GLOBAL WEAK SOLUTIONS TO THE STOCHASTIC ERICKSEN–LESLES SYSTEM IN DIMENSION TWO

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Abstract. We establish the global existence of weak martingale solutions to
the simplified stochastic Ericksen–Lesles system modeling the nematic liquid
crystal flow driven by Wiener-type noises on the two-dimensional bounded do-
mains. The construction of solutions is based on the convergence of Ginzburg–
Landau approximations. To achieve such a convergence, we first utilize the
concentration-cancellation method for the Ericksen stress tensor fields based
on a Pohozaev type argument, and then the Skorokhod compactness theorem,
which is built upon uniform energy estimates.

1. Introduction. In this article, we consider the following simplified stochastic
Ericksen–Lesles system on a two dimensional bounded domain $D$ with smooth
boundary:

$$
\begin{cases}
    du + (u \cdot \nabla u + \nabla P - \mu \Delta u)dt = -\lambda \nabla \cdot (\nabla d \circ \nabla d)dt + \xi_1 S(u)dW_1, \\
    \nabla \cdot u = 0, \\
    dd + u \cdot \nabla d dt = \gamma (\Delta d + |\nabla d|^2 d)dt + \xi_2 (d \times h) \circ dW_2,
\end{cases}
$$

(1)

where $u : D \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^2$, $d : D \times \mathbb{R}^+ \times \Omega \to S^2$ represent the fluid velocity field and the molecular director field, respectively, $P : D \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ stands for the hydro-static pressure. $(\nabla d \circ \nabla d)_{ij} = \langle \partial_i d, \partial_j d \rangle \ (1 \leq i, j \leq 2)$ represents the Ericksen stress tensor field. The multiplicative noise term $S(u)dW_1$ in (1) shall be understood in the Itô sense with a cylindrical Wiener process $W_1$ on a separable Hilbert space $K_1$. For a given $h : \mathbb{R}^2 \to \mathbb{R}^3$, $(d \times h) \circ dW_2$ is understood in the Stratonovich sense with a standard real-valued Brownian motion $W_2$. $\mu, \lambda, \gamma, \xi_1, \xi_2$ are positive physical constants.

We assume, further, $(u, d)$ satisfies the following initial-boundary conditions:

$$
(u, d)|_{t=0} = (u_0, d_0), \quad \text{in } D.
$$

(2)
\[ u|_{\partial D} = 0, \quad \frac{\partial d}{\partial n}|_{\partial D} = 0, \quad (\text{or } d|_{\partial D} = d_0). \]  

where \( n \) is the unit outward normal to \( \partial D \). In this paper, we use the Ginzburg–Landau type approximation which relaxes the condition \( |d| = 1 \) in (1) by introducing a penalized term. More specifically, we have a family of solutions \((u^\varepsilon, d^\varepsilon)_{0 < \varepsilon < 1}\) to

\[
\begin{aligned}
&d u^\varepsilon + (u^\varepsilon \cdot \nabla u^\varepsilon + \nabla P^\varepsilon - \mu \Delta u^\varepsilon) dt \\
&\quad = -\lambda \nabla \cdot (\nabla d^\varepsilon \circ \nabla d^\varepsilon) dt + \xi_1 S(u^\varepsilon) dW_1, \\
&\nabla \cdot u^\varepsilon = 0, \\
&d d^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon dt = \gamma \left( \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon) \right) dt + \xi_2 (d^\varepsilon \times h) \circ dW_2,
\end{aligned}
\]

where \( f_\varepsilon(d^\varepsilon) = \nabla_d F_\varepsilon(d^\varepsilon) = \frac{1}{2\varepsilon}(|d^\varepsilon|^2 - 1)d^\varepsilon \) with \( F_\varepsilon(d) = \frac{1}{4\varepsilon^2}(1 - |d|^2)^2 \).

In the deterministic case \((\xi_1 = \xi_2 = 0)\), the global existence of the weak solutions to the Ginzburg–Landau type Ericksen–Leslie system (4), which is a simplified version of the full Ericksen–Leslie system \([11, 12, 19, 20]\), was first investigated by Lin–Liu \([22]\). For the simplified Ericksen–Leslie system (1), motivated by Struwe \([27]\) on harmonic map heat flows in dimension two, the existence of a unique global weak solution with partial regularity was established Lin–Liu–Wang \([21]\) and Lin–Wang \([23]\), which was generalized by Huang–Lin–Wang \([17]\) for the full Ericksen–Leslie system. See also Hong \([15]\) and Hong–Xin \([16]\) for related works. We refer the readers to \([24]\) for a comprehensive survey for the recent developments. The question that whether one can obtain a weak solution of (1) via sending \( \varepsilon \to 0 \) in (4) remains open due to the difficulty with possible defect measures appearing in the Ericksen stress tensor field. In a very recent paper \([18]\), Kortum applied a concentration-cancellation method initiated by Diperna–Majda \([9]\) on the 2-D incompressible Euler equation to show that \( \text{div}(\nabla d \circ \nabla d^\varepsilon) \rightharpoonup \text{div}(\nabla d \circ \nabla d) \) in the torus \( T^2 \). For general domains and full Ericksen–Leslie system, the weak compactness result was shown in \([10]\) via the Hopf differential and the Pohozaev technique. We also want to point out that in 3-D, the Ginzburg–Landau approximation was implemented in \([25]\) to construct a global weak solutions to the simplified Ericksen–Leslie system (1) with the half-sphere assumption imposed on directors \((d \in S^2_+)\).

On the other hand, there is a growing number of research studies that are devoted to the simplified stochastic Ericksen–Leslie system (4) with various types of random noises \((\xi_1^2 + \xi_2^2 > 0)\). See for instance, \([3, 5, 6, 7]\). For the mathematical modeling, taking the stochastic terms into account reflects the influence of environmental noises, the measurement uncertainties as well as the thermal fluctuations. Analogously, Bouard–Hocquet–Prohl obtained the Struwe-like global solution to (1) in \([8]\) by a bootstrap argument together with Gyöngy–Krylov \( L^p \) estimates \([14]\). Very recently, Brzeźniak, Deugoué, and Razafimandimby in \([2]\) proved the existence of short time strong solutions to the simplified stochastic Ericksen–Leslie system. The main goal of this paper is to obtain a global weak solution to (1) by extending the compactness argument from \([10]\) into the stochastic setting.

