THE SECOND TYPE SINGULARITY OF SYMPLECTIC AND
LAGRANGIAN MEAN CURVATURE FLOWS

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Abstract. In this paper we mainly study the type II singularities of the mean curvature flow from a symplectic surface or from an almost calibrated Lagrangian surface in a Kähler surface. We study the relation between the maximum of the Kähler angle and the maximum of $|H|^2$ on the limit flow.

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1. Introduction

In this paper, we continue to study the symplectic mean curvature flow and Lagrangian mean curvature flow ([1], [2], [3] [8], [9], [12], [14]) in a Kähler surface. Suppose $M$ is a compact Kähler surface. Let $\Sigma$ be a smooth surface in $M$ and $\omega$, $\langle \cdot, \cdot \rangle$ be the Kähler form and the Kähler metric on $M$ respectively. The Kähler angle $\alpha$ of $\Sigma$ in $M$ is defined by [5]

$$\omega|_{\Sigma} = \cos \alpha d\mu_{\Sigma}$$

where $d\mu_{\Sigma}$ is the area element of $\Sigma$ of the induced metric from $\langle \cdot, \cdot \rangle$. We call $\Sigma$ a symplectic surface if $\cos \alpha > 0$, a Lagrangian surface if $\cos \alpha = 0$, a holomorphic curve if $\cos \alpha = 1$. In addition, we assume that $M$ is a Calabi-Yau manifold of complex dimension 2 with a complex structure $J$, i.e., a K3 surface. We consider a parallel holomorphic $(2,0)$ form,

$$\Omega = dz_1 \wedge dz_2.$$

If a surface $\Sigma$ is Lagrangian then (see [7])

$$\Omega|_{\Sigma} = e^{i\theta} d\mu_{\Sigma},$$

where $\theta$ is a multivalued function called Lagrangian angle. If $\cos \theta > 0$, then $\Sigma$ is called almost calibrated. If $\theta = \text{costant}$, then $\Sigma$ is called special Lagrangian.

It is proved in [1] and [14] that, if the initial surface is symplectic, then along the mean curvature flow, at each time $t$ the surface is still symplectic. Thus we speak of symplectic mean curvature flow. It is proved in [12], [13] that, if the initial surface is Lagrangian, then along the mean curvature flow, at each time $t$ the surface is still Lagrangian. Thus we speak of Lagrangian mean curvature flow.

In [8] we showed that, if the scalar curvature of the compact Kähler-Einstein surface $M$ is positive and the initial surface is sufficiently close to a holomorphic curve, then the mean curvature flow has a global solution and converges to a holomorphic curve.

Key words and phrases. Symplectic surface, lagrangian surface, mean curvature flow.
In general, the mean curvature flow may produce singularities. The beautiful results on the nature of singularities of the mean curvature flow of convex hypersurfaces have been obtained by Huisken-Sinestrari [10], [11] and White [15]. For symplectic mean curvature flow, Chen-Li [1] and Wang [14] proved that there is no Type I singularity. At a Type II singular point, Chen-Li [2], [3] proved that, the rescaled surfaces converge weakly (in the sense of measure) to a stationary tangent cone which is flat.

If we consider the strong convergence of the rescaled surfaces $\Sigma^k_s$ in $B_R(0)$ around a type II singular point, let $|A_k|$ be the second fundamental forms of $\Sigma^k_s$ in $B_R(0)$, then we have that $|A_k|^2 \leq 4$ in $B_R(0)$ during the rescaling process. Thus by Arzela-Ascoli theorem, $\Sigma^k_s \to \Sigma^\infty_s$ in $C^2(B_R(0) \times [-R,R])$ for any $R > 0$ and any $B_R(0) \subset \mathbb{C}^2$. By the definition of the type II singularity, we know that $\Sigma^\infty_s$ is defined on $(-\infty, +\infty)$ and $\Sigma^\infty_s$ also evolves along the mean curvature flow in $\mathbb{C}^2$ with the Euclidean metric. We call $\Sigma^\infty_s$ the limit flow at $X_0$. See Section 2 for details.

In this paper, we mainly study the nature of the limit flow $\Sigma^\infty_s$. For this purpose, we consider a general mean curvature flow $\Sigma_t$ in $\mathbb{R}^4$ which exists globally with bounded second fundamental forms. In particular, translating soliton to the mean curvature flow is a special case. Recently in [9] we proved that there is no translating soliton with $\cos \alpha \geq \delta$ to the symplectic mean curvature flow or to the almost calibrated Lagrangian mean curvature flow where $\delta > 0$ is a constant depending only on the speed of the soliton. Since $\Sigma_t$ come from the blow up, it is natural to assume that on $\Sigma_t$, we have

$$cR^2 \leq \mu_t(\Sigma_t \cap B_R(0)) \leq CR^2,$$

where $0 < c < C < \infty$ are constants which are independent of $t$ and $R$.

