Graph-Based Depth Denoising & Dequantization for Point Cloud Enhancement

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Abstract—A 3D point cloud is typically constructed from depth measurements acquired by sensors at one or more viewpoints. The measurements suffer from both quantization and noise corruption. To improve quality, previous works denoise a point cloud a posteriori after projecting the imperfect depth data onto 3D space. Instead, we enhance depth measurements directly on the sensed images a priori, before synthesizing a 3D point cloud. By enhancing near the physical sensing process, we tailor our optimization to our depth formation model before subsequent processing steps that obscure measurement errors. Specifically, we model depth formation as a combined process of signal-dependent noise addition and non-uniform log-based quantization. The designed model is validated (with parameters fitted) using collected empirical data from a representative depth sensor. To enhance each pixel row in a depth image, we first encode intra-view similarities between available row pixels as edge weights via feature graph learning. We next establish inter-view similarities with another rectified depth image via viewpoint mapping and sparse linear interpolation. This leads to a maximum a posteriori (MAP) graph filtering objective that is convex and differentiable.

This work: Explicit forward modeling + denoising + de-quantization

Fig. 1. Depth measurements from a consumer sensor suffer from signal-dependent noise and variable-size quantization. We enhance multi-view depth measurements as close to the physical sensing process as possible, resulting in higher-quality subsequent 3D point cloud construction.
here we mean performing joint denoising and dequantization based on our proposed depth formation model. In our case, by enhancing near the physical sensing process before steps in a PC synthesis pipeline including projection, registration, stitching, and filtering [17], we can tailor our optimization by parameterizing our proposed depth formation model to a specific sensor. As illustrated in Fig. 1, this means fitting model parameters for signal-dependent noise and variable-size quantization that are unique for individual sensors. This precise modeling of sensed depth measurements is in stark contrast to the grossly inaccurate i.i.d. Gaussian noise typically assumed in the PC denoising literature [7], [8], [9], [10], [11], [12].

Mathematically, we first model depth pixel acquisition as a combined process of signal-dependent noise addition and non-uniform log-based quantization. We verify the validity of this model and fit model parameters through an empirical study using the Intel RealSense™ D435 camera—a representative depth sensor that employs stereo correspondence technology on a pair of IR images to improve depth quality. We parameterize the signal-dependent noise variance as a function of depth from empirical data using the golden-section (GS) search [18] in a maximum likelihood (ML) formulation.

To enhance a row of available pixels in a left depth image—often interleaved with missing pixels— as shown in Fig. 2—we first encode intra-view similarities between neighboring available pixels as edge weights in a graph via feature graph learning [9]. Specifically, estimated 3D features per depth pixel—i.e., 3D coordinate and surface normal—are used to compute feature distance \( d_{ij} \) between pixel pair \((i, j)\), so that edge weight \( w_{ij} = \exp(-d_{ij}) \) can be computed using rich 3D structure information to construct a similarity graph.

We next establish inter-view similarities with a rectified right depth image via viewpoint mapping [20] and sparse linear interpolation. Appropriate approximations via Taylor series expansion lead to a maximum a posteriori (MAP) graph filtering objective that is convex and differentiable. We minimize the objective efficiently using accelerated gradient descent (AGD) [21], [22], where the optimal step size is efficiently approximated via Gershgorin circle theorem (GCT) [23]. Experimental results show that by enhancing depth measurements in the image domain prior to PC synthesis, our method significantly outperformed several recent PC denoising algorithms [7], [10], [11], [12], [24], [25] and representative image denoising schemes [26], [27], [28] in two commonly used PC quality metrics [29], [30] for three different types of depth datasets.

We summarize our technical contributions as follows:

(i) To improve PC construction quality near the physical sensing process, we enhance depth measurements in the image domain, where we can parameterize our depth formation model for a specific sensor.

(ii) We design the depth formation model by combining both signal-dependent noise addition and non-uniform log-based quantization process, validated with collected empirical data from a representative depth sensor.

(iii) We encode intra-view similarities for available pixel pairs in a depth image row as edge weights via feature graph learning, and inter-view similarities via viewpoint mapping and sparse linear interpolation.

(iv) We minimize the resulting convex and differentiable MAP graph filtering objective via AGD, where the important step size is approximated speedily via GCT.

We stress that, while the general approach to denoise/restore observed data as close to the physical layer as possible is not new, our specialization for 3D point cloud is novel. In particular, we are the first in the 3D point cloud literature to propose a joint non-uniform quantization/signal-dependent additive noise model, and a corresponding restoration algorithm designed for the proposed model.

The outline of the paper is as follows. We first overview related works in Section II. We then describe our depth capturing system and the depth formation model in Section III. We also present our empirical study based on collected depth data. The formulation of our optimization problem and the corresponding algorithm to solve it are discussed in Section IV. We discuss the graph learning procedure to capture inter-pixel similarities in Section V. Experimental results and conclusion are presented in Section VI and VII, respectively.

II. RELATED WORKS

We review two categories of related works: depth image denoising/enhancement and 3D point cloud denoising.

A. Depth Image Denoising/Enhancement

1) Depth Image Denoising: Unlike natural image denoising [28], [31], [32], existing literature in depth image denoising is dominated by model-based schemes [26], [33], [34], [35], [36], [37], [38], [39], [40]. Specifically, it is common to perform depth denoising using an assumed signal prior or filter [33], [34], [35], [36], [37], [38]. For example, [33] used a sparsity prior for each pixel patch in the graph Fourier domain.

2) Depth Image Enhancement: A representative image enhancement scheme is the GCT [23].
while [35] employed a regularized shock filter. Both methods exploited the known piecewise smooth (PWS) characteristic of depth images [41] for better performance. When the color image associated to the depth map is available, denoising performance can be further boosted. For example, [39] built neighborhood graphs from color images for depth denoising.

However, these works modeled the depth degradation process with signal-independent noise—e.g., i.i.d. additive Gaussian noise [33], [37], [38]—for simplicity. As discussed in [42], [43] and also verified in our work, depth measurements are noisier for larger depth values; i.e., the noise variance is signal-dependent. Moreover, existing depth image denoising algorithms offer no explicit procedures to handle missing depth values that are common in real-world sensed depth images, as shown in Fig. 2. In contrast, our proposed depth formation model accounts for noise variance’s signal dependency, which we parameterize accurately using collected empirical data of an actual depth sensor in Section III-C. Further, our graph-based optimization enhances available pixels around missing ones by capturing inter-pixel similarity via feature graph learning [9]. We compare against several representative image-denoising schemes [26], [27], [28] in Section VI.

2) Depth Image Dequantization: Depth images are quantized by depth sensors for storage or transmission purposes [44]. To enhance depth precision, previous works adopted different strategies. Given several depth maps from different views, [45] used the Projection On Convex Set (POCS) procedure to enhance precision of two depth images simultaneously, while [26], [41], [46] employed a graph-signal smoothness prior to enhance a single depth image.

