Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$ based on Clifford Fourier Transform

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1 Introduction

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in geometric algebra.

Brackx et al. [1] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied approach for hypercomplex Clifford Fourier Transformations in $Cl_{0,n}$ was followed by Bülow et. al. [2]. In [3], Li et. al. extended the Fourier Transform holomorphically to a function of $m$ complex variables.

In this paper we adopt and expand the generalization of the Fourier transform in Clifford geometric algebra recently suggested by Ebling and Scheuermann [5]. We explicitly show detailed properties of the real Clifford geometric algebra Fourier transform (CFT), which we subsequently use to define and prove the uncertainty principle for $G_3$ multivector functions.

2 Clifford’s Geometric Algebra $G_3$ of $\mathbb{R}^3$

Let us consider an orthonormal vector basis $\{e_1, e_2, e_3\}$ of the real 3D Euclidean vector space $\mathbb{R}^3$. The geometric algebra over $\mathbb{R}^3$ denoted by $G_3$ then has the graded $2^3 = 8$-dimensional basis

$$\{1, e_1, e_2, e_3, e_{12}, e_{31}, e_{23}, e_{123}\}.$$ (1)

The grade selector is defined as $\langle M \rangle_k$ for the $k$-vector part of $M$, especially $\langle M \rangle_0 = \langle M \rangle$. Then $M$ can be expressed as

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3.$$ (2)
The reverse of $M$ is defined by the anti-automorphism
\[ \tilde{M} = \langle M \rangle + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3. \] (3)

The square norm of $M$ is defined by
\[ \|M\|^2 = \langle M \tilde{M} \rangle, \] (4)

where
\[ \langle M \tilde{N} \rangle = M * \tilde{N} = \sum_A \alpha_A \beta_A \] (5)
is a real valued (inner) scalar product for any $M, N$ in $G_3$ with $M = \sum_A \alpha_A e_A$ and $N = \sum_A \beta_A e_A$, $A \in \{0, 1, 2, 3, 12, 23, 31, 123\}$, $\alpha_A, \beta_A \in \mathbb{R}$. As a consequence we obtain the multivector Cauchy-Schwarz inequality
\[ |\langle M \tilde{N} \rangle|^2 \leq \|M\|^2 \|N\|^2 \] for all $M, N \in G_3$. (6)

3 Multivector Functions, Vector Differential and Vector Derivative

Let $f = f(x)$ be a multivector-valued function of a vector variable $x$ in $G_3$. For an arbitrary vector $a$ we define the vector differential in the $a$ direction as
\[ a \cdot \nabla f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon a) - f(x)}{\epsilon} \] (7)

provided this limit exists and is well defined. The basis independent vector derivative $\nabla$ defined in [6, 7] obeys equation (7) for all vectors $a$ and can be expanded as
\[ \nabla = e_k \partial_k = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3, \] (8)

**Proposition 3.1** $\nabla(f + g) = \nabla f + \nabla g$ (linearity).

**Proposition 3.2** $\nabla(fg) = (\tilde{\nabla} f)g + \tilde{\nabla} fg = (\tilde{\nabla} f)g + \sum_{k=1}^n e_k f(\partial_k g)$.

(Multivector functions $f$ and $g$ do not necessarily commute.)

**Proposition 3.3** For $f(x) = g(\lambda(x))$, $\lambda(x) \in \mathbb{R}$,
\[ a \cdot \nabla f = \{a \cdot \nabla \lambda(x)\} \frac{\partial g}{\partial \lambda} \]

**Proposition 3.4** $\nabla f = \nabla_a (a \cdot \nabla f)$ (derivative from differential)

Differentiating twice with the vector derivative, we get the differential Laplacian operator $\nabla^2$. We can write $\nabla^2 = \nabla \cdot \nabla + \nabla \wedge \nabla$. But for integrable functions $\nabla \wedge \nabla = 0$. In this case we have $\nabla^2 = \nabla \cdot \nabla$.

