On ergodic control problem for viscous Hamilton–Jacobi equations for weakly coupled elliptic systems

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ABSTRACT. In this article, we study ergodic problems in the whole space $\mathbb{R}^n$ for weakly coupled systems of viscous Hamilton-Jacobi equations with coercive right-hand sides. The Hamiltonians are assumed to have a fairly general structure, and the switching rates need not be constant. We prove the existence of a critical value $\lambda^*$ such that the ergodic eigenvalue problem has a solution for every $\lambda \leq \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover, the existence and uniqueness of non-negative solutions corresponding to the value $\lambda^*$ are also established. We also exhibit the implication of these results to the ergodic optimal control problems of controlled switching diffusions.

1. INTRODUCTION

In this article we study the existence and uniqueness of solution $(u, \lambda) = (u_1, u_2, \lambda)$ to the equation

$$
-\Delta u_1(x) + H_1(x, \nabla u_1(x)) + \alpha_1(x)(u_1(x) - u_2(x)) = f_1(x) - \lambda \quad \text{in } \mathbb{R}^n,
$$

$$
-\Delta u_2(x) + H_2(x, \nabla u_2(x)) + \alpha_2(x)(u_2(x) - u_1(x)) = f_2(x) - \lambda \quad \text{in } \mathbb{R}^n,
$$

where $H_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denote the Hamiltonians, and $\alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are the switching rate parameters for $i = 1, 2$. We make the following set of assumptions

Assumption 1.1. The functions $\alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are continuously differentiable and for some constant $\alpha_0 > 0$ we have

$$
\alpha_0^{-1} \leq \alpha_i(x) \leq \alpha_0, \quad \sup_x |\nabla \alpha_i(x)| \leq \alpha_0 \quad \text{for } i = 1, 2.
$$

Also, the following hold.

(A1) There exist $\ell_i \in C(\mathbb{R}^n \times \mathbb{R}^n)$, $\xi \mapsto \ell_i(x, \xi)$ strictly convex, and

$$
H_i(x, p) = \sup_{\xi \in \mathbb{R}^n} \{\xi \cdot p - \ell_i(x, \xi)\}, \quad i = 1, 2,
$$

are the Legendre transformation of $\ell_i$, $i = 1, 2$. Moreover, $H_i \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ and the functions $\xi \mapsto H_i(x, \xi)$ are strictly convex, $i = 1, 2$.

(A2) For some constants $\gamma_i > 1, i = 1, 2$, we have

$$
C_1^{-1}|p|^{\gamma_i} - C_1 \leq H_i(x, p) \leq C_1(|p|^{\gamma_i} + 1), \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n,
$$

$$
|\nabla_x H_i(x, p)| \leq C_1(1 + |p|^{\gamma_i}) \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n,
$$

for some positive constant $C_1$ and $i = 1, 2$.

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Since $\xi \mapsto H_i(x,\xi)$ is convex, it follows from (1.2) that
\[|\nabla_p H_i(x,p)| \leq \tilde{C}_1(1 + |p|^{\gamma_1-1}) \quad (x,p) \in \mathbb{R}^N \times \mathbb{R}^N, \quad i = 1, 2, \tag{1.4}\]
for some positive constant $\tilde{C}_1$. In fact, for $|p| > 0$ we see that
\[|\nabla_p H_i(x,p)| = \max_{|e|=1} \nabla_p H_i(x,p) \cdot e = \max_{|z|=|p|} \frac{1}{|p|} \nabla_p H_i(x,p) \cdot z \leq \frac{1}{|p|} \max_{|z|=|p|} (H_i(x,p + z) - H_i(p)),\]
using convexity. Now (1.4) follows from (1.2).

Typical examples of $H_i$ satisfying the above assumptions would be
\[H_i(x,p) = \frac{1}{\gamma_i} (p, a_i(x)p)^{\gamma_i/2} + b_i(x) \cdot p,\]
where $a_i : \mathbb{R}^N \to \mathbb{R}^{N \times N}, b_i : \mathbb{R}^N \to \mathbb{R}^N$ are bounded functions with bounded derivatives, and $a_i$ are uniformly elliptic for $i = 1, 2$. In this case,
\[\ell_i(x,\xi) = \frac{1}{\gamma_i} \langle (\xi - b_i(x), a_i^{-1}(x)(\xi - b_i(x))) \rangle^{\gamma_i/2} \quad \text{where} \quad \frac{1}{\gamma_i} + \frac{1}{\gamma_i} = 1,\]
for $i = 1, 2$. The source terms $f_i, i = 1, 2$, are assumed to satisfy the following

**Assumption 1.2.** The functions $f_i : \mathbb{R}^N \to \mathbb{R}, i = 1, 2$, are continuously differentiable and for some positive constant $C_2$ we have
\[|\nabla f_i(x)| \leq C_2 (1 + |f_i(x)|^{2 - \frac{1}{\gamma_i}}) \quad x \in \mathbb{R}^N, \tag{1.5}\]
for $i = 1, 2$. We also assume that for some $r > 0$ we have
\[\left(1 + \sup_{B_r(x)} |f_i(x)| \right) \leq C_3, \quad \text{for} \quad x \in \mathbb{R}^N, \tag{1.6}\]
for some constant $C_3$ and $i = 1, 2$.

Without any loss of generality, we would assume that $r = 1$. Note that (1.5)-(1.6) hold if we have sup$_{x \in \mathbb{R}^N} |\nabla \log f_i(x)| < \infty, i = 1, 2$, and $f_1, f_2$ are positive outside a compact set. Some other type of examples include $f_i(x) = |x|^\beta_i (2 + \sin((1 + |x|^2)^{\beta_i}))$ for $\beta_i > 0$ and $(\beta_1 + 2\beta_2 - 1) \frac{\gamma_i}{2\gamma_i - 1} \leq \beta_1, i = 1, 2$. From (1.6) we also see that
\[|f_i(x)| \leq C_3 (|f_i(y)| + 1) \quad \text{whenever} \quad |x - y| \leq 1,\]
which readily gives
\[|f_i(x)| \leq C_3 \left( \inf_{B_1(x)} |f_i(y)| + 1 \right) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{1.7}\]

(1.6) will be used to obtain certain estimate on the gradient of $u$ (see Lemma 2.1).

Throughout the paper, if $\mathcal{X}(\mathbb{R}^N)$ is a subspace of real-valued functions on $\mathbb{R}^N$ then we define the corresponding space $\mathcal{X}(\mathbb{R}^N \times \{1, 2\}) := (\mathcal{X}(\mathbb{R}^N))^2$, and endow it with the product topology, if applicable. Thus, a function $g \in \mathcal{X}(\mathbb{R}^d \times \{1, 2\})$ is identified with the vector-valued function
\[g := (g_1, g_2) \in (\mathcal{X}(\mathbb{R}^d))^2, \quad \text{where} \quad f_k(\cdot) := f(\cdot, k), \quad k = 1, 2. \tag{1.8}\]

With a slight abuse in notation we write $g \in \mathcal{X}(\mathbb{R}^N \times \{1, 2\})$. 
1.1. **Background and Motivation.** The system of equations (EP) arise as the Hamilton-Jacobi equations (HJE) in certain ergodic control problems of diffusions in a switching environment. To be more precise, consider the controlled dynamics pair \((X, S)\) where \(\{X_t\}\) denotes the continuous part governed by a controlled diffusion

\[
dX_t = b(X_t, S_t) \, dt - U_t \, dt + dW_t,
\]

where \(W\) is a standard \(N\)-dimension Brownian motion, \(U\) is an admissible control, and \(\{S_t\}\) is a two state Markov process, taking values in \(\{1, 2\}\), responsible for random switching. The functions \(\alpha_1, \alpha_2\) corresponds to the switching rates which is also allowed to be state dependent, that is,

\[
\mathbb{P}(S_{t+\delta t} = j | S_t = i, X_s, S_s, s \leq t) = \begin{cases} 
\alpha_1(X_t) \delta t + o(\delta t) & \text{if } j = 2, i = 1, \\
\alpha_2(X_t) \delta t + o(\delta t) & \text{if } j = 1, i = 2.
\end{cases}
\]

We consider the minimization problem

\[
\lambda^* = \inf_{U \in \mathcal{A}} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (f(X_t) + \ell(X_t, S_t)) \, dt \right],
\]

where \(\mathcal{A}\) denotes the set of all admissible controls. Then the HJE equation associated to this optimal control problem is given by (EP) where

\[
H_i(x, p) = -b_i(x) \cdot p + \sup_{\xi \in \mathbb{R}^N} \{ p \cdot \xi - \ell_i(x, \xi) \} \quad i = 1, 2.
\]

For a more precise description see **Section 2.3**. Because of the presence of both continuous dynamics and discrete jumps, regime-switching systems are capable of describing complex systems and the randomness of the environment. We refer to the book of Yin and Zhu [25] for more detail on regime-switching dynamics and its application to the theory of stochastic control. Note that our equations (EP) includes the stochastic LQ ergodic control problem (that is, \(\gamma_1 = \gamma_2 = 2\)) for regime-switching dynamics which are quite popular models in portfolio selection problems (cf. [26, Chapter 6]). One of our main results establishes the existence of a unique optimal stationary Markov control (see Theorem 2.5) for the above optimization problem.

The ergodic control problems for scalar second order elliptic equations have been studied extensively by several mathematicians and therefore, it is almost impossible to list all the important works in this direction. Nevertheless, we mention some of them that, in our opinion, are milestones in this topic. Ergodic control problems with quadratic Hamiltonian are first studied by Bensoussan and Souganidis [8, 9] where the authors establish the existence and uniqueness of unbounded solutions in \(\mathbb{R}^N\). For space-time periodic Hamiltonians, the existence and uniqueness are considered by Barles and Souganidis [4]. Ichihara [16–18] considers the problem for a general class of Hamiltonians and recurrence/transience properties of the optimal feedback controls are also discussed. We also mention the work of Cirant [12] who investigates the ergodic control problem in \(\mathbb{R}^N\) for a fairly general family of Hamiltonians. It is shown in [12] that the problem in \(\mathbb{R}^N\) can be approximated by the ergodic control problems in bounded domains with Neumann boundary condition. Recently, the uniqueness of unbounded solutions for a general family of source terms are established by Barles and Meireles [5], which is then further improved by the first two authors and Caffarelli [3] in the subcritical case. There are also several important works studying long-time behaviour of the solutions to certain parabolic equations and its convergence to the solutions to the ergodic control problems: see for instance, Barles-Souganidis [4], Fujita-Ishii-Loreti [13], Tchamba [24], Ichihara [17], Barles-Porretta-Tchamba [6], Barles-Quaas-Rodríguez [7].

On the other hand, number of works on the ergodic control problem for second-order weakly coupled elliptic systems are very few. All existing results in this direction consider the domain to be a torus. See, for instance, Cagnetti-Gomes-Mitake-Tran [11], Ley-Nguyen [20] and references therein. We point out that [11, 20] also study the large-time asymptotics for the solutions to certain
systems of parabolic equations, which we do not consider in this article. However, if one assumes the action set to be compact then similar problems have been addressed in detail, see Ghosh-Arapostathis-Marcus [14], Arapostathis-Borkar-Ghosh [2, Chapter 5]. One of the main challenges in studying the weakly coupled systems lies in establishing appropriate gradient estimates of $u$ and bounds on the term $|u_1 - u_2|$ (see Proposition 2.1 below).

1.2. Main results. Our chief goal in this article is to find solutions corresponding to the critical value $\lambda^*$ defined by

$$\lambda^* = \sup \{ \lambda \in \mathbb{R} : \exists u \in C^2(\mathbb{R}^N \times \{1, 2\}) \text{ such that } (u, \lambda) \text{ is a subsolution to (EP)} \}. \quad (1.9)$$

The above definition is quite standard and has been used before by several authors [5, 6, 16, 24]. Our first main result is the following.

**Theorem 1.1.** Let Assumption 1.1 hold. Assume also that $\inf_{x \in \mathbb{R}^N} f_i(x) > -\infty$ for $i = 1, 2$. Then for every $\lambda \leq \lambda^*$ there exists $u \in C^2(\mathbb{R}^N \times \{1, 2\})$ such that $(u, \lambda)$ solves (EP).

For proof see Theorem 2.3 below. We should mention that the proof of Theorem 1.1 relies on an appropriate gradient estimate and bounds on the quantity $|u_1 - u_2|$ (see Proposition 2.1). In fact, these estimates are crucial for most of our proofs.

We say a function $g: \mathbb{R}^N \to \mathbb{R}$ is coercive if

$$g(x) \to \infty, \quad \text{as } |x| \to \infty.$$

Given a set $\mathcal{Y}$ and two functions $g_1, g_2 : \mathcal{Y} \to \mathbb{R}$, we say $g_1 \asymp g_2$ in $\mathcal{Y}$ if there exist positive constants $\kappa_1, \kappa_2$ satisfying

$$\kappa_1 g_1 \leq g_2 \leq \kappa_2 g_1 \quad \text{in } \mathcal{Y}.$$

Next we show that there exists a solution $u$, bounded from below, corresponding to the eigenvalue $\lambda^*$.

**Theorem 1.2.** Suppose that Assumption 1.1 holds. Also, assume that $f_i, i = 1, 2$, are coercive. Then there exists a solution $(u, \lambda^*)$ to (EP) where $\inf_{\mathbb{R}^N} u_i > -\infty$ for $i = 1, 2$.

For proof see Theorem 2.4. Our next result concerns the uniqueness of solutions.

