Some issues related to conformal anomaly induced effective action

Sergio Zerbini

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare
Gruppo Collegato di Trento, Italia

Abstract

Some issues related to quantum anomaly induced effects due to matter are considered. Explicit examples corresponding to quantum creation of d4 dilatonic AdS Universe and of d2 dilatonic AdS Black Hole (BH) are discussed. Motivated by holographic RG, in a similar approach, it is shown that, starting from a 5 dimensional AdS Universe, 4-dimensional de Sitter or AdS world is generated on the boundary of such Universe as a result of quantum effects. A 5-dimensional braneworld cosmological scenario is also considered, where the brane tension is not longer a free parameter, but its role is taken by quantum effects induced by the 4-dimensional conformal anomaly associated with conformal coupled matter. As a result, consistent quantum creations of De Sitter or AdS curved branes are possible.

*zerbini@science.unitn.it
1 Introduction

In this paper, we will review the role of quantum anomaly induced effective action (see [1]) in the dynamical realization of dilatonic AdS backgrounds in various dimensions. First, the 4d dilatonic classical gravity with $N$ dilaton coupled quantum fermions is considered. Solving the effective equations, it follows that quantum corrected dilatonic AdS Universe may occur, even in situation where such classical solution was absent. However, the probability of generation of AdS Universe is significantly less than of quantum creation of de Sitter Universe.

Making use of the same techniques, the dilatonic gravity with dilaton coupled massless quantum fermions in two dimensions is investigated. The complete anomaly induced effective action is found. The quantum 2d dilatonic AdS BH, which was not existing on classical level, is constructed as the solution of the effective equations, namely the creation of quantum 2d AdS BH is shown to occur.

From another side, within the well known AdS/CFT correspondence, where AdS background is introduced from the very beginning, we discuss the holographic RG action leading to warped compactification (RS-like Universe) in the region where both sides of AdS/CFT correspondence (i.e. bulk and boundary) are still relevant. Then, using the same anomaly induced effective action, previously constructed, we suggest the way it should appear in the dynamics of five-dimensional world, on equal footing with 5d gravity action. As a result, the dynamical effective equations could be solved realizing 5d AdS Universe with warp factor. On the boundary of such Universe, the de Sitter (inflationary) world occurs, namely it is actually induced by quantum effects, without introducing the four-dimensional cosmological constant on brane by hands.

2 Quantum instability of 4d AdS dilaton universe

Let us begin with the 4-dimensional dilaton gravity theory, described by

$$S = \int d^4 x \sqrt{-g} \left[ -\frac{1}{16\pi G} (R + 6\Lambda) + \alpha (\nabla_\mu \phi)(\nabla^\mu \phi) \right],$$

where $\Lambda$ is the cosmological constant and $\alpha$ some suitable parameter. For constant dilaton $\phi$ and negative cosmological constant, the classical background solution corresponds to the 4d AdS space. Even for non-constant dilaton, there are solutions interpolating between asymptotically AdS and
flat space with singular dilaton \[2\]. Our primary interest will be the issue of the stability of the (classical) AdS background in the above theory, under the quantum fluctuations of the conformal matter. As matter Lagrangian, we consider the one corresponding to \(N\) massless dilaton coupled Dirac spinors

\[
L_M = \exp (A \phi) \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \nabla_\mu \psi^i .
\]  

(2)

Here \(A\) is some constant parameter. The above strong matter-dilaton coupling is typical for any matter- Brans-Dicke theory in the Einstein frame.

The quantum effective action for dilaton coupled spinor has been found in ref. [3] by integrating the conformal anomaly. This quantum effective action should be added to the classical one \(S\) (there is only the dilaton-gravitational background under consideration).