**Main Theorem 1** Suppose that $\Sigma_t$, $t \in (-\infty, 0]$ is a complete symplectic mean curvature flow in $\mathbb{C}^2$ which satisfies (1.1). Assume that $\sup_{t \in (-\infty, 0]} \sup_{\Sigma_t} |A|^2 = 1$. If $h^2 = \sup_{t \in (-\infty, 0]} \sup_{\Sigma_t} |H|^2$ and $\delta = \inf_{t \in (-\infty, 0]} \inf_{\Sigma_t} \cos \alpha$, then $\delta e^{\frac{h^2}{2}} \leq 1$.

Analogously in the almost calibrated Lagrangian mean curvature flow, we have

**Main Theorem 2** Suppose that $\Sigma_t$, $t \in (-\infty, 0]$ is a complete almost calibrated Lagrangian mean curvature flow in $\mathbb{C}^2$ which satisfies (1.1). Assume further that $\sup_{t \in (-\infty, 0]} \sup_{\Sigma_t} |A|^2 = 1$. If $h^2 = \sup_{t \in (-\infty, 0]} \sup_{\Sigma_t} |H|^2$ and $\delta = \inf_{t \in (-\infty, 0]} \inf_{\Sigma_t} \cos \theta$, then $\delta e^{\frac{h^2}{2}} \leq 1$.

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## 2. Preparations

In this section we define the rescaled surfaces and study the strong convergence of the rescaled sequence at a type II singular point, which is more or less standard. However we can not find it in a reference, so we give all details here. It may be interesting in its own right. Suppose that $T$ is discrete singular time, that means...
there exists $\varepsilon > 0$ such that the mean curvature flow is smooth in $[T - \varepsilon, T)$. Assume that $(X_0, T)$ is a type II singular point of the mean curvature flow in $M$. Since this is type II singularity, then for any sequence $\{r_k\}$ with $r_k \to 0$,

$$
\max_{\sigma \in [0, r_k/2]} \max_{[T - (r_k - \sigma)^2, T - (r_k/2)^2]} \max_{\Sigma_t \cap B_{r_k - \sigma}(X_0)} |A|^2
\geq (r_k/2)^2 \max_{\Sigma_{T - (r_k/2)^2} \cap B_{r_k/2}(X_0)} |A|^2
= (T - (T - (r_k/2)^2)) \max_{\Sigma_{T - (r_k/2)^2} \cap B_{r_k/2}(X_0)} |A|^2
\to +\infty
$$

We choose $\sigma_k \in (0, r_k/2]$ such that

$$
\sigma_k^2 \max_{[T - (r_k - \sigma_k)^2, T - (r_k/2)^2]} \max_{\Sigma_t \cap B_{r_k - \sigma_k}(X_0)} |A|^2 = \max_{\sigma \in (0, r_k/2]} \sigma^2 \max_{[T - (r_k - \sigma)^2, T - (r_k/2)^2]} \max_{\Sigma_t \cap B_{r_k - \sigma}(X_0)} |A|^2.
$$

Let $t_k \in [T - (r_k - \sigma_k)^2, T - (r_k/2)^2]$ and $F(x_k, t_k) = X_k \in B_{r_k - \sigma_k}(X_0)$ satisfy

$$
\lambda_k^2 = |A|^2(x_k) = |A|^2(x_k, t_k) = \max_{[T - (r_k - \sigma_k)^2, T - (r_k/2)^2]} \max_{\Sigma_t \cap B_{r_k - \sigma_k}(X_0)} |A|^2.
$$

Obviously, we have $(X_k, t_k) \to (X_0, T)$ and $\lambda_k^2 \sigma_k^2 \to \infty$. In particular,

$$
\max_{[T - (r_k - \sigma_k)^2, T - (r_k/2)^2]} \max_{\Sigma_t \cap B_{r_k - \sigma_k}(X_0)} |A|^2 \leq 4\lambda_k^2,
$$

and hence

$$
\max_{[t_k - (\sigma_k/2)^2, t_k]} \max_{\Sigma_t \cap B_{r_k - \sigma_k/2}(X_0)} |A|^2 \leq 4\lambda_k^2.
$$

We now describe the rescaling process around $(X_0, T)$ in details. The argument is discussed with J. Chen. In the following we denote the points of the image of $F$ or $F_k$ in $M$ by capital letters. We choose a normal coordinates in $B_r(X_0)$ using the exponential map, where $B_{r}(X_0)$ is a metric ball in $M$ centered at $X_0$ with radius $r \in (0, r_0, r_0)$ such that the mean curvature flow is smooth in $[0, t_0)$. In its coordinates functions. Consider the following sequences,

$$
F_k(x, s) = \lambda_k(F(x_k + x, t_k + \lambda_k^{-2}s) - F(x_k, t_k)), \quad s \in [-\lambda_k^2 \sigma_k^2/4, \lambda_k^2(T - t_k)].
$$

(2.3)

