Comments on "Single meson production in photon-photon collisions and infrared renormalons"[arXiv:0911.5226v1][hep-ph]

S. S. Agaev

Institute for Physical Problems, Baku State University
Z. Khalilov St. 23, Az-1148 Baku, Azerbaijan
(Dated: February 8, 2010)

This paper contains our comments on the work of the authors A. I. Ahmadov et al. "Single meson production in photon-photon collisions and infrared renormalons" [arXiv:0911.5226v1 [hep-ph] 27 Nov 2009]. We draw attention to errors made in this work in calculation of the subprocess cross sections within the frozen and running coupling approaches, in deriving the Borel transforms and Borel resummed expressions for the cross sections in the running coupling approach. We show that the same fatal errors were made by these authors also in their work published in Phys. Rev. D80, 016003 (2009).

In the work "Single meson production in photon-photon collisions and infrared renormalons" [arXiv:0911.5226v1 [hep-ph] 27 Nov 2009] authors A. I. Ahmadov, Coskun Aydin, E. A. Dadashov and Sh. M. Nagiyev in accordance with the title of the paper intended to compute the cross section of the single meson production in the photon-photon collision $\gamma + \gamma \rightarrow M + X$.

In the framework of the perturbative QCD (pQCD) this process proceeds through leading and higher twist mechanisms. At the leading twist (LT) order the meson $M$ is produced due to fragmentation of the final quark or gluon (appearing in the hard-scattering subprocess) into the meson $M$. The relevant LT cross section can be computed using the standard pQCD methods, applicable to inclusive hadronic processes. The higher twist (HT) mechanism implies production of the meson $M$ directly at the hard-scattering subprocess. In the HT subprocess emergence of the meson in the final state becomes possible in the result of the hard gluon exchange between meson constituents. The cross section of the meson production via HT mechanism can be found employing the methods of the pQCD elaborated to compute hard exclusive processes and relevant factorization theorems. By this way HT corrections to some inclusive processes were obtained.

In order to find the amplitude of the HT hard-scattering subprocess one should use the distribution amplitude of the meson and perform integrations over the longitudinal momentum fractions $x_1$ and $x_2$ carried by quark and antiquark of the meson. In the case under discussion it takes the following form

$$M = \int_0^1 \int_0^1 dx_1 dx_2 \delta(1 - x_1 - x_2)\alpha_s(\mu_R^2)T_H(x_1, x_2, \mu_F^2)\Phi_M(x_1, x_2, \mu_F^2),$$

where $\mu_R^2$ and $\mu_F^2$ are the renormalization and factorization scales, $T_H(x_1, x_2, \mu_F^2)$ - hard-scattering function, $\Phi_M(x_1, x_2, \mu_F^2)$ is the distribution amplitude of the meson. One of the important questions to be solved in order to calculate Eq. 11 is a proper choice for the scales $\mu_R^2$ and $\mu_F^2$. It has been advocated that to reduce the higher order corrections to a physical quantity and improve the convergence of the corresponding perturbation series, the renormalization scale, i.e., the argument of the QCD coupling in a Feynman diagram should be set equal to the hard gluon’s squared four-momentum. This idea was proposed more than 25 years ago, and used in numerous works. Nevertheless, A.I. Ahmadov et al. write: "According to Ref. [26] [A.I.Ahmadov et al., Int. J. Mod. Phys. E15, 1209 (2006)] should be noted that in PQCD calculations the argument of the running coupling constant in both, the renormalization and factorization scale $\hat{Q}^2$ should be taken equal to the square of the momentum transfer of a hard gluon in a corresponding Feynman diagram!"

