Structure and generation of crossing-critical graphs

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Abstract
We study \( c \)-crossing-critical graphs, which are the minimal graphs that require at least \( c \) edge-crossings when drawn in the plane. For \( c = 1 \) there are only two such graphs without degree-2 vertices, \( K_5 \) and \( K_{3,3} \), but for any fixed \( c > 1 \) there exist infinitely many \( c \)-crossing-critical graphs. It has been previously shown that \( c \)-crossing-critical graphs have bounded path-width and contain only a bounded number of internally disjoint paths between any two vertices. We expand on these results, providing a more detailed description of the structure of crossing-critical graphs. On the way towards this description, we prove a new structural characterisation of plane graphs of bounded path-width. Then we show that every \( c \)-crossing-critical graph can be obtained from a \( c \)-crossing-critical graph of bounded size by replicating bounded-size parts that already appear in narrow “bands” or “fans” in the graph. This also gives an algorithm to generate all the \( c \)-crossing-critical graphs of at most given order \( n \) in polynomial time per each generated graph.

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1 Introduction
Minimizing the number of edge-crossings in a graph drawing in the plane (the crossing number of the graph, cf. Definition 2.1) is considered one of the most important attributes of a “nice drawing” of a graph, and this question has found numerous other applications (for example, in VLSI design [12] and in discrete geometry [18]). Consequently, a great deal of research work has been invested into understanding what forces the graph crossing number to be high. There exist strong quantitative lower bounds, such as the famous Crossing Lemma [1, 12]. However, the quantitative bounds show their strength typically in dense graphs, and hence they do not shed much light on the structural properties of graphs of high crossing number.

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The reasons for sparse graphs to have many crossings in any drawing are structural – there is a lot of “nonplanarity” in them. These reasons can be understood via corresponding minimal obstructions, the so called \(c\)-crossing-critical graphs (cf. Section 2 and Definition 2.2), which are the subgraph-minimal graphs that require at least \(c\) crossings. There are only two 1-crossing-critical graphs without degree-2 vertices, the Kuratowski graphs \(K_5\) and \(K_{3,3}\), but it has been known already since Sirán’s [19] and Kochol’s [11] constructions that the structure of \(c\)-crossing-critical graphs is quite rich and non-trivial for any \(c \geq 2\). Already the first nontrivial case of \(c = 2\) shows a dramatic increase in complexity of the problem. Yet, Bokal, Oporowski, Richter, and Salazar recently succeeded in obtaining a full description [3] of all the 2-crossing-critical graphs up to finitely many “small” exceptions.

To our current knowledge, there is no hope of extending the explicit description from [3] to any value \(c > 2\). We, instead, give for any fixed positive integer \(c\) an asymptotic structural description of all sufficiently large \(c\)-crossing-critical graphs.

Contribution outline. We refer to subsequent sections for the necessary formal concepts. On a high level of abstraction, our contribution can be summarized as follows:

1. There exist three kinds of local arrangements—a crossed band of uniform width, a twisted band, or a twisted fan—such that any optimal drawing of a sufficiently large \(c\)-crossing-critical graph contains at least one of them.
2. There are well-defined local operations (replacements) performed on such bands or fans that can reduce any sufficiently large \(c\)-crossing-critical graph to one of finitely many base \(c\)-crossing-critical graphs.
3. A converse—a well-defined bounded-size expansion operation—can be used to iteratively construct each \(c\)-crossing-critical graph from a \(c\)-crossing-critical graph of bounded size. This yields a way to enumerate all the \(c\)-crossing-critical graphs of at most given order \(n\) in polynomial time per each generated graph. More precisely, the total runtime is \(O(n)\) times the output size.

To give a closer (but still informal) explanation of these points, we should review some of the key prior results. First, the infinite 2-crossing-critical family of Kochol [11] explicitly showed one basic method of constructing crossing-critical graphs—take a sequence of suitable small planar graphs (called tiles, cf. Section 3), concatenate them naturally into a plane strip and join the ends of this strip with the Möbius twist. See Figure 1. Further constructions of this kind can be found, e.g., in [2, 13, 19]. In fact, [3] essentially claims that such a Möbius twist construction is the only possibility for \(c = 2\); there, the authors give an explicit list of 42 tiles which build in this way all the 2-crossing-critical graphs up to finitely many exceptions.

The second basic method of building crossing-critical graphs was invented later by Hliněný [9]; it can be roughly described as constructing a suitable planar strip whose ends are now joined without a twist (i.e., making a cylinder), and adding to it a few edges which then have to cross the strip. See again Figure 1 for an illustration. Furthermore, diverse
crossing-critical constructions can easily be combined together using so called \textit{zip product} operation of Bokal \cite{2} which preserves criticality. To complete the whole picture, there exists a third, somehow mysterious method of building \(c\)-crossing-critical graphs (for sufficiently high values of \(c\)), discovered by Dvořák and Mohar in \cite{5}. The latter can be seen as a degenerate case of the Möbius twist construction, such that the whole strip shares a central high-degree vertex, and we skip more details till the technical parts of this paper.

As we will see, the three above sketched construction methods roughly represent the three kinds of local arrangements mentioned in point (1). In a sense, we can thus claim that no other method (than the previous three) of constructing infinite families of \(c\)-crossing-critical graphs is possible, for any fixed \(c\). Moving on to point (2), we note that all three mentioned construction methods involve long (and also “thin”) planar strips, or \textit{bands} as subgraphs (which degenerate into \textit{fans} in the third kind of local arrangements; cf. Definition \ref{def:mapping}). We will prove, see Corollary \ref{cor:planarization}, that such a long and “thin” planar band or fan must exist in any sufficiently large \(c\)-crossing-critical graph, and we analyse its structure to identify elementary connected tiles of bounded size forming the band. We then argue that we can reduce repeated sections of the band while preserving \(c\)-crossing-criticality. Regarding point (3), the converse procedure giving a generic bounded-size expansion operation on \(c\)-crossing-critical graphs is described in Theorem \ref{thm:expansion} (for a quick illustration, the easiest case of such an expansion operation is edge subdivision, that is replacing an edge with a path, which clearly preserves \(c\)-crossing-criticality).

\textbf{Paper organization.} After giving the definitions and preliminary results about crossing-critical graphs in Section \ref{sec:graph-drawing}, we show a new structural characterisation of plane graphs of bounded path-width which forms the cornerstone of our paper in Section \ref{sec:structural-characterisation}. Then, in Section \ref{sec:structure-of-crossing-critical-graphs} we deal with the structure and reductions/expansions of crossing-critical graphs, presenting our main results. Some final remarks are presented in Section \ref{sec:final-remarks}.

\section{Graph drawing and the crossing number}

In this paper, we consider multigraphs by default, even though we could always subdivide parallel edges (with a slight adjustment of definitions) in order to make our graphs simple. We follow basic terminology of topological graph theory, see e.g. \cite{13}.

A \textit{drawing} of a graph \(G\) in the plane is such that the vertices of \(G\) are distinct points and the edges are simple curves joining their end vertices. It is required that no edge passes through a vertex, and no three edges cross in a common point. A \textit{crossing} is then an intersection point of two edges other than their common end. A drawing without crossings in the plane is called a \textit{plane drawing} of a graph, or shortly a \textit{plane graph}. A graph having a plane drawing is \textit{planar}.

The following are the core definitions of our research.

\textbf{Definition 2.1 (crossing number).} The \textit{crossing number} \(cr(G)\) of a graph \(G\) is the minimum number of crossings of edges in a drawing of \(G\) in the plane.

\textbf{Definition 2.2 (crossing-critical).} Let \(c\) be a positive integer. A graph \(G\) is \textit{\(c\)-crossing-critical} if \(cr(G) \geq c\), but every proper subgraph \(G'\) of \(G\) has \(cr(G') < c\).

Furthermore, suppose \(G\) is a graph drawn in the plane with crossings. Let \(G'\) be the plane graph obtained from this drawing by replacing the crossings with new vertices of degree 4. We say that \(G'\) is the plane graph associated with the drawing, shortly the \textit{planarization} of \(G\), and the new vertices are the \textit{crossing vertices} of \(G'\).
Preliminaries. Structural properties of crossing-critical graphs have been studied for more than two decades, and we now briefly review some of the previous important results which we shall use. First, we remark that a \( c \)-crossing-critical graph may have no drawing with only \( c \) crossings (examples exist already for \( c = 2 \)). Richter and Thomassen \cite{15} proved the following upper bound:

\[
\text{Theorem 2.3 (15). Every } c \text{-crossing-critical graph has a drawing with at most } \left\lceil \frac{5c}{2} + 16 \right\rceil \text{ crossings.}
\]

Interestingly, although the bound of Theorem 2.3 sounds rather weak and we do not know any concrete examples requiring more than \( c + O(\sqrt{c}) \) crossings, the upper bound has not been improved for more than two decades. We not only use this important upper bound, but also hope to be able to improve it in the future using our results.

Our approach to dealing with “long and thin” subgraphs in crossing-critical graphs relies on the folklore structural notion of path-width of a graph, which we recall in Definition 3.4. Hliněný \cite{7} proved that \( c \)-crossing-critical graphs have path-width bounded in terms of \( c \), and he and Salazar \cite{8} showed that \( c \)-crossing-critical graphs can contain only a bounded number of internally disjoint paths between any two vertices.

\[
\text{Theorem 2.4 (7). Every } c \text{-crossing-critical graph has path-width (cf. Definition 3.4) at most } \left\lceil \frac{2\cdot 72 \log_2 c + 248}{c^3 + 1} \right\rceil.
\]

Another useful concept for this work is that of nests in a drawing of a graph (cf. Definition 3.3), implicitly considered already in previous works \cite{7,8}, and explicitly defined by Hernandez-Velez et al. \cite{6} who concluded that no optimal drawing of a \( c \)-crossing-critical graph can contain a 0-, 1-, or 2-nest of large depth compared to \( c \).

Lastly, we remark that by trivial additivity of the crossing number over blocks, we may (and will) restrict our attention only to 2-connected crossing-critical graphs. We formally argue as follows. For \( c, \delta > 0 \), let us say a graph is \((c, \delta)\)-crossing-critical if it has crossing number exactly \( c \) and all proper subgraphs have crossing number at most \( c - \delta \).

\[
\text{Proposition 2.5 (folklore). A graph } H \text{ is } c \text{-crossing-critical if and only if there exist positive integers } c_1, \ldots, c_b \text{ and } \delta \text{ such that } c \leq c_1 + \cdots + c_b \leq c + \delta - 1, \text{ and } H \text{ is exactly } b \text{ 2-connected blocks } H_1, \ldots, H_b, \text{ and the block } H_i \text{ is } (c_i, \delta)\text{-crossing-critical for } i = 1, \ldots, b.
\]

Hence, strictly respecting Proposition 2.5, we should actually study 2-connected \((c, \delta)\)-crossing-critical graphs. To keep the presentation simpler, we stick with \( c \)-crossing-critical graphs, but we remark that our results also hold in the more refined setting.

3 Structure of plane tiles

The proof of our structural characterisation of crossing-critical graphs can be roughly divided into two main parts. The first one, presented in this section, establishes the existence of specific plane bands (resp. fans) and their tiles in crossing-critical graphs. The second part will then, in Section 4, closely analyse these bands and tiles. Unlike a more traditional “bottom-up” approach to tiles in crossing number research (e.g., \cite{3}), we define tiles and deal with them “top-down”, i.e., describing first plane bands or fans and then identifying tiles as their small elementary parts. Our key results are summarized below in Theorem 3.5 and Corollary 3.13.
Definition 3.1 (band and fan). Let $G$ be a 2-connected plane graph. Let $F_1$ and $F_2$ be distinct faces of $G$ and let $v_1, v_2, \ldots, v_m$, and $u_1, u_2, \ldots, u_m$ be some of the vertices incident with $F_1$ and $F_2$, respectively, listed in the cyclic order along the faces. If $P_1, \ldots, P_m$ are pairwise vertex-disjoint paths in $G$ such that $P_i$ joins $v_i$ with $u_{m+1-i}$, for $1 \leq i \leq m$, then we say that $(P_1, \ldots, P_m)$ forms an $(F_1,F_2)$-band of length $m$. Note that $P_i$ may consist of only one vertex $v_i = u_{m+1-i}$.