We denote the rescaled surfaces by $\Sigma_k^s$ in which $d\mu_k^s$ is the induced area element from $M$. For any $R > 0$, let $B_R(0)$ be a ball in $\mathbb{R}^4$ with radius $R$ in the Euclidean metric and centered at 0. Then

$$
\Sigma_k^s \cap B_R(0) = \{|F_k(x, s)| \leq R\},
$$

it is clear that for any fixed $R > 0$, $\lambda_k^{-1} R < r_k^2, r_k < r_k$ as $k$ sufficiently large, then the surface $\Sigma_k^s$ is defined on $B_R(0)$ because

$$
\exp_{X_0}(\lambda_k^{-1} |F_k(x, s)| \leq R) \subset \exp_{X_0}(|F - X_0| \leq \lambda_k^{-1} R + r_k)
\subset B_{\lambda_k^{-1} R + r_k}(X_0) \subset B_r(X_0).
$$
Moreover, we pull back the metric on $B_r(X_0) \subset M$ via $\exp_{X_0}$ so that we get a metric $h$ on the Euclidean ball $B_r(0)$. Then for any fixed $R > 0$ such that $\lambda_k^{-1}R < r/2$, we can define a metric $h_{k,R}$ on $B_R(0)$,

\[(h_{k,R})_{ij}(X) = \lambda_k^2 h(\lambda_k^{-1}X + X_k).\]

With respect to this metric $\Sigma^k_s$ evolves along the mean curvature flow, which will be derived as follows.

If $g^k_s$ is the metric on $\Sigma^k_s$ which is induced from the metric $g(\cdot, t_k + \lambda_k^{-1}s)$ on $\Sigma_{t_k + \lambda_k^{-1}s}$, it is clear that

\[(g^k_s)_{ij}(X) = \lambda_k^2 g_{ij}(\lambda_k^{-1}X + X_k, t_k + \lambda_k^{-2}s),\]

and

\[(g^k_s)_{ij}(X) = \lambda_k^{-2}g^{ij}(\lambda_k^{-1}X + X_k, t_k + \lambda_k^{-2}s).\]

In this setting $(\Sigma^k_s, g^k_s)$ is an isometric immersion in $(B_R(0), h_{k,R})$. Let $A_k, H_k$ be the second fundamental form and the mean curvature vector of $(\Sigma^k_s, g^k_s)$ in $(B_R(0), h_{k,R})$ respectively. Let $\bar{\Gamma}^k, \Gamma^k_s$ be the Christoffel symbols of $h_{k,R}$ on $B_R(0)$ and the Christoffel symbols of $g^k_s$ on $\Sigma^k_s$. Since $F_k$ is an isometric immersion in $(B_R(0), h_{k,R})$ with respect to the induced metric, hence by the Gauss equation we have,

\[
\begin{align*}
(A_k)_{ij} &= \sum_{\alpha=1,2} (h_k)_{ij}^\alpha \nu^k_{\alpha} \\
&= -\partial^2_{ij} F_k + \sum_{l=1,2} (\Gamma^k_s)_{ij}^l \partial_l F_k - \sum_{\alpha,\beta,\gamma=1,4} (\bar{\Gamma}^k)^\alpha_{\beta\gamma} \partial_\beta F_k^\gamma \partial_\gamma F_k^\alpha \nu^k_{\alpha},
\end{align*}
\]

(2.4)

where $\{\nu^k_{\alpha}, \alpha = 1, 2\}$ are bases of the normal space of $\Sigma^k_s$ in $(B_R(0), h_{k,R})$. Let $\Gamma^k\varepsilon_{t_k + \lambda_k^{-2}s}$ be the Christoffel symbols on $\Sigma_{t_k + \lambda_k^{-2}s}$, and $\bar{\Gamma}$ be the Christoffel symbols on $M$. It is not hard to check that

\[
\bar{\Gamma}^k(X) = \bar{\Gamma}(\lambda_k^{-1}X + X_k), \quad \Gamma^k_s(X) = \Gamma_{t_k + \lambda_k^{-2}s}(\lambda_k^{-1}X + X_k).
\]

Thus from (2.4), we get that,

\[
\begin{align*}
(A_k)_{ij} &= \lambda_k(-\partial^2_{ij} F + \sum_{l=1,2} (\Gamma^k_{t_k + \lambda_k^{-2}s})_{ij}^l \partial_l F_k - \sum_{\alpha,\beta,\gamma=1,4} \bar{\Gamma}^\alpha_{\beta\gamma} \partial_\beta F_k^\gamma \partial_\gamma F_k^\alpha \nu^k_{\alpha}) \\
&= \lambda_k A_{ij},
\end{align*}
\]

(2.5)

where $\{\nu^k_{\alpha}, \alpha = 1, 2\}$ are bases of the normal space of $\Sigma_{t_k + \lambda_k^{-2}s}$ in $M$. Therefore,

\[
\begin{align*}
|A_k|^2 &= \lambda_k^{-2} |A|^2, \\
H_k &= \lambda_k^{-1} H, \\
|H_k|^2 &= \lambda_k^{-2} |H|^2.
\end{align*}
\]

Set $t = t_k + \lambda_k^{-2}s$, it is easy to check that

\[
\frac{\partial F_k}{\partial s} = \lambda_k^{-1} \frac{\partial F}{\partial t}.
\]
Therefore, it follows that the scaled surface also evolves by a mean curvature flow

$$\frac{\partial F_k}{\partial s} = H_k$$

(2.6)
in $B_{\lambda_k \sigma_k}(0)$, where $s \in [-\lambda_k^2 \sigma_k^2/4, \lambda_k^2(T - t)]$.

