Isospin breaking in pion and $K_{e4}$ form factors

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Abstract. Isospin breaking in the $K_{e4}$ form factors induced by the difference between charged and neutral pion masses is discussed within a framework built on suitably subtracted dispersion representations. The $K_{e4}$ form factors are constructed in an iterative way up to two loops in the low-energy expansion by implementing analyticity, crossing, and unitarity due to two-meson intermediate states. Analytical expressions for the phases of the two-loop form factors of the $K^\pm \rightarrow \pi^+\pi^-e^\pm\nu_e$ channel are presented, allowing one to connect the difference of form-factor phase shifts measured experimentally (out of the isospin limit) and the difference of $S$- and $P$-wave $\pi\pi$ phase shifts studied theoretically (in the isospin limit). The dependence with respect to the two $S$-wave scattering lengths $a_{0}^0$ and $a_{1}^0$ in the isospin limit is worked out in a general way, in contrast to previous analyses based on one-loop chiral perturbation theory. The results on the phases of the $K^\pm \rightarrow \pi^+\pi^-e^\pm\nu_e$ form factors obtained by the NA48/2 collaboration at the CERN SPS are reanalysed including isospin-breaking correction to extract values for the scattering lengths $a_{0}^0$ and $a_{1}^0$.

Keywords: Isospin breaking, pion form factors, kaon form factors, dispersion relations, low-energy expansion, pion scattering lengths

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1. INTRODUCTION

Very accurate information on the $\pi\pi$ $S$-wave scattering lengths in the isospin limit $a_{0}^0$ and $a_{1}^0$ is now available from several experimental processes, $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$ [1], pionic atoms [2], and the $K_{e4}$ semi-leptonic decay $K^\pm \rightarrow \pi^+\pi^-e^\pm\nu_e$. In the last case, progress has been particularly impressive in recent years. The NA48/2 Collaboration [3, 4] at the CERN SPS has collected $\sim$ 1100000 events. This represents more than twice the statistics obtained by the previous experiment at the Brookhaven AGS, where the BNL-E865 collaboration [5, 6] had collected $\sim$ 400000 events, and an improvement by a factor of more than 35 with respect to the Geneva-Saclay experiment [7], the first high-controls experiment of this type, which, almost 40 years ago, had collected $\sim$ 30000 events.

Standard angular analysis of the $K_{e4}$ decay amplitude [8, 9] shows that information on $\pi\pi$ scattering is contained in the phases of the form factors that describe it. The interference term that two of these form factors produce in the scattering amplitude, as a consequence of Watson’s theorem [10]. The values of the scattering lengths can then be extracted upon fitting the experimentally measured phase difference with the corresponding solution $\delta_{R_{Roy}}^{S-P}(s; a_{0}^0, a_{1}^0)$ of the Roy equations:

$$[\delta_{S}(s) - \delta_{P}(s)]_{\text{exp}} = \delta_{R_{Roy}}^{S-P}(s; a_{0}^0, a_{1}^0).$$

(1)

The Roy equations [11] rely on fixed-$t$ dispersion relations (i.e. analyticity, unitarity, crossing, the Froissard bound) for the $\pi\pi$ amplitudes, $\pi\pi$ at higher energies $s \gtrsim 800$ MeV, and isospin symmetry. Numerical solutions for these equations exist and can be constructed for arbitrary values of the scattering lengths $a_{0}^0$ and $a_{1}^0$ belonging to the so-called universal band, see Ref. [12] for details. In the real world, isospin is not an exact symmetry. It is explicitly broken by electromagnetic corrections, and by the small effects induced by the quark-mass difference $m_u - m_d$. While radiative corrections are considered in the analysis performed by the NA48/2 Collaboration [3, 4], there remain small isospin-breaking (IB) effects related to the difference of the masses of charged ($M_\pi$) and neutral ($M_{\pi^0}$) pion, $M_\pi \neq M_{\pi^0}$. As
emphasized in [13], it is important to account for these effects in extracting the values of the scattering lengths from the
data, given the level of precision achieved. The evaluation of the relevant IB correction to the $K_{e4}$ matrix element and differential
decay rate were subsequently worked out at one-loop precision in the chiral expansion [14]. This allows one to replace Eq. (1) by the more appropriate relation
\[ [\Delta S(s) - \Delta P(s)]_{\text{exp}} = \delta_{\text{Roy}}^P(s; a_0^2, a_2^2) + \delta_{\text{IB}}^1(s; (a_0^2)_{\text{CA}}, (a_2^2)_{\text{CA}}), \] (2)
where $\delta_{\text{IB}}^1(s; (a_0^2)_{\text{CA}}, (a_2^2)_{\text{CA}})$ denotes the correction factor to the phase difference computed in Ref. [14]. Before commenting on it, let us quote the values [4] obtained from the fit using solutions of the Roy equations provided by
Refs. [12, 15], and the correction from Ref. [14] [we quote here the result from “Model B”]
\[ a_0^0 = 0.2220(128)_{\text{stat}}(50)_{\text{syst}}(37)_{\text{th}}, \quad a_2^0 = -0.0432(86)_{\text{stat}}(34)_{\text{syst}}(28)_{\text{th}}. \] (3)
As already mentioned, the correction $\delta_{\text{IB}}^1(s; (a_0^2)_{\text{CA}}, (a_2^2)_{\text{CA}})$ is evaluated at next-to-leading (one-loop) order only, which raises the issue of the possible sensitivity of the analysis to higher order corrections, given the high accuracy of the experimental data. In addition, the NLO correction computed in chiral perturbation theory necessarily involves the scattering lengths fixed at their tree level (current algebra) values [16], $(a_0^0)_{\text{CA}} = 7M_K^2/32\pi F_\pi^2$ and $(a_2^0)_{\text{CA}} = -M_K^2/16\pi F_\pi^2$. Actually, higher-order effects were estimated in Ref. [14], but from a NNLO calculation of the scalar form factor of the pion. This estimate accounts for almost all the theory error in Eq. (3). The same drawbacks are shared by other studies devoted to IB in $K_{e4}$ decays [17, 18, 19]. The situation is then that one extracts the scattering lengths from a fit to solutions of the Roy equations, which depend parametrically on the scattering lengths, after having applied IB corrections evaluated for fixed and predefined values of $a_0^0$ and $a_2^0$! This limitation may induce a bias in the extraction of the scattering lengths from data based on Eq. (2), and it is important to be able to quantify this effect. It is therefore necessary to develop a computational framework of isospin-breaking corrections in the phases of the form factors where the values of the scattering lengths are not unnecessarily restricted from the outset. The outcome of such a construction should result in the replacement of Eq. (2) by
\[ [\Delta S(s) - \Delta P(s)]_{\text{exp}} = \delta_{\text{Roy}}^P(s; a_0^0, a_2^0) + \delta_{\text{IB}}(s; a_0^2, a_2^2), \] (4)
where $\delta_{\text{IB}}(s; a_0^0, a_2^0)$ is evaluated at least at NNLO, and where $a_0^0$ and $a_2^0$ appear as free parameters. How this goal can be achieved will be described in the sequel. Further details may be found in Refs. [20] and [21], on which the present report is based.

