CAUCHY PROBLEM FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATION WITH AN EXTERNAL FORCE

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Abstract. In this paper we focus on the Cauchy problem for the incompressible Navier-Stokes equation with a rough external force. If the given rough external force is small, we prove the local-in-time existence of this system for any initial data belonging to the critical Besov space $\dot{B}^{1+\frac{3}{p}}_{p,q}$, where $3 < p < \infty$. Moreover, we show the long-time behavior of the priori global solutions constructed by us. Also, we give three kinds of uniqueness results of the forced Navier-Stokes equations.

1. Introduction

We study the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

$$
\begin{aligned}
\frac{\partial u_f}{\partial t} - \Delta u_f + u_f \cdot \nabla u_f &= -\nabla p + f, \\
\nabla \cdot u_f &= 0, \\
u_f|_{t=0} &= u_0.
\end{aligned}
$$

Here $u_f$ is a three-component vector field $u_f = (u_{1,f}, u_{2,f}, u_{3,f})$ representing the velocity of the fluid, $p$ is a scalar denoting the pressure, and both are unknown functions of the space variable $x \in \mathbb{R}^3$ and of the time variable $t > 0$. Finally $f = (f_1, f_2, f_3)$ denotes a given external force defined on $[0, T] \times \mathbb{R}^3$ for some $T \in \mathbb{R} \cup \{\infty\}$. We recall the Navier-Stokes scaling : $\forall \lambda > 0$, the vector field $u_f$ is a solution to $(NSf)$ with initial data $u_0$ if $u_{\lambda,f_{\lambda}}$ is a solution to $(NSf_{\lambda})$ with initial data $u_{0,\lambda}$, where

$$
\begin{aligned}
u_{\lambda,f_{\lambda}}(t,x) &:= \lambda u_f(\lambda^2t, \lambda x), \\
f_{\lambda}(t,x) &:= \lambda^3 f(\lambda^2t, \lambda x), \\
p_{\lambda}(t,x) &:= \lambda^2 p(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda} := \lambda u_0(\lambda x).
\end{aligned}
$$

Spaces which are invariant under the Navier-Stokes scaling are called critical spaces for the Navier-Stokes equation. Examples of critical spaces of initial data for the Navier-Stokes equation in 3D are:

$$
L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)(p < \infty, q \leq \infty) \hookrightarrow \text{BMO}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}.
$$

(See below for definitions).

To put our results in perspective, let us first recall related results concerning the Cauchy problem for the classical (the case $f \equiv 0$) Navier-Stokes equation with possibly irregular initial data:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u &= -\nabla p, \\
\nabla \cdot u &= 0, \\
u|_{t=0} &= u_0.
\end{aligned}
$$

In the pioneering work [22], J. Leray introduced the concept of weak solutions to $(NS)$ and proved global existence for datum $u_0 \in L^2$. However, their uniqueness has remained an open problem. In 1984, T. Kato [20] initiated the study of $(NS)$ with initial data belonging to the space $L^3(\mathbb{R}^3)$ and obtained global existence in a subspace of $C([0, \infty), L^3(\mathbb{R}^3))$ provided the norm $\|u_0\|_{L^3(\mathbb{R}^3)}$ is small enough. The existence result for initial data small in the Besov space $\dot{B}^{1+\frac{3}{p}}_{p,q}$ for $p, \in [1, \infty)$ and $q \in [1, \infty]$ can be found in [10, 16]. The function spaces $L^3(\mathbb{R}^3)$ and $\dot{B}^{1+\frac{3}{p}}_{p,q}$ for $(p, q) \in [1, \infty)^2$ both guarantee the existence of local-in-time solution for any initial data. In 2001, H. Koch and

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D. Tataru \cite{21} showed that global well-posedness holds as well for small initial data in the space $BMO^{-1}$. This space consists of vector fields whose components are derivatives of $BMO$ functions. On the other hand, it has been shown by J. Bourgain and N. Pavlović \cite{6} that the Cauchy problem with initial data in $B_{\infty,1}^{-1}$ is ill-posed no matter how small the initial are. Also P. Germain showed the ill-posedness for initial data in $\dot{B}_{\infty,q}^{-1}$ for any $q > 2$, see \cite{19}.

However, the situation is more subtle when it comes to forced Navier-Stokes equations. In 1999, M. Cannone and F. Planchon \cite{11} worked on constructing global mild solutions in $C([0,T), L^3(\mathbb{R}^3))$ to the Cauchy problem for the Navier-Stokes equations with an external force. They showed the local-in-time wellposedness for any initial data $u_0 \in L^3(\mathbb{R}^3)$, if the external force $f$ can be written as $f = \nabla \cdot V$ and $\sup_{0 < t < T} t^{1-\frac{3}{p}} \|V\|_{L^p}$ is small enough for some $3 < p \leq 6$ and $T > 0$. Also they showed there exists a unique global solution to $(NSf)$, provided $T = \infty$ and $u_0$ is small enough in $\dot{B}_{q,\infty}^{-1+\frac{3}{q}}$ with $3 < q < \frac{3p}{6-p}$. Later in 2005, M. Cannone and G. Karch \cite{9} proved that there exists a solution $u_f \in C_w(\mathbb{R}_+, L^3(\mathbb{R}^3))$ to $(NSf)$, if the initial data $u_0 \in L^3(\mathbb{R}^3)$ is small enough and the external force $f$ satisfies that

\begin{equation}
\sup_{t > 0} \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} f ds \right\|_{L^3(\mathbb{R}^3)} < \infty
\end{equation}

is small enough depending on the norm of the bilinear operator $B$ (defined in \cite{9}) in $L^\infty(\mathbb{R}_+, L^3(\mathbb{R}^3))$.

The basic approach to obtain the above results is, in principle, always the same. One first transforms the Navier-Stokes equations $(NSf)$ into an integral equation,

\begin{equation}
u_f(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds + B(u_f, u_f)(t)
\end{equation}

where

\begin{equation}B(u, v) := -\frac{1}{2} \int_0^t e^{(t-s)\Delta} \mathbb{P} (\nabla \cdot (u \otimes v + v \otimes u)) ds,
\end{equation}

$\mathbb{P}$ being the projection onto divergence free vector fields. It is customary to obtain the existence of a strong global ($T = \infty$) or local ($T < \infty$) solution $u_f \in X_T$ of \cite{2}, with $X_T$ being an abstract critical Banach space, by means of the standard contraction lemma. For example, in \cite{10} the terms $e^{t\Delta} u_0$ and $\int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds$ are treated as the first point of the iteration and they require that $e^{t\Delta} u_0$, $\int_0^t e^{(t-s)\Delta} \mathbb{P} f(s) ds$ belong to the corresponding Banach space $X_T$. That is why in \cite{10} $V$ needs to have a suitable decay in time and in \cite{9} the smallness is measured in $L^3(\mathbb{R}^3)$ and the initial data $u_0$ is restricted to $L^{3,\infty}$. The big difference between \cite{11} and \cite{9} is the following: in \cite{11} the external force has good regularity and $e^{t\Delta} u_0$ belongs to Kato’s space for initial data belonging to $\dot{B}_{p,\infty}^{s_p}$ for some $p > 3$ (see Definition \ref{2}), which allows the fixed point lemma to work in Kato’s space. Therefore the solutions in \cite{11} belong to $C([0, T^*), L^3)$; however in \cite{9}, the external force is rough, which limits the regularity of solution. Therefore in \cite{9} the solutions to \cite{9} only belong to $L^\infty_t(L^{3,\infty})$, even for small smooth initial data. That is the reason why these solutions lack uniqueness, unless the solution is small in $L^\infty_t(L^{3,\infty})$.

In this paper we consider $(NSf)$ with an external force given in \cite{9}, however the class of initial data is different. More precisely, we consider the force $f$ satisfying : $f \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^3))$ with for any $t > 0$

\begin{equation}\int_0^t e^{(t-s)\Delta} \mathbb{P} f ds \in L^\infty(\mathbb{R}_+, L^{3,\infty}),
\end{equation}

which belongs to $C_w(\mathbb{R}_+, L^{3,\infty}(\mathbb{R}^3))$, see \cite{9}. Under a smallness assumption on $f$ (controlled by a universal small positive constant depending on the singularity of initial data), we first show the local and global existence to $(NSf)$ for initial data $u_0$ belonging to $\dot{B}_p^{s_p}$. Moreover we obtain that the above solution belongs to $L^\infty_t(L^{3,\infty})$ when its initial data is in $L^{3,\infty} \cap \dot{B}_p^{s_p}$ for $p > 3$. Then we show the long-time behavior and stability of the above priori global solutions with initial data in
\[ L^3,\infty \cap \dot{B}^s_{p,p}. \]

We need to mention that the uniqueness of solutions in \( L_t^\infty (L^3,\infty) \), even for smooth initial data, is still an open problem. However, we show that if the difference between the above solution and another solution to \((NSf)\) with the same initial data belongs to \( C([0,T],L^3,\infty) \) or has finite energy on some interval \([0,T]\), then they are equal on \([0,T]\).

The rest of the paper is organized as follows. In Section 2 we give some notations and the main results of this paper. Section 3 addresses the proof of the existence and uniqueness of solutions to \((NSf)\) with initial data \( u_0 \) belonging to \( \dot{B}^s_{p,p} \). Section 4 is devoted to the long-time behavior and stability of a priori global solution to \((NSf)\) described in Section 2. The last section is devoted to a regularity result via an iteration. In the appendix, we recall several known results and properties of solutions in Besov spaces.

2. Notations and Main Results

Let us first recall the definition of Besov spaces, in dimension \( d \geq 1 \).

**Definition 2.1.** Let \( \phi \) be a function in \( S(\mathbb{R}^d) \) such that \( \hat{\phi} = 1 \) for \( |\xi| \leq 1 \) and \( \hat{\phi} = 0 \) for \( |\xi| > 2 \), and define \( \phi_j := 2^j \hat{\phi}(2^j x) \). Then the frequency localization operators are defined by

\[ S_j := \phi_j \ast \cdot, \quad \Delta_j := S_{j+1} - S_j. \]

Let \( f \) be in \( S'(\mathbb{R}^d) \). We say \( f \) belongs to \( \dot{B}^s_{p,q} \) if

1. the partial sum \( \sum_{j=-m}^{m} \Delta_j f \) converges to \( f \) as a tempered distribution if \( s < \frac{d}{p} \) and after taking the quotient with polynomials if not, and
2. \( \|f\|_{\dot{B}^s_{p,q}} := \|2^j \Delta_j f\|_{L^p_x} < \infty. \)

We refer to [14] for the introduction of the following type of space in the context of the Navier-Stokes equations.

**Definition 2.2.** Let \( u(\cdot,t) \in \dot{B}^s_{p,q} \) for a.e. \( t \) in \((t_1, t_2)\) and let \( \Delta_j \) be a frequency localization with respect to the \( x \) variable (see Definition 2.1). We shall say that \( u \) belongs to \( \mathcal{L}^p([t_1, t_2], \dot{B}^s_{p,q}) \) if

\[ \|u\|_{\mathcal{L}^p([t_1, t_2], \dot{B}^s_{p,q})} := \|2^j \Delta_j u\|_{L^p((t_1, t_2] \times \mathbb{R}^d)} < \infty. \]

Note that for \( 1 \leq \rho_1 \leq q \leq \rho_2 \leq \infty \), we have

\[ \mathcal{L}^{\rho_1}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1, t_2], \dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^{\rho_2}([t_1, t_2], \dot{B}^s_{p,q}). \]

Let us introduce the following notation (also used in [17]): we define \( s_p := -1 + \frac{d}{p} \) and

\[ \mathbb{L}^{a,b}_p(t_1, t_2) := \mathcal{L}^{a}((t_1, t_2]; \dot{B}^s_{p,q}) \cap \mathcal{L}^{b}((t_1, t_2]; \dot{B}^s_{p,q}); \mathbb{L}^{a,b}_p(t_1, t_2) := \mathbb{L}^{a,b}_p(t_1, t_2) \cap \mathbb{L}^{a,b}_p(t_1, t_2) \]

\[ \mathbb{L}^a_p(t_1, t_2) := \mathbb{L}^a_p(a(t_1, t_2), \mathbb{L}^a_p(T) := \mathbb{L}^{a,b}_p(0, T) \text{ and } \mathbb{L}^a_p(T < T^*) := \cap_{T < T^*} \mathbb{L}^{a,b}_p(T). \]

**Remark 2.3.** We point out that according to our notations, \( u \in \mathbb{L}^{a,b}_p(T < T^*) \) merely means that \( u \in \mathbb{L}^{a,b}_p(T) \) for any \( T < T^* \) and does not imply that \( u \in \mathbb{L}^{a,b}(T^*) \) (the notation does not imply any uniform control as \( T \nearrow T^* \)).

**Definition 2.4.** Let \( p \geq 3 \). Kato’s space is defined as follows,

\[ K_p := \{ u \in C(\mathbb{R}_+, L^p(\mathbb{R}^3)) : \|u\|_{K_p} := \sup_{t > 0} t^{\frac{3}{p} - \frac{3}{2}} \|u(t)\|_{L^p(\mathbb{R}^3)} < \infty \}. \]

In this paper we are also interested in the weak-strong uniqueness of solutions to \((NSf)\). We introduce the following notations. We define that for any \( T \in \mathbb{R}_+ \cup \{+\infty\} \)

\[ E(T) = L^\infty([0, T^*], L^2) \cap L^2([0, T^*], H^1) \]
where 1

Then the re-arrangement function $f^*$ of $f$ is defined by:

$$f^*(t) := \inf \{ s : \lambda_f(s) \leq t \}.$$ 

And for any $1 < p < \infty$, the Lorentz spaces $L^{p,q}$ is defined by:

$$L^{p,q}(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C}, \|f\|_{L^{p,q}} < \infty \},$$

where

$$\|f\|_{L^{p,q}} = \left\{ \begin{array}{ll}
\frac{q}{p} \left[ \int_0^\infty (t^p f^*(t)) dt \right]^{\frac{1}{q}}, & q < \infty,
\sup_{t \geq 0} \{ \frac{1}{t^q} f^*(t) \}, & q = \infty.
\end{array} \right.$$ 

We note that it is standard to use the above as a norm even if it does not satisfy the triangle inequality since one can find an equivalent norm that makes the space into a Banach space. In particular, $L^{p,\infty}$ agrees with the weak-$L^p$ (or Marcinkiewicz space):

$$L^{p,\infty}(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{C}, \|f\|_{L^{p,\infty}} < \infty \},$$

which is equipped the following quasi-norm

$$\|f\|_{L^{p,\infty}} := \sup_{t > 0} [\lambda_f(t)]^{\frac{1}{p}}.$$ 

To deal with external forces and for simplicity of notation we introduce the following space (introduced in [9]),

$$\mathcal{Y} = \{ f \in C(\mathbb{R}_+, S'(\mathbb{R}^3)) : \int_0^t e^{(t-s)\Delta} f(s) ds \in C_w(\mathbb{R}_+, L^{3,\infty}) \}$$

equipped with the norm

$$\|f\|_{\mathcal{Y}} := \sup_{t > 0} \left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L^{3,\infty}}.$$ 

**Remark 2.6.** We mention that $\mathcal{Y}$ contains many rough external forces.

For example,

(1) For every $g \in C_w(\mathbb{R}_+, L^{3,\infty})$, $f := \nabla g \in \mathcal{Y}$ and $\|f\|_{\mathcal{Y}} \lesssim \|g\|_{L^{\infty}(\mathbb{R}_+,L^{3,\infty})}$ (see Lemma 3.2 in [9]).

