On Itô differential equation in rough path theory

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Abstract

The Itô solution can be recovered pathwisely by concatenating a mean of Stratonovich solutions. Itô differential equation in rough path is a transformation between Itô signatures, which are group-valued continuous local martingales. As a consequence, we get a pathwise Itô’s lemma.

1 Introduction

Itô calculus [11], [12] is a transformation between martingales, and lies at the bottom of various mathematical models. However, Itô calculus is not pathwise and lacks stability. Itô calculus is not pathwise, because it respects the probabilistic structure, and problem will occur if one tries to solve differential equation driven by a selected sample path. On the other hand, as demonstrated by Wong and Zakai [23], the solutions of ordinary differential equations, driven by piecewise-linear approximations of Brownian motion, converge uniformly in probability to the Stratonovich solution as the mesh of partitions tends to zero. As a result, the solution of Itô stochastic differential equation is not stable with respect to perturbations of the driving signal, even when the perturbations are very natural.

There have been sustained interests in developing pathwise Itô calculus. Bichteler [1], by using factorization of operators, proved that, the Itô integral can be defined pathwisely outside a null set depending on the integrand function. Based on Bichteler’s approach, Karandikar [13] got similar result by using random time change. Föllmer [5] proposed a deterministic approach to integrate closed one-forms. He proved that, for a semi-martingale X, if the quadratic variation of X converge pathwisely (along a sequence of finite partitions), then the Itô Riemann sums of ∫ f(X) dX also converge pathwisely (along the same sequence of finite partitions). Russo and Vallois [22] developed an almost pathwise approach by regularizing integrals, but their method is not truly pathwise because their convergence is in probability.

Rough path [15], [16], [18], [7] is close in spirit to Föllmer’s approach [5], but a far more systematic methodology which is stable under a large class of approximations, and applies not only to semi-martingales but also to much wider classes of processes [19], [2], [6], [10], [20] etc.. As a natural generalization of classical calculus, rough path is essentially pathwise, and can provide a stable solution which is continuous with respect to the driving signal (in rough path metric).

However, there is some innate non-geometric property of the Itô integral which impedes a direct application of rough path theory. The set of (geometric) rough paths is defined as the closure of continuous bounded variation paths in rough path metric. Thus, Stratonovich integral, as the limit of piecewise-linear approximations, is very natural in rough path. On the other hand, Itô integral generally is not the limit of continuous bounded variation paths. Indeed, for 2-dimensional Brownian motion B = B1e1 + B2e2 on [0, 1], there does not exist a sequence of continuous bounded variation paths Bn = B1n e1 + B2n e2, n ≥ 1, such that Bn−1 converge to B1 (assuming B0n = B0 = 0) and ∫10 Bn u ⊗ dBn u

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by any sequence of $\{B^n\}_n$. The reason is that, for any $B^n$, the Riemann-Stieltjes integral satisfies,

$$\int_0^1 B^n_u \otimes dB^n_u - \frac{1}{2} (B^n_1)^{\otimes 2}$$

$$= \int_0^1 (B^{1,n}_u e_1 + B^{2,n}_u e_2) \otimes d (B^{1,n}_u e_1 + B^{2,n}_u e_2) - \frac{1}{2} (B^{1,n}_1 e_1 + B^{2,n}_1 e_2)^{\otimes 2}$$

$$= \frac{1}{2} \left( \int_0^1 B^{1,n}_u dB^{2,n}_u - \int_0^1 B^{2,n}_u dB^{1,n}_u \right) (e_1 \otimes e_2 - e_2 \otimes e_1).$$

While the Itô integral satisfies,

$$\int_0^1 B_u \otimes dB_u - \frac{1}{2} (B_1)^{\otimes 2}$$

$$= \frac{1}{2} \left( \int_0^1 B^1_u dB^2_u - \int_0^1 B^2_u dB^1_u \right) (e_1 \otimes e_2 - e_2 \otimes e_1) + \frac{1}{2} (e_1 \otimes e_1 + e_2 \otimes e_2).$$

Thus, there is a non-negligible symmetric part $2^{-1} (e_1 \otimes e_1 + e_2 \otimes e_2)$, which can not be approximated by any sequence of $\{B^n\}_n$. To put it in a more abstract way, it is because that, the Itô signature of Brownian motion, as will be defined afterwards, is not a geometric rough process, i.e. it does not take value in the nilpotent Lie group where the normal rough paths take value.

Since Stratonovich integral is well-defined in rough path, one possible approach to defining Itô integral in rough path is to define the Itô integral as Stratonovich integral plus a drift. This idea is adopted in Lejay and Victoir [14], where they combine a geometric $p$-rough path when $p \in [2,3)$, as the product of a weakly geometric $p$-rough path and another continuous path with finite $2^{-1}p$-variation. Similar idea is used in Friz and Victoir [7], where they interpret a geometric rough path with a continuous path with finite $q$-variation for $p^{-1} + q^{-1} > 1$, and get very concrete estimates of solution of rough differential equations driven by $(p,q)$-rough paths. In the more recent [9], by using similar approach as in [8], the authors embed a non-geometric rough path in a geometric rough path, extend the result in [14]. In this manuscript, we will not try to define rough differential equation driven by $p$-rough paths, because there is a canonical choice when $p \in [2,3)$. We interpret a $p$-rough path when $p \in [2,3)$ as a $(p, 2^{-1}p)$-rough path, and focus on interpreting the Itô solution in rough path.

With concrete convergence result, we want to convey the viewpoint that Itô solution can be recovered by taking expectation of Stratonovich solution over a selected noise. By Itô solution, we mean the solution of a rough differential equation driven by the Itô signature (of a $d$-dimensional continuous local martingale $Z$):

$$\mathcal{I}_2 (Z)_t = \left( 1, Z_t - Z_0, \int_0^t (Z_u - Z_0) \otimes dZ_u \right), \quad t \geq 0,$$

comparing with the Stratonovich solution driven by the Stratonovich signature

$$S_2 (Z)_t = \left( 1, Z_t - Z_0, \int_0^t (Z_u - Z_0) \otimes dZ_u \right), \quad t \geq 0.$$
general vector field, and the concatenated Stratonovich solutions converge uniformly to the Itô solution as the mesh of partitions tends to zero.

Moreover, when the vector field is $\text{Lip}(\beta)$ for $\beta > 1$, for almost every sample path of a $d$-dimensional continuous local martingale, the solution of rough differential equation (driven by the Itô signature of the sample path) exists uniquely, and the solution is (the Itô signature of) a sample path of the classical Itô solution. As a consequence, Itô differential equation in rough path is pathwise, and is a transformation between group-valued continuous local martingales (i.e. the Itô signatures), with the first level of its solution coincides almost surely with the solution of classical stochastic differential equation. We also get a pathwise Itô’s lemma, which decomposes the Stratonovich signature as the sum of two rough paths: one is a group-valued continuous local martingale and the other is constructed from continuous bounded variation paths.

We first develop the averaging result for general 2-dimensional $p$-rough process $(\gamma, \int \phi dB)$, $p \in [2, 3)$, with $\gamma$ a deterministic $d$-dimensional path, $\phi$ a deterministic path taking value in $d \times d$ matrices, and $B$ a $d$-dimensional Brownian motion. Then when $\gamma$ is a sample path of a continuous local martingale satisfying $(\gamma) = \int \phi^T \phi dt$, we recover Itô calculus in rough path.

2 Definitions and Notations

Notation 2.1 $(T^{(n)}(\mathbb{R}^d), \| \|)$ For integer $n \geq 1$, we denote $T^{(n)}(\mathbb{R}^d) := 1 \oplus \mathbb{R}^d \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes n}$, and denote $\pi_k$ as the projection of $T^{(n)}(\mathbb{R}^d)$ to $(\mathbb{R}^d)^{\otimes k}$, $k = 0, 1, \ldots, n$. We equip $T^{(n)}(\mathbb{R}^d)$ with the norm

$$
\|g\| := \sum_{k=1}^{n} |\pi_k(g)|^\frac{1}{p}, \forall g \in T^{(n)}(\mathbb{R}^d).
$$

We define product and inverse for $g, h \in T^{(n)}(\mathbb{R}^d)$ as

$$
g \otimes h := \left(1, \pi_1(g) + \pi_1(h), \ldots, \sum_{k=0}^{n} \pi_k(g) \otimes \pi_{n-k}(h)\right),
$$

$$
g^{-1} := \left(1, -\pi_1(g), \ldots, \sum_{k_1+\cdots+k_n=n,1 \leq k_i \leq n} (-1)^{k_1} \pi_{k_1}(g) \otimes \cdots \otimes \pi_{k_n}(g)\right).
$$

Then $(T^{(n)}(\mathbb{R}^d), \| \|)$ is a nilpotent topological group with identity $(1, 0, \ldots, 0)$.

Definition 2.2 ($p$-variation) Suppose $\gamma$ is a continuous path defined on $[0, T]$ taking value in topological group $(G, \| \|)$. For $1 \leq p < \infty$, define the $p$-variation of $\gamma$ as

$$
\|\gamma\|_{p\text{-var},[0,T]} := \left(\sup_{D \subseteq [0,T]} \sum_{k, t_k \in D} \|\gamma_{t_k}^{-1}\gamma_{t_{k+1}}\|^p\right)^\frac{1}{p} < \infty,
$$

where we take supremum over all finite partitions $D = \{t_k\}_{k=0}^n$ satisfying $0 = t_0 < \cdots < t_n = T$.

The definition of $p$-variation at (1) applies to continuous path taking value in Banach spaces, with $\gamma_{t_k}^{-1}\gamma_{t_{k+1}}$ replaced by $\gamma_{t_{k+1}} - \gamma_{t_k}$.

Definition 2.3 ($p$-rough path) Suppose $\gamma$ is a continuous path defined on $[0, T]$ taking value in $(T^{(2)}(\mathbb{R}^d), \| \|)$. We say $\gamma$ is a $p$-rough path for some $p \in [2, 3)$ if $\|\gamma\|_{p\text{-var},[0,T]} < \infty$.

Definition 2.4 ($p$-rough process) Process $X$ on $[0, T]$ is said to be a $p$-rough process for some $p \in [2, 3)$, if $X(\omega)$ is a $p$-rough path for almost every $\omega$. 

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Theorem 2.5 (Lyons) Suppose $\gamma$ is a $p$-rough path for $p \in [2, 3]$. Then for any integer $n \geq 3$, there exists a unique continuous path $\tilde{\gamma}$ taking value in $(T^{(n)}(\mathbb{R}^d), \|\cdot\|)$ satisfying $\pi_k(\tilde{\gamma}) = \pi_k(\gamma)$, $k = 1, 2,$ and $\|\gamma\|_{p-\text{var},[0,T]} < \infty$.

