ANOMALOUS PRIMES AND THE ELLIPTIC KORSELT CRITERION

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Abstract. We introduce the notion of a Bachet anomalous number and show that, conditional on a special case of the Tijdeman-Zagier conjecture, the Bachet anomalous numbers are exactly the prime powers of the form $3n^2 + 3n + 1$. We then examine Type I elliptic Korselt numbers, introduced in [19], and their connection to anomalous primes, generalizing some results in [19].

1. Introduction

The efficient generation of cryptographic parameters is a prerequisite for any cryptographic system. A specific example is the requirement of efficiently generating “random” prime numbers $p$ and $q$ for an RSA modulus $n = pq$. In this paper we study two notions that are on one hand of particular interest for cryptographic purposes, and on the other hand related to some fundamental mathematical problems.

Elliptic curve primality testing techniques are among the most efficient and most widely used methods in primality proving [4]. The idea of using elliptic curves in primality testing was introduced by Goldwasser and Kilian in 1986 [8] and in that same year their result was extended by Atkin [1]. In 1989, Gordon [9, 10] took a different approach to the problem of primality testing: instead of searching for a curve and then computing its order as in [8] he considered elliptic curves $E(\mathbb{Q})$ with complex multiplication for which the order is known to be $p + 1$ if $p$ is a prime and then used the case of $p$ not splitting in $\mathbb{K}$ to test compositeness instead of primality of a given number $n$. However, elliptic curves with complex multiplication define a necessary but not sufficient test for primality and so define a new class of pseudoprimes known as elliptic pseudoprimes.

An example of curves suitable for Gordon’s “compositeness” test is the set of curves of the form $y^2 = x^3 + D$ with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ and with $p \equiv 2 \pmod{3}$. In the case when $p \equiv 1 \pmod{3}$ the orders of these curves are $p + 1 \pm 2a$ or $p + 1 \mp a \pm 3b$ where $a \equiv 1 \pmod{3}$ and $b$ are positive integers and $p = a^2 + 3b^2$. A good discussion of these curves is given in the introduction of [15] where the notion of an anomalous prime for a given elliptic curve $E$ (not necessarily of the form $y^2 = x^3 + D$) is defined as a prime $p$ such that $E$ has a good reduction at $p$ and $\#E(\mathbb{F}_p) = p$. In the rest of our paper we will call anomalous primes for curves of the form $y^2 = x^3 + D$, i.e. curves given by the celebrated Bachet equation, Bachet anomalous primes. In [15] Mazur asked whether there is an elliptic curve that possesses an infinite number of anomalous primes. This question is still open.

2010 Mathematics Subject Classification. 14H52, 14K22, 11G07, 11G20, 11B99.
Key words and phrases. elliptic curves, anomalous primes, anomalous elliptic curves, elliptic Korselt numbers.

Supported by National Science Foundation under the Grant number DMS-1062857.

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in general. There are several results related to the number of Bachet anomalous primes. Namely, in [15] it was conjectured that the number of Bachet anomalous primes less than $N$ should be given asymptotically by $c\sqrt{N}/\log N$ ($c$ is a positive constant). In [14] it was shown that the elliptic curve $E$ has infinitely many Bachet anomalous primes if and only if a corresponding collection of quadratic polynomials represents infinitely many primes.

In this paper we extend the notion of Bachet anomalous prime to *Bachet anomalous number* as a number $q$ of the form $p^r$ ($p$ is prime and $r$ is a positive integer) for which there exists an elliptic curve $E : y^2 = x^3 + D$ with $#E(\mathbb{F}_{p^q}) = p^r$. We show that the Tijdeman-Zagier conjecture implies that for $p > 3$ and $r > 2$ there are no Bachet anomalous numbers. We also show that for $p > 3$, if $q$ is a Bachet anomalous number then it must be of the form $3n^2 + 3n + 1$ for some $n > 0$. For a prime $p$ and numbers $q = p^2$ of the form $3n^2 + 3n + 1$ we establish conditions under which $q$ is a Bachet anomalous number.

In 2012, Silverman [19] extended Gordon’s definition of elliptic pseudoprimes and elliptic Carmichael numbers by considering arbitrary elliptic curves $E(\mathbb{Q})$. Namely, for a given elliptic curve $E(\mathbb{Q})$ and point $P \in E(\mathbb{Z}/\mathbb{Z}_n)$ he defines $n$ to be an *elliptic pseudoprime* if $n$ has at least two distinct prime factors, $E$ has good reduction at every prime $p$ dividing $n$ and $(n + 1 - a_n)P = 0 \mod n$ where $a_n$ denotes the coefficients of the $L$-series of $E(\mathbb{Q})$. The *elliptic Carmichael numbers* for a given elliptic curve $E(\mathbb{Q})$ are defined as composite numbers which are elliptic pseudoprimes for all points of infinite order on $E$. Analogous to the classical case of the Korselt criterion of Carmichael number, in [19] the author gives two Korselt-type criteria for elliptic Carmichael numbers. He introduces the notion of an elliptic Korselt number of Type I and an elliptic Korselt number of Type II. The elliptic Korselt numbers of Type I give a practical one-way criterion for determining if a given integer $n$ is an elliptic Carmichael number and they are the focus of our paper. There are several reasons to study these numbers: they are interesting in their own right as a special set of numbers similar to the elliptic Korselt numbers introduced in [7], but different enough to require further analysis.

We show that for a given elliptic curve $E$ every number $n$ which is a product of distinct anomalous primes is a Type I elliptic Korselt number for $E$, but the converse need not to be true. However, under certain conjecture we show that every Type I elliptic Korselt number of the form $n = pq$ is a product of anomalous primes. Furthermore, we generalize a result from [19] for Type I elliptic Korselt numbers of the form $n = p_1p_2 \cdots p_m$.

Using a result from [14] we show that assuming the Hardy-Littlewood conjecture, there are infinitely many Type I elliptic Korselt numbers for the curve $E : y^2 = x^3 + D$, where $D \in \mathbb{Z}$ is neither a square nor a cube in $\mathbb{Q}(\sqrt{-3})$ and $D \neq 80d^6$ for any $d \in \mathbb{Z}[(1 + \sqrt{-3})/2]$.

**2. Preliminaries**

Let $E$ be an elliptic curve defined over the rational number field $\mathbb{Q}$ and for every prime power $q$, let $\mathbb{F}_q$ be the finite field of $q$ elements. We generally write $q = p^r$, where $p$ is prime and $r$ is a positive integer, so that $p$ is the characteristic of $\mathbb{F}_q$. For the rest of this paper we work with $q \neq 2^r, 3^2$, i.e. we only look at finite fields with characteristics not 2 or 3. A classical theorem of Hasse from 1933 states that for an elliptic curve $E(\mathbb{F}_q)$, the order of the group $E(\mathbb{F}_q)$ is an integer in the Hasse interval

$$\mathcal{H}_q = \left[ q + 1 - 2\sqrt{q}, q + 1 + 2\sqrt{q} \right]$$
around \( q + 1 \). For a prime \( p \) where \( E \) has good reduction at \( p \) we use \( t \) to denote the trace of Frobenius, or trace, of \( E \), that is, \( t = 1 + q - \#E(\mathbb{F}_q) \).

This section particularly deals with elliptic curves of the form \( E(\mathbb{F}_q) : y^2 = x^3 + D \) where \( q \equiv 1 \mod 3 \), because if \( q \equiv 2 \mod 3 \) then \( E \) has no anomalous primes, as shown in the following proposition.

**Proposition 2.1.** If \( q \equiv 2 \mod 3 \), then for any \( E(\mathbb{F}_q) : y^2 = x^3 + D \), the trace of Frobenius is \( t = 0 \).

**Proof.** For \( x \in \mathbb{F}_q \) we have have \( x^{2n-1} = x \) if \( q = 3n + 2 \), then \( x = x^{6n+3} = (x^{2n+1})^3 \), so \( x \) is a cube in \( \mathbb{F}_q \). Since there are \( q \) elements of \( \mathbb{F}_q \), each of which is a cube, each element of \( \mathbb{F}_q \) is a cube of exactly one element of \( \mathbb{F}_q \). Thus, for each \( y \in \mathbb{F}_q \), there is a unique solution to the equation \( x^3 = y^2 - D \). Thus, there are \( q \) solutions to the equation \( y^2 = x^3 + D \); including the point at infinity we have that \( \#E(\mathbb{F}_q) = q + 1 \) which means that the trace \( t = 0 \). \( \square \)

We present a technique used in [16] to describe the traces of \( E(\mathbb{F}_q) \). Although this is an already known result, we give correct proof of the claim presented in Appendix D of [16], using ideas similar to that in [16]. We first note that for any \( k \in \mathbb{F}_q^\times \), the curves \( E_1(\mathbb{F}_q) : y^2 = x^3 + D \) and \( E_2(\mathbb{F}_q) : y^2 = x^3 + k^6D \) have the same order, as the mapping \( \phi : (x_1, y_1) \mapsto (k^2x_1, k^3y_1) \) is a bijection from \( E_1 \) to \( E_2 \). It follows that there are six possible traces of the curves \( E(\mathbb{F}_q) : y^2 = x^3 + D \) corresponding to the six sextic residue classes of \( D \).

Let \( g \in \mathbb{F}_q \) be a quadratic and cubic non-residue. For \( 0 \leq i \leq 5 \), define \( E_i(\mathbb{F}_q) : y^2 = x^3 + g^i \), and let \( t_i \) be the trace of \( E_i \). We will show how to obtain the traces of these curves by computing only one trace among the six curves. We will make use of the following two well-known lemmas as well as Weil’s theorem (Theorem 2.4).

**Lemma 2.2.** For any \( g \in \mathbb{F}_q^\times \) that is not a quadratic residue, the sum of the traces of \( E_1(\mathbb{F}_q) : y^2 = x^3 + D \) and \( E_2(\mathbb{F}_q) : y^2 = x^3 + g^3D \) is equal to zero [16, Section 2.3].

**Lemma 2.3.** For any \( g \in \mathbb{F}_q^\times \) that is not a cubic residue, the sum of the traces of \( E_1(\mathbb{F}_q) : y^2 = x^3 + D \), \( E_2(\mathbb{F}_q) : y^2 = x^3 + g^2D \), and \( E_3(\mathbb{F}_q) : y^2 = x^3 + g^4D \) is equal to zero [16, Appendix B].

