ALGEBRAIC ANSWER SET PROGRAMMING

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Abstract. Non-monotonic reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via answer set programming. In this paper, we introduce the sequential composition of answer set programs. We show that the notion of composition gives rise to a family of finite program algebras, baptized ASP algebras in this paper. Interestingly, we can derive algebraic formulas for the syntactic representation of the well-known Faber-Leone-Pfeifer- and Gelfond-Lifschitz reducts. On the semantic side, we show that the immediate consequence operator of a program can be represented via composition, which allows us to compute the least model semantics of Horn programs without any explicit reference to operators. As a result, we can characterize answer sets algebraically, which bridges the conceptual gap between the syntax and semantics of an answer set program in a mathematically satisfactory way, and which provides an algebraic characterization of strong and uniform equivalence. Moreover, it gives rise to an algebraic meta-calculus for answer set programs. In a broader sense, this paper is a further step towards an algebra of logic programs first envisioned by Richard A. O’Keefe in 1985 and in the future we plan to lift the methods of this paper to wider classes of programs, most importantly to higher-order and disjunctive programs and extensions thereof.

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1. Introduction

Non-monotonic reasoning is an essential part of human intelligence prominently formalized in artificial intelligence research via answer set programming (cf. Brewka, Eiter, & Truszczynski, 2011; Lifschitz, 2019; Baral, 2003; Eiter, Ianni, & Krennwallner, 2009). Answer set programs are rule-based systems with the rules and facts being written in a sublanguage of predicate or propositional logic extended by a unary non-monotonic operator “not” denoting negation as failure (or default negation) (Clark, 1978). While each monotone (i.e., negation-free) answer set program has a unique least Herbrand model (with the least model semantics (van Emden & Kowalski, 1976) being the accepted semantics for this class of programs), for general answer set programs a large number of different purely declarative semantics exist. Many of it have been introduced some 30 years ago, among them the answer set semantics (Gelfond & Lifschitz, 1991) and the well-founded semantics (van Gelder, Ross, & Schlipf, 1991). The well-founded semantics, because of its nice computational properties (computing the unique well-founded model is tractable), plays an important role in database program. However, with the emergence of efficient solvers such as DLV (Leone, Pfeifer, Faber, Eiter, Gottlob, Perri, & Scarcello, 2006), Smodels (Simons, Niemelä, & Soininen, 2002), Cmodels (Giunchiglia, Lierler, & Maratea, 2006), and Clasp (Gebser, Kaufmann, & Schaub, 2012), programming under answer set semantics led to a predominant declarative problem solving paradigm, called answer set programming (or ASP) (Marek & Truszczynski, 1999; Lifschitz, 2002). Answer set programming has a wide range of applications and has been successfully applied to various AI-related subfields such as planning and diagnosis (for a survey see Brewka et al. (2011), Eiter et al. (2009), Baral (2003)). Driven by this practical needs, a large number of extensions of classical answer set programs have been proposed, e.g. aggregates (cf. Faber, Leone, & Pfeifer, 2004; Faber, Pfeifer, & Leone, 2011; Pelov, 2004), choice rules (Niemelä, Simons, & Soininen, 1999), dl-atoms (Eiter, Ianni, Lukasiewicz, Schindlauer, & Tompits, 2008), and general external atoms (Eiter, Ianni, Schindlauer, & Tompits, 2005). For excellent introductions to the field of answer set programming we refer the reader to Brewka et al. (2011), Baral (2003), Eiter et al. (2009).

Describing complex objects as the composition of elementary ones is a common strategy in computer science and science in general. Antić (2021a) introduced the sequential composition of Horn logic programs for syntactic program composition and decomposition in the context of logic-based analogical reasoning and learning. Antić (2021b) studies the propositional case in detail. In this paper, we lift Antić’s (2021b) results from Horn to (propositional) answer set programs containing negation as failure. This task turns out to be non-trivial due to the intricate algebraic properties of composing negation as failure occurring in rule bodies. The rule-like structure of answer set programs naturally induces the compositional structure of a unital magma on the space of all answer set programs as we will demonstrate in this paper. This extends Antić’s (2021b) results from the Horn to the non-monotonic case. Specifically, we show that the notion of composition gives rise to a family of finite magmas and algebras of answer set programs (Theorem 7), which we will call ASP magma and ASP algebra in this paper. We also show that the restricted class of proper Krom-Horn programs, which only contain rules with exactly one body atom, yields an idempotent semiring (Theorem 13). On the semantic side, we show that the van Emden-Kowalski immediate consequence operator of a program can be represented via composition (Theorem 31), which allows us to compute the answer set semantics of programs without any explicit reference to operators (Theorem 37). This bridges the conceptual gap between the syntax and semantics of an answer set program.
in a mathematically satisfactory way, and it provides an algebraic characterization of strong (Corollary 38) and uniform equivalence (Corollary 39).

From an artificial intelligence perspective, we obtain a novel and useful algebraic operation for the composition and decomposition of answer set programs, which in combination with Antić’s (2022) abstract algebraic framework of analogical proportions can be applied for learning answer set programs via logic program proportions of the form $P : Q :: R : S$ in the vein of Antić (2021a), which remains a promising line of future work. From a logical point of view, we obtain a meta-calculus for reasoning about answer set programs. From an algebraic point of view, this paper establishes a bridge between answer set programming and algebra, which enables us to transfer algebraic concepts from the literature to the setting of answer set programming and extensions thereof. In a broader sense, this paper is a further step towards an algebra of logic programs first envisioned by O’Keefe (1985) and in the future we plan to adapt and generalize the methods of this paper to wider classes of programs, most importantly for higher-order logic programs (Chen, Kiřík, & Warren, 1993; Miller & Nadathur, 2012; Eiter et al., 2005) and disjunctive answer set programs (Eiter, Gottlob, & Mannila, 1997) and extensions thereof (cf. Lifschitz, 2019; Brewka et al., 2011; Eiter et al., 2009; Baral, 2003).

2. Preliminaries

This section recalls the syntax and semantics of answer set programming, and some algebraic structures occurring in the rest of the paper.

2.1. Algebraic Structures. We define the composition $f \circ g$ of two functions $f$ and $g$ by $(f \circ g)(x) := f(g(x))$. Given two sets $A$ and $B$, we will write $A \subseteq B$ in case $A$ is a subset of $B$ with $k$ elements, for some non-negative integer $k$. We denote the identity function on a set $A$ by $I_A$.

A partially ordered set (or poset) is a set $L$ together with a reflexive, transitive, and antisymmetric binary relation $\leq$ on $L$. A prefixed point of an operator $f$ on a poset $L$ is any element $x \in L$ such that $f(x) \leq x$; moreover, we call any $x \in L$ a fixed point of $f$ if $f(x) = x$.

A magma is a set $M$ together with a binary operation $\cdot$ on $M$. We call $(M, \cdot, 1)$ a unital magma if it contains a unit element $1$ such that $1x = x1 = x$ holds for all $x \in M$. A semigroup is a magma $(S, \cdot)$ in which $\cdot$ is associative. A monoid is a semigroup containing a unit element $1$ such that $1x = x1 = x$ holds for all $x$. A group is a monoid which contains an inverse $x^{-1}$ for every $x$ such that $xx^{-1} = x^{-1}x = 1$. A left (resp., right) zero is an element $0$ such that $0x = 0$ (resp., $x0 = 0$) holds for all $x \in S$. An ordered semigroup is a semigroup $S$ together with a partial order $\leq$ that is compatible with the semigroup operation, meaning that $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $x, y, z \in S$. An ordered monoid is defined in the obvious way. A non-empty subset $I$ of $S$ is called a left (resp., right) ideal if $SI \subseteq I$ (resp., $IS \subseteq I$), and a two-sided ideal if it is both a left and right ideal.

A seminearring is a set $S$ together with two binary operations $+$ and $\cdot$ on $S$, and a constant $0 \in S$, such that $(S, +, 0)$ is a monoid and $(S, \cdot)$ is a semigroup satisfying the following laws:

1. $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in S$ (right-distributivity); and
2. $0 \cdot x = 0$ for all $x \in S$.

The seminearring $S$ is called idempotent if $x + x = x$ holds for all $x \in S$. A semiring is a seminearring $(S, +, \cdot, 0, 1)$ such that $(S, \cdot)$ is a monoid, $+$ is commutative, and additionally to the laws of a seminearring the following laws are satisfied:
2.2. **Answer Set Programs.** We shall recall the syntax and semantics of answer set programming by mainly following the lines of Baral (2003).

2.2.1. **Syntax.** In the rest of the paper, \( A \) denotes a finite alphabet of propositional atoms not containing the special symbols \( t \) (true) and \( f \) (false). A literal is either an atom \( a \) or a negated atom \( \neg a \), where \( \text{"not"} \) denotes \textit{negation as failure} (Clark, 1978).

