59) that \( \|S'_h(\cdot)\|_{\tilde{L}(X, U)} \) will be uniformly bounded on \( X \), whenever \( \|A(\cdot)\|_{\tilde{L}(\tilde{\tilde{Y}}, \tilde{\tilde{V}}^*)} \) and \( \|A'(\cdot)\| \) are uniformly bounded on \( X \), as well as \( \alpha_h^{-1}(\cdot) \).

(iv) Let us prove Lipschitzness. Using (56), observe that for any \( \xi_1, \xi_2, \eta \in X \) we have
\[
A(\xi_2)|_\tilde{K} [R'_h(\xi_1) - R'_h(\xi_2)]\eta = [A'(\xi_2) - A'(\xi_1)]|_\tilde{K} R_h(\xi_2) + [A(\xi_2) - A(\xi_1)]|_\tilde{K} R'_h(\xi_1)\eta
\]
\[
+ A'(\xi_1)\eta|_\tilde{K} [R_h(\xi_2) - R_h(\xi_1)].
\]
Hence,
\[
\| R_h'(\xi_1) - R_h'(\xi_2) \|_{\mathcal{L}(X,V)} \leq \frac{\| R_h(\xi_2) \|_{V}}{\alpha_h(\xi_2)} \| A'(\xi_1) - A'(\xi_2) \| + \frac{\| R_h'(\xi_1) \|_{\mathcal{L}(X,V)}}{\alpha_h(\xi_2)} \| A(\xi_1) - A(\xi_2) \|_{\mathcal{L}(V,V')} \leq \frac{\| A'(\xi_1) \|}{\alpha_h(\xi_2)} \| R_h(\xi_1) - R_h(\xi_2) \|_V .
\]

(60a)

Recall that under our hypothesis, \( a^{-1}(\cdot), R_h(\cdot), R_h'(\cdot), \) and \( A'(\cdot) \), they are all uniformly bounded on \( X \). Therefore, the first term on the right hand side (expression (60a)) is Lipschitz by the Lipschitz assumption on \( A'(\cdot) \); the second term (expression (60b)) is Lipschitz as a consequence of the mean value theorem on \( A(\cdot) \) and the uniform boundedness of \( A'(\cdot) \); while the last term (expression (60c)) is Lipschitz by the mean value theorem on \( R_h(\cdot) \) and the uniform boundedness of \( R_h'(\cdot) \).

Finally, to prove the Lipschitzness of \( S_h'(\cdot) \), we use (58) to write
\[
B(S_h'(\xi_1)\eta - S_h'(\xi_2)\eta) = [A'(\xi_2)\eta](R_h(\xi_2) - R_h(\xi_1)) + A(\xi_2)[R_h'(\xi_2)\eta - R_h'(\xi_1)\eta] + [(A'(\xi_2) - A'(\xi_1))\eta]R_h(\xi_1) + [A(\xi_2) - A(\xi_1)]R_h'(\xi_1)\eta.
\]

Hence,
\[
\| S_h'(\xi_1) - S_h'(\xi_2) \|_{\mathcal{L}(X,U)} \leq \frac{\| A'(\xi_2) \|}{\beta_h} \| R_h(\xi_1) - R_h(\xi_2) \|_V + \frac{\| A(\xi_2) \|_{\mathcal{L}(V,V')}}{\beta_h} \| R_h'(\xi_1) - R_h'(\xi_2) \|_{\mathcal{L}(X,V')} \left( \| A'(\xi_1) - A'(\xi_2) \| + \| A(\xi_1) - A(\xi_2) \|_{\mathcal{L}(V,V')} \right) .
\]

(61a)

(61b)

(61c)

(61d)

We recall again that \( R_h(\cdot), R_h'(\cdot), A(\cdot), \) and \( A'(\cdot) \), they are all uniformly bounded on \( X \). Therefore, the Lipschitzness of \( S_h'(\cdot) \) is implied by the following facts: the Lipschitzness of the first term on right hand side (expression (61a)) is a consequence of the mean value theorem applied to \( R_h(\cdot) \) and the uniform boundedness of \( R_h'(\cdot) \); the Lipschitzness of the second term (expression (61b)) is due to the previously proved Lipschitzness of \( R_h'(\cdot) \); the Lipschitzness of the third term (expression (61c)) is implied by the assumed Lipschitzness of \( A'(\cdot) \); and the Lipschitzness of the last term (expression (61d)) is consequence of the mean value theorem applied to \( A \) and the uniform boundedness of \( A'(\cdot) \).

