Comparison of Different Height Systems

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Abstract  Geopotential, dynamic, orthometric and normal height systems and the corrections related to these systems are evaluated in this paper. Along two different routes, with a length of about 5 kilometers, precise leveling and gravity measurements are done. One of the routes is in an even field while the other is in a rough field. The magnitudes of orthometric, normal and dynamic corrections are calculated for each route. Orthometric, dynamic, and normal height differences are acquired by adding the corrections to the height differences obtained from geometric leveling. The magnitudes of the corrections between the two routes are compared. In addition, by subtracting orthometric, dynamic, and normal heights from geometric leveling, deviations of these heights from geometric leveling are counted.

Keywords  geopotential; dynamic; orthometric; normal; ellipsoidal height; gravity; potential; geoid

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Introduction

To determine heights of earth points, an origin surface with zero height must be described and the vertical distance of points from the surface must be determined. For heights, a variety of surfaces can be taken as reference. Of these, the most important surface is the geoid.

Reference surfaces and vertical distances of points from the surfaces have different physical and geometric meanings. Geodetic heights also have different physical and geometric meanings. Thus, several height systems have been introduced. Different height systems were examined in this study.

1 Height systems

1.1 Geopotential and dynamic heights

Geopotential height or geopotential number is a surface point’s potential difference to the geoid potential. Put differently, it is the difference between a surface point’s $W_A$ potential with $O$ geoid point’s $W_O$ potential. The difference

$$C_d = W_O - W_A = \int_0^A \frac{dW}{d} = \int_0^A g \, dn$$

(1)

can be obtained by the integral from Eq.(1). $C_d$ is known as the geopotential number and is path-independent; $W_O$ is the geoid potential; $W_A$ is a point’s potential; $dW$ is the potential difference in differential meaning; $dn$ is the height difference in differential meaning; $g$ is gravity.

The geopotential number is measured in kgal·m and $C$ is equal in the same equipotential surface. Although it has no distance dimension, it is the natural criterion for heights.

Dynamic heights

$$H^{dn} = C / \gamma_0$$

(2)

have been obtained from Eq.(2). The geopotential number is divided by the $\gamma_0$ constant where $\gamma_0$ is the normal gravity value at 45° latitude. The value is
\( \gamma_{45} = 980.629 \text{ gal for international ellipsoids}^{[1,2]} \).

### 1.2 Orthometric height

The distance of a surface point along the plumb line to the geoid, which is taken as the reference surface, is named the orthometric height. It can be obtained as follows:

\[
H^o = C / \bar{g} \tag{3}
\]

For the computation of actual average gravity \( \bar{g} \) along the plumb line, actual gravity is required between the geoid and earth surface. However, gravity inside the earth cannot be measured so a hypothesis regarding the mass distribution must be formed and \( \bar{g} \) is computed on this basis. The orthometric height cannot be determined without a hypothesis.

Orthometric heights are the natural “heights above sea level”, that is, heights above the geoid. Therefore, they have an unequalled geometrical and physical significance\(^[3]\).

### 1.3 Normal height

The ellipsoid is the reference surface best resembling the geoid, which is the earth’s basic shape. An ellipsoid’s gravity field is called a normal gravity field. The Earth’s actual gravity field is slightly different from the normal gravity field. The difference is called the disturbing potential.

We are going to assume that the actual gravity field is equal to the normal gravity field. In other words, we assume \( W = U, g = \gamma, T = 0 \). Hence, orthometric heights which correspond to this approximation are named normal heights and are shown by \( H^N \). They can be obtained as follows:

\[
H^N = C / \gamma \tag{4}
\]

where \( W \) is the actual gravity potential; \( U \) is the normal gravity potential; \( T \) is the disturbing potential; \( \bar{\gamma} \) is the normal average gravity along a normal plumb line.

A surface point \( P \) has \( W_P \) actual and \( U_P \) normal gravity potential and these potentials are not equal \((W_P \neq U_P)\). But on the plumb line that crosses point \( P \), there is a \( Q \) point. In this \( Q \) point, the normal potential is equal to the actual potential \((W_P = U_Q)\). The surface for which \( W_P = U_Q \) holds for every point is called a “Telluroid”. The normal \( H^N \) of a point \( P \) is equivalent to the height of the corresponding telluroid point \( Q \) above the ellipsoid (Fig.1).

