Light-Front QCD and Heavy Quark Systems

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Abstract. In this series of lectures, I shall begin with the current investigations on phenomenology of hadron dynamics to demonstrate the importance of solving hadronic bound states within the framework of light-front (LF) QCD. Then, I will describe the basic procedure how to formulate the canonical theory of LFQCD, including light-front quantization of QCD, light-front gauge singularity, and light-front two-component formalism. I will also present a complete one-loop QCD calculation in terms of the light-front time-ordering perturbation theory, in comparison with the usual covariant perturbative QCD calculation. Following thereby I will discuss the development of heavy-quark effective theory and the manifestation of heavy quark symmetry on the light-front. Finally, by applying recently developed similarity renormalization group approach to light-front heavy quark effective theory, I will show a rigorous derivation of quark confinement interaction from LFQCD and its application to solve heavy hadron bound states.

1 Hadronic Phenomenology in the LF Formulation

1.1 An Overview

Simply speaking, the main task in the investigation of hadronic physics is how to provide a QCD description of hadronic structure. More specifically, how can we compute directly from QCD the fruitful hadronic properties, such as the hadronic structure functions in lepton-nucleon deep inelastic scatterings, the partonic fragmentation functions in high energy hadron-hadron or $e^+e^-$ collisions, and many hadronic form factors in various hadronic decay processes. However, although QCD has been accepted as a fundamental theory of the strong interaction that governs the underlying dynamics of hadronic constituents, a complete QCD description to hadronic structure is still lacking. In this series of lectures, I will attempt to show you that the light-front formulation of field theory may provide a natural and systematic QCD description to all the processes mentioned above [Zhang (1994)].

Historically, light-front dynamics played a very important role in every step of the development of the strong interaction theory. The most important application of light-front dynamics to hadronic physics is perhaps the parton phenomena in the lepton-nucleon deep inelastic scatterings (DIS). As it is well-known,
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DIS probes hadronic dynamics near the light-cone. Physically, the DIS phenomena can be understood in terms of Feynman’s parton picture [Feynman (1972)]. While, only the light-front formulation of field theory can provide a natural quantum field theory description of parton dynamics. For examples, the leading contributions of the unpolarized structure function \( F_2(x, Q^2) \) and the polarized structure function \( g_1(x, Q^2) \) in DIS are simply written in terms of hadronic matrix elements on the light-front surface \( \xi^- = 0 \):

\[
F_2(x, Q^2) = \frac{1}{2\pi P^+} \int d\eta e^{-i\eta x} \langle ps | \psi_+^\dagger (\xi^-) Q^2 \psi_+ (0) - h.c. | ps \rangle, \quad (1)
\]

\[
g_1(x, Q^2) = \frac{1}{4\pi S^+} \int d\eta e^{-i\eta x} \langle ps | \psi_+^\dagger (\xi^-) \gamma_5 Q^2 \psi_+ (0) + h.c | ps \rangle. \quad (2)
\]

Here, \( \psi_+(\xi) = \frac{1}{2} \gamma^0 \gamma^+ \psi(\xi) \) is the light-front quark field operator, \( \eta = \frac{1}{2} p^+ \xi^-, \) \( \xi^- = \xi^0 - \xi^3, \) \( Q \) the quark charge operator, and \( | ps \rangle \) the hadronic states. It can be shown that in the light-front field theory, Eqs.(1) and (2) are proportional to the momentum and helicity distributions of partons (quarks and gluons) inside hadrons respectively. Other structure functions \( (F_L \text{ and } g_2) \) also have a similar but a bit complicated expressions. Nevertheless, it is obvious that if we knew the hadronic bound states \( | ps \rangle \) from QCD on the light-front, we could completely understand the QCD dynamics of DIS.

Another measurement of hadronic structure in terms of light-front hadronic matrix elements is the parton fragmentation functions in hadron-hadron and other collisions. During high-energy collisions, many hard partons are produced and then are hadronized. Hadronization processes can be characterized by the so-called fragmentation functions which is also introduced initially by Feynman [Feynman (1972)]. Physically, quark and gluon fragmentation functions are probabilities of finding hadrons in a hard parton produced in collisions. These fragmentation functions can be defined as matrix elements of quark and gluon operators at light-front separations. For examples, the unrenormalized quark fragmentation function is given by [Collins and Soper (1982)]

\[
f_{A/q}(z) = \frac{z}{18\pi} \int dx^- e^{-i p^+ x^-/z} Tr(0 | \psi_+ (0) | ps \rangle \langle ps | \psi_+^\dagger (x^-) | 0 \rangle, \quad (3)
\]

and the gluon fragmentation function is defined as

\[
f_{A/g}(z) = \frac{-z}{36\pi k^+} \int dx^- e^{-i k^+ x^-/z} \langle 0 | F^{+\mu} (0) | ps \rangle \langle ps | F_{\mu}^+ (x^-) | 0 \rangle. \quad (4)
\]

where \( Tr \) traces the color and Dirac components of quarks. Again, if we knew hadronic bound states from QCD on the light-front, we could directly study the QCD dynamics represented by these fragmentation functions.

In recent years, light-front formulation has also been widely used in the phenomenological study of hadronic form factors involving in various hadron elastic scatterings and decay processes, by the use of the so-called relativistic quark model or light-front quark model [Terent’ev et al. (1976)]. Simply speaking,
light-front quark model is based on truncated Fock space expansion of light-front bound states (upon only the valence quark states) and then phenomenologically determines the valence Fock states’ amplitude (the wavefunction). Unlike the study of the structure functions and the fragmentation functions where the use of light-front description can make the physical picture manifestation, the interesting feature of using light-front description to hadronic decay processes is that the simple boost operations and the transparent relativistic properties containing in light-front bound states may allow one to describe hadronic form factors for entire kinematic range of momentum transfer for these space-like processes. This is quite different from descriptions of other hadronic quark models, such as the nonrelativistic constituent quark model and the beg model, which are normally believed to be applicable only for the processes involving small momentum transfer. Very recently, applications of light-front quark model have also been extended to the description of various heavy meson decay processes, although most of the investigations are limited to the calculations of form factors at zero momentum transfer, due to the limitation of using the light-front quark model for time-like processes. Extending the light-front quark model incorporated with higher Fock space contribution (a more realistic light-front bound state description) may make the description of hadronic decay form factors become possible for the entire kinematic range of momentum transfer. Nevertheless, all the hadronic form factors are extracted from some hadronic matrix elements, such as,

\[ \langle H'(p')|\Gamma|H(p)\rangle, \]  

(5)

where \( \Gamma \) is a transition operator in the corresponding process. Again, if we knew the associated hadronic bound states that solved from QCD on the light-front, we would have a true QCD description of hadronic decay processes.

The above analysis indicates that once we know how to solve the hadronic bound states from QCD, especially for these defined on a surface of light-front, we can directly calculate various hadronic matrix elements involved in many hadronic processes. Then a true QCD description of hadronic physics may be realized. This series of lectures is devoted to the light-front formulation of QCD dynamics and the attempt of solving hadronic bound states, especially the heavy hadron bound states, directly from such a formulation. In the first lecture, I will mainly discuss the general structure of hadronic bound states on the light-front.

1.2 General Structure of Light-Front Bound States

In the standard language of field theory, relativistic bound states and resonances are identified by the occurrence of poles in Green functions. Although the information extracted from this approach provides a good definition of physical particles, the ordinary wave function structure of bound states in the usual quantum mechanics language is lacked. As a result, wave function amplitudes extracted from Green functions may not be universally valid in the calculations of various hadronic matrix elements that measured in experiments. In order to
understand hadronic structure in terms of hadronic bound state wavefunctions (which is the most transparent picture in quantum theory), the explicit form of hadronic bound states on some fixed time surface is wanted.

However, solving bound states in field theory as an eigenstate problem has not been well established. One may define the bound states as eigenstates of $P^0$ and determine these states by solving the eigenequation of $P^0$. But $P^0$ is a square root function of the momentum and mass operators which does not give us a clear picture of the Schrödinger’s eigenstate equation in quantum mechanics. The widely used framework of finding relativistic bound states is the Bethe-Salpeter equation. However, Bethe-Salpeter equation itself involve many unsolved problems, such as the physical interpretation of the Bethe-Salpeter amplitudes, and the numerical difficulty in solving the Bethe-Salpeter equation in space-time space, etc. Some approximations, such as instant-time approximation, may simplify the Bethe-Salpeter equation. But with such approximation, the main properties of relativistic dynamics, namely the boost dynamics, will be lacking. In other words, the results may be no longer relativistic.

Also, in principle, a relativistic bound state can always be written as an operator function of the particle creation operators acting on the vacuum of the theory. However, for many theories that we are interested in, especially for QCD, the vacuum is very complicated. With a complicated vacuum, formally writing down a relativistic bound state as a series of Fock space expansion also becomes very difficult.

However, these subtle problems may be removed when we look at the bound states on the light-front.

i). Light-Front Vacuum. In the equal-time framework, the vacuum of QCD is crucial for a realization of chiral symmetry breaking and color confinement. It is also a starting point in the construction of hadronic bound states. However, the understanding of the true QCD vacuum is still very limited, although a lot of informative work has been carried out in the past two decades based on the instanton phenomena [Hooft (1976)] and the QCD sum rule [Shifman et al. (1979)].

In the light-front coordinates, a particle’s momentum is divided into the longitudinal component and the transverse components. For a physical (on-mass-shell) particle, its longitudinal momentum, $k^+ = k^0 + k^3$, cannot be negative since the energy of a physical state always dominates its momentum. As a result, the light-front vacuum for any interacting field theory can only be occupied by the particles with zero-longitudinal momentum, namely

$$|\text{vac}\rangle_{LF} = f(a_{k^+ = 0}^\dagger)|0\rangle,$$

so that $P^+|\text{vac}\rangle_{LF} = 0$, where $P^+ = \sum_i k^+_i$. At this point, the light-front vacuum is still not simple. In the past several years, to obtain a nontrivial light-front vacuum, many tried to solve the so-called zero-mode (the particles with $k^+ = 0$) problem [Burkardt (1996)].

To construct hadronic bound states consisting of many quarks and gluons, one will naturally ask whether it is possible to express hadronic states in terms
of Fock space expansion with a trivial vacuum. It is obvious that if we could “remove” from the theory the basic constituents with zero longitudinal light-front momentum, the vacuum of the full interacting theory would be the same as the free field theory, namely

$$|\text{vac}\rangle_{LF} = |0\rangle.$$  

(7)

It must note that here “removing” from the theory the basic constituents with zero longitudinal light-front momentum does not mean to simply ignore dynamics of these constituents and their contributions to the bound states. Mathematically, one can remove these constituents with zero longitudinal momentum by either using a prescription that requires the field variables to satisfy the antisymmetric boundary condition in the light-front longitudinal direction [Zhang and Harindranath (1993a)] or dealing with a cutoff theory that imposing a cutoff, $$k^+ \geq \epsilon$$, on the momentum expansion of each field variable, where $$\epsilon$$ is a small number [Wilson et al. (1994)]. Thus, the positivity of longitudinal momentum with such a prescription or an explicit cutoff ensures that the light-front vacuum must be trivial. Now a relativistic bound state can be expressed as an ordinary Fock state expansion:

$$|\Psi\rangle = f(a^\dagger, b^\dagger, d^\dagger)|0\rangle.$$  

(8)

For QCD, $$a^\dagger, b^\dagger$$ and $$d^\dagger$$ are the gluon, quark and antiquark creation operators with nonzero longitudinal momentum, and $$f(a^\dagger, b^\dagger, d^\dagger)$$ must also be a color singlet operator as a polynomial function of $$\{a^\dagger, b^\dagger, d^\dagger\}$$.

ii). Light-Front Bound State Equation. Once the light-front vacuum becomes trivial and the light-front bound states for various hadrons are expanded in terms of the Fock space, the dynamic equation to determine these states is rather simple. Explicitly, a hadronic bound state labeled by $$\alpha$$ with total longitudinal and transverse momenta $$P^+$$ and $$P_\perp$$, and helicity (the total spin along the longitudinal direction) $$\lambda$$ can be expressed as follows:

$$|\alpha, P^+, P_\perp, \lambda\rangle = \sum_{n, \lambda_i} \int' \frac{dx_i d^2k_{\perp i}}{(2\pi)^3} |n, x_i P^+, x_i P_\perp + k_{\perp i}, \lambda_i\rangle \Phi_{n/\alpha}(x_i, k_{\perp i}, \lambda_i),$$

(9)

In Eq.(9), $$n$$ represents $$n$$ constituents contained in the state $$|n, x_i P^+, x_i P_\perp + k_{\perp i}, \lambda_i\rangle$$, $$\lambda_i$$ is the helicity of the $$i$$-th constituent, and $$\int'$$ denotes the integral over the space:

$$\sum_i x_i = 1, \quad \text{and} \quad \sum_i k_{\perp i} = 0,$$  

(10)

where $$x_i$$ is the fraction of the total longitudinal momentum that the $$i$$-th constituent carries, and $$k_{\perp i}$$ is its relative transverse momentum with respect to the center of mass frame:

$$x_i = \frac{p_i^+}{P^+}, \quad k_{i\perp} = p_{i\perp} - x_i P_\perp,$$  

(11)
with $p_i^+, p_{i\perp}$ being the transverse and longitudinal momentum of the $i$-th constituent. $\Phi_{n/\alpha}(x_i, k_{i\perp}, \lambda_i)$ is the amplitude of the Fock state $|n, x_i P^+, x_i P_\perp + k_{i\perp}, \lambda_i\rangle$ which satisfies the following normalization condition:

$$
\sum_{n, \lambda_i} \int \frac{dx_i d^2 k_{i\perp}}{2(2\pi)^3} |\Phi_{n/\alpha}(x_i, k_{i\perp}, \lambda_i)|^2 = 1.
$$ (12)

The eigenstate equation that the wave functions obey on the light-front is obtained from the operator Einstein equation $P^2 = P^+ P^- - P_\perp^2 = M^2$:

$$
H_{LF}|\alpha, P^+, P_\perp, \lambda\rangle = P_\perp^2 + M^2 \alpha P^+ |\alpha, P^+, P_\perp, \lambda\rangle,
$$ (13)

where $H_{LF} = P^-$ is the light-front Hamiltonian. Furthermore, since the boost on the light-front only depends on kinematics, boosting a bound state from one Lorentz frame to any other frame is quite simple, and is dynamically independent [Zhang (1994)]. Thus, if we found the bound state in the rest frame, we could completely understand the particle structure in any frame. This is not true in the instant form. In the instant form, the solutions in the rest frame are not easily boosted to other Lorentz frames due to the dynamical dependence of the boost transformation. Therefore, in each different Lorentz frame, one needs to solve the bound state equation of $P^0$ to obtain the corresponding wave functions. This is perhaps the reason why one has not established a reliable approach to construct relativistic wave functions in the instant field theory in terms of the Schrödinger picture. This obstacle is removed on the light-front.

