Noether Symmetry Analysis of Anisotropic Universe in Modified Gravity

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Abstract

In this paper, we study anisotropic universe using Noether symmetries in modified gravity. In particular, we choose locally rotationally symmetric Bianchi type-I universe for the analysis in $f(R, G)$ gravity, where $R$ is the Ricci scalar and $G$ is the Gauss-Bonnet invariant. Firstly, a model $f(R, G) = f_0 R^l + f_1 G^n$ is proposed and the corresponding Noether symmetries are investigated. Further, we have also recovered the Noether symmetries for $f(R)$ and $f(G)$ theories of gravity. Secondly, some important cosmological solutions are reconstructed. Exponential and power law solutions are reported for a well-known $f(R, G)$ model, i.e., $f(R, G) = f_0 R^n G^{1-n}$. Especially, the Kasner’s solution is recovered and it is anticipated that the familiar de-Sitter spacetime giving $ΛCDM$ cosmology may be reconstructed for some suitable value of $n$.

Keywords: Anisotropic Models; Exact Solutions; Noether Symmetry.

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1 Introduction

Experimental observations in the recent years have indicated that universe is expanding \[ 1\]–\[ 4\]. The candidate referred for the explanation of this expansion is known as Dark energy (DE). DE is thought to be the energy density reserved in the cosmological constant and its value is considered \( \rho_\Lambda \sim 10^{-3} eV^4 \) \[ 5\]. Modification of general relativity (GR) can also be one of the possibility to explain the existence of DE. The presence of some additional terms in the matter part of gravitational action which cause a minimum modification in GR can be helpful to explore the nature of DE \[ 6\]. Another method to explain the cosmological acceleration is to modify the geometrical part of the action, by coupling curvature scalars, topological invariants and their derivatives which results in modified theories like \( f(R) \) gravity, \( f(G) \) gravity, \( f(R, T) \) gravity etc., where \( R \) is the Ricci scalar and \( G \) is the Gauss-Bonnet invariant, and \( T \) is the trace of energy momentum tensor.

A new modified theory by considering a general function of \( R \) and \( G \) as \( f(R, G) \) has gained popularity in recent years \[ 7\]–\[ 9\]. There are many interesting aspects of Gauss-Bonnet theory which motivate the researchers to study modified theories of gravity involving Gauss-Bonnet term. In particular, it have been shown that Gauss-Bonnet gravity can address the DE problem without the need for any exotic matter components \[ 10\]. Gauss-Bonnet term is a specific combination of curvature invariants that includes Ricci scalar, Ricci and Riemann tensors. In fact, Gauss-Bonnet invariant naturally arises in the process of quantum field theory regularization and renormalization of curved spacetime. In particular, including \( G \) and \( R \) in a bivariate function provides a double inflationary scenario where the two acceleration phases are led by \( G \) and \( R \) respectively \[ 11\]. Moreover, the involvement of Gauss-Bonnet invariant may play an important role in the early time expansion of universe as it is connected with the string theory and the trace anomaly \[ 12\]. The viability of modified Gauss-Bonnet gravity has been studied by considering different realistic models using the weak energy condition and it was concluded that \( f(R, G) \) gravity models show consistency with the recent observational data \[ 13\].

Symmetry approach performs a pivotal part to find exact solutions or simply reduce a non-linear system of equations to a linear system of equations. In the literature (see \[ 16\]–\[ 22\] for references), Noether symmetries have been studied in the context of cosmology and astrophysics, in particular, to investigate the exact solutions of field equations. Noether symmetry
is a proficient method to calculate unknown variables of differential equations. Sharif and Waheed [23] studied Bardeen model and stringy charged black holes by using approximate symmetry methods. They also explored Noether symmetries of Friedmann-Robertson-Walker (FRW) and locally rotationally symmetric (LRS) Bianchi type-I (BI) universe models by adding an inverse curvature term in Brans-Dicke theory [24]. Capozziello et al. [25] discussed FRW universe model in \( f(R, \mathcal{G}) \) gravity using the Noether symmetry approach. In modified scalar-tensor gravity, Sharif and Shafique [26] studied BI model using Noether and Noether gauge symmetry. Sharif and Fatima [27] investigated Noether symmetry of flat FRW model for vacuum and non-vacuum cases in \( f(G) \) gravity.