In exclusive processes and in HT Feynman diagrams describing corrections to inclusive ones, the renormalization scale chosen this way inevitably depends on the longitudinal momentum fractions $x_1$ and $x_2$ carried by the meson’s quark and/or antiquark $\mu_R^2 \sim x_1(x_2)$. Then the pQCD factorization formula (in our case Eq. 11) diverges, since $\alpha_s(\mu_R^2)$ suffers from an end-point $x_1(x_2) \rightarrow 0, 1$ singularities. Therefore, the HT amplitude may be computed using two options: one of them is the standard “frozen” coupling approximation, when one fixes $x_1$ and $x_2$ equating them to their mean values $x_1 = x_2 = 1/2$, and remove $\alpha_s(\mu_R^2)$ as a constant from the integral Eq. 11. In the second approach one allows $x_1$ and $x_2$ to run in the argument of the $\alpha_s(\mu_R^2)$ and calculate the HT amplitude using the running coupling (RC) method, and removes divergences appearing in the perturbative expression with the help of

*Electronic address: agaev.shahin@yahoo.com*
the infrared (IR) renormalon calculus [11] (the Borel transformation, resummed expression for a physical quantity) and the principal value prescription [12]. It turns out that this method allows one to estimate power corrections arising from the end-point integration regions.

The RC method was suggested in our works [13, 14] (see also, [15]) in order to compute effects of the infrared renormalons on the pion and kaon electromagnetic form factors (FFs). Later the RC method was applied for calculation of power suppressed corrections to some exclusive processes, vertex function and HT corrections to semi-inclusive processes. Namely, it was used for computations of the pion, kaon electromagnetic FFs [12, 14, 16], \(\pi^0\gamma\) [17] and \(\eta'\), \(\eta'\gamma\) electromagnetic transition FFs [18], for evaluation of the power corrections to the gluon-gluon-\(\eta'\) vertex function [19] and to the two pion production in the process \(\gamma\gamma^* \rightarrow \pi\pi\) [20]. It was also employed to estimate HT correction to the semi-inclusive meson production \(\gamma + h \rightarrow M + X\) [21]. It is worth noting that the RC method deals with power-suppressed corrections to exclusive processes coming from the end-point integration regions. There are another source of the infrared renormalon effects in inclusive and exclusive processes, namely one appearing due to resummation of diagrams with quark vacuum insertions ("bubble chains") into a gluon line. It is known that resummation of infinite number of such diagrams corresponds to computation of one-loop diagrams with the running coupling \(\alpha_s\) not only due to loop integration, but also because of the integration in the process amplitude over the longitudinal momentum fractions of hadron constituents. Thus the exclusive processes have two independent sources of the infrared renormalon effects.

In Ref. [1] both the frozen and running coupling approaches were used. Employing the RC method elaborated in Refs. [13, 14, 17–21], authors of [1] nevertheless "forgot" to cite them - the real sources used in their calculations and in preparation of the text of the paper. Now we want to concentrate on "calculations" performed in Ref. [1]. The higher twist subprocesses considered in Ref. [1] are

\[\gamma q_1 \rightarrow M q_2, \quad \gamma \underline{q}_2 \rightarrow M \underline{q}_1, \quad M = (q_1 \underline{q}_2).\]  
\(\text{(2)}\)

In the "frozen" coupling approximation they were calculated by Bagger and Gunion [7]. The authors presented their results for the \(\gamma q_1 \rightarrow M q_2\) subprocess cross section in Eqs. (16) and (17) of Ref. [7]:

\[\frac{d\sigma}{dt} |_{\gamma q \rightarrow M q} (\hat{s}, \hat{t}, \hat{u}) = \frac{8\pi^2 C_F \alpha_E}{9} \left| \Delta(\hat{s}, \hat{u}) \right|^2 \frac{1}{\hat{s}^2 (-\hat{t})} \left[ \frac{1}{\hat{s}^2} + \frac{1}{\hat{u}^2} \right], \quad \pi, \rho, L,\]

\[\frac{d\sigma}{dt} |_{\gamma q \rightarrow M q} (\hat{s}, \hat{t}, \hat{u}) = \frac{8\pi^2 C_F \alpha_E}{9} \left| \Delta(\hat{s}, \hat{u}) \right|^2 \frac{8 (-\hat{t})}{\hat{s}^4 \hat{u}^2}, \quad \rho_T,\]

where

\[\Delta(\hat{s}, \hat{u}) = \left[ \hat{u} e_1 \alpha_s \left( \hat{s} \frac{1}{2} \right) I_M \left( \hat{s} \frac{1}{2} \right) \right] + \hat{s} e_2 \alpha_s \left[ - \hat{u} \frac{1}{2} \right] I_M \left( - \hat{u} \frac{1}{2} \right).\]

\(\text{(4)}\)