By (2.1) and (2.2) we know that,

$$|A_k|(0,0) = 1, \quad |A_k|^2 \leq 4$$
in $B_{\lambda_k \sigma_k}(0)$ and $s \in [-\lambda_k^2 \sigma_k^2/4, \lambda_k^2(T - t)]$. Since $(X_0, T)$ is a type II singularity, then $\lambda_k^2 \sigma_k^2 \rightarrow \infty$ and $\lambda_k^2(T - t_k) \rightarrow \infty$. Thus by Arzela-Ascoli theorem, $\Sigma^k_s \rightarrow \Sigma^\infty_s$ in $C^2(B_R(0) \times [-R, R])$ for any $R > 0$ and any $B_R(0) \subset \mathbb{C}^2$. By (2.3), we know that $\Sigma^\infty_s$ is defined on $(-\infty, +\infty)$. Since for each fixed $R > 0$, $\lambda_k^{-1}X + X_k \rightarrow X_0$ for $X \in B_R(0)$ as $k \rightarrow \infty$, then $h_{k,R}$ converges uniformly in $B_R(0)$ to the Euclidean metric as $k \rightarrow \infty$, and the Christoffel symbols $(\bar{\Gamma}^k)$ of $h_{k,R}$ converges uniformly in $B_R(0)$ to 0 as $k \rightarrow \infty$, we see that $\Sigma^\infty_s$ also evolves along the mean curvature flow in $\mathbb{C}^2$ with the Euclidean metric. We call $\Sigma^\infty_s$ the limit flow at $X_0$.

In the rest part of this section, we estimate the different of $A_k, H_k$ and $A_0^k, H_0^k$ where $A_0^k$ and $H_0^k$ are the second fundamental form and the mean curvature vector of $\Sigma^k_s$ in the Euclidean metric on $B_R(0)$ respectively. Although it is not needed in this paper, it is interesting in its own right.

Let $\Gamma^k_s$ be the Christoffel symbols of $\Sigma^k_s$ for the Euclidean metric on $B_R(0)$ and $\{\nu^0_s : \alpha = 1, 2\}$ be bases of the normal space of $\Sigma^k_s$ with respect to the Euclidean metric on $B_R(0)$. Similarly, considering $F_k$ as an isometric immersion in $B_R(0)$ with Euclidean metric, we have,

$$(A^0_k)_{ij} = \sum_{\alpha=1,2} (h_0)_{ij}^\alpha (\nu^0_s)_\alpha = -\partial^2_{ij} F_k + \sum_{l=1,2} (\Gamma^0_s)_l^{ij} \partial_l F_k.$$  

(2.7)

Note that the induced metric on $\Sigma^k_s$ from $h_{k,R}$ is given by $\langle \partial F_k, \partial F_k \rangle_{h_{k,R}}$, so it holds

$$|\partial F_k|_{h_{k,R}}^2 = 2,$$

which in turn implies that for $k$ sufficiently large and $R$ fixed $|\partial F_k^0|$ is uniformly bounded in $B_R(0)$ with Euclidean metric.

Using the Euclidean metric on $B_R(0)$, we decompose the tangent bundle of $B_R(0)$ along $\Sigma^k_s$ into the tangential component $T\Sigma^k_s$ and the normal component $T^\perp \Sigma^k_s$. Let $A^\perp_k : T\Sigma^k_s \times T\Sigma^k_s \rightarrow T^\perp \Sigma^k_s$ be the normal component of $A_k$. Notice that $A^\perp_k - A^0_k$ lies in $T^\perp \Sigma^k_s$ and $\partial_t F_k$ lies in $T\Sigma^k_s$, it follows from (2.4) and (2.7) that,

$$\sup_{B_R(0)} |A^\perp_k - A^0_k| \leq C \sup_{B_R(0)} |\bar{\Gamma}^k| \rightarrow 0$$
as $k \rightarrow \infty$ for any fixed $R > 0$. From the uniform convergence of the metrics $h_{k,R}$ to the Euclidean metric,

$$|A^\perp_k| \leq |A_k| \leq 2|A_k|_{h_{k,R}}$$

for any fixed $R > 0$ and sufficiently large $k$. Hence, there exist positive constants $\delta_{k,R}$ which tend to 0 as $k \rightarrow \infty$ such that

$$|A_k^0| = |A^\perp_k| + \delta_{k,R} \leq 2|A_k|_{h_{k,R}} + \delta_{k,R}$$
for all sufficiently large $k$ and any fixed $R > 0$; and similarly there exist constants $\delta'_{k,R} > 0$ with $\delta'_{k,R} \to 0$ as $k \to \infty$ such that

$$|H^0_k| \leq 2|H_{k,R}| + \delta'_{k,R}$$

for sufficiently large $k$ and for any given $R > 0$.

