Research Article

Multiple Attribute Decision-Making Problem Using Picture Fuzzy Graph

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In a picture fuzzy environment, almost all multiple attribute decision-making (MADM) methods have been discussed a type of problem in which there is no relationship among the attributes. Although the relationship among the attributes should be considered in the actual applications, so we need to pay attention to that important issue. This article applied graph theory to the picture fuzzy set (PFS) and obtained a new method, MADM, to solve complicated problems under a picture fuzzy environment. The developed method can capture the relationship among the attributes that cannot be handled well by any existing methods. This study introduces union, intersection, sum, Cartesian product, the composition of picture fuzzy graphs (PFGs), and their important properties. Finally, by considering the importance of relationships among attributes in the determination process, two algorithms, based on PFG, have developed to solve complicated problems using picture fuzzy information. Also, two numerical examples have introduced to explain how to deal with the MADM problem under picture fuzzy environment.

1. Introduction

At present, graphs do not disclose all the systems properly because of the uncertainty of the parameters within a system. For instance, a social network can be uttered as the graph, where nodes denote an account (such as institution or person) and edges express the connection between the accounts. If the connections among accounts are measurable as bad or good according to the recurrence rate of contacts among the accounts, fuzziness should be added to representation. In 1975, Rosenfeld first defined the fuzzy graph considering fuzzy relations on fuzzy sets [1]. A PFS is a generalization of intuitionistic fuzzy set (IFS) [2]. The picture fuzzy model gives more precision, flexibility, and compatibility than the intuitionistic fuzzy model.

The concept of PFS was first introduced by Coung [3] in 2013. In addition to IFS, Coung appended new components which determine the neutral membership degree. IFS gives an element’s membership and nonmembership degree, while PFS gives positive membership degree, neutral membership degree, and negative membership degree of an element. These memberships are almost independent and the sum of these three membership degrees is ≤1. Basically, PFS-based models may be adequate in situations, where we counter several opinions that involve more answers of types: yes, no, abstain, and refusal. If we take voting as an example, human voters may be separated into four possible groups with distinct opinions: vote for, vote against, abstain, and refusal of the voting. Picture fuzzy sets have several interesting applications in system analysis, operation research, economics, medicine, computer science, engineering,
mathematics, etc. Some properties of PFS and its operators have been studied in [4, 5].

1.1. Review of Literature. After invention of fuzzy graph, it develops with its different branches, such as fuzzy threshold graph [6], balanced interval-valued fuzzy graphs [7], cubic graph [8], m-step fuzzy competition graphs [9], fuzzy planar graphs [10,11], and fuzzy k-competition graph [12]. Pramanik et al. defined interval-valued fuzzy threshold graph and studied several properties [13]. They also have considered planarity in bipolar fuzzy graph, and they extended it to bipolar fuzzy planar graphs [14]. Also, Pramanik et al. have extended fuzzy planar graph to interval-valued fuzzy planar graph [15] and interval-valued fuzzy graph [16]. Voskoglou et al. [17] have discussed and characterized several fuzzy graph theoretic structure and fuzzy hypergraphs. Sahoo et al. [18] have studied the intuitionistic fuzzy competition graph. Balanced intuitionistic fuzzy graphs are discussed by Karunambigai et al. [19]. Also, Sahoo et al. have studied some problems regarding IFG [18, 20, 21]. Recently, many researchers have carried out study regarding picture hesitant fuzzy sets and their application in multiple attribute group decision-making process have been presented. In Section 2, PFGMADM problem with picture fuzzy information are presented, and the union and the intersection of the PFSs S and T are defined by

\[
(i) \quad (S \cup T)(t) = \{(t, p_S(t) \vee p_T(t), q_S(t) \vee q_T(t), r_S(t) \vee r_T(t)), t \in X\}
\]

\[
(ii) \quad (S \cap T)(t) = \{(t, p_S(t) \wedge p_T(t), q_S(t) \wedge q_T(t), r_S(t) \wedge r_T(t)), t \in X\}
\]

A picture fuzzy number is defined by \( f_n = (\alpha, \beta, \gamma) \).

Definition 1 (see [4]). Let \( S = (p_S, q_S, r_S) \) and \( T = (p_T, q_T, r_T) \) be two PFSs. Then, the union and the intersection of the PFSs S and T are defined by

\[
(i) \quad (S \cup T)(t) = \{(t, p_S(t) \vee p_T(t), q_S(t) \vee q_T(t), r_S(t) \vee r_T(t)), t \in X\}
\]

\[
(ii) \quad (S \cap T)(t) = \{(t, p_S(t) \wedge p_T(t), q_S(t) \wedge q_T(t), r_S(t) \wedge r_T(t)), t \in X\}
\]

A picture fuzzy number is defined by \( f_n = (\alpha, \beta, \gamma) \).

Definition 2 (see [4]). Let \( f_n = (\alpha, \beta, \gamma) \) be a picture fuzzy number; then, the score function of \( f_n \) is denoted by \( \text{scor} (f_n) \) and is defined by \( \text{scor} (f_n) = ((1 + \alpha - 2\beta - \gamma)/2) \).

Observation 1. Let \( f_1 \) and \( f_2 \) be two picture fuzzy numbers; then, \( \text{scor} (f_1) > \text{scor} (f_2) \Rightarrow f_1 \Rightarrow f_2 \).

Definition 3 (see [4]). A PF relation \( \rho \) in a universe \( A \times B \) is a PFS defined by \( \rho(a, b) = \{(a, b), p_a(a, b), q_a(a, b), r_p(a, b) \} \), where \( p_a: A \times B \rightarrow [0, 1] \), \( q_a: A \times B \rightarrow [0, 1] \), \( r_p: A \times B \rightarrow [0, 1] \), and \( 0 \leq p_a(a, b) + q_a(a, b) + r_p(a, b) \leq 1 \), for all \( (a, b) \in A \times B \).

Definition 4 (see [4]). Let \( S = (p_S, q_S, r_S) \) and \( T = (p_T, q_T, r_T) \) be two PFSs on a set X. If \( S \) be a PFR on X, then \( S \) is also a PFR on \( T \) if \( p_S(a, b) \leq p_T(a) \wedge p_T(b), q_S(a, b) \geq q_T(a) \vee q_T(b), \) and \( r_S(a, b) \geq r_T(a) \vee r_T(b), \) for all \( a, b \in X \).
3. Picture Fuzzy Graph

In this section, the PFG and some properties and theorems of PFG have been described.

Definition 5. A PFG of a graph \( G^* = (V, E) \) is a pair \( G = (S, T) \), where \( S = (p_S, q_S, r_S) \) is a PFS on \( V \) and \( T = (p_T, q_T, r_T) \) is a PFR on \( E \) such that \( p_T(a, b) \leq p_S(a) \wedge p_S(b) \), \( q_T(a, b) \geq q_S(a) \vee q_S(b) \), \( r_T(a, b) \geq r_S(a) \vee r_S(b) \), and \( 0 \leq p_T(a, b) + q_T(a, b) + r_T(a, b) \leq 1 \).

Here, \( S \) is the picture fuzzy node set of \( G \) and \( T \) is a picture fuzzy edge set on \( G \). Also, \( p_S(a), q_S(a), \) and \( r_S(a) \), respectively, denote the positive, neutral, and negative membership degree of the node \( a \), and \( p_T(a, b), q_T(a, b), \) and \( r_T(a, b) \) denote that of edge \( (a, b) \).

Now, we give some properties of PFG as composition, Cartesian product, union, and intersection.

