Coherent States for Unusual Potentials

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ABSTRACT

The program to construct minimum-uncertainty coherent states for general potentials works transparently with solvable analytic potentials. However, when an analytic potential is not completely solvable, like for a double-well or the linear (gravitational) potential, there can be a conundrum. Motivated by supersymmetry concepts in higher dimensions, we show how these conundrums can be overcome.

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1 Background

The Minimum-Uncertainty Method to obtain coherent states for general potentials harks back to Schrödinger’s discovery of the coherent states (of the harmonic oscillator) [1]. It has been applied to general Hamiltonian potential systems, to obtain both generalized coherent states and generalized squeezed states [2, 3].

One starts with the classical Hamiltonian problem in terms of $x_c$ and $p_c$, the classical position and momentum. Then one transforms it into the “natural classical variables,” $X_c$ and $P_c$, which vary as the sin and the cos of the classical $\omega t$. The classical Hamiltonian is therefore of the form $P_c^2 + X_c^2$. These natural classical variables are next changed into “natural” quantum operators. These quantum operators have a commutation relation and an associated uncertainty relation:

$$[X, P] = iG, \quad (\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle G \rangle^2, \quad (1)$$

where in general $G$ is an operator. The states that minimize this uncertainty relation are given by the solutions to the equation

$$Y\psi_{ss} \equiv \left( X + \frac{i\langle G \rangle}{2(\Delta P)^2} P \right) \psi_{ss} = \left( \langle X \rangle + \frac{i\langle G \rangle}{2(\Delta P)^2} \langle P \rangle \right) \psi_{ss}. \quad (2)$$

Note that of the four parameters $\langle X \rangle, \langle P \rangle, \langle P^2 \rangle$, and $\langle G \rangle$, only three are independent because they satisfy the equality in the uncertainty relation. Therefore,

$$(X + iBP)\psi_{ss} = C\psi_{ss}, \quad B = \frac{\Delta X}{\Delta P}, \quad C = \langle X \rangle + iB\langle P \rangle. \quad (3)$$

Here $B$ is real and $C$ is complex. These states, $\psi_{ss}(B, C)$, are the minimum-uncertainty states for general potentials [2, 3]. Using later parlance, they are the squeezed states for general potentials. $B$ can be adjusted to $B_0$ so that the ground eigenstate of the potential is a member of the set. Then these restricted states, $\psi_{ss}(B = B_0, C) = \psi_{cs}(B_0, C)$, are the minimum-uncertainty coherent states for general potentials.

It can be understood that $\psi_{ss}(B, C)$ and $\psi_{ss}(B_0, C)$ are the squeezed and coherent states by recalling the situation for the harmonic oscillator. The harmonic oscillator
coherent states are the displaced ground state. The harmonic oscillator squeezed states are Gaussians that have widths different than that of the ground state Gaussian, which are then displaced.

For the harmonic oscillator these coherent states are equivalent to those obtained from the ladder operator method:

\[ a|\alpha\rangle = \alpha|\alpha\rangle. \] (4)

In general the \( X \) and \( P \) operators can be given in terms of the raising and lowering operators (or their \( n \)-dependent generalizations):

\[ X = \frac{1}{\sqrt{2}}[a + a^\dagger], \quad P = \frac{1}{i\sqrt{2}}[a - a^\dagger]. \] (5)

Here \( a \) and \( a^\dagger \) are the lowering and raising operators of the system.

### 2 The double-well and linear potential conundrums

Although this procedure works well for exactly solvable systems, one nagging question has always been if one could, in principle, handle double-well potentials. This question was raised by a number of people, in particular by Rohrlich [4]. The problem was that no completely solvable double-well potential existed. Therefore, earlier techniques could not give a demonstration that a coherent-state procedure could analytically work for a double well. In the following Section 3 we discuss supersymmetry techniques that have now been developed and then apply them to a double-well system in Section 4.

Another specific problem has to do with the linear potential, \( V = mg|x| \). As was pointed out by Kienle and Straub [5], the standard method of solution for the natural classical variables [2, 3] breaks down here. Usually, solving for the natural variables amounts to solving the differential equation

\[ \frac{d}{dx} X_c(x) = \omega_c(E_c) \left( \frac{m}{2} \right)^{1/2} \left[ \frac{X^2_c(MAX) - X^2_c(x)}{E_c - V(x)} \right]^{1/2}. \] (6)
For normal systems, like $V \propto \{x^2, -1/\cosh^2 x\}$, respectively, Eq. (8) is easily solved; $X_c \propto \{x, \cosh x\}$, respectively. Here things get singular. In Section 5 we apply the same supersymmetric techniques of Section 3 to resolve this problem.

3 Supersymmetry for Volcano Potentials

We first remind the reader of another type of uncommon potential, volcano-shaped potentials. They turn out to be of current interest in theories of higher dimensions [6]-[12]. In these theories one can be trying to discover if volcano-shaped potentials have zero-energy bound states satisfying supersymmetry [13], in what amounts to a 1-dimensional Schrödinger equation [14].

An example is the volcano potential (shown in Figure 1)

$$V(z) = \frac{-\left(\sqrt{5} - \frac{1}{2}\right) + \frac{19}{4}z^2}{[1 + z^2]^2}. \quad (7)$$

Figure 1: The dashed and solid curves show the potentials of Eqs. (7) and (14), respectively, both of which are supersymmetric and have zero-energy ground states.
(For the rest of this paper we use the unitless quantities defined by \( \{ h, m \} \rightarrow 1 \), with factors such as 2 absorbed into \( x \rightarrow z \). Thus, the first term in the Schrödinger equation will always have the form \(-\partial^2\).