In Eqs. (3) and (4) \(\hat{s}, \hat{u}, \hat{t}\) are the Mandelstam invariants of the subprocess, \(e_1\) and \(e_2\) are the charges of the quark \(q_1\) and antiquark \(\underline{q}_2\). The function \(I_M\) in Ref. [7] is defined as

\[I_M(Q^{2\nu}) = \int_0^1 \frac{dx M(x, "Q^{2\nu})}{x (1 - x)},\]

\(\text{(5)}\)

where "\(Q^{2\nu}\) = \(t_g\)" and "\(t_g\)" is the average squared momentum transfer carried by the hard gluon in a given subprocess" equal to \(\hat{s}/2\) and \(-\hat{u}/2\) for two type of relevant Feynman diagrams.

In our work "Single meson photoproduction and IR renormalons" [21] it was demonstrated that namely the HT subprocesses [22] contribute to the photoproduction of the single meson \(\gamma + h \rightarrow M + X\) via the HT mechanism. Within the RC method the differential cross sections of these HT subprocesses were found in our work [21]. The differential cross section of the subprocess \(\gamma q_1 \rightarrow M q_2\), when \(M\) is a pseudoscalar or longitudinally polarized vector meson, is given by the following expression [Eq.(9) in Ref. [21]]:

\[\frac{d\sigma^{HT}(e_1, e_2)}{dt} = \frac{32\pi^2 C_F \alpha_E}{9s^2} \left\{ \frac{e_1^2}{s^2} \left[ \hat{t} I_1^2 - I_1 (I_1 \hat{s} + I_2 \hat{u}) \frac{\hat{u}}{t} + I_2 \frac{\hat{u}^2}{t} \right] - \frac{e_2^2}{\hat{u}^2} \left[ K_1^2 \hat{t} - 2K_1 (K_1 \hat{u} + K_2 \hat{s}) \frac{\hat{s}}{t} + K_2^2 \frac{\hat{s}^2}{t} \right] \right\} - \frac{2e_1 e_2}{s u t} \left[ I_1 K_1 \hat{t}^2 - I_1 (K_2 \hat{s} + K_1 \hat{u}) \hat{s} - K_1 (I_1 \hat{s} + I_2 \hat{u}) \hat{u} \right].\]

\(\text{(6)}\)
For the transversely polarized vector meson we get [Eq.(10) in Ref. [21]]:

\[
\frac{d\hat{\sigma}^{HT}(e_1, e_2)}{dt} = \frac{64\pi^2 C_F \alpha_E}{9\hat{s}^4} \frac{-\hat{t}}{\hat{u}^2} \left[ e_1 \hat{u} I_2 - e_2 \hat{s} K_2 \right]^2.
\]

(7)

In Eqs. (6), (7) the functions \( I_{1(2)} \) and \( K_{1(2)} \) encode an information on the meson distribution amplitude and have the forms:

\[
I_1 = \int_0^1 \int_0^1 dx_1 dx_2 \delta(1 - x_1 - x_2) \alpha_s(\mu_{R1}^2) \Phi_M(x_1, x_2; \mu_{R1}^2), \quad I_2 = \int_0^1 \int_0^1 dx_1 dx_2 \delta(1 - x_1 - x_2) \alpha_s(\mu_{R1}^2) \Phi_M(x_1, x_2; \mu_{F1}^2),
\]

(8)

\[
K_1 = \int_0^1 \int_0^1 dx_1 dx_2 \delta(1 - x_1 - x_2) \alpha_s(\mu_{R2}^2) \Phi_M(x_1, x_2; \mu_{R2}^2), \quad K_2 = \int_0^1 \int_0^1 dx_1 dx_2 \delta(1 - x_1 - x_2) \alpha_s(\mu_{R2}^2) \Phi_M(x_1, x_2; \mu_{F2}^2).
\]

(9)

Here

\[
\mu_{R1}^2 = x_2 \hat{s}, \quad \mu_{F1}^2 = \hat{s}/2; \quad \mu_{R2}^2 = -x_1 \hat{u}, \quad \mu_{F2}^2 = -\hat{u}/2,
\]

\( \hat{s}, \hat{u}, \hat{t} \) are the Mandelstam invariants of the subprocess, \( e_1 \) and \( e_2 \) are the charges of the quarks \( q_1 \) and \( q_2 \), respectively.

