A Note on the Classification of Permutation Matrix

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Abstract

This paper is concentrated on the classification of permutation matrix with the permutation similarity relation, mainly about the canonical form of a permutational similar equivalence class, the cycle matrix decomposition and the factorization of a permutation matrix or monomial matrix.

Key Words: permutation matrix, monomial matrix, permutation similarity, canonical form, cycle matrix decomposition, cycle factorization

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1 Introduction

The incidence matrix of a projective plane of order $n$ is a 0-1 matrix of order $n^2 + n + 1$. Two projective planes will be isomorphic if the incidence matrix of one projective plane could be transformed by permuting the rows and/or columns to the incidence matrix of the other one. After sorting the rows and columns, the incidence matrix of a projective plane could be reduced in standard form (not unique). In the reduced form, the incidence matrix could be divided into some blocks. Most blocks are permutation matrices (refer [4]). If we keep the incidence matrix reduced and keep the position of every block when permuting the rows and columns of the reduced incidence matrix, every permutation matrix will be transformed into another matrix that is permutationally similar to the original one (refer [5] sec. 1.2).
This paper will focus on the permutational similarity relation and the classification of the permutation matrices. It will demonstrate the standard structure of a general permutation matrix, the canonical form of a permutation similarity class, how to generate the canonical form and solve some other related issues. The main theorems will be proved by two different methods (linear algebra method and the combinatorial method). The number of permutational similarity classes of permutation matrices of order \( n \) will be mentioned on Section 5. A similar factorization proposition on monomial matrix will be introduced at the end.

2 Preliminary

Let \( n \) be a positive integer, \( P \) be a square matrix of order \( n \). If \( P \) is a binary matrix (every entry in it is either 0 or 1, also called 0-1 matrix or (0, 1) matrix) and there is a unique “1” in its every row and every column, then \( P \) is called a permutation matrix. If we substitute the units in a permutation matrix by other non-zero elements, it will be called a monomial matrix or a generalized permutation matrix.

There is a reason for the name “permutation matrix”. A permutation matrix \( P \) of order \( n \) multiplies a matrix \( T \) of size \( n \times r \) (from the left side of \( T \)) will result the permutation of rows of \( T \). If \( U \) is a matrix of size \( t \) by \( n \), that \( P \) acts on \( U \) from the right hand side of \( U \) will leads to the permutation of columns of \( U \).

Let \( k \) be a positive integer greater than 1, \( C \) be an invertible \((0, 1)\) matrix of order \( k \), if \( C^k = I_k \) (\( I_k \) is the identity matrix of order \( k \)) and \( C^i \neq I_k \) for any \( i \) (\( 1 < i < k \)), then \( C \) will be called a cycle matrix of order \( k \). A cycle matrix of order \( k \) in this form

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & \ldots \\
\ldots & \ldots \\
1 & 0 
\end{bmatrix}
\]

will be called a standard cycle matrix. Sometimes, the identity matrix of order 1 can also be considered as a cycle matrix of order 1.

If \( C_1 \) is a permutation matrix of order \( n \), and there are exact \( k \) entries in diagonal being 0, (here \( 2 \leq k \leq n \)), if \( C^k = I_n \) and \( C^i \neq I_n \) for any \( i \) (\( 1 \leq i < k \)), then \( C_1 \) will be called a Generalized Cycle Matrix of Type I with cycle order \( k \).

If \( C_2 \) is a \((0, 1)\) matrix of order \( n \), rank \( C_2 = k \), and there are exact \( k \) entries in \( C_2 \) is non-zero, (\( 2 \leq k \leq n \)), if \( C^k \) is a diagonal of rank \( k \), too, and \( C^i \) is non-diagonal (\( 1 \leq i < k \)), then \( C_2 \) will be called a Generalized Cycle Matrix of Type II with cycle order \( k \). Obviously, a generalized cycle matrix of type II plus some suitable diagonal \((0, 1)\) matrix will result a Generalized Cycle Matrix of Type I with the same cycle order.

