On the instability of Lorentzian Taub–NUT space

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Abstract
I show that there are no $SU(2)$-invariant (time-dependent) tensorial perturbations of Lorentzian Taub–NUT space. It follows that the spacetime is unstable at the linear level against generic perturbations. The difficulty of formulating a physically reasonable thermodynamics for Taub–NUT space is discussed.

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1. Introduction

There has been some interest recently in spacetimes admitting NUT charge and the thermodynamics associated with them (see, e.g., [1, 10, 11] and references therein). Inspired by the AdS–CFT conjecture, the entropy for many such spacetimes has been computed by counter-term methods but a satisfactory first law of thermodynamics has not yet been stated. Given the confusion about the thermodynamics of NUT spacetimes, and in fact about NUT charge in general, it is perhaps illuminating to perform a Lorentzian linear stability analysis for these spacetimes. In this paper the simplest case, Lorentzian Taub–NUT without a cosmological constant, is addressed, finding instability of the spacetime at the linear level.

I revisited a paper [17], which provides the methods to analyse linear stability of spacetimes admitting $SU(2)$ invariance by decomposition of the perturbation into tensor-harmonics using Wigner functions. In [17], the signature used was $(+, +, +, +)$ and the stability of gravitational instantons was analysed. These results have recently been generalized to a non-vanishing cosmological constant by Warnick [16]. However, the same techniques can be applied if the signature is Lorentzian. Looking only at the lowest tensor-harmonic, this leads to the result proven in this paper, namely that there exist no $SU(2)$-invariant tensor perturbations of Taub–NUT space. This means that possible perturbations behave either badly at infinity or at the horizon in a sense that will be made precise below. It is important to note that the perturbations under consideration are time-dependent, with a periodicity dictated by the $SU(2)$ symmetry of the metric. Having found this kind of instability in the lowest mode, it follows that the spacetime is unstable at the linear level against generic perturbations.
It is interesting to relate the result to the full nonlinear problem in the $SU(2)$-symmetry class, which corresponds to finding solutions to the Einstein equations for the following metric:

$$ds^2 = f(r)\, dr^2 + A(r)\sigma_1^2 + B(r)\sigma_2^2 - C(r)\sigma_3^2$$  \hspace{1cm} (1)

with the $SU(2)$ left-invariant 1-forms $\sigma_i$ defined in equation (15) and for positive functions $f, A, B$ and $C$. The metric (1) is related to the famous, thoroughly studied, Bianchi-IX cosmologies for which the $SU(2)$ acts on spatial hypersurfaces. The metric (1) has also been studied intensively with Euclidean signature in the search for gravitational instantons. In both cases the Einstein equations can be solved exactly only under restrictive conditions like $A(r) = B(r)$. In the case considered here, that is Lorentzian signature as given in (1), there will be important sign changes in the formulae compared to either Bianchi-IX cosmologies or Euclidean instantons. Hence an interesting problem, worth studying in its own right, is to resolve whether the assumption of $SU(2)$ invariance in the form of the metric ansatz (1) already forces the spacetime to be stationary, as might be suggested by the result of this paper. This would correspond to a Birkhoff-type theorem. Note that the standard Birkhoff theorem is a local statement about any piece of spacetime, with the symmetry group $SO(3)$ acting on a 2-sphere. Here, on the other hand, the $SU(2)$ acts on a 3-sphere and its action implies in particular the existence of closed timelike curves. Consequently, the global assumptions about spacetime made with the ansatz (1) might already force the spacetime to admit a timelike Killing vector field with closed orbits. Hence there would be no physically realistic dynamical processes that can be modelled within that class.

For Lorentzian Taub–NUT there is in fact a rigorous theorem for the nonlinear problem. The authors of [13] (see also [6]) prove that any spacetime admitting a compactly generated Cauchy horizon with closed generators necessarily has a Killing symmetry, which changes from being spacelike to timelike when crossing the Cauchy horizon. In other words, situations as they occur in Taub–NUT are essentially an artefact of symmetry. In particular, there is no dynamics of the Cauchy horizon (i.e. $dA = 0$), and hence no standard first law of thermodynamics with which physical meaning could be associated. It is worth mentioning how the result of [13] is related to the reasoning made in the previous paragraph. Namely if one could show that the symmetry assumption of (1) implied the existence of a compact Cauchy horizon with closed generators, one could apply the result of [13] directly to infer the existence of a timelike Killing symmetry.

