Effective actions in $\mathcal{N} = 1$, D5 supersymmetric gauge theories: harmonic superspace approach

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Abstract

We consider the off-shell formulation of the 5D, $\mathcal{N} = 1$ super Yang-Mills and super Chern-Simons theories in harmonic superspace. Using such a formulation we develop a manifestly supersymmetric and gauge invariant approach to constructing the one-loop effective action both in super Yang-Mills and super Chern-Simons models. On the base of this approach we compute the leading low-energy quantum contribution to the effective action on the Abelian vector multiplet background. This contribution corresponds to the $'F^4'$ invariant which is given in 5D superfield form.

Dedicated to the memory of Boris Zupnik

1 Introduction

The study of the quantum structure of the supersymmetric five-dimensional field theories attracts recently considerable attention, mainly due to attempts to find the effective world-volume action for multiple M5-branes [1]. It was conjectured in [2] that the six-dimensional (2,0) superconformal field theories on a stack of M5-branes are equivalent to five-dimensional super Yang-Mills (SYM) theories. To establish this correspondence, the Kaluza-Klein reduction of the general 6D (1,0) pseudo-action with a non-Abelian gauge group $G$ was performed in the series of papers [3] and the 5D effective action for the Kaluza-Klein zero-modes was derived. This duality serves as an important constraint on the models for multiple M5-branes. Of course, not all consistent five-dimensional theories arise in such a circle compactification.
On the other hand, the 5D theory has a global $U(1)$ symmetry and the current $j = \ast F \wedge F$ is always conserved. The corresponding conserved charge is the instanton number. Such a conserved current can be coupled to vector superfield what allows us to identify the scalar component $\phi$ of this vector superfield as the gauge coupling $<\phi> \sim \frac{1}{g_{5Y M}}$ [4]. Using this observation the authors of papers [5] proposed that the maximally extended 5D supersymmetric gauge theory describes the 6D, $(2,0)$ superconformal field compactified on circle without introducing the Kaluza-Klein reduction. This proposition was based on the observation that the Kaluza-Klein momentum along the circle can be identified with instanton charge in the 5D theory. The latter is a topological charge carried by soliton configurations, which are analogous to monopole and dyon configurations in 4D. Such an attractive hypothesis means, in fact, that adding the 1/2-BPS particle soliton states with instanton number $k$ and a mass formula $M \sim \frac{4\pi k}{g_{5YM}} \sim \frac{k}{R_5}$ to 5D SYM theory gives us the full nonperturbative particle spectrum and determines the nonperturbative completion of the theory under consideration. This might be an argument in favor of UV finiteness of 5D SYM perturbative theory and thus be an argument for the consistent quantum theory.

In the strong coupling limit the 5D SYM should define the fully decompactified 6D, $(2,0)$ theory, which, in its turn, is expected to describe the low-energy dynamics of multiple M5-branes.

An additional important motivation to study 5D supersymmetric gauge theories comes from the existence of the corresponding super Chern-Simons (SCS) theory. This theory is interesting since it has a conformal fixed point in five dimensions and can admit a holographic duality [4]. There are several reasons why the 5D supersymmetric Chern-Simons theory can be interesting in quantum domain. First of all, the Chern-Simons terms can be generated by integrating out the massive hypermultiplets in the SYM theory when the hypermultiplets transform in complex representations of the gauge group [2]. If we consider the masses of the hypermultiplets as the UV cut-off, then this leads to the generation of the Chern-Simons term in the one-loop correction to the classical theory. Hence inclusion of the SCS term into the action can be useful in some cases if we want to have a complete description of the theory. SCS theory can also be important in the relationship between 5D SYM and 6D, $(2,0)$ theories. In particular, one can argue [6] that the 5D Chern-Simons term can be generated by the anomaly terms in the six-dimensional theory. By focusing on a certain class of anomaly-free six-dimensional theories the authors of [6] formulated the explicit constraints on the spectrum and supersymmetry content of the six-dimensional theory in terms of the five-dimensional Chern-Simons couplings. Therefore it would be interesting to compute the perturbative quantum corrections in such 5D theories. In particular, it was demonstrated that massive fermions running in the loop generate constant corrections to the 5D Chern-Simons terms of the form $k_{ABC} A^A \wedge F^B \wedge F^C + \kappa_A A^A \wedge \text{tr}(R \wedge R)$, where $A^A$ denotes collectively the graviphoton and the vectors from the vector multiplet, $F^A$ are the corresponding field strengths, and $R$ is the curvature two-form.

Though 5D and 3D Chern-Simons theories share some interesting properties, such as quantization of the level $k$, nevertheless there are also some differences. The most important difference is the presence of local degrees of freedom in the higher dimensional
case. This peculiar fact makes it attractive and interesting to perform a more detailed analysis of 5D SCS theories.

It is known that non-trivial observables exist in supersymmetric gauge theories which are not very sensitive to details of the UV completion. Quantum effects which non-trivially contribute to such BPS observables are often highly constrained. With using the procedure of localization of the path integral were studied of the various observables in 5D supersymmetric theories [7]. It was shown that the partition function for the maximally extended SYM on $S^5$ captures the physical aspects of the 6D, $(2, 0)$ theory in a surprisingly accurate and detailed manner. In particular, the $N^3$ behavior of this partition function in 5D supersymmetric gauge theory is in agreement with the important results obtained for 6D, $(2, 0)$ theory from the supergravity duals and conformal anomaly [8].

The study of integral invariants in half-maximally and maximally extended supersymmetric theories such as supergravity and SYM attracts an attention because they can be viewed, on the one hand, as possible higher order corrections to the string or brane effective actions and, on the other hand, as quantum field counterterms. It is well known on the base of the power counting arguments that the 5D SYM is perturbatively non-renormalizable. Therefore we should expect an infinite number of divergent structures at any loop what leads to infinite number of counterterms. However, the superspace arguments and the requirements of on-shell supersymmetry rule out the first divergences in D-dimensional SYM theory. Actually the divergences can appear at L loops where $D=4+6/L^1$. Construction of the various supersymmetric, gauge invariant functionals in quantum field theory is conveniently formulated in the framework of the effective action. The low-energy effective action can be represented as a series in supersymmetric and gauge invariants with some coefficients. In general, the supersymmetry together with conformal symmetry imposes rigid constraints on these coefficients. In some cases, they can be determined exactly [11]. For example, the leading term in the low-energy effective action on the Abelian vector field background is $F^4$ which is generated only at one loop and is not renormalized at higher loops. A possible new non-renormalization theorems for Abelian $F^n$ was conjectured in [12]. Recently the authors of [13] systematically analyzed the effective action on the moduli space of $(2,0)$ superconformal field theories in six dimensions, as well as their toroidal compactification to maximally SYM theories in five and four dimensions. They presented an approach to non-renormalization theorems that constrain this effective action. The first several orders in its derivative expansion are determined by a one-loop calculation in five-dimensional SYM theory. In general, the functional form of the effective action at the first several orders in the derivative expansion can be obtained by integrating out the massive degrees of freedom in the path integral. However, it is difficult enough to perform exactly such an analysis for supersymmetric models in the component formulation.

Construction of the background field method in extended supersymmetric gauge theories faces a fundamental problem. The most natural and proper description of such theories should be formulated in terms of a suitable superspace and unconstrained super-

\footnote{For a discussion of this issue see [9], [10] and references therein.}
fields on it. Some time ago a systematic background field method to study the effective actions in 4D, $\mathcal{N} = 2$ supersymmetric field theories was developed in a series of papers [14], [15]. This method is based on formulation of $\mathcal{N} = 2$ theories in harmonic superspace [16], [17] and guarantees the manifest $\mathcal{N} = 2$ supersymmetry and gauge invariance at all stages of calculations. The method under consideration gives the possibility to calculate in a straightforward manner not only the holomorphic and non-holomorphic contributions to the low-energy effective action but also to study the full structure of the effective action. Evaluation of the effective action within the background field method is often accompanied by using the proper time or heat kernel techniques. These techniques allow us to sum up efficiently an infinite set of Feynman diagrams with increasing number of insertions of the background fields and to develop a background field derivative expansion of the effective action in a manifestly gauge covariant way.

The 5D SCS theories are superconformally invariant and, hence, their effective actions must be independent of any scale. Unlike the 4D, $\mathcal{N} = 2$, 4 supersymmetric theories where holomorphy allows one to get the chiral contributions to effective action [18], in the 5D case the contributions to the effective action can be written only either in full or in analytic superspaces. Then, taking into account the mass dimensions of the harmonic potential $V^{++}$ and superfield strength $\mathcal{W}$ as well as the dimensions of the superspace measures $d^5xd^8\theta$ and $d^5zd^8\theta$, one obtains that the most general low-energy effective U(1)-gauge invariant action in the analytic superspace is the SCS action [19], [20]. The next-to-leading effective Abelian 5D action can be written only at full superspace in terms of the manifestly gauge invariant functional $\Gamma = \int d^5xd^8\theta \mathcal{W} \mathcal{H}(\frac{\Lambda}{\mathcal{W}})$, where $\Lambda$ is some scale and $\mathcal{H}(\frac{\Lambda}{\mathcal{W}})$ is the dimensionless function of its argument. The requirement of scale invariance means an equation $\Lambda \frac{d}{d\Lambda} \int d^5zd^8\theta \mathcal{W} \mathcal{H}(\frac{\Lambda}{\mathcal{W}}) = 0$, where the only solution is $\mathcal{H} = c \ln \frac{\mathcal{W}}{\Lambda}$. Any perturbative or non-perturbative quantum corrections should be included into a single constant $c$. The component Lagrangian in the bosonic sector corresponding to the above effective action is $\frac{1}{\Lambda}(F^4 + (\partial \phi)^4 + \ldots)$, where $F$ is the Abelian strength of the component vector field from 5D, $\mathcal{N} = 1$ vector multiplet and $\phi$ is the corresponding scalar component.

In this paper we derive the leading contribution to the low-energy effective action in the 5D SCS theory using the harmonic superspace description of the theory and proper-time techniques. The result precisely corresponds to the above analysis and has the form $\int d^5xd^8\theta \mathcal{W} \ln \frac{\Lambda}{\mathcal{W}}$. Besides the effective action in the 5D SCS theory, we calculate the leading one-loop low-energy contribution in the 5D SYM theory. Although this theory is not superconformal and is characterized by the dimensional coupling constant, its leading contribution to the effective action has the same functional form as in the 5D SCS theory and does not depend on the scale and coupling constant. Also, we consider the effective action in the 5D hypermultiplet theory coupled to a background 5D vector multiplet. The leading low-energy contribution to effective action in this theory was calculated in paper [20], where it was shown that this contribution is the 5D SCS action. In the given paper we calculated the first next-to-leading term in the low-energy effective action for the theory under consideration and found that this term again has the same functional form as the leading term in 5D SCS theory.

