SINGULAR CONTINUOUS AND DENSE POINT SPECTRUM FOR SPARSE TREES WITH FINITE DIMENSIONS

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Abstract. Sparse trees are trees with sparse branchings. The Laplacian on some of these trees can be shown to have singular spectral measures. We focus on a simple family of sparse trees for which the dimensions can be naturally defined and shown to be finite. Generically, this family has singular spectral measures and eigenvalues that are dense in some interval.

1. Introduction

This paper extends and complements the paper [5] in which the notion of sparse trees was introduced. Sparse trees are trees which have arbitrarily long ‘one-dimensional’ segments (by which we mean - intervals of $\mathbb{Z}$), separated by occasional non-trivial branchings. It is shown in [5] that, when these trees are spherically symmetric, one may decompose the Laplacian as a direct sum of Jacobi matrices which have sparse ‘bumps’ off the diagonal. The spectral theory of these matrices is similar to that of one-dimensional Schrödinger operators with sparse potentials (see [11] and references therein). In particular, matrices of this type exist for which the spectral measures are singular with respect to Lebesgue measure. These ideas make it possible to construct simple examples of trees for which the Laplacian has interesting spectral behavior. Several examples with singular continuous spectrum were presented in [5].

In this paper we will be concerned with a family of sparse trees that ‘interpolates’ between $\mathbb{Z}^+$ and the Bethe lattice. These trees can be obtained from the Bethe lattice by replacing an edge, at a distance $n$ from the root, by a segment of length $\sim \gamma^n$ for some fixed $\gamma > 1$. While the Bethe lattice is infinite dimensional, a tree obtained in

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this manner can be shown to have dimensionality \( \frac{\log \gamma}{\log \gamma_k} \), where \( k \) is the connectivity of the original Bethe lattice. (For our definition of dimension see section 3). Thus, by letting \( \gamma \) vary from 1 to \( \infty \), one gets a family of trees corresponding at one end (\( \gamma = 1 \)) to the Bethe lattice, and at the other end (\( \gamma = \infty \)) to \( \mathbb{Z}^+ \).

We shall analyze the spectral properties of the Laplacian on these trees with the help of the decomposition described above and some tools from the spectral theory of Schrödinger operators with sparse potentials. The constant branching, however, turns out to be a technical difficulty. We will bypass this difficulty by using an idea from [21] - namely, we shall impose a certain probability measure on these trees and prove an ‘almost sure’ result.

It turns out that the situation for these finite dimensional structures is markedly different from the one for \( \mathbb{Z}^d \). These trees (generically) have purely singular spectral measures and some dense point spectrum.

In addition to the new result described above, we also use this opportunity to expand the discussion on the basic setting and on some of the examples presented in [5]. Some basic facts that were briefly mentioned in that paper (such as the self-adjointness of the Laplacian on normal sparse trees), will be explained here in greater detail.

We remark that graphs with singular continuous [18] and pure point spectrum [13] are known to exist. In this context, the family of sparse trees is interesting in that, when varying two parameter sequences (namely - the branching size and the distances between branchings), one encounters a rich spectrum of phenomena. We note, in particular, the existence of examples with spectral measures of fractional Hausdorff dimensions (see [5] and theorem 4.4 below).

This paper is structured as follows. The next section presents the notion of sparse trees and the decomposition theorem that is basic for all that follows. Some simple results concerning spectral measures for the Laplacian on sparse trees are given in section 3. Section 4 describes the construction of the finite dimensional trees mentioned above and our results for them. As mentioned above, this paper uses some ideas and tools from the spectral theory of discrete one-dimensional Schrödinger operators. Relevant notions and results are presented in the appendix.

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2. Sparse Trees

As noted in the introduction, basic to the analysis which follows is a certain decomposition of the Laplacian on a sparse tree. Since this is possible only when the tree has a certain spherical symmetry, we start with:

**Definition 2.1 (Spherically Homogeneous Rooted Tree).** A rooted tree is called spherically homogeneous (SH) (see [3]) if any vertex \( v \) of generation \( j \) is connected with \( \kappa_j \) vertices of generation \( j + 1 \). A locally finite spherically homogeneous tree is uniquely determined by the sequence \( \{\kappa_j\}_{j=0}^{\infty} \). By locally finite we mean that the valence of any vertex is finite.

![Figure 1. An example of a SH rooted tree with \( \kappa_0 = 1 \), \( \kappa_1 = 2 \), \( \kappa_2 = \kappa_3 = \kappa_4 = 1 \), \( \kappa_5 = 2 \ldots \)

In the definition above, a vertex is said to be of generation \( j \) if it is at a distance \( j \) from the root - \( O \) (where the distance between two vertices is defined as the number of edges of the unique path between them). Thus, for spherically homogeneous rooted trees, the valence of a vertex depends solely upon its location with respect to the root.

Let \( \{L_n\}_{n=1}^{\infty} \) and \( \{k_n\}_{n=1}^{\infty} \) be two sequences of natural numbers such that \( k_n \geq 2 \) for all \( n \), and \( \{L_n\}_{n=1}^{\infty} \) is strictly increasing. We say that \( \Gamma \) is a SH rooted tree of type \( \{L_n, k_n\}_{n=1}^{\infty} \) if

\[
\kappa_j = \begin{cases} 
  k_n & j = L_n \text{ for some } n \\
  1 & \text{otherwise}
\end{cases}
\]  

We say that \( \Gamma \) is *sparse* if \( (L_{n+1} - L_n) \to \infty \) as \( n \to \infty \). Since sparse trees are not regular (the coordination number is not constant), there are two natural choices for the Laplacian:

\[
(\Delta f)(x) = \sum_{y:d(x,y)=1} f(y),
\]  

where \( d(x,y) \) is the distance between vertices \( x \) and \( y \).
where \( \#A \) for a finite set \( A \) is the number of elements in \( A \) (\( d(x,y) \) denotes the distance on the tree). For simplicity, we shall restrict our attention to \( \Delta \), though we note that all our results hold (when properly modified) for \( \tilde{\Delta} \) as well.

It is clear that if \( \{k_n\}_{n=1}^\infty \) is a bounded sequence then both \( \Delta \) and \( \tilde{\Delta} \), on the tree, are bounded and self-adjoint. For unbounded coordination number, the issue of self-adjointness has to be addressed.

**Definition 2.2.** We call a SH rooted tree of type \( \{L_n, k_n\}_{n=1}^\infty \) - \( \Gamma \) - normal if \( \{k_n\} \) unbounded implies that \( \limsup_{n \to \infty} (L_{n+1} - L_n) > 1 \).

The appendix has a proof that the Laplacians on normal SH rooted trees are self-adjoint. Clearly, any sparse tree is normal.

