Introduction to end super dominating sets in graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A dominating set of $G$ is a subset $S \subseteq V$ such that every vertex not in $S$ is adjacent to at least one vertex in $S$. The cardinality of a smallest dominating set of $G$, denoted by $\gamma(G)$, is the domination number of $G$. In this paper, we define a new domination number, and call it end super domination number. We give some applications of this definition and obtain the exact value of that on specific graphs. We count the number of end super dominating sets of these graphs too. Also, we present some sharp bounds on the end super domination number, where graph is modified by vertex (edge) removal and contraction. Finally, we generalize our definition and present some results on that.

Keywords: Domination number, end super dominating set, end super domination number, networks, generalization

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Throughout this paper, we consider connected graphs without loops and directed edges. For each vertex $v \in V$, the set $N(v) = \{u \in V | uv \in E\}$ refers to the open neighbourhood of $v$ and the set $N[v] = N(v) \cup \{v\}$ refers to the closed neighbourhood of $v$ in $G$. The degree of $v$, denoted by $\deg(v)$, is the cardinality of $N(v)$. A set $S \subseteq V$ is a dominating set if every vertex in $\overline{S} = V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. Domination number is one of the most important aspects in graph theory since it solves many problems in real life. There are various domination numbers in the literature. For a detailed treatment of domination theory, the reader is referred to [7].

A dominating set $S$ is called a super dominating set of $G$, if for every vertex $u \in \overline{S}$, there exists $v \in S$ such that $N(v) \cap \overline{S} = \{u\}$. The cardinality of a smallest super
dominating set of $G$, denoted by $\gamma_{sp}(G)$, is the super domination number of $G$ \cite{9}. We refer the reader to \cite{3,4,6} for more details on super dominating set of a graph. Motivated by the definition of super dominating set, we define a new dominating set:

**Definition 1.1** A dominating set $S$ is called an end super dominating set of $G$, if for every vertex $u \in \overline{S}$, $\deg(u) \geq 2$ and there exists $v \in S$ such that $N(v) \cap \overline{S} = \{u\}$. The cardinality of a smallest end super dominating set of $G$, is the end super domination number of $G$ and is denoted by $\gamma_{esp}(G)$.

Let $G = (V, E)$ be a graph and $S \subseteq V$ is an end super dominating set of that. We say that vertex $v \in \overline{S}$ is end super dominated by vertex $u \in S$ (or equivalently $u$ is end super dominating vertex $v$) if $N(v) \cap \overline{S} = \{u\}$.

In this paper, we study the end super dominating sets of graphs. In Section 2, we present some applications of end super dominating sets. In Section 3, we obtain some results on the end super dominating sets of graphs. Also, we find exact value of end super domination number of some specific graphs. In Section 4, we present some bounds on the number of edges of a graph based on the end super domination number. In Section 5, we remind the definition of edge removal and contraction and obtain sharp upper and lower bound for end super domination number of a graph by these modifications. In Section 6, we mention the definition of vertex removal and contraction and find sharp bounds for the end super domination number of a graph which is modified by these operations. In Section 7, we compute the number of end super dominating sets with smallest cardinality of some specific graphs. Finally, in Section 8, we generalize our definition and obtain some results on that.

## 2 Application

In this section, we focus on the application of end super dominating sets. We begin with the following example:

**Example 2.1** we can consider an end super dominating set $E$ as a set of main servers and temporary servers of a network, and $\overline{E}$ as a set of backup servers of this network. All kinds of servers can be connected or not. A server is a main server if it is connected to one and only one backup server. A server from $E$ is a temporary server if it is connected to more than one backup server. By the definition of end super dominating set, each backup server in $\overline{E}$ is connected to at least two main servers or is connected to at least one main server and is connected to at least another backup or temporary server and this temporary server is connected to another backup server and they keep data safe for each other.

Suppose that the number of main servers in the Example 2.1 is $n$. Then by our definition, we need at most $n$ backup servers and can manage our system more properly. For example, suppose that we have a network of the users and servers. first remove all
users. Then find end super domination number of this network related to the minimal end super dominating set. Now, if two backup servers $b_i$ and $b_j$ are connected, then we can replace them with just a better backup server $b_k$ with less spending of money by connecting all the previous servers connected to one of the $b_i$ or $b_j$ to $b_k$. Also, if a backup server is so expensive and risky to keep, then we can do it conversely and put at least two backup servers in replacement of the expensive one and connect them to each other and then decide what main server is better to connect to which one of the new backup servers.

As we discussed in Example 2.1, it is also possible to discuss about the location of fire stations of a city and backup forces, hospitals and backup ambulance stations, and many other situations and examples. Here we find end super domination number of the graph $G$ in Figure 1 and discuss about this graph and what we mentioned before in Example 2.1.

**Example 2.2** Consider the graph $G$ in Figure 1. By the definition of end super dominating set, six vertices $a_4$, $b_4$, $c_4$, $d_4$, $e_4$, and $f_4$, should be in our end super dominating set. One can easily check that the set $S = \{a_1, b_1, c_1, d_1, e_1, f_1, a_4, b_4, c_4, d_4, e_4, f_4\}$ is an end super dominating set of $G$ and $\gamma_{esp}(G) = 12$. Now consider Example 2.1. If $G$ is our network, then the set of black vertices includes all main servers and the rest are backup servers.

Here we consider Figure 1 and Example 2.2. Suppose that this is our network and we want to reduce the number of backup servers by the idea that if there is a path between two backup servers without passing a main server, then just use only one and connect it to the corresponding main servers and name them as powerful backup servers. Also each main server can support at most 100 users. Then, one of the modifications to this network is the same as we see in Figure 2, where PBS is a powerful backup server, MS is a main server and $\lambda U$ is $\lambda$ users connecting to a main server ($1 \leq \lambda \leq 100$), respectively.
3 Basic results on the end super domination number of a graph

In this section, we present some results on the end super dominating sets of graphs. It is clear that a super dominating set is not necessarily an end super dominating set, since there might be a vertex with degree one in the complement of our super dominating set. As an immediate result of the Definition 1.1, if $G$ is a graph and $S$ is a super dominating set of that, then $S$ is an end super dominating set of $G$, if for all $u \in S$, $\deg(u) \geq 2$. Also, we have:

**Proposition 3.1** Let $G$ be a graph. Then

$$\gamma(G) \leq \gamma_{sp}(G) \leq \gamma_{esp}(G).$$

Before we continue, we state a known result on the super domination number of a graph:

**Theorem 3.2** [9] Let $G$ be a connected graph of order $n$. Then,

$$1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq n - 1.$$  

Here we state an immediate result of Definition 1.1, Proposition 3.1 and Theorem 3.2

**Remark 3.3** Let $G = (V, E)$ be a graph of order $n \geq 3$ and $S$ be an end super dominating set of $G$. Then,
(i) All vertices of degree at most one belong to every end super dominating set.

