A representation of $su_q(n)$, which diverges in the limit of $q \to 1$, is investigated. This is an infinite dimensional and a non-unitary representation, defined for the real value of $q$, $0 < q < 1$. Each irreducible representation is specified by $n$ continuous variables and one discrete variable. This representation gives a new solution of the Yang-Baxter equation, when the R-matrix is evaluated. It is shown that a continuous variable can be regarded as a spectral parameter.
When a physical realization of the quantum universal enveloping algebra (quantum algebra)\(^1\) is considered, its representation theory plays a crucial role. The representations of the quantum algebra \(U_q(g)\), where \(g\) denotes a simple Lie algebra, are classified into three categories according to the value of \(q\): 1) generic, that is, \(q\) can take any value except \(q = 0, \pm 1\) and a root of unity, 2) \(q\) is a root of unity, 3) \(q = 0\).

Rosso proved that for \(q\) generic, all finite dimensional representations of \(U_q(g)\) are completely reducible and the irreducible ones are classified in terms of highest weights.\(^2\) In particular, they can be regarded as deformation of the representations of the classical \(U(g)\). When \(q\) is a root of unity, the representations of \(U_q(g)\) become strikingly different from the classical case \(q \to 1\).\(^3\) They are not completely reducible. Some finite dimensional representations are not the highest weight ones. The \(q = 0\) case is called the crystal base.\(^4\)

An infinite dimensional representation of \(U_q(g)\) is still an open problem.

In this paper, the representation of \(su_q(n)\), which diverges in the classical limit, is investigated. We call it a classically singular representation (CSR). Such a representation was first found by Kulish for the q-(bosonic) oscillator.\(^5\) Zhedanov found a q-oscillator realization of \(su_q(2)\) and \(su_q(1, 1)\) which diverges in the classical limit.\(^6\) Using the Fock representation of q-oscillator, he derived an infinite dimensional representation of \(su_q(2)\) and \(su_q(1, 1)\). We shall show that another CSR of \(su_q(n)\) (infinite dimensional) is possible. This representation gives a new solution of the Yang-Baxter equation, when the R-matrix is evaluated.

Recently, in connection with \(q \leftrightarrow q^{-1}\) invariance of q-oscillator, a new Jordan-Schwinger type realization of \(su_q(2)\) was introduced.\(^7\) This realization reflects the fact that there exists a non-trivial central element in the q-oscillator algebra, while the first proposed q-analogue of the Jordan-Schwinger realization, ref.8), must assume the vanishing central element. We generalize this realization to \(su_q(n)\), and applying Kulish’s CSR to it, we can obtain a CSR of \(su_q(n)\). Because the eigenvalue of the central element is non-vanishing in Kulish’s CSR of q-oscillator,
the Jordan-Schwinger realization of ref.8) does not work.

The obtained CSR of $su_q(n)$ is infinite dimensional and non-unitary. Each irreducible representation is specified by $n$ continuous variables and one discrete one. The continuous variables can take any positive real values and the discrete one can take any integral (both positive and negative) numbers. These irreducible representations can be regarded as follows: If the discrete variable is fixed, we obtain a continuous family of representations parametrized by the continuous variables. In other words, a irreducible representation specified by the discrete variable contains $n$ parameters. By setting an appropriate condition, the number of parameters reduces to one. We can regard this as a spectral parameter, when the R-matrix is evaluated on the representation space, although the universal R-matrix of $su_q(n)$ contains no parameter except $q$. We shall discuss this point concretely for the case of $su_q(2)$.