Let \( G_1 = (S_1, T_1) \) and \( G_2 = (S_2, T_2) \) be two PFGs, where \( S_1 = (p_{S_1}, q_{S_1}, r_{S_1}) \), \( S_2 = (p_{S_2}, q_{S_2}, r_{S_2}) \), \( T_1 = (p_{T_1}, q_{T_1}, r_{T_1}) \), and \( T_2 = (p_{T_2}, q_{T_2}, r_{T_2}) \).

\[
\begin{align*}
(p_T \times p_T)((a, a_2), (a, b_2)) &= p_S(a) \wedge p_T(a, b) \\
&\leq p_S(a) \wedge (p_S(a_2) \wedge p_S(b_2)) \\
&= (p_S(a) \wedge p_S(a_2)) \wedge (p_S(a) \wedge p_S(b_2)) \\
&= (p_S \times p_S)(a, a_2) \wedge (p_S \times p_S)(a, b_2),
\end{align*}
\]
\[
\begin{align*}
(q_T \times q_T)((a, a_2), (a, b_2)) &= q_S(a) \vee q_T(a, b) \\
&\geq q_S(a) \vee (q_S(a_2) \vee q_S(b_2)) \\
&= (q_S(a) \vee q_S(a_2)) \vee (q_S(a) \vee q_S(b_2)) \\
&= (q_S \times q_S)(a, a_2) \vee (q_S \times q_S)(a, b_2),
\end{align*}
\]
\[
\begin{align*}
(r_T \times r_T)((a, a_2), (a, b_2)) &= r_S(a) \vee r_T(a, b) \\
&\geq r_S(a) \vee (r_S(a_2) \vee r_S(b_2)) \\
&= (r_S(a) \vee r_S(a_2)) \vee (r_S(a) \vee r_S(b_2)) \\
&= (r_S \times r_S)(a, a_2) \vee (r_S \times r_S)(a, b_2).
\end{align*}
\]

Again, let \( c \in V_2 \) and \( (a_1, b_1) \in E_1 \). Then, we obtain

\[
\begin{align*}
(p_T \times p_T)((a_1, c), (b_1, c)) &= p_T(a_1, b_1) \wedge p_S(c) \\
&\leq p_S(c) \wedge (p_S(a_1) \wedge p_S(b_1)) \\
&= (p_S(a_1) \wedge p_S(c)) \wedge (p_S(b_1) \wedge p_S(c)) \\
&= (p_S \times p_S)(a_1, c) \wedge (p_S \times p_S)(b_1, c).
\end{align*}
\]
(q_{T_1} \times q_{T_2})((a_1, c), (b_1, c)) = q_{T_1}(a_1, b_1) \lor q_{S_2}(c) \\
\geq q_{S_2}(c) \lor q_{S_2}(a_1) \lor q_{S_2}(b_1) \\
= (q_{S_2}(a_1) \lor q_{S_2}(c)) \lor q_{S_2}(b_1) \\
= (q_{S_2}(a_1) \lor q_{S_2}(c)) \lor q_{S_2}(b_1) \\
= (q_{S_2}(a_1) \lor q_{S_2}(c)) \lor q_{S_2}(b_1), \\
\mathbf{(i)} \\
(r_{T_1} \times r_{T_2})((a_1, c), (b_1, c)) = r_{T_1}(a_1, b_1) \lor r_{S_2}(c) \\
\geq r_{S_2}(c) \lor r_{S_2}(a_1) \lor r_{S_2}(b_1) \\
= (r_{S_2}(a_1) \lor r_{S_2}(c)) \lor r_{S_2}(b_1) \\
= (r_{S_2}(a_1) \lor r_{S_2}(c)) \lor r_{S_2}(b_1) \\
= (r_{S_2}(a_1) \lor r_{S_2}(c)) \lor r_{S_2}(b_1), \mathbf{(ii)} \\
(p_{T_1} \cdot p_{T_2})((a_1, b_1), (a_2, b_2)) = p_{T_1}(a_1, b_1) \land p_{T_2}(a_2, b_2) \\
\leq p_{T_1}(a_1, b_1) \land p_{T_2}(a_2, b_2) \\
= (p_{T_1}(a_1, b_1) \land p_{T_1}(a_2, b_2)) \\
= (p_{T_1}(a_1, b_1) \land p_{T_1}(a_2, b_2)) \\
= (p_{T_1}(a_1, b_1) \land p_{T_1}(a_2, b_2)) \\
= p_{T_1}(a_1, b_1) \land p_{T_1}(a_2, b_2), \mathbf{(iii)} \\
(q_{T_1} \cdot q_{T_2})((a_1, c), (b_1, c)) = q_{T_1}(a_1, b_1) \lor q_{T_2}(c) \\
\geq q_{T_1}(a_1, b_1) \lor q_{T_2}(c) \\
= (q_{T_1}(a_1, b_1) \lor q_{T_1}(c)) \lor q_{T_2}(c) \\
= (q_{T_1}(a_1, b_1) \lor q_{T_1}(c)) \lor q_{T_2}(c) \\
= (q_{T_1}(a_1, b_1) \lor q_{T_1}(c)) \lor q_{T_2}(c), \mathbf{(iv)} \\
(r_{T_1} \cdot r_{T_2})((a_1, c), (b_1, c)) = r_{T_1}(a_1, b_1) \lor r_{T_2}(c) \\
\geq r_{T_1}(a_1, b_1) \lor r_{T_2}(c) \\
= (r_{T_1}(a_1, b_1) \lor r_{T_2}(c)) \lor r_{T_2}(c) \\
= (r_{T_1}(a_1, b_1) \lor r_{T_2}(c)) \lor r_{T_2}(c) \\
= (r_{T_1}(a_1, b_1) \lor r_{T_2}(c)) \lor r_{T_2}(c).
Again, let $c \in V_2$ and $(a_1, b_1) \in E_1$. Then, we obtain

\[
\begin{align*}
(p_{\mathcal{T}_1} \cdot p_{\mathcal{T}_2})((a_1, c), (b_1, c)) &= p_{\mathcal{T}_1}(a_1, b_1) \land p_{\mathcal{S}_1}(c) \\
&\leq p_{\mathcal{S}_1}(c) \land (p_{\mathcal{S}_1}(a_1) \land p_{\mathcal{S}_1}(b_1)) \\
&= (p_{\mathcal{S}_1}(a_1) \land p_{\mathcal{S}_1}(c)) \land (p_{\mathcal{S}_1}(b_1) \land p_{\mathcal{S}_1}(c)) \\
&= (p_{\mathcal{S}_1} \cdot p_{\mathcal{S}_1})(a_1, c) \land (p_{\mathcal{S}_1} \cdot p_{\mathcal{S}_1})(b_1, c),
\end{align*}
\]

\[
\begin{align*}
(q_{\mathcal{T}_1} \cdot q_{\mathcal{T}_2})((a_1, c), (b_1, c)) &= q_{\mathcal{T}_1}(a_1, b_1) \lor q_{\mathcal{S}_1}(c) \\
&\geq q_{\mathcal{S}_1}(c) \lor (q_{\mathcal{S}_1}(a_1) \lor q_{\mathcal{S}_1}(b_1)) \\
&= (q_{\mathcal{S}_1}(a_1) \lor q_{\mathcal{S}_1}(c)) \lor (q_{\mathcal{S}_1}(b_1) \lor q_{\mathcal{S}_1}(c)) \\
&= (q_{\mathcal{S}_1} \cdot q_{\mathcal{S}_1})(a_1, c) \lor (q_{\mathcal{S}_1} \cdot q_{\mathcal{S}_1})(b_1, c),
\end{align*}
\]

\[
\begin{align*}
(r_{\mathcal{T}_1} \cdot r_{\mathcal{T}_2})((a_1, c), (b_1, c)) &= r_{\mathcal{T}_1}(a_1, b_1) \lor r_{\mathcal{S}_1}(c) \\
&\geq r_{\mathcal{S}_1}(c) \lor (r_{\mathcal{S}_1}(a_1) \lor r_{\mathcal{S}_1}(b_1)) \\
&= (r_{\mathcal{S}_1}(a_1) \lor r_{\mathcal{S}_1}(c)) \lor (r_{\mathcal{S}_1}(b_1) \lor r_{\mathcal{S}_1}(c)) \\
&= (r_{\mathcal{S}_1} \cdot r_{\mathcal{S}_1})(a_1, c) \lor (r_{\mathcal{S}_1} \cdot r_{\mathcal{S}_1})(b_1, c).
\end{align*}
\]

