Gauge Formalism for General Relativity and Fermionic Matter

by

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Summary: A new formalism for spinors on curved spaces is developed in the framework of variational calculus on fibre bundles. The theory has the same structure of a gauge theory and describes the interaction between the gravitational field and spinors. An appropriate gauge structure is also given to General Relativity, replacing the metric field with spin frames. Finally, conserved quantities and superpotentials are calculated under a general covariant form.

I Introduction

In our opinion, the history of spinor field theories may be split into two parts. The first part has generated a framework suitable to deal with special relativistic theories still used to describe Fermionic particles in quantum field theories; the Poincaré group, i.e. the isometry group of the Minkowski space, plays an important role in it, so that on general curved spaces it is hard to build a theory in a simple and unconditioned way as it happens instead in the case of tensor (Bosonic) matter (see [1]).

The second research trend has thence come up to deal with curved spaces, setting a first step towards a general relativistic theory. Solutions to the problem should be unrestricted or, at least, reasonably general to include all matter fields which have a physical relevance and all admissible space-times, preferably without any stringent symmetry requirement. A number of possible approaches have been proposed in the literature (see [2], [3], [4], [5]). Most of them rely on the following definition:

Definition (1.1): Let \((M, g)\) be a (pseudo)-Riemannian orientable manifold; a spin structure on \((M, g)\) is a pair \((\Sigma, \Lambda)\) where \(\Sigma\) is a principal fibre bundle with \(\text{Spin} (\eta)\) as structure group, \(\eta\) being the signature of \(g\), and \(\Lambda : \Sigma \to SO(M, g)\) is a bundle morphism such that:

\[
\begin{align*}
\Sigma & \xrightarrow{\bar{\Lambda}} SO(M, g) \\
M & \xrightarrow{id_M} M
\end{align*}
\]

\[
\begin{align*}
\Sigma & \xrightarrow{R_{\Lambda}} \Sigma \\
SO(M, g) & \xrightarrow{R_{\Lambda(S)}} SO(M, g)
\end{align*}
\]

where \(SO(M, g)\) is the \(g\)-orthonormal (equioriented) frame bundle, \(\Lambda : \text{Spin}(\eta) \to \)

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SO(\(\eta\)) is the epimorphism which exhibits Spin(\(\eta\)) as a two-fold covering of SO(\(\eta\)) and \(R_S\) and \(R_{\Lambda(S)}\) are the canonical right action respectively on \(\Sigma\) and SO(\(M, g\)).

It can be shown that also \(\bar{\Lambda}\) is an epimorphism and moreover a two-fold covering space map. The obstruction to the existence of spin structures on a manifold \(M\) has been solved by Haefliger, Milnor, Greub and Petry (see [2], [4], [6]) by the following theorem:

**Theorem (1.2):** A manifold \(M\) allows spin structures of signature \(\eta\) if and only if is orientable, it has a metric \(g\) with signature \(\eta\) and satisfies a topological condition which amounts to require that the second Stiefel-Whitney class vanishes.

Under these conditions, let us choose a family of trivializations on SO(\(M, g\)) and let \(g_{\alpha\beta}\) be its SO(\(\eta\))-cocycle of transition functions; we can then build a Spin(\(\eta\))-cocycle \(\gamma_{\alpha\beta}\) such that:

\[
\Lambda(\gamma_{\alpha\beta}(x)) = g_{\alpha\beta}(x) \quad \forall x \in U_{\alpha} \cap U_{\beta}
\]

(1.3)

This Spin(\(\eta\))-cocycle defines a principal bundle, which will be called \(\Sigma(M, g)\), having with Spin(\(\eta\)) as fibre and \(\gamma_{\alpha\beta}\) as transition functions. It also defines a morphism \(\bar{\Lambda}: \Sigma(M, g) \to SO(M, g)\) such that \((\Sigma(M, g), \bar{\Lambda})\) is a spin structure on \(M\) according to definition (1.1).

We remark that, at least in general, there is no canonical choice of \(\gamma_{\alpha\beta}\) since more than one inequivalent Spin(\(\eta\))-cocycles fulfilling condition (1.3) can exist.

At this point, most authors choose a linear representation of Spin(\(\eta\)) on a suitable real or complex vector space \(V\) to build a vector bundle associated to \(\Sigma(M, g)\), the sections of which are to be identified with spinor fields.

Theories of this last kind improve a lot the situation with respect to the Minkowski case. Still they are unsatisfactory, because of at least two reasons:

(1.4) They are adequate to describe spinor fields in interaction with a fixed gravitational field, but if we need a true relativistic theory i.e. a theory in which also the metric is dynamical, we must as well consider deformations of the spin structure which are related to deformations of the metric. However in this formalism it is hard to talk about spin structure deformations as we are actually working with a fixed background according to (1.1).

(1.5) If we are aimed to deal with physical applications we would like to cope with the problem of conserved quantities. To solve this problem in the most general situation by means of Nöther theorem, it is necessary to define Lie derivatives of spinor fields. As is well known (see, e.g., [7]) this is quite difficult because spinors are not natural objects so that they cannot be dragged along arbitrary vector fields on \(M\).
In this paper we aim to present an alternative and new viewpoint on the description of spinor fields on a (dynamical) curved space, by means of which we believe to have overcome both these problems and to have developed the tools necessary to analyse the relations between the different solutions proposed earlier.

Our approach relies on a new definition of spin structures which from the very beginning avoids any reference to a fixed metric. In most practical cases our definition turns out to be equivalent to the classical one but, in general, it requires stricter hypotheses.

**Definition (1.6):** Let $M$ be an orientable manifold which admits a (pseudo-)Riemannian metric. A free spin structure on $M$ is a pair $(\Sigma, \tilde{\Lambda})$ where $\Sigma$ is a principal fibre bundle with $\text{Spin}(\eta)$ as structure group and $\tilde{\Lambda} : \Sigma \to L(M)$ is a morphism such that:

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\tilde{\Lambda}} & L(M) \\
p \downarrow & & \downarrow \pi \\
M & \xrightarrow{id_M} & M
\end{array}
\quad
\begin{array}{ccc}
\Sigma & \xrightarrow{R_{\Sigma}} & \Sigma \\
\downarrow \tilde{\Lambda} & & \downarrow \tilde{\Lambda} \\
L(M) & \xrightarrow{R_{\Lambda(g)}} & L(M)
\end{array}
$$

We stress that under this new definition $\tilde{\Lambda}$ is not necessarily an epimorphism as in the case of definition (1.1) (see [8]).

