New fractional differential inequalities with their implications to the stability analysis of fractional order systems

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Abstract

It is well known that the Leibniz rule for the integer derivative of order one does not hold for the fractional derivative case when the fractional order lies between 0 and 1. Thus it poses a great difficulty in the calculation of fractional derivative of given functions as well as in the analysis of fractional order systems. In this work, we develop a few fractional differential inequalities which involve the Caputo fractional derivative of the product of continuously differentiable functions. We establish some of their properties and propose a few propositions. We show that these inequalities play a very essential role in the stability analysis of nonautonomous fractional order systems.

1 Introduction

Fractional calculus has been playing a great role in many areas of science, engineering and interdisciplinary subjects, and has a broad range of applications [1, 2, 3, 4]. Indeed, fractional calculus allows a new way to model or design or enhance many scientific and engineering systems, and that can be described by linear or nonlinear fractional differential equations or fractional order systems. Thus fractional calculus will provide a great scope for the advancement of the scientific studies in the modern era.

Fractional operators (fractional derivative and/or integral) are global operators [1, 2]. Unlike the usual local derivative operators (integer order derivative operators), which have easy properties (e.g. Leibniz rule, Chain rule and so on), but the properties of fractional derivative operators are not simple (e.g. Leibniz rule, Chain rule and so on). Indeed, fractional operators possess very rich and complex properties [1, 2]. In [5, 6, 7], it has been shown that the Leibniz rule, Chain rule, etc., do not hold for fractional derivatives.
whenever the fractional order lies between 0 and 1. In fact, these rules do not hold for
the Caputo fractional derivative too. Thus due to the global nature and the complicated
properties of Caputo fractional derivative operator, in practice, it is a very difficult task
for the estimation of fractional derivative of the given functions.

Fractional differential inequalities (inequalities which involve fractional derivatives)
exist in the theory of fractional order systems. Indeed, they play a very essential role
in the qualitative theory of fractional order systems. It may be noted that stability
is an important qualitative property of the solutions to fractional order systems. In
the literature, there has been a growing interest in the investigation of the stability or
asymptotic stability of the solutions to the autonomous and nonautonomous fractional
order systems [8, 9, 10, 11, 12, 13, 14]. This is because analysing any fractional order
system is one challenging problem in the theory of fractional order systems as well as in
fractional order control systems.

It may be noted that Lyapunov direct method [15], fractional Lyapunov direct method
[8, 9, 10, 11, 12], distributed order Lyapunov direct method [14] and fractional compar-
ison method [13] are powerful methods for the stability analysis of the fractional order
systems. However, the application of these methods demand some suitable candidate
functions or time-varying Lyapunov functions (which involve both state variables (de-
pendent variables) and time (independent variable)) that are continuously differentiable,
and the calculation of its fractional derivatives. Indeed, the application of these methods
require fractional differential inequalities.

Recently, in [16, 12, 17, 18, 19], the authors have developed some interesting fractional
differential inequalities which involve the Caputo fractional derivative. In fact, all the
works have been focused on the calculation of the fractional derivative of autonomous
Lyapunov functions (continuously differentiable functions which involve only the state
variables (dependent variables)) of the fractional order systems.

Here we raise a few interesting questions: Can we identify time-varying Lyapunov
functions for fractional order systems? Is it possible to calculate the fractional derivative
of such Lyapunov functions? How can one identify or discover such functions? How
can one calculate the fractional derivative of such functions? How can one ensure the
stability of fractional order systems based on such type of functions? To the best of our
knowledge, the answers to these questions are not available in the literature. Indeed,
we find no work has been reported in the literature for the construction or discovery of
time-varying Lyapunov functions for the stability analysis of fractional order systems.
Thus the issue remains open in the theory of fractional calculus.

In this paper, we attempt to find the answer to the raised questions. Indeed, we
provide positive answers to these questions. First, we present some new fractional dif-
ferential inequalities which involve the Caputo fractional derivative of the product of
continuously differentiable functions. Then, we propose some equivalence results and
also establish some of their interesting properties. The best part of these inequalities are
that they allow us to construct or discover potential candidate time-varying Lyapunov
functions as well as the calculation of their fractional derivatives at the same time for
the nonautonomous fractional order systems. By presenting a few interesting examples,
we demonstrate the usefulness of some appropriate fractional differential inequalities and
their importance in the application of fractional Lyapunov direct method. Indeed, we
show that the fractional differential inequalities are absolutely necessary in the stability
analysis of nonautonomous fractional order systems.