Again, let $(a_1, a_2), (b_1, b_2) \in E - E$, so $(a_1, b_1) \in E_1$, $a_2 \neq b_2$. Then, we obtain

\[
\begin{align*}
(p_{\mathcal{T}_1} \cdot p_{\mathcal{T}_2})((a_1, a_2), (b_1, b_2)) &= p_{\mathcal{S}_1}(a_2) \land p_{\mathcal{S}_1}(b_2) \land p_{\mathcal{T}_1}(a_1, b_1) \\
&\leq p_{\mathcal{S}_1}(a_2) \land p_{\mathcal{S}_1}(b_2) \land (p_{\mathcal{S}_1}(a_1) \land p_{\mathcal{S}_1}(b_1)) \\
&= p_{\mathcal{S}_1}(a_1) \land p_{\mathcal{S}_1}(a_2) \land (p_{\mathcal{S}_1}(b_1) \land p_{\mathcal{S}_1}(b_2)) \\
&= (p_{\mathcal{S}_1} \cdot p_{\mathcal{S}_1})(a_1, a_2) \land (p_{\mathcal{S}_1} \cdot p_{\mathcal{S}_1})(b_1, b_2),
\end{align*}
\]

\[
\begin{align*}
(q_{\mathcal{T}_1} \cdot q_{\mathcal{T}_2})((a_1, a_2), (b_1, b_2)) &= q_{\mathcal{S}_1}(a_2) \lor q_{\mathcal{S}_1}(b_2) \lor q_{\mathcal{T}_1}(a_1, b_1) \\
&\leq q_{\mathcal{S}_1}(a_2) \lor q_{\mathcal{S}_1}(b_2) \lor (q_{\mathcal{S}_1}(a_1) \lor q_{\mathcal{S}_1}(b_1)) \\
&= q_{\mathcal{S}_1}(a_1) \lor q_{\mathcal{S}_1}(a_2) \lor (q_{\mathcal{S}_1}(b_1) \lor q_{\mathcal{S}_1}(b_2)) \\
&= (q_{\mathcal{S}_1} \cdot q_{\mathcal{S}_1})(a_1, a_2) \lor (q_{\mathcal{S}_1} \cdot q_{\mathcal{S}_1})(b_1, b_2),
\end{align*}
\]

\[
\begin{align*}
(r_{\mathcal{T}_1} \cdot r_{\mathcal{T}_2})((a_1, a_2), (b_1, b_2)) &= r_{\mathcal{S}_1}(a_2) \lor r_{\mathcal{S}_1}(b_2) \lor r_{\mathcal{T}_1}(a_1, b_1) \\
&\leq r_{\mathcal{S}_1}(a_2) \lor r_{\mathcal{S}_1}(b_2) \lor (r_{\mathcal{S}_1}(a_1) \lor r_{\mathcal{S}_1}(b_1)) \\
&= r_{\mathcal{S}_1}(a_1) \lor r_{\mathcal{S}_1}(a_2) \lor (r_{\mathcal{S}_1}(b_1) \lor r_{\mathcal{S}_1}(b_2)) \\
&= (r_{\mathcal{S}_1} \cdot r_{\mathcal{S}_1})(a_1, a_2) \lor (r_{\mathcal{S}_1} \cdot r_{\mathcal{S}_1})(b_1, b_2).
\end{align*}
\]
The proves that \( G_1 [G_2] \) is a PFG. \( \square \)

**Definition 8.** Let \( G_1 \) and \( G_2 \) be two PFGs; then, the union of \( G_1 \) and \( G_2 \) is defined by \( G = G_1 \cup G_2 = (S_1 \cup S_2, T_1 \cup T_2) \), where

\[
\begin{align*}
(p_{S_1} \cup p_{S_2}) (a) &= \begin{cases} 
p_{S_1} (a), & \text{if } a \in V_1 \cap V_2, 
p_{S_2} (a), & \text{if } a \in V_2 \cap V_1, 
p_{S_1} (a) \lor p_{S_2} (a), & \text{if } a \in V_1 \cup V_2. 
\end{cases} \\
(q_{S_1} \cup q_{S_2}) (a) &= \begin{cases} 
q_{S_1} (a), & \text{if } a \in V_1 \cap V_2, 
q_{S_2} (a), & \text{if } a \in V_2 \cap V_1, 
q_{S_1} (a) \lor q_{S_2} (a), & \text{if } a \in V_1 \cup V_2.
\end{cases}
\end{align*}
\]

\( (p_{T_1} \cup p_{T_2}) (a, b) = p_{T_1} (a, b) \lor p_{T_2} (a, b) \leq (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) = (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) = (p_{S_1} \cup p_{S_2}) (a) \land (p_{S_1} \cup p_{S_2}) (b), \)

\( (q_{T_1} \cup q_{T_2}) (a, b) = q_{T_1} (a, b) \lor q_{T_2} (a, b) \geq (q_{S_1} (a) \lor q_{S_2} (b)) \lor (q_{S_1} (a) \land q_{S_2} (b)) = (q_{S_1} (a) \lor q_{S_2} (a)) \lor (q_{S_1} (b) \lor q_{S_2} (b)) = (q_{S_1} \cup q_{S_2}) (a) \lor (q_{S_1} \cup q_{S_2}) (b), \)

\( (r_{T_1} \cup r_{T_2}) (a, b) = r_{T_1} (a, b) \lor r_{T_2} (a, b) \geq (r_{S_1} (a) \lor r_{S_2} (b)) \lor (r_{S_1} (a) \land r_{S_2} (b)) = (r_{S_1} (a) \lor r_{S_2} (a)) \lor (r_{S_1} (b) \lor r_{S_2} (b)) = (r_{S_1} \cup r_{S_2}) (a) \lor (r_{S_1} \cup r_{S_2}) (b). \)

If \( (a, b) \in E_1 \cap E_2 \), then

\( (p_{T_1} \cup p_{T_2}) (a, b) = \begin{cases} 
p_{T_1} (a, b), & \text{if } (a, b) \in E_1 \cap E_2, 
p_{T_2} (a, b), & \text{if } (a, b) \in E_2 \cap E_1, 
p_{T_1} (a, b) \lor p_{T_2} (a, b), & \text{if } (a, b) \in E_1 \cap E_2, \end{cases} \)

\( (q_{T_1} \cup q_{T_2}) (a, b) = \begin{cases} 
q_{T_1} (a, b), & \text{if } (a, b) \in E_1 \cap E_2, 
q_{T_2} (a, b), & \text{if } (a, b) \in E_2 \cap E_1, 
q_{T_1} (a, b) \lor q_{T_2} (a, b), & \text{if } (a, b) \in E_1 \cap E_2.
\end{cases} \)

