Quantum Symmetry Reduction for Diffeomorphism Invariant Theories of Connections

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Abstract

Given a symmetry group acting on a principal fibre bundle, symmetric states of the quantum theory of a diffeomorphism invariant theory of connections on this fibre bundle are defined. These symmetric states, equipped with a scalar product derived from the Ashtekar-Lewandowski measure for loop quantum gravity, form a Hilbert space of their own. Restriction to this Hilbert space yields a quantum symmetry reduction procedure in the framework of spin network states the structure of which is analyzed in detail.

Three illustrating examples are discussed: Reduction of 3 + 1 to 2 + 1 dimensional quantum gravity, spherically symmetric quantum electromagnetism and spherically symmetric quantum gravity.

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1 Introduction

During the last years there have been many very active and partially successful attempts to quantize gravitational systems: One very ambitious approach is string theory (see, e.g. Ref. [1]) which tries to cover the quantum theory of all interactions “at once”. Another one is loop quantum gravity (see the reviews [2]) which uses a (non-perturbative) canonical framework (Cauchy surfaces, canonical coordinates and momenta, constraints etc.) for diffeomorphism invariant gravity formulated in terms of connection variables, without or with couplings to matter.

Among its achievements loop quantum gravity counts the following ones: The uncovering of a discrete structure of space [3, 4, 5, 6], a derivation of the Bekenstein-Hawking formula for the black hole entropy in terms of microscopic degrees of freedom [7, 8, 9, 10, 11] and a regularization and partial solution of the constraints of general relativity [12, 13, 14, 15]. All these results rely heavily on the fundamental assumption that holonomies (Wilson loops) of the $SU(2)$-gauge connection of general relativity in the Ashtekar formulation with real connections [16, 17] become densely defined operators in the quantum theory.

In order to test the basic assumptions of that approach and in order to check whether the theory can have the correct classical limit, mini- and midi-superspace models [18] are very appropriate because their (strongly) reduced number of degrees of freedom make them more transparent and sometimes even solvable.

These models are usually obtained by a classical symmetry reduction of the full theory and they are selected in order to facilitate quantization while keeping some of the basic problems faced in the quantization of general relativity. An example is the reduction of the $3 + 1$ dimensional theory to a $2 + 1$ dimensional one. Here a simplification occurs because $2 + 1$ dimensional gravity has only finitely many physical degrees of freedom and it is exactly soluble [19]. This model has already been investigated in the loop quantization approach [20] starting from a classical $2 + 1$ dimensional theory of gravity in terms of Ashtekar variables.

Our strategy will be different: We want to isolate symmetric states of the already loop quantized $3 + 1$ dimensional full theory. These states have to be exactly symmetric, not only symmetric at large distances compared to the Planck scale. Constraining the space of physical states to these symmetric ones amounts to a quantum symmetry reduction.
Because the solutions to all the constraints of loop quantum gravity are not known we will carry out this symmetry reduction procedure on an auxiliary Hilbert space. Therefore we will have to regularize and to solve the reduced constraints on our spaces of symmetric states. As always with reduced models, the hope is that this regularization and the search for solutions can be done with more ease. At the same time one hopes that the model under consideration can lead to new insights in the quantization of the full theory.

Our main motive for the analysis of the present paper has its origin in our interest in the reduction to spherical symmetry. The classically reduced Schwarzschild system has been quantized using Ashtekar variables (but not using loop quantization techniques) in Refs. [21, 22]. A corresponding analysis has been performed for the Reissner-Nordstrøm model [23]. These models are of physical interest because they are related to vacuum black holes. Similar to 2+1 dimensional gravity they have only finitely many physical degrees of freedom, and it is an interesting question how this reduction of infinitely many degrees of freedom of the full theory takes place in a loop quantization using spin network states.

Another motivation is to possibly find a way in order to calculate the degeneracy of energy levels. The levels are not degenerate in the quantization of the classically reduced theory; however, this approach makes use only of smooth fields whereas loop quantum gravity relates the black hole entropy to distributional configurations (generalized functions). Also, non-spherical fluctuations should not be ignored. The degeneracy plays a crucial role in the calculation [24, 25] of black hole entropy using a canonical partition function approach. The entropy obtained in this way is proportional to the horizon area, but the constant of proportionality is only known to be $O(1)$, so that a quantitative comparison with the semiclassical $A/4$-law is not immediate. A similar problem arises in the loop quantum gravity calculation [11] because of the so-called Immirzi parameter [26]. A comparison of the entropy of the full theory with that of the spherically symmetric one may shed some light on the origin of the degeneracies because degenerate states of the symmetric model may become non-degenerate for the non-symmetric one.

Another possible application is to study cosmological models within loop quantum gravity [27].

In order to attack these physical questions we have to define symmetric states in the Hilbert space of loop quantum gravity and have to determine their properties. This will be done in the present paper in a more general
setting: We will investigate the case of a Lie symmetry group $S$ acting on a principal fibre bundle with a compact Lie structure group $G$.

Before going to this general case let us dwell briefly on spherically symmetric quantum gravity: The states of the (non-spherically symmetric) full theory are given in terms of a polymer-like structure called spin network, lying in the spacelike section of space-time used to carry out a canonical quantization. We now have to face the problem how to establish a symmetry of a discrete structure under a continuous symmetry group. A possible approach is suggested by the well-known solution of the diffeomorphism constraint using group averaging methods [12]. However, this cannot lead to the desired result here: We would have to average not only over rotated subgraphs of the graph underlying a given spin network which is to be averaged, but it would be necessary to average over all rotations of an edge while keeping the other edges fixed because these give the same holonomy when evaluated in an invariant connection. Otherwise the holonomy of an edge as a multiplication operator would not commute with rotations. Therefore some averaging of parts of a given graph has to be done, which, however, runs into problems when gauge invariance has to be imposed. In some sense the rotation group is too rigid as compared to the diffeomorphism group: It acts only simply transitively on its orbits, whereas the diffeomorphism group acts $k$-transitively for any $k \in \mathbb{N}$, i.e. any two given sets of $k$ different points can be mapped one onto another by a single diffeomorphism.

A lesson, however, can nevertheless be drawn from group averaging: Symmetric states have to be distributions (generalized functions) on the function space over the quantum configuration space. This can be understood from the following intuitive picture: Constraining the support of a spin network function to only symmetric connections yields a singular distribution. This observation will guide us in our definition of symmetric states.

In order to achieve this we will use the theory of invariant connections on symmetric principal fibre bundles [28, 29, 30]. The essential properties and results will be recalled in the next section, together with some mathematical techniques of loop quantization. That section will also serve to fix our notation.

Section 3 deals with the definition of symmetric states and the analysis of their properties; it contains entirely new material. Our main result is Theorem 3.2, by means of which we can identify spaces of symmetric states with certain spin network spaces.

In Section 4 we will give some examples: Quantum symmetry reduction
to $2+1$ dimensional gravity and spherically symmetric electromagnetism as well as gravitation. These examples are intended to illustrate the “kinematical” framework of the symmetry reduction proposed here. Solving the corresponding constraints is another task.

As to $2+1$ dimensional gravity we shall show how our approach leads to the same results as that of Thiemann [20]. Spherically symmetric diffeomorphism invariant electromagnetism will be treated for a vanishing gravitational field only. It is nevertheless an interesting example for the symmetry reduction of a diffeomorphism invariant system, and it illustrates the reduction of degrees of freedom to finitely many ones and also the classifying role of the magnetic charge.

The discussion of spherically symmetric gravity is mainly restricted to kinematical aspects: the symmetry reduction is implemented, the Gauß constraint solved in the context of the appropriate spin networks and the solution of the diffeomorphism constraint by group averaging indicated. The problem of dealing with the more difficult Hamiltonian constraint (definition as a well-defined operator on spin-network states and the solution of the constraint) will be discussed elsewhere [31].

An interesting application of our results to the spectrum of the area operator acting in the spherically symmetric sector of loop quantum gravity will be published in a separate note [32].

2 Preparations

In this section we will recall some facts concerning loop quantization techniques and the theory of invariant connections on symmetric principal fibre bundles which will be used in the following sections.

2.1 Spin Networks with Higgs Field Vertices

Let $G$ be a compact Lie group, $\Sigma$ an analytic manifold and $P(\Sigma, G, \pi)$ a principal fibre bundle over the base manifold $\Sigma$ with structure group $G$. The affine space of connections on this fibre bundle will be denoted as $\mathcal{A}$, and the (local) gauge group as $\mathcal{G}$. Investigating invariant connections will lead us to the use of Higgs fields, which are sections of the adjoint bundle of $P$ (the associated vector bundle employing the adjoint representation). The space of all smooth Higgs fields will be called $\mathcal{U}$. These three function spaces will
have to be extended in the course of quantization. Their treatment is given in Ref. [12] for $A$ and $G$, and in Ref. [33] for $U$. In the following we will combine these procedures.

Let $T$ be the parallel transport algebra generated by elements of the matrix-valued parallel transporters associated with a fundamental representation of $G$ along all piecewise analytic paths in $\Sigma$, subject to appropriate relations ensuring the correct matrix multiplication under composition of paths and taking care of the group properties of $G$ (incorporating Mandelstam identities).

Similarly, let $P$ be the point holonomy algebra generated by elements of matrices, including the identity, in a fundamental representation of $G$ of point holonomies. According to Ref. [33] a point holonomy is a function on the space $U$ of classical Higgs fields obtained by exponentiating the value of a Higgs field at a given point.

The multiplication within the algebras is multiplication of $\mathbb{C}$-valued functions on $A$ and $U$, respectively. Let $\overline{T}$ and $\overline{P}$ be their completions in the sup norm. These two algebras are abelian $C^\star$-algebras with identity. We can build the product algebra $\overline{T} \otimes \overline{P}$ which is the completed tensor product space of the underlying vector spaces with pointwise multiplication. This, too, is an abelian $C^\star$-algebra with identity and we can use Gel’fand-Neumark theory (see, e.g. Refs. [34]) to obtain the following isometry of $C^\star$-algebras:

$$\overline{T} \otimes \overline{P} \cong C(\overline{A}) \otimes C(\overline{U}) \cong C(\overline{A} \times \overline{U}).$$

(1)

$\overline{A}$ and $\overline{U}$ are the Gel’fand spectra of the respective algebras consisting of all continuous $\star$-homomorphisms of the algebras to $\mathbb{C}$. The isometry is given by the Gel’fand transform $\hat{\cdot}: \overline{T} \to C(\overline{A})$ defined below, and similarly for $\overline{P}$. $\overline{A}$ and $\overline{U}$ are extensions of the spaces $A$ and $U$ which are densely embedded in the Gel’fand topology. This topology is uniquely defined by the following two conditions: $\overline{A}$ be compact and $\hat{T}: \overline{A} \to \mathbb{C}, A \mapsto \hat{T}(A) = A(T)$ be continuous for all $T \in \overline{T}$. The compact Hausdorff space $\overline{A} \times \overline{U}$ will serve as quantum configuration space.

There is, however, an alternative construction of these spaces which is better suited for calculations. Here the extensions are constructed as certain projective limit spaces. The partially ordered directed set used to define these limits is the set $\Gamma$ of all piecewise analytic graphs $\gamma$ in $\Sigma$. The projective families $(A_\gamma, p_{\gamma})$ and $(\mathcal{G}_\gamma, p_{\gamma}) = (U_\gamma, p_{\gamma})$ can be found in Refs. [33, 12]. $A_\gamma$ is the space of all functions which assign elements of the group $G$ to the
edges of \( \gamma \), and which obey certain relations ensuring the correct behavior under inversion and composition of edges. The elements of \( \mathcal{G}_\gamma \) and \( \mathcal{U}_\gamma \) assign group elements to the vertices of \( \gamma \). The projections \( p_{\gamma \gamma'}: \mathcal{A}_\gamma \to \mathcal{A}_{\gamma'} \), \( \gamma \subset \gamma' \) restrict the domain of definition of the connections in \( \mathcal{A}_{\gamma'} \) to the edges of \( \gamma \), and similarly for \( \mathcal{G}_\gamma \) and \( \mathcal{U}_\gamma \). \( \mathcal{G}_\gamma \) acts on both \( \mathcal{A}_\gamma \) and \( \mathcal{U}_\gamma \) by usual gauge transformations. The projective families define the projective limits \( \mathcal{A}, \mathcal{G} \) and \( \mathcal{U} \), where now \( \mathcal{G} \) acts on \( \mathcal{A} \) and \( \mathcal{U} \). These projective limit spaces are identical to the Gel’fand spectra constructed above, and their topology induced from the Tychonov topology is equivalent to the Gel’fand topology. This can be seen from the fact that \( \mathcal{A} \) is – as the projective limit of compact spaces – compact, and that the maps \( \hat{T} \) are continuous. In the proof of Theorem 3.1 we will make use of the equivalence of Gel’fand and Tychonov topologies.

