A CANONICAL COMPATIBLE METRIC FOR GEOMETRIC STRUCTURES ON NILMANIFOLDS

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Abstract. Let \((N, \gamma)\) be a nilpotent Lie group endowed with an invariant geometric structure (cf. symplectic, complex, hypercomplex or any of their ‘almost’ versions). We define a left invariant Riemannian metric on \(N\) compatible with \(\gamma\) to be minimal, if it minimizes the norm of the invariant part of the Ricci tensor among all compatible metrics with the same scalar curvature. We prove that minimal metrics (if any) are unique up to isometry and scaling, they develop soliton solutions for the ‘invariant Ricci’ flow and are characterized as the critical points of a natural variational problem. The uniqueness allows us to distinguish two geometric structures with Riemannian data, giving rise to a great deal of invariants.

Our approach proposes to vary Lie brackets rather than inner products; our tool is the moment map for the action of a reductive Lie group on the algebraic variety of all Lie algebras, which we show to coincide in this setting with the Ricci operator. This gives us the possibility to use strong results from geometric invariant theory.

1. Introduction

Invariant structures on nilpotent Lie groups, as well as on their compact versions, nilmanifolds (i.e. quotients by cocompact discrete subgroups), play an important role in symplectic and complex geometry. Such structures are described easily in terms of data arising from the Lie algebra, but can in turn be used to get very interesting and exotic examples such as, to mention one, metrics with exceptional holonomy. The aim of this paper is the search for the ‘best’ Riemannian metric compatible with a given geometric structure. Symplectic, complex and hypercomplex cases will be treated in some detail, and although we will always have these particular cases in mind, the main results will be proved in a more general setting.

1.1. Geometric structures and compatible metrics. Let \(N\) be a real \(n\)-dimensional nilpotent Lie group with Lie algebra \(\mathfrak{n}\), whose Lie bracket will be denoted by \(\mu: \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n}\). By an invariant geometric structure on \(N\) we mean a tensor on \(N\) defined by left translation of a tensor \(\gamma\) on \(\mathfrak{n}\) (or a set of tensors), usually non-degenerate in some way, which satisfies a suitable integrability condition

\[
\text{IC}(\gamma, \mu) = 0,
\]
involving only $\mu$ and $\gamma$. The pair $(N, \gamma)$ will be called a class-$\gamma$ nilpotent Lie group, and $N$ will be assumed to be simply connected for simplicity. A left invariant Riemannian metric on $N$ is said to be compatible with $(N, \gamma)$ if the corresponding inner product $\langle \cdot, \cdot \rangle$ on $n$ satisfies an orthogonality condition

\begin{equation}
OC(\gamma, \langle \cdot, \cdot \rangle) = 0,
\end{equation}

in which only $\langle \cdot, \cdot \rangle$ and $\gamma$ are involved. We denote by $C = C(N, \gamma)$ the set of all left invariant metrics on $N$ which are compatible with $(N, \gamma)$. The pair $(\gamma, \langle \cdot, \cdot \rangle)$ with $\langle \cdot, \cdot \rangle \in C$ will often be referred to as a class-$\gamma$ metric structure.

A natural question arises:

Given a class-$\gamma$ nilpotent Lie group $(N, \gamma)$, are there canonical or distinguished left invariant Riemannian metrics on $N$ compatible with $\gamma$?, where the meaning of 'canonical' and 'distinguished' is of course part of the problem. The Ricci tensor has always been a very useful tool to deal with this kind of questions, and since the answer should depend on the metric and on the structure under consideration, we consider the invariant Ricci operator $\operatorname{Ric}_\gamma(\langle \cdot, \cdot \rangle)$ (and the invariant Ricci tensor $\operatorname{ric}_\gamma(\langle \cdot, \cdot \rangle) = \langle \operatorname{Ric}_\gamma(\langle \cdot, \cdot \rangle), \langle \cdot, \cdot \rangle \rangle$), that is, the orthogonal projection of the Ricci operator $\operatorname{Ric}(\langle \cdot, \cdot \rangle)$ onto the subspace of those symmetric maps of $n$ leaving $\gamma$ invariant. D. Blair, S. Ianus and A. Ledger [BI, BL] have proved in the compact case that metrics satisfying

\begin{equation}
\operatorname{ric}_\gamma(\langle \cdot, \cdot \rangle) = 0
\end{equation}

are very special in symplectic (so called metrics with hermitian Ricci tensor) and contact geometry, as they are precisely the critical points of two very natural curvature functionals on $C$: the total scalar curvature functional $S$ and a functional $K$ measuring how far are the metrics of being Kähler or Sasakian, respectively (see also [B]).

We will show that for a non-abelian nilpotent Lie group, condition (3) cannot hold for the classes of structures we have in mind, and hence it is natural to try to get as close as possible to this unattainable goal. In this light, a metric $\langle \cdot, \cdot \rangle \in C(N, \gamma)$ is called minimal if it minimizes the functional $\|\operatorname{ric}_\gamma(\langle \cdot, \cdot \rangle)\|^2 = tr(\operatorname{Ric}_\gamma(\langle \cdot, \cdot \rangle))^2$ on the set of all compatible metrics with the same scalar curvature. It turns out that minimal metrics are the elements in $C$ closest to satisfy the ‘Einstein-like’ condition $\operatorname{ric}_\gamma(\langle \cdot, \cdot \rangle) = c \langle \cdot, \cdot \rangle$, $c \in \mathbb{R}$. We may also try to improve the metric via the evolution flow

\begin{equation}
\frac{d}{dt} \langle \cdot, \cdot \rangle_t = \pm \operatorname{ric}_\gamma(\langle \cdot, \cdot \rangle)_t,
\end{equation}

whose fixed points are precisely metrics satisfying (3). In the symplectic case, this flow is called the anticomplexified Ricci flow and has been recently studied by H-V Le and G. Wang [LW]. Of particular significance are then those metrics for which the solution to the normalized flow (under which the scalar curvature is constant in time) remains isometric to the initial metric. Such special metrics will be called invariant Ricci solitons. The main result in this paper can be now stated.

**Theorem 1.1.** Let $(N, \gamma)$ be a nilpotent Lie group endowed with an invariant geometric structure $\gamma$ (non-necessarily integrable). Then the following conditions on a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ which is compatible with $(N, \gamma)$ are equivalent:

(i) $\langle \cdot, \cdot \rangle$ is minimal.
(ii) \( \langle \cdot, \cdot \rangle \) is an invariant Ricci soliton.

(iii) \( \text{Ric} \gamma \langle \cdot, \cdot \rangle = c I + D \) for some \( c \in \mathbb{R}, D \in \text{Der}(n) \).

Moreover, there is at most one compatible left invariant metric on \((N, \gamma)\) up to isometry (and scaling) satisfying any of the above conditions.

A major obstacle to classify geometric structures is the lack of invariants. The uniqueness result in the above theorem gives rise to a useful tool to distinguish two geometric structures; indeed, if they are isomorphic then their respective minimal compatible metrics (if any) have to be isometric. One therefore can eventually distinguish geometric structures with Riemannian data, which suddenly provides us with a great deal of invariants. This will be used in Section 5 to find explicit continuous families of pairwise non-isomorphic geometric structures in low dimensions, mainly by using only one Riemannian invariant: the eigenvalues of the Ricci operator.

A weakness of this approach is however the existence problem; the theorem does not even suggest when such a distinguished metric does exist. How special are the symplectic or (almost-) complex structures admitting a minimal metric? So far, we know how to deal with this ‘existence question’ only by giving several explicit examples (see Remark 1.3), for which the neat ‘algebraic’ characterization (iii) is very useful. It turns out that in low dimensions the structures in general tend to admit a minimal compatible metric, and the only obstruction we know at this moment is when the subgroup of \( \text{GL}(n) \) preserving \( \gamma \) is in \( \text{SL}(n) \) (e.g. for symplectic structures) and the nilpotent Lie algebra does not have any nonzero symmetric derivation. In this way, the characteristically nilpotent Lie algebras (i.e. \( \text{Der}(n) \) is nilpotent) admitting a symplectic structure recently found by D. Burde in [Bu] can not be endowed with a minimal compatible metric. At the moment, these are the only counterexamples we have to the existence question, and the lowest dimension among them is 8.

1.2. Variety of compatible metrics and the moment map. A class-\( \gamma \) metric structure on a nilpotent Lie group is determined by a triple \((\mu, \gamma, \langle \cdot, \cdot \rangle) \) of tensors on \( n \). The proof of Theorem 1.1 is based on an approach which proposes to vary the Lie bracket \( \mu \) rather than the inner product \( \langle \cdot, \cdot \rangle \).

Let us consider as a parameter space for the set of all real nilpotent Lie algebras of a given dimension \( n \), the set

\[ N = \{ \mu \in V : \mu \text{ satisfies Jacobi and is nilpotent} \}, \]

where \( n \) is a fixed \( n \)-dimensional real vector space and \( V = \Lambda^2 n^* \otimes n \) is the vector space of all skew-symmetric bilinear maps from \( n \times n \) to \( n \). Since the Jacobi identity and the nilpotency condition are both determined by zeroes of polynomials, \( N \) is a real algebraic variety. We fix a tensor \( \gamma \) on \( n \) (or a set of tensors), and denote by \( G_\gamma \) the subgroup of \( \text{GL}(n) \) preserving \( \gamma \). The reductive Lie group \( G_\gamma \) acts naturally on \( V \) leaving \( N \) invariant and also the algebraic subset \( N_\gamma \subset N \) given by

\[ N_\gamma = \{ \mu \in N : \text{IC}(\gamma, \mu) = 0 \}, \]

that is, those nilpotent Lie brackets for which \( \gamma \) is integrable (see (1)).

For each \( \mu \in N_\gamma \), let \( N_\mu \) denote the simply connected nilpotent Lie group with Lie algebra \((n, \mu)\). Fix an inner product \( \langle \cdot, \cdot \rangle \) on \( n \) compatible with \( \gamma \), that is, such that (2) holds. We identify each \( \mu \in N_\gamma \) with a class-\( \gamma \) metric structure on a nilpotent
Lie group

\[ \mu \leftrightarrow (N_\mu, \gamma, \langle \cdot, \cdot \rangle), \]

where all the structures are defined by left invariant translation. The orbit \( G_\gamma . \mu \) parameterizes then all the left invariant metrics which are compatible with \((N_\mu, \gamma)\) and hence we may view \( N_\gamma \) as the space of all class-\( \gamma \) metric structures on nilpotent Lie groups of dimension \( n \). Two metrics \( \mu, \lambda \in N_\gamma \) are isometric if and only if they live in the same \( K_\gamma \)-orbit, where \( K_\gamma = G_\gamma \cap O(n, \langle \cdot, \cdot \rangle) \) is the maximal compact subgroup of \( G_\gamma \).

We now go back to our search for the best compatible metric. It is natural to consider the functional \( F : N_\gamma \mapsto \mathbb{R} \) given by \( F(\mu) = \text{tr}(\text{Ric}^{\gamma}_\mu)^2 \), which in some sense measures how far the metric \( \mu \) is from satisfying (3). The critical points of \( F/||\mu||^4 \) on the projective algebraic variety \( \mathbb{P}N_\gamma \subset \mathbb{P}V \) (which is equivalent to normalize by the scalar curvature since \( \text{sc}(\mu) = -\frac{1}{4}||\mu||^2 \)), may therefore be considered compatible metrics of particular significance.

