Stochastic Shell Model: a model of anomalous scaling and non-Gaussian distribution in turbulence

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We propose a simple stochastic model of cascading transport in wave number space to clarify the origin of intermittent behavior of fully-developed fluid turbulence. In spite of lack of nonlinearity and viscosity the model gives non-Gaussian fluctuations and multifractal scalings consistent with experimental data.

Spatio-temporal structure of fully-developed isotropic fluid turbulence is one of frontier topics in both fluid mechanics and statistical physics. Although several new concepts, for example, fractals, chaos, and intermittency, have been introduced to characterize fully-developed turbulence, our present understanding of fluid turbulence is still far from complete.

There are many difficulties both in experimental and numerical approaches. Experimentally, typical available data is time sequential data of velocity at a fixed point, which is obviously not sufficient to observe global spatio-temporal structures of fluid turbulence. Instantaneous local velocities at arbitrary points can be obtained by using direct numerical integration, however, fully-developed turbulence requires so many degrees of freedom that even a super computer is not powerful enough. To help our understanding of the behavior of turbulence we need simple models which are easily accessible and share some basic properties with fluid turbulence.

In the history of study on turbulence one of the most successful approaches is Kolmogorov’s scaling argument, in which the energy spectrum, $E(k)$, was predicted to have a power law wave number dependence as $k^{-5/3}$. His prediction was based on a very simple assumption of cascading process that conserved energy is transported continuously in the wave number space toward higher wave numbers. Although his simple assumption captured the most basic property of fully-developed turbulence, a modification was proposed by himself and others to include the effect of spontaneous spatial inhomogeneity so called the intermittency. In order to take into account the fluctuations of energy dissipation rates a log-normal distribution was assumed and the exponent of the energy spectrum was modified.

Another approach to the intermittency is based on geometrical models that an eddy breaks into smaller eddies forming a fractal configuration. This fractal approach is recently extended to apply the concept of multifractals and now the agreement with experiments becomes quantitatively acceptable.

Recently, much experimental interest is focused on observation of probability density function of various variables in turbulence. It is well-confirmed that the PDF of velocity differences between a pair of points separated by a distance clearly deviates from Gaussian and the deviation becomes more evident for closer pairs. In terms of Fourier space this means that higher wave number components of velocity field show larger deviation from Gaussian.

The multifractal concept and the non-Gaussian PDFs are consistent representations of the intermittency. Actually, by combining these two ingredients She and Leveque have derived an analytical formula of multifractal exponents which fits with experimental values very nicely. An interesting point in their derivation is that Navier-Stokes equation is not used explicitly like other theories on intermittency.

Generally speaking lack of dynamic equation in theory apparently shows incompleteness of the theory itself, however, in the present case there is a positive aspect that the theory’s applicability is not limited to Navier-Stokes turbulence. Actually, almost identical non-Gaussian PDFs and the multifractal exponents have been reported in an experiment on thermal convective turbulence which is governed by Boussinesq equation.

Similar intermittent behavior can also be found in a drastically simplified numerical model called GOY model. It belongs to so-called shell models as the model discards infinite degrees of freedom in Navier-Stokes equation and uses only finite numbers of representative wave number components called shells. It is described by the following set of nonlinear differential equations for complex velocity Fourier components, $u(j,t)$, for discrete wave numbers $k_j = b^j$, where $b$ is a positive constant and
$j$ is an integer.

\[
\frac{d}{dt}u(j,t) = -\nu k_j^2 u(j,t) + f\delta j,4 \\
+ i \{ak_j^2 u^*(j + 1, t)u^*(j + 2, t) \\
+ bk_j u^*(j - 1, t)u^*(j + 1, t) \\
+ ck_j u^*(j - 1, t)u^*(j - 2, t)\}.
\]

Here, the first term in right hand side shows a viscous dissipation effect, and the second term corresponds to external forcing. The remaining three terms are nonlinear terms which are designed to have similar nonlinearity with Navier-Stokes equation. By these nonlinear terms this deterministic system shows chaotic behavior and the resulting statistics agrees with real fluid intermittency [13].

