Space-time symmetries and simple superalgebras

S. Ferrara

Theoretical Physics Division, CERN
CH - 1211 Geneva 23
and
Laboratori Nazionali di Frascati, INFN, Italy

ABSTRACT

We describe spinors in Minkowskian spaces with arbitrary signature and their role in the classification of space-time superalgebras and their R-symmetries in any dimension.

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1 Introduction

We consider supersymmetry algebras in space-times with arbitrary signature and minimal number of spinor generators. The interrelation between super Poincaré and superconformal algebras is elucidated. Minimal superconformal algebras are seen to have as bosonic part a classical semisimple algebra naturally associated to the spin group. This algebra, the Spin($s,t$)-algebra, depends both on the dimension and on the signature of space time. We also consider superconformal algebras, which are classified by the orthosymplectic algebras.

We then generalize the classification to $N$-extended space-time superalgebras and notice that $R$-symmetries may become non-compact depending on the space-time signature. The latter applies to the case of Euclidean super Yang-Mills theories in four dimensions.

2 Properties of spinors of $\text{SO}(V)$

Let $V$ be a real vector space of dimension $D = s + t$ and $\{v_\mu\}$ a basis of it. On $V$ there is a non degenerate symmetric bilinear form which in the basis is given by the matrix

$$\eta_{\mu\nu} = \text{diag}(+\ldots(s \text{ times})\ldots,+,-\ldots(t \text{ times})\ldots,-).$$

We consider the group Spin($V$), the unique double covering of the connected component of $\text{SO}(s,t)$ and its spinor representations. A spinor representation of Spin($V$)$^\mathbb{C}$ is an irreducible complex representation whose highest weights are the fundamental weights corresponding to the right extreme nodes in the Dynkin diagram. These do not descend to representations of $\text{SO}(V)$. A spinor type representation is any irreducible representation that doesn’t descend to $\text{SO}(V)$. A spinor representation of Spin($V$) over the reals is an irreducible representation over the reals whose complexification is a direct sum of spin representations.

Two parameters, the signature $\rho \mod(8)$ and the dimension $D \mod(8)$ classify the properties of the spinor representation. Through this paper we will use the following notation,

$$\rho = s - t = \rho_0 + 8n, \quad D = s + t = D_0 + 8p,$$

where $\rho_0, D_0 = 0, \ldots, 7$. We set $m = p - n$, so

$$s = \frac{1}{2}(D + \rho) = \frac{1}{2}(\rho_0 + D_0) + 8n + 4m,$$
$$t = \frac{1}{2}(D - \rho) = \frac{1}{2}(D_0 - \rho_0) + 4m.$$

The signature $\rho \mod(8)$ determines if the spinor representations are real ($\mathbb{R}$), quaternionic ($\mathbb{H}$) or complex ($\mathbb{C}$) type. Also note that reality properties depend only on $|\rho|$ since Spin($s,t$) = Spin($t,s$).

The dimension $D \mod(8)$ determines the nature of the Spin($V$)-morphisms of the spinor representation $S$. Let $g \in \text{Spin}(V)$ and let $\Sigma(g) : S \rightarrow S$ and $L(g) : V \rightarrow V$ the spinor and vector representations of $l \in \text{Spin}(V)$ respectively. Then a map $A$

$$A : S \otimes S \rightarrow \Lambda^k,$$

where $\Lambda^k = \Lambda^k(V)$ are the $k$-forms on $V$, is a Spin($V$)-morphism if

$$A(\Sigma(g)s_1 \otimes \Sigma(g)s_2) = L^k(g)A(s_1 \otimes s_2).$$

In Tables 1 and 2, reality and symmetry properties of spinors are reported.