2 Fractional differential inequalities

In this section, first we recall some useful definitions, and then present the main results which involve the Caputo fractional derivative of continuously differentiable functions.

2.1 Notations and definitions

Let us denote by \( \mathbb{N} \) be the set of natural numbers, \( \mathbb{Z}^+ \) the set of positive integers, \( \mathbb{R}^+ \) the set of positive real numbers, \( \mathbb{R} \) the set of real numbers, \( \mathbb{C} \) the set of complex numbers, \( \Re(z) \) the real part of complex number \( z \), \( \mathbb{R}^n \) the \( n \)-dimensional Euclidean space.

**Definition 2.1.** \[1\] The Gamma function is defined as

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} dt
\]

where \( z \in \mathbb{C} \) and \( \Re(z) > 0 \).

**Definition 2.2.** \[1, 2\] The Riemann-Liouville fractional integral of order \( \alpha \) of a continuous function \( x: [t_0, T) \to \mathbb{R}, -\infty < t_0 < T \leq \infty \) is defined as

\[
RLD_{t_0,t}^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau)d\tau
\]

where \( \alpha \in \mathbb{R}^+ \).

**Definition 2.3.** \[1, 2\] The Caputo fractional derivative of order \( \alpha \) of a \( n \)th continuously differentiable function \( x: [t_0, T) \to \mathbb{R}, -\infty < t_0 < T \leq \infty \) is defined as

\[
CD_{t_0,t}^{\alpha}x(t) = \begin{cases} 
RLD_{t_0,t}^{-(n-\alpha)}\left(\frac{d^n x(t)}{dt^n}\right), & \text{if } \alpha \in (n-1, n) \\
\frac{d^n x(t)}{dt^n}, & \text{if } \alpha = n 
\end{cases}
\]

where \( \alpha \in \mathbb{R}^+ \) and \( n \in \mathbb{Z}^+ \).

2.2 Main inequalities

**Result 2.1.** Let \( w: \mathbb{R} \to \mathbb{R} \) be a monotonically increasing and continuously differentiable function. Let \( \phi: [t_0, \infty) \to \mathbb{R} \) be the monotonically decreasing and continuously differentiable function. Suppose \( x: [t_0, \infty) \to \mathbb{R} \) is a non-negative and continuously differentiable function. Then, the inequality

\[
CD_{t_0,t}^{\alpha}\{w(\phi(t))x(t)\} \leq w(\phi(t))CD_{t_0,t}^{\alpha}x(t), \ \forall t \geq t_0, \ \forall \alpha \in (0, 1],
\]

holds.
Proof. Let
\[ F_\alpha(t, t_0) = C^{D}_0 \{ w(\phi(t))x(t) \} - w(\phi(t))^{D}_0 x(t). \] (5)
By the definition of Caputo fractional derivative, we have
\[ C^{D}_0 \{ w(\phi(t))x(t) \} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \left[ \frac{dw(\phi(\tau))}{d\tau} \cdot x(\tau) + \frac{dx(\tau)}{d\tau} \cdot w(\phi(\tau)) \right] d\tau, \] (6)
and
\[ w(\phi(t))^{D}_0 x(t) = \frac{w(\phi(t))}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{dx(\tau)}{d\tau} d\tau. \] (7)
Substituting (6) and (7) into the right hand side of (5), gives
\[ F_\alpha(t, t_0) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{dx(\tau)}{d\tau} d\tau \{ (w(\phi(\tau)) - w(\phi(t))) x(\tau) \} d\tau. \] (8)
Let us define by \( u(\tau) = \{ w(\phi(\tau)) - w(\phi(t)) \} x(\tau) \). After substituting it into (8), and then integrating by parts, yield
\[ F_\alpha(t, t_0) = \left. \frac{u(\tau)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}} \right|_{\tau=t} - \frac{u(t_0)}{\Gamma(1-\alpha)(t-t_0)^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{u(\tau)}{(t-\tau)^{\alpha+1}} d\tau. \] (9)
Since the first term in the right hand side of (9) is in \( \frac{0}{0} \) form, then by L’Hôpital rule, it follows that the first term has limiting value 0. Due to the monotonically decreasing property of function \( \phi \), monotonically increasing property of function \( w \), and non-negativity of function \( x \), the second and third terms of (9) become negative. Therefore, we have \( F_\alpha(t, t_0) \leq 0 \) for \( t \geq t_0 \). As a result, the inequality (11) holds.

