The Fuzzy $S^4$ by Quantum Deformation

Shogo Aoyama\(^1\) and Takahiro Masuda\(^2\)

\(^1\) Department of Physics, Shizuoka University
Ohya 836, Shizuoka, Japan

\(^2\) High Energy Accelerator Research Organization (KEK),
Tsukuba, Ibaraki 305-0801, Japan

Abstract

The fuzzy algebra of $S^4$ is discussed by quantum deformation. To this end we embed the classical $S^4$ in the Kähler coset space $SO(5)/U(2)$. The harmonic functions of $S^4$ are constructed in terms of the complex coordinates of $SO(5)/U(2)$. Being endowed with the symplectic structure they can be deformed by the Fedosov formalism. We show that they generate the fuzzy algebra $\hat{A}_\infty(S^4)$ under the $\star$ product defined therein, by using the Darboux coordinate system. The fuzzy spheres of higher even dimensions can be discussed similarly. We give basic arguments for the generalization as well.

PACS:02.40.Gh, 04.62.+v, 11.10.Nx
Keywords:Noncommutative geometry, Deformation quantization, Fuzzy sphere

\(^*\)e-mail: spsaoya@ipc.shizuoka.ac.jp
\(^†\)e-mail: stmasud@post.kek.jp
1 Introduction

The fuzzy 4-sphere was discussed in [1] as the next simplest generalization of the 2-sphere which had been extensively studied in the literature. In the string context the fuzzy spheres of four and other dimensions appeared as classical solutions[2][3] in the Matrix Model[4]. They represent non-flat p-branes in the string theory. It has been argued that perturbation around such classical solutions provides us with non-commutative gauge theories on the fuzzy spheres[5][6][7]. The fuzzy spheres were discussed also as classical solutions of the DBI action which represent non-commutative backgrounds of D-string propagation[8]. Moreover non-commutativity of spheres was found in the string theory with the $AdS_n \times S^m$ geometry as well[9].

All the above arguments were developed for matrix realization of the fuzzy spheres. On the contrary in this paper we will discuss the fuzzy spheres by the Fedosov deformation quantization[10][11][12][13], as long as their dimensions are even. The key point to this end was found in [5][14]. Namely they gave a proper account of the relevance of the coset space $SO(2a + 1)/U(a)$ for the harmonics analysis of the fuzzy $S^{2a}$. Based on their findings we will study quantum deformation of the harmonic functions of $S^{2a}$, exploiting the Kähler structure of the coset space $SO(2a + 1)/U(a)$.

We shall briefly review the works [5][14]. The fuzzy 4-sphere is described by looking for an $N \times N$ matrix realization of the equation

$$\sum_{\mu=1}^{5} x^\mu \cdot x^\mu = \text{const.},$$

which classically describes the 4-sphere. The matrices $x^\mu$ transform as $\mathbf{5}$ of $SO(5)$. Hence they are operators in an $N$-dimensional irreducible (spinor) representation of $SO(5)$. For a generic $N$ they generate an infinite dimensional algebra under matrix multiplication. When $\mathbf{N}$ is the representation obtained by $n$-fold symmetric products of the spinor $\mathbf{4}$ of $Spin(5)$, we have an arithmetical identity

$$N^2 = \frac{1}{36} (n+1)^2 (n+2)^2 (n+3)^2 = \sum_{n \geq r_1 \geq r_2} D(r_1, r_2).$$

Here $D(r_1, r_2)$ is the dimension of the representation corresponding to the Young diagram of $SO(5)$, labelled by row length $(r_1, r_2)$.

For this special value of $N$, products of the matrices $x^\mu$ generate a finite dimensional algebra which is isomorphic to the full set of $N \times N$ matrices. They are decomposed
into sets of matrices which transform irreducibly under $SO(5)$ according to the Young diagrams $(r_1, r_2)$ relevant in the sum. The matrix algebra is called $\hat{A}_n(S^4)$. From $\hat{A}_n(S^4)$ we may project out the generators corresponding to the Young diagrams with $r_2 \neq 0$ to define a subalgebra, called $A_n(S^4)$. It is the classical analogue of the algebra generated by products of the harmonic functions on $S^4$, but clearly $A_n(S^4)$ is no longer an associative algebra. The associativity is recovered at the limit $n \to \infty$.

In this paper we will reverse the above arguments. Namely we start with explicitly giving the harmonic functions $x^\mu$ of $S^4$. Then we will deform them according to the Fedosov formalism and realize the algebraic equation (1.1) with the $\star$ product defined therein. However $S^4$ is a real 4-dimensional manifold with no symplectic structure. As such the deformation quantization by Fedosov does not work for $S^4$. A hint to overcome this difficulty is to consider a bundle over $S^4$ with fibre $S^2$, which is the Kähler coset space

$$SO(5)/U(2) = \{SO(5)/SO(4)\}{SO(4)/U(2)} = S^4 \times S^2.$$  

Then $S^4$ may be described by the complex coordinate system of $SO(5)/U(2)$, where a symplectic structure manifests. The Kähler coset space $SO(5)/U(2)$ has a set of Killing potentials $M^A$, $A = 1, 2, \cdots, 10$. By the Lie-variation of the isometry $SO(5)$ they transform as $10$:

$$\mathcal{L}_{RA} M^B = \sum_{C \in 10} f^{ABC} M^C,$$  

and satisfy

$$\sum_{A \in 10} M^A M^A = \text{const.},$$

with $f^{ABC}$ the structure functions of $SO(5)$. The existence of such Killing potentials is known for the general Kähler coset space$[13]$. But an unusual feature of $SO(5)/U(2)$ is that from these Killing potentials one can construct a fundamental vector $x^\mu$ of $SO(5)$ by the tensor product $10 \otimes 10 = 5 \oplus \cdots$. By the same Lie-variation as above it transforms as

$$\mathcal{L}_{RA} x^\mu = \sum_{\nu \in 2} f^{A\mu\nu} x^\nu,$$

in which $f^{A\mu\nu}$ are matrix elements of the $SO(5)$-generators in the 5-dimensional representation. We will then find $x^\mu$ to obey the algebraic equation (1.1). The existence of such a fundamental vector is characteristic for the class of the Kähler coset space $SO(2a + 1)/U(a)$. In contrast with the matrix realization, symmetric tensor products of $x^\mu$ generate the commutative subalgebra of the harmonic functions of $S^4$, $A_\infty(S^4)$. By the construction it is obvious that these harmonic functions are expressed by the complex coordinates of $SO(5)/U(2)$. Hence they can now be deformed by the Fedosov formalism to discuss the fuzzy $S^4$[11]. We will then examine the fuzzy algebra under the Fedosov $\star$ product by taking the Darboux coordinates[12]. It will be shown that

$$\sum_{\mu=1}^5 x^\mu \star x^\mu = d_0 + d_2 \hbar^2, \quad [x^\mu, x^\nu]_\star = i d_1 \hbar \sum_{A \in 10} f^{A\mu\nu} M^A,$$
with some constants \(d_0, d_1\) and \(d_2\). More generally we can show that the Fedosov \(\star\) product of \(x^\mu\) preserves the symmetry of \(SO(5)\). Therefore repeating the \(\star\) product generates the algebra isomorphic to \(\hat{A}_\infty(S^4)\).

The paper is organized as follows. In Section 2 we discuss the Kähler coset space \(SO(5)/U(2)\). The Killing vectors, Kähler potential and the Killing potentials of the coset space are explicitly given. The fundamental vector \(x^\mu\) of \(SO(5)\) is constructed from the Killing potentials. In Section 3 we discuss the harmonic functions of \(S^4\). In Section 4 they are deformed by the Fedosov formalism in the Darboux coordinates. They are shown to generate the non-commutative algebra \(\hat{A}_\infty(S^4)\) under the \(\star\) product defined therein. In Section 5 we explain the relation between the coset spaces \(SO(5)/U(2)\) and \(U(4)/U(3) \otimes U(1)\), which is useful to get better understanding of the former coset space. The whole arguments on the fuzzy 4-sphere can be straightforwardly generalized to the case of the fuzzy \(S^{2a}\). Appendix is devoted to give basic arguments for the generalization.