If $M$ satisfies the conditions of Greub and Petry (see theorem (1.2)) spin structures in the sense of (1.1) do exist for any metric $g$ and, letting $i_g : \text{SO}(M, g) \to L(M)$ be the canonical immersion, $(\Sigma(M, g), i_g \circ \tilde{\Lambda})$ turns out to be a free spin structure in the sense of (1.3). Therefore, these conditions guarantee also the existence of free spin structures.

If $M$ is parallelizable, the conditions of Greub and Petry certainly hold, and roughly speaking there is a bijection between free spin structures and the ordinary spin structures (see theorem (2.3) below); in this case we are therefore led to think about free spin structures just as a reformulated version of ordinary spin structures.

In general, manifolds which allow spin structures need not to be parallelizable; however, a remarkable result by Geroch (see [9]) asserts that in dimension four and signature $(1, 3)$ a noncompact manifold $M$ admits a spin structure if and only if it is parallelizable.

Since compact space-times are classically forbidden by causality (see, e.g., [10]) and globally hyperbolic space-times have necessarily a non-compact topology $\mathbb{R} \times M_3$ (again Geroch; see [11], [12]), Geroch theorems seem apparently to close the question. In reasonable space-times, in fact, the two notions defined by (1.1) and (1.6) do practically coincide. Let us however remark that the topological conditions of Greub and Petry are not enough to build a physically meaningful general relativistic field theory since, as we said, they do not allow to talk about deformations of spin structures in a fairly general way.
Moreover and more fundamentally, Fermionic theories find their justification only in view of quantization. As is well known, quantum techniques in gravitation often require the use of compactifications and possibly also of signature changes, so that having at one’s own disposal a definition which allows space to dynamical metrics and holds also in these cases seems to be rather important. As for the problem (1.5) of conserved quantities we envisage two alternative strategies:

(1.7) We can define a canonical (not natural) lift which associates a vectorfield up on Σ to each vectorfield down on the base M. This lift allows us to define the Lie derivatives of sections of Σ (and its associated bundles) along vectorfields of M ([13], [14]). In our opinion, this enables us to define the energy-momentum stress tensor (see [1], [15], [16], [17], [18], [19]).

(1.8) An alternative way is to implement a technique similar to the method used in gauge theory to deal with conserved quantities ([1], [16]), avoiding any reference to Lie derivatives with respect to vectorfields on M and replacing them with Lie derivatives with respect to projectable vectorfields on Σ. In other words, we can enlarge the symmetry group by adding the vertical transformations to obtain all automorphisms of Σ which are canonically representable on the configuration bundle, instead of using Diff(M) which has no such natural representation.

Although both approaches seem to be viable (at least a priori), we will herein develop the second approach. It is in our intention, however, to investigate also the remaining approach and to discuss and analyse its relations with other techniques in forthcoming papers. We remark that the second strategy we have chosen is, a priori, the most difficult to interpret. In fact by enlarging the symmetry group we add degrees of freedom to the conserved currents and consequently we expect to have more conserved quantities than those we are able to interpret in our case of Fermionic matter. However, we will see that the vertical contributions to the currents vanish identically off-shell (owing to covariance) so that no additional conserved quantities will at last be defined but the energy-momentum tensors.

We would finally like to stress that the reformulation of General Relativity in terms of free spin structures is in our opinion essential if we want to treat spinor theories as gauge theories, since in our formulation spinors interact directly with spin structures while ordinary Bosonic matter just interacts with the metric associated to it.

II Deformation of free spin structures and notation

Let us choose an orientable manifold M such that there exists on M a metric g
of signature $\eta$. We will call $\eta_{ab} = g(u_a, u_b)$ the metric components of $g$ with respect to an orthonormal (local) basis.

**Assumption (2.1):** Let $\Sigma$ be a principal bundle with fibre $\text{Spin}(\eta)$ called structure bundle and let us assume that there exists at least one morphism $\Lambda : \Sigma \to L(M)$ such that $(\Sigma, \tilde{\Lambda})$ is a free spin structure.

We remark that we are not fixing $\tilde{\Lambda}$ uniquely; we are rather asking that such a morphism exists. This condition is not always guaranteed depending on the topology of $M$. For example, if $M$ is not parallelizable one cannot choose $\Sigma$ to be a trivial bundle on $M$. In this case, in fact, if one morphism $\tilde{\Lambda}$ exists, then $L(M)$ should admit a global section and this is a contradiction. However, if $M$ satisfies the conditions of theorem (1.2), one can choose a metric $g$ and build $\Sigma(M, g)$ as explained above. This bundle allows, by construction, a free spin structure. As a consequence, conditions (1.2) are sufficient conditions for the existence of at least one structure bundle $\Sigma$.

We stress that, in general, there can be more than one structure bundle on $M$ (see [8]). This situation is not different from what one encounters in gauge theories when we fix the gauge bundle $P$ and, here as in that case, there is no point in looking for a canonical choice; different choices give rise to different theories. In other words fixing $\Sigma$ is part of the system specification.

**Definition (2.2)** A spin frame on $\Sigma$ is a morphism $\tilde{\Lambda} : \Sigma \to L(M)$ for which $(\Sigma, \tilde{\Lambda})$ is a free spin structure.

We are now able to state and prove the following:

**Theorem (2.3)** Let $M$ be parallelizable. Then there exists a bijection between spin frames and spin structures on $\Sigma$. Moreover, for each metric $g$ on $M$ there exists a spin frame $\tilde{\Lambda}$ such that all frames in $\text{Im}(\tilde{\Lambda}) \subset L(M)$ are $g$-orthonormal frames.

