On the Incompressible Limit for the Compressible Free-Boundary Euler Equations with Surface Tension in the Case of a Liquid

Marcelo M. Disconzi† Chenyun Luo‡

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Abstract

In this paper we establish the incompressible limit for the compressible free-boundary Euler equations with surface tension in the case of a liquid. Compared to the case without surface tension treated recently in [46, 48], the presence of surface tension introduces severe new technical challenges, in that several boundary terms that automatically vanish when surface tension is absent now contribute at top order. Combined with the necessity of producing estimates uniform in the sound speed in order to pass to the limit, such difficulties imply that neither the techniques employed for the case without surface tension, nor estimates previously derived for a liquid with surface tension and fixed sound speed, are applicable here. In order to obtain our result, we devise a suitable weighted energy that takes into account the coupling of the fluid motion with the boundary geometry. Estimates are closed by exploiting the full non-linear structure of the Euler equations and invoking several geometric properties of the boundary in order to produce some remarkable cancellations. We stress that we do not assume the fluid to be irrotational.

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*Vanderbilt University, Nashville, TN, USA. marcelo.disconzi@vanderbilt.edu
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‡Vanderbilt University, Nashville, TN, USA. chenyun.luo@vanderbilt.edu
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1 Introduction

We consider the motion of a compressible liquid with free surface boundary in \( \mathbb{R}^3 \). We use the notation \( D_t \) to represent the bounded domain occupied by the fluid at each time \( t \), whose boundary is advected by the fluid. The motion of the fluid is described by the compressible Euler equations

\[
\begin{align*}
\rho (\partial_t u + \nabla_u u) &= -\nabla p, & & \text{in } D, \\
\partial_t \rho + \nabla_u \rho + \rho \text{div } u &= 0, & & \text{in } D, \\
p &= p(\rho), & & \text{in } D.
\end{align*}
\]
Here, $\mathcal{D} = \cup_{0 \leq t \leq T} \{t\} \times \mathcal{D}_t$, $u = u(t,x)$ is the velocity of the fluid, whereas $p(t,x)$ and $\rho(t,x)$ are the pressure and density, respectively. The density is bounded from below away from zero, i.e., $\rho \geq \text{constant} > 0$. This condition on the density is what characterizes the fluid as a liquid. The initial and boundary conditions are

\[
\begin{align*}
\{x : (0,x) \in \mathcal{D}\} &= \mathcal{D}_0, \\
u = u_0, \rho = \rho_0 &\quad \text{in } \{0\} \times \mathcal{D}_0, \\
(\partial_t + \nabla u)|_{\partial \mathcal{D}} &\in T(\partial \mathcal{D}), \\
p|_{\partial \mathcal{D}} &= \sigma \mathcal{H},
\end{align*}
\]

(1.2)

where $\mathcal{H}$ is the mean curvature of $\partial \mathcal{D}_t$, $\sigma \geq 0$ is a constant, and $T(\partial \mathcal{D})$ is the tangent bundle of $\partial \mathcal{D}$ (the condition $(\partial_t + \nabla u)|_{\partial \mathcal{D}} \in T(\partial \mathcal{D})$ expresses the fact that the boundary moves with speed equal to the normal component of the velocity). Finally, the equation of state is assumed to be a strictly increasing function of the density, i.e.,

\[p = p(\rho), \quad p'(\rho) > 0.\]

The unknowns in (1.1)-(1.2) are $u, \rho$ and $\mathcal{D}_t$, and hence, $\mathcal{H}$ and $p$ are function of the unknowns, and therefore, are not known a priori.

Problem (1.1)-(1.2) behaves significantly different depending on whether $\sigma = 0$ or $\sigma > 0$. The former is known as the case without surface tension whereas the latter is the case with surface tension, which is the situation treated in this manuscript. Our goal is to show that, for $\sigma > 0$, the motion of a free-boundary incompressible fluid with surface tension (corresponding to the idealized situation of a constant density fluid) is well-approximated by (1.1)-(1.2) when an appropriate notion of compressibility is very small. It is well-known that solutions to the incompressible equations, written in section 1.2 below, cannot be obtained by simply setting $\rho$ to a constant in (1.1)-(1.2) (see, e.g., [46]). The correct way of setting the incompressible limit is via the fluid’s sound speed introduced in section 1.3.

The study of the incompressible limit has a long history in fluid dynamics, see section 1.2. For the case of a motion with free-boundary, the only results we are aware are the recent works [46, 48] by Lindblad and the second author, both treating the case $\sigma = 0$. In particular, to the best of our knowledge this is the first proof of the incompressible limit for the free-boundary compressible Euler equations with surface tension, i.e., $\sigma > 0$. Despite many new difficulties introduced by the presence of surface tension, which are discussed in section 1.5, it is important to consider the case $\sigma > 0$ because real fluids have surface tension. Thus, this feature has to be incorporated in the construction of more realistic models. We remark that we do not assume that the fluid is irrotational.

### 1.1 Lagrangian coordinate and the reference domain

We introduce Lagrangian coordinates, under which the moving domain becomes fixed. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$. Denoting coordinates on $\Omega$ by $y = (y_1, y_2, y_3)$, we define $\eta : [0,T] \times \Omega \rightarrow \mathcal{D}$ to be the flow of the velocity $u$, i.e.,

\[
\begin{align*}
\partial_t \eta(t,y) &= u(t,\eta(t,y)), \\
\eta(0,y) &= y.
\end{align*}
\]

We introduce the Lagrangian velocity, density and pressure, respectively, by $v(t,y) := u(t,\eta(t,y))$, $R(t,y) := \rho(t,\eta(t,y))$ and $q(t,y) := p(t,\eta(t,y))$. Therefore,

\[\partial_t \eta = v.\]
For the sake of simplicity and clean notation, here we consider the model case when $D_0 = \Omega = T^2 \times (0,1)$. We set

$$
\Gamma_0 := T^2 \times \{x_3 = 0\}, \quad \Gamma_1 := T^2 \times \{x_3 = 1\},
$$

so that $\Gamma := \partial \Omega = \Gamma_0 \cup \Gamma_1$. Using a partition of unity, as in, e.g., [9, 40], a general domain can be treated with the same tools we shall present. Choosing $\Omega$ as above, however, allows us to focus on the real issues of the problem without being distracted by the cumbersomeness of the partition of the unity. We also note that one might want to consider a situation more akin the finite-depth water waves problem, where the bottom boundary, $\Gamma_0$, remains fixed. This case requires only minor modifications from our presentation but, again, we believe that this would be a distraction from the main problem.

Let $\partial$ be the spatial derivative with respect to the spatial variable $y$. We introduce the matrix $a = (\partial \eta)^{-1}$. This is well-defined since $\eta(t, \cdot)$ is almost id (i.e., the identity diffeomorphism on $\Omega$) whenever $t$ is sufficiently small. Define the cofactor matrix

$$
A = Ja,
$$

where $J = \det(\partial \eta)$. Then, $A$ satisfies the Piola identity:

$$
\partial_\mu A^{\mu \alpha} = 0.
$$

Here, the summation convention is used for repeated upper and lower indices, and in above and throughout, we adopt the convention that the Greek indices range over 1, 2, 3, while the Latin indices range over 1 and 2.

In terms of $v, R, q$ and $a$, the system (1.1)-(1.2) becomes

$$
\begin{cases}
R \partial_t v^\alpha + a^{\mu \alpha} \partial_\mu q = 0, & \text{in } [0,T] \times \Omega \\
\partial_t R + Ra^{\mu \alpha} \partial_\mu v_\alpha = 0, & \text{in } [0,T] \times \Omega \\
q = q(R), & \text{in } [0,T] \times \Omega \\
A^{\mu \alpha} N_\mu q + \sigma \sqrt{g} \Delta g \eta^\alpha = 0, & \text{on } [0,T] \times \Gamma, \\
\eta(0, \cdot) = \text{id}, \quad R(0, \cdot) = R_0(= \rho_0), \quad v(0, \cdot) = v_0,
\end{cases}
$$

where $N$ is the unit outward normal to $\Gamma$, and $\Delta_g$ is the Laplacian of the metric $g_{ij}$ induced on $\Gamma(t) = \eta(t, \Gamma)$ by the embedding $\eta$, i.e.,

$$
g_{ij} = \partial_i \eta^\mu \partial_j \eta_\mu, \quad \Delta_g (\cdot) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j (\cdot)),
$$

where $g = \det g$. Since $\eta(0, \cdot) = \text{id}$, the initial Eulerian and Lagrangian velocities (i.e., $u_0$ and $v_0$) agree. In addition, we also have $a(0, \cdot) = I$, where $I$ is the identity matrix. Finally, $J = \det(\partial \eta)$ satisfies

$$
\partial_t J = Ja^{\mu \nu} \partial_\mu v_\nu, \quad [0,T] \times \Omega.
$$

This, together with the second equation of (1.3) imply

$$
RJ = \rho_0, \quad [0,T] \times \Omega,
$$

and hence the first equation in (1.3) is equivalent to

$$
\rho_0 \partial_t v^\alpha + A^{\mu \alpha} \partial_\mu q = 0, \quad \text{in } [0,T] \times \Omega.
$$
1.2 Background

The study of the motion of a fluid has a long history in mathematics. In particular, the study of free-boundary fluid problems has blossomed over the past decade or so. However, much of this activity has focused on the study of the incompressible free-boundary Euler equations, i.e.,

\[
\begin{align*}
\beta v_t^\alpha + a^{\mu\alpha} \partial_\mu q &= 0, & \text{in } [0, T] \times \Omega \\
\text{div } v &= 0, & \text{in } [0, T] \times \Omega \\
\mathfrak{A}^{\mu\alpha} N_\mu q + \sigma \sqrt{\mathfrak{A}} \Delta \tilde{\eta} &= 0, & \text{on } [0, T] \times \Gamma,
\end{align*}
\]

(1.7)

where \(\beta\) is a positive constant corresponding to the fluid’s constant density, \(v\) and \(q\) are the incompressible Lagrangian velocity and pressure, \(a = (\partial \tilde{\eta})^{-1}\), \(\mathfrak{A} = \det(\partial \tilde{\eta} a)\), where \(\tilde{\eta}\) is the Lagrangian map associated with \(v\).

It is well-known that for the incompressible equations, \(q\) is not determined by an equation of state. Rather, it is a Lagrange multiplier enforcing the constraint \(\text{div } v = 0\). The local well-posedness for the incompressible free-boundary Euler equations has been studied by many authors, see [7, 8, 11, 12, 16, 17, 19, 31, 41, 42, 45, 47, 52, 54, 55, 56, 57, 65, 66, 69] and references therein. It is worth mentioning here that when \(D_0\) is unbounded (with finite or infinite depth) and the velocity \(v_0\) is irrotational (i.e., \(\text{curl } v_0 = 0\), a condition that is preserved by the evolution), this problem is called the water-waves problem, which has received a great deal of attention [4, 5, 6, 15, 24, 25, 26, 27, 28, 29, 30, 35, 32, 33, 34, 59, 61, 62, 63, 67, 68].

However, the theory of the free-boundary compressible Euler equations is far less developed. It is known that for suitable initial data, the system (1.1) modeling a liquid admits a local (in time) solution, e.g., [9, 21, 20, 43, 44, 60], and for the gas model, the existence of a local solution was obtained in [10, 13, 14, 36, 37, 49].

In this paper we study how the solutions to (1.3) and (1.7) are related. Intuitively, one expects that the solution of (1.3) should converge to that of (1.7) when the “compressibility vanishes”. The proper way to define this problem is via the fluid’s sound speed (see (1.8) below), which corresponds to the speed of propagation of sound waves inside the fluid and captures the fluid’s compressibility in that stiffer fluids have larger sound speed\(^1\).

The incompressible limit problem consists in proving that if a sequence \((v_0, k, R_0, n)\) of well-prepared initial data for (1.3) converges to \((v_0, \beta)\), where \(v_0\) is the initial data for the incompressible problem (1.7), and the sound speed at time zero diverges to infinity, then the respective solution \((v, R)\) of (1.1) converges to \((v, \beta)\), where \(v\) solves (1.7). Here, well-prepared initial data means that, in addition to satisfying the compatibility conditions, the initial data has to be tailored to the above limit (see Theorem 1.3).

The incompressible limit for the compressible Euler equations in a fixed domain (i.e., \(D_t = D_0\) or the whole space) was established by several authors under different assumptions, see [2, 3, 18, 22, 23, 38, 39, 50, 53] and references therein. In addition, the incompressible limit for the compressible free-boundary Euler equations was solved by Lindblad and the second author in [46] with \(\sigma = 0\) in a bounded domain, and by the second author [48] in the same case but with unbounded domain. To our best knowledge, the aforementioned works [46, 48] are the only known results in the study of the incompressible limit for equations (1.3). In particular, no result is available for the case with \(\sigma > 0\). We will establish a priori estimates for (1.3) that are uniform in the sound speed (see Sections 3-4).

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\(^1\)This is an experimental fact, see, e.g., [64].
the convergence of the compressible solution to the incompressible one by an Arzelà-Ascoli-type theorem.

1.3 The sound speed

Physically, the sound speed is defined as $c = \sqrt{q \circ R}$. To set up the incompressible limit, it is convenient to view the sound speed as a parameter. As in [18, 23], we consider a family $\{q_\kappa(R)\}$ parametrized by $\kappa \in [0, \infty)$, where

$$c := q'_\kappa(R)|_{R=1}. \quad (1.8)$$

Here, $'= \frac{d}{dR}$, and

$$q_\kappa(R) = c_\gamma \kappa(R^\gamma - \beta), \quad c_\gamma > 0, \beta > 0, \gamma \geq 1. \quad (1.9)$$

We slightly abuse terminology and call $\kappa$ the sound speed. In order to consider the incompressible limit, we view the density as a function of the pressure, i.e., $R_\kappa = R_\kappa(q) = [(c_\gamma \kappa)^{-1} q + \beta]^{1/\gamma}$, and we see that $R'_\kappa(q)$ satisfies

$$\frac{1}{c_0} R_\kappa \leq R'_\kappa(q) \leq c_0 R_\kappa, \quad (1.10)$$

for some fixed constant $c_0 > 0$, where $R_\kappa = (C_\kappa \kappa)^{-\frac{1}{\gamma}}$. Also, for $0 \leq k \leq 4$, we have that:

$$|R^{(k)}_\kappa(q)| \leq c_0, \quad |R^{(k)}_\kappa(q)| \leq c_0 |R'_\kappa(q)||k| \leq c_0 |R'_\kappa(q)|, \quad (1.11)$$

$$|q^{(k)}_\kappa(R)| \leq c_0 |q'_\kappa(R)|,$$

hold uniformly in $\kappa$.

1.4 The main results

Notations. All notations will be defined as they are introduced. In addition, a list of symbols is given at the end of this section for a quick reference.

Definition 1.1. The $L^2$-based Sobolev spaces are denoted by $H^s(\Omega)$, with the corresponding norm denoted by $|||\cdot|||_s$; note that $|||\cdot|||_0 = |||\cdot|||_{L^2(\Omega)}$. We denote by $H^s(\Gamma)$ the Sobolev space of functions defined on $\Gamma$, with norm $|||\cdot|||_{s, \Gamma}$.

Theorem 1.1. Let $\Omega = \mathbb{T}^2 \times (0, 1)$ and $v_{0, \kappa}$ be a smooth vector field. Let $\rho_{0, \kappa}$ be a smooth function satisfying $\rho_{0, \kappa} \geq c > 0$ and $q_{0, \kappa}$ be the associated pressure given by (1.9). Suppose

$$||v_{0, \kappa}||_4, ||v_{0, \kappa}||_{4, \Gamma}, ||q_{0, \kappa}||_4, ||q_{0, \kappa}||_{4, \Gamma} \leq m, \quad \text{for all } \kappa > 0. \quad (1.12)$$

Then there exist a $T > 0$ and a constant $\mathfrak{M}$ such that any smooth solution $(v_\kappa, R_\kappa)$ to (1.3) defined on the time interval $[0, T]$ satisfies

$$\mathcal{N}(t) \leq \mathfrak{M},$$

where

$$\mathcal{N} = ||v_\kappa||^2 + |\mathfrak{R}_\kappa \partial_t v_\kappa|^2 + ||\mathfrak{R}_\kappa \partial^2_t v_\kappa||^2 + ||\mathfrak{R}_\kappa \partial^3_t v_\kappa||^2 + ||R_\kappa \partial_t v_\kappa||^2 + ||R_\kappa \partial^2_t v_\kappa||^2 + ||R_\kappa \partial^3_t v_\kappa||^2$$

$$+ ||R_\kappa \partial_t v_\kappa||^2 + ||\mathfrak{R}_\kappa \partial^2_t v_\kappa||^2 + ||\mathfrak{R}_\kappa \partial^3_t v_\kappa||^2 + ||\partial_t R_\kappa||^2 + ||\partial^2_t R_\kappa||^2 + ||\partial^3_t R_\kappa||^2 + E, \quad (1.13)$$

where $E$ is defined as Definition 3.1.

The next theorem is a direct consequence of Theorem 1.1 together with the Arzelà-Ascoli theorem.
Theorem 1.2. Let \( v_0 \in H^{6.5}(\Omega) \) be a divergence free vector field and let \( v \) be the solution to the incompressible free-boundary Euler equations (1.7) with data \( v_0 \) defined on a time interval \([0,T]\). Let \((v_{0,\kappa}, R_{0,\kappa}) \in H^4(\Omega) \times H^4(\Omega)\) be a sequence of initial data for the compressible free-boundary Euler equations (1.3) satisfying the compatibility conditions up to order 3 (see Section 5.1 for a statement of the compatibility conditions). Furthermore, assume that \((v_{0,\kappa}, R_{0,\kappa}) \to (v_0, \beta)\) in \(C^2(\Omega)\) as \(\kappa \to \infty\) and that (1.12) holds. Let \((v_{\kappa}, R_{\kappa})\) be the solution for (1.3) with the equation of state (1.9). Then:

1. For \(\kappa\) sufficiently large, \((v_{\kappa}, R_{\kappa})\) is defined on \([0,T]\).
2. \((v_{\kappa}, R_{\kappa}) \to (v, \beta)\) in \(C^0([0,T], C^2(\Omega))\) after possibly passing to a subsequence.

Remark. \(v_0 \in H^{6.5}(\Omega)\) is required so that the initial norms are uniformly bounded. We refer the proof of Theorem 5.1 for details.

Finally, we need the following theorem to show that the data required in Theorem 1.1 and Theorem 1.2 exists.

Theorem 1.3. Let \( v_0 \in H^{6.5}(\Omega) \) be a divergence free vector field in \(\Omega\). Then there exists initial data \((v_{0,\kappa}, R_{0,\kappa}) \in H^4(\Omega) \times H^4(\Omega)\) satisfying the compatibility conditions up to order 3 (see Section 5.1 for a statement of the compatibility conditions) such that \((v_{0,\kappa}, R_{0,\kappa}) \to (v_0, \beta)\) in \(C^2(\Omega)\) as \(\kappa \to \infty\), and (1.12) holds.

Notation 1.4. For the sake of clean notations, we will drop the \(\kappa\)-indices on \(v_{\kappa}, R_{\kappa}, q_{\kappa}\), i.e., we will denote \((v_{\kappa}, R_{\kappa}, q_{\kappa}) = (v, R, q)\) when no confusion can arise.

1.5 Strategy, organization of the paper, and discussion of the difficulties

In this section we overview the main arguments of the paper, summarize the main difficulties, and explain how they are confronted.

1.5.1 Special cancellations

As mentioned, having \(\sigma > 0\) leads to several new difficulties not present when \(\sigma = 0\). This can be immediately seen from the boundary terms appearing in the energy estimates (see Sections 3.4 and 3.5), since all these terms are proportional to \(\sigma\) and, therefore, automatically vanish when \(\sigma = 0\). (Incidentally, we do not set \(\sigma\) to 1 as it is customary but keep it explicit in order to highlight all the terms that would be absent had \(\sigma\) been zero.) Not only are these terms present but, as we discuss below, they are some of the most difficult terms to handle. As a consequence, the methods used in the second author’s previous papers to study the problem with \(\sigma = 0\) \([46, 48]\) cannot be applied when \(\sigma > 0\).

At first sight one might think that the surface tension should help with closing a priori estimates since it has a regularizing effect on the boundary. This regularization, however, it is not enough to produce control of the velocity on the boundary. After differentiating the equations with respect to \(D^k\), where \(D^k\) is a \(k^{th}\) order derivative, possibly mixing space and time derivatives, contracting with \(D^k v\) and integrating by parts, one is left with a boundary term that reads, schematically,

\[
\int_{\Gamma} D^k v D^k q \, dS.
\]
It is not difficult to see that we can only hope to control this term by employing the boundary condition so that (again, schematically)
\[
\int_D D^k v D^k q Ds \sim \int_D D^k v D^k (\Delta_g \eta) Ds.
\] (1.14)

The presence of the boundary Laplacian and the fact that \( v = \partial_t \eta \) suggest that we should integrate by parts in space and factor a \( \partial_t \). Although this is the strategy, we end up with a commutator term that is not of lower order. This is because the coefficients of \( \Delta_g \) involve one derivative of \( g \) which, in turn, involves one derivative of \( \eta \) (so that the coefficients depend on as many derivatives of \( \eta \) as the order of the equation). Thus, commuting \( D^k \) and \( \Delta_g \) still leaves a top order term that cannot be written as a perfect derivative (in time or space) to be integrated away. Moreover, this top order term does not seem to have any good structure. In fact, one should not expect such term to have a good structure, since differentiating the coefficients of \( \Delta_g \) corresponds to differentiate \( g^{ij} \), and, thus, to take derivatives of some non-linear combinations of the components \( g_{ij} \) and its determinant.

The above difficulties are overcome by observing some remarkable cancellations among the bad top order terms in (1.14). Such cancellations are not visible in any way in the expressions that appear by simply manipulating (1.14). Rather, they are identified after some judicious and lengthy analysis that relies heavily on some geometric properties, expressed in the form of several geometric identities, of the boundary. The first cancellation appears in (3.19). The reader can check that the terms that cancel out are top order and that there does not seem to be possible to bound them individually. The second cancellation happens between a term in (3.18) and (3.20). This second cancellation is even more remarkable because the terms involved come from completely different parts of \( D^k \Delta_g \eta \): one from when all derivatives fall on the coefficient \( \sqrt{g} g^{ij} \) of \( \Delta_g \), the other from when we integrate one derivative in \( \Delta_g \) by parts.

We also need a special cancellation for interior terms. This comes from when we take \( D^k \) of the first equation in (1.3) and all derivatives fall on \( a \). Since the matrix \( a \) already involves one derivative of \( \eta \), we find terms in \( D^{k+1} \eta \), which have one too many derivatives of the Lagrangian map. Exploiting the explicit structure of \( a \), however, we are able to show that, when appropriately grouped, these bad terms cancel each other after some careful integration by parts (see (3.14) and what follows).

As this point one may ask if all such cancellations are indeed necessary since a priori estimates for (1.3) have been derived in the literature. The relevant work in this regard is [9]. There, the authors construct initial data where \( \eta \) is everywhere one degree more differentiable than \( v \), and then prove that this extra regularity is propagated by the evolution. They rely on such extra regularity to close the estimates. However, this does not seem possible here because such an extra differentiability is not compatible with the weights we need to introduce in order to obtain estimates uniform in the sound speed (see Section 1.5.2).

A crucial aspect of all the cancellations mentioned above is that they require the derivatives \( D^k \) to contain at least one time derivative. As a consequence, only the Sobolev norms of time-derivatives of \( v \) on the boundary are controlled from the energy estimates (we remark that the energy does involve time derivatives of the variables; it does not seem possible to close the estimates without time-differentiating the equations). To obtain control of non-time differentiated \( v \) on the boundary, we rely directly on the boundary condition which, after a time derivative, produces an equation of the form \( \Delta_g v = \ldots \) which is amenable to elliptic estimates. (One might wonder why we do not take further time derivatives of the boundary condition to obtain estimates for \( D^k v \) on the boundary. The reason is that, as mentioned above, \( \Delta_g \) does not commute well with derivatives due to the dependence of the coefficients on two derivatives of \( \eta \), so that we obtain an equation of worsening
structure with each derivative. However, for only one time derivative, the resulting equation still has some good structure that can be used to derive estimates.)

1.5.2 Weighted estimates

Another difficulty to establish the the incompressible limit is that one has to derive estimates that are uniform in the sound speed, since the goal is to take the sound speed to infinity. This is substantially different than estimates for (1.3) (with $\sigma > 0$) currently available [9, 21]. Establishing the required uniform-in-$\kappa$ a priori estimate does not seem to be possible solely by the methods used to derive the currently available estimates. In particular, a crucial element to derive such uniform estimates is the use of a non-linear wave equation satisfied by the density, whereas non-uniform-in-$\kappa$ estimates have been proven without this wave equation. In fact, the known a priori energy bounds rely heavily on the fact that when $\mathcal{A}_{\kappa}$ is bounded from below (as $\kappa$ is bounded from above), $\partial q \approx \partial R$ and $||q||_\Gamma \approx ||R||_\Gamma$, which is a direct consequence of the equation of state. In particular, the energy used in [21] controls $||\partial^k q||_{3-k}$ for free as a lower order term. However, this fact no longer holds when $\mathcal{A}_{\kappa} \to 0$. Indeed, since $\partial R = R' \partial q$, $||R||_\Gamma$ is merely equivalent to $||\mathcal{A}_{\kappa} q||_\Gamma$; in other words, we have to take extra effort to control the full Sobolev norms of $\partial^k q$. In [46] and [48], where $\sigma = 0$, these norms are controlled by elliptic estimate. This relies on the fact that one is able to control $||q_t||_\Gamma$ by the $r$-th order energy $E_r$ since

$$\partial^r q_t \sim \overline{\partial}^r q_t + \partial^{r-2} q_t + \text{lower order terms},$$

where $\overline{\partial}$ denotes derivatives tangent to the boundary. The first term, $\overline{\partial}^r q_t$, vanishes due to $q|_{\Gamma} = 0$. However, this method does not work when $\sigma > 0$, which is simply due to the fact that $q \sim \Delta q \eta$ on $\Gamma$, and so $\overline{\partial}^r q_t \sim \overline{\partial}^{r+2} v$ on the boundary which has two derivatives too many.