Let us now define the space-time we are going to work with. We consider the 4d AdS background with the static metric

\[
ds^2 = e^{-2\lambda \bar{x}_3} \left[ dt^2 - (dx_1)^2 - (dx_2)^2 - 4 \right].
\]  

(3)

It has a negative cosmological constant \(\Lambda = -\lambda^2\). Making the coordinates transformations

\[
y = \frac{e^{\lambda \bar{x}_3}}{\lambda}
\]  

(4)

one can present \(3\) in the conformally flat form

\[
ds^2 = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu ,
\]  

(5)

where

\[
a = e^{-\lambda \bar{x}_3} = \frac{1}{\lambda y}.
\]  

(6)

In order to investigate the role of the quantum effects to the dilaton AdS universe, we shall consider the metric \(3\) with an arbitrary scale factor to be determined dynamically. The anomaly induced effective action of ref. [3] on such background may be written in the following form [4]:

\[
W = V_3 \int dy \left\{ 2b_1 \sigma_1 \sigma''' - 2(b + b_1) \left( \sigma'' - \sigma'_1 \right)^2 \right\} .
\]  

(7)

Here, \(V_3\) is the (infinite) volume of 3-dimensional flat space-time( time is included), \(\sigma = \ln a(y), \sigma_1 = \sigma + \frac{A \phi}{3}, ' = dy, and b = \frac{3N}{60(4\pi)^2}, b_1 = \)
\[-\frac{11N}{300(4\pi)^3}\] It should be noted that (7) may be regarded as the complete one-loop effective action. The classical gravitational action on the background (3) with non-trivial dilaton reads
\[
S = V_3 \int dy \left\{ \frac{1}{\chi} \left[ 6(\sigma'' + \sigma'^2)e^{2\sigma} - 6\Lambda e^{4\sigma} \right] + \alpha\phi'^2 e^{2\sigma} \right\}.
\]
with \(\chi = 16\pi G\). The quantum corrections of the conformal coupled matter field can be taken into account starting from the effective equations obtained from the effective action \(S + W\), thus implementing the standard semiclassical approach based on the Einstein Eqs. having as source the vacuum expectation value of the stress tensor of the quantum matter. These equations look similarly to the ones associated with the De Sitter quantum corrected universe [4], with the addition of the cosmological constant contribution and an opposite sign in the Einstein term: One has [5]
\[
\tilde{C}e^{(A\phi/3)} - \frac{12}{\chi} a'' - \frac{24\Lambda a^3}{\chi} + 2\alpha a\phi'^2 = 0
\]
\[
\frac{A}{3} \tilde{C}ae^{(A\phi/3)} - 2\alpha(a^2 \phi')' = 0,
\]
(8)
where
\[
\tilde{C} = -\frac{4b}{a} \left[ \frac{\tilde{a}'''' - 3\tilde{a}''}{\tilde{a}''} - \frac{\tilde{a}'''''}{\tilde{a}''} \right]
\]
(9)
\[
- \frac{24}{a^4} \left[ (b - b_1)\tilde{a}'' a'^2 + b_1 \frac{\tilde{a}'^2}{\tilde{a}} \right],
\]
(10)
and
\[
\tilde{a} = ae^{(A\phi/3)}.
\]
First, it is easy to show that the AdS Universe exists at classical level.

Furthermore, in the absence of the dilaton and vanishing cosmological constant, only the first of (8) survives. It may be solved via the special ansatz \(a = c/y\) with the constraint \(c^2 = b_1\chi\). Since \(b_1 < 0\), one gets an imaginary scale factor \(a\), namely a quantum annihilation of the AdS universe, as it was shown in detail in ref. [6]. Hence, classical AdS Universe is not stable under the action of quantum fluctuations. Note that dilaton is constant there.
The general solution of the system of differential equations (8) is very
difficult to find. However, there exist specific solutions [5]
a(y) = \frac{1}{Hy}, \quad \phi'(y) = \frac{1}{H_1y},
(12)
where $H$ and $H_1$ are constants such that
\[ \frac{12\Lambda}{\chi H^2} = \frac{\alpha}{H_1^2} - \frac{9\alpha}{AH_1} - \frac{12}{\chi}, \]
(13)
\[ \frac{81\alpha}{2AbH^2} = -(A - 3H_1) \left( \frac{18H_1^2 - 2\tilde{b}(A - 3H_1)^2}{H_1^2} \right), \]
(14)
with $\tilde{b} = 1 + \frac{b}{\alpha}$. As a result, one may eliminate $H$ from the first equation
and obtain a third order complete algebraic equation for the quantity $H_1$,
which in principle can be solved. However, for $\Lambda = 0$, there is the complete
decoupling of the two equations and one easily arrives at
\[ H_1 = \frac{1}{24} \left[ \frac{9\alpha\chi}{A} \pm \sqrt{\frac{81\alpha^2\chi^2}{A^2} + 48\alpha\chi} \right], \]
(15)
and the other quantity $H$ may be obtained from equation (14):
\[ \frac{81\alpha}{2bH^2} = \frac{2b}{81\alpha} \left( -18 + 81\tilde{b} \right) A^2 + \frac{24\tilde{b}A^4}{\alpha\chi} \]
\[ \pm \frac{\alpha\chi}{3bA} \sqrt{\frac{81\alpha^2\chi^2}{A^2} + 48\alpha\chi}. \]
(16)
(17)
Since $\tilde{b} = \frac{\pi^2}{12} > 0$ and $-18 + 81\tilde{b} > 0$, there is always a real solution for $H$
at least for positive $\alpha$ if we choose the sign $\pm$ in front of the square root properly.