A crucial fact of this approach is that the moment map \( m_\gamma : V \mapsto p_\gamma \) for the action of \( G_\gamma \) on \( V \) is proved to be\[
m_\gamma(\mu) = 8 \text{Ric}^{\gamma}_\mu, \quad \forall \mu \in N_\gamma,
\]
where \( p_\gamma \) is the space of symmetric maps of \((n, \langle \cdot, \cdot \rangle)\) leaving \( \gamma \) invariant (i.e. \( g_\gamma = \mathfrak{e}_\gamma \oplus p_\gamma \) is a Cartan decomposition). This allows us to use strong and well-known results on the moment map due to F. Kirwan \[K1\] and L. Ness \[N\], and proved by A. Marian \[M\] in the real case (see Section 3 for an overview). Indeed, since \( F \) becomes a scalar multiple of the square norm of the moment map, we obtain the following

**Theorem 1.2.** \[M\] Let \( F : \mathbb{P}N_\gamma \mapsto \mathbb{R} \) be defined by \( F([\mu]) = \text{tr}(\text{Ric}^{\gamma}_\mu)^2/||\mu||^4 \).

Then for \( \mu \in N_\gamma \) the following conditions are equivalent:

1. \([\mu]\) is a critical point of \( F \).
2. \( F|_{G_\gamma . [\mu]} \) attains its minimum value at \([\mu]\).
3. \( \text{Ric}^{\gamma}_\mu = cI + D \) for some \( c \in \mathbb{R} \), \( D \in \text{Der}(\mu) \).

Moreover, all the other critical points of \( F \) in the orbit \( G_\gamma . [\mu] \) lie in \( K_\gamma . [\mu] \).

The equivalence between (i) and (iii) in Theorem 1.1 as well as the uniqueness result, follow then almost directly from the above theorem. We note that Theorem 1.2 also gives a variational method to find minimal compatible metrics, by characterizing them as the critical points of a natural curvature functional (see Example 5.2 for an explicit application).

Most of the results obtained in this paper are still valid for general Lie groups, although some considerations have to be carefully taken into account (see Remark 4.6).

1.3. **Examples.** We first prove that a symplectic non-abelian nilpotent Lie group \((N, \omega)\) can never admit a compatible left invariant metric with hermitian Ricci tensor. We also exhibit a curve of pairwise non-isomorphic symplectic structures on the 6-dimensional nilpotent Lie group denoted by \((0, 0, 12, 14 + 23, 24 + 15)\) in \[S\]. Also, a curve of pairwise non-isomorphic non-abelian complex structures on the Iwasawa manifold is given. The initial point of such a curve is the bi-invariant complex structure, and after a finite period of time it becomes a curve of non-abelian complex structures on the group denoted by \((0, 0, 0, 12, 14 + 23)\) in \[S\].
By using results due to I. Dotti and A. Fino [DF1, DF3], we prove that any hypercomplex 8-dimensional nilpotent Lie group admits a minimal compatible metric. We actually show that the moduli space of all hypercomplex 8-dimensional nilpotent Lie groups up to isomorphism is 9-dimensional, and the moduli space of the abelian ones has dimension 5. We finally give a surface of pairwise non-isomorphic non-abelian hypercomplex structures on $\mathfrak{g}_3$, the 8-dimensional Lie algebra obtained as the direct sum of an abelian factor and the 7-dimensional quaternionic Heisenberg Lie algebra.

**Remark 1.3.** More evidence of the existence of minimal compatible metrics is showed in [L3], including all 4-dimensional symplectic structures and another curve in dimension 6, two curves of abelian complex structures on the Iwasawa manifold and several continuous families depending on various parameters of abelian and non-abelian hypercomplex structures in dimension 8. It is also showed in [L3] that if one considers no structure (i.e. $\gamma = 0$), then the ‘moment map’ approach proposed in this paper can be also applied to the study of Einstein solvmanifolds, obtaining many of the uniqueness and structure results proved by J. Heber in [Hb].

By taking advantage again of the interplay with invariant theory, we describe in [L4] the moduli space of all isomorphism classes of geometric structures on nilpotent Lie groups of a given class and dimension admitting a minimal compatible metric, as the disjoint union of semi-algebraic varieties which are homeomorphic to categorical quotients of suitable linear actions of reductive Lie groups. Such special geometric structures can therefore be distinguished by using invariant polynomials.

## 2. Geometric structures and compatible metrics

Let $N$ be a real $n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{n}$. An invariant geometric structure on $N$ is defined by left translation of a tensor $\gamma$ on $\mathfrak{n}$ (or a set of tensors), usually non-degenerate in some way, which satisfies a suitable integrability condition

$$IC(\gamma, \mu) = 0,$$

involving only $\mu$ and $\gamma$. In this paper, we will focus on the following classes of geometric structures: symplectic, complex and hypercomplex, as well as on their respective ‘almost’ versions, that is, when condition 5 is not required. In this way, IC($\gamma, \mu$) can be for instance the differential of a 2-form or the Nijenhuis tensor associated to some $(1, 1)$-tensor. The contact case is somewhat different because the condition is ‘open’, but it becomes an equation of the form 4 when one considers fixed the underlying almost-contact structure. We shall deal with contact and complex symplectic structures in a forthcoming paper.

The pair $(N, \gamma)$ will often be called a class-$\gamma$ nilpotent Lie group, and $N$ will be assumed to be simply connected for simplicity. The group $GL(n) := GL(\mathfrak{n}, \mathbb{R}) = GL(\mathfrak{n})$ of invertible maps of $\mathfrak{n}$ acts on the vector space of tensors on $\mathfrak{n}$ of a given class, preserving the non-degeneracy, and if $\gamma$ is integrable then $\varphi.\gamma$ is so for any $\varphi \in Aut(\mathfrak{n})$, the group of automorphisms of $\mathfrak{n}$. In view of this fact, two class-$\gamma$ nilpotent Lie groups $(N, \gamma)$ and $(N', \gamma')$ are said to be isomorphic if there exists a Lie algebra isomorphism $\varphi : \mathfrak{n} \rightarrow \mathfrak{n}'$ such that $\gamma' = \varphi.\gamma$. Also, given two geometric structures $\gamma, \gamma'$ of the same class on $N$, we say that $\gamma$ degenerates to $\gamma'$ if $\gamma' \in Aut(\mathfrak{n}).\gamma$, the closure of the orbit $Aut(\mathfrak{n}).\gamma$ relative to the natural topology.
A left invariant Riemannian metric on $N$ is said to be *compatible* with $(N, \gamma)$ if the corresponding inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ satisfies an orthogonality condition

$$OC(\gamma, \langle \cdot, \cdot \rangle) = 0,$$

in which only $\langle \cdot, \cdot \rangle$ and $\gamma$ are involved. We denote by $\mathcal{C} = \mathcal{C}(N, \gamma)$ the set of all left invariant metrics on $N$ which are compatible with $(N, \gamma)$. The pair $(\gamma, \langle \cdot, \cdot \rangle)$ will often be referred to as a *class-$\gamma$ metric structure*. It is clear from (6) that for an invariant geometric structure there always exist a compatible metric, since the condition is independent from $\mu$. Moreover, the space $\mathcal{C}$ is usually huge; recall for instance that the group $G_{\gamma} = \{ \varphi \in GL(n) : \varphi.\gamma = \gamma \}$ acts on $\mathcal{C}$, and it is easy to see that actually for any $\langle \cdot, \cdot \rangle \in \mathcal{C}$ we have that

$$\mathcal{C} = G_{\gamma}.\langle \cdot, \cdot \rangle = \{ \langle \varphi^{-1}.\cdot, \varphi^{-1}.\cdot \rangle : \varphi \in G_{\gamma} \}.$$

A natural question takes place:

Given a class-$\gamma$ nilpotent Lie group $(N, \gamma)$, are there canonical or distinguished left invariant Riemannian metrics on $N$ compatible with $\gamma$?

This problem may be (and it is) stated for differentiable manifolds in general, and does not only present some interest in Riemannian geometry; indeed, the existence of a certain nice compatible metric could eventually help to distinguish two geometric structures as well as to find privileged geometric structures on a given manifold.

The aim of this section is to propose two properties which make a compatible metric very distinguished, one is obtained by minimizing a curvature functional and the other as a soliton solution for a natural evolution flow. The Ricci tensor will be used in both approaches. In Appendix we have reviewed some well known properties of left invariant metrics on nilpotent Lie groups, which will be used constantly from now on.

Fix a class-$\gamma$ nilpotent Lie group $(N, \gamma)$. Let $\mathfrak{g}_{\gamma}$ be the Lie algebra of $G_{\gamma}$,

$$\mathfrak{g}_{\gamma} = \{ A \in \mathfrak{gl}(n) : A.\gamma = 0 \}.$$

**Definition 2.1.** For each compatible metric, we consider the orthogonal projection $\text{Ric}^\gamma(\cdot,\cdot)$ of the Ricci operator $\text{Ric}(\cdot,\cdot)$ on $\mathfrak{g}_{\gamma}$, called the *invariant Ricci operator*, and the corresponding *invariant Ricci tensor* given by $\text{ric}^\gamma = \langle \text{Ric}^\gamma \cdot, \cdot \rangle$.

The role of the Ricci tensor has always been crucial in defining privileged (compatible) metrics; we have for example Einstein metrics, extremal Kähler metrics in complex geometry, and more recently metrics with hermitian Ricci tensor (i.e. when the Ricci operator commutes with $J$) and $\phi$-invariant Ricci tensor in symplectic and contact geometry, respectively. These two last notions are equivalent to $\text{ric}^\gamma = 0$ and have been characterized in the compact case by D. Blair, S. Ianus and A. Ledger [BI, BL] as the critical points of two very natural curvature functionals on $\mathcal{C}$: the total scalar curvature functional $S$ and a functional $K$ for which the global minima are precisely Kähler or Sasakian metrics, respectively (see also [B]).

In this light, condition

$$\text{ric}^\gamma(\cdot,\cdot) = 0,$$

involves both the geometric structure and the metric, and seems to be very natural to require to a compatible metric. Nevertheless, if $RI \subset \mathfrak{g}_{\gamma}$, then $\text{tr} \text{Ric}^\gamma(\cdot,\cdot) = $
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sc(⌜·, ·⌝), and so it is forbidden for instance for non-abelian nilpotent Lie groups (where always \(\text{sc}(⌜·, ·⌝) < 0\)) in the complex and hypercomplex cases. We shall prove that this condition is forbidden in the symplectic case as well. We therefore have to consider \(\text{sc}(⌜·, ·⌝) < 0\) as an unreachable goal and try to get as close as possible, for instance, by minimizing \(\|\text{ric}^\gamma⌜·, ·⌝\|^2 = \text{tr}(\text{ric}^\gamma⌜·, ·⌝)^2\). In order to avoid homothetical changes, we must normalize the metrics some way. In the noncompact homogeneous case, the scalar curvature always appears as a very natural choice. We then propose the following

**Definition 2.2.** A left invariant metric \(⌜·, ·⌝\) compatible with a class-\(\gamma\) nilpotent Lie group \((N, \gamma)\) is called *minimal* if

\[
\text{tr}(\text{ric}^\gamma⌜·, ·⌝) = \min \{\text{tr}(\text{ric}^\gamma⌜·, ·⌝') : \langle ·, ·\rangle' \in C(N, \gamma), \text{sc}(⌜·, ·⌝') = \text{sc}(⌜·, ·⌝)\}.
\]

Recall that the existence and uniqueness (up to isometry and scaling) of minimal metrics is far to be clear from the definition. The uniqueness shall be proved in Section 3, but the ‘existence question’ is still nebulous. Minimal metrics are the compatible metrics closest to satisfy the ‘Einstein-like’ condition \(\text{ric}^\gamma⌜·, ·⌝ = c⟨·, ·⟩\), for some \(c \in \mathbb{R}\). Indeed,

\[
\|\text{Ric}^\gamma⟨·, ·⟩ - \frac{\text{tr}(\text{Ric}^\gamma⟨·, ·⟩)}{n} I\|^2 = \text{tr}(\text{Ric}^\gamma⟨·, ·⟩)^2 - \frac{(\text{tr}(\text{Ric}^\gamma⟨·, ·⟩))^2}{n}
\]

and \(\text{tr}(\text{Ric}^\gamma⟨·, ·⟩)\) equals either 0 or \(\text{sc}(⟨·, ·⟩)\), depending on \(\gamma\) contains or not \(\mathbb{R}I\).