The arguments of [13] require the full nonlinear Einstein equations. While they make no explicit connection with stability, they certainly suggest that the Taub–NUT metric is dynamically uninteresting. The result of the present paper demonstrates that the pathological behaviour of Lorentzian Taub–NUT space already shows up at the linear level by considering the stability of the metric under perturbations.

Let me finally comment on the issue of thermodynamics of NUT-charged spacetimes. The standard derivation of black hole thermodynamics [2] (see also [15]) assumes the existence of a linear perturbation linking two stationary states with slightly different parameters of mass and angular momentum. Since such perturbations are shown not to exist for NUT space, a first law cannot be derived by these methods. Moreover, the derivation of Hawking radiation [7] assumes gravitational collapse to a certain end state. NUT space does not qualify as a possible end state due to its instability. All these arguments indicate that even if a first law could be established formally by Euclidean quantum gravity (or other) arguments, its physical significance would be questionable. The reason that even a formal first law has not yet been stated is likely to be due to the unusual topology of NUT space. In particular there are no closed homologically trivial 2-cycles at infinity to define conserved quantities like energy in
the usual way—a fact that seems to be ignored in some treatments of the subject. These difficulties persist if time is not made periodic and a singularity is tolerated.

For completeness I outline a different (but also failing) approach to formulating a first law in an appendix to the paper. The procedure does not make use of any arguments from the Euclidean theory and directly leads to a Smarr formula for (Kerr–) Taub–NUT spacetime via the derivation of a NUT potential, which is constant on the horizon. However, a formal first law could not be derived from the Smarr formula in the usual fashion.

While the analysis of the present paper does not shed any light on the precise reason why Euclidean and other techniques fail to establish a first law, the results certainly question the physical relevance of any first law that might be derived for Taub–NUT space.

2. The geometry of Taub–NUT spacetime

The four-dimensional Taub–NUT spacetime [14] is given by the metric

\[ g^{TN} = -A(r)(dt + 2n \cos \theta \, d\phi)^2 + \frac{dr^2}{A(r)} + B(r)(d\theta^2 + \sin^2 \theta \, d\phi^2), \]

where

\[ A(r) = \frac{r^2 - 2mr - n^2}{r^2 + n^2}, \]

\[ B(r) = r^2 + n^2. \]

The metric has singularities where \( A(r) = 0 \) and where \( \theta = 0, \pi \). The former correspond to horizons at \( r_{\pm} = m \pm \sqrt{m^2 + n^2} \). The singularities at \( \theta = 0, \pi \) are not the usual coordinate degeneracies of the 2-sphere. In fact if we allow \(-\infty < t < \infty\) the metric (2) will have a line singularity along the \( z \)-axis, whose existence needs to be interpreted physically in one or the other way (see [3, 9] for two different suggestions). On the other hand, if the coordinate \( t \) is assumed to be periodic with period \( 8\pi n \), the singularities in \( \theta \) can be interpreted as the usual Euler-coordinate singularities of the 3-sphere, as was first observed by Misner [12].

In this paper we will follow [12] and assume periodic \( t \)-coordinate to avoid the line singularity. Our spacetime then has topology \( \mathbb{R} \times S^3 \), it is free of singularities but there are closed timelike curves for \( A(r) > 0 \).
3. Stability

Let us assume a perturbation of the Taub–NUT metric (2):

\[ g_{ab} = g^{TN}_{ab} + h_{ab}. \] (8)

Fixing the gauge we require the perturbation \( h_{ab} \) to be transverse with respect to the Taub–NUT derivative operator

\[ \nabla^a h_{ab} = 0, \] (9)

and traceless

\[ g^{TN}_{ab} h_{ab} = 0. \] (10)

Furthermore, it should, to first order, satisfy the vacuum Einstein equations

\[ R_{ab}(g) = 0. \] (11)

This implies the following equation for the perturbation \( h_{ab} \):

\[ \Delta_L h_{ab} = -\nabla^a \nabla^b h_{ab} - 2R_{acbd} h^{cd} = 0, \] (12)

where we have defined the Lichnerowicz operator \( \Delta_L \). All derivative operators are taken with respect to the background Taub–NUT metric. As a last requirement, the perturbation should be finite in the sense that the invariant expression

\[ \text{tr}(g) = g_{ab} g^{ab} = (g^{TN}_{ab} + h_{ab})(g^{TN}_{ab} - h_{ab}) = 4 - h_{ab} h^{ab} \] (13)

is bounded everywhere between the horizon and infinity. That is

\[ h_{ab} h^{ab} < \infty \quad \text{everywhere between the horizon and infinity.} \] (14)

Summarizing, we want perturbations \( h_{ab} \) satisfying (9), (10) and (12) and being finite in the sense of (14). The strategy below will be to show that perturbations satisfying (9), (10) and (12) will violate the condition (14).

