VASYLII PESTUN

Abstract. We prove conjecture due to Erickson-Semenoff-Zarembo and Drucker-Gross which relates supersymmetric circular Wilson loop operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with a Gaussian matrix model. We also compute the partition function and give a new matrix model formula for the expectation value of a supersymmetric circular Wilson loop operator for the pure $\mathcal{N} = 2$ and the $\mathcal{N} = 2^*$ supersymmetric Yang-Mills theory on a four-sphere. A four-dimensional $\mathcal{N} = 2$ superconformal gauge theory is treated similarly.

Contents

1. Introduction 1
2. Fields, action and symmetries 8
3. Localization 18
4. Determinant factor 24
5. Instanton corrections 40
Appendix A. Clifford algebra 45
Appendix B. Conformal killing spinors on $S^4$ 48
Appendix C. Off-shell supersymmetry 50
Appendix D. Index of transversally elliptic operators 54
References 59

1. Introduction

Topological gauge theory can be obtained by a twist of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [1]. The path integral of the twisted theory localizes to the moduli space of instantons and computes the Donaldson-Witten invariants of four-manifolds [1,3].

In a flat space the twisting does not change the Lagrangian. In [4] Nekrasov used a $U(1)^2$ subgroup of the $SO(4)$ Lorentz symmetry on $\mathbb{R}^4$ to define a $U(1)^2$-equivariant version of the topological partition function, or, equivalently, the partition function of the $\mathcal{N} = 2$ supersymmetric gauge theory in the $\Omega$-deformed background [5]. The integral over moduli space of instantons $\mathcal{M}_{\text{inst}}$ localizes at the fixed point set of a group which acts on $\mathcal{M}_{\text{inst}}$ by Lorentz rotations of the space-time and gauge transformations at infinity. The partition function $Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, a)$ depends on the parameters $(\varepsilon_1, \varepsilon_2)$, which generate $U(1)^2$ Lorentz rotations, and

Date: December, 2007.
On leave of absence from ITEP, Moscow, 117259, Russia.

1
the parameter \(a \in \mathfrak{g}\), which generates gauge transformations at infinity. By \(\mathfrak{g}\) we denote the Lie algebra of the gauge group. This partition function is finite because the \(\Omega\)-background effectively confines the dynamics to a finite volume \(V_{\text{eff}} = \frac{1}{\varepsilon_1 \varepsilon_2}\).

In the limit of vanishing \(\Omega\)-deformation (\(\varepsilon_1, \varepsilon_2 \to 0\)) the effective volume \(V_{\text{eff}}\) diverges as well as the free energy \(F = -\log Z_{\text{inst}}\). But the specific free energy \(F/V_{\text{eff}}\) has a well-defined limit, which actually coincides with Seiberg-Witten low-energy effective prepotential \(\mathcal{F}(a)\) of the \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory \([4, 7]\). In this way instanton counting gives a derivation of Seiberg-Witten prepotential from the first principles.

In this paper we consider another interesting situation where an analytical computation of the partition function is possible. We consider the \(\mathcal{N} = 2\), the \(\mathcal{N} = 2^*\) and the \(\mathcal{N} = 4\) Yang-Mills theory on a four-sphere \(S^4\) equipped with the standard round metric.

There are no zero modes for the gauge fields, because the first cohomology group of \(S^4\) is trivial. There are no zero modes for the fermions. This follows from the fact that the Laplacian operator on a compact space is semipositive and the formula \(\mathcal{D}^2 = \Delta + R^4\), where by \(\mathcal{D}\) we denote the Dirac operator, by \(\Delta\) the Laplacian, and by \(R\) the scalar curvature, which is positive on \(S^4\). There are no zero modes for the scalar fields, because there is a mass term in the Lagrangian proportional to the scalar curvature.

Observing that there are no zero modes at all, we can try to integrate over all fields in the path integral and to compute the full partition function of the theory. In addition, we would like to compute expectation values of certain interesting observables.

In this paper we are mostly interested in the observable defined by the supersymmetric circular Wilson loop operator (see Fig. 1)

\[
W_R(C) = \text{tr}_R \text{Pexp} \oint_C (A_\mu dx^\mu + i \Phi^\mu_0 ds).
\]

Here \(R\) is a representation of the gauge group, \(\text{Pexp}\) is the path-ordered exponent, \(C\) is a circular loop located at the equator of \(S^4\), \(A_\mu\) is the gauge field and \(i \Phi^\mu_0\) is one of the scalar fields of the \(\mathcal{N} = 2\) vector multiplet. We reserve notation \(\Phi^\mu_0\) for the scalar field in a theory obtained by dimensional reduction of a theory in Euclidean signature. Our conventions are that all fields take values in the real Lie algebra of the gauge group. For example, if the gauge group is \(U(N)\), then all fields can be represented by antihermitian matrices. The covariant derivative is \(D_\mu = \partial_\mu + A_\mu\) and the field strength is \(F_{\mu\nu} = [D_\mu, D_\nu]\).

In [4] Erickson, Semenoff and Zarembo conjectured that the expectation value \(\langle W_R(C) \rangle\) of the Wilson loop operator (1.1) in the four-dimensional \(\mathcal{N} = 4\) \(SU(N)\) gauge theory in the large \(N\) limit can be exactly computed by summing all rainbow diagrams in Feynman gauge. The combinatorics of rainbow diagrams can be represented by a Gaussian matrix model. In [9] the conjecture was tested at one-loop level in gauge theory. In [10] Drukker and Gross conjectured that the exact relation to the Gaussian matrix model holds for any \(N\) and argued that the expectation value of the Wilson loop operator (1.1) can be computed by a matrix

---

1What we call \(\mathcal{N} = 2\) supersymmetry on \(S^4\) is explained in section 2. It would be interesting to extend the analysis to more general backgrounds [4].
LOCALIZATION OF GAUGE THEORY ON $S^4$

Figure 1. Wilson loop on the equator of $S^4$

model. However, Drukker-Gross argument does not prove that this matrix model is Gaussian.

In the context of the $AdS/CFT$ correspondence \cite{11,13} the conjecture was relevant for many works; see for example \cite{14,39} and references therein. But there has been no direct gauge theory derivation of the conjecture beyond the two-loop level \cite{40,41} and some attempts to evaluate the first instanton corrections \cite{22}.

In this paper, we prove the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture for the $\mathcal{N}=4$ supersymmetric Yang-Mills theory formulated for an arbitrary gauge group. Let $r$ be the radius of $S^4$. The conjecture states that

$$\langle W_R(C) \rangle_{\mathcal{N}=4} \text{ on } S^4 = \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 g^2}{\mathcal{V}_M(a,a)} tr R e^{2\pi r i a}} \cdot$$ (1.2)

The finite dimensional integrals in this formula are taken over the Lie algebra $\mathfrak{g}$ of the gauge group, $a$ denotes an element of $\mathfrak{g}$. By $(a,a)$ for $a \in \mathfrak{g}$ we denote an invariant positive definite quadratic form on $\mathfrak{g}$. Our convention is that the kinetic term in the gauge theory is normalized as $\frac{1}{4g^2 \mathcal{V}_M} \int d^4x \sqrt{g} (F_{\mu\nu}, F^{\mu\nu})$. The formula (1.2) can be rewritten in terms of the integral over the Cartan subalgebra of $\mathfrak{g}$ with insertion of the usual Weyl measure $\Delta(a) = \prod_{\alpha \in \text{roots of } \mathfrak{g}} \alpha \cdot a$.

We also get a new formula for the $\langle W_R(C) \rangle$ in the $\mathcal{N}=2$ and the $\mathcal{N}=2^*$ supersymmetric Yang-Mills theory. As in the $\mathcal{N}=4$ case, the result can be written in terms of a matrix model. However, this matrix model is much more complicated than a Gaussian matrix model. We derive this matrix model action up to all orders in perturbation theory. Then we argue what is the non-perturbative contribution of all instanton/anti-instanton corrections.

Our main result is

$$Z_{S^4}^\mathcal{N} (W_R(C)) = \frac{1}{\text{vol}(G)} \int_{\mathfrak{g}} [da] e^{-\frac{4\pi^2 g^2}{\mathcal{V}_M(a,a)} Z^\mathcal{N}_{1\text{-loop}}(ia) Z^\mathcal{N}_{\text{inst}}(r^{-1}, r^{-1}, ia)^2 tr R e^{2\pi r i a}}.$$ (1.3)

Here $Z_{S^4}^\mathcal{N}$ is the partition function of the $\mathcal{N}=2$, the $\mathcal{N}=2^*$ or the $\mathcal{N}=4$ supersymmetric Yang-Mills theory on $S^4$, defined by the path integral over all fields in the theory, and $\langle W_R(C) \rangle_{\mathcal{N}}$ is the expectation value of $W_R(C)$ in the corresponding theory. In particular, if we take $R$ to be the trivial one-dimensional representation, the formula says that the partition function $Z_{S^4}^\mathcal{N}$ is computed by the following
finite-dimensional integral:
\[
Z^N_{4} = \frac{1}{\text{vol}(G)} \int [da] e^{-\frac{4\pi^2 r^2}{N} \text{tr}^2 (a,a)} Z^N_{\text{1-loop}}(ia) Z^N_{\text{inst}}(r^{-1}, r^{-1}, ia)^2. 
\] (1.4)

In other words, we show that the Wilson loop observable (1.1) is compatible with the localization of the path integral to the finite dimensional integral (1.3) and that

\[
\langle W_R(C) \rangle_{4d \text{ theory}} = \langle \text{tr}_R e^{2\pi ria} \rangle_{\text{matrix model}}.
\] (1.5)

where the matrix model measure \(\langle \ldots \rangle_{\text{matrix model}}\) is given by the integrand in (1.4).

The factor \(Z_{\text{1-loop}}(ia)\) is a certain infinite dimensional product, which appears as a determinant in the localization computation. It can be expressed in terms of a product of Barnes \(G\)-functions [42]. In the most general \(\mathcal{N} = 2^*\) case, the factor \(Z_{\text{1-loop}}(ia)\) is given by the formula (4.48). The \(\mathcal{N} = 2\) and \(\mathcal{N} = 4\) cases can be obtained by taking respectively limits \(m = \infty\) and \(m = 0\), where \(m\) is the hypermultiplet mass in the \(\mathcal{N} = 2^*\) theory. For the \(\mathcal{N} = 4\) theory we get \(Z_{\text{1-loop}} = 1\).

The factor \(Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, ia)\) is Nekrasov’s partition function [5] of point instantons in the equivariant theory on \(\mathbb{R}^4\). In the \(\mathcal{N} = 2^*\) case it is given by the formula (5.12). In the limit \(m = \infty\), one gets the \(\mathcal{N} = 2\) case (5.1), in the limit \(m = 0\) one gets the \(\mathcal{N} = 4\) case. In the \(\mathcal{N} = 4\) case, the instanton partition function (5.15) does not depend on \(a\). Therefore in the \(\mathcal{N} = 4\) case, instantons do not contribute to the expectation value \(\langle W_R(C) \rangle\).

Our claim about vanishing of instanton corrections for the \(\mathcal{N} = 4\) theory contradicts to the results of [22], where the first instanton correction for the \(SU(2)\) gauge group was found to be non-zero. In [22] the authors introduced a certain cut-off on the instanton moduli space, which is not compatible with the relevant supersymmetry of the theory and the Wilson loop operator. Our instanton calculation is based on Nekrasov’s partition function on \(\mathbb{R}^4\). This partition function is regularized by a certain non-commutative deformation of \(\mathbb{R}^4\) compatible with the relevant supersymmetry. Though we do not write down explicitly the non-commutative deformation of the theory on \(S^4\), we assume that such deformation can be well defined. We also assume that in a small neighbourhood of the North or the South pole of \(S^4\) this non-commutative deformation agrees with the non-commutative deformation used by Nekrasov [4] on \(\mathbb{R}^4\).

Since both \(Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, ia)\) and its complex conjugate enter the formula, this means that we count both instantons and anti-instantons. The formula is similar to Ooguri-Strominger-Vafa relation between the black hole entropy and the topological string partition function [43, 44]

\[
Z_{BH} \propto |Z_{\text{top}}|^2.
\] (1.6)

Actually the localization computation is compatible with more general observables than a single Wilson loop in representation \(R\) inserted on the equator (1.1). Let us fix two opposite points on the \(S^4\) and call them the North and the South poles. Then we can consider a class of Wilson loops placed on circles of arbitrary radius such that they all have a common center at the North pole, and such that they all can be transformed to each other by a composition of a dilation in the North-South direction and by an anti-self-dual rotation in the \(SU(2)_L\) left subgroup of the \(SO(4)\) subgroup of the \(S^4\) isometry group which fixes the North pole.
However, for Wilson loops of not maximal size, we need to change the relative coefficient between the gauge and the scalar field terms in (1.1). Let $C_θ$ be a circle located at an arbitrary polar angle $θ$ measured from the North pole (at the equator $\sin θ = 1$). Then we consider

$$W_R(C_θ) = \text{tr}_R \text{Pexp} \oint_{C_θ} \left( A_µ dx^µ + \frac{1}{\sin θ} (iΦ^E_0 + Φ_9 \cos θ) ds \right),$$

where $Φ^E_0$ and $Φ_9$ are the scalar fields of the $N = 2$ vector multiplet.

Equivalently this can be rewritten as

$$W_R(C_θ) = \text{tr}_R \text{Pexp} \oint_{C_θ} \left( A_µ dx^µ + (iΦ^E_0 + Φ_9 \cos θ) r dα \right),$$

where $α \in [0, 2\pi)$ is an angular coordinate on the circle $C$. Formally, as the size of the circle vanishes ($θ \to 0$) we get a “holomorphic” observable $W_R(C_{θ → 0}) = \text{tr}_R \exp 2πrΦ(N)$ where $Φ(N)$ is the complex scalar field $iΦ^E_0 + Φ_9$ evaluated at the North pole. In the opposite limit ($θ \to \pi$) we get an “anti-holomorphic” observable $W_R(C_{θ → π}) = \text{tr}_R \exp 2πrΦ(S)$, where $Φ(S)$ is the conjugated scalar field $-iΦ^E_0 + Φ_9$ evaluated at the South pole. However, in the actual computation of the path integral we will always assume a finite size of $C$, so that the operator $W_R(C)$ is well defined.

Then for an arbitrary set $\{W_{R_1}(C_{θ_1}), \ldots, W_{R_n}(C_{θ_n})\}$ of Wilson loops in the class described above we obtain

$$\langle W_{R_1}(C_{θ_1}) \ldots W_{R_n}(C_{θ_n}) \rangle_{4\text{d theory}} = \langle \text{tr}_{R_1} e^{2πr iα} \ldots \text{tr}_{R_n} e^{2πr iα} \rangle_{\text{matrix model}}$$

The Drukker-Gross argument only applies to the case of a single circle which can be related to a straight line on $\mathbb{R}^4$ by a conformal transformation, but in the present approach we can consider several circles simultaneously.

So far we described the class of observables which we can compute in the massive $N = 2^*$ theory. All these observables are invariant under the same operator $Q$ generated by a conformal Killing spinor on $S^4$ of constant norm. This operator $Q$ is a fermionic symmetry at quantum level.

Now we describe more general classes of circular Wilson loops which can be solved in $N = 4$ theory. Thanks to the conformal symmetry of the $N = 4$ theory there is a whole family of operators $\{Q(t)\}$ where $t$ runs from 0 to $\infty$, which we can use for the localization computation. The case $t = 1$ corresponds to the conformal Killing spinor of constant norm and to the observables which we study in the $N = 2^*$ theory. However, for a general $t$ in the $N = 4$ theory we can take

$$W_R(C_θ, t) = \text{tr}_R \text{Pexp} \oint_{C_θ} \left( A_µ dx^µ + \frac{1}{t \sin θ} \left( (\cos^2 \frac{θ}{2} + t^2 \sin^2 \frac{θ}{2}) iΦ^E_0 + Φ_9 (\cos^2 \frac{θ}{2} - t^2 \sin^2 \frac{θ}{2}) \right) ds \right).$$

At $t \sin \frac{θ}{2} = \cos \frac{θ}{2}$ we get the Wilson loop (1.1) with the same relative coefficient 1 between $A_µ$ and $iΦ^E_0$ but of arbitrary size. The $N = 4$ theory with insertion of the operator $W_R(C_θ, t)$ still localizes to the Gaussian matrix model.

The idea underlying localization is that in some situations the integral is exactly equal to its semiclassical approximation. For example, the Duistermaat-Heckman formula says [45]

$$\int_M (2\pi)^n n! e^{iH(φ)} = i^n \sum_{p ∈ E} e^{iH(φ)} \prod \alpha'_i(φ),$$

where $M$ is a compact manifold, $H(φ)$ is a Hamiltonian, and $E$ is a set of points in $M$. This formula provides a powerful tool for computing integrals in various physical systems.
where \((M, \omega)\) is a symplectic manifold, and \(H : M \to g^*\) is a moment map for a Hamiltonian action of \(G = U(1)^k\) on \(M\). The Duistermaat-Heckman formula is a particular case of a more general Atiyah-Bott-Berline-Vergne localization formula \([46, 47]\). Let an abelian group \(G\) act on a compact manifold \(M\). We consider the complex of \(G\)-equivariant differential forms on \(M\) valued in functions on \(g\) with the differential \(Q = d - \partial^a i_a\). The differential squares to a symmetry transformation \(Q^2 = -\partial^a \mathcal{L}_{e^a}\). Here \(\mathcal{L}_{e^a}\) represents the action of \(G\) on \(M\). Hence \(Q^2\) annihilates \(G\)-invariant objects. Then for any \(Q\)-closed form \(\alpha\), Atiyah-Bott-Berline-Vergne localization formula is

\[
\int_M \alpha = \int_F i_{e^a}^* \alpha e^a(N_F),
\]

where \(F \hookrightarrow M\) is the \(G\)-fixed point set, and \(e(N_F)\) is the equivariant Euler class of the normal bundle of \(F\) in \(M\). When \(F\) is a discrete set of points, the equivariant Euler class \(e(N_F)\) at each point \(f \in F\) is simply the determinant of the representation in which \(g\) acts on the tangent bundle of \(M\) at a point \(f\).

Localization can be argued in the following way \([1, 48]\). Let \(Q\) be a fermionic symmetry of a theory. Let \(Q^2 = \mathcal{L}_{\phi}\) be some bosonic symmetry. Let \(S\) be a \(Q\)-invariant action, so that \(QS = 0\). Consider a functional \(V\) which is invariant under \(\mathcal{L}_{\phi}\), so that \(Q^2 V = 0\). Deformation of the action by a \(Q\)-exact term \(QV\) can be written as a total derivative and does not change the integral up to boundary contributions

\[
\frac{d}{dt} \int e^{S + tQV} = \int \{Q, V\} e^{S + tQV} = \int \{Q, Ve^{S + tQV}\} = 0.
\]

As \(t \to \infty\), the one-loop approximation at the critical set of \(QV\) becomes exact. Then for a sufficiently nice \(V\), the integral is computed by evaluating \(S\) at critical points of \(QV\) and the corresponding one-loop determinant.

We apply this strategy to the \(\mathcal{N} = 2\), the \(\mathcal{N} = 2^*\) and the \(\mathcal{N} = 4\) supersymmetric Yang-Mills gauge theories on \(S^4\) and show that the path integral is localized to the constant modes of the scalar field \(\Phi_0\) with all other fields vanishing. In this way we also compute exactly the expectation value of the circular supersymmetric Wilson loop operator \([11]\).

Remark. Most of the presented arguments in this work should apply to an \(\mathcal{N} = 2\) theory with an arbitrary matter content. For a technical reasons related to the regularization issues, we limit our discussion to the \(\mathcal{N} = 2\) theory with a single \(\mathcal{N} = 2\) massive hypermultiplet in the adjoint representation, also known as the \(\mathcal{N} = 2^*\). By taking the limit of vanishing or infinite mass we can respectively recover the \(\mathcal{N} = 4\) or the \(\mathcal{N} = 2\) theory.

Still we will give in \([4, 57]\) a formula for the factor \(Z_{1\text{-loop}}\) for an \(\mathcal{N} = 2\) gauge theory with a massless hypermultiplet in such representation that the theory is conformal. Perhaps, one could check our result by the traditional Feynman diagram computations directly in the gauge theory. To simplify comparison, we will give an explicit expansion in \(g_{YM}\) up to the sixth order of the expectation value of the Wilson loop operator for the \(\mathcal{N} = 2\) theory with the gauge group \(SU(2)\) and \(4\)

\[\text{In other words, } i_\phi \omega = dH(\phi) \text{ for any } \phi \in g, \text{ where } i_\phi \text{ is a contraction with a vector field generated by } \phi.\]
localization of gauge theory on $S^4$ hypermultiplets in the fundamental representation (see (4.58))

$$\langle e^{2\pi na} \rangle_{\text{matrix model}} = 1 + \frac{3}{2} \cdot \frac{1}{2^2} n^2 g_Y^2 M + \frac{5}{8} \cdot \frac{1}{2^4} n^4 g_Y^4 M + \frac{7}{48} \cdot \frac{1}{2^6} n^6 g_Y^6 M - \frac{35}{12} \cdot \frac{\zeta(3)}{(4\pi)^2} n^2 g_Y^2 M + O(g_Y^8),$$

(1.11)

In this formula $a \in \mathbb{R}$ is a coordinate on the Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$. By an integer $n \in \mathfrak{h}^*$ we denote a weight. For example, if the Wilson loop is taken in the spin-$j$ representation, where $j$ is a half-integer, the weights are \{-2j, -2j + 2, \ldots, 2j\}.

Hence we get $\langle W_j(C) \rangle = \langle \sum_{m=-j}^j e^{4\pi ma} \rangle_{\text{MM}}$.

We shall note that the first difference between the $\mathcal{N} = 2$ superconformal theory and the $\mathcal{N} = 4$ theory appears at the order $g_Y^6$, up to which the Feynman diagrams in the $\mathcal{N} = 4$ theory were computed in [40, 41]. Therefore a direct computation of Feynman diagrams in the $\mathcal{N} = 2$ theory up to this order seems to be possible and would be a non-trivial test of our results.

Some unusual features in this work are: (i) the theory localizes not on a counting problem, but on a nontrivial matrix model, (ii) there is a one-loop factor involving an index theorem for transversally elliptic operators [49, 50].

In section 2 we give details about the $\mathcal{N} = 2$, the $\mathcal{N} = 2^*$ and $\mathcal{N} = 4$ SYM theories on a four-sphere $S^4$. In section 3 we make a localization argument to compute the partition function for these theories. Section 4 explains the computation of the one-loop determinant [41, 50], or, mathematically speaking, of the equivariant Euler class of the infinite-dimensional normal bundle in the localization formula. In section 5 we consider instanton corrections.

There are some open questions and immediate directions in which one can proceed:

(1) One can consider more general supersymmetric Wilson loops like studied in [28, 37, 51] and try to prove the conjectural relations of those with matrix models or two-dimensional super Yang-Mills theory. Perhaps it will be also possible to extend the analysis of those more general loops to (superconformal) $\mathcal{N} = 2$ theories like it is done in the present work.

(2) Using localisation, one can try to solve exactly for an expectation value of a circular supersymmetric 't Hooft-Wilson operator (this is a generalization of Wilson loop in which the loop carries both electric and magnetic charges) [52–54]. The expectation values of such operators should transform in the right way under the S-duality transformation which replaces the coupling constant by its inverse and the gauge group $G$ by its Langlands dual $^L G$. Perhaps this could tell us more on the four-dimensional gauge theory and geometric Langlands [52] where 't Hooft-Wilson loops play the key role.

(3) It would be interesting to find more precise relation between our formulas, and Ooguri-Strominger-Vafa [43] conjecture (1.6). There could be a four-dimensional analogue of the tt*-fusion [52].

Acknowledgment. I would like to thank E. Witten and N. Nekrasov for many stimulating discussions, important comments and suggestions. I thank M. Atiyah, N. Berkovits, A. Dymarsky, D. Gaiotto, S. Gukov, J. Maldacena, I. Klebanov, H. Nakajima, A. Neitzke, I. Singer, M. Rocek, K. Zarembo for interesting discussions and remarks. Part of this research was done during my visit to Physics Department of Harvard University in January 2007. The work was supported in part
by Federal Agency of Atomic Energy of Russia, grant RFBR 07-02-00645, grant for support of scientific schools NSh-8004.2006.2 and grant of the National Science Foundation PHY-0243680. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of these funding agencies.

2. Fields, action and symmetries

To write down the action of the $\mathcal{N} = 4$ SYM on $S^4$, we use dimensional reduction of the $\mathcal{N} = 1$ SYM on $\mathbb{R}^{9,1}$. By $G$ we denote the gauge group. By $A_M$ with $M = 0, \ldots, 9$ we denote the components of the gauge field in ten dimensions, where we take the Minkowski metric $ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_9^2$. When we write formulas in Euclidean signature so that the metric is $ds^2 = dx_0^2 + dx_1^2 + \cdots + dx_9^2$, we use notation $A_0^E$ for the zero component of the gauge field.

By $\Psi$ we denote a sixteen real component ten-dimensional Majorana-Weyl fermion valued in the adjoint representation of $G$. (In Euclidean signature $\Psi$ is not real, but its complex conjugate does not appear in the theory.) The ten-dimensional action $S = \int d^{10}x \mathcal{L}$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2g_{YM}^2} \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi \right)$$

(2.1)

is invariant under the supersymmetry transformations

$$\delta_\varepsilon A_M = \varepsilon \Gamma_M \Psi$$

$$\delta_\varepsilon \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon.$$

Here $\varepsilon$ is a constant Majorana-Weyl spinor parameterizing the supersymmetry transformations in ten dimensions. (See appendix A for our conventions on the algebra of gamma-matrices.)

We do not write explicitly the color and spinor indices. We also assume that in all bilinear terms the color indices are contracted using some invariant positive definite bilinear form (Killing form) on the Lie algebra $\mathfrak{g}$ of the gauge group. Sometimes we denote this Killing form by $(\cdot, \cdot)$. In Euclidean signature we integrate over fields which all take value in the real Lie algebra of the gauge group. For example, for the $U(N)$ gauge group all fields are represented by the antihermitean matrices, and we can define the Killing form on $\mathfrak{g}$ as $(a, b) = -\text{tr}_F ab$, where $\text{tr}_F$ is the trace in the fundamental representation.

We take $(x_1, \ldots, x_4)$ to be the coordinates along the four-dimensional space-time, and we make dimensional reduction in the remaining directions: $0, 5, \ldots, 8, 9$. Note that the four-dimensional space-time has Euclidean signature.

Now we describe the symmetries of the four-dimensional theory if we start from Minkowski signature in ten dimensions. Note that we make dimensional reduction along the time-like coordinate $x_0$. Therefore we get the wrong sign for the kinetic term for the scalar field $\Phi_0$, where $\Phi_0$ denotes the 0-th component of the gauge field $A_M$ after dimensional reduction. To make sure that the path integral is well defined and convergent, in this case in the path integral for the four-dimensional theory we integrate over imaginary $\Phi_0$. Actually this means that the path integral is the same as in the Euclidean signature with all bosonic fields taken real.

The ten-dimensional $\text{Spin}(9,1)$ Lorentz symmetry group is broken to $\text{Spin}(4) \times \text{Spin}(5,1)^R$, where the first factor is the four-dimensional Lorentz group acting on
LOCALIZATION OF GAUGE THEORY ON $S^4$ (x_1, \ldots, x_4)$ and the second factor is the R-symmetry group acting on $(x_5, \ldots, x_9, x_0)$. It is convenient to split the four-dimensional Lorentz group as $Spin(4) = SU(2)_L \times SU(2)_R$, and brake the $Spin(5, 1)_R$-symmetry group into $Spin(4)_R \times SO(1, 1)_R = SU(2)_L \times SU(2)_R \times SO(1, 1)_R$. The components of the ten-dimensional gauge field, which become scalars after the dimensional reduction are denoted by $\Phi_A$ with $A = 0, 5, \ldots, 9$. Let us write the bosonic fields and the symmetry groups under which they transform:

$$SU(2)_L \times SU(2)_R \quad SU(2)_L^R \times SU(2)_R^R \quad SO(1, 1)_R$$

Using a certain Majorana-Weyl representation of the Clifford algebra $Cl(9, 1)$ (see appendix A for our conventions), we write $\Psi$ in terms of four four-dimensional chiral spinors as

$$\Psi = \begin{pmatrix} \psi^L \\ \chi^R \\ \psi^R \\ \chi^L \end{pmatrix}.$$ 

Each of these spinors $(\psi^L, \chi^R, \psi^R, \chi^L)$ has four real components. Their transformation properties are summarized in the table:

| $\varepsilon$ | $\Psi$ | $SU(2)_L$ | $SU(2)_R$ | $SU(2)_L^R$ | $SU(2)_R^R$ | $SO(1, 1)_R$ |
|----------------|--------|------------|------------|---------------|---------------|---------------|
| *              | $\psi^L$ | 1/2        | 0          | 1/2           | 0             | +             |
| 0              | $\chi^R$ | 0          | 1/2        | 0             | 1/2           | +             |
| *              | $\psi^R$ | 0          | 1/2        | 1/2           | 0             | -             |
| 0              | $\chi^L$ | 1/2        | 0          | 0             | 1/2           | -             |

Let the spinor $\varepsilon$ be the parameter of the supersymmetry transformations. We restrict the $\mathcal{N} = 4$ supersymmetry algebra to the $\mathcal{N} = 2$ subalgebra by taking $\varepsilon$ in the +1-eigenspace of the operator $\Gamma^{5678}$. Such spinor $\varepsilon$ has the structure

$$\varepsilon = \begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix},$$

transforms in the spin-$\frac{1}{2}$ representation of the $SU(2)_L^R$ and in the trivial representation of the $SU(2)_R^R$.

With respect to the supersymmetry transformation generated by such $\varepsilon$, the $\mathcal{N} = 4$ gauge multiplet splits in two parts

- $(A_1 \ldots A_4, \Phi_9, \phi^L, \phi^R)$ is the $\mathcal{N} = 2$ vector multiplet
- $(\Phi_5 \ldots \Phi_8, \chi^L, \chi^R)$ is the $\mathcal{N} = 2$ hypermultiplet.

So far we considered dimensional reduction from $\mathbb{R}^{9,1}$ to the flat space $\mathbb{R}^4$. Now we would like to put the theory on a four-sphere $S^4$. We denote by $A_\mu$ with $\mu = 1, \ldots, 4$ the four-dimensional gauge field and by $\Phi_A$ with $A = 0, 5, \ldots, 9$ the four-dimensional scalar fields. The only required modification of the action is a coupling of the scalar fields to the scalar curvature of space-time. Namely, the kinetic term must be changed as $(\partial \Phi)^2 \rightarrow (\partial \Phi)^2 + \frac{R}{4} \Phi^2$, where $R$ is the scalar curvature. One way to see why this is the natural kinetic term for the scalar fields is to use the argument of the conformal invariance. Namely, one can check that $\int d^4x \sqrt{g} (\partial \Phi)^2 + \frac{R}{4} \Phi^2$ is invariant under Weyl transformations of the metric.
$g_{\mu\nu} \rightarrow e^{2\Omega}g_{\mu\nu}$ and scalar fields $\Phi \rightarrow e^{-\Omega}\Phi$. Then the action on $S^4$ of the $\mathcal{N} = 4$ SYM is
\[ S_{N=4} = \frac{1}{2g_Y^2} \int_{S^4} \sqrt{-g} d^4x \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \gamma^M D_M \Psi + 2 \frac{2}{r^2} \Phi^A \Phi_A \right), \tag{2.2} \]
where we used the fact that the scalar curvature of a $d$-sphere $S^d$ of radius $r$ is $\frac{d(d-1)}{r^2}$.