Our quantum configuration space now is the space \( \mathcal{A} \times \mathcal{U} \) and the auxiliary Hilbert space will consist of functions on this space. An important class of such functions is that of cylindrical functions which depend on the connection and Higgs field only via a finite number of edges and vertices in \( \Sigma \). A function \( f \), cylindrical with respect to the underlying graph \( \gamma \), can be written as

\[
f(A, U) = f_\gamma(A(e_1), \ldots, A(e_n), U(v_1), \ldots, U(v_m))
\]

where \( e_1, \ldots, e_n \) are the edges of \( \gamma \), and \( v_1, \ldots, v_m \) its vertices. The functions \( f_\gamma \) representing a cylindrical function \( f \) have to obey certain consistency conditions. The auxiliary Hilbert space is \( L_2(\mathcal{A} \times \mathcal{U}, d\mu_{AL}) \) obtained from the space of cylindrical functions by completion with respect to the Ashtekar-Lewandowski measure \( d\mu_{AL} \), which is, on a cylindrical subspace, the finite product of Haar measure on \( G \) for each edge and vertex of the respective graph \( \gamma \).

An orthogonal basis is given by the set of spin network functions with Higgs field vertices. These are cylindrical functions given by a graph \( \gamma \) together with a labeling \( j \) of its edges, and labelings \( j'_e, j''_v \) of its vertices with equivalence classes of irreducible representations of \( G \), and a third labeling \( C \) of the vertices with certain intertwining operators. Given a vertex \( v \), \( C_v \) is given by an intertwining operator from the tensor product of the representations \( j_e \) labeling incoming edges \( e \) and the representation \( j'_v \) to the tensor product of the representations \( j'_e \) labeling outgoing edges and the representations \( j''_v \) and \( j''_v \). The value of a spin network function on an element \( (A, U) \in \mathcal{A} \times \mathcal{U} \) is found by taking for each edge \( e \subset \gamma \) the element \( A(e) \) in the representation \( j_e \) and for each vertex \( v \in \gamma \) the element \( U(v) \) in the representation \( j'_v \), and then contracting these matrices according to the intertwining
operators in the vertices. The resulting function will transform according to the representation \(j_v''\) in each vertex \(v\). In particular, the spin network function will be gauge invariant if all the representations \(j_v''\) are trivial.

### 2.2 Invariant Connections on Symmetric Principal Fibre Bundles

It is well known [28, 29, 30] that an invariant connection on a manifold \(\Sigma\) can be decomposed into a reduced connection of a reduced gauge group on a submanifold \(B \subset \Sigma\) plus some scalar fields on \(B\) acted on by a group action determined by the symmetry reduction (a representation of the reduced structure group). The multiplet of scalar fields will be called “Higgs field” in the following. It arises because in general an invariant connection is not manifestly invariant, but only invariant up to gauge transformations.

E.g. the authors of [36, 37] make an ansatz for a spherically symmetric connection using the fact that a symmetry transformation can be compensated by a gauge transformation if the Lie algebra of the structure group contains a \(su(2)\) subalgebra.

In this paper we will use a more general and more systematic approach which yields a complete classification of invariant connections on symmetric principal fibre bundles. It can be found in Refs. [28, 29, 30], and its main elements will be recalled in the present subsection. The method has the following advantages:

- The structure of the reduction of the gauge group and the appearance of Higgs fields becomes clearer.

- All partial gauge fixings (selections of a certain homomorphism \(\lambda \in [\lambda]\) defined below) can be treated on the same footing (and eventually be relaxed), whereas the ansatz of Refs. [36, 37] amounts to selecting one special \(\lambda\), i.e. a partial gauge fixing.

- A possible topological charge given by gauge inequivalent actions of the symmetry group in the fibres can be taken into account. This is excluded by the ansatz of [36, 37] from the outset by using trivial bundles and a fixed action of the symmetry group on the bundle only.

Whereas the first two points will be essential for constructing symmetry reductions in the spin network context (Section 3), the last point is needed for generality and allows to describe e.g. a magnetic charge.
Now let $S < \text{Aut}(P)$ be a Lie symmetry subgroup of bundle automorphisms acting on the principal fibre bundle $P(\Sigma, G, \pi)$ defined above. Using the projection $\pi: P \to \Sigma$ we get a symmetry operation of $S$ on $\Sigma$. For simplicity we will assume that all orbits of $S$ are of the same type; if necessary we will have to decompose the base manifold in several orbit bundles $\Sigma_{(F)} \subset \Sigma$, where $F = S_x$ is the isotropy subgroup of $S$ consisting of elements fixing a point $x$ of the orbit bundle $\Sigma_{(F)}$. This amounts to a special treatment of possible symmetry axes or centers, respectively.

By restricting ourselves to one fixed orbit bundle we fix an isotropy subgroup $F \leq S$ and we require that the action of $S$ on $\Sigma$ is such that the orbits are given by $S(x) \cong S/F$ for all $x \in \Sigma$. This will be the case if $S$ is compact. Moreover, we have to assume that the coset space $S/F$ is reductive \cite{28}, i.e. $\mathcal{L}S$ can be decomposed as a direct sum $\mathcal{L}S = \mathcal{L}F \oplus \mathcal{L}F_\perp$ with $\text{Ad}_F(\mathcal{L}F_\perp) \subset \mathcal{L}F_\perp$. If $S$ is semisimple, $\mathcal{L}F_\perp$ is the orthogonal complement of $\mathcal{L}F$ with respect to the Cartan-Killing metric on $\mathcal{L}S$. The base manifold can then be decomposed as $\Sigma \cong \Sigma/S \times S/F$ where $\Sigma/S \cong B \subset \Sigma$ is the base manifold of the orbit bundle and it can be realized as a submanifold $B$ of $\Sigma$ via a section in this bundle.

Given a point $x \in \Sigma$, the action of the isotropy subgroup $F$ yields a map $F: \pi^{-1}(x) \to \pi^{-1}(x)$ of the fibre over $x$ commuting with the right action of the bundle. To each point $p \in \pi^{-1}(x)$ we can assign a group homomorphism $\lambda_p: F \to G$ defined by $f(p) =: p \cdot \lambda_p(f)$ for all $f \in F$. For a different point $p' = p \cdot g$ in the same fibre we get, using commutativity of the action of $S < \text{Aut}(P)$ with right multiplication of $G$ on $P$, the conjugated homomorphism $\lambda_{p'} = \text{Ad}_g^{-1} \circ \lambda_p$. This construction yields a map $\lambda: P \times F \to G, (p, f) \mapsto \lambda_p(f)$ obeying the relation $\lambda_{p \cdot g} = \text{Ad}_g^{-1} \circ \lambda_p$.

Given a fixed homomorphism $\lambda: F \to G$, we can build the principal fibre subbundle

$$Q_{\lambda}(B, Z_{\lambda}, \pi_Q) := \{p \in P|_B : \lambda_p = \lambda\}$$

over the base manifold $B$ the structure group of which is the centralizer $Z_{\lambda} := Z_G(\lambda(F))$ of $\lambda(F)$ in $G$. $P|_B$ is the restricted fibre bundle over $B$. A conjugated homomorphism $\lambda' = \text{Ad}_g^{-1} \circ \lambda$ will lead to an isomorphic fibre bundle.

The structure elements $[\lambda]$ and $Q$ classify symmetric principal fibre bundles according to the following theorem \cite{30}:

**Theorem 2.1** A $S$-symmetric principal fibre bundle $P(\Sigma, G, \pi)$ with the isotropy subgroup $F \leq S$ of the action of $S$ on $\Sigma$ is uniquely characterized by
a conjugacy class $[\lambda]$ of homomorphisms $\lambda: F \to G$ together with a reduced bundle $Q(\Sigma/S, Z_G(\lambda(F)), \pi_Q)$.

Given two groups $F$ and $G$ we can make use of the relation

$$\text{Hom}(F,G)/\text{Ad} \cong \text{Hom}(F,T(G))/W(G)$$

in order to determine all conjugacy classes of homomorphisms $\lambda: F \to G$. Here $T(G)$ is a standard maximal torus and $W(G)$ the Weyl group of $G$.

Now let $\omega$ be a $S$-invariant connection on the bundle $P$ classified by $([\lambda], Q)$. The connection $\omega$ induces a connection $\tilde{\omega}$ on the reduced bundle $Q$. Because of the $S$-invariance of $\omega$ the reduced connection $\tilde{\omega}$ is a one-form on $Q$ with values in the Lie algebra of the reduced structure group. Furthermore, by using $\omega$ we can construct the linear map $\Lambda_p: \mathcal{L}S \to \mathcal{L}G, X \mapsto \omega_p(X)$ for any $p \in P$. Here $\tilde{X}$ is the vector field on $P$ given by $\tilde{X}(f) := d(\exp(tX)^*f)/dt|_{t=0}$ for any $X \in \mathcal{L}S$ and $f \in C^1(P, \mathbb{R})$. For $X \in \mathcal{L}F$ the vector field $\tilde{X}$ is a vertical vector field, and we have $\Lambda_p(X) = d\lambda_p(X)$ where $d\lambda: \mathcal{L}F \to \mathcal{L}G$ is the derivative of the homomorphism defined above. This component of $\Lambda$ is therefore already given by the classifying structure of the principal fibre bundle. Using a suitable gauge, $\lambda$ can be held constant along $B$. The remaining components $\Lambda_p|_{\mathcal{L}F_{\perp}}$ yield information about the invariant connection $\omega$. They are subject to the condition

$$\Lambda_p(\text{Ad}_f(X)) = \text{Ad}_{\lambda(f)}(\Lambda_p(X)) \quad \text{for } f \in F, X \in \mathcal{L}S$$

which follows from the transformation of $\omega$ under the adjoint representation and which provides a set of equations which determine the Higgs field.

Keeping only the information characterizing $\omega$ we have, besides $\tilde{\omega}$, the Higgs field $\phi: Q \to \mathcal{L}G \otimes \mathcal{L}F_{\perp}^*$ determined by $\Lambda_p|_{\mathcal{L}F_{\perp}}$. The reduced connection and the Higgs field suffice to characterize an invariant connection. This is the assertion of the following theorem:

**Theorem 2.2 (Generalized Wang theorem)** Let $P(\Sigma, G)$ be a $S$-symmetric principal fibre bundle classified by $([\lambda], Q)$ according to Theorem 2.1 and let $\omega$ be a $S$-invariant connection on $P$.

Then the connection $\omega$ is uniquely classified by the reduced connection $\tilde{\omega}$ on $Q$ and the Higgs field $\phi: Q \times \mathcal{L}F_{\perp} \to \mathcal{L}G$ obeying Eq. (5). In general, $\phi$ will transform under some representation of the reduced structure group $Z_\lambda$: The Higgs field lies in the subspace of $\mathcal{L}G$ determined
by Eq. (5). It forms a representation space of all group elements of \( G \) (which act on \( \Lambda \)) whose action preserves the Higgs subspace. These are precisely elements of the reduced group by definition.