Before starting, let us illustrate the issue with a simple example, leaving aside, for the sake of demonstration, violations of isospin symmetry. In the isospin limit, the one-loop expressions of the $K_{e4}$ form factors are well documented in the literature [22, 23], and one finds for one of the form factors involved in the decay channel of interest, $K^+ \to \pi^+ \pi^- \ell^+ \nu_\ell$,
\[ F^-(s, t, u) = \frac{M_K}{\sqrt{2}F_\pi} \left[ 1 + \frac{2s - M_K^2}{2F_\pi^2} J_{\pi\pi}^r(s) + \cdots \right], \] (5)
where the ellipses stand for additional contributions that play no role in the present discussion, $s$ denotes the square of the invariant mass of the dipion system, $F_\pi$ is the pion decay constant, and $J_{\pi\pi}^r$ is the renormalized one-loop two-point function. In this expression of the one-loop form factor, no dependence on the scattering lengths is visible, neither in this term nor in the omitted ones. However, in the computation of the form factors, the actual expression in terms of the low-energy constants of the $\chi$PT Lagrangian [24] reads
\[ F^-(s, t, u) = \frac{M_K}{\sqrt{2}F_\pi} \left[ 1 + \frac{2s - 2\tilde{m}B_0}{2F_\pi^2} J_{\pi\pi}^r(s) + \cdots \right], \]
\[ \tilde{m} = (m_u + m_d)/2 \] (6)
which agrees with the previous expression (5) if the leading-order relations $F_\pi = F_0$ and $M_K^2 = 2\tilde{m}B_0$ are used [this is the appropriate order to consider in this example], explaining why the expression (5) is usually quoted. However, it is not straightforward to reinterpret the expression (6) in terms of the $\pi\pi$ scattering lengths $a_0^0$ and $a_2^0$: they are both proportional to $2\tilde{m}B_0$ at lowest order [24], but there are infinitely many combinations of $M_K^2$, $a_0^0$, and $a_2^0$ that sum up to $2\tilde{m}B_0$ at this order. Even if a contribution from the $l = 2$ channel is forbidden by the $\Delta l = 1/2$ rule of the corresponding weak charged current, the question still remains how to determine the combination that gives the correct dependence
on $a_0^0$. Obviously, the information provided by Eq. (6) alone does not allow for an unambiguous answer. As can easily be guessed, the missing link is provided by unitarity. The function $J^K_{\pi\pi}$ encodes the discontinuity of the form factor $F^{\pi\pi}(s,t,u)$ along the positive real $s$-axis, which involves the $l = 0 \pi\pi$ partial wave in the channel with zero angular momentum as a final-state interaction effect [23]. A careful analysis shows that, at one-loop order, Eqs. (5) and (6) actually read
\[ \begin{align*}
F^{\pi\pi}(s,t,u) &= \frac{M_K}{\sqrt{2}F_\pi} \left[ 1 + \cdots + \left( \frac{s - 4M_\pi^2}{F_0^2} + 16\pi a_0^0 \right) J^K_{\pi\pi}(s) \right]. \tag{7}
\end{align*} \]
Let us stress that, barring higher-order contributions presently not under discussion, the three representations are strictly identical. However, if one considers the scattering lengths $a_0^0$ and $a_0^2$ as free variables that have to be adjusted from a fit to experimental data, only the third form is actually suitable. It is certainly conceivable to use the existing one-loop expressions of $K_{\pi4}$ form factors, now including isospin-violating effects [17, 18], and to repeat the above analysis for each separate contribution. But this would represent a rather cumbersome exercise, and would anyway only give a result to one-loop precision. Instead, we will develop a more global approach, where the relevant unitarity properties are put forward explicitly from the start, and which, in addition, holds at two-loop precision. This approach proceeds along the same lines as those followed in order to establish the “reconstruction theorem” for the $\pi\pi$ scattering amplitude (in the isospin limit) in Ref. [25] and then implemented in order to construct an explicit two-loop representation of this amplitude in Ref. [26].

2. TWO-LOOP REPRESENTATION OF PION FORM FACTORS WITH IB

In order to dispense, at a first stage, with some of the kinematical complexities that beset the discussion of the $K_{\pi4}$ form factors, we describe the general method using the simpler framework provided by the neutral and charged scalar form factors of the pion. These form factors are defined as $[\hat{m} \equiv (m_u + m_d)/2]$
\[ \begin{align*}
&\langle \pi^0(p_1)p_2|\hat{m}(\bar{u}d + d\bar{u})|\Omega \rangle = +F^0_S(s), \\
&\langle \pi^+(p_+)|\pi^-(p_-)|\hat{m}(\bar{u}d + d\bar{u})|\Omega \rangle = -F^\mp_S(s), \tag{8}
\end{align*} \]
and
\[ \begin{align*}
\frac{1}{2}(\pi^+\pi^-|[(\pi\gamma_\mu u - \bar{d}\gamma_\mu d)(0)|\Omega) = (p_+ - p_-)\mu F^\mu_\pi(s). \tag{9}
\end{align*} \]

The starting point of the construction is provided by dispersive representations of the form factors and of the $\pi\pi$ scattering amplitudes. For the former, they write [27]
\[ \begin{align*}
F^\pi_S(s) &= F^\pi_S(0) \left[ 1 + \frac{1}{6}(r^2)^0 r^0 s + c^r_0 s^2 + U^\pi_S(s) \right], \\
F^\mp_S(s) &= F^\mp_S(0) \left[ 1 + \frac{1}{6}(r^2)^\mp r^\mp s + c^r_\pi s^2 + U^\mp_S(s) \right], \\
F^\mu_\pi(s) &= 1 + \frac{1}{6}(r^2)^\mu r^\mu s + c^r_\mu s^2 + U^\mu_\pi(s), \tag{10}
\end{align*} \]
with
\[ \begin{align*}
U^\pi_S(s) &= \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im}F^\pi_S(x)/F^\pi_S(0)}{x - s - i\delta}, \\
U^\mp_S(s) &= \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im}F^\mp_S(x)/F^\mp_S(0)}{x - s - i\delta}, \\
U^\mu_\pi(s) &= \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im}F^\mu_\pi(x)}{x - s - i\delta}. \tag{11}
\end{align*} \]
For the scattering amplitudes, we start from fixed-$t$ dispersion relations with three subtractions [25]
\[ \begin{align*}
A(s,t) = P(t|s,u) + \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{1}{x - s - i\delta} \text{Im}_r A(x,t) + \frac{u^3}{\pi} \int \frac{dx}{x^3} \frac{1}{x - u - i\delta} \text{Im}_u A(x,t). \tag{12}
\end{align*} \]
In the case where \( M_\pi \neq M_{\pi^0} \), one has several amplitudes to consider [21], according to the number of charged pions involved. Eq. (12) merely displays the general structure of the corresponding dispersion relations. The absorptive parts in the \( s \) and \( u \) channels are related by crossing.

The second ingredient consists of the partial wave expansions of the \( \pi \pi \) amplitudes [this is one instance where the case of the pion form factors is simpler: the \( K_{l4} \) form factors, which depend on an angular variable, are also subject to a decomposition into partial-wave projections, see below]

\[
A(s,t) = 16\pi \sum_{l \geq 0} (2l+1)P_l(\cos \theta) f_l(s), \quad f_l(s) = \frac{1}{32\pi} \int_{-1}^{+1} dz A(s,t) P_l(z). \tag{13}
\]

The third ingredient is provided by chiral counting for the partial waves and the form factors. If \( E \) denotes a pion momentum or a pion mass, the leading behaviour of the various quantities is given by

\[
\begin{align*}
\text{Re} F_S^{\pi(\pi^0)}(s) &\sim \mathcal{O}(E^2), & \text{Im} F_S^{\pi(\pi^0)}(s) &\sim \mathcal{O}(E^4), \\
\text{Re} F_V^{\pi}(s) &\sim \mathcal{O}(E^0), & \text{Im} F_V^{\pi}(s) &\sim \mathcal{O}(E^2),
\end{align*}
\tag{14}
\]

and

\[
\begin{align*}
\text{Re} f_l(s) &\sim \mathcal{O}(E^2), & \text{Im} f_l(s) &\sim \mathcal{O}(E^4), \quad l = 0, 1, \\
\text{Re} f_l(s) &\sim \mathcal{O}(E^4), & \text{Im} f_l(s) &\sim \mathcal{O}(E^8), \quad l \geq 2.
\end{align*}
\tag{15}
\]

These properties allow us to decompose the real parts of the \( l = 0, 1 \) partial waves as

\[
\text{Re} f_l(s) = \varphi_l(s) + \psi_l(s) + \mathcal{O}(E^6), \quad \varphi_l(s) \sim \mathcal{O}(E^2), \quad \psi_l(s) \sim \mathcal{O}(E^4),
\tag{16}
\]

so that

\[
|f_l(s)|^2 = |\text{Re} f_l(s)|^2 + \mathcal{O}(E^8) = [\varphi_l(s)]^2 + 2\varphi_l(s)\psi_l(s) + \mathcal{O}(E^8), \quad l = 0, 1. \tag{17}
\]