Hence, we demonstrated that in presence of non-constant dilaton
the quantum AdS Universe solution in dilatonic gravity is less unstable. At
least, it may be realized while it did not exist on classical level! The above
mechanism may serve as the one corresponding to quantum creation of pri-
mordial AdS Black Holes in early Universe. However, for 4d AdS BH one
should calculate the extra piece of effective action which is non-local and
very complicated. The complete calculation of it is not known, presumably it
could be found only as expansion on theory parameters. That is the reason
we prefer to present such analysis only in two dimensions.
3 Quantum creation of 2d AdS black hole

Here we investigate the possibility for quantum creation of 2d AdS BHs using methods developed in previous section. Motivated by the 4-dimensional case, we may assume that the classical action for the 2-dimensional dilaton gravity theory reads

\[ S = \int d^2x \sqrt{-g(x)} \left[ -\frac{R + 6\Lambda}{\chi} + \frac{1}{2}(\nabla_\mu \phi)(\nabla^\mu \phi) \right] + S_M, \tag{18} \]

where \( A \) is a constant parameter and the matter Lagrangian is the one of two-dimensional Majorana spinors:

\[ S_M = \int d^2x \sqrt{-g(x)} \exp(A\phi)L_M \]

\[ L_M = \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \nabla_\mu \psi^i. \tag{19} \]

Let us neglect the classical matter contribution since we are interested only in the one-loop EA induced by the conformal anomaly of the quantum matter.

In two dimensions, a general static metric may be written in the form

\[ ds^2 = V(r)dt^2 - \frac{1}{W(r)}dr^2. \tag{20} \]

It is well know that introducing the new radial coordinate \( r^* \), defined by

\[ r^* = \int \frac{dr}{\sqrt{V(r)W(r)}}, \tag{21} \]

one gets a conformally flat space-time,

\[ g_{\mu\nu} = V(r(r^*))\eta_{\mu\nu} = e^{2\sigma(r^*)}\eta_{\mu\nu}. \tag{22} \]

We also assume that the field \( \phi \) depends only on \( r^* \).

Since the conformal anomaly for the dilaton coupled spinor is (see \([3]\))

\[ T = \frac{c}{2} \bar{R}, \tag{23} \]
where $c = N/(12\pi)$ and $R$ is calculated on the metric $\tilde{g}_{\mu\nu} = e^{2A\phi}g_{\mu\nu}$, the anomaly induced EA in the local, non-covariant form reads
\begin{equation}
W = \frac{c}{2} \int d^2 x \tilde{\sigma} \tilde{\sigma}''.
\end{equation}

Here $\tilde{\sigma} = \sigma + A\phi$ and $' = \frac{d}{dr}\text{*}$. Note that this is, up to a non–essential constant, an exact one-loop expression. The total one-loop effective action is $S + W$, i.e.
\begin{equation}
S + W = V_1 \int dr^* \left[ \frac{k_G}{2} \left( \sigma'' - 6\Lambda e^{2\sigma} \right) + \frac{1}{2}(\phi')^2 \right] \\
+ V_1 \int dr^* \left[ \frac{c}{2} (\sigma'' + A\phi'') (\sigma'' + A\phi'') \right],
\end{equation}

where $k_G = \frac{1}{8\pi}$, and $V_1$ the (infinite) temporal volume. Since 2d Einstein theory is trivial, the whole dynamics appears as a result of quantum effects. The equations of motion given by the variations of $\phi$ and $\sigma$ are
\begin{align}
0 &= -\left(1 - cA^2\right) \phi'' + cA\sigma'' \\
0 &= c (\sigma'' + A\phi'') - 6k_G\Lambda e^{2\sigma}.
\end{align}

From (25) and (26), we obtain
\begin{equation}
E = -3k_G\Lambda e^{2\sigma} + \frac{c}{2 (1 - cA^2)} (\phi')^2.
\end{equation}

where $E$ is a constant of integration.

