Linear stability of the new relativistic theory of modified Newtonian dynamics

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We have recently proposed a simple relativistic theory which reduces to Modified Newtonian Dynamics (MOND) for the weak-field quasistatic situations applied to galaxies, and to cosmological behaviour as in the ΛCDM model, yielding a realistic cosmology in line with observations. A key requirement of any such model is that Minkowski space is stable against linear perturbations. We expand the theory action to 2nd order in perturbations on a Minkowski background and show that it leads to healthy dispersion relations involving propagating massive modes in the vector and the scalar sector. We use Hamiltonian methods to eliminate constraints present, demonstrate that the massive modes have positive Hamiltonian and show that a non-propagating mode with a linear time dependence may have negative Hamiltonian for wavenumbers \( k < k_\ast \), and positive otherwise. The scale \( k_\ast \) is estimated to be around \( \lesssim \) Mpc\(^{-1}\) so that the low momenta instability may only play a role on cosmological scales.

I. INTRODUCTION

The Dark Sector (DS) - dark matter and dark energy - plays a pivotal role in cosmology and astrophysics. As yet, the evidence for the DS comes exclusively via its inferred contribution to the gravitational fields that known matter is observed to experience. Thus, it is possible that the phenomena of dark matter and/or dark energy may arise from a modification to the gravitational interaction.

Typically, non-GR theories of gravity introduce new degrees of freedom into the gravitational sector beyond the metric tensor present in General Relativity \cite{1, 2}. Whilst these degrees of freedom may have an important role to play in explaining aspects of the DS, it is crucial that they do not also introduce instabilities that are incompatible with observation.

Observational constraints suggest that there exist regions of spacetime that can be approximated by highly symmetric solutions (for example geometry in the solar system can be described as a perturbed Minkowski spacetime, whereas the late universe on the largest scales can be described as perturbed de Sitter spacetime) and that these approximations persist for a proper time at least of the order \( \tau_\ast \) (for example lower bounds on the age of the solar system or the period of A-domination in cosmology).

It is vital then that new degrees of freedom do not introduce instabilities that grow on timescales \( \tau_\ast \ll \tau_s \). To probe this question, one can consider the propagation of small perturbations to the aforementioned highly symmetric solutions. Classically, some theories of gravity allow perturbative modes that grow exponentially, where the timescale \( \tau_\ast \) of growth may depend on basic parameters in the theories which can lead to significant constraints on their viability \cite{3, 4}. Another possibility is that around some backgrounds, some perturbative modes can carry negative energy - either via wrong-sign kinetic term (ghosts) or wrong-sign mass term (tachyons). The former especially can signal pathological behaviour in the quantum theory of these perturbations, signalling at the least that the background solution cannot be considered stable\textsuperscript{\dagger}. If experimental constraints suggest that approximations to the background are long-lived then this suggests that the theory of gravity in question is not healthy. Such considerations are therefore vital when considering the viability of a gravitational theory \cite{6, 10}.

We have recently proposed a new relativistic theory which introduces additional fields in the gravitational sector in order to account for the dark matter phenomenon \cite{17}. The theory depends on the metric tensor \( g_{\mu\nu} \) but also introduces a unit time-like vector field \( A_\mu \) and a scalar field \( \phi \) into the gravitational sector. These new degrees of freedom combine with the metric to produce Modified Newtonian Dynamics (MOND) phenomenology \cite{18, 19} in the quasistatic, weak-field limit relevant to galaxies whilst accounting for precision cosmological data \cite{20} comparably well to the cold dark matter paradigm \cite{21}.

Our goal is to study the linear stability of this theory on a Minkowski background and to establish that the theory is free of propagating ghost instabilities. In section \textsuperscript{II} we introduce the theory in detail; in section \textsuperscript{III} we consider small fluctuations of the fields around a Minkowski background, and expand the action to quadratic order. There, we also discuss gauge transformations and separately compute the dispersion relations for tensor, vector, and scalar modes, determining at the same time the conditions on the theory parameters for these relations to be healthy. The case of scalar perturbations requires further treatment and in Section \textsuperscript{IV} we consider their
Hamiltonian formulation. We discuss our findings and their interpretation in a cosmological setting in section \( \mathbf{V} \) and conclude in section \( \mathbf{VI} \).

We use a metric signature \(-+\) and curvature conventions of Wald [38]. We use brackets to denote antisymmetrization with the convention that \([A, B] = \frac{1}{2}(AB - BA)\).