(2) Every time-independent $f$ satisfying $\Delta^{-1} f \in L^{3,\infty}$ belongs to $\mathcal{Y}$ (see Theorem 4.3 in [9]).

(3) By Lemma 3.4 in [9], $\mathcal{Y}$ contains some really rough external force: $f := (c_1 \delta_0, c_2 \delta_0, c_3 \delta_0)$, where $\delta_0$ stands for the dirac mass.

According to Theorem 2.1 in [9], there exists a constant $\varepsilon_0 > 0$ such that if $f \in \mathcal{Y}$ and $u_0 \in L^{3,\infty}$ satisfy that $\|u_0\|_{L^{3,\infty}} + \|f\|_{\mathcal{Y}} < \varepsilon_0$, there exists a unique solution to $(NSf)$ with initial data $u_0$ and external force $f$, denoted by $NSf(u_0)$, which belongs to $C_w(\mathbb{R}_+, L^{3,\infty})$ such that

$$\|NSf(u_0)\|_{L^{\infty}(\mathbb{R}_+,L^{3,\infty})} \leq 2\|u_0\|_{L^{3,\infty}} + \|f\|_{\mathcal{Y}}.$$ 

In particular, we have $NSf(0) \in C_w(\mathbb{R}_+, L^{3,\infty})$ satisfying

$$\|NSf(0)\|_{L^{\infty}(\mathbb{R}_+,L^{3,\infty})} \leq 2\|f\|_{\mathcal{Y}} < 2\varepsilon_0.$$ 

From now on, we denote $U_f := NSf(0)$.
Now let us state our main results. We first state a local in time existence result for \((NSf)\) for initial data belonging to \(\dot{B}^{s_p}_{p,p}\) for any \(p > 3\) under a smallness assumption on \(f\) depending on \(p\) (it is no loss of generality to set \(p,p\) rather than \(p,q\), which deduces some technical difficulties). Moreover we obtain a local in time existence result for \((NSf)\) in \(L^\infty_t(L^{3,\infty})\) for initial data belonging to \(\dot{B}^{s_p}_{p,p} \cap L^{3,\infty}\).

**Theorem 2.7** (Existence). Let \(p > 3\). There exists a small universal constant \(c(p) > 0\) with the following properties:

Suppose that \(f \in \mathcal{Y}\) is a given external force satisfying that \(\|f\|_\mathcal{Y} < c(p)\). Then

1. For any initial data \(u_0 \in \dot{B}^{s_p}_{p,p}\), a unique maximal time \(T^*(u_0,f) > 0\) and a unique solution \(u_f \in L^r_T[0,T^*] + C_w([0,T^*],L^{3,\infty})\) to \((NSf)\) with initial data \(u_0\) exist such that
   \[ u_f - U_f \in L^r_T[0,T^*]. \]
   with \(r_0 = \frac{2p}{p+1}\). And if \(T^* < \infty\),
   \[ \limsup_{T \to T^*} \|u - U_f\|_{L^r_T[0,T]} = \infty. \]
   Moreover there exists a small constant \(\eta > 0\) depending on \(p\) such that
   \[ \|u_0\|_{\dot{B}^{s_p}_{p,p}} < \eta \Rightarrow T^* = \infty \text{ and } \|u_f - U_f\|_{L^r_T[0,\infty]} \leq C(f)\|u_0\|_{\dot{B}^{s_p}_{p,p}}. \]
2. If \(u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}\), the above solution \(u_f\) to \((NSf)\) with initial data \(u_0\) belongs to \(C_w([0,T^*],L^{3,\infty})\).
3. If \(u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{2}(\mathbb{R}^3)\), the above solution \(u_f\) to \((NSf)\) with initial data \(u_0\) satisfies
   \[ u_f - U_f \in E_{loc}(T^*). \]

Our method is to transform \((NSf)\) into the perturbation equation,
\[
(PNS_{U_f}) \\quad \begin{cases}
\partial_t v - \Delta v + v \cdot \nabla v + U_f \cdot \nabla v + v \cdot \nabla U_f = -\nabla \pi, \\
\nabla \cdot v = 0, \\
v|_{t=0} = v_0 := u_0,
\end{cases}
\]
The corresponding integral form of \((PNS_{U_f})\) is
\[
(4) \quad v = e^{t\Delta} v_0 + B(v,v) + 2B(U_f,v),
\]
where \(B\) is defined as \((3)\). The reason why we focus on \((PNS_{U_f})\) is that \((3)\) allows us to use the classical contraction lemma in the Besov space \(L^r([0,T],\dot{B}^{-1+\frac{3}{p}+\frac{2}{r}}_{p,p})\) with any \(p > 3\) and some \(r > 2\).

Also in order to control the energy estimate, we adopt the argument about the trilinear form \(\int_0^T \int_{\mathbb{R}^3} (a \cdot \nabla b) \cdot c(t) dx dt\) in [18].

From Theorem 2.7 for any \(u_0 \in \dot{B}^{s_p}_{p,p} \cap L^{3,\infty}\), there exists a solution \(u_f \in C_w([0,T^*],L^{3,\infty})\). Actually \(C_w([0,T^*],L^{3,\infty})\) is the highest regularity of solutions to \((NSf)\), as the singularity of \(f\) limits it. Therefore the uniqueness of solutions to \((NSf)\) in \(C_w([0,T^*],L^{3,\infty})\) is crucial. We point out that we cannot prove that the above solution is unique in \(L^\infty_t(L^{3,\infty})\) without the smallness assumption on the solution. Actually even if for \((NS)\) the uniqueness in \(L^\infty_t(L^{3,\infty})\) is still open (the uniqueness just holds when solution is small in \(L^\infty_t(L^{3,\infty})\)). However, we obtain that the above solution is unique in the following sense:

**Theorem 2.8** (Uniqueness). Let \(p > 3\). There exists a universal small constant \(0 < c_1(p) \leq c(p)\) with the following properties:

Suppose that \(f \in \mathcal{Y}\) is a given external force satisfying that \(\|f\|_\mathcal{Y} < c_1\) and \(u_f \in C_w([0,T^*],L^{3,\infty})\) is a solution to \((NSf)\) constructed in Theorem 2.7 with initial data \(u_0 \in L^{3,\infty} \cap \dot{B}^{s_p}_{p,p}\). Then \(u_f\) is unique in the following sense: Assume that \(\tilde{u}_f \in C_w([0,T],L^{3,\infty})\) for some \(T < T^*\) is another solution to \((NSf)\) with same initial data \(u_0\).
If \( u_f - \bar{u}_f \in L^{r,\infty}_T + \{ U(t, x) \in C_w(\mathbb{R}^+, L^{3,\infty}) : \| U \|_{L^{\infty}(\mathbb{R}^+, L^{3,\infty})} < 2c_1 \} \) for some \( 2 < r < \frac{2p}{p-3} \), then \( u_f \equiv \bar{u}_f \) on \([0, T] \).

- if \( u_f - \bar{u}_f \in C([0, T], L^{3,\infty}) \), then \( u_f \equiv \bar{u}_f \) on \([0, T] \).
- if \( 3 < p < 5 \) and \( u_f - \bar{u}_f \in L^{\infty}([0, T], L^{2}) \cap L^2([0, T], H^1) \), then \( u_f \equiv \bar{u}_f \) on \([0, T] \).

We prove Theorem 2.7 and 2.8 in Section 3. Our method depends on an iteration regularity result developed in Section 5.

The global existence of the solutions described in Theorem 2.7 for large initial data \( u_0 \in \dot{B}^{p,p}_{3} \) is still open, even for \( f = 0 \). We mention that even if a solution \( u_f \in C_w([0, T^*], L^{3,\infty}) \) to \((NSf)\) is global, which just means its corresponding life span \( T^* = \infty \), one cannot obtain that \( u_f(t) \) has a uniform bound in \( L^{3,\infty} \) as \( t \) goes to infinity in general. However, if \( u_f \) is a global solution to \((NSf)\) with initial data \( u_0 \in \dot{B}^{p,p}_{3} \) described in Theorem 2.7, the next theorem shows the solution belongs to \( L^{\infty}(\mathbb{R}^+, L^{3,\infty}) \).

Comparing with previous results of long-time behavior, our assumptions on \( u_f \) and \( f \) are all in critical spaces, but the class of initial data is smaller. For example, in [5] C. Bjorland, L. Brandolese, D. Iftimie & M. E. Schonbek proved that if the external force \( f \) is time-independent satisfying that \( \Delta^{-1}f \in L^{3,\infty} \cap L^{4} \) and \( \| \Delta^{-1}f \|_{L^{3,\infty}} \) is small, then for any priori global solution \( u_f \in C_w(\mathbb{R}^+, L^{3,\infty}) \cap L^{1}_{100}(\mathbb{R}^+, L^{4}) \) with initial data \( u_0 = \bar{u}_0 + v_0 \) satisfying that \( v_0 \in L^{4} \) and \( \| v_0 \|_{L^{3,\infty}} \) is smaller than a fixed small constant \( \epsilon \), then \( u_f \in L^{\infty}(\mathbb{R}^+, L^{3,\infty}) \). It clear that the space of initial data they are working on is larger than \( L^{3,\infty} \cap \dot{B}^{p,p}_{3} \) and \( \Delta^{-1}f \in L^{3,\infty} \cap L^{4} \) implies that \( f \in \mathcal{Y} \). However the condition \( \Delta^{-1}f \in L^{3,\infty} \cap L^{4} \) excludes some important singular force: \( \Delta^{-1}f \sim \frac{1}{|x|} \), which belongs to \( \mathcal{Y} \).

**Theorem 2.9** (Long-time behavior of global solutions). Let \( p > 3 \). Suppose that \( f \in \mathcal{Y} \) is a given external force such that \( \| f \|_{\mathcal{Y}} < c_1(p) \), where \( c_1(p) \) is the small constant in Theorem 2.8.

Suppose that \( u_f \in C_w([0, \infty), L^{3,\infty}) \) is an a priori global solution to \((NSf)\) described in Theorem 2.7, whose initial data \( u_0 \in L^{3,\infty} \cap \dot{B}^{p,p}_{3} \). Then there exists a constant \( M \) independent of \( u_f \) such that

\[
\limsup_{t \to \infty} \| u_f(t) \|_{L^{3,\infty}} \leq M.
\]

The idea of the proof of long-time behavior, as in [16, 5], consists in decomposing the initial velocity in a small part plus a square integrable part. The small part remains small by the small data theory and the square-integrable part will become small at some point by using some energy estimates. More precisely, we split the initial data \( u_0 = \bar{u}_0 + v_0 \), where \( \bar{u}_0 \) is small enough in \( L^{3,\infty} \) and \( v_0 \in L^{2}(\mathbb{R}^3) \cap L^{3,\infty} \). By the global existence of \((NSf)\) for small initial data (see [9]) we have \( NSf(\bar{u}_0) \in L^{\infty}(\mathbb{R}^+, L^{3,\infty}) \) and \( v := u_f - NSf(\bar{u}_0) \) satisfies the perturbation equation \( PNS_{NSf}(\bar{u}_0) \).

Compared to the unforced case, it is hard to obtain that \( v \) has finite energy on \([0, T]\) for any \( 0 < T < \infty \) in general, which is the reason why the restriction on external force: \( \Delta^{-1}f \in L^{3,\infty} \cap L^{4} \) is crucial in Theorem 4.7 in [5]. In our case, we have obtained that \( v \) has finite energy on \([0, T]\) for any \( 0 < T < \infty \) by Theorem 2.7.

We show the stability of priori global solutions constructed in Theorem 2.7 in the following theorem.

**Theorem 2.10** (Stability of global solutions). Let \( p > 3 \). Suppose that \( f \in C(\mathbb{R}^+, S'(\mathbb{R}^3)) \) satisfies the same conditions as Theorem 2.9 and that \( u_f \) is an a priori global solution to \((NSf)\) described in Theorem 2.7 with initial data \( u_0 \in \dot{B}^{p,p}_{3} \).

Then there exists an \( \delta \) (depending on \( u_f \)) with the following property.

For any initial data \( \bar{u}_0 \in \dot{B}^{p,p}_{3} \) satisfying \( \| u_0 - \bar{u}_0 \|_{\dot{B}^{p,p}_{3}} < \delta \), there exist a global solution \( \bar{u}_f \) to \((NSf)\) with initial data \( \bar{u}_0 \), and

\[
\| u_f(t) - \bar{u}_f(t) \|_{L^{r,\infty}_T(\mathbb{R})} \lesssim \| u_0 - \bar{u} \|_{\dot{B}^{p,p}_{3}}.
\]
The stability result for the solution introduced as above is an extension of Theorem 3.1 in [16]. We prove it with a similar proof to Theorem 3.1 in [16], the difference between these two cases is that there is a small bounded in time and no-decay in time drift part in our case. The proofs of Theorem 2.7 and 2.8 are presented in Section 4.