In general solutions to equations (9), (10), (12) are hard to find. The key is to exploit the SU(2)-invariance of the background by expanding the perturbations in terms of Wigner functions (‘modes’). If we then restrict ourselves to SU(2)-invariant perturbations (that is, zero modes, \( J = K = M = 0 \)), we can prove the following result:

There are no finite, transverse, traceless SU(2)-invariant perturbations of the Lorentzian Taub–NUT metric.

This result indicates that the Taub–NUT metric is unstable. As mentioned in the introduction this result regarding linear stability is related to the nonlinear result proved in [13]. We will make further comments about the type of the perturbation after presenting its precise form in the next section.

4. Decomposition of perturbations

The Taub–NUT metric has an SU(2) invariance, which can be made more manifest by introducing the left invariant 1-forms of SU(2) (in Euler coordinates \( \theta, \phi, t \))

\[ \sigma_1 = -\sin t \, d\theta + \sin \theta \cos t \, d\phi, \]
\[ \sigma_2 = \cos t \, d\theta + \sin \theta \sin t \, d\phi, \]
\[ \sigma_3 = dt + \cos \theta \, d\phi, \] (15)

and writing the metric (2) as

\[ ds^2 = \frac{1}{A(r)} \, dr^2 - 4n^2 A(r) \sigma_1^2 + B(r) (\sigma_1^2 + \sigma_2^2). \] (16)
Since the left invariant 1-forms $\sigma_i$ are invariant under the left action of SU(2) but transform as a triplet under the right action, the metric (16) is invariant under SU(2)$_L \times U(1)_R$. For future reference we also define

$$\gamma_{00} = \gamma_{rr} = \frac{1}{A(r)}, \quad \gamma_{11} = \gamma_{22} = B(r), \quad \gamma_{33} = -4n^2 A(r). \quad (17)$$

As mentioned in the last section, the key is an expansion of the perturbations in terms of SU(2)-Wigner functions $D_{KM}^J$, which are analogues of the spherical harmonics on the 2-sphere. The general perturbation can be decomposed as

$$h = \sum_{J,M,K} \sum_{K=-J}^{J} h_{ab}^{J,M,K}(r) D_{KM}^J(t, \theta, \phi) \sigma^a \otimes \sigma^b. \quad (18)$$

The tedious work of rephrasing equations (9), (10) and (12) in terms of modes is done in [17], where the Euclidean Taub–NUT metric is studied. In particular, the zero-mode equations for $h_{00,00,00}$ together with their transverse-traceless gauge conditions are derived. Taking care of the signs all these equations can be restated for the Lorentzian case. In both cases second-order ordinary differential equations can be decoupled for the components $h_{00}, h_{33}, h_{03}, X = h_{11} + h_{22}, Y = h_{11} - h_{22}, h_{12}, h_{13}, h_{23}$ of the perturbation $h_{ab}$. Here and in the following we suppress the $(0,0,0)$-mode indices. The strategy followed in this paper will be to analyse the equation for each component carefully and to prove that the solutions they admit will behave badly at the horizon or at infinity. To be more specific we will show that any solution would violate the boundedness condition (14).

Note that the zero mode under consideration will in general be time-dependent. This is because the zero mode of (18) is of the form

$$h = h_{ab}^{0,0,0}(r) \sigma^a \otimes \sigma^b, \quad (19)$$

using that $D_{00}^0 = \text{const.}$ Since the left invariant 1-forms (15) explicitly depend on time in a periodic fashion, the modes under consideration are periodic in time. It is this fact that renders the result non-trivial.

It is also worth pointing out that our non-existence result is different in style from the usual linear perturbation-theory arguments for spacetimes without closed timelike curves but an ordinary timelike Killing vector (see, e.g. [5]). There it is usually shown that regular perturbations can be defined on a spacelike hypersurface but that they will blow up in time. Here we show that in fact no SU(2)-invariant perturbation can be defined initially that behaves well at the horizon and at infinity.

5. Differential equations

In the following, we will constantly refer to equations derived in [17]. The differential equations for the metric perturbation components $h_{ab}$ stated in this section can be derived from the Wigner decomposition and are found explicitly (with Euclidean signature) in the appendix of that paper.

