Low-temperature thermodynamics of the two-leg ladder Ising model with trimer rungs: A mystery explained

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Recently, a surprising low-temperature behavior has been revealed in a two-leg ladder Ising model with trimer rungs (Weiguo Yin, arXiv:2006.08921). Motivated by these findings, we study this model from another perspective and demonstrate that the reported observations are related to a critical phenomenon in the standard Ising chain. We also discuss a related curiosity, namely, the emergence of a power-law behavior characterized by quasicritical exponents.

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Recently, Weiguo Yin considered several two-leg-ladder Ising models with short-range interactions which exhibit intriguing behavior at low temperatures which strongly resembles a finite-temperature phase transition [1, 2]. A simplest model is defined by the Hamiltonian

\[
H = -\sum_{i=1}^{N} [J (\sigma_{i,1} \sigma_{i+1,1} + \sigma_{i,2} \sigma_{i+1,2}) + J_1 (\sigma_{i,1} \sigma_{i+3} + \sigma_{i,2} \sigma_{i+3}) + J_2 \sigma_{i,1} \sigma_{i,2}]
\]

which is associated with the lattice shown in Fig. 1 top. Here \(\sigma = \pm 1\), \(N\) is the number of rungs, the positive/negative sign of the Ising exchange couplings corresponds to the ferromagnetic/antiferromagnetic interaction, and the exchange-coupling scheme is illustrated in Fig. 1 top. (Our \(J_1\) and \(J_2\) correspond to \(J'\) and \(J''\) of Ref. [1], respectively; we have changed the notations since in what follows we use the prime notation to denote the first and second derivatives with respect to temperature.) The model (1) is exactly solvable by the transfer-matrix method [1, 2]. Weiguo Yin found that for \(J_2 < 0\) and small positive values of the parameter \(\alpha = (|J_1| - |J_2|)/|J|\) the exact specific heat exhibits a sharp peak at certain (but finite) temperature, the exact entropy shows a waterfall behavior at this temperature etc [1]. He referred to such a phenomenon as marginal phase transition (MPT) [1, 2] or practically perfect phase transition (PPPT) [2] and provided an extensive mathematical analysis to explain this phenomenon. It is worthwhile noting that similar peculiar low-temperature thermodynamics was observed for other one-dimensional Ising-like models with short-range interactions [4–15].

In what follows, we will explain the astonishing low-temperature behavior of the model (1) using the approach suggested in Ref. [16]. Our present study is not restricted to model (1), but has a broader significance illustrating that the low-temperature peculiarities reported in Refs. [1, 2, 4–15] all are related to the criticality of the standard Ising chain at zero temperature [3].

The first step is to trace out in the partition function of the initial model (1) the spins \(\sigma_{i,3}, i = 1, \ldots, N\) at the middle of the rung, see Fig. 1 top, and to arrive at the effective model – the standard two-leg rail-road Ising ladder with the interaction along the legs \(-J\) and the temperature-dependent interaction that couples two legs \(-J_\perp\), see Fig. 1 bottom (decoration-iteration transformations [17]). The Hamiltonian of the effective model reads:

\[
H = -NT \ln C - \sum_{i=1}^{N} [J (\sigma_{i,1} \sigma_{i+1,1} + \sigma_{i,2} \sigma_{i+1,2}) + J_\perp \sigma_{i,1} \sigma_{i,2}],
\]

\[
C = 2 \sqrt{\frac{2J_1}{T}}; \quad J_\perp = J_\perp (T) = J_2 + \frac{T}{2} \ln \cosh \frac{2J_1}{T}.
\]

This representation was noticed in Ref. [1] but not used for further analysis.

