Off-Policy Risk Assessment in Markov Decision Processes

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Abstract

Addressing such diverse ends as safety alignment with human preferences, and the efficiency of learning, a growing line of reinforcement learning research focuses on risk functionals that depend on the entire distribution of returns. Recent work on off-policy risk assessment (OPRA) for contextual bandits introduced consistent estimators for the target policy’s CDF of returns along with finite sample guarantees that extend to (and hold simultaneously over) all risk. In this paper, we lift OPRA to Markov decision processes (MDPs), where importance sampling (IS) CDF estimators suffer high variance on longer trajectories due to small effective sample size. To mitigate these problems, we incorporate model-based estimation to develop the first doubly robust (DR) estimator for the CDF of returns in MDPs. This estimator enjoys significantly less variance and, when the model is well specified, achieves the Cramer-Rao variance lower bound. Moreover, for many risk functionals, the downstream estimates enjoy both lower bias and lower variance. Additionally, we derive the first minimax lower bounds for off-policy CDF and risk estimation, which match our error bounds up to a constant factor. Finally, we demonstrate the precision of our DR CDF estimates experimentally on several different environments.

1 Introduction

In off-policy evaluation (OPE), we aim to estimate the risk associated with a target policy given only offline datasets collected from a deployed behavior policy. Broadly, a risk functional maps from the distribution of returns to a real value. While OPE research has historically focused on estimating the expected return, a wave of recent works in reinforcement learning (RL) focus on other risk functionals, motivated by such diverse desiderata as risk aversion (and seeking) and alignment with human preferences. Examples of such alternative risk functionals include the expectation, variance, value at risk, conditional value at risk, and broader families such as distortion risk functionals, coherent risk functionals, and Lipschitz risk functionals. Notably, the Lipschitz risks, introduced by Huang et al. [2021b] subsume most common risk functionals.

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In a new line of work on off-policy risk assessment (OPRA), researchers have developed methods for simultaneously estimating arbitrarily many risk functionals, providing guarantees that hold simultaneously over an entire collection of risks [Huang et al., 2021b, Chandak et al., 2021a]. These results are achieved through a two-step process: (i) estimate the cumulative distribution function (CDF) of returns under the target policy; and (ii) calculate the risk functionals of interest on the estimated CDF. This approach has two advantages: (a) guarantees on the CDF estimation error yield corresponding guarantees on the risk estimates; and (b) because the risk estimates all share the same underlying CDF, the corresponding guarantees on any number of downstream risk estimates hold simultaneously without loss in statistical power.

To our knowledge, for MDPs, importance sampling (IS) is the only established method for estimating off-policy CDFs [Chandak et al., 2021a]. However, IS estimators are known to suffer from high variance, render them impractical over long trajectories. This variance can be exponential in the horizon [Liu et al., 2018], as weights vanish for most trajectories and explode for others, yielding tiny effective sample sizes. Consider, for example, that practitioners often evaluate a deterministic target policy given a dataset collected by a stochastic behavioral policy. In such cases, the IS weight is zero for all trajectories where the target policy places zero probability on any action that the behavioral policy takes, even if the policies agree on most actions. This renders most trajectories unusable, shrinking the effective size of an offline dataset [Sussex et al., 2018].

In off-policy evaluation addressing the standard expected return risk functional, doubly robust estimators and their variants have been shown to mitigate these problems [Dudik et al., 2011, Dudík et al., 2014, Jiang and Li, 2016, Thomas and Brunskill, 2016, Wang et al., 2017, Su et al., 2020]. In the contextual bandit setting, the paper introducing OPRA also proposed and analyzed a doubly robust estimator for the CDF [Huang et al., 2021b], but model-based and doubly robust risk estimation in MDPs, where it may be most impactful, remains an open problem. One primary challenge is that, in the expected return setting, extending the model-based and doubly robust estimators from contextual bandits to MDPs [Jiang and Li, 2016] relied on stepwise or recursive formulations of the importance sampling estimator, which lack clear analogs in CDF estimation.

In this paper, we define and analyze the first doubly robust CDF estimator for off-policy risk assessment in MDPs. Specifically, we contribute the following:

1. A recursive formulation of the CDF of returns in MDPs (§ 3.2), which enables us to derive importance sampling (§ 4), and model-based CDF estimators (§ 5).
2. Rate-matching upper and minimax lower bounds (§ 4.2) for the IS CDF estimator.
3. Variance-reduced IS estimators leveraging unique properties of the return distribution (§ 4.4).
4. The first doubly robust off-policy estimator for the CDF of returns (§ 6), and analysis demonstrating its potential for significant variance reduction.
5. The first Cramer-Rao lower bound for off-policy CDF estimation and analysis demonstrating that the doubly robust estimator achieves this lower bound in certain cases (§ 6.3).
6. A demonstration that the risk estimates inherit the reduced variance and error in CDF estimation (§ 7).
7. Experimental evidence that the doubly robust CDF and risk estimates exhibit significantly...
lower error and variance than importance sampling estimators on benchmark problems, with applications to safe policy evaluation and improvement (§ 8).

2 Related Work

Several estimators have been studied for off-policy estimation of the expected return. A seminal work by Dudik et al. [2011] proposed the doubly robust estimator for the expected return in contextual bandits, which was extended to the MDP setting by Jiang and Li [2016]. The doubly robust estimator leverages both importance sampling weights and models to reduce variance while retaining unbiasedness. Researchers have proposed several variants of the doubly robust estimator, aiming to improve its performance, with Thomas and Brunskill [2016], Wang et al. [2017] optimizing the balance between importance sampling and model-based estimators and Su et al. [2020] optimizing the weights used in the estimator. A Cramer-Rao variance lower bound [Jiang and Li, 2016] and minimax lower bounds [Li et al., 2015, Wang et al., 2017, Ma et al., 2021] for off-policy estimation of the mean have been derived.

While the expected return has been the focus of most previous off-policy evaluation research, other risk functionals are increasingly of interest. In the RL literature, the variance [Sani et al., 2013, Tamar et al., 2016], variance-constrained mean [Mannor and Tsitsiklis, 2011], and conditional value-at-risk [Rockafellar et al., 2000, Keramati et al., 2020, Huang et al., 2021a] are often employed. Other risk functionals, including exponential utility [Denardo et al., 2007], distortion risk functionals [Dabney et al., 2018], and cumulative prospect weighting [Gopalan et al., 2017, Prashanth et al., 2016, 2020] have also been investigated.

Several recent works have aimed to estimate risk functionals from off-policy datasets. Of these, Chandak et al. [2021b] estimates the variance, while more recent works [Huang et al., 2021b, Chandak et al., 2021a] tackle the estimation of more general risks and are the closest works of comparison. Both Huang et al. [2021b], Chandak et al. [2021a] take a two-step approach of first estimating the off-policy CDF of returns; and then estimating their risks via a plug-in approach. Chandak et al. [2021a] proposed an IS CDF estimator in both stationary and nonstationary MDPs and derives confidence intervals for a number of risk estimates on a per-risk basis. Huang et al. [2021b] proposed both IS and DR estimators for the CDF of returns in contextual bandits, establishing the first finite sample guarantees for off-policy CDF estimation and showing that the estimated CDFs converge uniformly, achieving $O(1/\sqrt{n})$ rates. Additionally, they introduced the broad class of Lipschitz risk functionals, where the errors in risk estimation are upper-bounded by the error in CDF estimation multiplied by each functional’s Lipschitz constant.

While Chandak et al. [2021a] proposes only IS CDF estimates for MDPs, Huang et al. [2021b] proposes variance-reduced DR CDF estimates in only contextual bandits. Thus variance-reduced risk estimation in MDPs, where high variance from IS weights is most problematic [Liu et al., 2018], is an open problem. Towards this end, we (1) develop provably variance-reduced CDF and risk estimators in MDPs, and, (2) demonstrate their optimality by deriving the first matching upper and lower bounds.
3 Preliminaries

3.1 Problem Setting

An MDP is defined by \( \mathcal{M} = (S, A, P, R, \gamma, \mu) \), where \( S \) is the state space, \( A \) is the action space, \( P : S \times A \times S \to \mathbb{R} \) is the transition function, \( R : S \times A \to \mathbb{R} \) is the reward function, \( \gamma \in (0, 1) \) is the discount factor, and \( \mu \) is the fixed starting state distribution. We assume that rewards are bounded on the support \([0, D]\). A stationary policy \( \pi : S \to \Delta(A) \) maps a state to a probability distribution over actions.

In the off-policy evaluation problem, we are concerned with estimating risk functionals of a policy \( \pi \) given a dataset of trajectories \( D = \{\tau_i\}_{i=1}^n \) collected by the behavioral policy \( \beta \) interacting with MDP \( \mathcal{M} \). Each \( \tau = (S_1, A_1, R_1, \ldots, S_H, A_H, R_H, S_{H+1}) \) is an \( H \)-step trajectory, with \( S_1 \sim \mu \), \( R_h \sim R(S_h, A_h) \), and \( S_{h+1} \sim P(\cdot|S_h, A_h) \). In general, we will refer to random variables with capital letters, and for convenience, we denote \( \tau_j = (S_1, A_1, R_1, \ldots, S_j, A_j, R_j) \), or a sub-trajectory of \( \tau \) from horizon 1 to \( j \).

Further, let \( w(A_h, S_h) := \frac{\pi(A_h|S_h)}{\beta(A_h|S_h)} \) denote the importance weight, and let \( w_{\max} := \max_{h,A_h,S_h} w(A_h, S_h) \) be the maximum importance weight over all horizons, states, and actions. For convenience, by \( w_j := \prod_{h=1}^j w(A_h, S_h) \), we denote the importance weight of subtrajectory \( \tau_j \). Thus, \( w_H \) is the importance weight of the trajectory \( \tau \).

We denote by \( \mathbb{P} \) the distribution of trajectories induced by \( \pi \) on \( \mathcal{M} \), and by \( \mathbb{P}_\beta \) the distribution induced by \( \beta \). The random variable of returns of an \( H \)-step trajectory induced by \( \pi \) in \( \mathcal{M} \) is given by \( Z^\pi = \left( \sum_{h=1}^{H} \gamma^{h-1} R_h \right| \pi \). We also write the random variable of returns starting from a state \( s_h \) at horizon \( h \), summed until the end of horizon \( H \), to be \( Z^\pi_{s_h} = \left( \sum_{k=h}^{H} \gamma^{k-h} R_k \right| \pi, s_h \) \). The random variable of returns conditioned on a state and action at horizon \( h \), \( Z^\pi_{s_h,a_h} \), is defined similarly.

The cumulative distribution function (CDF) of \( Z^\pi \) is \( F^\pi(t) = \mathbb{E}_\mathbb{P} [\mathbb{1}\{Z^\pi \leq t\}] \), where \( \mathbb{1}\{\cdot\} \) is the indicator function. Similarly, the CDF of conditional returns \( Z^\pi_{s_h} \) is defined as \( F^\pi_{s_h}(t) = \mathbb{E}_\mathbb{P} [\mathbb{1}\{Z^\pi_{s_h} \leq t\} | s_h] \), and the state-action dependent distribution \( F_{s_h,a_h}(t) \) is defined similarly. We also define the complementary cumulative distribution function (CCDF) \( S^\pi(t) := 1 - F^\pi(t) = \mathbb{E}_\mathbb{P} [\mathbb{1}\{Z^\pi > t\}] \), and \( S^\pi_{s_h}(t) := 1 - F^\pi_{s_h}(t) \).

For short, we write \( \mathbb{E}_{\beta}^h[\cdot] := \mathbb{E}_{\mathbb{P}_\beta}[\cdot|\tau_h] \) to be the expectation under behavioral policy \( \beta \), conditioned on a trajectory up until time \( h \). Similarly, for \( \pi \), we write the conditional expectation \( \mathbb{E}_{\pi}^h[\cdot] := \mathbb{E}_{\mathbb{P}_\pi}[\cdot|\tau_h] \). We define the conditional variances \( \nabla_h, \nabla^\beta_h \) similarly, e.g., \( \nabla^\beta_h = \nabla_{\beta,h}[\cdot|\tau_{h-1}] \).

3.2 CDF Bellman Equation

Our off-policy CDF estimators rely on a novel CDF Bellman equation that provides a recursive formulation of the CDF of returns \( F \). In the off-policy setting, for state \( s_h \) at horizon \( h \), we have:

\[
F_{s_h}(t) := \mathbb{E}_{\mathbb{P}_\beta} \left[ w(A_h, s_h) F_{S_{h+1}} \left( \frac{t - R_h}{\gamma} \right) \Big| s_h \right] \tag{1}
\]

where \( F_{s_{H+1}} = \mathbb{1}\{0 \leq t\} \) for all \( s_{H+1} \) at the end of the horizon. Note that the CDF of returns under \( \pi \) can also be written as \( F^\pi(t) = \mathbb{E}_{\mathbb{P}_\beta} \left[ F^\pi_{s_0}(t) \right] \), i.e. the expectation under the starting state.
distribution $s_0 \sim \mu$ of the CDFs $F_{s_0}^\pi$.

This recursion was first introduced in Sobel [1982] for deterministic rewards in the value iteration setting, which we have extended to the more general stochastic setting. Using the definition of equivalence in distribution ($D$), it can be seen that the CDF Bellman operator is equivalent to the distributional Bellman operator [Bellemare et al., 2017, Dabney et al., 2018], $Z_{s_h} \overset{D}{=} R(s_h, A_h) + \gamma Z_{s_{h+1}}$.