The paper is organized as follows. Section 2 is devoted to a brief review of harmonic
superspace formulation of the 5D, $\mathcal{N} = 1$ supersymmetric field models such as the SYM theory, the hypermultiplet theory and the SCS theory. In section 3 we consider the Abelian 5D SCS theory and develop the manifestly supersymmetric and gauge invariant procedure for calculating the effective action. This procedure is based on the background field method and proper-time technique. We find the exact expression for one-loop effective action in terms of functional determinants of differential operators in analytic subspace of harmonic superspace and calculate the leading low-energy contribution to this effective action. In section 4 we develop the analogous procedure for 5D SYM theory and calculate the leading low-energy contribution to one-loop effective action. Section 5 is devoted to the study of the first next-to-leading contribution to effective action in the 5D hypermultiplet theory coupled to a 5D vector multiplet background. The last section is devoted to the summary of the results.

2 Review of the 5D, $\mathcal{N} = 1$ harmonic superspace approach

Various supersymmetric theories with eight supercharges admit the off-shell superfield formulations in terms of formalism of the harmonic superspace. The harmonic superspace approach for the 4D, $\mathcal{N} = 2$ theories was originally developed in [16]. The formulation for the 5D, $\mathcal{N} = 1$ models has been given in [19], [21], [22]. The harmonic superspace approach for the 6D, $(1,0)$ SYM theories was considered in [23], [24], [25] and for the 6D, $(1,1)$ SYM in [26]. The construction the super-de Rham complex in five-dimensional, $\mathcal{N} = 1$ superspace and its relationship to the complex of six-dimensional, $\mathcal{N} = (1, 0)$ superspace via dimensional reduction was considered in [27]. All these harmonic superspace formulations in space-time dimensions 4, 5 and 6 look almost analogous modulo some details. In a series of papers [28], [29], [30] an extensive program of constructing the manifestly supersymmetric formulation for the 5D, $\mathcal{N} = 1$ supergravity-matter models was realized and the universal procedures to construct manifestly 5D supersymmetric action functionals for a different supermultiples were developed. These superfield results are in agreement with the earlier component considerations of the same models [31], [32].

The study of the structure of the low-energy effective action in the 5D superconformal theories looks useful from the point of view of the classification of theories consistent at the quantum level. In ref [20] a manifestly 5D, $\mathcal{N} = 1$ supersymmetric, gauge covariant formalism for computing of the one-loop effective action for a hypermultiplet coupled to a background vector multiplet was developed. It was demonstrated, as a simple application, that the SCS action is generated at the quantum level on the Coulomb branch. The above paper was the only one where explicit one-loop harmonic superspace calculations were done. We believe that until now the possibilities of the covariant 5D harmonic superspace methods were non-sufficiently explored to study the effective action in 5D supersymmetric gauge theories. The aim of this paper is to develop the general manifestly supersymmetric and gauge invariant methods for 5D quantum supersymmetric gauge theories and apply these methods to calculate the low-energy effective action in 5D SCS and SYM theories.
In this section we briefly review the superspace description of a vector multiplet in 5D supersymmetric gauge theories. Our aim here is to fix our basic notation and conventions (for details see [21]).

The 5D gamma-matrices $\Gamma_\dot{a}$ are defined as follows $\{\Gamma_\dot{a}, \Gamma_\dot{b}\} = -2\eta_{\dot{a}\dot{b}}1$, with $\eta_{\dot{a}\dot{b}} = \{-1, 1, \ldots, 1\}$. The matrices $\{1, \Gamma_\dot{a}, \Sigma_{\dot{a}\dot{b}}\}$ form a basis in the space of $4 \times 4$ matrices. The charge conjugation matrix $C = (\varepsilon^{\dot{a}\dot{b}})$ and its inverse $C^{-1} = (\varepsilon_{\dot{a}\dot{b}})$ are used to raise and lower the spinor indices. The matrices $\varepsilon_{\dot{a}\dot{b}}$ and $(\Gamma_\dot{a})_{\dot{a}\dot{b}}$ are antisymmetric, while the matrices $(\Sigma_{\dot{a}\dot{b}})_{\dot{a}\dot{b}}$ are symmetric. The anticommuting variables $\theta_\dot{a}\dot{a}$ are assumed to obey the pseudo-Majorana reality condition $(\theta_\dot{a}^*)^* = \theta_\dot{a} = \varepsilon_{\dot{a}\dot{b}}\varepsilon^{\dot{i}\dot{j}}\theta_\dot{j}$. 5D, $\mathcal{N} = 1$ superspace is parameterized by the coordinates $z^M = (x^\alpha, \theta_\dot{a})$ where $i = 1, 2$ and $x^\alpha = (\Gamma_\dot{a})_{\dot{a}\dot{b}}x_\dot{b}$. The basic spinor covariant derivatives of the 5D, $\mathcal{N} = 1$ superspace are $D_\dot{a}^i = \frac{\partial}{\partial \theta_\dot{a}^i} - i\partial_\dot{a}\dot{b}\theta_\dot{b}$. They obey the anti-commutation relations

$$\{D_\dot{a}^i, D_\dot{b}^j\} = -2i\varepsilon^{ij}\partial_\dot{a}\dot{b}. \quad (1)$$

The $\mathcal{N} = 1$ harmonic superspace $\bar{R}^{5|8} \times S^2$ extends the conventional 5D, $\mathcal{N} = 1$ Minkowski superspace $\bar{R}^{5|8}$ with the coordinates $z^M = (x^\alpha, \theta_\dot{a})$, by the two-sphere $SU(2)/U(1)$ parameterized by harmonics, i.e., group elements

$$u_i^\pm \in SU(2), \quad u^{*i} = u_i^-, \quad u^{i+}u_i^- = 1.$$  

The main conceptual advantage of harmonic superspace is that the $\mathcal{N} = 1$ vector multiplet as well as hypermultiplets can be described by unconstrained superfields on the analytic subspace of $\bar{R}^{5|8} \times S^2$ parameterized by the coordinates $\zeta^M = (x_A^\alpha, \theta^{\dot{a}}, u_i^\pm)$, where

$$x_A^\alpha = x^\alpha + i\theta^{\dot{a}}\Gamma_{\dot{a}\dot{b}}\theta^{\dot{b}}, \quad \theta^\pm = u^\pm u^\pm i.$$  

The important property of the coordinates $\zeta^M$ is that they form a subspace closed under $\mathcal{N} = 1$ supersymmetry transformations. In the coordinates $\zeta^M$ the spinor covariant derivatives $D_\dot{a}^i = u_i^+ D_\dot{a}^i$ have a short form (see Eqs (5)) and therefore the superfield $\Phi(x^\alpha, \theta^{\dot{a}}, u_i^\pm)$ satisfying the constraints $D_\dot{a}^i\Phi = 0$ is an analytic superfield $\Phi(\zeta, u)$. It leads to reducing the number of the anticommuting coordinates and, hence, to reducing the number of independent components in comparison with general superfields. However, all component fields depend now on extra bosonic coordinates $u_i^\pm$.

In harmonic superspace a full set of gauge covariant derivatives includes the harmonic derivatives which form a basis in the space of left-invariant vector fields of $SU(2)$:

$$D^{++} = u^{i+} \frac{\partial}{\partial u^{i-}}, \quad D^{-i} = u^{-i} \frac{\partial}{\partial u^{i+}}, \quad D^0 = u^{i+} \frac{\partial}{\partial u^{i+}} - u^{-i} \frac{\partial}{\partial u^{-}}, \quad (3)$$

$$[D^0, D^{\pm}] = \pm 2D^{\pm}, \quad [D^{++}, D^{--}] = D^0. \quad (4)$$

The generator of the $SU(2)$ algebra $D^0$ is an operator of harmonic charge, $D^0 \Phi^{(q)} = q\Phi^{(q)}$. In the analytic basis, parameterized by the coordinates $\zeta^M$ (2) the spinor covariant
derivatives: \( D_\alpha^+ = u_i^+ D_\alpha^i \) and the harmonic derivatives take the form

\[
D_\alpha^+ = \frac{\partial}{\partial \theta^{-\alpha}}, \quad D_\alpha^- = -\frac{\partial}{\partial \theta^{+\alpha}} - 2i \partial \tilde{\alpha} \theta^{-\tilde{\beta}},
\]

\[
D^{++} = u^+ \frac{\partial}{\partial u^{-i}} + i \theta^{+\tilde{\alpha}} \partial_{\tilde{\alpha} \tilde{\beta}} \theta^{+\tilde{\beta}} + \theta^{+\tilde{\alpha}} \frac{\partial}{\partial \theta^{-\tilde{\alpha}}}, \quad D^{--} = u^- \frac{\partial}{\partial u^{+i}} + i \theta^{-\tilde{\alpha}} \partial_{\tilde{\alpha} \tilde{\beta}} \theta^{-\tilde{\beta}} + \theta^{-\tilde{\alpha}} \frac{\partial}{\partial \theta^{+\tilde{\alpha}}},
\]

\[
D^0 = u^+ \frac{\partial}{\partial u^{+i}} - u^- \frac{\partial}{\partial u^{-i}} + \theta^+ \frac{\partial}{\partial \theta^+} - \theta^- \frac{\partial}{\partial \theta^-}.
\]

In this basis Eqs (1), (4) leads to

\[
\{D_\alpha^+, D_\beta^+\} = 0, \quad [D^{\pm \pm}, D^{\pm \pm}_\alpha] = 0, \quad [D^{\pm \pm}, D^{\pm \mp}_\alpha] = D^\mp_\alpha,
\]

\[
\{D_\alpha^+, D_\beta^-\} = 2i \partial \tilde{\alpha} \tilde{\beta}.
\]

These relations are necessary integrability conditions for the existence of the analytic superfields. Since the field in the harmonic superspace depend on the additional bosonic variable \( u^i \), we must define rules of integration over harmonics (that is, over the group manifold \( SU(2)/U(1) \)). The basic harmonic integrals have the form [17]

\[
\int du = 1, \quad \int du u^+_1 \ldots u^+_n u^-_1 \ldots u^-_m = 0, \quad n + m > 0.
\]

It means that the harmonic integrals are nonzero only if the integrand has the zero \( U(1) \) charge.