To see the explicit form of the light-front bound state equation, let us consider a meson wave function (for instance, a pion). The light-front bound state equation can be expressed as:

$$
\begin{pmatrix}
\frac{m^2}{2} - \sum_i \frac{k_{i\perp}^2 + m_i^2}{x_i} \\
\psi_{q\bar{q}} \\
\vdots \\
\psi_{q\bar{q}g}
\end{pmatrix} = 
\begin{pmatrix}
\langle q\bar{q}| H_{int}| q\bar{q}\rangle \\
\langle q\bar{q}g| H_{int}| q\bar{g}\rangle \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
\frac{m^2}{2} - \sum_i \frac{k_{i\perp}^2 + m_i^2}{x_i} \\
\psi_{q\bar{q}} \\
\vdots \\
\psi_{q\bar{q}g}
\end{pmatrix}.
$$ (14)

Of course, to exactly solve the above equation for the whole Fock space is still impossible. Practically, one has to truncate the Fock space to only include these Fock states with a small number of particles. For example, one may truncate all the high order Fock space sectors (approximately) from the valence constituent space. Then the light-front bound state equation is reduced to the light-front Bethe-Salpeter equation:

$$
\left(m^2 - \frac{k_{\perp}^2 + m^2}{x(1-x)}\right)\psi_{q\bar{q}}(x, k_{\perp}) = \int \frac{dy d^2 k'_{\perp}}{2(2\pi)^3} V_{eff}(x, k_{\perp}, y, k'_{\perp})\psi_{q\bar{g}}(y, k'_{\perp}).
$$ (15)
Note that in Eq. (15), $V_{\text{eff}}$ denotes an effective two-body interaction kernel. In other words, by “truncating” the Fock space to only keep the valence quark states, the complicated Eq. (14) is reduced to the manageable Eq. (15) but the dominant contribution of higher Fock space to the bound states must be now described effectively by $V_{\text{eff}}$. The residual effect should be manageable in the framework of perturbation theory. A true nonperturbative QCD solution to the hadronic bound states is if one were able to derive these effective interactions directly from QCD rather than that phenomenologically are put by hand. This will be discussed in the last lecture.

1.3 Phenomenological Hadronic Bound States on the Light-Front

At the present time, how to solve for the bound states discussed above from QCD is still unclear. Hence, it may be useful to have some insights into the light-front behavior of the meson and baryon wave functions which have been constructed phenomenologically in describing hadrons. In fact, the phenomenological light-front meson and baryon bound states have been studied extensively in the last few years, based on the light-front quark model or light-front wave-function description. The motivation of light-front quark model is to provide a simple relativistic constituent quark model for mesons and baryons that can yield a consistent description of the hadronic processes for both low and high momentum transfer.

The general construction of the phenomenological wave functions is motivated by that of the non-relativistic constituent quark model. The constituent quark model has been very successful in the description of hadronic spectroscopy with a very simple structure, namely that all mesons consist of a quark and antiquark pair and the baryons are made of three constituent quarks, their wave functions satisfy the $SU(6)$ classification and Zweig’s rule which suppresses particle production in favor of rearrangement of constituents for hadrons [Close (1979)]. However, such a simple picture is very difficult to be understood within QCD, due to its nonrelativistic assumption and due to our belief that QCD vacuum must be very complicated so that hadrons must contain an infinite number of quark-antiquark pairs and gluons.

Light-front bound states describe the relativistic hadronic structure with a nonrelativistic form. Furthermore, the simple vacuum state on the light-front ensures the validity of the Fock state expansion of hadronic states. With the assumption of existence of constituent quarks (of masses of hundreds of MeVs), the leading approximation to hadronic states that consist of a quark-antiquark pair for mesons and a three-quark cluster for baryons should be a reasonable starting point. More theoretical discussion for such a assumption from low energy QCD will be given later.

However, it must note that there is a subtle problem in the description of hadronic structure in terms light-front bound states. That is, it is not easy to identify the light-front hadronic bound states with hadronic states which are commonly characterized by spin as a good quantum number. On the light-front,
we are unable to kinematically construct the hadronic bound states with fixed spin. The light-front bound states discussed in the last section are labeled by helicity rather than spin. In these calculations of the parton distribution and fragmentation functions, the hadronic bound states are defined or classified in terms of the helicity. However, when we use the light-front bound states to compute the hadronic structural quantities, such as hadronic decay form factors and coupling constants, we must have states with a definite spin. A general solution to the spin problem on the light-front has not been found. However, phenomenologically, the helicity part of the bound states on the light-front can be transformed to a light-front spin part via the so-called Melosh transformation (which is exact only for free quark theory) such that the hadronic states may be projected (approximately but no necessary to be correct) from the set of light-front bound states labeled with helicities. Here, I list some meson and baryon light-front bound states that have been used to calculate various hadronic quantities in the past few years.

The general form of the phenomenological light-front hadronic bound states has a similar structure to the constituent quark model states: for meson states (with only the $q\bar{q}$ Fock space sector),

$$|P^+, P_\perp, SS_3\rangle = \int \frac{dxd^2k_\perp}{16\pi^3} \sum_{\lambda_1\lambda_2} \psi_m^{SS_3}(x, k_\perp, \lambda_1, \lambda_2)|x, k_\perp, \lambda_1; 1 - x, -k_\perp, \lambda_2\rangle,$$

and for baryon states (with the three quark Fock space sector),

$$|P^+, P_\perp, SS_3\rangle = \sum_{\lambda_i} \int \prod_i^2 dx_i d^2k_{i\perp} \frac{16\pi^3}{\lambda_i} \psi^{SS_3}_b(x_i, k_{i\perp}, \lambda_i)$$

$$\times |x_1, k_{1\perp}, \lambda_1; x_2, k_{2\perp}, \lambda_2; 1 - x_1 - x_2, -(k_{1\perp} + k_{2\perp}), \lambda_3\rangle,$$

where $\psi^{SS_3}$ is the amplitude of the corresponding $q\bar{q}$ or three quark sector (the wave function of the quark model):

$$\psi^{SS_3} = F \Xi^{SS_3}(k_{i\perp}, \lambda_i)\Phi(x_i, k_{i\perp}),$$

with $F$ the flavor part of the wave function which is the same as in the constituent quark model, and $\Xi$ and $\Phi$ are the spin and space parts that depend on the dynamics. By ignoring the dynamic dependence of the spin configuration and by using the Melosh transformation [Melosh (1974)],

$$R_M(k_{i\perp}, m_i) = \frac{m_i + x_i M_0 - i\sigma \cdot (n \times k_{i\perp})}{\sqrt{(m_i + x_i M_0)^2 + k_{i\perp}^2}},$$

where $n = (0,0,1)$, $\sigma$ is the Pauli spin matrix, $m_i$ the $i$-th constituent quark mass, and $M_0$ satisfies

$$M_0^2 = \sum_i \frac{k_{i\perp}^2 + m_i^2}{x_i},$$

(19)

(20)
the light-front spin wave function can be given by
\[
\Xi_m^{SS}(k_\perp, \lambda_1, \lambda_2) = \sum_{s_1, s_2} \langle \lambda_1 | R_{k_\perp}^f (k_\perp, m_1) | s_1 \rangle \langle \lambda_2 | R_{k_\perp}^f (-k_\perp, m_2) | s_2 \rangle \langle 1/2s_1, 1/2s_2 | SS \rangle ,
\]
(21)
for mesons; for baryons the spin part is rather complicated for a detailed construction, see for example Ref. [Schlumpf (1993)]. The momentum part of the wave function may be written as
\[
\Phi_m(x_i, k_i \perp) = N_m \sqrt{\frac{dx}{dz}} \exp(-k^2/2\omega^2_m)
\]
(22)
for meson, where \( k = (k_\perp, k_z) \), \( k_z = (x - \frac{1}{2})M_0 - \frac{m^2 - m_0^2}{2M_0} \); and for baryons
\[
\Phi_b(x_i, k_i \perp) = N_b \frac{1}{(1 + M^2_b/\omega^2_b)^{3.5}} ,
\]
(23)
where \( N \) is a normalization constant and \( \omega \) is a parameter fixed by the data. Other phenomenological light-front wave functions have also been used.

These phenomenological light-front wave functions have been widely used to calculate hadronic form factors and coupling constants [Chung et al. (1988)]; the results look pretty good for a very broad range of momentum transfer, and should provide a much better description than the nonrelativistic constituent quark model and other phenomenological descriptions.

Nevertheless, all these are just some phenomenological examinations of light-front hadronic wave functions. The true strong interaction description of hadronic structure is the solution of the bound state equation, Eq.(14) or approximately Eq.(15), from QCD. This is the main task of the recently development of QCD formulated on the light-front. In the remaining lectures, I will discuss the QCD formulation on the light-front and then explore its application to heavy quark systems, based heavily on the works which have been done with my collaborators in the last few years [Zhang (1993), Zhang and Harindranath (1993a), Zhang and Harindranath (1993b), Harindranath and Zhang (1993), Wilson et al. (1994), Cheung et al. (1995), Zhang (1996)].

2 Canonical Light-Front QCD

2.1 Introduction

Light-front QCD that I am going to discuss is the theory of QCD formulated on a light-front surface with the light-front gauge \( A_+^\perp = 0 \). Before start the discussion on LFQCD, I would like to make a few remarks: First of all, I would like to claim that any problem of QCD that can be solved in the instant formulation should be undoubtedly solved on the light-front. This is not surprise at all! However, the importance of LFQCD is that we hope to solve the subtle problems
in QCD that have not been solved in the instant form, such as color confinement and dynamical chiral symmetry breaking problems. To reach this goal, one may need to have some relatively complete knowledge on the canonical formulation of LFQCD and from which to find the key problem associated with these subtleties. Hence, in this lecture, I will introduce the canonical form of light-front QCD, then discuss the origin of the light-front gauge singularity and the light-front two-component formulation of QCD which has some very special structure for field theory that are only manifested on the light-front.

The QCD Lagrangian is defined by

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \overline{\psi}(i\gamma_\mu D^\mu - m)\psi,$$  

(24)

where $F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$, $A^\mu = \sum_a A^a T^a$ is a 3 × 3 gluon field color matrix and the $T^a$ are the generators of the SU(3) color group: $[T^a, T^b] = if^{abc} T^c$ and $\text{Tr}(T^aT^b) = \frac{1}{2} \delta_{ab}$. The field variable $\psi$ describes quarks with three colors and $N_f$ flavors, $D^\mu = \partial^\mu - igA^\mu$ is the symmetric covariant derivative, and $m$ is an $N_f \times N_f$ diagonal quark mass matrix. The Lagrange equations of motion are well-known:

$$\left(i\gamma_\mu \partial^\mu - m + g\gamma_\nu T^a A^\nu\right)\psi = 0,$$  

(25)

$$\partial_\mu F^\mu_\perp + g f^{abc} A_\mu^b F^\mu_\perp + g\overline{\psi}\gamma^\nu T^a \psi = 0,$$  

(26)

The following discussion and also that of the next lecture are mainly based on the work in collaboration with A. Harindranath [Zhang and Harindranath (1993a), Zhang and Harindranath (1993b), Harindranath and Zhang (1993)].

2.2 Light-front (phase space) Quantization

To formulate the QCD on the light-front, the following light-front notations will be adopted: The space-time coordinate is denoted by $x^\mu = (x^+, x^- , x_\perp)$, where $x^+ = x^0 + x^3$ is the light-front time-like component, $x^- = x^0 - x^3$ and $x^\perp (i = 1, 2)$ are respectively the light-front longitudinal and transverse components. The light-front derivatives are given by $\partial^+ = 2\frac{\partial}{\partial x^+}$, $\partial^- = 2\frac{\partial}{\partial x^-}$ and $\partial^\perp_i = \frac{\partial}{\partial x^\perp_i}$. The product of two four-vectors is written as $a \cdot b = \frac{1}{2}(a^+ b^- + a^- b^+) - a_\perp \cdot b_\perp$.

The canonical theory of QCD on the light-front is constructed with the choice of the light-front gauge $A^+_a = 0$. The first question you may ask is why we choose the light-front gauge. The answer is as follows:

On the light-front, the quark (more generally the fermion) fields can be decoupled into $\psi(x) = \psi_+(x) + \psi_-(x)$ with $\psi_\pm(x) = \frac{1}{2} \gamma^0 \gamma^\pm \psi(x)$. Then the Dirac equation (25) can be separated into:

$$\left(i\partial^- + gT^a A^+_a\right)\psi_+ = \left(i\alpha_\perp \cdot D_\perp + \beta m\right)\psi_-, \quad \quad \quad (27)$$

$$\left(i\partial^+ + gT^a A^+_a\right)\psi_- = \left(i\alpha_\perp \cdot D_\perp + \beta m\right)\psi_+,$$  

(28)
where $\alpha = \gamma^0 \gamma_\perp, \beta = \gamma^0$. It shows that the component $\psi^-$ is a constraint field variable, which can be solved nonperturbatively from the about equation ONLY if we take $A^+_a = 0$.

$$\psi^- = \frac{1}{i \partial^+} \left( i \alpha \cdot D \perp + \beta m \right) \psi^+.$$  \hspace{1cm} (29)

Secondly, due to gauge symmetry among the four components of the vector gauge field, only two of them are the physically independent variables. By taking $A^+_a = 0$, the equation of motions (26) for $\nu = +$ is simply reduced to

$$\frac{1}{2} \left( \partial^+ \right)^2 A^-_a = \partial^+ \partial^j A^j_a + g \rho_a,$$  \hspace{1cm} (30)

where $\rho_a = f^{abc} A^i_b \partial^+ A^i_c + 2 \psi^+ T^a \psi^+$ is the light-front color charge density. This is indeed the light-front Gauss Law which can be used to determines $A^-_a$ in terms of $\psi^+$ and $A^i_a$:

$$A^-_a = 2 \left\{ \left( \frac{1}{\partial^+} \right) \left( \partial^j A^j_a \right) + \left( \frac{1}{\partial^+} \right)^2 \rho_a \right\},$$  \hspace{1cm} (31)

where the operator $\frac{1}{\partial^+}$ will be defined later. It shows that with the light-front gauge, we can explicitly eliminate all the unphysical gauge degrees of freedom.

Now, we can write a simple close form for the LFQCD Lagrangian in terms of the pure physical degrees of freedom, $\psi^+$ and $A^i_a (i = 1, 2)$, with the choice of the light-front gauge:

$$\mathcal{L}_{QCD} = \frac{1}{2} \left( \partial^+ A^i_a \right) \left( \partial^- A^i_a \right) + i \psi^+ \partial^- \psi^+ - \mathcal{H},$$  \hspace{1cm} (32)

where $\mathcal{H}$ is the LFQCD Hamiltonian density:

$$\mathcal{H} = \frac{1}{2} \left( \partial^i A^i_a \right)^2 + g f^{abc} A^i_a \partial^+ A^i_b \partial^i A^j_c + \frac{g^2}{4} f^{abc} f^{ade} A^i_b A^j_c A^k_d A^i_e$$

$$+ \left[ \psi^+ \left\{ \alpha \perp \cdot \left( i \partial \perp + g A \perp \right) + \beta m \right\} \left( \frac{1}{i \partial^+} \right) \left\{ \alpha \perp \cdot \left( i \partial \perp + g A \perp \right) + \beta m \right\} \psi^+ \right]$$

$$+ g \partial^+ A^i_a \left( \frac{1}{\partial^+} \right) \rho_a + \frac{g^2}{2} \left( \frac{1}{\partial^+} \right) \rho_a \left( \frac{1}{\partial^+} \right) \rho_a.$$  \hspace{1cm} (33)

Next we discuss the light-front quantization. A self-consistent canonical quantization requires that the resulting Hamiltonian must generate the correct equations of motion for the physical degrees of freedom $(A^i_a, \psi^+, \psi^+_\perp)$. To reproduce the Lagrangian equations of motion, we need to find consistent commutators for physical field variables. In the light-front gauge, the LFQCD phase space is spanned by the field variables, $A^i_a, \psi^+, \psi^+_\perp$ and their canonical momenta, $\mathcal{E}^i_a = \frac{1}{2} \partial^+ A^i_a, \pi_{\psi^+} = \frac{1}{2} \psi^+_\perp, \pi_{\psi^+_\perp} = -\frac{i}{2} \psi^+$. The phase space structure which
determines the Poisson brackets of its variables can be found by the Lagrangian one-form $Ldx^+$ (apart from a total light-front time derivative),

\[
Ldx^+ = \frac{1}{2} \{ \mathcal{E}_a^i dA^i_a + \pi_\psi^i d\psi_+ + d\psi_+^i \pi_\psi^i + A_0^i d\mathcal{E}_a^i - d\pi_\psi^i \psi_+ - \psi_+^i d\pi_\psi^i \} - \mathcal{H} dx^+
\]

\[
= \frac{1}{2} q^\alpha T_{\alpha\beta} dq^\beta - \mathcal{H} dx^+, \tag{34}
\]

where the first term on the right-hand side is called the canonical one-form of the phase space (note that quark fields are anticommuting c-numbers (Grassmann variables)). Correspondingly, the canonical equal-$x^+$ commutation relations are then given by:

\[
[q^\beta(x), q^\alpha(y)]_{x^+ = y^+} = i \Gamma^{-1}_{\alpha\beta} . \tag{35}
\]

Explicitly, we have

\[
\left\{ \psi_+(x), \psi_+^\dagger(y) \right\}_{x^+ = y^+} = i A_+ \delta^3(x - y) , \tag{36}
\]

\[
[A_+^i(x), A_+^j(y)]_{x^+ = y^+} = i \delta_{ab} \delta^{ij} \left( \frac{1}{\partial y} \right) \delta^3(x - y) . \tag{37}
\]

From these commutation relations it is straightforward to verify that the Hamiltonian equations of motion are consistent with Eqs.(25) and (26). As we see in the above light-front quantization of QCD one does not need to introduce the ghost field. However, this canonical formulation does not completely define theory for practical computations due to existence of gauge singularity.

