Extensions of a theorem of Erdős on nonhamiltonian graphs

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Abstract

Let \( n, d \) be integers with \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), and set \( h(n, d) := \left( \frac{n-d}{2} \right) + d^2 \). Erdős proved that when \( n \geq 6d \), each nonhamiltonian graph \( G \) on \( n \) vertices with minimum degree \( \delta(G) \geq d \) has at most \( h(n, d) \) edges. He also provides a sharpness example \( H_{n,d} \) for all such pairs \( n, d \). Previously, we showed a stability version of this result: for \( n \) large enough, every nonhamiltonian graph \( G \) on \( n \) vertices with \( \delta(G) \geq d \) and more than \( h(n, d) + 1 \) edges is a subgraph of \( H_{n,d} \).

In this paper, we show that not only does the graph \( H_{n,d} \) maximize the number of edges among nonhamiltonian graphs with \( n \) vertices and minimum degree at least \( d \), but in fact it maximizes the number of copies of any fixed graph \( F \) when \( n \) is sufficiently large in comparison with \( d \) and \( |F| \). We also show a stronger stability theorem, that is, we classify all nonhamiltonian \( n \)-graphs with \( \delta(G) \geq d \) and more than \( h(n, d+2) \) edges. We show this by proving a more general theorem: we describe all such graphs with more than \( \left( \frac{n-(d+2)}{k} \right) + (d+2)^{(d+2)}(k+1) \) copies of \( K_k \) for any \( k \). Mathematics Subject Classification: 05C35, 05C38.

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1 Introduction

Let \( V(G) \) denote the vertex set of a graph \( G \), \( E(G) \) denote the edge set of \( G \), and \( e(G) = |E(G)| \). Also, if \( v \in V(G) \), then \( N(v) \) is the neighborhood of \( v \) and \( d(v) = |N(v)| \). If \( v \in V(G) \) and \( D \subset V(G) \) then for shortness we will write \( D + v \) to denote \( D \cup \{v\} \). For \( k, t \in \mathbb{N} \), \( (k)_t \) denotes the falling factorial \( k(k-1)\ldots(k-t+1) = \frac{k!}{(k-t)!} \).

The first Turán-type result for nonhamiltonian graphs was due to Ore [11]:

**Theorem 1** (Ore [11]). If \( G \) is a nonhamiltonian graph on \( n \) vertices, then \( e(G) \leq \left( \frac{n-1}{2} \right) + 1 \).

This bound is achieved only for the \( n \)-vertex graph obtained from the complete graph \( K_{n-1} \) by adding a vertex of degree 1. Erdős [4] refined the bound in terms of the minimum degree of the graph:

\[\delta(G) \geq \frac{n}{2}, \text{ then } e(G) \leq \frac{n^2 - 3n + 4}{4} + 1.\]

\[\delta(G) \geq \frac{n}{2}, \text{ then } e(G) \leq \frac{n^2 - 3n + 4}{4} + 1.\]
Theorem 2 (Erdős [4]). Let \( n, d \) be integers with \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), and set \( h(n, d) := (\frac{n}{2}) + d^2 \). If \( G \) is a nonhamiltonian graph on \( n \) vertices with minimum degree \( \delta(G) \geq d \), then

\[
e(G) \leq \max \left\{ h(n, d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\} =: e(n, d).
\]

This bound is sharp for all \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

To show the sharpness of the bound, for \( n, d \in \mathbb{N} \) with \( d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \), consider the graph \( H_{n,d} \) obtained from a copy of \( K_{n-d} \), say with vertex set \( A \), by adding \( d \) vertices of degree \( d \) each of which is adjacent to the same \( d \) vertices in \( A \). An example of \( H_{11,3} \) is on the left of Fig 1.

![Graphs H_{11,3} (left) and K'_{11,3} (right).](image)

By construction, \( H_{n,d} \) has minimum degree \( d \), is nonhamiltonian, and \( e(H_{n,d}) = (\frac{n}{2}) + d^2 = h(n, d) \). Elementary calculation shows that \( h(n, d) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \) in the range \( 1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \) if and only if \( d < (n + 1)/6 \) and \( n \) is odd or \( d < (n + 4)/6 \) and \( n \) is even. Hence there exists a \( d_0 := d_0(n) \) such that

\[
e(n, 1) > e(n, 2) > \cdots > e(n, d_0) = e(n, d_0 + 1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor),
\]

where \( d_0(n) := \left\lceil \frac{n+1}{6} \right\rceil \) if \( n \) is odd, and \( d_0(n) := \left\lceil \frac{n+4}{6} \right\rceil \) if \( n \) is even. Therefore \( H_{n,d} \) is an extremal example of Theorem 2 when \( d < d_0 \) and \( H_{n,\left\lfloor (n-1)/2 \right\rfloor} \) when \( d \geq d_0 \).

In [10] and independently in [6] a stability theorem for nonhamiltonian graphs with prescribed minimum degree was proved. Let \( K'_{n,d} \) denote the edge-disjoint union of \( K_{n-d} \) and \( K_{d+1} \) sharing a single vertex. An example of \( K'_{11,3} \) is on the right of Fig 1.

Theorem 3 ([10] [6]). Let \( n \geq 3 \) and \( d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \). Suppose that \( G \) is an \( n \)-vertex nonhamiltonian graph with minimum degree \( \delta(G) \geq d \) such that

\[
e(G) > e(n, d + 1) = \max \left\{ h(n, d + 1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}.
\]

Then \( G \) is a subgraph of either \( H_{n,d} \) or \( K'_{n,d} \).

One of the main results of this paper shows that when \( n \) is large enough with respect to \( d \) and \( t \), \( H_{n,d} \) not only has the most edges among \( n \)-vertex nonhamiltonian graphs with minimum degree at least \( d \), but also has the most copies of any \( t \)-vertex graph. This is an instance of a generalization of the Turán problem called subgraph density problem: for \( n \in \mathbb{N} \) and graphs \( T \) and \( H \), let \( ex(n, T, H) \) denote the maximum possible number of (unlabeled) copies of \( T \) in an \( n \)-vertex \( H \)-free graph. When \( T = K_2 \), we have the usual extremal number \( ex(n, T, H) = ex(n, H) \).
Some notable results on the function $ex(n, T, H)$ for various combinations of $T$ and $H$ were obtained in [5, 2, 1, 8, 9, 7]. In particular, Erdös [5] determined $ex(n, K_s, K_t)$, Bollobás and Győri [2] found the order of magnitude of $ex(n, C_3, C_5)$, Alon and Shikhelman [1] presented a series of bounds on $ex(n, T, H)$ for different classes of $T$ and $H$.

In this paper, we study the maximum number of copies of $T$ in nonhamiltonian $n$-vertex graphs, i.e., $ex(n, T, C_n)$. For two graphs $G$ and $T$, let $N(G, T)$ denote the number of labeled copies of $T$ that are subgraphs of $G$, i.e., the number of injections $\phi : V(T) \rightarrow V(G)$ such that for each $xy \in E(T)$, $\phi(x)\phi(y) \in E(G)$. Since for every $T$ and $H$, $|Aut(T)|ex(n, T, H)$ is the maximum of $N(G, T)$ over the $n$-vertex graphs $G$ not containing $H$, some of our results are in the language of labeled copies of $T$ in $G$. For $k \in \mathbb{N}$, let $N_k(G)$ denote the number of unlabeled copies of $K_k$’s in $G$. Since $|Aut(K_k)| = k!$, we have $N_k(G) = N(G, K_k)/k!$.

2 Results

As an extension of Theorem [2], we show that for each fixed graph $F$ and any $d$, if $n$ is large enough with respect to $|V(F)|$ and $d$, then among all $n$-vertex nonhamiltonian graphs with minimum degree at least $d$, $H_{n,d}$ contains the maximum number of copies of $F$.

**Theorem 4.** For every graph $F$ with $t := |V(F)| \geq 3$, any $d \in \mathbb{N}$, and any $n \geq n_0(d, t) := 4dt + 3d^2 + 5t$, if $G$ is an $n$-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then $N(G, H) \leq N(H_{n,d}, F)$.

On the other hand, if $F$ is a star $K_{1,t-1}$ and $n \leq dt - d$, then $H_{n,d}$ does not maximize $N(G, F)$. At the end of Section 4, we show that in this case, $N(H_{n,[(n-1)/2]}, F) > N(H_{n,d}, F)$. So, the bound on $n_0(d, t)$ in Theorem 4 has the right order of magnitude when $d = O(t)$.