3. Existence and uniqueness of \((NSf)\)

The aim of this section is to prove Theorem 2.7 and Theorem 2.8. Let us recall the situation: Let \(p > 3\) be fixed and the external force \(f \in \mathcal{Y}\) and \(\|f\|_{\mathcal{Y}} < c(p)\), where \(c(p)\) is a small universal constant smaller than the constant \(\varepsilon\) in Theorem 2.1 of [3]. The class of initial data is \(\hat{B}_{p,p}^{s}\).

3.1. Existence of \((NSf)\). By Theorem 2.1 in [5], there exists a unique solution \(U_{f} := NSf(0) \in L^{\infty}(\mathbb{R}_{+}, L^{3, \infty})\) such that

\[
\|U_{f}\|_{L^{\infty}(\mathbb{R}_{+}, L^{3, \infty})} \leq 2\|f\|_{\mathcal{Y}} < 2c(p).
\]

Then we can transform the Cauchy problem of \((NSf)\) into the Cauchy problem of \((PNS_{U_{f}})\):

\[
(PNS_{U_{f}}) \left\{ \begin{array}{l}
\partial_{t} v - \Delta v + v \cdot \nabla v + U_{f} \cdot \nabla v = -\nabla \pi, \\
v|_{t=0} = u_{0},
\end{array} \right.
\]

whose integral form is

\[
v(t, x) = e^{t\Delta} u_{0} + B(v, v) + B(2U_{f}, v),
\]

where \(B\) is defined in [3]. We use a standard fixed point lemma to solve the above system: We first recall without proofs the following lemma.

**Lemma 3.1.** Let \(X\) be a Banach space, \(L\) a linear operator from \(X \rightarrow X\) such that a constant \(\lambda < 1\) exists such that

\[
\forall x \in X, \quad \|L(x)\|_{X} \leq \lambda \|x\|_{X},
\]

and \(B\) a bilinear operator such that for some \(\gamma\),

\[
\forall (x, y) \in X^{2}, \quad \|B(x, y)\|_{X} \leq \gamma \|x\|_{X} \|y\|_{X}.
\]

Then for all \(x_{1} \in X\) such that

\[
\|x_{1}\|_{X} < \frac{(1 - \lambda)^{2}}{4\gamma},
\]

the sequence defined by

\[
x^{(n+1)} = x_{1} + L(x^{(n)}) + B(x^{(n)}, x^{(n)})
\]

with \(x^{(0)} = 0\) converges in \(X\) and towards the unique solution of

\[x = x_{1} + L(x) + B(x, x)\]

such that

\[2\gamma \|x\|_{X} \leq (1 - \lambda)\].

In the proof of Theorem 2.7, we first show the local in time existence of \((NSf)\) with initial data in \(\hat{B}_{p,p}^{s}\). Next, we show the propagation of the regularity of the solution constructed above with initial data, in addition, belonging to \(L^{3, \infty}\) or \(L^{2}\).

**Proof of Theorem 2.7.** Let \(u_{0} \in \hat{B}_{p,p}^{s}\) be a divergence-free vector field. We note that \(v\) is the solution satisfying system \((PNS_{U_{f}})\) with initial data \(u_{0}\).

**Existence:** It is clear that if there exists a solution \(v\) to \((PNS_{U_{f}})\) with initial data \(u_{0}\) on \([0, T]\), then \(v + U_{f}\) is a solution to \((NSf)\) with initial data \(u_{0}\). Hence to prove the first statement in Theorem 2.7, it is enough to prove that for any initial data \(u_{0} \in \hat{B}_{p,p}^{s}\), there exists a unique \(T^{*} > 0\) and a unique solution \(v \in L^{3, \infty}_{p}\) to \((PNS_{U_{f}})\) with initial data \(u_{0}\).
Cauchy problem for the incompressible Navier-Stokes equation with an external force

Now we start to prove the above statement by applying Lemma 6.3.

We choose $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ as the Banach space in Lemma 3.1 where $r_0 = \frac{2p}{p-1}$. It is easy to check that $s_p = \frac{2}{r_0} > 0$. To apply Lemma 3.1, we need to obtain that $B(u,v)$ defined in (3) is a continuous bilinear operator from $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}}) \times L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ to $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ and the linear operator $L(v) := B(2U_f,v)$ is continuous on $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ with its norm strictly smaller than 1.

In fact, according to the first statement in Lemma 6.2 and the first statement of Proposition 6.3, we have that $B$ is a continuous operator from $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}}) \times L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ to $L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ and hence, for some $\gamma > 0$

$$\|B(v_1,v_2)\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \leq \gamma \|v_1\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \|v_2\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})}.$$  

According to the third statement in Proposition 6.3 replacing $w$ by $U_f$, we have for any $v \in L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$,

$$\|B(2U_f,v)\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \leq 2C(p)\|U_f\|_{L^{\infty}(\mathbb{R}_+,L^{3,\infty})} \|v\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})},$$

where $C(p) \to \infty$ as $p \to \infty$ and $\frac{1}{p} = \frac{1}{3} + \frac{1}{5}$. By taking $c(p) \leq (4C(p))^{-1}$, then by the above estimate and (3), we have

$$\lambda := 2\gamma_1 C(p) \|U_f\|_{L^{\infty}(\mathbb{R}_+,L^{3,\infty})} < 1,$$

and

$$\|B(2U_f,v)\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \leq \lambda \|v\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})}.$$  

Therefore according to Lemma 3.1 and the fact that

$$\|e^{t\Delta}u_0\|_{L^{0}(\mathbb{R}_+,\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \lesssim \|u_0\|_{\dot{B}_{p,p}^{s+\frac{2}{r_0}}},$$

one can find a small enough number $\eta(p,f)$ such that, for any $u_0 \in \dot{B}_{p,p}^{s+\frac{2}{r_0}}$ with $\|u_0\|_{\dot{B}_{p,p}^{s+\frac{2}{r_0}}} < \eta$, there exists a unique global solution $v \in L^{0}(\mathbb{R}_+,\dot{B}_{p,p}^{s+\frac{2}{r_0}})$ with initial data $u_0$ satisfying that

$$\|v\|_{L^{0}(\mathbb{R}_+,\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \leq \frac{1 - \lambda}{2\gamma}. $$

Moreover we notice that for any given $u_0 \in \dot{B}_{p,p}^{s+\frac{2}{r_0}}$ and any $T > 0$,

$$\|e^{t\Delta}u_0\|_{L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} = \left(\sum_{j \in \mathbb{Z}} \left(2^{j(s_p+\frac{2}{r_0})}\|\Delta_j e^{t\Delta}u_0\|_{L^{0}(\{0,T\};L^{\infty})}\right)^{\frac{p}{p}}\right)^{\frac{1}{p}} = \left\|(1 - e^{-r_0 T \gamma^{22s}})^{\frac{1}{r_0}} 2^{j(s_p+\frac{2}{r_0})}\|\Delta_j u_0\|_{L^p}\right\|_{L^p} = 0.$$  

Next, an application of Lebesgue’s dominated convergence theorem shows that

$$\lim_{t \to 0} \left\|(1 - e^{-r_0 T \gamma^{22s}})^{\frac{1}{r_0}} 2^{j(s_p+\frac{2}{r_0})}\|\Delta_j u_0\|_{L^p}\right\|_{L^p} = 0.$$  

It follows that for any given $u_0 \in \dot{B}_{p,p}^{s+\frac{2}{r_0}}$, there exists $T_0$ such that

$$\|e^{t\Delta}u_0\|_{L^{0}(\{0,T_0\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})} \leq \frac{(1 - \lambda)^2}{4\gamma}. $$

Therefore we have $v \in L^{0}(\{0,T\},\dot{B}_{p,p}^{s+\frac{2}{r_0}})$. 

Hence for any \( u_0 \in \dot{B}^{s_p}_{p,p} \), there exists a \( T^* (u_0, f) > 0 \) such that \( v \in \mathcal{L}^{r_0}([0, T^*), \dot{B}^{s_p}_{p,p}) \). And according to Lemma 6.2, we obtain that \( v \in \mathcal{L}^r([0, T^*); \dot{B}^{s_p+\frac{2}{3}}_{p,p}) \) for any \( r \in [r_0, \infty) \), which implies that \( v \in \mathbb{L}^{r_0, \infty}([T < T^*].)

When \( T^* < \infty \), we claim that

\[
\lim_{T \to T^*} \|\cdot\|_{\mathcal{L}^{r_0, \infty}}(T) = \infty,
\]

by a similar argument in [13]. In fact, if

\[
\lim_{T \to T^*} \|\cdot\|_{\mathcal{L}^{r_0, \infty}}(T) < \infty,
\]
in particular,

\[
v \in \mathcal{L}^\infty([0, T^*), \dot{B}^{s_p}_{p,p})
\]

which implies that for any \( \varepsilon > 0 \), there exists \( N(\varepsilon) \) such that for any \( t' \in [0, T^*) \)

\[
\left( \sum_{|j| > N(\varepsilon)} 2^{2js_p} \left\| \Delta_j v(t') \right\|^2_{L^p} \right)^{\frac{1}{p}} < \varepsilon.
\]

Therefore for any fixed \( t' \in [0, T^*), \)

\[
\| e^{t\Delta} v(t') \|_{\mathcal{L}^{r_0}([0,T];\dot{B}^{s_p+\frac{2}{3}}_{p,p})} \leq \| (1 - e^{-r_0 T\varepsilon_r^{2j}}) \right\|^2_{L^p} 2^{js_p} \| \Delta_j v(t') \|_{L^p} < \varepsilon + 2N(\varepsilon)s_p (1 - e^{-r_0 T}) \| v \|_{L^\infty([0,T^*), \dot{B}^{s_p}_{p,p})},
\]

which implies for any \( t' \in [0, T^*), \)

\[
\| e^{\Delta} v(t') \|_{\mathcal{L}^{r_0}([0,\tau];\dot{B}^{s_p+\frac{2}{3}}_{p,p})} < \left( \frac{1 - \lambda)^2}{4\gamma} \right).
\]

Hence we obtain that \( v \in \mathcal{L}^{r_0}([0, T^* + \tau/2), \dot{B}^{s_p+\frac{2}{3}}_{p,p}), \) which contradicts the maximality of \( T^* \).

To finish the proof of the first statement in Theorem 2.1, we need to prove \( v \) is the unique solution to \((PNS_{U_f})\) with initial data \( u_0 \in \dot{B}^{s_p}_{p,p} \) in \( L^{r_0, \infty}[T < T^*]\). We suppose that \( \tilde{v} \in \mathbb{L}^{r_0, \infty}(T) \) for some \( T < T^* \) is another solution to \((PNS_{U_f})\) with the same initial data \( u_0 \) and set \( w := \tilde{v} - v \). It is easy to check that \( w \) satisfies

\[
w = B(w, w) + B(2(U_f + v), w).
\]

A similar argument as above implies that

\[
\| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} \leq K_0 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} + K_1 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} + \lambda \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})},
\]

for some \( K_0 > 0 \). This fact implies that one can find a \( K_1 > K_0 > 0 \) such that

\[
\| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} \leq K_1 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} + K_1 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})}.
\]

We infer that

\[
(6) \quad \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} (K_1 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} + K_1 \| w \|^2_{L^0_t(\dot{B}^{s_p+\frac{2}{3}}_{p,p})} - 1) \geq 0.
\]
By continuity of the norm of $\mathcal{L}_t^{\gamma_{0}}(\dot{B}^{s_p + \frac{2}{p}}_{p,p})$ with respect to the time, there exists $\bar{T}$ such that for all $t \in [0, \bar{T}]$

$$K_1\|w\|_{\mathcal{L}_t^{\gamma_{0}}(\dot{B}^{s_p + \frac{2}{p}}_{p,p})} + K_1\|v\|_{\mathcal{L}_t^{\gamma_{0}}(\dot{B}^{s_p + \frac{2}{p}}_{p,p})} - 1 < 0.$$  

Therefore, for $t \in [0, \bar{T}]$ relation (6) can hold only if $\|w\|_{\mathcal{L}_t^{\gamma_{0}}(\dot{B}^{s_p + \frac{2}{p}}_{p,p})} = 0$, that is $w \equiv 0$ on $[0, \bar{T}]$, by continuity again, $w \equiv 0$ on $[0, T]$ for any $T < T^*$.

The first statement of the theorem is proved.

**Propagation of perturbations:**

Next we turn to show the propagation of $v$. According to Theorem 3.1 by choosing $w = U_f$ and $\bar{w} = 0$, we have that $v$ can be written as, for any $T \in [0, T^*)$

$$v = v^H + v^S,$$

where $v^H = H_{N_0} \in L^1_p(\infty)$ and $v^S = W_{N_0} + Z_{N_0} \in L^\infty([0, T], L^{3,\infty})$ with $N_0$ being the largest integer such that $3(N_0 - 1) < p$. We first notice that in the case when $\bar{w} = 0$, $H_N$ is a sum of a finite number of multilinear operators of order at most $N - 1$, acting on $e^{t\Delta}u_0$ only. Hence according to Lemma 6.6 and an inductive argument, we obtain for any $N \geq 2$,

$$H_N \in L^\infty([\mathbb{R}_+, L^{3,\infty}),$$

which implies that $v^H \in L^\infty([\mathbb{R}_+, L^{3,\infty}).$

To prove the second statement of the theorem, we are left with the proof of $v \in C_w([0, T^*), L^{3,\infty})$. We notice that by Lemma 2 & 3 in [2], $e^{t\Delta}u_0 \in C_w([0, \infty), L^{3,\infty})$. This fact combined with Lemma 3.6 implies that for any $T \in [0, T^*)$

$$v = v^H + v^S = H_{N_0} + W_{N_0} + Z_{N_0} \in C_w([0, T^*), L^{3,\infty}).$$

The second statement of Theorem 2.7 is proved.

**Finite energy of perturbations:**

In the last part of the proof, we show that $v$ has finite energy on $[0, T]$ for any $T < T^*$, if $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2$.

Now we suppose that $u_0 \in \dot{B}^{s_p}_{p,p} \cap L^2$ and $T \in [0, T^*)$ is fixed. We recall that

$$v = e^{t\Delta}u_0 + B(v, v) + B(2U_f, v).$$

It is clear that $e^{t\Delta}u_0 \in E(\infty)$. Hence we only need to prove $B(v, v) + B(2U_f, v) \in E(T)$.