The action (2.2) is invariant under the $\mathcal{N} = 4$ superconformal transformations
\[ \delta_\varepsilon A_M = \varepsilon \Gamma_M \Psi \]
\[ \delta_\varepsilon \Phi = \frac{1}{2} F_{MNP} \Gamma^{NP} \varepsilon + \frac{1}{2} \Gamma_{\mu A} \Phi^A \nabla^\mu \varepsilon, \tag{2.4} \]
where $\varepsilon$ is a conformal Killing spinor solving the equations
\[ \nabla_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon} \]
\[ \nabla_\mu \tilde{\varepsilon} = -\frac{1}{4r^2} \Gamma_\mu \varepsilon. \tag{2.6} \]

(See e.g. [57] for a review on conformal Killing spinors, and for the explicit solution of these equations on $S^4$ see appendix C.) To get intuition about the meaning of $\varepsilon$ and $\tilde{\varepsilon}$ we can take the flat space limit $r \rightarrow \infty$. In this limit $\tilde{\varepsilon}$ becomes covariantly constant spinor $\tilde{\varepsilon} = \tilde{\varepsilon}_c$, while $\varepsilon$ becomes a spinor with at most linear dependence on flat coordinates $x^\nu$ on $\mathbb{R}^4$: $\varepsilon = \varepsilon_s + x^\nu \Gamma_\mu \tilde{\varepsilon}_c$. By $\varepsilon_s$ and $\tilde{\varepsilon}_c$ we denote some constant spinors. Then $\varepsilon_s$ generates supersymmetry transformations, while $\tilde{\varepsilon}_c$ generates special superconformal symmetry transformations.

The superconformal algebra closes only on-shell. Let $\delta_\varepsilon^2$ be the square of the fermionic transformation $\delta_\varepsilon$ generated by a spinor $\varepsilon$. After some algebra (see appendix C) we obtain
\[ \delta_\varepsilon^2 A_\mu = - (\varepsilon \Gamma^\nu \varepsilon) F_{\nu\mu} - [(\varepsilon \Gamma^B \varepsilon) \Phi_B, D_\mu] \]
\[ \delta_\varepsilon^2 \Phi_A = - (\varepsilon \Gamma^\nu \varepsilon) D_\nu \Phi_A - [(\varepsilon \Gamma^B \varepsilon) \Phi_B, \Phi_A] + 2 (\varepsilon \Gamma_{AB} \varepsilon) \Phi^B - 2 (\varepsilon \tilde{\varepsilon} \varepsilon) \Phi_A \]
\[ \delta_\varepsilon^2 \Psi = - (\varepsilon \Gamma^\nu \varepsilon) D_\nu \Psi - [(\varepsilon \Gamma^B \varepsilon) \Phi_B, \Psi] - \frac{1}{2} (\varepsilon \tilde{\varepsilon} \varepsilon) \Gamma^{\mu \nu} \Psi + \frac{1}{2} (\varepsilon \Gamma_{AB} \varepsilon) \Gamma^{AB} \Psi - 3 (\varepsilon \tilde{\varepsilon} \varepsilon) \Psi + \text{com[}\Psi]. \]
\[ \delta_\varepsilon^2 \Phi = \frac{1}{2} (\varepsilon \Gamma_N \varepsilon) \Gamma^N \Phi - (\varepsilon \Phi) \varepsilon. \tag{2.8} \]

Here the term denoted by $\text{com[}\Psi]$ is proportional to the Dirac equation of motion for fermions $\Psi$
\[ \text{com[}\Psi] = \frac{1}{2} (\varepsilon \Gamma_N \varepsilon) \Gamma^N \Phi - (\varepsilon \Phi) \varepsilon. \tag{2.8} \]

The square of the supersymmetry transformation can be written as
\[ \delta_\varepsilon^2 = - \mathcal{L}_v - R - \Omega. \tag{2.9} \]

The first term is the gauge covariant Lie derivative $\mathcal{L}_v$ in the direction of the vector field
\[ v^M = \varepsilon \gamma^M \varepsilon. \tag{2.10} \]

For example, $\mathcal{L}_v$ acts on scalar fields as follows: $\mathcal{L}_v \Phi_A = v^M D_M \Phi = v^\mu D_\mu \Phi_A + v^B [\Phi_B, \Phi]$. Here $D_\mu$ is the usual covariant derivative $D_\mu = \partial_\mu + A_\mu$.

To explain what the gauge covariant Lie derivative means geometrically, first we consider the situation when the gauge bundle, say $E$, is trivial. We fix some flat background connection $A^{(0)}_\mu$ and choose a gauge such that $A^{(0)}_\mu = 0$. For any connection $A$ on $E$ we define $\hat{A} = A - A^{(0)}$. The field $\hat{A}$ transforms as a one-form valued in the adjoint representation of $E$. The path integral over $A$ is equivalent
Yang-Mills theories on \( S^4 \) adjoint valued scalar field \( \Phi \) where \( N \) conformally invariant. We will be able to give a precise definition of the quantum Spin transform with weight 1, and the fermions transform with weight 3 classically the supersymmetry groups for the \( N \) mass term to the hypermultiplet, which we will send to the infinity in the end.

The gauge transformation \( G \) generates the \( \varepsilon \) and restrict \( N \) parameter 2(\( \varepsilon \)) set to zero. The second term \( G_\Phi \) is the gauge transformation generated by the adjoint valued scalar field \( \Phi \) where

\[
\Phi = v^M \tilde{A}_M. \tag{2.12}
\]

The gauge transformation \( G_\Phi \) acts on the matter and the gauge fields in the usual way: \( G_\Phi \Phi_A = [\Phi, \Phi_A], \) \( G_\Phi \cdot A_\mu = [\Phi, D_\mu] = -D_\mu \Phi. \)

The term denoted by \( R \) in (2.9) is a \( Spin(5,1)^R \)-symmetry transformation. It acts on scalar fields as \( (R \cdot \Phi)_A = R_{AB} \Phi^B, \) and on fermions as \( R \cdot \Psi = \frac{1}{4} R_{AB} \Gamma^{AB} \Psi, \) where \( R_{AB} = 2\varepsilon \tilde{\Gamma}_{AB} \varepsilon. \) When \( \varepsilon \) and \( \tilde{\varepsilon} \) are restricted to the \( \mathcal{N} = 2 \) subspace of \( \mathcal{N} = 4 \) algebra, \( (\Gamma^{5678} \varepsilon = \varepsilon \) and \( \Gamma^{5678} \tilde{\varepsilon} = \tilde{\varepsilon} \), the matrix \( R_{AB} \) with \( A, B = 5, \ldots, 8 \) is an anti-self-dual (left) generator of \( SO(4)^R \) rotations. In other words, when we restrict \( \varepsilon \) to the \( \mathcal{N} = 2 \) subsalgebra of the \( \mathcal{N} = 4 \) algebra, the \( SO(4)^R \) \( R \)-symmetry group restricts to its \( SU(2)^R \) subgroup. The fermionic fields of the \( \mathcal{N} = 2 \) vector multiplet (we call them \( \psi \)) transform in the trivial representation of \( R \), while the fermionic fields of the \( \mathcal{N} = 2 \) hypermultiplet (we call them \( \chi \)) transform in the spin-\( \frac{1}{2} \) representation of \( R \).

Finally, the term denoted by \( \Omega \) in (2.9) generates a local dilatation with the parameter 2(\( \varepsilon, \tilde{\varepsilon} \)), under which the gauge fields do not transform, the scalar fields transform with weight 1, and the fermions transform with weight \( \frac{1}{2} \). (In other words, if we make Weyl transformation \( g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}, \) we should scale the fields as \( A_\mu \rightarrow A_\mu, \Phi \rightarrow e^{-\Omega} \Phi, \Psi \rightarrow e^{-\frac{1}{2}\Omega} \Psi \) to keep the action invariant.)

Classically, it is easy to restrict the fields and the symmetries of the \( \mathcal{N} = 4 \) SYM to the pure \( \mathcal{N} = 2 \) SYM: one can discard all fields of the \( \mathcal{N} = 2 \) hypermultiplet and restrict \( \varepsilon \) by the condition \( \Gamma^{5678} \varepsilon = \varepsilon. \) The resulting action is invariant under \( \mathcal{N} = 2 \) superconformal symmetry. On quantum level the pure \( \mathcal{N} = 2 \) SYM is not conformally invariant. We will be able to give a precise definition of the quantum \( \mathcal{N} = 2 \) theory on \( S^4 \), considering it as the \( \mathcal{N} = 4 \) theory softly broken by giving a mass term to the hypermultiplet, which we will send to the infinity in the end.

If we start from Minkowski signature in the ten dimensional theory, then classically the supersymmetry groups for the \( \mathcal{N} = 4 \), the \( \mathcal{N} = 2 \), and the \( \mathcal{N} = 2^* \) Yang-Mills theories on \( S^4 \) are the following.

In the \( \mathcal{N} = 2 \) case, \( \varepsilon \) is a Dirac spinor on \( S^4 \). The equation (2.13) has 16 linearly independent solutions, which correspond to the fermionic generators of the \( \mathcal{N} = 2 \) superconformal algebra. Intuitively, 8 generators out of these 16 correspond to 8 charges of \( \mathcal{N} = 2 \) supersymmetry algebra on \( \mathbb{R}^4 \), and the other 8 correspond to the remaining generators of \( \mathcal{N} = 2 \) superconformal algebra. The full \( \mathcal{N} = 2 \) superconformal group on \( S^4 \) is \( SL(1|2, \mathbb{H}) \). Its bosonic subgroup is \( SL(1, \mathbb{H}) \times SL(2, \mathbb{H}) \times SO(1, 1) \). The first factor \( SL(1, \mathbb{H}) \simeq SU(2) \) generates the \( R \)-symmetry \( SU(2)^L \) transformations. The second factor \( SL(2, \mathbb{H}) \simeq SU^*(4, \mathbb{C}) \simeq Spin(5, 1) \) generates conformal transformations of \( S^4 \). The third factor \( SO(1, 1)^R \) generates the \( SO(1, 1)^R \) symmetry transformations. The fermionic generators of

\footnote{By \( SL(n, \mathbb{H}) \) we mean group of general linear transformation \( GL(n, \mathbb{H}) \) over quaternions factored by \( \mathbb{R}^* \), so that the real dimension of \( SL(n, \mathbb{H}) \) is \( 4n^2 - 1 \).}
SL(1, 2|\mathbb{H}) transform in the $2 + 2'$ of the $SL(2, \mathbb{H})$, where $2$ denotes the fundamental representation of $SL(2, \mathbb{H})$ of quaternionic dimension two. This representation can be identified with the fundamental representation $4$ of $SU^*(4)$ of complex dimension four, or with chiral (Weyl) spinor representation of the conformal group $Spin(5, 1)$. The other representation $2'$ corresponds to the other chiral spinor representation of $Spin(5, 1)$ of the opposite chirality.

In the $\mathcal{N} = 4$ case we do not impose the chirality condition on $\varepsilon$. Hence a sixteen component Majorana-Weyl spinor $\varepsilon$ of $Spin(9, 1)$ reduces to a pair of the four-dimensional Dirac spinors $(\varepsilon_\psi, \varepsilon_\chi)$, where $\varepsilon_\psi$ and $\varepsilon_\chi$ are elements of the $+1$ and $-1$ eigenspaces of the chirality operator $\Gamma_{5678}$ respectively. Each of the Dirac spinors $\varepsilon_\psi$ and $\varepsilon_\chi$ independently satisfies the conformal Killing spinor equation (2.5) because the operators $\Gamma_\mu$ do not mix the $+1$ and $-1$ eigenspaces of $\Gamma_{5678}$. Then we get $16+16 = 32$ linearly independent conformal Killing spinors. Each of these spinors corresponds to a generator of the $\mathcal{N} = 4$ superconformal symmetry. One can check that the full $\mathcal{N} = 4$ superconformal group on $S^4$ is $PSL(2|2, \mathbb{H})$.

To describe the $\mathcal{N} = 2^*$ theory on $S^4$, which is obtained by giving mass to the hypermultiplet, we need some more details on Killing spinors on $S^4$. Because mass terms break conformal invariance, we should expect the $\mathcal{N} = 2^*$ theory to be invariant only under 8 out of 16 fermionic symmetries of the $\mathcal{N} = 2$ superconformal group $SL(1, 2|\mathbb{H})$. In other words, we should impose some additional restrictions on $\varepsilon$. Let us describe this theory in more details.

First we explicitly give a general solution for the conformal spinor Killing equation on $S^4$. Let $x^\mu$ be the stereographic coordinates on $S^4$. The origin corresponds to the North pole, the infinity corresponds to the South pole. If $r$ is the radius of $S^4$, then the metric has the form

$$g_{\mu\nu} = \delta_{\mu\nu}e^{2\Omega}, \quad \text{where} \quad e^{2\Omega} := \frac{1}{(1 + \frac{x^2}{4r^2})^2}. \quad (2.13)$$

We use the vielbein $e^i_\mu = \delta^i_\mu e^\Omega$ where $\delta^i_\mu$ is the Kronecker delta, the index $\mu = 1, \ldots, 4$ is the space-time index, the index $i = 1, \ldots, 4$ enumerates vielbein elements. The solution of the conformal Killing equation (2.5) is (see appendix B)

$$\varepsilon = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_s + x^i \Gamma_i \hat{\varepsilon}_c) \quad (2.14)$$

$$\tilde{\varepsilon} = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_c - x^i \Gamma_i \hat{\varepsilon}_s), \quad (2.15)$$

where $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ are Dirac spinor valued constants.

Classically, the action of $\mathcal{N} = 2$ SYM on $\mathbb{R}^4$ with a massless hypermultiplet is invariant under the $\mathcal{N} = 2$ superconformal group, which has 16 fermionic generators. Turning on non-zero mass of the hypermultiplet breaks 8 superconformal fermionic symmetries, but preserves the other 8 fermionic symmetries which generate the $\mathcal{N} = 2$ supersymmetry. These 8 charges are known to be preserved on quantum level [7]. The $\mathcal{N} = 2$ supersymmetry algebra closes to the scale preserving transformations: the translations on $\mathbb{R}^4$. These scale preserving transformations are symmetries of the massive theory as well.
Following the same logic, we would like to find a subgroup, which will be called \( S \), of the \( \mathcal{N} = 2 \) superconformal group on \( S^4 \) with the following properties. The super-group \( S \subset SL(1|2, \mathbb{H}) \) contains 8 fermionic generators, the bosonic transformations of \( S \) are the scale preserving transformations and are compatible with mass terms for the hypermultiplet. The group \( S \) is what we call the \( \mathcal{N} = 2 \) supersymmetry group on \( S^4 \).

The conformal group of \( S^4 \) is \( SO(5, 1) \). The scale preserving subgroup of the \( SO(5, 1) \) is the \( SO(5) \) isometry group of \( S^4 \). We require that the space-time bosonic part of \( S \) is a subgroup of this \( SO(5) \). This means that for any conformal Killing spinor \( \epsilon \) that generates a fermionic transformation of \( S \), the dilatation parameter \( (\epsilon \epsilon) \) in the \( \delta^2 \) vanishes.

For a general \( \epsilon \) in the \( \mathcal{N} = 2 \) superconformal group, the transformation \( \delta^2 \) contains \( SO(1, 1)^R \) generator. Since the \( SO(1, 1)^R \) symmetry is broken explicitly by hypermultiplet mass terms, and since it is broken on quantum level in the usual \( \mathcal{N} = 2 \) theory in the flat space\(^4\) we require that \( S \) contains no \( SO(1, 1)^R \) transformations. In other words, the conformal Killing spinors \( \epsilon \) which generate transformations of \( S \) are restricted by the condition that the \( SO(1, 1)^R \) generator in \( \delta^2 \) vanishes. By equation (2.14) this means \( \tilde{\epsilon} \Gamma^{09} \epsilon = 0 \).

Using the explicit solution (2.14) we rewrite the equation \((\tilde{\epsilon} \epsilon) = (\tilde{\epsilon} \Gamma^{09} \epsilon) = 0 \) in terms of \( \tilde{\epsilon}_s \) and \( \tilde{\epsilon}_c \)

\[
\tilde{\epsilon}_s \tilde{\epsilon}_c = \tilde{\epsilon}_s \tilde{\epsilon}_c \Gamma^{09} \epsilon_c = 0
\]

\[
\tilde{\epsilon}_c \Gamma^\mu \tilde{\epsilon}_c - \frac{1}{4r^2} \tilde{\epsilon}_c \Gamma^\mu \tilde{\epsilon}_s = 0.
\]

To solve the second equation, we take chiral \( \tilde{\epsilon}_s \) and \( \tilde{\epsilon}_c \) with respect to the four-dimensional chirality operator \( \Gamma^{1234} \). Since the operators \( \Gamma^\mu \) reverse the four-dimensional chirality, both terms in the second equation vanish automatically. There are two interesting cases: (i) the chirality of \( \tilde{\epsilon}_s \) and \( \tilde{\epsilon}_c \) is opposite, (ii) the chirality of \( \tilde{\epsilon}_s \) and \( \tilde{\epsilon}_c \) is the same. The main focus of this work is on the second case.

1. In the first case we can assume that

\[
\epsilon^L_s = 0, \quad \epsilon^R_c = 0.
\]

Here by \( \epsilon^L_s \) and \( \epsilon^R_c \) we denote left/right four-dimensional chiral components. They are respectively defined as the \(-1/1\) eigenspaces of the chirality operator \( \Gamma^{1234} \). In this case the first equation in (2.10) is also automatically satisfied. Moreover, the spinors \( \epsilon \) and \( \tilde{\epsilon} \) also have opposite chirality over the whole \( S^4 \). Hence we have 8 generators, say \( \tilde{\epsilon}_s^R \) and \( \tilde{\epsilon}_c^L \), which anticommute to pure gauge transformations generated by the scalar field \( \Phi := (\tilde{\epsilon} \Gamma^A \epsilon) \Phi_A \). The \( \delta_c \)-closed observables are the gauge invariant functions of \( \Phi \) and their descendants. One could try to interpret such \( \delta_c \) as a cohomological BRST operator \( Q \) and to relate in this way the physical \( \mathcal{N} = 2 \) gauge theory on \( S^4 \) with the topological Donaldson-Witten theory. That does not work, because in the present case the conformal Killing spinor \( \epsilon \), generated by such \( \tilde{\epsilon}_s \) and \( \tilde{\epsilon}_c \) necessary vanishes somewhere on \( S^4 \). Of course, in the twisted theory \( \mathcal{N} \) the problem does not arise, since \( \epsilon \) is a scalar and can be set to be a non-zero constant everywhere. However, our goal is to treat the non-twisted theory.

\(^4\)See e.g. \([1, 5]\) keeping in mind that if we start from the Euclidean signature in ten dimensions, the \( SO(1, 1)^R \) group is replaced by the usual \( U(1)^R \) symmetry of \( \mathcal{N} = 2 \) theory.
Moreover, the circular Wilson loop operator $W_R(C)$ is not closed under such $\delta_\varepsilon$. Thus we turn to the second case.

2. The spinors $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ have the same chirality, say left, and the first equation restricts them to be orthogonal

$$\hat{\varepsilon}_s^R = 0, \quad \hat{\varepsilon}_c^R = 0, \quad (\hat{\varepsilon}_s^L \hat{\varepsilon}_c^L) = 0.$$ 

The Killing vector field $u^\mu = \varepsilon \Gamma^\mu \varepsilon$, associated with the $\delta_\varepsilon^2$, generates an anti-self-dual (left) rotation of $S^4$ around the North pole. In addition, $\delta_\varepsilon^2$ generates a $SU(2)^R_L$-symmetry transformation and a gauge symmetry transformation. The spinor $\varepsilon$ is chiral only at the North and the South poles of $S^4$, but not at any other point. At the North pole $\varepsilon$ is left, at the South pole $\varepsilon$ is right. We can find circular Wilson loop operators of the form (1.1) which are invariant under such $\delta_\varepsilon$.

Conversely, for any given circular Wilson loop $W_R(C)$ of the form (1.1) we can find a suitable conformal Killing spinor $\delta_\varepsilon$ which annihilates $W_R(C)$. (The North pole is picked up at the center of the $W_R(C)$.) If the spinors $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ are both non zero, then $\varepsilon$ is a nowhere vanishing spinor on $S^4$. We can use such $\delta_\varepsilon$ to relate the physical $N = 2$ gauge theory on $S^4$ to a somewhat unusual equivariant topological theory, and apply localization methods developed for topological theories [1, 2] to solve for $\langle W_R(C) \rangle$. The relation has the simplest form if the norm of $\varepsilon$ is constant.

Before proceeding to this equivariant topological theory, we would like to finish our description of the supersymmetry group $S$ of the $N = 2^*$ theory on $S^4$. First we find the maximal set of linearly independent conformal Killing spinors $\{\varepsilon^i\}$ that simultaneously satisfy the equations

$$\varepsilon^{(ij)} = \varepsilon^{(i\Gamma^{09} \varepsilon^{j})} = 0,$$ 

and then we find what superconformal group is generated by this set. One can show that the equivalent way to formulate the conformal Killing spinor equation for the spinors in the $+1$ space of the chirality operator $\Gamma^{0978}$ is the following

$$D_\mu \varepsilon = \frac{1}{2r} \Gamma_\mu \Lambda \varepsilon,$$ 

(2.18)

where $\Lambda$ is a generator of $SU(2)^R_L$-symmetry. For example, if we start from the ten-dimensional Minkowski signature we can take $\Lambda = \Gamma^9 \Gamma_{ij}$ where $5 \leq i < j \leq 8$. If we start from the ten-dimensional Euclidean signature we can take $\Lambda = -i \Gamma^9 \Gamma_{ij}$ where $5 \leq i < j \leq 8$. Equivalently, $\Lambda$ is a real antisymmetric matrix, which acts in the $+1$ eigenspace of $\Gamma^{0978}$, satisfies $\Lambda^2 = -1$ and commutes with $\Gamma^m$ for $m = 1, \ldots, 4, 0, 9$. The equation (2.18) has 8 linearly independent solutions. Let $V_\Lambda$ be the vector space that they span. Then the space of solutions of the conformal Killing spinor equations (2.5) is $V_\Lambda \oplus V_{-\Lambda}$, where we take $\bar{\varepsilon} = \frac{1}{\sqrt{r}} \Lambda \varepsilon$.

The spinors in the space $V_\Lambda$ satisfy our requirement (2.17), because $\Lambda$ is antisymmetric and commutes with $\Gamma^9$. The generators $\{\delta_\varepsilon | \varepsilon \in V_\Lambda\}$ anticommute to generators of $Spin(5) \times SO(2)^R$, where $Spin(5)$ rotates $S^4$, and $SO(2)^R$ is a subgroup of the $SU(2)^R_L$-symmetry group. This $SO(2)^R$ subgroup is generated by $\Lambda$. The space $V_\Lambda$ transforms in the fundamental representation of $Sp(4) \simeq Spin(5)$. We conclude that restricting the fermionic generators to the space $V_\Lambda$ of (2.18) breaks the full $N = 2$ superconformal group $SL(1|2, \mathbb{H})$ to the supergroup $OSp(2|4)$, where the choice of the $SU(2)^R_L$ generator $\Lambda$ defines the embedding of the $SO(2)_R$ in the $SU(2)^R_L$. 
Besides the spaces $V_{\Lambda}$, obtained as solutions of (2.18), we can find other half-dimensional fermionic subspaces of the $\mathcal{N} = 2$ superconformal group satisfying (2.17). These spaces can be obtained by $SO(1,1)_R$ twisting of $V_{\Lambda}$. Indeed, if the spinors $\varepsilon$ and $\bar{\varepsilon}$ satisfy (2.17), then so do the spinors $\varepsilon' = e^{\frac{i}{2}\Gamma^0}\varepsilon$ and $\bar{\varepsilon}' = e^{-\frac{i}{2}\Gamma^0}\bar{\varepsilon}$, where $\Gamma^0$ generates $SO(1,1)_R$, and $\beta$ is a parameter of the twisting. The $SO(1,1)_R$ twisted space $V_{\Lambda,\beta}$ is equivalently a space of solutions to the twisted Killing equation

$$D_\mu \varepsilon = \frac{1}{2r} \Gamma_\mu e^{-\beta \Gamma^0} \Lambda \varepsilon. \quad (2.19)$$

We summarize, that restriction to the half-dimensional fermionic subspace by equation (2.17) breaks the $\mathcal{N} = 2$ superconformal group $SL(1,\mathbb{H})$ down to $OSp(2|4)$. The choice of $OSp(2|4)$ is defined by the generator of $SU(2)_R$ symmetry $\Lambda$, and the generator of $SO(1,1)_R$ symmetry $\beta$.

If we require that the Wilson loop operator is closed with respect to $\delta_\varepsilon$, then the parameter $\beta$ is related to the radius of the Wilson loop. In the ten-dimensional Minkowski conventions, the Wilson loop operator has the form

$$W_R(\rho) = tr_R \exp \oint_C \left( (A_\mu \frac{dx^\mu}{ds} + \Phi_0)ds \right). \quad (2.20)$$

Let the circular contour $C$ be $(x^1,x^2,x^3,x^4) = t(\cos \alpha, \sin \alpha, 0, 0)$ in the stereographic coordinates. Here $t = 2r \tan \frac{\theta_0}{2}$ for the Wilson loop located at the polar angle $\theta_0$. The combination $v^M A_M = v^\mu A_\mu + v^A \phi_A$ is annihilated by $\delta_\varepsilon$, since $(\varepsilon \Gamma^M \varepsilon)(v \Gamma_M \varepsilon)$ vanishes because of the triality identity $[\Lambda,0]$. Then the Wilson loop (2.20) is $\delta_\varepsilon$-closed if $(v^4, v^9, v^0) = (\frac{dx^3}{ds}, 0, 1)$. Using $\Gamma^0 = 1$ and the explicit form (2.16) for $\varepsilon$ we get

$$\hat{\varepsilon}_c = \frac{1}{t} \Gamma_{12} \hat{\varepsilon}_s. \quad (2.21)$$

To satisfy (2.18) we must have

$$\hat{\varepsilon}_c = \frac{1}{2r} e^{-\beta \Gamma^0} \Lambda \hat{\varepsilon}_s. \quad (2.22)$$

Let chirality of $\hat{\varepsilon}_s, \hat{\varepsilon}_c$ be positive at $x = 0$. Then $\beta = \log \frac{1}{2r}$, and $(\Lambda - \Gamma_{12}) \hat{\varepsilon}_s = 0$. This equation has a non-zero solution for $\hat{\varepsilon}_s$ only when $\det(\Lambda - \Gamma_{12}) = 0$. That determines $\Lambda$ uniquely up to a sign. In other words, the choice of the position of the Wilson loop on $S^4$ determines the way the $SU(2)_R$ symmetry group breaks to $SO(2)$, and the size of the Wilson loop determines the $SO(1,1)$ twist parameter $\beta$. For the Wilson loop located at the equator $t = 2r$.

A very nice property of the conformal Killing spinor $\varepsilon$ generating $OSp(2|4)$ is that it has a constant norm over $S^4$, similarly to a supersymmetry transformation on flat space. Since $OSp(2|4)$ has 8 fermionic generators, contains only scale preserving transformations, and it is generated by spinors of constant norm on $S^4$, we call it $\mathcal{N} = 2$ supersymmetry on $S^4$. So we have found that $S = OSp(2|4)$.

Now we show that it is possible to add a mass term for the hypermultiplet fields and preserve the $OSp(2|4)$ symmetry. From now we will assume that the Wilson loop is located at the equator, so that $\varepsilon$ has a constant norm. To generate such mass term in four dimensions we use Scherk-Schwarz reduction of ten-dimensional $\mathcal{N} = 1$ SYM. Namely, we turn on a Wilson line in the $SU(2)_R$ symmetry group along the coordinate $x_0$. The $\mathcal{N} = 2$ vector multiplet fields $A_\mu, \Phi_0, \Phi_9, \Psi$ are not charged under $SU(2)_R$, therefore their kinetic terms are not changed. The hypermultiplet fields $\chi$ and $\Phi_i$ with $i = 5, \ldots, 8$ transform in the spin-$\frac{1}{2}$ representation.
under $SU(2)_R^0$. Explicitly it means that we should replace $D_0\Phi_i$ by $D_0\Phi_i + M_{ij}\Phi_j$, and $D_0\chi$ by $D_0\chi + \frac{1}{4}M_{ij}\Gamma_{ij}\chi$, where an antisymmetric $4 \times 4$ matrix $M_{ij}$ with $i, j = 5, \ldots, 8$ is a generator of the $SU(2)_R^0$ symmetry. Since $F_{0i}$ is replaced by $[\Phi_0, \Phi_i] + M_{ij}\Phi_j$, the $F_0F^{0i}$ term in the action generates mass for the scalars of the hypermultiplet.

On the flat space, the resulting action is still invariant under the usual $\mathcal{N} = 2$ supersymmetry. However, on $S^4$ we need to be more careful with the $\varepsilon$-derivative terms in the supersymmetry transformations. Let us explicitly compute variation of the Scherk-Schwarz deformed $\mathcal{N} = 4$ theory on $S^4$. We use the conformal Killing spinor $\varepsilon$ in the $\mathcal{N} = 2$ superconformal subsector, i.e. $\Gamma^{5\ldots 8}\varepsilon = \varepsilon$. Then $\varepsilon$ is not charged under $SU(2)_R^0$, so $D_0\varepsilon = 0$. Variation of (2.2) by (2.3) gives us (we write variation of the Lagrangian up to total derivative terms since they vanish after integration over the compact space $S^4$)

$$\delta_\varepsilon \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A \right) =$$

$$= 2 D_M (\varepsilon \Gamma_N \Psi) F_{MN} + 2 \Psi \Gamma^M D_M \left( \frac{1}{2} F_{PQ} \Gamma^{PQ} \varepsilon - 2 \Phi_A \tilde{\Gamma}^A \varepsilon \right) + \frac{4}{r^2} (\varepsilon \Gamma^A \psi) \Phi_A =$$

$$= -2 (\varepsilon \Gamma_N \Psi) D_M F^{MN} + \Psi D_M F_{PQ} \Gamma^M \Gamma^{PQ} \varepsilon + \Psi \Gamma^M \Gamma^{PQ} F_{PQ} D_M \varepsilon - 4 \Psi \Gamma^M \tilde{\Gamma}^A \varepsilon D_M \Phi_A +$$

$$+ \frac{1}{r^2} \Psi \Gamma^\mu \tilde{\Gamma}^A \Phi_A \Gamma_{\mu} \varepsilon + \frac{4}{r^2} (\varepsilon \Gamma^A \psi) \Phi_A = \ldots$$

Using

$$\Gamma^M \Gamma^{PQ} = \frac{1}{3} (\Gamma^M \Gamma^{PQ} + \Gamma^P \Gamma^{QM} + \Gamma^M \Gamma^{PQ}) + 2g^{M[P} \Gamma^{Q]}$$

(2.23)

and the Bianchi identity, we see that the first term cancels the second, and that the last two terms cancel each other. Then

$$\cdots = \Psi \Gamma^\mu \Gamma^{PQ} \Gamma_\mu \varepsilon F_{PQ} - 4 \Psi \Gamma^M \tilde{\Gamma}^A \varepsilon D_M \Phi_A = 4 \Psi \tilde{\Gamma}^M \varepsilon F_{MA} - 4 \Psi \Gamma^M \tilde{\Gamma}^A \varepsilon D_M \Phi_A$$

where we use the index conventions $M, N, P, Q = 0, \ldots, 9$, $\mu = 1, \ldots, 4$, $A = 5, \ldots, 9, 0$. In the absence of Scherk-Schwarz deformation we have $F_{MA} = D_M \Phi_A$ for all $M = 0, \ldots, 9$ and $A = 5, \ldots, 9, 0$, hence the two terms cancel. After the deformation, we have $F_{0i} = D_0 \Phi_i$, but $F_{00} = -D_0 \Phi_0 = -[\Phi_0, \Phi_1] - M_{ij} \Phi_j = D_0 \Phi_0 - M_{ij} \Phi_j$. Therefore, the naively Scherk-Schwarz deformed $\mathcal{N} = 4$ theory on $S^4$ is not invariant under arbitrary $\mathcal{N} = 2$ superconformal transformation:

$$\delta_\varepsilon \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A \right) = -4 \Psi \Gamma^i \tilde{\Gamma}^0 \varepsilon M_{ij} \Phi_j. \quad (2.24)$$

This is the natural consequence of adding mass terms to the Lagrangian. Nevertheless, we can add some other terms to the action in such a way to make the action invariant under the $OSp(2|4)$ subgroup of $\mathcal{N} = 2$ superconformal group on $S^4$. We use the fact that $\varepsilon$ generating a transformation in the $OSp(2|4)$ subgroup satisfies the conformal Killing equation with $\bar{\varepsilon} = \frac{1}{2} \Lambda \varepsilon$, where $\Lambda$ is a generator of $SU(2)_L^0$ group normalized as $\Lambda^2 = -1$. Let us take $\Lambda = \frac{1}{2} \Gamma_{kl} R_{kl}$ where $R_{kl}$ is an anti-self-dual matrix normalized as $R_{kl} R^{kl} = 4$, where $k, l = 5, \ldots, 8$. Then we get

$$\delta_\varepsilon \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi_A \Phi^A \right) = \frac{1}{2r} \Psi \Gamma^i \Gamma^{kl} \varepsilon R_{kl} M_{ij} \Phi_j =$$

$$= \frac{1}{2r} (\Psi \Gamma^i \varepsilon) R_{kl} M_{kj} \Phi_j = \frac{1}{2r} (\delta_\varepsilon \Phi^i)(R_{kl} M_{kj}) \Phi_j. \quad (2.25)$$
Hence, the addition of $\frac{1}{4r}(R_{ki}M_{kj})\Phi^i\Phi^j$ term to the Scherk-Schwarz deformed action on $S^4$ makes the action invariant under the $OSp(2|4)$.