An immediate corollary of Theorem 4 is the following generalization of Theorem [1].

**Corollary 5.** For every graph $F$ with $t := |V(F)| \geq 3$ and any $n \geq n_0(t) := 9t + 3$, if $G$ is an $n$-vertex nonhamiltonian graph, then $N(G, H) \leq N(H_{n,1}, F)$.

We consider the case that $F$ is a clique in more detail. For $n, k \in \mathbb{N}$, define on the interval $[1, [(n-1)/2]]$ the function

$$h_k(n, x) := \binom{n-x}{k} + x \binom{x}{k-1}.$$  

(2)

We use the convention that for $a \in \mathbb{R}$, $b \in \mathbb{N}$, $\binom{n}{k}$ is the polynomial $\frac{1}{b!} a \times (a-1) \times \ldots \times (a-b+1)$ if $a \geq b-1$ and 0 otherwise.

By considering the second derivative, one can check that for any fixed $k$ and $n$, as a function of $x$, $h_k(n, x)$ is convex on $[1, [(n-1)/2]]$, hence it attains its maximum at one of the endpoints, $x = 1$ or $x = [(n-1)/2]$. When $k = 2$, $h_2(n, x) = h(n, x)$. We prove the following generalization of Theorem [2].

**Theorem 6.** Let $n, d, k$ be integers with $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$ and $k \geq 2$. If $G$ is a nonhamiltonian graph
on n vertices with minimum degree $\delta(G) \geq d$, then the number $N_k(G)$ of k-cliques in $G$ satisfies

$$N_k(G) \leq \max \left\{ h_k(n, d), h_k\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor \right) \right\}$$

Again, graphs $H_{n,d}$ and $H_{n,\lfloor (n-1)/2 \rfloor}$ are sharpness examples for the theorem.

Finally, we present a stability version of Theorem 6. To state the result, we first define the family of extremal graphs.

Fix $d \leq \lfloor (n-1)/2 \rfloor$. In addition to graphs $H_{n,d}$ and $K'_{n,d}$ defined above, define $H'_{n,d}$: $V(H'_{n,d}) = A \cup B$, where $A$ induces a complete graph on $n-d-1$ vertices, $B$ is a set of $d+1$ vertices that induce exactly one edge, and there exists a set of vertices $\{a_1, \ldots, a_d\} \subseteq A$ such that for all $b \in B$, $N(b) - B = \{a_1, \ldots, a_d\}$. Note that contracting the edge in $H'_{n,d}[B]$ yields $H_{n-1,d}$. These graphs are illustrated in Fig. 2.

![Figure 2: Graphs $H_{n,d}$ (left), $K'_{n,d}$ (center), and $H'_{n,d}$ (right), where shaded background indicates a complete graph.](image)

We also have two more extremal graphs for the cases $d = 2$ or $d = 3$. Define the nonhamiltonian $n$-vertex graph $G'_{n,2}$ with minimum degree 2 as follows: $V(G'_{n,2}) = A \cup B$ where $A$ induces a clique or order $n-3$, $B = \{b_1, b_2, b_3\}$ is an independent set of order 3, and there exists $\{a_1, a_2, a_3, x\} \subseteq A$ such that $N(b_i) = \{a_i, x\}$ for $i \in \{1, 2, 3\}$ (see the graph on the left in Fig. 3).

The nonhamiltonian $n$-vertex graph $F_{n,3}$ with minimum degree 3 has vertex set $A \cup B$, where $A$ induces a clique of order $n-4$, $B$ induces a perfect matching on 4 vertices, and each of the vertices in $B$ is adjacent to the same two vertices in $A$ (see the graph on the right in Fig. 3).

![Figure 3: Graphs $G'_{n,2}$ (left) and $F_{n,3}$ (right).](image)

Our stability result is the following:

**Theorem 7.** Let $n \geq 3$ and $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Suppose that $G$ is an $n$-vertex nonhamiltonian graph.
Suppose there exist \( x, y \in V(G) \) such that there exists \( k \geq 2 \) for which

\[
N_k(G) > \max \left\{ h_k(n, d + 2), h_k(n, \left\lfloor \frac{n - 1}{2} \right\rfloor) \right\}.
\]

Let \( H_{n,d} := \{ H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d} \} \).

(i) If \( d = 2 \), then \( G \) is a subgraph of \( G'_{n,2} \) or of a graph in \( H_{n,2} \);
(ii) if \( d = 3 \), then \( G \) is a subgraph of \( F_{n,3} \) or of a graph in \( H_{n,3} \);
(iii) if \( d = 1 \) or \( 4 \leq d \leq \left\lfloor \frac{n - 1}{2} \right\rfloor \), then \( G \) is a subgraph of a graph in \( H_{n,d} \).

The result is sharp because \( H_{n,d+2} \) has \( h_k(n, d + 2) \) copies of \( K_k \), minimum degree \( d + 2 > d \), is nonhamiltonian and is not contained in any graph in \( H_{n,d} \cup \{ G'_{n,2}, F_{n,3} \} \).

The outline for the rest of the paper is as follows: in Section 3 we present some structural results for graphs that are edge-maximal nonhamiltonian to be used in the proofs of the main theorems, in Section 4 we prove Theorem 4 in Section 5 we prove Theorem 6 and give a cliques version of Theorem 3 and in Section 6 we prove Theorem 7.

### 3 Structural results for saturated graphs

We will use a classical theorem of Pósa (usually stated as its contrapositive).

**Theorem 8** (Pósa [12]). Let \( n \geq 3 \). If \( G \) is a nonhamiltonian \( n \)-vertex graph, then there exists \( 1 \leq k \leq \left\lfloor \frac{n - 1}{2} \right\rfloor \) such that \( G \) has a set of \( k \) vertices with degree at most \( k \).

Call a graph \( G \) saturated if \( G \) is nonhamiltonian but for each \( uv \notin E(G) \), \( G + uv \) has a hamiltonian cycle. Ore’s proof [11] of Dirac’s Theorem [3] yields that

\[
d(u) + d(v) \leq n - 1
\]

for every \( n \)-vertex saturated graph \( G \) and for each \( uv \notin E(G) \).

We will also need two structural results for saturated graphs which are easy extensions of Lemmas 6 and 7 in [9].

**Lemma 9.** Let \( G \) be a saturated \( n \)-vertex graph with \( N_k(G) > h_k(n, \left\lfloor \frac{n - 1}{2} \right\rfloor) \) for any \( k \geq 2 \). Then for some \( 1 \leq r \leq \left\lfloor \frac{n - 1}{2} \right\rfloor \), \( V(G) \) contains a subset \( D \) of \( r \) vertices of degree at most \( r \) such that \( G - D \) is a complete graph.

**Proof.** Since \( G \) is nonhamiltonian, by Theorem 6 there exists some \( 1 \leq r \leq \left\lfloor \frac{n - 1}{2} \right\rfloor \) such that \( G \) has \( r \) vertices with degree at most \( r \). Pick the maximum such \( r \), and let \( D \) be the set of the vertices with degree at most \( r \). Since \( h_k(G) > h(n, \left\lfloor \frac{n - 1}{2} \right\rfloor) \), \( r < \left\lfloor \frac{n - 1}{2} \right\rfloor \). So, by the maximality of \( r \), \( |D| = r \).

Suppose there exist \( x, y \in V(G) - D \) such that \( xy \notin E(G) \). Among all such pairs, choose \( x \) and \( y \) with the maximum \( d(x) \). Since \( y \notin D, d(y) > r \). Let \( D' := V(G) - N(x) - \{ x \} \) and \( r' := |D'| = n - 1 - d(x) \). By (4),

\[
d(z) \leq n - 1 - d(x) = r' \quad \text{for all} \quad z \in D'.
\]
Lemma 11. For any extremal graphs of Lemma 10, if \( F \) is large then \( (n-r')/2 \) \( k' \)-cliques in \( D' \) can be in at most \( (n-r')/2 \) \( k' \)-cliques. Therefore \( N_k(G) \leq (n-r')/2 + r'(k-1) \leq h_k(n, [n-1]/2) \), a contradiction.

Also, repeating the proof of Lemma 7 in [6] gives the following lemma.