By replacing $v$ of $B(v, v) + B(2U_f, v)$ by $v^H + v^S$, we have

$$B(v, v) + B(2U_f, v) = B(v^H, v^H + 2v^S + 2U_f) + B(v^S, v^S + 2U_f).$$

By applying Lemma 6.8 and the fact that $e^{t\Delta}u_0 \in E(\infty)$, we first obtain $v^H = H_{N_0} \in E(\infty)$. Again by Lemma 6.8 we obtain that

$$B(v^H, v^H + 2v^S + 2U_f) \in E(T),$$

provided that $v^H \in L^1_p(\infty)$ and $v^S + U_f \in L^\infty([0, T], \dot{B}^{s_p}_{p,p})$ where $q = \frac{3p}{p-2}$.

Now we turn to the proof of $B(v^S, v^S + 2U_f) \in E(T)$. We recall that

(7) \[ v^S + 2U_f \in L^\infty([0, T], L^{3,\infty}). \]

On the other hand, by $v^S \in L^p_{p,p}([0, \infty))$ with some $r_0 = \frac{2p}{p-1}$ and some $\bar{p} < 3$, we have

$$v^S \in L^{3,\infty}([0, \bar{T}) \hookrightarrow L^{3,\infty}(T),$$

provided that $\frac{2p}{p-1} < 3$ for any $p > 3$ and standard embedding $L^{3,\infty}([0, \bar{T}) \hookrightarrow L^{3,\infty}(T)$. Hence by Lemma 6.9 we obtain

(8) \[ v^S \in L^2([0, T], L^{6,2}). \]
Thanks to (7) and (8), applying Lemma 6.6, we obtain
\[B(v^S, v^S + 2U_f) \in E(T).\]
Therefore we obtain \(v \in E(T)\). Theorem 2.7 is proved. \(\square\)

3.2. **Uniqueness of** \((NSf)\). Although the solutions in Theorem 2.7 need not be unique in \(L_t^\infty(L^{3,\infty})\), the following argument shows that the gap between two different solutions has infinite energy.

**Proof of Theorem 2.8.** Let \(u_f \in C_w([0, T^*], L^{3,\infty})\) be a solution to \((NSf)\) constructed in Theorem 2.7 with initial data \(u_0 \in L^{3,\infty} \cap B_{r,p}^{s_p}\).

We now prove the first statement in Theorem 2.8: Assume that \(\bar{u}_f \in C_w([0, T], L^{3,\infty})\) for some \(T < T^*\) is another solution to \((NSf)\) with initial data \(u_0\) and satisfies \(w := \bar{u}_f - u_f = w_1 + w_2\), where
\[w_1 \in L_p^r(T) \quad \text{and} \quad \|w_2\|_{L_t^\infty(L^{3,\infty})} < 4c_1\]
for some \(p > 3, 2 < r < \frac{2p}{p-3}\). According to Theorem 2.7, \(u_f\) can be decomposed as
\[u_f = v + U_f,\]
where \(v \in L_p^r(T)\) and \(U_f \in C_w(\mathbb{R}_+, L^{3,\infty})\) with \(\|U_f\|_{L_t^\infty(\mathbb{R}_+, L^{3,\infty})} < 2c_1\).

We notice that \(w\) satisfies:
\[w = B(w, w) + 2B(u_f, w) = B(w_1 + w_2, w) + B(2u_f, w) = B(w_1 + 2v, w) + B(w_2 + U_f, w).\]

On the other hand, we notice that for any \(q < 3,\)
\[\mathcal{L}^\infty([0, T], B_{q,\infty}^{s_q}) \to L^\infty([0, T], L^{3,\infty})\]
combining with \(w_1, v \in L_p^r(T)\) and \(w \in L_t^\infty([0, T], L^{3,\infty})\), using Proposition 6.3 we obtain that, for any \(\tau \in [0, T]\)
\[\|B(w_1 + 2v, w)\|_{L^\infty([0, \tau], L^{3,\infty})} \lesssim \|B(w_1 + 2v, w)\|_{L_t^\infty([0, \tau], B_{r,p}^{s_p})}\]
\[\leq K\|w_1 + 2v\|_{\mathcal{L}^r([0, \tau], B_{r,p}^{s_p} + \frac{2}{r})} \|w\|_{L_t^\infty([0, \tau], L^{3,\infty})}.\]

And according to Lemma 6.6 we obtain that
\[\|B(w_2 + U_f, w)\|_{L_t^\infty([0, \tau], L^{3,\infty})} \lesssim \|w_2 + U_f\|_{L_t^\infty(\mathbb{R}_+, L^{3,\infty})} \|w\|_{L_t^\infty([0, \tau], L^{3,\infty})}.\]
From the smallness of \(w_2\) and \(U_f\), which is
\[\|w_2\|_{L_t^\infty(\mathbb{R}_+, L^{3,\infty})} + \|U_f\|_{L_t^\infty(\mathbb{R}_+, L^{3,\infty})} < 6c_1,\]
we obtain that
\[\|B(w_2 + U_f, w)\|_{L_t^\infty([0, \tau], L^{3,\infty})} \lesssim \|w\|_{L_t^\infty([0, \tau], L^{3,\infty})},\]
provided that \(c_1\) is small enough.

By (7) and (10), we obtain that for any \(\tau \in [0, T]\),
\[\|w\|_{L_t^\infty([0, \tau], L^{3,\infty})} \leq K\|w_1 + 2v\|_{\mathcal{L}^r([0, \tau], B_{r,p}^{s_p} + \frac{2}{r})} \|w\|_{L_t^\infty([0, \tau], L^{3,\infty})}.\]
By continuity of the norm of \(\mathcal{L}_t^r(B_{r,p}^{s_p} + \frac{2}{r})\) with respect to time, there exists \(N\) real numbers \((T_i)_{1 \leq i \leq N}\) such that \(T_1 = 0\) and \(T_N = T\), satisfying that
\[[0, T] = \bigcup_{i=1}^{N-1} [T_i, T_{i+1}] \text{ and } \|w_1 + 2v\|_{\mathcal{L}^r([T_i, T_{i+1}], B_{r,p}^{s_p} + \frac{2}{r})} \leq \frac{1}{2K},\]
for all \(i \in \{1, \ldots, N - 1\}\).
Now we prove that \( w \equiv 0 \) on \([T_i, T_{i+1}]\) for all \( i \in \{1, \ldots, N - 1\} \) by induction. We first notice that

\[
\|w\|_{L^\infty([0,T_2],L^3,\infty)} \leq K\|w_1 + 2v\|_{L^r([0,T_2];L^{p,p+\frac{q}{q-1}})} \|w\|_{L^\infty([0,T_2],L^3,\infty)}
\]

\[
\leq \frac{1}{2}\|w\|_{L^\infty([0,T_2],L^3,\infty)},
\]

which implies that

\[
w \equiv 0 \quad \text{on} \quad [0,T_2].
\]

Now we assume that \( w \equiv 0 \) on \([0,T_k]\) for some \( k \geq 2 \). Hence

\[
1_{[T_k,T]}(t)w = w = B(w_1 + 2v, 1_{[T_k,T]}(t)w) + B(w_2 + U_f, 1_{[T_k,T]}(t)w).
\]

Therefore we have the following bounds for \( w \),

\[
\|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)} = \|w_1\|_{L^\infty([T_k+1],L^3,\infty)}
\]

\[
\leq \frac{1}{2}\|B(w_1 + 2v, w)\|_{L^\infty([0,T_{k+1}],L^3,\infty)} + \frac{1}{2}\|B(w_2 + U_f, w)\|_{L^\infty([0,T_{k+1}],L^3,\infty)}.
\]

Combining with (11), we have

(11) \[
\|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)} \leq \|B(1_{[T_k,T]}(w_1 + 2v), w)\|_{L^\infty([0,T_{k+1}],L^3,\infty)}.
\]

On the other hand, we notice that

\[
B(w_1 + 2v, w) = B(w_1 + 2v, 1_{[T_k,T]}w) = B(1_{[T_k,T]}(w_1 + 2v), w),
\]

again by Lemma 6.3, we obtain that

\[
\|B(w_1 + 2v, w)\|_{L^\infty([0,T_{k+1}],L^3,\infty)} = \|B(1_{[T_k,T]}(w_1 + 2v), w)\|_{L^\infty([0,T_{k+1}],L^3,\infty)}
\]

\[
\leq K\|1_{[T_k,T]}(w_1 + 2v)\|_{L^r([0,T_{k+1}],L^{p,p+\frac{q}{q-1}})} \|w\|_{L^\infty([0,T_{k+1}],L^3,\infty)}
\]

\[
= K\|w_1 + 2v\|_{L^r([T_k,T_{k+1}],L^{p,p+\frac{q}{q-1}})} \|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)}
\]

\[
\leq \frac{1}{2}\|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)}.
\]

Hence, by the above estimate and (11), we have

\[
\|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)} \leq \frac{1}{2}\|w\|_{L^\infty([T_k,T_{k+1}],L^3,\infty)},
\]

which implies that

\[
w \equiv 0 \quad \text{on} \quad [T_k,T_{k+1}].
\]

Then we have \( w \equiv 0 \) on \([0,T]\). The first statement in Theorem 2.8 is proved.

Now we turn to prove the second statement in Theorem 2.8.

Assume that \( \overline{u}_f \in C_w([0,T],L^3,\infty) \) for some \( T < T^* \) is another solution to \((NSf)\) with same initial data \( u_0 \). We denote \( w := \overline{u}_f - u_f \). By the assumption of the theorem, \( w := \overline{u}_f - u_f \in C([0,T],L^3,\infty) \) with \( w(0) = 0 \). We notice that \( w \) satisfies the following equation on \([0,T]\)

\[
w(t) = B(w + 2U_f, w) + B(2v, w),
\]

where \( v := u_f - U_f \in L^p_{T_0}\infty[T < T^*] \). According to Lemma 6.6 we have

\[
\|B(w + 2U_f, w)\|_{L^\infty([0,T],L^3,\infty)}
\]

\[
\leq C\|w\|^2_{L^\infty([0,T],L^3,\infty)} + C\|w\|_{L^\infty([0,T],L^3,\infty)}\|U_f\|_{L^\infty([0,T],L^3,\infty)}
\]

\[
\leq C\|w\|_{L^\infty([0,T],L^3,\infty)}(\|U_f\|_{L^\infty([R_+,L^3,\infty])} + \|w\|_{L^\infty([0,T],L^3,\infty)}).
\]

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According to the continuity of \( w \) in \( L^{3,\infty} \) and the fact that \( w(0) = 0 \), one can choose a \( T_1 \) such that, combined with the smallness of \( U_f \),
\[
\|U_f\|_{L^\infty([0,t],L^{3,\infty})} + \|w\|_{L^\infty([0,t],L^{3,\infty})} \leq \frac{1}{3C},
\]
which implies that
\[
\|B(w + 2U_f, w)\|_{L^\infty([0,T_1],L^{3,\infty})} \leq \frac{1}{3}\|w\|_{L^\infty([0,T_1],L^{3,\infty})}.
\]
By Lemma 6.3 by a similar argument as the above paragraph, we have that for any \( t \in [0, T] \)
\[
\|B(2v, w)\|_{L^\infty([0,t],L^{3,\infty})} \leq C\|v\|_{L^0([0,t],B^{s_p + \frac{2}{p}}_{p,p})} \|w\|_{L^\infty([0,t],L^{3,\infty})}.
\]
By continuity of the norm of \( L^0([0,t],B^{s_p + \frac{2}{p}}_{p,p}) \) with respect to the time, there exists \( T_2 > 0 \) such that
\[
C\|v\|_{L^0([0,T_2],B^{s_p + \frac{2}{p}}_{p,p})} < \frac{1}{3},
\]
which implies that
\[
\|B(2v, w)\|_{L^\infty([0,t],L^{3,\infty})} \leq \frac{1}{3}\|w\|_{L^\infty([0,T_2],L^{3,\infty})}.
\]
According to (13) and (14), taking \( T_0 = \min\{T_1, T_2\} \), we have
\[
\|w\|_{L^\infty([0,T_0],L^{3,\infty})} \leq \|B(w + 2U_f, w)\|_{L^\infty([0,T_0],L^{3,\infty})} + \|B(2v, w)\|_{L^\infty([0,T_0],L^{3,\infty})} \leq \frac{2}{3}\|w\|_{L^\infty([0,T_0],L^{3,\infty})},
\]
which implies \( \|w\|_{L^\infty([0,T_1],L^{3,\infty})} \equiv 0 \) on \([0,T_1]\) and, by continuity, \( w \equiv 0 \) on \([0,T]\) too. Therefore we proved the second result in the theorem.

Now we are left with the proof of the last statement of the theorem. Since we need to apply Lemma 6.7 to obtain a uniform energy bound, we set \( 3 < p < 5 \) to make sure that \( v \in L^p([0,T],B^{s_p + \frac{2}{p}}_{p,p}) \) with \( s_p + \frac{2}{p} > 0 \).