Let us summarize. The action

$$S_{N=2} = \frac{1}{2g^2} \int d^4x \sqrt{g} \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{r^2} \Phi^A \Phi_A - \frac{1}{4r}(R_{ki}M_{kj})\Phi^i\Phi^j \right),$$

(2.26)

where $D_0\Phi^i = [\Phi_0, \cdot] + M_{ij}\Phi^j$ and $D_0\Psi = [\Phi_0, \Psi] + \frac{1}{4}\Gamma^{ij}M_{ij}\Psi$, is invariant under the $OSp(2|4)$ transformations, generated by conformal Killing spinors solving $D_\mu \varepsilon = \frac{1}{8r} \Gamma_\mu \Gamma^{kl} R_{kl} \varepsilon$ with $\varepsilon$ restricted to $N = 2$ subspace $\Gamma^{5678} \varepsilon = \varepsilon$.

Since $\delta^2 \varepsilon$ generates a covariant Lie derivative along the vector field $-v^M = -\varepsilon \Gamma^M \varepsilon$, in particular it is contributed by the gauge transformation along the 0-th direction. After we turned on mass for the hypermultiplet by Scherk-Schwarz mechanism, $\delta^2 \varepsilon$ gets new contributions on the hypermultiplet

$$\delta^2 \varepsilon \Phi^i = \delta^2 \varepsilon_{M=0} \Phi^i - v^0 M_{ij} \Phi^j,$$

$$\delta^2 \varepsilon \chi = \delta^2 \varepsilon_{M=0} \chi - \frac{1}{4} v^0 M_{ij} \Gamma^{ij} \chi.$$

(2.27)

So far we computed $\delta^2 \varepsilon$ on-shell. To use the localization method we need an off-shell closed formulation of the fermionic symmetry of the theory. The pure $N = 2$ SYM can be easily closed by means of three auxiliary scalar fields, but it is well known that the off-shell closure of $N = 2$ hypermultiplet is impossible with a finite number of auxiliary fields. For our purposes we do not need to close off-shell the whole $OSp(2|4)$ symmetry group. Since the localization computation uses only one fermionic generator $Q_\varepsilon$, it is enough to close off-shell only the symmetry generated by this $\varepsilon$.

To close off-shell the relevant supersymmetry of the $N = 4$ theory on $S^4$ we make the dimensional reduction of Berkovits method [59] used for the ten-dimensional $N = 1$ SYM, see also [60, 61]. The number of auxiliary fields compensates the difference between the number of fermionic and bosonic off-shell degrees of freedom modulo gauge transformations. In the $N = 4$ case we add $16 - (10 - 1) = 7$ auxiliary fields $K_i$ with free quadratic action and modify the superconformal transformations to

$$\delta \varepsilon A_M = \Psi \Gamma_M \varepsilon,$$

$$\delta \varepsilon \Psi = \frac{1}{2} \gamma^{MN} F_{MN} + \frac{1}{2} \gamma^M A D_M \varepsilon + K^i \nu_i,$$

$$\delta \varepsilon K_i = -\nu_i \gamma^M D_M \Psi,$$

(2.28)

where spinors $\nu_i$ with $i = 1, \ldots, 7$ are required to satisfy

$$\varepsilon \Gamma^M \nu_i = 0,$$

$$\frac{1}{2} (\varepsilon \Gamma_N \varepsilon) \Gamma^N_{\alpha\beta} = \nu^i_\alpha \nu^i_\beta + \varepsilon_\alpha \varepsilon_\beta,$$

$$\nu_i \Gamma^M \nu_j = \delta_{ij} \varepsilon \Gamma^M \varepsilon.$$

(2.29)

For any non-zero Majorana-Weyl spinor $\varepsilon$ of $Spin(9,1)$ there exist seven linearly independent spinors $\nu_i$, which satisfy these equations [59]. They are determined up to an $SO(7)$ transformations. The equation (2.29) ensures closure on $A_M$, the

---

5The author thanks N.Berkovits for communications.
equation (2.31) ensures closure on $\Psi$, and the equations (2.29) and (2.31) ensure closure on $K$

$$\delta_s^2 K_i = - (\varepsilon^M \varepsilon) D_M K^i - (\nu_i \gamma^M D_M \nu_j) K^j - 4(\varepsilon \varepsilon) K_i. \quad (2.32)$$

If $E_K$ is an $SO(7) \otimes \text{ad}(G)$ vector bundle over $S^4$ whose sections correspond to the auxiliary fields $K_i$, then (2.32) can be interpreted as a covariant Lie derivative action along the vector field $v^\mu$, or in other words as a lift of the $L_v$ action on $S^4$ to the action on the vector bundle $E_K \to S^4$. A conformal Killing spinor $\varepsilon$ generating a transformation of the $OSp(2|4)$ subgroup can be represented in the following form (see appendix B for details)

$$\varepsilon(x) = \exp \left( \frac{\theta}{2} n_1(x) \Gamma^9 \right) \bar{\varepsilon}_s, \quad (2.33)$$

where $x^i$ are the stereographic coordinates on $S^4$, $n_1$ is the unit vector in the direction of the vector field $v_i = \frac{1}{2} x^2 \omega_{ij}$. We use the conformal Killing spinor $\varepsilon(x)$ such that $(\varepsilon(x), \varepsilon(x)) = 1$ and $\Gamma^9 \varepsilon_s = \bar{\varepsilon}_s$. The matrix $\omega_{ij}$ is the anti-self-dual generator of $SU(2)_L \subset SO(4)$ rotation around the North pole in $\delta_s^2$. We see that the conformal Killing spinor $\varepsilon(x)$ at an arbitrary point $x$ is obtained by $Spin(5)$ rotation $\exp(\frac{\theta}{2} n_1(x) \Gamma^9)$ of its value at the origin $\varepsilon(0) = \bar{\varepsilon}_s$.

For the closure of $N = 4$ symmetry we need seven spinors $\nu_i$ which satisfy (2.29)-(2.31). Following [59], at the origin we can take $\hat{\nu}_i = \Gamma^{is} \bar{\varepsilon}_s$ for $i = 1 \ldots 7$, and then transform $\hat{\nu}_i$ to an arbitrary point on $S^4$ as

$$\nu_i(x) = \exp \left( \frac{\theta}{2} n_1(x) \Gamma^{is} \right) \bar{\varepsilon}_s. \quad (2.34)$$

Finally, we conclude that the action

$$S_{N=2^*} = \frac{1}{2 g_{YM}^2} \int d^4 x \sqrt{g} \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{2}{x^2} \Phi A \Phi^A - \frac{1}{4r} (R_{ki} M_{kj}) \Phi^i \Phi^j - K_i K_i \right), \quad (2.35)$$

is invariant under the off-shell supersymmetry $Q_\varepsilon$ given by (2.29) with $\nu_i$ defined by (2.31). Though we will not need this fact, we remark that it is possible to simultaneously close four fermionic symmetries generating the $OSp(2|2)$ subgroup of $OSp(2|4)$. The space-time part of this $OSp(2|2)$ subgroup consists of anti-self-dual rotations around the North pole on $S^4$.

3. Localization

As explained in the introduction, to localize the theory we deform the action by a $Q$-exact term

$$S \to S + t Q V. \quad (3.1)$$

Since we use $Q$ which squares to a symmetry of the theory, and since the action and the Wilson loop observable are $Q$-closed, we can use the localization argument. For $Q^2$-invariant $V$, the deformation (3.1) does not change the expectation value of $Q$-closed observables. Hence, when we send $t$ to infinity, the theory localizes to some set $F$ of critical points of $QV$, over which we will integrate in the end. The measure in the integral over $F$ comes from the restriction of the action $S$ to $F$ and the determinant of the kinetic term of $QV$ which counts fluctuations in the normal directions to $F$. 
To ensure convergence of the four-dimensional path integral, we compute it for a theory obtained by dimensional reduction from a theory in ten-dimensional Euclidean signature. To technically simplify the description of the symmetries in the previous section, we used ten-dimensional Minkowski signature. We can keep Minkowski metric $g_{MN}$ and Minkowski gamma-matrices $\Gamma_M$ and still get the same partition function as in Euclidean signature by making Wick rotation of the $\Phi_0$ field. In other words, the path integral, computed with Minkowski metric $g_{MN}$ but with $\Phi_0$ substituted by $i\Phi_0$ where $\Phi_0$ is real, is convergent and is equal to the Euclidean path integral. We also integrate over imaginary contour for the auxiliary fields $K_i$, so that $K_i = iK_i^E$, where $K_i^E$ is real.

For localization computation we will take the following functional
\[ V = (\Psi, \overline{Q}\Psi). \] (3.2)

Then the bosonic part of the $QV$-term is a positive definite functional
\[ S^Q|_{\text{bos}} = (Q\Psi, \overline{Q}\Psi). \] (3.3)

Explicitly we have
\[
\begin{align*}
Q\Psi &= \frac{1}{2} F_{MN}\Gamma^{MN}\varepsilon + \frac{1}{2} \Phi_A \Gamma^{\mu A} \nabla_\mu \varepsilon + K^i \nu_i \\
\overline{Q}\Psi &= \frac{1}{2} F_{MN} \tilde{\Gamma}^{MN} \varepsilon + \frac{1}{2} \Phi_A \tilde{\Gamma}^{\mu A} \nabla_\mu \varepsilon - K^i \nu_i,
\end{align*}
\] (3.4)
where $\tilde{\Gamma}^0 = -\Gamma^0, \tilde{\Gamma}^M = \Gamma^M$ for $M = 1, \ldots, 9$, and $\Gamma^{MN} = \tilde{\Gamma}^{[M} \Gamma^{N]}, \tilde{\Gamma}^{MN} = \Gamma^{[M} \tilde{\Gamma}^{N]}$.

Before proceeding to technical details of the computation, let us explicitly define the conformal Killing spinor $\varepsilon$ which we will use, and find the vector field $v^M = \varepsilon \Gamma^M \varepsilon$ generated by the corresponding $\delta^2_\varepsilon$. We take $\varepsilon$ in the form (2.14), where $\hat{\varepsilon}_s$ is any spinor such that

1. The chirality operator $\Gamma^{5678}$ acts on $\hat{\varepsilon}_s$ by 1
2. The chirality operator $\Gamma^{1234}$ acts on $\hat{\varepsilon}_s$ it by $-1$
3. $\hat{\varepsilon}_s \hat{\varepsilon}_s = 1$

The first condition means that $\varepsilon$ generates transformation inside the $\mathcal{N} = 2$ superconformal subgroup of $\mathcal{N} = 4$ superconformal group. The second condition ensures that $\varepsilon$ is a four-dimensional left chiral spinor on the North pole of $S^4$. The third condition is a conventional normalization. In our conventions for the gamma-matrices (appendix A) we can take $\hat{\varepsilon}_s = (1, 0, \ldots, 0)^t$. Let the Wilson loop be located at the equator and invariant under anti-self-dual rotations in the $SO(4)$ group of rotations around the North pole. To be concrete, let the Wilson loop be placed in the $(x_1, x_2)$ plane. Then we take $\hat{\varepsilon}_c = \frac{1}{2r} \Gamma^{12} \hat{\varepsilon}_s$. The conformal Killing spinor $\varepsilon$ defined by such $\hat{\varepsilon}_s$ and $\hat{\varepsilon}_c$ has a constant unit norm over the whole four-sphere $((\varepsilon\varepsilon) = 1)$. At the North pole the spinor $\varepsilon$ is purely left, at the South pole the spinor $\varepsilon$ is purely right.

Now we compute the components of the vector field $v^M = \varepsilon \Gamma^M \varepsilon$. If we assume ten-dimensional Minkowski signature, then we get
\[
\begin{align*}
v_t &= \sin \theta \\
v^0 &= 1 \\
v^9 &= \cos \theta \\
v^i &= 0 \quad \text{for} \quad i = 5, \ldots, 8,
\end{align*}
\] (3.5)
where $\theta$ is the polar angle on $S^4$ such that the Wilson loop is placed at $\theta = \frac{\pi}{2}$, the North pole is at $\theta = 0$, and the South pole is at $\theta = \pi$. The four-dimensional space-time component $v_i$ of $v^M$ has length $\sin \theta$ and is directed along the orbits of the $U(1) \subset SU(2)_L \subset SO(4)$ group which rotates the $(x_1, x_2)$ plane. If we switch to the ten-dimensional Euclidean signature, then $v^0 = i$ while the other components are the same as in Minkowski signature.

To simplify $S^Q\vert_{bos}$ we use the Bianchi identity for $F_{MN}$, the gamma-matrices algebra and integration by parts. The principal contribution to $S^Q\vert_{bos}$ is the curvature term

$$S_{FF} = \frac{1}{4} \epsilon \tilde{\Gamma}^M \Gamma^\rho \Gamma^Q \epsilon F^{MN} F^{PQ} \tag{3.6}$$

The $F_{MN} K_i$ cross-terms vanish because $\nu_i \Gamma^0 \Gamma^M \epsilon = \nu_i \Gamma^M \epsilon = 0$. Then we have a simple contribution from auxiliary $K K$-term

$$S_{KK} = -K_i K^i. \tag{3.7}$$

In the flat space limit, $r \to \infty$ the spinor $\epsilon$ is covariantly constant $\nabla_\mu \epsilon = 0$. Therefore, in the flat space we simply have $S^Q\vert_{bos} = S_{FF} + S_{KK}$. Up to the total derivatives and $\nabla_\mu \epsilon$-terms, using the Bianchi identity and the gamma-matrices algebra, we can see that $S_{FF}$ is equivalent to the usual Yang-Mills action $\frac{1}{4} F^{MN} F_{MN}$. When the space is curved and $\nabla_\mu \epsilon \neq 0$ we shall make more careful computation. Using (2.23) we get

$$S_{FF} = \frac{1}{2} F^{MN} F_{MN} + \frac{1}{4} \epsilon \tilde{\Gamma}^M \Gamma^\rho \Gamma^Q \epsilon \frac{1}{3} (F_{MN} F_{PQ} + F_{PN} F_{QM} + F_{QN} F_{MP}). \tag{3.8}$$

To simplify the last term, first we break the indices into two groups: $M, N, P, Q = (1, \ldots, 4, 9, 0)$ and $M, N, P, Q = (5, \ldots, 8)$ describing respectively the fields of the vector and hyper multiplet. Using $\Gamma^{0678} \epsilon = \epsilon$ we can see that the nonvanishing terms have only zero, two or four of indices in the hypermultiplet range $(5, \ldots, 8)$. We call the resulting terms as vector-vector, vector-hyper and hyper-hyper respectively. First we consider vector-vector terms. For vector-vector terms we split indices to the gauge field part $(1, \ldots, 4)$ and to the scalar part $(0, 9)$. The nonvanishing gauge field terms all have different values of $M, N, P, Q$. Then their contribution is simplified to

$$\frac{1}{4} \frac{1}{3} 24 \epsilon \Gamma^{1234} \epsilon (F^{21} F^{34} + F^{31} F^{42} + F^{41} F^{23}) = -\frac{1}{2} \epsilon \Gamma^{1234} \epsilon (F, * F) = -\frac{1}{2} \cos \theta (F, * F), \tag{3.9}$$

where $* F$ is the Hodge dual of $F$. All terms in which one of the indices is 0 vanish because $\Gamma^{MPQ}$ is antisymmetric matrix, hence $\epsilon \Gamma^{0} \Gamma^{MPQ} \epsilon = 0$. Then the remaining vector-vector terms have $D_\mu \Phi_9 F$ structure. Integrating by parts and using Bianchi identity we get

$$-\frac{1}{3} D_\mu (\epsilon \Gamma^0 \Gamma^{\mu \nu \rho} \epsilon) \Phi_9 F_{\nu \rho} + \text{cyclic}(\mu \nu \rho) = 4 (\epsilon \Gamma^0 \Gamma^{\mu \nu} \epsilon) \Phi_9 F_{\mu \nu}. \tag{3.10}$$

Doing similar algebra we get the contribution to the vector-hyper mixing terms in $S_{FF}$

$$-8 \epsilon \Gamma^{ij} \epsilon \Phi_i [\Phi_9, \Phi_j] - 6 \epsilon \Gamma^{ij} \epsilon \Phi_i D_\mu \Phi_j \tag{3.11}$$
We consider separately the cases when the index $i$ is in the set $\{0, 9\}$ and in the set $\{5, \ldots, 8\}$. The terms with index $A = 0$ all vanish because $\Gamma^0 = -\Gamma^0$ and because $\bar{\Gamma}^M \varepsilon = 0$ for our choice of $\varepsilon$ in $OSp(2|4)$. Next we take index $A = 9$. The only nonvanishing terms are

$$-2\bar{\varepsilon} \Gamma^\mu \varepsilon \Phi_9 F_{\mu \nu} - 2\bar{\varepsilon} \Gamma^9 \varepsilon \Phi_9 \Phi_9,$$

where $\mu, \nu = 1, \ldots, 4$ and $i, j = 5, \ldots, 8$. Finally, we consider the case when the index $A$ is in the hypermultiplet range $5, \ldots, 8$. The result is

$$4\bar{\varepsilon} \Gamma^\mu \Gamma^{ij} \varepsilon \Phi_i D_\mu \Phi_j + 6\bar{\varepsilon} \Gamma^9 \varepsilon \Phi_i \Phi_j.$$

Then

$$S_{F \Phi} = -2\bar{\varepsilon} \Gamma^9 \varepsilon \Phi_9 F_{\mu \nu} + 4\bar{\varepsilon} \Gamma^\mu \Gamma^{ij} \varepsilon \Phi_i D_\mu \Phi_j + 6\bar{\varepsilon} \Gamma^9 \varepsilon \Phi_i \Phi_j.$$

The $\Phi \Phi$ term is easy

$$S_{\Phi \Phi} = 4\Phi^A \Phi^B \bar{\varepsilon} \Gamma^A \bar{\varepsilon} = 4\bar{\varepsilon} \Phi^A \Phi_A.$$

Finally, we need the $\Phi K$ cross-term. Only $\Phi_0$ contributes

$$S_{\Phi K} = 2K_i \Phi_0 \nu_i \Gamma^0 \bar{\varepsilon} - 2K_i \Phi_0 \nu_i \Gamma^0 \bar{\varepsilon} = -4K_i \Phi_0 \nu_i \bar{\varepsilon}.$$

The total result is

$$S^Q_{\text{bos}} = S_{FF} + S_{F \Phi} + S_{\Phi \Phi} + S_{\Phi K} + S_{KK} =$$

$$\frac{1}{2} F_{MN} F^{MN} - \frac{1}{2} \cos \theta F_{\mu \nu} (\ast F)^{\mu \nu} + 2\bar{\varepsilon} \Gamma^\mu \varepsilon \Phi_9 F_{\mu \nu} + 4\bar{\varepsilon} \varepsilon \Phi_9 \Phi^9 =$$

$$\cos^2 \frac{\theta}{2} (F^-_{\mu \nu} + w^-_{\mu \nu} \Phi_9)^2 + \sin^2 \frac{\theta}{2} (F^+_{\mu \nu} + w^+_{\mu \nu} \Phi_9)^2. \quad (3.14)$$

where

$$w^-_{\mu \nu} = \frac{1}{\cos^2 \frac{\theta}{2}} \bar{\varepsilon}^L \Gamma^\mu \varepsilon^L,$$

$$w^+_{\mu \nu} = \frac{1}{\sin^2 \frac{\theta}{2}} \bar{\varepsilon}^R \Gamma^\mu \varepsilon^R. \quad (3.15)$$

Next we make a full square with the terms

$$D_m \Phi_i D^n \Phi^i - 2\bar{\varepsilon} \Gamma^{ij} \varepsilon \Phi_i \Phi_j + 2\bar{\varepsilon} \Gamma^9 \varepsilon \Phi_i D_\mu \Phi_j = (D_m \Phi_j - \bar{\varepsilon} \Gamma_m \Gamma_i \varepsilon \Phi^i)^2 - \Phi^i (\tilde{\varepsilon} \varepsilon) \varepsilon \varepsilon).$$
Finally we absorb the mixing term $K_i \Phi_0$ as follows

$$-4(\bar{\epsilon} \tilde{\epsilon}) \Phi_0 \Phi_0 - 4 \Phi_0 K_i (\nu^i \tilde{\epsilon}) - K_i K^i = -(K_i + 2 \Phi_0 (\nu_i \tilde{\epsilon}))^2.$$ 

We use the following relations throughout the computation

$$(\varepsilon \varepsilon) = 1, \quad (\varepsilon L \varepsilon L) = \cos^2 \frac{\theta}{2}, \quad (\varepsilon H \varepsilon H) = \sin^2 \frac{\theta}{2}, \quad (\varepsilon \tilde{\epsilon}) = \frac{1}{4 \pi^2}, \quad w_{\mu \nu} w^{- \mu \nu} = w_{\mu \nu} w^{\mu \nu} = \frac{1}{r^2}.$$ 

The final result is

$$S^Q_{\text{bos}} = S^Q_{\text{vect,bos}} + S^Q_{\text{hyper,bos}}.$$ 

Here

$$S^Q_{\text{vect,bos}} = \cos^2 \frac{\theta}{2} (F_{\mu \nu} + w_{\mu \nu} \Phi_0)^2 + \sin^2 \frac{\theta}{2} (F_{\mu \nu} - w_{\mu \nu} \Phi_0)^2 + (D_\mu \Phi_0)^2 + \frac{1}{2} [\Phi_0, \Phi_0] [\Phi^a, \Phi^b] + (K_i + w_i \Phi_0^E)^2$$

(3.16)

where the indices $a, b = 0, 9$ run over the scalars of the vector multiplet, the index $i = 5, 6, 7$ runs over the three auxiliary fields for the vector multiplet, and $w_i = 2(\nu_i \tilde{\epsilon})$ has norm $w_i w^i = \frac{1}{4 \pi}$. 

At this moment we also switched to the fields $\Phi_0^E, K_i^E$ which are related to the original fields in Minkowski signature as $\Phi_0 = i \Phi_0^E, K_i = i K_i^E$. Equivalently, we could make the computation in the Euclidean signature from the very beginning keeping all fields real. In this case some imaginary coefficients would appear in the supersymmetry transformations: we would write down $i$ in front of the fields $K_i$ and would replace the $\Gamma^0$ matrix by $i \Gamma^0$. 

One could worry then that such supersymmetry transformations spoil the reality conditions on the fields. However, our localization computation is not affected. The Lagrangian and the theory is still invariant under such transformations if we understand the action as an analytically continued functional to the space of complexified fields. The path integral is understood as an integral of a holomorphic functional of fields over a certain real half-dimensional “contour of integration” in the complexified space of fields. Strictly speaking, the bar in the formula for $Q \bar{\Psi}$ literally means complex conjugation only if we assume that we use that contour of integration which we described before: all fields are real except $\Phi_0$ and $K_i$ which are imaginary. For a general contour of integration in the path integral we just use the functional $V$ where $Q \bar{\Psi}$ is defined by the second line of (3.14). This means that the functional $V$ holomorphically depends on all complexified fields. The bosonic part of $QV$ is positive definite after restriction to the correct contour of integration.

From any point of view, we should stress that $\delta \varepsilon$ squares to a complexified gauge transformation, whose scalar generator is $i \Phi_0^E + \cos \theta \Phi_9 + \sin \theta A^v$, where $\Phi_0^E$, $\Phi_9$ and $A^v$ take value in the real Lie algebra of the gauge group, and where $A^v$ is the component of the gauge field in the direction of the vector field $v^u$. The theory is similar to the Donaldson theory near the North pole where this generator becomes $i \Phi_0^E + \Phi_9$, and anti-Donaldson theory near the South pole where this generator becomes $i \Phi_0^E - \Phi_0$.

The hypermultiplet contribution is

$$S^Q_{\text{hyper,bos}} = (D_0 \Phi_i)^2 + (D_m \Phi_j - f_{mi j} \Phi_1)^2 + \frac{1}{2} [\Phi_i, \Phi_j][\Phi^i, \Phi^j] + \frac{3}{4 \pi^2} \Phi^i \Phi_i + K_i^E K_i^E,$$

where $m = 1, \ldots, 5$, $i = 5, \ldots, 8$, $I = 1, \ldots, 4$ and $f_{mi j} = \varepsilon \Gamma_m \Gamma_{ij} \varepsilon$. We see that with our choice of the “integration contour” in the space of complexified fields (all fields are real except $K_i, K_I$ and $\Phi_0$ which are pure imaginary), all terms in the
action $S^Q_{bos}$ are semi-positive definite. Therefore, in the limit $t \to \infty$ we need to care in the path integral only about the locus at which all squares vanish and small fluctuations in the normal directions.

For the hypermultiplet action we get a simple “vanishing theorem”: because of the quadratic term $\frac{3}{8\pi} \Phi^i \Phi_i$, the functional $S^Q_{hyper,bos}$ vanishes if all fields $\Phi_i$ vanish.

Next consider zeroes of the term $S^Q_{vect,bos}$. The term $(D_\mu \Phi_9)^2$ ensures that the field $\Phi_9$ must be covariantly constant. Away from the North and the South poles and requiring that the curvature terms vanish, we get the equations

$$F_{\mu\nu} = -w_{\mu\nu} \Phi_9$$

where $w_{\mu\nu} = w^+_{\mu\nu} + w^-_{\mu\nu}$. The curvature $F_{\mu\nu}$ satisfies Bianchi identity, hence we must have

$$d[\lambda w_{\mu\nu}] \Phi_9 = 0.$$ 

(3.17)

It is easy to check that away from the North and the South poles, $d[\lambda w_{\mu\nu}]$ does not vanish, hence $\Phi_9$ and $F_{\mu\nu}$ must vanish. The kinetic term $(D_\mu \Phi^E_9)^2$ ensures that $\Phi^E_9$ is covariantly constant. Since $F_{\mu\nu} = 0$ we can assume that the gauge field vanish, then $\Phi^E_9$ is a constant field over $S^4$. We call this constant $a_E$ and conclude, that up to a gauge transformation, at smooth configurations we must have

$$S^Q_{bos} = 0 \Rightarrow \begin{cases} A_\mu = 0 & \mu = 1, \ldots 4 \\ \Phi_i = 0 & i = 5, \ldots 9 \\ \Phi^E_9 = a_E \quad \text{constant over } S^4 \\ K_i = -w_i a_E \\ K_I = 0 \end{cases}$$

(3.18)

This is the key step in the localization procedure and in the proof of the Erickson-Semenoff-Zarembo/Drukker-Gross conjecture about circular Wilson loop operators. The infinite-dimensional path integral localizes to the finite dimensional locus (3.18), and the integral over $a_E \in g$ is the resulting matrix model.

Let us evaluate the $S_{YM}$ action (2.35) at (3.18). The nonvanishing terms are only

$$S_{YM}[a] = \frac{1}{2g_{YM}^2} \int d^4x \sqrt{g} \left( \frac{2}{r^2} (\Phi^E_9)^2 + (K^E_i)^2 \right) = \frac{1}{2g_{YM}^2} \text{vol}(S^4) \frac{3}{2} r^2 a_E^2 = \frac{4\pi^2 r^2}{g_{YM}^2} a_E^2$$

(3.19)

where we used $w_i w^i = \frac{1}{27}$ and the volume of the four-sphere $\frac{8\pi^2 r^4}{3}$. We obtained precisely the Drukker-Gross matrix model.

Let us check that the coefficient is correct. Recall, that the original action has the following propagators in Feynman gauge on $\mathbb{R}^4$

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{g^2_{YM}}{4\pi^2} \frac{g_{\mu\nu}}{(x-x')^2}$$

$$\langle \Phi^E_9(x) \Phi^E_9(x') \rangle = \frac{g^2_{YM}}{4\pi^2} \frac{1}{(x-x')^2}.$$

Hence, the correlator functions which appear in the perturbative expansion of the Wilson loop operator, have the structure

$$\langle A_\mu(\alpha) \dot{x}^\mu A_\nu(\alpha') \dot{x}^\nu + i \Phi^E_9(\alpha) i \Phi^E_9(\alpha') \rangle = -\frac{g^2_{YM}}{4\pi^2 r^2} \frac{\cos(\alpha - \alpha') - 1}{4 \sin^2 \left( \frac{\alpha - \alpha'}{2} \right)} = -\frac{g^2_{YM}}{8\pi^2 r^2},$$
where $\alpha$ denotes an angular coordinate on the loop. That was the original motivation for Erickson-Semenoff-Zarembo conjecture \[9\]. We see that the first order perturbation theory agrees with the matrix model action derived by localization. The power of the localization computation is that it actually proves the relation between the field theory and the matrix model in all orders in perturbation theory. It is also capable of taking into account instanton effects, which we describe shortly after computing the fluctuation determinant near the locus \[3.18\] and confirming the exact solution.

We remark that for the $\mathcal{N} = 2^*$ theory, the same argument about zeroes of $S_Q|_{\text{bos}}$ holds. To ensure that all terms are positive definite, we take the mass parameter $M_{ij}$ in the Scherk-Schwarz reduction to be pure imaginary antisymmetric self-dual matrix. Then the action of the mass deformed $\mathcal{N} = 2^*$ theory at configurations \[3.18\] reduces to the same matrix model action. However, as we will see shortly, when the mass parameter $M_{ij}$ is non zero, the matrix model measure for the $\mathcal{N} = 2^*$ theory is corrected by a non-trivial determinant.

4. **Determinant factor**

4.1. **Gauge-fixing complex.** Because of the infinite-dimensional gauge symmetry of the action we need to work with the gauge-fixed theory. We use the Faddeev-Popov ghost fields and introduce the following BRST like complex with the differential $\delta$:

\[
\delta X = -[c, X] \quad \delta c = -a_0 - \frac{1}{2}[c, c] \quad \delta \bar{c} = b \quad \delta \bar{c}_0 = \bar{c}_0 \quad \delta b_0 = c_0
\]

\[
\delta a_0 = 0 \quad \delta b = [a_0, \bar{c}] \quad \delta \bar{c}_0 = [a_0, \bar{a}_0] \quad \delta c_0 = [a_0, b_0].
\]

(4.1)

Here $X$ stands for all physical and auxiliary fields entering \[2.35\]. All other fields are the gauge-fixing fields. By $[c, X]$ we denote a gauge transformation with a parameter $c$ of any field $X$. (For the gauge fields $A_\mu$ we have $\delta A_\mu = -[c, \nabla_\mu]$. The gauge transformation of $\Phi = v^A A_A$ is $\delta \Phi = [v^A D_\mu + v^A \Phi A, c] = \text{ad}(\Phi) c + L_c c$, where $\text{ad}(\Phi) c$ is the pointwise adjoint action of $\Phi$ on $c$ involving no differential operators). The fields $c$ and $\bar{c}$ are the usual Faddeev-Popov ghost and anti-ghost. The bosonic field $b$ is the standard Lagrange multiplier used in $R_\xi$-gauge, where the gauge fixing is done by adding terms like $(b, id^a A + \frac{1}{2} b)$ and $(\bar{c}, d^a \nabla_A c)$ to the action. The fields $c$ and $\bar{c}$ actually have zero modes. To treat them systematically we add constant fields $c_0, \bar{c}_0, a_0, \bar{a}_0, b_0$ to the gauge-fixing complex. The field $a_0$ is interpreted as a ghost field for the ghost $c$. The fields $a_0, \bar{a}_0, b_0$ are bosonic, and the fields $c_0, \bar{c}_0$ are fermionic. The operator $\delta$ squares to the gauge transformation by the constant bosonic field $a_0$

\[
\delta^2 = [a_0, \cdot].
\]

The gauge invariant action and observable are $\delta$-closed

\[
\delta S_{YM}[X] = 0,
\]

therefore their correlation functions are not changed when we add the $\delta$-exact gauge-fixing term.