Let \( A \) and \( B \) be two monomial matrix of order \( n \), if there is a permutation matrix \( T \) such that \( B = T^{-1}AT \), then \( A \) and \( B \) will be called permutationally similar. Of course the permutation similarity relation is an equivalence relation. Hence the set of the permutation matrices (or monomial matrix) or order \( n \) will be split into some equivalence classes.
3 Main Result

There are some problems presented naturally:

1. What’s the canonical form of a permutation similarity class?
2. How to generate the canonical form of a given permutation matrix?
3. If $B$ is the canonical form of the permutation matrix $A$, how to find the permutation matrix $T$, such that $B = T^{-1}AT$?

**Theorem 1.** (Decomposition Theorem) For any permutation matrix $A$ of order $n$, if $A$ is not identical, then there are some generalized cycle matrices $Q_1, Q_2, \cdots, Q_r$ of type II and a diagonal matrix $D_t$ of rank $t$, such that, $A = Q_1 + Q_2 + \cdots + Q_r + D_t$, where the non-zero elements in $D_t$ are all ones, $\sum_{i=1}^{r} \text{rank}Q_i + t = n; 1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor, r, P_i (i = 1, 2, \cdots, r)$ and $D_t$ are determined by $A$.

If the cycle order of $Q_i$ is $k_i, (i = 1, 2, \cdots, r)$, then $2 \leq \sum_{i=1}^{r} k_i \leq n$. When $A$ is a cycle matrix, $t = 0, r = 1, k_1 = n$.

**Theorem 2.** (Factorization Theorem) For any permutation matrix $A$ of order $n$, if $A$ is not identical, then there are some generalized cycle matrices $P_1, P_2, \cdots, P_r$ of type I, such that, $A = P_1P_2\cdots P_r$, where $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor; r, P_i (i = 1, 2, \cdots, r)$ are determined by $A$. $P_{i_1}$ and $P_{i_2}$ commute $(1 \leq i_1 \neq i_2 \leq r)$.

If the cycle order of $P_i$ is $k_i, (i = 1, 2, \cdots, r)$, then $2 \leq \sum_{i=1}^{r} k_i \leq n$.

**Theorem 3.** (Similarity Theorem) For any permutation matrix $A$ of order $n$, there is a permutation matrix $T$, such that, $T^{-1}AT = \text{diag} \{I_t, N_{k_1}, \cdots, N_{k_r}\}$, where $N_{k_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & \ddots \\ & & \ddots \\ & & & 1 \\ & & & & 0 \end{bmatrix}$ is a cycle matrix of order $k_i$ in standard form, $(i = 1, 2, \cdots, r)$,

$2 \leq k_1 \leq k_2 \cdots \leq k_r, 0 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor, 0 \leq t \leq n, \text{ and } \sum_{i=1}^{r} k_i + t = n$. $T, t, r, k_r$ are determined by $A$.

If $A$ is a identity matrix, then $t = n, r = 0$. When $A$ is a cycle matrix, $t = 0, r = 1, k_1 = n$. In this theorem, the quasi-diagonal matrix (or block-diagonal matrices) $\text{diag} \{I_t, N_{k_1}, \cdots, N_{k_r}\}$ will be called the canonical form of a permutation matrix in permutational similarity relation.
Here a proof by linear algebra method is presented.

**Proof of Theorem 3**

For any permutation matrix $A$ of order $n$, let $\mathcal{A}$ be a linear transformation defined on the vector space $\mathbb{R}^n$ with bases $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$, $i = 1, 2, \cdots, n$, such that $A$ is the matrix of the transformation $\mathcal{A}$ in the basis $\mathcal{B}$, or for any vector $\alpha \in \mathbb{R}^n$ with the coordinates $x$ (in the basis $\mathcal{B}$), the coordinates of $\mathcal{A}\alpha$ is $Ax$, i.e., $\mathcal{A}\alpha = B\alpha$. Here the coordinates are written in column vectors.

It is clear that the coordinates of $e_i$ in the basis $\mathcal{B}$ is $(0, \cdots, 0, 1, 0, \cdots, 0)^T$. Since $A$ is a permutation matrix, $Ae_i$ is the $i$th column of $A$.

