MARKOV ELEMENTS IN AFFINE TEMPERLEY-LIEB ALGEBRAS

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Abstract. We define a tower of affine Temperley-Lieb algebras of type $\tilde{A}_n$ and we define Markov elements in those algebras. We prove that any trace over an affine Temperley-Lieb algebras of type $\tilde{A}_2$ is uniquely defined by its values on the Markov elements.

1. Introduction

In [5] we define a tower $(\tilde{TL}_{n+1}(q))_{n\geq 0}$ of affine Temperley-Lieb algebras of type $\tilde{A}_n$ and we prove that there exists a unique Markov trace on this tower. Crucial in the proof is the definition of Markov elements and the following Theorem:

Theorem 1.1. Any trace over $\tilde{TL}_{n+1}(q)$ for $2 \leq n$ is uniquely defined by its values on the Markov elements in $\tilde{TL}_{n+1}(q)$.

The proof of this Theorem for $3 \leq n$ is given in [3], where we have omitted the case $n = 2$, long and technical. We thus present it here for completeness.

2. Notations

Let $K$ be an integral domain of characteristic 0. Suppose that $q$ is a square invertible element in $K$ of which we fix a root $\sqrt{q}$. For $x, y$ in a given ring we define $V(x, y) := xyx + xy + yx + x + y + 1$. We mean by algebra in what follows $K$-algebra.

We denote by $B(\tilde{A}_n)$ (resp. $W(\tilde{A}_n)$) the affine braid (resp. affine Coxeter) group with $n + 1$ generators of type $\tilde{A}$, while we denote by $B(A_n)$ (resp. $W(A_n)$) the braid (resp. Coxeter) group with $n$ generators of type $A$, where $n \geq 0$. Let $W^c(\tilde{A}_n)$ (resp. $W^c(A_n)$) be the set of fully commutative elements in $W(\tilde{A}_n)$ (resp. $W(A_n)$).

Let $n \geq 2$. We define $\tilde{TL}_{n+1}(q)$ to be the algebra with unit given by a set of generators $\{g_{\sigma_1}, ..., g_{\sigma_n}, g_{a_{n+1}}\}$, with the following relations [1]:

- $g_{\sigma_i}g_{\sigma_j} = g_{\sigma_j}g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i - j| \geq 2$.
- $g_{\sigma_i}g_{a_{n+1}} = g_{a_{n+1}}g_{\sigma_i}$, for $2 \leq i \leq n - 1$.
- $g_{\sigma_i}g_{\sigma_i+1}g_{\sigma_i} = g_{\sigma_{i+1}}g_{\sigma_i}g_{\sigma_{i+1}}$, for $1 \leq i \leq n - 1$.
- $g_{\sigma_i}g_{a_{n+1}}g_{\sigma_i} = g_{a_{n+1}}g_{\sigma_i}g_{a_{n+1}}$, for $i = 1, n$. 


For $V$ the algebra of $\tau_2$ and it is a $K$-basis. We set $T_{a_{n+1}}$ (resp. $T_{\sigma_i}$ for $1 \leq i \leq n$) to be $\sqrt{q}a_{n+1}$ (resp. $\sqrt{q}\sigma_i$ for $1 \leq i \leq n$). Hence, $T_w$ is well defined for $w \in W^c(\tilde{A}_n)$, it equals $q^{-\frac{1}{2}}g_w$. The multiplication associated to the basis $\{T_w : w \in W^c(\tilde{A}_n)\}$, is given as follows:

$$T_uT_v = T_{uv} \quad \text{whenever } l(uv) = l(u) + l(v).$$

$$T_sT_w = \sqrt{q}(q-1)T_w + q^2T_{sw} \quad \text{whenever } l(sw) = l(w) - 1,$$

for $w,v$ in $W^c(\tilde{A}_n)$ and $s$ in $\{\sigma_1, \ldots, \sigma_n, a_{n+1}\}$.

In what follows we suppose that $q+1$ is invertible in $K$, we set $\delta = \frac{1}{2+q+q^{-1}} = \frac{q}{(1+q)^2}$ in $K$. In view of [2], for $1 \leq i \leq n$ we set $f_{\sigma_i} := \frac{g_{\sigma_i}+1}{q+1}$ and $f_{a_{n+1}} := \frac{g_{a_{n+1}}+1}{q+1}$. In other terms $g_{\sigma_i} = (q+1)f_{\sigma_i} - 1$, and $g_{a_{n+1}} = (q+1)f_{a_{n+1}} - 1$. The set $\{f_w : w \in W^c(\tilde{A}_n)\}$ is well defined and it is a $K$-basis for $\overline{T}L_{n+1}(q)$.

We define the Temperley-Lieb algebra of type $A$ with $n$ generators $TL_n(q)$, as the subalgebra of $\overline{T}L_{n+1}(q)$ generated by $\{g_{\sigma_1}, \ldots, g_{\sigma_n}\}$, with $\{g_w : w \in W^c(A_n)\}$ as $K$-basis.

Now for $TL_0(q) = K$, we consider the following tower:

$$TL_0(q) \subset TL_1(q) \subset \ldots \subset TL_{n-1}(q) \subset TL_n(q) \ldots$$

**Theorem 2.1.** [6] There is a unique collection of traces $\{\tau_{n+1}\}_{0 \leq n}$ on $(TL_n)_{0 \leq n}$, such that:

1. $\tau_1(1) = 1$.
2. For $1 \leq n$, we have $\tau_{n+1}(hT_{\sigma_n}^\pm) = \tau_n(h)$, for any $h$ in $TL_{n-1}(q)$.

