Theoretical analysis of super–Bloch oscillations

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Several recent studies have investigated the dynamics of cold atoms in optical lattices subject to AC forcing; the theoretically predicted renormalization of the tunneling amplitudes has been verified experimentally. Recent observations include global motion of the atom cloud, such as giant “Super–Bloch Oscillations” (SBOs). We show that, in order to understand unexplained features of SBOs, in addition to the renormalization of the tunneling, a new and important phase correction must be included. For Fermionic systems with strong attractive interactions, one may engineer different types of collisions and recollisions between bound-pairs and unpaired atoms.

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I. INTRODUCTION

Recent experiments on cold atoms in optical lattices subject to time-periodic perturbations [1][2][6] have provided clean realizations of a range of different types of AC control of coherent matter waves proposed by earlier theoretical studies [7][12]. Even neglecting the effects of inter-particle interactions, these studies identified non-trivial dynamical effects. In particular, for an oscillating potential of strength \( F \) and frequency \( \omega \), it was found that the tunneling amplitudes \( J \) of the driven atoms take an effective, renormalized, value:

\[
J_{\text{eff}} \propto J_n \left( \frac{F_d}{\hbar \omega} \right),
\]

where \( d \) is the lattice constant and \( J_n \) denotes an ordinary Bessel function and \( n = 0, 1, 2, \ldots \). Values of \( n > 0 \) arise if an additional static (in \( x \), the position) field \( F_0 x \) is applied, satisfying a resonance condition \( F_0 d = n \hbar \omega \). The inter-site transport is completely suppressed at parameters corresponding to the zeros of the Bessel functions.

The oscillating potential can be implemented with ultracold atoms in shaken optical lattices, which yield potentials of the form \( H_F(t) = F x \sin(\omega t + \phi) \), where \( \phi \) is a phase which can be controlled. Thus, Eq. (1) was demonstrated and investigated experimentally in Refs. [1][2]. The observed suppression of transport is variously termed Dynamic Localization (DL) or Coherent Destruction of Tunneling (CDT), depending on the transport regime [6][2][11]. They are both effects of the one-particle dynamics; comparatively less work has been done in the strongly interacting regime though, for example, in [6][13][14] they have investigated interactions-driven effects. Recent studies consider even triangular, shaken lattices [10].

The experiments in Refs. [1][2] investigated the spreading or local tunneling of an atomic wavepacket, without global motion. But an earlier theoretical study [12] had proposed also the possibility of global transport of the atomic wavepacket in the presence of the additional static field and assuming \( F_0 d \approx \hbar \omega \) (i.e., \( n = 1 \)). Recent experiments using a static field [4][6] were able to realize both directed motion as well as large oscillations, occurring over hundreds of sites, which were termed “Super–Bloch Oscillations” (SBOs) [5]. These SBOs were analyzed in Ref. [17], including the effects of weak interactions (mean-field regime). The general assumption was that the group velocity in these cases \( v_g \propto J_1(F_\omega) \), i.e., the dependence of the dynamics on the oscillating field \( F_\omega \equiv F_d/(\hbar \omega) \), is entirely contained in the Bessel function argument.

We show here that, in order to explain the new experiments, a phase correction \( F_\omega \cos \phi - n(\phi + \frac{\pi}{2}) \) must—additionally—be considered either in an effective dispersion relation or in the average group velocities. We show that inclusion of the appropriate terms can account for several unexplained experimental features in the experiments [4][6]. A notable example is a field dependent shift observed in the phase of SBOs, not accounted for by the usual analysis. Other features include the different field dependence of the speed of directed motion, on resonance, relative to the amplitude of SBOs.

