ON THE SPECTRUM OF MARKOV SEMIGROUPS 
VIA SAMPLE PATH LARGE DEVIATIONS.

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Abstract. The essential spectral radius of a sub-Markovian process is defined 
as the infimum of the spectral radiuses of all local perturbations of the process. 
When the family of rescaled processes satisfies sample path large deviation 
principle, the spectral radius and the essential spectral radius are expressed 
in terms of the rate function. The paper is motivated by applications to 
reflected diffusions and jump Markov processes describing stochastic networks 
for which the sample path large deviation principle has been established and 
the rate function has been identified while essential spectral radius has not 
been calculated.

1. Introduction

For a sub-Markovian process \((X(t))\) on a locally compact set \(E\) endowed with a 
non-negative Radon measure \(m\), spectral radius \(r^*\) is defined as the infimum over 
all those \(r > 0\) for which the resolvent function

\[ R_r 1_W(x) = \int_0^\infty r^{-t} \mathbb{P}_x(X(t) \in W) \, dt \]

is \(m\)-integrable on compact subsets of \(E\) for every compact set \(W \subset E\). Under 
some general assumptions, the quantity \(r^*\) can be described in several ways:

(i) For an irreducible discrete time Markov chain \((X(n))\) on a countable set 
\(E\), \(1/r^*\) is a common radius of convergence of the series

\[ \sum_{n=1}^\infty z^n p(n, x, y), \quad x, y \in E, \]

where \(\{p(n, x, y)\}, x, y \in E\) are the transition probabilities of \((X(n))\) (see 
Seneta [14] and Vere-Jones [17]).

(ii) \(\log r^* = \sup_{W, V} \lim_{t \to +\infty} \frac{1}{t} \log m(1_W \mathbb{P}^t 1_V)\)

where the supremum is taken over all compact subsets \(V, W \subset E\) and 
\(\{\mathbb{P}^t, t > 0\}\) is the sub-Markovian semi-group associated to \((X(t))\).

(iii) \(r^*\) is the infimum over all \(r > 0\) for which there is a positive measurable 
function \(f\) which is \(m\)-integrable on compact subsets of \(E\) and such that 
\(\mathbb{P}^t f \leq r^* f\) for all \(t > 0\). A function \(f\) satisfying the inequality 
\(\mathbb{P}^t f \leq r^* f\) is usually called \(r\)-superharmonic. A dual description of \(r^*\) can be given by

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using $r$-superharmonic Radon measures on $E$ (see Seneta [14], Stroock [16] and Vere-Jones [17] for example).

(iv) When the sub-Markovian semi-group $\{P^t, t > 0\}$ is generated by a symmetric linear operator $A$ in $L^2(E, m)$, the value $-\log r^*$ is the bottom of the $L^2$-spectrum of $A$ (see Stroock [16] and also Liming Wu [21] for a similar result for discrete time Markov chains).

(v) \[ r^* = \sup_K \{ r > 0 : \mathbb{E}(r^{-\tau_K}) \in L^1(K, m) \} \]

where the supremum is taken over all compact subsets $K \subset E$ and for every compact set $K$, $\tau_K$ denotes the first exit time from the set $K$ (see Stroock [16]).

The last description of the quantity $r^*$ shows that $r^*$ provides the rate at which the process $(X(t))$ leaves compact sets. This quantity is of interest for transient Markov processes, because it shows how fast the process goes to infinity.

Because of the first property, the value $1/r^*$ is usually called convergence parameter of $(X(t))$. In the present paper, Woess’s terminology is used: we call the quantity $r^*$ spectral radius of the process $(X(t))$, see Woess’s book [20]. While this terminology may be misleading (in a non-symmetrical case, the correspondence between the value $r^*$ and operator properties of $\{P^t, t > 0\}$ is more intricate, see Vere-Jones [18]), it stresses the importance of the correspondence between the value $r^*$ and the spectral radius in $L^2(E, m)$ of a symmetric case.

The value $r^*$ is clearly sensitive to changing the transition probabilities on compacts. The quantity $r^*_e$ is defined as the infimum of $r^*$ over all such changes:

\[ r^*_e = \inf_K r^*_K \]

$r^*_K$ denotes here spectral radius of the sub-Markovian process $(X(t))$ killed on the set $K$ and the infimum is taken over all compact subsets $K \subset E$.

For symmetric Markov processes, by Persson’s principle (see Grillo [6] for symmetric diffusions and Liming Wu [21] for symmetric Markov chains), the quantity $r^*_e$ is related to the $L^2$-essential spectral radius of the corresponding Markov semi-group. It is of interest for recurrent Markov processes: given a compact set $K \subset E$, let $\tau(K)$ denote the first hitting time of $K$, then under some general assumptions, the number $r^*_e$ equals the infimum over all those $r > 0$ for which the function

\[ R_{K,r}1(x) = \int_0^\infty r^{-t}P_x(\tau(K) > t) \, dt \]

is $m$-integrable on compact subsets of $E \setminus K$ (see Proposition 3.6 below). The quantity $r^*_e$ provides therefore the rate at which the process returns to compacts.

For some positive recurrent countable Markov chains, the quantity $r^*_e$ gives an accurate bound to the rate of convergence to equilibrium: Malyshev and Spieksma [11] have shown that this is the best geometric convergence rate when the transitions of the Markov chain are changed on a finite subset of $E$. For more details concerning a relationship between the quantity $r^*_e$ and the rate of convergence to equilibrium see Liming Wu [21].

Unfortunately, in practice, outside of some particular examples, an explicit representation of $r^*_e$ is very difficult to obtain.

\[ ^1 \text{For the definition of convergence parameter for general state space Markov chains, see Nummelin [12].} \]
In the present paper, we consider a Markov process \((X(t))\) on \(\mathbb{R}^d\) for which the family of rescaled processes \((Z_a(t) = X(at)/a, \ t \in [0, 1])\) satisfies sample path large deviation principle with a good rate function \(I_{[0,1]}\). The quantities \(r^*\) and \(r^*_e\) are represented in terms of the rate function: we show that

\[
\log r^* = - \inf_{\phi: \phi(0) = \phi(1), \phi(t) \neq 0, \ \forall \ 0 < t < 1} I_{[0,1]}(\phi) \quad \text{and} \quad \log r^*_e = - \inf_{\phi(0) = \phi(1), \phi(t) \neq 0, \ \forall \ 0 < t < 1} I_{[0,1]}(\phi)
\]

where the first infimum is taken over all continuous functions \(\phi\) with \(\phi(0) = \phi(1)\) and the second infimum is taken over all continuous functions \(\phi\) with \(\phi(0) = \phi(1)\) such that \(\phi(t) \neq 0\) for all \(0 < t < 1\).

The first result in this domain was obtained by Malyshev and Spieksma \cite{MalyshevSpieksma} for discrete time partially homogeneous random walks in \(\mathbb{N}\) and in \(\mathbb{Z}^2\). Unfortunately, their proofs use particular properties of the processes and it is not usually possible to extend them to a more general situation (see section 2 for more details).

Our results are motivated by applications to reflected diffusions considered by Varadhan and Williams \cite{VaradhanWilliams} and jump Markov processes describing stochastic networks. For these processes, the sample paths large deviation principle has been established and an explicit representation of the corresponding rate function has been obtained (see \cite{MalyshevSpieksma, Orsingher, FerrariVares, Horvath, Last} for example) while the essential spectral radius \(r^*_e\) has not been identified.

An example of Jackson networks illustrates our results. Using Relation 1 we obtain an explicit representation of the quantities \(r^*\) and \(r^*_e\) for two-dimensional Jackson networks. In the forthcoming paper, we apply our results for reflected diffusions in \(\mathbb{R}^2_+\).

2. The main results

We consider a strong Markov process \((X(t))\) on \(E \subset \mathbb{R}^d\) whose sample paths are right continuous with left limits. The set \(E\) is endowed by a non-negative Radon measure \(m\). We assume that the set \(E\) is closed and unbounded and that \(m(O) \neq 0\) for every open non-empty subset \(O \subset E\). Given a closed set \(V \subset E\), \(\tau(V)\) denotes the hitting time of \(V\):

\[\tau(V) = \inf\{t > 0 : X(t) \in V\},\]

by convention \(\tau(\emptyset) = +\infty\). It is assumed that for every real bounded measurable function \(\varphi\) on \(E\), the mapping

\[(t, x) \mapsto \mathbb{E}_x(\varphi(X(t)); \ \tau(V) > t)\]

is \(\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(E)\)-measurable from \(\mathbb{R}_+ \times E\) to \(\mathbb{R}\).

**Definition 2.1.** Spectral radius \(r^*\) is the infimum of all those \(r > 0\) for which the resolvent function

\[R_r 1_W(x) = \int_0^\infty r^{-t} \mathbb{P}_x(X(t) \in W) \, dt\]

is \(m\)-integrable on compact subsets of \(E\) for every compact subset \(W \subset E\).

**Definition 2.2.** Essential spectral radius \(r^*_e\) is the infimum of all those \(r > 0\) for which there is a compact set \(K \subset E\) such that the truncated resolvent function

\[R_{K,r} 1_W(x) = \int_0^\infty r^{-t} \mathbb{P}_x(X(t) \in W, \ \tau(K) > t) \, dt\]
is $m$-integrable on compact subsets of $E \setminus K$ for every compact subset $W \subset E \setminus K$.

The Markov process $(X(t))$ is assumed to satisfy the following large deviation conditions.

**Assumption (A) : Large deviations.** Let $E$ be the set of all possible limits $\lim_{n \to \infty} x_n/a$ with $x_n \in E$, and let $D([0, T], E)$ denote the Skorohod space of all functions $\phi$ from $[0, T]$ to $E$ which are right continuous and have left limits. We assume that

(a$_0$) $E \subset E \neq \{0\}$ and the set $E \setminus \{0\}$ is convex;
(a$_1$) for every $T > 0$, the family of rescaled processes

$$(Z_n(t), t \in [0, T]) \overset{\text{def}}{=} (X(at)/a, t \in [0, T])$$

satisfies sample path large deviation principle in $D([0, T], E)$ with a good rate functions $I_{[0,T]}$ (see section 3 for a precise definition);
(a$_2$) the rate function $I_{[0,T]}$ has an integral form: there is a local rate function $L : E \times \mathbb{R}^d \to \mathbb{R}_+$ such that

$$I_{[0,T]}(\phi) = \int_0^T L(\phi(t), \dot{\phi}(t)) \, dt$$

if the function $\phi : [0, 1] \to E$ is absolutely continuous, and $I_{[0,1]}(\phi) = +\infty$ otherwise.
(a$_3$) there are two convex functions $l_1$ and $l_2$ on $\mathbb{R}^d$ such that
- $0 \leq l_1(v) \leq L(x, v) \leq l_2(v)$ for all $x \in E$ and for all $v \in \mathbb{R}^d$,
- the function $l_2$ is finite in a neighborhood of zero
- and

$$\lim_{n \to \infty} \inf_{|v| \geq n} l_1(v)/|v| > 0.$$

For $x, y \in E$ and $t > 0$, we denote by $I(t, x, y)$ the infimum of the rate function $I_{[0,t]}(\phi)$ over all continuous functions $\phi : [0, t] \to E$ with $\phi(0) = x$ and $\phi(t) = y$, $\hat{I}(t, x, y)$ denotes the infimum of $I_{[0,t]}(\phi)$ over all continuous functions $\phi : [0, t] \to E$ for which $\phi(0) = x$, $\phi(t) = y$ and the set $\{ s \in [0, t] : \phi(s) = 0 \}$ has Lebesgue measure zero, $I_0(t, x, y)$ denotes the infimum of $I_{[0,t]}(\phi)$ over all continuous functions $\phi : [0, t] \to E$ such that $\phi(0) = x$, $\phi(t) = y$ and $\phi(s) \neq 0$ for all $0 < s < t$. $\overline{I} : \mathbb{R}_+ \to \mathbb{R}^d$ denotes the constant function $\overline{I}(t) \equiv 0 \in \mathbb{R}^d$. The quantities $I^*$ and $I^*_0$ are defined by

$$I^* = I(1, 0, 0) \quad \text{and} \quad I^*_0 = \hat{I}(1, 0, 0).$$

Using classical large deviation techniques we obtain the following result.

**Theorem 1.** Under the hypotheses (A), for any $x, y \in E$,

$$\log r^* = -I^* = -I_{[0,1]}(\overline{I}) = -\lim_{t \to +\infty} \sup_{I_0(t, x, y)} I(t, x, y)/t = -\inf_{\phi(0) = \phi(1)} I_{[0,1]}(\phi)$$

where the infimum is taken over all continuous functions $\phi : [0, 1] \to E$ with $\phi(0) = \phi(1)$.

Our main technical result is the following theorem.
Theorem 2. Under the hypotheses (A), for any \( x, y \in \mathcal{E} \),

\[
\log r^*_e = -I^*_0 = -\lim_{t \to +\infty} \sup I_0(t, x, y)/t = -\lim_{t \to +\infty} \tilde{I}(t, x, y)/t
\]

\[
= -\inf_{\phi(0) = \phi(1)} I_{[0,1]}(\phi)
\]

where the infimum is taken over all continuous functions \( \phi : [0, 1] \to \mathcal{E} \) with \( \phi(0) = \phi(1) \) such that \( \phi(t) \neq 0 \) for all \( 0 < t < 1 \).

Theorem 2 extends the result obtained earlier by Malyshev and Spieksma [11] for discrete time homogeneous random walks in \( \mathbb{N} \) and in \( \mathbb{Z}_+^2 \). The main difficulty consists here in the proof of the upper bound \( \log r^*_e \leq -I^*_0 \). To get this inequality, one has to analyze the behavior of the rescaled processes \( (Z_n(t)) \) in a neighborhood of infinity where truncations on compact sets are not sufficient. In such a situation, Freidlin-Wentzel method can not be applied.

The proof of the upper bound \( \log r^*_e \leq -I^*_0 \) proposed by Malyshev and Spieksma in [11] uses particular properties of the processes: they considered discrete time random walks with uniformly bounded jumps for which the sets

\[ K_x = \{ y : I(1, x, x + y) < +\infty \} \]

are compact and bounded uniformly in \( x \). For continuous time Markov processes, the sets \( K_x \) are usually not bounded. Moreover, their method required that for any \( \varepsilon > 0 \) there exist \( \delta > 0 \) and \( n_\varepsilon > 0 \) such that for all \( n > n_\varepsilon \),

\[
\sup_{|x' - x| < \varepsilon} \frac{1}{n} \log P_{x'}(|X(n) - ny| < \delta n) \leq -I(1, x, y) + \varepsilon
\]

uniformly in \((x, y)\) on the set of all \((x, y)\) for which the quantity \( I(1, x, y) \) is finite. Such an uniform convergence is very difficult to check in practice and is sometimes wrong: this implies the uniform continuity of the mapping

\[
(x, y) \to I(1, x, y) = \inf_{\phi(0) = x, \phi(1) = y} I_{[0,1]}(\phi).
\]

For the standard Brownian motion in \( \mathbb{R} \), this infimum is achieved by the function \( \phi(t) = x + t(y - x) \) and equals \((y - x)^2/2\). The function \((y - x)^2/2\) is not uniformly continuous on \( \mathbb{R}^2 \).