We shall start with the definition of the $q$-oscillator algebra. It is generated by three elements: $q$-creation operator $a^\dagger$, $q$-annihilation operator $a$ and number operator $N$. They satisfy the following relations:

$$
[N, a^\dagger] = a^\dagger,
[N, a] = -a,
aa^\dagger - qa^\dagger a = q^{-N}.
$$

There exists a non-trivial central element,

$$
C = q^{-N}([N] - a^\dagger a),
$$

where $[N] \equiv (q^N - q^{-N})/(q - q^{-1})$. This allow us to express $a^\dagger a$ in terms of the operators $N$ and $C$,

$$
a^\dagger a = [N] - q^N C.
$$

It should be noted that this is a direct consequence of the defining relations (1), so more general than the standard prescription in which the relation $a^\dagger a = [N]$ is assumed.
Let us generalize the realization of $su_q(2)$ given in ref.7) to $su_q(n)$. Introducing $n$ independent (mutually commuting) q-oscillators, $a_i, a_i^\dagger, N_i, i = 1, 2, \cdots, n$, we construct the following $3(n-1)$ operators,

$$
X_i^+ = F_i^{-1/4} a_i^\dagger F_i^{-1/4} a_{i+1}, \quad X_i^- = F_i^{-1/4} a_i F_i^{-1/4} a_{i+1},
$$

$$
H_i = \frac{1}{2}(M_i - M_{i+1}),
$$

(4)

where $i = 1, 2, \cdots, n-1$, and the operator $F_i$ and $M_i$ are defined by

$$
F_i \equiv 1 - (q - q^{-1})C,
$$

$$
M_i \equiv N_i + \frac{\ln \sqrt{F_i}}{\ln q}.
$$

Using the relation (3), it is easy to verify that these operators form a q-analogue of the Chevalley basis of $su_q(n)$, that is, they satisfy the following commutation relations.

$$
[H_i, X_j^\pm] = \pm \frac{1}{2} A_{ij} X_j^\pm, \quad [X_i^+, X_j^-] = \delta_{ij} [2H_i],
$$

(6)

where $A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$ is a element of the Cartan matrix of $su(n)$.

Next, let us consider the CSR. In the following, we assume that $q$ is a real number. Kulish constructed two CSR’s of the q-oscillator, one of them is suitable for our purpose. It is given by,[5]

$$
N|m> = m|m>, \quad a|0> = L_0|m>, \quad a|m> = L_{m+1}|m+1>, \quad a|m> = L_m|m-1>,
$$

(7.a)

$$
a^\dagger|m> = L_{m+1}|m+1>, \quad a|m> = L_m|m-1>,
$$

(7.b)

$$
C|m> = -q(v + \Delta)|m>,
$$

(7.c)

where

$$
L_m \equiv (q^{-m+1}v + q^{m+1}\Delta)^{1/2}, \quad v \equiv (1 - q^2)^{-1}.
$$

(8)

The possible values of $q$ and $\Delta$ are restricted from the requirement that $L_m^2$ is
The peculiarity of this representation is, 1) the matrix elements of \( a^\dagger \), \( a \) and \( C \) diverge in the classical limit, since \( v \to \infty \) as \( q \to 1 \), 2) \( a^\dagger \) is the hermite conjugate of \( a \), 3) there exist no states which are annihilated by the q-annihilation or the q-creation operator, 4) non-vanishing eigenvalue of the central element, 5) the CSR (7) is irreducible for a fixed value of \( \Delta \), so we have an infinite numbers of irreducible representations parametrized by \( \Delta \).

In order to express the bases \( \{|m>\} \) of (7) in terms of the q-oscillator, we assume the existence of the state \(|0>\) such that,

\[
N|0> = 0, \quad C|0> = -q(v + \Delta)|0>,
\]

and for the definition of the norm

\[
<0|0> = 1, \quad <0|a^m|0> = <0|(a^\dagger)^m|0> = 0,
\]

where \( m \) is a positive integer. The bases \( \{|m>\} \) can be written as,

\[
|m> = (\prod_{k=1}^{m} L_k)^{-1}(a^\dagger)^m|0>, \quad \text{for } m \in \mathbb{Z}_+
\]

