WEAK KAM THEORY IN HIGHER-DIMENSIONAL
HOLONOMIC MEASURE FLOWS

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Abstract. We construct a weak KAM theory for parameterized cobordisms and their relaxation, holonomic measures. We find a weak KAM solution in that context, and we show that in many cases it corresponds to an exact form that satisfies a version of the Hamilton-Jacobi equation. Along the way, we give a characterization of minimizable Lagrangians, as well as some abstract weak KAM machinery.

1. Introduction

In this paper we construct a weak KAM theory for Lagrangian action functionals on a relaxation of the space of parameterized cobordisms. Let us briefly explain how this works. Given a manifold $M$ of dimension $d$ and two submanifolds $N_1$ and $N_2$ of dimension $n - 1$, a cobordism joining $N_1$ and $N_2$ is a submanifold $B$ of dimension $n$ whose boundary is exactly the two submanifolds $N_1$ and $N_2$. Let us say that the manifold $B$ can be parameterized by a set of maps $\varphi_i: U_i \subseteq \mathbb{R}^n \to M$, $i = 1, \ldots, k$, so that the derivative $d\varphi_i$ is a map from $U_i$ to $T^n M = TM \oplus \cdots \oplus TM$, the Whitney sum of $n$ copies of $TM$. Then the action of $B$ with respect to the Lagrangian function $L: T^n M \to \mathbb{R}$ is defined to be

$$\sum \int_{U_i} L(d\varphi_i(x)) d\mathcal{L}_{U_i},$$

where $\mathcal{L}_{U}$ denotes the Lebesgue measure on $U$. This integral is, in general, dependent on the parameterization $\{\varphi_i\}$ of $B$. An example of such a functional is the surface area, which corresponds to setting $L(x, v_1, \ldots, v_n) = \text{vol}(x, v_1, \ldots, v_n) = \sqrt{\det(g_{x}(v_i, v_j))}_{i,j=1}^n$. Another way to think about the action of the Lagrangian function $L$ is as the integral of $L$ over the measure

$$\mu_B = \sum_i (d\varphi_i)_*, \mathcal{L}_{U_i}$$

on $T^n M$. One can also integrate differential forms with respect to $\mu_B$, since they define real-valued functions on $T^n M$; thus $\mu_B$ induces a current $T_{\mu_B}: \Omega^n(M) \to \mathbb{R}$ with boundary equal to the difference of the currents similarly induced for the submanifolds $T_{\mu_{N_1}}$ and $T_{\mu_{N_2}}$, $\partial T_{\mu_B} = T_{\mu_{N_2}} - T_{\mu_{N_1}}$. Note that, up to a sign, $T_{\mu_X}$ is independent of the parameterization of the corresponding manifold $X$. Thus we consider the following relaxation: we replace the space $\mu_B$ of measures corresponding to parameterized cobordisms by the space of measures $\mu$ on $T^n M$ such that their induced current $T_{\mu}$ has the right boundary, $\partial T_{\mu} = T_{\mu_{N_2}} - T_{\mu_{N_1}}$; we term these measures holonomic measures.
The weak KAM theory was originally discovered \cite{9} in a very different context, namely, in the study of geodesics of Lagrangian and Hamiltonian dynamical systems. In a sense, that would correspond to the $n = 1$ case in our description above: instead of cobordisms, one simply considers curves joining points on a manifold $M$. The theory then gives a function $u : M \to \mathbb{R}$ that is a viscosity solution of the Hamilton–Jacobi equation, $H(du) = c$, where $H$ is the Hamiltonian and $c \in \mathbb{R}$. This gives a precise description of the asymptotic dynamics of many of the geodesics of the system. A relaxation similar to the one described above was also considered in that context (see for example \cite{6,13}).

The weak KAM theory we construct in this paper has many similarities and many differences with the one obtained in the original context. Among the differences, we can note that the function $u$ produced by the main theorem is no longer a function on the manifold $M$, but on a certain space of objects that can be understood as slices of holonomic measures: just like we can consider a cylinder as a set of circles glued together, and the cylinder then as a curve in the space of circles, a curve in the space of argute slices will give a holonomic measure, thus giving meaning to the notion of flow in the space of slices of holonomic measures. As was the case in the original context, the theory we describe can be interpreted as giving a description of the dynamics of the geodesic flow induced by the Lagrangian action, now in the space of argute slices. Instead of geodesics, we have action-minimizing holonomic measures, which are akin to action-minimizing cobordisms. The function $u$, which will be, in an appropriate sense, a weak KAM solution, will also in certain circumstances correspond to a differential form $\omega$ on $M$ that will satisfy a sort of Hamilton–Jacobi equation, $H(d\omega) = c$.

The organization of the paper is as follows. Section 2 develops a point of view of what a weak KAM theory is, and it introduces the concepts of Lagrangian category, the Lax-Oleinik semigroup, a weak KAM solution, and a finely KAM-amenable category; it then gives a few examples from the literature, before turning to the proof of a general weak KAM theorem with mild assumptions on the class of Lagrangian categories it applies to. Section 3 is devoted to the particular example of a Lagrangian category that we are interested in, in which the objects are $(n - 1)$-dimensional currents that arise as time slices of holonomic measures; in this section we give sufficient conditions for such a Lagrangian category to be weak KAM-amenable and we show that in that case there are weak KAM solutions. Section 4 gives the characterization of minimizable Lagrangian actions and the connection of weak KAM solutions on the Lagrangian category of currents to exact forms on a manifold, and explains in which sense they satisfy a Hamilton–Jacobi equation.

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2. What is a weak KAM theory?

2.1. Abstract formulation and examples. In this section, as a sort of preface, we give a point of view of what constitutes a weak KAM theory, for which we do an abstraction of some of the material in [10], and we relate it to the original construction of Fathi–Siconolfi [9].

We use the language of categories, as it appears to fit naturally and to appropriately generalize the weak KAM theories that we know of. However, the reader unfamiliar with categories can do the following substitutions without losing much: instead of the collection of objects of the category, think of a topological space, and instead of the morphisms of the category, think of paths joining the points of the topological space.

2.1.1. Abstract formulation. A category $C$ with morphisms

$$
\text{Hom} = \bigcup_{x, y \in C} \text{Hom}(x, y)
$$

is Lagrangian if it is endowed with two functions

$A: \text{Hom} \to \mathbb{R}$ and $t: \text{Hom} \to [0, +\infty)$

associating to each morphism $\gamma \in \text{Hom}(x, y)$ the action $A(\gamma)$ and the internal time duration $t(\gamma)$ satisfying the relations

$$
A(\gamma \circ \eta) = A(\gamma) + A(\eta) \quad \text{and} \quad t(\gamma \circ \eta) = t(\gamma) + t(\eta)
$$

for all objects $x, y, z \in C$ and all morphisms $\eta \in \text{Hom}(x, y)$ and $\gamma \in \text{Hom}(y, z)$ between them, as well as

$$
A(\text{id}_x) = 0 = t(\text{id}_x)
$$

for the identity morphism $\text{id}_x \in \text{Hom}(x, x)$.

Given a Lagrangian category $C$, we define, for $t \in [0, +\infty)$ and $x, y \in C$, the finite time action potential $h_t: C \times C \to \mathbb{R}$ by

$$
h_t(x, y) = \inf_{\gamma \in \text{Hom}(x, y)} A(\gamma). \quad t \geq 0.
$$

It satisfies, for all $s, t \geq 0$, and $x, z \in C$,

$$
h_{s+t}(x, z) = \inf_{y \in C} h_s(x, y) + h_t(y, z).
$$

Associated to the maps $h_t$, we have a set of maps $\varphi_t$ that operate on functions $u: C \to \mathbb{R}$ by

$$
\varphi_t u(x) = \inf_{y \in C} u(y) + h_t(y, x), \quad t \geq 0.
$$

The set $\{\varphi_t : t \geq 0\}$ is known as the Lax-Oleinik semigroup and it satisfies

$$
\varphi_s \circ \varphi_t = \varphi_{s+t}, \quad s, t \geq 0.
$$

The central result of a typical weak KAM theory, a version of which is presented below in Theorem 2, is that, under certain technical assumptions on a Lagrangian category $C$, there exists a function $u: C \to \mathbb{R}$ satisfying

$$
u = \varphi_t u + c_0 t.
$$

The significance of this result depends on the context of the specific category $C$, so the reader will be well-served with a some examples, and we give some in the next subsections.
2.1.2. Weak KAM solutions. Before passing to the examples, however, let us give a useful alternative characterization of a function $u$ satisfying (1) that applies to many cases of interest.

A function $u: \mathcal{C} \to \mathbb{R}$ is a weak KAM solution (of negative type) if there is some $c_0 \in \mathbb{R}$ such that, for all objects $x$ and $y$ of $\mathcal{C}$ and every $\gamma \in \text{Hom}(y, x)$ we have

\begin{equation}
    u(x) - u(y) \leq A(\gamma) + c_0 t(\gamma)
\end{equation}

and if, for every object $x$ in $\mathcal{C}$, there is a subcategory $\Gamma_x$ of $\mathcal{C}$ such that $x$ is an object of $\Gamma_x$, $\Gamma_x$ contains an object $y_t$ for each $t \leq 0$ with $x = y_0$, the set of morphisms $\text{Hom}_{\Gamma_x}(y_s, y_t)$ is nonempty for all $s \leq t \leq 0$, and for all $\gamma \in \text{Hom}_{\Gamma_x}(y_s, y_t)$,

\begin{equation}
    t(\gamma) = t - s \quad \text{and} \quad u(y_t) - u(y_s) = A(\gamma) + c_0(t - s).
\end{equation}

Analogously to the classical case, we refer to the subcategory $\Gamma_x$ as a geodesic of the Lagrangian category $\mathcal{C}$.

