Focusing light in a bianisotropic slab with negatively refracting materials

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Abstract

We investigate the electromagnetic response of a pair of complementary bianisotropic media, which consist of a medium with positive refractive index ($\epsilon_+, \mu_+, \xi_+$) and a medium with negative refractive index ($\epsilon_-, \mu_-, \xi_-$). We show that this idealized system has peculiar imaging properties in that it reproduces images of a source, in principle, with unlimited resolution. We then consider an infinite array of line sources regularly spaced in a 1D photonic crystal (PC) consisting of $2n$ layers of bianisotropic complementary media. Using coordinate transformations, we map this system into 2D corner chiral lenses of $2n$ heterogeneous anisotropic complementary media sharing a vertex, within which light circles around closed trajectories. Alternatively, one can consider corner lenses with homogeneous isotropic media and map them into 1D PCs with heterogeneous bianisotropic layers. Interestingly, such complementary media are described by scalar, or matrix valued, sign-shifting parameters, which satisfy a new version of the generalized lens theorem of Pendry and Ramakrishna. This theorem can be derived using Fourier series solutions of the Maxwell–Tellegen equations, or from space–time symmetry arguments. Also of interest are 2D periodic checkerboards consisting of alternating rectangular cells of complementary media which are such that one point source in one cell gives rise to an infinite set of images with an image in every other cell. Such checkerboards can themselves be mapped into a class of 3D corner lenses of complementary bianisotropic media. These theoretical results are illustrated by finite element computations.

(Some figures may appear in colour only in the online journal)

1. Introduction

Five years after Pendry’s seminal paper [1] which established that negative refraction (i.e. simultaneously negative permittivity and permeability [2]) makes a perfect lens, Jin and He proposed to use a slab of chiral medium in order to achieve a refocusing effect [3]. The latter authors noted that, at the interface between a conventional medium (vacuum with permittivity $\epsilon_0$ and permeability $\mu_0$) and a chiral medium (with permittivity $\epsilon_1$, permeability $\mu_1$ and chirality $\kappa$), the refractive indices for circularly polarized waves associated with wavenumbers $k_{\pm} = \pm \omega (\sqrt{\mu_1 \epsilon_1} \pm \sqrt{\mu_0 \epsilon_0})$ are $n_{\pm} = \sqrt{(\mu_1 \epsilon_1)/(\mu_0 \epsilon_0)} \pm \kappa$. Indeed, when a wave is obliquely incident upon such an interface, and if $\kappa \sqrt{\mu_0 \epsilon_0} > \sqrt{\epsilon_1 \mu_1}$, then $n_-$ is clearly negative, so that circularly polarized waves with wavenumber $k_-$ undergo some form of negative refraction [4–6], as shown in figure 1(a) (rays in red). The electromagnetics of so-called bianisotropic media [7] is of paramount importance in metamaterials [8]. However, the work of Jin and He [3] would have remained an academic curiosity if Wiltshire, Pendry and Hajnal had not shown that, in chiral Swiss rolls, the magneto-electric coupling can be larger than either the permittivity or the permeability [9].

However, it is also legitimate to ask whether one can achieve similar focusing effects with more general bianisotropic media with sign shifting in permittivity, permeability and bianisotropy. Furthermore, one can also consider heterogeneous anisotropic (matrix valued) parameters. One
to answer such questions is to consider complementary media which cancel out optical space, as first studied by Pendry and Ramakrishna [10], and to extend their analysis to chiral complementary media. Using the powerful tools of transformation optics, one can also extend the design of generalized perfect lenses and corners to chiral and bianisotropic media. Finite element methods shall be used to numerically explore such optical systems. These analytical solutions of the singular systems we consider in this paper represent important benchmarks for the validation of non-trivial numerical calculations.

2. Pendry–Ramakrishna generalized lens theorem for chiral media

In this section, we propose two demonstrations of the extension of the generalized lens theorem of Pendry and Ramakrishna [10] to bianisotropic media for scalar isotropic chirality and matrix valued general bianisotropy parameters.

2.1. Extension of Pendry–Ramakrishna result via modal analysis

Pendry’s perfect lens, envisioned by Victor Veselago in 1968 [2], is defined by a slab of material with \( \varepsilon = -1, \mu = -1 \). However, a wider class of generalized perfect lenses was proposed by Pendry and Ramakrishna in 2003 [10]. Any system described by

\[
\begin{align*}
\varepsilon_1 &= +\varepsilon(x, y), & \mu_1 &= +\mu(x, y), \\
\varepsilon_2 &= -\varepsilon(x, y), & \mu_2 &= -\mu(x, y),
\end{align*}
\]

makes a perfect lens.

Within this framework, a generalized lens theorem can be stated for perfect lensing: To prove this theorem, Pendry and Ramakrishna pointed out [10] that two complementary media have a vanishing optical path (the media in the region \(-d < z < d\) behave as though they had zero thickness).

However, when one assumes chirality, or more generally a bianisotropy, in the complementary media, do these still make a perfect lens? An extension of the generalized lens theorem of Pendry and Ramakrishna should be considered. In what follows, we provide a positive answer for such complementary bianisotropic media, including isotropic chiral media.

A typical system with two complementary bianisotropic media is shown in figure 2:

\[
\begin{align*}
\varepsilon_1 &= +\varepsilon(x, y), & \mu_1 &= +\mu(x, y), \\
\xi_1 &= +\xi(x, y), & -d &< z < 0, \\
\varepsilon_2 &= -\varepsilon(x, y), & \mu_2 &= -\mu(x, y), \\
\xi_2 &= -\xi(x, y), & 0 &< z < d.
\end{align*}
\]

Let us assume constitutive relations in a isotropically chiral medium as follows:

\[
\begin{align*}
D &= \varepsilon E + i\xi H, & B &= \mu H - i\xi E
\end{align*}
\]

with \( \xi \) the chirality parameter (equal to \( \kappa \sqrt{\mu_0 \varepsilon_0} \) in the work of Jin and He [3]).

Maxwell’s equations take the form (known as Maxwell–Tellegen equations),

\[
\begin{align*}
\nabla \times E &= \mu \frac{\partial H}{\partial t} - i\xi \frac{\partial E}{\partial t}, \\
\nabla \times H &= -\varepsilon \frac{\partial E}{\partial t} - i\xi \frac{\partial H}{\partial t}.
\end{align*}
\]

One can show by decomposing the electric field into Fourier series (see appendix A) that

\[
\begin{align*}
E(x, y, z)_{|z=d} &= E(x, y) \exp(-ik_d d) \\
&= E(x, y) \exp(ik_d (-d)) \\
&= E(x, y, z)_{|z=-d}, & d > 0.
\end{align*}
\]
Figure 2. A pair of complementary bianisotropic media whereby the permittivity, permeability and chirality change sign simultaneously.

Equation (5) is valid for each \( k \), and hence for the sums over \( k \). The same result holds for the magnetic fields. Thus, not only do the two slabs need to possess opposite signs for their dielectric permittivities and magnetic permeabilities, they also need to possess opposite chiralities, albeit with the same strength tensors of permittivity, permeability and bianisotropy parameters with non-zero off-diagonal elements. This can be done using the same kind of technique as above, but the calculations become tedious. We therefore propose another, more subtle, but concise, derivation of the generalized lens theorem using space–time symmetries.