(ii) If $G$ has no pendant vertices, then $\gamma_{esp}(G) = \gamma_{sp}(G)$.

(iii) If $G$ has no isolate vertices, then

$$1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq \gamma_{esp}(G) \leq n - 1.$$ 

Here we show that if $G$ has $r$ pendant vertices, then there is no reason to have $\gamma_{esp}(G) = r$.

Example 3.4 Consider the graph $H$ as shown in Figure 3. One can easily check that the set of black vertices is an end super dominating set of minimum size. So $\gamma_{esp}(H) = 6$, but we have only 5 pendant vertices.

Now we present the end super domination number of some graph classes. Lemańska et al. found the super domination number of complete, complete bipartite and cycle graphs in [9]. Since these graphs have no pendant vertices, then by Remark 3.3, the end super domination number of these graphs are equal to the super domination number of them. So we have the following result:

**Proposition 3.5**  
(i) For the complete graph $K_n$,

$$\gamma_{esp}(K_n) = n - 1.$$ 

(ii) Suppose that $\min\{n, m\} \geq 2$. For the complete bipartite graph $K_{n,m}$,

$$\gamma_{esp}(K_{n,m}) = n + m - 2.$$ 

(iii) For the cycle graph $C_n$,

$$\gamma_{esp}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 0, 3 \pmod{4}, \\ \left\lceil \frac{n+1}{2} \right\rceil & \text{otherwise.} \end{cases}$$ 

Except one vertex in star graph $S_n = K_{1,n-1}$, all vertices are pendant vertices. So by Remark 3.3 we have:
Proposition 3.6 \textit{For the star graph }$S_n$, $\gamma_{esp}(S_n) = n - 1$.

Now we consider the path graph of order $n$, $P_n$, and find its end super domination number.

Theorem 3.7 \textit{Suppose that }$n \in \mathbb{N} \setminus \{1\}$ \textit{and }$k \in \mathbb{N} \cup \{0\}$. \textit{For the path graph }$P_n$,$$
\gamma_{esp}(P_n) = \begin{cases} 
2k & \text{if } n = 4k, \\
2k + 1 & \text{if } n = 4k + 1, \\
2k + 2 & \text{if } n = 4k + 2 \text{ or } n = 4k + 3.
\end{cases}
$$

\textbf{Proof.} It is easy to check that $\gamma_{esp}(P_2) = 2$, $\gamma_{esp}(P_3) = 2$, $\gamma_{esp}(P_4) = 2$, $\gamma_{esp}(P_5) = 3$, $\gamma_{esp}(P_6) = 4$ and $\gamma_{esp}(P_7) = 4$. So the equality holds for $\gamma_{esp}(P_n)$, where $2 \leq n \leq 7$. Consider $P_n$, where $n \geq 8$, with the vertex set $V = \{v_1, \ldots, v_n\}$ (see Figure 4). To find the end super domination number, we consider the following cases:

(i) $n = 4k$, for some $k \in \mathbb{N} \setminus \{1\}$. Let $$S = \{v_1, v_4, v_5, \ldots, v_{4i}, v_{4i+1}, \ldots, v_{4k}\}.$$ Clearly, $S$ is a dominating set of $P_{4k}$ and for every $u \in S$, deg$(u) = 2$. Also for every $v_{4i+2} \in S$, where $0 \leq i \leq k - 1$, $N(v_{4i+1}) \cap S = \{v_{4i+2}\}$. On the other hand, for every $v_{4i+3} \in S$, where $0 \leq i \leq k - 1$, $N(v_{4i+4}) \cap S = \{v_{4i+3}\}$. So, $S$ is an end dominating set of $P_{4k}$ with size $2k$. By Remark 3.3, $\gamma_{esp}(P_{4k}) \geq 2k$. Hence, $S$ is an end dominating set of minimum size and $\gamma_{esp}(P_{4k}) = 2k$.

(ii) $n = 4k + 1$, for some $k \in \mathbb{N} \setminus \{1\}$. Let $$S = \{v_1, v_4, v_5, \ldots, v_{4i}, v_{4i+1}, \ldots, v_{4k-4}, v_{4k-3}, v_{4k-1}, v_{4k+1}\}.$$ By a similar argument as Part (i), $S$ is an end super dominating set for $P_{4k+1}$ of size $2k + 1$. On the other hand, by Remark 3.3, $\gamma_{esp}(P_{4k+1}) \geq 2k + \frac{1}{2}$. Hence $\gamma_{esp}(P_{4k+1}) = 2k + 1$.

(iii) $n = 4k + 2$, for some $k \in \mathbb{N} \setminus \{1\}$. By Remark 3.3, $\gamma_{esp}(P_{4k+2}) \geq 2k + 1$. We show that there is no end super dominating set with size $2k + 1$. First, by Remark 3.3, $v_1$ and $v_{4k+2}$ should be in our dominating set (say $S$), since they have degree 1. Now we consider $v_2$ and $v_{4k+1}$ in $S$. Then, $N(v_1) \cap S = \{v_2\}$ and $N(v_{4k+2}) \cap S = \{v_{4k+1}\}$. Now we have $4k - 2$ vertices. No matter how we choose $2k - 1$ vertices among these vertices to put in $S$, then one can easily check that it is not a super dominating set. So we need at least $2k + 2$ vertices. Let $$S = \{v_1, v_4, v_5, \ldots, v_{4i}, v_{4i+1}, \ldots, v_{4k-4}, v_{4k-3}, v_{4k-1}, v_{4k}, v_{4k+2}\}.$$ By a similar argument as Part (i), $S$ is an end super dominating set for $P_{4k+2}$ with size $2k + 2$. Hence $\gamma_{esp}(P_{4k+2}) = 2k + 2$. 


(iv) $n = 4k + 3$, for some $k \in \mathbb{N} \setminus \{1\}$. Let $$S = \{v_1, v_4, v_5, \ldots, v_{4i}, v_{4i+1}, \ldots, v_{4k}, v_{4k+1}, v_{4k+3}\}.$$ By a similar argument as Part (i), $S$ is an end super dominating set for $P_{4k+3}$ of size $2k + 2$. On the other hand, by Remark $\gamma_{esp}(P_{4k+3}) \geq 2k + 1 + \frac{1}{2}$. Hence $\gamma_{esp}(P_{4k+3}) = 2k + 2$.