3.3 Risk Functionals

Let $Z \in L_\infty(\Omega, \mathcal{F}_Z, \mathbb{P}_Z)$ denote a real-valued random variable that admits a CDF $F_Z \in L_\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A risk functional $\rho : L_\infty(\Omega, \mathcal{F}_Z, \mathbb{P}_Z) \rightarrow \mathbb{R}$ is said to be law-invariant if $\rho(Z)$ depends only on the distribution of $Z$ [Kusuoka, 2001], i.e., if for any pair of random variables $Z$ and $Z'$, $F_Z = F_{Z'} \implies \rho(Z) = \rho(Z')$. When clear from the context, we write $\rho(F_Z)$ in place of $\rho(Z)$. In this paper, we restrict our focus to law invariant risk functionals, as it may not be practical to estimate risk functionals that are not law invariant from data [Balbás et al., 2009].

Examples of popularly studied law-invariant risk functionals include the expected value, variance, mean-variance, coherent risks (including conditional value-at-risk (CVaR) and proportional hazard), distortion risks, and cumulative prospect theory risks, which we discuss more extensively in Appendix A due to space constraints. We give particular attention to distortion risk functionals due to their widespread usage and general form. Distortion risks apply a distortion function $g : [0, 1] \rightarrow [0, 1]$ to the CDF $F_Z$ and have the formula $\rho(Z) = \int_0^\infty g(1 - F_Z(t)) dt = \int_0^\infty g(S(t)) dt$.

Notably, expected value is a special case of distortion risk with $g(x) = x$, while CVaR at level $\alpha$ (CVaR$_{\alpha}$) has a distortion function $g(x) = \min\{\frac{x}{1-\alpha}, 1\}$. A large number of the aforementioned risk functionals satisfy a notion of smoothness in the CDF, which Huang et al. [2021b] define as the class of Lipschitz risk functionals:

**Definition 3.1 (Lipschitz Risk Functionals).** A law invariant risk functional $\rho$ is $L$-Lipschitz if there exists $L \in [0, \infty)$ such that for any pair of CDFs $F_Z$ and $F_{Z'}$, it satisfies

$$|\rho(F_Z) - \rho(F_{Z'})| \leq L \|F_Z - F_{Z'}\|_\infty.$$

Lipschitz risk functionals include distortion risk functionals with Lipschitz distortion functions $g$. As special cases, expected value is $D$-Lipschitz and CVaR at level $\alpha$ is $D/(1-\alpha)$-Lipschitz. Additionally, variance is $(3D^2)$-Lipschitz and so is mean-variance [Huang et al., 2021b].

As we analyze the theoretical properties of CDF and risk estimators later in our paper, we show that estimates of Lipschitz risk functionals retain many desirable properties, including characterizable finite sample error and consistency.

3.4 Off-Policy Risk Assessment

In the remainder of this paper, we will take the general approach of Algorithm 1 (OPRA) for off-policy risk assessment. OPRA takes as input a set of risk functionals of interest, and outputs
estimates of their values under the target policy $\pi$. It first estimates the CDF of returns under $\pi$ from off-policy data, then generates plug-in risk estimates on the CDF. In Sections 4, 5 and 6, we define and analyze CDF estimators. In Section 7, we turn to plug-in risk estimation.

**Algorithm 1: Off-Policy Risk Assessment (OPRA)**

**Input:** Dataset $\mathcal{D}$, policy $\pi$, probability $\delta$, risk functionals $\{\rho_p\}_{p=1}^P$ with Lipschitz constants $\{L_p\}_{p=1}^P$.

1. Estimate the CDF $\hat{F}$ with error $\epsilon$;
2. For $p = 1...P$, estimate risk $\hat{\rho}_p = \rho_p(\hat{F})$;

**Output:** Estimates with errors $\{\hat{\rho}_p \pm L_p \epsilon\}_{p=1}^P$.

4 Importance Sampling CDF Estimation

Off-policy evaluation faces the unique challenge that only rewards for actions taken by the behavioral policy are observed. Importance sampling (IS), which reweights samples according to the ratio between target and behavioral policy probabilities, can be used to account for this distribution shift. While the IS estimator for off-policy estimation was previously introduced in Huang et al. [2021b], Chandak et al. [2021a], finite sample convergence in MDPs and lower bounds for off-policy CDF estimation (in general) remained open questions. In this section, we derive the first minimax lower bound for off-policy CDF estimation, and show that it matches the IS estimator upper bound up to a constant factor. Further, we leverage unique properties of the return distribution to develop new variance-reduced IS estimators.

4.1 Importance Sampling Estimator

The IS CDF estimator forms an empirical estimate of the recursion in (1):

$$
\hat{F}_{IS}(t) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_{s_i^h}(t),
$$

where $\hat{F}_{s_i^h}(t) = w(a_i^h, s_i^h) \hat{F}_{s_{i+1}^h} \left( \frac{t - r_i^h}{\gamma} \right)$.

At each horizon $h$, the recursive IS estimator applies the importance weight to the estimated CDF at the next state $s_{h+1}$ convolved with the reward $r_h$ and discount factor $\gamma$. Unrolling the recursion recovers the more intuitive form of the IS estimator, also introduced in Chandak et al. [2021a], that reweights each sample by the importance weight of the entire trajectory:

$$
\hat{F}_{IS}(t) = \frac{1}{n} \sum_{i=1}^{n} w_i \mathbb{1} \left\{ \sum_{h=1}^{H} \gamma^{h-1} r_h^i \leq t \right\}.
$$
Lemma 4.1. The IS CDF estimator (3) is unbiased and its variance is recursively given by

\[ \forall_h \mathbb{E}_h \left[ \hat{F}_{S_h}(t) \right] = \mathbb{E}_h \left[ \forall_h \left[ w(A_h, S_h) F_{S_h, A_h}(t) \left| S_h \right. \right] \right] 
+ \mathbb{E}_h \left[ w(A_h, S_h)^2 \forall_{h+1} \left[ \hat{F}_{S_{h+1}} \left( \frac{t-R_h}{\gamma} \right) \left| S_h, A_h \right. \right] \right] 
+ \forall_h \left[ F_{S_h}(t) \right]. \]  

The first term expresses variance from importance-weighted actions sampled as \( A_h \sim \beta(\cdot \left| S_h \right. ) \), while the third term expresses variance from transitions. The second term encompasses variance from random rewards and transitions, and recurses to the next time step. At horizon \( h \), each term in the variance is multiplied by the product of importance weights from the trajectory thus far, \( w_{h-1} \).

4.2 Finite Sample Error Bound

While the IS estimator is known to be consistent, obtaining finite sample error bounds in MDPs has remained an important open problem. Such bounds characterize the convergence rate of the estimate towards the true \( F \), and provide confidence intervals around estimates in the finite sample regimes of practical scenarios.

First, however, it is important to note that, given finite samples, the IS estimator may not be a valid CDF due to the use of importance weighting, e.g., it can be greater than 1. To address this issue, we can use a weighted importance sampling (WIS) estimator instead:

\[ \hat{F}_{WIS}(t) = \frac{1}{\sum_{j=1}^n w_j} \hat{F}_{IS}(t), \]

which normalizes the estimate using the sum of importance weights instead of the sample size, guaranteeing that \( \hat{F}_{WIS} \in [0, 1] \). As Chandak et al. [2021a] demonstrates, the WIS estimator is biased but uniformly consistent.

Alternatively, the IS estimator can be clipped to the unit range, a strategy also employed by Huang et al. [2021b] for off-policy CDF estimation in contextual bandits, shown below:

\[ \hat{F}_{IS-clip}(t) = \min \left\{ \hat{F}_{IS}(t), 1 \right\}, \]

The IS-clip estimator, too, is biased but uniformly consistent, and its error is uniformly bounded in the following lemma:

Lemma 4.2. With probability at least \( 1 - \delta \), for universal constants \( c_1, c_2 > 0 \)

\[ \| \hat{F}_{IS-clip} - F \|_\infty \leq c_1 \sqrt{\frac{\mathbb{E}_P[w_H^2]}{n}} + c_2 \frac{w_{max}}{n}. \]  

The IS estimator converges at a rate of \( O(\mathbb{E}_P[w_H^2]/\sqrt{n}) \), subsuming previous results on convergence in contextual bandits [Huang et al., 2021b]. For readability throughout the paper, we leave constants implicit and give their exact form in the proof (Appendix B).

How close to optimal is this convergence rate? For large enough sample size \( n \), the IS estimator is in fact minimax rate-optimal, as the following lower bound demonstrates:
Theorem 4.1. For a universal constant $c$, when $n \geq c\frac{w_{\max}^2}{E_{P_\beta}[w_H]}$, we have

$$\inf_{\hat{F}} \sup_{F \in \mathcal{F}} E\left[\|\hat{F} - F^\pi\|_\infty\right] \geq c \sqrt{E_{P_\beta}[w_H^2] / n} \quad (7)$$

4.3 Variance-Reduced IS Estimation

Unlike the IS estimator of the expected return [Dudik et al., 2011, Jiang and Li, 2016], the IS estimator of the CDF has the unique property that its variance differs across its support as a function of $t$. Examining the variance for $\hat{F}_{IS}$ (5) at a fixed $t$, the first term is small if $F_{S_{h,A_h}}(t)$ is close to 0, which can occur at small $t$. However if $F_{S_{h,A_h}}(t)$ is close to its maximal value of 1, which occurs when $t$ is close to the maximum return $D$, the first term is close to its largest possible value.

4.3.1 Estimating the Complementary Cumulative Distribution Function (CCDF)

Rather than estimating the CDF $F$, another possibility is to estimate the CCDF $S = 1 - F$, which is directly used in the expression for many risk expressions (e.g. distortion risk functionals, see § 3). IS estimators for $S$ take a similar form as those for $F$, but with the indicator function inequality swapped to $1\{ \cdot > t \}$. For example,

$$\hat{S}_{IS}(t) = \frac{1}{n} \sum_{i=1}^{n} w_i^H \mathbb{1}\left\{ \sum_{h=1}^{H} \gamma^{h-1} r_{h}^i > t \right\}, \quad (8)$$

Going forward, we refer to the estimator in (4) as $\hat{F}_{F-IS}$, and $\hat{F}_{S-IS} := 1 - \hat{S}_{IS}$ from (8) to avoid confusion. Perhaps non-intuitively, $\hat{F}_{F-IS}$ and $\hat{F}_{S-IS}$ have different theoretical properties, namely its variance:

$$\mathbb{V}_{h}^{\beta} \left[ \hat{F}_{F-IS,S_{h}}(t) \right] = \mathbb{V}_{h}^{\beta} \left[ 1 - \hat{S}_{S_{h}}(t) \right] = \mathbb{E}_{h}^{\beta} \left[ \mathbb{V}_{h}^{\beta} \left[ w(A_h,S_{h})(1 - F_{S_{h},A_h}(t)) | S_{h} \right] \right] + \mathbb{E}_{h}^{\beta} \left[ w(A_h,S_{h})^2 \mathbb{V}_{h+1}^{\beta} \left[ \hat{S}_{S_{h+1}}(t - R_{h} / \gamma) | S_{h}, A_{h} \right] \right] + \mathbb{V}_{h} \left[ S_{S_{h}}(t) \right]$$

The first term of the variance expression for $\hat{F}_{S-IS}$ takes the exact opposite trend as that of $\hat{F}_{F-IS}$: it is low where $F_{S_{h,A_h}}(t)$ is high, and vice versa. For example, if $\hat{F}_{F-IS}$ has low variance in estimating the lower tails but high variance in the upper tails, $\hat{F}_{S-IS}$ will have low variance in the upper tails but high variance in the lower tails. Lower variance leads to lower error of the CDF estimate on certain portions of the distribution. To formalize this argument, we present the following bounds on the error of the CDF estimate in terms of its variance at each $t$. As we later show, this will also have important consequences for downstream estimation of different risks, which up-weight different parts of the distribution.
Lemma 4.3. If \( \mathbb{V}[\hat{F}_{IS}] \) and \( \mathbb{V}[\hat{S}_{IS}] \) have bounded variation (see Assumption B.1), there exists universal constants \( c_1, c_2 \) such that with probability at least \( 1 - \delta \), \( \forall t, \)

\[
|F(t) - \hat{F}_{F-IS}(t)| \leq \frac{c_1 \log(\sqrt{n}/\delta)}{n} + \sqrt{\frac{c_2 \mathbb{V}[\hat{F}_{IS}(t)] \log(\sqrt{n}/\delta)}{n}},
\]

\[
|F(t) - \hat{F}_{S-IS}(t)| \leq \frac{c_1 \log(\sqrt{n}/\delta)}{n} + \sqrt{\frac{c_2 \mathbb{V}[\hat{S}_{IS}(t)] \log(\sqrt{n}/\delta)}{n}}.
\]

This difference has important implications for downstream risk estimation. The choice of whether to estimate \( S \) or \( F \) can reduce the variance of the plug-in risk estimator, especially for risk functionals such as CVaR that heavily up-weight one tail of the distribution. In general, under reward distributions, \( \hat{F}_{S-IS} \) will lead to lower variance estimators for risk-averse risk functionals that upweight the lower tails of the distribution. \( \hat{F}_{F-IS} \) will lead to lower-variance risk estimators for risk-seeking risk functionals that upweight the upper tails.

