Abstract

We present first results for $B_c$ spectroscopy using Lattice Non-Relativistic QCD (NRQCD). For the NRQCD action the leading order spin-dependent and next to leading order spin-independent interactions have been included with tadpole-improved coefficients. We use multi-exponential fits to multiple correlation functions to extract ground and excited $S$ states and give accurate values for the $S$ state hyperfine splitting and the $P$ state ($B_c^{**}$) fine structure, including the effects of $^1P_1/^3P_1$ mixing.

PACS: 12.38.Gc, 12.39.Hg, 14.40.Lb, 14.40.Nd

1 Introduction

There is much current interest in the spectrum of mixed bound states of bottom and charm quarks with the recent appearance of experimental candidates\[1\] Lattice QCD is the best method of calculating this spectrum from first principles, although attention must be paid to the control of systematic errors from a variety of sources. We have recently given accurate results for bottomonium and charmonium spectroscopy on the lattice\[2, 3\], exploiting the non-relativistic nature of these systems by using a non-relativistic effective theory, NRQCD\[4\]. This can be systematically matched to full QCD, order by order in $\alpha_s$ and $v^2/c^2$, where $v$ is the velocity of the heavy quark in the bound state. Here we present results for $B_c$ spectroscopy using the same techniques and use our previous results to give good indications of the size and direction of systematic errors. We compare our results to recent calculations in potential models\[5\].
2 Lattice NRQCD

The starting point of NRQCD is to expand the original QCD Lagrangian in powers of $v^2$, the typical quark velocity in a bound state. For the $J/\Psi$ system $v^2 \sim 0.3$ and for $\Upsilon$ $v^2 \sim 0.1$. For $B_c$ $v^2 \sim 0.15$ for the single particle with mass equal to the reduced mass of the $b, c$ system, but the kinetic energy will be shared unequally between the $b$ and $c$. $v^2 \sim 0.5$ for the $c$ quark so relativistic corrections will be more important than for $c\bar{c}$ systems. NRQCD enables these corrections to be added systematically.

The action used here is the same one used in refs. [2] and [3]. Relativistic corrections $O(M_Qv^4)$ have been included for both quarks - errors are then at the level of $O(M_c v^6)$. Other sources of systematic error are discretisation errors and errors from the absence of virtual quark loops because we use quenched configurations generated with the standard plaquette action. Finite volume errors should be negligible because of the relatively small size of heavy-heavy systems.

To calculate masses for $b\bar{c}$ bound states we define $b$ and $c$ quark Green functions on the lattice. The NRQCD Lagrangian involves a simple difference equation in the temporal direction, which allows the evolution of the quark Green function as an initial value problem. We start on the first time slice with

$$G_1 = \left( 1 - \frac{aH_0}{2n} \right)^n U_4^\dagger \left( 1 - \frac{aH_0}{2n} \right)^n \delta_{\vec{x},0}$$

and then continue to evolve in the temporal direction using

$$G_{t+1} = \left( 1 - \frac{aH_0}{2n} \right)^n U_4^\dagger \left( 1 - \frac{aH_0}{2n} \right)^n (1-a\delta H) G_t \quad (t > 1).$$

$H_0$ is the lowest order piece of the non-relativistic Hamiltonian, i.e. the kinetic energy operator. On the lattice, this is

$$H_0 = -\frac{\Delta^{(2)}}{2M_c^0}.$$ 

The higher order terms in the Hamiltonian that we have included are [9]

$$\delta H = -c_1 \frac{(\Delta^{(2)})^2}{8(M_c^0)^3} + c_2 \frac{ig}{8(M_c^0)^2} (\Delta \cdot E - E \cdot \Delta)$$

$$-c_3 \frac{g}{8(M_c^0)^2} \sigma \cdot (\Delta \times E - E \times \Delta) - c_4 \frac{g}{2M_c^0} \sigma \cdot B$$

$$+c_5 \frac{a^2\Delta^{(4)}}{24M_c^0} - c_6 \frac{a(\Delta^{(2)})^2}{16n(M_c^0)^2}.$$ 