2.2. Extension of Pendry–Ramakrishna result via space–time symmetries

Rewrite the Maxwell–Tellegen equations (4) for a bianisotropic medium as

\[
\nabla \times \mathbf{E} + i \frac{\partial (\xi \mathbf{E})}{\partial t} = \frac{\partial (\mu \mathbf{H})}{\partial t},
\]

\[
\nabla \times \mathbf{H} + i \frac{\partial (\xi \mathbf{H})}{\partial t} = -\frac{\partial (\varepsilon \mathbf{E})}{\partial t}.
\]

Let us list some transformations which leave the equation (6) invariant (see [11] for the case when \( \xi = 0 \):)

S1 (Generalized conformal invariance): \( \mathbf{E} \rightarrow A \mathbf{E}, \mathbf{H} \rightarrow A \mathbf{H}, \mu^{-1} \rightarrow A \mu^{-1} A^{-1}, \varepsilon \rightarrow A \varepsilon A^{-1} \), \( \xi \rightarrow A \xi A^{-1} \), where \( A \) is invertible and an element of \( GL_3(\mathbb{R}) \) (a group of \( 3 \times 3 \) linear operators).

S2 (Generalized duality): \( \mathbf{E} \rightarrow -\mathbf{H}, \mathbf{H} \rightarrow \mathbf{E} \) (iff \( \mu = \varepsilon \)).

S3 (Parity invariance): \( \mathbf{r} \rightarrow -\mathbf{r} \), where \( \mathbf{r} = [x, y, z] \); the Maxwell–Tellegen equations in (6) take the form

\[
\nabla \times \mathbf{E} + i \frac{\partial (\xi \mathbf{E})}{\partial t} = \frac{\partial (\mu \mathbf{H})}{\partial t},
\]

\[
\nabla \times \mathbf{H} + i \frac{\partial (\xi \mathbf{H})}{\partial t} = -\frac{\partial (\varepsilon \mathbf{E})}{\partial t}.
\]

In order to preserve the form of equation (6), \( \mathbf{E} \rightarrow -\mathbf{E}, \mathbf{H} \rightarrow -\mathbf{H}, \varepsilon \rightarrow -\varepsilon, \mu \rightarrow -\mu, \xi \rightarrow -\xi \) should be chosen to cancel out the minus sign introduced by \( -\mathbf{r} \).

S4 (Time reversal): \( t \rightarrow -t \), then \( \mathbf{E} \rightarrow -\mathbf{E} \) (or \( \mathbf{H} \rightarrow -\mathbf{H} \)) and \( \xi \rightarrow -\xi \) (similar idea as parity invariance).

S5 Any additional space–time symmetries.

The combination of any of these symmetries is again a symmetry of the system of equations (6). In other words, if the fields in a particular region of space can be mapped onto another region of space through the symmetry transformations S1–S5 while preserving the respective boundary conditions, then the transformed fields solve the field equations whenever the original fields do. This general law will also be established in the next section with coordinate transformations.

Let us consider a homogeneous slab in figure 2 with permittivity, permeability and chirality tensors in the region \(-d < z < 0\),

\[
v_1 = \begin{bmatrix} v_{xx} & v_{xy} & v_{xz} \\ v_{yx} & v_{yy} & v_{yz} \\ v_{zx} & v_{zy} & v_{zz} \end{bmatrix}, \quad \nu = \varepsilon, \mu, \xi.
\]

For a source at the interface and propagation along the \( z \) direction, we use the symmetry operations S3, followed by S1 with

\[
A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

We shall call this sequence of operations a mirror operation. The choice of \( A \) preserves the continuity of \( \mathbf{E} \) and \( \mathbf{H} \) (tangential components of the electric and magnetic fields) across the boundary. Then the complementary medium in \( 0 < z < d \) is

\[
v_2 = \begin{bmatrix} -v_{xx} & -v_{xy} & -v_{xz} \\ -v_{yx} & -v_{yy} & -v_{yz} \\ v_{zx} & v_{zy} & v_{zz} \end{bmatrix}, \quad \nu = \varepsilon, \mu, \xi.
\]

Similarly to the illustrative diagrams in [10], a schematic diagram for the bianisotropic complementary media is drawn in figure 3, where the media have a vanishing optical path. The salient consequence is that the optical response on the two sides of a structure with complementary media can be calculated by cutting out the complementary media and closing the gap between the two sides.

The two derivations of the generalized lens theorem are in fact underpinned by the invariance of the Maxwell–Tellegen equations under geometric changes. If one wants to study other classes of generalized lenses, say in polar or spherical coordinates, some more elaborate mathematical tools are needed, as we shall now see.

3. Transformation optics applied to the design of corner lenses for bianisotropic media

3.1. General coordinate transformations

Consider a new coordinate system described by [12],

\[
q_1(x, y, z), \quad q_2(x, y, z), \quad q_3(x, y, z),
\]

where lines of constant \( q_2, q_3 \) define the generalized \( q_1 \) axis, and so forth. It should be noted that we can thus produce any desired mesh by choosing the proper coordinate...
transformation. For example, if we define a set of points by equal increments along the new coordinate axes $q_1$, $q_2$ and $q_3$, these will make the mesh in the original coordinate system look distorted, as shown in figure 4.

Since the Maxwell–Tellegen equations have the structure of (4) in a Cartesian coordinate system, one may wonder what will happen to these equations for a new coordinate system $(q_1, q_2, q_3)$ (which can be non-orthogonal), i.e. will they preserve their form

\[
\nabla_q \times \hat{E} = \hat{\mu} \frac{\partial \hat{H}}{\partial t} - i\hat{\xi} \frac{\partial \hat{E}}{\partial t}, \quad \nabla_q \times \hat{H} = -\hat{\varepsilon} \frac{\partial \hat{E}}{\partial t} - i\hat{\xi} \frac{\partial \hat{H}}{\partial t},
\]

(12)

where $\hat{E}$, $\hat{H}$ are re-normalized electric and magnetic fields, and $\hat{\varepsilon}$, $\hat{\mu}$ and $\hat{\xi}$ are in general tensors (i.e. spatially varying matrices)? While this has been discussed by Ward and Pendry for anisotropic systems [12], we present here a generalization of those results to fully bianisotropic systems.

The proof of the form invariance of the Maxwell–Tellegen equations is detailed in appendix B, and follows closely that of Ward and Pendry [12].

3.2. 1D PCs to 2D corner lenses

In the previous section, we have analyzed a system with two complementary media: the optical path through these two regions is zero due to the fact that they are filled with media that are inverted mirror images of one another. This translates into the fact that the media also are required to have opposite helicities in their chiral properties. In the following sections, we should like to give a further generalization of the cancellation principle combined with coordinate transformations.

Using a general method of coordinate mapping, one can express the Maxwell–Tellegen equations in other geometries and obtain a class of generalized perfect lenses in curvilinear coordinates. For instance, two negative 2D corners, within which light radiating from a line source is bent around a closed trajectory and is refocused back onto itself [16, 10], can be mapped onto a layered structure consisting of layers of complementary bianisotropic media with a periodic set of line sources, as shown in figure 5. Conversely, the inverse transformation from a layered structure onto 2D corners is also possible.

