STABILITY AND COERCIVITY FOR TORIC POLARIZATIONS

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Abstract. The purpose of this note is to explain that a toric polarized manifold is uniformly K-stable in the toric sense if and only if the K-energy functional is coercive modulo the maximal torus action.

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Introduction

The idea of \textit{uniform K-stability} was first introduced by the thesis \cite{Sz} and developed as it can be seen in \cite{Der14a}, \cite{Der14b}, \cite{BBJ15}, \cite{BHJ15}, \cite{DR15}, \cite{BHJ16}, and \cite{BDL16}. Especially it was shown by \cite{BDL16} that if the automorphism group is discrete, any polarized manifold \((X, L)\) admitting a constant scalar curvature Kähler metric is uniformly K-stable.

In analytic point of view the counterpart of the uniform K-stability should be the coercivity property of the K-energy. Let \(\mathcal{H}\) be the collection of positively curved fiber metrics on \(L\) and denote by \(\varphi_g\) the pull-back of \(\varphi \in \mathcal{H}\) by the bundle automorphism \(g \in \text{Aut}(X, L)\).

\textbf{Definition.} Let \(G\) be a closed algebraic subgroup of the identity component \(\text{Aut}^0(X, L)\). We say that K-energy functional \(M : \mathcal{H} \to \mathbb{R}\) is \(G\)-coercive (with respect to Aubin’s \(J\)-functional) if there exists a constant \(\delta, C \in \mathbb{R}_{>0}\) such that for any \(\varphi \in \mathcal{H}\)

\[ M(\varphi) \geq \delta \inf_{g \in G} J(\varphi_g) - C \]

holds.

This growth condition for the K-energy originates from Aubin’s strong Moser-Trudinger inequality on the two-sphere. It assures the critical point in a certain completion of \(\mathcal{H}\) and in fact in the Fano case the obtained a priori singular metric defines a smooth

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Kähler-Einstein metric (see \cite{BBGZ13}). As well, the uniform estimate could help seeking a constant scalar curvature Kähler metric. See also \cite{BBJ15}.

This note is devoted to establish one expected correspondence between the stability and the coercivity, for toric polarized manifolds. Instead we restrict ourselves to the space of torus-invariant metrics $H^S$ and toric test configurations, according to the symmetry of this class of manifolds. A toric test configuration is represented by a convex, rational piecewise-linear function $f$ on the moment polytope $P \subset M_\mathbb{R}$. Following \cite{Don02} we define

$$L(f) := \int_{\partial P} f - \frac{\text{area}(\partial P)}{\text{vol}(P)} \int_P f.$$  \hfill (0.1)

In addition, let us introduce the J-norm as

$$\|f\|_J := \inf \left\{ \frac{1}{\text{vol}(P)} \int_P (f + \ell) - \min_{P} \{f + \ell\} \right\},$$  \hfill (0.2)

where $\ell$ runs through all the affine functions. This gives the toric correspondence of the non-Archimedean J-functional introduced in \cite{BHJ15}. See Proposition 7.8 of \cite{BHJ15}.

**Main Theorem.** For any toric polarized manifold with the maximal torus $T$, K-energy functional is $T$-coercive on $H^S$ if and only if there exists a constant $\delta > 0$ such that

$$L(f) \geq \delta \|f\|_J$$  \hfill (0.3)

holds for any convex, rational piecewise-linear function $f : P \to \mathbb{R}$.

We call the algebraic condition *uniform K-stability in the toric sense*. Whenever $f$ is affine, the condition yields $L(f) = 0$ hence it includes classical Futaki’s obstruction. The proof of the coercivity is essentially due to \cite{ZZ08} where they adopt the larger “boundary norm” to measure the uniformity of stability. At the same time \cite{ZZ08}, Theorem 0.2 assures a lot of example of uniformly K-stable toric polarizations. The converse implication needs a much newer argument, however, as the both proofs show our main declaration is that $J$-norm more naturally fits into the coercivity concept and it could work for general polarizations. In fact we derive the stability from $T$-coercivity for general polarized manifolds. This is done by the slope of energy formula in \cite{BHJ16} and the generalization of Hilbert-Mumford type argument to the stability of pairs, developed in \cite{Paul12}. Lastly we refer the paper \cite{CLS14} where they claim that a toric polarization with a constant scalar curvature Kähler metric satisfies our uniform K-stability. It combined with our result yields that $T$-coercivity follows from the existence of a constant scalar curvature Kähler metric. In one conclusion the problem of finding a constant scalar curvature Kähler metric on a toric polarized manifold is reduced to the regularity of a weak solution.

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1. Preliminary toric materials

We start from quickly reviewing the toric setting, as convenience for the readers. See [Gui94], [Abr98], [Don02], and [SZ12] for the detail. Let $M$ be an integral lattice of rank $n$. A toric polarized manifold is defined by a Delzant polytope $P$ in $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. In our notation $P$ contains its boundary and we denote the interior by $P^\circ$. Let us denote the dual lattice by $N$. Then the complex torus $\mathbb{T} := N \otimes \mathbb{C}^*$ naturally acts on $X$, having an open dense orbit. Notice that in the setting there exists a natural real form $S := N \otimes_\mathbb{Z} S^1$.