**Result 2.2.** Let \( \phi: [t_0, \infty) \to \mathbb{R} \) be a monotonically decreasing and continuously differentiable function. Suppose \( x: [t_0, \infty) \to \mathbb{R} \) is a non-negative and continuously differentiable function. Then, the inequality
\[ C^{D}_0 \{ \phi(t)x(t) \} \leq \phi(t)^{D}_0 x(t), \quad \forall t \geq t_0, \quad \forall \alpha \in (0, 1], \] (10)
holds.

**Proof.** Let us define the function \( w: \mathbb{R} \to \mathbb{R} \) by \( w(y) = y \). Then, the inequality (10) holds by Result 2.1.

**Result 2.3.** Let \( \phi: [t_0, \infty) \to \mathbb{R} \) be a monotonically increasing and continuously differentiable function. Suppose \( x: [t_0, \infty) \to \mathbb{R} \) is a non-negative and continuously differentiable function. Then, the inequality
\[ C^{D}_0 \{ \phi(t)x(t) \} \geq \phi(t)^{D}_0 x(t), \quad \forall t \geq t_0, \quad \forall \alpha \in (0, 1], \] (11)
holds.
Proof. Note that the function \(-\phi\) is monotonically decreasing and continuously differentiable. Thus the inequality \((11)\) holds by Result 2.2.

**Result 2.4.** Let \(\phi : [t_0, \infty) \to \mathbb{R}\) be a monotonically decreasing and continuously differentiable function. Suppose \(x : [t_0, \infty) \to \mathbb{R}\) is a non-negative continuously differentiable function. Then, \(\forall t \geq t_0, \forall \alpha \in (0, 1]\), the following inequality holds

\[
C_{t_0,t}^\alpha \{ \phi^{2n+1}(t)x^\beta(t) \} \leq \phi^{2n+1}(t)C_{t_0,t}^\alpha x^\beta(t),
\]

where \(n \in \mathbb{N}\) and the constant \(\beta\) is a non-negative real number.

Proof. Let \(w = y^{2n+1}\), where \(y \in \mathbb{R}\) and \(n \in \mathbb{N}\). Then, the application of the Result 2.1 gives the inequality \((12)\).

**Result 2.5.** Let \(\phi : [t_0, \infty) \to \mathbb{R}\) be a non-negative, monotonically decreasing and continuously differentiable function. Suppose \(x : [t_0, \infty) \to \mathbb{R}\) is a non-negative continuously differentiable function. Then, \(\forall t \geq t_0, \forall \alpha \in (0, 1]\), the following inequality holds

\[
C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)C_{t_0,t}^\alpha x^\beta(t),
\]

where the constant \(\beta\) is a non-negative real number.

Proof. The proof follows from the Result 2.2.

**Result 2.6.** Let \(\phi : [t_0, \infty) \to \mathbb{R}\) be a non-negative, monotonically decreasing and continuously differentiable function. Suppose \(x : [t_0, \infty) \to \mathbb{R}\) is a continuously differentiable function. Then, \(\forall t \geq t_0, \forall \alpha \in (0, 1]\), the following inequality holds

\[
C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)C_{t_0,t}^\alpha x^\beta(t),
\]

where the constant \(\beta = 2k\), and \(k \in \mathbb{N}\).

Proof. The proof directly follows from the Result 2.5.

### 2.3 Main propositions

**Result 2.7.** Let \(\phi : [t_0, \infty) \to \mathbb{R}\) be a non-negative, monotonically decreasing and continuously differentiable function. Suppose \(x : [t_0, \infty) \to \mathbb{R}\) is a non-negative continuously differentiable function. Then, \(\forall t \geq t_0, \forall \alpha \in (0, 1]\), the inequality

\[
C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)\beta x^{\beta-1}(t)C_{t_0,t}^\alpha x(t),
\]

holds, if the inequality

\[
C_{t_0,t}^\alpha \{ x^\beta(t) \} \leq \beta x^{\beta-1}(t)C_{t_0,t}^\alpha x(t),
\]

holds, where the real constant \(\beta \geq 1\).