Let $S = \{1, 2, \cdots, n\}$, $C = \{e_i \mid i \in S\}$, $a_{11} = \min S$, $F_1 = [a_{11}]$, $G_1 = [e_{a_{11}}]$. (Here $F_1$ and $G_1$ are sequences, or sets equipped with orders) Of course $Ae_{a_{11}} \in C$. If $Ae_{a_{11}} \neq e_{a_{11}}$, assume $e_{a_{12}} = Aa_{11}$, then $a_{12}$ and $e_{a_{12}}$ to the ends of the sequences $F_1$ and $G_1$, respectively. If $Ae_{a_{1,j}} \neq e_{a_{11}}$, suppose $e_{a_{1,(j+1)}} = Aa_{1,j}$, (i.e., $A^j e_{a_{11}} = e_{a_{1,(j+1)}}$), then add $a_{1,(j+1)}$ and $e_{a_{1,(j+1)}}$ to the ends of sequences $F_1$ and $G_1$, respectively ($j = 1, 2, \cdots$). Since $Ae_i \in C \forall i \in S$, there will be a $k'_i$ such that $Ae_{a_{1,k'_i}} = e_{a_{11}}$ (otherwise the sequence $e_{a_{11}}, A^2 e_{a_{11}}, A^3 e_{a_{11}}, \cdots$ will be infinite). Suppose that $k'_1$ is the minimal integer satisfying this condition (1 $\leq k'_1 \leq n$). It is clear that $A^{k'_1} e_{a_{11}} = e_{a_{11}}, A^{k'_1} e_{a_{11}} = e_{a_{11}}, (1 \leq j \leq k'_1)$. It is possible that $k'_1 = 1$ or $n$. So, at last $|F_1| = |G_1| = k'_1$. Then remove the elements in $G_1$ from $C$, remove the elements in $F_1$ from $S$.

Now, if $S \neq \emptyset$, let $a_{21} = \min S$, $F_2 = [a_{21}], G_2 = [e_{a_{21}}]$. It is clear that $Ae_{a_{21}} \in C$. \(1 \leq k'_1 \leq n\) If $A^{-1} e_{a_{21}} \neq e_{a_{21}}$, suppose $A^{-1} e_{a_{21}} = e_{a_{21}}$ ($i = 2, 3, \cdots$), then add $a_{2i}$ and $e_{a_{2i}}$ to the ends of the sequences $F_2$ and $G_2$, respectively. There will be a $k_2$, such that $A^{k_2} e_{a_{21}} = e_{a_{21}}$ (let $k_2$ be the minimal integer satisfying this condition. It is possible that $k_2 = 1$ or $n - k'_1$). Obviously, $A^{k_2} e_{a_{21}} = e_{a_{21}}$, $1 \leq i \leq k'_2$). Then remove the elements in $G_2$ from $C$, remove the elements in $F_2$ from $S$. If $S \neq \emptyset$, continue this step and constructing $F_3, F_4, \cdots$ and $G_3, G_4, \cdots$. It will stop since $n$ is finite.

Assume that we have $F_1, F_2, \cdots, F_u$ and $G_1, G_2, \cdots, G_u$, such that $\bigcup_{i=1}^{u} F_i = \{1, 2, \cdots, n\}$, $\bigcup_{i=1}^{u} G_i = \{e_1, e_2, \cdots, e_n\}$, $F_i \cap F_j = G_i \cap G_j = \emptyset$, ($1 \leq i \neq j \leq u$). There is a possibility that $u = 1$ (when $A$ is a cycle matrix of order $n$) or $n$ (when $A$ is an identity matrix). Sort $F_1, F_2, \cdots, F_u$ by cardinality, we will have $|F'_1| \leq |F'_2| \leq \cdots \leq |F'_u|$. Then

1 Here the regular letter “T” in the upper index means transposition.
2 Otherwise $Ae_{a_{21}} \in G_1$, since all the elements in $G_1$ are removed from $C$, then there is a $k_0$, s. t. $A^{k_0} e_{a_{11}} = A e_{a_{21}}$, of course $k_0 \neq 0$, so, $A^{k_0 - 1} e_{a_{11}} = e_{a_{21}}$ as $A$ is invertible, which means that $e_{a_{21}} = A^{k_0 - 1} e_{a_{11}}$ is in the set $G_1$. Contradiction.
Proof of Theorem 1

