An Algorithmic Argument for Nonadaptive Query Complexity Lower Bounds on Advised Quantum Computation

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Abstract. This paper employs a powerful argument, called an algorithmic argument, to prove lower bounds of the quantum query complexity of a multiple-block ordered search problem in which, given a block number \( i \), we are to find a location of a target keyword in an ordered list of the \( i \)th block. Apart from much studied polynomial and adversary methods for quantum query complexity lower bounds, our argument shows that the multiple-block ordered search needs a large number of nonadaptive oracle queries on a black-box model of quantum computation that is also supplemented with advice. Our argument is also applied to the notions of computational complexity theory: quantum truth-table reducibility and quantum truth-table autoreducibility.

Keywords: algorithmic argument, query complexity, nonadaptive query, advice, quantum computation, ordered search

1 An Algorithmic Argument for Query Complexity

A major contribution of this paper is the demonstration of a powerful argument, which we refer to algorithmic argument, to prove a lower bound of the nonadaptive query complexity for a multiple-block ordered search problem on advised quantum computation. In the literature, quantum query complexity lower bounds have been proven by classical adversary methods [11], polynomial methods [8], and quantum adversary methods [5, 7, 27]. Each method has its own strength and advantages over its simplicity, clarity, and dexterity. An algorithmic argument, however, is essentially different from these methods in its constructive manner. A basic scheme of our algorithmic argument is illustrated as follows: we (i) commence with the faulty assumption that a quantum algorithm \( \mathcal{A} \) of low query complexity exists, (ii) define a compression scheme \( E \) that encodes each input \( s \) of fixed length into a shorter string \( E(s) \), and (iii) prove the one-to-oneness of \( E \) by constructing a decoding algorithm from \( \mathcal{A} \) that uniquely extracts \( s \) from \( E(s) \), which leads to a conspicuous contradiction against the pigeonhole principle\(^\dagger\). In this paper, we build a classical algorithm that decodes \( E(s) \) by simulating \( \mathcal{A} \) in a deterministic fashion. We therefore reach the conclusion that any quantum algorithm should require high query complexity. When \( \mathcal{A} \) further models “uniform” quantum computation, we can use resource-bounded Kolmogorov complexity\(^\ddagger\) as a technical tool in place of the pigeonhole principle.

We apply our algorithmic argument to obtain a new nonadaptive query complexity lower bound on a query computation model, known as a black-box quantum computer (sometimes called a quantum network [5]), in which a query is an essential method to access information stored outside of the computer. The minimal number of such queries, known as the query complexity, measures the smallest amount of information necessary to finish the desired computation. Query complexity lower bounds on various quantum computational models have been studied for numerous problems, including ordered search [11, 18, 24, 27], unordered search [6, 8, 9, 11], element distinctness [5, 17, 36], and collision [1, 9, 36].

A black-box quantum computer starts with a fixed initial state (e.g., \(|0 \cdots 0\rangle \)), accesses a given source \( x \) (which is called an “oracle”) by way of queries—“what is the binary value at location \( i \) in \( x \)?”—and outputs a desired solution with small error probability. If any query (except the first one) is chosen according to the answers to its previous queries, such a query pattern is conventionally referred to as adaptive. Adaptive

\(^\dagger\)The pigeonhole principle formalizes the intuition that, when \( n \) pigeons rest in fewer than \( n \) nests, at least two pigeons share the same nest.

\(^\ddagger\)The use of Kolmogorov complexity unintentionally adds an extra constant additive term and thus gives a slightly weaker lower bound.
oracle quantum computation has been extensively studied and have given rise to useful quantum algorithms, e.g., [12, 19, 22, 26, 35]. An adaptive computation in general requires a large number of times of interactions between the computer and a given oracle. Since a quantum computer is known to be sensitive to any interaction with other physical systems, such as an oracle, it would be desirable to limit the number of times of interactions with the oracle. In contrast, the query pattern in which all the query words are prepared before the first query is referred to as nonadaptive queries (including parallel queries and truth-table queries). Recently, Buhrman and van Dam [16] and Yamakami [39] extensively studied the nature of parallel queries on quantum computation. By visiting the results in [13, 16, 20, 22, 35, 39], we can find that quantum nonadaptive queries are still more powerful than classical adaptive queries. This paper pays its attention to the truth-table query model in which the query words produced in a prequery quantum state are answered all at once.

Our black-box quantum computation is further equipped with advice, which was first discussed by Karp and Lipton [28], to provide an additional source of information that boosts the computational power. Such advice supplements the information drawn from an oracle and therefore the advice reduces the query complexity. Lately, time-bounded advised quantum computation was introduced by Nishimura and Yamakami [33] and discussed later by Aaronson [2]. Their notion has a close connection to nonuniform computation [33] and also one-way communication [2]. Our interest in this paper lies in the relationship between the size of advice and the nonadaptive quantum query complexity.

Based on a black-box model of quantum computation with advice, we employ our algorithmic argument to prove a lower bound of the query complexity of a so-called ordered search problem. For simplicity, we focus our interest on the ordered search problem of the following kind: given an $N$-bit string $x$ of the form $0^{j-1}1^{N-j+1}$ for certain positive integers $N$ and $j$, we are to find the leftmost location $s$ of 1 (which equals $j$). Such a unique location $s$ is called the step of $x$ (since the input $x$ can be viewed as a so-called step function). This ordered search problem is one of the well-studied problems for their quantum adaptive query complexity. Naturally, we can expand this problem into a “multiple-block” ordered search problem, in which we are to find the step (called the $i$th step) in each block $i$ when the block number $i$ is given as an input. To avoid the reader’s confusion, we call the standard ordered search problem the single-block ordered search problem. This paper presents a new query complexity lower bound for the multiple-block ordered search problem on a nonadaptive black-box quantum computer with the help of advice.

Independently, Laplante and Magniez [29] also found a similar algorithmic argument to demonstrate general lower bounds of randomized and quantum query complexity. Their argument, nonetheless, is meant for the adaptive query complexity without advice strings and is different in its nature of query computation from our argument.

Finally, we note that an algorithmic argument is not new in classical complexity theory. Earlier, Feigenbaum, Fortnow, Laplante, and Naik [29] applied an algorithmic argument to show that the multiple-block ordered search problem is hard to solve on a classical Turing machine using nonadaptive queries. Their proof, nonetheless, cannot be directly applied to the case of black-box quantum computation since their proof exploits the fact that a probabilistic polynomial-time Turing machine with polynomial advice can be simulated by a certain deterministic polynomial-time Turing machine with polynomial advice. Our technique developed in this paper, to the contrary, enables us to show a desired quantum query complexity lower bound for the multiple-block ordered search problem.

In the subsequent sections, we will give the formal definition of nonadaptive black-box query computation and of the multiple-block ordered search problem. We will also present an overview of our lower bound results before proving the main theorems. Finally, we hope that an algorithmic argument would find more useful applications in other fields of theoretical computer science in the future.

## 2 A Model of Nonadaptive Query Computation

We formally describe a black-box model of quantum nonadaptive query computation. The reader may refer to [8] for the formal description of quantum “adaptive” query computation. In particular, we use a “truth-table” query model rather than the “parallel” query model given in [16, 39] to simplify our algorithmic argument although our results still hold in the parallel query model. Note that these two models are fundamentally equivalent in the classical setting since there is no timing problem as it occurs in the quantum case (see, e.g., [39] for more details). In our truth-table query model, all queries are made at once after the first phase of computation and the second phase leads to a desirable solution without any query. This truth-table query model can be seen as a special case of the parallel query model of Yamakami [39].
In the rest of this paper, we assume the reader’s familiarity with the fundamental concepts in computational complexity theory and quantum computing (see, e.g., [24] for computational complexity theory and [31] for quantum computing). Hereafter, we fix our alphabet \( \Sigma \) to be \( \{0, 1\} \) for simplicity. Let \( N \) be the set of all natural numbers (i.e., nonnegative integers) and set \( \mathbb{N}^+ = \mathbb{N} - \{0\} \). For any two integers \( m \) and \( n \) with \( m < n \), the notation \([m, n]\) denotes the set \( \{m, m+1, m+2, \ldots, n\} \). For any \( n \in \mathbb{N}^+ \) and \( i \in [1, 2^n] \), let \( \text{bin}_n(i) \) represent the lexicographically \( i \)th binary string in \( \Sigma^n \) (e.g., \( \text{bin}_n(1) = 0^n \) and \( \text{bin}_n(2^n) = 1^n \)). For any finite set \( A \), write \( |A| \) for the cardinality of \( A \). The tower of 2s is inductively defined by \( 2_0 = 1 \) and \( 2^n = 2^{2^{n-1}} \) for each \( n \in \mathbb{N}^+ \).

For convenience, let \( \text{Tower}2 = \{2^n \mid n \in \mathbb{N}\} \). All logarithms are to base two and all polynomials have integer coefficients. For convenience, we set \( \log 0 = 0 \).

Fix \( N \) as a power of 2, say, \( N = 2^n \) for a certain number \( n \in \mathbb{N}^+ \). A (black-box) problem \( F_N \) over alphabet \( \Sigma \) is a function that maps each instance \( f \) to its value \( F_N(f) \), where \( f \) is any function from \( \Sigma^n \) to \( \Sigma \). For convenience, \( f \) is often identified with its characteristic sequence \( \{f(x) \mid x \in \Sigma^n\} \) of length \( N \).

To solve such a problem \( F_N \), we carry out a nonadaptive query quantum computation on a black-box quantum computer. A nonadaptive black-box quantum computer is formally described as a series of pairs of unitary operators, say \( \{(U_n, V_n)\}_{n \in \mathbb{N}} \). Let \( n \) be any fixed size of our instances. The quantum computer \((U_n, V_n)\) consists of three registers. The first register is used to generate query words. The second register is used to receive the oracle answers to these query words, and the third register is used to perform non-query computation. Assume that an instance \( x \in \Sigma^N \) of the problem \( F_N \) is given to the computer as an oracle and all the registers are initially set to be the quantum state \(|000\rangle\). In the first phase, the computer changes the initial quantum state \(|000\rangle\) to the query quantum state \( \sum_{i_1, \ldots, i_r} \langle i_1, \ldots, i_r | (0^T \phi_{i_1, \ldots, i_r}) \) by applying the unitary operator \( U_n \), where \( T \in \mathbb{N} \) and \( i_1, \ldots, i_r \in [1, N] \). Each index \( i_j \) is called a query word and the list \( \vec{i} = (i_1, i_2, \ldots, i_T) \) is called a query list. Strictly speaking, each query word \( i_j \) in a query list \( \vec{i} \) should be generated as the \( n \)-bit string bin\(_n\)(\( i \)) in the first register. In this way, we often identify \( |1, N| \) with \( \Sigma^{\log N} \). We next prepare the unitary operator \( O_x \) that represents \( x \). The application of \( O_x \) then results in the postquery quantum state \( \sum_{i_1, \ldots, i_T} \langle i_1, \ldots, i_T | (x_{i_1} \cdots x_{i_T}) \phi_{i_1, \ldots, i_T} \rangle \). In the final phase, the computer applies the unitary operator \( V_n \) and then halts. When the computer halts, its output state becomes \( V_n O_x U_n(|000\rangle) \). The first \( |F_N(x)\rangle \) qubits of the third register are measured on the computational basis to obtain the outcome of the computation.