In fact, if $M$ is parallelizable for each pair of metrics $(g, \tilde{g})$ on $M$ there exists an isomorphism $\Phi : \text{SO}(M, g) \to \text{SO}(M, \tilde{g})$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\text{SO}(M, g) & \xrightarrow{\Phi} & \text{SO}(M, \tilde{g}) \\
\downarrow & & \downarrow \\
M & \xrightarrow{id_M} & M
\end{array}
$$

(2.4)

Accordingly, we can choose $\Sigma = \Sigma(M, g) = \Sigma(M, \tilde{g})$. Thence for each spin structure $(\Sigma, \tilde{\Lambda})$ we can build a spin frame on $\Sigma$ by composition with the canonical injection:
On the other hand, for each spinor frame $\tilde{\Lambda} : \Sigma \to L(M)$ there exists one and only one metric $\tilde{g}$ having the frames of $\text{Im}(\tilde{\Lambda})$ as orthonormal frames. If we build $\text{SO}(M, \tilde{g})$ from diagram (2.5) we infer that $\tilde{\Lambda}$ induces $\bar{\Lambda}$. These two maps are inverse of each other, and define the bijection as we claimed. ■

To summarize, if $M$ is parallelizable we can consider spin frames instead of spin structures on a structure bundle $\Sigma$ fixed once for all; moreover, every metric can be associated to some spin frame.

If we want to consider a field theory in which spin frames are dynamical we must first construct a fibre bundle the sections of which represent spin frames.

Let us then consider the following action on the manifold $\text{GL}(m)$:

\[
\rho : (\text{GL}(m) \times \text{Spin}(\eta)) \times \text{GL}(m) \to \text{GL}(m) : ((A^\mu_\nu, S), e^\nu_a) \mapsto A^\mu_\nu e^\nu_a \Lambda^b_a (S^{-1})
\]

Together with the associated bundle $\Sigma_\rho = ((L(M) \times_M \Sigma) \times \text{GL}(m))/\rho$, where $m$ is the dimension of $M$. According to the theory of gauge-natural bundles and gauge-natural operators (see [20]) $L(M) \times_M \Sigma$ is nothing but the principal prolongation of the principal fibre bundle $\Sigma$, also denoted by $W^{1,0}(\Sigma)$, with structure group $\text{GL}(m) \times \text{Spin}(\eta)$. It turns out that $\Sigma_\rho$ is a fibre bundle associated to $W^{1,0}(\Sigma)$, i.e. a gauge-natural bundle of order $(1,0)$. The bundle $\Sigma_\rho$ will be called the bundle of spin tetrads or simply (by the following result) the bundle of spin frames.

**Theorem (2.7)** Sections of $\Sigma_\rho$ are in one-to-one correspondence with spin frames on $\Sigma$.

The bijection is the following. Let $\tilde{\Lambda} : \Sigma \to L(M)$ be a spin frame, $\sigma^{(\alpha)}(x)$ be the identity (local) sections with respect to a trivialization of $\Sigma$ and $u^{(\alpha)}_a(x) = \tilde{\Lambda}(\sigma^{(\alpha)}(x))$ the corresponding (local) sections of $L(M)$. To these objects we can associate (local) sections:

\[
s^{(\alpha)}(x) = [\sigma^{(\alpha)}(x), u^{(\alpha)}_a(x), \mathbb{1}] \in \Sigma_\rho
\]

which glue together to generate a global section $s$ on $\Sigma_\rho$, which is said to represent our spin frame. ■

If we choose an automorphism $\Phi \in \text{Aut}(\Sigma)$ of the structure bundle, it can be represented on the bundle $\Sigma_\rho$ in the following way:
(2.8) \[ \Phi_\rho : \Sigma_\rho \rightarrow \Sigma_\rho : [u_a, p, e^\mu_a] \mapsto [L(f)(u_a), \Phi(p), e^\mu_a] \]

where \( f : M \rightarrow M \) is the projection of \( \Phi \) on \( M \) and \( L(f) \) is the natural lift of \( f \) to \( L(M) \). It can be easily checked that this is a good definition.

Letting \( \Xi \) be the infinitesimal generator of a 1-parameter subgroup \( \{ \Phi_t \} \) of automorphisms on \( \Sigma \), let us denote by \( \Xi_\rho \) the generator of the subgroup induced on \( \Sigma_\rho \) by (2.8). The flow of \( \Xi_\rho \) drags any section of \( \Sigma_\rho \), thus defining a family \( \tilde{\Lambda}_t \) of spin frames which will be for us, by definition, an infinitesimal deformation of the spin frame \( \tilde{\Lambda}_0 \) (see [8]).

Finally let us remark that, whenever we choose a trivialization of \( \Sigma \) and on \( L(M) \) induced respectively by local sections \( \sigma^{(\alpha)} \) and \( \partial^{(\alpha)} \), we can locally choose standard representatives on \( \Sigma_\rho \) as follows:

(2.9) \[ [\sigma^{(\alpha)}, \partial^{(\alpha)}_\mu, e^\mu_a] \]

so that \( (x^\mu, e^\mu_a) \) are (local) coordinates in \( \Sigma_\rho \). The generic automorphism \( \Phi(x, S) = (f(x), \phi(x) \cdot S) \) induces then the following automorphism on \( \Sigma_\rho \):

(2.10) \[ \Phi_\rho(x^\mu, e^\mu_a) = (f^\mu(x), J^\mu_\nu e^\nu_b \Lambda_a^b(\phi^{-1}(x))) \quad J^\mu_\nu := \partial_\nu f^\mu(x) \]

For other standard notation see, e.g., [15], [16], [21] and [22].