Thus, from (27) one gets
\begin{equation}
e^{2\sigma} = \frac{(r - r_0)^2}{b} - a \\
b \equiv \frac{6(1 - cA^2)k_G\Lambda}{c}, \quad a \equiv \frac{E}{3k_G\Lambda}.
\end{equation}

Here $r_0$ is another constant of the integration. If we further redefine,
\begin{equation}
\frac{1}{l^2} \equiv b, \quad cM \equiv \frac{2r_0}{b}, \quad k \equiv -a + \frac{r_0}{b}^2,
\end{equation}

we obtain a generic 2d AdS black hole solution,
\begin{equation}
W(r) = V(r) = e^{2\sigma} = k - cMr + \frac{r^2}{l^2}.
\end{equation}
where $M$ may be interpreted as the mass of the BH. We may take $k = \pm 1$, or $k = 0$. In general, we have a simple positive root, interpreted as horizon radius. In the case $k = 1$ one must have $cM > 2$. It is easy to show that the above metric has a negative constant scalar curvature and for large $r$, $V(r) \simeq \frac{c^2}{r^2}$, namely one gets the AdS asymptotic behavior. For the sake of simplicity, let us consider the case $k = 0$. In this case the horizon radius and the Hawking temperature read

$$r_H = cMl^2, \beta_H = \frac{4\pi}{cM} = \frac{4\pi l^2}{r_H}. \quad (31)$$

Thus, using anomaly induced effective action for dilaton coupled spinor we proved the possibility of quantum realization of 2d AdS BH which was not existing at classical level. It is interesting that, unlike to 4d case where EA for AdS BH is not completely known, 2d case is exactly solvable. Our solution may be interpreted as quantum creation of dilatonic AdS BH.

## 4 Holographic Renormalization Group and Dynamical Gravity

In the standard AdS/CFT correspondence, one can think about the simultaneous incorporation of string compactification with exponential warp factor (Randall-Sundrum compactification and the holographic map between 5d supergravity and 4d boundary (gauge) theory. Moreover, it could be extremely interesting to do it in such a way that dynamical gravity would appear on the boundary side. One possibility to realize such a mechanism is presumably related with holographic renormalization group (RG), see [7, 8] for an introduction.

There was very interesting suggestion in this respect in ref. [8] to consider low-energy effective action (EA) in the region where field theoretical quantities and analogous supergravity quantities could be considered on equal foot. In other words, this is the way to match two dual descriptions into the global picture of some, more universal RG flow. Immediate consequence of such point of view is the possible explanation of smallness of cosmological constant, the stability of flat spacetime along the RG flow and possible understanding of 4d gravity appearance in standard AdS/CFT set-up.

The explicit realization of ideas of ref. [8] via the construction of the corresponding phenomenological model has been presented in [9]. We summarize the results.
In the calculation of complete low-energy EA in AdS/CFT correspondence, one can divide it into a high energy and low energy pieces, separated by some given RG scale (fixed value of radial coordinate):

\[ S = S_{\text{UV}} + S_{\text{IR}}. \]  

(32)

Here \( S_{\text{UV}} \) is obtained from the original stringy action as a result of specific compactification. \( S_{\text{IR}} \) may be identified with the quantum effective action of (gauge and matter) low energy theory.

Now let us consider 5d warped AdS metric:

\[ ds^2 = -dr^2 + a_1^2(r)\tilde{g}_{\alpha\beta}dx^\alpha dx^\beta \]  

(33)

where \( r \) is radius of d5 AdS, or RS Universe, i.e., \( a_1 = a_1(r) \) is scale factor of d5 AdS and \( a_1(r) \) usually depends exponentially on the radial coordinate. \( \tilde{g}_{\alpha\beta} \) is 4d metric of boundary, time dependent FRW Universe. We assume that \( \tilde{g}_{\alpha\beta} = a^2(\eta)\eta_{\alpha\beta} \) where \( \eta \) is conformal time and \( \eta_{\alpha\beta} \) is 4d Minkowski tensor. As \( \tilde{g}_{\mu\nu} \) corresponds to conformally flat space, it is defined by conformal time dependent scale factor \( a \). Hence we have two scale factors. One can discuss now the structure of low-energy effective action. Truncation of \( S_{\text{UV}} \) gives basically the bosonic sector of 5d gauged supergravity and, for sake of simplicity, we consider the situation with only a constant scalar (dilaton).

