Examples of infinitesimal non-trivial accumulation of secants in dimension three

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Abstract

We present non-trivial examples of accumulation of secants for orbits (of real analytic three dimensional vector fields) having the origin as only $\omega$-limit point. These non-trivial sets have the structure of a proper algebraic variety of $\mathbb{S}^2$ intersected with a cone.

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1 Introduction

Consider a real analytic vector-field $X$ and a regular orbit $γ(t)$ of $X$. In this work we are interested in the study of how the orbit $γ(t)$ “converges”. In a first level, this study concerns the $ω$-limit of the orbit $γ(t)$, i.e. the set:

$$ω(γ(t)) = \{ p \in M; \exists (t_n)_{n \in \mathbb{N}}, t_n \to \infty, γ(t_n) \to p \}$$

But the study of “convergence” may go a little farther. For example, even in the simple case that the orbit $γ(t)$ is converging to an isolated singularity, one can ask how the orbit is converging. This question is in the spirit of Poincaré, Lyapunov and Lefschetz (see [Le, Ly, P]) on the qualitative behavior of trajectories of vector-fields.

More precisely, suppose that the origin is an isolated singularity of the vector-field $X$ and the $ω$-limit of the orbit $γ(t)$ is also the origin. The secants curve of $γ(t)$ is the curve $\frac{γ(t)}{∥γ(t)∥} \in \mathbb{S}^{n-1}$, where $n$ is the dimension of the ambient space. We want to study the set of accumulation of secants:

$$Sec(γ(t)) = \bigcap_s \{ γ(t)/∥γ(t)∥; t \geq s \} \subset \mathbb{S}^{n-1}$$

To exemplify this study, let us consider planar vector-fields with isolated singularities. The Poncaré-Bendixson Theorem gives a good description of the $ω$-limit of a regular orbit. Moreover, by an argument of Khovanskii (see [Kh, CMoSa1]), there exists a dichotomy for analytic vector-fields:

- Either, the accumulation of secants $Sec(γ(t))$ is an unique point for all regular orbits converging to the origin, or;

- The accumulation of secants $Sec(γ(t))$ is all the circle $\mathbb{S}^1$ for all the regular orbits converging to the origin.

For dimension bigger than 2, the problem is much more complicated. For example, a particular case of this study is the famous Thom’s gradient conjecture, proved in 2000 by Kurdyka, Mostowski and Parusinski (see [KuMP]). The result states that if $X$ is an analytic gradient vector-field (i.e. $X = \nabla f$, where $f$ is an analytic function) and the origin is an isolated
singularity of $X$, then for all regular orbits $\gamma(t)$ converging to 0, the accumulation of secants $Sec(\gamma(t))$ is an unique point of $S^{n-1}$.

In this work, we will focus on three dimensional vector-fields. In this case, the dichotomy presented for planar vector-fields is not true, and the “fauna” of possible accumulation of secants sets is not yet classified. We can mention the following recent contributions:

- Alonso-González, Cano and Rosas in [GCR] give a description for the accumulation of secants $Sec(\gamma(t))$ under the hypotheses that the origin is a generic absolutely isolated singularity (see definition in the same article);
- Cano, Mossu and Sanchez in [CMoSa1, CMoSa2, CMoSa3] and Cano, Mossu and Rolin in [CMoR] studies properties of orbits under some additional oscillating hypotheses (as, the so-called, sub-analytically non-oscillating property).

In this work, we present examples of polynomial three dimensional vector-fields with a regular orbit $\gamma(t)$ such that its accumulation of secants $Sec(\gamma(t))$ is a non-trivial set.

In order to be precise, consider a non-zero real-polynomial $h \in \mathbb{R}[x, y]$ and let $\Gamma$ be the semi-algebraic set $V(h) \cap B_1(0)$, where $B_1(0)$ is the closed ball with radius one and center at the origin and $V(h) = \{p \in \mathbb{R}^2; h(p) = 0\}$ is the algebraic variety given by the zeros of the polynomial $h$. We say that the polynomial $h$ is adapted if:

- The set $\Gamma$ is connected and non-empty;
- The variety $V(h, \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h) = V(h, h_x, h_y)$ is equal to a finite set of points;
- The set $V(h) \cap \partial B_1(0)$ is a finite set of points such that $< \nabla h(p), p > \neq 0$, where $\partial B_1(0)$ stands for the border of $B_1(0)$.

Now, consider the following chart of $S^2$:

$$\alpha : \mathbb{R}^2 \to S^2 \quad (x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, \frac{1}{\sqrt{x^2+y^2+1}}\right)$$

With this notation, the main result of this work can be finally enunciated:
Theorem 1.0.1. Let \( h \in \mathbb{R}[x, y] \) be an adapted polynomial. There exists a real three dimensional polynomial vector-field \( X \) and a regular orbit \( \gamma(t) \) of \( X \) such that:

i ) The \( \omega \)-limit of the orbit \( \gamma(t) \) is an isolated singularity of \( X \);

ii ) The accumulation of secants \( \text{Sec}(\gamma(t)) \) is equal to \( \alpha(\Gamma) \).

In this work we are actually going to prove a slightly stronger result (intended for more general sets \( \Gamma \)). The proof of the above Theorem is a direct consequence of Theorem 2.4.1 and Proposition 2.4.2. Nevertheless, it is worth remarking that Propositions 2.1.3 and 2.3.1 are enough to prove property \([ii]\) of the Theorem.

This paper is divided in two sections. In section 2 we give the explicit expression of the vector-field of Theorem 1.0.1 and prove the Theorem assuming a topological result (Proposition 2.2.2 below). In section 3 we prove this topological result with a desingularization process.

We would also like to remark that the ideas behind this work may lead to a stronger result. More precisely, we believe that it should be possible to prove that any semi-algebraic variety of \( \mathbb{S}^2 \) (maybe satisfying some generic condition) may be realized as an accumulation of secants set.

2 Explicit construction of the vector-field

In this section we will construct the explicit expression of the vector-field enunciated in Theorem 1.0.1. To simplify the treatment, we will regard the convergence of the orbit \( \gamma(t) \) after the projective blowing-up of the origin:

\[
\tau : \mathbb{R} \times \mathbb{P}^2 \rightarrow \mathbb{R}^3
\]

instead of the secants of \( \gamma(t) \). We remark that this treatment will lead to the desired result because \( \mathbb{P}^2 \) is a double cover of \( \mathbb{S}^2 \).
2.1 The $\Gamma$-convergence

Consider a real polynomial $H \in \mathbb{R}[x, y, z]$ and a compact and connected sub-set $\Gamma$ of $\mathbb{R}^2$. Let $L_{z_0}$ be the variety $V(H, z - z_0)$ for any constant $z_0$ in $\mathbb{R}$. We say that the polynomial $H$ is $\Gamma$-convergent if the varieties $L_{z_0}$, with positive constants $z_0$, converge (in the Hausdorff topology) to $\Gamma$ when $z_0$ converges to zero, i.e. $L_{z_0} \xrightarrow{z_0 \to 0^+} \Gamma \times \{0\}$.