### III Covariant Lagrangians

Let \( M \) be a parallelizable and orientable manifold which admits (pseudo)-Riemannian metrics of signature \( \eta \). Let \( \Sigma \) be our structure bundle and \( \lambda \) be a linear representation of \( \text{Spin}(\eta) \) on a suitable vector space \( V \). We can then construct the associated vector bundle \( \Sigma_\lambda := \Sigma \times_\lambda V \). Any \( \Phi \in \text{Aut}(\Sigma) \) can be represented on \( \Sigma_\lambda \) as follows:

(3.1) \[ \Phi_\lambda : \Sigma_\lambda \rightarrow \Sigma_\lambda : [p, v] \mapsto [\Phi(p), v] \]

Moreover, since we aim to describe a spinor field (not subject to any further gauge symmetry) in interaction with the gravitational field, our configuration space will be assumed to be the following bundle:

(3.2) \[ B = \Sigma_\rho \times_M \Sigma_\lambda \]
and the Lagrangian will be chosen in the following form:

\( L : J^2 \Sigma^p \times J^1 \Sigma^\lambda \rightarrow A^0_m(M) \)

According to the principle of minimal coupling, the Lagrangian \( L \) is assumed to split into two parts \( L = L_H + L_D \), with:

\( L_H : J^2 \Sigma^p \rightarrow A^0_m(M) \) \hspace{1cm} \text{(gravitational Lagrangian)}

\( L_D : J^1(\Sigma^p \times \Sigma^\lambda) \rightarrow A^0_m(M) \) \hspace{1cm} \text{(spinor Lagrangian)}

For \( L_H \) we can take the standard Hilbert Lagrangian \( L_H = -(1/2\kappa) R(j^2 g)\sqrt{g} \, ds \) written in metric coordinates

\[
g = \hat{e}^\alpha_{\mu} \hat{e}_{\alpha}^\nu \hat{e}^\lambda_{\mu} \eta_{ab} \hat{e}^b_{\nu} dx^\nu \otimes dx^\nu \text{ where } \|\hat{e}^\alpha_{\mu}\| = \|e^\alpha_{\mu}\|^{-1} \text{ being ds = dx}^1 \wedge \ldots \wedge dx^m \text{ the local volume element and } g = |\det \|g_{\mu\nu}\||.
\]

We require that \( L \) be covariant with respect to any generalised spinor transformation, i.e. with respect to any element of Aut(\( \Sigma \)). Since locally we have the following transformations rules:

\[
e_{\alpha}^{\prime \mu} = J^\mu_{\nu} e_{\nu}^{\prime} \Lambda_{\alpha}^{b}(\phi^{-1}(x))
\]

\[
v^{\prime} A = \lambda^A_B(\phi(x)) v^B
\]

\[
g^{\prime}_{\mu\nu} = \Lambda^\alpha_{\nu}(\phi(x)) e^\sigma_{\alpha} \bar{J}^\sigma_{\mu} \eta_{ab} \Lambda_{\alpha}^{b}(\phi(x)) e^d_{\rho} \bar{J}^d_{\nu} = \bar{J}^\sigma_{\mu} g_{\sigma\rho} \bar{J}^d_{\nu}
\]

the Hilbert Lagrangian is covariant with respect to all these transformations. As for the rest, we have to seek conditions which must to be satisfied by \( L_D \). To this purpose let us choose local coordinates in \( J^1B \) as follows:

\[
(x^\mu, e^\rho_{\alpha}, e^\rho_{\alpha\sigma}, v^A, \Omega^A_{\alpha}, \bar{v}^\dagger_A, \Omega^{\dagger}_{\alpha a})
\]

where

\[
\Omega^A_{\alpha} \equiv e^\mu_{\alpha} \Omega^A_{\mu} := e^\mu_{\alpha}(v^A_{\mu} + \lambda^A_{\alpha j} \Gamma^j_{ij} v^B_{i})
\]

\[
\lambda^A_{\alpha j} := \frac{1}{8} \partial^\alpha \lambda^A_B(e)[\gamma_i, \gamma_j]^{\alpha}_B, \quad \partial^\alpha := \frac{\partial}{\partial S^\alpha}
\]

\[
\Gamma^i_{ij} := \bar{e}^\rho_{\rho}(\Gamma^i_{\sigma \mu} e^\sigma_{\rho} + \eta^{\rho}_{k\nu} \eta^{k\mu})
\]

\[
\Gamma^{\rho}_{\sigma \mu} := \frac{1}{2} g^{\rho\nu}(-d_{\nu} g_{\sigma\mu} + d_{\sigma} g_{\mu\nu} + d_{\mu} g_{\nu\sigma})
\]

being \( \gamma_i \) a set of Dirac matrices fixed to define the two-fold covering \( \Lambda : \text{Spin}(\eta) \rightarrow \text{SO}(\eta) \).
Let us notice that, if we denote by
\[ \vec{\sigma}_{ij} := \frac{1}{8} ([\gamma_i, \gamma_j] S_\alpha^\beta \partial_\alpha \partial_\beta) \]
a system of right-invariant vector fields over Spin(\(\eta\)), the quantities \(\Gamma^{ij}_\mu\) defined by (3.12) are nothing but the coefficients of the spinor connection induced canonically on \(\Sigma\) by the Levi–Civita connection:

\[ (3.14) \quad \omega = dx^\mu \otimes (\vec{\sigma}_{ij} - \Gamma^{ij}_\mu \vec{\sigma}_{ij}) \]

while the expression \(\Omega^A_a\) defined by (3.10) is the formal covariant derivative of the field \(v^A\) with respect to the connection on \(\Sigma_\lambda\) induced by this (principal) spinor connection.

Taking into account the transformation rule:

\[ (3.15) \quad \Gamma'^{ij}_\mu = \check{J}_\mu \Lambda^k_i(\phi(x)) \left( \Gamma^{kh}_\nu \Lambda^j_h(\phi(x)) + d_\nu \Lambda^k_i(\phi^{-1}(x)) \eta^{ij} \right) \]

we obtain the transformation rule:

\[ (3.17) \quad \Omega'^A_a = \check{\lambda}_B^A(\phi(x)) \Omega^B_b \Lambda^b_a(\phi^{-1}(x))\]

In the chosen coordinates the Lagrangian (3.5) has, in general, the following form:

\[ (3.18) \quad L_D = \mathcal{L}_D(x^\mu, e^\mu_a, e^{\mu\sigma} a, v^A, v^A, \Omega^A_a, \Omega^A_{ab}) \sqrt{\gamma} ds \]