The \( \epsilon \)-error bounded quantum nonadaptive query complexity of the problem \( F_N \), denoted by \( Q^\epsilon_{\text{ut}}(F_N) \), is defined to be the minimal number \( T \) of the nonadaptive queries made by any nonadaptive black-box quantum computer with oracle \( x \) such that \( F_N(x) \) is observed with error probability at most \( \epsilon \) by the measurement of the output state. Throughout this paper, we restrict all the amplitudes of a bounded-error black-box quantum computer to the amplitude set \( \{0, \pm 3/5, \pm 4/5, \pm 1\} \). This restriction does not affect our results since bounded-error quantum algorithms are known to be robust against the choice of an amplitude set [3].

In the case where an advice string \( h_x \) is given as a supplemental input, a nonadaptive black-box quantum computer starts with the initial quantum state \(|000\rangle| h_x \rangle \) instead of \(|000\rangle\). We denote by \( Q^\epsilon_{\text{ut}}(F_N) \) the \( \epsilon \)-error bounded quantum nonadaptive query complexity of \( F_N \) given advice of length \( k \). For convenience, we often suppress the subscript \( \epsilon \) if \( \epsilon = 1/3 \).

We further expand the problem \( F_N \) into a “multiple-block” problem in the following section.
The \(M\)-block ordered search problem \(G_{M,N}\), where \(N = 2^n\), is formally defined as follows. The domain of \(G_{M,N}\) is the set \(\{(i, x_1 \cdots x_M) \in [1, M]^2 \times \Sigma^M \mid \forall j \exists s_j \forall k \{ (x_j)_k = 1 \text{ if } k \geq s_j \text{ and } (x_j)_k = 0 \text{ if } k < s_j\}\}\), where each \(x_j\) is taken from \(\Sigma^N\). Each \(s_j\) is called the step of \(x_j\). The output value \(G_{M,N}(i, x_1 x_2 \cdots x_M)\) is the \(i\)th step \(s_i\). For a later use, we define the modified problem \(G_{M,N,p}\) for each number \(p \in [1, n]\) as follows. The domain of \(G_{M,N,p}\) is the same as that of \(G_{M,N}\) but the outcome \(G_{M,N,p}(i, x_1 x_2 \cdots x_M)\) is instead the last \(p\) bits of \(s_i\). Obviously, \(G_{M,N,n}\) coincides with \(G_{M,N}\).

To solve the multiple-block ordered search problem \(G_{M,N}\), our nonadaptive black-box quantum computer \((U_n, V_n)\) operates in the following fashion. Given a pair \((i, x)\) of a number \(i \in [1, M]\) and an \(M\)-bit string \(x = x_1 x_2 \cdots x_M\), where each \(x_i\) is in \(\Sigma^n\), \((U_n, V_n)\) starts with a block number \(i\) and an advice string \(h\) (which is given independent of \(i\)) of length \(k\) and attempts to compute the value \(G_{M,N}(i, x)\) with small error probability. It is desirable in practice to minimize the number of queries and also the length of an advice string.

### 4 Query Complexity Lower Bounds: Overview

This section presents an overview of the adaptive and nonadaptive query complexity bounds for the multiple-block ordered search problem \(G_{M,N}\). This problem can separate the power of quantum adaptive query computation and that of quantum nonadaptive query computation.

For its quantum adaptive query complexity, the single-block ordered search problem \(G_{1,N}\) is one of the well-studied problems. In this adaptive query case, a simple binary search algorithm provides a trivial adaptive query complexity upper bound of \(\log N\). The lower bound of the adaptive query complexity \(Q(G_{1,N})\) was explored in [18, 24], and the \(\Omega(\log N)\)-lower bound was recently given by Ambainis [4] and Høyer, Neerbek, and Shi [27].

| problem   | \(G_{1,N}\)       | \(G_{M,N}\)       |
|-----------|-------------------|-------------------|
| advice    | no advice         | advice length \(k\) |
| upper bound | \(N - 1\)       | \(N/2^k - 1\)     |
| lower bound | \(\Omega(N)\)   | \(\Omega(N)/2^k\)  |

Table 1: Quantum nonadaptive query complexity bounds of \(G_{1,N}\) and \(G_{M,N}\)

On the contrary, the nonadaptive query complexity has a trivial upper bound of \(N - 1\) for the multiple-block problem \(G_{1,N}\). Similarly, in the presence of advice of length \(k\), the query complexity \(Q^{k,tt}(G_{1,N})\) is upper-bounded by \(N/2^k - 1\). As for the lower bound, using our algorithmic argument, we can show in Theorem 5.1 a lower bound \(Q^{k,tt}(G_{1,N}) \geq \Omega(N)/2^k\), which almost matches the above trivial upper bound for \(G_{1,N}\). Turning to the multiple-block problem \(G_{M,N}\), we can show in Theorem 5.1 a lower bound \(Q^{k,tt}(G_{M,N}) \geq \Omega(p(N, M, k))\), where \(p(N, M, k) = \min\left\{\frac{N}{M^2/2^{k/2} + M^{1/3}}, \frac{M - M^{1/3}}{(2M^{1/3} \log M + k + 2)^2}\right\}\). These two lower bounds of the nonadaptive query complexity will be proven in Sections 5 and 6. Moreover, a large gap between \(Q(G_{M,N})\) and \(Q^{k,tt}(G_{M,N})\) will be used in Section 7 to separate adaptive and nonadaptive complexity classes.

The aforementioned upper and lower bounds of the quantum nonadaptive query complexity of the multiple-block and single-block ordered search problems are summarized in Table 1.

### 5 Query Complexity for Multiple Block Ordered Search

We demonstrate how to use an algorithmic argument to obtain a new query complexity lower bound for the multiple-block ordered search problem \(G_{M,N}\). As a special case, the query complexity for the single-block ordered search problem will be discussed later in Section 6. Now, we prove the following lower bound of \(Q^{k,tt}(G_{M,N})\) as a main theorem.

**Theorem 5.1** \(Q^{k,tt}(G_{M,N}) \geq \Omega\left(\min\left\{\frac{N}{M^2/2^{k/2} + M^{1/3}}, \frac{M - M^{1/3}}{(2M^{1/3} \log M + k + 2)^2}\right\}\right)\).

Theorem 5.1 intuitively states that multiple-block ordered search requires a large number of nonadaptive queries even with the help of a relatively large amount of advice. To show the desired lower bound, we employ
an algorithmic argument that revolves around the incompressibility of instances.

We first prove a key proposition from which our main theorem follows immediately. For convenience, for any constant \( \epsilon \in [0, 1/2] \), we define \( d(\epsilon) = 1/(2\epsilon) - 1 \) if \( \epsilon > 0 \) and \( d(\epsilon) = 1 \) otherwise. Letting \( c \) be any constant satisfying \( 0 < c < d(\epsilon) \), we further define \( \epsilon' = (1 + c/\epsilon) \) and \( C_\epsilon = (1 - 2\epsilon')^2/16 \). For any string \( y \) and any number \( a \in [0, |y|] \), \( \text{First}_a(y) \) (\( \text{Last}_a(y) \), resp.) denotes the first (last, resp.) \( a \)-bit segment of \( y \). Clearly, \( y = \text{First}_y[|y| - a](y) \text{Last}_a(y) \).

To describe the key proposition, we need the notion of the *weight* of a query word and the function \( C_{U,V} \).

Fix \( n, M \in \mathbb{N} \) and \( p \in [1, n] \) and set \( N = 2^n \). Consider the multiple-block problem \( G_{M,N,p} \). Assume that a nonadaptive black-box quantum computer \((U,V)\) solves \( G_{M,N,p} \) with error probability \( \leq \epsilon \) with advice of length \( k \) using \( T \) nonadaptive queries. Let \( s \) be any string of length \( Mn \) and assume that \( s = s_1 s_2 \cdots s_M \), where each \( s_i \) is the \( i \)-th block segment of \( s \) with \( |s_i| = n \). Let \( f \) be its corresponding \( k \)-bit advice string. For any \( i,j \in [1, M] \) and any \( z \in \Sigma^n \), the weight of the query word \((j,z)\), denoted \( wt(i : j, z) \), is the sum of all the squared magnitudes of amplitudes of \( |\tilde{y}(i: j, f)\rangle \) such that the list \( \tilde{y} = (y_1, \ldots, y_T) \) of query words contains \((j, z)\) in the prequery quantum state \( U(|0\rangle|0\rangle|i,f)\) = \( \sum_y |\tilde{y}\rangle|0\rangle|\phi_{i,y,f}\rangle \). Moreover, for each \( i,j \in [1, M] \) and \( z', z'' \in \Sigma^{n-p} \), let \( wt_p(i : j, z') \) be the sum of the values \( wt(i : j, z) \) over all \( z \in \Sigma^n \) satisfying \( z' = \text{First}_{n-p}(z) \). An index \( i \) is called **good** if \( wt_p(i : i, \text{First}_{n-p}(s_i)) > C_\epsilon \). Any index that is not good is called **bad**.

Let \( l' \) denote the total number of good indices; i.e., \( l' = \{ i \in [1, M] \mid wt_p(i : i, \text{First}_{n-p}(s_i)) > C_\epsilon \} \). Note that \( 0 \leq l' \leq M \). At length \( l \), the function \( C_{U,V} \) is introduced in the following way:

\[
C_{U,V}(M,N,k,p,s,l) = \begin{cases} 
\frac{C_N}{M^{2^p+1+(k+2)/l}} & \text{if } l \leq l', \\
\frac{C_N}{C(M-1)p^2} & \text{if } l > l',
\end{cases}
\]

where \( l \geq 1 \). The key proposition below relates to a relationship between the query complexity and the function \( C_{U,V} \).