**Lemma 10** (Lemma 7 in [6]). Under the conditions of Lemma 7, if \( r = \delta(G) \), then \( G = H_{n,\delta(G)} \) or \( G = K'_{n,\delta(G)} \).

## 4 Maximizing the number of copies of a given graph and a proof of Theorem 4

In order to prove Theorem 4, we first show that for any fixed graph \( F \) and any \( d \), of the two extremal graphs of Lemma 10 if \( n \) is large then \( H_{n,d} \) has at least as many copies of \( F \) as \( K'_{n,d} \).

**Lemma 11.** For any \( d,t,n \in \mathbb{N} \) with \( n \geq 2dt + d + t \) and any graph \( F \) with \( t = |V(F)| \) we have \( N(K'_{n,d}, F) \leq N(H_{n,d}, F) \).

**Proof.** Fix \( F \) and \( t = |V(F)| \). Let \( K'_{n,d} = A \cup B \) where \( A \) and \( B \) are cliques of order \( n - d \) and \( d + 1 \) respectively and \( A \cap B = \{v^*\} \), the cut vertex of \( K'_{n,d} \). Also, let \( D \) denote the independent set of order \( d \) in \( H_{n,d} \). We may assume \( d \geq 2 \), because \( H_{n,1} = K'_{n,1} \). If \( x \) is an isolated vertex of \( F \) then for any \( n \)-vertex graph \( G \) we have \( N(G,F) = (n-t+1)N(G,F-x) \). So it is enough to prove the case \( \delta(F) \geq 1 \), and we may also assume \( t \geq 3 \).

Because both \( K'_{n,d}[A] \) and \( H_{n,d} - D \) are cliques of order \( n - d \), the number of embeddings of \( F \) into \( K'_{n,d}[A] \) is the same as the number of embeddings of \( F \) into \( H_{n,d} - D \). So it remains to compare only the number of embeddings in \( \varphi : V(F) \to V(K'_{n,d}) \) such that \( \varphi(F) \) intersects \( B - v^* \) to the number of embeddings in \( \psi : V(F) \to V(H_{n,d}) \) such that \( \psi(F) \) intersects \( D \).

Let \( C \cup \overline{C} \) be a partition of the vertex set \( V(F) \), \( s := |C| \). Define the following classes of \( \Phi \) and \( \Psi \):
- \( \Phi(C) := \{ \varphi : V(F) \to V(K'_{n,d}) \text{ such that } \varphi(C) \text{ intersects } B - v^*, \varphi(C) \subseteq B, \text{ and } \varphi(\overline{C}) \subseteq V - B \} \); 
- \( \Psi(C) := \{ \psi : V(F) \to V(H_{n,d}) \text{ such that } \psi(C) \text{ intersects } D, \psi(C) \subseteq (D \cup N(D)), \text{ and } \psi(\overline{C}) \subseteq V - (D \cup N(D)) \} \).

By these definitions, if \( C \neq C' \) then \( \Phi(C) \cap \Phi(C') = \emptyset \), and \( \Phi(C) \cap \Phi(C') = \emptyset \). Also \( \bigcup_{\emptyset \neq C \subseteq V(F)} \Phi(C) = \Phi \). We claim that for every \( C \neq \emptyset \), 
\[
|\Phi(C)| \leq |\Psi(C)|. \tag{6}
\]

Summing up the number of embeddings over all choices for \( C \) will prove the lemma. If \( \Phi(C) = \emptyset \), then (6) obviously holds. So from now on, we consider the cases when \( \Phi(C) \) is not empty, implying \( 1 \leq s \leq d + 1 \).

**Case 1:** There is an \( F \)-edge joining \( \overline{C} \) and \( C \). So there is a vertex \( v \in C \) with \( N_F(v) \cap \overline{C} \neq \emptyset \). Then for every mapping \( \varphi \in \Phi(C) \), the vertex \( v \) must be mapped to \( v^* \) in \( K'_{n,d} \). Let \( \varphi(v) = v^* \). So this
vertex \( v \) is uniquely determined by \( C \). Also, \( \varphi(C) \cap (B - v^*) \neq \emptyset \) implies \( s \geq 2 \). The rest of \( C \) can be mapped arbitrarily to \( B - v^* \) and \( \overline{C} \) can be mapped arbitrarily to \( A - v^* \). We obtained that \( |\Phi(C)| = (d)_{s-1}(n - d - 1)_{t-s} \).

We make a lower bound for \( |\Psi(C)| \) as follows. We define a \( \psi \in \Psi(C) \) by the following procedure. Let \( \psi(v) = x \in N(D) \) (there are \( d \) possibilities), then map some vertex of \( C - v \) to a vertex \( y \in D \) (there are \( (s-1)d \) possibilities). Since \( N+y \) forms a clique of order \( d+1 \) we may embed the rest of \( C \) into \( N-v \) in \( (d-1)s-2 \) ways and finish embedding of \( F \) into \( H_{n,d} \) by arbitrarily placing the vertices of \( \overline{C} \) to \( V-(D \cup N(D)) \). We obtained that \( |\Psi(C)| \geq d^2(s-1)(d-1)_{s-2}(n-2d)_{t-s} = d(s-1)(d)_{s-1}(n-2d)_{t-s} \).

Since \( s \geq 2 \) we have that

\[
\frac{|\Psi(C)|}{|\Phi(C)|} \geq \frac{d(s-1)(d-1)_{s-2}(n-2d)_{t-s}}{(d)_{s-1}(n - d - 1)_{t-s}} \geq d(2-1)\left(\frac{n-2d+1-t+s}{n-d-t+s}\right)^{t-s} = d\left(1 - \frac{d-1}{n-d-t+s}\right)^{t-s} \geq d\left(1 - \frac{(d-1)(t-s)}{n-d-t+s}\right) \geq d\left(1 - \frac{(d-1)t}{n-d-t}\right) > 1 \text{ when } n > dt + d + t.
\]

**Case 2:** \( C \) and \( \overline{C} \) are not connected in \( F \). We may assume \( s \geq 2 \) since \( C \) is a union of components with \( \delta(F) \geq 1 \). In \( K'_{n,d} \) there are at exactly \( (d+1)s(n-d-1)_{t-s} \) ways to embed \( F \) into \( B \) so that only \( C \) is mapped into \( B \) and \( \overline{C} \) goes to \( A - v^* \), i.e., \( |\Phi(C)| = (d+1)s(n - d - 1)_{t-s} \).

We make a lower bound for \( |\Psi(C)| \) as follows. We define a \( \psi \in \Psi(C) \) by the following procedure. Select any vertex \( v \in C \) and map it to some vertex in \( D \) (there are \( sd \) possibilities), then map \( C - v \) into \( N(D) \) (there are \( (d)_{s-1} \) possibilities) and finish embedding of \( F \) into \( H_{n,d} \) by arbitrarily placing the vertices of \( \overline{C} \) to \( V - (D \cup N(D)) \). We obtained that \( |\Psi(C)| \geq ds(d)_{s-1}(n-2d)_{t-s} \). We have

\[
\frac{|\Psi(C)|}{|\Phi(C)|} \geq \frac{ds(d)_{s-1}(n-2d)_{t-s}}{(d+1)s(n - d - 1)_{t-s}} \geq \frac{ds}{d+1} \left(1 - \frac{(d-1)t}{n-d-t}\right) \geq \frac{2d}{d+1} \left(1 - \frac{(d-1)t}{n-d-t}\right) \text{ because } s \geq 2 \geq 1 \text{ when } n > 2dt + d + t.
\]

\[ \square \]

We are now ready to prove **Theorem 4**

**Theorem 4** For every graph \( F \) with \( t := |V(F)| \geq 3 \), any \( d \in \mathbb{N} \), and any \( n \geq n_0(d,t) := 4dt + 3d^2 + 5t \), if \( G \) is an \( n \)-vertex nonhamiltonian graph with minimum degree \( \delta(G) \geq d \), then \( N(G,H) \leq N(H_{n,d}, F) \).

**Proof.** Let \( d \geq 1 \). Fix a graph \( F \) with \( |V(F)| \geq 3 \) (if \( |V(F)| = 2 \), then either \( F = K_2 \) or \( F = \overline{K}_2 \)). The case where \( G \) has isolated vertices can be handled by induction on the number of isolated
vertices, hence we may assume each vertex has degree at least 1. Set
\[ n_0 = 4dt + 3d^2 + 5t. \] (7)

Fix a nonhamiltonian graph $G$ with $|V(G)| = n \geq n_0$ and $\delta(G) \geq d$ such that $N(G, F) > N(H_{n,d}, F) \geq (n - d)_t$. We may assume that $G$ is saturated, as the number of copies of $F$ can only increase when we add edges to $G$.