Here, the parameters of the bianisotropic media are considered in the most general case. Specifically, the permittivity and permeability of the original perfect lens proposed by Pendry [1] are supposed to be $-1$ (i.e. negative definite tensors proportional to the identity), while for a perfect lens made of a chiral medium the refractive index is given by $n_\pm = \sqrt{\varepsilon \mu / \varepsilon_0 \mu_0} \pm c_0 \xi$, where $c_0$ is the speed of light in vacuum, $\varepsilon_0 \mu_0 = c_0^{-2}$ and $\xi$ is the chiral (or magneto-electric coupling) parameter. Hence, by taking proper positive values of the permittivity, permeability and
chirality, we can obtain a negative refractive index \( n \), as already noted in the introduction. If we now consider a stack of such layers, as shown in figure 5(c), and a periodic set of line sources in all layers with a positive refractive index, we shall end up with an alternation of complementary media with refractive indices of opposite signs, and a source placed in any one of them leads to an image point in every other medium in the most unusual optical system shown in figure 5.

Here, we should like to give a proof of the mapping from 2D corners onto periodic layered systems. For this, let us consider the following homogeneous bianisotropic media in the four regions delimited by the corner in figure 5(a):

\[
\begin{align*}
\varepsilon(\phi) &= +\varepsilon(\phi), & \mu(\phi) &= +\mu(\phi), \\
\tilde{\varepsilon}(\phi) &= +\tilde{\varepsilon}(\phi), & \tilde{\mu}(\phi) &= +\tilde{\mu}(\phi), \\
\epsilon(\phi) &= -\epsilon(\phi), & \mu(\phi) &= -\mu(\phi), \\
\tilde{\epsilon}(\phi) &= -\tilde{\epsilon}(\phi), & 0 &< \phi < \pi/2.
\end{align*}
\]  

(13)

Following [10], let us now introduce a mapping of coordinates that takes the structure of the left panel of figure 5 into the structure of its right panel; see appendix C. This leads us to

\[
\begin{align*}
\pi/2 < \phi < \pi &: \\
\tilde{\varepsilon}_l &= \tilde{\varepsilon}_\phi = \epsilon(\phi), & \tilde{\varepsilon}_z &= r_0^2 e^{2i\epsilon(\phi)} \\
\tilde{\mu}_l &= \tilde{\mu}_\phi = \mu(\phi), & \tilde{\mu}_z &= r_0^2 e^{2i\mu(\phi)} \\
\tilde{\xi}_l &= \tilde{\xi}_\phi = \xi(\phi), & \tilde{\xi}_z &= r_0^2 e^{2i\xi(\phi)} \\
0 < \phi < \pi/2 &: \\
\varepsilon_l &= \varepsilon_\phi = -\varepsilon(\phi), & \varepsilon_z &= -r_0^2 e^{2i\epsilon(\phi)} \\
\mu_l &= \mu_\phi = -\mu(\phi), & \mu_z &= -r_0^2 e^{2i\mu(\phi)} \\
\xi_l &= \xi_\phi = -\xi(\phi), & \xi_z &= -r_0^2 e^{2i\xi(\phi)}
\end{align*}
\]  

(14)

which are heterogeneous fully bianisotropic layers as shown in figure 5(c). The salient consequence is that the electromagnetic field radiated by line sources in figures 5(a) and (c) can be mapped onto one another from our extended version of the Pendry–Ramakrishna theorem. Note that inclusion of chiral media in the systems would demand spatially varying chiral media for such focusing effects. First numerics corresponding to this configuration in the absence of chirality have been published in [17] with further theoretical and numerical investigations in [18]. Subsequent theoretical [19, 20] and experimental [21] investigations have shown that such checkerboards already have highly unusual spectral [19] and effective [20] properties, including the feature of broadband extraordinary transmission [21], in the absence of chirality. We expect that novel effects will be made possible by considering chirality in the studies [19–21].

3.3. From 2D checkerboards to 3D corner lenses

Figure 6 shows a 2D checkerboard consisting of an alternation of rectangular blocks of complementary bianisotropic media. The axes along the checkerboard directions are defined by \( \theta \) and \( \phi \), along which the periodicity of the checkerboard is \( \pi \); the direction normal to this plane is denoted by \( l \). The structure is defined by

\[
\begin{align*}
e(\theta, \phi) &= +e(\theta, \phi), & \mu(\theta, \phi) &= +\mu(\theta, \phi), \\
\xi(\theta, \phi) &= +\xi(\theta, \phi), & \forall (\theta, \phi) &\in \text{positive}, \\
\varepsilon(\theta, \phi) &= -e(\theta, \phi), & \mu(\theta, \phi) &= -\mu(\theta, \phi), \\
\xi(\theta, \phi) &= -\xi(\theta, \phi), & \forall (\theta, \phi) &\in \text{negative},
\end{align*}
\]

(15)

where

\[
\begin{align*}
\text{positive} &= (0, \pi/2) \times (0, \pi/2) \cup (\pi/2, \pi) \times (\pi/2, \pi), \\
\text{negative} &= (0, \pi/2) \times (\pi/2, \pi) \cup (\pi/2, \pi) \times (0, \pi/2).
\end{align*}
\]

(16)

From the ray picture, a refocusing can be observed in some cells, when we consider a set of rays from a single source (filled dot) in a positive cell as shown in figure 6. However, the ray picture cannot describe the localized fields at the corners [22], hence this does not necessarily mean images would not form in all checkerboard cells. Indeed, a possible perfect lensing phenomenon in every single cell is suggested would not form in all checkerboard cells. Hence, this does not necessarily mean images would not form in all checkerboard cells. Indeed, a possible perfect lensing phenomenon in every single cell is suggested; see appendix C. This leads us to

\[
\begin{align*}
\pi/2 < \phi < \pi &: \\
\tilde{\varepsilon}_l &= \tilde{\varepsilon}_\phi = \epsilon(\phi), & \tilde{\varepsilon}_z &= r_0^2 e^{2i\epsilon(\phi)} \\
\tilde{\mu}_l &= \tilde{\mu}_\phi = \mu(\phi), & \tilde{\mu}_z &= r_0^2 e^{2i\mu(\phi)} \\
\tilde{\xi}_l &= \tilde{\xi}_\phi = \xi(\phi), & \tilde{\xi}_z &= r_0^2 e^{2i\xi(\phi)} \\
0 < \phi < \pi/2 &: \\
\varepsilon_l &= \varepsilon_\phi = -\varepsilon(\phi), & \varepsilon_z &= -r_0^2 e^{2i\epsilon(\phi)} \\
\mu_l &= \mu_\phi = -\mu(\phi), & \mu_z &= -r_0^2 e^{2i\mu(\phi)} \\
\xi_l &= \xi_\phi = -\xi(\phi), & \xi_z &= -r_0^2 e^{2i\xi(\phi)}
\end{align*}
\]

(14)
Figure 6. An infinite checkerboard with complementary bianisotropic cells. When a source (filled dot) is placed in one cell of the checkerboard, then, according to the ray routes, images (empty dots) should form in many other cells, but not in all of them.

set of images along the $\phi = \phi_1$ line at $\theta = \pm n\pi \pm \theta_1$ (the optical path cancels), with $n$ a positive integer. Similarly, a set of images along the $\phi$ direction can be obtained. Finally, an image in every cell of the checkerboard would thus be achieved, corresponding to the source placed in one cell of the checkerboard. However, this is not a mathematical proof, hence some numerics would be in order to support this claim; see section 4.