We now consider an evolution approach. Motivated by the famous Ricci flow introduced by R. Hamilton [H1], we consider the *invariant Ricci flow* for our left invariant metrics on \(N\), given by the following evolution equation

\[
\frac{d}{dt}⟨·, ·⟩_t = ± \text{ric}^\gamma⟨·, ·⟩_t,
\]

which coincides for example with the anticomplexified Ricci flow studied in [LW] in the symplectic case. The choice of the best sign might depend on the class of structure. This is just an ordinary differential equation and hence the existence for almost all \(t\) and uniqueness of the solution is guaranteed. It follows from (7) that

\[
T⟨·, ·⟩_0 \subseteq C = \{ \alpha \in \text{sym}(n) : A_\alpha, \gamma = 0 \},
\]

and therefore, if \(⟨·, ·⟩_0 \in C\) then the solution \(⟨·, ·⟩_t \in C\) for all \(t\) since \(\text{ric}^\gamma⟨·, ·⟩_t \in T⟨·, ·⟩_0 \), \(C\) (see Appendix 6).

**Remark 2.3.** If we had however the uniqueness of the solution for the flow (9) in the non-compact general case, then we would not need to restrict ourselves to left invariant metrics. Indeed, if \(f\) is an isometry of the initial metric \(⟨·, ·⟩_0\) which also leaves \(\gamma\) invariant, then since \(f^*⟨·, ·⟩_0\) is also a solution and \(f^*⟨·, ·⟩_0 = ⟨·, ·⟩_0\) we would get by uniqueness of the solution that \(f\) is an isometry of all the metrics \(⟨·, ·⟩_t\) as well. Left invariance of the starting metric would be therefore preserved along the flow.

When \(M\) is compact, a normalized Ricci flow is often considered, under which the volume of the solution metric is constant in time. Actually, the normalized equation differs from the Ricci flow only by a change of scale in space and a change of parametrization in time (see [H2, CC]). In our case, where the manifold is non-compact but the metrics are homogeneous, it seems natural to do the same thing but normalizing by the scalar curvature, which is just a single number associated
to the metric. We recall that a left invariant metric \( \langle \cdot, \cdot \rangle \) on a nilpotent Lie group \( N \) has always \( \text{sc}(\langle \cdot, \cdot \rangle) < 0 \), unless \( N \) is abelian (see (28)).

**Proposition 2.4.** The solution to the normalized invariant Ricci flow

\[
\frac{d}{dt} \langle \cdot, \cdot \rangle_t = \pm \text{ric}^\gamma(\langle \cdot, \cdot \rangle)_t \pm \frac{\text{tr}(\text{Ric}^\gamma(\langle \cdot, \cdot \rangle)_t)}{\text{sc}(\langle \cdot, \cdot \rangle)_t} \langle \cdot, \cdot \rangle_t
\]

satisfies \( \text{sc}(\langle \cdot, \cdot \rangle_t) = \text{sc}(\langle \cdot, \cdot \rangle_0) \) for all \( t \). Moreover, this flow differs from the invariant Ricci flow (4) only by a change of scale in space and a change of parametrization in time.

**Proof.** It follows from the formula for the gradient of the scalar curvature functional

\[
\text{sc} : \mathcal{P} \to \mathbb{R}
\]

that if \( \langle \cdot, \cdot \rangle_t \) is a solution of (11), then the function

\[
f(t) = \text{sc}(\langle \cdot, \cdot \rangle_t)
\]

satisfies

\[
f'(t) = g(\langle \cdot, \cdot \rangle_t) \left( \frac{\text{d}}{dt} \langle \cdot, \cdot \rangle_t, -\text{ric}(\langle \cdot, \cdot \rangle)_t \right)
\]

\[
= \mp g(\langle \cdot, \cdot \rangle_t) (\text{ric}^\gamma(\langle \cdot, \cdot \rangle)_t, \text{ric}(\langle \cdot, \cdot \rangle)_t) \pm \frac{\text{tr}(\text{Ric}^\gamma(\langle \cdot, \cdot \rangle)_t)}{\text{sc}(\langle \cdot, \cdot \rangle)_t} \text{tr}(\text{Ric}(\langle \cdot, \cdot \rangle)_t)
\]

\[
= \mp \text{tr}(\text{Ric}^\gamma(\langle \cdot, \cdot \rangle)_t)^2 (1 - \frac{f(t)}{\text{sc}(\langle \cdot, \cdot \rangle)_t}) = 0, \quad \forall t,
\]

and thus \( f(t) \equiv f(0) = \text{sc}(\langle \cdot, \cdot \rangle_0) \). The last assertion follows as in (12) in a completely analogous way.

The fixed points of this normalized flow (11) are those metrics satisfying \( \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) \in \mathbb{R} I \), and so in particular, if \( g_\gamma \subset \mathfrak{sl}(n) \), then this is equivalent to \( \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) = 0 \). Indeed, if \( \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) = \mp \frac{\text{tr}(\text{Ric}^\gamma(\langle \cdot, \cdot \rangle)_0)^2 I}{\text{sc}(\langle \cdot, \cdot \rangle_0)} \) then \( \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) = 0 \) since \( \text{tr} \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) = 0 \). We should also note that for the flow (11), \( \frac{d}{dt} \langle \cdot, \cdot \rangle_t \in T(\langle \cdot, \cdot \rangle)_t \mathbb{R} I \) for all \( t \), which implies that the solution \( \langle \cdot, \cdot \rangle_t \) stays in the set of all scalar multiples of compatible metrics. Recall that if \( \mathbb{R} I \subset \mathfrak{g}_\gamma \), then the solution stays anyway in \( C \).

In these evolution approaches always appear naturally the soliton metrics, which are not fixed points of the flow but are close to, and they play an important role in the study of singularities (see the surveys (12) [CC] for further information). The idea is that if one is trying to improve a metric via an evolution equation, then those metrics for which the solution remains isometric to the initial point may be certainly considered as very distinguished.

**Definition 2.5.** A metric \( \langle \cdot, \cdot \rangle \) compatible with \( (N, \gamma) \) is called an **invariant Ricci soliton** if the solution \( \langle \cdot, \cdot \rangle_t \) to the normalized invariant Ricci flow (11) with initial metric \( \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle \) is given by \( \varphi^*_t \langle \cdot, \cdot \rangle \), the pullback of \( \langle \cdot, \cdot \rangle \) by a one parameter group of diffeomorphisms \( \{ \varphi_t \} \) of \( N \).

We now give a neat characterization of invariant Ricci soliton metrics, which will be very useful in Section 4 to prove the equivalence with the property of being minimal (see Definition 2.2), and to find explicit examples in the subsequent sections.

**Proposition 2.6.** Let \( (N, \gamma) \) be a class-\( \gamma \) nilpotent Lie group. A compatible metric \( \langle \cdot, \cdot \rangle \) is an invariant Ricci soliton if and only if \( \text{Ric}^\gamma(\langle \cdot, \cdot \rangle) = c I + D \) for some \( c \in \mathbb{R} \) and \( D \in \text{Der}(n) \). In such a case, \( c = \frac{\text{tr}(\text{Ric}^\gamma(\langle \cdot, \cdot \rangle))}{\text{sc}(\langle \cdot, \cdot \rangle)} \).
Proof. We first note that the assertion on the value of the number \( c \) follows from \([64]\); in fact,
\[
\text{tr}(\text{Ric}^\gamma(\cdot,\cdot))^2 = \text{tr}(\text{Ric}(\cdot,\cdot) \text{Ric}^\gamma(\cdot,\cdot)) = c \text{tr} \text{Ric}(\cdot,\cdot) + \text{tr}(\text{Ric}(\cdot,\cdot) D) = \text{csc}(\langle \cdot, \cdot \rangle).
\]
Assume that there exists a one-parameter group of diffeomorphisms \( \psi_t \) on \( N \) such that \( (\cdot,\cdot)_t = \psi_t^*(\cdot,\cdot) \) is a solution to the flow \([11]\). By the uniqueness of the solution we have that \( \psi_t^* \) is also \( N \)-invariant for all \( t \) (see Remark \([23]\)). Thus \( \psi_t \) normalizes \( N \) and so it follows from \([Wl, \text{Thm} 2, 4)\] that \( \varphi_t \in \text{Aut}(N) \cdot N \). This implies that there exists a one-parameter group \( \psi_t \) of automorphisms of \( N \) such that \( \varphi_t^* = \psi_t^* \cdot \cdot \cdot \) for all \( t \). Now, if \( \psi_t = e^{-\frac{t}{2} D} \) with \( D \in \text{Der}(n) \) then \( \frac{d}{dt} |_{t=0} \psi_t^*(\cdot,\cdot) = \langle D \cdot, \cdot \rangle \), and using that \( \psi_t^*(\cdot,\cdot) \) is a solution of \([11]\) in \( t = 0 \) we obtain that \( \text{Ric}^\gamma(\cdot,\cdot) = c(\langle \cdot, \cdot \rangle + \langle D \cdot, \cdot \rangle) \) for some \( c \in \mathbb{R} \), or equivalently, \( \text{Ric}^\gamma = cI + D \).

Conversely, if \( \text{Ric}^\gamma = cI + D \) then we will show that the curve \( \langle \cdot, \cdot \rangle_t = e^{-\frac{t}{2} D} \cdot \langle \cdot, \cdot \rangle \) is a solution of the flow \([11]\). For any \( t \), it follows from \( \frac{d}{dt} |_{t=0} \text{Ric}^\gamma(\cdot,\cdot) = \frac{t}{2} \text{Ric}^\gamma(\cdot,\cdot) - \frac{t}{2} cI \in g, \gamma + \mathbb{R} I \) that
\[
\gamma = b(t)e^{-\frac{t}{2} D} \cdot \gamma.
\]
for some \( b(t) \in \mathbb{R} \). This implies that
\[
\text{Ric}^\gamma(\cdot,\cdot)_t = e^{-\frac{t}{2} D} \text{Ric}^\gamma(\cdot,\cdot) e^{\frac{t}{2} D} = e^{-\frac{t}{2} D} (cI + D) e^{\frac{t}{2} D} = cI + D
\]
for all \( t \). Therefore
\[
\frac{d}{dt} |_{0} \langle \cdot, \cdot \rangle_t = \langle D \cdot, \cdot \rangle_t = \langle (\text{Ric}^\gamma(\cdot,\cdot)_t - cI) \cdot, \cdot \rangle_t
\]
\[
= \text{Ric}^\gamma(\cdot,\cdot)_t, -c(\cdot,\cdot)_t = \text{Ric}^\gamma(\cdot,\cdot)_t + \frac{\text{tr}(\text{Ric}^\gamma(\cdot,\cdot)_t)}{\text{csc}(\langle \cdot, \cdot \rangle_0)} \langle \cdot, \cdot \rangle_t,
\]
as was to be shown. \( \square \)

Recall that the condition in the above proposition can be replaced by
\[
\text{Ric}^\gamma(\cdot,\cdot) - \frac{\text{tr}(\text{Ric}^\gamma(\cdot,\cdot))^2}{\text{csc}(\langle \cdot, \cdot \rangle)} I \in \text{Der}(n),
\]
which gives a computable method to check whether a metric is an invariant Ricci soliton or not.

3. Real geometric invariant theory and the moment map

In this section, we overview some results from (geometric) invariant theory over the real numbers. We refer to \([RS]\) for a detailed exposition. These will be our tools to study metrics compatible with geometric structures on nilpotent Lie groups.