When we combine the gauge-fixing terms with the physical action, we will see that the convergence of the path integral requires the imaginary contour of integration for the constant field $a_0$. This field $a_0$ later will be identified with the zero
mode of the physical field $\Phi_0$ which is integrated over imaginary contour. To have consistent notations we set $a_0 = ia_0^E$ and assume that $a_0^E$ is integrated over the real contour.

The $\delta$-exact term

$$S_{\gamma.f.} = \delta ((\bar{c}, i d^* A + \frac{\xi_1}{2} b + ib_0) - (c, \bar{a}_0 - \frac{\xi_2}{2} a_0)) =$$

$$= (b, i d^* A + \frac{\xi_1}{2} b + ib_0) - (\bar{c}, i d^* \nabla A c + ic_0 + \frac{\xi_1}{2} [a_0, \bar{c}]) + (\bar{c}, i d^* A + \frac{1}{2} [c, \bar{c}], \bar{a}_0 - \frac{\xi_2}{2} i a_0^E) + (c, i \bar{c}_0)$$

(4.2)

properly fixes the gauge.

Assuming that all bosonic fields are real, the bosonic part of gauge-fixed action has strictly positive definite quadratic term for all fields and ghosts at $\xi_1, \xi_2 > 0$.

By general arguments the partition function does not depend on the parameters $\xi_1, \xi_2$ in the $\delta$-exact term. Let us fix $\xi_1 = 0$ and demonstrate explicitly independence on $\xi_2$ and equivalence with the standard gauge-fixing procedure. First we do Gaussian integral integral over $\tilde{a}_0$ and get

$$(ia_0^E + \frac{1}{2} [c, \bar{c}], i \bar{a}_0 - \frac{\xi_2}{2} i a_0^E) \rightarrow + \frac{1}{2\xi_2} (\bar{a}_0 - \frac{\xi_2}{4} [c, \bar{c}])^2.$$ 

Then we do Gaussian integral over $\bar{a}_0$ and the above term goes away completely. The determinant coming from the Gaussian integral over $\bar{a}_0$ is inverse to the determinant coming from the Gaussian integral over $a_0$. Then we integrate the zero mode of $b$ against $b_0$. Then integral over non-zero modes of $b$ gives Dirac delta-functional inserted at the gauge-fixing hypersurface $d^* A = 0$. The remaining terms are

$$(\bar{c}, i d^* \nabla A c) + i(\bar{c}, c_0) + i(c, \bar{c}_0).$$

We can integrate out $c_0$ with the zero mode of $\bar{c}$, and $\bar{c}_0$ with the zero mode of $c$. Then we are left with the integral over $c$ and $\bar{c}$ with the zero modes projected out and the gauge-fixing term

$$(\bar{c}, i d^* \nabla A c).$$

This reproduces the usual Faddeev-Popov determinant $\det'(d^* \nabla A)$ which we need to insert into the path integral for the partition function after restricting to the gauge-fixing hypersurface $d^* A = 0$. The symbol $'$ means that the determinant is computed on the space without the zero modes.

We summarize the gauge fixing procedure by the formula

$$Z = \frac{1}{\text{vol}(\mathcal{G}, g_{YM})} \int [D X] e^{-S_{YM}[X]} = \frac{1}{\text{vol}(\mathcal{G})} \int [D X] e^{-S_{YM}[X]} \int_{g \in \mathcal{G}'} [D g] \delta_{\text{Dirac}}(d^* A g) \det'(d^* \nabla_A) =$$

$$= \frac{\text{vol}(\mathcal{G}', g_{YM})}{\text{vol}(\mathcal{G}, g_{YM})} \int [D X D b D c D \bar{c}] e^{-S_{YM}[X] - \int_{\mathcal{G}} \sqrt{\text{det}[g_{YM}]} (\bar{c} c, i d^* \nabla_A c))} =$$

$$= \frac{1}{\text{vol}(\mathcal{G}, g_{YM})} \int [D X D b D b_0 D c D c_0 D \bar{c} D \bar{c}_0 D a_0 D \bar{a}_0] e^{-S_{YM}[X] - S_{g.f.}[X, \text{ghosts}]},$$

(4.3)

where $\mathcal{G}' = \mathcal{G}/G$ is the coset of the group of gauge transformations by constant gauge transformations. We shall note that in our conventions for the gauge theory Lagrangian $\frac{1}{4g_{YM}^2} (F, F)$, where $F = dA + A \wedge A$, we need to take the volume of the group of gauge transformations with respect to the measure which is rescaled
by a power of the coupling constant $g_{YM}$. In other words, we take $\text{vol}(G, g_{YM}) = g_{YM}^{\text{dim}G} \text{vol}(G)$, where $\text{vol}(G)$ is the volume of the gauge group computed with respect to the Haar measure induced by the coupling constant independent Killing form $(,)$ on the Lie algebra.

4.2. Supersymmetry complex. To compute the path integral, it is convenient to bring the supersymmetry transformations to a cohomological form by a change of variables. (This change of variables involves no Jacobian, one can think about it as a change of notations.) We use the fact that conformal Killing spinor $\epsilon$ in (2.28) has constant unit norm at any point on $S^4$. Then the set of sixteen spinors consisting of $\{\Gamma_M \epsilon\}$ for $M = 1, \ldots, 9$ and $\{\nu_i\}$ for $i = 1, \ldots, 7$ form an orthonormal basis for the space of $\text{Spin}(9,1)$ Majorana-Weyl spinors reduced on $S^4$. We expand $\Psi$ over this basis

$$\Psi = \sum_{M=1}^9 \Psi_M \Gamma_M \epsilon + \sum_{i=1}^7 \Upsilon_i \nu^i.$$  

In new notations $(\Psi_M, \Upsilon_i)$, the supersymmetry transformations (2.28) take the following form:

$$\begin{cases} 
\delta A_M = \Psi_M \\
\delta \Psi_M = -(L_v + R + M + G\Phi)A_M \\
\delta \Upsilon_i = H^i \\
\delta H^i = -(L_v + R + M + G\Phi)\Upsilon_i,
\end{cases} \tag{4.4}$$

where

$$H^i \equiv K^i + w_i \Phi_0 + s_i(A_M). \tag{4.5}$$

Now $\delta$ denotes $\delta_\epsilon$ to distinguish it from the differential $\delta$ of the Faddeev-Popov complex. By $L_v$ we denote the Lie derivative in the direction of the vector field $v^\mu$, $R$ denotes the $R$-symmetry transformation in $SU_R^R$, $M$ denotes the mass-term induced transformation by $M_{ij}$ in $SU_R^R$, and $G\Phi$ denotes the gauge transformation by $\Phi$. The functions $s_i(A_M)$ with $i = 1, \ldots, 7$ are the “equations” of the equivariant theory

$$s_i(A_M) = \frac{1}{2} F_{MN} \nu_i \Gamma^{MN} \epsilon + \frac{1}{2} \Phi_A \nu_i \Gamma^A \nu M \nabla^A \epsilon \quad \text{for} \quad M, N = 1, \ldots, 9 \quad A = 5, \ldots, 9. \tag{4.6}$$

Even shorter, we can write the supersymmetry complex like

$$\begin{cases} 
\delta X = X' \\
\delta X' = [\phi + \epsilon, X],
\end{cases} \tag{4.7}$$

and $\delta \phi = 0$, where we denoted $\phi = -\Phi$, $[\phi, X] = -G\Phi X$ and $[\epsilon, X] = -(L_v + R + M)X$.

All fields except $\Phi$ (2.12) are grouped in $s$-doublets $(X, X')$, where the fields $X$ and $X'$ have opposite statistics. We can think about fields $X$ as coordinates on some infinite-dimensional supermanifold $\mathcal{M}$, on which group $G$ acts. The fields $X'$ can be interpreted as de Rham differentials $X' \equiv dX$, if we identify the operator $s$ with the differential in the Cartan model of $G$-equivariant cohomology on $\mathcal{M}$

$$s = d + \phi^a i_v^a \tag{4.8}$$

where $\phi^a$ are the coordinates on the Lie algebra $\mathfrak{g}$ of the group $G$ with respect to some basis $\{e_a\}$, and $i_v^a$ is the contraction with a vector field $v^a$ representing action
of $e_a$ on $\mathcal{M}$. The differential $s$ squares to the Lie derivative $\mathcal{L}_\phi$. In the present case, the group $\mathcal{G}$ is a semi-direct product

$$
\mathcal{G} = \mathcal{G}_{\text{gauge}} \ltimes U(1) \tag{4.9}
$$

of the infinite-dimensional group of gauge transformations $\mathcal{G}_{\text{gauge}}$ and the $U(1)$ subgroup of the $OSp(2|4)$ symmetry group generated by the conformal Killing spinor $\varepsilon$.

In the path integral (4.3) for the partition function $Z_{\text{phys}}$, we integrate $s$-equivariantly closed form $e^S$ over $\mathcal{M}$ and then over $\phi$. See [52, 53, 62] for twisted $\mathcal{N} = 4$ SYM related theories which have similar cohomological structure, and [63] where similar integration over the parameter of the equivariant cohomology is performed.

4.3. The combined $Q$-complex. So far we constructed separately the gauge-fixing complex with the differential $\delta$ and the supersymmetry complex with the differential $s$:

$$
\begin{align*}
\delta a_0 &= 0 \quad \delta X = -[c, X] \\
\delta c &= -a_0 - \frac{1}{2}[c, c] \\
\delta \bar{c} &= b \\
\delta \bar{a}_0 &= \bar{c}_0 \\
\delta b_0 &= c_0 \\
\delta X' &= -[c, X'] \\
\delta \phi &= -[c + \varepsilon, \phi] \\
\delta \bar{c} &= [a_0, \bar{c}] \\
\delta \bar{c}_0 &= [a_0, \bar{a}_0] \\
\delta c_0 &= [a_0, b_0].
\end{align*}
$$

Then we get

$$
\begin{align*}
Qa_0 &= 0 \\
QX = X' - [c, X] \\
Qc &= \phi - a_0 - \frac{1}{2}[c, c] \\
Q\bar{c} &= b \\
Q\bar{a}_0 &= \bar{c}_0 \\
Qb_0 &= c_0 \\
QX' &= [\phi + \varepsilon, X] - [c, X'] \\
Q\phi &= -[c, \phi + \varepsilon] \\
Qb &= [a_0 + \varepsilon, \bar{c}] \\
Q\bar{c}_0 &= [a_0, \bar{c}_0] \\
Qc_0 &= [a_0, b_0].
\end{align*}
$$

This means that $Q$ satisfies on all fields

$$
Q^2 = [a_0 + \varepsilon, \cdot]
$$

In other words, $Q$ squares to a constant gauge transformation generated by $a_0$ and the $U(1)$ anti-self-dual Lorentz rotation around the North pole generated by $\varepsilon$.

Now, since $sS_{\text{phys}} = 0$ and $\delta S_{\text{phys}} = 0$ we have

$$
QS_{\text{phys}} = 0.
$$

We would like to make sure that the gauge-fixing term (4.2) is also $Q$-closed so that we could use the localization argument.
We will take the following $Q$-exact gauge-fixing term:

$$S_{g.f.}^Q = (\delta+s)((\hat{c}, id^* A + \frac{\xi_1}{2} b + ib_0) - (c, \tilde{a}_0 - \frac{\xi_2}{2} a_0)) = S_{g.f.}^\delta - \hat{c}, s(id^* A + \frac{\xi_1}{2} b + ib_0) - (\phi, \tilde{a}_0) = S_{g.f.}^\delta - (\hat{c}, d^* \psi + \frac{\xi_1}{2} [\varepsilon, \hat{c}]) - (\phi, \tilde{a}_0 - \frac{\xi_2}{2} a_0) \quad (4.13)$$

The replacement of $S_{g.f.}^\delta$ by $S_{g.f.}^Q$ does not change the partition function $Z_{phys} (4.13)$. We can easily see this at $\xi_1 = 0$. Integrating over $a_0$ we get

$$(ia_0^E + \frac{1}{2} [\varepsilon, c] - \phi, \tilde{a}_0 - \frac{\xi_2}{2} ia_0^E) \to \frac{1}{2\xi_2} \left( \frac{\xi_2}{2} \frac{1}{2} [\varepsilon, c] - \phi \right) + i\tilde{a}_0)^2.$$

After we integrate over $\tilde{a}_0$ the above term goes away completely. The determinants for the Gaussian integrals over $a_0$ and $\tilde{a}_0$ cancel. Then we are left with the following gauge-fixing terms

$$i(b, d^* A + b_0) - i(\hat{c}, d^* \nabla c + c_0) + i(c, \hat{c}) - (\hat{c}, d^* \psi),$$

where $\psi$ is the fermionic one-form which is the superpartner of the gauge field $A$. Then we note that the term $(\hat{c}, d^* \psi)$ does not change the fermionic determinant arising from the integral over $c$, $\hat{c}$, $c_0$ and $\tilde{c}_0$. The reason is that all modes of $c$ are coupled to $\hat{c}$ by this quadratic action

$$i(\hat{c}, d^* \nabla c + c_0) + i(c, \hat{c}),$$

and that there are no other terms in the gauge-fixed action which contain modes of $c$. In other words, if treat the term $(\hat{c}, d^* \psi)$ as a perturbation to the usual gauge fixed action, all diagrams with it vanish because $\hat{c}$ can be connected by a propagator only to $c$, but there are no other terms which generate vertices with $c$.

In other words we did the following. The action of the theory gauge-fixed in the standard way (4.3) is $\delta$-closed, but not $Q$-closed. We make the action $Q$-closed by adding such terms to it which do not change the path integral. The fact that the partition function does not change can be also shown by making a change of variables which has trivial Jacobian.

We conclude that the total gauge-fixed action

$$S_{phys} = S_{phys} + S'_{g.f.} \quad (4.14)$$

is $Q$-closed

$$Q S_{phys} = 0, \quad (4.15)$$

and that the partition function defined by the path integral over all fields and ghosts with the action $S_{phys}$ is equal to the standard partition function with the usual gauge-fixing (4.3).

It is possible to write the operator $Q$ in the canonical form; namely $Q$ is the equivariant differential in the Cartan model for the $\tilde{G} = G \ltimes U(1)$ cohomology generated by $a_0$ and $\varepsilon$ on the space of all other fields over which we integrate in the path integral (4.3). The multiplets $(\hat{c}, b)$, $(\tilde{a}_0, \tilde{c}_0)$ and $(b_0, c_0)$ are already in the canonical form. To bring the transformations of $(X, X')$ and $(c, \phi)$ to the canonical form we make a change of variables

$$\tilde{X}' = X' - [c, X]$$

$$\tilde{\phi} = \phi - a_0 - \frac{1}{2}[c, c]. \quad (4.16)$$
Such change of variables has trivial Jacobian and does not change the path integral. In terms of new fields, the $Q$-complex is canonical: all fields are grouped in doublets $(Field, Field')$, while $Q$ acts as

\begin{align*}
Q(Field) &= (Field') \\
Q(Field') &= [a_0 + \varepsilon, Field].
\end{align*}

(4.17)

Moreover, $Qa_0 = Q\varepsilon = 0$.

Now recall Atiyah-Bott-Berline-Vergne localization formula for the integrals of the equivariantly closed differential forms $[46, 47]$.

\begin{align*}
\int_M \alpha &= \int_{F \subset M} i_{F*}\alpha e(N) \\
&(4.18)
\end{align*}

The numerator corresponds to the physical action evaluated at the critical locus of the $tQV$ term. The equivariant Euler class of the normal bundle in the denominator is just a determinant, coming from the Gaussian integral using quadratic part of $tQV$ in the normal directions $N$. We will argue that this determinant can be expressed as a product of weights for the group action on $N$ defined by (4.17). The basic difference with the usual localization formula (4.18) is that the manifold $M$ in our problem is not a usual manifold, but an (infinite-dimensional) supermanifold. Hence, the equivariant Euler class must be understood in a super-formalism [64, 65]. In our case it is just a super-determinant. If we split the normal bundle to the bosonic and the fermionic subspaces, the resulting determinant is the product of weights on the bosonic subspace divided by the product of weights on the fermionic subspace.

Before making gauge-fixing procedure we argued previously that the theory localizes to the zero modes of the field $\Phi_0$. The localization argument for the gauge-fixed theory remains the same, except that now we can identify the zero mode of the field $\Phi_0$ with $a_0$. Indeed, if we first integrate over $\tilde{a}_0$ using gauge fixing terms at $\xi_2 = 0$

\begin{align*}
(i\Phi_0^E + \frac{1}{2}[c, c] - i\Phi^E, \tilde{a}_0),
\end{align*}

we get the constraint that the zero mode of $\Phi^E$ is equal to $a_0^E$.

\subsection*{4.4. Computation of the determinant by the index theory of transversally elliptic operators.}

We write the linearization of the $Q$-complex in the form

\begin{align*}
QX_0 &= X_0' \\
QX_1 &= X_1' \\
QX_0' &= R_0X_0 \\
QX_1' &= R_1X_1
\end{align*}

(4.19)

where all bosonic and fermionic fields in the first line of (4.17) are denoted as $X_0$ and $X_1$ respectively, and their $Q$-differentials are denoted as $X_0'$ and $X_1'$. So $X_0, X_1'$ are bosonic, and $X_0', X_1$ are fermionic fields.

The quadratic part of the functional $V$ is

\begin{align*}
V^{(2)} &= \left( \begin{array}{c}
X_0' \\
X_1'
\end{array} \right)^t \left( \begin{array}{cc}
D_{00} & D_{01} \\
D_{10} & D_{11}
\end{array} \right) \left( \begin{array}{c}
X_0 \\
X_1'
\end{array} \right),
\end{align*}

(4.20)

where $D_{00}, D_{01}, D_{10}, D_{11}$ are some differential operators. Then we have

\begin{align*}
QV^{(2)} &= (X_{bos}, K_{bos}X_{bos}) + (X_{ferm}, K_{ferm}X_{ferm}),
\end{align*}
where the kinetic operators $K_{bos}, K_{ferm}$ are expressed in terms of $D_{00}, D_{01}, D_{10}, D_{11}$ and $R_0, R_1$ in a certain way. The Gaussian integral gives

$$Z_{1\text{-loop}} = \left( \frac{\det K_{bos}}{\det K_{ferm}} \right)^{-\frac{1}{2}}. \quad (4.21)$$

Let $E_0$ and $E_1$ denote the vector bundles whose sections can be identified with fields $X_0, X_1$. Some linear algebra shows that the ratio of the determinants depends only on the representation structure $R$ on the kernel and cokernel spaces of the operator $D_{10}: \Gamma(E_0) \to \Gamma(E_1)$. Namely we have

$$\frac{\det K_{bos}}{\det K_{ferm}} = \frac{\det_{\ker D_{10}} R}{\det_{\coker D_{10}} R}. \quad (4.22)$$

The operator $D_{10}$ in our problem is not an ordinary elliptic operator, but a transversally elliptic operator with respect to the $U(1)$ rotation of $S^4$.

This means the following. Let $E_0$ and $E_1$ be vector bundles over a manifold $X$ and $D: \Gamma(E_0) \to \Gamma(E_1)$ be a differential operator. (In our problem $X = S^4$.) Let a compact Lie group $\tilde{G}$ act on $X$ such that its action preserves all structures. Let $\pi: T^*X \to X$ be the cotangent bundle of $X$. Then pullback $\pi^*E_i$ is a bundle over $T^*X$. By definition, a symbol of the differential operator $D: \Gamma(E_0) \to \Gamma(E_1)$ is a vector bundle homomorphism $\sigma(D): \pi^*E_0 \to \pi^*E_1$, such that in local coordinates $x_i$, the symbol is defined by replacing all partial derivatives in the highest order component of $D$ by momenta, so that $\frac{\partial}{\partial x_i} \to ip_i$, and then taking $p_i$ to be coordinates on fibers of $T^*X$. The operator $D$ is called elliptic if its symbol $\sigma(D)$ is invertible on $T^*X \setminus 0$, where $0$ denotes the zero section. The kernel and cokernel of an elliptic operator are finite dimensional vector spaces. Using the Atiyah-Singer index theory \cite{atiyah1968index, asmus1974index} one can find a formal difference of representations in which $\tilde{G}$ acts on these spaces, as we will see in a moment. However, we will see as well that the operator $D_{10}$ is not elliptic, so the ordinary Atiyah-Singer index theory does not apply. There is a generalization of Atiyah-Singer index theory for operators which are elliptic only in directions transverse to the $\tilde{G}$-orbits \cite{atiyah1974index, felix1995cohomology}. Such operators are called transversally elliptic. In other words, for any point $x \in X$ we consider the subspace $T^*_G X_x$ of the $T^*X_x$, which consists of elements which are orthogonal to the $\tilde{G}$-orbit through $x$. We have

$$T^*_G X_x = \{ p \in T^*X_x \text{ such that } p \cdot v(\tilde{g}) = 0 \forall \tilde{g} \in \text{Lie}(\tilde{G}) \},$$

where $v(\tilde{G})$ denotes a vector field on $X$ generated by an element $\tilde{g}$ of the Lie algebra of $\tilde{G}$. Then the family of the vector spaces $T^*_G X$ over $X$ is defined as the union of $T^*_G X_x$ for all $x \in X$. The notion of a family of vector spaces over some base is similar to the notion of a vector bundle, except that dimension of fibers can jump. The operator $D$ is called transversally elliptic if its symbol $\sigma(D)$ is invertible on $T^*_G X \setminus 0$. Computing the symbol of $D_{10}$, we will see explicitly in \cite{pes1, pes2} that $D_{10}$ is not an elliptic operator, but a transversally elliptic one. The kernel and the cokernel of such an operator are not generally finite dimensional vector spaces, but if we decompose them into irreducible representations, then each irreducible representation appears with a finite multiplicity \cite{atiyah1974index, felix1995cohomology}. So we have

$$\ker D_{10} = \oplus_{\alpha} m^{(0)}_{\alpha} R_{\alpha}$$

$$\coker D_{10} = \oplus_{\alpha} m^{(1)}_{\alpha} R_{\alpha}. \quad (4.23)$$
where $\alpha$ runs over irreducible representations of $\tilde{G}$, and $m_\alpha$ denotes the multiplicity of the irreducible representation $R_\alpha$. Then

$$\frac{\det K_{bos}}{\det K_{ferm}} = \prod_\alpha (\det R_\alpha)^{m_\alpha(0) - m_\alpha(1)}.$$ (4.24)

Thus we need to know only the difference of multiplicities $m_\alpha(0)$ and $m_\alpha(1)$ of irreducible representations into which the kernel and cokernel of $D_{10}$ can be decomposed. To find this difference we use Atiyah-Singer index theory for transversally elliptic operators, which generalizes the usual theory. In our problem, $R_\alpha$ is an irreducible representation of the group $\tilde{G} = U(1) \times G$. We also denote this $U(1)$ group by $H$, so that $G = H \times G$. The relevant representations of $G$ are those in which the physical fields transform (we will consider only the adjoint representation), but all representations of $H = U(1)$ arise. Let $q \in \mathbb{C}, |q| = 1$ denote an element of $U(1)$. Irreducible representations of $U(1)$ are labeled by integers $n$, so that the character of representation $n$ is $q^n$. The $U(1)$-equivariant index of $D_{10}$ is defined as

$$\text{ind}(D_{10}) = \text{tr}_{\ker D_{10}} R(q) - \text{tr}_{\text{coker} D_{10}} R(q) = \sum_n (m_n(0) - m_n(1)) q^n.$$ (4.25)

Hence, if we compute the equivariant index of $D_{10}$ as a series in $q$, we will know $m_n(0) - m_n(1)$ and will be able to evaluate (4.24).

To compute the index of $D_{10}$, first we need to describe the bundles $E_0, E_1$ and the symbol of the operator $D_{10} : \Gamma(E_0) \to \Gamma(E_1)$.

The collective notation $X_0, X_0', X_1, X_1'$ corresponds to the original fields in the following way

$$X_0 = (A_M, \tilde{a}_0, b_0) \quad X_1 = (\Upsilon_i, c, \tilde{c})$$

$$X_0' = (\tilde{\Psi}_M, \tilde{c}_0, c_0) \quad X_1' = (\tilde{H}_i, \tilde{\phi}, b).$$ (4.26)

The space of all fields decomposes in a way compatible with $Q$-action into direct sum of two subspaces: the fields of vector multiplet and hypermultiplet. The vector subspace also includes fields of the gauge fixing complex. The vector subspace consists of

$$X_0^{\text{vect}} = (\Phi_9, A_M, \tilde{a}_0, b_0) \quad X_1^{\text{vect}} = (\Upsilon_i, c, \tilde{c}) \quad i = 5, \ldots, 7$$ (4.27)

and their $Q$-superpartners. The hyper subspace consists of

$$X_0^{\text{hyper}} = (A_M) \quad \text{for } M = 5, \ldots, 8 \quad X_1^{\text{hyper}} = (\Upsilon_i) \quad \text{for } i = 1, \ldots, 4$$ (4.28)

and their $Q$-superpartners. The operator $D_{10}$ does not mix the vector and hyper subspaces. So the vector bundles split as $E_0 = E_0^{\text{vect}} \oplus E_0^{\text{hyper}}$, and $E_1 = E_1^{\text{vect}} \oplus E_1^{\text{hyper}}$, as well as the operator $D_{10} = D_{10}^{\text{vect}} + D_{10}^{\text{hyper}}$, where $D_{10}^{\text{vect}} : \Gamma(E_0^{\text{vect}}) \to \Gamma(E_1^{\text{vect}})$ and $D_{10}^{\text{hyper}} : \Gamma(E_0^{\text{hyper}}) \to \Gamma(E_1^{\text{hyper}})$.

First we consider the index of $D_{10}^{\text{vect}}$. The constant fields $(\tilde{a}_0, b_0)$ are in the kernel of $D_{10}^{\text{vect}}$ and have zero $U(1)$ weights, hence their contribution to the index is 2. The remaining fields, denoted by $X_0^{\text{vect}}$, are identified with sections of bundle $(T^* \oplus \mathcal{E}) \otimes \text{ad } E$, where $T^*$ is the cotangent bundle, and $\mathcal{E}$ is the rank one trivial bundles over $S^4$. The fields $X_1^{\text{vect}}$ are identified with sections of $(\mathcal{E}^3 \oplus \mathcal{E}^2) \otimes \text{ad } E$, where $\mathcal{E}^3$ is the rank three trivial bundle of auxiliary scalar fields, and $\mathcal{E}^2$ is the
rank two trivial bundle of the gauge fixing fields \(c\) and \(\tilde{c}\). Because of the difference due to \((\tilde{a}_0, b_0)\) contribution we have

\[
\text{ind}(D_{10}^{\text{rect}}) = \text{ind}'(D_{10}^{\text{rect}}) + 2.
\]  

(4.28)

Now we compute the symbol of the operator \(D_{10}^{\text{rect}}\). The relevant terms are

\[
V^{(2)} = (\tilde{c}, d^*A) + (c, \nabla_\mu L_c A_\mu) + (\nabla_i, (\ast \Phi_{0i}) - F_{0i} \cos \theta + \nabla_i \Phi_9 \sin \theta),
\]  

(4.29)

where index \(i\) runs over vielbein elements on \(S^4\).

We chose a vielbein in such a way that \(i = 1\) is the direction of the \(U(1)\) vector field, and \(i = 2, 3, 4\) are the remaining orthogonal directions. The term \((c, \nabla_\mu L_c A_\mu)\) comes from the term \((\psi_\mu, L_c A_\mu)\) and the relation \(\psi_\mu = \tilde{\psi}_\mu - \nabla_\mu c\). Then the symbol \(\sigma(D_{10}^{\text{rect}}): \pi^* E_0^{\text{rect}} \to \pi^* E_1^{\text{rect}}\), where \(\pi\) denotes the projection of the cotangent bundle \(\pi: T^*X \to X\), is represented by the following matrix

\[
\begin{pmatrix}
\Phi_9 \\
-\Phi_9 \\
\pi_2 \\
\pi_3
\end{pmatrix} = \begin{pmatrix}
c_0 p^2 & s_0 \bar{p}^2 & -s_0 p_2 p_1 & -s_0 p_3 p_1 & -s_0 p_4 p_1 \\
0 & p_1 & p_2 & p_3 & p_4 \\
s_0 p_2 & -c_0 p_2 & c_0 p_1 & -p_4 & p_3 \\
s_0 p_3 & -c_0 p_3 & p_4 & c_0 p_1 & -p_2 \\
s_0 p_4 & -c_0 p_4 & -p_3 & p_2 & c_0 p_1
\end{pmatrix} \begin{pmatrix}
\Phi_9 \\
A_1 \\
A_2 \\
A_3 \\
A_4
\end{pmatrix}.
\]  

(4.30)

Here \(p_i\) for \(i = 1, \ldots, 4\) denotes coordinates on fibers of \(T^*X\), \(\bar{p} = (p_2, p_3, p_4)\) denotes coordinate on fibers of \(T_T^*X\), and \(c_0 \equiv \cos \theta, s_0 \equiv \sin \theta\). In other words, \(\bar{p}\) is a momentum orthogonal to the direction of the \(U(1)\) vector field on \(S^4\). After a change of coordinates on fibers of bundles \(E_0 \to T^*X\) and \(E_1 \to T^*X\)

\[
c \to c + s_0 \Phi_9 \tilde{c}
\]

\[
\Phi_9 \to c_0 \Phi_9 + s_0 A_1
\]

\[
A_1 \to -s_0 \Phi_9 + c_0 A_1,
\]  

(4.31)

the matrix of the symbol of \(D_{10}^{\text{rect}}\) takes the form

\[
\begin{pmatrix}
p^2 & 0 & 0 & 0 & 0 \\
0 & s_0 p_1 & c_0 p_1 & p_2 & p_3 & p_4 \\
0 & c_0 p_2 & c_0 p_2 & -p_4 & -p_3 & p_3 \\
0 & -p_3 & p_3 & c_0 p_2 & -p_2 & -p_4 \\
0 & -p_4 & -p_3 & p_2 & c_0 p_2 & c_0 p_2
\end{pmatrix}.
\]  

(4.32)

The term \(s_0 p_1\) in the first column of the second line can be also removed by subtracting the first line multiplied by \(s_0 p_1/p^2\). Then the nontrivial part of the symbol is represented by the following 4 \(\times\) 4 matrix

\[
\sigma = \begin{pmatrix}
c_0 p_1 & p_2 & p_3 & p_4 \\
-p_2 & c_0 p_2 & -p_4 & p_3 \\
-p_3 & c_0 p_2 & -p_2 & c_0 p_2 \\
-c_0 p_1 & -p_3 & -p_2 & c_0 p_2
\end{pmatrix}.
\]  

(4.33)

The determinant of this matrix is \((\cos^2 \theta p_1^2 + \bar{p}^2)^2\). First of all, we see that the symbol is not elliptic at the equator of \(S^4\), since if \(\cos \theta = 0\) we can take \((p_1 \neq 0, \bar{p} = 0)\) and the determinant will vanish. But the symbol is transversally elliptic with respect to the \(H = U(1)\) group, since its determinant is always non-zero whenever \(\bar{p} \neq 0\). Indeed, to check if the symbol is transversally elliptic, we need to consider only non-zero momenta orthogonal to the \(U(1)\) orbits. In our notations that means \(p_1 = 0, \bar{p} \neq 0\).
In a neighborhood of the North pole \( (c_\theta = 1) \) the symbol is equivalent to the elliptic symbol of the standard anti-self-dual complex \((d, d^-)\)
\[
\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^-} \Omega^{2-},
\]
while in a neighborhood of the South pole \((c_\theta = -1)\), the symbol is equivalent to the elliptic symbol of the standard self-dual complex \((d, d^+)\)
\[
\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}.
\]
Intuitively one can see that from the structure of the \(QV\)-action \((3.14)\).

In the elliptic case, one could use Atiyah-Bott formula \([70, 71]\) to compute the index as a sum of local contributions from \(H\)-fixed points on \(X\). In the transversally elliptic case the situation is more complicated. By definition, the index is a sum of characters of irreducible representations. We have
\[
\text{ind}(D) = \sum_{n=-\infty}^{\infty} a_n q^n,
\]
where \(a_n = m_n^{(0)} - m_n^{(1)}\) is a difference of multiplicities in which irreducible representation \(n\) appears in the kernel and cokernel of \(D\). In the elliptic case, only a finite number of \(a_n\) does not vanish, so that the index is a finite polynomial in \(q\) and \(q^{-1}\). This also means that the index is a regular function on the group \(H = U(1)\).