**Theorem 1.3.** Let Assumptions 1.1 and 1.2 hold. In addition, we also assume that $f_1 \asymp f_2$ outside a compact set, and $f_i, i = 1, 2$, are coercive. Let $(u, \lambda)$ and $(\tilde{u}, \tilde{\lambda})$ be two solutions to (EP) with $\inf_{\mathbb{R}^N} u_i > -\infty$, $\inf_{\mathbb{R}^N} \tilde{u}_i > -\infty$ for $i = 1, 2$. Then we must have $\lambda = \tilde{\lambda} = \lambda^*$ and $u_i = \tilde{u}_i + c$ for some constant $c$ and $i = 1, 2$.

Proof of Theorem 1.3 follows from Theorem 2.1. As can be seen from above that Assumption 1.2 is a bit stronger than the usual hypotheses used to establish uniqueness in the super-critical regime (that is, $\gamma_i \geq 2$) for scalar model (cf. [5]). In the scalar case, one generally uses an exponential transformation together with the coercive property of the solutions to establish uniqueness [5, 8]. Similar transformation does not seem to work in the present setting because of the presence of the coupling terms. So for the uniqueness we rely on the convex analytic approach of [3] and the estimates in Proposition 2.1. Also, the condition $f_1 \asymp f_2$ can be relaxed provided $f_i, i = 1, 2$, satisfy certain polynomial growth hypothesis. See Theorem 2.2 for further detail.

**Remark 1.1.** The above results correspond to a switching Markov process having two states, that is, the solution $u$ is given by a tuple $(u_1, u_2)$ of length 2. All the results of this article continue to hold for weakly coupled systems with any finite number of states, provided Assumptions 1.1 and 1.2 are modified accordingly.
The rest of the article is organized as follows. Section 2 contains the proofs of our main results and their implication to the optimal control problems. The proof of Proposition 2.1 is presented in Appendix A, whereas Appendix B contains few results about the existence of solutions in bounded domains which are used in the proofs of Theorems 1.1 and 1.2.

2. Proofs of main results

In this section we prove Theorems 1.1 to 1.3. We start by proving a gradient estimate which is a key ingredient for most of the proofs below.

**Proposition 2.1.** Let Assumption 1.1 hold. Let \( \varepsilon \in [0,1] \). Suppose \( B_1 \subseteq B_2 \subseteq D \) be two given concentric balls, centered at \( z \), in \( \mathbb{R}^N \). Consider a solution \( u \in C^2(D \times \{1,2\}) \) to the system of equations

\[
\begin{align*}
-\Delta u_1(x) + H_1(x, \nabla u_1) + \alpha_1(x) (u_1(x) - u_2(x)) + \varepsilon u_1(x) &= f_1(x) \quad \text{in } D, \\
-\Delta u_2(x) + H_2(x, \nabla u_2) + \alpha_2(x) (u_2(x) - u_1(x)) + \varepsilon u_2(x) &= f_2(x) \quad \text{in } D.
\end{align*}
\]

Then there exists a constant \( C > 0 \), dependent only on \( \text{dist}(B_1, \partial B_2), \gamma_i, C_1, N \) and \( \sup_{B_2}(|\alpha_i| + |\nabla \alpha_i|) \) for \( i = 1,2 \), satisfying

\[
\sup_{B_1}\{|\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2}\} \leq C \left( 1 + \sup_{B_2} \sum_{i=1}^{2} (f_i)^2 + \sup_{B_2} \sum_{i=1}^{2} |\nabla f_i|^{2\gamma_i/(2\gamma_i-1)} + \sup_{B_2} \sum_{i=1}^{2} (\varepsilon u_i)^2 \right). \tag{2.2}
\]

Furthermore, for some positive constant \( \tilde{C} \), dependent only on \( \text{dist}(B_1, \partial B_2), \gamma_i, C_1, N, \alpha_0 \), we have

\[
|u_1(z) - u_2(z)|^2 \leq \tilde{C} \left( 1 + \sup_{B_2} \sum_{i=1}^{2} (f_i)^2 + \sup_{B_2} \sum_{i=1}^{2} |\nabla f_i|^{2\gamma_i/(2\gamma_i-1)} + \sup_{B_2} \sum_{i=1}^{2} (\varepsilon u_i)^2 \right). \tag{2.3}
\]

The proof of this Proposition is quite long and therefore, is deferred to Appendix A.

Next, we show that any solution of \( \text{(EP)} \) which is bounded from below, is actually coercive. This lemma should be compared with [5, Proposition 3.4] and [3, Lemma 2.1]. Our proof does not use Harnack’s inequality like these previous works. Our proof is based on the comparison principle.

**Lemma 2.1.** Grant Assumptions 1.1 and 1.2. Let \( u = (u_1, u_2) \) be a non-negative solution to

\[
\begin{align*}
-\Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x) (u_1 - u_2) &= f_1 \quad \text{in } \mathbb{R}^N, \\
-\Delta u_2 + H_2(x, \nabla u_2) + \alpha_2(x) (u_2 - u_1) &= f_2 \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

Also, assume that \( f_i, i = 1,2 \), are coercive. Then for some positive constants \( M_1, M_2 \) we have

\[
u_i(x) \geq M_1 |f_i(x)|^{1/\gamma_i} - M_2 \quad x \in \mathbb{R}^N, \ i = 1,2. \tag{2.5}
\]

Moreover, if \( f_1 \neq f_2 \) outside a compact set, then \( \frac{1}{u_i(x)} |\nabla u_i|^2 \leq M_3 |f_i(x)|^{1/\gamma_i} \) outside a compact set, for some positive constant \( M_3 \).

**Proof.** Choose \( R > 0 \) so that \( f_i(x) > 1 \) for \( |x| \geq R \). Fix a point \( x_0 \in B_R^c(0) \) and define

\[
\psi_i(y) = \theta |f_i(x_0)|^{1/\gamma_i} (1 - |y - x_0|^2),
\]

where \( \theta > 0 \) is to be chosen later and \( i = 1,2 \). Then, using (1.1)-(1.2), we have in \( B_1(x_0) \)

\[
\begin{align*}
\Delta \psi_1(y) - H_1(y, \nabla \psi_1(y)) + \alpha_1(y) (\psi_2 - \psi_1) + f_1(y) \\
\geq \Delta \psi_1(y) - C_1 |\nabla \psi_1|^{\gamma_1} - C_1 + \alpha_1(y) (\psi_2 - \psi_1) + f_1(y) \\
\geq -2N \theta |f_1(x_0)|^{1/\gamma_1} - 2^\gamma_1 \theta \gamma \gamma C_1 |f_1(x_0)| |y - x_0|^{\gamma_1} - C_1 - \alpha_1(y) \theta |f_1(x_0)|^{1/\gamma_1} + f_1(y) \\
\geq f_1(x_0) \left[ -2N \theta |f_1(x_0)|^{1/\gamma_1 - 1} - 2^\gamma_1 \theta \gamma \gamma C_1 - C_1 |f_1(x_0)|^{\gamma_1} - \alpha_0 \theta |f_1(x_0)|^{1/\gamma_1} + \kappa \right], \tag{2.6}
\end{align*}
\]
where
\[
\left[ \inf_{|x| \geq R+1} \inf_{y \in B_1(x)} f(y) \right] (|f(x)| + 1)^{-1} \geq \kappa > 0 \quad \text{for } R \text{ large enough, by (1.7)}.
\]

Since \( f_1 \) is coercive, we can choose \( \theta \) small and \( R \) large so that the rhs of (2.6) is positive. Similarly, we can also show that for some small \( \theta \) and large \( R \)
\[
\Delta \psi_2(y) - H_2(y, \nabla \psi_2) + \alpha_2(x)(\psi_1 - \psi_2) + f_2(y) \geq 0 \quad \text{in } B_1(x_0),
\]
whenever \( |x_0| > R \). We can now apply comparison principle, Theorem B.1, in \( B_1(x_0) \) to conclude that \( (u_1, u_2) \geq (\psi_1, \psi_2) \) in \( B_1(x_0) \) implying \( u_i(x_0) \geq \theta f_i(x_0) \) for \( i = 1, 2 \) and for all \( |x_0| > R \). This gives (2.5). Again, from (1.5)-(1.6) and (2.2) we have
\[
\max\{|Du_1(x)|^{2\gamma_1}, |Du_2(x)|^{2\gamma_2}\} \leq C(1 + |f_1(x)|^2 + |f_2(x)|^2),
\]
for some constant \( C \) and for all \( x \) outside a compact set. Since \( f_1 \times f_2 \) outside a compact set, the second conclusion follows from the above display and (2.5). Hence this completes the proof. \( \square \)

We now first establish the uniqueness and then discuss the existence results, that is, we assume Theorems 1.1 and 1.2 and prove Theorem 1.3 first, and then we prove Theorems 1.1 and 1.2.

2.1. Uniqueness. We begin by introducing a few notations. By \( g = (g_1, g_2) \in C^2(\mathbb{R}^N \times \{1, 2\}) \) we mean \( g_i \in C^2(\mathbb{R}^N) \) for \( i = 1, 2 \). Define the operator \( \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : C^2(\mathbb{R}^N \times \{1, 2\}) \rightarrow C^2(\mathbb{R}^N \times \mathbb{R}^N \times \{1, 2\}) \) by
\[
\mathcal{A}_k g(x, \xi) := \Delta g_k(x) - \xi \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^{2} (g_j(x) - g_k(x)), \quad (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N, \quad k = 1, 2, \quad (2.7)
\]
with \( g = (g_1, g_2) \in C^2(\mathbb{R}^N \times \{1, 2\}) \). Also, \( C^2(\mathbb{R}^N \times \{1, 2\}) \) denotes the class of functions in \( C^2(\mathbb{R}^N \times \{1, 2\}) \) with compact support. Let \( \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N \times \{1, 2\}) \) denotes the set of Borel probability measures \( \mu = (\mu_1, \mu_2) \), with \( \mu_i = \mu(\cdot \times \{i\}) \) being a sub-probability measure. For a function \( h : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) we use the notation
\[
\mu(h) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \langle h(x, \xi), \mu(dx, d\xi) \rangle = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} h_k(x, \xi) \mu_k(dx, d\xi).
\]
We define
\[
\mathcal{M} := \left\{ \mu \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N \times \{1, 2\}) : \mu(\mathcal{A} g) = 0 \quad \forall \ g \in C^2(\mathbb{R}^N \times \{1, 2\}) \right\}.
\]
Let
\[
F_k(x, \xi) := f_k(x) + \ell_k(x, \xi) \quad k = 1, 2, \quad (2.8)
\]
where \( \ell_k \) is given by Assumption 1.1. Now define
\[
\mathcal{M}_F := \left\{ \mu \in \mathcal{M} : \mu(F) < \infty \right\}, \quad (2.9)
\]
and
\[
\bar{\lambda} := \inf_{\mu \in \mathcal{M}} \mu(F) = \inf_{\mu \in \mathcal{M}_F} \mu(F). \quad \text{(LP)}
\]
In Lemma 2.3 below we show that \( \mathcal{M}_F \) is non-empty. Our next result shows that \( \lambda^* \) in (1.9) is smaller than \( \bar{\lambda} \).

**Lemma 2.2.** Consider the setting of Theorem 1.3. Then we must have \( \lambda^* \leq \bar{\lambda} \).
Theorem 1.2

Lemma 2.1

we also know that

2.10

2.11

From

\[ \chi \] we select

Existence of \( u \) coercive. We would modify \( u \) suitably so that it can be used in (2.10) as a test function. To do so, we consider a family of concave functions.