The main technical tool in the spectral analysis of sparse trees is the following theorem:

**Theorem 2.3** (Theorem 2.4 in [5]). Let \( \Gamma \) be a normal rooted SH tree of type \( \{L_n, k_n\}_{n=1}^\infty \). Let

\[
M_n = \begin{cases} 
\prod_{j=1}^n k_j - \prod_{j=1}^{n-1} k_j & n > 1 \\
\frac{1}{k_1 - 1} & n = 1 \\
1 & n = 0.
\end{cases} \tag{2.4}
\]

Furthermore, let \( R_0 = 0 \) and \( R_n = L_n + 1 \), for \( n \geq 1 \). Then \( \Delta \) is unitarily equivalent to a direct sum of Jacobi matrices, each operating on a copy of \( \ell^2(\mathbb{Z}^+) \): \n
\[
\Delta \cong \bigoplus_{n=0}^\infty (J_n \oplus J_n \oplus \cdots \oplus J_n)^{M_n \text{ times}} \tag{2.5}
\]

where \( J_n = J(\{a_n(j)\}_{n=1}^\infty, \{b_n(j)\}_{n=1}^\infty) \) with

\[
a_n(j) = \begin{cases} 
\sqrt{k_m} & j = R_m - R_n \text{ for some } m > n \\
1 & \text{otherwise}
\end{cases} \tag{2.6}
\]

and

\[
b_n(j) \equiv 0. \tag{2.7}
\]
Remarks. 1. The term - Jacobi matrix, with the notation $J(\{a(j)\}_{j=1}^{\infty}, \{b(j)\}_{j=1}^{\infty})$, stands for the semi-infinite matrix

$$J(\{a(j)\}_{j=1}^{\infty}, \{b(j)\}_{j=1}^{\infty}) = \begin{pmatrix} b(1) & a(1) & 0 & 0 & \cdots \\ a(1) & b(2) & a(2) & 0 & \cdots \\ 0 & a(2) & b(3) & a(3) & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (2.8)$$

with

$$b(j) \in \mathbb{R}, \; a(j) > 0.$$ 

2. For the case of a regular tree, a similar decomposition was discussed in [2, 8, 16] (see also [20] for a related result in the case of a metric tree).

3. As noted above, this theorem holds for $\tilde{\Delta}$ as well, with the decomposition:

$$\tilde{\Delta} \cong \bigoplus_{n=0}^{\infty} (\tilde{J}_n \oplus \tilde{J}_n \oplus \cdots \oplus \tilde{J}_n) \quad (2.9)$$

where $\tilde{J}_n = J(\{\tilde{a}_n(j)\}_{n=1}^{\infty}, \{\tilde{b}_n(j)\}_{n=1}^{\infty})$ with

$$\tilde{a}_n(j) = a_n(j) \quad (2.10)$$

and

$$\tilde{b}_n(j) = \begin{cases} -k_m - 1 & j = R_m - R_n \text{ for some } m > n \\ -2 & \text{otherwise} \end{cases} \quad (2.11)$$

4. Note that each $J_n$ is a 'tail' of $J_{n-1}$ in the sense that one can get $J_n$ by deleting a finite number of rows from the top, and the same number of columns from the left, of $J_{n-1}$.

3. Singular Measures on Sparse Trees

In this and the next section we freely use terms (such as ‘transfer matrices’) associated with the spectral theory of Jacobi matrices. The reader is referred to the appendix for their definitions, the notation we use, some basic results, and further references.

We start with a remark about the essential spectrum. Let $\Gamma$ be a sparse tree of type $\{L_n, k_n\}_{n=1}^{\infty}$. If $k_n \to \infty$, perturbation theory arguments show that the essential spectrum of $\Delta$ on $\Gamma$ is $[-2, 2]$. If $\{k_n\}$ is bounded, then from remark 4 after theorem 2.3, it is easy to see that the essential spectrum of $\Delta$ is contained in the essential spectrum of $J_0$. Since the reverse inclusion is immediate, we have the following
Proposition 3.1. Let $\Gamma$ be a sparse tree of type $\{L_n, k_n\}_{n=1}^\infty$ and let $J_0 = J_0(\Gamma)$ be the corresponding Jacobi matrix appearing in theorem 2.3. Let $\sigma_{\text{ess}}(\Delta)$ be the essential spectrum of $\Delta$ on $\Gamma$ and $\sigma_{\text{ess}}(J_0)$ be the essential spectrum of $J_0$. Then, if either $k_n \to \infty$ or $\{k_n\}$ is bounded, then

$$\sigma_{\text{ess}}(\Delta) = \sigma_{\text{ess}}(J_0). \quad (3.1)$$

Now, let $H$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and $\psi \in \mathcal{H}$. The spectral measure associated with $\psi$ and $H - \mu$, is the unique measure on $\mathbb{R}$ satisfying

$$\langle \psi, (H - z)^{-1} \psi \rangle = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

(see e.g. [15]).

Theorem 2.3 reduces the spectral analysis of $\Delta$ on a sparse tree to the spectral analysis of Jacobi matrices with sparse ‘bumps’ off the diagonal. (For $\tilde{\Delta}$, we get ‘bumps’ on the diagonal as well). An application of (suitably adapted) methods from the spectral theory of one-dimensional Schrödinger operators with sparse potentials (see [11] for a review of the relevant theory) to the situation at hand, allows us to establish interesting spectral behavior for the Laplacian on certain sparse trees.

The following basic lemma was proven in the appendix of [1] for the case of the Bethe lattice. It holds for any SH rooted tree.

Lemma 3.2. Let $\Gamma$ be a normal rooted SH tree with root $-O$. For any vertex $v$ in $\Gamma$, let $\delta_v \in \ell^2(\Gamma)$ be the delta function at $v$, and $\mu_v$ - the spectral measure associated with $\Delta$ and $\delta_v$. Let $(a,b)$ be an interval on which the absolutely continuous part of $\mu_O$ vanishes. Then, for any vertex $v$ of $\Gamma$, the absolutely continuous part of $\mu_v$ vanishes on $(a,b)$.

Remark. Throughout this paper, ‘absolutely continuous’ means absolutely continuous with respect to Lebesgue measure.

Proof. The proof is a simple consequence of the identification of the essential support of the absolutely continuous spectrum with the set of energies for which the Green’s function has positive imaginary part, combined with the recursion relation (see [1]) for the diagonal elements of the forward resolvents (these are the resolvents of $\Delta$ restricted to the various forward subtrees of $\Gamma$).

For a normal rooted SH tree - $\Gamma$, let $\{J_n(\Gamma)\}_{n=0}^\infty$ be the Jacobi matrices appearing in the decomposition of $\Delta$ on $\Gamma$, given by theorem 2.3. Let $\mu_n$ be the spectral measure associated with $\delta_1 \in \ell^2(\mathbb{Z}^+)$ and $J_n$. Then lemma 3.2 above, says that, if we want to prove that all spectral
measures associated with the Laplacian on $\Gamma$ are singular, it suffices to prove that $\mu_0$ is singular. This will be useful later on.

The following lemma is another simple tool for proving singularity of all the spectral measures. Its proof features the ‘bump’ transfer matrix which will prove itself useful throughout the rest of this paper. (See equations (B.10)-(B.13) for the definitions of the transfer matrices that we use below).

**Lemma 3.3.** Let $R_m$ be a strictly increasing sequence of natural numbers such that, for $m$ large enough, $R_{m+1} - R_m \geq 2$. Let

$$J = J(\{a(j)\}_{j=1}^\infty, \{b(j)\}_{j=1}^\infty)$$

be a Jacobi matrix such that

$$a(j) = \begin{cases} \rho_m & j = R_m \text{ for some } m \\ 1 & \text{otherwise} \end{cases} \quad (3.2)$$

and

$$b(j) \equiv 0, \quad (3.3)$$

where $\rho_m > \delta > 0$ for all $m$. Then, if $\{\rho_m\}_{m=1}^\infty$ is unbounded, then the spectral measure, $\mu$, associated with $\delta_1 \in \ell^2(\mathbb{Z}^+)$ and $J$, is singular with respect to Lebesgue measure.

