UNIFORM LIE ALGEBRAS AND UNIFORMLY COLORED GRAPHS

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ABSTRACT. Uniform Lie algebras are combinatorially defined two-step nilpotent Lie algebras which can be used to define Einstein solvmanifolds. These Einstein spaces often have nontrivial isotropy groups. We derive basic properties of uniform Lie algebras and we classify uniform Lie algebras with five or fewer generators. We define a type of directed colored graph called a uniformly colored graph and establish a correspondence between uniform Lie algebras and uniformly colored graphs. We present several methods of constructing infinite families of uniformly colored graphs and corresponding uniform Lie algebras.

1. INTRODUCTION

A Riemannian manifold is Einstein if its Ricci form is a scalar multiple of its metric. One approach to the study of Einstein manifolds is to focus on classes of Einstein manifolds with symmetry, such as homogeneous spaces and spaces of cohomogeneity one (see [32, 33]). We are interested in a special class of Einstein homogeneous spaces: Einstein solvmanifolds. A solvmanifold is a simply connected solvable group endowed with a left-invariant Riemannian metric.

In [10], a representation of a compact Lie group is used to define a two-step metric nilpotent Lie algebra, which can then be used to define an Einstein solvmanifold. In [10, 9, 17], and [14], the geometry of these solvmanifolds was studied. Such nilpotent Lie algebras have large automorphism groups and the affiliated Einstein solvmanifolds have isotropy groups with positive dimension. In his earlier study of Einstein solvmanifolds [7],
DeLoff defined a class of nilpotent Lie algebras called uniform Lie algebras enjoying combinatorial symmetry instead of algebraic symmetry (see Definition 2.1). Every uniform Lie algebra can be used to define an Einstein solvmanifold, and due to the combinatorial regularity in the definition of uniform Lie algebra, the Einstein solvmanifolds often have nontrivial isotropy groups which may now be finite or infinite. The category of uniform Lie algebras includes many of the Lie algebras defined by representations of compact Lie groups, and well-known classes of two-step nilpotent Lie algebras, such as Heisenberg Lie algebras, two-step free nilpotent Lie algebras, Iwasawa type nilpotent Lie algebras for rank one symmetric spaces of noncompact type, and Lie algebras of Heisenberg type.

In this work, we study uniform Lie algebras. For the sake of economy, we do not explicitly discuss the Einstein solvmanifolds that they determine. Uniform Lie algebras may be defined over any field, and may be of interest over general fields from a purely algebraic perspective, but throughout this work, because of our geometric motivation, we assume that the field of definition is \( \mathbb{R} \).

Let \((V, E)\) be a simple digraph with edge coloring \( c : E \to S \). Let \( V = \{v_i\}_{i=1}^q \) and \( S = \{z_k\}_{k=1}^p \), and let \( v \) and \( z \) be the vector spaces of \( \mathbb{R} \)-linear combinations of \( V \) and \( S \), respectively. Setting \([v_i, v_j] = \sum_{k=1}^p \alpha_{ij}^k z_k\), where

\[
\alpha_{ij}^k = \begin{cases} 
1 & e_{ij} = (v_i, v_j) \in E \text{ and } c(e_{ij}) = z_k \\
-1 & e_{ji} = (v_j, v_i) \in E \text{ and } c(e_{ji}) = z_k \\
0 & \text{otherwise}
\end{cases}
\]

defines a two-step nilpotent Lie algebra structure on \( v \oplus z \).

We define a class of edge colored digraphs, called uniformly colored graphs, and show that any uniformly colored graph defines a uniform Lie algebra, and any uniform Lie algebra may be encoded as a uniformly colored graph. Uniform colorings of graphs are (up to a choice of orientation) equivalent to \( H \)-decompositions of regular graphs with \( H = K_2 + \cdots + K_2 \); that is, decompositions of regular graphs into subgraphs all isomorphic to the same disjoint sum \( K_2 + \cdots + K_2 \), where \( K_2 \) is the complete graph on two vertices.

Algebraic objects are often used to analyze combinatorial and topological problems, such as in the case of Orlik-Solomon algebras or free partially commutative monoids, and conversely, graphs or simplices have been used to define algebraic objects, as with rooted tree algebras and in Stanley-Reisner theory. Closer to the topic at hand, Dani and Mainkar defined a class of two-step nilpotent Lie algebras associated to graphs that they called nilpotent Lie algebras of graph type ([6]), and Mainkar showed that two
nilpotent Lie algebras of graph type are isomorphic if and only if the graphs they arise from are equivalent ([19]).

Einstein solvmanifolds defined by solvable extensions of nilpotent Lie algebras of graph type have been studied in [16, 13, 22], and the geometry of metric nilpotent Lie algebras defined by Schreier graphs was addressed in [27]. (Although both of these classes of nilpotent Lie algebras, graph type and Schreier type, overlap with the class of uniform Lie algebras studied here, there are no containment relations between the class of uniform Lie algebras and either of these classes. See Remark [4.11]) Pseudo H-type algebras were analyzed using combinatorial and orthogonal designs in [11]. In [24], it was shown that if \( rp - q - p + 1 > 0 \), an Einstein solvmanifold defined by a uniform metric Lie algebra of type \((p, q, r)\) admits nonisometric Einstein deformations.

We derive basic properties of uniform Lie algebras and uniformly colored graphs. We show how algebraic properties of a uniform Lie algebra translate to graph theoretic properties of the corresponding uniformly colored graph. We show in Proposition 4.14 how totally geodesic subalgebras of a uniform Lie algebra may be found using the corresponding uniform graph. In Proposition 4.17 we show that graph unions correspond to concatenations of Lie algebras, and in Propositions 4.18 and 4.20 we determine when graph unions of uniform graphs are again uniform.

We give many examples of uniform Lie algebras and uniformly colored graphs, some in infinite families. We show uniform Lie algebras can be found from well-known combinatorially defined graphs, such as Kneser graphs, and decompositions of familiar regular graphs, such as one-factorizations and near-one-factorizations of complete graphs. We present a general method of constructing uniform Lie algebras from Cayley graphs. These examples indirectly give a wealth of new examples of Einstein solvmanifolds.

Many of the examples we give come from graphs with symmetries, and the corresponding Einstein solvmanifolds inherit those symmetries. We show in Proposition 4.16 that the symmetry group of a uniformly colored digraph embeds in the automorphism group of the corresponding uniform Lie algebra. Thus, the automorphism group of a uniform Lie algebra may contain a nontrivial finite subgroup, and the corresponding Einstein solvmanifold will have that finite group as a subgroup of its isotropy group. In addition, the corresponding simply connected Lie groups may have infranilmanifold quotients. (For all uniform Lie algebras, the corresponding simply connected nilpotent Lie group admits a lattice).

In Theorem 6.3 we classify all uniform Lie algebras with five or fewer generators; a complete list appears in Table 1. There are only 12 Lie algebras in the list, demonstrating that uniform Lie algebras are not so common.
in low dimensions. However, there are many in higher dimensions (See Remark 5.9).

In addition to the examples that we present here, many more uniform edge colorings can be found on well-known graphs and families of graphs. We leave it to the interested reader to construct uniform edge colorings on the Heawood graph and the Desargues graph; one-skeletons of symmetric polyhedra; some (but not all) circulant graphs, some (but not all) bipartite graphs, and graphs arising from incidence geometries. All the resulting uniformly colored graphs define Einstein solvmanifolds, with the symmetry groups of the colored graphs embedding into the isometry groups of the manifolds.

We pose two problems. The same strategy we used for the proof of Theorem 6.3 can be used to classify uniform Lie algebras of type $(p, q, r)$ with $q \geq 6$. To assist in this goal, and of independent interest, one could analyze infinite classes of uniform Lie algebras of type $(p, q, r)$ which exist for all possible $q \geq 6$, or in the cubic case, all even $q \geq 6$.

**Problem 1.1.** What are the structure and algebraic properties of uniform Lie algebras that have an underlying graph which is a cycle? a bipartite graph? a complete graph? a cubic graph?

This involves a simply stated problem in graph theory.

**Problem 1.2.** What are the structure and properties of $K_2 + \cdots + K_2$-decompositions of regular graphs?

The rest of the paper is organized as follows. In Section 2, we give the formal definition of a uniform (metric) Lie algebra and present some illuminating examples. In Section 3, we derive basic properties of uniform Lie algebras. Then, in Section 4, we define uniformly colored graphs, give the correspondences between uniform Lie algebras and uniformly colored graphs, and translate between algebraic properties of Lie algebras and properties of uniformly colored graphs. In Section 5, we present a variety of constructions of uniformly colored graphs. Finally in Section 6, we classify uniform type Lie algebras with five or fewer generators.

### 2. Definition and Some Examples

The first thing we do is define Lie algebras of uniform type and give some examples. The definition may seem complicated at first, but once a connection is made with Definition 2.1, it will seem quite simple.

**Definition 2.1.** Let $p, q,$ and $r$ be positive integers. A real nilpotent Lie algebra $n$ is said to be of *uniform of type $(p, q, r)$* if there exists a basis $B = \{v_i\}_{i=1}^p \cup \{z_j\}_{j=1}^q$ of $n$ and a positive integer $s$, called the *degree*, such that the following properties hold.

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The uniform basis has rational structure constants, so by Mal’cev’s Criterion, the simply connected Lie groups corresponding to uniform Lie algebras and free nilpotent algebras.

(1) For all $i, j = 1, \ldots, q$ and all $l, m = 1, \ldots, p$,
   (a) $[v_i, z_j] = 0$ and $[z_l, z_m] = 0$,
   (b) $[v_i, v_j] \in \{0, \pm z_1, \ldots, \pm z_p\}$.

(2) If $[v_i, v_j]$ is nonzero and $[v_i, v_j] = \pm [v_i, v_k]$, then $v_j = v_k$.

(3) For all $l = 1, \ldots, p$, there exist exactly $r$ disjoint pairs $\{v_i, v_j\}$ with $[v_i, v_j] = z_l$.

(4) For all $j = 1, \ldots, q$, there exist exactly $s$ basis vectors $v_i$ with $[v_i, v_j] \neq 0$.

The basis $\mathcal{B}$ is called a uniform basis for $\mathfrak{n}$. When $\mathbb{F} = \mathbb{R}$, it is natural to endow $\mathfrak{n}$ with the inner product $Q$ which makes $\mathcal{B}$ orthonormal; then we call $(\mathfrak{n}, Q)$ a uniform metric Lie algebra.