Equation (7) is a (Schrödinger-like factorization) supersymmetric potential \([13]\) of the form

\[
V(z) = [W'(z)]^2 - W''(z).
\]

\[
\psi_0(z) = N \exp[-W(z)],
\]

\[
W(z) = \left[ \frac{\sqrt{5}}{2} - \frac{1}{4} \right] \ln \left( 1 + z^2 \right).
\]

The Hamiltonian can be written as

\[
H = -\partial^2 + V(z) = A^\dagger A,
\]

\[
A = \partial + W'(z).
\]

Any potential that is supersymmetric has a ground state with zero-energy. Sometimes a constant must be added to a potential to make it supersymmetric. For example, the hydrogen atom and simple harmonic oscillator potentials can be made supersymmetric by subtracting the original ground-state energies from the potentials \([15]\).

But playing with the form of the above \( W(z) \), one can quickly convince oneself that various shaped potentials can be obtained. For example, a volcano potential with a plug in the center is given by (also shown in Figure \([\text{Fig. 1}]\)\)

\[
W(z) = \frac{1}{4} \ln \left[ 1 - \frac{1}{2} x^2 + x^4 \right],
\]

\[
V(z) = \frac{1 - \frac{45}{4} x^2 - 3 x^4 + 8 x^6}{4 (1 - x^2/2 + x^4)^2}.
\]

Note that although we have analytic potentials and analytic ground states, here we do not have the excited spectra and their wave functions. However, because of the properties of supersymmetry, it will turn out that such a situation will be sufficient to resolve our conundrums.
4 Demonstrating coherent states for a double-well potential

Stimulated by the results of the last section, consider the function

\[ W(z) = -\frac{1}{2}z^2 + \frac{1}{4}z^4. \]  \hspace{1cm} (15)

This yields the supersymmetric potential

\[ V(z) = 1 - 2z^2 - 2z^4 + z^6 \]  \hspace{1cm} (16)

with normalized zero-energy ground state wave function

\[ \psi_0(z) = N_0 \exp[-W(z)] = [2.0410\ldots] \exp\left[\frac{1}{4}z^2 - \frac{1}{4}z^4\right]. \]  \hspace{1cm} (17)

These quantities are shown in Figure 2. In particular, as it should, the ground state wave packet has no zero but double humps centered at the potential’s minima.

![Figure 2: The thick curve plots the potential of Eq. (16). To maintain the same scale, 5 times the mod-squares of various coherent-state wave functions are shown. The ground-state wave function is given by the medium-thick double-humped curve. The coherent state with \( \alpha = 1/2 \) is the thin curve, and the coherent state with \( \alpha = 2 \) is the dashed curve.](image-url)
Now consider the supersymmetry (factorized) annihilation operator for this system

\[ A(z) = \partial_z + W'(z). \quad (18) \]

Using this in the ladder-operator definition of coherent states,

\[ A(z)\psi_\alpha(z) = \alpha \psi_\alpha(z) \quad (19) \]

yields

\[ \psi_\alpha(z) = N_\alpha \exp[\alpha z - W(z)]. \quad (20) \]

In Figure 2 we also show the coherent-state wave packets for \( \alpha = \{1/2, 2\} \). As \( \alpha \) becomes larger the wave packet moves further to the right and assumes a more peaked form, as coherent states should. (For negative \( \alpha \) the parity-reversed situation occurs.) Eq. (20) shows that these states partially resemble displacement-operator states. Also, if one defines one’s \( X \) and \( P \) in terms of the sums and differences of the \( A \) and \( A^\dagger \) then they also have a minimum-uncertainty characteristic. So, these coherent states obtained from supersymmetry/factorization are well behaved. This is because \( A \) is the ground-state annihilation operator.

5. Coherent states for the linear (gravitational) potential

Using units where \( (2m^2g/\hbar^2)^{1/3} x \to z \), the Schrödinger equation is

\[ \left[-\frac{d^2}{dz^2} + |z| \right] \psi_n(z) = \lambda_n \psi_n(z). \quad (21) \]

This is Airy’s equation and, with foresight, we subtract off the ground state eigenenergy to make the system supersymmetric:

\[ \left[-\frac{d^2}{dz^2} + (|z| - \lambda_0) \right] \psi_n(z) = (\lambda_n - \lambda_0) \psi_n(z) \equiv \Lambda_n \psi_n(z). \quad (22) \]

\( \lambda_0 = 1.018.... \) It and the other \( \lambda_n \) are well known numerically [16]. The ground state-solution is the Airy function [17]

\[ \psi_0 = N_0 \ Ai(|z| - \lambda_0). \quad (23) \]
In Figure 3 we show the supersymmetric potential and the zero-energy ground state wave packet.

![Figure 3: The thick curve shows the supersymmetric linear potential. To maintain the same scale, 15 times the mod-squares of various coherent-state wave functions are shown. The ground-state wave packet is given by the medium-thick curve. The coherent state with $\alpha = 2$ is the thin curve.](image)

But now we can give the supersymmetric function as simply being

$$W(z) = -\ln [Ai(|z| - \lambda_0)],$$

and the formalism follows through. This means we can write the coherent states as

$$A(z)\psi_\alpha(z) = [\partial_z + W'(z)]\psi_\alpha(z) = \alpha \psi_\alpha(z),$$

$$\psi_\alpha(z) = N_\alpha \exp[\alpha z]Ai(|z| - \lambda_0).$$

In Figure 3 we also shown the coherent state for $\alpha = 2$.

Once again the properties of supersymmetry have allowed us to solve the problem.

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