For \( r > 0 \), we let \( \chi_r \) be a concave function in \( C^2(\mathbb{R}) \) such that \( \chi_r(t) = t \) for \( t \leq r \), and \( \chi_r(t) = 0 \) for \( t \geq 3r \). Then \( \chi'' \) and \(-\chi''\) are nonnegative, and the latter is supported on \([r, 3r]\). In addition, we select \( \chi_r \) so that

\[ |\chi''(t)| \leq \frac{2}{r} \quad \forall t > 0. \]  

(2.12)

In particular, we may define \( \chi_r \) by specifying

\[ \chi''(t) = \begin{cases} \frac{4 - t}{3} & \text{if } r \leq t \leq \frac{3r}{2}, \\ -\frac{2}{3r} & \text{if } \frac{3r}{2} \leq t \leq \frac{5r}{3}, \\ \frac{4}{3}(\frac{4}{3r} - \frac{2}{3}) & \text{if } \frac{5r}{3} \leq t \leq 3r. \end{cases} \]

Using (2.11) we now compute

\[
\Delta \chi_r(u_k) - \xi \cdot \nabla \chi_r(u_k) + \alpha_k \sum_{j=1}^{2} (\chi_r(u_j) - \chi_r(u_k)) \\
= \chi''(u_k)|\nabla u_k|^2 + \chi'(u_k)(\Delta u_k - \xi \cdot \nabla u_k) + \alpha_k \sum_{j=1}^{2} (\chi_r(u_j) - \chi_r(u_k)) \\
= \chi''(u_k)|\nabla u_k|^2 + \chi'(u_k)\left(\lambda^* + H_k(x, \nabla u_k) - f_k - \xi \cdot \nabla u_k\right) \\
+ \alpha_k \sum_{j=1}^{2} (\chi_r(u_j) - \chi_r(u_k) - \chi'(u_k)(u_j - u_k)) \\
= \chi''(u_k)|\nabla u_k|^2 + \chi'(u_k)\left(\lambda^* - f_k - \ell_k(x, \xi)\right) \\
+ \chi'(u_k)\left(\ell_k(x, \xi) - \xi \cdot \nabla u_k + H_k(x, \nabla u_k)\right) + \alpha_k \sum_{j=1}^{2} (\chi_r(u_j) - \chi_r(u_k) - \chi'_r(u_k)(u_j - u_k)).
\]  

(2.13)

Thus, defining

\[ G_{r,k}[u](x) := \alpha_k \sum_{j=1}^{2} (\chi_r(u_j) - \chi_r(u_k) - \chi'_r(u_k)(u_j - u_k)), \]
and integrating (2.13) with respect to a $\mu$, we obtain
\[
\sum_{k=1}^{n} \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi''_r(u_k(x)) \left( f_k(x) + \ell_k(x, \xi) - \lambda^* \right) \mu_k(dx, d\xi) = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi''_r(u_k(x)) \left( \ell_k(x, \xi) - \xi \cdot \nabla u_k + H_k(x, \nabla u_k) \right) \mu_k(dx, d\xi) + \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi''_r(u_k(x)) |Du_k(x)|^2 + G_{r,k}[u](x) \mu_k(dx, d\xi). \tag{2.14}
\]

Next we show that the last term on the rhs of (2.14) goes to 0 as $r \to \infty$. Since $f_1 \asymp f_2$ outside a compact set and $\mu(f) = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} f_k(x) \mu_k(dx, d\xi) < \infty$, we obtain
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} (|f_1(x)| + |f_2(x)|) \mu_1(dx, d\xi) < \infty, \quad \text{and} \quad \int_{\mathbb{R}^N \times \mathbb{R}^N} (|f_1(x)| + |f_2(x)|) \mu_2(dx, d\xi) < \infty. \tag{2.15}
\]

Therefore, using Lemma 2.1 and (2.12), we get
\[
\sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} |\chi''_r(u_k(x))| |Du_k(x)|^2 \mu_k(dx, d\xi) \leq \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} 1_{\{r<u_k(x)<3r\}} \frac{2}{u_k(x)} |Du_k(x)|^2 \mu_k(dx, d\xi)
\]
\[
\leq \kappa \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} 1_{\{r<u_k(x)<3r\}} |f_k(x)|^{1/\gamma} \mu_k(dx, d\xi),
\]
for some constant $\kappa$. Since $u_k$, $k = 1, 2$, are coercive, using dominated convergence theorem it follows that the rhs of the above display tends to 0 as $r \to \infty$. Again, since $\chi' \leq 1$, it follows that
\[
|G_{r,k}[u](x)| \leq 2\alpha_0 1_{A_r(x)}|u_1(x) - u_2(x)| \quad \text{for all } x \in \mathbb{R}^N, \ k = 1, 2,
\]
where $A_r = \{ x : u_2(x) \vee u_1(x) \leq r \}$. Using (1.5)-(1.6) and (2.3) we then have
\[
|G_{r,k}[u](x)| \leq \kappa_1 1_{A_r(x)} (|f_1(x)| + |f_2(x)|) \quad \text{for all } x \in \mathbb{R}^N, \ k = 1, 2,
\]
for some constant $\kappa_1$. Again using (2.15) and dominated convergence theorem we thus get
\[
\lim_{r \to \infty} \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} G_{r,k}[u](x) \mu_k(dx, d\xi) = 0.
\]

From our construction, it also follows that $\chi_{3^n}$ is an increasing sequence. Therefore, letting $r = 3^n \to \infty$ in (2.14) and applying monotone convergence theorem we obtain
\[
\mu(F) - \lambda^* = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \left( \ell_k(x, \xi) - \xi \cdot \nabla u_k + H_k(x, \nabla u_k) \right) \mu_k(dx, d\xi) \geq 0. \tag{2.16}
\]

Since $\mu$ is arbitrary, this proves the lemma. \hfill \Box

Next we show that $\mathcal{M}_F$ is non-empty.

**Lemma 2.3.** Suppose that $u$ is a coercive, nonnegative solution to (EP) with eigenvalue $\lambda$. Define $\xi_k(x) = \nabla_p H_k(x, \nabla u_k(x)) \quad k = 1, 2.$

Then there exists a Borel probability measure $\nu = (\nu_1, \nu_2)$ on $\mathbb{R}^N \times \{1, 2\}$ so that
\[
\mu_u = (\mu_{1,u}, \mu_{2,u}) \in \mathcal{M}_F \quad \text{where} \quad \mu_{k,u} := \nu_k(dx) \delta_{\xi_k(x)}(d\xi).
\]

Furthermore, $\bar{\lambda} \leq \lambda$. 

Proof. Since $H_k$ is the Fenchel-Legendre transformation of $\ell_k$, it is well known that
$$H_k(x, p) = p \cdot \xi - \ell_k(x, \xi) \quad \text{for} \quad \xi = \nabla_p H_k(x, p),$$
for $k = 1, 2$. Therefore, we can rewrite (EP) as
$$\begin{cases}
\Delta u_1(x) - \xi_1(x) \cdot \nabla u_1(x) - \alpha_1(x)(u_1(x) - u_2(x)) = \lambda - F_1(x, \xi_1(x)) & \text{in } \mathbb{R}^N, \\
\Delta u_2(x) - \xi_2(x) \cdot \nabla u_2(x) - \alpha_2(x)(u_2(x) - u_1(x)) = \lambda - F_2(x, \xi_2(x)) & \text{in } \mathbb{R}^N,
\end{cases}$$
where $F$ is given by (2.8). We define the extended generator $A_u = (A_{1,u}, A_{2,u}) : C^2(\mathbb{R}^N \times \{1, 2\}) \rightarrow C^2(\mathbb{R}^N \times \{1, 2\})$ by
$$A_{k,u}g(x) := \Delta g_k(x) - \xi_k(x) \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^2 (g_j(x) - g_k(x)), \quad (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N, \ k = 1, 2.$$ 

Since $u, F$ are coercive, there exists a switching diffusion $(X_t, S_t)$ associated to the generator $A_u$ (cf. [2, Chapter 5]). Furthermore, the mean empirical measures of $(X_t, S_t)$ will be tight and therefore, should have a limit point (cf. [2, Lemma 2.5.3]). Let $\nu = (\nu_1, \nu_2)$ be one such limit points. It is also standard to show that
$$\sum_{k=1}^2 \int_{\mathbb{R}^N} A_{k,u}g(x)u_k(dx) = 0$$
for all $g \in C^2(\mathbb{R}^N \times \{1, 2\})$. Hence it follows that $\mu_u \in M$.

To prove the second part, we consider the concave function $\chi_r$ from Lemma 2.2. Since $\chi_r$ is concave we have $\chi'_r \leq 0$ and
$$\chi_r(u_j) - \chi_r(u_k) - \chi'_r(u_k)(u_j - u_k) \leq 0.$$ 

Thus, the calculation of (2.13) and (2.17)-(2.18) gives
$$\Delta \chi_r(u_k) - \xi_k \cdot \nabla \chi_r(u_k) + \alpha_k \sum_{j=1}^2 (\chi_r(u_j) - \chi_r(u_k)) \leq \chi'_r(u_k)(\lambda - F_k(x, \xi_k(x))).$$
Integrating both sides with $\nu_k$ and summing over $k$, we obtain from (2.20) that
$$\sum_{k=1}^2 \int_{\mathbb{R}^N} \chi'_r(u_k)F_k(x, \xi_k(x))\nu_k(dx) \leq \lambda.$$ 

Now letting $r \to \infty$ and using Fatou’s lemma we obtain
$$\mu_u(F) \leq \lambda.$$ 
Thus, $\mu_u \in M_F$ and $\tilde{\lambda} \leq \lambda$. \hfill \Box

We note that the proof of Lemma 2.3 also works for non-negative $C^2$ super-solutions. Combining the above result with Lemma 2.2 we get the following corollary.

**Corollary 2.1.** Under the setting of Theorem 1.3 we have
$$\lambda^* = \inf\{\lambda \in \mathbb{R} : \exists \ \text{nonnegative } u \in C^2(\mathbb{R}^N \times \{1, 2\}) \text{ such that } (u, \lambda) \text{ is a super-solution to (EP)}\}.$$ 
Note that the existence of a non-negative solution $u$ for the value $\lambda^*$ follows from Theorem 1.2.

Now we are ready to establish our uniqueness result.

**Theorem 2.1.** Assume the setting of Theorem 1.3. Let $(u, \lambda)$ be a solution to (EP) and $u$ is non-negative. Then
Lemma and \( \mu \) is given by Lemma 2.3
(b) Suppose that \( \tilde{u}, \tilde{\lambda} \) is another solution to (EP) and \( \tilde{u} \) is non-negative, then \( \tilde{\lambda} = \lambda^* \) and \( \tilde{u} = u + c \) for some constant \( c \).

Proof. (a) follows from Lemmas 2.2 and 2.3 and (2.16). So we consider (b). Using Lemma 2.3, we find a Borel probability measure \( \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \) such that for
\[
\tilde{\mu} \tilde{u} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{u}) \quad \text{with} \quad \tilde{\mu}_k : = \tilde{\nu}_k(dx)\delta_{\tilde{\xi}_k}(d\xi), \quad \tilde{\xi}_k(x) = \nabla_p H_k(x, \nabla \tilde{u}_k),
\]
we have \( \tilde{\lambda} = \tilde{\mu} \tilde{u} (F) = \lambda^* \). Again, by [2, Theorem 5.3.4], there exist strictly positive Borel measurable functions \( \rho = (\rho_1, \rho_2) \) and \( \tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) \) satisfying
\[
\nu_k(dx) = \rho_k(x) dx, \quad \tilde{\nu}_k(dx) = \tilde{\rho}_k(x) dx \quad \text{for } k = 1, 2.
\]
(2.21)
Let us now define
\[
\zeta_k = \frac{\rho_k}{\rho_k + \tilde{\rho}_k}, \quad \tilde{\zeta}_k = \frac{\tilde{\rho}_k}{\rho_k + \tilde{\rho}_k}, \quad v_k(x) = \xi_k(x)\zeta_k(x) + \tilde{\xi}_k(x)\tilde{\zeta}_k(x),
\]
\[
\tilde{\mu}_k(dx, d\xi) = \frac{1}{2}(\nu_k(dx) + \tilde{\nu}_k(dx))\delta_{\nu_k}(d\xi) \quad \text{for } k = 1, 2.
\]
We claim that \( \tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2) \in \mathcal{M} \). Consider \( g = (g_1, g_2) \in C^2_c(\mathbb{R}^N \times \{1, 2\}) \). We note that
\[
\frac{1}{2}(\nu_k(dx) + \tilde{\nu}_k(dx)) = \frac{1}{2}(\rho_k(x) + \tilde{\rho}_k(x)) dx \quad \text{for } k = 1, 2.
\]
A simple computation then yields
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} A_k(x, \xi) \tilde{\mu}_k(dx, d\xi)
\]
\[
= \int_{\mathbb{R}^N} \left( \Delta g_k(x) - v_k(x) \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^2 (g_j(x) - g_k(x)) \right) \frac{1}{2}(\nu_1(dx) + \tilde{\nu}_1(dx))
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \left( (\rho_k(x) + \tilde{\rho}_k(x))\Delta g_k(x) - (\xi_k(x)\rho_k(x) + \tilde{\xi}_k(x)\tilde{\rho}_k(x)) \cdot \nabla g_k(x) \right.
\]
\[
\left. + (\rho_k(x) + \tilde{\rho}_k(x))\alpha_k(x) \sum_{j=1}^2 (g_j(x) - g_k(x)) \right) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} A_k(x, u) g(x) v_k(dx) + \frac{1}{2} \int_{\mathbb{R}^N} A_k(x, u) g(x) \tilde{v}_k(dx).
\]
Therefore
\[
\sum_{k=1}^2 \int_{\mathbb{R}^N \times \mathbb{R}^N} A_k(x, \xi) \tilde{\mu}_k(dx, d\xi) = \frac{1}{2}[\mu_u(A_u u) + \mu_u(A_u g)]
\]
This proves the claim. Using the convexity of \( \ell_k \) in \( \xi \) it is also easily seen that \( \tilde{\mu}(F) < \infty \). Now from Lemmas 2.2 and 2.3 we see that \( \mu_u \) and \( \mu_{\tilde{u}} \) are optimal for (LP). Thus we have
\[
0 \leq \tilde{\mu}(F) - \frac{1}{2} \mu_u(F) - \frac{1}{2} \mu_{\tilde{u}}(F)
\]
\[
= \frac{1}{2} \sum_{k=1}^2 \left[ \int_{\mathbb{R}^N} \ell_k(x, v_k(x))(\rho_k(x) + \tilde{\rho}_k(x)) dx - \int_{\mathbb{R}^N} \ell_k(x, \xi_k(x))\rho_k(x) dx - \int_{\mathbb{R}^N} \ell_k(x, \tilde{\xi}_k(x))\tilde{\rho}_k(x) dx \right]
\]
\[
= \frac{1}{2} \sum_{k=1}^2 \left[ \int_{\mathbb{R}^N} \left( \ell_k(x, v_k(x)) - \ell_k(x, \xi_k(x))\zeta_k(x) - \ell_k(x, \tilde{\xi}_k(x))\tilde{\zeta}_k(x) (\rho_k(x) + \tilde{\rho}_k(x)) \right) dx \right] \leq 0,
\]
where the last line follows from the convexity of \( \ell_k \) in \( \xi \). Therefore,
\[
\sum_{k=1}^{2} \left[ \int_{\mathbb{R}^N} \left( \ell_k(x, v_k(x)) - \ell_k(x, \xi_k(x)) - \ell_k(x, \tilde{\xi}_k(x)) \right) \rho_k(x) + \tilde{\rho}_k(x) \, dx \right] = 0.
\]

Since \( \rho_k, \tilde{\rho}_k \) are strictly positive, and \( \ell_k \) is strictly convex, it the follows that \( \xi_k = \tilde{\xi}_k \) for \( k = 1, 2 \). Since \( H_k(x, \cdot) \) is strictly convex, by (A1), given \( \xi \) there exists a unique \( p \) satisfying
\[
H_k(x, p) = p \cdot \xi - \ell_k(x, \xi).
\]

Thus, from (2.17), we obtain \( \nabla u_k = \nabla \tilde{u}_k \) in \( \mathbb{R}^N \), for \( k = 1, 2 \). This, of course, implies \( u_k = \tilde{u}_i + c_i \) for some constant \( c_i, i = 1, 2 \). Again, subtracting the equations of \( u \) from the equations of \( \tilde{u} \) we see that \( \alpha_1(c_1 - c_2) = 0 \) implying \( c_1 = c_2 \). This completes the proof. \( \square \)

The proof of uniqueness in Theorem 2.1 requires \( f_1 \) to be comparable to \( f_2 \) outside a compact set. This property is crucially used in Lemmas 2.1 and 2.2. However, if we impose more structural assumption on \( f \) then we could relax the requirement of \( f_1 \asymp f_2 \).

