RANKINGS IN DIRECTED CONFIGURATION MODELS WITH HEAVY TAILED IN-DEGREES

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ABSTRACT. We consider the extremal values of the stationary distribution of sparse directed random graphs with given degree sequences and their relation to the extremal values of the in-degree sequence. The graphs are generated by the directed configuration model. Under the assumption of bounded \((2 + \eta)\)-moments on the in-degrees and of bounded out-degrees, we obtain tight comparisons between the maximum value of the stationary distribution and the maximum in-degree. Under the further assumption that the order statistics of the in-degrees have a power-law behavior, we show that the extremal values of the stationary distribution also have a power-law behavior with the same index. In the same setting, we prove that these results extend to the PageRank scores of the random digraph, thus confirming a version of the so-called power-law hypothesis. Along the way, we establish several facts about the model, including the mixing time cutoff and the characterization of the typical values of the stationary distribution, which were previously obtained under the assumption of bounded in-degrees.

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1. INTRODUCTION

The stationary distribution of the simple random walk on a directed graph (digraph) provides a natural measure of the ranking of its nodes. The potentially non-local nature of the stationary distribution in directed networks makes the analysis of ranking a challenging task, and it is of interest to relate the ranking statistics to much simpler local statistics such as the in-degrees of the nodes. In this paper we compare the maximum values of the stationary distribution to the maximum values of the in-degrees in the setting of directed configuration models with sparse degree sequences. We start with a presentation of the model and the main results, and then return to a general discussion of the problems involved, main motivations and relations to previous work.

1.1. The model and the assumptions. Let \([n] := \{1, \ldots, n\}\) be a set of \(n\) nodes. Let \(d_n = ((d^-_1, d^+_1), \ldots, (d^-_n, d^+_n)) \in \mathbb{N}^{2 \times n}\) be a bi-degree sequence with

\[
  m := \sum_{v \in [n]} d^+_v = \sum_{v \in [n]} d^-_v. \tag{1.1}
\]
The directed configuration model (DCM) is the random directed multigraph (digraph) on \([n]\), \(G = G_n\), generated as follows: Assign \(d^-_v\) heads and \(d^+_v\) tails to vertex \(v\), match the \(m\) heads and the \(m\) tails with a uniformly random bijection, and finally add a directed edge \((u, v)\) for each tail from \(u\) that is matched to a head from \(v\).

For a node \(v \in [n]\), \(d^-_v\) and \(d^+_v\) are called the in-degree and the out-degree of \(v\) respectively. Let \(\Delta^+_n = \Delta_n^+ := \max_{v \in [n]} d^+_v\) denote the maximum in/out-degree. Unless otherwise specified, we will always assume that the sequence of bi-degree sequences \((d_n, n \in \mathbb{N})\) satisfies the following condition:

**Assumption 1.1.** There exist constants \(\eta, C > 0\) and \(K \geq 2\) such that for all \(n \in \mathbb{N}\)

(i) minimum out-degree: \(\min_{v \in [n]} d^+_v \geq 2\);
(ii) bounded maximum out-degree: \(\Delta^+_n \leq K\)
(iii) bounded \((2 + \eta)\)-moment for in-degrees:

\[
\sum_{v \in [n]} (d^-_v)^{2+\eta} \leq Cn. \tag{1.2}
\]

Note that we do not assume any lower bound on the in-degrees. For the sake of brevity, we will use \(d_n\) to refer to the sequence \((d_n, n \in \mathbb{N})\).

### 1.2. Definitions and notations.

We recall some standard definitions and fix some notations. We write \(1(E) = 1_E\) for the indicator function of an event \(E\). A sequence of events \((E_n)_{n \geq 0}\) occurs with high probability (whp) if \(\mathbb{P}(E_n) = \mathbb{E}[1(E_n)] \to 1\) as \(n \to \infty\). We write \(X_n \xrightarrow{p} X\) whenever a sequence of random variables \(X_n\) converges in probability to the random variable \(X\), i.e., when \(\mathbb{P}(|X - X_n| > \varepsilon) \to 0\) for all \(\varepsilon > 0\). We also use \(o_p(1)\) to denote an implicit sequence random variables which converges to 0 in probability [34]. To avoid repetitions, it is often understood that our inequalities hold provided that \(n\) is sufficiently large.

Under **Assumption 1.1**, the probability that \(G\) is simple (neither loops nor multiple edges) is bounded away from zero [11, 33]. Furthermore, conditional on being simple, \(G\) has the uniform distribution over simple digraphs on \([n]\) with degree sequence \(d_n\). Thus, all results in this paper that hold whp can be transferred to uniform simple digraphs.

Under an assumption weaker than **Assumption 1.1**, it is known that whp the resulting digraph has a unique strongly connected component which is globally attractive [15]; this is false in general if vertices of out-degree at most one are allowed. In particular, there exists a unique stationary distribution \(\pi\) characterized by the equations

\[
\pi(x) = \sum_{y \in [n]} \pi(y) P(y, x), \quad x \in [n], \tag{1.3}
\]

with the normalization \(\sum_{x \in [n]} \pi(x) = 1\). Here \(P = P_G\) is the transition matrix of the simple random walk on the multigraph \(G\), defined as

\[
P(y, x) = \frac{m(y, x)}{d^+_y}, \tag{1.4}
\]

where we write \(m(y, x)\) for the multiplicity of the directed edge \((y, x)\) in \(G\).

We write \((X_t, t \geq 0)\) for the simple random walk on \(G\). Thus, \(P^t(v, \cdot)\) denotes the distribution of \(X_t\) on \([n]\) conditioned on \(X_0 = v\). If \(t \geq 0\) is not an integer, for simplicity
we often write \( t \) instead of \( \lfloor t \rfloor \) so that for example \( P^t(v, \cdot) \) represents the distribution of the walk after \( \lfloor t \rfloor \) steps. A standard measure of the distance to stationarity is \( \| P^t(v, \cdot) - \pi \|_{TV} \), where the total variation distance between two probability measures \( \mu, \nu \) on \([n]\) is defined by
\[
\| \mu - \nu \|_{TV} = \max_{A \subseteq [n]} | \mu(A) - \nu(A) |.
\] (1.5)

The mixing time of the random walk \((X_t, t \geq 0)\) is defined, for \( \varepsilon \in (0, 1) \), by
\[
T_{\text{mix}}(\varepsilon) = \inf \{ t \in \mathbb{N} : \max_{v \in [n]} \| P^t(v, \cdot), \pi \|_{TV} < \varepsilon \}. \tag{1.6}
\]

The in-degree distribution \( \mu_{\text{in}} \) and out-degree distribution \( \mu_{\text{out}} \) on \([n]\) are defined by
\[
\mu_{\text{in}}(v) = \frac{d_v^-}{m}, \quad \mu_{\text{out}}(v) = \frac{d_v^+}{m}, \quad \forall v \in [n]. \tag{1.7}
\]

Following [13], we define the entropic time by
\[
T_{\text{ent}} = \log \frac{n}{H}, \quad \text{where} \quad H = \sum_{v \in [n]} \mu_{\text{in}}(v) \log(d_v^+).
\] (1.8)

1.3. Results.

1.3.1. Mixing time. Our first result concerns the mixing time and the cutoff phenomenon.

**Theorem 1.1.** Let \( d_n \) be a bi-degree sequence satisfying Assumption 1.1. Then, for all \( \rho \neq 1 \),
\[
\max_{x \in [n]} \| P^{\rho T_{\text{ent}}}(x, \cdot) - \pi \|_{TV} - 1(\rho < 1) \xrightarrow{\text{whp}} 0. \tag{1.9}
\]

In particular, for any \( \varepsilon \in (0, 1) \), whp \( T_{\text{mix}}(\varepsilon) = (1 + o(1)) T_{\text{ent}} \).

**Remark 1.2.** Theorem 1.1 is an extension of the cutoff results from [12] which were obtained in the case of bounded degrees \( \Delta^{\pm} = O(1) \). It shows that \( T_{\text{mix}}(\varepsilon) \) is, to leading order, independent of \( \varepsilon \), that is the Markov chain satisfies the cutoff phenomenon. Theorem 2 in [12] considers also the cutoff window, namely the behavior of the function \( t \mapsto \| P^t(\cdot) - \pi \|_{TV} \) on the finer scale \( t = T_{\text{ent}} + aw_n \), where \( a \in \mathbb{R} \) and \( w_n = O(\sqrt{\log n}) \), and showed that it approaches a universal Gaussian shape for all \( x \in [n] \). One can check that the techniques we use here to prove Theorem 1.1 are sufficient to obtain this refinement in our more general setting.

1.3.2. Typical values of the stationary distribution. Our second result addresses the convergence of the empirical distribution
\[
\psi_n := \frac{1}{n} \sum_{v \in [n]} \delta_{n\pi(v)}, \tag{1.10}
\]
where \( \delta_a \) is the Dirac distribution centered at \( a \in \mathbb{R} \). Note that \( \psi_n \) represents the law of \( n\pi(v) \) when \( v \) is picked uniformly at random in \([n]\). We recall that the 1-Wasserstein (or Kantorovich-Rubinstein) distance between two probability measures \( \mu, \nu \) on \( \mathbb{R} \) is defined by
\[
W_1(\mu, \nu) = \sup_{f} \left| \int_{\mathbb{R}} f \, d\mu - \int_{\mathbb{R}} f \, d\nu \right|, \tag{1.11}
\]
where the supremum runs over of \( f : \mathbb{R} \to \mathbb{R} \) such that \( |f(x) - f(y)| \leq |x - y| \) (see, for example, [49, Chapter 6]).

Consider the sequence of deterministic probability measures on \( \mathbb{R}_+ := [0, \infty) \), \( (\mathcal{L}_n, n \in \mathbb{N}) \), where for \( n \in \mathbb{N} \), \( \mathcal{L}_n \) is defined as the law of the random variable \( X_n \) that satisfies \( \mathbb{E}[X_n] = 1 \), and

\[
X_n \overset{d}{=} \frac{n}{m} \sum_{k=1}^{d_I^+} Z_k, \tag{1.12}
\]

where \( I \) is a uniformly sampled vertex in \([n]\) and \( \{Z_k\}_{k \geq 1} \) are independent and identically distributed (iid) random variables satisfying the stochastic fixed point equation (SFPE)

\[
Z_1 \overset{d}{=} \frac{1}{d_J^-} \sum_{k=1}^{d_J^-} Z_k, \tag{1.13}
\]

where \( J \) is a random vertex in \([n]\) distributed as \( \mu_{\text{out}} \). Existence and uniqueness of solutions to recursive distributional equations of the type Eq. (1.12) is well known; see for example [2].

**Theorem 1.3.** Let \( d_n \) be a bi-degree sequence satisfying Assumption 1.1. We have

\[
\mathcal{W}_1(\psi_n, \mathcal{L}_n) \xrightarrow{p} 0. \tag{1.14}
\]

1.3.3. Extremal values of the stationary distribution. Our main concern in this paper will be the behavior of the extremal values of the stationary distribution. We start with the maximum \( \pi_{\text{max}} := \max_{v \in [n]} \pi(v) \) and its relation to the maximum in-degree \( \Delta^- \).

**Theorem 1.4.** Let \( d_n \) be a bi-degree sequence satisfying Assumption 1.1. Then,

(i) There exists an absolute constant \( C > 0 \) such that, whp

\[
\pi_{\text{max}} \leq C \log(n) \frac{\Delta^-}{m}. \tag{1.15}
\]

(ii) If \( \Delta^- \to \infty \), then \( \forall \varepsilon > 0 \), whp

\[
\pi_{\text{max}} \geq (1 - \varepsilon) \frac{\Delta^-}{m}. \tag{1.16}
\]

**Remark 1.5.** The bounds Eq. (1.15) and Eq. (1.16) on \( \pi_{\text{max}} \) are essentially optimal under this generality. Clearly, if the graph is Eulerian, that is if \( d_v^+ = d_v^- \) for all \( v \in [n] \), then \( \pi = \mu_{\text{in}} = \mu_{\text{out}} \) and thus \( \pi_{\text{max}} = \frac{\Delta^-}{m} \). On the other hand, Theorem 1.6 in [18] shows the existence of bounded degree sequences for which \( \pi_{\text{max}} = \log^{1-o(1)}(n) \frac{\Delta^-}{m} \). We remark that the assumption \( \Delta^- \to \infty \) in Eq. (1.16) is not really restrictive since by [18] we already know that the bound Eq. (1.16) is always satisfied if \( \Delta^- = O(1) \). It is an interesting open question to determine whether the logarithmic term is necessary given that \( \Delta^- \) diverges sufficiently fast. In Section 9, we refine Eq. (1.15) for a wide class of sequences, called extremal, proving that in these cases

\[
\pi_{\text{max}} = (1 + o(1)) \frac{\Delta^-}{m}. \tag{1.17}
\]
In particular, this proves in a strong sense the asymptotic tightness of Eq. (1.16). Moreover, we will see that for such extremal sequences the vertex with the maximum in-degree coincides with the vertex with maximum stationary value.

1.3.4. Power-law behavior. We turn to the analysis of the order statistics of the stationary distribution, in the case where the in-degrees have an approximate power-law behavior. We will consider the following notion of heavy tails. Let \( \mathcal{M}_n \) denote the set of empirical distributions of size \( n \) on \( \mathbb{R}_+ \), that is the set of probability measures of the form

\[
\mu_n = \frac{1}{n} \sum_{v \in [n]} \delta_{x_v},
\]

(1.18)

for some fixed vector \((x_1, \ldots, x_n) \in \mathbb{R}_+^n\). For any \( \mu_n \in \mathcal{M}_n \) and \( t \geq 0 \), let

\[
\mu_n(t, \infty) = \frac{1}{n} |\{ i \in [n] : x_i > t \}|
\]

(1.19)

denote the right tail of \( \mu_n \).

**Definition 1.6.** Given a constant \( \kappa > 0 \), and a sequence of measures \( \mu_n \in \mathcal{M}_n \), we say that \( \mu_n \) has power-law behavior with index \( \kappa \) if for all \( \varepsilon > 0 \) and for all \( a \in (0, 1/\kappa) \),

\[
n^{-a\kappa-\varepsilon} \leq \mu_n(n^a, \infty) \leq n^{-a\kappa+\varepsilon}, \quad \mu_n(n^{1/\kappa+\varepsilon}, \infty) = 0,
\]

(1.20)

for all sufficiently large \( n \). If the measures \( \mu_n \) are random elements in \( \mathcal{M}_n \), we say that \( \mu_n \) has power-law behavior with index \( \kappa \) with high probability, if for all \( \varepsilon > 0 \), whp Eq. (1.20) holds for any \( a \in (0, 1/\kappa) \).

Since \( \mu_n \in \mathcal{M}_n \) has minimal mass \( 1/n \), the upper bounds in Eq. (1.20) are equivalent to the requirement that \( \mu_n(n^a, \infty) \leq n^{-a\kappa+\varepsilon} \) for all \( \varepsilon > 0 \) and all \( a > 0 \). Notice that if \((x_1, x_2, \ldots)\) are iid random variables with probability density \( f(t) \propto \min\{1, t^{-1-\kappa}\}, t \in \mathbb{R}_+ \), for some \( \kappa > 0 \), then the sequence of random empirical measures \((\mu_n)_{n \in \mathbb{N}}\) in Eq. (1.18) has power-law behavior with index \( \kappa \) with high probability (see, e.g., [45]).

We apply this notion to our degree sequence. Let \( \phi \) be the empirical in-degree distribution; that is, for \( k \geq 0 \),

\[
\phi(k) = \phi_n(k) := \frac{1}{n} \sum_{v \in [n]} 1(d^-_v = k).
\]

(1.21)

Both \( \phi, \psi \) define sequences of distributions \( \phi_n, \psi_n \in \mathcal{M}_n \), but for simplicity we often drop the subscript \( n \) from our notation. The distribution \( \phi \) has mean value \( m/n \), while \( \psi \) has mean value \( 1 \) for all \( n \).

**Theorem 1.7.** Let \( d_n \) be a bi-degree sequence satisfying Assumption 1.1 and assume that its empirical in-degree distribution \( \phi \) has power-law behavior with index \( \kappa > 2 \). Then, whp the distribution \( \psi \) has power-law behavior with the same index \( \kappa \), that is for all \( \varepsilon > 0 \), whp, \( \psi(n^{1/\kappa+\varepsilon}, \infty) = 0 \) and for all \( a \in (0, 1/\kappa) \),

\[
n^{-a\kappa-\varepsilon} \leq \psi(n^a, \infty) \leq n^{-a\kappa+\varepsilon}.
\]

(1.22)

**Remark 1.8.** If we only assume Assumption 1.1, setting \( \kappa_0 = 2 + \eta \), where \( \eta > 0 \) is such that Eq. (1.2) holds, then we will see that for all \( \varepsilon > 0 \), \( \psi \) satisfies the following upper bound whp: for all \( a > 0 \),

\[
\psi(n^a, \infty) \leq n^{-a\kappa_0+\varepsilon},
\]

(1.23)
that is the right tail of $\psi$ is dominated by a heavy tail with index $\kappa_0$. In some sense, this indicates that among all in-degree distributions with bounded $\kappa_0$-moment, the ones with power-law behavior with index $\kappa_0$ “maximize” the upper tail of the stationary distribution. Eq. (1.23) will be proved in Section 7 together with Theorem 1.7, as a consequence of a more general upper bound on $\psi(n^a, \infty)$.

1.3.5. PageRank surfer. Next, we discuss the power-law behavior of PageRank. Fix $\alpha \in (0, 1)$ and let $\lambda$ be a probability distribution on $[n]$, which we refer to as the teleporting probability and the teleporting distribution respectively. The factor $1 - \alpha$ is also referred in the literature as the damping factor. Consider the $(\alpha, \lambda)$-PageRank surfer, that is the Markov chain with transition matrix

$$P_{\alpha, \lambda}(x, y) = (1 - \alpha)P(x, y) + \alpha\lambda(y).$$

(1.24)

We call $(\alpha, \lambda)$-PageRank score the stationary distribution $\pi_{\alpha, \lambda}$ of this Markov chain, which is known to always be unique and to satisfy

$$\pi_{\alpha, \lambda} = \sum_{k=0}^{\infty} \alpha(1 - \alpha)^k \lambda^k,$$

(1.25)

see, e.g., [19]. We will further assume that the teleporting distribution $\lambda$ is uniform up to multiplicative sub-polynomial factors, that is, for all $\varepsilon > 0$,

$$n^{-1-\varepsilon} \leq \lambda(x) \leq n^{-1+\varepsilon},$$

(1.26)

for all sufficiently large $n$, uniformly in $x \in [n]$.

Let $\psi_{\alpha, \lambda} \in \mathcal{M}_n$ be the empirical distribution in Eq. (1.18) corresponding to $x_v = n\pi_{\alpha, \lambda}(v)$.

**Theorem 1.9.** Let $d_n$ be a bi-degree sequence satisfying Assumption 1.1 and assume that its empirical in-degree distribution $\phi$ has power-law behavior with index $\kappa > 2$. For any constant $\alpha \in (0, 1)$, and probability distribution $\lambda$ satisfying Eq. (1.26), whp $\psi_{\alpha, \lambda}$ has power-law behavior with the same index $\kappa$, that is, for all $\varepsilon > 0$, whp, $\psi_{\alpha, \lambda}(n^{1/\kappa + \varepsilon}, \infty) = 0$ and for all $\alpha \in (0, 1/\kappa)$,

$$n^{-\alpha \kappa - \varepsilon} \leq \psi_{\alpha, \lambda}(n^a, \infty) \leq n^{-\alpha \kappa + \varepsilon}.$$

(1.27)

**Remark 1.10.** We will actually show a stronger result which holds for non-constant $\alpha$. Namely, that the upper bound $\psi_{\alpha, \lambda}(n^a, \infty) \leq n^{-\alpha \kappa + \varepsilon}$ holds uniformly for arbitrary sequences $\alpha = \alpha_n \in [0, 1]$. Indeed, as far as the upper bounds on the stationary distribution are concerned, it turns out that the presence of the parameter $\alpha$ can only make our analysis simpler; see Remark 8.2. Moreover, the lower bound $\psi_{\alpha, \lambda}(n^a, \infty) \geq n^{-\alpha \kappa - \varepsilon}$ holds under the only assumption that $\limsup_{n \to \infty} \alpha_n < 1$; see Section 8.

**Remark 1.11.** Concerning the maximum PageRank score we will see that the following bounds hold whp for any bi-degree sequence satisfying Assumption 1.1, for all $\alpha \in [0, 1]$ and any probability $\lambda$ on $[n]$:

$$\alpha (1 - \alpha) \lambda_{\min} \frac{\Delta^-}{\Delta^+} \leq \max_{x \in [n]} \pi_{\alpha, \lambda}(x) \leq C \log(n) \left( \lambda_{\max} + \frac{\Delta^-}{m} \right),$$

(1.28)

where $C$ is an absolute constant, $\lambda_{\min} = \min_{x \in [n]} \lambda(y)$, and $\lambda_{\max} = \max_{x \in [n]} \lambda(x)$. These bounds will be a simple consequence of our main results; see Remark 8.3.
1.4. Motivation and related work. Random walks on random undirected graphs have attracted a lot of attention in the last decade [9, 10, 29, 39]. Contrarily, much less is known in random digraphs. The non-reversible nature of random walks in directed environments poses the challenge of developing new techniques to study their properties.

One of the most natural models for random digraphs is the directed configuration model (DCM), which has been introduced in the literature as a directed analogue of the configuration model [21, 25, 41]. We refer the interested reader to [15, 25] for results on its component structure and to [16, 18, 48] for the study of its distance profile. Bordenave, the second author and Salez [12] recently initiated the study of random walks on the DCM. Provided that the minimum out-degree is at least 2 and the maximum in-degree and out-degree are bounded, they showed that the mixing time coincides with the entropic time, defined in Eq. (1.8), exhibits cutoff and has a Gaussian behavior inside the cutoff window. Moreover, they showed that the stationary distribution of a uniformly random vertex \( I \) converges (in the \( 1-Wasserstein \) sense) to the solution of the stochastic fixed point equation (SFPE) displayed in Eq. (1.12). These results are extended to other models of non-reversible sparse random Markov chains in [13]. Our results in Theorem 1.1 and Theorem 1.3 show that the hypothesis on the degree sequence in [12] can be further relaxed to Assumption 1.1.

One of the questions left open in [12] is the determination of the extremal behavior of the stationary values. The second and fourth authors [18] showed that, in the bounded degree setting, the extremal (minimum and maximum) values of the stationary distribution exhibit logarithmic fluctuations around the average value, the exponents of the logarithm being essentially determined by the minimum and maximum in- and out-degrees. In particular, regarding \( \pi_{\text{max}} \), Theorem 1.6 in [18] shows that if \( d_n \) is a bi-degree sequence satisfying \( 2 \leq d_n^\pm = O(1) \), and such that there are linearly many vertices with degrees \((\Delta^-, \delta^+)\), where \( \delta^+ = \min_v d_v^+ \), then there exists a constant \( C > 1 \) such that whp

\[
\frac{n \pi_{\text{max}}}{\log^1 \kappa_0 n} \in [C^{-1}, C],
\]

where \( \kappa_0 = \frac{\log \delta^+}{\log \Delta^-} \). Theorem 1.4 shows that if we allow the in-degrees to grow with the order of the digraph then \( \pi_{\text{max}} \) will have much larger fluctuations.

In a similar spirit, the first and third authors [17] proved that by dropping the condition on the minimum in-degree, the minimum stationary value may become polynomially smaller than the average, with the exponent given by the solution of an optimization problem involving subcritical branching processes and large deviation rate functions of the bi-degree distribution. Moreover, their results also give an implicit description of the lower tail of \( \psi \), complementing Remark 1.8. In both works [17, 18], controlling the minimum stationary values allows us to estimate the cover time of a random walk in DCM.

Stationary measures have also been studied for other random digraphs models. Cooper and Frieze [26] determined the stationary distribution of the directed Erdős-Rényi random graph in the strong connectivity regime, motivated by their systematic study of the cover time in random graph models. Addario-Berry, Balle and the third author [1] provided estimations for the extremal values of the stationary distribution in random out-regular digraphs, with applications to random deterministic finite automata.
While the analysis of random walks on DCM only started recently, its stationary distribution has in fact received a lot of attention under the framework of the PageRank algorithm. PageRank was introduced in [43] as a ranking measure for the webgraph and is a core element in Google’s search engine. The PageRank score is simply defined as the stationary distribution of the PageRank surfer defined in Eq. (1.24). We refer to [19] for the mixing properties of the PageRank surfer on DCM. Compared to the in-degree ranking, the PageRank score is less susceptible to assign high priority to spam pages [32]. Nevertheless, empirical observations give a high average correlation between in-degrees and PageRank [3]. The so-called power-law hypothesis ventures a more precise description for scale-free networks: if the in-degree of a network is power-law distributed, then its PageRank score also follows a power-law distribution with the same exponent. This has been experimentally confirmed in several real-world networks [27, 44, 46], in the particular case of the webgraph, the in-degree and PageRank are both approximately power-law distributions with index $\kappa \approx 1.1$. The effect of the teleporting factor $\alpha$ has also been studied in [8], observing that the top 10% ranked elements follow a power-law distribution regardless of $\alpha$.

The abundance of empirical evidence has motivated the mathematical analysis of the power-law hypothesis. A series of papers [38, 50, 51] proposed an idealized stochastic model proving that the power-law distributions of the in-degree and of the PageRank score of a uniformly random vertex $I$ only differ by a multiplicative factor. Chen, Litvak and Olvera-Cravioto [22, 23] initiated the rigorous analysis of PageRank on DCM, proving that the score of $I$ can be approximated by the PageRank score of the root of certain infinite random tree, under the assumption that the in- and out-degrees are independently distributed. In particular, if the in-degree distribution of $I$ is a power-law, so is its PageRank. Olvera-Cravioto has recently extended these results to degree-degree correlated distributions [42]. In particular, the distribution of the PageRank of $I$ weakly converges to the attractive endogenous solution of an SFPE that generalizes Eq. (1.12). The asymptotic properties of the solution imply that upper tail of the PageRank of $I$ is asymptotically distributed as power-law with the right exponent.

The PageRank has also been studied in other directed random networks such as inhomogeneous random graph (IRG) [36, 42] and the directed preferential attachment model (DPA) [5, 7]. Remarkably, the power-law hypothesis is only partially true in DPA: PageRank exhibits a power-law distribution with different index than the index of the in-degree distribution. An approach based on local weak convergence was given in [31], yielding lower bounds for the PageRank of a random vertex for any sequence of digraphs that has a local weak limit.