Therefore we have the result. \qed

Let $G = (V, E)$ be a graph, $S \subseteq V$, and $u \in S$. A vertex $v \in V \setminus S$ is an external private neighbor of $u$ if $N(v) \cap S = \{u\}$. Bollobás and Cockayne in [1] proved the following result.

**Theorem 3.8** [1] Any graph with no isolated vertices has a minimum dominating set $S$ in which every element of $V \setminus S$ has an external private neighbor.

Now we have the following corollary.

**Corollary 3.9** If $G$ is a graph, then $G$ has a minimum dominating set $S$ such that $\overline{S}$ is an end super dominating set.

Nordhaus and Gaddum found sharp bounds on the sum and product of the chromatic numbers of a graph and its complement [10]. Since then such results have been given for several parameters. Here we end this section by similar inequalities for the end super domination number which is an immediate result of Remark 3.3. One can easily check that the upper bounds are sharp for complete graph $K_n$.

**Theorem 3.10** For any graph $G$ of order $n \geq 3$, we have:

(i) $n \leq \gamma_{esp}(G) + \gamma_{esp}(\overline{G}) \leq 2n - 1$.

(ii) $\frac{n^2}{4} \leq \gamma_{esp}(G) \cdot \gamma_{esp}(\overline{G}) \leq n(n - 1)$.

## 4 Some bounds on the number of edges of a graph based on the end super domination number

In this section, we present some bounds on the number of edges of a graph based on the end super domination number. Given a graph $G = (V, E)$, a matching $M$ in $G$ is a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share common vertices. A vertex is matched (or saturated) if it is an endpoint of one
of the edges in the matching. Otherwise the vertex is unmatched (or unsaturated). A maximum matching (also known as maximum-cardinality matching \cite{8}) is a matching that contains the largest possible number of edges. There may be many maximum matchings. The matching number $\nu(G)$ of a graph $G$ is the size of a maximum matching. Here, we find an upper bound for the number of edges of a graph based on a matching of graph and its end super domination number.

**Theorem 4.1** Let $G = (V, E)$ be a graph of order $n \geq 3$ and size $m$ with the end super domination number $\gamma$. Then we have

$$m \leq \binom{n}{2} - (n - \gamma)(n - \gamma - 1),$$

and the equality holds if and only if $G$ is isomorphic to $K_n \setminus M$, where $M$ is a matching of size $n - \gamma$.

**Proof.** Since $\gamma_{esp}(G) = \gamma$, $G$ has a matching of size $n - \gamma$. Let this matching be $M = \{a_1b_1, \ldots, a_{n-\gamma}b_{n-\gamma}\}$ such that $a_ib_j \notin E$, for each $i \neq j, 1 \leq i, j \leq n - \gamma$. Therefore

$$m \leq \binom{n}{2} - ((n - \gamma)^2 - (n - \gamma)),$$

and we are done. If the equality holds, then clearly $G = K_n \setminus M$, where $M$ is a matching of size $n - \gamma$. \hfill \Box

Now, we find a lower bound for the number of edges of a graph based on its end super domination number. This bound works for the number of edges of a graph based on its super domination number too.

**Theorem 4.2** Let $G = (V, E)$ be a graph of order $n$ and size $m$ with the end super domination number $\gamma$. Then we have

$$m \geq 2(n - \gamma) - 1.$$

**Proof.** By Remark 3.3, we have $\gamma \geq \frac{n}{2}$. So we have $2(n - \gamma) - 1 \leq n - 1$. Since every connected graph on $n$ vertices has at least $n - 1$ edges, therefore we have the result. \hfill \Box

We end this section by showing that the upper bound in Theorem 4.2 is tight.

**Remark 4.3** Consider path graph $P_{4k}$ where $k \in \mathbb{N}$. By Theorem 3.7, we have $\gamma_{esp}(P_{4k}) = 2k$, and clearly the equality in Theorem 4.2 holds.

### 5 End super domination number of $G - e$ and $G/e$

The graph $G - e$ is a graph that obtained from $G$ by simply removing the edge $e$. In a graph $G$, contraction of an edge $e$ with endpoints $u, v$ is the replacement of $u$ and $v$ with a single vertex such that edges incident to the new vertex are the edges other
than $e$ that were incident with $u$ or $v$. The resulting graph $G/e$ has one less edge than $G$ \( (2) \). We denote this graph by $G/e$. We refer the reader for more results about $G/e$ to \( [5] \). In this section, we examine the effects on $\gamma_{esp}(G)$ when $G$ is modified by an edge removal and edge contraction.

First, we consider the edge removal of a graph and find upper and lower bound for the end super domination number of that.

**Theorem 5.1** Let $G = (V, E)$ be a graph and $e = uv \in E$. Then,

$$
\gamma_{esp}(G) - 1 \leq \gamma_{esp}(G - e) \leq \gamma_{esp}(G) + 2.
$$

**Proof.** First we find the upper bound for $\gamma_{sp}(G - e)$. Suppose that $S$ is a super dominating set for $G$. Here, by removing $e$ and letting $S' = S \cup \{u, v\}$, by the definition of the end super dominating sets, $S'$ is an end super dominating set for $G - e$ and we have $\gamma_{esp}(G - e) \leq \gamma_{esp}(G) + 2$. Now we find a lower bound for $\gamma_{sp}(G - e)$. Suppose that $S$ is an end super dominating set for $G - e$. We have the following cases:

(i) $u, v \in S$. In this case, $S$ is an end super dominating set for $G$ too. So $\gamma_{esp}(G) \leq \gamma_{esp}(G - e)$.

(ii) $u, v \notin S$. Then, there are vertices $w, x \in S$ such that $N(w) \cap \overline{S} = \{u\}$ and $N(x) \cap \overline{S} = \{v\}$. Also $\deg(u) \geq 2$ and $\deg(v) \geq 2$. Clearly $w \neq x$. So $S$ is an end super dominating set for $G$ too, and $\gamma_{esp}(G) \leq \gamma_{esp}(G - e)$.

(iii) $u \in S$ and $v \notin S$. In this case, if for any $x \in \overline{S}$, there exists a vertex $x' \in S \setminus \{u\}$ such that $N(x') \cap \overline{S} = \{x\}$, then $S$ is an end super dominating set for $G$ and we are done. Otherwise, there exists $v' \in \overline{S}$ such that $N(u) \cap \overline{S} = \{v'\}$ in $G - e$. Now, $S \cup \{v'\}$ is an end super dominating set for $G$ and the proof is complete.