### A.6 Proof of Proposition 3.1

Let us prove item by item.

(i) Observe that in this case, the bilinear form \( a(\xi,\cdot,\cdot) \) defines a weighted inner product in \( L^2(\Omega) \), for which its induced norm \( \| v \|_{V,\xi} := \sqrt{\langle v, v \rangle_{\xi}} \) satisfies
\[
\sqrt{\omega_{\min}} \| v \|_{L^2} \leq \| v \|_{V,\xi} \leq \sqrt{\omega_{\max}} \| v \|_{L^2} , \quad \forall v \in V = L^2(\Omega).
\]

Thus, the first inf-sup condition in (15) is satisfied with \( \alpha_h = \omega_{\min} \); see Proposition 2.9(iii) and Footnote 19.
On the other hand, we are under the assumption that the operator \( B : H_B \to V^* \) is boundedly invertible. Hence, there must be a uniform constant \( \beta > 0 \) such that
\[
\sup_{v \in V} \frac{b(w_h, v)}{\|v\|_{V^*}} = \|Bw_h\|_{V^*} \geq \beta \|w_h\|_{H_B}, \quad \forall w_h \in U_h,
\]
which implies the second inf-sup condition in (15).

(ii) Uniform boundedness of \( S_h(\cdot) \) is a consequence of Proposition 2.9(iii). Indeed, in our particular case we get
\[
\|S_h(\xi)\|_{H_B} \leq \frac{\omega_{\text{max}}}{\omega_{\text{min}}} \|f\|_{L^2}, \quad \forall \xi \in L^2(\Omega).
\]
To show differentiability of \( S_h(\cdot) \), let us recall the operator \( A : X \to \mathcal{L}(V, V^*) \) defined in section 2.4, which in this particular case, given \( \xi \in L^2(\Omega) \), it takes the form
\[
A(\xi)v := (\varpi(\xi)v, \cdot)_{L^2}, \quad \forall v \in L^2(\Omega).
\]
Furthermore, we have the uniform bound
\[
\|A(\xi)\| = \sup_{v \in L^2(\Omega)} \frac{\|\varpi(\xi)v\|_{L^2}}{\|v\|_{L^2}} = \|\varpi(\xi)\|_{L^\infty} \leq \omega_{\text{max}}.
\] (62)

Since \( \varpi(\cdot) \) is differentiable, it is straightforward to check that \( A(\cdot) \) is also differentiable, and given \( \xi, \eta \in L^2(\Omega) \), we have
\[
[A'(\xi)\eta]v = ([\varpi'(\xi)(\eta)v, \cdot]_{L^2}, \quad \forall v \in L^2(\Omega).
\]
Moreover, we can verify
\[
\|A'(\xi)\| = \sup_{\eta \in L^2(\Omega)} \frac{\|\varpi'(\xi)\eta\|_{L^\infty}}{\|\eta\|_{L^2}} = \|\varpi'(\xi)\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))} \leq \omega'_{\text{min}}.
\] (63)

Thus, the differentiability of \( S_h(\cdot) \) is a consequence of Proposition 2.11(ii).

(iii) Uniform boundedness of \( S_h^*(\cdot) \) is a consequence of Proposition 2.11(iii), using the fact that \( A(\cdot), A'(\cdot), \) and \( \alpha_{h}^{-1} \equiv \varpi_{\text{min}}^{-1} \) are all uniformly bounded (see the above expressions (62) and (63)).

On the other hand, the Lipschitz-continuity of \( S_h^*(\cdot) \) relies on the Lipschitz-continuity of \( A'(\cdot) \) (by Proposition 2.11(iv)). The latter is true since
\[
\|A'(\xi_1) - A'(\xi_2)\| = \sup_{\eta \in L^2(\Omega)} \frac{\|\varpi'(\xi_1)\eta - \varpi'(\xi_2)\eta\|_{L^\infty}}{\|\eta\|_{L^2}} \leq \omega L \|\xi_1 - \xi_2\|_{L^2}.
\]

A.7 Proof of Proposition 3.4

(i) We verify the hypothesis of Proposition 2.9(i). Since \( V \) is infinite dimensional, we first need to show that
\[
\{v_2 \in K : b(\varpi(v_2), v_1) = 0, \forall v_1 \in K\} = \{0\},
\]
which is an immediate consequence of (33) taking \( v_1 = v_2 \). To show the inf-sup conditions (15), from one hand observe that
\[
\sup_{v_2 \in K} \frac{b(\varpi(v_2), v_1)}{\|v_2\|_V} \geq \frac{b(\varpi(v_1), v_1)}{\|v_1\|_V} \geq \alpha_h(\xi)\|v_1\|_V, \quad \forall v_1 \in K.
\]
A.8 Proof of Proposition 3.8