An \textit{(answer set) program} over \( A \) is a finite set of rules of the form
\[
(1) \quad a_0 \leftarrow a_1, \ldots, a_\ell, \neg a_{\ell+1}, \ldots, \neg a_k, \quad k \geq \ell \geq 0,
\]
where \( a_0, \ldots, a_k \in A \) are atoms, and we denote the set of all answer set programs over \( A \) by \( \mathcal{P}_A \) or simply by \( \mathcal{P} \) in case \( A \) is understood. It will be convenient to define, for a rule \( r \) of the form (1), \( \text{head}(r) := \{a_0\} \) and \( \text{body}(r) := \{a_1, \ldots, a_\ell, \neg a_{\ell+1}, \ldots, \neg a_k\} \), extended to sequences of programs via
\[
\text{head}(P_1, \ldots, P_n) := \bigcup_{r \in P_1 \cup \ldots \cup P_n} \text{head}(r)
\]
and
\[
\text{body}(P_1, \ldots, P_n) := \bigcup_{r \in P_1 \cup \ldots \cup P_n} \text{body}(r), \quad n \geq 1.
\]
In this case, the \textit{size} of \( r \) is \( k \) denoted by \( \text{sz}(r) \). A rule \( r \) of the form (1) is \textit{positive} or \textit{Horn} if \( \ell = k \), and \textit{negative} if \( \ell = 0 \). A program is \textit{positive} or \textit{Horn} (resp., \textit{negative}) if it contains only positive (resp., negative) rules. Define the positive and negative part of \( r \) by
\[
\text{pos}(r) := a_0 \leftarrow a_1, \ldots, a_\ell \quad \text{and} \quad \text{neg}(r) := a_0 \leftarrow \neg a_{\ell+1}, \ldots, \neg a_k,
\]
extended to programs rule-wise via \( \text{pos}(P) := \{\text{pos}(r) \mid r \in P\} \) and \( \text{neg}(P) := \{\text{neg}(r) \mid r \in P\} \).

Moreover, define the \textit{hornification} of \( r \) by
\[
\text{horn}(r) := a_0 \leftarrow a_1, \ldots, a_k,
\]
extended to programs rule-wise via \( \text{horn}(P) := \{\text{horn}(r) \mid r \in P\} \). A \textit{fact} is a rule with empty body and a \textit{proper} rule is a rule which is not a fact. The facts and proper rules of a program \( P \) are denoted by \( \text{facts}(P) \) and \( \text{proper}(P) \), respectively. A Horn rule \( r \) is called \textit{Krom} if it has at most one body atom. A Horn program is \textit{Krom} if it contains only Krom rules. We call a program \textit{minimalistic} if it contains at most one rule for each rule head.

An \textit{\( \vee \)-program} over \( A \) is a finite set of \( \vee \)-\textit{rules} of the form
\[
a \leftarrow B_1 \vee \ldots \vee B_m, \quad m \geq 0,
\]
where \( a \in A \) is an atom and \( B_1, \ldots, B_m \) are sets of literals. \( \vee \)-Programs containing disjunction in rule bodies will only occur as intermediate steps in the computation of composition below.

Define the \textit{dual} of a Horn program \( H \) by
\[
H^d := \text{facts}(H) \cup \{a \leftarrow \text{head}(r) \mid r \in \text{proper}(P) : a \in \text{body}(r)\}.
\]
Roughly, we obtain the dual of a Horn program by reversing all the arrows of its proper rules.

\[^1\text{Krom-Horn rules where first introduced and studied by Krom (1967).}\]
2.2.2. Semantics. An interpretation is any subset of $A$. The entailment relation with respect to an interpretation $I$ and a program $P$ is defined inductively as follows: (i) for an atom $a \in A$, $I \models a$ if $a \in I$; (ii) for a rule $r$ of the form $[\mathbf{I}]$, $I \models r$ if $I \models \text{head}(r)$ or $body(pos(r)) \not\subseteq I$ or $I \cap \text{horn}(body(neg(r))) \neq \emptyset$; finally, (iii) $I \models P$ if $I \models r$ holds for each rule $r \in P$. In case $I \models P$, we call $I$ a model of $P$.

It is well-known that the space of all models of a Horn program $H$ forms a complete lattice containing a least model denoted by $LM(H)$.

Define the Gelfond-Lifschitz reduct of $P$ with respect to $I$ by the Horn program

$$gP^I := \{ \text{pos}(r) \mid r \in P : I \models \text{body}(\text{neg}(r)) \}.$$ 

Moreover, define the left and right reduct of $P$, with respect to some interpretation $I$, by

$$I^P := \{ r \in P \mid I \models \text{head}(r) \} \quad \text{and} \quad P^I := \{ r \in P \mid I \models \text{body}(r) \}.$$ 

Of course, our notion of right reduct is identical to the Faber-Leone-Pfeifer reduct (Faber et al., 2011) well-known in answer set programming.

The following definition is due to Gelfond and Lifschitz (1991).

**Definition 1.** An interpretation $I$ is an answer set of $P$ if $I$ is the least model of $gP^I$.

Define the van Emden-Kowalski operator of $P$, given an interpretation $I$, by

$$T_P(I) := \{ \text{head}(r) \mid r \in P : I \models \text{body}(r) \}.$$ 

It is well-known that the least model semantics of a Horn program coincides with the least fixed point of its associated van Emden-Kowalski operator (van Emden & Kowalski, 1976). We call an interpretation $I$ a supported model of $P$ if $I$ is a fixed point of $T_P$.

The following results are answer set programming folklore.

**Theorem 2.** Let $H$ be a Horn program and let $P$ be an answer set program.

1. An interpretation $I$ is a model of $P$ iff $I$ is a prefixed point of $T_P$.
2. The least model of $H$ coincides with the least fixed point of $T_H$.
3. An interpretation $I$ is a model of $P$ iff $I$ is a prefixed point of $T_P$.
4. An interpretation $I$ is an answer set of $P$ iff $I$ is the least fixed point of $T_{gP^I}$.
5. An interpretation $I$ is an answer set of $P$ iff $I$ is a subset minimal model of $P^I$.

We say that $P$ and $R$ are (i) equivalent if $P$ and $R$ have the same answer sets; (ii) subsumption equivalent if $T_P = T_R$; (iii) uniformly equivalent (Eiter & Fink, 2003) if $P \cup I$ is equivalent to $R \cup I$ for any interpretation $I$; and (iv) strongly equivalent (Lifschitz, Pearce, & Valverde, 2001) if $P \cup Q$ is equivalent to $R \cup Q$ for any program $Q$.

3. Composition

This is the main section of the paper. Here we define the sequential composition of answer set programs and prove in the Main Theorem [7] that it induces the structure of a unital magma on the space of all answer set programs.

Before we give the formal definition of composition below, we shall first introduce some auxiliary constructions. The goal is to extend the “not” operator from atoms to programs, which will be essential in the definition of composition below.

**Notation 3.** In the rest of the paper, $P$ and $R$ denote answer set programs, and $I$ denotes an interpretation over some joint finite alphabet of propositional atoms $A$. 


First, define the \( \text{tf-operator} \) by
\[
\text{tf}(P) := \text{proper}(P) \cup \{ a \leftarrow t \mid a \in P \} \cup \{ a \leftarrow f \mid a \in A - \text{head}(P) \}.
\]

Roughly, the \( \text{tf} \)-operator replaces every fact \( a \) in \( P \) by \( a \leftarrow t \), and it makes every atom \( a \) not occurring in any rule head of \( P \) explicit by adding \( a \leftarrow f \). This transformation will be needed in the treatment of negation in the definition of composition. The reader should interpret \( a \leftarrow t \) as “\( a \) is true” and \( a \leftarrow f \) as “\( a \) is false”, similar to truth value assignments in imperative programming. Notice that the \( \text{tf} \)-operator depends implicitly on the underlying alphabet \( A \).

Second, define the \( \overline{\cdot} \)-operator by
\[
\overline{P} := \{ \text{head}(r) \leftarrow (\text{body}(r) - \{ t \}) \mid r \in P \} - \{ r \mid f \in \text{body}(r) \}.
\]

Roughly, the \( \overline{\cdot} \)-operator removes every occurrence of \( t \) from rule bodies and eliminates every rule containing \( f \) and is therefore “dual” to the \( \text{tf} \)-operator.

Third, define the \( \lor \)-operator by\(^2\)
\[
P' := \left\{ \text{head}(r) \leftarrow \bigvee \text{body}(\text{head}(r) P) \mid r \in P \right\}.
\]

Intuitively, the \( \lor \)-program \( P' \) contains exactly one \( \lor \)-rule for each head atom in \( P \) containing the disjunction of all rule bodies with the same head atom.