A Lagrangian category $\mathcal{C}$ is finely KAM-amenable if the morphisms can be realized as curves in the collection of objects, the action can be realized as an integral over these curves, the minimal action is always attained, and the curves realizing it can be extended indefinitely; more precisely:

A1. For every pair of objects $x$ and $y$ in $\mathcal{C}$ and every morphism $\gamma \in \text{Hom}(x, y)$ between them, there are $T > 0$ and a curve $\bar{\gamma}: [0, T] \to \mathcal{C}$ such that $\bar{\gamma}(0) = x$, $\bar{\gamma}(T) = y$.

A2. The association $\gamma \mapsto \bar{\gamma}$ is such that, for every two morphisms $\gamma \in \text{Hom}(x, y)$ and $\eta \in \text{Hom}(y, z)$, the curve $\eta \circ \bar{\gamma}$ associated to the composition $\eta \circ \gamma \in \text{Hom}(x, z)$ is the concatenation $\bar{\gamma} \ast \bar{\eta}$ of $\bar{\gamma}: [0, T_\gamma] \to \mathcal{C}$ followed by $\bar{\eta}: [0, T_\eta] \to \mathcal{C}$, that is,

\[ \bar{\gamma} \ast \bar{\eta} = \bar{\gamma} \ast \bar{\eta}(s) := \begin{cases} \bar{\gamma}(s), & s \in [0, T_\gamma], \\ \bar{\eta}(s - T_\gamma), & s \in [T_\gamma, T_\gamma + T_\eta]. \end{cases} \]

A3. For each $\gamma \in \text{Hom}$, if $\bar{\gamma}$ is defined on $[0, T]$, then for each $t \in (0, T)$ there exist morphisms $\xi \in \text{Hom}(\bar{\gamma}(0), \bar{\gamma}(t))$ and $\eta \in \text{Hom}(\bar{\gamma}(t), \bar{\gamma}(T))$ such that $\bar{\gamma} = \bar{\xi} \ast \bar{\eta}$.

A4. For every $x \in \mathcal{C}$ and $t > 0$ and each function $u: \mathcal{C} \to \mathbb{R}$ satisfying (1), the function

\[ (y, \gamma) \mapsto u(y) + A(\gamma) + c_0 t \]

attains its minimum in the set of pairs $(y, \gamma)$ with $\gamma \in \text{Hom}(y, x)$ and $t(\gamma) = t$.

A5. Let $x$ be an object in $\mathcal{C}$. For each $T > 0$, let $\mathcal{G}^x_T$ be the collection of morphisms $\gamma \in \bigcup_{y \in \mathcal{C}} \text{Hom}(y, x)$ such that $t(\gamma) = T$ and $A(\gamma) + u(\bar{\gamma}(0)) \leq A(\xi) + u(\bar{\xi}(0))$ for all $\xi \in \bigcup_{y \in \mathcal{C}} \text{Hom}(y, x)$ with $t(\xi) = T$; by A1, $\mathcal{G}^x_T$ is nonempty. For $0 < t \leq T$, let $\mathcal{G}^x_{T|t} \subseteq \mathcal{G}^x_T$ be the set of morphisms $\gamma \in \mathcal{G}^x_T$ such that there is some $\xi \in \mathcal{G}^x_T$ with $\bar{\gamma}(s) = \bar{\xi}(s + T - t)$ for all $s \in [0, t]$. Then, for each $t > 0$,

\[ \bigcap_{T \in [t, \infty)} \mathcal{G}^x_{T|t} \text{ is nonempty.} \]

\footnote{The classical theory \cite{9} includes the concept of weak KAM solutions of positive type obtained by appropriately changing the signs of time parameters $s, t$, and the subsequent idea of conjugate solutions, which are both very interesting. For simplicity we do not discuss those topics here.}
Lemma 1. On a finely KAM-amenable Lagrangian category $\mathcal{C}$, a function $u$ satisfies (1) if, and only if, $u$ is a weak Kam solution.

Proof. Let us first assume that $u$ satisfies (1); from the definitions this clearly implies (2). Let $x$ be an object in $\mathcal{C}$ and let us show that the subcategory $\Gamma_x$ exists. We will use the notations of [A5]. Inductively, pick $\gamma_i \in \bigcap_{T \geq i} G_{T, i}$, with $\bar{\gamma}_i(s + 1) = \bar{\gamma}_i(s)$ for $s \in [0, i - 1)$. For $t \leq 0$, we let $y_t = \bar{\gamma}_i(i + t)$ for any $i \geq -t$. Given $s < t \leq 0$, by [A3] there is $\gamma \in \text{Hom}(y_s, y_t)$ such that $\bar{\gamma}(r) = y_{s + r}$ for all $r \in [0, t - s]$.

Since $\mathcal{C}$ is finely KAM-amenable, there is also, for $s \leq t \leq 0$, $\gamma_{st} \in \text{Hom}(y_s, y_t)$ with $t(\gamma_{st}) = t - s$, $A(\gamma_{st}) = h_{t-s}(y_s, y_t)$, so we let $\text{Hom}_{\Gamma_x}(y_s, y_t) = \{\gamma_{st}\}$, and with this definition $\mathcal{B}$ is verified because we have

$$u(y_t) = u(\bar{\gamma}_{st}(t - s)) = \varphi_{t-s} u(\bar{\gamma}_{st}(t - s)) + c_0(t - s)$$

$$= \inf_{y \in \mathcal{C}} u(y) + h_t(y, \bar{\gamma}_{st}(t - s)) + c_0(t - s)$$

$$= \inf_{y \in \mathcal{C}} u(y) + \inf_{\eta \in \text{Hom}(y, \bar{\gamma}_{st}(t - s))} A(\eta) + c_0(t - s)$$

$$= u(\bar{\gamma}_{st}(0)) + A(\gamma_{st}) + c_0(t - s) = u(y_s) + A(\gamma_{st}) + c_0(t - s)$$

by the definition of $G_{T, i}$, [A4] and [A5]

For the converse, assume that $u$ is a weak Kam solution. From (2) it follows that $u \leq \varphi_t u + c_0t$ for all $t > 0$. To prove the opposite inequality, let $t > 0$ and $x$ be an object in $\mathcal{C}$. Let $y_{t}$ be the object in $\Gamma_x$ and $\gamma_{t, 0}$ the morphism in $\text{Hom}_{\Gamma_x}(y_t, x)$ verifying (2), that is,

$$u(x) - u(y_{t}) = A(\gamma_{t, 0}) - c_0(0 - t).$$

Then we have

$$\varphi_t u(x) + c_0t = \inf_{y \in \mathcal{C}} + h_t(y, x) + c_0t$$

$$\leq u(y_{t}) + h_t(y_{t}, x) + c_0t$$

$$= u(y_{t}) + A(\gamma_{t, 0}) - c_0(0 - t) = u(x).$$

\[\square\]

2.1.3. Weak Kam for Lagrangian dynamics. Our first example concerns the setting originally treated by Fathi–Siconolfi [9]. We take a compact Riemannian manifold $(M, g)$ and we let the objects of the category $\mathcal{C}_{FS}$ the points of the manifold $M$, whose distance function is the one induced by the Riemannian metric $g$. The morphisms $\text{Hom}(x, y)$ of $\mathcal{C}$ are simply the sets of absolutely continuous curves $\gamma: [0, T] \to M$ with $\gamma(0) = x$ and $\gamma(T) = y$. Their composition $\gamma \circ \eta$ is defined to be the concatenation $\gamma \ast \eta$, that is, the curve traversing $\gamma$ and then $\eta$. The tangent bundle $TM$ is endowed with a Lagrangian function $L: TM \to \mathbb{R}$ that satisfies the Tonelli conditions of strict convexity and superlinearity on the fibers of $TM$. The action $A: \text{Hom} \to \mathbb{R}$ is defined by

$$A(\gamma) = \int_0^T L(\gamma(t), \gamma'(t)) \, dt$$

for a curve $\gamma: [0, T] \to M$, and the time function $t: \text{Hom} \to [0, +\infty)$ is given by $t(\gamma) = T$. The function $u$ rendered by Theorem [2] can then be interpreted as a
function on $M$, and

$$u(y) = \varphi_T u(y) + c_0 T$$

$$= \inf_{y \in C_{FS}} u(x) + h_T(x, y) + c_0 T$$

$$= \inf_{x \in C_{FS}} u(x) + \inf_{\gamma \in \text{Hom}(x, y)} A(\gamma) + c_0 T$$

$$= \inf_{x \in M} u(x) + \inf_{\gamma: [0, T] \to M} \int_0^T L(\gamma(t), \gamma'(t)) dt + c_0 T,$$

we have

$$u(y) - u(x) \leq \int_0^T L(\gamma(t), \gamma'(t)) dt + c_0 T$$

for all $\gamma \in \text{Hom}(x, y)$ with $t(\gamma) = T$. Denoting by $du$ the differential of $u$, it follows that

$$du_x v \leq L(x, v) + c_0$$

for $x \in M$ a point where $u$ is differentiable, and $v \in T_x M$. If we let $H$ be the Hamiltonian function associated to $L$ by the Legendre–Fenchel transform,

$$H(x, p) = \inf_{v \in T^*_x M} p(v) - L(x, v),$$

this can be written

$$H(du) \leq c_0,$$

which is to say that $u$ is a subsolution of the Hamilton–Jacobi equation. As the category $C_{FS}$ is finely KAM-amenable, as a consequence of [3] the Fathi–Siconolfi [9] weak KAM theory actually gives us a viscosity solution $u$ of the Hamilton–Jacobi equation, $H(du) = c_0$. Here, “viscosity” means that, while $u$ is not a classical solution of this partial differential equation, it does enjoy certain desirable regularity properties; see [7, 9].