**Theorem 2.4.** [20, Theorem 4.12] Let \( \#E(\mathbb{F}_q) = q + 1 - t \) and let \( \alpha, \beta \in \mathbb{C} \) be complex numbers satisfying \( \alpha + \beta = t \) and \( \alpha \beta = q \). Then

\[
\#E(\mathbb{F}_q^\ast) = q^n + 1 - (\alpha^n + \beta^n).
\]

**Theorem 2.5.** Let \( g \in \mathbb{F}_q \) be a quadratic and cubic non-residue. For \( 0 \leq i \leq 5 \), define \( E_i(\mathbb{F}_q) : y^2 = x^3 + g^i \), and let \( t_i \) be the trace of \( E_i \). Then the following hold:

(a) The traces \( t_3 = -t_0 \), \( t_4 = -t_1 \), and \( t_5 = -t_2 \).

(b) The sums \( t_0 + t_2 + t_4 = 0 \) and \( t_1 + t_3 + t_5 = 0 \).

(c) The traces \( t_0, t_2, t_4 \) are the three roots of the polynomial \( x^3 - 3qx - t_0^3 + 3qt_0 \).

**Proof.** The statements in part (a) and part (b) follow directly from the two lemmas above. By Lemma 2.2, we have \( t_3 = -t_0 \), \( t_4 = -t_1 \), and \( t_5 = -t_2 \). By Lemma 2.3, we have \( t_0 + t_2 + t_4 = 0 \) and \( t_1 + t_3 + t_5 = 0 \).

\(^1\)We ignore the case of \( D = 0 \), for then the resulting curve is singular.
To prove part (c) of the theorem we will first show that $t_0, t_2, t_4$ are roots of the polynomial $x^3 - 3qx - t_0^3 + 3qt_0$. Note that $x \in \mathbb{F}_q$ is a cubic residue in $\mathbb{F}_{q^3}$. This is because $x^{q-1} = 1$ (since $x \in \mathbb{F}_q$), which means that over $\mathbb{F}_{q^3}$, 

$$x^{q^3-1} = (x^{q-1})^{q^2+1} = 1.$$ 

It follows that $g^2$ and $g^4$ are sextic residues in $\mathbb{F}_{q^3}$, and so the curves $E_0, E_2,$ and $E_4$ have the same trace over $\mathbb{F}_{q^3}$. Thus, by Theorem 2.4 we have

$$t_0^3 - 3qt_0 = t_2^3 - 3qt_2 = t_4^3 - 3qt_4,$$

and so $t_0, t_2,$ and $t_4$ must be the roots of the polynomial $x^3 - 3qx - t_0^3 + 3qt_0$. We will now show that $t_0, t_2,$ and $t_4$ are exactly the roots of this polynomial.

We first show that $t_0$ is even. Consider the solutions to $E(\mathbb{F}_q) : y^2 = x^3 + 1$. These solutions come in pairs $(x, y)$ and $(x, -y)$, except for the solutions $(x, 0)$. Note that $x^3 + 1$ has an odd number of roots in $\mathbb{F}_q$, since $-1$ is a root and it is impossible for there to be exactly two roots (for then the two roots could be factored out of $x^3 + 1$, leaving a linear polynomial). Thus, the points on $E(\mathbb{F}_q)$ come in pairs, along with an odd number of points of the form $(x, 0)$ and the point at infinity. It follows that $\#E(\mathbb{F}_q)$ is even, and so $t_0 = q + 1 - \#E(\mathbb{F}_q)$ is even.

Next we show that $t_2$ is odd. Consider the solutions to $E(\mathbb{F}_q) : y^2 = x^3 + g^2$. These solutions come in pairs $(x, y)$ and $(x, -y)$; note that $x^3 + g^2 = 0$ has no solutions, since $-g^2$ is not a cubic residue. If it were, then by taking $k^3 = -g^2$ we get

$$g = \frac{-g^3}{-g^2} = \left(\frac{-g}{k}\right)^3,$$

which contradicts with the fact that $g$ is not a cubic residue. Thus, the points of $E(\mathbb{F}_q)$ come in pairs, along with the point at infinity. It follows that $\#E(\mathbb{F}_q)$ is odd, and so $t_2 = q + 1 - \#E(\mathbb{F}_q)$ is odd.

Since $t_0$ and $t_2$ have different parities, they are distinct roots of $x^3 - 3qx - t_0^3 + 3qt_0$. Thus, we can write

$$x^3 - 3qx - t_0^3 + 3qt_0 = (t - t_0)(t - t_2)(t - r)$$

for some $r \in \mathbb{Z}$.

We will show that $r = t_4$. Since the coefficient of $x^2$ in the cubic polynomial is 0, we have $t_0 + t_2 + r = 0$. But by Lemma 2.3 we also have $t_0 + t_2 + t_4 = 0$. Therefore, $r = t_4$, and

$$x^3 - 3qx - t_0^3 + 3qt_0 = (t - t_0)(t - t_2)(t - t_4).$$

This completes the proof of part (c) and the theorem. \(\square\)

Since $t_0, t_2$ and $t_4$ are distinct roots of the polynomial $x^3 - 3qx - t_0^3 + 3qt_0$, we can divide the polynomial by $x - t_0$ to obtain

$$(x - t_2)(x - t_4) = x^2 + t_0x + t_0^2 - 3q,$$

and then use a simple algebra to express $t_2, t_4$ in terms of $t_0$ as follows

$$t_2, t_4 = -\frac{t_0 \pm \sqrt{3(4q - t_0^2)}}{2}.$$

Conveniently, $E_0(\mathbb{F}_q) : y^2 = x^3 + 1$ is a curve that is well-defined over $\mathbb{F}_p$ and using Theorem 2.4 we can find the trace of $E_0(\mathbb{F}_q)$ from the trace of $E_0(\mathbb{F}_p)$. The possible orders (and
Theorem 2.6. Let \( p > 3 \) be prime and let \( D \neq 0 \mod p \). Then the order of \( E(\mathbb{F}_p) : y^2 = x^3 + D \) is given by

- If \( p \equiv 2 \mod 3 \), then \( \#E(\mathbb{F}_p) = p + 1 \); and
- If \( p \equiv 1 \mod 3 \), write \( p = a^2 + 3b^2 \) for integers \( a \) and \( b \), where \( b \) is positive and \( a \equiv 1 \mod 3 \). Then

\[
\#E(\mathbb{F}_p) = \begin{cases} 
  p + 1 + 2a & \text{if } D \text{ is a sextic residue in } \mathbb{F}_p \\
  p + 1 - 2a & \text{if } D \text{ is a cubic residue and quadratic non-residue in } \mathbb{F}_p \\
  p + 1 - a \pm 3b & \text{if } D \text{ is a quadratic residue and cubic non-residue in } \mathbb{F}_p \\
  p + 1 + a \pm 3b & \text{if } D \text{ is a quadratic and cubic non-residue in } \mathbb{F}_p.
\end{cases}
\]

The \( \pm \) signs in the last two cases are determined by the sextic residue class of \( D \) in \( \mathbb{F}_p \).

The fact that \( p \equiv 1 \mod 3 \) can be written uniquely as \( a^2 + 3b^2 \) as specified is due to Gauss and according to [5] can be found in his work *Disquisitiones Arithmeticae*.

3. Bachet Anomalous Numbers

Let \( y^2 = x^3 + D \) be an elliptic curve defined over the rational number field \( \mathbb{Q} \). In this section we extend the notion of anomalous prime defined by Mazur in [15] by introducing the notion of anomalous number. We show that the anomalous numbers satisfy certain properties in the case when the elliptic curve is of the form \( y^2 = x^3 + D \). The Diophantine equation \( y^2 = x^3 + D \) was first studied by Claude-Gaspar Bachet de Mézirac in the 1600s and is known as Bachet’s Equation. In his honor we call those anomalous numbers Bachet anomalous numbers.

Definition 3.1. The number \( q = p^r \) is called an anomalous number if there exists an elliptic curve \( E : y^2 = x^3 + Cx + D \) with good reduction at \( p \) such that \( \#E(\mathbb{F}_q) = q \). If the curve is of the form \( E : y^2 = x^3 + D \) we say that \( q \) is a Bachet anomalous number.

If \( r = 1 \), we will refer to such Bachet anomalous numbers as Bachet anomalous primes.

Theorem 3.2. If \( q > 3 \) is a Bachet anomalous number, then \( q = 3n^2 + 3n + 1 \) for some positive integer \( n \).

Proof. Let \( q > 3 \) be a Bachet anomalous number. This means that there is a curve \( E(\mathbb{F}_q) : y^2 = x^3 + D \) such that \( \#E(\mathbb{F}_q) = q \) i.e. trace \( t = 1 \). By Proposition 2.1 we have \( q \equiv 1 \pmod{3} \). Let \( t \) be the trace of \( y^2 = x^3 + 1 \) over \( \mathbb{F}_q \). From Proposition 2.5 we have that \( t \) is even, as is the trace of \( y^2 = x^3 + g^3 \) for a quadratic and cubic non-residue \( g \). Thus one of the other four possible traces of \( E(\mathbb{F}_q) \) is equal 1. From Theorem 1 and Lemma 2.2 we have that

\[
- t \pm 1 \sqrt{3(4q - t^2)} = \pm 2, \quad \text{where } \pm_1 \text{ and } \pm_2 \text{ represent independent } \pm \text{ signs. Using simple algebra we get}
\]

\[
3(4q - t^2) = t^2 \pm 4t + 4
\]

\[
4t^2 \pm 4t + 4 = 12q
\]
Theorem 3.6. If \( p = 3n^2 + 3n + 1 \) is prime, then \( p \) is a Bachet anomalous prime.

Proof. Let \( p = 3n^2 + 3n + 1 \) be a prime for a positive integer \( n \). Suppose first that \( n \) is even. Let \( b = \frac{n}{2} \) and \( a = -3b - 1 = -\frac{3n-2}{2} \). Then

\[
a^2 + 3b^2 = \frac{1}{4} ((3n + 2)^2 + 3n^2) = 3n^2 + 3n + 1 = p.
\]

Clearly \( b > 0 \) and \( a \equiv 2 \mod 3 \), so \( a \) and \( b \) satisfy the conditions of Theorem 2.6 and so there is an elliptic curve \( E(\mathbb{F}_p) \) with order \( p + 1 + a + 3b = p \).