Remark 2.1.1. The fact that a polynomial $H$ is $\Gamma$-convergent does not imply that the variety $L_0$ is equal to $\Gamma$.

We will further say that $H$ is:

- **connected** $\Gamma$-convergent if there exists a positive real number $\epsilon$ such that $L_z$ is a connected variety for $0 < z < \epsilon$;

- **smooth** $\Gamma$-convergent if there exists a positive real number $\epsilon$ such that $L_z$ is a smooth variety for $0 < z < \epsilon$.

Remark 2.1.2. If $H$ is connected $\Gamma$-convergent then, for a small enough positive real number $\epsilon$, the semi-analytic variety $\mathcal{C}_\epsilon = V(H) \cap \{0 < z < \epsilon\}$ is homeomorphic to a topological cylinder.

If $\Gamma$ is equal to a semi-analytic variety $V(h) \cap \{x^2 + y^2 \leq 1\}$, where $h$ is an adapted polynomial (see definition in the introduction), then Propositions [2.1 − 2.2] of [Bel] guarantees the existence of a polynomial $H$ that is smooth connected $\Gamma$-convergent. More explicitly, in [Bel] it is consider the family of polynomials:

$$H_\alpha(x, y, z) = h(x, y)^2 + g(x, y, z) + \alpha z^4$$

where the parameter $\alpha$ takes values in the interval $[0, 1]$ and the polynomial $g(x, y, z)$ has the form $z(\tilde{g}(x, y, z) + \sum_{i=1}^{N} \binom{N}{i} x^2 y^{2(N-i)})$ for some $N \in \mathbb{N}$, where $\tilde{g}(x, y, z)$ is a polynomial with degree strictly smaller than $2N$. We remark that the explicit expression of the polynomial $g(x, y, z)$ can be found in [Bel], but is not important for this work. Finally, Propositions [2.1 − 2.2] of [Bel] are sufficient to prove the following:
Proposition 2.1.3. If \( \Gamma \) is equal to a semi-analytic variety \( V(h) \cap \{ x^2 + y^2 \leq 1 \} \), where \( h \) is an adapted polynomial, then there exists a polynomial \( H \) smooth connected \( \Gamma \)-convergent. Furthermore, such a polynomial can be chosen as the polynomials \( H_\alpha(x, y, z) \) for almost all \( \alpha \) fixed.

In what follows we will work with any polynomial \( H(x, y, z) \) that is smooth connected \( \Gamma \)-convergent.

2.2 Construction in a chart

We say that a real three dimension vector-field \( X \) is \( \Gamma \)-convergent if the variety \( \{ z = 0 \} \) is invariant by \( X \), i.e. \( X \) is tangent to \( \{ z = 0 \} \), and there exists an orbit \( \gamma(t) \) of \( X \) contained in \( \{ z > 0 \} \) such that the \( \omega \)-limit of \( \gamma(t) \) is equal to \( \Gamma \times \{ 0 \} \).

In this section, we prove that the existence of a polynomial smooth connected \( \Gamma \)-convergent implies the existence of a vector-field \( \Gamma \)-convergent. More precisely, let \( H \) be smooth connected \( \Gamma \)-convergent and consider:

- The \( H \)-horizontal vector-field \( \mathcal{H}(H) = -H_y \frac{\partial}{\partial x} + H_x \frac{\partial}{\partial y} \);
- The \( H \)-vertical vector-field \( \mathcal{V}(H) = \mathcal{V}(H) := H_x \frac{\partial}{\partial x} + H_y \frac{\partial}{\partial y} - (H_x^2 + H_y^2) \frac{\partial}{\partial z} \);
- The \( H \)-perturbed vector-field \( \mathcal{P}(H) := H \frac{\partial}{\partial z} \).

Notice that, for a small enough \( \epsilon > 0 \):

- The \( H \)-horizontal vector-field \( \mathcal{H}(H) \) is the Hamiltonian of the function \( H \) in respect to the variables \( (x, y) \);
- The \( H \)-horizontal vector-field \( \mathcal{H}(H) \) and the \( H \)-vertical vector-field \( \mathcal{V}(H) \) forms a base of \( T_p V(H) \) for all points \( p \) in the intersection of \( V(H) \) and \( \{ 0 < z < \epsilon \} \).
- The \( H \)-perturbed vector-field \( \mathcal{P}(H) \) is identically zero in \( V(H) \).

Now, consider the polynomial family of vector-fields:

\[
X_{\beta_1, \beta_2}(H) = \mathcal{H}(H) + z^{\beta_1} \mathcal{V}(H) + z^{\beta_2} \mathcal{P}(H)
\]
for constants $\beta_1$ and $\beta_2$ in $\mathbb{N} \setminus \{0\}$. The main result of this section can now be formulated precisely:

**Proposition 2.2.1.** Suppose that there exists a polynomial $H$ smooth connected $\Gamma$-convergent. Then, for any fixed constants $\beta_1$ and $\beta_2$ in $\mathbb{N} \setminus \{0\}$, the vector-field $X_{\beta_1,\beta_2}(H)$ constructed above is $\Gamma$-convergent.

**Proof.** Fix constants $\beta_1$ and $\beta_2$ in $\mathbb{N} \setminus \{0\}$ and let $X$ denote the vector-field $X_{\beta_1,\beta_2}(H)$.

**Claim 1:** The vector-field $X$ is tangent to $V(H)$.

**Proof.** Indeed, notice that the derivations $\mathcal{H}(H)$ and $\mathcal{V}(H)$ applied to the polynomial $H$ is equal to zero and that the derivation $\mathcal{P}(H)$ is zero everywhere over the variety $V(H)$. □

**Claim 2:** The vector-field $X$ is tangent to $V(z)$.

**Proof.** Indeed, notice that the constants $\beta_1$ and $\beta_2$ are positive and the derivation $\mathcal{H}(H)$ applied to the polynomial $z$ is equal to zero. □

**Claim 3:** Given $\gamma(t)$ an orbit of $X$ passing through the semi-analytic set $C_\epsilon = V(H) \cap \{0 < z < \epsilon\}$, with $\epsilon$ small enough, the $\omega$-limit of $\gamma(t)$ is a subset of $\Gamma \times \{0\}$.

**Proof.** Indeed, notice that for a small enough positive real number $\epsilon$:

I Since $H$ is smooth $\Gamma$-convergent, the intersection of the variety $V(H, H_x, H_y)$ with the set $\{0 < z < \epsilon\}$ is empty. Thus, the vector-field $X$ is non-singular in the semi-analytic set $C_\epsilon$;

II The function $L := z$ is a strict negative Lyapunov function of $X$ over the semi-analytic set $C_\epsilon$. Indeed, for a point $p$ in $C_\epsilon$:

$$X(L)(p) = [-z^{\beta_1}(H_x^2 + H_y^2) + z^{\beta_2}H](p) = [-z^{\beta_1}(H_x^2 + H_y^2)](p) < 0$$

So, if $\gamma(t)$ is an orbit of $X$ passing through $C_\epsilon$, we conclude that:

- By [I], $\gamma(t)$ is a regular orbit totally contained in $C_\infty$;
• By [II], the \( \omega \)-limit of \( \gamma(t) \) must be contained in the intersection of \( \{ z = 0 \} \) and the topological adherence of \( C_\epsilon \), which is \( \Gamma \times \{ 0 \} \) (because \( H \) is \( \Gamma \)-convergent).

