Explicit Generation of Integer Solutions via CY manifolds

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Abstract

Metrics on Calabi-Yau manifolds are used to derive a formula that finds the existence of integer solutions to polynomials. These metrics are derived from an associated algebraic curve, together with its anti-holomorphic counterpart. The integer points in the curve coincide with points on the manifold, and the metric form around these points are used to find their existence. The explicit form of the metrics can be found through a solution to the D-terms in a non-linear sigma model.
The metrics on general CY manifolds have recently been computed in [1], [2]. This task was accomplished by using a field theoretic count of classical tree graphs in scalar field theories\(^1\) In prior work, the metrics of these Calabi-Yau manifolds was shown to provide means to finding both solutions to systems of algebraic equations and non-linear partial differential equations [3], [4] (with related work in [5]-[7]). In this work explicit formulae are given that generate the integer solutions to polynomial equations using the metric form of these manifolds.

**Metric Expansions and Integer Solutions**

The starting point is the algebraic equation

\[
\sum a_\sigma \prod z_{\sigma(i)}^{\rho(i)} = 0 ,
\]

with \(\rho(i)\) labeling the exponent of the coordinate \(z_{\sigma(i)}\), whose index is labeled by the set \(\sigma(i)\). To every one of these equations, or sets of these equations, there is a Calabi-Yau metric. These metrics can be formulated by a quotient construction using for example a non-linear \(\mathcal{N} = 2\) sigma model. The D-term solution of these models generates the Kähler potential, and the D-terms were solved in [2].

The Kähler potential on these metrics, in the patch containing the coordinates at \(\phi_i = 0\) has the expansion,

\[
K = \sum b_\omega \prod \phi_{\omega(i)} .
\]

Due to the polynomial form in (1), there is a resolvable singularity at the origin at \(\phi_i = 0\) in the metric derived from the Kähler potential. This is not apparent in the solution obtained to [2] due to the classical field tree diagram count in [1], [2], but it can be obtained by resumming the series with the numbers \(b_\omega\). The resummations of these \(b_\omega\) seems complicated, but there is an alternative to obtaining the analytic continuation that extracts the branch cuts.

The expansion to the metric about any integer sets \(z_i = p_i\) to the equation in (1) can be obtained by following the same classical graph count in [1]. Expand the form \(\phi_i = y_i + p_i\), and reexpress the metric as an expansion

\(^{1}\)A subtlety associated with a deformation parameter can be circumvented with a variant of the procedure involving background \(\phi\) line dependence.
\[ \mathcal{K} = \sum b_\omega(p_i) \prod y_{\omega(i)}. \tag{3} \]

The coefficients \( b_\omega(p_j) \) can be obtained, and they have a similar form as to \( b_\omega(i) \) but with dependence on the integers \( p_j \). An important feature is that the solution to the ‘metric’ even in the presence of a set of integers \( p_j \) that do not solve the algebraic equation can be found. The metric around these points is not expected to be Ricci-flat, and it will not contain the required branch cut representing the resolvable singularity at the origin; in this case the origin is at \( y_i = 0 \).

Because the metric can be found at any possible sets of integers \( p_j \), the presence of an actual integer solution can be determined by finding whether or not there is a branch cut at the origin about the sets of integers. The analytic continuation of the sums in \( b_\omega(p_j) \) are required in order to determine this presence. However, the explicit form of these coefficients is known due to the solution of the D-terms, which is found by the classical graph count \([1]-[2]\).

The analytic solution to the metric can also be used to obtain the solution to sets of arbitrary algebraic equations, including those of Fermat.

**Analytic Continuation**

There are various ways to analytically continue the infinite sum in the coordinates to find the branch cuts at the origin. A direct sum is a bit problematic due to the complicated form of the coefficients \( b_\omega(p_j) \), although this could be done. The Kähler potential, and the metric, form about the origin can be obtained by setting all coordinates to \( \phi_i = \phi \), with \( \phi \) about the origin at \( \phi = 0 \).

This analytic continuation is simplified with the notation

\[ g = \sum a_n(p_i)x^n, \tag{4} \]

with \( x^n \) representing the terms \( \phi^n \) after substituting \( \phi_i = \phi \). The coefficients \( a_n \) are found from the coefficients \( b_\omega(p_i) \), which have been computed with the D-term solution.

The same series in (4) has the expansion in terms of logarithms

\[ g = \sum c_n(p_i) \ln^n(x), \tag{5} \]
which manifests the branch at the origin. The coefficients $c_n$ can be found from those $a_n$. Two explicit contour integrals around the origin will show whether or not there is a branch cut in the series $\Pi$.

The coefficients $c_n$ in terms of $a_n$ are found by differentiation at $x = 1$. In terms of $x$, the Kähler potential derivatives evaluated at $x = 1$ are,

$$\partial^n_x g(x) = \sum_n a_n(p_i) \frac{n!}{(n - b)!} ,$$

and in terms of $\ln(x)$, the derivatives are,

$$\partial^n_x g(\ln(x))|_{x=1} = \sum_n c_n(p_i) \partial^n_x \ln(x) .$$

The identification

$$\partial^n_x g(x)|_{x=1} = \partial^n_x g(\ln(x))|_{x=1} ,$$

generates the identification of the coefficients. These identifications are found in closed form.