The first two terms in $\delta H$ are spin-independent relativistic corrections and the next two are spin-dependent terms which contribute the leading order pieces to
the P and S spin splittings respectively. The last two terms come from discretisation corrections to the lattice Laplacian and the lattice time derivative. $\Delta$ is the symmetric lattice derivative, $\Delta^{(2)}$ is the lattice form of the Laplacian and $\Delta^{(4)}$ is a lattice version of the continuum operator $\sum D^4_i$. We use the standard traceless cloverleaf operators for the chromo-electric and magnetic fields, $E$ and $B$. The parameter $n$ is introduced to remove instabilities in the heavy quark propagator caused by the highest momentum modes of the theory[4]. We require $n > 3/Ma$.

To calculate we must fix both the bare coupling and the bare quark masses. We used 200 quenched gluon field configurations of size $12^3 \times 24$ at $\beta = 6/g^2 = 5.7$ generously supplied by the UKQCD collaboration [5], and fixed to Coulomb gauge, using a Fourier accelerated steepest descents algorithm [7]. The bare masses of the $c$ and $b$ quarks were chosen from fits to the kinetic mass of the $\eta_c$ and the $\Upsilon$ [3, 8]. At $\beta = 5.7$ we used a mass for the $c$ quark in lattice units of 0.8 with $n = 4$, and a mass for the $b$ quark of 3.15 with $n = 2$.

The coupling constants $c_i$ appearing in equation (4) can be calculated by matching NRQCD to full QCD [9, 10]. At tree level all the coefficients are one. The largest radiative corrections are believed to be tadpole contributions which can be removed by redefining the gluon fields $U: [11]

$$U_\mu(x) \rightarrow \frac{U_\mu(x)}{u_0}$$

with $u_0$ the fourth root of the plaquette (at $\beta=5.7$ we use $u_0 = 0.861$). Since the cloverleaf expression involves the evaluation of a plaquette this renormalization will have a large effect on $E$ and $B$ fields and thereby on spin-dependent splittings[2]. With the dominant tadpole contributions thus removed, we use the tree level values for the $c_i$’s.

Given the quark propagators in equation (2) it is relatively straightforward to combine them appropriately to form meson propagators with specific quantum numbers. This is outlined in [2]. Here we must combine a $b$ quark propagator with a $\bar{c}$ propagator, but the methods are identical. We use ‘smearing functions’ as sources for the $b$ quark propagator and a delta function source for the $c$ quark. Various different smearing functions are included to study $p$ states and excited $s$ states. It is more efficient, from equation (2), to have numerous $b$ quark propagators with a small value of $n$ and only one $c$ quark propagator. It also enabled us to simultaneously calculate the $b\bar{b}$ spectrum on these lattices, and this will be reported separately [8]. The radius of the smearing functions (taken as wavefunctions from a $1/r$ potential) was kept fixed at an optimal value for $b\bar{b}$ correlation functions. This meant that the effective mass in the $B_c$ correlation functions did not plateau quite as early as it would have done with more optimal smearing but it did not significantly detract from the accuracy of our results.

Local meson operators are tabulated in [4]. Using the notation $2S+1L_J$, we have looked at meson propagators for the following states: $^1S_0$, $^3S_1$, $^1P_1$, $^3P_0$, $^3P_1$ and $^3P_2$ for both the E and T representation. Since $C$ is not a good quantum
number for the $B_c$ system the $^1P_1$ and $^3P_1$ mesons will mix and so in addition we calculated the cross-correlation between these two. For the $s$ states, smearing functions both for the ground and first radially excited state were used as well as a local $\delta$ function. From this all possible combinations of smearing at the source with a local sink were formed. For the $p$ states only the ground state smearing function was used at the source. We calculated the dispersion relation for the $^1S_0$ ($B_c$) by looking at the meson propagator for small non-zero momenta. Because of the relatively small size of these systems it is possible to use more than one starting site for the mesons. We used 8 different spatial origins and 2 different starting times to increase our statistics, but we bin correlation functions over each configuration.