Let \( G \) be a real reductive Lie group acting on a real vector space \( V \) (see \([RS]\) for a precise definition of the situation). Let \( \mathfrak{g} \) denote the Lie algebra of \( G \) with Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{k} \) is the Lie algebra of a maximal compact subgroup \( K \) of \( G \). Endow \( V \) with a fixed from now on inner product \( \langle \cdot, \cdot \rangle \) such that \( \mathfrak{k} \) and \( \mathfrak{p} \) act by skew-symmetric and symmetric transformations, respectively. Let \( \mathcal{M} = \mathcal{M}(G,V) \) denote the set of minimal vectors, that is
\[
\mathcal{M} = \{ v \in V : ||v|| \leq ||g.v|| \quad \forall g \in G \}.
\]
For each \( v \in V \) define
\[
\rho_v : G \mapsto \mathbb{R}, \quad \rho_v(g) = ||g.v||^2 = \langle g.v, g.v \rangle.
\]
In [RS], R. Richardson and P. Slodowy showed that the nice interplay between closed orbits and minimal vectors found by G. Kempf and L. Ness for actions of complex reductive algebraic groups, is still valid in the real situation.

**Theorem 3.1.** [RS] Let $V$ be a real representation of a real reductive Lie group $G$, and let $v \in V$.

(i) $v \in \mathcal{M}$ if and only if $\rho_v$ has a critical point at $e \in G$.

(ii) If $v \in \mathcal{M}$ then $G.v \cap \mathcal{M} = K.v$.

(iii) The orbit $G.v$ is closed if and only if $G.v$ meets $\mathcal{M}$.

Let $(d\rho_v)_e : \mathfrak{g} \rightarrow \mathbb{R}$ denote the differential of $\rho_v$ at the identity $e$ of $G$. It follows from the $K$-invariance of $\langle \cdot, \cdot \rangle$ that $(d\rho_v)_e$ vanishes on $\mathfrak{k}$, and so we can assume that $(d\rho_v)_e \in \mathfrak{p}^*$, the vector space of real-valued functionals on $\mathfrak{p}$. We therefore may define a function called the moment map for the action of $G$ on $V$ by

$$m : V \rightarrow \mathfrak{p}, \quad \langle m(v), A \rangle_p = (d\rho_v)_e(A),$$

where $\langle \cdot, \cdot \rangle_p$ is an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}$. Since $m(tv) = t^2m(v)$ for all $t \in \mathbb{R}$, we also may consider the moment map

$$m : \mathbb{P}V \rightarrow \mathfrak{p}, \quad m(x) = \frac{m(v)}{|v|^2}, \quad 0 \neq v \in V, \ x = [v],$$

where $\mathbb{P}V$ is the projective space of lines in $V$. If $\pi : V \setminus \{0\} \rightarrow \mathbb{P}V$ denotes the usual projection map, then $\pi(v) = x$. In the complex case, under the natural identifications $\mathfrak{p} = \mathfrak{p}^* = (1\mathfrak{t})^* = \mathfrak{t}^*$, the function $m$ is precisely the moment map from symplectic geometry, corresponding to the Hamiltonian action of $K$ on the symplectic manifold $\mathbb{P}V$ (see for instance the survey [K2] or [MFK, Chapter 8] for further information).

Consider the functional square norm of the moment map

$$F : V \rightarrow \mathbb{R}, \quad F(v) = ||m(v)||^2 = \langle m(v), m(v) \rangle_p,$$

which is easily seen to be a 4-degree homogeneous polynomial. Recall that $\mathcal{M}$ coincides with the set of zeros of $F$. It then follows from Theorem 3.1 parts (i) and (iii), that an orbit $G.v$ is closed if and only if $F(w) = 0$ for some $w \in G.v$, and in that case, the set of zeros of $F|_{G.v}$ coincides with $K.v$. A natural question arises: what is the role played by the remaining critical points of $F : \mathbb{P}V \rightarrow \mathbb{R}$ (i.e. those for which $F(x) > 0$) in the study of the $G$-orbit space of the action of $G$ on $V$, as well as on other real $G$-varieties contained in $V$?. This was studied independently by F. Kirwan [K1] and L. Ness [N], and it is shown in the complex case that the non-minimal critical points share some of the nice properties of minimal vectors stated in Theorem 3.1. In the real case, which is actually the one we need to apply in this paper, the analogous of some of these results have been proved by A. Marian [M].

**Theorem 3.2.** [M] Let $V$ be a real representation of a real reductive Lie group $G$, $m : \mathbb{P}V \rightarrow \mathfrak{p}$ the moment map and $F = ||m||^2 : \mathbb{P}V \rightarrow \mathbb{R}$.

(i) If $x \in \mathbb{P}V$ is a critical point of $F$ then the functional $F|_{G.x}$ attains its minimum value at $x$.

(ii) If nonempty, the critical set of $F|_{G.x}$ consists of a unique $K$-orbit.

We endow $\mathbb{P}V$ with the Fubini-Study metric defined by $\langle \cdot, \cdot \rangle$ and denote by $x \mapsto A_x$ the vector field on $\mathbb{P}V$ defined by $A \in \mathfrak{g}$ via the action of $G$ on $\mathbb{P}V$, that is, $A_x = \frac{\partial}{\partial t}|_0 \exp(tA).x$. 
Lemma 3.3. The gradient of the functional $F = \| m \|_2: \mathbb{P}V \mapsto \mathbb{R}$ is given by
\[
\text{grad}(F)_x = 4m(x)_x, \quad x \in \mathbb{P}V,
\]
and therefore $x$ is a critical point of $F$ if and only if $m(x)_x = 0$, if and only if $\exp tm(x)$ fixes $x$.

We now develop some examples which are far to be the natural ones, but they are those ones will be considered in this paper to study left invariant structures on nilpotent Lie groups.

Example 3.4. Let $\mathfrak{n}$ be an $n$-dimensional real vector space and $V = \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}$ the vector space of all skew-symmetric bilinear maps from $\mathfrak{n} \times \mathfrak{n}$ to $\mathfrak{n}$. There is a natural action of $\text{GL}(n) := GL(n, \mathbb{R})$ on $V$ given by
\[
g \cdot (X,Y) = g^{-1}X, g^{-1}Y, \quad X, Y \in \mathfrak{n}, \quad g \in \text{GL}(n), \quad \mu \in V.
\]
Any inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ defines an $O(n)$-invariant inner product on $V$, denoted also by $\langle \cdot, \cdot \rangle$, as follows:
\[
\langle \mu, \lambda \rangle = \sum_{ijk} \langle \mu(X_i, X_j), X_k \rangle \langle \lambda(X_i, X_j), X_k \rangle,
\]
where $\{X_1, ..., X_n\}$ is any orthonormal basis of $\mathfrak{n}$. A Cartan decomposition for the Lie algebra of $\text{GL}(n)$ is given by $\mathfrak{gl}(n) = \mathfrak{so}(n) \oplus \mathfrak{sym}(n)$, that is, in skew-symmetric and symmetric transformations respectively, and we consider the following $\text{Ad}(O(n))$-invariant inner product on $\mathfrak{p} := \mathfrak{sym}(n)$,
\[
\langle A, B \rangle_\mathfrak{p} = \text{tr} AB, \quad A, B \in \mathfrak{p}.
\]
The action of $\mathfrak{gl}(n)$ on $V$ obtained by differentiation of (15) is given by
\[
\mu(A, \cdot) := A \mu(\cdot, \cdot) - \mu(\cdot, A), \quad A \in \mathfrak{gl}(n), \quad \mu \in V.
\]
If $\mu \in V$ satisfies the Jacobi condition, then $\delta_\mu : \mathfrak{gl}(n) \mapsto V$ coincides with the cohomology coboundary operator of the Lie algebra $(\mathfrak{n}, \mu)$ from level 1 to 2, relative to cohomology with values in the adjoint representation. Recall that $\text{Ker} \delta_\mu = \text{Der}(\mu)$, the Lie algebra of derivations of the algebra $\mu$. Let $A^t$ denote the transpose relative to $\langle \cdot, \cdot \rangle$ of a linear transformation $A : \mathfrak{n} \mapsto \mathfrak{n}$ and consider the adjoint map $\text{ad}_\mu X : \mathfrak{n} \mapsto \mathfrak{n}$ (or left multiplication) defined by $\text{ad}_\mu X(Y) = \mu(X,Y)$.

Proposition 3.5. The moment map $m : V \mapsto \mathfrak{p}$ for the action (16) of $\text{GL}(n)$ on $V = \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}$ is given by
\[
m(\mu) = -4 \sum_i (\text{ad}_\mu X_i)^t \text{ad}_\mu X_i + 2 \sum_i \text{ad}_\mu X_i (\text{ad}_\mu X_i)^t,
\]
where $\{X_1, ..., X_n\}$ is any orthonormal basis of $\mathfrak{n}$, and it is a simple calculation to see that
\[
\langle m(\mu)X, Y \rangle = -4 \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(Y_i, X_j), X \rangle + 2 \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle, \quad \forall X, Y \in \mathfrak{n}.
\]
Proof. For any $A \in \mathfrak{p}$ we have that

$$(d \rho_p)_I(A) = \left. \frac{d}{dt} \right|_0 \langle e^{tA} \mu, e^{tA} \mu \rangle = -2\langle \mu, \delta_p(A) \rangle$$

$$= -2 \sum_{p_{ij}} \langle \mu(X_p, X_i), X_j \rangle \langle \delta_p(A)(X_p, X_i), X_j \rangle$$

$$= -2 \left( \sum_{p_{ij}} \langle \mu(X_p, X_i), X_j \rangle \langle \mu(A X_p, X_i), X_j \rangle + \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_p, A X_i), X_j \rangle \right)$$

$$- \langle \mu(X_p, X_i), X_j \rangle \langle A \mu(X_p, X_i), X_j \rangle \right)$$

$$= -2 \left( \sum_{p_{ijr}} \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_r, X_i), X_j \rangle \langle A X_p, X_r \rangle + \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_p, X_r), X_j \rangle \langle A X_i, X_r \rangle \right)$$

$$- \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_p, X_i), X_r \rangle \langle A X_j, X_r \rangle \right)$$

By interchanging the indexes $p$ and $i$ in the second line, and $p$ and $j$ in the third one, we get

$$(d \rho_p)_I(A) = -4 \sum_{p_{ijr}} \langle \mu(X_p, X_i), X_j \rangle \langle \mu(X_r, X_i), X_j \rangle \langle A X_p, X_r \rangle$$

$$+ 2 \sum_{p_{ijr}} \langle \mu(X_i, X_j), X_p \rangle \langle \mu(X_i, X_j), X_r \rangle \langle A X_p, X_r \rangle.$$

If we call $M$ the right hand side of (19), then we obtain from (20) that

$$(d \rho_p)_I(A) = \sum_{pr} \langle MX_p, X_r \rangle \langle A X_p, X_r \rangle = \text{tr} MA = \langle M, A \rangle_p,$$

which implies that $m(\mu) = M$ (see (21)).

**Example 3.6.** We keep the notation as in Example 3.4. Let $\gamma$ be a tensor on $\mathfrak{n}$ and let $G_\gamma \subset GL(n)$ denote the subgroup leaving $\gamma$ invariant, with Lie algebra $\mathfrak{g}_\gamma$. The group $G_\gamma$ is reductive and $K_\gamma = G_\gamma \cap O(n)$ is a maximal compact subgroup of $G_\gamma$, whose Lie algebra will be denoted by $\mathfrak{k}_\gamma$. A Cartan decomposition is given by

$$\mathfrak{g}_\gamma = \mathfrak{k}_\gamma \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \mathfrak{p} \cap \mathfrak{g}_\gamma.$$

If $p : \mathfrak{p} \mapsto \mathfrak{p}_\gamma$ is the orthogonal projection relative to $\langle \cdot, \cdot \rangle_p$, then it is easy to see that

$$m_\gamma : V \mapsto \mathfrak{p}_\gamma, \quad m_\gamma = p \circ m,$$

is precisely the moment map for the action of $G_\gamma$ on $V$.