With \( F'_t \) generated above, construct a 0-1 matrix \( D_t \) of order \( n \), such that the \( j \)’th column of \( D_t \) is the \( j \)’th column of \( A (j \in \bigcup_{i=1}^{t} F'_i) \) and the other columns of \( D_t \) are 0 vectors. Of course, \( D_t \) is a diagonal matrix of rank \( t \), as the \( j \)’th column of \( A \) is \( e_j \) (by definition, \( A e_j = e_j \)). Construct a 0-1 matrix \( Q_i \) \((i = 1, 2, \ldots, r)\) of order \( n \), such that the \( j \)’th column of \( Q_i \) is the \( j \)’th column of \( A (j \in F'_{t+i}) \) and the other columns of \( Q_i \) are 0 vectors. As \( \bigcup_{i=1}^{t} F'_i \bigcup \bigcup_{i=1}^{r} F'_{t+i} = \bigcup_{i=1}^{u} F_i = \{1, 2, \ldots, n\} \), and \( F_{i_1} \cap F_{i_2} = \emptyset \), \((1 \leq i_1 \neq i_2 \leq u)\), so every column of \( A \) appears exact once in a matrix in the expression \( \sum_{i=1}^{r} Q_i + D_t \) in the same position as it appears in \( A \), besides, the columns in the same position in all the other addend matrices in the expression \( \sum_{i=1}^{r} Q_i + D_t \) are all 0 vectors, so,

\[
\sum_{i=1}^{r} Q_i + D_t = A.
\]

Now we prove that \( Q_i \) is a generalized cycle matrix of type II with cycle order \( k_i \).

Suppose the members in \( F'_{t+i} \) \((i = 1, 2, \ldots, r)\) are \( a'_{i,1}, a'_{i,2}, \ldots, a'_{i,k_i} \). Suppose \( F'_{t+i} = F_s \) for some \( s \) \((1 \leq s \leq u)\). By the definition of \( F_s \), we know that \( A e_{a'_{i,v}} = e_{a'_{i,v+1}} \) \((v = 1, 2, \ldots, k_i)\),
Therefore \( Q_i = 0 \), so the \( a_{i,v} \)th column of \( A \).

As \( Q_i \) is made of the 0 vector and \( k_i \) columns of \( A \). The columns of \( A \) are linear independent, so the rank of \( Q_i \) is \( k_i \). While the \( a_{i,v} \)th column of \( Q_i \) is the \( a_{i,v} \)th column of \( A \), so \( Q_i e_{a_{i,v}} = e_{a_{i,v+1}} \) (\( v = 1, 2, \cdots, k_i - 1 \)), \( Q_i e_{a_{i,k_i}} = e_{a_{i+1}} \), \( Q_i e_1 = 0 \) (\( \forall e_1 \in D \setminus G'_{i+1} \)).

Therefore \( Q_i e_{a_{i,v}} = e_{a_{i,v+1}} \) (\( v = 1, 2, \cdots, k_i - 1 \)), \( Q_i e_{a_{i,1}} = e_{a_{i,1}} \), (so \( Q_i \) is not diagonal as \( Q_i e_{a_{i,v}} = e_{a_{i,v+1}} \neq e_{a_{i,1}} \)). Then \( Q_i e_{a_{i,v}} = Q_i^{v-1} \left( Q_i^{k_i-v+1} e_{a_{i,v}} \right) = Q_i^{v-1} \left( e_{a_{i,1}} \right) = e_{a_{i,v}} \), (\( v = 1, 2, \cdots, k_i \)). Hence \( Q_i \) is a diagonal matrix of rank \( k_i \). Therefore \( Q_i \) is a generalized cycle matrices of type II with cycle order \( k_i \).