In our paper, we prove the inequality \( \log r^*_e \leq -I^*_0 \) by using a method of statistical physics called cluster expansions, see for example Malyshev and Minlos [10] or Rivasseau [13]. In the present setting, this method consists in bounding the quantity \( \log r^*_e \) by a limit of a sum of the terms indexed by geometrical objects called clusters where the number of terms can be estimated and for each term, a large deviation technique with an appropriate scaling can be applied. The main steps of our proof can be summarized as follows.

1. Proof of the inequality

\[
\log r^*_e \leq \lim_{a \to +\infty} \sup_{b \to +\infty} \lim_{T \to +\infty} \sup_{z : a \in E, N < |z| \leq bN} \frac{1}{aT} \log P_{az}(|Z_0(T)| \leq bN, |Z_0(s)| \geq N, \forall 0 \leq s \leq T)
\]

for \( N > 0 \) large enough. \( \mathbb{P}_{az}(\cdot) \) denotes here the conditional probability given that \( Z_0(0) = z \) (or equivalently \( X(0) = az \)). If the order of the limits in \( a \) and \( T \) could
be reversed, a large deviation upper bound would give directly a good estimate of the right hand side of this inequality. Unfortunately, such an inversion of the limits seems very difficult to prove: for this one should be able to perform large deviation estimates simultaneously for all $T$ large enough with the same large deviation parameter $a$ large enough. An alternative approach consists in sub-dividing the interval $[0, T]$ on smaller disjoint intervals $[t_{i-1}, t_i]$ is such a way that for every of these intervals, the desired large deviation estimates can be performed with the same parameter $a$ large enough. For this we use the following preliminary results.

2. Some preliminary results and constructions. Let $M_N(t, x, y)$ denote the infimum of the rate function $I_{[0, t]}(\phi)$ over all $\phi : [0, t] \to \mathcal{E}$ such that $\phi(0) = x$, $\phi(t) = y$ and

$$\sup_{0 \leq s \leq t} |\phi(s)| \geq N.$$ 

Using the upper large deviation bound we show that for given $\varepsilon > 0$ and $N > 0$ there exists a finite set $V(\varepsilon, N) \subset \{(x, y) : |x| \leq 2N, |y| \leq 2N\}$ and there are strictly positive real numbers $\delta(x, y)$ and $a(x, y)$ such that

$$\{(x, y) : |x| \leq 2N, |y| \leq 2N\} \subset \bigcup_{(x, y) \in V(\varepsilon, N)} B(x, \delta(x, y)) \times B(y, \delta(x, y))$$

and for any $(x, y) \in V(\varepsilon, N)$, for all $a \geq a(x, y)$, and for any $z \in \frac{1}{a} E$ satisfying the inequality $|z - x| \leq \delta(x, y)$, the following inequalities hold

$$\log \mathbb{P}_a \left( \sup_{s \in [0, 1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta(x, y)) \right) \leq -aM_N(1, x, y) + a\varepsilon$$

when $M_N(1, x, y) < +\infty$, and

$$\log \mathbb{P}_a \left( \sup_{s \in [0, 1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta(x, y)) \right) \leq -aI_0^* N/\varepsilon$$

when $M_N(1, x, y) = +\infty$. Here and throughout, $B(x, \delta)$ denotes the open ball centered at $x$ and having radius $\delta$.

3. Change of scale. Using the above estimates and the identity $Z_a(s) = tZ_{at}(s/t)$ we conclude that for any $(x, y) \in V(\varepsilon, N)$, for all $\sigma > 0$, $a \geq \sigma a(x, y)$ and $t \geq 1/\sigma$, and for any $z \in \frac{1}{\sigma} E \cap tB(x, \delta(x, y))$, the following inequalities hold

$$\log \mathbb{P}_a \left( \sup_{s \in [0, t]} |Z_a(s)| \geq Nt \text{ and } Z_a(t) \in tB(y, \delta(x, y)) \right) \leq -aM_N(t, tx, ty) + at\varepsilon$$

when $M_N(t, tx, ty) < +\infty$, and

$$\log \mathbb{P}_a \left( \sup_{s \in [0, t]} |Z_a(s)| \geq Nt \text{ and } Z_a(t) \in tB(y, \delta(x, y)) \right) \leq -atI_0^* N/\varepsilon$$

otherwise. These inequalities hold simultaneously for all $(x, y) \in V(\varepsilon, N)$ and $t \geq 1/2$ when $a > 2 \max_{(x, y) \in V(\varepsilon, N)} a(x, y)$. Such a change of scale is a key point of our proof.

4. Cluster expansion. For given $\varepsilon > 0$, $N > 0$ and $T$, and for every function $\phi \in D([0, T], \mathcal{E})$ with $|\phi(t)| \geq N$ for all $0 \leq t \leq T$ and such that $|\phi(0)| \leq 2TN$ and
This algorithm terminates because \( |\phi(T)| \leq 2TN \), we define a partition \( 0 = t_0 < t_1 < \ldots < t_n = T \) and a sequence \( (x_i, y_i) \in V(\varepsilon, N), \ i = 1, \ldots, n \), as follows.

- If \( \sup_{0 \leq t \leq T} |\phi(t)| > NT \) we let \( n = 1 \) and we choose \( (x_1, y_1) \in V(\varepsilon, N) \) such that \( \phi(0) \in TB(x_1, \delta(x_1, y_1)) \) and \( \phi(T) \in TB(y_1, \delta(x_1, y_1)) \).

- Otherwise, we divide the interval \( [0, T] \) in two intervals \( [0, T/2] \) and \( [T/2, T] \) and we restart our construction for the restriction of \( \phi \) on each of them.

This algorithm terminates because \( |\phi(t)| \geq N \) for all \( 0 \leq t \leq T \). The resulting sequence \( \Gamma(\phi) = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) is called \((T, \varepsilon, N)-\) cluster corresponding to \( \phi \) (see section 6 for a more careful definition of \((T, \varepsilon, N)-\) cluster).

In statistical physics, the notion of cluster is usually associated with a connected graph. In our context, the cluster \( \Gamma(\phi) \) is connected in the following sense:

\[
\phi(t_i) \in (t_i - t_{i-1})B(y_i, \delta(x_i, y_i)) \cap (t_{i+1} - t_i)B(x_{i+1}, \delta(x_{i+1}, y_{i+1})) \neq \emptyset
\]

for every \( 1 \leq i \leq n \). For each cluster \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) we consider the quantity

\[
\chi_a(\Gamma) = \sup_z P_{a \varepsilon}(\Gamma(Z_a) = \Gamma)
\]

where the supremum is taken over all \( z \in \frac{1}{a}E \cap t_1 B(x_1, \delta(x_1, y_1)) \). Using inequality (4) we obtain

\[
\log r^*_a \leq \limsup_{a \to +\infty} \limsup_{b \to +\infty} \frac{1}{aT} \log \left( \sum_\Gamma \chi_a(\Gamma) \right)
\]

where the summation is taken over all clusters \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) for which the sets \( t_1 B(x_1, \delta(x_1, y_1)) \cap B(0, bN) \) and \( (t_n - t_{n-1})B(y_n, \delta(x_n, y_n)) \cap B(0, bN) \) are non-empty.

4. Cluster estimates. We show that for given \( \varepsilon > 0 \), \( N > 0 \) and \( T \geq 1 \), there are at most \( (2V(\varepsilon, N))^2T \) clusters. For every cluster \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) we obtain a good estimate of the quantity \( \chi_a(\Gamma) \) by using the inequalities (3), (6) with \( x = x_i, \ y = y_i \) and \( t = t_i - t_{i-1} \) for every \( i = 1, \ldots, n \) (see Lemma 6 below) and we deduce from them the desired inequality \( \log r^*_a \leq -I^*_0 \).

Our paper is organized as follows. Section 3 is devoted to general properties of the quantities \( r^* \) and \( r^*_a \). In section 4 the definition and some useful properties of sample path large deviations are recalled and different representations of the quantities \( I^* \) and \( I^*_0 \) are derived. Section 5 is devoted to the proof of Theorem 1. Theorem 2 is proved in section 6. In section 7 we apply our results to calculate the quantities \( r^* \) and \( r^*_a \) for two dimensional Jackson networks.

To simplify the notations, we consider continuous time Markov processes. For discrete time Markov processes our results can be extended in a straightforward way.

3. SOME GENERAL PROPERTIES

In this section, we recall general properties of the quantities \( r^* \) and \( r^*_a \).
3.1. **Spectral radius.** For $r > 0$ and for a real bounded measurable function $\varphi$ on $E$, the resolvent function is defined by

$$R_r\varphi(x) = \int_0^\infty r^{-t} P^t \varphi(x) \, dt$$

where $P^t$ denotes Markov semi-group corresponding to the process $(X(t))$:

$$P^t \varphi(x) = E_x(\varphi(X(t))).$$

Recall that by definition, $r^*$ is the infimum of all those $r > 0$ for which the function $R_r 1_V$ is $m$-integrable on compact subsets of $E$ for every compact set $V \subset E$.

It is clear moreover that $r^*$ is the infimum of all those $r > 0$ for which the function $R_r \varphi$ is $m$-integrable on compact subsets of $E$ for every non-negative continuous function $\varphi$ having a compact support.

The following property of the quantity $r^*$ immediately follows from the definition.

**Proposition 3.1.**

(7) \[ \log r^* \leq \sup_{W,V} \limsup_{t \to \infty} \sup_{x \in W} \frac{1}{t} \log P^t 1_V(x) \]

where the supremum $\sup_{W,V}$ is taken over all compact subsets $W, V \subset E$. Moreover, if for a compact set $V \subset E$, there exists a compact set $V' \subset E$ and there are real numbers $t > 0$ and $\varepsilon > 0$ such that $P^s 1_{V'} \geq \varepsilon 1_V$ for all $0 < s < t$, then for any compact subset $W \subset E$,

(8) \[ \log r^* \geq \limsup_{t \to \infty} \inf_{x \in W} \frac{1}{t} \log P^t 1_V(x). \]

**Proof.** Indeed, Relation (7) holds because by Fubini’s theorem

$$m(1_W R_r 1_V) = \int_0^\infty r^{-t} m(1_W P^t 1_V(x)) \, dt \leq m(W) \int_0^\infty r^{-t} \sup_{x \in W} P^t 1_V(x) \, dt.$$

Suppose moreover that for a compact set $V \subset E$ there are real numbers $t > 0$ and $\varepsilon > 0$ and a compact set $V' \subset E$ such that $P^s 1_{V'}(x) \geq \varepsilon$ for all $0 < s < t$ and for all $x \in V$. Then for any increasing sequence of real positive numbers $t_n$ with $\inf_n (t_{n+1} - t_n) \geq t$, the following inequalities hold

$$m(1_W R_r 1_{V'}) \geq \sum_n \int_{t_n}^{t_{n+1}} r^{-s} m(1_W P^s 1_{V'}) \, ds$$

$$\geq \sum_n \int_0^t r^{-t_n - s} m(1_W P^{t_n} 1_V P^s 1_{V'}) \, ds$$

$$\geq \varepsilon \sum_n r^{-t_n} m(1_W P^{t_n} 1_V) \int_0^t r^{-s} \, ds.$$

The last inequality shows that for $x \in W$, $m(1_W R_r 1_{V'}) = +\infty$ whenever

$$0 < r < \limsup_n \frac{1}{t_n} \log m(1_W P^{t_n} 1_V)$$

and consequently, Inequality (8) is verified. \qed

The following proposition shows that for a large class of Markov processes, the quantity $r^*$ can be represented in terms of $r$-superharmonic functions.
Definition 3.1. A measurable function \( f : E \to \mathbb{R}_+ \) is called \( r \)-superharmonic with \( r > 0 \) if the inequality \( P^t f \leq r^t f \) holds for all \( t \in \mathbb{R}_+ \). We say that a function \( f \) is locally \( m \)-integrable on \( E \) if it is \( m \)-integrable on the compact subsets of \( E \).

Proposition 3.2. Suppose that there exists a non-negative continuous function \( \varphi_0 \) on \( E \) having a compact support such that for every \( t > 0 \) the function \( P^t \varphi_0 \) is continuous on \( E \) and for every \( x \in E \) there exists \( t > 0 \) such that \( P^t \varphi_0(x) > 0 \). Then \( r^* \) is the infimum of all those \( r > 0 \) for which there exists a non-negative locally \( m \)-integrable \( r \)-superharmonic function \( f \) with \( \inf_{x \in W} f(x) > 0 \) for every compact subset \( W \subset E \).

Proof. Indeed, for \( r > r^* \), the function \( R_r \varphi_0 \) is non-negative locally \( m \)-integrable and \( r \)-superharmonic. Moreover, the sample paths of the Markov process \( (X(t)) \) being right continuous, the mapping \( t \to P^t \varphi_0(x) \) is right continuous and hence, under the hypotheses of our proposition, \( R_r \varphi_0(x) > 0 \) for every \( x \in E \). Finally, by Fatou's lemma, the function \( R_r \varphi_0 \) is lower semi-continuous on \( E \) and consequently, for any compact set \( W \subset E \),

\[
\inf_{x \in W} R_r \varphi_0(x) > 0.
\]

Conversely, suppose that for \( r > 0 \) there exists a non-negative locally \( m \)-integrable \( r \)-superharmonic function \( f \) such that \( \inf_{x \in W} f(x) > 0 \) for every compact subset \( W \subset E \). Then for every compact set \( W \subset E \), and for any \( t > 0 \) and \( x \in W \),

\[
P^t 1_W(x) \inf_{y \in W} f(y) \leq P^t f(x) \leq r^t f(x).
\]

\[ \text{From this it follows that for any } r' > r, \text{ the function } R_{r'} 1_W \text{ is locally } m \text{-integrable and consequently, } r^* \leq r. \text{ Proposition 3.2 is therefore verified.} \]

3.2. Essential spectral radius. Recall that by definition, the quantity \( r^*_e \) is the infimum over all those \( r > 0 \) for which there is a compact set \( K \subset E \), such that the truncated resolvent function

\[
R_{K,1} 1_W(x) = \int_0^\infty r^{-t} \mathbb{P}_x \{ X(t) \in W, \tau(K) > t \} \, dt
\]

is \( m \)-integrable on the compact subsets of \( E \setminus K \) for every compact set \( W \subset E \setminus K \).