Notation 2.6 ($S_n(\gamma)$) Suppose $\gamma$ is a $p$-rough path for $p \in [2, 3]$. For $n \geq 3$, denote $S_n(\gamma)$ as the enhancement of $\gamma$ to the path taking value in $T^{(n)}(\mathbb{R}^d)$ as in Theorem 2.5. For $n = 1, 2$, denote $S_1(\gamma) := (1, \pi_1(\gamma))$ and $S_2(\gamma) := \gamma$.

Definition 2.7 ($\text{Lip}(\beta)$ vector field) $f : \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$ is said to be $\text{Lip}(\beta)$ for $\beta > 0$, if $f$ is $[\beta]$-times Fréchet differentiable ($[\beta]$ is the largest integer which is strictly less than $\beta$) with derivatives $\{D^k f\}_{k=1}^{[\beta]}$ and $|f|_{\text{Lip}(\beta)} := \max_{k=0, \ldots, [\beta]} \|D^k f\|_{\text{var}} \vee \|D^k f\|_{(\beta-[\beta])\text{-Fréchet}} < \infty$.

Based on [14], any $p$-rough path, $p \in [2, 3]$, can be interpreted as the product of a weak geometric $p$-rough path and another continuous path with finite $2^{-1}p$-variation. We will use this equivalence and define solution of rough differential equations driven by $p$-rough paths, $p \in [2, 3]$, in the sense of $(p, 2^{-1}p)$-rough path in [7].

Notation 2.8 Suppose $\gamma : [0, T] \to (T^{(2)}(\mathbb{R}^d), \|\cdot\|)$ is a $p$-rough path for $p \in [2, 3]$. Then we denote $\gamma' := (\gamma^1, \gamma^2)$ with $\gamma^1 : [0, T] \to (T^{(2)}(\mathbb{R}^d), \|\cdot\|)$ and $\gamma^2 : [0, T] \to (\mathbb{R}^d)^{\otimes 2}$ continuous paths defined as

$$
\gamma^1_t := 1 + \pi_1(\gamma_t) + \text{Anti}(\pi_2(\gamma_t)) + \frac{1}{2} (\pi_1(\gamma_t))^{\otimes 2}, \quad t \in [0, T],
$$

and

$$
\gamma^2_t := \text{Sym}(\pi_2(\gamma_t)) - \frac{1}{2} (\pi_1(\gamma_t))^{\otimes 2}, \quad t \in [0, T],
$$

where $\text{Anti}(\cdot)$ denotes the projection of $(\mathbb{R}^d)^{\otimes 2}$ to span $\{e_i \otimes e_j - e_j \otimes e_i\}_{1 \leq i < j \leq d}$ and $\text{Sym}(\cdot)$ denotes the projection of $(\mathbb{R}^d)^{\otimes 2}$ to span $\{e_i \otimes e_j + e_j \otimes e_i\}_{1 \leq i < j \leq d}$.

Then based on Definition 12.2 in [7], we define solution of rough differential equation driven by $p$-rough path for $p \in [2, 3]$.

Definition 2.9 (solution of RDE) Suppose $\gamma = (\gamma^1, \gamma^2)$ is a $p$-rough path for $p \in [2, 3]$, and $f : \mathbb{R}^e \to L(\mathbb{R}^d, \mathbb{R}^e)$ is $\text{Lip}(\beta)$ for some $\beta \geq 1$. Then continuous path $Y : [0, T] \to (T^{(n)}(\mathbb{R}^e), \|\cdot\|)$ is said to be a solution of the rough differential equation

$$
dY = f(\pi_1(Y)) \, d\gamma, \quad Y_0 = \xi \in T^{(n)}(\mathbb{R}^e),
$$

if there exists a sequence $\{\gamma^{1,m}, \gamma^{2,m}\}_m$ in $C^{1-\text{var}}([0, T], \mathbb{R}^d) \times C^{1-\text{var}}([0, T], (\mathbb{R}^d)^{\otimes 2})$ satisfying

$$
\sup_m \left(\|\gamma^{1,m}\|_{p-\text{var},[0,T]} + \|\gamma^{2,m}\|_{q-\text{var},[0,T]}\right) < \infty,
$$

and the solution of the ordinary differential equations

$$
dy^m = f(y^m) \, d\gamma^{1,m} + (Df \, f)(y^m) \, d\gamma^{2,m}, \quad y^m_0 = \pi_1(\xi) \in \mathbb{R}^e,
$$

converge to $Y$ uniformly:

$$
\lim_{m \to \infty} \sup_{1 \leq k \leq n \atop 0 \leq t \leq T} \left|\pi_k \left(\xi \otimes S_n(y^m)_{0,t} - Y_t\right)\right| = 0.
$$
Then based on the uniqueness of enhancement (i.e. Theorem [2.5]), \( Y \) is a solution of (2) when \( n \geq 2 \), if and only if \( Y_t = \xi \otimes S_n (\bar{Y})_{0,t} \) with \( \bar{Y} \) a solution to the rough differential equation
\[
d\bar{Y} = f (\pi_1 (\bar{Y})) d\gamma, \quad \bar{Y}_0 = (1, \pi_1 (\xi), 0) \in T^{(2)} (\mathbb{R}^e).
\]

Based on Theorem 12.6 and Theorem 12.11 in [7], we have

**Theorem 2.10 (existence and uniqueness)** The solution of (2) exists when \( f \) is \( \text{Lip} (\beta) \) for \( \beta > p - 1 \), and is unique when \( \beta \geq p \).

**Definition 2.11 (integration of 1-form)** Suppose \( \gamma : [0, T] \rightarrow (T^{(2)} (\mathbb{R}^d), \| \cdot \|) \) is a \( p \)-rough path for some \( p \in [2, 3) \), and \( f : \mathbb{R}^d \rightarrow L (\mathbb{R}^d, \mathbb{R}^e) \) is \( \text{Lip} (\beta) \) for some \( \beta \geq 1 \). Then continuous path \( Y : [0, T] \rightarrow (T^{(2)} (\mathbb{R}^e), \| \cdot \|) \) is said to be the rough integral of \( f \) against \( \gamma \) and denoted as
\[
Y_t = \int_0^t f (\pi_1 (\gamma_u)) d\gamma_u, \quad t \in [0, T],
\]
if there exists a continuous path \( \Gamma : [0, T] \rightarrow (T^{(2)} [\mathbb{R}^{d+\epsilon}], \| \cdot \|) \) satisfying \( \pi_{T^{(2)} [\mathbb{R}^e]} (\Gamma) = \gamma, \pi_{T^{(2)} (\mathbb{R}^e)} (\Gamma) = Y \), and \( \Gamma \) is a solution to the rough differential equation:
\[
d\Gamma = (1, f (\pi_{\mathbb{R}^d} (\Gamma))) d\gamma, \quad \Gamma_0 = (1, (\pi_1 (\gamma_0), 0)), 0) \in T^{(2)} (\mathbb{R}^{d+\epsilon}).
\]

### 3 Formulation and Results

As mentioned in the introduction, we want to recover Itô solution by concatenating a mean of Stratonovich solutions. The idea is elegant, but the concrete formulation needs some care.

Suppose \( \gamma : [0, T] \rightarrow (T^{(2)} (\mathbb{R}^d), \| \cdot \|) \) is a fixed \( p \)-rough path for some \( p \in [2, 3) \) and \( M \) a continuous martingale taking value in \( \mathbb{R}^d \). Further assume that
\[
S_n (\gamma + M)_t := \left( 1, \pi_1 (\gamma_t) + M_t, \pi_2 (\gamma_t) + \int_0^t \pi_1 (\gamma_u) \otimes dM_u + \int_0^t M_u \otimes d\pi_1 (\gamma_u) + \int_0^t M_u \otimes dM_u \right)
\]
is a \( p \)-rough process for some \( p \in [2, 3) \). We want to get the mathematical expression of the discrete Itô increment on small interval \([s, t]\).

Suppose \( f : \mathbb{R}^e \rightarrow L (\mathbb{R}^d, \mathbb{R}^e) \) is \( \text{Lip} (\beta) \) for \( \beta \geq p \) and \( \xi \in T^{(n)} (\mathbb{R}^e) \). Suppose \( z^i \) is the solution to the rough differential equations:
\[
\begin{align*}
dz^1 &= f (\pi_1 (z^1)) d\gamma, \quad z^1_s = \xi \in T^{(n)} (\mathbb{R}^e), \\
dz^2 &= f (\pi_1 (z^2)) dS_2 (\gamma + M), \quad z^2_s = \xi \in T^{(n)} (\mathbb{R}^e).
\end{align*}
\]

Then by using uniqueness of enhancement (i.e. Theorem [2.5]), it can be checked that, if denote \( y^1, y^2 \) as the solution to the rough differential equation:
\[
\begin{align*}
dy^1 &= f (\pi_1 (y^1)) d\gamma, \quad y^1_s = (1, \pi_1 (\xi), 0) \in T^{(2)} (\mathbb{R}^e), \\
dy^2 &= f (\pi_1 (y^2)) dS_2 (\gamma + M), \quad y^2_s = (1, \pi_1 (\xi), 0) \in T^{(2)} (\mathbb{R}^e).
\end{align*}
\]

then (with \( S_n (\cdot) \) in Notation [2.10]) \( z^i = \xi \otimes S_n (y^i)_{s,u}, \quad i = 1, 2 \), and \( z^i \) is the solution to the rough differential equation driven by \( y^i \):
\[
dz^i = z^i \otimes dy^i, \quad z^i_s = \xi \in T^{(n)} (\mathbb{R}^e).
\]

Suppose \( y^i \) are known, we want to modify the initial value of \( z^2 \) at time \( s \) to \( \delta^{s,t} \) in such a way that, the solution of the rough differential equation
\[
dz^2 = z^2 \otimes dy^2, \quad z^2_s = \delta^{s,t} \in T^{(n)} (\mathbb{R}^e), \quad (3)
\]
satisfies \((z_{s,t}^γ)^{-1} \otimes z^1_t)\)

\[ E\left( z_t^2 \right) = z_t^1 \quad \text{i.e.} \quad E\left( \delta^{s,t} \otimes z_{s,t}^2 \right) = \xi \otimes z_{s,t}^1. \]

We let

\[ \delta^{s,t} := \xi \otimes z_{s,t}^1 \otimes E\left( z_{s,t}^2 \right)^{-1} \quad (\gamma \text{ is a fixed path}), \]

and define the discrete Itô increment on \([s,t]\) as

\[ \xi \otimes z_{s,t}^1 \otimes E\left( z_{s,t}^2 \right)^{-1} \otimes z_{s,t}^1, \quad (4) \]

which is the value at time \(t\) of the solution to the rough differential equation:

\[ dz = z \otimes dy^1, \quad z_s = \delta^{s,t} \in T^{(n)}(\mathbb{R}^e). \quad (5) \]

For each sample path \(\gamma\) of the underlying and any interval \([s,t]\), we have two solutions: \(y^1\) driven by the underlying \(\gamma\) and \(y^2\) driven by the underlying with noise \(S_2(\gamma + M)\). We want to discount \(z_t^2\) in such a way that one is expected to get the real value \(z_t^1\). More specifically, when the underlying is polluted with noise, we consider \((\ref{eq:discreteItô})\) instead of \((\ref{eq:forward-backward-forward-equation})\), and we have \(E(z_t^2) = z_t^1\). We will concatenate the discrete Itô increments in the form of \((\ref{eq:discreteItô})\), let the mesh of partitions tends to zero, and recover the Itô solution.