All aforementioned results describe the PageRank score of a vertex $I$ picked uniformly at random in $[n]$, or of a fixed given vertex, as in the case of the oldest vertex in DPA obtained in [7]. However, in most of the applications (such as web indexing), it is of foremost importance to identify the top ranked elements [6]. To our best knowledge, Theorem 1.9 is the first result that establishes the power-law hypothesis in the large deviation sense, providing the shape of the upper tail of the PageRank distribution in DCM, not only the upper tail for a typical vertex in the bulk of the digraph.

We refer to Section 10 for a discussion of open problems and future research directions.
2. Preliminary results

2.1. Bounded moments. We will use frequently the following deterministic property of degree sequences with bounded \(2 + \eta\) moment of in-degrees.

**Lemma 2.1.** Let \(d_n\) be a bi-degree sequence satisfying Eq. (1.2) with \(\eta \in (0, 1)\). Then

\[
\Delta^- = O(n^{\frac{1}{2} - \frac{\eta}{4}}). \tag{2.1}
\]

Moreover, for any \(S \subset [n]\) with \(|S| \leq n^{1-\eta}\),

\[
\sum_{v \in S} d^-_v = O\left((|S|n)^{\frac{1}{2} - \frac{\eta}{8}}\right). \tag{2.2}
\]

**Proof.** By Hölder’s inequality [28, Theorem 1.5.2] with \(p = 2 + \eta\),

\[
\sum_{v \in S} d^-_v \leq \left(\sum_{v \in S} (d^-_v)^p\right)^{\frac{1}{p}} |S|^{\frac{1}{2} - \frac{1}{p} \eta} = O\left((n/|S|)^{\frac{1}{2} \eta}\right)|S| = O\left(n^{\frac{1}{2} - \frac{\eta}{8}} |S|^{\frac{1}{2} + \frac{\eta}{8}}\right), \tag{2.3}
\]

where in the last inequality we used that \(\frac{1}{p} \leq \frac{1}{2} - \frac{\eta}{6}\) for \(\eta \in (0, 1)\). Taking \(|S| = 1\) we obtain Eq. (2.1). Finally, using \(|S| \leq n^{1-\eta}\) we obtain Eq. (2.2) from Eq. (2.3). \(\square\)

2.2. Local structure. Let \(d_n = ((d^-_1, d^+_1), \ldots, (d^-_n, d^+_n))\) be a bi-degree sequence. For each \(v \in [n]\), assign a set \(E^-_v\) of \(d^-_v\) labeled heads, and a set \(E^+_v\) of \(d^+_v\) labeled tails, and let \(E^\pm = \bigcup_{v \in [n]} E^\pm_v\). Throughout the paper, we will use \(f\) to denote heads in \(E^-\) and \(e\) to denote tails in \(E^+\). Denote by \(v_e\) (or \(v_f\)) the vertex incident to \(e\) (or \(f\)). Every bijection \(\omega : E^+ \to E^-\) induces a multi-digraph \(G = G_n(\omega)\) with vertex set \([n]\) and bi-degree sequence \(d_n\) by assigning a directed edge to every pair of vertices \((v_e, v_f)\) such that \(\omega(e) = f\). For simplicity, the multi-digraph \(G\) will be often referred to as the digraph.

For \(h \in \mathbb{N}\), a **path** of length \(h\) is a sequence of edges

\[
p = \{(e_0, f_1), (e_1, f_2), \ldots, (e_{h-1}, f_h)\}, \tag{2.4}
\]

where \(e_{j-1} \in E^+, f_j \in E^\omega(e_{j-1}) = f_j, v_{f_j} = v_{e_j}\) for all \(j \in [h]\). If \(x = v_{e_0}\) and \(y = v_{f_h}\), we say that \(p\) is a path starting at \(x\) and ending at \(y\). The **weight** of the path \(p\) is the product of the inverse of the out-degrees of all vertices along \(p\) except the last one, that is

\[
w(p) = \prod_{j=0}^{h-1} \frac{1}{d^+_{v_{e_j}}}. \tag{2.5}
\]

Let \(\mathcal{P}(x, y, h, G)\) denote the set of all paths of length \(h\) starting at \(x\) and ending at \(y\) in the multi-digraph \(G\). A path is called **simple** if it never visits the same vertex more than once.

For any \(x \in [n]\) and \(h \in \mathbb{N}\), the out-neighborhood of \(x\) of depth \(h\), \(B^+_x(h)\), is the subgraph induced by all paths of length at most \(h\) starting at \(x\). Similarly, for any \(y \in [n]\) the in-neighborhood of \(y\) of depth \(h\), \(B^-_y(h)\), is the subgraph induced by all paths of length at most \(h\) ending at \(y\). We often identify \(B^+_x(h)\) with its vertex set. The boundary of \(B^+_x(h)\), that is the set of vertices \(v\) such that the shortest path starting at \(x\) and ending \(v\) has length \(h\), is denoted by \(\partial B^+_x(h)\). Similarly \(\partial B^-_y(h)\) represents the set of vertices \(v\) such that the shortest path starting at \(v\) and ending at \(y\) has length \(h\).
2.2.1. **Sequential generation.** For each $n \in \mathbb{N}$ the digraph $G = G_n(\omega)$ can be generated by matching tails and heads one at a time as follows. Given a priority rule $\mathcal{R}$,

(i) choose an unmatched head $f \in E^-$ (if any) according to $\mathcal{R}$;
(ii) choose an unmatched tail $e \in E^+$ uniformly at random;
(iii) set $\omega(e) = f$, and proceed.

Observe that the roles of tails and heads can be reversed.

To explore an in-neighborhood $B^-_y(h)$, we run the previous procedure with the priority rule given by the breadth-first search (BFS) order. In other words, at each time we choose a head closest from $y$ that has not been matched yet, and pair it with a uniformly random unmatched tail. We halt the procedure whenever all unmatched heads are at distance at least $h$ from $y$. Similarly, reversing tails and heads, one can explore out-neighborhoods.

As in many sparse random models, one may expect that the neighborhoods are locally tree-like. It will be important to see how much they differ from a tree, motivating the following definition. The **tree-excess** of a multi-digraph $G = (V, E)$ is the number of additional edges it has with respect to a tree; that is,

$$Tx(G) := 1 + |E(G)| - |V(G)|. \quad (2.6)$$

A step of the generating procedure is called a **collision** if the vertex $v_f$ of the head $f$ such that $\omega(e) = f$ had been exposed during one of the previous pairings. Collisions indicate the appearance of additional edges in neighborhoods. In particular, the number of collisions in the BFS generation of $B^+_y(h)$ is $Tx(B^+_y(h))$.

2.2.2. **Coupling with marked Galton-Watson trees.** For $x \in [n]$, let $T^-_y$ be the marked random tree with marks $\ell : V(T^-_y) \to [n]$, where $V(T^-_y)$ is the set of vertices of the tree, having root $a_0$ with $\ell(a_0) = 0$ and constructed iteratively with the following procedure, starting with $a = a_0$:

(i) Attach $d^-_{\ell(a)}$ children to $a$.
(ii) Assign to each child $b$ of $a$ independently at random the mark $\ell(b) = z \in [n]$ with probability $d^+_z/m$.
(iii) Choose the next $a$ to be the element in the tree which is one of the closest to the root among elements whose children have not been exposed. Terminate if no such element exists; otherwise go to step (i).

Reversing the roles of in-degrees and out-degrees, we construct the random tree $T^+_x$. Denote by $T^-_y(h)$ the subtree of $T^-_y$ containing the elements at distance at most $h$ from the root, and $\partial T^-_y(h)$ the subtree containing those at distance exactly $h$; similarly for $T^+_x(h)$ and $\partial T^+_x(h)$. Notice that the random tree $T^-_y$ is obtained by gluing $d^-_y$ independent copies of a Galton-Watson tree with offspring distribution given by

$$p^-(k) = \frac{1}{m} \sum_{v \in [n]} d^-_v 1_{d^-_v = k}, \quad \forall k \geq 0. \quad (2.7)$$

There is a natural coupling between the generating process of $B^+_v(h)$ and the construction of $T^+_x(h)$. We now describe the coupling of $B^-_y(h)$ and $T^-_y(h)$. The corresponding coupling of $B^+_x(h)$ and $T^+_x(h)$ can be obtained by reversing the role of heads and tails.

Clearly, step (ii) in the construction from Section 2.2.1 can be modified by picking $e$ uniformly at random among all (matched or unmatched) tails in $E^+$ and rejecting the
proposal if the tail was already matched. The tree can then be generated by iteration of
the same sequence of steps with the difference that at step (ii) we never reject the proposal
and at step (iii) we add a new leaf to the current tree, with mark \( v \) if \( e \in E^+_v \), together with
a new set of \( d^-_v \) unmatched heads attached to it.

Call \( \tau \) the first time that a uniform random choice among all tails gives \( e \in E^+_v \) for some
mark \( v \) already in the tree. By construction, the in-neighborhood and the tree coincide up
to time \( \tau \). At the \( k \)-th iteration, the probability of picking a tail with a mark already used
is at most \( k \Delta^+ / m \). Therefore, by a union bound, for any \( k \in \mathbb{N} \),

\[
P(\tau \leq k) \leq \frac{k^2 \Delta^+}{m}.
\] (2.8)

2.3. In-neighborhoods. We start with an estimate of the size of the in-neighborhoods
and then proceed with the analysis of the coupling with random trees described above.

For all \( \varepsilon > 0 \) define

\[
h_\varepsilon := \frac{\varepsilon \log n}{20 \log \Delta^+},
\] (2.9)

and the event

\[
S^-_\varepsilon := \{ \forall y \in [n], |B^-_y(h_\varepsilon)| \leq n^{1/2+\varepsilon} \}.
\] (2.10)

Lemma 2.2. For all \( \varepsilon > 0 \), \( \mathbb{P}(S^-_\varepsilon) = 1 - o(1) \).

Proof. Fix \( y \in [n], \varepsilon > 0 \), and let \( h = h_\varepsilon \) and \( B^\pm_y = B^\pm_y(h_\varepsilon) \). It is enough to show that, for all
\( n \) large enough

\[
\mathbb{E}[|B^-_y|^2] \leq (d^-_y)^2 n^\varepsilon.
\] (2.11)

Indeed, Eq. (2.11) and Markov’s inequality imply

\[
P(|B^-_y| > n^{1/2+\varepsilon}) \leq \frac{\mathbb{E}[|B^-_y|^2]}{n^{1+2\varepsilon}} \leq \frac{(d^-_y)^2}{n^{1+\varepsilon}}.
\] (2.12)

Therefore, by taking a union bound over \( y \in [n] \) and applying Eq. (1.2)

\[
P(S^-_\varepsilon) \geq 1 - \sum_{y \in [n]} \frac{(d^-_y)^2}{n^{1+\varepsilon}} = 1 - o(1).
\] (2.13)

To prove Eq. (2.11), note that

\[
|B^-_y| = \sum_{v \in [n]} 1_{y \in B^+_v},
\] (2.14)

and therefore

\[
\mathbb{E}[|B^-_y|^2] = \sum_{v \in [n]} \sum_{z \in [n]} \mathbb{P}(y \in B^+_v, z \in B^+_z).
\] (2.15)

Observe that the out-neighborhood \( B^+_v \) has at most \( (\Delta^+)^h \) edges and at each step of
the generation of \( B^+_v \) one has a probability of matching a head of \( y \) bounded above by
\( \frac{d^-_v}{m - (\Delta^+)^h} \leq \frac{d^-_v}{n} \) for \( n \) large enough. Thus, by a union bound over all steps of the generation
of \( B^+_v \), for all \( v \in [n] \),

\[
P(y \in B^+_v) \leq (\Delta^+)^h \frac{d^-_v}{n}.
\] (2.16)
Next, for all $z \neq v$,
\[ \Pr \left( y \in B_v^+, y \in B_z^+ \right) \leq \Pr \left( y, z \in B_v^+ \right) + \Pr \left( y \in B_v^+, y \in B_z^+, z \notin B_v^+ \right). \tag{2.17} \]
For the event $y, z \in B_v^+$ to occur, one must match a head of $y$ and a head of $z$ during the generation of $B_v^+$. Since there are at most $(\Delta^+)^h$ steps during the generation of $B_v^+$, and as in Eq. (2.16) one has a probability at most $\frac{d_z d_y^{-}}{n^2}$ to match a head of $y$ at any given step (and $\frac{d_z}{n}$ for $z$), a union bound gives
\[ \Pr \left( y, z \in B_v^+ \right) \leq (\Delta^+)^h \frac{d_z d_y^{-}}{n^2}. \tag{2.18} \]

Let us bound the second term in Eq. (2.17). We generate first $B_v^+$ and then $B_z^+$. Given the realization of $B_v^+$, and assuming $y \in B_v^+$ and $z \notin B_v^+$, the event $y \in B_z^+$ can be obtained in two ways: either we match a fresh head of $y$ during the generation of $B_z^+$, or we match a fresh head of another vertex $w$ and then during the generation of $B_z^+$ we match a fresh head of $w$. This event then satisfies
\[ \Pr(E_{v,z,w}) \leq \frac{d_y^-(\Delta^+)^h}{n} \left( \frac{d_w^- (\Delta^+)^h}{n} \right)^2 = \frac{d_y^- (d_w^-)^2}{n^3} (\Delta^+)^{3h}. \tag{2.19} \]
Summing over all possible choices of $w$ and using Assumption 1.1, we have
\[ \Pr \left( y \in B_v^+, y \in B_z^+, z \notin B_v^+ \right) \leq (\Delta^+)^{2h} \frac{(d_y^-)^2}{n^2} + \sum_{w \neq y} \Pr(E_{v,z,w}) \leq (\Delta^+)^{2h} \frac{(d_y^-)^2}{n^2} + (\Delta^+)^{3h} d_y^- \sum_{w \neq y} \frac{(d_w^-)^2}{n^3} \leq (\Delta^+)^{4h} \frac{(d_y^-)^2}{n^2}. \tag{2.20} \]
Combining Eqs. (2.16)–(2.18) and (2.20) we obtain,
\[ \Pr \left( y \in B_v^+, y \in B_z^+ \right) \leq (\Delta^+)^h \frac{d_y^{-}}{n} 1(v = z) + (\Delta^+)^{2h} \frac{d_z^- d_y^-}{n^2} + (\Delta^+)^{4h} \frac{(d_y^-)^2}{n^2}. \tag{2.21} \]
Inserting the above estimates into Eq. (2.15), we obtain Eq. (2.11). \hfill \Box

In what follows $\Pr$ denotes the probability under the coupling defined in Section 2.2.2. We shall often take the parameter $\varepsilon > 0$ smaller than some $\varepsilon_0 = \varepsilon_0(\eta)$ where $\eta > 0$ is the parameter appearing in Assumption 1.1. To avoid repetitions, we will simply say that our statements hold for $\varepsilon > 0$ sufficiently small.

**Lemma 2.3.** Fix $\varepsilon > 0$ sufficiently small and $h = h_\varepsilon$ as in Eq. (2.9). For any $y \in [n]$,
\[ \Pr \left( B^+_y(h) \neq T^-_y(h) \right) \leq (d_y^-)^2 n^{-\frac{\delta}{\varepsilon}}. \tag{2.22} \]
Moreover, for any $a > 0$
\[ \Pr \left( |B^+_y(h)| > d_y^- n^a \right) \leq n^{\frac{\delta}{n} - 2a} + (d_y^-)^2 n^{-\frac{\delta}{\varepsilon}}. \tag{2.23} \]
Moreover, for any \( y \in [n] \) the coupling of \( B_y^-(h) \) and \( T_y^-(h) \) succeeds whp and there exists \( \eta' > 0 \) such that
\[
\mathbb{P}\left(|B_y^-(h)| > n^{1/2-\eta'}\right) = o(1). \tag{2.24}
\]

**Proof.** Call \( C_y = \{B_y^-(h) = T_y^-(h)\} \). Then, for all \( a > 0 \)
\[
\mathbb{P}\left(C_y^c\right) \leq \mathbb{P}\left(|T_y^-(h)| > d_y^-n^a\right) + \mathbb{P}\left(|T_y^-(h)| \leq d_y^-n^a, C_y^c\right). \tag{2.25}
\]
Reasoning as in Eq. (2.8)
\[
\mathbb{P}\left(|T_y^-(h)| \leq d_y^-n^a, C_y^c\right) \leq \frac{\Delta^+(d_y^-n^a)^2}{m}. \tag{2.26}
\]
Notice that for each \( j \geq 1 \),
\[
|\partial T_y^-(j)| = \sum_{i=1}^{d_y^-} Z_j^{(i)}, \tag{2.27}
\]
where the \( Z_j^{(i)} \), \( i = 1, \ldots, d_y^- \) are iid random variables representing the size of the \( j \)-th generation of a Galton-Watson process with offspring distribution given by Eq. (2.7). The latter has expected value and variance
\[
\nu = \frac{1}{m} \sum_{v \in [n]} d_v^+ d_v^- = O(1), \quad \sigma^2 = \frac{1}{m} \sum_{v \in [n]} d_v^+(d_v^-)^2 - \nu^2 = O(1), \tag{2.28}
\]
where the estimates follow from Assumption 1.1.

Setting \( W_j = \nu^{-j} Z_j^{(1)} \), standard martingale computations (see [4, Chapter I.4]) show that
\[
\mathbb{E}[Z_j^{(1)}] = \nu^j, \quad \text{Var}(Z_j^{(1)}) = \nu^{2j} \text{Var}(W_j) = \nu^{2j} \sum_{\ell=0}^{j-1} \nu^{-\ell-2} \sigma^2 \leq C \nu^{2j}, \tag{2.29}
\]
for some constant \( C > 0 \). It follows that
\[
\mathbb{E}[|\partial T_y^-(j)|^2] = d_y^- \nu^j, \quad \text{Var}(|\partial T_y^-(j)|) = d_y^- \text{Var}(Z_j^{(1)}) \leq C \nu^{2j} d_y^- . \tag{2.30}
\]
By Markov’s inequality, uniformly in \( j \leq h \), for all \( s > 0 \),
\[
\mathbb{P}\left(|\partial T_y^-(j)| > s\right) \leq \frac{1}{s^2} \mathbb{E}[|\partial T_y^-(j)|^2] = \frac{1}{s^2} ((d_y^- \nu^j)^2 + C \nu^{2j} d_y^-) \leq \frac{1}{s^2} 2C((d_y^-)^2 \nu^{2j} . \tag{2.31}
\]
By a union bound over \( j \leq h \) and setting \( s = d_y^- n^a / h \), it follows that
\[
\mathbb{P}\left(|T_y^-| > d_y^- n^a\right) \leq 2Ch^3 \nu^{2h} \frac{n^{2a}}{n^{2a}} \leq n^{\varepsilon/4 - 2a} . \tag{2.32}
\]
for \( n \) large enough. By Eqs. (2.25), (2.26) and (2.32) and choosing \( a = \frac{1}{4} \) we conclude that
\[
\mathbb{P}(C_y^c) \leq (d_y^-)^2 n^{-1/3}. \tag{2.33}
\]
The proof of Eq. (2.23) is an immediate consequence of Eq. (2.22) and Eq. (2.32). Finally, by setting \( a = \frac{\eta}{4} \) and using Lemma 2.1, we obtain that the left-hand-side of Eqs. (2.26) and (2.32) is \( o(1) \) for \( \varepsilon < \eta \).

**Corollary 2.4.** Fix \( \varepsilon > 0 \) sufficiently small and \( h = h_\varepsilon \). Let \( I \) be a uniformly random vertex in \([n] \). Then,
\[
\mathbb{P}\left(B_I^- (h) \neq T_I^- (h)\right) \leq n^{-1/4}. \tag{2.34}
\]
Proof. The estimate follows from Lemma 2.3, summing over \( y \in [n] \) and using
\[
\frac{1}{n} \sum_{y \in [n]} (d_y) 2n^{-\frac{1}{3}} \leq n^{-\frac{1}{4}},
\]
which holds for all \( n \) large enough because of Assumption 1.1. \( \square \)

2.4. Out-neighborhoods. In this section we focus on tree-excesses of out-neighborhoods. Recall the definition of \( h_\varepsilon \) in Eq. (2.9). For \( h \in \mathbb{N} \), consider the event
\[
\mathcal{G}^+(h) := \bigcap_{x \in [n]} \{ \text{Tx}(B_x^+(h)) \leq 1 \}.
\]

Lemma 2.5. For \( \varepsilon > 0 \) sufficiently small,
\[
\mathbb{P} \left( \mathcal{G}^+(2h_\varepsilon) \right) = 1 - o(1),
\]
where \( h_\varepsilon \) is defined as in Eq. (2.9).

Proof. Fix \( x \in [n] \) and \( h = h_\varepsilon \). To generate \( B_x^+(2h) \), we match at most \( K := (\Delta^+)^{2h} = n^{\varepsilon/10} \) tails. The probability that at any given step we choose a head incident to an already revealed vertex is at most
\[
q := \frac{K \Delta^-}{m - K}.
\]

Therefore,
\[
\mathbb{P} \left( \text{Tx}(B_x^+(2h)) \geq 2 \right) \leq \mathbb{P}(\text{Bin}(K, q) \geq 2) \leq (Kq)^2,
\]
where Bin\((N, p)\) denotes a binomial random variable with parameters \( N, p \) and we use the simple bound \( \mathbb{P}(\text{Bin}(N, p) \geq \ell) \leq (\ell) \leq (Np) \ell \), valid for all \( N, \ell \in \mathbb{N}, p \in [0, 1] \). By Lemma 2.1, if \( \varepsilon < \frac{\eta}{2} \) it follows that \( (Kq)^2 = o(n^{-1}) \), and the conclusion follows by a union bound over \( x \in [n] \). \( \square \)

While the previous lemma cannot be improved substantially, the set of vertices with positive tree-excess is small, as the next result shows. Define
\[
V_\varepsilon := \{ x \in [n] : \text{Tx}(B_x^+(h_\varepsilon)) = 0 \}.
\]

Lemma 2.6. For any \( \varepsilon > 0 \),
\[
\mathbb{P} \left( |[n] \setminus V_\varepsilon| \leq n^{\frac{\varepsilon}{2}} \right) \geq 1 - n^{-\frac{1}{2}},
\]

Proof. We may assume that \( \varepsilon \) is sufficiently small. As in the proof of Lemma 2.5 we know that
\[
\mathbb{P} \left( \text{ Tx}(B_x^+(h_\varepsilon)) \geq 1 \right) \leq \mathbb{P}(\text{Bin}(K, q) \geq 1) \leq Kq,
\]
where now
\[
K = (\Delta^+)^{h_\varepsilon}, \quad q = \frac{K \Delta^-}{m - K}.
\]

Therefore, for any \( x \in [n] \), \( \mathbb{P} (x \notin V_\varepsilon) \leq Kq \). As a consequence, \( \mathbb{E} |[n] \setminus V_\varepsilon| \leq nKq \leq K^2 \Delta^- \). By Markov’s inequality and Eq. (2.1)
\[
\mathbb{P} \left( |[n] \setminus V_\varepsilon| \geq n^{\frac{\varepsilon}{2}} \right) \leq K^2 \Delta^- n^{-\frac{\varepsilon}{2}} = O \left( n^{\frac{\varepsilon}{20} - \frac{\varepsilon}{2} - \frac{\varepsilon}{10}} \right).
\]

Since \( \varepsilon < \eta \), the lemma follows for large enough \( n \). \( \square \)
3. Random Walk

In this section we introduce the random walk on $G$ and prove our main results concerning convergence to stationarity. For a given realization of the digraph $G = G_n$, and a probability distribution $\mu$ on $[n]$, we will denote by $P_\mu$ the quenched law of the random walk on $G$ with initial distribution $\mu$, i.e., the law of the Markov chain on $[n]$ with transition matrix $P$ as in Eq. (1.4). When $\mu$ is concentrated on a single vertex $x \in [n]$, i.e., $\mu = \delta_x$, we write $P_x$. We write $X_t$ for the position of the random walk at any given time $t \geq 0$. Notice that, for any $A \subset [n]$, $P_\mu(X_t \in A) = \mu P^t(A)$ is itself a random variable. When we want to emphasize its dependence on the realization of the bijection $\omega$ which induces the digraph $G$, we write $P_\mu^\omega$. Thanks to Lemma 2.5 and Assumption 1.1(i), we can immediately infer the following property of the quenched law.

**Lemma 3.1.** Fix $\varepsilon > 0$ sufficiently small and let $V_\varepsilon$ be defined as in Eq. (2.40). For all $t \in \mathbb{N}$ such that $t \leq h_\varepsilon$,

$$P \left( \max_{x \in [n]} P_x(X_t \notin V_\varepsilon) \leq 2^{-t} \right) = 1 - o(1).$$

(3.1)

**Proof.** From Lemma 2.5, it is sufficient to prove that the event $G^+ (2h_\varepsilon)$ implies $P_x(X_t \notin V_\varepsilon) \leq 2^{-t}$ for all $x \in [n]$ and for all $t \leq h_\varepsilon$. Under the event $\mathbb{T} x (B^+(2h_\varepsilon)) \leq 1$ there is at most one path of length $t \leq h_\varepsilon$ from $x$ to any point $y \notin V_\varepsilon$. Since each path of length $t$ has weight at most $2^{-t}$ the conclusion follows.

A key object in our analysis is the so-called annealed law, obtained by averaging the quenched law over the environment:

$$\mathbb{P}_x^{\text{an}} = \mathbb{E} \left[ P_x \right] = \frac{1}{m!} \sum_\omega P_\omega^x, \quad \mathbb{P}_\mu^{\text{an}} = \sum_{x \in [n]} \mu(x) \mathbb{P}_x^{\text{an}},$$

(3.2)

where $\omega$ ranges over all bijections from $E^+$ to $E^-$. It will also be important to consider the average over the environment of the quenched law of $K$ independent walks. As observed in [12, 13] the corresponding annealed law is a very powerful tool in estimating high order moments of random variables such as $P_\mu(X_t \in A)$ for $A \subset [n]$ and $t \geq 0$. More precisely, for any probability distribution $\mu$ on $[n]$, for any subset $A \subset [n]$ and for all $t, K \in \mathbb{N}$,

$$\mathbb{E} \left[ (\mu P^t(A))^K \right] = \mathbb{E} \left[ (P_\mu(X_t \in A))^K \right] = \mathbb{P}_\mu^{\text{an}, K} \left( X_t^{(k)} \in A, \forall k \in [K] \right),$$

(3.3)

where the annealed law $\mathbb{P}_\mu^{\text{an}, K}$ is the deterministic law of a non-Markovian process

$$\left\{ X_s^{(k)}, \ s \in \{0, \ldots, t\}, \ k \in \{1, \ldots, K\} \right\},$$

(3.4)

which can be described as follows. Start with an empty matching. For each $k$, given the first $k - 1$ walks $(X_s^{(\ell)})_{s \leq t, \ell \leq k-1}$, to generate the $k$-th walk $X^{(k)}$,

(i) start the $k$-th walk at a random vertex $X_0^{(k)} \sim \mu$.