Equations (6), (7) are general expressions valid for mesons with both the symmetric (under exchange \( x_1 \leftrightarrow x_2 \)) and non-symmetric distribution amplitudes. In the frozen coupling approximation, i.e., when

\[
\mu_{R1}^2 = \hat{s}/2; \quad \mu_{R2}^2 = -\hat{u}/2,
\]

from Eqs. (6), (7) for mesons with symmetric distributions the Bagger-Gunion results can be obtained. Indeed, in this approximation for the functions \( I_{1(2)} \) and \( K_{1(2)} \) we find

\[
I_0^0 \left( \frac{\hat{s}}{2} \right) = 2I_1^0 \left( \frac{\hat{s}}{2} \right) = \alpha_s \left( \frac{\hat{s}}{2} \right) \int_0^1 dx \frac{\Phi_M(x, \hat{s}/2)}{x(1 - x)},
\]

\[
K_2^0 \left( \frac{-\hat{u}}{2} \right) = 2K_2^0 \left( \frac{-\hat{u}}{2} \right) = \alpha_s \left( \frac{-\hat{u}}{2} \right) \int_0^1 dx \frac{\Phi_M(x, -\hat{u}/2)}{x(1 - x)}.
\]

(10)

Here the superscript "0" indicates that the functions are found in the frozen coupling approximation. Having performed some analytical calculations we get

\[
\frac{d\hat{\sigma}^{HT}(e_1, e_2)}{dt} = \frac{8\pi^2 C_F \alpha_E}{9\hat{s}^4} \frac{1}{\hat{t}} \left[ e_1 \hat{u} I_2^0 - e_2 \hat{s} K_2^0 \right]^2 \left[ \frac{1}{\hat{s}^2} + \frac{1}{\hat{u}^2} \right],
\]

(11)

\[
\frac{d\hat{\sigma}^{HT}(e_1, e_2)}{dt} = \frac{64\pi^2 C_F \alpha_E}{9\hat{s}^4} \frac{-\hat{t}}{\hat{u}^2} \left[ e_1 \hat{u} I_2^0 - e_2 \hat{s} K_2^0 \right]^2.
\]

(12)

These expressions coincide with the Bagger-Gunion results, if to take into account that in Ref. [2] \( e_1 \) and \( e_2 \) are the charges of the quark \( q_1 \) and antiquark \( \bar{q}_2 \), whereas in our work they denote the charges of the quarks \( q_1 \) and \( q_2 \).

In Ref. [1] authors presented results of their calculations of the subprocess \( \gamma q_1 \rightarrow M q_2 \) cross section. In accordance with Eqs.(2.10) and (2.11) of Ref. [1] these cross sections are given by the following expressions:

\[
\frac{d\sigma}{dt}(\gamma q \rightarrow M q) = \frac{8\pi C_F \alpha_E}{9} \left[ D(\hat{s}, \hat{u}) \right]^2 \frac{1}{\hat{s}^2(\hat{-t})} \left[ \frac{1}{\hat{s}^2} + \frac{1}{\hat{u}^2} \right], \quad M = \pi, \rho_L,
\]

(13)

\[
\frac{d\sigma}{dt}(\gamma q \rightarrow M q) = \frac{8\pi C_F \alpha_E}{9} \left[ D(\hat{s}, \hat{u}) \right]^2 \frac{8(\hat{-t})}{\hat{s}^4 \hat{u}^2} \left[ \frac{1}{\hat{s}^2} + \frac{1}{\hat{u}^2} \right], \quad M = \rho_T,
\]
where
\[ D(\hat{t}, \hat{u}) = e_1 \hat{t} \int_0^1 dx_1 \left[ \frac{\alpha_s(Q_1^2)\Phi_M(x, Q_1^2)}{1 - x_1} \right] + e_2 \hat{u} \int_0^1 dx_1 \left[ \frac{\alpha_s(Q_2^2)\Phi_M(x, Q_2^2)}{1 - x_1} \right] \] (14)
and \( Q_1^2 = \hat{s}/2, Q_2^2 = -\hat{u}/2 \).