### 2.1 5D SYM theory in harmonic superspace

To describe a Yang-Mills supermultiplet in 5D conventional superspace we introduce the gauge covariant derivatives \( \mathcal{D}_A = D_A + i A_A \) where \( D_A = (\partial_a, D_\alpha^a) \) are the flat covariant derivatives and \( A_A \) is the gauge connection taking values in the Lie algebra of the gauge group. The operator \( \mathcal{D}_A \) satisfies the gauge transformation law \( \mathcal{D}_A \rightarrow e^{i \tau(z)} \mathcal{D}_A e^{-i \tau(z)} \), \( \tau^\dagger = \tau \) with a superfield gauge parameter \( \tau(z) \). The gauge covariant derivatives are required to obey some constraints [33]

\[
\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = -2i \varepsilon^{ij} (\mathcal{D}_{\tilde{\alpha} \tilde{\beta}} + \varepsilon_{\tilde{\alpha} \tilde{\beta}} i \mathcal{W}),
\]

\[
[\mathcal{D}_\alpha^i, \mathcal{D}_\alpha] = -i (\Gamma_{\tilde{\alpha} \tilde{\beta}})^\tilde{\beta} \mathcal{D}_\beta^i \mathcal{W}, \quad [\mathcal{D}_\alpha, \mathcal{D}_\beta] = -\frac{1}{4} (\Sigma_{\tilde{a} \tilde{b}})^{\tilde{a} \tilde{b}} \mathcal{D}_\alpha^i \mathcal{D}_\beta^j \mathcal{W} = i F_{\tilde{a} \tilde{b}},
\]

with the matrices \( \Gamma_{\tilde{\alpha}} \) and \( \Sigma_{\tilde{a} \tilde{b}} \) defined above\(^2\). Here the field strength \( \mathcal{W} \) is Hermitian, \( \mathcal{W}^\dagger = \mathcal{W} \) and obeys the Bianchi identity

\[
\mathcal{D}_\alpha^{(i} \mathcal{D}_\beta^{j)} \mathcal{W} = \frac{1}{4} \varepsilon_{\tilde{a} \tilde{b}} \mathcal{D}_\alpha^{(i} \mathcal{D}_\beta^{j)} \mathcal{W}.
\]

\(^2\)The properties of these matrices are discussed in [21], [29].
In harmonic superspace the full set of gauge covariant derivatives includes the harmonic derivatives. In this basis Eqs. (8), (5) leads to

\[ \{ \mathcal{D}_a^+, \mathcal{D}_b^- \} = 0, \quad [D^\pm, \mathcal{D}_a^\pm] = 0, \quad [D^\pm, \mathcal{D}_a^T] = \mathcal{D}_a^\pm, \quad (10) \]

\[ \{ \mathcal{D}_a^+, \mathcal{D}_b^+ \} = 2i\mathcal{D}_{a\dot{\beta}} - 2\varepsilon_{a\dot{\beta}}W, \quad [D_7^\dot{\gamma}, \mathcal{D}_a^\pm] = i\{ \varepsilon_{a\dot{\beta}}D_7^{\dot{\gamma}}W + 2\varepsilon_{\dot{\gamma}\dot{\alpha}}D_7^\dot{\alpha}W - 2\varepsilon_{\dot{\gamma}\dot{\beta}}D_7^\dot{\beta}W \}, \]

\[ i\mathcal{F}_{\tilde{a}\tilde{b}} = \frac{1}{2}\mathcal{D}_a^-(\Sigma_{a\tilde{b}})\mathcal{D}_\tilde{b}^+W. \]

The integrability condition \( \{ \mathcal{D}_a^+, \mathcal{D}_b^+ \} = 0 \) is solved by \( \mathcal{D}_a^+ = e^{-iv}\mathcal{D}_a^- e^{iv} \) with some Lie-algebra valued harmonic superfield \( v(z, u) \) which is called the bridge. The bridge possesses a richer gauge freedom than the original \( \tau \)-group

\[ e^{iv(z,u)} = e^{i\lambda(z,u)} e^{iv} e^{-ir(z)}, \quad D_a^+ \lambda = 0. \quad (11) \]

The \( \lambda \)- and \( \tau \)-transformations generate, respectively, the so-called \( \lambda \)- and \( \tau \)-groups. In the \( \lambda \)-frame the spinor covariant derivatives \( \mathcal{D}_a^\pm \) coincide with the flat ones, while the harmonic covariant derivatives acquire connections \( \mathcal{D}^\pm = D^\pm + iV^\pm \). The real connection \( V^++ \) is an analytic superfield, \( D_a^+V^++ = 0 \) and its transformation law is

\[ V^{++} \prime = e^{i\lambda}V^{++}e^{-i\lambda} - ie^{i\lambda}D^{++}e^{-i\lambda}. \quad (12) \]

The gauge freedom \( -\delta V^+ = \mathcal{D}^{++} \lambda \) can be used to impose the Wess-Zumino gauge in the form

\[ V^{++} = (\theta^+)^2 i\phi + i\theta^+\lambda A_{\alpha\dot{\alpha}}\theta^+\dot{\alpha} + 4(\theta^+)^2 \theta^+\lambda^\alpha - \frac{3}{2}(\theta^+)^2(\theta^+)^2 Y^{--}. \quad (13) \]

In this gauge, the superfield \( V^{++} \) contains the real scalar field \( \phi \), the Maxwell field \( A_{\dot{m}} \), the isodoublet of spinors \( \lambda_\alpha \) and the auxiliary isotriplet \( Y^{(ij)} \). The analytic superfield \( V^{++} \) turns out to be the single unconstrained potential of the pure 5D, \( \mathcal{N} = 1 \) SYM theory and all other quantities, associated with this theory, are expressed in its terms. In particular, the other harmonic connection \( V^{--} \) turns out to be uniquely determined in terms of \( V^{++} \) using the zero-curvature condition \( [\mathcal{D}^{++}, \mathcal{D}^{--}] = 0 \). The result looks like [35]

\[ V^{--}(z,u) = \sum_{n=1}^{\infty} (-i)^{n+1} \int du_1 \ldots du_n \frac{V^{++}(\zeta, u_1) \ldots V^{++}(\zeta, u_n)}{(u^+ u_1^+)(u_1^+ u_2^+)(u_n^+ u_n^+)\ldots(u_n^+ u^+)}, \quad (14) \]

where \( (u_1^+ u_2^+) = u_1^+ u_2^+ \). The details of the harmonic analysis on \( S^2 = SU(2)/U(1) \) were designed in the pioneering works [16], [34], [35], [17]. In the \( \lambda \)-basis the connections \( A_{\dot{a}} \), \( A_{\dot{m}} \) and the field strength can be expressed in terms of \( V^{--} \) using the relations (10):

\[ A_{\dot{a}} = -D_{\dot{a}}^- V^{--}, \quad W_\lambda = i\frac{1}{8} D_\dot{a}^+ D_{\dot{a}}^+ V^{--}, \quad \mathcal{F}_{\tilde{a}\tilde{b}} = -i\frac{1}{2}\mathcal{D}_{\dot{a}}^- D_{\dot{a}}^+ W. \quad (15) \]

The superfield strength \( W_\lambda \) satisfies the following constraints \( \mathcal{D}^{++} W_\lambda = D^{++} W_\lambda + i[V^{++}, W_\lambda] = 0 \). In the Abelian case, the superfield \( W_\lambda = \frac{1}{8} \int du(D^-)^2 V^{++} \) does not
depend on harmonics and moreover, in this case there is no distinction between $\mathcal{W}_\lambda$ and $\mathcal{W}$. The Bianchi identity (9) takes the form
\[ D_\alpha^+ D_\beta^+ \mathcal{W} = \frac{1}{4} \varepsilon_{\alpha \beta} (D^+)^2 \mathcal{W}, \]
where we have used the notation $(D^+)^2 = D^{+\dot{\alpha}} D^{+\dot{\alpha}}$. Using these identities, the authors of [21] built an important covariantly analytic superfield $\mathcal{G}^{++}$:
\[ -i \mathcal{G}^{++} = D^{+\dot{\alpha}} \mathcal{W} D_{\dot{\alpha}}^+ \mathcal{W} + \frac{1}{4} \{ \mathcal{W}, (D^+)^2 \mathcal{W} \}, \quad D_{\dot{\alpha}}^+ \mathcal{G}^{++} = 0, \quad D^{++} \mathcal{G}^{++} = 0. \]
This superfield can be transformed to the form
\[ \mathcal{G}^{++} = (D^+)^4 (V^- \mathcal{W}), \]
where $(D^\pm)^4 = -\frac{1}{32} (D^\pm)^2 (D^\pm)^2$ and $\frac{1}{2} (D^+)^4 (D^-)^2 \Phi(\zeta) = -\frac{1}{4} D^{\dot{\alpha} \dot{\beta}} D_{\dot{\alpha} \dot{\beta}} \Phi(\zeta)$.

Unlike the 4D, $\mathcal{N} = 2$ case, the chiral superspaces are not Lorentz-covariant in the case of 5D, so that to construct the superfield actions we can use only the full superspace or else the analytic superspace. In the full harmonic superspace the 5D SYM action has the universal form [35]
\[ S_{\text{SYM}}(V^{++}) = \frac{1}{g_{\text{SYM}}^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{tr} \int d^3 z d u_1 \ldots d u_n \frac{V^{++}(z, u_1) \ldots V^{++}(z, u_n)}{(u_1^1 u_2^2) \ldots (u_n^1 u_1^1)}, \]
where $g_{\text{SYM}}^2$ is the coupling constant of dimension $[g_{\text{SYM}}^2] = 1$. The SYM action in terms of the component fields defined by (13) is
\[ S_{\text{SYM}} = \frac{1}{g_{\text{SYM}}^2} \int d^5 x \{ -\frac{1}{4} F^{a b} F_{a b} - \frac{1}{2} D^a \phi D_a \phi + \frac{1}{4} V^{i j} Y_{i j} + \frac{i}{2} \lambda^I P \lambda_i - \frac{1}{2} \lambda^I [\phi, \lambda_i] \}. \]
The SYM equations of motions $(D^+)^2 \mathcal{W} = 0, \quad \Box \mathcal{W} = 0$ have a vacuum Abelian solution $V^{\pm \pm} = i (\theta^{\pm \pm})^2 Z$ where $Z$ is the linear combination of the Cartan generators of the gauge group [19]. This vacuum solution spontaneously breaks the gauge symmetry, but it conserves the 5D supersymmetry with the central charge and produces BPS masses of the $Z$-charged fields. In addition, let us note that on-shell, the degrees of freedom $(1+3)_B + 4_F$ in $\mathcal{W}$ match the degrees of freedom in a 5D tensor multiplet described by the superfield $\Phi$ subject to the constraint $D_{\dot{\alpha}}^i D_{\dot{\beta}}^j \Phi = 0$. The $\theta$-expansion of $\Phi$ is
\[ \Phi = \varphi + \theta^\dot{\alpha} i \lambda^i_{\dot{\alpha}} + \theta^\dot{\alpha} i \theta^\dot{\beta} H_{\dot{\alpha} \dot{\beta}} + \ldots, \]
where $H_{\dot{\alpha} \dot{\beta}} = \frac{1}{2} (\Sigma_{\dot{a} \dot{b}})_{\dot{\alpha} \dot{\beta}} H_{ab}$ is dual to the 3-form field strength $F_{a b c}$ of the 2-form gauge field $B_{\dot{a} \dot{b}}$. In this situation the vector massless representation is equivalent under duality to a tensor representation.