3. Proof of the Main Theorems

Now we begin to prove our Main Theorems. We first prove Main Theorem 2. Let $H(X, X_0, t, t_0)$ be the backward heat kernel on $\mathbb{R}^4$. Let $\Sigma_t$ be a smooth family of surfaces in $\mathbb{R}^4$ defined by $F_t : \Sigma \to \mathbb{R}^4$. Define

$$\rho(X, t) = \frac{1}{4\pi(t_0 - t)} H(X, X_0, t, t_0) = \exp -\frac{|X - X_0|^2}{4(t_0 - t)}$$

for $t < t_0$, such that

$$\frac{d}{dt}\rho = -\Delta \rho - \rho \left( \left| H + \frac{(X - X_0)^\perp}{2(t_0 - t)} \right|^2 - |H|^2 \right).$$

where $(X - X_0)^\perp$ is the normal component of $X - X_0$.

Define

$$\Psi_{X_0,t_0}(X, t) = \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(X, t) d\mu_t.$$

Proposition 3.1. Along the almost calibrated Lagrangian mean curvature flow $\Sigma_t$ in $\mathbb{C}^2$, we have,

$$\frac{\partial}{\partial t} \Psi_{X_0,t_0}(X, t) = - \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, t) H - \frac{(F - X_0)^\perp}{2(t_0 - t)} \right|^2 d\mu_t$$

$$+ \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, t)|H|^2 d\mu_t + \int_{\Sigma_t} \frac{2}{\cos^3 \theta} |\nabla \cos \theta|^2 \rho(F, t) d\mu_t.$$

Proof. From the evolution equation of Lagrangian angle ([12], [13]),

$$\left( \frac{\partial}{\partial t} - \Delta \right) \cos \theta = |H|^2 \cos \theta, \quad (3.1)$$

we know that

$$\left( \frac{\partial}{\partial t} - \Delta \right) \frac{1}{\cos \theta} = - \frac{|H|^2}{\cos^2 \theta} - 2 \frac{2 |\nabla \cos \theta|^2}{\cos^3 \theta}. \quad (3.2)$$

Recall the general formula (7) in [6], for a smooth function $f = f(x, t)$ on $\Sigma_t$ with polynomial growth at infinity,

$$\frac{d}{dt} \int_{\Sigma_t} f \rho d\mu_t = \int_{\Sigma_t} \left( \frac{d}{dt} f - \Delta f \right) \rho d\mu_t - \int_{\Sigma_t} f \rho \left| H + \frac{(X - X_0)^\perp}{2(t_0 - t)} \right| d\mu_t. \quad (3.3)$$

Choosing $f = \frac{1}{\cos \theta}$ in (3.3) and putting (3.2) into (3.3), we get our monotonicity formula.
Proof of Main Theorem 2. If \( h = 0 \), or \( \delta = 0 \), or \( \delta = 1 \), it is evident that the result holds. Now we assume that \( h > 0 \), \( 0 < \delta < 1 \) and argue it by contradiction. Suppose that \( \delta > e^{-\frac{h^2}{2}} \). Fix \( R > 0 \). First we claim that there exists a sequence \( \{s_i\} \) such that \( s_i \to -\infty \) as \( i \to \infty \) and \( \lim_{s_i \to \infty} \max_{\Sigma_t \cap B_R(X_0)} |H|^2 = 0 \). Without loss of generality, we assume \( X_0 = 0 \). Integrating the monotonicity formula in Proposition 2.1 with \( t_0 = 0 \) from \( 2s \) to \( s \) for \( s < 0 \), we get that,

\[
\int_{\Sigma_{2s}} \frac{1}{\cos \theta(x, 2s)} e^{\frac{|F|^2}{2s}} d\mu_{2s} - \int_{\Sigma_s} \frac{1}{\cos \theta(x, s)} e^{\frac{|F|^2}{s}} d\mu_s \\
\geq \int_{2s}^s \int_{\Sigma_t} \frac{1}{\cos \theta} \rho(F, t)|H|^2 d\mu_t dt.
\]

By Proposition 3.1, we know that \( \int_{\Sigma_s} \frac{1}{\cos \theta} \rho(F, s) \) is nonincreasing as \( s \). Since \( \cos \theta \) is bounded below by \( \delta \), for any \( t < 0 \),

\[
\int_{\Sigma_t} \frac{1}{\cos \theta} \rho(X, t) d\mu_t \leq \frac{1}{\delta} \int_{\Sigma_t} \rho(X, t) d\mu_t \\
\leq C/\delta \int_0^\infty \int_{\Sigma_t \cap B_\rho(0)} \frac{1}{2} e^{\frac{\rho^2}{t}} d\sigma_t d\rho \\
\leq \frac{C}{-t} \int_0^\infty e^{\frac{\rho^2}{t}} \frac{d}{d\rho} \text{vol}(B_\rho(0) \cap \Sigma_t) d\rho \\
\leq \frac{C}{-t} \left[ e^{\frac{\rho^2}{t}} \text{vol}(B_\rho(0) \cap \Sigma_t) \right]_0^\infty - \int_0^\infty \text{vol}(B_\rho(0) \cap \Sigma_t) e^{\frac{\rho^2}{t}} \frac{2\rho}{t} d\rho,
\]

where we denote by \( C > 0 \) the constants which does not depend on \( t \) and may change from one line to another line. Since we have assumed that \( cR^2 \leq B_R(0) \cap \Sigma_t \leq CR^2 \) in (1.1), thus we have,

\[
\int_{\Sigma_t} \frac{1}{\cos \theta} \rho(X, t) d\mu_t \leq C \left[ \frac{1}{-t} e^{\frac{\rho^2}{t}} \rho^2 \right]_0^\infty + \int_0^\infty \frac{2\rho^3}{t^2} e^{\frac{\rho^2}{t}} d\rho \\
\leq C \left[ \frac{1}{-t} e^{\frac{\rho^2}{t}} \rho^2 + e^{\frac{\rho^2}{t}} \rho^2 \frac{2\rho}{t} - e^{\frac{\rho^2}{t}} \right]_0^\infty \\
\leq C.
\]