For simplicity, we assume \( \lambda = \xi_1 = \gamma = \xi_2 = 1 \). We introduce some function spaces:

\[
\begin{align*}
H &= \text{closure of } C^\infty_0(D, \mathbb{R}^2) \cap \{ f | \nabla \cdot f = 0 \} \text{ in } L^2(D, \mathbb{R}^2), \\
J &= \text{closure of } C^\infty_0(D, \mathbb{R}^2) \cap \{ f | \nabla \cdot f = 0 \} \text{ in } H^1_0(D, \mathbb{R}^2), \\
H^1(D, S^2) &= \{ f \in H^1(D, \mathbb{R}^3) | |f| = 1 \text{ a.e. } x \in D \}.
\end{align*}
\]
For a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), let \(K_1\) be an infinite dimensional separable Hilbert space and \(W_1 = \{W_1(t)\}_{t \geq 0}\) be a \(K_1\)-cylindrical Wiener process such that it is formally written as a series

\[
W_1(t) = \sum_{i=1}^{\infty} B_i(t)e_i, \forall t \geq 0,
\]

where \(\{B_i(t)\}_{i=1}^{\infty}\) is a family of i.i.d. standard Brownian motions and \(\{e_i\}_{i=1}^{\infty}\) is an orthonormal base of \(K_1\). The above series does not converge in \(K_1\), but it does converge in \(K_2\) if \(K_2\) is a larger Hilbert space containing \(K_1\) such that the inclusion map \(J: K_1 \to K_2\) is Hilbert-Schmidt. It is always possible to construct a space \(K_2\) with this property. For example, we can define \(K_2\) to be the closure of \(K_1\) under the norm

\[
\|x\|_{K_2}^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} \langle x, e_i \rangle_{K_1}^2.
\]

Then we can view \(W_1\) as a \(K_2\)-valued Wiener process. Let \(W_2 = \{W_2(t)\}_{t \geq 0}\) be a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) adapted to \(\{\mathcal{F}_t\}_{t \geq 0}\). \(S\) is a map from \(H\) to \(L_2(K_1, J)\), where \(L_2(K_1, J)\) denotes the space of all Hilbert–Schmidt operators from \(K_1\) to \(J\), i.e.,

\[
\sum_{i=1}^{\infty} \|S^i(\cdot)\|_J^2 < \infty, \text{ if } \{e_i\}_{i=1}^{\infty} \text{ is an orthonormal base of } K_1.
\]

We now introduce the notion of a weak martingale solution to (1).

**Definition 1.1.** A weak martingale solution to (1), (2), (3) is a system consisting of a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), and \(\mathcal{F}_t\) adapted stochastic processes \((u(t), d(t), W_1(t), W_2(t))_{t \geq 0}\) such that for any \(0 < T < \infty\)

1. \(\{W_1(t)\}_{t \geq 0}\) (or \(\{W_2(t)\}_{t \geq 0}\)) is a \(K_1\)-cylindrical (resp. real-valued) Wiener process.
2. \((u, d) : \Omega \times \mathbb{R}_+ \to H \times H^1(D, S^2)\) is progressively measurable with respect to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) such that for almost surely \(\omega \in \Omega\),

\[
u \in L^\infty([0, T], H) \cap L^2([0, T], J), \quad d \in L^2([0, T], H^1(D, S^2)).
\]

3. We have

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_{D \times \{t\}} |\nu|^2 + |\nabla d|^2 + \int_0^T \int_D (|\nabla \nu|^2 + |\Delta \nu + |\nabla d|^2 |d|^2) dx ds \right] < \infty.
\]

(5)

Here \(\mathbb{E}\) stands for the expectation.

4. For almost surely \(\omega \in \Omega\), for every \(t \in [0, T]\), for any \(\varphi \in C^\infty(D, \mathbb{R}^2)\), \(\text{div} \varphi = 0\), we have

\[
- \int_{D \times \{t\}} \langle u, \varphi \rangle dx - \int_0^t \int_D (\langle u \otimes u, \nabla \varphi \rangle + \langle u, \Delta \varphi \rangle) dx ds
\]

\[
= - \int_D \langle u_0, \varphi \rangle dx + \int_0^T \int_D (\langle \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_2, \nabla \varphi \rangle) dx ds
\]

\[
+ \int_0^T \int_D \langle \varphi, S(u) dW_1(s) \rangle dx,
\]

and for any \(\psi \in C^\infty(D, \mathbb{R}^3)\),
1.2 to dimension three, if, in addition, the initial data estimates for the approximate solutions \((u_0, d_0)\) are such that after passing to a subsequence, \(u_0 \rightharpoonup u_0^\varepsilon, d_0 \rightharpoonup d_0^\varepsilon\) weakly in \(L^2((0, T]; H^1(\Omega; \mathbb{H}))\). Furthermore, we assume \(((u_0^\varepsilon, d_0^\varepsilon))_{0 < \varepsilon < 1} \subseteq J \times H^2(D; \mathbb{S}^2)\) and satisfies \((u_0^\varepsilon, d_0^\varepsilon) \to (u_0, d_0)\) in \(H^2(D; \mathbb{S}^2)\).

We introduce the following assumptions that are required by our main theorem.

**Assumption 1.** Let \(S : \mathbb{H} \to \mathcal{L}_2(K_1, J)\) be a global Lipschitz map. In particular, there exists \(C > 0\) such that \(\|S(u)\|_{\mathcal{L}_2(K_1, J)}^2 \leq C(1 + \|u\|_H^2)\) for all \(u \in \mathbb{H}\), \(h \in H^2(\mathbb{R}^2, \mathbb{R}^3)\) and \((u_0, d_0) \in H \times H^1(D; \mathbb{S}^2)\). Furthermore, we assume \(\{(u_0^\varepsilon, d_0^\varepsilon)\}_{0 < \varepsilon < 1} \subseteq \mathcal{J} \times H^2(D; \mathbb{S}^2)\) and satisfies \((u_0^\varepsilon, d_0^\varepsilon) \to (u_0, d_0)\) in \(H \times H^1(D; \mathbb{S}^2)\).