To resolve the above difficulties, our energy is defined using the weighted derivatives $\partial^r$ $(1 \leq r \leq 4)$, where

$$\mathcal{D} = \overline{\partial}, \partial_t; \quad \mathcal{D}^2 = \overline{\partial}^2, \overline{\partial} \partial_t, \sqrt{\mathcal{A}_{\kappa}} \partial_t^2; \quad \mathcal{D}^3 = \overline{\partial}^3, \overline{\partial} \partial_t^2, \mathcal{A}_{\kappa} \partial_t^3; \quad \mathcal{D}^4 = \mathcal{A}_{\kappa} (\overline{\partial} \partial_t), \mathcal{A}_{\kappa} (\overline{\partial}^2 \partial_t), (\mathcal{A}_{\kappa})^2 \partial_t^4.$$

The energy $E = E(t)$ is defined by employing these weighted derivatives, which is of the form:

$$E = \sum_{1 \leq \ell \leq 4} ||\mathcal{D}^\ell v||^2_{L^2(\Omega)} + \sum_{1 \leq \ell \leq 4} \sqrt{\mathcal{A}_{\kappa}} ||\mathcal{D}^\ell q||^2_{L^2(\Omega)} + \sigma \sum_{1 \leq \ell \leq 4} ||\Pi \mathcal{D}^\ell q||^2_{L^2(\Gamma)} + W,$$

where $\Pi$ is the projection onto the normal to the moving boundary (see Lemma 2.2) and $W$ stands for the energy of the wave equation satisfied by $q$, which is defined in Section 2.3-2.4.

The energy estimate for $E$ cannot be closed by itself; in fact, the energy estimate requires control of

$$||v||_4, ||\mathcal{A}_{\kappa} v_t||_3, ||\mathcal{A}_{\kappa} v_{tt}||_2, ||(\mathcal{A}_{\kappa})^2 v_{ttt}||_1,$$

and

$$||R||_4, ||R_t||_3, ||\sqrt{\mathcal{A}_{\kappa}} R_{tt}||_2, ||\mathcal{A}_{\kappa} R_{ttt}||_1.$$

These quantities are not part of the energy since $\mathcal{D}^\ell$ for $\ell = 1, 2, 3, 4$ do not involve non-tangential derivatives, nor the full tangential spatial derivative $\overline{\partial}^4$. Such missing derivatives, however, cannot be included in the energy because they would lead to the presence of non-tangential derivatives.
on the boundary. As a consequence, we need to estimate $E$ together with the quantities above in order to close the a priori estimate. This is done with the help of elliptic estimates.

We now schematically show how to get the correct weights for our energy, since they are crucial for the desired uniform-in-$\kappa$ estimates. We differentiate the equations

$$R\partial_t v_\alpha + q'(R)a^{\alpha\mu}\partial_\mu R = 0,$$

and

$$\partial_t R + Ra^{\alpha\mu}\partial_\mu v_\alpha = 0,$$

with respect to time. Since $R' = R'(q) = \frac{1}{q(R)}$, equation (1.17) implies

$$\partial^k_t R \sim R'\partial_t^{k+1}v;$$

in other words, we can trade one (full) spatial derivative on $R$ by one time derivative of $v$ multiplied by $R'$. On the other hand, in view of the standard div-curl estimate (i.e., (A.2) in Appendix), $\partial_t^k v$ is estimated via $\div \partial_t^k v$, $\curl \partial_t^k v$ and $\partial_t^k v \cdot N$. While in the reference domain $\Omega = \mathbb{T}^2 \times (0, 1)$, $\partial_t^k v \cdot N = \pm \partial_t^k v^3$, which is almost $\Pi\partial_t^k v$, where $\Pi$ denotes the projection to the normal direction, and hence this can be controlled by $E$. In addition, $\curl \partial_t^k v$ is estimated via Cauchy invariance which can be treated by adapting the method introduced in [21]. Finally, the equation (1.18) yields

$$a^{\mu\alpha}\partial_\mu \partial_t^k v_\alpha \sim \partial_t^{k+1} R;$$

in other words, we can estimate $\div \partial_t^k v$ using $\partial_t^{k+1} R$. Hence,

$$\partial^4 v \xrightarrow{\div} \partial^3 R_t \xrightarrow{(1.19)} R'\partial^2 \partial_t^2 v \xrightarrow{\div} R'\partial\partial_t^3 R \xrightarrow{(1.19)} (R')^2\partial_t^4 v,$$

where $(R')^2\partial_t^4 v$ is part of $E$. In addition, we have

$$R'\partial^3 \partial_t v \xrightarrow{\div} R'\partial^2 \partial_t^2 R \xrightarrow{(1.19)} (R')^2\partial^3 \partial_t^3 v \xrightarrow{\div} (R')^2\partial_t^4 v.$$

This algorithm also provides

$$R'\partial^2 \partial_t^3 v \xrightarrow{\div} R'\partial\partial_t^3 R \xrightarrow{(1.19)} (R')^2\partial_t^4 v,$$

$$(R')^3 \partial_t^4 R \xrightarrow{\div} (R')^2 \partial_t^4 R.$$

Here, $(R')^3 \partial_t^4 R$ can be controlled directly by $E$ since it is equal to $(R')^2 \partial_t^4 q$ up to lower order terms. On the other hand, applying this algorithm starting from $\partial^4 R$, we get

$$\partial^4 R \xrightarrow{(1.19)} R'\partial^3 \partial_t v \xrightarrow{\div} R'\partial^2 \partial_t^2 R \xrightarrow{(1.19)} (R')^2\partial^2 \partial_t^3 v \xrightarrow{\div} (R')^2\partial_t^4 R,$$

$$\partial^3 \partial_t R \xrightarrow{(1.19)} R'\partial^2 \partial_t^2 v \xrightarrow{\div} R'\partial\partial_t^3 R \xrightarrow{(1.19)} (R')^2\partial_t^4 v,$$

$$\sqrt{R'}\partial^2 \partial_t^2 R \xrightarrow{(1.19)} (R')^2 \partial_t^4 R \xrightarrow{\div} (R')^2 \partial_t^4 R,$$

$$R'\partial\partial_t^3 R \xrightarrow{(1.19)} (R')^2 \partial_t^4 v.$$
Remark. The condition (1.10) allows us to define the weighted Sobolev norms (e.g., (1.13)) with constant $R_\kappa$-weight. It is convenient to have constant weights for the boundary estimates in Section 3.4 to avoid derivatives falling on $R'$. In addition, the condition (1.10) allows us to distribute $R_\kappa$-weight in order to obtain an uniform control in $\kappa$.

The definition of the weighted derivative $D^r$ allows us to control the highest order (i.e., 4th order) mixed norms of $q$ directly by the energy. However, in order to pass to the incompressible limit, we have to control $||v||_4$ directly without $R_\kappa$-weight, and this requires the control of $||q_t||_2$. In Section 3.2, we control $||q_t||_2$ by the elliptic estimate, which requires the control of $||q_t||_1$ first. This is indeed of lower order but we need to take extra effort to prove that they can be controlled uniformly as $R_\kappa \to 0$. In addition, we remark here that in [21], the authors were able to close the a priori energy estimate in $H^3$. However, in our case, the bound for $||q_t||_1$ require the control of $||v||_4$ and $||\eta||_4$. This is because control of $||\partial q_t||_0^2$ requires integration by parts, which yields $||\overline{D}_\eta||_{1.5,\Gamma}$ and $||\overline{\partial^2} v||_{1.5,\Gamma}$ at the top order, and these quantities require $H^4$ control of $v, \eta$.

1.5.3 The initial data

As with the estimates themselves, the initial data has to be constructed uniform in the sound speed in order to allow the passage to the limit $\kappa \to 0$. This was done for $\sigma = 0$ in [46], but that method relied heavily on the fact that $q$ vanishes on the boundary when surface tension is absent. Instead, we employ the method used in [9]: For each $1 \leq k \leq 3$, the data that satisfies the $k$-th order compatibility condition is obtained via solving an elliptic equation of order $2k$, which is acquired by time differentiating the boundary condition $q = \sigma \mathcal{H}$ for $k$ times and then restrict at $t = 0$, where the previous $0, \cdots, k - 1$-th compatibility conditions are served as the boundary conditions. This construction process allows one to show that the initial data is uniformly bounded for all sound speed $\kappa$, so that one can take the limit $\kappa \to \infty$.

1.6 List of notations

- $\nabla$: Eulerian spatial derivative.
- $\partial$: Lagrangian spatial derivative.
- $\overline{\partial}$: Tangential spatial derivative. In particular, $\overline{\partial} = (\partial_1, \partial_2)$ in $\Omega$ and we will emphasize that these derivatives are tangential by denoting $(\partial_1, \partial_2) = (\overline{\partial}_1, \overline{\partial}_2)$.
- $D$: Either $\overline{\partial}$ or $\partial_t$.
- $\Omega$ and $\Gamma$: The reference domain $(0, 1) \times \mathbb{T}^2$ in Lagrangian coordinate, whose boundary $\partial \Omega = \Gamma$.
- The matrices $a$ and $A$: $a = (\partial \eta)^{-1}$, and $A = J a$, where $J = \det(\partial \eta)$.
- $\kappa$: The sound speed.
- $\mathfrak{R}_\kappa$: $\mathfrak{R}_\kappa \approx R'_\kappa \to 0$ as $\kappa \to \infty$.
- $||\cdot||_s = ||\cdot||_{H^s(\Omega)}$ and $||\cdot||_{s, \Gamma} = ||\cdot||_{H^s(\Gamma)}$.
- $P(\cdot)$: A smooth function expression in its arguments.
- $\overset{L}{\equiv}$: Equality modulo lower order terms that can be controlled appropriately.
2 Preliminary results

In this section, we give some auxiliary results providing the bounds on the flow map \( \eta \) and the matrix \( a \). In addition, we record several facts, expressions and inequalities that will come in handy in the later sections. These results will be employed in the proof of Theorem 1.1.

**Lemma 2.1.** Assume that \( ||v||_{L^\infty([0,T],H^4(\Omega))}+||R||_{L^\infty([0,T],H^4(\Omega))} \leq M \). Let \( p \in [1,\infty) \), then there exists a sufficiently large constant \( C > 0 \), such that if \( T \in \left[0, \frac{1}{C^2M^2}\right] \) and \((v,q)\) is defined on \([0,T]\), the following statements hold:

1. \( ||\eta||_4 \leq C \).
2. \( ||a||_3 \leq C \).
3. \( ||a_t||_{L^p(\Omega)} \leq C||\partial v||_{L^p(\Omega)} \), and \( ||a_t||_s \leq C||\partial v||_s \), \( 0 \leq s \leq 3 \).
4. \( ||\partial a_t||_{L^p(\Omega)} \leq C||\partial v||_{L^p(\Omega)} |||\partial a_t||_{L^p(\Omega)} + C||\partial a_t||_{L^p(\Omega)} \), where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \).
5. \( ||a_{tt}||_s \leq C||\partial v||_s |||\partial v||_s + C||\partial v||_s \), \( 0 \leq s \leq 2 \).
6. \( ||a_{ttt}||_s \leq C||\partial v||_s |||\partial v||_s + C||\partial v||_s \), \( 0 \leq s \leq 1 \).
7. \( ||\partial^4 a||_{L^p(\Omega)} \leq C||\partial v||_{L^p(\Omega)} |||\partial v||_{L^p(\Omega)} |||\partial v||_{L^p(\Omega)} |||\partial v||_{L^p(\Omega)} + C||\partial v||_{L^p(\Omega)} |||\partial v||_{L^p(\Omega)} |||\partial v||_{L^p(\Omega)} \).
8. \( J \geq \frac{1}{2} \).
9. If \( \epsilon \) is sufficiently small and for \( t \in \left[0, \frac{1}{C^2M^2}\right] \), we have \( ||a^{\alpha\beta} - \delta^{\alpha\beta}||_3 \leq \epsilon \), and \( ||a^{\alpha\mu}a^{\beta}_\mu - \delta^{\alpha\beta}||_3 \leq \epsilon \).
10. \( C^{-1} \leq R \leq C \).

Proof. We refer [21] and [31] for the detailed proof. We point out that the proof follows directly from the equations, interpolation, and the fundamental theorem of calculus.

We record here the explicit form of the matrix \( a \) which will be needed.

\[
a = J^{-1} \begin{pmatrix}
\partial_2 \eta^2 \partial_3 \eta^3 - \partial_3 \eta^2 \partial_2 \eta^3 \\
\partial_3 \eta^3 \partial_1 \eta^3 - \partial_1 \eta^3 \partial_3 \eta^3 \\
\partial_1 \eta^3 \partial_2 \eta^3 - \partial_2 \eta^3 \partial_1 \eta^3 \\
\partial_1 \eta^2 \partial_2 \eta^2 - \partial_2 \eta^2 \partial_1 \eta^2 \\
\partial_2 \eta^1 \partial_3 \eta^2 - \partial_3 \eta^2 \partial_2 \eta^1 \\
\partial_3 \eta^1 \partial_2 \eta^2 - \partial_2 \eta^1 \partial_3 \eta^2 \\
\partial_1 \eta^1 \partial_3 \eta^1 - \partial_3 \eta^1 \partial_1 \eta^1 \\
\partial_3 \eta^2 \partial_2 \eta^1 - \partial_2 \eta^2 \partial_3 \eta^1 \\
\partial_2 \eta^1 \partial_1 \eta^1 - \partial_1 \eta^1 \partial_2 \eta^1 \\
\partial_1 \eta^1 \partial_3 \eta^2 - \partial_3 \eta^1 \partial_1 \eta^2
\end{pmatrix}
\]

(2.1)

Moreover, since \( A = Ja \), and in view of (2.1), we can write

\[
A^{1\alpha} = \epsilon^{\alpha\lambda\tau} \partial_2 \eta_\lambda \partial_3 \eta_\tau, \quad A^{2\alpha} = -\epsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_3 \eta_\tau, \quad A^{3\alpha} = \epsilon^{\alpha\lambda\tau} \partial_1 \eta_\lambda \partial_2 \eta_\tau.
\]

(2.2)

Here, \( \epsilon^{\alpha\lambda\tau} \) is the fully antisymmetric symbol with \( \epsilon^{123} = 1 \). This representation will be used to create a special cancellation scheme that leads to control of the energy when all derivatives fall on the cofactor matrix (recall the discussion in Section 1.5.1).

We also need some geometric identities to treat the boundary terms in the energy estimate. We record these identities in the next lemma.

**Lemma 2.2.** Let \( n \) be the outward unit normal to \( \eta(\Gamma) \). Let \( \tau \) be the tangent bundle of \( \eta(\Omega) \) and \( \nu \) be the normal bundle of \( \eta(\Gamma) \), the canonical projection is given by

\[
\Pi_\beta = \delta_\beta^\alpha - g^{kl} \bar{\partial}_k \eta^\alpha \bar{\partial}_l \eta_\beta,
\]

and on \( \Gamma \) it holds that:
1. \(-\Delta_g \eta^\alpha = \mathcal{H} \circ \eta \eta^\alpha \circ \eta\).
2. \(n \circ \eta = \frac{\rho}{|a^TN|}\).
3. \(J|a^TN| = \sqrt{g}\).

Above, \(a^T\) is the transpose of \(a\). Furthermore, setting \(\hat{n} = n \circ \eta\), the following identities hold on \(\Gamma\):

4. \(\Pi^\beta_n = \hat{n}_\beta \hat{n}^\alpha\).
5. \(\Pi^\alpha_n \Pi^\lambda_n = \Pi^\alpha_n\).
6. \(\hat{n}_\alpha = \hat{n}_n \Pi^\alpha_n\).
7. \(\sqrt{g} \Delta_g \eta^\alpha = \sqrt{g} g^{ij} \Pi^\alpha_n \vec{\partial}_{ij} \eta^\mu\).
8. \(\partial_t \hat{n}_\mu = -g^{kl} \partial_k \eta^\tau \hat{n}_\tau \partial_l \eta_\mu\).
9. \(\partial_t \hat{n}_\mu = -g^{kl} \partial_k \eta^\tau \hat{n}_\tau \partial_l \eta_\mu\).
10. \(\vec{\partial}_i(\sqrt{g} g^{ik}) = -\sqrt{g} g^{ij} g^{kl} \vec{\partial}_j \eta^\mu \vec{\partial}_l \eta_\mu\).
11. \(\vec{\partial}_i(\sqrt{g} g^{ij}) = \sqrt{g} (g^{ij} g^{kl} - 2g^{ij} g^{ik}) \vec{\partial}_k \eta^\lambda \vec{\partial}_l \eta_\lambda\).

\textbf{Proof.} These identities are well-known. The interested reader can consult, e.g., [21] for their proof.

The equation of state \(q = q(R)\) allows us to control \(R'q\) and \(R\) interchangeably:

\textbf{Lemma 2.3.} Suppose \(R' := R'(q)\) satisfies (1.11), and let \(\partial\) be either \(\partial_t\) or \(\partial_\alpha\), then for each \(1 \leq r \leq 4\), we have:

\[ |R' \partial^\tau q| \lesssim |\partial^\tau R| + \sum_{2 \leq k \leq r} |\partial^{j_1} R| \cdots |\partial^{j_k} R| \]  \hspace{1cm} (2.3)

\textbf{Proof.} A direct computation yields:

\[ R' \partial^\tau q = \partial^\tau R + \sum_{2 \leq k \leq r} |\partial^{j_1} R| \cdots |\partial^{j_k} R| \]

and invoking (1.11) and the fact \(R' \partial q = \partial R\), (2.3) then follows.

\section{2.1 The boundary condition}

The identities of Lemma 2.2 imply that the boundary condition

\[ A^{\mu\alpha} N_\mu q + \sigma \sqrt{g} \Delta_g \eta^\alpha = 0, \quad \text{on} \ \Gamma, \]  \hspace{1cm} (2.4)

can be expressed in the following equivalent ways:

1. \(\sqrt{g} g^{ij} \vec{\partial}_{ij} \eta^\alpha - \sqrt{g} g^{ij} g^{kl} \vec{\partial}_l \eta^\mu \vec{\partial}_j \eta_\mu = -\frac{1}{\sigma} A^{\mu\alpha} N_\mu q\), where \(g^{kl} \vec{\partial}_l \eta^\mu \vec{\partial}_j \eta_\mu = \Gamma^{lk}_{ij}\).
2. \(\sqrt{g} g^{ij} \Pi^\alpha_n \vec{\partial}_{ij} \eta^\mu = -\frac{1}{\sigma} A^{\mu\alpha} N_\mu q\).
3. \(q = -\sigma (A^{3\alpha} \hat{n}_\alpha)^{-1} \sqrt{g} g^{ij} \hat{n}_\mu \vec{\partial}_{ij} \eta^\mu = -\sigma g^{ij} \hat{n}_\mu \vec{\partial}_{ij} \eta^\mu\), since \((A^{3\alpha} \hat{n}_\alpha)^{-1} \sqrt{g}\) simplifies to 1.

These identities follow directly from the definition. Interested readers can consult [21] for their proof. The above expressions will be frequently used to deal with the boundary estimates.
2.2 The interpolation inequality

Besides standard interpolation, we will also use the following interpolation inequality throughout this paper.

**Theorem 2.4.** Let $u : \Omega \to \mathbb{R}$ be a $H^1$ function. Then:

$$||u||_{L^4(\Omega)} \lesssim ||u||^\frac{3}{2}_0 ||u||^\frac{1}{2}_1.$$  

**Proof.** See Theorem 5.8 in [1].

2.3 The wave equations of order 3 or less

The second equation in (1.3) can be re-expressed as

$$a^{\mu \alpha} \partial_\mu v_\alpha = -\frac{R' \partial q}{R}, \quad (2.5)$$

where $R' = R'_\kappa(q) \sim \mathcal{R}_\kappa$ via assumption (1.11). Identity (2.5) together with (1.6) yield, after commuting $\partial^r_t - 1$ for $1 \leq r \leq 3$ and then $a^{\nu \alpha} \partial_\nu$, that:

$$JR' \partial^{r+1}_t q - a^{\nu \alpha} A^\mu_\alpha \partial_\nu \partial_\mu \partial^{r-1}_t q = F_r, \quad (2.6)$$

where

$$F_r = -\sum_{j_1 + j_2 = r} (\partial^{j_1}_t (JR'))(\partial^{j_2+1}_t q) + a^{\nu \alpha} (\partial_\nu \rho_0) \partial^{r}_t v_\alpha$$

$$-\rho_0 \sum_{j_1 + j_2 = r-1} (\partial^{j_1+1}_t a^{\nu \alpha}) (\partial^{j_2}_t \partial_\nu v_\alpha) + a^{\nu \alpha} (\partial_\nu A^\mu_\alpha) \partial_\mu \partial^{r-1}_t q. \quad (2.7)$$

The wave equation (2.6) yields an energy identity which is essential when estimating $||q||_2$ and $||q_t||_2$ in Section 3.2:

**Theorem 2.5.** For $1 \leq r \leq 3$, let

$$W_r^2 = \frac{1}{2} \int_\Omega \rho_0^{-1}(JR'\partial^r_t q)^2 \, dy + \frac{1}{2} \int_\Gamma \rho_0^{-1} R'(A^{\mu \alpha} \partial_\nu \partial^r_t q) (A^\mu_\alpha \partial_\mu \partial^{r-1}_t q) \, dy$$

$$+ \frac{\sigma}{2} \int_\Gamma \mathcal{R}_\kappa \sqrt{g} g^{ij} \Pi^{\alpha}_{\mu}(\partial_\nu \partial^{r} q^\mu) (\partial_\nu \partial^{r} q^\mu) \, ds. \quad (2.8)$$

Then,

$$\sum_{1 \leq r \leq 3} W_r^2 \leq \epsilon P(N) + \epsilon(||q||_2^2 + ||q_t||_2^2) + P_0 + P \int_0^t \mathcal{P}, \quad t \in [0, T], \quad (2.9)$$

where $T > 0$ is sufficiently small.

**Proof.** See Appendix B.  \[ \square \]
2.4 The $\mathcal{R}_\kappa$-weighted wave equations

We consider the following $\mathcal{R}_\kappa$-weighted derivatives:

$$\mathcal{R}_\kappa \partial_t^3, \ \sqrt{\mathcal{R}_\kappa \partial_t^2}, \ \partial_t \mathcal{R}_\kappa^2.$$  

Writing these derivatives as $\mathcal{R}_\kappa^\ell D^3$ ($\ell = 1, \frac{1}{2}, 0$), and the identity (2.5) together with (1.6) yield, after commuting $\mathcal{R}_\kappa^\ell D^3$ and then $a^{\nu\alpha} \partial_\nu$, that:

$$\mathcal{R}_\kappa^\ell R' JD^3 \partial_t^2 q - \mathcal{R}_\kappa^\ell a^{\nu\alpha} A^\nu_\alpha \partial_\nu \partial_\mu D^3 q = \bar{F},$$

where

$$\bar{F} = -\mathcal{R}_\kappa^\ell [D^3 \partial_t, JR'] \partial_t q + \mathcal{R}_\kappa^\ell [D^3, \rho_0] \partial_t (R^{-1} R' \partial_t q)$$

$$+ \mathcal{R}_\kappa^\ell a^{\nu\alpha} (\partial_\nu \rho_0) D^3 \partial_t v_\alpha + \mathcal{R}_\kappa^\ell a^{\nu\alpha} \partial_\nu ([D^3, A^\mu_\alpha] \partial_\mu q) + \mathcal{R}_\kappa^\ell a^{\nu\alpha} \partial_\nu ([D^3, \rho_0] \partial_t v_\alpha)$$

$$- \mathcal{R}_\kappa^\ell \rho_0 [D^3 \partial_t, a^{\nu\alpha}] \partial_\nu v_\alpha + \mathcal{R}_\kappa^\ell a^{\nu\alpha} (\partial_\nu A^\mu_\alpha) \partial_\mu D^3 q.$$  

We need these $\mathcal{R}_\kappa$-weighted wave equations since their energies yield a better control of certain $\mathcal{R}_\kappa$-weighted energy terms.

**Theorem 2.6.** Let

$$W_1^2 = \frac{1}{2} \int_0^1 \rho_0^{-1} \mathcal{R}_\kappa^{2t} (JR' D^3 \partial_t q)^2 dy + \frac{1}{2} \int_0^1 \rho_0^{-1} \mathcal{R}_\kappa^{2t} R' (A^{\nu\alpha} \partial_\nu D^3 q) (A^\mu_\alpha \partial_\mu D^3 q) dy$$

$$+ \frac{\sigma}{2} \int_\Gamma \mathcal{R}_\kappa^{2t+1} \sqrt{gg^{ij} \Pi^3_{\alpha\mu} (\partial_\alpha D^3 \partial_t q) (\partial_j D^3 \partial_\alpha q)} dS. \quad (2.10)$$

Then,

$$W_4^2 \leq \epsilon P(\mathcal{N}) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad t \in [0, T],$$

where $T > 0$ is sufficiently small.