3 Simulation results

For $b\overline{b}$ and $c\overline{c}$ the correlation functions are explicitly real because of charge conjugation symmetry - for $b\overline{c}$ they are not. We find, however, that the imaginary parts of all our correlation functions are very small and much smaller than the real parts. We fit simultaneous multi-exponential fits to the real parts of several correlation functions with different sources and local sinks. This is the ‘row’ fit of refs. 2, 3:

$$G_{\text{meson}}(n_{sc}, loc; t) = \sum_{k=1}^{N_{exp}} b(n_{sc}, k) e^{-E_k t}.$$  \hspace{1cm} (6)

$n_{sc}$ denotes the source smearing, $loc$ denotes a local sink. It enables us to extract masses in lattice units for ground and excited $s$ states. The $p$ states are much noisier and for them we have used single exponential fits to the correlation function with ground state smearing at both source and sink. For spin splittings it is most accurate to perform a single exponential fit to a ratio of correlation functions. We generate a bootstrap ensemble of ratios (including both real and imaginary parts of the correlation functions) and then fit to the real part.

To find masses for the eigenvectors of the $^1P_1$ and $^3P_1$ mixing matrix we fit a 2-exponential form to the $2 \times 2$ matrix formed from these correlation functions smeared at source and sink. Figure 1 shows effective amplitude plots and fits for the four correlation functions in the $2 \times 2$ matrix. The physical states are called the $1^+$ and the $1^{+'}$. To calculate the splitting between these two states and the mixing angle with the $^1P_1/3P_1$ basis accurately, we did matrix fits to an ensemble of matrices generated by bootstrap. This allows the correlation between the states on a given configuration to be taken into account. We checked these fits against those obtained by diagonalising the matrix time-slice by time-slice and fitting single exponential forms to the eigenvalues. There was good agreement between the two methods. We did not find it possible to extract the masses of the $1^+$ and $1^{+'}$ purely from the $^1P_1$ and $^3P_1$ correlation functions (i.e. the diagonal
terms of the matrix) in the absence of the cross-correlation terms. Although theoretically possible this obviously requires very high statistics or a more finely grained lattice in time.

Table 1 shows our results for masses and splittings in lattice units. The errors quoted are statistical errors only.

To convert the masses and splittings in lattice units to physical results, we need a value for $a^{-1}$, the inverse lattice spacing. Heavy-heavy systems provide a particularly good way of fixing $a^{-1}$ because the spin-averaged 1P-1S splitting is insensitive to the other parameter in the theory, i.e. the heavy quark mass, and also to any systematic errors in spin-dependent terms. Thus $a^{-1}$ can be fixed from a comparison of either the $c\bar{c}$ or the $b\bar{b}$ spectrum to experiment. Experimentally the spin averaged splitting is 452 MeV from the $b\bar{b}$ spectrum and almost the same, 458 MeV, from the $c\bar{c}$ spectrum. Here the spin-averaged 1P mass is taken as the average of the $\chi_b$ states, since only a preliminary value for the mass of the $h_c$ is known and no $h_b$ has been seen [12]. Potential models also lead to the belief that
Simulation Results

|        |                  |                |
|--------|------------------|----------------|
| $1^1S_0$ | 0.6052(8)        |                |
| $1^3S_1$ | 0.6353(9)        |                |
| $2^1S_0$ | 1.12(6)          |                |
| $2^3S_1$ | 1.14(6)          |                |
| $1^3P_2$ | 0.986(7)         |                |
| $1^3P_0$ | 0.944(7)         |                |
| $1\, 1^+$ | 0.956(6)         |                |
| $1\, 1^{+'}$ | 0.973(7)     |                |
| $3^1S_1-^1S_0$ | 0.0307(2)   |                |
| $3^1P_2-^3P_0$ | 0.045(4)    |                |
| $3^3P_2-1^+$ | 0.023(2)     |                |
| $1^{+'}-1^+$ | 0.0172(25)   |                |
| $\tan(\theta)$ | 0.66(3)      |                |
| $|\psi(0)|_{B_c}$ | 0.2118(4)    |                |