### 2.3 Light-Front Gauge Singularity

The gauge singularity is perhaps the most difficult problem in non-abelian gauge theory that has not been completely solved since it was developed. In LFQCD, it arises from the elimination of the unphysical gauge degrees of freedom. To eliminate the unphysical degrees of freedom on the light-front, we need to solve the constraint equations which depend on the definition of the operator $1/\partial^+$. In our formulation, we define this operator by

\[
\left( \frac{1}{\partial^+} \right) f(x^- , x^+ , x_{\perp}) = \frac{1}{4} \int_{-\infty}^{\infty} dx^- \varepsilon(x^- - x^-_1) f(x^-_1 , x^+ , x_{\perp}) , \tag{38}
\]

where $\varepsilon(x) = -1 , 0 , 1$ for $x < 0 , = 0 , > 0$.

In perturbation theory, the gauge singularity manifests itself clearly in momentum space. The momentum representation of Eq.(38) is

\[
\left( \frac{1}{k^+} \right)^n f(x^-) = \left( \frac{1}{4} \right)^n \int_{-\infty}^{\infty} dx_1 \cdots dx_n \varepsilon(x^- - x^-_1) \cdots \varepsilon(x^-_{n-1} - x^-_n) f(x^-_n) \rightarrow \left[ \frac{1}{2} \left( \frac{1}{k^+ + i\varepsilon} + \frac{1}{k^+ - i\varepsilon} \right) \right]^n f(k^+) = \frac{1}{(k^+)^n} f(k^+) . \tag{39}
\]
As we see the $k^+ = 0$ modes are removed with this definition. In other words, the singularity of $\frac{1}{k^+}$ is regularized. However, with such an infrared regularization, many infrared divergences from the small longitudinal momentum, surrounding the $k^+ = 0$ region, will occurs in the perturbative calculation. We will discuss these divergences in the next lecture.

On the other hand, with the definition of Eq.(38), we have

$$\left[ A^i_a(x), A^j_b(y) \right]_{x^+ = y^+} = -i \frac{1}{4} \delta^{ij} \epsilon(x^- - y^-) \delta^2(x_{\perp} - y_{\perp}). \quad (40)$$

This leads to the fact that $A^i_a$ satisfies an antisymmetric boundary condition:

$$A^i_a(x^- = -\infty) = -A^i_a(x^- = +\infty). \quad (41)$$

It also shows that the zero-mode (the longitudinal momentum is zero) in $A^i_a$ is removed. Meanwhile, quarks in QCD should always be massive, namely their longitudinal momentum is not really zero. Thus, with the definition of Eq.(38), the theory of light-front QCD does not contain zero-modes. Therefore the LFQCD vacuum in this formulation is always trivial! Now you may ask where is the nontrivial properties of QCD with such a trivial vacuum in your formulation?

Apparently, after solving $A^-_a$ component from the light-front Gauss law in the $A^+ a = 0$ gauge, the gauge freedom should be completely fixed. However, a careful check shows that there is still a residual gauge transformation in the above formulation. It is given by

$$U_r = \exp \left\{ - \frac{i}{g} \int d^2 x_{\perp} \theta^a(x_{\perp}) R_a(x_{\perp}) \right\}, \quad (42)$$

which is associated with the field $A^-_a$ at the longitudinal infinity:

$$R_a = \frac{1}{2} \partial^+ A^-_a |_{x^- = \infty} = \frac{1}{2} \int_{-\infty}^{\infty} dx^- \left[ 2 \partial^+ \partial^i A^i_a(x) + g \rho_a(x) \right]. \quad (43)$$

This gauge freedom can be further fixed for physical states. This is because the operator $R_a = E^a_\perp |_{x^- = \infty}$ which is the longitudinal component of color electric field strength at longitudinal infinity. For physical states, finite energy density requires that the color electric field strength must vanish at the longitudinal boundary: $E^a_\perp |_{x^- = \infty} = 0$. This condition canonically removes the residual gauge freedom and leads to a constraint on the $A^i_a$ at longitudinal infinity:

$$\partial^i A^i_a |_{x^- = \pm \infty} = \pm \frac{g}{2} \int_{-\infty}^{\infty} dx^- \rho_a(x^- , x). \quad (44)$$

The nontrivial properties of QCD in our formulation are indeed hidden in this condition. The main effect of this equation should be only manifested in nonperturbative dynamics, i.e., in physical bound states. An explicit nontrivial effect
can be seen from the axial anomaly of QCD, for example. Consider the axial current equation (for zero quark mass)

\[ \partial_\mu j_5^\mu = N_f \frac{g^2}{8\pi^2} \text{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}), \]  

(45)

where the axial current is \( j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \), and the dual field strength is \( \tilde{F}^{\mu\nu} = \frac{i}{2}F^{\mu\nu}\sigma^\rho F_{\sigma\rho} \). The winding number in LFQCD is defined as the net charge between \( x^+ = -\infty \) and \( x^+ = \infty \),

\[ \Delta Q_5 = N_f \frac{g^2}{8\pi^2} \int_M d^4x \text{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}). \]  

(46)

The integration on the r.h.s. of the above equation is defined in Minkowski space \( (M) \) and can be replaced by a surface integral. It has been found [Zhang (1993)] that

\[ \Delta Q_5 = -N_f \frac{g^2}{\pi^2} \int dx^+ d^2x_\perp \text{Tr} (A^-[A^1, A^2]) \bigg|_{x^-=-\infty}^{x^-=\infty}, \]  

(47)

namely, a non-vanishing \( \Delta Q_5 \) is generated from the asymptotic fields of \( A^a_\mu \) and their antisymmetric boundary conditions at longitudinal infinity.

From the above canonical analysis, we can see that nontrivial features in LFQCD are induced by the gauge singularity and are manifested at the longitudinal infinity on the light-front. They are also associated with the light-front longitudinal infrared divergence in momentum space when the zero-modes are removed in our canonical quantization. This analysis gives us some hint where we should look for the problems in the study of nonperturbative QCD with a trivial vacuum on the light-front.

### 2.4 Two-Component Formulation

When QCD is formulated on the light-front, the theory can be expressed in terms of a pure two-component form. This is another useful feature of LFQCD. After the elimination of the unphysical gauge degrees of freedom, the QCD gauge field has already been reduced to the two transverse components, \( A^1_\mu \) and \( A^2_\mu \). While, as we will see soon that the quark field can also be written in terms of a two-component field (rather than the four-component field in instant form) [Zhang and Harindranath (1993b)].

To do so, we should introduce the following \( \gamma \) matrix representation:

\[ \gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} -ie^{ij}\sigma_j & 0 \\ 0 & ie^{ij}\sigma_j \end{bmatrix}. \]  

(48)

Then, one can find that the light-front project operators become

\[ \Gamma^+ = \frac{1}{2}\gamma^0\gamma^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^- = \frac{1}{2}\gamma^0\gamma^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]  

(49)
and the light-front quark field have the two-component form:

\[
\psi = \begin{bmatrix} \varphi \\ \nu \end{bmatrix}, \quad \psi_+ = \begin{bmatrix} \varphi \\ 0 \end{bmatrix}, \quad \psi_- = \begin{bmatrix} 0 \\ \nu \end{bmatrix} = \frac{1}{\sqrt{2}} \left( (D_\perp \times \sigma_\perp)^3 + m \right) \varphi,
\]

where \( \varphi(x) \) is a two-component spinor field. In the above expressions, \( \sigma_i \) are the Pauli matrix. Thus, the relativistic fermion particles can be described as a nonrelativistic spin \( \frac{1}{2} \) particle on the light-front. The canonical commutation relation is also reduced to:

\[
\{ \varphi(x), \varphi^\dagger(y) \}_{x^+ = y^+} = \delta^3(x - y).
\]

With the above formulation, the LFQCD Hamiltonian can be rewritten as

\[
H = \int dx^- d^2x_\perp (H_0 + H_{int}) = H_0 + H_1,
\]

where

\[
H_0 = \frac{1}{2} (\partial^i A^i_a) (\partial^j A^j_a) + \varphi\varphi^\dagger \frac{-\nabla^2 + m^2}{i\partial^+} \varphi,
\]

\[
H_{int} = H_{qqq} + H_{ggg} + H_{qqgg} + H_{qqqq} + H_{gggg},
\]

and

\[
H_{qqq} = g \varphi^\dagger \left\{ -2 \left( \frac{1}{\partial^+} \right)(\partial \cdot A_\perp) + \sigma \cdot A_\perp \left( \frac{1}{\partial^+} \right)(\sigma \cdot \partial_\perp + m) \\
+ \left( \frac{1}{\partial^+} \right)(\sigma \cdot \partial_\perp - m)\sigma \cdot A_\perp \right\} \varphi,
\]

\[
H_{ggg} = gf abc \left\{ \partial^i A^i_a A^i_b A^i_c + (\partial^i A^i_a) \left( \frac{1}{\partial^+} \right)(A^i_b \partial^+ A^i_c) \right\},
\]

\[
H_{qqgg} = g^2 \left\{ \varphi^\dagger \sigma \cdot A_\perp \left( \frac{1}{i\partial^+} \right)\sigma \cdot A_\perp \varphi \\
+ 2 \left( \frac{1}{\partial^+} \right)(f abc A^a_i \partial^+ A^a_j) \left( \frac{1}{\partial^+} \right)(\varphi^\dagger T^a \varphi) \right\}
= H_{qqgg1} + H_{qqgg2},
\]

\[
H_{qqqq} = 2g^2 \left\{ \left( \frac{1}{\partial^+} \right)\varphi^\dagger T^a \varphi \left( \frac{1}{\partial^+} \right)(\varphi^\dagger T^a \varphi) \right\},
\]

\[
H_{gggg} = \frac{g^2}{4} f abc ade \left\{ A^a_i A^c_a A^d_i A^e_i + 2 \left( \frac{1}{\partial^+} \right)(A^a_i \partial^+ A^a_j) \left( \frac{1}{\partial^+} \right)(A^e_i \partial^+ A^e_j) \right\}
= H_{gggg1} + H_{gggg2}.
\]

The above two-component formulation simplify the relativistic field theory structure, especially in the study of the relativistic bound state problems.
3 LF Time-Ordered Perturbation Theory for QCD

3.1 About Light-Front Perturbative QCD

Time-ordered perturbation theory, especially the light-front time-ordered perturbation theory, provides a natural perturbative description for parton phenomena [Drell et al. (1970), Mueller (1989)]. The current attempts of solving nonperturbative QCD dynamics on light-front is also based on the analysis of time-ordered approach in Hamiltonian formulation. However, the light-front gauge singularity discussed above will lead to severe infrared divergences in such perturbation theory, although Eq.(39) provides a well-defined regulator (a generalized principal value prescription) for the small $k^+$ momentum.

In covariant perturbation theory, the use of the principal value prescription still leads to the so-called “spurious” poles in the light-front Feynman integrals, which prohibit any continuation to Euclidean space (Wick rotation) and hence the use of standard power counting arguments for Feynman loop integrals. This causes difficulties in addressing renormalization of QCD in covariant perturbation theory with the light-front gauge. In the last decade there are many investigations attempting to solve this problem. One excellent solution is given by Mandelstam and Leibbrandt, i.e., the Mandelstam-Leibbrandt (ML) prescription [Mandelstam (1983) and Leibbrandt (1984)], which allows continuation to Euclidean space and hence power counting. It has also been shown that, with the ML prescription, the multiplicative renormalization in the two-component LFQCD Feynman formulation is restored [Lee and Milgram (1986)].

Unfortunately, the ML prescription cannot be applied to equal-$x^+$ quantization because the ML prescription is defined by a boundary condition which depends on $x^+$ itself and is not allowed in equal-$x^+$ canonical theory. Yet, as we pointed out recently [Wilson et al. (1994)], light-front power counting differs completely from the power counting in equal-time quantization that noncanonical counterterms are allowed in light-front field theory. In other words, multiplicative renormalization is not required in LFQCD. Furthermore, the current attempts to understand nonperturbative QCD in light-front coordinates is based on the $x^+$-ordered diagrams in which no Feynman integral is involved. Thus the power counting criterion for Feynman loop integrals is no longer available in LFQCD Hamiltonian calculations. In $x^+$-ordered perturbation theory with the principle value prescription, LFQCD contains severe linear and logarithmic infrared divergences. Here I will give some results from the $x^+$-ordered perturbative loop calculations and renormalization of LFQCD Hamiltonian theory up to one-loop [Zhang and Harindranath (1993b), Harindranath and Zhang (1993)], where the infrared divergences are systematically analyzed. Since light-front power counting allows noncanonical counterterms, a complete understanding of renormalized LFQCD may not be worked out within perturbation theory; new renormalization and regularization approaches need to be developed, as we will see later.
### 3.2 LF $x^+$-Ordered Perturbation Theory

The $x^+$-ordered perturbation theory can be obtained from the familiar perturbation expansion in quantum mechanics. The perturbation expansion of a bound state is given by (in the Rayleigh-Schrödinger perturbation theory):

$$|\Psi\rangle = \sum_{n=0}^{\infty} \left( \frac{Q}{E_0 - H_0} (H'_I) \right)^n |\Phi\rangle,$$

where $|\Phi\rangle$ is a unperturbative state, $Q$ and $H'_I$ are defined by:

$$Q = \langle \Phi | \langle \Psi |, \quad H'_I = H_I - \Delta E, \quad \Delta E = \langle \Phi | H_I | \Psi \rangle.$$

With this perturbative expansion formula, the mass, the wave functions, and the coupling constants renormalizations can be expressed as follows. For the convenience of practical calculations, we consider the expressions in momentum space.