To complete the proof of Proposition 2.2.1 we need to prove the following result:

**Proposition 2.2.2.** Given \( \gamma(t) \) an orbit of \( X \) passing through the semi-analytic set \( C_\epsilon \) for some \( \epsilon \) small enough, the \( \omega \)-limit of \( \gamma(t) \) is equal to \( \Gamma \times \{ 0 \} \).

Section 3 is entirely dedicated to the proof of this Proposition. The proof of Proposition 2.2.1 is now over.

### 2.3 Globalization

In this section, we prove that the existence of a polynomial \( H \) smooth connected \( \Gamma \)-convergent implies the existence of a vector-field with an orbit \( \gamma(t) \) such that its accumulation of secants set \( \text{Sec}(\gamma(t)) \) is \( \alpha(\Gamma) \) (see the definition of the morphism \( \alpha : \mathbb{R}^2 \rightarrow S^2 \) in the introduction).

More precisely, let \( H \) be a smooth connected \( \Gamma \)-convergent and consider the subset of polynomials:

\[
\mathcal{O}_{\mathbb{R}^3}' = \{ f \in \mathcal{O}_{\mathbb{R}^3}; f(0,0,0) = 0 \text{ and } f(x,y,0) \neq 0 \}
\]

and the function:

\[
\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}^3
\]

\[
(x, y, z) \mapsto (xz, yz, z)
\]

which is also equal to a blowing-up restricted to one of its charts. Then, there exists two unique functions:

\[
\phi : \mathcal{O}_{\mathbb{R}^3}' \rightarrow \mathbb{N}^*, \quad \psi : \mathcal{O}_{\mathbb{R}^3} \rightarrow \mathcal{O}_{\mathbb{R}^3}'
\]

such that:

\[
z^{\phi(f)} f = \psi(f) \circ \sigma
\]

Now, consider the constants:

• \( \alpha = \max\{ \phi(H_y), \phi(H_x) \} \);

• \( \beta_1 = 2[\phi(H_x) + \phi(H_y)] + \phi(H_z) - \alpha + 2 \) (in particular \( \beta_1 \geq 1 \));
\[ \beta_2 = \phi(H) - \alpha + 1 \text{ (in particular } \beta_2 \geq 1) \]

And the vector-field:
\[
Y = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z})
\]

where:

- \( A = -z^{a_1} \psi(H_y) + z^{a_2} \psi(H_x) \psi(H_z) \) where:
  \[ a_1 = \alpha + 1 - \phi(H_y) \geq 1 \]
  \[ a_2 = \alpha + \beta_1 + 1 - \phi(H_x) - \phi(H_z) \geq 1 \]

- \( B = z^{b_1} \psi(H_x) + z^{b_2} \psi(H_y) \psi(H_z) \) where:
  \[ b_1 = \alpha + 1 - \phi(H_x) \geq 1 \]
  \[ b_2 = \alpha + \beta_1 + 1 - \phi(H_y) - \phi(H_z) \geq 1 \]

- \( C = -z^{c_1} \psi(H_x)^2 - z^{c_2} \psi(H_y)^2 + \psi(H) \) where:
  \[ c_1 = \alpha + \beta_1 - 2 \phi(H_x) \geq 1 \]
  \[ c_2 = \alpha + \beta_1 - 2 \phi(H_y) \geq 1 \]

The main result of this section can now be formulated precisely:

**Theorem 2.3.1.** Suppose that there exists a polynomial \( H \) smooth connected \( \Gamma \)-convergent. Then, the vector-field \( Y \) constructed above has a regular orbit \( \gamma(t) \) such that the accumulation of secants \( \text{Sec}(\gamma(t)) \) is equal to \( \alpha(\Gamma) \) (see the definition of the morphism \( \alpha : \mathbb{R}^2 \to S^2 \) in the introduction).

**Proof.** Consider the blowing-up of \( \mathbb{R}^3 \) with the origin as center:
\[
\tau : \mathbb{R} \times \mathbb{P}^2 \to \mathbb{R}^3
\]

and let \( Y' \) be the strict transform of the vector-field \( Y \). Then, in the \( z \)-chart, the strict transform \( Y' \) is equal to \( X_{\beta_1, \beta_2}(H) \). Indeed:

\[
Y' = z^{-\alpha} Y^* = z^{-(\alpha+1)} A^* \frac{\partial}{\partial x} + z^{-(\alpha+1)} B^* \frac{\partial}{\partial y} + z^{-\alpha} C^*(z \frac{\partial}{\partial z})
\]

where:
\[ z^{-(\alpha+1)}A^\ast = -z^{\alpha_1} H_y + z^{\alpha_3} H_x H_z \text{ where:} \]
\[ a_1^\ast = \alpha + 1 - \phi(H_y) + \phi(H_y) - \alpha - 1 = 0 \]
\[ a_2^\ast = \alpha + \beta_1 + 1 - \phi(H_x) + \phi(H_z) + \phi(H_z) - \alpha - 1 = \beta_1 \]

\[ z^{-(\alpha+1)}B^\ast = -z^{b_1^\ast} H_y + z^{b_2^\ast} H_x H_z \text{ where:} \]
\[ b_1^\ast = \alpha + 1 - \phi(H_x) + \phi(H_x) - \alpha - 1 = 0 \]
\[ b_2^\ast = \alpha + \beta_1 + 1 - \phi(H_y) + \phi(H_y) + \phi(H_z) - \alpha - 1 = \beta_1 \]

\[ z^{-\alpha}C^\ast = -z^{c_1^\ast} H_x^2 - z^{c_2^\ast} H_y^2 + z^{c_3^\ast} H \text{ where:} \]
\[ c_1^\ast = \alpha + \beta_1 - 1 - 2\phi(H_x) + 2\phi(H_x) - \alpha = \beta_1 - 1 \]
\[ c_2^\ast = \alpha + \beta_1 - 1 - 2\phi(H_y) + 2\phi(H_y) - \alpha = \beta_1 - 1 \]
\[ c_3^\ast = \phi(H) - \alpha = \beta_2 - 1 \]

Since, by Proposition 2.2.1, the vector-field \( X_{\beta_1,\beta_2}(H) \) is \( \Gamma \)-convergent, the thesis clearly follows.

\[ \square \]

### 2.4 Isolated singularity

In this subsection, we want to prove that the vector-field \( Y \) of Theorem 2.3.1 can be chosen so that the origin is an isolated singularity. For this end, we need to strengthen the hypotheses over the polynomial \( H \Gamma \)-convergent.