The spatially homogeneous but largely anisotropic nature of early universe was disclosed after the discovery of cosmic microwave background radiation (CMBR) [28]. Bianchi type universe models can be considered to quantify the change of anisotropy in the early universe through recent observations. These universe model indicates that the anisotropy of early universe determines the acceleration rate of the universe. If the primary anisotropy is less than this rate of acceleration would lead to a highly isotropic universe [29]. Akarsu and Kilinc [30] considered BI model to study different equation of state (EoS) models which coincide with de-Sitter universe. Sharif and Zubair [31] studied the solutions of BI universe model by using power-law and exponential expansions in scalar-tensor gravity. Shamir [32] explored exact solutions of LRS BI universe model and investigated physical behavior of cosmological parameters in \( f(R, T) \) gravity. Shamir and Ahmad [33] has discussed the Noether symmetry approach for FRW universe model in \( f(\mathcal{G}, T) \) gravity. In another paper [34], the same authors explored some important cosmological solutions in \( f(\mathcal{G}, T) \) gravity using Noether symmetries. In particular they found some interesting results by considering LRS BI spacetime and recovered \( \Lambda CDM \) model universe for some specific choice of \( f(\mathcal{G}, T) \) gravity model. Thus it seems interesting to investigate Noether symmetry of anistropic universe in modified gravity.

In this paper, we have explored Noether symmetries of BI cosmological model. The interesting physical forms of \( f(R, \mathcal{G}) \) can be determined by existence of Noether symmetry which allows to reduce the dynamics. The Noether symmetry approach has been broadly used for modified theories which gave some applicable results for cosmological systems. In this work, exponential and power law solutions are reported for a well-known \( f(R, \mathcal{G}) \) model, i.e., \( f(R, \mathcal{G}) = f_0 R^n \mathcal{G}^{1-n} \). Especially, the Kasner’s solution is recov-
ered and it is anticipated that the familiar de-Sitter spacetime giving \( \Lambda CD M \) cosmology may be reconstructed for some suitable value of \( n \). The layout of paper is as follows: In section 2, we give gravitational action and Einstein field equations for modified Gauss-Bonnet gravity and derive a point like canonical Lagrangian for configuration space. We explore the Noether symmetries for some cosmological models in section 3. In section 4, we have discussed some examples of exact solution in cosmological context. Final remarks are given in last section.

## 2 Modified Field Equations and Lagrangian Framework

For \( f(R,G) \) gravity the most general action in 4-dimensions is \[ 25 \]

\[
\mathcal{A} = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} f(R,G) + \mathcal{L}_m \right], \tag{1}
\]

where \( \kappa \) is the coupling constant, \( \mathcal{L}_m \) is the matter Lagrangian and \( g \) denotes the determinant of the metric tensor. \( G \) indicates the Gauss-Bonnet invariant

\[
G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}, \tag{2}
\]

where \( R_{\mu\nu} \) is Ricci tensor, \( R_{\mu\nu\lambda\sigma} \) represents Riemann tensor and \( R = g^{\mu\nu}R_{\mu\nu} \) is called the Ricci scalar. From now onwards, we will use \( f(R,G) \equiv f, \frac{\partial f(R,G)}{\partial G} \equiv f_G, \frac{\partial f}{\partial R} \equiv f_R \), etc. Variation of \( 1 \) with respect to the metric tensor \( g_{\mu\nu} \), leads to the modified field equations \[ 35 \]

\[
0 = \kappa^2 T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} f - 2f_G R R^{\mu\nu} + 4f_G R^\rho R^{\nu\rho} - 2f_G R^{\mu\rho\sigma\tau} R_{\rho\sigma\tau} - 4f_G R^{\mu\rho\sigma\nu} R_{\rho\sigma} \]
\[
+ 2R \nabla^\mu \nabla^\nu f_G - 2g^{\mu\nu} R \nabla^2 f_G - 4\nabla_\rho \nabla^\mu f_G R^{\rho\nu} - 4R^{\mu\rho} \nabla_\rho \nabla^\nu f_G + 4R^{\mu\nu} \nabla^2 f_G \]
\[
+ 4g^{\mu\nu} R^{\rho\sigma} \nabla_\rho \nabla_\sigma f_G - 4R^{\mu\rho\sigma\nu} \nabla_\rho \nabla_\sigma f_G - f_G R_R^{\mu\nu} + \nabla^\mu \nabla^\nu f_G - g^{\mu\nu} \nabla^2 f_G, \tag{3}
\]