Proposition 3.3.

\[ \log r^*_e \leq \inf_{K \subset E \setminus K} \sup_{W \subset E \setminus K} \limsup_{t \to \infty} \sup_{x \in W} \frac{1}{t} \log \mathbb{P}_x \{ X(t) \in V, \tau(K) > t \} \]

where the infimum is taken over all compact subsets \( K \subset E \) and the supremum is taken over all compact subsets \( W, V \subset E \setminus K \). Suppose moreover that for any compact subset \( K \subset E \) there are compact sets \( V_K, V'_K \subset E \setminus K \) and there is a real number \( t > 0 \) such that

\[ \inf_{0 < s \leq t} \inf_{x \in V_K} \mathbb{P}_x \{ X(s) \in V'_K, \tau(K) > t \} > 0. \]

Then

\[ \log r^*_e \geq \inf_{K \subset E \setminus K} \sup_{W \subset E \setminus K} \limsup_{t \to \infty} \inf_{x \in W} \frac{1}{t} \log \mathbb{P}_x \{ X(t) \in V_K, \tau(K) > t \} \]

where the infimum \( \inf_K \) is taken over all compact subsets \( K \subset E \) and the supremum is taken over all compact subsets \( W \subset E \setminus K \).
Proof. Indeed, let \( r^*_K \) be the infimum over all those \( r > 0 \) for which the function \( f \) is \( m \)-integrable on the compact subsets of \( E \setminus K \) for every compact set \( W \subset E \setminus K \). Using the same arguments as for the proof of Proposition 3.1, we obtain that for every compact set \( K \subset E \),

\[
\log r^*_K \leq \sup_{W, V \subset E \setminus K} \limsup_{t \to \infty} \sup_{x \in W} \frac{1}{t} \log \mathbb{P}_x(X(t) \in V, \tau(K) > t)
\]

and moreover, for compact subsets \( W, V \subset E \setminus K \),

\[
\log r^*_K \geq \limsup_{t \to \infty} \inf_{x \in W} \frac{1}{t} \log \mathbb{P}_x(X(t) \in V, \tau(K) > t)
\]

if there are a compact set \( V' \subset E \setminus K \) and a real numbers \( t > 0 \) such that

\[
\inf_{0 < s \leq t} \inf_{x \in V} \mathbb{P}_x(X(s) \in V', \tau(K) > s) > 0.
\]

Using relation \( r^*_c = \inf_K r^*_K \), this proves Proposition 3.3. \( \square \)

The next proposition describes the quantity \( r^*_c \) in terms of Lyapunov functions which are superharmonic outside of compact sets.

**Definition 3.2.** A measurable function \( f : E \to \mathbb{R}_+ \) is called \( r \)-superharmonic outside of a compact set \( K \subset E \) with \( r > 0 \) if the inequality

\[
\mathbb{E}_x(f(X(t)) ; \tau(K) > t) \leq r^t f(x)
\]

holds for all \( x \in E \setminus K \) and for all \( t \in \mathbb{R}_+ \).

**Proposition 3.4.** Suppose that there exists an increasing sequence of open relatively compact sets \( U_n \subset E \) such that \( \bigcup_n U_n = E \) and let for every \( n \in \mathbb{N} \) there exist a non-negative continuous function \( \varphi_n \) on \( E \) having a compact support in \( E \setminus \overline{U}_n \) such that for every \( t > 0 \) the function \( x \to E_x(\varphi_n(X(t)) ; \tau(\overline{U}_n) > t) \) is continuous on \( E \setminus \overline{U}_n \) and for every \( x \in E \setminus \overline{U}_n \) there exists \( t > 0 \) such that

\[
E_x(\varphi_n(X(t)) ; \tau(\overline{U}_n) > t) > 0.
\]

Then \( r^*_c \) is the infimum of all those \( r > 0 \) for which there exists a compact set \( K \subset E \) and a non-negative locally \( m \)-integrable \( r \)-superharmonic outside of \( K \) function \( f \) with \( \inf_{x \in W} f(x) > 0 \) for every compact subset \( W \subset E \setminus K \).

Proof. Given a compact subset \( K \subset E \), let \( r^*_K \) be the infimum over all those \( r > 0 \) for which the function \( f \) is \( m \)-integrable on the compact subsets of \( E \setminus K \) for every compact set \( W \subset E \setminus K \), and let \( \rho_K \) be the infimum of all those \( r > 0 \) for which there exists a non-negative locally \( m \)-integrable \( r \)-superharmonic outside of \( K \) function \( f \) with \( \inf_{x \in W} f(x) > 0 \) for every compact subset \( W \subset E \setminus K \). The same arguments as in the proof of Proposition 3.2 show that for \( K_n = \overline{U}_n \), \( r^*_K \cdot r^*_K = r^*_K \cdot \rho_K \) for every \( n \in \mathbb{N} \). The quantities \( r^*_K \) and \( \rho_K \) being decreasing with respect to \( K \), this implies that

\[
r^* = \inf_n r^*_K = \inf_n \rho_K = \inf_K \rho_K
\]

where the last infimum is taken over all compact subsets \( K \subset E \). Proposition 3.3 is therefore proved. \( \square \)
Definition 3.3. Let $\sigma^*_e$ denote the infimum over all those $r > 0$ for which there is a compact set $K \subset E$ such that the function
\begin{equation}
(12) \quad x \to \int_0^{+\infty} r^{-t} \mathbb{P}_x(\tau(K) > t) \, dt
\end{equation}
is $m$-integrable on the compact subsets of $E \setminus K$.

The following proposition represents the number $\sigma^*_e$ in terms of Lyapunov functions.

Proposition 3.5. $\sigma^*_e$ is the infimum of all those $r > 0$ for which there exists a compact set $K \subset E$ and a non-negative locally $m$-integrable $r$-superharmonic outside of $K$ function $f$ with $\inf_{x \in E \setminus K} f(x) > 0$.

Proof. Suppose that for $\hat{r} > 0$, there is a non-negative $\hat{r}$-superharmonic outside of $K$ function $f$ which is $m$-integrable on the compact subsets of $E \setminus K$ and such that $\inf_{x \in E \setminus K} f(x) > 0$. Then for any $t > 0$, and for any $x \in E \setminus K$,
\[ \mathbb{P}_x(\tau(K) > t) \times \inf_{y \in E \setminus K} f(y) \leq \mathbb{E}_x(f(X(t) \mid \tau(K) > t) \leq \hat{r}^t f(x). \]

For any $r > \hat{r}$, the function $(12)$ is therefore $m$-integrable on the compact subsets of $E \setminus K$ and consequently, $\hat{r} \geq \sigma^*_e$. The function $(12)$ being $r$-superharmonic outside of $K$, this proves Proposition 3.5. \qed

The next proposition shows that for a large class of recurrent Markov processes, the quantities $r^*_e$ and $\sigma^*_e$ are equal.

Proposition 3.6. Suppose that the hypotheses of Proposition 3.4 are satisfied and let for every $n > 0$, and $\tau(U_{n+1}) < \tau(U_n) < +\infty$, $\mathbb{P}_x$-almost surely for every $x \in E \setminus U_{n+1}$. Then $r^*_e = \sigma^*_e$.

Proof. If for $r > 0$ there is a compact set $K \subset E$ such that the function $(12)$ is $m$-integrable on the compact subsets of $E \setminus K$, then for every compact set $W \subset E$, the function $(10)$ is $m$-integrable on the compact subsets of $E \setminus K$ and therefore, $\sigma^*_e \geq r^*_e$.

Let us prove that $\sigma^*_e \leq r^*_e$. Because of Proposition 3.5, it is sufficient to show that for any $r > r^*_e$ there is $n \in \mathbb{N}$ and a non-negative $r$-superharmonic outside of $K_n = \overline{U}_n$ function $f$ which is $m$-integrable on the compact subsets of $E \setminus K_n$ and such that $\inf_{x \in E \setminus K_n} f(x) > 0$. It is sufficient moreover to consider the case when $r^*_e < r < 1$ because for $r \geq 1$, the function $f \equiv 1$ is $r$-superharmonic.

Given a compact subset $K \subset E$, let $r^*_K$ be the infimum over all those $r > 0$ for which the function $(10)$ is $m$-integrable on the compact subsets of $E \setminus K$ for every compact set $W \subset E \setminus K$. The quantity $r^*_K$ being decreasing with respect to $K$,
\[ r^*_e = \inf_n r^*_K. \]

Given $r^*_e < r < 1$ let us choose $k \in \mathbb{N}$ such that $r > r^*_K$ and let $n > k$ be such that $K_k \subset U_{n-1}$. Under the hypotheses of Proposition 3.4, the function
\[ f(x) = \int_0^{+\infty} r^{-t} \mathbb{E}_x(\varphi_k(X(t)) \mid \tau(K_k) > t) \, dt \]
is $m$-integrable on the compact subsets of $E \setminus K_k$ and $r$-superharmonic outside of $K_k$. Moreover the same arguments as for the proof of Proposition 3.2 show that
for any compact subset \( W \subset E \setminus K_k \), there is \( \varepsilon(W) > 0 \) such that
\[
\inf_{x \in W} f(x) \geq \varepsilon(W).
\]
Consider a compact set \( W_n = K_n \setminus U_{n-1} \). Using strong Markov property, we obtain
\[
\inf_{x \in E \setminus K_n} f(x) \geq \inf_{x \in E \setminus K_n} \int_0^{+\infty} r^{-t} \mathbb{E}_x (\varphi_k(X(t)); \tau(K_k) > t \geq \tau(W_n)) \, dt
\]
\[
= \inf_{x \in E \setminus K_n} \mathbb{E}_x \left( r^{-\tau(W_n)} f(X(\tau(W_n))); \tau(K_k) > \tau(W_n) \right)
\]
\[
\geq \varepsilon(W_n) \inf_{x \in E \setminus K_n} \mathbb{P}_x \left( \tau(K_k) > \tau(W_n) \right) = \varepsilon(W_n)
\]
The last equality holds here because under the hypotheses of our proposition, for all \( x \in E \setminus K_n \), \( \mathbb{P}_x \)-almost surely \( \tau(K_n) = \tau(W_n) < \tau(K_{n-1}) < \tau(K_k) \). The function \( f \) being \( m \)-integrable on the compact subsets of \( E \setminus K_n \) and \( r \)-superharmonic outside of \( K_n \) the last relation shows that \( \sigma_r^* \leq r \). Letting finally \( r \to r^*_c \) we conclude that \( \sigma_r^* \leq r^*_c \). The equality \( \sigma_r^* = r^*_c \) is therefore proved. \( \square \)

4. Sample path large deviations

4.1. Definitions and general properties. Let \( D([0, T], \mathbb{R}^d) \) denote the set of all right continuous with left limits functions from \( [0, T] \) to \( \mathbb{R}^d \) endowed with Skorohod metric \( d_S(\cdot, \cdot) \). Recall that Skorohod metric \( d_S(\phi, \psi) \) is defined as the infimum of those positive \( \varepsilon \) for which there exists a strictly increasing continuous mapping \( \lambda \) from \([0, T]\) onto itself satisfying inequalities
\[
\sup_{t > s} \left| \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq \varepsilon \quad \text{and} \quad \sup_t |\phi(t) - \psi(\lambda(t))| \leq \varepsilon
\]
where the metric is induced by the Euclidean norm \(| \cdot |\) on \( \mathbb{R}^d \).

The space \( D([0, T], \mathbb{R}^d) \) endowed with Skorohod metric is complete. A sequence \( \phi_n \in D([0, T], \mathbb{R}^d) \) converges to a limit \( \phi \in D([0, T], \mathbb{R}^d) \) in the Skorohod metric if and only if there exist strictly increasing continuous mappings \( \lambda_n : [0, T] \to [0, T] \) such that \( \lambda_n(0) = 0, \lambda_n(T) = T, \lambda_n(t) \to t \) as \( n \to \infty \) uniformly in \( t \in [0, T] \) and \( \phi_n \circ \lambda_n(t) \to \phi(t) \) as \( n \to \infty \) uniformly in \( t \in [0, T] \). When \( \phi \) is continuous on \([0, T]\), Skorohod convergence \( \phi_n \to \phi \) implies uniform convergence. For non-continuous \( \phi \), Skorohod convergence \( \phi_n \to \phi \) implies \( \phi_n(0) \to \phi(0), \phi_n(T) \to \phi(T) \) and \( \phi_n(t) \to \phi(t) \) for continuity points \( t \in [0, T] \) of \( \phi \). For more details about Skorohod metric, we refer the reader to Billingsley [2].

We consider a Markov process \( (X(t)) \) on \( E \subset \mathbb{R}^d \). The trajectories of the Markov process \( X(t) \) are assumed to be almost surely right continuous and to have left limits.

**Definition 4.1.** 1) A mapping \( I_{[0,T]} : D([0, T], \mathbb{R}^d) \to [0, +\infty] \) is a good rate function on \( D([0, T], \mathbb{R}^d) \) if for any \( c > 0 \) and for any compact set \( V \subset \mathbb{R}^d \), the set
\[
\{ \varphi \in D([0, T], \mathbb{R}^d) : \phi(0) \in V \text{ and } I_{[0,T]}(\varphi) \leq c \}
\]
is compact in \( D([0, T], \mathbb{R}^d) \). According to this definition, a good rate function is lower semi-continuous.

2) The family of scaled Markov processes
\[
(Z_a(t), t \in [0, T]) \text{ def} = (X(at)/a, t \in [0, T])
\]
is said to satisfy sample path large deviation principle in $D([0, T], \mathbb{R}^d)$ with a rate function $I_{[0,T]}$ if for any $x \in \mathbb{R}^d$

$$\lim_{\varepsilon \to 0} \liminf_{a \to \infty} \frac{1}{a} \log \mathbb{P}_{x,y} (Z_a(\cdot) \in O) \geq - \inf_{\phi \in \mathcal{O} : \phi(0) = x} I_{[0,T]}(\phi),$$

for every open set $O \subset D([0, T], \mathbb{R}^d)$, and

$$\lim_{\varepsilon \to 0} \limsup_{a \to \infty} \sup_{y : |y - x| < \varepsilon} \frac{1}{a} \log \mathbb{P}_{x,y} (Z_a(\cdot) \in F) \leq - \inf_{\phi \in \mathcal{F} : \phi(0) = x} I_{[0,T]}(\phi),$$

for every closed set $F \subset D([0, T], \mathbb{R}^d)$.

$\mathbb{P}_{x,y}$ denotes here and throughout a conditional probability given that $Z_a(0) = y$ (or equivalently $X(0) = ay$), the infimum at the left hand side of the inequality (13) and the supremum at the left hand side of the inequality (14) are taken over all $y \in \frac{1}{a} E$ satisfying the inequality $|x - y| < \varepsilon$.

We refer to sample path large deviation principle as SPLD principle. Inequalities (13) and (14) are referred as lower and upper SPLD bounds respectively.

The following statement is a consequence of contraction principle (see Dembo and Zeitouni [4]) and the identity $Z_a(t) = T_a(t/T)$.

**Proposition 4.1.** Suppose that the family of scaled processes $Z_a(t)$ satisfies SPLD principle in $D([0, 1], \mathbb{R}^d)$ with a good rate functions $I_{[0,1]}$. Then for any $T > 0$, this family of processes satisfies SPLD principle in $D([0, T], \mathbb{R}^d)$ with the good rate functions

$$I_{[0,T]}(\phi) = TI_{[0,1]}(G_T \phi)$$

where $G_T$ denotes a mapping from $D([0, T], \mathbb{R}^d)$ to $D([0, 1], \mathbb{R}^d)$ defined by

$$G_T \phi(t) = \phi(Tt)/T, \quad t \in [0, 1].$$

When the SPLD principle holds, the corresponding rate function is unique and hence, under the hypotheses (A), relation (15) is satisfied. A good rate function satisfies moreover the following properties.

**Lemma 4.1.** For any closed set $F \subset D([0, T], \mathbb{R}^d)$, the mapping

$$(x, y) \to \inf_{\phi \in F : \phi(0) = x, \phi(T) = y} I_{[0,T]}(\phi)$$

is lower semi-continuous on $\mathbb{R}^{2d}$.