**Proof of Theorem 2**

Let \( D_i = I_n - D_t \). Construct a 0-1 matrix \( J_i^a \) (\( i = 1, 2, \cdots, r \)) of order \( n \), such that the \( j \)th column of \( J_i^a \) is the \( j \)th column of \( I_n \) (\( j \in F'_{i+t} \)) and the other columns of \( J_i^a \) are 0 vectors. Let \( J_i^b = I_n - J_i^a \). So \( \sum_{i=1}^r J_i^a + D_t = I_n \). It is obvious that \( D_i D_t = D_t D_i \).

\[
J_i^a J_i^b = J_i^b J_i^a = 0, \quad Q_i J_i^a = J_i^a Q_i = 0, \quad \text{and} \quad Q_i J_i^a = Q_i J_i^b = J_i^b Q_i = 0 \quad (1 \leq i_1 \neq i_2 \leq r).
\]

Although \( J_i^a J_i^a = J_i^a J_i^a = 0 \), but \( J_i^a J_i^b = J_i^b J_i^a = J_i^a J_i^a = 0 \). It is not difficult to prove that \( J_i^a J_i^b = J_i^b J_i^a = I_n - J_i^a - J_i^a \).

Let \( P_i = Q_i + J_i^a \), so rank \( P_i = n \). \( I_n + Q_i = P_i + J_i^a \).

\[
P_{11} P_{12} = \left( Q_1 + I_n - J_i^a \right) \left( Q_1 + I_n - J_i^a \right) = Q_1 + Q_2 + I_n - J_i^a - J_i^a,
\]

then

\[
\prod_{i=1}^r P_i = \prod_{i=1}^r \left( Q_i + I_n - J_i^a \right) = \sum_{i=1}^r Q_i + I_n - \sum_{i=1}^r J_i^a = \sum_{i=1}^r Q_i + D_t = A.
\]

We can prove the equality above in another way.

It is clear that \( Q_i Q_{12} = 0 \). \( Q_i D_t = D_t Q_i = 0 \) (\( 1 \leq i_1 \neq i_2 \leq r \)). So

\[
(I_n + D_t) \prod_{i=1}^r (I_n + Q_i) = I_n + \sum_{i=1}^r Q_i + D_t = I_n + A. \tag{4.1}
\]

Since \( D_t Q_i = Q_i D_t = 0 \), so \( D_t J_i^{a} = J_i^{a} D_t = 0 \). By definition, when \( 1 \leq i \leq r \), the \( v \)th column or the \( v \)th row of \( P_j \) with \( v \in F'_{i+t} \cup \bigcup_{i=1}^t F_i \) are same as the \( v \)th column or the \( v \)th row of \( I_n \), respectively, (since the \( v \)th column or the \( v \)th row of \( P_j \) (\( 1 \leq j \leq r \), \( j \neq i \)) with \( v \in F'_{i+t} \cup \bigcup_{i=1}^t F_i \) are same as the \( v \)th column or the \( v \)th column or the \( v \)th row of \( I_n \), respectively, where \( e_{a_{i,v}} \in G_s \).
row of \( I_n \), so when \( J_i^{(a)} \) is multiplied by \( \prod_{1 \leq j \leq r} P_j \) (no matter from the left or right), the

\( v' \)th column and the the \( v' \)th row will not change, while the other columns and rows of

\( J_i^{(a)} \) is 0 vector, so \( J_i^{(a)} \prod_{1 \leq j \leq r, j \neq i} P_j = J_i^{(a)} \). For the same reason, \( D_l \prod_{i=1}^r P_i = D_l \). As

\[
(I_n + D_l) \prod_{i=1}^r (I_n + Q_i) = (I_n + D_l) \prod_{i=1}^r (P_i + J_i^{(a)})
\]

\[
= (I_n + D_l) \left( \prod_{i=1}^r P_i + \sum_{i=1}^r \left( J_i^{(a)} \prod_{1 \leq j \leq r, j \neq i} P_j \right) \right) = (I_n + D_l) \left( \prod_{i=1}^r P_i + \sum_{i=1}^r J_i^{(a)} \right)
\]

\[
= \prod_{i=1}^r P_i + \sum_{i=1}^r J_i^{(a)} + D_l \prod_{i=1}^r P_i + D_l \sum_{i=1}^r J_i^{(a)} = \prod_{i=1}^r P_i + \sum_{i=1}^r J_i^{(a)} + D_l + 0 = \prod_{i=1}^r P_i + I_n,
\]

that is,

\[
(I_n + D_l) \prod_{i=1}^r (I_n + Q_i) = \prod_{i=1}^r P_i + I_n. \tag{4.2}
\]

By equations (4.1) and (4.2), we will have \( \prod_{i=1}^r P_i + I_n = I_n + A \),

so \( \prod_{i=1}^r P_i = A \).