where \( \nabla \) represents the covariant derivative and the energy momentum tensor \( T_{\mu\nu} \) is defined as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}.
\]
Let us note that Einstein equations for GR are recovered by putting \( f(R, \mathcal{G}) = R \) and by replacing the \( f(R, \mathcal{G}) \) with \( f(\mathcal{G}) \), we obtain the field equations for \( f(\mathcal{G}) \) gravity. We consider an LRS BI universe model, defined by the line element \[ ds^2 = dt^2 - A(t)^2 dx^2 - B(t)^2(dy^2 + dz^2), \]

where \( A \) and \( B \) are known as cosmic scale factors of the universe. We need to find out a point-like canonical Lagrangian \( L(q^i, \dot{q}^i) \) from the gravitational action, defined on the configuration space \( Q \) and the corresponding tangent space \( TQ \) where \( q^i \) represents \( n \) generalized positions and dot denotes time derivative. By using the technique of the Lagrange multipliers, we can deduce \( R \) and \( \mathcal{G} \) as constraints for dynamics. In order to reduce the order of derivatives in Lagrangian \( L \), we use the Lagrange multiplier approach and integrate by parts so that the Lagrangian \( L \) becomes canonical \[ L = 2\pi^2 \int dt AB^2 \left[ f(R, \mathcal{G}) - \chi_1 \left( R + 2 \left( \frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} + 2 \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right) \right) - \chi_2 \left( \mathcal{G} - 8 \left( \frac{\dot{A}\dot{B}^2}{AB^2} + 2 \frac{\dot{A}\dot{B}\dot{B}}{AB^2} \right) \right) \right], \]

where dot indicates the derivative with respect to time \( t \) and \( \chi_1, \chi_2 \) are the Lagrange multipliers. The Ricci scalar and the Gauss-Bonnet invariant for LRS BI metric is \[ R = -2 \left( \frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} + 2 \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right), \quad \mathcal{G} = 8 \left( \frac{\dot{A}\dot{B}^2}{AB^2} + 2 \frac{\dot{A}\dot{B}\dot{B}}{AB^2} \right). \]

The Lagrange multipliers \( \chi_1, \chi_2 \) are obtained by varying the action (5) with respect to \( R \) and \( \mathcal{G} \) respectively as \[ \chi_1 = f_R, \quad \chi_2 = f_\mathcal{G}. \]

Using Eq.(7), the action (5) becomes \[ A = 2\pi^2 \int dt AB^2 \left[ f_R - \chi_1 \left( R + 2 \left( \frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} + 2 \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right) \right) - \chi_2 \left( \mathcal{G} - 8 \left( \frac{\dot{A}\dot{B}^2}{AB^2} + 2 \frac{\dot{A}\dot{B}\dot{B}}{AB^2} \right) \right) \right], \]

After an integration by parts, the Lagrangian turns out to be \[ L = 2AB^2 f + 4\dot{A}\dot{B}B f + 2\dot{A}B^2 \frac{d}{dt} f + 2AB^2 \frac{d}{dt} f + 8\dot{A}\dot{B}^2 \frac{d}{dt} f + AB^2(f - R f_R - \mathcal{G} f_\mathcal{G}). \]
This is a point-like canonical Lagrangian whose configuration space is $Q \equiv \{A, B, R, \mathcal{G}\}$ and tangent space is $QT \equiv \{A, B, R, \dot{A}, \dot{B}, \dot{R}, \dot{\mathcal{G}}\}$. Due to highly nonlinear nature of Lagrangian, it is complicated to deal with it. For the sake of simplicity, we assume $B = A^m$. The physical importance of this assumption is that it gives constant ratio of shear and expansion scalar [38]. Thus the Lagrangian (9) takes the form

$$\mathcal{L} = 2m^2 A^{2m-1} \dot{A}^2 f_R + 4mA^{2m-1} A^2 f_R + 2A^{2m} \dot{A} \frac{d}{dt} f_R + 4 mA^{2m} \dot{A} \frac{d}{dt} f_R$$

$$- 8m^2 A^{2m-2} \dot{A}^3 \frac{d}{dt} f_G + A^{2m+1} [f - R f_R - G f_G], \quad (10)$$

which is now a function of $A$, $R$ and $\mathcal{G}$.