The thermodynamics of the model (2) can be found

![Diagram of the initial two-leg ladder Ising model with trimer rungs](image-url)
by the transfer-matrix method \[3\]. The Helmholtz free energy (per rung) \( f \) reads:

\[
f = -T \ln C - T \ln \lambda_1,
\]

\[
\lambda_1 = 2 \left( \cosh \frac{J_\perp}{T} \cosh \frac{2J}{T} + \sqrt{1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}} \right).
\]

The peculiarity of the effective ladder model \[2\] stems from a dependence of the interleg coupling \(-J_\perp\) on the temperature \( T \). Therefore, the internal energy \( e \), the entropy \( s \), and the specific heat \( c \) are given by the formulas:

\[
e = -T^2 \frac{\partial}{\partial T} \frac{f}{T} - T \frac{\partial f}{\partial J} J_\perp = \varepsilon^{(1)} + \varepsilon^{(2)},
\]

\[
s = -\frac{\partial f}{\partial T} - \frac{\partial f}{\partial J} J_\perp = s^{(1)} + s^{(2)},
\]

\[
c = -T \frac{\partial^2 f}{\partial T^2} - 2T \frac{\partial^2 f}{\partial T \partial J_\perp} J_\perp - T \frac{\partial^2 f}{\partial J_\perp^2} \left( J_\perp^2 - T \frac{\partial f}{\partial J} J_\perp \right) = \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)} + \varepsilon^{(4)}.
\]

where according to Eq. \[3\]

\[
\frac{\partial f}{\partial J_\perp} = -\frac{\sinh \frac{J_\perp}{T} \cosh \frac{2J}{T}}{1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}},
\]

\[
\frac{\partial^2 f}{\partial T \partial J_\perp} = J_\perp \frac{\cosh \frac{J_\perp}{T} \cosh \frac{3J}{T} + 2J \sinh \frac{J_\perp}{T} \sinh \frac{3J}{T}}{T^2 \left(1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}\right)^{\frac{3}{2}}},
\]

\[
\frac{\partial^2 f}{\partial J_\perp^2} = -\frac{\cosh \frac{J_\perp}{T} \sinh \frac{3J}{T}}{T \left(1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}\right)^{\frac{3}{2}}},
\]

and \( J_\perp = \partial J \perp / \partial T \), \( J_\perp' = \partial^2 J_\perp / \partial T^2 \). It is worthy noting that \( \partial f / \partial J_\perp = -\sum_{j=1}^N (\sigma_{j-1}\sigma_2) / N \equiv -C_{12}(0) \), where \( C_{12}(0) \) is the on-rung correlation between two outer spins on the legs, see Ref. \[1\]. Moreover, for the correlation length \( \xi \) we have

\[
1 = \ln \frac{\cosh \frac{J_\perp}{T} \cosh \frac{3J}{T} + \sqrt{1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}}}{\cosh \frac{J_\perp}{T} \cosh \frac{3J}{T} - \sqrt{1 + \sinh^2 \frac{J_\perp}{T} \cosh^2 \frac{2J}{T}}}.
\]

Equations \[2\] - \[6\] explain the enigmatic low-temperature behavior of the initial model \[1\]. Assume that the effective interleg interaction in Eq. \[2\] is ferromagnetic for \( 0 \leq T < T_p \), vanishes at \( T = T_p \), and becomes antiferromagnetic for \( T_p < T \), see Fig. \[2\] for examples. The equation

\[
J_\perp(T_p) = 0
\]

has a simple analytical solution for \( T_p > 0 \) if \( J_2 = -|J_2| < 0 \) and \( (|J_1| - |J_2|)/|J_1| \to +0 \). Namely, \( T_p = 2(|J_1| - |J_2|)/\ln 2 \approx 2.885(|J_1| - |J_2|) \). This is the MPT temperature reported in Ref. \[1\]. At \( T = T_p \) the legs decouple, see Eq. \[7\], and the correlation length \( \xi \) becomes \( 1/\xi(T_p) = \ln[(\cosh(2J/T_p) + 1)/\cosh(2J/T_p) - 1] \), cf. the correlation length for the standard Ising chain \[3\]. It is not surprising that \( \xi(T_p) \) is extremely large if \( T_p/|J| \) is small enough, i.e., the correlation length for the standard Ising chain at the temperature \( T_P \) which is low in the scale of \( |J| \) is obviously large [in this limit \( \xi(T_p) \propto \exp(2|J|/T_p) \)]. Hence, if