Note that the parameters $p, q, r$ and $s$ in the definition are dependent: the number of nonzero structure constants relative to the basis $\mathcal{B}$ is $sq = 2rp$. The uniform basis has rational structure constants, so by Mal’cev’s Criterion, the simply connected Lie groups corresponding to uniform Lie algebras always admit compact quotients.

The most simple examples of uniform Lie algebras are Heisenberg algebras and free nilpotent algebras.

**Example 2.2.** Let $\mathfrak{h}_{2n+1}$ be the Heisenberg Lie algebra with basis $\mathcal{B} = (\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^n) \cup \{z\}$ and Lie bracket defined by $[x_i, y_i] = z$ for $i = 1, \ldots, n$. With respect to the basis $\mathcal{B}$, $\mathfrak{h}_{2n+1}$ is a uniform Lie algebra of type $(p, q, r) = (1, 2n, n)$ with degree $s = 1$.

Note that the uniform basis for $\mathfrak{h}_{2n+1}$ is not unique; in fact, there are uncountably many different uniform bases.

**Example 2.3.** Let $\mathfrak{f}_{n, 2}$ be the free two-step nilpotent Lie algebra on $n$ generators with basis $\{x_i\}_{i=1}^n \cup \{x_i \wedge y_j\}_{1 \leq i < j \leq n}$ and Lie bracket defined by $[x_i, x_j] = x_i \wedge x_j$ for $1 \leq i < j \leq n$. With respect to this basis, $\mathfrak{f}_{n, 2}$ is a uniform Lie algebra. The values of the associated parameters are $(p, q, r) = (\binom{n}{2}, n, 1)$ and $s = n - 1$.

Algebras of Heisenberg type are two-step metric nilpotent Lie algebras which have been studied extensively by geometers ([11]). Crandall and Dodziuk have shown that all algebras of Heisenberg type are uniform ([5]). Here is an example of a metric Lie algebra of Heisenberg type.

**Example 2.4.** Let $(\mathfrak{n}, Q)$ be the nilpotent metric Lie algebra with orthonormal basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\} \cup \{z_1, z_2\}$ and Lie bracket

\[(1) \quad [v_1, v_2] = [v_3, v_4] = z_1, [v_2, v_3] = [v_1, v_4] = z_2.\]

This Lie algebra is uniform of type $(2, 4, 2)$. 
We may generalize Example 2.4 to higher dimensions, although the examples in dimension $2r + 2 > 6$ are no longer Heisenberg type.

**Example 2.5.** For $r \geq 2$, define the Lie algebra $n(2r + 2)$ of dimension $2r + 2$ to have basis $\mathcal{B} = \{v_i\}_{i=1}^{2r} \cup \{z_1, z_2\}$ and Lie bracket determined by

\[
[v_{2i-1}, v_{2i}] = z_1 \quad \text{for } i = 1, 2, \ldots, r,
\]
\[
[v_{2i}, v_{2i+1}] = z_2 \quad \text{for } i = 1, 2, \ldots, r - 1,
\]
\[
[v_1, v_r] = z_2.
\]

With respect to the basis $\mathcal{B}$, $n(2r + 2)$ is a uniform Lie algebra of type $(2, 2r, r)$ with $s = 2$.

We may also change one sign in the defining bracket relations for $n(2r + 2)$ to define the uniform Lie algebra $n'(2r + 2)$ with uniform basis $\mathcal{B} = \{v_i\}_{i=1}^{2r} \cup \{z_1, z_2\}$ and Lie bracket

\[
[v_{2i-1}, v_{2i}] = z_1 \quad \text{for } i = 1, 2, \ldots, r,
\]
\[
[v_{2i}, v_{2i+1}] = z_2 \quad \text{for } i = 1, 2, \ldots, r - 1,
\]
\[
[v_1, v_1] = z_2.
\]

The Heisenberg type Lie algebra in the next example is isomorphic to the Iwasawa type nilpotent Lie algebra in the Iwasawa decomposition of the Lie algebra of the isometry group of quaternionic hyperbolic space of dimension 8.

**Example 2.6.** Let $(n, Q)$ be the metric nilpotent Lie algebra with orthonormal basis $\mathcal{B} = \{v_i\}_{i=1}^{4} \cup \{z_j\}_{j=1}^{3}$ and Lie bracket

\[(2) \quad [v_1, v_2] = [v_3, v_4] = z_1, [v_1, v_3] = -[v_2, v_4] = z_2, [v_1, v_4] = [v_2, v_3] = z_3.
\]

With respect to $\mathcal{B}$, $n$ is a uniform Lie algebra of type $(3, 4, 2)$.

As already seen in Example 2.5, it is possible that there are two or more uniform Lie algebras which have the same set of indices of nonzero structure constants relative to a uniform basis; these may or may not be isomorphic. In the next example we present a uniform Lie algebra which, with respect to the uniform basis, has the same indices for nonzero structure constants as the Lie algebra in Example 2.6.

**Example 2.7.** Let $(n, Q)$ be the metric nilpotent Lie algebra with orthonormal basis $\mathcal{B} = \{v_i\}_{i=1}^{4} \cup \{z_j\}_{j=1}^{3}$ and Lie bracket

\[(3) \quad [v_1, v_2] = [v_3, v_4] = z_1, [v_1, v_3] = [v_2, v_4] = z_2, [v_1, v_4] = -[v_2, v_3] = z_3.
\]

This metric Lie algebra is not of Heisenberg type; therefore $n$ is not isomorphic to the Lie algebra in Example 2.6.
This presentation of \( \mathfrak{n} \) is not the most efficient one, in the sense that more nontrivial brackets of basis vectors appear than is necessary. With respect to a different basis \( \mathcal{C} = \{ u_i \}_{i=1}^4 \cup \{ y_j \}_{j=1}^3 \), the Lie algebra is given by

\[
[u_1, u_2] = y_1, [u_1, u_3] = [u_2, u_4] = y_2, [u_3, u_4] = y_3.
\]

The bases \( \mathcal{B} \) and \( \mathcal{C} \) are related by

\[
v_1 = u_1 + u_3, v_2 = u_2 + u_4, v_3 = u_3 - u_1, v_4 = u_4 - u_2,
\]

\[
z_1 = y_1 + y_3, z_2 = 2y_2, z_3 = -y_1 + y_3.
\]

Note that the basis \( \mathcal{C} \) is not orthogonal with respect to \( Q \).

The following family \( m(q) \) of nilpotent Lie algebras arose in the study of Anosov Lie algebras (Example 4.4, [23]). Because a cyclic group acts transitively on it, it is among a class of nilpotent Lie algebras called cyclic Lie algebras.

**Example 2.8.** Let \( m(q) \) be the nilpotent Lie algebra with basis \( \mathcal{B} = \{ v_i \}_{i=1}^q \cup \{ z_j \}_{j=1}^p \) and Lie bracket

\[
[v_1, v_2] = z_1, [v_2, v_3] = z_2, \ldots, [v_q, v_1] = z_q.
\]

With respect to the given basis, the Lie algebra \( m(q) \) is uniform of type \( (q, q, 1) \).

### 3. Properties of Uniform Lie Algebras

In this section we derive some basic properties of uniform Lie algebras. First we review some standard definitions. Let \( (\mathfrak{n}, Q) \) be a two-step nilpotent metric Lie algebra with center \( z \), and let \( \mathfrak{v} \) be the orthogonal complement to the center. For all \( z \in z \), the map \( J_z \) in \( \text{End}(\mathfrak{v}) \) is defined by \( J_z v = \text{ad}_v^* z \) for all \( v \in \mathfrak{v} \), where \( \text{ad}_v^* \) is the metric dual of the linear map \( \text{ad}_v \).

If \( \mathfrak{n} \) is uniform with respect to \( \mathcal{B} = \{ v_i \}_{i=1}^q \cup \{ z_j \}_{j=1}^p \), we can use the identity \( Q(J_{z_l} v_i, v_j) = Q([v_i, v_j], z_l) \) to show that

\[
J_{z_l}(v_i) = \sum_{j=1}^p \epsilon_{ji}^l v_j, \quad \text{where} \quad \epsilon_{ji}^l = \begin{cases} 1 & \text{if } [v_i, v_j] = z_l \\ -1 & \text{if } [v_i, v_j] = -z_l \\ 0 & \text{otherwise} \end{cases}
\]

Since \( \mathfrak{n} \) is uniform with respect to \( \mathcal{B} \), for fixed \( i \) and \( l \), there is at most one \( j \) so that \( [v_i, v_j] = \pm z_l \). Thus we obtain simple formula for \( J_{z_l} \), when \( z_l \) is in a uniform basis for a uniform Lie algebra.
Proposition 3.1. Let $(n, Q)$ be a uniform metric Lie algebra with uniform basis $B = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p$. Then

$$J_{z_j}(v_i) = \begin{cases} v_j & \text{if there exists } v_j \text{ so that } [v_i, v_j] = z_l \\ -v_j & \text{if there exists } v_j \text{ so that } [v_i, v_j] = -z_l \\ 0 & \text{otherwise} \end{cases}$$

Fundamental properties of uniform Lie algebras are collected in the next theorem. Recall that the Frobenius inner product on $\text{End}(\mathbb{R}^q)$ is defined by $\langle A, B \rangle = \text{trace}(AB^T)$ for $A, B \in \text{End}(\mathbb{R}^q)$.

Theorem 3.2. Let $n$ be a uniform Lie algebra of type $(p, q, r)$ with uniform basis $B = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p$ and degree $s$. Let $v = \text{span}\{v_i\}_{i=1}^q$ and let $\mathfrak{z} = \text{span}\{z_j\}_{j=1}^p$. Then $n$ is a two-step nilpotent Lie algebra with the following properties.

1. The center of $n$ is $\mathfrak{z}$, and $[n, n] = \mathfrak{z}$.
2. The centralizer $C(v_i)$ of any vector $v_i$ in $B$ is spanned by $\{v_j : [v_i, v_j] = 0\} \cup \{z_j\}_{j=1}^p$, and it has dimension $p + q - s$.
3. For all $v_i$ in $B$, the rank of $\text{ad}_{v_i}$ is $s$.
4. For all $z_j$ in $B$, the rank of $J_{z_j}$ is $2r$.
5. The set of maps $\{J_{z_j}\}_{j=1}^p$ is orthogonal with respect to the Frobenius inner product on $v$. For all $z_j$ in $B$, the Frobenius norm of $J_{z_j}$ is

$$\sqrt{-\text{trace}J_{z_j}^2} = \sqrt{2r}.$$ 

To prove the theorem we require the lemma below.