We say that a polynomial \( H \) is isolated \( \Gamma \)-convergent if:

i ) There exists a positive real number \( \epsilon \) such that \( L_z \) is a smooth variety for \( -\epsilon < z < 0 \);

ii ) The origin of \( \mathbb{R}^2 \) is an isolated solution of the equation in two variables \( \{ \psi(H)(x, y, 0) = 0 \} \).

With this definition, we can prove the following result:

**Theorem 2.4.1.** Suppose that there exists a polynomial \( H \) isolated smooth connected \( \Gamma \)-convergent. Then, the origin is an isolated singularity of the vector-field \( Y \) constructed in Theorem 2.3.1.

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Proof. Notice that the vector-field \( Y \) constructed in Theorem 2.3.1 is such that:

\[
Y(x, y, 0) = C(x, y, 0)(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})
\]

and, by construction:

\[
C(x, y, 0) = [-z^{c_1} \psi(H_x)^2 - z^{c_2} \psi(H_y)^2 + \psi(H)](x, y, 0) = \psi(H)(x, y, 0)
\]

because the natural constants \( c_1 \) and \( c_2 \) are not 0. Since \( H \) is isolated \( \Gamma \)-convergent, we conclude that there exists a positive real number \( \delta \) such that the intersection of the singularities of \( Y \) intersected with the variety \( V(z) \), with the ball of radius \( \delta \) and center in the origin is just the origin, i.e.:

\[
\text{Sing}(Y) \cap \{z = 0\} \cap B_\delta(0) = \{(0, 0, 0)\}
\]

Thus, we only need to search singularities of \( Y \) outside the plane \( V(z) \), which can be done in the \( z \)-chart of the blowing-up of the origin. By construction, after blowing-up the origin the vector-field \( Y \) is transformed into the vector-field \( X_{\beta_1, \beta_2}(H) \). We now study the singularities of this vector-field:

Claim: The set of singularities \( \text{Sing}(X_{\beta_1, \beta_2}) \) is equal to the variety \( V(zH, H_x, H_y) \).

Proof. By construction of the vector-field \( X_{\beta_1, \beta_2}(H) \), it is clear that \( V(zH, H_x, H_y) \subset \text{Sing}(X_{\beta_1, \beta_2}) \). So consider a point \( p \) in \( \text{Sing}(X_{\beta_1, \beta_2}(H)) \). We have that:

\[
\begin{align*}
[H_y - z^{\beta_1} H_x H_x] (p) &= 0 \\
[H_x + z^{\beta_1} H_x H_y] (p) &= 0 \\
[-z^{\beta_1}(H_x^2 + H_y^2) + z^{\beta_2} H] (p) &= 0
\end{align*}
\]

From the first equation, we have that \( H_y(p) = z^{\beta_1} H_x H_x(p) \). Replacing this expression in the second equation, we get that \( [H_x(1 + z^{2\beta_1} H_x^2)](p) = 0 \), which implies that \( H_x(p) = 0 \). Thus, \( H_y(p) = 0 \). Replacing this in the last equation, we finally get that either \( z(p) = 0 \) or \( H(p) = 0 \).

Notice that the hypotheses over the polynomial \( H \) implies that the varieties \( L_z \) are all smooth for \( z \) small enough (positive or negative) different from zero. This clearly implies
that the intersection of the variety $V(H, H_x, H_y)$ with the set $\{-\epsilon < z < \epsilon\}$ is all contained in $\{z = 0\}$. Furthermore, by the Claim, taking a smaller $\delta > 0$ if necessary, the singularities of the vector-field $X_{\beta_1, \beta_2}(H)$ intersected with the set $\{-\epsilon < z < \epsilon\}$ is all contained in $\{z = 0\}$ (which is the exceptional divisor). This finally implies that:

$$\text{Sing}(Y) \cap B_\delta(0) = \{(0, 0, 0)\}$$

and the origin is an isolated singularity.

To finish the proof of Theorem 1.0.1 we need to prove the existence of polynomials $H$ isolated smooth connected $\Gamma$-convergent.

**Proposition 2.4.2.** If $\Gamma$ is equal to a semi-analytic variety $V(h) \cap \{x^2 + y^2 \leq 1\}$, where $h$ is an adapted polynomial, then there exists a polynomial $H$ isolated smooth connected $\Gamma$-convergent.

**Proof.** We recall that:

$$H_\alpha(x, y, z) = h(x, y)^2 + g(x, y, z) + \alpha z^4$$

where the parameter $\alpha$ takes values in the interval $[0, 1]$ and the polynomial $g(x, y, z)$ has the form $z(\bar{g}(x, y, z) + \sum_{i=1}^{N} \binom{N}{i} x^{2i} y^{2(N-i)})$ for some $N \in \mathbb{N}$, where $\bar{g}(x, y, z)$ is a polynomial with degree strictly smaller than $2N$.

In particular, notice that $(0, 0)$ is an isolated solution of the equation $\psi(g)(x, y, 0) = 0$. So, if we can choose $g$ such that $\phi(g) \geq \phi(h^2)$ and $\phi(g) > 4$, we have that $(0, 0)$ is an isolated solution of $\psi(H)(x, y, 0) = 0$.

But this property is not generally true for the polynomial $g(x, y, z)$ given in [Bel]. So consider $f(x, y) = ((x - x_0)^2 + (x - y_0)^2)^M$ where $(x_0, y_0)$ is a point outside the ball with radius one and the variety $V(h)$, i.e. $p \notin V(h) \cup B_1(0)$, and $M$ is a natural number sufficiently big. It is clear that:

- The polynomial $g(x, y, z)f(x, y)$ is equal to $z(\bar{g}(x, y, z) + \sum_{i=1}^{N+M} \binom{N+M}{i} x^{2i} y^{2(N-i)})$, where $\bar{g}(x, y, z)$ is a polynomial with degree strictly smaller than $2(N + M)$;
• For $M$ sufficiently big, we have that: $\phi(gf) \geq \phi(f^2)$ and $\phi(hg) > 4$.

So, we consider:

$$\tilde{H}_\alpha(x, y, z) = h(x, y)^2 + f(x, y)g(x, y, z) - \alpha z^4$$

It is now clear that property $[ii]$ of the definition of isolated $\Gamma$-convergent is satisfied. The proof that this polynomials (for almost all $\alpha$ in $[0, 1]$) is isolated smooth connected $\Gamma$-convergent now follows, mutatis mutandis, the same proof of Propositions [2.1 − 2.2] of [Bel]. Nevertheless, we remark two important details for the necessary adaptations:

• Lemma 3.3 of [Bel] contains the proof of property $[i]$ of the definition of isolated $\Gamma$-convergent, although it does not enunciate it;

• The polynomial $f(x, y)$ is a positive unity over any small enough neighborhood of $B_1(0)$.

3 Proof of Proposition 2.2.2

The proof of Proposition 2.2.2 is given in the last subsection of this section. In the first three subsections we present three important back-grounds. We hope that this presentation will help readers that are not used with desingularization algorithms.