In the transversally elliptic case, the series \((4.36)\) can be infinite, so that index is generally not a regular function. However, Atiyah and Singer showed \([41, 50]\) that in the transversally elliptic case, all coefficients \(a_n\) are finite, and that the index is well defined as a distribution (a generalized function) on the group.

For example, consider the zero operator acting on functions on a circle \(X = S^1\), so \(D : C^\infty(S^1) \to 0\). This is a transversally elliptic operator with respect to the canonical \(U(1)\) action on \(S^1\). The kernel of the zero operator is the space of all functions on \(S^1\), the cokernel is zero. Then \(m_n^{(0)} = 1, m_n^{(1)} = 0\) for all \(n\), so the index is \(\sum_{n=\infty}^{\infty} q^n\), which is the Dirac delta-function supported at \(q = 1\).

The equivariant index theory can be generalized to the transversally elliptic case \([41, 50, 70, 72]\). The idea is that we can cut a \(H\)-manifold \(X\) into small neighborhoods of \(H\)-fixed points and the remaining subspace \(Y \subset X\) on which \(H\) acts freely. By definition, at each \(H\)-fixed point the symbol of transversally elliptic operator is actually elliptic, so the ordinary equivariant index theory applies. Since \(H\) acts freely on \(Y\), we can consider the quotient \(Y/H\). A \(H\)-transversally elliptic operator on \(Y\) gives us an elliptic operator on \(Y/H\). Then we can combine the representation theory of \(G\) and the usual index theory on the quotient \(Y/H\) to find the index of transversally elliptic operator on \(Y\) \([50]\).

Let \(R(H)\) be the space of regular functions on \(H\) (the space of finite polynomials in \(q\) and \(q^{-1}\)). Let \(D'(H)\) denote the space of distributions (generalized functions) on \(H\) (not necessarily finite series in \(q\) and \(q^{-1}\)). The space of distributions \(D'(H)\) is a module over the space of regular functions \(R(H)\), since there is a well defined term by term multiplication of series in \(q\) and \(q^{-1}\) by finite polynomials in \(q\) and \(q^{-1}\).

Some singular generalized functions such as the Dirac delta-function \(\sum_{n=-\infty}^{\infty} q^n\) can be annihilated by non-zero regular functions. For example, Dirac delta-function \(\sum_{n=-\infty}^{\infty} q^n \in D'(H)\) vanishes after multiplication to \((1-q)\). Such elements of \(D'(H)\) which can be annihilated by non-zero regular functions in \(R(H)\) are called torsion elements.
To find the index of transversally elliptic operator up to a distribution supported at \( q = 1 \) (a torsion element of \( D'(H) \)), we can use the usual Atiyah-Bott formula \([63-71]\) (see appendix (D)). This formula gives a contribution to the index from each fixed point as a rational function of \( q \). This function is generally singular at \( q = 1 \).

For example, if \( H = U(1) \) acts on \( \mathbb{C} \) as \( z \to qz \), then the Atiyah-Bott formula for the index of the \( \bar{\partial} \)-operator at the fixed point \( z = 0 \) gives

\[
\text{ind}(\bar{\partial})|_0 = \frac{1}{1 - q^{-1}}. \tag{4.37}
\]

To get a distribution associated with this rational function, we need to expand it in series in \( q \) and \( q^{-1} \). Of course, the result is not unique, but different expansions differ only by a distribution supported at \( q = 1 \). For \( H = U(1) \), there are two basic ways, or regularizations, which fix the singular part \([50]\). The regularization \([f(q)]_+\) is defined by taking expansion at \( q = 0 \). This gives us a series infinite in positive powers of \( q \). The regularization \([f(q)]_-\) is defined by taking expansion at \( q = \infty \). This gives us a series infinite in negative powers of \( q \). These two regularizations differ by a distribution supported at \( q = 1 \). For example, for the \( \bar{\partial} \)-operator we get as the difference the Dirac delta-function \([\{1 - q^{-1}\}^+ - \{1 - q^{-1}\}^-] = -\sum_{n=-\infty}^{\infty} q^n\).

Let \( X = \mathbb{C}^n \) be a \( H = U(1) \) module with positive weights \( m_1, \ldots, m_n \), so that \( U(1) \) acts as \( z_i \to q^{m_i}z_i \), and let \( Y = \{0\} \) be the \( H \)-fixed point set. Let \( v \) be the vector field generated by the \( U(1) \) action on \( X \). Let \( \sigma(D) \) be an elliptic symbol defined on \( T^*X|_Y \), i.e. defined on the fiber of the cotangent bundle to \( X \) at the origin. Atiyah showed \([50]\) that we can use the vector field \( v \) in two different ways, called \([\cdot]_+ \) and \([\cdot]_- \), to construct a transversally elliptic symbol \( \tilde{\sigma} = [\sigma]_\pm \) on the whole space \( T^*_H X \) such that \( \tilde{\sigma} \) is an isomorphism outside of the origin \( Y \). (If \( (x, p) \) are coordinates on \( T^*X \), then, loosely speaking, we take \( \tilde{\sigma}(x, p) = \sigma(0, p \pm v) \). See appendix (D) for more precise details). Then the index of the transversally elliptic symbol \( \tilde{\sigma} \) is well defined as a distribution on \( H \). Moreover, if \( \text{ind}(\sigma) \) is a rational function of \( q \) associated at the fixed point \( Y \) to the elliptic symbol \( \sigma \) by Atiyah-Bott formula, then

\[
\text{ind}([\sigma]_\pm) = |\text{ind}(\sigma)|_\pm. \tag{4.38}
\]

We apply this procedure to our problem. Namely, we use the vector field generated by the \( H = U(1) \)-action on \( X = S^4 \) to trivialize the symbol \( \sigma(D^\text{vect}_1) \) everywhere on \( T^*_H X \) except at the North and the South pole. Then the index is equal to the sum of contributions from the fixed points, where each contribution is expanded in positive or negative powers of \( q \) according to the (4.38). More concretely, we trivialize the transversally elliptic symbol \( \sigma = \sigma(D^\text{vect}_1) \) everywhere outside the North and the South poles on \( T^*_H X \) by replacing \( cqp_1 \) by \( cqp_1 + v \) on the diagonal in (4.38), where \( v = \sin \theta \). In other words, we deform the operator by adding the Lie derivative in the direction of the vector field \( v \). The resulting symbol

\[
\tilde{\sigma} = \begin{pmatrix}
    cqp_1 + s\theta & p_2 & p_3 & p_4 \\
    -p_2 & cqp_1 + s\theta & -p_4 & p_3 \\
    -p_3 & p_4 & cqp_1 + s\theta & -p_2 \\
    -p_4 & -p_3 & p_2 & cqp_1 + s\theta
\end{pmatrix}.
\tag{4.39}
\]

has determinant \((\tilde{\partial}^2 + (cqp_1 + s\theta)^2)^2\) which is non-zero everywhere outside the North and the South poles at \( T^*_H X \). (To check this, take \( p_1 = 0 \) and \( s\theta > 0 \).) The index of \( \tilde{\sigma} \) is equal to the index of \( \sigma \), since \( \tilde{\sigma} \) is a continuous deformation of \( \sigma \). On the other hand, since \( \tilde{\sigma} \) is an isomorphism outside of the North and the
South pole, to get the index of $\tilde{\sigma}$ we sum up contributions from the North and the South pole. At the North pole $\cos \theta = 1$. Therefore, in a small neighborhood of the North pole, the transversally elliptic symbol $\tilde{\sigma}$ coincides with the symbol associated to the elliptic symbol $\tilde{\sigma}_{\theta=0}$ by the $[\cdot]_+$ regularization. At the South pole $\cos \theta = -1$. Therefore, in a small neighborhood of the South pole, the transversally elliptic symbol $\tilde{\sigma}$ coincides with the symbol associated to the elliptic symbol $\tilde{\sigma}_{\theta=\pi}$ by the $[\cdot]_-$ regularization.

Finally we obtain

$$\text{ind}'(D^{\text{vect}}_{10}) = [\text{ind}(d, d^-)|_{\theta=0}]_+ + [\text{ind}(d, d^+)|_{\theta=\pi}]_-. \quad (4.40)$$

One could probably also derive this result following the procedure in [74], where the index theorem for the Dirac operator was obtained using the deformation $\Gamma^\mu D_\mu \rightarrow \Gamma^\mu D_\mu + t \Gamma^\nu v_\mu$.

Let $z_1, z_2$ be complex coordinates in a small neighborhood of the South pole, such that the $U(1)$ action is $z_1 \rightarrow qz_1, z_2 \rightarrow qz_2$. With respect to this action the complexified self-dual complex is isomorphic to the Dolbeault $\bar{\partial}$-complex twisted by the bundle $O \oplus \Lambda^2 T_{1,0}^*$. Using the fact that the index of $\bar{\partial}$ operator is $(1-q^{-1})^{-2}$, we get

$$\text{ind}'(D^{\text{vect}}_{10}) = \left[-\frac{1+q^2}{(1-q)^2}\right]_+ + \left[-\frac{1+q^2}{(1-q)^2}\right]_-, \quad (4.41)$$

where $[f(q)]_\pm$ respectively means to take expansion of $f(q)$ in positive or negative powers of $q$. In our conventions $E_0$ corresponds to the middle term of the standard (anti)-self dual complex (4.35), therefore we get an extra minus sign.

Finally,

$$\text{ind}(D^{\text{vect}}_{10}) = 2 + \text{ind}'(D^{\text{vect}}_{10}) =$$

$$= 2 - (1 + q^2)(1 + 2q + 3q^2 + \ldots) - (1 + q^{-2})(1 + 2q^{-1} + 3q^{-2} + \ldots) =$$

$$= - \sum_{n=-\infty}^{\infty} |2n|q^n. \quad (4.42)$$

Note that in the computation of the index for the vector multiplet, the chirality of the complex coincides with the chirality of the $U(1)$ rotation near each of the fixed points.

Now we proceed to the hypermultiplet contribution to the index. The computation is similar to the vector multiplet. The transversally elliptic operator $D^{\text{hyper}}_{10} : \Gamma(E^{\text{hyper}}_0) \rightarrow \Gamma(E^{\text{hyper}}_1)$ can be trivialized everywhere over $T^*_G X$ except fixed points, where it is isomorphic to the self-dual complex at the North pole, or anti-self-dual complex at the South pole. For the hypermultiplet the chirality of the complex is opposite to the chirality of the $U(1)$ rotation near each of the fixed points. Then, using that the index of the twisted Dolbeault operator is $(1 + q^{-1})/((1-q)(1-q^{-1}))$, we get

$$\text{ind}_q(D^{\text{hyper}}_{10}) = \left[-\frac{2}{(1-q)(1-q^{-1})}\right]_+ + \left[-\frac{2}{(1-q)(1-q^{-1})}\right]_-, \quad (4.43)$$

which results in

$$\text{ind}_q(D^{\text{hyper}}_{10}) = + \sum_{n=-\infty}^{\infty} |2n|q^{-n}. \quad (4.44)$$
So far we considered the massless hypermultiplet. In this case its contribution to the index exactly cancels the vector multiplet. Hence, the determinant factor in the $\mathcal{N} = 4$ theory is trivial. This finishes the proof that the Erickson-Semenoff-Zarembo matrix model is exact in all orders of perturbation theory.

In the $\mathcal{N} = 2^*$ case the situation is more interesting. Now the hypermultiplet is massive. In the transformations (4.19) the action of $SU(2)^R_H$ generator $M_{ij}$. We normalize it as $M_{ij}M^{ij} = 4m^2$. The hypermultiplet fields transform in the spin-$\frac{1}{2}$ representation of $SU(2)_H$. Therefore, in the massive case the index is multiplied by the spin-$\frac{1}{2}$ character relative to the massless case: $\frac{1}{2}(e^{im} + e^{-im})$. Hence all $U(1)$-eigenspaces split into half-dimensional subspaces with eigenvalues shifted by $\pm m$.

Finally, all fields transform in the adjoint representation of gauge group. Making a constant gauge transformation we can assume that the generator $a_0$ is in the Cartan subalgebra of the Lie algebra $g$ of the gauge group. Then non-zero eigenvalues of $a_0$ in the adjoint representation are $\{\alpha \cdot a_0\}$, where $\alpha$ runs over all roots of $g$. Hence, combining all contributions to the index, we obtain for the $\mathcal{N} = 2^*$ theory

$$\left(\frac{\det K_{bos}}{\det K_{ferm}}\right)_{\mathcal{N}=2^*} = \prod_{\text{roots } \alpha} \prod_{n=-\infty}^{\infty} \left[ (\alpha \cdot a_0 + n\varepsilon + m)(\alpha \cdot a_0 + n\varepsilon - m) \right]^{n^2} \cdot \frac{(\alpha \cdot a_0 + n\varepsilon)^2}{(\alpha \cdot a_0 + n\varepsilon)^2}.$$  

Here we denote $\varepsilon = r^{-1}$. The term $n\varepsilon$ comes from a weight $n$ representation of the $U(1)$, the term $\alpha \cdot a_0$ is an eigenvalue of $a_0$ acting on the eigensubspace of the adjoint representation corresponding to root $\alpha$.

We argued before that to ensure convergence of the path integral the mass parameter and the scalar field $\Phi_0$ should be taken imaginary if we work with ten-dimensional Minkowski signature. The parameter $a_0$ is also imaginary since it is identified with the zero mode of $\Phi_0$. Let us denote $m = \text{im} E, a_0 = i a_E \equiv \text{ia}_E^R$. Then, recalling (4.21) we get

$$Z_{1\text{-loop}}^{\mathcal{N}=2^*}(ia_E) = \prod_{\text{roots } \alpha} \prod_{n=1}^{\infty} \left[ \frac{((\alpha \cdot a_E)^2 + \varepsilon^2 n^2)^2}{((\alpha \cdot a_E + mE)^2 + \varepsilon^2 n^2)((\alpha \cdot a_E - mE)^2 + \varepsilon^2 n^2)} \right]^{rac{1}{2}}.$$  

(4.45)

This product requires some regularization which we explain in a moment.

Recall the product formula for the Barnes $G$-function (see e.g. [42])

$$G(1+z) = (2\pi)^{z/2} e^{-(1+\gamma z^2)} e^{z(1+\gamma z^2)} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{n} e^{-\frac{z^2}{2n}},$$  

(4.46)

where $\gamma$ is the Euler constant. Then we introduce a function $H(z) = G(1+z)G(1-z)$ and obtain

$$H(z) = e^{-(1+\gamma)z^2} \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right)^n \prod_{n=1}^{\infty} e^{\frac{z^2}{n^2}}.$$  

(4.47)

Using this relation we obtain formally

$$Z_{1\text{-loop}}^{\mathcal{N}=2^*}(ia_E) = \exp \left( \frac{m^2}{\varepsilon^2} \left( 1 + \frac{1}{2} \gamma - \sum_{n=1}^{\infty} \frac{1}{n} \right) \right) \times \prod_{\text{roots } \alpha} \frac{H(ia \cdot a/E/\varepsilon)}{H((ia \cdot a + im E)/\varepsilon) H((ia \cdot a - im E)/\varepsilon)}^{1/2}.$$  

(4.48)
The first factor $\exp(\ldots)$ is divergent, but it does not depend on $a_E$. Therefore it cancels when we compute expectation value of the operators which localize to functions of $a_E$, such as the circular supersymmetric Wilson loop operator. Therefore we can remove this factor from the partition function. The resulting product of the $G$-functions is a well defined analytic function of $a_E$.

Our result is consistent with the renormalization properties of the gauge theory. To check that the $\beta$-function comes out right, we need asymptotic expansion of the $G$-function at large $z$

$$
\log G(1+z) = \frac{1}{12} - \log A + \frac{z}{2} \log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z - \frac{3}{4} z^2 + \sum_{k=1}^{\infty} \frac{B_{2k+2}}{4k(k+1)} z^{2k},
$$

(4.49)

where $A$ is a constant and $B_n$ are Bernoulli numbers. Then

$$
\frac{1}{2} \left(\log G(1+iz_E) + \log G(1-iz_E)\right) = \frac{1}{12} - \log A + \left(\frac{-iz_E}{2} - \frac{1}{12}\right) \log z + \frac{3}{4} z^2 + \ldots
$$

(4.50)

If we take a limit of very large mass of the hypermultiplet, we expect to get the minimal $\mathcal{N} = 2$ theory at the energy scales much lower than the mass of the hypermultiplet. At large $m$, we expand the denominator in (4.48), corresponding to the hypermultiplet contribution to $Z_{1\text{-loop}}$, and get

$$
Z_{1\text{-loop}}^{\text{hyper}} = \text{const}(m_E) + \left(\text{const} + \log \frac{m_E}{\varepsilon} \sum_{a} \frac{(\alpha \cdot a_E)^2}{\varepsilon^2}\right) + \mathcal{O}(\frac{1}{m^2}).
$$

(4.51)

The important dependence on $a_E$ can be combined with the classical Gaussian action in the matrix model

$$
\frac{4\pi^2 r^2}{g_Y^2 M} (a_E, a_E) \rightarrow \left(\frac{4\pi^2 r^2}{g_Y^2 M} - \frac{C_2}{\varepsilon^2} \log \frac{m_E}{\varepsilon}\right) (a_E, a_E),
$$

(4.52)

where $C_2$ denotes the proportionality constant of the second Casimir $\text{tr}_{A_d} T_a T_b = C_2 \delta_{a,b}$. We can write that as

$$
\frac{1}{g_Y^2 M} = \frac{1}{g_Y^2 M} - \frac{C_2}{4\pi^2} \log \frac{m_E}{\varepsilon}
$$

(4.53)

where $g_Y^2 M$ has a simple meaning of the renormalized coupling constant. In other words, the bare microscopical constant $g_Y^2 M$ is defined at the UV scale $m_E$ and higher (in that region it does not run because of restored $\mathcal{N} = 4$ supersymmetry). At scales less than $m_E$, the coupling constant runs by beta-function of pure $\mathcal{N} = 2$ theory. Recall that the one-loop beta function for a gauge theory with $N_f$ Dirac fermions and $N_s$ complex scalars in adjoint representation is

$$
\frac{\partial g(\mu)}{\partial \log \mu} = \beta(g) = -\frac{C_2 g^3}{(4\pi)^2} \left(\frac{11}{3} - \frac{4}{3} N_f - \frac{1}{3} N_c\right).
$$

(4.54)

Taking $N_f = N_s = 1$ for a pure $\mathcal{N} = 2$ theory we get precisely the relation (4.53), which says that $g_Y^2 M$ is the running coupling constant at the IR scale $\varepsilon = r^{-1}$, which is the lowest scale for the theory on $S^4$ of radius $r$. This is also the scale of the Wilson loop operator, since it is placed on the equator.

We can check that the resulting integral over $a_E$ is always convergent as long as the bare coupling constant $g_Y^2 M$ is positive, in other words as long as the original action is positive definite. First of all, the Barnes function $G(1+z)$ does not have
poles or zeroes on the imaginary contour \( \text{Re} z = 0 \) over which we integrate. To see that the integral also behaves nicely at infinity we use the asymptotic expansion (4.50).

In the pure \( \mathcal{N} = 2 \) case the leading term in the exponent comes from the numerator of \( Z_{1\text{-loop}} \) and is equal to \(-{1 \over 2} z_E^2 \log z_E \). This is a negative function which grows in absolute value faster than any other terms including the renormalized quadratic term (4.52) even if \( \tilde{g}_{YM}^2 \) formally becomes negative.

In the \( \mathcal{N} = 2^* \) case we need to take asymptotic expansion at large \( z_E \) of both the numerator and denominator of (4.48) to check convergence at infinity. The leading terms \( \alpha \cdot (a_E \cdot a_E) \) cancel, and the next order term is proportional to \( m^2_{\mathcal{E}} \log(\alpha \cdot a_E) \). This does not spoil the convergence insured by the Gaussian classical factor \( \exp(-{4 \pi^2 \over 2} r^2 g_{YM}^2(a_E, a_E)) \).

To summarize, in the pure \( \mathcal{N} = 2 \) theory we need to insert the factor

\[
Z_{1\text{-loop}}^{\mathcal{N}=2} = \prod_{\text{roots } \alpha} H \left( i\alpha \cdot a_E / \varepsilon \right),
\]

under the integral in the matrix model and to substitute \( g_{YM} \) by the renormalized coupling constant \( \tilde{g}_{YM} \) in the Gaussian classical action.

When we set \( m = 0 \) we get the \( \mathcal{N} = 4 \) theory. The numerator coming from the vector multiplet exactly cancels the denominator coming from the hypermultiplet in the formula (4.48) and we get

\[
Z_{1\text{-loop}}^{\mathcal{N}=4} = 1.
\]

We shall note that most of the above computations are generalized easily for the \( \mathcal{N} = 2 \) theory with a massless hypermultiplet taken in an arbitrary representation. Let us denote this representation by \( W \). Analogously to the case of the adjoint representation, one can get a formula

\[
Z_{1\text{-loop}}^{\mathcal{N}=2, W} (ia_E) = \prod_{\alpha \in \text{weights(Ad)}} H \left( i\alpha \cdot a_E / \varepsilon \right) \prod_{w \in \text{weights(W)}} H \left( iw \cdot a_E / \varepsilon \right),
\]

Strictly speaking, this formula is valid in the situations when the infinite product of weights for the vector multiplet and hypermultiplet is proportional to the product of Barnes \( G \)-functions with the same divergent factor. That happens for such representations \( W \) when \( \sum_{\alpha} (\alpha \cdot a)^2 = \sum_{w} (w \cdot a)^2 \) for any \( a \in g \). This is actually the condition of vanishing \( \beta \)-function for the \( \mathcal{N} = 2 \) theory with a hypermultiplet in representation \( W \). Therefore we claim that the formula (4.57) literally holds for all \( \mathcal{N} = 2 \) superconformal theories. In a general \( \mathcal{N} = 2 \) case, the one-loop determinant requires regularization similarly to what we did for the pure \( \mathcal{N} = 2 \) theory.

It would be interesting to combine the factor \( Z_{1\text{-loop}} \) with the partition function of instanton corrections \( |Z_{\text{inst}}|^2 \) in an arbitrary \( \mathcal{N} = 2 \) superconformal case, integrate over \( a_E \) and check predictions of the S-duality for these theories (see e.g.\([53, 75]\)).

4.5. Example. Before turning to the instanton corrections, let us give a simplest example of a non-trivial prediction of the formula (4.57), which perhaps can be checked using the traditional methods of the perturbation theory.

Take the \( \mathcal{N} = 2 \) theory with with the \( SU(2) \) gauge group and 4 hypermultiplets in the fundamental representation. We choose coordinate \( a \) on the Cartan subalgebra
of the real Lie algebra of the gauge group $SU(2)$, such that an element $a$ is represented by an anti-hermitian matrix $	ext{diag}(ia,-ia)$. Let the invariant bilinear form on the Lie algebra be minus the trace in the fundamental representation, and let the kinetic term of the Yang-Mills action be normalized as $\frac{1}{4 g_Y^2} \int d^4x \sqrt{g} (F_{\mu \nu} F^{\mu \nu})$. The weights $w$ in the spin-$j$ representation run from $-2j$ to $2j$. In the adjoint representation $(j = 1)$ we have $\{\alpha \cdot a\} = \{-2a, 0, 2a\}$. In the fundamental representation $(j = \frac{1}{2})$ we have $\{w \cdot a\} = \{a, -a\}$. We also have $(a, a) = 2a^2$. The matrix model for the expectation value of the Wilson loop in the spin-$j$ representation is
\[
\langle \text{tr}_j \text{Pexp}(\int A dx + i \Phi g ds) \rangle = Z^{-1} \int_{-\infty}^{\infty} da e^{-\frac{M^2}{4 \lambda} a^2} (2a)^2 \frac{H(2ia)H(-2ia)}{H(ia)H(-ia)} \sum_{m=-j}^{j} e^{4\pi ma},
\]
where $Z$ is a constant independent of the inserted Wilson loop operator. The extra factor $(2a)^2$ is the usual Vandermonde determinant appearing when we switch to the integral over the Cartan subalgebra from the integral over the whole Lie algebra. At the weak coupling $g_Y \to 0$ we can evaluate this integral as a series in $g_Y$. For the Barnes G-function we use Taylor series expansion at small $z$
\[
\log G(1 + z) = \frac{1}{2} \log(2\pi) - (1 + \gamma) \frac{z^2}{2} + \sum_{n=3}^{\infty} \frac{(-1)^{n-1} \zeta(n-1) z^n}{n}.
\]
After some algebra one gets the following perturbative result for the expectation value of $e^{2\pi na}$ in the matrix model (we write here $g = g_Y$)
\[
\langle e^{2\pi na} \rangle = 1 + \frac{3}{2} x n^2 g^2 + \frac{5}{8} n^4 g^4 + \frac{7}{48} \cdot \frac{1}{26} n^6 g^6 + \frac{35}{2^6 (4\pi)^2} t_2 n^2 g^6 + O(g^8),
\]
where $t_2$ is the coefficient coming from the expansion of the Barnes G-function. It is expressed in terms of Riemann zeta-function
\[
t_2 = -12\zeta(3).
\]
To get this result we expanded the determinant factor in powers of $a$: \[
\log \left( \frac{H(2ia)H(-2ia)}{H(ia)H(-ia)} \right) = -8 \sum_{k=2}^{\infty} \sum_{k=2}^{\infty} \frac{\zeta(2k-1)}{k} (2^{2k-2} - 1)(-1)^k a^{2k} = \sum_{k=2}^{\infty} t_k a^{2k}.
\]
Then for a Gaussian measure $\int da e^{-\frac{1}{2\sigma^2} a^2}$ with $\sigma^2 = \frac{9}{16\pi^2}$ we have
\[
\langle e^{a^2} \text{exp} \left( \sum_{k=2}^{\infty} t_k a^{2k} \right) \text{e}^{q a} \rangle_{\text{gauss}} = \left( \frac{\partial}{\partial q} \right)^2 \text{exp} \left( \sum_{k=2}^{\infty} t_k \left( \frac{\partial}{\partial q} \right)^2 \right) e^{\frac{1}{2} q^2 \sigma^2}.
\]
The perturbative result for the $N = 4$ $SU(2)$ theory is given by the same formula but with $t_k = 0$:
\[
\langle e^{a^2} \rangle_{N=4} = (1 + \sigma^2 q^2)^{1/2} e^{(1/2) \sigma^2 q^2} + \frac{1}{2} (\sigma q)^2 + \frac{5}{8} (\sigma q)^4 + \frac{7}{48} (\sigma q)^6 + O((\sigma q)^8).
\]
Taking $g = 2\pi n$ and $\sigma = \frac{2\sqrt{\pi}}{2\pi n}$ we get the result (4.58) for the $N = 4$ theory with $t_2 = 0$. For a superconformal $N = 2$ theory the Gaussian matrix model action is corrected by the terms $t_k a^{2k}$. The first correction is quartic $t_2 a^2$, and at the lowest order it gives the result (4.58) for the $SU(2)$ theory with 4 hypermultiplets in the fundamental representation.

The first difference for $\langle W_R(C) \rangle$ between the $N = 2$ $SU(2)$ gauge theory with 4 fundamental hypermultiplets and the $N = 4$ $SU(2)$ gauge theory appears at the
order $g_Y^6$. This is the order of the two-loop level Feynman diagram computations which have been done in the gauge theory for the $N = 4$ case [40,41].

In the matrix model it is very easy to get the higher terms in the expansion over $g_Y$. On the other hand, the complexity of the Feynman diagram computations done directly in the gauge theory grows enormously with the number of loops.

Now we will argue that we can improve the matrix model by taking into account all instanton corrections of the theory, so that the result becomes non-perturbatively exact.

5. Instanton corrections

When we argued by (3.17) that the theory localizes to the trivial gauge field configurations, we used the fact that $d_A w_{\mu\nu}$ does not vanish everywhere except at the North and the South poles and we assumed smooth gauge fields. Dropping the smoothness condition, we can only say that the gauge field strength can be supported at the poles and still be consistent with vanishing $tQV$-term. From (3.16) we see that $F^+$ might be non zero at the North pole, where $\sin^2 \theta$ vanish, while $F^-$ might be non zero at the South pole, where $\cos^2 \theta$ vanish. Thus, if we allow non-smooth gauge fields in the path integral, we should count configurations with point anti-instantons ($F^- = 0$) localized at the North pole, and point instantons ($F^+ = 0$) localized at the South pole. The $Q$-complex on $S^4$ in our problem in a neighborhood of the South/the North pole coincides with the $Q$-complex of the topological ($F^+ = 0$)/anti-topological ($F^- = 0$) gauge theory on $\mathbb{R}^4$ in the $\Omega$-background studied by Nekrasov [4]. There the moduli space of solutions to $F^+ = 0$ modulo gauge transformations was taken equivariantly under the $U(1)^2$ action on $\mathbb{R}^4 \simeq \mathbb{C}^2$ by $z_1 \rightarrow e^{i\epsilon_1} z_1, z_2 \rightarrow e^{i\epsilon_2} z_2$, and gauge transformations at infinity with generator $a \in \mathfrak{g}$. Making the correspondence between the theory on $S^4$ in a local neighborhood of the North pole and the theory on $\mathbb{R}^4$ we should take $\epsilon_1 = \epsilon_2 = r^{-1}$, since for the problem on $S^4$, the chirality of the equations at the North pole coincides with the chirality of the generator of the Lorentz rotations $d_{[\mu} v_{\nu]}$. The same applies to the South pole: the chirality of the equations is reversed as well as the chirality of the generator of the Lorentz rotations.

In this section we consider only the case of the $U(N)$ gauge group. We use the following conventions. The solutions of the equations $F^+ = 0$ are called instantons. The solutions of the equations $F^- = 0$ are called anti-instantons.

We define the instanton charge as the second Chern class\footnote{For $U(N)$ bundles we have the total Chern class $c = \det(1+\frac{1}{2}F) = \prod (1+x_i) = c_0 + c_1 + \ldots$, where $F$ is the curvature which takes value in the Lie algebra of the gauge group, $x_i$ are the Chern roots, and $c_k$ is polynomial of degree $k$ in $x_i$. We have $c_2 = \sum_{i<j} x_i x_j = \frac{1}{2} \left( \sum x_i \right)^2 - \frac{1}{2} \sum x_i^2$. If $c_1 = \sum x_i$ vanishes, we get $c_2 = -\frac{1}{2} \int tr \left( \frac{1}{2} F \wedge F \right) = \frac{1}{2} \int tr F \wedge F = -\frac{1}{8\pi^2} \int tr \left( F \wedge F \right)$.}:

$$k = c_2 = -\frac{1}{8\pi^2} \int F \wedge F,$$

and modify the action by the $\theta$-term

$$S_{YM} \rightarrow S_{YM} + \frac{i\theta}{8\pi^2} \int F \wedge F,$$

$$|w_{\mu\nu}| \prod (1+ x_i) \prod (1- x_i),$$

$$c_1 = \sum x_i = 0.$$

This is the order of the two-loop level Feynman diagram computations which have been done in the gauge theory for the $N = 4$ case [40,41].

In the matrix model it is very easy to get the higher terms in the expansion over $g_Y$. On the other hand, the complexity of the Feynman diagram computations done directly in the gauge theory grows enormously with the number of loops.

Now we will argue that we can improve the matrix model by taking into account all instanton corrections of the theory, so that the result becomes non-perturbatively exact.

5. Instanton corrections

When we argued by (3.17) that the theory localizes to the trivial gauge field configurations, we used the fact that $d_A w_{\mu\nu}$ does not vanish everywhere except at the North and the South poles and we assumed smooth gauge fields. Dropping the smoothness condition, we can only say that the gauge field strength can be supported at the poles and still be consistent with vanishing $tQV$-term. From (3.16) we see that $F^+$ might be non zero at the North pole, where $\sin^2 \theta$ vanish, while $F^-$ might be non zero at the South pole, where $\cos^2 \theta$ vanish. Thus, if we allow non-smooth gauge fields in the path integral, we should count configurations with point anti-instantons ($F^- = 0$) localized at the North pole, and point instantons ($F^+ = 0$) localized at the South pole. The $Q$-complex on $S^4$ in our problem in a neighborhood of the South/the North pole coincides with the $Q$-complex of the topological ($F^+ = 0$)/anti-topological ($F^- = 0$) gauge theory on $\mathbb{R}^4$ in the $\Omega$-background studied by Nekrasov [4]. There the moduli space of solutions to $F^+ = 0$ modulo gauge transformations was taken equivariantly under the $U(1)^2$ action on $\mathbb{R}^4 \simeq \mathbb{C}^2$ by $z_1 \rightarrow e^{i\epsilon_1} z_1, z_2 \rightarrow e^{i\epsilon_2} z_2$, and gauge transformations at infinity with generator $a \in \mathfrak{g}$. Making the correspondence between the theory on $S^4$ in a local neighborhood of the North pole and the theory on $\mathbb{R}^4$ we should take $\epsilon_1 = \epsilon_2 = r^{-1}$, since for the problem on $S^4$, the chirality of the equations at the North pole coincides with the chirality of the generator of the Lorentz rotations $d_{[\mu} v_{\nu]}$. The same applies to the South pole: the chirality of the equations is reversed as well as the chirality of the generator of the Lorentz rotations.