These are orthonormal: \(<m|n> = \delta_{mn} \) for \( m, n, \in \mathbb{Z} \). It is necessary to assume (10.c) to ensure the orthogonality of (11). These bases disappear in the classical limit, since \( L_n \) diverges as \( q \to 1 \).
It is now obvious that the CSR of $su_q(n)$ can be obtained from the realization (4) and the CSR of the q-oscillator (7). First, note that the fourth root of $F_i$ appears in (4) and eigenvalue of $F_i$ is negative:

$$F_i|m_i> = -\frac{\Delta_i}{v}|m_i>.$$ 

To avoid the multi-valuedness of the fourth root, we take the first branch. It is easily verified that this choice do not lose generality. The representation space is the direct product of the ones of the CSR (7). Denoting its basis by

$$|\vec{m}> = |m_1, m_2, \ldots, m_n> = |m_1> \otimes |m_2> \otimes \cdots \otimes |m_n>,$$

simple calculation gives the following representation of $su_q(n)$

$$X^+|\vec{m}> = i \left(\frac{v^2}{\Delta_i\Delta_{i+1}}\right)^{1/4} L_{m_i+1}L_{m_{i+1}}|m_1, \ldots, m_i+1, m_{i+1} - 1, \ldots, m_n> \quad (12.a)$$

$$X^-|\vec{m}> = i \left(\frac{v^2}{\Delta_i\Delta_{i+1}}\right)^{1/4} L_{m_i}L_{m_{i+1}+1}|m_1, \ldots, m_i - 1, m_{i+1} + 1, \ldots, m_n> \quad (12.b)$$

$$H_i|\vec{m}> = \frac{1}{2}(m_i - m_{i+1} + \frac{\ln(\Delta_i/\Delta_{i+1})}{\ln q})|\vec{m}>. \quad (12.c)$$

In this representation, $X^+_i$ and $X^-_i$ are not hermite conjugate each other, which stems from the choice of branch when the fourth root of $F_i$ is evaluated. It is possible, by taking the appropriate branch, to make $X^+_i$ and $X^-_i$ be hermite conjugate each other. In this case, however, $H_i$ become a non-hermitian matrix. In this sense, this representation is non-unitary.

The representation (12) is reducible. Because the action of $X^\pm_i$ and $H_i$ preserves the sum $\sum_{i=1}^n m_i$, the set \{|$\vec{m}> | \sum_{i=1}^n m_i = m$\} forms an invariant subspace for each value of $m = \cdots, -1, 0, 1, 2, \cdots$. The action of $X^\pm_i$ and $H_i$ also do not change the value of $\{\Delta_i, i = 1, 2, \cdots, n\}$, so each irreducible representation is specified by $m$ and $\Delta_i$’s. Since the values of $m_i$’s are not bounded, each irreducible representation is infinite dimensional. We can regard this situation as follows: if the value of $m$ is fixed, we have a continuous family of irreducible representations parametrized by $\Delta_i$’s.
Let us investigate the case of $su_q(2)$ in more detail. The generators of $su_q(2)$ are, $J_+ = X_1^+$, $J_- = X_1^-$, and $J_z = H_1$. The central element is given by,

$$C[su_q(2)] = [J_z][J_1 - 1] + J_+J_-.$$  \hfill(13)

Introducing the following notations,

$$J = \frac{1}{2}(m_1 + m_2), \quad M = \frac{1}{2}(m_1 - m_2),$$ \hfill(14.a)

$$\kappa = (\Delta_1\Delta_2)^{1/4}, \quad \lambda = \Delta_1/\Delta_2,$$ \hfill(14.b)

and

$$|J, M, \kappa, \lambda >= |m_1 > \otimes |m_2 >,$$ \hfill(14.c)

the CSR of $su_q(2)$ is rewritten as

$$J_z|J, M, \kappa, \lambda >= (M + \frac{1}{2}\ln\lambda)\ln_q)|J, M, \kappa, \lambda >,$$

$$J_+|J, M, \kappa, \lambda >= -i\sqrt{\kappa} L^{(+)}_{J+M+1} L^{(-)}_{J-M}|J, M + 1, \kappa, \lambda >,$$ \hfill(15)

$$J_-|J, M, \kappa, \lambda >= -i\sqrt{\kappa} L^{(+)}_{J+M} L^{(-)}_{J-M+1}|J, M - 1, \kappa, \lambda >,$$

where

$$L^{(\pm)}_N = (q^{-N+1}v\kappa^{-1} + q^{N+1}\kappa\lambda\pm\frac{1}{2})^{1/2}.$$  