The RHS of the error bound in Lemma 4.3 gives a uniform confidence band on the estimated CDF that contains \( F \) with high probability. However, the variance of \( \hat{F}_{F-IS} \) or \( \hat{F}_{S-IS} \) cannot be computed without knowledge of the underlying MDP. Instead, we can use the empirical variance of \( \hat{F}_{F-IS} \) or \( \hat{F}_{S-IS} \):

**Lemma 4.4** (Empirical Bernstein Bound). Let \( \mathbb{V}_n \) denote the sample variance. For any choice of \( M \) points \( \{t_j\}_{j=1}^M \) where \( t_j \in [0, D] \), with probability at least \( 1 - \delta \) we have that \( \forall j \in [M], \)

\[
|F(t_j) - \hat{F}_{F-IS}(t_j)| \leq \frac{c_1' \log(2M/\delta)}{n} + \sqrt{\frac{c_2' \mathbb{V}_n[\hat{F}_{IS}(t_j)] \log(2M/\delta)}{n}},
\]

\[
|F(t_j) - \hat{F}_{S-IS}(t_j)| \leq \frac{c_1' \log(2M/\delta)}{n} + \sqrt{\frac{c_2' \mathbb{V}_n[\hat{S}_{IS}(t_j)] \log(2M/\delta)}{n}}.
\]

Note that Lemma 4.4 is a bound for a finite number of points, while Lemma 4.3 is a uniform bound over all \( t \). In practice, the sample variance of only a finite number of points can be calculated.

### 4.4 Combining IS Estimators

Lemma 4.3 also indicates that the variance of the resulting estimator can be further reduced by combining the \( F \) and \( S \) estimators pointwise, such that the estimator with the lower variance is used at each \( t \). Such a combined estimator may be especially effective for reducing the variance of the plug-in estimators for distortion risk functionals that place weight on both upper and lower tails of the distribution, such as the expected return.

How should the choice between \( \hat{F}_{F-IS}(t) \) or \( \hat{F}_{S-IS}(t) \) be made for each \( t \)? Lemma 4.4 suggests that one effective method is to estimate the empirical variance of \( \hat{F}_{F-IS}(t) \) and \( \hat{F}_{S-IS}(t) \) at each \( t \), and to simply use the estimate with lower empirical variance:

\[
\hat{F}_{C-IS}(t) = \begin{cases} 
\hat{F}_{F-IS}(t), & \text{if } \mathbb{V}_n[\hat{F}_{F-IS}(t)] < \mathbb{V}_n[\hat{F}_{S-IS}(t)], \\
\hat{F}_{S-IS}(t), & \text{otherwise.}
\end{cases}
\]
The error bound of such a combined estimator is given below.

**Proposition 4.1.** Let \( V_n^{\text{min}}(t) = \min \{ V_n[\hat{F}_{\text{F-IS}}(t)], V_n[\hat{F}_{\text{S-IS}}(t)] \} \). For any choice of \( M \) points \( \{ t_j \}_{j=1}^M \) for \( t_j \in [0, D] \) and \( \delta \in (0, 1) \), define

\[
\Delta(V_n) := \sqrt{\frac{2 \log(2M/\delta)}{n-1}} + \max_{j \in [M]} |V_n[\hat{F}_{\text{F-IS}}(t_j)] - V_n[\hat{F}_{\text{S-IS}}(t_j)]|.
\]

Then, for \( n \geq \frac{8 \log(2n/\delta)}{\Delta(V_n)^2} \), we have with probability at least \( 1 - \delta \) that \( \forall j \in [M] \),

\[
|F(t_j) - \hat{F}_{\text{C-IS}}(t_j)| \leq c_1' \frac{\log(2M/\delta)}{n} + c_2' \frac{V_n^{\text{min}}(t_j) \log(2M/\delta)}{n}.
\]

5 Model-Based CDF Estimation

In the expected value OPE literature, model-based estimation can avoid the high variance associated with importance sampling by directly learning the expected return under \( \pi \) using a model of the underlying MDP learned from data [Jiang and Li, 2016]. Taking inspiration from such approaches, we now develop the first model-based off-policy CDF estimator. First, a model of the MDP \( \mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma, \mu) \) is formed from data, upon which a model \( \hat{F}_{s_h,a_h} \) of the return distribution under \( \pi \) can be directly computed using the recursion in (1) for all horizons \( h \), states \( s_h \), and \( a_h \).

Formally, for horizons \( h = H, \ldots, 1 \), \( F \) can be computed recursively:

\[
\hat{F}_{s_h,a_h}(t) = \mathbb{E}_{\mathcal{P},\mathcal{R}} \left[ F_{s_{h+1}} \left( \frac{t - R_h}{\gamma} \right) \right],
\]

\[
\hat{F}_{s_h}(t) = \mathbb{E}_{\pi} \left[ \hat{F}_{s_h,a_h}(t) \right],
\]

with \( \hat{F}_{s_{H+1}}(t) = 1 \{ 0 \leq t \} \) for all \( s_{H+1} \) at the end of the horizon. The models \( \hat{F} \) can then be used for the direct method (DM) of estimation, which simply averages \( \hat{F} \) for the starting states observed in the data:

\[
\hat{F}_{\text{DM}}(t) = \frac{1}{n} \sum_{i=1}^n \hat{F}_{s_i}(t)
\]

Because this direct method of estimation directly deploys the target policy \( \pi \) in the estimated model, the error of the estimators \( \hat{F}_{\text{DM}} \) and \( \hat{F} \) are directly related to the misspecification of the MDP model \( \mathcal{M} \). The model \( \mathcal{M} \) and thus the direct estimator is frequently prone to uncharacterizable and possibly high bias, e.g., when the dataset coverage is inadequate or the state space is high dimensional.

6 Doubly Robust CDF Estimation

We now define a new doubly robust (DR) estimator for the distribution of returns in MDPs (§ 6.1), which takes advantage of both importance sampling (§ 4) and model-based (§ 5) estimators to retain the unbiasedness of the IS CDF estimator, with potentially significant reduction in variance. In § 6.3, we derive the first Cramer-Rao lower bound on off-policy CDF estimation, and show that the DR estimator achieves this lower bound when the model \( \hat{F} \) is equal to the true distribution.
6.1 Doubly Robust (DR) Estimator

The recursive formulation in (3) is key for defining the doubly robust CDF estimator. At each time step $h$ in the recursive formulation, we use the model $F$ as a baseline and apply an IS-weighted data-dependent correction to obtain the doubly robust estimator. Specifically, the DR estimator has the following recursive form:

$$
\hat{F}_{\text{DR}}(t) = \frac{1}{n} \sum_{i=1}^{n} \hat{F}_{s_i}^t(t),
$$

where $\hat{F}_{s_i}^t(t) = P_{s_i}^t(t) + w(a_i^t, s_i^t) \left( \hat{F}_{s_{h+1}}^t \left( \frac{t-r_i^h}{\gamma} \right) - P_{s_i}^t(a_i^t) \right)$ \hspace{1cm} (11)

The model $F$ can be obtained using the methods described in § 5, and as before, $\hat{F}_{s_{H+1}}^t(t) = \mathbb{1} \{0 \leq t\}$ for all $s_{H+1}$. Note that when $H = 1$, (11) recovers the DR estimator for contextual bandits previously derived in Huang et al. [2021b]. Letting $z_i^h = \sum_{k=1}^{h} \gamma^{k-1} r_i^k$ be the return of trajectory $i$ up until step $h$, the recursion in (11) can be unrolled over the $H$ timesteps of the trajectory to obtain

$$
\hat{F}_{\text{DR}}(t) = \frac{1}{n} \sum_{i=1}^{n} \left( w_H^i \{ z_H^i \leq t \} + \sum_{h=1}^{H} \frac{w_{h-1}}{w_{h-1}} P_{s_h}^t \left( \frac{t-z_h^i}{\gamma_h} \right) - \sum_{h=1}^{H} \frac{w_h}{w_h} P_{s_h^t, a_h^t} \left( \frac{t-z_h^i}{\gamma_h} \right) \right).
$$

The bias and variance of the DR estimator is derived below.

**Lemma 6.1.** The doubly robust estimator is unbiased, and letting $\hat{F}_{s_h}$ be given as in (11), its variance is:

$$
\mathbb{V}^2 \left[ \hat{F}_{s_h}(t) \right] = \mathbb{E} \left[ w(A_h, S_h) \Delta_{s_h, A_h}(t) | S_h \right] + \mathbb{E} \left[ w(A_h, S_h)^2 \mathbb{V}_{h+1} \left[ \hat{F}_{s_{h+1}}^t \left( \frac{t-R_h}{\gamma} \right) | S_h, A_h \right] \right] + \mathbb{V}_h \left[ F_{s_h}(t) \right]
$$

where $\Delta_{s,a}(t) = F_{s,a}(t) - F_{s,a}(t)$.

Comparing the variance expression above with that of the IS estimator (Lemma 4.1), we see that the first term in the variance of the DR estimator contains a difference $\Delta_{s,a} = F_{s,a} - F_{s,a}$, while the variance of the IS estimator contains only $F$. The variance in the first term is over stochastic actions given the current state.

When $\Delta_{s,a} < F_{s,a}$, which is often the case in practice, the variance contributed by the first term can be significantly reduced in the DR estimator. Consequently, the DR estimator can be seen to reduce variance contributed by importance sampling for stochastic actions, but not variance from rewards or transitions, which arise from the last two terms in Lemma 6.1.

To further reduce variance, taking inspiration from Thomas et al. [2015], we can combine weighted importance sampling with the DR estimator (11). Letting $\hat{w}_h = \sum_{i=1}^{n} w_i^h$ and $\hat{z}_h^i = \sum_{k=1}^{h} \gamma^{k-1} r_i^k$,

$$
\hat{F}_{\text{WDR}}(t) = \sum_{i=1}^{n} \left( \frac{w_H^i}{w_H} \{ z_H^i \leq t \} + \sum_{h=1}^{H} \frac{w_{h-1}}{w_{h-1}} P_{s_h}^t \left( \frac{t-z_h^i}{\gamma_h} \right) - \sum_{h=1}^{H} \frac{w_h}{w_h} w(a_i^t, s_i^t) P_{s_h^t, a_h^t} \left( \frac{t-z_h^i}{\gamma_h} \right) \right).
$$
Neither the DR nor WDR estimators are guaranteed to valid CDFs; they may be outside the $[0, 1]$ interval or lose monotonicity due to the subtracted terms (particularly at lower sample sizes). In practice, we transform the DR (and WDR) estimators to valid CDFs using:

$$\hat{F}_{M-DR}(t) = \text{Clip}\left(\max_{t' \leq t} \hat{F}_{DR}(t'), 0, 1\right),$$

which simply does not allow the CDF to decrease, and clips the value to the unit interval.

### 6.2 Error Bound and Consistency

The WDR estimator is difficult to analyze, and we empirically demonstrate its efficacy in § 8. The advantage of the M-DR estimator in (12), however, is that it obeys the following finite sample error bound.

**Lemma 6.2.** With probability at least $1 - \delta$,

$$\left\| \hat{F}_{M-DR} - F \right\|_{\infty} \leq w_{\text{max}}^H \frac{\sqrt{72}}{n} \log \frac{8n^{1/2}}{\delta},$$

and thus $\hat{F}_{M-DR}$ is uniformly consistent.

We achieve a convergence rate of $O\left(w_{\text{max}}^H / \sqrt{n}\right)$. However, we note that the above bound is conservative in the sense that it assumes worst-case model misspecification, under which the DR estimator can perform worse than IS. This contributes looseness to the result, and a tighter bound remains an open problem.

### 6.3 Cramer-Rao Lower Bound

Though we have demonstrated the variance reduction potential of the DR estimator, a natural question to ask is whether there exists an unbiased estimator of the CDF that can even further reduce variance. Inspired by Jiang and Li [2016], Huang and Jiang [2020], we derive the first Cramer-Rao lower bound on the variance of off-policy CDF estimation in MDPs, and show that the DR estimator can achieve this lower bound.

**Theorem 6.1.** For unique directed acyclic graphs (Definition C.1), the variance of any unbiased off-policy CDF estimator is pointwise lower bounded by

$$\sum_{h=1}^{H} \mathbb{E}_{P_{\beta}} \left[ w_{h-1}^2 \mathbb{V}_{h} \left[ F_{S_h} \left( t - \sum_{k=1}^{h} R_k \right) \right] \right].$$

This lower bound on the variance shows that difficulties introduced by the intrinsic state transition stochasticity of the underlying MDP cannot be eliminated. In fact, the variance of the DR estimator in unique directed acyclic graphs is equal to the Cramer Rao lower bound when $F$ is a perfect model, that is, $F_{s_h, a_h}(t) = F_{s_h, a_h}(t)$ for all $s_h, a_h$ and $t$. Further, as the error between $F$ and $F$ decreases, the variance approaches the Cramer-Rao lower bound.
7 Risk Estimation

We have developed several methods for CDF estimation and their error bounds, which we now translate to off-policy risk assessment. Returning to Algorithm 1, we have the CDF estimator $\hat{F}$ and the CDF estimation error $\epsilon$ (Line 2). We form the plug-in estimate of a risk functional $\rho$ as:

$$\hat{\rho} = \rho(\hat{F})$$  \hspace{1cm} (13)

In this section, we characterize the theoretical properties of the risk estimate $\hat{\rho}$ and show that the properties of the CDF estimators translate directly to the risk estimators built upon them. Notably, the reduced variance of the DR CDF estimator results in risk estimators with reduced bias and variance, and for Lipschitz risk functionals (Definition 3.1), the error of $\hat{\rho}$ is proportional $\epsilon$. For uniformly consistent CDF estimators, such as the previously introduced IS (4) and DR (11) estimators, this implies that their downstream risk estimates will be consistent as well.