Previous research all assumed that depth images are quantized uniformly. However, in practical sensors, larger quantization bin are used as depth values increase, as confirmed by our empirical data in Section III-D. Thus, we propose a non-uniform log-based quantization process, so that larger depth values have coarser quantization.

B. 3D Point Cloud Denoising

PC denoising [47] is necessary prior to 2D viewpoint image rendering or another down-stream task. Related model-based methods can be roughly categorized into four types: moving least square (MLS)-based methods [25], [48], [49], locally optimal projection (LOP)-based methods [50], [51], sparsity-based methods [52], [53] and nonlocal-based methods [7], [54], [55]. Among these works, with the exception of [53] that related the noise variance to surface normal, all assumed independent additive noise. For instance, [7], [52], [55] adopted the additive i.i.d. Gaussian noise model to simplify analysis. As discussed earlier, we model noise on each measurement in a depth image as signal-dependent.

Beyond the signal-dependent nature of acquisition noise, another drawback of existing methods is the assumption that the 3D coordinates of each PC point are omnidirectionally corrupted by additive noise. This is a gross oversimplification: depth measurements are first corrupted by additive noise and quantization perpendicular to the image plane, before 3D projection to synthesize a PC—the resulting distortion in the projected 3D space is far from omnidirectional. In contrast, we enhance depth measurements on an image before 3D projection.

Recent developments of deep neural networks (DNN) have revolutionized different areas in computer vision and image processing, including PC denoising [11], [12]. Beyond the inaccurate assumption of each PC corrupted by i.i.d. Gaussian noise, these learning-based approaches are purely data-driven, and a large amount of labeled data is required for training [56]. However, it is difficult to collect noiseless ground-truth data for supervised learning, since the PCs acquired from depth sensors are always noise-corrupted. One option to circumvent this issue is to synthesize training data via computer graphics [57]. However, it still suffers from inaccurate modeling of noise and missing pixels of actual depth sensors. Different from the learning-based methods, by optimizing only a small set of parameters, our model-based method can easily adapt to new depth sensors with different specifications. Moreover, compared to complex deep models acting like black boxes, the derived filters of our methods are interpretable [58].

III. FORWARD MODEL AND VALIDATION

A. System Overview

We consider a depth-sensing system consisting of two depth sensors from different viewpoints, which are separated by a distance $D$. For systems with two sensors (or one sensor placed at two locations at different time instants), there exists an overlapping field of view (FoV), where the same object surface is observed twice from two different viewpoints as illustrated in Fig. 3. The output of each depth sensor is a depth map of resolution $H \times N$. Each pixel is a noise-corrupted observation of the physical distance between the camera and the object, and is non-uniformly quantized to a finite $B$-bit

Fig. 3. Example of the depth camera system. Two depth cameras with focal length $f$ are placed distance $D$ apart capturing the same 3D scene. Assuming that the corresponding depth images are rectified, a left pixel row $x_l$ maps to a right pixel row $x_r$ via a view-to-view mapping described in Section IV-B, $y_l$ and $y_r$ are corrupted observations.
has coarser quantization, as desired. See Fig. 4(a) for an illustration.

We note that the use of log function in (1) is similar to the μ-law companding algorithm in ITU-T’s G.711 standard for PCM digital communication,\(^6\) where non-uniform quantization is used for audio encoding.

The parameter \(\phi\) is chosen so that \(R(x) \in \{1, \ldots, 2^B\}\) can represent \(2^B\) quantization bins given available \(B\)-bit representation per pixel. Specifically, \(\phi\) is computed as

\[
\phi = \frac{2^B}{\ln(\theta x_{\text{max}} + \rho) - \ln(\theta x_{\text{min}} + \rho)}.
\]

In contrast, \(\theta\) and \(\rho\) are fitting parameters to fit characteristics of a particular depth sensor (to be discussed in Section III-D).

Fig. 4(a) shows \(R(x)\) when \(B = 2\), and there are \(2^B = 4\) quantization bins, i.e., \(R(x) \in \{1, 2, 3, 4\}\).

Given quantization mapping \(R(x)\) in (1), we can now define quantization function \(Q(x)\).

Definition 2 (Quantization Function): We define the quantization function \(Q(x)\) as

\[
Q(x) = \frac{1}{\theta} \left( \exp \left( \frac{R(x) + R_0 - 0.5}{\phi} \right) - \rho \right).
\]

The quantization function in (3) essentially reverses the operations in quantization mapping \(R(\cdot)\) in (1) to recover input \(x\). Given that \(R(x)\) is piecewise constant, \(Q(x)\) is also piecewise constant, as shown in Fig. 4(b). We verify that \(Q(x)\) is indeed a quantization function—the centers of each quantization step coincide with the identity function \(f(x) = x\)—while the quantization bin size is increasing with \(x\) as intended.

C. Signal-Dependent Noise Model

Before quantization \(Q(\cdot)\), we assume that each measurement \(x\) is first corrupted by signal-dependent noise \(n \sim N(0, \sigma^2)\) following a zero-mean Gaussian distribution. Noise variance \(\sigma^2 \in \mathbb{R}^+\) is dependent on signal \(x\). Specifically, as done in [42] and [43], we assume that the standard deviation (SD) \(\sigma\) is a quadratic function of signal \(x\), i.e.,

\[
\sigma = \alpha(x + \mu)^2 + \kappa, \quad \text{for } x \geq -\mu
\]

where parameters \(\alpha > 0, \mu < 0\) and \(\kappa > 0\). It means that \(\sigma\) increases quadratically with signal magnitude.

D. Validation of the Model

To provide a concrete example of the quantization and noise model we just presented, we use the Intel RealSense™ D435 camera to validate this model and fit model parameters through an empirical study. Our chosen Intel sensor is representative of popular depth sensors in the market and shares similar characteristics. Although the working principles of various types of 3D sensors, stereo, structured light, and time of flight (ToF), are slightly different, existing works [60], [61], [62] commonly assumed non-uniform quantization and [43], [63], [64] assumed signal-dependent noise in their depth sensor models. We stress, however, that we are
observations can be made. First, the variance of measurements in the top half of Fig. 6. The obtained measurements are shown as a histogram ranging from 615mm to 1525mm in steps of approximately 100mm. This process was repeated for multiple clusters of the frame until 100 measurements for a single distance were collected from a 5×5 window around the center of each measurement cluster. Second, the quantization step size increases as distance increases. Specifically, at 0.6m, measurements varied in step of 1mm, while at 1.5m, measurements varied in step of 3mm. This justifies the use of a non-uniform quantization function.

