Triangle-free graphs of tree-width $t$ are $\left\lceil \frac{(t + 3)}{2} \right\rceil$-colorable

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Abstract

We prove that every triangle-free graph of tree-width $t$ has chromatic number at most $\left\lceil \frac{(t + 3)}{2} \right\rceil$, and demonstrate that this bound is tight. The argument also establishes a connection between coloring graphs of tree-width $t$ and on-line coloring of graphs of path-width $t$.

While there exist triangle-free graphs of arbitrarily large chromatic number, forbidding triangles improves bounds on the chromatic number in certain graph classes. For example, planar graphs are 4-colorable [3, 4] and even deciding their 3-colorability is an NP-complete problem [6], while all triangle-free planar graphs are 3-colorable [7]. A graph on $n$ vertices may have chromatic number up to $n$, while Ajtai et al. [11] and Kim [10] proved a tight upper bound $O\left(\frac{\sqrt{n}}{\log n}\right)$ on the chromatic number of triangle-free $n$-vertex graphs. The chromatic number of graphs of maximum degree $\Delta$ may be as large as $\Delta + 1$, while Johansson [9] proved that triangle-free graphs of maximum degree $\Delta$ have chromatic number $O(\Delta/\log \Delta)$.

On the other hand, no such improvement is possible for graphs with bounded degeneracy. A graph $G$ is $d$-degenerate if each subgraph of $G$ contains a vertex of degree at most $d$. A straightforward greedy algorithm colors every $d$-degenerate graph using $d + 1$ colors; and a construction by Blanche Descartes [5] gives for every positive integer $d$ a $d$-degenerate triangle-free

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graph that is not \(d\)-colorable, as observed by Kostochka and Nešetřil [11]. A construction of triangle-free graphs with high chromatic number by Zykov [12] also has this property, and the same construction was given (and possibly independently rediscovered) by Alon, Krivelevich, and Sudakov [2].

In this note, we consider the chromatic number of graphs of given tree-width \(t\). Note that every graph of tree-width \(t\) is \(t\)-degenerate, but on the other hand, the clique \(K_{t+1}\) has tree-width \(t\), establishing \(t+1\) as the tight upper bound on the chromatic number of graphs of tree-width \(t\). We show that the bound can be improved by a constant factor if triangles are forbidden.

**Theorem 1.** For any positive integer \(t\), every triangle-free graph of tree-width at most \(t\) has chromatic number at most \(\lceil (t + 3)/2 \rceil\).

We also show that this result is tight. Actually, we give the following stronger result on graphs with bounded clique number. Let \(g(t,2) = g(0,k) = 1\) for all integers \(t \geq 0\) and \(k \geq 2\). For \(t \geq 1\) and \(k \geq 3\), let us inductively define \(g(t,k) = \lceil (t + 1)/2 \rceil + g(\lfloor (t-1)/2 \rfloor, k-1)\).

**Theorem 2.** For all integers \(t \geq 0\) and \(k \geq 2\), there exists a \(K_k\)-free graph of tree-width at most \(t\) with chromatic number at least \(g(t,k)\).

Note that \(g(t,3) = \lceil (t + 3)/2 \rceil\) for all \(t \geq 1\) and that \(g(t,k) > (1 - \frac{1}{2^{k-2}})t\) for all \(t \geq 0\) and \(k \geq 2\). Theorem 2 motivates the following question, which we were not able to resolve.

**Problem 3.** For integers \(k \geq 4\) and \(t \geq k - 1\), what is the maximum chromatic number of \(K_k\)-free graphs of tree-width at most \(t\)?

Let us remark that in the list coloring setting, the question is much easier to settle—the complete bipartite graph \(K_{t,t}\) has tree-width \(t\), but there exists an assignment of lists of size \(t\) to its vertices from that it cannot be colored, showing that no improvement analogous to Theorem 1 is possible.