i). Wavefunction renormalization: In momentum space, the perturbative expansion of a state is given by

$$|\Psi\rangle = \left\{ |\Phi\rangle + \sum_{n_1} \frac{\langle n_1 | H'_I | n_1 \rangle |\Phi\rangle}{p - p_{n_1}} \right. + \sum_{n_1 n_2} \frac{\langle n_1 | H'_I | n_2 \rangle \langle n_2 | H'_I | \Phi \rangle}{(p - p_{n_1})(p - p_{n_2})} + \cdots \right\},$$

which has not been normalized, where $|n_1\rangle, |n_2\rangle, \cdots$ are properly symmetrized (antisymmetrized) states with respect to identical bosons (fermions) in the states and $\sum'$ in Eq.(62) sums over all intermediate states except the initial state $|\Phi\rangle$. The normalized wave function is defined by

$$|\Psi'\rangle = \sqrt{Z_\Phi} |\Psi\rangle,$$

where the factor $Z_\Phi$ is the wavefunction renormalization constant:

$$Z_\Phi^{-1} = \langle \Psi' | \Psi \rangle = 1 + \sum_{n_1} \frac{\langle n_1 | H'_I | \Phi \rangle^2}{(p - p_{n_1})^2} + \cdots. \tag{63}$$

ii). Mass renormalization. The mass correction can then be computed from the “energy-level” shift, i.e., the correction to the energy of an on-mass-shell particle. It is obvious that the perturbative correction to the light-front energy ($p^-$) is given by

$$\delta p^- = \langle \Phi | (H - H_0) | \Psi \rangle = \langle \Phi | H_I | \Psi \rangle = \langle \Phi | H_I | \Phi \rangle$$

$$= \langle \Phi | H_I | \Phi \rangle + \sum_{n_1} \frac{\langle n_1 | H_I | \Phi \rangle^2}{p - p_{n_1}} + \cdots. \tag{64}$$
Using the mass-shell equation $m^2 = p^+ p^- - p_\perp^2$, and recalling that $p^+$ and $p_\perp$ are the conserved light-front kinematical momenta, we obtain the mass renormalization in the old-fashioned perturbative light-front field theory:

$$
\delta m^2 = p^+ \delta p^- = p^+ \langle \Phi | H_I | \Phi \rangle + p^+ \sum_{n_1}^i \frac{|\langle n_1 | H_I | \Phi \rangle|^2}{p^- - p_{n_1}} + \cdots .
$$

(iii). Coupling constant renormalization. The coupling constant renormalization is obtained by the perturbative calculation of various matrix elements of the vertices in $H_I$. Consider a vertex $H_I^j$ that is proportional to the coupling constant $g$, we have

$$
\langle \Psi_f^j | H_I^j | \Psi_i \rangle \equiv Z_g \sqrt{Z_i Z_f} \langle \Psi_f | H_I^j | \Psi_i \rangle
$$

$$
= \langle \Phi_f | H_I^j | \Phi_i \rangle + \sum_{n_1}^i \frac{\langle \Phi_f | H_I^j | n_1 \rangle \langle n_1 | H_I^j | \Phi_i \rangle}{p_f^+ - p_{n_1}}
$$

$$
+ \sum_{n_1, n_2}^i \frac{\langle \Phi_f | H_I^j | n_1 \rangle \langle n_1 | H_I^j | n_2 \rangle \langle n_2 | H_I^j | \Phi_i \rangle}{(p_f^+ - p_{n_1})(p_f^+ - p_{n_2} + i\epsilon)}
$$

$$
+ \sum_{n_1, n_2}^i \frac{\langle \Phi_f | H_I^j | n_1 \rangle \langle n_1 | H_I^j | n_2 \rangle \langle n_2 | H_I^j | \Phi_i \rangle}{(p_f^+ - p_{n_1})(p_f^+ - p_{n_2})}
$$

$$
+ \sum_{n_1, n_2}^i \frac{\langle \Phi_f | H_I^j | n_1 \rangle \langle n_1 | H_I^j | n_2 \rangle \langle n_2 | H_I^j | \Phi_i \rangle}{(p_f^+ - p_{n_1})(p_f^+ - p_{n_2})} + \cdots ,
$$

where $Z_g$ is the multiplicative coupling constant renormalization, and $Z_i$ and $Z_f$ are the wavefunction renormalization constants of the initial and final states.

It is also convenient to express the above perturbation expansion in terms of the diagrammatic approach. The rules for writing the expression of perturbative expansions from diagrams for QCD are as follows:

- Draw all topologically distinct $x^+$-ordered diagrams.
- For each internal line, sum over helicity and integrate using $\int \frac{dk^+ d^2 k_\perp}{16\pi^3} \theta(k^+)$ for quarks and $\int \frac{dk^+ d^2 k_\perp}{16\pi^3} \theta(k^+)$ for gluons.
- For each vertex, include a factor of $16\pi^3 \delta^3(p_f - p_i)$ and a simple matrix element listed in Ref. [Zhang and Harindranath (1993b)].
- Include a factor $(p_i^- - \sum_n p_n^- + i\epsilon)^{-1}$ or $(p_f^- - \sum_n p_n^- + i\epsilon)^{-1}$ for each intermediate state, where $\sum_n p_n^-$ sum over all on-mass-shell intermediate particle energies.
- Add a symmetry factor $S^{-1}$ for each gluon loop coming from the symmetrized boson states.
3.3 Perturbative Calculation of Light-Front QCD

To illustrate the above computation scheme and to explore the severe light-front infrared divergences, let me list some calculations up to one-loop based on the $x^+$-ordered diagrammatical approach. Note that besides the infrared divergence, which is regularized by Eq. (39), there are also ultraviolet divergences for which we use a transverse cut-off: $A_1^2 \geq \kappa_1^2 \geq \mu^2$. Here I have also introduced a mass scale $\mu$ for the minimum cut-off of the transverse momentum in order to avoid the several complicated pure infrared divergences and mass singularity from the massless gluon, $\mu$ should be much larger than all other masses in the theory, and is considered as a renormalization scale here.

i). Quark wavefunction and mass renormalization. The one-loop light-front quark energy corrections (for the three diagrams in Fig. 1, respectively) are given by

$$\delta p_1^- = -\frac{g^2}{8\pi^2} C_f \left\{ \frac{p^2 - m^2}{p^+} \left( 2 \ln \frac{p^+}{\epsilon} - \frac{3}{2} \right) \ln \frac{A_1^2}{\mu^2} + \frac{m^2}{p^+} \left( -2 \ln \frac{A_1^2}{\mu^2} \right) + \frac{A_1^2 - \mu^2}{p^+} \left( \frac{\pi p^+}{2\epsilon} - 1 + \ln \frac{p^+}{\epsilon} \right) \right\} ,$$

$$\delta p_2^- = \frac{g^2}{8\pi^2} C_f \frac{A_1^2 - \mu^2}{p^+} \ln \frac{\mu^2}{\epsilon} ,$$

$$\delta p_3^- = \frac{g^2}{8\pi^2} C_f \frac{A_1^2 - \mu^2}{p^+} \left( \frac{\pi p^+}{2\epsilon} - 1 \right) .$$

This shows that, in the one-loop quark energy correction, one-gluon exchange gives rise to both linear and logarithmic infrared divergences. The instantaneous fermion interaction contribution (see $\delta p_2^-$ in Fig. 1b) contains only one logarithmic divergence which cancels the logarithmic divergence in $\delta p_1^-$. The instantaneous gluon interaction contribution ($\delta p_3^-$ of Fig. 1c) has a linear infrared divergence which precisely cancels the same divergence in $\delta p_1^-$. This cancellation of linear infrared divergences is based on the use of the regularization for $k^+ \to 0$ in Eq. (39) [Zhang and Harindranath (1993b)].

The quark mass correction (dropping the finite part) is then given by

$$\delta m^2 = p^+ \delta p^- |_{p^2 = m^2} = \frac{g^2}{4\pi^2} C_f m^2 \ln \frac{A_1^2}{\mu^2} ,$$

which is longitudinally infrared divergence free; and the quark wavefunction renormalization constant is

$$Z_2 = 1 + \left. \frac{\partial \delta p^-}{\partial p^-} \right|_{p^2 = m^2} = 1 + \frac{g^2}{8\pi^2} C_f \left( \frac{3}{2} - 2 \ln \frac{p^+}{\epsilon} \right) \ln \frac{A_1^2}{\mu^2} .$$

The wavefunction renormalization contains an additional type of divergence, the mixing of infrared and ultraviolet divergences, that does not occur in covariant calculations. This is the ‘spurious’ mixing associated with the gauge singularity.
It corresponds to the so-called light-front double pole problem in the Feynman theory with the use of the light-front gauge and the principal value prescription that prohibits any continuation to Euclidean space and power counting in Feynman loop integrals. In the \( x^+ \)-ordered Hamiltonian perturbation theory the power counting is different. The above argument of power counting for Feynman loop integrals may be irrelevant. Furthermore, since the second order correction to wavefunctions must be negative, the above result shows that it is the additional infrared divergence that gives a consistent answer for wavefunction renormalization.

\( ii) \). Gluon wave function and mass correction. Similar calculation to the one-loop light-front gluon energy corrections leads to the solution:

\[
\delta \mu_G^2 = -\frac{g^2}{4\pi^2} \left\{ T_f N_f m^2 \ln \frac{A^2}{\mu^2} (A^2 - \mu^2) \left( \frac{C_A}{2} - T_f N_f \right) \left( 1 - \ln \frac{k^+}{\epsilon} \right) \right\},
\]

\[
Z_3 = 1 + \frac{g^2}{8\pi^2} \left\{ C_A \left( \frac{11}{6} - 2 \ln \frac{q^+}{\epsilon} \right) - \frac{2}{3} T_f N_f \right\} \ln \frac{A^2}{\mu^2}. \tag{71}
\]

In the gluon sector, more severe divergences appear. It contains the quadratic and logarithmic UV divergences, linear and logarithmic IR divergences, and an unusual large longitudinal momentum logarithmic divergence. Only the linear infrared divergences are cancelled with the principal value prescription. The gluon mass correction is not zero. The non-zero gluon mass correction of Eq.(71) is not surprising because it has the same divergence feature as the photon mass correction in light-front QED [Eq.(71) will be reduced to the photon mass correction when we set \( T_f = 1 \), \( C_A = 0 \) and \( N_f = 1 \)]. In a covariant calculation, the zero gluon mass correction is true only for dimensional regularization which “removes” or drops the mass correction. In the present calculation, maintaining zero gluon mass requires a mass counterterm, as is known in QED. The difference between QED and QCD is only manifest in the gauge boson wavefunction renormalization. For wavefunction renormalization, again there is an additional mixing of UV and IR divergences, which again provides the correct sign for the wavefunction renormalization constant.
iii). Coupling constant renormalization. For convenience, we set the external gluon momentum \( q(q^+, q_\perp) = 0 \). The quark-gluon vertex is then reduced:

\[
V_0 = 2gT^{a}_{\beta\alpha} \frac{p^i}{p^+} \delta_{\lambda_1 \lambda_2} \varepsilon_\sigma^i.
\]  

(72)

In \( x^+ \)-ordered perturbation theory, the one-loop vertex correction is given by

\[
\delta V_0 = \{ V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \} V_0,
\]  

(73)

where \( V_n, n = 1 - 6 \) are represented the contributions from different time-ordered diagrams (see Ref. [Harindranath and Zhang (1993)]:

\[
V_1 = \frac{g^2}{2\pi^2} \left( \frac{3}{2} - 2 \ln \frac{p^+}{\epsilon} \right) C_f \ln \frac{A}{\mu},
\]

\[
V_2 = \frac{g^2}{8\pi^2} \left( \frac{11}{3} C_A - \frac{4}{3} N_f T_f \right),
\]

\[
V_3 = -\frac{g^2}{4\pi^2} \left( \frac{3}{2} - 2 \ln \frac{p^+}{\epsilon} \right) \left( \frac{1}{2} C_A + C_f \right) \ln \frac{A}{\mu},
\]

\[
V_4 = -\frac{g^2}{8\pi^2} \left( \frac{3}{2} - 2 \ln \frac{p^+}{\epsilon} \right) C_A \ln \frac{A}{\mu},
\]

\[
V_5 = 0, \quad V_6 = 0.
\]

(74)

To evaluate the contributions to the coupling constant we have to multiply \( V_1 \) and \( V_2 \) by \( \frac{1}{2} \) in order to take into account the proper correction due to the renormalization of initial and final states. Thus adding the contributions together, we have,

\[
\delta V'_0 = \left( \frac{1}{2} V_1 + \frac{1}{2} V_2 + V_3 + V_4 + V_5 + V_6 \right) V_0
\]

\[
= V_0 \frac{g^2}{2\pi^2 \epsilon} \left( \frac{11}{6} C_A - \frac{2}{3} N_f T_f \right) \ln \frac{A}{\mu} = \delta g V_0.
\]  

(75)

Note that all mixed divergences cancel now. The correction to the coupling constant is given by

\[
g_R = g(1 + \delta g) = g \left( 1 + \frac{g^2}{8\pi^2} \left( \frac{11}{6} C_A - \frac{2}{3} N_f T_f \right) \ln \frac{A}{\mu} \right).
\]  

(76)

By redefining the bare coupling constant \( g \) such that \( g_R \) is finite. Thus we have given all canonical renormalization quantities in QCD up to one-loop order based on the \( x^+ \)-ordered perturbation theory.

From these results, the anomalous dimensions for quarks and gluons and the \( \beta \) function up to one-loop can be easily calculated. The anomalous dimension of the quark field to order \( g^2 \) is

\[
\gamma_F = \frac{1}{2Z_2} \frac{\partial Z_2}{\partial \ln \mu} = \frac{g^2}{8\pi^2} C_f \left( 2 \ln \frac{p^+}{\epsilon} - \frac{3}{2} \right).
\]  

(77)
The momentum-dependent term implies that the quark anomalous dimension is gauge dependent. The anomalous dimension for the gluon field is

\[
\gamma_G \equiv \frac{1}{2Z_3} \frac{\partial Z_3}{\partial \ln \mu} = \frac{g^2}{8\pi^2} \left\{ C_f \left( 2 \ln \frac{q^+}{\epsilon} - \frac{11}{6} \right) + \frac{2}{3} T_f N_f \right\},
\]

(78)

which is also gauge-dependent. In the case of \( q^+ = 0 \), the gauge dependent term can be removed, and Eq.(78) is reduced to Gross and Wilczek’s result in their Feynman calculation with \( A_+^a = 0 \) and \( q^+ = 0 \) [Gross and Wilczek (1974)]. The \( \beta \) function is

\[
\beta(g) = \frac{\partial g_R}{\partial \ln \mu} = - \frac{g^3}{16\pi^2} \left( \frac{11}{3} C_A - \frac{4}{3} N_f T_f \right),
\]

(79)

which is the well-known result to one loop order and is infrared divergence free, as we expected.

From the above result, we see that there are severe light-front divergences in LFQCD. Systematic control of these divergences is required *a priori* before we perform any practical numerical calculation in light-front coordinates for QCD bound states. From the basic one-loop calculations, one can see that, in the \( x^+ \)-ordered perturbation theory, light-front QCD involves various UV and IR divergences. Some of the divergences have not even been encountered in covariant and noncovariant Feynman calculations to the same order. Among various light-front divergences, there are two severe divergences one has to deal with in the \( x^+ \)-ordered theory for light-front QCD. The first is the mixing of UV and IR logarithmic divergences in wavefunction renormalization. The occurrence of the mixing divergences may not be a severe problem. The mixing divergences should be cancelled completely for physical quantities, as we have seen from the coupling constant renormalization. We expect that the problem of mixing divergences may not exist when we consider real physical processes. The second problem is the infinite gluon mass correction. In the time-ordered perturbation theory dimensional regularization is not available to avoid the nonzero gluon mass correction. To have a massless gluon in perturbation theory, we have to introduce a gluon mass counterterm. In the leading order (one-loop) calculation, there is no difficulty arising from a gluon mass counterterm. However, when we go to the next order, it has been found that the gluon mass counterterm leads to a noncancellation of infrared divergences. The non-vanishing infrared divergences could introduce non-local counterterms in both the longitudinal and transverse directions. In instant quantization, such non-local counterterms are forbidden for a renormalizable theory. Here, these non-local counterterms are allowed by the light-front power counting. This is a special feature of LFQCD. One speculation from this property is that the non-local counterterms for infrared divergences may also provide a source for quark confinement [Wilson et al. (1994)].

In summary, renormalization in LFQCD Hamiltonian theory is very different from conventional Feynman theory and it is an entirely new subject where investigations are still in their preliminary stage. In perturbative calculations,
careful treatment could remove all severe infrared divergences for interesting physical quantities in LFQCD. For nonperturbative studies, the cancellation of severe infrared divergences may not work because certain approximations (e.g., Fock space truncation) might be used. These approximations may also break many important symmetries such as gauge invariance and rotational invariance. It is the hope of the current investigation of light-front renormalization theory that the counterterms for the light-front infrared divergences may restore the broken symmetries and also provide an effective confining LFQCD Hamiltonian for hadronic bound states.