3.1 Back-ground: Length of a curve

An Euclidean metric over $\mathbb{R}^n$ is a Riemannian metric $g$ over $\mathbb{R}^n$ such that there exists a coordinate system $x = (x_1, ..., x_n)$ where $g = \sum_{i=1}^n dx_i \otimes dx_i$.

Now, consider an analytic curve $\gamma(t)$ over $\mathbb{R}^n$ defined for $t \in [0, +\infty[$ and $g$ any Riemannian metric over $\mathbb{R}^n$. The length of $\gamma(t)$ by $g$ is the number $\text{length}_g(\gamma(t)) = \int_0^\infty \|\dot{\gamma}(t)\|_g^g dt$, where $\|\cdot\|_g^g$ stands for the norm associated to $g$.

Given $F : \mathbb{R}^n \to \mathbb{R}^m$ a smooth function and $g$ a Riemannian metric over $\mathbb{R}^m$, the pull-back of
the metric $g$ by $F$, denoted by $F^*g$, is defined by the formula $F^*g_p(v, w) = g_{F(p)}(dF.v, dF.w)$. We remark that the pull-back of a Riemannian metric may not be Riemannian. In what follows, we will need the following result:

**Lemma 3.1.1.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function, $g$ an Euclidean metric over $\mathbb{R}^n$ and $\gamma(t)$ a curve such that $\lim_{t \to +\infty} \gamma(t)$ is an unique point $q$ in $\mathbb{R}^n$. Let $\tilde{\gamma}(t)$ be the image of $\gamma(t)$ by $F$, i.e. $\tilde{\gamma}(t) := F \circ \gamma(t)$. Then, if $\text{length}_g(\tilde{\gamma}(t)) = +\infty$, we have that $\text{length}_g(\gamma(t)) = +\infty$.

**Proof.** We choose a coordinate system $x = (x_1, ..., x_n)$, such that $q = (0, ..., 0)$ and $g = \sum dx_i \otimes dx_i$. Furthermore, without loss of generality, we assume that $F(q) = (0, ..., 0)$ (otherwise, make a translation).

In this coordinates, $F(x) = (f_1(x), ..., f_n(x))$, where each $f_i : \mathbb{R}^n \to \mathbb{R}$ is smooth. This implies that:

$$F^*(g) = \sum_i \sum_j \left( \sum_k \frac{\partial}{\partial x_i} f_k \frac{\partial}{\partial x_j} f_k \right) dx_i \otimes dx_j$$

In particular, for a point $p$ in $\mathbb{R}^n$ and a vector $v = (v_1, ..., v_n)$ in $T_p \mathbb{R}^n$:

$$\|v\|_{F^*(g)}^2 = \left( \sum_i \left( \sum_k \frac{\partial}{\partial x_i} f_k \right)^2 \right) v_i^2 + 2 \left( \sum_i \sum_k \frac{\partial}{\partial x_i} f_k \frac{\partial}{\partial x_j} f_k \right) v_i v_j$$

Now, let $U$ be a sufficiently small locally compact open neighborhood of $q$. There exists real numbers $\epsilon_{i,j} > 0$ for $i, j \leq n$ such that, for all points $p$ in $U$:

$$\left\| \sum_k \frac{\partial}{\partial x_i} f_k \frac{\partial}{\partial x_j} f_k \right\| \leq \epsilon_{i,j}$$

Furthermore, since $2ab \leq a^2 + b^2$, there exists $\epsilon > 0$ such that:

$$\|v\|_{F^*(g)}^2 \leq \epsilon \|v\|_{g}^2$$

Now, without loss of generality we can suppose that $\gamma(t)$ is contained in $U$. We also remark that, by the definition of pull-back $\|dF.v\|_{F^*(g)}^2 = \|v\|^2_{F^*(g)}$. So, if $\text{length}_g(\tilde{\gamma}(t)) = +\infty$:

$$+\infty = \int_0^\infty \|\tilde{\gamma}(t)\|_{\tilde{\gamma}(t)}^g dt = \int_0^\infty \|dF_{\gamma(t)} \gamma(t)\|_{F_{\gamma(t)}}^g dt = \int_0^\infty \|\gamma(t)\|_{\gamma(t)}^g dt \leq \epsilon \int_0^\infty \|\gamma(t)\|_{\gamma(t)}^g dt$$

which finally implies that $\text{length}_g(\gamma(t)) = +\infty$. \qed
3.2 Back-ground: Elementary vector-fields

We follow closely the presentation of [I1]. Two vector fields are *orbitally equivalent* in a neighborhood of a singular point 0 if there exists a smooth diffeomorphism carrying one neighborhood of zero into another that leaves 0 fixed and carries the phase curves of one field into phase curves of the other (perhaps reversing the direction of motion along the phase curves).

A singularity $p$ is called *elementary* if the linear part of $X$ over $p$ has a non-zero eigenvalue. These singularities are well-studied for planar real-analytic vector-fields $X$:

**Theorem 3.2.1.** (see section 1.E of [I1]) An analytic vector-field in some neighborhood of an isolated elementary singularity on the real plane is orbitally equivalent to one of the vector fields in the table below:

| Type of singularity                        | Normal Form                                                                 |
|--------------------------------------------|------------------------------------------------------------------------------|
| 1 - Non-resonant linear part               | $\omega(\bar{x}) = \Lambda \bar{x}$                                       |
| 2 - A linear center                        | $\omega(\bar{x}) = I(\bar{x}) + \epsilon (r^{2k} + ar^{4k})$               |
| 3 - A resonant node                        | $\omega(x, y) = (-kx + \epsilon y^{k}) \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ |
| 4 - A resonant saddle where $\lambda = \frac{m}{n}$ | $\omega(x, y) = x[1 + \epsilon (u^{k} + au^{2k})] \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y}$ where $u = x^{m}y^{n}$ is the resonant monomial |
| 5 - A degenerated elementary singularity   | $\omega(x, y) = x^{k}(1 + ax^{k})^{-1} \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ |

Here the numbers $k$, $m$ and $n$ are positive integers, $a$ is a real number, $\bar{x} = (x, y) \in \mathbb{R}^2$, $r^2 = x^2 + y^2$, $I$ is the operator of rotation through the angle $\frac{\pi}{2}$, the fraction $\frac{m}{n}$ is irreducible, and $\epsilon \in \{0, 1, -1\}$.

A singularity of a planar real-analytic vector-field is called *monodromic* if it is orbitally equivalent to a focus or a center.

**Remark 3.2.2.** If $\Sigma$ is a continuous curve passing thought a monodromic singularity $p$ of $X$, then every orbit of $X$ close enough to $p$ cuts $\Sigma$ an infinite number of times.

**Lemma 3.2.3.** Let $X$ be a planar real-analytic vector-field, $p$ a non-monodromic isolated singularity of $X$, $g$ an Euclidean metric at $p$, and $\gamma(t) : [0, +\infty[ \rightarrow \mathbb{R}^2$ an orbit of $X$ converging to...
Then \( \text{length}_g(\gamma(t)) < +\infty \).