1.1. Torus invariant metrics. An arbitrary moment map $\mu : X \to M_\mathbb{R}$ gives a homeomorphism $\mu^{-1}(P^\circ) \simeq S \times P^\circ$ so that $N_\mathbb{R} \times P^\circ$ gives the universal cover of $X_0 := \mu^{-1}(P^\circ)$. In a fixed coordinate $(x_1, \ldots, x_n)$ of $M_\mathbb{R}$ we have the complex coordinate $(z_1, \ldots, z_n)$ of $X_0 \simeq (\mathbb{C}^*)^n$ such that $\log z_i =: \xi_i + \sqrt{-1} \eta_i$ defines the dual coordinate $(\eta_1, \ldots, \eta_n)$ of $N_\mathbb{R}$. Then the Kähler metric defining $\mu$ is written in $X_0$ as

$$\omega = \sum dx_i \wedge d\eta_i.$$ 

Any $S$-invariant Kähler metric $\omega_\varphi = dd^c \varphi$ is represented by the local potential $\varphi$ on $X_0$, which is a function in $\xi_i$. If one denotes it by $\psi(\xi_1, \ldots, \xi_n)$ the gradient

$$(z_1, \ldots, z_n) \mapsto \left( \frac{\partial \psi}{\partial \xi_1}, \ldots, \frac{\partial \psi}{\partial \xi_n} \right)$$

gives the moment map for $\omega_\varphi$. Notice that the image $P$ is independent of the choice of moment maps.

The Legendre transform

$$u(x_1, \ldots, x_n) := \sup \left\{ \sum x_i \xi_i - \psi(\xi) \right\}$$

$$= \sum x_i \xi_i - \psi(\xi) \quad \text{when} \quad x_i = \frac{\partial \psi}{\partial \xi_i}$$

is called the symplectic potential. By the standard Delzant construction, one can prove that $u$ defines a convex function on $P^\circ$ such that if $P$ is written in the form

$$P = \{ x \in M_\mathbb{R} \mid \ell_k(x) := \langle x, \alpha_k \rangle - \beta_k \geq 0 \text{ for any } 1 \leq k \leq d \},$$

then $u - \frac{1}{2} \sum_{k=1}^d \ell_k \log \ell_k$ is smooth up to the boundary. Conversely, such a convex function defines an $S$-invariant Kähler metric.

If one rescales $u$ to $u - \sum a_i x_i - b$ by an affine function, $\psi$ changes into $\psi(\xi_i + a_i) + b$. In particular we can rescale $u$ to have a point $x^0 \in P^\circ$ as a minimizer. Then rescaling again by

$$\sum a_i x_i + b = \sum \frac{\partial u}{\partial x_i}(x^0)(x_i - x^0_i) + u(x^0)$$

one obtains the symplectic potential $u$ satisfying

(i) $\inf_P u = u(x^0) = 0$ for some $x^0 \in P^\circ$ and

(ii) $\frac{\partial u}{\partial x_i}(x^0) = 0$ for all $1 \leq i \leq n$. 


We say $u$ is normalized at this end. Note that the change of variables $\xi_i \mapsto \xi_i + a_i$ just corresponds to the $\mathbb{T}$-action.

The scalar curvature is given by a fourth-order differential of $u$.

**Theorem 1.1 (Abreu).**

(i) Hessian is transformed as

$$(u_{ij}) := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \left(\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j}\right)^{-1} \text{ when } x_i = \frac{\partial \psi}{\partial \xi_i}.$$  

(ii) Denoting $(u^{ij}) := (u_{ij})^{-1}$, (twice of) the scalar curvature is given by

$$S_{\omega_\varphi} := \frac{n \text{Ric} \omega_\varphi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} = -\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} \text{ when } x_i = \frac{\partial \psi}{\partial \xi_i}. \quad \square$$

1.2. Toric test configurations. Let us continuously fix the coordinate of $M_\mathbb{R}$. By the definition of moment map, Hamilton functions associated with the vector fields in $N_\mathbb{R}$ are translated to affine functions on $P$. A general toric test configuration correspond to a convex, rational piecewise-linear function $f : P \to \mathbb{R}$. This is a torus-equivariant morphism $(X, \mathcal{L}) \to (\mathbb{P}^1, \mathcal{O}(1))$ constructed by the big polytope

$$\mathfrak{P} := \left\{ (u, \lambda) \in M_\mathbb{R} \times \mathbb{R} \mid f(u) \leq \lambda \leq \max_P f \right\}. \quad (1.2)$$