**Proof.** Indeed, for any $c > 0$ and for any compact set $V \subset \mathbb{R}^d$,

$$\{ (x, y) \in V \times \mathbb{R}^d : \inf_{\phi \in F : \phi(0) = x, \phi(T) = y} I_{[0,T]}(\phi) \leq c \} = \bigcap_n \xi(K_n)$$

where $K_n = \{ \phi \in F : \phi(0) \in V, I_{[0,T]}(\phi) \leq c + 1/n \}$ and the mapping $\xi : D([0, T], \mathbb{R}^d) \to \mathbb{R}^2$ is defined by $\xi(\phi) = (\phi(0), \phi(T))$. The sets $K_n$, being compact and the mapping $\xi$ being continuous, the sets $\xi(K_n)$ are compact. This proves that the set (17) is also compact and consequently, the mapping (16) is lower semi-continuous.

$\square$
4.2. Different representations of the quantities \( I^* \) and \( I^0 \). Throughout this section, \( I_{[0,T]} \) denotes a good rate function on \( D([0,T], \mathbb{R}^d) \) satisfying Assumption (A) and Relation (15). Recall that \( \mathcal{E} \) denotes the set of all possible limits \( \lim_{a \to \infty} x_a/a \) with \( x_a \in E \). According to Assumption (A),

- the set \( \mathcal{E} \setminus \{0\} \) is convex and non-empty;
- the rate function \( I_{[0,T]} \) has an integral form: there is a local rate function \( L : \mathcal{E} \times \mathbb{R}^d \to \mathbb{R}_+ \) such that

\[
I_{[0,T]}(\phi) = \int_0^T L(\phi(t), \dot{\phi}(t)) \, dt
\]

if the function \( \phi : [0,1] \to \mathcal{E} \) is absolutely continuous, and \( I_{[0,1]}(\phi) = +\infty \) otherwise.

- there is a convex function \( l_2 \) on \( \mathbb{R}^d \) that is finite in a neighborhood of zero and such that \( L(x,v) \leq l_2(v) \) for all \( x \in \mathcal{E} \) and for all \( v \in \mathbb{R}^d \).

Here and throughout, for \( x,y \in \mathcal{E} \) and \( t > 0 \), \( I(t,x,y) \) denotes the infimum of the rate function \( I_{[0,t]}(\phi) \) over all continuous functions \( \phi : [0,t] \to \mathcal{E} \) with \( \phi(0) = x \) and \( \phi(t) = y \), \( I_0(t,x,y) \) denotes the infimum of the rate function \( I_{[0,t]}(\phi) \) over all continuous functions \( \phi : [0,t] \to \mathcal{E} \) such that \( \phi(0) = x \), \( \phi(t) = y \) and \( \phi(s) \neq 0 \) for all \( s \in ]0,t[ \), and \( \tilde{I}(t,x,y) \) is the infimum of the rate function \( I_{[0,t]}(\phi) \) over all those \( \phi : [0,t] \to \mathcal{E} \) for which \( \phi(0) = x \), \( \phi(t) = y \) and the set \( \{ s \in [0,t] : \phi(s) = 0 \} \) has Lebesgue measure zero. By definition,

\[
I^* = I(1,0,0) \quad \text{and} \quad I_0^* = \tilde{I}(1,0,0).
\]

**Remark 1.** Because of Relation (15), for any \( x, y \in \mathcal{E} \) and \( t > 0 \),

\[
I(t,tx,ty) = tI(1,x,y), \quad \tilde{I}(t,tx,ty) = t\tilde{I}(1,x,y),
\]

and

\[
I_0(t,tx,ty) = tI_0(1,x,y).
\]

**Remark 2.** The integral representation of the rate function implies that for any \( \phi \in D([0,T], \mathbb{R}^d) \) and \( 0 < t < T \),

\[
I_{[0,T]}(\phi) = I_{[0,t]}(\phi) + I_{[t,T]}(\phi)
\]

where \( I_{[t,T]}(\phi) = I_{[0,T-t]}(\phi(t+\cdot)) \). From this it follows that for any \( x, y, z \in \mathcal{E} \),

\[
I(T,x,y) \leq I(t,x,z) + I(T-t,z,y) \quad \text{and} \quad \tilde{I}(T,x,y) \leq \tilde{I}(t,x,z) + \tilde{I}(T-t,z,y).
\]

Moreover, for \( z \neq 0 \),

\[
I_0(T,x,y) \leq I_0(t,x,z) + I_0(T-t,z,y).
\]

**Remark 3.** Since the function \( l_2 \) is convex and finite in a neighborhood of zero, there are \( c > 0 \) and \( C > 0 \) such that \( l_2(v) \leq C \) for all \( v \in \mathbb{R}^d \) with \( |v| \leq c \). Hence, for \( \phi(t) = x + t(y-x) \) with \( x, y \in \mathcal{E} \) such that \( |x-y| \leq c \),

\[
I(1,x,y) \leq \tilde{I}(1,x,y) \leq I_0(1,x,y) \leq I_{[0,1]}(\phi) \leq l_2(y-x) \leq C.
\]

Using Relations (18) and (19) this implies that

\[
I(t,x,y) \leq \tilde{I}(t,x,y) \leq I_0(t,x,y) \leq Ct
\]

for all \( t > |y-x|/c \).

Using Relations (15) and (20) we obtain the following statement.
Lemma 4.2. For any continuous function \( \phi : [0, T] \to \mathcal{E} \) with \( \phi(0) = \phi(T) = 0 \),
\begin{equation}
I_{[0,T]}(\phi) \geq I_{[0,T]}(0)
\end{equation}
and
\begin{equation}
I_{[0,T]}(\phi) \geq I_0(1,0,0) \text{ mes}\{t \in [0,T] : \phi(t) \neq 0\}
\end{equation}
where \( \text{mes}\{t \in [0,T] : \phi(t) \neq 0\} \) denotes Lebesgue measure of the set \( \{t \in [0,T] : \phi(t) \neq 0\} \).

Proof. Indeed, given a continuous function \( \phi : [0, T] \to \mathcal{E} \) with \( \phi(0) = \phi(T) = 0 \), consider a sequence of functions \( \phi_n : [0, T] \to \mathcal{E} \) defined by \( \phi_n(t + k/n) = \phi(nt)/n \) for all \( t \in [0, T/n] \) and for all \( k = 0, \ldots, n \). It is clear that \( \phi_n \to \phi \) as \( n \to \infty \) uniformly on \([0, T]\). The rate function \( I_{[0,T]}(\phi) \) being lower semi-continuous, this implies that
\[
\liminf_{n \to \infty} I_{[0,T]}(\phi_n) \geq I_{[0,T]}(0)
\]
because uniform convergence implies Skorohod convergence. Relations (15) and (20) show that \( I_{[0,T]}(\phi_n) = n I_{[0,T/n]}(\phi_n) = I_{[0,T]}(\phi) \) and hence, the last inequality proves relation (22).

Furthermore, for such a function \( \phi \), the set \( \{t \in [0, T] : \phi(t) \neq 0\} \) is a union of countable family of open disjoint intervals \[t_i, t'_i[ \subset [0, T], i \in \mathbb{N} \]. Using relation (20) we obtain
\[
I_{[0,T]}(\phi) \geq \sum_i I_{[t_i, t'_i]}(\phi) \geq \sum_i I_0(t'_i - t_i, 0, 0) = I_0(1, 0, 0) \sum_i (t'_i - t_i)
\]
where the second relation holds because for every \( i, \phi(t_i) = \phi(t'_i) = 0 \) and \( \phi(t) \neq 0 \) for all \( t \in [t_i, t'_i[ \), and the third relation follows from relation (15). The last relation proves inequality (23). \( \square \)

The following proposition gives several equivalent representations of \( I^* \) and \( I_*^0 \).

Proposition 4.2. 1) For any \( x, y \in \mathcal{E} \) and \( T > 0 \),
\begin{equation}
I^* = I_{[0,1]}(0) = \limsup_{t \to +\infty} I(t, x, y)/t = \inf_{\phi(0) = \phi(T)} I_{[0,T]}(\phi)/T
\end{equation}
where the infimum is taken over all continuous functions \( \phi : [0, T] \to \mathcal{E} \) with \( \phi(0) = \phi(T) \), and
\begin{equation}
I_*^0 = I_0(1, 0, 0) = \limsup_{t \to +\infty} I_0(t, x, y)/t = \limsup_{t \to +\infty} I(t, x, y)/t
\end{equation}
where the infimum is taken over all continuous functions \( \phi : [0, T] \to \mathcal{E} \) with \( \phi(0) = \phi(T) \) such that \( \phi(t) \neq 0 \) for all \( 0 < t < T \).

Proof. The equalities \( I^* = I_{[0,1]}(0) \) and \( I_*^0 = I_0(1, 0, 0) \) hold because of Lemma 1.1. Furthermore, Relation (21) and the inequality
\[
I(t_1 + t_2 + t, x, y) \leq I(t_1, x, x') + I(t, x, y) + I(t_2, y', y)
\]
show that the limits \( \liminf_{t \to +\infty} I(t, x, y)/t \) and \( \limsup_{t \to +\infty} I(t, x, y)/t \) do not depend on \( x, y \in \mathcal{E} \). Since by (13), \( I(t, 0, 0) = t I(1, 0, 0) = I^* \), we conclude that for any \( x, y \in \mathcal{E} \)
\[
I(t, x, y)/t \to I(1, 0, 0) \quad \text{as} \quad t \to \infty.
\]
Moreover, Relation (20) shows that $I(nT, x, x) \leq nI(T, x, x)$ for all $x \in \mathcal{E}$, $T > 0$ and $n \in \mathbb{N}$ and consequently,

$$\frac{1}{T} I(T, x, x) \geq \frac{1}{nT} I(nT, x, x) = I^*. $$

The last relation combined with the inequality

$$I^* = \inf_{x \in \mathcal{E}} \frac{I(T, x, x)}{T} \geq \inf_{\phi(0) = \phi(T)} I_{[0, T]}(\phi)/T$$

proves that

$$I^* = \inf_{x \in \mathcal{E}} I(T, x, x)/T = \inf_{\phi(0) = \phi(T)} I_{[0, T]}(\phi)/T. $$

Relation (24) is therefore proved. The same arguments show that for all $x \in \mathcal{E}$ and $T > 0$

$$I_0^* = \limsup_{t \to +\infty} \frac{I(t, x, y)}{t} = \inf_{\phi(0) = \phi(T), \phi(x) \neq \phi(T), \phi(t) \neq \phi(t)} I_{[0, T]}(\phi)/T$$

where the infimum is taken over all continuous functions $\phi : [0, T] \to \mathcal{E}$ with $\phi(0) = \phi(T)$ for which the set $\{ t : \phi(t) = 0 \}$ has Lebesgue measure zero, that the limits $l = \liminf_{t \to +\infty} I_0(t, x, y)/t$, $l_1 = \liminf_{t \to +\infty} I_0(t, 0, y)/t$ and $l_2 = \liminf_{t \to +\infty} I_0(t, 0, 0)/t$ do not depend on $x, y \in \mathcal{E} \setminus \{0\}$ and that for every $T > 0$,

$$I_0(1, 0, 0) = I_0(T, 0, 0)/T = \lim_{t \to +\infty} I_0(t, 0, 0)/t$$

$$\leq \inf_{x \neq 0} I_0(T, x, x)/T = \inf_{\phi(0) = \phi(T), \phi(x) \neq \phi(T)} I_{[0, T]}(\phi), \quad i = 1, 2. $$

Hence, to complete the proof of Relation (26) it is sufficient to show that the infimum at the right hand side of the above relation does not exceed $I_0(T, 0, 0)/T$, or equivalently that

$$I_{[0, T]}(\phi) \geq \inf_{x \neq 0} I_0(T, x, x). $$

for any continuous function $\phi : [0, T] \to \mathcal{E}$ with $\phi(0) = \phi(T) = 0$ and $\phi(t) \neq 0$ for all $t \in [0, T]$. For such a function $\phi$, for every $0 < \delta < T/2$ there is a function $\psi_\delta : [0, T] \to \mathcal{E}$ with $\psi_\delta(0) = \phi(T) = 0$ and $\psi_\delta(t) \neq 0$ for $t \in [0, T]$ and $I_{[0, T]}(\psi_\delta) \leq C\delta$ (see Remark 3). Define the function $\psi_\delta$ by setting $\psi_\delta(t) = \phi(t + \delta)$ for $t \in [0, T - 2\delta]$ and $\psi_\delta(t) = \psi_\delta(t - T + 2\delta)$ for $t \in [T - 2\delta, T - 2\delta + \delta]$. Then, using relation (24), we get

$$I_{[0, T-2\delta + t_\delta]}(\psi_\delta) = I_{[0, t_\delta]}(\psi_\delta) + I_{[\delta, T-\delta]}(\phi) \leq C\delta + I_{[0, T]}(\phi)$$

and using relation (11) it follows that

$$I_{[0, T-2\delta + t_\delta]}(\psi_\delta) \geq I_0(T - 2\delta + t_\delta, \phi(\delta), \phi(\delta))$$

$$\geq \inf_{x \neq 0} I_0(T - 2\delta + t_\delta, x, x) = \frac{1}{T} (T - 2\delta + t_\delta) \inf_{x \neq 0} I_0(T, x, x). $$

This proves that

$$I_{[0, T]}(\phi) \geq \frac{1}{T} (T - 2\delta + t_\delta) \inf_{x \neq 0} I_0(T, x, x) = C\delta. $$

Letting at the last inequality $\delta \to 0$ we obtain Relation (26).
5. Proof of Theorem 1

Recall that \( I(t, x, y) \) denotes the infimum of the rate functions \( I_{[0,t]}(\phi) \) over all continuous functions \( \phi : [0, t] \to \mathcal{E} \), with \( \phi(0) = x \) and \( \phi(t) = y \). By Proposition 1.2

\[
I^* = I_{[0,1]}(0) = \limsup_{t \to +\infty} I(t, x, y) / t = \inf_{\phi(0) = \phi(T)} I_{[0,T]}(\phi) / T
\]

where the infimum is taken over all continuous functions \( \phi : [0, T] \to \mathcal{E} \) with \( \phi(0) = \phi(T) \). To complete the proof of Theorem 1 it is sufficient therefore to show that \( \log r^* \leq -I^* \).

To prove the upper bound

\[(27) \quad \log r^* \leq -I^*.\]

we use Proposition 3.1 and the SPLD upper bound. The SPLD upper bound implies that for all compact sets \( V, W \in \mathcal{K} \),

\[
\limsup_{t \to \infty} \sup_{x \in W} \frac{1}{t} \log P^t 1_V(x) \leq \lim_{\delta \to 0} \lim \inf_{a \to \infty} \sup_{x : |x| < \varepsilon} \inf_{y : |y| < \varepsilon} \mathbb{P}_{a y} (|Z_a(1)| \leq \delta) \leq - \lim_{\delta \to 0} \inf_{\phi(0) = 0, |\phi(1)| \leq \delta} I_{[0,1]}(\phi)
\]

where the infimum is taken over all continuous functions \( \phi : [0, 1] \to \mathcal{E} \) with \( \phi(0) = 0 \) and \( |\phi(1)| \leq \delta \). The right hand side of the last inequality equals

\[- \lim_{\delta \to 0} \inf_{\phi(0) = 0, |\phi(1)| \leq \delta} I(1, 0, y).
\]

The mapping \( (x, y) \to I(1, x, y) \) being lower semi-continuous (see Lemma 4.1), using Proposition 3.1 we conclude that

\[
\log r^* \leq \sup_{W, V} \limsup_{t \to \infty} \sup_{x \in W} \frac{1}{t} \log P^t 1_V(x) \leq -I(1, 0, 0) = -I^*.
\]

Inequality (27) is therefore proved.