**Proof.** See Appendix C. \hfill \Box

**Remark.** The energy (2.10) yields a better control of $q$ with 1/2 less $\mathcal{R}_\kappa$-weight, e.g., when $D = \partial_t$, $W_4$ controls $||\mathcal{R}_\kappa^2 \partial_t^3 q||_0$ and $||\mathcal{R}_\kappa^2 \partial_t \partial_\alpha^2 q||_0$. The corresponding terms in $E$ control merely $||\mathcal{R}_\kappa^2 \partial_t q||_0$ and $||\mathcal{R}_\kappa^2 \partial_\alpha^2 \partial_t^3 q||_0$. This observation is crucial to control $\mathcal{L}_3$ in Section 3.3 when $\mathcal{D}^4 = \mathcal{R}_\kappa^2 \partial_\alpha \partial_t^3$ or $\mathcal{R}_\kappa \partial_\alpha \partial_t^2$.

2.5 The Cauchy invariance

We conclude this section with a compressible version of the Cauchy invariance, which was introduced in [21].

**Theorem 2.7.** Let $(v, R)$ be a smooth solution to (1.3), then

$$e^{\alpha\beta\gamma} \partial_\beta v^\mu \partial_\gamma \eta_\mu = \omega^0_\alpha + \int_0^t e^{\alpha\beta\gamma} \partial_\lambda q \partial_\gamma \eta_\mu \frac{\partial_\beta R}{R^2},$$

for $t \in [0, T)$. Here, $e^{\alpha\beta\gamma}$ is the totally antisymmetric symbol with $e^{123} = 1$ and $\omega_0$ us the vorticity at $t = 0$.  

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3 Energy estimates

In this section we provide estimates for \((v,q)\) and their time derivatives. We shall make frequent use of the assumptions (1.10)-(1.11) and of the two preliminary lemmas (i.e., Lemma 2.1 and Lemma 2.2) in Section 2 throughout this section without mentioning them every time.

Notation 3.1. Let \(E\) be defined as in Definition 3.1, and let

\[
P = P(||v||_4, ||\mathcal{R}_v v_t||_3, ||v_t||_2, ||\mathcal{R}_v v_{tt}||_2, ||\sqrt{\mathcal{R}_v} v_t||_1, ||(\mathcal{R}_v)^{3/2} v_{ttt}||_1, ||\mathcal{R}_v v_{ttt}||_0, ||R||_4, ||R_t||_3, ||\sqrt{\mathcal{R}_v} R_{tt}||_2, ||R_{tt}||_1, ||\mathcal{R}_v R_{tt}||_1, ||\sqrt{\mathcal{R}_v} R_{ttt}||_0, ||\mathcal{R}_v \Pi \mathcal{D}^2 v_t||_{0,\Gamma}, ||\mathcal{R}_v \Pi \mathcal{D}^2 v_{tt}||_{0,\Gamma}, ||(\mathcal{R}_v)^{3/2} \Pi \mathcal{D} v_{ttt}||_{0,\Gamma}, ||\Pi \mathcal{D}^2 v_{tt}||_{0,\Gamma}, ||\sqrt{\mathcal{R}_v} \Pi \mathcal{D} v_{tt}||_{0,\Gamma})
\]

and \(P_0 = P(||\eta||_7, ||v||_4, ||v_t||_{4,\Gamma}, ||q||_4, ||q_t||_{4,\Gamma}, ||\text{div} v_0||_3, ||\Delta v_0||_{2,\Gamma}, ||\Delta v_0||_2)\), where we abbreviate

\[
||\Pi w||^2_{0,\Gamma} = \int_G \Pi^\beta w^\mu \Pi^\beta w_\alpha.
\]

Here (and throughout this paper), we use \(P(\cdot)\) to denote a smooth function in its arguments. In addition, we define \(\mathcal{N}\) to be

\[
\mathcal{N}(t) = ||v||_2^2 + ||\mathcal{R}_v v_t||_3^2 + ||\mathcal{R}_v v_{tt}||_2^2 + ||(\mathcal{R}_v)^{3/2} v_{ttt}||_1^2 + ||R||_4^2 + ||R_t||_3^2 + ||\sqrt{\mathcal{R}_v} R_{tt}||_2^2 + ||\mathcal{R}_v R_{ttt}||_2^2 + ||v_t||_2^2 + \sqrt{\mathcal{R}_v} v_{tt} ||_2^2 + ||R_{tt}||_1^2 + E.
\]

The rest of this section is devoted to prove:

Theorem 3.2. (Energy estimate for \(E\)) For sufficiently large \(\kappa > 0\), we have

\[
E(t) \leq \epsilon P(\mathcal{N}(t)) + P_0 + P \int_0^t \mathcal{P}, \quad (3.1)
\]

where \(t \in [0,T]\) for some \(T > 0\) chosen sufficiently small, provide that the a priori assumption

\[
||\partial \eta||_{L^\infty} + ||\partial^2 \eta||_{L^\infty} + ||\eta^2||_{L^\infty} \leq M, \quad (3.2)
\]

hold.

Notation 3.3. Here and thereafter, we use \(\epsilon\) to denote a small positive constant which may vary from expression to expression. Typically \(\epsilon\) comes from choosing the time sufficiently small (e.g., Lemma 2.1 (9)) and the Young’s inequality with \(\epsilon\). When all estimates are obtained, we can fix \(\epsilon\) sufficiently small in order to close the estimates.

3.1 The energy identity for the Euler equations

Notation 3.4. (Weighted tangential mixed derivatives) We let \(\mathcal{D}^r, r = 1, 2, 3, 4\) to be the mixed tangential differential operator defined as

\[
\begin{align*}
\mathcal{D} &= \overline{\partial}_k \partial_t, \\
\mathcal{D}^2 &= \overline{\partial}_k \overline{\partial}_t, \sqrt{\mathcal{R}_v} \partial^2_t, \\
\mathcal{D}^3 &= \overline{\partial}_k \overline{\partial}_t, \sqrt{\mathcal{R}_v} (\overline{\partial} \partial^2_t), \mathcal{R}_v \partial_t^3, \\
\mathcal{D}^4 &= \mathcal{R}_v (\overline{\partial} \partial_t), \mathcal{R}_v (\overline{\partial} \partial^2_t), (\mathcal{R}_v)^{3/2} (\overline{\partial} \partial_t^2), (\mathcal{R}_v)^2 \partial_t^4.
\end{align*}
\]
Notation 3.5. Here and in sequel, we use $\mathcal{R}$ to denote lower order terms whose time integral $\int_0^t \mathcal{R}$ can be controlled by the right hand side of (3.1).

Definition 3.1. For each fixed $1 \leq r \leq 4$, let $E = \sum_{r=1,2,3,4} (E_r + W_r^2)$, where

$$E_r = \frac{1}{2} \int_\Omega \rho_0 (\mathcal{D}^r v_\alpha) (\mathcal{D}^r v^\alpha) \, dy + \frac{1}{2} \int_\Omega JR'R^{-1} (\mathcal{D}^r q)^2 \, dy$$

$$+ \frac{\sigma}{2} \int_\Gamma \sqrt{g} q^{ij} \Pi^0 \left( \partial_i \mathcal{D}^r \eta^\mu \right) \left( \partial_j \mathcal{D}^r \eta_\alpha \right) \, dS.$$

Here, $W_r^2 (1 \leq r \leq 4)$ is defined as (2.8) and (2.10), and $\Pi$ is the normal projection operator defined in Lemma 2.2.

Remark. We use throughout that $|| \mathbf{R}_0^0 \Pi \mathbf{D}^m \partial^j \eta ||^2_{0,\Gamma}$ is comparable with the coercive term coming from the boundary part of the energy. We use that $g^{ij}$ is almost the Euclidean metric to make this comparison. For example, in the boundary estimates (Section 3.4) we control $|| \mathbf{R}_0^0 \Pi \mathbf{D}^m \mathbf{v} ||^2_{0,\Gamma}$ by $E$.

The energy defined above is derived by differentiating $\frac{1}{2} \int_\Omega R(\mathcal{D}^r v_\alpha) (\mathcal{D}^r v^\alpha) \, dy$ in time, invoking (1.3), (1.4), (1.5), (1.6), (1.11), (2.5) and the Piola’s identity

$$\partial_\mu A^\mu = \partial_\mu (Ja^\mu) = 0,$$

which follows from a direct computation using (2.1), we have:

$$\frac{d}{dt} \frac{1}{2} \int_\Omega \rho_0 (\mathcal{D}^r v_\alpha) (\mathcal{D}^r v^\alpha) \, dy = - \int_\Omega JR(\mathcal{D}^r v_\alpha) (\mathcal{D}^r (\epsilon^{\alpha \mu} \partial_\mu q)) \, dy$$

$$= - \int_\Omega (\mathcal{D}^r v_\alpha) (\mathcal{D}^r (A^\mu \partial_\mu q)) \, dy + \int_\Omega (\mathcal{D}^r v_\alpha) \left( \left[ \mathcal{D}^r, JR \right] \left( \epsilon^{\alpha \mu} \partial_\mu q \right) \right) \, dy$$

$$= \int_\Omega (\mathcal{D}^r \partial_\mu v_\alpha) (\mathcal{D}^r (A^\mu q)) \, dy - \int_\Gamma (\mathcal{D}^r v_\alpha) \left( N_\mu \mathcal{D}^r (A^\mu q) \right) \, dy + T_1$$

$$= \int_\Omega A^\mu (\mathcal{D}^r \partial_\mu v_\alpha) (\mathcal{D}^r q) \, dy + \int_\Omega \left( \left[ \mathcal{D}^r, A^\mu \right] \partial_\mu v_\alpha \right) \mathcal{D}^r q \, dy + BD + T_1. \tag{3.3}$$

The term $\int_\Omega (\mathcal{D}^r \partial_\mu v_\alpha) (A^\mu \mathcal{D}^r q) \, dy$ is equal to

$$\int_\Omega \mathcal{D}^r (A^\mu \partial_\mu v_\alpha) \mathcal{D}^r q \, dy + \int_\Omega \left( \left[ \mathcal{D}^r, A^\mu \right] \partial_\mu v_\alpha \right) \mathcal{D}^r q \, dy,$$

where, after invoking (2.5), we obtain

$$\int_\Omega \mathcal{D}^r (A^\mu \partial_\mu v_\alpha) \mathcal{D}^r q \, dy = - \int_\Omega \mathcal{D}^r \left( \frac{JR' \partial q}{R} \right) \mathcal{D}^r q \, dy$$

$$= - \int_\Omega JR'R^{-1} (\partial_i \mathcal{D}^r q) \mathcal{D}^r q \, dy + \int_\Omega \left( \left[ \mathcal{D}^r, JR'R^{-1} \partial_j q \right] \mathcal{D}^r q \, dy \right). \tag{3.4}$$
Thus, Theorem 3.2 follows if the terms control || of (3.1), which shall be treated in sections 3.3-3.4 below. However, before doing this, we need to consider the seventh identity in Lemma 2.2, \( BD \) is equal to:

\[
BD = -\int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (A^\mu_\alpha N_\mu q) \, dy
\]

where the main term is moved to the left hand side of (3.3). On the other hand, invoking the boundary condition \( A^\mu_\alpha N_\mu q = -\sigma \sqrt{g} \Delta_g \eta^\alpha \), as well as the second term in the second line of (3.4) is equal to

\[
\int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (\sqrt{g} \Delta_g \eta^\alpha) \, dy = \sigma \int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (\sqrt{g} g^{ij} \Pi^\mu_\mu \mathcal{D}' \eta^\mu) \, dS
\]

Where the main term is moved to the left hand side of (3.3). Summing things up, we have shown:

\[
\int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (\sqrt{g} g^{ij} \Pi^\mu_\mu (\mathcal{D}_i \mathcal{D}_j \eta^\mu)) \, dS = -\sigma \int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (\sqrt{g} g^{ij} \Pi^\mu_\mu (\mathcal{D}_i \mathcal{D}_j \eta^\mu)) \, dS + \frac{1}{2} \int_{\Gamma} \mathcal{D}_i (\sqrt{g} g^{ij} \Pi^\mu_\mu (\mathcal{D}_i \mathcal{D}_j \eta^\mu)) \, dS, \quad (3.5)
\]

The first term on the right hand side of (3.6) is equal to

\[
-\frac{d}{dt} \int_{\Gamma} \mathcal{D}' v_\alpha \mathcal{D}' (\sqrt{g} g^{ij} \Pi^\mu_\mu (\mathcal{D}_i \mathcal{D}_j \eta^\mu)) \, dS + \frac{1}{2} \int_{\Gamma} \mathcal{D}_i (\sqrt{g} g^{ij} \Pi^\mu_\mu (\mathcal{D}_i \mathcal{D}_j \eta^\mu)) \, dS, \quad (3.6)
\]

where the main term is moved to the left hand side of (3.3). Summing things up, we have shown:

\[
\frac{dE_r}{dt} = \sum_{1 \leq j \leq 4} \mathcal{I}_j + \sum_{j=1,2,3} B_j + \mathcal{R}.
\]

Thus, Theorem 3.2 follows if the terms \( \mathcal{I}_{1,2,3,4} \) and \( B_{1,2,3} \) can be controlled by the right hand side of (3.1), which shall be treated in sections 3.3-3.4 below. However, before doing this, we need to control \( ||q||_2 \) and \( ||q_t||_2 \).

### 3.2 Bounds for \( ||q||_2 \) and \( ||q_t||_2 \)

Since \( \mathcal{D}' \) symbolizes both weighted and non-weighted derivatives, we need to bound \( ||q||_2 \) and \( ||q_t||_2 \) in order to control \( \mathcal{I}_{3} \). Also, the bound for \( ||q_t||_2 \) is required to control \( ||v||_4 \) in Section 4. Taking \( X = \partial q \) and \( X = \partial q_t \), \( s = 1 \), the standard div-curl estimate (A.2) yields that we need to control the lower order terms \( ||\partial q||_0 \) and \( ||\partial q_t||_0 \). We remark here that in the case when \( \sigma = 0 \) (e.g., [46]), these terms are controlled via \( ||\Delta q||_0 \) and \( ||\Delta q_t||_0 \), respectively, after integrating by parts and applying the Poincaré’s inequality. However, we need to work a bit harder in order to control these quantities when \( \sigma > 0 \).

**Notation 3.6.** We write \( X \lesssim Y \) to mean \( X \leq CY \), where \( C > 0 \) is a large constant.
Note 3.7. We are going to identify $\mathcal{P}^n = \mathcal{P}$ ($n \geq 1$) by a slight abuse of notations. Also, when $0 \leq t < 1$, $(\int_0^t \mathcal{P})^n = t^{n-1} \int_0^t \mathcal{P}$, via Jensen’s inequality.

Lemma 3.8. Let $F_r$ be defined as (2.7). Assuming the a priori assumption (3.2) holds, then for sufficiently large $\kappa > 0$ (i.e., $\mathcal{R}_\kappa \ll 1$), we have:

$$||F_1||_0 \leq \epsilon N + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Proof. First, invoking (1.4) and the assumption (1.11), we have:

$$||\partial_t (JR)(\partial_t q)||_0 \leq \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Second, invoking Lemma 2.1(1-4), since $\partial_t q = R(a^{-1})_{\mu\beta} \partial_t v^\beta$ and $\partial_t \rho_0 \leq \mathcal{R}_\kappa |\partial_t \rho_0| \leq \epsilon |\partial_t \rho_0|$ for sufficiently small $\mathcal{R}_\kappa$, we get:

$$||a^{\mu\alpha}(\partial_t v_\alpha)|| + ||a^{\mu\alpha}(\partial_t \rho_0)\partial_t v_\alpha|| + ||a^{\mu\alpha}(\partial_t A_\alpha^\mu)\partial_t q||_0 
\leq \epsilon N + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Lemma 3.9. Let $F_r$ be defined as (2.7). Assuming the a priori assumption (3.2) holds, then for sufficiently large $\kappa > 0$ (i.e., $\mathcal{R}_\kappa \ll 1$), we have:

$$||F_2||_0 \leq \epsilon ||q_t||_2 + \epsilon \sqrt{N} + \mathcal{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Proof. First, there is no problem to control $\sum_{j_1+j_2=2} ||\partial_{j_1} (JR)(\partial_{j_2} q)||_0$ appropriately when $j_1 = 1$ using (1.4) and the assumption (1.11). Moreover, when $j_1 = 2$, one writes $J = \rho_0 R^{-1}$ and then $||\partial_t (\rho_0 R^{-1} R') q||_0 = ||\rho_0 R' \partial_t^2 (R^{-1}) q||_0$ modulo controllable terms, where

$$||\rho_0 R' \partial_t^2 (R^{-1}) q||_0 \leq ||\partial_t^2 (R^{-1}) ||_1 ||\partial_t (R^{-1}) ||_1 ||R_t||_0 \leq \epsilon (\sqrt{N} + \mathcal{N}) + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Here, we have applied the interpolation inequality (i.e., Theorem 2.4) and the fact $R' \partial_t q = \partial_t R$.

Second, invoking Lemma 2.1(1-6) we get:

$$\sum_{j_1+j_2=1} ||(\partial_t^{j_1} a^{\mu\alpha})(\partial_t^{j_2} \partial_t v_\alpha)||_0 \leq \epsilon N + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P},$$  

and since $\partial_t A_\alpha^\mu = O(\epsilon)$ for small time and $\partial_t \rho_0 \leq \mathcal{R}_\kappa |\partial_t \rho_0| \leq \epsilon |\partial_t \rho_0|$ for sufficiently small $\mathcal{R}_\kappa$, we have:

$$||a^{\mu\alpha}(\partial_t \rho_0)\partial_t^2 v_\alpha||_0 + ||a^{\mu\alpha}(\partial_t A_\alpha^\mu)\partial_t q||_0 \leq \epsilon ||q_t||_2 + \epsilon \sqrt{N} + \mathcal{P}_0 + \mathcal{P}\int_0^t \mathcal{P}.$$  

Third, since $\partial_t q = R(a^{-1})_{\mu\beta} \partial_t v^\beta$, $||a^{\mu\alpha} \partial_t (\partial_t a^\mu_\alpha \cdot \partial_t q)||_0$ can be controlled appropriately by interpolation.

\hfill \Box
Lemma 3.10. We have
\[ ||\partial q(t, \cdot)||^2_0 + ||\partial q(t, \cdot)||^2_0 \leq \epsilon||q(t, \cdot)||^2_0 + \epsilon P(N) + W_3^2 + P_0 + P \int_0^t P, \]
for \( t \in [0, T] \) where \( T > 0 \) is chosen sufficiently small.

Proof. It suffices to consider \( ||\partial q_t||_0 \) only. Integrating by parts yields:
\[ ||\partial q_t||_0^2 = \int_\Omega (\partial_{\mu} q_t)(\partial_{\mu} q_t) = -\int_\Omega q_t \Delta q_t + \int_\Gamma (N^\mu \partial_{\mu} q_t) q_t, \]
and so we need to bound \( \int_\Omega q_t \Delta q_t \) and \( \int_\Gamma (N^\mu \partial_{\mu} q_t) q_t \), respectively.

Bound for \( \int_\Omega q_t \Delta q_t \): Since \( t \in [0, T] \) and \( T > 0 \) is small, as well as
\[ \Delta q_t = (\delta^{\mu \nu} - a^\mu a^\nu) \partial_{\mu} \partial_{\nu} q_t + a^\mu a^\nu \partial_{\mu} \partial_{\nu} q_t, \]
Lemma 2.1 implies that
\[ \int_\Omega q_t (\Delta q_t) \leq \epsilon ||q_t||^2_0 + \int_\Omega q_t (a^\mu a^\nu \partial_{\mu} \partial_{\nu} q_t). \]
Now, invoking the wave equation (2.6) and Lemma 3.9, we have:
\[ \int_\Omega q_t (a^\mu a^\nu \partial_{\mu} \partial_{\nu} q_t) = \int_\Omega R^\mu q_t Y^\mu - \int_\Omega (q_t F_2) J^{-1} \lesssim ||q_t||_0 ||R^\mu q_t||_0 + ||F_2||_0 \]
\[ \leq ||q_t||_0 (W_3 + \epsilon ||q_t||_2 + \epsilon(\sqrt{N} + N) + P_0 + P \int_0^t P). \]

On the other hand, since
\[ ||q_t||_0 \leq ||\partial q_t||_0 + \int_\Omega q_t, \]
by the Poincaré’s inequality, if we let \( Y = (0, 0, y^3) \), then
\[ ||q_t||_0 \leq ||\partial q_t||_0 + \int_\Omega \partial_{\mu} Y^\mu q = ||\partial q_t||_0 - \int_\Omega Y^\mu \partial_{\mu} q_t + \int_\Gamma N^\mu Y^\mu q_t \]
\[ \leq C_{\text{vol}} ||q_t||_0 + \int_\Gamma y^3 q_t. \]
(3.8)

To control the last integral \( \int_\Gamma y^3 q_t \), time differentiating the boundary condition
\[ q = -\sigma g^{ij} n^\mu \partial_{ij} \eta^\mu, \quad \text{on} \quad \Gamma \]
gives
\[ q_t = -\sigma g^{ij} n^\mu \partial_{ij} v^\mu + R_{q_t}, \quad \text{on} \quad \Gamma \]
(3.9)
where \( R_{q_t} \) consists of terms of the form either
\[ \sigma g^{ij} g^{kl} (\partial_{k} v^\gamma n^\mu \partial_{ij} \eta_{\gamma \mu}) \partial_{ij} \eta^\mu \]
or \( \sigma (\partial_{\nu} v^\gamma) (\partial_{i} \eta^\gamma) n^\mu \partial_{ij} \eta^\mu \).

Now, invoking Lemma 2.1, Lemma 2.2, and the a priori assumption (3.2), we have:
\[ \int_\Gamma y^3 q_t \lesssim \epsilon N + P_0 + P \int_0^t P. \]
(3.10)
Wrapping these up, we get:
\[ \int_\Omega q_t \Delta q_t \lesssim \epsilon ||q_t||^2_0 + \epsilon ||\partial q_t||^2_0 + \epsilon P(N) + W_3^2 + P_0 + P \int_0^t P. \]
Bound for $\int_{\Gamma}(N^\mu\partial_\mu q_t)q_t$: We have
\[
\int_{\Gamma}(N^\mu\partial_\mu q_t)q_t \leq ||q_t||_{0,\Gamma}||\partial_\mu q_t||_{0,\Gamma} \leq C(\epsilon^{-1})||q_t||_{0,\Gamma}^2 + \epsilon||\partial q_t||_{0,\Gamma}^2.
\]
Here, we bound $\epsilon||\partial q_t||_{0,\Gamma}^2$ by $\epsilon||q_t||_{0}^2$ using the trace lemma, which is part of the right hand side of (3.7). On the other hand, invoking (3.9), we have:
\[
||q_t||_{0,\Gamma}^2 \lesssim \epsilon(N^2 + N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]
To see this, note that in $||q_t||_{0,\Gamma}^2$, the top order term is $\sqrt{g}g^{ij}\hat{n}^\mu \hat{\partial}_ij v_\mu$. Using the trace inequality, it suffices to bound $||\sqrt{g}g^{ij}\hat{n}^\mu \hat{\partial}_ij v_\mu||_{0,\Gamma}^2$. We control this top order term by the Young’s inequality, which leads to the appearance of $\epsilon N^2$. In addition, the lower order terms are controlled by $\epsilon N + P_0 + \mathcal{P} \int_0^t \mathcal{P}$ using the interpolation.

Hence,
\[
\int_{\Gamma}(N^\mu\partial_\mu q_t)q_t \lesssim \epsilon||q_t||_{0}^2 + \epsilon(N^2 + N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

Therefore,
\[
||\partial q_t||_{0,\Gamma}^2 = -\int_{\Omega} q_t \Delta q_t + \int_{\Gamma} q_t(N^\mu\partial_\mu q_t) \lesssim \epsilon||q_t||_{0}^2 + \epsilon(N + N^2) + W^2_3 + P_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

In addition, we are able to control $||\partial q_t||_{0,\Gamma}^2$ appropriately by integrating $||\partial q_t||_{0,\Gamma}^2$ in time, which, together with the estimate for $||\partial q_t||_{0,\Gamma}^2$, conclude the proof of (3.7).

In fact, the above proof implies the control for the lowest order norms $||q||_{0}$ and $||q_t||_{0}$.

**Corollary 3.11.** We have
\[
||q||_{0}^2 + ||q_t||_{0}^2 \lesssim ||\partial q||_{0}^2 + ||\partial q_t||_{0}^2 + \epsilon N + P_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

**Proof.** Let $Y = (0,0,y^3)$, the Poincaré’s inequality implies
\[
||q||_{0} + ||q_t||_{0} \lesssim ||\partial q||_{0} + ||\partial q_t||_{0} + \int_{\Omega} \partial_\mu Y^\mu q + \int_{\Omega} \partial_\mu Y^\mu q_t.
\]

Now, we proceed as in (3.8)-(3.10) and get
\[
\int_{\Omega} \partial_\mu Y^\mu q + \int_{\Omega} \partial_\mu Y^\mu q_t \lesssim ||\partial q||_{0} + ||\partial q_t||_{0} + \epsilon \sqrt{N} + P_0 + \mathcal{P} \int_0^t \mathcal{P},
\]
and hence (3.11) follows after squaring the above estimate.