Similar to Definition 1.1, a weak martingale solution \((u^\varepsilon(t), d^\varepsilon(t), W^\varepsilon_1(t), W^\varepsilon_2(t))\) adapted to a family of complete filtered probability spaces \((\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon, \{\mathcal{F}^\varepsilon_t\}_{t \geq 0})\) to (4), (2), (3) can be defined. Under Assumption 1, the existence of weak martingale solutions \((u^\varepsilon, d^\varepsilon, W^\varepsilon_1, W^\varepsilon_2)\) with respect to \((\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon, \{\mathcal{F}^\varepsilon_t\}_{t \geq 0})\) was established in [6, Theorem 3.2] via the Faedo–Galerkin approximation and the weak compactness method, together with the path-wise uniqueness in 2-D [6, Theorem 3.4]. It has been proved in the recent work [4, Theorem 3.17] that (4) possesses a unique strong solution, that is, given \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W_1, W_2)\), there exists a unique pair of stochastic processes \((u^\varepsilon, d^\varepsilon)\) which solves (4) with respect to \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W_1, W_2)\) for initial data \((u_0^\varepsilon, d_0^\varepsilon)\) in \(J \times H^2(D; \mathbb{S}^3)\).

Our main result asserts the existence of a global weak martingale solution to (1) via passing the limit of solutions \((u^\varepsilon, d^\varepsilon)\) to (4):

**Theorem 1.2.** Under Assumption 1, there exist a completed filtered probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) and a sequence of weak martingale solutions \((u^\varepsilon, d^\varepsilon, W^\varepsilon_1, W^\varepsilon_2)\) to (4), (2), (3) on \((\Omega', \mathcal{F}', \mathbb{P}')\) and a weak martingale solution \((u, d, W_1, W_2)\) to (1), (2), (3) such that after passing to a subsequence,

\[
\begin{align*}
  u^\varepsilon & \to u \text{ in } L^2(\Omega'; L^2([0, T], H^1(\Omega; \mathbb{H}))), \\
  d^\varepsilon & \to d \text{ in } L^2(\Omega'; L^2([0, T], H^1(\Omega; \mathbb{H})))
\end{align*}
\]

as \(\varepsilon \to 0\).

We would like to remark that it is feasible to extend the conclusion of Theorem 1.2 to dimension three, if, in addition, the initial data \(d_0(\Omega) \subset S^2_+\) and \(h\) satisfies \(h \times e_3 = 0\), where \(e_3 = (0, 0, 1)^T\). In this case, one can first show that the third component of \(d^\varepsilon\), \((d^\varepsilon)^3 \geq 0\) in \(\Omega \times \mathbb{R}_+\), and then modify the argument by Lin-Wang [25] from the deterministic to stochastic case accordingly.

The paper is organized as follows. In section 2 we establish some uniform energy estimates for the approximate solutions \((u^\varepsilon, d^\varepsilon)\) by Itô’s formula. The convergence of the approximate system, in particular, the Ericksen stress tensor field and martingale terms will be discussed in section 3. In Appendix A, we provide the computation of Itô’s formula for two functionals of \(d\).

2. Uniform estimates on approximated solutions. In this section, we will derive an uniform energy estimate for (4), (2), (3) via Itô’s calculus.

For simplicity, we denote \(\| \cdot \| := \| \cdot \|_{L^2(D)}\). First, applying Itô’s formula to \(\frac{1}{2}\|u^\varepsilon(t)\|^2\) yields
\[
\frac{1}{2} |u^\varepsilon(t)|^2 - \frac{1}{2} |u_0^\varepsilon|^2 + \int_0^t \int_D |\nabla u^\varepsilon|^2 dxds = \int_0^t \int_D (\nabla d^\varepsilon \cdot \nabla u^\varepsilon) dxds \\
+ \frac{1}{2} \int_0^t \int_D \nabla d^\varepsilon \cdot \nabla (u^\varepsilon, \nabla u^\varepsilon) dxds + \int_0^t \int_D (u^\varepsilon, S(u^\varepsilon)dW_1(s)) dx,
\]
where we have used the cancellation property
\[
\int_0^t \int_D (u^\varepsilon \cdot \nabla u^\varepsilon, u^\varepsilon) dxds = 0.
\]
From the relation between Stratonovich and Itô's integral, we have that
\[
(d \times h) \circ dW_2 = \frac{1}{2}((d \times h) \times h) dt + (d \times h)dW_2.
\]
Therefore (4)_3 can be written as
\[
dd d^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon dt = \left( \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon) + \frac{1}{2}(d^\varepsilon \times h) \right) dt + (d^\varepsilon \times h)dW_2.
\]
Now we apply the Itô formula to \(\Phi_\varepsilon(d^\varepsilon) := \frac{1}{2} \|\nabla d^\varepsilon\|^2 + \int_D F_\varepsilon(d^\varepsilon) dx\) (see Appendix A) to get
\[
\Phi_\varepsilon(d^\varepsilon)(t) - \Phi_\varepsilon(d_0^\varepsilon) = \int_0^t \int_D (u^\varepsilon \cdot \nabla d^\varepsilon, \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)) dxds - \int_0^t \int_D |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2 dxds \\
+ \frac{1}{2} \int_0^t \int_D (\nabla d^\varepsilon, \nabla (d^\varepsilon \times h) + |\nabla(d^\varepsilon \times h)|^2) dxds \\
+ \frac{1}{2} \int_0^t \int_D (-\Delta d^\varepsilon + f_\varepsilon(d^\varepsilon), d^\varepsilon \times h) dxdsdW_2(s).
\]
Using the fact that
\[
\int_0^t \int_D (u^\varepsilon \cdot \nabla d^\varepsilon, f_\varepsilon(d^\varepsilon)) dxds = \int_0^t \int_D u^\varepsilon \cdot \nabla F_\varepsilon(d^\varepsilon) dxds = 0,
\]
and
\[
- \int_0^t \int_D \langle u^\varepsilon \cdot \nabla d^\varepsilon, \Delta d^\varepsilon \rangle dxds \\
= \int_0^t \int_D \langle \nabla d^\varepsilon \circ \nabla d^\varepsilon, \nabla u^\varepsilon \rangle dxds + \int_0^t \int_D u^\varepsilon \cdot \nabla \left( \frac{|\nabla d^\varepsilon|^2}{2} \right) dxds \\
= \int_0^t \int_D \langle \nabla d^\varepsilon \circ \nabla d^\varepsilon, \nabla u^\varepsilon \rangle dxds,
\]
we can add (8) and (10) together to obtain
\[
\frac{1}{2} |u^\varepsilon(t)|^2 + \frac{1}{2} |\nabla d^\varepsilon(t)|^2 \\
+ \int_{D \times \{t\}} F_\varepsilon(d^\varepsilon) dx + \int_0^t \int_D (|\nabla u^\varepsilon|^2 + |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2) dxds \\
= \frac{1}{2} |u_0^\varepsilon|^2 + \frac{1}{2} |\nabla d_0^\varepsilon|^2
\]
We can derive from taking the expectation of (12) that