2 The Kähler coset space \(SO(5)/U(2)\)

The coset space \(SO(5)/U(2)\) is a Kähler manifold according to the Borel theorem[16]. We shall study on an explicit construction of this manifold. The Lie-algebra of \(SO(5)\) is given as

\[
[t^{\mu\nu}, t^{\rho\sigma}] = i\delta^{\mu\rho}t^{\nu\sigma} - i\delta^{\nu\rho}t^{\mu\sigma} - i\delta^{\mu\sigma}t^{\nu\rho} + i\delta^{\nu\sigma}t^{\mu\rho}
\]  

(2.1)

where \(t^{\mu\nu} = -t^{\nu\mu}\) with \(\mu, \nu = 1, 2, \cdots, 5\). We will decompose the generators \(t^{\mu\nu}\) into the broken generators, denoted by \(X_i\) and \(\bar{X}_i\), \(i = 1, 2, 3\) and the ones of the homogeneous group \(U(2)\), denoted by \(S_I, Y, I = 1, 2, 3\):

\[
\{T^A\} \equiv \{X^i, X^\bar{i}, S^I, Y\}.
\]  

(2.2)

By noting the \(SU(2) \otimes SU(2)\)-subalgebra formed by

\[
S^1 = \frac{1}{2}(t^{23} + t^{14}), \quad S^2 = \frac{1}{2}(t^{31} + t^{24}), \quad S^3 = \frac{1}{2}(t^{12} + t^{34}),
\]

\[
P^1 = \frac{1}{2}(t^{23} - t^{14}), \quad P^2 = \frac{1}{2}(t^{31} - t^{24}), \quad P^3 = \frac{1}{2}(t^{12} - t^{34}),
\]

they are identified as

\[
X^1 = \frac{1}{2}(t^{15} + it^{25}), \quad X^2 = \frac{1}{2}(-t^{35} + it^{45}), \quad X^3 = \frac{1}{\sqrt{2}}(P^1 + iP^2),
\]

\[
S^I = (S^1, S^2, S^3), \quad Y = P^3.
\]

In this basis the Casimir of \(SO(5)\) takes the form

\[
T^AT^A = X^i\bar{X}^i + \bar{X}^iX^i + S^+S^+ + S^-S^- + (S^3)^2 + (Y)^2,
\]

3
with $S^\pm = \frac{1}{\sqrt{2}}(S^1 \pm iS^2)$. The non-trivial part of the Lie-algebra (2.1) reads

\[
[Y, \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}] = \frac{1}{2} \begin{pmatrix} X^1 \\ -X^2 \end{pmatrix}, \quad [Y, X^3] = X^3, \\
[S^I, \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}] = \frac{1}{2} \sigma^I \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \quad [S^I, X^3] = 0,
\]

(2.3)

\[
[X^1, X^1] = \frac{1}{2}(Y + S^3), \quad [X^2, X^2] = \frac{1}{2}(Y - S^3), \quad [X^3, X^3] = Y \\
[X^1, X^2] = \frac{1}{\sqrt{2}} S^+, \quad [X^1, X^3] = -\frac{1}{\sqrt{2}} X^2, \quad [X^2, X^3] = \frac{1}{\sqrt{2}} X^1, \\
[X^1, X^3] = 0, \quad [X^2, X^3] = 0.
\]

The Kähler coset space $SO(5)/U(2)$ is parametrized by the coordinates corresponding to the broken generators $X^i$ and $X^1$. From (2.3) we find that under the homogeneous group $SU(2)$ the broken generators $X^1$ and $X^2$ transform as $\frac{1}{2}$, while $X^3$ as $\frac{1}{2}$. Therefore the Kähler coset space $SO(5)/U(2)$ is reducible. For an explicit construction we have to be involved in the general arguments given in [17]. But the homogeneous group contains a single $U(1)$ so that the construction is relatively easier.

### 2.1 The Killing vectors

First of all we discuss the holomorphic Killing vectors $R^{A\alpha}(z)$ and $R^{A\alpha}(\bar{z})$ in the basis of the decomposition (2.2). The standard application of the CCWZ formalism [19] does not give the holomorphic Killing vectors $R^{A\alpha}$ satisfying the Lie-algebra (2.3). Hence we extend the isometry group $SO(5)$ to the complex one $SO(5)^c$ and consider a coset space $SO(5)^c/\hat{U}(2)$ with the complex subgroup $\hat{U}(2)$ generated by $X^i, S^I, Y$ [17]. As will be explicitly shown later, there is an isomorphism between this complex coset space $SO(5)^c/\hat{U}(2)$ and $SO(5)/U(2)$:

\[
SO(5)/U(2) \cong SO(5)^c/\hat{U}(2).
\]

(2.4)

The holomorphic Killing vectors are obtained by applying the CCWZ formalism to the complex coset space $SO(5)^c/\hat{U}(2)$. It is parametrized by complex coordinates $z^\alpha, \alpha = 1, 2, 3$ corresponding to the broken generators $X^i$. Consider a holomorphic quantity

\[
\xi(z) = e^{z \cdot X} \in SO(5)^c/\hat{U}(2)
\]

(2.5)

with

\[
z \cdot X = z^1 X^1 + z^2 X^2 + z^3 X^3.
\]

By left multiplication of $g = e^{i\epsilon T^A} \in SO(5)$, we can find the relation

\[
g\xi(z) = \xi(z')\hat{h}(z, g),
\]

(2.6)

appropriately choosing the holomorphic compensator $\hat{h}(z, g) = e^{(\lambda(z, g)\cdot H)} \in \hat{U}(2)$. Here $\epsilon^A$ and $\lambda$s are global and local parameters parametrizing $g$ and $\hat{h}$ respectively as

\[
\epsilon^A T^A = \epsilon^i X^i + \epsilon^i X^i + \epsilon^S S^+ + \epsilon^S S^- + \epsilon^S S^3 + \epsilon^Y Y, \\
\lambda \cdot H = \lambda^1 \cdot X^1 + \lambda^2 S^+ + \lambda^2 S^- + \lambda^2 S^3 + \lambda^3 Y.
\]

(2.7)
This defines a holomorphic transformation of the coordinates $z^\alpha$ which realizes the isometry group non-linearly. When the parameters $\epsilon^A$ are infinitesimal, (2.6) yields the holomorphic Killing vectors $R^{A\alpha}(z)$ as
\[ \delta z = z'^\alpha(z) - z^\alpha = \epsilon^A R^{A\alpha}(z), \] (2.8)
which satisfy the Lie-algebra (2.3).