The connection \( \omega \) can be reconstructed from its classifying structure as follows: According to the decomposition \( \Sigma \cong B \times S/F \) we have \( \omega = \tilde{\omega} + \omega_{S/F} \), where \( \omega_{S/F} \) is given by \( \Lambda \circ \iota^* \theta_{MC} \) in a gauge depending on the (local) embedding \( \iota: S/F \hookrightarrow S \). Here \( \theta_{MC} \) is the Maurer-Cartan form on \( S \). For example, in the generic case (not in a symmetry center) of spherical symmetry we have \( S = SU(2), F = U(1) = \exp \langle \tau_3 \rangle \) (\( \langle \cdot \rangle \) denotes the linear span), and the connection form can be gauged to be

\[
A_{S/F} = (\Lambda(\tau_2) \sin \vartheta + \Lambda(\tau_3) \cos \vartheta) d\varphi + \Lambda(\tau_1) d\vartheta.
\]

Here \((\vartheta, \varphi)\) are coordinates on \( S/F \cong S^2 \). The \( \tau_j \) build a basis of \( \mathcal{L}S \) and are given by \( \tau_j := -\frac{i}{2} \sigma_j \), \( \sigma_j \) being the Pauli matrices. \( \Lambda(\tau_3) \) is given by \( d\lambda \), whereas \( \Lambda(\tau_{1,2}) \) are the Higgs field components.

Eq. (6) contains as special cases the invariant connections found in Ref. [37]. These are gauge equivalent by gauge transformations depending on the angular coordinates \((\vartheta, \varphi)\), i.e., they correspond to homomorphisms \( \lambda \) which are not constant on the orbits of the symmetry group.

## 3 Symmetric States

Before describing the rather abstract construction of symmetric spin network states we will present the general idea in a first subsection. The following subsections deal with the construction of symmetric states as generalized states of the unreduced theory and proofs of some of their properties.

### 3.1 Principal Idea

The principal idea of our construction [38] described in this section is to make use of the reconstruction of an invariant connection from its classifying structure, namely by means of the pull back of a function on the space of connections on \( \Sigma \) to a function on the space of connections plus Higgs fields on the reduced manifold \( B \) which in the context of analytic spin networks will be assumed to be an analytic submanifold of \( \Sigma \).

But some complications arise because of the classical partial gauge fixing by selecting a special homomorphism \( \lambda \in [\lambda] \). The reduced gauge group and
the space of Higgs fields depend on this selection. Moreover, the Higgs field
does in general not transform under the adjoint representation of the reduced
structure group, which would be helpful in spin network quantization. In
contrast, before imposing the constraint (5) it transforms in general under
the adjoint representation of the unreduced structure group. Such a Higgs
field can easily be implemented in the spin network context using the rules
recalled in Subsection 2.1.

The interrelation of partial gauge fixings and reductions of the gauge
group makes it possible to eliminate partial gauge fixings by using the full
gauge group on the reduced manifold. This is the essence of definition 3.3
below.

### 3.2 Construction

Let us now define the notion of symmetric states. The Hilbert space under
consideration is the auxiliary Hilbert space \( \mathcal{H}_\Sigma := L_2(\mathcal{A}_\Sigma, d\mu_{AL}) \) on which the
constraints have to be solved. Because of the singular character of symmetric
states mentioned in the introduction we will have to use the rigged Hilbert
space \( \Phi_\Sigma \subset \mathcal{H}_\Sigma \subset \Phi'_\Sigma \), where \( \Phi_\Sigma \) denotes the space of cylindrical functions
on the space \( \mathcal{A}_\Sigma \) of connections over \( \Sigma \) and \( \Phi'_\Sigma \) its topological dual.

**Definition 3.1 (Symmetric States)** Let \( P \) be a \( S \)-symmetric principal fi-
bre bundle, classified by \( ([\lambda], Q) \).

A \( [\lambda] \)-symmetric state is a distribution \( \psi \in \Phi'_\Sigma \) on \( \Phi_\Sigma \) whose support con-
tains only connections that are invariant under the \( S \)-action on \( P \) classified
by \([\lambda]\).

Although Definition 3.1 catches the intuitive notion of a symmetric state,
it is not well suited for a calculus. We have to develop some tools in analogy
to the spin network calculus. This will be done in the remaining part of this
section by combining techniques collected in the last section. Application of
these techniques will lead to several spaces of connections which are defined
in

**Definition 3.2** Let \( P(\Sigma, G) \) be a principal fibre bundle acted on by a sym-
metry group \( S \) according to the classification \( ([\lambda], Q) \), where \( Q \) is the reduced
bundle over the manifold \( B \subset \Sigma \).

\( \mathcal{A}_\Sigma \) and \( \mathcal{G}_\Sigma \) are the space of generalized connections on \( P \) and the ex-
tended local gauge group, respectively. \( \mathcal{A}_B \times \mathcal{U}_B \) is the space of generalized
$G$-connections and Higgs fields in the adjoint representation of $G$ over $B$. $\mathcal{G}_B$ is the extended local gauge group of generalized $G$-gauge transformations over $B$.

For any $\lambda' \in [\lambda]$, $(\mathcal{A} \times \mathcal{U})^{\lambda'}$ is defined to be the subset of $\mathcal{A}_B \times \mathcal{U}_B$ subject to the following constraints: The generalized connections take values in the structure group $Z_{\lambda'}$ of the reduced bundle $Q$, and the generalized Higgs fields take values in the submanifold of $G$ obtained by exponentiating the linear solution space of Eq. (5). Here we have to use a separate Higgs field component for every element of a basis of $\mathcal{L}F_{\perp}$.

Remark: The space $\mathcal{A}_B \times \mathcal{U}_B$ is independent of the reduction of the structure group and the constraints (5) on the Higgs field. These affect only the definition of $(\mathcal{A} \times \mathcal{U})^{\lambda'}$ which depends explicitly on the homomorphism $\lambda'$, not only on its conjugacy class. Therefore, for any $\lambda' \in [\lambda]$ we have a separate space of connections and Higgs fields because already the reduced structure group may depend on $\lambda'$. We can eliminate this redundancy by factoring out the gauge group, but this has to be done with care due to the classical reduction of the gauge group.

In order to achieve our goal, we will make use of the classifying structure $(\tilde{\omega}, \phi)$ of a $[\lambda]$-invariant connection $\omega$. Below Theorem 2.2, we described the reconstruction of $\omega$ from its classifying structure. This reconstruction defines a continuous map

$$r^{(\iota)}_{\lambda'}: (\mathcal{A} \times \mathcal{U})^{\lambda'} \to \mathcal{A}_{\Sigma}$$

which can be continued uniquely to a continuous map

$$r^{(\iota)}_{\lambda'}: (\mathcal{A} \times \mathcal{U})^{\lambda'} \to \mathcal{A}_{\Sigma}/G_{\Sigma}.$$ 

As the notation indicates, this map depends not only on the homomorphism $\lambda' \in [\lambda]$, but also on the embedding $\iota: S/F \hookrightarrow S$.

Because a different $\iota$ would reconstruct a gauge equivalent connection form, the dependence on $\iota$ can be eliminated by factoring out the gauge group on $P$. This leads us to the family of maps

$$r_{\lambda'}: (\mathcal{A} \times \mathcal{U})^{\lambda'} \to \mathcal{A}_{\Sigma}/\mathcal{G}_{\Sigma}. \quad (7)$$

The dependence on $\lambda'$ (as opposed to $[\lambda]$) is pure gauge: $\lambda'$ can be changed arbitrarily in its conjugacy class $[\lambda]$ by applying a global transformation with a $g \in G$, $g \not\in Z_{\lambda}$. This shows that the domains of definition of all the maps $r_{\lambda'}$ are in fact different, but that they are related by gauge transformations. This observation motivates the following
Definition 3.3 Let $[\lambda]$ be a conjugacy class of homomorphisms.

Then $(\mathcal{A} \times \mathcal{U}/\mathcal{G})^{[\lambda]}$ is the subset of $\mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B$ consisting of all $\mathcal{G}_B$-gauge equivalence classes containing a representative which lies in some $(\mathcal{A} \times \mathcal{U})^{\lambda'}$, $\lambda' \in [\lambda]$.

Remark: Because we allow here for any local gauge transformation, we relax the condition that $\lambda'$ be constant on $B$, which is imposed in the classical symmetry reduction procedure.

$\mathcal{G}$-Equivariance of $r_{\lambda'}$, which means that for any $\mathcal{G}_B$-gauge transformation $g_B$ there is a $\mathcal{G}_\Sigma$-gauge transformation $g_{\Sigma'}$ with $r_{\lambda'} \circ g_B = g_{\Sigma'} \circ r_{\lambda'}$, now allows to factor out gauge transformations in the domains of definition of the maps $r_{\lambda'}$. Thereby we obtain a further map

$$r_{[\lambda]}: (\mathcal{A} \times \mathcal{U}/\mathcal{G})^{[\lambda]} \rightarrow \mathcal{A}_\Sigma/\mathcal{G}_\Sigma$$

which depends only on the conjugacy class $[\lambda]$.

We then have the

Lemma 3.1 The subset of generalized gauge invariant, $[\lambda]$-invariant connections in $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$ is given by $\text{Im} \ (r_{[\lambda]})$, the image of the map $r_{[\lambda]}$.

Proof: This is clear from the construction of $r_{[\lambda]}$. \hfill \Box

Recall that our goal is to develop a calculus on the manifold of $[\lambda]$-invariant connections. Given a continuous function on this manifold, we can pull it back via $r_{[\lambda]}$ and so obtain a continuous function on $(\mathcal{A} \times \mathcal{U}/\mathcal{G})^{[\lambda]}$. If this function can be continued to a function on $\mathcal{A}_B \times \mathcal{U}_B$, we will have the desired calculus on that space at our disposal. An extension can indeed be achieved by expanding the pulled back function in the spin network basis of $C(\mathcal{A}_B \times \mathcal{U}_B)$, the space of continuous functions on the space of connections and Higgs fields over $B$.

In order to achieve uniqueness of this expansion we have to truncate the spin network basis $\mathcal{B}$ of $C(\mathcal{A}_B \times \mathcal{U}_B)$ if necessary in such a way that

$$\hat{\mathcal{B}} := \{ T |_{(\mathcal{A} \times \mathcal{U})^{[\lambda]}} \}_{T \in \mathcal{B}}$$

is a set of independent functions. E.g. we have to use only spin network states with trivial Higgs vertices if Eq. (5) does not allow any nonvanishing Higgs field.
This extension procedure following the pull back with \( r_{[\lambda]} \) finally yields the map

\[
\tau_{[\lambda]}: C(\mathcal{A}_\Sigma/\mathcal{G}_\Sigma) \rightarrow C(\mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B)
\]  

which will provide the key element in our investigation of symmetric states.

However, a function pulled back in such a way is quite singular on the space \( \mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B \). Even if we constrain the domain of definition of \( \tau_{[\lambda]} \) to \( \Phi_\Sigma \), the space of cylindrical functions, the pull back may in general not lead to a cylindrical or \( AL \)-integrable function on \( \mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B \): The holonomy to a generic edge depends on all components of an unreduced connection in all its points, which leads to a continuous distribution of Higgs vertices. We, therefore, have again to use a rigged Hilbert space, this time

\[
\Phi_B \subset \mathcal{H}_B \subset \Phi'_B.
\]  

Here, \( \Phi_B \) is the space of cylindrical functions on \( \mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B \), \( \Phi'_B \) its topological dual, and \( \mathcal{H}_B := L_2(\mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B, d\mu_{AL}) \) (again modulo relations which solve the Higgs constraint \((\mathbb{F})\), and which will be dealt with in more detail elsewhere; examples can be found in the last section).

The restriction of \( \tau_{[\lambda]} \) to \( \Phi_\Sigma \) can now be interpreted as an antilinear map

\[
\rho_{[\lambda]}: \Phi_\Sigma \rightarrow \Phi'_B,
\]

reminiscent of a group averaging map: The pull back of a cylindrical function \( f \in \Phi_\Sigma \) is interpreted as a distribution on \( \Phi_B \) according to \((g \in \Phi_B)\)

\[
\rho_{[\lambda]}(f)(g) := \int_{\mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B} d\mu_{AL} \bar{r}_{[\lambda]} f \ g.
\]  

In a similar way, we can interpret a cylindrical function \( g \in \Phi_B \) as a distribution on \( \Phi_\Sigma \) according to

\[
\sigma_{[\lambda]}(g)(f) := \int_{\mathcal{A}_B \times \mathcal{U}_B/\mathcal{G}_B} d\mu_{AL} \bar{\tau}_{[\lambda]} g \ r_{[\lambda]} f
\]  

where \( \sigma_{[\lambda]}: \Phi_B \rightarrow \Phi'_\Sigma \) is the antilinear map given by this interpretation.