Because $n \geq 4dt + t$ by [7],
\[ \frac{(n - d)_t}{(n)_t} \geq \left( \frac{n - d - t}{n - t} \right)^t = \left( 1 - \frac{d}{n - t} \right)^t \geq 1 - \frac{dt}{n - t} \geq 1 - \frac{1}{4} = \frac{3}{4}. \]

So, $(n - d)_t \geq \frac{3}{4}(n)_t$.

After mapping edge $xy$ of $F$ to an edge of $G$ (in two labeled ways), we obtain the loose upper bound,
\[ 2e(G)(n - 2)_{t - 2} \geq N(G, F) \geq (n - d)_t \geq \frac{3}{4}(n)_t, \]

therefore
\[ e(G) \geq \frac{3}{4} \binom{n}{2} > h_2(n, \lfloor (n - 1)/2 \rfloor). \] (8)

By Pósa’s theorem (Theorem [8]), there exists some $d \leq r \leq \lfloor (n - 1)/2 \rfloor$ such that $G$ contains a set $R$ or $r$ vertices with degree at most $r$. Furthermore by [8], $r < d_0$. So by integrality, $r \leq d_0 - 1 \leq (n + 3)/6$. If $r = d$, then by Lemma [10] either $G = H_{n,d}$ or $G = K'_{n,d'}$. By Lemma [11] and [7], $G = H_{n,d}$, a contradiction. So we have $r \geq d + 1$.

Let $I$ denote the family of all nonempty independent sets in $F$. For $I \in \mathcal{I}$, let $i = i(I) := |I|$ and $j = j(I) := |N_F(I)|$. Since $F$ has no isolated vertices, $j(I) \geq 1$ and so $i \leq t - 1$ for each $I \in \mathcal{I}$. Let $\Phi(I)$ denote the set of embeddings $\varphi : V(F) \to V(G)$ such that $\varphi(I) \subseteq R$ and $I$ is a maximum independent subset of $\varphi^{-1}(R \cap \varphi(F))$. Note that $\varphi(I)$ is not necessarily independent in $G$. We show that
\[ |\Phi(I)| \leq (r)_i r(n - r)_{t - i - 1}. \] (9)

Indeed, there are $(r)_i$ ways to choose $\varphi(I) \subseteq R$. After that, since each vertex in $R$ has at most $r$ neighbors in $G$, there are at most $r^j$ ways to embed $N_F(I)$ into $G$. By the maximality of $I$, all vertices of $F - I - N_F(I)$ should be mapped to $V(G) - R$. There are at most $(n - r)_{t - i - j}$ to do it. Hence $|\Phi(I)| \leq (r)_i r^j(n - r)_{t - i - j}$. Since $2r + t \leq 2(d_0 - 1) + t < n$, this implies [9].

Since each $\varphi : V(F) \to V(G)$ with $\varphi(V(F)) \cap R \neq \emptyset$ belongs to $\Phi(I)$ for some nonempty $I \in \mathcal{I}$, [5] implies
\[ N(G, F) \leq (n - r)_t + \sum_{\emptyset \neq I \in \mathcal{I}} |\Phi(I)| \leq (n - r)_t + \sum_{i=1}^{t-1} \binom{t}{i} (r)_i r(n - r)_{t - i - 1}. \] (10)
Hence
\[
\frac{N(G,F)}{N(H_{n,d},F)} \leq \frac{(n-r)_t}{(n-d)_t} + \sum_{i=1}^{t-1} \left( \frac{1}{i} \right) r (n-r)_{t-i} - 1 \\
\leq \frac{(n-r)_t}{(n-d)_t} + \frac{1}{n-r-t+2} \sum_{i=1}^{t-1} \left( \frac{1}{i} \right) r (n-r)_{t-i} \\
= \frac{(n-r)_t}{(n-d)_t} + \frac{(n)_t - (n-r)_t}{(n-d)_t} \times \frac{r}{n-r-t+2} \\
\leq \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2}{n-t+2} + \frac{(n)_t}{(n-d)_t} \times \frac{r}{n-t+2} := f(r).
\]

Given fixed \(n,d,t\), we claim that the real function \(f(r)\) is convex for \(0 < r < (n-t+2)/2\).

Indeed, the first term \(g(r) := \frac{(n-r)_t}{(n-d)_t} \times \frac{n-t+2-2r}{n-t+2} r\) is a product of \(t\) linear terms in each of which \(r\) has a negative coefficient (note that the \(n-t+2-r\) term cancels out with a factor of \(n-r-t+2\) in \((n-r)_t\)). Applying product rule, the first derivative \(g'\) is a sum of \(t\) products, each with \(t-1\) linear terms. For \(r < (n-t+2)/2\), each of these products is negative, thus \(g'(r) < 0\). Finally, applying product rule again, \(g''\) is the sum of \(t(t-1)\) products. For \(r < (n-t+2)/2\) each of the products is positive, thus \(g''(r) > 0\).

Similarly, the second factor of the second term (as a real function of \(r\), of the form \(r/(c-r)\)) is convex for \(r < n-t+2\).

We conclude that in the interval \([d+1,(n+3)/6]\) the function \(f(r)\) takes its maximum either at one of the endpoints \(r = d+1\) or \(r = (n+3)/6\). We claim that \(f(r) < 1\) at both end points.

In case of \(r = d+1\) the first factor of the first term equals \((n-d-t)/(n-d)\). To get an upper bound for the first factor of the second term one can use the inequality \(\prod (1+x_i) < 1+2 \sum x_i\) which holds for any number of non-negative \(x_i\)'s if \(0 < \sum x_i \leq 1\). Because \(dt/(n-d-t+1) \leq 1\) by (7), we obtain that

\[
f(d+1) < \frac{n-d-t}{n-d} \times \frac{n-t-2d}{n-t-d+1} + \left( 1 + \frac{2dt}{n-d-t+1} \right) \times \frac{d+1}{n-t-d+1} \\
= \left( 1 - \frac{t}{n-d} \right) \times \left( 1 - \frac{d+1}{n-t-d+1} \right) + \left( \frac{d+1}{n-t-d+1} \right) \times \left( \frac{2dt(d+1)}{(n-t-d+1)^2} \right) \\
= 1 - \frac{t}{n-d} \times \frac{d+1}{n-t-d+1} + \frac{t}{n-d} \times \frac{2dt(d+1)}{n-t-d+1} \times \frac{n-d}{n-t-d+1} \\
= 1 - \frac{t}{n-d} \times \left( \frac{d+1}{n-t-d+1} - \frac{2dt(d+1)}{n-t-d+1} \times \left( 1 + \frac{t}{n-t-d+1} \right) \right) \\
< 1 - \frac{t}{n-d} \times \left( 1 - \frac{1}{4t} - \frac{2}{3} \left( 1 + \frac{1}{4d} \right) \right) \\
\leq 1 - \frac{t}{n-d} \times \left( 1 - 1/12 - 2/3 \times 5/4 \right) \\
< 1.
\]

Here we used that \(n \geq 3d^2 + 2d + t\) and \(n \geq 4dt + 5t + d\) by (7), \(t \geq 3\), and \(d \geq 1\).
To bound $f(r)$ for other values of $r$, let us use $1 + x \leq e^x$ (true for all $x$). We get

$$f(r) < \exp\left\{ -\frac{(r-d)t}{n-d-t+1} \right\} + \frac{r}{n-r-t+2} \times \exp\left\{ \frac{dt}{n-d-t+1} \right\}. $$

When $r = (n + 3)/6$, $t \geq 3$, and $n \geq 24d$ by (7), the first term is at most $e^{-18/46} = 0.676\ldots$. Moreover, for $n \geq 9t$ (therefore $n \geq 27$) we get that $\frac{r}{n-r-t+2}$ is maximized when $r$ is maximized, i.e., when $t = n/9$. The whole term is at most $(3n+9)/(13n+27) \times e^{1/4} \leq 5/21 \times e^{1/4} = 0.305\ldots$, so in this range, $f((n+3)/6) < 1$. By the convexity of $f(r)$, we have $N(G, F) < N(H_{n,d}, F)$. \qed

When $F$ is a star, then it is easy to determine max $N(G, F)$ for all $n$.