The analytic continuation of the potential about the origin should have the form,

$$g(x) = ax^\delta + \ldots ,$$

which follows from the removable singularity occurring in the quotient description of the metric. Multiple branch cuts at points $|x| \leq 1$ would not be physical in view of the quotient of the polynomial $Z_i \rightarrow G \cdot Z_i$, which describes the singularity. The coefficients $a$ and $\delta$ can be found by two successive contour integrals around the origin.

One contour integration follows from,

$$I_1 = \oint dx \ ax^\delta + \ldots = \int_{\omega = 0}^1 d\omega \ ae^{2\pi i \delta \omega} = \frac{a}{2\pi i \delta} \left(e^{2\pi i \delta} - 1\right) .$$

$$= \frac{a}{2\pi i \delta} \left(\cos(2\pi \delta) + i \sin(2\pi \delta) - 1\right) .$$

The real and imaginary parts of this integral generate the $a$ and $\delta$, involving the inversion of the sin and cos function. All of the terms in the integrand which have integral powers of $x$ integrate to zero. The solution to $a$ and $\delta$ follows from,
\[
\frac{I_1^R}{I_1 - 1} = -\tan(2\pi\delta),
\]
which can be used to find \( \delta \). Substituting this parameter into (11) determines \( a \). Unfortunately, the inversion of a \( \tan \) function is required, which slightly complicates the determination of \( \delta \). The determination of \( I_1 \) follows from integrating the logarithmic form of the potential with the same contour.

A second contour integral is,

\[
I_2 = \oint dx \, a x^\delta + \ldots = \frac{a}{2\pi i \delta} \int_{\omega=0}^1 d\omega \left( e^{2\pi i \delta \omega} - 1 \right),
\]

\[
= \frac{a}{2\pi i \delta} \left( \frac{1}{2\pi i \delta} e^{2\pi i \delta} - \frac{1}{2\pi i \delta} - 1 \right).
\]

The remaining terms in the series integrate to,

\[
\oint d\omega \, a e^{2\pi in\omega} = 0,
\]

and

\[
\oint \oint d\omega \, a e^{2\pi in\omega} = -\frac{a}{2\pi in}.
\]

The second integral could be useful with further information of the analytic continuation.

The determination of \( I_1 \) follows from the same contour integration as used to determine \( a \) and \( \delta \). The individual terms integrate as

\[
\oint d\omega \, \ln(e^{2\pi i \omega})^n = \int_0^1 d\omega \, (2\pi i \omega)^n = (2\pi i)^n \frac{1}{n+1},
\]

and gives the form,

\[
I_1 = \sum_{n=0}^{\infty} c_n(p_i) \frac{(2\pi i)^n}{n+1}.
\]

The evaluation of \( I_1 \) is simple, but the coefficients of \( c_n(p_i) \) are found from the more complicated Calabi-Yau data. These coefficients can be found from the classical graph count.
**Solution to Polynomials**

Given the solution to the integral $I_1$, which leads to $a$ and $\delta$ through its real and imaginary parts, the counting of the solutions to the polynomials follows from the non-integrality of the parameter $\delta$. The function $\tan(2\pi \delta)$ vanishes whenever $\delta = n$; the vanishing of the function indicates the non-presence of the polynomial solution.

The singularity in $\delta$ can be found from the algebraic curve. With this value, the summation of the integers $p_i$ generates the allowed solutions to the polynomials; normalizing the $\tan(2\pi \delta)$ in the sum generates unity and a direct count of the integer solutions. Also, the individual polynomial solutions are found by a non-vanishing of the number $\tan(2\pi \delta)$, which is unity after normalization. A Heaviside step function would work also without normalization, which involves a Fourier transformation, but this is more complicated.

The existence of a polynomial solution is found from

$$C(p_i) = -\tan(2\pi \delta)^{-1} \arctan\left( \frac{I^R(p_i)}{I^I(p_i) - 1} \right), \quad (19)$$

which is either one or zero, and $I = I_1$. The complete sum,

$$N = \sum_{p_i} C(p_i), \quad (20)$$
generates the total number of solutions. These functions $C(p_i)$ and $N$ depend on the curve, and are quite explicit due to the explicit form of the Calabi-Yau metric.

**Discussion**

The explicit form of the Calabi-Yau metrics permits a closed form solution to the existence of integer solutions to polynomials. This closed form requires some complicated sums, due to the form of the metric expanded around integer points.

A counting function is given that allows the determination of the solutions. The sum over the integers generates the totality of these integer solutions to the polynomial equation $P(z) = 0$, or systems of polynomial equations $P_i(z_j) = 0$. The formulae are quite explicit in terms of the metric data on the associated Calabi-Yau metric, expanded about integer points. The well known example of Fermat's equations, or their generalizations, are an example. The counting functions are derived from the explicit form of the metrics associated to the curves, with help from summations.
References

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