Proof. We can write

\[
C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} = C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} - \phi(t)\beta x^{\beta-1}(t)C_{t_0,t}^\alpha x(t) = C_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} - \phi(t)C_{t_0,t}^\alpha x^\beta(t) + \phi(t) \left[ C_{t_0,t}^\alpha \{ x^\beta(t) \} - \beta x^{\beta-1}(t)C_{t_0,t}^\alpha x(t) \right] .
\]

Thus the result follows by using the Result 2.5 and the inequality \((16)\) in the equation \((17)\).
Result 2.8. Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a continuously differentiable function. Then, $\forall t \geq t_0$, $\forall \alpha \in (0, 1]$, the inequality
\[
^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)\beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (18)
holds, if the inequality
\[
^cD_{t_0,t}^\alpha \{ x^\beta(t) \} \leq \beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (19)
holds, where the real constant $\beta = 2n$, and $n \in \mathbb{N}$.

Proof. The proof is a consequence of the Result 2.7.

Result 2.9. Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a non-negative, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a positive continuously differentiable function. Then, $\forall t \geq t_0$, $\forall \alpha \in (0, 1]$, the inequality
\[
^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)\beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (20)
holds, if the inequality
\[
^cD_{t_0,t}^\alpha \{ x^\beta(t) \} \leq \beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (21)
holds, where the real constant $\beta \geq 0$.

Proof. The proof is similar to the proof of the Result 2.7.

Result 2.10. Let $\phi : [t_0, \infty) \to \mathbb{R}$ be a positive, monotonically decreasing and continuously differentiable function. Suppose $x : [t_0, \infty) \to \mathbb{R}$ is a positive continuously differentiable function. Then, $\forall t \geq t_0$, $\forall \alpha \in (0, 1]$, the inequality
\[
^cD_{t_0,t}^\alpha \{ x^\beta(t) \} \leq \beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (22)
holds, if the inequality
\[
^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \leq \phi(t)\beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t),
\] (23)
holds, where the real constant $\beta \geq 0$.

Proof. Note that
\[
^cD_{t_0,t}^\alpha \{ x^\beta(t) \} - \beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t) = \frac{1}{\phi(t)} \left[ ^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} - \phi(t)\beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t) \right]
\]
\[+ \frac{1}{\phi(t)} \left[ \phi(t)^cD_{t_0,t}^\alpha x^\beta(t) - ^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \right].
\] (24)

Let us denote by
\[
f(t) = \frac{1}{\phi(t)} \left[ ^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} - \phi(t)\beta x^{\beta-1}(t)^cD_{t_0,t}^\alpha x(t) \right],
\] (25)
and
\[
g(t) = \frac{1}{\phi(t)} \left[ \phi(t)^cD_{t_0,t}^\alpha x^\beta(t) - ^cD_{t_0,t}^\alpha \{ \phi(t)x^\beta(t) \} \right].
\] (26)

Clearly $f(t) \leq 0$, by inequality (23). Let $\psi(t) = \frac{1}{\phi(t)}$. Since $\psi(t)$ is positive, monotonically increasing and continuously differentiable function, it follows from Result 2.3 that $g(t) \leq 0$. This completes the proof.
2.4 Main properties

Lemma 2.1. [17] Suppose \( x : [t_0, \infty) \rightarrow \mathbb{R} \) is a non-negative continuously differentiable function. Then, \( \forall t \geq t_0, \forall \alpha \in (0, 1) \), the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ x^{\beta} (t) \right\} \leq \beta x^{\beta - 1} (t) CD_{t_0,t}^{\alpha} x(t),
\]  

(27)

where \( \beta \geq 1 \).

Lemma 2.2. [17] Suppose \( x : [t_0, \infty) \rightarrow \mathbb{R} \) is a continuously differentiable function. Then, \( \forall t \geq t_0, \forall \alpha \in (0, 1) \), the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ x^{\beta} (t) \right\} \leq \beta x^{\beta - 1} (t) CD_{t_0,t}^{\alpha} x(t),
\]  

(28)

where \( \beta = \frac{p}{q} \geq 1, p = 2n, \) and \( n, q \in \mathbb{N} \).

Result 2.11. Let \( \phi : [t_0, \infty) \rightarrow \mathbb{R} \) be a non-negative, monotonically decreasing and continuously differentiable function. Suppose \( x : [t_0, \infty) \rightarrow \mathbb{R} \) is a continuously differentiable function. Then, \( \forall t \geq t_0, \forall \alpha \in (0, 1) \), the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \phi(t) x^{\beta} (t) \right\} \leq \phi(t) \beta x^{\beta - 1} (t) CD_{t_0,t}^{\alpha} x(t),
\]  

(29)

where \( \beta = \frac{p}{q} \geq 1, p = 2k, \) and \( k, q \in \mathbb{N} \).

Proof. The proof follows by using the Lemma 2.2. Also it can be easily observed that the inequality \((29)\) holds from the Result 2.7 where the Lemma 2.2 is used.