If instead \( n \) is odd, let \( b = \frac{n+1}{2} \) and \( a = 3b - 1 = \frac{3n+1}{2} \). Then

\[
a^2 + 3b^2 = \frac{1}{4} ((3n + 1)^2 + 3(n + 1)^2) = 3n^2 + 3n + 1 = p.
\]

Again, clearly \( b > 0 \) and \( a \equiv 2 \mod 3 \), so \( a \) and \( b \) satisfy the conditions of Theorem 2.6 and so there is an elliptic curve over \( \mathbb{F}_p \) with order \( p + 1 + a - 3b = p \). Therefore, \( p \) is Bachet anomalous, as desired.

\[q = \frac{1}{3}(t^2 + t + 1)\]

Since \( q \) is an integer, it is clear that \( t \neq 0 \mod 3 \). If \( t \equiv 1 \mod 3 \), then \( q = \frac{1}{3}(t^2 + t + 1) \), since \( t^2 - t + 1 \not\equiv 0 \mod 3 \). Thus, we have

\[
q = \frac{1}{3}(t^2 + t + 1) = \frac{t^2 - 2t + 1}{3} + (t - 1) + 1 = 3 \left( \frac{t - 1}{3} \right)^2 + 3 \left( \frac{t - 1}{3} \right) + 1.
\]

If \( t \equiv 2 \mod 3 \), then we have \( q = \frac{1}{3}(t^2 - t + 1) \), since \( t^2 + t + 1 \not\equiv 0 \mod 3 \). Thus, we have

\[
q = \frac{1}{3}(t^2 - t + 1) = \frac{t^2 - 4t + 4}{3} + (t - 2) + 1 = 3 \left( \frac{t - 2}{3} \right)^2 + 3 \left( \frac{t - 2}{3} \right) + 1.
\]

In both cases, we showed that \( q \) can be written in the form \( 3n^2 + 3n + 1 \) for some \( n > 0 \). □

Proposition 3.3. Let \( q = p^r \), where \( p > 3 \) is a prime and \( r \equiv 0 \mod 3 \). Then \( q \) is not a Bachet anomalous number.

Proof. Suppose for contradiction that \( q \) is a Bachet anomalous number. Then \( q = 3n^2 + 3n + 1 = (n + 1)^3 - n^3 \) for some \( n > 0 \). Since \( 3 \mid r \) we must have

\[(p^3)^3 + n^3 = (n + 1)^3\]

However, the equality doesn’t hold by Fermat’s last theorem. □

The following conjecture, of which Fermat’s last theorem is a special case, suffices (though is not necessary) to show that no Bachet anomalous numbers exist for \( r > 2 \).

Conjecture 3.4 (Tijdeman-Zagier conjecture). Let \( A, B, C, x, y, \) and \( z \) be positive integers, with \( x, y, z > 2 \). If \( A^x + B^y = C^z \), then \( A, B, \) and \( C \) have a common prime factor. [15]

Proposition 3.5. Let \( p > 3 \) be a prime and \( r > 2 \) an integer. The Tijdeman-Zagier conjecture implies that \( q = p^r \) is not Bachet anomalous number.

Proof. Suppose for contradiction that \( q \) is Bachet anomalous number. Let \( q = 3n^2 + 3n + 1 = (n + 1)^3 - n^3 \). We have \( p^r + n^3 = (n + 1)^3 \), but the Tijdeman-Zagier conjecture implies that this is not possible, since \( \gcd(n, n + 1) = 1 \). □
In [15], Mazur conjectured that the number of Bachet anomalous primes less than \( N \) should be given asymptotically by \( c\sqrt{N}/\log N \) (\( c \) is a positive constant), and in particular there should be infinitely many Bachet anomalous. In [14] the author showed the following

**Theorem 3.7.** Let \( D \in \mathbb{Z} \) be an integer that is neither a square nor a cube in \( \mathbb{Q}(\sqrt{3}) \), and let \( E(\mathbb{Q}) \) be the elliptic curve defined by \( y^2 = x^3 + D \). Then Hardy-Littlewood conjecture implies the Mazur conjecture except for \( D = 80d^6 \), where \( 0 \neq d \in \mathbb{Z}[(1 + \sqrt{-3})/2] \) with \( d^6 \in \mathbb{Z} \).

This result combined with a result from Section 4 will be used to show that under certain conditions there are infinitely many Type I elliptic Korselt numbers, notion introduced in [19] and analyzed in this paper.

Next, we consider numbers of the form \( p^2 = 3n^2 + 3n + 1 \), where \( p \) is prime. For each such number, we show that there exists an elliptic curve \( E(\mathbb{F}_p) : y^2 = x^3 + D \) of order \( p^2 \), i.e. \( p^2 \) is a Bachet anomalous number. We begin by finding a recurrence relation for pairs of nonnegative integers \((p, n)\) satisfying \( p^2 = 3n^2 + 3n + 1 \).

**Proposition 3.8.** Let \((p_i, n_i)\) be the \( i^{th} \) pair of nonnegative integers \((p, n)\) satisfying \( p^2 = 3n^2 + 3n + 1 \). Then \((p_1, n_1) = (1, 0), (p_2, n_2) = (13, 7)\), and for \( k > 1 \)

\[
(p_{k+1}, n_{k+1}) = (14p_k - p_{k-1}, 14n_k - n_{k-1} + 6).
\]

**Proof.** Note that we can rewrite \( p^2 = 3n^2 + 3n + 1 \) as \((2p)^2 - 3(2n + 1)^2 = 1\).

Thus, each pair \((p_i, n_i)\) of non-negative integers that satisfy \( p^2 = 3n^2 + 3n + 1 \) is a solution to the Pell’s equation \( x^2 - 3y^2 = 1 \) where \( x \) is even and \( y \) is odd.

Let \((x_i, y_i)\) be the \( i^{th} \) non-negative integer solution to \( x^2 - 3y^2 = 1 \). We see that the pairs \((x_1, y_1) = (1, 0), (x_2, y_2) = (2, 1)\) are solutions to the equation. From [3] we obtain the recurrence relation

\[
(x_{k+1}, y_{k+1}) = (2x_k + 3y_k, x_k + 2y_k)
\]

for \( k > 1 \). It is clear from this recurrence relation that if \( x_i \) is even and \( y_i \) is odd, then \( x_{i+1} \) is odd and \( y_{i+1} \) is even, and that if \( x_i \) is odd and \( y_i \) is even, then \( x_{i+1} \) is even and \( y_{i+1} \) is odd. This means that \((x_i, y_i)\) is of the form (even, odd) if and only if \( i \equiv 0 \) (mod 2). Since \((2p_1, 2n_1 + 1) = (x_2, y_2)\), it follows that \((2p_i, 2n_i + 1) = (x_{2i}, y_{2i})\). It will be useful, in finding the recurrence relation for \((p_i, n_i)\), to express \((x_{k+1}, y_{k+1})\) in terms of \((x_k, y_k)\). From (2), for \( k > 1 \) we have

\[
(x_{k+1}, y_{k+1}) = (2x_k + 3y_k, x_k + 2y_k)
= (2(2x_{k-1} + 3y_{k-1}) + 3(x_{k-1} + 2y_{k-1}), 2x_{k-1} + 3y_{k-1} + 2(x_{k-1} + 2y_{k-1}))
= (7x_{k-1} + 12y_{k-1}, 4x_{k-1} + 7y_{k-1}).
\]

Thus, for \( k \geq 1 \) we have

\[
(p_{k+1}, n_{k+1}) = \left( \frac{1}{2}x_{2(k+1)} + 1, \frac{1}{2}(y_{2(k+1)} - 1) \right) = \left( \frac{7x_{2k} + 12y_{2k}}{2}, \frac{4x_{2k} + 7y_{2k} - 1}{2} \right)
= \left( \frac{7 \cdot 2p_k + 12(2n_k + 1)}{2}, \frac{4 \cdot 2p_k + 7(2n_k + 1) - 1}{2} \right)
= (7p_k + 12n_k + 6, 4p_k + 7n_k + 3).
\]
From this we have, for $k > 1$, that
\[ p_{k+1} = 7p_k + 12n_k + 6 = 7p_k + 6 + 12(4p_{k-1} + 7n_{k-1} + 3) = 7p_k + 48p_{k-1} + 42 + 7 \cdot 12n_{k-1} \]
\[ = 7p_k + 48p_{k-1} + 42 + 7(7p_{k-1} + 12n_{k-1} + 6 - 7p_{k-1} - 6) \]
\[ = 7p_k + 48p_{k-1} + 42 + 7(p_k - 7p_{k-1} - 6) = 14p_k - p_{k-1} \]
and that
\[ n_{k+1} = 4p_k + 7n_k + 3 = 7n_k + 3 + 4(7p_{k-1} + 12n_{k-1} + 6) = 7n_k + 48n_{k-1} + 27 + 7 \cdot 4p_{k-1} \]
\[ = 7n_k + 48n_{k-1} + 27 + 7(4p_{k-1} + 7n_{k-1} + 3 - 7n_{k-1} - 3) \]
\[ = 7n_k + 48n_{k-1} + 27 + 7(n_k - 7n_{k-1} - 3) = 14n_k - n_{k-1} + 6. \]
Therefore,
\[ (p_{k+1}, n_{k+1}) = (14p_k - p_{k-1}, 14n_k - n_{k-1} + 6), \]
with $(p_1, n_1) = (1, 0)$ and
\[ (p_2, n_2) = \left( \frac{x_4}{2}, \frac{2y_4 - 1}{2} \right) = \left( \frac{7x_2 + 12y_2}{2}, \frac{4x_2 + 7y_2 - 1}{2} \right) = (13, 7), \]
as desired. \qed

**Corollary 3.9.** If $p$ is a positive integer and $p^2$ is of the form $3n^2 + 3n + 1$, then $p \equiv 1 \pmod{3}$.

**Proof.** Every $p$ such that $p^2$ is of the form $3n^2 + 3n + 1$ is in the sequence $\{p_i\}$. We have $p_1 = 1$ and $p_2 = 13$, and so $p_1 \equiv p_2 \equiv 1 \pmod{3}$. Assume that all $p_i$ with $i \leq k$ satisfy $p_i \equiv 1 \pmod{3}$. Then $p_{k+1} = 14p_k - p_{k-1} \equiv 14 - 1 \equiv 1 \pmod{3}$. Thus, every $p_i \equiv 1 \pmod{3}$, and so we are done. \qed

**Lemma 3.10.** For $k \geq 1$, let $g_k = \frac{2p - 3n - 2}{12}$ and $h_k = \frac{2p + 3n + 1}{12}$. Then $g_1 = 0$, $g_2 = \frac{1}{4}$, and for all $k > 1$, $\sqrt{g_{k+1}} = 4\sqrt{g_k} - \sqrt{g_{k-1}}$. Furthermore, $h_k = g_{k+1}$ for all $k \geq 1$.