(ii) for all $s \in \{0, \ldots, t - 1\}$: select one of the tails of $X_s^{(k)}$ uniformly at random, and call it $e$:

- If $e$ was already matched by one of the previous walks, or by $X^{(k)}$ itself at a previous step, to some head $f$, then let $X_{s+1}^{(k)} = v_f$.
If $e$ is still unmatched select a uniformly random head, $f$, among the unmatched ones, match it to $e$, and let $X_{s+1}^{(k)} = v_f$.

We may view the walks as generating the environment (the digraph) as they activate new matchings (the edges). In particular, for any $K, t \geq 1$, the digraph $G$ may be sampled using the edges revealed by the $K$ walks up to time $t$ and then by completing with a uniform matching of the remaining heads and tails.

### 3.1. Law of large numbers

Using the annealed process as a computational tool, we prove a quenched law of large numbers for the weight of the path determined by the random walk trajectory, thus extending results in [12] previously obtained in the case of bounded in-degrees. In particular, we show that for any $t$ of order $\log n$, whp the quenched law of the random walk is concentrated on trajectories which have weight $e^{-Ht(1+o(1))}$, where $H$ is the entropy in Eq. (1.8). The proof follows very closely the original argument in [12], while some minor technical difficulties due to the unbounded in-degree setting are overcome using the $2 + \eta$ moment condition in Assumption 1.1.

**Proposition 3.2.** Let $t = \Theta(\log(n))$ and fix a sequence $\theta = \theta_n \in (0, 1)$. Recall the definitions of path $p$ in Eq. (2.4), of weight $w(p)$ in Eq. (2.5) and $P(x, y, t, G)$. Call

$$Q_{x,t}(\theta) := \sum_{y \in [n]} \sum_{p \in P(x, y, t, G)} w(p) 1_{w(p) > \theta}.$$  \hspace{1cm} (3.5)

If for $\lambda \neq H$ we have

$$-\frac{\log(\theta)}{t} \rightarrow \lambda,$$  \hspace{1cm} (3.6)

then

$$\max_{x \in [n]} |Q_{x,t}(\theta) - 1(\lambda > H)| \rightarrow^p 0.$$  \hspace{1cm} (3.7)

**Proof.** Fix sequences $t = \Theta(\log(n))$ and $\theta = \theta_n$ such that Eq. (3.6) holds. Let $\ell = 3 \log \log(n)$. Let us consider the averaged probability

$$\bar{Q}_{x,t}(\theta) := \sum_{y \in [n]} P^\ell(x, y) Q_{y,t}(\theta).$$  \hspace{1cm} (3.8)

We will show that $\bar{Q}_{x,t}(\theta)$ is well approximated by

$$q_t(\theta) := \mathbb{P}\left( \prod_{k=1}^{t} \frac{1}{D_k^+} > \theta \right) = \mathbb{P}\left( \frac{1}{t} \sum_{k=1}^{t} \log(D_k^+) < -\frac{\log(\theta)}{t} \right),$$  \hspace{1cm} (3.9)

where $(D_k^+)_{k \in \mathbb{N}}$ is an iid sequence of random variables with law

$$\mathbb{P}(D_k^+ = \ell) = \sum_{v \in [n]} \mu_n(v) 1(d_v^+ = \ell) \text{ for } \ell \geq 2.$$  \hspace{1cm} (3.10)

Using Eq. (1.8) and the law of large numbers for the bounded sequence $(\log(D_k^+))_{k \in \mathbb{N}}$ we have

$$q_t(\theta) \rightarrow \begin{cases} 1 & \text{if } \lambda > H, \\ 0 & \text{if } \lambda < H. \end{cases}$$  \hspace{1cm} (3.11)
We first show that Proposition 3.2 is implied by the following convergence

$$\max_{x \in V_\varepsilon} |\bar{Q}_{x,t} - q_t| \xrightarrow{\mathbb{P}} 0,$$  \hspace{1cm} (3.12)

where $V_\varepsilon$ is defined in Eq. (2.40) and $\varepsilon \in (0, \eta/2)$, where $\eta \in (0, 1)$ is as in Assumption 1.1.

Since the weight of a path of length $\ell$ is always in $[(\Delta^+)/2, 2\Delta^+]$ we may estimate

$$\max_{x \in [n]} Q_{x,t}(\theta) \leq \max_{x \in [n]} P^\ell(x, [n] \setminus V_\varepsilon) + \max_{x \in V_\varepsilon} \bar{Q}_{x,t-2\ell}(\theta2^\ell)$$

$$\leq 2^{-\ell} + o_\mathbb{P}(1) + \max_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta2^\ell(\Delta^+)^{-2\ell})$$

$$\leq q_t(\theta2^\ell(\Delta^+)^{-2\ell}) + o_\mathbb{P}(1),$$

where the second line follows from Lemma 3.1 and the third line from Eq. (3.12). Similarly,

$$\min_{x \in [n]} Q_{x,t}(\theta) \geq \min_{x \in [n]} P^\ell(x, V_\varepsilon) \min_{x \in V_\varepsilon} \bar{Q}_{x,t-2\ell}(\theta(\Delta^+)^{2\ell})$$

$$\geq (1 - 2^{-\ell} - o_\mathbb{P}(1)) \min_{x \in V_\varepsilon} \bar{Q}_{x,t-2\ell}(\theta(\Delta^+)^{2\ell})$$

$$\geq \min_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta(\Delta^+)^{2\ell}) + o_\mathbb{P}(1)$$

$$\geq q_t(\theta(\Delta^+)^{2\ell}) + o_\mathbb{P}(1).$$

Moreover, notice that if for some $\theta'$ it holds $\log(\theta') = \log(\theta) + O(\log \log(n))$ then

$$|q_t(\theta) - q_t(\theta')| \to 0.$$  \hspace{1cm} (3.15)

Thus, Eqs. (3.11) and (3.12) imply Proposition 3.2.

To prove Eq. (3.12), we show that for all $\delta > 0$

$$\mathbb{P}\left(1_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta) \geq q_t(\theta) + \delta\right) = o(n^{-1}).$$  \hspace{1cm} (3.16)

This, together with a union bound over $x \in V_\varepsilon$, establishes one half of Eq. (3.12); the other half can be obtained in the same fashion replacing $\bar{Q}_{x,t}(\theta)$ by $1 - \bar{Q}_{x,t}(\theta)$ and $q_t(\theta)$ by $1 - q_t(\theta)$, which amounts to inverting the inequality signs in the definition of $Q_{x,t}(\theta)$ and $q_t(\theta)$.

We now prove Eq. (3.16). By Markov's inequality, for all $K \geq 1$,

$$\mathbb{P}\left(1_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta) \geq q_t(\theta) + \delta\right) \leq \frac{\mathbb{E}\left[ \left(1_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta)\right)^K \right]}{(q_t(\theta) + \delta)^K}. $$  \hspace{1cm} (3.17)

Hence, it is enough to show that if $K = \lfloor \log^2(n) \rfloor$, for all $\delta > 0$ and $n$ large enough one has

$$\mathbb{E}\left[ \left(1_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta)\right)^K \right] \leq (q_t(\theta) + \frac{\delta}{2})^K.$$  \hspace{1cm} (3.18)

Notice that

$$\mathbb{E}\left[ \left(1_{x \in V_\varepsilon} \bar{Q}_{x,t}(\theta)\right)^K \right] \leq \mathbb{P}_{\mathbb{X}}^{n,K}(B_K), $$  \hspace{1cm} (3.19)

where $\mathbb{P}_{\mathbb{X}}^{n,K}$ is the law of $K$ annealed walks of length $\ell + t$ all started at $x$, see Eq. (3.3), and for all $j \leq K$, $B_j$ is the event that

(i) the union of the first $j$ trajectories up to time $\ell$, that is $(X_s^{(1)}, \ldots, X_s^{(j)})_{s \leq \ell}$, forms a directed tree.
(ii) for each \( i \leq j \), the last \( t \) steps of the \( i \)-th walk, that is \((X^{(i)}_s)_{s \in [\ell+1, \ell+t]}\), define a path \( p \) of weight \( w(p) > \theta \).

Since
\[
\mathbb{P}^{\text{pan},K}(B_K) = \mathbb{P}^{\text{pan},K}(B_1) \prod_{j=2}^{K} \mathbb{P}^{\text{pan},K}(B_j \mid B_{j-1}),
\]
(3.20)

it is enough to show that, uniformly in \( j \leq K \),
\[
\mathbb{P}^{\text{pan},K}(B_j \mid B_{j-1}) \leq q_t(\theta) + \frac{\delta}{2}.
\]
(3.21)

In order to check that Eq. (3.21) holds, we note that, given \( B_{j-1} \):

(i) either the \( j \)-th walk attains length \( \ell \) before reaching an unmatched tail: thanks to the tree structure, there are at most \( K-1 \) possible paths of length \( \ell \) to follow starting at \( x \), and each has weight at most \( 2^{-\ell} \). Thus, the conditional probability of this scenario is less than \( K 2^{-\ell} = o(1) \).

(ii) or the \( j \)-th walk has reached an unmatched tail by time \( \ell \): then, the remainder of the path after the first unmatched tail can be coupled with an iid sample from the in-degree distribution on \([n] \) at a total-variation cost less than \( \Delta \sum_{\ell=0}^{m} K(t+\ell) \), and the latter is deterministically \( o(1) \) thanks to Lemma 2.1. Indeed, there are at most \( t+\ell \) heads matched in the generation of the \( j \)-th walk and the probability that the coupling fails at a given step is bounded uniformly by \( \Delta \sum_{\ell=0}^{m} K(t+\ell) \), as all heads can be chosen. Thus, the conditional probability that the walk meets the requirement in that case is at most \( q_t(\theta) + o(1) \).

\[\Box\]

3.2. A weighted out-neighborhood construction. Following an idea introduced in [12, 13], we now consider a further construction of the out-neighborhood of a given vertex that reveals only the directed paths which have a sufficiently large probability to be followed by the random walk.

Fix \( \gamma \in (0,1) \). All parameters defined below depend implicitly on \( \gamma \). Let
\[
\overline{H} := (1 + \gamma)H,
\]
(3.22)

where \( H \) is the entropy in Eq. (1.8). For any integer \( s \geq 1 \) and every constant \( \gamma > 0 \), let \( \mathcal{G}_x(s) \) be the weighted directed graph spanned by the set of paths of length at most \( s \), starting from \( x \), and having weight \( w(p) \geq e^{-\overline{H}s} \). As in [13, Section 4.1], we construct a sequence \((\mathcal{G}^\ell, T^\ell)_{\ell \geq 0}\), where \( \mathcal{G}^\ell \) is a subgraph of \( \mathcal{G}_x(s) \) with \( \ell \) edges, constructing \( \mathcal{G}_x(s) \) edge by edge, and such that \( T^\ell \) is a spanning tree of \( \mathcal{G}^\ell \) for each \( \ell \). We call \( \kappa_x = \kappa_x(s) \) the random number of edges needed to construct the whole digraph \( \mathcal{G}^\kappa_x = \mathcal{G}_x(s) \). We now explain the detailed construction of the sequence \((\mathcal{G}^\ell, T^\ell)_{\ell \geq 0}\). As initialization, let \( \mathcal{E}_0 \) be the set of tails of \( x \), \( \mathcal{F}_0 = \emptyset \) and let \( \mathcal{G}^0 = T^0 \) be the digraph containing only \( x \) and no edges. Then, for all \( \ell \geq 1 \):

(i) Let \( \mathcal{E}_{\ell-1} \) be the set of unmatched tails which are incident to a node in \( \mathcal{G}^{\ell-1} \). For a tail \( e \in \mathcal{E}_{\ell-1} \), define its cumulative weight by
\[
\bar{w}(e) := \frac{w(p)}{d^+_e},
\]
(3.23)
where \( p \) is the unique path from \( x \) to \( v_e \) in \( T^{\ell-1} \). In words, \( \hat{w}(e) \) is the probability for the random walk to follow \( p \) and then the edge containing \( e \).

(ii) Pick \( e_\ell \in E_{\ell-1} \) such that
(a) \( v_{e_\ell} \) is at distance at most \( s - 1 \) from \( x \),
(b) \( \hat{w}(e_\ell) \geq w_{\text{min}} := e^{-\Pi s} \),
(c) \( e_\ell \) has maximum cumulative weight among all tails in \( E_{\ell-1} \) which satisfy (a) and (b), and use some arbitrary rule to break ties if needed.

(iii) If no such \( e_\ell \) exists, then terminate and let \( \kappa_x = \ell - 1 \).

(iv) Otherwise, pair \( e_\ell \) with a head \( f_\ell \) chosen uniformly at random in \( E - \mathcal{F}_{\ell-1} \) and set \( \mathcal{F}_\ell = \mathcal{F}_{\ell-1} \cup \{f_\ell\} \). Let \( G^\ell \) be the resulting partial pairing. If \( v_{f_\ell} \in V(G^{\ell-1}) \), the set of vertices of \( G^{\ell-1} \), then let \( T^\ell = T^{\ell-1} \); otherwise let \( T^\ell = T^{\ell-1} \cup (e_\ell, f_\ell) \). Return to step (i).

When the process terminates, the construction yields \( G_x(s) := G^{\kappa_x} \) and \( T_x(s) := T^{\kappa_x} \).

By [12, Lemma 11], one has the deterministic bound
\[
\frac{w_{\text{min}}}{2} \leq \hat{w}(e_\ell) \leq 2 + \ell,
\] \hspace{1cm} (3.24)
for all \( \ell \leq \kappa_x \). From this, it follows that,
\[
\kappa_x(s) \leq \frac{2}{w_{\text{min}}} = 2e^{\Pi s}.
\] \hspace{1cm} (3.25)

The main motivation for the above construction is the fact that with high probability the random walk trajectories up to time \( s \leq (1 - o(1))T_{\text{ent}} \) are concentrated on the tree \( T_x(s) \) provided the starting point \( x \) is locally tree-like, as we now explain. Note that the probability that a walk trajectory \( (X_0, \ldots, X_s) \) with \( X_0 = x \) stays on the tree \( T_x(s) \) may be written as
\[
p(x, s) = \sum_{y \in [n]} \sum_{p \in \mathcal{P}(x, y, s, T_x(s))} w(p),
\] \hspace{1cm} (3.26)
where \( \mathcal{P}(x, y, s, T_x(s)) \) denotes the set of all paths of length \( s \) from \( x \) to \( y \) in \( T_x(s) \).

**Lemma 3.3.** Recall the definition of \( V_\varepsilon \) in Eq. (2.40). For all \( \varepsilon > 0 \), \( \gamma > 0 \), and for any \( s \leq (1 - \gamma)T_{\text{ent}} \),
\[
\min_{x \in V_\varepsilon} p(x, s) \xrightarrow{p} 1.
\] \hspace{1cm} (3.27)

**Proof.** Let \( \mathcal{P}^{*}_{x,y} = \mathcal{P}(x, y, s, G) \setminus \mathcal{P}(x, y, s, T_x(s)) \) denote the set of paths starting at \( x \) and ending at \( y \) of length \( s \) which are not on the tree \( T_x(s) \). We need to show that
\[
\max_{x \in V_\varepsilon} \sum_y \sum_{p \in \mathcal{P}^{*}_{x,y}} w(p) \xrightarrow{p} 0.
\] \hspace{1cm} (3.28)
If a path \( p \) is in \( \mathcal{P}^{*}_{x,y} \), then at least one of the following conditions is satisfied:
(i) the weight of the path satisfies \( w(p) < e^{-\Pi s} \);
(ii) \( p \) contains an edge in \( G_x(s) \setminus T_x(s) \).
We call $P_{x,y}^1$ the set of paths in case (i), and $P_{x,y}^2$ the set of paths in case (ii) but not in case (i). From the law of large numbers in Proposition 3.2 we know that

$$\max_{x \in [n]} \sum_{y \in [n]} \sum_{p \in P_{x,y}^1} w(p) \xrightarrow{\mathbb{P}} 0. \quad (3.29)$$

It remains to show that

$$\max_{x \in V_\varepsilon} \sum_{y \in [n]} \sum_{p \in P_{x,y}^2} w(p) \xrightarrow{\mathbb{P}} 0. \quad (3.30)$$

Define a process $(M_\ell)_{\ell \geq 0}$ by $M_0 = 0$ and for $\ell \geq 1$

$$M_\ell = M_{\ell-1} + \hat{w}(e_\ell) 1(\ell \leq \kappa_x) 1(v_{f_\ell} \in V(G^{\ell-1})). \quad (3.31)$$

Notice that

$$M_{\kappa_x} = \sum_{y \in [n]} \sum_{p \in P_{x,y}^1} w(p). \quad (3.32)$$

We will show that $\mathbb{P}(M_{\kappa_x} > \delta) = o(n^{-1})$, uniformly in $x \in V_\varepsilon$, and for all $\delta > 0$. By a union bound over $x \in V_\varepsilon$, this implies the desired claim.

Since $x \in V_\varepsilon$, it follows that $G_x(h_\varepsilon)$ is a tree, where $h_\varepsilon$ is defined as in Eq. (2.9). Therefore, setting $\ell_\varepsilon = 2^h_\varepsilon$, one must have $M_\ell = 0$ for all $\ell < \ell_\varepsilon$. Thus, the desired conclusion follows once we prove

$$\mathbb{P}(M_{\kappa_x} - M_{\ell_\varepsilon} > \delta) = o(n^{-1}). \quad (3.33)$$

To prove it we use a martingale version of Bennett’s inequality obtained by Freedman [30].

Since $G_x(h_\varepsilon)$ is a tree of size at least $\ell_\varepsilon$ and $\hat{w}(e_\ell)$ is decreasing in $\ell$, we have that $M_\ell = 0$ for all $\ell \leq \ell_\varepsilon$ and, by Eq. (3.24), that

$$0 \leq M_\ell - M_{\ell-1} \leq 2^{-h_\varepsilon + 1} = o(1), \quad (3.34)$$

for all $\ell \geq \ell_\varepsilon + 1$. Let $F_\ell$ be the filtration associated to $(G^\ell, \mathcal{T}^\ell)$. Call

$$g_\ell = \sum_{v \in V(G^\ell)} d_v, \quad (3.35)$$

and note that by Eq. (3.25), the vertices in $V(G^\ell)$ are at most $\ell \leq \kappa_x \leq 2e^{H(1-\gamma)T_{ent}} = 2n^{1-\gamma^2}$.

Thus, by Lemma 2.1 we have $g_\ell = o(\sqrt{\ell n})$. Using Eq. (3.24),

$$\mathbb{E}[M_\ell - M_{\ell-1} \mid F_{\ell-1}] \leq 1_{\ell \leq \kappa_x} \frac{\hat{w}(e_\ell) g_{\ell-1}}{m - (\ell - 1)} = o\left(\frac{1}{\sqrt{n} \ell}\right);$$

$$\mathbb{E}[(M_\ell - M_{\ell-1})^2 \mid F_{\ell-1}] \leq 1_{\ell \leq \kappa_x} \frac{(\hat{w}(e_\ell))^2 g_{\ell-1}}{m - (\ell - 1)} = o\left(\frac{1}{n \ell^2}\right). \quad (3.36)$$

Adding over all steps until $\kappa_x$, we get

$$a := \sum_{\ell=1}^{\kappa_x} \mathbb{E}[M_\ell - M_{\ell-1} \mid F_{\ell-1}] \leq o\left(n^{-1/2}\right) \sum_{\ell=1}^{\kappa_x} \frac{1}{\ell^{1/2}} = o\left(n^{-\gamma^2/2}\right), \quad (3.37)$$

$$b := \sum_{\ell=1}^{\kappa_x} \mathbb{E}[(M_\ell - M_{\ell-1})^2 \mid F_{\ell-1}] \leq o\left(n^{-1/2}\right) \sum_{\ell=1}^{\kappa_x} \frac{1}{\ell^{3/2}} = o\left(n^{-1}\right).$$
where we used $\sum_{\ell=1}^k \ell^{-1/2} = O(\sqrt{k})$, $\sum_{\ell \geq 1} \ell^{-3/2} = O(1)$, and $\kappa_x \leq 2n^{1-\gamma^2}$. For $\ell \geq \ell_\varepsilon$, define
\[ Z_{\ell+1} = \frac{5}{\delta} (M_{\ell+1} - M_\ell - \mathbb{E}[M_{\ell+1} - M_\ell \mid \mathcal{F}_\ell]), \tag{3.38} \]
which satisfies $|Z_{\ell+1}| \leq 1$ by Eq. (3.34). Let $\phi_u = \sum_{i=\ell_\varepsilon}^u Z_{i+1}$. Then $(\phi_u)_{u \geq \ell_\varepsilon}$ is a martingale and $M_{\kappa_x} = a + \frac{\delta}{5} \phi_{\kappa_x}$. When $n$ is large enough, we have $a \leq \delta/5$. So
\[ \mathbb{P} \left( M_{\kappa_x} \geq \frac{4}{5} \delta \right) \leq \mathbb{P} (\phi_{\ell} \geq 3, \text{ for some } \ell \geq \ell_\varepsilon). \tag{3.39} \]
The conditional variance of $\phi_{\ell}$ is $b' := \sum_{i=1}^\ell \text{Var}(Z_i \mid \mathcal{F}_i) \leq (\delta/5)^{-2b} = o(n^{-1/3})$. Then by [30, Theorem 1.6],
\[ \mathbb{P} (\phi_{\ell} \geq 3, \text{ for some } \ell \geq \ell_\varepsilon) \leq o^3 \left( \frac{b'}{3 + b'} \right)^{3+b'} = o(n^{-1}). \tag{3.40} \]
\[ \Box \]

3.3.3. Mixing time. In this section we prove Theorem 1.1. We adapt the arguments in [12], which established the same result under a bounded degree assumption. As we will see, the role played by the boundedness of the in-degrees in [12] will be replaced by Lemma 2.1. We prove separately the lower and the upper bound on the total variation distance.

3.3.1. Proof of the upper bound of Theorem 1.1. Let us first explain the overall strategy of the proof, which is based on ideas introduced in [12, 13]. For each pair of vertices $x, y \in [n]$, we shall define a set $\tilde{P}(x, y, t, G) \subset \mathcal{P}(x, y, t, G)$ of nice paths of length $t$ starting at $x$ and ending at $y$. We will also let $\tilde{P}t(x, y) \leq P^t(x, y)$ denote the probability to go from $x$ to $y$ in $t$ steps following a nice path, i.e.,
\[ \tilde{P}^t(x, y) := \sum_{p \in \tilde{P}(x, y, t, G)} w(p). \tag{3.41} \]
Suppose that for some $\delta > 0$ and some probability distribution $\bar{\pi}$ on $[n]$ it holds that
\[ \tilde{P}^t(x, y) \leq (1 + \delta)\bar{\pi}(y) + \frac{\delta}{n}, \quad \forall x, y \in [n]. \tag{3.42} \]
For $x \in \mathbb{R}$, define $[x]_+ = \max\{x, 0\}$. If Eq. (3.42) holds, then
\[ \left[ \bar{\pi}(y) - P^t(x, y) \right]_+ \leq \left[ \bar{\pi}(y) - \tilde{P}^t(x, y) \right]_+ \leq (1 + \delta)\bar{\pi}(y) + \frac{\delta}{n} - \tilde{P}^t(x, y). \tag{3.43} \]
Therefore,
\[ \|P^t(x, \cdot) - \bar{\pi}\|_{\text{tv}} = \frac{1}{2} \sum_{y \in [n]} |\bar{\pi}(y) - P^t(x, y)| = \sum_{y \in [n]} \left[ \bar{\pi}(y) - P^t(x, y) \right]_+ \]
\[ \leq \sum_{y \in [n]} \left( (1 + \delta)\bar{\pi}(y) + \frac{\delta}{n} - \tilde{P}^t(x, y) \right) = 2\delta + \bar{q}(x), \tag{3.44} \]
where \( q(x) \) is the probability that the walk starting at \( x \) follows a path of length \( t \) which is not nice, i.e.,

\[
\tilde{q}(x) := 1 - \sum_{y \in [n]} \sum_{p \in \tilde{P}(x,y,t,G)} w(p).
\]  

(3.45)

Next, we define the set of nice paths.

**Definition 3.4.** Let \( \eta \in (0, 1) \) such that Assumption 1.1 is satisfied. Fix \( \varepsilon \in (0, \eta/6) \), \( h = h_\varepsilon \), \( \gamma = \frac{\varepsilon}{80 \log(\Delta^+)} \) and set

\[
s := (1 - \gamma) T_{\text{ent}}, \quad t := s + h + 1.
\]  

(3.46)

Notice that \( s = T_{\text{ent}} - \frac{h}{4H} \), and that our choice of parameters \( \varepsilon, h, s, t \) and \( \gamma \) is such that \( t = (1 + \delta') T_{\text{ent}} + 1 \) for some \( \delta' = \delta'(\varepsilon) > 0 \) such that \( \delta'(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). We define the set of nice paths as follows. Given \( x, y \in [n] \), a path \( p \) of length \( t \) starting at \( x \) and ending at \( y \) is nice, if

(i) the first \( s \) steps are contained in the tree \( T_x(s) \) defined in Section 3.2;

(ii) the first \( s + 1 \) steps of \( p \) form a path \( p_{s+1} \) such that

\[
w(p_{s+1}) \leq n^{-1+2\gamma};
\]  

(3.47)

(iii) the last \( h \) steps of \( p \) form the unique path in \( G \) of length at most \( h \) from its origin to its destination.