Fourth, define the negation of an \( \lor \)-program inductively as follows. First, define \( \textit{not } t := f \) and \( \textit{not } f := t \), extended to a literal \( L \) by
\[
\text{not } L := \begin{cases} a & \text{ if } L = \text{not } a, \\ \text{not } a & \text{ if } L = a. \end{cases}
\]

Then, for an \( \lor \)-rule \( r \) of the form
\[
\text{head}(r) \leftarrow \{L_1^1, \ldots, L_{k_1}^1\} \lor \ldots \lor \{L_1^n, \ldots, L_{k_n}^n\},
\]
where each \( L_i^j \) is a literal, \( 1 \leq j \leq n, n \geq 1 \)\(^3\) \( 1 \leq i \leq k_j, k_j \geq 1 \), define
\[
\text{not } r := \text{head}(r) \leftarrow \text{not}\{L_1^1, \ldots, L_{k_1}^1\}, \ldots, \text{not}\{L_1^n, \ldots, L_{k_n}^n\}
\]
\[
:= \text{head}(r) \leftarrow \{\text{not } L_1^1 \lor \ldots \lor \text{not } L_{k_1}^1\}, \ldots, \{\text{not } L_1^n \lor \ldots \lor \text{not } L_{k_n}^n\}
\]
\[
:= \bigcup_{1 \leq i_1 \leq k_1} \cdots \bigcup_{1 \leq i_n \leq k_n} \{\text{head}(r) \leftarrow \text{not } L_{i_1}^1, \ldots, \text{not } L_{i_n}^n\}
\]
\[
\text{head}(r) \leftarrow \text{not } L_1^1, \ldots, \text{not } L_1^n
\]
\[
\vdots
\]
\[
\text{head}(r) \leftarrow \text{not } L_{k_1}^1, \ldots, \text{not } L_{k_n}^n.
\]

In the first two steps, we have applied De Morgan’s law and in the third step we have applied the distributive law of Boolean logic. Now define the \( \textit{negation} \) of a program \( P \) rule-wise by
\[
\text{not } P := \{ \text{not } r \mid r \in \text{tf}(P)' \}.
\]

Notice that the \( \text{not} \)-operator depends implicitly on the underlying alphabet \( A \).

We are now ready to introduce the main notion of the paper.

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\(^2\)Recall from Section 2.2.2 that \( \text{head}(r) P \) is the left reduct of \( P \) with respect to \( \text{head}(r) \).

\(^3\)We assume here that the \( \lor \)-rule \( r \) contains at least one body literal possibly consisting only of a truth value among \( t \) and \( f \). This is consistent with our use of negation below where each atom \( a \) is first translated via the \( \text{tf} \)-operator into \( a \leftarrow t \).
Definition 4. Define the (sequential) composition of $P$ and $R$ by

$$P \circ R := \begin{cases} 
\text{head}(r) \leftarrow \text{body}(S, N) \\
\text{head}(S) = \text{body}(\text{pos}(r)) \\
\text{head}(N) = \text{body}(\text{horn}(\text{neg}(r))) \\
r \in P \\
S \subseteq_{sz(\text{pos}(r))} R \\
N \subseteq_{sz(\text{neg}(r))} \neg R 
\end{cases}.$$  

We will often write $PR$ instead of $P \circ R$ in case composition is understood. Roughly, the composition of $P$ and $R$ is computed by resolving all body literals in $P$ with ‘matching’ rule heads of $R$. Before proceeding with formal constructions and results, we first want to illustrate composition with the following simple example.

Example 5. Consider the rule $r$ and program $R$ given by

$$r := a \leftarrow \neg b \quad \text{and} \quad R := \begin{cases} 
b \leftarrow \neg c, \neg d \\
b \leftarrow c, d 
\end{cases}.$$  

We wish to compute $\{r\} \circ R$. For this, we first compute

$$\text{tf}(R) = \begin{cases} 
a \leftarrow f \\
b \leftarrow \{\neg c, \neg d\} \lor \{c, d\} \\
c \leftarrow f \\
d \leftarrow f 
\end{cases}$$  

and then

$$\neg R = \begin{cases} 
a \leftarrow \neg f \\
b \leftarrow \neg (\{\neg c, \neg d\} \lor \{c, d\}) \\
c \leftarrow \neg f \\
d \leftarrow \neg f 
\end{cases}$$  

$$= \begin{cases} 
a \\
b \leftarrow \neg \{\neg c, \neg d\}, \neg \{c, d\} \\
c \\
d 
\end{cases}$$  

$$= \begin{cases} 
a \\
b \leftarrow \{c \lor d\}, \{\neg c \lor \neg d\} \\
c \\
d 
\end{cases} = \begin{cases} 
a \\
b \leftarrow c, \neg c \\
b \leftarrow d, \neg c \\
b \leftarrow c, \neg d \\
b \leftarrow d, \neg d \\
c \\
d 
\end{cases}.$$  

We can now compute

$$\{r\} \circ R = \begin{cases} 
a \leftarrow c, \neg c \\
a \leftarrow d, \neg c \\
a \leftarrow c, \neg d \\
a \leftarrow d, \neg d 
\end{cases}.$$  

Notice that we can reformulate composition as

$$P \circ R = \bigcup_{r \in P} (\{r\} \circ R),$$  

(3)
which directly implies right-distributivity of composition, that is,

\[(P \cup R) \circ Q = (P \circ Q) \cup (R \circ Q)\]

holds for all Horn programs \(P, Q, R\).

However, the following counter-example shows that left-distributivity fails in general:

\[\{a \leftarrow b, c\} \circ (\{b\} \cup \{c\}) = \{a\}\]

whereas

\[\{(a \leftarrow b, c) \circ \{b\} \cup (a \leftarrow b, c) \circ \{c\}\} = \emptyset.\]

The situation changes for Krom-Horn programs (see Theorem 13).

We can write \(P\) as the union of its facts and proper rules, that is,

\[P = \text{facts}(P) \cup \text{proper}(P).\]

Hence, we can rewrite the composition of \(P\) and \(R\) as

\[P \circ R = (\text{facts}(P) \cup \text{proper}(P))R \equiv \text{facts}(P)R \cup \text{proper}(P)R\]

\[= \text{facts}(P) \cup \text{proper}(P)R,\]

which shows that the facts in \(P\) are preserved by composition, that is, we have

\[\text{facts}(P) \subseteq \text{facts}(P \circ R).\]

The following example shows that, unfortunately, composition is not associative even in the Horn case (but see Theorem 17).

**Example 6.** Consider the Horn rule

\[r := a \leftarrow b, c,\]

and the Horn programs

\[P := \begin{cases} b \leftarrow b \\ c \leftarrow b, c \end{cases}\quad \text{and} \quad R := \begin{cases} b \leftarrow d \\ b \leftarrow e \\ c \leftarrow f \end{cases}.\]

A simple computation yields

\[\{r\}(PR) = \begin{cases} a \leftarrow d, f \\ a \leftarrow e, f \\ a \leftarrow d, e, f \end{cases} \neq \begin{cases} a \leftarrow d, f \\ a \leftarrow e, f \end{cases} = \{(r)P\}R.\]

Define the *unit program* by the Krom-Horn program

\[1_A := \{a \leftarrow a \mid a \in A\}.\]

We will often omit the reference to the underlying alphabet \(A\).

We are now ready to prove the main structural result of the paper.

**Theorem 7.** The space of all answer set programs over some fixed alphabet forms a finite unital magma with respect to composition ordered by set inclusion with the neutral element given by the unit program. Moreover, the empty program is a left zero and composition distributes from the right over union, that is, for any answer set programs \(P, Q, R\) we have

\[\begin{aligned}
(P \cup R) \circ Q &= (P \circ Q) \cup (R \circ Q) \\
\end{aligned}\]

\[4\text{In the rest of the paper, all statements about spaces of programs are always with respect to some fixed underlying finite alphabet } A.\]
Proof. The composition of two programs is again a program, which is not completely obvious since \( \lor \)-programs possibly containing \( t \) and \( f \) occur in intermediate steps in the computation of composition. The reason is that the not-operator, which is defined rule-wise, translates every \( \lor \)-rule into a collection of ordinary rules not containing truth values in rule bodies. Hence, the space of all programs is closed under composition.