The collection $\{\tau_{n+1}\}_{0 \leq n}$ is called a Markov trace. Moreover, for any $a, b$ and $c$ in $TL_n(q)$ and for $n \geq 1$, every $\tau_{n+1} : TL_n(q) \rightarrow K$ verifies:

$$\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc) \quad \text{and} \quad \tau_{n+1}(a) = -\frac{1+q}{\sqrt{q}}\tau_n(a).$$
3. THE TOWER OF AFFINE TEMPERLEY-LIEB ALGEBRAS AND AFFINE MARKOV TRACE

In this section we define a tower of affine Temperley-Lieb algebras, we show that this tower "surjects" onto the tower of Temperley-Lieb algebras mentioned in the introduction, and we define the affine Markov trace.

We consider the Dynkin diagram of the group $B(\tilde{A}_n)$. We denote the Dynkin automorphism $(\sigma_1 \mapsto \sigma_2 \mapsto ... \sigma_n \mapsto a_{n+1} \mapsto \sigma_1)$ by $\psi_{n+1}$. Notice that $\sigma_n \sigma_{n-1} ... \sigma_1 a_{n+1} \mapsto \sigma_1$ acts on $B(\tilde{A}_{n-1})$ as $\psi_n$ as follows $(\sigma_1 \mapsto a_n \mapsto \sigma_{n-1} \mapsto \sigma_{n-2} \mapsto ... \sigma_2 \mapsto \sigma_1)$. We write $(\sigma_n \sigma_{1} a_{n+1})^d h = \psi^d [h] (\sigma_n \sigma_1 a_{n+1})^d$, for any $h$ in $B(\tilde{A}_{n-1})$, we keep same convention for the affine Temperley-Lieb algebra.

We have the following injection

$$G_n : K[\tilde{B}(A_{n-1})] \longrightarrow K[\tilde{B}(\tilde{A}_n)]$$

$$\sigma_i \mapsto \sigma_i \quad \text{for } 1 \leq i \leq n - 1$$

$$a_n \mapsto \sigma_n a_{n+1} \sigma_n^{-1}$$

We prove in [5], to which we refer for details, the following two propositions:

**Proposition 3.1.** The injection $G_n$ induces the following morphism of algebras:

$$F_n : \hat{TL}_n(q) \longrightarrow \hat{TL}_{n+1}(q)$$

$$t_{\sigma_i} \mapsto g_{\sigma_i} \quad \text{for } 1 \leq i \leq n - 1$$

$$t_{a_n} \mapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}.$$  

**Proposition 3.2.** The following map is a surjection of algebras

$$E_n : \hat{TL}_{n+1}(q) \longrightarrow TL_n(q)$$

$$g_{\sigma_i} \mapsto g_{\sigma_i} \quad \text{for } 1 \leq i \leq n$$

$$g_{a_{n+1}} \mapsto g_{\sigma_1} g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-1}}^{-1} ... g_{\sigma_1}^{-1}.$$
Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
\widehat{TL}_n(q) & \xrightarrow{F_n} & \widehat{TL}_{n+1}(q) \\
E_{n-1} & \downarrow & E_n \\
TL_{n-1}(q) & \leftarrow & TL_n(q)
\end{array}
\]

Moreover, it is immediate that \(E_n\) composed with the natural inclusion of \(TL_n(q)\) into \(\widehat{TL}_{n+1}(q)\), gives \(\text{Id}_{\widehat{TL}_n(q)}\).

In view of proposition 3.1 we can consider the tower of affine T-L algebras (it is not known whether it is a tower of faithful arrows or not):

\[
\begin{array}{ccccccc}
\widehat{TL}_1(q) & F_1 & \widehat{TL}_2(q) & F_2 & \widehat{TL}_3(q) & \cdots & \widehat{TL}_n(q) & F_n & \widehat{TL}_{n+1}(q) & \cdots
\end{array}
\]

**Definition 3.3.** We call \((\check{\tau}_n)_{1\leq n}\) an affine Markov trace, if every \(\check{\tau}_n\) is a trace function on \(\widehat{TL}_n(q)\) with the following conditions:

- \(\check{\tau}_1(1) = 1\), (here \(\widehat{TL}_1(q) = K\)).
- \(\check{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \check{\tau}_n(h)\), for all \(h \in \widehat{TL}_n(q)\) and for \(n \geq 1\).
- \(\check{\tau}_n\) is invariant under the Dynkin automorphism \(\psi_n\) for all \(n\).

**Remark 3.4.** We notice that the second condition gives us that \(\check{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{-1}) = \check{\tau}_n(h)\), which means that:

\[
\check{\tau}_{n+1}(F_n(h)[\frac{1}{q^2}T_{\sigma_n} - \frac{q - 1}{q\sqrt{q}}]) = \check{\tau}_n(h). \text{ Thus } \check{\tau}_{n+1}(F_n(h)) = -\frac{q + 1}{\sqrt{q}}\check{\tau}_n(h).
\]

**Remark 3.5.** The third condition of definition 3.3 is, in fact, not independent, i.e., it results from the first and second conditions (see [3]). Nevertheless, we will keep viewing it as a condition.

**Remark 3.6.** This affine Markov trace does the job topologically, i.e., it gives an invariant for *affine oriented knots* and generalizes, in fact, the Jones invariant, noticing that the set of oriented knots in \(S^3\) injects naturally into the set of *affine oriented knots*. For further details see [5].

Now, consider the following commutative diagram:
Set $\rho_{n+1}$ to be the trace over $\widehat{TL}_{n+1}(q)$ induced by $\tau_{n+1}$ over $TL_n(q)$ for $0 \leq n$. We prove in [3] that $(\rho_i)_{1 \leq i}$ is an affine Markov trace over $\left(\widehat{TL}_i(q)\right)_{1 \leq i}$ and we prove the following Theorem:

**Theorem 3.7.** [3]

There exists a unique affine Markov trace over the tower of $\tilde{A}$-type Temperley-Lieb algebras, namely $(\rho_i)_{1 \leq i}$.