(F) Suppose that there exist \( \beta_1, \beta_2 > 1 \) satisfying
\[
C_4^{-1} |x|^{\beta_1} - C_4 \leq f_i(x) \leq C_4 (|x|^{\beta_1} + 1), \quad x \in \mathbb{R}^N,
\]
for some \( C_4 > 0 \), where
\[
\beta_2 \leq \beta_1 \frac{\gamma_1 + 1}{2}, \quad \beta_1 \leq \beta_2 \frac{\gamma_2 + 1}{2}, \quad \max \left\{ \frac{\beta_1(\gamma_1 + 1)}{2\gamma_1}, \frac{\beta_2(\gamma_2 + 1)}{2\gamma_2} \right\} \leq \beta_1 \wedge \beta_2 - 1.
\]

As a consequence of (F) it follows that
\[
|f_2(x)|^{2/\gamma_1} \leq \kappa (1 + |f_1(x)|^{1+\gamma_1^{-1}}) \quad \text{and} \quad |f_1(x)|^{2/\gamma_2} \leq \kappa (1 + |f_2(x)|^{1+\gamma_2^{-1}}) \quad (2.22)
\]
for some \( \kappa > 0 \). Theorem 2.1 can be improved as follows.

**Theorem 2.2.** Suppose that Assumption 1.1, Assumption 1.2 and (F) hold. Then the conclusions of Theorem 2.1 hold true.

**Proof.** We only need to modify Lemmas 2.1 and 2.2. Note that (2.5) holds. Using (1.5), (1.6), (2.2) and (2.22) it follows that
\[
|\nabla u_i(x)| \leq \kappa_1 (1 + |f_i(x)|^{1+\gamma_i^{-1}}) \quad (2.23)
\]
for some constant \( \kappa_1 \). Therefore, for some compact set \( \mathcal{K} \) and a constant \( \kappa_3 \), we obtain from (2.5) that
\[
|\nabla u_i(x)|^2 / u_i(x) \leq \kappa_3 |f_i(x)| \quad x \in \mathcal{K}^c. \quad (2.24)
\]

Again, using (F) and (2.23) we see that
\[
|\nabla u_i(x)| \leq \kappa_4 \left( 1 + |x|^{\frac{\beta_i(1+\gamma_i)}{\gamma_i}} \right) \quad \text{for some} \ \kappa_4, \quad i = 1, 2.
\]

Using (F) this also implies
\[
\max\{u_1(x), u_2(x)\} \leq \kappa_5 \min\{1 + |f_1(x)|, 1 + |f_2(x)|\} \quad (2.25)
\]
for some \( \kappa_5 \). Using (2.24) and (2.25) we can complete the proof of Lemma 2.2. Rest of the argument of Theorem 2.1 follows without any change. \( \square \)
2.2. **Existence.** First we establish Theorem 1.1. We see that if inf_{R^N} f_i > -\infty, then set of subsolution in (1.9) is nonempty. In particular, if we set \( \lambda = \min_{i} \inf_{R^N} f_i \), then \( u = (1,1) \) is a subsolution to (EP) with eigenvalue \( \lambda \).

**Lemma 2.4.** Let Assumption 1.1 hold and also assume that \( f \in C^1(R^N \times \{1,2\}) \). Suppose that \( u \) is a \( C^2 \) subsolution to (EP) with some eigenvalue \( \lambda_1 \). Then (EP) has a \( C^2 \) solution for every \( \lambda \leq \lambda_1 \).

**Proof.** Since \( u \) is also a subsolution for any \( \lambda \leq \lambda_1 \), it is enough to show that there exists a solution \( w \) to (EP) with eigenvalue \( \lambda_1 \). For a \( n \in \mathbb{N} \), fix \( D = B_n(0) \). Applying Theorem B.3, we can find a function \( w^n = (w^n_1, w^n_2) \in C^2(D \times \{1,2\}) \) that satisfies

\[
\begin{align*}
-\Delta w^n_1(x) + H_1(x, \nabla w^n_1(x)) + \alpha_1(x)(w^n_1(x) - w^n_2(x)) &= f_1(x) - \lambda_1 \quad \text{in } B_n(0), \\
-\Delta w^n_2(x) + H_2(x, \nabla w^n_2(x)) + \alpha_2(x)(w^n_2(x) - w^n_1(x)) &= f_2(x) - \lambda_1 \quad \text{in } B_n(0).
\end{align*}
\]

(2.26)

We translate \( w^n \) to satisfy \( w^n_1(0) = 0 \). Let \( K \) be a compact subset of \( R^N \). Then, by Proposition 2.1, we get \( \sup_n \{|w^n_1(0)|, |w^n_2(0)|\} \) bounded and

\[
\sup_{K}\{|\nabla w^n_1|, |\nabla w^n_2|\} < C_K,
\]

for all \( n \) satisfying \( B_n(0) \supseteq K \). Thus, \( \{w^n\} \) is locally bounded in \( W^{2,p}_{\text{loc}} \), uniformly in \( n \). Applying a diagonalization argument, we can find a subsequence of \( \{w^n\} \), converging to some \( w \in W^{2,p}_{\text{loc}}(R^N \times \{1,2\}) \) for \( p > N \). Passing limit in (2.26) gives

\[
\begin{align*}
-\Delta w_1(x) + H_1(x, \nabla w_1(x)) + \alpha_1(x)(w_1(x) - w_2(x)) &= f_1(x) - \lambda_1 \quad \text{in } R^N, \\
-\Delta w_2(x) + H_2(x, \nabla w_2(x)) + \alpha_2(x)(w_2(x) - w_1(x)) &= f_2(x) - \lambda_1 \quad \text{in } R^N.
\end{align*}
\]

We can now bootstrap the regularity of \( w \) to \( C^2 \) using standard elliptic regularity theory (cf. [15]).

Now we can complete the proof of Theorem 1.1.

**Theorem 2.3.** Let Assumption 1.1 hold. Suppose that \( f_1, f_2 \in C^1(R^N) \) are bounded below. Then \( \lambda^* \) is finite and (EP) has solution for the eigenvalue \( \lambda^* \). In particular, by Lemma 2.4, (EP) has a solution for every \( \lambda \leq \lambda^* \).

**Proof.** From the discussion preceding Lemma 2.4 we see that

\[ \lambda^* \geq \min_{i=1,2} \inf_{R^N} f_i. \]

We first show that \( \lambda^* < \infty \). Suppose, on the contrary, that \( \lambda^* = \infty \). Then, in view of Lemma 2.4, there exists a sequence of solutions \( \{(\phi^k, \lambda_k)\} = \{(\phi^k_1, \phi^k_2, \lambda_k)\} \) of (EP) satisfying \( \lambda_k \to \infty \), as \( k \to \infty \). We can translate \( \phi^k \) to satisfy \( \phi^k_1(0) = 0 \). Since

\[
\begin{align*}
-\Delta \phi^k_1(x) + H_1(x, \nabla \phi^k_1(x)) + \alpha_1(x)(\phi^k_1(x) - \phi^k_2(x)) &= f_1(x) - \lambda_k \quad \text{in } R^N, \\
-\Delta \phi^k_2(x) + H_2(x, \nabla \phi^k_2(x)) + \alpha_2(x)(\phi^k_2(x) - \phi^k_1(x)) &= f_2(x) - \lambda_k \quad \text{in } R^N, 
\end{align*}
\]

(2.27)

and \( (f_i - \lambda_k)_{+} \leq (f_i)_{+} \) for large \( k \), it follows from Proposition 2.1 that

\[
\sup_k \sup_{K}\{|H_1(x, \nabla \phi^k_1)|, |H_2(x, \nabla \phi^k_2)|\} < \infty, \quad \sup_k \sup_{K}\{|\phi^k_1|, |\phi^k_2|\} < \infty,
\]

(2.28)

for every compact set \( K \) in \( R^N \). Setting

\[ \psi^k_i := \lambda_k^{-1} \phi^k_i \quad \text{for } i = 1, 2, \]
we see from (2.27) that 
\[ -\Delta \psi^k_i(x) + \lambda^k_i H_1(x, \nabla \phi^k_i(x)) + \alpha_1(x)(\psi^k_i(x) - \psi^k_2(x)) = \lambda^k_i f_1(x) - 1 \quad \text{in } \mathbb{R}^N, \]
\[ -\Delta \psi^k_2(x) + \lambda^k_i H_2(x, \nabla \phi^k_2(x)) + \alpha_2(x)(\psi^k_i(x) - \psi^k_2(x)) = \lambda^k_i f_2(x) - 1 \quad \text{in } \mathbb{R}^N. \]

Using (2.28) we see that \( \{\psi^k_i\} \) is locally bounded in \( W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) for \( p > N \). Therefore, we can find a convergence subsequence, converging to some \( \psi \). (2.28) also shows that \( |\nabla \psi_i| = 0 \) implying \( \psi \) to be a constant. Then passing limit in the above display we get a contradiction. Hence \( \lambda^* \) must be finite.

Now choose \( \lambda_n < \lambda^* \) such that \( \lambda_n \to \lambda^* \) as \( n \to \infty \). Then, using Lemma 2.4, we get a solution \((u^i_1, u^i_2, \lambda_n)\) to (EP). Applying an argument, similar to above, we can extract a convergent subsequence, converging locally to \( u = (u_1, u_2) \) and \( u \) solves (EP) with the eigenvalue \( \lambda^* \). This completes the proof.

The rest of this section is devoted to the proof of Theorem 1.2, that is, we construct a nonnegative solution to (EP) corresponding to the eigenvalue \( \lambda^* \). The broad idea of the proof is the following: We solve the ergodic control problem (EP) on an increasing sequence of balls \( B_n \) and find solution pairs \((u^n, \lambda_n)\) in the balls. We then show that \( \lambda_n \) decreases to \( \lambda^* \) and \( u^n \to u \). Using the coercivity of \( f \), we can confine the minimizer of \( u^n \) inside a fixed compact set, independent of \( n \). This also makes \( u \) bounded from below. For this idea to work it is important that \( u^n \) attends its minimum inside \( B_n \). This can be achieved if we set \( u^n = +\infty \) on \( \partial B_n \). For \( \gamma_i \leq 2 \), this can be done using the arguments of Lasry-Lions in [19]. But for \( \gamma_i > 2 \), we need to modify \( f \) to attend the boundary data.

Let \( f \) be a \( C^1 \) function. Let \( B = B_r(0) \) be the ball of radius \( r \geq 1 \) around 0. Let \( \varrho : (0, \infty) \to (0, \infty) \) be a smooth, nonnegative function satisfying
\[ \varrho(x) = \begin{cases} x^{-1} & \text{for } x \in (0, \frac{r}{2}), \\ 0 & \text{for } x \geq 1. \end{cases} \]

Define
\[ f_i,\alpha(x) = f_i(x) + [\varrho(r^2 - |x|^2)]^\alpha \quad x \in B, \quad i = 1, 2, \]
for some \( \alpha \) to be fixed later. Let \( \beta > \max\{2, \gamma_1, \gamma_2\} \) be such that \( (\beta + 1)(\gamma_i \wedge 2) > \beta + 2 \). Choose \( \alpha > 0 \) to satisfy \( \beta < \alpha < (\beta + 1)(\gamma_i \wedge 2) \) for \( i = 1, 2 \). With no loss of generality, we also assume that \( 1 < \gamma_2 \leq \gamma_1 \). Our next result concerns discounted problem in \( B \).