We proceed by showing that \( 1 \) is neutral with respect to composition. We first compute

\[
\text{not } 1 = \{ a \leftarrow \text{not } a \mid a \in A \}.
\]

Next, by definition of composition, we have

\[
P \circ 1 = \left\{ \begin{array}{l}
\text{head}(r) \leftarrow \text{body}(S, N) \\
S \subseteq \text{sz}(\text{pos}(r)) \text{ } 1 \\
N \subseteq \text{sz}(\text{neg}(r)) \text{ not } 1 \\
\text{head}(S) = \text{body}(\text{pos}(r)) \\
\text{head}(N) = \text{body}(\text{horn}(\text{neg}(r)))
\end{array} \right\}.
\]

Due to the simple structure of \( 1 \) and not \( 1 \), \( S \subseteq 1 \) and \( N \subseteq \text{not } 1 \) imply

\[
\text{head}(S) = \text{body}(S) \quad \text{and} \quad \text{head}(N) = \text{body}(\text{horn}(N)).
\]

Together with

\[
\text{head}(S) = \text{body}(\text{pos}(r)) \quad \text{and} \quad \text{head}(N) = \text{body}(\text{horn}(\text{neg}(r)))
\]

we further deduce

\[
\text{body}(S) = \text{body}(\text{pos}(r)) \quad \text{and} \quad \text{body}(\text{horn}(N)) = \text{body}(\text{horn}(\text{neg}(r)))
\]

which finally implies

\[
\text{body}(S, N) = \text{body}(r), \quad \text{for each rule } r \text{ in } P.
\]

As composition is defined rule-wise by (3), this shows

\[
P \circ 1 = P.
\]

The identity \( 1 \circ P = P \) follows from the fact that since \( 1 \) is Krom and Horn, we can omit every reference to \( N \) in the definition of composition and \( S \) amounts to a single rule \( s \in P \):

\[
1 \circ P = \{ \text{head}(r) \leftarrow \text{body}(s) \mid s \in P : \text{head}(s) = \text{body}(r) \} = P.
\]

Hence, we have established that composition gives rise to a unital magma with neutral element 1. That the magma is ordered by set inclusion is obvious. We now turn our attention to the operation of union. In (3), we argued for the right-distributivity \( \text{(9)} \) of composition. That the empty set is a left zero is obvious.

We will call magmas and arising from compositions of answer set programs as above \( \text{ASP magmas} \).

3.1. Cup. Here we introduce the cup as an associative commutative binary operation on programs with identity (Theorem \( \text{[10]} \)), which will allow us to decompose the bodies of rules and programs into its positive and negative parts (cf. \( \text{[11]} \) and \( \text{[20]} \)).

Definition 8. We define the cup of \( P \) and \( R \) by

\[
P \cup R := \{ \text{head}(r) \leftarrow \text{body}(r, s) \mid r \in P, s \in R : \text{head}(r) = \text{head}(s) \}.
\]
For instance, we have

\[
\begin{align*}
\{ a \leftarrow b \\ a \leftarrow c \} \sqcup \{ a \leftarrow b \\ a \leftarrow c \} &= \{ a \leftarrow b \\ a \leftarrow c \} \\
\{ a \leftarrow b \\ a \leftarrow c \} \sqcup \{ a \leftarrow b, c \} &= \{ a \leftarrow b \} \sqcup \{ a \leftarrow c \} \sqcup \{ a \leftarrow b, c \}
\end{align*}
\]

which shows that cup is not idempotent.

We can now decompose a rule \( r \) of the form (1) in different ways, for example, as

\[
\{ r \} = \{ a_0 \leftarrow a_1 \} \sqcup \ldots \sqcup \{ a_0 \leftarrow a_\ell \} \sqcup \{ a_0 \leftarrow \text{not } a_{\ell+1} \} \sqcup \ldots \sqcup \{ a_0 \leftarrow \text{not } a_k \}
\]

and as

\[
\{ r \} = \{ \text{pos}(r) \} \sqcup \{ \text{neg}(r) \}.
\]

As cup is defined rule-wise, we have

\[
P \sqcup R = \bigcup_{r \in P} (\{ r \} \sqcup \{ s \}).
\]

**Notation 9.** We make the notational convention than composition binds stronger than cup.

The next result shows that cup and union are compatible.

**Theorem 10.** The space \((P, \sqcup, A)\) of all answer set programs forms a finite commutative monoid with respect to cup with the neutral element given by the alphabet \( A \), and the space \((P, \sqcup, \sqcup, \emptyset, A)\) forms a finite idempotent semiring with respect to union and cup with the zero given by the empty program. That is, we have the following identities:

\[
\begin{align*}
(13) & \quad P \sqcup (Q \sqcup R) = (P \sqcup Q) \sqcup R \\
(14) & \quad P \sqcup R = R \sqcup P \\
(15) & \quad P \sqcup A = A \sqcup P = P \\
(16) & \quad \emptyset \sqcup P = P \sqcup \emptyset = \emptyset \\
(17) & \quad (P \sqcup R) \sqcup Q = (P \sqcup Q) \sqcup (R \sqcup Q) \\
(18) & \quad Q \sqcup (P \sqcup R) = (Q \sqcup P) \sqcup (Q \sqcup R).
\end{align*}
\]

Finally, given two rules \( r \) and \( s \), in case \( \text{body}(r) \cap \text{body}(s) = \emptyset \), we have

\[
(19) & \quad (\{ r \} \sqcup \{ s \})Q = \{ r \}Q \sqcup \{ s \}Q.
\]

**Proof.** The first four identities hold trivially. The identities (17) and (18) follow from (12).

We proceed with proving (19) as follows. We distinguish two cases: (i) If \( \text{head}(r) \neq \text{head}(s) \) then

\[
(\{ r \} \sqcup \{ s \})Q = \emptyset Q = \emptyset
\]

and since

\[
\text{head}(\{ r \}Q) \subseteq \text{head}(\{ r \}) = \{ \text{head}(r) \}
\]

and

\[
\text{head}(\{ r \}Q) \subseteq \text{head}(\{ s \}) = \{ \text{head}(s) \},
\]

\footnote{We can omit parentheses as cup is associative according to the forthcoming Theorem 10.}
we have

\[ \text{head}(\{r\}Q) \neq \text{head}(\{s\}Q) \]

which implies

\[ \{r\}Q \cup \{s\}Q = \emptyset. \]

(ii) If \(\text{head}(r) = \text{head}(s)\) then we have

\[
\begin{aligned}
\left\{ \text{head}(r) \leftarrow \text{body}(S, N) \right\} &:\ \\
&\begin{aligned}
& S \subseteq_{Sz} \text{pos}(r, s) \text{ Q} \\
& N \subseteq_{Sz} \text{neg}(r, s) \text{ not Q} \\
& \text{head}(S) = \text{body}(\text{pos}(r, s)) \\
& \text{head}(N) = \text{body}(\text{horn}(\text{neg}(r, s))) \\
\end{aligned}
\end{aligned}
\]

Now \(\text{body}(r) \cap \text{body}(s) = \emptyset\) implies

\[ \text{sz}(\text{pos}(r, s)) = \text{sz}(\text{pos}(r)) + \text{sz}(\text{pos}(s)) \]

and

\[ \text{sz}(\text{neg}(r, s)) = \text{sz}(\text{neg}(r)) + \text{sz}(\text{neg}(s)). \]

Hence, the above expression is equivalent to

\[
\begin{aligned}
\left\{ \text{head}(r) \leftarrow \text{body}(S_r, S_s, N_r, N_s) \right\} &:\ \\
&\begin{aligned}
&S_r \subseteq_{Sz} \text{pos}(r) \text{ Q} \\
&S_s \subseteq_{Sz} \text{pos}(s) \text{ Q} \\
&N_r \subseteq_{Sz} \text{neg}(r) \text{ not Q} \\
&N_s \subseteq_{Sz} \text{neg}(s) \text{ not Q} \\
&\text{head}(S_r) = \text{body}(\text{pos}(r)) \\
&\text{head}(S_s) = \text{body}(\text{pos}(s)) \\
&\text{head}(N_r) = \text{body}(\text{horn}(\text{neg}(r))) \\
&\text{head}(N_s) = \text{body}(\text{horn}(\text{neg}(s))) \\
\end{aligned}
\end{aligned}
\]

= \left\{ \text{head}(r) \leftarrow \text{body}(S, N) \right\} \cup \left\{ \text{head}(r) \leftarrow \text{body}(S, N) \right\}

\[
\begin{aligned}
&\begin{aligned}
&S \subseteq_{Sz} \text{pos}(r) \text{ Q} \\
&N \subseteq_{Sz} \text{neg}(r) \text{ not Q} \\
&\text{head}(S) = \text{body}(\text{pos}(r)) \\
&\text{head}(N) = \text{body}(\text{horn}(\text{neg}(r))) \\
\end{aligned}
\end{aligned}
\]

\[= \{r\}Q \cup \{s\}Q. \]

\[\square\]

The following counter-example shows why we require \(\text{body}(r) \cap \text{body}(s) = \emptyset\) in (19):

\[
\left(\{a \leftarrow b\} \cup \{a \leftarrow b, e\}\right) \circ \left\{ \begin{array}{c} b \leftarrow d \\ b \leftarrow e \\ c \leftarrow f \end{array} \right\} = \left\{ \begin{array}{c} a \leftarrow d, f \\ a \leftarrow e, f \end{array} \right\}
\]
whereas
\[
\begin{align*}
\{a ← b\} \circ \left(\begin{array}{l}
b ← d \\
b ← e \\
c ← f \\
\end{array}\right) \cup \left(\begin{array}{l}
a ← b, c \\
b ← d \\
c ← f \\
\end{array}\right) &= \left\{\begin{array}{l}
a ← d, f \\
a ← e, f \\
a ← d, e, f \\
\end{array}\right\}.
\end{align*}
\]

As rules can be decomposed into a positive and a negative part according to (11), composition splits into a positive and a negative part as well.