Before we move to section 4, we should like to stress that we have seen that, using some coordinate transformation, a 2D corner can be mapped onto a periodic array of 1D perfect lenses which satisfies the required complementary conditions. In a similar way, let us consider the mapping algorithm from a 2D checkerboard onto a 3D cubic corner as shown in figure 7. In the left panel of this figure, we have drawn the checkerboard consisting of a doubly periodic set of point sources with period $2\pi$ along $\phi$ and period $\pi$ along $\theta$, the coordinates of which are denoted by $(l, \theta, \phi)$. The right panel corresponds to a 3D cubic corner extending to infinity, its Cartesian coordinates being denoted by $(x_1, x_2, x_3)$. According to the spherical coordinate system indicated in the middle panel of figure 7, the two coordinates can be mapped onto one another via

$$l = l_0 \frac{1}{2} \ln \left( \frac{x_2^2 + x_2^2 + x_2^2}{r_0^2} \right),$$

$$\theta = \arccos \left( \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right),$$  

$$\phi = \arctan \left( \frac{x_2}{x_1} \right),$$

(17)

where $r_0$ is a scale factor and can be considered as the radial position of the source. $l$ denotes the radial (logarithmic) coordinate and $\theta, \phi$ are the longitudinal and azimuthal coordinates. We can then generate the corresponding cubic corner reflector in the coordinate system $(x_1, x_2, x_3)$, as shown in appendix D.

4. Numerical illustration of finite element method

4.1. Diffraction problem for fully bianisotropic media

In this section, we should like to give a numerical illustration of the optical systems consisting of complementary bianisotropic media using the finite element method (FEM implemented in COMSOL Multiphysics). First, we shall derive the diffraction equations for a general bianisotropic medium. We assume parameters in equation (4) such as permittivity, permeability and chirality are all tensors written as follows:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix},$$

(18)

We also assume that the electric field $E = (e_1, e_2, e_3)^T$, and the magnetic field $H = (h_1, h_2, h_3)^T$, which leads to the following

Figure 7. (a) 2D checkerboard with optically complementary cells. (b) The spherical coordinates. (c) Cubic corner lens.
set of coupled partial differential equations (PDEs):

\[
\begin{align*}
\nabla \cdot \left( \mu_T + \xi T \xi^{-1} \nabla E \right) &+ i \nabla \cdot \left( \varepsilon_T + \xi T \xi^{-1} \nabla H \right) = \omega^2 \varepsilon_{333} E + i \omega^2 \xi_{333} H \\
i \nabla \cdot \left( \xi T + \mu_T \xi^{-1} \nabla E \right) &+ \nabla \cdot \left( \varepsilon_T + \xi T \xi^{-1} \nabla H \right) = \omega^2 \xi_{333} E - \omega^2 \mu_{333} H
\end{align*}
\]

with \( E \) and \( H \) the longitudinal components of \( \mathbf{E} \) and \( \mathbf{H} \), respectively, and

\[
\begin{align*}
\varepsilon_T &= \begin{bmatrix} \varepsilon_{22} & -\varepsilon_{21} \\ -\varepsilon_{21} & \varepsilon_{11} \end{bmatrix}, & \mu_T &= \begin{bmatrix} -\mu_{22} & \mu_{21} \\ \mu_{12} & -\mu_{11} \end{bmatrix}, \\
\xi_T &= \begin{bmatrix} -\xi_{22} & \xi_{21} \\ \xi_{12} & -\xi_{11} \end{bmatrix}.
\end{align*}
\]

This set of PDEs (19) with coefficients as in (20) is the general form adopted in the COMSOL Multiphysics package for a bianisotropic medium with permittivity, permeability and chirality tensors as in (18). When the sign of \( \varepsilon_{ij}, \mu_{ij} \) and \( \xi_{ij} \) changes in (18), one has to change their sign accordingly in (20). Note that, when chirality vanishes, equations (19) decouple into two PDEs for s and p polarized fields. Based on this, we can set up any bianisotropic model with tensors (18) implementing (19) for instance in COMSOL Multiphysics by using the PDE module. We note that such a model is required, e.g., in order to implement perfectly matched layers (for which \( \varepsilon, \mu \) and \( \xi \) are all tensors) for diffraction problems in chiral media. In the following, we apply this numerical model to various types of complementary media.

4.2. Complementary media with isotropic \( \varepsilon, \mu \) and \( \xi \)

In this section, we should like to study complementary bianisotropic media described by isotropic (sign-shifting) parameters \( \varepsilon, \mu \) and \( \xi \). We set up our simulation models by using the PDE module in COMSOL Multiphysics.

Firstly, we analyze a periodic array of chiral lenses, consisting of an alternation of two layers as shown in figure 8(a), and a line source lying in the upper bianisotropic layer. In order to model the infinite periodic structure, we set Floquet–Bloch boundary conditions at the upper and lower interfaces, and perfect matched layers [23] at the left- and right-hand sides of the structure. We consider a multilayer consisting of an alternation of vacuum and chiral slabs that...
extends the original proposal of Jin and He [3] to periodic structures. The parameters in (18) are chosen as \( \epsilon_1 = \epsilon_2 = \text{diag} [\epsilon_0, \epsilon_0, \epsilon_0] \), \( \mu_1 = \mu_2 = \text{diag} [\mu_0, \mu_0, \mu_0] \), \( \xi_1 = 0 \) and \( \xi_2 = \text{diag} [1.99/\epsilon_0, 1.99/\epsilon_0, 1.99/\epsilon_0] \), which leads to a small mismatch in the refractive index (\( n_2 \approx -1 \)); a refocusing occurs in the FEM result, as shown in panel (b). Some asymmetry is observed, which we can attribute to the chirality, but otherwise this plot is very similar to the earlier configuration of an alternation of non-chiral layers with refractive indices \( \pm 1 \) in [17]. Furthermore, we also assume \( \epsilon_1 = -\epsilon_2 = \text{diag} [\epsilon_0, \epsilon_0, \epsilon_0] \), \( \mu_1 = -\mu_2 = \text{diag} [\mu_0, \mu_0, \mu_0] \), and \( \xi_1 = -\xi_2 = \text{diag} [0.99/\epsilon_0, 0.99/\epsilon_0, 0.99/\epsilon_0] \) for complementary bianisotropic layers, and find similar plots, which are shown in panels (c) and (d). Note that here the frequency of the source is \( f = 7 \times 10^{14} \) Hz. In order to improve the convergence of the COMSOL Multiphysics solver, a small absorption was added to the media with negative parameters, i.e. \( \epsilon_2 = \text{diag} [-\epsilon_0 + i\delta, -\epsilon_0 + i\delta, -\epsilon_0 + i\delta] \), in (d) where \( \delta \leq 10^{-17} \). We can see that absorption governs the physics of surface waves at interfaces between positive and negative media. We have checked that the field at the interfaces between complementary media presents some spatial oscillations with a period that is proportional to absorption as \( 1/\ln(\delta) \), in a way similar to what was first reported by Merlin in [24]. A good refocusing of the source can be observed in each layer, as predicted in the previous theoretical analysis.