4 Light-Front Heavy Quark Effective Theory (HQET)

4.1 About Heavy Quark Symmetry and HQET

The rich information about electroweak and strong interactions that can be extracted from various heavy hadron decays has led to the extensive exploration of the QCD based and model-independent description of heavy hadrons in the past few years. This is mainly due to the discovery of heavy quark spin-flavor symmetry (HQS) in heavy meson decays by Isgur and Wise [Isgur and Wise (1989), Neubert (1994)]. For a typical example, with the HQS, all six form factors in \( B \to D \) and \( B \to D^* \) decays are reduced to an universal function, called the Isgur-Wise function, and the normalization of this universal function at the zero-recoil point provides a model-independent determination of the Kabayshi-Makawa matrix element \( |V_{cb}| \). Similarly in heavy-baryon decays, the application of HQS also leads to tremendous simplifications.

On the other hand, heavy quark symmetry can be derived from QCD in heavy mass limit \( m_Q \to \infty \), via the so-called heavy quark effective theory (HQET) [Eichten and Hill (1990), Georgi (1990)]. The later is an effective theory of QCD for heavy quark expanded in inverse powers of heavy quark mass \( m_Q \). In fact, HQET provides us with a systematical expansion of QCD dynamics in terms of the dimensionless parameter \( \Lambda_{QCD}/m_Q \), and it serves as a theoretical framework for the systematical computation of the \( 1/m_Q \) corrections to the limit \( m_Q \to \infty \). Thus, the HQET offers us a new channel to explore the intrinsic properties of hadronic structure from QCD.

In order to actually compute any physical observables and make definite predictions, one still has to confront the non-perturbative QCD dynamics. Currently, except for the lattice approach, the main physical quantities, such as Isgur-Wise function, can only be computed in various hadronic models, such as the constituent quark model, the bag model, and QCD sum rules. It would be very interesting if one could calculate the Isgur-Wise function, or any hadronic form factors, directly from QCD. This requires to construct explicitly the heavy hadron bound states within the HQET, which is also necessary for a complete understanding of heavy hadron dynamics. We are motivated by such requirement to reformulate HQET on the light-front, from which we hope to consistently study the heavy hadron bound state problem [Cheung et al. (1995)]. Meanwhile, as we
know for light quark systems, both quark confinement and spontaneously chiral symmetry breaking play an essential role to the quark dynamics in hadrons. In order to provide a nonperturbative QCD description for light quark systems, it is necessary to understand the underlying mechanism for quark confinement as well as for chiral symmetry breaking. This will certainly make the problem most complicated. However, for heavy quark systems, chiral symmetry is explicitly broken so that confinement is the sole nontrivial feature influencing heavy quark dynamics. Choosing the heavy hadron systems should be a good starting point in the study of nonperturbative QCD. The light-front HQET discussed here is mainly based on the works collaborated with C. Y. Cheung and G. L. Lin [Cheung et al. (1995)].

4.2 $1/m_Q$ Expansion of the Heavy Quark Lagrangian on the LF

Let us begin with the QCD Lagrangian for a heavy quark:

$$\mathcal{L} = \bar{Q} (i\slashed{D} - m_Q) Q,$$

(80)

where $Q$ is the heavy quark field operator, $m_Q$ the heavy quark mass and $D^\mu$ the QCD covariant derivative.

In the instant formalism, HQET is obtained by redefining the heavy quark field as:

$$Q(x) = e^{-im_Q v \cdot x} [h_v(x) + H_v(x)],$$

(81)

where $v$ is the four velocity of the heavy quark, such that $v^2 = 1$; $h_v(x)$ and $H_v(x)$ are respectively the so-called large and small components of the heavy quark field, satisfying $\not\!v h_v(x) = h_v(x)$ and $\not\!v H_v(x) = -H_v(x)$. From the QCD equation of motion, one can express $H_v(x)$ in terms of $h_v(x)$ and show that the former is suppressed by $1/m_Q$ compared to the latter. Using Eq.(81) and the relation between $h_v(x)$ and $H_v(x)$, one can systematically expand the QCD Lagrangian in powers of $1/m_Q$, and arrive at an effective theory for the heavy quark.

In the framework of light-front quantization, the situation is quite different. Before taking the heavy quark mass limit, the quark field is already divided into two parts: $Q(x) = Q_+(x) + Q_-(x)$, with $Q_\pm(x) = A_\pm Q(x) = \frac{1}{2}(\gamma_0 \gamma_\perp Q(x) + \gamma_\perp).$ The Dirac equation for $Q$ can then be rewritten as two coupled equations for $Q_\pm$:

$$iD^- Q_+(x) = (i\alpha_\perp \cdot D_\perp + \beta m_Q) Q_-(x),$$

(82)

$$iD^+ Q_-(x) = (i\alpha_\perp \cdot D_\perp + \beta m_Q) Q_+(x),$$

(83)

where $\alpha_\perp = \gamma_0 \gamma_\perp$ and $\beta = \gamma^0$. As we known only the plus-component $Q_+(x)$ is the dynamical field. The minus-component $Q_-(x)$ is a light-front constraint that can be determined from $Q_+(x)$. In terms of $Q_+(x)$, the QCD Lagrangian (1) for the heavy quark can be rewritten as

$$\mathcal{L} = Q_+^\dagger iD^- Q_+ - Q_+^\dagger (i\alpha_\perp \cdot D_\perp + \beta m_Q) Q_-, $$

(84)
where $Q_-$ can be eliminated by Eq. (83).

To derive the light-front HQET, we use the same redefinition of the heavy quark field as in the covariant case,

$$Q(x) = e^{-imQv^+}Q_+(x),$$

but without imposing any constraint on the new variable $Q_+$ to separate the large and small components. It follows that $Q_\pm(x) = e^{-imQv^+}Q_{\pm}(x)$. Substituting this result into Eq. (83), we obtain

$$Q_\pm(x) = \frac{1}{mQv^+ + iD^+} \left[ i\alpha_\perp \cdot D_\perp + mQ (\alpha_\perp \cdot v_\perp + \beta) \right] Q_\pm(x).$$

It is worth noting that in the ordinary light-front formulation of field theory, the elimination of the dependent component $Q_-$ requires the choice of the light-front gauge $A^+ = 0$, and a specification of the operator $1/\partial^+$ which leads to severe light-front infrared problem that has still not been completely understood [Zhang and Harindranath (1993a)]. However, for the heavy quark field with the redefinition of Eq. (85), the above problem does not occur since the elimination of the dependent component $Q_{\pm}$ now depends on the operator $1/(mQv^+ + iD^+)$ which has no infrared problem. Moreover, it has a well defined series expansion in powers of $iD^+/mQ$:

$$\frac{1}{mQv^+ + iD^+} = \frac{1}{v^+} \sum_{n=1}^{\infty} \left( \frac{1}{mQ} \right)^n \left( -i \frac{D^+}{v^+} \right)^{n-1}. \tag{87}$$

Thus, the heavy quark QCD Lagrangian (80) can be expressed in terms of $Q_{v+}$ alone. The complete $1/mQ$ expansion is given by

$$\mathcal{L} = \frac{1}{v^+} \left\{ 2Q_{v+}^\dagger (iv \cdot D) Q_{v+} \right. \\
\left. - \sum_{n=1}^{\infty} \left( \frac{1}{mQ} \right)^n Q_{v+}^\dagger \left\{ (i\alpha \cdot D) \left( -i \frac{D^+}{v^+} \right)^{n-1} (i\alpha \cdot D) \right\} Q_{v+}(x) \right\} \\
= \mathcal{L}_0 + \sum_{n=1}^{\infty} \mathcal{L}_n, \tag{88}$$

where

$$\alpha \cdot D = \alpha_\perp \cdot D_\perp - \frac{\alpha_\perp \cdot v_\perp + \beta}{v^+} D^+. \tag{89}$$

This is the light-front effective heavy quark Lagrangian.
4.3 Properties of the Light-Front HQET

In the symmetry limit, the light-front HQET reduces to

$$\mathcal{L}_0 = 2v^+ Q_v^\dagger (iv \cdot D) Q_v^+, \quad (90)$$

which clearly exhibits the flavor and spin symmetries, because it is independent of Dirac $\gamma$-matrices and the heavy quark mass, as in the covariant formulation.

However, beyond the symmetry limit, the light-front HQET has several advantages over the instant formulation. In the instant HQET, the non-leading terms contain high order time-derivatives; consequently it is difficult to perform a consistent canonical quantization beyond the limit $m_Q \to \infty$ [Lebed and Suzuki (1991)]. It is remarkable to see that in the light-front HQET, only linear time-derivative appears, and it resides in $\mathcal{L}_0$. The presence of the matrix $\not{\!n}$ in the non-leading terms eliminates all light-front time derivative terms. This can be seen more clearly in Eq.(88). Thus the canonical quantization of light-front HQET is straightforward: First of all, the canonical conjugate of the dynamical variable $Q_v^+$ is given by

$$\Pi_{Q_v^+} = \frac{\partial \mathcal{L}}{\partial (\partial^- Q_v^+)} = iQ_v^+, \quad (91)$$

which does not involve any $1/m_Q$ corrections. Then using the light-front phase space quantization [Zhang and Harindranath (1993a)], we obtain the basic anticommutation relation:

$$\{Q_v^+(x), Q_{v'}^\dagger(y)\}_{x^+ = y^+} = A_+ \delta^3(x-y), \quad (92)$$

which is valid to all orders in $1/m_Q$.

The second very useful property of the light-front HQET is that the heavy quark effective Hamiltonian is well defined on the light-front. From Eqs.(88) and (91), we obtain the light-front heavy quark effective Hamiltonian,

$$H = \int dx^- d^2x_\perp \mathcal{H}(x) \quad (93)$$

with the Hamiltonian density $\mathcal{H}$ given by

$$\mathcal{H} = \frac{1}{iv^+} Q_v^\dagger (v^- \partial^+ - 2v_\perp \cdot \partial_\perp) Q_{v^+} - \frac{2g}{v^+} Q_{v^+}^\dagger (v \cdot A) Q_{v^+} + \mathcal{H}_{m_Q} \quad (94)$$

and

$$\mathcal{H}_{m_Q} = \sum_{n=1}^{\infty} \mathcal{H}_n = -\sum_{n=1}^{\infty} \mathcal{L}_n. \quad (95)$$

This light-front heavy quark effective Hamiltonian can serve as a basis for constructing heavy hadron bound states, as we will see in the next lecture. It is also useful for the study of the $1/m_Q$ corrections in heavy quark dynamics. Specifically, suppose we choose the light-front gauge ($A^+ = 0$) in the light-front
HQET, we see immediately from Eq.(88) that, in the symmetry breaking terms, the power of the gluon field does not increase with that of \(1/m_Q\). This property, which is unique to the light-front formulation, may greatly simplify our treatment of \(1/m_Q\) corrections. Meanwhile, note that the non-leading light-front effective Hamiltonian \(\mathcal{H}_n\) is precisely the minus of the corresponding effective Lagrangian \(\mathcal{L}_n\) given by Eq.(88). This simple relation is not valid in the instant HQET, due to appearance of the high-order time-derivative terms.

Furthermore, since we have not chosen any specific gauge, and also there is no light-front infrared divergent problem for the heavy quark sector, short-distance QCD corrections to the heavy quark current and the effective Lagrangian must be the same as those calculated in the covariant formulation. Of course, an explicit calculation of the short-distance effects in the light-front HQET is needed to confirm the above statement, which has not been done as I known.

4.4 Isgur-Wise Function

The heavy quark current can also be systematically expanded in \(1/m_Q\) on the light-front. In the heavy mass limit, it reduces to the following familiar form:

\[
\mathcal{Q}^i(x)\Gamma Q^i(x) = e^{-i(m_Qv'-m_Qv)v} \mathcal{P}^{IL}_v(x)\Gamma h^{IL}_v(x),
\]

where \(h^L_v = \left\{1 + \frac{\alpha_{\perp}v_{\perp} + \beta}{v^+}\right\}Q_v\). Consequences of the spin symmetry can be readily derived using this zeroth order heavy quark current. As an example, consider the matrix elements

\[
\langle P\mathcal{Q}^i(v')|\mathcal{P}^{IL}_v\Gamma h^{IL}_v|P\mathcal{Q}^i(v)\rangle \quad \text{and} \quad \langle P\mathcal{Q}^i(v')|\mathcal{P}^{IL}_v\Gamma h^{IL}_v|P\mathcal{Q}^i(v)\rangle,
\]

where \(\Gamma\) stands for any arbitrary gamma matrix, \(P\mathcal{Q}\) and \(P\mathcal{Q}^*\) represent respectively a pseudoscalar meson and a vector meson containing a single heavy quark \(Q\). The quantum numbers of the heavy mesons can be efficiently accounted for by the interpolating fields: \(|P\mathcal{Q}^i(v)\rangle = \mathcal{P}^{IL}_v\gamma_5\ell_v|0\rangle\), \(|P\mathcal{Q}^*_{i}(v)\rangle = \mathcal{P}^{IL}_v\not\epsilon_{v}\ell_v|0\rangle\), where \(\epsilon\) is the polarization vector of the vector meson, and \(\ell_v\) represents the fully interacting light quark (or brown muck). From \(|0\rangle\mathcal{Q}_{v+}\mathcal{Q}^{\dagger}_{v+}|0\rangle = \frac{v^+}{2}A_+\), it is easy to show that

\[
h^L_v\mathcal{P}^{IL}_v = \left(1 + \frac{\alpha_{\perp}v_{\perp} + \beta}{v^+}\right)\frac{v^+}{2}A_+\left(1 + \frac{\alpha_{\perp}v_{\perp} + \beta}{v^+}\right)\beta = \frac{1+\not\epsilon}{2}.
\]

Hence, in the heavy mass limit, the heavy meson decay matrix elements on the light-front take the familiar forms:

\[
\langle P\mathcal{Q}^i(v')|\mathcal{P}^{IL}_v\Gamma h^{IL}_v|P\mathcal{Q}^i(v)\rangle = Tr\left\{\gamma_5\left(\frac{1+\not\epsilon}{2}\right)\Gamma\left(\frac{1+\not\epsilon}{2}\right)\gamma_5M\right\}
\]

\[
\langle P\mathcal{Q}^*_{i}(v')|\mathcal{P}^{IL}_v\Gamma h^{IL}_v|P\mathcal{Q}^i(v)\rangle = Tr\left\{\not\epsilon\left(\frac{1+\not\epsilon}{2}\right)\Gamma\left(\frac{1+\not\epsilon}{2}\right)\gamma_5M\right\}.
\]
where $M$ is the transition matrix element for the light quark \cite{Wise1991},

$$M = \langle 0 | \ell_v \ell_v' | 0 \rangle \rightarrow \xi (v \cdot v') I. \quad (101)$$

Thus spin symmetry implies that the transition matrix elements (97) are described by a single form factor $\xi (v \cdot v')$, which is just the famous Isgur-Wise function. An explicit calculation of the Isgur-Wise function from light-front bound state wave function will be given in the next lecture.

5 Quark Confinement and Heavy Hadron Bound States

5.1 A Weak-Coupling Treatment to Nonperturbative QCD

There are two fundamental problems in QCD for hadronic physics, the quark confinement and the spontaneous breaking of chiral symmetry. These two problems are the basis for solving the low-energy hadronic bound states from QCD but none of them has been completely understood. Recently, Wilson et al. proposed a new approach to determine hadronic bound states from nonperturbative QCD on the light-front with a weak-coupling treatment (WCT) \cite{Wilson1994}. The key to eliminating necessarily nonperturbative effects is to construct an effective QCD Hamiltonian in which quarks and gluons have nonzero constituent masses rather than the zero masses of the current picture. The use of constituent masses cuts off the growth of the running coupling constant and makes it conceivable that the running coupling never leaves the perturbative domain. The WCT approach potentially reconciles the simplicity of the constituent quark model with the complexities of QCD. The penalty for achieving this weak-coupling picture is the necessity of formulating the problem in light-front coordinates and of dealing with the complexities of renormalization.