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1 The content of this report is based on Refs. 1 and 2.
\[ \rho_0(\text{odd}) \quad | \quad \text{real dim}(S) \quad | \quad \text{reality} \quad | \quad \rho_0(\text{even}) \quad | \quad \text{real dim}(S^\pm) \quad | \quad \text{reality} \]

|   |   | \(2^{(D-1)/2}\) | \(\mathbb{R}\) | 0 | \(2^{D/2}-1\) | \(\mathbb{R}\) |
|---|---|---|---|---|---|---|
| 1 |   | \(2^{(D+1)/2}\) | \(\mathbb{H}\) | 2 | \(2^{D/2}\) | \(\mathbb{C}\) |
| 3 |   | \(2^{(D+1)/2}\) | \(\mathbb{H}\) | 4 | \(2^{D/2}\) | \(\mathbb{H}\) |
| 5 |   | \(2^{(D-1)/2}\) | \(\mathbb{R}\) | 6 | \(2^{D/2}\) | \(\mathbb{C}\) |

Table 1: Reality properties of spinors

| \(D\) | \(k\) even | morphism | symmetry | \(k\) odd | morphism | symmetry |
|---|---|---|---|---|---|---|
| 0 | \(S^\pm \otimes S^\pm \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S^\pm \otimes S^\pm \rightarrow \Lambda^k\) |   |   |   |
| 1 | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S^\pm \otimes S^\pm \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 2 | \(S^\pm \otimes S^\mp \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 3 | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 4 | \(S^\pm \otimes S^\mp \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 5 | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 6 | \(S^\pm \otimes S^\mp \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |
| 7 | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) | \(S \otimes S \rightarrow \Lambda^k\) | \((-1)^{k(k-1)/2}\) |

Table 2: Properties of morphisms.

3 Orthogonal, symplectic and linear spinors

We consider now morphisms
\[ S \otimes S \rightarrow \Lambda^0 \simeq \mathbb{C}. \]

If a morphism of this kind exists, it is unique up to a multiplicative factor. The vector space of the spinor representation has then a bilinear form invariant under \(\text{Spin}(V)\). Looking at Table 2, one can see that this morphism exists except for \(D_0 = 2, 6\), where instead a morphism
\[ S^\pm \otimes S^\mp \rightarrow \mathbb{C} \]
occurs.

We shall call a spinor representation orthogonal if it has a symmetric, invariant bilinear form. This happens for \(D_0 = 0, 1, 7\) and \(\text{Spin}(V)^\mathbb{C}\) (complexification of \(\text{Spin}(V)\)) is then a subgroup of the complex orthogonal group \(\text{SO}(n, \mathbb{C})\), where \(n\) is the dimension of the spinor representation (Weyl spinors for \(D\) even). The generators of \(\text{SO}(n, \mathbb{C})\) are \(n \times n\) antisymmetric matrices. These are obtained in terms of the morphisms
\[ S \otimes S \rightarrow \Lambda^k, \]
which are antisymmetric. This gives the decomposition of the adjoint representation of \(\text{SO}(n, \mathbb{C})\) under the subgroup \(\text{Spin}(V)^\mathbb{C}\). In particular, for \(k = 2\) one obtains the generators of \(\text{Spin}(V)^\mathbb{C}\).

A spinor representation is called symplectic if it has an antisymmetric, invariant bilinear form. This is the case for \(D_0 = 3, 4, 5\). \(\text{Spin}(V)^\mathbb{C}\) is a subgroup of the symplectic group \(\text{Sp}(2p, \mathbb{C})\), where \(2p\) is the dimension of the spinor representation. The Lie algebra \(\text{sp}(2p, \mathbb{C})\) is formed by all the symmetric matrices, so it is given in terms of the morphisms \(S \otimes S \rightarrow \Lambda^k\) which are symmetric. The generators of \(\text{Spin}(V)^\mathbb{C}\) correspond to \(k = 2\) and are symmetric matrices.

For \(D_0 = 2, 6\) one has an invariant morphism
\[ B : S^+ \otimes S^- \rightarrow \mathbb{C}. \]

The representations \(S^+\) and \(S^-\) are one the contragradient (or dual) of the other. The spin representations extend to representations of the linear group \(\text{GL}(n, \mathbb{C})\), which leaves the pairing \(B\) invariant. These spinors are called linear. \(\text{Spin}(V)^\mathbb{C}\) is a subgroup of the simple factor \(\text{SL}(n, \mathbb{C})\).
These properties depend exclusively on the dimension \( n \). When combined with the reality properties, which depend on \( \rho \), one obtains real groups embedded in \( SO(n, \mathbb{C}) \), \( Sp(2p, \mathbb{C}) \) and \( GL(n, \mathbb{C}) \) which have an action on the space of the real spinor representation \( S^\sigma \). The real groups contain as a subgroup \( \text{Spin}(V) \).