3 Noether Symmetry and $f(R, \mathcal{G})$ Gravity

In the presence of Noether symmetry, the constants of motion can be selected by the reduction of dynamical system [39]. We consider the vector field and its first prolongation respectively as

$$X = \xi(t, q^i) \frac{\partial}{\partial t} + \eta^i(t, q^i) \frac{\partial}{\partial q^i}, \quad (11)$$

$$X^{[1]} = X + (\eta^i_t + \eta^j \dot{q}^j - \xi_t \dot{q}^i - \eta^i \dot{q}^j \dot{q}^j) \frac{\partial}{\partial \dot{q}^i}, \quad (12)$$

where $\xi$ and $\eta$ are the coefficients of the generators, $q^i$ provides the $n$ number of positions and dot gives the derivative with respect to time $t$ [40]. The vector field $X$ produces Noether gauge symmetry provided the condition

$$X^{[1]} \mathcal{L} + (D\xi) \mathcal{L} = DG(t, q^i) \quad (13)$$

is preserved. Here $G(t, q^i)$ denotes gauge term and $D$ is an operator defined as

$$D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}. \quad (14)$$

The Euler-Lagrange equations are given by [41]

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = 0. \quad (15)$$
Contraction of Eq. (15) with some unknown function \( \psi^i \equiv \psi_i(q^j) \) yields

\[
\psi^i \left( \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) \right) = 0,
\]

(16)

It is easy to verify that

\[
\frac{d}{dt} \left( \psi^i \frac{\partial L}{\partial \dot{q}^i} \right) - \left( \frac{d}{dt} \psi^i \right) \frac{\partial L}{\partial \dot{q}^i} = \psi^i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right).
\]

(17)

Putting this value in Eq. (16) provides us with

\[
L_X L = \left( \frac{d}{dt} \psi^i \right) \frac{\partial L}{\partial \dot{q}^i} + \psi^i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{d}{dt} \left( \psi^i \frac{\partial L}{\partial \dot{q}^i} \right),
\]

(18)

where \( L_X \) is the Lie derivative with respect to the Noether vector \( X \). Lagrangian generates a Noether symmetry if Lie derivative, for a vector field \( X \), vanishes

\[
L_X L = 0.
\]

(19)

The energy condition is given as

\[
\sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = E_L.
\]

(20)

We derive the Euler-Lagrange equations for \( A, R, G \) respectively as,

\[
\begin{align*}
2m(2m^2 + 3m - 2)A^{2m-2} \dot{A}^2 f_R + 4m(m + 2)A^{2m-1} \dot{A} f_R + 4m^2 A^{2m-1} \dot{A} \frac{d}{dt} \dot{A}^2 f_R - 32m^2(m - 1)A^{2m-3} \dot{A}^3 \frac{d}{dt} f_R + 2(2m + 1)A^{2m} \frac{d}{dt} \dot{A} f_R - 48m^2 A^{2m-2} \dot{A} \dot{A} \frac{d}{dt} f_R + 4(m + 1)A^{2m-1} \dot{A} \frac{d}{dt} f_R - 24m^2 A^{2m-2} \dot{A} \frac{d}{dt} f_R - (2m + 1)A^{2m} \left[ f - Rf_R - Gf_G \right] = 0,
\end{align*}
\]

(21)

\[
\begin{align*}
\left[ R + 2 \left( \frac{\dot{A}}{A} + 3m^2 \left( \frac{\dot{A}}{A} \right)^2 + 2m \frac{\ddot{A}}{A} \right) \right] f_{RR} + \left[ G + 8 \left( 2m^2 \left( \frac{\dot{A}}{A} \right)^4 - 2m^3 \left( \frac{\ddot{A}}{A} \right)^4 - 3m^2 \frac{\dot{A}^2 \dddot{A}}{A^3} \right) \right] f_{RG} = 0,
\end{align*}
\]

(22)
\[
\left[ R + 2\left( \frac{\dot{A}}{A} + 3m^2\left( \frac{\dot{A}}{A} \right)^2 + 2m\frac{\ddot{A}}{A} \right) \right] f_{RG} + \left[ G + 8\left( 2m^2\left( \frac{\dot{A}}{A} \right)^4 - 2m^3\left( \frac{\dot{A}}{A} \right)^4 \right) - 3m^2\frac{\dddot{A}}{A^3} \right] f_{GG} = 0. \tag{23}
\]

This is important to note that (22) and (23) are symmetric. The energy condition (20) takes the form
\[
\left( \frac{\dot{A}}{A} \right)^2 f_R + \frac{\dot{A}}{A} \frac{df_R}{dt} - 4\left( \frac{\dot{A}}{A} \right)^3 f_{\dot{G}} - \frac{1}{6} \left[ f - R f_R - G f_G \right] = 0. \tag{24}
\]

The corresponding vector field becomes
\[
X = \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial G} + \hat{\alpha} \frac{\partial}{\partial \dot{A}} + \hat{\beta} \frac{\partial}{\partial \dot{R}} + \hat{\gamma} \frac{\partial}{\partial \dot{G}}, \tag{25}
\]