\[ \frac{1}{2} \int_0^t \left( \| S(u^t) \|_{L_2(K_i, H)}^2 + \| \nabla (d^t \times h) \|^2 \right) ds \]

\[ + \frac{1}{2} \int_0^t \int_D (\nabla d^t, \nabla ((d^t \times h) \times h)) dx ds \]

\[ + \int_0^t \int_D (u^t, S(u^t)dW_1(s)) dx \]

\[ + \int_0^t \int_D (d^t \times h, \Delta d^t - f_\epsilon(d^t)) dx dW_2(s). \]

It has been shown in [6, Theorem 5.1] that \( d^t \) satisfies the maximum principle, i.e., \( |d^t| \leq 1 \) for almost all \((\omega, t, x) \in \Omega \times [0, T] \times D \) provided \( |d_0^t| \leq 1 \). Hence we have that

\[ \int_0^t \| S(u^t) \|_{L_2(K_i, H)}^2 ds \leq \int_0^t \| S(u^t) \|_{L_2(K_i, J)}^2 ds \leq C \int_0^t \int_D (1 + |u^t|^2) dx ds, \]

\[ \int_0^t \int_D |\nabla (d^t \times h)|^2 dx ds \leq C \int_0^t \int_D (|\nabla d^t|^2 + |\nabla h|^2) dx ds, \]

\[ \int_0^t \int_D (\nabla d^t, \nabla ((d^t \times h) \times h)) dx ds \leq C \int_0^t \int_D (|\nabla d^t|^2 + |\nabla h|^2) dx ds. \]

Combine all these estimates above, we arrive at

\[ \frac{1}{2} \| u^t(t) \|^2 + \frac{1}{2} \| \nabla d^t(t) \|^2 \]

\[ + \int_{D \times \{t\}} F_\epsilon(d^t) dx + \int_0^t \int_D (|\nabla u^t|^2 + |\Delta d^t - f_\epsilon(d^t)|^2) dx ds \]

\[ \leq \frac{1}{2} \| u_0^t \|^2 + \frac{1}{2} \| \nabla d_0^t \|^2 \]

\[ + C \int_0^t \int_D (|u^t|^2 + |\nabla d^t|^2 + |\nabla h|^2) dx ds \]

\[ + \int_0^t \int_D (u^t, S(u^t)dW_1(s)) dx + \int_0^t \int_D (d^t \times h, \Delta d^t - f_\epsilon(d^t)) dx dW_2(s). \]

We can derive from taking the expectation of (12) that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left[ \| u^t(t) \|^2 + \| \nabla d^t(t) \|^2 + \int_{D \times \{t\}} F_\epsilon(d^t) dx \right] \]

\[ + \mathbb{E} \int_0^T \int_D (|\nabla u^t|^2 + |\Delta d^t - f_\epsilon(d^t)|^2) dx ds \]

\[ \leq C \mathbb{E} \int_0^T \int_D (|u^t|^2 + |\nabla d^t|^2 + |\nabla h|^2) dx ds \]

\[ + C \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_D (u^t(s), S(u^t(s))dW_1(s)) dx \right| \]

\[ + C \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_D (d^t \times h, \Delta d^t - f_\epsilon(d^t)) dx dW_2(s) \right| \]

\[ + C(1 + \| u_0 \|^2 + \| \nabla d_0 \|^2). \]
Now we use the Burkholder–Davis–Gundy inequality, Cauchy–Schwarz inequality and Hölder inequality to show that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_D \langle \mathbf{d}^\varepsilon \times \mathbf{h}, \Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon) \rangle dx \right| ds \leq C \mathbb{E} \left[ \int_0^T \left| \int_D \langle \mathbf{d}^\varepsilon \times \mathbf{h}, \Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon) \rangle dx \right|^2 ds \right]^{\frac{1}{2}} \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \mathbf{d}^\varepsilon \times \mathbf{h} \|_{L^\infty(D)} \left( \int_0^T \int_D |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2 dx ds \right)^{\frac{1}{2}} \right] \leq C \mathbb{E} \sup_{0 \leq t \leq T} \| \mathbf{d}^\varepsilon \times \mathbf{h} \|_{L^\infty(D)}^2 + \frac{1}{4} \mathbb{E} \int_0^T \int_D |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2 dx ds \leq C(\| \mathbf{h} \|_{L^\infty}, T, D).
\]

Similarly, we can show

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_D \langle \mathbf{u}^\varepsilon(s), S(\mathbf{u}^\varepsilon(s)) \rangle dx ds \right| \leq \frac{1}{4} \mathbb{E} \sup_{0 \leq t \leq T} \| \mathbf{u}^\varepsilon(t) \|^2 + C \mathbb{E} \int_0^T \int_D |\mathbf{u}^\varepsilon|^2 dx ds.
\]

Now we can substitute (14) and (15) into (13) to get

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left[ \| \mathbf{u}^\varepsilon(t) \|^2 + \| \nabla \mathbf{u}^\varepsilon(t) \|^2 + \int_{D \times \{ t \}} F_\varepsilon(\mathbf{d}^\varepsilon) dx \right] + \mathbb{E} \int_0^T \int_D (|\nabla \mathbf{u}^\varepsilon|^2 + |\Delta \mathbf{d}^\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}^\varepsilon)|^2) dx ds \leq C \mathbb{E} \int_0^T \int_D (|\mathbf{u}^\varepsilon|^2 + |\nabla \mathbf{d}^\varepsilon|^2 + |\nabla \mathbf{h}|^2) dx ds + C(\| \mathbf{u}_0, \nabla \mathbf{d}_0 \|, \| \mathbf{h} \|_{L^\infty}, T, D).
\]