One might be tempted to replace the discrete increment \((\ref{eq:discreteItô})\) by the expectation of the solution of forward-backward-forward equation, which, however, does not work, even on the first level.

### 3.1 Averaging Stratonovich solutions

Our averaging process can be applied when \(\gamma\) is a fixed \(p\)-rough path, \(p \in [2, 3]\), and \(M = \int \phi dB\) with \(\phi\) a fixed path taking value in \(d \times d\) matrices and \(B\) a \(d\)-dimensional Brownian motion. When \(\gamma = S_2(Z)\) is the Stratonovich signature of a sample path of continuous local martingale \(Z\), by setting \(\phi = (Z)^{\frac{1}{2}}\) we can recover the Itô solution in rough path.

#### 3.1.1 Rough path underlying

**Definition 3.1 (perturbed rough path)** Suppose \(\gamma : [0, T] \to (T^{(2)}(\mathbb{R}^d), \|\cdot\|)\) is a fixed \(p\)-rough path on \([0, T]\) for some \(p \in [2, 3]\), \(\phi\) is a fixed path defined on \([0, T]\) taking value in \(d \times d\) matrices satisfying \(\max_{1 \leq i,j \leq d} \int_0^T (\phi_u^{ij})^2 du < \infty\), and \(B\) a \(d\)-dimensional Brownian motion. Define continuous \(d\)-dimensional martingale \(M\) as the Itô integral:

\[ M_t := \int_0^t \phi_u dB_u, \quad t \in [0, T]. \quad (6) \]

We assume that \(\gamma + S_2(M)\) \((S_2(M)\) denotes the step-2 Stratonovich signature of \(M\)) can be enhanced to a \(p\)-rough process \(S_2(\gamma + M) : [0, T] \to (T^{(2)}(\mathbb{R}^d), \|\cdot\|)\) with the explicit expression \((S_2(\gamma + M))_{s,t} := S_2(\gamma + M)^{-1} \otimes S_2(\gamma + M)_{t}, \gamma_{s,t} := \gamma_s^{-1} \otimes \gamma_t)\)

\[ S_2(\gamma + M)_{s,t} = (1, \pi_1(\gamma_{s,t}) + M_t - M_s, \pi_2(\gamma_{s,t}) + \int_{s \leq u_1 < u_2 \leq t} \circ dM_{u_1} \otimes \circ dM_{u_2} + R(\gamma, M)^{s,t}, \forall 0 \leq s \leq t \leq T, \]

and we assume that

\[ E\left( R(\gamma, M)^{s,t} \right) = 0, \forall 0 \leq s \leq t \leq T. \quad (7) \]

Since \(\gamma\) is fixed, the condition \((\ref{eq:boundary-condition})\) is satisfied e.g. when the cross integral \(R(\gamma, M)\) is defined in classical Itô sense or Stratonovich sense, or when \(R(\gamma, M)\) is defined as the \(L^1\) limit of piecewise-linear approximations.
The proof of Theorem 3.3 starts from page 11.

Remark 3.4 Based on the proof of Theorem 3.3, $E\left(\|S_2(\gamma + M)\|_p^{1/p/\text{var},[0,T]}\right) < \infty$ for some $q > p$ is sufficient for the convergence of the first level in (12).

As a specific example where $\gamma$ is not a sample path of a martingale, suppose $(X,B)$ is a 2$d$-dimensional continuous Gaussian process with independent components, and $B$ is a $d$-dimensional Brownian motion. Further assume that the covariance function of $(X,B)$ has finite $p$-variation for some $p \in [1,\frac{d}{2}]$ (see Section 15.3.2 [7]). Then based on Thm 15.33 [7], our Theorem 3.3 applies e.g. to $(\gamma,B)$ with $\gamma = (\gamma_1,\gamma_2)$ (Notation 2.8), where $\gamma_1$ is the step-2 Stratonovich signature of a sample path of fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$ and $\gamma_2$ is a fixed continuous path with finite $2^{-1}p$-variation for some $p \in [2,3)$.

### 3.1.2 Itô signature and martingale underlying

**Definition 3.5 (Itô signature $I_n(Z)$)** Suppose $Z$ is a continuous local martingale on $[0,\infty)$ taking value in $\mathbb{R}^d$. For integer $n \geq 1$, denote $I_n(Z) : [0,\infty) \to (T^{(n)}(\mathbb{R}^d),\|\cdot\|)$ as

\[
I_n(Z)_t := \left(1, Z_t - Z_0, \int_{0 < u_1 < u_2 < t} dZ_{u_2} \otimes dZ_{u_2}, \ldots, \int_{0 < u_1 < \cdots < u_n < t} dZ_{u_1} \otimes \cdots \otimes dZ_{u_n} \right), \forall t \in [0,\infty).
\]
Then we study the (pathwise and probabilistic) regularity of the Itô signature.

**Theorem 3.6** Suppose $Z$ is a continuous local martingale taking value in $\mathbb{R}^d$. Then $\mathcal{I}_n(Z) : [0, \infty) \to (T^{(n)}(\mathbb{R}^d), \|\cdot\|)$ is a group-valued continuous local martingale which satisfies, for any $T > 0$,

$$\|\mathcal{I}_n(Z)\|_{p\text{-var}, [0, T]} < \infty, \text{ a.s., } \forall p > 2.$$ 

Moreover, for any integer $d \geq 1$, any integer $n \geq 1$, any moderate function $F$, and any $p > 2$, there exists constant $C(d, n, F, p)$ such that

$$C^{-1} E\left( F\left(\|\mathcal{I}_n(Z)\|_{p\text{-var}, [0, \infty)}\right)\right) \leq E\left( F\left(\|\mathcal{I}_n(Z)\|_{p\text{-var}, [0, T]}\right)\right) \leq CE\left( F\left(\|\mathcal{I}_n(Z)\|_{\infty}\right)\right), \quad (14)$$

holds for any continuous local martingale $Z$ taking value in $\mathbb{R}^d$ starting from 0.

**Proof.** That $\mathcal{I}_n(Z)$ is a group-valued continuous local martingale can be proved e.g. by taking stopping times $\tau_n := \tau_n \wedge \inf\{t \mid |Z_t| \geq n\}$ with $\{\tau_n\}_n$ the stopping times of $Z$. Based on Lemma 4.5 on p. 17, we have, (S, (i)) in Notation 2.6

$$S_n(\mathcal{I}_2(Z)) = \mathcal{I}_n(Z), \forall t \in [0, \infty), \forall n \geq 1, \text{ a.s.}$$

Then based on Theorem 3.7 in [16] and Theorem 14.9 [7], we have, for any $T > 0$ and any $p > 2$,

$$\|\mathcal{I}_n(Z)\|_{p\text{-var}, [0, T]} \leq C_{p, n} \|\mathcal{I}_2(Z)\|_{p\text{-var}, [0, T]}$$

$$\leq C_{p, n} \left(\|S_2(Z)\|_{p\text{-var}, [0, T]} + \|\langle Z\rangle\|_{1\text{-var}, [0, T]}^{\frac{3}{2}}\right) < \infty, \text{ a.s.}$$

Based on Theorem 3.7 in [16] and Theorem 14.12 in [7], we have (14) holds. \(\blacksquare\)

When $\gamma$ (in Definition 3.1) is a sample path of a continuous martingale, by choosing the right noise, we can recover Itô solution (i.e. solution of rough differential equation driven by $\mathcal{I}_2(Z)$ defined at (13)). The following definition gives the explicit construction of the noise.

**Definition 3.7** Suppose $Z$ is a continuous $d$-dimensional martingale in $L^2$ on $[0, T]$, and there exists a $d \times d$-matrices valued adapted process $\psi$ in $L^2$ on $[0, T]$ such that

$$\langle Z\rangle_t = \int_0^t \psi_s^T \psi_s ds, \forall t \in [0, T], \text{ a.s..}$$

Suppose $B$ is a $d$-dimensional Brownian motion, independent from $Z$. For a fixed sample path of $Z$, define process $\tilde{Z}$ as the Itô integral:

$$\tilde{Z}_t := \int_0^t \psi_s dB_s, \, t \in [0, T]. \quad (15)$$

The Corollary below follows from Theorem 3.3 and Remark 3.4 with proof on p. 16

**Corollary 3.8** Suppose $Z$ is a continuous $d$-dimensional martingale on $[0, T]$ in $L^{2n+\epsilon}$ for some $\epsilon > 0$ and integer $n \geq 1$, and $\tilde{Z}$ as defined at (15). Suppose $f : \mathbb{R}^\epsilon \to L(\mathbb{R}^d, \mathbb{R}^\epsilon)$ is Lip($\beta$) for $\beta > 2$. Denote $Y$ as the solution to the rough differential equation: ($\mathcal{I}_2(Z)$ defined at (13))

$$dY = f(\pi_1(Y)) \, d\mathcal{I}_2(Z), \, Y_0 = \xi \in T^{(n)}(\mathbb{R}^\epsilon).$$

Then for almost every sample path of $Z$, $y^{n,D}(S_2(Z), \tilde{Z})$ (defined at (8) with $S_2(Z)$ denotes the step-2 Stratonovich signature of $Z$) satisfies

$$\lim_{\|D\| \to 0} \max_{1 \leq k \leq n} \sup_{0 \leq t \leq T} \left| \pi_k \left( y^{n,D}(S_2(Z), \tilde{Z}) \right) - \pi_k (Y_t) \right| = 0.$$
3.2 Relation with Itô stochastic differential equation

Then we investigate the pathwise property of the Itô solution (i.e. solution of rough differential equation driven by the Itô signature $I_2(Z)$ defined at (13)).