Let $F_1$ and $v_1, v_2, \ldots, v_m$ be as above. If $u$ is a vertex of $G$ and $P_1, \ldots, P_m$ are paths in $G$ such that $P_i$ joins $v_i$ with $u$, for $1 \leq i \leq m$, and the paths are pairwise vertex-disjoint except for their common end $u$, then we say that $(P_1, \ldots, P_m)$ forms an $(F_1,u)$-fan of length $m$. The $(F_1,u)$-fan is proper if $u$ is not incident with $F_1$.

Definition 3.2 (tiles and support). Let $(P_1, \ldots, P_m)$ be either an $(F_1,F_2)$-band or an $(F_1,u)$-fan of length $m \geq 3$. For $1 \leq i \leq m - 1$, let $\alpha_i$ be an arc between $v_i$ and $v_{i+1}$ drawn inside $F_1$, and let $\alpha'_i$ be an arc drawn between $u_i$ and $u_{i+1}$ in $F_2$ in the case of the band; $\alpha'_i$ are null when we are considering a fan. Furthermore, choose the arcs to be internally disjoint. Let $\theta_i$ be the closed curve consisting of $P_i$, $\alpha_i$, $P_{i+1}$, and $\alpha'_{m-i}$. Let $\lambda_i$ be the connected part of the plane minus $\theta_i$ that contains none of the paths $P_j$ ($1 \leq j \leq m$) in its interior.

The subgraphs of $G$ drawn in the closures of $\lambda_1, \ldots, \lambda_{m-1}$ are called tiles of the band or fan (and the tile of $\lambda_i$ includes $P_i \cup P_{i+1}$ by this definition). The union of these tiles is the support of the band or fan. The union of the arcs $\alpha_i$ is the $F_1$-span of the band or fan, and in the case of a band, the union of the arcs $\alpha'_i$ is the $F_2$-span of the band.

Definition 3.3 (nests). Let $G$ be a 2-connected plane graph. For an integer $k \geq 0$, a $k$-nest in $G$ of depth $m$ is a sequence $(C_1, C_2, \ldots, C_m)$ of pairwise edge-disjoint cycles such that for some set $K$ of $k$ vertices and for every $i < j$, the cycle $C_i$ is drawn in the closed disk bounded by $C_j$ and $V(C_i) \cap V(C_j) = K$.

Let $F$ be a face of $G$ and let $v_1, v_2, \ldots, v_{2m}$ be some of the vertices incident with $F$ listed in the cyclic order along the face. Let $P_1, \ldots, P_m$ be pairwise vertex-disjoint paths in $G$ such that $P_i$ joins $v_i$ with $v_{2m+1-i}$, for $1 \leq i \leq m$. Then, we say that $(P_1, \ldots, P_m)$ forms an $F$-nest of depth $m$. Similarly, let $v_1, v_2, \ldots, v_m, u$ be some of the vertices incident with $F$, let $P_1, \ldots, P_m$ be paths in $G$ such that $P_i$ joins $v_i$ with $u$, for $1 \leq i \leq m$, and the paths intersect only in $u$. Then, we say that $(P_1, \ldots, P_m)$ form a degenerate $F$-nest of depth $m$.

See Figure 3. Note that degenerate $F$-nests are the same as non-proper $(F,u)$-fans.
3.1 Plane graphs of bounded path-width

Our cornerstone claim, interesting on its own, is a structure theorem for plane graphs of bounded path-width. Before stating it, we recall the definition of path-width.

Definition 3.4 (path decomposition). A path decomposition of a graph $G$ is a pair $(P, \beta)$, where $P$ is a path and $\beta$ is a function that assigns subsets of $V(G)$, called bags, to nodes of $P$ such that

- for each edge $uv \in E(G)$, there exists $x \in V(P)$ such that $\{u, v\} \subseteq \beta(x)$, and
- for every $v \in V(G)$, the set $\{x \in V(P) : v \in \beta(x)\}$ induces a non-empty connected subpath of $P$.

The width of the decomposition is the maximum over all vertices $x$ of $P$, and the path-width of $G$ is the minimum width over all path decompositions of $G$.

Let $s$ denote the first node and $t$ the last node of $P$ in a path decomposition $(P, \beta)$. For $x \in V(P) \setminus \{s\}$, let $l(x)$ be the node of $P$ preceding $x$, and let $L(x) = \beta(l(x)) \cap \beta(x)$. For $x \in V(P) \setminus \{t\}$, let $r(x)$ be the node of $P$ following $x$, and let $R(x) = \beta(r(x)) \cap \beta(x)$. The path decomposition is proper if $\beta(x) \not\subseteq \beta(y)$ for all distinct $x, y \in V(P)$. The interior width of the decomposition is the maximum over $|\beta(x)| - 1$ over all nodes $x$ of $P$ distinct from $s$ and $t$. The path decomposition is $p$-linked if $|L(x)| = p$ for all $x \in V(P) \setminus \{s\}$ and $G$ contains $p$ vertex-disjoint paths from $R(s)$ to $L(t)$. The order of the decomposition is $|V(P)|$.

Theorem 3.5. Let $w$, $m$, and $k_0$ be non-negative integers, and $g : \mathbb{N} \to \mathbb{N}$ be an arbitrary non-decreasing function. There exist integers $w_0$ and $n_0$ such that the following holds. Let $G$ be a 2-connected plane graph and let $Y$ be a set of at most $k_0$ vertices of $G$ of degree at most 4. If $G$ has path-width at most $w$ and $|V(G)| \geq n_0$, then one of the following holds:

- $G$ contains a 0-nest, a 1-nest, a 2-nest, an $F$-nest, or a degenerate $F$-nest for some face $F$ of $G$, of depth $m$, and with all its cycles or paths disjoint from $Y$, or
- for some $w' \leq w_0$, $G$ contains an $(F_1, F_2)$-band or a proper $(F_1, u)$-fan (where $F_1$ and $F_2$ are distinct faces and $u$ is a vertex) of length at least $g(w')$ and with support disjoint from $Y$, such that each of its tiles has size at most $w'$.

We pay close attention to explaining Theorem 3.5 because of its great importance in this paper. Comparing it to Definition 3.4, one may think that there is not much difference—the bags $\beta(x)$ of a path decomposition of $G$ of width at most $w'$ might perhaps play the role of tiles of the band or fan in the second conclusion. Unfortunately, this simple idea is quite far from the truth. The subgraphs induced by the bags may not be “drawn locally”, that is, its edges may be geometrically far apart in the plane graph $G$. As an example, consider the width 2 path decomposition of a cycle where one of the vertices of the cycle appears in all the bags.

The main message of Theorem 3.5 thus is that in a plane graph of bounded path-width we can find a long band which is “drawn locally” and decomposes into well-defined small
and connected tiles (cf. Definition 3.2). Otherwise, such a graph must contain some kind of a deep nest or fan. However, as we will see in Corollary 3.13 the latter structures are impossible in the planarizations of optimal drawings of crossing-critical graphs.

The proof of Theorem 3.5 requires some preparatory work, and it uses tools of structural graph theory and of semigroup theory in algebra, which we present now.

**Lemma 3.6.** Let $a$ and $w$ be non-negative integers, and let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary non-decreasing function. There exist integers $w_0$ and $n_0$ such that the following holds. If a graph $G$ has a proper path decomposition of interior width at most $w$, adhesion at most $a$, and order at least $n_0$, then for some $w' \leq w_0$ and $p \leq a$, $G$ also has a $p$-linked proper path decomposition of interior width at most $w'$ and order at least $f(w')$.

**Proof.** Let $(P, \beta)$ be a proper path decomposition of $G$ of interior width at most $w$ and adhesion at most $a$. We prove the claim by induction on $a$. If $a = 0$, then $(P, \beta)$ is 0-linked, and thus the claim holds with $w_0 = w$ and $n_0 = f(w)$. Hence, assume that $a \geq 1$. Let $w'_0$ and $n'_0$ be the integers from the statement of the lemma for $a - 1$ and the interior width bounded by $2wf(w)$. Let $w_0 = \max(w'_0, w)$ and $n_0 = 2n'_0f(w)$.

We say that a node $x$ of $P$ distinct from its endpoints is unbroken if $|L(x)| = |R(x)| = a$ and $G[\beta(x)]$ contains a pairwise vertex-disjoint paths from $L(x)$ to $R(x)$, and broken otherwise. By Menger's theorem, if $x$ is broken, then there exist sets $A_x, B_x \subseteq \beta(x)$ such that $A_x \cup B_x = \beta(x)$, $L(x) \subseteq A_x$, $R(x) \subseteq B_x$, $|A_x \cap B_x| \leq a - 1$, and there is no edge from $A_x \setminus B_x$ to $B_x \setminus A_x$ in $G$. If $P$ contains a subpath $Q$ of $f(w)$ consecutive unbroken nodes, then the restriction of $(P, \beta)$ to $Q$ is an $a$-linked proper path decomposition of interior width at most $w$ and order at least $f(w)$. Otherwise, one of each $f(w)$ consecutive nodes of $P$ is broken, and thus $G$ has a proper path decomposition of interior width at most $2wf(w)$, adhesion at most $a - 1$, and order at least $n_0/(2f(w)) \geq n'_0$. Hence, the claim follows by the induction hypothesis. □

A crucial technical step in the proof of Theorem 3.5 is to analyse a topological structure of the bags of a path decomposition $(P, \beta)$ of a plane graph $G$, and to find many consecutive subpaths of $P$ on which the decomposition repeats the same “topological behavior”. For this we are going to model the bags of the decomposition $(P, \beta)$ as letters of a string over a suitable finite semigroup (these letters present an abstraction of the bags), and to apply the following algebraic tool, Lemma 3.7.

**Applying Simon’s factorisation forest.** Let $T$ be a rooted ordered tree (i.e., the order of children of each vertex is fixed). Let $f$ be a function that to each leaf of $T$ assigns a string of length 1, such that for each non-leaf vertex $v$ of $T$, $f(v)$ is the concatenation of the strings assigned by $f$ to the children of $v$ in order. We say that $(T, f)$ yields the string assigned to the root of $T$ by $f$. If the letters of the string are elements of a semigroup $A$, then for each $v \in V(T)$, let $f_A(v)$ denote the product of the letters of $f(v)$ in $A$. Recall that an element $e$ of $A$ is idempotent if $e^2 = e$. A tree $(T, f)$ is an $A$-factorization tree if for every vertex $v$ of $T$ with more than two children, there exists an idempotent element $e \in A$ such that $f_A(x) = e$ for each child $x$ of $v$ (and hence also $f_A(v) = e$). Simon [17] showed existence of bounded-depth $A$-factorization trees for every string; the improved bound in the following lemma was proved by Colcombet [4].

**Lemma 3.7.** For every finite semigroup $A$ and each string of elements of $A$, there exists an $A$-factorization tree of depth at most $3|A|$ yielding this string.

We will combine Lemma 3.7 with the following observation.
Lemma 3.8. Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an arbitrary non-decreasing function and let \( d \) be a positive integer. There exist integers \( k_0 \) and \( n_0 \) such that if \( T \) is a rooted tree of depth at most \( d \) with at least \( n_0 \) leaves, then for some \( k \leq k_0 \), there exists a vertex \( v \) of \( T \) that has at least \( f(k) \) children, and the subtree of \( T \) rooted at each child of \( v \) has at most \( k \) leaves.

Proof. We prove the claim by induction on \( d \). For \( d = 1 \), it suffices to set \( k_0 = 1 \) and \( n_0 = f(1) \). Suppose that \( d \geq 2 \) and the claim holds for \( d - 1 \), with \( k_0' \) and \( n_0' \) playing the role of \( k_0 \) and \( n_0 \). Let \( k_0 = \max(k_0', n_0') \) and \( n_0 = n_0'f(n_0') \). If the subtree rooted at some child of the root has at least \( n_0' \) leaves, then the claim follows by the induction hypothesis applied to this subtree. Otherwise, the root has at least \( n_0/n_0' \geq f(n_0') \) children, and the subtree rooted in each of them has at most \( n_0' \) leaves. Hence, we can let \( v \) be the root and \( k = n_0' \).

Combining Lemmas 3.7 and 3.8 we obtain the following.