There is also currently much interest in pairing phenomena, motivated by many ground-breaking experiments with ultracold Fermionic atoms in optical lattices [18][19]. We have previously shown [20] that in the case of attractive interactions, where the atoms can form bound-pairs, the second-order tunneling mechanism implies a different renormalization by the field and thus develop a global velocity relative to unpaired atoms. We investigate here for the first time the dynamics of pairs in the SBOs regime. We find the paired/unpaired components can be made to re-collide repeatedly; the novelty here is that the new \( F_\omega \)-dependent phase term enables control of re-collisions: the two components can either touch and reflect or can be forced through each other. Experimental demonstration is a matter of combining the techniques used to study of dynamics of bound pairs [19] with the
AC driving in Refs. [1–5]; the re-collisions in two component ultracold gases suggest the possibility of other applications in cold chemistry.

The central objective of the present work is the effect of the phase of the AC driving acting on an initially undisturbed atomic wavepacket. Thus, the phase $\phi$ must be well defined over every cycle, and the driving field must be switched on initially over a time that is small relative to the driving period. But provided this constraint is satisfied, the switch-on protocol is immaterial. The observed effects are never due to a discontinuous “switch-on” or an abrupt jump in amplitude of the potential at $t = 0$; in fact the strongest experimental effects occur for $\phi = 0, \pi$ where the potential ramps up smoothly from zero at $t = 0$ (i.e., a pure $\sin \omega t$ drive). The authors of Ref. [21] studied the effect of adiabatically switching on AC driving over many driving periods on the Floquet quasi-energy bands, where the phase is thus of no consequence. The authors of Ref. [22] considered slow linear ramping of the driving $f(t) = K t \sin(\omega t + \phi)$ by means of a perturbative treatment valid for high $\omega$. For a ramp period lasting an integer number of periods $t = nT$, a net ratchet current was obtained. Other forms of driving like amplitude modulated lattices that do not show dynamic localization can also produce directed motion [23].

In Section II below, we review key aspects of the theory of renormalization of tunneling for atoms subject to AC forcing. In Section III we show that the phase corresponds to directed motion and compare with other systems. In Section IV we analyze Super–Bloch Oscillations and obtain an expression which we show explains important features of recent experiments.

II. RENORMALIZATION OF TUNNELING

As in Refs. [3, 11], we consider the dynamics in a spatially periodic potential, subject to an additional time-periodic driving term. The total Hamiltonian is $H(t) = H_0 + H_F(t)$. Here, $H_0$ corresponds to the non-interacting limit of a variety of Hamiltonians with nearest-neighbor hopping (Hubbard, Bose-Hubbard, magnons in Heisenberg spin chains, etc.). It represents any spatially periodic potential characterized by energy eigenfunctions $\phi_{mk}$, with band index $m$ and wavenumber $k$, thus $H_0 \phi_{mk} = E_m(k) \phi_{mk}$. We restrict our one-particle problem, i.e.

$$H_0 = -J \sum_j (c_j^\dagger c_{j+1} + \text{H.c.}),$$

where $c_j^\dagger$ and $c_j$ are the creation and annihilation operators of a Fermion or Boson, to the lowest band $m = 1$; taking $E(k) \equiv E_{m=1}(k)$, the energy dispersion

$$E(k) = -J \cos kd,$$

so this corresponds to the group velocity

$$v_g = \frac{1}{\hbar} \frac{\partial E(k)}{\partial k} = \frac{Jd}{\hbar} \sin kd.$$

Assuming that the external driving is linear in position, we have $H_F(t) = -f(t)x$ where $f(t) = F_0 + F \sin(\omega t + \phi)$; in general it comprises both a static field and a sinusoidally oscillating field with an arbitrary phase $\phi$. The result of the driving is a time-dependent wavenumber:

$$q_k(t) = k + \frac{1}{\hbar} \int_0^t d\tau f(\tau).$$

The stationary states of the system are its Floquet states, the analogues of Bloch waves in a temporally periodic system. They are given by