5.1. $h_{21}$

The $h_{21}$ equation is ((B.1) of [17])

$$a(r)h_{21}''(r) + b(r)h_{21}'(r) + c(r)h_{21}(r) = 0, \quad (20)$$
where

\[ a(r) = A(r), \]

\[ b(r) = A' - \frac{A'B'}{B}, \]

\[ c(r) = \frac{1}{n^2} - \frac{A'B'}{B} + A \left( \frac{B'}{B} \right)^2 - A \left( \frac{B''}{B} \right) + \frac{2}{B}. \]

Near infinity the equation reads

\[ h''_{21} + \left( -\frac{2}{r} + \mathcal{O}\left( \frac{1}{r^2} \right) \right) h'_{21} + \left( \frac{1}{n^2} + \mathcal{O}\left( \frac{1}{r} \right) \right) h_{21} = 0, \]

signalling an irregular singularity near infinity. Using techniques from ordinary differential equations we can compute an asymptotic expansion near infinity. The two linearly independent asymptotic solutions are

\[ h_{21} = -\sqrt{\frac{2}{n}} \pi \left( r \cos \left( \frac{r}{n} \right) - n \sin \left( \frac{r}{n} \right) \right) \left( 1 + \mathcal{O}\left( \frac{1}{r} \right) \right), \]

\[ h_{21} = -\sqrt{\frac{2}{n}} \pi \left( n \cos \left( \frac{r}{n} \right) + r \sin \left( \frac{r}{n} \right) \right) \left( 1 + \mathcal{O}\left( \frac{1}{r} \right) \right). \]

None of these solutions decays at infinity, which is a necessary requirement for the perturbation. Hence \( h_{21} = 0 \). Note that it is the Lorentzian signature which is responsible for the oscillatory behaviour. For the Euclidean version one would have to replace \( \frac{2}{r} \) by \( -\frac{2}{r} \) and exponentially decaying solutions will exist. (Whether they are well behaved at the horizon remains to be checked.) This behaviour, that is the asymptotics of the perturbation being governed by the signature of the background, will appear in almost all of the following differential equations we are about to consider.

5.2. \( h_{20} \) and \( h_{10} \)

The components \( h_{20} \) and \( h_{10} \) satisfy the same differential equation. We spell out the formulae for \( h_{20} \) here but all conclusions are of course valid for \( h_{10} \) as well. (Inserting equations (B.6) into (B.4) of [17], respectively (B.9) into (B.7) yields)

\[ a(r)h''_{20}(r) + \left( b(r) - \frac{A'}{2B} + \frac{A'}{8n^2A} \right) h'_{20}(r) + \left( c(r) - d(r) \frac{A' + AB'}{2B} + \frac{A'}{8n^2A} \right) h_{20}(r) = 0, \]

with

\[ a(r) = -A, \]

\[ b(r) = -2A', \]

\[ c(r) = \frac{3}{2} A \left( \frac{B'}{B} \right)^2 - \frac{1}{4n^2A} - \frac{2n^2A}{B^2} - \frac{1}{2} A \left( \frac{B''}{B} \right) - \frac{1}{B} - \frac{A''}{2}, \]

\[ d(r) = \frac{B'}{B^2} - \frac{A'}{2AB} + \frac{A'}{8n^2A^2}. \]

Near infinity the differential equation looks like

\[ h''_{20}(r) + \left( 6m^2 \frac{1}{r^3} + \frac{8n^2 + 28n^2}{r^3} + \mathcal{O}\left( \frac{1}{r^2} \right) \right) h'_{20}(r) + \left( \frac{1}{4n^2} + \mathcal{O}\left( \frac{1}{r} \right) \right) h_{20}(r) = 0. \]
and we can construct the asymptotic solutions by applying theorems about irregular singular points at infinity. The two solutions are

\[ h_{20}(r) = \exp \left( \frac{i}{2n} r \right) \left[ 1 + O \left( \frac{1}{r} \right) \right], \]  

(33)

and its complex conjugate. Note that the series in the square brackets may have complex coefficients. We can easily construct two real solutions and it is immediate that they will be oscillatory near infinity. Computing the Riemann tensor generated by such an oscillatory perturbation one finds that it decays slower (namely with \( \frac{1}{r^2} \)) than the background curvature of Taub–NUT, which decays like \( \frac{1}{r^3} \). This violates the assumptions of perturbation theory. Hence we can discard the perturbation, \( h_{20} = h_{10} = 0 \).