Corollary 3.9. Let \( a \) be a non-negative integer and let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an arbitrary non-decreasing function. There exist integers \( k_0 \) and \( n_0 \) such that if \( A \) is a semigroup of order at most \( a \) and \( s \) is a string of elements of \( A \) of length at least \( n_0 \), then \( s \) is a concatenation of strings \( s_0, s_1, \ldots, s_m, s_{m+1} \) for some integer \( m \), such that

- for some \( k \leq k_0 \) with \( f(k) \leq m \), the strings \( s_1, \ldots, s_m \) have length at most \( k \) and
- the product of elements of \( A \) in each of the strings \( s_1, \ldots, s_m \) is the same idempotent element of \( A \).

We further need to formally define what we mean by a “topological behavior” of bags and subpaths of a path decomposition of our \( G \). This will be achieved by the following term of a \( q \)-type.

In this context we consider multigraphs (i.e., with parallel edges and loops allowed – each loop contributes 2 to degree of the incident vertex, and not necessarily connected) with some of its vertices labelled by distinct unique labels. A plane multigraph \( G \) is irreducible if \( G \) has no faces of size 1 or 2, and every unlabelled vertex of degree at most 2 is an isolated vertex incident with one loop (this loop, hence, cannot bound a 1-face).

Two plane multigraphs \( G_1 \) and \( G_2 \) with some of the vertices labelled are homeomorphic if there exists a homeomorphism \( \varphi \) of the plane mapping \( G_1 \) onto \( G_2 \) so that for each vertex \( v \in V(G_1) \), the vertex \( \varphi(v) \) is labelled iff \( v \) is, and then \( v \) and \( \varphi(v) \) have the same label. For \( G \) with some of its vertices labelled using the labels from a finite set \( \mathcal{L} \), the \( q \)-type of \( G \) is the set of all non-homeomorphic irreducible plane multigraphs labelled from \( \mathcal{L} \) and with at most \( q \) unlabelled vertices, and whose subdivisions are homeomorphic to subgraphs of \( G \).

Note that for every finite set of labels \( \mathcal{L} \) and every integer \( n \), there exist only finitely many irreducible non-homeomorphic plane multigraphs which are labelled with labels from \( \mathcal{L} \) and have at most \( q \) unlabelled vertices.

The definition of a \( q \)-type is going to be applied to graphs induced by subpaths of a path decomposition above (Lemma 3.6).

Let \( G \) be a plane graph and let \( (P, \beta) \) be its \( p \)-linked path decomposition. Let \( s \) and \( t \) be the endpoints of \( P \). Fix pairwise vertex-disjoint paths \( Q_1, \ldots, Q_p \) between \( R(s) \) and \( L(t) \). Consider a subpath \( P' \) of \( P - \{s, t\} \), and let \( G_{P'} \) be the subgraph of \( G \) induced by \( \bigcup_{x \in V(P')} \beta(x) \). If \( s' \) and \( t' \) are the (left and right) endpoints of \( P' \), we define \( L(P') = L(s') \) and \( R(P') = R(t') \). Let us label the vertices of \( G_{P'} \) using (some of) the labels \( \{l_1, \ldots, l_p, r_1, \ldots, r_p, c_1, \ldots, c_p \} \) as follows: For \( i = 1, \ldots, p \), let \( u \) and \( v \) be the vertices in which \( Q_i \) intersects \( L(P') \) and \( R(P') \), respectively. If \( u \neq v \), we give \( u \) the label \( l_i \) and \( v \) the label \( r_i \). Otherwise, we give \( u = v \) the label \( c_i \). For an integer \( q \), the \( q \)-type of \( P' \) is the \( q \)-type of \( G_{P'} \) with this labelling. If \( P' \) contains just one node \( x \), then we speak of the \( q \)-type of \( x \).
The \( q \)-types of subpaths of a linked path decomposition naturally form a semigroup with concatenation of the subpaths, as detailed next.

Let \( A \) be the set of \( q \)-types of subpaths of \( P - \{s,t\} \), together with a special element \( \epsilon \). Let \( \circ : A \times A \to A \) be defined as follows. If for \( t_1, t_2 \in A \setminus \{\epsilon\} \), there exist paths \( P_1, P_2 \subseteq P - \{s,t\} \) such that the first node of \( P_2 \) immediately follows the last node of \( P_1 \) in \( P \), the \( q \)-type of \( P_1 \) is \( t_1 \), and the \( q \)-type of \( P_2 \) is \( t_2 \), then \( t_1 \circ t_2 \) is defined as the \( q \)-type \( t \) of the path obtained from \( P_1 \) and \( P_2 \) by adding the edge of \( P \) joining them. For any other \( t_1, t_2 \in A \), we define \( t_1 \circ t_2 = \epsilon \).

\[ \text{Observation 3.10.} \quad (A, \circ) \text{ is a semigroup.} \]

\[ \text{Proof.} \quad \text{As associativity is obvious, it suffices to observe that if } P_1' \text{ and } P_2' \text{ are any other consecutive subpaths of } P \text{ with } q \text{-types } t_1 \text{ and } t_2, \text{ the path obtained by joining } P_1' \text{ with } P_2' \text{ via an edge of } P \text{ also has } q \text{-type } t. \quad \square \]

Applying Corollary 3.9, we have the following.

\[ \text{Corollary 3.11.} \quad \text{Let } f : \mathbb{N} \to \mathbb{N} \text{ be an arbitrary non-decreasing function and let } p \text{ and } q \text{ be non-negative integers. There exist integers } k_0 \text{ and } n_0 \text{ such that the following holds. Let } G \text{ be a plane graph and let } (P, \beta) \text{ be its } p \text{-linked path decomposition of order at least } n_0. \text{ Then } P \text{ can be split by removal of its edges into subpaths } P_0, P_1, \ldots, P_{m+1} \text{ in order for some integer } m, \text{ such that}
\]

- there exists \( k \leq k_0 \) with \( f(k) \leq m \) such that each of the paths \( P_1, \ldots, P_m \) has length at most \( k \), and
- the \( q \)-type of each of the paths \( P_1, \ldots, P_m \) is the same idempotent element in the semigroup \( (A, \circ) \).

\[ \text{Deconstructing plane graphs of bounded path-width.} \quad \text{A path decomposition } (P', \beta') \text{ of } G \text{ is a coarsening of } (P, \beta) \text{ if } P' = y_1 \ldots y_m \text{ and } P \text{ can be expressed as a concatenation of paths } P_1, \ldots, P_m \text{ such that } \beta'(y_i) = \bigcup_{x \in V(P_i)} \beta(x) \text{ for } i = 1, \ldots, m. \text{ For a subpath } Q \subseteq P, \text{ the restriction of the decomposition } (P, \beta) \text{ to } Q \text{ is the coarsening } (Q, \beta') \text{ of } (P, \beta) \text{ such that } \beta'(x) = \beta(x) \text{ for all nodes } x \text{ of } Q \text{ distinct from its endpoints.}
\]

If \( \Theta \) is a subgraph of a graph \( H \), a \( \Theta \)-bridge of \( H \) is either an edge of \( H \) not belonging to \( \Theta \) and with both ends in \( \Theta \), or a connected component of \( H - V(\Theta) \) together with all the edges from this component to \( \Theta \).

By first applying Lemma 3.6 (setting \( a = w \)), then using Corollary 3.11, and finally taking the coarsening of the decomposition according to the subpaths \( P_0, P_1, \ldots, P_{m+1} \), we finally obtain the desired:

\[ \text{Theorem 3.12.} \quad \text{Let } w \text{ and } q \text{ be non-negative integers, and let } f : \mathbb{N} \to \mathbb{N} \text{ be an arbitrary non-decreasing function. There exist integers } w_0 \text{ and } n_0 \text{ such that, for any plane graph } G \text{ that has a proper path decomposition of interior width at most } w \text{ and order at least } n_0, \text{ the following holds. For some } w' \leq w_0 \text{ and } p \leq w, G \text{ also has a } p \text{-linked proper path decomposition } (P, \beta) \text{ of interior width at most } w' \text{ and order at least } f(w'), \text{ such that for each node } x \text{ of } P \text{ distinct from its endpoints, the } q \text{-type of } x \text{ is the same idempotent element of the semigroup } (A, \circ). \]

In other words, we can find a decomposition in which all topological properties of the drawing that hold in one bag repeat in all the bags. So, for example, if for some node \( x \), the vertices of \( L(x) \) are separated in the drawing from vertices of \( R(x) \) by a cycle contained in the bag of \( x \), then this holds in every bag, and we conclude that the drawing contains
a large 0-nest. Other outcomes of Theorem 3.3 naturally correspond to other possible local properties of the drawings of the bags, and so we are ready to finish the main proof now.

Proof of Theorem 3.5. Let \( f(z) = \max(11, 4n + 5, 2g(3z) + 3)(2k_0 + 1) \), and let \( w_0' \) and \( n_0' \) be the corresponding integers from Theorem 3.12 applied with \( q = 1 \). Let \( w_0 = 3w_0' \) and \( n_0 = n_0'w_0' \).

Since \( G \) has path-width at most \( w \) and \( |V(G)| \geq n_0 \), \( G \) has a proper path decomposition of (interior) width at most \( w \) and order at least \( n_0/w = n_0' \). By Theorem 3.12 there exist integers \( w'' \leq w_0 \) and \( p \leq w \) such that \( G \) has a \( p \)-linked proper path decomposition \( (P, \beta) \) of interior width at most \( w'' \) and order at least \( f(w'') \), such that for each node \( x \) of \( P \) distinct from its endpoints, the 1-type of \( x \) is the same idempotent element. Note that the chosen labelling used to define the 1-type determines which vertices belong to \( L(x) \cap R(x) \); hence, there exists \( C \subseteq V(G) \) such that \( L(x) \cap R(x) = C \) for all nodes \( x \) of \( P \) distinct from its endpoints.

Since \( |Y| \leq k_0 \), there exists a restriction \( (P', \beta') \) of this path decomposition of order at least \( f(w'')/(2k_0 + 1) \), such that if \( s \) and \( t \) are the endpoints of \( P' \), all vertices of \( Y \) belong to \( \beta'(s) \cup \beta'(t) \). Let \( s' \) and \( t' \) be the neighbors of \( s \) and \( t \) in \( P' \), respectively, and let \( P'' = P' - \{s, t\} \) (so, \( P'' \) has ends \( s' \) and \( t' \)). For any \( x \in V(P'') \setminus \{s', t'\} \), we have \( \beta'(x) \cap Y \subseteq (L(s') \cap R(s')) \cup (L(t') \cap R(t')) = C \).

Let \( K_0 \) be a connected component of the graph \( G[\bigcup_{x \in V(P'')} - C] \). This graph is non-null, since the path decomposition \( (P', \beta') \) is proper. Let \( K \) be the induced subgraph of \( G \) consisting of \( K_0 \) and all vertices of \( C \) that have a neighbor in \( K_0 \). Let \( (P'', \beta'') \) be the path decomposition of \( K \) with \( \beta''(x) = \beta'(x) \cap V(K) \) for each \( x \in V(P'') \). Note that with respect to the drawing of \( K \) inherited from \( G \), all the nodes of \( P'' \) have the same idempotent 2-type. By idempotency, \( P'' \) has the same 2-type. Let \( x \in V(P'') \). Since \( K - C \) is connected, every vertex in \( \beta''(x) \) is connected in \( K[\beta''(x)] \) to \( S := L(x) \cup R(x) \). Since the 2-type of \( x \) is the same as the 2-type of \( P'' \), any two vertices of \( S \) are connected by a sequence of paths in \( K[\beta''(x)] - C \) with internal vertices in \( \beta''(x) \setminus S \), which implies that \( K[\beta''(x)] - C \) is connected. Similarly, each vertex of \( V(K) \cap C \) has a neighbor in \( K[\beta''(x)] - C \). Since the decomposition \( (P'', \beta'') \) has order at least \( f(w'')/(2k_0 + 1) - 5/2 \geq 9 \) and \( \beta''(x) \cap \beta''(y) \subseteq C \) for all non-consecutive nodes \( x \) and \( y \) of \( P'' \), it follows that all vertices in \( V(K) \cap C \) have degree at least 5 in \( G \), and thus they do not belong to \( Y \). Consequently, \( Y \cap \beta''(x) = \emptyset \) for all \( x \in V(P'') \setminus \{s', t'\} \).