The proof relies on Theorem 4.2 below, the proof of which separates into two cases: $n = 2$ and $n \geq 3$. The latter case is included in [3] while the former appears in the present note.

4. **Markov elements and traces on $\widehat{TL}_{n+1}(q)$**

4.1. **Markov elements.** We consider $F : \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q)$ of proposition 3.1. In this subsection we set $F := F_n$. We give a definition of Markov elements in $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then we show that any trace over $\widehat{TL}_{n+1}(q)$ is uniquely determined by its values on those elements.

**Definition 4.1.** For $F$ as above, and $n \geq 2$, a Markov element in $\widehat{TL}_{n+1}(q)$ is any element of the form $Ag_{\epsilon_n} B$, where $A$ and $B$ are in $F(\widehat{TL}_n(q))$ and $\epsilon \in \{0, 1\}$.

The aim of this subsection is to prove the following theorem for $n = 2$.

**Theorem 4.2.** [3] Let $\tau_{n+1}$ be any trace over $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then, $\tau_{n+1}$ is uniquely defined by its values on the Markov elements in $\widehat{TL}_{n+1}(q)$.

The proof of theorem 4.2 for $n = 2$ is divided into two parts. In the first we show some general facts, in the second we give the proof for $n = 2$. 

\begin{align*}
\widehat{TL}_1(q) & \longrightarrow \widehat{TL}_2(q) \longrightarrow \cdots \longrightarrow \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q) \\
TL_0(q) & \leftarrow TL_1(q) \leftarrow \cdots \leftarrow TL_{n-1}(q) \leftarrow TL_n(q)
\end{align*}
Part 1

In this part, we suppose that $\tau_{n+1}$ is any trace on $\mathcal{TL}_{n+1}(q)$. We will apply $\tau_{n+1}$ to $\mathcal{TL}_{n+1}(q)$ assuming that $2 \leq n$, and show that $\tau_{n+1}$ is uniquely determined on $\mathcal{TL}_{n+1}(q)$ by its values on the positive powers of $g_{\sigma_n\sigma_{n-1}..\sigma_{2}a_{n+1}}$, in addition to its values on Markov elements. From now on we denote by $w$: an arbitrary element in $W^c(\hat{A}_n)$.

Lemma 4.3. In $\mathcal{TL}_{n+1}(q)$ we have:

$$
(1) \ g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k + \sum_{i=1}^{i=k-1} f_i(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^i \\
+ A(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^k g_{\sigma_n} \prod_{i=0}^{i=k-1} \psi^j[F((t_{a_n})^{-1})],
$$

$$
(2) \ (g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n} = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k + \sum_{i=1}^{i=k-1} h_i(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^i \\
+ A \prod_{j=0}^{j=k-1} \phi^j[(g_{\sigma_{n-1}})^{-1}] g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^k,
$$

with $A$ in the ground field, $f_i, h_i$ in $F(\mathcal{TL}_n(q))$ and $\phi^{-1} = \psi$.

Proof.

$$
g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k \\
+ qg_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})g_{\sigma_n} F((t_{a_n})^{-1})(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1} \\
= (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k \\
+ qg_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^{k-1} \psi^{k-1}[F((t_{a_n})^{-1})].
$$

So, by induction on $k$, (1) follows. In the very same way we deal with (2), by noticing that: $g_{a_n+1}g_{\sigma_n} = g_{a_n}^{-1}F(t_{a_n})g_{\sigma_n}^2 = (q-1)g_{a_n+1} + qg_{\sigma_n}^{-1}F(t_{a_n})$.

A main result in [4] is to give a general form for “fully commutative braids”, from which we deduce that any element of the basis of $\mathcal{TL}_{n+1}(q)$ (where we have the convention $\sigma_{n+1} = 1$ in $W(\hat{A}_n)$ thus $g_{\sigma_n\sigma_{n-1}..\sigma_i} = 1$ when $i = n + 1$), is either of the form

$$
c(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}
$$

or of the form

$$
g_{\sigma_0..\sigma_2\sigma_1a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k dg_{\sigma_n\sigma_{n-1}..\sigma_i}
$$
where $c$ and $d$ are in $F(\overline{T\mathcal{L}_n(q)})$, $1 \leq i \leq n + 1$ and $0 \leq i_0 \leq n - 1$.

By lemma 4.3 $c(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$ is of the form:

$$\sum_{j=h}^{j=h} c_j(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^j + M.$$ 

Where $h \leq k$, $c_j$ is in $F(\overline{T\mathcal{L}_n(q)})$ for any $j$ and $M$ is a Markov element.

Now we deal with the second form:

$$\tau_{n+1}(g_{\sigma_{i_0}..\sigma_2\sigma_1a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}) = \tau_{n+1}(g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k).$$

For any possible value for $i_0$ or $i$, we see that:

$$g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k = c' g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}^s c'';$$

where $c'$, $c''$ are in $F(\overline{T\mathcal{L}_n(q)})$ and $s \leq k + 1$. By lemma 4.3 we see that this element is of the form:

$$\sum_{j=h}^{j=h} f_j(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^j + M,$$

where $h \leq k + 1$, $f_j$ is in $F(\overline{T\mathcal{L}_n(q)})$ for any $j$ and $M$ is a Markov element.

Hence, we see that in order to define $\tau_{n+1}$ uniquely it is enough to have its values on Markov elements and its values on $\Omega(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k$, where $1 \leq k$ (since if $k$ is equal to 0 then we are again in the case of a Markov element) and $\Omega$ is in $F(\overline{T\mathcal{L}_n(q)})$.

**Lemma 4.4.** Let $2 \leq n$ then $\tau_{n+1}$ is uniquely defined by its values on Markov elements, in addition to its values on $(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k$, with $0 \leq k$.

