AN UPPER BOUND ON THE NUMBER OF (132, 213)-AVOIDING CYCLIC PERMUTATIONS

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Abstract. We show a \( n^2 \cdot 2^{n/2} \) upper bound on the number of (132, 213) avoiding cyclic permutations. This is the first nontrivial upper bound on the number of such permutations. We also construct an algorithm to determine whether a (132, 213) avoiding permutation is cyclic that references only the permutation’s layer lengths.

1. Introduction

The theory of pattern avoidance in permutations has been widely studied since its introduction by Knuth in 1968 [10]. The classical form of this problem asks to count the number of permutations of \( [n] = \{1, \ldots, n\} \) avoiding a given pattern \( \sigma \). Since then, many variations of this problem have been proposed and studied in the literature.

We will focus on the problem of pattern avoidance among permutations consisting of a single cycle. This problem was first posed by Stanley in 2007 at the Permutation Patterns Conference, and was subsequently studied by Archer and Elizalde [1] and Bóna and Cory [4].

Let us first recall the definition of pattern avoidance. Let \( \mathfrak{S}_n \) denote the set of permutations of \( [n] = \{1, \ldots, n\} \).

Definition 1.1. Let \( \sigma \in \mathfrak{S}_k \). A permutation \( \pi \in \mathfrak{S}_n \) contains \( \sigma \) if there exist indices \( 1 \leq i_1 < \cdots < i_k \leq n \) such that the sequence

\[ \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} \]

is in the same relative order as \( \sigma_1 \sigma_2 \cdots \sigma_k \). Otherwise, \( \pi \) avoids \( \sigma \).
In either case, $\sigma$ is the pattern that $\pi$ contains or avoids.

**Example 1.1.** The permutation 12534 contains the pattern 132 and avoids the pattern 321.

In classical pattern avoidance, the central objects of study are the numbers $S_n(\sigma)$, defined as follows.

**Definition 1.2.** Let $\sigma \in \mathfrak{S}_k$ be a pattern. Then, $S_n(\sigma)$ is the number of permutations $\pi \in \mathfrak{S}_n$ avoiding the pattern $\sigma$.

In a classical result, Knuth [10] showed that $S_n(\sigma) = C_n$ for any pattern $\sigma \in \mathfrak{S}_3$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$th Catalan number. In 1985, Simion and Schmidt [12] proved an analogous result for permutations avoiding two patterns. Let $S_n(\sigma_1, \sigma_2)$ denote the number of permutations of length $n$ avoiding both patterns $\sigma_1$ and $\sigma_2$. They compute the value of $S_n(\sigma_1, \sigma_2)$ for any pair of patterns $\sigma_1, \sigma_2 \in \mathfrak{S}_3$. For an overview of related results in classical pattern avoidance, see the book by Linton, Rušck, and Vatter [11].

**Definition 1.3.** Let $\sigma \in \mathfrak{S}_k$ be a pattern. Then, $C_n(\sigma)$ is the number of permutations $\pi \in \mathfrak{S}_n$ avoiding $\sigma$ that consist of a single $n$-cycle.

In 2007, Richard Stanley asked for the determination of the value of $C_n(\sigma)$ for any $\sigma \in \mathfrak{S}_3$. All cases of this problem remain open; this problem is difficult because it requires looking at permutations both as linear orders and in terms of their cyclic structure.

Work on the analogous problem for cyclic permutations avoiding two patterns was begun by Archer and Elizalde in 2014. They show the following result.

**Theorem 1.1.** [1] Let $\mu$ be the number-theoretic Möbius function. Then,

$$C_n(132, 321) = \frac{1}{2n} \sum_{d | n \text{ odd}} \mu(d)2^{n/d}.$$  

Bóna and Cory [4] enumerate $C_n(\sigma_1, \sigma_2)$ for several other pairs $\sigma_1, \sigma_2 \in \mathfrak{S}_3$. Their most significant result is as follows.