Assume that \( \tilde{u}_f \in C_w([0,T],L^{3,\infty}) \) for some \( T < T^* \) is another solution to \((NSf)\) with same initial data \( u_0 \). We denote \( \omega = \tilde{u}_f - u_f \). By the assumption of the theorem, \( \omega \in L^\infty([0,T],L^2) \cap L^2([0,T], \dot{H}^1) \) satisfies the following system:
\[
\begin{cases}
\partial_t \omega - \Delta \omega + \omega \cdot \nabla \omega + u_f \cdot \nabla \omega + \omega \cdot \nabla u_f = -\nabla \pi, \\
\nabla \cdot \omega = 0, \\
\omega|_{t=0} = 0.
\end{cases}
\]
Therefore we have the following energy equation, for any \( t \in (0,T) \),
\[
\|\omega(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \omega(s)\|_{L^2}^2 ds = -2 \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla u_f \cdot \omega dx ds.
\]
According to Theorem 2.4, \( u_f \) can be written as \( u_f = U_f + v \), where \( U_f := NSf(0) \) is the solution to \((NSf)\) with initial data \( 0 \) and \( v \in L^p_{t,\infty}[T < T^*] \) is the solution to \((PNSu_f)\) with initial data \( u_0 \). Therefore we have that
\[
\int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla u_f \cdot \omega dx ds \\
\leq \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla (U_f) \cdot \omega dx ds + |\int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla v \cdot \omega dx ds|.
\]
By Young’s inequality in Lorentz spaces, the first term on the right can be controlled by:

\[
\left| \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla (U_f) \cdot \omega dx ds \right| = \left| \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla \omega \cdot U_f dx ds \right| \\
\leq \int_0^t \| \omega(s) \|_{L^{6,2}} \| \nabla \omega(s) \|_{L^2} \| U_f(s) \|_{L^{3,\infty}} ds.
\]

We observe now that \( \dot{H}^1(\mathbb{R}^3) \rightarrow L^{6,2}(\mathbb{R}^3) \). This embedding follows from the Young inequality for Lorentz spaces after noticing that \((-\Delta)^{-\frac{1}{2}}\) is a convolution operator with a function bounded by \(\frac{1}{|x|^2}\), which therefore belongs to \(L^{6,\infty} \). Hence

\[
\int_0^t \| \omega(s) \|_{L^{6,2}} \| \nabla \omega(s) \|_{L^2} \| U_f(s) \|_{L^{3,\infty}} ds \leq \| U_f \|_{L^\infty(\mathbb{R}_+,L^{3,\infty})} \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds.
\]

Since \(U_f\) is small enough in \(L^\infty(\mathbb{R}_+,L^{3,\infty})\), we obtain

\[
\left| \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla (U_f) \cdot \omega dx ds \right| \leq \frac{1}{3} \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds.
\]

We recall that \(v \in \mathbb{L}^{r_0,\infty}(T)\) with \(3 < p < 5\) and one can take \(r_0 = \frac{2p}{p-\frac{2}{5}}\). This implies \(v \in \mathbb{L}^p([0,T], \hat{B}^\gamma_{p,p})\) with \(\frac{2}{3} + \frac{2}{p} > 1\). Applying Lemma 6.4, we obtain

\[
\left| \int_0^t \int_{\mathbb{R}^3} \omega \cdot \nabla v \cdot dx ds \right| \leq C \int_0^t \| \omega(s) \|^2_{L^2} \| v(s) \|^p_{\dot{B}^\gamma_{p,p}} ds + \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds.
\]

Then \(w\) satisfies the following energy inequality,

\[
\| \omega(t) \|^2_{L^2} + \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds \leq C \int_0^t \| \omega(s) \|^p_{\dot{B}^\gamma_{p,p}} ds + \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds.
\]

By Gronwall’s inequality and the fact that \(w|_{t=0} = 0\), we get

\[
\| \omega(t) \|^2_{L^2} + \int_0^t \| \nabla \omega(s) \|^2_{L^2} ds \leq 0.
\]

Then \(\omega \equiv 0\) on \([0,T]\), which implies that \(u_f \equiv \bar{u}_f\) on \([0,T]\). Hence we have proved the second statement of Theorem 2.8.

Theorem 2.8 is proved. \(\square\)

4. Long-time Behavior and Stability of Global Solutions

Let \(f\) be a given external force satisfying the assumption of Theorem 2.7. We consider a global in time solution \(u_f\) to \((NSf)\) constructed in Theorem 2.7 with initial data \(u_0 \in L^{3,\infty} \cap \hat{B}^\gamma_{p,p}\). Also we are interested in the stability of this kind of global solutions.

4.1. Long-time behavior of global solutions. Now let us start to prove Theorem 2.9. In order to apply a weak-strong argument, we need to use the regularity result in Theorem 3.1 to obtain the local in time part has a local in time finite energy by a similar argument to the proof of the third statement of Theorem 2.8. However, we need to deal with a more complicated drift term than before.

**Proof.** Let \(u_0 \in L^{3,\infty} \cap \hat{B}^\gamma_{p,p}\). Suppose that \(u_f \in C_w(\mathbb{R}_+,L^{3,\infty})\) is a solution to \((NSf)\) with initial data \(u_0\) such that

\[
v := u_f - U_f \in L^p_{\text{loc}}[T < \infty],
\]

where \(U_f := NSf(0)\) and \(r_0 = \frac{2p}{p-\frac{2}{5}}\). By the smallness assumption on \(f\), we have \(U_f \in L^\infty(\mathbb{R}_+,L^{3,\infty})\). Therefore to prove the theorem, we need to prove \(v \in L^\infty(\mathbb{R}_+,L^{3,\infty})\). To achieve this goal, we only
need to prove $v \in L^0_{p;\infty}(\infty)$. More precisely, if $v \in L^0_{p;\infty}(\infty)$, by choosing $T = \infty$, $w = U_f$ and $\tilde{w} = 0$, Theorem 5.1 implies $v$ can be written as

$$v = v^H + v^S,$$

where $v^H = H_{N_0} \in L^1_p(\infty)$ and $v^S = W_{N_0} + Z_{N_0} \in L^\infty(\mathbb{R}^+, L^{3,\infty})$ with $N_0$ being the largest integer such that $3(N_0 - 1) < p$. We recall that in the case when $\tilde{w} = 0$, $v^H = H_{N_0}$ is a sum of a finite number of multilinear operators of order at most $N_0 - 1$, acting on $e^{\Delta} u_0$ only.

Hence according to $u_0 \in L^{3,\infty}$, Lemma 6.6 implies $H_{N_0} \in L^\infty(\mathbb{R}^+, L^{3,\infty})$. Thus $v \in L^\infty(\mathbb{R}^+, L^{3,\infty})$. Now we start to prove that $v \in L^0_p(\infty)$.

We use the method introduced by C. Calderón in [7] to prove results on weak solutions in $L^p$ spaces, and used in [18] in the context of 2D Navier-Stokes equations: we split the initial data into a finite number of multilinear operators of order at most $N_0$, acting on $\tilde{u}$.

We recall that in the case when $\tilde{w} = 0$, the Cauchy problem for the incompressible Navier-Stokes equation with an external force [15]

$$\begin{align*}
\begin{cases}
\partial_t \omega - \Delta \omega + \omega \cdot \nabla \omega + (U_f + \tilde{v}) \cdot \nabla \omega + \omega \cdot \nabla (U_f + \tilde{v}) = -\nabla \pi, \\
\n|\omega|_{t=0} = \omega_0.
\end{cases}
\end{align*}$$

Also $\omega$ can be written as the following integral form

$$\omega = e^{t\Delta} \omega_0 + B(\omega, v + \tilde{v} + 2U_f),$$

**Step 1:** We first show that for any $T \in (0, \infty)$, $\omega \in E(T)$. Suppose that $T > 0$ is fixed. We notice that $e^{t\Delta} \omega_0 \in E(T)$ provided $\omega_0 \in L^2$. Applying Theorem 5.1 by taking $w = U_f$ and $\tilde{w} = v$, we obtain that $\omega$ can be written as

$$\omega = \omega^H + \omega^S,$$

where $\omega^H \in L^1_p(\infty)$ and $\omega^S \in \mathbb{L}^{r_0;\infty}_{p,p}$ for some $2 < \tilde{p} < 3$. Therefore we obtain

$$\omega^S \in \mathbb{L}^{3,\infty}_{0,\infty}(T),$$

provided that $r_0 = \frac{2p}{p+1} < 3$ for any $p > 3$. Hence by Lemma 6.8, we have

$$B(\omega^S, v + \tilde{v} + 2U_f) \in E(T),$$

as $v + \tilde{v} + 2U_f \in L^\infty([0,T], L^{3,\infty})$.

We recall that $\omega^H = H_{N_0}^E$, where $H_{N_0}^E$ can be written as

$$H_{N_0}^E = H_{N_0-1}^E + \sum_{M=0}^{N_0-2} B_{N_0-1,N_0-1}^M (v \otimes M, v_L \otimes (N_0 - 1 - M)),$$

where $B_{N_0-1,N_0-1}^M$ are $(N_0 - 1)$-linear operators and $v_L = e^{t\Delta} \omega_0$. We recall that

$$H_2^E = e^{t\Delta} \omega_0 \quad \text{and} \quad H_3^E = H_2^E + B(e^{t\Delta} \omega_0, e^{t\Delta} \omega_0) + B(\tilde{v}, e^{t\Delta} \omega_0).$$

Therefore by Lemma 6.8 and an inductive argument, we obtain that

$$H_{N_0}^E \in E(T),$$

provided that $\omega_0 \in L^2$ and $\tilde{v} \in \mathbb{L}^{p;\infty}_{p}(T)$. Applying Lemma 6.8 again, we have

$$B(\omega^H, v + \tilde{v} + 2U_f) \in E(T),$$

as $v + \tilde{v} + 2U_f \in L^\infty([0,T], \mathbb{L}^{p;\infty}_{p})$ deduced by Lemma 6.3. Therefore we obtain that for any $T \in (0, \infty), \omega \in E(T)$.
\textbf{Step 2:} In this step we show a global energy estimate for \( \omega \). Let us write an energy estimate in \( L^2 \), starting at some \( t_0 \in (0, \infty) \). We get
\[
\|\omega(t)\|_{L^2}^2 + 2 \int_{t_0}^{t} \|\omega(s)\|_{L^2}^2 ds = \|\omega(t_0)\|_{L^2}^2 - 2 \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla (\tilde{v} + U_f)) \cdot \omega \, dx \, ds.
\]
We notice that
\[
| \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla U_f) \cdot \omega \, dx \, ds | \leq \|U_f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)} \int_{t_0}^{t} \|\omega\|_{L^{6,2}} \|\nabla \omega\|_{L^2} \, ds.
\]
We recall that \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3) \), which combined with the above relation implies that
\[
\|U_f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)} \leq 2c_1(p) \text{ with } c_1(p) \text{ is small enough, hence we obtain }
(15) \quad | \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla U_f) \cdot \omega \, dx \, ds | \leq \frac{1}{4} \int_{t_0}^{t} \|\nabla \omega(s)\|_{L^2}^2 \, ds.
\]
On the other hand, by a similar argument as above, we have that \( \bar{v} \) can be written as,
\[\bar{v} = \bar{v}^H + \bar{v}^S,\]
where \( \bar{v}^H \in L^{0,\infty}_p(\mathbb{R}) \) and \( \bar{v}^S \in L^{0,\infty}_{\tilde{p},p}(\mathbb{R}) \) for some \( 2 < \tilde{p} < 3 \). Hence
\[
| \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}) \cdot \omega \, dx \, ds | \leq | \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}^H) \cdot \omega \, dx \, ds | + | \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}^S) \cdot \omega \, dx \, ds |
\]
We recall that \( \bar{v}^H \) is a sum of a finite number of multilinear operators of order at most \( N_0 - 1 \), acting on \( e^{i \Delta} u_0 \) only, as \( \bar{v} \in L^{0,\infty}_p(\mathbb{R}) \) is the small global solution to \((PNSU_f)\), which is the case of \( \bar{v} = 0 \). Then by Lemma \( 5.3 \) (for details see \( 10 \)), we obtain that there exists \( K \) only depending on \( p \),
\[
\sup_{t > 0} \frac{1}{t} \| \bar{v}^H \|_{L^\infty} \lesssim \| \bar{v}_0 \|_{B^p_{p,p}} \leq K_1 \varepsilon(p)
\]
Therefore
\[
| \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla \bar{v}^H) \cdot \omega \, dx \, ds | \leq \int_{t_0}^{t} \|\omega(s)\|_{L^2} \|\nabla \omega\|_{L^2} \|\bar{v}^H\|_{L^\infty} \frac{ds}{\sqrt{s}}
\]
\[
\leq \frac{1}{4} \int_{t_0}^{t} \|\nabla \omega(s)\|_{L^2}^2 \, ds + K_1 \varepsilon^2 \int_{t_0}^{t} \|\omega(s)\|_{L^2}^2 \frac{ds}{\sqrt{s}}.
\]
Again by Theorem \( 5.1 \) we also notice that there exists \( K_1 \) only depending on \( p \)
\[
\| \bar{v}^S \|_{L^{0,\infty}_p(\mathbb{R})} \lesssim \| W_{N_0} \|_{L^{r_0,\infty}_{p_p}(\mathbb{R})} + \| \tilde{Z}_{N_0} \|_{L^{r_0,\infty}_{p_p}(\mathbb{R})} \lesssim \| \bar{v}_0 \|_{B^p_{p,p}} \leq K_1(\varepsilon(p))
\]
Hence we obtain
\[
| \int_{t_0}^{t} \int_{\mathbb{R}^3} (\omega \cdot \nabla (\bar{v}^S)) \cdot \omega \, dx \, ds |
\]
\[
\leq \| \bar{v}^S \|_{L^{0,\infty}_p(\mathbb{R})} \int_{t_0}^{t} \|\omega(s)\|_{L^{6,2}} \|\nabla \omega(s)\|_{L^2} \, ds
\]
\[
\leq \| \bar{v}^S \|_{L^{0,\infty}_p(\mathbb{R})} \int_{t_0}^{t} \|\nabla \omega(s)\|_{L^2}^2 \, ds
\]
\[
\leq K_1(\varepsilon(p)) \int_{t_0}^{t} \|\nabla \omega(s)\|_{L^2}^2 \, ds.
\]
Since $\varepsilon(p)$ is small enough, we have

\begin{equation}
|\int_{t_0}^t \int_{\mathbb{R}^3} (\omega \cdot \nabla (\bar{v}^S) \cdot \omega) dx ds| \leq \frac{1}{4} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2}^2 ds.
\end{equation}

According to \[14\], \[15\] and \[16\], we have the following energy estimate for $w$,

$$
\|w(t)\|_{L^2}^2 + \frac{1}{2} \int_{t_0}^t \|w(s)\|_{L^2}^2 ds \leq \|w(t_0)\|_{L^2}^2 + K^2 \varepsilon^2 \int_{t_0}^t \|\nabla w(s)\|_{L^2}^2 ds.
$$

We now use Gronwall’s Lemma, which yields

$$
\|w(t)\|_{L^2}^2 + \frac{1}{2} \int_{t_0}^t \|w(s)\|_{L^2}^2 ds \leq \|w(t_0)\|_{L^2}^2 \left(\frac{t}{t_0}\right)^{K^2 \varepsilon^2}.
$$

Now by Sobolev embedding and interpolation we have

$$
\int_{t_0}^t \|w(s)\|_{B^{p,p}_{r^p}}^{2} ds \lesssim \int_{t_0}^t \|w(s)\|_{H^{2,r}}^{1} ds \leq \int_{t_0}^t \|w(s)\|_{L^2}^{2} \|\nabla w(s)\|_{L^2}^{2} ds,
$$

which by the above estimate yields

$$(t-t_0) \inf_{s \in [0,t]} \|w(s)\|_{B^{p,p}_{r^p}}^{2} \lesssim \|w(t_0)\|_{L^2}^{2} \left(\frac{t}{t_0}\right)^{2K^2 \varepsilon^2} (t-t_0)^{-\frac{1}{2}}.
$$

Hence we obtain

$$
\inf_{s \in [0,t]} \|w(s)\|_{B^{p,p}_{r^p}}^{2} \lesssim \|w(t_0)\|_{L^2}^{2} \left(\frac{t}{t_0}\right)^{2K^2 \varepsilon^2} (t-t_0)^{-\frac{1}{2}}.
$$

In particular we can write, for all $t \geq t_0 + 1$,

$$
\inf_{s \in [0,t]} \|w(s)\|_{B^{p,p}_{r^p}}^{2} \lesssim \|w(t_0)\|_{L^2}^{2} (t-t_0)^{-\frac{1}{2}}.
$$

which can be made arbitrarily small for $\varepsilon(p)\frac{1}{2K}$ and $t$ large enough. It follows that one can find a time $\tau_0$ such that

$$
\|v(\tau_0)\|_{B^{p,p}_{r^p}} \leq \eta(p).
$$

By Theorem 4.7, we have $v \in \mathbb{L}^{r_0,\infty}_p(\infty)$.

