1. Introduction

We observe in this note that the proof of the Bogomolov stable restriction theorem \cite{B} can be adapted to give an ampleness criterion for globally generated rank 2 vector bundles on certain surfaces. This applies to the Lazarsfeld-Mukai bundles, to congruences of lines in \( \mathbb{P}^3 \), and possibly to the construction of surfaces with ample cotangent bundle.

2. Main result

Throughout the note, \( S \) will be a smooth projective surface over \( \mathbb{C} \). We denote by \( N^1(S) \) the group of divisors on \( S \) modulo numerical equivalence; this is a free, finitely generated abelian group, quotient of \( \text{NS}(S) = H^2(S, \mathbb{Z})_{\text{alg}} \) by its torsion subgroup.

Proposition 1. Let \( E \) be a globally generated rank 2 vector bundle on \( S \), with \( h^0(E) \geq 4 \). Assume that \( N^1(S) = \mathbb{Z} \cdot c_1(E) \). Then either \( E \) is ample, or \( E = \mathcal{O}_S \oplus \det(E) \).

We will need the following lemma:

Lemma. Let \( S \) be a smooth projective surface, and let \( E \) be a globally generated rank 2 vector bundle on \( S \), with \( h^0(E) \geq 4 \) and \( H^1(S, \det(E)^{-1}) = 0 \). Then \( c_1^2(E) > c_2(E) \).

Proof: Let \( V \) be a general 4-dimensional subspace of \( H^0(S, E) \). Then \( V \) generates \( E \) globally, giving rise to an exact sequence

\[
0 \to N \to V \otimes \mathcal{O}_S \to E \to 0.
\]

Since \( N^* \) is globally generated, the zero locus of a general section \( s \) of \( N^* \) is finite, of length \( c_2(N^*) = c_1^2(E) - c_2(E) \). Thus this number is \( \geq 0 \); if it is zero, \( s \) does not vanish, so we have an exact sequence

\[
0 \to \mathcal{O}_S \xrightarrow{s} N^* \to \det(E) \to 0.
\]

Since \( H^1(S, \det(E)^{-1}) = 0 \), this sequence splits, so that \( N \cong \mathcal{O}_S \oplus \det(E)^{-1} \). Thus the exact sequence (1) reduces to

\[
0 \to \det(E)^{-1} \to \mathcal{O}_S^3 \to E \to 0;
\]

but using again \( H^1(S, \det(E)^{-1}) = 0 \) this implies \( h^0(E) \leq 3 \), contradicting the hypothesis.

Proof of the Proposition: We denote by \( c_1 \) and \( c_2 \) the Chern classes of \( E \) in \( H^*(S, \mathbb{Z}) \), and by \( \Delta_E := 4c_2 - c_1^2 \) its discriminant. Assume that \( E \) is not ample. By Gieseker’s lemma \cite{L} Proposition 6.1.7], there exists an irreducible curve \( C \) in \( S \) and a surjective homomorphism \( u : E \to \mathcal{O}_C \). The kernel \( F \) of \( u \) is a vector bundle, with total Chern class \( c(F) = c(E)c(\mathcal{O}_C)^{-1} = (1 + c_1 + c_2)(1 - [C]) \), hence

\[
c_1(F) = c_1 - [C], \quad c_2(F) = c_2 - c_1 \cdot [C], \quad \text{and} \quad \Delta_F = \Delta_E - 2c_1 \cdot [C] - [C]^2.
\]

The curve \( C \) is numerically equivalent to \( rc_1 \) for some integer \( r \geq 1 \). Therefore

\[
\Delta_F = 4c_2 - (r+1)^2c_1^2 \leq 4(c_2 - c_1^2).
\]
Because of our hypotheses \( \det(E) \) is ample, so \( H^1(S, \det(E)^{-1}) = 0 \) and we can apply the Lemma, which gives \( \Delta_F < 0 \). By Bogomolov’s theorem (see [Ra] Théorème 6.1), we have an exact sequence

\[
0 \to L \to F \to \mathcal{F}_Z M \to 0
\]

where \( Z \) is a finite subscheme of \( S \), \( L \) and \( M \) are line bundles on \( S \), with \( c_1(L) = ac_1, c_1(M) = bc_1 \) for some integers \( a, b \) such that \( a \geq b \).

From that exact sequence we get \( c_1(F) = (a + b)c_1 \), hence \( a + b = 1 - r \), and \( c_2(F) = \deg(Z) + abc_1^2 \), hence \( \Delta_F = 4\deg(Z) - (a - b)^2c_1^2 \). Comparing with the previous expression for \( \Delta_F \) and using the Lemma again we find

\[
(a - b)^2c_1^2 \geq -\Delta_F = (r + 1)^2c_1^2 - 4c_2 > (r^2 + 2r - 3)c_1^2 \geq (r^2 - 1)c_1^2,
\]

hence \( a - b \geq r \), and \( a \geq 1 \).