7.1 Bias, Variance, and Error

In general, the risk estimator $\hat{\rho}$ (13) can be biased even if the CDF estimator is unbiased. Intuitively, however, $\hat{\rho}$ may have lower variance (and thus error) when it is estimated from a CDF estimate $\hat{F}$ with lower variance, because it is computed directly from $\hat{F}$ itself.

For Lipschitz risk functionals, we formalize this intuition as a finite-sample convergence guarantee (Corollary 7.1), which demonstrates that the error of any Lipschitz risk estimator is upper bounded by its Lipschitz constant (from Definition 3.1), and the error (upper bound) of the CDF estimate it utilizes.

**Corollary 7.1.** For all Lipschitz risk functionals simultaneously, given a CDF estimator $\hat{F}$ with error $\epsilon$, we have with probability at least $1 - \delta$ that

$$|\rho(F) - \rho(\hat{F})| \leq L\epsilon,$$  \hspace{1cm} (14)

where $L$ is the Lipschitz constant of $\rho$. Further, if $\hat{F}$ is uniformly consistent, then $\rho(\hat{F}) \xrightarrow{p} \rho(F)$.

As we have seen in Sections 4 and 6, the error guarantees for $\hat{F}$ are functions of its variance. Thus, Corollary 7.1 demonstrates that we can expect to have faster convergence for risk estimates estimated on lower-variance CDFs. As an additional consequence, downstream risk estimators built from the IS or DR CDF estimators, which have error that scale with $O(1/\sqrt{n})$, are guaranteed to be consistent.

7.2 Lower Bound

We have previously demonstrated that the IS CDF estimator is minimax rate-optimal, and demonstrated that the DR CDF estimator can achieve the variance lower bound for off-policy CDF estimation. However, the natural question to ask is: are their downstream risk estimators also rate-optimal? For general risk functionals this remains an open problem, but we show that for the Lipschitz risk functionals of CVaR$_{\alpha}$ and expected return (which is equivalent to CVaR$_{0}$), their off-policy risk estimator $\hat{\rho}$ has the following lower bound.
Theorem 7.1. Let $\mathcal{F}$ be the family of CDFs with bounded support and $\rho = \text{CVaR}_\alpha$. For $\alpha \in [0,1)$ and $n \geq c \frac{w_{max}^2}{\mathbb{E}_{\rho}[w_H]}$ for a universal constant $c > 0$,

$$\inf_{\hat{\rho}} \sup_{F \in \mathcal{F}} \mathbb{E}[|\hat{\rho} - \rho(F)|] \geq \frac{cD}{1 - \alpha} \sqrt{\frac{\mathbb{E}_{\rho}[w_H^2]}{n}}.$$ 

For large sample sizes, also matches existing results on off-policy mean estimation [Li et al., 2015, Wang et al., 2017, Ma et al., 2021]. As the Lipschitz constants for CVaR$_\alpha$ and expected return are $\frac{D}{1-\alpha}$ and $D$, respectively, the convergence rate of Corollary 7.1 with IS estimation is indeed minimax rate-optimal.

8 Experiments

We now provide empirical support for our theoretical results, and compare the performance of the CDF and plug-in risk estimators in Figure 1. We compare the DR and WDR estimators against the importance sampling estimators—F-IS (4), S-IS (8), C-IS (9), WIS (§4.2)—and model-based DM estimator. Our primary baselines of comparison are the F-IS and WIS estimators, which are
the only two off-policy CDF/risk estimators in existing work [Chandak et al., 2021a]. As we will see, the DR and WDR estimators outperform the IS-based estimators in all studied cases.

**Environment** We utilize two experimental domains: a tabular Cliffwalk [Sutton and Barto, 2018], and continuous diabetes treatment simulator [Xie, 2018]. The horizons of both are \( H = 200 \), and the behavioral policy \( \beta \) is a mixture between the target \( \pi \) and a uniform policy \( \text{UNIF} \), i.e., \( \beta = \lambda \pi + (1 - \lambda) \text{UNIF} \). Appendix E includes the full setup and additional experiments. Both environments have risk-related interpretations: in Cliffwalk the agent must take the shortest path while avoiding a slippery, high-cost cliff, while in Simglucose the agent must maintain a patient’s glucose levels within a healthy range.

**Risk Functionals** As such, we estimate one each of risk-neutral, risk-averse, and risk-seeking risk functionals using OPRA: the expected return, CVaR\(_{\alpha}\), and a risk functional we call CCaR\(_{\alpha}\) (conditional cost-at-risk), respectively. While CVaR\(_{\alpha}\) is the expectation of the worst-case \( \alpha \) tail of the distribution, CCaR is the expectation of the best-case \( \alpha \) tail.

**Results** In both environments, the DR and WDR estimators outperform the IS estimators and their variants in CDF and risk estimation error (Figure 1). These environments present challenges for IS estimators because of their long horizons, but DR and WDR effectively overcome them by incorporating model information. At very low sample sizes, however, the DM estimator can outperform the other estimators.

Our results also demonstrate the variability of the CDF estimator performance for different risk functionals; the gap between F-IS and S-IS is especially obvious. Under the Cliffwalk cost distribution, S-IS far outperforms F-IS for both the risk-neutral expected return and risk-sensitive CVaR\(_{\alpha}\). For the risk-seeking CCaR\(_{\alpha}\), however, F-IS outperforms S-IS. The combined estimator C-IS never does worse than either F-IS or S-IS, and as \( n \) increases can do better than both, as Proposition 4.1 implied. The DM estimator has low error under CCaR\(_{\alpha}\) because the model estimates the lower tail well, but the upper tail poorly. The DR and WDR estimator straddle best of both worlds; they perform well under all evaluated risk functionals.

9 Discussion

In this work, we have introduced and analyzed variance-reduced off-policy CDF and risk estimators for MDPs, including the first doubly robust estimator for return CDF estimation. Several future research directions are of interest. First, it is possible that estimators and bounds can be tailored and tightened for individual risk functionals, and a thorough comparison of different risk estimators is an important avenue of future work. Second, our results have also highlighted the risk estimation performance disparity between different types of CDF estimators. This disparity is more severe for risk-averse or risk-seeking risk functionals than expected value. As such, adaptive estimator selection (based on risk functionals of interest) may prove to be of both theoretical and practical interest. Lastly, in off-policy evaluation of the mean, marginalized importance sampling [Xie et al., 2019] has been shown to achieve the Cramer-Rao lower bound, and model-free regression [Duan et al., 2020] has been proven to be minimax optimal. One important future direction is to investigate whether
these methods can be extended to off-policy CDF or risk estimation, which, unlike the mean, are nonlinear functions of the return.

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A Preliminaries: Risk Overview

Let $Z$ be a random variable. A risk functional $\rho$ is a mapping from a space of random variables to the space of real numbers $\rho : \mathcal{L}_\infty(\Omega, \mathcal{F}_Z, \mathbb{P}_Z) \to \mathbb{R}$. Risk functionals are law-invariant if they depend only on the distribution of $Z$ [Kusuoka, 2001]. Formally $\rho(Z)$ is law invariant if for any pair of random variables $Z$ and $Z'$, $F_Z = F_{Z'} \implies \rho(Z) = \rho(Z')$.

Lipschitz risk functionals are a subset of law-invariant risk functionals, and we now delineate popular classes of risks and their associated Lipschitz constants, where possible, adapted from Huang et al. [2021a].

Distorted Risk Functionals For $Z \geq 0$, distortion risk functionals are defined to be Denneberg [1990], Wang [1996], Wang et al. [1997], Balbás et al. [2009] $$\rho(F_Z) = \int_0^\infty g(1 - F_Z(t))dt = \int_0^\infty g(S_Z(t))dt$$ where the distortion $g : [0, 1] \to [0, 1]$ has $g(0) = 0$ and $g(1) = 1$, and is increasing. When $g(x) = x$, the expected value is recovered. When $g(x) = \min\{x^{1-\alpha}, 1\}$ for $\alpha \in (0, 1)$, CVaR at level $\alpha$ is recovered.

When $g(x) = F(F^{-1}(x) - F^{-1}(\alpha))$, the Wang risk functional at level $\alpha$ [Wang, 1996] is recovered, and the proportional hazard risk functional can be obtained by setting $g(x) = x^\alpha$ for $\alpha < 1$. Distorted risk functionals are coherent if and only if $g$ is concave [Wirch and Hardy, 2001]. Not all distorted risk functionals are coherent. For example, setting $g(s) = \mathds{1}_{s \geq 1 - \alpha}$ recovers the value-at-risk (VaR), which is not coherent.

Coherent Risk Functionals Coherent risk functionals satisfy properties called monotonicity, subadditivity, translation invariance, and positive homogeneity Artzner et al. [1999], Delbaen [2002]. Examples include expected value, conditional value-at-risk (CVaR), entropic value-at-risk, and mean semideviation, proportional hazard, and Wang transform [Chang et al., 2020, Tamkin et al., 2019, Tamar et al., 2015, Shapiro et al., 2014, Wirch and Hardy, 2001, Wang, 1996].

Lemma A.1 (Lipschitzness of Coherent and Distorted Risk Functionals Huang et al. [2021b]). On the space of random variables with support in $[0, D]$, the distorted risk functional of any $\frac{D}{\alpha}$-Lipschitz distortion function is a $D$-Lipschitz risk functional.

The expected value risk functional is $D$-Lipschitz and CVaR$_\alpha$ is $\frac{D}{\alpha}$-Lipschitz.

Cumulative Prospect Theory (CPT) Risk Functionals CPT risks Prashanth et al. [2016] are defined as: $$\rho(F_Z) = \int_0^{+\infty} g^+(1 - F_{u^+(Z)}(t)) dt - \int_0^{+\infty} g^-(1 - F_{u^-(Z)}(t)) dt,$$ where $g^+, g^- : [0, 1] \to [0, 1]$ are continuous with $g^+/(0) = 0$ and $g^-/(1) = 1$. The functions $u^+, u^- : \mathbb{R} \to \mathbb{R}_+$ are continuous, with $u^+(z) = 0$ when $z \geq c$ and $u^-(z) = 0$ when $z < c$ for some constant $c \in \mathbb{R}$. The CPT functional handles gains and losses separately, applying distortion
Lemma A.2 (Lipschitzness of CPT Functional). If the CPT distortion functions $g^+$ and $g^-$ are both $\frac{L}{2}$-Lipschitz, then the CPT risk functional is $L$-Lipschitz.

Other Risk Functionals Many other risk functionals, though not members of a specific class, can be shown to be Lipschitz. This includes the variance and mean-variance. The variance is defined as $
abla(F) = 2 \int_{0}^{\infty} t(1 - F_Z(t)) dt - (\int_{0}^{\infty} (1 - F_Z(t)) dt)^2$.

Lemma A.3 (Lipschitzness of Variance). On the space of random variables with support in $[0, D]$, variance is a $3D^2$-Lipschitz risk functional. For mean-variance given by $\rho(Z) = \mathbb{E}[Z] + \lambda \mathbb{V}(Z)$ for some $\lambda > 0$, it is $(1 + 3\lambda D^2)$-Lipschitz.

B Proofs for Importance Sampling CDF Estimation (Section 4)

B.1 Bias and Variance (Proof of Lemma 4.1)

As the IS estimator can be seen as a special case of the DR estimator with $F = 0$, we defer the proof of bias and variance to Appendix C.1.

