SPECTRAL INEQUALITIES FOR A CLASS OF INTEGRAL OPERATORS

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We prove inequalities for the Riesz means for the discrete spectra of selfadjoint compact integral operators in some class. Such bounds imply inequalities for the counting function of the Dirichlet boundary problem for the Laplace operator. Bibliography: 7 titles. Illustrations: 2 figures.

To Nina Nikolaevna with respect and admiration

1 Introduction

This paper extends the results obtained earlier in [1]. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain of finite measure, $|\Omega| < \infty$, and let $K(x), x \in \mathbb{R}^d,$ be a homogeneous function of order $\alpha - d$ such that $0 < \alpha < d$,

$$K(tx) = t^{\alpha-d}K(x), \quad t > 0.$$ 

Assuming $K(x) = \overline{K(-x)}$, we consider the selfadjoint integral operator $\mathcal{K}$ defined in $L^2(\Omega)$ by

$$\mathcal{K}u(x) = \int_{\Omega} K(x-y)u(y)\,dy. \quad (1.1)$$

Let us introduce the Fourier transform $\widehat{K}$ of $K$ in the sense of the theory of distributions

$$\widehat{K}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}K(x)\,dx. \quad (1.2)$$

The general theory of homogeneous distributions (cf., for example, [2, 3]) says that, if $u \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree $q$, then $\widehat{u}$ is a homogeneous distribution of degree $-q-d$. In addition,

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if \( u \in C^\infty(\mathbb{R}^d \setminus \{0\}) \), then also \( \hat{u} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \). Therefore, \( \hat{K} \) is a real-valued homogeneous of order \(-\alpha\) function.

The operator \( \mathcal{K} \) is a compact selfadjoint operator in \( L^2(\Omega) \) that might have positive and negative eigenvalues \( \{s_k^\pm\}_{k=1}^\infty \) accumulating at zero. The Riesz means of the operator \( \mathcal{K} \) are defined by

\[
\sum_k (|s_k^+| - s)_+.
\]

The aim of this paper is to give both upper and lower bounds for the Riesz means of the operator \( \mathcal{K} \).

The structure of the paper is as follows. In Section 2, we deduce and prove the upper bound for the Riesz means of an arbitrary compact integral convolution type operator in \( L^2(\Omega) \), where \( \Omega \subset \mathbb{R}^d \) is a domain of finite measure. In Section 3, we obtain the lower bound which is more involved only for homogeneous kernels and strictly convex \( \Omega \). Section 4 presents some special cases and applications.

2 The Upper Bound

Let \( Q \) be a distribution from \( \mathcal{S}'(\mathbb{R}^d) \) such that its Fourier transform \( \hat{Q} \in L^1_{\text{loc}}(\mathbb{R}^d) \) satisfies

\[
\hat{Q}(\xi) = \int_{\mathbb{R}^d} Q(x) e^{-ix\xi} dx \to 0, \quad |\xi| \to \infty,
\]

and its convolution kernel generates a compact operator in \( L^2(\Omega) \)

\[
\mathcal{D}u(x) = \int_{\Omega} Q(x - y) u(y) \, dy.
\]

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^d, \; d \geq 1, \) be a domain of finite measure. Then the following inequality holds for the Riesz means of the eigenvalues \( \{s_k^\pm\} \) of the operator \( \mathcal{D} : 

\[
\sum_k (|\lambda_k^\pm| - \lambda)_+ \leq (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} (|\hat{Q}(\xi)| - \lambda)_+ d\xi. \tag{2.1}
\]

**Proof.** Let \( \{\psi_k^\pm\} \) be the orthonormal system of eigenfunctions of the operator \( \mathcal{D} \) corresponding to the eigenvalues \( \lambda_k^\pm \). By definition,

\[
\sum_k (|\lambda_k^\pm| - \lambda)_+ = \sum_k \left( (|\mathcal{D}\psi_k^\pm, \psi_k^\pm|) - \lambda \|\psi_k^\pm\|^2 \right)_+ \\
= \sum_k \left( \left| \int_{\Omega} \int_{\Omega} Q(x - y) \psi_k^\pm(y) \psi_k^\pm(x) \, dy \, dx \right| - \lambda \|\psi_k^\pm\|^2 \right)_+.
\]

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Extending $\psi_k^\pm$ by zero outside $\Omega$ and using the Plancherel theorem, we get

$$\sum_k (|\lambda_k^\pm| - \lambda) = \sum_k (2\pi)^{-d} \left( \int_{\mathbb{R}^d} \hat{Q}(\xi) |\hat{\psi}_k^\pm(\xi)|^2 \, d\xi \right) - \lambda \int_{\mathbb{R}^d} |\hat{\psi}_k^\pm(\xi)|^2 \, d\xi$$

$$\leq \sum_k (2\pi)^{-d} \left( \int_{\mathbb{R}^d} \left( |\hat{Q}(\xi)| - \lambda \right) |\hat{\psi}_k^\pm(\xi)|^2 \, d\xi \right)$$

$$\leq \sum_k (2\pi)^{-d} \int_{\mathbb{R}^d} \left( |\hat{Q}(\xi)| - \lambda \right) |\hat{\psi}_k^\pm(\xi)|^2 \, d\xi.$$