Many properties of the weak KAM solution $u$ and the curves realizing [3] have been studied in this context; see [9]. Among many other generalizations, the case in which $M$ is $\sigma$-compact instead of compact was treated in [10].

2.1.4. Weak KAM for optimal control problems. Other, more general, versions of weak KAM arise in the contexts of various optimal control problems; see for example [1]. For instance, if we consider the case in which $C_{OC}$ is given by the points of a $\sigma$-compact manifold $M$, and we also have a set $\Omega$ of controls and a function $v: M \times \Omega \to TM$ associating a velocity $v(x, \omega) \in T_x M$ to each pair $(x, \omega) \in M \times \Omega$. We also assume that we have a Lagrangian density $L: M \times \Omega \to \mathbb{R}$ satisfying some technical conditions.

The morphisms $\gamma \in \text{Hom}(x, y)$ are curves $\gamma: [0, T] \to M \times \Omega$ such that, if $\pi_M: M \times \Omega \to M$ is the projection, $\pi_M \circ \gamma$ is absolutely continuous and $(\pi_M \circ \gamma)'(t) = v(\gamma(t))$ for almost every $t \in [0, T]$, and $L \circ \gamma$ is integrable. For such $\gamma$, we define

$$A(\gamma) = \int_0^T L(\gamma(t)) dt$$

and we set $t(\gamma) = T$.

Then, in a manner similar to our explanation of Section 2.1.3, the function $u$ obtained by the weak KAM machinery can be identified with the value function and is a viscosity solution of the Hamilton–Jacobi–Bellman equation. See for example [2] and the references therein.
2.1.5. Weak KAM for mass transportation. This context has been explored for example in \cite{5}, whose conclusions we now try to paraphrase.

Let $M$ be a compact, connected manifold, and let the objects of $\mathcal{C}_{\text{MT}}$ be the set of compactly supported Radon probability measures on $M$. We define, for any two $\nu_1, \nu_2 \in \mathcal{C}_{\text{MT}}$, $\text{Hom}(\nu_1, \nu_2)$ to be the set of measures $\mu$ on $M \times M$ with marginals $\pi_1^*\mu = \nu_1$ and $\pi_2^*\mu = \nu_2$, where $\pi_1$ and $\pi_2$ are the projections of $M \times M$ onto the corresponding copies of $M$. In this context we have a cost function $c : M \times M \to \mathbb{R}$ that allows us to define the action $A(\mu) = \int_{M \times M} c(x, y) \, d\mu(x, y)$, and if we have a Riemannian metric on $M$ inducing the distance, we can define the time function $t(\mu) = \int_{M \times M} \text{dist}(x, y) \, d\mu(x, y)$. The function $u$ given by the weak KAM theorem satisfying (1) can be better interpreted in an equivalent context.

With some technical assumptions, this context can be shown to be equivalent to the following \cite{5}: Let $\tilde{\mathcal{C}}_{\text{MT}}$ be the category with the same objects as $\mathcal{C}_{\text{MT}}$, but now we define, for compactly supported probability measures $\nu_1$ and $\nu_2$ on $M$, $\text{Hom}(\nu_1, \nu_2)$ to be the set of compactly supported measures $\mu$ on $TM$ such that, for all $f \in C^\infty(M)$,

$$\int_{TM} df_x \, d\mu = \int_M f \, d\nu_1 - \int_M f \, d\nu_2.$$ 

Assume that $A$ can be written as $A(\mu) = \int_{TM} L \, d\mu$ for some function $L : TM \to \mathbb{R}$ and $t(\mu) = \int_{TM} \|v\| \, d\mu(v)$. This would be for example the case if $c(x, y) = \text{dist}(x, y)$ on a Riemannian manifold $(M, g)$, in which case $L(x, v) = \sqrt{g_x(v, v)}$ is the norm induced by the Riemannian metric $g$ of the corresponding tangent vector $v$. Technical assumptions on the function $L$ are necessary for the theory to hold.

The function $u$ given by the weak KAM theorem now turns out to be a viscosity solution of the Hamilton–Jacobi equation for the Hamiltonian associated to $L$.

2.1.6. Weak KAM on the space of slices. In the theory we want to develop in this paper, we will consider submanifolds of a manifold $M$, and $\mathcal{C}_{\text{HM}}$ and $M$ will not coincide. Roughly speaking, the role of $\mathcal{C}_{\text{HM}}$ will be played by the set of $(n-1)$-dimensional slices without boundary of $n$-dimensional submanifolds of $M$. The details of this approach will be developed in Section 3 and the significance of the function $u$ will be explored in Section 4.

2.2. Weak KAM machinery for noncompact metric spaces. The purpose of this section is to present and prove Theorem 2 that powers a rich range of weak KAM theories. We will use the definitions and notations stated in the abstract setting presented in Section 2.1.1.

Recall a topological space is $\sigma$-compact if it can be covered by countably-many compact sets.

**Theorem 2.** Let $\mathcal{C}$ be a Lagrangian category with action $A$ and time function $t$. Assume that the set of objects of $\mathcal{C}$ has the structure of a $\sigma$-compact metric space with distance function $\text{dist}$, and that the finite action potential $h$ satisfies the technical hypotheses that follow:

1. for every compact subset $K \subseteq \mathcal{C}$, there is some constant $P > 0$ such that $h_{\text{dist}(x,y)}(x, y) \leq P \, \text{dist}(x, y)$ for all objects $x, y \in K$. 


K2. There are a locally Lipschitz function \( u : C \to \mathbb{R} \) and a number \( c_1 > 0 \) such that
\[
u(y) - u(x) \leq h_t(x, y) + c_1 t
\]
for all \( t > 0 \) and all \( x, y \in C \).

Then there are a locally Lipschitz function \( u : C \to \mathbb{R} \) and a number \( c_0 \in \mathbb{R} \) such that, for all \( t > 0 \),
\[
u = \varphi_t u + c_0 t.
\]

Remark 3. By Lemma 1 on a finely KAM-amenable category the conclusion of Theorem 2 is equivalent to \( u \) being a weak KAM solution, as defined in Section 2.1.2.

Theorem 2 is a categorical formulation of a very slight generalization of the statement proved in [10]. To prove it, we need a definition and an auxiliary lemma.

Let \( u \in C^0(C) \). For \( c \in \mathbb{R} \), we say that \( u \) is \( c \)-dominated if, for every \( t > 0 \),
\[
u(y) - u(x) \leq h_t(x, y) + ct.
\]

We denote \( \mathcal{H}(c) \) the set of \( c \)-dominated functions.

Lemma 4. Under the assumptions of Theorem 2 we have:
i. \( \varphi_s \circ \varphi_t = \varphi_{s+t} \).
ii. The functions \( u \in \mathcal{H}(c) \) are locally uniformly Lipschitz, meaning that for each compact subset \( K \) of \( C \) there is a constant \( C > 0 \) such that, for all \( u \in \mathcal{H}(c) \), \( |u(x) - u(y)| \leq C \text{dist}(x, y) \) for all \( x, y \in K \).
iii. The set \( \mathcal{H}(c) \) is nonempty for \( c \) large enough.
iv. \( \varphi_s(\mathcal{H}(c)) \subseteq \mathcal{H}(c) \).

Proof of Lemma 4. To prove the first assertion, we observe that
\[
\varphi_{s+t}(x) = \inf_{y \in C} u(y) + h_{s+t}(y, x)
\]
\[
= \inf_{y \in C} \left[ u(y) + \inf_{z \in C} h_t(y, z) + h_s(z, x) \right]
\]
\[
= \inf_{y \in C} \left[ u(y) + \inf_{z \in C} h_t(y, z) + h_s(z, x) \right]
\]
\[
= \inf_{z \in C} \left[ \inf_{y \in C} [u(y) + h_t(y, z)] + h_s(z, x) \right]
\]
\[
= \inf_{z \in C} \varphi_t u(z) + h_s(z, x) = \varphi_s(\varphi_t u)(x).
\]

The second assertion follows immediately from assumption \([K1] \) and the third follows from assumption \([K2] \).