The symmetry generators \( \alpha, \beta \) and \( \gamma \) are functions of \( A, R \) and \( G \). Using Lagrangian (10) and Noether equation (25), we get a system of partial differential equations (PDEs)
\[
2m^3\alpha f_R + 3m^2\alpha f_R - 2m\alpha f_R + m^2\beta A f_{RR} + 2m\beta A f_{RR} + m^2\gamma A f_{RG}
+ 2m\gamma A f_{RG} + 2m^2 A \frac{\partial \alpha}{\partial A} f_R + 4mA \frac{\partial \alpha}{\partial A} f_R + A^2 \frac{\partial \beta}{\partial A} f_R + 2mA^2 \frac{\partial \beta}{\partial A} f_{RR}
+ A^2 \frac{\partial \gamma}{\partial A} f_{RG} + 2mA^2 \frac{\partial \gamma}{\partial A} f_{RG} = 0, \tag{26}
\]

\[
2m\alpha f_{RR} + 4m^2\alpha f_{RR} + 2m\beta A f_{RR} + \beta A f_{RR} + \gamma A f_{RR} + 2m\gamma A f_{RR}
+ A \frac{\partial \alpha}{\partial A} f_{RR} + 2mA \frac{\partial \alpha}{\partial A} f_{RR} + 2m^2 \frac{\partial \alpha}{\partial R} f_R + 4mA \frac{\partial \alpha}{\partial R} f_R + A \frac{\partial \beta}{\partial R} f_{RR}
+ 2mA \frac{\partial \beta}{\partial R} f_{RR} = 0, \tag{27}
\]

\[
2m\alpha f_{RG} + 4m^2\alpha f_{RG} + \beta A f_{RG} + 2m\beta A f_{RG} + \gamma A f_{RG} + 2m\gamma A f_{RG}
+ A \frac{\partial \alpha}{\partial A} f_{RG} + 2mA \frac{\partial \alpha}{\partial A} f_{RG} + 2m^2 \frac{\partial \alpha}{\partial G} f_R + 4mA \frac{\partial \alpha}{\partial G} f_R + 2mA \frac{\partial \beta}{\partial G} f_{RR}
+ A \frac{\partial \beta}{\partial G} f_{RR} + 2mA \frac{\partial \gamma}{\partial G} f_{RG} = 0. \tag{28}
\]
\[ 2m \alpha f_{RG} - 2 \alpha f_{RG} + \beta A f_{RGG} + \gamma A f_{GGG} + 3A \frac{\partial \alpha}{\partial A} f_{RG} + A \frac{\partial \beta}{\partial R} f_{RG} + A \frac{\partial \gamma}{\partial G} f_{GG} = 0, \]  
\quad (29)

\[ 2m \alpha f_{GG} - 2 \alpha f_{GG} + \beta A f_{RGG} + \gamma A f_{GGG} + 3A \frac{\partial \alpha}{\partial A} f_{GG} + A \frac{\partial \beta}{\partial R} f_{RG} + A \frac{\partial \gamma}{\partial G} f_{GG} = 0, \]  
\quad (30)

\[ \frac{\partial \alpha}{\partial R} f_{RG} + 2m \frac{\partial \alpha}{\partial R} f_{RG} + \frac{\partial \alpha}{\partial G} f_{RR} + 2m \frac{\partial \alpha}{\partial G} f_{RR} = 0, \]  
\quad (31)

\[ \frac{\partial \alpha}{\partial R} f_{RG} = 0, \quad \frac{\partial \alpha}{\partial G} f_{GG} + \frac{\partial \alpha}{\partial G} f_{RG} = 0, \]  
\quad (32)

\[ \frac{\partial \alpha}{\partial R} f_{RR} + 2m \frac{\partial \alpha}{\partial R} f_{RR} = 0, \quad \frac{\partial \alpha}{\partial G} f_{RG} + 2m \frac{\partial \alpha}{\partial G} f_{RG} = 0, \]  
\quad (33)

\[ \frac{\partial \alpha}{\partial G} f_{GG} = 0, \quad \frac{\partial \beta}{\partial A} f_{RG} + \frac{\partial \gamma}{\partial A} f_{GG} = 0, \]  
\quad (34)

\[ (2m + 1)\alpha [f - R f_R - G f_G] - \beta A [R f_{RR} + G f_{RG}] - \gamma A [R f_{RG} + G f_{GG}] = 0. \]  
\quad (35)