It follows from Gronwall’s lemma that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left[ \| \mathbf{u}^\varepsilon(t) \|^2 + \| \nabla \mathbf{u}^\varepsilon(t) \|^2 + \int_{D \times \{ t \}} F_\varepsilon(\mathbf{d}^\varepsilon) dx \right] \leq C(\| \mathbf{u}_0, \nabla \mathbf{d}_0 \|, \| \mathbf{h} \|_{L^\infty}, \| \nabla \mathbf{h} \|, T, D).
\]

Furthermore, if we raise both sides of (12) to the power \( p \) \( (p > 1) \) and take the expectation, we arrive at

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left[ \| \mathbf{u}^\varepsilon(t) \|^2 + \| \nabla \mathbf{u}^\varepsilon(t) \|^2 + \int_{D \times \{ t \}} F_\varepsilon(\mathbf{d}^\varepsilon) dx \right]^p \]
A similar argument yields

\[
E \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \left( \| \nabla u^\varepsilon \|^2 + |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2 \right) dx ds \right)^p \right] \leq C(\| u_0 \|, \| \nabla d_0 \|, p) + CTE \left( \int_0^T \left[ \| u^\varepsilon(t) \|^2 + \| \nabla d^\varepsilon(t) \|^2 + \| \nabla h \|^2 \right] dt \right)^p
\]

Now we apply the Burkholder–Davis–Gundy, Cauchy–Schwarz, and Hölder inequalities to the last two terms in the right hand side to get

\[
E \sup_{0 \leq t \leq T} \left| \int_0^t \int_D \langle u^\varepsilon(s), S(u^\varepsilon(s)) \rangle dx dW_1(s) \right|^p \leq C \left( \int_0^T \| u^\varepsilon(s) \|^2 \| S(u^\varepsilon(s)) \|^2 ds \right)^{\frac{p}{2}} \leq C \left( \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|^p \left( \int_0^T (1 + \| u^\varepsilon(s) \|^2) ds \right)^{\frac{p}{2}} \right) \leq \frac{1}{4} E \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|^{2p} + C \int_0^T (1 + \| u^\varepsilon(s) \|^2)^p ds.
\]

A similar argument yields

\[
E \sup_{0 \leq t \leq T} \left| \int_0^t \int_D \langle d^\varepsilon \times h, \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon) \rangle dx dW_2(s) \right|^p \leq \frac{1}{4} E \left[ \int_0^T \int_D |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2 dx ds \right]^p + C \sup_{0 \leq t \leq T} \| d^\varepsilon \times h \|^{2p}.
\]

Combine (17), (18) and (19), by Gronwall’s inequality we obtain that for \( p \geq 1 \), it holds

\[
E \sup_{0 \leq t \leq T} \left[ \| u^\varepsilon(t) \|^2 + \| \nabla d^\varepsilon(t) \|^2 + \int_{D \times \{ t \}} F_\varepsilon(d^\varepsilon) dx \right]^p + E \left[ \int_0^T \int_D (|\nabla u^\varepsilon|^2 + |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2) dx ds \right]^p \leq C(\| u_0 \|, \| \nabla d_0 \|, \| h \|_{L^\infty}, \| \nabla h \|, T, D, p).
\]

Similar to the Aubin-Lions lemma in the deterministic case, we need some fractional Sobolev estimates in \( t \) variable as in [13] for stochastic Navier-Stokes equations. Write

\[
u^\varepsilon(t) = u_0^\varepsilon + \int_0^t P \Delta u^\varepsilon(s) ds - \int_0^t P \nabla \cdot (u^\varepsilon \otimes u^\varepsilon)(s) ds
\]