B.2 Error Upper Bound (Proof of Lemma 4.2)

The proof of the upper bound closely follows the proof of Theorem 5.1 from Huang et al. [2021b]. For any $\lambda > 0$, we have

$$\nabla \left[ \left\| \hat{F} - F \right\| \right] = \nabla \left[ \log \exp \left( \lambda \left\| \hat{F} - F \right\| \right) \right]
\leq \lambda \nabla \left[ \exp \left( \lambda \left\| \hat{F} - F \right\| \right) \right]
$$

To upper bound the RHS, define the following function class:

$$\hat{F}(n) := \left\{ f(g) := \frac{1}{n} \mathbb{1}_{(g \leq t)} : \forall t \in \mathbb{R}; \forall g \in \mathbb{Q}, g \in \{-1, +1\} \right\}
$$

Note that this is a countable set. Denote $w(\tau) = \prod_{h=1}^{H} w(a_h, s_h)$, and $g^i = \sum_{h=1}^{H} \gamma^h r_h^i$ the return of trajectory $i$ for short. Using this definition, we have

$$\sup_{t \in \mathbb{R}} \left| \hat{F}_{IS}(t) - F(t) \right| = \sup_{f \in \hat{F}(n)} \left| \frac{1}{n} \sum_{i=1}^{n} w(\tau^i) f(g^i) - \mathbb{E}_{\beta} \left[ \frac{1}{n} \sum_{i=1}^{n} w(\tau^i) f(g^i) \right] \right|
$$

Then adapting the proof of Theorem 5.1 in Huang et al. [2021b], we have for any $\lambda \in (0, \frac{n}{n w_{max}})$ that,

$$\nabla \left[ \exp \left( \lambda \left\| \hat{F} - F \right\| \right) \right] \leq 4 \exp \left( \frac{n \lambda^2 \mathbb{E}_{\beta}[w(\tau^2)]}{2 \left( 1 - \lambda^2 n w_{max} \right)} \right)
$$
Then
\[
\mathbb{E} \left[ \| \hat{F} - F \|_\infty \right] \leq \frac{1}{\lambda} \log \left( 4 \exp \left( \frac{n\lambda^2 4\mathbb{E}_\beta [w(\tau)^2]}{2 \left( 1 - \frac{\lambda^2}{n} w_{\text{max}} \right)} \right) \right) \\
\leq \frac{\log 4}{\lambda} + \frac{2\lambda\mathbb{E}_\beta [w(\tau)^2]}{n(1 - \frac{\lambda^2}{n} w_{\text{max}})}
\]
(15)

Since this holds for any \( \lambda \in (0, \frac{n}{2w_{\text{max}}}) \), it also holds for \( \min \lambda \) of the RHS. To find the minimizer, we solve
\[
\frac{d}{d\lambda} \left( \frac{\log 4}{\lambda} + \frac{2\lambda\mathbb{E}_\beta [w(\tau)^2]}{n - 2\lambda w_{\text{max}}} \right) = 0
\]
which WolframAlpha tells us is solved by
\[
\lambda^* = \frac{n}{\sqrt{\mathbb{E}_\beta [w(\tau)^2] n \log 2 + 2w_{\text{max}}}}
\]
if \( \mathbb{E}_\beta [w(\tau)^2] n \neq w_{\text{max}}^2 \log 16 \). We observe that \( \lambda^* \in (0, \frac{n}{2w_{\text{max}}}) \) satisfies the constraints on \( \lambda \). Plugging \( \lambda^* \) back into (15) we have,
\[
\mathbb{E} \left[ \| \hat{F} - F \|_\infty \right] \leq \frac{\log 4}{\sqrt{\mathbb{E}_\beta [w(\tau)^2] n \log 2 + 2w_{\text{max}}}} + \frac{2n\mathbb{E}_\beta [w(\tau)^2]}{n - \sqrt{\mathbb{E}_\beta [w(\tau)^2] n \log 2 + 2w_{\text{max}}}}
\]
\[
= \log 4\sqrt{\log 2} \sqrt{\frac{\mathbb{E}_\beta [w(\tau)^2]}{n}} + \frac{2 \log 4 w_{\text{max}}}{n} + 2 \sqrt{\frac{\mathbb{E}_\beta [w(\tau)^2]}{n \log 2}}
\]
\[
\leq c_1 \sqrt{\frac{\mathbb{E}_\beta [w(\tau)^2]}{n}} + c_2 \frac{w_{\text{max}}}{n}
\]

### B.3 CDF Estimation Lower Bound (Proof of Theorem 4.1)

We begin by deriving a CDF estimation lower bound in multi-armed bandits (MABs). We can reduce the estimation problem in the MDP case to the MAB case by treating entire trajectories \( \tau \) as “actions”.

Let a multi-armed bandit be defined by \( K \) arms, and suppose the behavioral policy is given by \( \beta(a) \) for \( a \in [K] \), and the target policy is \( \pi(a) \) for \( a \in [K] \), with importance weight \( w(a) := \beta(a)/\pi(a) \). Let \( w_{\text{max}} = \max_a w(a) \).

**Lemma B.1 (CDF Lower Bound in MABs).** When \( n \geq c \frac{w_{\text{max}}^2}{\sum_a \beta(a) w(a)} \) for a universal constant \( c \), the minimax lower bound on off-policy CDF estimation in MABs is
\[
\inf_{\hat{F}} \sup_{F \in \mathcal{F}} \mathbb{E} \left[ \| \hat{F} - F^\pi \|_\infty \right] \geq \sqrt{\frac{\sum_{a \in [K]} \beta(a) w^2(a)}{n}}.
\]
(16)
We now lift this result to the MDP case. For a horizon of length $H$, the space of possible actions of size $|\tau| = |S|^{H+1}|A|^H$. Recall that in MDPs, $w_{\max} = \max \tau w(\tau)$, and $w(\tau) = \frac{\pi(\tau)}{\beta(\tau)} = \prod_{h=1}^{H} \frac{\pi(a_h|s_h)}{\beta(a_h|s_h)}$.

Thus using Lemma B.1, when $n \geq c\sum_\tau \mathbb{P}(\beta(\tau)w^2(\tau))$, we have

$$\inf_{\hat{F}} \sup_{F \in \mathcal{F}} \mathbb{E}\left[\|\hat{F} - F^\pi\|_\infty\right] \geq c\sqrt{\frac{\sum_\tau \mathbb{P}(\beta(\tau)w^2(\tau))}{n}}.$$  \hspace{1cm} (17)

This matches our Bernstein-style bound on the error of IS estimation up to a constant:

$$\mathbb{E}\left[\|\hat{F} - F\|_\infty\right] \leq c_1 \sqrt{\frac{\sum_\tau \mathbb{P}(\beta(\tau)w^2(\tau))}{n}} + c_2 w_{\max}\frac{n}{n}.$$  \hspace{1cm} (18)

**Proof of Lemma B.1.** We use Le Cam’s method to derive the lower bound. We first need to select two problem MAB problem instances, by defining their reward distributions $R_1, R_2$. Then define distributions $P_1 = \beta \circ R_1$, and $P_2 = \beta \circ R_2$.

Intuitively, we want to select them so that the difference $\|F_1^\pi - F_2^\pi\|_\infty$ is large enough that an estimator can distinguish between them, but with the distance $D_{\text{KL}}(P_1\|P_2)$ to be small enough that the testing problem is not too easy.

For each action $a \in [K]$, let $R_2(\cdot|a)$ be a Bernoulli distribution on the set $\{0,1\}$ with probability $p = 1/2$, and let $R_1(\cdot|a)$ be a Bernoulli distribution with $p = 1/2 + \delta(a)$ for some $\delta(a) \in [0,1/2]$. Then

$$\|F_1^\pi - F_2^\pi\|_\infty = \max_{t \in \{0,1\}} |F_1^\pi(t) - F_2^\pi(t)|$$

$$= |F_1^\pi(t = 0) - F_2^\pi(t = 0)|$$

$$= \left|\sum_a \pi(a)\delta(a)\right|$$

At the same time,

$$D_{\text{KL}}(P_1\|P_2) = \sum_a \beta(a)D_{\text{KL}}(R_1\|R_2) \leq 4 \sum_a \beta(a)\delta^2(a)$$

Since we want the probability of error in our test to be a constant, we want to upper bound the above by $\frac{1}{2n}$. Thus to obtain as tight a lower bound as possible, we want to solve the optimization problem

$$\max_{a \in [K]} \sum_a \pi(a)\delta(a) \quad \text{s.t.} \quad 4 \sum_a \beta(a)\delta^2(a) \leq \frac{1}{2n}, \quad \delta \in [0,1/2]$$  \hspace{1cm} (19)

Using Lemma 3 from Ma et al. [2021], the Fenchel dual of the optimization problem in the above panel is

$$\min_{\nu \in \mathbb{R}^K} \sqrt{\frac{1}{8n} \sum_a (\pi(a) - \nu(a))^2} \beta(a) + \frac{1}{2} \sum_a |\nu(a)|$$  \hspace{1cm} (20)
Let $\nu^*$ be the solution to (20) and define the set of actions

$$S^* = \{a \in [K] | \nu^*(a) > 0\}$$

Then Lemma 2 from Ma et al. [2021] states that

$$\min_{\nu \in \mathbb{R}^K} \frac{1}{8n} \sum_a (\pi(a) - \nu(a))^2 + \frac{1}{2} \sum_a |\nu(a)| \asymp \pi(S^*) + \sqrt{n \sum_{a \notin S^*} \beta(a)w^2(a)}$$

Thus we have that for some universal constant $c$,

$$\inf_{\hat{F}} \sup_{F \in \mathcal{F}} \mathbb{E} \left[ \|\hat{F} - F^p\|_\infty \right] \geq c \left\{ \pi(S^*) + \sqrt{n \sum_{a \notin S^*} \beta(a)w^2(a)} \right\}$$

(21)

When we have sample size $n$ such that $n \geq c \frac{w^2_{\max}}{\sum_n \beta(a)w^2(a)}$, $S^* = \emptyset$ and the lemma statement follows.

### B.4 Proofs for Section 4.3

#### B.4.1 Bernstein Bounds for $S$ and $F$ (Proof of Lemma 4.3)

**Definition B.1 (Bounded Variation).** A function $f$ has bounded variation if there exists $C < \infty$ such that

$$\sup_{N,\{t_i\}_{i=1}^N} \sum_{i=1}^N |f(t_{i+1}) - f(t_i)| \leq C$$

**Assumption B.1.** Suppose that the random variable of returns takes values on a compact set. Assume the variance of the IS estimator $\hat{F}(t)$ has bounded variation for all $t \in \mathbb{R}$.

**Lemma B.2.** If a function $f$ has bounded variation, then there exists monotone functions $f^+, f^-$ with bounded variation such that $\forall t$,

$$f(t) = f^+(t) - f^-(t)$$

**Proof.** Using Lemma B.2, let $\hat{F} = \mathbb{V}^+ - \mathbb{V}^-$, and note that $\mathbb{V}^+, \mathbb{V}^- \in [0, d_\infty]$ and are monotone functions. As they are monotone functions, let $\{s_j^+\}_{j=1}^{M}$ and $\{s_j^-\}_{j=1}^{M}$ be the respective stepping points for their stepwise approximations. Further, let $\{s_k^F\}_{k=1}^{M'}$ be the stepping points for the CDF $F$. Finally, overriding notation slightly, let $\{s_j\}_{j=1}^{M'+2M}$ be the union of these three sets of stepping points. We know that between adjacent stepping points, $F$, $\mathbb{V}^+$, and $\mathbb{V}^-$ do not change much. Let $\zeta^F, \zeta^+, \zeta^-$ be the step functions constructed using the stepping points for each function, respectively,
and let $\xi^+ = \xi^+ - \xi^-$. Formally, this means that

\[
\sup_{t \in \mathbb{R}} |F(t) - \xi^+(t)| \leq 1/2M \\
\sup_{t \in \mathbb{R}} |V^+(t) - \xi^+(t)| \leq 1/2M \\
\sup_{t \in \mathbb{R}} |V^-(t) - \xi^-(t)| \leq 1/2M \\
\sup_{t \in \mathbb{R}} |\hat{V}(t) - \xi^+(t)| \leq 1/M
\]

At a given $s_j$, Bernstein’s inequality gives us

\[
|F(s_j) - \hat{F}(s_j)| \leq \frac{c_1 \log(1/\delta)}{n} + \sqrt{\frac{c_2 V^+(s_j) \log(1/\delta)}{n}}
\]

Taking a union bound, this holds for all $\{s_j\}_{j=1}^{M'+2M}$ with probability at least $1 - \delta(M' + 2M)$.

For each $j$ and $t \in [s_j, s_{j+1}]$, if $F(t) \geq \hat{F}(t)$,

\[
F(t) - \hat{F}(t) \leq F(s_{j+1}) - \hat{F}(s_j) \\
\leq \frac{1}{M'} + F(s_j) - \hat{F}(s_j) \\
\leq \frac{1}{M'} + \frac{c_1 \log(1/\delta)}{n} + \sqrt{\frac{c_2 V^+(s_j) \log(1/\delta)}{n}}
\]

If $\hat{F}(t) > F(t)$,

\[
\hat{F}(t) - F(t) \leq \hat{F}(s_{j+1}) - F(s_j) \\
\leq \frac{1}{M'} + |\hat{F}(s_{j+1}) - F(s_{j+1})| \\
\leq \frac{1}{M'} + \frac{c_1 \log(1/\delta)}{n} + \sqrt{\frac{c_2 V^+(s_{j+1}) \log(1/\delta)}{n}}
\]

Choosing $M, M' \propto \sqrt{n}$ gives the resulting bound.

Similarly, for $\hat{F}_{S-IS}$, we have

\[
F - \hat{F}_{S-IS} = 1 - S - (1 - \hat{S}_{IS}) = \hat{S}_{IS} - S
\]

The remainder of the proof occurs in the same manner, but with $\mathbb{V}[\hat{S}_{IS}]$ on the RHS.
B.4.2 Empirical Bernstein Bound (Proof of Lemma 4.4)

Proof. We begin with the Bernstein bounds for \( \hat{F}_{F-IS}(t_j) \), \( \hat{F}_{S-IS}(t_j) \) at a given \( t_j \). That is,

\[
|F(t_j) - \hat{F}_{F-IS}(t_j)| \leq \frac{c_1 \log(1/\delta)}{n} + \sqrt{\frac{c_2 \mathbb{V}[\hat{F}_{IS}(t_j)] \log(1/\delta)}{n}}
\]

and a similar statement holds for \( \hat{F}_{S-IS} \). From Maurer and Pontil [2009], we have the following bound on estimation of the variance:

Lemma B.3 (Theorem 10 from Maurer and Pontil [2009]). With probability at least \( 1 - \delta \), for random variables \( X \in [0, 1] \),

\[
\left| \sqrt{\mathbb{V}_n} - \sqrt{\mathbb{V}} \right| \leq \sqrt{\frac{2 \ln 1/\delta}{n-1}} \tag{22}
\]

Taking a union bound between Lemma B.3 and Lemma 4.3 at all \( M \) points, we have that with probability at least \( 1 - \delta \) that for all \( t_j \),

\[
|F(t_j) - \hat{F}_{F-IS}(t_j)| \leq \frac{c_1 \log(2M/\delta)}{n} + \sqrt{\frac{c_2 \mathbb{V}_n[\hat{F}_{IS}(t)] \log(2M/\delta)}{n}} \]

\[
+ \sqrt{\frac{2c_2 \log^2(2M/\delta)}{n(n-1)}} + \sqrt{\frac{c_2 \log(2M/\delta)}{Mn}}
\]

Similar steps give the bound for \( \hat{F}_{S-IS} \), with \( \mathbb{V}_n[S_{IS}(t)] \) on the RHS.