**Corollary 11.** For any answer set programs \(P\) and \(R\), we have
\[
P \circ R = \bigcup_{r \in P} \{\{pos(r)\} \cup \{\text{horn}(neg(r))\}\} \cup \{\{not\}\} R.
\]

**Proof.** We compute
\[
P R = \bigcup_{r \in P} \{\{r\} R\}
\]
\[
(11) \quad \bigcup_{r \in P} \{\{pos(r)\} \cup \{pos(r)\}\} R
\]
\[
(19) \quad \bigcup_{r \in P} \{\{pos(r)\} \cup \{neg(r)\}\} R
\]
\[
(26) \quad \bigcup_{r \in P} \{\{pos(r)\} \cup \{\text{horn}(neg(r))\}\} \cup \{\{not\}\} R.
\]

\[\square\]

4. **Restricted Classes of Programs**

In this section, we shall study the basic properties of composition in the important classes of Horn, negative, and Krom programs, showing that the notion of composition simplifies in each of these classes.

4.1. **Krom-Horn Programs.** Recall that we call a Horn program Krom if it contains only rules with at most one body atom. This includes interpretations, unit programs, and permutations.

**Proposition 12.** For any Krom-Horn program \(K\) and answer set program \(R\), composition simplifies to
\[
K \circ R = \text{facts}(K) \cup \{a ← B \mid a ← b \in K, b ← B \in R\}.
\]

We have the following structural result as a specialization of Theorem 7.

**Theorem 13.** The space of all Krom-Horn programs forms a monoid with the neutral element given by the unit program. Moreover, Krom-Horn programs distribute from the left, that is, for any answer set programs \(P\) and \(R\), we have
\[
K \circ (P \cup R) = (K \circ P) \cup (K \circ R).
\]

This implies that the space of proper\(^6\) Krom-Horn programs forms a finite idempotent semiring. Moreover, For any Krom-Horn program \(K\) and answer set programs \(P\) and \(R\), we have
\[
K \circ (P \circ R) = (K \circ P) \circ R.
\]

\(^6\)If a Krom-Horn program \(K\) contains facts then \(K \circ \emptyset = \text{facts}(K) \neq \emptyset\) violates the axiom \(a \cdot 0 = 0\) of a semiring.
Proof. For the proof of the first three statements see Antić (2021b, Theorem 6). □

To prove (22), by (3) and (4) it suffices to prove
\[
\{r\}(PR) = \{r\}P \quad \text{for any Krom-Horn rule } r.
\]

We distinguish two cases: (i) in case \(r\) is a fact \(a \in A\), we have
\[
\{r\}(PR) = \{r\}P \quad \text{for any Krom-Horn rule } a \leftarrow b, \quad a \leftarrow a_1, \ldots, a_\ell, \text{not } a_{\ell+1}, \ldots, \text{not } a_k \in \{a \leftarrow b\}(PR), \quad k \geq 0,
\]
iff there is a rule
\[
b \leftarrow a_1, \ldots, a_\ell, \text{not } a_{\ell+1}, \ldots, \text{not } a_k \in PR
\]
iff there is a rule
\[
b \leftarrow b_1, \ldots, b_m, \text{not } b_{m+1}, \ldots, \text{not } b_n \in P
\]
and there are subprograms
\[
\begin{align*}
\{ b_1 \leftarrow B_1 \\
\vdots \\
{ b_m } \leftarrow B_m 
\} \subseteq R \quad \text{and} \quad \{ b_1 \leftarrow B_1 \\
\vdots \\
{ b_n } \leftarrow B_n 
\} \subseteq \text{not } R
\end{align*}
\]
such that
\[
B_1 \cup \ldots \cup B_n = \{a_1, \ldots, a_\ell, \text{not } a_{\ell+1}, \ldots, \text{not } a_k\}
\]
iff there is a rule
\[
a \leftarrow b_1, \ldots, b_m, \text{not } b_{m+1}, \ldots, \text{not } b_n \in \{a \leftarrow b\}P
\]
and
\[
a \leftarrow (B_1 \cup \ldots \cup B_n) = a \leftarrow a_1, \ldots, a_\ell, \text{not } a_{\ell+1}, \ldots, \text{not } a_k \in (\{a \leftarrow b\}P)R.
\]

Hence, we have shown (23), from which we deduce for any Krom-Horn program \(K\):
\[
K(PR) = \bigcup_{r \in K} \{r\}(PR) = \bigcup_{r \in K} [\{r\}P]R = \bigcup_{r \in K} ([rP]R)R = (KP)R.
\]
□

4.1.1. Interpretations. Formally, interpretations are Krom programs containing only rules with empty bodies (i.e. facts), which gives interpretations a special compositional meaning.

Proposition 14. Every interpretation is a left zero with respect to composition which means that for any answer set program \(P\), we have
\[
I \circ P = I.
\]
Consequently, the space of interpretations forms a right ideal. □

Proof. See Antić (2021b, Proposition 5). □
4.1.2. Permutation Programs. With every permutation \( \pi : A \to A \) we associate a Krom-Horn program
\[ \pi = \{ \pi(a) \leftarrow a \mid a \in A \}. \]
We adopt here the standard cycle notation for permutations. For instance, we have
\[ \pi_{(ab)} := \{ a \leftarrow b \} \quad \text{and} \quad \pi_{(ab)(c)} := \{ a \leftarrow b, b \leftarrow a, c \leftarrow c \}. \]
Notice that the inverse \( \pi^{-1} \) of a permutation \( \pi \) translates into the dual of a program. Interestingly, we can rename the atoms occurring in a program via permutations and composition by
\[ (\pi \circ P) \circ \pi^d = \{ \pi(\text{head}(r)) \leftarrow \pi(\text{body}(r)) \mid r \in P \}. \]

We have the following structural result as a direct instance of a more general result for permutations.

**Proposition 15.** The space of all permutation programs forms a subgroup of the space of all answer set programs.

4.2. Horn Programs. Recall that a program is called Horn if it contains only positive rules not containing negation as failure. The composition of Horn programs has been studied by Antić (2021b) and we shall recall here some basic results.

As the syntactic structure of Horn programs is much simpler than the structure of general answer set programs, one can expect the composition to simplify for Horn programs. In fact, the next result shows that even in the case where only the left program in the composition is Horn, the definition of composition simplifies substantially.

**Proposition 16.** For any Horn program \( H \) and answer set program \( R \), composition simplifies to
\[ (25) \quad H \circ R = \{ \text{head}(r) \leftarrow \text{body}(S) \mid r \in P, S \subseteq \text{sz}(r) : (\text{head}(S) = \text{body}(r)) \}. \]

*Proof.* Since \( H \) is Horn, we can omit every expression containing \( N \) in Definition 4. □

We have the following structural result.

**Theorem 17.** The space of all Horn programs forms a unital submagma of the space of all answer set programs with the neutral element given by the unit program.

*Proof.* See Antić (2021a, Theorem 3). □

4.3. Negative Programs. Recall that negative programs contain only rules with negated body atoms including facts and interpretations. As negative programs are not closed under composition, the space of negative programs does not form a submagma of the space of all answer set programs, which is in contrast to the situation for Horn programs (cf. Theorem 17). This is witnessed, for example, by the identity
\[ (\text{not } 1)(\text{not } 1) = 1. \]

However, computing the composition with respect to negative programs still simplifies compared to the general case as we can reduce the composition with a negative program to the Horn case as follows.

\[ (\text{not } a)(\text{not } b) = \text{not } (a \leftarrow b) \]

This is reasonable as we can interpret every fact \( a \) as the negated “rule” \( a \leftarrow \text{not } f \) (see Section 5).
**Proposition 18.** For any negative program $N$ and answer set program $R$, composition simplifies to

\[(26)\quad N \circ R = \text{horn}(N) \circ \text{not } R.\]

**Proof.** We compute

\[N \circ R = \{\text{head}(r) \leftarrow \text{body}(M) \mid r \in N, M \subseteq_{sz(r)} \text{not } R, \text{head}(M) = \text{body}(\text{horn}(r))\}\]

\[= \{\text{head}(r) \leftarrow \text{body}(M) \mid r \in \text{horn}(N), M \subseteq_{sz(r)} \text{not } R, \text{head}(M) = \text{body}(r)\}\]

\[\overset{\text{n18}}{=} \text{horn}(N) \circ \text{not } R.\]

Interestingly enough, composition is compatible with negation as failure in the following sense.

