On the Hankel transform of C-fractions

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Abstract
We study the Hankel transforms of sequences whose generating function can be expressed as a C-fraction. In particular, we relate the index sequence of the non-zero terms of the Hankel transform to the powers appearing in the monomials defining the C-fraction. A closed formula for the Hankel transforms studied is given. As every power-series can be represented by a C-fraction, this gives in theory a closed form formula for the Hankel transform of any sequence. The notion of multiplicity is introduced to differentiate between Hankel transforms.

1 Introduction

Given a sequence $a_n$, we denote by $h_n$ the general term of the sequence with $h_n = |a_{i+j}|_{0 \leq i,j \leq n}$. The sequence $h_n$ is called the Hankel transform of $a_n$ [7, 8, 10]. If the sequence $a_n$ has generating function $g(x)$, then by an abuse of language we can also refer to $h_n$ as the Hankel transform of $g(x)$.

A well known example of Hankel transform is that of the Catalan numbers, $C_n = \frac{1}{n+1} \binom{2n}{n}$, where we find that $h_n = 1$ for all $n$. Hankel determinants occur naturally in many branches of mathematics, from combinatorics [1] to number theory [12] and to mathematical physics [17].

We shall be interested in characterizing the Hankel transform of sequences whose generating functions can be expressed as the following type of C-fraction:

$$g(x) = \frac{1}{1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \frac{a_3 x^{q_3}}{1 + \ldots}}}}.$$  \hspace{1cm} (1)

for appropriate values of coefficients $a_1, a_2, a_3, \ldots$ and exponents $q_1, q_2, q_3, \ldots$. The results will depend on making explicit the relationship between this type of C-fraction, and $h(1/x)$,
where $h(x)$ is the following type of continued fraction:

$$h(x) = \frac{x^{p_0}}{b_1 x^{p_1} + \frac{1}{b_2 x^{p_2} + \frac{1}{b_3 x^{p_3} + \cdots}}}.$$  \hspace{1cm} (2)

We will then be able to use classical results \[5\] to conclude our study and to examine interesting examples.

2 Review of known results

The first part of this section reviews the close link between power series and C-fractions. Note that the “C” comes from the word “corresponding”.

We commence with a power series

$$f_0(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots.$$ \hspace{1cm} (3)

We form the family of power series \{\(f_n(x)\)\} by the relations

$$f_{n+1}(x) = \frac{a_{n+1} x^{q_{n+1}}}{f_n(x) - 1}, \quad n = 0, 1, 2, \ldots,$$ \hspace{1cm} (4)

where the \(q_n\) are positive integers chosen together with complex numbers \(a_n\) in such a way that if \(f_n(x) \neq 1\), \(f_{n+1}(0) = 1\). If no \(f_n(x) = 1\), this process yields an infinite sequence of power series \(f_0(x), f_1(x), f_2(x), \ldots\). If some \(f_n(x) = 1\), the process terminates and yields a finite set of power series \(f_0(x), f_1(x), \ldots, f_n(x)\). The continued fraction

$$1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \frac{a_3 x^{q_3}}{1 + \frac{a_4 x^{q_4}}{1 + \cdots}}}},$$ \hspace{1cm} (5)

formed with these \(a_n\) and \(q_n\) is said to correspond to the power series (3) \[6, 11\]. Conversely, if we begin with a continued fraction of the form (5), we can form the \(n\)-th approximant \(\frac{A_n(x)}{B_n(x)}\) by means of the recurrence relations

\[
\begin{align*}
A_0 &= 1, & B_0 &= 1, \\
A_1 &= 1 + a_1 x^{q_1}, & B_1 &= 1, \\
A_n &= A_{n-1} + a_n x^{q_n} A_{n-2}, & B_n &= B_{n-1} + a_n x^{q_n} B_{n-2}, \\
& \quad n = 2, 3, \ldots
\end{align*}
\]
We have
\[ \frac{A_n(x)}{B_n(x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} = \frac{(-1)^{n-1}a_1a_2a_3 \cdots a_n x^{s_n}}{B_{n-1}(x)B_{n-2}(x)}, \]
where
\[ s_n = q_1 + q_2 + \cdots + q_n. \]

By equation (6) the Taylor development of the rational function \( \frac{A_n(x)}{B_n(x)} \) about the origin agrees with the development of \( \frac{A_{n-1}(x)}{B_{n-1}(x)} \) up to but not including the term in \( x^{s_n} \). Hence if (5) is nonterminating, the C-fraction (5) determines uniquely a corresponding power series.