The other universal procedure to construct 5D manifestly supersymmetric actions is based on ideas developed in [36]. Let us consider two vector multiplets, namely a $U(1)$
vector multiplet $V^{++}_\Delta$ and a Yang-Mills vector multiplet $V^{++}_{SYM}$. They can be coupled in a gauge-invariant way, using the interaction

$$S_{SYM} = \int d\zeta (-4) V^{++}_\Delta \text{tr} G^{++}_{SYM},$$

(22)

where $G^{++}_{SYM}$ corresponds to a non-Abelian multiplet and is defined in eq. (17). If we assume that the physical scalar field in $V^{++}_\Delta$ possesses a non-vanishing expectation value $\langle V^{++}_\Delta(\zeta, u) \rangle = i(\theta^+)^2 \frac{1}{g^2_{YM}}$, then one gets

$$S_{SYM} = \frac{1}{g^2_{YM}} \int d\zeta (-4) i(\theta^+)^2 \text{tr} G^{++}_{SYM}.$$ 

(23)

The integration in (22), (23) is carried out over the analytic subspace of the harmonic superspace

$$\int d\zeta (-4) \equiv \int d^5 x d\hat{u}(\mathcal{D}^-)^4.$$ 

(24)

2.2 The super Chern-Simons model

Let us consider the 5D harmonic superspace formulation of a supersymmetric Chern-Simons (SCS) model. The off-shell non-Abelian SCS action in five dimensions was first constructed in [32]. SCS model possesses remarkable properties, which makes this theory very interesting for various applications. First of all, in five dimensions the pure $U(N)$ and $SU(N)$ Chern-Simons theories have superconformal fixed points [4]. Another reason to study the SCS models is that they can be generated by integrating out the massive hypermultiplets in the SYM theory [2], [6] when the hypermultiplets transform in complex representations of the gauge group. For example, if we consider the masses of the hypermultiplets as the UV cut-off, we obtain the Chern-Simons term in the one-loop correction to the classical theory. Hence inclusion of this term can be important in some cases to get a complete description of the theory. In a manifestly supersymmetric setting, where the entire vector supermultiplet is taken into account, the corresponding one-loop calculation was given in [20], both in the Coulomb and non-Abelian phases. Using the covariant harmonic supergraphs and the heat kernel techniques in harmonic superspace [14], [15], it was shown that the hypermultiplet effective action contains the SCS term. Finally Chern-Simons theory can be important in the relation between the 5D SYM and 6D $(2,0)$ theories. In particular, one can argue [6] that the 5D Chern-Simons term can be generated by the anomaly terms in the six-dimensional theory.

In general, in space-time with dimension $(2n - 1)$, the action of the Chern-Simons theory can be constructed using the Chern-Simons form $\Sigma_{2n-1}$, defined as $d\Sigma_{2n-1} = \text{tr} [F^n]$ where $F = dA + iA \wedge A$ is the gauge field strength two-form and its powers $F^n$ are defined by the wedge product. In three space-time dimensions this form gives rise to the famous 3D Chern-Simons action. In the 5D Chern-Simons theory the action is given by

$$S_{CS} = \frac{k}{12} \int d^5 x \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} \text{tr} \{ A_{\hat{a}} F_{\hat{b}\hat{c}} F_{\hat{d}\hat{e}} - i A_{\hat{a}} F_{\hat{b}\hat{c}} F_{\hat{d}\hat{e}} - \frac{2}{5} A_{\hat{a}} A_{\hat{b}} A_{\hat{c}} A_{\hat{d}} A_{\hat{e}} \},$$

where $k$ is the level of the Chern-Simons theory.
where $k$ is the Chern-Simons level which plays the role of the coupling constant. Here $A_\hat{a}$ is the gauge field with the gauge group $SU(N)$ or $U(N)$. The field $A_\hat{a}$ transforms under the gauge transformation $g$ as $A_\hat{a} \rightarrow g^{-1}A_\hat{a}g - ig^{-1}\partial_\hat{a}g$. Under this transformation, $S_{CS}$ gets an additional term $\delta S_{CS}$ given, modulo a total derivative, by $\delta S_{CS} = 2\pi ik Q(g)$, where $Q(g)$ takes only integer values. Like in the case of three-dimensional Chern-Simons theory, gauge invariance of the partition function leads to $k \in \mathbb{Z}$. However, despite the fact that the action of the theory $CS$ does not depend on the metric and thus the theory is topological, the five (and, generally, any higher odd) dimensional case admits local propagating degrees of freedom [37].

Unlike the component construction of [32], a closed-form expression for the non-Abelian SCS action has never been given in terms of the superfields. Here there exists only a unique definition of the variation of the SCS action with respect to an infinitesimal deformation of the gauge potential $V^{++}$. However, as it was noted in [30], it is unknown how to integrate this variation in a closed form to obtain the action as an integral over the superspace. The component formulation of the non-Abelian SCS model can be constructed in the framework of the superform approach [30], where a closed-form expression for the non-Abelian SCS action was given. The superfield analysis in the Abelian case is more transparent and the SCS action was derived in the 5D $\mathcal{N} = 1$ harmonic [19], [21] and projective superspaces [29].

The approach of constructing the manifestly supersymmetric actions developed in [19], [21] leads to Abelian SCS action in the form

$$S_{SCS} = \frac{1}{12g^2} \int d\zeta^{(-4)} V^{++} G^{++}, \quad D_\hat{a}^+ G^{++} = D^{++} G^{++} = 0.$$ (25)

The action (25) is invariant under the gauge transformations $-\delta V^{++} = D^{++} \lambda$. The equations of motions for the model with such an action are

$$-iG^{++} = D^{++} \mathcal{W} \dot{D}_\hat{a}^+ \mathcal{W} + \frac{1}{2} \mathcal{W}(D^+)^2 \mathcal{W} = 0.$$ (26)

The Abelian SCS theory with the superfield action (25) leads to the following component action:

$$S_{SCS} = \frac{1}{2g^2} \int d^5 x \{ \frac{1}{3} \varepsilon^{\hat{a} \hat{b} \hat{c} \hat{d}} A_\hat{a} F_{\hat{b} \hat{c}} F_{\hat{d}} - \frac{1}{2} \phi F_{\hat{a} \hat{b}} F_{\hat{a} \hat{b}} - \phi \dot{\phi} \partial_\hat{a} \phi + \frac{1}{2} \phi Y_{ij} Y_{ij} \}.$$ (27)

The action (27) clearly shows that the five dimensional Abelian SCS theory is a non-trivial interacting field model (see e.g. [38]).

The theory (27) is superconformal at the classical level and the coupling constant $g^2$ is dimensionless. The latter can mean a renormalizability of the theory. However, the action (27) does not involve a quadratic part and perturbative calculations, based on the weakness of the interaction term with respect to the free part, are impossible. However,
we can use the presence of the vacuum moduli space $\langle \phi \rangle = m$ in the Lagrangian (27). This allows one to decompose the Lagrangian into the free and interaction parts and construct the $S$ matrix in a conventional way. However in this case, conformal symmetry of the original Lagrangian is broken spontaneously and the mass parameter in the action makes the theory nonrenormalizable.

2.3 5D $\mathcal{N} = 1$ hypermultiplet in harmonic superspace

Here we briefly discuss the hypermultiplet formulation in harmonic superspace.

On mass-shell, the 5D, $\mathcal{N} = 1$ massless hypermultiplet contains two complex scalar fields forming an isodoublet of the automorphism group $SU(2)_A$ of the supersymmetry algebra (1) and an isosinglet Dirac spinor field. The description of the off-shell hypermultiplet in terms of the analytical subspace of harmonic superspace is completely analogous to the description of a 4D, $\mathcal{N} = 2$ hypermultiplet. Like in the four-dimensional 4D, $\mathcal{N} = 2$ supersymmetric theory [16], the off-shell hypermultiplet coupled to the vector supermultiplet is described by a superfield $q^+(\zeta)$ and its conjugate $\bar{q}^+(\zeta)$ with respect to the analyticity preserving conjugation [17]. The classical action for a massless hypermultiplet coupled to the background 5D, $\mathcal{N} = 1$ vector multiplet is

$$S_{\text{hyper}} = -\int d\zeta (-4)\bar{q}^+ D^{++} q^+. \quad (28)$$

The hypermultiplet that transforms in a real representation of the gauge group, on-shell has scalars in the representations $(1 + 3)$ of $SU(2)_A$ and an $SU(2)_A$ doublet of pseudo-Mayorana fermions. It can be described by a real analytic superfield $\omega(\zeta)$. Such a superfield is called the $\omega$-hypermultiplet. The action describing an interaction of this hypermultiplet with a vector supermultiplet is written in the form

$$S_{\omega} = -\frac{1}{2} \int d\zeta (-4)(D^{++}\omega)^2. \quad (29)$$

3 The background field formulation for quantum $\mathcal{N} = 1$ super Chern-Simons

In this section we will construct the background field method for the superfield theory with action (25).