\( (r_{T_1} \cup r_{T_2}) (a, b) = \begin{cases} 
r_{T_1} (a, b), & \text{if } (a, b) \in E_1 \cap E_2, 
r_{T_2} (a, b), & \text{if } (a, b) \in E_2 \cap E_1, 
r_{T_1} (a, b) \lor r_{T_2} (a, b), & \text{if } (a, b) \in E_1 \cap E_2. \end{cases} \)

\[ \begin{align*}
(p_{T_1} \cup p_{T_2}) (a, b) &= p_{T_1} (a, b) \lor p_{T_2} (a, b) \\
&\leq (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) \\
&= (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) \\
&= (p_{S_1} \cup p_{S_2}) (a) \land (p_{S_1} \cup p_{S_2}) (b), \]

\[ \begin{align*}
(q_{T_1} \cup q_{T_2}) (a, b) &= q_{T_1} (a, b) \lor q_{T_2} (a, b) \\
&\geq (q_{S_1} (a) \lor q_{S_2} (b)) \lor (q_{S_1} (a) \land q_{S_2} (b)) \\
&= (q_{S_1} (a) \lor q_{S_2} (a)) \lor (q_{S_1} (b) \lor q_{S_2} (b)) \\
&= (q_{S_1} \cup q_{S_2}) (a) \lor (q_{S_1} \cup q_{S_2}) (b), \]

\[ \begin{align*}
(r_{T_1} \cup r_{T_2}) (a, b) &= r_{T_1} (a, b) \lor r_{T_2} (a, b) \\
&\geq (r_{S_1} (a) \lor r_{S_2} (b)) \lor (r_{S_1} (a) \land r_{S_2} (b)) \\
&= (r_{S_1} (a) \lor r_{S_2} (a)) \lor (r_{S_1} (b) \lor r_{S_2} (b)) \\
&= (r_{S_1} \cup r_{S_2}) (a) \lor (r_{S_1} \cup r_{S_2}) (b). \]
\]

This shows that \( G_1 \cup G_2 \) is a PFG. \( \square \)

**Theorem 3.** Let \( G_1 \) and \( G_2 \) be two PFGs; then, \( G_1 \cup G_2 \) is also a PFG.

**Proof.** If \( (a, b) \in E_1 \cap E_2 \), then we obtain

\( \begin{align*}
(p_{T_1} \cup p_{T_2}) (a, b) &= p_{T_1} (a, b) \lor p_{T_2} (a, b) \\
&\leq (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) \\
&= (p_{S_1} (a) \land p_{S_2} (b)) \lor (p_{S_1} (a) \lor p_{S_2} (b)) \\
&= (p_{S_1} \cup p_{S_2}) (a) \land (p_{S_1} \cup p_{S_2}) (b), \]

\( (q_{T_1} \cup q_{T_2}) (a, b) = q_{T_1} (a, b) \lor q_{T_2} (a, b) \geq (q_{S_1} (a) \lor q_{S_2} (b)) \lor (q_{S_1} (a) \land q_{S_2} (b)) = (q_{S_1} (a) \lor q_{S_2} (a)) \lor (q_{S_1} (b) \lor q_{S_2} (b)) = (q_{S_1} \cup q_{S_2}) (a) \lor (q_{S_1} \cup q_{S_2}) (b), \)

\( (r_{T_1} \cup r_{T_2}) (a, b) = r_{T_1} (a, b) \lor r_{T_2} (a, b) \geq (r_{S_1} (a) \lor r_{S_2} (b)) \lor (r_{S_1} (a) \land r_{S_2} (b)) = (r_{S_1} (a) \lor r_{S_2} (a)) \lor (r_{S_1} (b) \lor r_{S_2} (b)) = (r_{S_1} \cup r_{S_2}) (a) \lor (r_{S_1} \cup r_{S_2}) (b). \)

This shows that \( G_1 \cup G_2 \) is a PFG. \( \square \)

**Corollary 1.** Let \( \{ G_\lambda : \lambda \in \Lambda \} \) be a family of PFGs; then, \( \bigcup_{\lambda \in \Lambda} G_\lambda \) is a PFG.
Definition 9. Let $G_1$ and $G_2$ be two PFGs; then, the intersection of $G_1$ and $G_2$ is defined by $G = G_1 \cap G_2 = (S_1 \cap S_2, T_1 \cap T_2)$, where

\begin{align*}
(i) \quad & (p_{S_1} \cap p_{S_2})(a) = p_{S_1}(a) \land p_{S_2}(a), \quad a \in V_1 \cap V_2, \\
& (q_{S_1} \cap q_{S_2})(a) = q_{S_1}(a) \lor q_{S_2}(a), \quad a \in V_1 \cap V_2, \\
& (r_{S_1} \cap r_{S_2})(a) = r_{S_1}(a) \lor r_{S_2}(a), \quad a \in V_1 \cap V_2.
\end{align*}

\begin{align*}
(ii) \quad & (p_{S_1} \cap p_{S_2})(a, b) = p_{T_1}(a, b) \land p_{T_2}(a, b), \quad (a, b) \in E_1 \cap E_2, \\
& (q_{S_1} \cap q_{S_2})(a, b) = q_{T_1}(a, b) \lor q_{T_2}(a, b), \quad (a, b) \in E_1 \cap E_2, \\
& (r_{S_1} \cap r_{S_2})(a, b) = r_{T_1}(a, b) \lor r_{T_2}(a, b), \quad (a, b) \in E_1 \cap E_2.
\end{align*}

Theorem 4. Let $G_1$ and $G_2$ be two PFGs; then, $G_1 \cap G_2$ is also a PFG.

Proof. For $u, v \in V$, we obtain

\begin{align*}
(p_{T_1} \cap p_{T_2})(a, b) &= p_{T_1}(a, b) \land p_{T_2}(a, b) \\
& \leq (p_{S_1}(a) \land p_{S_2}(b)) \land (p_{S_1}(b) \land p_{S_2}(b)) \\
& = (p_{S_1}(a) \land p_{S_2}(a)) \land (p_{S_1}(b) \land p_{S_2}(b)) \\
& = (p_{S_1} \cap p_{S_2})(a) \land (p_{S_1} \cap p_{S_2})(b),
\end{align*}

\begin{align*}
(q_{T_1} \cap q_{T_2})(a, b) &= q_{T_1}(a, b) \lor q_{T_2}(a, b) \\
& \geq (q_{S_1}(a) \lor q_{S_2}(b)) \lor (q_{S_1}(b) \lor q_{S_2}(b)) \\
& = (q_{S_1} \lor q_{S_2})(a) \lor (q_{S_1} \lor q_{S_2})(b).
\end{align*}

\begin{align*}
(r_{T_1} \cap r_{T_2})(a, b) &= r_{T_1}(a, b) \lor r_{T_2}(a, b) \\
& \geq (r_{S_1}(a) \lor r_{S_2}(b)) \lor (r_{S_1}(b) \lor r_{S_2}(b)) \\
& = (r_{S_1} \lor r_{S_2})(a) \lor (r_{S_1} \lor r_{S_2})(b).
\end{align*}

This shows that $G_1 \cap G_2$ is a PFG.

Corollary 2. Let $\{G_\lambda; \lambda \in \Lambda\}$ be a family of PFGs; then, $\bigcap_{\lambda \in \Lambda} G_\lambda$ is a PFG.

Definition 10. Let $G_1$ and $G_2$ be two PFGs; then, the sum of $G_1$ and $G_2$ is defined by $G = G_1 + G_2 = (S_1 + S_2, T_1 + T_2)$, where

\begin{align*}
(i) \quad & (p_{S_1} + p_{S_2})(a) = (p_{S_1} \cup p_{S_2})(a), \\
& (q_{S_1} + q_{S_2})(a) = (q_{S_1} \cup q_{S_2})(a), \\
& (r_{S_1} + r_{S_2})(a) = (r_{S_1} \cup r_{S_2})(a), \quad \text{if } a \in V_1 \cup V_2.
\end{align*}

\begin{align*}
(ii) \quad & (p_{T_1} + p_{T_2})(a, b) = (p_{T_1} \cup p_{T_2})(a, b) = p_{T_1}(a, b), \\
& (q_{T_1} + q_{T_2})(a, b) = (q_{T_1} \cup q_{T_2})(a, b) = q_{T_1}(a, b), \\
& (r_{T_1} + r_{T_2})(a, b) = (r_{T_1} \cup r_{T_2})(a, b) = r_{T_1}(a, b), \quad \text{if } (a, b) \in E_1 \cap E_2.
\end{align*}

\begin{align*}
(iii) \quad & (p_{T_1} + p_{T_2})(a, b) = p_{S_1}(a) \lor p_{S_2}(a), \\
& (q_{T_1} + q_{T_2})(a, b) = q_{S_1}(a) \land q_{S_2}(a), \\
& (r_{T_1} + r_{T_2})(a, b) = (r_{S_1}(a) \land r_{S_2})(a) \quad \text{if } (a, b) \in E', \quad \text{where } E' \text{ is the set of edges connecting the nodes of } V_1.
\end{align*}
Theorem 5. Let \( G_1 \) and \( G_2 \) be two PFGs; then, \( G_1 \) + \( G_2 \) is also a PFG.