Therefore we have the result. $\Box$

**Remark 5.2** Bounds in Theorem 5.1 are sharp. For the upper bound, it suffices to consider $G = P_4$ and $e$ the edge with endpoints of degree 2. Then by Theorem 3.4, we have $\gamma_{esp}(P_4) = 2$ and $\gamma_{esp}(P_4 - e) = 4$. For the lower bound let $H = K_4 - e$ where $e$ is an edge of $K_4$. One can easily check that $\gamma_{esp}(H) = 3$. Now, let $e'$ be the edge in $H$ with endpoints of degree 3. Then $H - e' = C_4$ and by Proposition 3.3, we have $\gamma_{esp}(H - e') = 2$.

As we see in Remark 5.2, there is an example which shows that the equality for the upper bound in Theorem 5.1 holds. In the following, we add some initial conditions to Theorem 5.1 and improve the upper bound as it holds for super domination number (see \([4]\)):

**Theorem 5.3** Let $G = (V, E)$ be a graph and $e = uv \in E$ such that $\deg(u) \geq 3$. Then,

$$
\gamma_{esp}(G - e) \leq \gamma_{esp}(G) + 1.
$$
Proof. Let \( S \) be an end super dominating set of size \( \gamma_{esp}(G) \) for \( G \). If \( u, v \in S \), then obviously \( \gamma_{esp}(G - e) \leq \gamma_{esp}(G) \). If \( u \in S \) and \( v \in S' \), then clearly \( S \cup \{v\} \) is an end super dominating set for \( G - e \), which implies that \( \gamma_{esp}(G - e) \leq \gamma_{esp}(G) + 1 \). Now, assume that \( u, v \in S \). Since \( \text{deg}(u) \geq 3 \), then clearly \( S \cup \{v\} \) is an end super dominating set for \( G - e \), and we are done. Therefore we have the result. \( \square \)

In remark 5.2 to show that there is a graph \( G \) which equality holds for upper bound in Theorem 5.1, we considered path \( P_4 \) and after removing edge \( e \) we had a disconnected graph. This raises the question that can we improve the upper bound in Theorem 5.1 if \( G - e \) is connected? The following example shows that it is not possible.

Example 5.4 Consider the graph \( G \) as shown in Figure 5, and \( e = v_{10}v_{16} \). Let \( S = \{v_1, v_4, v_5, v_8, v_9, v_{11}, v_{14}, v_{15}, v_{17}, v_{20}\} \). One can easily check that \( S \) is an end super dominating set of \( G \), and by Remark 5.3 it has the minimum size. So, \( \gamma_{esp}(G) = 10 \). Now, by the definition of the end super dominating sets, for \( G - e \) we should have six vertices \( v_1, v_4, v_{10}, v_{16}, v_{17} \) and \( v_{20} \) in our end super dominating set. One can easily check that there is no way to choose five other vertices to have an end super dominating set for \( G - e \). Here, let \( S' = S \cup \{v_{10}, v_{16}\} \). Then, \( S' \) is an end super dominating set for \( G - e \), and \( \gamma_{esp}(G - e) = 12 \).

In [4], we showed that if \( G = (V, E) \) be a graph and \( e \in E \), then for \( G/e \) we have \( \gamma_{sp}(G) - 1 \leq \gamma_{sp}(G/e) \leq \gamma_{sp}(G) \). Also, we showed that these bounds are sharp. Since edge contraction does not have effect on creating new vertices with degree 1, then this argument holds for the end super domination number too. So we have the following result:

Proposition 5.5 Let \( G = (V, E) \) be a graph and \( e \in E \). Then,

\[
\gamma_{esp}(G) - 1 \leq \gamma_{esp}(G/e) \leq \gamma_{esp}(G).
\]

We end this section by an immediate result of Theorem 5.1 and Proposition 5.5:

Corollary 5.6 Let \( G = (V, E) \) be a graph and \( e \in E \). Then,

\[
\frac{\gamma_{esp}(G - e) + \gamma_{esp}(G/e)}{2} - 1 \leq \gamma_{esp}(G) \leq \frac{\gamma_{esp}(G - e) + \gamma_{esp}(G/e)}{2} + 1.
\]
6 End super domination number of $G - v$ and $G/v$

Let $v$ be a vertex in graph $G$. The graph $G - v$ is a graph that is made by deleting the vertex $v$ and all edges connected to $v$ from the graph $G$. The contraction of $v$ in $G$ denoted by $G/v$ is the graph obtained by deleting $v$ and putting a clique on the open neighbourhood of $v$. Note that this operation does not create parallel edges; if two neighbours of $v$ are already adjacent, then they remain simply adjacent (see [11]). In this section we examine the effects on $\gamma_{sp}(G)$ when $G$ is modified by a vertex removal and vertex contraction. First, we consider vertex removal of a graph and find upper and lower bound for super domination number of that.

**Theorem 6.1** Let $G = (V, E)$ be a graph and $v \in V$. Then,

$$\gamma_{esp}(G) - 1 \leq \gamma_{esp}(G - v) \leq \gamma_{esp}(G) + \deg(v) - 1.$$  

**Proof.** First we find the upper bound for $\gamma_{sp}(G - v)$. Suppose that $S$ is an end super dominating set of $G$. We consider the following cases:

(i) $v \notin S$. Then removing $v$ does not have effect on our super dominating set unless it is connected to vertices with degree 2. Now one can easily check that $S \cup N(v)$ is an end super dominating set for $G - v$. Since $v \notin S$, then atleast one of its neighbours is in $S$. So $|S \cup N(v)| \leq \gamma_{esp}(G) + \deg(v) - 1$. and we have the result.

(ii) $v \in S$. By the same argument as case (i), $S \cup N(v)$ is an end super dominating set for $G - v$. Since $v \in S$, then $|S \cup N(v)| \leq \gamma_{esp}(G) + \deg(v) - 1$, and we have the result.