Another natural question concerns graphs with larger girth. The construction of Blanche Descartes [5] actually produces \(d\)-degenerate graphs of girth six that are not \(d\)-colorable, and Kostochka and Nešetřil [11] generalize this result to graphs of arbitrarily large girth.

**Problem 4.** For integers \(g \geq 4\) and \(t \geq g - 1\), what is the maximum chromatic number of graphs of tree-width at most \(t\) and girth at least \(g\)?
1 Tree-width, path-width, and on-line coloring

To prove Theorems 1 and 2, it is convenient to establish a connection between chromatic number of graphs with given tree-width $t$ and an on-line variant of the chromatic number for graphs of path-width $t$. In on-line coloring [8], the graph to be colored is revealed vertex by vertex and a color has to be assigned to each revealed vertex immediately, with no knowledge regarding the rest of the graph. We need a variation on this idea, revealing the vertices in the order given by a path decomposition of the graph.

Let us first recall some definitions. A tree decomposition of a graph $G$ is a pair $(T, \beta)$, where $T$ is a tree and $\beta$ is a function assigning to each vertex of $T$ a set of vertices of $G$, such that for every $uv \in E(G)$ there exists $z \in V(T)$ with $\{u, v\} \subseteq \beta(z)$, and such that for every $v \in V(G)$, the set $\{z : v \in \beta(z)\}$ induces a non-empty connected subtree in $T$. The width of the decomposition is the maximum of $1 + |\beta(z)|$ over all $z \in V(T)$, and the tree-width of $G$ is the minimum possible width of its tree-decomposition. A path decomposition is a tree decomposition $(T, \beta)$ where $T$ is a path, and path-width of $G$ is the minimum possible width of its path decomposition.

Suppose $(P, \beta)$ is a path decomposition of a graph $G$, where $P = z_0z_1 \ldots z_n$. We say that the path decomposition is nice if $\beta(z_0) = \emptyset$ and $|\beta(z_i) \setminus \beta(z_{i-1})| = 1$ for $i = 1, \ldots, n$. Observe that each path decomposition can be transformed into a nice one of the same width. A nice path decomposition gives a natural way to produce the graph $G$ by adding one vertex at a time: the construction starts with the null graph, and in the $i$-th step for $i = 1, \ldots, n$, the unique vertex $v_i \in \beta(z_i) \setminus \beta(z_{i-1})$ is added to $G$ and joined to its neighbors in $\beta(z_i)$. An on-line coloring algorithm $A$ at each step of this process assigns a color to the vertex $v_i$ distinct from the colors of the neighbors of $v_i$ in $\beta(z_i)$. Note that the assigned color cannot be changed later, and that the algorithm does not in advance know the graph $G$, only its part revealed till the current step of the process. Let $\chi_A(G, P, \beta)$ denote the number of colors $A$ needs to color $G$ when $G$ is presented to the algorithm $A$ vertex by vertex according to the nice path decomposition $(P, \beta)$.

Let us remark that we misuse the term “algorithm” a bit, as we are not concerned with the question of efficiency or even computability in any model of computation. Formally, an on-line coloring algorithm $A$ is a function from the set of triples $(H, Q, \gamma)$, where $H$ is a graph with a nice path decomposition.
(Q, γ), to the integers, such that the following holds. Consider any graph
G with a nice path decomposition (P, β), where P = z₀z₁ . . . zₙ for some
n ≥ 1. For i = 1, . . . , n, let vᵢ be the unique vertex in β(zᵢ) \ β(zᵢ⁻¹), let
Pᵢ = P − {zᵢ⁺¹, . . . , zₙ}, Gᵢ = G − {vᵢ⁺¹, . . . , vₙ}, and let βᵢ be the restriction
of β to V(Pᵢ). If ϕ : V(G) → Z is defined by ϕ(vᵢ) = A(Gᵢ, Pᵢ, βᵢ), then ϕ is
a proper coloring of G. And, χ₄(G, P, β) is defined as the number of colors
used by this coloring ϕ.