**Theorem 3.9 (Relation with Itô SDE)** Suppose $Z$ is a continuous local martingale on $[0, \infty)$ taking value in $\mathbb{R}^d$. Suppose $f : \mathbb{R}^c \to L(\mathbb{R}^d, \mathbb{R}^e)$ is Lip($\beta$) for $\beta > 1$. Then for almost every sample path of $Z$, the solution of the rough differential equation

\[ dY = f(\pi_1(Y)) dI_2(Z), \quad Y_0 = \xi \in T^{(n)}(\mathbb{R}^c), \tag{16} \]

exists uniquely, and the solution $Y$ has the explicit expression:

\[ Y_t = \xi \otimes I_n(y)_{0,t}, \quad \forall t \in [0, \infty), \forall n \geq 1, \]

with $I_n(y)$ defined at (13) and $y$ denotes the unique strong continuous solution of the Itô stochastic differential equation

\[ dy = f(y) dZ, \quad y_0 = \pi_1(\xi) \in \mathbb{R}^c. \tag{17} \]

The proof of Theorem 3.9 starts from page 19.

When the vector field $f$ in Theorem 3.9 is Lip($\beta$) for $\beta > 2$, the solution $Y$ in (16) is continuous w.r.t. the driving rough path (Thm 12.10 [7]).

Based on Thm 17.3 [7], the authors identified a relationship between the classical Itô solution and the (first level) rough Itô solution when the vector field is Lip($\beta$) for $\beta > 2$. In Prop 4.3 [3], the author proved that, when the vector field is Lip($\beta$) for $\beta > 1$, the (first level) solution of (16) driven by the Itô signature of Brownian motion exists uniquely a.s.. Based on Theorem 3.9, the result in [3] is applicable to the whole rough path solution and to all continuous local martingales.

The Corollary below follows from Definition 2.11 and Theorem 3.9.

**Corollary 3.10 (Integration of one-forms)** Suppose $Z$ is a continuous local martingale on $[0, \infty)$ taking value in $\mathbb{R}^d$, and $f : \mathbb{R}^c \to L(\mathbb{R}^d, \mathbb{R}^e)$ is Lip($\beta$) for $\beta > 1$. Then for almost every sample path of $Z$, the rough integral has the explicit expression:

\[ S_n \left( \int_0^t f(Z) dI_2(Z) \right)_t = I_n(y)_t, \quad t \in [0, \infty), \forall n \geq 1, \]

with $y$ denotes the classical Itô integral

\[ y_t := \int_0^t f(Z_u) dZ_u, \quad t \in [0, \infty). \]

When vector field $f$ in Corollary 3.10 is Lip($\beta$) for $\beta > 1$, the rough integral $\int_0^t f(Z) dI_2(Z)$ is continuous w.r.t. the driving rough path (Thm 12.10 [16]).

Based on Corollary 3.10 we have a pathwise Itô’s lemma, which decomposes the Stratonovich signature as the sum of two rough paths: one is a group-valued continuous local martingale and the other is constructed from continuous bounded variation paths.

Theorem 3.11 below follows from Lyons and Qian [17] (p244), only that the rough integral $t \mapsto \int_0^t Df(Z_u) dI_2(Z)_u$ has the explicit expression as seen in Corollary 3.10.

**Theorem 3.11 (Itô’s lemma)** Suppose $Z$ is a continuous local martingale on $[0, \infty)$ taking value in $\mathbb{R}^d$, and suppose $f : \mathbb{R}^c \to \mathbb{R}^e$ is Lip($\beta$) for $\beta > 2$. Denote $S_2(f(Z))$ and $S_2(Z)$ as the step-2 Stratonovich signature of $f(Z)$ and $Z$. Then the rough integral equation holds for almost every sample path of $Z$:

\[ S_2(f(Z))_{0,t} = \int_0^t Df(Z_u) dS_2(Z)_u = \int_0^t (Df(Z_u) dI_2(Z)_u + dH_u), \quad \forall t \in [0, \infty), \tag{18} \]
where $H : [0, \infty) \to \left( T^{(2)}(\mathbb{R}^d), \|\cdot\| \right)$ is defined as

$$H_t := \left( 1, x_1^t, \int_0^t x_u^1 \otimes dx_u^1 + x_u^2 \right), \quad t \in [0, \infty),$$

with $x_t^1 := \frac{1}{2} \int_0^t (D^2 f)(Z_u) \, d(Z)_u$ and $x_t^2 := \frac{1}{2} \int_0^t (D f)(Z_u) \otimes (D f)(Z_u) \, d(Z)_u$.

and $\int_0^t (D f)(Z_u) \, dI_2(Z)_u + dH_u) : [0, \infty) \to \left( T^{(2)}(\mathbb{R}^d), \|\cdot\| \right)$ is defined as

$$\begin{align*}
\pi_1 \left( \int_0^t (D f)(Z_u) \, dI_2(Z)_u + dH_u \right) & := \pi_1 \left( \int_0^t D f(Z_u) \, dI_2(Z)_u \right) + \pi_1 \left( H_t \right), \\
\pi_2 \left( \int_0^t (D f)(Z_u) \, dI_2(Z)_u + dH_u \right) & := \pi_2 \left( \int_0^t D f(Z_u) \, dI_2(Z)_u \right) + \pi_2 \left( H_t \right)
\end{align*}$$

$$+ \int_0^t \pi_1 \left( \int_0^t D f(Z) \, dI_2(Z) \right) \otimes d\pi_1 (H_u)$$

$$+ \int_0^t \pi_1 (H_u) \otimes d\pi_1 \left( \int_0^u D f(Z) \, dI_2(Z) \right),$$

where the cross integrals between $\pi_1 (\int D f(Z) \, dI_2(Z))$ and $\pi_1 (H)$ are defined as Young integrals.

4 Proofs

Our constants may implicitly depend on dimensions ($d$ and $e$). We specify the dependence on other constants (e.g. $C_p$), but the exact value of constants may change from line to line.

4.1 Results from rough path

We state the results in rough path theory that will be used in our proofs.

Suppose $\gamma$ is a $p$-rough path for $p \in [2, 3)$, denote $S_n(\gamma)$ as in Notation 2.6 (on page 4). Based on Thm 3.7 in [10], for any integer $k \geq 1$, there exists constant $C_{p,k}$ such that,

$$\left| \pi_k \left( S_n(\gamma)_{s,t} \right) \right| \leq C_{p,k} \|\gamma\|_{p-var,[s,t]}^k, \quad \forall 0 \leq s \leq t \leq T. \quad (19)$$

As a consequence, $S_n(\gamma) : [0, T] \to \left( T^{(n)}(\mathbb{R}^d), \sum_{n=1}^\infty |\cdot|^{\frac{1}{n}} \right)$ satisfies

$$\|S_n(\gamma)\|_{p-var,[s,t]} \leq C_{p,n} \|\gamma\|_{p-var,[s,t]}, \quad \forall 0 \leq s \leq t \leq T. \quad (20)$$

Moreover, for any integer $n \geq 2$ and any finite partition $D = \{t_k\}_{k=0}^K$ of $[s, t] \subseteq [0, T]$, we have (with $0 \in (\mathbb{R}^d)^{\otimes (n+1)}$)

$$\left| \pi_{n+1} \left( \left( S_n(\gamma)_{t_0,t} \oplus 0 \right) \otimes \cdots \otimes \left( S_n(\gamma)_{t_K,t} \oplus 0 \right) \right) - \pi_{n+1} \left( S_{n+1}(\gamma)_{s,t} \right) \right| \leq C_{p,n} \sum_{k=0}^{K-1} \|\gamma\|_{p-var,[t_k,t_{k+1}]}^{n+1}. \quad (21)$$

Based on Prop 14.9 [7], we have

**Theorem 4.1** Suppose $M : [0, \infty) \to \mathbb{R}^d$ is a continuous local martingale. Then the step-2 Stratonovich signature of $M$ is a geometric $p$-rough process on $[0, T]$ for any $p > 2$ and any $T > 0$. 

Suppose $\gamma = (\gamma^1, \gamma^2)$ (Notation 2.8 on p.141) is a $p$-rough path for some $p \in [2, 3)$ on $[0, T]$ taking value in $(T^{(2)}(\mathbb{R}^d), \|\cdot\|)$. Then it is clear (or based on (13)) that,

$$\|\gamma\|_{p\text{-var},[s,t]}^p \leq \|\gamma^1\|_{p\text{-var},[s,t]}^p + \|\gamma^2\|_{2^{-1}p\text{-var},[s,t]}^p \leq C_d \|\gamma\|_{p\text{-var},[s,t]}^p, \forall 0 \leq s \leq t \leq T.$$ 

Based on Theorem 12.6 in [7] and Corollary 12.8 in [7], we have: (the second level estimation can be obtained by considering the rough differential equation of the signature of the solution)

**Theorem 4.2** Suppose $\gamma$ is a $p$-rough path for some $p \in [2, 3)$ on $[0, T]$ taking value in $(T^{(2)}(\mathbb{R}^d), \|\cdot\|)$ and $f : \mathbb{R}^x \rightarrow L(\mathbb{R}^d, \mathbb{R}^e)$ is Lip($\beta$) for $\beta > p - 1, 2$. If for integer $n \geq 2$, $Y : [0, T] \rightarrow (T^{(n)}(\mathbb{R}^e), \|\cdot\|)$ is a solution to the rough differential equation

$$dY = f(\pi_1(Y)) d\gamma, \ Y_0 = \xi \in T^{(n)}(\mathbb{R}^e),$$

then for any $0 \leq s \leq t \leq T$, we have $(Y_{s,t} := Y_s^{-1} \otimes Y_t, \gamma_{s,t} := \gamma_s^{-1} \otimes \gamma_t)$

$$\|Y\|_{p\text{-var},[s,t]} \leq C_{p,\beta, f, n} \left(\|\gamma\|_{p\text{-var},[s,t]} \vee \|\gamma\|_{p\text{-var},[s,t]}^{\beta+1}\right),$$

$$\left|\pi_1(Y_{s,t}) - f(\pi_1(Y_s)) \pi_1(\gamma_{s,t}) - (Df f)(\pi_1(Y_s)) \pi_2(\gamma_{s,t})\right| \leq C_{p,\beta, f} \|\gamma\|_{p\text{-var},[s,t]}^{\beta+1},$$

$$\left|\pi_2(Y_{s,t}) - f(\pi_1(Y_s)) \otimes f(\pi_1(Y_s)) \pi_2(\gamma_{s,t})\right| \leq C_{p,\beta, f} \|\gamma\|_{p\text{-var},[s,t]}^{\beta+1} \vee \|\gamma\|_{p\text{-var},[s,t]}^{2p}.$$ 

The Theorem below follows from Theorem 12.10 in [7] and Theorem 3.1.3 in [18].