**Theorem 1.2.** [4] Let $\phi$ be the Euler totient function. Then,

$$C_n(123, 231) = \begin{cases} \phi\left(\frac{n}{2}\right) & n \equiv 0 \pmod{4}, \\ \phi\left(\frac{n+2}{2}\right) + \phi\left(\frac{n}{2}\right) & n \equiv 2 \pmod{4}, \\ \phi\left(\frac{n+1}{2}\right) & n \equiv 1 \pmod{2}. \end{cases}$$

Other results regarding pattern avoiding cyclic permutations can be found in [8, 13, 14].
As a consequence of the results in [4], formulae for \( C_n(\sigma_1, \sigma_2) \) are known for each pair \( \sigma_1, \sigma_2 \in \mathfrak{S}_3 \) except the pair \((132, 213)\). This motivates the following problem.

**Problem 1.1.** Determine an explicit formula for \( C_n(132, 213) \).

Solving this problem would complete the enumeration of cyclic permutations avoiding pairs of patterns of length 3.

The main result of this paper is the following bound.

**Theorem 1.3.** For all \( n \geq 1 \), \( C_n(132, 213) \leq n^2 \cdot 2^n/2 \).

To our knowledge, this is the first nontrivial upper bound of \( C_n(132, 213) \), though computer experiments suggest this bound is not asymptotically tight.

A *composition* of \( n \) is a tuple \((a_1, \ldots, a_k)\) of positive integers with sum \( n \). The \((132, 213)\)-avoiding permutations of size \( n \) are the so-called *reverse layered permutations*, parametrized by the compositions of \( n \). The second main result of this paper is the following algorithm, which determines, without reference to the underlying permutation, whether a composition \((a_1, \ldots, a_k)\) corresponds to a cyclic permutation.

**Algorithm 1.1.** On input a composition \( C \), apply the *repeated reduction algorithm* \( \text{Rred} \) to the *equalization* \( \text{Eq}(C) \). If the resulting composition is \((1, 1)\), \( C \) is cyclic; otherwise, \( C \) is not cyclic.

The operations \( \text{Rred} \) and \( \text{Eq} \) are defined in Sections 3 and 4, respectively. This result is interesting in its own right, and is the key step in the proof of Theorem 1.3. It implies that any cyclic composition has one or two odd terms, which gives the bound in Theorem 1.3.

The rest of this paper is structured as follows. In Section 2 we formalize the connection between \((132, 213)\)-avoiding permutations and compositions of \( n \) via reverse layered permutations. We introduce the notions of *balanced compositions* and *wiring diagrams*, tools that will be useful in our analysis. In Section 3 we prove our results for balanced permutations. In Section 4 we generalize our results to all permutations. Finally, in Section 5 we present some directions for further research.

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2. Preliminaries

2.1. Reverse Layered Permutations. A reverse layered permutation is a permutation of the form

$$(n - a_1 + 1) \ldots (n) (n - a_1 - a_2 + 1) \ldots (n - a_1) \ldots (1)(2) \ldots (a_k)$$

for some positive integers $a_1, \ldots, a_k$ summing to $n$. It is known that the $(132, 213)$ avoiding permutations are the reverse layered permutations [5]. We can bijectively identify these permutations with the compositions $(a_1, \ldots, a_k)$ of $n$.

As an immediate consequence, there are $2^{n-1}$ reverse layered permutations of length $n$, corresponding to the $2^{n-1}$ compositions of $n$. It remains, therefore, to determine the number of these permutations that are also cyclic.

We say a composition of $n$ is cyclic if its associated reverse layered permutation is cyclic. Enumerating $C_n(132, 213)$ is therefore equivalent to counting the cyclic compositions of $n$.

2.2. Balanced Compositions. Suppose $n$ is even. We say a composition $C = (a_1, \ldots, a_k)$ is balanced if some prefix $a_1, \ldots, a_j$ has sum $\frac{n}{2}$, and unbalanced otherwise. We denote such compositions with the notation $(a_1, \ldots, a_j|a_{j+1}, \ldots, a_k)$.