Therefore in all cases, $\gamma_{sp}(G - v) \leq \gamma_{esp}(G) + \deg(v) - 1$. Now we find a lower bound for $\gamma_{sp}(G - v)$. First we form $G - v$ and find an end super dominating set for that. Suppose that this set is $S$. Now let $S' = S \cup \{v\}$ and add all removed edges connected to $v$ to form $G$. Then all vertices in $S'$ dominated by the same vertex as they dominated by vertices in $S$. Hence $\gamma_{sp}(G) \leq \gamma_{sp}(G - v) + 1$ and therefore we have the result. \qed

**Remark 6.2** Bounds in Theorem 6.1 are sharp. For the upper bound, it suffices to consider $G$ as we see in Figure 6. One can easily check that the set of all black vertices is an end super dominating set with minimum cardinality. Now if we form $G - v$ then clearly we have $\gamma_{esp}(G - v) \leq \gamma_{esp}(G) + \deg(v) - 1$. For the lower bound, it suffices to consider $H = P_5$ and $v$ is a pendant vertex. Then by Theorem 3.7, $\gamma_{esp}(H) = 3$ and $\gamma_{esp}(H - v) = 2$ and therefore $\gamma_{esp}(H) - 1 = \gamma_{esp}(H - v)$.

Now we find upper and lower bound for the end super domination number of a graph when it is modified by vertex contraction. First we consider pendant vertices.

**Theorem 6.3** Let $G = (V, E)$ be a graph and $v \in V$ is a pendant vertex. Then,

$$\gamma_{esp}(G) - 1 \leq \gamma_{esp}(G/v) \leq \gamma_{esp}(G).$$
Proof. If \( v \) be a pendant vertex, then \( G/v = G - v \). Therefore by Theorem 6.1, we have the result. \( \square \)

Now we consider vertex \( v \) of a graph \( G \) which is not pendant. In [4], we showed that if \( G = (V, E) \) be a graph and \( v \in V \) is not a pendant vertex, then \( \gamma_{sp}(G) - 1 \leq \gamma_{sp}(G/v) \leq \gamma_{sp}(G) + \left\lfloor \frac{\deg(v)}{2} \right\rfloor - 1 \). Also, we showed that these bounds are sharp. Since vertex contraction does not have effect on creating new vertices with degree 1, then this argument holds for end super domination number too. So we have the following result:

Proposition 6.4 Let \( G = (V, E) \) be a graph and \( v \in V \) is not a pendant vertex. Then,

\[
\gamma_{esp}(G) - 1 \leq \gamma_{esp}(G/v) \leq \gamma_{esp}(G) + \left\lfloor \frac{\deg(v)}{2} \right\rfloor - 1.
\]

We end this section by an immediate result of Theorem 6.1 and Proposition 6.4.

Corollary 6.5 Let \( G = (V, E) \) be a graph and \( v \in V \) is not a pendant vertex. Then,

\[
\gamma_{esp}(G) \leq \frac{\gamma_{esp}(G - v) + \gamma_{esp}(G/v)}{2} + 1,
\]

and

\[
\gamma_{esp}(G) \geq \frac{\gamma_{esp}(G - v) + \gamma_{esp}(G/v) - \left\lfloor \frac{\deg(v)}{2} \right\rfloor - \deg(v)}{2} + 1.
\]

7 Number of end super dominating sets with smallest cardinality of some graphs

In this section, we study the number of end super dominating sets with the smallest cardinality of a graph. Let \( N_{esp}(G) \) be the family of super dominating sets of a graph \( G \) with cardinality \( \gamma_{esp}(G) \) and let \( N_{esp}(G) = |N_{esp}(G)| \). In the following we consider some special graph classes and compute \( N_{esp} \) of them. Following Proposition 3.5, by an easy argument, we have the following easy example for \( N_{esp} \) of the complete graph, the complete bipartite graph and the star graph.

Example 7.1 (i) If \( K_n \) is the complete graph, then \( N_{esp}(K_n) = n \).

(ii) If \( K_{n,m} \) is the complete bipartite graph, where \( \min\{n, m\} \geq 2 \), then \( N_{esp}(K_{n,m}) = nm \).
(iii) If $K_{1,n}$ is the star graph, then $N_{esp}(K_{1,n}) = 1$.

By Remark 6.3 if $G$ has no pendant vertices, then $\gamma_{esp}(G) = \gamma_{sp}(G)$. So, if $G$ has no pendant vertices, then $N_{esp}(G)$ is equal to the number of super dominating sets with the smallest cardinality of cycles. Here we present it again for the $N_{esp}$ of the cycles and prove it by a different approach.

**Theorem 7.2** If $C_n$ be the path graph of order $n \geq 3$, then

$$N_{esp}(C_n) = \begin{cases} 4 & \text{if } n = 4k, \\ 2n & \text{if } n = 4k + 1, \\ \frac{5n^2 - 10n}{8} & \text{if } n = 4k + 2, \\ n & \text{if } n = 4k + 3. \end{cases}$$

**Proof.** Suppose that $V = \{v_1, \ldots, v_n\}$ is the vertex set of $C_n$ and $S$ is an end super dominating set for that with minimum cardinality. We consider the following cases:

1. $n = 4k$. By Proposition 3.5 we have $\gamma_{esp}(C_{4k}) = 2k$. First, suppose that $n \geq 12$. We show that for every three consecutive vertices in $V$, we can not have the middle one in $S$ and the other two in $\overline{S}$. Without loss of generality, suppose that $v_i \in S$ and $\{v_{i-1}, v_{i+1}\} \subseteq \overline{S}$. Then, by the definition of end super dominating set, we should have $\{v_{i-3}, v_{i-2}\} \subseteq S$ and $\{v_{i+2}, v_{i+3}\} \subseteq S$. Even if $v_{i+4}$ is end super dominated by $v_{i+3}$ and $v_{i-4}$ is end super dominated by $v_{i-3}$, then we are allowed to use only $2k - 5$ vertices. Suppose that we can do that. Then $C_{4k} - \{v_{i-4}, v_{i-3}, \ldots, v_{i+4}\} = P_{4k-9}$ and we have an end super domination set for that with cardinality $2k - 5$ which is a contradiction with $\gamma_{esp}(P_{4k-9}) = 2k - 4$.

By the same argument, it holds for $C_4$ and $C_8$ too. Similarly, for every three consecutive vertices in $V$, we can not have all of them in $S$. So the only possible way to have an end super dominating set is having two consecutive vertices in $S$ and no more or less. The followings are the only end super dominating sets for $C_{4k}$ with cardinality $2k$:

- $S_1 = \{v_1, v_2, v_3, v_6, \ldots, v_{4i+1}, v_{4i+2}, \ldots, v_{2k-3}, v_{2k-2}\}$,
- $S_2 = \{v_2, v_3, v_6, v_7, \ldots, v_{4i+2}, v_{4i+3}, \ldots, v_{2k-2}, v_{2k-1}\}$,
- $S_3 = \{v_3, v_4, v_7, v_8, \ldots, v_{4i+3}, v_{4i+4}, \ldots, v_{2k-1}, v_{2k}\}$,
- $S_4 = \{v_1, v_4, v_5, v_8, \ldots, v_{4i+1}, v_{4i+4}, \ldots, v_{2k-3}, v_{2k}\}$.