The situation can be summarized in the diagram.
with the duality relation
\[ \sigma_{\lambda}(\mathcal{F})(f) = \rho_{\lambda}(\mathcal{F})(g) \quad \text{for } f \in \Phi_\Sigma, g \in \Phi_B \] (14)

between the maps \( \sigma_{\lambda} \), \( \rho_{\lambda} \) connecting the two Gel’fand triples.

In general (see the remarks preceding Eq. (11)) \( \rho_{\lambda}(\Phi_\Sigma) \) is not contained in \( \Phi_B \), and we cannot compose \( \rho_{\lambda} \) and \( \sigma_{\lambda} \) to obtain a map from \( \Phi_\Sigma \) to \( \Phi'_\Sigma \). This is the main difference to a group averaging map, which is aimed to solve a gauge constraint. Only in very special situations can the symmetry reduction be formulated analogous to a group averaging (Subsection 4.2 and [39]). But in general we have the two maps \( \rho_{\lambda} \), which restricts an unsymmetric state to its symmetric part, and \( \sigma_{\lambda} \), which identifies symmetric states with spin network states over \( B \) thereby equipping the space of symmetric states with a calculus.

### 3.3 Properties of the Symmetric States

The goal of this subsection is to prove that the construction of the previous subsection yields all symmetric states. In order to achieve this we need some preparations.

**Lemma 3.2** Let \( G \) be a compact topological group which is Hausdorff and \( H \) be a subgroup of \( G \). The centralizer \( Z_G(H) := \{ g \in G : gh = hg \text{ for all } h \in H \} \) is a compact subgroup of \( G \).

**Proof:** It is well known that \( Z_G(H) \) is a subgroup [40]. It can be written as

\[ Z_G(H) = \bigcap_{h \in H} G_h \]
where $G_h$ is the isotropy subgroup of $h \in H$ under the adjoint action $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$ of $G$ on itself. Because the intersection of an arbitrary set of closed sets is closed, it suffices to prove that all isotropy subgroups $G_h$ are closed.

The action of $G$ on itself leads, restricted to a fixed element $h \in G$, to the continuous map $c_h: G \to G, g \mapsto ghg^{-1}$. $G_h = c_h^{-1}(h)$ is the preimage of a closed set (because $G$ is Hausdorff) under a continuous map, and hence closed. Now, $Z_G(H)$ is a closed subset of a compact group and therefore compact.

**Lemma 3.3** Let $P$ be a $S$-symmetric principal fibre bundle and $[\lambda]$ be a conjugacy class of homomorphisms classifying $P$ together with the reduced bundle $Q$.

The set of generalized gauge invariant $[\lambda]$-invariant connections on $P$ is closed in $\mathcal{A}(\Sigma)/\mathcal{G}(\Sigma)$.

**Proof:** According to Lemma 3.1 we have to show that the image of $r_{\lambda}$ – which is identical to the image of $r_{[\lambda]}$ – is closed.

We will start by showing that the domain of definition of $r_{\lambda}$, i. e. $(\mathcal{A} \times \mathcal{U})^{\lambda}$, is compact. The elements of this space take values in a compact set, because we know in the first place from Lemma 3.2 that $Z_{\lambda}$ is compact. The generalized Higgs fields take values in a compact manifold, too. Such a Higgs field has several components given by the linear map $\phi: LF_\perp \to LG$ subject to the linear Eq. (5). Therefore, the values of $\phi$ lie in a linear subspace of $LG$, and exponentiating yields generalized Higgs fields taking values in a compact submanifold of $G$.

$(\mathcal{A} \times \mathcal{U})^{\lambda}$ can be constructed as a projective limit along the lines described in Subsection 2.1. The projective family consists of compact spaces, because the maps involved assign elements of compact spaces to a finite number of edges and vertices of graphs. Therefore, the projective limit is compact in its induced Tychonov topology which is equivalent to the induced topology as a subset of $\mathcal{A}_B \times \mathcal{U}_B$.

Now, as stated in Subsection 2.1, the Tychonov topology is equivalent to the Gel’fand topology. But $r_{\lambda}$ is continuous in the Gel’fand topology (being by construction a continuation of a continuous function on the dense subspace $(\mathcal{A} \times \mathcal{U})^{\lambda}$), and so the image of $r_{\lambda}$ is compact. As a compact subset of a Hausdorff space it is closed. \qed
Theorem 3.1 Let $[\lambda]$ be a conjugacy class of homomorphisms.
The image of $\sigma_{[\lambda]}$ contains only $[\lambda]$-symmetric states.

Proof: According to Definition 3.1 we have to prove that the support of a distribution in the image of $\sigma_{[\lambda]}$ contains only $[\lambda]$-invariant connections. This amounts to showing that for any non-invariant generalized connection $\overline{A} \in (\overline{A}_B / G_B) \backslash \text{Im } (r_{[\lambda]})$ there is a neighborhood $U$ of $\overline{A}$ so that the restriction of any distribution $\psi \in \text{Im } (\sigma_{[\lambda]})$ to $U$ is the zero distribution.

Because of Lemma 3.3 $\overline{A}$ has a neighborhood which is entirely contained in $(\overline{A}_B / G_B) \backslash \text{Im } (r_{[\lambda]})$. If we restrict $\psi$ to this neighborhood, it will be the zero distribution due to its very definition in Eq. (13): The pull back with $r_{[\lambda]}$ of any function supported on the neighborhood of $\overline{A}$ will be zero.

This theorem provides us with a rich class of $[\lambda]$-symmetric states. Even better, we have a calculus on this class of states, because they are identified by Eq. (13) with the space $\Phi_B$ of cylindrical functions. Elements of $\Phi_B'$ can be regarded as generalized symmetric states.

The following lemma states that we have found enough symmetric states.

Lemma 3.4 Let $[\lambda]$ be a conjugacy class of homomorphisms.
There is no cylindrical function $f \in \Phi_\Sigma$ that is non-trivial on $\text{Im } (r_{[\lambda]})$ and annihilated by all distributions in $\sigma_{[\lambda]}(\Phi_B)$. In particular, the space $\sigma_{[\lambda]}(\Phi_B)$ separates the elements of $\Phi_\Sigma$ that differ when restricted to $[\lambda]$-invariant connections.

Proof: Let $\psi \in \Phi_\Sigma'$ be a symmetric state. If $f, g \in \Phi_\Sigma$ are two cylindrical functions that are identical when restricted to $[\lambda]$-invariant connections, then $\psi(f) = \psi(g)$. Let us, therefore, introduce the following equivalence relation which respects the algebraic and topological structure of $\Phi_\Sigma$. Two cylindrical functions $f \sim g$ are equivalent if and only if $\overline{r}_{[\lambda]} f = \overline{r}_{[\lambda]} g$. Symmetric states can be seen as functions on the space of equivalence classes, and we need to show that, if $[f] \neq [g]$, there is a distribution $\psi \in \sigma_{[\lambda]}(\Phi_B)$ with $\psi(f) \neq \psi(g)$.

Let us look now at the function $\overline{r}_{[\lambda]} f \in C(\overline{A}_B \times \mathcal{U}_B / \mathcal{G}_B)$ where $[f]$ is not trivial. As noted earlier, it may not be cylindrical, not even integrable. But if it is cylindrical, it will correspond to a symmetric state obeying $\sigma_{[\lambda]}(\overline{r}_{[\lambda]} f)(f) \neq 0$. If it is not, we can approximate it by a sequence of cylindrical functions which is obtained by projecting it onto cylindrical subspaces, cylindrical with respect to graphs of an increasing net constructed.
as follows: If $\mathfrak{r}_{[\lambda]} f$ will lie in $\Phi_B^\prime$, but not in $\Phi_B$, then it will be a countably infinite sum of terms, each being cylindrical with respect to a finite graph, the union of all these graphs being an infinite graph. The projections will be obtained by truncating to a finite number of these graphs, their number tending to infinity in the sequence mentioned. (This is reminiscent of the well known approximating sequences of the $\delta$-distribution or, more generally, of an approximate identity in an algebra without identity.) All the functions $f_i$ in the sequence will fulfill $\sigma_{[\lambda]}(f_i)(f) \neq 0$ proving our assertion.

Separation now follows from linearity, but can also be proved directly: We pick a representative for each equivalence class of cylindrical functions in $\Phi_\Sigma$, and for each representative $f$ the cylindrical function $\mathfrak{r}_{[\lambda]} f$, if it lies already in $\Phi_B$, or else an appropriate element of the sequence approximating $\mathfrak{r}_{[\lambda]} f$. The term appropriate means, that any linear relation between the chosen functions reflects a linear relation between equivalence classes. An appropriate selection can always be done, because the sequences approximate the functions $\mathfrak{r}_{[\lambda]} f$ which are different for different classes. Thereby, we obtain a class of distributions separating the equivalence classes. ☐

The results of this lemma can be interpreted as (over-)completeness of the set $\sigma_{[\lambda]}(\Phi_B)$ of generalized functions on $\{ f|_{\text{Im}(r_{[\lambda]})} : f \in \Phi_\Sigma \}$ in the sense that $\sigma_{[\lambda]}(g)(f) = 0$ for all $g \in \Phi_B$ implies $f|_{\text{Im}(r_{[\lambda]})} = 0$. Together with Theorem 3.1 this can be summarized in

**Theorem 3.2 (Quantum Symmetry Reduction)** Let $P$ be a $S$-symmetric principal fibre bundle classified by $(\lambda, Q)$, $Q$ being the reduced bundle over $B$.

The space of $[\lambda]$-symmetric states on $\overline{A}_\Sigma/\overline{G}_\Sigma$ can, by means of the mapping $\sigma_{[\lambda]}$, be identified with the space $\Phi_B$ of cylindrical functions on $\overline{A}_B \times \overline{U}_B/\overline{G}_B$.

**Proof:** Let $\Phi_{\text{symm}} := \Phi_\Sigma/\sim$ be the space of functions on the space $\text{Im}(r_{[\lambda]})$ of generalized gauge invariant and $[\lambda]$-invariant connections, and $\Phi_{\text{symm}}^\prime$ its topological dual. Due to the definition in Lemma 3.4 of $\sim$, $\Phi_{\text{symm}}^\prime$ can be identified with the space of $[\lambda]$-symmetric states as defined in Definition 3.1.

According to Lemma 3.4, $\sigma_{[\lambda]}(\Phi_B)$ is in separating duality with $\Phi_{\text{symm}}$. This implies that $\sigma_{[\lambda]}(\Phi_B)$ is dense in $\Phi_{\text{symm}}^\prime$ in the weak topology [11]. Chapter II, §6.2, Corollary 4], and, therefore, in the space of $[\lambda]$-symmetric states. (Note, however, that the topology on $\Phi_B$ induced from the weak topology by
the continuous map $\sigma_{[\lambda]}$ is coarser than the topology in which $\Phi_B$ is completed to $\mathcal{H}_B$.

In conclusion, the map $\sigma_{[\lambda]}$ is injective (by construction of $\Phi_B$ – where the Higgs constraint (5) is assumed to be solved – and by the construction of $\sigma_{[\lambda]}$), and has a dense image in the space of $[\lambda]$-symmetric states. 

This theorem allows us to trade the space of symmetric states according to Definition 3.1, which is not well suited for establishing a calculus, for the space $\Phi_B$. This space and the given calculus thereon are only a bit more difficult to deal with than the space $\Phi_\Sigma$ of all states of the full theory, because we may have to use spin networks with Higgs field vertices.

The Gauß constraint is solved by using gauge invariant functions, i.e. cylindrical functions on $\mathcal{A}_B \times \mathcal{U}_B / \mathcal{G}_B$. The diffeomorphism constraint can be solved by group averaging. After imposing $S$-symmetry there are only those diffeomorphisms to average that respect this symmetry. These are precisely the diffeomorphisms of $B$, and the constraint can be solved by averaging over the diffeomorphism group of $B$ acting on $\Phi_B$.