**Fig.1  Orthometric and Normal Heights**

The physical and geometric meanings of normal heights are less obvious; they are dependent on the reference ellipsoid used\(^[2]\).

### 2 Computation of heights and corrections

Dynamic, orthometric and normal heights are obtained by using the geopotential number or by adding a correction magnitude to the result of leveling. In practice, these heights are generally obtained by making corrections to the result of leveling. In this study, these heights have also been computed by adding corrections.

#### 2.1 Computation of geopotential heights

The geopotential height difference between two points

\[
C_B - C_A = D_{AB} = W_A - W_B = \int_A^B g \, dn \equiv \sum_A^B g \, dn \tag{5}
\]

is computed from Eq.(5). Here, \( \Delta C_{AB} \) is the geopotential height difference between points \( A \) and \( B \); \( g \) is the measured actual gravity along the route; \( dn \) is the height difference in the meaning of the differential.

#### 2.2 Dynamic height correction and dynamic height computation

In practice, height differences \( \Delta n_{AB} \) measured by leveling are converted into dynamic heights by adding a correction magnitude. This correction is called a dynamic correction \( DC_{AB} \). It is calculated as follows:
2.3 Orthometric correction and orthometric height computation

Height differences (\(\Delta n_{ab}\)) measured by leveling are converted into orthometric heights by adding a correction magnitude. This correction is called an orthometric correction (\(OC_{ab}\)). It is calculated as follows:

\[
OC_{ab} = \sum_{A}^{B} \left( g - \frac{g_{0}}{\gamma_{0}} \right) \delta n + \left( \frac{\delta A}{\gamma_{0}} - \frac{g_{0}}{\gamma_{0}} \right) H_{A} + \left( \frac{\delta B}{\gamma_{0}} - \frac{g_{0}}{\gamma_{0}} \right) H_{B} \tag{8}
\]

where \(g\) is the actual gravity measured along the route; \(\delta A, \delta B\) are the actual average gravity.

Actual average gravity values can be computed either by using this equation

\[
\overline{g} = g + 0.042 \times 4 \times H^{*} \tag{9}
\]

(The unit of \(g\) is gals; \(H^{*}\) is km) or by using the reduction of Prey

\[
\overline{g} = \frac{1}{2} (g + g_{0}) \tag{10}
\]

where \(g\) is the gravity measured on point \(A\); \(g_{0}\) is the gravity measured on point \(A_{0}\) on the geoid corresponding to point \(A\). Eq.(10) was put forward by Mader in 1954. Here, \(g\) is accepted to be linearly changing along the plumb line.

Consequently, the \(\Delta H^{*}_{ab}\) orthometric height difference is given by

\[
\Delta H^{*}_{ab} = \Delta n_{ab} + OC_{ab} \tag{11}
\]

In correction computations, \(\gamma_{0}\) is taken as a constant gravity value (usually normal gravity \(\gamma_{0}^{5}\) for the geographic latitude \(\varphi = 45^\circ\))\[^{[1]}\].

2.4 Normal correction and normal height computation

Height differences (\(\Delta n_{ab}\)) measured by leveling are converted into normal heights by adding a correction magnitude. This correction is called a normal correction (\(NC_{ab}\)). The normal correction can be obtained by changing \(g\) with \(\gamma\) and \(H^{*}\) with \(H\) in

\[
DC_{ab} = \int_{A}^{B} \left( g - \frac{g_{0}}{\gamma_{0}} \right) \delta n = \sum_{A}^{B} \left( g - \frac{g_{0}}{\gamma_{0}} \right) \delta n \tag{6}
\]

\(\Delta H^{dn}_{ab}\) dynamic height difference

\[
\Delta H^{dn}_{ab} = \Delta n_{ab} + DC_{ab} \tag{7}
\]

is obtained from Eq.(7)\[^{[1,4]}\].

\[\text{Eq.}(8). \text{ It is calculated as follows;}
\]

\[
NC_{ab} = \sum_{A}^{B} \left( g - \frac{g_{0}}{\gamma_{0}} \right) \delta n + \frac{\delta A}{\gamma_{0}} H_{A} + \frac{\delta B}{\gamma_{0}} H_{B} \tag{12}
\]

Where \(g\) is the actual gravity measured along the route; \(\gamma_{0}\) is the normal gravity (\(\gamma_{0}^{5}\)); \(\gamma\) is the normal average gravity.