**Lemma 2.5.** Let Assumption 1.1 hold. Then, for any \( \varepsilon \in (0, 1) \), the system
\[ -\Delta w^1_\varepsilon + H_1(x, \nabla w^1_\varepsilon) + \alpha_1(x)(w^1_\varepsilon - w^2_\varepsilon) + \varepsilon w^1_\varepsilon = f_{1,\alpha} \quad \text{in } B, \]
\[ -\Delta w^2_\varepsilon + H_2(x, \nabla w^2_\varepsilon) + \alpha_2(x)(w^2_\varepsilon - w^1_\varepsilon) + \varepsilon w^2_\varepsilon = f_{2,\alpha} \quad \text{in } B, \]
\[ (2.29) \]
admits a solution \((w^1_\varepsilon, w^2_\varepsilon)\) in \( C^2(B \times \{1, 2\}) \) with \( w^i_\varepsilon \to \infty \) as \( x \to \partial B \). Moreover, the set \( \{\varepsilon w^i_\varepsilon(0) : \varepsilon \in (0, 1)\} \) is bounded for \( i = 1, 2 \).

**Proof.** To find a solution to (2.29), first we find appropriate sub and super-solutions to (2.29). Define \( \xi^\varepsilon(x) = -\log(\varepsilon^2 - \delta|x|^2) \) and let \( (\xi^1_\varepsilon, \xi^2_\varepsilon) = (\kappa_1 \xi^\varepsilon, \kappa_2 \xi^\varepsilon) \). It can be easily checked that, for some \( \delta_0 > 0 \) and \( \delta \in (\delta_0, 1) \),
\[ -\Delta \xi^1_\varepsilon + C_1(|\nabla \xi^1_\varepsilon|^{\gamma_1} + 1) + \alpha_1(x)(\xi^1_\varepsilon - \xi^2_\varepsilon) + \varepsilon \xi^1_\varepsilon \leq f_{1,\alpha} \quad \text{for } r - \delta_1 \leq |x| < r, \]
\[ -\Delta \xi^2_\varepsilon + C_1(|\nabla \xi^2_\varepsilon|^{\gamma_2} + 1) + \alpha_2(x)(\xi^2_\varepsilon - \xi^1_\varepsilon) + \varepsilon \xi^2_\varepsilon \leq f_{2,\alpha} \quad \text{for } r - \delta_1 \leq |x| < r, \]
for some appropriate constant \( \kappa_1 \), dependent on \( \gamma_1, \gamma_2 \). \( \kappa_1, \delta_1 \), and \( \delta \) can be chosen independent of \( \varepsilon \). Now choose \( M \) suitably large, independent of \( \varepsilon, \delta \), so that \( (\kappa_1 \xi^1_\varepsilon - \frac{M}{\varepsilon}, \kappa_2 \xi^2_\varepsilon - \frac{M}{\varepsilon}) \) forms a subsolution to (2.29).
Next we construct a super-solution. To this end, we consider the approximating function $\psi_n$ from Lemma B.1. More precisely, we consider a sequence of functions $\psi_n = (\psi_1^1, \psi_2^2)$ where $\psi_n^i(x) = x$ if $\gamma_i \leq 2$, otherwise $\psi_n^i = \psi_n$ from Lemma B.1.

We define $(\xi_1^\delta, \xi_2^\delta) = (\kappa\zeta, \kappa\zeta)$ where

$$
\zeta = (r^2 - \delta|x|^2)^{-\beta} \quad \text{for} \quad i = 1, 2.
$$

Using the condition $\beta < \alpha < (\beta + 1)(\gamma_1 \wedge 2)$, and choosing $M$ large, independent of $n, \varepsilon, \delta$, we see that $(\kappa\zeta_1^\delta + \frac{M}{\varepsilon}, \kappa\zeta_2^\delta + \frac{M}{\varepsilon})$ forms a supersolution to the equation

\begin{align*}
-\Delta \psi_1^1 + \psi_1^1(H_1(x, \nabla \psi_1^1)) + \alpha_1(x)(\psi_1^1 - \psi_2^1) + \varepsilon \psi_1^1 &= f_{1,\alpha} \quad \text{in } B, \\
-\Delta \psi_2^1 + \psi_2^1(H_2(x, \nabla \psi_2^1)) + \alpha_1(x)(\psi_2^1 - \psi_1^1) + \varepsilon \psi_2^1 &= f_{2,\alpha} \quad \text{in } B,
\end{align*}

for all $n$. From the argument of Theorem B.3, we find a solution $w^\delta = (w_1^\delta, w_2^\delta)$ of

\begin{align*}
-\Delta w_1^\delta + H_1(x, \nabla w_1^\delta) + \alpha_1(x)(w_1^\delta - w_2^\delta) + \varepsilon w_1^\delta &= f_{1,\alpha} \quad \text{in } B, \\
-\Delta w_2^\delta + H_2(x, \nabla w_2^\delta) + \alpha_1(x)(w_2^\delta - w_1^\delta) + \varepsilon w_2^\delta &= f_{2,\alpha} \quad \text{in } B,
\end{align*}

and

$$
\kappa_1^\delta \frac{M}{\varepsilon} \leq w_{i,n}^\delta \leq \kappa_2^\delta \frac{M}{\varepsilon} \quad \text{in } B, \quad i = 1, 2.
$$

Using the estimates in Proposition 2.1, we can now let $\delta \to 1$ and find a solution to

\begin{align*}
-\Delta \psi_1^\delta + H_1(x, \nabla \psi_1^\delta) + \alpha_1(x)(\psi_1^\delta - \psi_2^\delta) + \varepsilon \psi_1^\delta &= f_{1,\alpha} \quad \text{in } B, \\
-\Delta \psi_2^\delta + H_2(x, \nabla \psi_2^\delta) + \alpha_1(x)(\psi_2^\delta - \psi_1^\delta) + \varepsilon \psi_2^\delta &= f_{2,\alpha} \quad \text{in } B,
\end{align*}

satisfying

$$
- \kappa_1 \log(r^2 - |x|^2) - \frac{M}{\varepsilon} \leq \psi_i^\delta \leq \kappa_2(r^2 - |x|^2)^{-\beta} + \frac{M}{\varepsilon} \quad \text{in } B, \quad i = 1, 2. \tag{2.30}
$$

From (2.30) we also obtain

$$
\sup_{\varepsilon \in (0,1)} \sup_{B_{1/2}(0)} |\varepsilon \psi_i^\delta| < \infty.
$$

This completes the proof.

Now we can provide proof of Theorem 1.2.

**Theorem 2.4.** Suppose that Assumption 1.1 holds and $f_i, i = 1, 2$, are coercive. Then there exists a nonnegative solution to (EP) corresponding to the eigenvalue $\lambda^*$.

**Proof.** First we find a pair $(u^n, \lambda_n)$ solving

\begin{align*}
-\Delta u_1^n + H_1(x, \nabla u_1^n) + \alpha_1(x)(u_1^n - u_2^n) &= f_{1,\alpha} - \lambda_n \quad \text{in } B_n(0), \\
-\Delta u_2^n + H_2(x, \nabla u_2^n) + \alpha_2(x)(u_2^n - u_1^n) &= f_{2,\alpha} - \lambda_n \quad \text{in } B_n(0), \tag{2.31}
\end{align*}

with $u^n \to \infty$, as $x \to \partial B_n(0)$, where

$$
f_{i,\alpha} = f_i + [\varphi(n^2 - |x|^2)]^\alpha,
$$

and $\alpha$ is same as in Lemma 2.5. Fix $n \in \mathbb{N}$ and denote by $B = B_n(0)$. Consider the solution $w^\varepsilon$ from Lemma 2.5. We set $v_1^\varepsilon = \psi_1^\varepsilon(x) - \psi_1^\varepsilon(0)$ and $v_2^\varepsilon(x) = \psi_2^\varepsilon(x) - \psi_2^\varepsilon(0)$. From (2.29) we then find

\begin{align*}
-\Delta v_1^\varepsilon + H_1(x, \nabla v_1^\varepsilon) + \alpha_1(x)(v_1^\varepsilon - v_2^\varepsilon) + \varepsilon v_1^\varepsilon &= f_{1,\alpha} \quad \text{in } B, \\
-\Delta v_2^\varepsilon + H_2(x, \nabla v_2^\varepsilon) + \alpha_2(x)(v_2^\varepsilon - v_1^\varepsilon) + \varepsilon v_2^\varepsilon &= f_{2,\alpha} \quad \text{in } B. \tag{2.32}
\end{align*}

From our choice of $\alpha$ and (2.30) we see that $f_{i,\alpha} - \varepsilon v_i^\varepsilon \geq \frac{1}{2} f_{i,\alpha}$ near the boundary, and since $\max_{B_{1/2}} \{|v_1^\varepsilon|, |v_2^\varepsilon|\}$ is bounded uniformly in $\varepsilon$ (by Proposition 2.1), we can see that $v_i^\varepsilon \geq \kappa_3^\delta - M$ for some $\kappa_3$, using Theorem B.1, where $\xi^\delta$ is same as in Lemma 2.5. Now let $\delta \to 1$ to get a lower
bound that blows up at the boundary. Using Proposition 2.1 and the fact \( \{ \varepsilon w^\varepsilon(0) \} \) is bounded, we let \( \varepsilon \to 0 \) in (2.32) to find a solution to (2.31).

Now consider the sequence of solutions \( \{ u^n, \lambda_n \} \) solving (2.31). We claim that \( \lambda_n \geq \lambda_{n+1} \geq \lambda^* \). Suppose, on the contrary, that \( \lambda_n < \lambda_{n+1} \). Choose a constant \( \kappa \) so that \( u^{n+1} + \kappa \) touches \( u^n \) from below in \( B_n \). This is possible as \( u^n \) blows up at the boundary. Let \( v^n = u^n - u^{n+1} \). Also, note that

\[
f_{i,\alpha}^{n+1}(x) = f_i(x) - f_{i,\alpha}^{n+1} \quad \text{in } B_n.
\]

Choose \( D \subset B_n \), so that \( v^n \) vanishes at some point inside \( D \). From (2.31) we then have

\[
\begin{cases}
-\Delta v^n_1 + h_1^n \cdot \nabla v^n_1 + \alpha_1(x)(v^n_1 - v^n_2) & \geq \lambda_{n+1} - \lambda_n > 0 \quad \text{in } D, \\
-\Delta v^n_2 + h_2^n \cdot \nabla v^n_1 + \alpha_2(x)(v^n_2 - v^n_1) & \geq \lambda_{n+1} - \lambda_n > 0 \quad \text{in } D,
\end{cases}
\]

where

\[
h_i^n(x) = \int_0^1 \nabla_p H_i(x, \nabla u_{i+1}^n + t(\nabla u_1^n - \nabla u_{i+1}^n)) \, dt, \quad i = 1, 2.
\]

By strong maximum principle we obtain \( v^n = 0 \) in \( D \). Since \( D \) is arbitrary, we must have \( v^n = 0 \) in \( B_n \), which is a contradiction. Thus we have \( \lambda_n \geq \lambda_{n+1} \). An analogous argument also shows \( \lambda_n \geq \lambda^* \).

Using the estimates in Proposition 2.1, we can now find a subsequence of \( \{ u^n \} \) converging weakly in \( W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) to some \( u \). Passing limit in (2.31) we see that \( u \) solves (EP) with the eigenvalue \( \lambda^* \) (since \( \lim_{n \to \infty} \lambda_n \) is equal to \( \lambda^* \)). To see that \( u \) is bounded from below, we consider a point \( (x_n, i_n) \in B_n \times \{ 1, 2 \} \) so that \( u_n^i(x_n) \) is the minimum of \( u^n \) in \( B_n \). From (2.31) we then obtain

\[
\lambda_1 \geq \lambda_n \geq f_{i_n}^n(x_n) \geq f_{i_n}(x_n) \geq \min\{ f_1(x_n), f_2(x_n) \}.
\]

Since \( f_i \) is coercive, we can find a compact set \( K \), independent of \( n \), so that \( x_n \in K \). Thus \( u^n \geq \min_K \{ u_1^n, u_2^n \} \). This, of course, implies that \( u \) is bounded from below. We can now translate \( u \) to make it nonnegative. This completes the proof. \( \square \)

We complete the section by mentioning few properties of \( \lambda^* = \lambda^*(f) \).

**Proposition 2.2.** Let \( f, \tilde{f} \) be two \( C^1 \) functions. Then

(i) For any \( c \in \mathbb{R} \) we have \( \lambda^*(f + c) = \lambda^*(f) + c \).

(ii) \( f \mapsto \lambda^*(f) \) is concave, that is, for \( t \in [0, 1] \) we have

\[
\lambda^*(tf + (1 - t)\tilde{f}) \geq t\lambda^*(f) + (1 - t)\lambda^*(\tilde{f}).
\]

(iii) If \( f \leq \tilde{f} \), then \( \lambda^*(f) \leq \lambda^*(\tilde{f}) \). Furthermore, if we assume the setting of Theorem 2.1 or Theorem 2.2, then for \( f \leq \tilde{f} \) we have \( \lambda^*(f) < \lambda^*(\tilde{f}) \).