**Proposition 3.5** (integration of parts)
\[ \int_{\mathbb{R}^3} g(x)[a \cdot \nabla h(x)]d^3x = \int_{\mathbb{R}^3} g(x)h(x)d^3x \Big|_{a \cdot \partial = -\infty}^{a \cdot \partial = \infty} - \int_{\mathbb{R}^3} [a \cdot \nabla g(x)]h(x)d^3x \]

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Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in G_{p,q}$. The vector variable index $x$ of the vector derivative is dropped: $\nabla x = \nabla$ and $a \cdot \nabla x = a \cdot \nabla$, but not when differentiating with respect to a different vector variable (compare e.g. proposition [5, 4]).
Assume \( \int \) and its Clifford Fourier transform are related. A point of geometric algebra an uncertainty principle gives us information about how a multivector valued function has been devoted to extending the uncertainty principle to a function and its Fourier transform. From the view of Fourier transform which cannot both be simultaneously sharply localized. Furthermore much work (e.g. \([8, 9]\)) is central for information processing \([8]\). In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. In Fourier analysis such conjugate entities correspond to a function and its Fourier transform which are scalar valued (\( (f \ast g) (x) \)). Theorem 4.2 Let \( f \) be a multivector valued function in \( \mathbb{G}_3 \) which has the Clifford Fourier transform \( \mathcal{F}\{ f \} (\omega) \). Assume \( \int_{\mathbb{R}^3} \| f (x) \|^2 \, d^3 x = F < \infty \), then the following inequality holds for arbitrary constant vectors \( \alpha, \beta \):

\[
\int_{\mathbb{R}^3} (\alpha \cdot x)^2 \| f (x) \|^2 \, d^3 x \int_{\mathbb{R}^3} (\beta \cdot \omega)^2 \| \mathcal{F}\{ f \} (\omega) \|^2 \, d^3 \omega \geq (\alpha \cdot \beta)^2 \frac{(2\pi)^3}{4} F^2
\]

### Table 1 Properties of the Clifford Fourier transform (CFT)

| Property            | Multivector Function | CFT |
|---------------------|----------------------|-----|
| Linearity           | \( \alpha f (x) + \beta g (x) \) | \( \alpha \mathcal{F}\{ f \} (\omega) + \beta \mathcal{F}\{ g \} (\omega) \) |
| Delay               | \( f (x - a) \)      | \( e^{-i \omega \cdot a} \mathcal{F}\{ f \} (\omega) \) |
| Shift               | \( e^{-i \omega \cdot \beta} f (x) \) | \( \mathcal{F}\{ f \} (\omega - \omega_0) \) |
| Scaling             | \( f (\alpha x) \)   | \( \frac{1}{|\alpha|^3} \mathcal{F}\{ f \} (\frac{\omega}{|\alpha|^3}) \) |
| Convolution         | \( (f \ast g) (x) \) | \( \mathcal{F}\{ f \} (\omega) \mathcal{F}\{ g \} (\omega) \) |
| Vec. diff.          | \( a \cdot \nabla f (x) \) | \( i_3 a \cdot \omega \mathcal{F}\{ f \} (\omega) \) |
|                    | \( a \cdot x f (x) \) | \( i_3 a \cdot \nabla \mathcal{F}\{ f \} (\omega) \) |
|                    | \( x f (x) \)        | \( i_3 \nabla \mathcal{F}\{ f \} (\omega) \) |
| Vec. deriv.         | \( \nabla^m f (x) \) | \( (i_3 \omega)^m \mathcal{F}\{ f \} (\omega) \) |
| Plancherel T.       | \( \langle f_1 (x) f_2 (x) \rangle_V = \frac{1}{(2\pi)^3} \mathcal{F}\{ f_1 \} (\omega) \mathcal{F}\{ f_2 \} (\omega) \) |
| sc. Parseval T.     | \( \int_{\mathbb{R}^3} \| f (x) \|^2 \, d^3 x = \int_{\mathbb{R}^3} \| \mathcal{F}\{ f \} (\omega) \|^2 \, d^3 \omega \) |

### 4 Clifford Fourier Transform (CFT)

**Definition 4.1** The Clifford Fourier transform of \( f (x) \) is the function \( \mathcal{F}\{ f \} : \mathbb{R}^3 \rightarrow \mathbb{G}_3 \) given by

\[
\mathcal{F}\{ f \} (\omega) = \int_{\mathbb{R}^3} f (x) e^{-i \omega \cdot x} \, d^3 x,
\]

where we can write \( \omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \), \( x = x_1 e_1 + x_2 e_2 + x_3 e_3 \) with \( e_1, e_2, e_3 \) the basis vectors of \( \mathbb{R}^3 \). Note that

\[
d^3 x = \frac{d x_1 \wedge d x_2 \wedge d x_3}{i_3}
\]

is scalar valued (\( d x_k = d x_k e_k \), \( k = 1, 2, 3 \), no summation). Because \( i_3 \) commutes with every element of \( \mathbb{G}_3 \), the Clifford Fourier kernel \( e^{-i \omega \cdot x} \) will also commute with every element of \( \mathbb{G}_3 \).