In the cases considered in detail in this paper we will have $(G_\gamma, K_\gamma)$ equal to $(\text{Sp}(\mathfrak{n}, \mathbb{R}), U(\mathfrak{n}))$ (symplectic), $(GL(n, \mathbb{C}), U(\mathfrak{n}))$ (complex), $(GL(n, \mathbb{H}), Sp(\mathfrak{n}))$ (hypercomplex) and $(GL(n, \mathbb{R}), O(n))$ ($\gamma = 0$).
4. Variety of compatible metrics

Let us consider as a parameter space for the set of all real nilpotent Lie algebras of a given dimension \( n \), the set \( \mathcal{N} \) of all nilpotent Lie brackets on a fixed \( n \)-dimensional real vector space \( \mathfrak{n} \). If

\[
V = \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n} = \{ \mu : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{n} : \mu \text{ skew-symmetric bilinear map} \},
\]

then

\[
\mathcal{N} = \{ \mu \in V : \mu \text{ satisfies Jacobi and is nilpotent} \}
\]
is an algebraic subset of \( V \). Indeed, the Jacobi identity and the nilpotency condition are both determined by zeroes of polynomials.

We fix a tensor \( \gamma \) on \( \mathfrak{n} \) (or a set of tensors), and let \( G_\gamma \) denote the subgroup of \( GL(n) \) preserving \( \gamma \). These groups act naturally on \( V \) by and leave \( \mathcal{N} \) invariant. Consider the subset \( \mathcal{N}_\gamma \subset \mathcal{N} \) given by

\[
\mathcal{N}_\gamma = \{ \mu \in \mathcal{N} : IC(\gamma, \mu) = 0 \},
\]

that is, those nilpotent Lie brackets for which \( \gamma \) is integrable (see \( \text{[15]} \)). \( \mathcal{N}_\gamma \) is also an algebraic variety since \( IC(\gamma, \mu) \) is always linear on \( \mu \). Recall that

\[
W_\gamma = \{ \mu \in V : IC(\gamma, \mu) = 0 \}
\]
is a \( G_\gamma \)-invariant linear subspace of \( V \), and \( \mathcal{N}_\gamma = \mathcal{N} \cap W_\gamma \).

For each \( \mu \in \mathcal{N}_\gamma \), let \( N_\mu \) denote the simply connected nilpotent Lie group with Lie algebra \( (\mathfrak{n}, \mu) \). We now consider an identification of each point of \( \mathcal{N}_\gamma \) with a compatible metric. Fix an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{n} \) compatible with \( \gamma \), that is, such that \( \text{[3]} \) holds. We identify each \( \mu \in \mathcal{N}_\gamma \) with a class-\( \gamma \) metric structure

\[
(21) \quad \mu \leftrightarrow (N_\mu, \gamma, \langle \cdot, \cdot \rangle),
\]

where all the structures are defined by left invariant translation. Therefore, each \( \mu \in \mathcal{N}_\gamma \) can be viewed in this way as a metric compatible with the class-\( \gamma \) nilpotent Lie group \( (N_\mu, \gamma) \), and two metrics \( \mu, \lambda \) are compatible with the same geometric structure if and only if they live in the same \( G_\gamma \)-orbit. Indeed, the action of \( G_\gamma \) on \( \mathcal{N}_\gamma \) has the following interpretation: each \( \varphi \in G_\gamma \) determines a Riemannian isometry preserving the geometric structure

\[
(N_{\varphi \cdot \mu}, \gamma, \langle \cdot, \cdot \rangle) \mapsto (N_\mu, \gamma, \langle \varphi \cdot \cdot, \varphi \cdot \cdot \rangle)
\]

by exponentiating the Lie algebra isomorphism \( \varphi^{-1} : (\mathfrak{n}, \varphi, \mu) \mapsto (\mathfrak{n}, \mu) \). We then have the identification \( G_\gamma \cdot \mu = \mathcal{C}(N_\mu, \gamma) \), and more in general the following

**Proposition 4.1.** Every class-\( \gamma \) metric structure \( (N', \gamma', \langle \cdot, \cdot \rangle') \) on a nilpotent Lie group \( N' = (N', \gamma', \langle \cdot, \cdot \rangle') \) on a nilpotent Lie group \( N' \) of dimension \( n \) is isometric-isomorphic to a \( \mu \in N_\gamma \).

**Proof.** We can assume that the Lie algebra of \( N' \) is \( (\mathfrak{n}, \lambda) \) for some \( \lambda \in \mathcal{N} \). There exist \( \varphi \in GL(n) \) and \( \psi \in O(n, \langle \cdot, \cdot \rangle) \) such that \( \varphi \cdot \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \) and \( \psi(\varphi \cdot \cdot) = \varphi \cdot \cdot \). Thus the Lie algebra isomorphism \( \varphi \psi : \mathfrak{n}(\lambda) \mapsto (\mathfrak{n}, \mu) \), where \( \mu = \psi \varphi \cdot \lambda \), satisfies \( \psi \varphi \cdot \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \) and \( \psi \varphi \cdot \cdot = \varphi \cdot \cdot \) and so it defines an isometry

\[
(N', \gamma', \langle \cdot, \cdot \rangle') \mapsto (N_\mu, \gamma, \langle \cdot, \cdot \rangle)
\]

which is also an isomorphism between the class-\( \gamma \) nilpotent Lie groups \( (N', \gamma', \langle \cdot, \cdot \rangle') \) and \( (N, \gamma) \), concluding the proof. \( \square \)
According to the above proposition and identification (21), the orbit $G_\gamma . \mu$ parameterizes all the left invariant metrics which are compatible with $(N_\mu, \gamma)$ and hence we may view $N_\gamma$ as the space of all class-\(\gamma\) metric structures on nilpotent Lie groups of dimension $n$. Since two metrics $\mu, \lambda \in N_\gamma$ are isometric if and only if they live in the same $K_\gamma$-orbit, where $K_\gamma = G_\gamma \cap O(n, \langle \cdot, \cdot \rangle)$ (see Appendix B), we have that $N_\gamma/K_\gamma$ parameterizes class-\(\gamma\) metric nilpotent Lie groups of dimension $n$ up to isometry and $G_\gamma . \mu/K_\gamma$ do the same for all the compatible metrics on $(N_\mu, \gamma)$.

We now recall a crucial fact which is the link between the study of left invariant compatible metrics for geometric structures on nilpotent Lie groups and the results from real geometric invariant theory exposed in Section 3. The interplay is based on the identification given in (21), and it will be used in the proofs of the remaining results of this section. The proof of the following proposition follows just from a simple comparison between formulas (28) and (20).

**Proposition 4.2.** Let $m : V \mapsto p$ and $m_\gamma : V \mapsto p_\gamma$ be the moment maps for the actions of $GL(n)$ and $G_\gamma$ on $V = \Lambda^2 n^* \otimes n$, respectively (see Examples 5.4 and 5.6), where $p$ is the space of symmetric maps of $(n, \langle \cdot, \cdot \rangle)$ and $p_\gamma$ the subspace of those leaving $\gamma$ invariant.

(i) For each $\mu \in N_\gamma \subset V$,

$$m(\mu) = 8 \operatorname{Ric}_\mu,$$

where $\operatorname{Ric}_\mu$ is the Ricci operator of the Riemannian manifold $(N_\mu, \langle \cdot, \cdot \rangle)$.

(ii) For each $\mu \in N_\gamma \subset V$,

$$m_\gamma(\mu) = 8 \operatorname{Ric}^\gamma_\mu,$$

where $\operatorname{Ric}^\gamma_\mu$ is the invariant Ricci operator of $(N_\mu, \gamma, \langle \cdot, \cdot \rangle)$, that is, the orthogonal projection of the Ricci operator $\operatorname{Ric}_\mu$ on $p_\gamma$.

Let us now go back to our search for the best compatible metric. The identification (21) allows us to view each point of the variety $N_\gamma$ as a class-\(\gamma\) metric structure on a nilpotent Lie group of dimension $n$. In this light, it is natural to consider the functional $F : N_\gamma \mapsto \mathbb{R}$ given by $F(\mu) = \operatorname{tr}(\operatorname{Ric}_\mu)^2$, which in some sense measures how far is the metric $\mu$ from having $\operatorname{Ric}_\mu = 0$, which is the goal proposed in Section 4 (see (28)). The critical points of $F$ may be therefore considered compatible metrics of particular significance. However, we should consider some normalization since $F(t\mu) = t^4 F(\mu)$ for all $t \in \mathbb{R}$.

For any $\mu \in N_\gamma$ we have that $\operatorname{sc}(\mu) = -\frac{1}{4} ||\mu||^2$ (see 28), which says that normalizing by scalar curvature and by the spheres of $V$ is equivalent:

$$\{ \mu \in N_\gamma : \operatorname{sc}(\mu) = s \} = \{ \mu \in N_\gamma : ||\mu||^2 = -4s \}, \quad \forall s < 0.$$

The critical points of $F : PV \mapsto \mathbb{R}$, $F([\mu]) = \operatorname{tr}(\operatorname{Ric}_\mu)^2 / ||\mu||^4$, which lie in $\mathbb{P}N_\gamma = \pi(N_\gamma)$ appears then as very natural candidates, since it is like we are restricting $F$ to the subset of all class-\(\gamma\) metric structures having a given scalar curvature.

It follows from Proposition 4.2 (ii), that

$$F([\mu]) = \frac{1}{64} ||m_\gamma([\mu])||^2,$$

where $m_\gamma : PV \mapsto p_\gamma$ is the moment map for the action of $G_\gamma$ on $PV$. We then obtain from Lemma 3.3 and 18 that

$$\operatorname{grad}(F)|_{[\mu]|} = -\frac{1}{16} \pi^* \delta_\mu(\operatorname{Ric}^\gamma_\mu), \quad ||\mu|| = 1,$$
where $\pi^* : V \mapsto T_{[\mu]} \mathbb{P}V$ denotes the derivative of the projection map $\pi : V \mapsto \mathbb{P}V$.

This shows that $[\mu] \in \mathbb{P}V$ is a critical point of $F$ if and only if $\text{Ric}_\gamma^\mu = cI + D$ for some $c \in \mathbb{R}$ and $D \in \text{Der}(\mu) (= \text{Ker} \delta_\mu)$. By applying Theorem 3.2 to our situation we obtain the main result of this paper.

**Theorem 4.3.** Let $F : \mathbb{P}V \mapsto \mathbb{R}$ be defined by $F([\mu]) = \text{tr}(\text{Ric}_\gamma^\mu)^2/||\mu||^4$. Then for $\mu \in V$ the following conditions are equivalent:

(i) $[\mu]$ is a critical point of $F$.
(ii) $F|_{G,\gamma}^\mu$ attains its minimum value at $[\mu]$.
(iii) $\text{Ric}_\gamma^\mu = cI + D$ for some $c \in \mathbb{R}$, $D \in \text{Der}(\mu)$.

Moreover, all the other critical points of $F$ in the orbit $G,\gamma [\mu]$ lie in $K,\gamma [\mu]$.

We now rewrite the above result in geometric terms, by using the identification (21), Proposition 2.6 and Definition 2.2.

**Theorem 4.4.** Let $(N,\gamma)$ be a nilpotent Lie group endowed with an invariant geometric structure $\gamma$. Then the following conditions on a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ which is compatible with $(N,\gamma)$ are equivalent:

(i) $\langle \cdot, \cdot \rangle$ is minimal.
(ii) $\langle \cdot, \cdot \rangle$ is an invariant Ricci soliton.
(iii) $\text{Ric}_\gamma^\langle \cdot, \cdot \rangle = cI + D$ for some $c \in \mathbb{R}$, $D \in \text{Der}(n)$.

Moreover, there is at most one compatible left invariant metric on $(N,\gamma)$ up to isometry (and scaling) satisfying any of the above conditions.