Claim 12. Suppose $F = K_{1,t-1}$ with $t := |V(F)| \geq 3$, and $t \leq n$ and $d$ are integers with $1 \leq d \leq [(n-1)/2]$. If $G$ is an $n$-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$, then

$$N(G, F) \leq \max \left\{ H_{n,d}, H_{n,[(n-1)/2]} \right\}, \quad (11)$$

and equality holds if and only if $G \in \left\{ H_{n,d}, H_{n,[(n-1)/2]} \right\}$.

Proof. The number of copies of stars in a graph $G$ depends only on the degree sequence of the graph: if a vertex $v$ of a graph $G$ has degree $d(v)$, then there are $(d(v))_{t-1}$ labeled copies of $F$ in $G$ where $v$ is the center vertex. We have

$$N(G, F) = \sum_{v \in V(G)} \left( \frac{d(v)}{t-1} \right). \quad (12)$$

Since $G$ is nonhamiltonian, Pósa’s theorem yields an $r \leq [(n-1)/2]$, and an $r$-set $R \subset V(G)$ such that $d_G(v) \leq r$ for all $v \in R$. Take the minimum such $r$, then there exists a vertex $v \in R$ with $\deg(v) = r$. We may also suppose that $G$ is edge-maximal nonhamiltonian, so Ore’s condition (4) holds. It implies that $\deg(w) \leq n-r-1$ for all $w \notin N(v)$. Altogether we obtain that $G$ has $r$ vertices of degree at most $r$, at least $n-2r$ vertices (those in $V(G) - R - N(v)$) of degree at most $(n-r-1)$. This implies that the right hand side of (12) is at most

$$r \times (r)_{t-1} + (n-2r) \times (n-r-1)_{t-1} + r \times (n-1)_{t-1} = N(H_{n,r}, F).$$

(Here equality holds only if $G = H_{n,r}$). Note that $r \in [d, [(n,1)/2]]$. Since for given $n$ and $t$ the function $N(H_{n,r}, F)$ is strictly convex in $r$, it takes its maximum at one of the endpoints of the interval. \qed

Remark 13. As it was mentioned in Section 3, $O(dt)$ is the right order for $n_0(d,t)$ when $d = O(t)$.

To see this, fix $d \in \mathbb{N}$ and let $F$ be the star on $t \geq 3$ vertices. If $d < [(n-1)/2]$, $t \leq n$ and $n \leq dt - d$, then $H_{n,[(n-1)/2]}$ contains more copies of $F$ than $H_{n,d}$ does, the maximum in (11) is reached for $r = [(n-1)/2]$. We present the calculation below only for $2d + 7 \leq n \leq dt - d$, the case $2d + 3 \leq n \leq 2d + 6$ can be checked by hand by plugging $n$ into the first line of the formula below. We can proceed as follows.
In general, it is difficult to calculate the exact value of $N(H_{n,d}, F)$ for a fixed graph $F$. However, when $F = K_k$, we have $N(H_{n,d}, K_k) = h_k(n, d)k!$. Recall Theorem 6:

**Theorem 6**

Let $n, d, k$ be integers with $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $k \geq 2$. If $G$ is a nonhamiltonian graph on $n$ vertices with minimum degree $\delta(G) \geq d$, then

$$N_k(G) \leq \max \left\{ h_k(n, d), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}.$$  

**Proof of Theorem 6** By Theorem 8, because $G$ is nonhamiltonian, there exists an $r \geq d$ such that $G$ has $r$ vertices of degree at most $r$. Denote this set of vertices by $D$. Then $N_k(G - D) \leq \binom{n-r}{k-1}$, and every vertex in $D$ is contained in at most $\binom{r}{k-1}$ copies of $K_k$. Hence $N_k(G) \leq h_k(n, r)$. The theorem follows from the convexity of $h_k(n, x)$.

Our older stability theorem (Theorem 3) also translates into the language of cliques, giving a stability theorem for Theorem 6.

**Theorem 14.** Let $n \geq 3$, and $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Suppose that $G$ is an $n$-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ and there exists a $k \geq 2$ such that

$$N_k(G) > \max \left\{ h_k(n, d + 1), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}.$$  

(13)

Then $G$ is a subgraph of either $H_{n,d}$ or $K'_{n,d}$.

**Proof.** Take an edge-maximum counterexample $G$ (so we may assume $G$ is saturated). By Lemma 9, $G$ has a set $D$ of $r \leq \left\lfloor (n-1)/2 \right\rfloor$ vertices such that $G - D$ is a complete graph. If $r \geq d + 1$, then $N_k(G) \leq \max \left\{ h_k(n, d + 1), h_k(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\}$. Thus $r = d$, and we may apply Lemma 10.
6 Discussion and proof of Theorem 7

One can try to refine Theorem 3 in the following direction: What happens when we consider \( n \)-vertex nonhamiltonian graphs with minimum degree at least \( d \) and less than \( e(n, d+1) \) but more than \( e(n, d+2) \) edges?

Note that for \( d < d_0(n) - 2 \),
\[
e(n, d) - e(n, d + 2) = 2n - 6d - 7,
\]
which is greater than \( n \). Theorem 7 answers the question above in a more general form—in terms of \( s \)-cliques instead of edges. In other words, we classify all \( n \)-vertex nonhamiltonian graphs with more than \( \max \{ h_s(n, d+2), h_s(n, \lfloor \frac{n-1}{2} \rfloor) \} \) copies of \( K_s \).

As in Lemma 14, such \( G \) can be a subgraph of \( H_{n,d} \) or \( K'_{n,d} \). Also, \( G \) can be a subgraph of \( H_{n,d+1} \) or \( K'_{n,d+1} \). Recall the graphs \( H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2} \), and \( F_{n,3} \) defined in the first two sections of this paper and the statement of Theorem 3.

![Graphs](image)

**Figure 4:** Graphs \( H_{n,d}, K'_{n,d}, H'_{n,d}, G'_{n,2} \), and \( F_{n,3} \).

**Theorem 7** Let \( n \geq 3 \) and \( 1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor \). Suppose that \( G \) is an \( n \)-vertex nonhamiltonian graph with minimum degree \( \delta(G) \geq d \) such that exists a \( k \geq 2 \) for which
\[
N_k(G) > \max \{ h_k(n, d + 2), h_k(n, \lfloor \frac{n-1}{2} \rfloor) \}.
\]

Let \( \mathcal{H}_{n,d} := \{ H_{n,d}, H_{n,d+1}, K'_{n,d}, K'_{n,d+1}, H'_{n,d} \} \).

(i) If \( d = 2 \), then \( G \) is a subgraph of \( G'_{n,2} \) or of a graph in \( \mathcal{H}_{n,2} \);
(ii) if \( d = 3 \), then \( G \) is a subgraph of \( F_{n,3} \) or of a graph in \( \mathcal{H}_{n,3} \);
(iii) if \( d = 1 \) or \( 4 \leq d \leq \lfloor \frac{n-1}{2} \rfloor \), then \( G \) is a subgraph of a graph in \( \mathcal{H}_{n,d} \).

**Proof.** Suppose \( G \) is a counterexample to Theorem 7 with the most edges. Then \( G \) is saturated. In particular, degree condition 1 holds for \( G \). So by Lemma 9 there exists an \( d \leq r \leq \lfloor (n-1)/2 \rfloor \) such that \( V(G) \) contains a subset \( D \) of \( r \) vertices of degree at most \( r \) and \( G - D \) is a complete graph.

If \( r \geq d + 2 \), then because \( h_k(n, x) \) is convex, \( N_k(G) \leq h_k(n, r) \leq \max \{ h_k(n, d + 2), h_k(n, \lfloor \frac{n-1}{2} \rfloor) \} \). Therefore either \( r = d \) or \( r = d + 1 \). In the case that \( r = d \) (and so \( r = \delta(G) \)), Lemma 10 implies that \( G \subseteq H_{n,d} \). So we may assume that \( r = d + 1 \).

If \( \delta(G) \geq d + 1 \), then we simply apply Theorem 3 with \( d + 1 \) in place of \( d \) and get \( G \subseteq H_{n,d+1} \) or
\( G \subseteq K'_{n,d+1} \). So, from now on we may assume

\[
\delta(G) = d. \tag{14}
\]

Now \((14)\) implies that our theorem holds for \( d = 1 \), since each graph with minimum degree exactly 1 is a subgraph of \( H_{n,1} \). So, below \( 2 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Let \( N := N(D) - D \subseteq V(G) - D \). The next claim will be used many times throughout the proof.