4. Picture Fuzzy Graph-Based Multiple Attribute Decision-Making

PFS is an important tool to solve real-world problems. PFS deals with inconsistent, incomplete, and indeterminate information or fact. Nowadays, PFS has become an exciting topic for its wide applications. So, PFG can efficiently solve such type of real-world problem.

Here, the concept of the graph is applied to MADMP with a picture fuzzy environment, and we proposed two algorithms. Also, to illustrate our proposed decision-making algorithm, we have given two examples. Let \( A = \{ A_1, A_2, A_3, \ldots, A_m \} \) be an arrangement of alternatives and \( C = \{ C_1, C_2, C_3, \ldots, C_n \} \) be the arrangement of attributes. \( w = \{ w_1, w_2, w_3, \ldots, w_n \} \) be the weight vector of the attribute \( C_i \), \( i = 1, 2, \ldots, n \), where \( w_i \geq 0 \), for \( i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} w_i = 1 \).

Let \( M = (b_{kj})_{mn} = (f_{pkj,qkj,rkj})_{mn} \) be a picture fuzzy decision matrix, where \( p_{kj} \) is the positive membership degree for which alternative \( A_j \) satisfies the attribute \( C_k \), which was given by the decision makers. \( q_{kj} \) is the neutral membership degree so that alternative \( A_j \) does not satisfy the attribute \( C_k \), and \( r_{kj} \) is the degree that the alternatives \( A_k \) does not fulfill the attribute \( C_j \) which was given by the decision maker, where \( p_{kj} \in [0, 1], q_{kj} \in [0, 1], r_{kj} \in [0, 1] \), and \( 0 \leq p_{kj} + q_{kj} + r_{kj} \leq 1, k = 1, 2, \ldots, m \). The picture fuzzy relation between two attributes \( C_i = (p_i, q_i, r_i) \) and \( C_j = (p_j, q_j, r_j) \) is defined by \( f_{ij} = (p_{ij}, q_{ij}, r_{ij}) \), where \( p_{ij} \leq p_i \land p_j \), \( q_{ij} \geq q_i \lor q_j \), and \( r_{ij} \geq r_i \lor r_j \), \( i, j = 1, 2, \ldots, m \), otherwise, \( f_{ij} = (0, 0, 1) \).

We proposed two algorithms to develop the graph structure and solve multiattribute decision-making (MADM) problems using PFG (Algorithms 1 and 2).

Let \( A = (p_j, q_j, r_j) \) be a decision solution, for \( j = 1, 2, \ldots, n \). Now, we develop an algorithm that is based on PFG and the similarity measure between picture fuzzy numbers. Here, the main advantage is that it can compute the relationship among multiple-input arguments through the graph theory approach.

5. Numerical Example

In this part, numerical examples for the PFGMADM problem with picture fuzzy information are used to present the application of the proposed algorithms. Here, we consider a MADMP taken from S. Ashraf et al. [46].

Example 1. An investment company wants to invest money in the best choice. There are four measurable alternatives:

- \( A_1 \): a car company
- \( A_2 \): a food company
- \( A_3 \): a computer company
- \( A_4 \): an arms company

The investment company makes a decision based on the three attributes:

- \( C_1 \): the risk analysis
- \( C_2 \): the growth analysis
- \( C_3 \): the environmental impact analysis

The growth vector of the attribute is given by \( w = (0.35, 0.25, 0.40) \).

The four possible alternatives are to be measured under the three attributes and are given in the form of picture fuzzy information by decision-making according to three attributes \( C_1, C_2, \) and \( C_3 \) and the evaluation information on the alternative \( A_1, A_2, A_3, \) and \( A_4 \) under the factors \( C_1, C_2, \) and \( C_3 \) can be shown in the following picture fuzzy decision matrix \( M \):

\[
M = \begin{pmatrix}
(0.6, 0.2, 0.2) & (0.8, 0.1, 0.1) & (0.6, 0.1, 0.3) \\
(0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.8, 0.1, 0.1) \\
(0.4, 0.2, 0.4) & (0.6, 0.3, 0.1) & (0.4, 0.2, 0.4) \\
(0.3, 0.1, 0.6) & (0.7, 0.1, 0.2) & (0.7, 0.1, 0.2)
\end{pmatrix}
\]  

(21)

Also, we assume that the relationship among the attribute \( C_1, C_2, \) and \( C_3 \) can be described by a complete graph \( G = (V,E) \), where \( V = \{ C_1, C_2, C_3 \} \) and \( E = \{(C_1, C_2), (C_1, C_3), (C_2, C_3)\} \), see Figure 1.

From equation (1), we get all the impact coefficients to find out the relationships among the attribute. Now, the picture fuzzy edges denoting the connections among the attributes are described as

\[
f_{12} = (p_{12}, q_{12}, r_{12}) = (0.3, 0.4, 0.5),
\]

\[
f_{13} = (p_{13}, q_{13}, r_{13}) = (0.3, 0.3, 0.4),
\]

\[
f_{23} = (p_{23}, q_{23}, r_{23}) = (0.2, 0.4, 0.4).
\]

Notice that here \( G = (V,E) \) is a PFG according to the relationship among the attribute for every alternatives. To find the best alternatives, we perform the following steps:

Step 1: the impact coefficient between the attribute \( C_j \), \( j = 1, 2, 3 \), are as follows:

\[
\eta_{12} = \frac{p_{12} + (1 - q_{12})(1 - r_{12})}{3} = \frac{0.3 + (1 - 0.4)(1 - 0.4)}{3} = 0.22,
\]

\[
\eta_{13} = \frac{p_{13} + (1 - q_{13})(1 - r_{13})}{3} = \frac{0.3 + (1 - 0.3)(1 - 0.4)}{3} = 0.24,
\]

\[
\eta_{23} = \frac{p_{23} + (1 - q_{23})(1 - r_{23})}{3} = \frac{0.2 + (1 - 0.4)(1 - 0.4)}{3} = 0.187.
\]
Step 1: calculate the impact coefficient between the attributes \( C_i \) and \( C_j \) by
\[
\eta_{ij} = ((p_{ij} + (1 - q_{ij})(1 - r_{ij}))/3)
\]
for \( i, j = 1, 2, \ldots, n \), where \( \eta_{ij} = (p_{ij}, q_{ij}, r_{ij}) \) is the picture fuzzy edge between the nodes \( C_i \) and \( C_j \) for \( i, j = 1, 2, \ldots, n \). We have \( \eta_{ij} = 1 \) and \( \eta_{ij} = \eta_{ji} \) if \( i = j \).

Step 2: find the attribute of the alternative \( A_k \) by
\[
\bar{A}_k = (\bar{p}_{k}, \bar{q}_{k}, \bar{r}_{k}) = (1/3) \sum_{j=1}^{n} w_j (\sum_{i=1}^{n} \eta_{ij} b_{ki}),
\]
where \( f_{ij} = (p_{ij}, q_{ij}, r_{ij}) \).