**Lemma 19.** For any answer set programs $P$ and $R$, we have

\[(27)\quad \text{not } P = (\text{not } 1)P \quad \text{and} \quad \text{not}(PR) = (\text{not } P)R.\]

**Proof.** As a direct consequence of Proposition 18 and since $\text{not } 1$ is negative, we have

\[(\text{not } 1) \circ P \overset{\text{n18}}{=} \text{horn}(\text{not } 1) \circ \text{not } P = 1 \circ \text{not } P = \text{not } P,\]

which further implies

\[\text{not}(PR) = (\text{not } 1)(PR) \overset{\text{n22}}{=} ((\text{not } 1)P)R = (\text{not } P)R.\]

\[\square\]

5. **Algebraic Transformations**

In this section, we study algebraic transformations of programs expressible via composition and other operations.

5.1. **Reducts.** Reducing the rules of a program to a restricted alphabet is a fundamental operation on programs and in this section we will show how reducts can be computed via composition, cup, and union (cf. Theorem 24).

5.1.1. **Horn Programs.** We first recall some results of Antić (2021b, Section 4.1) on computing the reducts of Horn programs.

**Proposition 20.** The left and right reducts of a Horn program $H$ with respect to some interpretation $I$ can be expressed as

\[(28)\quad I^H = 1^I \circ H \quad \text{and} \quad H^I = H \circ 1^I.\]

Consequently, we obtain the reduction of $H$ to the atoms in $I$, denoted by $H|_I$, via

\[(29)\quad H|_I = 1^I(H^I) = (1^H)^I.\]

**Proof.** See the proof of Antić (2021b, Proposition 11). \[\square\]
We can compute the facts of a Horn program $H$ via

$$H^\emptyset \overset{28}{=} H \circ \emptyset = H \circ \emptyset = \text{facts}(H).$$  

Moreover, for any interpretations $I$ and $J$, we have

$$^J I = I \cap J \quad \text{and} \quad I^J = I.$$

**Proposition 21.** For any Horn programs $H$ and $G$, we have

(32) 
$$gH^I = H$$

(33) 
$$^I (H \cup G) = ^I H \cup ^I G \quad \text{and} \quad ^I (H \circ G) = ^I H \circ G$$

(34) 
$$(H \cup G)^I = H^I \cup G^I \quad \text{and} \quad (H \circ G)^I = H \circ G^I.$$

**Proof.** The first identity holds trivially. For the identities in the last two lines, see the proof of Antić (2021b, Proposition 4.2). □

5.1.2. **Negative Programs.** Reducts of negative programs are in a sense “dual” to reducts of Horn programs studied above. In the rest of this section, $N$ denotes a negative program.

Our first observation is a dual of (28).

**Proposition 22.** The left and right reducts of a negative program $N$ with respect to an interpretation $I$ can be expressed as

$$^I N = 1^I \circ N \quad \text{and} \quad N^I = N \circ 1^{A-I}.$$  

Moreover, the Gelfond-Lifschitz reduct of $N$ with respect to $I$ can be expressed as

$$gN^I = N \circ I.$$

**Proof.** The proof of the first identity is analogous to the proof of (28). For the second identity, we compute

$$N^I = \text{horn}(N) \circ \text{not} \quad 1^{A-I} \overset{26}{=} N \circ 1^{A-I}.$$  

For the last identity, we compute

$$N \circ I = \{ \text{head}(r) \leftarrow \text{body}(M) \mid r \in N, M \subseteq_{\text{sz}} \text{neg}(r) \} \not\vdash I : \text{head}(M) = \text{body}(\text{horn}(\text{neg}(r))) \}
= \{ \text{head}(r) \leftarrow \text{body}(M) \mid r \in N, M \subseteq_{\text{sz}} r \} \not\vdash I : \text{head}(M) = \text{body}(\text{horn}(r)) \}
= \{ \text{head}(r) \mid r \in N, M \subseteq_{\text{sz}} r \} \not\vdash A-I
= \{ \text{head}(r) \mid r \in N : I \cap \text{body}(\text{horn}(r)) = \emptyset \}
= \{ \text{head}(r) \mid r \in N : I = \text{body}(r) \}
= gN^I,$$

where the second equality follows from $\text{neg}(r) = r$ as $r$ is negative, and the third equality follows from $\text{body}(M) = \emptyset$ as $M$ is a subset of $A-I$. □
5.1.3. Answer Set Programs. We now focus on reducts of arbitrary answer set programs. Our first observation is that the first identity in (28) can be lifted to the general case as

\[ I_P = 1^I \circ P. \]

The next lemma shows that reducts are compatible with cup and union.

**Lemma 23.** For any answer set program \( P \) and interpretation \( I \), we have

\[
\begin{align*}
(37) & \quad g(P \cup R)^I = gP^I \cup gR^I \\
(38) & \quad I(P \cup R) = I_P \cup I_R \quad \text{and} \quad I(P \cup R) = I_P \cup I_R \\
(39) & \quad (P \cup R)^I = P^I \cup R^I \quad \text{and} \quad (P \cup R)^I = P^I \cup R^I.
\end{align*}
\]

**Proof.** The identities on the left-hand side are immediate consequences of the rule-wise definition of reducts.

For the identities on the right-hand side, we first show that for any rules \( r \) and \( s \), we have

\[
(40) \quad g(\{r\} \cup \{s\})^I = g\{r\}^I \cup g\{s\}^I.
\]

For this, we distinguish two cases: (i) if \( \text{head}(r) \neq \text{head}(s) \) then

\[
g(\{r\} \cup \{s\})^I = g\emptyset^I = \emptyset = g\{r\}^I \cup g\{s\}^I;
\]

(ii) if \( \text{head}(r) = \text{head}(s) \) then

\[
g(\{r\} \cup \{s\})^I = g(\text{head}(r) \leftarrow \text{body}(r, s))^I = \{\text{pos}(r, s) \mid I \models \text{body}(\text{neg}(r, s))\}
\]

\[
= \{\text{pos}(r) \mid I \models \text{body}(\text{neg}(r))\} \cup \{\text{pos}(s) \mid I \models \text{body}(\text{neg}(s))\}
\]

\[
= g\{r\}^I \cup g\{s\}^I.
\]

Now we have

\[
g(P \cup R)^I = \bigcup_{r \in P, s \in R} (\{r\} \cup \{s\})^I = \bigcup_{r \in P, s \in R} g(\{r\} \cup \{s\})^I
\]

\[
= \bigcup_{r \in P \cup R} (g\{r\}^I \cup g\{s\}^I) = \bigcup_{r \in P \cup R} (\{r\} \cup \{s\})^I = gP^I \cup gR^I.
\]

The proofs of the remaining identities are analogous. \( \square \)

We are now ready to express the Gelfond-Lifschitz and Faber-Leone-Pfeifer reducts via composition, cup, and union.

**Theorem 24.** For any answer set program \( P \) and interpretation \( I \), we have

\[
gP^I = \bigcup_{r \in P} \{\text{pos}(r)\} \cup \{\text{neg}(r)\}^I \quad \text{and} \quad P^I = \bigcup_{r \in P} \{\text{pos}(r)\}^I \cup \{\text{horn}(\text{neg}(r))\}^I A^I.
\]

**Proof.** We first compute, for any rule \( r \),

\[
g\{r\}^I = g\{\text{pos}(r)\}^I \cup g\{\text{neg}(r)\}^I \quad \text{and} \quad \{\text{pos}(r)\} \cup \{\text{neg}(r)\}^I
\]

extended to any answer set program \( P \) by

\[
gP^I = \bigcup_{r \in P} g\{r\}^I = \bigcup_{r \in P} \{\text{pos}(r)\} \cup \{\text{neg}(r)\}^I.
\]
For the Faber-Leone-Pfeifer reduct, we have
\[ P^I = \bigcup_{r \in P} \{ r \}^I = \bigcup_{r \in P} \{ \{ \text{pos}(r) \} \cup \{ \text{neg}(r) \} \}^I \]
\[ \overset{39}{\Leftrightarrow} \bigcup_{r \in P} \{ \{ \text{pos}(r) \}^I \cup \{ \text{neg}(r) \}^I \} \]
\[ \overset{28}{\Leftrightarrow} \bigcup_{r \in P} \{ \{ \text{pos}(r) \}^I \cup \{ \text{horn}(\text{neg}(r)) \}^{1A-I} \}. \]

5.2. Adding and Removing Body Literals. We now want to study algebraic transformations of rule bodies.

5.2.1. Horn Programs. Antić (2021b, Section 4.2) has studied the Horn case and we shall first recall here some basic constructions and results concerning the manipulation of Horn programs.