**Lemma 15.** (a) If there exists a vertex \( v \in D \) such that \( d(v) = d + 1 \), then \( N(v) = D \).

(b) If there exists a vertex \( u \in N \) such that \( u \) has at least 2 neighbors in \( D \), then \( u \) is adjacent to all vertices in \( D \).

**Proof.** If \( v \in D \), \( d(v) = d + 1 \) and some \( u \in N \) is not adjacent to \( v \), then \( d(v) + d(u) \geq d + 1 + (n - d - 2) + 1 = n \). A contradiction to \((4)\) proves (a).

Similarly, if \( u \in N \) has at least 2 neighbors in \( D \) but is not adjacent to some \( v \in D \), then \( d(v) + d(u) \geq d + (n - d - 2) + 2 = n \), again contradicting \((4)\). \( \square \)

Define \( S := \{ u \in V(G) - D : u \in N(v) \text{ for all } v \in D \} \), \( s := s \), and \( S' := V(G) - D - S \). By Lemma 15 (b), each vertex in \( S' \) has at most one neighbor in \( D \). So, for each \( v \in D \), call the neighbors of \( v \) in \( S' \) the private neighbors of \( v \).

We claim that

\( D \) is not independent. \( \tag{15} \)

Indeed, assume \( D \) is independent. If there exists a vertex \( v \in D \) with \( d(v) = d + 1 \), then by Lemma 15 (b), \( N(v) = D \). So, because \( D \) is independent, \( G \subseteq H_{n,d+1} \). Assume now that every vertex \( v \in D \) has degree \( d \), and let \( D = \{ v_1, \ldots, v_{d+1} \} \).

If \( s \geq d \), then because each \( v_i \in D \) has degree \( s = d \) and \( N = S \). Then \( G \subseteq H_{n,d+1} \). If \( s \leq d - 2 \), then each vertex \( v_i \in D \) has at least two private neighbors in \( S' \); call these private neighbors \( x_{v_i} \) and \( y_{v_i} \). The path \( x_{v_1}v_1y_{v_1}x_{v_2}v_2y_{v_2} \ldots x_{v_{d+1}}v_{d+1}y_{v_{d+1}} \) contains all vertices in \( D \) and can be extended to a hamiltonian cycle of \( G \), a contradiction.

Finally, suppose \( s = d - 1 \). Then every vertex \( v_i \in D \) has exactly one private neighbor. Therefore \( G = G'_{n,d} \) where \( G'_{n,d} \) is composed of a clique \( A \) of order \( n - d - 1 \) and an independent set \( D = \{ v_1, \ldots, v_{d+1} \} \), and there exists a set \( S \subseteq A \) of size \( d - 1 \) and distinct vertices \( z_1, \ldots, z_{d+1} \) such that for \( 1 \leq i \leq d + 1 \), \( N(v_i) = S \cup z_i \). Graph \( G'_{n,d} \) is illustrated in Fig. 6.

For \( d = 2 \), we conclude that \( G \subseteq G'_{n,2} \), as claimed, and for \( d \geq 3 \), we get a contradiction since \( G'_{n,d} \) is hamiltonian. This proves \((15)\).

Call a vertex \( v \in D \) open if it has at least two private neighbors, half-open if it has exactly one private neighbor, and closed if it has no private neighbors.

We say that \( \text{paths } P_1, \ldots, P_q \) partition \( D \), if these paths are vertex-disjoint and \( V(P_1) \cup \ldots \cup V(P_q) = D \). The idea of the proof is as follows: because \( G - D \) is a complete graph, each path with endpoints in \( G - D \) that covers all vertices of \( D \) can be extended to a hamiltonian cycle of \( G \). So such a path does not exist, which implies that too few paths cannot partition \( D \):
Lemma 16. If \( s \geq 2 \) then the minimum number of paths in \( G[D] \) partitioning \( D \) is at least \( s \).

Proof. Suppose \( D \) can be partitioned into \( \ell \leq s - 1 \) paths \( P_1, \ldots, P_\ell \) in \( G[D] \). Let \( S = \{z_1, \ldots, z_\ell\} \). Then \( P = z_1P_1z_2 \ldots z_\ell P_\ell z_{\ell+1} \) is a path with endpoints in \( V(G) - D \) that covers \( D \). Because \( V(G) - D \) forms a clique, we can find a \( z_1, z_{\ell+1} \) path \( P' \) in \( G - D \) that covers \( V(G) - D - \{z_2, \ldots, z_\ell\} \). Then \( P \cup P' \) is a hamiltonian cycle of \( G \), a contradiction.

Sometimes, to get a contradiction with Lemma 16 we will use our information on vertex degrees in \( G[D] \):

Lemma 17. Let \( H \) be a graph on \( r \) vertices such that for every nonedge \( xy \) of \( H \), \( d(x) + d(y) \geq r - t \) for some \( t \). Then \( V(H) \) can be partitioned into a set of at most \( t \) paths. In other words, there exist \( t \) disjoint paths \( P_1, \ldots, P_t \) with \( V(H) = \bigcup_{i=1}^t V(P_i) \).

Proof. Construct the graph \( H' \) by adding a clique \( T \) of size \( t \) to \( H \) so that every vertex of \( T \) is adjacent to each vertex in \( V(H) \). For each nonedge \( x, y \in H' \),

\[
d_{H'}(x) + d_{H'}(y) \geq (r - t) + t + t = r + t = |V(H')|.
\]

By Ore’s theorem, \( H' \) has a hamiltonian cycle \( C' \). Then \( C' - T \) is a set of at most \( t \) paths in \( H \) that cover all vertices of \( H \).

The next simple fact will be quite useful.

Lemma 18. If \( G[D] \) contains an open vertex, then all other vertices are closed.

Proof. Suppose \( G[D] \) has an open vertex \( v \) and another open or half-open vertex \( u \). Let \( v', v'' \) be some private neighbors of \( v \) in \( S' \) and \( u' \) be a neighbor of \( u \) in \( S' \). By the maximality of \( G \), graph \( G + vu' \) has a hamiltonian cycle. In other words, \( G \) has a hamiltonian path \( v_1v_2 \ldots v_n \), where \( v_1 = v \) and \( v_n = u' \). Let \( V' = \{v_i : vv_{i+1} \in E(G)\} \). Since \( G \) has no hamiltonian cycle, \( V' \cap N(u') = \emptyset \).

Since \( d(v) + d(u') = n - 1 \), we have \( V(G) = V' \cup N(u') + u' \). Suppose that \( v' = v_i \) and \( v'' = v_j \). Then \( v_{i-1}, v_{j-1} \in V' \), and \( v_{i-1}, v_{j-1} \notin N(u') \). But among the neighbors of \( v_i \) and \( v_j \), only \( v \) is not adjacent to \( u' \), a contradiction.

Now we show that \( S \) is non-empty and not too large.
Lemma 19. $s \geq 1$.

Proof. Suppose $S = \emptyset$. If $D$ has an open vertex $v$, then by Lemma 18 all other vertices are closed. In this case, $v$ is the only vertex of $D$ with neighbors outside of $D$, and hence $G \subseteq K_{n,d}$, in which $v$ is the cut vertex. Also if $D$ has at most one half-open vertex $v$, then similarly $G \subseteq K'_{n,d}$.

So suppose that $D$ contains no open vertices but has two half-open vertices $u$ and $v$ with private neighbors $z_u$ and $z_v$ respectively. Then $\delta(G[D]) \geq d - 1$. By Pósa’s Theorem, if $d \geq 4$, then $G[D]$ has a hamiltonian $v, u$-path. This path together with any hamiltonian $z_u, z_v$-path in the complete graph $G - D$ and the edges $uz_u$ and $vz_v$ forms a hamiltonian cycle in $G$, a contradiction.