**Proof.** Assume $\lim_{l \to \infty} \rho_{ml} = \infty$, $R_{ml} - R_{ml-1} \geq 2$, $R_{ml+1} - R_{ml} \geq 2$ and fix $E \in \mathbb{R}$. Then

$$S_{ml} = T_{R_{ml}+1,R_{ml}-1}(E) = S_{R_{ml}+1}(E)S_{R_{ml}}(E)$$

$$= \left( \begin{array}{cc} \frac{E^2}{\rho_{ml}} - \rho_{ml} & -\frac{E}{\rho_{ml}} \\ \frac{E}{\rho_{ml}} & -\frac{1}{\rho_{ml}} \end{array} \right). \quad (3.4)$$

It follows that

$$\max(1, \rho_{ml} - E^2) \leq \| T_{R_{ml}+1,R_{ml}-1}(E)^{-1} \|, \quad (3.5)$$

and so, applying proposition B.3 with $m_j = R_{ml} + 1$ and $l_j = R_{ml} - 1$, (note that $a_{R_{ml}-1} = 1$), we see that $\mu$ is singular on $\mathbb{R}$. \hfill \Box

**Corollary 3.4.** Let $\Gamma$ be a sparse tree of type $\{L_n, k_n\}_{n=1}^\infty$, with $\{k_n\}_{n=1}^\infty$ unbounded. Then all the spectral measures for $\Delta$ on $\Gamma$ are singular with respect to Lebesgue measure.

**Proof.** The statement follows from theorem 2.3, lemma 3.3 and the fact that for a general Jacobi matrix, the vector $\delta_1$ is a cyclic vector. \hfill \Box

On the other hand, a simple consequence of proposition B.4 is the following
Lemma 3.5. Let $R_m$ be a strictly increasing sequence of natural numbers. Let
\[ J = J(\{a(j)\}_{j=1}^{\infty}, \{b(j)\}_{j=1}^{\infty}) \]
be a Jacobi matrix such that
\[ a(j) = \begin{cases} \rho_m & j = R_m \text{ for some } m \\ 1 & \text{otherwise} \end{cases} \tag{3.6} \]
and
\[ b(j) \equiv 0, \tag{3.7} \]
where $\rho_m > 1$ for all $m$. Let $\{\beta_m\}_{m=1}^{\infty}$ be a sequence such that $\beta_m \geq \rho_m$ and $\lim_{m \to \infty} \beta_m = \infty$ and let $A_m = \prod_{l=1}^{m} \beta_l^2$. If for some $\varepsilon > 0$,
\[ \limsup_{m \to \infty} \frac{(R_{m+1} - R_m)}{A_m^{1+\varepsilon}} > 0 \tag{3.8} \]
then the spectral measure, $\mu$, associated with $\delta_1 \in \ell^2(\mathbb{Z}^+)$ and $J$, is continuous on $(-2, 2)$.

Proof. Consider an arbitrary closed interval $I \subseteq (-2, 2)$. We will show that $\mu(I \cap \cdot)$ is continuous. From this it will follow that $\mu((-2, 2) \cap \cdot)$ is continuous. For $R_m + 1 < j < R_{m+1}$ we have that
\[ S_j(E) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} \]
so that $\det(S_j(E)) = 1$. Furthermore, define, as in the proof of lemma 3.3
\[ S'_m(E) \equiv T_{R_m+1,R_m-1}(E) = S_{R_m+1}(E)S_{R_m}(E) \]
\[ = \begin{pmatrix} \frac{E^2}{\rho_m} - \rho_m & -E \\ \frac{E}{\rho_m} & -1 \end{pmatrix}. \tag{3.9} \]
Then we also have $\det(S'_m(E)) = 1$. Thus, if $j_2 \neq R_m$ and $j_1 \neq R_n$ for any $m, n$, we have that
\[ \det(T_{j_1,j_2}(E)) = 1. \tag{3.10} \]
Note that if $R_m + 1 \leq j_2 < j_1 < R_{m+1}$, then $T_{j_1,j_2}(E)$ is just the transfer matrix for the free Laplacian, so that there is a constant $C_1$, depending only on the interval $I$, such that $1 \leq \| T_{j_1,j_2}(E) \| < C_1$ for any such $j_1$, $j_2$ and $E \in I$. In addition, we have from (3.9)
\[ \| S'_{R_m}(E) \| \leq \rho_m + 5. \tag{3.11} \]
Thus, for $R_m < j < R_{m+1}$ we have
\[ \| T_j(E) \| \leq C^m \prod_{j=0}^{m} \rho_j \tag{3.12} \]
for some constant \( C \) depending on \( I \).

Let \( m_l \to \infty \) be a sequence for which
\[
\frac{R_{m_l+1} - R_{m_l}}{A_{m_l}^{(1+\varepsilon)}} > \delta
\]
for some \( \delta > 0 \). Let \( M \) be chosen so that for all \( m > M \), \( \beta_m^\varepsilon > C \).

Now, for sufficiently large \( m_l > N \), we have, from (3.12),
\[
\sum_{j=R_{m_l}+1}^{R_{m_l+1}-1} \| T_j(E) \|^{-2} \geq \frac{\delta}{2} A_m^\varepsilon C^{-2m_l} \geq \frac{\delta}{2} C^{-2N}.
\]
Thus, the tail of the sum in (B.15) does not converge to zero. Therefore
the sum is divergent and \( \mu \) has no eigenvalues in \( I \). This proves the lemma. \( \square \)

**Corollary 3.6.** Let \( \Gamma \) be a rooted SH tree of type \( \{L_n, k_n\}_{n=1}^\infty \). Assume
that, for some \( \varepsilon > 0 \),
\[
\limsup_{n \to \infty} \frac{(L_{n+1} - L_n)}{A_n^{(1+\varepsilon)}} > 0,
\]
where \( A_n = \prod_{l=1}^n \beta_l \) for some sequence \( \beta_n \geq k_n \), such that \( \beta_n \to \infty \).
Then any spectral measure for \( \Delta \) on \( \Gamma \), is continuous on \( (-2,2) \).

**Proof.** The statement follows from theorem 2.3, lemma 3.5 and the fact
that for a general Jacobi matrix, the vector \( \delta_1 \) is a cyclic vector. \( \square \)

A simple consequence of corollaries 3.4, 3.6 and proposition 3.1 is
the following theorem from [5]:

**Theorem 3.7** (Theorem 4.1 in [5]). Let \( \{k_n\}_{n=1}^\infty \) be a sequence of natural numbers such that \( k_n \to \infty \) as \( n \to \infty \). Let \( A_n = \prod_{j=1}^n k_j \). Assume
that \( (L_{n+1} - L_n) \to \infty \) and let \( \Gamma \) be a SH rooted tree of type \( \{L_n, k_n\}_{n=1}^\infty \).
Then the spectrum of \( \Delta \) on \( \Gamma \) consists of the interval \( [-2,2] \) along with
some discrete point spectrum outside this interval. If for some \( \varepsilon > 0 \),
\[
\limsup_{n \to \infty} \frac{(L_{n+1} - L_n)}{A_n^{(1+\varepsilon)}} > 0,
\]
then any spectral measure for \( \Delta \) is purely singular continuous on
\( (-2,2) \).

Since the next section discusses trees with bounded \( k_n \), we quote the
corresponding result from [5]. We sketch its proof here since some of
the ideas will appear in the sequel:
Theorem 3.8 (Theorem 2.2 in [5]). Let $k_0 \geq 2$ be a natural number and let $k_n \equiv k_0$. Assume that $(L_{n+1} - L_n) \to \infty$ and let $\Gamma$ be a SH rooted tree, of type $\{L_n, k_n\}_{n=1}^{\infty}$. Then the essential spectrum of $\Delta$ on $\Gamma$ contains the interval $[-2, 2]$ and, provided $(L_{n+1} - L_n)$ increase sufficiently rapidly, any spectral measure for $\Delta$ is purely singular continuous on $(-2, 2)$. By ‘sufficiently rapidly’ we mean that $(L_{n+1} - L_n)$ has to be made sufficiently large with respect to $\{(L_{i+1} - L_i)\}_{i<n}$.