Hence, $N_{esp}(C_{4k}) = 4$. 

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(2) $n = 4k + 1$. By Proposition 3.5, we have $\gamma_{esp}(C_{4k+1}) = 2k + 1$. We can choose three consecutive vertices in $V$ by $4k + 1$ cases. Without loss of generality, suppose that we choose $v_1, v_2$ and $v_3$ and $S$ is our end super dominating set with $|S| = 2k + 1$. We consider two cases. First, $v_2 \in S$ and $\{v_1, v_3\} \subseteq \overline{S}$. Then we should have $\{v_4, v_5, v_{4k}, v_{4k+1}\} \subseteq S$ to have an end super dominating set. By similar argument as Case (1), we can not have $v_6 \in S$ or $v_7 \in S$. By continuing this process we have

$$S = \{v_1, v_4, v_5, \ldots, v_{4k}, v_{4k+1}\},$$

and $|S| = 2k + 1$. Second, $\{v_1, v_2, v_3\} \subseteq S$. Then by similar argument, we should have $\{v_4, v_5, v_{4k}, v_{4k+1}\} \subseteq \overline{S}$, and $\{v_6, v_7, v_{4k-2}, v_{4k-1}\} \subseteq \overline{S}$. So by continuing our argument,

$$S = \{v_1, v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\},$$

and $|S| = 2k + 1$. Now, we have $4k + 1$ cases to choose three consecutive vertices in $V$. So we have $2(4k + 1)$ end super dominating sets with size $2k + 1$. Note that choosing two vertices among these three vertices is counted, and choosing none of them is not possible.

(3) $n = 4k + 2$. By Proposition 3.5, we have $\gamma_{esp}(C_{4k+2}) = 2k + 2$. Since choosing any 4 vertices from $C_6$ gives us an end super dominating set, then $\binom{6}{4}$ is the number of super dominating sets of it, and it shows that the formula holds for $n = 6$. So, suppose that $n \geq 10$. First we show that it is not possible to have 5 consecutive vertices in $S$. Suppose that it is possible and without loss of generality suppose that $\{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\} \subseteq S$. Then $v_{4k+2}$ can be end super dominated by $v_{4k+1}$, and $v_{4k-4}$ can be end super dominated by $v_{4k-3}$. So we have $2k - 3$ other vertices in $S$ which is a contradiction, because then we have a cycle of order $4k - 5$ which has an end super dominating set with cardinality $2k - 3$, and by Proposition 3.5 we know that it is not possible. By similar argument, we conclude that it is not possible to have $A_1 \subseteq S$, $A_2 \subseteq S$ and $A_3 \subseteq S$ and $A_i \cap A_j = \emptyset$ for $i, j = 1, 2, 3$ and $i \neq j$, and $A_k$ for $k = 1, 2, 3$ has consecutive vertices and $|A_k| \geq 3$. So it remains to consider cases with only one subset of consecutive vertices of $S$ with cardinality 4, two subsets of consecutive vertices of $S$ with cardinality 3, one subset of consecutive vertices of $S$ with cardinality 3 and all subsets of consecutive vertices of $S$ with cardinality less than 3. We have two cases:

(i) $n \equiv 2 \pmod{8}$ which means $k$ is even. All cases happen to $\{v_1, v_2\} \subseteq S$ are shown in Figure 7. One can easily check that if we shift the indices, then we have all cases of end super dominating sets for $C_n$ and we have no more cases (consider movements of blue vertices in Figure 7 where filled ones are in $S$ and empty ones are in $\overline{S}$). All cases can be shifted $4k + 2$ times except the rows $\frac{k}{2}$ and $\frac{k}{2} + 2k$ which can be shifted $2k + 1$ times. So in general we have $\frac{k}{2} - 2$ cases which can be shifted $n$ times and 2 cases which can be shifted $\frac{3n}{2}$ times. Note that other cases which happen to $\{v_1, v_2\} \subseteq S$, $v_1 \in S$ and $v_2 \in \overline{S}$ and $\{v_1, v_2\} \subseteq \overline{S}$ can be found in the shifted cases.
Figure 7: Making end super dominating sets of cycle graph of order $n \equiv 2 \pmod{8}$

(ii) $n \equiv 6 \pmod{8}$ which means $k$ is odd. All cases happen to $\{v_1, v_2\} \subseteq S$ are shown in Figure 8. By similar argument as previous case, in general we have $2k + \frac{k-1}{2} = \frac{5k-1}{2}$ cases which can be shifted $n$ times and one case which can be shifted $\frac{7}{2}$ times.

Hence, in both cases we have $N_{\text{esp}}(C_{4k+2}) = \frac{7}{8}(n-2)(n)$.