Theorem 4.10 is proved. \qed

4.2. Stability of global solutions. We are now in a position to show the stability of an a priori global solution constructed in Theorem 4.7 let us prove Theorem 4.10.

**Proof.** Suppose that a divergence free vector field $u_0 \in \mathbb{B}^{p,p}_{r^p}$ generating a global solution $u_f \in \mathbb{L}^{r_0,\infty}_p[T < \infty] + C_w(\mathbb{R}_+,L^{3,\infty})$ with $r_0 = \frac{2p}{p-1}$ such that $v := u_f - U_f \in \mathbb{L}^{r_0,\infty}_p[T < \infty]$, where $U_f := NSf(0)$. According to Theorem 4.9 we obtain that actually

$$
v \in \mathbb{L}^{r_0,\infty}_p(\infty).
$$

Now let $\bar{u}_0 \in \mathbb{B}^{p,p}_{r^p}$ be another divergence free vector field. By Theorem 4.7 there exist a $T^*(\bar{u}_0)$ and a solution $\bar{u}_f \in \mathbb{L}^{r_0,\infty}_p[T < T^*(\bar{u}_0)] + C_w(\mathbb{R}_+,L^{3,\infty})$ such that $\bar{u}_f - U_f \in \mathbb{L}^{r_0,\infty}_p[T < T^*(\bar{u}_0)]$. We mention that the life span $T^*(\bar{u}_0)$ is priori finite.

We denote $w := \bar{u}_f - u_f$, then it is enough to prove that for $\|w\|_{t=0} \in \mathbb{B}^{p,p}_{r^p}$ small enough $w \in \mathbb{L}^{r_0,\infty}_p(\infty)$.

The function $w$ satisfies the following system:

$$
\begin{cases}
\partial_t w - \Delta w + w \cdot \nabla w + (v + U_f) \cdot \nabla w + w \cdot \nabla (v + U_f) = -\nabla \pi, \\
\nabla \cdot w = 0, \\
w|_{t=0} = w_0.
\end{cases}
$$
We deduce from Proposition 4.1 in [16] and Lemma [25] & [8] that \( w \) satisfies the following estimate:
\[
\sup_{t \in [0, T]} \| w(t) \|_{\dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta])} + \| w \|_{\dot{L}^{\infty}([\alpha, \beta])} 
\leq K \| w(\alpha) \|_{\dot{B}^{s_p}_{p,p}} + K \| w \|^2_{\dot{L}^{\infty}([\alpha, \beta])} + K \| v \|_{\dot{L}^{\infty}([\alpha, \beta])} \| w \|_{\dot{L}^{\infty}([\alpha, \beta])}
\]
for some constant \( K > 1 \) and all times \( \alpha, \beta \in [0, T] \). Then there exists \( N \) real numbers \( (T_i)_{1 \leq i \leq N} \) such that \( T_1 = 0 \) and \( T_N = \infty \), satisfying
\[
\mathbb{R}_+ = \bigcup_{i=1}^{N} [T_i, T_{i+1}] \text{ and } \| v \|_{\dot{L}^{\infty}([T_i, T_{i+1}], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} \leq \frac{1}{4K}, \quad \forall i \in \{1, \ldots, N-1\}.
\]

Suppose that
\[
\| w_0 \|_{\dot{B}^{s_p}_{p,p}} \leq \frac{1}{8KN(2K)^N}.
\]
Then there exists a maximal time \( T_0 \in \mathbb{R}_+ \cup \{\infty\} \) such that
\[
\| w \|_{\dot{L}^{\infty}([0, T_0], \dot{B}^{s_p}_{p,p})} \leq \frac{1}{4K}.
\]
If \( T = \infty \) then the theorem is proved. Suppose now that \( T_0 < \infty \). Then we can find an integer \( k \in \{1, \ldots, N_1\} \) such that
\( T_k \leq T_0 < T_{k+1} \).
Then we have
\[
\| w \|_{\dot{L}^{\infty}([T_i, T_{i+1}], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} \leq 2K \| w(T_i) \|_{\dot{B}^{s_p}_{p,p}}
\]
which implies that
\[
\sup_{t \in [T_i, T_{i+1}]} \| w(t) \|_{\dot{B}^{s_p}_{p,p}} \leq 2K \| w(T_i) \|_{\dot{B}^{s_p}_{p,p}}.
\]
By induction, we have for all \( i \in \{1, \ldots, k-1\} \),
\[
\| w(T_i) \|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^{i-1} \| w_0 \|_{\dot{B}^{s_p}_{p,p}}.
\]
We conclude from the above two results that
\[
\| w \|_{\dot{L}^{\infty}([T_i, T_{i+1}], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} \leq (2K)^i \| w_0 \|_{\dot{B}^{s_p}_{p,p}}
\]
and
\[
\sup_{t \in [T_i, T_{i+1}]} \| w(t) \|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^i \| w_0 \|_{\dot{B}^{s_p}_{p,p}}.
\]
for all \( i \leq k-1 \). The same arguments as above also apply on the interval \([T_k, T_0]\) and yield
\[
\| w \|_{\dot{L}^{\infty}([T_k, T_0], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} \leq (2K)^k \| w_0 \|_{\dot{B}^{s_p}_{p,p}}
\]
and
\[
\sup_{t \in [T_k, T_0]} \| w(t) \|_{\dot{B}^{s_p}_{p,p}} \leq (2K)^k \| w_0 \|_{\dot{B}^{s_p}_{p,p}}.
\]
On the other hand,
\[
\| w \|_{\dot{L}^{\infty}([0, T_0], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} \leq \sum_{i=1}^{k-1} \| w \|_{\dot{L}^{\infty}([T_i, T_{i+1}], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))} + \| w \|_{\dot{L}^{\infty}([T_k, T_0], \dot{B}^{s_p}_{p,p} + \dot{L}^\infty([\alpha, \beta]))}
\]
\[
\leq N(2K)^k \| w_0 \|_{\dot{B}^{s_p}_{p,p}} < \frac{1}{4K}.
\]
Under assumption (21) this contradicts the maximality of \( T_0 \). Then the theorem is proved.
5. Regularity via iteration

Consider the following equation,

\[
v(t, x) = e^{t \Delta} v_0 + B(v, v) + B(w, v) + B(\bar{w}, v),
\]

where \( B \) is defined in \[3\]. This section is devoted to showing the regularity of the solution to (22) by using an iteration method introduced in \[16, 17\] and we adopt a similar notation in \[17\].

**Theorem 5.1** (Regularity). Let \( p > 3 \) and \( 2 < r_0 < \frac{2p}{p-3} \). And let \( w \in L^\infty(\mathbb{R}_+, L^3, \infty) \) and \( \bar{w} \in L^{r_0, \infty}(\mathbb{R}) \). Suppose that \( v \in L^{r_0, \infty}(T) \) for some \( T > 0 \) satisfies (22) with initial data \( v_0 \in \tilde{B}^{p}_{p,p} \).

Then for any integer \( N \geq 2 \) such that \( 3(N-1) < p \), there are \( H_N \in L^{1, \infty}(\mathbb{R}) \), \( W_N \in L^{r_0, \infty} \) for some \( 2 < \bar{p} < 3 \) and \( Z_N \in L^{r_0, \infty}(T) \) with \( p_N := \frac{p}{N} \) and \( r_N = \max\{1, \frac{r_0}{N}\} \) such that,

\[
v = H_N + W_N + Z_N.
\]

In particular, by taking \( N_0 := \max\{N \in \mathbb{N} : N \geq 2, 3(N-1) < p\} \), we obtain that \( v \) can be written as

\[
v = v^H + v^S,
\]

where \( v^H := H_{N_0} \in L^{1, \infty}(\mathbb{R}) \) and \( v^S := W_{N_0} + Z_{N_0} \in L^{r_0, \infty}(T) \) with \( \bar{p} := \max\{\bar{p}, p_{N_0}\} \).

The argument leading to a similar result to the above theorem in the case \( w = \bar{w} = 0 \) can be found in \[16\] and \[17\] (in turn inspired by \[24\]). The idea of proving Theorem 5.1 is nearly the same as the idea in \[16\] and \[17\]. However, since in our case we need to handle two kinds of drift terms, the decomposition via iteration becomes much more complicated than those results. More precisely, there are two main difference with previous results:

- the fact that one of the drift terms \( w \) does not have decay in time and cannot be approximated by smooth functions limits the decay in time and the regularity of \( W_N \). That is no matter how many times we iterate, there is at least one term of \( W_N \) only belonging to \( L^{r_0, \infty}(T) \).
- Compared with the previous results in the case when \( \bar{w} = 0 \) (for details, see \[16\]), we cannot obtain that \( H_N \) belongs to Kato’s spaces in general.

In the following, we adapt most of the notations in the proof of Lemma 3.3 in \[17\].

**Proof.** Let \( v \in L^{r_0, \infty}(T) \) for some \( T > 0 \) satisfies (22). We can write \( v \) as

\[
v = v_L + B(v, v) + B(w, v) + B(\bar{w}, v),
\]

where

\[
v_L := e^{t \Delta} v_0.
\]

This gives the desired expansion when \( N = 2 \): We note that

\[
v = H_2 + W_2 + Z_2,
\]

where

\[
H_2 = v_L, \quad W_2 = B(w, v) \quad \text{and} \quad Z_2 = B(v, v) + B(\bar{w}, v).
\]

Lemma 3.1 implies that \( H_2 \in L^{1, \infty}(\mathbb{R}) \). According to the second and last statement in Proposition 3.3 we have

\[
\|B(v, v)\|_{L^{r_0, \infty}(T)} \lesssim \|v\|_{L^{r_0, \infty}(T)}^2
\]

and

\[
\|B(\bar{w}, v)\|_{L^{r_0, \infty}(T)} \lesssim \|\bar{w}\|_{L^{r_0, \infty}(T)} \|\bar{w}\|_{L^{r_0, \infty}(\mathbb{R})}.
\]
which implies \( Z_2 \in L^{\frac{q_0\infty}{\bar{p}+1}}(T) \).

Note that the fact that the bilinear term \( B(v, v) \) and linear \( B(\bar{w}, v) \) allow to pass from an \( L^p \) to an \( L^{\frac{q_0\infty}{\bar{p}+1}} \) integrability is a key feature in this proof.

We recall the embedding property \( L^{3,\infty} \hookrightarrow \tilde{B}_{q_0,\infty}^t \) for any \( q > 3 \). Combining with the above property with the last statement of Proposition \([5,3]\) by taking \( q = \frac{3p}{p-2} \), we obtain that

\[
\|B(u, v)\|_{L^{\frac{q_0\infty}{\bar{p}+1}}(T)} \lesssim \|w\|_{L^{\infty}([0,+, L^{3,\infty})} \|v\|_{L^{\frac{q_0\infty}{\bar{p}+1}}(T)}. 
\]

Hence \( W_2 \in L^{\frac{q_0\infty}{\bar{p}+1}}(T) \) with \( \bar{p} = \frac{6p}{2p+1} < 3 \). Therefore we prove Theorem \([5,1]\) in the case \( N = 2 \).

Next we plug the expansion \([24]\) in to the term \( Z_2(v) := B(v, v) + B(\bar{w}, v) \), to find

\[
u = v_L + B(w, v) + B(v, v) + B(\bar{w}, v)
= v_L + B(w, v) + B(\bar{w}, v_L) + B(w, v) + B(v, v) + B(\bar{w}, v)
+ B(u_L + B(w, v) + B(v, v) + B(\bar{w}, v) + v_L + B(w, v) + B(v, v) + B(\bar{w}, v))
= v_L + B(u_L, v_L) + B(\bar{w}, v_L) + B(w, B(\bar{w}, v_L)) + B(w, v) + 2B(v_L, B(w, v))
+ 2B(B(w, v), B(\bar{w}, v)) + 2B(B(w, B(\bar{w}, v) + B(w, v), B(\bar{w}, v)) + B(w, v), B(\bar{w}, v))
+ 2B(v_L, B(v, v)) + B(\bar{w}, B(v, v)) + B(\bar{w}, B(\bar{w}, v)) + 2B(v_L, B(v, v))
+ 2B(B(v, v), B(\bar{w}, v)) + B(B(v, v), B(\bar{w}, v)) + B(B(\bar{w}, v), B(\bar{w}, v)).
\]

This gives the expansion for \( N = 3 \):

\[
v = H_3 + W_3 + Z_3 \text{ with } H_3 = H_2 + B(v_L, v_L) + B(\bar{w}, v_L), \\
W_3 = B(w, v) + B(\bar{w}, B(w, v)) + 2B(v_L, B(w, v))
+ 2B(B(w, v), B(v, v)) + 2B(B(w, B(\bar{w}, v)) + B(w, v), B(\bar{w}, v))
\]

and

\[
Z_3 = 2B(v_L, B(v, v)) + B(\bar{w}, B(v, v)) + B(\bar{w}, B(\bar{w}, v)) + 2B(v_L, B(v, v))
+ 2B(B(v, v), B(\bar{w}, v)) + B(B(\bar{w}, v), B(\bar{w}, v)).
\]

The first statement of Proposition \([5,3]\) implies that \( H_3 \in L^{3,\infty}(T) \) and the expected bounds of \( Z_3 \) follow again from product laws as soon as \( \frac{2}{p} > 3 \). Now we need to check that \( W_3 \in L^{3,\infty}(T) \).