Secondly, a 2D chiral corner lens consisting of eight triangular corners is considered as shown in figure 9(a), with a line source located in one corner. Here, we consider an alternation of vacuum and chiral media of the Jin and He type with negative refractive index [3]; the parameters are \( \epsilon_1 = \epsilon_2 = \text{diag} [\epsilon_0, \epsilon_0, \epsilon_0] \), \( \mu_1 = -\mu_2 = \text{diag} [\mu_0, \mu_0, \mu_0] \), \( \xi_1 = 0 \) and \( \xi_2 = \text{diag} [1.99/\epsilon_0, 1.99/\epsilon_0, 1.99/\epsilon_0] \). The source is refocused in the other corners, a fact which is less apparent in figure 9(b) than in figure 9(c), wherein similar plots are obtained for complementary bianisotropic media with sign-shifting parameters; see also figure 9(d). The parameters of the media are \( \epsilon_1 = -\epsilon_2 = \text{diag} [\epsilon_0, \epsilon_0, \epsilon_0] \), \( \mu_1 = -\mu_2 = \text{diag} [\mu_0, \mu_0, \mu_0] \), \( \xi_1 = -\xi_2 = \text{diag} [0.99/\epsilon_0, 0.99/\epsilon_0, 0.99/\epsilon_0] \), respectively. Here the frequency of the source is \( f = 7 \times 10^{14} \) Hz. From extension of the Pendry–Ramakrishna result, we already knew that some perfect images should appear in all corners, which is what the numerical package COMSOL Multiphysics finds.

Finally, we should like to discuss the computational setup of a checkerboard and its numerical results by COMSOL Multiphysics. First, figure 10(a) shows a configuration of an infinite checkerboard; Floquet–Bloch...
boundary conditions are set on either side of the cell (taking a vanishing Bloch vector ensures periodicity as
a particular case). For a Jin and He type checkerboard
with sign-shifting refractive index, the permittivity and
permeability are set to be \( \varepsilon_1 = \varepsilon_2 = \text{diag}[\varepsilon_0, \varepsilon_0, \varepsilon_0] \), \( \mu_1 = \mu_2 = \text{diag}[\mu_0, \mu_0, \mu_0] \) in every medium, while the chirality
is \( \xi_2 = \text{diag}[1.99/\varepsilon_0, 0.99/\varepsilon_0, 0.99/\varepsilon_0] \) in one medium in
two (and \( \xi_1 = 0 \) elsewhere). The frequency of the source
is \( f = 3.5 \times 10^{15} \) Hz. Images can be observed in the other
three cells as shown in figure 10(b), and appear to be slightly
asymmetric in the upper right and lower left corners, a fact
which we attribute to the presence of chirality in these two
regions. This result is unlike previous studies, wherein corner
reflectors alternating refractive indices of \( \pm 1 \) displayed a
fourfold symmetry.

Moreover, let us consider such a checkerboard consisting of
complementary bianisotropic media having the same
optical parameters as in figures 8(c) and 9(c), with frequency
\( f = 2 \times 10^{12} \) Hz. This computation is actually the most
challenging one, since this type of checkerboard is an
extremely singular system [11], as also pointed out in [19],
wherein the non-chiral case was considered. The band
structure of this checkerboard (not shown here for lack of
space) appears to display completely flat dispersion curves,
corresponding to a network of Floquet–Bloch eigenfields
with support on the interfaces between media (surface
plasmons) and at corners (what we might call line plasmons).
All modes are infinitely degenerate in this most unusual
checkerboard.

One should note again that the absence of ab-
sorption in sign-shifting bianisotropic media means the
Maxwell–Tellegen operator is no longer elliptic and hence
the Lax–Milgram lemma ensuring existence and uniqueness
of the solution to the diffraction problem is no longer
applicable [25–28]. Nevertheless, the Fredholm alternative
ensures that, if there is a solution for any applied electro-
static field whose tangential component is continuous
through interfaces separating complementary media (hence
exhibiting two anti-parallel wavevectors on both sides of such
interfaces) and satisfying periodic conditions on the unit cell
edges, it will be an eigenfunction for a problem very similar
to the one shown in figure 10(c). However, the convergence
of the finite element algorithm depends crucially upon the
regularity of the boundaries between complementary media,
with some ill-posedness occurring in certain corner cases for
a refractive index contrast of \(-1\) when the coercive plus

Figure 10. An infinite checkerboard. (a) Schematic view of the computational model with Floquet–Bloch boundary conditions set on either
side of the unit cell. (b) Plot of \( \sqrt{\varepsilon^2 + \mu^2} \) with a line source in vacuum; parameters are \( \varepsilon_1 = \varepsilon_2 = \text{diag}[\varepsilon_0, \varepsilon_0, \varepsilon_0] \),
\( \mu_1 = \mu_2 = \text{diag}[\mu_0, \mu_0, \mu_0] \), \( \xi_1 = 0 \) and \( \xi_2 = \text{diag}[1.99/\varepsilon_0, 1.99/\varepsilon_0, 1.99/\varepsilon_0] \). (c) and (d) Plots
\( \sqrt{\varepsilon^2 + \mu^2} \) for a line source radiating in an
infinite checkerboard of complementary bianisotropic media \( \varepsilon_1 = \text{diag}[\varepsilon_0, \varepsilon_0, \varepsilon_0] \), \( \mu_1 = \mu_2 = \text{diag}[\mu_0, \mu_0, \mu_0] \), and
\( \xi_1 = -\xi_2 = \text{diag}[0.99/\varepsilon_0, 0.99/\varepsilon_0, 0.99/\varepsilon_0] \); a small absorption \( \delta \) is added to the negative permittivity
\( \varepsilon_2 = \text{diag}[-\varepsilon_0 + i\delta, -\varepsilon_0 + i\delta, -\varepsilon_0 + i\delta] \) to preserve a nice convergence of the COMSOL Multiphysics solver. (c) \( \delta = 0 \), and (d)
\( \delta \leq 10^{-30} \).
Figure 11. A cylindrical lens consisting of complementary bianisotropic media. (a) Schematic view of the computational model with PML on either side of the unit cell. (b) Plot of $\sqrt{\varepsilon^2 + \mu^2}$ for a line source radiating in region 1 and $r = 0.13 \, \mu\text{m} \, (r > r_1^2/r_2)$. (c) Plot of $\sqrt{\varepsilon^2 + \mu^2}$ for a line source radiating in region 1 and $r = 0.095 \, \mu\text{m} \, (r < r_1^2/r_2)$. (d) Plot of $\sqrt{\varepsilon^2 + \mu^2}$ for a line source radiating in the annulus with $r = r_1 + (r_2 - r_1)/2$ along the horizontal axis from the origin.

compact mathematical framework breaks down; see [25] for the case when $\xi = 0$. Here, we add a little absorption, typically an imaginary part for either the permittivity, permeability or chirality parameter, to preserve the nice convergence of the COMSOL Multiphysics solver; see panel (d) in figure 10.