Now we prove that \( P_i \) is a generalized cycle matrix of type II with cycle order \( k_i \).

Since \( P_i = Q_i + J_i^{(b)} \), \( Q_i J_i^{(b)} = J_i^{(b)} Q_i = 0 \), so \( P_i^m = Q_i^m + \left(J_i^{(b)}\right)^m = Q_i^m + J_i^{(b)} (\forall m \in \mathbb{Z}^+) \),

then \( P_i^m = Q_i^m + J_i^{(b)} \). As \( Q_i^k e_{a', v} = A^k e_{a', v} = e_{a', v} \), \( (v = 1, 2, \cdots, k_i) \); \( Q_i e_l = 0 \implies Q_i^k e_l = 0 \) \((\forall e_l \in D \setminus G_{t+i}^c)\). On the other hand, \( J_i^{(b)} e_{a', v} = 0 \) \((v = 1, 2, \cdots, k_i) \), \( J_i^{(b)} e_l = e_l \) \((\forall e_l \in D \setminus G_{t+i}^c)\). So for any \( e_l \in B \), \( e_l \in C_i^c \), \( Q_i^k e_l = J_i^{(b)} e_l = e_l \), if \( e_l \notin G_i^c \), \( \left(Q_i^k + J_i^{(b)}\right) e_l = J_i^{(b)} e_l = e_l \). That means \( P_i^k e_l = \left(Q_i^k + J_i^{(b)}\right) e_l = e_l \) \((\forall e_l \in B)\). So \( P_i^k (e_1, e_2, \cdots, e_n) = (e_1, e_2, \cdots, e_n) \), or \( P_i^k I_n = I_n \); hence \( P_i^k I_n = I_n \). (Actually, \( Q_i^k = J_i^{(a)} \), so \( P_i^k = Q_i^k + J_i^{(b)} = J_i^{(a)} + J_i^{(b)} = I_n \).) When \( 1 \leq m < k_i \), \( Q_i^m \) is not diagonal, neither is \( Q_i^m + J_i^{(b)} = P_i^m \). So \( P_i \) is a generalized cycle matrix of type II with cycle order \( k_i \).
For instance, for the matrix $P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $P_1 e_1 = e_5$, $P_1 e_5 = e_1$, so $F_1 = [1, 5]$, $|F_1| = 2$; $P_1 e_2 = e_3$, $P_1 e_3 = e_4$, $P_1 e_4 = e_2$, so $\bar{F}_2 = [2, 3, 4]$, $|F_2| = 3$; $P_1 e_6 = e_6$, $F_3 = [6]$, $|F_3| = 1$. So $P_1$ is permutationally similar to the canonical form $B_1 = \text{diag}\{I_1, N_2, N_3\} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, or $P_1 \{e_6; e_1, e_5; e_2, e_3, e_4\} = \{e_6; e_5, e_1; e_3, e_4, e_2\} = \{e_6; e_1, e_5; e_2, e_3, e_4\}$.

As $\{e_6; e_1, e_5; e_2, e_3, e_4\} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, let $T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then $T_1^{-1} = T_1^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, so $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $(P_1 = T_1 B_1 T_1^{-1})$. 

4 Proof
5 On the Number of Permutation Similarity Classes

The number of permutation similarity classes of permutation matrices of order \(n\) is the partition number \(p(n)\). There is a recursion for \(p(n)\),

\[
    p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots + \\
    (-1)^{k-1}p\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots, \\
    = \sum_{k=1}^{k_1} (-1)^{k-1}p\left(n - \frac{3k^2 + k}{2}\right) + \sum_{k=1}^{k_2} (-1)^{k-1}p\left(n - \frac{3k^2 - k}{2}\right),
\]

where

\[
    k_1 = \left\lfloor \frac{\sqrt{24n + 1} - 1}{6} \right\rfloor, \quad k_2 = \left\lfloor \frac{\sqrt{24n + 1} + 1}{6} \right\rfloor,
\]

and assume that \(p(0) = 1\). Here \(\lfloor x \rfloor\) is the floor function, it stands for the maximum integer that is less than or equal to the real number \(x\).