**Proof.** The fact that $g_1 = 0$ and $g_2 = \frac{1}{4}$ follows immediately from Proposition 3.8. Furthermore, Proposition 3.8 gives us
\[ g_{k+1} = \frac{2p_{k+1} - 3n_{k+1} - 2}{12} = \frac{2(14p_k - p_{k-1}) - 3(14n_k - n_{k-1} + 6) - 2}{12} \]
\[ = 14 \cdot \frac{2p_k - 3n_k - 2}{12} - \frac{2p_{k-1} - 3n_{k-1} - 2}{12} + \frac{1}{2} = 14g_k - g_{k-1} + \frac{1}{2}. \]
This gives us that $g_3 = 14 \cdot \frac{1}{4} + \frac{1}{2} = 4$. We also have $h_1 = \frac{1}{4}$, $h_2 = 4$, and for $k > 1$,
\[ h_{k+1} = \frac{2p_{k+1} + 3n_{k+1} + 1}{12} = \frac{2(14p_k - p_{k-1}) + 3(14n_k - n_{k-1} + 6) + 1}{12} \]
\[ = 14 \cdot \frac{2p_k + 3n_k + 1}{12} - \frac{2p_{k-1} + 3n_{k-1} + 1}{12} + \frac{1}{2} = 14h_k - h_{k-1} + \frac{1}{2}. \]
Thus, $h_k = g_{k+1}$ for all $k \geq 1$. Finally, we show that for all $k > 1$, $\sqrt{g_{k+1}} = 4\sqrt{g_k} - \sqrt{g_{k-1}}$. We proceed by induction on the statement that $\sqrt{g_{k+1}} = 4\sqrt{g_k} - \sqrt{g_{k-1}}$ and $(g_k - g_{k-1})^2 - 2\sqrt{g_k} \sqrt{g_{k-1}} = \frac{1}{4}$.

For the base case we have $\sqrt{g_1} = 0$ and $\sqrt{g_2} = \frac{1}{2}$, so
\[ (4\sqrt{g_2} - \sqrt{g_1})^2 = \left( 4 \cdot \frac{1}{2} - 0 \right)^2 = 4 = g_3. \]
and
\[(\sqrt{g_2} - \sqrt{g_1})^2 - 2\sqrt{g_2}\sqrt{g_1} = \frac{1}{2} - 2 \cdot \frac{1}{4} \cdot 0 = \frac{1}{4} \]

Assume \(\sqrt{g_k} = 4\sqrt{g_{k-1}} - \sqrt{g_{k-2}}\) and \((g_{k-1} - g_{k-2})^2 - 2\sqrt{g_{k-1}}\sqrt{g_{k-2}} = \frac{1}{4}\) for some integer \(k\). We show that \(\sqrt{g_{k+1}} = 4\sqrt{g_k} - \sqrt{g_{k-1}}\) and \((g_k - g_{k-1})^2 - 2\sqrt{g_k}\sqrt{g_{k-1}} = \frac{1}{4}\). We expand:
\[
(4\sqrt{g_k} - \sqrt{g_{k-1}})^2 = 16g_k + g_{k-1} - 8\sqrt{g_{k-1}}\sqrt{g_k}
\]
\[
= 14g_k - g_{k-1} + 2g_k + 2g_{k-1} - 8\sqrt{g_{k-1}}\sqrt{g_k}
\]
\[
= 14g_k - g_{k-1} + 2(g_k + g_{k-1} - 4\sqrt{g_{k-1}}\sqrt{g_k})
\]
\[
= 14g_k - g_{k-1} + 2(\sqrt{g_k} - \sqrt{g_{k-1}})^2 - 2\sqrt{g_{k-1}}\sqrt{g_k}
\]
\[
= g_{k+1} - \frac{1}{2} + 2(\sqrt{g_k} - \sqrt{g_{k-1}})^2 - 2\sqrt{g_{k-1}}\sqrt{g_k}.
\]

Therefore, it is enough to show that \((\sqrt{g_k} - \sqrt{g_{k-1}})^2 - 2\sqrt{g_{k-1}}\sqrt{g_k} = \frac{1}{4}\). By the inductive hypothesis, we have \(\sqrt{g_k} = 4\sqrt{g_{k-1}} - \sqrt{g_{k-2}}\), so substituting gives:
\[
(\sqrt{g_k} - \sqrt{g_{k-1}})^2 - 2\sqrt{g_{k-1}}\sqrt{g_k} = (4\sqrt{g_{k-1}} - \sqrt{g_{k-2}})^2 - 2\sqrt{g_{k-1}}(4\sqrt{g_{k-1}} - \sqrt{g_{k-2}})
\]
\[
= (3\sqrt{g_{k-1}} - \sqrt{g_{k-2}})^2 - 8g_{k-1} + 2\sqrt{g_{k-1}}\sqrt{g_{k-2}}
\]
\[
= g_{k-1} - 4\sqrt{g_{k-1}}\sqrt{g_{k-2}} + g_{k-2}
\]
\[
= g_{k-1} - 4\sqrt{g_{k-1}}\sqrt{g_{k-2}} + g_{k-2}.
\]

where the last step follows from the inductive hypothesis. \(\square\)

**Lemma 3.11.** For \(k \geq 1\), let \(c_k = 2p_k - 3n_k + 1\) and \(d_k = 2p_k + 3n_k + 2\). Then \(c_1 = 1\), \(c_2 = 4\), and for all \(k > 1\), \(\sqrt{c_{k+1}} = 4\sqrt{c_k} - \sqrt{c_{k-1}}\). Furthermore, \(d_k = c_{k+1}\) for all \(k \geq 1\).

**Proof.** The fact that \(c_1 = 1\) and \(c_2 = 4\) follows immediately from Proposition 3.8. Furthermore, from Proposition 3.8 we have
\[
c_{k+1} = 2p_{k+1} - 3n_{k+1} - 1 = 2(14p_k - p_{k-1}) - 3(14n_k - n_{k-1} + 6) - 1
\]
\[
= 14(2p_k - 3n_k - 1) - 2p_{k-1} - 3n_{k-1} - 1 - 6 = 14c_k - c_{k-1} - 6.
\]

This means that \(c_3 = 14 \cdot 4 - 1 - 6 = 49\), \(d_1 = 4\), \(h_2 = 49\), and for \(k > 1\), we have
\[
d_{k+1} = 2p_{k+1} + 3n_{k+1} + 2 = 2(14p_k - p_{k-1}) + 3(14n_k - n_{k-1} + 6) + 2
\]
\[
= 14(2p_k + 3n_k + 2) - 2p_{k-1} + 3n_{k-1} + 2) - 6 = 14d_k - d_{k-1} - 6.
\]

Thus, \(d_k = c_{k+1}\) for all \(k \geq 1\). Finally, we show that for all \(k \geq 3\), \(\sqrt{c_{k+1}} = 4\sqrt{c_k} - \sqrt{c_{k-1}}\). We proceed by induction on the statement that \(\sqrt{c_{k+1}} = 4\sqrt{c_k} - \sqrt{c_{k-1}}\) and \((c_k - c_{k-1})^2 - 2\sqrt{c_k}\sqrt{c_{k-1}} = 1\). We expand:
\[
(\sqrt{c_2} - \sqrt{c_1})^2 = (4 \cdot 2 - 1)^2 = 49 = c_3
\]
and
\[
(\sqrt{c_2} - \sqrt{c_1})^2 - 2\sqrt{c_2}\sqrt{c_1} = (2 - 1)^2 - 2 \cdot 2 \cdot 1 = -3.
\]
Thus, we have

\((4\sqrt{c_k} - \sqrt{c_{k-1}})^2 = 16c_k + c_{k-1} - 8\sqrt{c_{k-1}}\sqrt{c_k}\)

\[= 14c_k - c_{k-1} + 2c_k + 2c_{k-1} - 8\sqrt{c_{k-1}}\sqrt{c_k}\]

\[= 14c_k - c_{k-1} + 2(c_k + c_{k-1} - 4\sqrt{c_{k-1}}\sqrt{c_k})\]

\[= 14c_k - c_{k-1} + 2((\sqrt{c_k} - \sqrt{c_{k-1}})^2 - 2\sqrt{c_{k-1}}\sqrt{c_k})\]

\[= c_{k+1} - 6 + 2((\sqrt{c_k} - \sqrt{c_{k-1}})^2 - 2\sqrt{c_{k-1}}\sqrt{c_k}).\]

Therefore, it is enough to show that \(\sqrt{c_k} = 4\sqrt{c_{k-1}} - \sqrt{c_{k-2}}\). By the inductive hypothesis, we have \(\sqrt{c_k} = 4\sqrt{c_{k-1}} - \sqrt{c_{k-2}}\). If we substitute we get

\((\sqrt{c_k} - \sqrt{c_{k-1}})^2 = 2\sqrt{c_{k-1}}\sqrt{c_k} = (4\sqrt{c_{k-1}} - \sqrt{c_{k-2}})^2 - 2\sqrt{c_{k-1}}(4\sqrt{c_{k-1}} - \sqrt{c_{k-2}})

\[= (3\sqrt{c_{k-1}} - \sqrt{c_{k-2}})^2 - 8c_{k-1} + 2\sqrt{c_{k-1}}\sqrt{c_{k-2}}\]

\[= 9c_{k-1} - 6\sqrt{c_{k-1}}\sqrt{c_{k-2}} + c_{k-2} - 8c_{k-1} + 2\sqrt{c_{k-1}}\sqrt{c_{k-2}}\]

\[= c_{k-1} - 4\sqrt{c_{k-1}}\sqrt{c_{k-2}} + c_{k-2}\]

\[= (\sqrt{c_{k-1}} - \sqrt{c_{k-2}})^2 - 2\sqrt{c_{k-1}}\sqrt{c_{k-2}} = -3,\]

where the last step follows from the inductive hypothesis. 

Lemma 3.12. Let \(q = p^2 = 3n^2 + 3n + 1\), where \(p\) is prime and \(n > 0\) is an integer. Suppose that there are \(a, b \in \mathbb{Z}\) that satisfy the following conditions

1. One of the following two equations holds

\[b = \sqrt{\frac{2p - 3n - 2}{12}} \text{ or } \sqrt{\frac{2p + 3n + 1}{12}},\]

2. In addition, one of the following four equations holds

\[a = \pm_1 3b \pm_2 \sqrt{12b^2 + 1},\]

where \(\pm_1\) and \(\pm_2\) denote independent \(\pm\) signs.