The obtained system of PDEs is over-determined. Hence one can solve it by assigning the suitable values to the unknown function \( f(R, G) \). Here we propose \( f(R, G) \) as linear combination of power law forms of \( R \) and \( G \) as

\[ f(R, G) = f_0 R^l + f_1 G^n, \]  
\quad (36)

where \( f_0, f_1 \) are the arbitrary constants and \( l, n \) are any non-zero real numbers. We get a number of solutions as follows:

\[ l = 1, \quad n = 1, \quad \alpha = c_1 A^{-m+\frac{1}{2}}, \quad \beta = 0, \quad \gamma = 0, \]  
\quad (37)
\begin{align*}
l \neq 1, \quad l &= \frac{4m^2 + 4m + 1}{3m^2 + 2m + 1}, \quad n = 1, \quad \alpha = c_2 A^{-\frac{3n^2}{2n^2+3n^2}}, \\
\beta &= -\frac{c_2 A^{-\frac{3n^2}{2n^2+3n^2}} R(3m^2 + 2m + 1)}{A(2m + 1)}, \quad \gamma = 0. \quad (38)
\end{align*}

Here we have explored some additional symmetries and it would be worthwhile to mention here that Eq. (37) agree with [25] for special case when \( m = 1 \). The corresponding Lagrangian for the particular functional form \( f(R, G) = f_0 R + f_1 G \) becomes

\[ \mathcal{L} = 2m(m + 2)f_0 A^{2m-1} \dot{A}^2. \quad (39) \]

The Euler Lagrange equations and energy equation are calculated as

\[ 2\frac{\ddot{A}}{A} + (2m - 1)\left(\frac{\dot{A}}{A}\right)^2 = 0, \quad (40) \]

\[ R + 2\left(2m + 1\right)\left(\frac{\ddot{A}}{A} + 3m^2 \frac{\dot{A}^2}{A^2}\right) = 0, \quad (41) \]

\[ \mathcal{G} - 8\left(m(2m + 1)\frac{\dddot{A}^2}{A^3} + 2m^2 (m - 1)\frac{\dot{A}^4}{A^4}\right) = 0, \quad (42) \]

\[ \left(\frac{\dot{A}}{A}\right)^2 = 0. \quad (43) \]

In these equations the Gauss-Bonnet term \( \mathcal{G} \) disappears so this theory becomes nothing but the General Relativity. If we consider vacuum case then we obtain Minkowski spacetime.

### 3.1 Recovering Noether Symmetries in \( f(R) \) and \( f(G) \) Theories of Gravity

Here we investigate the solution of determining system of equations mainly for two different cases, for \( f(R) \) gravity and \( f(G) \) gravity. First we explore solutions for \( f(R, G) = f_0 R^m \). The solutions of determining equations are given as

\[ m = \pm\frac{n - 2 + \sqrt{3n - 2n^2}}{3n - 4}, \quad \alpha = c_3 A^{-\frac{5n \pm 2\sqrt{3n - 2n^2} + 3n^2}{3n^2 - 4n}}, \]

\[ \beta = -\frac{c_3 A^{-\frac{5n \pm 2\sqrt{3n - 2n^2} + 3n^2}{3n^2 - 4n}} R(n \pm 2\sqrt{3n - 2n^2})}{(3n - 4)An}. \quad (44) \]
It is worthwhile to mention here that for a special case when $n = \frac{3}{2}$, we obtain

$$m = 1, \quad \alpha = \frac{c_2}{A}, \quad \beta = -\frac{2c_3 R}{A^2},$$  \hspace{1cm} (45)

and the results agree with [19, 42]. To recover Noether symmetries in $f(G)$ gravity, we choose $f(R, G) = f_0 G^n$ and the solution in this case turn out to be as

$$m = -\frac{1}{2}, \quad \alpha = c_4 A, \quad \gamma = 0.$$  \hspace{1cm} (46)

### 4 Some Exact Cosmological Solutions

Here we reconstruct some important cosmological solution using an interesting $f(R, G)$ model, i.e., $f(R, G) = f_0 R^n G^{1-n}$ [25]. For the simplest non-trivial case, we choose $n = 2$. In this case, the point-like Lagrangian [10] takes the form

$$\mathcal{L} = \frac{4f_0 \dot{A}}{G} \left[ (m^2 + 2m)A^{2m-1} \dot{A} \ddot{R} + (2m + 1)A^{2m} \ddot{R} - (2m + 1)A^{2m} \dot{G} \frac{R}{G} \right]
+ 4m^2 A^{2m-2} \dot{A}^2 \dot{R} \frac{R}{G} - 4m^2 A^{2m-2} \dot{A}^2 \dot{G} \left( \frac{R}{G} \right)^2 \right].$$  \hspace{1cm} (47)