(4) $n = 4k + 3$. It is easy to see that $N_{\text{esp}}(C_3) = 3$. Now suppose that $n \geq 7$. By Proposition 3.5, $\gamma_{\text{esp}}(C_{4k+3}) = 2k + 2$. So we need to choose $2k + 2$ vertices in a proper way to have an end super dominating set. First, we show that it is not possible to have 3 consecutive vertices in $S$. Suppose that it is possible, and without loss of generality suppose that $\{v_{4k}, v_{4k+1}, v_{4k+2}\} \subseteq S$. Then, $v_{4k+3}$ can be end super dominated by $v_{4k+2}$, and $v_{4k-1}$ can be end super dominated by $v_{4k}$. So, by our assumption, we can find $2k - 1$ vertices among $v_1, v_2, v_3, \ldots, v_{4k-2}$, let this set be $S'$, and have an end super dominating set for $C_{4k+3}$, which is $S = S' \cup \{v_{4k}, v_{4k+1}, v_{4k+2}\}$. But this is a contradiction, because if we can find $S'$, then we have an end super dominating set for $C_{4k-2}$ with size $2k - 1$ which is not possible by Proposition 3.5 since $\gamma_{\text{esp}}(C_{4k-2}) = 2k$. Second, we show that it is not possible among 3 consecutive vertices, the middle one be in $S$ and
two others be in \( \overline{S} \). Suppose that it is possible and without loss of generality, \( v_{4k-1} \in S \) and \( \{v_{4k-2}, v_{4k}\} \subseteq \overline{S} \). Then, because of the definition of end super dominating sets, we should have \( \{v_{4k+1}, v_{4k+2}\} \subseteq S \) and \( \{v_{4k-4}, v_{4k-3}\} \subseteq \overline{S} \). Now, suppose that \( \{v_{4k-5}, v_{4k+3}\} \subseteq \overline{S} \). Until now, we have chosen 5 vertices in \( S \). So, by our assumption, we can find \( 2k-3 \) vertices among \( v_1, v_2, v_3, \ldots, v_{4k-6} \), let this set be \( S'' \), and have an end super dominating set for \( C_{4k+3} \), which is \( S = S'' \cup \{v_{4k-4}, v_{4k-3}, v_{4k-1}, v_{4k+1}, v_{4k+2}\} \). But this is a contradiction too, because if we can find \( S'' \), then we have an end super dominating set for \( C_{4k-6} \) with size \( 2k-3 \) which is not possible by Proposition \( 3.5 \) since \( \gamma_{es}(C_{4k-6}) = 2k-2 \). Hence, we can only have 2 consecutive vertices in \( S \). Since \( \gamma_{es}(C_{4k+3}) = 2k + 2 \), then we have \( k+1 \) sets of size two with consecutive vertices in \( S \). So, for counting the number of super dominating sets of \( C_{4k+3} \), we should choose two consecutive vertices \( v_i, v_{i+1} \) and put them in our super dominating set and then put \( v_{i+2} \) in \( \overline{S} \) and possibly \( v_{i+3} \). Hence, among \( 2k+1 \) vertices in \( \overline{S} \), we should have \( k \) sets of size two with consecutive vertices in \( S \) and one set of size 1. Without loss of generality suppose that for the only set of size 1, this vertex is \( v_j \in \overline{S} \). Since we have only \( 4k+3 \) sets of 5 consecutive vertices \( v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2} \) such that \( v_j \in S \) and \( \{v_{j-2}, v_{j-1}, v_{j+1}, v_{j+2}\} \subseteq S \), we have \( N_{es}(C_{4k+3}) = 4k + 3 \).

**Figure 8:** Making end super dominating sets of cycle graph of order \( n \equiv 6 \) (mod 8)
Therefore we have the result.

We end this section by counting the $N_{esp}$ of the paths.

**Theorem 7.3** If $P_n$ be the path graph of order $n$, then

$$N_{esp}(P_n) = \begin{cases} 
1 & \text{if } n = 4k, \\
2k + 1 & \text{if } n = 4k + 1, \\
\frac{5k^2 + 5k + 2}{2} & \text{if } n = 4k + 2, \\
k + 1 & \text{if } n = 4k + 3.
\end{cases}$$

**Proof.** Suppose that $V = \{v_1, \ldots, v_n\}$ is the vertex set of $P_n$ and $S$ is an end super dominating set for that with minimum cardinality. We consider the following cases:

1. $n = 4k$. By Proposition 3.5 and Theorem 3.7, we have $\gamma_{esp}(P_{4k}) = \gamma_{esp}(C_{4k})$. In Theorem 7.2, Case (1), we found all end super dominating sets for $C_{4k}$. If we remove edge $v_1v_{4k}$, then only $S = S_4$ is an end super dominating set for $P_{4k}$ and we have $N_{esp}(P_{4k}) = 1$.

2. $n = 4k + 1$. We prove this case by induction on $k$. Let $k = 1$. Then by Theorem 3.7 we have $\gamma_{esp}(P_5) = 3$. We have three end super dominating sets of size 3 for $P_5$, which are $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_1, v_4, v_5\}$. So it holds for $k = 1$. Now suppose that for $k = i$, we have $N_{esp}(P_{4i+1}) = 2i + 1$. Here, by using this, we find the number of end super dominating sets for $P_{4(i+1)+1}$. Since $\gamma_{esp}(P_{4i+5}) = 2i + 3$, then we need two more vertices in $S$ to have an end super dominating set. Clearly, in the last four vertices we should have more than 1 vertex in $S$, so these two vertices should be picked from $v_{4i+2}$, $v_{4i+3}$, $v_{4i+4}$ and $v_{4i+5}$. In all cases, $v_{4i+5}$ should be in $S$. Now consider Figure 9 where black vertices belong to $S$. In both rows (I) and (II) in Figure 9 we can only add $v_{4i+2}$ and $v_{4i+5}$ to $S$, but in row (III), we have three different choices:

$$S_1 = S \cup \{v_{4i+2}, v_{4i+5}\},$$

$$S_2 = S \cup \{v_{4i+3}, v_{4i+5}\},$$

$$S_3 = S \cup \{v_{4i+4}, v_{4i+5}\}.$$ 

So, in total, we have $N_{esp}(P_{4i+5}) = N_{esp}(P_{4i+1}) + 2$, and the induction is completed. One can easily check that there is no possibility that we have $v_{4i+1} \in S$ and then find an end super dominating set for $P_{4i+5}$.

3. $n = 4k + 2$. By Proposition 3.5 and Theorem 3.7, we have $\gamma_{esp}(P_{4k+2}) = \gamma_{esp}(C_{4k+2})$. So, we do the same as Case (1). We have two cases:
Figure 9: Making end super dominating sets of path graph of order \(4k+1\)

(i) \(k\) is odd. Here we consider Figure 8 and find all end super dominating sets of this cycle. Then, remove edge \(v_1v_{4k+2}\) to have a path. Now, we choose among end super dominating sets of this cycle the proper ones. First, consider row 1. In this row, we have a set of 4 consecutive vertices in \(S\), and \(k - 1\) sets of 2 consecutive vertices in \(S\), which in total gives us \(4k + 2\) sets after shifting vertices, as we did in proof of Theorem 7.2. Among these sets, we have 3 sets which 4 consecutive vertices in \(S\), cover both of \(v_1\) and \(v_{4k+2}\), and \(k - 1\) sets which sets of 2 consecutive vertices cover them. So, in total for row 1, we have \(k + 2\) end super dominating sets for \(P_{4k+2}\). In the next step, we do the same for all rows in Figure 8. So, in general, we have:

(a) 1 row (row 1) with a set of 4 consecutive vertices and \(k - 1\) sets of 2 consecutive vertices in \(S\), which gives us \(k + 2\) end super dominating sets.