Lemma 3.3. Let $n$ be a uniform Lie algebra of type $(p, q, r)$ with uniform basis $B = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p$ and degree $s$. Let $v_i \in B$. Then the set

$$A_i = \{[v_i, v_j] : v_j \in B \text{ and } [v_i, v_j] \neq 0\}$$

is a linearly independent subset of $\{z_j\}_{j=1}^p$ of cardinality $s$.

Proof. Let $v_j \in B$, and let $A_i$ be as in the statement of the lemma. Since the degree $s$ in Part (4) of the definition of uniform Lie algebra is assumed to be positive, $A_i$ is nonempty.

From Part (1) of the definition of uniform Lie algebra we see that $A_i \subseteq \{\pm z_1, \pm z_2, \ldots, \pm z_p\}$. As $\{z_j\}_{j=1}^p$ is a subset of a basis, it is linearly independent. Therefore, in order to show that $A_i$ is linearly independent, it suffices to show that if $z_l \in A_i$, then $-z_l \notin A_i$. Assume to the contrary that $z_l$ and $-z_l$ are both in $A_i$. Then there exist $v_j$ and $v_k$ in $\{v_i\}_{i=1}^q$, necessarily distinct, so that $[v_i, v_j] = z_l$ and $[v_i, v_k] = -z_l$. We then have $[v_i, v_j] = -[v_i, v_k]$, and by Part (2) of the definition of uniform Lie algebra, $v_j = v_k$, a contradiction. Hence $z_l$ and $-z_l$ are not both in $A_i$, and the set $A_i$ is linearly independent.
By Part (4) of the definition of uniform Lie algebra, the cardinality of \( A_i \) is \( s \).

Now we are ready to prove Theorem 3.2.

Proof. Let \( (n, Q) \) be a uniform metric Lie algebra with uniform basis \( B = \{ v_i \}_{i=1}^q \cup \{ z_j \}_{j=1}^p \). It follows easily from Part (1) of Definition 2.1 that \( z = \text{span}\{z_j\}_{j=1}^p \) is contained in the center of \( n \). To show that the center is no larger, assume that \( x \) is in the center. Write \( x \) with respect to the uniform basis, so \( x = \sum_{i=1}^q \alpha_i v_i + \sum_{j=1}^p \beta_j z_j \). We must show that \( \alpha_i = 0 \) for all \( i \). Fix \( i_0 \). By Part (2) of the definition of uniform Lie algebra, there is some \( v_k \in B \) so that \( [v_{i_0}, v_k] \neq 0 \). Then

\[
0 = [v_k, x] = \sum_{i=1}^q \alpha_i [v_k, v_i] + \sum_{j=1}^p \beta_j [v_k, z_j] = \sum_{i=1}^q \alpha_i [v_k, v_i].
\]

Since \( [v_k, v_{i_0}] \neq 0 \), Lemma 3.3 implies that \( \alpha_{i_0} = 0 \). Hence, the center of \( n \) is \( z \).

Now we show that the commutator subalgebra \( [n, n] \) is equal to \( z \). It follows from Part (1) of the definition of uniform Lie algebra that \( [n, n] \subseteq z \). By Part (3) of the definition, \( z_l \in [n, n] \) for all \( z_j \in B \). Hence \( [n, n] = z \). From the properties of the bracket relations, we have \( [n, [n, n]] = \{0\} \), so \( n \) is a two-step nilpotent Lie algebra.

We omit the proof that \( C(v_i) = \text{span}\{v_j : [v_i, v_j] = 0 \} \cup \{ z_j \}_{j=1}^p \) because it is similar to the proof of Part (1). Since the spanning vectors are linearly independent, the dimension of \( C(v_i) \) is equal to the cardinality of the spanning set. By Part (4) of the definition of uniform Lie algebra, \( \{v_j : [v_i, v_j] = 0 \} \) has cardinality \( q - s \). Hence the dimension of \( C(v_i) \) is \( p + q - s \).

Part (3) of the proposition follows immediately from Lemma 3.3.

For Part (4) of the proposition, we use Proposition 3.1. Fix \( z_l \). The definition of uniform Lie algebra says that there are exactly \( r \) disjoint pairs \( \{v_i, v_j\} \) with \( [v_i, v_j] = z_l \). By Proposition 3.1, the image of \( J_{z_l} \) is the span of all \( v_i \) and \( v_j \) so that \( [v_i, v_j] = z_l \). Since the cardinality of that set is \( 2r \), the map \( J_{z_l} \) has rank \( 2r \).

Last we show that Part (5) holds. Let \( z_k, z_l \in B \). The number \( \text{trace}(J_{z_k}J_{z_l}^T) \) is nonzero if and only if there are \( i \) and \( j \) so that \( J_{z_k}v_i = \pm v_j \) and \( J_{z_l}v_i = \pm v_j \). But then \( \pm z_k = [v_i, v_j] = \pm z_l \). This is not possible for distinct \( k \) and \( l \), because \( z_k \) and \( z_l \) are linearly independent. By Proposition 3.1 after a change of basis, \( J_{z_l} \) is block diagonal, with nonzero blocks of form \( (0 \quad -1) \). There are precisely \( r \) of these blocks. Hence \( \text{trace}(J_{z_k}J_{z_l}^T) = -\text{trace}(J_{z_l}^2) = 2r \).
Remark 3.4. Statement (5) in the theorem may not be improved to say that trace $J_z^2 = -2r$ for all unit $z \in \mathbb{Z}$. For example, this fails in Example 2.8.

As a corollary to the previous theorem, since $p$ is the dimension of the center, and $p + q$ equals the total dimension, both of the parameters $p$ and $q$ are isomorphism invariants for uniform Lie algebras.

Corollary 3.5. Let $n_1$ be a uniform Lie algebra of type $(p_1, q_1, r_1)$ and let $n_2$ be a uniform Lie algebra of type $(p_2, q_2, r_2)$. If $n_1$ and $n_2$ are isomorphic, then $p_1 = p_2$ and $q_1 = q_2$.

The parameter $r$ is not an isomorphism invariant for uniform Lie algebras, as seen in the following example taken from [20].

Example 3.6. Let $n = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ with basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\} \cup \{z_1, z_2\}$, where $[x_1, x_2] = z_1$ and $[x_3, x_4] = z_2$. The basis $\mathcal{B}$ is a uniform basis of type $(2, 4, 1)$.

With respect to the basis $\mathcal{C} = \{u_i\}_{i=1}^4 \cup \{w_j\}_{j=1}^2$ given by

$u_1 = x_1 + x_3, u_2 = x_2 + x_4, u_3 = x_1 - x_3, u_4 = x_2 - x_4, w_1 = z_1 + z_2, w_2 = z_2 - z_1$,

the Lie bracket is given by

$[u_1, u_2] = [u_3, u_4] = w_1, [u_2, u_3] = [u_4, u_1] = w_2$.

Hence $n = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ is isomorphic to $n'(2r + 2)$ with $r = 2$ as in Example 2.5.

With respect to the basis $\mathcal{C}$, $n$ is uniform of type $(2, 4, 2)$.

Two Lie algebras $m$ and $n$ with bases $\mathcal{B}$ and $\mathcal{C}$ respectively are said to be associates with respect to bases $\mathcal{B}$ and $\mathcal{C}$ if their structure constants agree up to sign. That is, if $\alpha^k_{ij}$ denotes a structure constant for $m$ with respect to $\mathcal{B}$, and $\beta^k_{ij}$ denotes a structure constant for $n$ with respect to $\mathcal{C}$, then $(\alpha^k_{ij})^2 = (\beta^k_{ij})^2$ for all $(i, j, k)$. The Lie algebras in Examples 2.6 and 2.7 are nonisomorphic associates, as are the two families in Example 2.5. It is clear from the definition that uniform Lie algebras come in classes of associates.

Proposition 3.7. Suppose that the Lie algebras $m$ and $n$ with bases $\mathcal{B}$ and $\mathcal{C}$ respectively are associates with respect to bases $\mathcal{B}$ and $\mathcal{C}$. Then $m$ is a uniform Lie algebra with respect to the uniform basis $\mathcal{B}$ if and only if $n$ is a uniform Lie algebra with respect to the uniform basis $\mathcal{C}$.

Remark 3.8. Given a family of associate uniform Lie algebras (relative to given bases) as in Proposition 3.7, how can we determine which members of the family are isomorphic? In general, it can be very difficult to find an isomorphism between two Lie algebras. However, there is a simple computational method for finding isomorphic associate Lie algebras when the two presentations are related by a change of basis which simply changes signs.
of basis vectors. This procedure is described in Theorem B of [24]; one simply needs to find the orbits of a $\mathbb{Z}_2^n$ action on $\mathbb{Z}_2^m$ (defined in Definition 5 in [24]). We will use this method to find these kinds of isomorphic Lie algebras within classes of associate uniform Lie algebras, but because the set-up is somewhat technical, we do not reproduce the theorem statement here and refer the reader to [24] for details. Worked out examples of the method are given in [25]; see Examples 3.6, 3.7 and 4.5 there.

When the parameter $r$ of a uniform Lie algebra with uniform basis $\mathcal{B}$ is equal to one, then it is isomorphic to all of its associates relative to $\mathcal{B}$.

**Proposition 3.9.** Let $\mathfrak{n}$ be a uniform Lie algebra of type $(p, q, 1)$ with uniform basis $\mathcal{B}$. If $\mathfrak{m}$ is an associate to $\mathfrak{n}$ with respect to the basis $\mathcal{B}$, then $\mathfrak{m}$ and $\mathfrak{n}$ are isomorphic.

**Proof.** Let $\alpha_{ij}^k$ denote structure constants relative to the uniform basis $\{v_i\}_{i=1}^p \cup \{z_k\}_{k=1}^q$, so $[v_i, v_j] = \sum_{k=1}^q \alpha_{ij}^k z_k$. Fix $k$. But since $r = 1$, for all $k$ there is only one pair of indices $(i, j)$ so that structure constants $\alpha_{ij}^k$ and $\alpha_{ji}^k$ are nonzero. A change of basis sending $z_k$ to $-z_k$ while leaving all other basis vectors fixed changes the signs of $\alpha_{ij}^k$ and $\alpha_{ji}^k$ while leaving all remaining structure constants the same. By making all possible combinations of such changes of basis we obtain all possible sign choices for the structure constants.

Finally, we give some useful relationships the parameters $p, q, r$ and $s$ that we will use in Section 6.