\section{II. The Theory}

The theory depends on a metric \( g_{\mu \nu} \) universally coupled to matter so that the Einstein equivalence principle is obeyed, a scalar field \( \phi \) and a unit time-like vector field \( A^\mu \). The action is

\[
S = \int d^4x \sqrt{-g} \left\{ R - 2\Lambda - \frac{K_B}{2} F_{\mu \nu} F^{\mu \nu} + 2(2 - K_B) J^\mu \nabla_\mu \phi - (2 - K_B) \mathcal{Y} - \mathcal{F}(\mathcal{Y}, \mathcal{Q}) - \lambda(A^\mu A_\mu + 1) \right\} + S_m[g] \tag{1}
\]

where \( g \) is the metric determinant, \( \nabla_\mu \) the covariant derivative compatible with \( g_{\mu \nu} \), \( R \) is the Ricci scalar, \( \Lambda \) is the cosmological constant, \( \mathcal{G} \) is the bare gravitational strength, \( K_B \) is a constant and \( \lambda \) is a Lagrange multiplier imposing the unit time-like constraint on \( A_\mu \). The matter action \( S_m \) is assumed not to depend explicitly on \( \phi \) or \( A^\mu \). The theory has a function \( \mathcal{F}(\mathcal{Y}, \mathcal{Q}) \) which depends on the scalars \( \mathcal{Q} = A^\mu \nabla_\mu \phi \) and \( \mathcal{Y} = (g^{\mu \nu} + A^\mu A^\nu) \nabla_\mu \phi \nabla_\nu \phi \), while \( J^\mu = A^\nu \nabla_\nu A^\mu \) and \( F_{\mu \nu} = 2 \nabla_\mu A_\nu \). The function \( \mathcal{F} \) is subject to conditions so that the cosmology of the theory is compatible with \( \Lambda \)CDM on FRLW spacetimes and a MOND limit emerges in quasistatic situations.

On a flat FLRW background the metric takes the form \( ds^2 = -dt^2 + a^2 \gamma_{ij} dx^i dx^j \) where \( a(t) \) is the scale factor and \( \gamma_{ij} \) is a flat spatial metric. The vector field reduces to \( A^\mu = (1, 0, 0, 0) \) while \( \phi \rightarrow \phi(t) \) leading to \( \mathcal{Q} \rightarrow \tilde{\mathcal{Q}} = \dot{\phi} \) and \( \mathcal{Y} \rightarrow 0 \), so that we may define \( \mathcal{K}(\mathcal{Q}) \equiv -\mathcal{F}(0, \mathcal{Q}) \). We require that \( \mathcal{K}(\mathcal{Q}) \) has a minimum at \( \mathcal{Q}_0 \) (a constant) so that we may expand it as \( \mathcal{K} = \mathcal{K}_2 (\tilde{\mathcal{Q}} - \mathcal{Q}_0)^2 + \ldots \), where the \((\ldots)\) denote higher terms in this Taylor expansion. This condition leads to \( \tilde{\phi} \) contributing energy density scaling as dust \( \sim a^{-3} \) akin to \([39, 40]\); plus small corrections which tend to zero when \( a \rightarrow \infty \).

In principle, \( \mathcal{K} \) could be offsetted from zero at the minimum \( \mathcal{Q}_0 \), i.e. \( \mathcal{K}(\mathcal{Q}_0) = \mathcal{K}_0 \), however, such an offset can always be absorbed into the cosmological constant \( \Lambda \) and thus we choose \( \mathcal{K}_0 = 0 \) by convention, implying the same on the parent function \( \mathcal{F} \).

In the quasistatic weak-field limit we may set the scalar time derivative to be at the minimum \( \mathcal{Q}_0 \), as is expected to be the case in the late universe. This means that we may expand \( \phi = \mathcal{Q}_0 t + \varphi \). Moreover, in this limit \( \mathcal{F} \rightarrow (2 - K_B) \mathcal{J}(\mathcal{Y}) \), with \( \mathcal{J} \) defined appropriately as \( \mathcal{J}(\mathcal{Y}) = \frac{K_B}{2} \mathcal{F}(\mathcal{Y}, \mathcal{Q}_0) \). It turns out that MOND behaviour emerges if \( \mathcal{J} \rightarrow \frac{2K_B}{\lambda^2 + K_B^2} |\mathcal{Y}|^{3/2} \) where \( \lambda_0 \) is Milgrom’s constant and \( \lambda_0 \) is a constant which is related to the Newtonian/GR limit. Specifically, there are two ways that GR can be restored: (i) screening and (ii) tracking. In the former, the scalar is screened at large gradients

\( \nabla \varphi \), where \( \nabla \leftrightarrow \nabla_i \) is the spatial gradient on a flat background \( \gamma_{ij} \), and in the latter, \( \lambda_0 \varphi \) becomes proportional to the Newtonian potential, leading to an effective Newtonian constant

\[
G_N = \frac{1 + \frac{\lambda_s}{\lambda_0}}{1 - \frac{\lambda_s}{\lambda_0}} \tilde{G}.
\] (2)

Screening may be achieved either through terms in \( \mathcal{J} \sim \mathcal{Y}^p \) with \( p > 3/2 \) or through galileon-type terms which must be added to \([1]\). Either way, for our purposes in this article, we may model screening as \( \lambda_s \rightarrow \infty \).