5.3. \( Y = h_{11} - h_{22} \)

An equation for the difference \( Y = h_{11} - h_{22} \) can be decoupled from equations (B.10) and (B.11) of [17]. It reads

\[ a(r)Y''(r) + b(r)Y'(r) + (d(r) - e(r))Y(r) = 0, \]  

(34)

where

\[ a(r) = A(r), \]  

(35)

\[ b(r) = A' - A \frac{B'}{B}, \]  

(36)

\[ d(r) = \frac{2}{B} - \frac{AB''}{B} - \frac{A'B'}{B} + \frac{1}{2} A \left( \frac{B'}{B} \right)^2 + \frac{4n^2 A}{B^2} + \frac{1}{n^2 A}. \]  

(37)

\[ e(r) = -\frac{1}{2n^2 A} + \frac{4n^2 A}{B^2} - \frac{1}{2} A \left( \frac{B'}{B} \right)^2. \]  

(38)

Near infinity the differential equation reads

\[ Y''(r) + \left( -\frac{2}{r} + O \left( \frac{1}{r^2} \right) \right) Y'(r) + \left( \frac{3}{2n^2} + O \left( \frac{1}{r} \right) \right) Y(r) = 0. \]  

(39)

The asymptotic structure is very similar to that of the \( h_{21} \) equation considered above. In fact, the arguments used in that subsection can be brought forward to show that solutions of this equation have to oscillate (and blow up) at infinity. This shows that \( Y = 0 \) and therefore \( h_{11} = h_{22} \).

5.4. \( h_{30} \)

The solution for \( h_{30} \) can be obtained explicitly from the constraint equation ((B.2) of [17])

\[ h_{30}' = -\left( \frac{A'}{A} + \frac{B'}{B} \right) h_{30}. \]  

(40)

It is

\[ h_{30}(r) = \frac{1}{A(r)B(r)} = \frac{1}{r^2 - 2mr + n^2}. \]  

(41)
5.5. $h_{00}$

The $h_{00}$ equation that can be decoupled from (B.14), (B.15), (B.12) of [17] is the most complicated one. It reads

$$a(r)h''_{00} + b(r)h'_{00} + c(r)h_{00} = 0,$$

(42)

with the coefficients given by

$$a(r) = A,$$

$$b(r) = 3A' + \frac{AB'}{B} + B\left(\frac{A^2 - 2AA''}{BA' - AB'}\right) - \frac{A}{B}\left(\frac{2AB'^2 - BA'B' - 2ABB''}{BA' - AB'}\right),$$

$$c(r) = \frac{A^2}{2A} + \frac{A'B'}{B} - \frac{AB'^2}{B^2} + A'' + \frac{3}{2A}\left(\frac{B'A' + AB'}{BA' - AB'}\right) + \frac{BA'B' + 2BAB'' - 2AB'^2}{B^2}\left(\frac{2BA' + AB'}{BA' - AB'}\right).$$

Inserting the functions $A$ and $B$ the equation becomes

$$h''_{00}(r) + p(r)h'_{00}(r) + q(r)h_{00}(r) = 0,$$

(43)

where now

$$p(r) = \frac{8(m - r)}{n^2 + (2m - r)r} + \frac{2r}{n^2 + r^2} + \frac{6(n^2 + (2m - r)r)}{-3n^2r + r^3 + m(n^2 - 3r^2)},$$

$$q(r) = 2\left(\frac{4(m^2 + n^2)}{(n^2 + (2m - r)r)^2} + \frac{2n^2}{(n^2 + r^2)^2} + \frac{1}{n^2 + r^2}\right) + \frac{8}{-n^2 - 2mr + r^2} + \frac{9(m - r)}{-3n^2r + r^3 + m(n^2 - 3r^2)}.$$}

There are (regular) singular points at the horizon, $r_{hoz} = m + \sqrt{m^2 + n^2}$, and at infinity. There is also a regular singular point at

$$r = r_{in} = m + 2 \cos\left(\frac{\arctan\left(\frac{n}{m}\right)}{3}\right)\sqrt{m^2 + n^2}.$$

(44)

However, a Frobenius expansion around $r_{in}$ shows that the solution has always finite value of the first and second derivatives at that point.