Proof. The claim about the essential spectrum follows immediately from proposition 3.1.

We want to show that if $(L_{n+1} - L_n)$ grow sufficiently rapidly (in the sense described in the theorem), then $\mu_{\delta}$ (the spectral measure of the delta function at the root of $\Gamma$) is purely singular continuous on $(-2, 2)$. Lemmas 3.2 and 3.5 say that this suffices to imply that $(L_{n+1} - L_n)$ may be made to grow so fast as to make all spectral measures singular continuous.

Thus, our problem is reduced to the problem of studying a Jacobi matrix of the form $J(\{a(j)\}, \{b(j)\})$ with

$$a(j) = \begin{cases} \sqrt{k} & j = L_n + 1 \text{ for some } n \\ 1 & \text{otherwise} \end{cases} \quad (3.17)$$

and

$$b(j) \equiv 0. \quad (3.18)$$

The proof now follows closely Pearson’s classical proof [14]. For any $E \in (-2, 2)$, let $\phi \in (0, \pi)$ be defined by $2\cos(\phi) = E$, and let $u_{1,2\cos(\phi)} \equiv u_{1,E}$ as defined in the appendix. Define the EFGP variables [10] $r_\phi(j)$ and $\theta_\phi(j)$ through:

$$r_\phi(j) \cos(\theta_\phi(j)) = u_{1,E}(j) - \cos(\phi)u_{1,E}(j-1) \quad (3.19)$$

$$r_\phi(j) \sin(\theta_\phi(j)) = \sin(\phi)u_{1,E}(j-1). \quad (3.20)$$

It is easy to see that for $L_n + 3 < j \leq L_{n+1} + 1$, $r_\phi(j) = r_\phi(j-1)$ and $\theta_\phi(j) = \theta_\phi(j-1) + \phi$ (since the evolution equations for $u_{1,E}$ there coincide with those of the usual Laplacian on $\mathbb{Z}^+$), so that for all $n$,

$$r_\phi(L_n + 1) = r_\phi(L_{n-1} + 3) \quad (3.21)$$

and

$$\theta_\phi(L_n + 1) = \theta_\phi(L_{n-1} + 3) + (L_n - L_{n-1} - 2)\phi. \quad (3.22)$$

Since the map $g(\phi) = 2\cos \phi$ is continuously invertible on $(0, \pi)$, it is clear that, instead of studying $\mu_{\delta}$ on a given closed interval in $(-2, 2)$, we may study its push-forward (via $g^{-1}$) - $\nu$ - on the corresponding closed interval in $(0, \pi)$. Now, since $a(j)$ is bounded, standard methods
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(see e.g. [6] section III.3) show that $\nu$ on $(0, \pi)$ is equivalent to the measure defined by the limit

$$\lim_{n \to \infty} \int_I \frac{d\phi}{r_\phi (L_n + 3)^2}$$

(3.23)

where the integral is performed over closed subintervals $I \subseteq (0, \pi)$. Noting that

$$u_{1,2} \cos(\phi)(j) = \frac{\sin(\theta_\phi(j))}{\sin(\phi)} r_\phi(j),$$

$$u_{1,2} \cos(\phi)(j) = \frac{\sin(\phi + \theta_\phi(j))}{\sin(\phi)} r_\phi(j),$$

one can employ the transfer matrix

$$S_n'(2 \cos(\phi)) \equiv S_{L_n+2}(2 \cos(\phi))S_{L_n+1}(2 \cos(\phi))$$

and (3.21)-(3.22) to express $(r_\phi (L_n + 3))^2$ as a function of $(r_\phi (L_{n-1} + 3))^2$. Note that $S_n'(2 \cos(\phi))$ is unimodular. We get that

$$\int_I \frac{d\phi}{r_\phi (L_n + 3)^2} = \int_I \frac{d\phi}{r_\phi (L_{n-1} + 3)^2} \prod_{i=1}^n f_i(\phi, N_i \phi, \theta_{i-1}(\phi)),$$

(3.24)

where $N_i = L_i - L_{i-1} - 2$ and

$$f_i(\phi, y, \theta)^{-1} = A(\phi) + B(\phi) \cos(\theta + y) + C(\phi) \sin(\theta + y),$$

(3.25)

with $A^2 - B^2 - C^2 = \det(S_n'(E)) = 1$.

The situation described in (3.24)-(3.25) is exactly the same as that in section 3 of [14] (although the explicit expressions one gets for $A$, $B$ and $C$ are different). Note that the precise form of $S_n'(E)$ is of no importance. The unimodularity of this transfer matrix, together with the fact that $a(L_n + 1) = \text{const.} \neq 1$, suffice to imply that (as in [14]) the corollary to theorem 1 from [14], applies in this situation. This shows that if the $L_n$ are chosen to increase rapidly enough, $\nu$, and thus $\mu_{\delta_\phi}$, is singular continuous.

□

4. Finite Dimensional Trees

As noted in the introduction, aside from providing interesting examples for the Laplacian, sparse trees are interesting as objects interpolating between the one dimensional line ($\mathbb{Z}^+$) and the Bethe lattice (which is infinite dimensional in a natural sense). By tuning the sequences $\{L_n\}$ and $\{k_n\}$, one may construct trees with dimensions having any real value between one and infinity. A simple example is obtained as follows: Let $k_n \equiv k \geq 2$ and take $L_n = \lceil \gamma^n \rceil$ for some $\gamma > 1$. Denote
the SH rooted tree of type \( \{L_n, k_n\}_{n=1}^{\infty} \) by \( \Gamma_{k, \gamma} \). A simple calculation gives:

**Proposition 4.1.** Fix \( \gamma > 1 \) and \( N \ni k \geq 2 \). Let \( \Gamma = \Gamma_{k, \gamma} \) and let

\[ S_\Gamma(r) = \{ v_i \in \mathcal{V}(\Gamma) \mid d(v_i, O) \leq r \} \]

where \( \mathcal{V}(\Gamma) \) is the set of vertices of \( \Gamma \). Then

\[ \limsup_{r \to \infty} \frac{\log \#S_\Gamma(r)}{\log r} = \liminf_{r \to \infty} \frac{\log \#S_\Gamma(r)}{\log r} = \frac{\log \gamma k}{\log \gamma}. \]

Below, we shall refer to the quantity \( \frac{\log \gamma k}{\log \gamma} \) as the *dimension* of \( \Gamma_{k, \gamma} \).

In the context of the analogy between sparse trees and Schrödinger operators with sparse potentials, described in the previous section, \( \Gamma_{k, \gamma} \) is analogous to a Schrödinger operator with bumps of fixed height placed at the sites \([\gamma^n]\) of \( \mathbb{Z}^+ \). Zlatos deals with such operators in [21] and the analysis we present below is an adaptation of his methods (in particular - section 6 of [21]) to the case at hand. While a large part of the argument translates word for word, there are a few significant changes, mainly having to do with the fact that the transfer matrices for our case are not, in general, unimodular. This is important for some of the arguments and, therefore, has to be bypassed to get the same results here. We discuss the changes below and give a sketch of the proof. However, we refer the reader to [21] for a more detailed discussion.

First,

**Definition 4.2.** Let \( \Gamma \) be a rooted tree. For any self-adjoint operator \( H \) on \( \ell^2(\Gamma) \), and \( -\pi/2 < \varrho < \pi/2 \), let

\[ H_\varrho = H - \tan(\varrho)P_O \]

where \( P_O \) is the orthogonal projection onto the subspace spanned by the delta function at \( O \). We refer to \( H_\varrho \) as \( H \) with boundary condition \(- \varrho \).