\section{III. Linear Perturbations around Minkowski Space}

\subsection{A. Perturbative setup}

We are interested in spacetime regions which are well approximated by weak gravitational fields modeled as fluctuations on a Minkowski background \( \eta_{\mu \nu} \) and that these regions exist in the late universe where the time derivative of the background field has settled in its minimum \( \mathcal{Q}_0 \), i.e. \( \dot{\phi} \rightarrow 0 \). In addition, the size of these regions is taken to be much smaller than the size of the current cosmological horizon so that we may safely ignore the cosmological constant.

We expand the metric as \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where \( h_{00} = -1 \) and \( \eta_{ij} = \gamma_{ij} \), the vector field \( \vec{A} \) as \( A_{\mu} = (-1 + \frac{1}{2}h_{00}, \tilde{A}_i) \) and the scalar as \( \phi = \mathcal{Q}_0 t + \varphi \). Thus our degrees of freedom are the metric perturbation \( h_{\mu \nu} \), vector field perturbation \( \tilde{A}_i \) (only its 3-dimensional part remains free) and the scalar field perturbation \( \varphi \), all of

\footnote{Strictly speaking, to satisfy the Lagrange constraint we need \( A_0 \) to 2nd order, i.e. \( A_0 = -1 + \frac{1}{2}h_{00} - \frac{1}{4}(h_{00})^2 - \frac{1}{2}|\tilde{A}|^2 - h_{00} \tilde{A}_i \) and similarly for \( A^i \). However, for all the other terms in \([1]\), it is sufficient to expand \( A_0 \) and \( A^i \) to 1st order.}
which are in general functions of both space and time. We raise/lower spatial indices with the spatial metric $\gamma_{ij}$, i.e. $\vec{A}^i = \gamma^{ij} \vec{A}_j$ and set $|\vec{A}|^2 = \vec{A} \cdot \vec{A} = \vec{A}_i \vec{A}^i$ (and use similar notation for other spatial vectors).

### B. Gauge transformations

Our perturbative variables are amenable to gauge transformations generated by a vector field $\xi^\mu$. Generally, for a tensor $\mathbf{A}$, its perturbation transforms as $\delta \mathbf{A} \rightarrow \delta \mathbf{A} + \mathcal{L}_\xi \mathbf{A}$. Usually, on Minkowski only the metric has a non-zero background value ($\eta_{\mu\nu}$), so that other fields besides the metric perturbation are usually gauge-invariant on such a background; this is typical of dark fields, i.e. additional degrees of freedom which contribute to the energy density but do not mix with the metric perturbation through gauge transformations of this kind. In our case, however, both the vector field and the scalar field have non-zero background value: $\vec{A}_\mu = (-1, 0, 0, 0)$ and $\phi = Q_0 t$, hence, their perturbations do transform. Specifically, parametrizing $\xi^\mu$ as $\xi^\mu = (\xi_T, \vec{\xi})$, we have the usual metric gauge transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}$$

where $\nabla_{\mu}$ is the covariant derivative associated with the Minkowski metric $\eta_{\mu\nu}$. In $3 + 1$ form the above transformations are explicitly given as

$$h_{00} \rightarrow h_{00} - 2\xi_T \tag{4}$$

$$h_{0i} \rightarrow h_{0i} + \xi_i - \nabla_i \xi_T \tag{5}$$

$$h_{ij} \rightarrow h_{ij} + \nabla_i \xi_j + \nabla_j \xi_i. \tag{6}$$

The perturbations $\vec{A}$ and $\varphi$ transform as

$$\vec{A} \rightarrow \vec{A} - \nabla \xi_T \tag{7}$$

$$\varphi \rightarrow \varphi + Q_0 \xi_T. \tag{8}$$

Notice how the vector field transformation has the same form as gauge transformations in electromagnetism, however, the generator here is also a diffeomorphism.

With these gauge transformations at hand we can create the following gauge-invariant variables:

$$\{\vec{\nabla} \varphi + Q_0 \vec{A}, \vec{A} - \frac{1}{2} \vec{\nabla} h_{00}, \varphi + \frac{1}{2} Q_0 h_{00}\} \tag{9}$$

Hence, the fields $\varphi$ and $\vec{A}$ non-trivially mix with the metric perturbation through $\xi_T$.