To make the argument explicit we use the spinor representation of $SO(5)$. That is, the $SO(5)$-generators are given by
\[ t^{\mu\nu} = -\frac{i}{2} \gamma^\mu \gamma^\nu, \] (2.9)
with the $\gamma$-matrices
\[ \gamma^i = i \left( \begin{array}{cc} 0 & \sigma^i \\ -\sigma^i & 0 \end{array} \right), \quad \gamma^4 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \gamma^5 = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right). \]

In the basis of the decomposition (2.2) the broken generators become
\[ X^1 = -\frac{i}{4} (\gamma^1 + i\gamma^2) \gamma^5 = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \]
\[ X^2 = \frac{i}{4} (\gamma^3 - i\gamma^4) \gamma^5 = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \]
\[ X^3 = -\frac{1}{4\sqrt{2}} (\gamma^1 + i\gamma^2)(\gamma^3 - i\gamma^4) = \frac{1}{\sqrt{2}} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \] (2.10)
while the ones of the homogeneous part
\[ S^I = \frac{1}{2} \left( \begin{array}{cc} \sigma^I & 0 \\ 0 & 0 \end{array} \right), \quad Y = \frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma^3 \end{array} \right). \]

It is important to observe algebraic relations of the formers such that
\[ (X^i)^2 = 0, \quad i = 1, 2, 3, \]
\[ X^1 X^2 = -X^2 X^1 = \frac{\sqrt{2}}{4} X^3, \]
\[ X^1 X^3 = X^3 X^1 = 0, \quad X^2 X^3 = X^3 X^2 = 0. \]
Owing to these relations the holomorphic quantity (2.5) can be easily evaluated:

\[
\xi(z) = e^{z^2}X = \frac{1}{2} \begin{pmatrix} 2 & 0 & -z^2 & 0 \\ 0 & 2 & z^1 & 0 \\ 0 & 0 & 2 & 0 \\ z^1 & z^2 & z^3 & 2 \end{pmatrix}, \tag{2.11}
\]

Choose the local parameters of the holomorphic compensator \( \hat{h}(z, g) \) in (2.6) to be

\[
\begin{align*}
\lambda^1 &= \bar{e} + \frac{1}{\sqrt{2}} \bar{e}^3 z^2, \\
\lambda^2 &= \bar{e}^2 - \frac{1}{\sqrt{2}} \bar{e}^3 z^1, \\
\lambda^3 &= \bar{e}^3, \\
\lambda^+ &= \sqrt{2}e^+ + \frac{1}{2\sqrt{2}} \bar{e} z^1 - \frac{1}{4} \bar{e}^3 (z^1)^2, \\
\lambda^- &= \sqrt{2}e^- + \frac{1}{2\sqrt{2}} \bar{e}^1 z^2 - \frac{1}{4} \bar{e}^3 (z^2)^2, \\
\lambda^S &= \bar{e} + \frac{1}{2} \bar{e}^1 z^1 - \frac{1}{2} \bar{e}^2 z^2 + \frac{1}{2 \sqrt{2}} \bar{e}^3 z^1 z^2, \\
\lambda^Y &= e^Y + \frac{1}{2} \bar{e}^1 z^1 + \frac{1}{2} \bar{e}^2 z^2 + \bar{e}^3 z^3.
\end{align*}
\]  
\tag{2.12}

When the global parameters \( \epsilon^A \) are infinitesimal, we find the holomorphic Killing vectors from the relations (2.6) and (2.8)

\[
\begin{align*}
R^1 = i, & \quad R^{11} = -\frac{i}{4}(z^1)^2, \\
R^2 = 0, & \quad R^{21} = \frac{i}{4}(2\sqrt{2} z^3 - z^1 z^2), \\
R^3 = 0, & \quad R^{31} = -\frac{i}{2} z^1 z^3, \\
R^+ = 0, & \quad R^{-1} = -\frac{i}{2 \sqrt{2}} z^2, \\
R^2 = 0, & \quad R^{22} = -\frac{i}{4} (z^2)^2, \\
R^3 = 0, & \quad R^{32} = -\frac{i}{2} z^2 z^3, \\
R^+ = -\frac{i}{\sqrt{2}} z^1, & \quad R^{-2} = 0, \\
R^3 = -\frac{i}{4} z^1 z^3, & \quad R^{32} = -\frac{i}{4} z^2 z^3, \\
R^1 = -\frac{i}{4} \bar{e}^2 z^2, & \quad R^{13} = -\frac{i}{4} z^1 z^3, \\
R^2 = \frac{i}{4} \bar{e}^3 z^1, & \quad R^{23} = -\frac{i}{4} z^2 z^3, \\
R^3 = i, & \quad R^{33} = -\frac{i}{2} (z^3)^2, \\
R^+ = 0, & \quad R^{-3} = 0, \\
R^3 = 0, & \quad R^{33} = 0, \\
R^3 = -i z^3.
\end{align*}
\]  
\tag{2.13}
Next we will discuss the Kähler potential of $SO(5)/U(2)$. We have recourse to the generalized CCWZ formalism adapted for the Kähler coset space\cite{17}. Consider a quantity

$$U(z, \bar{z}) \in SO(5)/U(2),$$

(2.14)

with $U^\dagger U = UU^\dagger = 1$. But the standard parametrization of $U$, i.e., $U(z, \bar{z}) = e^{\sigma \cdot \tau \cdot X}$ does not give the metric of the type $(1,1)$, i.e., $g_{\alpha\beta} = g_{\alpha\beta} = 0$. Therefore we employ the non-standard one, namely

$$U(z, \bar{z}) = \xi(z) \zeta(z, \bar{z}),$$

(2.15)

in which $\xi(z)$ is the holomorphic quantity defined by (2.5), while $\zeta(z, \bar{z})$ an element of the complex subgroup $\hat{U}(2)$. We parametrize the latter as

$$\zeta(z, \bar{z}) = e^{a(z, \bar{z}) \cdot X} e^{b(z, \bar{z}) \cdot S} e^{c(z, \bar{z}) \cdot Y},$$

(2.16)

with $a \cdot X = a^i X^i$ and $b \cdot S = b^I S^I$. Here $a^i$ are complex functions, while $b(z, \bar{z})$ and $c(z, \bar{z})$ are chosen to be real functions because the purely imaginary parts can be absorbed into an element of $H$. They are determined by the unitary condition $U^\dagger U = 1$ which reads

$$\xi^\dagger(z) \xi(z) = e^{-2a(z, \bar{z}) \cdot X} e^{-2b(z, \bar{z}) \cdot S} e^{-2c(z, \bar{z}) \cdot Y} e^{-a(z, \bar{z}) \cdot X}.$$  

(2.17)

We then remark that (2.15) is an concrete expression of the isomorphism (2.4) between the coset spaces $SO(5)/U(2)$ and $SO(5)/\hat{U}(2)$. In ref. \cite{17} it was shown that we may identify the local parameter $c(z, \bar{z})$ to be the Kähler potential of the manifold

$$-2c(z, \bar{z}) = K(z, \bar{z}),$$

(2.18)

because the transformation (2.8) induces the change

$$c(z, \bar{z}) \rightarrow c(z, \bar{z}) + i \frac{2}{2} (\lambda^Y(z) - \bar{\lambda}^Y(\bar{z})), 

(2.19)

in (2.17). Here $\lambda^Y(z)$ and $\bar{\lambda}^Y(\bar{z})$ are the holomorphic functions given in (2.12).

We will apply this argument to find an explicit form of the Kähler potential for $SO(5)/U(2)$. It is again convenient to work out in the spinor representation (2.10). The $(3,3)$-element of the r.h.s. in the unitary condition (2.17) reduces to

$$[\xi^\dagger(z) \xi(z)]_{33} = [e^{-2c(z, \bar{z}) \cdot Y}]_{33} = e^{-c(z, \bar{z})}.$$ 

By calculating the l.h.s. with (2.11) it yields

$$K(z, \bar{z}) = 2 \log(1 + \frac{1}{4} |z^1|^2 + \frac{1}{4} |z^2|^2 + \frac{1}{2} |z^3|^2).$$

(2.20)

We may check the transformation property of this Kähler potential by the Killing vectors (2.13). It indeed changes as (2.19) with the holomorphic function $\lambda^Y(z)$ given in (2.12). We observe that the form of the Kähler potential is almost the same as the one of $CP^4(= U(4)/U(3) \otimes U(1))$. But the isometries realized on both manifolds are clearly different. We will later come back to inspect a relationship between them.
2.3 Killing potentials