Table 1: Fitted dimensionless energies for $b\bar{c}$ states and mixing angle for the $1^+$ state. $\theta$ is defined so that $|1^+\rangle = \sin(\theta)|1^1P_1\rangle + \cos(\theta)|1^3P_1\rangle$. The final entry is the wavefunction at the origin for the $B_c$.

the $1^1P_1$ state should lie approximately at the spin-average of the $3^3P$ states, and so does not need to be included.

Unfortunately, in the quenched approximation, the value obtained for $a^{-1}$ depends on the experimental quantity you use to fix it, and in particular, the momentum scales important to that quantity. This is because the effective coupling constant does not run correctly between different momentum scales. In general you expect that a quantity which is more sensitive to large momenta will give a higher value for $a^{-1}$. This is what we find on comparing the $b\bar{c}$ and $c\bar{c}$ spectrum on quenched lattices. At $\beta = 5.7$ $a_{c\bar{c}}^{-1} = 1.20(4)$ GeV [3] and $a_{b\bar{c}}^{-1} = 1.35(4)$ GeV [8]. These results use the spin-average of all $1P$ states [13] because we find that the $1^1P_1$ does not lie at the spin average of the $3^3P$ states. This is likely to be, at least partly, a discretisation error [3].

This discrepancy in values obtained for $a^{-1}$ from the quenched approximation means that the only way to obtain results in good agreement with experiment for a particular set of hadrons is to fix $a^{-1}$ separately for that set using one experimental result for a ‘typical hadron’ from the set. For $b\bar{c}$ systems there are no experimental results so we must fix $a^{-1}$ on the basis of the numbers for $b\bar{b}$ and $c\bar{c}$. We assume that the spin-averaged $1P-1S$ splitting for $b\bar{c}$ will have a value between that for $b\bar{b}$ and that for $c\bar{c}$ (which hardly differ, as above). We must now define the spin-averaged $P$ state from our results to include all $1P$ states since there is no reason for either of the $1^+$ or the $1^{+'}$ to lie at the spin-average of the
other states. This gives $a^{-1} = 1.32(4)$ GeV, as expected between the values from $c\bar{c}$ and $b\bar{b}$. The error quoted is purely statistical.

The difference in $a^{-1}$ between these different systems leads to the effect that the quark mass in physical units will differ between $b\bar{c}$ and either $b\bar{b}$ or $c\bar{c}$. This problem has already been pointed out for heavy-light systems [14] and means that again, in principle, one should fix the quark mass, using experiment, separately within each system in the quenched approximation. For the $B_c$ system, this is not possible, so final meson masses will have an additional uncertainty associated with this $a^{-1}$ effect. It will be important particularly for spin splittings since these are strongly dependent on the quark mass. We have fixed the bare quark masses in lattice units from results for $b\bar{b}$ and $c\bar{c}$. Another reasonable possibility is that we should readjust the bare quark masses in $b\bar{c}$ so that they agree in physical units with those of $b\bar{b}$ and $c\bar{c}$. We have estimated what changes this might give where possible.

Using the value for $a^{-1}$ above, we can convert meson masses to physical units. We still need to set a zero of energy since we have removed it from our Hamiltonian. We do this by calculating $B_c$ correlation functions at small but non-zero values of the meson momentum, $P$, and fitting to

$$E_P - E_0 = \frac{P^2}{2M_{kin}},$$

(7)

$M_{kin}$ is the kinetic (absolute) mass of the $B_c$ in lattice units. We find a kinetic mass of 4.76(2), which, using $a^{-1}$ as above, gives a mass in physical units of 6.28(20) GeV, where the error is dominated by the uncertainty in $a^{-1}$. Errors in the $b$ and $c$ quark masses discussed above actually cancel in the kinetic mass of the $B_c$, since to match the bare physical masses from $b\bar{b}$ and $c\bar{c}$ we in fact have to adjust the $c$ mass down and the $b$ mass up by the same amount.