In what follows, \( H \) denotes a Horn program. For example, we have
\[ \{ a \leftarrow b, c \} \circ \left\{ \begin{array}{l} b \leftarrow b \\ c \end{array} \right\} = \{ a \leftarrow b \}. \]

The general construction here is that we add a tautological rule \( b \leftarrow b \) for every body atom \( b \) of \( H \) which we want to preserve, and we add a fact \( c \) in case we want to remove \( c \) from the rule bodies in \( H \).

**Definition 25.** For any interpretation \( I \), define
\[ I^\Theta := 1^{A-I} \cup I. \]

Notice that \( .^\Theta \) is computed with respect to some fixed alphabet \( A \).

For instance, we have \( A^\Theta = A \) and \( \emptyset^\Theta = 1 \).

Interestingly enough, we have
\[ I^\Theta I = (1^{A-I} \cup I)I \overset{41}{=} 1^{A-I} I \cup I^2 \overset{24}{=} (28, 31) ((A-I) \cap I) \cup I = I \]
and
\[ I^\Theta H = (1^{A-I} \cup I)H \overset{43}{=} 1^{A-I} H \cup IH \overset{24}{=} (28) A-I H \cup I. \]

In the example above, we have
\[ \{ c \}^\Theta = \left\{ \begin{array}{l} a \leftarrow a \\ b \leftarrow b \\ c \end{array} \right\} \]
and \( \{ a \leftarrow b, c \} \circ \{ c \}^\Theta = \{ a \leftarrow b \} \)
as desired. Notice also that the facts of a program are, of course, not affected by composition on the right (cf. 8), that is, we cannot expect to remove facts via composition on the right.

We have the following general result due to Antić (2021b, Proposition 13).

**Proposition 26.** For any Horn program \( H \) and interpretation \( I \), we have
\[ HI^\Theta = \{ \text{head}(r) \leftarrow (\text{body}(r) - I) \mid r \in H \}. \]

\[ ^{10} \text{However, notice that we can add facts via composition on the left via } P \cup I = (1 \cup I)P \text{ (cf. 12).} \]
In analogy to the above construction, we can add body literals to Horn programs via composition on the right. For example, we have
\[
\{a \leftarrow b\} \circ \{b \leftarrow b, \neg c\} = \{a \leftarrow b, \neg c\}.
\]
Here, the general construction is as follows.

**Definition 27.** For any set of literals \(B\), define
\[
B^\oplus := \{a \leftarrow (\{a\} \cup B) \mid a \in A\}.
\]
Notice that \(\cdot^\oplus\) is computed with respect to some fixed alphabet \(A\).

For instance, we have
\[
A^\oplus = \{a \leftarrow A \mid a \in A\} \quad \text{and} \quad \emptyset^\oplus = 1.
\]
Interestingly enough, we have for any interpretation \(I\),
\[
I^\oplus I = I^\ominus \quad \text{and} \quad I^\ominus I = I.
\]
Moreover, in the example above, we have
\[
\{\neg c\}^\oplus = \begin{cases} a \leftarrow a, \neg c \\ b \leftarrow b, \neg c \\ c \leftarrow \neg c \end{cases} \quad \text{and} \quad \{a \leftarrow b\} \circ \{\neg c\}^\oplus = \{a \leftarrow b, \neg c\}
\]
as desired. As composition on the right does not affect the facts of a program, we cannot expect to append body literals to facts via composition on the right. However, we can add arbitrary literals to all proper rule bodies simultaneously and, in analogy to Proposition 26, we have the following general result.

**Proposition 28.** For any Horn program \(H\) and set of literals \(B\), we have
\[
HB^\oplus = \text{facts}(H) \cup \{\text{head}(r) \leftarrow (\text{body}(r) \cup B) \mid r \in \text{proper}(H)\}.
\]

**Proof.** See the proof of Proposition 14 in Antić (2021b).

The following example illustrates the interplay between the above concepts.

**Example 29.** Consider the Horn programs
\[
H = \begin{cases} c \\ a \leftarrow b, c \\ b \leftarrow a, c \end{cases} \quad \text{and} \quad \pi_{(a,b)} = \begin{cases} a \leftarrow b \\ b \leftarrow a \end{cases}.
\]
Roughly, we obtain \(H\) from \(\pi_{(a,b)}\) by adding the fact \(c\) to \(\pi_{(a,b)}\) and to each body rule in \(\pi_{(a,b)}\). Conversely, we obtain \(\pi_{(a,b)}\) from \(H\) by removing the fact \(c\) from \(H\) and by removing the body atom \(c\) from each rule in \(H\). This can be formalized as
\[
H = (\{c\}^* \pi_{(a,b)})\{c\}^\oplus \quad \text{and} \quad \pi_{(a,b)} = (1^{\{a,b\}}H)\{c\}^\ominus.
\]

\[\text{Here, we define } \{c\}^* := 1 \cup \{c\} \text{ which yields } \{c\}^* \pi_{(a,b)} = \pi_{(a,b)} \cup \{c\}; \text{ see the forthcoming equation } 41.\]
5.2.2. Negative Programs. Removing body literals from negative programs is similar to the Horn case above. For example, if we want to remove the literal \( \text{not } c \) from the rule body of \( a \leftarrow \text{not } b, \text{not } c \), we compute

\[
\{ a \leftarrow \text{not } b, \text{not } c \} \circ 1^{\{a,b,c\}-\{c\}} \circ \text{horn}(\{ a \leftarrow \text{not } b, \text{not } c \}) \circ \text{not } 1^{\{a,b\}} \]

\[
= \{ a \leftarrow b, c \} \circ \left\{ \begin{array}{l}
 a \leftarrow a \\
 b \leftarrow \text{not } b \\
 c
\end{array} \right\} \\
= \{ a \leftarrow \text{not } b \}.
\]

We have the following general result.

**Proposition 30.** For any negative program \( N \) and interpretation \( I \), we have

\[
N \circ 1^{A-I} = \{ \text{head}(r) \leftarrow (\text{body}(r) - \{ \text{not } a \mid a \in I \}) \mid r \in N \}.
\]

**Proof.** We compute

\[
N \circ 1^{A-I} \overset{(26)}{=} \text{horn}(N) \circ \text{not } 1^{A-I} \\
= \text{horn}(N) \circ (\{ a \leftarrow a \mid a \in A - I \} \cup I) \\
= \{ \text{head}(r) \leftarrow ((\text{body}(r) - I) \cup \{ \text{not } a \mid a \in (A - I) \cap \text{body}(r) \}) \mid r \in \text{horn}(N) \} \\
= \{ \text{head}(r) \leftarrow \{ \text{not } a \mid a \in (A - I) \cap \text{body}(r) \} \mid r \in \text{horn}(N) \} \\
= \{ \text{head}(r) \leftarrow (\text{body}(r) - \{ \text{not } a \mid a \in I \}) \mid r \in N \}.
\]

Adding literals to bodies of negative programs via composition is not possible as composition with negative rules yields disjunctions in rule bodies as is demonstrated by the following simple computation:

\[
\{ a \leftarrow \text{not } b \} \circ \{ b \leftarrow b,c \} = \left\{ \begin{array}{l}
 a \leftarrow \text{not } b \\
 a \leftarrow \text{not } c
\end{array} \right\}.
\]

5.2.3. Answer Set Programs. Unfortunately, systematically transforming the bodies of arbitrary programs requires more refined algebraic techniques in which the positive and negative parts of rules and programs can be manipulated separately, and we shall leave this problem as future work (cf. Section 8).

6. Algebraic Semantics

In this section, we reformulate the fixed point semantics of answer set programs in terms of composition without any explicit reference to operators. Our key observation is that the van Emde-Kowalski immediate consequence operator of a program can be algebraically represented via composition (Theorem 31), which implies an algebraic characterization of the answer set semantics (Theorem 37).
6.1. The van Emden-Kowalski Operator. Theorem 2 emphasizes the central role of the van Emden-Kowalski operator in answer set programming and the next result shows that it can be syntactically represented in terms of composition.