Here, the Euler-Lagrange equations become

$$m(2m^2 + 3m - 2)A^{2m-2} \ddot{A}^2 \frac{R}{G} + 2m(m + 2)A^{2m-1} \dot{A} \ddot{R} + 2m(m + 2)A^{2m-1} \dot{A} \dot{R} \frac{G}{G}
- 2m(m + 2)A^{2m-2} \ddot{A}^2 \dot{G} \frac{R}{G^2}
- (2m + 1)A^{2m-1} \ddot{A}^2 \frac{R}{G} - 2(2m + 1)A^{2m} \ddot{R} \frac{G}{G^2} - 2(2m + 1)A^{2m} \dot{R} \frac{G}{G^2}
+ 2(2m + 1)A^{2m} \dot{G} \frac{R}{G^2} + 16m^2 (m - 1)A^{2m-3} \ddot{A}^2 \frac{G}{G^2} + 24m^2 A^{2m-2} \dot{A}^2 \frac{\ddot{R} \dot{G}}{G^2}
+ 12m^2 (m - 1)A^{2m-3} \dot{A} \ddot{G} \frac{G}{G^3}
+ 12m^2 (m - 1)A^{2m-3} \ddot{A} \dot{G} \frac{G}{G^3} - 24m^2 A^{2m-2} \dot{A} \frac{\dot{R} \ddot{G}}{G^3}
- 12m^2 (m - 1)A^{2m-3} \dot{A} \frac{\ddot{G} \dot{R}}{G^3}
+ 36m^2 A^{2m-2} \dot{A} \frac{\dot{G} \dot{R}}{G^4}
= 0,$$  \hspace{1cm} (48)

$$(2m + 1)A^{2m} \ddot{A} + 3m^2 A^{2m-1} \dot{A}^2 + 8m^2 (m - 1)A^{2m-3} \ddot{A}^2 \frac{R}{G}
+ 12m^2 A^{2m-2} \dot{A} \ddot{A} \frac{R}{G} = 0.$$  \hspace{1cm} (49)
and the energy condition (20) takes the form

\[
  m(m + 2)A^{2m-1}A^2 R + (2m + 1)A^{2m} \dot{A} \dot{R} - (2m + 1)A^{2m} \dot{A} \frac{R \dot{G}}{G} + 12m^2 A^{2m-2} \dot{A}^3 \frac{R \dot{R}}{G} - 12m^2 A^{2m-2} \dot{A}^3 \frac{R \dot{G}}{G^2} = 0.
\]  

(50)

By putting the corresponding values of \( R \) and \( G \) and using Eq.(50), we get

\[
  12m^4(m^4 - 5m^3 + 2m + 2)\dot{A}^8 + m(132m^5 - 52m^4 - 41m^3 - 57m^2 \\
- 53m - 10)A^2 \dot{A}^4 \ddot{A}^3 + (16m^4 + 8m^3 - 12m^2 - 10m - 2)A^4 \ddot{A}^4 + \\
(80m^5 + 36m^4 + 40m^3 + 61m^2 + 24m + 2)A^3 \dot{A}^2 \ddot{A}^3 + m(4m^4 + 8m^3 \\
- 3m^2 - 7m - 2)A^3 \ddot{A}^3 \dot{A} \dot{A} + m^2(92m^5 - 110m^4 - 45m^3 - 44m^2 + 14m \\
+12)A \dot{A}^6 \dot{A} + m^2(10m^2 + 29m^3 + 24m^2 + 14m + 4)A^2 \ddot{A}^5 \dot{A} = 0.
\]

(51)

4.1 Exponential law solutions

Here we assume the metric coefficients in exponential law form, i.e.,

\[ A = e^{\varphi t}, \]

(52)

where \( \varphi \) is an arbitrary constant. The solution metric for this case is

\[ ds^2 = dt^2 - e^{2\varphi t} dx^2 - e^{2m\varphi t} (dy^2 + dz^2). \]

(53)

Using Eq.(52) in Eq.(51), we get the constraint equation

\[
  (12m^8 + 32m^7 + 32m^6 + 40m^5 + 23m^4 + 16m^3 + 5m^2 + 2m) = 0.
\]  

(54)

The real solutions for Eq.(50) are

\[ m = 0, \quad m = -2. \]