Equivalently, $C$ is balanced if its associated reverse layered permutation $\pi$ has the property that $\pi(i) > \frac{n}{2}$ if and only if $i \leq \frac{n}{2}$. We say a reverse layered permutation is balanced if it has this property, and unbalanced otherwise.

2.3. Wiring Diagrams. The wiring diagram of a permutation $\pi$ is obtained from the permutation matrix of $\pi$ by drawing vertical and horizontal lines, called wires, from each 1 to the main diagonal. We call the cells with 1’s the points of the wiring diagram. By slight abuse of notation, we say the wiring diagram of a composition $C$ is the wiring diagram of its associated permutation.

The following properties of wiring diagrams are clear.

Each layer $(n - a_1 - \cdots - a_i + 1) \ldots (n - a_1 - \cdots - a_{i-1})$ in a reverse layered permutation corresponds to a layer of $a_i$ diagonally-adjacent points in the wiring diagram running from top left to bottom right; successive layers are ordered from top right to bottom left.

A permutation is cyclic if and only if the wires in its wiring diagram make a single closed loop. Moreover, a permutation is balanced if and only if each point is, along the path of the wire, adjacent to two points on the opposite side of the main diagonal.
Example 2.1. Figure 1 shows the wiring diagrams of the balanced cyclic permutation 645312, the unbalanced cyclic permutation 53412, and the balanced noncyclic permutation 456321.

In the wiring diagram of 645312, the wire forms a single loop, and each point (marked with \( \times \)) is, along the wire, adjacent to two points on the opposite side of the main diagonal. In the wiring diagram of 53412, the wire still forms a single loop, but some the two points in the middle layer are adjacent to each other, and not to points on the opposite side of the main diagonal. In the wiring diagram of 456321, the wire does not form a single loop.

![Wiring diagrams for permutations 645312, 53412, and 456321](image)

**Figure 1.** Wiring diagrams for permutations 645312, 53412, and 456321, with compositions \((1, 2|1, 2), (1, 2), 2\), and \((3|1, 1, 1)\), respectively. The main diagonal is shown by a dashed line.

3. Determining if a Balanced Composition is Cyclic

In this section, we will prove specialized versions of our main results for balanced compositions. We will generalize these results to all compositions in the next section.

3.1. Reducing Cycles. The key result in this subsection is the Cycle Reduction Lemma, which is Lemma 3.2 below.

**Lemma 3.1.** Suppose the balanced composition

\[(a_1, \ldots, a_k|b_m, \ldots, b_1)\]

is cyclic, and \(a_k = b_m\). Then \(a_k = b_m = k = m = 1\) and \(n = 2\).

**Proof.** If \(a_k = b_m\), then the associated permutation \(\pi\) contains the 2-cycle \(\left(\frac{n}{2}, \frac{n}{2} + b_m\right)\). Thus \(n = 2\) and \(a_k = b_m = k = m = 1\). 

For a balanced composition

\[C = (a_1, \ldots, a_k|b_m, \ldots, b_1)\]
with $a_k \neq b_m$, we define the \textit{reduction} operation $\text{Red}(\cdot)$ as follows. Let
\begin{align*}
u &= a_k \mod |a_k - b_m| \\
v &= |a_k - b_m| - u.
\end{align*}
If $a_k > b_m$, define
$$\text{Red}(C) = (a_1, \ldots, a_{k-1}, u, v|b_{m-1}, \ldots, b_1).$$
Analogously, if $a_k < b_m$, define
$$\text{Red}(C) = (a_1, \ldots, a_{k-1}|v, u, b_{m-1} + \cdots + b_1).$$
In both cases, if $u = 0$, omit $u$ from the composition.

**Lemma 3.2 (Cycle Reduction Lemma).** Suppose $a_k \neq b_m$. Then, the composition $C = (a_1, \ldots, a_k|b_m, \ldots, b_1)$ is cyclic if and only if $\text{Red}(C)$ is cyclic.

Before proving this lemma, we give an example that captures the spirit of the proof.