Step 3: calculate the score function of the alternative \( \bar{A}_k \) by
\[
\text{score}(\bar{A}_k) = (1/2)[1 - \bar{p}_k - 2\bar{q}_k - \bar{r}_k].
\]

Step 4: rank all the alternative \( A_k \) depending on \( \text{score}(\bar{A}_k) \) and then select the best alternative.

Step 5: stop.

**Algorithm 1**: Computation of best alternative.

---

Step 1: calculate the impact coefficient between the attributes \( C_i \) and \( C_j \) by
\[
\eta_{ij} = ((p_{ij} + (1 - q_{ij})(1 - r_{ij}))/3)
\]
for \( i, j = 1, 2, 3, \ldots, n \), where \( \eta_{ij} = (p_{ij}, q_{ij}, r_{ij}) \) is the picture fuzzy edge between the nodes \( C_i \) and \( C_j \) for \( i, j = 1, 2, \ldots, n \). We have \( \eta_{ij} = 1 \) and \( \eta_{ij} = \eta_{ji} \) if \( i = j \).

Step 2: compute the associated weighted value of attribute \( C_j \), for \( j = 1, 2, 3, \ldots, n \), over the other criteria by
\[
\bar{b}_{kj} = (\bar{p}_{kj}, \bar{q}_{kj}, \bar{r}_{kj}) = (1/3) w_j \sum_{i=1}^{n} \eta_{ij} b_{ki}.
\]

Step 3: find the similarity measure between the decision solution \( A = (p_j, q_j, r_j) \), \( j = 1, 2, 3, \ldots, n \), and every alternative \( A_k \), \( k = 1, 2, 3, \ldots, m \), by
\[
\text{score}(A, A_k) = 1 - (1/3n) \sum_{j=1}^{n} |p_j - \bar{p}_k| + |q_j - \bar{q}_k| + |r_j - \bar{r}_k|.
\]

Step 4: rank all the alternative \( A_k \) according to \( \text{score}(A, A_k) \), \( k = 1, 2, 3, \ldots, m \).

Step 5: stop.

**Algorithm 2**: Computation of best alternative using similarity measure.

---

**Figure 1**: The graph relationship among the three attributes.

Step 2: the attribute of the alternative \( A_1 \) is calculated below:

\[
\bar{A}_1 = \frac{1}{3} [w_1(\eta_{13}b_{11} + \eta_{23}b_{12} + \eta_{33}b_{13}) + w_2(\eta_{13}b_{11} + \eta_{22}b_{12} + \eta_{32}b_{13}) + w_3(\eta_{13}b_{11} + \eta_{23}b_{12} + \eta_{33}b_{13})]
\]

\[
= \frac{1}{3} [0.35[1 \times (0.6, 0.2, 0.2) + 0.22 \times (0.8, 0.1, 0.1) + 0.24 \times (0.6, 0.1, 0.3)]
\]

\[
+ 0.25[0.22 \times (0.6, 0.2, 0.2) + 1 \times (0.8, 0.1, 0.1) + 0.187 \times (0.6, 0.1, 0.3)]
\]

\[
+ 0.4[0.24 \times (0.6, 0.2, 0.2) + 0.187 \times (0.8, 0.1, 0.1) + 1 \times (0.6, 0.1, 0.3)]
\]

\[
= (0.314, 0.065, 0.1),
\]

\[
\bar{A}_2 = \frac{1}{3} [w_1(\eta_{13}b_{21} + \eta_{23}b_{22} + \eta_{33}b_{23}) + w_2(\eta_{13}b_{21} + \eta_{23}b_{22} + \eta_{33}b_{23}) + w_3(\eta_{13}b_{21} + \eta_{22}b_{22} + \eta_{32}b_{23})]
\]

\[
= \frac{1}{3} [0.35[1 \times (0.5, 0.3, 0.2) + 0.22 \times (0.5, 0.2, 0.3) + 0.24 \times (0.8, 0.1, 0.1)]
\]

\[
+ 0.25[0.22 \times (0.5, 0.3, 0.2) + 1 \times (0.5, 0.2, 0.3) + 0.187 \times (0.8, 0.1, 0.1)]
\]

\[
= (0.337, 0.061, 0.1). \]
Step 3: now, we compute the score functions as follows:

\[
\text{scor}(\overline{A}_1) = \frac{1}{2} \left[ 1 + \overline{p}_1 - 2\overline{q}_1 - \overline{r}_1 \right] \\
= \frac{1}{2} \left[ 1 + 0.314 - 2 \times 0.065 - 0.1 \right] = 0.542,
\]

\[
\text{scor}(\overline{A}_2) = \frac{1}{2} \left[ 1 + \overline{p}_2 - 2\overline{q}_2 - \overline{r}_2 \right] \\
= \frac{1}{2} \left[ 1 + 0.292 - 2 \times 0.088 - 0.078 \right] = 0.519,
\]

\[
\text{scor}(\overline{A}_3) = \frac{1}{2} \left[ 1 + \overline{p}_3 - 2\overline{q}_3 - \overline{r}_3 \right] \\
= \frac{1}{2} \left[ 1 + 0.218 - 2 \times 0.109 - 0.151 \right] = 0.425,
\]

\[
\text{scor}(\overline{A}_4) = \frac{1}{2} \left[ 1 + \overline{p}_4 - 2\overline{q}_4 - \overline{r}_4 \right] \\
= \frac{1}{2} \left[ 1 + 0.264 - 2 \times 0.048 - 0.162 \right] = 0.505.
\]

Step 4: therefore, we rank these alternatives as \(A_1 \succ A_2 \succ A_4 \succ A_3\). From the above numerical observation we have, \(A_1\) is the best choice in the decision-making problem.

Example 2. In this example, we consider medical diagnosis problem adapted from Ye [47]. Let us consider a set of diagnosis as \(A = \{A_1\text{(viral fever)}, A_2\text{(malaria)}, A_3\text{(typhoid)}, A_4\text{(gastritis)}, A_5\text{(stenocardia)}\}\) and set of symptoms as \(C = \{C_1\text{(temperature)}, C_2\text{(headache)}, C_3\text{(stomach pain)}, C_4\text{(cough)}, C_5\text{(stenocardia)}\}\)
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Table 1: Performance values of the disease.

| $A_1$ (viral fever) | $A_2$ (malaria) | $A_3$ (typhoid) | $A_4$ (gastritis) | $A_5$ (stenocardia) |
|---------------------|-----------------|-----------------|-------------------|-------------------|
| $C_1$ (temperature) | (0.052, 0.037, 0.042) | (0.028, 0.014, 0.037) | (0.010, 0.011, 0.310) | (0.042, 0.023, 0.042) | (0.028, 0.024, 0.101) |
| $C_2$ (headache)    | (0.079, 0.016, 0.038) | (0.032, 0.010, 0.038) | (0.010, 0.005, 0.039) | (0.062, 0.027, 0.027) | (0.032, 0.008, 0.114) |
| $C_3$ (stomach pain) | (0.049, 0.026, 0.067) | (0.039, 0.015, 0.028) | (0.012, 0.006, 0.037) | (0.034, 0.015, 0.060) | (0.031, 0.010, 0.124) |
| $C_4$ (cough)       | (0.038, 0.021, 0.091) | (0.018, 0.019, 0.047) | (0.030, 0.006, 0.016) | (0.026, 0.017, 0.074) | (0.049, 0.020, 0.095) |
| $C_5$ (chestpain)   | (0.027, 0.012, 0.122) | (0.011, 0.007, 0.068) | (0.016, 0.002, 0.039) | (0.024, 0.005, 0.091) | (0.090, 0.013, 0.066) |

Figure 2: The graph relationship among the five attributes.

