Exponential Type Complex and non-Hermitian Potentials in $PT -$ Symmetric Quantum Mechanics

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Abstract

Using the NU method [A.F.Nikiforov, V.B.Uvarov, Special Functions of Mathematical Physics, Birkhauser, Basel, 1988], we investigated the real eigenvalues of the complex and/or $PT$-symmetric, non-Hermitian and the exponential type systems, such as Pöschl-Teller and Morse potentials.

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1 Introduction

Recently, an important feature of the \( PT \)-symmetric Hamiltonians is recognized. So that they have real spectra although they may be Hermitian or not. Initial studies of Bender and his co-workers on the \( PT \) - symmetric quantum mechanics which was restricted for those eigenvalues of Hermitian operators which are real. Physical motivation for the \( PT \)-symmetric but non-Hermitian Hamiltonians have been emphasized by many authors [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Following these detailed works, non-Hermitian Hamiltonians with real or complex spectra have been studied by using numerical and analytical techniques [4, 5, 6, 7, 12, 13]. The interesting results of non-Hermitian quantum mechanical \( PT \)-symmetric theories have been extended to the two-particle bound states of conventional \( g\phi^3 \) and \( g\phi^4 \) quantum field theories[14], and also to the two-point Green’s function representations[15].

There exist some physical reasons for such a generalization based on the time reversal operator \( T \) which is antilinear in the complex theory. Thus, the operator \( PT \) commutes with the Hamiltonian \( H \). There are well known physical parameters[16] saying that the Hilbert space , \( L_2(x_1,x_2) \), of quantum mechanics could be real or complex in the interval \( x_1 < x < x_2[17] \). In quantum mechanics, complexification of the energy can be used to describe resonance cases and scattering theory [18].

In this work, the Schrödinger equation is solved by using Nikiforov - Uvarov (NU) method[1] to get energy eigenvalues of bound states for real and complex form of the potentials. For the numerical application, general forms of Morse and Pöschl-Teller potentials are solved. One can also use Cauchy-Riemann equations in the complex space, with the additional degrees of freedom to study same complex Hamiltonian systems[19].

After a brief introductory discussion of Nikiforov - Uvarov (NU) method[1] in Section 2, we study complex and/or \( PT \)-symmetric non-Hermitian exponential type systems with the Pöschl-Teller and Morse potentials in Section 3 and 4. Results are discussed in Section 5.
2 Nikiforov-Uvarov Method

NU method[1] is based on the solutions of general second order linear equation with special functions. In this method, the Schrödinger equation in one dimension can be written in general as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma}(s)\psi' + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi = 0$$  \hspace{1cm} (1)

where $s = s(x)$, $\sigma(s)$, and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. In the NU method the new function $\pi$ and the parameter $\lambda$ are defined as

$$\pi = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}$$  \hspace{1cm} (2)

and

$$\lambda = k + \pi'.$$  \hspace{1cm} (3)

On the other hand, in order to find the value of $k$, the expression in the square root must be square of polynomial. Thus, a new eigenvalue equation for the Schrödinger equation becomes

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \hspace{1cm} (n = 0, 1, 2, ...$$  \hspace{1cm} (4)

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s),$$  \hspace{1cm} (5)

and it will have a negative derivative[1]. The wave function is constructed as a multiple of two independent parts,

$$\psi(s) = \phi(s)y(s),$$  \hspace{1cm} (6)

where $y(s)$ is the hypergeometric type function which is described with a weight function $\rho$ as

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma^n(s)\rho(s)\right],$$  \hspace{1cm} (7)

where $\rho(s)$ must satisfy the condition [1]

$$(\sigma\rho)' = \tau\rho.$$  \hspace{1cm} (8)

Other part is defined as a logarithmic derivative

$$\frac{\phi(s)'}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}.$$  \hspace{1cm} (9)
3 Generalized Morse Potential

We shall first study different types of the general Morse potential,
\[ V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x}. \]  (10)

Now, in order to apply the NU-method, we write the Schrödinger equation with the generalized Morse potential by using a new variable \( s = \sqrt{V_1 e^{-\alpha x}} \)
\[
\psi'' + \frac{1}{s} \psi' - \frac{1}{s^2} \left( \frac{2m}{\hbar^2 \alpha^2} s^2 - \frac{2m}{\hbar^2 \alpha^2} \frac{V_2}{\sqrt{V_1}} s + 4\epsilon^2 \right) \psi = 0. \]  (11)