**Theorem 3.12.** We have
\[
||q(t,\cdot)||_{\frac{3}{2}}^2 + ||q_t(t,\cdot)||_{\frac{3}{2}}^2 \lesssim \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P},
\]
for $t \in [0,T]$ where $T > 0$ is chosen sufficiently small.
Proof. It suffices to control $||q_t||^2$ by the right hand side of (3.12) since the control of $||q||^2$ follows from time integrating $||q_t||^2$. To control $||q_t||^2$, it suffices to consider $||\partial q_t||^2$ only thanks to Lemma 3.10 and Corollary 3.11. Now, invoking the div-curl estimate (A.2) with $X = \partial q_t$ and $s = 1$, we have

$$||\partial q_t||^2 \leq ||\Delta q_t||^2_0 + ||\partial q_t||^2_{0,5,\Gamma} + ||\partial q_t||^2_0.$$ 

**Bound for $||\Delta q_t||^2_0$:** Invoking Lemma 2.1, since $t \in [0, T]$ and $T$ is sufficiently small, we have

$$||\Delta q_t||^2_0 \leq ||a^{\mu a} a^\nu_\alpha \partial_\mu \partial_\nu q_t||^2_0 + ||(\delta^\mu_\nu - a^{\mu a} a^\nu_\alpha) \partial_\mu \partial_\nu q_t||^2_0 \leq ||a^{\mu a} a^\nu_\alpha \partial_\mu \partial_\nu q_t||^2_0 + \epsilon ||\partial^2 q_t||.$$ 

Furthermore, the wave equation (2.6), as well as Lemma 3.9 yield:

$$||a^{\mu a} a^\nu_\alpha \partial_\mu \partial_\nu q_t||^2_0 \leq ||\partial^t q_{ttt}||^2_{0} + ||\partial^2 q_{tt}||^2_{0} \leq W^2_2 + \epsilon (N + N^2) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} + \epsilon ||q_t||^2_{168}.$$ 

**Bound for $||\partial q_t||^2_{0,5,\Gamma}:** Invoking (3.9) and taking one more tangential derivative, we have

$$\partial q_t = \sigma g^{ij} n^k \partial \partial_{ij} v_{\mu} - \sigma g^{ij} g^{kl} (\partial k v^r \partial_\nu \eta_{\mu} \partial \partial_{ij} n^\nu) + \mathcal{R}_{\partial q_t},$$

where $\mathcal{R}_{\partial q_t}$ consists products of $\partial^k \eta$ and $\partial^k v$, $k = 1, 2$. To be more specific, $\mathcal{R}_{\partial q_t}$ consists terms of the forms

$$\sigma g^{ij} g^{kl} (\partial k \partial v^r \partial_\nu \eta_{\mu} \partial \partial_{ij} n^\nu), \quad \sigma g^{ij} g^{kl} (\partial k \partial \eta^r \partial_\nu \eta_{\mu} \partial \partial_{ij} n^\nu),$$

$$\sigma (\partial^r \eta^\mu) (\partial^j \eta_{\mu}) g^{kl} (\partial k \partial \eta^r \partial_\nu \eta_{\mu} \partial \partial_{ij} n^\nu), \quad \sigma (\partial^r \eta^\mu) (\partial^j \eta_{\mu}) (\partial k \partial \eta^r \partial_\nu \eta_{\mu} \partial \partial_{ij} n^\nu).$$

Given these, we have:

$$||\partial q_t||^2_{0,5,\Gamma} \leq \epsilon (N^2 + N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},$$

by interpolation and the Young’s inequality. Here, $\epsilon N^2$ appears since

$$||\sqrt{g} g^{ij} n^\mu \partial \partial^3 v_{\mu}||^2_0 \leq \epsilon ||v||^2_{168} + ||\sqrt{g} g^{ij} n^\mu ||^2 \leq \epsilon N^2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},$$

and we remark here that the interpolation cannot be applied since $\partial \partial^3 v$ is of the top order. Wrapping these up and invoking Lemma 3.10 and Corollary 3.11, we get

$$||q_t||^2_2 \leq W^2_2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} + \epsilon ||q_t||^2_2 + \epsilon (N^2 + N),$$

which proves the estimate for $||q_t||^2_2$ by invoking (2.9) and then absorbing $\epsilon ||q_t||^2_2$ to the left hand side. \qed

**Remark.** We are unable to control $||\partial q_{tt}||_1$ when surface tension is present. This is due to that the div-curl estimate yields the boundary term $||\partial q_{tt}||_{0,5,\Gamma}$, where $\partial q_{tt} \sim \partial^3 v_t$ on $\Gamma$, and hence $||\partial q_{tt}||_{0,5,\Gamma}$ has a loss of $1/2$ derivatives. Therefore, one has to define the energy using the $\kappa$-weighted derivatives and so the corresponding term can then be controlled by the energy.
3.3 Bounds for $\int_0^t \mathcal{I}_{1,2,3,4}$

This section is devoted to control $\int_0^t \mathcal{I}_{1,2,3,4}$. We recall

$$\mathcal{I}_1 = \int_{\Omega} (2^r v_{\alpha}) \left( |2^r, Rj| (a^{\mu\alpha} \frac{\partial q}{R}) \right), \quad \mathcal{I}_2 = \int_{\Omega} (2^r \partial_{\mu} v_{\alpha}) \left( |2^r, A^{\mu\alpha}| q \right),$$

$$\mathcal{I}_3 = \int_{\Omega} (2^r, A^{\mu\alpha} \partial_{\mu} v_{\alpha}) \partial_r q, \quad \mathcal{I}_4 = \int_{\Omega} \left( |2^r, J R^{-1}| \partial_t q \right) \partial_r q.$$

**Notation 3.13.** In what follows, we use $D$ to denote either $\overline{D}$ or $\partial_t$. This allows us to represent $\nabla r$ as $(\mathfrak{R}_n)^{\ell} D^r$, where $r = \frac{1}{2}, 1, \frac{3}{2}, 2$.

### 3.3.1 Control of $\int_0^t \mathcal{I}_1$

**For non-weighted $\nabla^r$:** We recall that there are four mixed derivatives which are not weighted, which are $\partial_0, \partial_r, \partial R$, and $\partial_t$. Hence, it suffices to consider only the case when $\nabla^r = \overline{D} \partial_t$. Invoking (1.5) and Theorem 3.12, we have:

$$\mathcal{I}_1 = \sum_{j_1+j_2=2}^{j_1 \geq 1} \int_{\Omega} (\partial_2 \partial_t v_{\alpha})(\partial_2^{j_1} \rho_0)(\partial_2^{j_2} \partial_t (a^{\mu\alpha} \frac{\partial q}{R})).$$

Since, to the highest order, the last term on the right hand side is $R^{-1} a^{\mu\alpha} \overline{D} \partial_t q$, which can be controlled by invoking Theorem 3.12. Therefore,

$$\int_0^t \mathcal{I}_1 \leq \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.$$

The $\epsilon \mathcal{P}(N)$ term introduced in Theorem 3.12 does not figure here since $\mathcal{I}_1$ is estimated under the time integral.

**For weighted $\nabla^r$:** It suffices to consider derivatives of the form $(\mathfrak{R}_n)^{\ell} D^{r-2} \overline{D} \partial_t$, where $\ell = \frac{1}{2}, 1, \frac{3}{2}$ and $r \leq 4$, since otherwise $\mathcal{I}_1$ would be 0 due to (1.5).

$$\mathcal{I}_1 = \sum_{j_1+j_2=r-2}^{j_1 \geq 1} \int_{\Omega} (\mathfrak{R}_n)^{2\ell} (D^{r-2} \overline{D} \partial_t v_{\alpha}) (\overline{D}^{j_1} \rho_0) (D^{j_2} \partial_t (a^{\mu\alpha} \frac{\partial q}{R})).$$

We henceforth adopt:

**Notation 3.14.** We use $\mathcal{L}$ to denote equality modulo lower order terms that can be controlled, i.e., $A \stackrel{\mathcal{L}}{=} B$ mean $A = B + \text{error terms}$, where the “error terms” can be controlled by the bound of $B$ plus $\mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}$.

Invoking (1.10) and (1.11) at $t = 0$ lead to

$$\int_0^t \mathcal{I}_1 = \sum_{j_1+j_2=r-2}^{j_1 \geq 1} \int_0^t \int_{\Omega} (\mathfrak{R}_n)^{2\ell+1} (D^{r-2} \overline{D} \partial_t v_{\alpha}) (\overline{D}^{j_1} \rho_0) (D^{j_2} \partial_t (a^{\mu\alpha} \frac{\partial q}{R})).$$

$$= \int_0^t \int_{\Omega} (\mathfrak{R}_n)^{\ell+\frac{3}{2}} (D^{r-2} \overline{D} \partial_t v_{\alpha}) (\mathfrak{R}_n)^{\ell+\frac{3}{2}} (D^{j_2} \partial_t (a^{\mu\alpha} \frac{\partial q}{R})) \leq \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.$$

**Remark.** The above expression yields a slightly better bound for $D^{r-2} \overline{D} \partial_t v$, since $\mathcal{P}$ requires only $\| (R')^{\ell} D^{r-2} \overline{D} \partial_t v \|_0$. 


3.3.2 Control of \( \int_0^t I_2 \)

For each \( r \), \( \int_0^t I_2 \) contains a term which is of the order \( r + 1 \), i.e.,

\[
\int_0^t \Xi = \int_\Omega (D^r \partial_\mu v_\alpha)(D^r A^{\mu\alpha})q.
\]

There is no problem to control \( \Xi \) when \( r \leq 2 \), and when \( r = 3 \), we need to put extra effort to control \( \Xi \) when \( D_3 = \overline{D}^3 \partial_3 \) since there are terms which cannot be controlled directly without \( R_\kappa \) weight, and one needs to integrate by parts in (tangential) spatial derivative and time derivative, respectively. On the other hand, when \( r = 4 \), this term is of above the top order, but it can be controlled using one of the special cancellations referred to in section 1.5.1, as we now show.

For non-weighted \( D^r \): As mentioned above, we consider only the case when \( r = 3 \) and \( D_3 = \overline{D}^3 \partial_3 \). In this case,

\[
\Xi = \int_\Omega (\overline{D}^3 \partial_\mu \partial_3 v_\alpha)(\partial_3 A^{\mu\alpha})q.
\]

Although this term is of the correct order, \( \overline{D}^3 \partial_\mu \partial_3 v \) cannot be controlled without \( R_\kappa \) weight. Hence, we integrate by parts with respect to the tangential derivative and get:

\[
\Xi = -\left( \int_\Omega (\partial_\mu \partial_3 v_\alpha)(\partial_3^3 A^{\mu\alpha})q + \int_\Omega (\partial_\mu \partial_\alpha)(\partial_3^2 A^{\mu\alpha})\partial q \right) \\
\leq ||\partial_3 v_\alpha||_0 ||\overline{D}^3 \partial_3 A||_0 ||q||_{L^\infty} + ||\partial_3 v_\alpha||_0 ||\overline{D}^2 \partial_3 A||_{L^4} ||\partial q||_{L^4}.
\]

Here, one adapts Theorem 3.12 to control \( ||q||_2 \). Integrating with respect to time, we obtain:

\[
\int_0^t \Xi \lesssim P_0 + P \int_0^t\mathcal{P}.
\]

We next consider \( I_2 - \Xi \). All terms involved in \( I_2 - \Xi \) can be controlled straightforwardly after integrating by part with respect to \( \overline{D} \) thanks to Theorem 3.12, except for

\[
\int_\Omega (\overline{D}^3 \partial_\mu \partial_3 v_\alpha)(\partial_3 A^{\mu\alpha})(\overline{D}^2 q).
\]

This is due to that integrating by part in \( \overline{D} \) yields \( \overline{D}^4 q \) which cannot be controlled without \( R_\kappa \) weight. To deal with this issue, we consider

\[
\int_0^t \int_\Omega (\overline{D}^2 \partial_\mu \partial_3 v_\alpha)(\partial_3 A^{\mu\alpha})(\overline{D}^2 q).
\]

Integrating by part in time, we get:

\[
\int_\Omega (\overline{D}^2 \partial_\mu \partial_3 v_\alpha)(\partial_3 A^{\mu\alpha})(\overline{D}^2 q)|_0^t - \int_0^t \int_\Omega (\overline{D}^2 \partial_\mu v_\alpha)(\partial_3^2 A^{\mu\alpha})(\overline{D}^2 q) - \int_0^t \int_\Omega (\overline{D}^2 \partial_\mu \partial_3 v_\alpha)(\partial_3 A^{\mu\alpha})(\overline{D}^2 \partial_3 q).
\]

The last two terms are bounded by \( P_0 + P \int_0^t\mathcal{P} \) thanks to Theorem 3.12, while the pointwise term at \( t = 0 \) by \( P_0 \). The pointwise term at \( t \) is bounded by

\[
||q||_3 ||\overline{D}^2 q||_0 ||\overline{D}^2 v||_{\frac{1}{2}} ||\overline{D}^2 v||_{\frac{1}{2}} ||v||_{\frac{1}{2}} ||v||_{\frac{1}{2}} \lesssim \epsilon P(N) + P_0 + P \int_0^t\mathcal{P},
\]

which is controlled by the right hand side of (3.1).
For weighted $\mathcal{D}^r$: $I_2$ contains a term above the top order, i.e.,

$$T = \int_{\Omega} (\mathcal{D}^r \partial_\mu v_\alpha)(\mathcal{D}^r A^{\mu\alpha}) q.$$ 

This term is controlled using the aforementioned special cancellation (see Section 1.5.1). For weighted derivatives, it suffices to consider only the case when $r = 4$, i.e., the derivatives are of the form $(\mathfrak{R}_\kappa)\ell D^3 \partial_t$, for $\ell = 1, \frac{5}{2}, 2$. Then the “tricky” term to be bounded is:

$$\mathcal{L} = \int_0^t T = \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} (D^3 \partial_t \partial_\mu v_\alpha)(D^3 \partial_t A^{\mu\alpha}) q. \quad (3.13)$$

In view of (2.2), expanding the index $\mu$ in (3.13), we have

$$\mathcal{L} = \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_2 D^3 v_\lambda \partial_3 \eta_\tau \partial_1 D^3 \partial_t v_\alpha + \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_2 \eta_\lambda \partial_3 D^3 v_\tau \partial_1 D^3 \partial_t v_\alpha$$

$$- \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_2 D^3 v_\lambda \partial_3 \eta_\tau \partial_1 D^3 \partial_t v_\alpha - \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_2 \eta_\lambda \partial_3 D^3 v_\tau \partial_1 D^3 \partial_t v_\alpha + \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_1 \partial_3 D^3 v_\tau \partial_2 D^3 \partial_t v_\alpha + L_{low} \quad (3.14)$$

where $L_{low}$ are lower order terms, which are all of the form

$$\sum_{j_1+j_2=3} \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q (\partial D^{j_1} v)(\partial D^{j_2} \eta)(\partial D^3 \partial_t v) - \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q (\partial D^{j_1} v)(\partial D^{j_2} \eta)(\partial D^3 \partial_t v).$$

Invoking Theorem 3.12, it is easy to see that the last three terms are controlled by $P_0 + P \int_0^t P$, while the pointwise term at $t$ is treated similar to (3.15)-(3.16), after distributing correct amount of $\mathfrak{R}_\kappa$ weight to each term. We omit the detail here. But it is worth noting that there are more than enough $\mathfrak{R}_\kappa$ weight for the pointwise term since there is one time derivative less.

Next, integrating by part in time in $L_3$, we find

$$- \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_1 D^3 v_\lambda \partial_3 \eta_\tau \partial_2 D^3 \partial_t v_\alpha = \int_0^t \int_{\Omega} (\mathfrak{R}_\kappa)^{2\ell} q e^{\alpha \lambda \tau} \partial_1 D^3 v_\lambda \partial_3 \eta_\tau \partial_2 D^3 \partial_t v_\alpha.$$
Adding $L_1$, we get:

$$L_1 + L_3 = \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_2 D^3 \eta \partial_2 \nabla_1 D^3 \partial_t v_\alpha$$

$$+ \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 \partial_t v_\alpha $$

$$- \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 \partial_t \eta \partial_2 D^3 v_\alpha |_{t=0}$$

$$= - \int_0^t (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 \partial_t v_\alpha \partial_2 \nabla_2 D^3 v_\alpha |_{t=0} = L_{13},$$

since first and the second term cancels with each other by the antisymmetry of $\epsilon^{\alpha\lambda\tau}$. Similarly, we have

$$L_4 + L_6 \equiv L_{46} = \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 \eta \partial_2 D^3 v_\alpha |_{t=0},$$

$$L_2 + L_5 \equiv L_{25} = \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 \partial_t v_\alpha.$$

**Bounds for $L_{13}$, $L_{46}$ and $L_{25}$** Since $L_{13}$ is pointwise in $t$, it suffices to consider

$$\int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 v_\alpha \partial_3 \eta \partial_2 D^3 v_\alpha |_{t=0}$$

only, since the other part is controlled directly by $\mathcal{P}_0$. In addition, since $D^3$ corresponds to $\partial_t^3$, $\overline{\partial}_3 \partial_t$, $\overline{\partial}_3 \partial_\ell$ and $\overline{\partial}_3$, associated with weights $(\mathcal{R}_\kappa)^{2\gamma}$, $(\mathcal{R}_\kappa)^{2\gamma}$, $\mathcal{R}_\kappa$ and $\mathcal{R}_\kappa$, respectively, we have:

$$\int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 v_\alpha \partial_3 \eta \partial_2 D^3 v_\alpha |_{t=0} \leq \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} q e^{\alpha\lambda\tau} ((\mathcal{R}_\kappa)^{2\gamma} \overline{\partial}_1 \partial_t^3 v_\alpha) \partial_3 \eta \partial_2 (D^3 v_\alpha) |_{t=0}$$

$$\leq ||R||^2 ||\eta||_3 ||(\mathcal{R}_\kappa)^{2\gamma} v_{ttt}||_3 \leq \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P},$$

where we have used $||\mathcal{R}_\kappa q||_2 \leq ||R'||_2 = ||R||_2$. Similarly, we have

$$\int_\Omega (\mathcal{R}_\kappa)^{3\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 v_\alpha \partial_3 \eta \partial_2 D^3 v_\alpha |_{t=0} + \int_\Omega (\mathcal{R}_\kappa)^{3\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 v_\alpha \partial_3 \eta \partial_2 D^3 v_\alpha |_{t=0}$$

$$+ \int_\Omega (\mathcal{R}_\kappa)^{3\gamma} q e^{\alpha\lambda\tau} \overline{\partial}_1 D^3 v_\alpha \partial_3 \eta \partial_2 D^3 v_\alpha |_{t=0} \leq \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}.$$

Moreover, this method can be adapted to control $L_{46}$ and $L_{25}$, and we omit the details. Therefore, we have

$$(L_1 + L_3) + (L_4 + L_6) + (L_2 + L_5) \leq \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}.$$

Now, we complete the treatment of $\mathcal{I}_2$ by estimating the rest of the terms, i.e., $\mathcal{I}_2 - \mathcal{I}$, for weighted forth order derivatives. Expressing:

$$\mathcal{I}_2 - \mathcal{I} = \int_\Omega (\mathcal{R}_\kappa)^{2\gamma} (D^3 \partial_t \eta \partial_\mu v_\alpha) \left(D^3 \partial_t (A^{\mu\alpha} q) - A^{\mu\alpha} D^3 \partial_t q - (D^3 \partial_t A^{\mu\alpha}) q \right).$$
and similar to the non-weighted case, we consider \( \int_0^t I_2 - \mathcal{I} \) and integrate by part in time to get

\[
\int_0^t I_2 - \mathcal{I} \leq \int_0^t (\mathcal{R}_k)^{2t} (D^3 \partial_\mu v_\alpha) \left( D^3 \partial_t (A^{\mu\alpha} q) - A^{\mu\alpha} D^3 \partial_t q - (D^3 \partial_t A^{\mu\alpha}) q \right) \bigg|_0^t \\
- \int_0^t \int_\Omega (\mathcal{R}_k)^{2t} (D^3 \partial_\mu v_\alpha) \partial_t \left( D^3 \partial_t (A^{\mu\alpha} q) - A^{\mu\alpha} D^3 \partial_t q - (D^3 \partial_t A^{\mu\alpha}) q \right).
\]

First, it is easy to check that

\[
\int_0^t \int_\Omega (\mathcal{R}_k)^{2t} (D^3 \partial_\mu v_\alpha) \partial_t \left( D^3 \partial_t (A^{\mu\alpha} q) - A^{\mu\alpha} D^3 \partial_t q - (D^3 \partial_t A^{\mu\alpha}) q \right) \leq \int_0^t \mathcal{P},
\]

Second, for the pointwise terms at \( t \), it suffices to consider the case when \( D^3 = \partial_\ell^3 \) and \( \ell = 2 \), since the bounds for the other (easier) cases follow from the same method. There are three terms, i.e.,

\[
\int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell^2 A) \partial_t q, \quad \int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell A) \partial_\ell^2 q, \quad \int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell A) \partial_\ell^2 q.
\] (3.15)

These terms are treated as

\[
\int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell^2 A) \partial_t q \approx \int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha) \left( (\partial v_t)(\partial \eta) + (\partial v_t)(\partial v) \right) \partial_t q
\]

\[
\lesssim \| (\mathcal{R}_k)^{\frac{3}{2}} v_{tt} \|_1 \| (\mathcal{R}_k)^{\frac{3}{2}} v_t \|_1 \| (\mathcal{R}_k)^{\frac{3}{2}} v \|_1 \| \eta \|_3 \| v \|_3 \| \eta \|_3 \| q \|_0 + \| \eta \|_3 \| q \|_0
\]

\[
\lesssim \epsilon P(N) + P_0 + P \int_0^t \mathcal{P},
\]

and

\[
\int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell A) \partial_\ell^2 q = \int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha) \left( (\partial v)^2 + (\partial v_t)(\partial \eta) \right) \partial_\ell^2 q
\]

\[
\lesssim \| (\mathcal{R}_k)^{\frac{3}{2}} v_{tt} \|_1 \left( \| v \|_2 \| \mathcal{R}_k v \|_3 + \| \mathcal{R}_k v_t \|_2 \| \mathcal{R}_k v \|_2 \| \eta \|_3 \right) \| (\mathcal{R}_k)^{\frac{3}{2}} v_t \|_0 \| \mathcal{R}_k v \|_1 \| q \|_1
\]

\[
\lesssim \epsilon P(N) + P_0 + P \int_0^t \mathcal{P}.
\]

Finally, we have

\[
\int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell A) \partial_\ell^2 q = \int_\Omega (\mathcal{R}_k)^{4}(\partial_\ell^3 \partial_\mu v_\alpha)(\partial_\ell v)(\partial \eta) \partial_\ell^2 q
\]

\[
\lesssim \| (\mathcal{R}_k)^{\frac{3}{2}} v_{tt} \|_1 \| v \|_1 \| v \|_2 \| \eta \|_3 \| (\mathcal{R}_k)^{\frac{3}{2}} v_t \|_0 \| (\mathcal{R}_k)^{\frac{3}{2}} v_t \|_1
\]

\[
\lesssim \epsilon P(N) + P_0 + P \int_0^t \mathcal{P}.
\] (3.16)

### 3.3.3 Control of \( \int_0^t I_3 \)

**For non-weighted \( \mathcal{D}^r \):** Expressing these derivatives as \( D^r \) where \( r \leq 3 \), we have:

\[
I_3 = \sum_{j_1 + j_2 = r} \int_\Omega (D^{j_1} A^{\mu\alpha})(\partial_\mu D^{j_2} v_\alpha)(D^r q) \leq \sum_{j_1 + j_2 = r} \| (D^{j_1} A^{\mu\alpha})(\partial_\mu D^{j_2} v_\alpha) \|_0 \| D^r q \|_0,
\]

and so \( \int_0^t I_3 \leq P_0 + P \int_0^t \mathcal{P} \) in light of Theorem 3.12.
For weighted $\mathcal{D}'$: It suffices to consider only the case when $r = 4$, i.e., the derivatives are of the form $(\mathcal{R}_\kappa)^\ell D^3 \partial_t$, for $\ell = 1, \frac{3}{2}, 2$. Now,
\[
\mathcal{I}_3 = \int_\Omega (\mathcal{R}_\kappa)^{2\ell} (\partial_t A^{\mu \alpha})(D^3 \partial_{\mu} v_\alpha)(D^3 \partial_t q) + \int_\Omega (\mathcal{R}_\kappa)^{2\ell} (DA^{\mu \alpha})(D^2 \partial_t \partial_\mu v_\alpha)(D^3 \partial_t q) \\
+ \int_\Omega (\mathcal{R}_\kappa)^{2\ell} (\partial_t D^3 A^{\mu \alpha})(\partial_\mu v_\alpha)(D^3 \partial_t q) + \text{error terms},
\]
where the main term is equal to
\[
\int_\Omega (\partial_t A^{\mu \alpha}) \left( (\mathcal{R}_\kappa)^{\ell - \frac{3}{2}} D^3 \partial_{\mu} v_\alpha \right) \left( (\mathcal{R}_\kappa)^{\ell + \frac{1}{2}} D^3 \partial_t q \right) \\
+ \int_\Omega (\partial A^{\mu \alpha}) \left( (\mathcal{R}_\kappa)^{\ell} D^2 \partial_t \partial_\mu v_\alpha \right) \left( (\mathcal{R}_\kappa)^{\ell} D^3 \partial_t q \right) \\
+ \int_\Omega (\partial_\mu v_\alpha) \left( (\mathcal{R}_\kappa)^{\ell - \frac{3}{2}} \partial_t D^3 A^{\mu \alpha} \right) \left( (\mathcal{R}_\kappa)^{\ell + \frac{1}{2}} D^3 \partial_t q \right) = \mathcal{I}_{3,1} + \mathcal{I}_{3,2} + \mathcal{I}_{3,3},
\]
where $\mathcal{I}_{3,2}$ does not appear when $\mathcal{D}' = \mathcal{R}_\kappa \partial_t^4$.