Analyticity and unitarity, which together make up the fourth ingredient, give us information about the cut singularities and their discontinuities. The absorptive parts of the dispersion relations we started with are given by unitarity. These discontinuities are restricted by power counting. Indeed, in the low-energy region, only two-pion intermediate states occur up to two loops. Making use of the counting rules in Eqs. (14) and (15), we are then led to

\[
\begin{align*}
\text{Im} F_S^{\pi(\pi^0)}(s) &= \Re \left\{ \frac{1}{2} \sigma_0(s) f_0^{00}(s) F_S^{\pi(\pi^0)}(s) \theta(s - 4M_\pi^2) - \sigma(s) f_0^{\pi}(s) F_S^{\pi(\pi^0)}(s) \theta(s - 4M_\pi^2) \right\} + \mathcal{O}(E^8), \\
\text{Im} F_V^{\pi}(s) &= \Re \left\{ \sigma(s) f_0^{-}(s) F_V^{\pi}(s) \theta(s - 4M_\pi^2) - \frac{1}{2} \sigma_0(s) f_0^{\pi}(s) F_V^{\pi}(s) \theta(s - 4M_\pi^2) \right\} + \mathcal{O}(E^8), \\
\text{Im} f_l(s) &= \Re \left\{ \sigma(s) f_l^{-}(s) F_V^{\pi}(s) \theta(s - 4M_\pi^2) \right\} + \mathcal{O}(E^6),
\end{align*}
\tag{18}
\]

Here,

\[
\sigma_0(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}, \quad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}} \tag{19}
\]

denote the neutral and charged two-pion phase spaces.

We possess now all the tools necessary to proceed towards the construction of the two-loop representations of the form factors and scattering amplitudes through an iterative two-step process that is described by Fig. 1 below. We start with the expressions of the \( \pi\pi \) amplitudes at lowest order [the superscript 00 stands for \( \pi^0\pi^0 \to \pi^0\pi^0 \), \( + \) for \( \pi^+\pi^- \to \pi^+\pi^- \), and \( x \) for the inelastic \( \pi^+\pi^- \to \pi^0\pi^0 \) channel]

\[
A^{00}(s,t) = 16\pi a_{00}, \quad A^x(s,t) = 16\pi \left[ a_+ + b_+ \frac{s - 4M_\pi^2}{F_\pi^2} \right], \quad A^- (s,t) = 16\pi \left[ a_- + b_- \frac{s - 4M_\pi^2}{F_\pi^2} + c_- \frac{t - u}{F_\pi^2} \right]. \tag{20}
\]
Injecting them into the dispersion relations, we obtain the one-loop expressions of the form factors as

\[ \phi_0^0(s) = a_0, \quad \phi_0^0(s) = a_s + b_s s - 4M^2_{\pi} F^2_{\pi}, \quad \phi_0^{-}(s) = a_- + b_- s - 4M^2_{\pi} F^2_{\pi}, \quad \phi_0^{+}(s) = c_+ s - 4M^2_{\pi} F^2_{\pi}. \]  

(21)

from which we obtain the partial-wave projections

\[ \phi_0^0(s) = a_0, \quad \phi_0^0(s) = a_s + b_s s - 4M^2_{\pi} F^2_{\pi}, \quad \phi_0^{-}(s) = a_- + b_- s - 4M^2_{\pi} F^2_{\pi}, \quad \phi_0^{+}(s) = c_+ s - 4M^2_{\pi} F^2_{\pi}. \]

This input then gives us the absorptive parts of the one-loop form factors

\[ \text{Im} F_S^0(s) = \frac{1}{2} \sigma_0(s) \phi_0^0(s) F_s^0(0) \theta(s - 4M^2_{\pi}) - \sigma(s) \phi_0^0(s) F_s^0(0) \theta(s - 4M^2_{\pi}) + O(E^6). \]

\[ \text{Im} F_S^{-}(s) = \sigma(s) \phi_0^{-}(s) F_s^0(0) \theta(s - 4M^2_{\pi}) - \frac{1}{2} \sigma_0(s) \phi_0^0(s) F_s^0(0) \theta(s - 4M^2_{\pi}) + O(E^6). \]

\[ \text{Im} F_S^{+}(s) = \sigma(s) \phi_0^{+}(s) \theta(s - 4M^2_{\pi}) + O(E^4). \]

(22)

Injecting them into the dispersion relations, we obtain the one-loop expressions of the form factors as

\[ F_S^0(s) = F_S^0(0) \left[ 1 + a^0 s + 16\pi \phi_0^0(s) 2 \tilde{J}_0(s) \right] - 16\pi F_S^0(0) \phi_0^0(s) \tilde{J}(s) \]

\[ F_S^{-}(s) = F_S^{-}(0) \left[ 1 + a^- s + 16\pi \phi_0^{-}(s) \tilde{J}(s) \right] - 16\pi F_S^0(0) \frac{1}{2} \phi_0^0(s) \tilde{J}_0(s) \]

\[ F_S^{+}(s) = 1 + a^+ s + 16\pi \phi_0^{+}(s) \tilde{J}(s), \]

(23)

where

\[ \tilde{J}_0(s) = \frac{s}{16\pi^2} \int_{4M^2_{\pi}}^{\infty} \frac{dx}{x-x-s-i0} \sigma_0(x) = -\frac{1}{16\pi^2} \int_0^1 dx \ln \left[ 1 - x(1-x) \frac{s}{M^2_{\pi}} \right] \]

\[ \tilde{J}(s) = \frac{s}{16\pi^2} \int_{4M^2_{\pi}}^{\infty} \frac{dx}{x-x-s-i0} \sigma(x) = -\frac{1}{16\pi^2} \int_0^1 dx \ln \left[ 1 - x(1-x) \frac{s}{M^2_{\pi}} \right]. \]

(24)

The scattering lengths, \( a_-, a_s, a_0 \), and the slope parameters \( b_-, b_s \) are related, at this order, to the scattering lengths \( a_0^0 \) and \( a_0^2 \) in the isospin limit by [28, 21, 20]

\[ a_- = \frac{2}{3} a_0^0 + \frac{1}{3} a_M^2 - 2a_0^0 \frac{\Delta_\pi}{M^2_{\pi}}, \quad b_- = c_- = \frac{1}{24 M^2_{\pi}} \left( 2a_0^0 - 5a_0^2 \right), \]

\[ a_s = -\frac{2}{3} a_0^0 + \frac{2}{3} a_0^2 + a_M^2 \frac{\Delta_\pi}{M^2_{\pi}}, \quad b_s = -\frac{1}{12 M^2_{\pi}} \left( 2a_0^0 - 5a_0^2 \right), \]

\[ a_{00} = \frac{2}{3} a_0^0 + \frac{4}{3} a_0^2 - \frac{2}{3} \left( a_0^0 + 2a_0^2 \right) \frac{\Delta_\pi}{M^2_{\pi}}, \quad \Delta_\pi = M^2_{\pi} - M^2_{\pi}. \]

(25)
The subtraction constants $\alpha^\pi_0$, $\alpha^\pi_3$, and $\alpha^\pi_4$ are related to the corresponding mean-square radii [21] in a calculable way, in terms of the scattering lengths. The same procedure can be applied to the $\pi\pi$ scattering amplitudes themselves. Let us just quote the result for the $\pi^0\pi^0 \to \pi^0\pi^0$ case,

$$A^{00}(s,t,u) = P^{00}(s,t,u) + W^{00}_0(s) + W^{00}_0(t) + W^{00}_0(u) + O(E^6).$$

Here the function $W^{00}_0(s)$ has a discontinuity starting at $s = 4M_{\pi^0}$ along the positive $s$ axis [the function $W^{00}_0(s)$ itself has only a right-hand cut; the left-hand cut of the amplitude $A^{00}(s,t,u)$ results from the two other contributions, involving $W^{00}_0(t)$ and $W^{00}_0(t)$]. At one-loop order it reads

$$\frac{1}{16\pi} \text{Im} W^{00}_0(s) = \frac{1}{2} \phi_0(s) \left[ \frac{\phi_0(s)}{s} \right]^2 \theta(s - 4M_{\pi^0}^2) + \sigma(s) \left[ \phi_0(s) \right]^2 \theta(s - 4M_{\pi}^2) + O(E^6),$$

so that

$$W^{00}_0(s) = \frac{1}{2} \left[ 16\pi \phi_0(s) \right]^2 J_0(s) + \left[ 16\pi \phi_0(s) \right]^2 J(s).$$