We have the following classical result [11]

**Proposition 1.** If the continued fraction (5) corresponds to the power series (3), then the power series (3) corresponds to the continued fraction (5), and conversely.

A division-free algorithm for the construction of the C-fraction (5) from the power series (3) is given by Frank [2, 3].

If we start with a power series \( f(x) = \sum_{i=0}^{\infty} t_i x^i \), then by considering the sequence \( 1 + xf(x) \), which is in the form (3), we see that \( f(x) \) corresponds to a C-fraction of the form
\[ \frac{a_0 x^{q_0}}{1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \cdots}}} \]

for appropriate values of \( a_0, a_1, a_2, a_3, \ldots \) and \( q_0, q_1, q_2, q_3, \ldots \).

We now recall known results concerning the Hankel transform of sequences whose generating functions are of the form \( f(1/x) \) where \( f(x) \) can be expressed as a continued fraction of the form
\[ f(x) = \frac{b_0 x^{p_0}}{b_1 x^{p_1} + \frac{1}{\frac{b_2 x^{p_2}}{b_3 x^{p_3} + \frac{1}{b_4 x^{p_4} + \ldots}}}} \]  

(7)

We have the following result [5].

**Proposition 2.** Let \( h_n \) denote the Hankel transform of the sequence \([x^n]f(1/x)\) where \( f(x) \) has the form (7) (give conditions on \( b_0 = 1 \) and \( p_0 = 0 \)). Then \( h_n \) is zero for all \( n \) unless \( n = p_1 + p_2 + \cdots + p_m \), for some \( m \), in which case
\[ h_n = \prod_{i=1}^{m} (-1)^{\frac{p_i(p_i-1)}{2}} \cdot (-1)^{\sum_{i=0}^{m-1} p_{i+1}} \prod_{i=1}^{m} \frac{1}{b_i^{p_i+2 \sum_{j=i+1}^{m} p_j}}. \]  

(8)
3 Main result

In order to obtain our main result, we need to relate C-fractions of the form

$$g(x) = \frac{1}{1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \frac{a_3 x^{q_3}}{1 + \ldots}}}}$$

to continued fractions of the form

$$f(x) = \frac{x^{p_0}}{b_1 x^{p_1} + \frac{1}{b_2 x^{p_2} + \frac{1}{b_3 x^{p_3} + \ldots}}}.$$  

We wish to find the conditions under which $f(1/x) = g(x)$. We look at the case of unit coefficients first. By equation (8), the corresponding Hankel transforms will then take on values from the set \{-1, 0, 1\}.

By successive divisions above and below the line, we can cast $f(x)$ in the form

$$f(x) = \frac{x^{p_0-p_1}}{1 + \frac{x^{-p_1-p_2}}{1 + \frac{x^{-p_2-p_3}}{1 + \ldots}}}$$

and hence we have

$$f(1/x) = \frac{x^{-p_0+p_1}}{1 + \frac{x^{p_1+p_2}}{1 + \frac{x^{p_2+p_3}}{1 + \ldots}}}.$$  

Starting from $g(x)$ and proceeding to $f(x)$ is more problematic, since it is not clear what to choose as $p_0$. The Hankel transforms that we will be concerned with determine that we require the condition $-p_0 + p_1 = 0$, and hence that $p_1 = p_0$. We choose to set $p_0 = 1$. Then starting from the C-fraction

$$\frac{1}{1 + \frac{x^{q_1}}{1 + \frac{x^{q_2}}{1 + \frac{x^{q_3}}{1 + \ldots}}}}$$
we find the following continued fraction of type (2):

\[
x^{p_0} + \frac{1}{x^{q_1 - p_0} + \frac{1}{x^{q_2 - q_1 + p_0} + \frac{1}{x^{q_3 - q_2 + q_1 - p_0} + \ldots}}}\]

By Proposition (2), the position of the non-zero terms of the corresponding Hankel transform will be given by the indexing sequence

\[
p_0, p_0 + (q_1 - p_0), p_0 + (q_2 - q_1 + p_0), p_0 + (q_2 - q_1 + p_0) + (q_3 - q_2 + q_1 - p_0), \ldots \text{ or } p_0, q_1, q_2 + p_0, q_3 + q_1, q_4 + q_2 + p_0, \ldots
\]