**Proof.** In order to determine $\tau_{n+1}(h(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k)$, with a positive $k$ and an arbitrary $h$ in $F(\overline{T\mathcal{L}_n(q)})$, it is enough to treat $\tau_{n+1}(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k)$, with $x$ in $W^c(A_{n-1})$, but the fact that $\tau_{n+1}$ is a trace, in addition to the fact that $g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}}$ acts as a Dynkin automorphism on $F(\overline{T\mathcal{L}_n(q)})$, authorizes us to suppose that $x$ has a reduced expression which ends with $\sigma_{n-1}$.

Now we show by induction on $l(x)$, that $\tau_{n+1}(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k)$ is a sum of values of $\tau_{n+1}$ over $(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})^k$, elements of the form $h_i(g_{\sigma_n\sigma_{n-1}..\sigma_1a_{n+1}})$ with $i < k$ and Markov elements, (of course with coefficients in the ground ring which might be zeros).
For \( l(x) = 0 \) the property is true. Take \( l(x) > 0 \), and let \( x = z\sigma_{n-1} \) be a reduced expression, hence:

\[
\tau_{n+1} \left( F(t_z)(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^k \right) = \tau_{n+1} \left( F(t_z)F(t_{\sigma_{n-1}}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
= \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right).
\]

Recalling that \( V(g_{\sigma_{n-1}}, g_{\sigma_n}) = 0 \), this is equal to the following sum:

\[
- \tau_{n+1} \left( F(t_z)(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^k \right)
\]

\[
- \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
- \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
- \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
- \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right).
\]

Now we apply the induction hypothesis to the first term. The second and the third terms are equal to:

\[
\tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

which is equal to:

\[
\tau_{n+1} \left( \psi^{1-k} \left[ F((t_{a_{n}})^{-1}) \right] F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
+ \tau_{n+1} \left( \psi^{1-k} \left[ F((t_{a_{n}})^{-1}) \right] F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right).
\]

The fourth and the fifth terms are equal to:

\[
\tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right)
\]

\[
+ \tau_{n+1} \left( F(t_z)g_{\sigma_{n-1}g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_{1}a_{n+1}})^{k-1}) \right).
\]
Thus, lemma 4.3 tells us that the property is true for those four terms. This step is to be applied repeatedly, to the powers of \( g_{\sigma_1\sigma_2\ldots\sigma_n} \) down to an element of the form \( \tau_{n+1}(h(g_{\sigma_1\sigma_2\ldots\sigma_n+1}^1)) \), arriving to the sum of:

\[
\tau_{n+1}(g_{\sigma_1\sigma_2\ldots\sigma_n+1})
\]

and

\[
\tau_{n+1}(h'g_{\sigma_1\sigma_2\ldots\sigma_n+1})
\]

which is the sum of values of \( \tau_{n+1} \) on Markov elements, since \( h, h' \in \mathcal{F}(\overline{T\mathcal{L}}_n(q)) \).

We end this part by the following lemma:

**Lemma 4.5.** Let \( 1 \leq k \). Then \((g_{\sigma_1\sigma_2\ldots\sigma_n+1})^k \) is a sum of two kinds of elements:

1. \( g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j) g_{\sigma_n}h \), with \( j \leq k \).
2. \( (g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j) g_{\sigma_n}f \), with \( i < k \);

with \( h, f \) in \( \mathcal{F}(\overline{T\mathcal{L}}_n(q)) \) and \( 2 \leq n \).

Moreover, in the first type we have one, and only one element, with \( j = k \), in which we have:

\[
h = \prod_{i=0}^{i=k-1} \phi^i[F(t_{a_n}^{-1})].
\]

**Proof.** Suppose that \( k = 1 \). Then,

\[
g_{\sigma_1\sigma_2\ldots\sigma_n+1} = g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}) g_{\sigma_n}F(t_{a_n})^{-1},
\]

the property is true.

Suppose the property is true for \( k - 1 \), then, with \( 2 \leq k \), we have:

\[
(g_{\sigma_1\sigma_2\ldots\sigma_n+1})^k = (g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})})^{k-1} g_{\sigma_1\sigma_2\ldots\sigma_n+1}.
\]

We apply the property to \((g_{\sigma_1\sigma_2\ldots\sigma_n+1})^{k-1}\), which gives two cases:

1. \( g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j') g_{\sigma_n}h g_{\sigma_1\sigma_2\ldots\sigma_n+1} \), with \( j' \leq k - 1 \) which is:

\[
g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j') g_{\sigma_1\sigma_2\ldots\sigma_n+1} \psi^{-1}[h],
\]

which is equal to:

\[
qg_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j'+1) g_{\sigma_n}F(t_{a_n})^{-1} \psi^{-1}[h]
\]

\[
+ (q-1)g_{\sigma_n}g_{\sigma_1\sigma_2\ldots\sigma_n+1} \psi^{-1}\left[(g_{\sigma_{n-1}\sigma_{n-2}F(t_{a_n})}j') \right] \psi^{-1}[h].
\]
Since, \( j' + 1 \leq k \), the first term is clear to be of the first type, while the second term is equal to:

\[
(q - 1)g_{\sigma_{n-1}..\sigma_1}F(t_{an})g_{\sigma_1}((t_{an})^{-1})^{-1}\psi^{-1}\left[(g_{\sigma_{n-1}..\sigma_1}F(t_{an}))^{j'}\right]^{-1}\psi^{-1}[h] + \\
(q - 1)^2g_{\sigma_{n-1}..\sigma_1\sigma_1a_{n+1}}\psi^{-1}\left[(g_{\sigma_{n-1}..\sigma_1}F(t_{an}))^{j'}\right]^{-1}\psi^{-1}[h].
\]

Here, the first term is of the second type (with \( i = 1 < k \)), and the second term is of the first type (with \( j = 1 \)).