The harmonic superfield background field method (see construction of this method for 4D, $\mathcal{N} = 2$ SYM theory in [14]) is based on the so-called background-quantum splitting of the initial gauge field into two parts: the background field $V^{++}$ and the quantum field $v^+$

$$V^{++} \to V^{++} + v^{++}. \quad (30)$$

To quantize the theory, one imposes the gauge fixing conditions only on the quantum field, introduces the corresponding ghosts and considers the background field as the functional
argument of the effective action. Then, the original infinitesimal gauge transformations (12) can be realized in two different ways: as the background transformations

\[ \delta V^{++} = - D^{++} \lambda - i[V^{++}, \lambda] = - D^{++} \lambda, \quad \delta v^{++} = i[\lambda, v^{++}], \]  

and as the quantum transformations

\[ \delta V^{++} = 0, \quad \delta v^{++} = - D^{++} \lambda - i[v^{++}, \lambda]. \]  

To construct an effective action as a gauge-invariant function of the background superfield \( V^{++} \) we will use another form of writing the action (25)

\[ S_{SCS} = \frac{1}{12 g^2} \text{tr} \int d\zeta (-4) V^{++} G^{++} = \frac{1}{12 g^2} \text{tr} \int d^5 x d^8 \theta du \ V^{++} V^{--} W_\lambda, \]  

and expand the action \( S[V^{++} + v^{++}] \) in powers of the quantum field \( v^{++} \):

\[ S = \sum_{n=1}^{\infty} \text{tr} \int d^{13} z_1 d u_1 \ldots d z_n d u_n \frac{1}{n!} \delta^{n} S \frac{\delta^n S}{\delta v^{++}(1) \ldots \delta v^{++}(n)} |_{v^{++}=0} v^{++}_\tau(1) \ldots v^{++}_\tau(n). \]  

Here \( W_\lambda \), and \( v^{++}_\tau \) denote the \( \lambda \)- and \( \tau \)-frame forms of \( W \), and \( v^{++} \) respectively

\[ W_\lambda = e^{i v} W, e^{-i v}, \quad v^{++}_\tau = e^{-i v} v^{++} e^{i v}. \]  

Each term in the action (34) is manifestly invariant with respect to the background gauge transformations. The first variation of the action is

\[ \delta S_{SCS} = \frac{1}{4 g^2} \text{tr} \int d^{13} z d u \delta V^{++} V^{--} W_\lambda = \frac{1}{4 g^2} \text{tr} \int d^{13} z d u v^{++}_\tau V^{--} W_\tau, \]  

where \( v^{++}_\tau = e^{-i v} \delta V^{++} e^{i v} \). It depends on \( V^{++} \) via the dependence of \( v^{++}_\tau \) on the bridge \( v \). The term linear in \( v^{++} \) determines the equations of motion. This term should be dropped when one considers the effective action.

To determine the second variation of the action it is necessary to express \( \delta V^{--} \) via \( \delta V^{++} \). A variation of \( V^{++} = -i e^{i v} (D^{++} e^{-i v}) \) can be represented as \( e^{-i v} \delta V^{++} e^{i v} = i D^{++} (e^{-i v} \delta e^{i v}) = \delta V^{++}_\tau \). This equation is solved in the form

\[ (e^{-i v} \delta e^{i v}) = -i \int d u_1 \frac{(u^+_1 u^-_1)}{(u^+_1 u^-_1)} \delta V^{++}_\tau(u_1). \]  

Here we use the identity

\[ -\frac{1}{32} (D^+)^2 (D^+)^2 [V^{--} W] = -\frac{1}{32} (D^+)^2 \{-8 i W W + 2 D^+ \beta V^{--} D^+ \beta W + V^{--} (D^+)^2 W\} \]

\[ = i (D^+ \beta W D^+ \beta W + \frac{1}{4} (W, (D^+)^2 W)) \]
Now
\[ \delta V^{-} = iD^{-} (e^{-iv} \delta e^{iv}) = D^{-} \int du \frac{(u^{+}u_{1}^-)}{(u^{+}u_{1}^+)^2} \delta V^{++}(u_{1}) = \int du \frac{\delta V^{++}(u_{1})}{(u^{+}u_{1}^+)^2}. \] (37)

Calculating the second variation of the action yields the result
\[ \delta^2 S_{SCS} = \frac{1}{2} \int d^5 x d^8 \theta \frac{du du_{2}}{(u^{+}u_{2}^+)^2} \frac{1}{g^2(\mathcal{W})} \delta V^{++}(u_{1}) \delta V^{++}(u_{2}). \] (38)

Here
\[ \frac{1}{g^2(\mathcal{W})} = \mathcal{W}. \] (39)

These two expressions illustrate the specific feature of 5D, \( \mathcal{N} = 1 \) SCS theory as a theory with local coupling constant (since there is a non-trivial background-dependent factor in the vector field kinetic operator) [14].

In the framework of the background field method, we should fix only the quantum field gauge transformations. As to the superfield \( \mathcal{W} \) it is invariant under these transformations. Let us introduce the gauge fixing function in the form
\[ F^{(+4)} = D^{++}v^{++}, \] (40)

which changes by the law
\[ \delta F^{(+4)} = (D^{++}(D^{++}\lambda + i[v^{++}, \lambda])), \] (41)

under the quantum gauge transformations. Eq. (41) leads to the Faddeev-Popov determinant and to the corresponding ghost action
\[ S_{FP} = \text{tr} \int d\zeta^{(-4)} du b (D^{++})^2 c. \] (42)

Next we average the \( \delta(F^{(+4)} - f^{(+4)}) \) with the weight
\[ 1 = \Delta[V^{++}] \exp\left\{ \frac{1}{2\alpha} \int d^3z du_{1} du_{2} f^{(+4)}(u_{1}) \mathcal{W} \left( \frac{u_{1}^- u_{2}^-}{(u_{1}^+ u_{2}^+)^3} \right) f^{(+4)}(u_{2}) \right\}. \] (43)

Here \( \alpha \) is an arbitrary gauge parameter, \( \Delta[V^{++}] \) is determinant of the Nielson-Kallosh ghosts and \( f^{(+4)} = e^{-iv} f^{(+4)} e^{iv} \) is a tensor of the \( \tau \)-group. Note that we need the insertion of the superfield \( \mathcal{W} \) to balance the dimensions. The functional \( \Delta[V^{++}] \) is chosen from the condition
\[ \Delta^{-1}[V^{++}] = \int D f^{(+4)} \exp\left\{ \frac{1}{2\alpha} \int d\zeta^{(-4)} d\zeta^{(-4)} f^{(+4)}(1) \hat{A}(1, 2) f^{(+4)}(2) \right\} = \text{Det}^{-1/2} \hat{A}. \]

The background field dependent operator \( \hat{A} \) has the form
\[ \hat{A} = \left( \frac{u_{1}^- u_{2}^-}{(u_{1}^+ u_{2}^+)^3} \right) (D_{1}^{+})^4 \mathcal{W}(D_{2}^{+})^4 \delta^{13}(z_{1} - z_{2}), \]
and act on the space of analytic superfields. Thus
\[ \Delta[V^{++}] = \text{Det}^{1/2} \hat{A}. \] (44)

To find \( \text{Det} \hat{A} \) we represent it by a functional integral over analytic superfields of the form
\[ \text{Det}^{-1}(\hat{A}) = \int \mathcal{D} \chi^{(4)} \mathcal{D} \rho^{(4)} \exp \left\{ -i \text{tr} \int d\zeta^{(-4)} d\zeta^{(-4)} \chi^{(4)}(1) \hat{A}(1|2) \rho^{(4)}(2) \right\}, \] (45)
and perform the following replacement of functional variables
\[ \rho^{(4)} = (\mathcal{D}^{++})^2 \sigma, \quad \text{Det} \frac{\delta \rho^{(4)}}{\delta \sigma} = \text{Det}(\mathcal{D}^{++})^2. \] (46)

Further, repeating the construction of the effective action from the work [34] we obtain
\[ \int d^{13}z du_1 du_2 \chi^{(4)}_1 \mathcal{W}(u_1^- u_2^-) (\mathcal{D}^{++})^2 \sigma_2 = \int d^{13}z du \chi^{(4)}_r \mathcal{W} \frac{1}{2} (\mathcal{D}^{--})^2 \sigma_r \]
\[ = \int d\zeta^{(-4)} \chi^{(4)} \hat{\Box} \mathcal{W} \sigma, \]
where
\[ \hat{\Box} = \frac{1}{2} (\mathcal{D}^{++})^4 \mathcal{W} (\mathcal{D}^{--})^2, \] (47)
is the deformed version of the covariant analytic d’Alembertian [21]
\[ \Box = \frac{1}{2} (\mathcal{D}^{++})^4 (\mathcal{D}^{--})^2. \] (48)

Eventually we get the representation of \( \Delta[V^{++}] \) by the following functional integral
\[ \Delta[V^{++}] = \text{Det}^{-1/2} (\mathcal{D}^{++})^2 \text{Det}^{1/2} (\hat{\Box}) = \text{Det}^{1/2} (\hat{\Box}) \int \mathcal{D} \varphi e^{iS_{NK} |\varphi, V^{++}|}. \] (49)

Here \( \text{Det}_{(q,4-q)}(H) \) is defined as follows
\[ \text{Det}_{(q,4-q)} \hat{H} = e^{\text{Tr}_{(q,4-q)} \ln \hat{H}}, \] (50)
and the functional trace of operators acting on the space of covariantly analytic superfields of \( U(1) \) charges \( q \) and \( 4 - q \) is
\[ \text{Tr}_{(q,4-q)} \hat{H} = \text{tr} \int d\zeta^{(-4)} \mathcal{H}^{(q,4-q)}(\zeta, \zeta), \] (51)
where ‘tr’ is the matrix trace and \( \mathcal{H}^{(q,4-q)}(\zeta_1, \zeta_2) \) is the kernel of the operator. Superfield \( \varphi \) in (49) is a bosonic real analytic superfield, the Nielsen-Kallosh ghost, with the action
\[ S_{NK} = -\frac{1}{2} \int d\zeta^{(-4)} \mathcal{D}^{++} \varphi \mathcal{D}^{++} \varphi. \] (52)
Upon averaging the effective action \( \Gamma_{SCS}[V^{++}] \) with the weight (43), one gets the following path integral representation of the one-loop effective action \( \Gamma_{SCS}^{(1)}[V^{++}] \) for the Abelian 5D SCS theory

\[
e^{i\Gamma_{SCS}^{(1)}[V^{++}] = e^{iS_{SCS}[V^{++}] \int Dv^{++} Db Dc D\phi e^{iS_{quant}[v^{++},b,c,\phi,V^{++}]}, (53)}
\]

where

\[
S_{quant} = \Delta S_{SCS}[v^{++}, V^{++}] + S_{gh}[v^{++}, V^{++}] + S_{FP}[b, c, v^{++}, V^{++}] + S_{NK}[\phi, V^{++}]. (54)
\]