\[
- \int_0^t P \nabla \cdot (\nabla u^\varepsilon \otimes \nabla u^\varepsilon)(s) ds + \int_0^t S(u^\varepsilon(s)) dW_1(s)
\]
Applying [13, Lemma 2.1] to $X$, it holds
\[
\begin{align*}
E \left[ \|I_2^2\|_{W^{1,2}(0, T; H^{-1}(D))}^2 + \|I_3^2\|_{W^{1,2}(0, T; H^{-1}(D))}^2 \right] \leq C,
\end{align*}
\]
and
\[
E \left[ \|I_4^2\|_{W^{1,2}(0, T; W^{-2,\bar{p}}(D))}^2 \right] \leq C, \quad \text{for some } \bar{p} > 2.
\]
Applying [13, Lemma 2.1] to $I_4$, we conclude that for any $\alpha \in (0, \frac{1}{2})$ and $p \in [2, \infty)$, it holds
\[
E \left[ \|I_4\|_{W^{\alpha, p}(0, T; L^2(D))}^p \right] = E \left[ \left\| \int_0^t S(u^\varepsilon(s))dW_1(s) \right\|_{W^{\alpha, p}(0, T; L^2(D))}^p \right]
\leq CE \int_0^T \|S(u^\varepsilon(t))\|_{L^p(K_1, H)}^p dt
\leq CE \int_0^T (1 + \|u^\varepsilon(t)\|_{L^2(D)}^p) dt \leq C.
\]
Now we define
\[
X := L^\infty([0, T]; L^2(D)) \cap L^2([0, T]; H^1(D))
\cap \left( W^{1,2}([0, T]; H^{-1}(D)) + W^{1,2}([0, T]; W^{-2,\bar{p}}(D)) + W^{\alpha, p}([0, T]; L^2(D)) \right).
\]
Let $\{ \mathcal{L}(u^\varepsilon) \}_{0 < \varepsilon < 1}$ be a family of probability measures define on $X$ as following:
\[
\mathcal{L}(u^\varepsilon)(B) := P(u^\varepsilon \in B)
\]
for any Borel set $B \subset X$. For a fix $R > 0$, we can derive from Chebyshev’s inequality that
\[
P(\|u^\varepsilon\|_X > R)
\leq \frac{3}{R} P\left( \|u^\varepsilon\|_{L^\infty([0, T]; L^2(D))} > \frac{R}{3} \right)
+ \frac{3}{R} P\left( \|u^\varepsilon\|_{L^2([0, T]; H^1(D))} > \frac{R}{3} \right)
\leq \frac{C}{R}.
\]
By a fractional version of Aubin-Lions lemma and the Sobolev interpolation inequality, $X$ is compactly embedded in $L^p([0, T]; L^p(D)) \cap C([0, T]; W^{-2,\bar{p}}(D))$ for $1 < p < 4$ (c.f. [13, 26]). Therefore $\{ \mathcal{L}(u^\varepsilon) \}_{0 < \varepsilon < 1}$ is tight in $L^p([0, T]; L^p(D)) \cap C([0, T]; W^{-2,\bar{p}}(D))$ for $1 < p < 4$. Similarly, we have
\[
d^\varepsilon(t) = d_0^\varepsilon - \int_0^t \nabla \cdot (u^\varepsilon \otimes d^\varepsilon)(s) ds + \int_0^t (\Delta d^\varepsilon - f^\varepsilon)(s) ds
+ \frac{1}{2} \int_0^t ((d^\varepsilon \times h) \times h)(s) ds + \int_0^t (d^\varepsilon \times h)(s) dW_2(s)
:= d_0^\varepsilon + \sum_{i=1}^4 J_i^\varepsilon(t).
\]
Then we have
\[
E \left[ \| J^1 \|^4_{W^{1,4}([0,T];L^6(D))} \right] < C,
\]
\[
E \left[ \| J^2 \|^2_{W^{1,2}([0,T];L^6(D))} + \| J^3 \|^2_{W^{1,2}([0,T];L^\infty(D))} \right] \leq C,
\]
and by an argument similar to that of \( I_4 \) we can show that for any \( \alpha \in (0, \frac{1}{2}) \) and \( p \in [2, \infty) \), it holds
\[
E \left[ \| J^4_p \|^p_{W^{\alpha,p}([0,T];L^2(D))} \right] = E \left[ \left\| \int_0^t \mathbf{d} \times \mathbf{h} dW_2(s) \right\|^p_{W^{\alpha,p}([0,T];L^2(D))} \right]
\leq C E \int_0^T \| \mathbf{d} \times \mathbf{h}(t) \|^p_{L^2(D)} dt
\leq C E \int_0^T \| \mathbf{h} \|^p_{L^\infty} \| \mathbf{d} \|^p_{L^2(D)} dt \leq C.
\]
Hence, the laws \( \{ \mathcal{L}(\mathbf{d}^\varepsilon) \}_{0 < \varepsilon < 1} \) are bounded in probability in
\[
Y := L^\infty([0,T];H^1(D)) \cap \left( W^{1,4}([0,T];L^4(D)) + W^{1,2}([0,T];L^6(D)) + W^{\alpha,p}([0,T];L^2(D)) \right).
\]
Since \( Y \) is compactly embedded into \( L^q([0,T];L^q(D)) \cap C([0,T];L^\frac{4}{3}(D)) \cap C([0,T];L^\frac{2}{3}(D)) \) for \( 1 < p < 4, 1 < q < \infty \). Now we apply the Prohorov’s theorem, there exists a probability measure \( \mu \) on \( L^p([0,T];L^p(D)) \cap C([0,T];W^{-2,q}(D)) \cap C([0,T];L^q(D)) \) such that after passing to a subsequence,
\[
\mathcal{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon, W_1, W_2) \rightarrow \mu.
\]
Then by Skorokhod’s embedding theorem, there exists a complete probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) and a sequence of random variables \( (\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon, W_1, W_2) \) on \((\Omega', \mathcal{F}', \mathbb{P}')\) such that
\[
\mathcal{L}(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon, W_1, W_2) = \mathcal{L}(\mathbf{u}, \mathbf{d}, W_1, W_2), \quad (21)
\]
and \((\mathbf{u}, \mathbf{d}, W_1, W_2)\) defined on \((\Omega', \mathcal{F}', \mathbb{P}')\) such that
\[
\begin{aligned}
\mathcal{L}(\mathbf{u}, \mathbf{d}, W_1, W_2) &= \mu, \\
\mathbf{u}^\varepsilon &\rightarrow \mathbf{u} \text{ in } L^p([0,T];L^p(D)) \cap C([0,T];W^{-2,q}(D)), 1 < p < 4, \mathbb{P}'-a.s., \\
\mathbf{u} &\rightarrow \mathbf{u} \text{ in } L^2(\Omega' \times [0,T];\mathbf{J}), \\
\mathbf{d}^\varepsilon &\rightarrow \mathbf{d} \text{ in } L^q([0,T];L^q(D)) \cap C([0,T];L^\frac{4}{3}(D)), 1 < q < \infty, \mathbb{P}'-a.s., \\
W_1^\varepsilon &\rightarrow W_1^\varepsilon \text{ in } C([0,T];K_2), \quad \mathbb{P}'-a.s., \\
W_2^\varepsilon &\rightarrow W_2^\varepsilon \text{ in } C([0,T];\mathbb{R}), \quad \mathbb{P}'-a.s.,
\end{aligned}
\]
And for \( \mathbb{P}'-a.s. \), \( \mathbf{u} \in L^\infty([0,T];\mathbf{H}) \cap L^2([0,T];\mathbf{J}) \), \( \mathbf{d} \in L^\infty([0,T];H^1(D)) \).
For martingale solutions, for each $0 < \varepsilon < 1$, we define $M_{u^\varepsilon}(t), M_{d^\varepsilon}(t)$ as

$$M_{u^\varepsilon}(t) = u^\varepsilon(t) - u_0 + \int_0^t [P \nabla \cdot (u^\varepsilon \otimes u^\varepsilon) - P \Delta u^\varepsilon + P \nabla \cdot (\nabla d^\varepsilon \otimes \nabla d^\varepsilon)](s)ds,$$

$$M_{d^\varepsilon}(t) = d^\varepsilon(t) - d_0 + \int_0^t [\nabla \cdot (u^\varepsilon \otimes d^\varepsilon) - \Delta d^\varepsilon + f_\varepsilon(d^\varepsilon) - \frac{1}{2}(d^\varepsilon \times h) \times h](s)ds,$$

for any $t \in (0, T]$. Also define $M_{\bar{u}^\varepsilon}, M_{\bar{d}^\varepsilon}$ by replacing $u^\varepsilon, d^\varepsilon$ in $M_{u^\varepsilon}, M_{d^\varepsilon}$ by $\bar{u}^\varepsilon, \bar{d}^\varepsilon$.