We have \( H^0(E \otimes L^{-1}) = H^0(E^* \otimes \det(E) \otimes L^{-1}) \neq 0 \). Since \( E \) is globally generated, the natural map \( E^* \to H^0(E^* \otimes \mathcal{O}_S) \) is injective, hence \( H^0(\det(E) \otimes L^{-1}) \neq 0 \). Since \( c_1(L) = ac_1 \) with \( a \geq 1 \), the only possibility is \( L \cong \det(E) \), and therefore \( H^0(E^*) \neq 0 \). Using again that \( E \) is globally generated, we obtain \( E = \mathcal{O}_S \otimes \det(E) \).

Remark. The condition \( h^0(E) \geq 4 \) is necessary: if \( E \) is ample and globally generated, the rational map \( \mathbb{P}(E) \to \mathbb{P}(H^0(E)) \) associated to the linear system \( |\mathcal{O}_{\mathbb{P}(E)}(1)| \) is a finite morphism, hence \( \dim \mathbb{P}(H^0(E)) \geq 3 \). On the other hand, the condition \( N^1(S) = \mathbb{Z} \cdot c_1 \) is quite restrictive, but it is not clear how it could be weakened. For instance, we will exhibit in Example 1 of §4 a globally generated rank 2 vector bundle \( E \) on \( \mathbb{P}^2 \) with \( h^0(E) \geq 4 \), \( \det E = \mathcal{O}_{\mathbb{P}^2}(2) \), which is not ample.

3. Application 1: Lazarsfeld-Mukai bundles

Let \( C \) be an irreducible curve in \( S \), \( L \) a line bundle on \( C \), and \( V \) a 2-dimensional subspace of \( H^0(L) \) which generates \( L \). The Lazarsfeld-Mukai bundle \( E_{C,V} \) is defined by the exact sequence

\[
0 \to E_{C,V}^* \to V \otimes \mathcal{O}_S \to L \to 0.
\]

Let \( N_C := \mathcal{O}_S(C)|_C \) be the normal of \( C \) in \( S \). By duality we get an exact sequence

\[
0 \to V^* \otimes \mathcal{O}_S \to E_{C,V} \to N_C \otimes L^{-1} \to 0.
\]

Proposition 2. Assume \( H^1(S, \mathcal{O}_S) = 0 \), \( N^1(S) = \mathbb{Z} \cdot [C] \), and that the line bundle \( N_C \otimes L^{-1} \) on \( C \) is globally generated and nontrivial. Then \( E_{C,V} \) is globally generated and ample.

Proof: We put \( E := E_{C,V} \). Since \( H^1(S, \mathcal{O}_S) = 0 \), we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & V^* \otimes \mathcal{O}_S & \to & H^0(S, E) \otimes \mathcal{O}_S & \to & H^0(C, N_C \otimes L^{-1}) \otimes \mathcal{O}_S & \to & 0 \\
0 & \to & V^* \otimes \mathcal{O}_S & \to & E & \to & N_C \otimes L^{-1} & \to & 0.
\end{array}
\]

This implies that \( E \) is globally generated, with \( h^0(E) = 2 + h^0(N_C \otimes L^{-1}) \geq 4 \). From the bottom exact sequence we get \( c_1(E) = [C] \) and \( c_2(E) = \deg(L) > 0 \). The conclusion follows from Proposition 2. 

4. Application 2: Congruences of Lines

Let $\mathbb{G}$ be the Grassmannian of lines in $\mathbb{P}^3$, which we view as a smooth quadric in $\mathbb{P}^5$; let $S \subset \mathbb{G}$ be a smooth surface. This defines a 2-dimensional family of lines in $\mathbb{P}^3$, classically called a congruence. A point $p \in \mathbb{P}^3$ through which pass infinitely many lines of the congruence is called a fundamental point (or, more classically, a singular point) of the congruence.

**Proposition 3.** Assume that $S$ has degree $> 1$ and that $N^1(S)$ is generated by the restriction of $\mathcal{O}_S(1)$. Then $S$ has no fundamental point.

**Proof:** Let $E$ be the restriction to $S$ of the universal quotient bundle $Q$ on $\mathbb{G}$. The projective bundle $\mathbb{P}(E)$ on $S$ parametrizes pairs $(\ell, p)$ in $S \times \mathbb{P}^3$ with $p \in \ell$, and the second projection $q : \mathbb{P}(S(E)) \to \mathbb{P}^3$ satisfies $q^* \mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)$. Thus $q$ is finite (that is, $S$ has no fundamental point) if and only if $E$ is ample.