**Example 3.1.** Let $a_k = 5, \ b_m = 2$, so $u = 2$ and $v = 1$. The Cycle Reduction Lemma states that a balanced composition
$$C = (a_1, \ldots, a_k|b_m, \ldots, b_1)$$
is cyclic if and only if
$$\text{Red}(C) = (a_1, \ldots, a_{k-1}, 2|v, u, b_{m-1} + \cdots + b_1)$$
is cyclic.

Let us focus on the layers corresponding to $a_k = 5$ and $b_m = 2$, which are the innermost layers of the wiring diagram on either side of the main diagonal.

The wires incident to these two layers connect with each other, leaving three pairs of loose ends, as shown in the left diagram in Figure 2. The wires induce a natural pairing on these loose ends: two loose ends are paired if they are connected by these wires.

These loose ends are paired the same way by two layers of size $u = 2$ and $v = 1$, as shown in the right diagram in Figure 2.
Consider any configuration of the remaining wires in the wiring diagram, which we call the outer wiring. These wires also form three pairs of loose ends, which connect with the three pairs of loose ends from the innermost wires. Thus, for a fixed outer wiring, whether the wiring diagram is cyclic depends only on how the pairing of the innermost wires’ loose ends.

Since the two above wirings of the innermost layers pair their loose ends the same way, the wiring diagram of $C$ is a single cycle if and only if the wiring diagram of $\text{Red}(C)$ is a single cycle.

**Proof of Lemma 3.2.** Let us assume $a_k > b_m$; the case $a_k < b_m$ is symmetric. Let $D = a_k - b_m$. Thus $u = a_k \mod D$ and $v = D - u$. Let us write $a_k = qD + u$.

Consider the wiring diagram of $C$. The terms $a_k$ and $b_m$ correspond to the two innermost layers of the wiring diagram on either side of the main diagonal. Call these layers $L_a$ and $L_b$. When we connect the wires incident to these layers, each point in $L_b$ is connected to two points in $L_a$, which are diagonally $D$ cells apart.

The leftward wires from the upper-left $a_k - b_m$ points in $L_a$ and the downward wires from the lower-right $a_k - b_m$ points in $L_a$ are loose ends. Call these sets the upper-left loose ends and the lower-right loose ends. The wiring of $L_a$ and $L_b$ induces a pairing on these loose ends, as described in Example 3.1.

Let us determine the pairing by starting at one of the upper-left loose ends and following its wire. Every time the wire winds around the center of the wiring diagram, it passes through one point in $L_a$ and one point in $L_b$; because each point
in $L_b$ is connected to two points in $L_a$ that are diagonally $D$ cells apart, successive points in $L_a$ and on this wire are diagonally $D$ cells apart.

Let us examine the points this wire passes through in $L_a$. If we started at one of the upper-leftmost $u$ loose ends, the wire passes through $q+1$ points in $L_a$; thus the last point in $L_a$ on the wire is diagonally $qD$ cells from the first point in $L_a$ on the wire. Moreover, this implies that the upper-leftmost $u$ loose ends are paired with the lower-rightmost $u$ loose ends. The remaining $D - u = v$ upper-left loose ends pass through $q$ points in $L_a$; thus the last point in $L_a$ on their wires is diagonally $(q - 1)D$ cells from the first point in $L_a$ on their wires.

This pairing is the same as the pairing produced by two consecutive layers of $u$ and $v$ cells, with the layer of size $v$ on the inside. Therefore, any fixed outer wiring makes a cyclic wiring when connected with opposing layers of size $a_k, b_m$ if and only it makes a cyclic wiring when connected with consecutive layers of size $u, v$. So, the wiring diagram of $C$ consists of a single cycle if and only if the wiring diagram of Red$(C)$ consists of a single cycle.

3.2. The Repeated Reduction Algorithm. Lemmas 3.1 and 3.2 imply the following algorithm to determine if a balanced composition is cyclic. This is Algorithm 1.1 specialized to balanced compositions.

Algorithm 3.1 (Repeated Reduction Algorithm). On input of balanced composition $C$, repeatedly apply Red to $C$ until $a_k = b_m$. If we stop at the balanced composition $(1|1)$, $C$ is cyclic; otherwise, $C$ is not cyclic.