We conclude this section with some remarks:

- We describe symmetric states by spin networks of the group $G$, not those of $Z_\lambda$, as might have been expected from the classical reduction. On the one hand, we are forced to do that because the Higgs field transforms in general according to the adjoint representation of $G$, not $Z_\lambda$. On the other hand, the reduced structure group $Z_\lambda$ has its origin in a partial gauge fixing imposed by fixing $\lambda$. Our quantum symmetry reduced theory does not make use of such a partial gauge fixing. Instead we will have to enforce gauge invariance under the full group $G$.

- In quantizing classical expressions we have nevertheless to start from those which might be invariant only under the reduced gauge group. The first step of quantization will yield an operator acting on functions on $(\mathcal{A} \times \mathcal{U})^\lambda$. It has then to be extended by gauge covariance to an operator on $\Phi_B$. This is parallel to our construction of the map $r_{[\lambda]}$ in the equations (7), (8) and (10). An example of the quantization of such an operator will be given in Subsection 4.3.2.

- In order to be actually able to quantize classical expressions in the Hamiltonian formalism used so far, we will have to claim validity of the symmetric criticality principle in the theory under consideration. This
means that critical points of the reduced action should correspond to critical points of the unreduced one. Validity of this principle cannot be taken for granted. It will be valid, if the symmetry group $S$ is compact, or if it is abelian (more generally, unimodular) and acts freely \[18\]. These two cases cover our examples given below.

- Until now we have confined ourselves to a fixed conjugacy class $[\lambda]$. In general there will be a family of such conjugacy classes. If it is not possible to select one by means of physical considerations, we will have to treat them on an equal footing. We will use the same space $\Phi_B$ for every class, but they will be differently represented as distributions on $\Phi_\Sigma$ by means of $\sigma_{[\lambda]}$. The different conjugacy classes give rise to different sectors, which are superselected as seen from the symmetric theory. They correspond to different subspaces of invariant connections embedded in $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$ via $r_{[\lambda]}$. We have to find a physical interpretation for the different conjugacy classes, the most natural being that of a topological charge. This is purely classical, because the different conjugacy classes arise already in the classical reduction. We will see in the next section that this case indeed occurs, namely for spherically symmetric electromagnetism where $[\lambda]$ gives the magnetic charge.

- Classically a symmetric principal fibre bundle is classified by a conjugacy class of homomorphisms and a reduced bundle, see Theorem 2.1. In the quantum theory, however, the conjugacy class of the homomorphisms will suffice, because the space $\mathcal{A}_B$ of generalized connections contains connections on all the bundles over $B$, as proven in 42.

- The emergence of the Higgs field can be understood as a reflection of the information which is contained in those edges associated with a cylindrical state which are located entirely in the orbits of $S$. E.g. in a spherically symmetric theory they represent edges in a $S^2$ orbit. This justifies even more the name “point holonomies”. From this picture one can see that in general the function $r_{[\lambda]}^*f$ will involve an infinite number of Higgs vertices, where $f \in \Phi_\Sigma$ is a cylindrical function. If the underlying graph contains an edge that lies neither entirely in an orbit nor entirely in the manifold $B$, then $f$ will depend on all components of a connection in points along the edge. In the symmetry reduced theory some of the components are given by the Higgs field, so the full
dependence of \( f \) on a connection requires a continuous set of Higgs vertices in the union of all graphs underlying \( \tau_{\lambda} f \).

- The procedure of the present section can be reversed, using a Kaluza-Klein construction of a Higgs field as components of a connection in a higher-dimensional, symmetric manifold. Suppose we want to quantize a diffeomorphism invariant theory of a connection and a Higgs field, for which there is a higher-dimensional Kaluza-Klein theory containing only connection fields. Then we can visualize the Higgs vertices arising in the quantization of the lower-dimensional theory as remnants of loops lying in the compactified extra dimensions. By “blowing up” a lower-dimensional spin network with Higgs vertices in this way in order to obtain a higher-dimensional ordinary spin network, we get a quantization of the Kaluza-Klein theory. This blowing up preserves, like our symmetry reduction, gauge invariance, because the Higgs vertices transform under the adjoint representation of the structure group as do loops based on the vertices.

4 Examples

In this section we will give examples in order to illustrate some of the ideas of the last section and to test them.

4.1 2 + 1 Dimensional Gravity

2 + 1 dimensional gravity can be obtained in terms of a symmetry reduction of 3 + 1 dimensional gravity by imposing the existence of one spacelike Killing vector field of constant norm. The extra condition on the norm of the Killing field is necessary in order to eliminate a scalar field which is related to its norm. Otherwise one would obtain 2 + 1 dimensional gravity coupled to a massless scalar field.

We therefore have the abelian symmetry group \( S = \mathbb{R} \) (or \( S = SO(2) \)), and we assume the space \( \Sigma \) to have the topology \( \Sigma = B \times \mathbb{R} \) (or \( \Sigma = B \times S^1 \)) where \( B \) is a two-dimensional manifold. The group \( S \) acts freely on the second component of \( \Sigma \) which means that \( F = \{0\} \). Hence, there can only be one homomorphism \( \lambda: F \to G, 0 \mapsto 1 \). Here \( G = SU(2) \) is the gauge group of gravity, formulated as a gauge theory using real Ashtekar variables.
In this case the reduced group $Z_\lambda = G$ is identical to the full structure group. Because $F$ is trivial, Eq. (3) is trivially true and we have a one-component Higgs field $\phi$ taking values in $\mathcal{L}G$. If we use the Maurer-Cartan form $\theta = dz$, where $z$ coordinatizes $\mathbb{R}$, and the embedding $\iota = \text{id}: \mathbb{R} \to \mathbb{R}$, the reconstruction is

$$r: (A^i_\alpha dx^\alpha, \phi^i \tau_i) \mapsto A^i_\alpha dx^\alpha + \phi^i \tau_i dz. \quad (15)$$

Here the matrices $\tau_i = -\frac{i}{2} \sigma_i$ form a basis of $\mathcal{L}G$ and $x^a$, $a = 1, 2$ are coordinates on a chart of $B$. This shows that the Higgs field gives the $z$-component of the connection form.

Following the steps of the last section, we obtain the configuration space $\mathcal{A}_B \times \mathcal{U}_B$ of the reduced theory which consists of the fields $A^i_\alpha$ and $\phi^i$ on $B$ appearing in (15).

Up to now we have imposed only the symmetry condition, not the condition on the norm of the Killing field. This second condition serves to remove the dynamical scalar field. An equivalent quantum condition can be formulated by allowing only trivial Higgs vertices. This will eliminate the dependence of the spin networks on generalized Higgs fields, thereby removing their local degrees of freedom. So we arrive at the quantum configuration space $\mathcal{A}_B$ of vacuum $2 + 1$ dimensional gravity.

We can now build an auxiliary Hilbert space of integrable cylindrical functions on $\mathcal{A}_B$, completely analogous to the unreduced theory. On this Hilbert space which is spanned by ordinary two-dimensional spin networks we can represent the constraints and search for solutions. This has already been done in Ref. [20] starting from the classical symmetry reduction and quantizing the classical configuration space. Our quantum symmetry reduction procedure gives the same results, but with a representation of the quantum states of the reduced theory as symmetric states of the full quantum theory.

The reduction of this subsection can be generalized straightforwardly to the case of an arbitrary gauge theory on $\Sigma = B \times \mathbb{R}^n$, where $B$ is an arbitrary $d$-dimensional manifold and $S = \mathbb{R}^n$ acts on the second factor of $\Sigma$. A symmetry reduction will involve $n$ components of a Higgs field, whereas a quantum dimensional reduction from $B \times \mathbb{R}^n$ to $B$ can be obtained by postulating trivial Higgs vertices of spin networks in $B$. 

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4.2 Spherically Symmetric Electromagnetism

We will now reduce electromagnetism to a spherically symmetric one. Although spherically symmetric electromagnetism is almost trivial it is nevertheless quite instructive for our purposes because the quantum symmetry reduction can be carried out explicitly.

First, let us explain why we treat electromagnetism as a diffeomorphism invariant theory. This can, of course, not be true for pure electromagnetism on, e.g. a Minkowski background. But we are interested in the electromagnetic field as a field of a Reissner-Nordstrøm black hole. We therefore have to couple electromagnetism to gravity, which ensures diffeomorphism invariance. The dynamics of the electromagnetic field is then encoded in the Hamiltonian constraint. The diffeomorphism constraint will only contain gravitational fields, because in the spherically symmetric case diffeomorphism invariance of the electromagnetic field is already imposed by the $U(1)$-Gauß constraint. As a test model this coupled theory as a whole would be too complicated, so we discard the gravitational field by constraining it to be zero. This amounts to treating the electromagnetic field on a degenerate background which renders the Hamiltonian ill-defined. But electric and magnetic fluxes will be well-defined, so that we, nevertheless, may study the kinematics of the fields.

4.2.1 Classical Symmetry Reduction

We now have $G = U(1)$, $S = SU(2)$ and in general $F = U(1)$. The topology of space is $\Sigma \cong \mathbb{R} \times S^2$ or $\Sigma \cong \mathbb{R}^+ \times S^2$, implying $B = \mathbb{R}$ or $B = \mathbb{R}^+$, respectively.

We first determine all conjugacy classes of the homomorphisms $\lambda: U(1) \to U(1)$. Because such a homomorphism is a one-dimensional unitary representation of $U(1)$, it is given by its character. Therefore we have the mappings $\text{Hom}(U(1), U(1))/\text{Ad} \cong \mathbb{Z}$ represented by the homomorphisms $\lambda_n: z \mapsto z^n$. For the abelian group $G$ the centralizers $Z_{\lambda_n} = U(1)$ are equal to the full group. Thus, a spherically symmetric principal fibre bundle with structure group $U(1)$ is classified by an integer $n$, and, of course, all reduced bundles are trivial, because $B$ is contractible, and they are not needed for the classification.

Let us now determine the type of Higgs field allowed by Eq. (5). This field is given by the map $\Lambda: \mathcal{L}S \to \mathcal{L}G$, where $\Lambda|_{\mathcal{L}F} = d\lambda_n$ is already fixed by the bundle classification. Here we have dim $\mathcal{L}G = 1$, leaving only the possibilities
dim ker Λ = 2 or dim ker Λ = 3. In the case \( n \neq 0 \), \( d\lambda_n \) is an isomorphism of vector spaces, forbidding \( \text{dim} \ker \Lambda = 3 \). So we have \( \text{dim} \ker \Lambda = 2 \) and the Higgs field \( \phi = \Lambda|_{\mathcal{L}_F} = 0 \) vanishes. This kind of reasoning can not be used in the case \( n = 0 \). But looking at Eq. (4) we see that \( n \) enters this equation only in connection with \( \text{Ad}_{\lambda_n(f)} \) for \( f \in F \). This occurrence is trivial in the abelian case, so Eq. (4) is independent of \( n \). Therefore, if it does not allow a Higgs field for \( n \neq 0 \) it cannot allow a Higgs field for \( n = 0 \), too. This proves that the reduced theory is a theory of a \( U(1) \)-connection only.

Let us recall the reduced phase space structure from Ref. [23]. The canonical variables of the symmetry reduced theory are given by the radial components \( p \) and \( i\omega \) of the electric field and the \( U(1) \)-connection form, respectively. They are subject to the Gauß constraint \( p' \approx 0 \) which enforces \( p \) to be constant. The reduced phase space is two-dimensional with the canonical variables \( p \) which is proportional to the electric charge and the canonically conjugate \( \Phi = -\int_B d\omega \). The electric charge is either a point charge sitting in \( x = 0 \) if \( B = \mathbb{R}^+ \), or the charge of the wormhole \( \Sigma = \mathbb{R} \times S^2 \).

**4.2.2 Quantum Symmetry Reduction**

According to Theorem 3.2 the physical Hilbert space of spherically symmetric electromagnetism is given by \( L^2(\mathcal{A}_B/\mathcal{G}_B, d\mu_{AL}) \), the space of gauge invariant, Ashtekar-Lewandowski square integrable functions on the space of generalized \( U(1) \)-connections over \( B \). It is spanned by \( U(1) \)-spin networks, which are, due to gauge invariance and the one-dimensional nature of \( B \), given by

\[
(\eta_B)^K, \quad \text{where } \eta_B := \exp i \int_B d\omega(x)
\]

is the holonomy along \( B \) of the \( U(1) \)-connection \( i\omega \) and \( K \in \mathbb{Z} \) is the only parameter, corresponding to irreducible \( U(1) \)-representations labeling the basis states. We see that the reduced theory is particularly simple, having only two canonical degrees of freedom. The physical Hilbert space can be identified with \( L^2(U(1), d\mu_H) \), where \( d\mu_H \) is the Haar measure on \( U(1) \).