$$\psi_k(x, t) = u_k(x, t) \exp \left[ -\frac{i}{\hbar} \epsilon(k) t \right],$$

where $u_k(x, t) = u_k(x, t + T)$, and the period $T = 2\pi/\omega$. The non-periodic phase term is characterized by the quasienergy $\epsilon(k)$. The evolution of a wavepacket projected onto its Floquet states is fully determined: the quasienergies play a role entirely analogous to the energy eigenvalues of a time-independent system; the (period-averaged) group velocity $v_g$ of a wavepacket is given from their dispersion:

$$v_g(k_0) = \frac{1}{\hbar} \frac{\partial \epsilon(k)}{\partial k} \big|_{k_0},$$

evaluated at the appropriate initial momentum, $k_0$, in analogy to Eq. (4) for the undriven system. Below we take $\hbar = 1$, which implies $F_0 = Fd/\omega$.

The presence of the static linear field, in general, destroys the band dynamics; however, here we consider the so-called resonant driving case, for which $F_0d = n\omega$, (where the driving compensates for the energy offset between neighboring wells in the lattice, restoring the band structure). In this case, it can be shown [11] that an effective quasienergy dispersion is obtained from the energy dispersion by a period-averaging, over one oscillation:

$$\epsilon(k) = \frac{1}{T} \int_0^T E(q_k(t)) dt.$$

We begin by considering the case $n = 0$ : in the first experimental studies on Dynamic Localization, $F_0 = 0$ and thus the static field was absent. In previous theoretical studies (Ref. [3]) a driving term of form $\sin(\omega t + \phi)$ was considered, but because of the particular objectives of that work, the effects of the phase $\phi$ were not retained. In that case, evaluating the integral in Eq. (8), the well-known renormalization expression (see also detailed derivation in Ref. [3]) was obtained:

$$\epsilon(k) = -J F_0(F_0 \cos kd).$$
It can be seen that the tunneling amplitude is multiplied by a Bessel function. Now the hopping can be completely suppressed at the zeros of the zero-th order Bessel function. This process was first demonstrated in Ref. 1. However, for the typical initial wavefunction corresponding to a zero-momentum ultracold atom cloud, for example, a Gaussian sharply peaked about \( k = k_0 = 0 \), Eq. (7) indicates an average group-velocity \( v_g = 0 \).

III. DIRECTED MOTION

The situation is quite different if the integral Eq. (8) is evaluated without disregarding \( \phi \). In Ref. 20 we found that even for the case \( \phi = 0 \), a momentum shift in the effective dispersion results. For the case of general \( \phi \), the effective dispersion relation becomes:

\[
\epsilon(k) = -J J_0(\omega x) \cos(kd + F_\omega \cos \phi),
\]

(10)

where one sees that there is an \( F_\omega \cos \phi \) shift, representing the average momentum over one cycle. This is no longer equal to zero. The result of this field-dependent shift is to introduce directed motion, at constant (period-averaged) group velocity \( v_g \).

Atoms in shaken lattices experience a homogeneous (position independent) force. Independently, theoretical \( 24 \) and experimental \( 25 \) studies considered the role of phase jumps in driven traps. These have inhomogeneous forces; for a driving field, \( V(x,t) = V'(x) \sin(\omega t + \phi) \) classically, the phase effects a position-dependent momentum shift \( \Delta P(x) \propto V'(x) \). The position dependence of \( V'(x) \) strongly couples the phase to the dynamics. \( \Delta P(x) \) can generate larger/smaller amplitude oscillations in the trap and has been proposed as a means to control the kinetic energy of the atoms oscillating in the trap.

Despite the different dynamics, for all the above systems, the effect of the phase vanishes for \( \phi = \pi/2 \).