In this letter, we are going to clarify why such a simplified model can produce intermittent behavior quite similar to those in real turbulence having infinitely many degrees of freedom. Our answer is simple: The intermittency can be a very general and universal behavior in a wide class of nonequilibrium systems not specified to fluid turbulence. The key point is a fluctuating local directional transfer in the wave number space, so the number of degrees of freedom, the details of nonlinearity and the types of dissipation are not crucial.

In order to validate this scenario we introduce a new model, the stochastic shell model, which is even much simpler than GOY model. Instead of the chaotic nonlinear interactions we introduce discrete stochastic processes using real variable $u$ as follows;

\[
u j, t + 1 = u(j, t) \\
+ \theta [R(j, j - 1, t)u(j - 1, t) - R(j + 1, j, t) | u(j, t)],
\]

Here, $\theta$ is a non-negative constant less than unity, and $R(j, j - 1)$ is a random number which expresses the momentum transfer from $j-1$ to $j$;

\[
R(j, j - 1, t) = \begin{cases} 
+1, & \text{Prob } p/2 \\
-1, & \text{Prob } p/2 \\
0, & \text{Prob } 1 - p
\end{cases}
\]

It should be noted that this model can be viewed as a multiplicative process of random matrices, thus it does not have any nonlinearity. Similar linear multiplicative stochastic process has been analyzed theoretically by Deutsch [15] in which an anomalous scaling relation like the multifractal scaling is reported.

In the following, we adopt the boundary condition that $u(N,t) = 0$ for $t > 0$ where $N$ is the total number of shells. Compared with the ordinary exponential decay by viscous dissipation in large wave number regions, this artificial boundary condition of a sudden cut off may look too rude. However, the boundary condition at the largest wave number does not affect the intermediate wave number components because our model introduces the one-way transport towards the larger wave numbers. Thus $u(N,t) = 0$ is simply for numerical convenience. As an energy injection effect by an external forcing we add uniform random numbers ($\in [-0.5, 0.5]$) to the 0-th shell at every time step.

Analytical treatment of this stochastic model can be done by introducing the characteristic function $Y_j(p, t) \equiv \langle \exp[i pu(j, t)] \rangle$ where $\langle O \rangle$ denotes the averaged value of the variable $O$ over realizations. Using this definition we get the following equation for the characteristic function from Eq.[14].

\[
Y_j(p, t + 1) = \{(1 - p)Y_j(p, t) + pY_j([1 - \theta]p, t)\} \\
\times \{(1 - p) + pY_{j-1}(\theta p, t)\}.
\]

We can show analytically that the system quickly converges to a statistically steady state where all averaged quantities become independent of time steps. This implies that the ensemble average and a time average gives the same result independent of initial conditions. By expanding the characteristic function in terms of $p$ up to the second order we have a rigorous steady state relation for ‘energy’, $E_j \equiv \langle |u(j, t)|^2 \rangle/2$, as $E_j = \theta E_{j-1}/(2 - \theta)$.

![FIG. 1. Symbols: Higher moment of $u(j, t)$. (a) $q = 2$ (b) $q = 6$ (c) $q = 10$ (x,y=16,+,-=64). Thick lines: $\zeta_q = q/3$, thin lines: $\zeta_q = q/9 + 2 \left[1 - \left(\frac{q}{4}\right)^{1/3}\right]$.](image-url)
The energy spectrum follows the familiar power law form, \( E_j = k_j^{-\zeta} \), by choosing \( \theta \) to be \( 2/(1 + b^r) \). We employ \( \zeta = 2/3 \) so as to satisfy the Kolmogorov spectrum. The parameter \( p \) does not appear in the energy relation and it turns out that \( p \) is irrelevant in the following discussion as far as \( 0 < p < 1 \), so we fix its value to be 0.5 hereafter. Thus, the model has only one free parameter \( b \).