Let $e_\xi(x) = e^{i\xi \cdot x}$. We now use that $\{\psi_k^\pm\}$ is the orthonormal system of functions in $L^2(\Omega)$ and, using the Parseval identity, get

$$\sum_k |\hat{\psi}_k^\pm(\xi)|^2 = \sum_k \left| \int_{\Omega} e^{-ix\xi} \psi_k^\pm(\xi) \, dx \right|^2 = \|e_\xi\|^2 = |\Omega|.$$

This finally implies

$$\sum_k (|\lambda_k^\pm| - \lambda) \leq (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} \left( |\hat{Q}(\xi)| - \lambda \right) \, d\xi.$$

The proof is complete.

Let now $\mathcal{Q} = \mathcal{K}$ be defined in (1.1). The next statement follows immediately from Theorem 2.1 by changing variables in the integral in (2.1) by substituting the homogeneous function $\hat{K}(\xi)$ given by (1.2).

**Corollary 2.1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain of finite measure, and let $0 < \alpha < d$. Then the following inequality holds for the Riesz means of the eigenvalues of the operator $\mathcal{K}$:

$$\sum_k (|\lambda_k^\pm| - \lambda) \leq (2\pi)^{-d} |\Omega| \lambda^{1 - \frac{d}{\alpha}} \int_{\mathbb{R}^d} \left( |\hat{K}(\xi)| - 1 \right) \, d\xi.$$

Similarly, we obtain the following result related to the Helmholtz operator.

**Corollary 2.2.** Let $\kappa \geq 0$, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a domain of finite measure, and let

$$\hat{Q}(\xi) = \frac{1}{|\xi|^2 + \kappa^2}.$$

Then the eigenvalues of the operator $\mathcal{Q}$ satisfy the inequality

$$\sum_k (|\lambda_k^\pm| - \lambda) \leq (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} \left( \frac{1}{|\xi|^2 + \kappa^2} - \lambda \right) \, d\xi.$$
3 The Lower Bound

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a strictly convex domain of finite measure, $|\Omega| < \infty$, and let there exist $P \in C^\infty(\mathbb{R}^d)$ such that the boundary $\partial\Omega$ is given by $P(x) = 0$ and $|\nabla P| = 1$ on $\partial\Omega$.

The proof of the lower bound is more involved and requires some geometric considerations about the domain $\Omega$. Due to convergence issues, we need to make the assumption $0 < \alpha < d - 1$ throughout this section.

Let $1_\Omega$ be the characteristic function of $\Omega$. We introduce the function

$$\eta(z) = \int 1_\Omega(z + y)1_\Omega(y)\,dy.$$ 

Geometrically, $\eta(z)$ measures the volume of the intersection of $\Omega$ with its translation by a vector $z$. We write $z = r\tilde{z}$, where $r = |z|$ and $\tilde{z} \in S^{d-1}$. So, we can consider $\eta$ as a function defined on $[0, \infty) \times S^{d-1}$. We want to compute the first terms in the Taylor expansion of $\eta$ (in the sense of the theory of distributions) around $(0, \tilde{z})$. We have

$$\eta(r, \tilde{z}) = \eta(0, \tilde{z}) + r\eta_\nu(0, \tilde{z}) + r^2 \int_0^1 \eta_\nu'((1 - t)r, \tilde{z})\,dt.$$ 

It is clear that $\eta(0, \tilde{z}) = |\Omega|$. In [1], the second term is computed by using the formula $\nabla 1_\Omega(z) = \delta(P)\nabla P(z)$ (cf. [3]):

$$\eta_\nu(r, \tilde{z}) = \int \delta(P(z + y))(\nabla P(z + y), \tilde{z})1_\Omega(y)\,dy.$$ 

In order to compute this, we use the following fact about the composition of the Dirac delta function with another function, which holds for $f, g : \mathbb{R}^d \to \mathbb{R}$

$$\int \delta(f(x))g(x)\,dx = \int_{f^{-1}(0)} \frac{g(x)}{|\nabla f(x)|}\,d\sigma,$$

where $\sigma$ is the surface measure on $f^{-1}(0)$. Then

$$\eta_\nu(r, \tilde{z}) = \int_{L_z} \frac{\nabla P(z + u)}{|\nabla P(z + u)|}\,d\sigma(u) = \int_{L_z} (\nabla P(z + u), \tilde{z})\,d\sigma(u),$$

where $L_z$ is the intersection of the domain $\Omega$ with the surface $P(z + y) = 0$ ($L_{(0, \tilde{z})}$ will be understood as a limit, and it will depend on the direction $\tilde{z}$) and $\sigma$ is the surface measure on $L_z$. For the second equality we used the fact that $|\nabla P| = 1$ on $\partial\Omega$.