To prove the lower bound \( \log r^* \geq -I^* \) we use Relation 8 of Proposition 3.1 and the SPLD lower bound. The hypotheses of Proposition 3.1 are satisfied for every compact set \( V \subset E \) with \( t = a \) and \( V' = V_{a \delta} = \{x \in E : |x| \leq a \delta\} \) for \( \delta > 0 \) and for \( a > 0 \) large enough because

\[
\inf_{0 < a < a} \inf_{y \in V} P^a 1_{V'}(y) \geq \inf_{y : |y| < \varepsilon} \mathbb{P}_{a y} (\|Z_a(s)\|_\infty < \delta)
\]

if \( V \subset \{x : |x| < a \varepsilon\} \), and because by the SPLD lower bound,

\[
\lim_{\varepsilon \to 0} \liminf_{a \to \infty} \inf_{y : |y| < \varepsilon} \inf_{a} \frac{1}{a} \log \mathbb{P}_{a y} (\|Z_a(s)\|_\infty < \delta) \geq - \inf_{\phi : |\phi(0)| = 0, |\phi(1)| \leq \delta} I_{[0,1]}(\phi) \geq -I_{[0,1]}(0) \geq -l_2(0) > -\infty.
\]

Using Relation 8 with \( W = V = \{x \in E : |x| \leq a \varepsilon\} \) and letting \( a \to +\infty \), we obtain

\[
\log r^* \geq \liminf_{a \to \infty} \liminf_{n \to \infty} \frac{1}{an} \inf_{|x| \leq \varepsilon} \log \mathbb{P}_{an} (|Z_a(n)| < \varepsilon).
\]
Hence, by Markov property,
\[
\log r^* \geq \lim_{a \to \infty} \inf_{n \to +\infty} \inf_{|x| < \varepsilon} \frac{1}{an} \log \mathbb{P}_{ax}(\{|Z_a(k)| < \varepsilon, \forall 1 \leq k \leq n\}) \geq \inf_{a \to \infty} \inf_{|x| < \varepsilon} \frac{1}{a} \log \mathbb{P}_{ax}(\{|Z_a(1)| < \varepsilon\}).
\] (28)

Furthermore, SPLD lower bound proves that for any \(\sigma > 0\) and for any \(x \in \mathbb{R}^d\), there exist \(\delta(x) > 0\) and \(a(x) > 0\) such that for all \(a \geq a(x)\) and for all \(x' \in \frac{1}{a}E\) satisfying inequality \(|x' - x| < \delta(x)|,
\[
\frac{1}{a} \log \mathbb{P}_{ax'}(|Z_a(1)| < \varepsilon) \geq -\inf_{\phi: \phi(0) = x, |\phi(T)| < \varepsilon} I_{[0,T]}(\phi) - \sigma \geq -\inf_{y:|y| < \varepsilon} I(1, x, y) - \sigma \geq -I(x, x, 0) - \sigma.
\]

Recall that \(E \subset \mathcal{E}\) and consequently, \(\frac{1}{a}E \subset \mathcal{E}\). The set \(\{x \in \mathcal{E} : |x| \leq \varepsilon\}\) being compact, there are \(x_1, \ldots, x_n \in \{x \in \mathcal{E} : |x| \leq \varepsilon\}\) such that
\[
\{x \in \mathcal{E} : |x| \leq \varepsilon\} \subset \bigcup_{i=1}^n \{x' : |x' - x_i| < \delta(x_i)\}.
\]

For \(a \geq \max\{a(x_1), \ldots, a(x_n)\}\), we obtain therefore
\[
\inf_{x \in \mathcal{E} : |x| < \varepsilon} \frac{1}{a} \log \mathbb{P}_{ax}(\{|Z_a(1)| < \varepsilon\}) \geq -\max I(1, x_i, 0) - \sigma \geq -\sup_{x:|x| \leq \varepsilon} I(1, x, 0) - \sigma.
\]

The last relation combined with Inequality (28) proves that
\[
\log r^* \geq -\sup_{x:|x| \leq \varepsilon} I(1, x, 0).
\] (29)

Finally, Relations (18) and (20) show that for any \(0 < t < 1\),
\[
I(1, x, 0) \leq I(t, x, 0) + I(1-t, 0, 0) = I(t, x, 0) + (1-t)I(1, 0, 0) \leq I(t, x, 0) + I^*\]
and consequently, for \(t = \varepsilon/c\), using Inequality (21) it follows that
\[
\log r^* \geq -\sup_{x:|x| \leq \varepsilon} I(1, x, 0) \geq -\sup_{x:|x| \leq \varepsilon} I(\varepsilon/c, x, 0) - I^* = -C\varepsilon/c - I^*.
\]

Letting at the last inequality \(\varepsilon \to 0\), we conclude \(\log r^* \geq -I^*\). The inequality \(\log r^* \geq -I^*\) combined with (24) proves that \(\log r^* = -I^*\).

6. Proof of Theorem (21)

Recall that for \(x, y \in \mathbb{R}^d\) and \(t > 0\), \(\tilde{I}(t, x, y)\) denotes the infimum of \(I_{[0,t]}(\phi)\) over all continuous functions \(\phi : [0, t] \to \mathcal{E}\) with \(\phi(0) = x, \phi(t) = y\) for which the set \(\{s \in [0, t] : \phi(s) = 0\}\) has Lebesgue measure zero. \(I_0(t, x, y)\) denotes the infimum of \(I_{[0,t]}(\phi)\) over all continuous functions \(\phi : [0, t] \to \mathcal{E}\) such that \(\phi(0) = x, \phi(t) = y\) and \(\phi(s) \neq 0\) for all \(0 < s < t\). By Proposition (1.2)
\[
I_0^* = I_0(1, 0, 0) = \lim_{t \to +\infty} \sup_{t \to +\infty} I_0(t, x, y)/t = \lim_{t \to +\infty} \sup_{t \to +\infty} \tilde{I}(t, x, y)/t = \inf_{\phi(0) = x, \phi(T)} I_{[0,T]}(\phi)/T\]
where the infimum is taken over all continuous functions \(\phi : [0, T] \to \mathcal{E}\) with \(\phi(0) = \phi(T)\) such that \(\phi(t) \neq 0\) for all \(0 < t < T\).
To prove Theorem 2 it is sufficient therefore to show that \( \log r^*_n = -I^*_n \).

### 6.1. Lower bound \( \log r^*_n \geq -I^*_0 \)

The proof of this inequality uses Relation (11) of Proposition 3.3 and SPLD lower bound. We begin our proof with the following lemma.

**Lemma 6.1.** Under the hypotheses (A), for any compact subset \( K \subset E \), for any \( x \in E \setminus \{0\} \), and for any \( \delta > 0 \) such that \( |x| > \delta \), the inequality

\[
(31) \quad \inf_{0 < s \leq t} \inf_{x \in V_K} P_x(X(s) \in V'_K, \tau(K) > s) > 0.
\]

holds with \( V_K = \{ y \in E : |ax - y| \leq a\varepsilon \} \) and \( V'_K = \{ y \in E : |ax - y| \leq a\delta \} \) for all \( a > 0 \) large enough.

**Proof.** Let \( x \in E \setminus \{0\} \), \( \sigma > 0 \) and \( \delta > \varepsilon > 0 \) be such that \( |x| > \delta + \sigma \). For a compact subset \( K \subset E \) are \( a_K > 0 \) such that for \( a > a_K \), \( K \subset \{ y : |y| < a\sigma \} \). For \( V_K = \{ y \in E : |ax - y| \leq a\varepsilon \} \) and \( V'_K = \{ y \in E : |ax - y| \leq a\delta \} \),

\[
(32) \quad \inf_{0 < s \leq a} \inf_{x \in V_K} P_x(X(s) \in V'_K, \tau(K) > s) \geq \inf_{y : |y - x| < \varepsilon} P_{ay} \left( \sup_{0 \leq s \leq 1} |Z_a(s) - x| < \delta \right)
\]

for all \( a > a_K \). Moreover because of SPLD lower bound and Assumption (a3),

\[
\lim_{\varepsilon \to 0} \liminf_{a \to \infty} \inf_{y : |y - x| < \varepsilon} \frac{1}{a} \log P_{ay} \left( \sup_{0 \leq s \leq 1} |Z_a(s) - x| < \delta \right) \geq -I_{[0,1]}(\phi) \geq -I_2(0) > -\infty
\]

where \( \phi(t) \equiv x \). The last inequality shows that for any \( \varepsilon > 0 \) small enough, there is \( a_\varepsilon > a_K \) such that for every \( a > a_\varepsilon \), the right hand side of Inequality (32) is strictly positive and hence, Inequality (31) holds. \( \square \)

Because of Lemma 6.1 and Proposition 3.3 Relation (11) holds for any compact set \( K \subset E \) with \( V_K = \{ y \in E : |ax - y| \leq a\varepsilon \} \) for any \( 0 < \varepsilon < |x| \) and for all \( a > 0 \) large enough. Using this relation together with Markov property we obtain

\[
\log r^*_n \geq \liminf_{K} \limsup_{t \to \infty} \inf_{x \in V_K} \frac{1}{t} \log P_x(X(t) \in V_K, \tau(K) > t)
\]

\[
\geq \liminf_{a \to \infty} \liminf_{n \to \infty} \inf_{y : |y - x| \leq \varepsilon} \frac{1}{anT} \log P_{ay} \left( |Z_a(nT) - x| < \varepsilon, \inf_{s \in [0,nT]} |Z_a(s)| > \sigma \right)
\]

\[
\geq \liminf_{a \to \infty} \liminf_{n \to \infty} \inf_{y : |y - x| \leq \varepsilon} \frac{1}{anT} \log P_{ay} \left( \max_{k \leq n} |Z_a(kT) - x| < \varepsilon, \inf_{s \in [0,nT]} |Z_a(s)| > \sigma \right)
\]

\[
\geq \liminf_{a \to \infty} \inf_{y : |y - x| \leq \varepsilon} \frac{1}{aT} \log P_{ay} \left( |Z_a(T) - x| < \varepsilon, \inf_{s \in [0,T]} |Z_a(s)| > \sigma \right)
\]

for any \( T > 0 \). The last inequality combined with the SPLD lower bound, the same arguments as in the proof of Theorem 1 (see the proof of inequality (28)) and Relation (20) show that for any \( T > 1 \),

\[
\log r^*_n \geq - \sup_{y \in E : |x - y| \leq \varepsilon} I_\sigma(T, y, x)/T
\]

\[
\geq - \sup_{y \in E : |x - y| \leq \varepsilon} I_\sigma(1, y, x)/T - I_\sigma(T - 1, x, x)/T
\]

(33)
Using this relation for the right hand side of Inequality (33) and letting \( \sigma \to 0 \) we get

\[
|\phi_y(t)| \geq |x| - |\phi_y(t) - x| \geq |x| - \varepsilon > \sigma
\]

for all \( t \in [0, 1] \) and consequently, by Assumption (A),

\[
I_\sigma(1, y, x) \leq I_{[0,1]}(\phi_y) = \int_0^1 L(\phi_y(t), \dot{\phi}_y(t)) dt \leq l_2(y - x).
\]

Using this relation for the right hand side of Inequality (34) and letting \( \sigma \to 0 \) we obtain

\[
\log r^*_e \geq - \sup_{v: |v| \leq \varepsilon} l_2(v)/T - I_0(T - 1, x, x)/T.
\]

The function \( l_2 \) being finite in a neighborhood of zero, this implies that

\[
\log r^*_e \geq - \limsup_{T \to \infty} I_0(T - 1, x, x)/T.
\]

The last inequality combined with Relation (30) proves that \( \log r^*_e \geq - I_0^* \).

6.2. Upper bound \( \log r^*_e \leq - I_0^* \). To prove this relation we use Proposition 6.2 and the SPLD upper bound. Because of Proposition 6.2 for any \( N > 0 \),

\[
(34) \quad \log r^*_e \leq \limsup_{\delta \to 0} \limsup_{a \to \infty} \log \mathbb{P}_{az} \left( \sup_{a \in [0,1]} |Z_a(s)| > N, \forall 0 \leq s \leq T \right).
\]

To estimate the right hand side of this inequality, we use cluster expansion method and SPLD upper bound. Before to introduce the notion of clusters, we consider the following preliminary results.

For \( x, y \in \mathcal{E} \) and \( t > 0 \), let \( M_N(t, x, y) \) denote the infimum of the rate function \( I_{[0,t]}(\phi) \) over all functions \( \phi \in D([0, t], \mathcal{E}) \) such that \( \phi(0) = x, \phi(t) = y \) and

\[
\sup_{0 \leq s \leq t} |\phi(s)| \geq N t.
\]

**Lemma 6.2.** For \( x, y \in \mathcal{E} \) and \( N > 0 \),

\[
(35) \quad \lim_{\delta \to 0} \limsup_{a \to \infty} \frac{1}{a} \log \mathbb{P}_{az} \left( \sup_{a \in [0,1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta) \right) \leq -M_N(1, x, y).
\]

**Proof.** Using SPLD upper bound it follows that for any \( x, y \in \mathcal{E} \) and for any \( \sigma > 0 \),

\[
\lim_{\delta \to 0} \limsup_{a \to \infty} \frac{1}{a} \log \mathbb{P}_{az} \left( \sup_{a \in [0,1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta) \right) \leq \lim_{\delta \to 0} \limsup_{a \to \infty} \frac{1}{a} \log \mathbb{P}_{az} \left( \sup_{a \in [0,1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in \overline{B}(y, \sigma) \right) \leq - \inf_{y' \in \overline{B}(y, \sigma)} M_N(1, x, y').
\]

By Lemma 6.2 the mapping \((x, y) \to M_N(1, x, y)\) is lower semi-continuous and hence, letting at the last relation \( \sigma \to 0 \) we get inequality (34). \( \square \)
By Lemma 6.2, for any $\varepsilon > 0$, $N > 0$ and $x, y \in \mathcal{E}$, there exist $\delta(x, y) > 0$ and $a(x, y) > 0$ such that for any $a > a(x, y)$ and for any $z \in \frac{1}{a}E$ satisfying the inequality $|z - x| \leq \delta(x, y)$, the following inequalities hold.

$$\log \mathbb{P}_{az} \left( \sup_{s \in [0,1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta(x, y)) \right) \leq -aM_N(1, x, y) + a\varepsilon$$

when $M_N(1, x, y) < +\infty$, and

$$\log \mathbb{P}_{az} \left( \sup_{s \in [0,1]} |Z_a(s)| \geq N \text{ and } Z_a(1) \in B(y, \delta(x, y)) \right) \leq -aI_0^* N/\varepsilon$$

when $M_N(1, x, y) = +\infty$. Moreover, the real number $a \geq a(x, y)$ at the above inequalities can be replaced by $at$ with $a \geq 2a(x, y)$ and $t \geq 1/2$. Using relation $Z_{at}(s) = Z_a(ts)/t$ this implies the following statement.