We need first some general facts about real forms of simple Lie algebras. Let \( S \) be a complex vector space of dimension \( n \) which carries an irreducible representation of a complex Lie algebra \( \mathcal{G} \). Let \( G \) be the complex Lie group associated to \( \mathcal{G} \). Let \( \sigma \) be a conjugation or a pseudoconjugation on \( S \) such that \( \sigma X \sigma^{-1} \in G \) for all \( X \in \mathcal{G} \). Then the map

\[
X \mapsto X^\sigma = \sigma X \sigma^{-1}
\]

is a conjugation of \( \mathcal{G} \). We shall write

\[
\mathcal{G}^\sigma = \{ X \in \mathcal{G} | X^\sigma = X \}.
\]

\( \mathcal{G}^\sigma \) is a real form of \( \mathcal{G} \). If \( \tau = h \sigma h^{-1} \), with \( h \in G \), \( \mathcal{G}^\tau = h \mathcal{G}^\sigma h^{-1} \). \( \mathcal{G}^\sigma = \mathcal{G}^{\sigma'} \) if and only if \( \sigma' = \epsilon \sigma \) for \( \epsilon \) a scalar with \( |\epsilon| = 1 \); in particular, if \( \mathcal{G}^\sigma \) and \( \mathcal{G}^{\sigma'} \) are conjugate by \( G \), \( \sigma \) and \( \tau \) are both conjugations or both pseudoconjugations. The conjugation can also be defined on the group \( G, g \mapsto \sigma g \sigma^{-1} \).

4 Real forms of the classical Lie algebras

We describe the real forms of the classical Lie algebras from this point of view. (See also Ref. [7]).

**Linear algebra, \( \text{sl}(S) \).**

(a) If \( \sigma \) is a conjugation of \( S \), then there is an isomorphism \( S \rightarrow \mathbb{C}^n \) such that \( \sigma \) goes over to the standard conjugation of \( \mathbb{C}^n \). Then \( \mathcal{G}^\sigma \simeq \text{sl}(n, \mathbb{R}) \). (The conjugation acting on \( \text{gl}(n, \mathbb{C}) \) gives the real form \( \text{gl}(n, \mathbb{R}) \)).

(b) If \( \sigma \) is a pseudoconjugation and \( \mathcal{G} \) doesn’t leave invariant a non degenerate bilinear form, then there is an isomorphism of \( S \) with \( \mathbb{C}^n, n = 2p \) such that \( \sigma \) goes over to

\[
(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z_{p+1}, \ldots, z_{2p}, -z_1, \ldots, -z_p).
\]

Then \( \mathcal{G}^\sigma \simeq \text{su}^*(2p) \). (The pseudoconjugation acting in on \( \text{gl}(2p, \mathbb{C}) \) gives the real form \( \text{su}^*(2p) \oplus \text{so}(1, 1) \).)

To see this, it is enough to prove that \( \mathcal{G}^\sigma \) does not leave invariant any non degenerate hermitian form, so it cannot be of the type \( \text{su}(p, q) \). Suppose that \( \langle \cdot, \cdot \rangle \) is a \( \mathcal{G}^\sigma \)-invariant, non degenerate hermitian form. Define \( (s_1, s_2) := \langle \sigma(s_1), s_2 \rangle \). Then \( \langle \cdot, \cdot \rangle \) is bilinear and \( \mathcal{G}^\sigma \)-invariant, so it is also \( \mathcal{G} \)-invariant.

(c) The remaining cases, following E. Cartan’s classification of real forms of simple Lie algebras, are \( \text{su}(p, q) \), where a non degenerate hermitian bilinear form is left invariant. They do not correspond to a conjugation or pseudoconjugation on \( S \), the space of the fundamental representation. (The real form of \( \text{gl}(n, \mathbb{C}) \) is in this case \( u(p, q) \)).