Finally, $P^{00}(s,t,u)$ represents a polynomial of at most second order (at one loop) in $s,t,u$, symmetric under any permutation of its variables [due to the fact that $A^{00}(s,t,u)$ transforms into itself under crossing]

$$P^{00}(s,t,u) = 16\pi a_{00} - w_{00} + \frac{3\lambda^{(1)}_{00}}{F_{\pi}} \left[ s(s - 4M_{\pi^0}^2) + t(t - 4M_{\pi^0}^2) + u(u - 4M_{\pi^0}^2) \right].$$

$\lambda^{(1)}_{00}$ denotes an additional subtraction constant, which can be related to two subtraction constants that describe the $\pi\pi$ amplitude in the isospin limit [26]

$$\lambda^{(1)}_{00} = \frac{1}{3} (\lambda_1 + 2\lambda_2),$$

and whose values are known [29, 15]. The quantity

$$w_{00} = \text{Re} \left[ W^{00}_0(4M_{\pi^0}^2) + W^{00}_0(0) + W^{00}_0(0) \right]$$

is then uniquely fixed by the requirement that $a_{00}$ retains its meaning as scattering length at next-to-leading order, i.e. $\text{Re} A^{00}(4M_{\pi^0}^2,0,0) = 16\pi a_{00}$. The structure of the other amplitudes is similar, and we refer the interested reader to [21] for details.

With the one-loop form factors and amplitudes at our disposal, we can now repeat the same procedure: compute the $S$ and $P$ partial-wave projections from the one-loop amplitudes, use them to express the discontinuities of the two-loop form factors and amplitudes, and eventually obtain the full two-loop form factors and amplitudes. Their expressions will involve a limited number of additional subtraction constants, which can however be related to the parameters that describe the same quantities in the isospin limit, the IB breaking corrections being expressed in terms of the scattering lengths [21]. The remarkable feature of this second iteration is that the partial-wave projections of the one-loop amplitudes can be obtained analytically, using the known expressions of the functions $J_0(s)$ and $J(s)$ in terms of elementary functions. However, it is in general not possible to perform all the corresponding dispersion integrals analytically if $M_\pi \neq M_{\pi^0}$, in contrast to the situation in the isospin limit, where analytical expressions are available [26]. Thus, the real parts of the two-loop amplitudes and form factors are partly known only as one-dimensional integrals, which have to be evaluated numerically. However, the expressions of the phases at two loops only involve the real parts at one loop, which are known analytically. We have therefore reached our goal, in this somewhat simpler setting, of obtaining expressions of the phases at two-loop precision, parameterized in terms of the scattering lengths in the isospin limit. We will now briefly explain how essentially the same procedure can be used in order to obtain two-loop expressions for the (phases of the) $K_{e4}$ form factors that depend parametrically on the scattering lengths.

### 3. TWO-LOOP REPRESENTATION OF $K_{e4}$ FORM FACTORS WITH IB

The construction of two-loop representations for the form factors describing the matrix elements for the $K_{e4}$ transitions $K^{\pm} \to \pi^+ \pi^- e^+ \nu_e$, $\ell = e, \mu$, proceeds essentially along the same lines. On the technical level, additional complications arise, due, on the one hand, to the fact that there are several form factors, related by crossing, to consider simultaneously, and, on the other hand, that these form factors depend on two energy variables and one angular variable. In
this Section, we will successively go through the list of ingredients listed in the preceding Section, and describe the changes that are induced by these two features.

In the Standard Model, the amplitudes corresponding to $K \to \pi\pi$ decays are defined by the matrix elements of the type $\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu(0)|K(k)\rangle$ and $\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu(0)|K(k)\rangle$ involving the $\Delta S = \Delta Q = +1$ axial and vector currents between a (charged or neutral) kaon state and the corresponding two-pion state, specifically $(K, a, b) \in \{(K^+, +, -), (K^+, 0, 0), (K^0, 0, -)\}$. In the present study, we will not consider the matrix element of the vector current, related to the axial anomaly, and described by a single form factor $H^{ab}(s, t, u)$. Since crossing is one of the ingredients of our construction, we also need to consider the matrix elements related to $\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu(0)|K(k)\rangle$ through this operation, namely $\langle \pi^a(p_a)\bar{K}(k)|iA_\mu(0)|\bar{K}(p_b)\rangle$ and $\langle \bar{K}(k)\pi^b(p_b)|iA_\mu(0)|\bar{K}(p_a)\rangle$. In order to be able to treat these matrix elements simultaneously and on a common footing, we consider general matrix elements of the type [20]

$$\mathcal{A}^{ab}(p_a, p_b, p_c) = (a(p_a)b(p_b)|iA_\mu(0)|\bar{c}(p_c)),$$

with $\{a, b, c\} = \{\pi^+, \pi^-, K^-, \pi^0, \pi^-, K^0\}$ or $\{\pi^0, \pi^-, K^0\}$. These matrix elements possess the general decompositions into invariant form factors

$$\mathcal{A}^{ab}(p_a, p_b, p_c) = (p_a + p_b)\mu F^{ab}(s, t, u) + (p_a - p_b)\mu G^{ab}(s, t, u) + (p_c - p_a - p_b)\mu R^{ab}(s, t, u).$$

They depend on the variables $s = (p_a + p_b)^2$, $t = (p_c - p_b)^2$, obeying the “mass-shell” condition $s + t + u = M_\pi^2 + M_\pi^2 + M_K^2 + s_t \equiv \Sigma_t$, with $s_t \equiv (p_c - p_a - p_b)^2$ being the square of the dilepton invariant mass. In the physical region of the $K\to\pi\pi$ decay, $s_t$ is strictly positive, $s_t \geq m_\pi^2$, and in what follows we will always assume this to be the case. Independent variables will conveniently be chosen as $s$, $s_t$, and the angle $\theta_{ab}$ made by the line of flight of particle $a$ in the $(a, b)$ rest frame with the direction of $\bar{p}_a + \bar{p}_b$ in the rest frame of particle $\bar{c}$.

$$\cos \theta_{ab} = \frac{(M_\pi^2 - M_K^2)(s_t - M_\pi^2) - s(t - u)}{\lambda_{ab}^2(s)\lambda_{ab}^2(s_t)}.
\lambda_{ab}^2(s) = \frac{(M_\pi^2 - M_K^2)(s_t - M_\pi^2) + s(s_t - 2t)}{\lambda_{ab}^2(s)\lambda_{ab}^2(s_t)}.
\lambda_{ab}(s) = \lambda(s, M_\pi^2, M_K^2)$$

The functions $\lambda_{ab}(s)$ and $\lambda_{ab}(s_t)$ are defined in terms of Kähler’s function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ by $\lambda_{ab}(s) = \lambda(s, M_\pi^2, M_K^2)$ and $\lambda_{ab}(s_t) = \lambda(s_t, M_\pi^2, M_K^2)$, respectively.

As in the case of the pion form factors discussed in the preceding Section, the starting point of the construction consists of suitably subtracted dispersion relations for fixed $t$ and $s_t$. Before writing down the relevant dispersion relations, let us briefly discuss under which form the other ingredients that were listed and used in the case of the pion form factors enter in the present case.