**Theorem 4.3** Suppose $\gamma$ is a $p$-rough path for some $p \in [2, 3)$ on $[0, T]$ taking value in $(T^{(2)}(\mathbb{R}^d), \|\cdot\|)$, and $f : \mathbb{R}^e \rightarrow L(\mathbb{R}^d, \mathbb{R}^e)$ is Lip($\beta$) for $\beta > p$. Suppose $Y^i$, $i = 1, 2$, is the solution to the rough differential equation

$$dY^i = f(\pi_1(Y^i)) d\gamma, \ Y^i_0 = \xi^i \in T^{(n)}(\mathbb{R}^e).$$

Then $(Y^i_{s,t} := (Y^i_s)^{-1} \otimes Y^i_t)$

$$\max_{1 \leq k \leq n} \sup_{0 \leq s \leq t \leq T} \left|\pi_k(Y^1_{s,t}) - \pi_k(Y^2_{s,t})\right| \leq C_{p,\beta, f} \|\gamma^1(\xi^1) - \pi_1(\xi^2)\| \exp \left(C_{p,\beta, f} \left(\|\gamma\|_{p\text{-var},[0,T]}\right)\right).$$

**4.2 Rough path perturbed by martingale**

**4.2.1 Rough Path Underlying**

**Proof of Theorem 3.3** Define $\omega_i : \{(s, t)|0 \leq s \leq t \leq T\} \rightarrow \mathbb{R}^+$, $i = 1, 2$, as, for any $0 \leq s \leq t \leq T$,

$$\omega_1(s, t) := \|S_2(\gamma + M)\|_{p\text{-var},[s,t]},$$

$$\omega_2(s, t) := \|\gamma\|_{p\text{-var},[s,t]} + \|(M)\|_{\text{1-var},[s,t]}^{\beta}.$$ 

Then $\omega_2$ is deterministic and $\omega_2(0, T) < \infty$. Based on our assumption (10) on p.7, for some integer $n \geq 2$,

$$E(\omega_1(0, T)^n) < \infty. \quad (22)$$

Denote $\pi_f(s, \eta, I_2(\gamma, M))$ as the solution to the rough differential equation:

$$dY = f(\pi_1(Y)) dI_2(\gamma, M), \ y_s = \eta \in T^{(n)}(\mathbb{R}^e).$$
For the selected integer \( n \geq 2 \) and finite partition \( D = \{ t_j \} \) of \([0, T]\), denote \( y^{n,D} := y^{n,D} (\gamma, M) \) (defined at (8) on p.7) and denote \( y^{s,t,D} := (y^{n,D})^{-1} \otimes y^{n,D} \) for \( 0 \leq s \leq t \leq T \). Recall \( \{ y^{i,j} \}_{i=1,2} \) defined at (9) on p.7
\[
\begin{align*}
  dy^{1,j}_u &= f (\pi_1 (y^{1,j}_u)) \, d\gamma_u, \quad y^{1,j}_t = y^D (\gamma, M)_{t_j} \in T(n) (\mathbb{R}^e), \\
  dy^{2,j}_u &= f (\pi_1 (y^{2,j}_u)) \, dS_2 (\gamma + M)_u, \quad y^{2,j}_t = y^D (\gamma, M)_{t_j} \in T(n) (\mathbb{R}^e).
\end{align*}
\]

Then, based on the definition of \( y^{n,D} \), we have
\[
y^{n,D}_{t_j, t_{j+1}} = y^{1,j}_{t_j, t_{j+1}} \otimes E \left( y^{2,j}_{t_j, t_{j+1}} \right)^{-1} \otimes y^{1,j}_{t_j, t_{j+1}}, \quad j \geq 0. \tag{23}
\]

Since \( f \) is Lip(\( \beta \)) for \( \beta > p \geq 2 \), \( f \) is Lip(2). Based on error estimates of rough Taylor expansion (Theorem 22), we have, on any \([t_j, t_{j+1}]\),
\[
\begin{align*}
  &\left| \pi_1 \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_1 \left( \pi_f \left( t_j, y^{n,D}_{t_j, t_{j+1}}, I_2 (\gamma, M) \right) \right) \right| \\
  &\leq C_{p,f} \left( E \left( \omega_1 (t_j, t_{j+1})^{\beta} \right) + \omega_2 (t_j, t_{j+1})^{\beta} \right) \\
  &\quad + |Dfx| \left( \pi_1 (Y_{t_j}) \right) \left( E \left( \int_{t_j}^{t_{j+1}} (M_u - M_{t_j}) \otimes \circ dM_u \right) - \frac{1}{2} (M)_{t_j, t_{j+1}} \right).
\end{align*}
\]

Since \( M \) is in \( L^2 \),
\[
E \left( \int_{t_j}^{t_{j+1}} (M_u - M_{t_j}) \otimes \circ dM_u - \frac{1}{2} (M)_{t_j, t_{j+1}} \right) = E \left( \int_{t_j}^{t_{j+1}} (M_u - M_{t_j}) \otimes dM_u \right) = 0,
\]
and we have \( (M) = \int \phi dB \) with \( \phi \) a fixed path taking value in \( d \times d \) matrices
\[
E \left( \int_{t_j}^{t_{j+1}} (M_u - M_{t_j}) \otimes \circ dM_u \right) = \frac{1}{2} E \left( (M)_{t_j, t_{j+1}} \right) = \frac{1}{2} (M)_{t_j, t_{j+1}}.
\]

Thus, for any \( t_j \in D \),
\[
\begin{align*}
  &\left| \pi_1 \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_1 \left( \pi_f \left( t_j, y^{n,D}_{t_j, t_{j+1}}, I_2 (\gamma, M) \right) \right) \right| \\
  &\leq C_{p,f} \left( E \left( \omega_1 (t_j, t_{j+1})^{\beta} \right) + \omega_2 (t_j, t_{j+1})^{\beta} \right). \tag{24}
\end{align*}
\]

For the second level, based on (23), we have
\[
\begin{align*}
  \pi_2 \left( y^{n,D}_{t_j, t_{j+1}} \right) &= \pi_2 \left( y^{1,j}_{t_j, t_{j+1}} \otimes E \left( y^{2,j}_{t_j, t_{j+1}} \right)^{-1} \otimes y^{1,j}_{t_j, t_{j+1}} \right) \\
  &= 2 \pi_2 \left( y^{1,j}_{t_j, t_{j+1}} \right) - \pi_2 \left( E \left( y^{2,j}_{t_j, t_{j+1}} \right) \right) + \pi_1 \left( y^{1,j}_{t_j, t_{j+1}} \right) - \pi_1 \left( E \left( y^{2,j}_{t_j, t_{j+1}} \right) \right)^{\otimes 2}.
\end{align*}
\]
Then, combined with Theorem 4.2 (denote $\xi_j := \pi_1 \left( y^D (\gamma, M)_{t_j} \right)$)

\[
|\pi_2 \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_2 \left( \pi_f \left( t_j, y^{n,D}_{t_j}, I_2 (\gamma, M) \right)_{t_j, t_{j+1}} \right) |
\]

\[
\leq \left| f (\xi_j) \otimes f (\xi_j) \right| \left( 2 \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) - \left( \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) + \frac{1}{2} (M)_{t_j, t_{j+1}} \right) - \left( \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) - \frac{1}{2} (M)_{t_j, t_{j+1}} \right) \right) 
\]

\[
+ C_{p,f} \left( E \left( \omega_1 (t_j, t_{j+1}) \right) \right) \left( 2 \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) - \left( \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) + \frac{1}{2} (M)_{t_j, t_{j+1}} \right) - \left( \pi_2 \left( \gamma_{t_j, t_{j+1}} \right) - \frac{1}{2} (M)_{t_j, t_{j+1}} \right) \right) 
\]

\[
+ \frac{1}{2} (D f f) (\xi_j) (M)_{t_j, t_{j+1}} \right| \left( E \left( \omega_1 (t_j, t_{j+1}) \right) \right) + \omega_2 (t_j, t_{j+1}) \right| \right) 
\]

\[
\left( E \left( \omega_1 (t_j, t_{j+1}) \right) \right) + \omega_2 (t_j, t_{j+1}) \right| \right) .
\]

Therefore,

\[
\left| \pi_2 \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_2 \left( \pi_f \left( t_j, y^{n,D}_{t_j}, I_2 (\gamma, M) \right)_{t_j, t_{j+1}} \right) \right| \leq C (p, f, E (\omega_1 (0, T)^2), \omega_2 (0, T)) 
\]

\[
\times \left( E \left( \omega_1 (t_j, t_{j+1}) \right) + \omega_2 (t_j, t_{j+1}) \right) .
\]

For the higher levels (i.e. $k \geq 3$), by using Young’s inequality, we have

\[
\left| \pi_k \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_k \left( \pi_f \left( t_j, y^{n,D}_{t_j}, I_2 (\gamma, M) \right)_{t_j, t_{j+1}} \right) \right| \leq C (p, f, k) \left( E \left( \omega_1 (t_j, t_{j+1}) \right) + \omega_2 (t_j, t_{j+1}) \right) \times \left( E \left( \omega_1 (t_j, t_{j+1}) \right) + \omega_2 (t_j, t_{j+1}) \right) .
\]

Combine (24), (25) and (26), if we define $\bar{\omega}_k : \{(s, t) \mid 0 \leq s \leq t \leq T \} \rightarrow \mathbb{R}^T$ as

\[
\bar{\omega}_k (s, t) := \left\{ \begin{array}{ll}
E \left( \omega_1 (s, t)^{\frac{k}{p}} \right) + \omega_2 (s, t)^{\frac{k}{p}}, & k = 1 \\
E \left( \omega_1 (s, t)^{\frac{k}{2}} \right) + \omega_2 (s, t)^{\frac{k}{2}}, & k = 2 \\
E \left( \omega_1 (s, t)^{\frac{k}{2}} \right) + \omega_2 (s, t)^{\frac{k}{2}}, & k \geq 3
\end{array} \right.
\]

then

\[
\left| \pi_k \left( y^{n,D}_{t_j, t_{j+1}} \right) - \pi_k \left( \pi_f \left( t_j, y^{n,D}_{t_j}, I_2 (\gamma, M) \right)_{t_j, t_{j+1}} \right) \right| \leq C (p, f, k) \left( E \left( \omega_1 (0, T)^2 \right) + \omega_2 (0, T) \right) \bar{\omega}_k (t_j, t_{j+1}) , \forall j \geq 0, k = 1, \ldots, n.
\]