**Theorem 4.2** The Clifford Fourier transform \( \mathcal{F}\{ f \} \) of \( f \in L^2 (\mathbb{R}^3, \mathbb{G}_3) \), \( \int_{\mathbb{R}^3} \| f \|^2 \, d^3 x < \infty \) is invertible and its inverse is calculated by

\[
\mathcal{F}^{-1}\{ \mathcal{F}\{ f \} (\omega) \} = f (x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathcal{F}\{ f \} (\omega) e^{i \omega \cdot x} \, d^3 \omega.
\]

A number of properties of the CFT are listed in table 1.

### 5 Uncertainty Principle

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing \([8]\). In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. In Fourier analysis such conjugate entities correspond to a function and its Fourier transform which cannot both be simultaneously sharply localized. Furthermore much work (e.g. \([8, 9]\)) has been devoted to extending the uncertainty principle to a function and its Fourier transform. From the view point of geometric algebra an uncertainty principle gives us information about how a multivector valued function and its Clifford Fourier transform are related.

**Theorem 5.1** Let \( f \) be a multivector valued function in \( \mathbb{G}_3 \) which has the Clifford Fourier transform \( \mathcal{F}\{ f \} (\omega) \). Assume \( \int_{\mathbb{R}^3} \| f (x) \|^2 \, d^3 x = F < \infty \), then the following inequality holds for arbitrary constant vectors \( \alpha, \beta \):

\[
\int_{\mathbb{R}^3} (\alpha \cdot x)^2 \| f (x) \|^2 \, d^3 x \int_{\mathbb{R}^3} (\beta \cdot \omega)^2 \| \mathcal{F}\{ f \} (\omega) \|^2 \, d^3 \omega \geq (\alpha \cdot \beta)^2 \frac{(2\pi)^3}{4} F^2
\]
Choosing $b = \pm a$, with $a^2 = 1$ we get the following uncertainty principle, i.e.

$$\int_{\mathbb{R}^3} (a \cdot x)^2 \|f(x)\|^2 \, d^3x \int_{\mathbb{R}^3} (a \cdot \omega)^2 \|F\{f\}(\omega)\|^2 \, d^3\omega \geq \frac{(2\pi)^3}{4} F^2.$$  \hspace{1cm} (13)

In (13) equality holds for Gaussian multivector valued functions

$$f(x) = C_0 e^{-kx^2}$$  \hspace{1cm} (14)

where $C_0 \in \mathcal{G}_3$ is a constant multivector, $0 < k \in \mathbb{R}$.

**Theorem 5.2** For $a \cdot b = 0$, we get

$$\int_{\mathbb{R}^3} (a \cdot x)^2 \|f(x)\|^2 \, d^3x \int_{\mathbb{R}^3} (b \cdot \omega)^2 \|F\{f\}(\omega)\|^2 \, d^3\omega \geq 0.$$  \hspace{1cm} (15)

**Theorem 5.3** Under the same assumptions as in theorem 5.1 we obtain

$$\int_{\mathbb{R}^3} x^2 \|f(x)\|^2 \, d^3x \int_{\mathbb{R}^3} \omega^2 \|F\{f\}(\omega)\|^2 \, d^3\omega \geq 3 \frac{(2\pi)^3}{4} F^2.$$  \hspace{1cm} (16)

### 6 Conclusions

The (real) Clifford Fourier transform extends the traditional Fourier transform on scalar functions to multivector functions. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems are demonstrated. We then presented an uncertainty principle in the geometric algebra $\mathcal{G}_3$ which describes how a multivector-valued function and its Clifford Fourier transform relate. The formula of the uncertainty principle in $\mathcal{G}_3$ can be extended to $\mathcal{G}_n$ using properties of the Clifford Fourier transform for geometric algebras with unit pseudoscalars squaring to -1.

It is known that the Fourier transform is successfully applied to solving physical equations such as the heat equation, wave equations, etc. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving scalar, vector, bivector and pseudoscalar fields.

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