Finally we calculate the Killing potentials $M^A(z, \bar{z})$ for $SO(5)/U(2)$. According to ref. [15] they are given by

$$-iM^A = K_\alpha R^\alpha_\alpha - F^A. \quad (2.21)$$

Here $F^A$ follow from the transformation property (2.19) of the Kähler potential, i.e.,

$$-i\lambda^Y = e^A F^A.$$

By using $\lambda^Y$ given in (2.12) together with (2.13) and (2.20), we calculate the r.h.s. of (2.21) to obtain the the Killing potentials $M^A(z, \bar{z})$

$$M^1 = -\frac{1}{2f}(z^1 - \frac{1}{\sqrt{2}}\bar{z}^2z^3), \quad c.c.,$$

$$M^2 = -\frac{1}{2f}(z^2 + \frac{1}{\sqrt{2}}\bar{z}^1z^3), \quad c.c.,$$

$$M^3 = -\frac{1}{f}z^3, \quad c.c.,$$

$$M^+ = \frac{1}{2\sqrt{2f}}\bar{z}^2z^1, \quad M^- = \frac{1}{2\sqrt{2f}}\bar{z}^1z^2,$$

$$M^S = \frac{1}{4f}(|z^1|^2 - |z^2|^2), \quad M^Y = -\frac{1}{2f}(2 - |z^3|^2),$$

in which

$$f = 1 + \frac{1}{4}|z^1|^2 + \frac{1}{4}|z^2|^2 + \frac{1}{2}|z^3|^2.$$

From these Killing potentials we calculate the fundamental vector $x^\mu$ by the formula

$$x^\mu = \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} M^\nu M^\rho M^\sigma M^\delta, \quad (2.23)$$

with $\varepsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric tensor of $SO(5)$. It reads

$$\frac{1}{2}(-ix^1 + x^2) = \sqrt{2}M^3 M^2 - \sqrt{2}M^+ M^2 - M^1(M^S - M^Y), \quad c.c.,$$

$$\frac{1}{2}(ix^3 + x^4) = -\sqrt{2}M^3 M^1 - \sqrt{2}M^- M^1 + M^2(M^S + M^Y), \quad c.c.,$$

$$x^5 = (M^S)^2 - (M^Y)^2 + 2M^+ M^- - 2M^3 M^3. \quad (2.24)$$

We then find

$$-ix^1 + x^2 = \frac{1}{f}(z^1 + \frac{1}{\sqrt{2}}\bar{z}^2z^3), \quad c.c.,$$

$$ix^3 + x^4 = \frac{1}{f}(z^2 - \frac{1}{\sqrt{2}}\bar{z}^1z^3), \quad c.c.,$$

$$x^5 = \frac{1}{f}(\frac{|z^1|^2}{4} + \frac{|z^2|^2}{4} - \frac{|z^3|^2}{2} - 1). \quad (2.25)$$
The respective transformation properties (1.3) and (1.5) of $M^A$ and $x^\mu$ are obvious by the construction. On the other hand the algebraic equation (1.4) and (1.1) follow from the theorem given in ref. [13]. But we have here checked them by direct calculations:

$$M^A M^A = 1, \quad x^\mu x^\mu = 1.$$  \hfill (2.26)

## 3 The harmonic functions of $S^4$

We now show that the fundamental vector $x^\mu$ generates harmonic functions of $S^4$. Define a “false” metric of $SO(5)/U(2)$ by

$$\hat{g}^{\alpha \beta} \equiv R^{A \alpha} R^{A \beta}, \quad \hat{g}^{\alpha \bar{\beta}} \equiv R^{A \alpha} R^{A \bar{\beta}}, \quad \hat{g}^{\bar{\alpha} \bar{\beta}} \equiv R^{A \bar{\alpha}} R^{A \bar{\beta}}.$$  \hfill (3.1)

They satisfy the Killing equations

$$\mathcal{L}_R \hat{g}^{\alpha \beta} = 0, \quad \text{etc.}.$$  

By (2.13) we find that

$$\hat{g}^{\alpha \bar{\beta}} = 0, \quad \text{c.c.},$$  \hfill (3.2)

and $\hat{g}^{\alpha \beta}$ is given by

$$\hat{g}^{11} = 1 + \frac{1}{2} |z^1|^2 + \frac{1}{2} |z^2|^2 + \frac{1}{4} |z^1|^2 |z^3|^2$$
$$\quad + \frac{1}{16} |z^1|^4 + \frac{1}{16} (2\sqrt{2} z^3 - z^1 z^2)(2\sqrt{2} \bar{z}^3 - \bar{z}^1 \bar{z}^2),$$

$$\hat{g}^{22} = 1 + \frac{1}{2} |z^1|^2 + \frac{1}{2} |z^2|^2 + \frac{1}{4} |z^2|^2 |z^3|^2$$
$$\quad + \frac{1}{16} |z^2|^4 + \frac{1}{16} (2\sqrt{2} z^3 + z^1 z^2)(2\sqrt{2} \bar{z}^3 + \bar{z}^1 \bar{z}^2),$$

$$\hat{g}^{33} = (1 + \frac{1}{2} |z^3|^2)(1 + \frac{1}{8} |z^1|^2 + \frac{1}{8} |z^2|^2 + \frac{1}{2} |z^3|^2),$$

$$\hat{g}^{12} = \frac{\sqrt{2}}{8} (z^1 z^3 - (z^2 z^3) + z^1 \bar{z}^2 \frac{1}{16} |z^1|^2 + \frac{1}{16} |z^2|^2 + \frac{1}{4} |z^3|^2),$$

$$\hat{g}^{13} = -\frac{\sqrt{2}}{4} z^2 (1 + \frac{1}{2} |z^3|^2) + z^1 \bar{z}^3 \frac{1}{16} |z^1|^2 + \frac{1}{16} |z^2|^2 + \frac{1}{4} |z^3|^2 + \frac{1}{2},$$

$$\hat{g}^{23} = \frac{\sqrt{2}}{4} z^1 (1 + \frac{1}{2} |z^3|^2) + z^2 \bar{z}^3 \frac{1}{16} |z^1|^2 + \frac{1}{16} |z^2|^2 + \frac{1}{4} |z^3|^2 + \frac{1}{2},$$

and their complex conjugates. Therefore they give a $(1, 1)$ metric, but the $\hat{g}^{\alpha \beta}$ is not the inverse of $g_{\alpha \beta}$ obtained from the Kähler potential (2.20). This discrepancy comes from the fact that

$$\hat{g}^{\alpha \beta} \mathop{\downarrow}_{z=\bar{z}=0} \neq g^{\alpha \beta} \mathop{\downarrow}_{z=\bar{z}=0}.$$
It is a quite general phenomenon when the Kähler coset space is reducible. The correct inverse metric is given by
\[ g^{\alpha \bar{\beta}} = R^{A\alpha} (U P U^{-1})^{AB} R^{B \bar{\beta}}, \tag{3.3} \]
as well as
\[ g^{\alpha \beta} = R^{A\alpha} (U P U^{-1})^{AB} R^{B \beta} = 0, \text{ c.c.} \]
Here \( U \) is the quantity defined by (2.15), but in the adjoint representation. \( P \) is a matrix which has non-vanishing elements only in the diagonal blocks corresponding to the broken generators \( X^a = (X^i, X^i) \) such that
\[ P^i_j = P^j_i = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
(For the details on this point the readers can refer to [18][20].) Nonetheless the Laplacian on \( SO(5)/U(2) \) with the “false” metric is a nice property, namely the Laplacian for scalar fields is given by
\[ \Delta = \frac{1}{\sqrt{\tilde{g}}} \partial_\alpha (\sqrt{\tilde{g}} \ g^{\alpha \bar{\beta}} \partial_{\bar{\beta}}) + \text{c.c.} = (R^{A\alpha} \partial_\alpha + R^{A\alpha} \partial_{\bar{\alpha}})(R^{A\beta} \partial_\beta + R^{A\beta} \partial_{\bar{\beta}}) = \mathcal{L}_{R^A} \mathcal{L}_{R^A}. \tag{3.4} \]
Here \( \tilde{g} = (\det \tilde{g}_{\alpha \bar{\beta}})^2 \). It can be easily shown by using (3.1) with (3.2) and the formulae following from them:
\[ R^{A\alpha} \partial_\alpha R^{A\beta} = 0, \quad R^{A\alpha} \partial_{\bar{\alpha}} R^{A\beta} = -\frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{\tilde{g}} \ g^{\alpha \bar{\beta}} \partial_{\bar{\beta}}). \]
Act the Laplacian on the fundamental vector \( x^\mu \) given by (2.25). Owing to (1.5) we find
\[ \Delta x^\mu = \mathcal{L}_{R^A} \mathcal{L}_{R^A} x^\mu = f^{A\mu} f^{A\rho} x^\rho = c_5 x^\mu, \tag{3.5} \]
in which \( c_5 \) is the Casimir of \( SO(5) \) in \( \mathbf{5} \) taking the value \(-2\) in the basis given by (2.3). This equation implies that the fundamental vector \( x^\mu \) is an eigenvector of the Laplacian (3.4) and gives a basis of the harmonic functions of \( S^4 \).