If \( |V(K) \cap C| \geq 2 \), then since \( P'' \) has at least \( f(w'')/(2k_0 + 1) - 5/2 \geq 2m \) nodes \( x_1, \ldots, x_{2m} \in V(P'') \setminus \{s', t'\} \) forming an independent set in \( P'' \), we conclude that \( K \) contains \( 2m \) paths between any two vertices \( u, v \in V(K) \cap C \). The \( j \)-th path goes through \( \beta''(x_j) \setminus C \), so they are pairwise disjoint except for their endpoints, and disjoint from \( Y \). Such paths form a 2-nest of depth \( m \) in \( G \). Hence, we can assume that \( |V(K) \cap C| \leq 1 \).

Since \( G \) is 2-connected, it follows that \( L(s') \not\subseteq C \) and \( R(t') \not\subseteq C \).

Let \( s'' \) and \( t'' \) be the neighbors of \( s' \) and \( t' \) in \( P'' \), respectively. Consider a node \( x \) of \( P'' \) distinct from \( s'', t'', s', t' \), and \( t' \). Note that the subgraph \( K[\beta''(s')] - C \) is connected and vertex-disjoint from \( K[\beta''(x)] \). Let \( l_x \) denote the face of \( K[\beta''(x)] \) in which \( K[\beta''(s')] - C \) is drawn. Similarly, let \( r_x \) denote the face of \( K[\beta''(x)] \) in which \( K[\beta''(t')] - C \) is drawn. Suppose that \( l_x \neq r_x \). Then there exists a cycle \( C_x \) in the drawing of \( K[\beta''(x)] \) separating \( l_x \) from \( r_x \). Note that \( C_x \) separates \( L(s') \setminus C \) from \( R(t') \setminus C \) and is disjoint from both of these sets. The existence of such a separating cycle is determined by the 1-type of \( P'' \), and by the idempotency, \( x \) has the same 1-type. Consequently, we can actually choose \( C_x \) to be disjoint from \( L(x) \setminus C \) and \( R(x) \setminus C \) (and thus to separate these sets). But then the cycles \( C_x \) for
We now continue with an application of Theorem 3.5 in the study of crossing-critical graph structure, as a strengthening of Theorem 2.4.

\textbf{Corollary 3.13.} Let \( c \) be a positive integer, and let \( g : \mathbb{N} \to \mathbb{N} \) be an arbitrary non-decreasing function. There exist integers \( w_0 \) and \( n_0 \) such that the following holds. Let \( G \) be a 2-connected \( c \)-crossing-critical graph, and let \( G' \) be the planarization of a drawing of \( G \) with the smallest number of crossings. Let \( Y \) denote the set of crossing vertices of \( G' \). If \( |V(G)| \geq n_0 \), then \( G' \) contains an \((F_1,F_2)\)-band or a proper \((F_1,u)\)-fan for some distinct faces \( F_1 \) and \( F_2 \) or a vertex \( u \), such that for some \( w' \leq w_0 \), all the tiles of the band or fan have size at most \( w' \) and are disjoint from \( Y \), and the length of the band or fan is at least \( g(w') \).
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Proof. Let $k_0 = [5c/2 + 16]$, $w = 2^{(72 \log_2 c + 248)c^3 + 1} + k_0$ and $m = 15c^2 + 105c + 17$. Let $w_0$ and $w_0$ be the corresponding integers from Theorem 3.5.

By Theorem 3.14, each $c$-crossing-critical graph has a drawing with at most $k_0$ crossings, and thus $|Y| \leq k_0$. By Theorem 3.3, $G$ has path-width at most $w - k_0$, and thus $G'$ has path-width at most $w$. Hliněný and Salazar [8] and Hernandez-Velez et al. [6] proved the graph $G'$ obtained from a $c$-crossing-critical graph $G$ as described does not contain a 0-, 1-, and 2-nests of depth $m$ with cycles disjoint from $Y$. Lemmas 3.14 and 3.15 (presented next) imply that no face $F$ of $G'$ has an $F$-nest or a degenerate $F$-nest of depth $m$ with paths disjoint from $Y$.

Furthermore, note that $G'$ is 2-connected—$G$ is 2-connected, and vertices of $Y$ cannot form 1-cuts in $G'$, as otherwise it would be possible to obtain a drawing of $G$ with fewer crossings. Hence, Corollary 3.14 is implied by Theorem 3.5.

Lemma 3.14. Let $G$ be a 2-connected $c$-crossing-critical graph drawn in the plane and $F$ its face. Then every $F$-nest of $G$ has depth at most $15c^2 + 105c + 16$.

Proof. Let $k_0 = [5c/2 + 16]$. Let $(P_1, \ldots, P_t)$ be an $F$-nest. We may assume that $F$ is the outer face of the embedding. Let $a$ and $b$ be indices such that the paths $P_a$ and $P_b$ contain no crossed edge, the subgraph of $G$ drawn between $P_a$ and $P_b$ contains no crossings and $q = b - a + 1$ is maximum. Since $G$ is 2-connected, we conclude that $t \leq k_0(q + 1) + q$.

For $a + 1 \leq i \leq b - 2$, let $G_i$ be the maximal 2-connected subgraph of $G - V(P_{i-1} \cup P_{i+2})$ that contains $P_i \cup P_{i+1}$; this graph exists, since $G$ is 2-connected. Furthermore, it is easy to see that $G - V(G_i)$ has exactly two components. Let $C_i$ be the cycle bounding the outer face of $G_i$. Let $m$ be maximal such that $6m + 1 \leq q$; then, $q \leq 6m + 6$. If $m \leq c - 1$, then we obtain $t \leq 2.5 \cdot 6c^2 + 17 \cdot 6c + [2.5c] + 16 \leq 15c^2 + 105c + 16$, as required. For contradiction assume that $m \geq c$. Consider the cycles $C_{a+3i-2}$ for $1 \leq i \leq 2m$ (since $6m + 1 \leq q$, these cycles exist). Let $e$ be an edge of $P_{a+3m}$. Since $G$ is crossing-critical, $G - e$ has a drawing with at most $c - 1$ crossings, and thus there exist indices $x \leq m$ and $y \geq m + 1$ such that the edges of $K_1 = C_{x+3x-2}$ and of $K_2 = C_{y+3y-2}$ are not crossed in this drawing.

Let $H_1$ be the subgraph of $G$ consisting of $K_1$ and the component $Z_1$ of $G - V(G_{a+3x-2})$ that does not contain $e$ and of the edges between them. Let $H_2$ be the subgraph of $G$ consisting of $K_2$ and the component $Z_2$ of $G - V(G_{a+3y-2})$ that does not contain $e$ and of the edges between them. Let $K$ be the cycle consisting of a path in $K_1$, a path in $K_2$ and of two subpaths of the boundary of $F$ whose interior in the drawing of $G$ is disjoint from $K_1 \cup K_2$. Consider the drawings of $H_1$ and $H_2$ induced by the drawing of $G - e$. For $i \in \{1, 2\}$, since $Z_i$ is connected, we can assume that it is drawn outside of $K_i$ in $H_i$, and furthermore, that the path $K \cap K_i$ is incident with the outer face of $H_i$.

Denote by $\bar{K}_1$ and $\bar{K}_2$ the subdrawings induced by the drawing of $G$ in the regions bounded by the closed simple curves $K_1$ and $K_2$, and by $\bar{K}$ the subdrawing induced by the drawing of $G$ in the region bounded by $K$. The natural composition of $H_1$, $H_2$, $\bar{K}_1$, $\bar{K}_2$ and $\bar{K}$ is a drawing of $G$ such that each crossing belongs to $H_1$ or $H_2$. This drawing has at most $c - 1$ crossings, which is a contradiction.

Lemma 3.15. Let $G$ be a 2-connected $c$-crossing-critical graph drawn in the plane and $F$ its face. Then every degenerate $F$-nest of $G$ has depth at most $15c^2 + 105c + 16$.

Proof. Let $(P_1, \ldots, P_t)$ be a degenerate $F$-nest such that all the paths $P_1, \ldots, P_t$ share a common vertex $u$ incident lemma-dwith $F$. We can follow exactly the same proof as in Lemma 3.14 except at the following two points. First, $G_i$ is defined as the maximal
2-connected subgraph of $G - (V(P_{i-1} \cup P_{i+2}) \setminus \{u\})$ that contains $P_i \cup P_{i+1}$. Then, again, $G - V(G)$ has exactly two components. Second, $K$ is defined as the cycle consisting of a path in $K_1$ and a path in $K_2$, both starting in $u$, and of one subpath of the boundary of $F$. The rest of the proof follows.

4 Removing and inserting tiles

In the second part of the paper, we study an arrangement of bounded tiles in a long enough plane band or fan (as described by Corollary 3.13), focusing on finding repeated subsequences which then could be shortened. Importantly, this shortening preserves $c$-crossing-criticality. In the opposite direction we then manage to define the converse operation of “expansion” of a plane band which also preserves $c$-crossing-criticality. These findings will imply the final outcome—a construction of all $c$-crossing-critical graphs from an implicit list of base graphs of bounded size. The formal statement can be found in Theorem 4.15.

Again, we start with a few relevant technical terms. Recall Definition 3.1.

Definition 4.1 (subband, necklace and shelled band). Let $\mathcal{P} = (P_1, \ldots, P_n)$ be an $(F_1, F_2)$-band or an $(F_1, u)$-fan in a 2-connected plane graph. A subband or subfan consists of a contiguous subinterval $(P_i, P_{i+1}, \ldots, P_j)$ of the band or fan (and its support is a subset of the support of the original band or fan).

We say that the band $\mathcal{P}$ is a necklace if each of its paths consists of exactly one vertex. A tile (cf. Definition 3.2) of the band or fan $\mathcal{P}$ is shelled if it is bounded by a cycle, consisting of two consecutive paths $P_i$ and $P_{i+1}$ of $\mathcal{P}$ and parts of the boundary of $F_1$ and $F_2$ (respectively, $u$), and the two paths $P_i, P_{i+1}$ delimiting the tile have at least two vertices each. The band or fan $\mathcal{P}$ is shelled if each of its tiles is shelled. See Figure 4.

One can easily show that, regarding the outcome of Corollary 3.13, there are only the following two refined subcases that have to be considered in further analysis:

Lemma 4.2. Let $w$ be a positive integer and $f: \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary non-decreasing function. There exist integers $n_0$ and $w'$ such that the following holds. Let $G$ be a 2-connected plane graph, and let $\mathcal{P} = (P_1, \ldots, P_m)$ be an $(F_1, F_2)$-band or a proper $(F_1, u)$-fan in $G$ of length $m \geq n_0$, with all tiles of size at most $w$. Then either $G$ contains a shelled subband or subfan of $\mathcal{P}$ of length $f(w)$, or $G$ contains a necklace of length $f(w')$ with tiles of size at most $w'$ whose support is contained in the support of $\mathcal{P}$.

Proof. If $\mathcal{P}$ is a proper $(F_1, u)$-fan, then since $u$ is not incident with $F_1$, we conclude that $\mathcal{P}$ is shelled. Thus, taking $n_0 \geq f(w)$ will work. Hence, we may assume that $\mathcal{P}$ is an $(F_1, F_2)$-band. Let $w' = w(2f(w) - 1)$ and $n_0 = \max(2f(w), f(w')f(w))$. For $i = 1, \ldots, \max(2, f(w'))$, let $\mathcal{P}_i$ be the subband of $\mathcal{P}$ between the paths $P_{i-1}f(w)+1$ and $P_{i}f(w)$. If one of the subbands...
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$\mathcal{P}_1, \ldots, \mathcal{P}_{f(w')}$ is shelled, the claim of the lemma holds. Otherwise, for $i = 1, \ldots, f(w')$, there exists a common vertex $v_i$ of the $F_1$-span and the $F_2$-span of $\mathcal{P}_i$. But then $v_1, \ldots, v_{f(w')}$ is a necklace of length $f(w')$ with support contained in the support of $\mathcal{P}$. Each of its tiles is contained in the union of two of the subbands and the tile separating them, and thus its size is at most $w'$.

4.1 Reducing a necklace

Among the two subcases left by Lemma 4.2, the easier one is that of a necklace which can be reduced simply to a bunch of parallel edges; see also Figure 6.