2) Theoretical Distribution: We synthesized the theoretical statistics using the image formation model described in (3) to match the corresponding experimental measurements. To generate these synthetic measurements, we estimated the ground truth to be equal to the mean of the corresponding distribution. Then, zero-mean Gaussian noise of variance \( \sigma^2 \) was added to the estimated ground truth followed by quantization using function \( Q(x) \). The SD of the noise \( \sigma \) varied quadratically as described in (4), and the parameters were set as \( \alpha = 1 \times 10^{-5}, \mu = -528 \) and \( \kappa = 1.4 \). Quantization mapping \( R(x) \) as defined in (1) was determined similarly to qualitatively match the bins of experimental distributions. After data fitting, the parameters were found to be \( \phi = 500, \theta = 500, \rho = 200, x_{min} = 10. \) Given \( R(x) \), the corresponding \( Q(x) \) was determined and was plotted in Fig. 7. The figure shows that the function remains close to the \( y = x \) line but on closer inspection, it is a staircase function with quantization bins of increasing size.

The lower half of Fig. 6 shows the histogram of the synthetic measurements. Based on the similarity between the top and bottom halves of Fig. 6, we can conclude that the image formation model described in previous subsection provides a good fit for measurements taken from a real depth sensor.

E. Parameter Estimation for Noise Variance

We next discuss how we compute noise variance \( \sigma^2 \) given observed depth measurements in a cluster from Fig. 6. Denote by \( \mathcal{Y} = \{y_1, y_2, \ldots, y_M\} \) a set of noise-corrupted and quantized observations of ground-truth depth \( x^* \). For each observation 7Note that the variance of measurements and noise variance \( \sigma^2 \) are fundamentally two different quantities, though the former is determined by the latter, and thus are positively correlated.
\( y_i \in \mathbb{R}^+, R(y_i) \in \mathbb{Z}^+ \) is the corresponding quantization mapping value. Thus, \( x^* + n_i \) must reside in the quantization bin indexed by \( R(y_i) \), i.e.,

\[
R(y_i) - \frac{1}{2} \leq \phi \ln(\theta(x^* + n_i) + \rho) - R_0 + \frac{1}{2} < R(y_i) + \frac{1}{2}
\]

(5)

Thus, we can derive that additive noise \( n_i \) is lower- and upper-bounded as follows:

\[
n^-(y_i, x^*) \leq n_i < n^+(y_i, x^*),
\]

\[
n^-(y_i, x^*) \triangleq \vartheta^{-1}(\exp\left(\frac{R(y_i) + R_0 - 1}{\phi}\right) - \rho) - x^*,
\]

\[
n^+(y_i, x^*) \triangleq \vartheta^{-1}(\exp\left(\frac{R(y_i) + R_0}{\phi}\right) - \rho) - x^*.
\]

(6)

Note that \( z^+_i \) and \( z^-_i \) are both function of \( y_i \).

Since the \( y_i \)'s in \( \mathcal{Y} \) are independent, we formulate a maximum likelihood estimation (MLE) problem for variance \( \sigma^2 \):

\[
\max_{\sigma^2} \Pr(\mathcal{Y}|\sigma^2) = \prod_{i=1}^{M} \Pr(y_i|\sigma^2)
\]

\[
= \prod_{i=1}^{M} \Pr\left(n^-(y_i, x^*) \leq n_i < n^+(y_i, x^*)| \sigma^2 \right).
\]

(7)

Since \( n_i \sim \mathcal{N}(0, \sigma^2) \), we can write

\[
\Pr(\mathcal{Y}|\sigma^2) = \prod_{i=1}^{M} \int_{R_i} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{n_i^2}{2\sigma^2}\right) d n_i \triangleq p(\sigma)
\]

(8)

where the region of integration \( R_i \) is

\[
R_i = \{n_i | n^-(y_i, x^*) \leq n_i < n^+(y_i, x^*)\}.
\]

(9)

The probability \( p(\sigma) \) in (8) is difficult to maximize over \( \sigma \) for two reasons. First, each term in the product is of the form \( \int_{R_i} e^{-n_i^2/2\sigma^2} d n_i \), which has no closed-form expression. Second, \( (8) \) involves a product of \( M \) terms, each integrating over a different region \( R_i \). Thus, we propose the following fast search procedure to find a near-optimal \( \sigma \).

For simplicity, first we assume there is only one observation, i.e., \( M = 1 \). Then the optimization problem becomes

\[
\sigma^*_1 = \arg \max_{\sigma} \left(\Phi\left(n^+(y_1, x^*)\right) - \Phi\left(n^-(y_1, x^*)\right)\right)
\]

\[
\approx \arg \max_{\sigma} \left(\left(\frac{n^-(y_1, x^*)}{(n^-(y_1, x^*))^2 + \sigma^2}\right) \exp\left(-\frac{(n^-(y_1, x^*))^2}{2\sigma^2}\right)
\]

\[
- \left(\frac{n^+(y_1, x^*)}{(n^+(y_1, x^*))^2 + \sigma^2}\right) \exp\left(-\frac{(n^+(y_1, x^*))^2}{2\sigma^2}\right)\right) \triangleq \tilde{g}(\sigma)
\]

(10)

where \( \Phi(x) \) is the cumulative distribution function (CDF) of \( n_1 \), and in (10) we use the approximation of \( \Phi(x) \) in [65], i.e.,

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left(-\frac{n_i^2}{2\sigma^2}\right) dn_i
\]

\[
\approx 1 - \frac{1}{\sqrt{2\pi}} \frac{x_\sigma}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right).
\]

(11)

One can show that \( \tilde{g}(\sigma) \) in (10) is a unimodal objective with a single local maximum at \( \sigma^*_1 \), as shown in Fig. 8(a). This means that \( \sigma^*_1 \) can be computed using numerical methods like the Newton’s method [18] to locate \( \sigma^*_1 \) where \( \tilde{g}'(\sigma^*_1) = 0 \).

More generally, when there are \( M > 1 \) observations, the approximate objective \( \tilde{g}(\sigma) \) becomes:

\[
\tilde{g}(\sigma) = \prod_{i=1}^{M} \left(\left(\frac{n^-(y_i, x^*)}{(n^-(y_i, x^*))^2 + \sigma^2}\right) \exp\left(-\frac{(n^-(y_i, x^*))^2}{2\sigma^2}\right)
\]

\[
- \left(\frac{n^+(y_i, x^*)}{(n^+(y_i, x^*))^2 + \sigma^2}\right) \exp\left(-\frac{(n^+(y_i, x^*))^2}{2\sigma^2}\right)\right).
\]

(12)

The optimal \( \sigma^* \) in this general \( M > 1 \) case is lower- and upper-bounded as follows

\[
\min_{i \in \{1, \ldots, M\}} (\sigma^*_i) \leq \sigma^* \leq \max_{i \in \{1, \ldots, M\}} (\sigma^*_i).
\]

(13)

Thus, we can first compute the lower/upper bounds, then use the Golden-section (GS) search [18] to find \( \sigma^* \). GS finds the optimal value \( \sigma^* \) that maximizes/minimizes \( \tilde{g}(\sigma^*) \) via an iterative search, given \( \tilde{g}(\sigma) \) is unimodal.\(^8\) See [18] for details.