Now using Eq. (14) [Eq.(2.11) in Ref.[1]] for the function \( D(\hat{s}, \hat{u}) \) we find
\[ D(\hat{s}, \hat{u}) = e_1 \hat{s} \int_0^1 dx_1 \left[ \frac{\alpha_s(\hat{s}/2)\Phi_M(x, \hat{s}/2)}{1 - x_1} \right] + e_2 \hat{u} \int_0^1 dx_1 \left[ \frac{\alpha_s(-\hat{u}/2)\Phi_M(x, -\hat{u}/2)}{1 - x_1} \right] \] (15)

It is evident that Eqs. (13) and (15) are the subprocess cross sections computed in the framework of the frozen coupling approximation. Having compared them with Bagger-Gunion expressions (33) and (44) one can fix errors made in Ref.[1]. Namely, ignoring error/misprint (instead of \( \pi \) should be \( \pi^2 \)) we see that the function \( D(\hat{s}, \hat{u}) \) does not coincide with \( \Delta(\hat{s}, \hat{u}) \) from Eq.(11): dependence on the subprocess invariants \( \hat{s} \) and \( \hat{u} \) are wrong, and normalization is also incorrect, because for the symmetric distribution amplitudes
\[ I_M(\hat{s}/2) = \int_0^1 dx \Phi_M(x, \hat{s}/2) \frac{1}{x(1 - x)} = \int \frac{dx}{(1 - x)} \Phi_M(x, \hat{s}/2) + \int 1 \int dx \Phi_M(x, \hat{s}/2) \frac{1}{1 - x} = 2 \int \int dx \frac{\Phi_M(x, \hat{s}/2)}{1 - x} . \]
In other words, the results of Ref.[1] contain additional numerical factor 1/4.

In Sec. III authors tried to compute the subprocess cross section in the RC method. Explaining the choice of the renormalization and factorization scales made in their work, they write: "As is seen from (2.11) [Eq.(14)], in general, one has to take into account not only dependence of \( \alpha(Q_{1,2}^2) \) on the scale \( Q_{1,2}^2 \), but also an evolution of \( \Phi(x, Q_{1,2}^2) \) with \( Q_{1,2}^2 \). The meson wave function evolves in accordance with a Bethe-Salpeter-type equation. Therefore, it is worth noting that, the renormalization scale (argument of \( \alpha_s \)) should be equal to \( Q_1^2 = -x_1, Q_2^2 = \hat{s}, \) whereas the factorization scale \( Q \) in \( \Phi(x, Q^2) \) is taken independent from \( x \), we take \( Q^2 = p_T^2 \). In the frozen coupling approximation, as is evident also from the Bagger-Gunion expression (14), the choice for the factorization scales is \( \hat{s}/2 \) and \( -\hat{u}/2 \), whereas in the RC method without any reasons authors put \( \mu_{1,2}^2 = \mu_{1,2}^2 = \hat{p}_T^2 \), \( \hat{p}_T^2 \) being the meson \( M \) transverse momentum square. Of course, such choice is not legitimate in the context of RC method. Actually authors did not compute the subprocess cross section using the RC method, but "generalized" their wrong expression obtained in the frozen coupling approximation by this way [the first line in Eq.(3.5) of Ref.[1]]:
\[ D(Q^2) = e_1 \hat{t} \int_0^1 dx \frac{\alpha_s(\lambda Q^2)\Phi_M(x, Q^2)}{1 - x} + e_2 \hat{u} \int_0^1 dx \frac{\alpha_s(\lambda Q^2)\Phi_M(x, Q^2)}{1 - x} , \]
where \( \lambda = 1 - x \) (as it follows later from Eq. (3.7)).

In our work [21] it was proved that within the RC method the cross section of the subprocess \( \gamma q_1 \rightarrow Mq_2 \) is given by Eqs.(3) [Eq.(10)]. Simple generalization of the Bagger-Gunion formula (33) for the pseudoscalar and longitudinally polarized vector meson in order to use it in the RC method leads to wrong results, because in the RC method even for mesons with symmetric distribution amplitudes relations (10) do not hold. The latter conclusion was made in Ref.[21] and proved by explicit calculations, results of which were presented in Eqs. (30) and (32) of this paper.