The difference between $E_0$ and $M_{kin}$ represents the shift of the zero of energy. This can be calculated per quark in perturbation theory and agreement with lattice results is very good for the $\Upsilon$ spectrum [2]. For $c$ quarks the perturbative results are not very reliable because the important scales for $\alpha_s$ are very low. However, we can still check that the non-perturbative energy shift for the $B_c$ is the average of those for $\Upsilon$ and $\Psi$. This should be the case if no non-perturbative effects specific to the mesons appear, and the results in Table 2 show that this is true. It is a requirement for the non-relativistic theory to make sense and is apparently not obeyed for heavy quark actions based on the Wilson action in current use [15].

Table 3 gives results for the meson splittings with the ground state ($B_c$) in GeV and the spectrum is plotted in Figure 2 (using the $B_c$ mass of 6.28 GeV), with a comparison to results from a recent potential model analysis [3].
Table 2: Results for static and kinetic masses and the difference between them for heavy-heavy mesons, using NRQCD at $\beta = 5.7$. Values are in lattice units.

|       | $E_0$    | $M_{kin}$ | shift |
|-------|----------|-----------|-------|
| $\Upsilon$ | 0.5030(5) | 6.97(8)   | 6.47(8) |
| $\eta_c$ | 0.618(1)  | 2.430(6)  | 1.812(6) |
| $B_c$   | 0.6052(8) | 4.76(2)   | 4.15(2)  |

Table 3: NRQCD results for splittings of $b\bar{c}$ states with $B_c$ for $a^{-1} = 1.32$ GeV. Errors shown are statistical only.

| Simulation Results [GeV] |
|---------------------------|
| $1^3S_1 - B_c$ | 0.0405(3) |
| $2^1S_0 - B_c$ | 0.68(8)   |
| $2^3S_1 - B_c$ | 0.71(8)   |
| $1^3P_0 - B_c$ | 0.447(9)  |
| $1^1P_1 - B_c$ | 0.463(8)  |
| $1^1P_{1'} - B_c$ | 0.485(8) |
| $1^3P_2 - B_c$ | 0.503(8)  |

4 Discussion

Figure 2 shows the lattice spectrum compared to recent potential model results [3]. The lattice results have systematic errors resulting from higher order relativistic corrections to the action, discretisation effects and errors from using the quenched approximation. We will attempt to quantify them below. The potential model results have systematic errors also from various sources including variations in the potential itself (compare, for example [16]). The lattice results have the advantage that the errors there can, and will, be systematically removed.

The main source of error for our lattice calculations is in higher order relativistic corrections to the charm quark propagator. The missing terms are at order $M_v^6$ and include $D^6/16M_c^5$, relevant for spin-independent splittings. A perturbative estimate of the size of this correction would be an energy shift of $<p^6>/16M_c^5$. A naïve evaluation of this expectation value in a lattice potential model for $B_c$ gives 100 MeV. However, there are several terms at this order that should be included and we would expect some cancellation between them. A similar analysis of terms of order $M_v^4$, which are included here, shows an almost complete cancellation between the $D^4$ correction and the Darwin term [17]. Our comparison of lattice results with and without these $M_v^4$ terms shows agreement with this analysis [2], giving us confidence that we can estimate these correc-
tions. The conclusion is then that terms of order $Mv^6$ not included could shift our masses by 100 MeV, but by cancellation it could be less than this.

The leading discretisation error is an $O(a^2)$ error from using gluon field configurations with the simple plaquette action. Again we can estimate perturbatively the shift induced by this correction \cite{18}. The shift is proportional to the square of the wave function at the origin, giving 15 MeV for the $B_c$, and zero for states with no wave function at the origin such as $p$ states. The quark propagators are already correct through $O(a^2)$ and residual $a^4$ errors should be smaller than this.