**Proposition 5.2** Let \( \epsilon \in [0, 1/2] \) and \( c \in (0, d(\epsilon)) \) and set \( \epsilon' = (1 + c/\epsilon) \) and \( C_\epsilon = (1 - 2\epsilon')^2/16 \). Let \( n, M \in \mathbb{N} \) and \( p \in [1, n] \) and set \( N = 2^n \). If a nonadaptive black-box quantum computer \((U,V)\) solves \( G_{M,N,p} \) with error probability \( \leq \epsilon \) by \( T \) queries with advice of length \( k \), then \( T \geq \max_{1 \leq l \leq M} \min_{s \in \Sigma^M} \{ C_{U,V}(M,N,k,p,s,l) \} \).

Clearly, Theorem 5.3 follows from Proposition 5.2 by setting \( p = 1 \) and \( l = M^{1/3} \) since \( Q^{k,H}(G_{M,N,1}) \leq Q^{k,H}(G_{M,N}) \) for any constant \( k \) in \( \mathbb{N} \).

Now, we detail the proof of Proposition 5.2 by employing our algorithmic argument. Assume to the contrary that Proposition 5.2 fails. Let \((U,V)\) be a nonadaptive black-box quantum computer that solves \( G_{M,N,p} \) with error probability \( \leq \epsilon \) with \( T \) nonadaptive queries using advice of length \( k \). By our assumption, there exists a number \( l \in [1, M] \) such that \( T < C_{U,V}(M,N,k,p,s,l) \) for all strings \( s \) in \( \Sigma^M \). It follows by a simple calculation that, for each \( s \in \Sigma^M \),

\[
l(2 \log M - n + \log(T/C_\epsilon) + p + 1) + k + 2 < 0 \text{ if } l \leq l',
\]

\[
2l \log M + k + 2 - p\sqrt{C(M-l)/T} < 0 \text{ if } l > l'.
\]

Our goal is to define a compression scheme \( E \) working on all strings in \( \Sigma^M \) such that (i) \( E \) is one-to-one and (ii) \( E \) is length-decreasing**. These two conditions clearly lead to a contradiction since any length-decreasing function from \( \Sigma^M \) to \( \Sigma^* \) cannot be one-to-one (by the pigeonhole principle). More precisely, we wish to define an “encoding” of \( s \), denoted \( E(s) \). We first show that \( |E(s)| < |s| \) using the definition of \( E \). To show the one-to-oneness of \( E \), we want to construct a “generic” deterministic decoding algorithm that takes \( E(s) \) and outputs \( s \) for any string \( s \) in \( \Sigma^M \) because this decoding algorithm guarantees the uniqueness of the encoding \( E(s) \) of \( s \). Therefore, we obtain a contradiction, as requested, and complete the proof.

Let \( s \) be any string in \( \Sigma^M \) and let \( f \) be its corresponding advice string of length \( k \). We split our proof into the following two cases: (1) \( l \leq l' \) and (2) \( l > l' \).

**Case 1: \( l \leq l' \)** The desired encoding \( E(s) \) contains the following four items: (i) the advice string \( f \), (ii) the \( 2l' \log M \)-bit string encoding in double binary all the \( l' \) good indices, (iii) a separator 01, and (iv) all the strings \( e(i) \) for each \( i \in [1, M] \), where \( e(i) \) is defined as follows. In case where \( i \) is good, \( e(i) \) is of the form \( (k_i, \text{Last}_p(s_i)) \) with \( k_i = \{a \in \Sigma^{n-p} \mid wt_p(i : a) > C_\epsilon \text{ and } a < \text{First}_{n-p}(s_i) \} \) (lexicographically). If \( i \) is bad, then \( e(i) = s_i \). These four items are placed in \( E(s) \) in order from (i) to (iv).

The following lemma shows that \( |\log(T/C_\epsilon)| \) bits are sufficient to encode \( k_i \) in binary.

**A function \( f \) from \( \Sigma^n \) to \( \Sigma^* \) is called length-decreasing if \( |f(x)| < |x| \) for all \( x \in \Sigma^n \).**
Lemma 5.3 For each good $i$, $k_i < T/C_e$.

Proof. Let $i$ be any good index and define the set $A_i = \{a \in \Sigma^{n-p} \mid wt_p(i : i, a) > C_e \}$. Obviously, $|A_i| \geq k_i$. It suffices to show that $|A_i| < T/C_e$. Recall first that the weight $wt_p(i : i, a)$ represents the value $\sum_{z \in \Sigma^p} \sum_{g(i, u) \in \tilde{g}} ||\phi_{i,f,g}||^2$, where “$(i, u) \in \tilde{g}$” means that the list $\tilde{g}$ contains query word $(i, u)$ in $\Gamma_{M,N}$. Since each query list $\tilde{y}$ contains at most $T$ query words,

$$\sum_{a \in \Sigma^{n-p}} wt_p(i : i, a) = \sum_{a \in \Sigma^{n-1}} \sum_{g(i, u) \in \tilde{g}} ||\phi_{i,f,g}||^2 = \sum_{g(i, u) \in \tilde{y}} \sum_{a(u, i, u) \in \tilde{g}} ||\phi_{i,f,g}||^2 \leq T \cdot \sum_{g(i, u) \in \tilde{g}} ||\phi_{i,f,g}||^2 \leq T.$$

The last inequality comes from the fact that $\sum_{g(i, u) \in \tilde{g}} ||\phi_{i,f,g}||^2 = 1$. It thus follows that $T \geq \sum_{a \in \Sigma^{n-p}} wt_p(i : i, a) \geq \sum_{a \in A_i} wt_p(i : i, a) > C_e|A_i|$, which implies that $|A_i| < T/C_e$, as requested. □

By Lemma 5.3, the representation of any pair $(k_i, \text{Last}_p(s_i))$ requires at most $\lceil \log(T/C_e) \rceil + p$ bits. The total length of the encoding $E(s)$ is thus bounded above by:

$$|E(s)| \leq k + 2l'_s \log M + 2 + l'_s(\log (T/C_e) + p + 1) + (M - l'_s)n \leq Mn + l(2 \log M - n + \log (T/C_e) + p + 1) + k + 2 < Mn,$$

where the second inequality is obtained from Eq. (1) and our assumption $l \leq l'_s$, and the last inequality comes from Eq. (1). Since $|s| = Mn$, it follows that $|E(s)| < |s|$.

We next show that the encoding $E(s)$ is uniquely determined from $s$. To show this, we give a deterministic decoding algorithm that extracts $s$ from $E(s)$ for all $s \in \Sigma^{Mn}$ with $l'_s \geq l$. The desired decoding algorithm $A$ is described as follows.

Decoding Algorithm $A$: For each $i \in [1, M]_\mathbb{Z}$, we compute $s_i$ in the following manner. First, check whether $i$ is good by examining item (ii) of $E(s)$. If $i$ is bad, then find $e(i) = s_i$ directly from item (iv).

The remaining case is that $i$ is good. Note that $wt_p(i : i, \text{First}_n-p(s_i)) > C_e$ and $e(i) = (k_i, \text{Last}_p(s_i))$. Define $A_i$ to be the set of all $(n-p)$-bit strings $a$ with $wt_p(i : i, a) > C_e$. Find the lexicographically $k_i+1$-st string in $A_i$ by preparing the query state classically. Obviously, this string equals $\text{First}_n-p(s_i)$ by the definition of $k_i$. Use $\text{Last}_p(s_i)$ to obtain the desired string $s_i = \text{First}_n-p(s_i)\text{Last}_p(s_i)$. Finally, output the decoded string $s = s_1s_2 \cdots s_M$.

Since $A$ does not involve the computation of $V$, it is easy to show that $A$ correctly outputs $s$ from $E(s)$.

(Case 2: $l > l'_s$) Different from Case (1), the encoding $E(s)$ includes the following six items: (i) the advice string $f$, (ii) the $2l'_s \log M$-bit string that encodes in double binary all $l'_s$ good indices, (iii) a separator 01, (iv) all the strings $s_i$ for each good index $i$, (v) all the strings $\text{First}_n-p(s_i)$ for each bad index $i$, and (vi) an additional string $r$ of length $\leq p((M-l'_s) - \sqrt{C_e(M-l'_s)/T})$, which will be defined later. These items are placed in $E(s)$ orderly from (i) to (vi).

We begin with the estimation of the length of $E(s)$. By summing up all the items of $E(s)$, we can upper-bound its length $|E(s)|$ by:

$$|E(s)| \leq k + 2l'_s \log M + 2 + l'_s n + (M - l'_s)(n-p) + p((M-l'_s) - \sqrt{C_e(M-l'_s)/T}) \leq Mn + 2\log M + k + 2 - p(\sqrt{C_e(M-l)/T}) < Mn,$$

where the second inequality comes from our assumption $l > l'_s$ and the fact that the derivative of the function $F(z) = -p(\sqrt{C_e(M-z)/T}) + 2z \log M$ satisfies $F'(z) \geq 0$ for any $z \in (0, M)$ and the last inequality follows from Eq. (2). Therefore, we obtain the desired inequality $|E(s)| < |s|$.

We still remain to define the string $r$. To describe it, we need to search for indices of light query weight. The following procedure, called the lightly weighted step search (abbreviated LWSS), selects a series of steps of light query weight. As we will show later, this series of steps are redundant and therefore, we can eliminate them, causing the compression of $s$. Let $m$ be the positive solution of the equation $(T/C_e)m^2 - (T/C_e - 1)m - (M - l'_s) = 0$. (In the case where $m$ is a non-integer, we need to round it down.)

Procedure LWSS: Let $R_1 = \emptyset$ and $L_1 = \{i \in [1, M]_\mathbb{Z} \mid i \text{ is bad} \}$. Repeat the following procedure by incrementing $i$ by one until $i = m$. At round $i$, choose the lexicographically smallest index $w_i$ in the difference $L_i - R_i$. Simulate $U$ deterministically on input $(w_i, f)$ to generate the query quantum state.
For each bad index $j \in [1, M]_\mathbb{Z}$, compute the weight $w_{tp}(w_i : j, \text{First}_{n-p}(s_i))$ in $|\gamma_f(w_i)|$.