In this section we consider only the case of the $U(N)$ gauge group. We use the following conventions. The solutions of the equations $F^+ = 0$ are called instantons. The solutions of the equations $F^- = 0$ are called anti-instantons.

We define the instanton charge as the second Chern class:

$$k = c_2 = -\frac{1}{8\pi^2} \int F \wedge F,$$

and modify the action by the $\theta$-term

$$S_{YM} \rightarrow S_{YM} + \frac{i\theta}{8\pi^2} \int F \wedge F.$$
At $F^+ = 0$ we have $\sqrt{g}F_{\mu\nu}F^{\mu\nu}d^4x = 2F \wedge *F = -2F \wedge F$. Then the Yang-Mills action of instanton of charge $k$ is

$$S_{YM}(k) = \frac{1}{4g_{YM}^2} \int \sqrt{g}d^4xF_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{8\pi^2} \int F \wedge F = \left( \frac{4\pi^2}{g_{YM}^2} - i\theta \right) k.$$ 

Its contribution to the partition function is proportional to

$$\exp(-S_{YM}(k)) = \exp(2\pi i k) = q^k,$$

where we introduced the complexified coupling constant

$$\tau = \frac{2\pi i}{g_{YM}^2} + \frac{\theta}{2\pi},$$

and the expansion parameter

$$q = \exp(2\pi i \tau).$$

(The expansion parameter $q$ in this section should not be confused with the formal generator of the $U(1)$ group used to compute the index of the transversally elliptic operator in the previous section).

Near the South pole the theory on $S^4$ looks like topological theory with the equations $F^+ = 0$, so that only point instantons contribute. Near the North pole the situation is opposite: the equations are replaced by $F^- = 0$, therefore we need to count anti-instantons. The generating function of anti-instantons is the same as the generating function of instantons with replacement of the expansion parameter $q$ by its complex conjugate $\bar{q}$.

For the $U(N)$ gauge group the explicit formula for the equivariant instanton partition function on $\mathbb{R}^4$ is [4, 74, 79]

$$Z_{\text{inst}}^{N=2}(\varepsilon_1, \varepsilon_2, \alpha) = \sum_{Y} \frac{q^{\text{Y}}}{\prod_{\alpha, \beta = 1}^{N} n_{\alpha, \beta}^Y(\varepsilon_1, \varepsilon_2, \vec{a})}, \quad (5.1)$$

where we sum over an ordered set of $N$ Young diagrams $\{Y_\alpha\}$ with $\alpha = 1 \ldots N$. By $|Y|$ we denote the total size of all diagrams in a set $|Y| = \sum |Y_\alpha|$. The total size is equal to the instanton number. The factor $n_{\alpha, \beta}^Y(\varepsilon_1, \varepsilon_2, \vec{a})$ denotes the equivariant Euler class of the tangent space to the instanton moduli space at the fixed point labeled by $Y$. It is given by

$$n_{\alpha, \beta}^Y(\varepsilon_1, \varepsilon_2, \vec{a}) = \prod_{s \in Y_\alpha} (-h_{Y_\beta}(s)\varepsilon_1 + (v_{Y_\alpha}(s) + 1)\varepsilon_2 + a_\beta - a_\alpha) \times$$

$$\times \prod_{t \in Y_\beta} ((h_{Y_\alpha}(t) + 1)\varepsilon_1 - v_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha). \quad (5.2)$$

(We assume that an element $a$ in the Cartan subalgebra of $\mathfrak{u}(N)$ is represented by a diagonal matrix $(ia_1, \ldots, ia_N)$.) Here $s$ and $t$ run over squares of Young diagrams $Y_\alpha$ and $Y_\beta$. Let $Y$ be a Young diagram $\nu_1 \geq \nu_2 \cdots \geq \nu_\ell$, where $\nu_i$ is the length of the $i$-th column, $\nu'_j$ is the length of the $j$-th row. If a square $s = (i, j)$ is located at the $i$-th column and the $j$-th row then $v_Y(s) = \nu_i(Y) - j$ and $h_Y(s) = \nu'_j(Y) - i$. In other words, $v_Y(s)$ and $h_Y(s)$ is respectively the vertical and horizontal distance from the square $s$ to the edge of the diagram $Y$. We can rewrite the product in the
denominator of (5.1) as
\[ \prod_{\alpha,\beta=1}^{N} n_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \bar{d}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_s} E_{\alpha\beta}(s)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s)), \tag{5.3} \]
where
\[ E_{\alpha\beta}(s) = (-h_{\alpha\beta}(s)\varepsilon_1 + (v_{\alpha\beta}(s) + 1)\varepsilon_2 + a_\beta - a_\alpha). \tag{5.4} \]
We will give a few simplest examples of evaluation of this formula. First consider \( U(1) \) case. Then we sum over all Young diagrams of one color. At one instanton level \( k = 1 \), there is only one diagram \( Y = (1) \). Then \( E_{11} = \varepsilon_2 \), so that
\[ Z_{k=1}^{N=2}(\varepsilon_1, \varepsilon_2, a) = \frac{1}{\varepsilon_2\varepsilon_1}. \tag{5.5} \]
At two instanton level \( k = 2 \), there are two diagrams \( Y = (2,0) \) and \( Y = (1,1) \). Their contribution is
\[ Z_{k=2}^{N=2}(\varepsilon_1, \varepsilon_2, a_1) = \frac{1}{(2\varepsilon_2)(\varepsilon_1 - \varepsilon_2)(\varepsilon_2)(\varepsilon_1)} + \frac{1}{(-\varepsilon_1 + \varepsilon_2)(2\varepsilon_1)(\varepsilon_2)(\varepsilon_1)} = \frac{1}{2(\varepsilon_1\varepsilon_2)^2} \tag{5.6} \]
At three instanton level \( k = 3 \), there are three diagrams \( Y = (3,0) \), \( Y = (2,1) \) and \( Y = (1,1) \). Their contribution is
\[ Z_{k=3}^{N=2}(a, \varepsilon_1, \varepsilon_2) = \frac{1}{(\varepsilon_2)(\varepsilon_1)(2\varepsilon_2)(\varepsilon_1 - \varepsilon_2)(3\varepsilon_2)(\varepsilon_1 - 2\varepsilon_2) + (\varepsilon_2)(\varepsilon_1)(2\varepsilon_2 - \varepsilon_1)(2\varepsilon_1 - \varepsilon_2)(\varepsilon_2)(\varepsilon_1)} = \frac{1}{6(\varepsilon_1\varepsilon_2)^3} \tag{5.7} \]
At an arbitrary instanton level \( k \), the sum of all Young diagrams of order \( k \) simplifies to
\[ Z_{k=1}^{N=2}(\varepsilon_1, \varepsilon_2, a) = \frac{1}{k!(\varepsilon_1\varepsilon_2)^k}, \tag{5.8} \]
hence
\[ Z_{U(1)}^{N=2}(\varepsilon_1, \varepsilon_2, a) = \sum_{k=1}^{\infty} \frac{q^k}{k!(\varepsilon_1\varepsilon_2)^k} = \exp \left( \frac{q}{\varepsilon_1\varepsilon_2} \right). \tag{5.9} \]
Now we consider a few instantons for the \( U(2) \) gauge group. At one instanton there are two colored Young diagrams \((1,0)\) and \((0,1)\) contributing
\[ Z_{k=1}^{N=2}(\varepsilon_1, \varepsilon_2, a_1, a_2) = \frac{1}{\varepsilon_1\varepsilon_2(a_2 - a_1 + \varepsilon_1 + \varepsilon_2)(a_1 - a_2)} + \frac{1}{(a_1 - a_2 + \varepsilon_1 + \varepsilon_2)(a_2 - a_1)\varepsilon_1\varepsilon_2} = \frac{2}{\varepsilon_1\varepsilon_2((\varepsilon_1 + \varepsilon_2)^2 - a^2)}, \tag{5.10} \]
where we denoted \( a = a_2 - a_1 \). As the instanton number grows, its contribution becomes more and more complicated rational function of \( a \). For example, at \( k = 2 \) we get (we set \( a = ia_E \), where \( a_E \) is real)
\[ Z_{k=2}^{N=2}(\varepsilon_1, \varepsilon_2, ia_E) = \frac{(2a_E^2 + 8\varepsilon_2^2 + 8\varepsilon_1^2 + 17\varepsilon_1\varepsilon_2)}{((\varepsilon_1 + 2\varepsilon_2)^2 + a_E^2)((2\varepsilon_1 + \varepsilon_2)^2 + a_E^2)((\varepsilon_1 + \varepsilon_2)^2 + a_E^2)(\varepsilon_1\varepsilon_2)^2 + a_E^4)\varepsilon_1\varepsilon_2}. \tag{5.11} \]
Generally, instanton contributions are certain rational functions of $a_i$ and $\varepsilon_i$. Contrary to the case $\varepsilon_1 = -\varepsilon_2 = h$, which is often taken in the literature to simplify the instanton partition function \[4, 5, 77\], in our problem we get the same signs: $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Looking at the examples above, one can note an important property of the instanton contributions at $\varepsilon_1 = \varepsilon_2$; they do not have poles at the integration contour for $a_i$. Recall that in the matrix integral we integrate over imaginary $a = iaE$, while $\varepsilon_1$ and $\varepsilon_2$ is real. Generally, the denominator contains factors $n_1\varepsilon_1 + n_2\varepsilon_2 + a$, where $n_1$ and $n_2$ are some numbers. There is a pole at the integration contour only if $n_1\varepsilon_1 + n_2\varepsilon_2 = 0$. Though it happens regularly at $\varepsilon_1 = -\varepsilon_2$, it never happens at $\varepsilon_1 = \varepsilon_2$. (This fact was checked explicitly up to $k = 5$ instantons for $U(2)$ gauge group and actually one can show it in general\[5\].) Therefore the integrand in (1.3) is a smooth function everywhere at the integration domain and it also decreases rapidly at infinity. Thus the integral is convergent and well defined.

In the $\mathcal{N} = 2^*$ case, each instanton contribution is multiplied by a new factor. This factor is equal to the product of the same weights as in the denominator, but shifted by the hypermultiplet mass. From \[4, 5, 77\] we get that the instanton partition function is given by

$$Z_{\text{inst}}^{\mathcal{N}=2^*} = \sum_\mathcal{Y} \prod_{e=1}^N \prod_{\alpha, \beta} \frac{(E_{\alpha\beta}(s) - \tilde{m})(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s) - \tilde{m})}{E_{\alpha\beta}(s)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s))},$$

where $\tilde{m}$ is the equivariant parameter used to introduce hypermultiplet mass in the Nekrasov’s theory in the $\Omega$-background \[4\] related to the mass of hypermultiplet used in the present work as $\tilde{m} = m + (\varepsilon_1 + \varepsilon_2)/2$ (see \[80\] for details). For example,

$$Z_{\text{inst}}^{\mathcal{N}=2^*} = \sum_\mathcal{Y} \prod_{e=1}^N \prod_{\alpha, \beta} \frac{(E_{\alpha\beta}(s) - \tilde{m})(\varepsilon_1 + \varepsilon_2 - \tilde{m})(a_2 - a_1 + \varepsilon_1 + \varepsilon_2 - \tilde{m})(a_1 - a_2 - \tilde{m})}{E_{\alpha\beta}(s)(\varepsilon_1 + \varepsilon_2 - E_{\alpha\beta}(s))(a_2 - a_1 + \varepsilon_1 + \varepsilon_2)(a_1 - a_2)} +$$

$$\frac{(a_1 - a_2 + \varepsilon_1 + \varepsilon_2 - \tilde{m})(a_2 - a_1 - \tilde{m})(\varepsilon_1 - \tilde{m})(\varepsilon_2 - \tilde{m})}{(a_1 - a_2 + \varepsilon_1 + \varepsilon_2)(a_2 - a_1)(\varepsilon_1 \varepsilon_2)} =$$

$$\frac{2(\tilde{m} - \varepsilon_2)(\tilde{m} - \varepsilon_1)(\tilde{m}^2 - a_2 - m)(\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 + \varepsilon_2)^2}{(\varepsilon_1 + \varepsilon_2^2 - a^2)\varepsilon_1 \varepsilon_2}.$$  (5.13)

The integrand is still a smooth function on the whole integration domain and decreases sufficiently fast at infinity.

Hence, we conclude that the matrix integral, with all instanton corrections included, is well defined in the $\mathcal{N} = 2$, the $\mathcal{N} = 2^*$ and the $\mathcal{N} = 4$ cases, and that it gives the exact partition function of these theories on $S^4$. The expectation value of a supersymmetric circular Wilson operator on $S^4$ in an arbitrary representation is equal to the expectation value of the operator $\text{tr}_R e^{2\pi i ra}$ in this matrix model.

For a generic $m$, the one-loop determinant factor $Z_{\text{inst}}^{\text{loop}}$ and the instanton factor $Z_{\text{inst}}$ are nontrivial. However, for $m = 0$, when $\mathcal{N} = 4$ symmetry, is recovered $Z_{\text{inst}}^{\text{loop}} = 1$ as well as $Z_{\text{inst}} = 1$ \[80\]. We conclude that in the $\mathcal{N} = 4$ theory there are no instanton corrections, and the the Gaussian matrix model conjecture \[12\] is exact.

\textsuperscript{7}The author thanks H. Nakajima for a discussion.

\textsuperscript{8}The correction $\tilde{m} = m + (\varepsilon_1 + \varepsilon_2)/2$ appeared in the v2 of this preprint, while in v1 it was erroneously assumed that $\tilde{m} = m$. The author thanks Takuya Okuda for pointing out this issue.
Another point in the parameter space of the $\mathcal{N} = 2^*$ theory, the point $\tilde{m} = 0$ is also interesting. It is easy to evaluate $Z_{1\text{-loop}}^\tilde{m}=0$ and $Z_{\text{inst}}^\tilde{m}=0$. The numerator and denominator cancel each other in each of the fixed point instanton contribution to $Z_{\text{inst}}$, hence in the $U(N)$ theory

$$Z_{U(N),\text{inst}}^{\tilde{m}=0} = \sum_{\mathcal{Y}} q^{\mathcal{Y}} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^N}$$

is the generating function for the number of $N$-colored partitions.

Using the definition of the Dedekind eta-function $\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1-q^k)$ we can write

$$Z_{U(N),\text{inst}}^{\tilde{m}=0} = \left(\frac{1}{q^{-1/24}\eta(\tau)}\right)^N.$$  

(5.15)

At $\tilde{m} = 0$ most of the factors in the infinite product (5.15) cancel each other, and we are left with

$$Z_{U(N),1\text{-loop}}^{\tilde{m}=0}(i a_E) = \prod_{\text{roots } \alpha} \frac{1}{|\alpha \cdot a_E|}$$

(5.16)

We see that the 1-loop contribution at $\tilde{m} = 0$ gives exactly the inverse Weyl measure in the reduction of the integral over $\mathfrak{g}$ to the Cartan algebra of $\mathfrak{g}$. Therefore, the total partition function at $\tilde{m} = 0$ for $U(N)$ theory is given by (we set $\varepsilon = \frac{i}{\tau} = 1$)

$$Z_{U(N)}^{\tilde{m}=0} = |Z_{U(N),\text{inst}}^{\tilde{m}=0}|^2 \frac{4\pi^2}{s_Y^2} a_E^2 = \left(\frac{1}{(qq)^{-1/24}\eta(\tau)\eta(\tau)\sqrt{2\tau_2}}\right)^N$$

(5.17)

This function does not transform well under $S$-duality $\tau \to -1/\tau$. However, it is possible to add to the theory $c$-number gravitational curvature terms which shift the action by a constant $58$, for example we can add the following $R^2$-term:

$$S_{YM} \to S_{YM} - 2\pi\tau_2 \frac{N}{24} \frac{1}{32\pi^2} \int_{S^4} R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}.$$  

(5.18)

Such $R^2$ terms generally appear as gravitational corrections to an effective action on a brane in string theory $58$. This $R^2$ term cancels the extra factor $q^{-1/24}$ in the partition function, and after such correction we get

$$Z_{U(N), R^2 \text{ background}}^{\tilde{m}=0} = \left(\frac{1}{\eta(\tau)\eta(\tau)\sqrt{2\tau_2}}\right)^N.$$  

(5.19)

So far we discussed instanton corrections only to the partition function. Now we consider corrections to the Wilson loop operator. One can show that the Wilson loop $W(C)$ which we consider is in the same $\delta_\varepsilon$ cohomology class as the operator $\text{tr}_R \exp(\frac{2\pi i}{\tau} \Phi)$ inserted at the North pole, where $\Phi = i\Phi_0 + \Phi_\eta$. Instanton corrections to the operator $\exp(\beta \Phi)$ in the $\mathcal{N} = 2$ equivariant theory on $\mathbb{R}^4$ for a given asymptotic of $\Phi$ at infinity were computed in $53, 70, 80, 82$. Using these results, one can actually see that if $\beta = \frac{2\pi m}{n}$ where $n$ is integer, there are no instanton corrections to the operator $\text{tr}_R \exp(\beta \Phi)$. In other words, the operator $\text{tr}_R \exp(2\pi i r a)$ in the field theory is replaced simply by the operator $\text{tr}_R \exp(2\pi i r a)$ in the matrix model.

This is exactly the case of Wilson loop operator which we consider. In other words, even after taking into account the instanton corrections, we still conclude that the Wilson loop operator $W(C)$ corresponds to the operator $\text{tr}_R \exp(2\pi i r a)$ in the matrix model. However, the expectation value of $W(C)$ in a generic $\mathcal{N} = ...
2 theory receives corrections because the measure in the matrix integral (1.3) is corrected by the insertion of the instanton factor $|Z_{\text{inst}}(ia, \varepsilon, \varepsilon)|^2$.

**Appendix A. Clifford algebra**

We use the following conventions to denote symmetrized and antisymmetrized tensors:

\[
a_{[i}b_{j]} = \frac{1}{2}(a_i b_j - a_j b_i)
\]

\[
a_{\{i}b_{j\}} = \frac{1}{2}(a_i b_j + a_j b_i),
\]

where $a$ and $b$ are any indexed variables.

Let us summarize here our conventions on gamma-matrices in ten dimensions. We start with Minkowski metric $ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_9^2$. Capital letters from the middle of the Latin alphabet normally are used to denote ten-dimensional space-time indices $M, N, P, Q = 0, \ldots, 9$. Let $\gamma^M$ for $M = 0, \ldots, 9$ be $32 \times 32$ matrices representing the Clifford algebra $\text{Cl}(9, 1)$. They satisfy the standard anticommutation relations

\[
\gamma^{\{M} \gamma^{N\}} = g^{MN},
\]

where $g^{MN}$ is the metric. The corresponding representation of $\text{Spin}(9, 1)$ has rank 32 and can be decomposed into irreducible spin representations $S^+$ and $S^-$ of rank 16. The chirality operator

\[
\gamma^{11} = \gamma^1 \gamma^2 \ldots \gamma^9 \gamma^0
\]

acts on $S^+$ and $S^-$ as multiplication by 1 and $-1$, respectively. The gamma-matrices $\Gamma^M$ reverse chirality, so $\Gamma^M : S^\pm \rightarrow S^\mp$. We can write $\gamma^M$ in the block form

\[
\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix},
\]

assuming that we write the rank 32 spin representation of $\text{Spin}(9, 1)$ as

\[
\begin{pmatrix} S^+ \\ S^- \end{pmatrix}.
\]

Let $\Gamma^M$ and $\tilde{\Gamma}^M$ be the chiral “half” gamma-matrices appearing in (A.3). Then

\[
\Gamma^{\{M} \Gamma^{N\}} = g^{MN}, \quad \tilde{\Gamma}^{\{M} \tilde{\Gamma}^{N\}} = g^{MN}.
\]

We define $\gamma^{MN}, \Gamma^{MN}$ and $\tilde{\Gamma}^{MN}$ as follows

\[
\gamma^{MN} = \gamma^{[M} \gamma^{N]} = \begin{pmatrix} \tilde{\Gamma}^{[M} \Gamma^{N]} & 0 \\ 0 & \Gamma^{[M} \tilde{\Gamma}^{N]} \end{pmatrix} = : \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & \tilde{\Gamma}^{MN} \end{pmatrix} :.
\]

Using anticommutation relations we get

\[
\Gamma^M \Gamma^{PQ} = 4g^{M[P} \Gamma^{Q]} + \tilde{\Gamma}^{PQ} \Gamma^M.
\]

For computations in the four-dimensional theory, we will often need to split the ten-dimensional space-time indices into two groups. The first group contains four-dimensional space-time indices in the range $1, \ldots, 4$, which we denote by Greek latter in the middle of the alphabet $\mu, \nu, \lambda, \rho$. The second group contains the indices for the normal directions, running over $5, \ldots, 9, 0$, which we denote by capital letters.
from the beginning of the Latin alphabet $A, B, C, D$. As usual, the repeated index means summation over it. Then we have the following identities

$$
\Gamma_\mu A \tilde{\Gamma}_A = -4 \tilde{\Gamma}_A \\
\Gamma^\mu \Gamma_\nu \tilde{\Gamma}_\mu = 0 \\
\Gamma^\mu \Gamma_\nu A \tilde{\Gamma}_\mu = 2 \tilde{\Gamma}_A \\
\Gamma^\mu \Gamma_{AB} \tilde{\Gamma}_A = 4 \tilde{\Gamma}_{AB}
$$

(A.8)

We choose matrices $\Gamma_M$ and $\tilde{\Gamma}_M$ to be symmetric:

$$(\Gamma_M)^T = \Gamma_M, \quad (\tilde{\Gamma}_M)^T = \tilde{\Gamma}_M.$$

Then we get $(\Gamma^{MN})^T = -\tilde{\Gamma}^{MN}$, so the representations $S^+$ and $S^-$ are dual to each other.

There is a very important “triality identity” which appears in the computations involving ten-dimensional supersymmetry:

$$
(\Gamma_M)_{\alpha_1}^{(\alpha_2} (\tilde{\Gamma}_M)_{\alpha_3}^{\alpha_4)} = 0,
$$

(A.9)

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, \ldots, 16$ are the matrix indices of $\Gamma_M$.

All gamma-matrices relations above are valid both for Minkowski and Euclidean signature.

The difference between gamma-matrices for Minkowski and Euclidean signature is the following. In Minkowski signature we choose $\Gamma_M$ to be real. In Euclidean signature we use the following matrices

$\{i\Gamma_0, \Gamma_1, \ldots, \Gamma_9\}$.

Therefore all Euclidean gamma-matrices are real except $\Gamma_0$, which is imaginary. In Euclidean signature the representation $S^+$ and $S^-$ are unitary. Since in Euclidean signature they are also dual to each other, we conclude that in Euclidean signature $S^+$ and $S^-$ are complex conjugate representations.

It is convenient to use octonions to explicitly write down $\Gamma^M$. In Minkowski signature we choose

$$
\Gamma^i = \begin{pmatrix} 0 & E_i^T \\ E_i & 0 \end{pmatrix}, \quad i = 1 \ldots 7 \\
\Gamma^9 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix}, \\
\Gamma^0 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & 1_{8 \times 8} \end{pmatrix},
$$

(A.10)

where $E_i$ for $i = 1 \ldots 8$ are $8 \times 8$ matrices representing left multiplication of the octonions.

Let $e_i$ with $i = 1 \ldots 8$ be the generators of the octonion algebra with the octonionic structure constants $c^k_{ij}$ defined by the multiplication table $e_i \cdot e_j = c^k_{ij} e_k$. Then $(E_i)^k_j = c^k_{ij}$. The element $e_1$ is the identity. To be concrete, we define the multiplication table by specifying the triples which have cyclic multiplication table: $(234), (256), (357), (458), (836), (647), (728)$ (e.g. $e_2 e_3 = e_4$, etc.). Then one can check that $E_i$ have the following form

$$
E_\mu = \begin{pmatrix} J_\mu & 0 \\ 0 & J_\mu \end{pmatrix}, \quad \mu = 1 \ldots 4 \\
E_A = \begin{pmatrix} 0 & -J_A^T \\ J_A & 0 \end{pmatrix}, \quad A = 5 \ldots 8,
$$

(A.11)
where $J_\mu$ for $\mu = 1 \ldots 4$ are the $4 \times 4$ matrices representing generators of quaternion algebra by the left action, while $\bar{J}_\mu$ are the $4 \times 4$ matrices representing generators of quaternion algebra by the right action. Concretely we obtain

\[(J_1, J_2, J_3, J_4) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \left( \begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \right),
\]

with the relations

\[J_i J_j = \varepsilon_{ijk} J_k, \quad i, j, k = 2 \ldots 4,
\]

and

\[(J_5, J_6, J_7, J_8) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \left( \begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \right),
\]

with the relations

\[\bar{J}_i \bar{J}_j = -\varepsilon_{ijk} \bar{J}_k, \quad i, j, k = 2 \ldots 4.
\]

Similarly,

\[(J_5, J_6, J_7, J_8) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \left( \begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array} \right), \left( \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \right),
\]

We choose orientation in the $(1 \ldots 4)$-plane and the $(5 \ldots 8)$-plane by saying that $1234$ and $5678$ are the the positive cycles.

Then the matrices $\Gamma^{\mu\nu}$ for $\mu, \nu = 1 \ldots 4$ and $\Gamma^{ij}$ for $i, j = 5 \ldots 8$ have the following block decomposition:

\[
\Gamma^{\mu\nu} = \begin{pmatrix} E^T_{[\mu} E_{\nu]} & 0 \\ 0 & E_{[\mu} E^T_{\nu]} \end{pmatrix} = \begin{pmatrix} J^{\mu\nu} & 0 & 0 & 0 \\ 0 & J^{\mu\nu}_+ & 0 & 0 \\ 0 & 0 & -J^{\mu\nu}_- & 0 \\ 0 & 0 & 0 & -\bar{J}^{\mu\nu}_- \end{pmatrix},
\]

\[
\Gamma^{ij} = \begin{pmatrix} E^T_{[i} E_{j]} & 0 \\ 0 & E_{[i} E^T_{j]} \end{pmatrix} = \begin{pmatrix} -J^{ij}_- & 0 & 0 & 0 \\ 0 & -J^{ij}_- & 0 & 0 \\ 0 & 0 & -J^{ij}_- & 0 \\ 0 & 0 & 0 & -J^{ij}_+ \end{pmatrix},
\]

where the ±-superscript denotes the self-dual and anti-self-dual tensors; $J_{12} = J_1^T J_2 = J_2$, etc.

Then we define the four-dimensional chirality operator acting in tangent directions to the four-dimensional space-time:

\[\Gamma^{(4)} = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4.
\]

It is represented by the matrix

\[\Gamma^{(4)} = \begin{pmatrix} 1_{4 \times 4} & 0 & 0 & 0 \\ 0 & -1_{4 \times 4} & 0 & 0 \\ 0 & 0 & -1_{4 \times 4} & 0 \\ 0 & 0 & 0 & 1_{4 \times 4} \end{pmatrix}.
\]

Similarly, we define the four-dimensional chirality operator

\[\Gamma^{(8)} = \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8,
\]
acting in four normal directions \( M = 5 \ldots 8 \). It is represented by the matrix

\[
\Gamma^{(58)} = \begin{pmatrix}
14 \times 4 & 0 & 0 & 0 \\
0 & -14 \times 4 & 0 & 0 \\
0 & 0 & 14 \times 4 & 0 \\
0 & 0 & 0 & -14 \times 4
\end{pmatrix}.
\] (A.17)

Finally, we define the eight-dimensional chirality operator

\[
\Gamma^9 = \Gamma^{(58)} \Gamma^{(14)}.
\]

It is represented by the matrix

\[
\Gamma^9 = \begin{pmatrix}
14 \times 4 & 0 & 0 & 0 \\
0 & 14 \times 4 & 0 & 0 \\
0 & 0 & -14 \times 4 & 0 \\
0 & 0 & 0 & -14 \times 4
\end{pmatrix}.
\] (A.18)

The representation \( 16 = S^+ \) (a sixteen component Majorana-Weyl fermion of \( \text{Spin}(9,1) \)) then splits as \( 16 = 8 + 8^\prime \) with respect to the \( \text{Spin}(8) \subset \text{Spin}(9,1) \) acting in the directions \( M = 1, \ldots, 8 \). Then we brake \( \text{Spin}(8) \) as \( \text{Spin}(8) \leftrightarrow \text{Spin}(4) \times \text{Spin}(4)^R \), where the group \( \text{Spin}(4) \) acts in the directions \( M = 1, \ldots, 4 \), while the group \( \text{Spin}(4)^R \) acts in the directions \( M = 5, \ldots, 8 \). We write the \( \text{Spin}(4) \) as \( \text{Spin}(4) = SU(2)_L \times SU(2)_R \) and the \( \text{Spin}(4)^R \) as \( \text{Spin}(4)^R = SU(2)_L^R \times SU(2)_R^R \). With respect to these \( SU(2) \)-subgroups, the representation \( 16 = S^+ \) of \( \text{Spin}(9,1) \) transforms as

\[
16 = (2, 1, 2, 1) + (1, 2, 1, 2) + (1, 2, 2, 1) + (2, 1, 1, 2).
\]

As we mentioned before, the only difference between the gamma-matrices in the Euclidean and Minkowski case is that we multiply the matrix \( \Gamma^0 \) by \( i \equiv \sqrt{-1} \), so the Euclidean gamma-matrices are:

\[
\Gamma^M = \begin{pmatrix} 0 & E^T_M \\ E_M & 0 \end{pmatrix}, \quad M = 1 \ldots 7
\]

\[
\Gamma^9 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix}
\]

\[
\Gamma^0 = \begin{pmatrix} i_{1_{8 \times 8}} & 0 \\ 0 & i_{1_{8 \times 8}} \end{pmatrix}
\] (A.19)

**Appendix B. Conformal Killing spinors on \( S^4 \)**

The explicit form of the Killing spinor on \( S^4 \) depends on the vielbein. For solution in spherical coordinates see \cite{34}. In stereographic coordinates the solution has simpler form and is easily related to the flat limit.

Pick up a point on \( S^4 \), call it the North pole, and call the opposite point the South pole. Let \( x^\mu \) be the stereographic coordinates on \( S^4 \) in the neighborhood of the North pole. The metric has the following form

\[
g_{\mu \nu} = \delta_{\mu \nu} e^{2\Omega}, \quad \text{where} \quad e^{2\Omega} := \frac{1}{(1 + \frac{x^2}{r^2})^2}.
\] (B.1)

By \( \theta \) we denote the polar angle in spherical coordinates measure from the North pole. In other words, \( \theta = 0 \) is the North pole, \( \theta = \frac{\pi}{2} \) is the equator, and \( \theta = \pi \)
is the South pole. We have $|x| = 2r \tan \frac{\theta}{2}$ and $e^\Omega = \cos^2 \frac{\theta}{2}$. Fix the vielbein $e_\lambda = \delta_\lambda^\mu e^{\mu}$. The spin connection $\omega^{\hat{\mu}}_{\hat{\lambda} \hat{\nu}}$ induced by the Levi-Civita connection can be computed using the Weyl transformation of the flat metric $\delta_{\mu \nu} \mapsto e^{2r} \delta_{\mu \nu}$. Under such transformation $\omega^{\hat{\mu}}_{\hat{\lambda} \hat{\nu}} \mapsto \omega^{\hat{\mu}}_{\hat{\lambda} \hat{\nu}} + (e_\lambda^\mu e_\nu^\nu \Omega_\nu - e_\lambda^\mu e^{\mu} \Omega_\nu)$. Since in the flat case $\omega^{\hat{\mu}}_{\hat{\lambda} \hat{\nu}} = 0$, we get

$$
\omega^{\hat{\mu}}_{\hat{\lambda} \hat{\nu}} = (e_\lambda^\mu e_\nu^\nu \Omega_\nu - e_\lambda^\mu e^{\mu} \Omega_\nu), \tag{B.2}
$$

where $\Omega_\nu \equiv \partial_\nu \Omega$.