$\int_0^t \mathcal{I}_{3,1} + \mathcal{I}_{3,3}$ can be controlled directly by $\mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}$. For $\mathcal{I}_{3,2}$, one requires the wave energy (2.10) to control $\| (\mathcal{R}_\kappa)^{\ell} D^3 \partial_t q \|_1$ when $D^3$ contains at least one $\partial_t$, and (2.3) to control this term when $D^3 = \mathcal{D}^2$ (i.e., $\mathcal{R}_\kappa \mathcal{D}^2 \partial_t q \sim \mathcal{D}^2 \partial_t R$), and so $\int_0^t \mathcal{I}_{3,2}$ can be controlled appropriately by $\mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}$. Furthermore, the (time integrated) error terms are of the form
\[
\sum_{j_1 + j_2 + j_3 = 3} \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{2\ell} (\partial D^{j_1} \eta)(\partial D^{j_2} v)(\partial D^{j_3} v)(D^4 q) = \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

3.3.4 Control of $\int_0^t \mathcal{I}_4$

$\mathcal{I}_4$ is the easiest one to control among the other $\mathcal{I}$ terms. This is due to the assumption (1.11), which implies that there are "sufficient" $\mathcal{R}_\kappa$ weights that can be distributed for all terms. In addition to this, we can also use the fact $DR = Rdq$ to get an extra $\mathcal{R}_\kappa$ weight if necessary.

For non-weighted $\mathcal{D}'$: By (1.11) and since $r \leq 3$, invoking Theorem 3.12, we have:
\[
\int_0^t \mathcal{I}_4 \leq \sum_{j_1 + j_2 = r} \int_0^t \int_\Omega (\mathcal{R}_\kappa)(D^{j_1}(\rho_0 R^{-2}))(D^{j_2} \partial_t q)(D^r q) \leq \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

For weighted $\mathcal{D}'$: For $\ell = \frac{1}{2}, 1, \frac{3}{2}, 2$, we have:
\[
\int_0^t \mathcal{I}_4 \leq \sum_{j_1 + j_2 = r} \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{2\ell + 1} (D^{j_1}(\rho_0 R^{-2}))(D^{j_2} \partial_t q)(D^r q) = \\
\sum_{j_1 + j_2 = r} \int_0^t \int_\Omega (\mathcal{R}_\kappa)^{\ell + \frac{1}{2}} (D^{j_1}(\rho_0 R^{-2}))(D^{j_2} \partial_t q)(D^r q) \leq \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},
\]
where the fact $DR = Rdq$ is used if $j_1 = 1$.

\footnote{This is explained in the remark after Theorem 2.6.}
3.4 Control of $\int_0^t B$ for non-weighted $\mathcal{D}^r$

This section is devoted to control the boundary terms

$$B_1 = \sigma \int_\Gamma (\mathcal{D}^r v_\alpha)([\mathcal{D}^r, \sqrt{g}g^{ij}\Sigma^\alpha_{\mu}](\partial_i \mathcal{D}^r \eta_\nu)) dS,$$

$$B_2 = -\sigma \int_\Gamma \partial_i (\sqrt{g}g^{ij}\Sigma_{\mu}^\alpha)(\partial_i \mathcal{D}^r \eta_\nu) dS,$$

$$B_3 = \frac{1}{2} \sigma \int_\Gamma \partial_i (\sqrt{g}g^{ij}\Sigma_{\mu}^\alpha)(\partial_j \mathcal{D}^r \eta_\nu) dS,$$

which appears in the energy estimate when $\mathcal{D}^r$ is non-weighted. The weighted cases are treated in section 3.5.

We recall that if $\mathcal{D}^r$ is non-weighted, then $r \leq 3$, i.e., the corresponding term is of lower order. Because of this, it would be suffice to consider the case when $\mathcal{D}^r = \partial^2 \partial_t$. Now, since $\Pi_{\mu}^\alpha = \hat{n}_\mu \hat{n}^\alpha$, we have:

$$B_1 = \sigma \sum_{j_1 + j_2 = 3} \int_\Gamma (\partial^{j_1} (\sqrt{g}g^{ij}\hat{n}_\mu \hat{n}^\alpha) (\partial^{j_2} \mathcal{D}^r \eta_\nu)) dS,$$

$$B_2 = \sigma \int_\Gamma \partial_i (\sqrt{g}g^{ij}\Sigma_{\mu}^\alpha)(\partial_j \mathcal{D}^r \eta_\nu)(\partial_i \mathcal{D}^r \eta_\nu) dS,$$

$$B_3 = \frac{1}{2} \sigma \int_\Gamma \partial_i (\sqrt{g}g^{ij}\Sigma_{\mu}^\alpha)(\partial_j \mathcal{D}^r \eta_\nu)(\partial_i \mathcal{D}^r \eta_\nu) dS.$$

Invoking Lemma 2.2, we get

$$\partial_t (\sqrt{g}g^{ij}\hat{n}_\mu \hat{n}^\alpha) = Q(\partial^2 \partial_t v_\nu), \quad \partial_t (\sqrt{g}g^{ij}\hat{n}_\mu \hat{n}^\alpha) = Q(\partial^2 \partial_t \eta),$$

where $Q$ is a rational function, and hence

$$\partial_t (\sqrt{g}g^{ij}\hat{n}_\mu \hat{n}^\alpha) = Q(\partial^2 \partial_t v_\nu)(\partial^2 \partial_t \eta) + Q(\partial^2 \partial_t v_\nu)(\partial^2 \partial_t \eta),$$

$$\partial_t (\sqrt{g}g^{ij}\hat{n}_\mu \hat{n}^\alpha) = Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t v_\nu)(\partial^2 \partial_t \eta).$$

In light of these, we have

$$\int_0^t B_2 = \sigma \int_0^t \int_\Gamma Q(\partial^2 \partial_t v_\nu)(\partial^2 \partial_t \eta) dS \leq P_0 + \mathcal{P} \int_0^t \mathcal{P},$$

via $(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$ duality. Moreover, $\int_0^t B_3$ is treated similarly. On the other hand,

$$B_1 \leq \sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta) + \sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta, \partial^2 \partial_t \eta, \partial^2 \partial_t \eta)(\partial^2 \partial_t \eta)$$

$$+ \sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta, \partial^2 \partial_t \eta)(\partial^2 \partial_t \eta) + \sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta).$$

The last three terms can be controlled in a routine fashion. However, $\sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta)$ cannot be controlled directly since $(H^{-\frac{1}{4}}, H^{\frac{1}{4}})$ duality requires the control $|\|v_\nu\|_3$ which is not part of $\mathcal{P}$, and so we consider

$$\sigma \int_0^t \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta)$$

and then integrate by parts in $t$. This yields

$$\sigma \int_0^t \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta) = \sigma \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta)\big|_0^t$$

$$- \sigma \int_0^t \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta) - \sigma \int_0^t \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta)$$

$$- \sigma \int_0^t \int_\Gamma (\partial^2 \partial_t v_\nu)Q(\partial^2 \partial_t \eta)(\partial^2 \partial_t \eta).$$
3.5 Control of \( B \) for weighted \( D^r \)

Here we show how to control \( B \) when \( D^2 = \sqrt{\kappa} \partial_{\ell}^2 \), \( D^3 = \sqrt{\kappa} (\partial^2 \partial_{\ell}) \), \( D^4 = \kappa (\delta^2 \partial_{\ell}) \), \( D^4 = \kappa (\delta^2 \partial_{\ell}) \), \( D^4 = \kappa (\delta^2 \partial_{\ell}) \), and \( D^4 = \kappa (\delta^2 \partial_{\ell}) \). The last three term on the right hand side can be controlled directly by \( P_0 + \mathcal{P} \int_0^t \mathcal{P} \) via \((H^{-\frac{1}{2}}, H^{\frac{1}{2}})\) duality. Moreover, the pointwise term is bounded by

\[
P_0 + \sigma Q(||\eta||_4)(|D^2 v|)|Dv|_1 \leq \epsilon N + P_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]

3.5.1 Case \( D^4 = (\kappa)^2 \partial_{\ell}^4 \)

The ensuing calculations produce a series of terms. In what follows we focus on the most delicate ones, in particular those leading to special cancellations. The remaining terms will either be of lower order or can be controlled by arguments similar to the ones presented for the aforementioned main terms. Therefore, all such remainders are collected and estimated at the very end in section 3.5.1.4. We note that certain cancellations are only visible after a series of manipulations have been made, requiring us to keep track of the explicit form of most terms in our calculations.

The following remark will be used throughout below. In view of identity Lemma 2.2–6, we have \( \hat{n} \rho \partial^m \partial_{\ell}^k v_\alpha = \hat{n}_\rho \Pi^m \partial^m \partial_{\ell}^k v_\alpha \), so that an estimate for \( \hat{n} \cdot \partial^m \partial_{\ell}^k v \) can controlled by \( \Pi \partial^m \partial_{\ell}^k v \).

We shall also need the following identity

\[
\partial_{\ell} v^\alpha \partial_{\ell} \eta_\alpha = -\frac{J}{\rho_0} \partial_{\ell} q, \quad \text{on } \Gamma,
\]

which is obtained upon contracting the first equation in (1.3) with \( \partial_{\ell} \eta_\alpha \), using the definition of \( a \), and (1.5).

3.5.1.1 Estimate for \( \int_0^t B_3 \) with \( D^4 = (\kappa)^2 \partial_{\ell}^4 \)

Using \( D^4 = \kappa \partial_{\ell}^4 \) in \( B_3 \) gives

\[
B_3 = \frac{1}{2} \int_\Gamma \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \Pi_{\mu}^a \Pi_{\nu}^a \partial_{\ell} v_\alpha) \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \nu^\mu) dS
\]

\[
= \frac{1}{2} \int_\Gamma \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \Pi_{\mu}^a \Pi_{\nu}^a \partial_{\ell} v_\alpha) \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \nu^\mu) dS + \sigma \int_\Gamma \sqrt{g} \partial_{\ell} (\kappa \partial_{\ell} \Pi_{\mu}^a \partial_{\ell} \partial_{\ell} v_\alpha) \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \nu^\mu) dS
\]

\[
= B_{31} + B_{32},
\]

where we used Lemma 2.2–5. It is immediate to estimate

\[
||B_{31}|| \leq \mathcal{P} ||\Pi \kappa \partial_{\ell} \partial_{\ell} v||_{0,\Gamma}.
\]

For \( B_{32} \), use Lemma 2.2–4 to find

\[
B_{32} = \sigma \int_\Gamma \sqrt{g} \partial_{\ell} (\kappa \partial_{\ell} \hat{n}^a \hat{n}_\alpha \partial_{\ell} \partial_{\ell} v_\alpha) \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \nu^\mu) dS + \sigma \int_\Gamma \sqrt{g} \partial_{\ell} (\kappa \partial_{\ell} \hat{n}^a \hat{n}_\alpha \partial_{\ell} \partial_{\ell} v_\alpha) \partial_{\ell} (\sqrt{g} \partial_{\ell}^m \nu^\mu) dS
\]

\[
= B_{321} + B_{322}.
\]
We have

\[ \|B_{322}\| \leq \mathcal{P}\|\Pi k\partial_t^3 v\|_{0,\Gamma}. \]

Using Lemma 2.2–8 we can write

\[ B_{321} = -\sigma \int_{\Gamma} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau \bar{\partial}_i \bar{\partial}_t^2 q \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS. \]

From (3.17) we have

\[ \bar{\partial}_t \eta^\alpha \bar{\partial}_i \bar{\partial}_t^3 v^\alpha = - \frac{J}{\rho_0} \bar{\partial}_i \bar{\partial}_t^2 q + \frac{J}{\rho_0} \bar{\partial}_i q - \frac{J}{\rho_0} \bar{\partial}_t^2 q - \bar{\partial}_i \bar{\partial}_t^2 \bar{\partial}_t [\bar{\partial}_t \eta^\alpha \bar{\partial}_t] v^\alpha. \]

Thus,

\[ B_{321} = \sigma \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ -\sigma \int_{\Gamma} \frac{1}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ +\sigma \int_{\Gamma} \frac{1}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ = B_{3211} + B_{3212} + B_{3213}. \]

Integrating \( \bar{\partial}_t \) by parts in \( B_{3211} \),

\[ B_{3211} = -\sigma \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ -\sigma \int_{\Gamma} \bar{\partial}_i (\frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS. \]

From Section 2.1, item 3, we have

\[ \bar{\partial}_i \bar{\partial}_t^2 q = -\sigma \bar{\partial}^m \bar{\partial}_m (\bar{\partial}_t v^\beta) - \bar{\partial}_i \bar{\partial}_t^2 q - \bar{\partial}_i \bar{\partial}_t^2 \bar{\partial}_t \bar{\partial}_t v^\beta, \]

so that

\[ B_{3211} = \sigma^2 \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \]

\[ +\sigma \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ -\sigma \int_{\Gamma} \bar{\partial}_i (\frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ = B_{3211} + B_{32112} + B_{32113}. \]

In \( B_{3211} \), we use Lemma 2.2–6 and factor a \( \bar{\partial}_t \) from \( \bar{\partial}_t^3 \) to obtain

\[ B_{3211} = \sigma^2 \bar{\partial}_t \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \]

\[ -\sigma^2 \int_{\Gamma} \frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \]

\[ -\sigma^2 \int_{\Gamma} \bar{\partial}_i (\frac{J}{\rho_0} \mathcal{A}^j_{k} \sqrt{g} g^{ij} g^{kl} \bar{n}_\lambda \bar{n}_\tau \bar{\partial}_k v^\tau (\bar{\partial}_i \bar{\partial}_t^2 q - \frac{J}{\rho_0} \bar{\partial}_t q) \Pi^\alpha_{\mu} \bar{\partial}_j \bar{\partial}_t^3 v^\mu \, dS \]

\[ = B_{32111} + B_{32112} + B_{32113}. \]

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For the first term, i.e., $B_{321111}$, we have

$$
\int_0^t B_{321111} \leq P_0 + \mathcal{M}_k^\alpha P(\mathcal{R}_k, \|\partial_t^3 \mathcal{D}_3^4 v\|_{0, \Gamma})(\|\mathcal{R}_k^\alpha \partial_t^2 \mathcal{D}_3^4 v\|_{0, \Gamma}).
$$

Using Young’s inequality and the fact that $\mathcal{R}_k$ can be made very small for large $\kappa$, we can bound the right-hand side by $P_0 + \epsilon P(N) + \epsilon N$. For $B_{321112}$, write

$$
g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 = \mathcal{D}_3^4 g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 + \mathcal{D}_3^4 g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu = \mathcal{D}_3^4 (g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu) - \mathcal{D}_3^4 g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu = \frac{1}{2} \mathcal{D}_3^4 (\partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu)^2 - \mathcal{D}_3^4 g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu,
$$

so that

$$
B_{321112} = -\frac{1}{2} \sigma^2 \int_\Gamma \frac{J}{P_0} \mathcal{R}_k^\alpha \sqrt{g} g^{kl} \partial_\alpha \mathcal{D}_3^4 v^\mu \mathcal{D}_3^4 \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 \partial_\gamma \mathcal{D}_3^4 v^\mu) \mathcal{D}_3^4 (g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu)^2 + \sigma^2 \int_\Gamma \frac{J}{P_0} \mathcal{R}_k^\alpha \sqrt{g} g^{kl} \partial_\alpha \mathcal{D}_3^4 v^\mu \mathcal{D}_3^4 \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 \partial_\gamma \mathcal{D}_3^4 v^\mu)
$$

Integrating $\mathcal{D}_3$ by parts in the first integral,

$$
B_{321112} = \frac{1}{2} \sigma^2 \int_\Gamma \mathcal{D}_3^4 (\frac{J}{P_0} \mathcal{R}_k^\alpha \sqrt{g} g^{kl} \partial_\alpha \mathcal{D}_3^4 v^\mu) (\partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu)^2 + \sigma^2 \int_\Gamma \frac{J}{P_0} \mathcal{R}_k^\alpha \sqrt{g} g^{kl} \partial_\alpha \mathcal{D}_3^4 v^\mu \mathcal{D}_3^4 \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 \partial_\gamma \mathcal{D}_3^4 v^\mu
$$

Writing

$$
B_{321112} = \mathcal{R}_k \frac{1}{2} \sigma^2 \int_\Gamma \mathcal{D}_3^4 (\frac{J}{P_0} \mathcal{R}_k^\alpha \sqrt{g} g^{kl} \partial_\alpha \mathcal{D}_3^4 v^\mu) (g^{\alpha \beta} \partial_\alpha \mathcal{D}_3^4 \partial_\beta \mathcal{D}_3^4 v^\mu)^2,
$$

we have

$$
B_{321112} \leq \epsilon P(N).
$$

This concludes the estimate for the most delicate terms in $\int_0^t B_3$. The remaining terms in $B_3$, i.e., $B_{321112}, B_{321113}, B_{321113}, B_{32112}, B_{3212},$ and $B_{3213}$, are treated in Section 3.5.1.4 below.

### 3.5.1.2 Estimate for $\int_0^t B_2$ with $\mathcal{D}_3^4 = (\mathcal{R}_k^\alpha)^2 \partial_t^4$ We now move to estimate $B_2$:

$$
B_2 = -\sigma \int_\Gamma \mathcal{D}_3^4 (\sqrt{g} g^{ij} \Pi^i_\mu (\mathcal{R}_k^\alpha \partial_t^4 v_\alpha) (\mathcal{R}_k^\alpha \partial_t^4 v_\mu) dS
$$

$$
= -\sigma \int_\Gamma \mathcal{D}_3^4 (\sqrt{g} g^{ij} \Pi^i_\mu \mathcal{R}_k^\alpha \partial_t^4 v_\alpha \mathcal{R}_k^\alpha \partial_t^4 v_\mu dS - \sigma \int_\Gamma \sqrt{g} g^{ij} \mathcal{D}_3^4 \Pi^i_\mu \mathcal{R}_k^\alpha \partial_t^4 v_\alpha \mathcal{R}_k^\alpha \partial_t^4 v_\mu dS
$$

$$
= B_{21} + B_{22}.
$$

(3.18)
We show below that $B_{21}$ exactly cancels with a term coming from $B_1$. Here we move to estimate $B_{22}$. Using Lemma 2.2–4,

$$
B_{22} = -\sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS - \int \nabla_{\kappa} \sqrt{g} g^{ij} \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
= B_{221} + B_{222}.
$$

We use Lemma 2.2–9 to write

$$
B_{221} = \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 \eta_\kappa \partial_t^2 \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS.
$$

From (3.17) we have

$$
\tilde{\partial}_i \eta_\kappa \bar{\partial}_i \partial_t^3 v_{\alpha} = -\frac{J}{\rho_0} \partial_i \tilde{\partial}_i \partial_t^2 q + [\partial_i \partial_t^2, -\frac{J}{\rho_0} \tilde{\partial}_i] q - [\partial_i \partial_t^2, \eta_\kappa \partial_t] v^\mu,
$$

whence

$$
B_{221} = -\sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 \eta_\kappa \partial_t^2 \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS + \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 \eta_\kappa \partial_t^2 \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
- \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^4 \eta_\kappa \partial_t^2 \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
= B_{2211} + B_{2212} + B_{2213}.
$$

In $B_{2211}$, we factor a $\partial_t$ in $\partial_i^3 v_{\alpha}$ to obtain

$$
B_{2211} = -\sigma \partial_t \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^3 \eta_\kappa \partial_t \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS + \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^3 \eta_\kappa \partial_t \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
= B_{22111} + B_{22112}.
$$

For $B_{22111}$, we integrate $\tilde{\partial}_j$ by parts to produce

$$
\int_0^t B_{22111} \leq \mathcal{P}(||R_{\kappa}^2 \Pi \tilde{\partial}_t^3 v||_{0, \Gamma})(||R_{\kappa}^2 \tilde{\partial}_t^2 q||_{0, \Gamma}) + \int_0^t \mathcal{P},
$$

where

$$
||R_{\kappa}^2 \Pi \tilde{\partial}_t^3 v||_{0, \Gamma} ||R_{\kappa}^2 \tilde{\partial}_t^2 q||_{0, \Gamma} \lesssim \epsilon ||R_{\kappa}^2 \Pi \tilde{\partial}_t^3 v||_{0, \Gamma} + R_{\kappa}^4 \epsilon^{-1} ||R_{\kappa}^2 \tilde{\partial}_t^3 q||_{1} \lesssim \epsilon (||R_{\kappa}^2 \Pi \tilde{\partial}_t^3 v||_{0, \Gamma} + ||R_{\kappa}^2 \tilde{\partial}_t^2 q||_{1}) \lesssim \epsilon \mathcal{P}(N),
$$

after choosing $R_{\kappa}$ sufficiently small and replacing $q$ by $R$.

For $B_{22112}$, write

$$
B_{22112} = \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^3 \eta_\kappa \partial_t \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
+ \sigma \int \nabla_{\kappa} \sqrt{g} g^{ij} g^{kl} \partial_k \partial_i \eta_\kappa \hat{n}_\alpha \partial_t^3 \eta_\kappa \partial_t \partial_i^3 v_{\alpha} \tilde{\partial}_j \bar{\partial}_i^3 v^\mu dS
$$

$$
= B_{221121} + B_{221122}.
$$
The term $B_{221121}$ can be handled with integration by parts with respect to $\partial_j$ (it yields a term in $||\Pi^{\alpha}_{\kappa} \partial_l^3 v||_0,1$). For $B_{221122}$, we use Section 2.1, item 3, to write

$$B_{221122} = -\sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$-\sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha (\partial_j \partial_l [\partial_l^3, g^{mn} \eta^\beta \partial_m \partial_n \eta^\beta]) \partial_l^3 v_\alpha$$

$$= B_{2211221} + B_{2211222}.$$ Integrating by parts $\partial_l$ in $B_{2211221}$,

$$B_{2211221} = \sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$+ \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$= \sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$+ \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

and then integrating by parts $\partial_l$ on the second integral,

$$B_{2211221} = \sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$- \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$- \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$+ \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$= B_{22112211} + B_{22112212} + B_{22112213} + B_{22112214} + B_{22112215}.$$ (3.19)

Note that the first and third terms, i.e., $B_{2211211}$ and $B_{2211212}$, cancel each other in view of the following identity, which can be verified by inspection,

$$\sum_{i,k,l=1}^{2} (g^{ij} g^{kl} - g^{ik} g^{lj}) = 0.$$

For the second term, $B_{22112212}$, integrate $\partial_l \partial_j$ by parts:

$$B_{22112212} = -\sigma^2 \int \partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$- \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha) (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$- \sigma^2 \int \partial_l (\partial_l^4 \sqrt{g} g^{ij} g^{kl} \partial_l \partial_l \eta^\tau \eta^\sigma \eta^\alpha) \partial_j (g^{mn} \eta^\beta \partial_m \partial_n \partial_l^2 v^\beta \partial_l^3 v_\alpha) dS$$

$$= B_{22112211} + B_{22112212} + B_{22112213}. $$
Factoring a $\partial_t$ from $\overline{\partial}_i \overline{\partial}_j \partial_t^3 v_\alpha$ in $B_{221122121}$, we find

$$
B_{221122121} = -\frac{1}{2} \sigma^2 \int \mathcal{N}_k^J \frac{1}{\rho_0} \sqrt{gg} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_j \eta^\tau \hat{n}_\tau g^{mn} \hat{n}_{\beta} \partial_n \partial_l \partial_t^2 v_\beta \hat{n}_\alpha \overline{\partial}_j \partial_t^3 v_\alpha \, dS
+ \frac{1}{2} \sigma^2 \int \partial_t \mathcal{N}_k^J \frac{1}{\rho_0} \sqrt{gg} g^{ij} g^{kl} \overline{\partial}_i \overline{\partial}_j \eta^\tau \hat{n}_\tau g^{mn} \hat{n}_{\beta} \hat{n}_\alpha \partial_m \partial_n \partial_l \partial_t^2 v_\beta \overline{\partial}_j \partial_t^2 v_\alpha \, dS
= B_{2211221211} + B_{2211221212}.
$$

The first term, $B_{2211221211}$, can be estimated by $\epsilon P(N)$. Here, the small number $\epsilon$ comes from estimating $\overline{\partial}_i \overline{\partial}_k \eta^\tau$ in $L^\infty$ and using that $\eta(0)$ is the identity diffeomorphism.

Now we move to $B_{222}$. Factoring a $\partial_t$ from $\partial_t^3 v_\alpha$, we find

$$
B_{222} = -\sigma \int \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v_\mu \, dS
= -\sigma \partial_t \int \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^3 \overline{\partial}_j \partial_t^3 v_\mu \, dS
+ \sigma \int \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v_\mu \, dS
+ \sigma \int \partial_t \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v_\mu \, dS.
$$

Integrating $\overline{\partial}_j$ by parts in the second integral,

$$
B_{222} = -\sigma \partial_t \int \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^4 v_\alpha \overline{\partial}_j \partial_t^3 v_\mu \, dS
- \sigma \int \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \overline{\partial}_j \partial_t^3 v_\alpha \partial_t \partial_t^3 v_\mu \, dS
+ \sigma \int \partial_t \mathcal{N}_k^J \sqrt{gg} g^{ij} \hat{n}_\mu \overline{\partial}_i \hat{n}_\alpha \partial_t^3 v_\alpha \overline{\partial}_j \partial_t^3 v_\mu \, dS.
$$

Note that $B_{2222} = B_{221}$, so this term is estimated as above. The term $B_{2221}$ can, after time integration, be estimated using Young’s inequality and interpolation.