We need the following geometric fact. Let $R_\Omega = \min \text{dist}(u_1, u_2)$ where the minimum is taken over all points $u_1, u_2 \in \partial\Omega$ such that $(\nabla P(u_1), \nabla P(u_2)) = -1$. In other words, $R_\Omega$ is the diameter of the largest sphere entirely contained in $\Omega$ or the maximum number with the property that $\Omega \cap (z + \Omega) \neq \emptyset$ for all $|z| < R_\Omega$. Then there exists a family of diffeomorphisms $T_z : S^{d-1}_+ \to L_z$ from a fixed hemisphere $S^{d-1}_+$ onto the surface $L_z$, for $|z| < R_\Omega$, which is infinitely differentiable in $z$.

This fact allows us to change variables and to obtain

$$\eta_\nu(r, \tilde{z}) = \int_{S^{d-1}_+} (\nabla P(z + T_z\theta), \tilde{z})J(T_z)\,d\sigma(\theta),$$

where $J(T_z)$ is the Jacobian of $T_z$. The integration is understood in the sense of distributions (as defined in [3]).
where $J(T_z)$ is the Jacobian determinant of $T_z$. Since the integrand above is a smooth function of $z$, this shows that $\eta$ is smooth on $[0, R_\Omega) \times S^{d-1}$.

We can then write down the Taylor expansion of $\eta$ around $r = 0$ in the form

$$\eta(r, \tilde{z}) = |\Omega| + r A_\Omega(\tilde{z}) + r^2 B_\Omega(r, \tilde{z}), \tag{3.1}$$

where $A_\Omega$ is a smooth function on $S^{d-1}$ and

$$B_\Omega(r, \tilde{z}) = \int_0^1 \eta''((1-t)r, \tilde{z}) t \, dt$$

is smooth on $[0, R_\Omega) \times S^{d-1}$.

**Remark 3.1.** Let $\Omega \subset \mathbb{R}^d$ be a ball of radius one. In this case, we have $P(x) = \frac{1}{2}(1 - |x|^2)$ and for any $z$, $L_{(0,\tilde{z})}$ is the hemisphere of $\Omega$ centered around the vector $-\tilde{z}$. Using our previous computations, we have

$$\eta'(r, \tilde{z}) = -\int_{L_{(0,\tilde{z})}} u \cdot \tilde{z} \, d\sigma_{d-1}(u).$$

Here, $\sigma_{d-1}$ is the surface measure of the sphere $S^{d-1}$. Due to the symmetry of $\Omega$, $A_\Omega(\tilde{z}) = \eta'(0, \tilde{z})$ does not depend on $\tilde{z}$, so it is a constant. Making a convenient choice, we can compute

$$A_\Omega = -\int_{S^{d-2}} \int_0^{\pi/2} \cos(\varphi) \sin^{d-2}(\varphi) \, d\varphi \, d\sigma_{d-2}(u) = -\frac{1}{d-1} |S^{d-2}|.$$

Let $F(z) = \frac{1}{|\Omega|^d} K(z) |z| A_\Omega(\tilde{z})$. So, $F$ is a homogeneous function of degree $d - \alpha + 1$. By the general theory, $F$ is a homogeneous function of degree $-\alpha - 1$. Since $\hat{K}$ is also homogeneous of degree $-\alpha$, there exist continuous functions $f, g : S^{d-1} \to \mathbb{C}$ such that

$$\hat{K}(\xi) = \frac{f(\xi)}{|\xi|^\alpha}, \quad \hat{F}(\xi) = \frac{g(\xi)}{|\xi|^{\alpha+1}}.$$

Let

$$\gamma = \int_{S^{d-1}} \text{sgn}(f(\theta)) |f(\theta)| \frac{d-\alpha-1}{\alpha} g(\theta) \, d\sigma(\theta).$$

We are now ready to state the main result of this section.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^d$ be a convex domain of finite measure, and let $0 < \alpha < d - 1$. Then we have the following lower bound for the Riesz means of the operator $K^*$ :

$$\sum_k (|\lambda_k \pm \lambda|) \geq \frac{|\Omega|}{(2\pi)^d} \lambda^{1-d-\alpha} \int_{\mathbb{R}^d} (|\hat{K}(\xi)| - 1)_+ \, d\xi + \frac{|\Omega|}{(2\pi)^d} \frac{\gamma}{d-\alpha-1} \lambda^{1-d-\alpha} + o(\lambda^{1-d-\alpha})$$

as $\lambda \to 0$.