To measure the intermittency quantitatively we calculate higher order moments of \( u(j, t) \) and generalize the definition of the scaling exponent \( \zeta \) as \( \langle |u(j, t)|^q \rangle \propto k_j^{-\zeta_q} \). Figure 2 shows the results for \( q = 2, 6, \) and 10 (\( N = 10, b = 16, 64 \)). Here averages are taken over \( 2 \times 10^5 \) steps. \( \zeta_2 \) is equal to 2/3 as expected by analytical treatment.

On the other hand, values of \( \zeta_q \) for \( q = 6 \) and 10 clearly deviate from Kolmogorov’s original scaling exponents, \( q/3 \). We can conclude that our model exhibits an anomalous scaling \([16]\). In order to compare the exponents with experimental values, we also plot the empirical multifractal relation, \( \zeta_q = q/9 + 2 \left[ 1 - \left( \frac{q}{3} \right)^{1/3} \right] \), which is known to be a good approximation to the experimental values \([11]\). Our data agree with this empirical values in the whole range, meaning that our model can reproduce the multifractal scaling quantitatively.

Next, we check our model whether the wave number dependence of PDFs observed in experiments can also be reproduced correctly or not. Figure 3 shows a comparison between experimentally observed PDFs and those of our model (\( b = 64, N = 5 \)). The experimental turbulence data are PDFs of frequency-band-pass-filtered velocity signals obtained by one of the authors TK [4]. In the case of our numerical model we plotted \( u(j, t) \) because the imaginary part follows the same statistics. Data are sampled every two steps over total \( 2 \times 10^5 \) steps. The experimental PDFs show a tendency that the deviation from Gaussian becomes more evident for larger wave numbers. The numerical data fit them very nicely in the whole range in each case. Namely, our model also reproduces the other aspect of intermittency, the non-Gaussian PDFs \([13]\).

In high wave number limit it is known that both GOY model and Navier-Stokes turbulence have almost identical power laws in PDF \( P(u) \) of velocity \( u \), i.e., \( P(u) \sim u^{-\beta} \) with \( \beta \approx 1.6 \) \([17]\). Thus, the appearance of power tails in PDFs at very high wavenumber may also be an important nature of fluid intermittency.

**FIG. 2.** Comparison of PDF between stochastic shell model (+) and experiment (○). All data are normalized so as to have variance of unity for comparison. Top: Experiment: 0.15kHz, model: \( j = 0 \). Middle: Experiment: 3kHz, model: \( j = 1 \). Bottom: Experiment: 10kHz, model: \( j = 2 \).

**FIG. 3.** Power law PDF obtained by the stochastic shell model \((j = 40)\). The straight line indicates \( P(u) \sim u^{-2} \).
with what we have reported. However, when we fix the sign of \( R(j, j - 1; t) \), independent of whether \( u \) is real or complex, we get considerably different state. We have a statistically steady result and the distributions deviate more from Gaussian for larger wave number components as in the case with sign randomization, but the energy spectrum is modified to a non-power law and the multifractal scalings are lost completely. This difference is expected to be caused by the strong correlations among velocity components in each shell. This result demonstrates that transport with sign (or phase for complex \( u \)) randomization is essential for the intermittency in real turbulence.

Concluding the paper, we have introduced a stochastic shell model to describe cascading transports in wave number space, which can be viewed as a kind of linear multiplicative stochastic process. The model with sign randomization reproduces the intermittent behavior correctly while its constant sign version loses all such behavior. This fact implies that the origin of intermittency is in the cascading process with sign (or phase for complex \( u \)). Nonlinearity and viscosity does not appear explicitly in our discussion, therefore, these effects may give only indirect contribution to the intermittency. Non-linearity in real turbulence is important as a origin of randomness. However, once randomness is introduced, it is not important whether randomness originates in non-linearity or white noise when we consider stochastic property like intermittency and anomalous scaling. Further investigation of the stochastic shell model will contribute on the problem of universality of intermittency.

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