**Proposition 3.10.** Let $\mathfrak{n}$ be a uniform Lie algebra of type $(p, q, r)$ with degree $s$. Then

1. $2rp = sq$,
2. $s \leq p \leq rp = \frac{1}{2}sq \leq \left(\frac{q}{2}\right)$,
3. $2 \leq 2r \leq q$.

Let $G = (V, E)$ be a uniformly colored graph of type $(p, q, r)$ with degree $s$. Then the same constraints hold, and $p$ divides $|E|$.

**Proof.** We have previously remarked that $2rp = sq$ because both numbers are equal to the number of nontrivial structure constants with respect to the uniform basis.

By Lemma 3.8, $s \leq p$. For each $v_i$ there are exactly $s$ basis vectors $v_j$ so that $[v_i, v_j] \neq 0$. The cardinality of $\{v_j\}_{j=1}^q$ is $q$, and $[v_i, v_i] = 0$, so $s \leq q - 1$. Hence $\frac{1}{2}sq \leq \frac{1}{2}(q-1)q = \left(\frac{q}{2}\right)$. Substituting $2rp$ for $sq$ gives $rp \leq \left(\frac{q}{2}\right)$. Hence Part (2) of the proposition holds.

Let $z_k$ be an element of the uniform basis. By Property (3) of a uniform Lie algebra, there are $r$ disjoint pairs $\{v_i, v_j\}$ drawn from $\{v_i\}_{i=1}^q$, a set of
cardinality \( q \), so that \([v_i, v_j] = z_k\). But then \( q \geq 2r \). Since \( r \geq 1 \), we get \( q \geq 2 \). □

4. uniformly colored graphs

4.1. Definition and examples of uniformly colored graph. Let \((V,E)\) be a graph with vertex set \( V \) and edge set \( E \). We always assume that all of our graphs are without loops or multiple edges. A graph is regular of degree \( s \) if each vertex has exactly \( s \) neighbors. An orientation of the graph \((V,E)\) is a map \( \sigma : E \rightarrow V \times V \) assigning to each edge \( \{v,w\} \) one of the ordered pairs \( (v,w) \) and \( (w,v) \). Then \((V,E)\), together with \( \sigma \), naturally defines a directed graph.

An edge coloring of a directed or undirected graph \( G = (V,E) \) is a map \( c : E \rightarrow S \). Let \( G = (V,E) \) be a graph with edge coloring \( c : E \rightarrow S \). A coloring is proper if no adjacent edges have the same color. We say that a mapping \( \phi : G \rightarrow G \) is a color-permuting automorphism or automorphism of the colored graph if \( \phi \) is an automorphism of the underlying graph, and there is a permutation \( \sigma \) of the color set \( S \) so that \( c \circ \phi(v,w) = \sigma \circ c(v,w) \) for all \( (v,w) \in E \). Two edge colorings \( b \) and \( c \) of a graph are called equivalent if there is a color-permuting automorphism \( \phi \) so that \( c = \phi \circ b \).

Definition 4.1. Let \((V,E)\) be an undirected graph with \( q \) vertices which is regular of degree \( s \). Let \( S \) be a set of cardinality \( p \), and let the surjective function \( c : E \rightarrow S \) define an edge coloring of \((V,E)\). Then the edge coloring is a uniform edge coloring of type \((p,q,r)\) with degree \( s \) if the edge-colored graph has the following properties.

1. The coloring is proper.
2. Each color occurs the same number, \( r \), of times; i.e. the cardinality of \( c^{-1}([s]) \) is \( r \), for all \( s \in S \).

When the undirected graph \((V,E)\) is endowed with such a coloring it is called a uniformly colored graph of type \((p,q,r)\). A directed graph is a uniformly colored graph of type \((p,q,r)\) if it is an undirected uniformly colored graph of type \((p,q,r)\) endowed with an orientation. A graph with a uniform edge coloring is said to be uniformly colored.

Note that if a uniformly colored graph \( G = (V,E) \) is directed, because \( G \) arises from imposing an orientation on an undirected graph, it is not possible for both \((v_i,v_j)\) and \((v_i,v_j)\) to be in \( E \). For the convenience of the reader, the next proposition summarizes how the parameters \( p,q,r \) and \( s \) for a uniformly colored graph are reflected in the graph. The proof is an easy application of definitions.

Proposition 4.2. A uniformly colored graph \( G = (V,E) \) of type \((p,q,r)\) with degree \( s \) has \( q \) vertices and \( rp \) edges. The degree of the underlying
undirected regular graph is $s$. There are $p$ colors, and each color occurs on exactly $r$ different edges.

A decomposition of a graph $G$ is a set of subgraphs $H_1, \ldots, H_k$ which partition the edges of $G$. Uniform edge colorings of undirected graphs of type $(p, q, r)$ and degree $s$ are equivalent to decompositions of $s$-regular graphs on $q$ vertices into $p$ copies of the $r$-fold disjoint sum $K_2 + \cdots + K_2$.

**Proposition 4.3.** Let $G$ be an undirected uniformly colored graph of type $(p, q, r)$ with degree $s$. For $k = 1, \ldots, p$, let $H_k$ be the subgraph of $G$ consisting of edges colored with the $k$th color. Then $H_1, H_2, \ldots, H_p$ defines a decomposition of $G$ into $p$ disjoint subgraphs, all isomorphic to the disjoint sum $K_2 + \cdots + K_2$.

Conversely, if $G = (V, E)$ is an undirected $s$-regular graph with $q$ vertices and $H_1, \ldots, H_p$ is a decomposition of $G$ into disjoint subgraphs each isomorphic to the disjoint sum $K_2 + \cdots + K_2$, then the coloring $c : E \to [p]$ defined by coloring an edge $k$ if it is in the factor $H_k$ is a uniform edge coloring of $G$.

(We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.) The proposition follows from the definition of uniformly colored graph. The next example shows that any regular graph admits a uniform edge coloring.

**Example 4.4.** Let $(V, E)$ be a regular graph with $q$ vertices and $p$ edges. Let the identity map $id : E \to E$ define a coloring. The resulting edge-colored graph is uniformly colored of type $(p, q, 1)$.

4.2. **Graph-algebra correspondences.** We associate to any uniform Lie algebra a uniform graph, and to any uniform graph a uniform Lie algebra.

**Definition 4.5.** Let $(n, Q)$ be a uniform Lie algebra of type $(p, q, r)$ with uniform basis $\mathcal{B} = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p$. Define a set of vertices by $V = \{v_i\}_{i=1}^q$, and let the set of directed edges be

$$E = \{(v_i, v_j) : [v_i, v_j] = z_k \text{ for some } k\}.$$ 

Define an edge coloring of the graph $(V, E)$ by mapping $E$ into the set $\{z_j\}_{j=1}^p$ so that the edge $(v_i, v_j) \in E$ is assigned to $z_k$ if $[v_i, v_j] = z_k$.

By Property (2) of the definition of uniform metric Lie algebras, the coloring function is well-defined.

**Proposition 4.6.** The graph associated to a uniform Lie algebra of type $(p, q, r)$ with degree $s$, as in Definition 4.5, is a uniformly colored graph of type $(p, q, r)$ with degree $s$.

**Proof.** The graph has no loops because $[v_i, v_i] = 0$ for all $i$. There are no multiple edges from $v_i$ to $v_j$ because $[v_i, v_j]$ is single-valued, and it is not
possible to have both \((v_i,v_j)\) and \((v_j,v_i)\) as an edge because \([v_i,v_j] = [v_j,v_i]\) implies that \([v_j,v_i] = 0\). Clearly there are \(q\) vertices and \(p\) colors. By Property (2) of Definition 2.1, the coloring is proper. Property (3) of Definition 2.1 insures that each color occurs the exactly \(r\) times and that the coloring function is surjective. □

Now we associate a uniform Lie algebra to any uniformly colored graph.

**Definition 4.7.** Let \((V,E)\) be a uniformly colored directed graph of type \((p,q,r)\) and degree \(s\). Let \(V = \{v_i\}_{i=1}^q\), let \(S = \{z_j\}_{j=1}^p\) and let \(c : E \rightarrow S\) be the surjective edge coloring function. Let \(n\) be the vector space with basis \(V \cup S\). Define a Lie bracket on \(n\) by setting

\[
[v_i,v_j] = \begin{cases} 
  c(v_i,v_j) & \text{if } (v_i,v_j) \in E \\
  -c(v_j,v_i) & \text{if } (v_j,v_i) \in E \\
  0 & \text{otherwise}
\end{cases}
\]

setting \([v_i,z_j] = 0\) and \([z_j,z_k] = 0\) for all \(i,j,k\), and extending bilinearly.

As the graph is uniform, it is not possible for both \((v_i,v_j)\) and \((v_j,v_i)\) to be in \(E\), so the Lie bracket in the definition is well-defined. By definition the Lie bracket is skew-symmetric. It is easy to see that \([n,n] = \text{span}\{z_j\}_{j=1}^p\) and \([n,[n,n]] = 0\), so the algebra is two-step nilpotent. Hence, the Jacobi Identity holds trivially. Thus, the product in Equation (6) truly does define a Lie algebra.

**Example 4.8.** The uniformly colored graph affiliated with the cyclic Lie algebra \(m(q)\) in Example 2.8 is the cycle graph \(C_q\) with \(q\) vertices (endowed with an orientation). All \(q\) edges have different colors.

The correspondences we have defined between uniformly colored graphs and uniform Lie algebras are inverse to one another.

**Proposition 4.9.** The correspondences between uniform Lie algebras (with fixed uniform bases) and uniformly colored graphs defined in Definitions 4.5 and Definition 4.7 are inverse to one another. The Lie algebra corresponding to a uniformly colored graph of type \((p,q,r)\) with degree \(s\) is a uniform Lie algebra with uniform basis of type \((p,q,r)\) and degree \(s\).

The proof is straightforward, so we leave it to the reader.

**Remark 4.10.** The correspondences in Definitions 4.5 and 4.7 between edge-colored graphs and algebras are defined more generally, even if not all of the properties of a uniform Lie algebra or uniformly colored graph hold. In the most general situation, there is a correspondence between edge-weighted directed graphs (possibly with loops or multiple edges) and two-step nilpotent Lie algebras. This is carefully described in Remark 3.2 of [27].
Remark 4.11. The two-step nilpotent Lie algebras of graph type from [6] can be defined as follows. Let $G = (V,E)$ be a graph on $q$ vertices without multiple edges or loops, where $V = \{v_i\}_{i=1}^q$. Let $\mathfrak{v}$ be the real span of $V$, so $\mathfrak{v} \cong \mathbb{R}^q$. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z} = \text{span}\{v_i \land v_j : \{v_i,v_j\} \in E\} \subseteq \mathfrak{so}(q)$, and define a Lie bracket for $\mathfrak{n}$ so that the only nonzero bracket relations of basis vectors are $[v_i,v_j] = v_i \land v_j$ if the edge $\{v_i,v_j\} \in E$. The Lie algebras in Examples 2.3 and 2.8 are of graph type.