Finally, given computed \( \sigma^* \)'s for different clusters with corresponding ground-truth depth \( x^* \)'s, we can parameterize \( \sigma = 1 \times 10^{-3}, \mu = -528 \) and \( \kappa = 1.4 \) in (4) via nonlinear least-squares algorithm [66] as shown in Fig. 8(b).

IV. APPROXIMATION AND OPTIMIZATION

The image formation model in Section III-B is difficult to optimize directly; we first present a practical approximation. We then introduce a mapping function from left-view pixels to right-view pixels assuming FoV overlapped. We next define a likelihood term and a signal prior for individual pixel rows in depth images. Finally, we formulate a MAP optimization problem and derive a corresponding algorithm based on AGD to reconstruct a target pixel row.

\(^8\)Though in general the multiplication of unimodal functions is not necessarily unimodal, in our experiments we found this to be the case.
A. Approximation of Image Formation Model

Additive noise in our image formation model, as discussed in Section III-C, is signal-dependent, which is difficult to address directly. Thus, as a pre-processing step, we first segment a depth image into non-overlapping layers, each of approximately the same depth, and then assume the same fixed noise variance in each layer. This constitutes a reasonable approximation in practice, as a depth image is typically composed of a static indoor background plus one or more foreground objects, each with roughly the same distance to the depth sensor.

To achieve robust image segmentation, we first segment the corrupted depth image into layers using the k-means algorithm, where the optimal number of layers is determined by the Elbow method [67]. We then pre-filter it using bilateral filter (BF) [27] layer by layer, where for each layer we compute the SD of the layer for the range filter parameter, while fixing the domain filter parameter at 3 for all layers. Finally, we perform segmentation again on the pre-filtered depth map. See Fig. 9 for an example.

After segmentation, for each layer \( \xi \) we compute an average depth \( \bar{x_\xi} \) given observed depth pixels of the layer, then use (4) to compute a noise variance \( \sigma^2_\xi \) for the layer. In the sequel, we will assume a constant and pre-computed noise variance \( \bar{\sigma}^2_\xi \) for each depth layer \( \xi \).

B. View-to-View Mapping

Consider two depth images of adjacent viewpoints with FoV overlapped, as shown in Fig. 3. Pixels in the left and right rectified images corresponding to overlapping FoV are projections of the same object surface onto two different camera planes, and thus are related. We optimize a left pixel row of a layer exploiting this inter-view redundancy as follows. (Optimization of a right pixel row is similar and thus omitted.) Specifically, denote by \( x_l \) a row with N available pixels in a layer of the left view, and by \( x_r \) a corresponding sub-row of \( M \) pixels, where \( M \leq N \), capturing the overlapping spatial region. For simplicity, we assume that there is no occlusion in the overlapping region between \( x_l \) and \( x_r \).

Given that the two depth images are rectified, we employ a known 1D warping procedure [20] to relate \( x_l \) and \( x_r \). For the \( i \)-th pixel in the left view, \( x_l(i) \), its (non-integer) horizontal position \( s(i, x_l(i)) \) in the right view after projection is

\[
s(i, x_l(i)) = i - \delta(x_l(i)),
\]

where \( \delta(x_l(i)) \) is the disparity of \( x_l(i) \). Assume \( \sigma_l \) is a constant and pre-computed noise variance to compute a noise variance \( \sigma^2_s \) for each right pixel.

Position of right pixel \( x_r(i) \) is large if the distance between location \( s(i, x_l(i)) \) of mapped left pixel \( x_l(i) \) and location \( j \) of target right pixel \( x_r(j) \) is small. To simplify (16),

\[
\end{align*}

where \( \sigma_l \) is a constant and pre-computed noise variance for each right pixel.


Each integration in (21) is over a Gaussian pdf, which has no closed-form expression. For ease of later optimization, we approximate each Gaussian function $\Pr(n_{l,i})$ as a linear function over the region of integration $\mathcal{R}_{l,i} = [n^-(y_{l,i}, x_{l,i}), n^+(y_{l,i}, x_{l,i})]$,

$$\begin{align*}
\Pr(n_{l,i}) & \approx a_{l,i} n_{l,i} + b_{l,i}, \quad n_{l,i} \in \mathcal{R}_{l,i}.
\end{align*}$$

(23)

$a_{l,i}$ and $b_{l,i}$ are constant scalars computed using Taylor series expansion at the initial estimate $n^0_{l,i} = y_{l,i} - \bar{x}_{l,i}$, where the recovered $\bar{x}_{l,i}$ is updated throughout the algorithm iterations (to be discussed in detail later). See Fig. 11 for an illustration.

The linear approximation (23) is good if i) the integration region $\mathcal{R}_i$ is narrow, or ii) the Gaussian function $\Pr(n_{l,i})$ is flat. When the captured depth pixel $x_{l,i}$ is near, the corresponding quantization bin is small, and hence $\mathcal{R}_i$ is narrow. On the other hand, when $x_{l,i}$ is far, the noise variance $\sigma^2_\xi$, is large, and hence $\Pr(n_{l,i})$ is flat. As we will demonstrate in Section VI, in either case the linear approximation is sufficiently accurate.

Denote by $\bar{l}$ a suitably long canonical vector of all zeros except entry $i$ is 1. We now rewrite (21) as

$$\begin{align*}
\Pr(y_i|x_i) & \approx \prod_{i=1}^{N} \int_{\mathcal{R}_{l,i}} (a_{l,i} n_{l,i} + b_{l,i}) \, d n_{l,i} \\
& = \prod_{i=1}^{N} \left[ \bar{a}_{l,i}^T x_i + \bar{b}_{l,i} \right]
\end{align*}$$

(24)

where $\bar{a}_{l,i}^T = a_{l,i} (z_{l,i}^+ - z_{l,i}^-) l_i^T$ and $\bar{b}_{l,i} = \frac{a_{l,i}}{2} ((z_{l,i}^+)^2 - (z_{l,i}^-)^2) + b_{l,i} (z_{l,i}^+ - z_{l,i}^-)$. (24) is proven in the Appendix.

Similarly, for noise $n_i \in \mathbb{R}^M$, we can write

$$\begin{align*}
\Pr(y_r|x_i) & \approx \prod_{j=1}^{M} \int_{\mathcal{R}_{r,j}} (a_{r,j} n_{r,j} + b_{r,j}) \, d n_{r,j} \\
& = \prod_{j=1}^{M} \left[ \bar{a}_{r,j}^T Hx_i + \bar{b}_{r,j} \right]
\end{align*}$$

(25)

where $\bar{a}_{r,j}^T = a_{r,j} (z_{r,j}^+ - z_{r,j}^-) l_j^T$ and $\bar{b}_{r,j} = \bar{a}_{r,j}^T e + \frac{a_{r,j}}{2} ((z_{r,j}^+)^2 - (z_{r,j}^-)^2) + b_{r,j} (z_{r,j}^+ - z_{r,j}^-)$. (25) is proven in the Appendix.