The corresponding toric polarization $(X, \mathcal{L})$ in fact forms a $\mathbb{C}^*$-equivariant flat family of $\mathbb{Q}$-polarized schemes with $(X \setminus X_0, \mathcal{L}) \simeq (X \times \mathbb{C}, p^*_1 \mathcal{L})$ so that it give a test configuration in the terminology of [Don02]. Toric version of the Donaldson-Futaki invariant is given by the form

$$\cdot L(f) = \int_{\partial P} f - \hat{S} \int_P f \quad (1.3)$$

with the volume $V = \text{vol}(P)$ and the mean of the scalar curvature

$$\hat{S} = \frac{\text{area}(\partial P)}{\text{vol}(P)}. \quad (1.4)$$

The integration on $P$ is for the Lebesgue measure of $M_\mathbb{R} \simeq \mathbb{R}^n$ and we adopt the area measure $d\sigma$ on $\partial P$ determined by

$$dx_1 \cdots dx_n = \pm d\sigma \wedge d\ell_k.$$

**Remark 1.2.** The above $L(f)$ is not precisely equivalent to the Donaldson-Futaki invariant introduced in [Don02]. Indeed $L(f)$ is homogeneous with respect to the finite base change of the affine line but the Donaldson-Futaki invariant is not. It should corresponds to homogeneous non-Archimedean K-energy $M_{\text{NA}}(X, \mathcal{L})$ defined by 3.3. All these invariants are equivalent when the central fiber $X_0$ is reduced. See [BHJ15] for the detail.
1.3. Energy functionals. Recall that classical K-energy $M$ and the Aubin-Mabuchi energy $E$ on $\mathcal{H}$ are characterized by their differential:

$$
\delta M(\varphi) = -\frac{1}{V} \int_X (\delta \varphi)(S_\varphi - \hat{S})\omega^n/ n! \quad \text{and} \quad (1.5)
$$

$$
\delta E(\varphi) = \frac{1}{V} \int_X (\delta \varphi)\omega^n/ n!. \quad (1.6)
$$

As a scale-free version of $E$, we use Aubin’s J-functional given by

$$
J(\varphi) = \frac{1}{V} \int_X \varphi\omega^n/ n! - E(\varphi), \quad (1.7)
$$

where $\omega$ is a reference metric to define $E$.

For a general polarized manifold $V$ denotes the self-intersection number $L^n$ and $\hat{S}$ denotes the mean of the scalar curvature $-\frac{n}{2}V^{-1}K_X L^{n-1}$. Restricted to $\mathcal{S}$-invariant metrics these functionals have the following description on $P$.

**Theorem 1.3.** Let $u$ be the symplectic potential of an $\mathcal{S}$-invariant Kähler metric $\varphi$ on toric $(X,L)$. Then the energies are written in the forms

(i)

$$
VE(\varphi) = -\int_P u \quad \text{and} \quad (1.8)
$$

(ii)

$$
VM(\varphi) = -\int_P \log \det(u_{ij}) + \int_{\partial P} u - \hat{S} \int_P u. \quad (1.9)
$$

It is remarkable in the toric case that the linear part is given by

$$
L(u) = \int_{\partial P} u - \hat{S} \int_P u. \quad (1.10)
$$

Abuse of notation we will write the energies like $M(u)$ or $J(u)$ as functions in $u$. For the $J$-functional we recall:

**Lemma 1.4 (ZZ08B, Lemma 2.1).** There exists a constant $C > 0$ such that for any normalized symplectic potential $u$

$$
\left| J(\varphi) - \frac{1}{V} \int_P u \right| \leq C. \quad \square
$$

2. Stability to Coercivity

In this section we prove the half of the main theorem, from stability to coercivity direction. Let $\mathcal{S}$ be the collection of convex functions with the growth condition:

$$
u - \frac{1}{2} \sum_k \ell_k \log \ell_k \in C^\infty(P). \quad (1.11)$$
By a standard approximation result (Proposition 5.2.8 and Corollary 5.2.5 of [Don02]), we may assume that \( L(u) \geq \delta \|u\|_J \) holds for any \( u \in \mathcal{S} \). We also need the space \( C_\infty \) which consists of continuous convex function smooth on \( P^o \).

Fix \( u_0 \in \mathcal{S} \) to set

\[
L_0(u) := \int_{\partial P} u - \int_P \left( -\frac{1}{2} \sum_{i,j} \frac{\partial^2 u_0^{ij}}{\partial x_i \partial x_j} \right) u \quad \text{and} \quad (2.1)
\]

\[
VM_0(u) := -\int_P \log \det(u_{ij}) + L_0(u).
\]

The idea of the proof originates from the following lemma.

**Proposition 2.1** ([Don02], Proposition 3.3.4). For any \( u \in C_\infty \), \( M_0(u) \geq M_0(u_0) \) holds.

If \( u \geq 0 \), it follows from Hölder’s inequality that

\[
|L(u) - L_0(u)| = \left| \int_P \left( \hat{S} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u_0^{ij}}{\partial x_i \partial x_j} \right) u \right| \leq C \int_P u.
\]