**Lemma 6.3.** For any $\varepsilon > 0$ and $N > 0$, there are strictly positive functions $\delta(\cdot, \cdot)$ and $a(\cdot, \cdot)$ on $\mathcal{E} \times \mathcal{E}$ such that for any $x, y \in \mathcal{E}$, $t > 1/2$, $a > 2a(x, y)$ and $z \in \frac{1}{a}E$ satisfying the inequality $|z - tx| \leq t\delta(x, y)$, the following inequalities hold

$$\log \mathbb{P}_{az} \left( \sup_{s \in [0,t]} |Z_a(s)| \geq Nt \text{ and } Z_a(t) \in tB(y, \delta(x, y)) \right) \leq -atM_N(1, x, y) + at\varepsilon$$

when $M_N(1, x, y) < +\infty$, and

$$\log \mathbb{P}_{az} \left( \sup_{s \in [0,t]} |Z_a(s)| \geq Nt \text{ and } Z_a(t) \in tB(y, \delta(x, y)) \right) \leq -atI_0^* N/\varepsilon$$

otherwise.

We are ready now to introduce the notion of cluster. For a given $\varepsilon > 0$, we choose $N \geq 1$ large enough so that

$$I_{[0,t]}^{(\varepsilon)}(\phi) > I_0^*/\varepsilon$$

for any function $\phi \in D([0,t], \mathcal{E})$ for which there are $0 < s_1 < s_2 \leq 1$ such that $|\phi(s_1) - \phi(s_2)| \geq N$. Such a choice is possible because under the hypotheses (A), for any absolutely continuous function $\phi : [0,1] \to \mathcal{E}$ for which there are $0 < s_1 < s_2 \leq 1$ such that $|\phi(s_1) - \phi(s_2)| \geq N$,

$$I_{[0,1]}(\phi) \geq \int_{s_1}^{s_2} L(\phi(s), \phi'(s))ds \geq \int_{s_1}^{s_2} \int_{s_1}^{s_2} L(\phi(s), \phi'(s))ds \geq (s_2 - s_1)I_1(\phi(s))ds \geq N \inf_{v : |v| \geq N} I_1(v)/|v| \to +\infty$$

when $N \to \infty$.

Given $\varepsilon > 0$ and $N \geq 1$, let $\delta(\cdot, \cdot)$ and $a(\cdot, \cdot)$ be the positive functions on $\mathcal{E} \times \mathcal{E}$ satisfying Lemma 6.3. Without any restriction of generality we will suppose that for any $x, y \in \mathcal{E}$,

$$\delta(x, y) \leq \varepsilon.$$
The set \( \overline{B}(0,2N) \times \overline{B}(0,2N) \) being compact there exists a finite subset \( V(N, \varepsilon) \subset \overline{B}(0,2N) \times \overline{B}(0,2N) \) such that
\[
\overline{B}(0,2N) \times \overline{B}(0,2N) \subset \bigcup_{(x,y) \in V(N, \varepsilon)} B(x, \delta(x,y)) \times B(y, \delta(x,y)).
\]

For a function \( \phi \in D([T_1, T_2], \mathcal{E}) \) such that \(|\phi(t)| \geq N \) for all \( T_1 \leq t \leq T_2 \) and \(|\phi(T_1)| \leq 2(T_2 - T_1)N \), \(|\phi(T_2)| \leq 2(T_2 - T_1)N \), we define a partition
\[
T_1 = t_0 < t_1 < \ldots < t_n = T_2
\]
and a sequence \(( (x_i, y_i) \in V(\varepsilon, N), i = 1, \ldots, n ) \) by induction:

- We choose \((x_1, y_1) \in V(\varepsilon, N) \) such that \( \phi(T_1) \in (T_2 - T_1)B(x_1, \delta(x_1, y_1)) \) and \( \phi(T_2) \in (T_2 - T_1)B(y_1, \delta(x_1, y_1)) \), and we let \( n = 1 \) if
  \[
  \sup_{T_1 \leq t \leq T_2} |\phi(t)| > N(T_2 - T_1).
  \]
- Otherwise, we divide the interval \([T_1, T_2] \) in two intervals \([T_1, (T_2 + T_1)/2] \) and \([(T_2 - T_1)/2, T_2] \) and we restart our construction for the restriction of \( \phi \) on each of them.

This algorithm terminates because \(|\phi(t)| \geq N \) for all \( t \in [T_1, T_2] \). The resulting sequence
\[
\Gamma(\phi) = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n))
\]
is called \((\varepsilon, N)\)-cluster corresponding to the function \( \phi \). Remark that for such a sequence,
\[
\phi(t_i) \in (t_i - t_{i-1})B(y_i, \delta(x_i, y_i)) \cap (t_{i+1} - t_i)B(x_{i+1}, \delta(x_{i+1}, y_{i+1})) \neq \emptyset
\]
for every \( i = 1, \ldots, n - 1 \) and \( (t_i - t_{i-1}) \geq 1/2 \) for all \( i = 1, \ldots, n \). Moreover, for \( T_1 = 0 \) and \( T_2 = 1 > 1 \) and for a natural number \( k \) such that \( 2^{k-1} < T \leq 2^k \), by construction, \( t_i 2^k/T \in \mathbb{N} \) for all \( i = 0, \ldots, n \).

**Definition 6.1.** A sequence \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) is called \((T, \varepsilon, N)\)-cluster if

- 0 = \( t_0 = t_1 = \ldots = t_n = T \) and \( t_i - t_{i-1} \geq 1/2 \) for all \( i = 1, \ldots, n \);
- \( t_i 2^k/T \in \mathbb{N} \) for all \( i = 0, \ldots, n \);
- the set \( (t_i - t_{i-1})B(y_i, \delta(x_i, y_i)) \cap (t_{i+1} - t_i)B(x_{i+1}, \delta(x_{i+1}, y_{i+1})) \) is non-empty for every \( i = 1, \ldots, n - 1 \).

For a given \((T, \varepsilon, N)\)-cluster \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \), we denote by \( D(\Gamma) \) the set of all functions \( \phi : [0, T] \to \mathcal{E} \) such that \( \Gamma(\phi) = \Gamma \). The quantity \( \chi_a(\Gamma) \) is defined by
\[
\chi_a(\Gamma) = \sup_x P_x \left( Z_a \in D(\Gamma) \right)
\]
where the supremum is taken over all \( x \in E \) such that \( |x - at_1x_1| \leq at_1\delta(x_1, y_1) \).

Remark that by construction,
\[
\sup_{N < |z| \leq kN} P_{az} \left( |Z_a(T)| \leq bN, \ |Z_a(s)| \geq N, \ \forall 0 \leq s \leq T \right) \leq \sum_{\Gamma} \chi_a(\Gamma),
\]
where the sum is taken over all \((T, \varepsilon, N)\)-clusters \( \Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)) \) for which the sets \( t_1B(x_1, \delta(x_1, y_1)) \cap B(0, bN) \) and \( (t_n - t_{n-1})B(y_n, \delta(x_n, y_n)) \cap B(0, bN) \) are non-empty.

To estimate the right hand side of the inequality \( \Box \) (and hence also the right hand side of the inequality \( \Box \)) we estimate the number of \((T, \varepsilon, N)\)-clusters and
Lemma 6.5. For any $I$ is a local rate function. Moreover, there is a finite in a neighborhood of zero convex function if the function possible choices for a sequence $(x_1, y_1), \ldots, (x_n, y_n) \in V(N, \varepsilon)$. To estimate the quantities $\chi_n(\Gamma)$ we use Assumption (A) and Lemma 6.3. Recall that under the hypotheses (A), the rate function $I_{[0,T]}$ has an integral form: there is a local rate function $L : \mathcal{E} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that

$$I_{[0,T]}(\phi) = \int_0^T L(\phi(t), \dot{\phi}(t)) \, dt$$

if the function $\phi : [0, 1] \rightarrow \mathcal{E}$ is absolutely continuous, and $I_{[0,1]}(\phi) = +\infty$ otherwise. Moreover, there is a finite in a neighborhood of zero convex function $l_2 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that $L(x, v) \leq l_2(v)$ for all $x \in \mathcal{E}$ and for all $v \in \mathbb{R}^d$. The function $l_2$ being convex and finite in a neighborhood of zero, there are real numbers $C > 0$ and $c > 0$ such that

$$\sup_{v : |v| \leq \varepsilon} l_2(v) \leq C.$$  

Lemma 6.5. For any $T > b > 1$, for any $a \geq 2 \max\{a(x, y) \mid (x, y) \in V(N, \varepsilon)\}$ and for any $(T, \varepsilon, N)$-cluster $\Gamma = \{(t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)\}$ for which the sets $t_i B(x_1, \delta(x_1, y_1)) \cap B(0, bN)$ and $(t_n - t_{n-1}) B(y_n, \delta(x_n, y_n)) \cap B(0, bN)$ are non-empty, the following inequality holds

$$\frac{1}{a} \log \chi_n(\Gamma) \leq -T(1 - 2\varepsilon) I_0^* + 4T\varepsilon C/c + T\varepsilon + 2bNC/c$$

Proof. Using Lemma 6.3 and Markov property, it follows that for any real number $a \geq 2 \max\{a(x, y) \mid (x, y) \in V(N, \varepsilon)\}$, for any $T \geq 1$ and for any $(T, \varepsilon, N)$-cluster $\Gamma = \{(t_1, x_1, y_1), \ldots, (t_n, x_n, y_n)\}$,

$$\frac{1}{a} \log \chi_n(\Gamma) \leq -\sum' (t_i - t_{i-1}) (M_N(1, x_i, y_i) - \varepsilon) - \sum'' (t_i - t_{i-1}) I_0^* N/\varepsilon$$

where $\sum'$ denotes the sum over all $i = 1, \ldots, n$ for which $M_N(1, x_i, y_i) < +\infty$ and $\sum''$ is the sum over all those $i = 1, \ldots, n$ for which $M_N(1, x_i, y_i) = +\infty$. If $\sum'' (t_i - t_{i-1}) \geq T\varepsilon/N$, because of Relation 12, \log $\chi_n(\Gamma)$ \leq -TI_0^* a + Ta\varepsilon$ and hence, inequality 12 holds.

Furthermore, let us consider for every $1 \leq i \leq n$, for which $M_N(1, x_i, y_i) < +\infty$, a continuous function $\phi_i : [0, 1] \rightarrow \mathcal{E}$ with $\phi_i(0) = x_i, \phi_i(1) = y_i$ and sup$_t |\phi(t)| \geq N$, such that

$$I_{[0,1]}(\phi_i) = M_N(1, x_i, y_i).$$

Such a function exists because the set of all continuous functions $\phi : [0, 1] \rightarrow \mathcal{E}$ with $\phi(0) = x_i, \phi(1) = y_i$ and sup$_t |\phi(t)| \geq N$ is closed in $D([0,1], \mathcal{E})$, and because the level sets $\{\phi : \phi(0) = x_i, I_{[0,1]}(\phi) \leq c\}$ are compact (see the definition of a good rate function in Section 4).
Let \( J(\Gamma) \) be the set of all those \( 1 \leq i \leq n \) for which the set \( \{ t : \phi_i(t) = 0 \} \) is empty and \( M_N(1, x_i, y_i) < +\infty \). Because of relation \((35)\), \( M_N(1, x_i, y_i) > I_0^a/\varepsilon \) for all \( i \notin J(\Gamma) \). When \( \sum_{i \in J(\Gamma)} (t_i - t_{i-1}) > T \varepsilon \), using Inequality \((33)\) we get 
\[ \log \chi_a(\Gamma) \leq -TI_0^a + T\varepsilon \] and hence, inequality \((12)\) holds.

To prove our lemma, it is sufficient now to verify Inequality \((12)\) in the case when

\[
\sum_{i \in J(\Gamma)} (t_i - t_{i-1}) < T \varepsilon \quad \text{and} \quad \sum''(t_i - t_{i-1}) < T \varepsilon /N.
\]

For this, we construct an absolutely continuous function \( \phi : [0, \tilde{t}] \to \mathcal{E} \) with \( \phi(0) = \phi(\tilde{t}) = 0 \) for which the Lebesgue measure of the set \( \{ s \in [0, \tilde{t}] : \phi(s) \neq 0 \} \) is greater than \( T(1-2\varepsilon) \) and

\[
I_{[0,\tilde{t}]}(\phi) \leq \sum(t_i - t_{i-1})M_N(1, x_i, y_i) + 2T \varepsilon C/c + 2(bN + T \varepsilon)C/c.
\]

\textbf{Construction of the function} \( \phi \): We let \( \tilde{x}_i = (t_i - t_{i-1})x_i \) and \( \tilde{y}_i = (t_i - t_{i-1})y_i \) for every \( i = 1, \ldots, n \) and we define \( 0 = t'_0 < t''_0 < t'_1 < t''_1 < \ldots < t''_n < t = \phi(\tilde{t}) \) by setting \( t''_0 = \Delta_0 \) and \( t''_n = t' + \Delta_1 \) for all \( i = 0, \ldots, n, \) where \( \Delta_0 = (t_1 \varepsilon + bN)/c \), \( \Delta_n = ((t_n - t_{n-1})\varepsilon + bN)/c \) and \( \Delta_1 = (t_{i+1} - t_i - t)\varepsilon /c \) for \( i = 1, \ldots, n - 1 \), and by letting \( t''_i = \sum_{l=0}^i t''_l + (t_i - t_{i-1}) \) if \( M_N(1, x_i, y_i) < +\infty \) and \( t''_i = \sum_{l=0}^i t''_l + 2N(t_i - t_{i-1})/c \) otherwise. The function \( \phi : [0, \tilde{t}] \to \mathcal{E} \) is defined then in the following way: for \( t \in [t''_i, t''_{i+1}], i = 1, \ldots, n, \) we let

\[
\phi(t) = \begin{cases} 
(t_i - t_{i-1})\phi_i((t - t''_{i-1})/(t_i - t_{i-1})) & \text{if } M_N(1, x_i, y_i) < +\infty, \\
\tilde{x}_i + c(y_i - \tilde{x}_i)(t - t''_{i-1})/(2N(t_i - t_{i-1})) & \text{if } M_N(1, x_i, y_i) = +\infty.
\end{cases}
\]

Then for every \( 1 < i < n, \) \( \phi(t''_i) = (t_i - t_{i-1})y_i = \tilde{y}_i \) and \( \phi(t''_n) = (t_{i+1} - t_i)x_{i+1} = \tilde{x}_{i+1}. \) For \( t \in [t''_{i-1}, t''_i], i = 0, \ldots, n, \) we let

\[
\phi(t) = \tilde{y}_i + ((\tilde{x}_{i+1} - \tilde{x}_i)(t - t''_i)/(t''_i - t'_i)
\]

where \( \tilde{y}_0 = \tilde{x}_{n+1} = 0. \) The resulting function \( \phi \) is absolutely continuous on \([0, \tilde{t}]\) and \( \phi(0) = \phi(\tilde{t}) = 0. \) Moreover, by construction, \( \phi(t) \neq 0 \) for all \( t \in \cup_{i \in J(\Gamma)}[t''_{i-1}, t''_i]. \)