### C. The 2nd order action

Our aim is to then expand the action (1) to 2nd order in these fields. With these considerations, and having in mind the discussion in the previous section, we then expand the function $\mathcal{F}$ as

$$\mathcal{F} = (2 - K_B) \lambda_s \mathcal{Y} - 2K_2 (Q - Q_0)^2 + \ldots \tag{10}$$

since $\mathcal{F}(0, Q_0) = 0$ by convention and $\frac{\partial \mathcal{F}}{\partial Q}|_{(0, Q_0)} = 0$ at the minimum. The terms denoted by $(\ldots)$ are higher order terms which do not contribute to the 2nd order action. We particularly note that one of these is the MOND-type term $\sim |J|^2/2$ as discussed in the previous section. This term does not contribute to the 2nd order action but we return it in the discussion section. Expanding (1) to 2nd order leads to

$$S = \int d^4 x \left\{ -\frac{1}{2} \nabla_{\mu} h \nabla^{\mu} h + \frac{1}{4} \nabla_{\rho} h \nabla^{\rho} h + \frac{1}{2} \nabla_{\mu} h_{\rho\sigma} \nabla_{\nu} h^{\rho\sigma} - \frac{1}{4} \nabla^{\rho} h_{\mu\nu} \nabla_{\rho} h^{\mu\nu} + K_B |\vec{A}| - \frac{1}{2} |\vec{\nabla} h_{00}|^2 - 2K_B \vec{\nabla} \cdot (\vec{A}_j (\vec{A}^j)) \right\}$$

where for convenience we have rescaled the action $S \rightarrow 16\pi G S$. We have also omitted the determinant $\sqrt{\gamma}$ in the measure since we are dealing with integrals on Minkowski, but can be understood to be present in all integrations.

$$S = \int d^4 x \left\{ -\frac{1}{2} \nabla_{\mu} h \nabla^{\mu} h + \frac{1}{4} \nabla_{\rho} h \nabla^{\rho} h + \frac{1}{2} \nabla_{\mu} h_{\rho\sigma} \nabla_{\nu} h^{\rho\sigma} - \frac{1}{4} \nabla^{\rho} h_{\mu\nu} \nabla_{\rho} h^{\mu\nu} + K_B |\vec{A}| - \frac{1}{2} |\vec{\nabla} h_{00}|^2 - 2K_B \vec{\nabla} \cdot (\vec{A}_j (\vec{A}^j)) \right\} \tag{11}$$

Expanding (11) to 2nd order leads to

$$h_{00} = -2\Psi \tag{12}$$

$$h_{0i} = -\nabla_i \zeta - W_i \tag{13}$$

$$h_{ij} = -2\Psi \gamma_{ij} + D_{ij} \nu + 2\nabla_{(i} V_{j)} + H_{ij} \tag{14}$$

$$\vec{A} = \vec{\nabla} \alpha + \vec{\beta} \tag{15}$$

where $D_{ij} = \nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2$ is a traceless derivative operator. The modes $\vec{\nabla}, \vec{V}$ and $\vec{\beta}$ are pure vector modes, that is, they are transverse: $\vec{\nabla} \cdot \vec{W} = \vec{\nabla} \cdot \vec{V} = \vec{\nabla} \cdot \vec{\beta} = 0$, while the mode $H_{ij}$ is a pure tensor mode, that is, trans-
verse and traceless: \[ \vec{\nabla}_i H^i_j = H^i_i = 0. \]

The matter stress-energy tensor \( T_{\mu \nu} \) is likewise decomposed as

\[
\begin{align*}
T_{00} &= \rho \\
T_{0i} &= \vec{\nabla}_i \theta + p_i \\
T_{ij} &= P \delta_{ij} + D_{ij} S^{(S)} + 2 \vec{\nabla}_i \Sigma_{ij}^{(V)} + \Sigma_{ij}^{(T)}
\end{align*}
\]

where the scalar modes are the matter density \( \rho \), momentum divergence \( \theta \), pressure \( P \) and scalar shear \( \Sigma^{(S)} \), the vector modes are the matter vortical momentum density \( p_i \) and vector shear \( \Sigma^{(V)}_i \), such that \( \vec{\nabla} \cdot \vec{\beta} = \vec{\nabla} \cdot \Sigma^{(V)}_i = 0 \), and the tensor mode is the tensor shear \( \Sigma^{(T)}_i \), such that \( \vec{\nabla}_i \Sigma^{(T)} j = \Sigma^{(T)}_j = 0 \).