Succinctly, this new approach of achieving a QCD description of hadronic bound states can be summarized as follows: Using a new renormalization scheme, called similarity renormalization group (SRG) scheme that is recently proposed by Glazek and Wilson \cite{GlazekWilson1994, Wilson1994}, one can obtain an effective QCD Hamiltonian $H_\lambda$ which is a series of expansion in terms of the QCD coupling constant, where $\lambda$ is a low energy scale. Then one may solve from $H_\lambda$ the strongly interacting bound states as a weak-coupling problem. The WCT scheme contains the following steps: (i) Compute explicitly from SRG the $H_\lambda$ up to the second order and denote it by $H_{\lambda 0}$ as a nonperturbative part of $H_\lambda$. The remaining higher order contributions in $H_\lambda$ are considered as a perturbative part $H_{\lambda I}$. (ii) Introduce a constituent picture which allows one to start the hadronic bound states with the valence constituent Fock space. The constituent quarks and gluons have masses of a few hundreds MeV, and these masses are functions of the scale $\lambda$ that must vanish when the effective theory goes back to the high energy region. (iii) Solve hadronic bound states with $H_{\lambda 0}$ nonperturbatively in the constituent picture and determine the scale dependence of the constituent masses and the coupling constant. The coupling constant $g$
now becomes an effective one, denoted by $g_\lambda$. If we could show that with a suitable choice of $\lambda$ at the hadronic mass scale, the effective coupling constant $g_\lambda$ can be arbitrarily small, then WCT could be applied to $H_\lambda$ such that the corrections from $H_{\lambda I}$ can be truly computed perturbatively. If everything listed above works well, we may arrive at a weak-coupling QCD theory of the strong interaction for hadronic bound states.

With the idea of SRG and the concept of coupling coherence [Perry and Wilson (1993)], Perry has shown that upon a calculation to the second order, there exists a logarithmic confining potential in the resulting LFQCD effective Hamiltonian [Perry (1994)]. This is a crucial finding to light-front nonperturbative QCD. However, the general strategy of solving hadrons through the WCT scheme is far to be completed. Very recently, I used SRG to analytically derive from the light-front HQET [Cheung et al. (1995)] a heavy quark QCD Hamiltonian which is responsible to heavy hadron bound states. The resulting Hamiltonian explicitly contains a confining interaction between a heavy quark and a heavy antiquark at long distance plus a Coulomb-type interaction at short distance. With this effective QCD Hamiltonian, I study the strongly interacting heavy hadronic bound states, from which I can provide a WCT to nonperturbative QCD on the light-front, at least for heavy quarkonia. The following discussion is mainly based on my recent work, Ref. [Zhang (1996)].

5.2 Light-front Similarity Renormalization Scheme

The basic idea of the SRG approach is to develop a sequence of infinitesimal unitary transformations that transform an initial bare Hamiltonian $H^B$ to an effective Hamiltonian $H_\lambda$ in a band-diagonal form relative to an arbitrarily chosen energy scale $\lambda$:

$$H_\lambda = S_\lambda H^B S_\lambda^\dagger. \quad (102)$$

Here the band-diagonal form means that the matrix elements of $H_\lambda$ involving energy jumps much larger than $\lambda$ will all be zero, while matrix elements involving smaller jumps or two nearby energies remain in $H_\lambda$. The similarity transformation should satisfy the condition that for $\lambda \to \infty$, $H_\lambda \to H^B$ and $S_\lambda \to 1$.

Here, I shall follow the formulation of SRG developed on the light-front [Wilson et al. (1994)]. The effective Hamiltonian we seek is $H_\lambda$ with $\lambda$ being of order a hadronic mass ($\sim 1$ GeV). We begin with a given bare Hamiltonian which can be written by $H^B = H_0 + H^B_I$, where $H_0$ is a bare free Hamiltonian and $E_i$ is its eigenvalue. Consider an infinitesimal transformation, then Eq.(102) is reduced to

$$\frac{dH_\lambda}{d\lambda} = [H_\lambda, T_\lambda], \quad (103)$$

which is subject to the boundary condition $\lim_{\lambda \to \infty} H_\lambda = H^B$.

To force the Hamiltonian $H_\lambda$ becoming a band-diagonal form in energy space, we need to specify the action of the generator operator $T_\lambda$. This can be done by introducing the scale $\lambda$ with $x_{\lambda ij} = \frac{E_i - E_j}{E_i + E_j + \lambda}$ into a smearing function $f_{\lambda ij} = \ldots$
\[ f(x_{i,j}) \text{ such that when } x < 1/3, f = 1; \text{ when } x > 2/3, f = 0; \text{ and } f \text{ may be a smooth function from 1 to 0 for } 1/3 \leq x \leq 2/3. \text{ We can write } H_\lambda = H_0 + H_{I\lambda} \text{ because } H_0 \text{ is invariant under transformations. Then Eq.}(103) \text{ can be reexpressed as}
\]
\[
\frac{dH_{\lambda ij}}{d\lambda} = f_{\lambda ij}[H_{I\lambda}, T_{\lambda}]_{ij} + \frac{d}{d\lambda}(\ln f_{\lambda ij})H_{\lambda ij},
\]
\[
T_{\lambda ij} = \frac{1}{E_j - E_i} \left\{ (1 - f_{\lambda ij})[H_{I\lambda}, T_{\lambda}]_{ij} - \frac{d}{d\lambda}(\ln f_{\lambda ij})H_{\lambda ij} \right\}. \tag{104}
\]

Here we have used the notation \(A_{ij} = \langle i | A | j \rangle\), and \(|i\rangle\) is an eigenstate of \(H_0\). Since \(f(x)\) vanishes when \(x \geq 2/3\), one can see that \(H_{\lambda ij}\) does indeed vanish in the far off-diagonal region. It also can be seen that \(T_{\lambda ij}\) is zero in the near-diagonal region. The solutions for \(H_{I\lambda}\) and \(T_{\lambda}\) are
\[
H_{I\lambda} = H_{I\lambda}^B + [H_{I\lambda'}, T_{\lambda'}]_{R}, \quad T_{\lambda} = H_{I\lambda T}^B + [H_{I\lambda'}, T_{\lambda'}]_{T}, \tag{105}
\]
where \(H_{I\lambda ij}^B = f_{\lambda ij} H_{ij}^B, H_{I\lambda T ij}^B = -\frac{1}{E_j - E_i} \left( \frac{d}{d\lambda} f_{\lambda ij} \right) H_{ij}^B\), and
\[
X_{\lambda ij} = -f_{\lambda ij} \int_{\lambda}^{\infty} d\lambda' X_{\lambda' ij}, \tag{106}
\]
\[
X_{\lambda' ij} = -\frac{1}{E_j - E_i} \left( \frac{d}{d\lambda} f_{\lambda ij} \right) \int_{\lambda}^{\infty} d\lambda' X_{\lambda' ij} + \frac{1 - f_{\lambda ij}}{E_j - E_i} X_{\lambda ij}. \tag{107}
\]

Finally, one obtains an iterated solution for \(H_\lambda\),
\[
H_\lambda = \left( H_0 + H_{I\lambda}^B \right) + \left[ H_{I\lambda'}, H_{I\lambda'}^T \right]_R + \left[ H_{I\lambda'}, H_{I\lambda'}^T \right]_T + \ldots
\]
\[
= H_\lambda^{(0)} + H_\lambda^{(2)} + H_\lambda^{(3)} + \ldots. \tag{108}
\]

Thus, through SRG, we eliminate the interactions between the states well-separated in energy and generate the effective Hamiltonian of eq.(108). The expansion of eq.(108) in terms of the interaction coupling constant brings in order by order the full theory corrections to this band diagonal low energy Hamiltonian.

Explicitly, the bare Hamiltonian \(H^B\) input in the above formulation can be obtained from the canonical Lagrangian with a high energy cutoff that removes the usual UV divergences. For LFQCD dynamics, the bare Hamiltonian has been constructed in lecture II (for detailed discussion, see [Zhang and Harindranath (1993a), Zhang and Harindranath (1993b)]). Instead of the cutoff on the field
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operators which is introduced in ref. [Wilson et al. (1994)], I shall use a vertex
cutoff to every vertex in the bare Hamiltonian [Zhang (1996)]:

$$\theta(\Lambda^2/P^+ - |p^-_i - p^-_j|),$$

(109)

where $p^-_i$ and $p^-_j$ are the initial and final state light-front energies respectively
between the vertex, $\Lambda$ is the UV cutoff parameter, and $P^+$ the total light-front
longitudinal momentum of the system we are interested in. Eq.(109) is also
also called the local cutoff in light-front perturbative QCD [Lepage and Brodsky
(1980)]. All the $\Lambda$-dependences in the final bare Hamiltonian are removed by the
counterterms. The use of eq.(109) largely simplifies the analysis on the cutoff
scheme in ref. [Wilson et al. (1994)].

Meanwhile, in SRG calculation, we should also give an explicit form of the
smearing function $f_{\lambda ij}$. One of the simplest smearing functions that satisfies the
requirements of SRG is a theta-function [Zhang (1996)]:

$$f_{\lambda ij} = \theta\left(\frac{1}{2} - x_{\lambda ij}\right).$$

(110)

On the light-front, it is convenient to redefine $x_{\lambda ij} = \frac{|p^-_i - p^-_j|}{P^-_i + P^-_j + \Lambda^2/P^+}$. Then we
can further replace the above smearing function by the following form:

$$f_{\lambda ij} = \theta\left(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|\right),$$

(111)

where $\Delta P^-_{ij} = P^-_i - P^-_j$ is the light-front free energy difference between the
initial and final states of the physical processes. The light-front free energies of
the initial and final states are defined as sums over the light-front free energies
of the constituents in the states.

With the definition of (111), Eq.(108) can be reduced to

$$H_{\lambda ij} = \theta\left(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|\right) \left\{ H_{ij}^B + \sum_k H_{iik}^B H_{kj}^B \left[ \frac{g_{\lambda jk}}{\Delta P^-_{ik}} + \frac{g_{\lambda ik}}{\Delta P^-_{jk}} \right] + \cdots \right\}$$

(112)

The front factor (the theta-function) in the above equation indicates that $H_{\lambda}$
only describes long distance interactions (with respect to the scale $\lambda$) which is
responsible to hadronic bound states. The function $g_{\lambda ij}$ in eq.(112) is given by

$$g_{\lambda ij} = \int_{\lambda^2/P^+}^{\infty} d(\lambda^2/P^+) \frac{d}{d(\lambda^2/P^+)} \frac{d}{\lambda^2/P^+} f_{\lambda ij k}$$

$$= \theta(|\Delta P^-_{jk}| - \lambda^2/P^+)\theta(|\Delta P^-_{jk}| - |\Delta P^-_{ik}|).$$

(113)
5.3 Heavy Quark Confining Interaction

Now we can use SRG to the light-front HQET to derive a heavy quark confining Hamiltonian, from which we may solve from QCD the heavy hadron bound states directly.

In the large $m_Q$ limit, only the leading (spin and mass independent) Hamiltonian is remained. The $1/m_Q^n$ terms ($n \geq 1$) in (88) can be regarded as perturbative corrections to the leading order operators and states. To determine confining interactions in heavy quark systems, the leading heavy quark Hamiltonian plays an essential role. With the light-front gauge $A^+ = 0$, the leading-order bare QCD Hamiltonian density is

$$H_{ld} = \frac{1}{iv^+} Q^\dagger v^+ (v^- \partial^+ - 2v_\perp \cdot \partial_\perp) Q_{v^+}$$

$$- \frac{2g}{v^+} Q^\dagger v^+ \left\{ v^+ \left[ \left( \frac{1}{\partial^+} \right) \partial_\perp \cdot A_\perp \right] - v_\perp \cdot A_\perp \right\} Q_{v^+}$$

$$+ 2g^2 \left( \frac{1}{\partial^+} \right) \left( Q^\dagger T^a Q_{v^+} \right) \left( \frac{1}{\partial^+} \right) \left( \psi^+ T^a \psi^+ \right),$$

where $\psi^+$ is either the heavy antiquark field or the light-front quark field operator in the present consideration. Note that besides the leading term in eq.(88), the above bare Hamiltonian has also already included the relevant terms from the gauge field part, $-\frac{1}{2} \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, of the QCD Lagrangian. These terms come from the elimination of the unphysical gauge degrees of freedom, the longitudinal component $A_-$ [Zhang and Harindranath (1993b)]. Eq.(114) has obviously the spin and flavour heavy quark symmetry, or simply the heavy quark symmetry.

The above leading Hamiltonian (or Lagrangian) is the basis of the QCD-based description for heavy hadrons containing a single heavy quark, such as $B$ and $D$ mesons. As recently pointed out by Mannel et al. [Mannel and Schuler (1995)] the purely heavy quark leading Lagrangian may be not appropriate to describe heavy quarkonia. This is because the anomalous dimension of QCD radiative correction to $Q\bar{Q}$ currents contains an infrared singularity in the limit of two heavy constituents having equal velocity. Such an infrared singularity is a long distance effect and should be absorbed into quarkonium states. To avoid this problem, they argued that one may incorporate the effective Hamiltonian with at least the first order kinetic energy term into the leading Hamiltonian [Mannel and Schuler (1995)]. The light-front kinetic energy can be obtained from eq.(95),

$$H_{kin} = -\frac{1}{m_Q v^+} Q^\dagger v^+ \left\{ \partial^2_\perp - \frac{2v_\perp \cdot \partial_\perp}{v^+} \partial^+ + \frac{v^-}{v^+} \partial^{+2} \right\} Q_{v^+}. \quad (115)$$

As a consequence, in the heavy mass limit, quarkonia have spin symmetry but no flavour symmetry.

i). Confining Hamiltonian for Heavy Quarkonia. Within light-front HQET, we now follow the procedure described above to find an effective QCD Hamiltonian for $Q\bar{Q}$ systems. The bare Hamiltonian for $Q\bar{Q}$ systems contains (114)
are given by \( M_Q \leq 0 \) for the first term in eq.(114) plus the free gluon Hamiltonian. Thus, the free Hamiltonian \( H_0 \) used in SRG is given only by the first term in eq.(114) plus the free gluon Hamiltonian.