The errors from the quenched approximation should be between those for $b\bar{b}$ and $c\bar{c}$ and can be estimated from the differences we find in simulating those systems with and without dynamical fermions. One source of error, that of uncertainties in the quark masses, has been discussed above.

From Figure 2 the absolute mass of the $B_c$ which we obtain agrees well with potential model estimates. Since it is close to the average of the $\eta_b$ and $\eta_c$ masses this is not surprising. The spin-averaged 1P-1S splitting also agrees by design, since we fixed it to be the same as that for $b\bar{b}$ and $c\bar{c}$ and this would come out of a potential model too. The possible 100 MeV (i.e. 25\%) systematic error in the 1P-1S splitting from higher order relativistic corrections should be borne in mind. The corresponding systematic error for $b\bar{b}$ and $c\bar{c}$ is 6 MeV and 40 MeV respectively.

The other spin-independent splitting is that between the 2S and 1S states. All our quenched calculations \cite{3, 2, 8} give a ratio for the 2S-1S splitting over 1P-1S splitting which is too large. This is expected because the 1S state suffers a bigger shift downwards under quenching than the other states. We would therefore expect our 2S states to appear too high. For $b\bar{b}$ we find the 2S-1S/1P-1S ratio (using the $^3S_1$ for S) to be 1.40(3) at $\beta = 6.0$ \cite{19} and 1.4(2) at $\beta = 5.7$ \cite{8} compared to the experimental value of 1.28 and for $c\bar{c}$ we get 1.4(2) \cite{3} against the experimental value of 1.38. So, we would expect that unquenching would correct this ratio for $B_c$ by roughly 10\%. Since we use the 1P-1S splitting to fix $a^{-1}$, this means that our 2S-1S splitting would be reduced by 10\%, i.e. our 2S level would fall by 70MeV. This brings it rather closer to the potential model prediction, which presumably already incorporates some effects from dynamical fermions through the phenomenological form of the potential. Large relativistic corrections for $B_c$ could distort this picture somewhat.

The spin splittings are perhaps of more immediate interest and particularly that between the $B_c^*$ and the $B_c$, since these states will be the first to be studied experimentally in detail. We find 40 MeV for this splitting in this simulation. Again we expect a sizeable error from the quenched approximation. For $b\bar{b}$ we have results both for dynamical flavors, $N_f = 0$ and 2 and find that the hyperfine splitting changes by a factor of 40 \% on unquenching. For $c\bar{c}$ the hyperfine splitting is known and our quenched simulation gives a result 20 \% too low. So we conclude that the splitting between the $B_c^*$ and $B_c$ might reasonably change to 55 MeV on unquenching. It will increase by an additional 10\% to 60 MeV.
if the charm quark mass in the $B_c$ is reduced so that it matches the bare value in physical units for the $J/\Psi$. Both of these effects again tend to improve the agreement with the potential model (perturbative) value.

For the $p$ fine structure quenching effects may not be very large because of the small wavefunction at the origin. More serious are discretisation errors since the $p$ fine structure is the result of a balance between short and long range effects. From work on $b\bar{b}$ [3, 8] and $c\bar{c}$ [3] we find a tendency for lattice NRQCD calculations to underestimate the overall size of the splittings $[M(3P_3) - M(3P_0)]$, a larger effect for $b\bar{b}$ at $\beta = 5.7$ than for $c\bar{c}$. We believe that this is because the short-range components of the spin-orbit and tensor forces are underestimated at this lattice spacing. If so, the $3P_2 - 3P_0$ splitting will also be underestimated here, from the same effect. The result for the splitting would additionally increase if the charm quark mass were reduced. Our value for this splitting is 60(5) MeV, already larger than some potential model results. It seems likely to us that the experimental value will exceed 60 MeV.

Another result from our $b\bar{b}$ and $c\bar{c}$ work was the ratio of fine structure splittings $[M(\chi_2) - M(\chi_1)]/[M(\chi_1) - M(\chi_0)]$. This exceeded the experimental values and was another indicator that the long-range spin-orbit force has undue dominance at this lattice spacing. For the perturbative potential model of ref. [8] this ratio comes out below experiment for charmonium (although not for bottomonium), indicating that perhaps the long-range spin-orbit term is not large enough.