With this decomposition, the 2nd order action splits into three distinct parts: one for the scalar modes \( S^{(S)} \), one for the vector modes \( S^{(V)} \) and one for the tensor modes \( S^{(T)} \). We consider each of these three one by one.

### D. Tensor modes

The perturbations to fields \( A_\mu \) and \( \phi \) do not contribute any tensor mode components and so the tensor mode action takes the form:

\[
S^{(T)} = \int d^4x \left\{ \dot{H}^{ij} H_{ij} - \vec{\nabla} k H^{ij} \vec{\nabla} k H_{ij} + 32 \pi \mathcal{G} \Sigma^{(T)}_{ij} H^{ij} \right\}
\]

This corresponds to the action for tensor modes present in General Relativity, a result consistent with the earlier, more general calculation that tensor modes in the superclass of theories of which \( \mathcal{S} \) is a special subset, propagate at the speed of light \( \mathcal{L} \).

### E. Vector modes

We now consider vector modes, which are described by the action

\[
S^{(V)} = \int d^4x \left\{ -\frac{1}{2} \left( \dot{V}_i + W_i \right) \vec{\nabla}^2 \left( \dot{V}^i + W^i \right) \\
+ K_B \left[ |\vec{\beta}|^2 - \vec{\nabla}_i \vec{\beta} \cdot \vec{\nabla}^i \vec{\beta} - M^2 |\vec{\beta}|^2 \right] \\
+ 16\pi \mathcal{G} \left( \vec{\beta} \cdot \vec{W} - \Sigma^{(V)} \vec{\nabla}^2 \phi \right) \right\}
\]

where

\[
M^2 = \frac{(2 - K_B)(1 + \lambda_s)Q_0^2}{K_B}
\]

The field \( \vec{\beta} \) decouples from the metric fields \( \vec{V} \) and \( \vec{\dot{W}} \) and describes two massive degrees of freedom with mass \( \mathcal{M} \). Clearly then we must require \( K_B > 0 \) to avoid ghosts and gradient instabilities. The mass term \( \mathcal{M} \) is also nontachyonic if both \( 0 < K_B < 2 \) and \( \lambda_s > -1 \). Hence, stability considerations for the vector modes imply the following constraints on the theory’s parameter space:

\[
0 < K_B < 2, \quad \lambda_s > -1
\]

Notice that to this order, the vector modes \( \vec{\beta} \) do not couple to matter and thus they are not expected to be generated by sources to leading order.

### F. Scalar modes

We now consider scalar perturbations. Considering only scalar modes in \( \mathcal{S} \) and after some integrations by parts we find the action \( S^{(S)} \):

\[
S^{(S)} = \int d^4x \left\{ 6 \left( \frac{1}{6} \vec{\nabla}^2 \dot{\phi} - \dot{\phi} \right) - \frac{1}{6} \left( \vec{\nabla}^2 \dot{\phi} + \dot{\phi} \right) \vec{\nabla}^2 \phi + 4 \left( \frac{1}{6} \vec{\nabla}^2 \dot{\phi} + \dot{\phi} \right) \vec{\nabla}^2 \phi + 2|\vec{\nabla} \phi|^2 - \frac{2}{3} \Phi \vec{\nabla}^4 \nu + 4 \left( \vec{\nabla}^2 \Phi + \frac{1}{6} \vec{\nabla}^4 \nu \right) \Psi \\
+ \frac{1}{18} \left( \vec{\nabla} \left( \vec{\nabla}^2 \nu \right) \right)^2 + 2K_2 \vec{\nabla}^2 \nu - 4K_2 \vec{\nabla}^2 \Psi + 2K_2 \vec{\nabla}^2 \Psi^2 + K_B \vec{\nabla} (\dot{\phi} + \Psi)^2 + 2 \left( 2 - K_B \right) \vec{\nabla} (\dot{\phi} + \Psi) \cdot \vec{\nabla} \chi \\
- (2 - K_B) \left( 1 + \lambda_s \right) |\vec{\nabla} \chi|^2 - 16\pi \mathcal{G} \rho \Psi - 16\pi \mathcal{G} \vec{\nabla}^2 \phi \cdot \vec{\nabla} \chi - 48\pi \mathcal{G} P \Phi + \frac{16\pi \mathcal{G}}{3} \vec{\nabla}^4 \nu \right\}
\]

where we have defined the gauge-invariant variable \( \chi \) as

\[
\chi \equiv \varphi + Q_0 \alpha
\]

that will be shown to play a prominent role in what follows.