Before coming to the observables of the theory we will investigate the quantum symmetry reduction \( \pi_{\lambda_n}^\star \). Let the symmetric principal fibre bundle be classified by the homomorphism \( \lambda_n \), and let \( F = U(1) \cong \exp\langle \tau_3 \rangle \subset SU(2) \) be the isotropy subgroup. We then have to set \( \Lambda(\tau_3) = d\lambda_n(\tau_3) = in \) and \( \Lambda(\tau_{1,2}) = 0 \) in Eq. (4). The reconstruction of a \( U(1) \)-connection with
$x$-component $i \omega_x : B \to i \mathbb{R} = \mathcal{L}U(1)$ on $Q$ is given by the $[\lambda_n]$-invariant connection
\[ i \omega := r^{(\lambda_n)}_{\lambda_n} (i \omega_x \, dx) = i \omega_x \, dx + i n \cos \vartheta \, d \varphi. \tag{17} \]

Here $x$ is a (local) coordinate of $B$ and $(\vartheta, \varphi)$ are the Killing parameters of the $S$-orbits. Together they yield a (local) coordinate system $(x, \vartheta, \varphi)$ of $B \times S^2$. We can calculate the curvature of this $[\lambda_n]$-invariant connection to obtain the (densitized) magnetic field of a Dirac monopole with magnetic charge $n$:
\[ B_x = -n \sin \vartheta, \quad B_\vartheta = B_\varphi = 0. \tag{18} \]

This confirms our claim that the conjugacy classes of homomorphisms correspond to the values of a topological charge.

Eq. (17) will now be used to pull back a spin network state to a function on the space of invariant connections. To that end let $e : [0, 1] \to \Sigma$ be an edge in $\Sigma$, running from $e(0) = (x_1, \vartheta_1, \varphi_1)$ to $e(1) = (x_2, \vartheta_2, \varphi_2)$, chosen such that a parameterization $\vartheta(\varphi)$ is possible along $e$ (otherwise we can cut $e$ in pieces and set $\vartheta(\varphi) = 0$ if $\varphi$ is constant along a piece without affecting the following). Then the $i \omega$-holonomy along $e$ is given by
\[ h_e(i \omega) = \exp\left[ \int_e dt \left( i \dot{x} \omega_x + i n \dot{\varphi} \cos \vartheta \right) \right] = \exp \left( i \int_{x_1}^{x_2} dx \omega_x \right) \exp \left( in \int_{\varphi_1}^{\varphi_2} d\varphi \cos \vartheta(\varphi) \right) =: \ h_{\pi(e)}(i \omega_x)(\beta_e)^n. \]

Here $\pi(e) := [x_1, x_2] = [x(e(0)), x(e(1))] \subset B$ is an edge in $B$, and $h_{\pi(e)}(i \omega_x)$ is the holonomy of the reduced connection $i \omega_x$ along it. $\beta_e$ is a phase factor depending only on the geometry of $e$.

If $T_{\gamma, k}$ is a spin network with a graph $\gamma \subset \Sigma$ and a labeling $k$ of its edges with irreducible $U(1)$-representations (a labeling of the vertices is not necessary for gauge invariant $U(1)$-spin networks because it is already given uniquely by the edge labeling), we can evaluate it on a $[\lambda_n]$-invariant connection: If $E(\gamma)$ denotes the edge set of $\gamma$ we have
\[ T_{\gamma, k}(i \omega) = \prod_{e \in E(\gamma)} h_e(i \omega)^{k_e} \tag{19} \]
\[ = \prod_{e \in E(\gamma)} h_{\pi(e)}(i \omega_x)^{k_e} \prod_{e \in E(\gamma)} (\beta_e)^{nk_e} = (\beta_{\gamma, k})^n \prod_{e \in E(\gamma)} h_{\pi(e)}(i \omega_x)^{k_e}. \]
Here \( \beta_{\gamma, k} := \prod_{e \in E(\gamma)} \beta_e^{k_e} \) is a phase factor depending only on the geometry of \( T \) and its labeling.

The right hand side of the last equation can be written as the evaluation of a spin network state in \( B \) on the connection \( i \omega_x \). In order to do this we will define a projection \( \pi \) which assigns to a spin network \( T_{\gamma, k} \) in \( \Sigma \) a spin network \( \pi(T_{\gamma, k}) \) in \( B \). The set of vertices \( V(\gamma) \) of \( \gamma \) can be projected on a finite set

\[
\pi(V(\gamma)) := \{ x(v) : v \in V(\gamma) \} =: \{ x^{(i)} \}_{1 \leq i \leq |\pi(V(\gamma))|},
\]

where we have ordered the elements of \( \pi(V(\gamma)) \) such that \( x^{(i)} < x^{(j)} \) for \( i < j \). Their number is bounded by \( |\pi(V(\gamma))| \leq |V(\gamma)| \). Let \( \gamma \) be chosen – if necessary by inverting edges and splitting edges by introducing new vertices – such that each edge \( e \in E(\gamma) \) either lies entirely in an orbit of \( S \), in which case we define \( \pi(e) := \emptyset \), or it has a projection \( \pi(e) = [x^{(i)}, x^{(i+1)}] \) for some \( 1 \leq i \leq |\pi(V(\gamma))| \) and \( x(e(t)) \) increases monotonely in \( t \). To every projected edge \( \pi(e) \) we assign the point \( x_m(\pi(e)) := \frac{1}{2}(x(e(0)) + x(e(1))) \) in its interior. We can now define the projected spin network:

**Definition 4.1** Let \( T_{\gamma, k} \) be a \( U(1) \)-spin network in \( \Sigma = B \times S^2 \), the graph \( \gamma \) be chosen as above.

The projected graph \( \pi(\gamma) \subset B \) is given by its edge set

\[
E(\pi(\gamma)) := \{ \pi(e) : e \in E(\gamma), \pi(e) \neq \emptyset \} = \{ [x^{(i)}, x^{(i+1)}] : 1 \leq i \leq |\pi(V(\gamma))| \}
\]

and its vertex set

\[
V(\pi(\gamma)) := \pi(V(\gamma)).
\]

The labeling of the projected graph descending from the spin network \( T_{\gamma, k} \) is given by

\[
\pi(k)_{\pi(e)} := \sum_{|e' \cap S_{x_m(\pi(e))}|=1} k_{e'}
\]

for any edge \( \pi(e) \in E(\pi(\gamma)) \). Here \( S_{x_m(\pi(e))} \) is the \( S \)-orbit through \( x_m(\pi(e)) \), and the condition \( |e' \cap S_{x_m(\pi(e))}| = 1 \) in the sum ensures that we count only the charge of edges running transversally through \( S_{x_m(\pi(e))} \).

The projected spin network is given by \( \pi(T_{\gamma, k}) := T_{\pi(\gamma), \pi(k)} \).
By means of the projected spin network, we can write Eq. (19) as
\[ T_{\gamma,k}(i\omega) = (\beta_{\gamma,k})^n \prod_{e_B \in E(\pi(\gamma))} h_{e_B} (i\omega_x)^{k_{e'}} \]
\[ = (\beta_{\gamma,k})^n \prod_{e_B \in E(\pi(\gamma))} h_{e_B} (i\omega_x)^{\pi(k)_{e_B}} = (\beta_{\gamma,k})^n \pi(T_{\gamma,k})(i\omega_x). \]

This equation enables us to write
\[ \tilde{T}_{[\lambda_n]} T_{\gamma,k} = (\beta_{\gamma,k})^n \pi(T_{\gamma,k}). \] (20)

We can see that \( \tilde{T}_{[\lambda_n]} T \in \Phi_B \) is cylindrical on \( \tilde{A}_B \) for any spin network \( T \in \Phi_\Sigma \). This convenient circumstance is related to the vanishing of the Higgs field and cannot be taken for granted in cases of other structure groups. Here this fact allows us to compose the maps \( \rho_{[\lambda]} \) and \( \sigma_{[\lambda]} \) in the diagram at the end of Section 3.2 to obtain a map
\[ \sigma_{[\lambda]} \circ \rho_{[\lambda]} : \Phi_\Sigma \rightarrow \Phi_{\Sigma}' \]
reminiscent of a group averaging map. In fact the symmetry reduction considered here can be imposed by supplementing electrodynamics with a further constraint besides the Gauß constraint to arrive at an abelian BF-theory [39]. Thus, the symmetry can be reduced by means of a “rigging” map analogous to \( \sigma_{[\lambda]} \circ \rho_{[\lambda]} \). The vanishing Higgs field also causes the appearance of the topological charge \( n \) as a power of the phase factor only. Furthermore, the above calculations exhibit the fact that a projected spin network state is gauge invariant if and only if the original spin network state in \( \Sigma \) is gauge invariant. This follows from the fact that gauge invariance in \( \Sigma \) forces the spin network labeling to fulfill
\[ \sum_{|e \cap S_1| = 1} k_e = \sum_{|e \cap S_2| = 1} k_e \]
for any two \( S \)-orbits \( S_1 \) and \( S_2 \) not containing a vertex of the graph \( \gamma \) (for simplicity we assume that all edges are oriented outwards). Therefore a gauge invariant spin network will project to a spin network with labeling \( k_{e_B} = K \) for each edge \( e_B \) of the projected graph. This yields the gauge invariant spin network \( (\eta_B)^K \) defined above as projected spin network.
Let us now write down the action of a $\lambda_n$-symmetric state as distribution on $\Phi_\Sigma$. For a spin network $T_{\gamma,k}$ it is given by

$$\sigma_{[\lambda_n]}((\eta_B)^K)(T_{\gamma,k}) = \int_{A_B} d\mu_{AL} (\eta_B)^K \tau_{[\lambda_n]} T_{\gamma,k}$$

$$= (\beta_{\gamma,k})^n \int_{A_B} d\mu_{AL} (\eta_B)^K T_{\pi(\gamma),\pi(k)}$$

$$= (\beta_{\gamma,k})^n \delta(B),\pi(\gamma) \delta(K),\pi(k).$$

It is non-vanishing only if $T_{\gamma,k}$ is gauge invariant implied by $\pi(\gamma) = \{B\}$, and if the charge $\pi(k) = \sum_{|e\cap S_x|=1} k_e$ equals the labeling $K$ of the symmetric state.

4.2.3 Observables

We will now quantize the observables $p$ and $\Phi$ found in the classical reduction. $p$ is canonically conjugate to the connection $\omega$, and standard quantization rules lead in the connection representation to the quantization $\hat{p} = -i\hbar \frac{\delta}{\delta \omega}$. Acting on a gauge invariant state (16) this yields $\hat{p}(\eta_B)^K = hK(\eta_B)^K$. We see that spin network states are eigenstates of $\hat{p}$. All the eigenvalues of $\hat{p}$ are real, which shows that it is essentially selfadjoint. Furthermore, they are discrete exhibiting electric charge quantization as integer multiples of an elementary charge (this fact has already been observed in Ref. 43 in the unsymmetric theory). The value of this elementary charge is, however, not fixed because $p$ is only proportional to the electric charge. The factor of proportionality ensures the correct dimension and scales the elementary charge. This factor is arbitrary and is related to the normalization of the electromagnetic action.

In the following we will denote the basis states of the Hilbert space $L_2(U(1),d\mu_H)$ as

$$|K\rangle := (\eta_B)^K.$$

These states span an orthonormal basis with respect to the Haar measure on $U(1) = \mathcal{A}_B/\mathcal{G}_B$ which derives from the Ashtekar-Lewandowski measure on $\mathcal{A}_B$. They are labeled by the charge eigenvalues according to $\hat{p} |K\rangle = hK |K\rangle$.