Lemma 4.3. Let $c$ be a non-negative integer. Let $G$ be a 2-connected $c$-crossing-critical graph, and let $G'$ be the planarization of a drawing of $G$ with the smallest number of crossings. Let $Y$ denote the set of crossing vertices of $G'$. Suppose that $\mathcal{P} = (v_1, \ldots, v_m)$, where $m \geq 2$, is a necklace in $G'$ whose support is disjoint from $Y$. Then for some $p \leq c$, the support of $\mathcal{P}$ consists of $p$ pairwise edge-disjoint paths from $v_1$ to $v_m$. Furthermore, the graph $G_0$ obtained from $G$ by removing the support of $\mathcal{P}$ except for $v_1$ and $v_m$ and by adding $p$ parallel edges between $v_1$ and $v_m$ is $c$-crossing-critical.

Proof. Let $G_1$ denote the subgraph of $G$ obtained by removing the support of $\mathcal{P}$ except for $v_1$ and $v_m$. Let $p$ be the maximum number of pairwise edge-disjoint paths from $v_1$ to $v_m$ in the support $S$ of $\mathcal{P}$. Suppose for a contradiction that either $p \geq c + 1$ or some edge $e$ of $S$ is not contained in an edge-cut of size $p$ separating $v_1$ from $v_m$. In the former case, let $e$ be an arbitrary edge of $S$. Let $c = 0$ if $p \geq c + 1$ and $q = p$ otherwise.

By criticality of $G$, the graph $G - e$ can be drawn in the plane with at most $c - 1$ crossings. Consider the drawing of $G_1$ induced by this drawing, and let $a$ be the minimum number of edges that have to be crossed by any curve in the plane from $v_1$ to $v_m$ and otherwise disjoint from $V(G_1)$. Note that $a \geq 1$, since otherwise we could draw $S$ without crossings between $v_1$ and $v_m$, obtaining a drawing of $G$ with fewer than $c$ crossings. Since $G - e$ contains $q$ pairwise edge-disjoint paths from $v_1$ to $v_m$ which are not contained in $G_1$, we conclude that $cr(G - e) \geq cr(G_1) + aq \geq q$. Since $cr(G - e) < c$, we have $q < c$. It follows that $q = p$ and $cr(G_1) < c - ap$. However, $S$ contains an edge-cut $C$ of order $p$ separating $v_1$ from $v_m$ by Menger's theorem, and we can add $S$ to the drawing of $G_1$ so that exactly the edges of $C$ are crossed, and each of them exactly $a$ times (by drawing the part of $S$ between $v_1$ and $C$ close to $v_1$, and the part of $S$ between $v_m$ and $C$ close to $v_m$). This way, we obtain a drawing of $G$ with $cr(G_1) + ap < c$ crossings. This is a contradiction, which shows that $p \leq c$ and that $S$ is the union of $p$ edge-disjoint paths from $v_1$ to $v_m$.

Any drawing of $G_0$ can be transformed into a drawing of $G$ with at most as many crossings in the same way as described in the previous paragraph. Thus $cr(G_0) \leq c$. Consider now any edge $e_0$ of $G_0$. If $e_0$ is one of the parallel edges between $v_1$ and $v_m$, then let $e'$ be any edge of $S$ and $p' = p - 1$, otherwise let $e' = e_0$ and $p' = p$. By the $c$-crossing-criticality of $G$, there exists a drawing of $G - e'$ with less than $c$ crossings. Consider the induced drawing of $G_1 - e'$, and let $a'$ denote the minimum number of edges in this drawing that have to be crossed by any curve in the plane from $v_1$ to $v_m$ and otherwise disjoint from $V(G_1)$. Since $S - e'$ contains $p'$ edge-disjoint paths from $v_1$ to $v_m$, we conclude that $cr(G - e') \geq cr(G_1 - e') + a'p'$. We can add $p'$ edges between $v_1$ and $v_m$ to the drawing of $G_1 - e'$ to form a drawing of $G_0$ with at most $cr(G_1 - e') + a'p' \leq cr(G - e') < c$ crossings. Consequently, $G_0$ is $c$-crossing-critical.

Observe that replacing a parallel edge of multiplicity $p$ between vertices $u$ and $v$ in a $c$-crossing-critical graph with any set of $p$ edge-disjoint plane paths from $u$ to $v$ gives another
4.2 Reducing a shelled band or fan

If we could follow the same proof scheme as with necklaces also in the remaining cases of shelled bands and fans, then we would already reach the final goal. Unfortunately, the latter cases are more involved, and require some preparatory work. Compared to the easier case of a necklace, the important difference in the case of a shelled band comes from the fact that the band may be drawn not only in the “straight way” but also in the “twisted way” (recall Figure 1). An indication that this is troublesome comes from the result of Hliněný and Derňár [10], who showed that determining the crossing number of a twisted planar tile is NP-complete (and thus it is not determined by a simple parameter such as the number of edge-disjoint paths between its sides). Consequently, the analysis of shelled bands is significantly more complicated than the relatively straightforward proof of Lemma 4.3. The same remark applies to the shelled fans.

Before we dive into technical details needed to at least formulate the final result, Theorem 4.15, we present an informal outline of our approach:

1. Having a very long shelled band $\mathcal{P}$ in our graph $G$, it is easy to see that the isomorphism types of bounded-size tiles in $\mathcal{P}$ must repeat. Moreover, even bounded-length subbands must have isomorphic repetitions. The first idea is to shorten the band between such repeated isomorphic subbands $\mathcal{P}_1$ and $\mathcal{P}_2$—by identifying the repeated pieces and discarding what was between (cf. Definition 4.8). If the repeated subband is long enough, we can use some rather easy connectivity properties of $\mathcal{P}$ to show that this yields a smaller graph $G_1$ of crossing number at least $c$.

2. Though, it is not clear that the reduced graph $G_1$ is $c$-crossing-critical. Analogously to Lemma 4.3, for any edge $e \in E(G_1)$, we would like to transform a drawing of $G - e$ with less than $c$ crossings to a drawing of $G_1 - e$ with less than $c$ crossings. However, if the drawing of $G - e$ uses some unique properties of the part $\mathcal{P}_{12}$ of the band between $\mathcal{P}_1$ and $\mathcal{P}_2$, we have no way how to mimic this in the drawing of $G_1 - e$ (this is especially troublesome if this part of $G - e$ is drawn in a twisted way, since there is no easy description of what these “unique properties” might be by the NP-completeness result [10]).

We overcome this difficulty by performing the described reduction only inside longer pieces which repeat elsewhere in the band (cf. Definition 4.12). Hence, in $G_1 - e$ we have many copies of $\mathcal{P}_{12}$, and by appropriate surgery, we can use one of them to mimic the drawing of $\mathcal{P}_{12}$ in $G - e$.

3. A further advantage of reducing within parts that repeat elsewhere is that we can more explicitly describe the converse expansion operation, as duplicating subbands which already exist elsewhere in the (reduced) band.
Figure 6 A scheme of a reducible subband $P'$ (in grey) with repetition $(P_1, P_2)$ of order 3 (darker grey), as in Definition 4.8 and the result of the reduction on $P'$ (on the right).

Let us remark that considering a shelled $(F, u)$-fan instead of a band is not different, all the arguments simply carry over. The following additional definitions are needed to formalize the outlined claims.

Let $P = (P_1, \ldots, P_m)$ be an $(F_1, F_2)$-band or an $(F_1, F)$-fan in a 2-connected plane graph $G$, and let $T_i$ be the tile of $P$ delimited by $P_i$ and $P_{i+1}$. We say that the band $P$ is $k$-edge-linked if $k \in \mathbb{N}$ and there exist $k$ pairwise edge-disjoint paths from $V(P_1)$ to $V(P_m)$ contained in the support of $P$, and for each $i = 1, \ldots, m-1$, the tile $T_i$ contains an edge-cut of size $k$ separating $V(P_1)$ from $V(P_{i+1})$.

Similarly, the fan $P$ is $k$-edge-linked if there exist $k$ pairwise edge-disjoint paths from $V(P_1) \setminus \{u\}$ to $V(P_m) \setminus \{u\}$ contained in the support of $P$ minus $u$, and for each $i = 1, \ldots, m-1$, the sub-tile $T_i - u$ contains an edge-cut of size $k$ separating $V(P_1) \setminus \{u\}$ from $V(P_{i+1}) \setminus \{u\}$. For a closer explanation, one may say that, modulo a trivial adjustment, the fan $P$ is $k$-edge-linked iff the corresponding band in $G - u$ is $k$-edge-linked.

Properties of shelled bands and fans. We now proceed with the (rather long) preparatory work towards handling shelled bands and fans in further Lemma 4.13 and Theorem 4.15.

Lemma 4.4. Let $w$ and $c$ be positive integers and $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary function. There exist integers $n_0$ and $w_0$ as follows. Let $G$ be a 2-connected plane multigraph in which each edge has multiplicity at most $c$, and let $P = (P_1, \ldots, P_m)$ be a shelled $(F_1, F_2)$-band or $(F_1, F)$-fan in $G$ with $m \geq n_0$, with all tiles of size at most $w$. Then for some $w' \leq w_0$ and $k \leq 3cw$, $G$ contains a $k$-edge-linked shelled $(F_1, F_2)$-band or $(F_1, F)$-fan of length $f(k, w')$ with tiles of size at most $w'$, whose support is contained in the support of $P$.

Proof. Let $k_0 = 3cw$. Let $w_{k_0} = w$ and $n_{k_0} = f(k_0, w)$, and for $i = k_0 - 1, \ldots, 0$, let $w_i = (n_{i+1} - 1)w$ and $n_i = \max(n_{i+1}, (f(i, w_i) - 1)(n_{i+1} - 2))$.

Assume first that $P$ is an $(F_1, F_2)$-band. For $1 \leq i < j \leq m$, let $P_{i,j}$ denote the subband of $P$ between $P_i$ and $P_j$, and let $p_{i,j}$ denote the maximum number of pairwise edge-disjoint paths from $V(P_i)$ to $V(P_j)$ in the support of $P_{i,j}$. Note that if $i \leq i' < j' \leq j$, then $p_{i',j'} \geq p_{i,j}$. Furthermore, each path from $V(P_1)$ to $V(P_j)$ in the support of $P_{i,j}$ contains an edge of the tile between $P_i$ and $P_{i+1}$, and there are at most $3cw$ such edges (since the tiles are planar and the factor $c$ comes from the maximum multiplicity of edges), and thus $p_{i,j} \leq k_0$. Let $k$ be the largest integer such that $p_{i,j} = k$ for some $1 \leq i < j \leq m$ satisfying $j \geq i + n_k - 1$ (such $k$ exists, since $m \geq n_0$ and $n_0 \geq n_1 \geq \cdots \geq n_{k_0}$).

If $k = k_0$, then the claim of the lemma holds with $w' = w$ and a subband of $P$ between $P_i$ and $P_j$; hence, assume that $k < k_0$. Let $i_t = i$, and for $t = 2, \ldots, f(k, w_k)$, let $i_t$ be the minimum index greater than $i_{t-1}$ such that $p_{i_{t-1},i_t} = k$. Note that $i_1 \leq i_{t-1} + n_{k+1} - 2$ by the maximality of $k$ and the fact that for $i \leq i' < j \leq j$ we have $p_{i',j} \geq k_0$. In particular, $i_{f(k, w_k)} \leq i + (f(k, w_k) - 1)(n_{k+1} - 2) \leq j$. Consider the $(F_1, F_2)$ band $P'$ formed by the
paths $P_1, \ldots, P_{t_j(w_k)}$. The tiles of this band have size at most $(n_{k+1} - 1)w = w_k$. Since the $F_1$- and $F_2$-spans of this band are subpaths of the $F_1$- and $F_2$-spans of $\mathcal{P}$, the band $\mathcal{P}'$ is shelled. Each tile of $\mathcal{P}'$ contains an edge-cut of size $k$ separating the paths bounding it, and since $p_{i,j(w_k)} = k$, the support of $\mathcal{P}'$ contains $k$ pairwise edge-disjoint paths from $V(P_{i,1})$ to $V(P_{i,j(w_k)})$; i.e., $\mathcal{P}'$ is $k$-edge-linked. Hence, $k$, $w' = w_k$ and $\mathcal{P}'$ satisfy the claim of the lemma.