For the fourth assertion, observe first that \( u \in \mathcal{H}(c) \) if, and only if, for all \( x, y \in C \) and \( t > 0 \),
\[
u(x) \leq u(y) + h_t(y, x) + ct
\]
which is true if, and only if, (taking the infimum on the right hand side)
\[
u(x) \leq \varphi_t u(x) + ct.
\]
Moreover, since by the definition of \( \varphi_t \) we clearly have that if \( u \leq v \) then \( \varphi_t u \leq \varphi_t v \), we see that applying \( \varphi_t \) to both sides of (4) and using that \( \varphi_t(u+k) = \varphi_t u + k \) for \( k \in \mathbb{R} \), we get

\[
\varphi_t u(x) \leq \varphi_t \varphi_t u(x) + ct,
\]
whence \( \varphi_t u \in \mathcal{H}(c) \) again. \( \square \)

**Proof of Theorem 2.** The proof is essentially the same as that in [10, Section 4].

We denote by \( \tilde{C}^0(\mathcal{C}) \) the quotient of the vector space \( C^0(\mathcal{C}) \) by its subspace of constant functions. If \( \tilde{q}: C^0(\mathcal{C}) \to \tilde{C}^0(\mathcal{C}) \) is the quotient map, then since \( \varphi_s(u+k) = k + \varphi_s(u) \) for \( k \in \mathbb{R} \), the semigroup \( \varphi_s \) (cf. item (i) in Lemma 4) induces a semigroup on \( \tilde{C}^0(\mathcal{C}) \) that we denote \( \tilde{\varphi}_s \) and satisfies \( \tilde{\varphi}_s \circ \tilde{q} = \tilde{q} \circ \varphi_s \) for all \( s > 0 \) as well as \( \tilde{\varphi}_{s+t} = \tilde{\varphi}_s + \tilde{\varphi}_t \).

The topology on \( \tilde{C}^0(\mathcal{C}) \) is the quotient of the compact open topology (i.e., the topology of uniform convergence on compact sets). With this topology, the space \( \tilde{C}^0(\mathcal{C}) \) becomes a locally convex topological vector space.

Denote by \( \tilde{\mathcal{H}}(c) \) the image \( \tilde{q}(\mathcal{H}(c)) \) in \( \tilde{C}^0(\mathcal{C}) \). The subset \( \tilde{\mathcal{H}}(c) \) of \( \tilde{C}^0(\mathcal{C}) \) is convex and compact. The convexity of \( \tilde{\mathcal{H}}(c) \) follows from that of \( \mathcal{H}(c) \). To prove that \( \tilde{\mathcal{H}}(c) \) is compact, we introduce \( C^0_x(\mathcal{C}) \) the set of continuous functions \( \mathcal{C} \to \mathbb{R} \) vanishing at some fixed \( x \in \mathcal{C} \). The map \( \tilde{q} \) induces a homeomorphism from \( C^0_x(\mathcal{C}) \) onto \( \tilde{C}^0(\mathcal{C}) \).

Since \( \mathcal{H}(c) \) is stable under addition of constants, its image \( \tilde{\mathcal{H}}(c) \) is also the image under \( \tilde{q} \) of the intersection \( \mathcal{H}_x(c) = \mathcal{H}(c) \cap C^0_x(\mathcal{C}) \). The subset \( \mathcal{H}_x(c) \) is closed in \( C^0(\mathcal{C}) \) for the compact-open topology. Moreover, it consists of functions that all vanish at \( x \) and are locally uniformly Lipschitz (because of item (iii) in Lemma 4).

Let us show that \( \mathcal{H}_x(c) \) is compact. Pick compact sets \( K_1 \subset K_2 \subset \cdots \subset \mathcal{C} \) such that \( x \in K_i \) for all \( i \) and \( \mathcal{C} = \bigcup_i K_i \), and let \( \{f_i\}_i \subset \mathcal{H}_x(c) \) be a sequence of functions. From the Lipschitz version of the Arzelà-Ascoli theorem, it follows that there is a subsequence \( \{f_{i_j}\}_j \) convergent in \( K_1 \). We iteratively apply the same theorem to pass to subsequent subsequences \( \{f_{i_{j_m}}\}_j \subset \{f_{i_{j_{m-1}}}\}_j \) that converge on \( K_m \) for each \( m \in \mathbb{N} \). Taking the diagonal sequence \( \{f_{i_j}\}_j \) we ensure convergence throughout \( \mathcal{C} \). This means that \( \mathcal{H}_x(c) \) is sequentially compact. Since the space \( C^0(\mathcal{C}) \) is metrizable by

\[
\text{dist}(f, g) = \sum_i \frac{1}{2^i} \frac{\sup_{K_i} \|f - g\|}{1 + \sup_{K_i} \|f - g\|},
\]

\( \mathcal{H}_x(c) \) is also compact.

The restriction of \( \tilde{q} \) to \( \mathcal{H}_x(c) \) induces a homeomorphism onto \( \tilde{\mathcal{H}}(c) \). As a first conclusion we conclude that if

\[
c_0 = \inf\{c \in \mathbb{R} | \mathcal{H}(c) \neq \emptyset\}
\]

then \( \bigcap_{c > c_0} \tilde{\mathcal{H}}(c) \neq \emptyset \) as the intersection of a decreasing family of compact nonempty subsets (cf. item (iii) in Lemma 4). It follows that \( \mathcal{H}(c_0) \) is also nonempty because it contains the nonempty subset \( \tilde{q}^{-1} \left[ \bigcap_{c > c_0} \tilde{\mathcal{H}}(c) \right] \).

We have that \( \tilde{\varphi}_s(\tilde{q}(u)) = \tilde{q}(\varphi_s u - \varphi_s u(x)) \) for \( u \in \mathcal{H}_x(c) \) because \( \varphi_s u(x) \) is a constant so \( \tilde{q} \) maps it to 0. Since the map

\[
[0, +\infty) \times \mathcal{H}_x(c) \to \mathcal{H}_x(c)
\]

\[
(s, u) \mapsto \varphi_s(u) - \varphi_s(u)(x)
\]
is continuous, we conclude that \( \hat{\varphi}_t \) induces a continuous seminguid group of \( \mathcal{H}(c) \) into itself (cf. item \([iv]\) in Lemma 4). Since this last subset is a nonempty convex compact subset of a locally convex topological vector space \( \mathcal{C}^0(C) \), we can apply the Schauder-Tychonoff theorem [5, pages 414–415] to conclude that, for each \( t > 0 \), \( \hat{\varphi}_t \) has a fixed point \( \hat{u}_t \) in \( \mathcal{H}(c) \) if \( \mathcal{H}(c) \neq \emptyset \), that is, for every \( c \geq c_0 \). Observing that for every rational \( \frac{t}{q} \in \mathbb{Q} \) we have

\[
\hat{u}_t = \hat{\varphi}_{\frac{t}{q}} \circ \hat{\varphi}_{\frac{t}{q}} \circ \cdots \circ \hat{\varphi}_{\frac{t}{q}}(\hat{u}_\frac{1}{q}) = \hat{\varphi}_t(\hat{u}_\frac{1}{q})
\]

and using a density argument, we conclude that there is in fact a fixed point \( \hat{u} \in \mathcal{H}(c) \) common to all the maps \( \hat{\varphi}_t \), \( t > 0 \).

If we let \( u \in \mathcal{H}(c) \) be such that \( \hat{q}(u) = \hat{u} \), then the fact that \( \hat{u} \) is a fixed point for \( \hat{\varphi}_s \) means that \( \varphi_t u = u + a(t) \). Using that \( \varphi_t \) is a semigroup, we get that \( a(t) = a(1)t \). The equality \( u = \varphi_t u - a(1)t \) shows that for all \( x, y \in C \)

\[
u(y) - u(x) < h_\nu(x, y) + a(1)t.
\]

Hence \(-a(1) \geq c_0 \). Since \( u \in \mathcal{H}(c_0) \), we must have \( u \leq \varphi_t u + c_0 t \), which gives \( \varphi_t u - a(1)t \leq \varphi_t u + c_0 t \) for all \( t \geq 0 \), and \(-a(1) \leq c_0 \). We conclude that \(-a(1) = c_0 \).

3. **Weak KAM in the space of slices**

Let \( M \) be a \( \sigma \)-compact, \( C^\infty \) manifold of dimension \( d > 0 \) without boundary, and let \( g \) be a Riemannian metric on \( M \). For \( 0 < n \leq d \) we let \( T^n M \) be the Whitney sum of \( n \) copies of the tangent bundle \( TM \), so that the fiber

\[
T^n_x M = T^n_x M = \bigoplus_{\nu = 0}^{n} T_x M
\]

is a vector space of dimension \( nd \), and we also define \( T^0 M = M \times \{0\} \). We will denote a point in \( T^n M \) by \( (x, v_1, \ldots, v_n) \) where \( x \in M \) and \( v_i \in T_x M \).

For vectors \( v_1, \ldots, v_n \in T_x M \), denote by \( v_1 \wedge \cdots \wedge v_n \) their antisymmetric product. Given local coordinates \( x_1, \ldots, x_n \) on an open set of \( M \) containing a point \( x \), the vectors \( \partial/\partial x_1, \ldots, \partial/\partial x_n \) form a basis of \( T_x M \) and the covectors \( dx_1, \ldots, dx_n \), satisfying \( dx_i(\partial/\partial x_j) = 0 \) for \( i \neq j \) and to 1 for \( i = j \), form a basis of \( T^*_x M \).