Notice that (ii) implies that for every nice path \( p \)

\[
w(p) \leq w(p_{s+1}) \cdot 2^{-h} \leq 2^{-h} n^{-1+2\gamma} \leq n^{-1-\delta'},
\]  

(3.48)

for some \( \delta' > 0 \), where we use the definition of \( h = h_\varepsilon \) and \( \gamma \), and the fact that \( \log(2) > 2/3 \).

With this definition at hand, we show that, whp, starting from any locally-tree-like vertex, the quenched probability to follow a nice path converges to 1.

**Proposition 3.5.** Recall the definition of \( V_\varepsilon \) in Eq. (2.40). The probability \( \tilde{q}(x) \) defined in Eq. (3.45) satisfies

\[
\max_{x \in V_\varepsilon} \tilde{q}(x) \xrightarrow{\text{p}} 0.
\]  

(3.49)

**Proof.** We are going to check that the three requirements in Definition 3.4 are satisfied whp by the quenched law uniformly in the starting state \( x \in V_\varepsilon \). Notice that requirement (i) follows, uniformly in \( x \in V_\varepsilon \), by Lemma 3.3. By our choice of the parameter \( \gamma \), \( n^{-1+2\gamma} \geq e^{-(1-\delta')H} \) if \( \delta' > 0 \) is small enough, and therefore the requirement in (ii) is a simple consequence of Proposition 3.2, which holds uniformly in \( x \in [n] \). Finally, the probability of the event in requirement (iii) can be bounded from above by the probability that at time \( s + 1 \) the walk is in \( V_\varepsilon \). Moreover, for any \( \ell \leq s \):

\[
\min_{x \in V_\varepsilon} \mathbb{P}_x(X_{s+1} \in V_\varepsilon) \geq 1 - \max_{v \in [n]} \mathbb{P}_v(X_{\ell} \not\in V_\varepsilon).
\]  

(3.50)

where the last step follows from Lemma 3.1. \( \square \)

The last ingredient we need is an approximation \( \tilde{\pi} \) for the stationary distribution \( \pi \), which satisfies Eq. (3.42). To this end, we define

\[
\tilde{\pi} = \mu_{\text{in}} P^h.
\]  

(3.51)
Proposition 3.6. For all $\delta > 0$,
\[
\mathbb{P}\left( \max_{x \in [n]} \tilde{P}^t(x, y) \leq (1 + \delta) \mu_{\text{in}} P^h(y) + \frac{\delta}{n} \quad \forall y \in [n] \right) = 1 - o(1). \tag{3.52}
\]

We first conclude the proof of the upper bound in Theorem 1.1, and then provide the proof of Proposition 3.6.

Proof of the upper bound of Theorem 1.1. Fix the parameters $\varepsilon, h, s, t$ and $\gamma$ as above. It follows from the argument in Eq. (3.44) and Propositions 3.5 and 3.6 that
\[
\max_{x \in V_\varepsilon} \| P^t(x, \cdot) - \mu_{\text{in}} P^h \|_{TV} \xrightarrow{\mathbb{P}} 0. \tag{3.53}
\]
Thus, the upper bound in Theorem 1.1, i.e., when $\rho > 1$, holds for all starting states $x \in V_\varepsilon$ with $\tilde{\pi}$ in place of $\pi$. Setting $t' = t + \ell$,
\[
\max_{v \in [n]} \| P^{t'}(v, \cdot) - \mu_{\text{in}} P^h \|_{TV} \leq \max_{x \in V_\varepsilon} \| P^t(x, \cdot) - \mu_{\text{in}} P^h \|_{TV} + \max_{v \in [n]} \mathbb{P}(X_t \not\in V_\varepsilon). \tag{3.54}
\]
Taking, e.g., $\ell = \log \log(n)$, by Lemma 3.1, the last term in Eq. (3.54) tends to zero in probability. Thus,
\[
\max_{x \in [n]} \| P^{t'}(x, \cdot) - \mu_{\text{in}} P^h \|_{TV} \xrightarrow{\mathbb{P}} 0. \tag{3.55}
\]
Since the latter convergence is uniform in the starting position $x$, it must hold for every initial distribution. Starting at stationarity, it follows by Eq. (3.55) that
\[
\| \pi - \tilde{\pi} \|_{TV} = \| \pi - \mu_{\text{in}} P^h \|_{TV} \xrightarrow{\mathbb{P}} 0. \tag{3.56}
\]
Hence by the triangular inequality
\[
\max_{x \in [n]} \| P^{t'}(x, \cdot) - \pi \|_{TV} \xrightarrow{\mathbb{P}} 0. \tag{3.57}
\]
By monotonicity of the total variation distance, this implies the same estimate for all times larger than $t'$. Moreover, since $\varepsilon$ can be taken arbitrarily small, the above defined $t'$ is sufficient to cover the whole range of times of the form $\rho T_{\text{ent}}$, with fixed $\rho > 1$. \qed

Proof of Proposition 3.6. Fix the parameters $\varepsilon, h, s, t$ and $\gamma$ as above. Given $x, y \in [n]$, we first generate the pair $(G_x(s), T_x(s))$ using the construction in Section 3.2, and then sample the in-neighborhood of $y$ up to height $h, B_y^-(h)$, using the procedure in Section 2.2 with the BFS rule. Some of the matching defining $B_y^-(h)$ may have been already revealed during the construction of $G_x(s)$. Call $\kappa_y$ the additional (random) number of matchings needed to complete the construction of $B_y^-(h)$. In total, thanks to Lemma 2.2 and Eq. (3.25), whp and for all $x, y \in [n]$, the total number of matchings revealed is bounded by
\[
\kappa_x + \kappa_y \leq 2e^{s\Pi} + n^{\frac{1}{2} + \varepsilon} = 2e^{(1-\gamma)T_{\text{ent}}\Pi} + n^{\frac{1}{2} + \varepsilon} \leq 3n^{1-\gamma^2}, \tag{3.58}
\]
for all sufficiently large $n$, where we used the definitions of $\gamma$ and $\Pi$.

Let $\sigma = \sigma(x, y)$ denote the partial environment obtained after the generation of the neighborhoods $(G_x(s), T_x(s))$ and $B_y^-(h)$. Let $W_{x, y}$ be the event that $\sigma$ satisfies Eq. (3.58). Let $F_{\sigma}$ denote the set of unmatched heads $f$ with $v_f \in \partial B_y^-(h)$ that admit a unique path of length $h$ ending at $y$, and with a slight abuse of notation, let $\hat{w}(f)$ denote the weight of
such a unique path. Similarly, call \( \mathcal{E}_\sigma \) the set of unmatched tails at height \( s \) in \( T_x(s) \). Notice that, as a consequence of these definitions,

\[
\sum_{e \in \mathcal{E}_\sigma} \tilde{w}(e) \leq 1, \quad \sum_{f \in \mathcal{F}_\sigma} \tilde{w}(f) \leq m \mu_{in} P^h(y). \tag{3.59}
\]

Moreover, our construction is such that the probability to follow a nice path of length \( t \) from \( x \) to \( y \) can be written as

\[
\tilde{P}_t(x, y) = \sum_{e \in \mathcal{E}_\sigma} \sum_{f \in \mathcal{F}_\sigma} \tilde{w}(e) \tilde{w}(f) 1_{\tilde{w}(e) \leq n^{-1+2\gamma}} 1_{\omega(e) = f}, \tag{3.60}
\]

where we use the fact that for a nice path \( p \) the first \( s+1 \) steps are such that \( w(p_{s+1}) = \tilde{w}(e) \) for a suitable \( e \in \mathcal{E}_\sigma \).

Given the partial environment \( \sigma \), the sampling of the full environment is completed by using a random permutation \( \omega \) of the remaining \( m - \kappa_x - \kappa_y \) heads and tails. In particular, conditionally on the partial environment \( \sigma \), for all \( (e, f) \in \mathcal{E}_\sigma \times \mathcal{F}_\sigma \), the random variable \( 1_{\omega(e) = f} \) is marginally distributed as a Bernoulli random variable with parameter \( \frac{1}{m - \kappa_x - \kappa_y} \).

Therefore, using Eq. (3.59), for each \( \sigma \in \mathcal{W}_{x,y} \) we estimate

\[
\mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] = \frac{1}{m - \kappa_x - \kappa_y} \sum_{e \in \mathcal{E}_\sigma} \sum_{f \in \mathcal{F}_\sigma} \tilde{w}(e) \tilde{w}(f) 1_{\tilde{w}(e) \leq n^{-1+2\gamma}} \leq \left( 1 + 3n^{-\gamma^2} \right) \mu_{in} P^h(y). \tag{3.61}
\]

It follows that, uniformly in \( x, y \) and \( \sigma \in \mathcal{W}_{x,y} \), for all fixed \( \delta > 0 \) and all \( n \) large enough, one has

\[
(1 + \delta/2) \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] \leq (1 + \delta) \mu_{in} P^h(y). \tag{3.62}
\]

A concentration result for functions of a random permutation due to Chatterjee (see [20, Proposition 1.1]) shows that for all \( a > 0 \),

\[
\mathbb{P} \left( \left| \tilde{P}_t(x, y) - \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] \right| \geq a \big| \sigma \right) \leq 2 \exp \left( - \frac{a^2}{2||e||^2 \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right]^2} \right). \tag{3.63}
\]

where, using Eq. (3.48),

\[
||e||^2 := \max_{(e, f) \in \mathcal{E}_\sigma \times \mathcal{F}_\sigma} \tilde{w}(e) \tilde{w}(f) 1_{\tilde{w}(e) \leq n^{-1+2\gamma}} \leq 2^h n^{-1+\gamma} \leq n^{-1-\delta'} \tag{3.64}
\]

Choosing \( a = \frac{\delta}{2} \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] + \frac{\delta}{n} \) in Eq. (3.63), and using Eq. (3.62), we get that, for all \( \delta > 0 \), uniformly in \( x, y \) in \( \mathcal{W}_{x,y} \) and \( \sigma \in \mathcal{W}_{x,y} \) one has

\[
\mathbb{P} \left( \tilde{P}_t(x, y) \geq (1 + \delta) \mu_{in} P^h(y) + \frac{\delta}{n} \big| \sigma \right) \leq \exp(- \log^2(n)), \tag{3.65}
\]

for all \( n \) large enough. Note that a lower bound of \( \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] \) is not needed here. In fact, this bound also holds if \( y \) has in-degree \( d_y^+ = 0 \), in which case \( \mathbb{E} \left[ \tilde{P}_t(x, y) \big| \sigma \right] = 0 \).

In order to get the desired conclusion, let \( \mathcal{W} := \cap_{x,y} \mathcal{W}_{x,y} \supset \mathcal{S}_{\delta} \); see Eq. (2.10). Define

\[
\mathcal{Z}_{x,y} := \left\{ \tilde{P}_t(x, y) \leq (1 + \delta) \mu_{in} P^h(y) + \frac{\delta}{n} \right\}. \tag{3.66}
\]
Then, using Lemma 2.2 one has
\[
\mathbb{P}\left(\bigcap_{x,y \in [n]} Z_{x,y}^c\right) = 1 - \mathbb{P}\left(\bigcup_{x,y \in [n]} Z_{x,y}^c\right) \geq 1 - \mathbb{P}\left(\bigcup_{x,y \in [n]} Z_{x,y}^c \cap W\right) - \mathbb{P}(W^c)
\]
\[
\geq 1 - n^2 \max_{x,y \in [n]} \mathbb{P}\left(Z_{x,y}^c \cap W_{x,y}\right) - o(1). 
\]
(3.67)
Moreover, Eq. (3.65) implies
\[
n^2 \max_{x,y \in [n]} \mathbb{P}\left(Z_{x,y}^c \cap W_{x,y}\right) \leq n^2 \max_{x,y \in [n]} \mathbb{P}\left(Z_{x,y}^c \mid \sigma\right) = o(1).
\]
(3.68)
In what follows, we will use the following corollary of the upper bound in Theorem 1.1. For \( t \in \mathbb{N} \) and \( y \in [n] \), define
\[
\mu_t(y) := \frac{1}{n} \sum_{x \in [n]} P^t(x,y). 
\]
(3.69)

**Corollary 3.7.** With high probability, for all \( t = \Omega(\log^3(n)) \),
\[
\max_{x \in [n]} \| P^t(x,\cdot) - \pi \|_{TV} \leq e^{-\log^{3/2}(n)}. 
\]
(3.70)
In particular, \( \| \mu_t - \pi \|_{TV} \leq e^{-\log^{3/2}(n)} \).

**Proof.** Let \( d(s) = \max_{x \in [n]} \| P^s(x,\cdot) - \pi \|_{TV} \). It is standard that \( d(ks) \leq 2^k d(s)^k \) for all \( k \in \mathbb{N} \); see [37, Section 4.4]. The upper bound in Theorem 1.1 implies that whp \( d(2T_{\text{ent}}) \leq 1/2e \). Therefore if \( t = \Omega(\log^3(n)) \) one may take \( k = \Omega(\log^2(n)) \) and \( s = 2T_{\text{ent}} \) to conclude. \( \square \)

### 3.3.2. Proof of the lower bound in Theorem 1.1.

We will use the fact that Proposition 3.2 implies that, uniformly on the starting point \( x \), the distribution of the location of the random walk at time \( t = (1 - \beta)T_{\text{ent}} \) is concentrated on a set of size \( O(n^{1-\beta^2}) \). More precisely, pick \( \beta \in (0,1) \) and \( t = (1 - \beta)T_{\text{ent}} \). For all \( x, y \in [n] \), call \( \mathcal{P}_{x,y}^\beta \) the set of paths of length \( t \) starting at \( x \) and ending at \( y \) having weight at least \( e^{-(1+\beta)Ht} = n^{-1+\beta^2} \). Clearly, for all \( x \in [n] \)
\[
\sum_{y \in [n]} \sum_{p \in \mathcal{P}_{x,y}^\beta} w(p) \leq 1. 
\]
(3.71)
Therefore, for all \( x \in [n] \)
\[
\sum_{y \in [n]} |\mathcal{P}_{x,y}^\beta| \leq n^{1-\beta^2}. 
\]
(3.72)
In particular, the set \( S_x := \{ y \in [n] \mid \mathcal{P}_{x,y}^\beta \neq \emptyset \} \) satisfies \( |S_x| \leq n^{1-\beta^2} \) and, by Proposition 3.2,
\[
\min_{x \in [n]} P^t(x,S_x) = 1 - o_\varepsilon(1). 
\]
(3.73)
Thus,
\[
\min_{x \in [n]} \| P^t(x,\cdot) - \pi \|_{TV} \geq P^t(x,S_x) - \pi(S_x) = 1 - o_\varepsilon(1) - \max_{x \in [n]} \pi(S_x). 
\]
(3.74)
Proposition 3.8 below with \( \delta = \beta^2/6 \) implies that \( \max_{x \in [n]} \pi(S_x) = o_\varepsilon(1) \), which concludes the proof of the lower bound in Theorem 1.1.

**Proposition 3.8.** For any \( \delta \in (0,\frac{1}{6}) \), we have
\[
\mathbb{P}\left(\forall S \subset [n], |S| \leq n^{1-6\delta}, \pi(S) \leq n^{-\delta/2}\right) = 1 - o(1). 
\]
(3.75)
Proof. It suffices to prove the statement for $S$ of size exactly $L := \lceil n^{1-6\delta} \rceil$. By Corollary 3.7, for $t = \log^3(n)$,

$$\max_{v \in [n]} |\pi(v) - \mu_t(v)| \leq e^{-\log^{3/2}(n)}, \quad (3.76)$$

Hence, it is enough to prove that

$$\max_{S : |S|=L} \mathbb{P}(\mu_t(S) \geq n^{-\delta}) = o(n^{-L}), \quad (3.77)$$

and then apply a union bound over all sets $S \subseteq [n]$ of cardinality $L$.

To prove Eq. (3.77), fix a set $S$ with cardinality $L$ and let $K = \delta^{-1}L$. Consider the annealed random walk construction described in the beginning of the section with $K$ walks of length $t$ starting at uniform and independent random vertices. Call $B_j$, $j \leq K$, the event that the first $j$ walks end at a vertex in $S$. Thanks to Lemma 2.1, for each $j \leq K$ and at each time $s \leq t$ there are $A = o(\sqrt{(Kt + L)n})$ unmatched heads incident to either $S$ or to the vertices visited by the first $j - 1$ walks, or by the $j$-th walk up to time $s$. Therefore, conditionally on the first $j - 1$ walks, the probability that the $j$-th walk ends at $S$ is at most

$$\mathbb{P}_{\text{unif}}^{S,K}(B_j \mid B_{j-1}) \leq \frac{Kt}{n} + \frac{tA}{m-Kt} \leq \sqrt{L/n} \log^5 n = o(n^{-2\delta}), \quad (3.78)$$

where $\mathbb{P}_{\text{unif}}^{S,K}$ is defined by Eq. (3.2) with $\mu$ uniform over $[n]$. Indeed, in order to end in $S$ the walk needs to visit at some $s \leq t$ a vertex which is either in $S$ or has already been visited by one of the previous walks. The probability that such an event occurs at the initialization step is bounded by $Kt/n$, while $tA/(m-Kt)$ bounds the probability that the event occurs at some later step. It follows that

$$\mathbb{E}[(\mu_t(S))^K] = \mathbb{P}_{\text{unif}}^{S,K}(B_K) = \mathbb{P}_{\text{unif}}^{S,K}(B_1) \prod_{j=2}^{K} \mathbb{P}_{\text{unif}}^{S,K}(B_j \mid B_{j-1}) = o(n^{-2L}). \quad (3.79)$$

By Markov’s inequality,

$$\mathbb{P}(\mu_t(S) \geq n^{-\delta}) \leq \frac{\mathbb{E}[(\mu_t(S))^K]}{n^{-L}} = o(n^{-L}). \quad (3.80)$$

This implies Eq. (3.77). \qed

\section{4. Bulk behavior}

In this section we prove Theorem 1.3. As we will see, the distribution $\mathcal{L}_n$ approximating the bulk values of the stationary distribution can be characterized as the almost sure limit of an $L^2$-bounded martingale.

\subsection{4.1. The martingale.}

Fix $y \in [n]$, an arbitrary $h \in \mathbb{N}$, and consider the random tree $T_y^- (h)$ constructed in Section 2.2.2 with marks $\ell (\cdot)$. For $a \in \partial T_y^- (h)$, define

$$w_T(a) := d^-_{\ell(a)} \prod_{i=1}^{h} \frac{1}{d^{+}_{\ell(a_i)}}, \quad (4.1)$$

where $(a_0, \ldots, a_h = a)$ is the unique path joining $a$ with the root $a_0$ for which $\ell(a_0) = y$. If $a = a_0$, then the empty product is interpreted as 1 and we define $w_T(a_0) = d^-_y$ in this case.
Define the random process
\[ M_y(h) = \sum_{a \in \partial T_y(h)} w_T(a), \quad \forall h \geq 0. \quad (4.2) \]

**Lemma 4.1.** Let \( F_h \) be the sigma algebra generated by the random tree \( T_y(h) \). Then \( (M_y(h))_{h \geq 0} \) is a martingale satisfying \( \mathbb{E} [M_y(h)] = d_y^- \) and, uniformly in \( h \in \mathbb{N} \), \( \text{Var}(M_y(h)) = O(d_y^-) \).

**Proof.** For simplicity, write \( M_h = M_y(h) \), and note \( M_0 = w_T(a_0) = d_y^- \). For each \( a \in T_y(h) \), let \( N(a) \) denote the set of its children. Then, for all \( h \geq 0 \),
\[ M_{h+1} - M_h = \sum_{a \in T_y(h)} \frac{w_T(a)}{d_{\ell(a)}^-} \left( \sum_{b \in N(a)} \left( \frac{d_{\ell(b)}^-}{d_{\ell(b)}^+} - 1 \right) \right). \quad (4.3) \]

Let \( J \) denote a random vertex in \( [n] \) distributed as \( \mu_{\text{out}}(x) \) defined in Eq. (1.7). Using
\[ \mathbb{E} \left[ d_J^- \left( d_J^+ + J \right)^2 \right] = \sum_{v \in [n]} \frac{d_v^-}{m} \frac{d_v^+}{d_v^+} = 1, \quad (4.4) \]
and the fact that the marks in the tree are distributed according to \( \mu_{\text{out}} \), we obtain
\[ \mathbb{E}[M_{h+1} \mid F_h] = M_h. \quad (4.5) \]

Hence, \( (M_h)_{h \geq 0} \) is a martingale with expectation \( d_y^- \). It remains to compute its variance. Let
\[ \Sigma_h := \text{Var}(M_{h+1} - M_h \mid F_h). \quad (4.6) \]

By Eq. (4.3), \( M_{h+1} - M_h \) is given by
\[ \sum_{a \in T_y(h)} \frac{w_T(a)}{d_{\ell(a)}^-} Y_a, \quad (4.7) \]
where, for each \( a \), \( Y_a \) is the sum of \( d_{\ell(a)}^- \) iid copies of the random variable \( \frac{d_{\ell(a)}^-}{d_{\ell(a)}^+} - 1 \). Therefore, by conditioning on \( F_h \), we obtain
\[ \Sigma_h = (A - 1) \sum_{a \in T_y(h)} \frac{w_T(a)^2}{d_{\ell(a)}^-}, \quad (4.8) \]
where, by Assumption 1.1,
\[ A := \mathbb{E} \left[ \left( \frac{d_J^-}{d_J^+} \right)^2 \right] = \sum_{v \in [n]} \frac{d_v^-}{m} \left( \frac{d_v^-}{d_v^+} \right)^2 = O(1). \quad (4.9) \]

Since
\[ \sum_{b \in T_y(h+1)} \frac{w_T(b)^2}{d_{\ell(b)}^-} = \sum_{a \in T_y(h)} \frac{w_T(a)^2}{(d_{\ell(a)}^-)^2} \sum_{b \in N(a)} \frac{d_{\ell(b)}^-}{(d_{\ell(b)}^+)^2}, \quad (4.10) \]
we conclude that
\[ \mathbb{E} [\Sigma_{h+1} \mid F_h] = \mathbb{E} \left[ \frac{d_J^-}{(d_J^+)^2} \right] \Sigma_h. \quad (4.11) \]
Using Eq. (4.4) and the fact that the out-degrees are at least 2 we see that \( \mathbb{E}[\Sigma_{h+1} | \mathcal{F}_h] \leq \frac{1}{2} \Sigma_h \). Thus, taking the expectation and applying induction on \( h \), we have
\[
\mathbb{E}[\Sigma_h] \leq 2^{-h} \Sigma_0, \quad \forall h \geq 0,
\]
(4.12)
where \( \Sigma_0 = \text{Var}(M_1) = (A - 1)d_y^- \). By orthogonality of the martingale increments, and using Eq. (4.12), it follows that
\[
\text{Var}(M_h) = \sum_{i=0}^{h-1} \mathbb{E}[\Sigma_i] \leq 2(A - 1)d_y^-.
\]
(4.13)

**Corollary 4.2.** Fix \( n \in \mathbb{N} \) and let \( I \) be a uniform random vertex in \([n]\). Define
\[
\Phi_h := \frac{1}{\langle d \rangle} M_I(h), \quad h \geq 0,
\]
(4.14)
where \( \langle d \rangle = m/n \) is the average degree. Then \( (\Phi_h)_{h \geq 0} \) is a martingale satisfying \( \mathbb{E}[\Phi_h] = 1 \) and its limit \( \Phi_\infty := \lim_{h \to \infty} \Phi_h \) exists almost surely and in \( L^2 \). Moreover, there exists \( C > 0 \) independent of \( n, h \) such that for all \( h \geq 0 \)
\[
\mathbb{E}[(\Phi_h - \Phi_\infty)^2] \leq C 2^{-h}.
\]
(4.15)

**Proof.** The first assertion follows from Lemma 4.1 by averaging over \( y \). We are left to show Eq. (4.15). Arguing as in Eqs. (4.11) and (4.12),
\[
\mathbb{E}[(\Phi_h - \Phi_\infty)^2] = \sum_{j=h}^{\infty} \mathbb{E}[(\Phi_{j+1} - \Phi_j | \mathcal{F}_j)]
\leq \mathbb{E}[(\Phi_1 - \Phi_0 | \mathcal{F}_0)] \sum_{j=h}^{\infty} 2^{-j} \leq 2^{-j+1} \frac{(A - 1)}{\langle d \rangle},
\]
(4.16)
where \( A \), defined as in Eq. (4.9), is uniformly bounded in \( n \) thanks to Assumption 1.1. \( \square \)

### 4.2. Proof of Theorem 1.3.
Fix \( n \in \mathbb{N} \) and consider the martingale \( (\Phi_h)_{h \geq 0} \) in Corollary 4.2. It follows from [12, Lemma 16] that the random variable \( \Phi_\infty \) has law \( \mathbb{L}_n \) as in Eq. (1.12). Hence, we are left to show that as \( n \to \infty \) the convergence in Eq. (1.14) takes place.