A Lie algebra of graph type need not come from a regular graph, so not all Lie algebras of graph type are uniform. Lie algebras of graph type correspond, in the sense of Proposition 4.7, to colored graphs with each edge colored a different color, as in Example 4.4; hence if they are uniform, the parameter $r$ is equal to one as in Proposition 3.9. Therefore uniformly colored graphs of type $(p,q,r)$ with $r \neq 1$ may fail to be of graph type.

Remark 4.12. An alternate point of view on the definitions of uniform Lie algebra and Lie algebras of graph type is from the perspective of ideals. Every two-step nilpotent Lie algebra with $n$ generators is the quotient of $\mathfrak{f}_{n,2}$, the free two-step nilpotent Lie algebra on $n$ generators, by a central ideal. Let $x_1, x_2, \ldots, x_n$ denote $n$ generators of $\mathfrak{f}_{n,2}$ and let $\mathcal{B} = \{x_i\}_{i=1}^n \cup \{x_i \land x_j\}_{1 \leq i < j \leq n}$ be the basis for $\mathfrak{f}_{n,2}$ as in Example 2.3. Lie algebras of graph type are quotients of free two-step nilpotent Lie algebras by monomially generated ideals, i.e., ideals spanned by subsets of $\{x_i \land x_j\}_{1 \leq i < j \leq n}$, and every monomially generated ideal defines a Lie algebra of graph type. In contrast, akin to toric ideals, ideals defining uniform Lie algebras are generated by elements of the form $x_i \land x_j$, with $i \neq j$, or $x_i \land x_j \pm x_k \land x_l$, where $i, j, k, l$ are distinct.

4.3. Translations between algebraic properties and graph properties. The $J_z$ maps defined in Section 3 completely encode the algebraic structure of a metric nilpotent Lie algebra. The skew-adjacency matrix of a directed graph $G$ is defined to be the matrix $A(G) = (a_{ij})$ with $a_{ij} = 1$ if $(v_i,v_j) \in E$, $a_{ij} = -1$ if $(v_j,v_i) \in E$, and $a_{ij} = 0$ otherwise. Skew-adjacency matrices for uniformly colored graphs are closely related to the $J_z$ maps for the corresponding uniform nilpotent Lie algebra.

Proposition 4.13. Let $\mathfrak{n}$ be a uniform metric Lie algebra with uniform basis $\{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p$ and let $G = (V,E)$ be the associated uniformly colored graph with coloring function $c : E \to [p]$. Then $J_{z_j} = -A(H_j)$, where $H_j$ is the subgraph consisting of the edges with color $j$, so for $z = \sum_{j=1}^p b_j z_j$, the endomorphism $J_z$ is given by $J_z = -J \left( \sum_{j=1}^p b_j z_j \right) = -\sum_{j=1}^p b_j A(H_j)$.

To prove the proposition, use Proposition 3.1.
In geometry, it is of interest to understand when submanifolds of homogeneous manifolds are totally geodesic. Totally geodesic subalgebras of metric nilpotent Lie algebras are tangent to totally geodesic submanifolds of the corresponding nilmanifolds (See [8]). A subalgebra \( m \) of a metric nilpotent Lie algebra is totally geodesic if it is invariant under the \( J_z \) map for all \( z \in m \). Recall that the uniformly colored graph \( G = (V, E) \) corresponding to the Lie algebra \( n \) with uniform basis \( \mathcal{B} = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p \) has vertex set \( V = \{v_i\}_{i=1}^q \) and edge colors in the set \( \{z_j\}_{j=1}^p \).

**Proposition 4.14.** Let \( (n, Q) \) be a uniform metric nilpotent Lie algebra with uniform basis \( \mathcal{B} = \{v_i\}_{i=1}^q \cup \{z_j\}_{j=1}^p \). Let \( G = (V, E) \) be the corresponding uniformly colored graph. Let \( V' \subseteq V \) be a subset of \( V \) and let \( S' \) be a subset of \( \{z_j\}_{j=1}^p \), where \( V' \cup S' \neq \emptyset \). Let \( m \) be the subspace of \( n \) spanned by \( V' \cup S' \).

1. The subspace \( m \) is a subalgebra of \( (n, Q) \) if and only if whenever \( v_i \) and \( v_j \) are in \( V' \) and \( (v_i, v_j) \in E \), the color \( c(v_i, v_j) \) is in \( S' \).
2. Suppose \( S' \neq \emptyset \). The subspace \( m \) is invariant under the \( J_z \) map for all \( z \in m \) if and only if for all \( z_k \in S' \), every edge with color \( z_k \) has neither or both of its vertices in \( V' \).

**Proof.** The subspace \( m \) is a subalgebra if and only if \( [v_i, v_j] \in \text{span} S' \) for all \( v_i, v_j \in V' \). But \([v_i, v_j]\) is nontrivial if and only if \( (v_i, v_j) \in E \) or \( (v_j, v_i) \in E \), and in that case \([v_i, v_j] = \pm z_k \), where \( z_k \) is the color of the edge between \( v_i \) and \( v_j \).

By Proposition 4.13 the \( J_{z_k} \) map for color \( z_k \) inverts edges with color \( z_k \); i.e., if \( (v_i, v_j) \in E \) and \( c(v_i, v_j) = z_k \), then \( J_{z_k}(v_i) = v_j \) and \( J_{z_k}(v_j) = -v_i \), and if there is no edge colored \( z_k \) incident to vertex \( v_i \), then \( J_{z_k}(v_i) = 0 \). Hence, if \( z_k \in S' \), \( m \) is \( J_{z_k} \)-invariant if and only if any \( z_k \)-colored edge with one vertex in \( V' \) has the other vertex in \( V' \).

The next example describes a type of totally geodesic subalgebra for a uniform metric nilpotent Lie algebra \((n, Q)\).

**Example 4.15.** Let \( H \) be the union of connected components of \( G \) and let \( V' = V_H \) be the set of vertices spanning \( H \). Let \( S' = S_H \) be the set of colors assigned to edges of \( H \). Then \( V_H \cup S_H \) spans a totally geodesic subalgebra of \((n, Q)\).

Recall that the basis for defining the Lie algebra \( n \) corresponding to a uniformly colored graph \( G = (V, E) \) with colors in \( S \) is \( \mathcal{B} = V \cup S \). Any color-permuting automorphism \( \phi \) of \( G \) with permutation \( \sigma \) of \( S \) extends to a function \( \hat{\phi} : n \to n \) of the corresponding uniform Lie algebra \( n \), by defining \( \hat{\phi}(v) = \phi(v) \) for \( v \in V \) and \( \hat{\phi}(z) = \sigma(z) \) for \( z \in S \), and extending \( \phi \) from \( \mathcal{B} \) bilinearly. We show that the function \( \hat{\phi} \) is an automorphism of \( n \).
Proposition 4.16. Let $\phi$ be a color-permuting automorphism of a uniformly colored directed graph $G$ with corresponding uniformly colored Lie algebra $\mathfrak{n}$. Then the map $\hat{\phi} : \mathfrak{n} \to \mathfrak{n}$ is an automorphism of $\mathfrak{n}$. Furthermore, the map $\phi \mapsto \hat{\phi}$ from group of color-permuting automorphisms of $G$ to the automorphism group of $\mathfrak{n}$ is injective.

Proof. Let $G = (V, E)$ be a uniformly colored graph with coloring function $c : E \to S$, and denote the vertices of $G$ by $V = \{v_i\}_{i=1}^n$. Let $\phi : G \to G$ and $\sigma : S \to S$ define a color-permuting automorphism of $G$. Recall that $[v_i, v_j] \neq 0$ if and only if $(v_i, v_j) \in E$ or $(v_j, v_i) \in E$. Hence $[v_i, v_j] = 0$ if and only if $[\hat{\phi}(v_i), \hat{\phi}(v_j)] = 0$, and we need only to check that $[\hat{\phi}(v_i), \hat{\phi}(v_j)] = \hat{\phi}([v_i, v_j])$ for $(v_i, v_j) \in E$. Assume that $(v_i, v_j) \in E$. Then $(\phi(v_i), \phi(v_j)) \in E$.

By definitions of the Lie bracket and $\hat{\phi}$,

$$[\hat{\phi}(v_i), \hat{\phi}(v_j)] = [\phi(v_i), \phi(v_j)] = c([\phi(v_i), \phi(v_j)]) = c((\phi(v_i), \phi(v_j))),$$

while

$$\hat{\phi}([v_i, v_j]) = \hat{\phi}(c(v_i, v_j)) = \sigma(c(v_i, v_j)).$$

Thus, $[\hat{\phi}(v_i), \hat{\phi}(v_j)] = \hat{\phi}([v_i, v_j])$ as desired. Clearly the map $\phi \mapsto \hat{\phi}$ is injective. \hfill \Box

Recall that the disjoint union $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph whose vertex set is the union of the vertex sets of $G_1$ and $G_2$, and whose edge set is the union of the edge sets of $G_1$ and $G_2$. If $G_1$ and $G_2$ are edge-colored by coloring functions $c_1 : E_1 \to S_1$ and $c_2 : E_2 \to S_2$, then it is natural to color $G_1 + G_2$ by defining $c : E_1 \cup E_2 \to S_1 \cup S_2$ by $c(e) = c_1(e)$ if $e \in E_1$, and $c(e) = c_2(e)$ if $e \in E_2$.