With exception of $B_{21}$, which, as said, involves a special cancellation showed below, this concludes the estimate of the most delicate terms in $\int t^l B_2$. The remaining terms $B_{2212}$, $B_{2213}$, $B_{2211222}$, $B_{22112214}$, $B_{22112215}$, $B_{22112212}$, $B_{221122123}$, $B_{221122112}$, $B_{2223}$, and $B_{2224}$, are treated in Section 3.5.1.4 below.
3.5.1.3 Estimate for $\int_0^t B_1$ with $\mathcal{D}^1 = (\mathcal{R}_\alpha)^2 \partial_t^4$. We now move to estimate $B_1$:

$$B_1 = \sigma \int_{\Gamma} (\mathcal{R}_\alpha^2 \partial_t^4 v_\alpha)([\mathcal{R}_\alpha^2 \partial_t^4 \rho^i, \sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha \bar{\eta}_j^\mu]) dS$$

$$= 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^4 (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 6\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^3 (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^2 (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ \sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$= B_{11} + B_{12} + B_{13} + B_{14}.$$

We have

$$B_{14} = \sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \sqrt{\sigma} g^{ij} \partial_t^4 \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^3 (\sqrt{\sigma} g^{ij} \rho^i) \partial_t^3 \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 6\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^2 (\sqrt{\sigma} g^{ij} \rho^i) \partial_t^2 \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t (\sqrt{\sigma} g^{ij} \rho^i) \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ \sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$= B_{141} + B_{142} + B_{143} + B_{144} + B_{145}.$$

Using Lemma 2.2–4, we have

$$B_{141} = \sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \sqrt{\sigma} g^{ij} \partial_t^4 \rho^i \partial_t^3 \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^3 (\sqrt{\sigma} g^{ij} \rho^i) \partial_t^2 \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 6\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t^2 (\sqrt{\sigma} g^{ij} \rho^i) \partial_t \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ 4\sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t (\sqrt{\sigma} g^{ij} \rho^i) \rho^i \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$+ \sigma \int_{\Gamma} \mathcal{R}_\alpha^4 \partial_t (\sqrt{\sigma} g^{ij} \Pi_{\mu}^\alpha) \Pi_{\mu}^\alpha \bar{\eta}_j^\mu \partial_t^4 v_\alpha$$

$$= B_{1411} + B_{1412} + B_{1413} + B_{1414} + B_{1415}.$$

From Lemma 2.2–8 we have

$$\partial_t^4 \rho^i = -g^{kl} \partial_l^3 \rho^i v^r \partial_t^4 \bar{\eta}_r^\mu_\gamma - [\partial_t^3, g^{kl} \partial_t^4 \bar{\eta}_r^\mu_\gamma] v^r$$,
and thus

\[ B_{1411} = -\sigma \int_{\Gamma} \mathcal{M}_k^4 \sqrt{g} g^{ij} \hat{n}^\alpha g^{kl} \overline{\partial}_{k} \overline{\partial}_{j} \nabla^{\alpha} \nabla_{\mu} \overline{\partial}_{i} \overline{\partial}_{j} \eta^\mu \partial_i \nu_{\alpha} \]

\[ -\sigma \int_{\Gamma} \mathcal{M}_k^4 \sqrt{g} g^{ij} n^a \left[ \partial_{i}^3 \right. + \cdots \left. + \partial_{i} n_{\tau} \overline{\partial}_{j} \eta^\mu \partial_i \overline{\partial}_{j} \eta^\mu \partial_i \nu_{\alpha} \right] \]

\[ = B_{14111} + B_{14112}. \]

We now invoke Lemma 2.2–10, to replace \( \sqrt{g} g^{kj} \overline{\partial}_{j} \overline{\partial}_{i} \eta^\mu \overline{\partial}_{i} \eta_{\mu} \) in \( B_{14111} \) by \( -\overline{\partial}_{i} (\sqrt{g} g^{ik}) \), obtaining

\[ B_{14111} = \sigma \int_{\Gamma} \mathcal{M}_k^4 \overline{\partial}_{i} (\sqrt{g} g^{ik}) \overline{\partial}_{k} \overline{\partial}_{i} \nabla^{\alpha} \nabla_{\mu} \overline{\partial}_{i} \overline{\partial}_{j} \eta^\mu \partial_i \nu_{\alpha}. \]  

We see that this term exactly cancels \( B_{21} \), as mentioned earlier. The other terms in the estimate of \( \int_0^t B_1 \) are treated in section 3.5.1.4.

### 3.5.1.4 Remainders in \( \int_0^t B \) with \( \mathcal{D}^4 = \mathcal{M}_k^2 \partial_i^4 \)

Above we have showed how to control the most delicate terms in the estimate for \( \int_0^t B \) when \( \mathcal{D}^4 = \mathcal{M}_k^2 \partial_i^4 \). In particular, we have showed how some top order terms, which seemingly cannot be individually bounded, cancel out when taken together. Now we consider the remaining terms, which we list here for the reader’s convenience. They are, for \( B_3 \),

\[ B_{321122}, B_{321113}, B_{321112}, B_{32112}, B_{3213} \]

from section 3.5.1.3; for \( B_2 \)

\[ B_{2212}, B_{2213}, B_{221122}, B_{22112214}, B_{22112215}, B_{22112212}, B_{2212223}, B_{2222} \]

from section 3.5.1.2; for \( B_1 \)

\[ B_{11}, B_{12}, B_{13}, B_{142}, B_{143}, B_{144}, B_{145}, B_{1412}, B_{1413}, B_{1414}, B_{1415}, B_{14112} \]  

from section 3.5.1.3. Not all these terms are immediately of lower order, but they can be estimated using the same kind of ideas that have already been employed. Therefore, it suffices to briefly indicate how this is done.

The terms \( B_{321122}, B_{321113}, B_{321112}, B_{321112} \) can be bounded directly. The term \( B_{32113} \) is bounded upon replacing \( q \) by \( R \) and estimating in routine fashion.

The terms \( B_{2212} \) and \( B_{2213} \) can be estimated with integration by parts in time. The terms \( B_{2211222}, B_{22112214}, B_{22112215}, B_{22112212}, B_{22112212} \) can be estimated directly. The term \( B_{22112212} \) requires integration by parts in space and then using arguments similar to above, with one extra step: after integrating \( \overline{\partial}_{j} \) by parts, we obtain a term with four derivatives of \( \eta \). This term, however, has the form \( \hat{n}_{\tau} g^{ij} \partial_{i} \partial_{i} \partial_{j} \partial_{j} \eta^{\tau} \), which allows us to use Section 2.1, item 3, to eliminate two derivatives of \( \eta \). (Alternatively, we can use elliptic estimates for equations with Sobolev coefficients, as, e.g., Theorem 4 and Remark 2 in [51]).

The terms listed in (3.21) are again handled by a repetition of ideas used above (without requiring special cancellations). In particular, Lemma 2.2–8 is used heavily and Lemma 2.2–11 is employed to estimate \( B_{145} \).

Combining these observations with the estimates of section 3.5.1.1, 3.5.1.2, and 3.5.1.1, we finally obtain

\[ \int_0^t (B_1 + B_2 + B_3) \leq P_0 + \epsilon N + \epsilon P(N) + P \int_0^t P, \]  

when \( \mathcal{D}^4 = \mathcal{M}_k^2 \partial_i^4 \).
3.5.2 Estimate of the remaining weighed boundary terms

It remains to carry out control of \( \int_0^t (B_1 + B_2 + B_3) \) when \( D^2 = \sqrt{\mathcal{R}_\kappa} \partial_t^2 \), \( D^3 = \sqrt{\mathcal{R}_\kappa} (\partial D^2) \), \( D^3 = \mathcal{R}_\kappa \partial_\kappa^3 \), \( D^4 = \mathcal{R}_\kappa (\partial^3 D^2) \), \( D^4 = \mathcal{R}_\kappa (\partial \mathcal{R}_\kappa^3) \), and \( D^4 = (\mathcal{R}_\kappa)^3 \). These cases are treated in an almost identical fashion as the case \( D^4 = (\mathcal{R}_\kappa)^2 \partial_t^4 \) from section 3.5.1. In this regard, we note that a crucial requirement to carry these estimates is that \( D \) contains at least one time derivative, which is the case for all the weighted derivatives we need to consider. We therefore conclude

\[
\int_0^t (B_1 + B_2 + B_3) \leq \mathcal{P}_0 + \epsilon N + \epsilon \mathcal{P}(N) + \mathcal{P} \int_0^t \mathcal{P},
\]

for \( D^2 = \sqrt{\mathcal{R}_\kappa} \partial_t^2 \), \( D^3 = \sqrt{\mathcal{R}_\kappa} (\partial D^2) \), \( D^3 = \mathcal{R}_\kappa \partial_\kappa^3 \), \( D^4 = \mathcal{R}_\kappa (\partial^3 D^2) \), \( D^4 = \mathcal{R}_\kappa (\partial \mathcal{R}_\kappa^3) \), and \( D^4 = (\mathcal{R}_\kappa)^3 \partial_t^4 \).

4 Closing the estimate

In this section, we prove:

**Theorem 4.1.** Let \( N(t) \) and \( \mathcal{P}(t) \) be defined as Notation 3.1, then for sufficiently large \( \kappa \) (i.e., \( \mathcal{R}_\kappa \ll 1 \)), we have:

\[
N(t) \leq C(M) \left( \epsilon \mathcal{P}(N(t)) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} \right), \quad t \in [0, T],
\]

where \( T > 0 \) is chosen sufficiently small, provided that:

\[
||\partial \eta||_{L^\infty} + ||\partial^2 \eta||_{L^\infty} \leq M, \quad (4.1)
\]

\[
||g^{ij}||_{L^\infty} + ||\Gamma_i^j||_{L^\infty} \leq M, \quad (4.2)
\]

hold a priori for some large constant \( M \).

Since the energy estimate for \( E \) is established in the previous section (i.e., Theorem 3.2), we only need to show

\[
||v||_2^2 + ||\mathcal{R}_\kappa v_t||_2^2 + ||\mathcal{R}_\kappa v_{tt}||_2^2 + ||(\mathcal{R}_\kappa)^{3/2} v_{ttt}||_2^2 + ||R||_2^2 + ||R_t||_2^2 + ||\sqrt{\mathcal{R}_\kappa} R_{tt}||_2^2 + ||\mathcal{R}_\kappa R_{ttt}||_2^2
\]

\[
+ ||v_t||_2^2 + ||v_t v_{tt}||_0^2 + ||\mathcal{R}_\kappa v_{ttt}||_0^2 + ||R_{tt}||_1 + ||\sqrt{\mathcal{R}_\kappa} R_{ttt}||_0
\]

\[
\leq C(M) \left( \epsilon \mathcal{P}(N(t)) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} \right), \quad (4.3)
\]

This is proved via an iterated argument using div-curl estimate (A.1). It suffices to consider the first line in (4.3), since the second line consists lower order terms and can be treated by the same method. Taking \( X = v \) and \( s = 4 \), (A.1) yields

\[
||v||_4^2 \leq ||\text{div } v||_2^2 + ||\text{curl } v||_2^2 + ||v^3||_{3,5,0}^2 + ||v||_0^2. \quad (4.4)
\]

On the other hand, taking \( X = \mathcal{R}_\kappa \partial_t v \) and \( s = 3 \), we have:

\[
||\mathcal{R}_\kappa v_t||_3^2 \leq ||\mathcal{R}_\kappa \text{div } v_t||_2^2 + ||\mathcal{R}_\kappa \text{curl } v_t||_2^2 + ||\mathcal{R}_\kappa v_{tt}||_2^2 + ||\mathcal{R}_\kappa v_t||_0^2. \quad (4.4)
\]

\[\text{Incidentally, this is why an estimate for the normal component of } v \text{ with no time derivatives has to be obtained in a different way, see Section 4.1}\]
Similarly, by taking $X = R' v_t$, $s = 2$ and $X = (R')^{3/2} v_{ttt}$, $s = 1$, we get
\begin{align}
||\mathcal{R}_\kappa v_t||^2_2 &\lesssim ||\mathcal{R}_\kappa \text{div } v_t||^2_2 + ||\mathcal{R}_\kappa \text{curl } v_t||^2_2 + ||\mathcal{R}_\kappa v^3_t||^2_{0,5,\Gamma} + ||\mathcal{R}_\kappa v_t||^2_0, \\
||\mathcal{R}_\kappa^{3/2} v_{ttt}||^2_2 &\lesssim ||\mathcal{R}_\kappa^{3/2} \text{div } v_{ttt}||^2_0 + ||\mathcal{R}_\kappa^{3/2} \text{curl } v_{ttt}||^2_0 + ||\mathcal{R}_\kappa^{3/2} v^3_{ttt}||^2_{0,5,\Gamma} + ||\mathcal{R}_\kappa^{3/2} v_{ttt}||^2_0,
\end{align}
respectively. In light of (4.4)-(4.6), in order to estimate $v$ and its time derivative, we need to bound $\text{div } \partial_t^k v$, $\text{curl } \partial_t^k v$ and $\partial_t^k v^3$, for $k = 0, 1, 2, 3$, respectively.

### 4.1 Bounds for the curl and the boundary term of $v$

In this section we prove:

**Theorem 4.2.**

\[ ||\text{curl } v||^2_3 + ||\mathcal{R}_\kappa \text{curl } v_t||^2_2 + ||\mathcal{R}_\kappa \text{curl } v_t||^2_1 + ||\mathcal{R}_\kappa^{3/2} \text{curl } v_{ttt}||^2_0 \lesssim \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}. \]  \hspace{1cm} (4.7)

**Proof.** The proof is almost identical to Section 4 of [21], and so we omit the details. The only modification is that the weights $\mathcal{R}_\kappa$ or $(\mathcal{R}_\kappa)^{3/2}$ are used to compensate $q'(R) \sim \mathcal{R}_\kappa^{-1}$, which allows us to get an uniform control.

On the other hand, we have:

**Theorem 4.3.**

\[ ||v^3||^2_{3,5,\Gamma} \lesssim \epsilon P(N) + P_0 + \mathcal{P} \int_0^t \mathcal{P}, \]  \hspace{1cm} (4.8)

and

\begin{align}
||\mathcal{R}_\kappa v^3_t||^2_{2,5,\Gamma} &\lesssim \epsilon N + ||\mathcal{R}_\kappa \Pi \overline{\Omega}^3 v_t||^2_{0,\Gamma} + P_0 + \mathcal{P} \int_0^t \mathcal{P}, \\
||\mathcal{R}_\kappa v^3_{tt||}||^2_{1,5,\Gamma} &\lesssim \epsilon N + ||\mathcal{R}_\kappa \Pi \overline{\Omega}^3 v_{ttt}||^2_{0,\Gamma} + P_0 + \mathcal{P} \int_0^t \mathcal{P}, \hspace{1cm} (4.9)
\end{align}

\begin{align}
||\mathcal{R}_\kappa^{3/2} v^3_{ttt||}||^2_{0,5,\Gamma} &\lesssim \epsilon N + ||(\mathcal{R}_\kappa)^{3/2} \Pi \overline{\Omega}^3 v_{tttt}||^2_{0,\Gamma} + P_0 + \mathcal{P} \int_0^t \mathcal{P}. \hspace{1cm} (4.10)
\end{align}

**Proof.** For any vector field $X$, the following identity allows one to compare $(\Pi \overline{\Omega} X)^3$ and $\overline{\partial} X^3$:

\[ (\Pi \overline{\Omega} X)^3 = \Pi \overline{\partial} X^3 = - g^{kl} \overline{\partial}_k \eta^3 \eta_{\lambda} \overline{\partial}_\lambda X^3. \]  \hspace{1cm} (4.12)

Invoking (4.12), let $X = \mathcal{R}_\kappa^{3/2} \partial_t^3 v$ and then taking $H^{-0.5}(\Gamma)$ norm yields

\begin{align}
||\mathcal{R}_\kappa^{3/2} \overline{\partial} \partial_t^3 v^3||^2_{0,5,\Gamma} &\lesssim ||\mathcal{R}_\kappa^{3/2} \Pi \overline{\partial} \partial_t^3 v^3||^2_{0,\Gamma} + ||g^{kl} \overline{\partial}_k \eta^3 \eta_{\lambda}||^2_{1,5,\Gamma} ||\mathcal{R}_\kappa^{3/2} \partial_t^3 v^3||^2_{0,5,\Gamma}.
\end{align}

We add $||\mathcal{R}_\kappa^{3/2} \partial_t^3 v^3||^2_{0,5,\Gamma}$ to both sides, use the fact that $||\mathcal{R}_\kappa^{3/2} \partial_t^3 v^3||^2_{0,5,\Gamma} + ||\mathcal{R}_\kappa^{3/2} \overline{\partial} \partial_t^3 v^3||^2_{0,5,\Gamma}$ is equivalent to $||\mathcal{R}_\kappa^{3/2} \partial_t^3 v^3||^2_{0,5,\Gamma}$, invoke $\overline{\partial} \eta^3 = \int_0^t \overline{\partial} \eta^3$ (which is true since $\eta^3(0) = 1$), to conclude (4.11), where the term $||\mathcal{R}_\kappa^{3/2} \partial_t^3 v^3||^2_{0,5,\Gamma}$ on the right hand side is estimated using interpolation, Young’s inequality, and the fundamental theorem of calculus.
Similarly, using (4.12) with \( X = \mathcal{R}_k \partial_t \partial^2 v \) and \( X = \mathcal{R}_k \partial^2 \partial_t v \), estimating in \( H^{-0.5}(\Gamma) \) yields (4.10) and (4.9), respectively. Now, we need to control \( \| v^3 \|_{3.5, \Gamma} \). This cannot be controlled using the above method since \( \| \prod \partial^3 v \|_{0, \Gamma} \) is not part of the energy \( E \). Nevertheless, we recall the boundary condition

\[
\sqrt{\gamma} \Delta g^{\alpha} = \sqrt{g} g^{ij} \partial_{ij} g^{\alpha} - \sqrt{g} g^{ij} \Gamma_k^i j \partial_k g^{\alpha} = -\sigma^{-1} A^{\mu \alpha} N_\mu q, \quad \text{on } \Gamma \quad (4.13)
\]

where \( \Gamma_k^i j = g^{kl} \partial_l g^{ij} \partial_k g^{\alpha} \). Time differentiating (4.13) with \( \alpha = 3 \) gives:

\[
\sqrt{g} g^{ij} \partial_{ij} v^3 - \sqrt{g} g^{ij} \Gamma_k^i j \partial_k v^3 = -\partial_t (\sqrt{g} g^{ij} \partial_{ij} g^{\alpha} - \sqrt{g} g^{ij} \Gamma_k^i j \partial_k g^{\alpha}) - \sigma^{-1} \partial_t A^{\mu \alpha} N_\mu q - \sigma^{-1} A^{\mu \alpha} N_\mu \partial_t q \quad (4.14)
\]

holds on \( \Gamma \). Because \( g^{ij} \in H^{2.5}(\Gamma) \) and \( \Gamma_k^i j \in H^{1.5}(\Gamma) \), invoking the elliptic estimate for rough coefficients (see, e.g., Theorem 4 and Remark 2 in Milani [51]), we obtain:

\[
\| v^3 \|_{3.5, \Gamma} \lesssim_{\Gamma} \| \partial_t (\sqrt{g} g^{ij} \partial_{ij} g^{\alpha}) \|_{1.5, \Gamma} + \| \partial_t (\sqrt{g} g^{ij} \Gamma_k^i j \partial_k g^{\alpha}) \|_{1.5, \Gamma} + \| \partial_t A^{\mu \alpha} N_\mu q \|_{1.5, \Gamma} + \| A^{\mu \alpha} N_\mu \partial_t q \|_{1.5, \Gamma},
\]

which can be controlled appropriately by the right hand side of (4.8), where the last two terms can be controlled by with the help of Theorem 3.12. \( \square \)

### 4.2 Bounds for \( v, R \) and their time derivatives

Let \( k = 1, 2, 3 \), commuting \( \partial_k^k \) to the second equation of (1.3), we get

\[
\partial^\alpha \partial_t^k v_\alpha = (\delta^{\mu \alpha} - a^{\mu \alpha}) \partial_\mu \partial_t^k v_\alpha - \sum_{j_1 + j_2 = k} R^{-1} \partial_t^{j_1} (Ra^{\mu \alpha})(\partial_\mu \partial_t^{j_2} v_\alpha) - R^{-1} \partial_t^{k+1} R. \quad (4.15)
\]

In addition, the first equation of (1.3) can be re-written as

\[
R'R\partial_t v^\alpha + a^{\mu \alpha} \partial_\mu R = 0.
\]

Commuting \( \partial_t^k \) to this equation and invoking (1.11), we get

\[
\partial^\alpha \partial_t^k R = (\delta^{\mu \alpha} - a^{\mu \alpha}) \partial_\mu \partial_t^k R - R'R\partial_t^{k+1} v^\alpha - \sum_{j_1 + j_2 = k} \{ (\delta_t^{j_1} a^{\mu \alpha})(\partial_\mu \partial_t^{j_2} R) + (\partial_t^{j_1} (R'R))(\partial_t^{j_2+1} v^\alpha) \}. \quad (4.16)
\]

When \( k = 3 \), multiplying \( (R')^{\frac{3}{2}} \) and then taking \( L^2 \) norm on both sides of (4.15), we get

\[
\| (R')^{\frac{3}{2}} \partial_t^3 v_\alpha \|_{0} \leq \epsilon \| (R')^{\frac{3}{2}} \partial_t^3 v_\alpha \|_{1} + C \sum_{j_1 + j_2 = 3} \| (R')^{\frac{3}{2}} \partial_t^{j_1} (Ra^{\mu \alpha})(\partial_\mu \partial_t^{j_2} v_\alpha) \|_{0} + C \| (R')^{\frac{3}{2}} Rttt \|_{0},
\]

where we have used Lemma 2.1(9)(10). The term

\[
\sum_{j_1 + j_2 = 3} \| (R')^{\frac{3}{2}} \partial_t^{j_1} (Ra^{\mu \alpha})(\partial_\mu \partial_t^{j_2} v_\alpha) \|_{0}
\]

is of lower order and can be controlled appropriately. Squaring and using Theorem 3.2, we have

\[
\| (\mathcal{R}_k)^{\frac{3}{2}} \text{div} v_{ttt} \|_{0} \lesssim \| (R')^{\frac{3}{2}} \text{div} v_{ttt} \|_{0} \lesssim \epsilon P(N) + P_0 + P \int_0^t \mathcal{P}.
\]
Now, in view of (4.6), invoking (4.7), (4.11) and Theorem 3.2 gives
\[
\| (\mathcal{R}_s)^2 v_{ttt} \|^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\] (4.17)

We now move to estimate \( |\mathcal{R}_s R_{ttt}| \). Invoking (4.16) for \( k = 3 \), multiplying \( R' \) on both sides and taking \( L^2 \) norm, we have:
\[
\| R' R_{ttt} \|_1 \lesssim \epsilon \| R' R_{ttt} \|_1 + \| (R')^2 v_{ttt} \|_0 + \epsilon N + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]
Here, \( \epsilon N \) appears when controlling the error term of (4.16)\(^4\). Squaring this provides:
\[
\| \mathcal{R}_s R_{ttt} \|^2 \lesssim \| R' R_{ttt} \|^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},
\] (4.18)
where Theorem 3.2 is also used.

Next, we estimate \( |\mathcal{R}_s \text{div} v_t| \). Invoking (4.15) with \( k = 2 \), multiplying \( R' \) and then applying \( H^1 \) norm on both sides, we get
\[
\| R' \partial^2 \partial_s^2 v_0 \|_1 \leq \epsilon \| R' v_t \|_2 + C \sum_{j_1 + j_2 = 2} \| R' \partial_t^{j_1} (R_0 a^\alpha)(\partial_t^{j_2} v_0) \|_1 + C \| R' R_{ttt} \|_1.
\]
Using (4.18), squaring the above estimate leads to
\[
\| \mathcal{R}_s \text{div} v_t \|^2 \lesssim \| R' \text{div} v_t \|^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\]
In light of (4.5), the above bound for \( |\mathcal{R}_s \text{div} v_t| \), together with (4.7), (4.10) and Theorem 3.2 give
\[
\| \mathcal{R}_s v_t \|^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\] (4.19)

Furthermore, invoking (4.16) for \( k = 2 \), multiplying \( \sqrt{R'} \) and taking \( H^1 \) norm and squaring, we get:
\[
\| \sqrt{R'} R_{ttt} \|^2 \lesssim \epsilon \| \sqrt{R'} R_{ttt} \|^2 + \| (R')^2 v_{ttt} \|^2 \lesssim \epsilon N + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},
\]
which implies, after invoking (4.17), that
\[
\| \sqrt{\mathcal{R}_s R_{ttt}} \|^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}.
\] (4.20)

In addition, this allows us to continue this procedure to get an estimate for \( R' \text{div} v_t \); let \( X = R' \partial_t v \) and \( s = 3 \) in (4.15), we get:
\[
\| R' \text{div} v_t \|_2 \lesssim \epsilon \| R' v_t \|_3 + \| R' R_{ttt} \|_2 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P},
\]
squearing, and invoking (4.7), (4.9) and (4.20) gives
\[ ||R_\kappa v_t||_3^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \] (4.21)
Now, invoking (4.16) for \( k = 1 \), squaring and taking \( H^2 \) norm yields
\[ ||R_t||_3^2 \lesssim \epsilon ||R_t||_3^2 + ||R' v_t||_2 + \epsilon N + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \] (4.22)
as a consequence of (4.19).

Finally, the above procedure yields
\[ ||\text{div} v||_3 \lesssim \epsilon ||v||_4 + ||R_t||_3, \]
and hence
\[ ||v||_4^2 \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \]
via (4.7), (4.8) and (4.22). Moreover, we have:
\[ ||R||_4^2 \lesssim \epsilon ||R||_4^2 + ||R' v_t||_3 + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} \lesssim \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \]
via (4.21).

4.3 The continuity argument, proof of Theorem 1.1

**Recovering the a priori assumptions:** We need to control the left hand side of (4.1)-(4.2) by \( \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P} \). The control for (4.1) is a direct consequence of the Sobolev embedding, i.e.,
\[ ||\partial \eta||_{L^\infty} + ||\partial^2 \eta||_{L^\infty} \lesssim ||\eta||_4 \leq \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}. \]
This also controls the left hand side of (4.2) by the definition of \( g^j i \) and \( \Gamma^k_{ij} \).