We denote this algorithm by Rred.

We can now prove a necessary condition for a balanced composition to be cyclic.

Proposition 3.3. Every cyclic balanced composition has exactly two odd entries.

Proof. The operation Red, and therefore the algorithm Rred, preserves the number of odd entries in a composition. The cyclic balanced compositions all reduce to $(1|1)$, so they must themselves have exactly two odd entries.

4. The General Setting

In this section, we will generalize the results of the previous section to all compositions.
4.1. **Equalization.** This subsection will develop the notion of *equalization*, an operation that transforms any composition into a balanced composition. The Equalization Lemma, which is Lemma 4.1 below, allows us to reduce to the results in the previous section.

In an unbalanced composition \( C = (a_1, \ldots, a_k) \), no prefix sums to \( \frac{n}{2} \). Therefore there is \( i \) such that
\[
    a_1 + \cdots + a_{i-1} < \frac{n}{2} \quad \text{and} \quad a_1 + \cdots + a_i > \frac{n}{2}.
\]
We call this \( i \) the *dividing index* of the composition.

For unbalanced compositions, we will use notions of *nearly-equal division* and *unequality*.

**Definition 4.1.** The *nearly-equal division* of an unbalanced composition \( C \) with dividing index \( i \) is defined as follows.

- If \( a_1 + \cdots + a_{i-1} \leq a_{i+1} + \cdots + a_k \), then the nearly equal division is \(( (a_1, \ldots, a_i), (a_{i+1}, \ldots, a_k) )\).
- If \( a_1 + \cdots + a_{i-1} > a_{i+1} + \cdots + a_k \), then the nearly equal division is \(( (a_1, \ldots, a_{i-1}), (a_i, \ldots, a_k) )\).

**Definition 4.2.** The *unequalness* of an unbalanced composition \( C \) with nearly-equal division \(( (a_1, \ldots, a_j), (a_{j+1}, \ldots, a_k) )\) is
\[
    U(C) = \left| \sum_{\ell=1}^{j} a_{\ell} - \sum_{\ell=j+1}^{k} a_{\ell} \right|.
\]

**Definition 4.3.** The *equalization* of an unbalanced composition \( C \) with nearly-equal division \(( (a_1, \ldots, a_j), (a_{j+1}, \ldots, a_k) )\) is
\[
    \text{Eq}(C) = (a_1, \ldots, a_j, U, a_{j+1}, \ldots, a_k),
\]
where \( U = U(C) \). The equalization of a balanced composition \( C \) is \( \text{Eq}(C) = C \).

Note that \( \text{Eq}(C) \) is a balanced composition; if \( C \) is unbalanced, the additional term \( U \) is defined to be on the smaller side of the nearly-equal division.

**Remark 4.1.** The nearly-equal division is defined non-symmetrically; when
\[
    a_1 + \cdots + a_{i-1} = a_{i+1} + \cdots + a_k,
\]
we arbitrarily define the nearly-equal division to be \(( (a_1, \ldots, a_i), (a_{i+1}, \ldots, a_k) )\). However, the definition of equalization is still symmetric. When the above equality holds, regardless whether we define the nearly-equal division as
\[
    ( (a_1, \ldots, a_i), (a_{i+1}, \ldots, a_k) )
\]
or
\[
    ( (a_1, \ldots, a_{i-1}), (a_i, \ldots, a_k) ),
\]
the equalization is

$$\text{Eq}(C) = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_k).$$

The following lemma allows us to reduce the question of whether an unbalanced composition is cyclic to the question of whether a balanced composition is cyclic.

**Lemma 4.1 (Equalization Lemma).** The unbalanced composition $C$ is cyclic if and only if $\text{Eq}(C)$ is cyclic.

Once again, we first demonstrate the lemma with an example.