For an arbitrary \( u \in \mathcal{S} \) one can take any affine function \( \ell \) for which \( (u + \ell) - \min_P \{u + \ell\} \) is applied to the above inequality so that

\[
|L(u) - L_0(u)| \leq C \|u\|_J
\]

holds. Let us decompose like

\[
\|u\|_J = (1 + k) \|u\|_J - k \|u\|_J
\]

and bound the first term by \( L(u) \geq \delta \|u\|_J \). This leads us

\[
|L(u) - L_0(u)| \leq \delta^{-1}C(1 + k)L(u) - Ck \|u\|_J
\]

to conclude

\[
VM(u) \geq -\int_P \log \det(u_{ij}) + \frac{1}{1 + \delta^{-1}C(1 + k)}L_0(u) + \frac{Ck}{1 + \delta^{-1}C(1 + k)} \|u\|_J
\]

\[
= VM_0(\frac{u}{1 + \delta^{-1}C(1 + k)}) - n \log(1 + \delta^{-1}C(1 + k)) + \frac{Ck}{1 + \delta^{-1}C(1 + k)} \|u\|_J.
\]

Then the value of \( M_0 \) in the first term is bounded from below by Proposition 2.1. Notice that as \( k \to \infty \) the coefficient of \( J \)-norm approaches to \( \delta \). In particular for normalized \( u \in \mathcal{S} \) we obtain

\[
M(u) \geq \delta' J(u) - C'
\]

with effective \( \delta' \) and \( C' \).

For any affine function \( \ell \), the non-linear term of \( M(u + \ell) \) is the same as that of \( M(u) \). It is equivalently to say

\[
M(u + \ell) = M(u) + L(\ell).
\]

(2.3)
The stability assumption yields $L(\ell) \geq \|\ell\|_J = 0$ and $L(-\ell) \geq 0$ since $-\ell$ is also an affine function. Thus we conclude

$$M(u) \geq \delta' \inf_{\ell} J(u + \ell) - C'.$$

As we noted, adding an affine function corresponds to the $T$-action so that we get $T$-coercivity.

To show the converse, it is natural to attach the “ray” emanating from $u \in S$, to a convex $f$. For example this is given in [SZ12] by the form $u + tf$ with $\tilde{f} := f - \frac{1}{V} \int_P f$ and $t \geq 0$. Since the non-linear term of $M(u + tf)$ does not depend on $t$ we expect the stability dividing the estimate

$$M(u + tf) \geq \delta' \inf_{\ell} J(u + tf + \ell) - C$$

by $t \to \infty$. In fact the affine function $\ell$ which attains the infimum depends on $t$. In the next section we will fix the problem by taking a natural filtration of $H$.

3. **Argument of Hilbert-Mumford criterion**

Let us now treat with a general complex polarized manifold $(X, L)$. In this section we would rather like to represent a test configuration by a one-parameter subgroup (1-PS for short). For an exponent $r \in \mathbb{N}$ global sections $H^0(X, L \otimes r)$ define the Kodaira embedding $X \to \mathbb{P}^{N_r-1}$. Any 1-PS $\lambda : \mathbb{C}^* \to \text{GL}(N_r; \mathbb{C})$ defines a test configuration as the Zariski closure

$$X_\lambda := \bigcup_{\tau \in \mathbb{C}^*} (\lambda(\tau)X \times \{\tau\}) \subseteq \mathbb{P}^{N_r-1} \times \mathbb{C}^*$$

with the $\mathbb{Q}$-line bundle $L_\lambda := O(1/r)|_X$. By the equivariant version of Kodaira embedding one can prove that any test configuration is obtained in this way from some (not unique) 1-PS.

Let $T \subseteq \text{Aut}^0(X, L)$ be a maximal torus. We say that a test configuration is $T$-equivariant if it is endowed with a $T$-action which preserves fiber, commutes the $\mathbb{C}^*$-action, and is compatible with the identification $(X_1, L_1) \simeq (X, L)$. For any representative 1-PS $\lambda$ it is equivalent to say that $\lambda$ is commutative with $T$. In other words there exists a maximal torus $D \subseteq \text{GL}(N_r; \mathbb{C})$ satisfying $\lambda(\mathbb{C}^*) \subseteq D$ and $T \subseteq D$. In case $(X, L)$ is toric this is equivalent to give a toric test configuration. In fact $T \times \mathbb{C}^*$ acts on $X$ with the open dense orbit. Let us denote the space of 1-PS to $D$ by $N(D)$.

A guideline for the stability from the coercivity should be the fact that the Donaldson-Futaki invariant of a test configuration is given by the slope of the K-energy along the associated ray $\varphi^t$ on $H$ ($t \in \mathbb{R}_{>0}$). Actually in [BHL15] the *non-Archimedean energy* $M_{NA}(X, L)$, $J_{NA}(X, L)$ of a test configuration, which is characterized by the property

$$M_{NA}(X, L) = \lim_{t \to \infty} M(\varphi^t)/t \quad \text{and} \quad J_{NA}(X, L) = \lim_{t \to \infty} J(\varphi^t)/t$$

were introduced. Here $\varphi^t$ is a smooth ray on $H$ with appropriate compatibility with $(X, L)$. For example one can take $\varphi^t(x) := \Phi_{FS}(\lambda(e^{-t})x, e^{-t})$; pull-back of the restriction
of some Fubini-Study metric in (3.1). The relation of $M^{NA}$ and the Donaldson-Futaki invariant is given by the explicit form
\[ DF(X, \mathcal{L}) = M^{NA}(X, \mathcal{L}) + ((\mathcal{X}_0 - \mathcal{X}_{0,\text{red}}) \mathbb{C}^n), \]
(3.3)
where the second term is the intersection of the non-reduced part of the central fiber and the extension of $\mathcal{L}$ to the trivial compactification of $X$ (which is obtained by (1.2) for a toric test configuration). What we need in the sequel is a consequence that $DF(X, \mathcal{L}) \geq M^{NA}(X, \mathcal{L})$ and the equality holds if $\mathcal{X}_0$ is reduced. Notice that in the toric case normality of $\mathcal{X}_0$ implies $X \simeq X \times \mathbb{C}$ (see the proof of [DS15], Theorem 17).