Based on our assumption (22) and that $n \geq 2$, we have

\[
\lim_{|D| \to 0} \sum_{t_j \in D} \bar{\omega}_k (t_j, t_{j+1}) = 0, \quad k = 1, \ldots, n.
\]

Since $f$ is Lip ($\beta$) for $\beta > p$, denote $Y$ as the unique solution to the rough differential equation

\[
dY = f (\pi_1 (Y)) dI_2 (\gamma, M) , \quad Y_0 = \xi \in T^{(n)} (\mathbb{R}^r).
\]

We want to prove

\[
\lim_{|D| \to 0} \sup_{1 \leq k \leq n} \sup_{0 \leq t \leq T} \left| \pi_k \left( y^{n,D}_t \right) - \pi_k (Y_t) \right| = 0.
\]
It is clear that

$$\pi_0 \left( y^{n,D} \right) = \pi_0 \left( Y_t \right) \equiv 1,$$

so (30) holds trivially at level 0. Then we use mathematical induction. Suppose (30) holds for level $l = 0, \ldots, k - 1$, we want to prove (30) at level $k$. Based on our inductive hypothesis, we have

$$\sup_{D \subset [0,T]} \max_{0 \leq l \leq k-1, 0 \leq t \leq T} |\pi_l \left( y^{n,D}_t \right)| < \infty. \quad (31)$$

For $t_j \in D$, when $j = 0$, $y^{n,D}_0 = Y_0 = \xi$. When $j = 1$, based on (28), we have

$$\left| \pi_k \left( y^{n,D}_{t_1} - Y_{t_1} \right) \right| = \sum_{l=0}^{k-1} |\pi_l \left( \xi \right)| \left| \pi_{k-l} \left( y^{n,D}_{0,t_1} - Y_{0,t_1} \right) \right|$$

$$\leq C \left( p, f, k, E \left( \omega_1 (0, T)^2 \right), \omega_2 (0, T) \right) \max_{0 \leq l \leq k-1} |\pi_l \left( \xi \right)| \sum_{l=1}^{k} \tilde{w}_l (0, t_1).$$

When $j \geq 2$, we have

$$\left| \pi_k \left( y^{n,D}_{t_j} - Y_{t_j} \right) \right|$$

$$= \sum_{i=0}^{j-1} \pi_k \left( \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} - \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right)$$

$$\leq \sum_{i=0}^{j-2} \pi_k \left( \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} - \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right) \right)$$

$$+ \pi_k \left( y^{n,D}_{t_j} - \pi_f \left( t_{j-1}, y^{n,D}_{j-1}, I_2 (\gamma, M) \right)_{t_j} \right).$$

Then for each $i = 0, 1, \ldots, j - 2$,

$$\pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} - \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right)$$

$$= y^{n,D}_{i+1} \otimes \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j}$$

$$- \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \otimes \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right)$$

$$= y^{n,D}_{i+1} \otimes \left( \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} - \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right) \right)$$

$$+ \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right)$$

$$\otimes \pi_f \left( t_{i+1}, \pi_f \left( t_{i+1}, y^{n,D}_{i+1}, I_2 (\gamma, M) \right)_{t_j} \right).$$
Then use (31), Theorem 4.3 on p.11 and (28) \( \bar{\omega}_1 \) defined at (27), we have

\[
\begin{align*}
\left| \pi_k \left( \gamma_{i_{i+1}} \otimes \left( \pi_f \left( t_{i+1}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) - \pi_f \left( t_{i+1}, \pi_f \left( t_{i}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) \right) \right| \\
\leq \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \times \left( \sum_{l=1}^{k} |\pi_l \left( y_{t_{i+1}}^{n,D} - \pi_f \left( t_{i}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) | \right) \\
\leq C \left( p, \beta, f, k, \omega_2 (0, T) \right) \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \left( \sum_{l=1}^{k} |\pi_l \left( y_{t_{i+1}}^{n,D} - \pi_f \left( t_{i}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) | \right) \cdot \bar{\omega}_1 (t_{i}, t_{i+1})
\end{align*}
\]

On the other hand, use (31), Theorem 4.2 on p.11 and (28) \( \bar{\omega}_1 \) defined at (27), we have

\[
\begin{align*}
\left| \pi_k \left( \gamma_{i_{i+1}} \otimes \left( \pi_f \left( t_{i+1}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) \otimes \pi_f \left( t_{i+1}, \pi_f \left( t_{i}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) \right) \right| \\
\leq C \left( p, \beta, f, k, \omega_1 (0, T)^2, \omega_2 (0, T) \right) \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \sum_{l=1}^{k} \bar{\omega}_l (t_{i}, t_{i+1}).
\end{align*}
\]

Therefore, we have, for any \( i = 0, 1, \ldots, j-2 \),

\[
\begin{align*}
\left| \pi_k \left( \pi_f \left( t_{i+1}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) - \pi_f \left( t_{i+1}, \pi_f \left( t_{i}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) \right| \\
\leq C \left( p, \beta, f, k, \omega_1 (0, T)^2, \omega_2 (0, T) \right) \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \sum_{l=1}^{k} \bar{\omega}_l (t_{i}, t_{i+1}).
\end{align*}
\]

As a result,

\[
\sum_{i=0}^{j-2} \left| \pi_k \left( \pi_f \left( t_{i+1}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{i+1}, t_j} \right) - \pi_f \left( t_{i+1}, \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right) \right| \leq C \left( p, \beta, f, k, \omega_1 (0, T)^2, \omega_2 (0, T) \right) \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \sum_{i=0}^{j-2} \left( \sum_{l=1}^{k} \bar{\omega}_l (t_{i}, t_{i+1}) \right).
\]

On the other hand, for the term left in (33),

\[
\begin{align*}
\left| \pi_k \left( y_{t_{j+1}}^{n,D} - \gamma_{i_{i+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{j+1}, t_j} \right| & = \left| \pi_k \left( y_{t_{j+1}}^{n,D} - y_{t_{j+1}, t_{j+1}}, \mathcal{I}_2 (\gamma, M) \right)_{t_{j+1}, t_j} \right| \\
& \leq C \left( p, \beta, f, k, \omega_1 (0, T)^2, \omega_2 (0, T) \right) \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \sum_{l=1}^{k} \bar{\omega}_l (t_{j+1}, t_j).
\end{align*}
\]

Therefore, combining (33), (34) and (35), we have

\[
\begin{align*}
\left| \pi_k \left( y_{t_{j+1}}^{n,D} - S_n (Y)_{t_{j+1}} \right) \right| & \leq C \left( p, \beta, f, k, \omega_1 (0, T)^2, \omega_2 (0, T) \right) \times \left( \sup_{D} \max_{0 \leq t \leq k-1} \left( \sup_{0 \leq t \leq T} |\pi_l \left( y_{t}^{n,D} \right)| \right) \right) \sum_{i=0}^{j-2} \left( \sum_{l=1}^{k} \bar{\omega}_l (t_{i}, t_{i+1}) \right)
\end{align*}
\]

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Then, based on (29) and the inductive assumption (31), we have
\[
\lim_{|D| \to 0} \max_{t_j \in D} \left| \pi_k \left( y^n_{t_j} \right) \right| = 0.
\] (35)

Since \( y^n_{t_j} \) is piecewise-constant, we have
\[
\sup_{0 \leq t \leq T} \left| \pi_k \left( y^n_{t_j} \right) - \pi_k \left( Y_t \right) \right| \leq \max_{t_j \in D} \left| \pi_k \left( y^n_{t_j} \right) - \pi_k \left( Y_{t_j} \right) \right| + \sup_{|t-s| \leq |D|} \left| \pi_k \left( Y_t \right) - \pi_k \left( Y_s \right) \right|.
\] (36)

For interval \([s, t]\) satisfying \( \| Y \|_{p, \var, [s, t]} \leq 1 \), we have,
\[
\left| \pi_k \left( Y_t \right) - \pi_k \left( Y_s \right) \right| = \sum_{j=1}^k \pi_{k-j} \left( Y_s \right) \otimes \pi_j \left( Y_{s,t} \right) \leq C \left( p, k, \sup_{t \in [0, T]} \| Y_t \| \right) \| Y \|_{p, \var, [s, t]}.
\]

Since \( Y \) is continuous and \( \| Y \|_{p, \var, [0, T]} < \infty \), we have
\[
\lim_{|D| \to 0} \sup_{|t-s| \leq |D|} \left| \pi_k \left( Y_t \right) - \pi_k \left( Y_s \right) \right| = 0.
\]

Combined with (35) and (36), we get
\[
\lim_{|D| \to 0} \sup_{0 \leq t \leq T} \left| \pi_k \left( y^n_{t_j} \right) - \pi_k \left( Y_t \right) \right| = 0.
\]

\[\blacksquare\]

4.2.2 Martingale underlying

The Theorem below follows from Thm 14.12 in [7] and Doob's maximal inequality.

**Theorem 4.4** Suppose \( M \) is a continuous \( d \)-dimensional martingale. Denote \( S_2 (M) \) as the step-2 Stratonovich signature of \( M \). Then for any \( q > 1 \) and any \( p > 2 \),
\[
E \left( |M_T - M_0|^q \right) \sim E \left( \| M \|_{1-\var, [0, T]}^{2-1} \right) \sim E \left( \| M \|_{1-\var, [0, T]}^{2-1 q} \right) \sim E \left( \| S_2 (M) \|_{p, \var, [0, T]}^q \right).
\] (37)

They are equivalent up to a positive constant depending on \( p, q \) and \( d \).