Using Relations \((35)\) this implies that

\[
\text{mes}\{ t : \phi(t) \neq 0 \} \geq \sum_{i \in J(\Gamma)} (t_i - t_{i-1})
\]

\[
\geq T - \sum_{i \in J(\Gamma)} (t_i - t_{i-1}) - \sum''(t_i - t_{i-1}) \geq T - 2T \varepsilon.
\]

The last inequality combined with Lemma \((12)\) and Proposition \((12)\) show that

\[
I_{[0,\tilde{t}]}(\phi) \geq T(1 - 2\varepsilon)I_0^a.
\]

\textbf{Proof of Inequality} \((16)\): Relations \((15)\) and \((14)\) imply that

\[
I_{[t''_{i-1}, t''_i]}(\phi) = (t_i - t_{i-1})I_{[0,\tilde{t}]}(\phi_i) = (t_i - t_{i-1})M_N(1, x_i, y_i)
\]

for all \( i = 1, \ldots, n \) for which \( M_N(1, x_i, y_i) < +\infty. \) When \( M_N(1, x_i, y_i) = +\infty, \) using Assumption (A) and Inequality \((11)\) we obtain

\[
I_{[t''_{i-1}, t''_i]}(\phi) \leq (t''_i - t'_i)l_2((y_i - x_i)c)/2N \leq 2NC(t_i - t_{i-1})/c.
\]

Because of Assumption (A),

\[
I_{[t''_{i-1}, t''_i]}(\phi) \leq (t''_i - t'_i)l_2((\tilde{x}_{i+1} - \tilde{y}_i)/\Delta_i)
\]
for all \( i = 0, \ldots, n \). Moreover, for \( 1 \leq i \leq n-1 \),
\[
|\hat{x}_{i+1} - \hat{y}_i| = |(t_{i+1} - t_i)x_{i+1} - (t_i - t_{i-1})y_i|
\leq (t_i - t_{i-1})\delta(x_i, y_i) + (t_{i+1} - t_i)\delta(x_{i+1}, y_{i+1}) \leq (t_i - t_{i-1})\varepsilon = c\Delta_i
\]
where the first inequality holds because by definition of \((T, \varepsilon, N)\)-cluster, the set
\[
(t_i - t_{i-1})B(y_i, \delta(x_i, y_i)) \cap (t_{i+1} - t_i)B(x_{i+1}, \delta(x_{i+1}, y_{i+1}))
\]
is non-empty and the second inequality follows from Relation (39). Similarly, for \( i = 0 \),
\[
|\hat{x}_1 - \hat{y}_0| = t_1|x_1| \leq t_1\delta(x_1, y_1) + bN \leq t_1\varepsilon + bN = c\Delta_0
\]
and for \( i = n \),
\[
|\hat{x}_{n+1} - \hat{y}_n| = (t_n - t_{n-1})|y_n| \leq (t_n - t_{n-1})\delta(x_n, y_n) + bN
\leq (t_n - t_{n-1})\varepsilon + bN = c\Delta_n
\]
because under the hypotheses of our lemma, the sets \( t_1B(x_1, \delta(x_1, y_1)) \cap B(0, bN) \) and \( (t_n - t_{n-1})B(y_n, \delta(x_n, y_n)) \cap B(0, bN) \) are non-empty. Using Inequality (41) this implies that \( I_2((\hat{x}_{i+1} - \hat{y}_i)/\Delta_i) \leq C \) and consequently,
\[
I_{[r', r'']}([\hat{t}_i, \hat{t}_i'')}([\phi]) \leq (t''_i - t'_i)C = \Delta_i C
\]
for every \( i = 0, \ldots, n \). The last inequality, Relations (43), (48), (49) and the equality
\[
\sum_{i=0}^{n} \Delta_i = (2bN + t_1\varepsilon + (t_n - t_{n-1})\varepsilon + \varepsilon\sum_{i=1}^{n-1}(t_{i+1} - t_{i-1})/c) = 2(bN + T\varepsilon)/c
\]
show that
\[
I_{[0,1]}([\phi]) = \sum_{i=1}^{n} I_{[r'_{i-1}, r_i]}([\phi]) + \sum_{i=0}^{n} I_{[r'_i, r'_{i+1}]}([\phi])
\leq \sum_i(t_i - t_{i-1})M_N(1, x_i, y_i) + 2NC\sum_i(t_i - t_{i-1})/c + \sum_i \Delta_i C
\leq \sum_i(t_i - t_{i-1})M_N(1, x_i, y_i) + 2TC\varepsilon + 2(bN + T\varepsilon)C/c.
\]
Relation (46) is therefore verified.

We are ready now to complete the proof of Lemma 6.5: Relations (43), (48), (49) and (47) imply that
\[
\frac{1}{a} \log \chi_a(\Gamma) \leq -\sum_i(t_i - t_{i-1})(M_N(1, x_i, y_i) - \varepsilon)
\leq -I_{[0,1]}([\phi]) + 2T\varepsilon C/c + 2(bN + T\varepsilon)C/c + \varepsilon\sum_i(t_i - t_{i-1})
\leq -T(1 - 2\varepsilon)I_0 + 4T\varepsilon C/c + 2bNC/c + T\varepsilon.
\]
The last inequality proves Relation (42). \( \square \)

Let us complete now the proof of the inequality \( \log r^*_e \leq -I_0^* \). Inequalities (44) and (49) prove that
\[
\log r^*_e \leq \limsup_{a \to +\infty} \limsup_{b \to +\infty} \limsup_{T \to +\infty} \frac{1}{Ta} \log \sum_{\Gamma} \chi_a(\Gamma).
\]
The sum is taken here over all \((T, \varepsilon, N)\)-clusters \(\Gamma = ((t_1, x_1, y_1), \ldots, (t_n, x_n, y_n))\) for which the sets \(t_i B(x_i, \delta(x_i, y_i)) \cap B(0, bN)\) and \((t_n - t_{n-1})B(y_n, \delta(x_n, y_n)) \cap B(0, bN)\) are non-empty. Using Lemma 6.4 and Lemma 6.5 we obtain therefore

\[
\log r^*_c \leq \limsup_{a \to +\infty} \limsup_{b \to +\infty} \limsup_{T \to +\infty} \left( \frac{2}{a} \log(2V(N, \varepsilon)) + \max \chi(I) \frac{1}{T} \log \chi(I) \right)
\]

\[
\leq \limsup_{a \to +\infty} \left( \frac{2}{a} \log(2V(N, \varepsilon)) - (1 - 2\varepsilon)I_0 + 4\varepsilon C/c + \varepsilon \right)
\]

\[
\leq -(1 - 2\varepsilon)I_0 + 4\varepsilon C/c + \varepsilon
\]

Letting finally \(\varepsilon \to 0\) we conclude that \(\log r^*_c \leq -I_0^*\).

### 7. Example: Applications to Jackson Networks

In this section, we apply our results to Jackson networks. Let us recall the definition and some well-known results concerning Jackson network. This is a network with \(d\) queues. For \(i = 1, \ldots, d\), the arrivals and the service times at the \(i\)-th queue are Poisson with parameters \(\lambda_i\) and \(\mu_i\) respectively. All the Poisson processes are independent. When the customer finish its service at queue \(i\), it goes to the queue \(j\) with probability \(p_{ij}\). The residual quantity \(p_0 = 1 - p_{11} + \cdots + p_{id}\) is the probability that the customer leaves definitely the network. We assume that \(p_{ii} = 0\) for all \(i = 1, \ldots, d\).

Let \(X_i(t)\) denote the length of the queue \(i\) at time \(t\), \(i = 1, \ldots, d\). Then \(X(t) = (X_1(t), \ldots, X_d(t))\) is a continuous time Markov chain on \(\mathbb{Z}_+^d\) with generator

\[
\mathcal{G}f(x) = \sum_{y \in \mathbb{Z}_+^d} q(x, y)(f(y) - f(x)), \quad x \in \mathbb{Z}_+^d,
\]

where

\[
q(x, y) = \begin{cases} 
\lambda_i & \text{if } y - x = \epsilon_i, \ i \in \{1, \ldots, d\}, \\
\mu_i p_{ij} & \text{if } y - x = -\epsilon_i, \ i \in \{1, \ldots, d\}, \\
\mu_i p_{0j} & \text{if } y - x = \epsilon_j - \epsilon_i, \ i, j \in \{1, \ldots, d\}, i \neq j, \\
0 & \text{otherwise,}
\end{cases}
\]

\(\epsilon_i\) is the \(i\)-th unit vector. We set \(p_{00} = 1\) and \(p_{0i} = 0\) for all \(i \neq 0\), the matrix \((p_{ij}; i, j = 0, \ldots, d)\) is then stochastic.

**Assumption (J)** We suppose that

1. the spectral radius of the matrix \((p_{ij}; i, j = 1, \ldots, d)\) is strictly less than unity,
2. for any \(1 \leq i \leq d\), there exists \(n \in \mathbb{N}\) and \(1 \leq j \leq d\) such that \(\lambda_j p^{(n)}_{ji} > 0\) where \(p^{(n)}_{ji}\) denotes the \(n\)-time transition probability of a Markov chain with \(d + 1\) states associated to the stochastic matrix \((p_{ij}; i, j = 0, \ldots, d)\).

The Markov process \((X(t))\) is then irreducible. The system of traffic equations

\[
\nu_i = \lambda_i + \nu_1 p_{1i} + \cdots + \nu_d p_{di}
\]

has the unique solution \((\nu_i)\) (see [9]). The Markov process \((X(t))\) is recurrent if and only if \(\nu_i \leq \mu_i\) for all \(i = 1, \ldots, d\), and it is ergodic (positive recurrent) if and only if \(\nu_i < \mu_i\) for all \(i = 1, \ldots, d\).
Recall that the spectral radius of the process \((X(t))\) is defined by
\[
r^* = \inf \left\{ r > 0 : \int_0^\infty r^{-t}\mathbb{P}_x(X(t) = y)\,dt < +\infty, \ \forall x, y \in \mathbb{Z}^d \right\}
\]
When the process \((X(t))\) is recurrent we have obviously \(r^* = 1\). For a transient Markov process, spectral radius \(r^*\) shows how fast the process goes to infinity.

Essential spectral radius \(r_e^*\) is defined as the infimum over all those \(r > 0\) for which there is a finite set \(K \subset \mathbb{Z}^d_+\) such that
\[
\int_0^\infty r^{-t}\mathbb{P}_x(X(t) = y, \tau(K) > t)\,dt < +\infty \quad \text{for all } x, y \in \mathbb{Z}^d_+ \setminus K
\]
where \(\tau(K)\) denotes the first time when the process \((X(t))\) hits the set \(K\).

When the Markov process \((X(t))\) is recurrent the quantity \(r_e^*\) is the infimum over all those \(r > 0\) for which there is a finite set \(K \subset \mathbb{Z}^d_+\) such that
\[
\int_0^\infty r^{-t}\mathbb{P}_x(\tau(K) > t)\,dt < +\infty \quad \text{for all } x \in \mathbb{Z}^d_+ \setminus K.
\]
The hypotheses of Proposition 3.6 are satisfied here because the Markov process \((X(t))\) has uniformly bounded jumps.

The Markov process \((X(t))\) satisfies the hypotheses (A) : the family of scaled Markov processes \((Z_{\alpha}(t), t \in [0,T]) \overset{\text{def.}}{=} (X(at)/a, t \in [0,T])\) satisfies sample path large deviation principle (see [1, 5, 8]) with a good rate function having an integral form:
\[
I_{[0,T]}(\phi) = \int_0^T L(\phi(t), \dot{\phi}(t))\,dt
\]
for every absolutely continuous function \(\phi : [0,T] \to \mathbb{R}^d_+\), and \(I_{[0,T]}(\phi) = +\infty\) otherwise. The local rate function \(L(x,v)\) can be represented in several ways, see [11 12 14]: for \(\Lambda \subset \{1, \ldots, d\}\) and \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+\) with \(x_i > 0\) for \(i \in \Lambda\) and \(x_i = 0\) for \(i \notin \Lambda\),
\[
L(x,v) = L_\Lambda(v) = \sup_{\alpha \in \mathcal{B}_\Lambda} \left( \langle \alpha, v \rangle - R(\alpha) \right) = \sup_{\alpha \in \mathbb{R}^d} \left( \langle \alpha, v \rangle - \max_{\Lambda' \subset \Lambda'} R_{\Lambda'}(\alpha) \right)
\]
for all \(v \in \mathbb{R}^d\), \(\langle \cdot, \cdot \rangle\) denotes here the usual scalar product in \(\mathbb{R}^d\),
\[
R(\alpha) = \sum_{j=1}^d \mu_j \left( \sum_{j \neq i} p_{ij} e^{\alpha_j - \alpha_i} + p_{ii} e^{-\alpha_i} - 1 \right) + \sum_{j=1}^d \lambda_i (e^{\alpha_i} - 1),
\]
\[
R_{\Lambda'}(\alpha) = \sum_{j \notin \Lambda'} \mu_j \left( \sum_{j \neq i} p_{ij} e^{\alpha_j - \alpha_i} + p_{ii} e^{-\alpha_i} - 1 \right) + \sum_{j=1}^d \lambda_i (e^{\alpha_i} - 1)
\]
and \(\mathcal{B}_\Lambda\) is the set of those all \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d\) for which
\[
\alpha_i \leq \log \left( \sum_{j=1}^d p_{ij} e^{\alpha_j} + p_{ii} \right) \quad \text{for all } i \notin \Lambda.
\]
The hypotheses \((a_3)\) are satisfied with
\[
l_1(v) = \sup_{\alpha \in \mathcal{B}_0} \left( \langle \alpha, v \rangle - R(\alpha) \right) = \sup_{\alpha \in \mathbb{R}^d} \left( \langle \alpha, v \rangle - \max_{\Lambda \subset \{1, \ldots, d\}} R_\Lambda(\alpha) \right)
\]
and

\[ l_2(v) = R^*(\alpha) = \sup_{\alpha \in \mathbb{R}^d} (\langle \alpha, v \rangle - R(\alpha)) . \]

Under Assumption (J), the convex conjugate \( R^* \) of the function \( R \) is finite everywhere on \( \mathbb{R}^d \) (see [7], Lemma 10.1), and for every \( \delta > 0 \),

\[ l_1(v) \geq \sup_{|\alpha| \leq \delta} \langle \alpha, v \rangle - \sup_{|\alpha| \leq \delta} \max_{\lambda \in \{1, \ldots, d\}} R_\lambda(\alpha) \geq \delta|v| - \sup_{|\alpha| \leq \delta} \max_{\lambda \in \{1, \ldots, d\}} R_\lambda(\alpha). \]

which implies that

\[ \lim_{|v| \to \infty} \frac{l_1(v)}{|v|} > 0. \]

Using Theorem 1 and Theorem 2 it follows that

**Corollary 7.1.** Under the hypotheses (J),

\[ \log r^* = -L_\emptyset(0) = -\inf_{\emptyset} I_{[0,1]}(\phi) \]

where the infimum is taken over all continuous functions \( \phi : [0, 1] \to \mathbb{R}_+ \) with \( \phi(0) = \phi(1) \) and

\[ \log r^*_\epsilon = -\inf_{\phi : \phi(t)\neq0, 0 \leq t < 1} I_{[0,1]}(\phi) \]

where the infimum is taken over all continuous functions \( \phi : [0, 1] \to \mathbb{R}_+ \) with \( \phi(0) = \phi(1) \) such that \( \phi(t) \neq 0 \) for all \( 0 < t < 1 \).