As a consequence

\[ S_{\text{UV}} = \int d^5x \sqrt{-g(5)} \left\{ -\frac{1}{H}R(5) - \frac{6\Lambda}{H} \right\} \]  

(34)

where \( V(\phi = \text{const}) \equiv \frac{6\Lambda}{H} \). We consider 5d AdS background with some 4d time-dependent conformally-flat boundary in this theory as vacuum state. The question is, can boundary quantum effects (instead of 4d cosmological constant) stabilize such space?

The 4d quantum effective action of low-energy theory on the conformally-flat space \( g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \) looks as (see, for example, [3])

\[ W = V_3 \int d\eta \left\{ 2b_1\sigma^\prime\prime\prime - 2(b + b_1) \left( \sigma'' - \sigma'^2 \right)^2 \right\} \]  

(35)

where \( \sigma = \ln a(\eta) \), \( V_3 \) is space volume, \( \sigma' = \frac{da}{d\eta} \), for \( \mathcal{N} = 4 \) SU(\( N \)) SYM theory \( b = \frac{N^2-1}{4(4\pi)^2} \), \( b_1 = -b \). In general, \( b > 0 \), \( b_1 < 0 \) and \( b \neq b_1 \).

One has to relate \( W \) with \( S_{\text{IR}} \). We consider d5 AdS background with the metric of the form:

\[ ds_5^2 = -dr^2 + a_1^2(r)a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu. \]  

(36)
One knows that $S_{\text{IR}} = W$ at $r = r_0$, cut-off scale. On the other side in AdS limit the description is completely from supergravity side. So, at $r \to r_A$, $S_{\text{IR}} \to 0$. Then one can adopt the phenomenological approach where

$$S_{\text{IR}} = \int f(a_1(r)) W dr$$

so that $f(a_1(r))$ satisfies above relations connecting $S_{\text{IR}}$ and $W$. A simple choice for it is $f(a_1) = a_1(r)/a_1(r_0)$. Then, one can solve Eqs. of motion from $S_{\text{UV}} + S_{\text{IR}}$ on the background (36). As result, the metric reads

$$ds^2 = -dr^2 + e^{2(r-r_0)/l} ds_{\text{wall}}^2,$$

where the metric on the wall of the brane is

$$ds_{\text{wall}}^2 = dt^2 - e^{2t} \sum_{i=1}^3 (dx^i)^2,$$

namely the metric of de Sitter space, which can be regarded as inflationary universe. It is interesting that Hubble parameter is depending from radial coordinate of 5d AdS Universe. Therefore we have obtained the time dependent solution in the form of warped compactification, which is caused by the quantum correction coming from the boundary QFT. In the above treatment, we have assumed that the wall lies at $r = r_0$ since $f = 1$ there. We need, however, to check the dynamics of the wall by solving junction equation coming from the surface counterterm, which should include $W$ as a quantum correction.

In the same way, when $b_1 < 0$, one obtains as a solution, the Eq. (38), but now the metric of the wall of the brane is given by

$$ds_{\text{wall}}^2 = \frac{1}{y^2} \left(dt^2 - dy^2 - (dx^1)^2 - (dx^2)^2\right).$$

The metric in (40) is nothing but that of 4d AdS. Hence, one can get 5d AdS Universe with warp scale factor a la Randall-Sundrum, where 4d AdS world is generated on the wall. Again, as in section 2 the probability of realization of 4d AdS is less than the one for de Sitter Universe.

In conclusion, we have exhibit a model where warped RS type scenario may be realized simultaneously with generation of inflationary Universe (or less stable AdS) on the wall. Dynamical 4d gravity is induced from background gravitational field on the boundary. The source for such mechanism is quantum effects due to boundary QFT. It is not quite clear how one
can estimate exactly these quantum effects. That is the reason we adopted
the phenomenological approach introducing some cut-off, interpolating, fifth
coordinate dependent function in such a way that near AdS the theory is
described by 5d SG. Far away of AdS, at some fixed radius it is described by
anomaly induced effective action of dual 4d QFT. There is, of course, some
ambiguity in the choice of this function. However, that may be considered
as kind of usual regularization dependence in frames of holographic RG.