**Proof.** (i) is obvious. (ii) follows from the convexity of \( H_t \) and the definition (1.9). Also, first part of (iii) follows from the definition (1.9). To prove the second part, we suppose, on the contrary, that \( \lambda^*(f) = \lambda^*(\tilde{f}) \). Let \( \tilde{u} \) be a non-negative solution to (EP) with right-hand side \( f \) and eigenvalue \( \lambda^*(f) \). Then \( \tilde{u} \) would be a supersolution to (EP) with right-hand side \( f \). From Lemma 2.3 we know that for

\[
\xi_k(x) = \nabla_p H_k(x, \nabla \tilde{u}(x)) \quad k = 1, 2,
\]

there exists a Borel probability measure \( \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \) so that

\[
\tilde{\mu}_{\tilde{u}} = (\tilde{\mu}_{1,u}, \tilde{\mu}_{2,u}) \quad \text{with} \quad \tilde{\mu}_{k,u} := \tilde{\nu}_k(dx)\delta_{\xi_k(x)}(d\xi) \in M_{\mathcal{F}}.
\]

Moreover, \( \tilde{\mu}_{\tilde{u}}(F) \leq \lambda^*(f) \). By Theorem 2.1 or Theorem 2.2 we must have \( \tilde{\mu}_{\tilde{u}}(F) = \lambda^*(f) \). Again, using (2.16), we obtain

\[
\sum_{k=1}^2 \int_{\mathbb{R}^N} \left( \ell_k(x, \xi_k(x)) - \xi_k(x) \cdot \nabla u_k + H_k(x, \nabla u_k) \right) \tilde{\nu}_k(dx) = 0.
\]
Since \( \tilde{v}_k \) has strictly positive densities (cf. [2, Theorem 5.3.4]), it follows that \( \nabla u_k = \nabla \tilde{u}_k \). Thus \( u_k = \tilde{u}_k + c_k \) for some constants \( c_k \) for \( k = 1, 2 \). Subtracting the equation satisfied by \( u \) and \( \tilde{u} \) we obtain

\[
\alpha_1(x)(c_2 - c_1) = \hat{f}_1(x) - f_1(x), \quad \text{and} \quad \alpha_2(x)(c_1 - c_2) = \hat{f}_2(x) - f_2(x),
\]

which implies

\[
\frac{\hat{f}_1(x) - f_1(x)}{\alpha_1(x)} + \frac{\hat{f}_2(x) - f_2(x)}{\alpha_2(x)} = 0.
\]

But this is not possible as \( f \leq \hat{f} \). Hence we must have \( \lambda^*(f) < \lambda^*(\hat{f}) \).

\( \Box \)

2.3. **Application to optimal ergodic control.** In this section, we describe the optimal ergodic control problem associated with the system of equations \((\text{EP})\). Denote by \( \mathcal{S} = \{1, 2\} \), the state space of the switching continuous time Markov process. We introduce the regime switching controlled diffusion process on a given complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). This is a process \((X_t, S_t)\) in \( \mathbb{R}^N \times \mathcal{S} \) governed by the following stochastic differential equations:

\[
dX_t = b(X_t, S_t)dt - U_t \, dt + dW_t,
\]

\[
dS_t = \int_{\mathbb{R}} h(X_t, S_t, z) \nu(dt, dz),
\]

for \( t \geq 0 \), where

(i) \((X_0, S_0)\) are prescribed deterministic initial data;
(ii) \( W \) is an \( N \)-dimensional standard Wiener process;
(iii) \( \nu(dt, dz) \) is a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity \( dt \times m(dz) \), where \( m \) is the Lebesgue measure on \( \mathbb{R} \);
(iv) \( \nu(\cdot, \cdot), W(\cdot) \) are independent;
(v) The function \( h: \mathbb{R}^d \times \mathcal{S} \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
h(x, i, z) := \begin{cases} j - i & \text{if } z \in \Delta_{ij}(x), \\ 0 & \text{otherwise}, \end{cases}
\]

where for \( i, j \in \mathcal{S}, i \neq j \), and fixed \( x \), \( \Delta_{ij}(x) \) are left closed right open disjoint intervals of \( \mathbb{R} \) having length \( m_{ij}(x) \), and

\[
m_{11}(x) = -\alpha_1(x), \quad m_{12} = \alpha_1(x), \quad m_{21}(x) = \alpha_2(x), \quad m_{22}(x) = -\alpha_2(x).
\]

Note that \( M(x) := (m_{ij}) \) can be interpreted as the rate matrix of the Markov chain \( S_t \) given that \( X_t = x \). In other words,

\[
P(S_{t+h} = j \mid X_t, S_t) = \begin{cases} m_{S_{ij}}(X_t)h + o(h) & \text{if } S_t \neq j, \\ 1 + m_{S_{ij}}(X_t)h + o(h) & \text{if } S_t = j, \end{cases}
\]

and \( X \) behaves like an ordinary diffusion process governed by (2.33) between two consecutive jumps of \( S \).

We assume \( b: \mathbb{R}^N \times \mathcal{S} \to \mathbb{R}^N \) to be a bounded \( C^1 \) function with bounded first derivatives. The process \( \{U_t\} \) takes values in \( \mathbb{R}^N \) and non-anticipative in nature, that is, the sigma fields

\[
\sigma\{X_0, S_0, W_s, U_s, \varphi(A, B) : A \in \mathcal{B}([0, s]), B \in \mathcal{B}(\mathbb{R}), s \leq t\},
\]

and

\[
\sigma\{W_s - W_t, \varphi(A, B) : A \in \mathcal{B}([s, \infty)), B \in \mathcal{B}(\mathbb{R}), s \geq t\},
\]

are independent. To introduce the admissible class of controls we set \( \gamma_1 = \gamma_2 = \gamma \) and define

\[
\mathcal{U} = \left\{ U : \mathbb{E} \left[ \int_0^T |U_t|^\gamma dt \right] < \infty \quad \text{for all } T > 0 \right\},
\]
where \( \gamma' \) is the Hölder conjugate of \( \gamma \). We also assume \( \tilde{\ell}_i \) to satisfy the following bound
\[
\kappa^{-1}|\xi|^{\gamma'} - \kappa \leq \tilde{\ell}_i(x, \xi) \leq \kappa(1 + |\xi|^{\gamma'}),
\]
for some \( \kappa > 0 \) and \( \xi \mapsto \ell_i(x, \xi) \) are strictly convex, \( i = 1, 2 \). We let
\[
H_i(x, p) = -b_i(x) \cdot p + \sup_{\xi \in \mathbb{R}^N} \{ p \cdot \xi - \tilde{\ell}_i(x, \xi) \} \quad i = 1, 2.
\]

Also, assume that \( H_i \in C^1(\mathbb{R}^N \times \mathbb{R}^N) \) and the functions \( \xi \mapsto H_i(x, \xi) \) are strictly convex for \( i = 1, 2 \). It can be easily shown that (2.33) has a unique strong solution for \( U \in \mathcal{U} \). Now we can state the main result of this section.

**Theorem 2.5.** Consider the setting of Theorem 2.1 or Theorem 2.2. We also assume that \( \gamma_1 = \gamma_2 = \gamma \). Then
\[
\inf_{U \in \mathcal{U}} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right] = \lambda^*.
\]
Furthermore, the stationary Markov control
\[
(\nabla_p H_1(x, \nabla u_1(x)), \nabla_p H_2(x, \nabla u_2(x)) + b
\]
is optimal where \( u \) is a non-negative solution to (EP) corresponding to the eigenvalue \( \lambda^* \). Furthermore, from (2.16), we also see that this is the only optimal stationary Markov control.

**Proof.** We only show that the lhs of (2.34) is larger than \( \lambda^* \). Rest of the proof follows from Theorem 2.1 or Theorem 2.2. Consider \( U \in \mathcal{U} \) so that
\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right] = \lim_{T_n \to \infty} \frac{1}{T_n} \mathbb{E} \left[ \int_0^{T_n} (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right] < \infty.
\]

We define the mean empirical measure as on \( \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S} \) as follows
\[
\mu^n(A_1 \times A_2 \times C) = \frac{1}{T_n} \mathbb{E} \left[ \int_0^{T_n} \mathbb{1}_{A_1 \times C \times A_2}(X_t, S_t, U_t)dt \right], \quad A_i \in \mathcal{B}(\mathbb{R}^N), \ C \subset \mathcal{S}.
\]

From the definition of \( \mu^n \) it follows that
\[
\mu^n(F) = \frac{1}{T_n} \mathbb{E} \left[ \int_0^{T_n} (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right],
\]
where \( F \) is given by (2.8). From the coercivity property of \( F \) it can be easily seen that \( \{\mu^n\} \) is tight. Let \( \mu \) be a sub-sequential limit of \( \{\mu^n\} \). Using [2, Lemma 2.5.3] and the lower-semicontinuity property of weak convergence we see that \( \mu \in \mathcal{M}_F \). Again, from (2.35), we get
\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right] \geq \mu(F).
\]

By Lemma 2.2 we obtain
\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (f(X_t, S_t) + \ell(X_t, S_t, U_t))dt \right] \geq \lambda^*.
\]
This completes the proof. \( \square \)
Appendix A. Proof of Proposition 2.1

Proof. With no loss of generality, we assume that \( z = 0, B_1 = B_1(0), \) and \( B_2 = B_2(0) \). We first show that

\[
\sup_{B_1} \{ |\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2} \} \leq C(1+\sup_{B_2} \sum_{i=1}^{2} (f_i)_+^2 + \sup_{B_2} \sum_{i=1}^{2} |\nabla f_i|^{2\gamma_i/(2\gamma_i-1)} + |u_1(0) - u_2(0)|^2 + \sup_{B_2} \sum_{i=1}^{2} (\varepsilon u_i)_+^2). \tag{A.1}
\]

Let \( \rho : B_2 \to [0,1] \) be smooth, radial function which is decreasing along the radius, \( \rho = 1 \) in \( B_1 \), and support(\( \rho \)) \( \subset B_2 \). We take \( \gamma = \min\{\gamma_1, \gamma_2\} \) and define \( \eta = \rho^\frac{\gamma}{\gamma_1} \). Without loss of generality we may assume that

\[
\max_{B_2} \{ |\eta| |\nabla u_1|^2 \eta, |\nabla u_2|^2 \} = |\eta(0)| |\nabla u_1(0)|^2 \quad \text{for some } x_0 \text{ in } B_2.
\]

Define \( \theta(x) = \eta(x) |\nabla u_1(x)|^2 \eta(x) w(x) \) where \( w(x) = |\nabla u_1(x)|^2 \). Then we have \( \nabla \theta(x_0) = 0 \) and \( \Delta \theta(x_0) \leq 0 \). We may also assume that \( \theta(x_0) > 1 \). Otherwise, if \( \theta(x_0) \leq 1 \), we get

\[
\max_{B_1} \{ |\eta| |\nabla u_1|^2 \eta, |\nabla u_2|^2 \} \leq \theta(x_0) \leq 1,
\]

and (A.1) follows. Therefore, we work with \( \theta(x_0) > 1 \). We see that

\[
0 = \nabla \theta(x_0) = \eta(x_0) \nabla w(x_0) + w(x_0) \nabla \eta(x_0). \tag{A.2}
\]

Now onward we shall evaluate everything at the point \( x = x_0 \), without explicitly mentioning the point \( x_0 \). Then

\[
\Delta w = \text{Tr}[(D^2 u_1)^2] + \nabla (\Delta u_1) \cdot \nabla u_1
= \text{Tr}[(D^2 u_1)^2] + \nabla (H_1(x, \nabla u_1) + \alpha_1(u_1 - u_2) + \varepsilon u_1 - f_1) \cdot \nabla u_1
= \text{Tr}[(D^2 u_1)^2] + [\nabla x H_1 + (\nabla_p H_1)D^2 u_1 + (u_1 - u_2)\nabla \alpha_1 + \alpha_1(\nabla u_1 - \nabla u_2) + \varepsilon \nabla u_1 - \nabla f_1] \cdot \nabla u_1.
\]

Using (A.2), we then obtain

\[
0 \geq \Delta \theta = \eta \Delta w + 2\nabla \eta \cdot \nabla w + w \Delta \eta
= \eta \left[ \text{Tr}[(D^2 u_1)^2] + \nabla x H_1 \cdot \nabla u_1 + (-2\eta \varepsilon)^{-1}\nabla \eta \cdot \nabla_p H_1 + (u_1 - u_2)\nabla \alpha_1 \cdot \nabla u_1
+ \alpha_1(\nabla u_1 - \nabla u_2) \cdot \nabla u_1 + \varepsilon \nabla u_1 - \nabla f_1 \cdot \nabla u_1 \right] - 2\eta^{-1} w|\nabla \eta|^2 + w \Delta \eta
\geq \eta \left[ \text{Tr}[(D^2 u_1)^2] - |\nabla x H_1||\nabla u_1| - 2\eta \varepsilon^{-1}|\nabla_p H_1||\nabla \eta| + (u_1 - u_2)\nabla \alpha_1 \cdot \nabla u_1
+ \alpha_1(\nabla u_1 - \nabla u_2) \cdot \nabla u_1 - |\nabla f_1||\nabla u_1| \right] - 2\eta^{-1} w|\nabla \eta|^2 - w|\Delta \eta|.
\]

Using (2.1), (1.2) and the inequality \( (t_1 + t_2 + t_3 + t_4)^2 \geq \frac{1}{4} t_1^2 - [(t_2)^2 + (t_3)^2 + (t_4)^2] \), we get (taking \( t_1 = H_1 + C_1 \geq 0 \))

\[
N \text{Tr}[(D^2 u_1)^2] \geq (\Delta u_1)^2 \geq \left( \frac{1}{4C_1^2} |\nabla u_1|^{2\gamma_1} - (f_1 + C_1)^2_+ - \alpha_1^2 (u_1 - u_2)^2 - (\varepsilon u_1)_+^2 \right).
\]
Since $N \geq 1$ and $\eta \leq 1$, we obtain
\[
\frac{1}{4NC_1^2}\eta|\nabla u_1|^{2\gamma_1} \leq \eta \text{Tr}[(D^2 u_1)^2] + (f_1 + C_1)^2 + \eta \alpha_1^2(u_1 - u_2)^2 + (\varepsilon u_1)^2
\]
\[
\leq (f_1 + C_1)^2 + \eta \alpha_1^2(u_1 - u_2)^2 + (\varepsilon u_1)^2 + \eta|\nabla x H_1||\nabla u_1| + 2w|\nabla^2 H_1||\nabla \eta|
\]
\[
- \eta(u_1 - u_2)\nabla \alpha_1 \cdot \nabla u_1 - \eta\alpha_1(\nabla u_1 - \nabla u_2) \cdot \nabla u_1
\]
\[
+ \eta|\nabla f_1||\nabla u_1| + 2\eta^{-1}w|\nabla \eta|^2 + w|\Delta \eta|.
\]  
(A.3)