Substituting \( \sigma(s), \, \tilde{\sigma}(s) \), and \( \tilde{\tau}(s) \) in Eq.(2) we can immediately obtain \( \pi \) function as
\[
\pi = \pm \sqrt{\frac{2m}{\hbar^2 \alpha^2} s^2 + (k - \frac{2m}{\hbar^2 \alpha^2} \frac{V_2}{\sqrt{V_1}}) s + 4\epsilon^2}. \]  (12)

According to the NU method, the expression in the square root must be square of polynomial. So, one can find new possible functions for each \( k \) as
\[
\pi = \begin{cases} 
\pm (\gamma s + 2\epsilon), & \text{for } k = \frac{\gamma^2 V_2}{\sqrt{V_1}} + 4\epsilon \gamma \\
\pm (\gamma s - 2\epsilon), & \text{for } k = \frac{\gamma^2 V_2}{\sqrt{V_1}} - 4\epsilon \gamma 
\end{cases}, \]  (13)

where \( \epsilon^2 = -\frac{mE}{2\hbar^2 \alpha^2} \) and \( \gamma^2 = \frac{2m}{\hbar^2 \alpha^2} \). After determining \( k \) and \( \pi \), we can write \( \tau \) as,
\[
\tau = 1 - 2\gamma s + 4\epsilon. \]  (14)

Therefore from Eqs. 4 and 14, we get exact energy eigenvalues in atomic units as,
\[
E_n = -\frac{\alpha^2}{4} \left( 2n + 1 - \frac{V_2}{\alpha \sqrt{V_1}} \right)^2. \]  (15)

Using \( \sigma(s) \), and \( \pi(s) \) in Eqs.(7 - 9), one can find the corresponding wave functions \( y(s) \) and \( \phi(s) \), then from Eq.(6) as
\[
\Psi_n(s) = C_n s^2 e^{-\gamma s} L_n^{4\epsilon}(2\gamma s). \]  (16)

Where \( L_n^{\mu}(x) \) stands for the associated Laguerre functions. One can easily see well behavior of the wave function at infinity. As an example, the ground state wave function behaves like
\[
\Psi_0(s \to 0) \sim \left( e^{-\alpha s} \right)^i E_n/\alpha e^{-\epsilon s}/\alpha. \]  (17)

Let us now consider different types of the generalized Morse potential.
3.1 Non-$PT$ symmetric and non-Hermitian Morse case

We have considered a more general form of the complexified Morse potential than that was previously studied [6, 7, 9, 12]. For a special value of its parameters, it reduces to the usual one. Thus, we may choose the parameters from two types.

Nov, let us take the potential parameters as in the Refs.[7] and [8], $V_1 = (A+iB)^2$, $V_2 = (2C + 1)(A + iB)$, and $\alpha = 1$, then the potential is

$$V(x) = (A + iB)^2 e^{-2x} - (2C + 1)(A + iB)e^{-x},$$

(18)

where $A$, $B$, and $C$ are arbitrary real parameters and $i = \sqrt{-1}$. Such potentials are non-$PT$- symmetric and also non-Hermitian but have real spectra. According to Solombrino[9], this type of the complex Morse potential is a pseudo-Hermitian and the corresponding Hamiltonian verifies the pseudo-Hermitian propositions weakly (Proposition 3. and 5.). More recently, the basic properties of pseudo-Hermitian operators, pseudo-supersymmetric quantum mechanics and diagonalizable pair of isospectral Hamiltonians with identical degeneracy structure are intensively studied[10].