As is seen from Eq.(10), in Ref. [1] even for the renormalization scales \( \mu_{1,2}^2, \mu_{1,2}^2 \) the choice \( \mu_{1,2}^2 = \mu_{1,2}^2 = (1 - x)p_T^2 \) was made, which contradicts to principles of the RC method and declarations of the authors. Therefore, Eqs.(3.9)-(3.13) are wrong themselves and do not describe situation with the two scales. The Borel transforms presented in Eqs.(3.14)-(3.17) and resummed expressions shown in Eqs.(3.18)-(3.21) are also wrong in the case of two scales. We think that we can stop at this point our analysis of the "investigation" performed in Ref.[1].

Unfortunately, exactly the same errors were made by these authors in their paper published in Phys. Rev. D80, 016003 (2009) and entitled "Infrared renormalons and single meson production in proton-proton collisions" [22]. Despite declarations of the authors the process that they tried to considere is not "single meson production"
\[ p + p \rightarrow M + X, \]
but "prompt photon and meson production"
\[ p + p \rightarrow \gamma + M + X. \]

In Eq.(2.10) of Ref. [22] authors present the diffential cross section of the HT subprocess \( q_1 \bar{q}_2 \rightarrow \pi^+(\pi^-) + \gamma \)
\[ \frac{d\sigma}{d\hat{t}}(\hat{s}, \hat{t}, \hat{u}) = \frac{8\pi^2 CF\alpha_E}{27} \left[ D(\hat{t}, \hat{u}) \right]^2 \left[ \frac{1}{\hat{t}^2} + \frac{1}{\hat{u}^2} \right], \] (17)
where
\[ D(t, \bar{u}) = e_1 t \int_0^1 dx_1 \left[ \frac{\alpha_s(Q_t^2)\Phi_M(x, Q_1^2)}{1-x_1} \right] + e_2 \bar{u} \int_0^1 dx_1 \left[ \frac{\alpha_s(Q_t^2)\Phi_M(x, Q_2^2)}{1-x_1} \right]. \] (18)

Here \( Q_1^2 = -(1-x_1)\bar{u} \) and \( Q_2^2 = -x_1t\).

In Sec.III of Ref. [22] authors tried to compute the subprocess \( q_1 g_2 \to \pi^+(\pi^-) + \gamma \) cross section in the context of the RC method. But instead of calculation of the Feynman diagrams of the subprocess within the RC method they "generalize" the expression obtained in the frozen coupling approach. The correct expression for the subprocess cross section in the RC approach can be obtained from our formula (6) for the pseudoscalar mesons, after exchange of the Mandelstam invariants and by taking into account numerical factors.

In this section the authors write: "As is seen from (2.11) [Eq.(18)], in general, one has to take into account not only dependence of \( \alpha(Q_{1,2}^2) \) on the scale \( \hat{Q}_{1,2}^2 \), but also an evolution of \( \Phi(x, Q_{1,2}^2) \) with \( Q_{1,2}^2 \). The meson wave function evolves in accordance with a Bethe-Salpeter-type equation. Therefore, it is worth noting that, the renormalization scale (argument of \( \alpha_s \)) should be equal to \( Q_t^2 = (x_1 - 1)\bar{u}, Q_2^2 = -tx_1 \), whereas the factorization scale \( Q^2 \) in \( \Phi(x, Q^2) \) is taken independently from \( x \); we take \( Q^2 = p_T^2 \)." Such choice for the factorization scales, i.e. equating them to the pion's transverse momentum square \( p_T^2 \), in the HT subprocess calculations is wrong. The factorization scales had to be chosen equal to \( -\bar{u}/2 \) and \( -t/2 \) in accordance with the standard prescriptions. As is seen from Eq.(3.5) of [22] and the text above of the Eq.(3.7), not only the factorizations scales, but also the renormalization scales were chosen in a way that contradicts prescriptions of the RC method and statements of the authors. Indeed, in the first line of Eq.(3.5) having corrected the misprint (should be \( x \) instead of \( x_1 \)) we find

\[ D(Q^2) = e_1 t \int_0^1 dx \frac{\alpha_s(\lambda Q^2)\Phi_M(x, Q^2)}{1-x} + e_2 \bar{u} \int_0^1 dx \frac{\alpha_s(\lambda Q^2)\Phi_M(x, Q^2)}{1-x}, \] (19)

where \( \lambda = 1-x \). It is evident that in this case \( \mu_{R1}^2 = \mu_{R2}^2 = (1-x)p_T^2 \), and they are not equal to \( Q_1^2 = (x_1 - 1)\bar{u}, \) \( Q_2^2 = -tx_1 \) as write authors in the paper. In other words, in Ref. [22] the factorization and renormalization scales were expressed using the transverse momentum square of the final particle, but not the momentum transfer carried by hard gluon. All computations in Ref [22] were carried out with these incorrect choices for both the factorization and renormalization scales, which led to wrong expressions.