Setting scalar matter sources to vanish and moving to Fourier space we have
where fields in (25) are understood to be the Fourier modes of those in (23) and (c.c) means complex conjugate.

We now find the normal modes. It is sufficient to work in the Newtonian gauge by setting \( \nu = \zeta = 0 \). We set the time dependence of all perturbations to \( e^{i\omega t} \) and rewrite (25) as \( \int dt \int \frac{d^3k}{(2\pi)^3} Z^{i} U^{j} + (h.c.) \), where \( Z = \{ \Psi, \Phi, \alpha, \varphi \} \) and \( U \) is a \( 4 \times 4 \) matrix of coefficients which depend on \( \omega, k \) and the other theory parameters. The determinant of \( U \) is found to be

\[
\det U = 4k^6\omega^2 \left\{ (2 - K_B) \left[ (2 + K_B\lambda_5)k^2 + 2K_2Q_0^2(1 + \lambda_5) \right] - 2K_2K_B\omega^2 \right\}
\]

so setting \( \det U = 0 \) gives the two dispersion relations

\[
\begin{align*}
\omega^2 &= 0 \\
\omega^2 &= c_s^2k^2 + \mathcal{M}^2
\end{align*}
\]

where the scalar speed of sound is

\[
c_s^2 = \frac{(2 - K_B)}{K_2K_B}(1 + \frac{1}{2}K_B\lambda_5)
\]

We notice that the 1st mode does not lead to a propagating wave but rather to a mode evolving as \( \sim A_0 + B_0t \) where \( A_0 \) and \( B_0 \) are \( k \)-dependent constants. Interestingly also, the 2nd mode is massive with the same mass as the vector mode \( \beta \).

Positivity of \( c_s^2 \) implies further stability conditions in addition to the ones found above for the vector modes. Specifically, since from (22) we have \( \lambda_5 > -1 \), then \( 1 + \)

\[
\mathcal{H} = -\frac{1}{6}\{P_\Phi|^2 + \frac{6}{k^4}|P_\nu|^2 + \frac{1}{2K_2}|P_\chi|^2 + \frac{1}{k^2K_B}|P_\alpha + Q_0P_\chi|^2 - 2k^2|\Phi - \frac{1}{6}k^2\nu|^2 + \frac{2}{K_B}k^2(2 + K_B\lambda_5)|\chi|^2
\]

\[
- \frac{2 - K_B}{K_B} \left[ (P_\alpha + Q_0P_\chi)\chi^* + (P_\alpha^* + Q_0P_\chi^*)\chi \right] + C_\Phi\Psi^* + C_\Psi^*\Phi + C_\chi^*\zeta + C_\zeta^*\chi
\]

Since the variables \( \Psi \) and \( \zeta \) are not dynamical, their function is to act as Lagrange multipliers imposing the constraints

\[
\begin{align*}
C_\Psi &\equiv 2k^2\Phi - \frac{k^4}{3}\nu - P_\alpha \approx 0 \\
C_\zeta &\equiv -2P_\nu - \frac{k^2}{3}P_\Phi \approx 0
\end{align*}
\]
define the Poisson brackets on phase space as
\[ \{ f, g \} = \int d^3k \left[ \sum_I \left( \frac{\delta f}{\delta \phi_I^*} \frac{\delta g}{\delta \phi_I} - \frac{\delta g}{\delta \phi_I^*} \frac{\delta f}{\delta \phi_I} \right) + c.c. \right] \]
where \( I \) runs over \( \{ \Phi, \nu, \chi, \alpha \} \). The time evolution of a variable \( f \) is
\[ \dot{f} = \{ f, H \}, \]
so we have
\[ \dot{C}_\Psi = C_\zeta, \]
\[ \dot{C}_\zeta = 0, \]
therefore, the constraints are preserved by time evolution on-shell. Therefore as one might expect, the stability of the primary constraints in the absence of gauge fixing does not create new constraints. Having ensured the stability of constraints in the Hamiltonian, we can now simplify the system by employing gauge fixing.