According to the previous arguments, we have \( B(w, v) \in L^{3,\infty}(T) \). Hence we obtain that

\[
B(w, v) \in L^{\infty}([0, T], \tilde{B}_{q_0,\infty}^t), \forall q > 3,
\]

provided that \( \bar{p} < 3 \). Again by the last statement of Proposition \([5,3]\) and taking \( q = \frac{3p}{p-2} \), we have the rest of terms in \( Z_3 \) belong to \( L^{3,\infty}(T) \), which implies that \( W_3 \in L^{3,\infty}(T) \).

Iterating further, the formulas immediately get very long and complicated, so let us argue by induction:

Assume that for any \( 2 \leq N \leq N_0 \), there is an integer \( K_N \geq 0 \), and for any \( 0 \leq k \leq K_N \) some \((N + k)\)-linear operators \( B^M_{N+k,N}(\text{the parameter } M \in \{1, \ldots, N+k\} \text{ measures the number of entries in which } v \text{ and } \bar{w}, \text{ rather than } u_L, \text{ appears and the second parameter in the subscript denotes that the operators are generated in } N\text{th step}) \) such that

\[
v = H_N + W_N + Z_N
\]

with for any \( N \geq 3 \)

\[
H_N = H_{N-1} + \sum_{M=0}^{N-2} B^M_{N-1,N-1}(\bar{w}^M, v^L, v^L)\]

(25)
$Z_N$ may be written as the form

$$Z_N = \sum_{M=1}^{N} \sum_{J+L=M, J \geq 1} B_{N,N}^{M} (v \otimes J, \bar{w} \otimes L, v_L^{(N-M)})$$

(26)

$$+ \sum_{k=1}^{K_N} \sum_{N+k}^{N} \sum_{M=0}^{J+L=M} B_{N+k,N}^{M} (v \otimes J, \bar{w} \otimes L, v_L^{(N+k-M)}),$$

and

$$W_N = \sum_{M=1}^{N-1} \sum_{J+L=M}^{J-1} \sum_{i+j=l+i+j+m=J}^{J-1} \sum_{m=0}^{J} B_{N-1,N-1}^{M} (B(v,v) \otimes i, B(w,v) \otimes m),$$

(27)

$$\sum_{J \text{ terms}} \sum_{L \text{ terms}} \sum_{N-M \text{ terms}} B_{N+1,N+1}^{M} (\bar{w} \otimes J, \bar{w} \otimes L, v_L^{(N+1-M)}),$$

we have used the following convention: for any $J + L = M$

$$B_{N+k,N}^{M} (u, \cdots , u, v, \cdots , v, w, \cdots , w) := B_{N+k,N}^{M} (u \otimes J, v \otimes L, w \otimes (N-M))$$

Now let us prove that for any $2 \leq N \leq N_0$

$$Z_N = \sum_{M=1}^{N} \sum_{J+L=M}^{J-1} \sum_{i+j=l+i+j+m=J}^{J-1} \sum_{m=0}^{J} B_{N,N}^{M} (B(v,v) \otimes i, B(w,v) \otimes m),$$

(28)

$$\sum_{M=0}^{N-1} B_{N,M}^{M} (\bar{w} \otimes M, v_L^{(N-M)}) + Z_{N+1}$$

where $Z_{N+1}$ can be written in the following way: there exists an integer $K_{N+1} \geq 0$ for all $0 \leq k \leq K_{N+1}$ and $0 \leq M \leq N + 1 + k$, some $N + 1 + k$-linear operators $B_{N+1+k,N+1}^{M}$, such that

$$Z_{N+1} = \sum_{M=1}^{N+1} \sum_{J+L=M, J \geq 1} B_{N+1,N+1}^{M} (v \otimes J, \bar{w} \otimes L, v_L^{(N+1-M)})$$

(29)

$$+ \sum_{k=1}^{K_{N+1}} \sum_{N+k}^{N} \sum_{M=0}^{J+L=M} B_{N+k+1,N+1}^{M} (v \otimes J, \bar{w} \otimes L, v_L^{N+1+k-M}).$$

In order to prove (28) and (29) we just need to use (24) again: replacing $v$ by $v_L + B(w,v) + B(w,v)$ in the argument of $B_{N,N}^{M}$ gives

$$B_{N,N}^{M} (v \otimes J, \bar{w} \otimes L, v_L^{(N-M)})$$

$$= B_{N,N}^{M} (v_L + B(w,v) + B(v,v) + B(\bar{w},v) \otimes J, \bar{w} \otimes L, v_L^{(N-M)})$$

$$= B_{N,N}^{M} (v_L \otimes J, \bar{w} \otimes L, v_L^{(N-M)})$$

$$+ \sum_{l=0}^{J-1} \sum_{i+j=l+i+j+m=J} B_{N+1,N}^{M} (B(v,v) \otimes i, B(w,v) \otimes m, B(\bar{w},v) \otimes i, \bar{w} \otimes L, v_L^{(N-L-i-m)})$$

$$+ \sum_{l=1}^{J} \sum_{i+j=l} B_{N+1,N}^{M} (v \otimes (2i+j), \bar{w} \otimes (L+j), v_L^{(N-L-i)})$$.
Proof. Let 

\[
\| e \|_{H^k} \text{, hence we have }
\]

Also we have

\[
\sum_{k=1}^{K_N} N + k \sum_{M=0}^{J+L=M} B_{N+k,N}^M (v \otimes J, w \otimes L, v_L \otimes (N+k-M))
\]

\[
+ \sum_{M=1}^{N} J+L=M \sum_{j=1+j=l}^{J} \tilde{B}_{N+l,N}^M (v \otimes (2i+j), w \otimes (L+j), v_L \otimes (N-L-l))
\]

after reordering, this proves (28) and (29). Moreover (28) and (29) imply that (25) and (27) hold for the case that \( N = N_0 + 1 \).

To conclude the proof it remains to prove that \( H_N \in L_{\tilde{p}}^{1,\infty}(\mathbb{R}), W_N \in L_{\tilde{p},p}^{0,\infty} \) for some \( 2 < \tilde{p} < 3 \) and \( Z_N \in L_p^{r_N,\infty}(T) \) with \( p_N := \frac{p}{N} \) and \( r_N = \max\{1, \frac{\tilde{p}}{pN} \} \). In fact, the above results again follow from estimates about the heat flow (see Lemma 6.1) and product laws in Proposition 6.3, which are based on a similar argument of the cases that \( N = 2, N = 3 \).

Now we take \( N_0 := \max\{N \in \mathbb{N} : N \geq 2, 3(N-1) < p\} \). It is obvious that \( p_{N_0} = \frac{p}{N_0} < 3 \), which implies that

\[
v^S = W_{N_0} + Z_{N_0} \in L_{\tilde{p},p}^{1,\infty}(T) \hookrightarrow L^{\infty}([0, T], L^{3,\infty})
\]

provided Lemma 6.4.

Theorem 5.1 is proved. \( \square \)

6. Appendix

6.1. Estimates on the heat equation. For the completeness of our proof, we give standard estimates for the heat kernel in Besov space. A similar result can be found in [12]. We first recall the long-time behavior of heat flow. We mention that the following lemmas only focus on critical Besov spaces.

**Lemma 6.1.** Let \( p, q \in [1, \infty] \) and \( g \in \dot{B}^{sp}_{p,q} \). Then we have that

\[
e^{t\Delta} g \in L_{p,q}^{1,\infty}(\mathbb{R})
\]

and

\[
\lim_{t \to \infty} \| e^{t\Delta} g \|_{\dot{B}^{sp}_{p,q}} = 0.
\]

**Proof.** Let \( g \in \dot{B}^{sp}_{p,q} \). We notice that for any \( j \in \mathbb{Z} \),

\[
\| e^{t\Delta} \Delta g \|_{L^p} \leq e^{-t^{2j}} \| \Delta g \|_{L^p} \leq 2^{-jsp} e^{-t^{2j}} c_{j,q} \| g \|_{\dot{B}^{sp}_{p,q}},
\]

where \( \|(c_{j,q}) \|_{\ell^q} = 1 \). Then for any \( r \in [1, \infty] \), we have

\[
\| e^{t\Delta} \Delta g \|_{L^r(\mathbb{R}, L^q_x)} \leq 2^{-jsp} 2^{-\frac{2j}{q}} c_{j,q} \| g \|_{\dot{B}^{sp}_{p,q}},
\]

which implies that

\[
\| e^{t\Delta} \Delta g \|_{L^r(\mathbb{R}, L^q_x)} \leq 2^{-jsp} e^{-t^{2j}} c_{j,q} \| g \|_{\dot{B}^{sp}_{p,q}}.
\]

Hence we have \( e^{t\Delta} g \in L_{p,q}^{1,\infty}(\mathbb{R}) \).

Moreover for any \( \varepsilon > 0 \), one can choose an integer \( N \) such that for any \( t \geq 0 \)

\[
\left( \sum_{|j| > N} 2^{jqsp} \| e^{t\Delta} \Delta g \|_{L^p}^q \right)^{\frac{1}{q}} < \frac{\varepsilon}{2}.
\]

Also we have

\[
\sum_{|j| \leq N} 2^{jqsp} \| e^{t\Delta} \Delta g \|_{L^p}^q \leq 2^{-jsp} e^{-t^{2N}} c_{j,q} \| g \|_{\dot{B}^{sp}_{p,q}}^q.
\]
hence for the fixed $N$, there exists a $T(\varepsilon) > 0$ such that for any $t > T$,

$$
\left( \sum_{|j| \leq N} 2^{j s_p} \| e^{t \Delta} \Delta_j g \|_{L^p} \right)^{1/2} < \frac{\varepsilon}{2}.
$$

Therefore we have that for any $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$, such that for any $t > T$

$$
\| e^{t \Delta} g \|_{B^{s_p}_{p,q}} < \varepsilon.
$$

The lemma is proved. \qed

**Lemma 6.2.** Let $p \in [1, \infty]$ and $r \in [1, \infty]$. Suppose that $f$ is a function belonging to $L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})$. We denote that, for any $t \in [0, T]$

$$
H(f) := \int_0^t e^{(t-s)\Delta} f(s, \cdot) ds.
$$

Then we have $H(f) \in L^\infty_T(B^{s_p+\frac{2}{p}}_{p,p})$ for any $\bar{r} \geq r$, and

$$
\| H(f) \|_{L^\infty_T(B^{s_p+\frac{2}{p}}_{p,p})} \lesssim \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})}.
$$

Moreover, if $r < \infty$,

$$
\lim_{t \to \infty} \| H(f) \|_{B^{s_p}_{p,p}} = 0.
$$

**Proof.** We first notice that

$$
\| \Delta_j H(f) \|_{L^1_T L^p_x} \leq \| \int_0^t e^{(t-s)\Delta} \Delta_j f(s, \cdot) \|_{L^p_x} ds \|_{L^1_T} \lesssim \| \int_0^t e^{-c(t-s)^2} \| \Delta_j f(s, \cdot) \|_{L^p_x} ds \|_{L^1_T} \lesssim \| e^{-c t^2} \|_{L^1_T} \| \Delta_j f \|_{L^p_x L^\infty_T},
$$

where $\frac{1}{p} + 1 = \frac{1}{\bar{r}} + \frac{1}{p}$. Since $f \in L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})$ we have

$$
\| \Delta_j f \|_{L^1_T L^p_x} \lesssim 2^{-j(s_p+\frac{2}{p}-2)} d_{j,p} \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})},
$$

where $(d_{j,p}) \in \ell^p$ and $\|(d_{j,p})\|_{\ell^p} = 1$. We also notice that

$$
\| e^{-c t^2} \|_{L^1_T} \lesssim 2^{-\frac{2}{\bar{r}}}. \n$$

Then we have

$$
\| \Delta_j H(f) \|_{L^1_T L^p_x} \lesssim 2^{-j(s_p+\frac{2}{p}+\frac{2}{\bar{r}}-2)} d_{j,p} \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})} = 2^{-j(s_p+\frac{2}{p})} d_{j,p} \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})},
$$

which implies that

$$
\left\| \left( 2^{j(s_p+\frac{2}{p})} \right) \| \Delta_j H(f) \|_{L^1_T L^p_x} \right\|_{\ell^p} \lesssim \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})}.
$$

Thus we proved that $H(f) \in L^\infty_T(B^{s_p+\frac{2}{p}}_{p,p})$ for any $\bar{r} \geq r$, and

$$
\| H(f) \|_{L^\infty_T(B^{s_p+\frac{2}{p}}_{p,p})} \lesssim \| f \|_{L^\infty_T(B^{s_p+\frac{2}{p}-2}_{p,p})}.
$$

Now we suppose that $r < \infty$. First we decompose $H(f)$ into two parts:

$$
H_1(f) := \int_0^\frac{t}{2} e^{(t-s)\Delta} f(s, \cdot) ds,
$$
We first take Proposition 6.3. We notice that \( H_1(f) \) can be written as

\[
H_1(f) = e^{it \Delta} \int_0^t e^{(t-s) \Delta} f(s, \cdot) ds = e^{it \Delta} H(f)(\frac{t}{2}).
\]

According the above argument, we have, for any \( t > 0 \), \( H(f)(\frac{t}{2}) \in \dot{B}^{s_p} \). Applying Lemma 6.1, we have

\[
\lim_{t \to \infty} \| e^{it \Delta} H(f)(\frac{t}{2}) \|_{\dot{B}^{s_p}} = 0.
\]

Now we turn to \( H_2(f) \), we have

\[
\| \Delta_j H_2(f) \|_{L^p} \lesssim \int_0^t e^{-2^{j}(t-s)} \| \Delta_j f(s) \|_{L^p} ds \lesssim 2^{j(\frac{1}{p}-1)} \| \Delta_j f \|_{L^r([t/2, \infty); L^p)},
\]

which implies that

\[
\| H_2(f)(t) \|_{\dot{B}^{s_p}} \lesssim \| f \|_{L^r([t/2, \infty); \dot{B}^{s_p+\frac{2}{p}-2})} \to 0, \text{ as } t \to \infty.
\]

Lemma 6.2 is proved. \( \square \)

6.2. Product laws in Besov spaces. In this paragraph we recall the following product laws in Besov spaces, which use the theory of paraproducts. We only elected to state the results we needed previously, but it should be clear that we have not stated all possible estimates in their greatest generality.