**Orthogonal algebra, \( \text{so}(S) \).** \( \mathcal{G} \) leaves invariant a non degenerate, symmetric bilinear form. We will denote it by \( \langle \cdot, \cdot \rangle \).

(a) If \( \sigma \) is a conjugation preserving \( \mathcal{G} \), one can prove that there is an isomorphism of \( S \) with \( \mathbb{C}^n \) such that \( \langle \cdot, \cdot \rangle \) goes to the standard form and \( \mathcal{G}^\sigma \) to \( \text{so}(p, q) \), \( p + q = n \). Moreover, all \( \text{so}(p, q) \) are obtained in this form.

(b) If \( \sigma \) is a pseudoconjugation preserving \( \mathcal{G} \), \( \mathcal{G}^\sigma \) cannot be any of the \( \text{so}(p, q) \). By E. Cartan’s classification, the only other possibility is that \( \mathcal{G}^\sigma \simeq \text{so}^*(2p) \). There is an isomorphism of \( S \) with \( \mathbb{C}^{2p} \) such that \( \sigma \) goes to

\[
(z_1, \ldots, z_p, z_{p+1}, \ldots, z_{2p}) \mapsto (z_{p+1}, \ldots, z_{2p}, -z_1, \ldots, -z_p).
\]

**Symplectic algebra, \( \text{sp}(S) \).** We denote by \( \langle \cdot, \cdot \rangle \) the symplectic form on \( S \).
(a) If $\sigma$ is a conjugation preserving $G$, it is clear that there is an isomorphism of $S$ with $\mathbb{C}^{2p}$, such that $G^{\sigma} \simeq \text{sp}(2p, \mathbb{R})$.

(b) If $\sigma$ is a pseudoconjugation preserving $G$, then $G^{\sigma} \simeq \text{usp}(p,q)$, $p+q = n = 2m$, $p = 2p'$, $q = 2q'$. All the real forms $\text{usp}(p,q)$ arise in this way. There is an isomorphism of $S$ with $\mathbb{C}^{2p}$ such that $\sigma$ goes to

$$(z_1, \ldots, z_m, z_{m+1}, \ldots, z_n) \mapsto J_m K_{p',q'} (z_1^*, \ldots, z_m^*, z_{m+1}^*, \ldots, z_n^*),$$

where

$$J_m = \begin{pmatrix} 0 & I_{m \times m} \\ -I_{m \times m} & 0 \end{pmatrix}, \quad K_{p',q'} = \begin{pmatrix} -I_{p' \times p'} & 0 & 0 & 0 \\ 0 & I_{q' \times q'} & 0 & 0 \\ 0 & 0 & -I_{p' \times p'} & 0 \\ 0 & 0 & 0 & I_{q' \times q'} \end{pmatrix}.$$

In Section 2 we saw that there is a conjugation on $S$ when the spinors are real and a pseudoconjugation when they are quaternionic [1] (both denoted by $\sigma$). We have a group, $\text{SO}(n, \mathbb{C})$, $\text{Sp}(2p, \mathbb{C})$ or $\text{GL}(n, \mathbb{C})$ acting on $S$ and containing $\text{Spin}(V)^\mathbb{C}$. We note that this group is minimal in the classical group series. If the Lie algebra $G$ of this group is stable under the conjugation

$$X \mapsto \sigma X \sigma^{-1}$$

then the real Lie algebra $G^{\sigma}$ acts on $S^{\sigma}$ and contains the Lie algebra of $\text{Spin}(V)$. We shall call it the $\text{Spin}(V)$-algebra.