D. Signal Prior

As done in graph-based image processing work [70], [71], [72], we model the similarities of pixel pairs in $x_i$ using a graph Laplacian matrix $L_d$, and thus prior $\Pr(x_i)$ can be written as

$$\Pr(x_i) = \exp \left( \frac{-x_i^T L_d x_i}{\sigma^2_p} \right)$$

(26)

where $\sigma_p > 0$ is a weight parameter that is a function of noise variance. $\sigma_p$ determines the importance of prior $\Pr(x_i)$ in a MAP formulation. In particular, we define $\sigma_p$ as

$$\sigma^2_p = \frac{1}{g_1 \sigma^2_\xi + g_2}$$

(27)

where $g_1 > 0$ and $g_2$ are empirically fitted constants for a dataset. Recall that $\sigma^2_\xi$ is the noise variance for the layer $\xi$ in which pixels in $x_i$ reside. In words, (27) states that a larger

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This text continues with further details and discussions on the linear approximations, likelihood terms, and signal priors, providing a comprehensive understanding of the methods and derivations used in the paper.
noise variance \( \sigma_n^2 \) leads to a smaller \( \sigma_p^2 \), making prior \( \Pr(x_i) \) more important in a noisy setting.

We assume that the previous \( K \) pixel rows in the left depth image have been enhanced first. Assuming in addition that the next row \( i \) follows a similar image structure, \( \mathbf{L}_i \) can be learned from the previous \( K \) rows. See Section V for details.

E. MAP Formulation

We now formulate a MAP problem for \( \tilde{x}_i \) as follows.

\[
\max_{\tilde{x}_i} \Pr(y_i, y_r | x_i, x_r) \Pr(x_i, x_r)
\]

\[
= \Pr(y_i | x_i, g(x_i)) \Pr(x_i, g(x_i))
\]

\[
= \Pr(y_i | x_i) \Pr(y_r | g(x_i)) \Pr(x_i) \Pr(g(x_i))
\]

\[
\approx \prod_{i=1}^{N} (\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j}) \prod_{j=1}^{M} (\tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j})
\]

\[
\times \exp \left( -\frac{x_i^T L_i x_i}{\sigma_p^2} \right) \exp \left( -\frac{g(x_i)^T L_r g(x_i)}{\sigma_p^2} \right)
\]

where in (29) we substituted \( g(x_i) \) for \( x_r \), and in (30) we split up the first term since left and right noise, \( n_l \) and \( n_r \), are independent. Note that parameters \( \tilde{a}_{i,j}, \tilde{b}_{i,j}, \tilde{a}_{r,j} \) and \( \tilde{b}_{r,j} \) are all computed from Taylor series expansion using the updated \( \tilde{x}_i \) in each algorithm iteration.

To ease optimization, we minimize the negative log of (31):

\[
\min_{x_i} -\ln \prod_{i=1}^{N} (\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j}) - \ln \prod_{j=1}^{M} (\tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j})
\]

\[
+ x_i^T L_i x_i + g(x_i)^T L_r g(x_i)
\]

\[
= -\sum_{i=1}^{N} \ln(\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j}) - \sum_{j=1}^{M} \ln(\tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j})
\]

\[
+ \frac{1}{\sigma_p^2} \left( x_i^T L_i x_i + x_r^T H^T L_r H x_r + 2 e^T L_r H x_r \right).
\]

(32) is an unconstrained convex and differentiable objective, but has no closed-form solution. Thus, we solve for its minimum efficiently using AGD [21], [22]. AGD is an extension of gradient descent (GD) that provably achieves a convergence rate of \( 1/t^2 \) after \( t \) steps in the convex scenario.

To execute AGD efficiently, selecting an appropriate step size is important. We next overview AGD and our choice of step size in our optimization.

F. Optimization of \( x_i \)

For notation simplicity, we rewrite the objective in (32) as

\[
\min_{x_i} f(x_i) = -\sum_{i=1}^{N} \ln(\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j}) - \sum_{j=1}^{M} \ln(\tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j})
\]

\[
+ \frac{1}{\sigma_p^2} \left( x_i^T L_i x_i + x_r^T H^T L_r H x_r + 2 e^T L_r H x_r \right)
\]

where \( \mathcal{L} = \mathbf{L}_i + \mathbf{H}^T \mathbf{L}_r \mathbf{H} \) and \( \mathbf{H} = \mathbf{e}^T \mathbf{L}_r \mathbf{H} \).

Algorithm 1 Accelerated Gradient Descent

Input: Convergence parameter \( \epsilon \), smooth parameter \( \beta \).

1: Initialize \( c_0^t \leftarrow x_i^0 \), \( t \leftarrow 1 \) and \( \eta_0 \leftarrow 0 \).

2: while \( \| \nabla f(x_i^t) \|_2 \geq \epsilon \) do

3: \( c_{i+1}^t \leftarrow x_i^t - \frac{1}{\beta} \nabla f(x_i^t) \).

4: \( \eta_t \leftarrow 1 + \frac{\eta_0}{\tau^2} \).

5: \( \gamma_t \leftarrow \frac{1 - \eta_t}{\eta_t} \).

6: \( x_i^{t+1} \leftarrow (1 - \gamma_t) c_{i+1}^t + \gamma_t c_i^t \).

7: \( t \leftarrow t + 1 \).

8: endwhile

Output: \( x_i^{t+1} \).

We summarize AGD in Algorithm 1. Suppose objective \( f(x_i) \) is \( \beta \)-smooth (i.e., \( \beta \)-Lipschitz),\(^9\) with gradient denoted by \( \nabla f(x_i) \). From (33), \( \nabla f(x_i) \) is

\[
\nabla f(x_i) = -\sum_{i=1}^{N} \tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j} - \sum_{j=1}^{M} \tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j}
\]

\[
+ \frac{2}{\sigma_p^2} (L x_i + h) \quad (34)
\]

At each iteration, AGD first takes a greedy step in the negative gradient direction \( -\nabla f(x_i) \) from previous solution \( x_i^t \) to \( c_i^{t+1} \) with step size \( 1/\beta \). Then, new solution \( x_i^{t+1} \) is a convex combination of \( c_i^{t+1} \) and previously computed \( c_i^t \) in iteration \( t - 1 \), given parameter \( \gamma^t \) determined by AGD.