Recall that the proof of this theorem does not use the integrability of $\gamma$, and so it is valid for the ‘almost’ versions as well.

We also note that part (i) of Theorem 4.3 makes possibly the study of minimal compatible metrics by a variational method. Indeed, the projective algebraic variety $\mathbb{P}N,\gamma$ may be viewed as the space of all class-$\gamma$ metric structures on $n$-dimensional nilpotent Lie groups with a given scalar curvature, and those which are minimal are precisely the critical points of $F : \mathbb{P}N,\gamma \mapsto \mathbb{R}$. This variational approach will be used quite often in the search for explicit examples in the following section and in [L3].

The theorems above propose then as privileged these compatible metrics called minimal, which have a neat characterization (see (iii)), are critical points of a natural curvature functional (square norm of Ricci), minimize such a functional when restricted to the compatible metrics for a given geometric structure, and are solitons for a natural evolution flow. Moreover, the uniqueness up to isometry of such special metrics holds. But a weakness of this approach is the existence problem. How special are the symplectic or (almost-) complex structures admitting a minimal metric?. So far, we know how to deal with this ‘existence question’ only by giving several examples, which is the goal of Section 5 and [L3]. The only obstruction we have found is in the case $\mathbb{R}I \not\subset \mathfrak{g}_\gamma$, namely when $\text{Der}(n)$ is nilpotent. These Lie algebras are called characteristically nilpotent and have been extensively studied in the last years. Examples of characteristically nilpotent Lie algebra admitting a symplectic structure have been recently given in [Bu], and these are the only counterexamples we know to the existence problem.

**Corollary 4.5.** Let $\gamma,\gamma'$ be two geometric structures on a nilpotent Lie group $N$, and assume that they admit minimal compatible metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$, respectively.
Then $\gamma$ is isomorphic to $\gamma'$ if and only if there exists $\varphi \in \text{Aut}(n)$ and $c > 0$ such that $\gamma' = \varphi \gamma$ and

$$\langle \varphi X, \varphi Y \rangle' = c \langle X, Y \rangle \quad \forall X, Y \in n.$$ 

In particular, if $\gamma$ and $\gamma'$ are isomorphic then their respective minimal compatible metrics are necessarily isometric up to scaling (recall that $c = 1$ when $\text{sc}(\langle \cdot, \cdot \rangle) = \text{sc}(\langle \cdot, \cdot \rangle')$).

We have here a very useful tool to distinguish two geometric structures. Indeed, the corollary allows us to do it by looking at their respective minimal compatible metrics, that is, with Riemannian data. This is a remarkable advantage since we suddenly have a great deal of invariants. This method will be used in the next section to find explicit continuous families of pairwise non-isomorphic geometric structures in low dimensions, mainly by using only one Riemannian invariant: the eigenvalues of the Ricci operator.

**Remark 4.6.** The Ricci curvature operator of a left invariant metric $\langle \cdot, \cdot \rangle$ on a Lie group $G$ is given by

$$\text{Ric}_{\langle \cdot, \cdot \rangle} = R_{\langle \cdot, \cdot \rangle} - \frac{1}{2} B_{\langle \cdot, \cdot \rangle} - D_{\langle \cdot, \cdot \rangle},$$

where $R_{\langle \cdot, \cdot \rangle}$ is defined by (28), $B_{\langle \cdot, \cdot \rangle}$ is the Killing form of the Lie algebra $g$ of $G$ in terms of $\langle \cdot, \cdot \rangle$, $D_{\langle \cdot, \cdot \rangle}$ is the symmetric part of $\text{ad} \ Z_{\langle \cdot, \cdot \rangle}$ and $Z_{\langle \cdot, \cdot \rangle} \in g$ is defined by $\langle Z_{\langle \cdot, \cdot \rangle}, X \rangle = \text{tr}(\text{ad} X)$ for any $X \in g$. Recall that $Z_{\langle \cdot, \cdot \rangle} = 0$ if and only if $g$ is unimodular, and that $\text{Ric}_{\langle \cdot, \cdot \rangle} = R_{\langle \cdot, \cdot \rangle}$ in the nilpotent case. If we consider the tensor $\mathcal{R}$ instead of the Ricci tensor, and the variety of all Lie algebras $\mathcal{L}$ rather than just the nilpotent ones, to define and state all the notions, flows, identifications and results in Sections 2 and 4, then everything is still valid for Lie groups in general, with the only exception of the first part of Corollary 4.5. We only have to consider the corresponding invariant part $\mathcal{R}^\gamma$ and replace $\text{sc}(\langle \cdot, \cdot \rangle)$ with $\text{tr} \mathcal{R}_{\langle \cdot, \cdot \rangle}$ each time it appears. The only detail to be careful with is that if two $\mu, \lambda \in \mathcal{L}_\gamma$ lie in the same $K_\gamma$-orbit then they are isometric, but the converse might not be true. Recall that the uniqueness result in Theorem 4.4 is nevertheless valid.

The reason why we decided to work only in the nilpotent case is that, at least at first sight, the use of this ‘unnamed’ tensor $\mathcal{R}$ make minimal and soliton metrics, as well as the functionals and evolution flows, into concepts lacking in geometric sense. For instance, we have found ourselves with the unpleasant fact that some Kähler metrics on solvable Lie groups would not be minimal viewed as compatible metrics for the corresponding symplectic structures, in spite of $\text{Ric}_{\langle \cdot, \cdot \rangle} = 0$.

For a compact simple Lie group, $-B$ is minimal for the case $\gamma = 0$, and if $g = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition for a non-compact semi-simple Lie algebra $g$, then it is easy to see that the metric $\langle \cdot, \cdot \rangle$ given by $\langle \mathfrak{k}, \mathfrak{p} \rangle = 0$, $\langle \cdot, \cdot \rangle|_{\mathfrak{k} \times \mathfrak{k}} = -B$ and $\langle \cdot, \cdot \rangle|_{\mathfrak{p} \times \mathfrak{p}} = B$ is minimal as well.

5. **Applications**

5.1. **Symplectic structures.** Let $N$ be a real $2n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$. An invariant *symplectic* structure on $N$ is defined by a 2-form $\omega$ on $\mathfrak{n}$ satisfying

$$\omega(X, \cdot) \equiv 0 \quad \text{if and only if} \quad X = 0 \quad (\text{non-degenerate}),$$
and for all $X, Y, Z \in \mathfrak{n}$

\begin{equation}
\omega(\mu(X, Y), Z) + \omega(\mu(Y, Z), X) + \omega(\mu(Z, X), Y) = 0 \quad \text{(closed, } d\omega = 0)\).
\end{equation}

Fix a symplectic nilpotent Lie group $(N, \omega)$. A left invariant Riemannian metric which is compatible with $(N, \omega)$ is determined by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ such that if $\omega(X, Y) = \langle X, JY \rangle \quad \forall X, Y \in \mathfrak{n}$ then $J^2 = -I$.

For the geometric structure $\gamma = \omega$ we have that

\[ G_\gamma = Sp(n, \mathbb{R}) = \{ g \in GL(2n) : g^t J g = J \}, \quad K_\gamma = U(n), \]

and the Cartan decomposition of $\mathfrak{g}_\gamma = \mathfrak{sp}(n, \mathbb{R}) = \{ A \in gl(2n) : A^t J + JA = 0 \}$ is given by

\[ \mathfrak{sp}(n, \mathbb{R}) = u(n) \oplus \mathfrak{p}_\gamma, \quad \mathfrak{p}_\gamma = \{ A \in \mathfrak{p} : AJ = -JA \}. \]

Thus the invariant Ricci tensor $\text{ric}^\gamma$ coincides with the anti-complexified Ricci tensor (see [LW]) and for any $\langle \cdot, \cdot \rangle \in \mathcal{C}$,

\begin{equation}
\text{Ric}^{\gamma}(\cdot, \cdot) = \text{Ric}^{ac}(\cdot, \cdot) = \frac{1}{2} (\text{Ric}(\cdot, \cdot) + J(\cdot, \cdot) \text{Ric}(\cdot, \cdot) J(\cdot, \cdot)).
\end{equation}

This implies that our ‘goal’ condition $\text{Ric}^{\gamma}(\cdot, \cdot) = 0$ (see [8]) is equivalent to have hermitian Ricci tensor. Also, the evolution flow considered in Section 2 is not other than the anti-complexified Ricci flow.

Concerning the search for the best compatible left invariant metric for a symplectic nilpotent Lie group, our first result is negative.

**Proposition 5.1.** Let $(N, \omega)$ be a symplectic nilpotent Lie group. Then $(N, \omega)$ does not admit any compatible left invariant metric with hermitian Ricci tensor, unless $N$ is abelian.

*Proof.* We first note that since $\mu$ is nilpotent the center $\mathfrak{z}$ of $(\mathfrak{n}, \mu)$ meets non-trivially the derived Lie subalgebra $\mu(\mathfrak{n}, \mathfrak{n})$, unless $\mu = 0$ (i.e. $\mathfrak{n}$ abelian). Assume that $\langle \cdot, \cdot \rangle \in \mathcal{C}(N, \omega)$ has hermitian Ricci tensor and consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mu(\mathfrak{n}, \mathfrak{n})$. If $Z \in \mathfrak{z}$ then $JZ \in \mathfrak{v}$. In fact, it follows from (22) that

\[ \langle \mu(X, Y), JZ \rangle = \omega(\mu(X, Y), Z) = 0 \quad \forall X, Y \in \mathfrak{n}. \]

Now, the above equation, the fact that $\text{Ric}(\cdot, \cdot) J = J \text{Ric}(\cdot, \cdot)$ and the definition of $\text{Ric}(\cdot, \cdot)$ (see [25]) imply that

\[ 0 \leq \langle \text{Ric}(\cdot, \cdot), Z \rangle \leq \frac{1}{4} \sum_{ij} \langle \mu(X_i, X_j), Z \rangle^2 = \langle \text{Ric}(\cdot, \cdot), Z \rangle = 0 \quad \forall Z \in \mathfrak{z}. \]

Thus $\mu(\mathfrak{n}, \mathfrak{n}) \perp \mathfrak{z}$ and so $\mathfrak{n}$ must be abelian by the observation made in the beginning of the proof. \qed

We now review the variational approach developed in Section 4. Fix a non-degenerate 2-form $\omega$ on $\mathfrak{n}$, and let $Sp(n, \mathbb{R})$ denote the subgroup of $GL(2n)$ preserving $\omega$, that is,

\[ Sp(n, \mathbb{R}) = \{ \varphi \in GL(2n) : \omega(\varphi X, \varphi Y) = \omega(X, Y) \quad \forall X, Y \in \mathfrak{n} \}. \]

Consider the algebraic subvariety $\mathcal{N}_s := \mathcal{N}_\gamma \subset \mathcal{N}$ given by

\[ \mathcal{N}_s = \{ \mu \in \mathcal{N} : d_\mu \omega = 0 \}, \]
that is, those nilpotent Lie brackets for which $\omega$ is closed (see (22)). By fixing an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ satisfying that
\[
\omega = \langle \cdot, J \cdot \rangle \quad \text{with} \quad J^2 = -I,
\]
identify each $\mu \in \mathcal{N}_s$ with the almost-Kähler manifold $(N_{\mu}, \omega, \langle \cdot, \cdot \rangle, J)$. The action of $\text{Sp}(n, \mathbb{R})$ on $\mathcal{N}_s$ has the following interpretation: each $\varphi \in \text{Sp}(n, \mathbb{R})$ determines a Riemannian isometry which is also a symplectomorphism
\[
(N_{\varphi, \mu}, \omega, \langle \cdot, \cdot \rangle, J) \mapsto (N_{\mu}, \omega, \langle \varphi \cdot, \varphi \cdot \rangle, \varphi^{-1} J \varphi)
\]
by exponentiating the Lie algebra isomorphism $\varphi^{-1} : (\mathfrak{n}, \varphi, \mu) \mapsto (\mathfrak{n}, \mu)$.