\[
(2) \left(g_{\sigma_{n-1}..\sigma_1}F(t_{an})\right)^{i'}g_{\sigma_1}f g_{\sigma_{n-1}..\sigma_1a_{n+1}}, \text{ with } i' < k - 1, \text{ which is:}
\]

\[
\left(g_{\sigma_{n-1}..\sigma_1}F(t_{an})\right)^{i'}g_{\sigma_1}g_{\sigma_{n-1}..\sigma_1a_{n+1}}\psi^{-1}[f] = \\
q\left(g_{\sigma_{n-1}..\sigma_1}F(t_{an})\right)^{i'+1}g_{\sigma_1}f((t_{an})^{-1})\psi^{-1}[f] + \\
(q - 1)g_{\sigma_1}\left(g_{\sigma_{n-1}..\sigma_1}F(t_{an})\right)g_{\sigma_1}f((t_{an})^{-1})\psi^{-1}\left[(g_{\sigma_{n-1}..\sigma_1}F(t_{an}))^{i'}\right]^{-1}[f].
\]

Since \( i' + 1 < k \), the first term is of the second type, while the second term is of the first type with \( j = 1 \). The lemma is proven.

(By induction over \( k \) again, the last formula is easy).

\[ \Box \]

**Part 2**

In this part we will consider a given trace \( \tau_3 \) over \( \widehat{TL}_3(q) \). The aim is to show that \( \tau_3 \) is uniquely defined by its values on Markov elements. consider

\[
F_2 : \widehat{TL}_2(q) \rightarrow \widehat{TL}_3(q)
\]

\[
t_{\sigma_1} \mapsto g_{\sigma_1}
\]

\[
t_{an} \mapsto g_{\sigma_2}g_{a_3}g_{\sigma_1}^{-1}.
\]

In this part we will denote \( F_2 \) by \( F \).

Lemma 4.4 tells that we can uniquely determine \( \tau_3 \) by its values over \( (g_{\sigma_2\sigma_1a_3})^k \) for a positive \( k \) beside its values on Markov elements. We know as well by lemma 4.5 that \( (g_{\sigma_2\sigma_1a_3})^k \) is a sum of two kinds of elements:

1. \( g_{\sigma_2}(g_{\sigma_1}F(t_{a_2}))^j g_{\sigma_2}h \) with \( j \leq k \).
2. \( (g_{\sigma_1}F(t_{a_2}))^i g_{\sigma_2}f \) with \( i < k \).
Here, $h$ and $f$ are in $F(\overline{T\Lambda}_2(q))$.

Moreover, in first type, only when $j = k$, we have:

$$h = \prod_{i=0}^{i=k-1} \psi^i \left[ (F(t_{a_2})^{-1}) \right].$$

In other terms:

$$g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} = \left( g_{\sigma_2 \sigma_1 a_3} \right)^k \prod_{i=0}^{i=k-1} \psi^i \left[ (F(t_{a_2})) \right]$$

$$- \sum_{r=1}^{r=k-1} \left( g_{\sigma_1} F(t_{a_2}) \right)^r g_{\sigma_2} f_r \prod_{i=0}^{i=k-1} \psi^i \left[ (F(t_{a_2})) \right]$$

$$+ \sum_{l=1}^{l=k-1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^l g_{\sigma_2} f_{r_l} \prod_{i=0}^{i=k-1} \psi^i \left[ (F(t_{a_2})) \right].$$

We repeat the same step on $g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^l g_{\sigma_2}$ for every $l$. We deduce the following:

**Corollary 4.6.** For every $h > 0$, we have:

$$g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^h g_{\sigma_2} = \sum_{j=0}^{j=h} c_j \left( g_{\sigma_2 \sigma_1 a_3} \right)^j + \sum_i M_i.$$

Here, $c_j$ is in $F(\overline{T\Lambda}_2(q))$ for every $j$, and $M_i$ is a Markov element for every $i$.

Our way to prove Theorem 4.2 for $n = 3$, is to show that $\tau_3 \left( (g_{\sigma_2 \sigma_1 a_3})^k \right)$ is a linear combination of values of $\tau_3$ on Markov elements and values on elements of the form $c(g_{\sigma_2 \sigma_1 a_3})^h$, where $h < k$ and $c$ in $F(\overline{T\Lambda}_2(q))$. Then, using the induction in the proof of Lemma 4.4, beside the fact that $\tau_3(g_{\sigma_2 \sigma_1 a_3})$ is a linear combination of some values of $\tau_3$ on Markov elements, we see that the work is done.

**Lemma 4.7.** Suppose that $r$ and $s$ are positives, such that $r \leq s$. Then:

$$\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^r g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} = \sum_{j=0}^{j=h} c_j \left( g_{\sigma_2 \sigma_1 a_3} \right)^j + \sum_i M_i,$$

where $h \leq s$, $c_j$ is in $F(\overline{T\Lambda}_2(q))$ for every $j$ and $M_i$ is a Markov element for every $i$. 
Proof.

\[ \tau_3 \left( (g_{\sigma_1} \ F \ (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} (g_{\sigma_1} \ F (t_{a_2}))^s \ g_{\sigma_1} g_{\sigma_2} \right) = \]

\[ = \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} (g_{\sigma_1} F (t_{a_2}))^s \ g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_2 \ a_3} \right)^{-1} g_{\sigma_2} \right) \]

\[ = \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} \ g_{\sigma_2 a_3} \ g_{\sigma_1}^{-1} g_{\sigma_2} \right) \]

\[ = \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} \ g_{\sigma_2 a_3} \ g_{\sigma_1}^{-1} g_{\sigma_2} \right) \]

\[ = \frac{1 - q}{q} \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} \ g_{\sigma_2 a_3} \ g_{\sigma_1}^{-1} g_{\sigma_2} \right) \]