- Crossing properties

In the previous Section, crossing was only relevant for the $\pi\pi$ scattering amplitudes. In the present case, the matrix elements (32) are also concerned. Their crossing properties are expressed through the relations

$$\mathcal{A}^{ac}(p_a, p_c; p_b) = \lambda_{abc}\mathcal{A}^{ab}(p_a, p_b; p_c), \quad \mathcal{A}^{cb}(p_c, p_b; p_a) = \lambda_{abc}\mathcal{A}^{ab}(p_a, p_b; p_c),$$

where the matrix elements on the right-hand sides are related through analytic continuations to the original matrix element $\mathcal{A}^{ab}(p_a, p_b; p_c)$, assuming that the usual analyticity properties hold. The coefficients $\lambda_{abc}$ are crossing phases, which are chosen such as to reduce the Condon-Shortley phase convention in the isospin limit,

$$\lambda_{K^+} = \lambda_{\pi^+} = -1, \quad \lambda_{\pi^0} = \lambda_{K^0} = +1.$$

At the level of the form factors themselves, these crossing relations become

$$\mathcal{A}^{ac}(s, t, u) = \lambda_{abc}\mathcal{A}^{ab}(s, t, u), \quad \mathcal{A}^{cb}(s, t, u) = \lambda_{abc}\mathcal{A}^{ab}(s, t, u), \quad \mathcal{A}^{ba}(s, t, u) = \mathcal{A}^{ab}(s, u, t),$$

with

$$\mathcal{A}^{X}(s, t, u) = \begin{pmatrix} F^X(s, t, u) \\ G^X(s, t, u) \\ R^X(s, t, u) \end{pmatrix}.$$
where $X$ stands for any one of the couples of indices $ab$ (and, in the present case, also $ba$), $ac$, or $cb$, and

$$
G_{st} = \begin{pmatrix}
-1 & +3 & 0 \\
+1 & +1 & 0 \\
-1 & -1 & +1
\end{pmatrix},
G_{us} = \begin{pmatrix}
-1 & +3 & 0 \\
-1 & -1 & 0 \\
+1 & -1 & +1
\end{pmatrix},
G_{iu} = \begin{pmatrix}
+1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & +1
\end{pmatrix}.
$$

(39)

Each of these crossing matrices squares to the identity matrix. In addition, they satisfy the relations

$$
G_{st}G_{us} = G_{us}G_{iu},
G_{us}G_{iu} = G_{st}G_{tu},
G_{iu}G_{tu} = G_{st}G_{us}.
$$

(40)

It is useful to notice that under crossing the form factors $F^X$ and $G^X$ transform into form factors $F^Y$ and $G^Y$, without mixing with the form factors $R^T$. In the following, we will omit the form factors $R^X$ from the discussion most of the time, writing

$$
A^X(s, t, u) = \begin{pmatrix}
F^X(s, t, u) \\
G^X(s, t, u)
\end{pmatrix},
$$

(41)

instead of Eq. (38). When it is the case, it is understood that the crossing matrices are reduced to their upper-left $2 \times 2$ blocks. All the previous relations between these matrices remain unaffected by this truncation.

- Partial-wave projections

  The form factors appearing in the decomposition (33) are free from kinematical singularities, but do not have simple decompositions into partial waves. For the latter, it is more convenient to introduce another set of form factors. To this effect, adapting the method of Ref. [9] to the more general situation at hand, we define

$$
F^{ab}_i(s, t, u) = \frac{M^2 - M^2_b}{s} + \frac{M^2 - s - s_i}{s} \frac{\lambda_{ab}^i(s) \cos \theta_{ab}(s)}{\lambda_{ab}^2(s)} G^{ab}(s, t, u),
$$

$$
G^{ab}_i(s, t, u) = G^{ab}(s, t, u),
$$

$$
R^{ab}_i(s, t, u) = R^{ab}(s, t, u) + \frac{M^2 - s - s_i}{2s_i} F^{ab}(s, t, u)
+ \frac{1}{2s_i} \left[ (M^2 - M^2_b)(M^2 - s - s_i) + \frac{1}{s_i} \lambda_{ab}^2(s) \lambda_{ab}^2(s) \cos \theta_{ab} \right] G^{ab}(s, t, u).
$$

(42)

Notice that the form factor $R^{ab}(s, t, u)$ describes the matrix element of the divergence of the current $A^\mu(x)$,

$$
\langle a(p_a) b(p_b) [\partial^\mu A_\mu(0)]^c(p_c) \rangle = -s_i R^{ab}(s, t, u).
$$

(43)

These form factors have the following partial-wave decompositions [9]

$$
F^{ab}_i(s, t, u) = \sum_{\ell \geq 0} f^{ab}_i(s, s_i) P_\ell(\cos \theta_{ab}),
$$

$$
G^{ab}_i(s, t, u) = \sum_{\ell \geq 0} g^{ab}_i(s, s_i) P_\ell(\cos \theta_{ab}),
$$

$$
R^{ab}_i(s, t, u) = \sum_{\ell \geq 0} r^{ab}_i(s, s_i) P_\ell(\cos \theta_{ab}).
$$

(44)

Since $\{F; G; R\}^{ab}(s, t, u) = \{F; -G; R\}^{ba}(s, u, t)$ and $\cos \theta_{ab} = -\cos \theta_{ba}$, one has the symmetry properties

$$
f^{ba}_i(s, s_i) = (-1)^{\ell} f^{ab}_i(s, s_i), \quad g^{ba}_i(s, s_i) = (-1)^{\ell} g^{ab}_i(s, s_i), \quad r^{ba}_i(s, s_i) = (-1)^{\ell} r^{ab}_i(s, s_i).
$$

(45)

Let us also note that the form factors $F^X$ and $G^X$ transform among themselves under crossing. On the other hand, and in contrast with the form factors $R^X$, the form factors $R^X$ transform into themselves, without mixing with $F^X$ and $G^X$,

$$
R^{ab}(s, t, u) = \lambda_{ab} A^c R^{ab}(s, t, u), \quad R^{ab}(s, t, u) = \lambda_{ab} A^c R^{ab}(s, t, u).
$$

(46)
This result follows from Eq. (43): the form factors $\mathcal{F}^X$ cannot mix under crossing with the other form factors, which correspond to the transverse components of axial current.

- **Chiral counting**
  The chiral counting is given by $M_P \sim \mathcal{O}(E)$, $s, t, u, s_1 \sim \mathcal{O}(E^2)$, where $M_P = M_{\pi}, M_{\rho}, M_K$, and $s_1$ is treated on the same footing as one of the masses squared, which is compatible with its allowed range inside the $K_{\ell 4}$ phase space. On the level of the partial waves, this gives [30] [the counting of the $\pi\pi$ partial waves remains of course unchanged]

\[
\begin{align*}
\text{Re} f_0^{ab}(s,s_1), \text{Re} f_1^{ab}(s,s_1), \text{Re} g_0^{ab}(s,s_1) & \sim \mathcal{O}(E^0), \quad \text{Im} f_0^{ab}(s,s_1), \text{Im} f_1^{ab}(s,s_1), \text{Im} g_0^{ab}(s,s_1) \sim \mathcal{O}(E^2) \\
\text{Re} f_1^{ab}(s,s_1), \text{Re} g_1^{ab}(s,s_1) & \sim \mathcal{O}(E^2), \quad l \geq 2, \quad \text{Im} f_1^{ab}(s,s_1), \text{Im} g_1^{ab}(s,s_1) \sim \mathcal{O}(E^6), \quad l \geq 2.
\end{align*}
\]

The $S$ and $P$ waves as therefore dominant at low energies, which makes them the central subject of study for $K_{\ell 4}$ decays. In terms of the form factors $F^{ab}(s,t,u)$ and $G^{ab}(s,t,u)$, the chiral counting of the partial waves translates into the decompositions

\[
\begin{align*}
F^{ab}(s,t,u) &= F_0^{ab}(s,s_1) + F_1^{ab}(s,s_1) \cos \theta_{ab} + F_2^{ab}(s,\cos \theta_{ab},s_1), \\
G^{ab}(s,t,u) &= G_0^{ab}(s,s_1) + G_1^{ab}(s,\cos \theta_{ab},s_1).
\end{align*}
\]