Before proving this theorem, we need some auxiliary results.
Proposition 3.1. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with support contained in the ball of radius $R$ centered at 0 for some $R > 0$, and let $v$ be a homogeneous function of order $\kappa - d$, $\kappa > 0$. Then the Fourier transform $\hat{v}h$ of the product $vh$ satisfies

$$\hat{v}h(\xi) = \hat{h}(0) + O(\lvert\xi\rvert^{\kappa - 1}), \quad \lvert\xi\rvert \to \infty.$$  

Proof. Since $h$ is smooth, we can consider its Taylor expansion around zero with a remainder term. Each term of the expansion is a homogeneous function that has a weaker singularity at zero than $\kappa$. The Fourier transform of the product of the remainder term and $v$ decays to zero as fast as we like depending on the number of term in the Taylor expansion.

We apply Proposition 3.1 in the context of the Taylor expansion of $\eta$, where the function $B$ is only smooth on $[0, R_\Omega) \times \mathbb{R}^{d-1}$. In order to avoid this problem, we introduce a smooth even function $\kappa : \mathbb{R}^d \to \mathbb{R}$ such that $0 \leq \kappa \leq 1$, $\kappa(x) = 1$ for $\lvert x \rvert \leq R_\Omega/2$, and $\kappa(x) = 0$ for all $\lvert x \rvert \geq R_\Omega$. Now, the function $h = \kappa B$ satisfies the assumptions of Proposition 3.1.

Let $K_0 = \kappa K$. Consider the operator

$$\mathcal{K}_0 u(x) = \int_\Omega K_0(x - y)u(y) \, dy$$

which is a compact selfadjoint operator on $L^2(\Omega)$ with positive and negative eigenvalues $\{ \mu_k^\pm \}_{k=1}^\infty$ accumulating at 0. The kernel of the operator $\mathcal{K}_0 - \mathcal{K}$ is smooth. Therefore, using [4] (cf. also [5]), we see that the eigenvalues $\nu_n^\pm$ of the operator $\mathcal{K}_0 - \mathcal{K}$ satisfy $\nu_n^\pm = o(n^{-l})$ for all $l > 0$. Therefore, it is sufficient to prove our result for $\mathcal{K}_0$.

Proof of Theorem 3.1. Let $\{ \varphi_k^\pm \}$ be an orthonormal set of eigenfunctions of $\mathcal{K}_0$ corresponding to the eigenvalues $\mu_k^\pm$. Fix $\lambda > 0$, and let $\varphi(x) := (\lvert x \rvert - \lambda)_+$. Then

$$\sum_k (\mu_k^\pm - \lambda)_+ = \sum_k \varphi(\mu_k^\pm) = \sum_k \varphi(\mu_k^\pm) \lVert \varphi_k^\pm \rVert_2 = \frac{1}{(2\pi)^d} \sum_k \varphi(\mu_k^\pm) \int_{\mathbb{R}^d} \lvert \hat{\varphi}_k^\pm(\xi) \rvert^2 \, d\xi$$

$$= \frac{1}{(2\pi)^d} \sum_k \varphi(\mu_k^\pm) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_k^\pm(x) \varphi_k^\pm(y) e^{-i(x - y) \cdot \xi} \, dx \, dy \, d\xi.$$  

Recall that $e_\xi(x) = e^{ix \cdot \xi}$. Using the spectral theorem for compact selfadjoint operators, we get

$$\sum_k (\mu_k^\pm - \lambda)_+ = \frac{1}{(2\pi)^d} \sum_k \varphi(\mu_k^\pm) \int_{\mathbb{R}^d} \lvert \varphi(x, e_\xi) \rvert^2 \, d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi(\mu) \, d(E_{\mu}e_\xi, e_\xi) \, d\xi,$$

where $E_{\mu}$ is the spectral measure of $\mathcal{K}_0$.

Since

$$\int_{\mathbb{R}^d} d(E_{\mu}e_\xi, e_\xi) = \lvert \Omega \rvert \quad \forall \xi \in \mathbb{R}^N,$$

we see that $\frac{1}{\lvert \Omega \rvert} d(E_{\mu}e_\xi, e_\xi)$ is a probability measure. Since $\varphi$ is convex, we can apply the Jensen inequality to obtain

$$\varphi \left( \int_{\mathbb{R}^d} \mu(1) d(E_{\mu}e_\xi, e_\xi) \right) \leq \frac{1}{\lvert \Omega \rvert} \int_{\mathbb{R}^d} \varphi(\mu) \, d(E_{\mu}e_\xi, e_\xi).$$

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Thus, we need to estimate $G$ where \[ \kappa \] parts, the integral in this relation is $|\rightarrow\infty$  as $\kappa$ Since $B$ Let \[ \hat{G}(\xi) = G(\xi) = \hat{K}(\xi) + \int K(z)(1 - \kappa(z))e^{-iz\xi} \, dz. \]