Here \( S_{gh}[v^{++}, V^{++}] \) is the gauge fixing contribution to the action of quantum superfields

\[
S_{gh}[v^{++}, V^{++}] = \frac{1}{2\alpha} \int d^{13}z du_1 du_2 D_1^{++} v_1^{++} W_1 (u_1^- u_2^-) (u_1^+ u_2^+) D_2^{++} v_2^{++} (55)
\]

The sum of the quadratic part \( \Delta S_{SCS} \) in quantum superfield \( v^{++} \) and \( S_{gh} \) (55) for special values \( \alpha = -1 \) has the form

\[
-\frac{1}{2} \int d\zeta (-4) du v^{++} (D^+) W(D^{--})^2 v^{++} = -\frac{1}{2} \int d\zeta (-4) du v^{++} \triangle_W v^{++}. (56)
\]

Eqs. (53)-(54) completely determine the structure of the perturbation expansion to calculate the one-loop effective action \( \Gamma_{SCS}^{(1)}[V^{++}] \) of the pure \( N = 1 \) SCS theory in a manifestly supersymmetric and gauge invariant form\(^\text{4}\). The complete one-loop effective action is given by the sum of the contributions coming from the ghost superfields and from the quantum superfields \( v^{++} \). It has the form

\[
\tilde{\Gamma}_{SCS}^{(1)} = i \left[ \frac{1}{2} \text{Tr} \ln(D^{++})^2 - \text{Tr} \ln(D^{++})^2 \right] + i \left\{ \text{Tr} \ln \triangle_W (2,2) - \text{Tr} \ln \triangle_W (4,0) \right\}. (57)
\]

Here the first term is the contribution from the Nielsen-Kallosh ghosts, the second term is the contribution from the Faddeev-Popov ghost and the third term is the contribution from the vector multiplet. In the next subsection we will consider the contribution to this effective action from the SCS multiplet, The contribution from the ghosts will be considered in section 5.

\(^{4}\)We restricted ourselves to the one loop approximation. However, the same construction will be valid at any loop if we replace in (53) the quadratic part of the action for the quantum superfields with the general non-quadratic action.
3.1 The proper-time representation of the contributions in (57) from the Chern-Simons vector multiplet

The contribution of the Chern-Simons vector multiplet to (57) can be written in the proper-time representation (see the analogous representation for 4D, \( N = 2 \) SYM theories in [15]):

\[
\Gamma_{SCS}^{(1)} = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \text{Tr}(e^{is\mathcal{W}} \Pi_T^{(2,2)}),
\]  

(58)

where \( \Pi_T^{(2,2)} \) is the five-dimensional gauge covariant version of the projector [34], [15] on the space of covariant analytic transverse superfields \( v^{++} \). The properties of the \( \Pi_T^{(2,2)}(\zeta_1, \zeta_2) \) were described in the paper [29].

For later, it is convenient to rewrite the projector \( \Pi_T^{(2,2)}(\zeta_1, \zeta_2) \) in a different form. Here we follow the work [15]. We have

\[
\Pi_T^{(2,2)}(\zeta_1, \zeta_2) = \delta_A^{(2,2)}(1, 2) - \Pi_L^{(2,2)}(1, 2)
\]

(59)

\[
= \delta_A^{(2,2)}(1, 2) - D_1^{++}D_2^{++} \frac{1}{\square_1} (D_1^+) (D_2^+) \delta^{13}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}.
\]

The operator \( \Box \) is defined by (48). As a result, \( \Pi_T^{(2,2)} \) can be expressed as

\[
\Pi_T^{(2,2)}(\zeta_1, \zeta_2) = \frac{1}{\Box_1} [D_1^{++}, \Box_1] \frac{1}{\Box_1} (D_1^+) (D_2^+) \delta^{13}(z_1 - z_2) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}
\]

(60)

\[
- \frac{1}{\Box_1} (D_1^+) (D_2^+) \delta^{13}(z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^2}.
\]

Note that the first term does not vanish since \( [D^{++}, \Box] \Phi(q) = \frac{1}{4} (D^{++2}\mathcal{W})(1 - q)\Phi(q) \).

The next step is a representation of a two-point function in the form

\[
(D_1^+) (D_2^+) \frac{1}{(u_1^+ u_2^+)^q} = \]

\[
= (D_1^+) (D_2^+) \left\{ (u_1^- u_2^+) (u_1^+ u_2^+) \right\}^2 \frac{1}{(u_1^+ u_2^+)^q} - \frac{1}{4} (u_1^+ u_2^+) (u_1^- u_2^-) \Delta^{-} - (u_1^+ u_2^+) (u_1^- u_2^-) \Box_1
\]

(61)

\[
+ \frac{1}{4} (q - 3) (u_1^+ u_2^+) (u_1^- u_2^-) \mathcal{D}^{++2}\mathcal{W}
\]

where

\[
\Delta^{-} = i \mathcal{D}^{\alpha\beta} \mathcal{D}_{\alpha} \mathcal{D}_{\beta} + \mathcal{W}(\mathcal{D}^-)^2 + 4 (\mathcal{D}^{-\alpha}\mathcal{W}) \mathcal{D}_{\alpha} + (\mathcal{D}^- \mathcal{D}^- \mathcal{W}).
\]

(62)

\[\footnote{Here we are using a manifestly analytic form of the delta function:}

\[
\delta_A^{(2,2)}(1, 2) = \frac{1}{2} \Box_1 (D_1^+) (D_2^+) \delta^{13}(z_1 - z_2) (D_2^-) (\mathcal{D}^{--}) (u_1, u_2)
\]

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In the case under consideration this operator plays a fundamental role in calculations of the effective action in the framework of the background field method. The operator in the quadratic part of the action for quantum fields plays a fundamental role.

### 3.2 The deformed covariantly analytic d’Alembertian $\Box_{\mathcal{W}}$

The operator in the quadratic part of the action for quantum fields plays a fundamental role in calculations of the effective action in the framework of the background field method. In the case under consideration this operator is $\Box_{\mathcal{W}}$ (47).

In the analytic subspace the operator $\Box_{\mathcal{W}}$ (47) is rewritten in the form

\[
\frac{1}{2} (\mathcal{D}^+)^4 \mathcal{W} (\mathcal{D}^-)^2 = (-\frac{1}{64}) \left\{ \mathcal{W} (\mathcal{D}^+)^4 + 4 (\mathcal{D}^{+\hat{a}} \mathcal{W}) \mathcal{D}^+_a (\mathcal{D}^+)^2 + ((\mathcal{D}^+)^2 \mathcal{W}) (\mathcal{D}^+)^2 \right\} (\mathcal{D}^-)^2.
\]

The first term here is, up to the multiplier $\mathcal{W}$, the standard covariant analytic d’Alembertian (48)

\[
-\frac{1}{64} \mathcal{W} (\mathcal{D}^+)^2 (\mathcal{D}^+)^2 (\mathcal{D}^-)^2 = \mathcal{W} [\mathcal{D}^{\hat{a}} \mathcal{D}_a - \frac{1}{4} (\mathcal{D}^{+\hat{a}} \mathcal{D}^+_a \mathcal{W})] (\mathcal{D}^-)^2 = \mathcal{W} \Box.
\]

The deformation of the operator defined by (47) is stipulated by the superfield $\mathcal{W}$ as follows

\[
(-\frac{1}{64})((\mathcal{D}^+)^2 \mathcal{W}) (\mathcal{D}^+)^2 (\mathcal{D}^-)^2 = (-\frac{1}{64})((\mathcal{D}^+)^2 \mathcal{W}) [2 (\mathcal{D}^-)^2 + 16 \mathcal{W} \mathcal{D}^-].
\]

\[
(-\frac{1}{64})4(\mathcal{D}^{+\hat{a}} \mathcal{W}) \times \mathcal{D}^{\hat{a}}_a (\mathcal{D}^+)^2 (\mathcal{D}^-)^2 = (-\frac{1}{64})4(\mathcal{D}^{+\hat{a}} \mathcal{W}) \times [16 (\mathcal{D}^+_\hat{a} \mathcal{W})] (\mathcal{D}^-) - 8 \mathcal{W} \mathcal{D}^- - 8i \mathcal{D}_{\hat{a}\hat{b}}^\beta \mathcal{D}^{-\hat{b}} - 16 (\mathcal{D}^- \mathcal{W})].
\]

Summing up all together we obtain

\[
\Box_{\mathcal{W}} = \mathcal{W} \left\{ \mathcal{D}^{\hat{a}} \mathcal{D}_a - \frac{D^{+\hat{a}} \mathcal{W} \mathcal{D}^+_a \mathcal{W}}{\mathcal{W}} + \frac{1}{2} (\mathcal{D}^{+\hat{a}} \mathcal{D}^+_a \mathcal{W}) (\mathcal{D}^-)^2 - \frac{1}{32} (\mathcal{D}^{+\hat{a}} \mathcal{W}) (\mathcal{D}^-)^2 - \frac{3}{2} (\mathcal{D}^{+\hat{a}} \mathcal{W}) \mathcal{D}^-_{\hat{a}} + \frac{1}{2} \mathcal{W} (\mathcal{D}^{+\hat{a}} \mathcal{W}) i \mathcal{D}^{-\hat{a}\beta}_{\hat{a}} \mathcal{D}^{\hat{b}} \mathcal{D}^{-\hat{b}} - \frac{D^{+\hat{a}} \mathcal{W} \mathcal{D}^- \mathcal{W}}{\mathcal{W}} + \frac{1}{4} (\mathcal{D}^{+\hat{a}} \mathcal{W}) (\mathcal{D}^- \mathcal{W}) - \mathcal{W}^2 \right\}.
\]
This is a final result for the operator $\hat{\Box}_W$ acting on analytic superfields.