Next we show that for $\mathbb{P}'$-a.s.,

$$M_{\bar{u}^\varepsilon}(t) = \int_0^t S(\bar{u}^\varepsilon)d\bar{W}_1(s), \quad (23)$$

$$M_{\bar{d}^\varepsilon}(t) = \int_0^t (\bar{d}^\varepsilon \times h)d\bar{W}_2(s) \quad (24)$$

for every $\varepsilon > 0$ and every $t \in [0, T]$. For any $z \in L^2(0, T; H^{-1})$ we set

$$\varphi(z) = \frac{\int_0^T \|z(s)\|_{H_{-1}}^2 ds}{1 + \int_0^T \|z(s)\|_{H_{-1}}^2 ds}.$$  

By a argument similar to that in [1, 6] we can show that

$$\mathbb{E}'\varphi\left(M_{\bar{u}^\varepsilon}(\cdot) - \int_0^\cdot S(\bar{u}^\varepsilon(s))d\bar{W}_1(s)\right) = \mathbb{E}\varphi\left(M_{u^\varepsilon}(\cdot) - \int_0^\cdot S(u^\varepsilon(s))dW_1(s)\right) = 0.$$  

This implies that for $\mathbb{P}'$-a.s. (23) holds for all $t \in (0, T]$. Similarly, we can show (24) is also true.

Let $M_u(t)$ and $M_d(t)$ be defined by

$$M_u(t) = u(t) - u_0 + \int_0^t [P \nabla \cdot (u \otimes u) - P \Delta u + P \nabla \cdot (\nabla d \otimes \nabla d)](s)ds,$$

$$M_d(t) = d(t) - d_0 + \int_0^t [\nabla \cdot (u \otimes d) - \Delta d - |\nabla d|^2 d - \frac{1}{2}(d \times h) \times h](s)ds.$$  

With (22), we have the almost surely convergence of every term in $M_{\bar{u}^\varepsilon}$ except the Ericksen stress tensor ($\nabla \bar{d}^\varepsilon \otimes \nabla \bar{d}^\varepsilon$). Now we claim that for $\mathbb{P}'$-a.s.

$$\lim_{\varepsilon \to 0} \int_0^T \int_D \langle \nabla \bar{d}^\varepsilon \otimes \nabla \bar{d}^\varepsilon - \frac{1}{2} |\nabla \bar{d}^\varepsilon|^2 \nabla \varphi \rangle dxd\xi$$

$$= \int_0^T \int_D \langle \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \nabla \varphi \rangle dxd\xi.$$  

For any $0 < \Lambda_1, \Lambda_2 < \infty$, define the set $X(\Lambda_1, \Lambda_2)$ consisting of solutions $d^\varepsilon$ to

$$\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon) = \tau^\varepsilon \text{ in } D$$  

such that the following properties hold:

1. $|\bar{d}^\varepsilon| \leq 1$ for a.e. $x \in D$.
2. \[ \sup_{0 < \varepsilon \leq 1} \mathcal{E}_\varepsilon(\bar{d}^\varepsilon) = \int_D \left( \frac{1}{2} |\nabla \bar{d}^\varepsilon|^2 + F_\varepsilon(\bar{d}^\varepsilon) \right) dx \leq \Lambda_1. \]
3. \[ \sup_{0 < \varepsilon \leq 1} \|\tau^\varepsilon\|_{L^2(D)} \leq \Lambda_2. \]

The following small energy regularity lemma [18, 25] plays a key role in our analysis.
Lemma 3.1. Suppose \( \{\tilde{d}^\varepsilon\}_{0 < \varepsilon \leq 1} \subset X(\Lambda_1, \Lambda_2) \) and \( \tau^\varepsilon \to \tau \) in \( L^2(D) \). Then there exists a \( \delta_0 > 0 \) such that if for \( x_0 \in D \) and \( 0 < r_0 < \text{dist}(x_0, \partial \Omega) \),

\[
\sup_{0 < \varepsilon \leq 1} \int_{B_{r_0}(x_0)} \left( \frac{1}{2} \|\nabla \tilde{d}^\varepsilon\|^2 + F_\varepsilon(\tilde{d}^\varepsilon) \right) \, dx \leq \delta_0^2,
\]

then there exists an approximated harmonic map \( d \in H^1(B_{2r}(x_0), S^2) \) with tensor field \( \tau \), i.e.,

\[\Delta d + |\nabla d|^2 d = \tau,\]

such that

\[\tilde{d}^\varepsilon \to d \text{ in } H^1(B_{2r}(x_0))\]

as \( \varepsilon \to 0 \).

This leads to the following \( H^1 \) precompactness result.

Lemma 3.2. Under the same assumption as Lemma 3.1,

\[\tilde{d}^\varepsilon \to d \text{ in } H^1_{\text{loc}}(D \setminus \Sigma),\]

where

\[\Sigma := \bigcap_{r > 0} \left\{ x \in D : \liminf_{\varepsilon \to 0} \int_{B_r(x)} \left( \frac{1}{2} \|\nabla \tilde{d}^\varepsilon\|^2 + F_\varepsilon(\tilde{d}^\varepsilon) \right) \, dx > \delta_0^2 \right\} .\]

Moreover, \( \Sigma \) is a finite set.