We also need:

**Definition 4.3.** Let \( \mu \) be a measure on \( \mathbb{R} \). We say that \( \mu \) has *exact local dimension* in \( I \subseteq \mathbb{R} \) if for any \( E \in I \) there is an \( \alpha(E) \) and for any \( \varepsilon > 0 \) there is \( \delta > 0 \) for which \( \mu((E - \delta, E + \delta) \cap \cdot) \) is both continuous with respect to \( (\alpha(E) - \varepsilon) \)-dimensional Hausdorff measure, and singular with respect to \( (\alpha(E) + \varepsilon) \)-dimensional Hausdorff measure. We call \( \alpha(E) \) the *local dimension* of the measure \( \mu \).

Let \( \omega_n \) be a random variable uniformly distributed over

\( [-n, -n + 1, \ldots, n - 1, n] \).
Let \((\Omega, \mathbb{P})\) be the product probability space for all \(\omega_n, n = 1, 2, \ldots\). Fix \(1 < k \in \mathbb{N}\) and \(\gamma > 2\) and for each \(\omega \in \Omega\) let \(\Gamma_{\omega}^k,\gamma\) be the SH rooted tree of type \(\{L_n^\omega, k_n\}_{n=1}^\infty\) for \(L_n^\omega = [\gamma^n] + \omega_n\) and \(k_n \equiv k\). Clearly, proposition 4.1 holds for any \(\Gamma_{\omega}^k,\gamma\). The main result of this section is

**Theorem 4.4.** For \(\mathbb{P}\)-a.e. \(\omega\), all the spectral measures for \(\Delta\) on \(\Gamma_{\omega}^k,\gamma\) are singular with respect to Lebesgue measure. Furthermore, let \(V(k) = \frac{(1+k)^2}{4k}\) and let

\[
I = \left( -\sqrt{\frac{8(\gamma - V(k))}{2\gamma - 1}}, \sqrt{\frac{8(\gamma - V(k))}{2\gamma - 1}} \right)
\]

if \(\gamma > V\), and \(I = \emptyset\) otherwise. Then for \(\mathbb{P}\)-a.e. \(\omega\) and for Lebesgue a.e. \(\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})\), the spectral measure, \(\mu_{\delta_1}\), associated with \(\Delta_{\vartheta}\) on \(\Gamma_{\omega}^k,\gamma\), and with the delta function at the root, is purely singular continuous in \(I\) with exact local dimension

\[
1 - \frac{\log(\frac{4V(k) - \vartheta^2}{4\vartheta^2})}{\log(\gamma)}
\]

and it is dense pure point in the rest of \([-2, 2]\).

**Corollary 4.5.** Assume \(\gamma \geq 4\). Then if the dimension of \(\Gamma_{\omega}^k,\gamma\) is at least 3, we have that \(I = \emptyset\) and so, for \(\mathbb{P}\)-a.e. \(\omega\) and for Lebesgue a.e. \(\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})\), the spectral measure \(\mu_{\delta_1}\) is dense pure point in \([-2, 2]\).

**Proof of the corollary.** This is a simple computation. \(\square\)

**Proof of theorem 4.4.** Theorem 2.3 and lemma 3.2 imply that the theorem is an immediate consequence of proposition 4.6 below. \(\square\)

**Proposition 4.6.** Fix \(k \geq 2\) and \(\gamma > 2\). For any \(\omega \in \Omega\) and \(-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}\), let \(J_{\vartheta}^\omega = J(\{a_{\omega}(j)\}, \{b_{\vartheta}(j)\})\) be a Jacobi matrix with

\[
a_{\omega}(j) = \begin{cases} \sqrt{k} & j = [\gamma^m] + \omega_m + 1 \text{ for some } m \\ 1 & \text{otherwise} \end{cases}
\]

and

\[
b_{\vartheta}(j) = \begin{cases} -\tan(\vartheta) & j = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Then, for \(\mathbb{P}\)-a.e. \(\omega\), and for any \(\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})\), the spectral measure \(\mu_{\vartheta}\), associated with the vector \(\delta_1 \in \mathbb{Z}^+\) and with the Jacobi matrix \(J_{\vartheta}^\omega\), is singular with respect to Lebesgue measure. Furthermore, for \(\mathbb{P}\)-a.e. \(\omega\) and for Lebesgue a.e. \(\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})\), the spectral measure \(\mu_{\vartheta}\) is purely singular continuous in \(I\) with exact local dimension given by (4.3), where \(I\) is as defined in theorem 4.4, and it is dense pure point in the rest of \([-2, 2]\).
A central role in the proof of the proposition will be played by the EFGP transform introduced in the proof of theorem 3.8:

Fix \( k \geq 2 \) and let \( J = J(\{a(j)\}, \{b(j)\}) \) be a Jacobi matrix satisfying

\[
a(j) = \begin{cases} \sqrt{k} & j = L_n + 1 \text{ for some } n \\ 1 & \text{otherwise} \end{cases} \quad \text{(4.6)}
\]

for a sequence \( \{L_n\} \) satisfying \( L_{n+1} - L_n \geq 2 \), and

\[
b(j) \equiv 0. \quad \text{(4.7)}
\]

For any \( E \in (-2, 2) \), let \( \phi \in (0, \pi) \) be defined by \( 2 \cos(\phi) = E \), and let \( u \) solve (B.4) for \( J \), namely

\[
a(j)u(j + 1) + a(j - 1)u(j - 1) = Eu(j), \quad j \geq 1 \quad \text{(4.8)}
\]

with \( a(0) = 1 \). Recall that the EFGP variables \([10]\) corresponding to \( u, r_\phi(j) \) and \( \theta_\phi(j) \), are defined through:

\[
r_\phi(j) \cos(\theta_\phi(j)) = u_E(j) - \cos(\phi)u_E(j - 1) \quad \text{(4.9)}
\]

\[
r_\phi(j) \sin(\theta_\phi(j)) = \sin(\phi)u_E(j - 1). \quad \text{(4.10)}
\]

First, note that there are positive constants, \( C_1(\phi), C_2(\phi) \), such that

\[
C_1(\phi)(|u(j - 1)|^2 + |u(j)|^2) \leq r_\phi(j)^2 \leq C_2(\phi)(|u(j - 1)|^2 + |u(j)|^2). \quad \text{(4.11)}
\]

We call \( r(j) \) the EFGP norm of \( u(j) \). Note, also, that for any \( j \)

\[
C_1(\phi)\left( \sum_{i=(j-1)}^{(j+2)} |u(i)|^2 \right) \leq r_\phi(j)^2 + r_\phi(j + 2)^2 \leq C_2(\phi)\left( \sum_{i=(j-1)}^{(j+2)} |u(i)|^2 \right). \quad \text{(4.12)}
\]

For a function \( f : \mathbb{Z}^+ \to \mathbb{C} \) and a sequence \( \mathcal{L} = \{L_n\}_{n=1}^\infty \) of natural numbers, define

\[
\|f\|_{L,\mathcal{L}} = \left( \sum_{1 \leq j \leq L, j \neq L_{n+2} \text{ for any } n} |f(j)|^2 \right)^{1/2}
\]

for any \( L \in \mathbb{N} \), and extend to \( L \in \mathbb{R} \) by linear interpolation. Then we have

**Lemma 4.7.** Let \( J = J(\{a(j)\}, \{b(j)\}) \) be as defined in (4.6)-(4.7) and let \( \mathcal{L} = \{L_n\} \). Assume that for some \( 1 \leq \beta < 2 \) and every \( E \) in some Borel set \( A \subseteq (-2, 2) \), the EFGP norm, \( r_u \), of every solution \( u \) of (4.8) obeys

\[
\limsup_{L \to \infty} \frac{\|r_u\|_{L,\mathcal{L}}^2}{L^\beta} < \infty. \quad \text{(4.13)}
\]
Then \( \mu(A \cap \cdot) \) is continuous with respect to \((2 - \beta)\)-dimensional Hausdorff measure, where \( \mu \) is the spectral measure associated with \( \delta_1 \) and \( J \).