Note that the Killing potentials \( M^A \) given by (2.22) are also eigenvectors of the Laplacian (3.4), i.e.,
\[ \Delta M^B = \mathcal{L}_{R^A} \mathcal{L}_{R^A} M^B = f^{A\beta} f^{A\rho} M^\rho = c_{10} M^B, \tag{3.6} \]
with \( c_{10} = -3 \) which is the Casimir of \( SO(5) \) in \( \mathbf{10} \). But \( M^A \notin A_\infty(S^4) \). In other words the Killing potentials \( M^A \) are not harmonic functions of \( S^4 \), but of \( SO(5)/U(2) \), since they cannot be obtained by symmetric products of \( x^\mu \).
4 Fuzzy algebrae

The Fedosov formalism \cite{10} for the deformation quantization provides us with \(*\) product of functions on symplectic manifolds. It may be applied for the Kähler manifold most effectively as shown in \cite{11}. When we change the coordinates \((z^\alpha, \bar{z}^\alpha)\) to \((q^\alpha, p_\alpha)\) as

\[
q^\alpha = z^\alpha, \quad p_\alpha = -iK_\alpha, \tag{4.1}
\]

with the Kähler potential, the Kähler two-form can be put in the form

\[
d\omega = dp_\alpha \wedge dq^\alpha.
\]

Hence \((q^\alpha, p_\alpha)\) are the Darboux coordinates. Then the Fedosov \(*\) product for the Kähler manifold reduces to

\[
a(p, q) \ast b(p, q) = \sum_n \frac{1}{n!} a(p, q) \left[ \frac{i\hbar}{2} \left( \frac{\partial^2}{\partial p_\alpha \partial q^{\alpha'}} - \frac{\partial^2}{\partial p_{\alpha'} \partial q^{\alpha}} \right) \right]^n b(p, q), \tag{4.2}
\]

which is the Moyal product \cite{12}. The Killing potentials are given by (2.21) for the general Kähler coset space. In terms of the Darboux coordinates the Killing potentials become

\[
-iM^A(q, p) = ip_\alpha R^{A\alpha}(q) - F^A(q). \tag{4.3}
\]

In ref. \cite{13} it was shown that with the \(*\) product (4.2) they satisfy the fuzzy algebrae

\[
[M^A, M^B]_* = -i\hbar f^{ABC} M^C, \quad M^A \ast M^A = c_0 + c_2 \hbar^2,
\]

in which \(c_0\) and \(c_2\) are constants. For \(SO(5)/U(2)\) we find that

\[
c_0 = 1 \quad c_2 = -1
\]

by the normalization of the \(SO(5)\)-algebra in (2.3). The fundamental vector \(x^\mu\) is also expressed by the Darboux coordinates, owing to the formula (2.24). Plugging the Killing potentials (4.3) into the formula we find \(x^\mu\) to take the simple form

\[
ix^1 + x^2 = \frac{1}{2} (4ip_1 + \sqrt{2}ip_2q^2),
\]

\[
-ix^1 + x^2 = -\frac{1}{2} \left[ ip_1(q^1)^2 + ip_2q^1q^2 + ip_3q^1q^3 - 2\sqrt{2}ip_2q^3 - 2q^1 \right],
\]

\[
ix^3 + x^4 = -\frac{1}{2} \left[ ip_1q^1q^2 + ip_2(q^2)^2 + ip_3q^2q^3 + 2\sqrt{2}ip_1q^3 - 2q^2 \right],
\]

\[
-ix^3 + x^4 = \frac{1}{2} (4ip_2 - \sqrt{2}ip_3q^1),
\]

\[
x^5 = ip_1q^1 + ip_2q^2 - 1.
\]

Then the fuzzy algebrae (1.6) can can be easily checked. We find that

\[
x^\mu \ast x^\mu = 1 - \hbar^2, \quad [x^\mu, x^\nu]_* = -2i\hbar f^{A\mu\nu} M^A. \tag{4.5}
\]

11
The coefficients $f^{A_{\mu\nu}}$ should be matrix elements of the $SO(5)$-generators in the 5-dimensional representation because the Jacobi identity of the commutator with the $\star$ product. The $\star$ product (4.2) preserves the symmetry of $SO(5)$. Namely we have

$$\mathcal{L}_{R^A} \left( \frac{\partial}{\partial q^\alpha} \frac{\partial}{\partial p_\alpha} \right) = \mathcal{L}_{R^A} \left( \frac{\partial}{\partial z^\alpha} i g^{\alpha\bar{\beta}} \frac{\partial}{\partial z^\bar{\beta}} \right) = 0,$$

due to the Killing equation $\mathcal{L}_{R^A} g^{\alpha\bar{\beta}} = 0$, in which $g^{\alpha\bar{\beta}}$ is the inverse metric of $g_{\alpha\bar{\beta}}$ or equivalently given by (3.3). Therefore the symmetric product $\{x^\mu, x^\nu\}_\star$ transforms as a tensor of the second rank. Subtracting the scalar component from this product by (4.5) one obtains the harmonic function in $SO(5)$ of the Killing vector $\mathcal{L}_{R^A} g_{\alpha\bar{\beta}} = 0$, in which $g_{\alpha\bar{\beta}}$ is the inverse metric of $g^{\alpha\bar{\beta}}$ or equivalently given by (3.3). Therefore the symmetric product $\{x^\mu, x^\nu\}_\star$ transforms as a tensor of the second rank. Subtracting the scalar component from this product by (4.5) one obtains the harmonic function in $SO(5)$. Thus repeating the symmetric or antisymmetric $\star$ product generates the fuzzy algebra $\hat{A}_\infty(S^4)$.