Ignoring for the moment that expression for the function \( D(Q^2) \) [19] in the RC method is wrong and does not describe pion production, let us nevertheless explain the correct approach to computation of such integrals. After correct choices for the scales it should have the form:

\[ D(t, \bar{u}) = e_1 t \int_0^1 dx \frac{\alpha_s(1-x)\bar{u}\Phi_M(x, -\bar{u}/2)}{1-x} + e_2 \bar{u} \int_0^1 dx \frac{\alpha_s(-x/t)\Phi_M(x, -\bar{t}/2)}{1-x}. \] (20)

Having expressed the running couplings \( \alpha_s(1-x)\bar{u} \) and \( \alpha_s(-x/t) \) in terms of \( \alpha_s(-\bar{u}), \alpha_s(-\bar{t}) \) with the aid of the renormalization group equation we find

\[ \alpha(\lambda Q^2) = \frac{\alpha(Q^2)}{1 + \ln \lambda/t}, \]

where \( \alpha = \alpha_s/\pi \) and \( t = 4\pi/\beta_0 \alpha_s(Q^2) \) (there is an error/misprint in corresponding Eq.(3.4) of Ref. [22]). Having used this expression the integrals in Eq. (20) can be recasted into the following forms:

\[ D(t, \bar{u}) = \frac{4\sqrt{3}\pi f_\pi e_1 t}{\beta_0} \int_0^\infty du e^{-t_1 u} A[u, \alpha_s(-\bar{u}/2)] + \frac{4\sqrt{3}\pi f_\pi v_2 \bar{u}}{\beta_0} \int_0^\infty du e^{-t_1 u} B[u, \alpha_s(-\bar{t}/2)]. \] (21)

Here
\[ t_1 = \frac{4\pi}{\beta_0 \alpha_s(-\bar{u})}, \ t_2 = \frac{4\pi}{\beta_0 \alpha_s(-\bar{t})}. \]

and functions \( A[u, \alpha_s(-\bar{u}/2)] \), \( B[u, \alpha_s(-\bar{t}/2)] \) contain IR renormalon poles, number and locations of which depend on the meson (pion) distribution amplitude. These functions depend also on the factorization scales through \( \alpha_s(-\bar{u}/2) \) and \( \alpha_s(-\bar{t}/2) \), respectively. In other words correct treatment leads to expressions with two scales and two integrals, but not to formulas like (3.10)-(3.13) in Ref [22].
with
\[
\frac{t}{\beta_0 \alpha_s(p_T^2)} = 4\pi.
\]

Therefore Eqs.(3.10)-(3.13) in Ref. [22] are wrong. Authors state that they found the relevant Borel transforms (3.10)-(3.13) of the resummed expressions. In the case of correct choice for the scales Eqs.(3.14)-(3.17) are wrong, because as it follows from our analysis and from Eq.(21) the corresponding Borel transforms have two components:
\[
B_1[D(\hat{t}, \hat{u})](u) = A[u, \alpha_s(-\hat{u}/2)], \quad B_2[D(\hat{t}, \hat{u})](u) = B[u, \alpha_s(-\hat{t}/2)].
\]