Let $\mathfrak{n}$ be a 6-dimensional vector space with basis $\{X_1, \ldots, X_6\}$ over $\mathbb{R}$, and consider the non-degenerate 2-form
\[
\omega = \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4,
\]
where $\{\alpha_1, \ldots, \alpha_6\}$ is the dual basis of $\{X_i\}$. For the compatible inner product $\langle X_i, X_j \rangle = \delta_{ij}$ we have that $\omega = \langle \cdot, J \cdot \rangle$ for
\[
J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.
\]

In the following example the symplectic structure will always be $\omega$, the almost-complex structure $J$ and the compatible metric $\langle \cdot, \cdot \rangle$. We will vary Lie brackets and use constantly identification (21).

**Example 5.2.** We shall take advantage of the variational nature of Theorem 4.3 to find explicit examples of minimal compatible metrics. Consider for each 6-upla $\{a, \ldots, f\}$ of real numbers the skew-symmetric bilinear form $\mu = \mu(a, b, c, d, e, f) \in V = \Lambda^2(\mathbb{R}^6)^* \otimes \mathbb{R}^6$ defined by
\[
\mu(X_1, X_2) = aX_3, \quad \mu(X_1, X_3) = bX_4, \quad \mu(X_1, X_4) = cX_5, \\
\mu(X_1, X_5) = dX_6, \quad \mu(X_2, X_3) = eX_5, \quad \mu(X_2, X_4) = fX_6.
\]

Our plan is to find first the critical points of $F$ restricted to the set $\{[\mu(a, \ldots, f)] : a, \ldots, f \in \mathbb{R}\}$ and after that to show by using the characterization given in part (iii) of the theorem that they are really critical points of $F : \mathbb{P}V \mapsto \mathbb{R}$. We can see by a simple computation that $\text{Ric}^{ac, \mu}$ is given by the diagonal matrix with entries
\[
\text{Ric}^{ac, \mu} = -\frac{1}{8} \begin{bmatrix} a^2 + b^2 + c^2 + d^2 + f^2 \\ a^2 + c^2 - d^2 + 2e^2 + f^2 \\ -a^2 + 2b^2 - c^2 + e^2 + f^2 \\ a^2 - 2b^2 + c^2 + 2e^2 - f^2 \\ -a^2 - c^2 + d^2 - 2e^2 - f^2 \\ -a^2 - b^2 - c^2 - 2d^2 - f^2 \end{bmatrix},
\]
and hence we are interested in the critical points of
\[
F(\mu) = \text{tr}(\text{Ric}^{ac, \mu})^2 = F(a, \ldots, f)
\]
\[
= \frac{1}{32} \left( (a^2 + b^2 + c^2 + d^2 + f^2)^2 + (a^2 + c^2 - d^2 + 2e^2 + f^2)^2 \\
+ (a^2 - b^2 - c^2 - 2d^2 - f^2)^2 \right)
\]
restricted to any leaf of the form $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \equiv \text{const.}$, which are easily seen to depend of three parameters. We still have to impose the Jacobi and
closeness conditions on these critical points (or equivalently to find the intersection with $N_x$), after which we obtain the following ellipse of symplectic structures:

$$\{\mu_{xy} = \mu(x, 1, x + y, 1, 1, y) : x^2 + y^2 + xy = 1\}.$$ 

It follows from the formula for $\text{Ric}^c \mu$ given above that

$$\text{Ric}^c \mu_{xy} = -\frac{1}{4} \begin{bmatrix} 5 & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 3 & -5 \end{bmatrix} = -\frac{7}{4} \mathbb{I} + \frac{1}{2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix} \in \mathbb{R} \mathbb{I} + \text{Der}(\mu_{xy}),$$

showing definitely that this is a curve of minimal compatible metrics. We furthermore have that the Ricci tensor of the metrics $\mu_{xy}$ is given by

$$\text{Ric}_{\mu_{xy}} = -\frac{1}{4} \begin{bmatrix} 4-y^2 & 2-x-y & 2-x \\ 2-x-y & 2-y^2 & 1-x-y \\ 2-x & 1-x-y & -1-y^2 \end{bmatrix},$$

which clearly shows that they are pairwise non-isometric for $x, y \geq 0$. It then follows from Corollary 5.3 that

$$\{(N_{\mu_{xy}}, \omega) : x^2 + y^2 + xy = 1, x, y \geq 0\}$$

is a curve of pairwise non-isomorphic symplectic nilpotent Lie groups. There are three 6-dimensional nilpotent Lie groups involved, $N_{\mu_{10}}, N_{\mu_{01}},$ and $N_{\mu_{xy}},$ $x, y > 0,$ denoted in Table A.1 by $(0, 0, 12, 13, 14, 23 + 15),$ $(0, 0, 0, 12, 14 - 23, 15 + 34)$ and $(0, 0, 12, 13, 14 + 23, 24 + 15),$ respectively.

5.2. Complex structures. Let $N$ be a real 2$n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n},$ whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{n}.$ An invariant almost-complex structure on $N$ is defined by a map $J : \mathfrak{n} \rightarrow \mathfrak{n}$ satisfying $J^2 = -I.$ If in addition $J$ satisfies the integrability condition

$$(24) \quad \mu(JX, JY) = \mu(X, Y) + J\mu(JX, Y) + J\mu(X, JY), \quad \forall X, Y \in \mathfrak{n},$$

then $J$ is said to be a complex structure. A left invariant Riemannian metric which is compatible with $(N, J),$ also called an almost-hermitian metric, is given by an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ such that

$$\langle JX, JY \rangle = \langle X, Y \rangle \quad \forall X, Y \in \mathfrak{n}.$$ 

As in the symplectic case, condition $\text{Ric}^c(\langle \cdot, \cdot \rangle) = 0$ is forbidden for non-abelian $N$ since $\text{tr} \text{Ric}^c(\langle \cdot, \cdot \rangle) = \text{sc}(\langle \cdot, \cdot \rangle) < 0.$

Let $\{X_1, \ldots, X_4, Z_1, Z_2\}$ be a basis for $\mathfrak{n}$. The complex structure and the compatible metric in the following example will always be defined by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \langle X_i, X_j \rangle = \langle Z_i, Z_j \rangle = \delta_{ij}.$$

Example 5.3. The following curve has been obtained via the variational method provided by Theorem 5.3 by using an approach very similar to that in Example 5.2. Let $\mu_t$ be the curve defined by

$$\mu(X_1, X_3) = -tsZ_2, \quad \mu(X_2, X_3) = szZ_1,$$

$$\mu(X_1, X_4) = sZ_1, \quad \mu(X_2, X_4) = s(2-t)Z_2, \quad s = \sqrt{2 + t^2 + (2-t)^2}, \quad t \in \mathbb{R}.$$ 

It is easy to check that $(N_{\mu_t}, J)$ is a non-abelian complex nilpotent Lie group for all $t \in \mathbb{R}.$ Moreover, $\text{Ric}_{\mu_t} |_{\mathfrak{n}_1}$ is diagonal and hence both $\text{Ric}^c_{\mu_t} |_{\mathfrak{n}_1}$ and $\text{Ric}^c_{\mu_t} |_{\mathfrak{n}_2}$
are scalar multiples of the identity (recall that the invariant Ricci operator is given in this case by $\text{Ric}^\gamma(\cdot,\cdot) = \text{Ric}^\epsilon(\cdot,\cdot) = \frac{1}{4}(\text{Ric}(\cdot,\cdot) - J\text{Ric}(\cdot,\cdot)J)$). Since $\mu$ is two-step nilpotent, this easily implies that $\langle \cdot, \cdot \rangle$ is a minimal compatible metric for all $(N_\mu, J)$. Indeed, if $\text{Ric}\mu|_{_1} = pI$ and $\text{Ric}\mu|_{_2} = qI$ for some $p, q \in \mathbb{R}$, we would have that

$$\text{Ric}_\mu = \begin{bmatrix} pI & qI \\ \end{bmatrix} = (2p - q)I + \begin{bmatrix} (q-p)^I \\ 2(q-p)I \end{bmatrix} \in \mathbb{R}I + \text{Der}(\mu).$$

It follows from

$$\text{Ric}_\mu|_{_2} = \frac{1}{2} \begin{bmatrix} 2s^2 & s^2(t^2 + (2-t)^2) \\ 0 & s^2(t^2 + (2-t)^2) \end{bmatrix},$$

that the hermitian manifolds $\{(N_\mu, J, \langle \cdot, \cdot \rangle) : 1 \leq t < \infty \}$ are pairwise non-isometric since

$$s^2(t^2 + (2-t)^2) - 2s^2 = (t^2 + (2-t)^2)^2 - 4$$

is a strictly increasing non-negative function for $1 \leq t$, which vanishes if and only if $t = 1$. We therefore obtain a curve $\{(N_\mu, J) : 1 \leq t < \infty \}$ of pairwise non-isomorphic non-abelian complex nilpotent Lie groups. A natural question is which are the nilpotent Lie groups involved. We have for all $t$ that

$$j_{\mu_t}(Z_1) = \begin{bmatrix} 0 & 0 \\ -s & 0 \end{bmatrix}, \quad j_{\mu_t}(Z_2) = \begin{bmatrix} -ts & 0 \\ 0 & -(2-t)s \end{bmatrix},$$

(see Appendix and hence $j_{\mu_t}(Z)$ is non-singular if and only if

$$-t(2-t)\langle Z, Z_1 \rangle^2 - \langle Z, Z_2 \rangle^2 \neq 0.$$  

This implies that $\mu_t$ is isomorphic to the complex Heisenberg Lie algebra (i.e. when $j_{\mu_t}(Z)$ is non-singular for any non-zero $Z \in \mathbb{n}_2$) if and only if $1 \leq t < 2$, providing a curve on the Iwasawa manifold. Furthermore, $(N_\mu, J)$ is the bi-invariant complex structure and it can be showed by computing $j_{\mu_t}(Z)^2$ that $(N_\mu, \langle \cdot, \cdot \rangle)$ is not modified H-type for any $1 < t$. We finally note that $\mu_2$ is isomorphic to the group denoted by $(0, 0, 0, 0, 12, 14 + 23)$ in $\mathbb{S}$, and one can easily see by discarding any other possibility that actually $\mu_t \simeq \mu_2$ for all $2 \leq t < \infty$, which gives rise a curve of pairwise non-isomorphic structures on such a group.

5.3. **Hypercomplex structures.** Let $N$ be a real $4n$-dimensional nilpotent Lie group with Lie algebra $\mathfrak{n}$, whose Lie bracket is denoted by $\mu : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{n}$. An invariant hypercomplex structure on $N$ is defined by a triple $\{J_1, J_2, J_3\}$ of complex structures on $\mathfrak{n}$ (see Section 5.2) satisfying the quaternion identities

$$J_i^2 = -I, \quad i = 1, 2, 3, \quad J_1J_2 = J_3 = -J_2J_1.$$  

An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ is said to be compatible with $\{J_1, J_2, J_3\}$, also called an hyper-hermitian metric, if

$$\langle J_iX, J_iY \rangle = \langle X, Y \rangle \quad \forall X, Y \in \mathfrak{n}, \quad i = 1, 2, 3.$$  

Two hypercomplex nilpotent Lie groups $(N, \{J_1, J_2, J_3\})$ and $(N', \{J'_1, J'_2, J'_3\})$ are said to be isomorphic if there exists an isomorphism $\phi : \mathfrak{n}' \rightarrow \mathfrak{n}$ such that

$$\phi J'_i \phi^{-1} = J_i, \quad i = 1, 2, 3.$$  

There are no non-abelian nilpotent Lie groups of dimension 4 admitting an hypercomplex structure. In dimension 8, hypercomplex nilpotent Lie groups have been determined by I. Dotti and A. Fino in [DF1] and [DF3]. They proved the following strong restrictions on an 8-dimensional nilpotent Lie algebra $\mathfrak{n}$ which admits an
hypercomplex structure: \( n \) has to be 2-step nilpotent, \( \dim \mu(n, n) \leq 4 \), there exists a decomposition \( n = n_1 \oplus n_2 \) such that \( \dim n_1 = 4 \), \( n_i \) is \( \{ J_1, J_2, J_3 \} \)-invariant and \( \mu(n, n) \subset n_2 \subset \mathfrak{g} \), where \( \mathfrak{g} \) is the center of \( n \). Thus the Lie bracket of \( n \) is just given by a skew-symmetric bilinear form \( \mu : n_1 \times n_1 \mapsto n_2 \), and those for which a fixed \( \{ J_1, J_2, J_3 \} \) is integrable are also completely described in [DF3] (see also [DF2]) as a 16-dimensional subspace \( W_h \) of the 24-dimensional vector space \( W = \Lambda^2 n_1^* \oplus n_2 \) of all such forms. If we ask in addition abelian (i.e. \( \mu = \mu(J_i, J_i) \)), then we get a subspace \( W_{ah} \) of dimension 12.