Assume now that $\mathcal{P}$ is an $(F_1, u)$-fan. This case can be handled analogously to the previous case by considering pairwise edge-disjoint paths in the support of $\mathcal{P}$ minus the vertex $u$.

Let $\mathcal{P} = (P_1, \ldots, P_m)$ be an $(F_1, F_2)$-band or an $(F_1, u)$-fan in a 2-connected plane graph $G$. Let $U = \{u\}$ if $\mathcal{P}$ is a fan and $U = \emptyset$ otherwise. For $2 \leq i \leq m - 2$, let $T_i$ be the tile of $\mathcal{P}$ delimited by $P_i$ and $P_{i+1}$. If $T_i$ is shelled, then we define the extended tile $X(T_i)$ of $T_i$ to be the maximal 2-connected subgraph $T$ of $G - (\cup_{i=1}^m V(P_{i,1})) \cup U)$ containing $P_i \cup P_{i+1}$. We define $C(T_i)$ to be the cycle of $X(T_i)$ bounding its face (in the drawing induced by $G$) containing $F_1$ as a subset, and we say that $C(T_i)$ bounds the extended tile. The subgraph $H$ of $G$ obtained by removing all vertices and edges of $X(T_i)$ not belonging to $C(T_i)$ is the complement of the extended tile. In case $G$ is the planarization of a drawing of some graph $G_0$ with crossings, and the crossing vertices of $G$ are disjoint from $X(T_i)$, let $H_0$ be the subgraph of $G_0$ obtained from $H$ by turning the vertices of $Y$ back into crossings; in this case, we also say that $H_0$ is the complement of the extended tile in $G_0$. We say that another drawing $G_1$ of $G_0$ is $T_i$-flat if $X(T)$ is drawn in the closed disk bounded by $C(T)$ in a way homeomorphic to the way it is drawn in $G_0$ and the complement of $X(T)$ is drawn in the unbounded face of $C(T)$. For a set $\mathcal{T}$ of tiles of $\mathcal{P}$, we say that a drawing is $\mathcal{T}$-flat if it is $T$-flat for all $T \in \mathcal{T}$.

**Lemma 4.5.** Let $G$ be a 2-connected graph drawn in the plane with crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Suppose that $G'$ is 2-connected, and let $\mathcal{P}$ be an $(F_1, F_2)$-band or an $(F_1, u)$-fan in $G'$ whose support is disjoint from $Y$. Let $T$ be a shelled tile of $\mathcal{P}$. Let $G_1$ be another drawing of $G$ with crossings, such that no edge of $C(T)$ is crossed. Then there exists a $T$-flat drawing $G_2$ of $G$ such that the $X(T)$-bridges of $G$ are drawn in the unbounded face of $C(T)$ in a way homeomorphic to the way they are drawn in $G_1$, and the drawings of distinct $X(T)$-bridges are disjoint except for their possible intersection in $C(T)$ (and in particular, $cr(G_2) \leq cr(G_1)$ and none of the edges of $X(T)$ is crossed).

**Proof.** Let $P_1, P_2, P_3$, and $P_4$ be consecutive paths of $\mathcal{P}$ such that $T$ is delimited by $P_2$ and $P_3$. Consider any $X(T)$-bridge $B$ of $G$. If $B$ contains neither $P_1$ nor $P_4$, then $B$ intersects $T$ in at most one vertex by the maximality of $X(T)$, contradicting the assumption that $G$ is 2-connected. Hence, each $X(T)$-bridge of $G$ contains $P_1$ or $P_4$, and thus there either is only one such $X(T)$-bridge, or two $X(T)$-bridges attaching to subpaths of $C(T)$ that are disjoint except possibly for $u$ when $\mathcal{P}$ is a fan. To obtain $G_2$, draw $X(T)$ in the disk bounded by $C(T)$ homeomorphically to the way it is drawn in $G$, apply circle inversion to the drawings of $X(T)$-bridges from $G_1$, if necessary, so that their intersections with $C(T)$ are incident with their outer face, and shift and distort them so that they attach disjointly (except possibly for $u$) to the appropriate subpath of $C(T)$.

Applying Lemma 4.5 repeatedly, we obtain the following.

**Corollary 4.6.** Let $G$ be a 2-connected graph drawn in the plane with crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Suppose that
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$G'$ is 2-connected, and let $\mathcal{P}$ be an $(F_1,F_2)$-band or an $(F_1,u)$-fan in $G'$ whose support is disjoint from $Y$. Let $\mathcal{T}$ be a set of shelled tiles of $\mathcal{P}$ whose extended tiles are disjoint except possibly for their common intersection in $u$ when $\mathcal{P}$ is a fan. Let $G_1$ be any drawing of $G$ with crossings, such that no edge of the boundary cycles of tiles from $\mathcal{T}$ is crossed. Then there exists a $\mathcal{T}$-flat drawing $G_2$ of $G$ with $\text{cr}(G_2) \leq \text{cr}(G_1)$.

Reductions and criticality

- **Definition 4.7** (isomorphic tiles). Two $(F_1,F_2)$-bands or $(F_1,u)$-fans $\mathcal{P}_1 = (P_1,\ldots,P_m)$ and $\mathcal{P}_2 = (P'_1,\ldots,P'_n)$ are isomorphic if there exists a homeomorphism mapping the support of $\mathcal{P}_1$ to the support of $\mathcal{P}_2$ and mapping the path $P_i$ to $P'_i$ for $i = 1,\ldots,m$, where the paths are taken as directed away from $F_1$ (i.e., the homeomorphism must map the vertex of $P_i$ incident with $F_1$ to the vertex of $P'_i$ incident with $F_1$).

- **Definition 4.8** (band or fan reduction). Let $G$ be a graph drawn in the plane with crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Let $\mathcal{P}$ be an $(F_1,F_2)$-band or an $(F_1,u)$-fan in $G'$ whose support is disjoint from $Y$. Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic subbands or subfans of $\mathcal{P}$, with disjoint supports, except for the vertex $u$ when $\mathcal{P}$ is a fan, and not containing the first and the last path of $\mathcal{P}$. Let $\mathcal{P}'$ be the minimal subband or subfan of $\mathcal{P}$ containing both $\mathcal{P}_1$ and $\mathcal{P}_2$. We then say that $\mathcal{P}'$ is a reducible subband or subfan with repetition $(\mathcal{P}_1,\mathcal{P}_2)$. See Figure 8. The order of this repetition $(\mathcal{P}_1,\mathcal{P}_2)$ equals the length of $\mathcal{P}_1$ (which is the same as the length of $\mathcal{P}_2$).

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the last paths of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Denote by $S$ the support of the subband or subfan between $\mathcal{P}_1$ and $\mathcal{P}_2$, excluding these two paths. Let $G_1'$ be obtained from $G'$ by removing $S$ and by identifying $\mathcal{P}_1$ with $\mathcal{P}_2$ (stretching the drawing of the support of $\mathcal{P}_1$ within the area originally occupied by $S$). Let $G_1$ be obtained from $G_1'$ by turning the vertices of $Y$ back into crossings. For clarity, note that the support of $\mathcal{P}'$ is disjoint from $Y$, and so $\mathcal{P}'$ is also a band or fan in a plane subgraph of $G$. We then say that $G_1$ is the reduction of $G$ on $\mathcal{P}'$.

We now thoroughly study properties of our reductions (cf. Definition 4.8).

- **Lemma 4.9.** Let $G$ be a drawing of a 2-connected graph in the plane with minimum number $k$ of crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Let $\mathcal{P}$ be a shelled $(F_1,F_2)$-band or $(F_1,u)$-fan in $G'$ whose support is disjoint from $Y$. Let $\mathcal{P}'$ be a reducible subband or subfan of $\mathcal{P}$ with repetition $(\mathcal{P}_1,\mathcal{P}_2)$. Let $G_1$ with associated planarization $G'_1$ be the reduction of $G$ on $\mathcal{P}'$. If $\mathcal{P}_1$ has length at least $6k - 4$, then $G_1$ has no drawing in the plane with less than $k$ crossings.

**Proof.** Let $\mathcal{P}''$ be the band or fan corresponding to $\mathcal{P}$ in $G'_1$. Clearly, $\mathcal{P}''$ is shelled and its support is disjoint from $Y$.

Suppose for a contradiction that $G_2$ is a drawing of $G_1$ with less than $k$ crossings. Since $\mathcal{P}_1$ has length at least $6k - 4$, it contains $2k - 1$ pairwise edge-disjoint extended tiles. Each crossing of $G_2$ belongs to at most two of these extended tiles, and thus there exists a tile $T$ of $\mathcal{P}_1$ such that the edges of $X(T)$ are not crossed in $G_2$. By Lemma 4.5, we can assume that the drawing of $G_2$ is $\mathcal{T}$-flat. Let $\mathcal{P}_T$ denote the subband of $\mathcal{P}'$ between the copies of the tile $T$ in $\mathcal{P}_1$ and $\mathcal{P}_2$. We can transform $G_2$ to a drawing of $G$ by replacing the tile $T$ by the support of $\mathcal{P}_T$ inside the area occupied by $T$. This creates no new crossings, giving a drawing of $G$ with less than $k$ crossings, which is a contradiction.

Let $G_1$ and $G_2$ be two drawings of the same graph in the plane, and let $C_1$ and $C_2$ be vertex-disjoint cycles in this graph, drawn with all edges uncrossed both in $G_1$ and $G_2$. 


Consider the drawing of $C_1 \cup C_2$ that is induced by $G_1$. Without loss of generality (by performing circle inversion of the plane if necessary), we can assume that the face of this drawing incident with both $C_1$ and $C_2$ is the unbounded one. Let us orient $C_1$ and $C_2$ in the clockwise direction. Consider now the drawing of the directed subgraph $C_1 \cup C_2$ induced by $G_2$. Without loss of generality (by performing circle inversion or reflection of the plane if necessary), we can assume that the face of this drawing incident with both $C_1$ and $C_2$ is the unbounded one, and that $C_1$ is oriented in the clockwise direction. If $C_2$ is oriented in the clockwise direction, we say that the drawing $G_2$ is $(C_1, C_2)$-straight with respect to $G_1$. If $C_2$ is oriented in the counterclockwise direction, we say that the drawing $G_2$ is $(C_1, C_2)$-twisted with respect to $G_1$. We need the following observation, whose simpler version we already used in the proof of Lemma 4.3.

Lemma 4.10. Let $G$ be a drawing of a 2-connected graph in the plane with crossings. Let $G'$ be the planarization of $G$, which we also assume to be 2-connected, and let $Y$ denote the set of crossing vertices of $G'$. Let $P$ be an $(F_1, F_2)$-band or an $(F_1, u)$-fan in $G'$ whose support is disjoint from $Y$. Let $P'$ be a subband or subfan of $P$ of length at least 5, not containing the first and the last path of $P$. Let $P_1$ and $P_2$ be the first and the last path of $P'$, oriented away from $F_1$, let $T_1$ and $T_2$ be the first and the last tile of $P'$, and assume that both $T_1$ and $T_2$ are shelled. Let $H$ be a connected graph drawn in the plane without crossings, and let $P'_1$ and $P'_2$ be paths of the same lengths as $P_1$ and $P_2$, contained in the boundary of the outer face of $H$ and oriented in the opposite directions along the curve tracing the boundary, and disjoint if $P'$ is a band and sharing their last vertices if $P'$ is a fan. Let $G'_i$ be the subdrawing of $G'$ obtained by removing the support of $P'$ except for $P_1 \cup P_2$, and let $G_H$ be the graph obtained from $G'_i$ by adding $H$ and identifying $P'_1$ with $P_1$ and $P'_2$ with $P_2$.

Let $p$ denote the maximum number of pairwise edge-disjoint paths from $P_1$ to $P_2$ in the support of $P'$, disjoint from $u$ when $P$ is a fan, and suppose that $H - V(P'_1 \cap P'_2)$ contains an edge-cut of size at most $p$ separating $V(P'_1) \setminus V(P'_2)$ from $V(P'_2) \setminus V(P'_1)$. Let $G_1$ be another $\{T_1, T_2\}$-flat drawing of $G$ with crossings. If the drawing $G_1$ is $(C(T_1), C(T_2))$-straight with respect to the drawing $G$, then $G_H$ has a drawing with with at most as many crossings as $G_1$ extending the drawing $G'_i$; and furthermore, all edges in $E(G'_i)$ that are uncrossed in the drawing $G_1$ are also uncrossed in the drawing of $G_H$.