Similarly, $\eta_{15} = 0.140$, $\eta_{23} = 0.053$, $\eta_{24} = 0.343$, $\eta_{25} = 0.140$, $\eta_{34} = 0.093$, $\eta_{35} = 0.280$, and $\eta_{45} = 0.140$.

Now, the associated weighted values of disease are obtained by $\tilde{b}_{kj} = \frac{1}{3} \sum_{s=1}^{5} \eta_{js} b_{ks}$, where $\tilde{b}_{kj} = (p_{kj}, q_{kj}, r_{kj})$ is a PFN.

Therefore,

\[
\begin{align*}
\tilde{b}_{11} &= \frac{w_1}{3} \sum_{s=1}^{5} \eta_{1s} b_{1s} \\
&= \frac{w_1}{3} \left( \eta_{11} b_{11} + \eta_{12} b_{12} + \eta_{13} b_{13} + \eta_{14} b_{14} + \eta_{15} b_{15} \right) \\
&= \frac{0.25}{3} \left[ 1 \times (0.4, 0.3, 0.0) + 0.277 \times (0.3, 0.1, 0.5) + 0.273 \times (0.1, 0.2, 0.7) \\
&\quad + 0.260 \times (0.4, 0.2, 0.3) + 0.140 \times (0.1, 0.1, 0.7) \right] \\
&= (0.052, 0.037, 0.042),
\end{align*}
\]

\[
\begin{align*}
\tilde{b}_{12} &= \frac{w_2}{3} \sum_{s=1}^{5} \eta_{2s} b_{1s} \\
&= \frac{w_2}{3} \left( \eta_{12} b_{11} + \eta_{13} b_{12} + \eta_{14} b_{13} + \eta_{15} b_{14} + \eta_{25} b_{15} \right) \\
&= \frac{0.15}{3} \left[ 0.277 \times (0.4, 0.3, 0.0) + 1 \times (0.3, 0.1, 0.5) + 0.053 \times (0.1, 0.2, 0.7) \\
&\quad + 0.343 \times (0.4, 0.2, 0.3) + 0.140 \times (0.1, 0.1, 0.7) \right] \\
&= (0.028, 0.014, 0.037),
\end{align*}
\]
\[
\tilde{b}_{13} = \frac{w_1}{3} \sum_{i=1}^{5} \eta_{i3} b_{1i}
= \frac{w_1}{3} (\eta_{13} b_{11} + \eta_{23} b_{12} + \eta_{33} b_{13} + \eta_{43} b_{14} + \eta_{53} b_{15})
= \frac{0.10}{3} [0.273 \times (0.4, 0.3, 0.0) + 0.053 \times (0.3, 0.1, 0.5) + 1 \times (0.1, 0.2, 0.7)
+ 0.093 \times (0.4, 0.2, 0.3) + 0.280 \times (0.1, 0.1, 0.7)]
= (0.010, 0.011, 0.031),
\]

\[
\tilde{b}_{14} = \frac{w_4}{3} \sum_{i=1}^{5} \eta_{i4} b_{1i}
= \frac{w_4}{3} (\eta_{14} b_{11} + \eta_{24} b_{12} + \eta_{34} b_{13} + \eta_{44} b_{14} + \eta_{54} b_{15})
= \frac{0.20}{3} [0.26 \times (0.4, 0.3, 0.0) + 0.034 \times (0.3, 0.1, 0.5) + 0.093 \times (0.1, 0.2, 0.7)
+ 1 \times (0.4, 0.2, 0.3) + 0.140 \times (0.1, 0.1, 0.7)]
= (0.042, 0.023, 0.042),
\]

\[
\tilde{b}_{15} = \frac{w_5}{3} \sum_{i=1}^{5} \eta_{i5} b_{1i}
= \frac{w_5}{3} (\eta_{15} b_{11} + \eta_{25} b_{12} + \eta_{35} b_{13} + \eta_{45} b_{14} + \eta_{55} b_{15})
= \frac{0.30}{3} [0.140 \times (0.4, 0.3, 0.0) + 0.140 \times (0.3, 0.1, 0.5) + 0.280 \times (0.1, 0.2, 0.7)
+ 0.140 \times (0.4, 0.2, 0.3) + 1 \times (0.1, 0.1, 0.7)]
= (0.028, 0.024, 0.101),
\]

\[
\tilde{b}_{21} = \frac{w_1}{3} \sum_{i=1}^{5} \eta_{i1} b_{2i}
= \frac{w_1}{3} (\eta_{11} b_{21} + \eta_{21} b_{22} + \eta_{31} b_{23} + \eta_{41} b_{24} + \eta_{51} b_{25})
= \frac{0.25}{3} [1 \times (0.7, 0.1, 0.0) + 0.277 \times (0.2, 0.1, 0.6) + 0.273 \times (0.0, 0.1, 0.9)
+ 0.260 \times (0.7, 0.2, 0.0) + 0.140 \times (0.1, 0.0, 0.8)]
= (0.079, 0.016, 0.038),
\]

\[
\tilde{b}_{22} = \frac{w_2}{3} \sum_{i=1}^{5} \eta_{i2} b_{2i}
= \frac{w_2}{3} (\eta_{12} b_{21} + \eta_{22} b_{22} + \eta_{32} b_{23} + \eta_{42} b_{24} + \eta_{52} b_{25})
= \frac{0.15}{3} [0.277 \times (0.7, 0.1, 0.0) + 1 \times (0.2, 0.1, 0.6) + 0.053 \times (0.0, 0.1, 0.9)
+ 0.343 \times (0.7, 0.2, 0.0) + 0.140 \times (0.1, 0.0, 0.8)]
= (0.032, 0.010, 0.038).
\]
Similarly, \( \overline{b}_{23} = (0.010, 0.005, 0.039), \quad \overline{b}_{24} = (0.062, 0.027, 0.027), \) and \( \overline{b}_{25} = (0.032, 0.008, 0.114). \)

\[
\overline{b}_{31} = \frac{w_1}{3} \sum_{i=1}^{5} \eta_{i1} b_{3i}
\]
\[
= \frac{w_1}{3} \left( \eta_{11} b_{31} + \eta_{21} b_{32} + \eta_{31} b_{33} + \eta_{41} b_{34} + \eta_{51} b_{35} \right)
\]
\[
= \frac{0.25}{3} \left[ (0.3, 0.2, 0.3) + 0.277 \times (0.6, 0.2, 0.1) + 0.273 \times (0.2, 0.1, 0.7) 
+ 0.260 \times (0.2, 0.1, 0.6) + 0.140 \times (0.1, 0.0, 0.9) \right]
\]
\[
= (0.049, 0.026, 0.067).
\]

Similarly, \( \overline{b}_{32} = (0.039, 0.015, 0.028), \quad \overline{b}_{33} = (0.012, 0.006, 0.037), \quad \overline{b}_{34} = (0.034, 0.015, 0.060), \) and \( \overline{b}_{35} = (0.031, 0.010, 0.124). \)

\[
\overline{b}_{41} = \frac{w_1}{3} \sum_{i=1}^{5} \eta_{i1} b_{4i}
\]
\[
= \frac{w_1}{3} \left( \eta_{11} b_{41} + \eta_{21} b_{42} + \eta_{31} b_{43} + \eta_{41} b_{44} + \eta_{51} b_{45} \right)
\]
\[
= \frac{0.25}{3} \left[ (0.1, 0.1, 0.7) + 0.277 \times (0.2, 0.3, 0.4) + 0.273 \times (0.8, 0.1, 0.0) 
+ 0.260 \times (0.2, 0.1, 0.7) + 0.140 \times (0.2, 0.1, 0.7) \right]
\]
\[
= (0.038, 0.021, 0.091).
\]