The regular singular point at infinity can also be analysed by a Frobenius series. Near infinity

$$p(r) = \frac{4}{r} + 2m \frac{1}{r^2} + O\left(\frac{1}{r^3}\right),$$

(45)

$$q(r) = -\frac{4m}{r^3} + O\left(\frac{1}{r^4}\right),$$

(46)

therefore the solution is asymptotically either $h_{00} \propto \frac{1}{r}$ or proportional to a constant ($\neq 0$) near infinity. Of course, only the first case will lead to a sensible perturbation. In fact, the solution $h_{00}$ that behaves like $\frac{1}{r}$ near infinity can be constructed explicitly and is unique by the theory of ODEs. It reads

$$h_{00} = \frac{2r^3 - m(n^2 + 3r^2)}{(r^2 + n^2)(r^2 - 2mr - n^2)^2}.$$
5.6. The other components

So far we found that for any finite, transverse, traceless perturbation, \( h_{21} = h_{20} = h_{10} = 0 \) must hold. Using the equations given in [17] one concludes that

- \( h_{20} = 0 \) implies \( h_{31} = 0 \) (by (B.6) of [17])
- \( h_{10} = 0 \) implies \( h_{32} = 0 \) (by (B.9) of [17]).

Thus there are only five possibly non-zero components left, of which we already know \( h_{00} \) and \( h_{30} \) explicitly. The remaining components could be calculated explicitly from what we already know but in fact there is no need to do this. Writing out the boundedness condition (14) yields

\[
 h_{ab} h^{ab} = h_{00} h^{00} + 2 h_{30} h^{30} + h_{11} h^{11} + h_{22} h^{22} + h_{33} h^{33}.
\] (48)

Taking into account that the metric used to raise and lower indices is the \( \gamma \) given in (17) we observe that only the second term of the right-hand side of (48) will be negative, whereas all the others are positive. Now it can easily be shown, using the explicit solutions (47) and (41), that the sum \( \lambda^2 h_{00} h^{00} + \mu^2 h_{30} h^{30} \) always admits a positive divergent \( \frac{1}{r - r_{\text{hor}}} \) term at the horizon \( r_{\text{hor}} = m + \sqrt{m^2 + n^2} \) unless \( \lambda = \mu = 0 \). (Remark: the sum may or may not have a divergent \( \frac{1}{(r - r_{\text{hor}})^2} \) term of whatever sign. The parameters \( \lambda \) and \( \mu \) could be chosen to cancel it.) Hence the perturbation \( h_{ab} \) is not finite unless we set \( h_{00} = h_{30} = 0 \). From \( h_{00} = 0 \) and equations (B.12)–(B.15) of [17] it follows that \( h_{11} = 0 \) and \( h_{22} = 0 \) which together with \( Y = h_{11} - h_{22} = 0 \) implies \( h_{11} = h_{22} = 0 \). This finally shows that \( h_{ab} = 0 \) for all \( a, b \). There are no \( SU(2) \) invariant, finite perturbations of Taub–NUT space.

6. Discussion

The fact that there are no \( SU(2) \)-invariant perturbations of Lorentzian Taub–NUT space is a clear indication that the space is unstable. In proving the result one clearly noted how the Lorentzian signature of the spacetime enforced possible perturbations to oscillate near infinity. A straightforward generalization would be to consider the cases with non-vanishing cosmological constant and electric charge, where I expect the same result to hold. Higher dimensional generalizations also seem possible with the advanced technical effort of finding the analogues of the Wigner functions.

The result is in accord with the result of [13], in that it shows pathological behaviour already appears at the level of a linear stability analysis.

It is unlikely that a physically sensible thermodynamics can be associated with Lorentzian Taub–NUT spaces and the possible attempts need to be rethought. In fact the so far unsuccessful efforts to establish a truly satisfying first law are very likely to be related to the instability of the space. Moreover, the linear analysis might help to gain a better understanding of why standard strategies fail.

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Appendix. NUT thermodynamics

In this appendix I want to show how a Smarr formula can be derived for Taub–NUT space via dimensional reduction along the $U(1)$ fibres of the metric and comment on unsuccessful attempts to deduce a first law from it.

A.1. Komar integrals

Let us compute the conserved quantity associated with the Killing vector $\frac{\partial}{\partial t}$. The corresponding Killing 1-form is

$$k = A(r)(dt + 2n \cos \theta \, d\phi).$$  \hfill (A.1)

Thus up to a sign depending on one’s convention for the Hodge operator $\star$:

$$\star dk = A'(r)(r^2 + n^2) \sin \theta \, d\theta \wedge d\phi + \frac{2n}{r^2 + n^2} dr \wedge (dt + 2n A(r) \cos \theta \, d\phi).$$  \hfill (A.2)