Let us consider the real spectrum for this case. Substituting the parameters into the energy expression, We simply get

$$E_n = -(n - C)^2.$$  

(19)

In this case the spectrum is completely real and independent from the potential parameters $A$ and $B$. However, there are degeneracy for $A$ and $B$. If $V_1$ is real, and $V_2 = A + iB$, and $\alpha = i\alpha$, $PT$– violation case, the Morse potential can be written in the following form

$$V(x) = V_1 e^{-2i\alpha x} - (A + iB)e^{-i\alpha x}.$$  

(20)

For $V_1 > 0$ case, there are real spectra if and only if $\text{Re}(V_2) = 0$. When $V_1 < 0$ there are real spectra if and only if $\text{Im}(V_2) = 0$. It can be shown that this is related by a pseudo-Hermitian transformation [7, 10].
3.2 \textit{PT} symmetric and non-Hermitian Morse case

When $\alpha = i\alpha$, and $V_1$ and $V_2$ are real, in this case potential takes

\[ V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x}, \tag{21} \]

with the $\text{Re}(V(x)) = V_1 \cos(2\alpha x) - V_2 \cos(\alpha x)$ and $\text{Im}(V(x)) = -V_1 \sin(2\alpha x) + V_2 \sin(\alpha x)$.

From the Eq. (15) for $V_1 > 0$, we get no real spectra of this kind of $PT$-symmetric Morse potentials. In order to compare our results with the ones obtained by Znojil [11], we simply take the parameters as $V_1 = -\omega^2$, $V_2 = D$, and $\alpha = 2$. In this particular case, we get $\pi$ function as

\[ \pi = \pm \frac{i}{2} \left\{ \begin{array}{ll} (\omega s - \alpha) & , \text{for } k = (-iD + 2\omega \alpha)/4 \\ (\omega s + \alpha) & , \text{for } k = (-iD - 2\omega \alpha)/4 \end{array} \right. \tag{22} \]

and after appropriate choice of $k$ and $\pi$, we can write $\tau$ as

\[ \tau = 1 - i(\omega s + \alpha). \tag{23} \]

Thus, the energy eigenvalues are reduced to the simple form

\[ E_n = \left( 2n + 1 + \frac{D}{2\omega} \right)^2. \tag{24} \]

More recently many interesting properties of such particular cases were studied by Znojil [11] and Bagchi and Quesne [12].

4 Pöschl-Teller potential

We shall consider the general form of the Pöschl-Teller potential

\[ V(x) = -4V_0 \frac{e^{-2ax}}{(1 + qe^{-2ax})^2}. \tag{25} \]

This potential has more flexible form. Because it has a couple of additional free parameters $\alpha$ and $q$ to the well known standard Pöschl-Teller form [20]. First of all if we fix the free parameters as $\alpha = 1$ and $q = 1$ the potential reduces to the well known standard Pöschl-Teller potential. This form of the potential was studied extensively by many authors [21, 22]. The standard Pöschl-Teller potential was applied in the framework of the $su(2)$ vibron model [23]. The special case of (25) for $q = 1$, the modified Pöschl-Teller potential

\[ V(x) = -\frac{D_0}{\cosh^2(\alpha x)}, \tag{26} \]
is used to derive the well known $su(2)$ spectrum-generating algebra of an infinite square well problem [24].

Let us solve the Schrödinger equation for the generalized Pöschl-Teller potential. One can get the following form with the new variable $s = -e^{-2\alpha x}$ as

$$
\psi''(s) + \frac{1 - qs}{s(1 - qs)} \psi' + \frac{1}{[s(1 - qs)]^2} [-\epsilon^2 q^2 s^2 + (2\epsilon^2 q - \beta^2) s - \epsilon^2] \psi = 0
$$

(27)

where $\epsilon^2 = -\frac{mE}{2\hbar^2 \alpha^2}$, and $\beta^2 = \frac{2mV_0}{\hbar^2 \alpha^2}$. Thus, one can easily get the energy eigenvalues in atomic units [25] as,

$$
E_n(q, \alpha) = -\frac{\alpha^2}{4} \left[ -(2n + 1) + \sqrt{1 + \frac{4V_0}{q\alpha^2}} \right]^2.
$$

(28)

The corresponding wave function becomes

$$
\psi_n(s) = s^{-\epsilon}(1 - s)^{\nu/2} P_n^{(2\kappa,\nu - 1)}(1 - 2qs).
$$

(29)

Where $\nu = 1 - \sqrt{1 + \frac{8nV_0}{q\alpha^2}}$ and $P_n^{\mu,\nu}(x)$ stands for Jacobi polynomials. One can easily get proper behavior of wave function at infinity.