**Lemma 4.8.** Let \( J = J(\{a(j)\}, \{b(j)\}) \) be as defined in (4.6)-(4.7) and let \( \mathcal{L} = \{L_n\} \). Let \( u_{1,E} \) be the solution of (4.8) that satisfies
\[
 u(0) = 0, \quad u(1) = 1,
\]
and let \( r_{1,E} \) denote the corresponding EFGP norm. If
\[
 \liminf_{L \to \infty} \frac{\| r_{1,E} \|_{L,\mathcal{L}}^2}{L^\alpha} = 0 \quad \text{(4.14)}
\]
for every \( E \) in some Borel set \( A \subseteq (-2, 2) \), then \( \mu(A \cap \cdot) \) is singular with respect to \( \alpha \)-dimensional Hausdorff measure, where \( \mu \) is the spectral measure associated with \( \delta_1 \) and \( J \).

**Proof of lemmas 4.7 and 4.8.** The proofs follow immediately from (4.12), from the fact that \( L_{n+1} - L_n \geq 2 \), and from propositions B.1 and B.2 respectively. \( \square \)

Our aim, therefore, is to control the growth of the EFGP norms of generalized eigenfunctions. We recall from the previous section that for \( L_n + 3 < j \leq L_{n+1} + 1 \), \( r_\phi(j) = r_\phi(j - 1) \) and \( \theta_\phi(j) = \theta_\phi(j - 1) + \phi \) and that, therefore,
\[
 r_\phi(L_n + 1) = r_\phi(L_{n-1} + 3) \quad \text{(4.15)}
\]
and
\[
 \theta_\phi(L_n + 1) = \theta_\phi(L_{n-1} + 3) + (L_n - L_{n-1} - 2)\phi. \quad \text{(4.16)}
\]
From lemmas 4.7 and 4.8, we see that \( r_\phi(L_n + 2) \) is irrelevant so all we need is to find the relation between \( r_\phi(L_n + 1) \) and \( r_\phi(L_n + 3) \). As before, using the relations
\[
 u_{2\cos(\phi)}(j - 1) = \frac{\sin(\theta_\phi(j))}{\sin(\phi)} r_\phi(j),
\]
\[
 u_{2\cos(\phi)}(j) = \frac{\sin(\phi + \theta_\phi(j))}{\sin(\phi)} r_\phi(j),
\]
and the transfer matrix
\[
 S'_n(2\cos(\phi)) \equiv S_{L_n+2}(2\cos(\phi)) S_{L_n+1}(2\cos(\phi))
\]
\[
 = \left( \begin{array}{cc}
 \left( \frac{4\cos^2(\phi)}{\sqrt{k}} - \sqrt{k} \right) & \frac{-2\cos(\phi)}{\sqrt{k}} \\
 \frac{2\cos(\phi)}{\sqrt{k}} & \frac{-1}{\sqrt{k}}
\end{array} \right),
\]
we get that
\[ r_\phi(L_n + 3)^2 = r_\phi(L_n + 1)^2. \]
\[ \cdot \left( A(\phi) + B(\phi) \cos(2\theta_\phi(L_n + 1)) + C(\phi) \sin(2\theta_\phi(L_n + 1)) \right), \] (4.17)
where
\[ A(\phi) = \frac{1}{\sin^2(\phi)} \left( \frac{1 + k^2}{2k} - \cos^2(\phi) \right) \geq 0 \] (4.18)
and
\[ A(\phi)^2 - B(\phi)^2 - C(\phi)^2 = 1. \] (4.19)
Explicit formulas for \( B(\phi) \) and \( C(\phi) \) can be derived but they are of no consequence. The last relation follows from the fact that \( \det(S_n'(2 \cos(\phi))) = 1 \).

Proof of proposition 4.6. For a fixed \( \phi \in (0, \pi) \) (and \( E = 2 \cos(\phi) \in (-2, 2) \)), let
\[ f(\theta) = \frac{1}{2} \log \left( A(\phi) + B(\phi) \cos(2\theta + C(\phi) \sin(2\theta) \right). \] (4.20)
Now, for a given \( \omega \in \Omega \), let \( L_\omega = \{ L_\omega^n \}_{n=1}^\infty \) (recall \( L_\omega^n = [\gamma^n] + \omega_n \)). Let \( u_{1,E}^\omega \) be a generalized eigenfunction for \( J_0^\omega \) such that \( u_{1,E}^\omega(0) = 0 \) and \( u_{1,E}^\omega(1) = 1 \), and let \( r_{1,\phi}^\omega \) be the corresponding EFGP norm. Then, the above implies that
\[ Y_n(\omega) \equiv \log r_{1,\phi}^\omega(L_n + 3) - \log r_{1,\phi}^\omega(L_{n-1} + 3) \]
\[ = \frac{1}{2} \log \left( A(\phi) + B(\phi) \cos(2\theta_\phi(L_n + 1)) + C(\phi) \sin(2\theta_\phi(L_n + 1)) \right). \] (4.21)
Note that, if \( \omega, \eta \in \Omega \) are such that \( \omega_j = \eta_j \) for \( j = 1, 2, \ldots, n - 1 \), and \( \eta_n = \omega_n + l \) for some \( l \), and \( Y_n(\omega) = f(\theta) \) for some \( \theta \), then \( Y_n(\eta) = f(\theta + l\phi) \).
Let
\[ Z = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)d\theta = \frac{1}{2} \log \frac{A(\phi) + \sqrt{A(\phi)^2 - B(\phi)^2 - C(\phi)^2}}{2} \]
\[ = \frac{1}{2} \log \frac{A(\phi) + 1}{2} = \frac{1}{2} \log \left( \frac{V(k)}{\sin^2(\phi)} - \frac{\cot^2(\phi)}{2} \right), \] (4.22)
and
\[ \tilde{f}(\theta) = f(\theta) - Z. \] (4.23)
Then \( \tilde{f}(\theta) = \tilde{f}(\theta + 2\pi) \), \( \int_0^{2\pi} \tilde{f}(\theta) d\theta = 0 \) and \( \tilde{f} \) is \( C^4 \), so \( \tilde{f} \) satisfies the conditions of lemma 6.2 in [21]. Thus, it follows that, for any \( \phi \) in a set of full Lebesgue measure - \( S \subseteq (0, \pi) \), and for \( \mathbb{P}\text{-a.e. } \omega \),
\[
\sum_{n=1}^{N}(Y_n(\omega) - Z) \xrightarrow{N} 0,
\]
(see section 6 of [21]). From this we get that for any \( \phi \in S \) and for a.e. \( \omega \in \Omega \), the EFGP norm \( r_{1,\phi}^\omega \) satisfies
\[
(L_n + 3)^{d_1} \leq r_{1,\phi}^\omega(L_n^\omega + 3) \leq (L_n^\omega + 3)^{d_2}
\]
for any
\[
d_1 < \frac{Z}{\log \gamma} < d_2
\]
provided \( n \) is large enough. By Fubini’s theorem, we get that (4.24) holds for a.e. \( \omega \) and Lebesgue almost every \( \phi \). Furthermore, lemma 4.9 below assures us that for each such \((\omega, \phi)\), there exists a subordinate solution \( u_{\text{sub}, \phi}^\omega \) to (4.8) whose EFGP norm \( r_{\text{sub}, \phi}^\omega \) satisfies
\[
r_{\text{sub}, \phi}^\omega(L_n^\omega + 3) \leq (L_n^\omega + 3)^{-d_1}
\]
for large enough \( n \). Since the absolutely continuous part of \( \mu_\varrho \) (for any \( \varrho \)) is supported on the set of energies with no subordinate solution, it follows that, for any \( \varrho \), \( \mu_\varrho \) is purely singular on \((-2, 2)\). Furthermore, since the singular part of \( \mu_\varrho \) is supported on the set of energies where the subordinate solutions obey the appropriate boundary conditions, rank-one perturbation arguments (see e.g. [17]), imply that for Lebesgue almost any \( \varrho \), \( \mu_\varrho \) is supported on the set of energies where the subordinate solutions behave as in (4.25). Noting that, in this case,
\[
||r_{\text{sub}, \phi}^\omega||_{L^2_{\omega}, L}^2 \leq CL^{1-2\frac{Z}{\log \gamma}}
\]
if \( \frac{Z}{\log \gamma} \leq \frac{1}{2} (\iff E \in I) \) and is square-summable otherwise, and that the EFGP norm - \( r_\phi^\omega \) of any other solution to (4.8) satisfies
\[
||r_\phi^\omega||_{L^2_{\omega}, L}^2 \leq CL^{1+2\frac{Z}{\log \gamma}}
\]
we finish the proof with the help of lemmas 4.7 and 4.8.