Define $R_{i+1} = R_i \cup \{w_i\}$ and $L_{i+1} = L_i \cap \{j \in [1, M]_\mathbb{Z} \mid w_{tp}(w_i : j, \text{First}_{n-p}(s_i)) < C_e/m\}$. Finally, set $W = R_{m+1}$. Output all the elements in $W$.

We can prove the following lemma regarding LWSS.

**Lemma 5.4** LWSS produces a unique series of $m$ distinct indices $w_1, w_2, \ldots, w_m$ such that, for any pair $i, j \in [1, m]_\mathbb{Z}$, if $i < j$ then $w_{tp}(w_i : w_j, \text{First}_{n-p}(s_{w_j})) < C_e/m$.

**Proof.** First, we show that $|L_i| \geq M - l'_s - (T/C_e)m(i - 1)$ by induction on $i$ with $1 \leq i \leq m$. In the basis case where $i = 1$, this is true because the number of bad indices is $M - l'_s$. For the induction step, we assume by our induction hypothesis that $|L_i| \geq M - l'_s - (T/C_e)m(i - 1)$. For convenience, set $L' = \{\mu \in [1, M]_\mathbb{Z} \mid w_{tp}(w_i : \mu, \text{First}_{n-p}(s_\mu)) < C_e/m\}$. Now, we claim that $|L_i - L'| \leq (T/C_e)m$. From the definition of $L'$, it holds that, for any $\mu \notin L'$, $w_{tp}(w_i : \mu, \text{First}_{n-p}(s_\mu)) \geq C_e/m$. It thus follows that $|L_i - L'| \cdot (C_e/m) \leq \sum_{\mu \notin L_i} w_{tp}(w_i : \mu, \text{First}_{n-p}(s_\mu)) \leq T$, where the last inequality comes from the fact that the total weight of query words must be at most $T$. Therefore, we obtain the claim $|L_i - L'| \leq (T/C_e)m$.

Using the fact that $L_{i+1} = L_i \cap L'$, $|L_{i+1}| = |L_i| - |L_i - L'| \geq |L_i| - |L_i - L'|$, which is further bounded by:

$$|L_i| - |L_i - L'| \geq M - l'_s - (T/C_e)m(i - 1) - (T/C_e)m = M - l'_s - (T/C_e)mi.$$  

Therefore, $|L_{i+1}| \geq M - l'_s - (T/C_e)mi$. This completes the induction step.

To guarantee the existence of the $m$ indices $w_1, \ldots, w_m$, we want to show that $L_m \neq \emptyset$. Since $m(m - 1) \leq C_e(M - l'_s - m)/T$ (even though $m$ is rounded down), we know that $|L_m| \geq M - l'_s - (T/C_e)m(m - 1) \geq m$. Because of $|R_m| = m - 1$, $L_m - R_m$ cannot be empty. This implies that $w_m$ truly exists. Note that, by the definition of $L_j$, $L_j = \{\mu \in [1, M]_\mathbb{Z} \mid \forall i < j \text{ } w_{tp}(w_i : \mu, \text{First}_{n-p}(s_\mu)) < C_e/m\}$. Procedure LWSS clearly ensures that $w_j$ belongs to $L_j$. Hence, for any $i < j$, $w_{tp}(w_i : w_j, \text{First}_{n-p}(s_{w_j})) < C_e/m$.

The key element $r$ in item (vi) is defined as follows. Let $v_i$ be the lexicographically $i$th element in the set $\{j \in [1, M]_\mathbb{Z} \mid j \text{ is bad and } j \notin W\}$ and let $r_i$ be the last $p$ bits of $s_{v_i}$. The string $r = r_1 r_2 \cdots r_M$ constitutes item (vi). The length $|r|$ is clearly at most $p(\frac{M - l'_s - \sqrt{C_e(M - l'_s)}}{T} + \sqrt{2(T/C_e)})$ since

$$m = \frac{(T/C_e - 1) + \sqrt{4(T/C_e)(M - l'_s) + (T/C_e - 1)^2}}{2(T/C_e)} \geq \sqrt{C_e(M - l'_s)/T}.$$  

Next, we want to show the uniqueness of our encoding $E(s)$ by constructing its decoding algorithm. First, we check which index $i$ is good by simply examining item (ii). For any good index $i$, we immediately obtain $s_i$ from item (iv). When $i$ is bad, however, we obtain only First$_{n-p}(s_i)$ from item (v). To obtain the last $p$ bits of $s_i$, we need to exploit item (vi) of $E(s)$ and simulate $(U, V)$ in a deterministic fashion. Since we cannot use the oracle $G_{M, N, p}$, we need to substitute its true oracle answers with their approximated values. The desired decoding algorithm $B$ is given as follows.

**Decoding Algorithm B:** 1) For any good index $i \in [1, M]_\mathbb{Z}$, obtain $s_i$ directly from item (iv) of $E(s)$. For the other indices, run LWSS to compute $W = \{w_1, \ldots, w_m\}$. For any bad index $e$ outside of $W$, item (vi) provides Last$_{n-p}(s_e)$. Combining it with First$_{n-p}(s_e)$ from item (v), we obtain $s_e$.

2) Let $e$ be any bad index in $W$. First, obtain First$_{n-p}(s_e)$ from item (v). The remaining part, Last$_{n-p}(s_e)$, is obtained as follows. Repeat the following procedure starting at round 1 up to $m$. At round $i$ ($1 \leq i \leq m$), assume that the last $p$ bits of $s_{w_1}, \ldots, s_{w_{i-1}}$ have been already obtained. Simulate $U$ deterministically on input $(w_{i-1}, f)$ to generate the prequery quantum state $|\gamma_f(w_i)\rangle = \sum_g |g\rangle |0\rangle |\phi_{w_i, f, g}\rangle$. Transform $|\gamma_f(w_i)\rangle$ into the state $|\gamma_f(w_i')\rangle = \sum_g |g\rangle |u_{1, g} u_{2, g} \cdots u_{r, g} \phi_{w_i, f, g}\rangle$ using the string $r$, where each bit $u_{i, g}$ is defined below. Choose any list $y$ of query words. Note that $y$ is of the form $(y_1, y_2, \ldots, y_T)$. Let $j \in [1, T]_\mathbb{Z}$ and assume that $y_j$ is of the form $(a, \epsilon z)$, where $a \in [1, M]_{\mathbb{Z}}, v \in \Sigma^{n-p}$, and $z \in \Sigma$. For simplicity, write $u_j$ for $u_{j, g}$.

a) If $v$ is lexicographically smaller (larger, resp.) than First$_{n-p}(s_a)$, then let $u_j = 0$ ($u_j = 1$, resp.). Next, assume $v =$ First$_{n-p}(s_a)$. In the case where either $a$ is good or $a \notin W$, obtain Last$_{n-p}(s_a)$ from items (iv) and (vi) and let $u_j = 1$ if $z \geq$ Last$_{n-p}(s_a)$ and let $u_j = 0$ otherwise.

b) Consider the case where $a$ is in $W$. Let $b$ be the index satisfying $a = w_b$. There are two cases to consider.
b-i) Assume that \( b < i \). Note that \( \text{Last}_p(s_a) \) has been obtained at an earlier round. Define \( u_j = 1 \) if \( z \geq \text{Last}_p(s_a) \) and \( u_j = 0 \) otherwise.

b-ii) If \( b \geq i \), then set \( u_j = 0 \). In particular, if \( b > i \), then Lemma 5.2 implies that \( wt_p(w_i : w_b, \text{First}_{n-p}(s_{w_i})) < C_{r} / m \) and \( wt_p(w_i : w_i, \text{First}_{n-p}(s_{w_i})) \leq C_{r} \).

3) Simulate \( V \) deterministically on input \( |\gamma_f(w_j)^r\rangle \). Find its output that is obtained with probability \( \geq 1/2 \). Such a string must be \( \text{Last}_p(s_{w_i}) \). With the known string \( \text{First}_{n-p}(s_i) \), this gives the entire string \( s_i \), as required.

4) Output the decoded string \( s = s_1 s_2 \cdots s_M \).

We need to verify that the decoding algorithm \( B \) correctly extracts \( s \) from \( E(s) \). If \( i \) is good, then \( s_i \) can be correctly obtained from item (iv). If \( i \) is bad and not in \( W \), then \( B \) computes \( \text{Last}_p(s_i) \). Henceforth, we assume that \( i \) is bad and in \( W \). Let \( j \) be the index satisfying \( i = w_j \). The operator \( U \) on input \( (w_j, f) \) generates the prequery quantum state \( |\gamma_f(w_j)^r\rangle \). We need to prove that our approximation of the true oracle answers from \( G_{M,N,p} \) suffices for the correct simulation of \( (U,V) \). Let \( |\gamma_f(w_j)^G\rangle \) be the true postquery quantum state; namely, \( \sum_{i} \langle \tilde{y} | G_{M,N,p}(y_1) \cdots G_{M,N,p}(y_T) | \phi_{w_j,f,\tilde{y}} \rangle \), where \( \tilde{y} = (y_1, \ldots, y_T) \). Now, we claim that \( |\gamma_f(w_j)^G\rangle \) is close to \( |\gamma_f(w_j)^r\rangle \).

**Lemma 5.5** \( \| |\gamma_f(w_j)^r\rangle - |\gamma_f(w_j)^G\rangle \| \leq 2 \sqrt{C_{r}} \).

**Proof.** Consider the set \( A \) of all query lists \( \tilde{y} \) that, for a certain number \( b \geq j \) and a certain string \( z \in \Sigma^p \), include a query word \( (w_b, \text{First}_{n-p}(s_{w_i}) z) \), which was dealt with at Step (b-ii). The value \( \| |\gamma_f(w_j)^r\rangle - |\gamma_f(w_j)^G\rangle \|^2 \) is estimated as follows:

\[
\| |\gamma_f(w_j)^r\rangle - |\gamma_f(w_j)^G\rangle \|^2 = \left\| \sum_{\tilde{y} \in A} \langle \tilde{y} | (u_1, \tilde{y}, \ldots, u_T, \tilde{y}) - | G_{M,N,p}(y_1) \cdots G_{M,N,p}(y_T) \rangle | \phi_{w_j,f,\tilde{y}} \rangle \right\|^2 \\
\leq \sum_{\tilde{y} \in A} \left( \| \langle \tilde{y} | (u_1, \tilde{y}, \ldots, u_T, \tilde{y}) | \phi_{w_j,f,\tilde{y}} \rangle \|^2 + \| \langle \tilde{y} | G_{M,N,p}(y_1) \cdots G_{M,N,p}(y_T) | \phi_{w_j,f,\tilde{y}} \rangle \|^2 \right),
\]

which equals \( \sum_{\tilde{y} \in A} 2 \| | \phi_{w_j,f,\tilde{y}} \rangle \|^2 \). This term is further bounded by:

\[
\sum_{\tilde{y} \in A} 2 \| | \phi_{w_j,f,\tilde{y}} \rangle \|^2 \leq 2 \sum_{b \geq j} wt_p(w_j : w_b, \text{First}_{n-p}(s_{w_b})) \\
\leq 2 |W| C_r / m + C_e = 4C_e.
\]

The second inequality follows from Lemma 5.2 and the bound \( wt_p(w_j : w_j, \text{First}_{n-p}(s_{w_j})) \leq C_e \). Therefore, we obtain \( \| |\gamma_f(w_j)^r\rangle - |\gamma_f(w_j)^G\rangle \|^2 \leq 4C_e \), which yields the lemma.