As a result, the resummed expressions should have also two pieces. Then, for example, for Eq.(21) we get
\[
[D(\hat{t}, \hat{u})]^{res} = \frac{4\sqrt{3}\pi f_e e_1 \hat{t}}{\beta_0} \tilde{A}[Li(\lambda_1^0)/\lambda_1^0, \alpha_s(-\hat{u}/2)] + \frac{4\sqrt{3}\pi f_e e_2 \hat{u}}{\beta_0} \tilde{B}[Li(\lambda_2^0)/\lambda_2^0, \alpha_s(-\hat{t}/2)],
\]

where
\[
\lambda_1 = -\frac{\hat{u}}{\Lambda^2}, \quad \lambda_2 = -\frac{\hat{t}}{\Lambda^2},
\]

and the logarithmic integral \( Li(\lambda) \) for \( \lambda > 1 \) is defined in its principal value
\[
Li(\lambda) = P.V. \int_0^\lambda \frac{dx}{\ln x}.
\]

(in Eq.(3.22) of Ref.[22] there is evident error/misprint). Consequently, Eqs.(3.18)-(3.21) presented in Ref.[22] are wrong as well. Our analysis of the Borel transforms and the resummed expressions presented in Eqs. (20)-(23) are valid also for the first process \( \gamma + \gamma \rightarrow M + X \) [1].

We think that after this analysis the remaining part of the works [1] and [22], namely the leading twist contribution to the process and numerical calculations do not deserve further discussions. Unfortunately, all these fatal errors, wrong statements, numerous misprints were overlooked by referees of such prestigious journal like Phys. Rev. D. Therefore, we decided to inform the HEP community on "methods" used by these authors in their "investigations".

[1] A. I. Ahmadov, Coskun Aydin, E. A. Dadashov and Sh. M. Nagiyev, arXiv:0911.5226v1 [hep-ph] 27 Nov 2009.
[2] E. L. Berger and S. J. Brodsky, Phys. Rev. Lett. 42, 940 (1979); E. L. Berger, Z. Phys. C 4, 289 (1980).
[3] G.P. Lepage and S.J. Brodsky, Phys. Rev. D 22, 2157 (1980).
[4] A.V. Efremov and A.V. Radyushkin, Phys. Lett. 94B, 245 (1980).
[5] A. Duncan and A.H. Mueller, Phys. Rev. D 21, 1636 (1980).
[6] V. N. Baier, A. G. Grozin, Phys. Lett. B96, 181 (1980); S. Gupta, Phys. Rev. D 24, 1169 (1981).
[7] J. A. Bagger, J. F. Gunion, Phys. Rev. D 25, 2287 (1982).
[8] J. A. Hassan, J. F. Storrow, Z. Phys. C 14, 65 (1982).
[9] S. S. Agaev, Phys. Lett. B283, 125 (1992); Z. Phys. C 57, 403 (1993).
[10] S. J. Brodsky, G. P. Lepage, P. B. Mackenzie, Phys. Rev. D 28, 228 (1983).
[11] M. Beneke, Phys. Rep. 317, 1 (1999).
[12] H. Contopanagos and G. Sterman, Nucl. Phys. B419, 77 (1994).
[13] S.S. Agaev, Phys. Lett. B360, 117 (1995) [E: B369, 379 (1996)]; S.S. Agaev, Mod. Phys. Lett. A10, 2009 (1995).
[14] S. S. Agaev, Mod. Phys. Lett. A11, 957; S.S. Agaev, ICTP Preprint IC/95/291, [hep-ph/9611215]
[15] A.I. Karanikas and N.G. Stefanis, Phys. Lett. B504, 225 (2001) [E: Phys. Lett. B636, 330 (2006)].
[16] S.S. Agaev, Mod. Phys. Lett. A13, 2637 (1998).
[17] S.S. Agaev, Phys. Rev. D69, 094010 (2004).
[18] S.S. Agaev, Phys. Rev. D 64, 014007 (2001); S.S. Agaev and N.G. Stefanis, Phys. Rev. D 70, 054020 (2004).
[19] S.S. Agaev and N.G. Stefanis, Phys. Rev. D 70, 054020 (2004).
[20] S.S. Agaev and N.G. Stefanis, Eur. Phys. J. C 32, 507 (2004); S.S. Agaev and M.A. Gomshi Nobary, Eur. Phys. J. C54, 219 (2008).
[21] S.S. Agaev, M. Guidal, B. Pire, Eur. Phys. J. C 37, 457 (2004).
[22] A. I. Ahmadov, Coskun Aydin, Sh. M. Nagiyev, Yilmaz A. Hakan, E. A. Dadashov, Phys. Rev. D 80, 016003 (2009).