In Eq. (10), the \( F_\omega \cos \phi \) phase shift coincides with the value of the lower bound of the integrand in Eq. (9) (here evaluated at \( t = 0 \)). This might lead one to conclude that the shift arises from the abrupt jump in the amplitude of the driving potential at \( t \approx 0 \). But this would be a misapprehension; the key physical significance of the shift is that it is the average momentum over each cycle, since \( \langle p(t) \rangle_T = F_\omega \cos \phi \) takes the same form. Below we find that the strongest experimental effects occur for \( \phi = 0, \pi \) where the driving grows linearly from zero for \( t \approx 0 \) (a pure \( \sin \omega t \) drive). The phase must be well defined over every cycle, so the switch-on time \( t_0 \) should satisfy the condition \( t_0 \ll T \). But provided this constraint is satisfied, the switch-on protocol is inammatural.

We now analyze recently discovered large scale oscillations for cold atoms in optical lattices. Although they are phase-dependent, we note that they are quite different from the atoms in traps: the dynamics are independent of initial position (the wavepacket is delocalized over several wells). We show also that the SBOs rely crucially on a specific quantum resonance (more precisely a slight detuning from it). We show below that their amplitude does not depend on \( \phi \) (unlike oscillations in traps); only their phase does.

IV. SUPER-BLOQUE OSCILLATIONS

Below we also consider the case of non-zero \( F_0 \) as well as slight detuning for which \( F_0d = (n + \delta)\omega \), with \( \delta \ll 1 \), associated with SBOs, for which the above relation still holds.

In order to calculate (period-averaged) group velocities, we first evaluate Eq. (5):

\[ q_k(t) = k + [(n + \delta)\omega - F_\omega \cos(\omega t + \phi) + F_\omega \cos \phi] / d, \]

(11)

(assuming \( q_k(t = 0) = k \)) then substitute the result in Eq. (3). However, for the slight-detuning case \( \delta \neq 0 \), we assume that the time-dependence due to the \( \delta \omega t/d \) remains negligible over one period \( T \). Thus, we take it out of the integral and Eq. (3) becomes (see Appendix A for the detailed derivation):

\[ \epsilon(k) \simeq -J J_n(\omega x) \cos \left[ kd + \delta \omega t + F_\omega \cos \phi - n(\phi + \pi/2) \right]. \]

(12)

The above represents an effective dispersion relation, but which oscillates slowly in time with a period \( T_{SBO} = 2\pi/(\delta \omega) \gg T_B \), where \( T_B \propto 1/F_0 \) is the Bloch period. They correspond to the SBOs investigated by Refs. 4, 5, 12, 17. Even at resonance \( \delta \omega = 0 \), Eq. (12) differs from previous expressions by the phase-shifts \( F_\omega \cos \phi - n(\phi + \pi/2) \).

Evaluating Eq. (12) for the \( n = 1 \) case, we obtain

\[ v_g = \frac{\partial \epsilon}{\partial k} \simeq -J d \cos(kd + F_\omega \cos \phi - \phi + \delta \omega t) J_1(\omega x). \]

(13)

However, experiments 5 measure the center-of-mass position \( x(t) = \int_0^t v_g(t')dt' \), where the inter-site spacing \( d \approx 0.533 \mu m \) is included if \( x(t) \) is obtained in \( \mu m \). In that case we obtain

\[ x(t) \simeq -\frac{Jd}{\delta \omega} J_1(\omega x) \sin[K_F + \delta \omega t] - \sin[K_F], \]

(14)

where \( K_F = kd + F_\omega \cos \phi - F_\omega \cos \phi - \phi \), since in experimental situations of interest here \( k \ll 0 \) initially.

Equation. (14) can largely account for the complex dependence of experimental results on \( \phi \). For instance, it was noted in Ref. 5 that the experimental phase of the SBOs depends on the sign of \( \delta \), for \( k = 0 \); this would not be expected without the \( F_\omega \cos \phi - \phi \) shifts, since \( \cos(\delta \omega t) = \cos(-\delta \omega t) \) [see Eq. (12)]; it was also noted that the SBO amplitudes scale as \( 1/(\delta \omega) \); also, for \( \phi = 0, \pi \), the SBO amplitude was close to a maximum for \( F_\omega = 1.52 \approx \pi/2 \). In contrast, we see that for \( \delta = 0 \) (directed motion) the peak velocity occurs wherever \( \cos(F_\omega) J_1(F_\omega) \) is a maximum (i.e., at \( F_\omega \simeq 1 \)). The directed motion is in fact almost zero for \( F_\omega = 1.52 \), the
point where the SBOs were near their maximum, but the highest experimental directed motion was given for \( F_\omega \approx 1 \).