Proof. Let $Q_1, \ldots, Q_p$ be pairwise edge-disjoint paths from $P_1$ to $P_2$ in the support of $P'$, disjoint from $P_1 \cup P_2$ except for their ends, and disjoint from $u$ when $P$ is a fan. Since the drawing $G_1$ is $\{T_1, T_2\}$-flat, the first edges of these paths are drawn in the same face $f_1$ of $G'_i$ and the last edges are drawn in the same face $f_2$ of $G'_i$, and if $P$ is a fan, then both of these faces are incident with $u$. By symmetry, we can assume that the curve $\gamma$ representing $Q_1$ in the drawing of $G_1$ has the least number $a$ of crossings with the edges of $G'_i$ among the paths $Q_1, \ldots, Q_p$, and thus $\text{cr}(G_1) \geq \text{cr}(G'_i) + ap$. Let $\gamma'$ be a simple curve contained in the support of $\gamma$ with the same endpoints ($\gamma$ does not have to be simple, since the drawing of the path $Q_1$ may intersect itself). Let $C$ be an edge-cut of size at most $p$ in $H - V(P'_1 \cap P'_2)$ separating $V(P'_1) \setminus V(P'_2)$ from $V(P'_2) \setminus V(P'_1)$. Draw the part of $H$ between $P'_1$ and $C$ in $f_1$ (identifying $P'_1$ with $P_1$) and the part of $H$ between $C$ and $P'_2$, and join them by the edges of $C$ drawn along $\gamma'$ to form a drawing of $G_H$ with the required properties. This is possible without creating any crossings among the edges of $C$ since the drawing $G_1$ is $(C(T_1), C(T_2))$-straight with respect to $G$. ▶

Lemma 4.11. Let $G$ be a 2-connected graph drawn in the plane with crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Let $P$ be a $k$-edge-linked shelled $(F_1, F_2)$-band or proper $(F_1, u)$-fan in $G'$ whose support is disjoint from
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Y. Let $\mathcal{P}'$ and $\mathcal{P}''$ be isomorphic reducible subbands or subfans of $\mathcal{P}$ with repetitions ($\mathcal{P}'_1, \mathcal{P}'_2$) and ($\mathcal{P}''_1, \mathcal{P}''_2$), with disjoint supports (except for $u$ when $\mathcal{P}$ is a fan). For some integer $j \geq 2$ smaller than the length of $\mathcal{P}'$, let $T'_1, T'_2, T'_3$, and $T'_4$ be the $j$-th tile of $\mathcal{P}'_1, \mathcal{P}'_2, \mathcal{P}''_1$, and $\mathcal{P}''_2$. Let $\mathcal{P}'''$ be the subband or subfan of $\mathcal{P}$ of length $k$ immediately following $\mathcal{P}''$. Let $T$ be a tile of $\mathcal{P}$ such that $\mathcal{P}'$, $\mathcal{P}''$, $\mathcal{P}'''$, and $T$ appear in $\mathcal{P}$ in order. Let $e$ be an edge of $G$ appearing neither in the support of the subband or subfan of $\mathcal{P}$ between $\mathcal{P}'$ and $\mathcal{P}'''$ (inclusive) nor in $X(T)$. Let $G_1$ be the reduction of $G$ on $\mathcal{P}'''$, and let $P_1$ be the $(F_1,F_2)$-band or $(F_1,u)$-fan in $G_1$ corresponding to $\mathcal{P}$. If $G_2$ is a $(T'_1, T'_2, T'_3, T'_4, T)$-flat drawing of $G-e$, then there exists a drawing of $G_1-e$ with at most as many crossings.

**Proof.** Let $U = \{u\}$ when $\mathcal{P}$ is a fan and $U = \emptyset$ when $\mathcal{P}$ is a band. Without loss of generality, we can assume that the graphs $G-e$ and $G_1-e$ are 2-connected—otherwise, note that the blocks of $G-e$ and $G_1-e$ are the same, except for the blocks $B$ and $B_1$ containing $\mathcal{P}'_1$ (this uses the assumption that if $\mathcal{P}$ is a fan, it is proper, otherwise $u$ could be a cutvertex in $G-e$ splitting $\mathcal{P}$); we apply the argument below to $B$ and $B_1$, and add to the resulting drawing of $B_1$ the rest of the blocks drawn in the same way as in $G-e$, without creating any additional crossings.

Suppose that the drawing $G_2$ is $(C(T'_1), C(T))$-straight with respect to $G$. Let $H$ and $H_1$ be the supports of the subband or the subfan of $\mathcal{P}$ and $P_1$ between $T'_1$ and $T$ (inclusive), respectively, and let $P_3$ and $P_4$ be their first and last paths. Let $P_3$ and $P_4$ be the first and the last path of $\mathcal{P}'''$. Let $p$ denote the maximum number of pairwise edge-disjoint paths in $H-U-e$ from $P_1-U$ to $P_2-U$. Since $\mathcal{P}$ is $k$-edge-linked, we have $p \in \{k-1,k\}$, and $H_1-U$ contains an edge cut of size $k$ separating $P_1-U$ from $P_2-U$. If $H_1-U-e$ does not contain an edge cut of size $k-1$ separating $P_1-U$ from $P_2-U$, then by Menger’s theorem, $H_1-U-e$ contains $k$ pairwise edge-disjoint paths from $P_1-U$ to $P_2-U$, and thus $H-U-e$ contains $k$ pairwise edge-disjoint paths $Q_1$ from $P_3-U$ to $P_2-U$. Since $\mathcal{P}$ is $k$-edge-linked, $H-U-e$ also contains $k$ pairwise edge-disjoint paths $Q_2$ from $P_1-U$ to $P_2-U$. No set of $k-1$ edges of $H-U-e$ can intersect all paths of $Q_1$, or all paths of $Q_2$, or all paths of $\mathcal{P}'''$; hence, even after removing any $k-1$ edges, $H-U-e$ contains a path from $P_1-U$ to $P_2-U$, and thus Menger’s theorem implies that $H-U-e$ contains $k$ pairwise edge-disjoint paths from $P_1-U$ to $P_2-U$, and $p=k$. In conclusion, $H_1-U-e$ contains an edge-cut of size $p$ separating $P_1-U$ from $P_2-U$. Therefore, we can use Lemma 4.10 to transform $G_2$ into a drawing of $G_1-e$ with at most as many crossings by replacing the subband or subfan corresponding to $H-e$ with $H_1-e$.

Hence, we can assume that $G_2$ is $(C(T'_1), C(T))$-twisted with respect to $G$. Similarly, we can assume that $G_2$ is $(C(T''_1), C(T))$-twisted and $(C(T''_2), C(T))$-twisted with respect to $G$. Consequently, $G_2$ is $(C(T'_1), C(T'_2))$-straight and $(C(T''_2), C(T))$-straight with respect to $G$. Let $H_2$ be the support of the subband or subfan of $P_1$ between $T'_2$ and $T$. Using Lemma 4.10 to replace the part of $G_2$ between $T'_1$ and $T'_2$ by a copy of the tile $T'_1$ (which is isomorphic to $T''_1$) and the part between $T''_2$ and $T$ by $H_2-e$, obtaining a drawing of $G_1-e$ with at most as many crossings.

We can now summarise a suitable conditions on a band or a fan, under which our reductions preserve crossing-criticality.

**Definition 4.12** ($t$-typical subband or subfan). We say that, in an $(F_1,F_2)$-band or an $(F_1,u)$-fan $\mathcal{P}$, a subband $\mathcal{Q}$ is $t$-typical if the following holds: there exist subbands or subfans $\mathcal{P}_1, \ldots, \mathcal{P}_{2t+1}$ of $\mathcal{P}$ appearing in this order, such that they are pairwise isomorphic, with pairwise disjoint supports except for the vertex $u$ when $\mathcal{P}$ is a fan, and $\mathcal{Q} = \mathcal{P}_{t+1}$.
Lemma 4.13. Let $G$ be a 2-connected c-crossing-critical graph drawn in the plane with the minimum number of crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Let $c_0 = \lceil 5c/2 + 16 \rceil$ and $k \in \mathbb{N}$. Let $\mathcal{P}$ be a $k$-edge-linked shelled $(F_1, F_2)$-band or proper $(F_1, u)$-fan in $G'$ whose support is disjoint from $Y$. Let $Q$ be a subband or subfan of $\mathcal{P}$ which is reducible with repetition of order at least $12c_0 + 2k$. If $Q$ is $c$-typical in $\mathcal{P}$, then the reduction $G_1$ of $G$ on $Q$ is a c-crossing-critical graph again.

Proof. By Theorem 2.3, each c-crossing-critical graph has a drawing with at most $c_0$ crossings, and thus $|Y| \leq c_0$. Let $Q$ be reducible with a repetition $(Q_1, Q_2)$. Since the length of $Q_1$ is at least $6c_0 - 4$, Lemma 4.9 implies that the crossing number of $G_1$ is at least $|Y| \geq c$. Let $P_1$ be the fan or band in $G_1$ corresponding to $\mathcal{P}$. Let $P$ be the middle path of $Q_1$ taken as a subband or subfan of $P_1$, and let $P_0$ be the subband or subfan of $P_1$ between the first path of $P_1$ and the path preceding $P$. Consider any edge $e \in E(G_1)$. By symmetry, we can assume that $e$ does belong to the support of $P_1$. Let $\mathcal{P}'_1$ and $\mathcal{P}'_2$ be the subbands of $\mathcal{P}$ whose support is disjoint from $Q_1$ and $Q_2$ consisting of the first $6c_0 - 4$ of their paths, and let $\mathcal{P}''$ be the subband or subfan of $\mathcal{P}$ between $\mathcal{P}'_1$ and $\mathcal{P}'_2$. Let $\mathcal{P}'''$ be the subband or subfan of $\mathcal{P}$ of length $k$ immediately following $\mathcal{P}''$. Note that $\mathcal{P}'''$ is reducible with repetition $(\mathcal{P}'_1, \mathcal{P}'_2)$, and $G_1$ is a reduction of $G$ on $\mathcal{P}''$, and $e$ does not appear in the support of the subband or subfan of $\mathcal{P}$ between $\mathcal{P}''$ and $\mathcal{P}'''$ (inclusive).

Consider a drawing $G_2$ of $G$ with the minimum number of crossings, which is smaller than $c$ since $G$ is c-crossing-critical. Since $Q$ is c-typical, there exists a subband or subfan $\mathcal{P}'$ of $\mathcal{P}$ isomorphic to $\mathcal{P}'''$ and strictly preceding $\mathcal{P}'''$ in $\mathcal{P}$, such that no edge of the support of $\mathcal{P}'''$ is crossed in $G_2$. Let $(\mathcal{P}'_1, \mathcal{P}'_2)$ be the repetition of $\mathcal{P}'$ of the same length as $\mathcal{P}'_1$. Since this length is at least $6c - 4$, there exists $j \geq 2$ smaller than the length of $\mathcal{P}'_1$ such that, denoting by $T'_1$, $T'_2$, $T''_1$, and $T''_2$ the $j$-th tile of $\mathcal{P}'_1$, $\mathcal{P}'_2$, $\mathcal{P}'_1$, and $\mathcal{P}'_2$, none of the edges of $X(T'_1)$, $X(T'_2)$, $X(T''_1)$, and $X(T''_2)$ is crossed. Furthermore, since $Q$ is c-typical, there exists a tile $T$ of $\mathcal{P}$ appearing after $\mathcal{P}'''$ such that $e \notin X(T)$ and none of the edges of $X(T)$ is crossed. By Corollary 1.16 we can assume that the drawing $G_2$ is $(T'_1, T'_2, T''_1, T''_2, T)$-flat.

By Lemma 4.11 there exists a drawing of $G_1 - e$ with at most as many crossings as $G_2$ has. Therefore, for every $e \in E(G_1)$, there exists a drawing of $G_1 - e$ with less than $c$ crossings, and thus $G_1$ is c-crossing-critical.