**Theorem 31.** For any answer set program \( P \) and interpretation \( I \), we have \( T_P(I) = P \circ I \).

**Proof.** We compute

\[
P \circ I = \begin{cases}
\text{head}(r) \leftarrow \text{body}(S,N) & r \in P \\
\text{S} \subseteq_{s2(\text{pos}(r))} I \\
\text{N} \subseteq_{s2(\text{neg}(r))} \text{not } I \\
\text{head}(S) = \text{body}(\text{pos}(r)) \\
\text{head}(N) = \text{body}(\text{horn}(\text{neg}(r)))
\end{cases}
\]

\[
= \{ \text{head}(r) \mid r \in P : \text{body}(\text{pos}(r)) \subseteq I, \text{body}(\text{horn}(\text{neg}(r))) \subseteq A - I \}
\]

\[
= \{ \text{head}(r) \mid r \in P : I = \text{body}(r) \}
\]

\[
= T_P(I)
\]

where the second equality follows from \( \text{not } I = A - I \) and \( \text{body}(S) = \text{body}(N) = \emptyset \).

\[\Box\]

As a direct consequence of Theorems 2 and 31, we have the following algebraic characterization of (supported) models in terms of composition.

**Corollary 32.** An interpretation \( I \) is a model of \( P \) iff \( P \circ I \subseteq I \), and \( I \) is a supported model of \( P \) iff \( P \circ I = I \).

**Corollary 33.** The space of all interpretations forms an ideal.

**Proof.** By Proposition 14, we know that the space of interpretations forms a right ideal and Theorem 31 implies that it is a left ideal—hence, it forms an ideal. \[\Box\]

**Corollary 34.** For any answer set program \( P \) and interpretation \( I \), we have \( P \circ I = I \circ P \) iff \( I \) is a supported model of \( P \).

**Proof.** A direct consequence of Proposition 14 and Theorem 31. \[\Box\]

6.2. Answer Sets. We interpret programs according to their answer set semantics and since answer sets can be constructively computed by bottom-up iterations of the associated van Emden-Kowalski operators (cf. Theorem 2), we can finally reformulate the fixed point semantics of answer set programs in terms of sequential composition (Theorem 37).

**Definition 35.** Define the Kleene star and plus of a Horn program \( H \) by

\[
H^* := \bigcup_{n \geq 0} H^n \quad \text{and} \quad H^+ := H^* H,
\]

where

\[
H^n := (((H \cdot H) \cdot H) \ldots H) \cdot H \quad (n \text{ times}).
\]

Moreover, define the omega operation by

\[
H^\omega := H^+ \circ \emptyset = \text{facts}(H^*).
\]
Notice that the unions in the computation of Kleene star are finite since \( H \) is finite. For instance, for any interpretation \( I \), we have as a consequence of Proposition \( 14 \)
\[
I^* = 1 \cup I \quad \text{and} \quad I^+ = I \quad \text{and} \quad I^\omega = I.
\]
Interestingly enough, we can add the atoms in \( I \) to \( P \) via
\[
P \cup I \overset{(24)}{=} P \cup IP \overset{(4)}{=} (1 \cup I)P \overset{(11)}{=} I^* P.
\]
Hence, as a consequence of \( (5) \) and \( (12) \), we can decompose \( P \) as
\[
P = \text{facts}(P)^* \circ \text{proper}(P),
\]
which, roughly, says that we can sequentially separate the facts from the proper rules in \( P \).

We have the following algebraic characterization of least models due to Antić (2021b).

**Theorem 36.** For any Horn program \( H \), we have
\[
\text{LM}(H) = H^\omega.
\]
Consequently, two Horn programs \( H \) and \( G \) are equivalent iff \( H^\omega = G^\omega \).

We have finally arrived at the following algebraic characterization of answer sets in terms of composition, cup, and union (cf. Theorem \( 24 \)).

**Theorem 37.** An interpretation \( I \) is an answer set of a program \( P \) iff \( I = (gP^I)^\omega \).

**Proof.** The interpretation \( I \) is an answer set of \( P \) iff \( I \) is the least model of \( gP^I \) iff \( I \) is the least fixed point of \( T_{gP^I} \) (Theorem \( 2 \)). By Theorem \( 31 \) we have
\[
T_{gP^I}(J) = gP^I \circ J \quad \text{for every interpretation} \ J.
\]
Hence, as \( gP^I \) is Horn, Theorem \( 36 \) implies that \( I \) is an answer set of \( P \) iff \( I = (gP^I)^\omega \). \( \square \)

**Corollary 38.** Two answer set programs \( P \) and \( R \) are strongly equivalent iff
\[
I = (gP^I \cup gQ^I)^\omega \iff I = (gR^I \cup gQ^I)^\omega
\]
holds for any interpretation \( I \) and program \( Q \).

**Proof.** A direct consequence of \( (37) \) and Theorem \( 37 \). \( \square \)

**Corollary 39.** Two answer set programs \( P \) and \( R \) are uniformly equivalent iff
\[
I = (g(J^* P)^I)^\omega \iff I = (g(J^* R)^I)^\omega \quad \text{for any interpretations} \ I \text{ and } J.
\]

**Proof.** A direct consequence of \( (42) \) and Theorem \( 37 \). \( \square \)

7. Future Work

In the future, we plan to extend the constructions and results of this paper to wider classes of answer set programs as, for example, higher-order programs (Miller & Nadathur, 2012) and disjunctive programs (Eiter et al., 1997). The former task is non-trivial since function symbols require most general unifiers in the definition of composition and give rise to infinite algebras, whereas disjunctive rules yield non-deterministic behavior which is more difficult to handle algebraically. Nonetheless, we expect interesting results to follow in all of the aforementioned cases.

From an artificial intelligence perspective, it is interesting to apply Antić’s (2022) abstract algebraic framework of analogical proportions to ASP magmas and algebras defined in this
paper for learning answer set programs via logic program proportions. More precisely, Antić (2022) provides a formal model for analogical proportions of the form \( a : b :: c : d \) in the generic setting of universal algebra, and proportions of the form \( P : Q :: R : S \) between programs provide a mechanism for deriving novel programs by analogy-making (Antić, 2021a). For this it will be of central importance to study (sequential) decompositions of various program classes. Specifically, we wish to compute decompositions of arbitrary answer set programs (and extensions thereof) into “prime” programs, where we expect permutations (Section 4.1.2) to play a fundamental role in such decompositions. For this, it will be necessary to resolve the issue of a “prime” or indecomposable answer set program. Algebraically, it will be of central importance to study Green’s relations (Green, 1951) in the finite ASP magmas and algebras introduced in this paper. From a practical point of view, a mathematically satisfactory theory of program decompositions is relevant to modular knowledge representation and optimization of reasoning.

Corollaries 38 and 39 are the entry point to an algebraization of the notions of strong (Lifschitz et al., 2001) and uniform equivalence (Eiter & Fink, 2003), related to Truszczyński’s (2006) operational characterization. This line of research is related to modular answer set programming (Oikarinen, 2006) and in the future we wish to express algebraic operations for modular program constructions within our framework.

Approximation Fixed Point Theory (AFT) (Denecker, Bruynooghe, & Vennekens, 2012; Denecker, Marek, & Truszczyński, 2004) is an operational framework based on Fitting’s (2002) work on fixed points in logic programming relating different semantics of logical formalisms with non-monotonic entailment (cf. Pelov, Denecker, & Bruynooghe, 2007; Antić, Eiter, & Fink, 2013; Antić, 2020). It is interesting to try to syntactically reformulate AFT within the ASP algebra introduced in this paper. For this, the first step will be to redefine approximations as pairs of programs instead of lattice elements satisfying certain conditions. In Section 5.2.3 we mentioned systematic algebraic transformations of arbitrary answer set programs for which the current tools are not sufficient. More precisely, manipulating rule bodies requires a finer separation of the positive and negative parts of rules and programs in the vein of approximations in “syntactic AFT” mentioned before. Finally, it is interesting to compare sequential composition to cascade products of answer set programs (Antić, 2014).

8. Conclusion

This paper contributed to the foundations of answer set programming and artificial intelligence by introducing and studying the sequential composition of answer set programs. We showed in our main structural result (Theorem 7) that the space of all programs forms a finite unital magma with respect to composition ordered by set inclusion, which distributes from the right over union. We called the magmas induced by sequential composition ASP magmas, and we called the algebras induced by sequential composition and union ASP algebras. Moreover, we showed that the restricted class of Krom programs is distributive and therefore its proper instance forms an idempotent semiring (Theorem 13). These results extended the results of Antić (2021b) from Horn to answer set programs. From a logical point of view, we obtained an algebraic meta-calculus for reasoning about answer set programs. Algebraically, we obtained a correspondence between answer set programs and finite magmas, which enables us to transfer algebraic concepts to the logical setting. In a broader sense, this paper is a further step towards an algebra of rule-based logical theories and we expect interesting concepts and results to follow.
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