### 5 Relation between $SO(5)/U(2)$ and $U(4)/U(3) \otimes U(1)$

As has been noted at the beginning of Section 2 the Kähler coset space $SO(5)/U(2)$ is reducible, but $U(4)/U(3) \otimes U(1) (\cong SO(6)/U(3))$ not. We will discuss on a relation between these Kähler coset spaces. The Lie-algebra of $U(4)$ are given by

$$[T^j_I, T^k_K] = -\delta^j_K T^k_I + \delta^k_I T^j_K,$$

where $(T^j_I)^\dagger = T^I_j, I, J = 1, 2, 3, 4$. Under the subgroup $U(3)$ the generators $T^j_I$ are decomposed as

$$\{T^j_I\} = \{T^4_i, T^i_4, T^j_i, T^i_4\}, \quad i, j = 1, 2, 3.$$

We parametrize the Kähler coset space $U(4)/U(3) \otimes U(1)$ by the coordinates $\phi^\alpha$ and $\bar{\phi}^\alpha, \alpha = 1, 2, 3$, which respectively correspond to the broken generators $T^4_i$ and $T^i_4$. Then the Killing vectors $R^4_i{}^\alpha(\phi)$ and the complex conjugates are given by

$$R^4_i{}^\alpha = i \delta^\alpha_i, \quad R^i_4{}^\alpha = -i \phi^i \phi^\alpha,$$

$$R^j_i{}^\alpha = -i \delta^\alpha_j \phi^i, \quad R^i_4{}^\alpha = i \phi^\alpha. \quad (5.1)$$

The Kähler potential $\tilde{K}(\phi, \bar{\phi})$ and the Killing potentials $M^4_I(\phi, \bar{\phi})$ of $U(4)/U(3) \otimes U(1)$ respectively are found to take the forms

$$\tilde{K} = \log(1 + |\phi^1|^2 + |\phi^2|^2 + |\phi^3|^2) \equiv \log \tilde{f},$$

and

$$M^4_i = -\frac{1}{f} \phi^i, \quad M^4_i = \frac{1}{f}, \quad (5.2)$$

$$M^4_4 = \frac{1}{f}, \quad M^4_i = \frac{1}{f}, \quad M^4_i = \frac{1}{f}.$$
From (5.1) we find that
\[
\begin{align*}
\tilde{g}^{\alpha\beta} & \equiv R^I J_{\alpha} R^J I_{\beta} = \tilde{f}(\delta^{\alpha\beta} + \phi^{\alpha} \bar{\phi}^{\beta}), \\
\tilde{g}^{\alpha\beta} & \equiv R^I J_{\alpha} R^J I_{\beta} = 0. \quad \text{c.c.}
\end{align*}
\] (5.3)

On the contrary to the case of \(SO(5)/U(2)\) it gives the correct inverse metric of \(\tilde{K}_{\alpha\beta}\). This is a fact which always holds when the \(\text{Kähler coset space is irreducible.}\)

The isometry group \(U(4)\) contains \(SO(5)\). Hence the generators \(T^J_I\) are decomposed also under this subgroup as \(16 \to 10 + \bar{5} + 1\). They are grouped into \(10\): \(\begin{align*}
X^1 &= \frac{1}{2}(T^1_4 + T^3_2), \\
X^2 &= \frac{1}{2}(T^2_4 - T^3_1), \\
X^3 &= \frac{1}{\sqrt{2}} T^3_4, \\
S^+ &= \frac{1}{\sqrt{2}} T^1_2, \\
S^- &= \frac{1}{\sqrt{2}} T^2_1, \\
S^3 &= \frac{1}{2}(T^1_1 - T^2_2),
\end{align*}\) \(\begin{align*}
Y &= \frac{1}{2}(T^3_3 - T^4_4), \\
\bar{5}: \begin{cases}
\frac{1}{2}(T^1_4 - T^3_2), & \text{h.c.}, \\
\frac{1}{2}(T^2_4 + T^3_1), & \text{h.c.}, \\
\frac{1}{2\sqrt{2}}(-T^1_1 - T^2_2 + T^3_3 + T^4_4),
\end{cases}\)
\(\begin{align*}
1: \frac{1}{2\sqrt{2}}(T^1_1 + T^2_2 + T^3_3 + T^4_4).
\end{align*}\) (5.4)

The generators in \(10\) satisfy the \(SO(5)\)-Lie-algebra in the form (2.3). Correspondingly the Killing potentials (5.2) are decomposed to yield those of \(SO(5)/U(2)\) and the fundamental vector, respectively given by (2.22) and (2.25). For the precise identification we should understand the scaling
\[
\begin{align*}
\phi^1 &= \frac{1}{2} z^1, \\
\phi^2 &= \frac{1}{2} z^2, \\
\phi^3 &= \frac{1}{\sqrt{2}} z^3, \\
\tilde{K} &= \frac{1}{2} K.
\end{align*}
\]
(Note also a slight difference between the normalization of the fundamental vector (2.25) and that of the corresponding generators (5.4).) The Killing vectors of \(SO(5)/U(2)\), given by (2.13), can be obtained by similarly decomposing those given by (5.1).

The Killing potentials of \(U(4)/U(3) \otimes U(1)\) may be decomposed under any other subgroup. The unusual feature of the decomposition under \(SO(5)\) is that the Killing
potentials in 10 and 5 each obey the constraints (2.26). For a representation \( n \) of a generic subgroup for the isometry group \( U(4) \) we find that
\[
\sum_{(i) \in \mathbb{N}} M_i^j M_j^i \neq \text{const.}
\]
For instance, take a set of \( M_i^j, i, j = 1, 4 \). The corresponding set of the Killing vectors \( R_i^\alpha \) is a non-linear realization of the subgroup \( U(2) \) generated by \( T_i^j, i, j = 1, 4 \). By the Lie-variation with respect to them \( M_i^j \) transform as the adjoint representation of \( U(2) \).

However we find that
\[
\sum_{i,j=1,4} M_i^j M_j^i = \frac{1}{f^2} (1 + |\phi^1|^2)^2 \neq \text{const.},
\]
and
\[
\sum_{i,j=1,4} R_i^\alpha \overline{R_i}^\beta = (1 + |\phi^1|^2)(\delta_\alpha^\alpha \delta_\beta^\beta + \phi^\alpha \overline{\phi}^\beta),
\]
\[
\sum_{i,j=1,4} R_i^\alpha R_j^\beta = -(\phi^\alpha - \delta_\alpha^\alpha \phi^1)(\phi^\beta - \delta_\beta^\beta \phi^1).
\]

We might say that the last two equations define a false metric of some manifold. But it is degenerate at \( \phi^\alpha = \overline{\phi}^\alpha = 0 \), and is no longer of \((1, 1)\) type.

### 6 Conclusions

One of the important ingredients of this paper is that we have found the harmonic functions of \( S^4 \) in the form (2.25). For this purpose we considered a bundle over \( S^4 \) with fibre \( S^2 \), which is the Kähler coset space \( SO(5)/U(2) \). We have constructed the Killing potentials for \( SO(5)/U(2) \) as (2.22). The harmonic functions (2.25) followed from them by the formula (2.23). Hence they were expressed by the complex coordinates \( z^\alpha \) and \( \overline{z}^\alpha, \alpha = 1, 2, 3 \) of the Kähler coset space \( SO(5)/U(2) \).

We can apply the deformation quantization by Fedosov\(^{[10][11]}\) for those harmonic functions and explore the fuzzy \( S^4 \) with the \( \star \) product defined therein. To do this most conveniently we changed the complex coordinates \( (z^\alpha, \overline{z}^\alpha) \) to the Darboux coordinates \( (q^\alpha, p_\alpha) \) defined by (4.1). Then the Fedosov \( \star \) product reduced to the usual Moyal product (4.2), and the deformation quantization was much simplified. Moreover the harmonic functions of \( S^4 \) (2.25) were readily expressed in the Darboux coordinates as (4.4). It consists of another important ingredient of this paper. As the result we were able to easily show the fuzzy algebrae (4.5).

In \([3]\) it was discussed that \( N \times N \) matrix obeying the constraint (1.1) generates the matrix algebra \( \hat{A}_n(S^4) \), when \( N \) takes the special value such as (1.2) given by an integer \( n \). Symmetric traceless products of the matrices up to order \( n \) form its subgroup \( A_n(S^4) \),
which is not associative. In the limit \( n \to \infty \) the associativity is recovered and \( A_\infty(S^4) \) becomes the algebra equivalent to the one generated by the commutative products of the harmonic functions. We have shown that this commutative algebra of the harmonic functions becomes the non-commutative one \( \hat{A}_\infty(S^4) \) by the deformation quantization by Fedosov. This is the main result of this paper.