\[
T_p \frac{|J|}{|J|} \ll 1
\]

holds, one observes the traces of the Ising-chain criticality in the behavior of the initial model \[1\] which were interpreted as very unusual and puzzling phenomena. For above mentioned specific case \( J_2 = -|J_2| < 0 \) and \( (|J_1| - |J_2|)/|J_1| \to +0 \), the inequality \[8\] becomes \( (|J_1| - |J_2|)/|J| = \alpha \ll 1 \), i.e., corresponds to the strong frustration regime of Ref. \[1\]. Substituting \( J_\perp(T_p) = 0 \) into Eq. \[5\] we obtain: \( \partial f / \partial J_\perp = \partial^2 f / \partial T \partial J_\perp = 0 \), and \( \partial^2 f / \partial J_\perp^2 = -\cosh(2J/T_p)/|J| \). If Eq. \[8\] holds, this results in a large value of the specific heat \( c(T_p) \), see the third term \( c^{(3)} \) in the formula for \( c \) in Eq. \[4\] which is \( \propto \exp(2|J|/T_p) \). Let us consider \( T \) in the vicinity of \( T_p \) when the inequality \( \sinh^2(J_\perp/T) \cosh^2(2J/T) \gg 1 \) holds. Then, according to Eq. \[5\], \( \partial f / \partial J_\perp \approx \text{sgn}(J_\perp) \) whereas \( \partial^2 f / \partial T \partial J_\perp \propto \text{sgn}(J_\perp)/|J_\perp| \) and \( \partial^2 f / \partial J_\perp^2 \propto 1/|J_\perp|^3 \). Moreover, according to Eq. \[6\], \( 1/\xi \propto |J_\perp| \). Bearing in mind that \( J_\perp \propto T - T_p \) around \( T_p \) (see Fig. \[2\]), one immediately concludes that while approaching \( T_p \) (i) \( e \) and \( s \) show a jump, see the second terms \( e^{(2)} \) and \( s^{(2)} \) in the formulas for \( e \) and \( s \) in Eq. \[4\], and (ii) \( c \) and \( \xi \) show power-law dependences: \( c \propto (T - T_p)^{-3} \) and \( \xi \propto |T - T_p|^{-1} \), see the third term \( c^{(3)} \) in the formula for \( c \) in Eq. \[4\] and Eq. \[8\]. Such quasicritical exponents
$\alpha = \alpha' = 3$ and $\nu = \nu' = 1$ were observed for similar one-dimensional Ising-like models in Ref. [12]. It is worthy noting that the exactly calculated specific heat $c$ [Eq. (4)] satisfies the sum rule: $\int_0^\infty dT c/T = s(\infty) - s(0)$, where $s(\infty) = 3 \ln 2 \approx 2.079$ and $s(0) = 0$ are the entropies at infinite and zero temperatures, respectively. Hence, the higher the peak of $c$ around $T_p$ is, a narrower it should be. Recall also that precisely at $T_p$ both $c$ and $\xi$ are finite, see above.