Now we have to quantize $\Phi = -\int_B \omega = i \log \eta_B$. Recall that holonomies are well-defined as operators in the loop quantization. This fact suggests to use $\eta_B$ instead of $\Phi$. $\eta_B$ can straightforwardly be promoted to the ‘creation’ operator

$$\hat{\eta}_B |K\rangle = (\eta_B)^{K+1} = |K+1\rangle.$$
which is unitary because it is invertible and preserves the norm of states it acts on. In view of $\eta_B = \exp(-i\Phi)$ the operator $\hat{\eta}_B$ has indeed to be unitary. Note that it is the Haar measure on $U(1)$, which derives from the Ashtekar-Lewandowski measure on the unconstrained space (before solving the Gauß constraint) that incorporates the classical reality conditions correctly.

We will now close this subsection by showing that $\hat{p}$ and $\hat{\eta}_B$ have the correct commutator.

Theorem 4.1 $(p, \exp(-i\Phi)) \mapsto (\hat{p}, \hat{\eta}_B)$ is a representation of the classical Poisson ⋆-algebra on the Hilbert space $L_2(U(1), d\mu_H)$.

Proof: We have already seen, that the ⋆-relations $p^* = p$ and $\exp(-i\Phi)^* = \exp(i\Phi)$ are represented properly.

Because $p$ and $\Phi$ are canonically conjugate, we have the Poisson bracket

$$\{p, \exp(-i\Phi)\} = \frac{d\exp(-i\Phi)}{d\Phi} = -i\exp(-i\Phi).$$

The only non-vanishing matrix elements of the commutator $[\hat{p}, \hat{\eta}_B]$ are given by

$$\langle K | [\hat{p}, \hat{\eta}_B] | K - 1 \rangle = \langle K | \hbar K | K \rangle - \langle K | \hbar(K - 1) | K \rangle = \hbar.$$

We, therefore, have the relation

$$[\hat{p}, \hat{\eta}_B] = \hbar\hat{\eta}_B = i\hbar\{p, \exp(-i\Phi)\}^\ast$$

implementing the correct representation of the Poisson structure.

\[22\]

4.3 Spherically Symmetric Quantum Gravity

Our last example deals with spherically symmetric quantum gravity. Undoubtedly this one of our three examples is of the greatest physical interest because it describes quantum properties of non-rotating black holes. The classical symmetry and constraint reduction reveals that there is only one physical configuration degree of freedom, the mass (or a canonical pair: mass and time). This is similar to spherically symmetric electromagnetism discussed above which has only the electric charge as physical degree of freedom. However, it can be anticipated that gravity will be more difficult since its constraints are more complicated.

In the present paper we treat in detail only the Gauß constraint and comment briefly on the solution of the diffeomorphism constraint. The more complicated Hamiltonian constraint will be dealt with elsewhere [31].
4.3.1 Symmetry Reduction

We now specialize our general framework to the case \( S = SU(2) \), \( F = U(1) \) and \( G = SU(2) \) (we will use real Ashtekar variables). The topology of the space \( \Sigma \) will be \( \Sigma = B \times S^2 \) with \( B = \mathbb{R} \) or \( B = \mathbb{R}^+ \).

First we have to find all conjugacy classes of homomorphisms \( \lambda : F = U(1) \rightarrow SU(2) = G \). In order to do that we can make use of Eq. (4). We need the following information about \( SU(2) \) (see, e.g. Ref. [40]).

**Lemma 4.1** The standard maximal torus of \( SU(2) \) is given by

\[
T(SU(2)) = \{ \text{diag} (z, z^{-1}) | z \in U(1) \} \cong U(1).
\]

The Weyl group of \( SU(2) \) is the permutation group of two elements,

\[
W(SU(2)) \cong S_2,
\]

its generator acting on \( T(SU(2)) \) by \( \text{diag} (z, z^{-1}) \mapsto \text{diag} (z^{-1}, z) \).

All homomorphisms in \( \text{Hom} (U(1), T(SU(2))) \) are then given by

\[
\lambda_k : z \mapsto \text{diag} (z^k, z^{-k})
\]

for any \( k \in \mathbb{Z} \), as in the electromagnetic example. However, here we have to divide out the action of the Weyl group, leaving only the maps \( \lambda_k, k \in \mathbb{N}_0 \), as representatives of all conjugacy classes of homomorphisms. We see that spherically symmetric gravity has a topological charge taking values in \( \mathbb{N}_0 \) (in the dreibein, not in the metric formulation, as we will see below).

We will represent \( F \) as the subgroup \( \exp \langle \tau_3 \rangle < SU(2) \) of the symmetry group \( S \), and use the homomorphisms \( \lambda_k : \exp t\tau_3 \mapsto \exp kt\tau_3 \), out of each conjugacy class. This amounts to a partial gauge fixing called \( \tau_3 \)-gauge in the following. As reduced structure group we obtain \( \mathbb{Z}_G(\lambda_k(F)) = \exp \langle \tau_3 \rangle \cong U(1) \) for \( k \neq 0 \) and \( \mathbb{Z}_G(\lambda_0(F)) = SU(2) \) (\( k = 0 \) leads to the manifestly symmetric connections of Ref. [36]). The map \( \Lambda|_{\mathcal{L}F} \) is then given by \( d\lambda_k : (\tau_3) \rightarrow \mathcal{L}G, \tau_3 \mapsto k\tau_3 \). The remaining components of \( \Lambda \) which give us the Higgs field, are determined by \( \Lambda(\tau_{1,2}) \in \mathcal{L}G \), and are subject to Eq. (5). This equation can here be written as

\[
\Lambda \circ \text{ad} \tau_3 = \text{ad} d\lambda_k(\tau_3) \circ \Lambda.
\]
Using \( \text{ad}_{\tau_3}\tau_1 = \tau_2 \) and \( \text{ad}_{\tau_3}\tau_2 = -\tau_1 \) we obtain the equation

\[
\Lambda(a_0\tau_2 - b_0\tau_1) = k(a_0[\tau_3, \Lambda(\tau_1)] + b_0[\tau_3, \Lambda(\tau_2)]),
\]

where \( a_0\tau_1 + b_0\tau_2, a_0, b_0 \in \mathbb{R} \) is an arbitrary element of \( \mathcal{LF}_\bot \). Since \( a_0 \) and \( b_0 \) are arbitrary this is equivalent to the two equations

\[
k[\tau_3, \Lambda(\tau_1)] = \Lambda(\tau_2) \quad \text{and} \quad k[\tau_3, \Lambda(\tau_2)] = -\Lambda(\tau_1).
\]

The general ansatz

\[
\Lambda(\tau_1) = a_1\tau_1 + b_1\tau_2 + c_1\tau_3, \quad \Lambda(\tau_2) = a_2\tau_1 + b_2\tau_2 + c_2\tau_3
\]

with arbitrary parameters \( a_i, b_i, c_i \in \mathbb{R} \) yields the equations

\[
k(a_1\tau_2 - b_1\tau_1) = a_2\tau_1 + b_2\tau_2 + c_2\tau_3, \quad (23)
k(-a_2\tau_2 + b_2\tau_1) = a_1\tau_1 + b_1\tau_2 + c_1\tau_3.
\]

These equations have a non-trivial solution only if \( k = 1 \), for which we get

\[
b_2 = a_1, \quad b_2 = -b_1 \quad \text{and} \quad c_1 = c_2 = 0.
\]

We shall discuss this main case first where we also will see how the present approach is related to that of Refs. [21, 22] and will comment on the physically uninteresting ones \( (k \neq 1) \) which have a vanishing Higgs field afterwards.

The configuration variables of the system are the above fields \( a, b, c : B \to \mathbb{R} \) of the \( U(1) \)-connection form \( A = c(x)\tau_3\,dx \) on the one hand and the two Higgs field components

\[
\Lambda|_{\langle (\tau_1) \rangle} : B \to \langle \tau_1, \tau_2 \rangle, \quad x \mapsto a(x)\tau_1 + b(x)\tau_2
\]

\[
= \frac{1}{2} \begin{pmatrix} 0 & -b(x) - ia(x) \\ b(x) - ia(x) & 0 \end{pmatrix} =: \begin{pmatrix} 0 & -w(x) \\ w(x) & 0 \end{pmatrix}
\]

on the other hand.

Under a local \( U(1) \)-gauge transformation \( z(x) = \exp(t(x)\tau_3) \) they transform as

\[
c \mapsto c + \frac{dt}{dx} \quad \text{and} \quad w(x) \mapsto \exp(-it)w
\]

which can be read off from

\[
A \mapsto z^{-1}Az + z^{-1}dz = A + \tau_3dt,
\]

\[
\Lambda(\tau_1) \mapsto z^{-1}\Lambda(\tau_1)z = \begin{pmatrix} 0 & -\exp(it)w \\ \exp(-it)w & 0 \end{pmatrix}.
\]
Comparing with Ref. [21] we see that the above variable $c$ transforms as the connection coefficient $A_1$ there and the variables $(a, b)$ as $(A_2, A_3)$. However, the reconstructed connection form

$$A(x, \vartheta, \varphi) = c(x)\tau_3 dx + (-a(x)d\vartheta + b(x)\sin \vartheta d\varphi)\tau_1 - (b(x)d\vartheta + a(x)\sin \vartheta d\varphi)\tau_2 + \cos \vartheta d\varphi \tau_3$$

is different from the connection form

$$\left( A_1 n_x^i dx + 2^{-\frac{1}{2}}(A_2 n_\vartheta^i + (A_3 - \sqrt{2})n_\varphi^i) d\vartheta + 2^{-\frac{1}{2}}(A_2 n_\varphi^i - (A_3 - \sqrt{2})n_\vartheta^i) \sin \vartheta d\varphi \right) \tau_i$$

used in Refs. [21], [22] (now expressed in terms of real Ashtekar variables) where $n_x^i$, $n_\vartheta^i$ and $n_\varphi^i$ are the standard unit vectors in polar coordinates. The two connection forms differ, however, only by a gauge rotation $g_1 := \exp(-\vartheta n_\varphi^i \tau_i)$ followed by a second gauge rotation $g_2 := \exp(-\varphi n_x^i \tau_i)$.

We note that the term $\cos \vartheta d\varphi \tau_3$ in Eq. (25) which is the only term independent of the fields $a$, $b$ and $c$, in the connection form $A(x, \vartheta, \varphi)$ derived above leads automatically to the appearance of $A_3$ as the combination $A_3 - \sqrt{2}$ in the connection form (26) of Refs. [21], [22]. This subtraction of $\sqrt{2}$ (which does not appear in Ref. [36]) has been added by hand in Ref. [21] in order to ensure the correct transformation of $(A_2, A_3)$ in the defining representation of $SO(2)$.

The calculation above shows that $c$ has to be identified with $A_1$ and (up to a rescaling with $\sqrt{2}$ used in Ref. [21] in order to get the standard symplectic structure) $(a, b)$ with $(A_2, A_3)$. We will here rescale the variables, too, and denote the rescaled ones by $(A_1, A_2, A_3)$. Their conjugate variables are $(E^1, E^2, E^3)$, and the symplectic structure is adapted to our notation using the Immirzi parameter $\iota$ - given by

$$\{ A_I(x), E^J(y) \} = \frac{\kappa L}{4\pi} \delta_i^j \delta(x,y)$$

where $\kappa = 8\pi G$ is the gravitational constant.

We recall [21], [22] that the metric on $\Sigma$ is given by

$$(g_{ab}) = \text{diag}(\frac{E}{2E^1}, E^1, E^1 \sin^2 \vartheta), \ E = (E^2)^2 + (E^3)^2.$$ 

The functions $E^1$ and $E$ are closely related to two important geometrical quantities:
The surface $A(x)$ of the 2-dimensional spherical orbit generated by the symmetry group $S$ at $x \in B$ is given by

$$A(x) = 4\pi E^1(x)$$

(29)

and the spherically symmetric three-dimensional volume element $dV$ “transverse” to those orbits is

$$dV = \frac{4\pi}{\sqrt{2}} \sqrt{E^1} E dx .$$

(30)

We finally turn briefly to the case $k \neq 1$ which is associated with vanishing Higgs fields $\Lambda(\tau_i) = 0, i = 1, 2$ or, equivalently, $A_2 = A_3 = 0$. The vanishing of the canonical variables $A_2$ and $A_3$ implies that their canonically conjugate momenta $E^2$ and $E^3$ become irrelevant, too, and we may put them to zero, $E^2 = E^3 = 0$. This means that the metric (28) and the volume (30) become degenerate and that we do not have a non-vanishing volume element.