Using this result we calculate explicitly the quantity \( r^*_\epsilon \) for \( d = 1 \) and for \( d = 2 \). This is a subject of the following propositions.

**Proposition 7.1.** For \( d = 1 \), \( \log r^*_\epsilon = \inf_{\alpha \in \mathbb{R}} R(\alpha) \).

**Proof.** Indeed, for any continuous function \( \phi : [0, 1] \to \mathbb{R}_+ \) with \( \phi(0) = \phi(1) \) and \( \phi(t) \neq 0 \) for all \( 0 < t < 1 \),

\[ I_{[0,1]}(\phi) = \int_0^1 R^*(\phi(t)) \, dt \geq R^*(0) = -\inf_{\alpha \in \mathbb{R}} R(\alpha) \]

because the convex conjugate \( R^* \) of the function \( R \) is convex. For a constant function \( \phi(t) \equiv x \) with \( x > 0 \),

\[ I_{[0,1]}(\phi) = R^*(0) = -\inf_{\alpha \in \mathbb{R}} R(\alpha). \]

This proves that the right hand side of Relation (52) equals \( \inf_{\alpha \in \mathbb{R}} R(\alpha) \) and hence,

\[ \log r^*_\epsilon = \inf_{\alpha \in \mathbb{R}} R(\alpha). \]

**Proposition 7.2.** For \( d = 2 \),

\[ \log r^*_\epsilon = -\inf_i L_{(i)}(0) = -(1 - p_{12}p_{21}) \min\{(\sqrt{\mu_1} - \sqrt{\nu_1})^2, (\sqrt{\mu_2} - \sqrt{\nu_2})^2\} \]

if the Markov process \( (X(t)) \) is ergodic, and

\[ \log r^*_\epsilon = -\inf_i L_{(i)}(0) = \log r^* = -L_\emptyset(0) \]

otherwise.

To prove Proposition 7.2 we consider the following lemmas.

**Lemma 7.1.** For all \( \alpha = (\alpha_1, \alpha_2) \in \mathcal{B}_{(1)} \) and \( \beta = (\beta_1, \beta_2) \in \mathcal{B}_{(2)} \) such that \( \alpha_1 \geq \beta_1 \) and \( \beta_2 \geq \alpha_2 \), the following inequality holds

\[ \log r^*_\epsilon \leq \max\{R(\alpha), R(\beta)\}. \]
Proof. The function $R(\cdot)$ being continuous, it is sufficient to prove our lemma for the case when $\alpha_1 > \beta_1$ and $\alpha_2 < \beta_2$.

Recall that $B_{(1)}$ is the set of all $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_2 \leq \log(p_{21} e^{\alpha_1} + p_{20})$, and $B_{(2)}$ is the set of all $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $\alpha_1 \leq \log(p_{12} e^{\alpha_2} + p_{10})$. For given $\alpha = (\alpha_1, \alpha_2) \in B_{(1)}$ and $\beta = (\beta_1, \beta_2) \in B_{(2)}$ satisfying the inequalities $\alpha_1 > \beta_1$ and $\beta_2 > \alpha_2$, let $f(x) = \exp \langle \alpha, x \rangle + \exp \langle \beta, x \rangle$. Then for $x = (x_1, x_2) \in \mathbb{Z}_+^2$ with $x_1 > 0$ and $x_2 > 0$, we have

$$Gf(x) = R(\alpha) \exp \langle \alpha, x \rangle + R(\beta) \exp \langle \beta, x \rangle \leq \max\{R(\alpha), R(\beta)\} f(x).$$

Furthermore, for $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, denote $c_1(\theta) = -\mu_2 (p_{21} e^{\theta_1 - \theta_2} + p_{20} e^{-\theta_2} - 1)$. The inequality $\alpha_2 \leq \log(p_{21} e^{\alpha_1} + p_{20})$ implies that $c_1(\alpha) \leq 0$ and therefore, for $x_1 > 0$ and $x_2 = 0$, we obtain

$$Gf(x) = (R(\alpha) + c_1(\alpha)) \exp(\alpha_1 x_1) + (R(\beta) + c_1(\beta)) \exp(\beta_1 x_1)$$

and

$$Gf(x)/f(x) \leq R(\alpha) + (R(\beta) + c_1(\beta)) \exp(\beta_1 x_1 - \alpha_1 x_1).$$

The right hand side of the last inequality tends to $R(\alpha)$ as $x_1 \to \infty$ because $\alpha_1 > \beta_1$, and hence, for any $\varepsilon > 0$ there is $N_1(\varepsilon) > 0$ such that for $x_1 > N_1(\varepsilon)$ and $x_2 = 0$,

$$Gf(x) \leq (R(\alpha) + \varepsilon) f(x).$$

The same arguments show that for any $\varepsilon > 0$ there is $N_2(\varepsilon) > 0$ such that for $x_2 > N_2(\varepsilon)$ and $x_1 = 0$,

$$Gf(x) \leq (R(\beta) + \varepsilon) f(x).$$

The last inequality combined with Relations (56) and (57) shows that

$$Gf(x) \leq \left(\max\{R(\alpha), R(\beta)\} + \varepsilon\right) f(x)$$

for all $x = (x_1, x_2) \in \mathbb{Z}_+^2$ with $x_1 + x_2 > N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. This implies that for all $t > 0$ and $x = (x_1, x_2) \in \mathbb{Z}_+^2$ with $x_1 + x_2 > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$,

$$E_x(f(X(t)); \tau(W) > 0) \leq \exp\left(t \max\{R(\alpha), R(\beta)\} + t\varepsilon\right) f(x)$$

where $\tau(W)$ denotes the first time when the process $(X(t))$ hits the set $W = \{(x_1, x_2) \in \mathbb{Z}_+^2 : x_1 + x_2 \leq N(\varepsilon)\}$. Using Proposition 3.4 we conclude that

$$\log r_*^x \leq \max\{R(\alpha), R(\beta)\} + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain inequality (55). \qed

Let $R$ denote the infimum of $\max\{R(\alpha_1, \alpha_2), R(\beta_1, \beta_2)\}$ over all $(\alpha_1, \alpha_2) \in B_{(1)}$ and $(\beta_1, \beta_2) \in B_{(2)}$ with $\alpha_1 \geq \beta_1$ and $\beta_2 \geq \alpha_2$.

**Lemma 7.2.** When the Markov process $(X(t))$ is ergodic,

$$R \leq -\min_i L_{i(i)}(0) = -(1 - p_{12}p_{21}) \min\{\sqrt{\mu_1 - \sqrt{\mu_1^2 - \mu_2^2}}, \sqrt{\mu_2 - \sqrt{\mu_1^2 - \mu_2^2}}\}.$$

**Proof.** Recall that the Markov process $(X(t))$ is ergodic if and only if $\nu_1 < \mu_1$ and $\nu_2 < \mu_2$ where $(\nu_1, \nu_2)$ is a unique solution of the traffic equations $\nu_1 = \lambda_1 + \nu_2 p_{21}$ and $\nu_2 = \lambda_2 + \nu_1 p_{12}$. 


Let $\partial B_{(i)}$ denote the boundary of the set $B_{(i)}$. Straightforward calculations show that for $(\alpha_1, \alpha_2) \in \partial B_{(i)}$,

$$R(\alpha_1, \alpha_2) = (1 - p_{12}p_{21})(\mu_i(e^{-\alpha_i} - 1) + \nu_i(e^{\alpha_i} - 1)), \quad i = 1, 2.$$ The minimum of the function $R(\cdot)$ on the boundary $\partial B_{(1)}$ is achieved at the point $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^* = \log \sqrt{\mu_1/\nu_1} > 0 \quad \alpha_2^* = \log(p_{21}\sqrt{\mu_1/\nu_1} + p_{20}) \geq 0,$$

and equals

$$R(\alpha^*) = -(1 - p_{12}p_{21})(\sqrt{\mu_1} - \sqrt{\nu_1})^2.$$ The minimum of the function $R(\cdot)$ on the boundary $\partial B_{(2)}$ is achieved at the point $\beta^* = (\beta_1^*, \beta_2^*)$ with

$$\beta_2^* = \log \sqrt{\mu_2/\nu_2} > 0, \quad \beta_1^* = \log(p_{12}\sqrt{\mu_2/\nu_2} + p_{10}) \geq 0,$$

and equals

$$R(\beta^*) = -(1 - p_{12}p_{21})(\sqrt{\mu_2} - \sqrt{\nu_2})^2.$$ Without any restriction of generality we will suppose that $R(\alpha^*) \geq R(\beta^*)$ which implies that

$$L_2(0) \geq -R(\beta^*) \geq -R(\alpha^*) = (1 - p_{12}p_{21}) \min\{(\sqrt{\mu_1} - \sqrt{\nu_1})^2, (\sqrt{\mu_2} - \sqrt{\nu_2})^2\}.$$ Hence, to prove the equality

$$\min_{i=1}^2 L_{(i)}(0) = (1 - p_{12}p_{21}) \min\{(\sqrt{\mu_1} - \sqrt{\nu_1})^2, (\sqrt{\mu_2} - \sqrt{\nu_2})^2\}$$

it is sufficient to show that $L_{(1)}(0) = -R(\alpha^*)$. For this, we consider the set $S = \{\alpha = (\alpha_1, \alpha_2) : R(\alpha) < R(\alpha^*)\}$ and we notice that $\beta^* \notin B_{(1)}$ because $\beta^* \notin B_{(2)}$ and $\beta_1^* > 0, \beta_2^* > 0$ while for all $\beta = (\beta_1, \beta_2) \in B_{(1)} \cap B_{(2)}$, $\beta_1 \leq 0$ and $\beta_2 \leq 0$. Since $\alpha^* \in \partial B_{(1)}$ this implies that $t\alpha^* + (1 - t)\beta^* \notin B_{(1)}$ for all $0 < t < 1$ because the set $\mathbb{R}^2 \setminus B_{(1)}$ is convex. The function $R(\cdot)$ being strictly convex, the inequality $R(\alpha^*) \geq R(\beta^*)$ implies that $t\alpha^* + (1 - t)\beta^* \notin S$ for all $0 < t < 1$. The set $S$ being connected we conclude that $S \cap B_{(1)} = \emptyset$ because the boundary of the set $B_{(1)}$ has no intersection with $S$. This proves that the point $\alpha^*$ achieves the infimum of the function $R(\cdot)$ on $B_{(1)}$ and consequently, $L_{(1)}(0) = -R(\alpha^*)$. Relation (59) is therefore proved.

Notice furthermore that

$$\beta_2^* = \sqrt{\mu_2/\nu_2} \geq 1 + (\sqrt{\mu_1/\nu_1} - 1)\sqrt{\nu_1/\nu_2} \geq p_{21}\sqrt{\mu_1/\nu_1} + p_{20} = \alpha_2^*$$

where the first inequality follows from the inequality $R(\alpha^*) \geq R(\beta^*)$ and the second inequality holds because $\nu_1 = \lambda_1 + \nu_2p_{21} \geq \nu_2p_{21}^2$. When $\alpha_1^* \geq \beta_1^*$ we obtain therefore $R \leq \max\{R(\alpha^*), R(\beta^*)\} = R(\alpha^*) = -L_{(1)}(0)$ and hence, relation (58) holds. Suppose now that $\alpha_1^* < \beta_1^*$ (see Figure 1), then $p_{12} \neq 0$ because otherwise, $\beta_1^* = 0 < \alpha_1^*$. For $(\beta_1, \beta_2) \in \partial B_{(2)}$ with $\beta_1 = \alpha_1^*$ and $\beta_2 = \log(e^{\alpha_1^*} - p_{10}) - \log p_{12}$, we have $\beta_2 \geq \alpha_1^* \geq \alpha_2^*$ which implies that $R \leq \max\{R(\alpha^*), R(\beta)\}$. Moreover, using
Figure 1.

the inequality $\hat{\beta}_2 \leq \log(e^{\beta_1} - p_{10}) - \log p_{12} = \beta_2^*$, we obtain

$$R(\hat{\beta}) = (1 - p_{12}p_{21})(v_2 - \mu_2 e^{-\hat{\beta}_2})(e^{\hat{\beta}_2} - 1)$$

$$\leq (1 - p_{12}p_{21})(v_2 - \mu_2 e^{-\beta_2^*})(e^{\beta_2^*} - 1)/p_{12}$$

$$\leq -(1 - p_{12}p_{21})(\sqrt{\mu_2/v_2} - 1)(\sqrt{\mu_1/v_1} - 1)v_2/p_{12}$$

$$\leq -(1 - p_{12}p_{21})(\sqrt{\mu_1/v_1} - 1)^2 = R(\alpha^*)$$

where the last inequality holds because $\sqrt{\mu_2/v_2} - 1 \geq (\sqrt{\mu_1/v_1} - 1)\sqrt{\nu_1/v_2} > 0$ and $\sqrt{\nu_2} \geq \sqrt{\nu_1}p_{12} \geq \sqrt{\nu_1}p_{12}$. This proves that $R \leq R(\alpha^*) = -L_{\{1\}}(0)$ and consequently, Relation (54) holds. □

Proof of Proposition 7.2

Relation (61) imply that

$$\log r^* = -L_0(0) = -\inf_{\phi(0)=\phi(1)} I_{[0,1]}(\phi)$$

where the infimum is taken over all continuous functions $\phi : [0,1] \to \mathbb{R}_+^2$ with $\phi(0) = \phi(1)$. Because of Relation (54),

$$\log r^*_c = -\inf_\phi I_{[0,1]}(\phi)$$

where the infimum is taken over all those continuous functions $\phi : [0,1] \to \mathbb{R}_+^2$ for which $\phi(0) = \phi(1)$ and $\phi(t) \neq 0$ for all $0 < t < 1$.

When the Markov process $(X(t))$ is non-ergodic, Theorem 2 of Ignatiouk [7] proves that

$$L_0(0) = \min L_{\{i\}}(0) = \min\{I_{[0,1]}(\phi), I_{[0,1]}(\psi)\}$$

where $\phi(t) = x' = (1,0) \in \mathbb{R}_+^2$ and $\psi(t) = x'' = (0,1) \in \mathbb{R}_+^2$ for all $t \in [0,1]$. Using Relations (59) and (61), this implies that

$$\log r^* = -L_0(0) = -\min_{i} L_{\{i\}}(0) = -\min\{I_{[0,1]}(\phi), I_{[0,1]}(\psi)\} \leq \log r^*_c \leq \log r^*.$$
Suppose now that the Markov process \((X(t))\) is ergodic. Then using Lemma 7.1 and Lemma 7.2 it follows that
\[
\log r^*_e \leq \mathcal{R} \leq -\min_i L_{(1)}(0) = -\min \{ I_{[0,1]}(\phi), I_{[0,1]}(\psi) \}.
\]
The last relation combined with Inequality (61) proves Relation (53).

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