Result 2.12. Let \( \phi : [t_0, \infty) \rightarrow \mathbb{R} \) be a non-negative, monotonically decreasing and continuously differentiable function. Suppose \( x : [t_0, \infty) \rightarrow \mathbb{R} \) is a non-negative continuously differentiable function. Then, \( \forall t \geq t_0, \forall \alpha \in (0, 1) \), the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \phi^{p} (t) x^{\beta} (t) \right\} \leq \phi^{p} (t) \beta x^{\beta - 1} (t) CD_{t_0,t}^{\alpha} x(t),
\]  

(30)

where the real positive constants \( p \geq 1 \) and \( \beta \geq 1 \).

Proof. It follows from that Result 2.5 that

\[
CD_{t_0,t}^{\alpha} \left\{ \phi^{p} (t) x^{\beta} (t) \right\} \leq \phi^{p} (t) CD_{t_0,t}^{\alpha} x(t),
\]  

(31)

where the real positive constants \( p \geq 1 \) and \( \beta \geq 1 \). Then, the application of the Result 2.7 or the Lemma 2.4 into the inequality \((31)\) yields the inequality \((30)\).

Result 2.13. Let \( \phi_i : [t_0, \infty) \rightarrow \mathbb{R} \) are non-negative, monotonically decreasing and continuously differentiable functions for \( i = 1, 2, \cdots, n \). Suppose \( x_i : [t_0, \infty) \rightarrow \mathbb{R} \) are non-negative continuously differentiable functions for \( i = 1, 2, \cdots, n \). Then, \( \forall t \geq t_0, \forall \alpha \in (0, 1) \), the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \sum_{i=1}^{n} \phi_i^{p_i} (t) x_i^{\beta_i} (t) \right\} \leq \sum_{i=1}^{n} \phi_i^{p_i} (t) \beta_i x_i^{\beta_i - 1} (t) CD_{t_0,t}^{\alpha} x_i(t),
\]  

(32)

where the real positive constants \( p_i \geq 1 \) and \( \beta_i \geq 1 \) for \( i = 1, 2, \cdots, n \).
Proof. The proof is similar to the proof of Result 2.12.

Result 2.14. Let $\phi_i : [t_0, \infty) \to \mathbb{R}$ are non-negative, monotonically decreasing and continuously differentiable functions for $i = 1, 2, \cdots, n$. Suppose $x_i : [t_0, \infty) \to \mathbb{R}$ are continuously differentiable functions for $i = 1, 2, \cdots, n$. Then, for all $t \geq t_0$, $\forall \alpha \in (0, 1]$, the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \sum_{i=1}^{n} \phi_i^p(t)x_i^\beta(t) \right\} \leq \sum_{i=1}^{n} \phi_i^p(t)\beta_ix_i^{\beta_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t),
\]

where the real positive constants $p_i \geq 1$ and $\beta_i = \frac{u_i}{v_i} \geq 1$, $u_i = 2k_i$, $k_i, v_i \in \mathbb{N}$ for $i = 1, 2, \cdots, n$.

Proof. The result follows by combining the Lemma 2.2 with the Result 2.13 where the Lemma 2.2 is used.

Result 2.15. Let $\phi_i : [t_0, \infty) \to \mathbb{R}$ are non-negative, monotonically decreasing and continuously differentiable functions for $i = 1, 2, \cdots, n$. Suppose $x_i : [t_0, \infty) \to \mathbb{R}$ are continuously differentiable functions for $i = 1, 2, \cdots, n$. Then, for all $t \geq t_0$, $\forall \alpha \in (0, 1]$, the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \sum_{i=1}^{n} c_i \phi_i^p(t)x_i^\beta(t) + \sum_{i=1}^{n} d_i x_i^\beta(t) \right\} \leq \sum_{i=1}^{n} c_i \phi_i^p(t)\beta_ix_i^{\beta_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t) + \sum_{i=1}^{n} d_i \beta_i x_i^{\beta_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t),
\]

where the real positive constants $c_i > 0$, $d_i > 0$, $p_i \geq 1$, and $\beta_i = \frac{u_i}{v_i} \geq 1$, $u_i = 2k_i$, $k_i, v_i \in \mathbb{N}$ for $i = 1, 2, \cdots, n$.

Proof. The result follows by combining the Lemma 2.2 with the Result 2.14.