The only remaining question is how to best estimate smoothness \( \beta \) of \( f(x_i) \) for step size \( 1/\beta \). As discussed in [22], \( \beta \) is the upper bound of the largest eigenvalue of the Hessian matrix \( \Psi \) of \( f(x_i) \). Thus, we first write the Hessian matrix \( \nabla^2 f(x_i) \) of \( f(x_i) \) as

\[
\nabla^2 f(x_i) = \sum_{i=1}^{N} (\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j})^2 + \sum_{j=1}^{M} (\tilde{a}_{r,j}^T H x_i + \tilde{b}_{r,j})^2
\]

\[
+ \frac{2}{\sigma_p^2} \mathcal{L} \quad (35)
\]

Recall that \( \tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j}, \forall i \) are linear functions in (24) that approximate Gaussian integrals over quantization bins. In contrast, matrix \( \tilde{a}_{i,j}, \tilde{a}_{r,j} \) in the numerator of the first summation has a single non-zero entry proportional to \( (z_{i,j}^+ - z_{i,j}^-)^2 \), which is small when compared to \( (\tilde{a}_{i,j}^T x_i + \tilde{b}_{i,j})^2 \). (A similar argument can be made for matrix \( \tilde{a}_{r,j}, \tilde{a}_{r,j} \) of the second summation.) Hence, the two summations in (35) are small compared to \( 2/\sigma_p^2 \mathcal{L} \), and we can approximate \( \nabla^2 f(x_i) \approx (2/\sigma_p^2) \mathcal{L} \).

Instead of using computation-expensive eigen-decomposition, we consider Geršgorin Circle Theorem (GCT) [23] to compute an upper bound of the largest eigenvalue \( \lambda_{\max} \) of \( \mathcal{L} \). By GCT, each eigenvalue \( \lambda \) of \( \Psi \) resides in at least one Geršgorin disc corresponding to row \( i \) of \( \mathcal{L} \), with center \( o_i = \mathcal{L}_{i,i} \) and radius \( r_i = \sum_{j \neq i} |\mathcal{L}_{i,j}| \).

\(^9\)Though log functions are used in (33), each argument is an approximation of Gaussian density integral over a quantization bin in (28), and thus is sufficiently larger than 0. Hence, the log function slopes are upper-bounded.
Thus, $\lambda_{\text{max}}$ must satisfy
\[
\lambda_{\text{max}} \leq \max_i (a_i + r_i)
\]
\[
= \max_i \left( L_{i,i} + \sum_{j \neq i} |L_{i,j}| \right) \triangleq \lambda_{\text{max}}^+(L). \quad (36)
\]
$\lambda_{\text{max}}^+(L)$ can be computed efficiently given $L$ is a sparse matrix. Thus, we can finally compute $\beta \triangleq (2/a_p^2)\lambda_{\text{max}}^+(L)$.

### G. Computation Complexity Analysis

We analyze the computation complexity of our algorithm. Assuming metric matrix $M$ is optimized via feature graph learning (see Section V for details), we compute sparse symmetric graph Laplacian matrices $L_d \in \mathbb{R}^{N \times N}$ and $L_r \in \mathbb{R}^{M \times M}$, where $M \leq N$, with non-zero entries $(i,j)$ iff $i-2 \leq j \leq i+2$. From Section IV-B, we know $H \in \mathbb{R}^{M \times N}$ is also sparse, with non-zero entries $(j,i)$ if $s(i,x_i,j) \in N_j$. Thus, assuming $h \geq 2$, matrix product $L_rH$ has non-zero entries $(i,j)$ if $s(i,x_i,j) \in \bigcup_{k=1}^{2} N_k = N^+$. Since each $i$ of $N$ left pixels is mapped to $O(1)$ neighborhoods $N_j$ of right pixels $j$, $L_rH$ is sparse and has $O(N)$ non-zero entries. Similarly, we can conclude $H^T L_rH$ is also sparse and contains $O(N)$ non-zero entries.

Thus, $L = L_d + H^T L_rH$ is also sparse and contains $O(N)$ non-zero entries, computed in $O(N)$. Similarly, $h^T = e^T L_rH$ contains $O(N)$ non-zero entries. To compute $\lambda_{\text{max}}^+(L)$ in (36), we compute all candidates $i$, which requires accessing each non-zero entry in $L$ exactly once, and thus complexity is $O(N)$. Thus, computing $\tilde{\beta}$ is also $O(N)$.

Given $L$, $h$ and $\tilde{\beta}$, we execute AGD iteratively. In each iteration, we calculate gradient $\nabla f(x_i)$ in (34). Recalling that $\tilde{a}_{i,j}$ has only one single non-zero entry and $H$ is sparse, the cost of computing $H^T \tilde{a}_{i,j}$ and $\tilde{a}_{i,j}Hx_i$ in the second summation can be omitted. Computing $Lx_i$ has complexity $O(N)$ since $L$ has $O(N)$ non-zero entries. Thus, computing $\nabla f(x_i)$ has complexity $O(N + M + N) = O(N)$. Since the number of iterations of AGD is $O(1/\sqrt{\tau})$ [22], executing AGD has $O\left(\frac{N}{\sqrt{\tau}}\right)$ computation. To summarize, combining the computation cost of $L$, $h$, $\beta$ and AGD, we get $O\left(\frac{N + N + N}{\sqrt{\tau}}\right) = O\left(\frac{N}{\sqrt{\tau}}\right)$, which is linear time. Linear-time complexity essentially means accessing each datum once, which constitutes a complexity lower bound for serial data processing. Thus, our depth enhancement algorithm is practical and can potentially be implemented in real-time.

### V. FEATURE GRAPH LEARNING

#### A. Learning Metric for Graph Construction

When pixel row $i$ of the left view is optimized, we assume that the previous $K$ rows, $i-1, \ldots, i-K$, have already been enhanced into $\tilde{x}_i^{-1}, \ldots, \tilde{x}_i^{-K}$. Using these $K$ enhanced rows, we compute graph Laplacian $L_d$ to define prior $Pr(x)$ in (26). Given $K < N$ in practice, estimating $L_d \in \mathbb{R}^{N \times N}$ reliably using only $K$ signal observations is difficult. In particular, statistical graph learning algorithms such as graphical lasso (GLASSO) [73] that compute a sparse precision matrix using as input a reliable empirical covariance matrix estimated from many observations do not work in our scenario.

Instead, we construct an appropriate similarity graph via metric learning [9]. We first assume that associated with each pixel (graph node) $i$ in $x_i$ is a length-$F$ relevant feature vector $f_i \in \mathbb{R}^F$ (to be discussed). The feature distance $d_{ij}$ between two nodes $i$ and $j$ is computed using a real, symmetric and PD metric matrix $M \in \mathbb{R}^{F \times F}$ as
\[
d_{ij} = (f_i - f_j)^T M (f_i - f_j). \quad (37)
\]
(37) is also called the Mahalanobis distance in the machine learning literature [74]. Since $M$ is PD, $d_{ij} > 0$ for $f_i - f_j \neq 0$. The edge weight $u_{ij}$ between nodes $i$ and $j$ is then computed using a Gaussian kernel:
\[
u_{ij} = \exp(-d_{ij}). \quad (38)
\]
Note that, to reduce computation complexity, we construct a sparse graph where each pixel $i$ is only connected to its four closest neighbors $i \pm 1$ and $i \pm 2$. Thus, Laplacian $L$ is sparse with $O(1)$ non-zero entries per row/column.