Now, the term corresponding to \( \frac{1 - q}{q} \) is \( \tau_3 \) evaluated on the sum of Markov element and an element of style \( c_j (g_{\sigma_2 \ a_3}) \). So, We are reduced to the second term, thus, reduced to:

\[ \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} g_{\sigma_3} \ g_{\sigma_1}^{-1} g_{\sigma_3} F (t_{a_2}) \right). \]

Obviously, we are in the case:

\[ q \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} g_{\sigma_3} \ g_{\sigma_1}^{-1} g_{\sigma_3} F (t_{a_2}) \right) = \]

\[ q \tau_3 \left( (g_{\sigma_1} F (t_{a_2}))^r \ g_{\sigma_1} g_{\sigma_2} g_{\sigma_3} \ g_{\sigma_1}^{-1} g_{\sigma_3} F (t_{a_2}) \right), \]

since \( g_{\sigma_3} F (t_{a_2}) = F (t_{a_2}) g_{\sigma_2} \).
Now,
\[
\tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 g_{a_3} (F(t_{a_2}) g_{\sigma_1})^s F(t_{a_2}^2) g_{\sigma_2} \right) = \\
(q-1) \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 F(t_{a_2}) g_{a_3} (g_{\sigma_1} F(t_{a_2}))^s g_{\sigma_2} \right) \\
+ q \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 F(t_{a_2}) g_{a_3} F(t_{a_2}) g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} g_{\sigma_2} \right).
\]

Which is equal to the sum:
\[
(q-1) \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 F(t_{a_2}) g_{a_3} (g_{\sigma_1} F(t_{a_2}))^s g_{\sigma_2} \right) \\
+ q \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 F(t_{a_2}) g_{a_3} F(t_{a_2}) g_{\sigma_1} F(t_{a_2})^{s-1} g_{\sigma_2} \right).
\]

Now, the first term is covered by corollary 4.6. Thus we are interested with the second term:
\[
\tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-1} g_{\sigma_1}^2 F(t_{a_2}) g_{a_3} (g_{\sigma_1} F(t_{a_2}))^s g_{\sigma_1} g_{\sigma_2} \right) = \\
q \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r} g_{a_2} (g_{\sigma_1} F(t_{a_2}))^{s-1} g_{\sigma_1} g_{\sigma_2} \right) \\
+ (q-1) \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-2} g_{\sigma_1} F(t_{a_2}^2) g_{a_3} (g_{\sigma_1} F(t_{a_2}))^{s-1} g_{\sigma_1} g_{\sigma_2} \right).
\]

Which is equal to:
\[
q \tau_3 \left( g_{a_2} (g_{\sigma_1} F(t_{a_2}))^{r} g_{a_2} (g_{\sigma_1} F(t_{a_2}))^{s-1} g_{\sigma_1} \right) + \\
(q-1) \tau_3 \left( (g_{\sigma_1} F(t_{a_2}))^{r-2} g_{\sigma_1} F(t_{a_2}^2) g_{a_3} (g_{\sigma_1} F(t_{a_2}))^{s-1} g_{\sigma_1} g_{\sigma_2} \right)
\]

The first term is covered by corollary 4.6. We are reduced to
\[
\left( \tau_3 (g_{\sigma_1} F(t_{a_2}))^{r-2} g_{\sigma_1} F(t_{a_2}^2) g_{a_2} (g_{\sigma_1} F(t_{a_2}))^{s-1} g_{\sigma_1} g_{\sigma_2} \right).
\]
which is equal to:
\[
(q - 1) \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{r-1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{s-1} g_{\sigma_1} \right) + \\
q \tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^{r-2} g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{s-1} g_{\sigma_1} g_{\sigma_2} \right).
\]

The first term is covered by corollary 4.6. Thus, we see that, in general, the value of \( \tau_3 \) over \( \left( g_{\sigma_1} F(t_{a_2}) \right)^r g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} \) can be shifted to its value over:
\[
\left( g_{\sigma_1} F(t_{a_2}) \right)^{r-2} g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{s-1} g_{\sigma_1} g_{\sigma_2}.
\]

After a finite number of repetitions of the computation above (with the possibility of exchanging \( r \) and \( s \)), we see that the lemma is proven modulo determining:
\[
\tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^m g_{\sigma_1} g_{\sigma_2} g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} g_{\sigma_2} \right).
\]

We see that the terms corresponding to \(-g_{\sigma_1}\) and \(-1\) correspond to Markov elements. While those who correspond to \(-g_{\sigma_1} g_{\sigma_2}\) and \(-g_{\sigma_2}\) are covered by corollary 4.6 for \( h = 1 \). Finally, the term corresponding to \(-g_{\sigma_2} g_{\sigma_1}\) is covered by corollary 4.6 for \( h = m \).

\[
\blacksquare
\]

**Lemma 4.8.** Suppose that \( r \) and \( s \) are positive such that \( r \leq s \). Then:
\[
\tau_3 \left( g_{a_3} F(t_{a_2}) \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{a_3} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right) = \sum_{j=0}^{j=h} c_j \left( g_{\sigma_2 \sigma_1 a_3} \right)^j + \sum_i M_i.
\]

Where \( h \leq s \), \( c_j \) is in \( F(\overline{TL}_2(q)) \) for every \( j \) and \( M_i \) is a Markov element for every \( i \).