**Proposition 6.3.**

1. Let \( p > 3 \) and \( 2 < r < \frac{2p}{p-3} \). Then there exists a constant \( \gamma > 0 \) such that for any \( v, w \in \mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p}) \), we have

\[
\| vw \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p})} \leq \gamma \| v \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p})} \| w \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p})}.
\]

2. Let \( p_1, p_2 \in (3, \infty) \), \( 2 < r < \frac{2p}{p-3} \), and \( T \in \mathbb{R}_+ \cup \{ \infty \} \). Suppose that \( v \in \mathbb{L}^r_{p_1}(T) \) and \( w \in \mathbb{L}^r_{p_2}(T) \). Then we have

\[
\| vw \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}-1}_{p, p})} \lesssim \| v \|_{\mathbb{L}^r_{p_1}(T)} \| w \|_{\mathbb{L}^r_{p_2}(T)},
\]

where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).

3. Let \( p > 3 \). Suppose that \( w \in \mathcal{L}^\infty([0, T], \dot{B}^{s_p+\frac{2}{p}}_{p, p}) \) and \( v \in \mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p}) \) for some \( T \in \mathbb{R}_+ \cup \{ +\infty \} \). Then we have

\[
\| vw \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}-1}_{p, p})} \leq C(p) \| w \|_{\mathcal{L}^\infty([0, T], \dot{B}^{s_p+\frac{2}{p}}_{p, p})} \| v \|_{\mathcal{L}^r([0, T]; \dot{B}^{s_p+\frac{2}{p}}_{p, p})},
\]

where \( \frac{1}{p} = \frac{1}{3} + \frac{1}{r} \) and \( C(p) \to \infty \) as \( p \to \infty \).

Since the first two results in the proposition are standard and well-known, which can be found in \[12\] \[10\], we only give the proof of the last of the proposition.

**Proof.** For simplicity, we treat \( w \) and \( v \) as functions. We have

\[
\Delta_j wv = \Delta_j T_w v + \Delta_j T_v w + \Delta_j R(u, v).
\]

We first take \( q_1 \) such that \( \frac{1}{p} = \frac{1}{p} + \frac{1}{q_1} = \frac{1}{3} + \frac{1}{q_1} \) implying that \( q_1 = \frac{6p}{2p-3} > 3 \).
About $\Delta_j T_w v$, we have

$$\|\Delta_j T_w v\|_{L^\infty(\mathbb{R}^p)} \lesssim \|(S_j w)(\Delta_j v)\|_{L^\infty(\mathbb{R}^p)} \lesssim \|S_j w\|_{L^\infty(\mathbb{R}^1)} \|v\|_{L^\infty(\mathbb{R}^p)}.$$

And we notice that

$$\|S_j w\|_{L^\infty(\mathbb{R}^1)} \lesssim \sum_{j' \leq j} \|\Delta_j v\|_{L^\infty(\mathbb{R}^1)} \lesssim \sum_{j' \leq j} 2^{-j's_1} C_{j,\infty} \|w\|_{L^\infty([0,T],\dot{B}_{q_1,1}^s)},$$

and

$$\|v\|_{L^\infty(\mathbb{R}^p)} \lesssim 2^{-j(s_p + \frac{2}{r_0})} C_{j,p'} \|v\|_{L^{\infty}([0,T];\dot{B}_{p,p}^{s_p + \frac{2}{r_0}})}.$$
we have that, by applying Lemma 6.4,
\[ 2^{(\sigma_p + \frac{3}{p} - 1)} \| \Delta_j R(w, v) \|_{L^p(\mathbb{R})} \|_{L^p} \lesssim \| w \|_{L^\infty([0,T]; \dot{B}^s_{p,q})} \| v \|_{L^\infty([0,T]; \dot{B}^s_{p,q})} \]
\[ \lesssim \| w \|_{L^\infty([0,T]; L^{3,\infty})} \| v \|_{L^\infty([0,T]; \dot{B}^s_{p,q})}, \]
which is \( R(w, v) \in L^\infty([0,T]; \dot{B}^s_{p,q}). \)

And we have \( R(w, v) \in L^\infty([0,T]; \dot{B}^s_{p,q}), \) as \( \tilde{p} < \bar{p}. \) Combining with (32) and (34) we get
\[ \| uv \|_{L^\infty([0,T]; \dot{B}^s_{p,q})} \leq C(p) \| w \|_{L^\infty([0,T], L^{3,\infty})} \| v \|_{L^\infty([0,T]; \dot{B}^s_{p,q})}, \]
where \( C(p) \to \infty \) as \( p \to \infty. \) The proposition is proved. \( \square \)

We also recall the following standard embedding without proof. For details of the proof, one can check [4] [24].

**Lemma 6.4.** Let \( q_1 < 3 < q_2. \) Then the following embeddings hold:
\( \dot{B}^s_{q_1,\infty} \hookrightarrow L^{3,\infty} \hookrightarrow \dot{B}^s_{q_2}. \)

### 6.3. Properties of the bilinear operator \( B. \)** We show a well-known results on the continuity of \( B(u,v) \) in Kato’s space by using the spatial decay of the convolution kernel appearing in \( B \) (see [21]).

**Lemma 6.5.** Let \( p > 3. \) Suppose that \( u, v \in K_p(\mathbb{R}^3), \)
\[ \| B(u,v) \|_{K_p} \lesssim \| u \|_{K_p} \| v \|_{K_p}. \]
Moreover if \( p > 6, \)
\[ \| B(u,v) \|_{K_\infty} \lesssim \| u \|_{K_\infty} \| v \|_{K_p}. \]

And we recall that \( B \) is a bounded operator from \( L^\infty([0,T], L^{3,\infty}) \times L^\infty([0,T], L^{3,\infty}) \) to \( L^\infty([0,T], L^{3,\infty}) \) for any \( T \in \mathbb{R}_+ \cup \{+\infty\} \) (see [5]).

**Lemma 6.6.** Suppose that \( u,v \in L^\infty([0,T], L^{3,\infty}) \) for some \( T \in \mathbb{R}_+ \cup \{+\infty\}. \) Then
\[ \| B(u,v) \|_{L^\infty([0,T], L^{3,\infty})} \lesssim \| u \|_{L^\infty([0,T], L^{3,\infty})} \| v \|_{L^\infty([0,T], L^{3,\infty})}. \]
Moreover, \( B(u,v) \in C_w([0,T], L^{3,\infty}). \)

The following lemma is a particular case of the result about the continuity of the trilinear form
\[ \int_0^T \int_{\mathbb{R}^3} (a \cdot \nabla b) \cdot c dx dt \] proved by I.Gallagher & F. Planchon in [18].

**Lemma 6.7.** Let \( d \geq 2 \) be fixed, and let \( r \) and \( q \) be two real numbers such that \( 2 \leq 2 < q < +\infty. \) Suppose \( a \in L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, H^1) \) and \( c \in L^q([0,T], \dot{B}^r_{r,q}). \) Then for every \( 0 \leq t \leq T, \)
\[ | \int_0^t \int_{\mathbb{R}^3} (a \cdot \nabla a) \cdot c dx ds | \leq |\nabla a|^2_{L^2(\mathbb{R}_+, L^2)} + C \int_0^t |a(s)|^2_{L^2} |c(s)|^q_{\dot{B}^r_{r,q}} ds. \]

Now we recall that for any \( T \in \mathbb{R}_+ \cup \{+\infty\} \)
\[ E(T) = L^\infty([0,T^*], L^2) \cap L^2([0,T^*], H^1). \]

**Lemma 6.8.** (1) Let \( p > 3 \) and \( T > 0. \) Suppose that \( v \in E(T) \) and \( \bar{v} \in L^\infty([0,T], \dot{B}^s_{p,q}). \) Then \( B(v, \bar{v}) \in E(T). \)
(2) Let \( T \in (0, \infty). \) Suppose that \( v \in L^\infty([0,T], L^{3,\infty}) \) and \( \bar{v} \in L^2([0,T], L^{0,2}). \) Then \( B(v, \bar{v}) \in E(T). \)
Proof. We denote \( w := B(v, \bar{v}) \), which satisfies the system

\[
\begin{align*}
\partial_t w - \Delta w + \bar{v} \cdot \nabla v + v \cdot \nabla \bar{v} &= - \nabla \pi, \\
\nabla \cdot w &= \nabla \cdot v = \nabla \cdot \bar{v} = 0,
\end{align*}
\]

For \( v \in E(T) \) and \( \bar{v} \in L^\infty([0, T], \dot{B}_6^{\alpha_6}) \), by Proposition 4.2 in [10], we obtain that \( B(v, \bar{v}) \in E(T) \).

Hence we are left with the proof of the second statement of the lemma. We now suppose that \( v \in L^\infty([0, T], L^{3, \infty}) \) and \( \bar{v} \in L^2([0, T], L^{6,2}) \).

First let \( J_\epsilon \) be a smoothing operator that multiplies in the frequency space by a cut-off function bounded by 1 which is a smoothed out version of the characteristic function of the annulus \( \{ \epsilon < |\xi| < \frac{1}{\epsilon} \} \). Then we have for any \( t \in [0, T] \)

\[
\| J_\epsilon w(t) \|_{L^2}^2 + 2 \int_0^t \| \nabla J_\epsilon w(s) \|_{L^2}^2 ds = \| w_0 \|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla J_\epsilon^2 w) \cdot \bar{v} dx ds + 2 \int_0^t \int_{\mathbb{R}^3} (\bar{v} \cdot \nabla J_\epsilon^2 w) \cdot v dx ds.
\]

Then for any \( t \in [0, T] \),

\[
\begin{align*}
\left| \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla J_\epsilon^2 w) \cdot \bar{v} + \int_t^T \int_{\mathbb{R}^3} (\bar{v} \cdot \nabla J_\epsilon^2 w) \cdot v dx ds \right| &\leq C \int_0^t \| \nabla J_\epsilon^2 w \|_{L^2} \| \bar{v} \|_{L^{6,2}} \| v \|_{L^{3,\infty}} \\
&\leq \frac{1}{2} \int_0^t \| \nabla J_\epsilon w \|_{L^2}^2 + \frac{C^2}{2} \| v \|_{L^{\infty}(\mathbb{R}_+, L^{3,\infty})}^2 \| \bar{v} \|_{L^2((0,T),L^{6,2})}^2,
\end{align*}
\]

which implies that for any \( t \in [0, T] \)

\[
\| J_\epsilon w(t) \|_{L^2}^2 + \int_0^t \| \nabla J_\epsilon w(s) \|_{L^2}^2 ds \lesssim \| w_0 \|_{L^2}^2 + \| v \|_{L^{\infty}(\mathbb{R}_+, L^{3,\infty})}^2 \| \bar{v} \|_{L^2((0,T),L^{6,2})}^2.
\]

By taking \( \epsilon \to 0 \), we have \( w \in E(T) \).

\[ \square \]

Lemma 6.9. Let \( p > 3 \). Suppose that \( g \in \mathbb{L}^{3,\infty}_{0,\infty}[T < T^*] \) for some \( T^* > 0 \). then we have \( g \in L^2([0, T], L^{6,2}(\mathbb{R}^3)) \) for any \( T < T^* \).

Proof. Suppose that \( g \) is a function belonging to \( \mathbb{L}^{3,\infty}_{0,\infty}[T < T^*] \) for some \( T^* > 0 \). Then for any fixed \( T < T^* \), we have that

\[
\| g \|_{L^3([0,T], \dot{B}_{6,\infty}^{\alpha_6})} \leq T^\frac{1}{3} \| g \|_{L^{\infty}(\mathbb{R}_+, \dot{B}_{6,\infty}^{\alpha_6})}.
\]

Hence we obtain \( g \in L^3([0, T], \dot{B}_{6,\infty}^{\alpha_6}) \cap L^3([0, T], \dot{B}_{6,\infty}^{\alpha_6+\frac{2}{3}}) \). Since that \( \alpha_6 < 0 \) and \( \alpha_6 + \frac{2}{3} > 0 \), Then by using Proposition 2.22 in [10], we have

\[
\| g \|_{L^3([0,T], \dot{B}_{6,1}^{\alpha_6})} \leq \| g \|_{L^3([0,T], \dot{B}_{6,\infty}^{\alpha_6})} \| g \|_{L^3([0,T], \dot{B}_{6,\infty}^{\alpha_6+\frac{2}{3}})} \leq T^\frac{4}{9} \| g \|_{\mathbb{L}^{3,\infty}_{0,\infty}(T)}.
\]

Now we are left with proving that

\[
L^3([0, T], \dot{B}_{6,1}^{0}) \hookrightarrow L^3([0, T], L^{6,2}).
\]

By Littlewood-Paley decomposition,

\[
\| g \|_{L^3([0,T], L^{6,2})} \leq \sum_{j \in \mathbb{Z}} \| \Delta_j g \|_{L^3([0,T], L^{6,2})}.
\]
And $\Delta_j g$ can be written as the following convolution form:

$$\Delta_j g = \Delta_j(\Delta_j g) = 2^{3j} \int_{\mathbb{R}^3} h(2^j (x - y)) \Delta_j g(y) dy.$$ 

By using Young’s inequality,

$$\|\Delta_j g\|_{L^3([0,T],L^{6,2})} \lesssim 2^{3j} \| h(2^j \cdot) \|_{L^3} \| \Delta_j g \|_{L^3([0,T],L^6)}$$

$$\lesssim \|\Delta_j g\|_{L^3([0,T],L^6)} \lesssim c_j \|g\|_{L^3([0,T],B^{0,1}_{6,1})},$$

where $\sum_{j \in \mathbb{Z}} |c_j| = 1$. Then we have

$$\|g\|_{L^3([0,T],L^{6,2})} \lesssim \|g\|_{L^3([0,T],B^{0,1}_{6,1})},$$

which combined with the fact that

$$\|g\|_{L^2([0,T],L^{6,2})} \leq T^{\frac{1}{2}} \|g\|_{L^3([0,T],L^{6,2})}.$$ 

The lemma is proved. □

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