To optimize $M$, we minimize the graph Laplacian regularizer (GLR) evaluated using $K$ previous pixel rows, i.e.,
\[
\min_{M>0} \sum_{k=1}^{K} \left( \tilde{x}_i^k \right)^T L_d^k (M) \tilde{x}_i^k = \sum_{k=1}^{K} \sum_{i,j} u_{ij}^k \left( \tilde{x}_i^k - \tilde{x}_j^k \right)^2 \quad (40)
\]
where edge weights $u_{ij}^k$ in Laplacian $L_d^k (M)$ is computed using features $\tilde{f}_i^k$ and $\tilde{f}_j^k$ of the $k$-th observation $\tilde{x}_i^k$ via (37) and (38).

To optimize $M$ in (39), [9] proposed a fast algorithm to optimize the diagonal and off-diagonal entries of $M$ alternately. See [9] for details.

#### B. Feature Selection for Metric Learning

To construct a feature vector $f_i$ for each pixel $i$ in $x_i$, we first compute the pixel’s corresponding surface normal $\nu_i \in \mathbb{R}^3$ by projecting it to 3D space and computing it using its neighboring points via a method in [53]. Then, together with depth value $x_i$ and location $l_i \in \mathbb{R}^2$ in the 2D grid, we construct $f_i \in \mathbb{R}^6$. Because $M$ is symmetric, the number of matrix entries we need to estimate is only 21.

### VI. EXPERIMENTATION

#### A. Experimental Setup

We conducted simulations with three types of datasets:

(i) Public synthetic dataset: in [57], three synthetic datasets, FlyingThings, Monkaa and Driving, were introduced. They are generated using Blender,\textsuperscript{10} and the ground-truth depth images of both views are provided. As a pre-processing, we projected the depth images to 3D space, leading to PCs with around 1 million points;

\textsuperscript{10}https://www.blender.org/
For dataset (iii) without ground-truths, we employed a no-scale-free point-to-point (C2C) error [29] and point-to-plane (C2P) error [30], where plane normal in C2P was used PC evaluation metrics for objective evaluation, i.e., the parameter-free point-to-point (C2C) error [29] and point-to-plane (C2P) error [30], where plane normal in C2P was.

We compared our 3D PC enhancement method against three representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR.

When learning the metric for graph construction, we considered the previous $K = 30$ pixel rows. To reduce computation complexity, the same optimized $\mathbf{M}$ was used for the right view when enhancing the left view, and vice versa. Given feature vector in $\mathbf{y}$ of the current row $i$, we computed the corresponding Laplacian $\mathbf{L}(\mathbf{M})$.

We compared our 3D PC enhancement method against three model-based image denoising schemes, BF [27], BM3D [28] and the early version of this work, SINUQ — short for Signal-Independent Noise Uniform Quantization [26]. Similar to our workflow, we applied these three schemes on the noise-corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR methods was applied to the PCs projected from the corrupted and quantized depth image pairs before the PC synthesis steps. We also compared our work with six representative PC denoising algorithms, APSS [24], RIMLS [25], MRPCA [10], GLR [7], PCN [11], and DMR [12]. This set of methods was applied to the PCs projected from the corrupted left and right views. Among these six methods, PCN and DMR.

To make different PCs comparable, we re-centered each PC to the origin then scaled it inside the unit sphere. For datasets (i) and (ii) with ground-truth PC's, we employed two commonly used PC evaluation metrics for objective evaluation, i.e., the parameter-free point-to-point (C2C) error [29] and point-to-plane (C2P) error [30], where plane normal in C2P was computed using six neighboring points (the only parameter). For dataset (iii) without ground-truths, we employed a no-reference metric (i.e., Pseudo MOS) using sparse convolutional neural network, designed specifically for quality assessment of 3D PCs [76]. We include also visual comparisons.

B. Experimental Results

1) Linear Approximation of the Gaussian Pdf: To verify the accuracy of the linear approximation (23) of the Gaussian pdf, we constructed two extreme cases, as shown in Fig. 12. We first consider a close depth pixel $x_{1,1} = 0.5$m and with a small quantization bin, which results in a very narrow region of integration $\mathcal{R}_1$, as shown in Fig. 12(a). We observe that the pdf $\text{Pr}(n_{1,1})$ within $\mathcal{R}_1$ is roughly linear. Specifically, the likelihood term $\text{Pr}(y_{1,1}|x_{1,1})$ in (21) is 0.0098, and its linear approximation in (23) is 0.0083. In contrast, for a far depth pixel $x_{1,1} = 4.5$m, and a large noise variance $\sigma^2_{n_{1,1}}$, the pdf $\text{Pr}(n_{1,1})$ becomes relatively flat, see Fig. 12(b). We observe that $\text{Pr}(n_{1,1})$ within $\mathcal{R}_1$ is roughly linear as well. In this case, the likelihood term $\text{Pr}(y_{1,1}|x_{1,1})$ in (21) is 0.0209, and its linear approximation in (23) is 0.0209. Thus, we can conclude that, in both cases, the linear approximation was sufficiently accurate.

2) Comparison for Public Dataset: In the case of the depth views corrupted by SDGN, quantitative results of different methods in terms of C2C and C2P errors are shown in Table I and II, where the former one provides results for the PCs of the synthetic datasets, and the latter one is for the PCs of the Middlebury dataset. Overall, our method achieved by far the best performance under both metrics for both datasets. For synthetic datasets in Table I, our proposal outperformed the second-best algorithm SINUQ by 0.54 in terms of C2C ($\times 10^{-3}$) and PCN by 0.7 in terms of C2P ($\times 10^{-5}$) on average, respectively. For the Middlebury dataset, ...

![Fig. 12. Gaussian pdf of (a) $x_{1,1} = 0.5$m and (b) $x_{1,1} = 4.5$m, where the red dots are pdf of the lower and upper bounds $n^u(y_{1,1}, x_{1,1})$ and $n^l(y_{1,1}, x_{1,1})$ in the region of integration $\mathcal{R}_1$, respectively, which is an empirical demonstration corresponding to Fig. 11.]