Then there is an elliptic curve over \(E(\mathbb{F}_q) : y^2 = x^3 + D\) with order \(\#E(\mathbb{F}_q) = q\).

Proof. We note that the \(\pm\) sign, wherever it is used in this proof, means that the stated equations are true for one of the signs (or combinations thereof), not both. We also note that by Corollary 3.9 \(p \equiv 1 \pmod{3}\). Choose \(a, b \in \mathbb{Z}\) that satisfy the conditions of the theorem. It can be verified that for either value of \(b\), we have

\[(1008b^4 + 84b^2 - 3n^2 - 3n)^2 = 144b^2(24b^2 + 1)^2(12b^2 + 1).

Thus, we have

\[1008b^4 + 84b^2 - 3n^2 - 3n = \pm 12b(24b^2 + 1)\sqrt{12b^2 + 1}.
\]

\[1008b^4 + 84b^2 + 1 \pm 12b(24b^2 + 1)\sqrt{12b^2 + 1} = 3n^2 + 3n + 1
\]

\[= p^2.
\]

We also have

\[(a^2 + 3b^2)^2 = ((\pm_1 3b \pm_2 \sqrt{12b^2 + 1})^2 + 3b^2)^2\]
Proof. Let \( q \) is a Bachet anomalous number. To show that one of \( 2p^2 \) or \( 3b^2 \) is prime and \( p = a^2 + 3b^2 \). It is easy to see that \( a \) and \( b \) satisfy the conditions of Theorem 2.6.

Recall that \( a = \pm 3b \pm \sqrt{12b^2 + 1} \). Equivalently, we have

\[
a = \frac{\pm_1 6b \pm_2 \sqrt{(6b)^2 - 4(-3b^2 - 1)}}{2}.
\]

These are the solutions for \( a \) to the quadratic equation \( a^2 \pm 6ab - 3b^2 - 1 = 0 \), so we have

\[
a^2 \pm 6ab - 3b^2 - 1 = 0
\]

\[
- a^2 \pm 6ab + 3b^2 = -1
\]

\[
a^2 \pm 6ab + 9b^2 - 2(a^2 + 3b^2) = -1
\]

\[
(a \pm 3b)^2 - 2p = -1.
\]

Now, by Theorem 2.6 we know that both \( a + 3b \) and \( a - 3b \) are traces of elliptic curves of the form \( E(\mathbb{F}_p) : y^2 = x^3 + D \). By Theorem 2.4 we know that if \( t \) is the trace of \( E(\mathbb{F}_p) \), then \( t^2 - 2p \) is the trace of \( E(\mathbb{F}_p^2) \). For \( \alpha \) and \( \beta \) as in Theorem 2.4 we have \( \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = t^2 - 2p \). This means that there is a curve \( E(\mathbb{F}_p^2) : y^2 = x^3 + D \) with trace \( t = -1 \). Then by Lemma 2.2 we have that the curve \( y^2 = x^3 + g^3D \), where \( g \) is a quadratic and cubic non-residue over \( \mathbb{F}_{p^2} \), must have a trace \( t = 1 \). This curve has order \( p^2 = q \), and so we are done.

We are now ready to prove the main theorem of this section.

**Theorem 3.13.** Let \( q = p^2 = 3n^2 + 3n + 1 \), where \( p \) is prime and \( n > 0 \) is an integer. Then \( q \) is a Bachet anomalous number.

**Proof.** Let \( q = p^2 = 3n^2 + 3n + 1 \), where \( p \) is prime and \( n > 0 \). By Lemma 3.12 it suffices to show that one of \( 2p - 3n - 2 \) and \( 2p + 3n + 1 \), as well as \( 12b^2 + 1 \) (as defined in Lemma 3.12), are perfect squares, because in that case the integers \( a \) and \( b \) in Lemma 3.12 exist.

We begin by showing that \( \frac{2p - 3n - 2}{12} \) or \( \frac{2p + 3n + 1}{12} \) is a perfect square. Note that \( (p, n) = (p_k, n_k) \) for some \( k \). It thus suffices to prove that for every \( k \), \( g_k = \frac{2p_k - 3n_k - 2}{12} \) or \( h_k = \frac{2p_k + 3n_k + 1}{12} \) is a perfect square. By Lemma 3.10 we have the recurrence relation \( \sqrt{g_k} = 4\sqrt{g_{k-1}} - \sqrt{g_{k-2}} \), with \( \sqrt{g_1} = 0 \) and \( \sqrt{g_2} = \frac{1}{2} \). It follows immediately from the recurrence relation that \( \sqrt{g_{2k-1}} \) is an integer and \( \sqrt{g_{2k}} \) is an integer plus \( \frac{1}{2} \). Since \( h_k = g_{k+1} \), \( \sqrt{h_{2k}} \) is an integer and \( h_{2k+1} \) is an integer plus \( \frac{1}{2} \). Thus, for all positive integer \( k \), exactly one of \( h_k = \frac{2p_k + 3n_k + 1}{12} \) is a perfect square, implying that one of the two values of \( b \) we found will always be an integer.

We will now show that \( 12b^2 + 1 \) is a perfect square. Note that \( 12b^1 + 1 \) equals either \( 2p - 3n - 1 \) or \( 2p + 3n + 2 \), if \( b \) is \( \sqrt{g_k} \) or \( \sqrt{h_k} \) for some \( k \), respectively. Since \( (p, n) = (p_k, n_k) \) for some \( k \), it suffices to prove that for every \( k \), \( c_k = 2p_k - 3n_k - 1 \) and \( d_k = 2p_k + 3n_k + 2 \) are
perfect squares, since one of these must be $12b^2 + 1$. By Lemma 3.11 we have the recurrence relation $\sqrt{c_k} = 4\sqrt{c_{k-1}} - \sqrt{c_{k-2}}$, with $\sqrt{g_1} = 1$ and $\sqrt{g_2} = 4$. Thus, $\sqrt{c_k}$ is an integer for all $k$. Since $d_k = c_{k+1}$, $\sqrt{d_k}$ is also an integer for every $k$. We showed that $a$ and $b$ are integers that satisfy the conditions of Lemma 3.12. Thus $q = p^2$ is a Bachet anomalous number. 

4. Type I Elliptic Korselt Numbers

4.1. Generalizations. The classical notions of pseudoprimes and Carmichael numbers are related to the orders of numbers in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$. These concepts were generalized to other algebraic structures such as elliptic curves. The notion of elliptic pseudoprime and elliptic Carmichael number were introduced in [10] for curves with complex multiplication. In [19] these notions were extended for arbitrary elliptic curves $E(\mathbb{Q})$. The definition of elliptic pseudoprime for arbitrary elliptic curves is follows: Let $n \in \mathbb{Z}$ and $E(\mathbb{Q})$ be an elliptic curve given by a minimal Weierstrass equation, and let $P \in E(\mathbb{Z}/n\mathbb{Z})$. Write the $L$-series of $E(\mathbb{Q})$ as $L(E(\mathbb{Q}), s) = \sum a_n/n^s$. Then $n$ is an elliptic pseudoprime for $(E, P)$ if $n$ has at least two distinct prime factors, $E$ has a good reduction at every prime dividing $n$ and $(n+1-a_n)P \equiv 0 \pmod{n}$. The definition of elliptic Carmichael for arbitrary elliptic curves is follows: Let $n \in \mathbb{Z}$ and $E(\mathbb{Q})$ be an elliptic curve. Then $n$ is an elliptic Carmichael number for $E$ if $n$ is an elliptic pseudoprime for $(E, P)$ for every point $P \in E(\mathbb{Z}/n\mathbb{Z})$.

In [19], the author also introduce two Korselt-type criteria (called Type I and Type II elliptic Korselt numbers) for elliptic Carmichael numbers.

Definition 4.1. A positive integer $n$ is called a Type I elliptic Korselt number if it has at least two distinct prime factors, such that for every prime dividing $n$, the following hold:

1. $E$ has good reduction at $p$
2. $p + 1 - a_p | n + 1 - a_n$
3. $\text{ord}_p(a_n - 1) \geq \text{ord}_p(n) - \begin{cases} 1 & a_p \not\equiv 1 \pmod{p} \\ 0 & a_p \equiv 1 \pmod{p} \end{cases}$.

Here, $a_p$ is the Frobenius trace of $E(\mathbb{F}_p)$ as usual, and $a_n$ is the $n^\text{th}$ coefficient of the $L$-series of $E(\mathbb{Q})$; for how to compute this coefficient, see [22]. In particular, $a_n$ is a multiplicative function when $n$ is square-free, in the sense that if $n = \prod_i p_i$ for distinct $p_i$, then $a_n = \prod_i a_{p_i}$. Finally, $\text{ord}_p(n)$ denotes the highest power of $p$ that appears in the prime factorization of $n$, with $\text{ord}_p(0) = \infty$. In [19] it has been shown that any number satisfying this elliptic Korselt criterion is an elliptic Carmichael number, but the converse need not be true.

Proposition 4.2. If $n$ is a Type I elliptic Korselt number for an elliptic curve $E$, then $n$ is an elliptic Carmichael number for $E$ [19, Proposition 11].

The second notion of Korselt-type criteria described in [19], Type II elliptic Korselt number, is bi-directional (Type II elliptic Korselt if and only if elliptic Carmichael), but less practical. Type I elliptic Korselt numbers is relatively easy to check in practice if one knows how to factor $n$. The focus of this section are Type I elliptic Korselt numbers, their properties and relation to anomalous primes (not necessarily Bachet anomalous primes).

Proposition 4.3. Let $E$ be an elliptic curve and $p_1, p_2, \ldots, p_m$ be Bachet distinct anomalous primes for $E$. Then $n = \prod_{i=1}^m p_i$ is a Type I elliptic Korselt number for $E$. 


Proof. The first condition of Definition 4.1 is clearly satisfied, by virtue of the curve being anomalous over each \( \mathbb{F}_p \), i.e. \(#E(\mathbb{F}_p) = p_i\). The second condition is satisfied since \( a_n = \prod_{i=1}^m a_{p_i} = 1 \), and each \( p_i \) divides \( n \). The third condition is satisfied because for each \( i \), \( \text{ord}_{p_i}(a_n - 1) = \text{ord}_{p_i}(0) = \infty \). \qed

The converse is not true i.e. not all Korselt numbers for an elliptic curve \( E \) are products of distinct primes \( p_i \) such that \( E(\mathbb{F}_p) \) is anomalous. However, this direction does hold for product of two primes \( n = pq \) when certain conditions are placed on \( p \) and \( q \). The product \( n = pq \) being a Type I elliptic Korselt number was studied in [19]. However, we give a counterexample to the claim of Proposition 17 stating that if \( n = pq \) is Type I elliptic Korselt for \( E \) with \( 17 < p < \sqrt{q} \), then \( a_p = a_q = 1 \) i.e. \( #E(\mathbb{F}_p) = p \) and \( #E(\mathbb{F}_q) = q \).