Thus the quantity \( \int_{\Sigma_s} \frac{1}{\cos \theta} \rho(F, s) \) is uniformly bounded above. Moreover, by the mean value theorem there is \( s' \in [2s, s] \) such that,

\[
\int_{2s}^s \int_{\Sigma_t} \frac{1}{\cos \theta} e^{\frac{|F|^2}{t}} \rho(F, t)|H|^2 d\mu_t \\
= -s \int_{\Sigma_{s'}} \frac{1}{\cos \theta} e^{\frac{|F|^2}{s'}} \rho(F, s)|H|^2 d\mu_{s'} \\
\geq Ce^{\frac{|F|^2}{s}} \int_{\Sigma_{s'} \cap B_R(0)} |H|^2 d\mu_{s'},
\]
where \( C \) is independent of \( s \). Thus we can find a sequence \( \{s_i\} \) such that \( s_i \to -\infty \) as \( i \to \infty \) and

\[
\int_{\Sigma_i \cap B_R(0)} |H|^2 d\mu_{s_i} \to 0 \quad \text{as} \quad i \to \infty.
\]

Since the second fundamental forms of \( \Sigma_{s_i} \) are bounded above and \( \Sigma_s \) satisfy the mean curvature flow equation, then \( \Sigma_{s_i} \) strongly converges to a smooth limit surface \( \Sigma_{-\infty} \) in \( B_R(0) \). Therefore,

\[
\lim_{i \to \infty} \max_{\Sigma_{s_i} \cap B_R(0)} |H|^2 = 0.
\] (3.4)

The identity can also be proved by Morse iteration.

Now we use gradient estimate to prove our theorem. For this purpose we introduce a new function \( f(X,t) = \frac{e^{p|H|^2}}{\cos^2 \theta} \), where \( t \in [s_i,0] \), \( \{s_i\} \) is the sequence in (3.4), and \( p \) is constant such that \( 1 - p > 0 \).

\[
(\Delta - \frac{\partial}{\partial t})f = \frac{1}{\cos^2 \theta}(\Delta - \frac{\partial}{\partial t})e^{p|H|^2} + e^{p|H|^2}(\Delta - \frac{\partial}{\partial t})\frac{1}{\cos^2 \theta} + 2\nabla e^{p|H|^2} \cdot \nabla \frac{1}{\cos^2 \theta}.
\]

Using the evolution equation for \( |H|^2 \) in \( \mathbb{R}^4 \):

\[
(\Delta - \frac{\partial}{\partial t})|H|^2 = 2|\nabla H|^2 - 2(H^\alpha h^\alpha_{ij})^2,
\]

we get

\[
(\Delta - \frac{\partial}{\partial t})e^{p|H|^2} = e^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H^\alpha h^\alpha_{ij}|^2)
\]
\[
\geq e^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H|^2|A|^2)
\]
\[
\geq e^{p|H|^2}(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 - 2p|H|^2).
\]

Since

\[
\nabla e^{p|H|^2} = \nabla (f \cos^2 \theta)
\]
\[
= \cos^2 \theta \nabla f + 2f \cos \theta \nabla \cos \theta,
\]

we have,

\[
\nabla e^{p|H|^2} \cdot \nabla \frac{1}{\cos^2 \theta} = \cos^2 \theta \nabla f \cdot \nabla \frac{1}{\cos^2 \theta} - \frac{4f}{\cos^2 \theta} |\nabla \cos \theta|^2.
\]

Using the evolution equation (3.1) we get,

\[
(\Delta - \frac{\partial}{\partial t})\frac{1}{\cos^2 \theta} = \frac{6|\nabla \cos \theta|^2}{\cos^4 \theta} + 2\frac{|H|^2}{\cos^2 \theta}.
\]

So,

\[
(\Delta - \frac{\partial}{\partial t})f \geq f(4p^2|H|^2|\nabla|H||^2 + 2p|\nabla H|^2 + 2(1 - p)|H|^2 - 2\frac{|\nabla \cos \theta|^2}{\cos^2 \theta})
\]
\[
+ 2\cos^2 \theta \nabla f \cdot \nabla \frac{1}{\cos^2 \theta}.
\] (3.5)
Let $\psi(r)$ be a $C^2$ function on $[0, \infty)$ such that

$$
\psi(r) = \begin{cases} 
1 & \text{if } r \in [0, \frac{1}{2}] \\
0 & \text{if } r \geq 1
\end{cases}
$$

$$
0 \leq \psi(r) \leq 1, \psi'(r) \leq 0, \psi''(r) \geq -C \text{ and } \frac{\left|\psi'(r)\right|^2}{\psi(r)} \leq C
$$

where $C$ is an absolute constant.