**Proof of Corollary 3.8** \((Z, \bar{Z})\) is a \( 2d \)-dimensional continuous martingale w.r.t. the filtration generated by \( Z \) and \( B \), so can be enhanced (by their Stratonovich integrals) to a \( p \)-rough process for any \( p > 2 \) (Theorem 1.1 on [10]). Suppose \( Z \) is in \( L^{2n+\epsilon} \) for some \( \epsilon > 0 \) and integer \( n \geq 1 \). Select \( p = 2 + n^{-\epsilon} \). Using inequality (37), we get
\[
E \left( \| S_2 (Z + \bar{Z}) \|_{p, \var, [0, T]}^{np} \right) \leq C_{d,p,n} E \left( \| \langle Z + \bar{Z} \rangle \|_{\infty-\var, [0, T]}^{2-1 np} \right) \leq C_{d,p,n} E \left( \| \langle Z \rangle \|_{\infty-\var, [0, T]}^{2-1 np} \right) \leq C_{d,p,n} E \left( |Z_T - Z_0|^{np} \right) = C_{d,p,n} E \left( |Z_T - Z_0|^{2n+\epsilon} \right) < \infty.
\]

Thus, we have
\[
E \left( \| S_2 (Z + \bar{Z}) \|_{p, \var, [0, T]}^{np} \right) = C_{d,p,n} E \left( \| S_2 (Z + \bar{Z}) \|_{p, \var, [0, T]}^{np} \right) < \infty \text{ a.s.}
\]

On the other hand, fix a sample path of \( Z \), we have that, the Stratonovich integrals satisfy:
\[
E \left( \int_s^t Z_{s,u} \otimes \partial \bar{Z}_u + \int_s^t \bar{Z}_{s,u} \otimes \partial Z_u |Z| \right) = 0, \forall 0 \leq s \leq t \leq T, \text{ a.s.}
\]

Thus, based on Theorem 3.3 Corollary holds. (When \( n = 1 \), it holds based on Remark 3.4) \[\blacksquare\]
4.3 Relation with Itô SDE

**Lemma 4.5** Suppose $Z = (Z^i)_{1 \leq i \leq d}$ is a continuous local martingale on $[0, \infty)$ taking value in $\mathbb{R}^d$. For integer $n \geq 1$, denote $\mathcal{I}_n (Z)$ as at (13) (on page 7) and $S_n (\cdot)$ as in Notation 2.6 (on page 4). Then

$$S_n (\mathcal{I}_2 (Z))_t = \mathcal{I}_n (Z)_t, \quad t \in [0, \infty), \forall n \geq 1, \ a.s.. \tag{38}$$

**Proof.** We prove (38) on $[0, T]$ for some $T > 0$, for fixed $n \geq 1$ and when $Z$ is a bounded continuous martingale. Then by properly stopping the process and unionizing countably many null sets, we can prove (38).

Denote the filtration of $Z$ as $(\mathcal{F}_t)$. Firstly, we prove that $S_n (\mathcal{I}_2 (Z))$ is a martingale. It is clear that, $S_1 (\mathcal{I}_2 (Z)) = Z$ and $S_2 (\mathcal{I}_2 (Z)) = \mathcal{I}_2 (Z)$ are martingales.

For integer $n \geq 2$, suppose $S_n (\mathcal{I}_2 (Z))$ is a continuous martingale, we want to prove that $S_{n+1} (\mathcal{I}_2 (Z))$ is also a continuous martingale. Based on (21) on page 10 for any $0 \leq s \leq t \leq T$, (with $0 \in (\mathbb{R}^d)^{(n+1)}$

$$\pi_{n+1} \left( S_{n+1} (\mathcal{I}_2 (Z))_{s,t} \right) = \lim_{|D| \to 0, D = \{t^i_k\} \subset [s,t]} \pi_{n+1} \left( \left( S_n (\mathcal{I}_2 (Z))_{t_0,t_1} \oplus 0 \right) \otimes \cdots \otimes \left( S_n (\mathcal{I}_2 (Z))_{t_{n-1},t_n} \oplus 0 \right) \right), \tag{39}$$

and the error in $L^1$ is bounded by, for any $p > 2$,

$$C_{p,n} \lim_{|D| \to 0} E \left( \sum_{k,t^i_k \in D} \|\mathcal{I}_2 (Z)\|_{p-var,[t_k,t_{k+1}]}^{n+1} \right) \leq C_{p,n} \lim_{|D| \to 0} E \left( \|\mathcal{I}_2 (Z)\|_{p-var,[0,T]}^{n+1} \right)^{1/p} \leq C_{p,d,n} E \left( |Z_T - Z_0|^{n+1} \right) < \infty.$$

Based on (37) and using that $Z$ is a bounded martingale, we have

$$E \left( \|\mathcal{I}_2 (Z)\|_{p-var,[0,T]}^{n+1} \right) \leq C_n \left( E \left( \|S_2 (Z)\|_{p-var,[0,T]}^{n+1} \right) + E \left( \|Z\|_{1-var,[0,T]}^{n+1} \right) \right) \leq C_{p,d,n} E \left( |Z_T - Z_0|^{n+1} \right) < \infty.$$

Thus, using dominated convergence theorem, the convergence at (39) is in $L^1$ and we have:

$$E \left( \pi_{n+1} \left( S_{n+1} (\mathcal{I}_2 (Z))_{s,t} \right) \mid \mathcal{F}_s \right) = \lim_{|D| \to 0, D = \{t^i_k\}_{k=0}^K \subset [s,t]} \pi_{n+1} \left( E \left( \left( S_n (\mathcal{I}_2 (Z))_{t_0,t_1} \oplus 0 \right) \otimes \cdots \otimes \left( S_n (\mathcal{I}_2 (Z))_{t_{K-1},t_K} \oplus 0 \right) \mid \mathcal{F}_s \right) \right) \ a.s..$$

While using the inductive hypothesis that $S_n (\mathcal{I}_2 (Z))$ is a martingale, we have

$$E \left( \left( S_n (\mathcal{I}_2 (Z))_{t_0,t_1} \oplus 0 \right) \otimes \cdots \otimes \left( S_n (\mathcal{I}_2 (Z))_{t_{K-1},t_K} \oplus 0 \right) \mid \mathcal{F}_s \right) = E \left( \left( S_n (\mathcal{I}_2 (Z))_{t_0,t_1} \oplus 0 \right) \otimes \cdots \otimes \left( S_n (\mathcal{I}_2 (Z))_{t_{K-1},t_K} \oplus 0 \right) \mid \mathcal{F}_{t_{K-1}} \right) \mid \mathcal{F}_s \right) = \cdots = 1, \ a.s..$$

Thus, for any $0 \leq s \leq t \leq T$,

$$E (S_{n+1} (\mathcal{I}_2 (Z))_t \mid \mathcal{F}_s) = S_{n+1} (\mathcal{I}_2 (Z))_s \otimes E (S_{n+1} (\mathcal{I}_2 (Z))_{s,t} \mid \mathcal{F}_s) = S_{n+1} (\mathcal{I}_2 (Z))_s.$$
Then we prove \( (38) \). It is clear that \( (38) \) holds for level 1 and level 2. For the higher levels, we use mathematical induction. Suppose for some \( k \geq 2 \), we have

\[
\left( 1, Z_t - Z_0, \int_{0<t_1<...<t_k} dZ_{u_1} \otimes dZ_{u_2}, \ldots, \int_{0<t_1<...<t_k} dZ_{u_1} \otimes \cdots \otimes dZ_{u_k} \right)
\]

(40)

Denote

\[
Z_{s,t}^{I\hat{t}_0,k} := \int_{s}^{t} \cdots \int_{s}^{t} dZ_{u_1} \otimes \cdots \otimes dZ_{u_k}, \forall 0 \leq s \leq t \leq T, \forall k \geq 1.
\]

Since \( Z \) is a bounded continuous martingale, \( t \mapsto Z_{0,t}^{I\hat{t}_0,k} \) is a continuous martingale adapted to the filtration of \( Z \), and the process \( t \mapsto \left( Z_{0,t}^{I\hat{t}_0,k}, Z_t \right) \) is a continuous martingale w.r.t. the filtration of \( Z \). Then, based on Theorem 4.1 (on page 10), \( \left( Z_{0,t}^{I\hat{t}_0,k}, Z \right) \) can be enhanced by their Stratonovich integrals to a \( p \)-rough process for any \( p > 2 \). As a result, we have

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} \left( Z_{0,t}^{I\hat{t}_0,k} - Z_{0,t_j}^{I\hat{t}_0,k} \right) \otimes dZ_t \right|^p < \infty \ a.s., \forall p > 2.
\]

(41)

Then using the relationship between Itô integral and Stratonovich integral, we have

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} \left( Z_{0,t}^{I\hat{t}_0,k} - Z_{0,t_j}^{I\hat{t}_0,k} \right) \otimes dZ_t \right|^p < \infty \ a.s., \forall p > 2.
\]

(42)

For \( i = 1, \ldots, k - 1 \), we have

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} Z_{t_j,t}^{I\hat{t}_0,k-i} \otimes Z_{t_j,t}^{I\hat{t}_0,i} \otimes dZ_t \right|^p \leq \sup_{t \in [0,T]} \left| Z_{t_j,t}^{I\hat{t}_0,k-i} \right|^p \sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} Z_{t_j,t}^{I\hat{t}_0,i} \otimes dZ_t \right|^p.
\]

Based on the inductive hypothesis \( (40) \), we have, for any \( p > 2 \) and \( i = 1, \ldots, k - 1 \), (since \( i + 1 \geq 2 \))

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} Z_{t_j,t}^{I\hat{t}_0,i} \otimes dZ_t \right|^p \leq \left( \sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| Z_{t_j,t}^{I\hat{t}_0,i+1} \right|^p \right)^{\frac{p}{p+1}} < \infty \ a.s., \forall p > 2.
\]

(43)

Thus, combine (41), (42) and (43), we have

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| Z_{t_j,t}^{I\hat{t}_0,k+1} \right|^p = \sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \int_{t_j}^{t_{j+1}} Z_{t_j,t}^{I\hat{t}_0,k} \otimes dZ_t \right|^p < \infty \ a.s., \forall p > 2.
\]

(44)

On the other hand, \( \pi_{k+1} \left( S_{k+1} (I_2 (Z))_{t_j,t_{j+1}} \right) \) satisfies (based on \( (19) \) on page 10)

\[
\sup_{D, D \subset [0,T]} \sum_{t_j \in D} \left| \pi_{k+1} \left( S_{k+1} (I_2 (Z))_{t_j,t_{j+1}} \right) \right|^p < \infty \ a.s., \forall p > 2.
\]

(45)
Since both \((1, Z, \ldots, Z^{k, k}, Z^{k, k+1})\) and \(S_{k+1}(\mathcal{I}_2(Z))\) are multiplicative and \((40)\) holds, there exists a process \(\varphi : [0, T] \to \mathbb{R}^d \otimes (k+1)\) such that
\[
Z_s^{k, k+1} - \pi_{k+1} \left( S_{k+1}(\mathcal{I}_2(Z))_{s, t} \right) = \varphi_t - \varphi_s, \quad \forall 0 \leq s \leq t < \infty, \text{ a.s.} \tag{46}
\]
Moreover, \(t \mapsto Z_t^{k, k+1}\) is a continuous martingale, and, as we proved above, \(t \mapsto \pi_{k+1} \left( S_{k+1}(\mathcal{I}_2(Z))_{0, t} \right)\) is also a continuous martingale. Then based on \((46)\), \(t \mapsto (\varphi_t - \varphi_0)\) is a continuous martingale vanishing at 0. Combined with \((44)\) and \((45)\) (with \(k \geq 2\)), we have
\[
\sup_{D, D \subset [0, T], t_j \in D} \sum_{t_j \in D} \varphi_{t_{j+1}} - \varphi_{t_j} < \infty \text{ a.s., for any } p > 2.
\]
Since a continuous martingale with finite \(q\)-variation for \(q < 2\) is a constant, we have \(\varphi_t \equiv \varphi_0\), and
\[
Z_t^{k, k+1} = \pi_{k+1} \left( S_{k+1}(\mathcal{I}_2(Z))_t \right), \quad \forall 0 \leq t \leq T, \text{ a.s.}
\]
**Proof of Theorem 3.9** We only prove the theorem when \(Z\) is a continuous bounded martingale. Then by properly stopping \(Z\) and unionizing countably many null sets, we can prove Theorem 3.9 for continuous local martingales.