Thanks to the characterization of \( \mathcal{W}_1 \) convergence via non-expansive functions (see [12, Lemma 19]), it is enough to show that, for all \( g : \mathbb{R} \to \mathbb{R} \) such that \( |g(x) - g(y)| \leq |x - y| \) and for all \( x, y \in \mathbb{R} \),
\[
\frac{1}{n} \sum_{v \in [n]} g(n\pi(v)) - \mathbb{E}[g(\Phi_\infty)] \xrightarrow{p} 0.
\]
(4.17)

Since \( |g(x) - g(y)| \leq |x - y| \), for all \( x, y \in \mathbb{R} \), for all \( h \in \mathbb{N} \),
\[
\left| \frac{1}{n} \sum_{v \in [n]} g(n\pi(v)) - \frac{1}{n} \sum_{v \in [n]} g(n\mu_{in} P^h(v)) \right| \leq 2 \| \pi - \mu_{in} P^h \|_{\text{TV}}.
\]
(4.18)
From now we fix $h = h_\varepsilon$ as in Eq. (2.9). For definiteness, we take $\varepsilon = \eta/10$, where $\eta \in (0, 1)$ is such that Assumption 1.1 holds. Hence, by Eq. (3.56),
\[
\frac{1}{n} \sum_{v \in [n]} g(n\pi(v)) = \frac{1}{n} \sum_{v \in [n]} g(n\mu_{in}P^h(v)) + o_P(1).
\] (4.19)

We now show that the first term on the right-hand-side (RHS) of Eq. (4.19) concentrates, that is
\[
\frac{1}{n} \sum_{v \in [n]} g(n\mu_{in}P^h(v)) = \frac{1}{n} \sum_{v \in [n]} \mathbb{E}[g(n\mu_{in}P^h(v))] + o_P(1).
\] (4.20)

For every realization of the matching $\omega$ inducing the digraph, define
\[
Z(\omega) := \frac{1}{n} \sum_{v \in [n]} g(n\mu_{in}P^h(v)).
\] (4.21)

Consider a realization $\omega'$ obtained from $\omega$ by switching two edges: there exists $e, e' \in E^+$ and $f, f' \in E^-$ such that $(\omega(e), \omega(e')) = (f, f')$, and $(\omega'(e), \omega'(e')) = (f', f)$, while $\omega = \omega'$ for all other tails in $E^+$. Then, as in Eq. (4.18)
\[
|Z(\omega) - Z(\omega')| \leq 2\|\mu_{in}P^h_\omega - \mu_{in}P^h_{\omega'}\|_{TV} =: b,
\] (4.22)

where $P_\omega$ denotes the transition matrix of the random walk in the digraph induced by $\omega$. Let $B_{v,\omega}$ denote the out-neighborhood of $v$ up to height $h$ in the digraph induced by $\omega$. Notice that the probability that a random walk starting with distribution $\mu_{in}$ reaches a vertex $v$ after $h$ steps coincides under $\omega$ and $\omega'$ for all vertices $v \notin Q(\omega, \omega')$, where
\[
Q(\omega, \omega') := B_{v_f,\omega}^+ \cup B_{v_f,\omega'}^+ \cup B_{v_f,\omega'}^- \cup B_{v_f,\omega}^-.
\] (4.23)

Therefore, we can bound
\[
b = \sum_{v \in [n]} |\mu_{in}P^h_\omega(v) - \mu_{in}P^h_{\omega'}(v)| 1_{v \in Q(\omega, \omega')} \leq 4(\Delta^+)^hW(\omega, \omega'),
\] (4.24)

where we use the simple uniform bound $\max_{v \in [n], \omega} |B_{v,\omega}^+| \leq (\Delta^+)^h$, and we define
\[
W(\omega, \omega') := \max_{\omega \in (\omega, \omega')} \max_{v \in [n]} \mu_{in}P^h_\omega(v).
\] (4.25)

Consider the event
\[
\mathcal{A} = \left\{ \omega : \max_{v \in [n]} \mu_{in}P^h_\omega(v) \leq n^{-\frac{1}{2}-\varepsilon} \right\}.
\] (4.26)

If $\omega, \omega' \in \mathcal{A}$ and since $(\Delta^+)^h = n^{\frac{\varepsilon}{10}}$, then $b \leq n^{-\frac{1}{2}-\varepsilon}$. By a generalization of Azuma’s inequality (see, e.g., Theorem 3.7 in [40]), we have that, for all $\delta > 0$,
\[
\mathbb{P}\left(|Z - \mathbb{E}[Z]| \geq \delta\right) \leq 2\left(\mathbb{P}(\mathcal{A}^c) + \exp\left(-\frac{\delta^2}{mb^2}\right)\right).
\] (4.27)

Since $mb^2 \to 0$ as $n \to \infty$, to conclude the proof of Eq. (4.20) it suffices to show that $\mathbb{P}(\mathcal{A}^c) = o(1)$.

Fix $v \in [n], K = 2/\varepsilon$ and notice that
\[
\mathbb{P}\left(\mu_{in}P^h(v) > n^{-\frac{1}{2}-\varepsilon}\right) \leq \mathbb{E}\left[\left(\mu_{in}P^h(v)\right)^K\right]\frac{n^{-K-K\varepsilon}}{n^{-K-K\varepsilon}}.
\] (4.28)
Using the annealed process as in Eqs. (3.78) and (3.79), replacing the uniform measure by $\mu_{in}$ and $t$ by $h$, we infer that, for $n$ large enough

$$
\mathbb{E} \left[ (\mu_{in} P^h(v))^K \right] \leq \left( \frac{K h \Delta^-}{n} \right)^K \leq n^{\frac{K}{2} - K \varepsilon},
$$

(4.29)

where in the last inequality we used Lemma 2.1 and the fact that $\varepsilon < \eta/6$. Therefore,

$$
\mathbb{P} \left( \mu_{in} P^h(v) > n^{\frac{1}{2} - \frac{\varepsilon}{5}} \right) \leq n^{-4 \varepsilon/5} = o \left( n^{-1} \right).
$$

(4.30)

By a union bound over $v \in \{n\}$ we get $\mathbb{P}(A^c) = o(1)$. This ends the proof of Eq. (4.20).

Since $|g(x) - g(y)| \leq |x - y|$, using Cauchy-Schwarz inequality and Eq. (4.15) we have

$$
|\mathbb{E}[Z] - \mathbb{E}[g(\Phi^\infty)]| \leq n^{-1/2} \varepsilon = o(1/4),
$$

(4.31)

Hence, by the triangular inequality, it suffices to show that $|\mathbb{E}[Z] - \mathbb{E}[g(\Phi_h)]| \to 0,$ $n \to \infty.$

Let $\tilde{P}$ denote the joint law of the uniform random choice of $I \in \{n\}$ and the coupled construction of the in-neighborhood of $I$ and the random tree in Section 2.2.2, with root having label $I$. Notice that if the coupling succeeds, then one has $\Phi_k = n \mu_{in} P^h(I)$ for all $k \leq h$. Call $C$ the event that the coupling succeeds. By Corollary 2.4 we know that $\tilde{P}(C^c) \leq n^{-1/4}$. Following the same argument as in Eq. (4.18) and by Cauchy-Schwarz inequality,

$$
|\mathbb{E}[Z] - \mathbb{E}[g(\Phi_h)]| \leq \tilde{E} \left[ |n \mu_{in} P^h(I) - \Phi_h| \cdot 1_{C^c} \right] \leq \sqrt{\tilde{P}(C^c) \tilde{E} \left[ (n \mu_{in} P^h(I) - \Phi_h)^2 \right]},
$$

(4.33)

where $\tilde{E}$ denote taking expectation under the law $\tilde{P}$. Therefore, it is enough to show that

$$
\tilde{E} \left[ (n \mu_{in} P^h(I) - \Phi_h)^2 \right] = o(n^{1/4}).
$$

(4.34)

To prove Eq. (4.34) we write

$$
\tilde{E} \left[ (n \mu_{in} P^h(I) - \Phi_h)^2 \right] \leq 2 \mathbb{E} \left[ (n \mu_{in} P^h(I))^2 \right] + 2 \mathbb{E} \left[ \Phi_h^2 \right].
$$

(4.35)

By Corollary 4.2, $\mathbb{E}[\Phi_h^2] = O(1)$. Concerning the first term above, notice that

$$
\mathbb{E} \left[ (n \mu_{in} P^h(I))^2 \right] = n \sum_{x \in \{n\}} \mathbb{E} \left[ (\mu_{in} P^h(x))^2 \right]
\leq \frac{1}{n} \sum_{x \in \{n\}} \mathbb{E} \left[ \left( \sum_{y \in \{n\}} d_y^{-1} 1_{x \in B^+_y} \right)^2 \right]
\leq \frac{1}{n} \sum_{x,y \in \{n\}} d_y^- d_z^- \mathbb{P}(x \in B^+_y, x \in B^+_z).
$$

(4.36)

Arguing as in Eq. (2.21), we obtain

$$
\mathbb{P}(x \in B^+_y, x \in B^+_z) \leq (\Delta^+)^h \frac{d_z^-}{n} 1(y = z) + (\Delta^+)^{2h} \frac{d_z^- d_x^-}{n^2} + (\Delta^+)^{4h} \frac{(d_x^-)^2}{n^2}.
$$

(4.37)
Recall that Assumption 1.1 implies that $\sum_{x \in [n]} (d^-_x)^2 = O(n)$. Thus, we obtain
\[
\mathbb{E} \left[ (n \mu_{in} P^h(I)) \right] = O \left( (\Delta^+)^{4h} \right).
\] (4.38)

By our choice of $\varepsilon$ and $h = h_\varepsilon$, we have $(\Delta^+)^{4h} = o(n^{1/4})$, which ends the proof of Eq. (4.34).

5. LOWER BOUNDS

5.1. Access probabilities to the maximum in-degree vertex.

Proposition 5.1. Assume that $\Delta^- = \Delta^-_n \to \infty$ as $n \to \infty$. For every sufficiently small $\varepsilon > 0$ and for any $y \in [n]$ with $d^-_y = \Delta^-$, we have
\[
\mathbb{P} \left( \min_{x \in V_\varepsilon} P^t(x, y) \geq (1 - \varepsilon) \frac{\Delta^-}{m} \right) = 1 - o(1),
\] (5.1)

where $t = (1 - \gamma) T_{ent} + h_\varepsilon + 1$, $\gamma = \frac{\varepsilon}{80 \log(\Delta^+)}$, and $T_{ent}$, $h_\varepsilon$ and $V_\varepsilon$ are defined as in Eqs. (1.8), (2.9) and (2.40).

Throughout Section 5.1 we write $h = h_\varepsilon$, and set $s$ and $t$ as in Eq. (3.46). Fix $x \in [n]$. Recall the out-neighborhood exploration defined in Section 3.2 which exposes $G_x(s)$ and $T_x(s)$ in at most $\kappa_x = \kappa_x(s) \leq 2n^{1-\gamma^2}$ steps (see Eq. (3.25)). We let $\sigma_x^+$ be the partial pairing obtained after the generation of $G_x(s)$.

Generate the in-neighborhood of $y$ using the sequential generation in Section 2.2.1 process according to the BFS rule. Conditional on $\sigma_x^+$, the in-neighborhood generation constructs another sequence $(H^t)_{t \geq 0}$ that exposes edge by edge the subgraph induced by $B^-_y(h)$. Let $\kappa_y$ be the number of edges that have been paired during the generation of the in-neighborhood. Let $\sigma = \sigma(x, y)$ be the partial pairing revealed after the two exploration processes. Let $\omega$ be a complete pairing of half-edges chosen uniformly at random among all extensions of $\sigma$.

Given $\sigma$, call $E_\sigma$ the set of unmatched tails at height $s$ in $T_x(s)$ and $F_\sigma$ the set of unmatched heads incident to $\partial B^-_y(h)$ that admit a unique path of length $h$ ending at $y$. Recall that, for $f \in F_\sigma$, $\hat{\omega}(f)$ is defined as before Eq. (3.59).

We now use the notion of nice path given in Definition 3.4. Let $\tilde{P}^t(x, y) \leq P^t(x, y)$ be the probability of following a nice path of length $t$ from $x$ to $y$. Conditional on $\sigma$, Eq. (3.60) holds:
\[
\tilde{P}^t(x, y) = \sum_{e \in E_\sigma} \sum_{f \in F_\sigma} \hat{w}(e) \hat{\omega}(f) 1_{\hat{\omega}(e) \leq n^{-1+2\gamma}} 1_{\omega(e) = f}.
\] (5.2)

Observe that
\[
\mathbb{E} \left[ \tilde{P}^t(x, y) \mid \sigma \right] \geq \frac{1}{m} A_{x,y}(\sigma) B_{x,y}(\sigma),
\] (5.3)

where
\[
A_{x,y}(\sigma) = \sum_{e \in E_\sigma} \hat{w}(e) 1_{\hat{\omega}(e) \leq n^{-1+2\gamma}} \quad \text{and} \quad B_{x,y}(\sigma) = \sum_{f \in F_\sigma} \hat{w}(f).
\] (5.4)

Choose $\delta > 0$ sufficiently small such that $(1 - \delta)^5 > 1 - \varepsilon$. Since $y$ is fixed, we define
\[
\mathcal{Y}_x = \left\{ \sigma : A_{x,y}(\sigma) \geq (1 - \delta)^2, B_{x,y}(\sigma) \geq (1 - \delta)^2 \Delta^- \right\}.
\] (5.5)
If $\sigma \in \mathcal{Y}_x$, we have
\[ \mathbb{E}[\tilde{P}^t(x,y) | \sigma] \geq \frac{(1-\delta)^4 \Delta^-}{m}. \] (5.6)
Define also the event
\[ \mathcal{Y} = \cap_{x \in [n]} \left( \{ x \notin V_x \} \cup \mathcal{Y}_x \right). \] (5.7)
We state the following fact that we will prove later.

**Lemma 5.2.** We have $\mathbb{P}(\mathcal{Y}) = 1 - o(1)$.

**Proof of Proposition 5.1.** Define the event
\[ \mathcal{Z}_x = \left\{ P^t(x,y) \geq (1-\varepsilon) \frac{\Delta^-}{m} \right\}. \] (5.8)
Let $\sigma \in \mathcal{Y}_x$. Swapping two pairings in $\omega$ which are not fixed by $\sigma$ can change $\tilde{P}^t(x,y)$ by at most $\|c\|_\infty \leq n^{-1-\delta}$ by Eq. (3.48). Chatterjee’s inequality in Eq. (3.63) implies that
\[ \mathbb{P}(\mathcal{Z}_x^c \mid \sigma) \leq \mathbb{P}\left( \tilde{P}^t(x,y) \leq (1-\delta)\mathbb{E}[\tilde{P}^t(x,y) \mid \sigma] \right) \leq 2 \exp\left( -\frac{(1-\delta)^4(2+\delta)^{-1}\delta^2\Delta^-}{2\|c\|_\infty m} \right) = o(n^{-1}). \] (5.9)
We then have
\[ \mathbb{P}\left( \cup_{x \in [n]} \left( \{ x \in V_x \} \cap \mathcal{Z}_x^c \right) \right) \leq \mathbb{P}\left( \cup_{x \in [n]} (\mathcal{Z}_x^c \cap \mathcal{Y}_x) \right) + \mathbb{P}(\mathcal{Y}^c) = o(1), \] (5.10)
where the first term is bounded by Eq. (5.9) and the second by Lemma 5.2.

**Proof of Lemma 5.2.** Define the events
\[ \mathcal{Y}^{(1)} = \cap_{x \in [n]} \left( \{ x \notin V_x \} \cup \{ A_{x,y}(\sigma) \geq (1-\delta)^2 \} \right), \]
\[ \mathcal{Y}^{(2)} = \cap_{x \in [n]} \{ B_{x,y}(\sigma) \geq (1-\delta)^2 \Delta^- \}, \] (5.11)
and note that $\mathcal{Y} = \mathcal{Y}^{(1)} \cap \mathcal{Y}^{(2)}$.

We first focus on $\mathcal{Y}^{(1)}$. Let $\mathcal{E}_x(s)$ be the set of tails at height $s$ in $\mathcal{T}_x(s)$. Write
\[ A_x(\sigma) = \sum_{e \in \mathcal{E}_x(s)} \hat{\omega}(e) 1_{\hat{\omega}(e) \leq n^{-1+2\gamma}} \quad \text{and} \quad A_{x,y}(\sigma) = A_x(\sigma) - A_{x,y}(\sigma). \] (5.12)
Let $\tilde{q}_0(x)$ and $\tilde{q}(x)$ be the probabilities that the walk starting at $x$ violates the condition of a nice path within the first $s$ and $t$ steps respectively. So $\tilde{q}_0(x) \leq \tilde{q}(x)$. Proposition 3.5 implies that whp,
\[ \min_{x \in V_x} A_x(\sigma) \geq 1 - \max_{x \in V_x} \tilde{q}_0(x) \geq 1 - \max_{x \in V_x} \tilde{q}(x) \geq 1 - \delta, \] (5.13)
for sufficiently large $n$.

We now turn our attention to the in-neighborhood exploration to bound $A_{x,y}(\sigma)$. Let $\mathcal{T}_y^-$ be a Galton-Watson tree as defined in Section 2.2.2. Consider the event
\[ \mathcal{C}_1 = \{ B_y^-(h) = \mathcal{T}_y^-(h), \kappa_y \leq n^{1/2} \}. \] (5.14)
That is, the coupling succeeds up to depth $h$, and not too many edges are revealed by it. Since under $\mathcal{C}_1$ we have $\kappa_y \leq |B_y^-(h)|$, Lemma 2.3 implies that $\mathbb{P}(\mathcal{C}_1) = 1 - o(1)$, provided that $\varepsilon < \eta$. 

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Recall that $\hat{w}(e) \leq n^{-1+2\gamma}$ for all $e \in \mathcal{E}_x(s)$. Moreover, $\mathbb{E}[1_{C_1} 1_{e \notin \mathcal{E}_x}] = \mathbb{P}(C_1, e \notin \mathcal{E}_x)$ is uniformly bounded from above by $\frac{\varepsilon_y}{m-\kappa_x-\kappa_y} \leq n^{-1/2}$. Therefore, the random variable $1_{C_1} \tilde{A}_{x,y}(\sigma)$ is stochastically dominated by $X = \sum_{i=1}^{m} X_i$, where the $X_i$ are independent random variables satisfying $X_i \in [0, n^{-1+2\gamma}]$ and $\mathbb{E}[X] \leq mn^{-3/2+2\gamma} \leq n^{-1/3}$ for $\varepsilon$ small enough. Using Hoeffding’s inequality for the sum of independent bounded random variables (see, e.g., Theorem 2.5 in [40]), we obtain

$$
\mathbb{P}(1_{C_1} \tilde{A}_{x,y}(\sigma) \geq \delta/2) \leq \mathbb{P}(X \geq \mathbb{E}[X] + \delta/3) \leq \exp\left(-\frac{2\delta^2}{9mn^{-2+4\gamma}}\right) = o(n^{-1}). \tag{5.15}
$$

By $\mathbb{P}(C_1) = 1 - o(1)$ and a union bound with Eq. (5.15), whp

$$
\max_{x \in [n]} \tilde{A}_{x,y}(\sigma) = \max_{x \in [n]} 1_{C_1} \tilde{A}_{x,y}(\sigma) \leq \delta/2. \tag{5.16}
$$

Combining Eqs. (5.13) and (5.16), whp

$$
\min_{x \in V_x} A_{x,y}(\sigma) \geq \min_{x \in V_x} A_x(\sigma) - \max_{x \in [n]} \tilde{A}_{x,y}(\sigma) \geq (1 - \delta) - \frac{\delta}{2} \geq (1 - \delta)^2. \tag{5.17}
$$

for $\delta \leq 1/2$, which implies that $\mathbb{P}(\{Y^{(1)}\}^c) = o(1)$.

It remains to bound the probability of $Y^{(2)}$. It will be convenient to interchange the order of the out- and in-neighborhood exploration processes. We first generate $B_{y}^{-}(h)$ which reveals the set of heads at distance $h$ from $y$, denoted by $\mathcal{F}_{y}(h)$, and then generate $G_x(s)$. Let $\sigma_y^-$ be the partial paring revealed by generating $B_{y}^{-}(h)$, and call again $\sigma = \sigma(x, y)$ the partial paring obtained after the generation of both $B_{y}^{-}(h)$ and $G_x(s)$.

For all $j \geq 0$, define

$$
\Gamma_y(j) = \sum_{z \in \partial B_{y}^{-}(j)} d_y^- P^y(z, y). \tag{5.18}
$$

Under $C_1$, there is a unique path of length $h$ from each head $f$ incident to $\partial B_{y}^{-}(h)$ to $y$, so we can write

$$
\Gamma_y(h) = \sum_{f \in \mathcal{F}_{y}(h)} \hat{w}(f) \quad \text{and} \quad \overline{B}_{x,y}(\sigma) = \Gamma_y(h) - B_{x,y}(\sigma). \tag{5.19}
$$

Recall the definition of $M_y(h)$ given in Eq. (4.2). By Chebyshev’s inequality and using Lemma 4.1, we have

$$
\mathbb{P}(M_y(h) \leq (1 - \delta)d_y^-) \leq \frac{\text{Var}(M_y(h))}{\delta^2(d_y^-)^2} \leq \frac{1}{\delta^2 \Delta^-} = o(1), \tag{5.20}
$$

as $\Delta^- \to \infty$. Moreover, under $C_1$, we have $\{\Gamma_y(h) = M_y(h)\}$.

Consider the event

$$
C_2 = \{\Gamma_y(h) \geq (1 - \delta)\Delta^-\}. \tag{5.21}
$$

Using Lemma 2.3 and Eq. (5.20), we have that

$$
\mathbb{P}(C_2^c) \leq \mathbb{P}(C_2^c, C_1^c) + \mathbb{P}(C_1^c) \leq \mathbb{P}(M_y(h) < (1 - \delta)\Delta^-) + o(1) = o(1). \tag{5.22}
$$

Fix $x \in [n]$. Conditional on $\sigma_y^-$, we now generate $G_x(s)$ obtaining the partial paring $\sigma = \sigma(x, y)$. Now we argue as in Eq. (5.15) to bound $\overline{B}_{x,y}(\sigma)$. On the one hand, for any
\( f \in \mathcal{F}_y(h) \) we have \( \hat{w}(f) \leq 2^{-h} \leq n^{-2\gamma} \) and

\[
\sum_{f \in \mathcal{F}_y(h)} (\hat{w}(f))^2 \leq \left( \max_{f \in \mathcal{F}_y(h)} \hat{w}(f) \right)^2 \sum_{f \in \mathcal{F}_y(h)} \hat{w}(f) \leq n^{-2\gamma} \Gamma_y(h). \tag{5.23}
\]

On the other hand, \( \mathbb{E}[1_{c_1}1_{f \in \mathcal{F}_y}] \leq \frac{\kappa_y}{m-\kappa_y} \leq 3n^{-2\gamma} \). Fix a realization \( \sigma_y^- \), let \( \ell = |\mathcal{F}_y(h)| \) and let \( f_1, f_2, \ldots, f_\ell \) be an arbitrary ordering of \( \mathcal{F}_y(h) \). Let \( (Y_i)_{i \in [\ell]} \) be independent random variables such that \( Y_i = \hat{w}(f_i) \) with probability \( 3n^{-2\gamma} \) and \( Y_i = 0 \) with probability \( 1-3n^{-2\gamma} \). Then, \( Y = \sum_{i \in [\ell]} Y_i \) stochastically dominates \( 1_{c_1} \mathcal{B}_{x,y}(\sigma) \) conditionally on \( \sigma_y^- \). Moreover, the expected value of \( Y \) satisfies \( \mathbb{E}[Y | \sigma_y^-] \leq 3n^{-2\gamma} \Gamma_y(h) = o(\Gamma_y(h)) \) and, by Eq. (5.23), the sum of the squared ranges of the random variables \( (Y_i)_{i \in [\ell]} \) is at most \( n^{-2\gamma} \Gamma_y(h) \). Applying Hoeffding’s inequality,

\[
\mathbb{P}\left( 1_{c_1}1_{c_2} \mathcal{B}_{x,y}(\sigma) \geq \delta \Gamma_y(h) \right) \leq \mathbb{E} \left[ 1_{c_1} \mathbb{P}\left( Y \geq \mathbb{E}[Y | \sigma_y^-] + \frac{\delta}{2} \Gamma_y(h) | \sigma_y^- \right) \right] \\
\leq \mathbb{E} \left[ 1_{c_2} \exp\left( -\frac{\delta^2 \Gamma_y(h)}{2n^{-2\gamma}} \right) \right] = o(n^{-1}). \tag{5.24}
\]

Since \( \mathbb{P}(C_1) = 1 - o(1) \), by Eq. (5.22) and a union bound with Eq. (5.24), whp

\[
\max_{x \in [n]} B_{x,y}(\sigma) = \max_{x \in [n]} 1_{c_1}1_{c_2} \mathcal{B}_{x,y}(\sigma) \leq \delta \Gamma_y(h). \tag{5.25}
\]

Combining Eqs. (5.22) and (5.25), whp we have,

\[
\min_{x \in [n]} B_{x,y}(\sigma) \geq \Gamma_y(h) - \max_{x \in [n]} \mathcal{B}_{x,y}(\sigma) \geq (1-\delta) \Gamma_y(h) \geq (1-\delta)^2 \Delta^- \tag{5.26}
\]

and we conclude that \( \mathbb{P}\left( (\Delta^{(2)})^c \right) = o(1) \).

### 5.2. Access probabilities to large in-degree vertices.

For any \( a \in (0,1) \), define the set

\[
V(a) = \{ y \in [n] : d_y^+ > n^a \}. \tag{5.27}
\]

Note that by Lemma 2.1, for any \( a > \frac{1}{2} - \frac{\gamma}{6} \), \( V(a) = \emptyset \).

**Proposition 5.3.** For all \( \varepsilon > 0 \) sufficiently small, \( \gamma = \gamma(\varepsilon) := \frac{\varepsilon}{80 \log(\Delta^+)} \) and \( a \in (2\gamma, 1) \),

\[
\mathbb{P}\left( \min_{x \in V_c} P^t(x, y) \geq (1-\varepsilon) \frac{d_y^+}{m} \forall y \in V(a) \right) = 1 - o(1), \tag{5.28}
\]

where \( t = (1-\gamma) T_{ent} + 1 \), and \( T_{ent} \) and \( V_c \) are defined as in Eqs. (1.8) and (2.40).