By Lemma 5.5 using the approximated oracle answer \( |\gamma_f(w_j)^r\rangle \) instead, the operator \( V \) produces a wrong solution with probability at most \( 2 \sqrt{C_e} + \epsilon \leq 1/2 - \epsilon \). In other words, \( V \) outputs the correct string \( \text{Last}_p(s_{w_i}) \) with probability at least \( 1/2 + \epsilon \). Since \( \epsilon \) is positive, the deterministic simulation of \( V \) correctly provides us with the true outcome of \( V \). This guarantees that \( B \) correctly outputs \( s \) from \( E(s) \). This ends the discussion of Case (2).

Combining Cases (1) and (2), we conclude that \( E \) is a length-decreasing one-to-one function from \( \Sigma^{Mn} \) to \( \Sigma^* \), contradicting the pigeonhole principle. This completes the proof of Proposition 5.2.

### 6 Query Complexity for Single Block Ordered Search

The single-block ordered search problem \( G_{1,N} \) has been extensively studied in the literature for the lower bound of its quantum adaptive query complexity. Upon quantum nonadaptive query computation, this section demonstrates a lower bound of the query complexity \( Q^{k,tt}(G_{1,M}) \) in the presence of advice. Our algorithmic argument again proves its usefulness. We follow the notations introduced in Section 5.

**Theorem 6.1** Let \( n \in \mathbb{N} \) and set \( N = 2^n \). For any \( \epsilon \in [0,1/2) \) and any \( c \in (0, d(\epsilon)) \), \( Q^{k,tt}_c(G_{1,N}) \geq C_c N / 2^{2k+2} \), where \( \epsilon' = (1 + c)\epsilon \) and \( C_c = (1 - 2\epsilon' )^2 / 16 \). 

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Proof. Letting \( p = k + 1 \), it suffices to show that \( Q^{k,t\ell}(G_{1,N,p}) \geq C_N^t N/2^{2k+2} \). Henceforth, we consider the case where \( k + 1 \leq n \) since, otherwise, the theorem trivially holds. Taking \( T < C_N^t N/2^{2k+2} \), we assume that there is a nonadaptive black-box quantum computer \( (U,V) \) that solves the problem \( G_{1,N,p} \) with error probability \( \leq \epsilon \) with advice of length \( k \) by \( T \) nonadaptive queries. Fix any string \( s \) of length \( n \) and let \( f \) be its corresponding advice string in \( \Sigma^k \). For simplicity, write \( w_p \) for \( wt_p(1:1, \text{First}_{n-p}(s)) \). We aim at defining an encoding scheme \( E \) that can be proven to be length-decreasing and one-to-one. We need to consider the following two cases separately: (1) \( w_p > C e \) and (2) \( w_p \leq C e \).

(Case 1: \( w_p > C e \)) The encoding \( E(s) \) consists of (i) the advice string \( f \), (ii) the \( p \)-bit string \( \text{Last}_p(s) \), and (iii) the number \( e = |\{a ∈ \Sigma^{n-p} | w_p(1:1,a) > C e \text{ and } a < \text{First}_{n-p}(s)\}| \) in binary. Similar to Lemma 5.8, \( e \) can be expressed with at most \( |\log(T/C e)| \) bits. Thus, the coding length \( |E(s)| \) is bounded above by:

\[
|E(s)| \leq k + p + \log(T/C e) + 1 = 2k + 2 + \log(T/C e) < n = |s|.
\]

The deterministic decoding algorithm for \( E(s) \) is given in a fashion similar to Case (1) in the proof of Proposition 6.2.

(Case 2: \( w_p \leq C e \)) In this case, the desired encoding \( E(s) \) includes two items: (i) \( f \) and (ii) \( \text{First}_{n-p}(s) \). The length of \( E(s) \) equals \( k + n - p \), which is obviously \( n - 1 \) and is clearly less than \( |s| \). In the following deterministic manner, we uniquely extract \( s \) from \( E(s) \). Simulate \( U \) deterministically to generate query lists. Using the information \( \text{First}_{n-p}(s) \), we can determine the true oracle answer to any query word whose first \( n - p \) bits are different from \( \text{First}_{n-p}(s) \). For any other query word, we can replace its true oracle answer by its estimation 0. Such a replacement may increase the error probability of \( V \). As in Case (2) of the proof of Proposition 6.2, nonetheless, the probability that \( V \) produces a wrong solution is bounded above by \( 2\sqrt{C e} + \epsilon = 1/2 - cc \) since \( w_p \leq C e \). Hence, the simulation of \( V \) in a deterministic manner helps find the right solution \( s \). We therefore extract \( s \) from \( E(s) \) successfully.

Cases (1) and (2) imply that \( E \) is length-decreasing and also one-to-one. This obviously leads to a contradiction against the pigeonhole principle and we thus obtain the theorem. \( \square \)

For the special case where \( k = 0 \) and \( \epsilon = 1/3 \), Theorem 6.1 gives a lower bound \( Q^{t\ell}(G_{1,N}) \geq \Omega(N) \), which is optimal if we ignore its constant multiplicative factor since \( Q^{t\ell}(G_{1,N}) \) is at most \( N - 1 \).

As for the single-block ordered search problem, we can also employ a quantum adversary argument to prove its quantum nonadaptive query complexity. Particularly, using an inner product method of Høyer et al. 24, we again obtain a similar lower bound of \( Q^{k,t\ell}(G_{1,N}) \).

**Proposition 6.2** For any constant \( \epsilon \in [0, 1/2) \), \( Q^{k,t\ell}(G_{1,N}) \geq (1 - 2\sqrt{\epsilon}(1-\epsilon))(N/2^k - 1) \).

**Proof.** Assume that a nonadaptive black-box quantum computer \((U,V)\) needs \( T \) queries to solve the problem \( G_{1,N} \) with error probability \( \leq \epsilon \) using advice of length \( k \). For each step \( s \), \( \hat{s} \) denotes the input string \( g^{s-1}1^{N-s+1} \) to the black-box quantum computer and let \( f(s) \) denote the advice string that minimizes the number of queries used for solving \( G_{1,M} \) on the input \( \hat{s} \). For each \( d ∈ Σ^k \), define the set \( A_d = \{ s ∈ [1,N] | f(s) = d \} \) of advice strings.

Now, consider an advice string \( a ∈ Σ^k \) whose cardinality is at least \( N/2^k \). Write \( b = |A_a| \) and assume that \( A_a = \{ s_1, s_2, \ldots, s_b \} \) with \( s_1 < s_2 < \cdots < s_b \). Recall that \( O_s \) denotes the unitary operator representing \( a \). Given an input \( \hat{s} \), the final quantum state of the black-box quantum computer \((U,V)\) is \( |ψ_s⟩ = VO_s U|a⟩ \). For any indices \( i,j ∈ [1,b] \), let \( I(i,j) \) be the inner product between \( |ψ_{s_i}⟩ \) and \( |ψ_{s_j}⟩ \). We focus on the variable \( \zeta = \sum_{t=1}^{b-1} |I(t,t+1)| \). Note that our assumption yields an upper bound \( |I(t,t+1)| ≤ 2\sqrt{\epsilon}(1-\epsilon) \) due to [3] 27. This gives an upper bound \( \zeta ≤ 2\sqrt{\epsilon}(1-\epsilon)(b - 1) \). We next show a lower bound of \( \zeta \). The prequery state of the machine is \( U(a) = \sum_i{\alpha_i}|i⟩|0⟩ \), where \( i = (i_1, \ldots, i_T) \) corresponds to \( T \) query words and \( z \) represents work bits. Hence, \( \zeta = \sum_{t=1}^{b-1} \sum_{i_t} |α_{i_t}|^2 \prod_{j=1}^{t-1} (|\hat{s}_{i_j}⟩_i)(|\hat{s}_{i_{t+1}}⟩_i) \). By the choice of \( s_t \) and \( s_{t+1} \), the inner product
\[ \langle \delta_t \rangle_{i_j} \langle \delta_{t+1} \rangle_{i_j} \text{ becomes 0 if } i_j \in [s_t, s_{t+1} - 1] \text{ and 1 otherwise.} \]

The term \( \zeta \) is then estimated as:

\[
\zeta = \sum_{t=1}^{b-1} \sum_{i_2 \in [s_t, s_{t+1} - 1]} \ldots \sum_{i_T \in [s_T, s_{T+1} - 1]} |\alpha_{1,t,z}^i|^2
\geq \sum_{t=1}^{b-1} \sum_{i_2 \in [s_t, s_{t+1} - 1]} \ldots \sum_{i_T \in [s_T, s_{T+1} - 1]} |\alpha_{1,t,z}^i|^2 \sum_{j=1}^{T} \left( \sum_{t=1}^{b-1} \sum_{u=0}^{s_t - s_{t+1} - 1} |\eta_{j,s_t+u,z}|^2 \right),
\]

where \( \eta_{j,w} \) denotes \( (i_1, i_2, \ldots, i_{j-1}, w, i_{j+1}, \ldots, i_T) \). Since \( \sum_{i,w} |\alpha_{1,t,z}^i|^2 = 1 \), the above inequality implies a lower bound \( \zeta \geq (b-1) - T \).

The above two bounds of \( \zeta \) derive the inequality \( 2\sqrt{\epsilon(1-\epsilon)}(b-1) \geq b-1 - T \), which immediately implies \( T \geq (1 - 2\sqrt{\epsilon(1-\epsilon)})(b-1) \geq (1 - 2\sqrt{\epsilon(1-\epsilon)})(N/2^k - 1) \).