The following lemma was used in the proof above. It is essentially the same as lemma 2.1 from [21], the only difference being in that not all transfer matrices for the general Jacobi case are unimodular. Since not all transfer matrices are relevant for the argument, this is inconsequential. In the proof below we only demonstrate this point. For the complete proof the reader is referred to [21].
Lemma 4.9. Let \( J = J(\{a(j)\}, \{b(j)\}) \) be a Jacobi matrix satisfying

\[
a(j) = \begin{cases} \sqrt{k} & j = L_n + 1 \text{ for some } n \\ 1 & \text{otherwise} \end{cases}
\]

(4.26)

for a sequence \( \{L_n\} \) satisfying \( L_{n+1} - L_n \geq 2 \), and

\[
b(j) \equiv 0.
\]

(4.27)

Assume that for some \( E \in (-2, 2) \), \( u \) is a solution of (4.8) whose EFGP norm satisfies

\[
r(L_n + 3) = e^{\sigma_n}
\]

(4.28)

where \( \sigma_n = \sum_{j=1}^n (Z_j + X_j) \) with \( 0 < d_1 \leq Z_j \leq d_2 < \infty \) and \( \sum_{j=1}^n X_j = o(n) \). Then there exists a subordinate solution \( v \) of (4.8) for \( E \), such that for any \( d < d_1 \) and for all sufficiently large \( n \), the corresponding EFGP norm \( p \) satisfies

\[
p(L_n + 3) \leq e^{-dn}.
\]

(4.29)

Proof. Let \( v \) be any solution of (4.8) different from \( u \) and let \( p \) be its EFGP norm. Since the transfer matrices \( T_{L_n+2}(E) \) are unimodular, the argument of theorem 2.3 from [10] applies to show that there exist \( E \)-dependent constants, \( c_1, c_2 \) such that

\[
c_1 \max(p_{L_n+3}, r_{L_n+3}) \leq \| T_{L_n+2}(E) \| \leq c_2 \max(p_{L_n+3}, r_{L_n+3}).
\]

(4.30)

Note, further, that there exists a constant \( B > 0 \) such that

\[
\| T_{n+1,n}(E) \| \equiv \| T_{L_n+1+2L_n+2}(E) \| \leq B
\]

(4.31)

and that \( \det(T_{n+1,n}(E)) = 1 \) as well. Thus, it follows that

\[
\sum_{n=1}^\infty \| T_{n,n-1}(E) \|^2 / \| T_{L_n+2}(E) \|^2 < \infty
\]

(4.32)

so one can apply theorem 8.1 from [12] to get a vector \( \overline{v} \in \mathbb{R}^2 \) such that

\[
\frac{\| T_{L_n+2}(E)\overline{v} \|}{\| T_{L_n+2}(E)\overline{v} \|} \to 0
\]

for any other vector \( \overline{v} \in \mathbb{R}^2 \). From this point, the proof follows the proof of lemma 2.1 from [21], word for word, to show that \( \overline{v} \) generates the claimed solution. \( \square \)
Appendix A. Self-Adjointness of the Laplacian on Normal SH Rooted Trees

Proposition A.1. Let $\Gamma$ be a rooted SH tree of type $\{L_n, k_n\}_{n=1}^\infty$. Then the operator $\Delta_1$ defined over

$$D(\Delta_1) = \{u \in \ell^2(\Gamma) \mid u \text{ is of compact support}\} \quad (A.1)$$

via the equation

$$(\Delta_1 u)(x) = \sum_{y : d(x, y) = 1} u(y), \quad (A.2)$$

is symmetric. If $\lim \sup_{n \to \infty} (L_{n+1} - L_n) > 1$ then $\Delta$ - the closure of $\Delta_1$ - is self-adjoint. The same statement holds for $\tilde{\Delta}_1$ and $\tilde{\Delta}$ (defined over the same domain), with equation (A.2) replaced by

$$(\tilde{\Delta}_1 u)(x) = \sum_{y : d(x, y) = 1} u(y) - \#\{y : d(x, y) = 1\} \cdot u(x). \quad (A.3)$$

Proof. Since the proof for $\Delta$ and $\tilde{\Delta}$ is precisely the same, we use $\Delta$.

It is trivial to see that $\Delta_1$ is symmetric, so in order to show that $\Delta$ is self-adjoint, all we have to show is that $\ker(\Delta^* \pm i) = \{0\}$.

Assume that $\Delta u = iu$, then it follows that $\Delta \overline{u} = -i\overline{u}$ (where $\overline{\cdot}$ for a complex number denotes complex conjugation). Let $n_j$ be a subsequence for which $L_{n_j} + 1 < L_{n_j+1}$. For a vertex $v$ with $|v| \equiv d(v, O) = L_{n_j} + 1$, let us denote its unique forward neighbor by $\hat{v}$. One can verify that an analogue of Green’s formula (see e.g. [4] Chapter VII, formula 1.4) holds and we have:

$$2i \sum_{|v| \leq L_{n_j} + 1} |u(v)|^2 = \sum_{|v| = L_{n_j} + 1} u(\hat{v}) \cdot \overline{u(v)} - \sum_{|v| = L_{n_j} + 1} \overline{u(\hat{v})} \cdot u(v)$$

so

$$\sum_{|v| \leq L_{n_j} + 1} |u(v)|^2 \leq \sum_{|v| = L_{n_j} + 1} |u(\hat{v})| \cdot |u(v)|.$$

From this it follows that if $u \neq 0$, then the RHS above does not converge to zero and therefore $u \notin \ell^2(\Gamma)$. This proves the proposition. \qed