Nevertheless, it is easier to think about on-line coloring in the adversarial
setting: an enemy is producing the graph G with its nice path decomposition
on the fly and the algorithm A is assigning colors to the vertices as they
arrive; and the enemy can construct further parts of the graph depending
on the coloring chosen by A so far. We now present the main result of this
section, which we use both to give upper bounds on the chromatic number
of graphs with bounded tree-width and to prove the existence of bounded
tree-width graphs with large chromatic number.

Lemma 5. Let t and k be positive integers, and let c be the maximum of
χ(G) over all Kₖ-free graphs G of tree-width at most t. There exists an on-
line coloring algorithm Aₜ such that every Kₖ-free graph H that has a nice
path decomposition (P, β) of width at most t satisfies χ₄ₜ(H, P, β) ≤ c.

Conversely, if an on-line coloring algorithm Aₜ satisfies χ₄ₜ(H, P, β) ≤ c'
for every Kₖ-free graph H having a path decomposition (P, β) of width at
most t, then χ₄(G) ≤ c' for every Kₖ-free graph G of tree-width at most t.

Proof. Let us start with the second claim. Let G be a Kₖ-free graph and let
(T, β) be its tree decomposition of width at most t. Without loss of generality,
we can assume that the tree T is rooted in a vertex r such that β(r) = ⊥
and that each vertex z ≠ r of T with parent z' satisfies |β(z \ β(z'))| = 1.
For each path P in T starting in r and ending in a leaf of T, let Gₚ be the
subgraph of G induced by ∪z∈V(P) β(z) and let βₚ be the restriction of β to
V(P). Then (P, βₚ) is a nice path decomposition of Gₚ of width at most
t, and thus the algorithm Aₜ can be used to color Gₚ by at most c' colors.
Furthermore, if P' is another root-leaf path in T and some vertex v belongs
both to Gₚ and Gₚ', then v ∈ β(z) for some vertex z belonging to the shared
initial subpath of P and P', and thus the algorithm Aₜ assigns the same color
to v in its run on (Gₚ, P, βₚ) and on (Gₚ', P', βₚ'). We conclude that the
colorings of the graphs Gₚ for all root-leaf paths P are consistent, and their
union gives a proper coloring of G using at most c' colors.
Conversely, let $c$ be as given in the statement of the lemma. Consider the "universal" infinite $K_k$-free graph $G$ of tree-width at most $t$. That is, $G$ is an infinite graph with rooted tree decomposition $(T, \beta)$ of width at most $t$ satisfying the following conditions. The root $r$ of $T$ has $\beta(r) = \emptyset$. For every vertex $z$ of $T$ and all sets $I \subseteq B \subseteq \beta(z)$ such that $G[I]$ is $K_{k-1}$-free and $|B| \leq t$, there exists a child $z'$ of $z$ in $T$ such that $\beta(z') = B \cup \{v\}$ for a vertex $v$ not belonging to $\beta(z)$, with $v$ adjacent in $G[\beta(z')]$ precisely to the vertices in $I$; and $z$ has no other children. A standard compactness argument shows that there exists a coloring $\phi$ of $G$ using $c$ colors. Now, for any $K_k$-free graph $H$ with a nice path decomposition $(P, \beta)$ of width at most $t$, we can identify $P$ with a path in $T$ and $H$ with an induced subgraph of $G$ in the natural way, and the algorithm $A_t$ works by assigning colors to vertices of $H$ according to the coloring $\phi$. Hence, $\chi_A(H, P, \beta) \leq c$. 

\section{Non-colorability}

In this section, we give the construction proving Theorem 2.

Let $H_0$ and $H$ be graphs with nice path decompositions $(P_0, \beta_0)$ and $(P, \beta)$, respectively. We say that $(H, P, \beta)$ extends $(H_0, P_0, \beta_0)$ if $P_0$ is an initial segment of $P$, $\beta_0$ is the restriction of $\beta$ to $V(P_0)$, and $H_0$ is an induced subgraph of $H$. Let $z$ be the last vertex of $P$. We say that a coloring $\phi$ of $H$ is $c$-forced if there exists an independent set $I \subseteq \beta(z)$ of size $c$ such that vertices of $I$ receive pairwise distinct colors according to $\phi$.