Since $\kappa = 1$ near 0, we conclude that $K(z)(1 - \kappa(z))$ is smooth on $\mathbb{R}^d$. So, by integration by parts, the integral in this relation is $O(|\xi|^{-k})$ as $|\xi| \to \infty$ for all $k > 0$. Similarly, \[ \hat{G}_1(\xi) = |\Omega|\hat{F}(\xi) + O(|\xi|^{-k}) \]
as $|\xi| \to \infty$ for all $k > 0$. Finally, by Proposition 3.1, we have $\hat{G}_2(\xi) = O(|\xi|^{-\alpha-2})$. Putting all these together, we obtain \[ \frac{1}{|\Omega|} \int_{\mathbb{R}^d} K_0(z)\eta(z)e^{-iz\xi} \, dz = \hat{K}(\xi) + \hat{F}(\xi) + G(\xi), \]
where $G(\xi) = O(|\xi|^{-\alpha-2})$ as $|\xi| \to \infty$.

Going back to the computations above, we have \[ \sum_k (|\mu_k^\pm| - \lambda)_+ \geq \frac{|\Omega|}{(2\pi)^d} \int_{\mathbb{R}^d} \varphi(\hat{K}(\xi) + \hat{F}(\xi) + G(\xi)) \, d\xi. \tag{3.2} \]
Thus, we need to estimate \[ I := \int_{\mathbb{R}^d} \left( |\hat{K}(\xi) + \hat{F}(\xi) + G(\xi) - \lambda \right)_+ \, d\xi \]
\[ = \int_{\mathbb{R}^d} \left( \hat{K}(\xi) + \hat{F}(\xi) + G(\xi) - \lambda \right)_+ \, d\xi + \int_{\mathbb{R}^d} \left( -\hat{K}(\xi) + \hat{F}(\xi) + G(\xi) - \lambda \right)_+ \, d\xi. \]

Denote by $I_1$ and $I_2$ the two integrals on the right-hand side respectively.

Since $G(\xi) = O(|\xi|^{-\alpha-2})$ as $|\xi| \to \infty$, there exist constants $M, C > 0$ such that \[ |G(\xi)| \leq M|\xi|^{-\alpha-2} \quad \forall \, |\xi| \geq C. \]

Let $B_C$ be the ball of radius $C$ centered at the origin. We can estimate the integral $I_1$ by splitting it into an integral over $B_C$ and an integral over its complement and treating each term separately. The integral over $B_C$ can be easily bounded by using the inequality $(X + Y)_+ \geq X_+ - |Y|$. Hence \[ \int_{B_C} \left( \hat{K}(\xi) + \hat{F}(\xi) + G(\xi) - \lambda \right)_+ \, d\xi \geq \int_{B_C} \left( \hat{K}(\xi) + \hat{F}(\xi) - \lambda \right)_+ \, d\xi - \int_{B_C} |G(\xi)| \, d\xi \tag{3.3} \]
Let \( m = \max \{ \sup_{\mathbb{R}^{d-1}} |f(\theta)|, \sup_{\mathbb{R}^{d-1}} |g(\theta)| \} \). It can be easily checked that for \( \lambda < 1 \)
\[
\tilde{K}(\xi) + \tilde{F}(\xi) = \frac{f(\tilde{\xi})}{|\tilde{\xi}|^\alpha} + \frac{g(\tilde{\xi})}{|\tilde{\xi}|^{\alpha+1}} < \lambda \quad \forall \ |\xi| > \left( \frac{2m}{\lambda} \right)^{1/\alpha}.
\]

Using this fact, we can also estimate the second term:
\[
\int_{\mathbb{R}^d \setminus \mathcal{B}_c} \left( \tilde{K}(\xi) + \tilde{F}(\xi) + G(\xi) - \lambda \right)_{+} d\xi \geq \int_{\mathbb{R}^d \setminus \mathcal{B}_c} \left( \tilde{K}(\xi) + \tilde{F}(\xi) - \frac{M}{|\xi|^{\alpha+2}} - \lambda \right)_{+} d\xi \\
\geq \int_{\mathbb{R}^d \setminus \mathcal{B}_c} \left( \tilde{K}(\xi) + \tilde{F}(\xi) - \lambda \right)_{+} d\xi - \int_{C \subseteq |\xi| < \left( \frac{2m}{\lambda} \right)^{1/\alpha}} \frac{M}{|\xi|^{\alpha+2}} d\xi.
\tag{3.4}
\]

Adding up (3.3) and (3.4), we find
\[
I_1 \geq \int_{\mathbb{R}^d} \left( \tilde{K}(\xi) + \tilde{F}(\xi) - \lambda \right)_{+} d\xi + O(\lambda^{-\frac{\alpha-2}{\alpha}}), \quad \lambda \to 0.
\tag{3.5}
\]

Similarly, we could bound \( I_1 \) from above. So, the inequality (3.5) is, in fact, an equality.