However, the most interesting experimental feature predicted by our Eq. (14) is that the SBOs begin with a field-dependent phase, a feature surprisingly evident even in measurements not looking for this behavior. Figure 1 demonstrates this for \( F_\omega = 0.15, 1.52, \pi/2 \) and \( \pi \), in the non-interacting Hubbard model. Figures 1 (c) and (d) show that Eq. (14) reproduces quite well the experimental values of Ref. [5], especially for small times. The disagreement with experimental data in large times can be attributed mostly to interactions. The graph shows clearly the displacement of the first maximum, seen in the experiment as well as the order of magnitude variation in amplitude. Such a field-dependent shift is also apparent in the results of Ref. [4]. Figures 1 (e) and (f) show that Eq. (14) reproduces quite well the experimental data (symbols) for the regime of the experiments (panel correspond to \( F_\omega = 0.15 \) and \( F_\omega = 0.15 \); both have \( \phi = \pi \). For illustration purposes, the theoretical amplitudes were equalized by choosing \( J \)-values which equalize \( J_{\text{eff}} \approx J_\text{F} \). (c) and (b) Shows that good agreement is obtained between experiment and the Hubbard numerics, as well as the analytical formula: Eq. (14) (dashed lines) reproduces well both the amplitude and phase of SBO experimental data (symbols) of Ref. [5], using \( \delta \omega = 2\pi/1000 \approx 1 \text{ Hz detuning} \) \( \phi = \pi \) and \( J/\hbar \omega = 90 \mu \text{m} \). This implies \( J/\hbar \approx 1.06 \text{ ms}^{-1} \).

FIG. 1: (Color online) (a) and (b) One-particle solutions of the Hubbard Hamiltonian for \( L = 30 \) lattice sites, showing the field-dependence of the phase of SBOs. Upper/lower panel correspond to \( F_\omega = 1.52 \) and \( 0.15 \); both have \( \phi = \pi \). For illustration purposes, the theoretical amplitudes were equalized by choosing \( J \)-values which equalize \( J_{\text{eff}} \approx J_\text{F} \). (c) and (b) Shows that good agreement is obtained between experiment and the Hubbard numerics, as well as the analytical formula: Eq. (14) (dashed lines) reproduces well both the amplitude and phase of SBO experimental data (symbols) of Ref. [5], using \( \delta \omega = 2\pi/1000 \approx 1 \text{ Hz detuning} \) \( \phi = \pi \) and \( J/\hbar \omega = 90 \mu \text{m} \). This implies \( J/\hbar \approx 1.06 \text{ ms}^{-1} \).

Thus, the order of the Bessel, its argument, and the additional phase are all doubled; so the zeros and maxima of the BP motion occur at different fields from the unpaired atoms. However, in the presence of the driving, neither the BPs nor unpaired states remain eigenstates of the system. For \( |U| \gtrsim 2(F_0 + F) \), though, the energy gap between BPs and unpaired atoms cannot be closed by the fields, and there is relatively little mixing. We tested this assumption for simulations for \( L \approx 30-50 \) sites with \( N = 2, 3, \) and \( 4 \) atoms. Numerics are presently unfortunately limited by \( N = 4 \) and \( L = 30 \); since \( N = 4 \) includes the residual BP-BP interactions (and BP’s interact like hard-core bosons), it is
expected that the essential physics is included.