The deformed covariantly analytic d’Alembertian $\hat{\Box}_W$ possesses a useful property $[\mathcal{D}^+_\alpha, \hat{\Box}_W] = 0$. It is important to note that the coefficient at the harmonic derivative $\mathcal{D}^-$ in (64) is $-\frac{1}{W}G^{++}$ and that on the equations of motion (26) this term vanishes! Also on the equations of motion $D^{--}G^{++} = 0$ and therefore

$$\frac{D^{+\hat{\alpha}}W\mathcal{D}_{\hat{\alpha}}W}{W} + \frac{1}{4}(D^-D^W) = -\frac{1}{4}(D^+D^-W).$$

This means that in this case the operator $\hat{\Box}_W$ takes the form

$$\hat{\Box}_W = W\left\{D^\hat{\alpha}D_{\hat{\alpha}} - \frac{1}{32}\frac{(D^{+2}W)}{W}(D^-)^2 + \frac{3}{2}(D^{+\hat{\alpha}}W)\mathcal{D}_{\hat{\alpha}} - \frac{1}{2W}(D^{+\hat{\alpha}}W)i\mathcal{D}_{\hat{\alpha}\beta}\mathcal{D}^{-\beta} - \frac{1}{4}(D^+D^-W) - W^2\right\}.$$  (65)

### 3.3 The leading contribution to effective action of the 5D SCS multiplet

Our next aim is to calculate the leading low-energy quantum contribution to the one-loop effective action. To do that it is sufficient to evaluate the one-loop effective action $\Gamma_{SCS}^{(1)}[\bar{V}^{++}]$ for an on-shell background vector multiplet. The representation (58) provides a simple and powerful scheme for computing the effective action in the framework of 5D, $\mathcal{N} = 1$ superfield proper-time technique.

If the background gauge multiplet is on-shell, $G^{++} = 0$, the analytic d’Alembertian $\hat{\Box}_W$ does not involve any harmonic derivative $\mathcal{D}^{--}$, but $\hat{\Box}$ contains the factor $\mathcal{D}^{+2}W\mathcal{D}^{--}$. Hence, we get

$$\frac{1}{\Box_1}(u_1^+u_2^+) = (u_1^+u_2^+)\frac{1}{\Box_1} + \frac{1}{4}(D^{+2}W)(u_1^-u_2^+)\frac{1}{\Box_1}.$$ 

Taking into account this relation we obtain that the second term in the operator $\Pi^{(2,2)}$ is absent. Therefore the projector is reduced only to $(\mathcal{D}^+)4\delta^{13}(z_1 - z_2)$. Then

$$\Gamma_{SCS}^{(1)}[V^{++}] = -\frac{i}{2}\int \frac{ds}{s} \int d\zeta^{(-4)}e^{i\zeta\hat{\Box}_W}(\mathcal{D}^+)4\delta^{13}(z_1 - z_2)|_{z_1 = z_2}. \quad (66)$$

Now we replace the delta-function in (66) by its Fourier representation

$$\delta^5(x_1 - x_2)(\mathcal{D}^+)4\delta^5(\theta_1 - \theta_2) = \int \frac{d^5p}{(2\pi)^5}e^{ip\rho_\theta}\delta^4(\theta_1^+ - \theta_2^+), \quad (67)$$

where $\rho_\theta = (x_1 - x_2)^\alpha - 2i(\theta_1^+ - \theta_2^+)\Gamma^\alpha\theta_2^+$. Then we push $e^{ip_\theta\rho_\theta}$ through all the operator factors in (66) to the left and then replace it by unity in the coincidence limit. This leads to the following transformation of the covariant derivatives

$$\mathcal{D}_\alpha \rightarrow \mathcal{D}_\alpha + ip_\alpha, \quad \mathcal{D}^-_\alpha \rightarrow \mathcal{D}^-_\alpha - 2ip_{\alpha\beta}(\theta_1 - \theta_2)^{-\beta}. \quad (68)$$

Note that $\text{Tr} \ln W$ does not contribute to (57).
Note that the shift $-2i\rho_{\dot{\alpha}\dot{\beta}}(\theta_1 - \theta_2)^{-\dot{\beta}}$ in $\mathcal{D}_{\dot{\alpha}}^-$ vanishes in the coincidence limit since there is no enough operators $\mathcal{D}_{\dot{\alpha}}^+$ to annihilate $(\theta_1^- - \theta_2^-)$. To get the expansion of the effective action in the background fields and their derivatives we should expand the exponent of the operator and calculate the standard momentum integrals. Thus, we expand the exponent $e^{i\Sigma W}$ in powers of spinor derivatives near $e^{-is\phi^2}$ up to the fourth order in spinor derivatives (it corresponds to expanding up to the fourth order in the proper time) and use the identity $-\frac{1}{2}(\mathcal{D}_-^+)^4\delta^4(\theta_1^+ - \theta_2^+)|_{\theta_1=\theta_2}=1$. It remains to perform a standard integration over the momentum variables and over the proper-time. The final result has the form

$$
\Gamma^{(1)}_{SCS}[V^{++}] = \frac{5}{(16\pi)^2} \int d^{13}z W \ln \frac{W}{\Lambda}. \quad (69)
$$

Here $\Lambda$ is some scale\textsuperscript{7}. One can show, using the methods developed in [22], that the action (69) is superconformal. A component form of (69) in the bosonic sector corresponds to the Lagrangian $F^4_{\phi^4}$.

The effective action (69) can be treated as the 5D analogue of the non-holomorphic potential in $4D, \mathcal{N} = 2$ supersymmetric gauge theories. This effective action is also similar to the action of the the so-called 4D improved tensor multiplet [39].

4 The leading contribution to the effective action of a 5D SYM multiplet

It is interesting and instructive to compare the one-loop leading low-energy effective action in 5D SCS theory and in 5D SYM theory.

5D, $\mathcal{N} = 1$ SYM theory is described by the action (see (14), (19)):

$$
S_{SYM} = \frac{1}{g^2_{SYM}} \int d^{13}z V^{++}V^{--}, \quad (70)
$$

where the coupling constant has dimension $[g^2_{SYM}] = 1$.

Let us consider the construction of the background field method for the theory (70). It is obvious that the second variation of the action has the form completely analogous to the 4D, $\mathcal{N} = 2$ case

$$
\delta^2 S = \frac{1}{g^2_{SYM}} \int d^{13}z du_1 du_2 \delta V^{++}(u_1) \delta V^{++}(u_2) \frac{1}{(u_1^+ u_2^+)^2}. \quad (71)
$$

Therefore we can simply repeat step by step all the stages of the construction of the effective action developed in [14], [15]. In particular, we can use the same choice of the\textsuperscript{5}

\textsuperscript{5}It is easy to see that the effective action (69) does not depend on the scale, since

$$
\int d^{13}z W \ln \Lambda = \ln \Lambda \int d\zeta (-4)(D^+)^4W = 0.
$$

20
gauge conditions on the quantum superfield $D^{++}v^{++} = 0$ and the same procedure to fix gauge as in [14]. It gives the sum of the quadratic part of action (71) and the corresponding gauge-fixed action in the form $S_{GF}$

$$
S_2 + S_{gh} = \frac{1}{2} (1 + \frac{1}{\alpha}) \int d^{13}zdu_1du_2 \frac{1}{g_{SYM}^2} \frac{v^{++}(1)v^{++}(2)}{(u_1^+u_2^+)^2}
$$

$$
- \frac{1}{2\alpha} \int d^{13}zdu \frac{1}{g_{SYM}^2} v^{++} \frac{1}{2}(D^{--})^2 v^{++}
$$

Further we put gauge parameter $\alpha = -1$.

The derivation of the general expression for the one-loop effective action in 5D SYM theory is completely analogous to derivation of the expression (57). The final result is

$$
\tilde{\Gamma}^{(1)}_{SYM} = -\frac{i}{2} \text{Tr} \ln(D^{++})^2 + \frac{i}{2} (\text{Tr}_{(2,2)} \ln \Box - \text{Tr}_{(0,4)} \ln \Box).
$$

The first term is the contribution of the ghosts and the second term is the contribution of a SYM multiplet. Here the covariantly analytic d’Alembertian $\Box$ is given by (48) and has the form [20]

$$
\Box = -\frac{1}{64}(D^+)^4(D^{-})^2 = D^\alpha D_\alpha + (D^{+\bar{\alpha}} W)D^{\bar{\alpha}} - \frac{1}{4}(D^{+\bar{\alpha}} D_{\bar{\alpha}} W)D^{-}
$$

$$
+ \frac{1}{4}(D^{\bar{\alpha}} D^{+\bar{\alpha}} W) - W^2.
$$

Now we consider a calculation of the contribution to the effective action (72) from a SYM multiplet. Contribution of the first term in (72) will be calculated in section 5. In the proper-time representation, the contribution of SYM multiplet has the form analogous to (58)

$$
\Gamma^{(1)}_{SYM} = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \text{Tr} \left(e^{is\Box} \Pi^{(2,2)}_T\right).
$$

The only difference with (58) is the operator $\Box$ instead of the operator $\Box_{\mathcal{W}}$. To find the leading low-energy contribution to effective action it is sufficient to consider the on-shell background superfield. Classical on-shell equation for the 5D SYM theory look like $(D^+)^2\mathcal{W} = 0$. In this case the covariantly analytic d’Alembertian $\Box$ and the projection operator $\Pi^{(2,2)}_T$ are simplified and we have the effective action in the following form

$$
\Gamma^{(1)}_{SYM} = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \text{Tr} e^{is\Box} \Pi^{(2,2)}_T = -\frac{i}{2} \int_0^\infty \frac{ds}{s} \int d\zeta^{(-4)} e^{is\Box} (D^+)^4 \delta^{13}(z_1 - z_2),
$$

Evaluation of this expression is realized completely the same way as it was done in the subsection 3.3 to obtain the (69). The final result has the form

$$
\Gamma^{(1)}_{SYM} = \frac{1}{24\pi^2} \int d^{13}z \mathcal{W} \ln \frac{\mathcal{W}}{\Lambda}.
$$
We see that the functional form of the effective action (75) generated by a SYM multiplet coincides with the one generated by a SCS multiplet (69). Although the on-shell conditions for 5D SCS theory and for 5D SYM theory are different and, moreover, 5D SYM theory is characterized by a dimensional coupling constant, the leading low-energy contributions in these two theories to the one-loop effective action turn out to be the same up to a numerical coefficient.