Throughout Section 5.2 we write \( h = h_{\varepsilon} \) and set

\[
t := s + 1, \quad s := (1-\gamma) T_{ent}. \tag{5.29}
\]

Note that \( s = T_{ent} - \frac{h_{\varepsilon}}{4} \) as in our previous proofs but this time the overall time \( t \) is smaller than the mixing time \( T_{ent} \). Fix \( x \in [n] \) and \( y \in V(a) \). Generate \( G_{\varepsilon}(s) \) and \( T_{\varepsilon}(s) \) as described in Section 3.2. In contrast to the previous section, here we do not generate the in-neighborhood of \( y \). Let \( \sigma = \sigma_x^+ \). Call \( E_{\varepsilon} \) the set of all tails at height \( s \) in \( T_{\varepsilon}(x) \), by definition they are all unmatched. Call \( F_{\sigma} \) the set of unmatched heads in \( E_{\varepsilon}^+ \). Let \( \omega \) be a complete pairing of half-edges chosen uniformly at random among all extensions of \( \sigma \).
We need to slightly adjust the notion of nice path in Definition 3.4, by letting \( t = s + 1 \). In particular, condition (3) is now void. Recall that \( \tilde{P}^t(x, y) \leq P^t(x, y) \) is the quenched probability of a random walk following a nice path of length \( t \) starting at \( x \) and ending at \( y \). Conditional on \( \sigma \) we have
\[
\tilde{P}^t(x, y) = \sum_{e \in E} \sum_{f \in F_x} \mathcal{w}(e) 1_{\mathcal{w}(e) \leq n^{-1+2\gamma}} 1_{\omega(e) = f} = \sum_{e \in E} \sum_{f \in F_y} \mathcal{w}(e) 1_{\mathcal{w}(e) \leq n^{-1+2\gamma}} 1_{\omega(e) = f}.
\]
(5.30)
Define,
\[
A_x(\sigma) = \sum_{e \in E} \mathcal{w}(e) 1_{\mathcal{w}(e) \leq n^{-1+\gamma}}, \quad \text{and} \quad B_{x,y}(\sigma) = |F_\sigma|.
\]
(5.31)
Choose \( \delta > 0 \) sufficiently small such that \( (1 - \delta)^3 > 1 - \varepsilon \). Let
\[
\mathcal{V}_{x,y} := \{ \sigma : A_x(\sigma) \geq 1 - \delta, B_{x,y}(\sigma) \geq (1 - \delta)d_y^{-}\}
\]
(5.32)
Then, for all \( \sigma \in \mathcal{V}_{x,y} \), we have
\[
\mathbb{E} \left[ \tilde{P}^t(x, y) \mid \sigma \right] \geq \frac{1}{m} A_x(\sigma) B_{x,y}(\sigma) \geq \frac{(1 - \delta)^2 d_y^{-}}{m}.
\]
(5.33)
Write
\[
\mathcal{Y} := \bigcap_{x \in V, y \in V(a)} \mathcal{V}_{x,y}.
\]
(5.34)

**Lemma 5.4.** We have \( \mathbb{P}(\mathcal{Y}) = 1 - o(1) \).

**Proof of Proposition 5.3.** Define
\[
\mathcal{Z}_{x,y} := \left\{ P^t(x, y) \geq (1 - \varepsilon) \frac{d_y^{-}}{m} \right\}.
\]
(5.35)
Let \( \sigma \in \mathcal{V}_{x,y} \). By definition, a nice path \( p \) has \( \mathcal{w}(p) \leq n^{-1+2\gamma} \). Therefore, swapping two pairings in \( \omega \) which are not fixed by \( \sigma \) can change \( \tilde{P}^t(x, y) \) by at most \( \|c\|_{\infty} \leq n^{-1+2\gamma} \).

Applying Chatterjee's inequality given in Eq. (3.63) and using \( y \in V(a) \) with \( a > 2\gamma \), we obtain
\[
\mathbb{P} \left( \mathcal{Z}_{x,y}^c \mid \sigma \right) \leq \mathbb{P} \left( \tilde{P}^t(x, y) \leq (1 - \delta) \mathbb{E} \left[ \tilde{P}^t(x, y) \mid \sigma \right] \mid \sigma \right)
\leq 2 \exp \left( -\frac{(1 - \delta)^2 (2 + \delta)^{-1} \delta^2 d_y^{-}}{2 \|c\|_{\infty} m} \right)
\leq \exp \left( -\Theta(n^{a-2\gamma}) \right) = o \left( n^{-2} \right).
\]
(5.36)
We then have
\[
\mathbb{P} \left( \bigcup_{x \in V, y \in V(a)} \mathcal{Z}_{x,y}^c \right) \leq \mathbb{P} \left( \bigcup_{x \in V, y \in V(a)} (\mathcal{Z}_{x,y}^c \cap \mathcal{Y}_{x,y}) \right) + \mathbb{P}(\mathcal{Y}^c) = o(1),
\]
(5.37)
where the first term is bounded using a union bound and Eq. (5.36), and the second term by Lemma 5.4.

**Proof of Lemma 5.4.** Define the events
\[
\mathcal{Y}^{(1)} = \cap_{x \in \mathcal{X}} \left\{ \{ x \notin V \} \cup \{ A_x(\sigma) \geq 1 - \delta \} \right\},
\]
\[
\mathcal{Y}^{(2)} = \cap_{x \in \mathcal{X}} \cap_{y \in V(a)} \left\{ B_{x,y}(\sigma) \geq (1 - \delta)d_y^{-} \right\},
\]
(5.38)
and note that
\[ \mathcal{Y} = \mathcal{Y}^{(1)} \cap \mathcal{Y}^{(2)}. \]  \hspace{1cm} (5.39)

Recall that \( \tilde{q}(x) \) is the probability of not following a nice path, as defined as in Eq. (3.45). Since the definition of nice path used in this section is less restrictive, we have \( A_x(\sigma) \geq 1 - \tilde{q}(x) \). Proposition 3.5 directly implies that \( \mathbb{P} \left( \mathcal{Y}^{(1)} \right) = 1 - o(1) \).

Let us now show that \( \mathbb{P} \left( \mathcal{Y}^{(2)} \right) = 1 - o(1) \). Fix \( x \in [n] \) and \( y \in V(a) \). Recall that \( \sigma \) has paired \( \kappa_x \leq n^{1-\delta'} \) tails (see Eq. (3.25)). For each such tail, the probability of pairing it to a head in \( E_y^- \) is uniformly bounded from above by \( q := \frac{d_y^-}{m-\kappa_x} \). So the number of heads in \( E_y^- \setminus F_\sigma \) is stochastically dominated by a Binomial random variable with parameters \( \kappa_x \) and \( q \). Since \( \kappa_x q = o(d_y^-) \) and \( y \in V(a) \), Chernoff’s inequality (e.g., Corollary 2.4 in [35]) implies, for all \( \delta > 0 \),
\[ \mathbb{P} \left( B_{x,y}(\sigma) < (1 - \delta)d_y^- \right) = \mathbb{P} \left( |E_y^- \setminus F_\sigma| > \delta d_y^- \right) \leq \exp \left( -\delta d_y^- \right) = o(n^{-2}). \]  \hspace{1cm} (5.40)

The desired bound follows from a union bound over \( x \in [n] \) and \( y \in V(a) \). \( \square \)

5.3. Lower bounds on stationary values. The following result implies that lower bound on the access probabilities give lower bounds on the stationary values.

**Lemma 5.5.** Let \( \varepsilon > 0 \), \( K = K_n > 0 \), \( t = t_n \in \mathbb{N} \) and \( Y = Y_n \subset [n] \). Suppose the following holds whp:
\[ P^t(x, y) \geq K, \quad \forall x \in V_\varepsilon, y \in Y. \]  \hspace{1cm} (5.41)

Then, the following holds whp:
\[ \pi(y) \geq (1 - \varepsilon)K, \quad \forall y \in Y. \]  \hspace{1cm} (5.42)

**Proof.** We may assume that \( \varepsilon \) is sufficiently small with respect to the constant \( \eta > 0 \) appearing in Eq. (1.2). By applying Lemma 2.6 and Proposition 3.8 with \( \delta = 1/18 \), we have that whp
\[ \pi_0 := \sum_{v \in V_\varepsilon} \pi(v) = 1 - \sum_{v \notin V_\varepsilon} \pi(v) \geq 1 - \max_{|S| \leq n^{2/3}} \pi(S) = 1 - o(1). \]  \hspace{1cm} (5.43)

Using Eq. (5.43), we can conclude that whp, for every \( y \in Y \)
\[ \pi(y) = \sum_{x \in [n]} \pi(x) P^t(x, y) \geq \sum_{x \in V_\varepsilon} \pi(x) P^t(x, y) \geq \pi_0 \min_{x \in V_\varepsilon} P^t(x, y) \geq (1 - \varepsilon)K. \]  \hspace{1cm} (5.44)

\( \square \)

**Proof of Eq. (1.16) in Theorem 1.4.** Again, we may assume that \( \varepsilon \) is sufficiently small with respect to the constant \( \eta > 0 \) appearing in Eq. (1.2). We apply Proposition 5.1 to \( y \in [n] \) with \( d_y^- = \Delta^- \). Let \( h = h_{\varepsilon, \gamma} = \frac{\varepsilon}{80 \log(\Delta^+)} \), \( s = (1 - \gamma)T_{\text{ent}} \) and \( t = s + h + 1 \). Then, whp and uniformly over \( x \in V_\varepsilon \)
\[ P^t(x, y) \geq (1 - \varepsilon) \frac{\Delta^-}{m}. \]  \hspace{1cm} (5.45)

By Lemma 5.5 with \( Y = \{y\} \), we conclude that whp \( \pi_{\text{max}} \geq \pi(y) \geq (1 - 2\varepsilon) \frac{\Delta^-}{m} \). As \( \varepsilon \) can be made arbitrarily small, Eq. (1.16) holds. \( \square \)

The following is a direct consequence of Proposition 5.3 and Lemma 5.5.
Corollary 5.6. Fix $\varepsilon, a > 0$ and let $V(a)$ as in Eq. (5.27). Then whp, for every $y \in V(a)$ we have $\pi(y) \geq (1 - \varepsilon) \frac{d_w}{m}$.

6. UPPER BOUNDS

This section is devoted to the proof of the upper bound in Theorem 1.4. The proof is based on the analysis of the annealed process introduced in Section 3. More precisely, we will need to control the high moments of the random distribution $\mu_t$ defined in Eq. (3.69) for $t = \log^3(n)$. Thanks to Corollary 3.7, the measure $\mu_t$ is a good approximation of the stationary distribution $\pi$, and this will give the desired result.

In what follows we will consider $\eta \in (0, 1)$ satisfying Assumption 1.1, $\varepsilon \in (0, \eta/6)$, $h = h_\varepsilon$ as in Eq. (2.9) and $G = G^+(h)$ as in Eq. (2.36).

Lemma 6.1. For any constant $C > 0$, taking $t = \log^3(n)$ and $K = C \log(n)$, one has

$$\mathbb{E}\left[1_G(\mu_t(y))^K\right] \leq \left(\frac{10K\Delta^-}{n}\right)^K,$$

for all $y \in [n]$ and all $n$ sufficiently large.

Proof. Consider the non-Markovian process of $K$ annealed walks, each of length $t$, starting at independent uniformly random vertices. Let $\mathbb{P}^{an} = \mathbb{P}_{unif}^{an,K}$ denote their joint law as defined by Eq. (3.2) with $\mu$ uniform over $[n]$. Call $D_t$ the digraph generated by the first $\ell$ walks and, for $s \geq 1$, we define $D_{\ell,s}$ as the union of $D_{\ell-1}$ and the edges generated by $X_{s-1}^{(\ell)}, \ldots, X_0^{(\ell)}$. Note that $D_{\ell-1} = D_{\ell,1}$. We also write $D_{\ell,0} = D_{\ell-1}$. As usual, with a slight abuse of notation, we identify a digraph $G' \subset G$ with the partial matching of heads and tails that defines it. Recall the definition of $\mathcal{P}(x, y, s, G')$ in Section 2.2. Let $\text{dist}_{G'}(x, y)$ be the length of the shortest path starting at $x$ and ending at $y$ in $G'$ and $\mathcal{B}_y^{G'}(h) (\partial \mathcal{B}_y^{G'}(h))$ the set of vertices $x \in [n]$ such that $\text{dist}_{G'}(x, y) \leq h$ ($\text{dist}_{G'}(x, y) = h$).

Notice that, by definition,

$$D_{\ell-1} \subset D_{\ell,s} \subset D_{\ell}, \quad \forall \ell \in \{1, \ldots, K\}, s \in \{1, \ldots, t\},$$

(6.2)

where we used the convention $D_0 = \emptyset$. Moreover, for all $\ell \in \{1, \ldots, K\}$ we have $|D_\ell| \leq \ell t$. We say that $D_\ell$ is compatible with an event $\mathcal{A}$, denoted by $D_\ell \sim \mathcal{A}$, if there exists a realization of the environment $\omega$ that contains $D_\ell$ and such that $\omega \in \mathcal{A}$. In particular, $D_\ell \sim \mathcal{G}$ if all out-neighborhoods of depth $h$ have tree-excess at most 1 in $D_\ell$. Fix $y \in [n]$ and let us consider the following events which implicitly depend on $y$: for every $\ell \in \{1, \ldots, K\}$

$$B_\ell := \{X_{\ell}^{(1)} = \cdots = X_{\ell}^{(\ell)} = y\},$$

$$F_\ell := \{D_\ell \sim \mathcal{G}\},$$

$$H_\ell := \{\forall j \leq h, |\partial \mathcal{B}_y^{D_\ell}(j)| \leq 2K\}.\quad (6.3)$$

Notice that

$$\mathbb{E}[1_G(\mu_t(y))^K] \leq \mathbb{P}^{an}(B_K \cap F_K \cap H_K) + \mathbb{P}(H_K^c). \quad (6.4)$$
Moreover, letting $B_0 = F_0 = H_0$ be sure events and using the monotonicity of the events defined above,

$$\mathbb{P}^\alpha(B_K \cap F_K \cap H_K) = \prod_{\ell=1}^K \mathbb{P}^\alpha(B_\ell \cap F_\ell \cap H_\ell \mid B_{\ell-1} \cap F_{\ell-1} \cap H_{\ell-1}).$$  \hspace{1cm} (6.5)

Therefore, in order to prove Eq. (6.1) it suffices to show that

$$\mathbb{P}^\alpha(B_\ell \cap F_\ell \cap H_\ell \mid B_{\ell-1} \cap F_{\ell-1} \cap H_{\ell-1}) \leq \frac{9K\Delta^-}{n}, \quad \forall \ell \in \{1, \ldots, K\},$$  \hspace{1cm} (6.6)

and, moreover,

$$\mathbb{P}^\alpha(H^c_K) = o\left(\left(\frac{\log(n)\Delta^-}{n}\right)^K\right).$$  \hspace{1cm} (6.7)

Plugging (6.5)–(6.7) into Eq. (6.4) we obtain Eq. (6.1).

We start by proving Eq. (6.7). For $j \in \{1, \ldots, h\}$, set

$$H_{K,j} = \{|\partial B_y^{-D_K}(j)| \leq 2K\},$$  \hspace{1cm} (6.8)

so we may write

$$\mathbb{P}^\alpha(H^c_K) = \sum_{j=1}^h \mathbb{P}^\alpha(H^c_{K,j} \cap (\cap_{i<j} H_{K,i})).$$  \hspace{1cm} (6.9)

Notice that $H^c_{K,1}$ is the event that the $K$ walks have matched more than $2K$ heads of $y$. Hence, for all $n$ large enough,

$$\mathbb{P}^\alpha(H^c_{K,1}) \leq \mathbb{P}\left(\text{Bin}\left(Kt, \frac{\Delta^-}{m-Kt}\right) > 2K\right) \leq \left(\frac{Kt\Delta^-}{n}\right)^{2K} \leq \left(\frac{\Delta^- \log(n)}{n}\right)^{\frac{3}{2}K}. \hspace{1cm} (6.10)$$

For $j \in \{2, \ldots, K\}$, under the event $\cap_{i<j} H_{K,i}$, there are at most $2K\Delta^-$ heads that can be matched to violate $H_{K,j}$, and therefore

$$\mathbb{P}^\alpha(H^c_{K,j} \cap (\cap_{i<j} H_{K,i})) \leq \mathbb{P}\left(\text{Bin}\left(Kt, \frac{2K\Delta^-}{m-Kt}\right) > 2K\right) \leq \left(\frac{2K\Delta^-(m-Kt)}{n}\right)^{2K} \left(\frac{\Delta^- \log(n)}{n}\right)^{\frac{3}{2}K}. \hspace{1cm} (6.11)$$

Plugging Eq. (6.10) and Eq. (6.11) into Eq. (6.9), we obtain Eq. (6.7).

We now turn to the proof of Eq. (6.6). We fix a realization $D_{t-1}$ of the partial matching generated by the first $\ell-1$ walks, and assume that it satisfies $B_{t-1} \cap F_{t-1} \cap H_{t-1}$. Let $V(D_{t,s})$ be the set of vertices previously visited by the other walks or by the $\ell$-th walk itself up to time $s-1$ for $\ell \geq 1$, together with $y$. For the event $B_\ell$ to occur, the $\ell$-th walk must enter at some time $s \in \{0, \ldots, t\}$ and then traverse only edges in $D_{t,s}$ from time $s$ up to time $t$. If $s \in \{1, \ldots, t\}$, the event that the $\ell$-th walk enters in $D_{t,s}$ at time $s$ in a given vertex $z \in V(D_{t,s})$ has probability bounded above by $d_z^-/(m-tt) \leq d_z^-/n$, uniformly in the realization of $D_{t,s}$. On the other hand, the probability that the $\ell$-th walk enters in $D_{t-1}$ at time $s = 0$ in $z \in D_{t-1}$ is $1/n$. 

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Given $D_{t,s} = D$ and $z \in V(D)$, let $q_D(z, z', r)$ denote the probability that a walk started at $z$ at time 0 arrives in $z'$ at time $r$ by traversing only edges in $D$. Note that $q_D(z, z', r) = 0$ if $z' \notin V(D)$, $q_D(z, z, 0) = 1$, and that

$$q_D(z, z', r) \leq 2^{-r}|\mathcal{P}(z, z', r, D)|,$$

(6.12)

since the out-degrees are at least 2. In conclusion, we can bound uniformly in $D_{t-1}$

$$\mathbb{P}^n(B_t \cap F_t \cap H_t \mid D_{t-1}) \leq \sum_{s=0}^{t} \max_{D \in \mathcal{A}_t(s)} \sum_{z \in V(D)} \frac{d^-_z}{n} q_D(z, y, t - s),$$

(6.13)

with the convention that $d^-_z$ must be replaced by $\max\{d^-_z, 1\}$ when $s = 0$, and where we define the set $\mathcal{A}_t(s)$ of all possible realizations $D$ of $D_{t,s}$ such that

$$\mathbb{P}^n(D_{t,s} = D, B_t \cap F_t \cap H_t \mid D_{t-1}) > 0.$$

(6.14)

We split the interval $s \in \{0, \ldots, t\}$ into two parts:

$$I_1 = \{0, \ldots, t - h\}, \quad I_2 = \{t - h + 1, \ldots, t\}.$$

(6.15)

Observe that if $s \in I_1$, for every realization $D$ of $D_{t,s}$:

$$q_D(z, y, t - s) = \sum_{z' \in V(D)} q_D(z, z', t - s - h)q_D(z', y, h) \leq \max_{z' \in V(D)} 2^{-h}|\mathcal{P}(z', y, h, D)|.$$

(6.16)

Thanks to the event $F_t$, we know that $|\mathcal{P}(z', y, h, D)| \leq 2$ for all $z' \in V(D)$, and $D \in \mathcal{A}_t(s)$. Moreover, $|V(D)| \leq (1 + Kt)$, and therefore

$$\sum_{s \in I_1} \max_{D \in \mathcal{A}_t(s)} \sum_{z \in V(D)} \frac{d^-_z}{n} q_D(z, y, t - s) \leq \sum_{s \in I_1} \max_{D \in \mathcal{A}_t(s)} \frac{|V(D)| \Delta^-}{n} 2^{-h+1} \leq |I_1| \frac{(1 + Kt) \Delta^-}{n} 2^{-h+1} = o\left(\frac{\Delta^-}{n}\right).$$

(6.17)

We now turn to bound the sum in Eq. (6.13) for $s \in I_2$. Notice that

$$\sum_{s \in I_2} \max_{D \in \mathcal{A}_t(s)} \sum_{z \in V(D)} \frac{d^-_z}{n} q_D(z, y, t - s) = \sum_{a=0}^{h-1} \max_{D \in \mathcal{A}_t(t-a)} \sum_{j=0}^{a} \sum_{z \in \partial B^-_y \cap D(j)} \frac{d^-_z}{n} q_D(z, y, a).$$

(6.18)

As the event $F_t$ holds, if $D \in \mathcal{A}_t(t-a)$, if $a \leq h$, we may estimate

$$q_D(z, y, a) \leq 2^{-a}|\mathcal{P}(z, y, a, D)| \leq 2^{-a+1}.$$

(6.19)

Therefore, using that $H_t$ holds and Eq. (6.19), for all $D \in \mathcal{A}_t(t-a)$, it follows that

$$\sum_{j=0}^{a} \sum_{z \in \partial B^-_y \cap D(j)} \frac{d^-_z}{n} q_D(z, y, a) \leq \frac{\Delta^-}{n} (2Ka + 1)2^{-a+1}.$$

(6.20)

Using Eq. (6.20) we see that Eq. (6.18) is bounded above by $\frac{8\Delta^-}{n}K$. Recalling Eqs. (6.13) and (6.17), we conclude the validity of Eq. (6.6).
Proof of the upper bound in Theorem 1.4. We show how the desired bound follows by the moment estimates in Lemma 6.1 and Corollary 3.7. Fix $C > 0$ and consider
\[ P \left( \max_{y \in [n]} \mu_t(y) > \frac{20C \log(n) \Delta^-}{n} \right) \leq P \left( \max_{y \in [n]} \mu_t(y) 1_G > \frac{20C \log(n) \Delta^-}{n} \right) + P(G^c) \]
\[ \leq n \max_{y \in [n]} P \left( \mu_t(y) 1_G > \frac{20C \log(n) \Delta^-}{n} \right) + o(1), \] (6.21)
where the second inequality follows by a union bound over $y \in [n]$ and by Lemma 2.5. By Markov’s inequality, for $K = C \log(n)$, and using Lemma 6.1,
\[ P \left( \mu_t(y) 1_G > \frac{20K \Delta^-}{n} \right) \leq \frac{\mathbb{E}[1_G \mu_t(y)^K]}{\left( \frac{20K \Delta^-}{n} \right)^K} \leq 2^{-K}. \] (6.22)
Choosing $C > 1/\log(2)$ we conclude that the probability in Eq. (6.22) is $o(n^{-1})$. Since we may take $C$ such that $20C < 29$, plugging Eq. (6.22) into Eq. (6.21) we obtain, whp
\[ \max_{y \in [n]} \mu_t(y) < \frac{29 \log(n) \Delta^-}{n}. \] (6.23)

Define the event $\mathcal{E} = \{ \forall y \in [n], \pi(y) \leq \mu_t(y) + 2e^{-\log^{3/2}(n)} \}$, with $t = \log^3(n)$, and observe that
\[ P \left( n \pi_{\text{max}} > 30 \log(n) \Delta^- \right) \leq P \left( n \pi_{\text{max}} > 30 \log(n) \Delta^-; \mathcal{E} \right) + P(\mathcal{E}^c) \]
\[ \leq P \left( \max_{y \in [n]} \mu_t(y) > \frac{30 \log(n) \Delta^-}{n} - 2e^{-\log^{3/2}(n)} \right) + P(\mathcal{E}^c) \]
\[ \leq P \left( \max_{y \in [n]} \mu_t(y) > \frac{29 \log(n) \Delta^-}{n} \right) + P(\mathcal{E}^c). \] (6.24)
Corollary 3.7 implies $P(\mathcal{E}^c) = o(1)$ and the desired conclusion follows from Eq. (6.23). \(\square\)

7. Power-law behavior: Proof of Theorem 1.7

In this section we prove Theorem 1.7. Recall the definition of $\phi(k)$ in Eq. (1.21) as the proportion of vertices having in-degree $k$. The lower bound in Eq. (1.22) is an immediate corollary of the results in Section 5.3. Indeed, from Corollary 5.6 it follows that whp for all $y \in [n]$ such that $d_y^- > n^a$, one has $n \pi(y) > \frac{n^a}{2d_y}$, where $\langle d \rangle$ is the average degree. Therefore, for all $a \in (0, 1/\kappa)$,
\[ \psi(n^a, \infty) \geq \phi(2 \langle d \rangle n^a, \infty) \geq n^{-a\kappa - \epsilon}, \] (7.1)
for all $\epsilon > 0$ by the assumed power-law behavior of the degree sequence.

The rest of this section is concerned with the proof of the upper bound in Theorem 1.7. As announced in Remark 1.8, we actually prove a slightly more general result; see Theorem 7.3 below.

Definition 7.1. A bi-degree sequence is $\kappa$-light if for all $\epsilon > 0$, for all $a > 0$ we have
\[ \sum_{k > n^a} k \phi(k) \leq n^{-a(\kappa - 1) + \epsilon}. \] (7.2)
Proposition 7.2. If \( d_n \) satisfies Assumption 1.1, then \( d_n \) is \((2 + \eta)\)-light, where \( \eta > 0 \) is such that Eq. (1.2) holds. Moreover, if \( d_n \) has power-law behavior with index \( \kappa > 2 \) as in Eq. (1.20), then \( d_n \) is \( \kappa \)-light.

Proof. For any \( a > 0 \) and using the bounded \((2 + \eta)\)-moment we have

\[
\sum_{k > n^a} k\phi(k) \leq n^{-a(1+\eta)} \sum_{k > n^a} k^{2+\eta}\phi(k) = O(n^{-a(1+\eta)}). \tag{7.3}
\]

This proves the first assertion.

To prove the second one, for any \( \varepsilon > 0 \) and using Eq. (1.20), we have

\[
\sum_{k \geq [n^a]} k\phi(k) = \sum_{r=0}^{[1/\varepsilon]} \sum_{k = [n^a + r\varepsilon]} k\phi(k) \leq \sum_{r=0}^{[1/\varepsilon]} [n^{a+(r+1)\varepsilon}] n^{-(a+r\varepsilon)\kappa+\varepsilon} = O(n^{-(\kappa-1)2\varepsilon}). \tag{7.4}
\]

Since \( \varepsilon \) can be arbitrarily small, this implies the claim.

From the previous facts, we see that the upper bound in Theorem 1.7 and the claim in Remark 1.8 both follow from the next result.

**Theorem 7.3.** Suppose \( d_n \) satisfies Assumption 1.1 and assume that \( d_n \) is \( \kappa \)-light for some \( \kappa > 2 \). Then for all \( \varepsilon > 0 \), whp for all \( a > 0 \),

\[
\psi(n^a, \infty) \leq n^{-a\kappa+\varepsilon}. \tag{7.5}
\]

7.1. The \( \alpha \)-skeleton. The proof of Theorem 7.3 is based on the following construction. For any \( a \in (0, 1/\kappa) \), the \( \alpha \)-skeleton \( \xi_a \) is the partial matching of heads and tails defined as follows. Recall the definition of \( V(a) \) in Eq. (5.27). For any \( z \in V(a) \), call \( W_{a,z} \) the set of \( \) paths \( p \) starting at \( z \) whose weight \( w(p) \) is at least \( w_{a,z} := n^a d_z^{-1} \).

\[
w_{a,z} = n^a d_z^{-1}. \tag{7.6}
\]

Note that we have not fixed the length of the paths in this definition, and that \( W_{a,z} \) covers a portion of the out-neighborhood of \( z \) with depth growing logarithmically as a function of \( d_z^{-1} \). The \( \alpha \)-skeleton is defined by

\[
\xi_a = \bigcup_{z \in V(a)} W_{a,z}. \tag{7.7}
\]

We call \( V(\xi_a) \) the set of \( y \in [n] \) such that at least one of the heads or tails of \( y \) is matched in \( \xi_a \). In particular, \( V(a) \subseteq V(\xi_a) \).