With the above consideration, it is easy to compute the effective Hamiltonian eq.(112) for \( Q\bar{Q} \) systems. Following the WCT ideas, we shall calculate \( H_\Lambda \) for \( Q\bar{Q} \) systems up to the second order in the initial and final states defined by \( |i\rangle = \delta_i^j(k_1, \lambda_1)d_i^0(k_2, \lambda_2)|0\rangle \) and \( |j\rangle = \delta_i^j(k_3, \lambda_3)d_i^0(k_4, \lambda_4)|0\rangle \), respectively, where \( k_i \) is the residual momentum of heavy quarks, \( p_i^\Lambda = m_Qv^\Lambda + k_i^\Lambda \), and \( \lambda_i \) its helicity. The result is

\[
H_{\Lambda ij} = H_{q\bar{q}ij}^{\text{free}} + V_{q\bar{q}ij},
\]

where

\[
H_{q\bar{q}ij}^{\text{free}} = \left\{ \frac{\Lambda}{m_Q}[2\kappa_1^2 + \Lambda^2(y^2 - 2y + 1)] - \Lambda^2 - 2\frac{g^2}{4\pi^2}C_f \frac{\Lambda^2}{K^+} \right\},
\]

\[
V_{q\bar{q}ij} = \frac{4q^2(T^u)(T^a)}{(K^+)^2} \left\{ \frac{1}{(y - y')^2} \left( 1 - \theta(A(y - y', \kappa_\perp - \kappa'_\perp, \Lambda) - \Lambda^2 \right) \right\}.
\]

Here we have introduced the longitudinal residual momentum fractions and the relative transverse residual momenta,

\[
y = k_1^+ / K^+, \quad \kappa_\perp = k_1^\perp - yK_\perp,
y' = k_2^+ / K^+, \quad \kappa'_\perp = k_2^\perp - y'K_\perp,
\]

where \( K^\mu \) is defined as the residual center mass momentum of the heavy quarkonia: \( K^\mu = \bar{T}_\mu v^\mu \), and \( \Lambda = M_H - m_Q - m_T \) is a residual heavy hadron mass. It follows that \( K^+ = k_1^+ + k_2^+ = k_3^+ + k_4^+ \), \( K_\perp = k_1^\perp + k_2^\perp = k_3^\perp + k_4^\perp \). Since \( 0 \leq p_1^+ = m_Qv^+ + k_1^+ \leq M_Hv^+ \), in the heavy quark mass limit, we have \( M_H \to 2m_Q \) so that \( -m_Qv^+ \leq k_1^+ \), \( k_3^+ \leq m_Qv^+ \). Hence, the range of \( y \) and \( y' \) are given by \( -\infty < y, y' < \infty \). We have also defined in eq.(118)

\[
A(y - y', \kappa_\perp - \kappa'_\perp, \Lambda) = \frac{(\kappa_\perp - \kappa'_\perp)^2}{|y - y'|} + |y - y'|\Lambda^2.
\]

Eq.(116) is the nonperturbative part of the effective Hamiltonian for heavy quarkonia in the WCT scheme, in which we have already let UV cutoff parameter \( \Lambda \to \infty \) and the associated divergence has been put in the mass correction.
The kinetic energy (115) is now included in the above effective Hamiltonian [the $1/m_Q$ term in eq.(117)]. Note that there is an infrared divergent term in eq.(117) which comes from the quark self-energy correction in SRG, where $\epsilon$ is an infrared cutoff of the momentum fraction $q^+/K^+$, and $q^+$ the longitudinal momentum carried by gluon in the quark self-energy loop. The usual mass correction $\delta m_Q^2 = \frac{2^3}{2\pi^2}C f^2 \ln \frac{q^2}{\Lambda^2}$, has been renormalized away in eq.(117). In the WCT scheme, by removing away this mass correction, we should assign the corresponding constituent quark mass in $H_{\lambda 0}$ being $\lambda$-dependent. But, the heavy quark mass is larger than the low energy scale. Its dependence on $\lambda$ should be very weak and could be neglected. While, the $Q\overline{Q}$ interaction (118) contains two contributions: the instantaneous interaction plus the second order contribution in eq.(112) [i.e. the terms proportional to the theta function in eq.(118)].

We shall show next that the above $V_{Q\overline{Q}}$ is indeed a combination of a confining interaction plus a Coulomb-type interaction.

ii). Quark Confinement on the Light-Front. In our framework, LFQCD vacuum is trivial. The nature of nontrivial QCD vacuum structure, the confinement as well as the chiral symmetry breaking, must made manifestly in $H_{\lambda}$ in terms of new effective interactions. We will see that $H_{\lambda 0}$ explicitly contains a confining interaction at long distances. The interactions associated with the chiral symmetry breaking may be manifested in the fourth order computation of $H_{\lambda}$ for light quark systems [Wilson et al. (1994)], but these interactions are not important in the study of heavy hadrons here.

The confining interaction can be easily obtained by applying the Fourier transformation to the first term in (118). It is convenient to perform the calculation in the frame $K_\perp = 0$, in which

$$\int \frac{dq^+ d^2q_{\perp}}{(2\pi)^2} e^{i(q^+ x^- + q_{\perp} \cdot x_\perp)} \left\{ -4g^2 \frac{1}{q^+} \theta(\lambda^2 - A(q^+/K^+, q_{\perp}, \Lambda)) \right\}$$

$$= -\frac{g^2}{2\pi^2} \int_0^{2\pi} dq^+ e^{iq^+ x^-} \frac{q_{\perp m}^2}{q^+} \frac{2J_1(|x_\perp|q_{\perp m})}{|x_\perp|q_{\perp m}}$$

(121)

where we have used the relation $q^+ = k^+_1 - k^+_3 = K^+(y - y')$, $q_{\perp} = k_{\perp 1} - k_{\perp 3} = \kappa_{\perp} - \kappa'_{\perp}$ for $K_\perp = 0$, while $q_{\perp m} \equiv \sqrt{x^2 \kappa^+ q^+ - \frac{x^2}{K^+} q^+ q^2}$, and $J_1(x)$ is a Bessel function. An analytic solution to the integral (121) may be difficult to carry out. However, the nature of confining interactions is a large distance QCD behavior. We may consider the integral for large $x^-$ and $x_\perp$. In this case, if $q^+ x^-$ and/or $|x_\perp|q_{\perp m}$ are large, the integration vanishes, yet $J_1(x) = \frac{\pi}{2} + \frac{\pi}{x} + \cdots$ for small $x$. The dominant contribution of the integral (121) for large $x^-$ and $x_\perp$ comes from the small $q^+$ such that $q^+ x^-$ and/or $|x_\perp|q_{\perp m}$ must remain small, which leads to $e^{iq^+ x^-} \frac{2J_1(|x_\perp|q_{\perp m})}{|x_\perp|q_{\perp m}} \simeq 1$. This corresponds to $q^+ < \frac{1}{x^2}$ and/or $q^+ < \frac{K^+}{|x_\perp|^2 \lambda^2}$. 

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If \( q^+ < \frac{1}{x^+} < \frac{K^+}{|x_\perp|^2 x^2} \), eq.(121) is reduced to

\[
-\frac{g_\lambda^2}{2\pi^2} \int_0^{\frac{1}{x^+}} dq^+ \frac{1}{q^+^2} \left( \frac{\lambda^2}{K^+} q^+ \right) = \frac{g_\lambda^2 \lambda^2}{2\pi^2 K^+} \left( \ln(K^+ |x^-|) + \ln \epsilon \right),
\]

where a term \( \sim \frac{1}{x^+} \) is neglected since \( x^- \) is large, and \( \epsilon \) is an infrared cutoff of the momentum fraction \( q^+/K^+ \). It is the same as the divergence occurs in the quark self-energy contribution so that the above infrared logarithmic divergence \( \sim \ln \epsilon \) exactly cancels the divergence in eq.(117) for color single states. What remains is a logarithmic confining interaction (except for a color factor):

\[
V_{\text{conf.}}(x^- , x_\perp) \sim \frac{g_\lambda^2 \lambda^2}{2\pi^2 K^+} \ln(K^+ |x^-|). \tag{123}
\]

Similarly, when \( q^+ < \frac{K^+}{|x_\perp|^2 x^2} < \frac{1}{x^+} \), we have

\[
-\frac{g_\lambda^2}{2\pi^2} \int_{\frac{K^+}{|x_\perp|^2 x^2}}^{\frac{1}{x^+}} dq^+ \frac{1}{q^+^2} \left( \frac{\lambda^2}{K^+} q^+ \right) = \frac{g_\lambda^2 \lambda^2}{2\pi^2 K^+} \left( \ln(\lambda^2 |x_\perp|^2) + \ln \epsilon \right),
\]

where the term \( \sim \frac{1}{x^+} \) has also been ignored because of the large \( x_\perp^2 \). Again, the infrared divergence \( \sim \ln \epsilon \) is cancelled in \( H_\lambda \) for physical states, and we obtain the following confining interaction:

\[
V_{\text{conf.}}(x^- , x_\perp) \sim \frac{g_\lambda^2 \lambda^2}{2\pi^2 K^+} \ln(\lambda^2 |x_\perp|^2). \tag{125}
\]

Hence, the effective Hamiltonian \( H_{\lambda 0} \) exhibits a logarithmic confining interaction between a heavy quark and a heavy antiquark in all the directions of \( x^- \) and \( x_\perp \) space.

The Coulomb interaction corresponds the second term in (118), its Fourier transformation (except for the color factor) is

\[
\frac{\Lambda^2}{(\kappa_\perp - \kappa'_\perp)^2 + (y - y')^2 \Lambda^2} \sim \frac{1}{4\pi} \int dx^- d^2 x_\perp e^{i(x^- q^+ + q_\perp \cdot x_\perp)} \left( \frac{\Lambda}{K^+} \right) \frac{1}{r_l}, \tag{126}
\]

where \( r_l \equiv \sqrt{x_\perp^2 + \left( \frac{\Lambda}{K^+} \right)^2 (x^-)^2} \) which is defined as a “radial” variable in the light-front space [Wilson et al. (1994)]. Eq.(126) shows that the Coulomb interaction on the light-front has the form

\[
V_{\text{Coul.}}(x^- , x_\perp) \sim \frac{g_\lambda^2 \Lambda}{4\pi K^+ r_l}. \tag{127}
\]

Thus, we have explicitly shown that \( H_{\lambda 0} \) contains a Coulomb interaction at short distances and a confining interaction at long distances.

Moreover, a clear light-front picture of quark confinement emerges here. To be specific, we define quark confinement as follows: i) There is a confining interaction
between quarks such that quarks cannot be well-separated; ii) No color non-
singlet bound states exist in nature, only color singlet states with finite masses
can be produced and observed; and iii) The conclusions of i–ii) are only true for
QCD but not for QED.

We have shown explicitly the existence of a confining interaction in $H_{\lambda 0}$. One
can also easily see from $H_{\lambda 0}$ the non-existence of color non-singlet bound states.
This is essentially related to the infrared divergences in $H_{\lambda 0}$. From eqs. (122) and
(124), we find that the uncancelled instantaneous interaction contains a logarith-
mic infrared divergence. Except for the color factor, this infrared divergence has
the same form as the divergence in eq. (117). Thus, we immediately obtain the
following conclusions.

(a). For a single (constituent) quark state, the interaction part of $H_{\lambda 0}$ does
not contribute to its energy. The remaining infrared divergence from quark self-
energy correction implies that the dynamical quark mass for a single quark state
is infinite (infrared divergent) and cannot be renormalized away in the spirit of
gauge invariance. Equivalently speaking, single quark states carry an infinitely
large mass and therefore they cannot be produced.

(b). For color non-singlet composite states, the color factor $\left( T^{a} \right)_{\alpha\beta} \left( T^{a} \right)_{\delta\gamma}$ in
the $QQ$ interaction is different from the color factor $C_f = \text{Tr} \left( T^{a} T^{a} \right)$. Therefore,
the infrared divergence in the self-energy correction also cannot be cancelled by
the corresponding divergence from the uncancelled instantaneous interaction. As
a result, color non-singlet composite states are infinitely heavy that they cannot
be produced as well.

(c). For color singlet $QQ$ states, the color factor $\left( T^{a} \right) \left( T^{a} \right) \to C_f$. Thus, the
infrared divergences are completely cancelled and the resulting effective Hamiltonian
is finite. In other words, only color singlet composite are physically ob-
servable.

Finally, we argue that the above mechanism of quark confinement is indeed
only true for QCD. As we have seen the light-front confinement interaction is
just an effect of the non-cancellation between instantaneous interaction and one
transverse gluon interaction generated in SRG. Such a non-cancellation arises in
SRG because we introduce the energy scale $\lambda$. Introducing the energy scale $\lambda$ in
SRG forces the transverse gluon energy involved in the $QQ$ effective interaction
never be less than a certain value (the energy scale $\lambda$). This implies that the
gluon may become massive at the hadronic mass scale. Of course, such a gluon
mass must be a dynamical mass generated from the highly nonlinear gluon in-
teractions. In other words, the above confining picture is indeed a dynamical
consequence of non-Abelian gauge theory. This confinement mechanism is not
valid in QED. In QED, since photon mass is always zero, the photon energy cov-
ers the entire range from zero to infinity. Thus, in QED, we can always choose
the energy scale $\lambda$ being zero. With $\lambda = 0$, the infrared divergences do not occur
in the electron self-energy correction. As a result, the renormalized single elec-
tron mass is finite, in contrast to the divergent mass of single quark states. For
the same reason, with $\lambda = 0$, the instantaneous interaction in the effective QED
Hamiltonian is also exactly cancelled by the same interaction from one trans-
verse photon exchange so that only one photon exchange Coulomb interaction remains. Thus, applying SRG to QED and let $\lambda = 0$ in the end of procedure, we obtain a conventional QED Hamiltonian which only contains the Coulomb interaction.

iii). Extension to Heavy-Light Quark Systems. We can also apply SRG to the heavy-light quark system (heavy hadrons containing one heavy quark). The bare cutoff Hamiltonian we begin with for heavy-light quark systems is the combination of the heavy quark effective Hamiltonian (114) and the full Hamiltonian for the light quarks and gluons. We may also introduce the residual center momentum for heavy-light systems, $K^+ = \lambda v^+ = p_1^+ + k_1^+ = p_2^+ + k_2^+$, $K_\perp = \lambda v_\perp = p_{1\perp} + k_{1\perp} = p_{2\perp} + k_{2\perp}$, where $\lambda = M_H - m_Q$, $p_1$ and $p_2$ are the light antiquark momenta and $k_1$ and $k_2$ the residual momenta of the heavy quarks in the initial and final $Q\bar{q}$ states respectively.

Following the general procedure, it is easy to find the nonperturbative part of the effective Hamiltonian for heavy-light quark systems,

$$H_{\lambda oij} = \theta(\frac{\lambda^2}{K^+} - |\Delta K^+|)\left\{H_{Q\bar{q}ij} + V_{Q\bar{q}ij}\right\},$$

where

$$H_{Q\bar{q}ij} = [2(2\pi)^3]^2 \delta^3(\vec{k}_1 - \vec{k}_2)\delta^3(\vec{p}_1 - \vec{p}_2)\delta_{\lambda_1\lambda_2}\delta_{\lambda_3\lambda_4}$$

$$\times \left\{(y - 1)\frac{\lambda^2}{K^+} + \frac{m_q^2}{y} - \frac{g^2}{2\pi^2}C_f \frac{\lambda^2}{K^+} \ln \epsilon\right\}$$

(128)

$$V_{Q\bar{q}ij} (y - y', \kappa_\perp - \kappa'_\perp) = 2(2\pi)^3 \delta^3(\vec{k}_1 + \vec{p}_1 - \vec{k}_2 - \vec{p}_2)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}$$

$$\times -2g^2(T^u)(T^a)\left\{\frac{2}{(y - y')^2} - 2\frac{(\kappa_\perp - \kappa'_\perp)^2}{(y - y')^2} - \frac{\kappa_\perp \cdot \kappa'_\perp}{y(y - y')^2}$$

$$\frac{\theta(B - \lambda^2)\theta(B - A)}{(\kappa_{\perp} - \kappa'_{\perp})^2 - (y - y')(\frac{\kappa_{\perp}^2}{y} - \frac{\kappa'_{\perp}^2}{y})}$$

$$+ \frac{\theta(A - \lambda^2)\theta(A - B)}{(\kappa_{\perp} - \kappa'_{\perp})^2 + (y - y')^2}\right\},$$

(130)

with $B \equiv \left|\frac{(\kappa_{\perp} - \kappa'_{\perp})^2}{y - y'} - \frac{\kappa^2_{\perp}}{y} + \frac{\kappa^2_{\perp}}{y}\right|$ and the function $A$ has the same form as in quarkonium case. Here we have also introduced $\frac{y}{p_1^+} = \frac{y}{K^+}$, $\kappa_\perp = \frac{p_{1\perp}}{y} - yK_\perp$, but the range of $y$ is now given by $0 < y = \frac{M_H}{m_Q} < \infty$.