In the next proposition we show how the disjoint union of two edge-colored directed graphs translates at the level of corresponding uniform Lie algebras. Actually, we do not need to assume that the graphs or Lie algebras in the proposition are uniform; even if regularity axioms involving $p, q$, and $r$ are dropped, Definition 2.1 and Definition 4.1 still make sense, and Propositions 4.5 and 4.7 still hold. (See Remark 4.10)

Proposition 4.17. Suppose that for $k = 1, 2$, $G_k = (V_k, E_k)$ is a colored graph of type $(p_k, q_k, r_k)$ with coloring function $c_k : E_k \to S_k$. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the corresponding uniform Lie algebras, and let $B_1 = \{v_i\}_{i=1}^{p_1} \cup \{z_j\}_{j=1}^{p_1}$ and $B_2 = \{x_i\}_{i=1}^{q_2} \cup \{w_j\}_{j=1}^{q_2}$ be the corresponding uniform bases of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively. Let $\mathfrak{n}$ be the Lie algebra associated to the colored graph $G_1 + G_2$. Let $i$ be the ideal in $\mathfrak{n}_1 \oplus \mathfrak{n}_2$ defined by

$$i = \text{span}\{(z_i - w_j) : z_i = w_j\}$$

if $S_1 \cap S_2 \neq \emptyset$ and let $i = \{(0, 0)\}$ otherwise. Then

(7) $\mathfrak{n}(G_1 + G_2) := (\mathfrak{n}_1 \oplus \mathfrak{n}_2)/i$. 
In particular, if $S_1$ and $S_2$ are disjoint, then $n \cong n_1 \oplus n_2$.

The idea of the proof is as follows. The center of $n_1 \oplus n_2$ is spanned by elements $(z_i, 0)$ and $(0, w_j)$ with $z_i$ in the first color set $S_1$ and $w_j$ in the second color set $S_2$. To get $n(G_1 + G_2)$, for every color $z_i = w_j$ occurring in both $S_1$ and $S_2$, we need to identify $(z_i, 0)$ and $(0, w_j)$. Hence the ideal $i$ must be spanned by $(z_i, -w_j)$, for all colors with $z_i = w_j$.

In the case that $S_1 \supseteq S_2$, the Lie algebra $n(G_1 + G_2)$ in the proposition is defined by a process that Jablonski calls concatenation of the “structure matrices” of $n_1$ and $n_2$ (Section 3, [12]). In the case that $S_1 \supseteq S_2$, the uniform basis for $n(G_1 + G_2)$ may be naturally identified with $\{v_i\}_{i=1}^{q_1} \cup \{z_j\}_{j=1}^{r_2}$.

We would like to know when the graph union of uniformly colored graphs defines a uniformly colored graph. In the next proposition we consider the case that the coloring sets are disjoint.

**Proposition 4.18.** For $k = 1$ and 2, let $G_k = (V_k, E_k)$ be a uniformly colored graph of type $(p_k, q_k, r_k)$ with degree $s_k$, and let $c_k : E_k \rightarrow S_k$ be the coloring function for $(V_k, E_k)$. Assume that the graphs $G_1$ and $G_2$ are disjoint, and the coloring sets $S_1$ and $S_2$ are disjoint. Define the coloring function $c : E_1 \cup E_2 \rightarrow S_1 \cup S_2$ by $c(e) = c_1(e)$ if $e \in E_1$, and $c(e) = c_2(e)$ if $e \in E_2$. Then the following are equivalent:

1. $G_1 + G_2$, endowed with the coloring function $c$, is a uniformly colored graph.
2. $s_1 = s_2$ and $r_1 = r_2$.

Furthermore, if $G_1 + G_2$, endowed with the coloring function $c$, is uniformly colored, then it is of type $(p_1 + p_2, q_1 + q_2, r_1 + r_2)$, where $r_1 = r_2$, and it has degree $s = s_1 = s_2$.

The proof is elementary. We leave it to the reader.

If we use Proposition 4.17 to re-interpret Proposition 4.18 in terms of algebras, we see how uniformly colored Lie algebras behave under direct sums.

**Corollary 4.19.** Let $n_1$ be a uniform Lie algebra of type $(p_1, q_1, r_1)$, with uniform basis $B_1$, and let $n_2$ be a uniform Lie algebra of type $(p_2, q_2, r_2)$, with uniform basis $B_2$. Let $n = n_1 \oplus n_2$ be the direct sum, and let $B$ be the union of $i_1(B_1)$ and $i_2(B_2)$ where $i_j : n_j \rightarrow n_1 \oplus n_2$, for $i = 1, 2$, are the natural inclusion maps. Then $B$ is a uniform basis of type $(p, q, r)$ if and only if $r_1 = r_2$ and $s_1 = s_2$. If $B$ is a uniform basis, then $n$ is uniform of type $(p_1 + p_2, q_1 + q_2, r_1 + r_2)$, where $r = r_1 = r_2$.

Next we look at the union of disjoint uniformly colored graphs having same coloring set to see when we get a uniformly colored graph. We leave the proof to the reader.
Proposition 4.20. For \( k = 1, 2 \), let \( G_k = (V_k, E_k) \) be a uniformly colored graph of type \( (p, q_k, r_k) \) with degree \( s_k \), and edge coloring function \( c_k : E_k \to S \). Suppose that the graphs \( G_1 \) and \( G_2 \) are disjoint. Define \( c : E_1 \cup E_2 \to S \) by \( c(e) = c_1(e) \) if \( e \in E_1 \), and \( c(e) = c_2(e) \) if \( e \in E_2 \). Then the following are equivalent:

1. \( G_1 + G_2 \), endowed with the coloring function \( c \), is a uniformly colored graph.
2. \( s_1 = s_2 \).

Furthermore, if \( G_1 + G_2 \) endowed with the coloring function \( c \) is uniformly colored, then it is of type \( (p, q_1 + q_2, r_1 + r_2) \), with degree \( s = s_1 = s_2 \).

Note that if \( G \) is a uniformly colored graph, it is not true that all of its connected components must be uniformly colored.

Now we translate Proposition 4.20 from graphs to algebras. This corollary overlaps with Proposition 3.4 of [12], which says that if \( n_1 = v_1 \oplus j \) and \( n_2 = v_2 \oplus j \) have Einstein extensions, then so does a concatenation of \( n_1 \) and \( n_2 \).

Corollary 4.21. Let \( n_1 \) be a uniform Lie algebra of type \( (p, q_1, r_1) \) of degree \( s \) with uniform basis \( B_1 = \{v_i\}_{i=1}^{q_1} \cup \{z_j\}_{j=1}^{s} \), and let \( n_2 \) be a uniform Lie algebra of type \( (p, q_2, r_2) \) of degree \( s \) with uniform basis \( B_2 = \{x_i\}_{i=1}^{q_1} \cup \{z_j\}_{j=1}^{s} \). Let \( n \) be the concatenation of \( n_1 \) and \( n_2 \) as in Equation (7). Then \( n \) is uniform of type \( (p, q_1 + q_2, r_1 + r_2) \).

5. Examples of uniform Lie algebras

5.1. Examples defined by colorings of notable graphs. Many well-known graphs from combinatorial, geometric or number-theoretic constructions admit uniform edge colorings. We give just one example here. Recall that for the Kneser graph \( K_{n,m} = (V, E) \), the vertex set \( V \) is the set of subsets of \([n]\) having cardinality \(m\). Two vertices (sets) are connected by an edge if and only if they are disjoint. The graph is regular with degree \( \binom{n-m}{m} \). From the symmetry of the roles of the vertices, any permutation of the \( n \) vertices is an automorphism.

Proposition 5.1. Let \( K_{n,m} = (V, E) \) be a Kneser graph, where \( m < n/2 \). Then \( K_{n,m} \) admits a uniform edge coloring. With this coloring, \( K_{n,m} \) is a uniformly colored graph of type \( (p, q, r) = (\binom{n}{2m}, \binom{n}{m}, \frac{1}{2}(\binom{2m}{m})) \).

Proof. First we define the coloring function \( c \). Two adjacent vertices have set union of cardinality \(2m\); we color the edge between them with the set \([n]\) \( \setminus (S_1 \cup S_2) \). Such a set is a subset of \([n]\) of cardinality \( \binom{n}{n-2m} = \binom{n}{2m} \), so there are \( p = \binom{n}{2m} \) colors. Clearly, from the \( S_n \) symmetry, the coloring map is surjective.
This coloring has the property that if the edge $e$ is incident to vertices $v$ and $w$, then the subsets $v, w$ and $c(e)$ of $[n]$ form a partition of $n$. Hence, for each edge, the edge’s color and one of the vertices it is incident to determine the other vertex incident to the edge. Hence, no vertex is incident to two edges with the same color, and the coloring is proper. Each color occurs the same number of times and each vertex has the same degree, again due to the $S_n$ symmetry of the graph.

The degree $s$ of vertex $v$ is the number of subsets of $[n]$ disjoint from $s$, which is $(\binom{n-m}{m})$. We compute $r$ using $r = \frac{sq}{2p} = \frac{1}{2}(\binom{2m}{m})$. □

As a special case of Proposition 5.1, we get a uniform edge coloring of the Petersen graph $K_{5,2}$. After assignments of orientation we may apply Proposition 4.9 to define uniform Lie algebras.

Example 5.2. A uniform Lie algebra of type $(5,10,3)$ defined by the uniform edge coloring of the Petersen graph $K_{5,2}$ has basis $\mathcal{B} = \{v_{ij}\}_{1 \leq i < j \leq 5} \cup \{z_k\}_{k=1}^5$ and Lie brackets determined by relations of form $[v_{ij}, v_{kl}] = \pm z_m$ for all $i, j, k, l, m$ with $i < j, k < l$ and $\{i, j, k, l, m\} = [5]$.

5.2. Examples defined by Cayley graphs. Let $G$ be a nontrivial group, and let $T \subseteq G$ be a nonempty set of elements of $G$ all of order 2. The set $T$ need not be a generating set. Let $Cay(G, T) = (G, E)$ be the Cayley graph of $G$ relative to $T$; it is a digraph with vertex set $G$ and arc set $\{(g, tg) : g \in G, t \in T\}$. Since the elements in $T$ all have order two, for all $t \in T$ and all $g \in G$, both $(g, tg)$ and $(tg, g)$ are edges. We identify such pairs with an undirected edge $\{g, tg\}$ to view $Cay(G, T)$ as an undirected graph. Because the identity of $G$ is not in $T$, $Cay(G, T)$ has no loops or multiple edges.

Define a coloring function $c : E \rightarrow T$ so that if $g \in G$ and $t \in T$, the edge $\{g, tg\}$ is assigned the color $t$.

Proposition 5.3. Let $G$ be a nontrivial group of cardinality $q$, and let $T \subseteq G$ be a nonempty set of $p \geq 1$ elements of $G$, all of order 2. Let $Cay(G, T)$ be the undirected Cayley graph of $G$ relative to $T$. The function $c$ defined above defines a uniform edge coloring on $Cay(G, T)$ with respect to which $Cay(G, T)$ is a uniformly colored graph of type $(p, q, q/2)$ and degree $p$.