These arguments on \( S^4 \) can be straightforwardly generalized to the case of \( S^{2a} \). This time we consider a bundle over \( S^{2a} \) with fibre \( SO(2a)/U(a) \), which is the Kähler coset space \( SO(2a+1)/U(a) \). In Appendix we show an explicit way to construct the harmonic functions of \( S^{2n} \) in the symplectic coordinates of the Kähler coset space \( SO(2a+1)/U(a) \). Although we do not discuss in details, it is obvious that we can find the fuzzy algebra \( \hat{A}_\infty(S^{2n}) \) by applying the deformation quantization for those harmonic functions similarly to the case of \( S^4 \).

The arguments in this paper were done by fully exploiting the the Kähler structure of \( SO(5)/U(2) \). It was noticed in \cite{[7][21]} that the Kähler structure is important for studying the Matrix Model on some non-commutative coset spaces. It is desired to extend their study to non-commutative backgrounds with the general Kähler coset space geometry following the works\cite{13}.

**Acknowledgements**

T.M. would like to thank Y. Kitazawa and Y. Kimura for discussions. The work of S.A. was supported in part by the Grant-in-Aid for Scientific Research No. 13135212.

**Appendix**

\( S^{2a} \) in \( SO(2a+1)/U(a) \)

The \( 2a \)-sphere is described by the coordinates of the Kähler coset space \( SO(2a+1)/U(a) \) as noted by

\[
SO(2a+1)/U(a) = \{SO(2a+1)/SO(2a)\} \times \{SO(2a)/U(a)\} = S^{2a} \times SO(2a)/U(a).
\]

To show this it suffices to explicitly construct \( SO(2a+1)/U(a) \). It is a reducible Kähler coset space. The direct construction following the arguments in Section 2 is rather involved. Instead we will do it via the irreducible Kähler coset space \( SO(2a+2)/U(a+1) \), as
was done in Section 5. \(SO(2a + 2)/U(a + 1)\) may be constructed according to the general method for the irreducible Kähler coset space discussed in refs [12]. The generators of \(SO(2a + 2)\) are decomposed under the subgroup \(U(a + 1)\) as

\[
\{T^A\} = \{Y_{IJ}, \overline{Y}^{IJ}, T_I^J\}, \quad I, J = 1, 2, \ldots, a + 1,
\]

in which \(Y_{IJ} = -Y_{JI}, (Y_{IJ})^\dagger = \overline{Y}^{IJ}\) and \((T_I^J)^\dagger = T_J^I\). They satisfy the Lie-Algebra

\[
\begin{align*}
[Y_{IJ}, Y_{KL}] &= 0, \quad \text{h.c.}, \\
[Y_{IJ}, Y_{KL}] &= \delta^I_K T_J^L - \delta^J_L T_K^I - \delta^J_L T_K^I + \delta^I_K T_J^L, \\
[T_I^J, Y_{KL}] &= -\delta^J_K Y_{IL} - \delta^J_L Y_{KI}, \quad \text{h.c.}, \\
[T_I^J, T_K^L] &= -\delta^J_K T_L^I + \delta^L_I T_J^K.
\end{align*}
\] (A.1)

\(T_I^J\) are the generators of \(U(a + 1)\), while \(Y_{IJ}\) and \(\overline{Y}^{IJ}\) the broken generators. The Casimir is given by

\[
\frac{1}{2}(Y_{IJ} Y^{IJ} + \overline{Y}^{IJ} Y_{IJ}) + T_I^J T_I^J.
\]

The local coordinates of the coset space \(SO(2a + 2)/U(a + 1)\) are denoted by \(\phi_{IJ}\) and \(\overline{\phi}^{IJ}\), correspondingly to the broken generators. Hereinafter upper or lower indices of the coordinates stand for complex conjugation. Therefore lowering or raising them should be done by writing the metric \(g_{IJ}^{KL}\) or \((g^{-1})^{IJ}_{KL}\) explicitly.

The Killing vectors \(R^A_{MN}(\phi)\) and \(R^A_{\overline{M}N}(\overline{\phi})\) are respectively non-linear realizations of the Lie-algebra (A.1) on \(\phi_{MN}\) and \(\overline{\phi}^{MN}\):

\[
R^A_{MN} \equiv -i[T^A, \phi_{MN}], \quad \text{c.c.}, \quad (A.2)
\]

They are given by

\[
\begin{align*}
R^{IJ}_{MN} &= i\delta^{IJ}_{MN} (\equiv \delta^I_M \delta^J_N - \delta^I_N \delta^J_M), \\
R_{IJ\overline{M}N} &= i(-\phi_{IM} \overline{\phi}^{NJ} + \phi_{IN} \overline{\phi}^{MJ}), \\
R^I_J^{\overline{M}N} &= i(\delta^I_M \phi_J N + \delta^J_N \phi_M I).
\end{align*}
\] (A.3)

Then the Kähler potential is found according to the formula (28) in [12]:

\[
K = \frac{1}{2} \log \det[1 + Q],
\]

where \(Q^N_M = \phi_{ML} \overline{\phi}^{NL}\). Indeed by the Lie-variation it transforms as

\[
\epsilon^A \mathcal{L}_{R^A} K \equiv \frac{1}{2} \epsilon^A (R^A_{MN} \frac{\partial}{\partial \phi_{MN}} + R^A_{MN} \frac{\partial}{\partial \overline{\phi}^{MN}}) K
\]

\[
= -\frac{i}{2} (\epsilon^{MN} \phi_{MN} + \epsilon_{MN} \overline{\phi}^{MN}). \quad (A.4)
\]

Note that for the case of \(a = 2\) the Kähler potential takes a simple form such that

\[
K = \log(1 + \phi_{12} \overline{\phi}^{12} + \phi_{23} \overline{\phi}^{23} + \phi_{31} \overline{\phi}^{31}). \quad (A.5)
\]
By using the formula (2.21) with (A.3) and (A.4) we obtain the Killing potentials

\[ M_{IJ} = -\left[ \frac{1}{1+Q} \right] L^L_{L^J}, \quad M_{I} = -\left[ \frac{1}{1+Q} \right] L^L_{L^J}, \]
\[ M_{I} = -\left[ \frac{Q}{1+Q} \right] L^L_{L^J} + \frac{1}{2} \delta_{I^J}. \]

(A.6)

One can check that they transform as the adjoint representation of \( SO(2a + 2) \) by the Lie-variation with respect to the Killing vectors (A.3) and satisfy

\[ M_{IJ}M_{IJ} + M_{I}M_{I} = \frac{1}{4}(a + 1). \]

The Kähler coset space \( SO(2a + 1)/U(a) \) can be constructed from the knowledge of the coset space \( SO(2a + 2)/U(a + 1) \), as has been done for the case of \( a = 2 \) in Section 5. To this end we decompose the generators of \( SO(2a + 2) \) under \( U(a) \) as

\[ \{ T^A \} = \{ Y_{ij}, \overline{Y}^{ij}, Y_{i \ a + 1}, \overline{Y}^{i \ a + 1}, T^j_{i}, T^a_{i}, \ldots \}. \]

with \( i, j = 1, 2, \ldots, a \). They are grouped in the irreducible representations of \( SO(2a + 1) \) as

**adjoint rep.:**

\[ \begin{align*}
Y_{i \ a + 1} &= T^a_{i} (= X_i), \\
\sqrt{2}Y_{ij} &= X_{ij}, \\
\sqrt{2}T^j_{i} &= H^j_{i}.
\end{align*} \]

(A.7)

**funda. rep.:**

\[ \begin{align*}
Y_{i \ a + 1} &= T_{a + 1}^{a + 1}, \\
\sqrt{2}T_{a + 1}^{a + 1}.
\end{align*} \]

(A.8)

In (A.7) the generators of \( SO(2a + 1) \) are given in the basis of \( U(a) \). The Casimir of \( SO(2a + 1) \) takes the form

\[ X_i \overline{X}^i + \overline{X}^i X_i + \frac{1}{2}(X_{ij} \overline{X}^{ij} + \overline{X}^{ij} X_{ij}) + H^j_{i} H^j_{i}. \]