**Example 4.1.** Suppose $a_1 + \cdots + a_{i-1} + 3 = a_{i+1} + \cdots + a_k$, and $a_i = 5$. Then the nearly-equal division of $C$ is

$$(((a_1, \ldots, a_{i-1}, 5), (a_{i+1}, \ldots, a_k)),$$

and

$$U(C) = \sum_{\ell=1}^{i} a_{\ell} - \sum_{\ell=i+1}^{k} a_{\ell} = 2.$$ 

Thus,

$$\text{Eq}(C) = (a_1, \ldots, a_{i-1}, 5|2, a_{i+1}, \ldots, a_k).$$

In the wiring diagram of $C$, the wires incident to the layer corresponding to $a_i$ are shown in Figure 3.

![Figure 3](image-url)

**Figure 3.** Wiring diagram for a central layer of size 5, in a composition with unequalness 2. The main diagonal is shown by a dotted line, and loose ends are circled.

The two innermost layers of the wiring diagram of $\text{Eq}(C)$ are shown in the left diagram in Figure 2. In both diagrams, there are three pairs of loose ends; the loose
ends are paired by connectivity in the same way. Therefore, $C$ is cyclic if and only if $\text{Eq}(C)$ is cyclic.

**Proof of Lemma 4.1.** If $C$ is balanced, there is nothing to prove. Thus, we can assume $C$ is unbalanced.

Let $C = (a_1, \ldots, a_k)$ have dividing index $i$. If

$$a_1 + \cdots + a_{i-1} = a_{i+1} + \cdots + a_k,$$

then

$$\text{Eq}(C) = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_k).$$

So, the middle $a_i$ entries of the permutation of $C$ are fixed points, and the middle $2a_i$ entries of the permutation of $\text{Eq}(C)$ form a $2a_i$-cycles. If $C = (1)$ these are simultaneously cyclic, and otherwise they are simultaneously not cyclic.

Otherwise, we further assume

$$a_1 + \cdots + a_{i-1} \neq a_{i+1} + \cdots + a_k.$$

The remaining proof is, once again, a matter of following loose ends. Let us assume, without loss of generality, that

$$a_1 + \cdots + a_{i-1} < a_{i+1} + \cdots + a_k.$$

Then the nearly-equal division of $C$ is $((a_1, \ldots, a_i), (a_{i+1}, \ldots, a_k))$. Let $U = U(C)$. Note that $U(C) < a_i$, because otherwise

$$\sum_{\ell=1}^{i} a_\ell < \sum_{\ell=i+1}^{k} a_\ell,$$

contradicting that $i$ is the dividing index.

Let us call the layer corresponding to $a_i$ in the wiring diagram of $C$ the *central layer* of the wiring diagram; we denote this layer $L_c$. Note that points not in $L_c$ are adjacent, along the wire, to two points on the opposite side of the main diagonal. This property may only be false for points in $L_c$.

Since

$$\sum_{\ell=i+1}^{k} a_\ell - \sum_{\ell=1}^{i-1} a_\ell = a_i - U,$$

all points in $L_c$ are $a_i - U$ cells above the main diagonal. Therefore, points in $L_c$ that are diagonally $a_i - U$ apart are adjacent along the wire. Moreover, these connections leave $a_i - U$ pairs of loose ends: $a_i - U$ leftward-pointing loose ends incident to the upper-leftmost $a_i - U$ points in $L_c$, and equally many downward-pointing loose ends incident to the lower-rightmost $a_i - U$ points in $L_c$.

In $\text{Eq}(C)$, an additional layer $L_u$ of size $U$ is added just below and to the left of $L_c$, resulting in a balanced composition where the two innermost layers have size $a_i$.
and $U$. Each point in $L_u$ is adjacent along the wire to two points in $L_c$; it is easy to see these points are diagonally $a_i - U$ apart.

Thus, if in the wiring diagram of $C$, two points in $L_c$ are adjacent along the wire, in the wiring diagram of $\text{Eq}(C)$, they are two apart along the wire, separated by a point in $L_u$.