If we require the spinor Lagrangian \(L_D\) to be covariant with respect to any automorphisms of the structure bundle \(\Sigma\) (see [23] and [24] for the natural case) its associated scalar density \(\hat{\mathcal{L}} = L_D \sqrt{\gamma}\) must satisfy the following identity:

\[ (3.19) \quad d_\sigma (\hat{\mathcal{L}} \xi^\sigma) = \frac{\partial \hat{\mathcal{L}}}{\partial v^A} L \xi v^A + \frac{\partial \hat{\mathcal{L}}}{\partial \Omega^A_a} L \xi \Omega^A_a + \frac{\partial \hat{\mathcal{L}}}{\partial v^A} L \xi v^A + \frac{\partial \hat{\mathcal{L}}}{\partial \Omega^A_{ab}} L \xi \Omega^A_{ab} + \frac{\partial \hat{\mathcal{L}}}{\partial e^\mu_a} L \xi e^\mu_a + \frac{\partial \hat{\mathcal{L}}}{\partial e^{\mu\sigma} a} L \xi e^{\mu\sigma}_a \]

where \(\Xi\) is the infinitesimal generator of a one–parameter subgroup of automorphisms of \(\Sigma\) and \(\xi\) is its projection on \(M\). The identity (3.19) holds if and only if \(\mathcal{L}_D\) does not depend on \((x^\mu, e^\mu_a, e^{\mu\sigma}_a)\) and moreover the following identity holds:

\[ (3.20) \quad \lambda^A_{Blm} \left( \frac{\partial \mathcal{L}_D}{\partial v^B} v^B_a + \frac{\partial \mathcal{L}_D}{\partial \Omega^A_\alpha} \Omega^B_\alpha \right) - \frac{\partial \mathcal{L}_D}{\partial \Omega^A_\alpha} \eta_{a[m} \Omega^A_{l]} + c.c. = 0 \]
where c.c. stands for the complex conjugate terms.

IV The Dirac Lagrangian on curved spaces

We intend here to define the generalization to curved spaces of the Dirac Lagrangian used in Quantum Field Theory to describe Fermionic fields. We content ourselves to discuss the case of Fermionic fields on a four-dimensional curved space–time \( M \) which admits a metric of Lorentzian signature \( \eta = (+, -, -, -) \). Anyhow we remark that we are not fixing a particular metric as background, but it is to be understood as determined by the spin-tetrad field. This generalization will also provide us an example of covariant spinor Lagrangian.

We first recall that the Dirac matrices are defined as follows:

\[
\begin{align*}
\gamma_0 &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\
\gamma_1 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \\
\gamma_2 &= \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
\gamma_3 &= \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}
\end{align*}
\]

where \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices defined by:

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

The group \( \text{Spin}(\eta) \) is defined to be the set of matrices \( S \in \text{GL}(4, \mathbb{C}) \) such that an element \( \Lambda \in \text{SO}(\eta) \) exists for which the following holds:

\[
S \gamma_i S^{-1} = \Lambda_i^j \gamma_j
\]

This matrix \( \Lambda \in \text{SO}(\eta) \) is by definition the image of \( S \) with respect to the homomorphism \( \Lambda : \text{Spin}(\eta) \to \text{SO}(\eta) \). The group \( \text{Spin}(\eta) \) acts canonically on \( V = \mathbb{C}^4 \) by a representation which we denote by \( \lambda \). We can then construct the associated vector bundle \( \Sigma_\lambda = \Sigma \times_\lambda \mathbb{C}^4 \) as in the general case explained above. The Dirac Lagrangian is then defined as follows:

\[
L_D = \left[ \frac{i}{2} \bar{v} \cdot \gamma^a \cdot \Omega_a - \frac{i}{2} \bar{\Omega}_a^i \gamma^a \cdot v - m \bar{v} \cdot v \right] \sqrt{g} \, ds
\]

where \( \bar{v} = v^\dagger \gamma_0 \), \( \Omega_a = \Omega_a^i \gamma_0 \), \( \gamma^a = \eta^{ab} \gamma_b \) and \( \cdot \) denotes the matrix product. It is easy to verify that the Lagrangian defined by (4.3) is covariant, i.e. it fulfills condition (3.20).

This Lagrangian is of particular importance because, on \( M = \mathbb{R}^4 \) it reduces to the standard Dirac Lagrangian which provides us the only spinor theory which is
truly understood and experimentally tested, and has therefore to be reproduced by any generalised spinor theory.

V Conserved Quantities and Superpotentials

We shall herein rely on Nöther theorem to generate conserved currents associated to a family of generalised spinor transformations $\Phi_t \in \text{Aut}(\Sigma)$. Let this family be generated by the vectorfield

$$\Xi = \xi^\mu \bar{\partial}_\mu + \xi^{ij} \bar{\sigma}_{ij}$$

where $\xi^\mu = \left. \frac{\partial}{\partial t} (f_t)^\mu \right|_{t=0}$ and $\xi^{ij} = \left. \frac{\partial^\beta \Lambda^i_k(c) \eta^{kj} \frac{\partial}{\partial t} (\phi_t)^\alpha}{\beta} \right|_{t=0}$. In metric coordinates $(x^\lambda, g_{\mu\nu})$ the Hilbert Lagrangian reads as:

$$L_H = L_H \sqrt{g} ds = -\frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu} (j^2 g) \sqrt{g} ds$$

Since both Lagrangians $L_H$ and $L_D$ are separately covariant, the following two identities hold:

(5.1) \hspace{1cm} d_\sigma (L_H^* \xi^\sigma) = p^{\mu\nu} L_{\Xi} g_{\mu\nu} + p^{\alpha\beta\gamma\delta} L_{\Xi} R_{\alpha\beta\gamma\delta}

(5.2) \hspace{1cm} d_\sigma (\hat{\mathcal{L}} \xi^\sigma) = p_A L_{\Xi} v^A + p^a_A L_{\Xi} \Omega^A_a + p^A \Xi_{\Xi v^A} + p^a_A \Xi_{\Xi v^a} + p^a_\mu \Xi_{\Xi e^\mu_a}

where we have defined the naive momenta of $L$ by:

(5.3) \hspace{1cm} p^{\mu\nu} = \frac{\partial \mathcal{L}_H^*}{\partial g^{\mu\nu}}, \hspace{1cm} p^{\alpha\beta\gamma\delta} = \frac{\partial \mathcal{L}_H^*}{\partial R_{\alpha\beta\gamma\delta}}, \hspace{1cm} p_A = \frac{\partial \hat{\mathcal{L}}}{\partial v_A}, \hspace{1cm} p^A = \frac{\partial \hat{\mathcal{L}}}{\partial v^A}, \hspace{1cm} p^a_A = \frac{\partial \hat{\mathcal{L}}}{\partial \Omega^A_a}, \hspace{1cm} p^a_\mu = \frac{\partial \hat{\mathcal{L}}}{\partial e^\mu_a}, \hspace{1cm} p^{\alpha\beta\gamma\delta} = p^{\alpha(\beta\gamma\delta)}.

Starting from (5.1), substituting the expressions of the Appendix and integrating covariantly by parts, we are finally led to the following formulas for the gravitational current and work:

(5.4) \hspace{1cm} \nabla_{\lambda} E^\lambda (L_H, \Xi) = W(L_H, \Xi)

(5.5) \hspace{1cm} E^\lambda (L_H, \Xi) = \frac{H}{\sigma} \nabla_{\lambda} \xi^\sigma + \frac{H}{\mu} \nabla_{\mu} \xi^\sigma + \frac{H}{\nu} \nabla_{\nu} \xi^\sigma = -4 p^{\theta\mu\nu}\epsilon^a_{\mu\nu} \nabla_{\theta} L_{\Xi} e^\sigma_a + 4 \nabla_{\theta} p^{\lambda\mu\nu}\epsilon^a_{\mu\nu} \nabla_{\Xi} e^\sigma_a - \mathcal{L}_H \xi^\lambda

(5.6) \hspace{1cm} W(L_H, \Xi) = 2 e^{\mu\nu} (L_H) \epsilon^a_{\mu\nu\sigma} L_{\Xi} e^\sigma_a = \mathcal{W}_\mu \nabla_{\mu} \xi^\sigma

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with \(\nabla_{\mu\nu}\xi^\sigma = \frac{1}{2}(\nabla_\mu \nabla_\nu + \nabla_\nu \nabla_\mu)\xi^\sigma\) and:

\[
e^{\mu\nu}(L_H) = p^{\mu\nu} + p^{(\mu|\beta\gamma\delta|} R_{\beta\gamma\delta}^{\nu)} + 2\nabla_\lambda \nabla_\theta \tilde{p}^{\lambda\mu\nu\theta}
\]

\[
W^H_\sigma = -2e^{\mu\nu}(L_H)g_{\nu\sigma}
\]

\[
T^\lambda_\sigma = -2\tilde{p}^{\nu\mu\theta} R_{\sigma\theta\nu\mu} - \mathcal{L}_H^\delta_\sigma
\]

\[
T^{\lambda\mu}_\sigma = -4\nabla_\theta \tilde{p}^{\lambda\mu\nu\theta} g_{\nu\sigma} \equiv 0
\]

As a consequence, the covariant conditions for the gravitational part are:

\[
\begin{cases}
\nabla^H_\lambda T^\lambda_\rho + \frac{1}{2} T^{\rho\mu}_\sigma R^{\sigma}_{\rho\mu} + \frac{1}{3} \nabla_\mu R^{\sigma}_{\rho\lambda\nu} T^\lambda_\mu = 0 \\
W^H_\sigma = -2e^{\nu\mu}(L_H) g_{\nu\sigma}
\end{cases}
\]

\[
T^{\lambda\mu}_\rho + \frac{1}{2} T^{\alpha\rho}_\sigma R^{\sigma}_{\rho\lambda\nu} T^{\lambda\mu}_\nu = W^H_\mu
\]

\[
T^{(\mu\nu)}_\rho + \nabla^H_\lambda T^{\lambda\mu}_\rho = 0
\]

\[
T^{(\lambda\mu)}_\rho = 0
\]

We remark that the contribution to the vertical current vanishes identically.

Taking now (5.2) into account, substituting the expressions of the Appendix and integrating again (covariantly) by parts, we obtain the following:

\[
\nabla_\lambda E^\lambda(L_D, \Xi) = W(L_D, \Xi)
\]

\[
E^\lambda(L_D, \Xi) = D^\lambda_\sigma \xi^\sigma + D^\lambda_\mu \nabla_\mu \xi^\sigma = p^A_{\sigma} e^\lambda_{a} \mathcal{L}_\Xi v^A + A^\sigma\lambda^0_{\rho} e^\rho_\sigma \xi^\rho + A^\sigma\lambda^\rho_{\rho} g_{\sigma\rho} + c.c. - \hat{\mathcal{L}}^\lambda
\]

\[
W(L_D, \Xi) = W^\rho_\sigma \xi^\sigma + W^\rho_\mu \nabla_\mu \xi^\sigma = -H^\rho_{\mu} \mathcal{L}_\Xi e^\mu_\sigma - e_A(L_D) \mathcal{L}_\Xi v^A + c.c.
\]

where

\[
A^{\rho\sigma\mu} = p^a_A e^\mu_{a} \lambda^A_{Bij} v^B e^{i\rho} e^{j\sigma}
\]

\[
H^\rho_{\mu} = p^\rho_{\mu} + p^A_{\mu} \Omega^A_{\rho} - \nabla_\rho A^\sigma_{\mu} e^\sigma_\rho e^\alpha + 2\nabla_\alpha A^\sigma_{(\rho\nu)} e^\sigma_\nu g_{\mu\rho} + c.c.
\]

\[
e_A(L_D) = p_A^a - e^\mu_\sigma = e_A(L_D) + c.c.
\]

\[
W^a_\mu = -e_A(L_D) \Omega^A_{\sigma}
\]
\[ D_\sigma = p^\sigma_a e^\mu_a + p^\sigma_A e^\mu_A - \nabla_\rho (-A^{\mu \rho \lambda} + A^{\lambda \mu \rho} + A^{\rho \lambda \mu})g_{\sigma \lambda} + \text{c.c.} \]  