Result 2.16. Let $\phi_i : [t_0, \infty) \to \mathbb{R}$ are non-negative, monotonically decreasing and continuously differentiable functions for $i = 1, 2, \cdots, n$. Suppose $x_i : [t_0, \infty) \to \mathbb{R}$ are continuously differentiable functions for $i = 1, 2, \cdots, n$. Then, for all $t \geq t_0$, $\forall \alpha \in (0, 1]$, the following inequality holds

\[
CD_{t_0,t}^{\alpha} \left\{ \sum_{i=1}^{n} c_i \phi_i^p(t)x_i^\beta(t) + \sum_{i=1}^{n} d_i x_i^\beta(t) \right\} + CD_{t_0,t}^{\alpha} \left\{ \sum_{i=1}^{n} a_i x_i^\gamma(t) \right\} \leq \sum_{i=1}^{n} c_i \phi_i^p(t)\beta_ix_i^{\beta_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t) + \sum_{i=1}^{n} d_i \beta_i x_i^{\beta_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t) + \sum_{i=1}^{n} a_i \gamma_i x_i^{\gamma_i-1}(t)CD_{t_0,t}^{\alpha}x_i(t),
\]

where the real constants $a_i \geq 0$, $c_i \geq 0$, $d_i \geq 0$, $p_i \geq 1$, $\beta_i = \frac{u_i}{v_i} \geq 1$, $u_i = 2k_i$, $k_i, v_i \in \mathbb{N}$, and $\gamma_i = \frac{m_i}{n_i} \geq 1$, $m_i = 2\ell_i$, $\ell_i, n_i \in \mathbb{N}$ for $i = 1, 2, \cdots, n$.

Proof. The proof is a consequence of the Result 2.15.
3 Examples

In this section, we discuss the fractional order generalizations or versions of a few interesting examples [15]. We apply the fractional Lyapunov direct method and utilize appropriate fractional differential inequalities, in order to analyse the stability of the zero solutions to nonautonomous fractional order systems. We use the numerical predictor-corrector method [20] in order to carry out the numerical solutions to the presented examples.

Example 3.1. Consider the nonautonomous linear fractional order system

\[\begin{align*}
C^{\alpha}_{0,t}x_1(t) &= -x_1(t) - \frac{1}{1+t}x_2(t), \quad x_1(0) = x_{10}, \\
C^{\alpha}_{0,t}x_2(t) &= x_1(t) - x_2(t), \quad x_2(0) = x_{20},
\end{align*}\]

where \(0 < \alpha \leq 1\).

Let us choose the function \(V(t, x) = x_1^2 + x_2^2 + \frac{1}{1+t}x_2^2\). Note that \(x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + 2x_2^2, \forall x = (x_1, x_2)^T \in \mathbb{R}^2\). Then, the application of the Result 2.16 allows us to calculate the Caputo fractional derivative of \(V(t, x)\) along the solution \(x(t)\) to (37) as follows

\[C^{\alpha}_{0,t}V(t, x(t)) \leq -\|x(t)\|^2.\]

Note that \(x^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x\) is a positive definite quadratic function. Then, it follows from (38) that

\[C^{\alpha}_{0,t}V(t, x(t)) \leq -\|x(t)\|^2.\]

Let \(\gamma_1(r) = r^2, \gamma_2(r) = 2r^2\) and \(\gamma_3(r) = r^2\), where \(r = \|x\|\). Then, all the assumptions of Theorem 6.2 of [9] are satisfied. This observation confirms that \(V(t, x)\) is indeed a time-varying Lyapunov function. Hence, by Theorem 6.2 of [9], we conclude that the zero solution is asymptotically stable.

Further, from (39), we deduce

\[V(t, x(t)) \leq E_{\alpha}\left( -\frac{1}{2}t^\alpha \right) V(0, x(0)), \quad \forall t \geq 0.\]

Thus, it follows that

\[\|x(t)\| \leq \sqrt{2E_{\alpha}\left( -\frac{1}{2}t^\alpha \right) \|x(0)\|^2}, \quad \forall t \geq 0.\]
As a result, the zero solution to the system (37) is Mittag-Leffler stable. Thus, the zero solution is asymptotically stable. The numerical solution shown in the FIG 1 indicates that the non-trivial solutions approach to the zero solution.

Figure 1: Numerical solution to (37) for the value of fractional order $\alpha = 0.9$ starting from initial values $x_1(0) = -10$ and $x_2(0) = 10$.