At points \(A\) and \(B\) normal average gravity values are obtained from normal gravity values \(\gamma\) at points \(A\) and \(B\)

\[
\gamma = \gamma_{0} \left[ 1 - (1 + f + m - 2 f \sin^{2} \varphi) \frac{H_{A} + H_{B}}{a} \right] \tag{13}
\]

\(\gamma\) is obtained as follows:

\[
\gamma = 978.049 \times (1 + 0.005288 \times \sin^{2} \varphi - 0.0000059 \times 9 \times \sin^{2} 2\psi) \tag{14}
\]

Where \(\varphi\) is the latitude; \(f\) is the geometric flattening; \(m = \frac{b}{a}\); \(\omega\) is the angular velocity of the earth’s rotation; \(a\) is the semi major axis of the ellipsoid, and \(b\) is the semi minor axis of the ellipsoid\[^{[3]}\].

\(\Delta H^{N}_{ab}\) normal height difference can also be obtained as follows:

\[
\Delta H^{N}_{ab} = H_{B} - H_{A} = \Delta n_{ab} + NC_{ab} \tag{15}
\]

3 Lang measures

To compute geopotential, dynamic, orthometric and normal heights, various measurements were done along two different routes. These routes’ lengths are nearly equal but one is relatively even while the other is rough land.

3.1 Study area

Application was done along two different routes. The first route is relatively even, approximately 5.25 km in length and has a 7.87 m height difference. This route lies between RS points. These RS points have been established by mapping authorities in Trabzon, Turkey (Fig.2).

The second route is relatively rougher, approximately 4.78 km in length, and has a 385 m height difference. The route’s initial and final points have been established by another project at the Karadeniz Technical University in Trabzon, Turkey (Fig.3).
3.2 Precision Leveling

Precision leveling was done according to a double run measure plan. In the measurements, the double run’s difference does not exceed $6\sqrt{K}$ mm ($K$ is the length of the route in km units). Leveling was done with a DL-101C electronic digital leveler. To improve both accuracy and precision, invar rods were operated with two ground plates. In addition, backward and foresight distances were taken to be equal while maximum sighting distances were 30 m for 30x magnification\(^5\).

The heights of the initial and final points are as follows:

$$
H_{DN\ 88-2}=43.712\ 8\ m, \quad H_{55-DN254}=35.835\ m \\
H_{G1}=8.961\ 9\ m, \quad H_{G2}=394.0201\ m
$$

3.3 Gravity measurements

For these measurements, a Worden Master Gravimeter was used. First, the instrument’s constant was checked in the port of Trabzon. The values read inland were multiplied by the instrument constant to arrive at values measured in mgal\(^6\).

Along the first route, gravity measurements were done in every other point established digital level with simultaneous leveling. The gravity values of the omitted points were determined by interpolating the gravity values at the points which were measured.

Since the second route is rougher, gravity measurements were done more frequently, approximately with 70~100 intervals.

The gravity values of the omitted points were also determined by interpolating the gravity values at the points which were measured.

3.4 GPS measurements

For determining the latitudes of the routes’ initial and final points, we have used a SPORTRAK Map (MAGELLAN) handheld GPS device. By using the latitude values in Eq.(14), we obtained the normal gravity values ($\gamma$) of the routes’ initial and final points.