Exactly the same method could be applied to \( I_2 \), and we obtain
\[
I_2 = \int_{\mathbb{R}^d} \left( -(\tilde{K}(\xi) + \tilde{F}(\xi)) - \lambda \right)_{+} d\xi + O(\lambda^{-\frac{\alpha-2}{\alpha}}),
\]

which implies
\[
I = \int_{\mathbb{R}^d} \left( |\tilde{K}(\xi) + \tilde{F}(\xi)| - \lambda \right)_{+} d\xi + O(\lambda^{-\frac{\alpha-2}{\alpha}}), \quad \lambda \to 0.
\]

It remains to compute the integral appearing in this expression. In the polar coordinates \((r, \theta)\), it can be written as
\[
\int_{\mathbb{R}^d} \left( |\tilde{K}(\xi) + \tilde{F}(\xi)| - \lambda \right)_{+} d\xi = \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \left( \left| \frac{f(\theta)}{r^\alpha} + \frac{g(\theta)}{r^{\alpha+1}} \right| - \lambda \right)_{+} r^{d-1} dr d\sigma(\theta).
\]

We use the following lemma to compute this integral.

**Lemma 3.1.** Let \( C_1, C_2 \in \mathbb{R} \) be constants, and let \( \mu > 0 \) be a variable which will be allowed to tend to 0. Then
\[
\int_{0}^{\infty} \left( \left| \frac{C_1}{r^\alpha} + \frac{C_2}{r^{\alpha+1}} \right| - \mu \right)_{+} r^{d-1} dr = \frac{\alpha}{d(d-\alpha)} |C_1|^\frac{d}{\alpha} \mu^{1-\frac{d}{\alpha}}
\]
\[+ \frac{1}{d-\alpha-1} \text{sgn} \ (C_1)|C_1|^{\frac{d-\alpha-1}{\alpha}} C_2 \mu^{1-\frac{d-1}{\alpha}} + o(\mu^{1-\frac{d-1}{\alpha}}), \quad \mu \to 0. \tag{3.6}
\]
Proof. Let \( h : (0, \infty) \to \mathbb{R} \) be defined by

\[
h(r) = \frac{C_1}{r^\alpha} + \frac{C_2}{r^{\alpha+1}}.
\]

We first need to find for which values of \( r \) we have \( h(r) \geq \mu \) and \( h(r) \leq -\mu \).

We distinguish a number of cases depending on the sign of the constants \( C_1 \) and \( C_2 \). The case \( C_1 = 0 \) is immediate.

If \( C_1 > 0 \) and \( C_2 < 0 \) (the case \( C_1 < 0 \) and \( C_2 > 0 \) is very similar), then the function \( h \) increases from \(-\infty\) up to a positive value and then decreases to \( 0 \). The equation \( h(r) = -\mu \) has one real solution \( r^-(-\mu) \) (cf. Figure 1). Then \( h^{-1}([-\infty, -\mu]) = (0, r^-(-\mu)] \) and \( h^{-1}([\mu, \infty)) = [r^+_1(\mu), r^+_2(\mu)] \). These roots can be estimated as follows:

\[
r^-(\mu) = -\frac{C_2}{C_1} - \frac{1}{C_1} \left(-\frac{C_2}{C_1}\right)^{\alpha+1} \mu + o(\mu),
\]

\[
r^+_1(\mu) = -\frac{C_2}{C_1} + \frac{1}{C_1} \left(-\frac{C_2}{C_1}\right)^{\alpha+1} \mu + o(\mu),
\]

\[
r^+_2(\mu) = C_1^{1/\alpha} \mu^{-1/\alpha} + \frac{C_2}{\alpha C_1} + o(1)
\]
as \( \mu \to 0 \). A straightforward (yet rather tedious) computation then gives (3.6).

![Figure 1. The case \( C_1 > 0, C_2 < 0 \).](image)

If \( C_1 > 0 \) and \( C_2 \geq 0 \) (and, similarly, if \( C_1 < 0 \) and \( C_2 \leq 0 \)), then the function \( h \) is strictly decreasing from \( \infty \) to \( 0 \), so the equation \( h(r) = \mu \) has a unique solution \( r^+(\mu) \), and \( h^{-1}([1, \infty)) = [r^+(\mu), \infty) \) (cf. Figure 2). We can estimate the root

\[
r^+(\mu) = C_1^{1/\alpha} \mu^{-1/\alpha} + \frac{C_2}{\alpha C_1} + o(1)
\]
as \( \mu \to 0 \), and (3.6) easily follows. \( \square \)
Using this lemma we have

\[
\int_{\mathbb{R}^d} \left( |\hat{K}(\xi) + \hat{F}(\xi)| - \lambda \right)_+ \, d\xi = \frac{\alpha}{d(d-\alpha)} \lambda^{1-\frac{d}{\alpha}} \int_{S^{d-1}} |f(\theta)|^{\frac{d}{\alpha}} \, d\sigma(\theta) \\
+ \frac{1}{d-\alpha-1} \lambda^{1-\frac{d-1}{\alpha}} \int_{S^{d-1}} \text{sgn}(f(\theta))|f(\theta)|^{\frac{d-\alpha-1}{\alpha}} g(\theta) \, d\sigma(\theta) + o(\lambda^{1-\frac{d-1}{\alpha}}).
\]