### 4.1 Non-PT symmetric and non-Hermitian Pöschl-Teller cases

Now, we are going to choose $V_0$ and $q$ as complex parameters $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$. Where $V_{0R}$, $V_{0I}$, $q_R$, and $q_I$ are arbitrary real parameters. In this case, although the potential is complex, and corresponding Hamiltonian is non-Hermitian and also non-PT-symmetric, there may be a real spectra if and only if $V_{0R}q_R = V_{0I}q_I$. When both parameters $V_0$ and $q$ are taken pure imaginary the potential turns out to be

$$
V(x) = -4V_0 \frac{2qe^{-4\alpha x} + i(1 - q^2e^{-4\alpha x})}{(1 + q^2e^{-4\alpha x})^2}.
$$

(30)

Here, we have simply used $V_0$ and $q$ instead of $V_{0I}$ and $q_I$. The energy eigenvalues are the same given in Eqn.(28).

If $q$ is arbitrary real parameter and $V_0 \Rightarrow iV_0$ also $\alpha \Rightarrow i\alpha$ completely imaginary, the potential becomes

$$
V(x) = -4V_0 \frac{(1 - q^2)\sin(2\alpha x) + i(2q + (1 + q^2)\cos(2\alpha x))}{(1 + q^2)^2 + 4q\cos(2\alpha x)(1 + q\cos(2\alpha x) + q^2)},
$$

(31)
and the corresponding energy eigenvalues become

\[ E_n = \frac{\alpha^2}{4} \left[ \text{sgn} \left( \left( 4V_0 + iq\alpha^2 \right) q\alpha^2 \right) \left( -\frac{B}{2} + (2n + 1) \right) iA - B(2n + 1) + 2 + 4n + 4n^2 \right]. \]  

(32)

where, we use the abbreviations as \( A = \sqrt{2\sqrt{1 + \left( \frac{4V_0}{q\alpha^2} \right)^2} - 2} \), \( B = \sqrt{2\sqrt{1 + \left( \frac{4V_0}{q\alpha^2} \right)^2} + 2} \). csgn is used for complex signum function in MAPLE Program. For a real spectrum we take \(-\frac{B}{2} + (2n + 1) = 0\). So this requires that \( V_0/(q\alpha^2) = \pm \sqrt{n(n + 1)(1 + 2n)^2} \) which is a constraint on the potential parameters.

When all three potential parameters are complex, Hamiltonian is non-Hermitian and also non-\( PT\)-symmetric having real spectra. For simplicity, let us take all three parameters are pure imaginary. That is \( \alpha \) replaced by \( i\alpha \), \( q \) replaced by \( iq \), and, \( V_0 \) replaced by \( iV_0 \). In this case the potential takes the form

\[ V(x) = -4V_0 \frac{(1 + q^2) \sin(2\alpha x) + 2q + i(1 - q^2) \cos(2\alpha x)}{(1 + q^2)^2 + 4q^2(1 - \cos^2(2\alpha x)) + 4q(1 + q^2) \sin(2\alpha x)}, \]  

(33)

and Hamiltonian is non-Hermitian and also non-\( PT\)-symmetric. The energy eigenvalues become

\[ E_n = \frac{\alpha^2}{4} \left[ \sqrt{\text{sgn}(A)} \sqrt{A(1 + 2n)(1 + i) - \sqrt{|A|(1 + 2n)(1 + i) - \frac{4V_0}{q\alpha^2}} + 2(1 + 2n + 2n^2)} \right]. \]  

(34)

Here there are a real spectra if and only if \( A \geq 0 \). sgn is used for signum function in MAPLE Program.

### 4.2 \( PT \) symmetric and non-Hermitian Pöschl-Teller case

For \( PT\)-symmetric and non-Hermitian potential case, we choose the parameters \( V_0 \), and \( q \) are arbitrarily real, and \( \alpha \Rightarrow i\alpha \). In this case, the potential becomes

\[ V(x) = -4V_0 \frac{(1 + q^2) \cos(2\alpha x) + 2q + i(q^2 - 1) \sin(2\alpha x)}{(1 + q^2)^2 + 4q \cos(2\alpha x)(1 + q \cos(2\alpha x) + q^2)}, \]  