In the Hamiltonian formulation, primary first class constraints generate gauge transformations. The infinitesimal change of a phase space quantity \( f \) under this gauge transformation generated by the constraint \( \epsilon_I C_I \) is given by:
\[ \Delta f = \{ f, \epsilon_I C_I^* \}. \]
Consider the following gauge transformations generated by the constraints \( C_\zeta \) and \( C_\Psi \):
\[ \Delta \nu = \{ \nu, \epsilon_\zeta C_\zeta^* \} = -2\epsilon_\zeta \]
\[ \Delta P_\nu = \{ P_\nu, \epsilon_\zeta C_\zeta^* \} = \frac{1}{3} k^4 \epsilon_\nu \]
Thus, we may set \( \nu \) and \( P_\nu \) to zero by a gauge transformation by choosing \( \epsilon_\zeta = \frac{1}{2} \nu \) and \( \epsilon_\Psi = -\frac{1}{3} P_\nu \). We then check what constraints are placed on the Lagrange multipliers \( \zeta, \Psi, \) by this gauge fixing. We invoke two new gauge fixing constraints:
\[ G_\nu \equiv \nu \approx 0 \]
\[ G_{P_\nu} \equiv P_\nu \approx 0 \]
and find
\[ \{ G_\nu, H \} = \frac{6}{k^4} G_{P_\nu} - 2\zeta \]
\[ \{ G_{P_\nu}, H \} = -\frac{1}{3} k^4 (\Phi - \Psi) + \frac{1}{18} k^6 G_\nu \]
Therefore the following gauge restrictions are placed on the Lagrange multipliers: \( \zeta = 0 \) and \( \Psi = \Phi \). We recognize these conditions, respectively, as a restriction to the conformal Newtonian gauge and the content of the Einstein equation here dictating equality between metric potentials in this gauge. We may adopt these conditions alongside the constraints \( G_\nu, G_{P_\nu} \) in the Hamiltonian and the primary constraints, yielding in addition
\[ P_\Phi \approx 0 \]
\[ \Phi \approx \frac{1}{2k^2} P_\alpha \]
so that the deconstrained Hamiltonian density is
\[ \mathcal{H}^{(\text{Dec})} = \frac{1}{2K_2} |P_\alpha|^2 + \frac{1}{k^2 K_B} |P_\alpha + Q_0 P_\chi|^2 - \frac{1}{2k^2} |P_\alpha|^2 
- 2 - \frac{K_B}{K_B} \left[ (P_\alpha + Q_0 P_\chi) P_\chi^* + (P_\chi^* + Q_0 P_\chi) P_\alpha \right] 
+ 2 - \frac{K_B}{K_B} k^2 (2 + K_B \lambda_0) |X|^2. \]
The Hamiltonian density \( \mathcal{H}^{(\text{Dec})} \) is free of constraints but its form remains rather complicated. We can make an additional simplification by making a canonical transformation to canonical pairs \( (P_X, X), (P_Y, Y) \) defined via
\[ \chi = \sqrt{K_B k^2 + (2 - K_B) \mu^2 Q_0} \mu \frac{Q_0}{(2 + K_B \lambda_0) k^2 + (2 - K_B)(1 + \lambda_0) \mu^2} \]
\[ P_\chi = \sqrt{K_B k^2 + (2 - K_B) \mu^2 Q_0} \mu \left[ \frac{(2 - K_B) Q_0}{K_B} X + P_X \right] - \frac{1}{Q_0} \frac{(2 - K_B)(1 + \lambda_0) \mu^2}{(2 + K_B \lambda_0) k^2 + (2 - K_B)(1 + \lambda_0) \mu^2} P_Y \]
\[ \alpha = \sqrt{K_B k^2 + (2 - K_B) \mu^2 Q_0} \mu \left[ \frac{Q_0}{K_B k^2} X + \frac{P_X}{(2 + K_B \lambda_0) k^2 + (2 - K_B)(1 + \lambda_0) \mu^2} \right] \]
\[ P_\alpha = P_Y \]
where we have also defined
\[ \mu^2 = \frac{2K_2Q_0^2}{2-K_B} \]  
(56)

This gives a Hamiltonian density
\[ \tilde{H} = |P_X|^2 + (c_s^2k^2 + \mathcal{M}^2)|X|^2 \]
+ \[ \frac{(2-K_B)^2\lambda_s - 1 - k^2}{4K_Bk_2c_s^2k^2 + \mathcal{M}^2} |P_Y|^2 \]  
(57)

where
\[ k_s^2 = \frac{1+\lambda_s}{\lambda_s} \mu^2 \]  
(58)

We see then that the system can be cast in terms of two decoupled fields, \( X \) and \( Y \), with canonical momenta \( P_X \) and \( P_Y \) respectively, and each field corresponds to one of the normal modes in \( \mathbb{C}^2 \). Specifically, the field \( X \) propagates the massive modes in \( \mathbb{C}^2 \) while the field \( Y \) corresponds to the non-propagating the \( \omega = 0 \) modes.