5 Quantum creation of a de Sitter (anti-de Sitter)
4 d universes

Developing further the study of warped compactifications with curved bound-
dary (inflationary brane) within AdS/CFT correspondence, the natural ques-
tion is about the role of quantum bulk effects in such scenario.

This has been done in Ref. 9, where (at least, qualitatively) the role of
bulk quantum effects to the scenario of refs. 5, 10, 11 has been considered
and, consequently, the brane-world cosmology has been investigated. Here
a summary of the results.

Let us start with a 5 dimensional bulk space whose boundary is 4-
dimensional sphere $S_4$ or 4-dimensional hyperboloid $H_4$. The bulk metric is
given by 5 dimensional Euclidean Anti-de Sitter space $AdS_5$

$$d_{AdS_5}^2 = dy^2 + \sinh^2 \frac{y}{l} d\Omega_4^2 .$$

Here $d\Omega_4^2$ is given by the metric of $S_4$ or $H_4$ with unit radius. One also
assumes the boundary (brane) lies at $y = y_0$ and the bulk space is given by
gluing two regions given by $0 \leq y < y_0$ (see 10 for more details.)

The starting point is the action $S$ which is the sum of the Einstein-
Hilbert action $S_{EH}$, the Gibbons-Hawking surface term $S_{GH}$, the surface
counter term $S_1$ and the trace anomaly induced action $W$

$$S = S_{EH} + S_{GH} + 2S_1 + W$$

$$S_{EH} = \frac{1}{16\pi G} \int d^5x \sqrt{g_5} \left( R_5 + \frac{12}{l^2} \right)$$

$$S_{GH} = \frac{1}{8\pi G} \int d^4x \sqrt{g_4} \nabla_\mu n^{\mu}$$

$$S_1 = -\frac{3}{8\pi G} \int d^4x \sqrt{g_4}$$
The factor 2 in front of $S_1$ in (42) is coming from that we have two bulk regions which are connected with each other by the brane. In (44), $n^\mu$ is the unit vector normal to the boundary. The expression for $W$ is omitted here and it is a complicated expression obtained from the 4 dimensional conformal anomaly, which depends on $G$ and $F$, the Gauss-Bonnet invariant and the square of the Weyl tensor

$$G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl},$$

$$F = \frac{1}{3} R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl},$$

(46)

and on the number of quantum fields by means of the coefficients $b$ and $b'$. For example, in the case of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory $b = -b' = \frac{N^2 - 1}{4(4\pi^2)}$. See ref. [11] for more details.

Motivated by (41), one assumes the following anzatz for the metric of 5 dimensional bulk space:

$$ds^2 = dy^2 + e^{2A(y,\sigma)}l^2(d\sigma^2 + d\Omega_3^2).$$

The actions in (43), (44), (45), and $W$, have the following forms:

$$S_{EH} = \frac{l^4V_3}{16\pi G} \int dyd\sigma \left\{ \left( -8\partial_y^2 A - 20(\partial_y A)^2 \right) e^{4A} + \left( -6\partial_{\sigma}^2 A - 6(\partial_{\sigma} A)^2 + 6 \right) e^{2A} + \frac{12}{l^2} e^{4A} \right\}$$

$$S_{GH} = \frac{3l^4V_3}{8\pi G} \int d\sigma e^{4A} \partial_y A$$

$$S_1 = -\frac{3l^4V_3}{8\pi G} \int d\sigma e^{4A}$$

$$W = V_3 \int d\sigma \left[ b' A \left( 2\partial_y^4 A - 8\partial_y^2 A \right) - 2(b + b') \left( 1 - \partial_y^2 A - (\partial_y A)^2 \right) \right].$$

Here $V_3 = \int d\Omega_3$ is the volume or area of the unit 3 sphere.