4.3 Expanding a band, fan or a necklace.

Finally, it is time to formally define what is a generic converse operation of the instances of reduction considered by Lemmas 4.13 and 4.2.

Definition 4.14 ((n-bounded expansion)). Let $G$ be a 2-connected c-crossing-critical graph drawn in the plane with the minimum number of crossings. Let $G'$ be the planarization of $G$ and let $Y$ denote the set of crossing vertices of $G'$. Let $c_0 = \lceil 5c/2 + 16 \rceil$. Assume $\mathcal{P}$ is a $k$-edge-linked shelled $(F_1, F_2)$-band or proper $(F_1, u)$-fan in $G'$ whose support is disjoint from $Y$. Let $Q$ be a c-typical subband or subfan of $\mathcal{P}$ which is reducible with repetition of order at least $12c_0 + 2k$. Let the number of vertices of the support of $Q$ be at most $n$, and let $G_1$ denote the reduction of $G$ on $Q$. In these circumstances, we say that $G$ is an $n$-bounded expansion of $G_1$.

Assume $\mathcal{P}'$ is a necklace in $G'$ whose support is disjoint from $Y$, and let $Q' = (v_1, v_2)$ be a 1-typical subband of $\mathcal{P}'$ of length $2$. Let $G_2$ be obtained from $G$ by replacing the support $S$ of $Q'$ by a parallel edge of multiplicity equal to the maximum number of pairwise edge-disjoint paths between $v_1$ and $v_2$ in $S$. Let the number of vertices of the support of $Q'$ be at most $n$. In these circumstances, we also say that $G$ is an $n$-bounded expansion of $G_1$. 
**Theorem 4.15.** For every integer \( c \geq 1 \), there exists a positive integer \( n_0 \) such that the following holds. If \( G \) is a 2-connected \( c \)-crossing-critical graph, then there exists a sequence \( G_0, G_1, \ldots, G_m \) of 2-connected \( c \)-crossing-critical graphs such that \( |V(G_0)| \leq n_0 \), \( G_m = G \), and for \( i = 1, \ldots, m \), \( G_i \) is an \( n_0 \)-bounded expansion of \( G_{i-1} \).

**Proof.** Let \( c_0 = \lceil 5c/2 + 16 \rceil \). For an integer \( w \), let \( t(w) \) denote the number of non-homeomorphic 2-connected plane graphs with at most \( w \) vertices and with two edge-disjoint oppositely directed subpaths selected in the boundary of their outer face. Let \( r(k, w) = (12c_0 + 2k)(t(w)^{12c_0 + 2k} + 1) \), and let \( r_i(k, w) = r(k, w)(2t(w)^{r(k, w)} + 1) \). For every positive integer \( w \), let \( w_0(w) \) and \( w_0(w) \) be the corresponding integers from Lemma 4.4 applied with \( f = r_i \). Let \( n_i(w) \) and \( n_i(w) \) be the corresponding integers from Lemma 4.2 applied with \( f(w) = \max(2t(w) + 2, n_i(w)) \). Let \( w_0 \) and \( n_0 \) be the integers from Corollary 3.13 applied with \( g = n_i \). Let \( w_b \) be the maximum of \( w^b(w) \) over \( 0 \leq w \leq w_b \). Let \( n_0 \) be the maximum of

\[
\begin{align*}
&= n^c_0, \\
&= (2t(w) + 2)w \text{ over all positive } w \leq w_b, \\
&= r_i(k, w)w \text{ over all positive } k \leq \left( \frac{w_b}{2} \right) \text{ and } w \leq w_a.
\end{align*}
\]

Clearly, it suffices to prove that if \( |V(G)| > n_0 \), then \( G \) is an \( n_0 \)-bounded expansion of some 2-connected \( c \)-crossing-critical graph. The rest follows by induction on the number of vertices.

Let \( G' \) be the planarization of a drawing of \( G \) with the smallest number of crossings, and let \( Y \) be the set of its crossing vertices. Since \( |V(G)| > n_0 \geq n^c_0 \), Corollary 3.13 implies that \( G' \) contains an \((F_1, F_2)\)-band or a proper \((F_1, u)\)-fan \( P \) for some distinct faces \( F_1 \) and \( F_2 \) or a vertex \( u \), such that for some \( w \leq w^c_0 \), all the tiles of the band or fan have size at most \( w \) and are disjoint from \( Y \), and the length of the band or fan is at least \( n^c_0(w) \).

Hence, we can apply Lemma 4.2 to \( P \), obtaining either a shelled subband or subfan \( P_b \) of length at least \( n^c_0(w) \), or for some \( w \leq w_b \), a necklace \( P'_b \) of length exactly \( 2t(w) + 2 \) and tiles of size at most \( w' \). In the latter case, since there are at most \( t(w') \) distinct tiles of size \( w' \), the pigeonhole principle implies that \( P'_b \) contains a 1-typical subband \( Q' = (v_1, v_2) \) of length two. Note also that the support of \( P'_b \) has size at most \( (2t(w') + 2)w' \leq n_0 \). Let \( G_2 \) be obtained from \( G \) by replacing the support \( S \) of \( Q' \) by a parallel edge of multiplicity equal to the maximum number of pairwise edge-disjoint paths between \( v_1 \) and \( v_2 \) in \( S \), so that \( G \) is an \( n_0 \)-bounded expansion of \( G_2 \). Note that \( G_2 \) is 2-connected, and it is \( c \)-crossing-critical by Lemma 4.3, as required.

Hence, assume that the former holds. We now apply Lemma 4.4 to \( P_b \), obtaining for some \( w'' \leq w_a \) and \( k \leq \left( \frac{w_b}{2} \right) \) a \( k \)-edge-linked shelled band or fan \( P \) of length \( r_i(k, w'') \) with tiles of size at most \( w'' \) and with support contained in the support of \( P_b \).

Note that there exist at most \( t(w'')^{12c_0 + 2k} \) non-isomorphic subbands or subfans of \( P \) of length \( 12c_0 + 2k \), and by pigeonhole principle, any subband or subfan of \( P \) of length at least \( r(k, w'') \) contains a reducible subband or subfan with repetition of order \( 12c_0 + 2k \). Similarly, pigeonhole principle implies that since the length of \( P \) is \( r_i(k, w'') \), it contains a \( c \)-typical reducible subband or subfan \( Q \) with repetition of order at least \( 12c_0 + 2k \). Let \( G_1 \) be the reduction of \( G \) on \( Q \). By Lemma 4.13, \( G_1 \) is \( c \)-crossing-critical. Note that the support of \( P \) has at most \( r_i(k, w'')w'' \leq n_0 \) vertices, and thus \( G \) is an \( n_0 \)-bounded expansion of the 2-connected \( c \)-crossing-critical graph \( G_1 \), as required.

We remark on the following natural algorithmic consequence: the generating sequences claimed by Theorem 4.15 can be turned into an efficient enumeration procedure to generate...
all 2-connected $c$-crossing-critical graphs of at most given order $n$, for each fixed $c$. See below for details.

We start with an immediate corollary:

**Corollary 4.16.** For every $c$, there exist $n_0$ and $s$ as follows. If $G$ is 2-connected and $c$-crossing-critical and $|V(G)| > n_0$, then there exist two isomorphic induced subgraphs $J_1$ and $J_2$ of $G$ with at most $s$ vertices intersecting in at most one vertex $z$ fixed by their isomorphism, and disjoint sets $L_i, R_i \subseteq V(J_i) \setminus \{z\}$ for $i = 1, 2$, where $L_1$ is mapped to $L_2$ and $R_1$ to $R_2$ by the isomorphism, such that

- $J_1$ and $J_2$ are 2-connected and all their vertices except for $z$ have degree at most $s$ in $G$,
- $G - V(J_1 \cup J_2)$ has exactly two components $C$ and $K$, with $|C| \leq n_0$,
- $C$ has only neighbors in $R_1 \cup L_2 \cup \{z\}$ and $K$ has only neighbors in $L_1 \cup R_2 \cup \{z\}$, and
- the graph $G'$ obtained from $G - C$ by identifying $J_1$ with $J_2$ according to their isomorphism is 2-connected and $c$-crossing-critical.

**Enumerating $c$-crossing-critical graphs.** Let $H$ be a graph with at most $n_0 + 2s$ vertices, let $J_1$ and $J_2$ be two isomorphic induced 2-connected subgraphs with at most $s$ vertices intersecting in at most one vertex $z$ fixed by the isomorphism $\theta$, and let $L_i, R_i \subseteq V(J_i) \setminus \{z\}$ be disjoint sets for $i = 1, 2$, where $L_1$ is mapped to $L_2$ and $R_1$ to $R_2$ by the isomorphism, such that $H - V(J_1 \cup J_2)$ has exactly one component and this component only has neighbors in $R_1 \cup L_2 \cup \{z\}$. We say that $(H, J_1, J_2, \theta, L_1, R_1, L_2, R_2)$ is a template.

Fix such a template $T$. Let $G'$ be a graph, let $S$ be a set of vertices of $G'$ such that $G'[S]$ is isomorphic to $J_1$, let $z'$, $L$, and $R$ correspond to $z$, $L_1$ and $R_1$ via this isomorphism $\theta'$. If $G' - V(J)$ has only one component which only has neighbors in $L \cup R \cup \{z'\}$, then let $G'_{J, \theta', T}$ be the graph obtained from $G' - V(J)$ by adding $H$ and joining vertices of $G' - V(J')$ to appropriate vertices in $L_1 \cup R_2 \cup \{z\}$.

For each graph $G'$ with at most $n$ vertices, there are only $O(n)$ graphs of form $G'_{G'[S], \theta', T}$ for sets $S \subseteq V(G)$ such that all vertices of $S$ other than $z'$ have degree at most $s$ in $G'$; the subgraph $G'[S \setminus \{z'\}]$ is connected and all its vertices have degree at most $s$ in $G'$; hence, once we choose one of the vertices of $S \setminus \{z'\}$, there are less than $s^{2s}$ ways to choose the rest of the vertices of $S$, by repeatedly selecting one of already chosen vertices and adding one of its neighbors (with the vertex $z'$ chosen in the last step). The choice of $\theta'$ only adds a multiplicative factor of at most $s!$. Furthermore, since $c$-crossing-critical graphs have bounded treewidth and we can express the property “the graph is 2-connected and $c$-crossing-critical” by a MSOL formula, we can test for such graph whether it is 2-connected and $c$-crossing-critical in time $O(|V(G')|^2) = O(n|V(G')|)$.

Consider the auxiliary directed multigraph whose vertices are 2-connected $c$-crossing-critical graphs with at most $n$ vertices, and edge joins $G'$ to $G_1$ if $G_1 = G'_{G'[S], \theta', T}$ for some template $T$, some $S \subseteq V(G')$ of size at most $s$ and with all vertices other than $z$ having degree at most $s$ in $G'$, and an isomorphism $\theta'$. By Corollary 4.16, all vertices of this multigraph are reachable from the vertices corresponding to the graphs with at most $n_0$ vertices. Furthermore, according to the previous paragraph, the outneighbours of each vertex $G'$ can be enumerated in time $O(|V(G')|^2) = O(n|V(G')|)$. Hence, we can enumerate all 2-connected $c$-crossing-critical graphs by searching this auxiliary graph, in time $O(n)$ times the output size.
5 Conclusion

To summarize, we have shown a structural characterisation and an enumeration procedure for all 2-connected $c$-crossing-critical graphs, using bounded-size replication steps over an implicit finite set of base $c$-crossing-critical graphs. The characterisation can be reused to describe all $c$-crossing-critical graphs (without the connectivity assumption) since all their proper blocks must be $c_i$-crossing-critical for some $c_i < c$.

With this characterisation at hand, one can expect significant progress in the crossing number research, both from mathematical and algorithmic perspectives. For example, one can quite easily derive from Theorem 4.15 that, for no $c$ there is an infinite family of 3-regular $c$-crossing-critical graphs, a claim that has been so far proved only via the Graph minors theorem of Robertson and Seymour. One can similarly expect a progress in some long-time open questions in the area of crossing-critical graphs, such as to improve the bound of Theorem 2.3 or to decide possible existence of an infinite family of 5-regular $c$-crossing-critical graphs for some $c$.

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