Let $B$ be the space of $\text{Spin}(V)^\mathbb{C}$-invariant bilinear forms on $S$. Since the representation on $S$ is irreducible, this space is at most one dimensional. Let it be one dimensional and let $\sigma$ be a conjugation or a pseudoconjugation and let $\psi \in B$. We define a conjugation in the space $B$ as

$$
\begin{array}{ccc}
B & \rightarrow & B \\
\psi & \mapsto & \psi^{\sigma} \\
\end{array}
$$

$$\psi^{\sigma}(v, u) = \psi(\sigma(v), \sigma(u))^*.$$

It is then immediate that we can choose $\psi \in B$ such that $\psi^\sigma = \psi$. Then if $X$ belongs to the Lie algebra preserving $\psi$, so does $\sigma X \sigma^{-1}$.

One can determine the real Lie algebras in each case [1]. All the possible cases must be studied separately. All dimension and signature relations are mod(8). In the following, a relation like $\text{Spin}(V) \subseteq G$ for a group $G$ will mean that the image of $\text{Spin}(V)$ under the spinor representation is in the connected component of $G$. The same applies for the relation $\text{Spin}(V) \simeq G$. For $\rho = 0, 1, 7$ spin algebras commute with a conjugation, for $\rho = 3, 4, 5$ they commute with a pseudoconjugation. For $\rho = 2, 6$ they are complex. The complete classification is reported in Table 3.

5 \textbf{Spin}(V) superalgebras

We now consider the embedding of $\text{Spin}(V)$ in simple real superalgebras. We require in general that the odd generators are in a real spinor representation of $\text{Spin}(V)$. In the cases $D_0 = 2, 6$, $\rho_0 = 0, 4$ we have to allow the two independent irreducible representations, $S^+$ and $S^-$ in the superalgebra, since the relevant morphism is

$$S^+ \otimes S^- \rightarrow \Lambda^2.$$

The algebra is then non chiral.

We first consider minimal superalgebras [10, 11] i.e. those with the minimal even subalgebra. From the classification of simple superalgebras [10, 11, 12] one obtains the results listed in Table 4.

We note that the even part of the minimal superalgebra contains the $\text{Spin}(V)$ algebra obtained in Section 4 as a simple factor. For all quaternionic cases, $\rho_0 = 3, 4, 5$, a second simple factor $\text{SU}(2)$ is present. For the linear cases there is an additional non simple factor, $\text{SO}(1,1)$ or $\text{U}(1)$, as discussed in Section 4.
### Table 3: Spin($s, t$) algebras.

| Orthogonal | $D_0 = 1, 7$ | Real, $\rho_0 = 1, 7$ | $\text{so}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ if $D = \rho$ | $\text{so}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ |
|-------------|--------------|----------------------|-------------------------------------------------|-------------------------------------------------|
|             | Quaternionic, $\rho_0 = 3, 5$ | $\text{so}^*(2^{\frac{D_0-1}{2}})$ | | |
| Symplectic  | $D_0 = 3, 5$ | Real, $\rho_0 = 1, 7$ | $\text{sp}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ | $\text{sp}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ |
|             | Quaternionic, $\rho_0 = 3, 5$ | $\text{usp}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ if $D = \rho$ | $\text{usp}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ | |
| Orthogonal  | $D_0 = 0$ | Real, $\rho_0 = 0$ | $\text{so}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ if $D = \rho$ | $\text{so}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ |
|             | Quaternionic, $\rho_0 = 4$ | $\text{so}^*(2^{\frac{D_0-1}{2}})$ | | |
|             | Complex, $\rho_0 = 2, 6$ | $\text{so}(2^{\frac{D_0-1}{2}}, \mathbb{C})$ | | |
| Symplectic  | $D_0 = 4$ | Real, $\rho_0 = 0$ | $\text{sp}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ | |
|             | Quaternionic, $\rho_0 = 4$ | $\text{usp}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ if $D = \rho$ | $\text{usp}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ | |
|             | Complex, $\rho_0 = 2, 6$ | $\text{sp}(2^{\frac{D_0-1}{2}}, \mathbb{C})$ | | |
| Linear      | $D_0 = 2, 6$ | Real, $\rho_0 = 0$ | $\text{sl}(2^{\frac{D_0-1}{2}}, \mathbb{R})$ | |
|             | Quaternionic, $\rho_0 = 4$ | $\text{su}^*(2^{\frac{D_0-1}{2}})$ | | |
|             | Complex, $\rho_0 = 2, 6$ | $\text{su}(2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ | $\text{su}(2^{\frac{D_0-1}{2}}, 2^{\frac{D_0-1}{2}})$ if $D \neq \rho$ | |