Counterexample 4.4. Let \( E : y^2 = x^3 + 1 \), \( p = 53 \) and \( q = 2971 \). We have \( a_p = 0 \) and \( a_q = 56 \), and yet \( pq \) is Type I elliptic number Korselt for \( E \), despite \( 17 < p < \sqrt{q} \) being satisfied.

Proposition 4.3 and Propositions 11 and 12 from [19] give the following implications.

Bachet Anomalous \( \xrightarrow{\text{Prop. 13}} \) Korselt Type I \( \xrightarrow{\text{Prop. 11}} \) elliptic Carmichael \( \xleftarrow{\text{Prop. 12}} \) Korselt Type II

The above implications together with Theorem 3.7 from [14] imply the following result.

Corollary 4.5. Assuming the Hardy-Littlewood Conjecture, there are infinitely many Type I elliptic Korselt number for the curve \( E : y^2 = x^3 + D \), where \( D \in \mathbb{Z} \) is neither a square nor a cube in \( \mathbb{Q}(\sqrt{-3}) \) and \( D \neq 80d^6 \) for any \( d \in \mathbb{Z} \{(1 + \sqrt{-3})/2 \} \).

In the rest of this section we present two results. First, we correct [19] Proposition 17 by giving accurate conditions on when a Korselt number for \( E \) must be a product of two anomalous primes for \( E \) (Theorem 4.8), and generalize these conditions to a product of arbitrarily many distinct primes (Theorem 4.6). However, these conditions, just like the ones in [19] Proposition 17, hold rarely: if two primes \( p, q < N \) are chosen at random, the probability that they satisfy these conditions goes to zero as \( N \) becomes large. Our second result is probabilistic, and shows, conditional on a conjecture, that in most cases, if \( n = pq \) is Korselt for a random curve \( E \), then \( p, q \) are anomalous primes for \( E \) (Theorem 4.2).

Theorem 4.6. Let \( E \) be an elliptic curve and \( n = p_1 p_2 \cdots p_m \) be a Type I elliptic Korselt number for \( E \) such that \( 5 \leq p_1 < p_2 < \cdots < p_m \), for \( m \geq 2 \). Then one of the following conditions is satisfied:

(1) \( p_1 \cdots p_{m-1} \leq 4^m \)

(2) \( a_{p_m} = 1 \), and for \( 1 \leq i \leq m-1 \), \( a_{p_i} = -1 \) for an even number of values of \( i \) and the remaining traces are equal to 1

(3) \( p_1 \cdots p_{m-1} \geq \sqrt[n]{2^m} \).

Proof. Assume \( p_1 \cdots p_{m-1} = \frac{n}{p_m} > 4^m \). We show that one of the two remaining conditions of the theorem are satisfied. We have

\[
n + 1 - a_n = p_1 \cdots p_m + 1 - a_{p_1} \cdots a_{p_m}
= \frac{n}{p_m} (p_m + 1 - a_{p_m}) - \frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n.
\]
By the divisibility criterion of Type I elliptic Korselt numbers, we have $p_m + 1 - a_{p_m} \mid n + 1 - a_n$, so

$$p_m + 1 - a_{p_m} \mid -\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n.$$

We now consider two cases: $-\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n = 0$ and $-\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n \neq 0$.

**Case 1:** $-\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n = 0$.

In this case, we have

$$\frac{n}{p_m} (a_{p_m} - 1) = a_n - 1.$$  
(3)

Suppose for contradiction that $a_{p_m} \neq 1$. Then we have

$$\frac{n}{p_m} = \frac{a_n - 1}{a_{p_m} - 1} = \frac{(a_{p_1} \cdots a_{p_{m-1}}) a_{p_m} - 1}{a_{p_m} - 1}.$$

We claim that $\frac{n}{p_m} \leq 4^m$. For simplicity of notation, let $r$ denote $a_{p_1} \cdots a_{p_{m-1}}$. Since $a_{p_m} \neq 1$ is an integer, the possible values of $\frac{n}{p_m}$ in terms of $r$ are

$$\ldots, \frac{2r + 1}{3}, \frac{r + 1}{2}, 1, 2r - 1, \frac{3r - 1}{2}, \frac{4r - 1}{2}, \ldots,$$

where $a_{p_m} = \ldots, -2, -1, 0, 2, 3, 4, \ldots$, respectively. If $r < 0$ then the maximum of these values is 1, so the desired inequality is clear. Assume instead that $r$ is positive. Then $\frac{n}{p_m}$ is maximized when $a_{p_m} = 2$, in which case $p_m = 2r - 1$. Now by Hasse’s theorem, $r \leq 2^{m-1} \sqrt{\frac{n}{p_m}}$, and so

$$\frac{n}{p_m} \leq 2 \cdot 2^{m-1} \sqrt{\frac{n}{p_m}} - 1 < 2^m \sqrt{\frac{n}{p_m}},$$

so $\frac{n}{p_m} \leq 4^m$, as desired. However, by assumption, $\frac{n}{p_m} > 4^m$. Thus, we have a contradiction: if $a_{p_m} - p_1 \cdots p_{m-1} - a_{p_1} \cdots a_{p_{m-1}} = 0$, then $a_{p_m}$ must be 1. We can say more: if $a_{p_m} = 1$, then by (3), $a_{p_1} \cdots a_{p_{m-1}} = 1$. Thus, an even number of traces $a_i$ for $1 \leq i \leq m - 1$ must be equal to $-1$, while the rest of these traces must be equal to 1.

**Case 2:** $-\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n \neq 0$.

Since $p_m + 1 - a_{p_m} \mid -\frac{n}{p_m} + a_{p_m} \frac{n}{p_m} + 1 - a_n$, we have

$$|p_m + 1 - a_{p_m}| \leq \left| \frac{n}{p_m} a_{p_m} - \frac{n}{p_m} - a_{p_1} \cdots a_{p_m} + 1 \right|$$

$$p_m + 1 - 2\sqrt{p_m} \leq |p_m + 1 - a_{p_m}| \leq 2 \frac{n}{p_m} \sqrt{p_m} + \frac{n}{p_m} + 2^m \sqrt{p_m} \sqrt{\frac{n}{p_m}} - 1.$$

Subtracting the left-most quantity from the right-most gives:

$$(2\sqrt{p_m} + 1) \frac{n}{p_m} + 2^m \sqrt{p_m} \sqrt{\frac{n}{p_m}} - p_m - 2 + 2\sqrt{p_m} \geq 0.$$

Solving the quadratic equation for $\sqrt{\frac{n}{p_m}}$ yields:

$$\sqrt{\frac{n}{p_m}} \geq \frac{2^m \sqrt{p_m} + \sqrt{4^m p_m - 4(2\sqrt{p_m} + 1)(-p_m + 2 + 2\sqrt{p_m})}}{2(2\sqrt{p_m} + 1)}.$$  
(4)
Using the following claim, we will show the right-hand side of (4) is at least $\frac{1}{2^m}p_m^{1/4}$.

**Claim.**  
\[(5) \quad (8 - \frac{16}{4^m})p_m^{3/2} - 8p_m^{5/4} - \left(12 + \frac{16}{4^m}\right)p_m - 4p_m^{3/4} + \left(8 - \frac{4}{4^m}\right)p_m^{1/2} + 8 \geq 0.\]

We now prove the above claim. For $m = 2$ it can be verified with a computer algebra system that (5) is true when $p_2 \geq 19$. By assumption, $p_1 > 16$, so this is always the case. For $m = 3$ it can be verified with a computer algebra system that (5) is true when $p_3 \geq 13$. Note that $p_3 > 11$ because otherwise we have $p_1p_2 \leq 5 \cdot 7 = 35$, contradicting the initial assumption. Thus, the claim holds for $m = 3$.

Now, let $f(m, p_m)$ be the left-hand side of (5). Observe that if $p_m > 0$ is held constant and $m$ is increased, then $f(m, p_m)$ increases. This is because we may write

$$f(m, p_m) = g(p_m) - \frac{16}{4^m}p_m^{3/2} - \frac{16}{4^m}p_m - \frac{4}{4^m}p_m^{1/2};$$

where

$$g(p_m) = 8p_m^{3/2} - 8p_m^{5/4} - 12p_m - 4p_m^{3/4} + 8p_m^{1/2} + 8.$$

and as $m$ increases, $-\frac{16}{4^m}$ and $-\frac{4}{4^m}$ increase. Thus, since $f(3, p_m) \geq 0$ for $p_m \geq 13$, $f(m, p_m) \geq 0$ for $p_m \geq 13$ for all $m \geq 3$. Since 13 is the fourth prime greater than or equal to 5, $p_m \geq 13$ for all $m > 3$. This completes the proof of the claim.