(5.21) \[ T^\lambda_\sigma = p^\sigma_A e^\lambda A^\sigma_a + \text{c.c.} - \hat{\delta}_\sigma \]  

(5.22) \[ T^{\lambda \mu}_\sigma = (A^{\mu \lambda \rho} - A^{\rho \mu \lambda} - A^{\lambda \rho \mu})g_{\rho \sigma} + \text{c.c.} \]

We remark once again that the contribution to the vertical current vanishes identically. The covariant conditions are therefore the following:

\[
\begin{align*}
\nabla_\lambda T^\lambda_\rho + \frac{1}{2} T^{\nu \mu}_\sigma R^\sigma_{\rho \nu \mu} &= \frac{D}{W} \\
\frac{D}{T} T^\mu_\rho + \nabla_\lambda T^{\lambda \mu}_\rho &= \frac{D}{W} \\
\frac{D}{T} T^{(\mu \nu)}_\rho &= 0
\end{align*}
\]

(5.23)

We can now define the total current \( E^\lambda (L, \Xi) \) and the total work \( W(L, \Xi) \) by setting:

(5.24) \[ E^\lambda (L, \Xi) = E^\lambda (L_H, \Xi) + E^\lambda (L_D, \Xi) \]

(5.25) \[ W(L, \Xi) = W(L_H, \Xi) + W(L_D, \Xi) \]

Once again we have:

(5.26) \[ \nabla_\lambda E^\lambda (L, \Xi) = W(L, \Xi) \]

where:

(5.27) \[ E^\lambda (L, \Xi) = T^\lambda_\sigma \xi^\sigma + T^{\lambda \mu}_\sigma \nabla_\mu \xi^\sigma + T^{\lambda \mu \nu}_\sigma \nabla_{\mu \nu} \xi^\sigma = (T^\lambda_\sigma + \frac{D}{T} T^\sigma_\lambda) \xi^\sigma + (T^{\lambda \mu}_\sigma + \frac{D}{T} T^{\lambda \mu}_\sigma) \nabla_\mu \xi^\sigma + \frac{H}{T} T^{\lambda \mu \nu}_\sigma \nabla_{\mu \nu} \xi^\sigma \]

(5.28) \[ W(L, \Xi) = W^\sigma_\sigma \xi^\sigma + W^\sigma_\mu \nabla_\mu \xi^\sigma = (W^\sigma_\sigma + \frac{D}{T} W^\sigma_\sigma) \xi^\sigma + (W^H_\sigma + \frac{D}{T} W^H_\sigma) \nabla_\mu \xi^\sigma \]

and the total covariant conditions are the following:

\[
\begin{align*}
\nabla_\lambda T^\lambda_\rho + \frac{1}{2} T^{\nu \mu}_\sigma R^\sigma_{\rho \nu \mu} + \frac{1}{3} \nabla_\mu R^\sigma_{\rho \lambda \nu} T^{\lambda \nu \mu}_\sigma &= W_\rho \\
T^\mu_\rho + \nabla_\lambda T^{\lambda \mu}_\rho + T^{\lambda \mu \nu}_\sigma R^\sigma_{\rho \lambda \nu} - \frac{2}{3} T^{\lambda \nu}_\rho R^\mu_{\lambda \nu} &= W_\rho \\
T^{(\mu \nu)}_\rho + \nabla_\lambda T^{\lambda \mu \nu}_\rho &= 0 \\
T^{(\lambda \mu \nu)}_\rho &= 0
\end{align*}
\]

(5.29)
The following weak conservation law is then valid on-shell (i.e., along solutions of field equations)

\[(5.30) \quad \nabla \lambda E^\lambda (L, \Xi) = d \lambda E^\lambda (L, \Xi) = 0\]

Hence \(E(L, \Xi) = E^\lambda (L, \Xi)ds_\lambda\) is a conserved current on-shell, i.e.: \n
\[(5.31) \quad E(L, \Xi, \rho) = (j^3 \rho)^* E(L, \Xi)\]

is a closed form if and only if \(\rho\) is a critical section. Suitably manipulating on (5.29) we get the so-called generalised Bianchi identities:

\[(5.32) \quad W_\sigma - \nabla_\mu W^\mu_\sigma = 0\]

Following the standard procedure of [14] and [21] define now a \((m - 2)\)–form, called a superpotential, by:

\[(5.33) \quad U(L, \Xi) = \frac{1}{2} \left[ \left( T^\lambda_\mu |^{\lambda_\mu}_\rho - \frac{2}{3} \nabla_\sigma T^\lambda_\rho |^{\lambda_\mu}_\sigma \right) \xi^\rho + \left( \frac{4}{3} T^\lambda_\rho |^{\lambda_\mu}_\nu \right) \nabla_\nu \xi^\rho \right] ds_\lambda_\mu\]

and a \((m - 1)\)–form \(\tilde{E}(L, \Xi)\), called the reduced energy-density, by:

\[(5.34) \quad \tilde{E}(L, \Xi) = W^\mu_\sigma \xi^\sigma ds_\mu\]

For the energy-density flow the following representation is thence true

\[(5.35) \quad E(L, \Xi) = \tilde{E}(L, \Xi) + \text{Div} U(L, \Xi)\]

where Div denotes the formal divergence, defined for any global section \(\rho\) and any \(p\)–form \(\omega\) over \(J^2B\) by:

\[(5.36) \quad (j^3 \rho)^* (\text{Div} \omega) = d \left[ (j^2 \rho)^* \omega \right]\]

We finally consider the global forms \(U(L, \Xi, \rho) = (j^2 \rho)^* U(L, \Xi)\) and \(\tilde{E}(L, \Xi, \rho) = (j^3 \rho)^* \tilde{E}(L, \Xi)\) obtained by pull back along any section \(\rho\); the second one vanishes on shell (i.e. if \(\rho\) is critical). Therefore, the energy-density flow \(E(L, \Xi)\) is an exact form along critical sections.