**Lemma 7.4.** Under the assumptions of Theorem 7.3, for all \( a \in (0, 1/\kappa) \), \( \varepsilon > 0 \), \( \xi_a \) has at most \( n^{1-a\kappa+\varepsilon} \) edges for \( n \) large enough. In particular,

\[
|V(\xi_a)| \leq n^{1-a\kappa+\varepsilon}. \tag{7.8}
\]

**Proof.** Fix some \( z \in V(a) \) and notice that the set \( W_{a,z} \) can be generated with the weighted out-neighborhood construction given in Section 3.2, with the only difference that here we impose no constraint on the distance to the root \( z \), and the minimal weight \( w_{\text{min}} \) is now replaced by \( w_{a,z} \). Thus, it follows from the argument used in Eq. (3.25), that the number
of edges in \( W_{a,z} \) is at most \( 2/w_{a,z} = 2n^{-a}d_z^- \). Since the degree sequence is \( \kappa \)-light, the total number of edges in \( \xi_a \) is deterministically bounded by
\[
\sum_{z \in V(a)} 2n^{-a}d_z^- \leq 2n^{-a} \sum_{k>n^a} kn\phi(k) \leq n^{1-\alpha\kappa+\varepsilon}.
\] (7.9)
This finishes the proof. \( \square \)

**Remark 7.5.** Let us observe that whp if \( y \in V(\xi_a) \) then \( n\pi(y) \geq \frac{n^a}{2\langle d \rangle} \). Indeed, if \( y \in V(a) \) then this is a consequence of Corollary 5.6. If instead \( y \in V(\xi_a) \setminus V(a) \), then by definition of \( a \)-skeleton there exists a vertex \( z \in V(a) \) and a path \( p \) starting at \( z \) and ending at \( y \) with \( w(p) \geq w_{a,z} = n^a/d_z^- \). Thus, if \( s \) denotes the length of \( p \), then
\[
n\pi(y) \geq n\pi(z)P^s(z,y) \geq n\pi(z)w_{a,z} \geq n\frac{d_z^-}{2m}n^a = \frac{n^a}{2\langle d \rangle},
\] (7.10)
where we used again Corollary 5.6 to lower bound \( \pi(z) \). The heart of the proof of Theorem 7.3 will consist in showing that for all \( y \notin V(\xi_a) \) except for at most \( n^{o(1)} \) of them one has \( n\pi(y) \leq n^{a+o(1)} \).

### 7.2. Proof of Theorem 7.3.
Recall that
\[
\psi(n^a, \infty) = \frac{1}{n} \sum_{y \in [n]} 1(\pi(y) > n^{a-1}).
\] (7.11)
We need to prove that for any \( \varepsilon > 0 \),
\[
\mathbb{P}(\exists a \in (0, \infty) : \psi(n^a, \infty) > n^{-\alpha\kappa+\varepsilon}) = o(1).
\] (7.12)
Let \( t = \log^3(n) \) and consider \( \mu_t \) as in Eq. (3.69). Define
\[
Z_a = \sum_{y \in [n]} 1(\mu_t(y) > \frac{1}{2} n^{a-1}).
\] (7.13)
The next lemma is the main technical estimate in this section.

**Lemma 7.6.** For any \( \varepsilon > 0 \), for any fixed \( a \in (0, 1/\kappa) \),
\[
\mathbb{P}(Z_a > n^{1-\alpha\kappa+\varepsilon}) = o(1).
\] (7.14)
Before proving Lemma 7.6, let us show that this estimate implies Eq. (7.12). We start by proving that
\[
\mathbb{P}(\psi(n^a, \infty) > 0) = o(1),
\] (7.15)
for all \( a > 1/\kappa \). The maximum in-degree \( \Delta^- \) is the largest \( \ell \) such that \( \phi(\ell) \geq 1/n \), or equivalently, such that \( n \sum_{k \geq \ell} k\phi(k) \geq \ell \). If \( \Delta^- > n^{1/\kappa} \), by definition of \( \kappa \)-light degree sequence, for all \( \varepsilon > 0 \) we have
\[
\Delta^- \leq n \sum_{k > n^{1/\kappa}} k\phi(k) \leq n^{1/\kappa+\varepsilon}.
\] (7.16)
By the upper bound in Theorem 1.4, we have whp \( n\pi_{\max} \leq \Delta^- n^\varepsilon \leq n^{1/\kappa+2\varepsilon} \), and therefore \( \psi(n^a, \infty) = 0 \) if \( a \geq \frac{1}{\kappa} + 2\varepsilon \). Since \( \varepsilon \) is arbitrarily small, this proves Eq. (7.15) for all \( a > 1/\kappa \).

To prove Eq. (7.12), thanks to the monotonicity of \( a \mapsto \psi(n^a, \infty) \) and Eq. (7.15) one can replace \( \exists a \in (0, \infty) \) with \( \exists a \in (0, b) \) for any fixed \( b > 1/\kappa \) in that statement. For simplicity,
we take $b = 2/\kappa$. Moreover, since we have restricted to a bounded interval of exponents $a$, it is actually sufficient to prove that for any $\varepsilon > 0$

$$\mathbb{P}\left(\psi(n^a, \infty) > n^{-a\kappa + \varepsilon}\right) = o(1), \quad (7.17)$$

for each fixed $a \in (0, 2/\kappa)$. Indeed, let $I_\varepsilon$ denote the set of integers in the interval $[0, 4/\varepsilon]$. If $a \in (0, 2/\kappa)$, there exists $i \in I_\varepsilon$ such that $i\varepsilon/2 \leq a\kappa \leq (i + 1)\varepsilon/2$ and $\psi(n^{i\varepsilon/(2\kappa)}, \infty) \geq \psi(n^a, \infty)$. Therefore, if $\psi(n^a, \infty) > n^{-a\kappa + \varepsilon}$ for some $a \in (0, 2/\kappa)$, there must exist $i \in I_\varepsilon$ such that

$$\psi(n^{i\varepsilon/(2\kappa)}, \infty) > n^{-a\kappa + \varepsilon} \geq n^{-i(\varepsilon/2) + \varepsilon/2}. \quad (7.18)$$

Using Eq. (7.17), a union bound over the finite set $I_\varepsilon$ then allows us to conclude Eq. (7.12).

Finally we observe that it is sufficient to establish Eq. (7.17) for all fixed $a \in (0, 1/\kappa)$. Indeed, the case $a > 1/\kappa$ is covered by Eq. (7.15), and the case $a = 1/\kappa$ follows by the arbitrariness of $\varepsilon$.

By Corollary 3.7, we know that whp $\mu_t(y) \geq \pi(y) - 1/n$ for all $y \in [n]$, which implies $n\psi(n^a, \infty) \leq Z_a$ for all $a > 0$. Therefore, in order to prove Eq. (7.17) it suffices to show Lemma 7.6.

### 7.3. Proof of Lemma 7.6.

To prove Eq. (7.14), we fix $\eta \in (0, 1)$ such that Assumption 1.1 applies, and define $h = c \log n$, where $c = c(\eta, \Delta^+)$ can be taken, e.g., as $c = \eta/(200 \log \Delta^+)$. As $h \leq 2\varepsilon_\eta$, with, e.g., $\varepsilon = \eta/4$, by Lemma 2.5 the event $\mathcal{G} = \mathcal{G}^+(h)$ has probability $1 - o(1)$. The key to our proof will be the following estimate on the moments of $\mu_t(y)$ for $y \notin \xi_a$.

**Lemma 7.7.** Fix $t = \log^3(n)$. For all constants $\varepsilon > 0$, $a \in (0, 1/\kappa)$, $K > 1$,

$$\max_{y \in [n]} \mathbb{E}\left[1_{\mathcal{G}}1_{y \notin V(\xi_a)}\mu_t(y)^K\right] \leq n^{K(a+\varepsilon-1)}, \quad (7.19)$$

for all $n$ large enough.

**Proof.** Fix $a \in (0, 1/\kappa)$ and $y \in [n]$. We may assume that $y \notin V(a)$, since otherwise $y \in V(\xi_a)$ and the estimate becomes trivially satisfied.

We use the same construction based on the $K$ annealed walks as in Lemma 6.1. With the notation used in that proof, recall that $D_\ell$ is the digraph generated by the first $\ell$ trajectories and $D_{\ell,z}$ is the union of $D_{\ell-1}$ and the edges generated by the $\ell$-th walk up to time $t-1$. For any $z \in V(D_\ell) \cap V(a)$, define $W_{a,z}(D_\ell)$ as the union of all paths $p$ contained in $D_{\ell,z}$ starting at $z$, and such that $w(p) \geq w_{a,z}$. Recall the definition of an event being $D_\ell$ compatible given in Section 6. In particular, $D_\ell \sim \{y \notin V(\xi_a)\}$, if $y \notin V(W_{a,z}(D_\ell))$ for all $z \in V(D_\ell) \cap V(a)$.

For $\ell \in \{1, \ldots, K\}$, we consider the events

$$E_\ell = \{D_\ell \sim \{y \notin V(\xi_a)\}\}, \quad F_\ell = \{D_\ell \sim \mathcal{G}\}, \quad B_\ell = \{X_1^{(1)} = \cdots = X_\ell^{(\ell)} = y\}. \quad (7.20)$$

We may write

$$\mathbb{E}\left[1_{\mathcal{G}}1_{y \notin \xi_a}\mu_t(y)^K\right] \leq \mathbb{P}^{\text{an}}(B_K \cap E_K \cap F_K)
= \mathbb{P}^{\text{an}}(B_1 \cap E_1 \cap F_1) \prod_{\ell=2}^{K} \mathbb{P}^{\text{an}}(B_\ell \cap E_\ell \cap F_\ell | B_{\ell-1} \cap E_{\ell-1} \cap F_{\ell-1}). \quad (7.21)$$

Bounding the first term is simple. Indeed, the event that the first walk visits $y$ up to time $t$ has probability bounded by $1/n + \frac{t d_0}{n} \leq n^{a+\varepsilon-1}$, since it has probability $1/n$ of hitting
y at time 0 and probability at most \( \frac{d^u}{m-t} \leq \frac{d^u}{n} \leq n^{a-1} \) of visiting y for the first time at any subsequent step. It follows that

\[
\mathbb{P}(B_1 \cap E_1 \cap F_1) \leq \mathbb{P}(B_1) \leq n^{a+\varepsilon-1}.
\] (7.22)

Hence, it is suffices to show that

\[
\mathbb{P}^{an}(B_{\ell} \cap E_{\ell} \cap F_{\ell} | B_{\ell-1} \cap E_{\ell-1} \cap F_{\ell-1}) \leq n^{a+\varepsilon-1}, \quad \forall \ell \in \{2, \ldots, K\}.
\] (7.23)

Let \( D_{\ell-1} \) denote a realization of the partial matching generated by the first \( \ell - 1 \) walks, and assume that \( D_{\ell-1} \) satisfies \( B_{\ell-1} \cap E_{\ell-1} \cap F_{\ell-1} \). For the event \( B_{\ell} \) to occur, the \( \ell \)-th walk must enter at some time \( s \in \{0, \ldots, t\} \) in \( D_{\ell,s} \). Arguing exactly as in Eq. (6.13) we estimate

\[
\mathbb{P}^{an}(B_{\ell} \cap F_{\ell} \cap E_{\ell} | D_{\ell-1}) \leq \sum_{s=0}^{t} \max_{D \in A_\ell(s)} \sum_{z \in V(D)} \frac{d^{-}}{n} q_D(z, y, t-s),
\] (7.24)

where we define the set \( A_\ell(s) \) as in Eq. (6.14) with the only difference that the event \( H_\ell \) is replaced by \( E_\ell \).

We split the last sum in Eq. (7.24) according to whether \( z \) is in \( V(a) \). If \( z \notin V(a) \), for all \( s \leq t \), one has

\[
\sum_{z \in V(D) \setminus V(a)} \frac{d^{-}}{n} q_D(z, y, t-s) \leq n^{-a-1}|V(D)|.
\] (7.25)

Since \( |V(D)| \leq Kt + 1 \) and \( t = \log^3(n) \), the contribution of this term to the RHS of Eq. (7.24) is at most \( (t+1)(Kt+1)n^{a-1} \leq n^{a-1+\varepsilon} \). Therefore, we may restrict to estimating the contribution of the terms corresponding to \( z \in V(a) \).

We now show that, for every \( D \sim \mathcal{G} \) it is unlikely that a walk stays on \( D \) for \( 3 \log(n) \) steps. First observe that

\[
\max_{z \in V(D)} \sum_{x \in V(D)} q_D(z, x, h) \leq 2|V(D)|2^{-h} \leq n^{-\frac{c}{2}},
\] (7.26)

where the first inequality is a consequence of \( D \sim \mathcal{G} \), as in Eq. (6.16), and the second inequality follows from \( |V(D)| \leq Kt + 1 = n^{o(1)} \) and \( h = c \log n \). By the Markov property, for all \( t \in \mathbb{N} \)

\[
\max_{z \in V(D)} \sum_{x \in V(D)} q_D(z, x, t) \leq n^{-\frac{t}{2}}.
\] (7.27)

Therefore, if \( j \geq 3 \log(n) \),

\[
\max_{z \in V(D)} q_D(z, y, j) \leq \max_{z \in V(D)} \sum_{x \in V(D)} q_D(z, x, j) \leq n^{-\lfloor j/2 \rfloor} \leq n^{-1}.
\] (7.28)

By Eq. (7.28),

\[
\sum_{s=0}^{t-3\log n} \sum_{z \in V(D)} \frac{d^{-}}{n} q_D(z, y, t-s) \leq \frac{t}{n} \sum_{z \in V(D)} \frac{d^{-}}{n} \leq \frac{(d)}{n} \leq n^{a+\varepsilon-1}.
\] (7.29)

Hence we can restrict to the case \( z \in V(a) \) and \( s \in \{t - 3 \log(n), \ldots, t\} \) in Eq. (7.24). If the event \( E_\ell \) holds, and the entry vertex \( z \) of \( D_{\ell,s} \) is in \( V(a) \), then each path from \( z \) to \( y \) in \( D \) has weight smaller than \( w_{a,z} \). Therefore,

\[
q_D(z, y, t-s) \leq w_{a,z} |P(z, y, t-s, D)|.
\] (7.30)
Since \( \frac{d_n}{n} w_{a,z} = n^{a-1} \), it remains to show that under the event \( D \sim G \) and for all fixed \( \varepsilon > 0 \),
\[
\max_{z \in D} \max_{j \leq \log(n)} \left| P(z, y, j, D) \right| \leq n^\varepsilon,
\]
for all \( n \) large enough.

Let \( L = \lfloor \frac{n}{k} \rfloor \). To prove Eq. (7.31), we split each path of length \( j \) in \( D \) from any \( z \) to \( y \) into \( L + 1 \) consecutive paths of which the first \( L \) have length \( h \) and the last one has length \( h' \leq h \). Note that
\[
L \leq \frac{j}{h} \leq \frac{3}{c}.
\]
(7.32)

Letting \( u_i \) denote the end vertex of the \( i \)-th sub-path, we have
\[
\left| P(z, y, j, D) \right| \leq \sum_{u_1 \in V(D)} \cdots \sum_{u_L \in V(D)} \left| P(z, u_1, h, D) \right| \left| P(u_1, u_2, h, D) \right| \cdots \left| P(u_L, y, h', D) \right|.
\]
(7.33)

Since \( D \sim G \), for every \( u, u' \in V(D) \) the number of paths of length \( h' \leq h \) from \( u \) to \( u' \) in \( D \) is bounded by 2. Thus,
\[
\left| P(z, y, j, D) \right| \leq \sum_{u_1 \in V(D)} \cdots \sum_{u_L \in V(D)} 2^{L+1} \leq 2 (2|V(D)|)^L \leq 2 (2(Kt + 1))^L \leq n^\varepsilon,
\]
for any \( \varepsilon > 0 \), if \( n \) is large enough. \( \square \)

**Proposition 7.8.** Fix any \( a \in (0, 1/\kappa) \) and consider \( Z_a \) as in Eq. (7.13). Then, for all \( \varepsilon > 0 \),
\[
\mathbb{E} [1_G Z_a] \leq n^{1-a\kappa+\varepsilon},
\]
(7.35)
for all \( n \) large enough.

**Proof.** Fix \( a \in (0, 1/\kappa) \), \( \varepsilon \in (0, a/4) \), and \( \delta = \varepsilon/\kappa \). Let \( V_k \) denote the set of \( y \in [n] \) with \( d_y = k \). Let \( \mathcal{E}(y) = \{1_G \mu_k(y) \geq \frac{1}{2} n^{a-1}\} \). We have
\[
\mathbb{E} [1_G Z_a] = \sum_{y \in [n]} \mathbb{P} (\mathcal{E}(y)) = \sum_{k \geq 0} \sum_{y \in V_k} \mathbb{P} (\mathcal{E}(y)).
\]
(7.36)

First note that
\[
\sum_{k \geq n^{a-\delta}} \sum_{y \in V_k} \mathbb{P} (\mathcal{E}(y)) \leq \sum_{k \geq n^{a-\delta}} n^\delta \sum_{k \geq n^{a-\delta}} k \phi(k) \leq n^{1-\kappa + 2\varepsilon},
\]
(7.37)
where the last bound follows from the assumption that the degree sequence is \( \kappa \)-light. Thus, in the rest of the proof we restrict to \( k \leq n^{a-\delta} \), i.e., \( y \notin V(a-\delta) \).

For all \( y \in V \), let
\[
\mathcal{E}_1(y) = \mathcal{E}(y) \cap \{y \in V(\xi_{a-\delta})\}, \quad \mathcal{E}_2(y) = \mathcal{E}(y) \cap \{y \notin V(\xi_{a-\delta})\}.
\]
(7.38)

Therefore, by the arbitrariness of \( \varepsilon \), the desired statement follows if we prove
\[
\sum_{k \leq n^{a-\delta}} \sum_{y \in V_k} \mathbb{P}(\mathcal{E}_i(y)) \leq n^{1-a\kappa+3\varepsilon}, \quad i = 1, 2.
\]
(7.39)
To prove Eq. (7.39) for \( i = 1 \), we use the rough bound \( \mathbb{P}(\mathcal{E}_1(y)) \leq \mathbb{P}(y \in V(\xi_{a-\delta})) \). Since \( y \notin V(a-\delta) \), in the generation of \( \xi_{a-\delta} \) at least one head incident to \( y \) has been matched. By Lemma 7.4 and the choice of \( \delta = \varepsilon/\kappa \), we know that \( \xi_{a-\delta} \) contains at most \( n^{1-a\kappa+2\varepsilon} \) edges. Thus, the probability that during the generation of \( \xi_{a-\delta} \) one of the heads of a given
estimate Eq. (7.14). We write
\[ P(y \in V(\xi_{a-\delta})) \leq Np \leq d_y^{-n^{-\alpha+2\varepsilon}}. \] (7.40)
Summing over \( k \leq n^{a-\delta} \) and \( y \in V_k \), we obtain Eq. (7.39) for \( i = 1 \).

We actually prove a stronger estimate than Eq. (7.39) for \( i \geq 2 \). Indeed, for every \( K > 0 \):
\[ P \left( \mathcal{E}_2(y) \right) = P \left( 1_G 1_y \notin \xi_{a-\delta} \mu_t(y) > \frac{1}{2} n^{-\alpha-1} \right) \leq \frac{2^K E \left[ 1_G 1_y \notin \xi_{a-\delta} \mu_t(y)^K \right]}{n^{K(\alpha-1)}}. \] (7.41)
By Lemma 7.7, for all \( \varepsilon' > 0 \), taking \( n \) large enough,
\[ E \left[ 1_G 1_y \notin \xi_{a-\delta} \mu_t(y)^K \right] \leq n^{K(\alpha-\varepsilon')}. \] (7.42)
Choosing \( \varepsilon' = \delta/3 \), the right hand side of Eq. (7.41) can be bounded by \( n^{-\delta K/2} \). Therefore
\[ \sum_{k \leq n^{a-\delta}} \sum_{y \in V_k} P(\mathcal{E}_2(y)) \leq n^{-\delta K/2}. \] (7.43)
The desired estimate follows by choosing, e.g., \( K = \lceil \frac{2}{\delta} \rceil \). \( \square \)

We are now able to conclude the proof of Lemma 7.6. Recall that all we needed is the estimate Eq. (7.14). We write
\[ P \left( Z_a > n^{1-\alpha+\varepsilon} \right) \leq P \left( 1_G Z_a > n^{1-\alpha+\varepsilon} \right) + P(G^c). \] (7.44)
By Lemma 2.5, \( P(G^c) = o(1) \). By Proposition 7.8, \( E[1_G Z_a] \leq n^{-\alpha+\varepsilon/2} \) and therefore Eq. (7.14) is a consequence of Markov’s inequality.

8. **Power-law for PageRank: Proof of Theorem 1.9**

8.1. **Lower bound.** Let us take \( \alpha = \alpha_n \) a sequence in \((0, 1)\) and assume that
\[ \limsup_{n \to \infty} \alpha_n \leq 1 - \delta, \] (8.1)
for some \( \delta > 0 \). For \( x \in [n] \), it follows from Eq. (1.25) that
\[ \pi_{\alpha, \lambda}(x) = \sum_{k=0}^{\infty} \alpha(1 - \alpha)^k \lambda^k P^k(x) \geq \alpha(1 - \alpha) \lambda_{\min} \sum_{y \in [n]} P(y, x) \] (8.2)
\[ \geq \alpha(1 - \alpha) \lambda_{\min} \frac{d_x^-}{\Delta^+} \geq \alpha \frac{\delta n^{-\varepsilon}}{2 \Delta^+} \frac{d_x^-}{n}, \]
where \( \lambda_{\min} = \min_{y \in [n]} \lambda(y) \), and we have used \( 1 - \alpha \geq \delta/2 \), \( \lambda_{\min} \geq n^{-1-\varepsilon} \) by Eq. (1.26), for all \( \varepsilon > 0 \), and \( n \) large enough.

On the other hand, by the definition of total variation distance (see Eq. (1.5)) and the monotonicity of distance to equilibrium, for all \( x \in [n] \) and \( k \geq t \in \mathbb{N} \),
\[ \lambda^k(x) \geq \pi(x) - \| \lambda^k - \pi \|_{TV} \geq \pi(x) - \| P^t - \pi \|_{TV}. \] (8.3)
Thus, for any $t \in \mathbb{N}$,
\[
\pi_{\alpha, \lambda}(x) \geq \sum_{k=t}^{\infty} \alpha (1 - \alpha)^k \lambda P^k(x) \geq (1 - \alpha)^t \left( \pi(x) - \| \lambda P^t - \pi \|_{TV} \right).
\]
Taking $t = \log^3(n)$, by Corollary 3.7, whp $\| \lambda P^t - \pi \|_{TV} \leq n^{-1}$. Thus, whp for all $x \in [n]$
\[
n\pi_{\alpha, \lambda}(x) \geq \max \left\{ \frac{\delta n^{-\varepsilon}}{2\Delta^+}, (1 - \alpha)^t (n\pi(x) - 1) \right\} =: u(x, \alpha).
\]

By the lower bound in Corollary 5.6, whp for all $x \in V(a)$, $a > 0$, we have $n\pi(x) \geq \frac{d^e}{2(d_x)}$ where $\langle d \rangle = m/n$. If $\alpha \leq n^{-\varepsilon}$ then $(1 - \alpha)^t \geq 1/2$ for $n$ large enough. Thus, for all $\varepsilon > 0$, max $\{ \alpha, (1 - \alpha)^t \} \geq n^{-\varepsilon}$ for $n$ large enough. We obtain that whp
\[
u(x, \alpha) \geq n^{-2\varepsilon}d_x^e \max \{ \alpha, (1 - \alpha)^t \} \geq n^{-3\varepsilon}d_x^e.
\]

It follows that whp
\[
n
.\psi(\alpha, \lambda) n^a, \infty \geq \sum_{x \in V(a+3\varepsilon)} 1(u(x, \alpha) > n^a) \geq |V(a+3\varepsilon)| = n\phi(n^{a+3\varepsilon}, \infty).
\]

Hence, by the power-law assumption on the in-degree sequence, we conclude that for all $\varepsilon > 0$, whp for all $a \in (0, 1/\kappa)$, $\psi(\alpha, \lambda) n^a, \infty \geq n^{-a\kappa-(3\kappa+1)\varepsilon}$.

8.2. Upper bound. By Proposition 8 in [19] we have
\[
\max_{x \in [n]} \| P_{\alpha, \lambda}(x, \cdot) - \pi_{\alpha, \lambda} \|_{TV} \leq 2(1 - \alpha)^t \max_{x \in [n]} \| P^t(x, \cdot) - \pi \|_{TV}.
\]
Thus, by Corollary 3.7, for $t = \log^3(n)$, whp
\[
\max_{x \in [n]} \| P_{\alpha, \lambda}(x, \cdot) - \pi_{\alpha, \lambda} \|_{TV} \leq e^{-\log^{3/2}(n)}.
\]

Call $\mu_{\alpha, \lambda}$ the probability measure
\[
\mu_{\alpha, \lambda}(y) = \frac{1}{n} \sum_{x \in [n]} P_{\alpha, \lambda}(x, y).
\]

From Eq. (8.9) we have, whp, $|\mu_{\alpha, \lambda}(y) - \pi_{\alpha, \lambda}(y)| \leq n^{-1}$ for all $y \in [n]$. Let us also introduce
\[
Z_a = \sum_{y \in [n]} 1(\mu_{\alpha, \lambda}(y) > \frac{1}{2} n^{a-1}).
\]

Then, whp
\[
\psi_{\alpha, \lambda}(n^a, \infty) = \sum_{y \in [n]} 1(\pi_{\alpha, \lambda}(y) > n^{a-1}) \leq Z_a.
\]

Thus, the upper bound in Theorem 1.9 follows from Eq. (8.14) in the following lemma.