If $d = 3$, then by Dirac’s Theorem, $G[D]$ has a hamiltonian cycle, i.e. a 4-cycle, say $C$. If we can choose our half-open $v$ and $u$ consecutive on $C$, then $C - uv$ is a hamiltonian $v, u$-path in $G[D]$, and we finish as in the previous paragraph. Otherwise, we may assume that $C = vxuy$, where $x$ and $y$ are closed. In this case, $d_G(x) = d_G(y) = 3$, thus $xy \in E(G)$. So we again have a hamiltonian $v, u$-path, namely $vxuy$, in $G[D]$. Finally, if $d = 2$, then $|D| = 3$, and $G[D]$ is either a 3-vertex path whose endpoints are half-open or a 3-cycle. In both cases, $G[D]$ again has a hamiltonian path whose ends are half-open. \hfill \Box

Lemma 20. $s \leq d - 3$.

Proof. Since by (14), $\delta(G) = d$, we have $s \leq d$. Suppose $s \in \{d - 2, d - 1, d\}$.

Case 1: All vertices of $D$ have degree $d$.

Case 1.1: $s = d$. Then $G \subseteq H_{n,d+1}$.

Case 1.2: $s = d - 1$. In this case, each vertex in graph $G[D]$ has degree 0 or 1. By (15), $G[D]$ induces a non-empty matching, possibly with some isolated vertices. Let $m$ denote the number of edges in $G[D]$.

If $m \geq 3$, then the number of components in $G[D]$ is less than $s$, contradicting Lemma 16. Suppose now $m = 2$, and the edges in the matching are $x_1 y_1$ and $x_2 y_2$. Then $d \geq 3$. If $d = 3$, then $D = \{x_1, x_2, y_1, y_2\}$ and $G = F_{n,3}$ (see Fig 3 (right)). If $d \geq 4$, then $G[D]$ has an isolated vertex, say $x_3$. This $x_3$ has a private neighbor $w \in S'$. Then $|S + w| = d$ which is more than the number of components of $G[D]$ and we can construct a path from $w$ to $S$ visiting all components of $G[D]$.

Finally, suppose $G[D]$ has exactly one edge, say $x_1 y_1$. Recall that $d \geq 2$. Graph $G[D]$ has $d - 1$ isolated vertices, say $x_2, \ldots, x_d$. Each of $x_i$ for $2 \leq i \leq d$ has a private neighbor $u_i$ in $S'$. Let $S = \{z_1, \ldots, z_{d-1}\}$. If $d = 2$, then $S = \{z_1\}$, $N(D) = \{z_1, u_2\}$ and hence $G \subseteq H_{n,2}$. So in this case the theorem holds for $G$. If $d \geq 3$, then $G$ contains a path $u_d x_d z_{d-1} x_{d-1} z_{d-2} x_{d-2} \ldots z_2 x_1 y_1 z_1 x_2 u_2$ from $u_d$ to $u_2$ that covers $D$.

Case 1.3: $s = d - 2$. Since $s \geq 1$, $d \geq 3$. Every vertex in $G[D]$ has degree at most 2, i.e., $G[D]$ is a union of paths, isolated vertices, and cycles. Each isolated vertex has at least 2 private neighbors in $S'$. Each endpoint of a path in $G[D]$ has one private neighbor in $S'$. Thus we can find disjoint paths from $S'$ to $S'$ that cover all isolated vertices and paths in $G[D]$ and all are disjoint from $S$. Hence if the number $c$ of cycles in $G[D]$ is less than $d - 2$, then we have a set of disjoint paths from $V(G) - D$ to $V(G) - D$ that cover $D$ (and this set can be extended to a hamiltonian cycle in $G$).

Since each cycle has at least 3 vertices and $|D| = d + 1$, if $c \geq d - 2$, then $(d + 1)/3 \geq d - 2$, which
is possible only when \( d < 4 \), i.e. \( d = 3 \). Moreover, then \( G[D] = C_3 \cup K_1 \) and \( S = N \) is a single vertex. But then \( G = K_{n,3}' \).

**Case 2:** There exists a vertex \( v^* \in D \) with \( d(v^*) = d + 1 \). By Lemma 15 (b), \( N = N(v^*) - D \), and so \( G \) has at most one open or half-open vertex. Furthermore,

\[
\text{if } G \text{ has an open or half-open vertex, then it is } v^*, \text{ and by Lemma 15 there are no other vertices of degree } d + 1. \tag{16}
\]

**Case 2.1:** \( s = d \). If \( v^* \) is not closed, then it has a private neighbor \( x \in S' \), and the neighborhood of each other vertex of \( D \) is exactly \( S \). In this case, there exists a path from \( x \) to \( S \) that covers \( D \). If \( v^* \) is closed (i.e., \( N = S \)), then \( G[D] \) has maximum degree 1. Therefore \( G[D] \) is a matching with at least one edge (coming from \( v^* \)) plus some isolated vertices. If this matching has at least 2 edges, then the number of components in \( G[D] \) is less than \( s \), contradicting Lemma 16. If \( G[D] \) has exactly one edge, then \( G \subseteq H_{n,d}' \).

Case 2.2: \( s = d - 1 \). If \( v^* \) is open, then \( d_{G[D]}(v^*) = 0 \) and by (16), each other vertex in \( D \) has exactly one neighbor in \( D \). In particular, \( d \) is even. Therefore \( G[D - v^*] \) has \( d/2 \) components. When \( d \geq 3 \) and \( d \) is even, \( d/2 \leq s - 1 \) and we can find a path from \( S \) to \( D \) that covers \( D - v^* \), and extend this path using two neighbors of \( v^* \) in \( S' \) to a path from \( V(G) - D \) to \( V(G) - D \) covering \( D \). Suppose \( d = 2 \), \( D = \{v^*, x, y\} \) and \( S = \{z\} \). Then \( z \) is a cut vertex separating \( \{x, y\} \) from the rest of \( G \), and hence \( G \subseteq K_{n,2}' \). If \( v^* \) is half-open, then by (16), each other vertex in \( D \) is closed and hence has exactly one neighbor in \( D \). Let \( x \in S' \) be the private neighbor of \( v^* \). Then \( G[D] \) is 1-regular and therefore has exactly \( (d + 1)/2 \) components, in particular, \( d \) is odd. If \( d \geq 2 \) and is odd, then \( (d + 1)/2 \leq d - 1 = s \), and so we can find a path from \( x \) to \( S \) that covers \( D \).

Finally, if \( v^* \) is closed, then by (16), every vertex of \( G[D] \) is closed and has degree 1 or 2, and \( v^* \) has degree 2 in \( G[D] \). Then \( G[D] \) has at most \([d/2]\) components, which is less than \( s \) when \( d \geq 3 \). If \( d = 2 \), then \( s = 1 \) and the unique vertex \( z \) in \( S \) is a cut vertex separating \( D \) from the rest of \( G \). This means \( G \subseteq K_{n,3}' \).

**Case 2.3:** \( s = d - 2 \). Since \( s \geq 1 \), \( d \geq 3 \). If \( v^* \) is open, then \( d_{G[D]}(v^*) = 1 \) and by (16), each other vertex in \( D \) is closed and has exactly two neighbors in \( D \). But this is not possible, since the degree sum of the vertices in \( G[D] \) must be even. If \( v^* \) is half-open with a neighbor \( x \in S' \), then \( G[D] \) is 2-regular. Thus \( G[D] \) is a union of cycles and has at most \([(d + 1)/3]\) components. When \( d \geq 4 \), this is less than \( s \), contradicting Lemma 16. If \( d = 3 \), then \( s = 1 \) and the unique vertex \( z \) in \( S \) is a cut vertex separating \( D \) from the rest of \( G \). This means \( G \subseteq K_{n,4}' \).

If \( v^* \) is closed, then \( d_{G[D]}(v^*) = 3 \) and \( \delta(G[D]) \geq 2 \). So, for any vertices \( x, y \) in \( G[D] \),

\[
d_{G[D]}(x) + d_{G[D]}(y) \geq 4 \geq (d + 1) - (d - 2 - 1) = |V(G[D])| - (s - 1).
\]

By Lemma 17 if \( s \geq 2 \), then we can partition \( G[D] \) into \( s - 1 \) paths \( P_1, ..., P_{s-1} \). This would contradict Lemma 16. So suppose \( s = 1 \) and \( d = 3 \). Then as in the previous paragraph, \( G \subseteq K_{n,4}' \).

Next we will show that we cannot have \( 2 \leq s \leq d - 3 \).
Lemma 21.  $s = 1$.

Proof.  Suppose $s = d - k$ where $3 \leq k \leq d - 2$.