For $D = 7$ and $\rho = 3$ there is actually a smaller superalgebra, the exceptional superalgebra $f(4)$ with bosonic part $\text{spin}(5, 2) \times \text{su}(2)$. The superalgebra appearing in Table 3 belongs to the classical series and its even part is $\text{so}^*(8) \times \text{su}(2)$, being $\text{so}^*(8)$ the $\text{Spin}(5, 2)$-algebra.

Keeping the same number of odd generators, the maximal simple superalgebra containing $\text{Spin}(V)$ is an orthosymplectic algebra with $\text{Spin}(V) \subset \text{Sp}(2n, \mathbb{R})$, being $2n$ the real dimension of $S$. The various cases are displayed in the Table 5. We note that the minimal superalgebra is not a subalgebra of the maximal one, although it is so for the bosonic parts.

## 6 Extended Superalgebras

The present analysis can be generalized to the case of $N$ copies of the spinor representation of $\text{spin}(s, t)$-algebras. By looking at the classification of classical simple superalgebras, we find extensions for all $N$, where the number of supersymmetries is always even if spinors are quaternionic (because of reality properties) or orthogonal (because of symmetry properties).

In Table 6 the classification analogous to the one in Table 4 is given. SuperPoincaré algebras can be obtained from the simple superalgebras either by contraction $\text{Spin}(s, t) \rightarrow \text{InSpin}(s, t-1)$ or as subalgebras $\text{Spin}(s, t) \rightarrow \text{InSpin}(s-1, t-1)$. It is important to observe that the $R$-symmetry may be non-compact for different signatures of space-time.

In fact the conjugation properties of the $R$-symmetry algebra is the same of the space-time part.

As an example if we consider Euclidean four-dimensional $N = 2$ and $N = 4$ Yang-Mills theory, the $R$-symmetry becomes respectively $\text{SU}(2) \times \text{SO}(1, 1)$ and $\text{SU}^*(4)$. The first case was considered long ago by Zumino. These are the superalgebras appropriate for Yang-Mills instantons. On the other hand, if we consider a Minkowskian space with signature $(2, 2)$ the $R$-symmetry is $\text{GL}(2, \mathbb{R})$ (for $N = 2$) and $\text{SL}(4, \mathbb{R})$ for $N = 4$.

Compact $R$-symmetries occur for $q = 0$ in Table 6, including all cases when the conformal group $\text{SO}(D, 2)$ corresponds to ordinary Minkowski space $V_{(D-1, 1)}$. 
### Table 4: Minimal Spin(\(V\)) superalgebras.

| \(D_0\) | \(\rho_0\) | Spin(\(V\)) algebra | Spin(\(V\)) superalgebra |
|--------|-----------|----------------------|--------------------------|
| 1,7    | 1,7       | \(\text{so}(2^{(D-3)/2}, 2^{(D-3)/2})\) | \(\text{osp}(2^{(D-1)/2})*|2\) |
| 1,7    | 3,5       | \(\text{so}^*(2^{(D-1)/2})\) | \(\text{osp}(2^{(D-1)/2})|2\) |
| 3,5    | 1,7       | \(\text{sp}(2^{(D-4)/2}, \mathbb{R})\) | \(\text{osp}(1|2^{(D-1)/2}, \mathbb{R})\) |
| 3,5    | 3,5       | \(\text{usp}(2^{(D-3)/2}, 2^{(D-4)/2})\) | \(\text{osp}(1|2^{(D-2)/2}, \mathbb{R})\) |
| 0      | 0         | \(\text{so}(2^{(D-4)/2}, 2^{(D-4)/2})\) | \(\text{osp}(2^{(D-2)/2})|2\) |
| 0      | 2,6       | \(\text{so}(2^{(D-2)/2}, \mathbb{C})^\mathbb{R}\) | \(\text{osp}(1|2^{(D-2)/2}, \mathbb{C})\) |
| 0      | 4         | \(\text{so}^*(2^{(D-2)/2})\) | \(\text{osp}(1|2^{(D-2)/2})|2\) |
| 2,6    | 0         | \(\text{sl}(2^{(D-2)/2}, \mathbb{R})\) | \(\text{osp}(1|2^{(D-2)/2}), \mathbb{R})\) |
| 2,6    | 4         | \(\text{sp}(2^{(D-2)/2}, \mathbb{C})^\mathbb{R}\) | \(\text{osp}(1|2^{(D-2)/2}, \mathbb{C})\) |
| 4      | 0         | \(\text{su}(2^{(D-2)/2})\) | \(\text{osp}(1|2^{(D-2)/2})|2\) |
| 4      | 2,6       | \(\text{sp}(2^{(D-2)/2}, \mathbb{C})^\mathbb{R}\) | \(\text{osp}(1|2^{(D-2)/2}), \mathbb{C})\) |
| 4      | 4         | \(\text{usp}(2^{(D-2)/2}, 2^{(D-4)/2})\) | \(\text{osp}(1|2^{(D-2)/2})|2\) |