For \( 0 \leq k \leq d \), let \( \Omega^k(M) \) be the set of \( C^\infty \) differential forms of order \( k \) on \( M \). In particular \( \Omega^0(M) = C^\infty(M) \). A form \( \omega \in \Omega^k(M) \) can be written, locally on an open set \( U \subseteq M \) diffeomorphic to a ball, as \( \omega(x, v_1, \ldots, v_k) = \sum_I g_I(x)dx_I(v_1, \ldots, v_k) \), where the sum is taken over all subsets \( I \subseteq \{1, 2, \ldots, d\} \) of cardinality \( k \), \( g_I \in C^\infty(U) \) and \( dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \) for \( I = \{i_1, \ldots, i_k\} \). The topology of \( \Omega^k(M) \) is induced by the seminorms

\[
|\omega|_{U,m} = \sum_{|J| \leq m} \sup_{x \in U} |\partial^J(g_I(x))|_I
\]

where \( U \) is a precompact subset of \( M \), \( m > 0 \), and the sum is taken over all multiindices \( J = (j_1, \ldots, j_k) \) with \( |J| = \sum_j j_i \leq m \). Observe that a differential form of order \( k \) is a function on \( T^k M \).

A current \( T \) of dimension \( k \) is a continuous functional \( T : \Omega^k(M) \to \mathbb{R} \). The boundary \( \partial T \) of a \( k \)-dimensional current \( T \) is a \((k-1)\)-dimensional current defined
by
\[ \partial T(\omega) = T(d\omega), \quad \omega \in \Omega^{k-1}(M). \]
Since \( dd\omega = 0 \) for all \( \omega \in \Omega^k(M) \), we also have \( \partial \partial T = 0 \) for all currents \( T \). A compactly-supported Radon measure \( \mu \) on \( T^n M \) induces a current \( T_\mu \) by
\[ T_\mu(\omega) = \int_{T^n M} \omega \, d\mu, \quad \omega \in \Omega^n(M). \]

3.1. Skeleton theory. We will first discuss a stripped-down theory that essentially deals with bare cobordisms, and in the next subsection we will explain how this can be enriched with a set of objects, namely, the argute slices, that make it more evident how the slices fit together and allow for the construction of a finely KAM amenable Lagrangian category.

Let \( \mathcal{C}_{HM} \) be the category whose objects are the currents \( T \) of dimension \( n - 1 \) with null boundary \( \partial T = 0 \) and induced by a finite, nonnegative, compactly-supported Radon measure \( \nu \) on \( T^{n-1} M \), that is, \( T = T_\nu \) as defined by (5). The morphisms of \( \mathcal{C}_{HM} \) are, for each pair of currents \( T_1 \) and \( T_2 \), the collections \( \text{Hom}(T_1, T_2) \) of compactly-supported, nonnegative, Radon measures \( \mu \) on \( T^n M \) that satisfy
\[ \int_{T^n M} d\omega \, d\mu = T_2(\omega) - T_1(\omega) \quad \text{for all } \omega \in \Omega^{n-1}(M). \]
In other words, \( \partial T_\mu = T_2 - T_1 \). Composition of \( \mu_1 \in \text{Hom}(T_1, T_2) \) and \( \mu_2 \in \text{Hom}(T_2, T_3) \) is defined by
\[ \mu_2 \circ \mu_1 = \mu_1 + \mu_2. \]
We will refer to the objects of \( \mathcal{C}_{HM} \) as slices, and to the morphisms as holonomic measures.

Given a function \( L: T^n M \to \mathbb{R} \), we let, for \( \mu \in \text{Hom}(T_1, T_2) \),
\[ A(\mu) = \int_{T^n M} L \, d\mu \]
and
\[ t(\mu) = \int_{T^n M} 1 \, d\mu = \mu(T^n M). \]
With these definitions, \( \mathcal{C}_{HM} \) is a Lagrangian category.

We give the collection of objects of \( \mathcal{C}_{HM} \) the structure of a metric topological space by letting
\[ \text{dist}(T_1, T_2) = \inf_{\mu \in \text{Hom}(T_1, T_2)} \int_{T^n M} \text{vol} \, d\mu, \quad T_1, T_2 \in \mathcal{C}_{HM}, \]
where
\[ \text{vol}(x, v_1, \ldots, v_n) = \sqrt{\det (g_z(x, v_j))_{i,j=1}^n}. \]
In geometric measure theory, dist is known as the flat distance.

Lemma 5. With the topology induced by the metric dist, \( \mathcal{C}_{HM} \) is \( \sigma \)-compact.

Proof. Recall that a current is normal if it and its boundary are both representable by integration. Thus the objects and morphisms of \( \mathcal{C}_{HM} \) are normal currents. Let \( K_1 \subseteq K_2 \subseteq \cdots \subseteq M \) be a family of nested compact sets with \( K_i = \text{cl}(\text{int} K_i) \) and \( M = \bigcup_i K_i \). It follows from the Compactness Theorem for Normal Currents (see for example \( [4, \text{Theorem 1.4}] \), or \( [3, \text{Theorem 5.2}] \) for a very general version) that the closed set \( C_\ell \) of normal currents \( T_\nu \in \mathcal{C}_{HM} \) associated to measures \( \nu \) and
supported in $K_\ell$ and with mass $M(T_\nu) = \int \text{vol } dv$ bounded by $\ell$, is compact in the weak* topology. By [11, Corollary 7.3], the sets $C_\ell$ are also compact in the topology induced by the flat distance, so $C_{\text{HM}} = \bigcup \ell C_\ell$ is $\sigma$-compact.

In order to ensure we can find a weak KAM theory in this context through the application of Theorem 2 we need some technical assumption on $L$ to ensure that $h_t$ will satisfy the hypotheses of the theorem. Our choice of assumptions on $L$: $T^n M \to \mathbb{R}$ is the following:

L1. The function $L$ is measurable and bounded on compact sets.

L2. There are a locally Lipschitz differential form $\omega_L$ of order $n$ on $M$ and a number $c > 0$ such that

$$L(x, v_1, v_2, \ldots, v_n) + c \geq d\omega_L(x, v_1, v_2, \ldots, v_n)$$

for all $(x, v_1, v_2, \ldots, v_n) \in T^n M$ with $x$ in the differentiability set of $\omega$.

A differential form $\omega$ of order $k$ is locally Lipschitz if it can be written locally on compact charts $U \subseteq M$ as $\omega = \sum_I g_I dx_I$ with $g_I: U \to \mathbb{R}$ Lipschitz continuous, local coordinates $x_1, \ldots, x_n$ on $U$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}, I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, d\}$.

Theorem 18 implies that L2 is a reasonable assumption. Examples of Lagrangians $\gamma$ satisfying L1, L2 include the family $a + \text{vol}^p$ with $a: M \to [0, +\infty)$ bounded from below and $b \geq 1$.

With these assumptions, we have

**Lemma 6.** Assume $L$ satisfies L1. For each compact set $K \subseteq M$ there is a number $P > 0$ such that for every pair of currents $T_1, T_2 \in C_{\text{HM}}$ supported in $K$, we have

$$h_{\text{dist}(T_1, T_2)}(T_1, T_2) \leq P \text{dist}(T_1, T_2).$$

**Proof.** Let $\mu \in \text{Hom}(T_1, T_2)$ be supported solely on points $(x, v_1, \ldots, v_n) \in T^n M$ with $\text{vol}(v_1, \ldots, v_n) = 1$, so that $t(\mu) = \int_{T^n M} \text{vol } d\mu = \text{dist}(T_1, T_2)$. Let $P > 0$ be such that $L(x, v_1, \ldots, v_n) \leq P$ throughout $K$. Then

$$A(\mu) = \int_{T^n M} L \, d\mu \leq \int_{T^n M} P \, d\mu \leq P \text{dist}(T_1, T_2).$$

**Proposition 7 (Weak KAM theorem).** Assume $L$ satisfies L1, L2. Then there is a Lipschitz function $u: C_{\text{HM}} \to \mathbb{R}$ satisfying (1).

**Proof.** Lemma 6 and assumption L2 correspond to assumptions K1 and K2 respectively, of Theorem 2 which gives the proposition.

### 3.2. Enriched theory.

In this subsection we give definitions conceived with the intention of clarifying what a slice of a holonomic measure is, and how the slices fit together. To fix ideas, we first offer an example.

**Example 8.** Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-dimensional torus, and consider $\mathbb{T}^2 \subset \mathbb{T}^3$ parameterized by $\phi(x_1, x_2) = (x_1, x_2, 0) \bmod \mathbb{Z}^3, x_1, x_2 \in \mathbb{R}$. The torus $\mathbb{T}^2$ can be encoded as the holonomic measure $\mu_\phi$ on $T^2 \mathbb{T}^3$ given by

$$\mu_\phi = (\phi, \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2})_* \mathcal{L}_{[0,1] \times [0,1]},$$

where $\mathcal{L}_X$ is the Lebesgue measure on $X$. We want to think of $\mathbb{T}^2$ as being composed of the diagonal copies of $\mathbb{T}^1$ parameterized by $\gamma_t(s) = (s, t-s, 0) \bmod \mathbb{Z}^3, s, t \in [0, 1)$; we understand each $\gamma_t$ as a slice of $\mathbb{T}^2$. For our purposes, we cannot assume that
we know in advance how these slices fit together, because we want to construct a theory in which a family of slices makes up a submanifold, in the sense that we recover the associated holonomic measure $\mu_\phi$. Thus, although we can encode the circles $\gamma_t$ with the measures $\mu_{\gamma_t} = (\gamma_t, \gamma'_t)_\#L_{[0,1]}$, this is not a satisfactory answer because the vector $\gamma'_t(s) = (1, -1, 0)$ does not encode the information contained in the vectors $\frac{\partial \phi}{\partial x_1} = (1, 0, 0)$ and $\frac{\partial \phi}{\partial x_2} = (0, 1, 0)$.