Care is needed when integrating over $\theta$ and $\phi$ at constant $r$. One can formally perform the integration

$$\int \star dk = m \sqrt{r^2 - n^2} + \frac{2n^2 r}{r^2 + n^2},$$  \hfill (A.3)

finding that for $n \neq 0$ the integral depends on $r$. Strictly speaking, one has to take the Misner strings on the $z$-axis (respectively the two different charts) into account when performing the integration. Thus one should omit from the integral a small region, radius $\epsilon$ around the intersection of the two concentric spheres with the Misner strings. To get a closed 2-cycle, homologically trivial, one should add two small tubes surrounding the Misner strings. The integral over the tubes would make a non-vanishing contribution in the limit $\epsilon \to 0$. This limit can be attributed to the Misner strings. But this means that the difference of the above integral evaluated at infinity and at the horizon must be the tube contribution of the Misner strings. At infinity

$$\frac{1}{8\pi} \int \star dk \to m,$$  \hfill (A.4)

whereas at the Cauchy horizon, $r = r_+ = m + \sqrt{m^2 + n^2}$, we have

$$\frac{1}{8\pi} \int_{r_+} \star dk = \sqrt{m^2 + n^2},$$  \hfill (A.5)

The contribution due to the Misner strings is given by the difference

$$\int_{\text{Misner strings}} \star dk = m - \sqrt{m^2 + n^2}.$$  \hfill (A.6)

It is negative for $n > 0$. A possible interpretation is that the spacetime contains negative energy between the horizon at $r_+$ and infinity [9].

To make contact with a Smarr-type formula we write the result (A.6) in a different way. A calculation of the surface gravity of the horizon at $r = r_+$ yields

$$\kappa = \frac{1}{2} \frac{r_+ - r_-}{r_+^2 + n^2} = \frac{\sqrt{m^2 + n^2}}{r_+^2 + n^2}.$$  \hfill (A.7)

The area $\tilde{A}$ of the horizon (not to be confused with the $A(r)$ of equation (3)) is

$$\tilde{A} = 4\pi (r_+^2 + n^2).$$  \hfill (A.8)
Now obviously
\[ m = \frac{\kappa A}{4\pi} + \int_{\text{Misner strings}} \star dk. \] (A.9)

We can gain deeper insight into the term which is due to the Misner strings by using a dimensional reduction.

A.2. Dimensional reduction

The idea of the method of dimensional reduction is to project the Einstein equations down to the space \( \Sigma \) of orbits of the Killing field \( \partial_t \). Note that \( \Sigma \) should not be thought of as a submanifold of the spacetime manifold \( M \) but—locally at least—as the base space of an \( \mathbb{R} \)-bundle over \( \Sigma \). If one focuses on the timelike orbits, the space of orbits will have a boundary component \( \partial \Sigma^+ \) on which the orbits become null, that is the boundary consists of the space of null generators of the Cauchy horizon at \( r = r_* \). Of course, \( \Sigma \) will also have a boundary at infinity \( (r \to \infty) \), which we denote by \( \partial \Sigma_\infty \).

Outside the horizon, the four-dimensional metric (2) may be written in the Kaluza–Klein form
\[ ds^2 = -e^{2U} (dt + \omega)^2 + e^{-2U} \tilde{\gamma}_{ij} dx^i dx^j, \] (A.10)
with
\[ \omega = 2n \cos \theta \, d\phi, \] (A.11)
\[ U(r) = \frac{1}{2} \log A(r), \] (A.12)
\[ \tilde{\gamma}_{ij} = \text{diag}(1, e^{2U}(r^2 + n^2), e^{2U}(r^2 + n^2) \sin^2 \theta). \] (A.13)

Let \( \Omega = d\omega \). Then the vacuum Einstein equations imply that
\[ d \star (e^{4U} \Omega) = 0, \] (A.14)
where the Hodge operator \( \star \) is taken with respect to the metric \( \tilde{\gamma}_{ij} \). Thus one may introduce a NUT potential \( \psi \) such that
\[ \Omega = e^{-4U} \star d\psi \] (A.15)
whence
\[ \nabla_i (\tilde{\gamma}^{ij} e^{-4U} \nabla_j \psi) = 0. \] (A.16)

The Einstein equations may now be obtained from the three-dimensional Lagrangian density
\[ \mathcal{L} = \sqrt{\tilde{\gamma}} \left( R + 2 \tilde{\gamma}^{ij} \partial_i U \partial_j U + \frac{1}{2} e^{-4U} \tilde{\gamma}^{ij} \partial_i \psi \partial_j \psi \right) \] (A.17)
where \( R \) is the Ricci scalar of the metric \( \tilde{\gamma}_{ij} \). The equation of motion for \( U \),
\[ \nabla^2 U + \frac{1}{2} e^{-4U} (\partial \psi)^2 = 0, \] (A.18)
can be solved for the potential \( \psi \) with the result
\[ \psi = -2n \frac{m - r}{(r^2 + n^2)} = \frac{2n}{r} + O \left( \frac{1}{r^2} \right). \] (A.19)