Example 3.2. Consider the nonautonomous nonlinear fractional differential equation

$$C^\alpha_{0,t} x(t) = -x^3(t) - e^{t/2}x^3(t), \ x(0) = x_0, \ 0 < \alpha \leq 1.$$  

(42)

Let $V(t, x) = x^6 + e^{-t/2}x^6$ be the function, which depends on time $t$ and variable $x$. Then, by using the Result 2.16, we get the Caputo fractional derivative of $V(t, x)$ along the solution $x(t)$ to (3.2) as follows

$$C^\alpha_{0,t} V(t, x(t)) \leq \left[-6x^8(t) - 6e^{t/2}x^8(t)\right] + \left[-6e^{-t/2}x^8(t) - 6x^8(t)\right]$$

$$= -6 \left(1 + e^{-t/2}\right) x^8(t) - 6 \left(1 + e^{t/2}\right) x^8(t)$$

$$\leq -12x^8(t), \ \forall \ x \in \mathbb{R}, \ \forall \ t \geq 0.$$  

(43)

Note that $x^6 \leq V(t, x) \leq 2x^6, \ \forall x \in \mathbb{R}, \ \forall t \geq 0$. Let $\gamma_1(r) = r^6, \ \gamma_2(r) = 2r^6$ and $\gamma_3(r) = 12r^8$, where $r = |x|$. Then, we see that all the assumptions of the Theorem 6.2 of [9] are satisfied. Hence, it follows from the Theorem 6.2 of [9] that the zero solution to the equation (3.2) is asymptotically stable. The numerical solution is shown in the FIG 2.
Example 3.3. Consider the nonautonomous nonlinear fractional order system

\[
\begin{align*}
CD_{0,t}^\alpha x_1(t) &= -x_1(t) - x_2(t) + \sin(t) \left( x_1^2(t) + x_2^2(t) \right), \quad x_1(0) = x_{10}, \\
CD_{0,t}^\alpha x_2(t) &= x_1(t) - x_2(t) + \cos(t) \left( x_1^2(t) + x_2^2(t) \right), \quad x_2(0) = x_{20},
\end{align*}
\]

where \(0 < \alpha \leq 1\).

Let \(V(t, x) = \frac{x_1^2}{2} + \frac{x_2^2}{2} + \phi(t) \frac{x_1^2}{2} + \phi(t) \frac{x_2^2}{2}\) be the time-varying Lyapunov function, where \(\phi(t)\) is a non-negative, monotonically decreasing, bounded and continuously differentiable function. Then, by using the Result 2.16 or (the Result 2.6 where the Lemma 1 of 16 is used), we get the Caputo fractional derivative of \(V(t, x)\) along the solution \(x(t)\) to (4.3) as follows

\[
\begin{align*}
CD_{0,t}^\alpha V(t, x) &\leq \left[ -x_1^2(t) - x_1(t)x_2(t) + x_1(t) \sin(t) \left( x_1^2(t) + x_2^2(t) \right) \right] \\
&\quad + \left[ x_1(t)x_2(t) - x_2^2(t) + x_2(t) \cos(t) \left( x_1^2(t) + x_2^2(t) \right) \right] \\
&\quad + \phi(t) \left[ -x_1^2(t) - x_1(t)x_2(t) + x_1(t) \sin(t) \left( x_1^2(t) + x_2^2(t) \right) \right] \\
&\quad + \phi(t) \left[ x_1(t)x_2(t) - x_2^2(t) + x_2(t) \cos(t) \left( x_1^2(t) + x_2^2(t) \right) \right] \\
&= - \left[ (1 + \phi(t))(x_1^2(t) + x_2^2(t)) \right] \\
&\quad + \left[ (1 + \phi(t))(x_1^2(t) + x_2^2(t))(x_1(t) \sin(t) + x_2(t)\cos(t)) \right] \\
&\leq - \left[ (1 + \phi(t))(x_1^2(t) + x_2^2(t)) \right] + \left[ (1 + \phi(t))(x_1^2(t) + x_2^2(t))^3 \right] \\
&= -(1 + \phi(t))\|x(t)\|^2(1 - \|x(t)\|) \\
&\leq -(1 + \phi(t))(1 - r)\|x(t)\|^2, \quad \forall \|x(t)\| \leq r, \quad \forall r < 1, \\
&\leq -(1 - r)\|x(t)\|^2, \quad \forall \|x(t)\| \leq r, \quad \forall r < 1.
\end{align*}
\]

Hence, it follows from the Theorem 6.2 of [9] that the zero solution to the system (44) is asymptotically stable.
Figure 3: Numerical solution to (44), starting from the initial values $x_1(0) = -0.2$ and $x_2(0) = 0.3$, for the value of fractional order $\alpha = 0.85$. It indicates that the zero solution is asymptotically stable.