\( \square \)

B.4.3 Empirical Bernstein Bound for Combined Estimator (Proof of Proposition 4.1)

In order to correctly choose either \( \hat{F}_{F-IS} \) or \( \hat{F}_{S-IS} \) for each \( t_j \), we need that for all \( t_j \), the empirical variance estimates \( \mathbb{V}_n \) are separated by at least \( \epsilon_{\mathbb{V}}(t) = \frac{1}{2} |\mathbb{V}[\hat{F}_{F-IS}(t)] - \mathbb{V}[\hat{F}_{S-IS}(t)]| \leq \frac{1}{2} |\mathbb{V}_n[\hat{F}_{F-IS}(t)] - \mathbb{V}_n[\hat{F}_{S-IS}(t)]| + \sqrt{\frac{\ln 1/\delta'}{2(n-1)}} \). For this to be the case, we set

\[
\delta(t) = \frac{1}{2} \exp \left( -\frac{1}{2} (n-1) \epsilon_{\mathbb{V}}(t)^2 \right),
\]

and \( \delta = \max_j \delta(t_j) \). Thus if

\[
n \geq \frac{8 \log(2n/\delta)}{\max_j \epsilon_{\mathbb{V}}(t_j)^2},
\]

then we correctly choose \( \hat{F}_{F-IS} \) or \( \hat{F}_{S-IS} \) for each \( t_j \), which gives the result.
C Proofs for Doubly Robust CDF Estimation (Section 6)

C.1 Bias and Variance (Proof of Lemma 6.1)

We will prove the unbiasedness of the DR estimator by induction. As the base case, $F_{S_{h+1}}^0(t) = 0$ for all states $s$ and for all $t$, which is unbiased by definition.

As the induction assumption, we assume that the DR estimator at the step $h+1$ is unbiased, that is $E_{h+1}[^{\hat{F}}_{S_{h+1}}(t)] = E_{h+1}[F_{S_{h+1}}(t)]$.

We need to prove that it is also unbiased for the $h$th step. Recall for now we assume that rewards are deterministic. Then for any $t$:

$$E_h[^{\hat{F}}_{S_{h}}(t)] = E_h\left[ \frac{F_{S_h}^{H+1-h}(t) + w(A_h, S_h) \left( \frac{t - R(S_h, A_h)}{\gamma} \right)}{H+1-h}(t) \right]$$

$$= E_h\left[ \frac{F_{S_h}^{H+1-h}(t) - w(A_h, S_h)F_{S_{h},A_h}^{H+1-h}(t)}{\gamma} \right] + E_h\left[ w(A_h, S_h)F_{S_{h+1}}^{H-h}(t) \right]$$

$$= E_h\left[ \frac{F_{S_h}^{Z(S_h)}(t) - E_h[F_{S_{h},A_h}^{H+1-h}(t)|S_h]}{\gamma} \right]$$

$$= E_h\left[ \frac{F_{S_{h+1}}^{H-h}(t) - E_h[F_{S_{h+1}}^{H+1-h}(t)]}{\gamma} \right] + E_h\left[ E_{h+1}\left[ \frac{F_{S_{h+1}}^{H-h+1}(t) - E_{h+1}[F_{S_{h+1}}^{H-h+1}(t)]}{\gamma} \right] \right]$$

$$= E_h\left[ F_{S_{h+1}}(t) \right]$$

where the third equality incorporates the importance sampling weight $w$ into a change of measure from $\beta$ to $\pi$, the fourth equality uses the conditional expectation. The second to last equality uses the induction assumption and the last uses the recursive identity.

Next, we derive the variance of the DR estimator.

$$\forall_h \left[ \frac{^{\hat{F}}_{S_{h}}(t)}{2} \right] = E_h\left[ \left( \frac{^{\hat{F}}_{S_{h}}(t)}{2} \right)^2 \right] - E_h\left[ \frac{^{\hat{F}}_{S_{h}}(t)}{2} \right]^2$$

$$= E_h\left[ \left( \frac{F_{S_h}^{H+1-h}(t) + w(A_h, S_h) \left( \frac{t - R(S_h, A_h)}{\gamma} \right)}{H+1-h}(t) \right)^2 - F_{S_{h}}(t)^2 \right]$$

$$+ E_h\left[ F_{S_{h+1}}(t)^2 \right] - E_h\left[ F_{S_{h}}(t) \right]^2$$

$$= E_h\left[ \left( \frac{F_{S_h}^{H+1-h}(t) + w(A_h, S_h) \left( \frac{t - R(S_h, A_h)}{\gamma} \right)}{H+1-h}(t) \right)^2 - F_{S_{h}}(t)^2 \right]$$

$$+ \forall_h F_{S_{h}}(t)$$

$$= E_h\left[ \left( \frac{F_{S_h}^{H+1-h}(t) + w(A_h, S_h)F_{S_{h},A_h}^{H+1-h}}{2} - w(A_h, S_h)F_{S_{h},A_h}^{H+1-h} \right) \right]$$

$$+ w(A_h, S_h) \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - F_{S_{h+1}}(t) \right)^2 - F_{S_{h}}(t)^2 \right] + \forall_h F_{S_{h}}(t)$$

26
\[ \mathbb{E}_h \left[ (F_{S_h}^{H+1-h}(t) - w(A_h, S_h) \Delta_{S_h, A_h}(t) \\ + w(A_h, S_h) (\hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right])^2 - F_{S_h}(t)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{F_{S_h}^{H+1-h}(t) - w(A_h, S_h) \Delta_{S_h, A_h}(t)}{\gamma} \right)^2 - F_{S_h}(t)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{w(A_h, S_h) \hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right]}{\gamma} \right)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{w(A_h, S_h) \hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right]}{\gamma} \right)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{w(A_h, S_h) \hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right]}{\gamma} \right)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{w(A_h, S_h) \hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right]}{\gamma} \right)^2 \right] \\ + \mathbb{E}_h \left[ \left( \frac{w(A_h, S_h) \hat{F}_{S_h} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) - \mathbb{E}_{h+1} \left[ \hat{F}_{S_h+1} + \left( \frac{t - R(S_h, A_h)}{\gamma} \right) \right]}{\gamma} \right)^2 \right] \right] \\
\]

where \( \Delta_{s,a}(t) = F_{s,a}(t) - F_{s,a}(t) \).

### C.2 Error Bound on Doubly Robust Estimator (Proof of Lemma 6.2)

We separate the bound into two parts using the definition of \( \hat{F}_{\text{DR}} \):

\[ \left\| F - \hat{F}_{\text{DR}} \right\|_\infty \leq \left\| F - \frac{1}{n} \sum_{i=1}^{n} w(\tau^i) \mathbb{I} \left\{ \sum_{h=1}^{H} \gamma^h r^h \leq t \right\} \right\|_\infty \\
+ \left\| \frac{1}{n} \sum_{i=1}^{n} w(\tau^i) \sum_{j=1}^{m} w(s^h, a^h) \mathbb{I} \{ s^h, a^h \} \mathbb{I} \{ s^h, a^h \} \} \right\|_\infty \]

Lemma 4.2 upper bounds the first term, and we are left to bound the second.

Let \( \tau = (s^0, a^0, r^0, \ldots, s^h, a^h, r^h, \ldots, s^H, a^H, r^H, s^{H+1}) \) be a trajectory, and define \( w(\tau_{1:h}) = \prod_{k=1}^{h} w(s^h, a^h) \).

Define the function class

\[ \mathbb{F}(m, w) = \left\{ f \left( \{ x_j^{i,j} \}_{j=1}^{M}_h \}_{h=1}^{H} : \forall t \in \mathbb{R}, \forall \{ x_j^{i,j} \}_{j=1}^{M}_h \in \mathbb{Q}^{M \times H} \right\} \]

Note that (23) is

\[ \leq \sup_{\zeta \in \mathbb{F}(m, w)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \zeta \left( \{ x^i_{S_h, A_h} \}, \tau_i \right) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{F_{\beta}} \left[ \zeta \left( \{ x^i_{S_h, A_h} \}, \tau_i \right) \left\{ S^h_{i} \right\}_{h=1}^{H} \right] \right\} + \frac{1}{m} \]

Going forward, we refer to \( \zeta \left( \{ x^i_{S_h, A_h} \}, \tau_i \right) \) as \( \zeta(\tau_i) \) for short. For \( \lambda > 0 \) we have

\[ \mathbb{E}_{F_{\beta}} \left[ \exp \left( \lambda \sup_{\zeta \in \mathbb{F}(m, w)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \zeta(\tau_i) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{F_{\beta}} \left[ \zeta(\tau_i) \left\{ S^h_{i} \right\}_{h=1}^{H} \right] \right\} \right) \right] \]
And we have the result, \( \tau \) where

\[
\text{Then we have}
\]

\[
\text{Now for each } h, j, \text{ permute the indices of } i \text{ with a new indexing } (i) \text{ such that}
\]

\[
x_{\bar{x}^{h}j, A_{(i)}h}^{i} \leq \ldots \leq x_{\bar{x}^{h}j, A_{(i)}h}^{j} \leq \ldots \leq x_{\bar{x}^{h}j, A_{(n)}h}^{j}.
\]

Then we have

\[
\mathbb{E}_{\bar{P}_{\beta, \mathbb{R}}} \left[ \sup_{q, t} \left( 2\lambda \frac{\varrho}{nm} \sum_{i=1}^{n} \sum_{h=1}^{H} w(\tau_{i}^{1:h-1}) \sum_{j=1}^{m} w(A_{i}^{h}, S_{i}^{h}) \mathbb{1}_{\{x_{\bar{x}^{h}j, A_{(i)}h}^{i} \leq t\}} \right) \right]
\]

\[
= \mathbb{E}_{\bar{P}_{\beta, \mathbb{R}}} \left[ \max_{q, k} \left( 2\lambda \frac{\varrho}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{k} \xi_{(i)} w(\tau_{(i)}^{1:k-1}) w(A_{(i)}^{h}, S_{(i)}^{h}) \right) \right]
\]

\[
\leq \mathbb{E}_{\bar{P}_{\beta, \mathbb{R}}} \left[ \max_{q, j, k} \left( 2\lambda \frac{\varrho}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{k} \xi_{(i)} w(\tau_{(i)}^{1:k-1}) w(A_{(i)}^{h}, S_{(i)}^{h}) \right) \right]
\]

\[
\leq 2 \sum_{j=1}^{m} \mathbb{E}_{\bar{P}_{\beta, \mathbb{R}}} \left[ \exp \left( 2\lambda \frac{\varrho}{n} \sum_{i=1}^{n} \sum_{h=1}^{H} w(\tau_{(i)}^{1:h}) \right) \mathbb{1}_{\{\sum_{i=1}^{n} \xi_{(i)} \sum_{h=1}^{H} w(\tau_{(i)}^{1:h}) \geq 0\}} \right]
\]

\[
\leq 2m \mathbb{E}_{\bar{P}_{\beta, \mathbb{R}}} \left[ \exp \left( 2\lambda \frac{\varrho}{n} \sum_{i=1}^{n} \sum_{h=1}^{H} w(\tau_{(i)}^{1:h}) \right) \right]
\]

\[
\leq 2m \exp \left( \frac{2\lambda^2 w_{\max}(w_{\max}^H - 1)}{n(w_{\max} - 1)} \right)
\]

And we have the result,

\[
\mathbb{P}_{\beta} \left( \cdot \geq \epsilon + \frac{1}{m} \right) \leq 4m \exp \left( \frac{2\lambda^2 w_{\max}(w_{\max}^H - 1)}{n(w_{\max} - 1)} - \lambda \epsilon \right)
\]

\[
\leq 4m \min_{\lambda > 0} \exp \left( \frac{2\lambda^2 w_{\max}(w_{\max}^H - 1)}{n(w_{\max} - 1)} - \lambda \epsilon \right)
\]

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\[ = 4m \exp \left( -n \epsilon \left( \frac{w_{\text{max}} - 1}{w_{\text{max}}^2} \right)^2 \right) \]

where the second line results because the inequality holds for all \( \lambda > 0 \), so we can minimize over \( \lambda \).

Then we have

\[ P_{\beta} \left( \cdot \geq \sqrt{\frac{2}{n(w_{\text{max}} - 1)^2} \log \frac{4m}{\delta} + \frac{1}{m}} \right) \leq \delta \]

Combining this with Lemma 4.2 to bound the first term gives the result and choice of \( m \propto \sqrt{n} \) gives the result.

### C.3 Cramer-Rao Variance Lower Bound of Off-Policy CDF Estimation (Proof of Theorem 6.1)

We prove a Cramer-Rao lower bound for UDAGs, defined below.