**Estimates at \( t = 0 \):** As we have seen that \( \mathcal{P} \) involves quantities involving time derivatives, and so one needs to show that these quantities can be controlled by \( \mathcal{P}_0 \). More precisely, we show:
\[ ||\mathcal{R}_\kappa v_t(0)||_3 + ||\mathcal{R}_\kappa v_{tt}(0)||_2 + ||\mathcal{R}_\kappa \sqrt{v_{ttt}(0)}||_1 + ||R_t(0)||_3 + ||\sqrt{\mathcal{R}_\kappa R_{ttt}(0)}||_1 \leq \mathcal{P}_0. \] (4.23)
This estimate is straightforward, i.e., we use (1.6) to obtain \( ||\mathcal{R}_\kappa v_t(0)||_3 \leq ||\rho_0^{-1} \partial q(0)||_3 \leq \mathcal{P}_0 \). Moreover, we use (4.15) with \( k = 0 \) at \( t = 0 \) to obtain \( ||R_t(0)||_3 \leq ||\rho_0^{-1} \text{div} v(0)||_3 \leq \mathcal{P}_0 \). The other quantities in (4.23) can be controlled similarly. In addition, we also need
\[ ||\mathcal{R}_\kappa v_t(0)||_3 + ||\mathcal{R}_\kappa v_{tt}(0)||_2 + ||\sqrt{\mathcal{R}_\kappa v_{ttt}(0)}||_1, r \leq \mathcal{P}_0. \]
To control $||R\kappa v_t(0)||_{3,\Gamma}$, we use (4.16) to obtain $R'v_t^i(0) = -\delta^{ij}\partial_jR(0)$, which implies $||R\kappa v_t^i(0)||_{3,\Gamma} \leq ||R(0)||_{4,\Gamma} \leq P_0$. On the other hand, we control the normal component $v_t^3(0)$ using the elliptic estimate. Time differentiating (4.14) and then restricting at $t = 0$ yields:

$$\nabla v_t^3(0) = -\sigma^{-1}q_{ttt}(0) + F,$$

where $F$ satisfies $||R\kappa F||_{1,\Gamma} \leq P_0$. From the elliptic theory, the control of $||R\kappa v_t^3(0)||_{3,\Gamma}$ requires the control of $||R\kappa q_{ttt}(0)||_{1,\Gamma}$ and hence $||R'q_{ttt}(0)||_{1,\Gamma}$. Invoking the wave equation (2.6), this is bounded by $||\Delta q(0)||_{1,\Gamma} + ||\mathcal{F}_1||_{1,\Gamma}$. There is no problem to control $||\mathcal{F}_1||_{1,\Gamma}$ by $P_0$ in light of (2.7). Furthermore, invoking the compatibility condition in Section 5, i.e., $q_0 = \sigma\Delta \eta^3_0$, one controls $||\Delta q_{01}||_{1,\Gamma}$ by $||\eta^0_0||_{5.5}$.

The estimates for $||R\kappa v_t(0)||_{2,\Gamma}, ((R\kappa)^{\frac{3}{2}}v_{ttt}(0))||_{1,\Gamma}$ are treated in a similar way, upon time differentiating more times and proceeding as above. We omit the details, but explain the estimates up to the highest order in an expository way. First, to control the tangential component, we use (4.16) and (4.15) to get

$$R'v_t^i(0) \sim \delta^{ij}\partial_jR_t(0) \sim \delta^{ij}\partial_j\partial^\alpha v^\alpha(0),$$

$$R'v_t^i(0) \sim \sqrt{R}\delta^{ij}\partial_jR_{ttt}(0) \sim \sqrt{R}\delta^{ij}\partial_j\Delta q_0 \sim \sqrt{R}\delta^{ij}\partial_j\Delta \eta_0^3,$$

where $\sim$ means up to controllable terms. This yields that

$$||R\kappa v_t^i(0)||_{2,\Gamma}, ((R\kappa)^{\frac{3}{2}}v_{ttt}(0))||_{1,\Gamma}$$

are controlled by $||\text{div} v_0||_{3,\Gamma}$ and $||\eta_0^3||_{6.5}$, respectively. Second, to control the normal component, time differentiating (4.14) two times and restricting at $t = 0$ yields $\nabla v_t^3(0) \sim q_{ttt}(0)$. Therefore, from the elliptic theory, the control of $||R\kappa v_t^3(0)||_{2,\Gamma}$ requires that of $||R\kappa q_{ttt}(0)||_{0,\Gamma}$ and hence $||\Delta q_t(0)||_{0,\Gamma}$, in light of the wave equation. Invoking the compatibility condition $q_t(0) \sim \Delta v_t^3$, $||\Delta q_t(0)||_{0,\Gamma}$ is controlled by $||\Delta \eta_0^3||_{0,\Gamma}$. On the other hand, time differentiating (4.14) three times and restricting at $t = 0$ yields $\nabla v_{ttt}^3(0) \sim q_{tttt}(0)$. Therefore, from the elliptic theory, the control of $||(R\kappa)^{\frac{3}{2}}v_{ttt}^3(0)||_{1,\Gamma}$ requires that of $||(R\kappa)^{\frac{3}{2}}q_{tttt}(0)||_{-1,\Gamma}$ and hence $||\sqrt{R}\Delta q_{ttt}(0)||_{-1,\Gamma}$. Invoking the compatibility conditions $q_{ttt}(0) \sim \Delta \partial^\alpha q_t(0)$ and $q(0) \sim \Delta \eta_0^3$, we have that $||\sqrt{R}\Delta q_{ttt}(0)||_{-1,\Gamma}$ is bounded by $||\eta_0^3||_{6.5}$.

Hence, Theorem 4.1 implies

$$\mathcal{N}(t) \lesssim \epsilon P(\mathcal{N}(t)) + P(\mathcal{N}(0)) + P(\mathcal{N}(t)) \int_0^t P(\mathcal{N}(s)) \, ds.$$

Invoking the continuity-bootstrap argument in [58], this implies that there exists $\mathfrak{M} > 0$ such that

$$\mathcal{N}(t) \leq \mathfrak{M}, \quad \text{whenever } t \in [0, T],$$

(4.24)

for some $T > 0$.

4.4 Passing to the incompressible limit, proof of Theorem 1.2

Proof for statement 1: This is standard since we have an uniform a priori estimate.
Proof for statement 2: The bound (4.24) implies that \(|v_\kappa|_4 + |R_\kappa|_4 \leq \sqrt{M}\) uniformly as \(\kappa \to \infty\). Therefore, by the Sobolev embedding, we have:

\[
\sum_{\ell \leq 2} \left( ||\partial^\ell v_\kappa||_{L^\infty(\Omega)} + ||\partial^\ell R_\kappa||_{L^\infty(\Omega)} \right) \leq \sqrt{M}.
\]

This yields that for each fixed \(t \in [0,T]\), \(v_\kappa\) and \(R_\kappa\) are uniformly bounded and equicontinuous in \(C^2(\Omega)\), which implies the convergence of \(v_\kappa\) and \(R_\kappa\) in \(C^2(\Omega)\). Moreover, \(v_\kappa \to v\) since \(a^{\mu \alpha} \partial_\mu (v_\kappa)_\alpha \to 0\) in \(L^\infty(\Omega)\), which is a consequence of \(||\partial_t q_\kappa||_2\) being bounded independent of \(\kappa\) and \(R'_\kappa \to 0\) as \(\kappa \to \infty\).

5 The initial data

5.1 The compatibility conditions

The compatibility conditions for the initial data are necessary for construction of solutions, as well as for passing the solution to the incompressible limit. We recall that since

\[ q = \sigma g^{ij} \hat{n}_\mu \partial_i^2 \eta^\mu, \quad \text{on } \Gamma, \]

we have:

\[ q|_{t=0} = \left( \sigma g^{ij} \hat{n}_\mu \partial_i^2 \eta^\mu \right)|_{t=0} := H_0, \quad \text{on } \Gamma, \]

which is the zero-th order compatibility condition. In addition, for each \(j \geq 1\), the \(j\)-th order compatibility reads

\[ \partial^j_t q|_{t=0} = \partial^j_t \left( \sigma g^{ij} \hat{n}_\mu \partial_i^2 \eta^\mu \right)|_{t=0} := H_j, \quad \text{on } \Gamma. \] (5.1)

Our goal is to construct \((v_0, q_0)\) that verifies the compatibility condition (5.1) for \(j = 0, 1, 2, 3\). We shall focus on the case when \(\Omega = T^2 \times (0,1)\), whose boundary \(\Gamma\) is flat. Our method can easily be generalized to more general domains.

5.2 Formal construction

We shall describe our method formally which serves as a good guideline for readers. Since

\[ q \sim \overline{\Delta} \eta^3, \quad \text{on } \Gamma, \]

we get

\[ q_t \sim \overline{\Delta} v^3, \quad q_{tt} \sim \overline{\Delta} v^3_t, \quad q_{ttt} \sim \overline{\Delta} v^3_{tt}, \quad \text{on } \Gamma, \]
after taking time derivatives. Moreover, since the Euler equations imply

\[ v_t \sim \partial q, \quad q_t \sim \kappa \operatorname{div} v, \]

we have

\[ q_{tt} \sim \overline{\Delta} \partial_3 q, \quad q_{ttt} \sim \overline{\Delta} \partial_3 q_t \sim \kappa \overline{\Delta} \partial_3 \operatorname{div} v, \quad \text{on } \Gamma. \]
For each $\ell = 0, 1, 2, 3$, we obtain the $\ell$-th order compatibility condition after restricting the above expression at $t = 0$, i.e.,

$$
q|_{t=0} \sim \Delta^3 \eta^0_0, \quad \text{on } \Gamma, \\
q_t|_{t=0} \sim \Delta \nu^3_0, \quad \text{on } \Gamma, \\
q_{tt}|_{t=0} \sim \Delta \partial_3 q_0, \quad \text{on } \Gamma, \\
q_{ttt}|_{t=0} \sim \kappa \Delta \partial_3 \text{div } v, \quad \text{on } \Gamma.
$$

On the other hand, since

$$
q_t \sim \kappa \text{div } v, \quad q_{tt} \sim \kappa \text{div } v_t \sim \kappa \Delta q, \quad q_{ttt} \sim \kappa \Delta q_t \sim \kappa^2 \Delta \text{div } v,
$$

then

$$
q_0 \sim \Delta^3 \eta^0_0, \quad \text{on } \Gamma, \quad \text{(5.2)}
$$

$$
\text{div } v_0 \sim \kappa^{-1} \Delta \nu^3_0, \quad \text{on } \Gamma, \quad \text{(5.3)}
$$

$$
\Delta q_0 \sim \kappa^{-1} \Delta \partial_3 q_0, \quad \text{on } \Gamma, \quad \text{(5.4)}
$$

$$
\Delta \text{div } v_0 \sim \kappa^{-1} \Delta \partial_3 \text{div } v_0, \quad \text{on } \Gamma. \quad \text{(5.5)}
$$

In other words, the first order compatibility condition (i.e., (5.1) when $j = 1$), is expressed in $v_0$, and the second order compatibility condition is expressed in $q_0$, and finally the third order compatibility condition is expressed in $v_0$ again.

To construct initial data that satisfies the compatibility conditions up to order 3, our first step is to obtain $(u_0, p_0)$ that satisfies the (5.2). This is easy, since we can simply let $u_0$ to be velocity for the incompressible case, i.e., $u_0 = u_0$, and $p_0$

$$
-\Delta p_0 = (\partial \cdot u_0^\mu)(\partial \cdot u_0^\mu), \quad \text{in } \Omega, \\
p_0 = \Delta^3 \eta^0_0, \quad \text{on } \Gamma. \quad \text{(5.6)}
$$

Our next step is to construct a velocity vector field $w_0 = (w^1_0, w^2_0, w^3_0)$ that satisfies (5.3). To achieve this, we set $w^1_0 = u^1_0$ and $w^2_0 = u^2_0$, while we define $w^3_0$ via solving

$$
\Delta^2 w^3_0 = \Delta^2 u^3_0, \quad \text{in } \Omega, \\
w^3_0 = u^3_0, \quad \partial_3 w^3_0 \sim \kappa^{-1} \Delta u^3_0 - \partial_1 u^1_0 - \partial_2 u^2_0, \quad \text{on } \Gamma. \quad \text{(5.7)}
$$

We now construct $q_0$ that satisfies (5.4). We define $q_0$ by the solution of

$$
\Delta^3 q_0 = 0, \quad \text{in } \Omega, \\
q_0 = p_0, \quad \partial_3 q_0 = \partial_3 p_0, \quad \Delta q_0 \sim \kappa^{-1} \Delta \partial_3 p_0, \quad \text{on } \Gamma. \quad \text{(5.8)}
$$

Finally, we need to construct $v_0$ using (5.5). To achieve this, we set $v^1_0 = u^1_0$, $v^2_0 = u^2_0$, and we define $v^3_0$ by solving

$$
\Delta^4 v^3_0 = \Delta^4 w^3_0, \quad \text{in } \Omega, \\
v^3_0 = w^3_0, \quad \partial_3 v^3_0 \sim \kappa^{-1} \Delta w^3_0 - \partial_1 w^1_0 - \partial_2 w^2_0, \quad \text{on } \Gamma, \\
\partial_3^2 v^3_0 \sim \kappa^{-1} \partial_3 \Delta w^3_0 - \partial_3 \partial_1 w^1_0 - \partial_3 \partial_2 w^2_0, \quad \text{on } \Gamma, \\
\Delta \partial_3 v^3_0 \sim \kappa^{-1} \Delta \partial_3 \text{div } w_0 - \Delta \partial_1 w^1_0 - \Delta \partial_2 w^2_0, \quad \text{on } \Gamma. \quad \text{(5.9)}
$$
Remark. In fact, $\nabla \eta_0^3 = 0$ on the boundary of the reference domain $T^2 \times (0,1)$. But that we do not use this condition exactly because we want to keep the regularity of each argument as it should hold for the general domain.

**Theorem 5.1.** Let $u_0 \in H^{6.5}(\Omega)$ be a divergence free vector field in $\Omega$ and $p_0$ be the associated pressure. Then there exists initial data $(v_0, q_0) = (v_0^0, q_0^0)$ satisfying the compatibility conditions up to order 3, i.e., (5.2)-(5.5), such that $v_0^\kappa \to u_0$ in $C^2(\Omega)$ and $\text{div} v_0^\kappa \to 0$ in $C^1(\Omega)$ as $\kappa \to \infty$, and $P_0$ is uniformly bounded for all $\kappa$.

**Proof.** $(v_0, q_0)$ verifies (5.2)-(5.5) follows automatically from our construction. Since $p_0$ satisfies the elliptic equation (5.6), for $s \geq 4$, we have:

$$||p_0||_s \lesssim ||\Delta p_0||_{s-2} + ||p_0||_{s-0.5, \Gamma}, \quad (5.10)$$

which requires $||u_0||_{s-1}$ and $||\eta_0||_{s+2}$ to control. Moreover, by the poly-harmonic estimate applied to (5.8) we have:

$$||q_0||_s \lesssim ||\Delta q_0||_{s-2.5, \Gamma} + ||\partial_3 q_0||_{s-1.5, \Gamma} + ||q_0||_{s-0.5, \Gamma} \lesssim \kappa^{-1} ||\Delta \partial_3 p_0||_{s-2} + ||\partial_3 p_0||_{s-1} + ||p_0||_s,$$

Invoking (5.10), this requires $||u_0||_s$ and $||\eta_0||_{s+3}$ to control. On the other hand, invoking (5.7) and the poly-harmonic estimate, we get:

$$||w_0^3||_s \lesssim ||\Delta^2 u_0^3||_{s-4} + ||\partial_3^2 w_0^3||_{s-1.5, \Gamma} + ||w_0^3||_{s-0.5, \Gamma} \lesssim ||\Delta^2 u_0^3||_{s-4} + \kappa^{-1} ||\Delta \partial_3 \text{div} w_0^3||_{s-3} + ||\partial_1 w_0^1||_{s-1} + ||\partial_2 w_0^2||_{s-1} + ||w_0^3||_s,$$

which needs $||u_0^2||_{s+1}$ to control. In addition, since $w_0^3 = u_0^3$, one controls $||w_0||_s$ via $||u_0||_{s+1}$. Moreover, invoking (5.9) and the poly-harmonic estimate, we get:

$$||v_0^3||_s \lesssim ||\Delta^2 u_0^3||_{s-8} + ||\Delta \partial_3 v_0^3||_{s-3.5, \Gamma} + ||\partial_3 v_0^3||_{s-2.5, \Gamma} + ||\partial_3 v_0^3||_{s-1.5, \Gamma} + ||v_0^3||_{s-0.5, \Gamma} \lesssim ||\Delta^2 u_0^3||_{s-8} + \kappa^{-1} ||\Delta \text{div} v_0^3||_{s-3} + ||\partial_1 w_0^1||_{s-1} + ||\partial_2 w_0^2||_{s-1} + ||w_0^3||_s,$$

which requires $||w_0^3||_{s+1}$ and hence $||u_0^3||_{s+2}$ to control. Once again, since $v_0^3 = u_0^3$, one controls $||v_0||_s$ through $||u_0||_{s+2}$.

Next, since (5.7) implies

$$\Delta^2 (w_0^3 - u_0^3) = 0, \quad \Delta^2 (w_0^3 - u_0^3) = 0, \quad \partial_3 (w_0^3 - u_0^3) \sim \kappa^{-1} \Delta u_0^3,$$

we have that $||w_0^3 - u_0^3||_{s+1} \rightarrow 0$ as $\kappa \to \infty$, and hence $w_0 \to u_0$ in $H^s(\Omega)$ as $\kappa \to \infty$. Similarly, (5.9) implies $v_0 \to w_0$ in $H^s(\Omega)$ as $\kappa \to \infty$, and so we conclude that $v_0 \to u_0$ in $H^s(\Omega)$ as $\kappa \to \infty$. Furthermore, because $s \geq 4$ and $v_0$ is uniformly bounded in $H^s$, we have that $v_0 \to u_0$ in $C^2(\Omega)$ thanks to Arzelà-Ascoli and $\text{div} v_0 \to \text{div} u_0 = 0$ in $C^1(\Omega)$.

Finally, we recall that $P_0$ consists

$$||v_0||_4, ||v_0||_{4, \Gamma}, ||q_0||_4, ||q_0||_{4, \Gamma}, ||\text{div} v_0||_{3, \Gamma}, ||\text{div} v_0||_{3, \Gamma}, ||\Delta v_0||_{2, \Gamma},$$

which can all be controlled by $||u_0||_{s+2} = ||u_0||_{s+2}$ and $||\eta_0||_{s+3}$ with $s = 4.5$. \qed
Remark. The initial data constructed in Theorem 5.1 is given in terms of the initial pressure \( q_0 \) instead of the initial density \( R_0 \). This is because the boundary condition is more easily stated in terms of \( q \) and we need to make sure that the quantities \( ||q_0||_4 \) and \( ||q_0||_{4,\Gamma} \) are bounded uniformly in \( \kappa \). But we can compute \( R_0 \) through the equation of states \( R = R(q) \), i.e., \( R_0 = [(c,\kappa)^{-1}q_0 + \beta]^{1/\gamma} \).

The rest of this section is devoted to provide detailed construction, and for the sake of simple expositions, we assume the equation of state is taken to be
\[
q(R) = \kappa(R - 1).
\]
This allows us to exchange \( q \) and \( R \) in an explicit way. Also, throughout the rest of this section, we shall use \( Q \) to denote a rational function.

5.3 Construction for \((u_0, p_0, \Omega)\) that satisfies (5.1) while \( j = 0 \)
Let \( u_0 = v_0 \), where \( v_0 \) is the data for the incompressible Euler equations. Since \( H_0 = \sigma \Delta v_0^3 \), we define \( p_0 \) by solving
\[
\begin{cases}
-\Delta p_0 = (\partial_\nu u_0')(\partial_\nu u_0''), & \text{in } \Omega, \\
p_0 = H_0, & \text{on } \Gamma.
\end{cases}
\]

5.4 Construction for \( w_0 \) that satisfies (5.1) while \( j = 1 \)
We next consider the first order compatibility condition, i.e., \( \partial_t q ||_{t=0} = H_1 \). Since
\[
\partial_t \left( \sigma g^{ij} \hat{n}_\mu \overrightarrow{\partial}_{ij} \eta^\mu \right) = \sigma g^{ij} \hat{n}_\mu \overrightarrow{\partial}_{ij} \nu^\mu + \sigma Q(\hat{n}, \overrightarrow{\partial} \eta, \overrightarrow{\partial} \nu) \overrightarrow{\partial}^2 \eta,
\]
and thus
\[
H_1 = \sigma \Delta v_0^3 + \sigma Q(\overrightarrow{\partial} \eta_0, \overrightarrow{\partial} \nu_0) \overrightarrow{\partial}^2 \eta_0.
\]
On the other hand, since \( \partial_t q = -R\kappa a_\alpha \partial_\mu v_\alpha \), (5.1) with \( j = 1 \) becomes:
\[
\text{div} \, v_0 = \kappa^{-1}(\kappa^{-1}q_0 + 1)H_1, \quad \text{on } \Gamma,
\]
and so
\[
\partial_3 v_0^3 = \kappa^{-1}(\kappa^{-1}q_0 + 1)H_1 - \partial_1 v_0^1 - \partial_2 v_0^2, \quad \text{on } \Gamma.
\]
Furthermore, this suggests that \( w_0 \) should be constructed as follows: let \( w_0 = (u_0^1, u_0^2, w_0^3) \), where \( w_0^3 \) solves
\[
\begin{cases}
\Delta^2 w_0^3 = \Delta^2 u_0^3, & \text{in } \Omega, \\
w_0^3 = u_0^3, & \text{on } \Gamma, \\
\partial_3 w_0^3 = \kappa^{-1} \sigma (\kappa^{-1}p_0 + 1) \Delta u_0^3 - \kappa^{-1} \sigma (\kappa^{-1}p_0 + 1)Q(\overrightarrow{\partial} \eta_0, \overrightarrow{\partial} u_0) \overrightarrow{\partial}^2 \eta_0 - \partial_1 u_0^1 - \partial_2 u_0^2, & \text{on } \Gamma.
\end{cases}
\]

5.5 Construction for \( q_0 \) that satisfies (5.1) while \( j = 2 \)
The second order compatibility condition reads \( \partial_t^2 q ||_{t=0} = H_2 \), and we need to express this in terms of \( \eta_0, v_0 \) and \( q_0 \), which yields a system satisfied by \( p_0 \). Invoking (5.11), we have
\[
\partial_t^2 \left( \sigma g^{ij} \hat{n}_\mu \overrightarrow{\partial}_{ij} \eta^\mu \right) = \sigma g^{ij} \hat{n}_\mu \overrightarrow{\partial}_{ij} \nu^\mu + \sigma Q(\hat{n}, \overrightarrow{\partial} \eta, \overrightarrow{\partial} \nu) \overrightarrow{\partial}^2 v + \sigma Q(\hat{n}, \overrightarrow{\partial} \eta, \overrightarrow{\partial} \nu) \overrightarrow{\partial}^2 \eta (\overrightarrow{\partial} v + 1).
\]
In addition, since $Rv_t^\mu + a^{\mu \nu} \partial_\nu q = 0$, we get for $s = 1, 2$ that

$$\overline{\delta}^s(v_t^\mu) = -R^{-1} a^{\nu \mu} \overline{\delta}^s \partial_\nu q - \sum_{1 \leq k \leq s} \overline{\delta}^k (R^{-1} a^{\nu \mu}) \overline{\delta}^{s-k} \partial_\nu q.$$ 

This, together with (5.12) and the equation of state $R = \kappa^{-1} q + 1$ imply

$$H_2 = H_2(\eta_0, p_0, v_0) = -\sigma (\kappa^{-1} q_0 + 1)^{-1} \overline{\delta}_0 q_0 + \sigma Q(\overline{\delta}_0, \overline{\delta}^2 v_0 \overline{\delta}_0 \overline{\delta}^2 v_0)
-\sigma Q((\kappa^{-1} q_0 + 1)^{-1}, \partial \eta_0, \partial \overline{\delta} \partial \eta_0) \overline{\delta} q_0
-\sigma Q((\kappa^{-1} q_0 + 1)^{-1}, \partial q_0, \kappa^{-1} \overline{\delta}^2 q_0) \overline{\delta} q_0
+\sigma Q((\kappa^{-1} q_0 + 1)^{-1}, \overline{\delta} \eta_0, \overline{\delta} \overline{\delta} \eta_0, \overline{\delta} v_0, \overline{\delta} q_0)(\overline{\delta} \overline{\delta} q_0 + \overline{\delta}^2 \eta_0). \quad (5.13)$$

On the other hand, the continuity equation implies $Ra^{\alpha \beta} \partial_\mu v_\alpha = -\kappa^{-1} \partial_\mu q$, and hence

$$-\kappa^{-1} \partial^2_t q = \partial_t (Ra^{\alpha \beta}) \partial_\mu v_\alpha + Ra^{\alpha \beta} \partial_\mu \partial_\nu v_\alpha = \partial_t (Ra^{\alpha \beta}) \partial_\mu v_\alpha - Ra^{\alpha \beta} \partial_\mu (R^{-1} a^{\nu \alpha} \partial_\nu q)
=-a^{\alpha \beta} a^{\nu \alpha} \partial_\mu \partial_\nu q - Ra^{\alpha \beta} \partial_\mu (R^{-1} a^{\nu \alpha} \partial_\nu q) + \partial_t (Ra^{\alpha \beta}) \partial_\mu v_\alpha. \quad (5.14)$$

Restricting the above identity to the boundary $\Gamma$ and then taking $t = 0$, we get

$$\kappa^{-1} \partial^2_t q|_{t=0} = \Delta q_0 - Q((\kappa^{-1} q_0 + 1)^{-1}, \partial \eta_0, \partial^2 \eta_0, \overline{\delta} v_0, \kappa^{-1} \partial q_0) \partial q_0
+Q(\kappa^{-1} q_0, \partial \eta_0, \partial v_0) \partial v_0. \quad (5.15)$$

Invoking (5.13) and (5.15), we are able to rewrite (5.1) when $j = 2$ as

$$\Delta q_0 = Q((\kappa^{-1} q_0 + 1)^{-1}, \partial \eta_0, \partial^2 \eta_0, \overline{\delta} v_0, \kappa^{-1} \partial q_0) \partial q_0
-Q(\kappa^{-1} q_0, \partial \eta_0, \partial v_0) \partial v_0 + \kappa^{-1} H_2(\eta_0, p_0, v_0). \quad (5.16)$$

This yields that $q_0$ should solve:

$$\begin{cases}
\Delta^3 q_0 = 0, \quad \text{in } \Omega, \\
q_0 = p_0, \quad \text{on } \Gamma, \\
\frac{\partial q_0}{\partial N} = \partial_3 q_0 = \partial_3 p_0 = \frac{\partial p_0}{\partial N}, \quad \text{on } \Gamma, \\
\Delta q_0 = \varphi, \quad \text{on } \Gamma.
\end{cases}$$

Here,

$$\varphi = Q((\kappa^{-1} p_0 + 1)^{-1}, \partial \eta_0, \partial^2 \eta_0, \overline{\delta} w_0, \kappa^{-1} \partial p_0) \partial p_0 - Q(\kappa^{-1} p_0, \partial \eta_0, \partial w_0) \partial w_0 + \kappa^{-1} H_2(\eta_0, p_0, w_0),$$

which is obtained from (5.16).