The contributions of the partial waves with $l \geq 2$ are collected in $F_2^{ab}(s,\cos \theta_{ab},s_1)$ and in $G_2^{ab}(s,\cos \theta_{ab},s_1)$, with the counting $\text{Re} F_2^{ab}(s,\cos \theta_{ab},s_1), \text{Re} G_2^{ab}(s,\cos \theta_{ab},s_1) \sim \mathcal{O}(E^3)$ and $\text{Im} F_2^{ab}(s,\cos \theta_{ab},s_1), \text{Im} G_2^{ab}(s,\cos \theta_{ab},s_1) \sim \mathcal{O}(E^6)$, while the contributions from $S$ and $P$ waves are collected in

\[
\begin{align*}
F_2^{ab}(s,s_1) &= f_0^{ab}(s,s_1) - \frac{M_0^2 - M_1^2}{s} g_1^{ab}(s,s_1), \\
F_2^{ab}(s,s_1) &= f_1^{ab}(s,s_1) - \frac{M_1^2 - s - s_1}{s} \lambda_0 \frac{\lambda_1}{\lambda_0} g_1^{ab}(s,s_1), \\
G_2^{ab}(s,s_1) &= g_1^{ab}(s,s_1).
\end{align*}
\]

- **Analyticity and unitarity**
  We now assume that the form factors $F^{ab}(s,t,u)$ and $G^{ab}(s,t,u)$ have the usual analyticity properties with respect to the variable $s$, for fixed values of $t$ and of $u$, with a cut on the positive $s$-axis, whose discontinuity is fixed by unitarity, and a cut on the negative $s$-axis generated by unitarity in the crossed channel. The form factors are regular and real in the interval between $s = 0$ and the positive value of $s$ corresponding to the lowest-lying intermediate state. We can thus write the following dispersion relations

\[
A^{ab}(s,t) = \mathcal{P}^{ab}(t|s,u) + \frac{s^2}{\pi} \int \frac{dx}{x^2} \frac{1}{x - s - i0} \text{Im} A^{ab}(s,t) + \frac{u^2}{\pi} \int \frac{dx}{x^2} \frac{1}{x - u - i0} \lambda_0 \lambda_0 \mathcal{C}_{us} \text{Im} A^{ab}(x,t).
\]

Each integral runs slightly above or below the corresponding cut in the complex $s$-plane, from the relevant threshold, $s_{ab}$ or $u_{ab}$, to infinity. $\mathcal{P}^{ab}(t|s,u)$ denotes a pair of subtraction functions that are polynomials of the first degree in $s$ and $u$, with coefficients given by arbitrary functions of $t$. Using the decompositions Eqs. (34) and (48), we may write

\[
\text{Im} A^{ab}(s,t) = \left( \text{Im} F^{ab}_S(s) + \frac{s(s_i - s - 2t) - (M_0^2 - M_1^2)(M_1^2 - s)}{\lambda_0 \lambda_1} \text{Im} F^{ab}_P(s) \right) + \text{Im} A^{ab}(s,t)_{l \geq 2},
\]

where $F^{ab}_S(s)$ and $F^{ab}_P(s)$ are given in terms of the lowest partial waves by Eq. (49). Furthermore, $\text{Im} A^{ab}(s,t)_{l \geq 2}$ collects the contributions of the higher ($l \geq 2$) partial-wave projections in (44), so that at low energies, $\text{Im} A^{ab}(s,t)_{l \geq 2} = \mathcal{O}(E^6)$. The last property is relevant as long as $s$ and $u$ remain below a typical hadronic scale $\Lambda_H \sim 1 \text{ GeV}$, but one should remember that the integrals in Eq. (50) involving $\text{Im} A^{ab}(s,t)_{l \geq 2}$ run up to infinity. However, in the range of $x$ above $\Lambda_H$, $\text{Im} A^{ab}(x,t)_{l \geq 2} = \mathcal{O}(E^6)$, so that (see the similar discussion in Ref. [25])

\[
\frac{s^2}{\pi} \int \frac{dx}{x^2} \frac{1}{x - s - i0} \text{Im} A^{ab}(x,t)_{l \geq 2} = \left( \frac{s}{\Lambda_H} \right)^2 H^{ab} + \mathcal{O}(E^6),
\]
where $H^{ab}$ denotes a set of constants, whose precise definitions need not concern us here. We thus obtain the expression

$$A^{ab}(s,t,u) = P^{ab}(t|s,u) + \Phi^{ab}_c(s) - (t-u)\Phi^{ab}_c(s) + \lambda_a \lambda_c \Phi^{ab}_c[(s-u)\Phi^{ab}_c(t)] \pm O(E^6).$$

In this expression, the pair of functions $P^{ab}(t|s,u)$ differs from the one introduced initially in Eq. (50) in two respects. First, it contains a contribution that compensates the fourth term on the right-hand side of Eq. (53), which has been introduced to make the crossing properties manifest. Second, the terms of Eq. (52) generated by the higher partial waves have also been absorbed into these polynomials. Therefore, $P^{ab}(t|s,u)$ in Eq. (53) still represents a pair of arbitrary polynomials of at most second order in $s$ and $u$, whose coefficients are functions of $t$. As for the functions $\Phi^{ab}_c(s)$, they are defined by the fact that they have a cut singularity along the positive real axis, with discontinuities along this cut expressed in terms of the lowest partial waves as

$$\text{Im} \Phi^{ab}_c(s) = \left( \frac{\text{Im} f^{ab}_1(s)}{\text{Im} f^{ab}_s(s)} \right) \left( \frac{M_c^2 - M_b^2}{\lambda_{c}(s)} \left[ (s - M_c^2 - 3s_t)\text{Im} g^{ab}_1(s) + (M_c^2 - s_t)\frac{\lambda_{c}(s)}{\lambda_{ab}(s)}\text{Im} f^{ab}_1(s) \right] \right),$$

$$\text{Im} \Phi^{ab}_c(s) = \frac{s}{\lambda_{ab}(s)} \left( \frac{\text{Im} f^{ab}_1(s)}{\text{Im} f^{ab}_s(s)} \right) \left( \frac{s-M_c^2-s_t}{s} \frac{\lambda_{c}(s)}{\lambda_{c}(s)}\text{Im} g^{ab}_1(s) \right),$$

supplemented by $\Phi^{ab}_c(0) = 0$ and by the asymptotic conditions

$$\lim_{|s| \to \infty} s^{-3+\frac{1}{2}(1+1)} \Phi^{ab}_c(s) = 0.$$ \hfill (55)

These conditions define $\Phi^{ab}_c(s)$ ($\Phi^{ab}_c(s)$) only up to a polynomial ambiguity, which is of second (first) order in $s$. The contributions of these polynomials to $A^{ab}(s,t,u)$ can then be absorbed by the arbitrary subtraction functions $P^{ab}(t|s,u)$ already at hand. Let us stress once more that the functions $\Phi^{ab}_c(s)$ only possess right-hand cuts, with discontinuities specified in terms of those of the partial waves, whereas the partial-wave projections themselves in general have a more complicated analytical structure. Enforcing the crossing relations, one finds that the arbitrary subtraction functions $P^{ab}(t|s,u)$ boil down to a pair of polynomials $P^{ab}(s,t,u)$ of at most second order in all three variables $s$, $t$, and $u$, with arbitrary constant coefficients. These coefficients may depend on the masses and on $s_t$, in a way that is compatible with the chiral counting. The polynomials in the different channels are then related by

$$P^{ac}(s,t,u) = \lambda_a \lambda_c \Phi^{ab}_c(t|s,u), \quad P^{bc}(s,t,u) = \lambda_b \lambda_c \Phi^{ba}_c(u|t,s), \quad P^{bu}(s,t,u) = \Phi^{ab}_c(s,u,t).$$ \hfill (56)

Finally, unitarity provides us with the discontinuities of the functions $\Phi^{ab}_c(s)$ and $\Phi^{ab}_c(s)$:

$$\text{Im} f^{ab}_l(s,s_t) = \sum_{\langle d' b' \rangle} \frac{1}{s} \lambda_{d' b'}(s) \text{Re} \left\{ \frac{i^{l}_{d' b'}(s)}{s} \left[ f^{d' b'}_l(s,s_t) \right] \right\} \theta(s - s_{d' b'}) + O(E^8),$$

$$\text{Im} g^{ab}_l(s,s_t) = \sum_{\langle d' b' \rangle} \frac{1}{s} \lambda_{d' b'}(s) \lambda_{d' b'}(s) \lambda_{a b} \lambda_{c b} \text{Re} \left\{ \frac{i^{l}_{d' b'}(s)}{s} \left[ g^{d' b'}_l(s,s_t) \right] \right\} \theta(s - s_{d' b'}) + O(E^8),$$ \hfill (57)

where $l = 0, 1$, and $i^{l}_{d' b'}(s)$ denotes the $l$-th partial wave of the $d' b' \to ab$ scattering amplitude. $s_{d' b'}$ stands for the lowest invariant mass squared of the corresponding intermediate state, $s_{d' b'} = (M_{d'} + M_{b'})^2$ in terms of the masses $M_{d'}, M_{b'}$ of the particles in the intermediate state. The symmetry factor reads $\mathcal{S}_{d' b'} = 1$ in all cases of interest, except for $\{d', b'\} = \{\pi^0, \pi^0\}$ or $\{\eta, \eta\}$, where $\mathcal{S}_{d' b'} = 2$.