(b) \(\frac{k-1}{2}\) rows (rows 2 to \(\frac{k+1}{2}\)) with two sets of 3 consecutive vertices and \(k - 2\) sets of 2 consecutive vertices in \(S\), which in each row, gives us \(k + 2\) end super dominating sets.

(c) \(k\) rows (rows \(\frac{k+3}{2}\) to \(\frac{3k+1}{2}\)) with a set of 3 consecutive vertices, a set with single vertex and \(k - 1\) sets of 2 consecutive vertices in \(S\), which in each row, gives us \(k + 1\) end super dominating sets.

(d) \(\frac{k-1}{2}\) rows (rows \(\frac{3k+3}{2}\) to \(2k\)) with \(k + 1\) sets of 2 consecutive vertices in \(S\), which in each row, gives us \(k + 1\) end super dominating sets.

(e) 1 row (row \(2k + 1\)) with \(k + 1\) sets of 2 consecutive vertices in \(S\), which gives us \(k + \frac{1}{2}\) end super dominating sets. (Note that in this row which is separated by red color, half of cases occur two times.)

(f) \(\frac{k-1}{2}\) rows (rows \(2k + 2\) to \(2k + 1 + \frac{k-1}{2}\)) with two sets of single vertex and \(k\) sets of 2 consecutive vertices in \(S\), which in each row, gives us \(k\) end super dominating sets.

By summing all these cases, we have \(\frac{5k^2+5k+2}{2}\) end super dominating sets of size \(2k + 2\) for \(P_{4k+2}\).

(ii) \(k\) is even. By considering Figure 7 and same argument as previous case, we have the result.

Hence, in both cases, we have \(\frac{5k^2+5k+2}{2}\) end super dominating sets of size \(2k + 2\) and we are done.
(4) \( n = 4k + 3 \). By the same argument as Case (2) and using figure 10, we conclude that there is only one more set which is added to row (I), and we are done.

Therefore we have the result. □

8 Generalization

In this section, we introduce a new domination number which is a generalization of the end super domination number and present some preliminary results on that.

**Definition 8.1** Suppose that \( l \in \mathbb{N} \). A dominating set \( S \) is called a \( l \)-super dominating set of \( G \), if for every vertex \( u \in \mathcal{S} \), \( \deg(u) \geq l \) and there exists \( v \in S \) such that \( N(v) \cap \mathcal{S} = \{u\} \). The cardinality of a smallest \( l \)-super dominating set of \( G \), is the \( l \)-super domination number of \( G \) and is denoted by \( \gamma_{lsp}(G) \).

By this definition, it is clear that \( \gamma_{1sp}(G) = \gamma_{sp}(G) \) and \( \gamma_{2sp}(G) = \gamma_{esp}(G) \). So, in the following, we only consider the cases with \( l \geq 3 \). As an application of this definition, we can consider the Example 2.1 again and generalized it. In the following, we only mention the results in general form, like we did in the in Section 2.

**Proposition 8.2** let \( G \) be a graph. Then

\[
\gamma(G) \leq \gamma_{sp}(G) \leq \gamma_{lsp}(G).
\]

**Theorem 8.3** Let \( G = (V, E) \) be a connected graph of order \( n \) and \( S \) be an end super dominating set of \( G \). Also let \( l \geq 3 \). Then,

(i) \( \deg(v) \geq l \), for all \( v \in \mathcal{S} \).

(ii) \( 1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq \gamma_{lsp}(G) \leq n \).

**Proof.** Both cases are immediate results of Definition 8.1 Proposition 8.2 and Theorem 8.2 □

By an easy argument, we have:
Proposition 8.4  
(i) For the complete graph $K_n$,

$$
\gamma_{ls}(K_n) = \begin{cases} 
  n-1 & \text{if } l < n, \\
  n & \text{otherwise.}
\end{cases}
$$

(ii) Suppose that $\min\{n, m\} \geq 2$ and $n \geq m$. For the complete bipartite graph $K_{n,m}$,

$$
\gamma_{ls}(K_{n,m}) = \begin{cases} 
  n+m-2 & \text{if } l < m, \\
  n+m-1 & \text{if } m \leq l < n, \\
  n+m & \text{otherwise.}
\end{cases}
$$

(iii) For the cycle graph $C_n$,

$$
\gamma_{ls}(C_n) = n.
$$

(iv) For the path graph $P_n$,

$$
\gamma_{ls}(P_n) = n.
$$

(v) For the star graph $S_n$,

$$
\gamma_{ls}(S_n) = \begin{cases} 
  n-1 & \text{if } l < n, \\
  n & \text{otherwise.}
\end{cases}
$$

We end this paper by Nordhaus and Gaddum inequalities for the $l$-super domination number which is an immediate result of Theorem 8.3. One can easily check that the upper bounds are sharp for complete graph of order $n$ and $l \geq n$.

Theorem 8.5  Let $l \geq 3$. For any graph $G$ of order $n$, we have:

(i) $n \leq \gamma_{ls}(G) + \gamma_{ls}(\overline{G}) \leq 2n$.

(ii) $\frac{n^2}{4} \leq \gamma_{ls}(G) \cdot \gamma_{ls}(\overline{G}) \leq n^2$.

9 Conclusions

In this paper, we defined the end super domination number of graphs and its generalization. We presented some results on the $\gamma_{esp}(G)$ and $\gamma_{ls}(G)$, where $G$ is a graph. Also, we obtained exact value of the end super domination number and $l$-super domination number of path, cycle, star, complete and complete bipartite graphs. We found $N_{esp}$ of these graphs too, and presented Nordhaus-Gaddum type inequalities for the end super domination number and $l$-super domination number of a graph. Also, we found some tight bounds where $G$ is modified by operations on vertices and edges. Future topics of interest include the following suggestions:
(i) Finding the end super \((l\text{-super})\) domination number and \(N_{\text{esp}}\) of other specific graphs.

(ii) Figure 3 shows that even if we have \(t\) pendant vertices in a tree \(T\) of order \(n\) with \(t\) pendant vertices and \(t \geq \frac{n}{2}\), then we can not conclude that \(\gamma_{\text{esp}}(T) = t\). So, what can we say about \(\gamma_{\text{esp}}(T)\) in general, if it has \(t\) pendant vertices?

(iii) Finding the end super \((l\text{-super})\) domination number of a graph when it is modified by other unary operations like \(k\)-subdivision of a graph.

(iv) Finding the end super \((l\text{-super})\) domination number of binary operations between two graphs like join, corona, etc.

(v) Finding better lower bound for Nordhaus and Gaddum inequalities or show that these bounds are sharp.

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