### Table 5: Orthosymplectic Spin(\(V\)) superalgebras

| \(D_0\) | \(\rho_0\) | Orthosymplectic |
|--------|-----------|----------------|
| 3,5,   | 1,7       | \(\text{osp}(1|2^{(D-1)/2}, \mathbb{R})\) |
| 1,7    | 3,5       | \(\text{osp}(1|2^{(D+1)/2}, \mathbb{R})\) |
| 0      | 4         | \(\text{osp}(1|2^{D/2}, \mathbb{R})\) |
| 4      | 0         | \(\text{osp}(1|2^{(D-2)/2}, \mathbb{R})\) |
| 4      | 2,6       | \(\text{osp}(1|2^{D/2}, \mathbb{R})\) |
| 2,6    | 0         | \(\text{osp}(1|2^{D/2}, \mathbb{R})\) |
| 2,6    | 4         | \(\text{osp}(1|2^{(D+2)/2}, \mathbb{R})\) |
| 2,6    | 2,6       | \(\text{osp}(1|2^{D/2}, \mathbb{R})\) |

### Table 6: \(N\)-extended \(\text{Spin}(s, t)\) superalgebras

| \(D_0\) | \(\rho_0\) | R-symmetry | Spin(s, t) superalgebra |
|--------|-----------|------------|-------------------------|
| 1,7    | 1,7       | \(\text{sp}(2N, \mathbb{R})\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N, \mathbb{R})\) |
| 1,7    | 3,5       | \(\text{usp}(2N-2q, 2q)\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N-2q, 2q)\) |
| 3,5    | 1,7       | \(\text{so}(N-q, q)\) | \(\text{osp}(N-q, q, 2^{s/2}, 2^{t/2})\) |
| 3,5    | 3,5       | \(\text{so}^*(2N)\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N)\) |
| 0      | 0         | \(\text{sp}(2N, \mathbb{C})^\mathbb{R}\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N, \mathbb{C})^\mathbb{R}\) |
| 0      | 2,6       | \(\text{usp}(2N-2q, 2q)\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N-2q, 2q)\) |
| 2,6    | 0         | \(\text{sl}(N, \mathbb{R})\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N, \mathbb{R})\) |
| 2,6    | 2,6       | \(\text{su}(N-q, q)\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N-q, q)\) |
| 2,6    | 4         | \(\text{su}^*(2N, \mathbb{R})\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N^*)\) |
| 4      | 0         | \(\text{so}(N-q, q)\) | \(\text{osp}(N-q, q, 2^{s/2}, 2^{t/2})\) |
| 4      | 2,6       | \(\text{so}(N, \mathbb{C})^\mathbb{R}\) | \(\text{osp}(N, 2^{s/2}, 2^{t/2}, 2N, \mathbb{C})^\mathbb{R}\) |
| 4      | 4         | \(\text{so}^*(2N)\) | \(\text{osp}(2^{s/2}, 2^{t/2}, 2N)\) |
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