**Definition C.1** (Unique Directed Acyclic Graph). An MDP is unique directed acyclic graph (UDAG) if the state and action spaces are finite. A state only occurs at a particular horizon \( h \), that is, for any \( s \in S \), there exists a unique \( h \) such that \( \max_{\pi} P(s_h = s | \pi) > 0 \). In addition, each state has a unique reward function. For simplicity, assume \( \gamma = 1 \).

**Proof.** First, we transform the MDP of Definition C.1 into one with an augmented state space, including the history of rewards encountered. That is, a trajectory is now defined as

\[ \tau = (s_0, g_0, a_0, \ldots s_H, g_H, a_H, s_{H+1}, g_{H+1}) \]

where \( g_h = \sum_{k=0}^{h-1} r_h \). Note that \( P(g_{h+1} | s_h, g_h, a_h) = P(r_h | s_h, a_h) \) and \( P(s_{h+1} | s_h, g_h, a_h) = P(s_{h+1} | s_h, a_h) \).

For brevity, denote \( \tilde{s} = (s, g) \) to be the augmented state space. We now derive a pointwise Constrained Cramer-Rao Bound (CCRB) for \( F(t) \) in the augmented MDP,

\[ KU(UU)^{-1}U^\top K^\top \]

where \( K \) is the Jacobian of \( F(t) \) and \( I \) is the Fisher information matrix. \( U \) is a matrix whose column vectors consist of an orthonormal basis for the null space of a matrix \( N \), which is a block-diagonal matrix with \( N_{(\tilde{s}, a), (\tilde{s}', a')} = 1 \{ \tilde{s}, a = \tilde{s}', a' \} \). Further, define \( \eta_{\tilde{s}, a, \tilde{s}', a'} = P(\tilde{s}' | \tilde{s}, a) \), and note that \( N \eta = 1 \).

To compute the Fisher information matrix \( I \),

\[ I = E \left[ \left( \frac{\partial \log P_{\beta}(\tau)}{\partial \eta} \right) \left( \frac{\partial \log P_{\beta}(\tau)}{\partial \eta} \right)^\top \right] \]

where \( P_{\beta}(\tau) \) is the probability of a trajectory \( \tau \) under the behavioral policy \( \beta \),

\[ P_{\beta}(\tau) = \mu(s_1, g_1)\beta(a_1 | s_1)P(s_2 | s_1, a_1)P(g_2 | s_1, a_1, g_1) \ldots P(s_H, g_H | s_{H-1}, g_{H-1}a_{H-1})\beta(a_H | s_H)P(s_{H+1}, g_{H+1} | s_H, g_H, a_H) \]
We now need to calculate the Jacobian \( \frac{\partial \log P_h(\tau)}{\partial \eta} \). Define a new indicator vector \( g(\tau) \) with \( g(\tau)\bar{s}_h, a_h, \bar{s}_{h+1} = 1 \) if \( (\bar{s}_h, a_h, \bar{s}_{h+1}) \in \tau \). Then, letting \( \circ \) denote an elementwise operation, we have

\[
\frac{\partial \log P_h(\tau)}{\partial \eta} = \eta^{-1} \circ g(\tau)
\]

which makes

\[
I = \mathbb{E} \left[ \frac{1}{\eta \eta_j} \circ \left( g(\tau) g(\tau)^\top \right) \right] = \frac{1}{\eta \eta_j} \circ \mathbb{E} \left[ g(\tau) g(\tau)^\top \right]
\]

As a result, the elements of \( I \) are as follows, where \( P_M \) refers to the marginal probability:

- diagonal:
  \[
P_M(\bar{s}_h, a_h) / P(\bar{s}_{h+1}|\bar{s}_h, a_h)
  \]
- row = \( (\bar{s}_h, a_h, \bar{s}_{h+1}) \), column = \( (\bar{s}_h', a_h', \bar{s}_{h+1}') \):
  \[
P_M(\bar{s}_h, a_h) P_M(\bar{s}_h', a_h')
  \]
- otherwise: 0

We know calculate the term \( (U^\top IU)^{-1} \), and we use a similar strategy to Jiang and Li [2016], Huang and Jiang [2020] to avoid taking its inverse. Note that \( X \) is arbitrary, \( X \) was defined previously. Let \( D = N^\top X^\top + I + XN \). Our goal is to define \( X \) such that \( D \) is a diagonal matrix with diagonal identical to the diagonal of \( I \). Note that \( XN \) and \( N^\top X^\top \) are symmetric, so we can design the former to eliminate the upper triangle of \( I \), and the latter to eliminate the lower triangle. To verify we can do this, set

- row = \( (\bar{s}_h, a_h, \bar{s}_{h+1}) \), column = \( (\bar{s}_h', a_h', \bar{s}_{h+1}') \):
  \[
  -P_M(\bar{s}_h, a_h) P_M(\bar{s}_h', a_h') \mathbb{1} \{ h < h' \}
  \]
- otherwise: 0

Then with the proper choice of \( U \),

\[
U(\bar{U}^\top IU)^{-1}U^\top = \text{Diag}(\{ B(\bar{s}_h, a_h) \}^H_0)
\]

where Diag\( \{ \cdot \} \) is a block-diagonal matrix consisting of matrices in the set \( \{ \cdot \} \), and

\[
B(\bar{s}_h, a_h) = \frac{\text{Diag}(P(\cdot|\bar{s}_h, a_h)) - P(\cdot|\bar{s}_h, a_h) P(\cdot|\bar{s}_h, a_h)^\top}{P_M(\bar{s}_h, a_h)}
\]

where \( P(\cdot|\bar{s}_h, a_h) \) is the transition vector.

We now need to calculate the Jacobian \( K \). Recall that the estimation objective is

\[
F(t) = \sum_{\bar{s}_1} \mu(\bar{s}_1) \sum_{a_1} \pi(a_1|\bar{s}_1) \cdots \sum_{\bar{s}_H} P(\bar{s}_H|\bar{s}_{H-1}, a_{H-1}) \mathbb{1} \{ g_H \leq t \}
\]
Then $K(s_h, a_h, \tilde{s}_{h+1})$ is
\[
\frac{\partial F(t)}{\partial P(r_h, s_{h+1}|s_h, a_h)} = \frac{\partial F(t)}{\partial P(s_{h+1}, g_{h+1}|s_h, g_h, a_h)} = P(s_h, g_h, a_h)P(r_h|s_h, a_h)P(s_{h+1}|s_h, a_h)F_{s_{h+1}}(t - g_h - r_h)
\]
Denote by $K(s_h, a_h, \cdot)$ the vector fragment of $K$ whose index tuple starts with $s_h, a_h$. Then we have that
\[
KU(UI)^{-1}U^T = \sum_{h=1}^{H+1} \sum_{\tilde{s}_h, a_h} (K(s_h, a_h, \cdot))^\top B(s_h, a_h)K(s_h, a_h, \cdot)
\]
We have
\[
(K(s_h, g_h, a_h, \cdot))^\top B(s_h, g_h, a_h)K(s_h, g_h, a_h, \cdot)
\]
\[
= \frac{P(s_h, g_h, a_h)^2}{P_\beta(s_h, g_h, a_h)^2} \left( \sum_{r_h} P(r_h|s_h, a_h) \sum_{s_{h+1}} P(s_{h+1}|s_h, a_h)F_{s_{h+1}}(t - g_h - r_h)^2 \right)
\]
\[
- \left( \sum_{r_h} P(r_h|s_h, a_h) \sum_{s_{h+1}} P(s_{h+1}|s_h, a_h)F_{s_{h+1}}(t - g_h - r_h) \right)^2
\]
\[
= \frac{P(s_h, g_h, a_h)^2}{P_\beta(s_h, g_h, a_h)^2} \mathbb{V}_{r_h, s_{h+1}|s_h, a_h} [F_{s_{h+1}}(t - g_h + 1)]
\]
Under Definition C.1, we have that $P(g_h) = P(\tau_h)$ so the lower bound is
\[
\sum_{h=1}^{H} \mathbb{E}_{\overline{P}_\beta} \left[ w_{1,h-1}^2 \mathbb{V}_{r_h, s_{h+1}|s_h, a_h} \left[ F_{s_{h+1}} \left( t - \sum_{k=1}^{h} \tau_k \right) \right] \right].
\]

\[
\textbf{Comparison with DR Variance.}
\]

When expanded over $H$ horizons, the DR variance (6.1) in MDP of Definition C.1 is
\[
\sum_{h=1}^{H} \mathbb{E}_{\overline{P}_\beta} \left[ w_{1,h-1}^2 \mathbb{V}_h \left[ w(A_h, S_h) \Delta_{S_h, A_h} \left( t - \sum_{k=1}^{h} R_k \right) \right] \right] + \sum_{h=1}^{H} \mathbb{E}_{\overline{P}_\beta} \left[ w_{1,h-1}^2 \mathbb{V}_h \left[ F_{S_h} \left( t - \sum_{k=1}^{h} R_k \right) \right] \right]
\]
where $\Delta_{s,a}(t) = \overline{F}_{s,a}(t) - F(t)$. Thus when $F = \overline{F}$, the first term of the above expression goes to 0 and the DR variance becomes
\[
\sum_{h=1}^{H} \mathbb{E}_{\overline{P}_\beta} \left[ w_{1,h-1}^2 \mathbb{V}_h \left[ F_{S_h} \left( t - \sum_{k=1}^{h} R_k \right) \right] \right]
\]
which attains the Cramer-Rao lower bound.
D Proofs for Risk Estimation (Section 7)

D.1 CVaR Risk Lower Bound (Proof of Theorem 7.1)

To prove the lower bound for CVaR estimation, we make an adaptation of the proof in B.3. From Le Cam’s method, we have

\[
\inf_{\tilde{F}} \sup_{P_1, P_2 \in \mathcal{P}} \mathbb{E}_P [|\tilde{\rho} - \rho_{\tilde{F}}^P|] \geq \frac{1}{8} |\rho_1^\pi - \rho_2^\pi| e^{-nD_{KL}(P_1||P_2)}
\]

(25)

For a distortion risk functional \( \rho \),

\[
|\rho_1^\pi - \rho_2^\pi| = \left| \int_0^D g(1 - F_1^\pi(t)) - g(1 - F_2^\pi(t))dt \right|
\]

In the case of CVaR, with \( \alpha \in [0, 1) \), we have \( g(x) = \min\{\frac{x}{1-\alpha}, 1\} \) so

\[
|\text{CVaR}_\alpha(F_1^\pi) - \text{CVaR}_\alpha(F_2^\pi)| = \left| \int_0^D \min\left\{ \frac{1 - F_1^\pi(t)}{1 - \alpha}, 1 \right\} - \min\left\{ \frac{1 - F_2^\pi(t)}{1 - \alpha}, 1 \right\} dt \right|
\]

Note that, under this definition, the expected return is CVaR with level \( \alpha = 0 \).

Recall the Bernoulli problem instances we previously used to prove the lower bound for MABs. For the MAB \( P_2 \), we set the arm reward distribution to be \( R_2(0|a) = c \) for all \( a \in \mathcal{A} \). For the MAB \( P_1 \), we set the arm reward distribution to be \( R_1(0|a) = c + \delta(a) \) for some \( \delta(a) \in [0, 1 - c] \). Then

\[
|\text{CVaR}_\alpha(F_1^\pi) - \text{CVaR}_\alpha(F_2^\pi)| = \left| \int_0^1 \min\left\{ \frac{1 - F_1^\pi(t)}{1 - \alpha}, 1 \right\} - \min\left\{ \frac{1 - F_2^\pi(t)}{1 - \alpha}, 1 \right\} dt \right|
\]

\[
= \left| \min\left\{ \frac{1 - F_1^\pi(t = 0)}{1 - \alpha}, 1 \right\} - \min\left\{ \frac{1 - F_2^\pi(t = 0)}{1 - \alpha}, 1 \right\} \right|
\]

\[
= \left| \min\left\{ 1 - c - \sum_a \pi(a)\delta(a), 1 \right\} - \min\left\{ \frac{1 - c}{1 - \alpha}, 1 \right\} \right|
\]

Then note that the difference above has three possibilities:

\[
\begin{align*}
0, & \quad \text{if } 1 - \alpha \leq 1 - c - \sum_a \pi(a)\delta(a) \\
\frac{\alpha - c}{1 - \alpha}, & \quad \text{if } 1 - c - \sum_a \pi(a)\delta(a) \leq 1 - \alpha \leq 1 - c \\
\frac{\sum_a \pi(a)\delta(a)}{1 - \alpha}, & \quad \text{if } 1 - c \leq 1 - \alpha
\end{align*}
\]

By setting \( c = \alpha \) and \( \delta(a) \in [0, 1 - \alpha] \), it can be seen that the first scenario (a vacuous lower bound) is impossible, and that the third scenario encompasses the second scenario. That is, if \( \delta(a) = 0 \ \forall a \), then both the second and third cases become 0.