Appendix B. Eigenfunctions and Transfer Matrices for Jacobi Matrices

Let $J = J(\{a(j)\}, \{b(j)\})$ be a Jacobi matrix with $b(j) \in \mathbb{R}$ and $a(j) > 0$ satisfying $\sum_{j=1}^{\infty} \frac{1}{a(j)} = \infty$ (which suffices for $J$ to be self-adjoint [4]). A basic idea in the spectral theory of Jacobi matrices is
to relate spectral properties of $J$ as reflected by $\mu = \mu_{\delta_1}$ - the spectral measure of $\delta_1$ - to properties of formal eigenfunctions

$$Ju = Eu.$$  \hspace{1cm} \text{(B.1)}

By this term we mean functions $u : \mathbb{Z}^+ \to \mathbb{C}$ which satisfy

$$a(j)u(j + 1) + a(j - 1)u(j - 1) + b(j)u(j) = Eu(j), \quad j > 1 \hspace{1cm} \text{(B.2)}$$

$$a(1)u(2) + b(1)u(1) = Eu(1). \hspace{1cm} \text{(B.3)}$$

Since, for a given $E \in \mathbb{R}$, all solutions to (B.2)-(B.3) are linearly dependent (determined by $u(1)$), it suffices to study $u_{1,E}$ which is the solution satisfying $u_{1,E}(1) = 1$. It is convenient to define $a(0) = 1$ and to extend (B.2) to $j = 1$ by demanding $u_{1,E}(0) = 0$. Thus $u_{1,E}$ is the unique solution to

$$a(j)u(j + 1) + a(j - 1)u(j - 1) + b(j)u(j) = Eu(j), \quad j \geq 1 \hspace{1cm} \text{(B.4)}$$

with

$$u_{1,E}(0) = 0, \quad u_{1,E}(1) = 1. \hspace{1cm} \text{(B.5)}$$

We further define $u_{2,E}$ as the unique solution to (B.4) satisfying

$$u_{2,E}(0) = 1, \quad u_{2,E}(1) = 0. \hspace{1cm} \text{(B.6)}$$

Note that any solution, $u$, to (B.4) with $u(1) \neq 0$ can be viewed as $u_{1,E}$ for a slightly modified Jacobi matrix. Namely, $u$ solves (B.2)-(B.3) for the same set of parameters except with $b(1)$ changed to $(b(1) + \frac{u(0)}{u(1)})$.

This remark is basic for the analysis of section 4.

We say that $u_{1,E}$ is subordinate if

$$\lim_{L \to \infty} \left\| \frac{u_{1,E}}{L} \right\|_{L} = 0 \hspace{1cm} \text{(B.7)}$$

where

$$\| f \|_{L} = \left( \sum_{j=1}^{[L]} |f(j)|^2 + (L - [L])|f([L] + 1)|^2 \right)^{1/2}.$$  

The Gilbert-Pearson theory of subordinacy [7] says that the singular part of $\mu$ is supported on the set of energies where $u_{1,E}$ is subordinate, and that the absolutely continuous part of $\mu$ is supported off this set. The Jitomirskaya-Last extension of this theory [9] analyzes further the singular part of $\mu$ according to its singularity/continuity with respect to dimensional Hausdorff measures (see [9] for the concept of Hausdorff measures and dimensions). In section 4 of the paper we use the following results from [9]:
Proposition B.1 ([9]). Assume that \(1 \leq a(j) \leq M\) for some \(M > 1\). Assume that for some \(1 \leq \beta < 2\) and every \(E\) in some Borel set \(A\), every solution \(u\) of (B.4) obeys

\[
\limsup_{L \to \infty} \frac{\| u \|_L^2}{L^\beta} < \infty. \tag{B.8}
\]

Then \(\mu(A \cap \cdot)\) is continuous with respect to \((2 - \beta)\)-dimensional Hausdorff measure.

Proposition B.2 ([9]). Assume that \(1 \leq a(j) \leq M\) for some \(M > 1\). If

\[
\liminf_{L \to \infty} \frac{\| u_{1,E} \|_L^2}{L^\alpha} = 0 \tag{B.9}
\]

for every \(E\) in some Borel set \(A\), then \(\mu(A \cap \cdot)\) is singular with respect to \(\alpha\)-dimensional Hausdorff measure.

The next results we quote relate the properties of \(\mu\) to the properties of the transfer matrices corresponding to \(J\). These are the \(2 \times 2\) matrices

\[
T_j(E) = S_j(E) S_{j-1}(E) \cdots S_1(E), \tag{B.10}
\]

where

\[
S_j(E) = \begin{pmatrix} \frac{E-b(j)}{a(j)} & -\frac{a(j-1)}{a(j)} \\ \frac{1}{a(j)} & 0 \end{pmatrix}. \tag{B.11}
\]

It isn’t hard to see that

\[
T_j(E) = \begin{pmatrix} u_{1,E}(j+1) & u_{2,E}(j+1) \\ u_{1,E}(j) & u_{2,E}(j) \end{pmatrix} \tag{B.12}
\]

so it is not surprising to find that the behavior of \(T_j(E)\) is related to the behavior of the eigenfunctions. For any \(j_1, j_2\), we use the shorthand

\[
T_{j_1,j_2}(E) \equiv T_{j_1}(E) T_{j_2}(E)^{-1}. \tag{B.13}
\]

The following is a generalization of theorem 1.2 of [12] relating the behavior of \(T_j(E)\) with the existence of absolutely continuous spectrum.

Proposition B.3. Let \(m_j, l_j\) be arbitrary sequences of natural numbers and let

\[
A_1 = \left\{ E \mid \liminf_{j \to \infty} \frac{1}{a_{l_j}} \| T_{m_j,l_j}(E) \| < \infty \right\}. \tag{B.14}
\]

Then \(A_1\) supports the a.c. part of \(\mu\) in that \(\mu_{\text{ac}}(\mathbb{R} \setminus A_1) = 0\).
Proof. Note that \( \det(T_j(E)) = \frac{1}{a_{t_j}} \) so that
\[
\| T_j(E)^{-1} \| = a_{t_j} \| T_j(E) \|. 
\]
Thus, we have that
\[
\frac{1}{a_{t_j}} \| T_m(E)T_j(E)^{-1} \| \leq \frac{1}{a_{t_j}} \| T_m(E) \| \| T_j(E)^{-1} \| 
= \| T_m(E) \| \| T_j(E) \|. 
\]
From here one proceeds exactly as in the proof of theorem 3.4D of [12] to get the conclusion: We know that \( u_1,E(j) \), viewed as functions of \( E \), are orthonormal polynomials with respect to \( d\mu \), and that \( u_2,E(j) \) are orthonormal polynomials with respect to another measure - \( d\tilde{\mu} \), such that the measure \( d\nu = \min(d\mu, d\tilde{\mu}) \) is purely absolutely continuous and equivalent to the absolutely continuous part of \( d\mu \) (see [12]). Thus, from the characterization (B.12), we get that
\[
\int_{\mathbb{R}} \| T_j(E) \|^2 d\nu(E) \leq 4. 
\]
The theorem now follows from an application of the Cauchy-Schwarz inequality and Fatou’s lemma.

In order to show absence of eigenvalues, the following idea of Simon-Stolz [19] is useful:

**Proposition B.4** (See [19]). For a given \( E \in \mathbb{R} \), if
\[
\sum_{j, \det(T_j(E))=1} \| T_j(E) \|^2 = \infty, \tag{B.15}
\]
then there can be no eigenvalue at \( E \).

**Proof.** The proposition follows from the observation that if \( \det(T_m(E)) = 1 \) and \( u \) solves (B.2) then
\[
|u(m+1)|^2 + |u(m)|^2 \geq \frac{|u(1)|^2 + |u(0)|^2}{\| T_m(E) \|^2}. 
\]
\( \square \)
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