The conformal Killing spinor equation takes the explicit form

$$(\partial_\lambda + \frac{1}{4} \omega_{\hat{\mu} \hat{\lambda}} \Gamma^{\hat{\mu}} \hat{\nu}) \hat{\epsilon} = \Gamma_\lambda \hat{\epsilon} \tag{B.3}$$

$$(\partial_\lambda + \frac{1}{4} \omega_{\hat{\mu} \hat{\lambda}} \Gamma^{\hat{\mu}} \hat{\nu}) \hat{\epsilon} = - \frac{1}{4r^2} \Gamma_\lambda \hat{\epsilon};$$

At the flat limit $r = \infty$ the equations simplify as $\partial_\lambda \hat{\epsilon} = \Gamma_\lambda \hat{\epsilon}$ and $\partial_\lambda \hat{\epsilon} = 0$; hence the flat space solution is

$$
\hat{\epsilon} = \hat{\epsilon}_s + x^\hat{\mu} \Gamma_{\hat{\mu}} \hat{\epsilon}_c \\
\hat{\epsilon} = \hat{\epsilon}_c, \tag{B.4}
$$

where $\hat{\epsilon}_s, \hat{\epsilon}_c$ are constant spinors on $\mathbb{R}^4$. The spinor $\hat{\epsilon}_s$ generates usual supersymmetry transformations, the spinor $\hat{\epsilon}_c$ generates special superconformal transformations.

For an arbitrary $r$ the solution is

$$
\hat{\epsilon} = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\epsilon}_s + x^\hat{\mu} \Gamma_{\hat{\mu}} \hat{\epsilon}_c) \tag{B.5}
$$

$$
\hat{\epsilon} = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\epsilon}_c - \frac{x^\hat{\mu} \Gamma_{\hat{\mu}} \hat{\epsilon}_s}{4r^2}), \tag{B.6}
$$

where $\hat{\epsilon}_s$ and $\hat{\epsilon}_c$ are arbitrary spinor parameters.

Consider the case when $\epsilon$ is the conformal Killing spinors generating a transformation of an $OSp(2|4)$ subgroup. We take chiral $\hat{\epsilon}_s$ and $\hat{\epsilon}_c$, such that $\Gamma^\lambda \hat{\epsilon}_s = \hat{\epsilon}_s$ and $\Gamma^\lambda \hat{\epsilon}_c = \hat{\epsilon}_c$, so

$$
\epsilon = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\epsilon}_s + x^\hat{\mu} \Gamma_{\hat{\mu}} \Gamma^9 \hat{\epsilon}_c). \tag{B.7}
$$

Moreover, for such spinor $\epsilon$ we have $\hat{\epsilon}_c = \frac{1}{2} \frac{1}{2} \omega_{\hat{\mu} \hat{\nu}} \Gamma^{\hat{\mu}} \hat{\nu} \hat{\epsilon}_s$, where $\omega_{\hat{\mu} \hat{\nu}}$ is an anti self-dual generator of $SO(4)$ normalized $\omega_{\hat{\mu} \hat{\nu}} \omega^{\hat{\mu} \hat{\nu}} = 4$.

This means that $\delta_c$ squares to a rotation around the North pole generated by $\omega$. Then $(\hat{\epsilon}_s, \hat{\epsilon}_c) = \frac{1}{r^2} (\hat{\epsilon}_s, \hat{\epsilon}_c)$, and thus $(\epsilon, \hat{\epsilon})$ is constant over $S^4$.

Take $(\hat{\epsilon}_s, \hat{\epsilon}_c) = 1$. Then we get the vector field $\nu_\epsilon = \hat{\epsilon} \Gamma_{\hat{\mu}} \tilde{\epsilon} = 2 \hat{\epsilon}_s \Gamma_{\hat{\mu}} x^\hat{\mu} \hat{\epsilon}_c = \Gamma^\lambda \hat{\epsilon}_s = \frac{1}{r} x^\hat{\mu} \omega_{\hat{\mu} \hat{\nu}} (\hat{\epsilon}_s \hat{\epsilon}_c) = \frac{1}{r} x^\hat{\mu} \omega_{\hat{\mu} \hat{\nu}}$. Using this identity we can

---

9In this section we use the indices $\hat{\mu}, \hat{\nu} = 1, \ldots, 4$ to enumerate the vielbein basis elements, that is $e^{\hat{\mu}}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$ where $\delta^{\hat{\mu}}_{\hat{\nu}}$ is the four-dimensional Kronecker symbol. Then $\Gamma^\hat{\mu}$ are the four-dimensional gamma-matrices normalized as $\Gamma^{\hat{\mu}} \Gamma^{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$. 

---
rewrite conformal Killing spinor $\varepsilon \equiv \varepsilon (x)$ as

$$
\varepsilon (x) = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_s + \frac{1}{2r^4} x^2 \Gamma_\mu \omega_\rho \Gamma^\rho \Gamma^\mu \hat{\varepsilon}_s) = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_s + \frac{1}{2r} x^2 \Gamma_\mu \omega_\rho \Gamma^\mu \Gamma^\rho \hat{\varepsilon}_s) =
$$

(B.8)

\[= \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_s + \frac{1}{2} v_\lambda \Gamma^\lambda \hat{\varepsilon}_s) = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} (\hat{\varepsilon}_s + \frac{|x|}{2r} n_\lambda (x) \Gamma^\lambda \hat{\varepsilon}_s) = \]

(B.9)

\[= \left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} (n_\lambda (x) \Gamma^\lambda \Gamma^9) \right) \hat{\varepsilon}_s = \exp \left( \frac{\theta}{2} n_\lambda (x) \Gamma^\lambda \Gamma^9 \right) \hat{\varepsilon}_s,
\]

(B.10)

where $n_\lambda$ is the unit vector in the direction of the vector field $v_\lambda$. The aim of these manipulations was to represent the spinor $\varepsilon (x)$ at an arbitrary point $x$ by an explicit $Spin(5)$ rotation $R(x) = \exp \frac{\theta}{2} (n_\lambda (x) \Gamma^\lambda \Gamma^9)$ of its value at the origin $\varepsilon (0) = \hat{\varepsilon}_s$.

APPENDIX C. OFF-SHELL SUPERSYMMETRY

Let $\delta_\varepsilon$ be the supersymmetry transformation generated by a Killing spinor $\varepsilon$. Then the square of $\delta_\varepsilon$ is computed as follows

$$
\delta_\varepsilon^2 A_M = \delta_\varepsilon (\varepsilon \Gamma_M \Phi) = \varepsilon \Gamma_M (\frac{1}{2} \Gamma^{PQ} \varepsilon F_{PQ} + \frac{1}{2} \Gamma^{\mu A} \Phi_A D_\mu \varepsilon).
$$

(C.1)

Since

$$
\varepsilon \Gamma_M \Gamma_P \varepsilon = \varepsilon \Gamma_P \varepsilon \varepsilon = -\varepsilon \tilde{\Gamma}_P \varepsilon - \Gamma_P \varepsilon = 2 g_{MN} [\varepsilon \varepsilon],
$$

the first term for $\delta_\varepsilon^2 A_M$ gives $-\varepsilon \Gamma^N \varepsilon F_{NM}$. The second term is

$$
\frac{1}{2} \varepsilon \Gamma_M \Gamma^{\mu A} \Phi_A D_\mu \varepsilon = -2 \varepsilon \Gamma_M \tilde{\Gamma}_A \varepsilon \Phi^A.
$$

Then

$$
\delta_\varepsilon^2 A_M = - (\varepsilon \Gamma^N \varepsilon) F_{NM} - 2 \varepsilon \Gamma_M \tilde{\Gamma}_A \varepsilon \Phi^A.
$$

(C.2)

Restricting the index $m$ to the range of $\mu$ or $A$ we get respectively

$$
\delta_\varepsilon^2 A_\mu = -v^\nu F_{\nu \mu} - [v^B \Phi_B, D_\mu],
$$

(C.3)

$$\delta_\varepsilon^2 \Phi_A = -v^\nu D_\nu \Phi_A - [v^B \Phi_B, \Phi_A] - 2 \varepsilon \tilde{\Gamma}_A \varepsilon \Phi^B - 2 \varepsilon \tilde{\varepsilon} \Phi_A,
$$

where we introduced the vector field $v$

$$
v^\mu \equiv \varepsilon \Gamma^\mu \varepsilon, \quad v^A \equiv \varepsilon \Gamma^A \varepsilon.
$$

(C.4)

Therefore

$$
\delta_\varepsilon^2 = -L_v - G_{v A M} A_M - R - \Omega.
$$

(C.5)

Here $L_v$ is the Lie derivative in the direction of the vector field $v^\mu$. The transformation $G_{v A M} A_M$ is the gauge transformation generated by the parameter $v^M A_M$. On matter fields $G$ acts as $G_u \cdot \Phi \equiv [u, \Phi]$, on gauge fields $G$ acts as $G_u \cdot A_M = -D_\mu u$. The transformation $R$ is the rotation of the scalar fields $(R \cdot \Phi)_A = R_{AB} \Phi^B$ with the generator $R_{AB} = 2 \varepsilon \tilde{\Gamma}_{AB} \tilde{\varepsilon}$. Finally, the transformation $\Omega$ is the dilation transformation with the parameter $2(\varepsilon \tilde{\varepsilon})$. 
The $\delta_\varepsilon^2$ acts on the fermions as follows

$$\delta_\varepsilon^2 \Psi = D_M (\varepsilon \Gamma_N \Psi) \Gamma^{MN} \varepsilon + \frac{1}{2} \Gamma^{\mu A} (\varepsilon \Gamma_A \Psi) D_\mu \varepsilon =$$

$$= (\varepsilon \Gamma_N D_M \Psi) \Gamma^{MN} \varepsilon + ((D_\mu \varepsilon) \Gamma_N \Psi) \Gamma^{\mu N} \varepsilon + \frac{1}{2} \Gamma^{\mu A} (\varepsilon \Gamma_A \Psi) D_\mu \varepsilon. \quad (C.6)$$

From the “triality identity” we have $\Gamma_{N\alpha_2}^{\alpha_1} \Gamma_N^{\alpha_3} \xi = -\frac{1}{2} \Gamma_{N\alpha_2}^{\alpha_1} \Gamma_{N\alpha_3}^{\alpha_1}$. Then the first term gives

$$(\varepsilon \Gamma_N D_M \Psi)(\Gamma^{MN} \varepsilon)_{\alpha_4} = (\varepsilon \Gamma_N D_M \Psi)((\tilde{\Gamma}^M \Gamma_N \varepsilon)_{\alpha_4} - g^{MN} \varepsilon_{\alpha_4}) =$$

$$= \varepsilon^{\alpha_4} \Gamma_{N\alpha_2}^{\alpha_1} D_M \Psi \varepsilon_{\alpha_3} \Gamma^M_{\alpha_2 \xi} \xi_{\alpha_3} - (\varepsilon \Gamma_N D_M \Psi) \varepsilon_{\alpha_4} =$$

$$= -\frac{1}{2} (\varepsilon^{\alpha_4} \Gamma_{N\alpha_2}^{\alpha_1} \varepsilon_{\alpha_3})(\tilde{\Gamma}^M_{\alpha_2 \xi} \Gamma^N_{\alpha_3} D_M \Psi) - (\varepsilon \Gamma_N D_M \Psi) \varepsilon_{\alpha_4} =$$

$$= -\frac{1}{2} (\varepsilon \Gamma_N \xi)(\tilde{\Gamma}^M \Gamma_N D_M \Psi)_{\alpha_4} - (\varepsilon \Gamma_N D_M \Psi) \varepsilon_{\alpha_4} =$$

$$= -\frac{1}{2} (\varepsilon \Gamma_N \xi)(-\tilde{\Gamma}^N \Gamma^M D_M \Psi + 2 D_N \Psi)_{\alpha_4} - (\varepsilon \Gamma_N D_M \Psi) \varepsilon_{\alpha_4} =$$

$$= \frac{1}{2} (\varepsilon \Gamma_N \xi) \tilde{\Gamma}^N (\Phi \Psi)_{\alpha_4} - (\varepsilon \Gamma_N \xi)(D_N \Psi)_{\alpha_4} - (\varepsilon \Phi \Psi) \varepsilon_{\alpha_4}. \quad (C.7)$$

The first and the third term in the last line vanish on-shell. When we add auxiliary fields, they will cancel the first and the third term explicitly. Then we get

$$\delta_\varepsilon^2 \Psi = - (\varepsilon \Gamma^N \varepsilon) D_N \Psi + (\Psi \Gamma_N D_\mu \varepsilon) \Gamma^{\mu N} \varepsilon + \frac{1}{2} \Gamma^{\mu A} (\varepsilon \Gamma_A \Psi) D_\mu \varepsilon + \text{com}[\Psi], \quad (C.8)$$

where \text{com}[\Psi] stands for the terms proportional to the Dirac equation of motion for $\Psi$. Then we rewrite the last two terms as follows

$$\left(\Psi \Gamma_N \Gamma_\mu \varepsilon\right) \Gamma^{\mu N} \varepsilon + \frac{1}{2} \Gamma^{\mu A} (\varepsilon \Gamma_A \Psi) \Gamma_\mu \varepsilon =$$

$$= (\Psi \Gamma_N \Gamma_\mu \varepsilon)(\tilde{\Gamma}^\mu \Gamma^N - g^{MN} \varepsilon) - 2(\varepsilon \Gamma_A \Psi) \tilde{\Gamma}^A \varepsilon = (\Psi \Gamma_N \Gamma_\mu \varepsilon) \tilde{\Gamma}^\mu \Gamma^N \varepsilon - 4(\varepsilon \Gamma_A \Psi) \tilde{\Gamma}^A \varepsilon =$$

$$\text{triality} = -\left(\tilde{\Gamma}^\mu \Gamma_N \varepsilon\right) \tilde{\Gamma}^\mu \Gamma^N \Psi - (\varepsilon \Gamma_A \Psi) \tilde{\Gamma}^A \Gamma_N \varepsilon - 4(\varepsilon \Gamma_A \Psi) \tilde{\Gamma}^A \varepsilon =$$

$$= -\left(\tilde{\Gamma}^\mu \Gamma_\nu \varepsilon\right) \tilde{\Gamma}^\mu \Gamma^N \Psi - (\varepsilon \Gamma_A \varepsilon) \tilde{\Gamma}^A \Gamma^N \Psi + 2(\varepsilon \Gamma_\nu \varepsilon) \tilde{\Gamma}^A \varepsilon + 4(\varepsilon \Gamma_A \varepsilon) \tilde{\Gamma}^A \varepsilon - 2(\varepsilon \Gamma_A \varepsilon) \tilde{\Gamma}^A \varepsilon =$$

$$= -\frac{1}{2} (\varepsilon \Gamma_\mu \varepsilon) \Gamma^{\mu N} \Psi - \frac{1}{2} (\varepsilon \Gamma_\mu \varepsilon) \Gamma^{\mu A} \varepsilon - 4(\varepsilon \Gamma_\mu \varepsilon) \Psi - (\varepsilon \Gamma_{\mu A} \varepsilon) \Gamma^A \varepsilon - \frac{1}{2} (\varepsilon \Gamma_{AB} \varepsilon) \Gamma^{AB} \Psi +$$

$$+ \frac{1}{2} (\varepsilon \Gamma_A \varepsilon) \Gamma^{AB} \Psi + 2 (\varepsilon \Gamma_N \Psi) \tilde{\Gamma}^N \varepsilon - 4(\varepsilon \Psi) \varepsilon =$$

$$= \left(\left(-\frac{1}{2} (\varepsilon \Gamma_\mu \varepsilon) \Gamma^{\mu N} \varepsilon + \frac{1}{2} (\varepsilon \Gamma_{AB} \varepsilon) \Gamma^{AB} \Psi \right)\right)$$

$$+ \left(\frac{1}{2} (\varepsilon \Gamma_{MN}) \Gamma^{MN} \Psi - 4(\varepsilon \Psi) \varepsilon - 4(\varepsilon \Psi) \varepsilon + 2(\varepsilon \Gamma_N \Psi) \tilde{\Gamma}^N \varepsilon \right). \quad (C.9)$$

The first term is a part of the Lie derivative along the vector field $\nu^\mu = (\varepsilon \Gamma^\mu \varepsilon)$ acting on $\Psi$. The second term correspond to the rotations of the scalar fields $\Phi^A$ by the generator $R_{AB}$ and the properly induced action on the fermions.

In the $N = 4$ case we use Fierz identity for $\Gamma^{MN}_{\alpha_1 \alpha_2} \Gamma_{MN \alpha_3} \alpha_4$ in the last line of (C.9) to see that all term in the second pair of parentheses are canceled except...
This expression is identically zero because of (C.13). Hence, after inclusion of the derivative acting on $\Psi$, we get

$$
\delta^2 \Psi = -(\varepsilon \Gamma^N \varepsilon) D_N \Psi - \frac{1}{2} (\varepsilon \Gamma_{\mu \nu} \varepsilon) \Gamma^{\mu \nu} \Psi - \frac{1}{2} (\varepsilon \Gamma_{A B} \varepsilon) \Gamma^{A B} \Psi - 3(\varepsilon \varepsilon) \Psi + \text{eom}[\Psi].
$$

(C.10)

To achieve off-shell closure in the $N = 4$ case we add seven auxiliary fields $K_i$ with $i = 1, \ldots, 7$ and modify the transformations as

$$
\delta^2 \Psi = \frac{1}{2} \Gamma^{M N} F_{M N} + \frac{1}{2} \Gamma^{\mu A} \Phi_A D_\mu \varepsilon + K^i \nu_i
$$

$$
\delta \nu_i K_i = -\nu_i \Gamma^M D_M \Psi.
$$

(C.11)

Here we introduced seven spinors $\nu_i$. They depend on choice of the conformal Killing spinor $\varepsilon$ and are required to satisfy the following relations:

$$
\varepsilon \Gamma^M \nu_i = 0
$$

(C.12)

$$
\frac{1}{2} (\varepsilon \Gamma_N \varepsilon) \tilde{\Gamma}^N_{\alpha \beta} = \nu_\alpha \nu_\beta + \varepsilon \varepsilon_{\alpha \beta}
$$

(C.13)

$$
\nu_i \Gamma^M \nu_j = \delta_{ij} \varepsilon \Gamma^M \varepsilon.
$$

(C.14)

The equation (C.12) ensures closure on $A_M$, the equation (C.13) ensures closure on $\Psi$. The new term in the transformations for $\Psi$ modifies the last line of (C.7) as

$$
\delta \nu_i (K^i \nu_i) = -(\nu_i \mathcal{D} \Psi) \nu_i.
$$

Then the terms in $\delta^2 \Psi$ which were not taken into an account in (C.18) are

$$
-(\nu_i \mathcal{D} \Psi) \nu_i + \frac{1}{2} (\varepsilon \Gamma_N \varepsilon) \tilde{\Gamma}^N \mathcal{D} \Psi - (\varepsilon \varepsilon) \varepsilon.
$$

(C.15)

This expression is identically zero because of (C.13). Hence, after inclusion of the auxiliary fields $K_i$, the formula (C.11) for $\delta^2 \Psi$ is valid off-shell.

For the transformation $\delta^2 \nu_i K_i$ we get

$$
\delta^2 \nu_i K_i = -\nu_i \Gamma^M [(\varepsilon \Gamma_M \Psi), \Psi] - \nu_i \Gamma^M D_M (\frac{1}{2} \Gamma^{P Q} F_{P Q} \varepsilon + \frac{1}{2} \Gamma^\mu A \Phi_A D_\mu \varepsilon + K^i \nu_i).
$$

(C.16)

Using the gamma matrix “triality identity” the first term is transformed to $\frac{1}{2} (\varepsilon \Gamma^M \varepsilon) (M \varepsilon) [\Psi, \Gamma^M \Phi]$, which vanishes because of (C.12). The second term with derivative acting on $F$ is equal by Bianchi identity to $\nu_i \Gamma^M \varepsilon D_M F^{M N}$ and vanishes because of (C.12). Then we use (A.8) to simplify the remaining terms

$$
\delta^2 \nu_i K_i = -\frac{1}{2} \nu_i \Gamma^\mu \Gamma^{P Q} \Gamma_\mu \varepsilon F_{P Q} \frac{1}{2} (\nu_i \Gamma^M \varepsilon) D_M \Phi_A - \frac{1}{2} (\frac{1}{4 \mu^2}) \Phi_A \nu_i \Gamma^\mu A \Gamma^\mu B \varepsilon - \nu_i \Gamma^M \varepsilon D_M K^i \nu_j - (\nu_i \Gamma^\mu D_\mu \nu_j) K^j = -\frac{1}{2} (4 \nu_i \Gamma^M \varepsilon) D_M \Phi_B - \frac{1}{2} (\nu_i \Gamma^M \varepsilon) D_M K^j - (\nu_i \Gamma^M \varepsilon) D_M \nu_j K^j
$$

$$
= -\frac{1}{2} (\nu_i \Gamma^\mu A \Gamma^\mu B + (\frac{1}{2}) \nu_i A \varepsilon) \Phi_A +
$$

$$
-(\nu_i \Gamma^M \varepsilon) D_M K^j - (\nu_i \Gamma^\mu D_\mu \nu_j) K^j = -(\varepsilon \Gamma^M \varepsilon) D_M K^j - (\nu_i \Gamma^M \varepsilon) D_M \nu_j K^j = -(\varepsilon \Gamma^M \varepsilon) D_M K^j - (\nu_i \Gamma^\mu A \Gamma^\mu B) A \varepsilon +
$$

To get the last line we use the differential of (C.14), i.e. $\nu_i (\mathcal{D} \nu_j) = 4(\varepsilon \varepsilon) \delta_{ij}$. 


Now we consider separately the case of pure \( \mathcal{N} = 2 \) Yang-Mills. First we rewrite the last terms in (C.19) as follows (here \( d \) is the dimension of uncompactified theory)

\[
\begin{align*}
(\bar{\epsilon} \Gamma_{MN}\bar{\epsilon})\bar{\Gamma}^{MN}\Psi &= (\bar{\epsilon} \bar{\Gamma}_M \Gamma_N \bar{\epsilon})\bar{\Gamma}^{MN}\Psi = (\bar{\epsilon} \bar{\Gamma}_M \Gamma_N \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \Psi - d(\bar{\epsilon} \bar{\epsilon})\Psi^\text{tr}ality} \\
-(\epsilon \Gamma_N \Psi)\bar{\Gamma}^M \Gamma^N \bar{\Gamma}_M \bar{\epsilon} - (\Psi \Gamma_N \bar{\Gamma}_M \bar{\epsilon})\Gamma^M \Gamma^N - d(\epsilon \bar{\epsilon})\Psi &= (d-2)(\epsilon \Gamma_N \Psi)\bar{\Gamma}^N \bar{\epsilon} - (\Psi \Gamma_N \bar{\Gamma}_M \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N - d(\bar{\epsilon} \epsilon)\Psi.
\end{align*}
\]

(C.18)

For the pure \( \mathcal{N} = 2 \) theory in four-dimensions we take \( d = 6 \) and get

\[
\begin{align*}
&\left(-\frac{1}{2}(\bar{\epsilon} \Gamma_{MN}\bar{\epsilon})\bar{\Gamma}^{MN}\Psi - 4(\bar{\epsilon} \bar{\epsilon})\Psi - 4(\Psi \bar{\epsilon})\epsilon + 2(\epsilon \Gamma_N \Psi)\bar{\Gamma}^N \bar{\epsilon}\right) = \\
&-\frac{1}{2} \left(4(\epsilon \Gamma_N \Psi)\bar{\Gamma}^N \bar{\epsilon} - (\Psi \Gamma_N \bar{\Gamma}_M \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \epsilon - 6(\bar{\epsilon} \bar{\epsilon})\Psi\right) - \\
&- 4(\epsilon \bar{\epsilon})\Psi - 4(\Psi \bar{\epsilon})\epsilon + 2(\epsilon \Gamma_N \Psi)\bar{\Gamma}^N \bar{\epsilon} = \\
&= \frac{1}{2}(\Psi(-\Gamma_M \bar{\Gamma}_N + 2g_{MN})\bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \epsilon - (\epsilon \bar{\epsilon})\Psi - 4(\Psi \bar{\epsilon})\epsilon = \\
&= -\frac{1}{2}(\Psi \Gamma_M \bar{\Gamma}_N \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N + 6(\Psi \bar{\epsilon})\epsilon - (\epsilon \bar{\epsilon})\Psi - 4(\Psi \bar{\epsilon})\epsilon = \\
&= -\frac{1}{2}(\Psi \Gamma_M \bar{\Gamma}_N \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \epsilon + 2(\epsilon \bar{\epsilon})\Psi = \\
&= (\Psi \Gamma_M \bar{\Gamma}_N \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \epsilon - (\epsilon \bar{\epsilon})\Psi.
\end{align*}
\]

(C.19)

We express the first term in terms of the triplet of matrices \( \Lambda^i \), which are defined as a set of three antisymmetric matrices such that

\[
\begin{align*}
\Lambda^i_{\alpha_1 \alpha_2} \Lambda^j_{\alpha_2 \alpha_3} &= \epsilon^{ijk} \Lambda^k_{\alpha_1 \alpha_2}, \quad i, j, k = 1, \ldots, 3. \\
[\Lambda_i, \Gamma^M] &= 0 \\
\frac{1}{2} \Gamma^M_{\alpha_1 \alpha_2} \bar{\Gamma}_M \alpha_3 \alpha_4 &= \delta_{\alpha_2 \alpha_1} \delta_{\alpha_3 \alpha_4} - \Lambda^i_{\alpha_2 \alpha_1} \Lambda^i_{\alpha_3 \alpha_4}.
\end{align*}
\]

(C.20) (C.21) (C.22)

Then we get

\[
(\Psi \Gamma_M \bar{\Gamma}_N \bar{\epsilon})\bar{\Gamma}^{M} \Gamma^N \epsilon = 4(\Psi \bar{\epsilon})\epsilon + 4(\epsilon \bar{\epsilon})\Psi + 4(\epsilon \Lambda^i \bar{\epsilon})\Lambda^i \Psi,
\]

(C.23) and finally the equation (C.19) turns into

\[
-2(\Psi \bar{\epsilon})\epsilon - 2(\epsilon \bar{\epsilon})\Psi - 2(\epsilon \Lambda^i \bar{\epsilon})\Lambda^i \Psi + 2(\Psi \bar{\epsilon})\epsilon - (\epsilon \bar{\epsilon})\Psi = -2(\epsilon \Lambda^i \bar{\epsilon})\Lambda^i \Psi - 3(\bar{\epsilon} \bar{\epsilon})\Psi.
\]

(C.24)

That completes simplification of \( \delta^2_\epsilon \) acting on fermions

\[
\delta^2_\epsilon \Psi = -(\epsilon \Gamma^N \epsilon)D_N \Psi - \frac{1}{2}(\bar{\epsilon} \Gamma_{\mu \nu} \epsilon)\Gamma^{\mu \nu} \Psi - \frac{1}{2}(\bar{\epsilon} \bar{\Gamma}_{AB} \bar{\epsilon})\Gamma^{AB} \Psi - 2(\epsilon \Lambda^i \bar{\epsilon})\Lambda^i \Psi - 3(\bar{\epsilon} \bar{\epsilon})\Psi.
\]

(C.25)

It has the structure

\[
\delta^2_\epsilon \Psi = -L_\epsilon \Psi - G_{\epsilon \Lambda^i A} \Psi = R \Psi - R^i \Psi - \Omega \Psi,
\]

(C.26)

where the notations for the generators are the same as in the bosonic case. The only new generator here is \( R^i \), corresponding to the term \( \delta^2_\epsilon \Psi = -2(\epsilon \Lambda^i \bar{\epsilon})\Lambda^i \Psi \). It generates an \( SU(2)_4 \) R-symmetry transformation of \( \mathcal{N} = 2 \) which acts trivially on the bosonic fields of the theory, and as \( \Psi \mapsto e^{\epsilon \Lambda^i} \Psi \) on fermionic fields.

To achieve off-shell closure in \( \mathcal{N} = 2 \) case we add a triplet of auxiliary fields \( K_i \) and modify the transformations as

\[
\begin{align*}
\delta_\epsilon \Psi &= \frac{1}{2} \Gamma^{MN} F_{MN} + \frac{1}{2} \Gamma^{\mu A} \Phi^A D_\mu \epsilon + K^i \Lambda_i \epsilon \\
\delta_\epsilon K_i &= \epsilon \Lambda^i \Gamma^M D_M \Psi,
\end{align*}
\]

(C.27)
The new term in the transformations for $\Psi$ modifies the last line of (C.17) as
\[
\delta_\epsilon(K^i\Lambda_\epsilon) = (\epsilon\Lambda_\epsilon \mathcal{D}\Psi)\Lambda_\epsilon\epsilon.
\]
Then the terms in $\delta_\epsilon^2\Psi$ which were not taken into an account in (C.18) are
\[
(\epsilon\Lambda_\epsilon \mathcal{D}\Psi)\Lambda_\epsilon\epsilon + \frac{1}{2}(\epsilon\Gamma_N\epsilon)\hat{\Gamma}^N\mathcal{D}\Psi - (\epsilon\mathcal{D}\Psi)\epsilon.
\] (C.28)
This expression is identically zero because of the relation (C.20). Hence, after inclusion of the auxiliary fields $K_i$, the formula (C.10) for $\delta_\epsilon^2\Psi$ is valid off-shell.

Remark. The second equation (C.13) follows from the first equation (C.12) and the third equation (C.12) as follows. Let
\[
M_{\alpha\beta} = \nu^i_\alpha\nu^j_\beta + \epsilon_\alpha\epsilon_\beta.
\]
We want to show that $M_{\alpha\beta} = \frac{1}{2}v_N\Gamma^N_{\alpha\beta}$, that is the matrix $M_{\alpha\beta}$ can be expanded over the matrices $\Gamma^N_{\alpha\beta}$ with the coefficients $\frac{1}{2}v_N$. Fix the positive definite metric on the space $\mathbb{R}^{16 \times 16}$ of $16 \times 16$ matrices as $(M, M) := M_{\alpha\beta}M_{\alpha\beta}$. Since $\tilde{\Gamma}^N = \Gamma_N$ and $\Gamma^\alpha_\beta\Gamma^\alpha_\beta = 16\delta_N^N$, the set of 10 matrices $\frac{1}{2}\Gamma^N$ is orthonormal in $\mathbb{R}^{16 \times 16}$. Complete this set to the basis of $\mathbb{R}^{16 \times 16}$. Then the coefficient $m_N$ of $\frac{1}{4}\Gamma^N$ in the expansion of $M$ over this basis is given by the scalar product
\[
m_N = (M, \frac{1}{4}\Gamma^N) = \frac{1}{4}(\nu^i\Gamma^N\nu^i + \epsilon\Gamma_N\epsilon) = 2v_N.
\]
Therefore we have $M = 2v_N(\frac{1}{4}\Gamma^N) + (\ldots)$, where $(\ldots)$ stands for possible other terms in the expansion over the completion of the set $\{\frac{1}{4}\Gamma^N\}$ to the basis of $\mathbb{R}^{16 \times 16}$. To prove that all other terms vanish, compare the norm of $M$
\[
(M, M) = (\epsilon\epsilon)(\epsilon\epsilon) + (\nu_i\nu_j)(\nu_i\nu_j) = (\epsilon\epsilon) + \delta_{ij}(\epsilon\epsilon)\delta_{ij}(\epsilon\epsilon) = 8(\epsilon\epsilon)(\epsilon\epsilon)
\]
with the $\sum_N m_N^2$
\[
\sum_N m_N^2 = 4v_Nv_N = 4(\epsilon\Gamma_N\epsilon)(\epsilon\tilde{\Gamma}^N\epsilon) = 4((\epsilon\Gamma_N\epsilon)(\epsilon\Gamma^N\epsilon) + 2(\epsilon\epsilon)(\epsilon\epsilon)) = 8(\epsilon\epsilon)(\epsilon\epsilon).
\]
Since the norms are the same, $(M, M) = \sum_N m_N^2$, and the metric is positive definite, we conclude that all other coefficients vanish.