Similarly, \( \overline{b}_{42} = (0.018, 0.019, 0.047), \quad \overline{b}_{43} = (0.030, 0.006, 0.016), \quad \overline{b}_{44} = (0.026, 0.017, 0.074), \) and \( \overline{b}_{45} = (0.049, 0.020, 0.095). \)

\[
\overline{b}_{51} = \frac{w_1}{3} \sum_{i=1}^{5} \eta_{i1} b_{5i}
\]
\[
= \frac{w_1}{3} \left( \eta_{11} b_{51} + \eta_{21} b_{52} + \eta_{31} b_{53} + \eta_{41} b_{54} + \eta_{51} b_{55} \right)
\]
\[
= \frac{0.25}{3} \left[ (0.1, 0.1, 0.8) + 0.277 \times (0.0, 0.1, 0.8) + 0.273 \times (0.2, 0.0, 0.8) 
+ 0.260 \times (0.2, 0.0, 0.8) + 0.140 \times (0.8, 0.1, 0.1) \right]
\]
\[
= (0.027, 0.012, 0.0122).
\]

Similarly, \( \overline{b}_{52} = (0.011, 0.007, 0.068), \quad \overline{b}_{53} = (0.016, 0.002, 0.039), \quad \overline{b}_{54} = (0.024, 0.005, 0.091), \) and \( \overline{b}_{55} = (0.090, 0.013, 0.066). \)

Therefore, the results obtained are shown in Table 2: The similarity measure between the ideal solution \( A \) and each diseases \( A_k, k = 1, 2, 3, 4, 5, \) are calculated below:
scor \((A, A_1)\) = \(1 - \frac{1}{15} \left[ |p_1 - p_{11}| + |q_1 - q_{11}| + |r_1 - r_{11}| + |p_2 - p_{12}| + |q_2 - q_{12}| + |r_2 - r_{12}|\)
\[+ |p_3 - p_{13}| + |q_3 - q_{13}| + |r_3 - r_{13}| + |p_4 - p_{14}| + |q_4 - q_{14}| + |r_4 - r_{14}|\]
\[+ |p_5 - p_{15}| + |q_5 - q_{15}| + |r_5 - r_{15}|\]
\(= 1 - \frac{1}{15} \left[ 0.869 + 0.821 + 0.691 + 0.693 + 0.647 \right] = 0.7519, \)

scor \((A, A_2)\) = \(1 - \frac{1}{15} \left[ |p_1 - p_{21}| + |q_1 - q_{21}| + |r_1 - r_{21}| + |p_2 - p_{22}| + |q_2 - q_{22}| + |r_2 - r_{22}|\]
\[+ |p_3 - p_{23}| + |q_3 - q_{23}| + |r_3 - r_{23}| + |p_4 - p_{24}| + |q_4 - q_{24}| + |r_4 - r_{24}|\]
\[+ |p_5 - p_{25}| + |q_5 - q_{25}| + |r_5 - r_{25}|\]
\(= 1 - \frac{1}{15} \left[ 0.867 + 0.820 + 0.956 + 0.684 + 0.646 \right] = 0.7351, \)

scor \((A, A_3)\) = \(1 - \frac{1}{15} \left[ |p_1 - p_{31}| + |q_1 - q_{31}| + |r_1 - r_{31}| + |p_2 - p_{32}| + |q_2 - q_{32}| + |r_2 - r_{32}|\]
\[+ |p_3 - p_{33}| + |q_3 - q_{33}| + |r_3 - r_{33}| + |p_4 - p_{34}| + |q_4 - q_{34}| + |r_4 - r_{34}|\]
\[+ |p_5 - p_{35}| + |q_5 - q_{35}| + |r_5 - r_{35}|\]
\(= 1 - \frac{1}{15} \left[ 0.858 + 0.818 + 0.957 + 0.691 + 0.635 \right] = 0.7361, \)

scor \((A, A_4)\) = \(1 - \frac{1}{15} \left[ |p_1 - p_{41}| + |q_1 - q_{41}| + |r_1 - r_{41}| + |p_2 - p_{42}| + |q_2 - q_{42}| + |r_2 - r_{42}|\]
\[+ |p_3 - p_{43}| + |q_3 - q_{43}| + |r_3 - r_{43}| + |p_4 - p_{44}| + |q_4 - q_{44}| + |r_4 - r_{44}|\]
\[+ |p_5 - p_{45}| + |q_5 - q_{45}| + |r_5 - r_{45}|\]
\(= 1 - \frac{1}{15} \left[ 0.850 + 0.816 + 0.960 + 0.717 + 0.636 \right] = 0.7347, \)

scor \((A, A_5)\) = \(1 - \frac{1}{15} \left[ |p_1 - p_{51}| + |q_1 - q_{51}| + |r_1 - r_{51}| + |p_2 - p_{52}| + |q_2 - q_{52}| + |r_2 - r_{52}|\]
\[+ |p_3 - p_{53}| + |q_3 - q_{53}| + |r_3 - r_{53}| + |p_4 - p_{54}| + |q_4 - q_{54}| + |r_4 - r_{54}|\]
\[+ |p_5 - p_{55}| + |q_5 - q_{55}| + |r_5 - r_{55}|\]
\(= 1 - \frac{1}{15} \left[ 0.883 + 0.815 + 0.947 + 0.680 + 0.631 \right] = 0.7363. \)

Table 2: The associated weighted values of the disease.

| \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) | \(A_5\) |
|-----|-----|-----|-----|-----|
| \((0.052, 0.037, 0.042)\) | \((0.028, 0.014, 0.037)\) | \((0.010, 0.011, 0.030)\) | \((0.042, 0.023, 0.042)\) | \((0.028, 0.024, 0.101)\) |
| \((0.079, 0.016, 0.038)\) | \((0.032, 0.010, 0.038)\) | \((0.010, 0.005, 0.039)\) | \((0.062, 0.027, 0.027)\) | \((0.032, 0.008, 0.114)\) |
| \((0.049, 0.026, 0.067)\) | \((0.039, 0.015, 0.028)\) | \((0.034, 0.015, 0.060)\) | \((0.031, 0.010, 0.124)\) | \((0.031, 0.010, 0.124)\) |
| \((0.038, 0.021, 0.091)\) | \((0.018, 0.019, 0.047)\) | \((0.030, 0.006, 0.016)\) | \((0.026, 0.017, 0.074)\) | \((0.049, 0.020, 0.095)\) |
| \((0.027, 0.012, 0.122)\) | \((0.011, 0.007, 0.068)\) | \((0.016, 0.002, 0.039)\) | \((0.024, 0.005, 0.091)\) | \((0.090, 0.013, 0.066)\) |
Thus, the patient A can be diagnosed with the diseases $A_1$ (viral fever) according to the recognition principle. The ranking is the same as J. Ye [2011]. The above example indicates that this type of decision-making algorithm is well suited for picture fuzzy environment and is a useful technique that provides a different respective than others for picture fuzzy environment.

6. Conclusion and Future Directions

Graph theory is a needful tool for solving MADMP in different areas. PFG is a new dimension of graph theory which is a useful tool for solving real-world problems. Most of MADM algorithms with picture fuzzy environment discuss a type of problem with no relationship among attributes. Although this relationship should be considered in the actual applications, so we need to pay attention to that issue. This article applies graph theory to PFS and obtained a new method for solving complicated problems under picture fuzzy information. The proposed method can capture the relationship among the attributes that cannot be handled well by any available methods. In this study, we introduce union, intersection, sum, Cartesian product, and the composition of PFG. Finally, by considering the importance of relationships among attributes in the decision process, two new techniques based on single-valued PFG were developed to solve complicated problems using picture fuzzy information. Also, two numerical examples were presented to explain how to deal with the MADMP under a picture fuzzy environment. In the future, we can solve this type of MADM problem using soft sets, picture fuzzy hesitant fuzzy sets, and spherical and T-spherical fuzzy sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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