We have now all the elements in our hands to go through the procedure depicted in Fig. 1 and obtain first the one-loop expressions of the form factors, and, from there, through a second iteration, the two-loop expressions. As before, the phases of the various form factors can be obtained analytically, and the IB contributions can be expressed in terms of the scattering lengths $a_0^b$ and $a_0^b$. We will not provide further details here, though. They can be found in Ref. [20], to which we refer the interested reader.
4. EXTRACTING THE SCATTERING LENGTHS FROM DATA

In this Section, we describe how the previous results allow one to analyse the available phase shifts from $K^+\pi^-$ decays, as provided by the old Geneva-Saclay experiment [7], the BNL-E865 experiment [5], and finally the quite recent NA48/2 experiment [3, 4] at the CERN SPS. Actually, the high accuracy of the latter analysis dominates completely the discussion, and we will only consider the data coming from NA48/2 in the following. We may restrict the discussion to the two form factors $F(s, t, u)$ and $G(s, t, u)$ [In order to simplify the notation, we suppress the $+\pi$ superscript, since no confusion can arise] that occur in the description of the matrix element for the transition $K^\pm \to \pi^+\pi^- e^+ e^-$. The generic low-energy structure of the form factors can be written as in Eq. (48),

$$F(s, t, u) = F_S(s, s_t) e^{i\delta_S(s, s_t)} + F_P(s, s_t) e^{i\delta_P(s, s_t)} \cos \theta + \text{Re} F_P(s, \cos \theta, s_t) + O(E^6),$$
$$G(s, t, u) = G_P(s, s_t) e^{i\delta_P(s, s_t)} + \text{Re} G_P(s, \cos \theta, s_t) + O(E^6),$$

where we have introduced the real functions $F_S(s, s_t)$ ($\equiv \tilde{f}_0(s, s_t)$ for $M_\mu = M_\pi = M_{\pi}^0$), $F_P(s, s_t)$, and $G_P(s, s_t)$ ($\equiv \tilde{g}_1(s, s_t)$), which correspond to the quantities appearing in Eq. (48), but with their phases removed, $F_S(s, s_t) = e^{-i\delta_S(s, s_t)} F_S(s + i0, s_t)$, etc. Notice that we have assumed these phases to depend on $s_t$, and that we have assigned the same phase to $F_P(s, s_t)$ and $G_P(s, s_t)$. The quantity $[\delta_S(s) - \delta_P(s)]_{\text{exp}}$ appearing in Eqs. (1), (2), and (4) corresponds to the difference $\delta_S(s) - \delta_P(s)$. In terms of the chiral expansions

$$\text{Re} F_S(s, s_t) = F_S^{(0)} + F_S^{(2)}(s, s_t) + O(E^4), \quad \text{Re} F_P(s, s_t) = F_P^{(0)} + F_P^{(2)}(s, s_t) + O(E^4),$$

where $F_S^{(0)}, G_S^{(0)} \sim O(E^0)$ and $F_S^{(2)}(s, s_t), G_S^{(2)}(s, s_t) \sim O(E^2)$, and using the unitarity condition Eq. (57) for the imaginary parts, we obtain the expressions

$$\delta_S(s, s_t) = \sum_{\{a', b'\}} \frac{1}{\mathcal{F}_{a'b'}(s)} \left[ \varphi_{0}^{a'b';-} (s) \frac{F_S^{(0)} + F_S^{(2)}(s, s_t)}{F_S^{(0)} + F_S^{(2)}(s, s_t)} + \varphi_{1}^{a'b';-} (s) \frac{F_S^{(0)} + F_S^{(2)}(s, s_t)}{F_S^{(0)} + F_S^{(2)}(s, s_t)} \right] \theta(s - s_{a'b'}) + O(E^6),$$

and

$$\delta_P(s, s_t) = \sum_{\{a', b'\}} \frac{\lambda_{a'b'}(s)}{\lambda_{a'b}(s)} \left[ \varphi_{1}^{a'b';-} (s) \frac{G_P^{(0)} + G_P^{(2)}(s, s_t)}{G_P^{(0)} + G_P^{(2)}(s, s_t)} + \varphi_{1}^{a'b';-} (s) \frac{G_P^{(0)} + G_P^{(2)}(s, s_t)}{G_P^{(0)} + G_P^{(2)}(s, s_t)} \right] \theta(s - s_{a'b'}) + O(E^6).$$

We see that the phases $\delta_S(s, s_t)$ and $\delta_P(s, s_t)$ depend on $s_t$ through the order $O(E^2)$ corrections to the form factors, as soon as a second intermediate state $a'b' \neq + +$ is involved. In the case of the $P$-wave phase shift, there can be no contribution from states with two identical particles due to Bose symmetry, explaining the absence of the factor $1/\mathcal{F}_{a'b'}$ in $\delta_P(s, s_t)$. Hence, for $\delta_P$ in the specific case $ab = ++$ and for $s \leq M_K^2$, the sum boils down to the single $\pi^+\pi^-$ intermediate state, the contribution from form factors drops out altogether, and there is no $s_t$ dependence left. In other words, while Watson’s theorem does not apply to the case of the $\delta_S(s, s_t)$ phase shift due to the occurrence of two different possible intermediate states $[\pi^0\pi^0 \text{ and } \pi^+\pi^- \text{ for } s \leq M_K^2]$, it is still operative in the $l = 1$ channel. This explains both why the phases of $F_P(s, s_t)$ and of $G_P(s, s_t)$ are identical, and why this common phase $\delta_P(s)$ actually does not depend on $s_t$. In the isospin limit, the dependence on $s_t$ also drops out from $\delta_S(s, s_t)$, and Watson’s theorem is recovered, i.e. the phases tend towards

$$\delta_S(s, s_t) \to \mathcal{D}_0(s), \quad \delta_P(s) \to \mathcal{D}_1(s)$$

where $\mathcal{D}_0(s)$ and $\mathcal{D}_1(s)$ denote the $\pi\pi$ phases in the $l = 0, l = 0$ and $l = 1, l = 1$ channels, respectively. It appears that the available statistics has not allowed the NA48/2 experiment to identify a dependence of the phases on $s_t$ [3, 4].

Our formalism allows us to check that, from the theoretical side, the dependence on $s_t$ is indeed sufficiently small, as compared to other sources of error. The quantity $\varphi_{0}^{a'b'}(s, a_0^2)$ occurring in Eqs. (1), (2), and (4) is given by the difference $\delta_0^{\text{Roy}}(s) - \delta_1^{\text{Roy}}(s)$ of the solutions of the Roy equations. The correction factor obtained from Eqs. (60) and (61),

$$\delta_{1B}(s, a_0^2, a_0^2) = [\delta_S(s, s_t = 0) - \delta_0(s)] - [\delta_P(s) - \delta_1(s)],$$

(63)
is shown on Fig. 2 for several illustrative values of $a_0^0$ and $a_0^2$ allowed by the analysis of Roy equations. For the details, especially as far as the numerical input used for the various parameters is concerned, we refer the reader to the extensive discussion in Ref. [20]. Let us just notice that, despite the uncertainties attached to the other parameters that are involved, the correction for larger values of $s$ can depend significantly on the values of the scattering lengths, and can, in particular, be different from the one-loop estimate performed in Ref. [14], even if order of magnitude and sign are the same.