The first term on the right-hand side of this equation can be simplified by using

\[
\int_{\mathbb{R}^d} \left( |\hat{K}(z)| - 1 \right)_+ \, dz = \int_{S^{d-1}} \int_0^\infty \left( |f(s)| - 1 \right)_+ r^{d-1} \, dr \, d\sigma(s) \\
= \int_{S^{d-1}} \int_0^\infty (|f(s)|r^{d-\alpha} - r^{d-1}) \, dr \, d\sigma(s) = \frac{\alpha}{d(d-\alpha)} \int_{S^{d-1}} |f(s)|^{\frac{d}{\alpha}} \, d\sigma(s).
\]

This completes the proof of Theorem 3.1. \(\square\)

### 4 Applications

Let us consider a special case of spherically symmetric kernels \(\hat{K}(\xi) = |\xi|^{-\alpha}\), so \(K(z) = C|z|^{-(d-\alpha)}\), where

\[
C = \pi^{-d/2}2^{-\alpha} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.
\]

**Corollary 4.1.** Let \(\hat{K}(\xi) = |\xi|^{-\alpha}\), \(0 < \alpha < d\). Then

\[
\sum_k (|\lambda_k| - \lambda)_+ \leq \frac{\Omega}{(2\pi)^d d(d-\alpha)} |S^{d-1}| \lambda^{1-\frac{d}{\alpha}}.
\]

(4.1)
If, moreover, $\alpha < d - 1$, we also have the lower bound
\[
\sum_k (|\lambda_k| - \lambda)_+ \geq \frac{\Omega}{(2\pi)^d} \frac{\alpha}{d(d - \alpha)} |S^{d-1}| \lambda^{1 - \frac{d}{2}} 
+ \frac{1}{(2\pi)^d} \Gamma \left( \frac{\alpha+1}{2} \right) \Gamma \left( \frac{d-\alpha}{2} \right) \lambda^{1 - \frac{d-1}{2}} \int_{S^{d-1}} A_\Omega(\theta) \, d\sigma(\theta) + o(\lambda^{1 - \frac{d-1}{2}})
\]
as $\lambda \to 0$.

**Proof.** Using Theorem 2.1 we find
\[
\sum_k (|\lambda_k| - \lambda)_+ \leq (2\pi)^d |\Omega| \lambda^{1 - \frac{d}{2}} \int_{\mathbb{R}^d} (|\xi|^{\alpha} - 1)_+ \, d\xi = \frac{\Omega}{(2\pi)^d} \frac{\alpha}{d(d - \alpha)} |S^{d-1}| \lambda^{1 - \frac{d}{2}}.
\]
For the lower bound, keeping the notation from the previous section, we first need to compute the constant
\[
\gamma = \int_{S^{d-1}} g(\theta) \, d\theta.
\]
Consider the function $E(z) = e^{-|z|^2/2}$, so $\hat{E}(\xi) = (2\pi)^{d/2} E(\xi)$. By the Parseval theorem,
\[
(2\pi)^d \int_{\mathbb{R}^d} F(z) E(z) \, dz = \int_{\mathbb{R}^d} \hat{F}(z) \hat{E}(z) \, d\xi.
\]
Using the polar coordinates $z = r\theta$ on both sides, we get
\[
(2\pi)^{d/2} \frac{C}{|\Omega|} \int_{S^{d-1}} A_\Omega(\theta) \, d\sigma(\theta) \int_0^\infty r^\alpha e^{-r^2/2} \, dr = \int_{S^{d-1}} g(\theta) \, d\sigma(\theta) \int_0^\infty r^{d-\alpha-2} e^{-r^2/2} \, dr.
\]
Changing the variable $y = r^2/2$ and using the definition of the gamma function, we finally obtain
\[
\gamma = 2 \frac{\Gamma \left( \frac{\alpha+1}{2} \right) \Gamma \left( \frac{d-\alpha}{2} \right)}{|\Omega| \Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{d-\alpha-1}{2} \right)} \int_{S^{d-1}} A_\Omega(\theta) \, d\sigma(\theta),
\]
and the bound follows from Theorem 3.1.