What shall be studied here are the isomorphism classes of such structures and the existence of minimal compatible metrics.

Fix basis \( \{ X_1, X_2, X_3, X_4 \} \) and \( \{ Z_1, Z_2, Z_3, Z_4 \} \) of \( n_1 \) and \( n_2 \), respectively. The compatible metric will be \( \langle X_i, X_j \rangle = \langle Z_i, Z_j \rangle = \delta_{ij} \) and the hypercomplex structure will always act on \( n_1 \) by

\[
J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

**Proposition 5.4.** (i) Every hypercomplex 8-dimensional nilpotent Lie group admits a minimal compatible metric.

(ii) Two hypercomplex 8-dimensional nilpotent Lie groups \( (N_1, \{ J_1, J_2, J_3 \}) \) and \( (N_2, \{ J_1, J_2, J_3 \}) \) are isomorphic if and only if \( \mu \) and \( c \mu \) lie in the same \( \text{Sp}(1) \times \text{Sp}(1) \)-orbit for some non-zero \( c \in \mathbb{R} \).

(iii) The moduli space of all 8-dimensional hypercomplex nilpotent Lie groups up to isomorphism is parameterized by

\[
\mathbb{P} W_h / \text{Sp}(1) \times \text{Sp}(1).
\]

The representation \( W_h \) is equivalent to \( (\mathfrak{su}(2) \otimes \mathbb{R}^4) \oplus \mathbb{R}^4 \), where \( \mathfrak{su}(2) \) is the adjoint representation and \( \mathbb{R}^4 \) is the standard representation of \( \text{SU}(2) = \text{Sp}(1) \) viewed as real. Since the isotropy of an element in general position is finite, the dimension of this quotient is \( 15 - 6 = 9 \).

(iv) The moduli space of all 8-dimensional abelian hypercomplex nilpotent Lie groups up to isomorphism is parameterized by

\[
\mathbb{P} W_{ah} / \text{Sp}(1) \times \text{Sp}(1).
\]

The representation \( W_{ah} \) is equivalent to \( \mathfrak{su}(2) \otimes \mathbb{R}^4 \), and since the isotropy of an element in general position is again finite, the dimension of this quotient is \( 11 - 6 = 5 \).

**Proof.** Since the only symmetric transformations of \( n = \mathbb{R}^4 \) commuting with all the \( J_i \)'s are the multiplies of the identity, we obtain that the invariant Ricci operator

\[
\text{Ric}^\gamma_{\langle \cdot , \cdot \rangle} = \frac{1}{4} (\text{Ric}_{\langle \cdot , \cdot \rangle} - J_1 \text{Ric}_{\langle \cdot , \cdot \rangle} J_1 - J_2 \text{Ric}_{\langle \cdot , \cdot \rangle} J_2 - J_3 \text{Ric}_{\langle \cdot , \cdot \rangle} J_3),
\]

satisfies \( \text{Ric}^\gamma_{\langle \cdot , \cdot \rangle} \mid_{n} \in \mathbb{R} I \) for any \( \mu \in W_h \). By arguing as in [260], we obtain that any \( \mu \in W_h \) is minimal, or equivalently, \( \langle \cdot , \cdot \rangle \) is a minimal compatible metric for every \( (N_1, \{ J_1, J_2, J_3 \}) \), \( \mu \in W_h \). This proves part (i).

In this case \( K_\gamma = \text{Sp}(2) \), and for each \( \mu \in W \) we have that \( K_\gamma \mu \cap W = \text{Sp}(1) \times \text{Sp}(1), \mu \). Part (ii) follows then from the uniqueness result in Theorem [42] and part (i). Finally, parts (iii) and (iv) follow from an elementary analysis of \( W_h \) and \( W_{ah} \) as \( \text{Sp}(1) \times \text{Sp}(1) \)-modules. □
We will give now an explicit continuous family of non-abelian hypercomplex structures on \( g_3 \), the 8-dimensional Lie algebra obtained as the direct sum of an abelian factor and the 7-dimensional quaternionic Heisenberg Lie algebra. Such a curve was again obtained via the variational method as in Example 5.2.

**Example 5.5.** Consider the family defined by

\[
\begin{align*}
\mu_{rst}(X_1, X_2) &= rz, \\
\mu_{rst}(X_2, X_3) &= (1-t)z, \\
\mu_{rst}(X_1, X_3) &= sZ, \\
\mu_{rst}(X_2, X_4) &= -(1-s)Z, \\
\mu_{rst}(X_1, X_4) &= tZ, \\
\mu_{rst}(X_3, X_4) &= (1-r)Z,
\end{align*}
\]

which is easily seen to satisfy the integrability condition for all \( J_i \)'s, though is not abelian since \( \mu_{rst}(X_1, X_3) = sZ \neq -(1-s)Z = \mu_{rst}(J_1 X_1, J_1 X_3) \). The Ricci operator on the center is given by

\[
\text{Ric}_{\mu_{rst}} \big|_{n_2} = \frac{1}{2} \begin{bmatrix}
0 & r^2 + (1-r)^2 \\
st^2 + (1-s)^2 & (1-t)^2 \end{bmatrix},
\]

and hence the family

\[
\left\{ (N_{\mu_{rst}}, \{ J_1, J_2, J_3 \}, \langle \cdot, \cdot \rangle) : \frac{1}{2} \leq r \leq s \leq t, \quad r^2 + s^2 + t^2 - r - s - t = -\frac{1}{2} \right\}
\]

is pairwise non-isometric. This gives rise to a surface of pairwise non-isomorphic non-abelian hypercomplex structures on \( g_3 \) (see Corollary [4.5]), since \( j_{\mu_{rst}}(Z) \) is invertible for any non-zero \( Z \in n_2 \) which orthogonal to \( Z_1 \).

6. Appendix

We briefly recall in this appendix some features of Riemannian geometry of left invariant metrics on nilpotent Lie groups.

Consider the vector space \( \text{sym}(n) \) of symmetric real valued bilinear forms on \( n \), and \( P \subset \text{sym}(n) \) the open convex cone of the positive definite ones (inner products), which is naturally identified with the space of all left invariant Riemannian metrics on \( N \). Every \( \langle \cdot, \cdot \rangle \in P \) induces a natural inner product \( g_{\langle \cdot, \cdot \rangle} \) on \( \text{sym}(n) \) given by \( g_{\langle \cdot, \cdot \rangle}(\alpha, \beta) = \text{tr} A_\alpha A_\beta \) for all \( \alpha, \beta \in \text{sym}(n) \), where \( \alpha(X, Y) = g(A_\alpha X, Y) \). We endow \( P \) with the Riemannian metric \( g \) given by \( g_{\langle \cdot, \cdot \rangle} \) on the tangent space \( T_{\langle \cdot, \cdot \rangle} \mathcal{P} = \text{sym}(n) \) for any \( \langle \cdot, \cdot \rangle \in \mathcal{P} \). Thus \( \mathcal{P}, g \) is isometric to the symmetric space \( GL(n)/O(n) \). E. Wilson proved that \( (N, \langle \cdot, \cdot \rangle) \) and \( (N, \langle \cdot, \cdot \rangle') \) are isometric if and only if \( \langle \cdot, \cdot \rangle' = \varphi \cdot, \cdot \) for some \( \varphi \in \text{Aut}(n) \) (see the proof of [W] Theorem 3]). Therefore, although the Lie bracket \( \mu \) does not play any role in the definition of a compatible metric, it is crucial in the study of the moduli space of compatible metrics on \( (N, \gamma) \) up to isometry.

The Ricci curvature tensor \( \text{ric}_{\langle \cdot, \cdot \rangle} \) and the Ricci operator \( \text{Ric}_{\langle \cdot, \cdot \rangle} \) of \( (N, \langle \cdot, \cdot \rangle) \) are given by (see [BS 7.39]),

\[
\text{ric}_{\langle \cdot, \cdot \rangle}(X, Y) = \langle \text{Ric}_{\langle \cdot, \cdot \rangle} X, Y \rangle = -\frac{1}{2} \sum_{ij} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle
\]

\[
+ \frac{1}{2} \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle,
\]

for all \( X, Y \in n \), where \( \{X_1, \ldots, X_n\} \) is any orthonormal basis of \( (n, \langle \cdot, \cdot \rangle) \). Notice that always \( \text{sc}(N, \langle \cdot, \cdot \rangle) < 0 \), unless \( N \) is abelian. It is proved in [L] that the gradient
of the scalar curvature functional \( \text{sc} : \mathcal{P} \mapsto \mathbb{R} \) is given by
\[
\text{grad}(\text{sc}) = -\text{ric},
\]
and hence it follows from the properties of \( \mathcal{P} \) described above that
\[
\text{tr Ric} D = 0, \quad \forall \text{ symmetric } D \in \text{Der}(\mathfrak{n}),
\]
where \( \text{Der}(\mathfrak{n}) \) is the Lie algebra of derivations of \( \mathfrak{n} \) (see for instance [L2, (2)] for a proof of this fact).

Assume now that \( \mathfrak{n} \) is 2-step nilpotent, and let \( \langle \cdot, \cdot \rangle \) an inner product on \( \mathfrak{n} \). Consider the orthogonal decomposition \( \mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \), where \( \mathfrak{n}_2 \) is the center of \( \mathfrak{n} \). Thus the Lie bracket of \( \mathfrak{n} \) can be viewed as a skew-symmetric bilinear map \( \mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \mapsto \mathfrak{n}_2 \). For each \( Z \in \mathfrak{n}_2 \) we define \( j_\mu(Z) \) : \( \mathfrak{n}_1 \mapsto \mathfrak{n}_1 \) by
\[
\langle j_\mu(Z)X, Y \rangle = \langle \mu(X, Y), Z \rangle, \quad X, Y \in \mathfrak{n}_1.
\]
\((N, \langle \cdot, \cdot \rangle)\) is said to be a modified H-type Lie group if for any non-zero \( Z \in \mathfrak{n}_2 \)
\[
j_\mu(Z)^2 = c(Z)I \quad \text{for some } c(Z) < 0,
\]
and it is called H-type when \( c(Z) = -\langle Z, Z \rangle \) for all \( Z \in \mathfrak{n}_2 \). These metrics, introduced by A. Kaplan, play a remarkable role in the study of Riemannian geometry on nilpotent and solvable Lie groups (see for instance [BTV] for further information and [L1] for the ‘modified’ case).

If \( \mu' = \varphi.\mu \) for some \( \varphi = (\varphi_1, \varphi_2) \in GL(\mathfrak{n}_1) \times GL(\mathfrak{n}_2) \), then it is easy to see that
\[
j_\mu'(Z) = \varphi_1 j_\mu(\varphi_2^t Z) \varphi_1^t, \quad \forall Z \in \mathfrak{n}_2.
\]

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