**Proof.** By lemma 3.1.1, we can take any locally smooth coordinate system, so we can consider that we are in one of the normal forms given by Theorem 3.2.1. Furthermore, without loss of generality, we can suppose that \( \gamma(t) \) is all contained in this neighborhood.

We divide in the cases of the Theorem (and remark that case 2 can be excluded since it is always monodromic).

Case 1 ) Notice that, since the singularity is non-monodromic, both eigenvalues of \( \Lambda \) are real. Furthermore, the non-resonant hypotheses implies that we can diagonalize \( \Lambda \) (otherwise we are in case 3 or 4). Thus, we can work directly with the normal form \( \omega(x, y) = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y} \). At this case the solutions are given by the curves \( \gamma(t) = (x_0e^{\lambda t}, y_0e^{\mu t}) \). With this expressions, we need a simple calculation to conclude that \( \text{length}_g(\gamma(t)) < \infty \);

Case 3 ) We work with the normal form \( \omega(x, y) = (-kx + \epsilon y^k) \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \). At this case the solutions are given by the curves \( \gamma(t) = (e^{-kt}(x_0 + \epsilon y_0^k t), -y_0 e^t) \). With this expressions, we need a simple calculation to conclude that \( \text{length}_g(\gamma(t)) < \infty \);

Case 4 ) We work with the normal form \( \omega(x, y) = x[1 + \epsilon (u^k + au^2k)] \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y} \). At this case, there are only four orbits converging to \((0, 0)\). Since they are all contained in the axes \( V(xy) \), they clearly have finite length;

Case 5 ) We work with the normal form \( \omega(x, y) = x^k(1 + ax^k)^{-1} \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \). At this case, if \((x_0, y_0)\) is sufficiently close to \((0, 0)\), the projection of the curves on the axes are either injective, or an unique point. Thus, the \( \text{length}_g(\gamma(t)) \leq x_0 + y_0 < +\infty \).

\[\square\]

### 3.3 Back-ground: Desingularization

A manifold with divisor is a pair \((M, E)\) such that:

- \( M \) is an algebraic smooth variety;

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• $E$ is an ordered collection $E = (E^{(1)}, ..., E^{(l)})$, where each $E^{(i)}$ is a smooth divisor on $M$ such that $\sum E^{(i)}$ is a reduced divisor with SNC, i.e. at each point $p$ in $\sum E^{(i)}$ there exists a local coordinate system $(x_1, ..., x_n)$ at $p$ such that $\sum E^{(i)}$ is given by $\{\prod_{j=1}^k x_j = 0\}$ for some natural number $k$.

A blowing-up $\sigma : M' \longrightarrow M$ is admissible in respect to $(M, E)$ if the center of blowing-up $C$ is a closed and smooth sub-variety of $M$ that has SNC with the divisor $E$, i.e. at each point $p$ in $\sum E^{(i)} \cap C$ there exists a local coordinate system $(x_1, ..., x_n)$ at $p$ such that $\sum E^{(i)}$ is given by $\{\prod_{j=1}^k x_j = 0\}$ for some natural number $k$ and $C$ is given by $\cap_{\lambda \in \Lambda} \{x_\lambda = 0\}$ for some subset $\Lambda$ of $\{1, ..., n\}$. In particular, if the center of blowing-up is a point, then the blowing-up is always admissible.

The strict transform of a subset $S$ of $M$ is the subset $S^* = \sigma^{-1}(S \setminus C)$ where $\overline{Set}$ stands for the topological closure of $Set$.

Whenever a blowing-up $\sigma : M' \longrightarrow M$ is admissible, we set $E' = (E^*, F)$, where $E^*$ is the strict transform of the divisor $E$ and $F$ is the exceptional divisor of the blowing-up. In this case, we denote the blowing-up by $\sigma : (M', E') \rightarrow (M, E)$.

A sequence of admissible blowings-up $\hat{\sigma}$ is a sequence $(\sigma_1, ..., \sigma_r)$:

$$(M_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, E_1) \xrightarrow{\sigma_1} (M, E)$$

such that $\sigma_i : (M_i, E_i) \longrightarrow (M_{i-1}, E_{i-1})$ is an admissible blowing-up.

Consider $N$ a hypersurface of $M$. A singular point of $N$ is a point $p$ of $N$ which is not regular. We denote the set of singular points of $N$ by $Sing(N)$. The following Theorem is a simplified version of a landmark in desingularization of varieties (see [H, Kô, Mu]):

**Theorem 3.3.1. (Theorem of Hironaka)** Let $(M, N, E)$ be a triple as above. Then, there exists a sequence of admissible blowings-up:

$$(M_r, N_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, N_1, E_1) \xrightarrow{\sigma_1} (M, N, E)$$

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where each $N_i$ is the strict transform of $N_{i-1}$ by the blowing-up $\sigma_i : (M_i, E_i) \to (M_{i-1}, E_{i-1})$ such that:

- $i$) The variety $N_r$ is regular;
- $ii$) The variety $N_r$ has SNC with the divisor $E_r$;
- $iii$) The morphism $\sigma = \sigma_1 \circ ... \circ \sigma_r$ is an isomorphism outside $\sigma^{-1}(\text{Sing}(N) \cup E)$.

**Remark 3.3.2.** The usual Hironaka’s Theorem can be found in [Ko] (see Theorem 3.36). Here, we present a slightly different version, simplified in many ways but also adding property $[ii]$. Nevertheless, property $[ii]$ can be obtained in a classical way (see, for example, section 5.2.1 of [Mu]).

Consider now $X$ a vector-field defined over $M$ and denote by $\text{Sing}(X)$ the singular set of it. Given an admissible blowing-up $\sigma : (M', E') \to (M, E)$, the strict transform of $X$ is denoted by $X^s$ and given by:

$$X^s = \mathcal{O}(\alpha F) X^*$$

where $X^*$ is the pull-back of $X$ by $\sigma$, $F$ is the exceptional divisor of the blowing-up $\sigma$ and $\alpha$ is the smallest integer such that $X^s$ is algebraic and $F$ is not contained in the singular set of $X^s$. The following Theorem is a landmark in desingularization of vector-fields (see [Sel, Ben, I2]):

**Theorem 3.3.3.** (Theorem of Bendixson-Seidenberg) Given a triple $(M, X, E)$ as above such that $M$ is two-dimensional and $X$ has only isolated singularities, there exists a sequence of admissible blowings-up:

$$(M_r, X_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, X_1, E_1) \xrightarrow{\sigma_1} (M, X, E)$$

where each $X_i$ is the strict transform of $X_{i-1}$ by the blowing-up $\sigma_i : (M_i, E_i) \to (M_{i-1}, E_{i-1})$ such that:

- $i$) All singularities of $X_r$ are isolated and elementary.
ii) The morphism $\sigma = \sigma_1 \circ ... \circ \sigma_r$ is an isomorphism outside $\sigma^{-1}(\text{Sing}(X))$.