(35)

and the energy eigenvalue is

\[ E_n = -\frac{\alpha^2}{4} \left[ \sqrt{|A|(1 + 2n)(1 + i)} + \sqrt{\text{sgn}(A)} \sqrt{A(1 + 2n)(-1 + i) + \frac{4V_0}{q\alpha^2}} - 2(1 + 2n + 2n^2) \right], \]  

(36)

where \( A = 1 - 4V_0/(q\alpha^2) \), there are a real spectra if and only if \( A = 0 \), i.e., \( 4V_0 = q\alpha^2 \).
5 Conclusions

We have extended the PT-symmetric formulation, developed recently within the non-relativistic quantum mechanics, to the more general complex Morse and Pöschl-Teller potentials. We solved the Schrödinger equation in one dimension first time for the complex potentials by using Nikiforov-Uvarov method. We studied so many different complex forms of these potentials. Interesting features of quantum expectation theory for $PT$-violating potentials may be affected by changing from complex to real systems. We observed that there were some restrictions on the potential parameters for bound states in $PT$-symmetric or, more generally, in non-Hermitian quantum mechanics. Because of the restriction $V_0/(q\alpha^2) = \pm \sqrt{n(n+1)(1+2n)^2}$, there is no ground state of generalized Pöschl-Teller potential when the parameters $V_0$ and $\alpha$ are pure imaginary. Although the number of positive bound states decreases with increasing $\alpha$ and $q$ or decreasing $V_0$ for real family of the Pöschl-Teller potential, there are positive and negative bound states for $PT$-symmetric cases. we have pointed out that our exact results of complexified Morse and the Pöschl-Teller potentials may increase the applications in the study of different quantum systems.
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2 Nikiforov-Uvarov Method

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where $s = s(x)$, $\sigma(s)$, and $\tilde{\tau}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. In the NU method the new function $\pi$ and the parameter $\lambda$ are defined as

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On the other hand, in order to find the value of $k$, the expression in the square root must be square of polynomial. Thus, a new eigenvalue equation for the Schrödinger equation becomes

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where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s),$$  \hspace{1cm} (5)

and it will have a negative derivative[1]. A family of particular solutions for a given $\lambda$ has hypergeometric type of degree $n$. So, when $\lambda = 0$ the part of polynomial solution will have degree one and energy eigenvalue corresponds to the ground state, i.e. $n=0$.

The wave function is constructed as a multiple of two independent parts,

$$\psi(s) = \phi(s)y(s),$$  \hspace{1cm} (6)

where $y(s)$ is the hypergeometric type function which is described with a weight function $\rho$ as

$$y_n(s) = \frac{B_n}{\rho(s)} \int_0^s ds [\sigma^n(s)\rho(s)],$$  \hspace{1cm} (7)
where $\rho(s)$ must satisfy the condition [1]

$$(\sigma \rho)' = \tau \rho.$$  \hfill (8)

Other part is defined as a logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}.$$  \hfill (9)

3 Generalized Morse Potential

We shall first study different types of the general Morse potential,

$$V(x) = V_1 e^{-2ax} - V_2 e^{-ax}.$$  \hfill (10)

Now, in order to apply the NU-method, we write the Schrödinger equation with the generalized Morse potential by using a new variable $s = \sqrt{V_1} e^{-ax}$

$$\psi'' + \frac{1}{s} \psi' - \frac{1}{s^2} \left( \frac{2m}{\hbar^2 \alpha^2} s^2 - \frac{2m}{\hbar^2 \alpha^2} \sqrt{V_1} s + 4\epsilon^2 \right) \psi = 0.$$  \hfill (11)

Substituting $\sigma(s)$, $\tilde{\sigma}(s)$, and $\tilde{\tau}(s)$ in Eq.(2) we can immediately obtain $\pi$ function as

$$\pi = \pm \sqrt{\frac{2m}{\hbar^2 \alpha^2} s^2 + (k - \frac{2m}{\hbar^2 \alpha^2} \sqrt{V_1}) s + 4\epsilon^2}.$$  \hfill (12)

