Comment on “Self-Isospectral Periodic Potentials and Supersymmetric Quantum Mechanics”

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Abstract

We show that the formalism of supersymmetric quantum mechanics applied to the solvable elliptic function potentials
\[ V(x) = mj(j + 1)\text{sn}^2(x, m), \]
produces new exactly solvable one-dimensional periodic potentials.

In a recent paper, Dunne and Feinberg have systematically discussed various aspects of supersymmetric quantum mechanics (SUSYQM) as applied to periodic potentials. In particular, they defined and developed the concept of self-isospectral periodic potentials at length. Basically, a one dimensional potential \( V_-(x) \) of period \( 2K \) is said to be self-isospectral if its supersymmetric partner potential \( V_+(x) \) is just the original potential up to a discrete transformation - a translation by any constant amount, a reflection, or both. An example is translation by half a period, that is \( V_+(x) = V_-(x - K) \). In this sense, a self-isospectral potential is somewhat trivial, since application of the SUSYQM formalism to it yields nothing new. The main example considered in ref. is the class of elliptic function potentials

\[ V(x) = mj(j + 1)\text{sn}^2(x, m), \quad j = 1, 2, 3, \ldots \]

Here \( \text{sn}(x, m) \) is a Jacobi elliptic function of real elliptic modulus parameter \( m \) \((0 \leq m \leq 1)\). From now on, for simplicity, the argument \( m \) is suppressed. The Schrödinger equation of the
given elliptic potential is the well-known Lamé equation [3]. There are \( j \) bound bands (whose edges have known energies) followed by a continuum band. In ref. [1] it is claimed that the potentials given in eq. (1) are self-isospectral. The purpose of this comment is to point out that although the \( j = 1 \) potential is self-isospectral, this is not the case for higher values of \( j \). Indeed, for \( j \geq 2 \), we claim that SUSYQM generates new exactly solvable periodic problems.

Taking the case \( j = 2 \), and shifting the potential by a constant so that the ground state has zero energy gives

\[
V_-(x) = -2 - 2m + 2\delta + 6m\text{sn}^2(x), \quad \delta = \sqrt{1-m+m^2}. \tag{2}
\]

The band edge energies and Bloch wave functions \( \psi^{(-)}_n(x) \) [3] are given in Table 1. The superpotential is

\[
W \equiv -\frac{d}{dx}\log\psi^{(-)}_0(x) = \frac{6m\text{sn}(x)\text{cn}(x)\text{dn}(x)}{1 + m + \delta - 3m\text{sn}^2(x)}, \tag{3}
\]

The supersymmetric partner potentials \( V_\pm(x) \) are related to \( W(x) \) via \( V_\pm(x) = W^2(x) \pm dW/dx \). Hence, the potential \( V_+ \) is given by

\[
V_+(x) = -V_-(x) + \frac{72m^2\text{sn}^2(x)\text{cn}^2(x)\text{dn}^2(x)}{(1 + m + \delta - 3m\text{sn}^2(x))^2}. \tag{4}
\]

Using SUSYQM and the known eigenfunctions \( \psi^{(-)}_n(x) \) of \( V_-(x) \) one can immediately write down the corresponding un-normalized eigenfunctions \( \psi^{(+)}_n(x) \) of \( V_+(x) \).

\[
\psi^{(+)}_0(x) = 1, \quad \psi^{(+)}_n(x) = \left( \frac{d}{dx} + W(x) \right)\psi^{(-)}_n(x). \tag{5}
\]

We have computed the band edge eigenfunctions of \( V_+(x) \) and give them in Table 1. Our expression for \( V_+(x) \) [eq. (3)] does not agree with eq. (29) in ref. [1]. We have checked the correctness of our results by direct substitution into the Schrödinger equation, and by noting that in the limit of \( m \to 1 \), our \( V_+(x) \to 4 - 2 \text{sech}^2x \), which indeed is the supersymmetric partner of \( V_-(x, m = 1) = 4 - 6 \text{sech}^2x \) [2].

Proceeding in the same way, we have also obtained a new periodic potential \( V_+(x) \) corresponding to \( j = 3 \) case of eq. (1). Here, the ground state wave function is

\[
\psi^{(-)}_0(x) = \text{dn}(x)[1 + 2m + \delta_1 - 5m\text{sn}^2(x)]
\]

and the corresponding superpotential is

\[
W = \frac{\text{msn}(x)\text{cn}(x)}{\text{dn}(x)} \left[ \frac{2m + \delta_1 + 11 - 15m\text{sn}^2(x)}{2m + \delta_1 + 1 - 5m\text{sn}^2(x)} \right]. \tag{6}
\]
The partner potentials $V_{\pm}(x)$ turn out to be

$$V_-(x) = -2 - 5m + 2\delta_1 + 12msn^2(x), \quad \delta_1 \equiv \sqrt{1 - m + 4m^2},$$

and

$$V_+(x) = -V_-(x) + \frac{2m^2 sn^2(x) cn^2(x)}{dn^2(x)} \frac{[2m + \delta_1 + 11 - 15msn^2(x)]^2}{[2m + \delta_1 + 1 - 5msn^2(x)]^2}. \quad (7)$$

Clearly, the potential $V_-(x)$ is not self-isospectral. In fact, $V_-(x)$ and $V_+(x)$ are distinctly different periodic potentials which have the same seven band edges corresponding to three bound bands and a continuum band [3].

Although in this comment we have only focused on the $j = 2, 3$ cases, it is clear that SUSYQM provides a way of generating new solvable problems for all higher $j$ values. This is an exciting result given the extreme scarcity of analytically solvable periodic potentials. Indeed, a further extension to even more general potentials involving Jacobi elliptic functions [3] yields additional quasi exactly solvable periodic potentials [4]. Partial financial support from the U.S. Department of Energy is gratefully acknowledged.

**References**

[1] G. Dunne and J. Feinberg, Phys. Rev. **D57**, 1271 (1998).

[2] See, for example, F. Cooper, A. Khare and U.P. Sukhatme, Phys. Rev. **251**, 267 (1995).

[3] F.M. Arscott, *Periodic Differential Equations* (Pergamon, Oxford, 1981); W. Magnus and S. Winkler, *Hill’s Equation* (Wiley, New York, 1966).

[4] A. Khare and U. Sukhatme, UIC preprint UICHEP-TH/99-3 (1999), manuscript in preparation.

**Table 1: Band Edge Eigenstates of $V_\pm$ for $j = 2$ [}$\delta \equiv \sqrt{1 - m + m^2}$, $B \equiv 1 + m + \delta$]}

| n | $E_n$ | $\psi_n^{(-)}$ | $[B - 3m sn^2(x)] \psi_n^{(+)}$ |
|---|---|---|---|
| 0 | 0 | $m + 1 + \delta - 3msn^2(x)$ | 1 |
| 1 | $2\delta - 1 - m$ | $cn(x) dn(x)$ | $sn(x)[6m - (m + 1)B + m sn^2(x)(2B - 3 - 3m)]$ |
| 2 | $2\delta - 1 + 2m$ | $sn(x) dn(x)$ | $cn(x)[B + m sn^2(x)(3 - 2B)]$ |
| 3 | $2\delta + 2 - m$ | $sn(x) cn(x)$ | $dn(x)[B + sn^2(x)(3m - 2B)]$ |
| 4 | $4\delta$ | $m + 1 - \delta - 3m sn^2(x)$ | $sn(x) cn(x) dn(x)$ |