4 Results and evaluation

Height differences obtained by leveling along routes are shown in Table 1.

| Route number-direction | Geometric height difference $\Delta H_{ij}$/m | Average geometric height difference $\Delta H_{ij}$/m |
|------------------------|---------------------------------------------|--------------------------------------------------|
| 1. Going route         | 787 402                                     | 787 416                                          |
| 1. Turning route       | 787 430                                     |                                                  |
| 2. Going route         | 38 505 517                                  | 38 505 536                                      |
| 2. Turning route       | 38 505 555                                  |                                                  |

Actual average gravity $\overline{\gamma}$, normal gravity $\gamma_i$ and normal average gravity $\overline{\gamma}$ have been alternately obtained by Eqs.(9), (14) and (13) and these values are shown in Table 2.

Table 3 was prepared using Table 2 and by measurements done inland. In this table, dynamic, orthometric and normal corrections are shown. Here, dynamic corrections have been computed from Eq.(6), orthometric corrections have been computed from Eq.(8) and normal corrections have been computed from Eq.(12).
Table 2  Gravity values used in corrections

| Route number-direction | Point Number  | Measured gravity $g$ / mgal | Actual average gravity $\bar{g}$ / mgal | Normal gravity $\gamma$ / mgal | Normal average gravity $\bar{\gamma}$ / mgal |
|-------------------------|---------------|-----------------------------|----------------------------------------|---------------------------------|---------------------------------------------|
| 1. Going route          | DN 88-2       | 980 263.319                 | 980 265.172                            | 980 266.523                     | 980 266.994                                 |
|                         | 85-DN 254     | 980 261.673                 | 980 263.192                            | 980 266.523                     | 980 266.994                                 |
| 1. Turning route        | DN 88-2       | 980 262.170                 | 980 263.985                            | 980 266.523                     | 980 262.508                                 |
|                         | G1            | 980 279.844                 | 980 280.224                            | 980 269.925                     | 980 268.542                                 |
|                         | G2            | 980 213.142                 | 980 229.848                            | 980 267.271                     | 980 206.482                                 |
| 2. Going route          | G1            | 980 279.844                 | 980 280.224                            | 980 269.925                     | 980 268.542                                 |
|                         | G2            | 980 213.142                 | 980 229.848                            | 980 267.271                     | 980 206.482                                 |
| 2. Turning route        | G1            | 980 279.844                 | 980 280.224                            | 980 269.925                     | 980 268.542                                 |
|                         | G2            | 980 213.142                 | 980 229.848                            | 980 267.271                     | 980 206.482                                 |

Thus, we could compute geopotential, dynamic, orthometric and normal height differences for each route as double runs. Table 5 has been prepared from Table 4 by subtracting going and turning measurements from each other. These differences are very small, smaller than a mm unit. This is because the going and turning routes are partly the same.

Geopotential height differences were computed using Eq.(5). Computed corrections were added to the measured height differences like the markings in Eqs.(7), (11) and (15). Then, Table 4 was prepared.

Table 3  Corrections for added leveling

| Route number-direction | Dynamic correction $DC$ / m | Orthometric correction $OC$ / m | Normal correction $NC$ / m |
|------------------------|-----------------------------|---------------------------------|----------------------------|
| 1. Going Route         | −0.002 89                   | −0.000 03                        | 0.000 01                   |
| 1. Turning Route       | −0.002 81                   | −0.000 02                        | 0.000 08                   |
| 2. Going Route         | −0.149 27                   | 0.008 08                         | 0.017 36                   |
| 2. Turning Route       | −0.150 14                   | 0.007 21                         | 0.016 50                   |

Average heights were obtained by taking the average of going and turning height differences using Table 4. These average heights are shown in Table 6.

Table 4  Height differences

| Route number-direction | Geopotential height diff. $\Delta C / (\text{kgal} \cdot \text{m})$ | Dynamic height diff. $\Delta H^D / \text{m}$ | Orthometric height diff. $\Delta H^O / \text{m}$ | Normal height diff. $\Delta H^N / \text{m}$ |
|------------------------|---------------------------------------------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| 1. Going Route         | 7.718 66                                                      | 7.871 13                                   | 7.873 98                                   | 7.874 02                                    |
| 1. Turning Route       | 7.719 04                                                      | 7.871 49                                   | 7.874 28                                   | 7.874 38                                    |
| 2. Going Route         | 377.449 19                                                   | 384.905 89                                 | 385.063 25                                 | 385.072 53                                 |
| 2. Turning Route       | 377.449 56                                                   | 384.905 41                                 | 385.062 76                                 | 385.072 05                                 |