We note that it is not clear whether the above inner product method can be extended to the multiple-block ordered search problem.

7 Other Applications of an Algorithmic Argument

We have shown in the previous sections how our algorithmic argument proves query complexity lower bounds. Hereafter, we apply our algorithmic argument to two notions of computational complexity theory: quantum truth-table reducibility and quantum truth-table autoreducibility. Particularly for quantum truth-table autoreductions, we describe our algorithmic argument using the notion of space-bounded Kolmogorov complexity.

7.1 Quantum Truth-Table Reducibility

The first example is a nonadaptive oracle separation between \( P \) and \( BQP/poly \). Earlier, Buhrman and van Dam \cite{buhrman1995quantum} and Yamakami \cite{yamakami2001oracle} investigated quantum parallel query computations (i.e., all query words are pre-determined before the first oracle query). It is shown in \cite{yamakami2001oracle} that there exists an oracle relative to which polynomial-time classical adaptive query computation is more powerful than polynomial-time quantum parallel query computation.

We have introduced a black-box quantum computer as a nonuniform model of computation. To describe nonadaptive \( BQP \)-computations, we need a uniformity notion. For simplicity, we introduce a “uniform” model of quantum truth-table query computation by simply replacing a nonadaptive black-box quantum computer \((U, V)\) with a pair \((M, N)\) of polynomial-time multi-tape well-formed quantum Turing machines\(^{1}\) (QTMs, in short). The notion of a QTM was introduced by Deutsch \cite{deutsch1985quantum} and later reformulated by Bernstein and Vazirani \cite{bernstein1993quantum}. A \( k \)-tape QTM \( M \) is a 6-tuple \((Q, \Sigma, \Gamma, q_0, q_f, \delta)\), where \( Q \) is a finite set of inner states, \( \Sigma \) is a finite alphabet, \( \Gamma \) is a finite input tape alphabet, \( q_0 \) is the initial inner state in \( Q \), \( q_f \) is the final (or halting) inner state in \( Q \), and \( \delta \) is a quantum transition function dictating the behavior of the machine \( M \). This function \( \delta \) induces the unitary operator acting on the Hilbert space spanned by the basis set consisting of all configurations of \( M \). We assume the reader’s familiarity with QTMs (e.g., see \cite{buhrman1995quantum} \cite{deutsch1985quantum} \cite{yamakami2001oracle} for more details).

A pair \((M, N)\) of QTMs recognizes a language \( L \) with oracle \( A \) in the following fashion similar to a nonadaptive black-box quantum computer. The machine \( M \) is equipped with at least two input tapes: one of which carries an original input and another does an advice string. On any input \( x \), \( M \) generates a prequery quantum state \( |\phi\rangle = \sum_{i_1, \ldots, i_T} |i_1, \ldots, i_T\rangle |0^T\rangle |\psi_{i_1, \ldots, i_T}\rangle \), which may depend on \( x \). In a single step, the oracle \( A \) answers all the queries by transforming \( |\phi\rangle \) to the postquery quantum state \( |\phi'\rangle = \sum_{i_1, \ldots, i_T} |i_1, \ldots, i_T\rangle A(i_1) \cdots A(i_T) |\psi_{i_1, \ldots, i_T}\rangle \). Finally, \( N \) begins with \( |\phi'\rangle \) as its initial superposition and eventually produces \( L(x) \) in the output tape with probability \( \geq 2/3 \). For our convenience sake, we henceforth call such a pair \((M, N)\) a truth-table query QTM.

The relativized complexity class \( BQP_{tt}^A \) relative to oracle \( A \) is then defined as the collection of all sets recognized by polynomial-time truth-table query QTMs in the aforementioned manner using \( A \) as an oracle. The class \( BQP_{tt} / poly \) uses, in addition, polynomial advice. Note that \( BQP_{tt}^A \subseteq BQP^A \) for any oracle set \( A \).

Applying the result of Section 5, we can show the following theorem. For its proof, we fix an effective enumeration \( \{p_i\}_{i \in \mathbb{N}} \) of all polynomials and an effective enumeration \( \{(M_i, N_i)\}_{i \in \mathbb{N}} \) of such polynomial-time truth-

\(^{1}\) Alternatively, we can use a uniform family of polynomial-size quantum circuits. The equivalence of a multiple-tape QTM model and a quantum circuit model follows from \cite{buhrman1995quantum} \cite{yamakami2001oracle} \cite{deutsch1985quantum}.
table query QTMs, each of which $(M_i, N_i)$ runs in time at most $p_i(n)$ on any input of length $n$. Moreover, for any set $A$ and any number $n \in \mathbb{N}$, the notation $A[n]$ denotes the $2^n$-bit string $A(\text{bin}_n(1))A(\text{bin}_n(2)) \cdots A(\text{bin}_n(2^n))$.

**Theorem 7.1** There is a recursive oracle $A$ such that $P^A \not\subseteq \text{BQP}^A_{tt}/\text{poly}$.

**Proof.** We first define $\mathcal{A}$ as the collection of all oracles $A$ such that, for every $n \in \mathbb{N}$, there exist $2^n$ steps $s_1, s_2, \ldots, s_{2^n} \in [1, 2^{3n}]_2$ satisfying $A[4n] = s_1 s_2 \cdots s_{2^n}$, where each $s_i$ denotes $0^{s_i-1}1^{2^{3n} - s_i + 1}$. Using any oracle $A$ drawn from $\mathcal{A}$, we define the oracle-dependent set $L^A$ to be the collection of all strings of the form $\text{bin}_n(i)$, where $n \in \mathbb{N}$ and $i \in [1, 2^n]_2$, such that $A[4n] = s_1 s_2 \cdots s_{2^n}$ for certain $2^n$ steps $s_1, s_2, \ldots, s_{2^n} \in [1, 2^{3n}]_2$ and $s_i \equiv 1 \pmod{2}$. Given any oracle $A \in \mathcal{A}$ and any input $\text{bin}_n(i)$, we can easily find the step $s_i$ deterministically by binary search over the set $A[4n]$ in polynomial time. Note that binary search requires adaptive queries to $A$. It thus follows immediately that $L^A$ belongs to $P^A$ for an arbitrary oracle $A \in \mathcal{A}$.

To prove the theorem, we want to construct a special oracle $A \in \mathcal{A}$ that places $L^A$ outside of BQP$_{tt}/$poly by diagonalizing against all polynomial-time truth-table query QTMs. Such an oracle $A$ will contain strings of certain lengths in $A$, where $A = \{\eta_j\}_{j \in \mathbb{N}}$ is any fixed subset of $\mathbb{N}$ satisfying that $n_{j+1} > p_j(n_j)$ for any number $j$ in $\mathbb{N}$. By stages, we build the desired set $A = \bigcup_{j \in \mathbb{N}} A_j$. We set $A_0 = \emptyset$ at stage 0. At stage $j \geq 1$, we focus our attention on the $j$th machine $(M_j, N_j)$ and henceforth let $n = n_j$ for simplicity. Intuitively, Proposition 5.2 implies, by taking $p = 1, l = 2^n/2$ and $k = 2^{n/2}$, that any “truth-table query QTM” solving the multiple-block ordered search problem $G_{2^n, 2^{3n} - 1}$ requires at least $2^{n/3}$ nonadaptive queries even with the use of advice of length $2^{n/3}$. In other words, there exist a block number $i \in [1, 2^n]_2$ and a series of $2^n$ steps $s_1, s_2, \ldots, s_{2^n} \in [1, 2^{3n}]_2$ such that, given instance $(i, s_1 \cdots s_{2^n}) \in [1, 2^n]_2 \times \Sigma^{2^n}$, $(M_j, N_j)$ needs $2^{n/3}$ nonadaptive queries on input $\text{bin}_n(i)$ to compute the value $s_i \mod 2$ with high success probability even with the help of any advice string of length $2^{n/3}$. Choose the minimal such series $(s_1, s_2, \ldots, s_{2^n})$ and define $A_j$ to satisfy $A_j[4n] = s_1 \cdots s_{2^n}$. This intuitive argument, nevertheless, ignores the fact that $M_j$ may make queries of words of length less than or greater than $4n$. To deal with such queries, we need to modify the proof of Proposition 5.2 in the way described below. Since $n_{j+1} > p_j(n_j)$, we can answer 0 to all the queries of length $> n$. Notice that any future stage will not affect the machine’s behavior on input $\text{bin}_n(i)$. When $M_j$ queries a word $y$ of length $< n$, we deterministically re-construct the oracle $A \cap \Sigma^{\leq n-1}$ and compute the oracle answer $A(y)$ using $A \cap \Sigma^{\leq n-1}$. This modification adds only an extra additive constant term to the encoding size given in the proof of Proposition 5.2. Hence, the main assertion of Proposition 5.2 is still valid and $(M_j, N_j)$ cannot recognize $L^A \cap \Sigma^{\leq n}$.

The desired set $A = \bigcup_{j \in \mathbb{N}} A_j$ clearly belongs to $\mathcal{A}$. Our construction further guarantees that $L^A$ is not in BQP$_{tt}/$poly. Moreover, $A$ can be recursive since every $(M_i, N_i)$ is a uniform model and the proof of Proposition 5.2 is constructive.

**7.2 Quantum Truth-Table Autoreducibility**

As the second example, we focus our interest on the notion of autoreducible sets. After Trakhtenbrot 37 brought in the notion of autoreduction in recursion theory, the autoreducible sets have been studied in, e.g., program verification theory. In connection to the program checking of Blum and Kannan 44, Yao 40 is the first to study BPP-autoreducible sets under the name “coherent sets,” where a set $A$ is BPP-autoreducible if there is a polynomial-time oracle probabilistic Turing machine (PTM, in short) with oracle $A$ which determines whether any given input $x$ belongs to $A$ with probability $\geq 2/3$ without querying the query word $x$ itself. Let BPP-AUTO denote the class of all BPP-autoreducible sets. Yao showed that $\text{DSpace}(2^{\log \log n}) \not\subseteq \text{BPP-AUTO}$. Later, Beigel and Feigenbaum 10 presented a set in ESPACE which is not BPP-autoreducible even with polynomial advice. If only nonadaptive queries are allowed in the definition of a BPP-autoreducible set, it is specifically called nonadaptively BPP-autoreducible. Feigenbaum, Fortnow, Laplante, and Naik 25 gave an adaptively BPP-autoreducible set which is not nonadaptively BPP-autoreducible even with polynomial advice.