It follows that the loose ends in $L_c$ are paired the same way in the wiring diagrams of $C$ and $\text{Eq}(C)$. So, the wiring diagram of $C$ is a single cycle if and only if the wiring diagram of $\text{Eq}(C)$ is a single cycle. □

4.2. Completion of the Proof. Lemma 4.1, in conjunction with Lemma 3.2, immediately implies the validity of Algorithm 1.1.

We will now prove Theorem 1.3 restated below for clarity.

**Theorem 1.3.** For all $n \geq 1$, $C_n(132, 213) \leq n^2 \cdot 2^{n/2}$.

We first note the following structural result.

**Proposition 4.2.** All cyclic compositions of $n$ have exactly one odd term if $n$ is odd and two odd terms if $n$ is even.

**Proof.** Suppose $C$ is a cyclic composition of $n$. If $n$ is odd, the unequalness $U(C)$ is odd; if $n$ is even, the unequalness $U(C)$ is even. By Proposition 3.3, $\text{Eq}(C)$ has exactly two odd terms. Therefore, $C$ has exactly one and two odd terms in these cases, respectively. □

**Proof of Theorem 1.3.** Suppose $n$ is odd. By Proposition 4.2, cyclic compositions of $n$ have exactly one odd term. We can obtain any such composition by decrementing one term of a composition of $n + 1$ with only even terms. There are $2^{(n-1)/2}$ compositions of $n + 1$ with only even terms. Since each has at most $\frac{n + 1}{2}$ terms, the number of cyclic compositions of $n$ is bounded by

$$\frac{n + 1}{2} \cdot 2^{(n-1)/2} \leq n^2 \cdot 2^{n/2}.$$ 

Otherwise, suppose $n$ is even. By Proposition 4.2, cyclic compositions of $n$ have exactly two odd terms. We can obtain any such composition by decrementing two terms of a composition of $n + 2$ with only even terms. There are $2^{n/2}$ compositions of $n + 2$ with only even terms. Since each has at most $\frac{n + 1}{2}$ terms, we can decrement two terms in at most $\binom{n/2 + 1}{2}$ ways. So, the number of cyclic compositions of $n$ is bounded by

$$\binom{n/2 + 1}{2} \cdot 2^{n/2} \leq n^2 \cdot 2^{n/2}$$

□
5. Future Directions

This paper makes progress towards Problem 1.1, the enumeration of the \((132, 213)\)-avoiding cyclic permutations. This problem is still open.

Leveraging Algorithm 1.1, computer experiments done by the author have computed \(C_n(132, 213)\) for \(n\) up to 75. This data is shown in Table 1 below.

| \(n\) | \(C_n(132, 213)\) | \(n\) | \(C_n(132, 213)\) | \(n\) | \(C_n(132, 213)\) | \(n\) | \(C_n(132, 213)\) |
|------|-----------------|------|-----------------|------|-----------------|------|-----------------|
| 1    | 1               | 16   | 762             | 31   | 27892           | 46   | 8501562         |
| 2    | 1               | 17   | 440             | 32   | 138200          | 47   | 2570744         |
| 3    | 2               | 18   | 1548            | 33   | 49276           | 48   | 15140024        |
| 4    | 4               | 19   | 818             | 34   | 252632          | 49   | 4498100         |
| 5    | 6               | 20   | 3060            | 35   | 87276           | 50   | 26777982        |
| 6    | 12              | 21   | 1490            | 36   | 459102          | 51   | 7886792         |
| 7    | 14              | 22   | 5960            | 37   | 153586          | 52   | 47470826        |
| 8    | 32              | 23   | 2720            | 38   | 827884          | 53   | 13792064        |
| 9    | 30              | 24   | 11404           | 39   | 270876          | 54   | 83680928        |
| 10   | 76              | 25   | 4894            | 40   | 1494032         | 55   | 24162342        |
| 11   | 62              | 26   | 21596           | 41   | 475282          | 56   | 147821872       |
| 12   | 170             | 27   | 8790            | 42   | 2671006         | 57   | 422411704       |
| 13   | 122             | 28   | 40446           | 43   | 835998          | 58   | 259952664       |
| 14   | 376             | 29   | 15654           | 44   | 4784840         | 59   | 73959542        |
| 15   | 232             | 30   | 74906           | 45   | 1464206         | 60   | 457955944       |

Table 1. Values for \(C_n(132, 213)\) for \(n\) up to 75.