**Lemma 8.1.** Fix an arbitrary sequence $\alpha = \alpha_n \in [0, 1]$, and an arbitrary sequence of probability measures $\lambda = \lambda_n$ on $[n]$. Then, under Assumption 1.1,
\[
\max_{x \in [n]} \pi_{\alpha, \lambda}(x) \leq \frac{30 \log(n) \Delta_*}{n},
\]
where \( \Delta_s = \Delta^- + n\lambda_{\text{max}} \) and \( \lambda_{\text{max}} = \max_{x \in [n]} \lambda(x) \). Moreover, if the empirical in-degree distribution has power-law behavior with index \( \kappa > 2 \) and \( \lambda \) satisfies Eq. (1.26), then for any \( a \in (0, 1/\kappa) \)
\[
\mathbb{P} \left( Z_{a,\lambda}^n > n^{1-a\kappa+\varepsilon} \right) = o(1).
\] (8.14)

**Proof.** We introduce the annealed construction for PageRank walks with uniform starting vertices. Adapting the discussion in Eq. (3.3), we write, for all \( t, K \in \mathbb{N} \),
\[
\mathbb{E} \left[ (\mu_t^{\alpha,\lambda}(A))^K \right] = \mathbb{E} \left[ (\mathbb{P}_{\alpha,\lambda}(X_t \in A))^K \right] = \mathbb{P}_{\alpha,\lambda}^{\text{an},K} \left( X_t^{(k)} \in A, \forall k \in [K] \right),
\] (8.15)
where \( A \subset [n] \),
\[
\mathbb{P}_{\alpha,\lambda}(X_t \in A) = \frac{1}{n} \sum_{x \in [n]} \sum_{y \in A} P_{\alpha,\lambda}(x, y),
\] (8.16)
and \( \mathbb{P}_{\alpha,\lambda}^{\text{an},K} \) is the law of the non-Markovian process
\[
\left\{ X_s^{(k)}, \ s \in \{0, \ldots, t\}, \ k \in \{1, \ldots, K\} \right\},
\] (8.17)
which can be described as follows. Start with an empty matching. For each \( k \in [K] \), given the first \( k-1 \) walks \( X_i^{(t)} \) \( s \leq t, t \leq k-1 \), to generate the \( k \)-th walk,

(i) start the \( k \)-th walk, i.e., \( X^{(k)} \), at a uniformly random vertex \( X_0^{(k)} \in [n] \);

(ii) for all \( s \in \{0, \ldots, t-1\} \): select one of the tails of \( X_s^{(k)} \) uniformly at random, call it \( e \), and draw an independent Bernoulli(\( \alpha \)) random variable \( U \),

- If \( U = 0 \) and \( e \) was already matched by one of the previous walks, or by \( X^{(k)} \) itself at a previous step, to some head \( f \), then let \( X_{s+1}^{(k)} = v_f \);

- If \( U = 0 \) and \( e \) is still unmatched, then select a uniformly random head, \( f \), among the unmatched ones, match it to \( e \), and let \( X_{s+1}^{(k)} = v_f \);

- If \( U = 1 \), then select a random vertex \( Y \sim \lambda \) and set \( X_{s+1}^{(k)} = Y \).

To prove the upper bound on \( \pi_{\alpha,\lambda} \) Eq. (8.13), we are going to show that, under Assumption 1.1, if \( K = \Theta(\log n) \), then
\[
\mathbb{E} \left[ 1_G \left( \mu_t^{\alpha,\lambda}(y) \right)^K \right] \leq \left( \frac{10K\Delta_s}{n} \right)^K, \] (8.18)
Note that this is the statement of Lemma 6.1 with \( \mu_t \) replaced by \( \mu_t^{\alpha,\lambda} \) and \( \Delta^- \) replaced by \( \Delta_s \). Once this bound is established, then the same argument in Eqs. (6.21) and (6.23) yields the estimate Eq. (8.13).

Going over the proof of Lemma 6.1 step by step, we see that up to Eq. (6.13) nothing is changed in the argument, while Eq. (6.13) continues to hold provided we replace \( d_z^-/n \) with \( d_z^-/n + \lambda(z) \). This is obtained by considering the last time \( s \) such that the \( \ell \)-th walk enters the set \( D \) of previously activated edges \( s \) and, from then on, stays on \( D \) without undergoing any teleportation. The vertex \( z \in D \) where this last entry occurs can be reached either by activating a fresh edge, which contributes at most \( d_z^-/n \) or by a teleportation which has probability \( \lambda(z) \). Once this modification is made, all arguments can be repeated without any change, and Eq. (8.18) follows.

The same reasoning shows that the proof of Lemma 7.6 goes through with the only change that \( d_z^-/n \) must be replaced by \( d_z^-/n + \lambda(z) \) in Eq. (7.24). Using also the assumption \( \lambda_{\text{max}} \leq n^{\varepsilon-1} \) for all \( \varepsilon > 0 \), this implies Eq. (8.14). \( \square \)
To prove the upper bound in Theorem 1.9, we can use Eq. (8.9) and Lemma 8.1 and the claim follows exactly as in Section 7.2.

**Remark 8.2.** Note that the walk in the above proof differs from previously introduced annealed walks only in that it is now possible to teleport, which corresponds to the event $U = 1$. Whenever this event occurs, the walk does not activate any new matching and therefore the environment is left unchanged. Since the main challenge in the analysis of the annealed walk is represented by the presence of the previously activated matching, this feature makes the case of PageRank surfers actually simpler than the case with $\alpha = 0$. This also explains why the upper bound on $\psi_{\alpha, \lambda}(n^a, \infty)$ in Theorem 1.9 holds uniformly in the choice of $\alpha = \alpha_n \in [0, 1]$. Moreover, for the degree sequence it is sufficient to assume $\kappa$-lightness.

**Remark 8.3.** We observe that the estimates above immediately imply the bounds on the maximum PageRank score in Remark 1.11. The lower bound is a consequence of Eq. (8.2). The upper bound has been established in Lemma 8.1.

9. Tightness of estimates on $\pi_{\max}$

In this section we discuss the tightness of the bounds in Theorem 1.4, providing examples that show that both bounds in Eq. (1.15) and Eq. (1.16) cannot be substantially improved in general.

We first focus on Eq. (1.15). Fix $\varepsilon > 0$. Let $\Delta = \lceil e^{1/\varepsilon} \rceil$ and $\delta = 2$. Consider a degree sequence $d_n$ with half of the $n$ vertices having degrees $(\Delta, \delta)$, and the other half of degrees $(\delta, \Delta)$. Note that

$$\frac{\log \delta}{\log \Delta} \leq \varepsilon \log 2 \leq \varepsilon. \tag{9.1}$$

Then, by Theorem 1.6 in [18], there exists a constant $c = c(\varepsilon) > 0$ such that whp

$$\pi_{\max} \geq \frac{c \log^{1-\varepsilon} n}{n}, \tag{9.2}$$

proving that Eq. (1.15) is tight up to a sub-logarithmic multiplicative factor.

The rest of the section is devoted to provide a wide class of examples where Eq. (1.16) is tight. We call a bi-degree sequence $d_n$ extremal if there exists $w \in [n]$ such that $d_w = \Delta^-$ and for any $z \neq w$,

$$d_z = o \left( \frac{d_w}{\log n} \right). \tag{9.3}$$

The following can be seen as a refinement of Eq. (1.15) for extremal sequences.

**Proposition 9.1.** Let $d_n$ be an extremal bi-degree sequence satisfying Assumption 1.1. Then, for any $\varepsilon > 0$ whp

$$\frac{\pi_{\max}}{\Delta^-/m} - 1 \leq \varepsilon. \tag{9.4}$$

Moreover, the maximum stationary value is attained uniquely at the vertex of maximum in-degree.

Note that any non-trivial extremal sequence satisfies $\Delta^- \log^{-1}(n) \to \infty$ as $n \to \infty$. Requiring such condition is natural in view of the existence of sequences whose $\pi_{\max}$ exhibits $\log^{1-o(1)}(n)$ deviations with respect to $\Delta^-/m$ (see Eq. (9.2)).
The lower bound follows from Eq. (1.16). The idea for the upper bound is to mimic the proof of the upper bound Eq. (1.15) while removing the factor $C \log n$ from it. Let $w$ be the vertex attaining the maximum in-degree. Let $\eta$ be the constant appearing in Eq. (1.2). Recall the definitions of $h_\epsilon$ and $\mathcal{G}^+(h)$ in Eq. (2.9) and Eq. (2.36). Choose $\epsilon \in (0, \eta/6)$, write $h = h_\epsilon$ and $\mathcal{G} = \mathcal{G}^+(h)$, and define $\mathcal{H} := \{\text{Tx}(\mathcal{B}_w^-(h)) = \text{Tx}(\mathcal{B}_w^+(h)) = 0\}$.

For $t \in \mathbb{N}$, recall the definition of the measure $\mu_t$ on $[n]$ given in Eq. (3.69). Following the argument in the proof of the upper bound in Theorem 1.4, it suffices to prove the following strengthening of Lemma 6.1.

**Lemma 9.2.** Let $d_n$ be as in Proposition 9.1. For any $\gamma, C > 0$, $t = \log^3(n)$ and $K = C \log(n)$, one has

$$
\mathbb{E}[1_{\mathcal{G} \cap \mathcal{H}}(\mu_t(y))^K] \leq \left(\left(\frac{1}{2} + \frac{1}{2}(y = w) + \gamma\right) \frac{\Delta^-}{m}\right)^K,
$$

(9.5)

for all $y \in [n]$ and all sufficiently large $n$.

**Proof.** Lemma 2.3 ensures that the coupling between $\mathcal{B}_w^-(h)$ and $\mathcal{T}_w^-(h)$ succeeds whp, thus $\mathbb{P}(\text{Tx}(\mathcal{B}_w^-(h)) = 0) = 1 - o(1)$. A similar argument as the one in Lemma 2.5 but only for the out-neighborhood of $w$ implies that $\mathbb{P}(\text{Tx}(\mathcal{B}_w^+(h)) = 0) = 1 - o(1)$. Combining it with Lemma 2.5 to bound the probability of $\mathcal{G}$, we have $\mathbb{P}(\mathcal{G} \cap \mathcal{H}) = 1 - o(1)$.

The proof is very similar to that of Lemma 6.1. We reuse the notation defined there and omit the identical details. As in Eq. (6.7), we have

$$
\mathbb{P}^{an}(H_{Kt}^o) = o\left((\Delta^-/m)^K\right),
$$

(9.6)

Having defined $B_t, F_t, H_t$, it suffices to bound the terms in Eq. (6.5). As in the proof of Lemma 6.1, if the event $B_t$ holds then let $s$ be the time that the walk enters $D_t,s$ and traverses only edges in $D_t,s$ until reaching $y$. Write $I_1 = \{0, \ldots, t-h\}$ and $I_2 = [t] \setminus I_1$. As in Eq. (6.17), the contribution of $s \in I_1$ is $o(\Delta^-/n)$. To bound the contribution of $s \in I_2$, we split the left-hand-side of Eq. (6.18) into two parts depending on whether we enter at $w$ or not:

$$
\sum_{a=0}^{h-1} \max_{D \in A_t(a-t)} \sum_{z \in V(D)} \frac{d_z}{m-Kt} q_D(z, y, a) \\
\leq \sum_{a=0}^{h-1} \max_{D \in A_t(a-t)} \frac{d_w}{m-Kt} q_D(w, y, a) + \sum_{a=0}^{h-1} \max_{D \in A_t(a-t)} \sum_{z \in V(D) \setminus \{w\}} \frac{d_z}{n} q_D(z, y, a).
$$

(9.7)

Let us bound the first contribution in Eq. (9.7). If $y = w$, since $D \sim \mathcal{H}$, there is no path from $w$ to $w$ of length at least 1 and at most $h$ and the term is bounded by the contribution of $a = 0$, that is $\frac{\Delta^-}{m-Kt} = (1 + o(1)) \frac{\Delta^-}{m}$. If $y \neq w$, since $D \sim \mathcal{H}$, there is at most one path of length at most $h$ from $w$ to $y$. As the minimum out-degree is at least 2 by Assumption 1.1, we have $\sum_{a=0}^{h-1} q_D(w, y, a) \leq 1/2$ and the term is bounded by $(1 + o(1)) \frac{\Delta^-}{2m}$. Therefore, the first contribution in Eq. (9.7) is bounded by $(\frac{1}{2} + \frac{1}{2}(y = w) + o(1)) \frac{\Delta^-}{m}$.

For the second contribution in Eq. (9.7), the same bound as in Eq. (6.20) gives a total of

$$
\sum_{a=0}^{h-1} \frac{(2Ka + 1)2^{-a+1}}{n} \max_{z \neq w} d_z = O\left(\frac{K \max_{z \neq w} d_z}{n}\right) = o\left(\frac{\Delta^-}{n}\right),
$$

(9.8)

(51)
\[ K d_z = o(\Delta^-) \text{ for all } z \neq w \text{ by Eq. (9.3).} \]

Putting all the contributions together, we have that Eq. (9.5) holds for all \( y \in [n] \), and the lemma follows. \( \square \)

10. Future research directions

A number of open problems arise from empirical observations. Several papers have identified a consistent disagreement between the largest in-degree nodes and the ones attaining the maximum PageRank score in real-world networks (see, e.g., [24, 52]). Outliers in each ranking exhibit correlation but the top sets tend to disagree. This reinforces the idea that rankings based on stationary values are much more than the simple in-degree ranking and poses the question of determining under which conditions the top in-degree and top score nodes coincide. Notably, Eq. (1.16) tells us that the maximum stationary value is never asymptotically smaller than the maximum in-degree divided by \( m \) and these two asymptotically coincide for sequences with an outstanding maximum in-degree vertex (see Proposition 9.1).

In contrast, in Fig. 1 we display the results of a simulation done for the DCM with power-law in-degree distribution and constant out-degree, which suggests that asymptotically the two may only differ by a non-trivial multiplicative factor [14]. It would be interesting to determine under which conditions on the degree sequence, the largest in-degree and the largest stationary (or PageRank) value coincide in order, or asymptotically.

A possible extension of Theorem 1.7 is to study the upper tail of \( \psi \) in Eq. (1.10) for degree sequences satisfying Assumption 1.1 but not necessarily having power-law in-degrees. In such case, Remark 1.8 gives an upper bound, which can be possibly refined if additional information about the upper tail of the empirical in-degree distribution \( \phi \) is known. Our results in Section 7 suggest that \( \psi(n^a, \infty) \) could be approximated by the order of the \( a \)-skeleton as defined in Section 7.1.

An important and challenging open problem is the extension of our results to the case of in-degrees with bounded first moment, that is replacing the \( 2 + \eta \) in condition (iii) of Assumption 1.1 by \( 1 + \eta \), or even 1. The structure and distances in random graphs with infinite variance degrees is strikingly different [47] from the ones satisfying Assumption 1.1. It would be interesting to determine whether the vertices of large in-degree will have a non-negligible effect, speeding-up the mixing time. This case is central in applications, as many real-world networks are believed to have power-law behavior with index \( \kappa \in (1, 2) \) [44].

Another interesting open problem concerns the relaxation of condition (i) of Assumption 1.1. Minimum out-degree at least 2 is required to ensure that the random walk has no trivial attractive strongly connected components, and in particular avoids the existence of dangling nodes (i.e., nodes of out-degree 0). While this is a necessary requirement for the random walk without teleporting, it is interesting to study the PageRank surfer walk under the presence of dangling nodes. Condition (ii) is mainly technical, facilitating the exploration of out-neighborhoods and the existence of a law of large numbers (Proposition 3.2). It would be interesting to obtain a version of Theorem 1.9 that allowed dangling nodes and arbitrarily large out-degrees satisfying a suitable moment assumption. Research on related stochastic models suggests that the effect of the out-degree distribution and of dangling nodes is essentially negligible [51].
Figure 1. Simulation in DCM with power-law in-degree distribution with index $\kappa = \frac{5}{2}$ and out-degree 2. $\pi_{\text{max}}$ and $\pi_{\Delta}$ denote the maximum stationary value and the stationary value of the node with maximum in-degree. $\text{PR}_{\text{max}}$ and $\text{PR}_{\Delta}$ denote the maximum PageRank and the PageRank of the node with maximum in-degree, for teleporting probability $\alpha = \frac{1}{4}$ and the uniform teleporting distribution $\lambda$. We generated 500 samples for each $n$.

**REFERENCES**

[1] L. Addario-Berry, B. Balle, and G. Perarnau. Diameter and stationary distribution of random r-out digraphs. *The Electronic Journal of Combinatorics*, page P3.28, 2020. doi: 10/ghd74q.

[2] D. J. Aldous and A. Bandyopadhyay. A survey of max-type recursive distributional equations. *The Annals of Applied Probability*, 15(2):1047–1110, 2005. doi: 10/bsq6kw.

[3] B. Amento, L. Terveen, and W. Hill. Does “authority” mean quality? predicting expert quality ratings of Web documents. In *Proceedings of the 23rd Annual International ACM SIGIR Conference on Research and Development in Information Retrieval*, SIGIR ’00, pages 296–303, New York, NY, USA, 2000. Association for Computing Machinery. doi: 10/dmjgds.
[4] K. B. Athreya and P. E. Ney. Branching Processes. Grundlehren Der Mathematischen Wissenschaften. Springer-Verlag, Berlin Heidelberg, 1972. doi:10/dft4.

[5] K. Avrachenkov and D. Lebedev. PageRank of scale-free growing networks. Internet Mathematics, 3(2):207–231, 2006. doi:10/ffckmh.

[6] K. Avrachenkov, N. Litvak, D. Nemirovsky, E. Smirnova, and M. Sokol. Quick detection of top-k personalized PageRank lists. In International Workshop on Algorithms and Models for the Web-Graph, pages 50–61. Springer, 2011. doi:10/dt74jh.

[7] S. Banerjee and M. Olvera-Cravioto. PageRank asymptotics on directed preferential attachment networks. arXiv:2102.08894 [math], 2021. URL http://arxiv.org/abs/2102.08894.

[8] L. Becchetti, C. Castillo, D. Donato, S. Leonardi, and R. Baeza-Yates. Using rank propagation and probabilistic counting for link-based spam detection. In Proc. of WebKDD, volume 6, 2006.

[9] A. Ben-Hamou and J. Salez. Cutoff for nonbacktracking random walks on sparse random graphs. The Annals of Probability, 45(3):1752–1770, 2017. doi:10/gbhtxj.

[10] N. Berestycki, E. Lubetzky, Y. Peres, and A. Sly. Random walks on the random graph. The Annals of Probability, 46(1):456–490, 2018. doi:10/gjj266.

[11] J. Blanchet and A. Stauffer. Characterizing optimal sampling of binary contingency tables via the configuration model. Random Structures & Algorithms, 42(2):159–184, 2013. doi:10/f4mtxh.

[12] C. Bordenave, P. Caputo, and J. Salez. Random walk on sparse random digraphs. Probab. Theory Relat. Fields, 170(3):933–960, 2018. doi:10/gc8nxk.

[13] C. Bordenave, P. Caputo, and J. Salez. Cutoff at the “entropic time” for sparse markov chains. Probab. Theory Relat. Fields, 173(1):261–292, 2019. doi:10/ghcchr.

[14] X. S. Cai. DCM.wl: A Mathematica package for simulation of random walks in Directed Configuration Model, 2021. URL https://github.com/newptcai/DCM.wl.

[15] X. S. Cai and G. Perarnau. The giant component of the directed configuration model revisited. arXiv:2004.04998 [cs, math], 2020. URL http://arxiv.org/abs/2004.04998.

[16] X. S. Cai and G. Perarnau. The diameter of the directed configuration model. arXiv:2003.04965 [cs, math], 2020. URL http://arxiv.org/abs/2003.04965.

[17] X. S. Cai and G. Perarnau. Minimum stationary values of sparse random directed graphs. arXiv:2010.07246 [cs, math], 2020. URL http://arxiv.org/abs/2010.07246.

[18] P. Caputo and M. Quattropani. Stationary distribution and cover time of sparse directed configuration models. Probab. Theory Relat. Fields, 178(3):1011–1066, 2020. doi:10/ghd74v.

[19] P. Caputo and M. Quattropani. Mixing time of PageRank surfers on sparse random digraphs. Random Structures & Algorithms, 2021. doi:10/gjpxsk.

[20] S. Chatterjee. Stein’s method for concentration inequalities. Probab. Theory Relat. Fields, 138(1-2):305–321, 2007. doi:10/fm2x4r.

[21] N. Chen and M. Olvera-Cravioto. Directed random graphs with given degree distributions. Stochastic Systems, 3(1):147–186, 2013. doi:10/gjj27p.
[22] N. Chen, N. Litvak, and M. Olvera-Cravioto. PageRank in scale-free random graphs. In International Workshop on Algorithms and Models for the Web-Graph, pages 120–131. Springer, 2014. doi:10/gjj27n.
[23] N. Chen, N. Litvak, and M. Olvera-Cravioto. Generalized PageRank on directed configuration networks. Random Structures & Algorithms, 51(2):237–274, 2017. doi:10/gbrth6.
[24] P. Chen, H. Xie, S. Maslov, and S. Redner. Finding scientific gems with Google’s PageRank algorithm. Journal of Informetrics, 1(1):8–15, 2007. doi:10/fctrbr.
[25] C. Cooper and A. Frieze. The size of the largest strongly connected component of a random digraph with a given degree sequence. Combinatorics, Probability and Computing, 13(3):319–337, 2004. doi:10/cn8q5j.
[26] C. Cooper and A. Frieze. Stationary distribution and cover time of random walks on random digraphs. J. Comb. Theory Ser. B, 102(2):329–362, 2012. doi:10/cv9wbh.
[27] D. Donato, L. Laura, S. Leonardi, and S. Millozzi. Large scale properties of the webgraph. The European Physical Journal B, 38(2):239–243, 2004. doi:10/fhdgcd.
[28] R. Durrett. Probability: Theory and Examples, volume 31 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010. doi:10/gfj8hd.
[29] N. Fountoulakis and B. A. Reed. The evolution of the mixing rate of a simple random walk on the giant component of a random graph. Random Structures & Algorithms, 33 (1):68–86, 2008. doi:10/bkwdf9j.
[30] D. A. Freedman. On tail probabilities for martingales. The Annals of Probability, 3(1):100–118, 1975. doi:10/fdvpst.
[31] A. Garavaglia, R. van der Hofstad, N. Litvak, et al. Local weak convergence for PageRank. The Annals of Applied Probability, 30(1):40–79, 2020. doi:10/gjj27r.
[32] T. H. Haveliwala. Topic-sensitive PageRank: A context-sensitive ranking algorithm for web search. IEEE transactions on knowledge and data engineering, 15(4):784–796, 2003. doi:10/cwp6vw.
[33] S. Janson. The probability that a random multigraph is simple. Combinatorics, Probability and Computing, 18(1-2):205–225, 2009. doi:10/bg4m2c.
[34] S. Janson. Probability asymptotics: Notes on notation. arXiv:1108.3924 [math], 2011. URL http://arxiv.org/abs/1108.3924.
[35] S. Janson, T. Łuczak, and A. Rucinski. Random Graphs. John Wiley & Sons, 2011. doi:10/d8w6m8.
[36] J. Lee and M. Olvera-Cravioto. PageRank on inhomogeneous random digraphs. Stochastic Processes and their Applications, 130(4):2312–2348, 2020. doi:10/gjj27q.
[37] D. A. Levin and Y. Peres. Markov Chains and Mixing Times. American Mathematical Soc., second edition, 2017.
[38] N. Litvak, W. R. W. Scheinhardt, and Y. Volkovich. In-degree and PageRank: Why do they follow similar power laws? Internet Mathematics, 4(2-3):175–198, 2007. doi:10/d4zqj5.
[39] E. Lubetzky, A. Sly, et al. Cutoff phenomena for random walks on random regular graphs. Duke Mathematical Journal, 153(3):475–510, 2010. doi:10/fxd427.
[40] C. McDiarmid. Concentration. In Probabilistic Methods for Algorithmic Discrete Mathematics, Algorithms and Combinatorics, pages 195–248. Springer, Berlin, Heidelberg,
1998. doi:10/f58t.

[41] M. E. Newman, S. H. Strogatz, and D. J. Watts. Random graphs with arbitrary degree distributions and their applications. Physical review E, 64(2):026118, 2001. doi:10/fsvfnf.

[42] M. Olvera-Cravioto. PageRank’s behavior under degree-degree correlations. arXiv:1909.09744 [math], 2019. URL http://arxiv.org/abs/1909.09744.

[43] L. Page, S. Brin, R. Motwani, and T. Winograd. The PageRank citation ranking: Bringing order to the web. Technical report, Stanford InfoLab, 1999. URL http://ilpubs.stanford.edu:8090/422/.

[44] G. Pandurangan, P. Raghavan, and E. Upfal. Using PageRank to characterize web structure. In International Computing and Combinatorics Conference, pages 330–339. Springer, 2002. doi:10/czd5pm.

[45] S. I. Resnick. Heavy-Tail Phenomena: Probabilistic and Statistical Modeling. Springer Series in Operations Research and Financial Engineering. Springer-Verlag, New York, 2007. doi:10/fpr8zr.

[46] T. Upstill, N. Craswell, and D. Hawking. Predicting fame and fortune: PageRank or indegree? In Proceedings of the Australasian Document Computing Symposium, ADCS2003, pages 31–40, 2003.

[47] R. van der Hofstad, G. Hooghiemstra, D. Znamenski, et al. Distances in random graphs with finite mean and infinite variance degrees. Electronic Journal of Probability, 12:703–766, 2007. doi:10/fxp3kj.

[48] P. van der Hoorn and M. Olvera-Cravioto. Typical distances in the directed configuration model. Ann. Appl. Probab., 28(3):1739–1792, 2018. doi:10/ggh2ch.

[49] C. Villani. Optimal Transport: Old and New. Grundlehren Der Mathematischen Wissenschaften. Springer-Verlag, Berlin Heidelberg, 2009. doi:10/bgcxnm.

[50] Y. Volkovich and N. Litvak. Asymptotic analysis for personalized web search. Advances in applied probability, 42(2):577–604, 2010. doi:10/ft99bt.

[51] Y. Volkovich, N. Litvak, and D. Donato. Determining factors behind the PageRank log-log plot. In International Workshop on Algorithms and Models for the Web-Graph, pages 108–123. Springer, 2007. doi:10/bqhm9z.

[52] Y. Volkovich, N. Litvak, and B. Zwart. Extremal dependencies and rank correlations in power law networks. In International Conference on Complex Sciences, pages 1642–1653. Springer, 2009. doi:10/d6fkwf.