Proof. Clearly there are $q$ vertices. Fix a vertex $g$ and $t \in T$. Since $t$ has order two, $tg \neq g$, and there is an edge labelled $t$ between $t$ and $tg$. That makes a total of $p$ edges incident to $g$. Hence, $Cay(G, T)$ is regular of degree $s = p$, and the coloring map is surjective. Because $G$ is a group, there is exactly one edge colored $t$ that is incident to $g$. This gives a total of $r = q/2$ edges labelled $t$. □

Example 5.4. If we let $G$ be the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and let $T_1 = \{(1, 0), (0, 1), (1, 1)\}$, then the resulting uniformly colored complete graph
is the one corresponding to the uniform Lie algebras in Examples 2.6 and 2.7. If we instead take \( T_2 = \{(1,0),(0,1)\} \), then we get the uniformly colored cycle graph associated to Example 2.4. Finally, the set \( T_2 = \{(1,0)\} \) gives the graph \( K_2 + K_2 \) colored with one edge color; the corresponding Lie algebra is the five-dimensional Heisenberg algebra \( h_5 \) as in Example 2.2.

We may let \( G \) be any finite reflection group.

**Example 5.5.** If we let \( G = S_3 \) and \( T = \{(12),(13),(23)\} \), then Cay\((G,T)\) is the Thomsen graph and the coloring described in Proposition 5.3 makes Cay\((G,T)\) into a uniform Lie algebra of type \((3,6,3)\). The corresponding nine-dimensional uniform Lie algebra is of type \((3,6,3)\). Note that the Thomsen graph is isomorphic to the complete bipartite graph \( K_{3,3} \), and that elements of odd order are in one partite set of Cay\((G,T)\) while even elements are in the other.

We leave it to the reader to confirm that, more generally, if \( G \) is the dihedral group of order \( 2p \), by taking \( T \) to be the set of all reflections, we get a uniform edge coloring on the complete bipartite graph \( K_{p,p} \). The corresponding \( 3p \)-dimensional uniform Lie algebra is of type \((p,2p,p)\).

5.3. **Examples defined by one-factorizations and near-one-factorizations.**

Let \( G = (V,E) \) be a graph. A **factor** of \( G \) is a subgraph with vertex set \( V \) and edge set \( E' \subseteq E \). A **factorization** of \( G \) is a set of factors of \( G \) so that the factors are pairwise edge-disjoint and whose union is \( G \). A **one-factor** is a factor which is a regular graph of degree one. A **one-factorization** of \( G \) is a factorization for which each factor is a one-factor. (If \( G \) is oriented, these definitions extend in the obvious way.) Clearly, a graph on an odd number of vertices can not have a one-factorization. In this case, the closest thing to a one-factorization is a **near-one-factorization**, defined as follows. A set of edges which covers all but one vertex in a graph is called a **near-one-factor**. A decomposition of a graph as the union of near-one-factors is a **near-one-factorization**. We may use Proposition 4.3 to define a uniform edge coloring of a graph admitting a one-factorization or near-one factorization. Hence, every one-factorization or near-one-factorization of a graph defines a class of associate uniform metric Lie algebras.

**Proposition 5.6.** Let \( G \) be an \( s \)-regular graph \( G \) with \( q \) vertices and \( m \) edges.

1. If \( q \) is even, every one-factorization of \( G \) defines a uniformly colored graph of type \((s,q,q/2)\), and \( m = sq/2 \).
2. If \( q \) is odd, every near-one-factorization of \( G \) defines a uniformly colored graph of type \((\frac{sq}{q-1},q,\frac{q-1}{2})\), and \( m = sq/2 \).

**Proof.** Proposition 4.3 defines a uniform edge coloring for a one-factorization or near-one factorization of a graph. If \( q \) is even and \( G \) has a one-factorization,
each factor has $q$ vertices and $q/2$ edges, so each color occurs $r = q/2$ times in such the corresponding coloring. Substituting into $2rp = sq$ gives $p = s$. The number of edges is $m = rp = sq/2$. If $q$ is odd, and $G$ has a near-one-factorization, each graph in the decomposition has $(q - 1)/2$ edges, so each color occurs $r = (q - 1)/2$ times in the associated coloring. Substituting into $2rp = sq$ gives $p = sq/(q - 1)$. Then $m = rp = sq/2$. □

Let $K_n$ denote the complete graph on $n$ vertices. All complete graphs admit a one-factorization or a near one-factorization, depending on whether their order is even or odd. Up to equivalence $K_5$ has a unique near-one-factorization. This should be known, but we could not find a reference. See [29] for a detailed proof. Alternately, use the fact that by adding a point at infinity, every near-one-factorization of $K_5$ extends to a one-factorization of $K_6$, and apply Sylvester’s argument for the uniqueness of one-factorizations of $K_6$ (up to isomorphism) to get a canonical presentation of near-one-factorizations of $K_5$ (see [3]).

**Example 5.7.** Let $K_5 = (V, E)$ be the complete graph on 5 vertices in $\{v_i\}_{i=1}^5$. Define the edge coloring $c : E \to [5]$ by

\[
\begin{align*}
    c(\{e_2, e_3\}) &= c(\{e_3, e_4\}) = 1 \\
    c(\{e_4, e_5\}) &= c(\{e_1, e_3\}) = 2 \\
    c(\{e_1, e_5\}) &= c(\{e_2, e_4\}) = 3 \\
    c(\{e_1, e_2\}) &= c(\{e_3, e_5\}) = 4 \\
    c(\{e_1, e_4\}) &= c(\{e_2, e_3\}) = 5.
\end{align*}
\]

Then $K_5$ together with the coloring $c$ is a uniformly colored graph of type $(5, 5, 2)$ with degree 4.

We will need the next lemma for the proof of Theorem 6.3.

**Lemma 5.8.** Let $n_1$ and $n_2$ be two associate uniform Lie algebras defined by the uniformly colored graph in Example 5.7. Then $n_1$ is isomorphic to $n_2$.

**Proof.** One choice of orientation of the graph gives the Lie algebra $n_1$ with

\[
\begin{align*}
    [v_3, v_4] &= [v_2, v_5] = z_1, [v_4, v_5] = [v_3, v_1] = z_2, [v_5, v_1] = [v_4, v_2] = z_3, \\
    [v_1, v_2] &= [v_5, v_3] = z_4, [v_2, v_3] = [v_1, v_4] = z_5.
\end{align*}
\]

Theorem B of [24] can be used to show that all uniform Lie algebras with this set of nonzero structure constants are isomorphic to $n_1$ or the Lie algebra $n_2$ with uniform basis $\{x_j\}_{j=1}^5 \cup \{w_j\}_{j=1}^5$ and Lie bracket

\[
\begin{align*}
    [x_3, x_4] &= [x_3, x_5] = w_1, [x_4, x_5] = [x_3, x_1] = w_2, [x_5, x_1] = [x_4, x_2] = w_3, \\
    [x_1, x_2] &= [x_5, x_3] = w_4, [x_2, x_3] = -[x_1, x_4] = w_5.
\end{align*}
\]
(See Remark 3.8.) But these two Lie algebras are isomorphic through the isomorphism \( \phi : n_2 \rightarrow n_1 \) defined by
\[
\phi(x_1) = -v_2, \phi(x_2) = -v_5, \phi(x_3) = -v_3, \phi(x_4) = -v_1, \phi(x_5) = v_4, \text{ and}
\]
\[
\phi(w_1) = z_2, \phi(w_2) = -z_5, \phi(w_3) = -z_3, \phi(w_4) = z_1, \phi(w_5) = z_4.
\]
Hence all sign choices yield isomorphic Lie algebras. \( \square \)

There is an extensive body of research on graph factorizations, one-factorizations and near-one-factorizations.

**Remark 5.9.** Graphs may admit inequivalent one-factorizations. For \( n \leq 3 \), \( K_{2n} \) has a single one-factorization up to isomorphism. For all \( n \geq 4 \), \( K_{2n} \) has non-isomorphic one-factorizations. The number \( F(2n) \) of nonisomorphic one-factorizations of \( K_{2n} \) grows like \( \ln F(2n) \sim 2n^2 \ln(2n) \) ([4]), showing that there is a profusion of uniformly colored graphs in higher dimensions.

See [18, 30, 26, 31] for results on classes of graphs admitting one-factorizations, and counts of nonisomorphic factorizations of \( K_{2n} \) for small \( n \).

6. **Classification of Uniform Lie Algebras of Type \((p, q, r)\)**

   with \( q \leq 5 \)

In Theorem 6.3 of this section we classify up to isomorphism all uniform Lie algebras of type \((p, q, r)\) with \( q \leq 5 \). We begin by establishing some general classification results which will be useful in the proof of Theorem 6.3. First we show that any uniform Lie algebra of type \((p, q, r)\) with \( p = 1 \) must be Heisenberg.

**Proposition 6.1.** Let \( n \) be a uniform Lie algebra of type \((1, q, r)\). Then \( q = 2r \) and \( n \) is isomorphic to the \((2r + 1)\)-dimensional Heisenberg algebra.

The proposition may be proved by using the definition of uniform Lie algebra, or by observing that any two-step nilpotent Lie algebra with a one-dimensional center must be the direct sum of a Heisenberg algebra and an abelian factor. However, by Part (1) of Proposition 3.2 uniform Lie algebras do not have abelian factors.

Nikolayevsky has classified the two-step Einstein nilradicals with two-dimensional center ([21]). See also [15] in which nonsingular two-step Einstein nilradicals with two-dimensional center are classified. It is not hard to describe uniform Lie algebra of type \((2, q, r)\) arising from connected graphs.