At this point we assume the coercivity
\[ M(\varphi^t) \geq \delta \inf_{\sigma \in \mathbb{T}} J(\varphi^t_\sigma) - C \]
to show the stability. The trouble here is of course that the element $\sigma \in \mathbb{T}$ attaining the infimum of $J(\varphi^t_\sigma)$ depends on the time parameter $t$. We go through the collection of all Fubini-Study fiber metrics for $X \to \mathbb{P}^{N_r-1}$ denoted by $\mathcal{H}_r$ to avoid the point. Tian-Zeldich-Catlin’s Bergman kernel asymptotic indicates us to exploit the space by showing $\mathcal{H} = \bigcup_r \mathcal{H}_r$.

3.1. Function of log-norm singularities. We basically quote the formalism of [BHJ16]. In the sequel we fix a sufficiently large exponent $r$ and some $\varphi \in \mathcal{H}_r$ to consider the functions in $g \in \text{GL}(N_r; \mathbb{C})$:
\[ m(g) := M(\varphi_g), \quad j(g) := J(\varphi_g), \]
(3.4)
and $e(g) := E(\varphi_g)$ for the Aubin-Mabuchi energy $E$.

Let us start from the following which I learned from Professor S. Boucksom.

**Proposition 3.1.** The energies $e$ and $m$ are pluriharmonic and $j$ is plurisubharmonic along $\text{Aut}^0(X, L)$.

**Proof.** Let us take an arbitrary holomorphic map from the one-dimensional open disk $\Delta \to \text{Aut}^0(X, L)$ which sends $z \in \Delta$ to $g(z)$. We have the well-known formula which gives the curvature by fiber integration:
\[ dd^c_x E(\varphi_{g(z)}) = (n+1)^{-1}V^{-1} \int_X (dd^c_{x,x} \varphi_{g(z)}(x))^{n+1}. \]
Defining the holomorphic map $F : \Delta \times X \to X$ by $F(z, x) := g(z) \cdot x$ we may proceed as
\[ ((dd^c_{x,x} \varphi_{g(z)}(x))^{n+1} = (dd^c_{x,x} F^*\varphi)^{n+1} = F^*(dd^c_x \varphi)^{n+1} = 0, \]
where the last line vanishes for the big degree. This completes the proof for $e$. Plurisubharmonicity of $j$ is immediate. Let us focus on $M$. We regard $M$ as a metric on the Deligne pairing $\langle K_{\Delta \times X/\Delta} \mathcal{L}^n \rangle + (n+1)^{-1}S(\mathcal{L}^{n+1})$ with $\mathcal{L} := F^*L$ (see e.g. [BHJ16]), such that the proof reduces to the equality
\[ dd^c_{x,x} (\log(dd^c \varphi_{g(z)})^n, \varphi_{g(z)}, \ldots, \varphi_{g(z)}) = 0, \]
where the Monge-Ampère operator is taken over $X$. Set $\Psi := \log(dd^c \varphi_{g(z)})^n$ as a metric on $K_{\Delta \times X/\Delta}$ and $\psi := \log(dd^c \varphi)^n$ as a metric on $K_X$. Since the natural automorphism
$G: (z, x) \mapsto (z, g(z) \cdot x)$ of $\Delta \times X$ induces $F^* K_X \simeq p_2^* K_X = K_{\Delta \times X / \Delta}$ we are led to show

$$F^* \psi = \Psi.$$ 

This can be checked fiberwisely. Indeed for each fixed $z$ we have the inclusion $i_z : X \to \Delta \times X$ such that

$$e^{\Psi_z} = (ddc i_z^* F^* \varphi)^n = V^{-1} i_z^* (ddc_{z,x} F^* \varphi)^n = i_z^* F^* (ddc \varphi)^n = (g(z))^* (ddc \varphi)^n.$$ 

\[ \square \]

Plurisubharmonicity on $GL(N_r, \mathbb{C})$ is not the case. A fundamental result of [Paul12] tells us that asymptotic of these functions are controlled by the difference of two plurisubharmonic functions. More specifically it is described by the actions to two homogeneous polynomials; $X$-resultant $R$ and $X$-hyperdiscriminant $\Delta$ associated with the Kodaira embedding. To be precise, taking norms $\| \cdot \|$ of $GL(N_r, \mathbb{C})$-vector space where $R$ and $\Delta$ lives and the Hilbert-Schmidt norm $\| \cdot \|_{HS}$ of $GL(N_r, \mathbb{C})$ we have

$$m(g) = V^{-1} \log \| g \cdot \Delta \| - V^{-1} \frac{\deg \Delta}{\deg R} \log \| g \cdot R \| + O(1) \quad (3.5)$$

and

$$j(g) = V^{-1} \log \| g \|_{HS} - V^{-1} \frac{1}{\deg R} \log \| g \cdot R \| + O(1). \quad (3.6)$$