We may find a famous asymptotic formula for \(p(n)\) in references [11] or [1],

\[
    p(n) \sim \frac{1}{4n\sqrt{3}}\exp\left(\sqrt{\frac{2}{3}\pi n^{1/2}}\right).
\]

This formula is obtained by Godfrey H. Hardy and Srinivasa Ramanujan in 1918 in the famous paper [3]. (In [2] and [9], we can find two different proofs of this formula. The evaluation of the constants can be found in [8].)

Formula (5.3) is very important for analysis in theory. It is very convenient to estimate the value of \(p(n)\) especially for ordinary people not majoring in mathematics. \(^4\) But the accuracy is not so satisfying when \(n\) is small.

In [9], several other formulae modified from formula (5.3) is obtained (with high accuracy). Such as

\[
    p(n) \approx \left[\frac{\exp\left(\sqrt{\frac{2}{3}\pi \sqrt{n}}\right)}{4\sqrt{3} (n + C_3(n))} + \frac{1}{2}\right], \quad 1 \leq n \leq 80.
\]

\(^4\) Compared with another famous formula in convergent series found by Rademacher in 1937, based on the work of Hardy and Srinivasa Ramanujan, refer [7] or [10].
with a relative error less than 0.004\%, where

\[
C'_2(n) = \begin{cases} 
0.4527092482 \times \sqrt{n + 4.35278} - 0.05498719946, & n = 3, 5, 7, \ldots, 79; \\
0.4412187317 \times \sqrt{n - 2.01699} + 0.2102618735, & n = 4, 6, 8 \ldots, 80.
\end{cases}
\]

and

\[
p(n) \approx \left[ \exp \left( \sqrt{2} \pi \sqrt{n} \right) + \frac{1}{2} \right], \quad n \geq 80
\]

with a relative error less than $5 \times 10^{-8}$ when $n \geq 180$. Here $a_2 = 0.4432884566$, $b_2 = 0.1325096085$ and $c_2 = 0.274078$.

6 Result on Monomial Matrix

For any monomial matrix $M$, it can be written as the product of a permutation matrix $P$ and an invertible diagonal matrix $D$. For the permutation matrix $P$, there is a permutation matrix $T$ such that $T^{-1}PT = Y$ is in canonical form diag \{ $I_t$, $N_1$, \ldots, $N_r$ \} as mentioned in Theorem 3. In the expression $T^{-1}PT$, the permutation matrix $T^{-1}$ changes only the position of the rows, $T$ just changes the position of the columns, neither will change the values of the members, as the non-zero members in $M$ and $P$ share the same positions, so do $T^{-1}MT$ and $T^{-1}PT$. Suppose the unique non-zero element in the $i$'th row of $T^{-1}MT$ is $a_i$, the unique non-zero element in the $i$'th column of $T^{-1}MT$ is $b_i$, $i = 1, 2, \ldots, n$. Let $D_3 = \text{diag} \{ a_1, a_2, \ldots, a_n \}$, $D_4 = \text{diag} \{ b_1, b_2, \ldots, b_n \}$, then $T^{-1}MT = D_3 Y = Y D_4$.

So $M = D_4 T \begin{bmatrix} I_t & N_1 & \cdots & N_r \\ I_t & N_1 & \cdots & N_r \\ \vdots & \vdots & \ddots & \vdots \\ I_t & N_1 & \cdots & N_r \end{bmatrix} T^{-1} = T \begin{bmatrix} I_t & N_1 & \cdots & N_r \\ I_t & N_1 & \cdots & N_r \\ \vdots & \vdots & \ddots & \vdots \\ I_t & N_1 & \cdots & N_r \end{bmatrix} T^{-1} D_2 = D_4 T^{-1}$.

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5. Turn all the non-zero elements in $M$ into 1, then we will have a permutation matrix $P$. Suppose the unique non-zero elements in the $i$'th row of $M$ is $c_i$, the unique non-zero elements in the $i$'th column of $M$ is $d_i$, $i = 1, 2, \ldots, n$. Let $D_1 = \text{diag} \{ c_1, c_2, \ldots, c_n \}$, $D_2 = \text{diag} \{ d_1, d_2, \ldots, d_n \}$, $M = PD_2 = D_1 P$. 


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