The eigenvalue of the central element is specified by $J$ and $\kappa$ :

$$C[su_q(2)]|J, M, \kappa, \lambda >= -q^2v(q^{2J+1}\kappa^2 + q^{-2J-1}v^2\kappa^{-2} + [2]v)|J, M, \kappa, \lambda >. \quad (16)$$

It is always negative. The possible values of $J$ are (both positive and negative) integers or half-integers. When $J$ is an integer (half-integer), $M$ can take any (half) integral values. Each irreducible representation is specified by $J$, $\kappa$ and $\lambda$.  \hfill7
Here gives an interesting remark. A CSR of $su_q(1,1)$ is obtained from (15), since the generators of $su_q(1,1) \{K_\pm, K_z\}$ can be expressed in terms of $su_q(2)$'s,

$$K_\pm = i J_\pm, \quad K_z = J_z.$$ 

This procedure gives a unitary representation of $su_q(1,1)$. Another infinite dimensional representation, which connects smoothly to the classical one, is found in ref.7.

Finally, the R-matrix is evaluated. Our convention of coproduct is,

$$\Delta(J_z) = J_z \otimes 1 + 1 \otimes J_z, \quad \Delta(J_\pm) = J_\pm \otimes q^{-J_z} + q^{J_z} \otimes J_\pm.$$ (17)

The universal R-matrix is given by,

$$R = q^{2J_z \otimes J_z} \sum_{l \geq 0} \frac{(1-q^{-2})^l}{[l]!} q^{\frac{1}{2}l(l-1)} (q^{-J_z} J_-)^l \otimes (q^{J_z} J_+)^l.$$ (18)

Before evaluating the matrix elements, we set $\lambda = 1$ (i.e. $\Delta_1 = \Delta_2$) in (15). Each irreducible representation is specified by $J$ and $\kappa$. If $J$ is fixed, a continuous family of irreducible representation, which is parametrized by $\kappa$, is obtained. The parameter $\kappa$ can be regarded as a spectral parameter, when the R-matrix is evaluated. Indeed, a matrix element of (18) with respect to (15) reads,

$$(R(\kappa_1, \kappa_2))^{M_1, M_2}_{M_1 - l, M_2 + l} = \frac{1}{[l]!} q^{\frac{1}{2}(2M_1 - l)(2M_2 + l) + l/2} \prod_{k=1}^{l} \Gamma_k^{(1)} \Gamma_k^{(2)},$$ (19)

where

$$\Gamma_k^{(1)} = \{q^{-2J_1 - 1} v^{-2 \kappa_1^{-2}} + q^{2J_1 + 2 \kappa_1^2} + v(q^{-2M_1 - 1 + 2k} + q^{2M_1 + 2k})\}^{1/2},$$ (20.a)

$$\Gamma_k^{(2)} = \{q^{-2J_2 - 1} v^{-2 \kappa_2^{-2}} + q^{2J_2 + 2 \kappa_2^2} + v(q^{-2M_2 - 1 + 2k} + q^{2M_2 + 2k})\}^{1/2}.$$ (20.b)
This R-matrix satisfies the Yang-Baxter equation with spectral parameters,

\[ R_{12}(\kappa_1, \kappa_2) R_{13}(\kappa_1, \kappa_3) R_{23}(\kappa_2, \kappa_3) = R_{23}(\kappa_2, \kappa_3) R_{13}(\kappa_1, \kappa_3) R_{12}(\kappa_1, \kappa_2). \] (21)

The R-matrix is also infinite dimensional and the difference property of parameter:

\[ R(\kappa_1, \kappa_2) = R(\kappa_1 - \kappa_2) \]

does not hold in this case.

The representation discussed in this paper is peculiar to the quantum algebras, since it has no classical counterpart. It will be easy to construct CSR for other quantum algebras such as \( so_q(n) \), \( sp_q(n) \) etc. Another possibility of representations, which have no classical counterpart, is the one vanishing in the limit of \( q \to 1 \). It will be interesting to consider such kind of representation.

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