In this work, we are actually going to need the following slightly stronger version:

**Theorem 3.3.4. (Modified Bendixson-Seidenberg)** The Bendixson-Seidenberg Theorem still holds if the singularities of $X$ are not isolated.

*Proof.* Notice that $\text{Sing}(X)$ is a sub-variety of $M$ and let $\{q_1, ..., q_m\}$ be the isolated singularities of $X$. Consider $N$ the hypersurface of $M$ given by $\text{Sing}(X) \setminus \{q_1, ..., q_m\}$. By Hironaka’s Theorem 3.3.1 there exists a sequence of admissible blowings-up:

$$(M_r, N_r, X_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, N_1, X_1, E_1) \xrightarrow{\sigma_1} (M, N, X, E)$$

such that $N_r$ is regular and have SNC with $E_r$. Furthermore, it is clear that all non-isolated singularities of $X_r$ are contained in $N_r$. Thus, at each point $p$ in $N_r$, there exists a local coordinate system $(x, y)$ such that the vector-field $X_r$ can be written as:

$$X_r = y^mY$$

where $N_r$ is locally given by $\{y = 0\}$, $m$ is a positive natural number and $Y$ is a locally defined vector-field with isolated singularities. Thus, consider the blowing-up:

$$\tau : (M_{r+1}, X_{r+1}, E_{r+1}) \longrightarrow (M_r, X_r, E_r)$$

with admissible center $N_r$ (recall that $N_r$ is regular and has SNC with $E_r$). It is clear that $\tau$ is an isomorphism of $M_{r+1}$ into $M_r$ and that, by equation (1), the vector-field $X_{r+1}$ has only isolated singularities. One can now apply the Bendixson-Seidenberg Theorem 3.3.3 to conclude. \qed

### 3.4 Proof of Proposition 2.2.2

Let $\gamma(t)$ be an orbit of the vector-field $X$ contained in the topological cylinder $C$. By Claim 3 of the proof of Proposition 2.2.1, we know that the $\omega$-limit of $\gamma(t)$ is contained in $\Gamma$. Here we prove that this $\omega$-limit is actually equal to $\Gamma$. 20
Intuitively, we show that the orbit $\gamma(t)$ must be spiraling in $C_\epsilon$. This is done in three steps: first we prove that the orbit $\gamma(t)$ has infinite length; second we use desingularization to simplify the problem; third we construct a segment $\Sigma$ transversal to $X$, where the return of the flow is well-defined.

**First step:** Let $\gamma(t)$ be an orbit of the vector-field $X$ contained in $C_\epsilon$. Then:

**Claim 1:** The length of $\gamma(t)$ by the usual Euclidean norm $\|\cdot\|$ of $\mathbb{R}^3$ is infinite.

**Proof.** Since the curve $\gamma(t)$ is contained in the variety $V(H)$:

$$\|\dot{\gamma}(t)\|_{\gamma(t)} = \left[\sqrt{(H_x^2 + H_y^2)(1 + z^{2\beta_1}(H_x^2 + H_y^2 + H_z^2))}\right](\gamma(t))$$

Furthermore:

- Since $C_\epsilon$ is relatively compact for $\epsilon$ small enough, there exists a constant $M_\epsilon$ such that $\sqrt{H_x^2 + H_y^2}(p) \geq M_\epsilon(H_x^2 + H_y^2)(p)$ over points $p$ in $C_\epsilon$.

So, fixed $\epsilon > 0$ small enough, we have that:

$$\|\dot{\gamma}(t)\|_{\gamma(t)} \geq M_\epsilon[H_x^2 + H_y^2](\gamma(t))$$

We want to prove that $\int_0^\infty [H_x^2 + H_y^2](\gamma(t))dt = +\infty$, which clearly implies the result. Since $\gamma(t) = (x(t), y(t), z(t))$ is contained in $C_\epsilon$:

$$\dot{z}(t) = -z^{2\beta_1}(H_x^2 + H_y^2)$$

is strictly negative. So, if $\gamma(0) = (x_0, y_0, z_0)$, we have that:

$$z_0 = \int_0^\infty \|\dot{z}(t)\|_{\gamma(t)}dt = \int_0^\infty [z^{2\beta_1}(H_x^2 + H_y^2)](\gamma(t))dt$$

Furthermore after a time $T > 0$:

$$z(T) = z_0 - \int_0^T \|\dot{z}(t)\|_{\gamma(t)}dt = z_0 - \int_0^T [z^{2\beta_1}(H_x^2 + H_y^2)](\gamma(t))dt$$

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So, let \((T_n)_{n \in \mathbb{N}}\) be the partition of \([0, \infty[\) such that \(z(T_n) = \frac{2n}{2n+1}\). We have that:

\[
\int_{T_n}^{T_{n+1}} [z^{\beta_1}(H_x^2 + H_y^2)](\gamma(t))dt = \frac{z_0}{2n+1}
\]

and, for \(t \in [T_n, T_{n+1}]\) we have that \(z \leq \frac{2n}{2n+1}\). In particular, this implies that \(z^{\beta_1} \leq \frac{z_0}{2n}\). Thus:

\[
\frac{z_0}{2n+1} = \int_{T_n}^{T_{n+1}} [z^{\beta_1}(H_x^2 + H_y^2)](\gamma(t))dt \leq \frac{z_0}{2n} \int_{T_n}^{T_{n+1}} [H_x^2 + H_y^2](\gamma(t))dt
\]

which implies that:

\[
\int_{T_n}^{T_{n+1}} [H_x^2 + H_y^2](\gamma(t))dt \geq \frac{1}{2}
\]

And, thus:

\[
\int_0^\infty [H_x^2 + H_y^2](\gamma(t))dt = \sum_n \int_{T_n}^{T_{n+1}} [H_x^2 + H_y^2](\gamma(t))dt \geq \sum_n \frac{1}{2} = +\infty
\]

**Notation:** In what follows, whenever we talk about the adherence of \(C_\varepsilon\) (or of one of its transforms), we mean the limit of the topological adherence of \(C_\varepsilon\) when \(\varepsilon \to 0\). In particular, the adherence of \(C_\varepsilon\) is \(\overline{C_\varepsilon} \cap \{z = 0\}\).

**Second step:** Consider the triple \((M, N, E) = (\mathbb{R}^3, V(H), \{z = 0\})\). By Hironaka’s Theorem 3.3.1, there exists a sequence of admissible blowings-up of order one:

\[
(M_r, N_r, E_r) \xrightarrow{\sigma_r} \cdots \xrightarrow{\sigma_2} (M_1, N_1, E_1) \xrightarrow{\sigma_1} (M, N, E)
\]

such that:

i ) The variety \(N_r\) is regular and has SNC with \(E_r\);

ii ) The morphism \(\sigma = \sigma_1 \circ \cdots \circ \sigma_r\) is an isomorphism outside the set \(\sigma^{-1}(\text{Sing} I \cup E)\).