The heavy-light quark effective Hamiltonian is $m_Q$-independent. This is because in heavy-light quark systems the heavy quark kinetic energy can be treated as a perturbative correction to $H_{\lambda o}$. Obviously the above $H_{\lambda o}$ has the heavy quark spin and flavour symmetry. Compared to the $V_{Q\bar{q}}, V_{Q\bar{q}}$ interactions are
much more complicated. But it is not difficult to check that the above $V_{Q\bar{Q}}$ contains a confining interaction. The confining mechanism is the same for $Q\bar{Q}$ and $Q\bar{q}$ systems, as well as for $q\bar{q}$ systems, as one can show.

In conclusion, we have obtained the nonperturbative part of a confining QCD Hamiltonian for heavy-heavy and heavy-light quark systems. We are now ready to solve heavy hadron states on the light-front and to show how the WCT scheme works in the present formulation.

5.4 Heavy Hadron Bound States

As we mentioned the ideas of WCT to nonperturbative QCD is to begin with the effective QCD Hamiltonian $H_\lambda = H_{M0} + H_{M1}$. Then using the constituent picture to solve nonperturbatively the hadronic bound state equations governed by $H_{M0}$ and to determine the running coupling constant $g_\lambda$. If one could properly choose the nonperturbative $H_{M0}$ such that $g_\lambda$ is arbitrarily small, then the corrections from $H_{M1}$ could be computed perturbatively, and we would say that a WCT to nonperturbative QCD is realized. Now, I shall discuss such a WCT to heavy hadron bound states.

i). Heavy Hadron Bound State Equation Under WCT. As we have pointed out in the first lecture solving eq.(14) from QCD with the entire Fock space is impossible. A basic motivation of introducing the WCT scheme is to simplify the complexities in solving the above equation. In the present framework, $H_{LF} = H_\lambda$, where $H_\lambda$ has already decoupled from high energy states. Furthermore, the reseparation $H_\lambda = H_{M0} + H_{M1}$ is another crucial step in WCT, where only $H_{M0}$ is assumed to have the nonperturbative contribution to bound states through eq.(14), and $H_{M1}$ is supposed to be a perturbative term which should not be considered when we try to solve eq.(14) nonperturbatively.

The next important step in the WCT scheme is the use of a constituent picture. The success of the constituent quark model suggests that we may only consider the valence quark Fock space in determining the hadronic bound states from $H_{M0}$. In this picture, quarks and gluons must have constituent masses. This constituent picture can naturally be realized on the light-front [Wilson et al. (1994)]. However, an essential difference from the phenomenological constituent quark model description is that the constituent masses introduced here are $\lambda$ dependent. The scale dependence of constituent masses (as well as the effective coupling constant) is determined by solving the bound states equation and fitting the physical quantities with experimental data. But for heavy quark mess, this $\lambda$-dependence can be ignored. Once the constituent picture is introduced, we can truncate the general expression of the light-front bound states to only including the valence quark Fock space. The higher Fock space contributions can be recovered as a perturbative correction through $H_{M1}$. Thus, eq.(9) for heavy quarkonia can be approximately written as:

$$|\Psi(K^+, K_\perp, \lambda_s)\rangle = \sum_{\lambda_1, \lambda_2} \int [d^3 \tilde{k}_1][d^3 \tilde{k}_2] (2\pi)^3 \delta^3(\tilde{K} - \tilde{k}_1 - \tilde{k}_2)$$
where the wavefunction $\phi_{Q\bar{Q}}(y, \kappa_\perp)$ may be mass dependent due to the kinetic energy in $H_{\lambda 0}$ [see (117)] but it is spin independent in heavy mass limit. Also note that the heavy quarkonium states in heavy mass limit are labelled by the residual center mass momentum $K^\mu$. We may normalize eq.(131) as follows:

$$\langle \Psi(K^+, K'_\perp, \lambda'_s)|\Psi(K^+, K_\perp, \lambda_s) \rangle = 2(2\pi)^3 \delta^3(K - K') \delta_{\lambda'_s\lambda_s},$$

(132)

which leads to

$$\int \frac{dyd^2\kappa_\perp}{2(2\pi)^3} |\phi_{Q\bar{Q}}(y, \kappa_\perp)|^2 = 1.$$  

(133)

With the above analysis on the quarkonium states, it is easy to derive the corresponding bound state equation. Let $H_{LF} = H_{\lambda 0}$ of eq.(116), eq.(15) is reduced to

$$\left\{2\Lambda^2 - \frac{\Lambda}{m_Q} \left[2\kappa^2_\perp + \Lambda^2(2y^2 - 2y + 1)\right]\right\} \phi_{Q\bar{Q}}(y, \kappa_\perp)$$

$$= \left( - \frac{g^2}{2\pi^2} \chi^2 C_f \ln \epsilon \right) \phi_{Q\bar{Q}}(y, \kappa_\perp)$$

$$-4g^2(T^o)(T^a) \int \frac{dy'^2\kappa'_\perp}{2(2\pi)^3} \left\{ \frac{1}{(y - y')^2} \theta(\lambda^2 - A) \right. \right. \right.$$  

$$\left. \left. \left. + \frac{\Lambda}{(\kappa_\perp - \kappa'_\perp)^2 + (y - y')^2 \Lambda^2} \theta(A - \lambda^2) \right\} \phi_{Q\bar{Q}}(y', \kappa'_\perp). \right.$$  

(134)

This is the light-front bound state equation for heavy quarkonia in the WCT scheme.

For the heavy mesons containing one heavy quark, similar consideration leads to

$$\left(\bar{\Lambda}^2 + (1 - y)\Lambda^2 - \frac{\kappa^2_\perp + m_q^2(\lambda)}{y}\right) \Phi_{Q\bar{q}}(y, k_\perp, \lambda_1, \lambda_2)$$

$$= \left( - \frac{g^2}{2\pi^2} \chi^2 C_f \ln \epsilon \right) \Phi_{Q\bar{q}}(y, k_\perp, \lambda_1, \lambda_2)$$

$$+ (K^+)^2 \int \frac{dy'^2\kappa'_\perp}{2(2\pi)^3} V_{Q\bar{q}}(y - y', \kappa_\perp - \kappa'_\perp) \Phi_{Q\bar{q}}(y', \kappa'_\perp, \lambda_1, \lambda_2),$$

(135)

where $V_{Q\bar{q}}$ is given by eq.(130). Note that the light antiquark here is a brown muck, a current light antiquark surround by infinite gluons and $q\bar{q}$ pairs that results in a constituent quark mass $m_q$ which is a function of $\lambda$.

ii). A General Analysis of Light-Front Wavefunctions. A numerical computation to the bound state equations, eqs.(134) and (135), is actually not too
difficult. However, to have a deeper insight about the internal structure of light-front bound states, it is better to have an analytic analysis. For this propose, we would like to present a general analysis of light-front hadronic wave functions.

The heavy hadronic wavefunctions in the heavy mass limit are rather simple. First of all, the heavy quark kinematics have already added some constraints on the general form of the light-front wavefunction $\phi(x, \kappa_{\perp})$. When we introduce the residual longitudinal momentum fraction $y$ for heavy quarks, the longitudinal momentum fraction dependence in $\phi$ is quite different for the heavy-heavy, heavy-light and light-light mesons.

For the light-light mesons, such as pions, rhos, kaons etc., the wavefunction $\phi_{qq}(x, \kappa_{\perp})$ must vanish at the endpoint $x = 0$ or 1. This can be seen from the kinetic energy term in eq.(15), where $M_0^2 = \frac{x^2 + m_1^2}{2} - \frac{\kappa_{\perp}^2 + m_2^2}{2}$ for the valence Fock space. To ensure that the bound state equation is well defined in the entire range of momentum space, $|\phi_{qq}(x, \kappa_{\perp})|^2$ must fall down to zero in the longitudinal direction not slower than $1/x$ and $1/(1-x)$ when $x \to 0$ and 1, respectively. In other words, at least $\phi_{qq}(x, \kappa_{\perp}) \sim \sqrt{x(1-x)}$. For heavy-light quark mesons, namely the $B$ and $D$ mesons, the wavefunction $\phi_{Qq}(y, \kappa_{\perp})$ is required to vanish at $y = 0$, where $y$ is the residual longitudinal momentum fraction carried by the light quark. This is because the kinetic energy in eq.(135) only contains a singularity at $y = 0$. On the other hand, since $0 \leq y \leq \infty$, $\phi_{Qq}(y, \kappa_{\perp})$ should also vanish when $y \to \infty$. Hence, a possible simple solution is $\phi_{Qq}(y, \kappa_{\perp}) \sim \sqrt{ye^{-\alpha y^2}}$. For heavy quarkonia, $-\infty < y < \infty$, the normalization forces $\phi_{QQ}(y, \kappa_{\perp})$ to vanish as $y \to \pm \infty$. Thus, a simple solution may be $\phi_{QQ}(y, \kappa_{\perp}) \sim e^{-\alpha y^2}$.

On the other hand, the transverse momentum dependence in these light-front wavefunctions should be more or less similar. They all vanish at $\kappa_{\perp} \to \pm \infty$. A simple form of the $\kappa_{\perp}$ dependence for these wavefunctions is a Gaussian function: $e^{-\kappa_{\perp}^2/2\omega^2}$.

The above analysis of light-front wavefunctions is only based on the kinetic energy properties of the constituents. Currently, many investigations on the hadronic structures use phenomenological light-front wavefunctions. One of such wavefunctions that has been widely used in the study of heavy hadron structure is the BSW wavefunction [Wirbel et al. (1985)],

$$\phi_{BSW}(x, \kappa_{\perp}) = N \sqrt{x(1-x)} \exp \left( \frac{\kappa_{\perp}^2}{2\omega^2} \right) \exp \left[ -\frac{M_H^2}{2\omega^2} (x - x_0)^2 \right],$$

where $N$ is a normalization constant, $\omega$ a parameter of order $\Lambda_{QCD}$, $x_0 = (1/2 - m_1^2/m_H^2)$, and $M_H, m_1$, and $m_2$ are the hadron, quark, and antiquark masses respectively. In the heavy mass limit, the BSW wavefunction can be produced from our analysis based on the light-front bound state equations.

Explicitly, for heavy-light quark systems, such as the $B$ and $D$ mesons, one can easily find that in the heavy mass limit, $m_1 = m_Q \sim M_H$, $m_q << m_Q$ so that $x_0 = 0$. Meanwhile, we also have $M_H x = \Lambda y$. Furthermore, the factor
\( \sqrt{x(1-x)} \) can be rewritten by \( \sqrt{y} \) in accordance to the corresponding bound state equation discussed above. Thus, the BSW wavefunction is reduced to
\[
\phi_Q(y, \kappa_\perp) = N \sqrt{y} \exp \left( -\frac{\kappa_\perp^2}{2\omega^2} \right) \exp \left( -\frac{\Lambda^2}{2\omega^2 y^2} \right). \tag{137}
\]

This agrees with our qualitative analysis given above. Indeed, using such a wavefunction we have already computed the universal Isgur-Wise function in \( B \to D, D^* \) decays [Cheung et al. (1995)];
\[
\xi(v \cdot v') = \frac{1}{v \cdot v'}, \tag{138}
\]
and from which we obtained the slope of \( \xi(v \cdot v') \) at the zero-recoil point, \( \rho^2 = -\xi'(1) = 1 \), in excellent agreement with the recent CLCO result [Patterson (1995)] of \( \rho^2 = 1.01 \pm 0.15 \pm 0.09 \).

For heavy quarkonia, such as the \( b\bar{b} \) and \( c\bar{c} \) states, \( m_1 = m_2 = m_Q \) which leads to \( x_0 = 1/2 \) in eq.(136). Also note that \( M_H(x-1/2) = A y \), and the factor \( \sqrt{x(1-x)} \) must be totally dropped as we have discussed form the quarkonium bound state equation. Then the BSW wavefunction for quarkonia is reduced to
\[
\phi_{Q\bar{Q}}(y, \kappa_\perp) = N \exp \left( -\frac{\kappa_\perp^2}{2\omega^2} \right) \exp \left( -\frac{\Lambda^2}{2\omega^2 y^2} \right), \tag{139}
\]
which is a form as we expected from the qualitative analysis. Here we have not taken the limit of \( m_Q \to \infty \) for heavy quarkonia. Thus a possible \( m_Q \) dependence in wavefunction may be hidden in the parameter \( \omega \).

Using the variational approach with the above trial wave function, we can analytically solve the bound state equation Eq.(134). Under the consideration of SRG invariant for the binding energy \( \Lambda \), we find that the effective coupling constant [Zhang (1996)];
\[
\alpha_\lambda = \frac{g_\lambda^2}{4\pi} = \frac{\pi}{C_f} \left( \frac{\Lambda^2}{\lambda^2} \right) \frac{1}{a + b \ln \frac{\Lambda}{\lambda}}, \tag{140}
\]
where the coefficients \( a \) and \( b \) are determined by minimizing the binding energy from (134) with (139). The coefficient \( b \) is almost a constant (with a weak dependence on \( m_Q \) but independence on \( \Lambda \) and \( \lambda \)), while \( a \) depends on both \( \Lambda \) and \( m_Q \), and also slightly on \( \lambda \). For \( \lambda \geq 0.6 \) GeV, the \( \lambda \)-dependence in the parameter \( a \) is negligible. It is known that \( \Lambda \) is of the same order as \( \Lambda_{QCD} \) which is about 100 ~ 400 MeV. The charmed and bottom quark masses used here are \( m_c = 1.4 \) GeV and \( m_b = 4.8 \) GeV. From the particle data [Particle Data Group (1994)], the binding energy \( \Lambda \) should also be less than 400 MeV. For a qualitative consideration, we may take \( \Lambda = 0.2 \) GeV and \( \Lambda = 0.4 \) GeV for charmonium. Then we find that \( b = 1.15 \), and \( a = -0.25 \) and 1.1 respectively.
Numerically, with $\Lambda = 200$ MeV and $\lambda = 1$ GeV, we obtain that
\[
\alpha_\lambda = \begin{cases} 
0.02665 & \text{charmonium}, \\
0.06795 & \text{bottomonium},
\end{cases}
\] (141)
which is much smaller than that extrapolated from the canonical running coupling constant in the naive perturbative QCD calculation. In order to see how this weak coupling constant varies with the scale $\lambda$, we take $\Lambda = 200$ MeV and vary the value of $\lambda$ around 1 GeV. We find that the coupling constant is decreased very faster with increasing $\lambda$. In other words, with a suitable choice of the hadronic mass scale $\lambda$ in SRG, we can make the effective coupling constant $\alpha_\lambda$ in $H_\lambda$ arbitrarily small. Then the WCT of nonperturbative QCD can be achieved in terms of $H_\lambda$ such that the corrections from $H_\lambda$ can be truly computed perturbatively. This provides the first realization of the WCT to nonperturbative QCD dynamics on the light-front.

5.5 Perspectives

The applications of the present theory to heavy quarkonium spectroscopy and various heavy quarkonium annihilation and production processes can be simply achieved by numerically solving the bound state equations (134), and by further including the $1/m_Q$ corrections (which naturally leads to the spin splitting interactions). The extension of the computations to heavy-light quark systems is straightforward. The extension of the present work to light-light hadrons requires the understanding of chiral symmetry breaking in QCD which is a new challenge to nonperturbative QCD on the light-front. Nevertheless, we have provided a detailed analysis to the weak-coupling treatment of nonperturbative QCD proposed recently [Wilson et al. (1994)]. I believe that LFQCD opens a new research direction in the attempt of solving the most difficult problem in field theory, that is, the problem of the relativistic composite particle bound states governed by the nonperturbative dynamics of QCD.

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A. Harindranath in this school; For an extensive list of the reference on light-front dynamics, see “Sources for Light-front Physics” available via anonymous FTP from pacific.mps.ohio-state.edu in the subdirectory pub/infolight.

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