We observe that
\[
\eta(x_0)\alpha_1(x_0)(|\nabla u_1(x_0)|^2 - |\nabla u_2(x_0)\cdot \nabla u_1(x_0)|) \geq \eta(x_0)\alpha_1(x_0)(|\nabla u_1(x_0)|^2 - |\nabla u_2(x_0)\|\nabla u_1(x_0)|) \geq 0.
\]

Also, by Mean Value Theorem, there exist $\zeta \in B_2$, with $|\zeta| < |x_0|$, and a constant $\kappa_1 > 0$, dependent on $\sup_{B_2}|\alpha_1|$, such that
\[
\eta(x_0)\alpha_1^2(u_1(x_0) - u_2(x_0))^2 \leq \eta(x_0)\kappa_1(|\nabla u_1(\zeta) - \nabla u_2(\zeta)|^2 + |u_1(0) - u_2(0)|^2)
\]
\[
\leq \eta(\zeta)\kappa_1(|\nabla u_1(\zeta) - \nabla u_2(\zeta)|^2 + |u_1(0) - u_2(0)|^2)
\]
\[
\leq \kappa_1(4\theta(x_0) + |u_1(0) - u_2(0)|^2),
\]
where in the second line we use the fact that $\eta$ is radially decreasing. Another application of the Mean Value Theorem and a similar estimate as above gives us, for some $\zeta_1$ with $|\zeta_1| < |x_0|$, 
\[
-\eta(x_0)(u_1(x_0) - u_2(x_0))\nabla \alpha_1(x_0) \cdot \nabla u_1(x_0) \leq \eta(x_0)|u_1(x_0) - u_2(x_0)||\nabla \alpha_1(x_0)||\nabla u_1(x_0)|
\]
\[
\leq \kappa_2\sqrt{\eta(x_0)}(|\nabla u_1(\zeta_1)| + |u_1(0) - u_2(0)|)\sqrt{\theta(x_0)}
\]
\[
\leq \kappa_2\sqrt{\eta(x_0)}(|\nabla u_1(\zeta_1)| + |u_1(0) - u_2(0)|)\sqrt{\theta(x_0)}
\]
\[
\leq \kappa_2(2\theta(x_0) + |u_1(0) - u_2(0)|^2),
\]
for some constant $\kappa_2$ dependent on $\sup_{B_2}|\nabla \alpha_1|$, where in the last part we used $ab \leq 2^{-1}(a^2 + b^2)$.

Again, using (1.3)-(1.4) and above three estimates in (A.3) we deduce that for some constant $\kappa_3$, dependent only on the bounds of $\alpha_1$, it holds
\[
\frac{1}{4NC_1^2}\eta|\nabla u_1|^{2\gamma_1}
\]
\[
\leq 2(f_1)^2 + 2C_1^2 + (\varepsilon u_1)^2 + C_1\eta(1 + |\nabla u_1|^{\gamma_1})|\nabla u_1| = 2\tilde{C}_1(1 + |\nabla u_1|^{\gamma_1-1})|\nabla u_1|^2|\nabla \eta|
\]
\[
+ \kappa_3(\eta|\nabla u_1|^2 + |u_1(0) - u_2(0)|^2) + \eta|\nabla f_1||\nabla u_1| + |\nabla u_1|^2(2\eta^{-1}|\nabla \eta|^2 + |\Delta \eta|).
\]  
(A.4)

Using Young’s inequality for appropriate $\delta > 0$ to $|\nabla u_1||\nabla f_1|$, we obtain $\kappa_4 > 0$ satisfying
\[
|\nabla u_1||\nabla f_1| \leq \delta|\nabla u_1|^{2\gamma_1} + \kappa_4|\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}.
\]

Since $|\nabla u_1(x_0)| \geq 1$, and $\gamma_1 > 1$, we also have
\[
(1 + |\nabla u_1|^{\gamma_1})|\nabla u_1| \leq 2|\nabla u_1|^{\gamma_1+1}, \quad \text{and} \quad (1 + |\nabla u_1|^{\gamma_1-1})|\nabla u_1|^2 \leq 2|\nabla u_1|^\gamma_1+1.
\]

Thus, from (A.4) we obtain a constant $\kappa_4 > 0$, dependent on $N, C_1, \kappa_1, \kappa_2, \kappa_3$, and $\kappa_4$, such that
\[
\eta|\nabla u_1|^{2\gamma_1} \leq \kappa_4\left(1 + (f_1)^2 + |u_1(0) - u_2(0)|^2 + (\varepsilon u_1)^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}
\]
\[
+ |\nabla u_1|^{\gamma_1+1}|\nabla \eta| + |\nabla u_1|^2(2\eta^{-1}|\nabla \eta|^2 + |\Delta \eta|)\right).
\]
Now we define \( V(x_0) = \eta(x_0)|\nabla u_1(x_0)|^{2\gamma_1} \) and \( \beta = \frac{2\gamma_1 + 1}{2\gamma_1} \in (\frac{1}{\gamma_1}, 1) \). Then

\[
\eta|\nabla u_1|^{2\gamma_1} \leq \kappa_4 \left( 1 + (f_1)_+^2 + |u_1(0) - u_2(0)|^2 + (\varepsilon u_1)_+^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1 - 1)} + V^\beta \eta^{-\beta} |\nabla \eta| + V^{1/\gamma_1} \left( 2\eta^{-(\gamma_1 + 1)/\gamma_1} |\nabla \eta|^2 + \eta^{-1/\gamma_1} |\Delta \eta| \right) \right) \\
\leq \kappa_4 \left( 1 + (f_1)_+^2 + |u_1(0) - u_2(0)|^2 + (\varepsilon u_1)_+^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1 - 1)} \right) \\
+ \kappa_4 V^\beta \left( \eta^{-\beta} |\nabla \eta| + 2\eta^{-2\beta} |\nabla \eta|^2 + \eta^{-\beta} |\Delta \eta| \right),
\]

where in the last line we used \( V(x_0) \geq (\eta(x_0)|\nabla u_1|^2)^{\gamma_1} > 1, \eta \leq 1 \) and \( \frac{1}{\gamma_1} < \beta \). To conclude the proof of (A.1) it is enough to show that \( \eta^{-\beta} |\nabla \eta| \) and \( \eta^{-\beta} |\Delta \eta| \) are bounded quantities. Recall that \( \eta = \rho^\tau \) where \( \tau = \frac{4\gamma_1}{\gamma_1 - 1} \) with \( \gamma = \min\{\gamma_1, \gamma_2\} \). It is easily seen that \( \tau = \max\{\frac{4\gamma_1}{\gamma_1 - 1}, \frac{4\gamma_2}{\gamma_2 - 1}\} \). A simple calculation yields

\[
\eta^{-\beta} |\nabla \eta| = \tau \rho^{1-\tau-\beta} |\nabla \rho|, \\
\eta^{-\beta} |\Delta \eta| \leq \tau \{\rho^{1-\tau-\beta}|\Delta \rho| + (\tau - 1)\rho^{-2-\tau-\beta}|\nabla \rho|^2\}.
\]

We observe that \( 1 - \beta = \frac{2\gamma_1 - 1}{2\gamma_1} \), and thus,

\[
\tau(1 - \beta) - 1 \geq \frac{\gamma_1 - 1}{2\gamma_1} \frac{4\gamma_1}{\gamma_1 - 1} - 1 = 1, \quad \text{and} \quad \tau(1 - \beta) - 2 \geq 0.
\]

Hence, there exist constant \( C > 0 \) satisfying

\[
\eta(x_0)|\nabla u_1|^{2\gamma_1} \leq C \left( 1 + (f_1)_+^2 + |u_1(0) - u_2(0)|^2 + (\varepsilon u_1)_+^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1 - 1)} \right).
\]

Now taking supremum over \( B_2 \), we can write

\[
\sup_{B_1} \{ |\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2} \} \leq C(1 + \sup_{B_2} (f_1)_+^2 + \sup_{B_2} |\nabla f_1|^{2\gamma_1/(2\gamma_1 - 1)} + |u_1(0) - u_2(0)|^2 + \sup_{B_2} (\varepsilon u_1)_+^2).
\]

If the maximum is attained at the second component we can repeat an analogous argument. This gives us (A.1).

Next, we prove (2.3). Suppose, on the contrary, that there exists \( \{(u_i^n, f_i^n, \alpha_i^n, \varepsilon_i^n)\}_n \) with \( \alpha_i^n \) satisfying (1.1), and

\[
\begin{cases}
-\Delta u_i^n(x) + H_1(x, \nabla u_i^n) + \alpha_i^n(x)(u_i^n(x) - u_0^n(x)) + \varepsilon u_i^n(x) = f_i^n(x) & \text{in } D, \\
-\Delta u_2^n(x) + H_2(x, \nabla u_2^n) + \alpha_2^n(x)(u_2^n(x) - u_0^n(x)) + \varepsilon_n u_2^n(x) = f_2^n(x) & \text{in } D,
\end{cases}
\]

and

\[
|u_i^n(0) - u_0^n(0)|^2 > n \left( 1 + \sup_{B_2} \sum_{i=1}^2 (f_i^n)_+^2 + \sup_{B_2} \sum_{i=1}^2 |\nabla f_i^n|^{2\gamma_1/(2\gamma_1 - 1)} + \sup_{B_2} \sum_{i=1}^2 (\varepsilon_i^n)_+^2 \right). \tag{A.6}
\]

First of all note that we can always set \( u_1^n(0) = 0 \). Therefore, by (A.6), we see that \( |u_2^n(0)| \to \infty \). Suppose that there is a subsequence, denoted by the actual sequence, along which \( u_2^n(0) \to \infty \). Define \( u_i^n = \frac{1}{u_2^n(0)} u_i^n \). Since \( \alpha^2 \leq \kappa_i + a^{2\gamma_1} \) for some \( \kappa_i \), for all \( a \geq 0 \), using (A.1) and (A.6) we find that

\[
\sup_{B_1} \{ |\nabla u_i^n|^{2\gamma_1}, |\nabla u_2^n|^{2\gamma_2} \} < C \quad \text{for all } n.
\]
Since \((v_1^n(0), v_2^n(0)) = (0, 1)\), from above estimate if follows that \(\sup_{B_1}(|v_1^n| + |v_2^n|)\) uniformly bounded in \(n\). Using (1.2) and (A.6) we also get

\[
\sup_n \sup_{B_1} \left\{ \frac{1}{u_2(0)} |H_1(x, \nabla u_1)| + \frac{1}{u_2(0)} |H_2(x, \nabla u_1)| \right\} < \hat{C}. \tag{A.7}
\]

Therefore, it follows from (A.5) that \(\|v_1^n\|_{W^{2,p}(B_\frac{1}{2})}\) and \(\|v_2^n\|_{W^{2,p}(B_\frac{1}{2})}\) are uniformly bounded in \(n\) (cf. [15, Theorem 9.11]) for any \(p > N\), and hence we can extract a weakly convergence subsequence converging to some \(v = (v_1, v_2) \in W^{2,p}(B_1) \times W^{2,p}(B_1)\). From the Sobolev embedding we also see that \(v_2^n \to v_2\) in \(C^{1,\alpha}(B_\frac{1}{2})\). Since \(\|
abla v_1^n\| \to \|
abla v_1\|\) in \(B_\frac{1}{2}\) and \(\sup_n \sup_{B_\frac{1}{2}} \frac{1}{|u_2(0)|} |\nabla u_1^n|^\gamma\) is bounded, by (1.2) and (A.7), it follows that \(\nabla v_i = 0\) in \(B_\frac{1}{2}\). Thus, \(v = (0, 1)\) in \(B_\frac{1}{2}\). Now from the second equation of (A.5) we get

\[-\Delta v_2^n + \alpha_2^n (v_2^n - v_1^n) = \frac{1}{u_2^n(0)} f_2^n - \frac{1}{u_2^n(0)} H_2(x, \nabla u_2^n) \leq \frac{1}{u_2^n(0)} f_2^n + \frac{C_1}{u_2^n(0)},\]

by (1.2). Let \(\varphi\) be a nonzero, non-negative test function supported in \(B_\frac{1}{2}\). Multiplying the above equation by \(\varphi\), integrating over \(B_\frac{1}{2}\) and letting \(n \to \infty\) we obtain

\[
\alpha_0^{-1} \int_{B_{\frac{1}{2}}} \varphi(x)dx \leq \liminf_{n \to \infty} \int_{B_{\frac{1}{2}}} \alpha_2^n(x)v_2^n(x)\varphi(x)dx
\]

\[
\leq \liminf_{n \to \infty} \int_{B_{\frac{1}{2}}} \varphi [\Delta v_2^n + \frac{1}{u_2^n(0)} f_2^n + \alpha_2^n v_1^n + \frac{C_1}{u_2^n(0)}]dx = 0,
\]

where we use the fact that \(\sup_{B_{\frac{1}{2}}} |\alpha_2^n v_1^n| \leq \alpha_0 \sup_{B_{\frac{1}{2}}} |v_1^n| \to 0\). Thus we arrive at a contradiction. A similar contradiction is also arrived is \(u_2^n(0) \to -\infty\) along some subsequence. This establishes (2.3).