Let $g(X, t) = \psi(\frac{|X|^2}{R^2})$.

Using the fact that $|\nabla X|^2 = 2$, a straightforward computation shows that,

$$
(\Delta - \frac{\partial}{\partial t}) g = 4\psi'' \frac{(X, \nabla X)^2}{R^4} + 2\psi' \frac{(\nabla X, \nabla X)}{R^2} \geq -\frac{C_1}{R^2},
$$

$$
\frac{|\nabla g|^2}{g} \leq \frac{C_2}{R^2}.
$$

(3.6)

Let $(X(s_i), t(s_i))$ be the point where $g \cdot f$ achieves its maximum in $\overline{B_R(0)} \times [s_i, 0]$. If $\Sigma_{s_i} \cap B_R(0) = \emptyset$ as $i \to \infty$, then $g \cdot f \to 0$ as $i \to \infty$. If $\Sigma_{s_i} \cap B_R(0) \neq \emptyset$ as $i \to \infty$, by (3.4), we know that $f(X, s_i)$ is close to $\frac{1}{\cos^2 \vartheta(x, s_i)}$ as $i$ large enough, therefore $f(X, s_i) < e^{-h^2}$ for $i$ sufficiently large, since we are assuming $\delta^2 > e^{-h^2}$.

We choose $p$ such that $p$ is sufficiently close to 1 and keep the condition $1 - p > 0$. Thus $f(X, s_i) \leq e^{ph^2}$ as $i \to \infty$. This implies that the maximum of $g \cdot f$ can not be achieved at $s_i$ as $i \to \infty$. We can assume that $g \cdot f(X(s_i), t(s_i)) > 0$. By the maximum principle, at $(X(s_i), t(s_i))$ we have,

$$
\nabla (g \cdot f) = 0
$$

$$
\frac{\partial}{\partial t} (g \cdot f) \geq 0
$$

(3.7)

and

$$
\Delta (g \cdot f) \leq 0.
$$

Hence

$$
(\Delta - \frac{\partial}{\partial t}) g \cdot f \leq 0,
$$

(3.8)

$$
\nabla g = -\frac{g}{f} \nabla f.
$$

(3.9)

Substituting (3.5) and (3.6) into (3.8) and using (3.9) twice we get,

$$
0 \geq (\Delta - \frac{\partial}{\partial t}) g \cdot f = f(\Delta - \frac{\partial}{\partial t}) g + g(\Delta - \frac{\partial}{\partial t}) f + 2\nabla g \cdot \nabla f
$$

$$
\geq -\frac{C_1}{R^2} f - 2 \frac{|\nabla g|^2}{g} f + g(\Delta - \frac{\partial}{\partial t}) f
$$
\[
\geq -\frac{C_1 + 2C_2}{R^2} f + 2g \cdot f |H|^2 (1 - p) + g \cdot f (2p |\nabla H|^2 + 4p^2 |H|^2 |\nabla|H||^2 - 2\frac{\nabla \cos \theta |^2}{\cos^2 \theta})
\]
\[
+ 2g \cos^2 \theta \nabla f \cdot \nabla \frac{1}{\cos^2 \theta}
\]
\[
\geq -\frac{C_1 + 2C_2}{R^2} f + 2g \cdot f |H|^2 (1 - p) + g \cdot f (2p |\nabla H|^2 + 4p^2 |H|^2 |\nabla|H||^2 - 2\frac{\nabla \cos \theta |^2}{\cos^2 \theta})
\]
\[
-2 \cos^2 \theta f \nabla \frac{1}{\cos^2 \theta} \cdot \nabla g.
\]

Using the equation (3.9),
\[
\nabla g = g (2 \frac{\nabla \cos \theta}{\cos \theta} - p \nabla |H|^2).
\]

Thus,
\[
4gp^2 |\nabla|H||^2 |H|^2 = \frac{|\nabla g|^2}{g} + 4g \frac{|\nabla \cos \theta|^2}{\cos^2 \theta} - 4 \nabla g \cdot \frac{\nabla \cos \theta}{\cos \theta}.
\]

Putting this equation into (3.10), we get,
\[
0 \geq -\frac{C_1 + 2C_2}{R^2} f + 2gf(1 - p)|H|^2 + 2pgf|\nabla H|^2 + \frac{f}{g} |\nabla g|^2 + 2g \frac{f |\nabla \cos \theta|^2}{\cos^2 \theta}
\]
\[
\geq -\frac{C_3}{R^2} f + 2gf(1 - p)|H|^2.
\]

This implies that
\[
\frac{C_3}{R^2} \geq 2g(1 - p)|H|^2 = 2gf(1 - p) \frac{\cos^2 \theta |H|^2}{e^{|H|^2}} \geq 2gf \delta^2 e^{-ph^2} (1 - p)|H|^2.
\]

By the assumption that \( \sup_{t \leq 0} \sup_{t \Sigma} |A|^2 = 1 \), we have \( h^2 \leq 2 \), so
\[
\frac{C_4}{R^2} \geq \delta^2 2gf(1 - p)|H|^2.
\]