Thus, the sectors with $k \neq 1$ describe degenerate sectors which are, however, different from the one found in Ref. [21]. There a degenerate sector with vanishing volume was found in our ($k = 1$)-sector after solving the constraints. On our ($k \neq 1$)-sectors the diffeomorphism and Hamiltonian constraint will be trivially fulfilled, but the $k = 1$-sector has nontrivial constraints whose solution manifold has several sectors.

That the degenerate sectors for $k \neq 1$ here are different from the degenerate one of Ref. [21] can be seen as follows: For $k \neq 1$ the last term $(\cos \vartheta d\varphi \tau_3)$ in Eq. (25) gets multiplied by $k$ and as a consequence the subtracted constant $-\sqrt{2}$ in Eq. (26), too, leading to different connection forms even for $A_2 = A_3 = 0$.

Thus, if we are interested in geometrically interesting systems with non-vanishing volumes we have to stick to the sector $k = 1$.

4.3.2 Quantization of the Gauß Constraint

Although the Gauß constraint can be solved by restricting to gauge invariant spin networks we here give a regularization in order to exhibit more concretely the role of the reduced gauge group.

Classically, one uses a partial gauge fixing reducing the structure group $SU(2)$ to $U(1)$. This is explicit in the symmetry reduction of the Gauß constraint

$$G^E[\Lambda] = \frac{4\pi}{\kappa t} \int_B dx d\Lambda(x) \left((E^1)' + A_2 E^3 - A_3 E^2\right)$$

33
which is taken from [21] adapted to our notation. $\Lambda(x)$ is a Lagrange multiplier. In a pure, one-dimensional $U(1)$-gauge theory one would encounter the constraint $(E^1)\prime \approx 0$. Here, we have an additional term which provides coupling to the Higgs field. For the sake of clarity we will regularize these two terms one after another.

By standard methods we get

$$\frac{4\pi}{\kappa \ell} \int_B dx \Lambda(\hat{E}^1) f_\gamma =$$

$$= \frac{\hbar}{\ell} \int_B dx \Lambda \frac{d}{dx} \sum_e \int_e dt \hat{e}^x \delta(x,e(t)) \text{tr} \left( (h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e} \right) f_\gamma$$

$$= i\hbar \sum_e \int_e dt \hat{e}^x \int_B dx \frac{d\Lambda}{dx} \delta(x,e(t)) \text{tr} \left( (h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e} \right) f_\gamma$$

$$= i\hbar \sum_e \int_e dt \frac{d}{dt} \left( \Lambda(e(t)) \text{tr} \left( (h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e} \right) \right) f_\gamma$$

$$= i\hbar \sum_e (\Lambda(e(1))X^3_L(h_e) - \Lambda(e(0))X^3_R(h_e)) f_\gamma.$$

In the first row, we used the standard quantization rule

$$\hat{E}^1 = \frac{\kappa \hbar}{4\pi i} \frac{\delta}{\delta A^1}$$

taking into account the symplectic structure (27). The functional derivative acts on a cylindrical function $f_\gamma$ which depends on edge and point holonomies. As remarked in Section 3.1 we start the quantization procedure on the space $(A \times U)$ due to classical gauge fixing. With $\lambda$ chosen as in the present section a holonomy on this space takes the form

$$h_e = \exp(\int_e dx A_1(x) \tau_3)$$

giving rise to the above derivative.

In the second step above we integrated by parts to be able to integrate over the $\delta$-distribution in the following step. Because the holonomies are abelian, the trace does actually not depend on $t$ so that we can include it into the argument of the $t$-derivative in order to integrate over $t$. Because of the abelian nature the left and right invariant vector field components $X^3_L$
and $X^3_R$ are identical. However, we keep both of them to make possible a comparison with a $SU(2)$-theory.

The Higgs coupling term has to be rephrased before proceeding: We introduce polar coordinates $(A, \alpha)$ in the $(A_2, A_3)$-plane, i.e., $(A_2, A_3) = (A \cos \alpha, A \sin \alpha)$. These variables have the advantage that $A$ is gauge invariant, whereas $\alpha$ can be gauged arbitrarily. After replacing the variables $E^2$ and $E^3$ with the respective derivative operators in the course of quantization, we encounter the derivation

\[ A_2 \frac{\partial}{\partial A_3} - A_3 \frac{\partial}{\partial A_2} = \frac{\partial}{\partial \alpha} \]

the action of which on a point holonomy $h_v(A, \alpha) = \exp A(\cos \alpha \tau_1 + \sin \alpha \tau_2)$ (in our $\tau_3$-gauge) can be calculated, using

\[ h_v(A, \alpha) = \exp(\alpha \tau_3) h_v(A, 0) \exp(-\alpha \tau_3) , \]

to be

\[ \frac{\partial h_v}{\partial \alpha} = \tau_3 h_v - h_v \tau_3. \]

Now we are in a position to quantize the remaining part of the Gauß constraint:

\[ \frac{4\pi}{\kappa \ell} \int_B dx \Lambda(x)(A_2 E^3 - A_3 E^2) f_\gamma \]

\[ = \frac{\hbar}{i} \int_B dx \Lambda(x) \frac{\delta}{\delta \alpha(x)} f_\gamma \]

\[ = \frac{\hbar}{i} \int_B dx \Lambda(x) \sum_{v \in B} \delta(x, v) \text{tr} \left( (\tau_3 h_v - h_v \tau_3)^T \frac{\partial}{\partial h_v} \right) f_\gamma \]

\[ = i\hbar \sum_{v \in B} \Lambda(v) \left( X^3_L(h_v) - X^3_R(h_v) \right) f_\gamma . \]

We can now combine the operators to obtain the quantization

\[ \hat{G}^E[\Lambda] = i\hbar \sum_{v \in B} \Lambda(v) \left( \sum_{e(1)=v} X^2_L(h_e) - \sum_{e(0)=v} X^3_R(h_e) + X^3_L(h_v) - X^3_R(h_v) \right) \]

of the Gauß constraint which has a finite action on cylindrical functions and is therefore densely defined.
The structure of the operator is as expected: It is a sum over vertices weighted with the Lagrange multiplier. In each vertex it acts as the sum of a left invariant vector field for each incoming edge, a right invariant vector field for each outgoing edge, and a left as well as right invariant vector field for the point holonomy in the vertex. That the point holonomy contributes by left and right invariant vector fields represents the fact that the Higgs field transforms in the adjoint representation of $SU(2)$. The sum of vector fields is in complete agreement with the definition of gauge invariant spin networks with Higgs. However, $\hat{G}^E$ is a sum only of the third components of the vector fields in accordance with our $\tau_3$-gauge. Recall that $\hat{G}^E$ is just the quantization on the space of functions over $(\overline{A} \times \overline{U})^\lambda$, and that it has to be extended to our full reduced Hilbert space of functions on $(\overline{A} \times \overline{U})^{[\lambda]}$ by gauge covariance.

We can change the partial gauge fixing by performing a global $SU(2)$-gauge transformation with $g \in SU(2) \backslash \lambda(F)$. This transformation will conjugate the holonomies appearing in the vector fields. Using the formula

$$\frac{\partial}{\partial g^{-1}hg} = g^T \frac{\partial}{\partial h} (g^{-1})^T$$

which can be proved by using the chain rule, we get the transformation rule

$$X^i_L(g^{-1}hg) = \text{tr} \left( (g^{-1}hg \tau_i)^T \frac{\partial}{\partial (g^{-1}hg)} \right) = \text{tr} \left( (hg \tau_i g^{-1})^T \frac{\partial}{\partial h} \right) = \text{Ad}_{ij}(g)X^j_L(h)$$

where $\text{Ad}_{ij}(g)$ are matrix elements in the adjoint representation defined by $\text{Ad}_g \tau_i = g \tau_i g^{-1} =: \text{Ad}_{ij}(g) \tau_j$. The right invariant vector fields transform analogously. The transformed Gauß constraint in the $g \tau_3 g^{-1}$-gauge is now

$$\hat{G}^i_3[\Lambda] = i\hbar \sum_{v \in B} \Lambda(v) \left( \sum_{e(1) = v} X^3_L(g^{-1}h_v g) - \sum_{e(0) = v} X^3_R(g^{-1}h_e g) + X^3_L(g^{-1}h_v g) - X^3_R(g^{-1}h_v g) \right) = \text{Ad}_{3i}(g)\hat{G}_i[\Lambda] =: \hat{G}[\Lambda_i].$$
Here \( \mathcal{G}_i \) are the components of the full \( SU(2) \)-Gauß constraint smeared with a function \( \Lambda \), whereas \( \mathcal{G} \) is the full Gauß constraint smeared with a function \( \Lambda_i := \text{Ad}_{g_i}(g)\Lambda \). Allowing arbitrary \( SU(2) \)-gauge transformations \( g \), we can change \( \Lambda_i \) arbitrarily. This shows that the Gauß constraint on \( \Phi_B \) is the full \( SU(2) \)-constraint forcing \( SU(2) \)-spin networks with Higgs to be gauge invariant.

We want to stress here the necessity of the mathematical apparatus developed in Section 3: It enabled us to take the role of the reduced gauge group into account, it provides us with an interpretation of symmetric states as generalized states of the unreduced theory and Theorem 3.2 shows that all symmetric states can be obtained by using the one-dimensional spin networks used in the present subsection.

### 4.3.3 Quantization of the Diffeomorphism Constraint

We conclude by making a few remarks on the diffeomorphism constraint. It can be solved by group averaging where the diffeomorphism group acts by dragging the Higgs vertices. But we can alternatively regularize the diffeomorphism constraint and solve it infinitesimally. The no-go theorem of Ref. [12, Appendix C] is evaded by the one-dimensional nature of our graphs: Diffeomorphisms act only by dragging the ends of edges and the Higgs vertices attached to them. But they can not deform an edge transversally, and a longitudinal deformation can always be absorbed by a reparameterization of the edge. The quantized constraint is given by Lie derivatives on the Higgs vertex positions leading to the well known solutions in \( \Phi'_B \).

### 5 Conclusions

The main part of this paper is intended to define a space of symmetric states in a diffeomorphism invariant theory of connections, to investigate its properties and, especially, to equip it with a calculus. This is achieved in Theorem 3.2. The results show that there are some subtleties, mainly due to the classical partial gauge fixing, which should be relaxed in the quantum theory.

In order to illustrate different points of the general framework we discussed several concrete examples:

2 + 1 dimensional gravity and spherically symmetric electromagnetism are quite easy to deal with. Quantum symmetry reduction of these models
yields the expected results. Holonomy variables turned out to be well suited in order to quantize the electromagnetic model and its observables. Furthermore, the Ashtekar-Lewandowski measure incorporates the classical reality conditions correctly. The electromagnetic model exhibits a simple reduction of the degrees of freedom to finitely many ones corresponding to the classical theory.

Spherically symmetric gravity is the only example which has necessarily a Higgs field in physically meaningful applications. This property allows a non-trivial action of constraints. The Gauß and diffeomorphism constraint are as easy to deal with as in the full 3+1 dimensional theory. Unfortunately, the Hamiltonian constraint does not seem to be more easily solvable within the framework of loop quantum gravity with its spin network states than in the full theory. In Refs. [21, 22] it was solved, but it, together with the diffeomorphism constraint, had to be rephrased into a form not well suited for loop quantization.

But regularized analogously to Ref. [14] the Hamiltonian constraint operator might be defined and analyzed more easily, because of the simplicity of one-dimensional graphs it acts on. There is no place for it to create new edges and it will create only Higgs vertices in the neighborhood of a Higgs vertex it acts on, thereby changing the spin of the edge connecting the old and the new Higgs vertex. Note in this context that the appearance of “Simon’s subgraph” of Ref. [44] is generic in the symmetry reduced theory: The newly created vertices can always be shifted to their neighboring vertices. Details and further developments are left for forthcoming publications [31].

A first physical application of the above considerations in the case of gravity is to calculate the spectrum of the area operator in the spherically symmetric sector [32].

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