We remark that these results for conserved currents are completely general and they hold actually for every covariant Lagrangian. If we turn back to our choice (3.4) and (3.5) of the total Lagrangian we get the following explicit expression for the superpotential:

\[(5.37) \quad U(L, \Xi) = U(L_H, \Xi) + U(L_D, \Xi)\]

\(U(L_H, \Xi) = \frac{1}{2\kappa} \left[ \left( \nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \right) \sqrt{g} ds_{\mu\nu} \right] \)

\(U(L_D, \Xi) = \frac{i}{8} \sqrt{g} ds_{\mu\nu} \left[ \left( \gamma^{[\mu} \gamma^{\nu]} \gamma^\rho + 2g^{[\mu} \gamma^{\nu]} \right) \xi^\rho \right] \)}
where we have set \( \gamma^\mu = e^\mu_a \gamma^a \).

Since \( W(L, \Xi) \) vanishes on shell, pulling back \( U(L, \Xi) \) on a solution and integrating it on the border of a spatial domain \( D \), one gets a conserved quantity for each 1-parameter family of symmetries:

\[
Q_\Xi := \int_{\partial D} (j^1 \rho)^* U(L, \Xi)
\]

(5.38)

VI Conclusion and perspectives

The new formalism we have developed for spinor theories is interesting for some further reasons besides those we already mentioned in the Introduction.

First of all we see once again that the requirement of geometric coherence actually allows us to select the theory we are looking for within the much larger set of possible theories. In fact here, as well as in our previous work concerning Bosonic matter [1], we have not been concerned with the existence of solutions, but we have just cared that all concepts entering the theory could be well defined by a global and geometric point of view. The surprise arises from the fact that even under these general and formal requests we can manage to build physically admissible theories and above all to set aside lots of other theories.

The second reason is related to researches about a unifying paradigm for General Relativity and Quantum Mechanics. Here it is important to fix what spinor matter is in General Relativity, since the new paradigm must reproduce this formalism in the classical limit (= not Quantum) and spinors of Quantum Mechanics in the flat limit (= on Minkowski space). However since spinor theories on curved spaces are qualitatively much more complicated, we can perhaps expect that the first request is stronger than the second one.

We finally want to stress which are the differences between the classical spinor theories on curved spaces (see, e.g., [8]) and our approach. First of all, our approach is naturally formulated in the framework of variational calculus on fibred bundles. Second, our formalism allows a deep analogy among spinor theories, General Relativity as formulated here, so called natural theories and gauge theories as formulated in [1] and [16]. In each case we start by choosing a principal bundle (called structure bundle); the configuration bundle is then a gauge-natural bundle associated to some principal prolongation of it and the Lagrangian is required to be covariant with respect to any principal automorphism of the structure bundle represented on the configuration bundle. In the case of General Relativity this result is obtained in a way that we believe is important to stress. We have both enlarged the symmetry group from \( \text{Diff}(M) \) to \( \text{Aut}(\Sigma) \) and the number of dynamical fields from \( g^{\mu\nu} \) to \( e^\mu_a \). The covariance request on the Lagrangian lets \( e^\mu_a \) appear when we are coupling with spinor matter but allows just \( g^{\mu\nu} \) when coupling with Bosonic
matter. Further investigations about the unified formulation of the above theories should therefore follow and we hope to address this problem in the near future.

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Appendix

The following Lie derivatives are easily calculated:

\[
\begin{align*}
(A.1) & \quad \mathcal{L}_\xi e^\mu_a = e^\mu_b (\xi_V)^b_a - e^\nu_a \nabla_\nu \xi^\mu \\
(A.2a) & \quad \mathcal{L}_\xi v^A = \Omega^A_b \varepsilon^a_b \xi^\mu - \lambda^A_{bij} v^b (\xi_V)^{ij} \\
(A.2b) & \quad (\xi_V)^{ij} := \xi^{ij} - \Gamma^{ij}_\mu \xi^\mu \\
(A.3) & \quad \mathcal{L}_\xi \Omega^A_a = \xi^\mu \nabla_\mu \Omega^A_a + (\xi_V)^b_a \Omega^A_b - \lambda^A_{bij} \Omega^B_b (\xi_V)^{ij} \\
(A.4) & \quad \mathcal{L}_\xi \Gamma^{ij}_\mu = R^{ij}_{\rho \mu} \xi^\rho + \nabla_\mu (\xi_V)^{ij} \\
(A.5) & \quad \mathcal{L}_\xi \Gamma^\rho_{\sigma \mu} = \xi^\mu \nabla_\sigma \Gamma^\rho_{\sigma \mu} - \nabla_\sigma (\sigma \nabla_\mu) \xi^\rho \\
(A.6) & \quad \mathcal{L}_\xi g_{\mu \nu} = \xi^\mu \nabla_\nu g_{\mu \nu} + \nabla_\mu \xi^\sigma g_{\sigma \mu} \\
\end{align*}
\]

Moreover it is not difficult to prove the following identities:

\[
\begin{align*}
(A.7) & \quad \mathcal{L}_\xi \Omega^A_a = \Omega^A_b \varepsilon^b_\mu \mathcal{L}_\xi e^\mu_a + \varepsilon^\nu_a (d_\mu \mathcal{L}_\xi v^A + \lambda^A_{bij} \Gamma^{ij}_\mu \mathcal{L}_\xi v^B + \lambda^A_{bij} v^B \mathcal{L}_\xi \Gamma^{ij}_\mu) \\
(A.8) & \quad \mathcal{L}_\xi \Gamma^{ij}_\mu = \varepsilon^i_\rho (\xi^k_\Gamma^\rho_{\sigma \mu} + \varepsilon^b_\sigma \nabla_\mu \mathcal{L}_\xi e^k_b) \varepsilon^\nu_\rho \eta^{ij} \\
\end{align*}
\]

We finally observe that:

\[
(A.9) \quad \nabla_\mu e^\nu_a := d_\mu e^\nu_a + e^\sigma_a \Gamma^\nu_{\sigma \mu} - \Gamma^b_{a \mu} e^\nu_b \equiv 0
\]

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