It is worth noticing that the expected images of the source in other cells are intuitive in figure 10(d), wherein a small absorption is added to $\varepsilon_2$. This is consistent with our discussion in the previous section, which points out a possible perfect lensing phenomenon in such a kind of periodic checkerboard. However, one can clearly see that the numerical results are sensitive to the absorption added to the system (see figures 10(c)–(d)). A similar phenomenon has been discussed in the case of a finite checkerboard consisting of alternating triangular or rectangular cells with sign-shifting permittivity and permeability [29, 30].

We feel confident that we have shown that our computer calculations represent a benchmark for validation of numerical results involving bianisotropic materials in very singular conditions. However, we should like to further exploit the versatility of finite elements, by computing eigenfields in anisotropic sign-shifting structures, which is the worst scenario we have in mind. We do this in the next section.

4.3. Numerical results for anisotropic $\varepsilon$, $\mu$ and $\xi$

Let us consider the case of a cylindrical lens with tensors of permittivity, permeability and chirality in (18) as follows:

$$\nu_I = \text{diag}[\nu^{(I)}_x, \nu^{(I)}_y, \nu^{(I)}_z], \quad v = \varepsilon, \mu, \xi,$$

(21)

with the index $I = 1–3$ corresponding to the three regions defining the cylindrical lens

$$\nu^{(1)}_x = +v_0, \quad \nu^{(1)}_y = +v_0, \quad \nu^{(1)}_z = +v_0,$$

$$\nu^{(2)}_x = -v_0, \quad \nu^{(2)}_y = -v_0, \quad \nu^{(2)}_z = -v_0,$$

$$\nu^{(3)}_x = +v_0, \quad \nu^{(3)}_y = +v_0, \quad \nu^{(3)}_z = +v_0,$$

$$r_1 < r \leq r_2,$$

(22)

where $\xi_0 = 0.99/c_0$ was assumed to prevent numerical instabilities. Region 3 is the matrix consisting of a bianisotropic medium with $\varepsilon = \text{diag}[[\varepsilon_0, \varepsilon_0, \varepsilon_0], \mu = \text{diag}[[\mu_0, \mu_0, \mu_0]]$ and $\xi = \text{diag}[[\xi_0, \xi_0, \xi_0]]$ surrounding the cylindrical lens within a certain distance $r_3 = r_2^2/r_1$ diagrammatically shown in
figure 11(a), region 2 is filled with a negative bianisotropic medium, which is designed to optically cancel out region 3, and region 1 is the interior part of the lens; the whole structure acts in such a way that region 1 is optically equivalent to a disc of matrix of radius \( r_3 \). Hence, for an object lying in region 1, its image will appear to an external observer as if it were magnified by a factor of \( r_1/r_3 = r_2^2/r_3^2 \) filling the space within region 3, as already observed in [10, 31] for non-chiral media. This kind of mirage is sometimes called a superscatter [32]. Also worth mentioning is the earlier work of Nicorovici, McPhedran and Milton [33]. Such a cylindrical Pendry’s poor man’s lens [1] with \( \varepsilon = \text{diag}[\varepsilon_0, -\varepsilon_0, -\varepsilon_0] \), \( \mu = \text{diag}[\mu_0, \mu_0, \mu_0] \) and \( \xi = 0 \) in region 2 is considered for one light polarization in the quasistatic limit. Indeed, the spatially

\[
\mu \nabla \cdot \mathbf{J} = \frac{\varepsilon_0}{
\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3} \text{ and its image will appear to an external observer as if it were}
\]

So we have discussed a generalized perfect lens theorem [10] that includes bianisotropic and chiral media in addition to the usual dielectric and magnetic media. We proposed two different methods to derive a generalized lens theorem for isotropic (section 2.1) and anisotropic (section 2.2), sign-shifting permittivity, permeability and chirality admittance tensors. The geometric transformation technique of Ward and Pendry [12] (section 3.1) has been generalized to include bianisotropic media. These results are applied to analyze a variety of singular situations involving corners and wedges of such media. Our numerical simulations with finite element calculations using the COMSOL Multiphysics package represent a benchmark for the validation of computations involving bianisotropic media in extremely singular circumstances. A possible extension of this work would be to include some dispersion in the material parameters.

### Appendix A. Extension of Pendry–Ramakrishna result via modal analysis

Let us assume that the electromagnetic field can be decomposed on modes associated with the propagation constant in the \( z \) direction. This assumption is supported by all the modal methods established in grating theory [39, 40].

\[
\mathbf{E}_1(x, y, z) = \exp(\pm ik_1z) \sum_{k_x, k_y} \mathbf{E}_1(k_x, k_y) \exp(\pm ik_xx + ik_yy),
\]

\[
-k < z < 0
\]

\[
\mathbf{E}_2(x, y, z) = \exp(\pm ik_2z) \sum_{k_x, k_y} \mathbf{E}_2(k_x, k_y) \exp(\pm ik_xx + ik_yy),
\]

\[
0 < z < d.
\]

The same holds true for the magnetic fields in media 1 and 2. These fields should satisfy some boundary conditions: the tangential components of both electric and magnetic fields are continuous across the interface at \( z = 0 \) between media 1 and 2.

Plugging the Fourier series (A.1) for the electric and magnetic fields into (4), and using the frequency domain notation \( \partial/\partial t = \text{io} \), we get

\[
k_{1z} \mathbf{z} \times [E_{1z}(k_x, k_y) \mathbf{x} + E_{1y}(k_x, k_y) \mathbf{y}] + (k_x \mathbf{x} + k_y \mathbf{y}) \times E_{1z}(k_x, k_y) \mathbf{z}
\]

\[
= \omega \sum_{k_x', k_y'} \mu_1(k_x, k_y, k_x', k_y') \left[ E_{1x}(k_x', k_y') \mathbf{x} + E_{1y}(k_x', k_y') \mathbf{y} \right]
\]

\[
- \text{io} \sum_{k_x', k_y'} \xi_1(k_x, k_y, k_x', k_y') \left[ E_{1x}(k_x', k_y') \mathbf{x} + E_{1y}(k_x', k_y') \mathbf{y} \right]
\]

\[
(k_x \mathbf{x} + k_y \mathbf{y}) \times [E_{1z}(k_x, k_y) \mathbf{x} + E_{1y}(k_x, k_y) \mathbf{y}]
\]

\[
= \omega \sum_{k_x', k_y'} \mu_1(k_x, k_y, k_x', k_y') \left[ H_{1z}(k_x, k_y, k_x', k_y') \mathbf{z} \right]
\]

\[
- \text{io} \sum_{k_x', k_y'} \xi_1(k_x, k_y, k_x', k_y') \left[ H_{1x}(k_x, k_y, k_x', k_y') \mathbf{x} + H_{1y}(k_x, k_y, k_x', k_y') \mathbf{y} \right]
\]

\[
(k_x \mathbf{x} + k_y \mathbf{y}) \times [H_{1x}(k_x, k_y) \mathbf{x} + H_{1y}(k_x, k_y) \mathbf{y}]
\]

\[
= - \sum_{k_x', k_y'} \varepsilon_1(k_x, k_y, k_x', k_y') \left[ E_{1x}(k_x', k_y') \mathbf{x} + E_{1y}(k_x', k_y') \mathbf{y} \right]
\]