**Proposition 6.2.** Let \( n \) be a uniform Lie algebra of type \((2, q, r)\). Suppose that the corresponding uniformly colored graph is connected. Then \( q = 2r \), where \( r \geq 2 \), \( s = 2 \), and \( n \) and isomorphic to one of the Lie algebras \( n(2r + 2) \) and \( n'(2r + 2) \) defined in Example 2.5.
Table 1. Uniform Lie algebras of type $(p, q, r)$ with $q \leq 5$

| Case | $(p, q, r)$ | Defining bracket relations | Description |
|------|-------------|-----------------------------|-------------|
| 1    | (1, 2, 1)   | $[v_1, v_2] = z_1$          | $\mathfrak{h}_3$ |
| 2    | (1, 4, 2)   | $[v_1, v_2] = [v_3, v_4] = z_1$ | $\mathfrak{h}_5$ |
| 2    | (2, 4, 1)   | $[v_1, v_2] = z_1$, $[v_3, v_4] = z_2$ | $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \cong \mathfrak{h}_2$ |
| 4    | (2, 4, 2)   | $[v_1, v_3] = [v_2, v_4] = z_1$, $[v_1, v_4] = [v_2, v_3] = z_2$ | Example 3.6 |
| 3    | (3, 3, 1)   | $[v_i, v_j] = z_{ij}$, $1 \leq i < j \leq 3$ | $\mathfrak{f}_{3,2}$ |
| 4    | (4, 4, 1)   | $[v_1, v_2] = z_1$, $[v_2, v_3] = z_2$, $[v_3, v_4] = z_3$, $[v_4, v_1] = z_4$ | Example 2.8 |
| 4    | (2, 4, 2)   | $[v_1, v_2] = [v_3, v_4] = z_1$, $[v_2, v_3] = [v_4, v_1] = z_2$ | Example 2.4 |
| 5    | (5, 5, 1)   | $[v_1, v_2] = z_1$, $[v_2, v_3] = z_2$, $[v_3, v_4] = z_3$, $[v_4, v_5] = z_5$ | Example 2.8 |
| 6    | (3, 4, 2)   | $[v_1, v_2] = [v_3, v_4] = z_1$, $[v_1, v_3] = [v_2, v_4] = z_2$, $[v_1, v_4] = [v_2, v_3] = z_3$ | Example 2.6 |
| 6    | (3, 4, 2)   | $[v_1, v_2] = [v_3, v_4] = z_1$, $[v_1, v_3] = [v_2, v_4] = z_2$, $[v_1, v_4] = [v_2, v_3] = z_3$ | Example 2.7 |
| 6    | (6, 4, 1)   | $[v_i, v_j] = z_{ij}$, $1 \leq i < j \leq 4$ | $\mathfrak{f}_{4,2}$ |
| 7    | (5, 5, 2)   | $[v_1, v_3] = [v_2, v_4] = z_1$, $[v_2, v_5] = [v_3, v_4] = z_2$, $[v_1, v_3] = [v_4, v_5] = z_3$, $[v_3, v_5] = [v_1, v_2] = z_4$, $[v_1, v_4] = [v_2, v_3] = z_5$ | Example 5.7 |
| 7    | (10, 5, 1)  | $[v_i, v_j] = z_{ij}$, $1 \leq i < j \leq 5$ | $\mathfrak{f}_{5,2}$ |

**Proof.** Let $(V, E)$ be the uniformly colored graph as in the statement of the proposition. The universal cover of this graph is the infinite line graph $T_2$ uniformly colored with two colors. The only finite uniformly colored quotients are even cycle graphs with edges colored alternately in two colors.

Theorem B of [24] (see Remark 3.8) can be used to show that all choices of signs for the structure constants yield either $n(2r+2)$ or $n'(2r+2)$. □

Now we reach our main classification theorem.

**Theorem 6.3.** Suppose that $\mathfrak{n}$ is a uniform Lie algebra of type $(p, q, r)$. If $q \leq 5$, then up to Lie algebra isomorphism, $\mathfrak{n}$ occurs exactly once in Table 1.

**Proof.** Suppose that $\mathfrak{n}$ is a uniform Lie algebra of type $(p, q, r)$ with $q \leq 5$. Let $G = (V, E)$ be the associated uniformly colored regular graph as in Definition 4.5. By Proposition 4.6, $(V, E)$ is a regular graph with $q$ vertices.
Table 2. Regular graphs with $q$ vertices and degree $s$, for $q \leq 5$ and $s \geq 1$. The rightmost columns gives the number of inequivalent uniform edge colorings and the number $p$ of colors for each of those colorings.

| Case | Degree $s$ | $q$ | Graph   | Uniform colorings | $p$ |
|------|------------|-----|---------|-------------------|-----|
| 1    | 1          | 2   | $K_2$   | 1                 | 1   |
| 2    | 1          | 4   | $K_2 + K_2$ | 2             | 1, 2|
| 3    | 2          | 3   | $C_3$   | 1                 | 3   |
| 4    | 2          | 4   | $C_4$   | 2                 | 2, 4|
| 5    | 2          | 5   | $C_5$   | 1                 | 5   |
| 6    | 3          | 4   | $K_4$   | 2                 | 3, 6|
| 7    | 4          | 5   | $K_5$   | 2                 | 5, 10|

and $rp$ edges. Such graph are classified; a list of all such graphs is in Table 2.

None of the regular graphs in the table have the same values of $p$ and $q$, unless $(p, q) = (2, 4)$. By Corollary 3.5, $p$ and $q$ are algebraic invariants, so we only need to show that any two algebras in Table 1 arising from the same graph in Table 2 and having the same value of $p$ are nonisomorphic, and that any two graphs with $(p, q) = (2, 4)$ are nonisomorphic.

We do a case by case analysis of the graphs in Table 2. We know from Proposition 3.10 that $2rp = sq, s \leq p$, and $p$ divides $|E|$. The first thing we do in each case is to use these three constraints to determine a set of feasible values for $p$. We then find possible uniform edge colorings and the Lie algebras associated to those colorings.

Case 1. If $G = K_2$, then $s = 1$ and $q = 2$. The only possible value for $p$ is one. By Proposition 6.1, the corresponding uniform Lie algebra is three-dimensional Heisenberg algebra $h_3$.

Case 2. Suppose that $G = K_2 + K_2$, so $s = 1$ and $q = 4$. Then $p = 1$ or $p = 2$. If $p = 1$, then by Proposition 6.1, the corresponding uniform Lie algebra is five-dimensional Heisenberg algebra $h_5$. If $p = 2$, then $r = 1$ and we have the algebra with $[v_1, v_2] = \pm z_1$ and $[v_3, v_4] = \pm z_2$. By Proposition 3.9, all such algebras are isomorphic to $h_3 \oplus h_3$.

Case 3. If $G = C_3$, then $s = 2$ and $q = 3$, then $p = 3$ and $r = 1$. The underlying graph is complete and $p = (\frac{3}{2})$. By Proposition 3.9, all orientations of the graph define isomorphic Lie algebras, and $n$ is isomorphic to $f_{3, 2}$.

Case 4. When $G = C_4$, $s = 2$ and $q = 4$. It follows that $p = 2$ or $p = 4$. If $p = 2$, Proposition 6.2 tells us that we have either the six-dimensional Lie algebra in Example 2.4, or $h_3 \oplus h_3$ as in Example 3.6. These are not
isomorphic, because the Lie algebra in Example 2.4 is irreducible. If \( p = 4 \), then \( r = 1 \), and by Proposition 3.9, \( n \) is isomorphic to the eight-dimensional cyclic Lie algebra \( m(4) \) as in Examples 2.8 and 4.8.

**Case 5.** If \( G = C_5 \), then \( s = 2 \) and \( q = 5 \), then \( p = 5 \). By Proposition 3.9, all orientations of the graph yield isomorphic Lie algebras. Hence \( n \) is isomorphic to the cyclic Lie algebra \( m(5) \) defined in Example 2.8.

**Case 6.** If \( G = K_4 \), then \( s = 3 \) and \( q = 4 \). Either \( p = 3 \) or \( p = 6 \). If \( p = 3 \), then each color occurs \( r = 2 \) times, and the coloring defines a one-factorization of \( K_4 \). Up to equivalence, there is a unique one-factorization of \( K_4 \). It yields uniform Lie algebras with

\[
[v_1, v_2] = \pm z_1, [v_3, v_4] = \pm z_2, [v_1, v_3] = \pm z_2, [v_2, v_3] = \pm z_2,
\]

\[
[v_1, v_4] = \pm z_3, [v_2, v_3] = \pm z_3.
\]

as in the nonisomorphic Lie algebras from Examples 2.6 and 2.7.

We claim that any Lie algebra defined as above is isomorphic to either the Lie algebra in Example 2.6 or the Lie algebra in Example 2.7. We use the method described in Remark 3.8 to reduce the problem to considering four different sign choices which are encoded in the sextuples

\[
s_1 = (+, +, +, +, +, +), \quad s_2 = (+, +, +, +, +, -),
\]

\[
s_3 = (+, +, +, -, +, +), \quad s_4 = (+, +, +, -, +, -),
\]

where the order of signs matched with the order of the brackets in Equation (8). The signs in \( s_3 \) define the Heisenberg type Lie algebra in Example 2.6 while the sign in \( s_2 \) give the Lie algebra in Example 2.7.

The Lie algebras with signs as in \( s_1 \) and \( s_4 \) are both isomorphic to the Lie algebra in Example 2.7, which has signs as in \( s_2 \). The change of basis \( x_1 = v_1, x_2 = v_3, x_3 = v_2, x_4 = v_4, w_1 = z_2, w_2 = z_1, w_3 = z_3 \) converts from the basis \( \{v_i\}_{i=1}^4 \cup \{z_j\}_{j=1}^3 \) with signs as in \( s_1 \) to a new basis \( \{x_i\}_{i=1}^4 \cup \{w_j\}_{j=1}^3 \) with signs as in \( s_2 \). The change of basis

\[
x_1 = v_2, x_2 = -v_1, x_3 = v_3, x_4 = v_4, w_1 = z_1, w_2 = z_3, w_3 = z_2
\]

takes the basis \( \{v_i\}_{i=1}^4 \cup \{z_j\}_{j=1}^3 \) with signs as in \( s_1 \) to the basis \( \{x_i\}_{i=1}^4 \cup \{w_j\}_{j=1}^3 \) with signs as in \( s_4 \). Thus, if \( p = 3 \), then \( n \) is isomorphic to either the Lie algebra from Example 2.6 or the Lie algebra in 2.7.

If \( p = 6 \), then each edge of the graph is a different color. By Proposition 3.9, \( n \) is isomorphic to \( f_{4,2} \).

**Case 7.** If \( G = K_5 \), then \( s = 4 \) and \( q = 5 \), so that \( p = 5 \) or \( p = 10 \). If \( p = 5 \), then the coloring is a near-one-factorization of \( K_5 \). Up to equivalence \( K_5 \) has a unique near-one-factorization. Hence the corresponding undirected
colored graph is the same as the one described in Example 5.7. By Lemma 5.8, all possible sign choices yield isomorphic Lie algebras. If \( p = 10 \), then 
\[
p = \left( \frac{q}{2} \right),
\]
so each edge of the graph is a different color. By Proposition 3.9, \( n \) is isomorphic to \( f_{5,2} \).

\[\square\]

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