Finally (2.2) follows from (2.3) and (A.1). This completes the proof. \(\Box\)

Appendix B. Existence results in bounded domains

By \(D\) we denote a bounded \(C^{2,\delta}\) domain in \(\mathbb{R}^N\) for some \(\delta > 0\).

**Theorem B.1** (Comparison principle). Let \(H_i \in C^1(\mathbb{R}^N \times \mathbb{R}^N), i = 1, 2\) be given functions. Let \(u = (u_1, u_2) \in C^2(D \times \{1, 2\}) \cap C^1(\overline{D} \times \{1, 2\})\) be a subsolution to

\[
-\Delta u_1 + H(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) = f_1 \quad \text{in } D,
\]

\[
-\Delta u_2 + H(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) = f_2 \quad \text{in } D,
\]

and \(v = (v_1, v_2) \in C^2(D \times \{1, 2\}) \cap C^1(\overline{D} \times \{1, 2\})\) be a supersolution to (B.1). Moreover, assume that \(v \geq u\) on \(\partial D\). Then we have \(v \geq u\) in \(D\).

**Proof.** Write \(w_i = v_i - u_i\). Then it follows from (B.1) that

\[
-\Delta w_1 + h_1(x) \cdot \nabla w_1 + \alpha_1(x)(w_1 - w_2) \geq 0 \quad \text{in } D,
\]

\[
-\Delta w_2 + h_2(x) \cdot \nabla w_2 + \alpha_2(x)(w_2 - w_1) \geq 0 \quad \text{in } D,
\]

where

\[
h_i(x) = \int_0^1 \nabla_p H_i(x, u_i(x) + t(\nabla v_i(x) - \nabla u_i(x)))dt, \quad i = 1, 2.
\]

The result follows by applying the maximum principle, Busca-Sirakov [10, Theorem 3.1], Sirakov [22, Theorem 1]. \(\Box\)
We next recall an existence result from [1]. Let \( K_i : \bar{D} \times \mathbb{R}^N \to \mathbb{R}, i = 1, 2, \) be two continuous functions satisfying
\[
|K_i(x, \xi)| \leq \kappa (1 + |\xi|^2)
\]
for all \((x, \xi) \in \bar{D} \times \mathbb{R}^N, i = 1, 2,\) for some constant \(\kappa\). We also assume that \(\xi \mapsto K_i(x, \xi)\) is continuously differentiable.

**Theorem B.2.** Let \(\bar{v}, v \in C^2(\bar{D} \times \{1, 2\})\) be respectively a subsolution and supersolution to
\[
-\Delta u_1 + K_1(x, \nabla u_1) + \alpha_1(u_1 - u_2) = 0 \text{ in } D,

-\Delta u_2 + K_2(x, \nabla u_2) + \alpha_1(u_2 - u_1) = 0 \text{ in } D,

u_1, u_2 = 0 \text{ on } \partial D.
\]
Also, assume that \(v \leq \bar{v}\) in \(D\). Then there exists a solution \(u \in W^{2,p}(D \times \{1, 2\}) \cap C(\bar{D} \times \{1, 2\})\) of the above equations satisfying \(v \leq u \leq \bar{v}\).

**Proof.** This can be established by mimicking the arguments of Amann-Crandall [1, Theorem 1]. \(\square\)

Note that Theorem B.2 can be applied to find the solution for our model provided the Hamiltonian has at-most quadratic growth in the gradient. To apply the theorem for a general Hamiltonian we need to introduce certain approximations.

**Lemma B.1.** Suppose that \(\gamma > 2\). Given \(C_1 > 0\), there exists a sequence of increasing \(C^1, 1\) functions \(\psi_n : [-C_1, \infty) \to [-C_1, \infty)\) satisfying the following
\begin{enumerate}
\item \(\psi_n(x) \leq x\) for all \(x \geq -C_1,\)
\item \(\psi_n(x) \geq \eta_1 x^\gamma - \eta_2,\)
\item \(0 \leq \psi_n'(x) \leq 1,\)
\end{enumerate}
where \(\eta_1, \eta_2\) are positive constants independent of \(n\). Furthermore,
\[
\sup_x \frac{\psi_n(x)}{1 + |x|^2} < \infty,
\]
and \(\psi_n(x) \to x\) as \(n \to \infty,\) uniformly on compact sets.

**Proof.** Define for each \(n \in \mathbb{N},\)
\[
\psi_n(x) = \begin{cases} 
  x & \text{for } x \leq n, \\
  n - \frac{\gamma}{2} + \frac{\gamma}{2} (x - n + 1)^{\frac{2}{\gamma}} & \text{for } x > n.
\end{cases}
\]
Differentiating \(\psi_n\) we get that
\[
\psi_n'(x) = \begin{cases} 
  1 & \text{for } x \leq n, \\
  (x - n + 1)^{\frac{2}{\gamma} - 1} & \text{for } x > n.
\end{cases}
\]
(i) and (iii) are obvious. To see (ii), we note that \(\psi_n(x) \geq x^\gamma - (1 + C_1^{\frac{2}{\gamma}} + C_1)\) for \(x \in [-C_1, n]\). For \(x > n\) we also note that
\[
n - \frac{\gamma}{2} + \frac{\gamma}{2} (x - n + 1)^{\frac{2}{\gamma}} \geq (n - 1)^{\frac{2}{\gamma}} + (x - n + 1)^{\frac{2}{\gamma}} - \frac{\gamma}{2}
\geq x^\gamma - \frac{\gamma}{2}.
\]
This gives us (ii). \(\square\)

We also require the following gradient estimate which follows by repeating the arguments in the proof of Proposition 2.1.
Lemma B.2. Grant Assumption 1.1. Let $\epsilon \in [0, 1)$ and $f_1, f_2 \in C^1(\mathbb{R}^d)$. Let $u$ be a $C^2$ function satisfying
\[
-\Delta u_1(x) + \psi^1_n(H_1(x, \nabla u_1)) + \alpha_1(x)(u_1(x) - u_2(x)) + \varepsilon u_1(x) = f_1(x) \quad \text{in } \bar{B}_2,
\]
\[
-\Delta u_2(x) + \psi^2_n(H_2(x, \nabla u_2)) + \alpha_2(x)(u_2(x) - u_1(x)) + \varepsilon u_2(x) = f_2(x) \quad \text{in } \bar{B}_2,
\]
where $\psi^i_n$ is the approximating sequence in Lemma B.1 if $\gamma_i > 2$, otherwise $\psi^i_n(x) = x$. Suppose that $B_1 \subseteq B_2$ and $B_1, B_2$ are concentric. Then there exists a constant $C > 0$, dependent on $\text{dist}(B_1, \partial B_2), \gamma_i, d, \eta_1, \eta_2,$ and $\alpha_0$ but not on $n$ and $u$, satisfying
\[
\sup_{B_1}(\psi^1_n(H_1(x, \nabla u_1)))^2, [\psi^2_n(H_2(x, \nabla u_2))]^2 \leq C(1 + \sup_{B_2} \sum_{i=1}^2 (f_i)_+^2 + \sup_{B_2} \sum_{i=1}^2 |\nabla f_i|^{4/3} + |u_1(0) - u_2(0)|^2 + \sup_{B_2} \sum_{i=1}^2 (\varepsilon u_i)_+^2).
\]

Now we can prove our existence result.

Theorem B.3. Grant Assumption 1.1. Suppose $\varepsilon \in [0, 1]$ and $f = (f_1, f_2) \in C^1(\bar{D} \times \{1, 2\})$. Let $\mathfrak{u} \in C^2(\bar{D} \times \{1, 2\})$ be a subsolution to
\[
-\Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) + \varepsilon u_1 = f_1 \quad \text{in } D,
\]
\[
-\Delta u_2 + H_1(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) + \varepsilon u_2 = f_2 \quad \text{in } D.
\]
There exists a solution $u \in C^2(\bar{D} \times \{1, 2\})$ to (B.2) satisfying $u \geq \mathfrak{u}$ in $D$.

Proof. The main idea of the proof is to use the existence result from Theorem B.2 by making use of the approximation sequence in Lemma B.1. A similar method was also used by Lions in [21] for scalar equations. In fact, the method of Lions uses more sophisticated tools like the Bony maximum principle to obtain an up to the boundary bounds of the gradient. We do not use such results. We split the proof into two steps.

Step 1. Fix $n \geq 1$ and consider the system of equations
\[
-\Delta w_1 + \psi^1_n(H_1(x, \nabla u_1)) + \alpha_1(x)(w_1 - w_2) + \varepsilon w_1 = f_1 \quad \text{in } D,
\]
\[
-\Delta w_2 + \psi^2_n(H_1(x, \nabla u_2)) + \alpha_2(x)(w_2 - w_1) + \varepsilon w_2 = f_2 \quad \text{in } D,
\]
where $\psi^i_n$ is the approximating sequence from Lemma B.1 if $\gamma_i > 2$, otherwise $\psi^i_n(x) = x$. By Lemma B.1(i), we note that $\mathfrak{u}$ is a subsolution to (B.3). So to apply Theorem B.2 we need to find a super-solution. Denote by $\bar{M} = \max_{\partial D} \{|\mathfrak{w}_1|, |\mathfrak{w}_2|\}$. Let $\mathfrak{v} \in C^2(\bar{D} \times \{1, 2\})$ be the unique solution to
\[
-\Delta \mathfrak{v}_1 + \alpha_1(x)(\mathfrak{v}_1 - \mathfrak{v}_2) + \varepsilon \mathfrak{v}_1 = f_1 + \eta_2 \wedge C_1 \quad \text{in } D,
\]
\[
-\Delta \mathfrak{v}_2 + \alpha_2(x)(\mathfrak{v}_2 - \mathfrak{v}_1) + \varepsilon \mathfrak{v}_2 = f_2 + \eta_2 \wedge C_1 \quad \text{in } D,
\]
\[
\mathfrak{v}_1, \mathfrak{v}_2 = \bar{M} \quad \text{on } \partial D,
\]
where $\eta_2$ is given by Lemma B.1(ii). In fact, using Sweers [23, Theorem 1.1], we can find a unique solution of (B.4) in $W^{2,p}_{\text{loc}}(D) \times C(D)$ and then using a standard bootstrapping argument we can improve the regularity. Using Lemma B.1(ii) and (1.2) we then obtain from (B.4) that
\[
-\Delta \bar{v}_1 + \psi^1_n(H_1(x, \nabla \bar{v}_1)) + \alpha_1(x)(\bar{v}_1 - \bar{v}_2) + \varepsilon \bar{v}_1 \geq f_1 \quad \text{in } D,
\]
\[
-\Delta \bar{v}_2 + \psi^2_n(H_2(x, \nabla \bar{v}_2)) + \alpha_2(x)(\bar{v}_2 - \bar{v}_1) + \varepsilon \bar{v}_2 \geq f_2 \quad \text{in } D,
\]
\[
\bar{v}_1, \bar{v}_2 = \bar{M} \quad \text{on } \partial D.
\]
This gives us the super-solution. By Theorem B.1 we also have $\mathfrak{u} \leq \mathfrak{v}$ in $\bar{D}$. Now we can apply Theorem B.2 to find a solution $\mathfrak{w}_n = (\mathfrak{w}_n^1, \mathfrak{w}_n^2) \in C^2(\bar{D} \times \{1, 2\}) \cap C(\bar{D} \times \{1, 2\})$ to (B.3) satisfying $\mathfrak{v} \leq \mathfrak{w}_n \leq \mathfrak{v}$ in $\bar{D}$ for all $n$. It should also be noted that $\mathfrak{v}$ is independent of $n$. 
Step 2. We now pass to the limit in (B.3) with the help of the gradient estimate in Lemma B.2. From step 1 we notice that \( \sup_D |w^n_1 - w^n_2| < \infty \) uniformly in \( n \). Thus, for any compact \( K \subset D \) we have \( \max_K \{ |\nabla w^n_1|, |\nabla w^n_2| \} < \infty \) uniformly in \( n \), by Lemma B.2. Using (B.3) and standard elliptic estimates, we get

\[
\sup_n \left\{ \|w^n_1\|_{W^{2,p}(K)}, \|w^n_2\|_{W^{2,p}(K)} \right\} < \infty \quad \text{for every compact} \ K \subset D.
\]

Using a standard diagonalization argument we can find a subsequence, denoted by the actual one, so that \( w^n_i \to u_i \) in \( W^{2,p}_{\text{loc}}(D) \) for \( p > N \) and \( w^n_i \to u_i \) in \( C^1_{\text{loc}}(D) \), as \( n \to \infty \). Thus passing to the limit in (B.3) we obtain

\[
-\Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) + \varepsilon u_1 = f_1 \quad \text{in} \ D,
\]

\[
-\Delta u_2 + H_1(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) + \varepsilon u_2 = f_2 \quad \text{in} \ D,
\]

and \( \underline{u} \leq u \leq \bar{v} \) in \( D \). Moreover, using standard theory of elliptic pde we obtain \( u \in C^2(D \times \{1, 2\}) \).

This completes the proof. \( \square \)

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