Moreover the second term for $j(g)$ just corresponds to the Aubin-Mabuchi functional. The point here is that we can not expect to have a single log-norm term as in the classical GIT setting. In the terminology of [BHJ16], for a general reductive group $G$ a function $f : G \to \mathbb{R}$ of the form

$$f(g) = a \log \| g \cdot v \| - b \log \| g \cdot w \| + O(1)$$

is said to have log-norm singularities. Even in this generalized “pair of log-norm terms” setting, we have the correspondence of the Hilbert-Mumford weight for a given 1-PS.

**Theorem 3.2** (Theorem 4.4 of [BHJ16]). Let $f$ be a function on $G$ with log norm singularities. Then,

(i) For each 1-PS $\lambda : \mathbb{C}^* \to G$ there exists $f^{NA} \in \mathbb{Q}$ such that

$$f(\lambda(\tau)) = f^{NA}(\lambda) \log |\tau|^{-1} + O(1)$$

for $|\tau| \leq 1$.

(ii) $f$ is bounded below on $G$ iff $f^{NA}(\lambda) \geq 0$ for all 1-PS $\lambda$.

Applying the theorem to $f(g) = M(\varphi_g)$ (resp. $J(\varphi_g)$) with $\varphi$ a Fubini-Study weight, one obtains $f^{NA}(\lambda) = M^{NA}(\lambda)$ (resp. $J^{NA}(\lambda)$). In the proof of the main theorem we like to study how the above $O(1)$-term is determined. For the reason let us go back to the construction of $f^{NA}(\lambda)$. As a standard fact of $\mathbb{C}^*$-representation any $v \in V$ in the $GL(N_r; \mathbb{C})$-vector space has the decomposition

$$\lambda(\tau)v = \sum \tau^{\lambda_i} v_i \quad (3.7)$$
so that
\[
\log \| \lambda(\tau)v \| = \max \{ \lambda_i \log |\tau| + \log \| v_i \| \} + O(1)
\]
\[
= -\min \lambda_i \cdot \log |\tau|^{-1} + O(1).
\]
In the first line
\[
\log \| \lambda(\tau)v \| \geq \max \{ \lambda_i \log |\tau| + \log \| v_i \| \}
\]
is trivial and \( O(1) \) only depends on \( N_r \). To investigate the second \( O(1) \) we take a maximal complex torus \( D \subseteq GL(N_r; \mathbb{C}) \) which contains the image of \( \lambda \). We denote the character space by \( M(D) := \text{Hom}(D, \mathbb{C}^*) \simeq \mathbb{Z}^{N_r} \). The dual space \( N(D) := \text{Hom}(\mathbb{C}^*, D) \) is naturally identified with the collection of one-parameter subgroups. Then any \( GL(N_r; \mathbb{C}) \)-module \( V \) has the decomposition \( V = \bigoplus \chi_{V} \) such that \( d \cdot v = \chi(d)v \) for \( v \in V_{\chi} \). Set the weight space of a given \( v \in V \) as \( M_v := \{ \chi \in M \mid \text{\( V_{\chi} \)-component of \( v \) is non-zero} \} \) and denote its convex hull in \( M_{\mathbb{R}} := M \otimes \mathbb{R} \) by \( P_v \). Define the characteristic function \( p_v(\chi) \) whose value is identically zero if \( \chi \in P_v \) and \( \infty \) otherwise. We set the Legendre transform of the convex function \( p_v \) as \( h_v(\lambda) = \max_{\chi \in M_v} \langle \chi, \lambda \rangle \) and call it the support function.

Next we fix any maximal compact group \( U \subset GL(N_r; \mathbb{C}) \). Notice that \( U \) ensures the following property.

\begin{enumerate}
\item[(i)] \( D \cap U \subseteq U \) is a maximal real torus.
\item[(ii)] Any two maximal tori \( D_1 \) and \( D_2 \) are conjugate by an element of \( U \).
\item[(iii)] The Cartan(polar) decomposition \( GL(N_r; \mathbb{C}) = UDU \) holds.
\end{enumerate}

From the property (i) we have \( \text{Log}|\cdot| : D \to N_{\mathbb{R}} \) which yields \( D/(D \cap U) \simeq N_{\mathbb{R}} \). One observes for any \( \chi \in M \log |\chi(d)| = \langle \chi, \text{Log}|d| \rangle \) and for any \( \lambda \in N \log |\lambda(\tau)| = \log |\tau| \cdot \lambda \) holds. In this setting we repeat the formula in [BHJ16] that for any \( u, u' \in U \) and \( d \in D \)
\[
f(u'du) = h_{u,v}(\text{Log}|d|) - h_{u,v}(\text{Log}|d|) + O(1)
\]
holds and claim that \( O(1) \)-term is determined by \( U \) hence independent of \( D \). We may assume that the norm is \( U \)-invariant so
\[
\log \| (u'du)v \| = \log \| (du)v \| = \log \| \sum_{\chi \in M_{uv}} \chi(d)(uv)_{\chi} \|.
\]
The last term is written to be
\[
\max_{\chi \in M_{uv}} \log \{ \langle \chi, \text{Log}|d| \rangle + \log \| (uv)_{\chi} \| \} + O(1)
\]
with \( O(1) \) determined by the cardinality of \( M_{uv} \). On the other hand compactness of \( U \) yields a constant \( C \) such that
\[
-C \leq \log \| (uv)_{\chi} \| \leq C
\]
for any \( u \in U \) and \( \chi \in M_{uv} \). If we take another maximal torus \( D' = uD u^{-1} \) characters are naturally related in the manner