**Proof.**
\[
\tau_3 \left( g_{a_3} F(t_{a_2}) \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{a_3} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right) = \\
\tau_3 \left( g_{a_3} \left( g_{\sigma_2 \sigma_1 a_3} \right)^{-1} g_{\sigma_2 \sigma_1 a_3} F(t_{a_2}) \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{a_3} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right) = \\
\tau_3 \left( g_{a_3} \left( g_{\sigma_2 \sigma_1 a_3} \right)^{-1} \psi \left( F(t_{a_2}) \left( g_{\sigma_1} F(t_{a_2}) \right)^s \right) g_{\sigma_2 \sigma_1 a_3} g_{a_3} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right) = \\
\tau_3 \left( g_{a_3}^{-1} g_{\sigma_2} g_{\sigma_1} \left( F(t_{a_2}) g_{a_3} \right)^s g_{\sigma_2} g_{\sigma_1} g_{a_3} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right).
\]

Here, we see that this term is a sum of two terms coming from \( g_{a_3}^2 = (q - 1) g_{a_3} + q \). The term corresponding to \((q - 1) g_{a_3}\) is covered the same way as in the last lemma (with \( a_3 \)
instead of \( \sigma_2 \) above. Hence we treat the term corresponding to \( q \), that is:

\[
\tau_3 \left( g_{\sigma_1}^{-1} g_{\sigma_2}^{-1} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right).
\]

Before applying TL relations, we see in the same way as above, that we are reduced to the next value (otherwise, it is \( \tau \) evaluated on a Markov element):

\[
\tau_3 \left( g_{\sigma_1}^{-1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right).
\]

We see that the terms corresponding to \(-g_{\sigma_1}\) and \(-1\) correspond to Markov elements. And those who correspond to \(-g_{\sigma_2}g_{\sigma_1}\) and \(-g_{\sigma_2}\) are covered by corollary 4.6 for \( h = s \).

The term corresponding to \(-g_{\sigma_1}g_{\sigma_2}\) is:

\[
\tau_3 \left( g_{\sigma_1}^{-1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right),
\]

which is:

\[
\frac{1 - q}{q} \tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^r g_{\sigma_2} \right) + \frac{1}{q} \tau_3 \left( g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^s g_{\sigma_1} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^r \right)
\]

The first term is covered by corollary 4.6 for \( h = r \). The second follows by lemma 4.7.

\[
\square
\]

Let us go back to \( \tau_3(g_{\sigma_2\sigma_1 a_{n+1}})^k \). The aim is to show that:

\[
\tau_3(g_{\sigma_2\sigma_1 a_{n+1}})^k = \tau_3 \left( \sum_{j=0}^{h} c_j (g_{\sigma_2\sigma_1 a_3})^j + \sum_i M_i \right),
\]

where \( h < k \). By lemma 4.5, it is sufficient to deal with:

\[
g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \prod_{i=0}^{i=k-1} \left[ (F(t_{a_2})^{-1}) \right].
\]

It is clear that this element is written as a linear combination of four kind of elements:

1. \( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^h \).
2. \( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^h g_{\sigma_1} \).
3. \( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^k F(t_{a_2}) \).
4. \( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h \).
where \( h \leq \left[ \frac{k}{2} \right] < k \), since 1 < k.

1) We start by \( \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} \right) \). Which is equal to:

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_2} \right),
\]

follows directly, regarding corollary 4.6.

2) Now we consider

\[
(1) \quad \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_1} \right),
\]

which is equal to:

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_2} g_{\sigma_2} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{-1} \right) =
\]

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} g_{\sigma_1} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{-1} \right) =
\]

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} g_{\sigma_1} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{-1} \right) =
\]

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} g_{\sigma_1} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{-1} \right) =
\]

with the very same steps as used above, we see that we are reduced to:

\[
\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} F(t_{a_2}) g_{\sigma_1} F(t_{a_2}) \right),
\]

which is:

\[
\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_1} F(t_{a_2}) \right) =
\]

\[
\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_1} F(t_{a_2}) \right),\]

which is equal to:

\[
(q - 1) \tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_2} + q \tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h-1} g_{\sigma_1} g_{\sigma_2},
\]

we see that corollary 4.6 covers the first term. Thus we see that:

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h} g_{\sigma_1} \right) \text{ in (eq. 1), is shifted to:}
\]

\[
\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h-1} g_{\sigma_1} \right),
\]
going on in this manner, we arrive to:

\[ \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} \right), \]

with the same steps above, we see that we are reduced to

\[ \tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} \right), \]

which is equal to

\[ (q - 1) \tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} F(t_{a_2}) g_{\sigma_1} \right) + q \tau_3 \left( \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} g_{\sigma_2} \right), \]

corollary 4.6 and TL relations end the job.

3) Here we deal with

\[ \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h F(t_{a_2}) \right), \]

which is:

\[ \tau_3 \left( F(t_{a_2}) g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right) \]

\[ = \tau_3 \left( g_{a_3} F(t_{a_2}) \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right), \]

but, \( g_{\sigma_2} = F(t_{a_2}^{-1}) g_{a_3} F(t_{a_2}) \), thus:

\[ \tau_3 \left( g_{a_3} \left( F(t_{a_2}) g_{\sigma_1} \right)^k g_{a_3} F(t_{a_2}) \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right) \]

\[ = \tau_3 \left( g_{a_3} g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^k g_{a_3} F(t_{a_2}) \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right), \]

\[ = \tau_3 \left( g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h g_{a_3} \right), \]

as we have done above, using the quadratic relations, we see that we are reduced to:
The first term is covered by corollary 4.6. For the second we see that it is equal to:

\[ q\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h-1} \right), \]

which is equal to

\[ q(q-1)\tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{h-1} \right) + \]

\[ q^2 \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} F \left( t_{a_2} \right) \left( g_{\sigma_1} F(t_{a_2}) \right)^{h-2} \right), \]

the first term is obviously, covered by corollary 4.6, for the second one we see that it is case 3 itself, but with \( h - 2 \) instead of \( h \). Thus, we get two elements for \( \tau_3 \) to be evaluated on:

\[ \begin{bmatrix} a \end{bmatrix} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} F \left( t_{a_2} \right) g_{\sigma_1} F \left( t_{a_2} \right), \]

\[ \begin{bmatrix} b \end{bmatrix} g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^2. \]