APPENDIX D. INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

Here we collect some facts about indices of transversally elliptic operators mostly following Atiyah’s book [50]. See also [49].

Let $\cdots \rightarrow E^1 \xrightarrow{D_1} E^{i+1} \rightarrow \cdots$ be an elliptic complex of vector bundles over a manifold $X$. Let a Lie group $G$ act on $X$ and bundles $E^i$. This means that for any transformation $g : X \rightarrow X$, which sends a point $x \in X$ to $g(x)$, we are given a vector bundle homomorphisms $\gamma^i : g^*E^i \rightarrow E^i$. Then we have natural linear maps $\hat{\gamma}^i : \Gamma(E^i) \rightarrow \Gamma(E^i)$ defined by $\hat{\gamma}^i = \gamma^i \circ g^*$. On any section $s(x) \in \Gamma(E^i)$ the map $\hat{\gamma}^i$ acts by the formula $\langle \hat{\gamma}^i s \rangle(x) = \gamma_x s(g(x))$. We assume that $\hat{\gamma}$ commutes with the differential operators $D_i$ of the complex $E$. Then $\hat{\gamma}$ descends to a well-defined action on the cohomology groups $H^i(E)$.

The $G$-equivariant index is defined as
\[
\text{ind}_g(E) = \sum_i (-1)^i \text{tr}_{H^i} \hat{\gamma}^i.
\] (D.1)
In the case when the set of $G$-fixed points is discrete and the action of $G$ is nice in a neighborhood of each of the fixed point, the Atiyah-Bott fixed point formula says \cite{63, 71}

$$\text{ind}_g(E) = \sum_{x \in \text{fixed point set}} \frac{\sum (-1)^i \text{tr} \gamma_s^i}{\det(1 - dg(x))}. \quad (D.2)$$

This formula can be easily argued in the following way (see \cite{63} for a derivation using supersymmetric quantum mechanics). For an illustration we consider the case when the complex $E$ consists of two vector bundles $E^0 \rightarrow E^1$, and we assume that the bundles are equipped with a hermitian $G$-invariant metric. Let $D : \Gamma(E^0) \rightarrow \Gamma(E^1)$ be the differential. Then we consider the Laplacian $\Delta = DD^* + D^*D$. The zero modes of the Laplacian are identified with the cohomology groups of $E$, which are in this case: $H^0(E) = \ker D$ and $H^1(E) = \text{coker} D$. Hence, the index can be computed as

$$\text{ind}_g(E) = \lim_{t \rightarrow \infty} \text{str}_{\Gamma(E)} \hat{\gamma} e^{-t\Delta}. \quad (D.3)$$

Here the supertrace for operators acting on $\Gamma(E)$ is defined assuming even parity on $\Gamma(E^0)$ and odd parity on $\Gamma(E^1)$. However, the expression under the limit sign actually does not depend on $t$ because $[\Delta, \hat{\gamma}] = 0$. Taking the limit $t \rightarrow 0$ we get supertrace of $\hat{\gamma}$. The trace can be easily taken in the coordinate representation. By definition, the operator $\hat{\gamma}$ has kernel $\hat{\gamma}(x, y) = \gamma_x \delta(g(x) - y)$ if we write $(\gamma_s)(x) = \int_X \gamma(x, y)s(y)$). Here $\delta(x)$ is the Dirac delta-function. Taking the trace we get Atiyah-Bott result

$$\text{ind}_g(E) = \lim_{t \rightarrow 0} \text{str}_{\Gamma(E)} \hat{\gamma} e^{-t\Delta} = \int dx \text{str}_{E_x} \hat{\gamma}(x, x) = \int dx \text{str}_{E_x} \gamma_x \delta(g(x) - x) = \sum_{g(x) = x} \frac{\text{str}_{E_x} \gamma_x}{|\det(1 - dg(x))|}. \quad (D.3)$$

Let $X$ be a complex manifold of dimension $n$. Consider the complex of $(0, p)$-forms with the differential $\partial$. Let $G = U(1)$ acts on $X$ holomorphically. In a neighborhood of a fixed point we can choose such coordinates $z^1, \ldots, z^n$ that an element $g \in G$ acts by $z^i \rightarrow q_i z^i$. If $z^i$ transforms in a $U(1)$ representation $m_i \in \mathbb{Z}$, and we parameterize $U(1)$ by a unit circle $\{ |q| = 1, q \in \mathbb{C} \}$, then $q_i = q^{m_i}$. One-forms $f_i$ transform as $f_i \rightarrow q_i^{-1} f_i$. Since $|q| = 1$ we have $f_i \rightarrow q_i f_i$. Computing the supertrace for the numerator on external powers of the anti-holomorphic subspace of the fiber of the cotangent bundle at the origin, we get $\text{str}_{\psi^0 \bullet \psi} = \prod_{i=1}^n (1 - q_i)$. The denominator is $\prod_{i=1}^n (1 - q_i)(1 - q_i^{-1})$. Then contribution of a fixed point with weights $\{ q_1, \ldots, q_n \}$ to the index of $\partial$ is

$$\text{ind}_q(\partial)|_0 = \frac{1}{\prod_{i=1}^n (1 - q_i^{-1})}.$$ 

Let $\pi : T^*X \rightarrow X$ be the cotangent bundle. Then $\pi^* E_i$ are the bundles over $T^*X$. The symbol of the differential operator $D : \Gamma(E_0) \rightarrow \Gamma(E_1)$ is a vector bundle homomorphism $\sigma(D) : \pi^* E_0 \rightarrow \pi^* E_1$. In local coordinates $x_i$ it is defined by replacing all partial derivatives in the highest order component of $D$ by momenta, so $\frac{\partial}{\partial x_i} \rightarrow ip_i$, and then taking $p_i$ to be coordinates on fibers of $T^*X$. Let the family of the vector spaces $T^*_G X$ be a union of vector spaces $T^*_G X_x$ over all points $x \in X$, where $T^*_G X_x$ denotes a subspace of $T^*X$ transversal to the $G$-orbit through $x$. The
operator $D$ is transversally elliptic if its symbol $\sigma(D)$ is invertible on $T^*_G X \setminus 0$, where 0 denotes the zero section.

We need a few notions of $K$-theory [86]. Let Vect$(X)$ be the set of isomorphism classes of vector bundles on $X$. It is an abelian semigroup where addition is defined as the direct sum of vector bundles. For any abelian semigroup $A$ we can associate an abelian group $K(A)$ by taking all equivalence classes of pairs $(a, b) \sim (a+c, b+c)$, where $a, b, c \in A$. Taking Vect$(X)$ as $A$ we define the $K$-theory group $K(X)$. Its elements are pairs of isomorphism classes of vector bundles $(E_0, E_1)$ over $X$ up to the equivalence relation $(E_0, E_1) \sim (E_0 \oplus H, E_1 \oplus H)$ for all vector bundles $H$ over $X$. If $X$ is a space with a basepoint $x_0$, then we define $\tilde{K}(X)$ as a kernel of the map $i_*: K(X) \to K(x_0)$ where $i : x_0 \to X$ is the inclusion map. Next we define relative $K$-theory group $K(X, Y)$ for a compact pair of spaces $(X, Y)$. Let $X/Y$ be the space obtained by considering all points in $Y$ to be equivalent and taking this equivalence class as a basepoint. Then $K(X, Y)$ is defined as $\tilde{K}(X/Y)$.

Equivalently, $K(X, Y)$ consists of pairs of vector bundles $(E_0, E_1)$ over $X$ such that $E_0$ is isomorphic to $E_1$ over $Y$, and considered up to the equivalence relation $(E_0, E_1) \sim (E_0 \oplus H, E_1 \oplus H)$ for all vector bundles $H$ over $X$. For a non-compact space, such as a total space of vector bundle $V \to X$, we define $K(V)$ as $\tilde{K}(X^V)$, where $X^V$ is a one-point compactification of $V$, or equivalently $B(V)/S(V)$, where $B(V)$ and $S(V)$ is respectively a unit ball and unit sphere on $V$.

If a group $G$ acts on $X$ we can consider the set of isomorphism classes of $G$-vector bundles over $X$. It is an abelian semi-group, to which we associate an abelian group $K_G(X)$. All constructions above can be done in $G$-equivariant fashion.

Since the symbol of a transversally elliptic operator is an isomorphism $\sigma(D): \pi^* E \to \pi^* F$ of vector bundles over $T^*_G X$ outside of zero section, by definition it represents an element of $K_G(T^*_G X)$. One can show that the index of transversally elliptic operator does not depend on continuous deformations of it symbol, hence it depends only on the homotopy type of the symbol. The index vanishes for a symbol which is induced by an isomorphism of vector bundles $E$ and $F$. Therefore the index of $D$ depends only on an element of $K_G(T^*_G X)$ which represents symbol $\sigma(D)$.

The equivariant index was defined for any group element $g$ as an alternating sum of traces of $g$ in representations $R^i$ in which $G$ acts on the cohomology groups $H^i$ of the complex $E$. One can show that for transversally elliptic operators the representations $R^i$ can be decomposed into a direct sum of irreducible representations where each irreducible representation enters with a finite multiplicity. In the elliptic case the number of irreducible representations which appear is finite since cohomology groups $H^i$ have finite dimensions. Let $\chi_\alpha$ be a character for each irreducible representation $\alpha$. Then the index of transversally elliptic operator is $\sum_\alpha m_\alpha \chi_\alpha$ where $m_\alpha$ are finite integer multiplicities. Thus the index can be regarded as a distribution on $G$, so that the multiplicities $m_\alpha$ are coefficients in its Fourier series expansion. Let $D'(G)$ be the space of distributions on $G$.

Consider an example. Let $X$ be a circle $S^1$ on which group $G = U(1)$ acts in a natural way. Let $E_0$ be the trivial rank one bundle $E$ over $S^1$, and $E_1$ be the zero bundle. Let $D : \Gamma(E) \to 0$ be the zero operator. Then the cohomology group $H^0$ is the space of all functions on a circle, and $H^1$ vanishes. Functions on a circle can be decomposed into Fourier modes labeled by integers, so that each mode corresponds to an irreducible representation of $U(1)$. If $q = e^{i\alpha}$ for $\alpha \in [0, 2\pi)$ denotes an
element of $U(1)$, then we obtain the index

$$\text{ind } 0 = \sum_{-\infty}^{\infty} q^n = \sum_{-\infty}^{\infty} e^{in\alpha} = 2\pi \delta(\alpha).$$

We see that the index is not a smooth function on $U(1)$, but a distribution – the Dirac delta-function.

We learned that the index is a map from $K$-theory group of $T^*_G X$ to distributions on $G$

$$\text{ind } : K_G(T^*_G X) \to D'(G).$$

Moreover, the index is a group homomorphism with respect to the abelian group structure on $K_G(T^*_G X)$ and the addition operation on $D'(G)$. The abelian groups $D'(G)$ and $K_G(T^*_G X)$ are modules over the character ring $R(G)$. Indeed, $K_G(pt) = R(G)$ since elements of $R(G)$ are formal linear combinations of irreducible representations of $G$, and $K_G(X)$ has a module structure over $K_G(pt)$, since we can take tensor products of vector bundles representing $K_G(X)$ with trivial vector bundles representing $K_G(pt)$. The module $D'(G)$ has a torsion submodule. For example, the Dirac delta-function on a circle supported at $q = 1$ is a torsion element of $D'(U(1))$, because it is annihilated by $q - 1$. One can show that the support of the index is a subset of points $g \in G$ for which $X_g \neq \emptyset$, where $X^g \subset X$ is the $g$-fixed set. If $G$ acts freely on $X$ then the index is supported at the identity of $G$, hence it is a pure torsion element.

From now we consider the case $G = U(1)$. We can find torsion free part of the index if we know it as a function on a generic group element $g \neq \text{Id}$. If $X^g$ consists of non-degenerate points, then we can repeat the argument used in the elliptic case and obtain the formula (D.3). In the elliptic case, separate contributions from fixed points are not well defined at $q = 1$, but the total sum is well defined, since the index is a finite polynomial in $q$ and $q^{-1}$. In the transversally elliptic case, if we add contributions of fixed points formally defined by the formula (D.3), we will obtain correctly only the torsion free part of the index. In other words, we will obtain the index up to a singular distribution supported at $q = 1$.

To fix the torsion part, we should find a way in which we associate distributions to rational functions given by the formula (D.3). This procedure is explained in details in [50]. For example, the contribution to the index of $\bar{\partial}$ operator from the origin of $\mathbb{C}$ as a rational function is

$$\text{ind}_q(\bar{\partial})|_0 = \frac{1}{1 - q^{-1}}. \quad (D.4)$$

There are two basic ways to associate a distribution to it, which we call expansions in positive or negative powers of $q$:

$$\left[ \frac{1}{1 - q^{-1}} \right]_+ = -\frac{q}{1 - q} = -\sum_{n=1}^{\infty} q^n \quad (D.5)$$

$$\left[ \frac{1}{1 - q^{-1}} \right]_- = -\sum_{n=0}^{\infty} q^{-n}. \quad (D.6)$$
These two regularizations differ by a torsion element (a distribution supported at \( q = 1 \)):

\[
\left[ \frac{1}{1 - q^{-1}} \right]_+ - \left[ \frac{1}{1 - q^{-1}} \right]_- = - \sum_{n = -\infty}^{\infty} q^n = -2\pi i(q - 1).
\]

The decomposition of \( K_G(T^*_G X) \) to the torsion part and the torsion free part can be described by the exact sequence

\[
0 \to K_G(T^*_G(X \setminus Y)) \to K_G(T^*_G X) \to K_G(T^* X|_Y) \to 0, \tag{D.7}
\]

where \( Y \) is the fixed point set in \( X \). Since \( G \) acts freely on \( X \setminus Y \), the image of \( K_G(T^*_G(X \setminus Y)) \) under the index homomorphism is a torsion submodule of \( D'(G) \). The last term of the sequence is the torsion free quotient determined completely by the fixed point set \( Y \). Using a vector field \( v \) on \( X \) generated by action of \( G \), it is possible to construct two homomorphisms

\[
\theta^\pm : K_G(T^* X|_Y) \to K_G(T^*_G X),
\]

where \( \pm \) signs correspond to a choice of the direction of the vector field. First, given a symbol \( \sigma : \pi^* E_0 \to \pi^* E_1 \), representing an element of \( K_G(T^* X|_Y) \), we extend it to an open neighborhood \( U \) of \( Y \). It is an isomorphism of the zero section. Second, we define a symbol \( \tilde{\sigma} \), restricting symbol \( \sigma \) to fibers of \( T^*_G X \) shifted in the direction of the vector field \( v \)

\[
\tilde{\sigma}(x, p) = \sigma(x, p + ve^{-p^2}),
\]

where \( (x, p) \) are local coordinates on \( T^* X \) in a neighborhood of \( Y \). Outside of \( Y \) the symbol \( \tilde{\sigma} \) is an isomorphism for all points on fibers of \( T^*_G X \) (not only outside of zero section). In other words, \( \tilde{\sigma} \) is an isomorphism everywhere in the neighborhood \( U \) outside of the fixed point set \( Y \). Hence \( \tilde{\sigma} \) represents an element of \( K_G(T^*_G U) \). Since \( U \) is open in \( X \), using the natural homomorphism \( K_G(T^*_G U) \to K_G(T^*_G X) \) we get an element of \( K_G(T^*_G X) \).

Applying this construction to the space \( X = \mathbb{C}^n \) on which \( U(1) \) acts with positive weights \( m_1, \ldots, m_n \), and taking generator of \( K(T^* \mathbb{C}^n|_0) \) associated with \( \bar{\partial} \) operator, we get its images under \( \theta^\pm \) in \( K_G(T^*_G \mathbb{C}^n) \). A direct computation shows that

\[
\text{ind } \theta^\pm [\bar{\partial}] = \left[ \prod_{i=1}^{n} \left( \frac{1}{1 - q^{-m_i}} \right) \right]_\pm.
\]

Now assume that using the vector field \( v \) it is possible to trivialize a transversally elliptic operator everywhere on \( T^*_G X \) outside of the fixed point set \( Y \), and that in a neighborhood of the fixed point set the trivialization is isomorphic to just described with some choice of \( \pm \) signs for each fixed point. Then the index is computed by summing contributions from the set of fixed points, where each contribution is regularized by an expansion in positive or negative powers of \( q \), according to the choice of sign for the \( \theta \) homomorphism.

For example in this way we get the \( U(1) \) index of the following operator on \( \mathbb{C}P^1 \):

\[
\text{ind}(f(\theta)\bar{\partial} + (1 - f(\theta))\partial) = \left[ \frac{1}{1 - q^{-1}} \right]_+ + \left[ \frac{1}{1 - q^{-1}} \right]_- \tag{D.8}
\]

Here \( \theta \) denotes the polar angle on \( \mathbb{C}P^1 \) measured from the North pole, and \( f(\theta) = \cos^2(\theta/2) \), so that the operator is approximately \( \bar{\partial} \) at the North pole and \( \partial \) at the South pole. It fails to be elliptic at the equator, but it is transversally elliptic with
respect to the canonical $U(1)$ action on $\mathbb{CP}^1$ whose fixed points are the North and South poles.

**References**

[1] E. Witten, “Topological quantum field theory,” *Commun. Math. Phys.* **117** (1988) 353.

[2] M. F. Atiyah and L. Jeffrey, “Topological Lagrangians and cohomology,” *J. Geom. Phys.* **7** (1990), no. 1 119–136.

[3] S. K. Donaldson, “An application of gauge theory to four-dimensional topology,” *J. Differential Geom.* **18** (1983), no. 2 279–315.

[4] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7** (2004) 831–864. hep-th/0206161

[5] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” hep-th/0306238

[6] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in $\text{N}=2$ supersymmetric QCD,” *Nucl. Phys.* **B431** (1994) 484–550, hep-th/9408099

[7] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in $\text{N}=2$ supersymmetric Yang-Mills theory,” *Nucl. Phys.* **B426** (1994) 19–52. hep-th/9407087

[8] A. Karlhede and M. Rocek, “Topological quantum field theory and $\text{N}=2$ conformal supergravity,” *Phys. Lett.* **B212** (1988) 51.

[9] J. K. Erickson, G. W. Semenoff, and K. Zarembo, “Wilson loops in $\text{N} = 4$ supersymmetric Yang-Mills theory,” *Nucl. Phys.* **B582** (2000) 155–175, hep-th/0003055

[10] N. Drukker and D. J. Gross, “An exact prediction of $\text{N} = 4$ SUSYM theory for string theory,” *J. Math. Phys.* **42** (2001) 2896–2914. hep-th/0010274

[11] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291. hep-th/9802150

[12] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428** (1998) 105–114, hep-th/9802109

[13] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252. hep-th/9711200

[14] J. M. Maldacena, “Wilson loops in large N field theories,” *Phys. Rev. Lett.* **80** (1998) 4859–4862. hep-th/9803002

[15] D. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” *Phys. Rev.* **D59** (1999) 105023. hep-th/9809188

[16] S.-J. Rey and J.-T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” *Eur. Phys. J.* **C22** (2001) 379–394, hep-th/9803001

[17] N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” *Phys. Rev.* **D60** (1999) 125006. hep-th/9904191

[18] G. W. Semenoff and K. Zarembo, “More exact predictions of SUSYM for string theory,” *Nucl. Phys.* **B616** (2001) 34–46. hep-th/0106015
[19] K. Zarembo, “Supersymmetric Wilson loops,” *Nucl. Phys. B* **643** (2002) 157–171, [hep-th/0205160](http://arxiv.org/abs/hep-th/0205160)

[20] K. Zarembo, “Open string fluctuations in AdS(5) x S(5) and operators with large R charge,” *Phys. Rev. D* **66** (2002) 105021, [hep-th/0209095](http://arxiv.org/abs/hep-th/0209095)

[21] A. A. Tseytlin and K. Zarembo, “Wilson loops in N =4 SYM theory: rotation in S(5),” *Phys. Rev. D* **66** (2002) 125010, [hep-th/0207241](http://arxiv.org/abs/hep-th/0207241)

[22] M. Bianchi, M. B. Green, and S. Kovacs, “Instanton corrections to circular Wilson loops in N =4 supersymmetric Yang-Mills,” *JHEP* **04** (2002) 040, [hep-th/0202003](http://arxiv.org/abs/hep-th/0202003)

[23] G. W. Semenoff and K. Zarembo, “Wilson loops in SYM theory: From weak to strong coupling,” *Nucl. Phys. Proc. Suppl.* **108** (2002) 106–112, [hep-th/0202156](http://arxiv.org/abs/hep-th/0202156)

[24] V. Pestun and K. Zarembo, “Comparing strings in AdS(5)xS(5) to planar diagrams: an example,” *Phys. Rev. D* **67** (2003) 086007, [hep-th/0212296](http://arxiv.org/abs/hep-th/0212296)

[25] N. Drukker and B. Fiol, “All-genus calculation of Wilson loops using D-branes,” *JHEP* **02** (2005) 010, [hep-th/0501109](http://arxiv.org/abs/hep-th/0501109)

[26] G. W. Semenoff and D. Young, “Exact 1/4 BPS loop: Chiral primary correlator,” *Phys. Lett. B* **643** (2006) 195–204, [hep-th/0609158](http://arxiv.org/abs/hep-th/0609158)

[27] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “On the D3-brane description of some 1/4 BPS Wilson loops,” *JHEP* **04** (2007) 008, [hep-th/0612168](http://arxiv.org/abs/hep-th/0612168)

[28] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Wilson loops: From four-dimensional SYM to two-dimensional YM,” [arXiv:0707.2699 [hep-th]](http://arxiv.org/abs/0707.2699)

[29] S. Yamaguchi, “Semi-classical open string corrections and symmetric Wilson loops,” *JHEP* **06** (2007) 073, [hep-th/0701052](http://arxiv.org/abs/hep-th/0701052)

[30] K. Okuyama and G. W. Semenoff, “Wilson loops in N =4 SYM and fermion droplets,” *JHEP* **06** (2006) 057, [hep-th/0604209](http://arxiv.org/abs/hep-th/0604209)

[31] J. Gomis and F. Passerini, “Wilson loops as D3-branes,” *JHEP* **01** (2007) 097, [hep-th/0612022](http://arxiv.org/abs/hep-th/0612022)

[32] T.-S. Tai and S. Yamaguchi, “Correlator of fundamental and anti-symmetric Wilson loops in AdS/CFT correspondence,” *JHEP* **02** (2007) 035, [hep-th/0610279](http://arxiv.org/abs/hep-th/0610279)

[33] S. Giombi, R. Ricci, and D. Trancanelli, “Operator product expansion of higher rank Wilson loops from D-branes and matrix models,” *JHEP* **10** (2006) 045, [hep-th/0608077](http://arxiv.org/abs/hep-th/0608077)

[34] B. Chen and W. He, “On 1/2-BPS Wilson–t Hooft loops,” *Phys. Rev. D* **74** (2006) 126008, [hep-th/0607024](http://arxiv.org/abs/hep-th/0607024)

[35] S. A. Hartnoll, “Two universal results for Wilson loops at strong coupling,” *Phys. Rev. D* **74** (2006) 066006, [hep-th/0606178](http://arxiv.org/abs/hep-th/0606178)

[36] N. Drukker, “1/4 BPS circular loops, unstable world-sheet instantons and the matrix model,” *JHEP* **09** (2006) 004, [hep-th/0605151](http://arxiv.org/abs/hep-th/0605151)

[37] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “More supersymmetric Wilson loops,” [arXiv:0704.2237 [hep-th]](http://arxiv.org/abs/0704.2237)

[38] C.-S. Chu and D. Giataganas, “1/4 BPS Wilson loop in beta-deformed theories,” [arXiv:0708.0797 [hep-th]](http://arxiv.org/abs/0708.0797)

[39] K. Okuyama, “t Hooft expansion of 1/2 BPS Wilson loop,” *JHEP* **09** (2006) 007, [hep-th/0607131](http://arxiv.org/abs/hep-th/0607131)
[40] G. Arutyunov, J. Plefka, and M. Staudacher, “Limiting geometries of two circular Maldacena-Wilson loop operators,” *JHEP* **12** (2001) 014, [hep-th/0111290](https://arxiv.org/abs/hep-th/0111290) [4, 7, 40]

[41] J. Plefka and M. Staudacher, “Two loops to two loops in \( N = 4 \) supersymmetric Yang-Mills theory,” *JHEP* **09** (2001) 031, [hep-th/0108182](https://arxiv.org/abs/hep-th/0108182) [3, 7, 40]

[42] V. S. Adanchik, “Contributions to the theory of the Barnes function,” [math/0308086](https://arxiv.org/abs/math/0308086) [4, 36]

[43] H. Ooguri, A. Strominger, and C. Vafa, “Black hole attractors and the topological string,” *Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146](https://arxiv.org/abs/hep-th/0405146) [4, 7]

[44] Chris Beasley, Davide Gaiotto, Monica Guica, Lisa Huang, Andrew Strominger, Xi Yin, “Why \( Z_{BH} = |Z_{top}|^2 \),” [hep-th/0608021](https://arxiv.org/abs/hep-th/0608021) [4]

[45] J. J. Duistermaat and G. J. Heckman, “On the variation in the cohomology of the symplectic form of the reduced phase space,” *Invent. Math.* **69** (1982), no. 2 259–268. [5]

[46] M. F. Atiyah and R. Bott, “The moment map and equivariant cohomology,” *Topology* **23** (1984), no. 1 1–28. [5, 20]

[47] N. Berline and M. Vergne, “Classes caract´eristiques ´equivariantes. Formule de localisation en cohomologie ´equivariante,” *C. R. Acad. Sci. Paris S´er. I Math.* **295** (1982), no. 9 539–541. [6, 29]

[48] E. Witten, “Mirror manifolds and topological field theory,” [hep-th/9112056](https://arxiv.org/abs/hep-th/9112056) [6]

[49] I. M. Singer, “Recent applications of index theory for elliptic operators,” in *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pp. 11–31. Amer. Math. Soc., Providence, R.I., 1973. [7, 30, 31, 33, 54]

[50] M. F. Atiyah, *Elliptic operators and compact groups*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 401. [7, 30, 31, 33, 54, 55, 57]

[51] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, “Supersymmetric Wilson loops on \( S^3 \),” [arXiv:0711.3226 [hep-th]](https://arxiv.org/abs/0711.3226) [7]

[52] A. Kapustin and E. Witten, “Electric-magnetic duality and the geometric Langlands program,” [hep-th/0604151](https://arxiv.org/abs/hep-th/0604151) [7, 27]

[53] A. Kapustin, “Holomorphic reduction of \( N = 2 \) gauge theories, Wilson–’t Hooft operators, and S-duality,” [hep-th/0612119](https://arxiv.org/abs/hep-th/0612119) [7, 38]

[54] A. Kapustin, “Wilson–’t Hooft operators in four-dimensional gauge theories and S-duality,” *Phys. Rev.* **D74** (2006) 025005, [hep-th/0501015](https://arxiv.org/abs/hep-th/0501015) [7]

[55] S. Cecotti and C. Vafa, “Topological antitopological fusion,” *Nucl. Phys.* **B367** (1991) 359–461. [7]

[56] L. Brink, J. H. Schwarz, and J. Scherk, “Supersymmetric Yang-Mills theories,” *Nucl. Phys.* **B121** (1977) 77. [8]

[57] H. Baum, “Conformal killing spinors and special geometric structures in lorentzian geometry — a survey,” [math/0202008](https://arxiv.org/abs/math/0202008) [10]

[58] C. Vafa and E. Witten, “A Strong coupling test of S duality,” *Nucl. Phys.* **B431** (1994) 3–77, [hep-th/9408074](https://arxiv.org/abs/hep-th/9408074) [13, 27, 44]

[59] N. Berkovits, “A ten-dimensional superYang-Mills action with off-shell supersymmetry,” *Phys. Lett.* **B318** (1993) 104–106, [hep-th/9308128](https://arxiv.org/abs/hep-th/9308128) [17, 18]

[60] J. M. Evans, “Supersymmetry algebras and Lorentz invariance for \( d = 10 \) superYang-Mills,” *Phys. Lett.* **B334** (1994) 105–112, [hep-th/9404190](https://arxiv.org/abs/hep-th/9404190) [17]
[61] L. Baulieu, N. J. Berkovits, G. Bossard, and A. Martin, “Ten-dimensional super-Yang-Mills with nine off-shell supersymmetries,” arXiv:0705.2002 [hep-th].

[62] J. M. F. Labastida and C. Lozano, “Mathai-Quillen formulation of twisted N =4 supersymmetric gauge theories in four dimensions,” Nucl. Phys. B502 (1997) 741–790. hep-th/9702106.

[63] E. Witten, “Two-dimensional gauge theories revisited,” J. Geom. Phys. 9 (1992) 303–368. hep-th/9204083.

[64] P. Lavaud, “Equivariant cohomology and localization formula in supergeometry,” math/0402068.

[65] P. Lavaud, “Superpfaffian,” math/0402067.

[66] M. F. Atiyah and I. M. Singer, “The index of elliptic operators. I,” Ann. of Math. (2) 87 (1968) 484–530.

[67] M. F. Atiyah and G. B. Segal, “The index of elliptic operators. II,” Ann. of Math. (2) 87 (1968) 531–545.

[68] M. F. Atiyah and I. M. Singer, “The index of elliptic operators. III,” Ann. of Math. (2) 87 (1968) 546–604.

[69] M. F. Atiyah and R. Bott, “A Lefschetz fixed point formula for elliptic differential operators,” Bull. Amer. Math. Soc. 72 (1966) 245–250.

[70] M. F. Atiyah and R. Bott, “A Lefschetz fixed point formula for elliptic complexes. I,” Ann. of Math. (2) 86 (1967) 374–407.

[71] M. F. Atiyah and R. Bott, “A Lefschetz fixed point formula for elliptic complexes. II. Applications,” Ann. of Math. (2) 88 (1968) 451–491.

[72] N. Berline and M. Vergne, “L’indice équivariant des opérateurs transversalement elliptiques,” Invent. Math. 124 (1996), no. 1-3 51–101.

[73] N. Berline and M. Vergne, “The equivariant Chern character and index of G-invariant operators. Lectures at CIME, Venice 1992,” in D-modules, representation theory, and quantum groups (Venice, 1992), vol. 1565 of Lecture Notes in Math., pp. 157–200. Springer, Berlin, 1993.

[74] E. Witten, “Holomorphic Morse inequalities,” in Algebraic and differential topology—global differential geometry, vol. 70 of Teubner-Texte Math., pp. 318–333. Teubner, Leipzig, 1984.

[75] P. C. Argyres and N. Seiberg, “S-duality in N=2 supersymmetric gauge theories,” arXiv:0711.0054 [hep-th].

[76] R. Flume and R. Poghossian, “An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential,” Int. J. Mod. Phys. A18 (2003) 2541, hep-th/0208176.

[77] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, “Multi-instanton calculus and equivariant cohomology,” JHEP 05 (2003) 054, hep-th/0211108.

[78] H. Nakajima and K. Yoshioka, “Instanton counting on blowup. I. 4-dimensional pure gauge theory,” math/0306198.

[79] H. Nakajima and K. Yoshioka, “Lectures on instanton counting,” math/0311058.

[80] T. Okuda and V. Pestun, “On the instantons and the hypermultiplet mass of N=2* super Yang-Mills,”.
[81] C. P. Bachas, P. Bain, and M. B. Green, “Curvature terms in D-brane actions and their M-theory origin,” *JHEP* 05 (1999) 011, [hep-th/9903210](https://arxiv.org/abs/hep-th/9903210).

[82] R. Flume, F. Fucito, J. F. Morales, and R. Poghossian, “Matone’s relation in the presence of gravitational couplings,” *JHEP* 04 (2004) 008, [hep-th/0403057](https://arxiv.org/abs/hep-th/0403057).

[83] A. S. Losev, A. Marshakov, and N. A. Nekrasov, “Small instantons, little strings and free fermions,” [hep-th/0302191](https://arxiv.org/abs/hep-th/0302191).

[84] H. Lu, C. N. Pope, and J. Rahmfeld, “A construction of Killing spinors on $S^n$,” *J. Math. Phys.* 40 (1999) 4518–4526, [hep-th/9805151](https://arxiv.org/abs/hep-th/9805151).

[85] M. W. Goodman and E. Witten, “Global symmetries in four-dimensions and higher dimensions,” *Nucl. Phys.* B271 (1986) 21.

[86] M. F. Atiyah, *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.