(55)

Hence we obtain two cosmological solutions by considering \( n = 2 \) in the \( f(R, G) \) model. Many other solutions can be reconstructed by choosing some other values of the parameter. It is anticipated that a \( \Lambda CDM \) universe may be generated for \( m = 1 \) and some suitable value of \( n. \)
4.2 Power law solutions

We assume $A = t^\zeta$, to extract a power law solution, where $\zeta$ is any non-zero real number. In this case, we obtain the constraint equation as

\begin{align*}
12m^4(m^4 - 5m^3 + 2m + 2)\zeta^4 + 2(m - 1)(2m + 1)^3(\zeta^4 - 4\zeta^3 + 6\zeta^2 - 4\zeta + 1) \\
+ m(2m + 1)(-10 + m(-33 + m(9 + m(-59 + 66m)))(\zeta^4 - 2\zeta^3 + \zeta^2) \\
+ m^2(12 + m(14 + m(-44 + m(-45 + 2m(-55 + 46m)))))(\zeta^4 - \zeta^3) \\
+ (2m + 1)^2(2 + m(16 + m(-11 + 20m)))(\zeta^4 - 3\zeta^3 + 3\zeta^2 - \zeta) \\
+ m^2(2 + m)(2m + 1)(2 + m(2 + 5m))(\zeta^4 - 3\zeta^3 + 2\zeta^2) \\
+ m(2m + 1)^2(m^2 + m - 2)(\zeta^4 - 4\zeta^3 + 5\zeta^2 - 2\zeta) = 0,
\end{align*}

(56)

In order to get a particular solution, we put $\zeta = -\frac{1}{3}$, so that (56) gives

\begin{align*}
m = -2, \quad m = -1, \quad m = -\frac{2}{3},
\end{align*}

(57)

It is interesting to notice that for $m = -2$, the solutions metric turns out to be

\begin{align*}
ds^2 = dt^2 - t^{\frac{2}{3}} dx^2 - t^{\frac{4}{3}} (dy^2 + dz^2),
\end{align*}

(58)

which is same as the well-known Kasners metric \[43\].

5 Final Remarks

Specifically, we study $f(R, G)$ gravity, where the function $f$ consists of the Ricci scalar $R$ and of the Gauss-Bonnet invariant $G$. There are many interesting aspects of Gauss-Bonnet theory which make it interesting for the researchers. Gauss-Bonnet term is a specific combination of curvature invariants that includes Ricci scalar, Ricci and Riemann tensors. In fact, Gauss-Bonnet invariant naturally arises in the process of quantum field theory regularization and renormalization of curved spacetime. In particular, including $G$ and $R$ in a bivariate function provides a double inflationary scenario where the two acceleration phases are led by $G$ and $R$ respectively \[11\]. Moreover, the involvement of Gauss-Bonnet invariant may play an important role in the early time expansion of universe as it is connected with the string theory and the trace anomaly \[12\].

In this paper, the Noether symmetry approach has been considered for an anisotropic cosmological model in modified Gauss-Bonnet gravity. The
Lagrange multiplier approach allows us to deal with this difficulties related to $f(R, \mathcal{G})$ model and reduces the Lagrangian into a canonical form. In particular, we consider an LRS BI model and due to highly non-linear and complicated field equations, we use a physical assumption $B = A^m$. A system of PDEs has been constructed using Noether symmetries. A detailed analysis of the determining equations is presented. Firstly, a model $f(R, \mathcal{G}) = f_0 R^l + f_1 \mathcal{G}^m$ is proposed and the corresponding Noether symmetries are investigated. The results in this regard agree with [25] for special case when $m = 1$. More importantly, we have also recovered the Noether Symmetries for $f(R)$ and $f(\mathcal{G})$ theories of gravity and the results agree with [19, 42] for $f(R)$ gravity.

The last part of the paper deals with the reconstruction of some important cosmological solutions. Exponential and power law solutions are reported for a well-known $f(R, \mathcal{G})$ model, i.e., $f(R, \mathcal{G}) = f_0 R^m \mathcal{G}^{1-n}$ [25]. Especially, the Kasner’s solution is recovered and it is anticipated that the familiar de-Sitter spacetime giving $\Lambda CDM$ cosmology may be reconstructed for $m = 1$ and some suitable value of $n$. In this paper, we have just discussed a few examples where some important Noether symmetries are reported. Many other cases can be explored giving important cosmological solutions. It is worth mentioning that our results agree with [25] for a special case when $m = 1$.

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