According to the NU method, the expression in the square root must be square of polynomial. So, one can find new possible functions for each $k$ as

$$\pi = \begin{cases} 
\pm (\gamma s + 2\epsilon) & \text{for } k = \gamma^2 \frac{V_1}{\sqrt{V_1}} + 4\epsilon \gamma \\
\pm (\gamma s - 2\epsilon) & \text{for } k = \gamma^2 \frac{V_1}{\sqrt{V_1}} - 4\epsilon \gamma
\end{cases}$$  \hfill (13)

where $\epsilon^2 = -\frac{mE}{2\hbar^2 \alpha^2}$ and $\gamma^2 = \frac{2m}{\hbar^2 \alpha^2}$. After determining $k$ and $\pi$, we can write $\tau$ as,

$$\tau = 1 - 2\gamma s + 4\epsilon.$$  \hfill (14)

Therefore from Eqs. 4 and 14, we get exact energy eigenvalues in atomic units as,

$$E_n = -\frac{\alpha^2}{4} \left( 2n + 1 - \frac{V_2}{\alpha \sqrt{V_1}} \right)^2.$$  \hfill (15)
Using $\sigma(s)$, and $\pi(s)$ in Eqs. (7 - 9), one can find the corresponding wave functions $y(s)$ and $\phi(s)$, then from Eq. (6) as

$$\Psi_n(s) = C_n s^{2\epsilon} e^{-\gamma s} L_n^4 (2\gamma s).$$  \hspace{1cm} (16)

Where $L_n^4(x)$ stands for the associated Laguerre functions. One can easily see well behavior of the wave function at infinity. As an example, the ground state wave function behaves like

$$\Psi_0(s \to 0) \sim \left(e^{-\alpha x}\right)^{iE_0/\alpha} e^{-e^{-\alpha x}/\alpha}.$$

Let us now consider different types of the generalized Morse potential.

### 3.1 Non-$PT$ symmetric and non-Hermitian Morse case

We have considered a more general form of the complexified Morse potential than that was previously studied [6, 7, 9, 12, 20, 21, 22]. For a special value of its parameters, it reduces to the usual one. Thus, we may choose the parameters from two types.

Now, let us take the potential parameters as in the Refs. [7] and [8], $V_1 = (A+iB)^2$, $V_2 = (2C + 1)(A+iB)$, and $\alpha = 1$, then the potential is

$$V(x) = (A+iB)^2 e^{-2x} - (2C + 1)(A+iB)e^{-x},$$

where A, B, and C are arbitrary real parameters and $i = \sqrt{-1}$. Such potentials are non-$PT$-symmetric and also non-Hermitian but have real spectra. According to Solombrino [9], this type of the complex Morse potential is a pseudo-Hermitian and the corresponding Hamiltonian verifies the pseudo-Hermitian propositions weakly (Proposition 3. and 5.). More recently, the basic properties of pseudo-Hermitian operators, pseudo-supersymmetric quantum mechanics and diagonalizable pair of isospectral Hamiltonians with identical degeneracy structure are intensively studied [10].

Let us consider the real spectrum for this case. Substituting the parameters into the energy expression, we simply get

$$E_n = -(n - C)^2.$$  \hspace{1cm} (19)

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In this case the spectrum is completely real and independent from the potential parameters $A$ and $B$. However, there are degeneracy for $A$ and $B$. If $V_1$ is real, and $V_2 = A + iB$, and $\alpha = i\alpha$, $PT-$ violation case, the Morse potential can be written in the following form

$$V(x) = V_1 e^{-2i\alpha x} - (A + iB)e^{-i\alpha x}.$$  \hspace{1cm} (20)

For $V_1 > 0$ case, there are real spectra if and only if $\text{Re}(V_2) = 0$. When $V_1 < 0$ there are real spectra if and only if $\text{Im}(V_2) = 0$. It can be shown that this is related by a pseudo-Hermitian transformation [7, 10].