On the other hand, we have that \( b(\cdot, \cdot) \) satisfies (25a). Thus, in particular

\[
\sup_{v \in V} \frac{b(v_2, v)}{\|v\|_V} \geq \beta \|v_h\|_V, \quad \forall v_h \in V_h.
\]

(iii) We use the a priori bound of Proposition 2.9(ii). In this case \( \beta_h = \beta, \alpha_h^{-1}(\xi) \leq \alpha^{-1}, \) and 
\( a(\xi; v_1, v_2) = b(\varpi(\xi)v_2, v_1), \) for all \( v_1, v_2 \in V. \) It is easy to see that 
\( \|a(\xi; \cdot, \cdot)\|_{\mathcal{L}(V^2; \mathbb{R})} \leq \|b\|_{\mathcal{L}(V \times V; \mathbb{R})} \varpi(\xi) \|\|_W. \) Thus, we get

\[
\|S_h(\cdot)\|_V \leq \frac{1}{\beta} \left( 1 + \frac{\varpi_\infty}{\alpha} \right) \|f\|_{V^*}.
\]

(iii) Now we apply Proposition 2.11. The operator \( A : X \to \mathcal{L}(V, V^*) \) takes the form \( A(\xi)v = b(\varpi(\xi)\cdot, v) \in V^* \), for all \( \xi \in X = L^2(\Omega) \) and \( v \in V. \) Moreover,

\[
\|A(\xi)\|_{\mathcal{L}(V, V^*)} \leq \|b\|_{\mathcal{L}(V \times V; \mathbb{R})} \varpi(\xi) \|\|_W \leq \|b\|_{\mathcal{L}(V \times V; \mathbb{R})} \varpi_\infty.
\]

On the other hand, it is immediate to see that if \( \varpi \) is differentiable, then \( A \) is differentiable and 
\( [A'(\xi)\eta]v = b(\varpi'(\xi)\eta, v) \in V^*, \) for any direction \( \eta \in L^2(\Omega). \) Moreover,

\[
\|A'(\xi)\eta\|_{\mathcal{L}(V, V^*)} \leq \|b\|_{\mathcal{L}(V \times V; \mathbb{R})} \varpi'(\xi) \|\|_{L^2(\Omega)} \|\eta\|_2.
\]

Hence, \( A'(\cdot) \) is uniformly bounded and Lipschitz-continuous whenever \( \varpi'(\cdot) \) is. By Proposition 2.11, differentiability of \( S_h(\cdot) \) is implied by differentiability of \( A(\cdot); \) uniform boundedness of \( S_h'(\cdot) \) is implied by uniform boundedness of \( A(\cdot), A'(\cdot) \) and \( \alpha_h^{-1}(\cdot); \) while Lipschitzness of \( S_h'(\cdot) \) is implied by Lipschitzness of \( A'(\cdot). \)

A.8 Proof of Proposition 3.8

(i) Making the identification \( \hat{V} \equiv V_h \) and \( a(\xi; \cdot, \cdot) \equiv (\cdot, \cdot)_{V, \xi}, \) we observe that the well-
posedness of (37) is a direct consequence of Proposition 2.9, using the fact that \( (\cdot, \cdot)_{V, \xi} \) is an equivalent inner-product, together with assumption (35b).

(ii) Using the hypothesis of this statement and the estimate (17) in Proposition 2.9(iii), we get the uniform bound

\[
\|S_h(\xi)\|_U \leq \frac{1}{\beta_h} \frac{\tilde{C}_2}{\tilde{C}_1} \|f\|_{V^*}, \quad \forall \xi \in X.
\]

(iii) Direct application of Proposition 2.11, noticing also that \( \alpha_h^{-1}(\xi) \leq \tilde{C}_1^{-2} \) and

\[
\|A(\xi)\|_{\mathcal{L}(V, V^*)} = \sup_{v \in V} \frac{(v_1, v_2)_{V, \xi}}{\|v_1\|_V} \leq \frac{\tilde{C}_2^2}{\beta_h} \left( \sup_{v_1 \in V} \frac{|(v_1, v_2)_{V, \xi}|}{\|v_1\|_V} \right) = \tilde{C}_2^2.
\]
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