Using the claim, we have

$$\left(8 - \frac{16}{4^m}\right)p_m^{3/2} - 8p_m^{5/4} - \left(12 + \frac{16}{4^m}\right)p_m - 4p_m^{3/4} + \left(8 - \frac{4}{4^m}\right)p_m^{1/2} + 8 \geq 0,$$

which implies

$$8p_m^{3/2} - 12p_m + 8p_m^{1/2} + 8 \geq \frac{16}{4^m}p_m^{3/2} + \frac{16}{4^m}p_m + \frac{4}{4^m}p_m^{1/2} + 8p_m^{5/4} + 4p_m^{3/4}$$

$$+ 4^mp_m + 8p_m\sqrt{p_m} - 12p_m + 8\sqrt{p_m} + 8 \geq 4^mp_m + \frac{4}{4^m}(2\sqrt{p_m} + 1)^2\sqrt{p_m} + 2\cdot 2(2\sqrt{p_m} + 1)p_m^{3/4}$$

$$+ 4^mp_m + 8p_m\sqrt{p_m} - 12p_m + 8\sqrt{p_m} + 8 \geq \left(\frac{2}{2^m} (2\sqrt{p_m} + 1)p_m^{1/4} + 2^m \sqrt{p_m}\right)^2.$$ 

The right-hand side above is positive and smaller than the left-hand side, so the left-hand side is also positive. We take the square root of both sides.

$$\sqrt{4^mp_m + 8p_m\sqrt{p_m} - 12p_m + 8\sqrt{p_m} + 8} \geq \frac{2}{2^m} (2\sqrt{p_m} + 1)p_m^{1/4} + 2^m \sqrt{p_m}$$

$$\sqrt{4^mp_m - 4(2\sqrt{p_m} + 1)(-p_m - 2 + 2\sqrt{p_m})} \geq \frac{2}{2^m} (2\sqrt{p_m} + 1)p_m^{1/4} + 2^m \sqrt{p_m}$$

$$\frac{-2^m \sqrt{p_m} + \sqrt{4^mp_m - 4(2\sqrt{p_m} + 1)(-p_m - 2 + 2\sqrt{p_m})}}{2(2\sqrt{p_m} + 1)} \geq \frac{1}{2^m}p_m^{1/4}.$$ 

Thus, $\frac{a}{p_m} = p_1 \cdots p_{m-1} \geq \frac{\sqrt{p_m}}{2^m}$, concluding the proof of Theorem 4.6.\(\square\)

Note that for $m \geq 4$ the inequality $p_1 \cdots p_{m-1} \leq 4^m$, i.e. the first condition of Theorem 4.6 is never satisfied.
Remark 4.7. Theorem 4.6 can be restated as follows. Let $E$ be an elliptic curve and $n = p_1p_2 \cdots p_m$ be a Type I elliptic Korselt number for $E$ such that $5 \leq p_1 < p_2 < \cdots < p_m$, for $m \geq 2$. If $4^m < p_1 \cdots p_{m-1} < \sqrt{4m}$, then $a_{p_m} = 1$ and for $1 \leq i \leq m - 1$, $a_{p_i} = -1$ for an even number of values of $i$ and $a_{p_i} = 1$ for the remaining values. This gives us a condition under which $E$ must have trace $\pm 1$ over a list of primes $p_1, p_2, \cdots p_m$.

The following corollary of Theorem 4.6 corrects [19] Proposition 17, to which $n = 53 \cdot 2971$ for $E : y^2 = x^3 + 1$ serves as a counterexample.

Corollary 4.8. Let $E$ be an elliptic curve and let $n = pq$ be a Type I elliptic Korselt number for $E$ such that $p < q$. Then one of the following conditions holds:

- $p \leq 13$
- $p$ and $q$ are anomalous for $E$.
- $p \geq \sqrt{\frac{q}{16}}$

4.2. A Probabilistic Result Relating Korselt Numbers and Anomalous Primes.

In Proposition 4.3 we showed that all products of distinct anomalous primes for an elliptic curve are Type I elliptic Korselt numbers. Theorem 4.6 gives sufficient conditions for when a square-free Type I elliptic Korselt number is a product of anomalous primes. In this section, we show that, assuming the following conjecture, in most cases a Type I elliptic Korselt number is a product of anomalous primes. For empirical evidence in favor of this conjecture, see the appendix.

Conjecture 4.9. For $N \geq 7$, let $5 \leq p, q \leq N$ be randomly chosen distinct primes, and let $n = pq$. Let $E(\mathbb{Z}/n\mathbb{Z})$ be a randomly chosen elliptic curve with good reduction over $\mathbb{F}_p$ and $\mathbb{F}_q$ such that $\#E(\mathbb{F}_p)$ and $\#E(\mathbb{F}_q)$ divide $n + 1 - a_n$. Then

$$
\lim_{N \to \infty} \Pr[\#E(\mathbb{Z}/n\mathbb{Z}) = n + 1 - a_n] = 1.
$$

Note that $\#E(\mathbb{Z}/n\mathbb{Z}) = (p + 1 - a_p)(q + 1 - a_q)$ and $n + 1 - a_n = pq + 1 - a_pa_q$. An intuitive reason for believing this conjecture is that $p + 1 - a_p$ and $q + 1 - a_q$ are close in value to $p$ and $q$, respectively, and $pq + 1 - a_pa_q$ is close in value to $pq$. Thus, the only way for $p + 1 - a_p$ and $q + 1 - a_q$ to divide $n + 1 - a_n$ but for their product not to be $n + 1 - a_n$ is if $p + 1 - a_p$ and $q + 1 - a_q$ share many factors; this should happen rarely.

From the conditions listed in Conjecture 4.9, it is clear that $n$ satisfies the first two conditions of the Type I elliptic Korselt criterion for $E$. In other words, $n$ is “nearly” Type I elliptic Korselt number. The following lemma states that when $p, q \geq 7$, the third Korselt condition is a redundancy given the first and second. We will need the following results to prove the Theorem 4.15 of this section.

Lemma 4.10. For $N \geq 7$, let $5 \leq p, q \leq N$ be randomly chosen distinct primes, and let $n = pq$. Let $E(\mathbb{Z}/n\mathbb{Z})$ be a randomly chosen elliptic curve for which $n$ is a Type I elliptic Korselt number. Then

$$
\lim_{N \to \infty} \Pr[p \text{ is anomalous for } E \text{ and } q \text{ is not}] = 0.
$$

Proof. Let $N, p, q,$ and $E$ be as in the lemma statement. Assume that $a_p = 1$ and $a_q \neq 1$. By the Korselt divisibility condition, we have that $p$ and $q + 1 - a_q$ divide $pq + 1 - a_q$. Since $p | pq + 1 - a_q$, we have $p | 1 - a_q$, and note that $1 - a_q \neq 0$. Thus,

$$
p \leq |1 - a_q| \leq |a_q| + 1 \leq 2\sqrt{q} + 1 \leq 2\sqrt{N} + 1.
$$
The probability that a randomly chosen prime below \( N \) is at most \( 2\sqrt{N} + 1 \) goes to zero as \( N \to \infty \). Since \( p \leq 2\sqrt{N} + 1 \) is a necessary condition for \( a_p = 1 \) and \( a_q \neq 1 \) for \( E \), it follows that the desired probability approaches zero.

\[ \text{Proposition 4.11.} \text{ If } n = pq \text{ for distinct primes } p, q \geq 7 \text{ and } E \text{ is an elliptic curve with good reduction over } \mathbb{F}_p \text{ and } \mathbb{F}_q, \text{ then } n \text{ is Type I elliptic Korselt for } E \text{ if and only if } p + 1 - a_p \text{ and } q + 1 - a_q \text{ divide } n + 1 - a_n. \]

\[ \text{Proof.} \text{ By the definition of Type I elliptic Korselt, the “only if” direction holds. Suppose that } E \text{ has good reduction and satisfies the condition that } p + 1 - a_p \text{ and } q + 1 - a_q \text{ divide } n + 1 - a_n. \text{ Since } n = pq, \text{ ord}_p(n) = 1, \text{ so } a_p \equiv 1 \pmod{p} \text{ implies the third condition of the elliptic Korselt criterion is satisfied for } p. \]

Alternatively, if \( a_p \equiv 1 \pmod{p} \), then by Hasse’s theorem, \( p \) being at least 7 implies that \( a_p = 1 \). Thus, \( p + 1 - a_p = p \) and \( n + 1 - a_n = pq + 1 - a_q \), and so \( p \mid 1 - a_q \). Thus,

\[ \text{ord}_p(a_n - 1) = \text{ord}_p(a_q - 1) \geq 1 = \text{ord}_p(n), \]

and so \( p \) satisfies the third condition of the elliptic Korselt criterion. By analogy, \( q \) satisfies the third condition as well, and so we are done. \qed

\[ \text{Lemma 4.12.} \text{ For } N \geq 7, \text{ let } 5 \leq p, q \leq N \text{ be randomly chosen distinct primes, and let } n = pq. \text{ Let } E(\mathbb{Z}/n\mathbb{Z}) \text{ be a randomly chosen elliptic curve for which } n \text{ is a Type I elliptic Korselt number. Then assuming Conjecture 4.9,} \]

\[ \lim_{N \to \infty} \Pr[p \text{ and } q \text{ are not anomalous for } E \text{ and } (p + 1 - a_p)(q + 1 - a_q) \neq n + 1 - a_n] = 0. \]

\[ \text{Proof.} \text{ Let } N, \ p, \ q, \text{ and } E \text{ be as in the lemma statement. By Proposition 4.11, this is equivalent to saying that } p, q \text{ and } E \text{ are selected in such a way that } E(\mathbb{Z}/n\mathbb{Z}) \text{ has good reduction over } \mathbb{F}_p \text{ and } \mathbb{F}_q, \text{ and } \#E(\mathbb{F}_p) \text{ and } \#E(\mathbb{F}_q) \text{ divide } n + 1 - a_n. \text{ By Conjecture 4.9, the probability that} \]

\[ \#E(\mathbb{Z}/n\mathbb{Z}) = \#E(\mathbb{F}_p)\#E(\mathbb{F}_q) = (p + 1 - a_p)(q + 1 - a_q) \neq n + 1 - a_n \]

approaches zero as \( N \to \infty \). Thus, the probability that this condition is satisfied \( p \) and \( q \) are not anomalous for \( E \) also approaches zero, as desired. \qed

\[ \text{Proposition 4.13.} \text{ Let } n \text{ be a positive integer and let } S \text{ be a finite multiset of factors of } n. \text{ For each } d \mid n, \text{ let } m_d(S) \text{ be the number of multiples of } d \text{ in } S. \text{ Then} \]

\[ \sum_{k \in S} k = \sum_{d \mid n} m_d(S)\phi(d). \]

\[ \text{Proof.} \text{ This is an induction on the number of elements of } S. \text{ The theorem is clear for } |S| = 0; \text{ suppose it holds for } |S| = r. \text{ Now let } S \text{ have } r + 1 \text{ elements and choose } k \in S. \text{ Let } S' \text{ be } S \text{ with one fewer copy of } k; \text{ the theorem holds for } S'. \text{ Adding } k \text{ to } S' \text{ increments the left-hand sum by } k \text{ and the right-hand sum by } \sum_{d \mid k} \phi(d), \text{ since } m_d(S) = m_d(S') + 1 \text{ for } d \mid k \text{ and } m_d(S) = m_d(S') \text{ for all other } d. \text{ But } \sum_{d \mid k} \phi(d) = k \text{ [12, Proposition 2.2.4]}, \text{ so the theorem holds for } S. \]

\[ \text{If } p = 5 \text{ or } q = 5, \text{ this does not follow from Proposition 4.11 but the probability of this happening goes to 0 as } N \text{ goes to } \infty, \text{ so we are free to ignore this case.} \]
Lemma 4.14. For \( N \geq 7 \), let \( 5 \leq p, q \leq N \) be randomly chosen distinct primes, and let \( n = pq \). Let \( E(\mathbb{Z}/n\mathbb{Z}) \) be a randomly chosen elliptic curve for which \( n \) is a Type I elliptic Korselt number. Then
\[
\lim_{N \to \infty} \Pr[p \text{ and } q \text{ are not anomalous for } E \text{ and } (p+1-a_p)(q+1-a_q) = n+1-a_n] = 0.
\]