\[
\chi \in M(D) \longleftrightarrow \chi' := \chi(u^{-1} \cdot u) \in M(D').
\]

For \( v_\chi \in V_\chi \) and \( d' = udu^{-1} \) we have

\[
d'(uv_\chi) = udv_\chi = u\chi(d)v_\chi.
\]

The last one equals to \( \chi(d)(uv_\chi) = \chi'(d')(uv_\chi) \) by linearity, hence \( uv_\chi \in V_{\chi'} \). It concludes that the multiplication \( v \mapsto uv \) is compatible to the weight decomposition in the sense that

\[
(u \cdot v)_{\chi'} = u \cdot v_\chi
\]

holds for \( v \in V \). Therefore the bound in (3.11) is determined only by \( U \). By the definition of \( h_{uv} \) we then obtain the formula (3.10) with the desired \( O(1) \)-term.

For a fixed maximal torus \( D \) any 1-PS \( \lambda : \mathbb{C}^* \to \text{GL}(N_\tau; \mathbb{C}) \) has its image in some \( D' \) which is conjugate to \( D \). Finally we agree that \( O(1) \) of Theorem 3.2 is determined by a fixed maximal compact subgroup \( U \) of \( G = \text{GL}(N_\tau; \mathbb{C}) \).

3.2. Completion of the proof of the main theorem. Assume \( \mathbb{T} \)-coercivity, namely that

\[
M(\varphi) \geq \delta \inf_{\sigma \in \mathbb{T}} J(\varphi_\sigma) - C
\]

holds for any \( \varphi \in H^S \). Then

\[
m(g) \geq \delta \inf_{\sigma \in \mathbb{T}} j(\sigma g) - C
\]

(3.12)

for any \( g \in \text{GL}(N_\tau; \mathbb{C}) \). Note that \( \varphi_g(x) = \varphi(g \cdot x) \) hence \( (\varphi_g)_\sigma = \varphi_{\sigma g} \).

Let us take any \( \mathbb{T} \)-equivariant test configuration and its representative 1-PS \( \lambda \in N(D) \). For \( k \in \mathbb{N} \) set \( d_k := \lambda(e^{-k}) \in D \). In a fixed coordinate, \( d_k \) can be written as a diagonal matrix \( (e^{-k\lambda_1}, \ldots, e^{-k\lambda_n}) \) and \( \mathbb{T} \) may be regarded as the collection of diagonal matrices the last \( n \) components of which are just \( 1 \in \mathbb{C}^* \). Then each infimum of \( j(\sigma d_k) \) is approximated by some \( \sigma_k := (e^{a_{1k}}, \ldots, e^{a_{nk}}, 1, \ldots, 1) \) with \( a_{ik} \in \mathbb{Q} \) so that

\[
\inf_{\sigma \in \mathbb{N}} j(\sigma d_k) \geq j(\sigma_k d_k) - 1/k
\]

(3.13)

holds. Set \( \mu_{ik} := -k^{-1}a_{ik} \in \mathbb{Q} \) and \( q_k \in \mathbb{N} \) such that \( q_k\mu_{ik} \in \mathbb{Z} \) holds for any \( 1 \leq i \leq n \). Then \( \mu_{ik} \) defines \( \mu_k \in N_\mathbb{Q}(\mathbb{T}) \) for which \( q_k\mu_k \) is a 1-PS. As a direct consequence of (3.2), \( M^{\text{NA}} \) and \( J^{\text{NA}} \) are homogeneous in \( \lambda \in N(D) \). Therefore one can extend them to \( \hat{N}(D) \) so that \( f^{\text{NA}}(\mu_k) = q^{-1}f^{\text{NA}}(\mu q) \) holds for \( q \in \mathbb{N} \). In addition, \( \mu_k \) itself is not a 1-PS but some value e.g.

\[
(\mu_k + \lambda)(e^{-k}) := (e^{-k(\mu_{ik} + \lambda_1)}, \ldots, e^{-k(\mu_{nk} + \lambda_n)}, e^{-k\lambda_{n+1}}, \ldots, e^{-k\lambda_n})
\]

makes sense in \( \text{GL}(N_\tau, \mathbb{C}) \). Noting these points the argument of subsection 3.1 can be applied to \( q_k(\mu_k + \lambda) \in N(D) \) and \( \tau := e^{-k/q_k} \) so that

\[
j((\mu_k + \lambda)(e^{-k})) \geq J^{\text{NA}}(\mu_k + \lambda) \cdot k + O(1)
\]