We have $h^0(Q) = 4$, and a nonzero section of $Q$ vanishes along a linear plane; therefore $h^0(E) \geq 4$, and we can apply Proposition 1. If $E = \mathcal{O}_S(1)$, we have $c_2(E) = 0$, that is, $c_2(Q) \cdot [S] = 0$; this can only happen if $S$ is a linear plane, which we have excluded. Therefore $E$ is ample. $lacksquare$

**Corollary.** Let $d, e$ be two integers with $d > 1$, or $d = 1$ and $e \geq 3$; let $S \subset \mathbb{G}$ be the complete intersection of two general hypersurfaces of degree $d$ and $e$. Then $S$ has no fundamental point.

Indeed $\text{Pic}(S)$ is generated by $\mathcal{O}_{\mathcal{S}}(1)$ [3, Théorème 1.2].

**Examples.** 1) Perhaps the simplest example of a nontrivial congruence is the surface $S$ of lines bisecant to a twisted cubic $T \subset \mathbb{P}^3$; it is isomorphic to $\text{Sym}^2 T \cong \mathbb{P}^2$, embedded in $\mathbb{G} \subset \mathbb{P}^5$ by the Veronese map. In that case $N^1 = \mathbb{Z} \cdot [\mathcal{O}_{\mathbb{P}^2}(1)]$ but $\det E = \mathcal{O}_{\mathbb{P}^2}(2)$, and indeed the fundamental locus of $S$ is $T$, so $E$ is ample.

2) Let $A$ be an abelian surface such that $\text{NS}(A) = \mathbb{Z} \cdot [L]$, where $L$ is a line bundle with $L^2 = 10$. The linear system $|L|$ embeds $A$ into $\mathbb{P}^4$ [4], giving the famous Horrocks-Mumford abelian surface. The projection $\pi : \mathbb{G} \to \mathbb{P}^4$ from a general point of $\mathbb{P}^5$ is a double covering, and the surface $S := \pi^{-1}(A) \subset \mathbb{G}$ is smooth. The line bundle $\pi^* L$ is not divisible in $N^1(S)$: since $(\pi^* L)^2 = 20$, this could happen only if $\pi^* L$ is divisible by 2; but $\pi^* L = K_S$, so this would imply that $K_S^2$ is divisible by 8, a contradiction. It then follows from [4] that $N^1(S)$ is generated by $\pi^* L = \mathcal{O}_S(1)$, so Proposition 2 applies and $S$ has no fundamental point.

5. Application 3 (Virtual): Surfaces with Ample Cotangent Bundle

The original motivation of this work was to obtain new examples of surfaces with ample cotangent bundle – these surfaces have very interesting properties, but there are few concrete examples known. Applying Proposition 1 to $\Omega^1_S$, we get the following result; unfortunately we do not know any example of a surface satisfying the hypotheses (help welcome!).

**Proposition 4.** Assume that $\Omega^1_S$ is globally generated (for instance that $S$ is a subvariety of an abelian variety), $q(S) \geq 4$, and $N^1(S) = \mathbb{Z} \cdot [K_S]$. Then $\Omega^1_S$ is ample.

**Proof:** The hypotheses imply that $K_S$ is ample, hence $c_2(S) > 0$; therefore $\Omega^1_S$ is not isomorphic to $\mathcal{O}_S \oplus K_S$. The conclusion follows from Proposition 1. $lacksquare$
REFERENCES

[B] F. Bogomolov: Holomorphic tensors and vector bundles on projective varieties. Math. of the USSR, Izvestija 13 (1979), 499-555.

[Bu] A. Buium: Sur le nombre de Picard des revêtements doubles des surfaces algébriques. C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 8, 361-364.

[D] P. Deligne: Le théorème de Noether. Groupes de monodromie en géométrie algébrique II, Exposé XIX, 328-340. Lecture Notes in Math. 340, Springer, Berlin-Heidelberg-New York, 1973.

[L] R. Lazarsfeld: Positivity in algebraic geometry, II. Ergeb. Math. (3) 49. Springer-Verlag, Berlin, 2004.

[R] S. Ramanan: Ample divisors on abelian surfaces. Proc. London Math. Soc. (3) 51 (1985), no. 2, 231-245.

[Ra] M. Raynaud: Fibrés vectoriels instables – applications aux surfaces (d’après Bogomolov). Algebraic surfaces, 293-314; Lecture Notes in Math. 868, Springer, Berlin-New York, 1981.

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