Proof. Let \( N, p, q, \) and \( E \) be as in the lemma statement. We impose the additional restriction that \( q \geq 67 \); this does not affect our proof, since the probability of a randomly selected prime below \( N \) being less than 67 approaches 0 as \( N \) approaches \( \infty \). Assume that \( a_p \neq 1, a_q \neq 1 \), and \( (p+1-a_p)(q+1-a_q) = n+1-a_n \). We have
\[
(p+1-a_p)(q+1-a_q) = pq + 1 - a_pa_q
\]
\[
2a_pa_q + p + q = pa_q + qa_p + a_p + a_q
\]
\[
a_p = \frac{p+q-(p+1)a_q}{q+1-2a_q}.
\]
(Hasse’s theorem tells us that \( q+1 > 2a_q \), so the division is always a valid step.) Thus, \( q+1-2a_q \) divides \( p+q-(p+1)a_q \). Subtracting \( q+1-2a_q \) from the dividend, we have
\[
q+1-2a_q | p - pa_q + a_q - 1 = (p-1)(1-a_q).
\]
Letting \( x = a_q - 1 \), \( p' = p - 1 \), and \( q' = q - 1 \), we find that \( q' - 2x \) divides \( p'x \). It follows that
\[
\frac{q' - 2x}{\gcd(q' - 2x, x)} = \gcd(q', x) \mid p'.
\]
We claim that the probability for randomly chosen \( 5 \leq p, q \leq N \) that there exists \( x \in [-2\sqrt{q-1}, 2\sqrt{q-1}] \) such that the above property is satisfied approaches zero as \( N \to \infty \); this is sufficient to prove our lemma.

To prove this claim, fix \( q \) (and thus \( q' \)) and examine how many values of \( p' < N \) (and thus \( p \leq N \)) satisfy the condition in (6) for some \( x \) in the interval.

For a fixed \( x \), the number of values of \( p' \) divisible by \( \frac{q' - 2x}{\gcd(q', x)} \) is bounded above by
\[
\frac{N}{\gcd(q', x)} = \frac{N \gcd(q', x)}{q' - 2x} \leq 2N \frac{\gcd(q', x)}{q'}.
\]
(The last step is justified by the fact that \( q \geq 67 \).) Thus, the total number number of values of \( p' \) that are divisible by \( \frac{q' - 2x}{\gcd(q', x)} \) for some \( x \in [-2\sqrt{q-1}, 2\sqrt{q-1}] \) is at most
\[
\sum_{x \in [-2\sqrt{q-1}, 2\sqrt{q-1}] \atop x \neq 0} \frac{2N \gcd(q', x)}{q'} \leq 2 \sum_{x = 1}^{2\sqrt{q}+1} \frac{2N \gcd(q', x)}{q'} = \frac{4N}{q'} \sum_{x = 1}^{2\sqrt{q}+1} \gcd(q', x).
\]
Now, let \( g(k) = \sum_{x=1}^{k} \gcd(x, k) \). We claim that
\[
\sum_{x=1}^{2\sqrt{q}+1} \gcd(q', x) \leq g(q') \cdot \frac{2\sqrt{q} + 1}{q'}.
\]
For \( n \) implicit, define the multiset \( S_{a,k} = \{ \gcd(x, n) \mid x \in \{a, a+1, \ldots, a+k-1\} \} \). Observe that for all \( d \mid n \), holding \( k \) constant, \( m_d(S_{a,k}) \) is minimal for \( a = 1 \). It follows from
Proposition 4.13 that
\[
\sum_{x=a}^{a+k-1} \gcd(x, n)
\]
is minimized for \( a = 1 \). In particular, let \( h(a) \) be (7) with \( n = q' \) and \( k = \lfloor 2\sqrt{q} + 1 \rfloor \). Note that
\[
h(1) + h(2) + \cdots + h(q') = g(q') \cdot \lfloor 2\sqrt{q} + 1 \rfloor,
\]
since the fact that \( \gcd(q', q' + x) = \gcd(q', x) \) means that for every \( x \in \{1, 2, \ldots, q'\} \), \( x \) appears \( \lfloor 2\sqrt{q} + 1 \rfloor \) times in (8). Since \( h(1) \) is the smallest value among the \( q' \) values in (8), we obtain
\[
\sum_{x=1}^{\lfloor 2\sqrt{q} + 1 \rfloor} \gcd(q', x) \leq g(q') \cdot \frac{2\sqrt{q} + 1}{q'},
\]
as desired. Thus, we have
\[
\sum_{x \in [-2\sqrt{q}-1, 2\sqrt{q}]} \frac{2N \gcd(q', x)}{q'} \leq \frac{4N}{q'} \sum_{x=1}^{\lfloor 2\sqrt{q} + 1 \rfloor} \gcd(q', x) \leq \frac{4N}{q'} g(q') \cdot \frac{2\sqrt{q} + 1}{q'}.
\]
Now, the number of primes \( p \leq N \) is on the order of \( \frac{N}{\log N} \). Thus, the probability that \( p \) is chosen such that the above divisibility property is satisfied for some \( x \) is
\[
O \left( \frac{4\log N}{q'} g(q') \cdot \frac{2\sqrt{q} + 1}{q'} \right) = O \left( \log N \cdot g(q') q^{-3} \right).
\]
It is known that \( g(k) = O(k^{1+\epsilon}) \) for every positive \( \epsilon \) [2, Theorem 3.2]. Thus, the probability above is \( O \left( \log N \cdot q^{-\frac{3}{2}+\epsilon} \right) \) for every positive \( \epsilon \), as a function of \( q \) and \( N \).

Now we express the probability as a function of just \( N \), randomly choosing \( q \) to be a prime below \( N \). The probability that \( q \leq N^{\frac{1}{3}} \) is on the order of
\[
\frac{N^{1/2}}{\log N^{1/2}} = \frac{2N^{1/2}}{N},
\]
which is on the order of \( N^{-\frac{1}{2}} \). If \( q > N^{\frac{1}{3}} \), then the above probability is \( O \left( \log N \cdot N^{-\frac{1}{2}+\epsilon} \right) \) for all \( \epsilon \). Thus, the total probability is at most on the order of \( N^{-\frac{1}{2}} + \log N \cdot N^{-\frac{1}{2}+\epsilon} \), which approaches zero as \( N \to \infty \), and so we are done.

**Theorem 4.15.** For \( N \geq 7 \), let \( 5 \leq p, q \leq N \) be randomly chosen distinct primes, and let \( n = pq \). Let \( E(\mathbb{Z}/n\mathbb{Z}) \) be a randomly chosen elliptic curve for which \( n \) is a Type I elliptic Korselt number. Assuming Conjecture 4.9, we have
\[
\lim_{N \to \infty} \Pr[p \text{ and } q \text{ are anomalous primes for } E|] = 1.
\]

**Proof.** A result of Deuring [3] states that for all primes \( p \), for every integer \(-2\sqrt{p} \leq t \leq 2\sqrt{p}\), there is an elliptic curve over \( \mathbb{F}_p \) with order \( p + 1 - t \). In particular, for every \( p \) there is an elliptic curve that is anomalous over \( \mathbb{F}_p \). Thus, for any two primes \( p \) and \( q \), we may use the Chinese Remainder Theorem to construct a curve over \( \mathbb{Q} \) that is anomalous both when
reduced over \( \mathbb{F}_p \) and over \( \mathbb{F}_q \). It follows by Proposition 4.3 that for all \((p, q)\) there is a curve \( E \) that makes \( p \) and \( q \) anomalous and therefore makes \( n = pq \) Type I elliptic Korselt number.

Suppose now that \( n = pq \) is a Type I elliptic Korselt number for some elliptic curve \( E \). Then the cases in which \( p \) and \( q \) are not both anomalous primes for \( E \) are as follows:

1. Exactly one of \( p \) and \( q \) is anomalous for \( E \).
2. Neither \( p \) nor \( q \) is anomalous for \( E \), and \((p + 1 - a_p)(q + 1 - a_q) \neq n + 1 - a_n\).
3. Neither \( p \) nor \( q \) is anomalous for \( E \), and \((p + 1 - a_p)(q + 1 - a_q) = n + 1 - a_n\).

Lemmas 4.10, 4.12, and 4.14 show that the probability that \( p \), \( q \), and \( E \) satisfy cases (1), (2), (3), respectively, goes to zero as \( N \to \infty \). Therefore, as \( N \to \infty \), the probability that \( p \) and \( q \) are both anomalous for \( E \) approaches 1. This completes the proof of Theorem 4.15.

\[\square\]

Acknowledgment. The authors would like to thank Lawrence Washington for many helpful discussions and suggestions regarding the work presented in this paper.

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Appendix. Empirical evidence that in most cases a type I elliptic Korselt number of the form \( n = pq \) is a product of anomalous primes.

We ran code that took as input an integer \( N \), randomly chose two distinct primes \( 5 \leq p, q \leq N \), and chose a random elliptic curve whose order over \( \mathbb{F}_p \) and \( \mathbb{F}_q \) divided \( pq + 1 - a_p a_q \). We did this 1000 times, each time recording whether \((p + 1 - a_p)(q + 1 - a_q) = pq + 1 - a_p a_q\). Below, \( \Pr(N) \) represents the fraction of the 1000 times that this property held; Conjecture 4.9 states that the true probability (of which we are taking a sample of size 1000 for each \( N \) below) should approach 1. Indeed, we see that this very much seems to be the case.

| \( N \)  | \( \Pr(N) \) |
|--------|-------------|
| 64     | .614        |
| 128    | .725        |
| 256    | .823        |
| 512    | .869        |
| 1024   | .924        |
| 2048   | .939        |
| 4096   | .965        |
| 8192   | .976        |
| 16384  | .994        |
| 32768  | .995        |
| 65536  | .995        |

Table 1. \( \Pr(N) \) versus \( \log_2(N) \), rounded to two decimal places.

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