V. DISCUSSION

One notices that the sign of the \( |P_Y|^2 \) term is not positive definite but rather depends on the relevant wavenumber \( k \) and parameters \( \lambda_s \) and \( k_s \). Clearly as \( k \to \infty \), \( |P_Y|^2 \) comes with a positive sign provided \( \lambda_s > 0 \), and negative otherwise, which provides an additional condition to the one found for vector modes in \( \mathbb{C}^2 \). Meanwhile, when \( k > k_s \) defined by \( \frac{\mathcal{M}}{2} \), the Hamiltonian density is positive while when \( k < k_s \), negative Hamiltonian density can occur if the \( |P_Y|^2 \) term in \( \mathbb{C}^2 \) becomes significant. The solutions for \( \omega = 0 \) correspond to \( Y = A_0(\tilde{k})t + B_0(\tilde{k}) \) while \( P_Y = A_0(\tilde{k}) \). Thus the mode which could cause negative Hamiltonian densities is the one evolving linearly with \( t \). Such instabilities are likely akin to Jeans-type instabilities and do not cause quantum vacuum instability at low momenta \( \frac{\mathcal{M}}{2} \).

Taking both scalar and vector mode conditions on the theory parameters we require that
\[ 0 < K_B < 2 \]  
(59)
\[ K_2 > 0 \]  
(60)
\[ \lambda_s > 0 \]  
(61)

These conditions also imply that \( G_N / G \) always.

As discussed in \( \frac{\mathcal{M}}{2} \), for a spherically symmetric static source of mass \( M \), the transition between the MOND and an oscillatory \( \mu \)-dominated regime occurs at \( r_\mu \sim (r_M \mu^{-2})^{1/3} \) where \( r_M \sim \sqrt{\frac{2GM}{a_0}} \) is the MOND scale which signifies the transition between the Newtonian and MOND regimes on even smaller distances. Thus, on observational grounds \( \mu^{-1} \) must be larger than \( \sim \text{Mpc} \), otherwise, the MOND regime would not occur at the scales of galaxies at distances \( \sim \text{kpc} \) (for the Milky Way \( r_M \sim 8\text{kpc} \)). A system with a MOND scale of \( \sim \text{Mpc} \) occurs if its mass is \( \sim 10^{15}M_\odot \) which is much larger than typical masses of bound structures. Thus, for \( \lambda_s \geq 1 \), the scale \( k_s \) is always hidden inside the MOND regime (i.e. \( k_s < r_M^{-1} \)) so that the negative Hamiltonian does not occur in the GR limit for all systems of interest.

At smaller wavenumbers \( r_M^{-1} \), the theory enters the MOND regime (in which case \( \lambda_s = 0 \)) which would signify that the \( Y \)-mode always has a negative Hamiltonian. However, then there exists a higher order term \( \sim |Y|^3/\mathcal{A}_0 = |\nabla \phi|^3/a_0 \) that is not part of the analysis above, and which may stabilize the system, while on even larger scales (smaller wavenumbers) the Minkowski approximation breaks down as the relevant background to expand upon is Friedmannian (or even more specifically de Sitter). We speculate that the \( t \)-instability found here on Minkowski is nothing but the cosmological dark matter Jeans-type of instability discussed in \( \frac{\mathcal{M}}{2} \) and shown to lead to excellent fits to the cosmological power spectra as in ΛCDM.

VI. CONCLUSION

We have expanded the action of a newly proposed theory \( \frac{\mathcal{M}}{2} \) which has a MOND limit relevant for galactic systems and ΛCDM limit relevant for cosmology, to 2nd order on Minkowski spacetime. We have identified the normal modes of the fluctuations and shown that the propagating vector modes are massive with mass given by \( \frac{\mathcal{M}}{2} \) and speed of sound \( c_s \) equal to the speed of light while the propagating scalar modes also have the same mass, \( \frac{\mathcal{M}}{2} \), and speed of sound given by \( \frac{\mathcal{M}}{2} \). We identified in addition, non-propagating scalar modes with dispersion relation \( \omega = 0 \). We computed the deconstrained Hamiltonian of the scalar modes of the theory on this spacetime and via a canonical transformation have shown that it corresponds to a massive particle corresponding to the massive normal mode, and a massless particle corresponding to the mode \( \omega = 0 \). The latter may lead to negative Hamiltonian densities for wavenumbers \( k < k_s \) given by \( \frac{\mathcal{M}}{2} \). However, as was discussed above \( k_s \leq \text{Mpc}^{-1} \) so that such instabilities do not occur in the GR limit of the theory for all systems of interest. Performing the same analysis on de Sitter space is left for future investigation.

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\[ 4 \] This is not to say that they propagate with the speed of light. We have defined \( c_s^2 \) as the coefficient of \( k^2 \) in the dispersion relation. Only in the limit \( k \to \infty \) does the speed of sound equal to the propagation speed.
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