Table 5  Going and turning difference of routes

| Route number          | $\Delta C / (\text{kgal} \cdot \text{m})$ | $\Delta H^D / \text{m}$ | $\Delta H^O / \text{mm}$ | $\Delta H^N / \text{mm}$ |
|-----------------------|---------------------------------|-----------------|-----------------|-----------------|
| 1.Route               | −0.000 35                       | −0.357 46       | −0.296 15       | −0.355 00       |
| 2.Route               | −0.000 37                       | 0.484 30        | 0.486 19        | 0.480 00        |

Table 6  Average Height Differences

| Route number          | Geopotential height diff. $\Delta C / (\text{kgal} \cdot \text{m})$ | Dynamic height diff. $\Delta H^D / \text{m}$ | Orthometric height diff. $\Delta H^O / \text{m}$ | Normal height diff. $\Delta H^N / \text{m}$ |
|-----------------------|---------------------------------------------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| 1.Route               | 7.718 83                                                      | 7.871 31                                    | 7.874 13                                   | 7.874 20                                    |
| 2.Route               | 377.449 37                                                   | 384.905 65                                 | 385.063 00                                 | 385.072 29                                 |

By making use of the data in Table 6 and Table 1, all of the height differences were subtracted from average geometric height differences. Then, Table 7 was prepared. In this table, we can see the minimum deviation of height from geometric leveling.

According to Table 7, the dynamic correction is
one hundred fold bigger than the orthometric correction and the normal correction is twofold bigger than the orthometric height in the even field (first route).

Along the rough route (second route), the dynamic correction is twenty fold bigger than the orthometric correction and the normal correction is twofold bigger than the orthometric height.

For each route, the smallest correction is the orthometric correction. The normal correction is closer to the orthometric correction but the dynamic correction is much bigger than the other two.

### 5 Conclusions

1) In the even field, the orthometric correction is smaller than a mm level but in the rough field, it is at mm level. The normal correction is smaller than the mm level in the even field, but in the rough field it is at cm level. The dynamic correction is at mm level in the even field but in the rough field it is at dm level.

Therefore, we can conclude that in even fields all of the corrections are too small but in the rough fields these values increase.

2) The difference of dynamic, orthometric and normal heights from geometric leveling are as follows, respectively: along the first route, which is nearly 5 km in length and has a 7 m height difference, they are $-3 \text{ mm}$, $-0.03 \text{ mm}$ and $0.04 \text{ mm}$. Along the second route, which is nearly 5 km in length and has a 387 m height difference, they are $-15 \text{ cm}$, $8 \text{ mm}$ and $1.7 \text{ cm}$.

Here, the corrections can be arranged in order from small to large as the orthometric, normal and dynamic correction.

3) The smallest correction is the orthometric correction. The height type which deviates the least from geometric leveling is the orthometric height.

4) Orthometric heights have been found to be the best kind of height in geodesy. This is because their correction magnitudes are small, and they take the geoid as the reference surface as well as their geometric meaning.

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**The XXI Congress of the International Society for Photogrammetry and Remote Sensing (ISPRS) was Held in Beijing**

The XXI Congress of the International Society for Photogrammetry and Remote Sensing (ISPRS), the greatest conference in the community once every four years, was held in Beijing on July 3 to 11, 2008. With the main topic “Silk Road for Information from Imagery”, this congress attracted researchers, scientists and users from all over the world to meet Beijing to exchange their latest achievement, share their successful experience and study and discuss the technique development directions. Companies and manufacturers from all the world joined the congress and demonstrated their latest techniques and world-class instruments. In the congress, Chinese government and Chinese departments of surveying and mapping at all level displayed the great advances made by China during the past 20 years in photogrammetry, remote sensing and spatial information science, especially the great contribution made by Chinese surveying and mapping techniques in the recent Wenchuan earthquake.