A solution to this problem is to consider a richer object that encodes not only the information of the partial derivatives of $\phi$ but also the differential $dt = dx_1 + dx_2$ of the parameter $t$ that describes how they fit together. Hence, instead of $\gamma_t$ we take the measure $\nu_t$ on $T^2M \oplus T^*M$ defined by

$$\nu_t = (\gamma_t, \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right), (dx_1 + dx_2)_\#L_{[0,1]}).$$

Then we have, forgetting the last piece of information by pushing forward with the projection $\pi^1: T^2T^3 \oplus T^*T^3 \to T^2T^3$,

$$\mu_\phi = \int_0^1 \pi^1_\# \nu_t \, dt,$$

so the slices fit together correctly. At the same time, we may recover the 1-dimensional character of each slice; they act on differential forms of order 1, according to the following recipe: for each $\nu_t$, we set $\hat{T}_{\nu_t}$ to be the current given, for $\omega = g_1 dx_1 + g_2 dx_2 + g_3 dx_3 \in \Omega^1(T^3)$, by

$$\hat{T}_{\nu_t}(\omega) := \int_0^1 \omega \wedge \pi(x, v_1, v_2) \, d\nu_t(x, v_1, v_2, \xi)$$

$$= \int_0^1 \omega \wedge (dx_1 + dx_2)(\gamma_t(s), \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)) \, ds$$

$$= \int_0^1 (g_1 - g_2) \circ \gamma_t(s) \, ds$$

$$= \int_0^1 \omega_{\gamma_t}(\gamma'_t(s)) \, ds$$

$$= \int_{\gamma_t} \omega.$$

So $\hat{T}_{\nu_t}(\omega)$ indeed encodes $\gamma_t$. In conclusion, the measures $\nu_t$ contain enough information to simultaneously recover both an infinitesimal 2-dimensional slice of $\mu_\phi$ and the currents induced by $\gamma_t$, and in this way they have 1- and 2-dimensional character at the same time, and they carry information about how they can fit within the full holonomic measure $\mu_\phi$.

We will now give the full definition. Let $\mathcal{X} = T^nM \oplus T^*M$, and denote by $\pi^1$ and $\pi^2$ the canonical projections

$$\pi^1: \mathcal{X} \to T^nM,$$

$$\pi^2: \mathcal{X} \to T^*M.$$

We will denote a point in $\mathcal{X}$ by $(x, v_1, v_2, \ldots, v_n, \xi)$ for $x \in M$, $v_i \in T_x M$, $\xi \in T^*_x M$. 
Definition 9. We define argute slices to be those compactly-supported Radon probability measures $\nu$ on $X$ such that their induced $(n-1)$-dimensional currents $T_\nu$,

$$\langle T_\nu, \omega \rangle = \int_X (\omega \wedge \xi)(x, v_1, v_2, \ldots, v_n) \, d\nu(x, v_1, \ldots, v_n, \xi), \quad \omega \in \Omega^{n-1}(M),$$

have null boundary $\partial T_\nu = 0$. Denote the set of argute slices by $\mathcal{S}$.

Note that the measure $\pi_1^*\nu$ on $T^n M$ induces a current of dimension $n$ on $M$. As in Example 8, the $\xi$ factor is there to perform the transition between the $(n-1)$-dimensional slice and the $n$-dimensional slice it is a slice of.

Definition 10. For $T \geq 0$, a curve of argute slices is a family $(\nu_t)_{t \in [0, T]}$, such that

C1. $\nu_t \in \mathcal{S}$ for all $t \in [0, T]$,

C2. for $t \in [0, T]$, the measure $\mu_t = \int_0^T \pi_1^* \nu_t \, dt$ on $T^n M$ is such that its associated current $T_{\mu_t}$ has boundary

$$\partial T_{\mu_t} = T_{\nu_0} - T_{\nu_T}.$$  

In other words, $\mu_t \in \text{Hom}(T_{\nu_0}, T_{\nu_T})$ in $\mathcal{C}_{HM}$. In particular, $s \mapsto \int \omega \, d\pi_1^* \nu_s$ must be integrable on $[0, T]$ for all $\omega \in \Omega^n(M)$.

C3. Given Lebesgue-almost any $t \in (0, T)$ and sequences $\{r_i\}$ and $\{s_i\}$ with $0 \leq r_i < t < s_i \leq T$ and $\lim_i r_i = t = \lim_i s_i$, we have

$$\nu_t = \lim_i \left( \frac{1}{s_i - r_i} \int_{s_i}^{r_i} \nu_s \, ds \right).$$

For a curve of slices $\gamma = (\nu_t)_{t \in [0, T]}$, its associated holonomic measure $\mu_\gamma$, is defined by

$$\mu_\gamma = \int_0^T \pi_1^* \nu_t \, dt.$$  

With these definitions and given the same information we used for $\mathcal{C}_{HM}$, namely, a Riemannian manifold $(M, g)$ and a Lagrangian function $L : T^n M \to \mathbb{R}$ satisfying $\mathbf{14.12.1}$ we may define a Lagrangian category $\mathcal{C}_{HM}$ that is closely related to $\mathcal{C}_{HM}$.

Definition 11. Let $(M, g)$ be a Riemannian manifold and $L : T^n M \to \mathbb{R}$ be a measurable function. We let the collection of objects of $\mathcal{C}_{HM}$ be the set of argute slices $\mathcal{S}$, and we let $\text{Hom}(\eta_1, \eta_2)$ be the set of curves $\gamma = (\nu_t)_{t \in [0, T]}$ of argute slices joining $\nu_0 = \eta_1$ and $\nu_T = \eta_2$, and such that $L$ is integrable with respect to the associated holonomic measure $\mu_\gamma$. The action of $\gamma = (\nu_t)_{t \in [0, T]} \in \text{Hom}(\nu_1, \nu_2)$ is given by

$$A(\gamma) = \int_0^T \left[ \int_{T^n M} L(\pi_1^* \nu_t) \right] \, dt = \int_{T^n M} L \, d\mu_\gamma,$$

and the internal time duration

$$t(\gamma) = T = \int_{T^n M} d\mu_\gamma$$

coincides with $\int_0^T \nu_t(X) \, dt = \mu_T(T^n M)$ because each $\nu_t$ is a probability.

Remark 12. The category $\mathcal{C}_{HM}$ can basically be obtained from $\mathcal{C}_{HM}$ by forgetting the extra structure and information that the latter encodes. More precisely, the connection of $\mathcal{C}_{HM}$ with $\mathcal{C}_{HM}$ can be described as follows.
First off, we may choose a measurable map $\Xi : T^{n-1}M \to \mathcal{X}$ associating to each point $(x, v_1, \ldots, v_{n-1})$ a lift $(x, v_1, \ldots, v_n, w, w^*) \in \mathcal{X}$ using a vector $w = w(x, v_1, \ldots, v_{n-1}) \in TM$ linearly independent to $v_1, \ldots, v_{n-1}$, and its dual $w^* \in T^*M$. Then, each slice $T \in C_{HM}$ induced by a measure $\nu$ on $T^{n-1}M$, $T = T_\nu$, can be realized as $T_\nu$ for the argute slice $\hat{\nu} = \Xi_\nu$.

Accordingly, given a curve of argute slices $\gamma \in \text{Hom}(\hat{\nu}_1, \hat{\nu}_2)$ in $\hat{C}_{HM}$, we get a holonomic measure $\mu_\gamma \in \text{Hom}(\hat{T}_{\nu_1}, \hat{T}_{\nu_2})$ in $C_{HM}$, and in this case it remains an open question whether this map is surjective; we expect it to be.

To obtain a weak KAM theory, we will assume the following:

R1. The function $L : T^nM \to \mathbb{R}$ is continuous.

R2. There are a constant $c \in \mathbb{R}$ and a Lipschitz differential form $\omega \in \Omega^{n-1}(M)$ such that $L + c \geq d\omega$.

R3. The smooth manifold $M$ is compact.

R4. For every $B > 0$ there is $C > 0$ such that, for all $(x, v_1, \ldots, v_n) \in T^nM$,

$$L(x, v_1, \ldots, v_n) \geq B \text{vol}(x, v_1, \ldots, v_n) - C.$$ 

The main result of this section is

**Theorem 13.** If [R1]–[R4] hold, then there is a weak KAM solution $u : C_{HM} \to \mathbb{R}$.

**Remark 14.** Remark 13 explains the way in which $u$ can be connected to an exact differential form $d\omega$ on $M$ that satisfies a sort of Hamilton–Jacobi equation.

**Proof of Theorem 13** By Corollary 16, there is a function $u$ satisfying [11]. Lemma 17 shows that $\hat{C}_{HM}$ is weak KAM amenable. By Lemma 1, it follows that $u$ is a weak KAM solution. 