Some observations are apparent from this data. First, the values of \(C_n(132, 213)\) show different behavior for even and odd \(n\). For \(n \geq 8\), the values for even \(n\) are larger than the values for adjacent odd \(n\). This is expected, because cyclic compositions with odd sum have exactly one odd term, whereas cyclic compositions with even sum have exactly two odd terms; the former condition is more restrictive.

Second, the growth of \(C_n(132, 213)\), for both even and odd \(n\), appears to be asymptotically slower than \(2^{n/2}\). From fitting the data, we get an asymptotic estimate of

\[ C_n(132, 213) = 2^{\alpha n + o(n)}, \]

where \(\alpha \approx 0.38\). Thus, we do not believe the upper bound in Theorem 1.3 is asymptotically tight. Of course, this prompts the following problem.

**Problem 5.1.** What is the correct value of \(\alpha\) in the above asymptotic?

Theorem 1.3 implies \(\alpha \leq 0.5\). Both an improvement of this bound and a non-trivial lower bound would be interesting results.
One approach for future research is to study the number-theoretic properties of Algorithm 1.1, to obtain results in the spirit of Proposition 4.2. If one can determine more number-theoretic structure of cyclic compositions, it may be possible to refine the upper bound in Theorem 1.3.

Another more algebraic approach is to obtain recursive identities or inequalities for values of $C_n(132, 213)$. This approach involves breaking $C_n(132, 213)$ into subproblems, perhaps by suffixes of the sequence $a_1, \ldots, a_k$ in the balanced composition $\text{Eq}(C) = (a_1, \ldots, a_k | b_m, \ldots, b_1)$, and finding injective or bijective mappings among the subproblems. It may be possible to obtain a lower bound for $C_n(132, 213)$ in this manner.

Because the first step of Algorithm 1.1 reduces all compositions to a balanced composition, it would be of independent interest to enumerate the balanced cyclic compositions of $n$. These correspond to the balanced reverse layered permutations of length $n$. Let $C^B_n(132, 213)$ be the number of such compositions and permutations. For even $n$ up to 74, computer experiments give the values of $C^B_n(132, 213)$ in Table 2 below.

| $n$ | $C^B_n(132, 213)$ | $n$ | $C^B_n(132, 213)$ | $n$ | $C^B_n(132, 213)$ | $n$ | $C^B_n(132, 213)$ | $n$ | $C^B_n(132, 213)$ |
|-----|-----------------|-----|-----------------|-----|-----------------|-----|-----------------|-----|-----------------|
| 2   | 1               | 34  | 86572           | 50  | 8948694        | 66  | 821844316       |
| 4   | 2               | 29  | 1140            | 36  | 158146         | 52  | 15884762        | 68  | 1442300988      |
| 6   | 6               | 22  | 2182            | 38  | 281410         | 54  | 27882762        | 70  | 2525295380      |
| 8   | 14              | 24  | 4130            | 40  | 509442         | 56  | 49291952        | 72  | 4426185044      |
| 10  | 34              | 26  | 7678            | 42  | 901014         | 58  | 86435558        | 74  | 7747801190      |
| 12  | 68              | 28  | 14368           | 44  | 1618544        | 60  | 152316976       |
| 14  | 150             | 30  | 26068           | 46  | 2852464        | 62  | 266907560       |
| 16  | 296             | 32  | 48248           | 48  | 5089580        | 64  | 469232204       |

Table 2. Values for $C^B_n(132, 213)$ for even $n$ up to 74.

The data suggests the following conjecture.

**Conjecture 5.1.** For even $n$,

$$\frac{C^B_n(132, 213)}{C_n(132, 213)} = \Omega(1).$$

That is, the proportion of $(132, 213)$-avoiding cyclic permutations that are balanced is lower bounded by a constant, which is empirically about 0.33.

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