However, there is also the gravitational Casimir contribution due to bulk quantum fields. It is possible to show, making use of zeta-function regularization techniques (see, for example [13, 14]), that for bulk scalar field, it has typically the following form $S_{\text{Csmr}}$

$$S_{\text{Csmr}} = \frac{cV_3}{R^5} \int dyd\sigma e^{-A}$$

(48)
Note that role of (effective) radius of 4d constant curvature space is played by $R e^A$. Here $c$ is some coefficient whose value and sign depend on the type of bulk field (scalar, spinor, vector, graviton, ...) and on parameters of bulk theory (mass, scalar-gravitational coupling constant, etc). In the following discussion it is more convenient to consider this coefficient to be some parameter of the theory.

Adding quantum bulk contribution to the action $S$ in (42), the total action is

$$S_{\text{total}} = S + S_{\text{Csmr}}.$$  \hfill (49)

$R$ is the radius of $S_4$ or $H_4$.

In the bulk, one obtains the effective equations of motion from $S_{EH} + S_{\text{Csmr}}$ by the variation over $A$. When scale factor depends on both coordinates: $y, \sigma$, one can find the solution of these Eqs. of motion as an expansion with respect to $e^{-y/\tilde{R}}$ by assuming that $\frac{y}{\tilde{R}}$ is large, namely

$$e^A = \frac{\sinh \frac{y}{\tilde{R}}}{\cosh \sigma} - \frac{32 \pi G c^3}{15 R^3} B^4(\sigma) e^{-\frac{4y}{\tilde{R}}} + O \left( e^{-\frac{5y}{\tilde{R}}} \right)$$ \hfill (50)

for the perturbation from the solution where $B(\sigma) = \cosh \sigma$ for the $S_4$ brane and $B(\sigma) = \sinh \sigma$ for from $H_4$ brane.

On the brane at the boundary, one gets the following equation:

$$0 = \frac{48 l^4}{16 \pi G} \left( \partial_y A - \frac{1}{l} \right) e^{4A} + b' \left( 4 \partial^4_y A - 16 \partial^2_y A \right) - 4(b + b') \left( \partial^4_y A + 2 \partial^2_y A - 6(\partial_y A)^2 \partial^2_y A \right).$$ \hfill (51)

We should note that the contributions from $S_{EH}$ and $S_{GH}$ are twice from the naive values since we have two bulk regions which are connected with each other by the brane. Substituting $e^A$ into (51), we find

$$0 \sim \frac{1}{\pi G} \left( \frac{1}{R} \sqrt{\frac{k}{R} + \frac{R^2}{l^2}} + \frac{64 \pi G l^7 c}{3 R^{10}} B^5(\sigma) - \frac{1}{l} \right) R^4$$

$$+ 8b'.$$ where $k = 1$ and $B(\sigma) = \cosh \sigma$ for $S_4$ brane and $k = -1$ and $B(\sigma) = \sinh \sigma$ for $H_4$ brane. Here the radius $R$ of $S_4$ or $H_4$ is related with $A(y_0)$, if we assume the brane lies at $y = y_0$, by

$$\tilde{R} = l e^{\tilde{A}(y_0)}.$$ \hfill (52)
In the above Eq., only the leading terms with respect to $1/R$ are kept in the ones coming from $S_{\text{Cas}}$ (the terms including $c$). When $c = 0$, the previous result in [10, 11] is recovered. As a result, the Casimir force deforms the shape of $S_4$ or $H_4$ since $R$ becomes $\sigma$ dependent. The effect becomes larger for large $\sigma$. In case of $S_4$ brane, the effect becomes large if the distance from the equator becomes large. Thus, bulk quantum effects do not destroy the quantum creation of de Sitter (inflationary) or Anti-de Sitter brane-world Universe.

When $c = 0$, the solution can exist when $b' < 0$ for $S_4$ brane (in this case it is qualitatively similar to quite well-known anomaly driven inflation of refs. [15]) and $b' > 0$ for $H_4$. For $S_4$ brane, if $b' < 0$, the effect of Casimir force makes the radius smaller (larger) if $c > 0$ ($c < 0$). For $H_4$ brane, for small $R$ it behaves as

$$0 \sim \frac{64l^7c}{3R^{10}} \sinh^5 \sigma + 8b' . \quad (53)$$

Then one would have solution even if $b' < 0$.

Furthermore, it is possible to show that bulk quantum effects do not violate (in some cases, even support) the quantum creation of de Sitter or Anti-de Sitter brane living in d5 AdS world.

For a recent review and a complete list of references, see [16] and for recent developments see [17, 18].

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