For \(0 \leq s \leq t \leq T\), we denote
\[
Z_{s, t}^1 := Z_t - Z_s \quad \text{and} \quad Z_{s, t}^2 := \int_{s < u_1 < u_2 < t} \odt Z_{u_1} \odt Z_{u_2}.
\]
Replace \(\beta\) by \(\beta \wedge 2\), and fix \(p \in (2, \beta + 1)\). Define \(\omega : \{(s, t) | 0 \leq s \leq t \leq T\} \to \mathbb{R}^\infty\) as
\[
\omega(s, t) := \|S_2(Z)\|^p_{p-\text{var}, [s, t]} + \|(Z)\|_{1-\text{var}, [s, t]}.
\]
Since \(\beta > p - 1\), the rough differential equation
\[
dY = f(\pi_1(Y))d\mathcal{I}_2(Z), \quad Y_0 = \xi \in T^n(\mathbb{R}^e), \tag{47}
\]
has a solution. Denote \(Y\) as a solution to the RDE \((47)\), based on rough Taylor expansion (Theorem 4.2 on page 11), we have the pathwise estimate that, for any \(0 \leq s \leq t \leq T\),
\[
\left| \pi_1(Y_{s,t}) - f(\pi_1(Y_s))Z_{s,t}^1 - (Dff)(\pi_1(Y_s)) \left( Z_{s,t}^2 - \frac{1}{2}(Z_{s,t}^t) \right) \right| \leq C_{p, \beta, f, \omega}(s, t)^{\frac{1}{p-1}}.
\]
As a result, for almost every sample path of \(Z\), any solution \(Y\) to \((47)\) and any \(0 \leq s \leq t \leq T\),
\[
\pi_1(Y_{s,t}) = \lim_{|D| \to 0, D \subset [s, t]} \sum_{t_k \in D} \left( f(\pi_1(Y_{t_k}))Z_{t_k, t_{k+1}}^{1} + (Dff)(\pi_1(Y_{t_k})) \left( Z_{t_k, t_{k+1}}^{2} - \frac{1}{2}(Z_{t_k, t_{k+1}}^t) \right) \right). \tag{48}
\]
Moreover, we have
\[
\lim_{|D| \to 0, D \subset [s, t]} \sum_{t_k \in D} (Dff)(\pi_1(Y_{t_k})) \left( Z_{t_k, t_{k+1}}^{2} - \frac{1}{2}(Z_{t_k, t_{k+1}}^t) \right) = 0 \text{ in } L^2. \tag{49}
\]
Indeed, since RDE solution is the limit of ODE solutions, and solution of ODE can be recovered via Picard iteration (\(f\) is \(\text{Lip}(\beta)\) for \(\beta > 1\)), \(\pi_1(Y)\) is adapted to the filtration of \(Z\). Since \((Dff)(\pi_1(Y))\) is in \(L^4\) (actually bounded, here we use \(L^4\) for the convenience of second level) and \(Z\) is bounded, the
cross terms in $L^2$ norm of (19) vanish after taking conditional expectation. Thus, for any $1 \leq i, j \leq d$ and any $1 \leq s \leq e$, we have ($(Dff)^s$ denotes the projection to the $s$th coordinate of $\mathbb{R}^e$)

$$E \left( \left( \sum_{l_k \in D} (Dff)^s (\pi_1 (Y_{l_k})) \left( Z_{t_k, t_k+1}^{2, i,j} - \frac{1}{2} \langle Z^i, Z^j \rangle_{t_k, t_k+1} \right) \right)^2 \right)$$

$$= \sum_{t_k \in D} E \left( \left( (Dff)^s (\pi_1 (Y_{l_k})) \left( Z_{t_k, t_k+1}^{2, i,j} - \frac{1}{2} \langle Z^i, Z^j \rangle_{t_k, t_k+1} \right) \right)^2 \right)$$

$$\leq \sup_{t \in [0,T]} E \left( \left( (Dff)^s (\pi_1 (Y_1))^4 \right)^{\frac{1}{2}} \sum_{t_k \in D} E \left( \left( Z_{t_k, t_k+1}^{2, i,j} - \frac{1}{2} \langle Z^i, Z^j \rangle_{t_k, t_k+1} \right)^{4} \right)^{\frac{1}{2}} \right)$$

$$\leq C_d \sup_{t \in [0,T]} E \left( \left( (Dff)^s (\pi_1 (Y_1))^4 \right)^{\frac{1}{2}} \sum_{t_k \in D} E \left( \left\| \langle Z \rangle \right\|_{1-var, [t_k, t_k+1]}^4 \right)^{\frac{1}{2}} \right)$$

$$\leq C_d \left( \sup_{t \in [0,T]} E \left( \left( (Dff)^s (\pi_1 (Y_1))^4 \right)^{\frac{1}{2}} \right) E \left( \left\| \langle Z \rangle \right\|_{1-var, [0,T]}^4 \right)^{\frac{1}{2}} \sup_{|t-s| \leq |D|} \left\| \langle Z \rangle \right\|_{1-var, [s,t]}^4 \right)^{\frac{1}{2}}.$$  

Since $Z$ is bounded, by using (37) on p16 we have

$$E \left( \left\| \langle Z \rangle \right\|_{1-var, [0,T]}^4 \right) \leq E \left( |Z_T - Z_0|^4 \right) < \infty.$$  

Then by dominated convergence theorem, (19) holds.

On the other hand, since $f (\pi_1 (Y_t))$ is bounded and adapted, and $Z$ is bounded, by using (37) on p16 we have

$$E \left( \int_0^T |f (\pi_1 (Y_u))|^2 d\langle Z \rangle_u \right) \leq |f|_{Lip(\beta)}^2 E \left( \left\| \langle Z \rangle \right\|_{1-var, [0,T]}^4 \right) \leq |f|_{Lip(\beta)}^2 E \left( |Z_T - Z_0|^2 \right)^{\frac{1}{2}} < \infty.$$  

Thus,

$$\lim_{|D| \to 0, D \subset [s,t]} \sum_{t_k \in D} f (\pi_1 (Y_{l_k})) Z_{t_k, t_k+1}$$

converge in $L^2$ to the Itô integral $\int_s^t f (\pi_1 (Y_u)) dZ_u$. As a result, (18) converge a.s. and in $L^2$, with $\pi_1 (Y_{s,t})$ the a.s. limit and the Itô integral $\int_s^t f (\pi_1 (Y_u)) dZ_u$ the $L^2$ limit. Therefore, since the null set can be chosen to be independent from $s$ and $t$, we have

$$\pi_1 (Y_{s,t}) = \int_s^t f (\pi_1 (Y_u)) dZ_u, \quad \forall 0 \leq s \leq t \leq T, \text{ a.s.}.$$  

Thus, $\pi_1 (Y)$ is a strong continuous solution to the stochastic differential equation:

$$dy = f (y) dZ, \quad y_0 = \pi_1 (\xi) \in \mathbb{R}^e.$$  

(50)

On the other hand, since $f$ is $Lip (\beta)$, $\beta > 1$, based on Thm (2.1) on p375 21, the SDE (50) has a unique strong continuous solution (denoted as $y$). Thus, $\pi_1 (Y)$ exists uniquely a.s. and

$$\pi_1 (Y_t) = y_t, t \in [0,T], \text{ a.s.}$$  

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For \( \pi_2(Y) \), consider the rough differential equation on \([0, T] \):

\[
dU_t = \left( f(\pi_{\mathcal{R}^e}(U_t)) \right) dZ_t,
\]

\[U_0 = \left( (1, \pi_1(\xi), 0), 0 \right) \in T^{(2)} \left( \mathbb{R}^e \oplus (\mathbb{R}^e)^{\otimes 2} \right).
\]

Although the vector field for \( U \) in (51) only locally \( Lip(\beta) \), \( \beta > 1 \), its first level has a global solution

\[\pi_1(U_t) = (\pi_1(Y_t), \pi_2(Y_{0,t})), \ \ t \in [0, T].\]

By using Theorem 4.2 (on p11), (37) on p16 and that \( Z \) is a martingale w.r.t. the filtration of \( \pi \), therefore:

\[\text{Therefore, the solution } Y \text{ and } (y, \alpha) \text{ of the RDE (47) has the explicit expression:}
\]

\[
S_2(Y)_{0,t} = \left( 1, y_t - \pi_1(\xi), \int_0^{u_1 < u_2 < t} dy_{u_1} \otimes dy_{u_2} \right), \ \ t \in [0, T], \ \ a.s.
\]

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Using the Theorem of enhancement (Theorem 2.5 on p4) and Lemma 4.5 on p17, we have the explicit expression that

\[ S_n(Y)_t = \xi \otimes S_n(Y)_{0,t}, \quad t \in [0, T], \quad \forall n \geq 1, \text{ a.s.,} \]

where \( y \) denotes the solution to the Itô stochastic differential equation [50] and \( \mathcal{I}_n(y) \) is the Itô signature of \( y \) defined at (13) on p7. \( \blacksquare \)

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