Lemma 6. Let $k \geq 3$ and $c \geq 1$ be integers, and let $A$ be an on-line coloring algorithm. Let $H_0$ be a triangle-free graph with a nice path decomposition $(P_0, \beta_0)$ of width at most $2c - 2$. Let $c_0 \leq c - 1$ be a non-negative integer. If the coloring $\phi_0$ of $H_0$ produced by $A$ is $c_0$-forced, then there exists a triangle-free graph $H$ with a nice path decomposition $(P, \beta)$ of width at most $2c - 2$ such that $(H, P, \beta)$ extends $(H_0, P_0, \beta_0)$ and the coloring $\phi$ of $H$ produced by $A$ is $(c_0 + 1)$-forced.

Proof. Let $z_0$ be the last vertex of $P_0$ and let $I = \{v_1, \ldots, v_{c_0}\}$ be an independent set contained in $\beta_0(z_0)$ such that $\phi_0$ assigns pairwise distinct colors to vertices of $I$. Without loss of generality $\phi_0(v_1) = 1, \ldots, \phi_0(v_{c_0}) = c_0$. As $(H, P, \beta)$ will be chosen to extend $(H_0, P_0, \beta_0)$, the coloring $\phi$ produced by the algorithm $A$ will match $\phi_0$ on $V(H_0)$. 

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Proof. Let $z_0$ be the last vertex of $P_0$ and let $I = \{v_1, \ldots, v_{c_0}\}$ be an independent set contained in $\beta_0(z_0)$ such that $\phi_0$ assigns pairwise distinct colors to vertices of $I$. Without loss of generality $\phi_0(v_1) = 1, \ldots, \phi_0(v_{c_0}) = c_0$. As $(H, P, \beta)$ will be chosen to extend $(H_0, P_0, \beta_0)$, the coloring $\phi$ produced by the algorithm $A$ will match $\phi_0$ on $V(H_0)$. 

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Proof. Let $z_0$ be the last vertex of $P_0$ and let $I = \{v_1, \ldots, v_{c_0}\}$ be an independent set contained in $\beta_0(z_0)$ such that $\phi_0$ assigns pairwise distinct colors to vertices of $I$. Without loss of generality $\phi_0(v_1) = 1, \ldots, \phi_0(v_{c_0}) = c_0$. As $(H, P, \beta)$ will be chosen to extend $(H_0, P_0, \beta_0)$, the coloring $\phi$ produced by the algorithm $A$ will match $\phi_0$ on $V(H_0)$.
We will now append further vertices $x_1, x_2, \ldots$ at the end of $P_0$ to obtain the path $P$ as follows. Let $β(x_1) = \{v_1, \ldots, v_{c_0}, v'_1\}$ for a new vertex $v'_1$ with no neighbors, and use the algorithm $A$ to extend $φ$ to this vertex. If $φ(v'_1)$ is distinct from $1, \ldots, c_0$, then we stop the construction. Otherwise, we can by symmetry assume that $φ(v'_1) = 1$. We then let $β(x_2) = \{v_1, \ldots, v_{c_0}, v'_1, v'_2\}$ for a new vertex $v'_2$ adjacent to $v_1$. If $φ(v'_2)$ is distinct from $1, \ldots, c_0$, then we stop the construction (the independent set receiving $c_0 + 1$ distinct colors is $\{v'_1, v_2, \ldots, v_{c_0}, v'_2\}$). Otherwise, since $φ(v'_2) ≠ φ(v_1)$, we can assume that $φ(v'_2) = 2$. Similarly, we proceed for $j = 3, \ldots, c_0 + 1$: we set $β(x_j) = \{v_1, \ldots, v_{c_0}, v'_1, \ldots, v'_j\}$, with $v'_j$ adjacent to $v_1, \ldots, v_{j-1}$, and depending on the decision of $A$ regarding the color of $v'_j$, we either stop with the independent set $\{v'_1, v'_2, \ldots, v'_{j-1}, v_j, v_{j-1}, \ldots, v_{c_0}, v'_j\}$ using $c_0 + 1$ distinct colors, or we can assume that $φ(v'_j) = j$. The former happens necessarily at latest when $j = c_0 + 1$.