**Lemma 15.** Assume that the manifold $M$ is $\sigma$-finite, and [R1] and [R2] hold. The Lagrangian category $\hat{C}_{HM}$ is a $\sigma$-compact metric space that satisfies hypotheses [K1] and [K2] in Theorem 2.

**Proof of Lemma 15** Let $K_1 \subset K_2 \subset K_3 \subset \ldots M$ be a collection of nested compact sets such that $M = \bigcup_k K_k$. For $m > 0$, consider the subcategory $\hat{C}_m$ whose objects are those $\nu \in \hat{C}_{HM}$ with mass $\int_{\mathcal{X}} 1 \, d\nu \leq m$ and supported on the compact subset of $X$ consisting of points $(x, v_1, \ldots, v_n, t)$ with $x \in K_m$ and $\|v_1\|, \ldots, \|v_n\|, \|t\| \leq m$, with morphisms corresponding to the curves of those argute slices contained in in $\hat{C}_m$. It follows from the same argument as in Lemma 5 that $\hat{C}_m$ is compact. Since $\hat{C}_{HM} = \bigcup_m \hat{C}_m$, it is $\sigma$-compact.

Property [K1] follows from the same argument as in the proof of Lemma 1 because [R3] and [R1] imply that $L$ is bounded on compact subsets of $T^nM$.

Property [K2] follows from [R2]. To see how this works, let, for each argute slice $\nu$, $u(\nu) = \int_{\mathcal{X}} \omega \wedge t \, d\nu = \hat{T}_\nu(\omega)$. If $\nu_1$ and $\nu_2$ are two argute slices and $\gamma \in \text{Hom}(\nu_1, \nu_2)$, we then have, by [K2]

$$u(\nu_2) - u(\nu_1) = \hat{T}_{\nu_2}(\omega) - \hat{T}_{\nu_1}(\omega) = \partial T_{\mu_\gamma}(\omega)$$

$$= T_{\mu_\gamma}(d\omega) = \int_{T^*M} d\omega \, d\mu_\gamma \leq \int_{T^*M} L \, d\mu_\gamma = A(\gamma) + \text{ct}(\gamma).$$

Taking the infimum over all $\gamma \in \text{Hom}(\nu_1, \nu_2)$ with $t(\gamma) = t$, we get [K2].

**Corollary 16.** Assume that the manifold $M$ is $\sigma$-finite, and [R1] and [R2] hold. Then there is a Lipschitz function $u : \hat{C}_{HM} \to \mathbb{R}$ satisfying [11].
Lemma 17. Assume \( H \) hold. Then the category \( \dot{C}_{\mathrm{HM}} \) is finely KAM amenable.

Proof of Lemma 17. In the notation of Section 2.1.2 for \( \eta, \nu \in C_{\mathrm{HM}} \) and \( \gamma \in \text{Hom}(\eta, \nu) \) given by a family \( \gamma = (\nu_t)_t \), we let \( \gamma(t) = \nu_t \). Then it is clear that \( A1 \) hold.

Let \( t > 0 \) and \( \nu \in C_{\mathrm{HM}} \) and let us show that \( A4 \) holds. It follows from \( R2 \) that, for \( \gamma \in \text{Hom}(\eta, \nu) \) with \( t(\gamma) = \int_{T^nM} d\mu_\gamma = t \),

\[
A(\gamma) \geq B \int_{T^nM} \text{vol } d\mu_\gamma - Ct.
\]

This means that, given \( \epsilon > 0 \), there is a compact set \( K_\epsilon \subset T^nM \) such that any \( \gamma \) with

\[
A(\gamma) \leq \inf_{\eta \in \text{Hom}(\eta, \nu)} A(\eta) + \epsilon
\]

satisfies \( \text{supp } \mu_\gamma \subseteq K_\epsilon \), lest the volume integral term be too large. In the following, set \( K = K_1 \).

Endow \( H_{\nu,t} = \bigcup_{\gamma \in C_{\mathrm{HM}}} \text{Hom}(\eta, \nu) \cap t^{-1}(t) \) with the topology in which a sequence of curves \( \gamma_i = (\nu_{si})_{s \in [0, t]} \) converges to another curve \( \gamma = (\nu_s)_{s \in [0, t]} \) if, for all \( f \in C^0(\mathcal{X}) \) and all \( 0 \leq r \leq s \leq t \)

\[
(6) \int_r^s \int_{\mathcal{X}} f \, d\nu'_{si} \, dt \rightarrow \int_r^s \int_{\mathcal{X}} f \, d\nu_{si} \, dt.
\]

In this topology, \( H_{\nu,t} \) is complete and, if \( H_{\nu,t,K} \) is the closed subset of \( H_{\nu,t} \) consisting of curves \( \gamma \) with \( t(\gamma) = t \) and \( \text{supp } \mu_\gamma \subseteq K \), then \( H_{\nu,t,K} \) is compact by the Prokhorov theorem [14]. In particular \( (6) \) means that, if \( \gamma_i \to \gamma \) in \( H_{\nu,t,K} \),

\[
A(\gamma_i) = \int_{T^nM} L \, d\mu_{\gamma_i} \to \int_{T^nM} L \, d\mu_\gamma = A(\gamma),
\]

so \( A \) is continuous.

By Lemma 19, \( h_{\text{dist}(\nu_1, \nu_2)}(\nu_1, \nu_2) \leq P \, \text{dist}(\nu_1, \nu_2) \), and the same argument as in Lemma 14 gives that any function \( u \) satisfying (1) is Lipschitz.

From the continuity of \( u \) and \( A \) and the compactness of \( H_{\nu,t,K} \), we know that within \( H_{\nu,t,K} \subset H_\nu \) the minimum of the map \( (y, \gamma) \mapsto u(y) + A(\gamma) + c \, t \) is achieved. Since all curves \( \gamma \) in \( H_\nu \) with \( \text{supp } \mu_\gamma \) not contained in \( K \) have larger action, the minimum in \( H_{\nu,t} \) is the same as in \( H_{\nu,t,K} \), and is hence also reached there. This is the statement of \( A4 \).

That \( A5 \) is true also follows from the compactness of \( H_{\nu,t,K} \); let us see how. For each \( i \in \mathbb{N} \), \( 0 < t \leq i \), \( \mathcal{G}_i \) is nonempty by \( A5 \), take \( \gamma_i = (\nu_{si})_{s \in [0, t]} \in \mathcal{G}_i \). Consider the subfamilies \( \gamma_i = (\nu_{si})_{s \in [-t, 0]} \). These satisfy \( \gamma_i \in \mathcal{G}_i \) and \( \gamma_i \in H_{\nu,t,K} \), so there is a subsequence converging to some \( \gamma_i \in \cap_{i \geq t} \mathcal{G}_i \subset \cap_{i \geq t} H_{\nu,t,K} \), which is what \( A5 \) requires.

4. Characterization of minimizable Lagrangians

In this section we characterize the Lagrangian actions minimizable by holonomic measures in Theorem 13. We then connect the weak KAM solutions from Theorem 13 with exact differential forms in Corollary 21.
We will work in the context of the Lagrangian category $\mathcal{C}_\text{HM}$ defined in Section 3.1 so that we are given a manifold $M$ and a Borel-measurable function $L: T^n M \to \mathbb{R}$. The action $A$ is given by integration of $L$ with respect to the holonomic measures that constitute the morphisms of $\mathcal{C}_\text{HM}$.

We denote by $C^\infty(T^n M)$ the space of infinitely-differentiable functions on the bundle $T^n M$, and we let $\mathcal{E}'(T^n M)$ denote the space of compactly-supported distributions on $T^n M$, which is dual to $C^\infty(T^n M)$. The set $\mathcal{E}'(T^n M)$ contains, in particular, all compactly-supported, Radon measures $\mu$ on $T^n M$.

Fix two objects $T_1, T_2$ in $\mathcal{C}_\text{HM}$ such that the set $\text{Hom}(T_1, T_2)$ is not empty. In other words, $T_1$ and $T_2$ are normal $(n-1)$-dimensional currents without boundary, and there exists compactly-supported Radon measures $\mu$ on $T^n M$ such that the induced current $T_\mu$ has boundary $\partial T_\mu = T_2 - T_1$, so that $\mu \in \text{Hom}(T_1, T_2)$.

Recall a topological vector space $V$ is sequential if for every set $S \subset V$ and every element $s$ in the closure $\overline{S}$ there exists a sequence of points $\{s_i\}_i \subseteq S$ such that $s_i \to s$. This is verified if $V$ is normed, metric, or first countable (that is, if every point has a countable neighborhood basis).

Let $E$ be a complete, sequential, locally-convex topological vector space of Borel measurable functions on $T^n M$ that contains $C^\infty(T^n M)$ as a subspace. Assume that every element of $\text{Hom}(T_1, T_2)$ induces a continuous linear functional on $E$, i.e., $\text{Hom}(T_1, T_2) \subseteq E^*$, and that the topology of $C^\infty(T^n M)$ is finer than the one this subspace inherits from $E$, or, in other words, that every open set in the inherited topology is an open set in the topology induced by the seminorms

$$|f|_{k,k} = \sum \sup_{|I| \leq k} \left| \partial^I f(x) \right|,$$

where $k \geq 0$, $K \subset T^n M$ is compact, and the sum is taken over all multi-indices $I = (i_1, \ldots, i_n)$ with $|I| = \sum_i i_r \leq k$. This assumption implies that every continuous linear functional $\vartheta \in E^*$ defines a compactly-supported distribution when restricted to $C^\infty(T^n M)$. For example, $E$ can be the space $C^k(T^n M), k \in [0, +\infty]$, with the topology of uniform convergence on compact sets of the derivatives of order $\leq k$.