For $B_c$ the $\chi_1$ state mixes with the $1^+ P_1$. Our results show such a large amount of mixing that the physical states $1^+$ and $1^+$ are very close to the $jj$ coupled states expected in the $M_b \to \infty$ limit. For these states we couple $\vec{L} + \vec{s}_c = \vec{J}_c$ and $\vec{L} + \vec{s}_b = \vec{J}$. The $J_c = 1/2$ state has a mixing angle $\theta$ with the $LS$ coupled basis of $\tan(\theta) = 1/\sqrt{2} = 0.71$. In the $M_b \to \infty$ limit the $1^+$ state becomes the $J_c = 1/2$ state, degenerate with the $\chi_0$, and the $1^+$, the $J_c = 3/2$ state, degenerate with the $\chi_2$. Our result of $\tan(\theta) = 0.66$ is quite different to the (perturbative) potential model [8] in which the mixing (viewed from the $LS$ coupled basis) is very small.

The true result probably lies somewhere in between. In the $LS$ basis and in potential model language, the off-diagonal terms of the mixing matrix are provided purely by the long-range spin-orbit potential. The short-range spin-orbit and tensor terms contribute only to the diagonal pieces (as does the spin-spin term, although we expect it to vanish for $p$ states). For our calculation then the off-diagonal terms are too large compared to the on-diagonal, and we overestimate the mixing. For the perturbative potential models, the opposite may be true. The mixing is clearly very sensitive to these effects.

We can also estimate the decay constant $f_{B_c}$ from our result for the wavefunction at the origin. Using the standard formula $f^2 = 12|\psi(0)|^2/M$, we obtain $f_{B_c} = 440(20)$ MeV, where the error quoted is dominated by the uncertainty in $a^{-1}$. This result is only valid to leading order in the inverse quark masses and
Figure 2: NRQCD simulation results for the spectrum of the $B_c$ system using an inverse lattice spacing of 1.32 GeV. Error bars are shown where visible and only indicate statistical uncertainties. Dashed lines show results from a recent potential model calculation [5].

$\alpha_s$. Relativistic $1/M_c^2$ corrections should be applied as well as a perturbative renormalisation factor before a useful continuum value can be quoted [20].

5 Conclusions

This represents a first calculation of the $b\bar{c}$ spectrum on the lattice. We use NRQCD and include the leading relativistic and discretisation corrections with tadpole-improved coefficients. We give a spectrum including radially excited $s$ states as well as $p$ fine structure taking account of mixing between the $J = 1$ states. Agreement with potential model results is surprisingly good, given that they include, at least explicitly, no relativistic corrections and the velocity of the charm quark within a $B_c$ is actually higher than for charmonium. Analysis of the systematic errors in the lattice calculation tends to improve the agreement with potential models except for the overall scale of the $p$ fine structure, $M(^3P_2) - M(^3P_0)$. We believe this is underestimated at present. Our result for the mixed $J = 1$ states show strong mixing in the $LS$ coupled basis so that physical states are close to the $jj$ coupled limit. We believe that the improvement of systematic errors from discretisation will tend to reduce this mixing. Future calculations of the spectrum will work at smaller lattice spacings with relativistic charm quarks.
Acknowledgements This calculation was performed at the Atlas Centre under grant GR/J18927 from the UK PPARC and at NERSC. AJL is also grateful to PPARC for a studentship, CTHD for support under grant GR/J21231 and JSI for a Visiting Fellowship to Glasgow while this work was being completed. This work was supported in part by grants from the U.S. Department of Energy (DE-FC05-85ER250000, DE-FG05-92ER40742, DE-FG02-91ER40690), and the National Science Foundation. We thank UKQCD for making their configurations available to us, and in particular David Henty who helped us to read them. We thank Ian Knowles for useful discussions.

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