All the bags created throughout this process have size at most $2c_0 + 1 ≤ 2c - 1$, and thus the width of the resulting path decomposition is at most $2c - 2$. □

Iterating Lemma 6 we obtain the following.

**Corollary 7.** Let $t ≥ 0$ be an integer. For any on-line coloring algorithm $A$, there exists a triangle-free graph $H$ with a nice path decomposition $(P, β)$ of width at most $t$ such that the coloring $φ$ of $H$ produced by $A$ is $\lceil \frac{t + 1}{2} \rceil$-forced.

We are now ready to establish Theorem 2.

**Proof of Theorem 2.** By Lemma 6 it suffices to show that for every on-line coloring algorithm $A$, there exists a $K_k$-free graph $H$ with a nice path decomposition $(P, β)$ of width at most $t$ such that $χ_A(H, P, β) ≥ g(t, k)$. We prove the claim by induction on $k$. When $k = 2$ or $t = 0$, we have $g(t, k) = 1$ and the claim obviously holds, with $H$ being a single-vertex graph. Hence, we can assume that $k ≥ 3$ and $t ≥ 1$. Let $c_1 = \lceil \frac{t + 1}{2} \rceil$ and $c_2 = t - c_1 = \lfloor \frac{t + 1}{2} \rfloor$.

Let $H_0$ be a triangle-free graph with a nice path decomposition $(P_0, β_0)$ of width at most $t$ such that the coloring $φ_0$ of $H_0$ produced by $A$ is $c_1$-forced, obtained using Corollary 7. Let $z_0$ be the last vertex of $P_0$ and let $I$ be an independent set in $H_0[β_0(z_0)]$ of size $c_1$ whose vertices are colored by pairwise distinct colors in $φ_0$.

Let $A'$ be an on-line coloring algorithm defined as follows: given any graph $H_1$ with a nice path decomposition $(P_1, β_1)$, let $H'_1$ be the graph obtained from
Let $c'$ be a positive integer, let $F$ be a graph, and let $\varphi$ be a $c'$-coloring of $F$. We say that $\varphi$ is $F$-valid if $F$ does not contain any independent set on which $\varphi$ uses all $c'$ distinct colors. We say that a color $a$ is $(F, \varphi)$-forbidden if there exists an independent set $A_a \subseteq V(F)$ in $F$ such that $\varphi$ uses all colors except for $a$ on $A_a$. We need the following auxiliary claim.

**Lemma 8.** Let $c'$ be a positive integer and let $\varphi$ be a $c'$-coloring of a graph $F$. If $\varphi$ is $F$-valid, then at most $\max(|V(F)| - c' + 2, 0)$ colors are $(F, \varphi)$-forbidden.

**Proof.** We prove the claim by induction on $|V(F)|$, and thus we assume that Lemma 8 holds for all graphs with fewer than $|V(F)|$ vertices. For each $(F, \varphi)$-forbidden color $a$, let $A_a$ be an independent set such that $\varphi$ uses all colors except for $a$ on $A_a$.

If $|V(F)| \leq c' - 2$, then $F$ contains no independent set of size $c' - 1$, and thus no color is $(F, \varphi)$-forbidden. Hence, suppose that $|V(F)| \geq c' - 1$, and thus $|V(F)| - c' + 2 \geq 1$. If at most one color is $(F, \varphi)$-forbidden, then the lemma holds. Hence, we can by symmetry assume that colors 1 and 2 are
forbidden. Note that all $c'$ colors appear at least once on $A_1 \cup A_2$. If each $(F, \varphi)$-forbidden color is used on at least two vertices of $F$, then the number of $(F, \varphi)$-forbidden colors is at most $|V(F)| - c'$, and the lemma holds.