We consider a quantum analogue of nonadaptively BPP-autoreducible sets, called BQP-tt-autoreducible sets, where “tt” is used to emphasize the nature of “truth-table” queries rather than parallel queries. Formally, we obtain a BQP-tt-autoreducible set by replacing a polynomial-time PTM in the above definition by a polynomial-time truth-table query QTM $(M, N)$, provided that any prequery quantum state produced by $M$ on each input $x$ does not include the query word $x$ with nonzero amplitude. Let BQP$_{tt}$/poly-AUTO be the class of all BQP-tt-autoreducible sets with polynomial advice.

We prove the following separation, which extends the aforementioned result of Feigenbaum et al. 25.
Theorem 7.2 \( \text{BPP-AUTO} \notin \text{BQP}_{tt}/\text{poly-AUTO} \).

Proof. From the proof of Theorem \ref{thm:bpp-auto} we recall the collection \( \mathcal{A} \) of oracles. In addition, we introduce the oracle-dependent set \( K_A \) for any oracle \( A \in \mathcal{A} \) as follows. For any number \( n \in \mathbb{N} \) and any two indices \( i \in [1, 2^n] \) and \( j \in [1, 2^{3n}] \), write \( (i, j)_n \) to denote \((i - 1) \cdot 2^{3n} + j\), which indicates the th\( i \)th location in the th\( i \)th block. For any string \( x \) of the form \( \text{bin}_n((i, j)_n) \), where \( n \in \mathbb{N} \), \( i \in [1, 2^n] \), and \( j \in [1, 2^{3n}] \), \( x \) is in \( K_A \) if either (i) \( j = 1 \) and \( s_i \equiv 1 \pmod{2} \) or (ii) \( j \neq 1 \) and \( A(x) = 1 \).

We first claim that \( K_A \) belongs to BPP-AUTO for any choice of \( A \) from \( \mathcal{A} \). For any nonempty string \( x \in \Sigma^* \), let \( x^+ \) \((x^-)\) denote the lexicographic successor \((\) predecessor \()\) of \( x \). Let \( x \) be an arbitrary string of the form \( \text{bin}_n((i, j)_n) \) for a certain choice of \( n \in \mathbb{N} \), \( i \in [1, 2^n] \), and \( j \in [1, 2^{3n}] \). When \( j = 2^{3n} \), we immediately output 1. If \( j \) is in \([2^{3n} - 1, 2^{3n}] \), then we first make two queries \( x^- \) and \( x^+ \) to the set \( K_A \) given as an oracle. If \( K_A(x^-) = K_A(x^+) \), then we output \( K_A(x^+) \). On the contrary, if \( K_A(x^-) < K_A(x^+) \), then we make an additional query \( \text{bin}_n((i, 1)_n) \) to \( K_A \) and output its oracle answer. In the last case where \( j = 1 \), we perform binary search over the set \( K_A[4n] \) to determine the th\( i \)st step \( s_1 \) and output the value \( s_1 \mod 2 \), which equals \( K_A(x) \).

Next, we show that, for a certain choice of \( A \) from \( \mathcal{A} \), \( K_A \) does not belong to \( \text{BQP}_{tt}/\text{poly-AUTO} \). We wish to construct such an \( A \) by stages. At each stage, we choose a new polynomial-time truth-table query QT \( (M, N) \) and also take an input \((i, 1)_n \) which is large enough for the diagonalization below. Assume that \((M, N)\) computes \( K_A((i, 1)_n) = (s_i \mod 2) \) with high probability. First, recall the definition of the weight \( wt_1(i : i', d) \), which denotes the sum of the squared magnitudes of the amplitudes of all vectors \( |\vec{y}\rangle |0\rangle |\phi_i, \vec{y}\rangle \) in the prequery state \( \text{M}[x] \) such that \( \vec{y} \) contains either query word \( \text{bin}_n((i', 2d - 1)_n) \) or \( \text{bin}_n((i', 2d)_n) \) for a certain number \( d \in [1, 2^{3n - 1}] \). Since our oracle is \( K_A \) instead of \( A \), we need to modify \( wt_1(i : i', d) \) by adding the squared magnitudes of the amplitudes of all states \(|\vec{y}\rangle |0\rangle |\phi_i, \vec{y}\rangle \) in which \( \vec{y} \) contains \((i', 1)_n \). A proof similar to that of Proposition \ref{prop:qspace}, together with a slight modification given in the proof of Theorem \ref{thm:bpp-auto} works to prove a superpolynomial lower bound of the nonadaptive query complexity on the computation of \((M, N)\). Therefore, there exists an input \((i, 1)_n \) on which \((M, N)\) fails to compute \( K_A((i, 1)_n) \) with high probability. \( \square \)

Beigel and Feigenbaum \cite{BeigelFeigenbaum} showed that \( \text{ESPACE} \notin \text{BPP}/\text{poly-AUTO} \). It is also proven in \cite{Stern} that \( \text{ESPACE} \notin \text{BQP}/\text{poly} \). In addition to these results, we show the existence of a set in \( \text{ESPACE} \) which is not \( \text{BQP}_{tt}\)-autoreducible even with polynomial advice. To show this, we apply our bounded compression algorithm.

Theorem 7.3 \( \text{ESPACE} \notin \text{BQP}_{tt}/\text{poly-AUTO} \).

Note that Theorem \ref{thm:espace-bpp} is incomparable to the aforementioned results in \cite{BeigelFeigenbaum, Stern}. To prove Theorem \ref{thm:espace-bpp} we first prove a key lemma on the space-bounded Kolmogorov complexity of any set in \( \text{BQP}_{tt}/\text{poly-AUTO} \). We need to fix a universal \((\text{deterministic})\) Turing machine \( \mathcal{M}_U \) in the rest of this section. Let \( q \) be any function mapping from \( \mathbb{N} \) to \( \mathbb{N} \). The conditional \( q \)-space bounded Kolmogorov complexity of \( x \) conditional to \( s \), denoted \( \text{C}^q(x|s) \), is the minimal length of any binary string \( w \) such that, on input \((w, s)\), \( \mathcal{M}_U \) produces \( x \) in its output tape using space at most \( q(|x| + |s|) \) \((\) see, e.g., \cite{Levin} for more details \()\). We now present the following technical lemma.

Lemma 7.4 Let \( A \) be any set in \( \text{BQP}_{tt}/\text{poly-AUTO} \) with a polynomial advice function \( h \) such that \( A \subseteq \bigcup_{n \in \text{Tower2}} \Sigma^n \) and that the number of queries to \( A \) is \( t(n) \) on any input of length \( n \) for any \( n \in \mathbb{N} \). There exist a polynomial \( q \) and a constant \( c \geq 0 \) such that, for any sufficiently large number \( n \in \text{Tower2} \), the space-bounded Kolmogorov complexity \( \text{C}^q(A[n]|h(n)) \) is bounded above by \( 2^n - m + 2n + 2 \log n + c \), where \( m \) is the positive solution of \( 288t(n)m^2 - (288t(n - 1)m - 2^n) = 0 \).

Proof. Let \( A \) be any set in \( \text{BQP}_{tt}/\text{poly-AUTO} \) with a polynomial advice function \( h \). There exist a polynomial \( t \) and a polynomial-time truth-table query QT \( (M, N) \) such that (i) on input \((x, h(|x|))\), \( M \) outputs the prequery state \( |\gamma\rangle = \sum_{y} |\vec{y}\rangle |0^t(|x|)|\phi_{\vec{y}}\rangle \), (ii) \( wt(x : z) = 0 \), and (iii) \( N(|\gamma_A\rangle) \) outputs \( A(x) \) with error probability at most \( 1/3 \), where \( |\gamma_A\rangle = \sum_{\vec{y}} |\vec{y}\rangle |A(y_1)A(y_2)\cdots A(y_t(|x|))\rangle |\phi_{\vec{y}}\rangle \) for \( \vec{y} = (y_1, y_2, \ldots, y_t(|x|)) \). For any pair \( x, z \in \Sigma^n \), we write \( wt(x : z) \) for the sum of all squared magnitudes \( \| |\vec{y}\rangle |0^t(|x|)\rangle |\phi_{\vec{y}}\rangle \|^2 \) over all query lists \( \vec{y} \) that contain \( z \). Take any sufficiently large integer \( n \) and fix it.

Similar to the proof of Theorem \ref{thm:bpp-auto} we need to deal with \( M \)'s query words by simulating \( M \)'s computation in a deterministic manner. Note that, since \( M \)'s running time is polynomially bounded, for any sufficiently large number \( n \), \( M \) cannot make any query of length \( \geq 2^n \). Moreover, when \( M \) queries words of length between
log $n + 1$ and $n - 1$, since $A \subseteq \bigcup_{i=2}^{2^n} \Sigma^n$, we know that the oracle answers negatively. Only query words of length $\leq \log n$ need our attention.

Consider the following deterministic procedure \textsc{LWSS2}. For convenience, abbreviate $t(n)$ as $t$ in the rest of the proof. Let $m$ be the positive solution of $288tm^2 - (288t - 1)m - 2^n = 0$. Note that $\sqrt{2^{n+576t}} \leq m \leq 2^n$.

Procedure \textsc{LWSS2}: Initially, set $R_1 = 0$ and $L_1 = \Sigma^n$. Repeat the following procedure by incrementing $i$ by one while $i \leq m$. At round $i$ ($1 \leq i \leq m$), choose the lexicographically smallest string $w_i$ in $L_i - R_i$.

Simulate $M$ deterministically on input $(w_i, h(n))$ to generate $|\gamma_i| = \sum_y |\gamma_i|^0(\phi_y)$ for each query word $y$, compute its weight $wt(w_i : y)$. Define $R_{i+1} = R_i \cup \{w_i\}$ and $L_{i+1} = L_i \cap \{y \in \Sigma^n \mid wt(w_i : y) < \frac{1}{2^{288m}}\}$. Finally, set $W = R_{m+1}$. Output all the elements in $W$.

Note that procedure \textsc{LWSS2} uses space $2O(n)$ since it deterministically simulates all computation paths of $M$ one by one and computes the weights of query words along these paths and stores the contents of $L_i$ and $R_i$.

The following lemma can be proven similar to Lemma 7.4.