To illustrate the discussion, we consider a specific example conveniently setting $J_1 = 9$, $J_2 = -3$, and $J = 100$. In this case the frustration parameter $\alpha = 0.06$ whereas $T_p \approx 24.937$ yielding $T_p/|J| \approx 0.249$ in Eq. (8) since $\left(|J_1| - |J_2|\right)/|J| \approx 0.667$, the analytical formula underestimates the value of $T_p$. $J_\perp$ as a function of $T$ for this set can be seen in the main panel of Fig. 2. The entropy $s$ [Eq. (2)] exhibits a jump at $T_p$ due to the term $s^{(2)} = -\left(\partial f/\partial J_\perp\right)_T J_\perp^2$, $s(T_p + 0) - s(T_p - 0) \approx 0.2$, see the black curve in Fig. 3 top. The specific heat $c$ [Eq. (4)] exhibits a peak around $T_p$ due to the term $c^{(3)} = -T^2(\partial f/\partial T)(T/J)^2$, $c(T_p) \approx 16.3$, see the black curves in Fig. 3 middle and bottom. The correlation length $\xi$ [Eq. (5)] also exhibits a peak around $T_p$, $\xi(T_p) \approx 760$, see the blue curves in Fig. 3 middle and bottom. Since $|J_\perp(T_p)| < 1$, we have $c(T_p) < \xi(T_p)$. Finally, Fig. 4 demonstrates quasicritical behavior: There is a finite range of temperatures in the vicinity of $T_p$ on which a clear power-law behavior develops with $\alpha = \alpha' = 3$ for the specific heat (black curves) and $\nu = \nu' = 1$ for the correlation length (blue curves). Moreover, the smaller is $T_p/|J|$ in the left-hand side of the inequality in Eq. (8), the larger the region of quasicriticality is, cf. the top panel $(T_p/|J| \approx 0.249)$ and the bottom panel $(T_p/|J| \approx 0.125)$ in Fig. 4. Recall that the quasicritical behavior fails for temperatures in the immediate vicinity of $T_p$ where both $c$ and $\xi$ are finite, see the values of $c$ and $\xi$ at the smallest values of $\tau$ in Fig. 4.

Let us summarize our findings. First of all, we have to emphasize the following. The model at hand is exactly solvable one (as well as other models discussed in Refs. [1, 2, 4–15]) and thus there is no much room for speculations about the solutions: The performed transformations are rigorous and the derived results are exact. However, the final results are somewhat astonishing: For example, the specific heat per cell $c$ may exhibit an extra low-temperature peak of extremely large height, the correlation length $\xi$ at this temperature is unexpectedly large etc, see Figs. 3 and 4. Naturally, these features of the rigorously found quantities call for explanations.

As a rule, the authors sought for explanations at the level of the initial models. In contrast, in Ref. [16] we suggest to examine, instead of several diverse initial models, the effective model which shows up after summing over redundant degrees of freedom. The effective model that has emerged after this rigorous decoration-iteration transformation is the Ising chain with temperature-dependent parameters. In the present study, applying this way of thinking to a typical representative consid-
considered in Refs. [1, 2] – the two-leg ladder Ising model with trimer rungs – we have arrived at the two-leg Ising ladder with temperature-dependent rung couplings $J_\perp = J_\perp(T)$, Eq. (2). Again, the analysis of the properties of the obtained effective two-leg Ising ladder with temperature-dependent parameters explains the low-temperature thermodynamics of a whole class of the initial models which can be cast into that effective model after eliminating redundant spins.

Although we do not deny the usefulness of studies on the level of the initial model, however, we see several advantages of the study based on the effective model. First, we clearly see a universality of the phenomenon: Many initial models collapse to one effective model and, after all, the low-temperature peculiarities of two-leg-ladder Ising models [1, 2] as well as of other one-dimensional Ising-like models [10] are related to the criticality of the standard Ising chain at zero temperature [2]. From our consideration it looks that frustration is not vitally necessary (see also Ref. [10]); the only demand is to have a suitable $J_\perp(T)$. The characteristic temperature $T_p$ is determined by the condition when the effective model reduces to the standard Ising-chain model, see Eq. (4). Temperature-dependent parameters of the effective model result in more complex formulas for thermodynamic quantities, see Eq. (4). If $T_p$ is small in the scale of the effective Ising-chain model, Eq. (5), the large values of the specific heat or the correlation length are evident. Moreover, as it follows from Eqs. (4), (5), (6) and Eq. (2), the temperature dependences around $T_p$ (but not in the immediate vicinity of $T_p$) follow power laws with quasicritical exponents $\alpha = \alpha' = 3$ and $\nu = \nu' = 1$.

Finally, our work paves the path for the higher-dimensional cases. For example, one can consider a two-dimensional rectangular effective Ising model with the horizontal couplings $J$ and the vertical couplings $J_\perp$ given in Eq. (2). Obviously, if condition (6) holds, the two-dimensional system becomes a set of noninteracting Ising chains and it exhibits the behavior discussed above.

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