Nevertheless, we take a conservative approach and report below only on dynamics that are qualitatively insensitive to whether \( N = 2 \) or \( N = 3, 4 \). Thus, we consider the large \( U \) regime where BP and unpaired dynamics is separated and break-up rates of the BPs are modest: for \( |U| \lesssim 2(F_0 + F) \), the pairing rate falls rapidly and can oscillate in time. A further advantage of large \( |U| \) is that the BPs are essentially static, thus reducing another source of uncertainty.

In Fig. 2 we show the results of calculations slightly above the threshold where the field can strongly couple the Bound-Pair states and free atoms, and there is already a certain degree of interaction in evidence. It is seen that the BP component is essentially immobile, while the unpaired atoms perform large-scale oscillations; the unpaired atoms periodically return and re-collide with the BP packet. However, the \( F_\omega = \pi \) trajectories attempt to “push-through” the BPs, leading in general to a degree of beam-splitting, which depends on the kinetic energy (i.e., \( J \)) and effective \( h \propto 1/L \). The \( F_\omega = \pi/2 \) atoms, on the other hand, return with zero velocity and simply turn around.

Our analysis on BP dynamics is valid only for \( |U| \gg \omega \sim J \). In this regime, bound pairs remain in the BP band, and thus Eqs. (16) and (18) are likely to remain valid. If \( \omega \gg |U| \), new types of tunneling might be found. Such a high-frequency regime warrants further investigation.

VI. CONCLUSION

We have shown a phase-correction in an effective dispersion relation is essential for full understanding of current experiments on transport with AC forcing which showed Super–Bloch Oscillations.

In addition, we investigate Super–Bloch Oscillations for systems with bound pairs. We show that we can control collisions and re-collisions between unpaired atoms and bound pairs. This has potential implications for studies of AC control of two-component condensates, including Fermionic systems and molecular condensates, of relevance in cold chemistry.

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Appendix A: Effective dispersion relation

In order to derive Eq. (12) from Eq. (11), we assume that the Super–Bloch period represents a completely different timescale from the much faster Bloch period. Thus, we introduce two timescales, a rapid time \( t \) for the time-average over a period and a slow time \( t' \equiv \delta \omega t \) which describes the slow dynamics. Equation (12) is rewritten as

\[
q_\omega(t) = k + [t' + F_\omega \cos \phi + n\omega t - F_\omega \cos(\omega t + \phi)]/d. \quad (A1)
\]

Since \( \delta \ll 1 \), so \( t' \ll t \) and the change due to \( t' \) during one period \( T \) is negligible. Substituting (A1) into (8), we
have
\[
\epsilon(k) = -\frac{J}{T} \int_0^T dt \cos [kd + t' + F_\omega \cos \phi \\
+ n\omega t - F_\omega \cos(\omega t + \phi)]
\]
\[
= -\frac{J}{2\pi} \int_0^{2\pi+\phi} d\tau \cos [kd + t' + F_\omega \cos \phi - n\phi \\
+ n\tau - F_\omega \cos \tau]
\]
\[
\simeq -\frac{J}{2\pi} \cos(kd + t' + F_\omega \cos \phi - n\phi) \\
\times \int_0^{2\pi} d\tau \cos[n\tau - F_\omega \cos \tau] \\
+ \frac{J}{2\pi} \sin(kd + t' + F_\omega \cos \phi - n\phi) \\
\times \int_0^{2\pi} d\tau \sin[n\tau - F_\omega \cos \tau],
\]
where \( \tau = \omega t + \phi \). Evaluating both integrals above, one obtains Eq. (12):
\[
\epsilon(k) \simeq -J \mathcal{J}_n(F_\omega) \\
\times \cos[kd + \delta \omega t + F_\omega \cos \phi - n(\phi + \frac{\pi}{2})].
\]

The Bessel function in Eq. (12) is obtained from the (non-zero) integrals; the additional term \( n\pi/2 \) in the phase arises because the first integral vanishes for \( n \) odd, while the second integral vanishes for \( n \) even.

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