### 5.6 Construction for $v_0$ that satisfies (5.1) while $j = 3$

Our last step is to construct $v_0$ that satisfies third order compatibility condition, i.e., $\partial^3_t q|_{t=0} = H_3$ on $\Gamma$. Similar to what has been done for the previous cases when $j = 0, 1, 2$, we shall first compute
the compatibility condition explicitly. Invoking (5.12), as well as \( v^\mu_t = -R^{-1}a^{\nu\mu}\partial_\nu q \) and \( \partial q = -\kappa Ra^{\mu\alpha}\partial_\mu v_\alpha \), we have

\[
\partial_t^2 \left( \sigma (g^{ij}\dot{v}_i\dot{v}_j)^\mu \right) = -\partial_t \left( \sigma (g^{ij}\dot{v}_i(R^{-1}a^{\nu\mu}\partial_\nu q)) \right) + \sigma Q(\dot{v}, \partial q, v) + \sigma Q(g, \dot{v}, \partial v, \dot{v})\partial^2 v + \sigma Q(\dot{v}, \partial q, \partial v) (\partial v t + \partial^2 \eta) = -\kappa g^{ij}\dot{v}_i\partial_t (R^{-1}a^{\nu\mu}\partial_\nu) + \sigma Q(\dot{v}, \partial q, v)\partial^2 (R^{-1}a^{\nu\mu}\partial_\nu) q + \sigma Q(\dot{v}, \partial q, \partial v, \partial q, \partial^2 q) \partial q + \sigma Q(\dot{v}, \partial q, \partial v, \partial^2 q) \partial^2 q)
\]

where

\[
\sigma g^{ij}\dot{v}_i\partial_t (R^{-1}a^{\nu\mu}\partial_\nu) = \sigma g^{ij}\dot{v}_i R^{-1}a^{\nu\mu}\partial_\nu \partial^2 q
\]

Restricting (5.17) and (5.18) at \( t = 0 \), we get

\[
H_3 = H_3(\eta_0, q_0, v_0) = -\kappa \sigma \partial_3 \sum \text{div} v_0 - \sigma \sum_{\ell = 1, 2, 3} (\partial^\ell q_0)(\partial^{3-\ell} \text{div} v_0) + \kappa \sigma Q((\kappa^{-1} q_0 + 1)^{-1}, \partial v, \partial^2 v, \partial^3 v, \partial^4 v, \partial^5 v, \partial^6 v, \partial^7 v, \partial^2 q) \sum_{\ell = 0, 1} \partial^\ell \text{div} v_0.
\]

Next, invoking (5.14), we obtain

\[
\kappa^{-1} q_{ttt} = \partial_t \left( a^{\mu\alpha}a^\nu_\alpha \partial_\mu \partial_\nu q + Ra^{\mu\alpha} \partial_\mu (R^{-1}a^\nu_\alpha) \partial_\nu q - \partial_t (Ra^{\mu\alpha}) \partial_\mu v_\alpha \right) = a^{\mu\alpha}a^\nu_\alpha \partial_\mu \partial_\nu q + Ra^{\mu\alpha} \partial_\mu (R^{-1}a^\nu_\alpha) \partial_\nu q + Q(R, R, R, R, \partial^2 q) \partial q = -R Ka^{\mu\alpha}a^\nu_\alpha \partial_\mu \partial_\nu (a^\beta_\gamma \partial^\beta_\nu) - 2K a^{\mu\alpha}a^\nu_\alpha \partial_\mu (a^\beta_\gamma \partial^\beta_\nu)
\]

Restricting (5.20) to the boundary \( \Gamma \) and then taking \( t = 0 \), we have

\[
\kappa^{-1} q_{ttt}|_{t=0} = -\kappa R_0 \Delta \text{div} v_0 - \sum_{\ell = 1, 2} 2(\partial^\ell q_0)(\partial^{2-\ell} \text{div} v_0) + Q((\kappa^{-1} q_0 + 1)^{-1}, \kappa^{-1} q_0, \partial q, \partial^2 q) \sum_{\ell = 0, 1} \partial^\ell \text{div} v_0 + Q((\kappa^{-1} q_0 + 1)^{-1}, \kappa^{-1} q_0, v_0, \partial v, \partial^2 \eta_0) \partial^2 q.
\]

Invoking (5.19), the compatibility condition \( q_{ttl}|_{t=0} = H_3 \) can then be re-expressed as

\[
\Delta \text{div} v_0 = \psi(\eta_0, q_0, v_0)
\]
where
\[ \psi(\eta_0, q_0, v_0) = -\kappa^{-1}(\kappa^{-1}q_0 + 1) \sum_{\ell=1}^{2} \left( \partial^\ell q_0 \right) (\partial^{2-\ell} \text{div} v_0) + \kappa^{-1}Q \left( (\kappa^{-1}q_0 + 1)^{-1}, \kappa^{-1}q_0, \partial v_0, \partial \eta_0, \partial^2 \eta_0 \right) \sum_{\ell=0} Q \partial^\ell \text{div} v_0 + \kappa^{-1}Q \left( (\kappa^{-1}q_0 + 1)^{-1}, \kappa^{-1}q_0, v_0, \partial v_0, \partial \eta_0, \partial^2 \eta_0 \right) \partial^2 q_0 - \kappa^{-2}(\kappa^{-1}q_0 + 1)^{-1} H_3(\eta_0, q_0, v_0). \]

This implies that \( v_0 = (v_0^1, v_0^2, v_0^3) \) should be constructed such that \( v_0^1 = u_0^1 \) and \( v_0^2 = u_0^2 \), whereas \( v_0^3 \) solves
\[
\begin{align*}
    \Delta^4 v_0^3 &= \Delta^4 w_0^3, \quad \text{in } \Omega, \\
    v_0^3 &= w_0^3, \quad \text{on } \Gamma, \\
    \partial_3 v_0^3 &= \kappa^{-1} \sigma (\kappa^{-1}q_0 + 1) \Delta w_0^3 - \kappa^{-1} \sigma (\kappa^{-1}q_0 + 1) Q (\overline{\partial} \eta_0, \overline{\partial} w_0) \overline{\partial^2 \eta_0} - \partial_1 w_0^1 - \partial_2 w_0^2, \quad \text{on } \Gamma, \\
    \partial_2^2 v_0^3 &= \partial_2 \left( \kappa^{-1} \sigma (\kappa^{-1}q_0 + 1) \Delta w_0^3 - \kappa^{-1} \sigma (\kappa^{-1}q_0 + 1) Q (\overline{\partial} \eta_0, \overline{\partial} w_0) \overline{\partial^2 \eta_0} - \partial_1 w_0^1 - \partial_2 w_0^2 \right), \quad \text{on } \Gamma, \\
    \Delta \partial_3 v_0^3 &= \psi(\eta_0, q_0, u_0) - \Delta \partial_1 w_0^1 - \Delta \partial_2 w_0^2, \quad \text{on } \Gamma.
\end{align*}
\]

Appendix

A Basic estimates

**Theorem A.1.** (Standard div-curl estimates) Let \( X \) be a vector field on \( \Omega \) with sufficiently regular boundary \( \Gamma \). Define \( \text{div} X = \partial_j X^j \) and \( (\text{curl} X)_{ij} = \partial_i X_j - \partial_j X_i \), then for \( 1 \leq s \leq 4 \), we have
\[
\begin{align*}
|X|_s &\lesssim |\text{div} X|_{s-1} + |\text{curl} X|_{s-1} + |X \cdot N|_{s-0.5} + \|X\|_0, \quad (A.1) \\
|X|_s &\lesssim |\text{div} X|_{s-1} + |\text{curl} X|_{s-1} + |X \cdot T|_{s-0.5} + \|X\|_0, \quad (A.2)
\end{align*}
\]
where \( N \) is the outward unit normal to \( \Gamma \), whereas \( T \) is the unit vector which is tangent to \( \Gamma \).

**Proof.** We refer [47] for the detailed proof.

B The energy identity for the wave equations of order 3

We recall that for \( r = 1, 2, 3 \), the wave equation reads:
\[ JR' \partial_t^{r+1} q - a^{\nu \alpha} A^\mu_\alpha \partial_\nu \partial_\mu \partial_t^{r-1} q = G_r + S_r, \]
where
\[
\begin{align*}
G_r &= - \sum_{j_1+j_2=r} \left( \partial_t^{j_1} (JR') \right) (\partial_t^{j_2+1} q) + a^{\nu \alpha} (\partial_\nu \rho_0) \partial_t^{j_1} v_\alpha \\
&\quad + \sum_{j_1+j_2=r-1} a^{\nu \alpha} \partial_\nu (\partial_t^{j_1} A^\mu_\alpha \partial_\mu \partial_t^{j_2} q) - \rho_0 \sum_{j_1+j_2=r-1} (\partial_t^{j_1+1} a^{\nu \alpha}) (\partial_t^{j_2} \partial_\nu v_\alpha).
\end{align*}
\]
and
\[ S_r = a^{\nu \alpha} (\partial_\nu A^\mu_\alpha) \partial_\mu \partial_t^{r-1} q. \]
Theorem B.1. For \( r = 1, 2, 3, \) let
\[
W_r^2 = \frac{1}{2} \int_\Omega \rho_0^{-1} (JR' \partial_t^3 q)^2 \, dy + \frac{1}{2} \int_\Omega \rho_0^{-1} R' (A^{\nu\alpha} \partial_\nu \partial_\mu \partial_\mu \partial_t^2 q) \, dy + \frac{\sigma}{2} \int_\Gamma \mathbb{R}_\nu \sqrt{g} g^{ij} \Pi_\mu (\partial_i \partial_j^t \eta^\mu) (\partial_j \partial_t^t \eta_\alpha) \, dS.
\]
Then,
\[
\sum_{r \leq 3} W_r^2 \leq \epsilon P(\mathcal{N}) + \epsilon(||q||_2^2 + ||q_e||_2^2) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad t \in [0, T],
\]
where \( T > 0 \) is sufficiently small.

Proof of Theorem B.1  It suffices to consider the case when \( r = 3 \). Invoking (1.4) and (1.11), we have:
\[
\frac{d}{dt} \frac{1}{2} \int_\Omega \rho_0^{-1} (JR' \partial_t^3 q)^2 \, dy = \int_\Omega \rho_0^{-1} (JR' \partial_t^3 q) (a^{\nu\alpha} A_\nu^\alpha \partial_\nu \partial_\mu \partial_\mu \partial_t^2 q) \, dy + \int_\Omega \rho_0^{-1} (JR' \partial_t^3 q) (\mathcal{G}_3 + \mathcal{S}_3) \, dy + \mathcal{R}, \tag{B.1}
\]
where \( \mathcal{R} \) consists of error terms that are generated when \( \partial_t \) falls on either \( J \) or \( R' \), which we have no problem to control. In addition,
\[
\int_\Omega \rho_0^{-1} (JR' \partial_t^3 q) (a^{\nu\alpha} A_\nu^\alpha \partial_\nu \partial_\mu \partial_\mu \partial_t^2 q) \, dy
\]
\[
= \int_\Omega \rho_0^{-1} (R' \partial_t^3 q) (A^{\nu\alpha} \partial_\nu) (A_\nu^\alpha \partial_\mu \partial_\mu \partial_t^2 q) \, dy - \int_\Omega \rho_0^{-1} (JR' \partial_t^3 q) \mathcal{S}_3. \tag{B.2}
\]
The last term in (B.2) cancels with the corresponding term in (B.1), which is essential since \( ||\mathcal{S}_3||_0 \) cannot be controlled uniformly when \( R' \rightarrow 0 \). Moreover, the first term on the right hand side of (B.2) is treated as:
\[
\int_\Omega \rho_0^{-1} (R' \partial_t^3 q) (A^{\nu\alpha} \partial_\nu) (A_\nu^\alpha \partial_\mu \partial_\mu \partial_t^2 q) \, dy = - \int_\Omega \rho_0^{-1} R' (A^{\nu\alpha} \partial_\nu \partial_\mu \partial_\mu \partial_t^2 q) \, dy + \int_\Gamma \rho_0^{-1} R' (A^{\nu\alpha} N_\nu \partial_t^2 q) (A_\nu^\alpha \partial_\mu \partial_\mu \partial_t^2 q) \, dS + \mathcal{R}. \tag{B.3}
\]
The first term on the right hand side of (B.3) is equal to
\[
- \frac{d}{dt} \frac{1}{2} \int_\Omega \rho_0^{-1} R' (A^{\nu\alpha} \partial_\nu \partial_\mu \partial_\mu \partial_t^2 q) \, dy + \mathcal{R}
\]
and hence moved to the left. In addition,
\[
\int_\Gamma \rho_0^{-1} R' (A^{\nu\alpha} N_\nu \partial_t^2 q) (A_\nu^\alpha \partial_\mu \partial_\mu \partial_t^2 q) \, dS = \int_\Gamma \rho_0^{-1} R' \partial_t^3 (A^{\nu\alpha} N_\nu q) \partial_\mu \partial_\mu \partial_t^2 q + \mathcal{W}_B_1
\]
\[
- \int_\Gamma \rho_0^{-1} R' \partial_t^3 (A^{\nu\alpha} N_\nu q) (\partial_t A_\alpha^\mu) (\partial_\mu \partial_\mu q) - \sum_{j_1 + j_2 = 3, j_1 \geq 1} \int_\Gamma \rho_0^{-1} R' \partial_t^2 (A_\nu^\alpha \partial_\mu \partial_\mu q) (\partial_t^{j_1} A^{\nu\alpha}) (N_\nu \partial_t^{j_2} q) + \mathcal{W}_B_2
\]
\[
+ \sum_{j_1 + j_2 = 3, j_1 \geq 1} \int_\Gamma \rho_0^{-1} R' (\partial_t A_\alpha^\mu) (\partial_\mu \partial_\mu q) (\partial_t^{j_1} A^{\nu\alpha}) (N_\nu \partial_t^{j_2} q), \tag{W_B_3}
\]
51
which is due to

\[ A^{\alpha} N_{\nu} \partial_t^3 q = \partial_t^3 (A^{\alpha} N_{\nu} q) - \sum_{j_1+j_2 = 3} \sum_{j_1 \geq 1} \frac{1}{3} \partial_t^{j_1} A^{\alpha_1} N_{\nu} \partial_t^{j_2} q, \]

\[ A^{\mu} \partial_\mu \partial_t^2 q = \partial_t^2 (A^{\mu} \partial_\mu q) - (\partial_t A^{\mu}) \partial_\mu \partial_t q. \]

Next, invoking (1.6), (1.10) and (2.4), the main boundary term is equal to

\[ \sigma \int \Gamma \bar{R}_\kappa \sqrt{g} g^{ij} \Pi_{\mu}^\alpha (\partial_t^3 \bar{\nabla}^j \eta^\mu) (\partial_t \bar{\nabla}^j \eta^\alpha) + \sigma \sum_{j_1+j_2 = 3} \sum_{j_1 \geq 1} \frac{1}{3} \int \Gamma \bar{R}_\kappa (\partial_t^{j_1} \sqrt{g} g^{ij} \Pi_{\mu}^\alpha) (\partial_t^{j_2} \bar{\nabla}^j \eta^\mu) (\partial_t^{j_2} \bar{\nabla}^j \eta^\alpha), \]

where the main term is moved to the left, and this completes the construction for (2.8).

The proof of Theorem B.1 requires the bound for \( \int_0^t |G_3||\| \) and \( \sum_{1 \leq j \leq 6} \int_0^t \mathcal{W} B_j \). There is no problem to control \( \int_0^t |G_3||\| \). In addition, using the duality, we have:

\[ \mathcal{W} B_1 \lesssim P(\|v\|_3, ||\eta||_3) ||R' \partial_t^2 (A^{\alpha} q)||_0 ||\eta q||_2, \]

and

\[ \mathcal{W} B_2 \lesssim P(\|v\|_3, ||\eta||_3) \left( ||(\sqrt{R'} \partial_t^2 (A^{\alpha} \partial_\mu q))||_0 (\sqrt{R'} \partial_t \eta^\mu)||_0 ||q||_2 \right) + ||(\sqrt{R'} \partial_t^2 (A^{\alpha} \partial_\mu q))||_0 ||\eta \partial t q||_2 + ||(\sqrt{R'} \partial_t^3 (A^{\alpha} \partial_\mu q))||_0 ||\eta||_1. \]

Therefore, \( \int_0^t \mathcal{W} B_1 + \mathcal{W} B_2 \) can be controlled appropriately. Moreover, \( \int_0^t \mathcal{W} B_3 + \mathcal{W} B_5 \) is controlled in a routine fashion. On the other hand, \( \int_0^t \mathcal{W} B_4 + \mathcal{W} B_6 \) is treated in [21], where the \( \bar{R}_\kappa \)-weight is incorporated so that the estimates in [21] can go through.

C The energy identity for \( \bar{R}_\kappa \)-weighted wave equations

We recall that the \( \bar{R}_\kappa \)-weighted wave equation reads:

\[ \bar{R}_\kappa^\ell R' J D^3 \partial_t^2 q - \bar{R}_\kappa^\ell a^\nu \alpha A^{\mu} \partial_\nu \partial_\mu D^3 q = \bar{G}_4 + \bar{S}_4, \]

where

\[ \bar{G}_4 = -\bar{R}_\kappa^\ell [D^3 \partial_t, J R'] \partial_t q + \bar{R}_\kappa^\ell [D^3, \rho_0] \partial_t (R^{-1} R' \partial_t q) + \bar{R}_\kappa^\ell a^\nu \alpha (\partial_\nu \rho_0) D^3 \partial_t v_\alpha \]

\[ + \bar{R}_\kappa^\ell a^\nu \alpha \partial_\nu ([D^3, A^{\mu}] \partial_\mu q) + \bar{R}_\kappa^\ell a^\nu \alpha (\partial_\nu ([D^3, \rho_0] \partial_\nu q) - \bar{R}_\kappa^\ell \rho_0 [D^3 \partial_\nu, a^\nu \alpha] \partial_\nu q, \]

and

\[ \bar{S}_4 = \bar{R}_\kappa^\ell a^\nu \alpha (\partial_\nu A^{\mu}) \partial_\mu D^3 q. \]

Here, \( \ell = 1 \) when \( D^3 = \partial_t^3 \), \( \ell = \frac{1}{2} \) when \( D^3 = \partial_t^2 \bar{\nabla} \), and \( \ell = 0 \) when \( D^3 = \partial_t \bar{\nabla}^2 \).
Theorem C.1. Let
\[
W_4^2 = \frac{1}{2} \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q)^2 \, dy + \frac{1}{2} \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' (A^{\nu_\alpha} \partial_\nu D^3 q) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dy \\
+ \frac{\sigma}{2} \int_\Gamma \mathcal{R}_\kappa^{2 \ell+1} \sqrt{g} g^{ij} \Pi_\mu^\alpha \left( \partial_i D^3 \partial_t \eta^\mu \right) \left( \partial_j D^3 \partial_t \eta^\alpha \right) \, dS.
\]
Then,
\[
W_4^2 \leq \epsilon P(N) + \mathcal{P}_0 + \mathcal{P} \int_0^t \mathcal{P}, \quad t \in [0, T],
\]
where \(T > 0\) is sufficiently small.

Proof of Theorem C.1 Invoking (1.4) and (1.11), we have:
\[
\frac{d}{dt} \frac{1}{2} \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q)^2 \, dy = \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q) (a^{\nu_\alpha} A^{\mu_\alpha} \partial_\nu \partial_\mu D^3 q) \, dy \\
+ \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q) (\tilde{G}_4 + \tilde{S}_4) \, dy + \mathcal{R}, \quad \text{(C.1)}
\]
where \(\mathcal{R}\) consists error terms that are generated when \(\partial_t\) falls on either \(J\) or \(R'\), which we have no problem to control. In addition,
\[
\int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q) (a^{\nu_\alpha} A^{\mu_\alpha} \partial_\nu \partial_\mu D^3 q) \, dy \\
= \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (R'D^3 \partial_t q) (A^{\nu_\alpha} \partial_\nu) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dy - \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (JR'D^3 \partial_t q) \tilde{S}_4, \quad \text{(C.2)}
\]
The last term in (C.2) cancels with the corresponding term in (C.1), which is essential since \(\|\tilde{S}_3\|_0\) cannot be controlled uniformly when \(R' \to 0\). Moreover, the first term on the right hand side of (C.2) is treated as:
\[
\int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} (R'D^3 \partial_t q) (A^{\nu_\alpha} \partial_\nu) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dy = - \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' (A^{\nu_\alpha} \partial_\nu D^3 \partial_t q) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dy \\
+ \int_\Gamma \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' (A^{\nu_\alpha} N_\nu D^3 \partial_t q) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dS + \mathcal{R} \quad \text{(C.3)}
\]
The first term on the right hand side of (C.3) is equal to
\[
- \frac{d}{dt} \frac{1}{2} \int_\Omega \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' (A^{\nu_\alpha} \partial_\nu D^3 q) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dy + \mathcal{R}
\]
and hence moved to the left. In addition,
\[
\int_\Gamma \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' (A^{\nu_\alpha} N_\nu D^3 \partial_t q) (A^{\mu_\alpha} \partial_\mu D^3 q) \, dS = \int_\Gamma \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' D^3 \partial_t (A^{\nu_\alpha} N_\nu q) D^3 (A^{\mu_\alpha} \partial_\mu q) \, dS \\
- \int_\Gamma \rho_0^{-1} \mathcal{R}_\kappa^{2 \ell} R' D^3 \partial_t (A^{\nu_\alpha} N_\nu q) ([D^3, A^{\mu_\alpha}] \partial_\mu q) - \int_\Gamma \rho_0^{-1} R' \mathcal{R}_\kappa^{2 \ell} D^3 (A^{\mu_\alpha} \partial_\mu q) ([D^3 \partial_t, A^{\nu_\alpha}] N_\nu q)
\]

\[
+ \int_\Gamma \rho_0^{-1} R' \mathcal{R}_\kappa^{2 \ell} ([D^3 \partial_t, A^{\nu_\alpha}] N_\nu q) ([D^3, A^{\mu_\alpha}] \partial_\mu q),
\]

\[
53
\]
which is due to
\[ A^{\mu \nu} N_\mu D^3 \partial_\nu q = D^3 \partial_\nu (A^{\mu \nu} N_\nu q) - [D^3 \partial_\nu, A^{\mu \nu}] N_\nu q, \]
\[ A^{\mu \nu} D^3 q = D^3 (A^\mu_\nu \partial_\mu q) - [D^3, A^\mu_\nu] \partial_\mu q. \]

Next, invoking (1.6), (1.10) and (2.4), the main boundary term is equal to
\[
\sigma \int_\Gamma R^2 \ell + \kappa \sqrt{g} g^{ij} \Pi^\alpha_\mu (D^3 \partial_i \eta^\mu) (D^3 \partial_j \delta q) + \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\mu (D^3 \partial_i \eta^\mu) (D^3 \partial_j \delta q) + R + \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\mu (D^3 \partial_i \eta^\mu) (D^3 \partial_j \delta q) + R.
\]

The first term on the last line is equal to
\[
-\frac{d}{dt} \sigma \int_\Gamma R^{2\ell+1} \sqrt{g} g^{ij} \Pi^\alpha_\mu (D^3 \partial_i \eta^\mu) (D^3 \partial_j \delta q) + \sigma \int_\Gamma R^{2\ell+1} \sqrt{g} g^{ij} \Pi^\alpha_\mu (D^3 \partial_i \eta^\mu) (D^3 \partial_j \delta q) + R.
\]

where the main term is moved to the left, and this completes the construction for (2.8).

The proof of Theorem C.1 requires the bound for \( \int_0^t ||\tilde{G}_4||_0 \) and \( \sum_{1 \leq j \leq 6} \int_0^t ||\tilde{W}B_j|| \). First, \( \int_0^t ||\tilde{W}B_1 + \tilde{W}B_2 \) can be controlled similar to \( \int_0^t WB_1 + WB_2 \) in the previous section, after distributing correct \( \kappa \)-weight. Second, the control of \( \int_0^t ||\tilde{G}_4||_0 \) and \( \int_0^t \tilde{W}B_3 \) can be done in a routine fashion. Finally, \( \int_0^t \tilde{W}B_4 + \tilde{W}B_5 + \tilde{W}B_6 \) is treated similar to \( \int_0^t B \) in Section 3.4.

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