Now from (A.16) one may obtain the identity
\[ -8\pi n = \int_{\partial \Sigma_\infty} e^{-4U} \partial_i \psi \, d\sigma^i = \int_{\partial \Sigma_*} e^{-4U} \partial_i \psi \, d\sigma^i. \] (A.20)

1 The reader more interested in the techniques of Kaluza–Klein- and dimensional reductions is referred to the review article [4].
From (A.18) one may obtain the identity
\[ 4\pi m = \int_{\partial \Sigma} \partial_i U \, d\sigma^i = -\frac{1}{2} \int_{\Sigma} e^{-4U} (\partial \psi)^2 + \int_{\partial \Sigma} \partial_i U \, d\sigma^i. \] (A.21)

But
\[ \int_{\partial \Sigma} \partial_i U \, d\sigma^i = \kappa \bar{A}. \] (A.22)

Moreover, multiplication of (A.16) by $\psi$ and integration by parts yields
\[ -\int_{\Sigma} e^{-4U} (\partial \psi)^2 = \int_{\partial \Sigma} \psi e^{-4U} \partial_i \psi \, d\sigma^i - \int_{\partial \Sigma} \psi e^{-4U} \partial_i \psi \, d\sigma^i. \] (A.23)

Note that the boundary term at infinity vanishes. Putting this all together gives
\[ 4\pi m = \kappa \bar{A} - 4\pi n \psi_H, \] (A.24)

and hence
\[ m = \frac{\kappa \bar{A}}{4\pi} - n \psi_H, \] (A.25)

where $\psi_H$ is the value of the NUT potential on the horizon. Comparing (A.25) with (A.9) we have found an interpretation for the Misner term as a NUT potential, which is constant on the horizon, multiplied by the NUT charge. In fact it is straightforward to generalize (A.25) to the Kerr–Taub–NUT case [8].

### A.3. Failure of a first law

From (A.25) one would expect the first law
\[ \frac{dm}{dt} = \frac{\kappa d\bar{A}}{8\pi} - \psi_H \, dn. \] (A.26)

However, (A.26) does not hold unless a special relation between $m$ and $n$ is assumed, which the reader might wish to work out. Moreover, in the general Kerr-NUT case this condition is notably different from the regularity condition that might be imposed from regularity of the Euclideanized metric.

Finally, it can be shown for the Kerr case [8] that there exists no modification of the area that allows a first law with NUT potential to be valid.

### References

[1] Astefanesei D, Mann R B and Radu E 2005 *Phys. Lett.* B 620 1–8
[2] Bardeen J M, Carter B and Hawking S W 1973 *Commun. Math. Phys.* 31 161–70
[3] Bonnor W B 1969 *Proc. Camb. Phil. Soc.* 66 145
[4] Duff M J, Nilsson B E W and Pope C N 1986 *Phys. Rep.* 130 1–142
[5] Gibbons G W and Hartnoll S A 2002 *Phys. Rev.* D 66 064024
[6] Gibbons G W and Stewart J M 1984 *Classical General Relativity* (Proc. of the Conf. on Classical (non-quantum) GR (London, 1983)) (Cambridge: Cambridge University Press)
[7] Hawking S W 1975 *Commun. Math. Phys.* 43 199–220
[8] Holzegel G H 2006 unpublished notes
[9] Manko V S and Ruiz E 2005 *Class. Quantum Grav.* 22 3555
[10] Mann R B and Stelea C 2004 *Class. Quantum Grav.* 21 2937–61
[11] Mann R B and Stelea C 2005 *Phys. Rev.* D 72 084032
[12] Misner C W 1963 *J. Math. Phys.* 4 7
[13] Moncrief V and Isenberg J 1983 *Commun. Math. Phys.* 89 387–413
[14] Newman E, Tamburino L and Unti T 1963 *J. Math. Phys.* 4 915–23
[15] Wald R M 1993 *Phys. Rev.* D 48 3427–31
[16] Warnick C M 2006 *Preprint* hep-th/0602127 (*Class. Quantum Grav.* at press)
[17] Young R E 1983 *Phys. Rev.* D 28 2420–35