\( X_i, X_{ij} \) and their hermite conjugates are the broken generators. The coset space \( SO(2a + 1)/U(a) \) is parametrized by the coordinates corresponding to them, denoted respectively by \( z_i, z_{ij} \) and their complex conjugates. To obtain the Killing vectors of \( SO(2a + 1)/U(a) \) we take from (A.3) the subset of the Killing vectors which realize the isometry \( SO(2a + 1) \)

\[ \begin{align*}
R^i_{MN} &= -i[X_i, \phi_{MN}], \\
R_{i \ MN} &= -i[X_i, \phi_{MN}], \\
R^{ij}_{MN} &= -i[X_{ij}, \phi_{MN}], \\
R_{ij \ MN} &= -i[X_{ij}, \phi_{MN}], \\
R^j_{i \ MN} &= -i[H^j_{i}, \phi_{MN}].
\end{align*} \]

(A.9)
We identify the coordinates of $SO(2a + 2)/U(a + 1)$ with those of $SO(2a + 1)/U(a)$ as
\[ \phi_{i\, a+1} \equiv z_i, \quad \phi_{ij} \equiv \sqrt{2}z_{ij}, \quad c.c.. \quad (A.10) \]

Then from (A.9) we find
\[\begin{align*}
R^i_m &= i\delta^i_m, \\
R^{ij}_m &= 0, \\
R^i_{mn} &= \sqrt{2}i\delta^i_m z_i, \\
R^{ij}_{mn} &= \frac{i}{\sqrt{2}}(\delta^i_m z_n - \delta^i_n z_m), \\
R_{ij\, mn} &= 2i(-z_{im}z_j + z_{jm}z_i), \\
R_{ij\, mn} &= 2i(-z_{im}z_j + z_{jm}z_i), \\
R^i_{mn} &= \sqrt{2}i(\delta^i_m z_m + \delta^i_n z_m).
\end{align*}\quad (A.11)\]

They realize the Lie-algebra of $SO(2a + 1)$ non-linearly on the coordinates $z_{mn}$ and $z_m$, given in the basis (A.7). Therefore they are the Killing vectors of $SO(2a + 1)/U(a)$. The Kähler potential is given in the same form as (A.5)
\[ K = \frac{1}{2} \log \det[1 + Q], \]
but with the identification (A.10), i.e.,
\[ Q^N_M = \begin{pmatrix}
2z_{ml}z^{nl} + z_mz^n & -\sqrt{2}z_{ml}z_l^l \\
-\sqrt{2}z_lz^{nl} & z_lz_l^l
\end{pmatrix}. \]

One can check that it transforms according to (A.4) by the Lie-variation with respect to the Killing vectors (A.11) of $SO(2a + 1)/U(a)$. The Killing potentials of $SO(2a + 2)/U(a + 1)$, given by (A.6), decomposed into the two subsets corresponding to (A.7) and (A.8). With the identification (A.10) the subset in the adjoint representation gives the Killing potentials of $SO(2a + 1)/U(a)$, while the one in the fundamental representation the harmonic functions of $S^{2a}$.

References

[1] H. Grosse, C. Klimcik and P. Presnajder, “On Finite 4D Quantum Field Theory in Non-commutative Geometry”, Commun. Math. Phys. 180(1996)429, hep-th/9602115.
[2] O. Ganor, S. Ramgoolam and W. Taylor IV, “Branes, Fluxes and Duality in M(atrix)-Theory”, Nucl. Phys. B492(1997)191, hep-th/9611202.

[3] J. Castelino, S. Lee and W. Taylor IV, “Longitudinal 5-branes as 4-spheres in Matrix Theory”, Nucl. Phys. B526(1998)334, hep-th/9712105.

[4] Banks, W. Fischler, S.H. Shenker and L. Susskind, “M Theory as a Matrix Model: A Conjecture ”, Phys. Rev. 55(1997)5112, hep-th/9610043.
N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large-N Reduced Model as Superstring”, Nucl. Phys. B498(1997)467, hep-th/9612115.

[5] S. Ramgoolam, “On Spherical Harmonics for Fuzzy Spheres in Diverse Dimensions”, Nucl. Phys. B610(2001)461, Nucl.Phys. B610(2001)461, hep-th/0105006.

[6] S. Iso, Y. Kimura, K. Tanaka and K. Watatsuki, “Noncommutative Gauge Theory on Fuzzy Sphere from Matrix Model”, Nucl. Phys. B604(2001)121, hep-th/0101102.
Y. Kimura, “Noncommutative Gauge Theory on Fuzzy Four-sphere and Matrix Model”, Nucl.Phys. B637(2002)177, hep-th/0204256.

[7] Y. Kitazawa, “Matrix Model in Homogeneous Spaces”, Nucl.Phys. B642(2002)210, hep-th/0207115.

[8] Z. Guralnik and S. Ramgoolam, “On the Polarization of Unstable D0-branes into Non-commutative Odd Spheres”, JHEP 0102 (2001)032, hep-th/0101001.
N. Constable, R. Myers and O. Tafjord, “Non-abelian Brane Interactions”, JHEP 0106(2001)023, hep-th/0102080.

[9] J. Maldacena and A. Strominger, “AdS3 Black Holes and a Stringy Exclusion Principle”, JHEP 9812(1998)005, hep-th/9804083.
J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the Giant Gravitons from Anti de Sitter Space”, JHEP 0006(2000)008, hep-th/0003073.
P. Ho and M. Li, “Fuzzy Spheres in AdS/CFT Correspondence and Holography from Noncommutativity”, Nucl.Phys. B596(2001)259, hep-th/0010472.

[10] B.V. Fedosov, “A Simple Geometrical Construction of Deformation Quantization”, J. Diff. Geom. 40(1994)213;
B.V. Fedosov, “Deformation Quantization and Index theory”, Berlin, Germany : Akademie-Verl. (1996) (Mathematical Topics : 9).

[11] S. Aoyama and T. Masuda, “The Fuzzy Kähler Coset Space by the Fedosov Formalism”, Phys. Lett. 514B(2001)385, hep-th/0105271.

[12] S. Aoyama and T. Masuda, “The Fuzzy Kähler Coset Space with the Darboux Coordinates”, Phys. Lett. 521B(2001)376, hep-th/0109020.

[13] S. Aoyama and T. Masuda, “The Fuzzy Algebrae of the General Kähler Coset Space G/S ⊗ U(1)k”, hep-th/0209082.
[14] P. Ho and S. Ramgoolam, “Higher Dimensional Geometries from Matrix Brane Constructions”, Nucl. Phys. B627(2002)266, hep-th/0111278.

[15] J. Bagger and E. Witten, “The Gauge Invariant Supersymmetric Nonlinear Sigma Model”, Phys. Lett. 118B(1982)103.

[16] A. Borel, “Kählerian Coset Spaces of Semisimple Lie Groups”, Natl. Acad. Sci. 40(1954)1147.

[17] K. Itoh, T. Kugo and H. Kunitomo, “Supersymmetric Nonlinear Realization for Arbitrary Kählerian Coset Space $G/H$”, Nucl. Phys. B263(1986)295.

[18] S. Aoyama, “The Four-fermi Coupling of the Supersymmetric Non-linear $\sigma$-model on $G/S \otimes \{U(1)\}^k$ ”, Nucl. Phys. B578(2000)449, hep-th/0001160.

[19] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177(1969)2239; C.G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev.177(1969)2247.

[20] S. Aoyama and T. Masuda, in preparation.

[21] V. Nair and S. Randjbar-Daemi, “On Brane Solutions in M(atrix) Theory”, Nucl.Phys. B533(1998)333, hep-th/9802187.