4 Conclusions

We have established several new fractional differential inequalities which involve the Caputo fractional derivative of continuously differentiable functions. By presenting a few simple illustrative examples, applying fractional direct method and using appropriate fractional differential inequalities, we have shown that the presented fractional differential inequalities play a basic role in the analysis as well as the calculation of bounds of solutions to nonautonomous fractional order systems.

Interestingly, the beauty of these fractional differential inequalities lies in the fact that they open up an opportunity and show a way to construct or discover different types of potential candidate time-varying Lyapunov functions. They can also be useful for the calculation of their fractional derivatives, at the same time, for the stability analysis of fractional order systems.

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References

[1] Podlubny I. Fractional differential equations. San Diego, Academic Press, 1999.
[2] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam, Elsevier, 2006.

[3] Petráš I. Fractional order nonlinear systems: Modelling, analysis and simulation. Springer, 2010.

[4] Valério D, Machado JT, Kiryakova V. Some pioneers of the applications of fractional calculus. Fractional Calculus and Applied Analysis, 2014, 17, pp. 552–578.

[5] Tarasov VE. No violation of the Leibniz rule. No fractional derivative. Communications in Nonlinear Science and Numerical Simulation, 2013, 18, pp. 2945–2948.

[6] Tarasov VE. Leibniz rule and fractional derivatives of power functions. Journal of Computational and Nonlinear Dynamics, 2016, 11, 031014–4.

[7] Tarasov VE. On chain rule for fractional derivatives. Communications in Nonlinear Science and Numerical Simulation, 2016, 30, pp. 1–4.

[8] Li Y, Chen YQ, Podlubny I. Mittag-Leffler stability of fractional order nonlinear dynamic systems. Automatica, 2009, 45, pp. 1965–1969.

[9] Li Y, Chen YQ, Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. Computer Mathematics and Applications, 2010, 59, pp. 1810–1821.

[10] Delavari H, Baleanu D, Sadati J. Stability analysis of caputo fractional-order nonlinear systems revisited. Nonlinear Dynamics, 2012, 67, pp. 2433–2439.

[11] Yu J, Hu H, Zhou S, Lin X. Generalized Mittag-Leffler stability of multi-variables fractional order nonlinear systems. Automatica, 2013, 49, pp. 1798–803.

[12] Duarte-Mermoud MA, Aguila-Camacho N, Gallegos JA, Castro-Linares R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. Communications in Nonlinear Science and Numerical Simulation, 2015, 22, pp. 650–659.

[13] Lenka BK. Fractional comparison method and asymptotic stability of multivariable fractional order systems. Communications in Nonlinear Science and Numerical Simulation, 2019, 69, pp. 398–415.

[14] Fernández-Anayaa G, Nava-Antonio G, Jamous-Galante J, Muñoz-Vega R, Hernández-Martínez EG. Asymptotic stability of distributed order nonlinear dynamical systems. Communications in Nonlinear Science and Numerical Simulation, 2017, 48, pp. 541–549.

[15] Khalil HK. Nonlinear systems. Prentice Hall, 2002.

[16] Aguila-Camacho N, Duarte-Mermoud MA, Gallegos JA. Lyapunov functions for fractional order systems. Communications in Nonlinear Science and Numerical Simulation, 2014, 19, pp. 2951–2957.
[17] Dai H, Chen W. New power law inequalities for fractional derivative and stability analysis of fractional order systems. Nonlinear Dynamics, 2017, 87, pp. 1531–1542.

[18] Fernández-Anaya G, Nava-Antonio G, Jamous-Galante J, Muñoz-Vega R, Hernández-Martínez EG. Lyapunov functions for a class of nonlinear systems using Caputo derivative. Communications in Nonlinear Science and Numerical Simulation, 2017, 43, pp. 91–99.

[19] Fernández-Anaya G, Nava-Antonio G, Jamous-Galante J, Muñoz-Vega R, Hernández-Martínez EG. Corrigendum to “Lyapunov functions for a class of nonlinear systems using Caputo derivative” [Commun Nonlinear Sci Numer Simulat 43 (2017) 91–99]. Communications in Nonlinear Science and Numerical Simulation, 2018, 56, pp. 596–597.

[20] Diethelm K, Ford NJ, Freed AD. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics, 2002, 29, pp. 3–22.