Case 1:  $G[D]$ has an open vertex $v$.  By Lemma 18 every other vertex in $D$ is closed.  Let $G' = G[D] - v$.  Then $\delta(G') \geq k - 1$ and $|V(G')| = d$.  In particular, for any $x, y \in D - v$,

\[ d_{G'}(x) + d_{G'}(y) \geq 2k - 2 \geq k + 1 = d - (d - k - 1) = |V(G')| - (s - 1). \]

By Lemma 17 we can find a path from $S$ to $S$ in $G$ containing all of $V(G')$.  Because $v$ is open, this path can be extended to a path from $V(G) - D$ to $V(G) - D$ including $v$, and then extended to a hamiltonian cycle of $G$.

Case 2:  $D$ has no open vertices and $4 \leq k \leq d - 2$.  Then $\delta(G[D]) \geq k - 1$ and again for any $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \geq 2k - 2$.  For $k \geq 4$, $2k - 2 \geq k + 2 = (d + 1) - (d - k - 1) = |D| - (s - 1)$.

Since $k \leq d - 2$, by Lemma 17 $G[D]$ can be partitioned into $s - 1$ paths, contradicting Lemma 16.

Case 3:  $D$ has no open vertices and $s = d - 3 \geq 2$.  If there is at most one half-open vertex, then for any nonadjacent vertices $x, y \in D$, $d_{G[D]}(x) + d_{G[D]}(y) \geq 2 + 3 = 5 \geq (d + 1) - (d - 3 - 1)$, and we are done as in Case 2.

So we may assume $G$ has at least 2 half-open vertices.  Let $D'$ be the set of half-open vertices in $D$.  If $D' \neq D$, let $v^* \in D - D'$.  Define a subset $D^- = D'$, otherwise, let $D^- = D' + v^*$.  Let $G'$ be the graph obtained from $G[D]$ by adding a new vertex $w$ adjacent to all vertices in $D^-$.  Then $|V(G')| = d + 2$ and $\delta(G') \geq 3$.  In particular, for any $x, y \in V(G')$, $d_{G'}(x) + d_{G'}(y) \geq 6 \geq (d + 2) - (d - 3 - 1) = |V(G')| - (s - 1)$.  By Lemma 17 $V(G')$ can be partitioned into $s - 1$ disjoint paths $P_1, \ldots, P_{s-1}$.  We may assume that $w \in P_1$.  If $w$ is an endpoint of $P_1$, then $D$ can also be partitioned into $s - 1$ disjoint paths $P_1 - w, P_2, \ldots, P_{s-1}$ in $G[D]$, a contradiction to Lemma 16.

Otherwise, let $P_1 = x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k$ where $x_i = w$.  Since every vertex in $(D^- - v^*)$ is half-open and $N_{G'}(w) = D^-$, we may assume that $x_{i-1}$ is half-open and thus has a neighbor $y \in S'$.  Let $S = \{z_1, \ldots, z_{d-3}\}$.  Then

\[ yx_{i-1}x_{i-2} \ldots x_1z_1x_{i+1} \ldots x_kz_2P_2z_3 \ldots z_{d-4}P_{d-4}z_{d-3} \]

is a path in $G$ with endpoints in $V(G) - D$ that covers $D$.  \hfill \Box

Now we may finish the proof of Theorem 7.  By Lemmas 19, 21 $s = 1$, say, $S = \{z_1\}$.  Furthermore, by Lemma 20

\[ d \geq 3 + s = 4. \quad (17) \]

Case 1:  $D$ has an open vertex $v$.  Then by Lemma 18 every other vertex of $D$ is closed.  Since $s = 1$, each $u \in D - v$ has degree $d - 1$ in $G[D]$.  If $v$ has no neighbors in $D$, then $G[D] - v$ is a clique of order $d$, and $G \subseteq K'_{d_d}$.  Otherwise, since $d \geq 4$, by Dirac’s Theorem, $G[D] - v$ has a hamiltonian cycle, say $C$.  Using $C$ and an edge from $v$ to $C$, we obtain a hamiltonian path $P$ in $G[D]$ starting with $v$.  Let $v' \in S'$ be a neighbor of $v$.  Then $v'Pz_1$ is a path from $S'$ to $S$ that covers $D$, a contradiction.
Case 2: \( D \) has a half-open vertex but no open vertices. It is enough to prove that

\[
G[D] \text{ has a hamiltonian path } P \text{ starting with a half-open vertex } v, \tag{18}
\]

since such a \( P \) can be extended to a hamiltonian cycle in \( G \) through \( z_1 \) and the private neighbor of \( v \). If \( d \geq 5 \), then for any \( x, y \in D \),

\[
d_{G[D]}(x) + d_{G[D]}(y) \geq d - 2 + d - 2 = 2d - 4 \geq d + 1 = |V(G[D])|.
\]

Hence by Ore’s Theorem, \( G[D] \) has a hamiltonian cycle, and hence \( \tag{18} \) holds.

If \( d < 5 \) then by \( \tag{17} \), \( d = 4 \). So \( G[D] \) has 5 vertices and minimum degree at least 2. By Lemma \( \tag{17} \) we can find a hamiltonian path \( P \) of \( G[D] \), say \( v_1v_2v_3v_4v_5 \). If at least one of \( v_1, v_5 \) is half-open or \( v_1v_5 \in E(G) \), then \( \tag{18} \) holds. Otherwise, each of \( v_1, v_5 \) has 3 neighbors in \( D \), which means \( N(v_1) \cap D = N(v_5) \cap D = \{v_2, v_3, v_4\} \). But then \( G[D] \) has hamiltonian cycle \( v_1v_2v_5v_4v_3v_1 \), and again \( \tag{18} \) holds.

Case 3: All vertices in \( D \) are closed. Then \( G \subseteq K'_{n,d+1} \), a contradiction. This proves the theorem.

\[ \square \]

7 A comment and a question

- It was shown in Section \( \tag{2} \) that the right order of magnitude of \( n_0(d, t) \) in Theorem \( \tag{4} \) when \( d = O(t) \) is \( dt \). We can also show this when \( d = O(t^{3/2}) \). It could be that \( dt \) is the right order of magnitude of \( n_0(d, t) \) for all \( d \) and \( t \).

- Is there a graph \( F \) and positive integers \( d, n \) with \( n < n_0(d, t) \) and \( d \leq \lfloor (n - 1)/2 \rfloor \) such that for some \( n \)-vertex nonhamiltonian graph \( G \) with minimum degree at least \( d \),

\[
N(G, F) > \max\{N(H_{n,d}, F), N(K'_{n,d}, F), N(H_{n,\lfloor (n - 1)/2 \rfloor}, F)\}?
\]

References

[1] N. Alon and C. Shikhelman, Many \( T \) copies in \( H \)-free graphs, J. of Combin. Theory Ser. B. 121 (2016), 146–172.

[2] B. Bollobás, E. Győri, Pentagons vs. triangles, Discrete Math. 308 (2008) 4332–4336.

[3] G. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) 2 (1952), 69–81.

[4] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962), 227–229.

[5] P. Erdős, On the number of complete subgraphs contained in certain graphs, Magy. Tud. Akad. Mat. Kutató Intéz. Közl. 7 (1962) 459–474.

[6] Z. Füredi, A. Kostochka, and R. Luo, A stability version for a theorem of Erdős on nonhamiltonian graphs, to appear in Discrete Math.

Also see: arXiv:1608.05741, posted on August 19, 2016, 4 pp.
[7] Z. Füredi and L. Özkahya, On 3-uniform hypergraphs without a cycle of a given length, *Discrete Applied Mathematics* **216** (2017), 582–588.

[8] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, *J. Combin. Theory Ser. B* **102** (2012) 1061–1066.

[9] H. Hatami, J. Hladký, D. Král’, S. Norine, and A. Razborov, On the number of pentagons in triangle-free graphs, *J. Combin. Theory Ser. A*. **120** (2013), 722–732.

[10] B. Li and B. Ning, Spectral analogues of Erdős’ and Moon-Moser’s theorems on Hamilton cycles, *Linear Multilinear Algebra*, **64** (2016), no.11, 1152–1169.

[11] O. Ore, Arc coverings of graphs, *Acta Math. Acad. Sci. Hung.* **10** (1959), 337–356.

[12] L. Pósa, A theorem concerning Hamilton lines, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 225–226.

[13] L. Pósa, On the circuits of finite graphs, *Magyar. Tud. Akad. Mat. Kutató Int. Közl.* **8** (1963/1964), 355–361.