Let \(\tilde{N}\) be the two-dimensional regular variety \(N_r\), \(\tilde{C}_r\) be the semi-algebraic variety given by the inverse image of \(C_\varepsilon\) by \(\sigma\) and \(X_r\) be the strict transform of the vector-field \(X\) under the sequence of blowings-up \(\tilde{\sigma}\). We have that:
I) Since $C_\epsilon$ is everywhere regular and has empty intersection with $E$, by $[ii]$, the morphism $\sigma$ induces an isomorphism between $\tilde{C}_\epsilon$ and $C_\epsilon$;

II) By $[i]$, the intersection of $E_r$ with the regular variety $\tilde{N}$ is a SNC divisor on $\tilde{N}$ denoted by $\tilde{E}$. Furthermore, since the adherence of $C_\epsilon$ is contained in $E$, the adherence of the semi-analytic variety $\tilde{C}_\epsilon$ is contained in $\tilde{E}$;

III) Since $X$ is everywhere tangent to the variety $N$, the vector-field $X_r$ is everywhere tangent to the variety $\tilde{N}$. Thus, the restriction of the vector-field $X_r$ to $\tilde{N}$ is well-defined and denoted by $\tilde{X}$.

Now, we can apply the modified Bendixson-Seidenberg Theorem 3 to the triple $(\tilde{N}, \tilde{X}, \tilde{E})$ in order to obtain a sequence of admissible blowings-up:

$$(\tilde{N}_s, \tilde{X}_s, \tilde{E}_s) \xrightarrow{\tau_s} \cdots \xrightarrow{\tau_2} (\tilde{N}_1, \tilde{X}_1, \tilde{E}_1) \xrightarrow{\tau_1} (\tilde{N}, \tilde{X}, \tilde{E})$$

such that:

iii) The singularities of the vector-field $\tilde{X}_s$ are isolated and elementary;

iv) The morphism $\tau = \tau_1 \circ \cdots \circ \tau_s$ is an isomorphism outside the set $\tau^{-1}(Sing(\tilde{X}))$.

Let $X'$ be the vector-field $\tilde{X}_s$, $E'$ be the simple normal crossing divisor $\tilde{E}_s$ and $C'_\epsilon$ be the semi-algebraic variety given by the inverse image of $\tilde{C}_\epsilon$ by $\tau$. We have that:

IV) Since $\tilde{C}_\epsilon$ is everywhere regular and has empty intersection with $\tilde{E}$, by $[iv]$, the morphism $\tau$ induces an isomorphism between $C'_\epsilon$ and $\tilde{C}_\epsilon$;

V) Since the adherence of $\tilde{C}_\epsilon$ is contained in $\tilde{E}$, the adherence of the semi-analytic variety $C'_\epsilon$ is contained in $\tilde{E}_r$. We denote the adherence of $C'_\epsilon$ by $\Gamma'$. We remark that $$(\sigma \circ \tau)(\Gamma') = \Gamma.$$  

Third step: Consider an orbit $\gamma : [0, \infty[ \rightarrow M$ of $X$ totally contained in $C_\epsilon$. We abuse notation and denote the image of the orbit by $\gamma$ (i.e. $\gamma = \gamma([0, \infty[))$. Let $\gamma'$ the pre-image of $\gamma$ by the morphism $\sigma \circ \tau$ (i.e. $\gamma' = (\sigma \circ \tau)^{-1}(\gamma)$). By $[I]$ and $[IV]$, $\gamma'$ is also the image of
an orbit of $X'$ totally contained in $C'_e$. We abuse notation and denote this orbit by $\gamma'(t)$.

Since the $\omega$-limit of $\gamma(t)$ is contained in the adherence of $C_e$ denoted by $\Gamma$, by $[II]$ and $[V]$, the $\omega$-limit of $\gamma'(t)$ is contained in the adherence of $C'_e$, denoted by $\Gamma'$.

Claim 2: The $\omega$-limit of $\gamma'(t)$ is equal to $\Gamma'$.

Proof. If $\Gamma'$ is an unique point, the result is trivially true. So, we assume that $\Gamma'$ is not an unique point. Since $\Gamma$ is compact and $\Gamma' \subset (\sigma \circ \tau)^{-1}(\Gamma)$ is closed, the set $\Gamma'$ is compact. Furthermore, since $\Gamma'$ is the adherence of a topological cylinder and is compact, $\Gamma'$ is connected.

Suppose by absurd that the $\omega$-limit of $\gamma'(t)$ is an unique point $p'$ of $\Gamma'$. Since, by Claim 1, the orbit $\gamma(t)$ has infinite length at the usual Euclidean metric of $\mathbb{R}^3$, using Lemma 3.1.1 for each blowing-up of the sequence $\bar{\sigma} \circ \bar{\tau}$, we conclude that $\gamma'(t)$ has infinite length at any Euclidean metric defined in a neighborhood of $p'$.

By the Flow-box Theorem, the vector-field $X'$ must be singular at $p'$, otherwise the orbit converging to $p$ would have finite length. In particular this also shows that $\Gamma'$ must be invariant by $X'$: otherwise there would exists a regular orbit cutting $\Gamma'$ which implies the existence of an orbit $\gamma(t)$ of $X$ finite length. Furthermore, by Lemma 3.2.3, the singularity of $X'$ over $p'$ must be monodromic, otherwise any orbit converging to $p'$ would have finite length. Since $\Gamma'$ is compact, connected and one-dimensional, we conclude that the orbit $\gamma'(t)$ cuts an infinite number of times $\Gamma'$ (see remark 3.2.2). But this contradicts the fact that $\Gamma'$ is invariant by the vector-field $X'$.

Thus, the $\omega$-limit of $\gamma'(t)$ is not an unique point of $\Gamma'$. In particular, there must exists a point $p'$ in the $\omega$-limit of $\gamma'(t)$ such that the vector-field $X'$ is regular at it. By the Flow-box Theorem, there exists a coordinate system $(x, y)$ in a neighborhood of $p'$ such that $X' = \frac{\partial}{\partial x}$ and $\Gamma' \subset \{y = 0\}$ (because $\Gamma'$ is invariant by $X'$). So there exists a differential segment $\Sigma'$ contained in $C'_e$, with topological adherence in $p'$, witch is everywhere transversal to $X'$. Since the point $p'$ is contained in the $\omega$-limit of $\gamma'(t)$, the flux of $X'$ gives rise to a
return map $\rho^\prime : \Sigma^\prime \rightarrow \Sigma^\prime$. We remark that the orbit of any point in $\Sigma^\prime$ is an infinite set of points converging to $p^\prime$.

Now, by the identification of $C^\prime_\epsilon$ with a topological cylinder, the fact that a return map can be constructed for any orbit contained in $C^\prime_\epsilon$ and the fact that different orbits have empty intersection, we conclude that the adherence of the topological curve $\gamma^\prime(t)$ must coincide with the adherence of $C^\prime_\epsilon$.

It is now clear that, since $\sigma \circ \tau(\Gamma^\prime) = \Gamma$, the $\omega$-limit of the orbit $\gamma(t)$ must be $\Gamma$.

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