\[
- \text{io} \sum_{k_x', k_y'} \xi_1(k_x, k_y, k_x', k_y') \left[ H_{1x}(k_x, k_y, k_x', k_y') \mathbf{x} + H_{1y}(k_x, k_y, k_x', k_y') \mathbf{y} \right]
\]

\[
(k_x \mathbf{x} + k_y \mathbf{y}) \times [H_{1x}(k_x, k_y) \mathbf{x} + H_{1y}(k_x, k_y) \mathbf{y}]
\]

\[
= - \omega \sum_{k_x', k_y'} \varepsilon_1(k_x, k_y, k_x', k_y') \left[ H_{1z}(k_x, k_y, k_x', k_y') \mathbf{z} \right]
\]

\[
- \text{io} \sum_{k_x', k_y'} \xi_1(k_x, k_y, k_x', k_y') \left[ H_{1x}(k_x, k_y, k_x', k_y') \mathbf{x} + H_{1y}(k_x, k_y, k_x', k_y') \mathbf{y} \right].
\]
These equations can be rearranged as
\[ k_2 \hat{z} \times \left[ H_{2z}(k_x, k_y) \hat{x} + H_{2y}(k_x, k_y) \hat{y} \right] \]
\[ + (k_\hat{x} + k_\hat{y}) \times H_{2z}(k_x, k_y) \hat{z} \]
\[ = - \omega \sum_{k_x, k_y} \varepsilon_2(k_x, k_y, k_x', k_y') \left[ E_{2z}(k_x', k_y') \hat{x} + E_{2y}(k_x', k_y') \hat{y} \right] \]
\[ - i \omega \sum_{k_x, k_y} \xi_2(k_x, k_y, k_x', k_y') \left[ E_{2x}(k_x', k_y') \hat{x} + E_{2y}(k_x', k_y') \hat{y} \right] \]
\[ (k_\hat{x} + k_\hat{y}) \times \left[ H_{2z}(k_x, k_y) \hat{x} + H_{2y}(k_x, k_y) \hat{y} \right] \]
\[ = - \omega \sum_{k_x, k_y} \varepsilon_1(k_x, k_y, k_x', k_y') \left[ E_{1z}(k_x', k_y') \hat{x} + E_{1y}(k_x', k_y') \hat{y} \right] \]
\[ - i \omega \sum_{k_x, k_y} \xi_1(k_x, k_y, k_x', k_y') \left[ E_{1x}(k_x', k_y') \hat{x} + E_{1y}(k_x', k_y') \hat{y} \right] \]

(A.3)

Equations (A.2) and (A.5) hold for the fields in chiral media with parameters $\varepsilon_1, \mu_1, \xi_1$ and $\varepsilon_2, \mu_2, \xi_2$ respectively. The fields in the two regions match across the interface due to the continuous boundary condition, while there is a sign-shifting between components along the $z$ direction.

**Appendix B. Extension of the proof by Ward and Pendry to Maxwell–Téllegen equations**

Following [12], we first define three unit vectors. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be along the axes of $q_1, q_2, q_3$. The transformation relation between coordinate systems $(x, y, z)$ and $(q_1, q_2, q_3)$ is

\[
\begin{pmatrix}
\frac{dx}{dq_1} & \frac{dx}{dq_2} & \frac{dx}{dq_3} \\
\frac{dy}{dq_1} & \frac{dy}{dq_2} & \frac{dy}{dq_3} \\
\frac{dz}{dq_1} & \frac{dz}{dq_2} & \frac{dz}{dq_3}
\end{pmatrix}
\]

The length of a line element in the new coordinate system can be most simply derived from

\[ ds^2 = dx^2 + dy^2 + dz^2 \]

\[ = Q_{11} dq_1^2 + Q_{22} dq_2^2 + Q_{33} dq_3^2 + 2 Q_{12} dq_1 dq_2 + 2 Q_{13} dq_1 dq_3 + 2 Q_{23} dq_2 dq_3. \]  

(B.2)

with

\[ Q_{ij} = \frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \frac{\partial z}{\partial q_j}. \]  

(B.3)

Let us now note that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ can be expressed in terms of contravariant components,

\[ \mathbf{U} = U^1 \mathbf{u}_1 + U^2 \mathbf{u}_2 + U^3 \mathbf{u}_3, \quad U = E, H. \]  

(B.4)

Conversely, they can be expressed in terms of the covariant components via

\[ g^{-1} \begin{bmatrix} U^1 \\ U^2 \\ U^3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \mathbf{u}_1 \cdot \mathbf{u}_3 \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \mathbf{u}_2 \cdot \mathbf{u}_3 \\ \mathbf{u}_3 \cdot \mathbf{u}_1 & \mathbf{u}_3 \cdot \mathbf{u}_2 & \mathbf{u}_3 \cdot \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} U^1 \\ U^2 \\ U^3 \end{bmatrix}. \]  

(B.5)
with \( g \) a \( 3 \times 3 \) matrix (a representation of the metric tensor) and

\[
U_1 = U \cdot u_1, \quad U_2 = U \cdot u_2, \quad U_3 = U \cdot u_3.
\]

(B.6)

According to (B.5), we note that

\[
U^j = \sum_{j=1}^{3} g^{ij} U_j.
\]

(B.7)

We take a small element which resembles a parallelepiped (left panel in figure B.1), and consider the projection of \( \nabla \times \mathbf{E} \) onto the normal to the \( u_1-u_2 \) plane by taking a line integral along the \( u_1-u_2 \) parallellogram (right panel in figure B.1) \[12\]. On the one hand, invoking the Kelvin–Stokes theorem,

\[
\oint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_l \mathbf{E} \cdot d\mathbf{l},
\]

(B.8)

where \( S \) is the oriented surface defined by the normal to the \( u_1-u_2 \) plane and \( l \) is its boundary. We deduce that

\[
(\nabla \times \mathbf{E}) \cdot (u_1 \times u_2)Q_1 dQ_1 dQ_2 dQ_3
\]

\[
= dq_1 \frac{\partial}{\partial q_1} (E_2 Q_2 dQ_3) - dq_2 \frac{\partial}{\partial q_2} (E_1 Q_1 dQ_1).
\]

(B.9)

We now note that the normalized electric field in the new coordinate system is defined as

\[
\hat{E}_i = Q_i E_i, \quad (i = 1, 2, 3).
\]

(B.10)

Hence equation (B.9) can be simplified as

\[
(\nabla \times \mathbf{E}) \cdot (u_1 \times u_2)Q_1 Q_2
\]

\[
= \frac{\partial \hat{E}_1}{\partial q_1} - \frac{\partial \hat{E}_1}{\partial q_2} = (\nabla \times \hat{\mathbf{E}})^3.
\]

(B.11)

The term on the right-hand side of this equation is the third component of curl in the new coordinate system. Then applying (4) to \( \nabla \times \mathbf{E} \) on the left-hand side of equation (B.11) we obtain

\[
(\nabla \times \mathbf{E})(u_1 \times u_2)Q_1 Q_2
\]

\[
= \mu \frac{\partial}{\partial t} (u_1 \times u_2)Q_1 Q_2 - i \xi \frac{\partial}{\partial t} (u_1 \times u_2)Q_1 Q_2.
\]