Since \( 1 - p > 0 \), we get that,
\[
|H|^2 (X(s_i), t(s_i))(g \cdot f)(X(s_i), t(s_i)) \leq \frac{C_4}{(1 - p)R^2}.
\]

So,
\[
|H|^2 (X(s_i), t(s_i)) f(0, 0) \leq |H|^2 (X(s_i), t(s_i))(g \cdot f)(X(s_i), t(s_i)) \leq \frac{C_4}{(1 - p)R^2}.
\]
Notice that \( f(0, 0) \neq 0 \), thus,
\[
|H|^2(X(s_i), t(s_i)) \leq \frac{C_5}{R^2}.
\]
Therefore,
\[
\sup_{B_{R^2}(s_i, t(s_i))} f(X, t) \leq \frac{1}{\delta^2} e^{p|H|^2(x(s), t(s))} \leq \frac{1}{\delta^2} e^{\frac{pc_5}{R^2}}.
\]
Let \( i \to \infty \) then \( R \to \infty \) we get that
\[
\frac{1}{\delta^2} \geq \sup f \geq e^{ph^2},
\]
which contradicts our assumption because \( p \) can be chosen so that it is close to 1. This completes the proof of Theorem 2.

Q. E. D.

Now we turn to the proof of Main Theorem 1. Recall the evolution equation of the Kähler angle in \( \mathbb{C}^2 \) (see [1]),
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha,
\]
where \( J_{\Sigma_t} \) is an almost complex structure in a tubular neighborhood of \( \Sigma_t \) in \( \mathbb{C}^2 \) with
\[
\begin{cases}
J_{\Sigma_t} e_1 &= e_2 \\
J_{\Sigma_t} e_2 &= -e_1 \\
J_{\Sigma_t} v_1 &= v_2 \\
J_{\Sigma_t} v_2 &= -v_1.
\end{cases}
\]
(3.12)

It is showed in [4] and [1] that,
\[
|\nabla J_{\Sigma_t}|^2 \geq \frac{1}{2} |H|^2,
\]
which implies that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \cos \alpha \geq \frac{1}{2} |H|^2 \cos \alpha.
\]
Using the equation (3.11) we can prove one monotonicity formula along the symplectic mean curvature flow in \( \mathbb{R}^4 \) by the same argument as the one used in the proof of Proposition 2.1.

**Proposition 3.2.** Along the symplectic mean curvature flow \( \Sigma_t \) in \( \mathbb{C}^2 \), we have,
\[
\frac{\partial}{\partial t} \left( \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, t) d\mu_t \right)
= - \left( \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, t) \left| H + \frac{(F - X_0)}{2(t_0 - t)} \right|^2 d\mu_t \right)
+ \int_{\Sigma_t} \frac{1}{\cos \alpha} \rho(F, t) |\nabla J_{\Sigma_t}|^2 d\mu_t + \int_{\Sigma_t} \frac{2}{\cos^3 \alpha} |\nabla \cos \alpha|^2 \rho(F, t) d\mu_t.
\]
By this monotonicity formula we can find a sequence \( \{s_i\} \) such that \( s_i \to -\infty \) and
\[
\int_{\Sigma_{s_i} \cap B_R(0)} |\nabla J_{\Sigma_t}|^2 \to 0 \text{ as } i \to \infty.
\]
By (3.13) we get that,
\[
\lim_{i \to \infty} \max_{\Sigma_{s_i} \cap B_R(0)} |H|^2 = 0. \tag{3.14}
\]
We still argue it by contradiction. We assume \( \delta > e^{-\frac{\epsilon^2}{4}} \) and construct the function
\[
f = e^{p|H|^2} \cos^2 \alpha \text{ where } t \in [s_i, 0].
\]
Due to the inequality (3.13), here \( p \) should be chosen so that \( p \) is sufficiently close to \( 1/2 \) and keeps the condition \( 1/2 - p > 0 \). Using the equation
\[
(\Delta - \frac{\partial}{\partial t}) \frac{1}{\cos^2 \alpha} \geq 6 \frac{|\nabla \cos \alpha|^2}{\cos^4 \alpha} + 2 \frac{|\nabla J_{\Sigma_t}|^2}{\cos^2 \alpha} \geq 6 \frac{|\nabla \cos \alpha|^2}{\cos^4 \alpha} + \frac{|H|^2}{\cos^2 \alpha},
\]
we obtain that,
\[
(\Delta - \frac{\partial}{\partial t}) f \geq f (4p^2|H|^2|\nabla|H|^2| + 2p|\nabla H|^2 + 2(1/2 - p)|H|^2 - 2 \frac{|\nabla \cos \theta|^2}{\cos^2 \theta}) + 2 \cos^2 \theta \nabla f \cdot \nabla \frac{1}{\cos^2 \theta} \tag{3.15}
\]
Similarly we can get,
\[
\frac{1}{\delta^2} \geq \sup f \geq e^{\rho \delta^2},
\]
which contradicts our assumption that \( \delta > e^{-\frac{\epsilon^2}{4}} \) because \( p \) is close to \( 1/2 \). We leave the details to the reader.

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