Lemma 7.5 \textsc{LWSS2} produces a unique series of $m$ distinct strings $w_1, w_2, \ldots, w_m$ such that, for any pair $i, j \in [1, m]$, $j > i$ implies $wt(w_i : w_j) < \frac{1}{2^{288m}}$.

For each $i \in [1, 2^n - m]$, let $v_i$ be the lexicographically $i$th element in the difference set $\Sigma^n - W$ and set $r = A(v_1)A(v_2) \cdots A(v_{2^n - m})$. Recall the notation $A[n] = A(\text{bin}_n(1)) \cdots A(\text{bin}_n(2^n))$ and define $z = A[n]A[1] \cdots A[\log n]$, which contains all the information on $A \cap \Sigma^{\leq \log n}$. Note that $|z| = 2n - 1$ if $n \geq 1$.

Consider the following deterministic algorithm $C$ that produces $A[n]$ on input $h(n)$ and extra information on $n, r$, and $z$.

Algorithm $C$: 1) On input $h(n)$, retrieve the hardwired number $n$ and the strings $r = r_1r_2 \cdots r_{2^n - m}$ and $z = z_1z_2 \cdots z_{\log n}$, where each $r_i$ is in $\{0, 1\}$ and each $z_i$ is in $\Sigma'$. First, run \textsc{LWSS2} to obtain $W = \{w_1, w_2, \ldots, w_m\}$.

2) Choose every string $y$ in $\Sigma^n - W$ lexicographically one by one and find the number $k$ such that $y = v_k$. Clearly, $r_k$ matches $A(y)$.

3) In this phase, we compute all the values $A(w_i)$ for $i \in [1, m]$. Repeat the following procedure. At round $i$ ($1 \leq i \leq m$), assume that the $i - 1$ values $A(w_1), A(w_2), \ldots, A(w_{i-1})$ have been already computed. Simulate $M$ on input $(w_i, h(n))$ deterministically to generate $|\gamma_i| = \sum_y |\gamma_i|^0(\phi_y)$. Using the extra information $r$, generate the vector $|\gamma|^r = \sum_{y} |\gamma|^y u_1 u_2 \cdots u_t(\phi_y)$ as follows. Let $\vec{y} = 1 y_1 y_2 \cdots y_h$ be any query list in $|\gamma_i|$. For each $j \in [1, t]$, we determine the value $u_j$ as follows.

i) In the case where $y_j \in W$, first find the index $k$ such that $y_j = v_k$. If $k < i$, let $u_j$ be the value $A(w_k)$ and otherwise, set $u_j = 0$. Note that $wt(w_i : w_k) < \frac{1}{2^{288m}}$ by Lemma 7.5.

ii) If $y_j \not\in W$, then we choose $k$ such that $y_j = v_k$ and define $u_j = r_k$.

iii) Assume that $|y_j| \neq n$. As noted before, since $|y_j| < 2^n$, if $|y_j| > n$ then let $u_j = 0$. If $\log n < |y_j| < n$, then let $u_j = 0$. Assume that $|y_j| \leq \log n$. Assume that $y_j$ is the lexicographically $k$th string in $\Sigma^{\leq \log n}$. In this case, let $u_j$ be the $k$th bit of $z$.

Finally, simulate $N$ on input $|\gamma|^r$ deterministically. There exists the unique output $z$ that is obtained by $N$ with probability $> 1/2$. This must be $A(w_i)$.

4) Finally, output the $2^n$-bit string $A(\text{bin}_n(1))A(\text{bin}_n(2)) \cdots A(\text{bin}_n(2^n))$ and halt.

Algorithm $C$ uses $2O(n)$ space on input $(h(n), n, r, z)$ since \textsc{LWSS2} requires only $2O(n)$ space. Hence, we can choose an appropriate polynomial $q$ satisfying that $C$ runs using space at most $q(2^n)$ for any $n \in \mathbb{N}$.

Now, we wish to prove that $C$ correctly produces $A[n]$. Let $i \in [1, 2^n]$. If $\text{bin}_n(i) \notin W$, then $A(\text{bin}_n(i))$ is directly obtained from $r$. Assuming $\text{bin}_n(i) \in W$ and let $j$ satisfy $w_j = \text{bin}_n(i)$. On input $(w_j, h(n))$, $C$ generates the quantum state $|\gamma|^r$. By a calculation similar to Lemma 5.8, it follows that $|||\gamma|^r - |\gamma|^A||^2 \leq 2 \cdot |W| \cdot \frac{1}{2^{288m}} = 1/144$. Thus, the error probability of $N$ is at most $\sqrt{1/144 + 1/3} = 5/12$. This implies that the output bit obtained by $N$ with probability $> 1/2$ matches the true value $A(\text{bin}_n(i))$. Therefore, $C$ correctly outputs $A(\text{bin}_n(i))$ since $C$ is deterministic.

Recall that $C$ uses the hardwired information $(n, r, z)$, which is given as the concatenation of the following four items: (i) the string expression of $n$ in double binary, (ii) a separator 01, (iii) the string $r$, and (iv) the string $z$. Simulating $C$ on the universal machine $M_U$, we obtain $C^q(A[n]h(n)) \leq 2\log n + |r| + |z| + c \leq 2\log n + |r| + |z| + c \leq 2\log n$.
$2^n - m + 2n + 2\log n + c$, where $c$ is a certain nonnegative constant independent of the choice of $n$. This completes the proof. □

Using Lemma 4.3 and a diagonalization method, we finally prove the desired theorem. In the following proof, we assume that a standard pairing function $(\cdot, \cdot)$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ with $(0,0) = 0$.

Proof of Theorem 5.3 We want to construct a set $A$ in ESPACE by stages. To simplify the proof, we fix the set $\{p_i\}_{i \in \mathbb{N}}$ of polynomials that satisfy $p_i(n) = n^{i+1} + i$ for all $n \in \mathbb{N}$. Note that, for any two indices $i,j \in \mathbb{N}$ and any string $y \in \Sigma^*$ (i) $C^p(y|z) \geq C^{p_{i+1}}(y|z)$ for any string $z$ and (ii) $\min_{z : |z| \leq p_j(n)} C^p(y|z) \geq \min_{z' : |z'| \leq p_{i+1}(n)} C^p(y|z')$ for any number $n \in \mathbb{N}$. Initially, set $n_0 = 1$ and $A_0 = \emptyset$ at stage 0. At stage $k = (i,j) \geq 1$, take the minimal number $n_k$ in Tower2 such that $n_k > n_{k-1}$, $2^{n_k/4} \geq p_j(n_k)+1$, and $2^{n_k/3} > 6n_k$. Take also the lexicographically minimal $2^{n_k}$-bit string $y$ such that $C^p(y|z) \geq 2^{n_k} - 2^{n_k/4}$ for all $z \in \Sigma^{\leq p_j(n_k)}$. We then define $A_k$ so that $A_k[n_k] = y$. Such a $y$ exists because of the following lemma.

Lemma 7.6 There exists a string $y \in \Sigma^{2^{n_k}}$ such that $C^p(y|z) \geq 2^{n_k} - 2^{n_k/4}$ for every $l \leq i$, $m \leq j$, and $z \in \Sigma^{\leq p_m(n_k)}$.

Proof. By our choice of polynomials, it suffices to prove the lemma for $l = i$ and $m = j$. Let $g = 2^{n_k} - p_j(n_k) - 1$. The definition of $n_k$ implies that $g \geq 2^{n_k} - 2^{n_k/4}$. Now, we assume otherwise that, for every $y \in \Sigma^{2^{n_k}}$, there exist a string $z \in \Sigma^{\leq p_j(n_k)}$ and a program $w \in \Sigma^g$ such that $M_U(w,z)$ outputs $y$ using space at most $p_i(|y| + |z|)$. For each $y$, define $B_y$ as the collection of all pairs $(w,z) \in \Sigma^g \times \Sigma^{\leq p_j(n_k)}$ such that $M_U(w,z)$ outputs $y$ using space $\leq p_i(|y| + |z|)$. Since $|B_y| \geq 1$ for every $y \in \Sigma^{2^{n_k}}$, we obtain the inequality $2^{2^{n_k}} \leq \sum_{y : |y| = 2^{n_k}} |B_y|$. Note that, for any pair $y, y' \in \Sigma^{2^{n_k}}$, $B_y \cap B_{y'} = \emptyset$ if $y \neq y'$. A simple estimation thus shows that:

$$\sum_{y : |y| = 2^{n_k}} |B_y| = \bigcup_{y : |y| = 2^{n_k}} B_y \leq \left| \Sigma^g \times \Sigma^{\leq p_j(n_k)} \right| = (2^g - 1)(2^{p_j(n_k)+1} - 1) < 2^{p_j(n_k)+g+1} = 2^{2^{n_k}},$$

which leads to a contradiction. Therefore, the lemma holds. □

Finally, the set $A$ is defined as the union $\bigcup_{i \in \mathbb{N}} A_i$. By our construction, at each stage $k$, we need only $2^{O(n_k)}$ space to compute $A_k$ because $M_U$ uses $2^{O(n_k)}$ space to find the minimal string $y$ that satisfies Lemma 7.6. It thus follows that $A$ belongs to ESPACE.

Next, we want to show that $A$ is not in BQP_{hit}/poly-AUTO. Assume to the contrary that $A$ is in BQP_{hit}/poly-AUTO. There exists a polynomial-time truth-table query QTM $(M,N)$ that recognizes $A$ by polynomially-many nonadaptive queries with a polynomial advice function $h$. Lemma 4.6 yields the existence of a polynomial $p$ and a constant $c$ satisfying that $C^p(A[|n|]|h(n)) \leq 2^n - 2^{n_k/3} + 2n + 2\log n + c$ for any sufficiently large number $n$ in Tower2. Choose two indices $l$ and $m$ such that $p(n) \leq p_l(n)$ and $|h(n)| \leq p_m(n)$ for all numbers $n \in \mathbb{N}$. Now, take any numbers $i$ and $j$ such that $i \geq l$, $j \geq m$, $n_k \geq 2\log n_k + c$, and the number $k = (i,j)$ is sufficiently large. By the choice of $n_k$, there exists a string $z \in \Sigma^{\leq p_m(n_k)}$ satisfying that $C^p(A[|n_k|]|z) \leq 2^{n_k} - 2^{n_k/3} + 3n_k < 2^{n_k} - 2^{n_k/4}$. This clearly contradicts Lemma 7.6. □

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