(3.14)
holds with $O(1)$ which is not only for $k$ but also for $\mu$ and $\lambda$. The right-hand side is obviously bounded from below by
\[
\inf_{\mu \in N_Q(T)} J^{NA}(\mu + \lambda) \cdot k + O(1).
\]
It follows that
\[
m(\lambda(e^{-k}))/k \geq \delta \inf_{\mu \in N_Q(T)} J^{NA}(\mu + \lambda) + O(1/k). \tag{3.15}
\]
Letting $k \to \infty$ we obtain the stability:
\[
M^{NA}(\lambda) \geq \delta \inf_{\mu \in N_Q(T)} J^{NA}(\mu + \lambda). \tag{3.16}
\]
In the toric case $N_Q(T)$ consists of rational linear functions on $M_\mathbb{R}$ so that we conclude the main theorem.

\[\square\]

**Remark 3.3.** Note that $\mathbb{T}$-coercivity in particular implies $m(g)$ bounded from below. Since Proposition 3.1 shows that $m(g)$ is pluriharmonic on the quasi-projective variety $\text{Aut}^0(X, L)$, it is constant along the automorphism group. It follows from the slope formula (3.2) that $DF(X, L) = M^{NA}(X, L) = 0$ if $(X, L)$ is product.

More generally one can consider a reductive subgroup $G = K_C \subseteq \text{Aut}^0(X, L)$ and the coercivity condition
\[
M(\varphi) \geq \delta \inf_{g \in C(G)} J(\varphi_g) - C, \tag{3.17}
\]
restricted to any $K$-invariant positively curved metrics. Note that $\varphi_g$ is $K$-invariant for any $g \in C(G)$. From this coercivity, the above argument actually derives uniform $K$-stability relative to $G$ in the sense of [His16].
References

[Abre98] M. Abreu: *Kähler geometry of toric varieties and extremal metrics*. Internat. J. Math. 9 (1998), no. 6, 641–651.

[BBGZ13] R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi: *A variational approach to complex Monge-Ampère equations*. Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245.

[BBJ15] R. J. Berman, S. Boucksom, and M. Jonsson: *A variational approach to the Yau-Tian-Donaldson conjecture*. arXiv:1509.04561.

[BDL16] R. J. Berman, T. Darvas, and C. H. Lu: *Regularity of weak minimizers of the K-energy and applications to properness and K-stability*. arXiv:1602.03114.

[BHJ15] S. Boucksom T. Hisamoto and M. Jonsson: *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*. arXiv:1603.01026.

[BHJ16] S. Boucksom T. Hisamoto and M. Jonsson: *Uniform K-stability and asymptotics of energy functionals in Kähler geometry*. arXiv:1506.07495.

[CLS14] B. Chen, A. M. Li, and S. Sheng: *Uniform K-stability for extremal metrics on toric varieties*. J. Differential Equations 257 (2014), 1487–1500.

[DR15] T. Darvas and Y. A. Rubinstein: *Tian’s properness conjectures and Finsler geometry of the space of Kähler metrics*. arXiv:1506.07129.

[DS15] V. Datar and G. Székelyhidi: *Kähler-Einstein metrics along the smooth continuity method*. arXiv:1506.07495.

[Der14a] R. Dervan: *Uniform stability of twisted constant scalar curvature Kähler metrics*. arXiv:1412.0648. To appear in Int. Math. Res. Notices.

[Der14b] R. Dervan: *Alpha invariants and coercivity of the Mabuchi functional on Fano manifolds*. Int. Math. Res. Notices 26 (2015), 7162–7189.

[Don02] S. K. Donaldson: *Scalar curvature and stability of toric varieties*. J. Differential Geom. 62 (2002), no. 2, 289–349.

[Gui94] V. Guillemin: *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*. Birkhäuser Boston, 1994. ISBN: 0-8176-3770-2.

[His16] T. Hisamoto: *Orthogonal projection of a test configuration to vector fields*. arXiv:1610.07158.

[Paul12] S. T. Paul: *Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics*. Ann. of Math. (2) 175 (2012), no. 1, 255–296.

[Paul13] S. T. Paul: *Stable Pairs and Coercive Estimates for The Mabuchi Functional*. arXiv:1308.4377.

[SS12] J. Song and S. Zelditch: *Test configurations, large deviations and geodesic rays on toric varieties*. Adv. Math. 229 (2012), no. 4, 2338–2378.

[Szé06] G. Székelyhidi: *Extremal metrics and K-stability*. arXiv:0611002. Ph.D Thesis.

[WN10] D. Witt Nyström: *Test configurations and Okounkov bodies*. Compositio Math. 148 (2012), 1736–1756.

[ZZ08a] B. Zhou and X. Zhu: *K-stability on toric manifolds*. Proc. Amer. Math. Soc. 136 (2008), no. 9, 3301–3307.

[ZZ08b] B. Zhou and X. Zhu: *Relative K-stability and modified K-energy on toric manifolds*. Adv. in Math. 219 (2008), 1327–1362.