### 3.2 $PT$ symmetric and non-Hermitian Morse case

When $\alpha \Rightarrow i\alpha$, and $V_1$ and $V_2$ are real, in this case potential takes

$$V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x},$$  \hspace{1cm} (21)

with the $\text{Re}(V(x)) = V_1 \cos(2\alpha x) - V_2 \cos(\alpha x)$ and $\text{Im}(V(x)) = -V_1 \sin(2\alpha x) + V_2 \sin(\alpha x)$. From the Eq. (15) for $V_1 > 0$, we get no real spectra of this kind of $PT$-symmetric Morse potentials. In order to compare our results with the ones obtained by Znojil [11], we simply take the parameters as $V_1 = -\omega^2$, $V_2 = D$, and $\alpha = 2$. In this particular case, we get $\pi$ function as

$$\pi = \pm \frac{i}{2} \begin{cases} 
\left(\omega s - \alpha\right), & \text{for } k = (-iD + 2\omega \alpha)/4 \\
\left(\omega s + \alpha\right), & \text{for } k = (-iD - 2\omega \alpha)/4
\end{cases}$$  \hspace{1cm} (22)

and after appropriate choice of $k$ and $\pi$, we can write $\tau$ as

$$\tau = 1 - i(\omega s + \alpha).$$  \hspace{1cm} (23)

Thus, the energy eigenvalues are reduced to the simple form

$$E_n = \left(2n + 1 + \frac{D}{2\omega}\right)^2.$$  \hspace{1cm} (24)

More recently many interesting properties of such particular cases were studied by Znojil [11] and Bagchi and Quesne [12].
4 Pöschl-Teller potential

We shall consider the general form of the Pöschl-Teller potential

\[ V(x) = -4V_0 \frac{e^{-2\alpha x}}{(1 + q e^{-2\alpha x})^2}. \] (25)

This potential has more flexible form. Because it has a couple of additional free parameters \( \alpha \) and \( q \) to the well known standard Pöschl-Teller form[23]. First of all if we fix the free parameters as \( \alpha = 1 \) and \( q = 1 \) the potential reduces to the well known standard Pöschl-Teller potential. This form of the potential was studied extensively by many authors [24, 25]. The standard Pöschl-Teller potential was applied in the framework of the \( su(2) \) vibron model[26]. The special case of (25) for \( q = 1 \), the modified Pöschl-Teller potential

\[ V(x) = -\frac{D_0}{\cosh^2(\alpha x)}, \] (26)

is used to derive the well known \( su(2) \) spectrum-generating algebra of an infinite square well problem[27].

Let us solve the Schrödinger equation for the generalized Pöschl-Teller potential. One can get the following form with the new variable \( s = e^{-2\alpha x} \) as

\[ \psi''(s) + \frac{1 - qs}{s(1 - qs)} \psi' + \frac{1}{[s(1 - qs)]^2} [-e^2 q^2 s^2 + (2e^2 q - \beta^2) s - \epsilon^2] \psi = 0 \] (27)

where \( \epsilon^2 = -\frac{mE}{2\hbar^2 \alpha^2} \), and \( \beta^2 = \frac{2mV_0}{\hbar^2 \alpha^2} \). Thus, one can easily get the energy eigenvalues in atomic units [28]as,

\[ E_n(q, \alpha) = -\frac{\alpha^2}{4} \left[ -(2n + 1) + \sqrt{1 + \frac{4V_0}{q \alpha^2}} \right]^2. \] (28)

The corresponding wave function becomes

\[ \psi_n(s) = s^{-\nu} (1 - s)^{\nu/2} P_{n}^{(2\nu, -1)}(1 - 2qs). \] (29)

Where \( \nu = 1 - \sqrt{1 + \frac{2mV_0}{\hbar^2 q \alpha^2}} \) and \( P_n^{(\nu, \mu)}(x) \) stands for Jacobi polynomials. One can easily get proper behavior of wave function at infinity.
4.1 Non-PT symmetric and non-Hermitian Pöschl-Teller cases

We are now going to choose $V_0$ and $q$ as complex parameters $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$, where $V_{0R}$, $V_{0I}$, $q_R$, $q_I$ and $\alpha$ arbitrary real parameters. In this case, although the potential is complex, and corresponding Hamiltonian is non-Hermitian and also non-PT-symmetric, there may be a real spectra if and only if $V_{0I}q_R = V_{0R}q_I$. When both parameters $V_0$, and $q$ are taken pure imaginary the potential turns out to be

$$V(x) = -4V_0 \frac{2q e^{-4ax} + i(1 - q^2 e^{-4ax})}{(1 + q^2 e^{-4ax})^2}. \quad (30)$$

Here, we have simply used $V_0$, and $q$ instead of $V_{0I}$ and $q_I$. The energy eigenvalues are the same given in Eqn.(28).