**Remark 4.1.** Note that if $\alpha = 2$, $d \geq 3$, then $0 < \alpha < d$, the kernel $K(x)$ is the fundamental solution for the Laplacian in $\mathbb{R}^d$. We have
\[
\sum_k (|\lambda_k| - \lambda)_+ \leq (2\pi)^{-d} |\Omega| |S^{d-1}| \lambda^{1 - \frac{d}{4}} \left( \frac{2}{d(d - 2)} \right).
\]
In particular, if $d = 3$, then
\[
\sum_k (|\lambda_k| - \lambda)_+ \leq \frac{1}{\sqrt{\lambda}} \frac{1}{12 \pi^3} |\Omega| |S^2|,
\]
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This inequality allows us to obtain a bound on the number of eigenvalues greater than $\lambda$.

**Corollary 4.2.** Let $\hat{K}(\xi) = |\xi|^{-\alpha}$, $0 < \alpha < d$. Then $\mathcal{K} \geq 0$, $\lambda_k \geq 0$, and the number of eigenvalues greater than $\lambda$ of the operator $\mathcal{K}$ satisfies the relation

$$n(\lambda) = \# \{ k : \lambda_k > \lambda \} \leq (2\pi)^{-d} \lambda^{-\frac{d}{\alpha}} |\Omega|^\frac{d}{d-\alpha} \frac{d^{d/\alpha}}{d(d-\alpha)^{d/\alpha}}.$$  \hspace{1cm} (4.2)

**Proof.** Let

$$\chi_\lambda(t) = \begin{cases} 1, & t \geq \lambda, \\ 0, & 0 \leq t < \lambda. \end{cases}$$

Let $\tau < \lambda$. Then it is clear that $\chi_\lambda(t) \leq (t-\tau)^+/(\lambda-\tau)$, and

$$n(\lambda) = \sum_k \chi(\lambda_k) \leq \sum_k (\lambda_k - \tau)^+/(\lambda-\tau) \leq (2\pi)^{-d} |\Omega| |S^{d-1}| \frac{\tau^{1-\frac{d}{\alpha}}}{\lambda-\tau} \left( \frac{\alpha}{d(d-\alpha)} \right)^{\frac{d}{2}}.$$ Minimizing with respect to $\tau$, we find $\tau = \lambda(1 - \alpha/d)$ and thus arrive at (4.2).

Let us consider the spectrum of the operator of the Dirichlet boundary value problem $-\Delta_D$ acting in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a domain of finite measure:

$$-\Delta u(x) = \nu u(x), \quad u(x)|_{x \in \partial \Omega} = 0.$$

The best known estimate for the number $N(\nu)$ of the eigenvalues $\{ \nu_k \}$ below $\nu$ of this operator follows from the sharp semiclassical inequality for the Riesz means

$$\sum_k (\nu - \nu_k)^+ \leq (2\pi)^{-d} |\Omega| \nu^{1+d/2} \int_{|\xi| < 1} (1 - |\xi|^2) d\xi,$$

which implies (cf. [6])

$$N(\nu) = \# \{ k : \nu_k < \nu \} \leq (2\pi)^{-d} |\Omega| \nu^{d/2} |S^{d-1}| \frac{1}{d} \left( \frac{d + 2}{d} \right)^{\frac{d}{2}}.$$ \hspace{1cm} (4.3)

We can compare the last estimate with the semiclassical constant that is the still open Pólya conjecture stated for all domains of finite measure

$$N(\nu) \leq (2\pi)^{-d} |\Omega| \nu^{d/2} \int_{|\xi|^2 < 1} d\xi = (2\pi)^{-d} |\Omega| \nu^{d/2} |S^{d-1}| \frac{1}{d}.$$ 

Note that if $\alpha = 2$ and $\hat{K}(\xi) = |\xi|^{-2}$, then the operator $\mathcal{K}$ is inverse to $-\tilde{\Delta}$ with some nonlocal boundary conditions and, since the eigenvalues of $-\Delta^D$ are larger than the eigenvalues of $-\tilde{\Delta}$, we have

$$N(\nu, -\Delta^D) \leq N(\nu, -\tilde{\Delta}) \leq n(1/\nu).$$

Thus, we obtain the following assertion.
Theorem 4.1. Let $d \geq 3$, and let $\Omega \subset \mathbb{R}^d$ be a domain of finite measure. Then for the number of the eigenvalues below $\nu$ of the Dirichlet Laplacian we have

$$N(\nu, -\Delta^D) \leq (2\pi)^{-d} \nu^\frac{d}{2} |\Omega| |\mathbb{S}^{d-1}| \frac{1}{d} \left( \frac{d}{d-2} \right)^{d/2}.$$ (4.4)

Remark 4.2. The constant appearing on the right-hand side of (4.4) is not as good as in (4.3). It must be related to the fact that, when considering the Dirichlet boundary problem, the Green function for the Laplacian in the whole space has a negative compensating term that is responsible for the Dirichlet boundary conditions. The integral operator $K$ with $\hat{K}(\xi) = 1/|\xi|^2$ in $L^2(\Omega)$ is the inverse to the Laplacian with some more complicated nonlocal boundary conditions (cf. [7]).

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