(B.12)

Let us now substitute (B.4) and combine it with expression (B.7) for both \( \mathbf{E} \) and \( \mathbf{H} \) to (B.12). We obtain

\[
(\nabla \times \mathbf{E}) \cdot (u_1 \times u_2)Q_1 Q_2
\]

\[
= \mu \sum_{j=1}^{3} g^{ij} \frac{\partial H_j}{\partial t} u_3 \cdot (u_1 \times u_2)Q_1 Q_2
\]

\[
= -i \xi \sum_{j=1}^{3} g^{ij} \frac{\partial E_j}{\partial t} u_3 \cdot (u_1 \times u_2)Q_1 Q_2.
\]

(B.13)

Let us now define

\[
\tilde{\mathbf{E}} = v g^{ij} (u_1 \times u_2)Q_1 Q_2 Q_3 (Q_1 Q_2)^{-1},
\]

(B.14)

and

\[
\hat{\mathbf{H}} = \mu \mathbf{H}.
\]

(B.15)

With this, (B.13) can be recast as

\[
(\nabla \times \mathbf{E}) \cdot (u_1 \times u_2)Q_1 Q_2
\]

\[
= \sum_{j=1}^{3} \hat{\mathbf{E}}_j \frac{\partial}{\partial t} - i \sum_{j=1}^{3} \hat{\mathbf{E}}_j \frac{\partial}{\partial t}.
\]

(B.16)

Comparing this equation with (B.11), we deduce that

\[
\nabla_q \times \hat{\mathbf{E}} = \mu \frac{\partial \hat{\mathbf{H}}}{\partial t} - i \xi \frac{\partial \hat{\mathbf{E}}}{\partial t}.
\]

(B.17)

Similarly, due to the symmetry between \( \mathbf{E} \) and \( \mathbf{H} \) fields, we have

\[
\nabla_q \times \hat{\mathbf{H}} = -\hat{\mathbf{E}} = -\mu \frac{\partial \hat{\mathbf{E}}}{\partial t} - i \xi \frac{\partial \hat{\mathbf{H}}}{\partial t}.
\]

(B.18)

where \( \hat{\mathbf{E}}, \hat{\mathbf{H}} \) also satisfies (B.14).

These results indicate that the form of Maxwell–Tellegen equations is preserved under a coordinate transformation, while the expressions of \( \varepsilon, \mu, \) and \( \xi \) need simply be replaced by \( \hat{\varepsilon}, \hat{\mu}, \) and \( \hat{\xi} \). In other words, no matter what is the coordinate system, we can always analyze the electromagnetic property of a structure through the Maxwell–Tellegen equations with redefined (often anisotropic heterogeneous) parameters. We note some analogy between (B.14) and equations (14a)–(14d) in a paper by Teixeira and Chew [41] on perfectly matched layers.

Appendix C. Mapping 1D periodic set of perfect lenses onto 2D corner reflectors

The coordinate transformation is

\[
x = r_0 \cos \psi e^{i/\lambda}, \quad y = r_0 \sin \psi e^{i/\lambda}, \quad z = Z
\]

(C.1)
where \( r_0, \varphi \) and \( Z \) are the cylindrical coordinates, in which the layered structure is defined [10]. In this frame,

\[
\tilde{e}_i = \xi_i \frac{Q_i Q_2 Q_3}{Q_i^2}, \quad \tilde{\mu}_i = \mu_i \frac{Q_i Q_2 Q_3}{Q_i^2}, \quad \tilde{\xi}_i = \xi_i \frac{Q_i Q_2 Q_3}{Q_i^2},
\]

(C.2)

where

\[
Q_i = \frac{r_0}{l_0} \sqrt{e^{2 l_i} \cos^2 \varphi + e^{2 l_i} \sin^2 \varphi} = \frac{r_0}{l_0} e^{l_i}
\]

\[
Q_\varphi = \frac{r_0}{l_0} \sqrt{e^{2 l_i} \cos^2 \varphi + e^{2 l_i} \sin^2 \varphi} = \frac{r_0}{l_0} e^{l_i}
\]

(C.3)

and hence,

\[
\tilde{e}_1 = \tilde{l}_0 \xi_1, \quad \tilde{e}_\varphi = \tilde{l}_0 \mu_1, \quad \tilde{e}_Z = \tilde{l}_0 \xi_1, \quad \tilde{\mu}_1 = \tilde{l}_0 \xi_1, \quad \tilde{\mu}_\varphi = \tilde{l}_0 \mu_1, \quad \tilde{\mu}_Z = \tilde{l}_0 \xi_1
\]

(C.4)

On the other hand, we have (C.2) as the new set of parameters, hence let us substitute (D.3) into (C.2). We can deduce

\[
\tilde{e}_1 = \frac{v_1}{r} \left( \frac{\sin^2 \theta \sin^2 \varphi + \cos^2 \varphi}{\sin^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right),
\]

\[
\tilde{\mu}_1 = \frac{v_2}{r} \left( \frac{\sin^2 \theta \cos^2 \varphi + \sin^2 \varphi}{\sin^2 \theta \sin^2 \varphi + \cos^2 \varphi} \right),
\]

\[
\tilde{\xi}_1 = \frac{v_3}{r \sin \theta} \sqrt{\frac{\sin^2 \theta \cos^2 \varphi + \sin^2 \varphi}{\sin^2 \theta \sin^2 \varphi + \cos^2 \varphi}},
\]

(D.4)

where \( v = e, \mu, \xi \). It is noted that the transverse \( r \) is irrelevant to the imaging. Now, choosing

\[
e_1 = \mu_1 = \xi_1 = r \left( \frac{\sin^2 \theta \cos^2 \varphi + \sin^2 \varphi}{\sin^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right),
\]

\[
e_2 = e_1^{-1} = \mu_2 = \mu_1^{-1} = \xi_2 = \xi_1^{-1},
\]

\[
e_3 = \mu_3 = \xi_3
\]

(D.5)

we obtain that \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\), \((\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)\) and \((\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)\) are identical to the checkerboard made of homogeneous, isotropic materials. So far, we have proved that we can achieve a cubical corner lens made up of anisotropic, inhomogeneous materials by mapping from a homogeneous checkerboard.

**Appendix D. Mapping doubly periodic checkerboard onto 3D corner reflectors**

Considering a checkerboard with homogeneous and isotropic cells, then according to the coordinate transformation algorithm we have

\[
Q_1 = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{r}, \quad Q_2 = \frac{x_1^2 + x_2^2 + x_3^2}{r}, \quad Q_3 = \frac{x_1^2 + x_2^2}{r},
\]

(D.1)

with \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \). Assuming \( l_0 = 1, r_0 = 1 \) and

\[
x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta,
\]

(D.2)

(D.1) can be recast as

\[
Q_1 = \frac{1}{r} \sqrt{\cos^2 \varphi + \frac{\sin^2 \varphi}{\sin^2 \theta}}, \quad Q_2 = \frac{1}{r} \sqrt{\sin^2 \varphi + \frac{\cos^2 \varphi}{\sin^2 \theta}}, \quad Q_3 = \frac{1}{r},
\]

(D.3)

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