Hence, we can assume that the color $c'$ is $(F, \varphi)$-forbidden and used on exactly one vertex $v$ of $F$. Let $F'$ be the graph obtained from $F$ by removing $v$ and all the neighbors of $v$, and let $\varphi'$ be the restriction of $\varphi$ to $F'$. Note that $\varphi'$ is a $(c' - 1)$-coloring of $F'$. We claim that $\varphi'$ is $F'$-valid. Indeed, if all colors $1, \ldots, c' - 1$ were used on an independent set $A' \subseteq V(F')$, then all colors $1, \ldots, c$ would be used on the independent set $A' \cup \{v\}$ in $F$, contradicting the assumption that $\varphi$ is $F$-valid. If a color $a \neq c'$ is $(F, \varphi)$-forbidden, then note that $A_a$ contains $v$ since $v$ is the only vertex of $F$ of color $c'$, and does not contain any of the neighbors of $v$ since $A_a$ is an independent set. Hence, $A_a \setminus \{v\}$ is an independent set in $F'$ on that all colors except for $a$ appear, and thus $a$ is $(F', \varphi')$-forbidden. Denoting by $f'$ the number of $(F', \varphi')$-forbidden colors, we conclude that at most $f' + 1$ colors are $(F, \varphi)$-forbidden.

Since $\varphi$ is $F$-valid, the set $A_{c'} \cup \{v\}$ is not independent, and thus $v$ has degree at least one. Consequently, $|V(F')| \leq |V(F)| - 2$. By the induction hypothesis, we conclude that the number of $(F, \varphi)$-forbidden colors is at most

$$\max(|V(F)| - (c' - 1) + 2, 0) + 1 \leq \max(|V(F)| - c' + 2, 1) = |V(F)| - c' + 2,$$

as required. 

We are now ready to bound the chromatic number of triangle-free graphs of tree-width at most $t$.

**Proof of Theorem 1.** We need to show that every triangle-free graph $G$ of tree-width at most $t$ can be colored using $c' = \lceil (t + 3)/2 \rceil$ colors. Note that $2c' - 2 > t$. By Lemma 5 it suffices to design an on-line coloring algorithm $A'$ that colors every triangle-free graph $H$ with a nice path decomposition $(P, \beta)$ of width at most $t$ using at most $c'$ colors.

The algorithm $A'$ maintains the invariant that the restriction of the $c'$-coloring $\varphi$ produced by this algorithm to $\beta(z)$ is $H[\beta(z)]$-valid for every $z \in V(P)$. Consider any vertex $z'$ with predecessor $z''$ in $P$, let $v$ be the unique vertex in $\beta(z') \setminus \beta(z'')$, and let $N$ be the set of neighbors of $v$ in $\beta(z')$. Since $H$ is triangle-free, $N$ is an independent set, and thus at most $\min(|N|, c' - 1)$ colors appear on $N$ by the invariant. To get a proper coloring maintaining the invariant, it suffices to assign $v$ an arbitrary color that does
not appear on \( N \) and that is not \((H[\beta(z') \setminus (N \cup \{v\})], \varphi)\)-forbidden. Since \(|\beta(z')| \leq t + 1\), Lemma \( \Box \) implies that the number of such colors is at least

\[
c' - \min(|N|, c' - 1) - \max(t - |N| - c' + 2, 0).
\]

If \(|N| \geq c'\), this is at least \(c' - (c' - 1) - \max(t - 2c' + 2, 0) = 1\), using the fact that \(2c' - 2 > t\). If \(|N| \leq c' - 1\), this is at least \(c' - |N| - \max(t - |N| - c' + 2, 0) = \min(2c' - t - 2, c' - |N|) \geq 1\). In either case it is possible to extend the coloring. \( \square \)

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