**TABLE I**

| Method | FlyingThings | Monkaas | Driving | Average |
|--------|--------------|---------|---------|---------|
| BF [27] | 1.83 | 0.88 | 2.06 | 1.29 |
| BMSD [28] | 4.77 | 1.13 | 4.11 | 3.34 |
| SINUQ [26] | 1.44 | 0.95 | 1.90 | 1.32 |
| APSS [24] | 2.67 | 1.00 | 2.94 | 2.20 |
| RIMLS [25] | 2.68 | 0.99 | 2.92 | 2.20 |
| MRPCA [10] | 2.69 | 1.00 | 2.93 | 2.19 |
| GLR [7] | 2.77 | 0.92 | 3.02 | 2.27 |
| PCN [11] | 2.13 | 0.98 | 1.65 | 1.58 |
| DMR [12] | 2.70 | 0.98 | 2.98 | 2.21 |
| Proposed | 1.12 | 0.59 | 1.22 | 0.98 |
dataset in Table II, in terms of C2C($\times 10^{-3}$), our method outperformed the second-best algorithm SINUQ by 1.23, while in terms of C2P($\times 10^{-5}$), our method outperformed the second-best algorithm BF by 2.5 on average.

The synthetic datasets had relatively small depth values, meaning that the signal-dependent additive noise was small. Thus, the performance of the six PC denoising algorithms were comparable to that of the three image denoising schemes. However, for the Middlebury dataset capturing real-world scenes, they had relatively large depth values, and thus the additive noise was large.

Table III shows C2C and C2P results of the PCs from both the public synthetic dataset and real dataset, corrupted by SDLN. Similarly, our proposal outperformed nine competitors in both metrics, with C2C($\times 10^{-3}$) error reduced by 0.91 compared with the second-best algorithm SINUQ, and C2P($\times 10^{-5}$) error reduced by 2.0 compared with the second-best algorithm PCN, respectively.

Visual results of the denoised PCs Recycle and ArtL from the Middlebury dataset, and a PC Driving from the synthetic dataset for the additive SDGN case are shown in Fig. 13, where we colored the point clouds according to the C2C absolute distances between the ground truth points and their closest denoised points. For the result of Driving, we only show its background for better visualization. From Fig. 13, it is obvious that our proposal achieved smaller C2C errors compared to the competitors.

Overall, one can observe that the six PC denoising algorithms were worse than the image denoising schemes. The reason is that these PC denoising algorithms assume that i.i.d. Gaussian noise are added to the point coordinates in 3D space. This assumption is inaccurate in practice, because a corrupted depth pixel causes errors only in the depth dimension perpendicular to the image plane, which is quite different from i.i.d. noise in 3D space. Thus, enhancing depth measurements before synthesizing a PC is more sensible. In contrast, the three selected image denoising schemes performed poorly under signal-dependent noise and non-uniform quantization. Further, it was difficult for BF [27] and BM3D [28] to handle existing holes throughout the images, leading to locally poor performance.

In contrast, our proposal enhances depth measurements before projecting to 3D space to synthesize a PC. It allows us to tailor the optimization specifically for our depth formation model which complies with the physical acquisition process. Particularly, we model the combination of signal-dependent noise addition and non-uniform log-based quantization. Further, by employing feature graph learning, we can
to the target pixels, as shown in the enlarged regions in the second column. Due to the i.i.d. Gaussian noise assumption on PCs, distortion in 3D points stemming from the real formation process were challenging for the two deep learning based methods, PCN and DMR, leading to poor restoration results.

In contrast, our proposal provided restoration with noticeably better visual quality, which has smoother surface and fewer noisy points.

VII. CONCLUSION AND DISCUSSION

Point clouds are typically synthesized from finite-precision depth measurements that are noise-corrupted. In this paper, we improve the quality of a synthesized point cloud by jointly enhancing multiview depth images—the “rawest” signal we can acquire from an off-the-shelf sensor—prior to steps in a typical point cloud synthesis pipeline that obscure acquisition noise. We formulate a graph-based MAP optimization that specifically targets an image formation model accounting for both signal-dependent noise addition and non-uniform log-based quantization. We validate the designed model using collected empirical data from an actual depth sensor. We optimize the objective efficiently using AGD with GCT-aided optimal step size determination. Simulation results show that our proposed scheme outperforms competing schemes that denoise point clouds after the synthesis pipeline and representative image denoising schemes.

APPENDIX

PROOF OF MULTIPLE INTEGRAL

We first prove (24) by induction.

\[
\text{Pr}(y_i|x_i) \approx \prod_{i=1}^{N} \int_{R_{i,i}} \left( a_{i,i} n_{i,i} + b_{i,i} \right) d n_{i,i} = \prod_{i=1}^{N} \int_{z_{i,i} - x_{i,i}} \left( a_{i,i} n_{i,i} + b_{i,i} \right) d n_{i,i}
\]
\[
\begin{align*}
&= \sum_{i=1}^{N} \frac{a_{l,i}}{2} r_{l,i}^2 + b_{l,i} n_{i,l} |z_{l,i}^+ - z_{l,i}^-| \\
&= \sum_{i=1}^{N} a_{l,i}(z_{l,i}^- - z_{l,i}^+) x_{i,l} + \frac{a_{l,i}}{2} (z_{l,i}^+)^2 - (z_{l,i}^-)^2 \\
&+ b_{l,i} (z_{l,i}^+ - z_{l,i}^-).
\end{align*}
\]

Denote by \( \mathbf{1} \) an one-hot vector, which is a suitably long canonical vector of all zeros except entry \( i \) is 1, \( \bar{a}_{l,i} = a_{l,i}(z_{l,i}^- - z_{l,i}^+) \mathbf{1}_i \) and \( \bar{b}_{l,i} = \frac{a_{l,i}}{2} (z_{l,i}^+)^2 - (z_{l,i}^-)^2 + b_{l,i} (z_{l,i}^+ - z_{l,i}^-) \). We now rewrite (41) as

\[
\Pr(y_i | x_i) \approx \prod_{i=1}^{N} (\bar{a}_{l,i}^T x_i + \bar{b}_{l,i}).
\]

Given (41), (42) and (19), we now prove (25) for noise \( n_i \).

\[
\begin{align*}
\Pr(y_i | x_i) &\approx \prod_{j=1}^{M} \int_{R_{r,j}} (a_{r,j} n_{r,j} + b_{r,j}) \, d n_{r,j} \\
&= \prod_{j=1}^{M} (\bar{a}_{r,j}^T x_i + \bar{b}_{r,j}) \approx \prod_{j=1}^{M} (\bar{a}_{r,j}^T (H x_i + e) + \bar{b}_{r,j}).
\end{align*}
\]

Denote by \( \bar{a}_{r,j} = a_{r,j} (z_{r,j}^+ - z_{r,j}^-) \mathbf{1}_i \) and \( \bar{b}_{r,j} = \bar{a}_{r,j}^T e + \bar{b}_{r,j} = \bar{a}_{r,j}^T e + \bar{b}_{r,j} (z_{r,j}^+ - z_{r,j}^-) \). We now rewrite (43) as

\[
\begin{align*}
\Pr(y_i | x_i) &\approx \prod_{j=1}^{M} (\bar{a}_{r,j}^T H x_i + \bar{b}_{r,j}).
\end{align*}
\]
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