This sequence can be realised by

\[
\begin{pmatrix}
m_0 \\
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 1 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 1 & 0 & 1 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
\vdots
\end{pmatrix}
\]

The \(n\)-th term of this sequence \(m_n\) is given by

\[
m_n = \sum_{k=0}^{n} \frac{1 + (-1)^{n-k}}{2} \tilde{q}_k = \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{k-i} \tilde{q}_i = \sum_{k=0}^{n} p_k,
\]

where \(\tilde{q}_0 = p_0, \tilde{q}_n = q_n\) for \(n > 0\), and \(p_n = \sum_{k=0}^{n} (-1)^{n-k} \tilde{q}_k\). Note that since the above matrix is \((\frac{1}{1-x^2}, x)\) as a Riordan array, then if the g.f. of the sequence \(q_1, q_2, q_3, \ldots\) is \(G(x)\), then the g.f. of the index set is

\[
\frac{1}{1-x^2}(1+xG(x)).
\]

We next note that if

\[
f(x) = \frac{b_0 x^{p_0}}{b_1 x^{p_1} + 1}
\]

is to be such that \(f(1/x)\) can be represented as

\[
g(x) = \frac{a_0 x^{q_0}}{1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \frac{a_3 x^{q_3}}{1 + \ldots}}}}
\]
then we must have
\[ a_k = \frac{1}{b_k b_{k+1}}. \]  
(9)

Reversing this set of equations, beginning with \( b_0 = 1 \), we find that
\[ b_{2n} = \frac{a_0 a_2 \cdots a_{2n-2}}{a_1 a_3 \cdots a_{2n-1}}, \]
and
\[ b_{2n+1} = \frac{a_1 a_3 \cdots a_{2n-1}}{a_0 a_2 \cdots a_{2n}}. \]
(See also [9], Theorem 3.6 and its corollaries). Substituting these values into Equation (8) and simplifying (where we take \( a_0 = 1, p_0 = 1 \)), gives us the main result of this note.

**Proposition 3.** The non-zero elements of the Hankel transform of the sequence with generating function given by the C-fraction

\[
\frac{1}{1 + \frac{a_1 x^{q_1}}{1 + \frac{a_2 x^{q_2}}{1 + \frac{a_3 x^{q_3}}{1 + \cdots}}}}
\]

are given by

\[ h_n = \prod_{i=1}^{m} (-1)^{\frac{p_i (p_i+1)}{2}} \cdot (-1)^{1+\sum_{i=0}^{m-i} p_{i+1}} \prod_{k=1}^{m} a_{\sum_{i=k}^{m} p_i}, \]

where

\[ p_i = \sum_{j=0}^{i} (-1)^{i-j} \tilde{q}_j \quad \text{and} \quad n = \sum_{k=0}^{m} p_i, \]

and the sequence \( \tilde{q}_n \) is given by \( 1, q_1, q_2, q_3, \ldots \).

**Example 4.** We consider the Fibonacci-inspired C-fraction

\[
\frac{1}{1 + \frac{x}{1 + \frac{2x^2}{1 + \frac{3x^3}{1 + \cdots}}}}
\]

where \( \tilde{q}_n = F_n + 0^n \) and \( a_n = F_n \). Then we find that the non-zero terms of the Hankel transform are indexed by

\[ \sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{k-i} (F_i + 0^i) = F_{i+1}. \]
The non-zero terms, calculated as

\[
\prod_{i=1}^{m}(-1)^{F_i(F_i+1)} \cdot (-1)^{1+\sum_{i=0}^{m-1} iF_i+1} \cdot \prod_{k=1}^{m} F_{\sum_{i=k}^{m} p_i},
\]

begin

1, 1, 1, −2, 72, 1944000, . . .

We see emerging here an interesting phenomenon, which we will naturally term “multiplicity”. In this case we note that \(F_2 = F_3 = 1\), corresponding to the first two 1’s of the non-zero Hankel elements above. The Hankel transform is thus given by

1, 1, −2, 0, 72, 0, 0, 0, 0, 0, 1547934105600000000, 0, 0, . . .

where the initial 1 has multiplicity 2.