For \[ b \] we can repeat what we have done until arriving to:

\[ \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} F \left( t_{a_2} \right) \right), \]

which is the following sum:

\[ (q-1)\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} g_{\sigma_1} F(t_{a_2}) g_{\sigma_2} + q\tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} F \left( t_{a_2} \right) g_{\sigma_2} \]

obviously, the first term is covered by corollary 4.6, the second term is a Markov element.
For [ a ] we see that:

\[ \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} F(t_{a_2}) g_{\sigma_1} F(t_{a_2}) \right) \]

\[ = \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} F(t_{a_2}) g_{\sigma_2} F(t_{a_2}) g_{\sigma_1} F(t_{a_2}) \right) \]

\[ = (q - 1) \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} g_{\sigma_2} a_3 g_{\sigma_1} F(t_{a_2}) \right) + \]

\[ q \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} a_3 g_{\sigma_1} F(t_{a_2}) \right), \]

the first term is covered by corollary 4.6, since it is equal to:

\[ (q - 1) \tau_3 \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} F(t_{a_2}) g_{\sigma_2} F(t_{a_2}) g_{\sigma_2} a_2 \).

For the second term, we see that:

\[ q \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} a_3 g_{\sigma_1} F(t_{a_2}) \right) \]

\[ = q \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} a_3 g_{\sigma_1} F(t_{a_2}) g_{\sigma_2} \right) \]

\[ = q \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} a_3 g_{\sigma_1} g_{\sigma_2} F(t_{a_2}) \right) \]

\[ = q \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^{k-1} g_{\sigma_1} a_3 g_{\sigma_1} F(t_{a_2}) \right), \]

which is a Markov element, since \( g_{\sigma_3} = F(t_{a_2}) g_{\sigma_2} F(t_{a_2}^{-1}) \).

4) We deal with \( \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right) \), using the same techniques:

\[ \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h \right) \]

\[ = \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} \left( F(t_{a_2}) g_{\sigma_1} \right)^h g_{\sigma_2} a_3 g_{\sigma_2} a_3^{-1} \right) \]

\[ = \tau_3 \left( g_{\sigma_2} \left( g_{\sigma_1} F(t_{a_2}) \right)^k g_{\sigma_2} a_3 g_{\sigma_1} \left( g_{\sigma_1} F(t_{a_2}) \right)^h g_{\sigma_2} a_3^{-1} \right), \]
so, we are reduced to:

\[ \tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-1} g_{\sigma_1} g_{a_3} (g_{\sigma_1} F(t_{a_2}))^h g_{a_3} F(t_{a_2}) \right). \]

Which is equal to:

\[ \tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-1} \frac{g_{\sigma_1} g_{a_3} g_{\sigma_2}}{V(g_{\sigma_1}, g_{a_3})} F(t_{a_2}) (g_{\sigma_1} F(t_{a_2}))^{h-1} F(t_{a_2}) g_{\sigma_2} \right), \]

for -1 and \(-g_{\sigma_1}\), it is a Markov element. For \(-g_{a_3} g_{\sigma_1}\) we see that:

\[
\tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-1} g_{a_3} g_{\sigma_1} F(t_{a_2}) (g_{\sigma_1} F(t_{a_2}))^{h-1} F(t_{a_2}) g_{\sigma_2} \right) \\
= \tau_3 \left( g_{a_3} F(t_{a_2}) (g_{\sigma_1} F(t_{a_2}))^{k-1} g_{a_3} (g_{\sigma_1} F(t_{a_2}))^h \right),
\]

which is covered by lemma 4.8.

For \(-g_{a_3}\), we see that:

\[
\tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-2} g_{\sigma_1} F(t_{a_2}^2) g_{a_2} (g_{\sigma_1} F(t_{a_2}))^{h-2} g_{\sigma_1} F(t_{a_2}^2) g_{\sigma_2} \right) \\
= (q - 1) \tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-2} g_{\sigma_2} (g_{\sigma_1} F(t_{a_2}))^{h-2} g_{\sigma_1} F(t_{a_2}^2) g_{\sigma_2} \right) + \\
q \tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-2} g_{\sigma_1} g_{\sigma_2} (g_{\sigma_1} F(t_{a_2}))^{h-2} g_{\sigma_1} F(t_{a_2}^2) g_{\sigma_2}, \right),
\]

the first term is covered by corollary 4.6. We do the same thing with \(F(t_{a_2}^2)\) in the second term, we arrive to:

\[ q^2 \tau_3 \left( g_{\sigma_1} F(t_{a_2})^{k-2} g_{\sigma_1} g_{\sigma_2} (g_{\sigma_1} F(t_{a_2}))^{h-2} g_{\sigma_1} g_{\sigma_2}, \right), \]

which is the case of lemma 4.7.
For $-g_\sigma g_\alpha$ we see that:

$$
\tau_3 \left( (g_\sigma F(t_{a_2}))^{h-1} g_\sigma g_\alpha F(t_{a_2}) (g_\sigma F(t_{a_2}))^{h-1} F(t_{a_2}) g_\sigma \right)
$$

$$
= \tau_3 \left( (g_\sigma F(t_{a_2}))^{k} g_\sigma g_\alpha (g_\sigma F(t_{a_2}))^{h-1} F(t_{a_2}) g_\sigma \right)
$$

$$
= (q-1) \tau_3 \left( (g_\sigma F(t_{a_2}))^{k} g_\sigma g_\alpha (g_\sigma F(t_{a_2}))^{h-1} g_\sigma \right) +
$$

$$
q \tau_3 \left( (g_\sigma F(t_{a_2}))^{k} g_\sigma g_\alpha (g_\sigma F(t_{a_2}))^{h-2} g_\alpha g_\sigma \right),
$$

corollary 4.6 covers the first term, while the second term is covered by (1) from our four cases.

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