**Theorem 18.** If $L$ is an element of $E$ such that its action functional $A$ reaches its minimum within $\text{Hom}(T_1, T_2)$ at some measure $\mu$, then there exist differential forms $\omega_1, \omega_2, \ldots$ in $\Omega^{n-1}(M)$, and nonnegative functions $g_1, g_2, \ldots$ in $E$ such that

$$\lim_{i \to +\infty} \int_{T^n M} g_i \, d\mu = 0,$$

and

$$L = \lim_{i \to +\infty} d\omega_i + g_i,$$

where the limit is taken in $E$. In particular,

$$\int_{T^n M} L \, d\mu = \lim_{i} \partial T_\mu(\omega_i) = T_2(\omega_i) - T_1(\omega_i).$$

**Remark 19.** In many cases, like in the situation of Corollary 21 below, using the version of the Arzelà–Ascoli theorem given in Lemma 22, one can extract a Lipschitz limit $\omega$ of the forms $\omega_i$. It then satisfies

$$d\omega - L \leq 0,$$

with equality $\mu$-almost everywhere. Here one can think of $H(d\omega) = d\omega - L$ as the Hamiltonian, so that we are looking at a sort of Hamilton-Jacobi equation. In this
sense, we can say that $\omega$ is a critical subsolution of the Hamilton-Jacobi equation; cf. [9].

In order to prove the theorem, we need

**Lemma 20.** In the setting of Theorem 18 let

$Q = \{ \ell : E^* \to \mathbb{R} \mid \ell \text{ is affine and continuous, } \ell(\mu) \geq 0 \text{ for all } \mu \in \text{Hom}(T_1, T_2) \}$,

$R = \{ \ell : E^* \to \mathbb{R} \mid \ell(\xi) = \xi(d\omega + g) - T_2(\omega) - T_1(\omega), \omega \in \Omega^{n-1}(M), g \in E, g \geq 0 \}$.

Then we have $Q = \overline{R}$ in $E$.

**Proof of Lemma 20.** For a convex subset $A$ of a topological vector space, we will denote by $A'$ the set of real-valued continuous affine functionals in that are non-negative on $A$.

We first observe that $Q' = \text{Hom}(T_1, T_2) = R'$. To see why, note that the set of functionals induced by nonnegative elements of $C^\infty(T^n M)$ is a subset both of $Q$ and of $R$ (in the latter case, take $\omega = 0$ and $g \in C^\infty(T^n M) \subseteq E$), so by [12, §6.22] all elements of $Q'$ and $R'$ can be represented as integration over compactly-supported, nonnegative, Radon measures. Also, if $\omega \in \Omega^{n-1}(M)$, then the affine functional

$$\ell_\omega(\xi) = \xi(d\omega) - T_2(\omega) + T_1(\omega)$$

belongs to both $Q$ and $R$, so it is nonnegative throughout $Q'$ and $R'$. Since its negative $\ell_{-\omega} = -\ell_\omega$ is, for the same reason, nonnegative throughout $Q'$ and $R'$, we conclude that

$$0 = \ell_{\omega}(\xi) = \xi(d\omega) - T_2(\omega) + T_1(\omega) = \partial T_\xi(\omega) - T_2(\omega) + T_1(\omega)$$

for all $\xi \in Q' \cup R'$ and all $\omega \in \Omega^{n-1}(M)$. In other words, the current $T_\xi$ induced by the measure representing $\xi$ has boundary $\partial T_\xi = T_2 - T_1$. So indeed $Q' = \text{Hom}(T_1, T_2) = R'$.

Since $Q' = R'$, we have $\overline{Q} = \overline{R}$. Indeed, if there were some $g \in \overline{Q} \setminus \overline{R}$, then the Hahn–Banach separation theorem would produce a continuous affine functional $\ell : E \to \mathbb{R}$ with $\ell(r) < a < b < \ell(q)$ for some $a, b \in \mathbb{R}$ and all $r \in \overline{R}$, whence the continuous affine functional $\ell - a$ would be positive on $R$ and not on $Q$, contradicting $Q' = R'$; a similar situation would arise if $\overline{R} \setminus \overline{Q} \neq \emptyset$.

The claim of the lemma follows from the fact that $Q$ is closed.

**Proof of Theorem 18** The functional $\ell(\xi) = \int L d\xi - \int L d\mu$ belongs to the set $Q$ in the statement of Lemma 20 and by the lemma it also belongs to $\overline{R}$. The sequentiality of $E$ implies that the topological closure equals the sequential closure, so there exists a sequence of affine functions $\ell_i \in R$ of the form

$$\ell_i(\xi) = \xi(g_i + d\omega_i) - T_2(\omega_i) - T_1(\omega_i), \quad i = 1, 2, \ldots,$$

with $g_i \in E$, $\omega_i \in \Omega^{n-1}(M)$, $g_i \geq 0$, and converging to $\ell_i \to \ell$. Comparing the linear and constant parts of the functionals $\ell$ and $\ell_i$, we conclude that $\int L d\mu = \lim_i T_2(\omega_i) - T_1(\omega_i)$. We also have that

$$0 = \ell(\mu) = \lim_i \ell_i(\mu) = \lim_i \int_{\Gamma^n M} g_i d\mu,$$

where the last equality is true because the boundary of $T_\mu$ is $T_2 - T_1$, that is $\int d\omega_i d\mu = T_2(\omega) - T_1(\omega_i)$. 

\[\square\]
Corollary 21. Let $L: T^n M \to \mathbb{R}$ be a $C^0$ function satisfying the hypotheses of Theorem 13. Assume that there is a family $\{\gamma_\alpha\}_{\alpha \in I}$ of curves of argute slices such that the support of the holonomic measures $\mu_{\gamma_\alpha}$ covers almost all of $M$, that is, such that if $\gamma_\alpha = (\nu_s^\alpha)_{s \in [0,T_\alpha]}$ for $\alpha \in I$, then the complement of the set
\[
\bigcup_{\alpha \in I} \pi_M(\text{supp } \mu_{\gamma_\alpha}) = \bigcup_{\alpha \in I} \bigcup_{s \in [0,T_\alpha]} \pi_M(\text{supp } \nu_s^\alpha) \subseteq M
\]
has Lebesgue measure zero on $M$.

Moreover, assume that the curves $\gamma_\alpha$ minimize the action $A$ simultaneously, in the sense that every convex combination of the associated holonomic measures $\mu_{\gamma_\alpha}$ minimizes $A$ with respect to all positive, compactly-supported, Radon measures with the same boundary.

Then the function $u$ in the conclusion of Theorem 13 corresponds to a Lipschitz form $\omega \in \Omega^{n-1}(M)$, and for all $\alpha \in I, t \in [0,T_\alpha]$ we have
\[
T^{\gamma_\alpha}_t(\omega) = u(\nu_t^\alpha).
\]

Proof. Form a convex combination of all the the associated holonomic measures $\mu_{\gamma_\alpha}$, using a probability measure supported throughout $I$. Apply Theorem 13 to obtain forms $\{\omega_j\}_{j=1}^\infty \subseteq \Omega^{n-1}$ as in the statement of that result, and then apply also Lemma 22 to obtain a subsequence of the forms $\omega_j$ converging to a Lipschitz differential form $\omega$ that corresponds to $u$ as statement of the corollary. □

Lemma 22 (Arzelà–Ascoli for sections of a vector bundle). Let $\alpha_1, \alpha_2, \ldots$ be a sequence of smooth, uniformly bounded as sections of a vector bundle $F \to M$ on the compact manifold $M$, and assume that their derivatives $d\alpha_1, d\alpha_2, \ldots$ are also uniformly bounded. Then there is a subsequence $\{i_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that the sequence $\alpha_{i_1}, \alpha_{i_2}, \ldots$ converges uniformly to a Lipschitz section $\alpha$ of $F$.

Proof. Take a finite set of sections $\beta_1, \beta_2, \ldots, \beta_N$ of $F$ such that, for each $x \in M$, the set $\{\beta_1(x), \ldots, \beta_N(x)\}$ is a basis for the fiber of $F$ at $x$. Express each $\alpha_i$ as $\alpha_i(x) = \sum_{j=1}^N \phi_i^j(x) \beta_j$, for some smooth functions $\phi_i^j$. Note that the uniform boundedness of $d\alpha_i$ implies the uniform boundedness of the derivatives $d\phi_i^j$. Apply the classical Arzelà–Ascoli result for Lipschitz functions to the sequences $\{\phi_i^j\}_{i=1}^\infty$, $j = 1, \ldots, N$, successively so as to obtain a subsequence for which all $N$ sequences converge simultaneously to Lipschitz functions $\phi^j$, which gives the statement of the lemma with $\alpha(x) = \sum_{j=1}^N \phi^j(x) \beta_j$. □

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