If $q$ is an arbitrary real parameter and $V_0 \Rightarrow iV_0$ also $\alpha \Rightarrow i\alpha$ completely imaginary, the potential becomes

$$V(x) = -4V_0 \frac{(1 - q^2) \sin(2\alpha x) + i(2q + (1 + q^2) \cos(2\alpha x))}{(1 + q^2)^2 + 4q \cos(2\alpha x)(1 + q \cos(2\alpha x) + q^2)}, \quad (31)$$

and the corresponding energy eigenvalues become

$$E_n = \frac{\alpha^2}{4} - B(2n + 1) + 2 + 4n + 4n^2 \quad (32)$$

where, we use the abbreviations as $A = \sqrt{2(1 + (\frac{4V_0}{q\alpha^2})^2) - 2}$, $B = \sqrt{2(1 + (\frac{4V_0}{q\alpha^2})^2) + 2}$. csgn is used for complex signum function in MAPLE Program. For a real spectrum we take $-\frac{B}{2} + (2n + 1) = 0$. So this requires that $V_0/(q\alpha^2) = \pm \sqrt{n(n+1)(1+2n)}$ which is a constraint on the potential parameters.

When all three potential parameters are complex, Hamiltonian is non-Hermitian and also non-PT-symmetric having real spectra. For simplicity, let us take all three parameters are pure imaginary. That is $\alpha$ replaced by $i\alpha$, $q$ replaced by $iq$, and, $V_0$ replaced by $iV_0$. In this case the potential takes the form

$$V(x) = -4V_0 \frac{(1 + q^2) \sin(2\alpha x) + 2q + i(1 - q^2) \cos(2\alpha x)}{(1 + q^2)^2 + 4q^2(1 - \cos^2(2\alpha x)) + 4q(1 + q^2) \sin(2\alpha x)}, \quad (33)$$

and Hamiltonian is non-Hermitian and also non-PT-symmetric. The energy eigenvalues become

$$E_n = \frac{\alpha^2}{4} - \frac{4V_0}{q\alpha^2} + 2(1 + 2n + 2n^2) \quad (34)$$

Here there is a real spectra if and only if $A \geq 0$. sgn is used for signum function in MAPLE Program.
4.2 PT symmetric and non-Hermitian Pöschl-Teller case

For PT-symmetric and non-Hermitian potential case, we choose the parameters $V_0$, and $q$ are arbitrarily real, and $\alpha \Rightarrow ia$. In this case, the potential becomes

$$V(x) = -4V_0 \frac{(1 + q^2) \cos(2\alpha x) + 2q + i(q^2 - 1) \sin(2\alpha x)}{(1 + q^2)^2 + 4q \cos(2\alpha x)(1 + q \cos(2\alpha x) + q^2)},$$  \hspace{1cm} (35)

and the energy eigenvalue is

$$E_n = -\frac{\alpha^2}{4} + \frac{4V_0}{q\alpha^2} - 2(1 + 2n + 2n^2)$$  \hspace{1cm} (36)

where $A = 1 - 4V_0/(q\alpha^2)$, there is a real spectra if and only if $A = 0$, i.e., $4V_0 = q\alpha^2$.

5 Conclusions

We have extended the PT-symmetric formulation, developed recently within the non-relativistic quantum mechanics, to the more general complex Morse and Pöschl-Teller potentials. We solved the Schrödinger equation in one dimension first time for the complex potentials by using Nikiforov-Uvarov method. We studied so many different complex forms of these potentials. Interesting features of quantum expectation theory for PT-violating potentials may be affected by changing from complex to real systems. We observed that there were some restrictions on the potential parameters for bound states in PT-symmetric or, more generally, in non-Hermitian quantum mechanics. Because of the restriction $V_0/(q\alpha^2) = \pm \sqrt{n(n + 1)(1 + 2n)^2}$, there is no ground state of generalized Pöschl-Teller potential when the parameters $V_0$ and $\alpha$ are pure imaginary. Although the number of positive bound sates decreases with increasing $\alpha$ and $q$ or decreasing $V_0$ for real family of the Pöschl-Teller potential, there are positive and negative bound states for PT-symmetric cases. We have pointed out that our exact resuls of complexified Morse and the Pöschl-Teller potentials may increase the applications in the study of different quantum systems.
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