Then since

\[
D_{KL}(P_1||P_2) = \sum_a \beta(a)D_{KL}(R_1(a)||R_2(a)) \leq 4 \sum_a \beta(a)\delta(a)^2,
\]

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Our optimization problem boils down to

\[
\max_{\delta \in \mathbb{R}^4} \frac{\sum_a \pi(a) \delta(a)}{1 - \alpha} \text{ s.t. } \sum_a \beta(a) \delta(a)^2 \leq \frac{1}{8n}, \ 0 \leq \delta \leq 1 - \alpha
\]

Thus we have the Lagrangian

\[
L(\delta, \lambda, \nu) = -\frac{\sum_a \pi(a) \delta(a)}{1 - \alpha} + \lambda \left( \sum_a \beta(a) \delta(a)^2 - \frac{1}{8n} \right) + \sum_a \nu(a) (\delta(a) - 1 + \alpha)
\]

\[
= -\frac{\lambda}{8n} + \sum_a \left\{ \lambda \beta(a) \delta^2(a) + \delta(a) \left( \nu(a) - \frac{\pi(a)}{1 - \alpha} \right) - (1 - \alpha) \nu(a) \right\}
\]

The optimal solution is

\[
\delta^*(a) = \frac{\pi(a)}{1 - \alpha} - \nu(a)
\]

When we plug this in, we have

\[
-\frac{\sum_a \pi(a) \delta^*(a)}{1 - \alpha} = \max_{\lambda \geq 0, \nu \geq 0} g(\lambda, \nu) = \max_{\nu \geq 0} \left\{ -\frac{1}{8n} \sum_a \left( \frac{\pi(a)}{1 - \alpha} - \nu(a) \right) \right\} - (1 - \alpha) \sum_a \nu(a)
\]

\[
= -\min_{\nu \geq 0} \left\{ \frac{1}{8n} \sum_a \left( \frac{\pi(a)}{1 - \alpha} - \nu(a) \right) \right\} + (1 - \alpha) \sum_a \nu(a)
\]

Finally, we have from Lemma D.1 that

\[
\min_{\nu \geq 0} \left\{ \frac{1}{8n} \sum_a \left( \frac{\pi(a)}{1 - \alpha} - \nu(a) \right)^2 \beta(a) \right\} + (1 - \alpha) \sum_a \nu(a) \propto \frac{1}{1 - \alpha} \left\{ \sum_{a \in S^*} \pi(a) + \sqrt{\sum_{a \in S^*} \beta(a) w(a)^2} \right\}
\]

\[
\min_{\nu \geq 0} \left\{ \frac{1}{8n} \sum_a \left( \frac{\pi(a)}{1 - \alpha} - \nu(a) \right)^2 \right\} + (1 - \alpha) \sum_a \nu(a) \propto \sum_{a \in S^*} \pi(a) + \frac{1}{1 - \alpha} \sqrt{\sum_{a \in S^*} \beta(a) w(a)^2} \quad (27)
\]

**Proof.** First, note that \(0 \leq \nu^* \leq \pi(a)\) from the optimization problem (26). Further, WLOG we can assume that \(\pi(a) > 0\) and \(\beta(a) > 0\). This is because if \(\pi(a) = 0\), then \(w(a) = 0\), which contributes 0 to both sides of the equation. Similarly, if \(\beta(a) = 0\), then \(\nu^*(a) = \pi(a)\), which means \(a \in S^*\), and \(\pi(a)\) is contributed to both sides of the equation.

We separate the proof into three cases: \(\nu^* = \frac{\pi}{1 - \alpha}\), \(\nu^* = 0\), \(0 \neq \nu^* \neq \frac{\pi}{1 - \alpha}\).
\( \mathbf{v}^* = \pi \). In this case, \( S^* = [k] \), and the LHS of (27) is

\[
(1 - \alpha) \sum_a \frac{\pi(a)}{1 - \alpha} = 1
\]

while the RHS is

\[
\sum_a \pi(a) = 1.
\]

\( \mathbf{v}^* = 0 \). In this case, \( S^* = \{ \} \), and the LHS of (27) is

\[
\sqrt{\frac{1}{8n} \sum_a \frac{\pi(a)^2}{\beta(a)}} = \frac{1}{1 - \alpha} \sqrt{\frac{1}{8n} \sum_a \beta(a)w(a)^2},
\]

while the RHS is

\[
\frac{1}{1 - \alpha} \sqrt{\sum_a \beta(a)w(a)^2}.
\]

\( 0 \neq \mathbf{v}^* \neq \pi \). First, we write the optimality conditions of (26):

\[
\left( \frac{w(a)}{1 - \alpha} - \frac{\nu^*(a)}{\beta(a)} \right)^2 = 8n(1 - \alpha)^2 \sum_a \frac{(\pi(a) - \nu^*(a))^2}{\beta(a)} \quad \text{for } a \in S^* \quad (28)
\]

\[
w(a)^2 \leq 8n(1 - \alpha)^4 \sum_a \frac{(\pi(a) - \nu^*(a))^2}{\beta(a)} \quad \text{for } a \notin S^* \quad (29)
\]

Denote

\[
T_1 := \sum_{a \in S^*} \frac{(\pi(a) - \nu^*(a))^2}{\beta(a)}
\]

\[
T_2 := \sum_{a \notin S^*} \frac{(\pi(a) - \nu^*(a))^2}{\beta(a)} = \frac{1}{(1 - \alpha)^2} \sum_{a \notin S^*} \beta(a)w(a)^2
\]

Next, we will show that \( T_1 = \frac{1 - \epsilon}{\epsilon} T_2 \) for some \( \epsilon \in (0, 1) \) such that \( \beta(S^*) = (1 - \epsilon)/8(1 - \alpha)^2 n \).

Summing (28) over \( a \in S^* \), we have

\[
T_1 = \sum_{a \in S^*} \beta(a) \left( \frac{w(a)}{1 - \alpha} - \frac{\nu^*(a)}{\beta(a)} \right)^2
\]

\[
= 8n(1 - \alpha)^2 \beta(S^*) \sum_a \frac{(\pi(a) - \nu^*(a))^2}{\beta(a)}
\]
\[ = 8n(1 - \alpha)^2 \beta(S^*)(T_1 + T_2) \]

which implies that \( S^* \neq [k] \). Then since \( w(a) > 0 \) and \( \nu^* \neq 0 \), both \( T_2 > 0 \) and \( \beta(S^*) > 0 \). Then we have

\[ T_1 > 8n(1 - \alpha)^2 \beta(S^*)T_1 \]

which implies that

\[ \frac{1}{8n(1 - \alpha)^2} > \beta(S^*) > 0. \]

This means that for some \( \epsilon \in (0, 1) \),

\[ \beta(S^*) = \frac{1 - \epsilon}{8n(1 - \alpha)^2}. \]

Then combining this with the previous relationship,

\[ T_1 = 8n(1 - \alpha)^2 \frac{1 - \epsilon}{8n(1 - \alpha)^2} (T_1 + T_2) \]

\[ = \frac{1 - \epsilon}{\epsilon} T_2. \]

Next, we show that (26) has optimal value

\[ \pi(S^*) + \sqrt{\frac{T_2}{8\epsilon}}. \]

From (28), for \( a \in S^* \) we have that

\[ \nu^*(a) = \frac{\pi(a)}{1 - \alpha} - \beta(a) \sqrt{8(1 - \alpha)^2 n(T_1 + T_2)}. \]

Then

\[
\sqrt{\frac{1}{8n} \sum_a \left( \frac{\pi(a)}{1 - \alpha} - \nu^*(a) \right)^2 \beta(a)} + (1 - \alpha) \sum_a |\nu^*(a)| = \sqrt{\frac{1}{8n} (T_1 + T_2)}
\]

\[ + (1 - \alpha) \sum_{a \in S^*} \left( \frac{\pi(a)}{1 - \alpha} - \beta(a) \sqrt{8(1 - \alpha)^2 n(T_1 + T_2)} \right) \]

\[ = \pi(S^*) + (1 - \beta(S^*)8n(1 - \alpha)^2) \sqrt{\frac{1}{8n} (T_1 + T_2)} \]

\[ = \pi(S^*) + \epsilon \sqrt{\frac{1}{8n} (T_1 + T_2)} \]

\[ = \pi(S^*) + \epsilon \sqrt{\frac{\epsilon}{8n} T_2} \]

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Going back to our original problem, note that for $\epsilon \in [1/2, 1]$ we achieve the desired equivalence. Thus we focus on the case where $\epsilon \in (0, 1/2)$, which means that $\frac{1}{16(1-\alpha)^2 n} < \beta(S^*) < \frac{1}{8(1-\alpha)^2 n}$. Assume WLOG that the actions are ordered according to likelihood ratios $w(1) \leq \ldots \leq w(k)$. From (28) and (29), $S^* = \{t + 1, \ldots, k\}$ for some $t \in [k]$. Then applying (28), we have that for action $a = t + 1$,

$$\frac{w(t + 1)^2}{(1 - \alpha)^2} \geq \left(\frac{w(t + 1) - \nu^*(t + 1)}{\beta(t + 1)}\right)^2 = 8n(1 - \alpha)^2 (T_1 + T_2) = \frac{8n(1 - \alpha)^2 T_2}{\epsilon}$$

Using the fact that $\frac{\pi(S^*)}{\beta(S^*)} \geq w(t + 1)$, we have that

$$\frac{\pi(S^*)}{\beta(S^*)} \geq (1 - \alpha)^2 \sqrt{\frac{8nT_2}{\epsilon}}$$

Then since $\beta(S^*) > \frac{1}{16(1-\alpha)^2 n}$,

$$\pi(S^*) \geq \sqrt{T_2 \frac{\epsilon}{8n}} \geq \sqrt{T_2 \frac{\epsilon}{4n}}$$

Then

$$\sqrt{\frac{1}{8n} \sum_a \left(\frac{\pi(a)}{1-\alpha} - \nu^*(a)\right)^2} + (1 - \alpha) \sum_a |\nu^*(a)| = \pi(S^*) \geq \sqrt{\frac{\epsilon T_2}{8n}} \times \pi(S^*) \times \pi(S^*) + \sqrt{\frac{T_2}{n}}$$

which achieves the target equivalence using the definition of $T_2$.

Finally, examining the optimality condition for $a \notin S^*$ (29) shows that $\nu^* = 0$ when

$$n \geq \frac{w_{\text{max}}^2}{8(1-\alpha)^2 \sum a \beta(a)w^2(a)}$$

which means that the lower bound is

$$\frac{cD}{1 - \alpha} \sqrt{\frac{\sum a \beta(a)w(a)^2}{n}}$$

for some universal constant $c$, giving the theorem statement.
E Experiments

E.1 Implementation

We describe each environment and their modeling methods below. For both environments we use a maximum horizon $H = 200$ and $\gamma = 1$.

**Cliffwalk.** Cliffwalk Sutton and Barto [2018] is a $4 \times 12$ tabular environment where the agent must travel from the start state (lower left corner) to the goal state (lower right corner). The agent can take one of four cardinal actions (left, down up, right). The bottom row of cells represents a cliff, and moving into the cliff incurs a cost of 100. Each other state incurs a cost of 1, thus incentivizing the agent to take the shortest path between start a goal. As the original Cliffwalk is deterministic, we introduce a random transition towards the cliff with probability $p = 0.05$ in each state.

The target policy $\pi$ is learned via Q-learning, and the behavioral policy is a mixture between the target and a uniform policy, that is for a constant $\lambda \in [0, 1]$, $\beta = \lambda \pi + (1 - \lambda)\text{UNIF}$. For estimators (DM, DR, WDR) involving models $F$, half of the dataset is used to construct the model and the other half is used for estimation. This process is then repeated with the halves switched, and the resulting estimate is an average of the two. First, an estimate of the MDP $\hat{M}$ is constructed using empirical averages for the transitions and rewards. Following Thomas et al. [2015], we assume that the model is given imperfect information of the horizon as $H = 201$. $\hat{F}$ is then computed recursively via the relations from Section 5:

$$ F_{s_h, a_h}^{H+1-h}(t) = \mathbb{E}_{\mathcal{P}, \mathcal{R}} \left[ F_{s_{h+1}}^{H-h} \left( \frac{t - R_h}{\gamma} \right) \right] $$

$$ F_{s_h}^{H+1-h}(t) = \mathbb{E}_\pi \left[ F_{s_{h}, A_h}^{H+1-h}(t) \right] $$

The Cliffwalk results shown in Figure 1 were averaged over 1000 repetitions.

**Simglucose.** In Simglucose, the agent must control insulin bolus injections to a patient with type 1 diabetes. The state is a continuous vector with the patient’s blood glucose levels and the carbohydrate intake from the last meal. We discretize the space of possible bolus injections into 6 actions. The agent receives a reward according to whether the patient’s blood glucose levels are within acceptable limits or not. If the patient’s blood glucose exceeds 180, a condition called hyperglycemia, the agent receives a reward of -1. If it falls under 70, called hypoglycemia, the agent receives a reward of -2. Otherwise, the agent receives a reward of +1.

The target policy is the built-in controller for the patient, and the behavioral policy is defined as the same way for Cliffwalk. As the state space is continuous, the model was built by first discretizing the state space, then using empirical averages to estimate the MDP $\hat{M}$ in the discretized space, upon which $\hat{F}$ is calculated.

The Simglucose results shown in Figure 1 were averaged over 100 repetitions.

E.2 Mean vs Plug-in + CDF

As an additional experiment, we compare the plug-in mean estimate on the estimated CDFs with direct mean estimation for existing IS, WIS and DR Jiang and Li [2016] estimates in the Cliffwalk
environment. For WIS, at low sample sizes the direct mean estimates can outperform the CDF estimates, but reach the same error as $n$ increases. For the IS and DR estimates, however, the CDF version of the estimates perform just as well, if not better, than direct mean estimation.

Figure 2: Normalized MSE for different $\lambda$ in the Cliffwalk environment. Dashed lines show mean analogues of CDF estimates.