The results of our analyses of the NA48/2 data are shown in Fig. 3, and summarised in Tab. 1. Note that the NA48/2 data alone lead to a strong correlation between the values of $a_0^0$ and $a_0^2$. In order to circumvent this problem, we have considered two possible fitting procedures. The first fit, called extended fit, supplements the NA48/2 data with low-energy data on the $\pi\pi$ scattering phases in the isospin 2 channel, as described in [15]. The second fit, called the scalar fit, adds a theoretical constraint on the scalar radius of the pion [31]. We have performed the analysis both in presence and in absence of the isospin-breaking correction terms, and we obtain

$$a_0^0 = 0.222 \pm 0.013, \quad a_0^2 = -0.043 \pm 0.009,$$

(64)

Our result is in good agreement with the one in Eq. (3) obtained by the NA48/2 collaboration for the fit corresponding to the so-called Model B in Ref. [4], but with slightly larger errors once isospin-breaking corrections are included. This is not surprising since our isospin-breaking correction varies with $a_0^0$ and $a_0^2$. In addition, we notice that the outcome of our fit provides values of $\lambda_1$ and $\lambda_2$ which are compatible with our inputs, $\lambda_4 = (-4.18 \pm 0.63) \cdot 10^{-3}$, $\lambda_2 = (8.96 \pm 0.12) \cdot 10^{-3}$ – in agreement with the fact that the determination of these two subthreshold parameters has remained very stable over time [29, 32, 15]. We see that in absence of isospin breaking, larger values of $a_0^0$ are preferred.

Once the scattering lengths in the isospin limit have been determined, we can test $N_f = 2$ $\chi$PT by comparing the dispersive and chiral descriptions of the low-energy $\pi\pi$ amplitude in the isospin limit. First, the solutions of the Roy equations are used to reconstruct the $\pi\pi$ amplitude in the unphysical (subthreshold) region where $\chi$PT should converge particularly well. As explained in Refs. [25, 26] and recalled in Ref. [15], in the isospin limit, one can describe the $\pi\pi$ amplitude in terms of only six parameters ($\alpha, \beta, \lambda_1, \lambda_2, \lambda_3, \lambda_4$) up to and including terms of order $(E/\Lambda_H)^6$ in the low-energy expansion. These subthreshold parameters yield the $N_f = 2$ chiral low-energy constants $\ell_3, \ell_4$, or equivalently the two-flavour quark condensate and pion decay constant measured in physical units

$$X(2) = \frac{2m\Sigma(2)}{F_\pi^2 M_\pi^2}, \quad Z(2) = \frac{F^2(2)}{F_\pi^2}, \quad \Sigma(2) = -\lim_{m_u, m_d \to 0} \langle 0 | \bar{u}u | 0 \rangle, \quad F(2) = \lim_{m_u, m_d \to 0} F_\pi,$$

(65)
by matching the chiral expansions to the subthreshold parameters in the estimate of the $\delta_{NN}(s)$ scattering amplitude far from singularities. The corresponding values of $\chi^2/N$ between the three fits, once isospin-breaking corrections are included, are gathered in Tab. 1. For comparison, we also show the results obtained without including the isospin corrections. One should emphasize that the minor difference in the estimate of the $N_f=2$ chiral low-energy constants and of the subthreshold parameters and of the chiral low-energy constants are gathered in Tab. 1. For comparison, we also show the results obtained without including the isospin corrections. One should emphasize that the minor differences in $\chi^2/N$ between the three fits once isospin-breaking corrections are included, are sufficient to yield significant differences in $\delta_{NN}(s)$. These expansions are expected to exhibit a good convergence since they involve the subthreshold parameters and of the chiral low-energy constants are gathered in Tab. 1. For comparison, we also show the results obtained without including the isospin corrections. One should emphasize that the minor differences in $\chi^2/N$ between the three fits once isospin-breaking corrections are included, are sufficient to yield significant differences in $\delta_{NN}(s)$.

### TABLE 1. Scattering lengths, subthreshold parameters and chiral low-energy constants for the different fits considered, with and without the isospin-breaking correction $\delta_{NN}(s;\alpha,\beta)$.  

| | With isospin-breaking corrections | Without isospin-breaking corrections |
|---|---|---|
| | $S$-$P$ | Extended | Scalar | $S$-$P$ | Extended | Scalar |
| $\alpha$ | $0.221 \pm 0.018$ | $0.232 \pm 0.009$ | $0.226 \pm 0.007$ | $0.247 \pm 0.014$ | $0.247 \pm 0.008$ | $0.242 \pm 0.006$ |
| $\beta$ | $0.0453 \pm 0.0106$ | $-0.0383 \pm 0.0040$ | $-0.0431 \pm 0.0019$ | $-0.0357 \pm 0.0096$ | $-0.0349 \pm 0.0038$ | $-0.0396 \pm 0.0015$ |
| $\rho_{\alpha\beta}$ | 0.5964 | 0.881 | 0.914 | 0.945 | 0.842 | 0.855 |
| $\theta_i$ | (82.3 ± 3.4)° | (82.3 ± 3.4)° | 82.3° | (82.3 ± 3.4)° | (82.3 ± 3.4)° | 82.3° |
| $\theta_{s}$ | (108.9 ± 2)° | (108.9 ± 2)° | 108.9° | (108.9 ± 2)° | (108.9 ± 2)° | 108.9° |
| $\chi^2/N$ | 7.6/6 | 16.6/16 | 7.8/8 | 7.2/6 | 15.7/16 | 7.3/8 |
| $\ell_2$ | 0.009 | 0.0019 | 0.0106 | 0.007 | 0.006 | 0.006 |
| $\ell_4$ | 0.009 | 0.0019 | 0.0106 | 0.007 | 0.006 | 0.006 |
| $X(2)$ | 0.38 ± 0.05 | 0.80 ± 0.06 | 0.38 ± 0.02 | 0.72 ± 0.05 | 0.71 ± 0.05 | 0.75 ± 0.03 |
| $Z(2)$ | 0.38 ± 0.05 | 0.80 ± 0.06 | 0.38 ± 0.02 | 0.72 ± 0.05 | 0.71 ± 0.05 | 0.75 ± 0.03 |
FIGURE 3. Results of the fits to the NA48/2 data in the \((a_0^0, a_2^0)\) plane. The two black solid lines indicate the universal band where the two \(S\)-wave scattering lengths comply with dispersive constraints (Roy equations) and high-energy data on \(\pi\pi\) scattering. The orange band is the constraint coming from the scalar radius of the pion, cf. Ref. [31]. The small dark (purple) ellipse represents the prediction based on \(N_f = 2\) chiral perturbation theory described in Ref. [31]. The three other ellipses on the left represent, in order of increasing sizes, the \(1-\sigma\) ellipses corresponding to the scalar (orange ellipse), the extended (blue ellipse) and \(S-P\) (green ellipse) fits, respectively, when isospin-breaking corrections are included. The light-shaded ellipses on the right represent the same outputs, but obtained without including isospin-breaking corrections.

5. SUMMARY - CONCLUSION

The high-precision data for \(\delta_S(s) - \delta_P(s)\) obtained by the NA48/2 experiment require that isospin-breaking corrections be taken into account. Since the ultimate goal is to extract the values of \(a_0^0\) and \(a_2^0\), the \(\pi\pi\) scattering lengths in the isospin limit, the corrections should not be computed at fixed values of the scattering lengths, but should be parameterized in terms of them.

We have shown that general properties (analyticity, unitarity, crossing, chiral counting) provide the necessary tools to do this in a model independent way. The phases of the two-loop form factors can be computed analytically, and the isospin-breaking correction \(\delta_{IB}(s; a_0^0, a_2^0)\) can be obtained as a function of the scattering lengths in the isospin limit. We have thus extended the analysis of IB correction in Ref. [14] in two respects: by going to two loops in the low-energy expansion, and by keeping the scattering lengths as free parameters.

We have redone the fit to the NA48/2 data using our determination of \(\delta_{IB}(s; a_0^2, a_2^0)\). The results we obtain are compatible with those published by NA48/2 within errors.

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