5 The leading and next-to-leading contributions of the ghosts and matter superfields

The ghost contribution to the one-loop effective action in both SCS and SYM theories is defined by the expression

$$\Gamma_{\text{ghost}}^{(1)} = -\frac{i}{2} \text{Tr} \ln \left( D^{++} \right)^2.$$  

(76)

The contribution from matter hypermultiplet superfields differs only by a sign and by the choice of the representation of the gauge group. Hence, to find the leading contribution to (76) we can use the results from the previous analysis [20] of the effective action for a $q$-hypermultiplet coupled to a background vector multiplet. It was shown in that paper that the leading quantum correction is the SCS action. Therefore, further we will focus only on the first next-to-leading correction.

For calculation of the first next-to-leading contribution to the effective action (76) we will follow the procedure proposed in the papers [15], [20]. This procedure is based on calculating the variation of the effective action with its subsequent restoration given the obtained variation.

Let $\Gamma_{\text{hyper}}^{(1)}$ be the one-loop contribution to the effective action from either matter hypermultiplets or from ghosts

$$\Gamma_{\text{hyper}}^{(1)} = (\pm)i \text{Tr} \ln D^{++} = \mp i \text{Tr} \ln G^{(1,1)},$$  

(77)

where the upper sign corresponds to the contribution from matter and the lower sign to that of ghosts. Further, for simplicity, we will consider for only the plus sign. Here $G^{(1,1)}(\zeta_1, \zeta_2)$ is the hypermultiplet Green function satisfying the equation

$$D^{++}_1 G^{(1,1)}(\zeta_1, \zeta_2) = \delta_A^{(3,1)}(\zeta_1, \zeta_2),$$  

(78)

where $\delta_A^{(3,1)}(\zeta_1, \zeta_2)$ is the appropriate covariantly analytic delta-function

$$\delta_A^{(3,1)}(\zeta_1, \zeta_2) = (D^+_1)^4 \delta^{13}(z_1 - z_2) \delta^{(-1,1)}(u_1, u_2) 1.$$

---

8From a formal point of view, the effective action (76) corresponds to a so called $\omega$-hypermultiplet [17]. It was pointed out some time ago [14] that this effective action is equal, up to the coefficient 2, to the effective action for a $q$ hypermultiplet. Therefore, further we will take into account just the $q$ hypermultiplet effective action.
This equation is similar to the equation for the hypermultiplet Green function in 4D, $\mathcal{N} = 2$ theories and we can use methods developed in [14]. The Green function $G^{(1,1)}(\zeta_1, \zeta_2)$ can be written in the form

$$G^{(1,1)}(\zeta_1, \zeta_2) = -\frac{1}{\Box_1} (\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4 \delta^{13} (z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^3} \mathbf{1}. \quad (79)$$

Here $\Box$ is the covariantly analytic d’Alembertian (73) and $(u_1^+ u_2^+)^{-3}$ is a special harmonic distribution [34].

The variation of the effective action (77) under the background superfield $V^{++}$ is written as follows (see [15], [20] for details)

$$\delta \Gamma_{\text{hyper}}^{[1]} = - \int d\zeta (-4) \left\{ \delta V^{++} G^{(1,1)} \right\}. \quad (80)$$

To evaluate this expression one can use the proper-time technique. The leading low-energy contribution goes from the terms without derivatives of $W$. It was calculated in [20] and has the form

$$\delta \Gamma_{\text{hyper}}^{[1]} = - \frac{1}{(4\pi)^2} \text{sign}(W) \int d^{13} z d\nu \delta V^{++} V^{--} W. \quad (81)$$

The variation of (81) corresponds precisely to the classical action (25) of the SCS gauge theory.

Our main purpose in this section is to find the first next-to-leading correction to the SCS action. First of all one considers the Dyson type equation relating the free and full hypermultiplet propagators [14]

$$G^{(1,1)}(1|2) = G^{(1,1)}_0(1|2) - \int d\zeta (\Box_1)^4 (\mathcal{D}_2^+)^4 \delta^{13} (z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^3} (1|2) G^{(1,1)}(2|3) i V^{++}(3) G^{(1,1)}_0(3|2). \quad (82)$$

Substituting eq.(82) into the variation (80), one finds

$$\delta \Gamma_{\text{hyper}}^{[1]} = \int d\zeta (\Box_1)^4 (\mathcal{D}_2^+)^4 \delta V^{++}(1) i V^{++}(2) G^{(1,1)}_0(1|2) G^{(1,1)}(2|1). \quad (83)$$

Taking into account the explicit form of the propagator (79), we rewrite this expression as follows

$$\delta \Gamma_{\text{hyper}}^{[1]} = \int d\zeta (\Box_1)^4 (\mathcal{D}_2^+)^4 \delta V^{++}(1) \frac{1}{\Box_1} (\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4 \delta^{13} (z_1 - z_2) (u_1^+ u_2^+)^3$$

$$\times i V^{++}(2) \frac{1}{\Box_2} (\mathcal{D}_2^+)^+ (\mathcal{D}_1^+)^4 \delta^{13} (z_2 - z_1) (u_2^+ u_1^+)^3. \quad (83)$$

Now we use the spinor derivatives from the first delta-function to restore the full $\mathcal{N} = 1$ superspace measure according to the rule $\int d\zeta (\mathcal{D}^+)^i = \int d^{13} z$. So far, we did not consider any restrictions for the background superfield, therefore (83) is the exact representation for the one-loop hypermultiplet effective action. In principle it can be a starting point for calculations of different contributions to the one-loop effective action.
To compute the first next-to-leading quantum corrections to (81) it is sufficient to expand in (83) the covariant analytic d’Alembertian in powers of the derivatives of the background superfields. Let us remember that the Green function of the hypermultiplet is antisymmetric with respect to the permutation of its arguments. This allows us to rewrite the expression (83) in the form

$$\delta \Gamma^{[1]}_{\text{hyper}} = -\int d^{13}z d\bar{u}_{1} d\bar{u}_{2} \frac{\delta V^{++}(1)}{(u_{1}^{+} u_{2}^{+})^{2}} (\slashed{D}_{1}^{+})^{2} i V^{++}(2) \frac{1}{\Box_{1}} \frac{1}{(\Box - W^{2})^{3}} (-\frac{1}{32})(\slashed{D}_{1}^{+})^{2} (\slashed{D}_{2}^{+})^{4} \delta^{13} (z_{2} - z_{1})^{3}.$$  

(84)

The next step is the expansion of the operator $\slashed{\Box}$ in the denominator in power series of the derivatives of $W$. In this expansion we keep only the leading terms with two derivatives. It leads to

$$\delta \Gamma^{[1]}_{\text{hyper}} = -\int d^{13}z d\bar{u}_{1} d\bar{u}_{2} \frac{\delta V^{++}(1)}{(u_{1}^{+} u_{2}^{+})^{2}} (\slashed{D}_{2}^{+})^{2} i V^{++}(2) \frac{1}{\Box_{1}} \frac{1}{\Box - W^{2}} W \slashed{D} \slashed{D} + W^{2} \frac{1}{(\Box - W^{2})^{3}} (-\frac{1}{4})(\slashed{D}_{2}^{+})^{4} (\slashed{D}_{2}^{+})^{4} \delta^{13} (z_{2} - z_{1}).$$

Now the usual steps lead to

$$\delta \Gamma^{[1]}_{\text{hyper}} = \frac{i}{6(4\pi)^{2}} \int d^{13}z d\bar{u} \delta V^{---} \frac{\slashed{D} + W \slashed{D} + W}{W^{2}}$$

$$= -\frac{i}{6(4\pi)^{2}} \int d^{13}z d\bar{u} \delta V^{---} (\slashed{D}^{+})^{2} \ln W = -\frac{1}{12\pi^{2}} \int d^{13}z \delta W \ln W.$$

As a result we obtain the first next-to-leading contribution to the one-loop hypermultiplet effective action in the form

$$\Gamma^{[1]}_{\text{hyper}} = c_{\text{hyper}} \int d^{13}z W \ln \frac{W}{\Lambda}.$$  

(85)

Here $c_{\text{hyper}}$ is a numerical coefficient depending on details of the hypermultiplet action (such as the number of components, the representation, whether hypermultiplet superfields are commuting or anticommuting). We see that the effective action (85) has the same functional structure as the effective actions generated by the SCS multiplet (69) and by the SYM multiplet (75).

6 Summary

We have considered the five-dimensional $\mathcal{N} = 1$ supersymmetric field models such as the Abelian Chern-Simons theory, the Yang-Mills theory and the hypermultiplet theory coupled to a background vector multiplet, formulated in the harmonic superspace approach. In all these models we calculated the universal four derivative contributions to the one-loop effective action.

In the 5D SCS theory we have developed the background field method in harmonic superspace and represented the one-loop effective action in terms of functional determinants of the operators acting in the analytic subspace of harmonic superspace (Eq. (57)).
The above expression contains the contributions from the ghost superfields and from a SCS vector multiplet superfield. We studied the structure of the latter contribution to the effective action and evaluated the leading low-energy effective action (Eq. (69)).

The same consideration has been realized in 5D SYM theory as well. We developed the background field method for this theory, found the one-loop effective action in terms of the functional determinants of the differential operators acting in the analytic subspace of harmonic superspace (Eq. (72)). We studied the structure of the one-loop contributions to the effective action from a SYM vector multiplet and calculated the leading low-energy contribution of this multiplet. Although 5D SYM theory is not superconformal, its coupling constant is dimensional, and the expressions (57) and (72) are different, the leading contribution to the effective action in 5D SYM theory (Eq. (75)) has the same functional form as in 5D SCS theory.

In 5D hypermultiplet theory in a vector multiplet background field we have calculated the first next-to leading contribution to the effective action (Eq. (85)). The corresponding leading contribution is the SCS action and was found in the paper [20]. We have shown that this first next-to leading contribution (85) again has the same functional form as the leading contribution in SCS theory.

We have found the manifestly 5D, \( \mathcal{N} = 1 \) superconformal form of the term \( \sim F^4 \) in the effective actions of the SCS theory, SYM theory and hypermultiplet theory. The next step of studying of the one-loop effective action in the theories under consideration is a construction of the full low-energy one-loop effective actions where all the powers of the Abelian strength are summed up. In other words, the next purpose is to construct the 5D superfield Heisenberg-Euler type of the effective action. Like in 4D, 3D superconformal gauge theories [40] it is reasonable to expect that this effective action will be expressed in the terms of so called superconformal invariants which transform as scalars under the 5D, \( \mathcal{N} = 1 \) superconformal group. For example, such a scalar invariant can be a superfield \( \Psi^2 = \frac{1}{16 \mathcal{W}} D^4 \ln \mathcal{W} \). We hope to study this issue in the forthcoming work.

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