**Example 5.** The Catalan numbers \(C_n = \frac{1}{n+1} \binom{2n}{n}\) provide an interesting example of the notion of multiplicity. They have generating function

\[
\frac{1}{1 - x} = \frac{1}{1 - \frac{x}{1 - \ldots}}
\]

and thus \(a_n = -1\) for all \(n > 0\) and \(q_n = 1\) for all \(n > 0\) (and hence \(\tilde{q}_n = 1\) for all \(n \geq 0\)). We then have

\[
\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \sum_{i=0}^{n} (-1)^{k-i} = \left\lfloor \frac{n + 2}{2} \right\rfloor.
\]

Thus the non-zero terms of the Hankel transform of \(C_n\) are indexed by

1, 1, 2, 2, 3, 3, 4, 4, 5, 5, . . .

These terms are all calculated to equal 1, as is well-known. Thus we can write the Hankel transform of \(C_n\) as

12, 12, 12, . . .

where the sub-index 2 indicates that each 1 occurs with “multiplicity” 2. This is a shorthand way of saying that the index set is 1, 1, 2, 2, 3, 3, . . .

**Example 6.** It is well known that the Hankel transform of the aerated Catalan numbers \(C_{\frac{n}{2}} = \frac{1+(-1)^n}{2}\) is also the all-1’s sequence. This sequence has generating function

\[
\frac{1}{1 - \frac{x^2}{1 - \ldots}}.
\]
where now \( a_n = -1 \) for all \( n > 0 \) and \( q_n = 2 \) for all \( n > 0 \) (and hence \( \tilde{q}_n = 2 - 0^n \) for all \( n \geq 0 \)). Now

\[
\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \sum_{i=0}^{n} (-1)^{k-i} (2 - 0^i) = n + 1,
\]

and hence the indexing set for this Hankel transform is \( 1, 2, 3, 4, 5, 6, \ldots \). That is, each \( 1 \) appears with multiplicity \( 1 \). Thus in a sense this is the original sequence with Hankel transform of all 1’s.

**Example 7.** The generalized Rogers-Ramanujan continued fraction. We consider the continued fraction

\[
\frac{1}{1 + \frac{\gamma x}{1 + \frac{\gamma x^2}{1 + \frac{\gamma x^3}{1 + \ldots}}}}.
\]

Here, \( \tilde{q}_n = n + 0^n \), and \( a_n = \gamma - \gamma 0^n = \gamma(1 - 0^n) \). We then have that \( p_n \) is the sequence

\[
1, 0, 2, 1, 3, 2, 4, 3, 5, 4, 6, \ldots,
\]

and \( \sum_{k=0}^{n} p_k \) is the sequence that begins

\[
1, 1, 3, 4, 7, 9, 13, 16, 21, 25, 31, \ldots
\]

The non-zero terms of the Hankel transform are, in order,

\[
1, 1, -\gamma^6, \gamma^{12}, \gamma^{32}, \gamma^{52}, -\gamma^{94}, \gamma^{136}, \gamma^{208}, \gamma^{280}, -\gamma^{390}, \ldots
\]

The exponent sequence

\[
0, 0, 6, 12, 32, 52, 94, \ldots
\]

can be shown to have generating function

\[
\frac{2x^2(x^3 + 3)}{(x + 1)^2(x - 1)^3}.
\]

### 4 Conclusion

Since to each sequence \( a_n \) there corresponds the power series \( \sum_{k=0}^{\infty} a_n x^n \), and to each power series there corresponds a C-fraction, the foregoing gives, in theory, a closed form formula for the Hankel transform of each sequence. Of course, this presupposes that the passage from generating function to C-fraction can be effected easily. The Q-D algorithm is one method for this.
We note that Heilermann’s formula [7, 8] for the Hankel transform of a sequence with generating function of the form

\[ \frac{1}{1 - \alpha_1 x - \frac{\beta_1 x^2}{1 - \alpha_2 x - \frac{\beta_2 x^2}{1 - \ldots}}} \]

can be derived from the above result, due to the fact that \( p_i = 1 \) in this case, and the fact that although in this note Equation (8) has been expressed in the case of monomials \( b_i x^{p_i} \), the result continues to be true for polynomials \( Q_{p_i}(x) = b_i x^{p_i} + \cdots \) of degree \( p_i \).

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2010 *Mathematics Subject Classification*: Primary 30B70; Secondary 11B83, 11C20, 15A15, 30B10

*Keywords*: C-fraction, continued fraction, power series, Hankel determinant, Hankel transform, sequences

Concerns sequences

A000045, A000108