From (16) and (21), we have

\[
\mathbb{E}' \sup_{0 \leq t \leq T} \left[ \|\tilde{u}^\varepsilon(t)\|^2 + \|\nabla \tilde{d}^\varepsilon(t)\|^2 + \int_{D \times \{t\}} F_\varepsilon(\tilde{d}^\varepsilon) \, dx \right] + \mathbb{E} \int_0^T (\|\nabla \tilde{d}^\varepsilon\|^2 + \|\Delta \tilde{d}^\varepsilon - f_\varepsilon(\tilde{d}^\varepsilon)\|^2) \, dt = \mathbb{E} \sup_{0 \leq t \leq T} \left[ \|\tilde{u}^\varepsilon(t)\|^2 + \|\nabla \tilde{d}^\varepsilon(t)\|^2 + \int_{D \times \{t\}} F_\varepsilon(\tilde{d}^\varepsilon) \, dx \right] + \mathbb{E} \int_0^T (\|\nabla \tilde{d}^\varepsilon\|^2 + \|\Delta \tilde{d}^\varepsilon - f_\varepsilon(\tilde{d}^\varepsilon)\|^2) \, dt \\
\leq C.
\]

Hence, there exists \( \mathcal{N} \subset \Omega' \) such that \( \mathbb{P}'(\mathcal{N}) = 0 \), and it holds for \( \omega \in \Omega' \setminus \mathcal{N} \) that

\[
\liminf_{\varepsilon \to 0} \int_0^T \int_D (\|\nabla \tilde{d}^\varepsilon\|^2 + |\Delta \tilde{d}^\varepsilon - f_\varepsilon(\tilde{d}^\varepsilon)|^2) \, dx \, dt = C_1(\omega) < \infty,
\]

and

\[
\liminf_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \int_{D \times \{t\}} (\|\tilde{u}^\varepsilon\|^2 + \|\nabla \tilde{d}^\varepsilon\|^2 + F_\varepsilon(\tilde{d}^\varepsilon)) \, dx = C_2(\omega) < \infty.
\]

Now fix \( \omega \in \Omega' \setminus \mathcal{N} \), by Fatou’s lemma, we have

\[
\int_0^T \liminf_{\varepsilon \to 0} \int_D (\|\nabla \tilde{d}^\varepsilon\|^2 + |\Delta \tilde{d}^\varepsilon - f_\varepsilon(\tilde{d}^\varepsilon)|^2) \, dx \, ds \\
\leq \liminf_{\varepsilon \to 0} \int_0^T \int_D (\|\nabla \tilde{d}^\varepsilon\|^2 + |\Delta \tilde{d}^\varepsilon - f_\varepsilon(\tilde{d}^\varepsilon)|^2) \, dx \, ds < \infty.
\]
Hence there exists $A \subset [0, T]$ with full Lebesgue such that for any $t \in A$,

$$\liminf_{\varepsilon \to 0} \int_{D \times \{t\}} (|\nabla \mathbf{d}\varepsilon|^2 + |\Delta \mathbf{d}\varepsilon - f_{\varepsilon}(\mathbf{d}\varepsilon)|^2)dx < \infty.$$  

For $t \in A$, we set

$$\Sigma_t := \bigcap_{r>0} \left\{ x \in D : \liminf_{\varepsilon \to 0} \int_{B_r(x) \times \{t\}} \frac{1}{2} |\nabla \mathbf{d}\varepsilon|^2 + F_{\varepsilon}(\mathbf{d}\varepsilon)dx > \delta_0^2 \right\}.$$  

By Lemma 3.2, it holds that $\#(\Sigma_t) \leq C_3(\omega) < \infty$ and $d\varepsilon(t) \to d(t)$ in $H^1_{\text{loc}}(D \setminus \Sigma_t)$. 

Hence we get (25) holds for $\varphi$ with $\text{supp} \varphi \subset D \setminus \Sigma_t$. Now we consider the case $\Sigma_t \cap \text{supp} \varphi \neq \emptyset$. Since $\Sigma_t$ is finite, we may assume $(0,0) \in \text{supp} \varphi$. Write

$$\nabla \mathbf{d}\varepsilon \cdot \nabla \mathbf{d}\varepsilon - \frac{1}{2} |\nabla \mathbf{d}\varepsilon|^2 \mathbb{I}_2 = \frac{1}{2} \begin{pmatrix} |\partial_x \mathbf{d}\varepsilon|^2 - |\partial_x \mathbf{d}\varepsilon|^2 & 2\langle \partial_x \mathbf{d}\varepsilon, \partial_y \mathbf{d}\varepsilon \rangle \\ 2\langle \partial_y \mathbf{d}\varepsilon, \partial_x \mathbf{d}\varepsilon \rangle & |\partial_y \mathbf{d}\varepsilon|^2 - |\partial_x \mathbf{d}\varepsilon|^2 \end{pmatrix}. \quad (33)$$  

We can now assume that there exists two real number $\alpha, \beta$ such that

$$\left(\nabla \mathbf{d}\varepsilon \cdot \nabla \mathbf{d}\varepsilon - \frac{1}{2} |\nabla \mathbf{d}\varepsilon|^2 \mathbb{I}_2 \right) dx 
- \left(\frac{1}{2} \nabla \mathbf{d} \cdot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \mathbb{I}_2 \right) dx + \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \delta_{(0,0)}$$
as convergence of Radon measures. (25) is true if we can show $\alpha = \beta = 0$.

We apply the same Pohozaev argument as that in [10]. Set $\tau^\varepsilon, e^\varepsilon$ to be

$$\Delta \mathbf{d}\varepsilon - f_{\varepsilon}(\mathbf{d}\varepsilon) =: \tau^\varepsilon \quad (34)$$
and

$$e^\varepsilon(\mathbf{d}\varepsilon) := \frac{1}{2} |\nabla \mathbf{d}\varepsilon|^2 + F_{\varepsilon}(\mathbf{d}\varepsilon).$$

For any $X \in C^\infty(D, \mathbb{R}^3)$, multiplying (34) by $X \cdot \nabla \mathbf{d}\varepsilon$ and integrating over $B_r(0)$ we get

$$\int_{\partial B_r(0)} (X \cdot \nabla \mathbf{d}\varepsilon) \frac{x}{|x|} d\sigma - \int_{B_r(0)} (\nabla \mathbf{d}\varepsilon \cdot \nabla \mathbf{d}\varepsilon, \nabla X)dx 
+ \int_{B_r(0)} \text{div} X e^\varepsilon(\mathbf{d}\varepsilon)dx - \int_{\partial B_r(0)} e^\varepsilon(\mathbf{d}\varepsilon) (X, \frac{x}{|x|}) d\sigma 
= \int_{B_r(0)} (X \cdot \nabla \mathbf{d}\varepsilon, \tau^\varepsilon)dx. \quad (35)$$

If we choose $X(x) = x$, then (35) becomes

$$r \int_{\partial B_r(0)} \left| \frac{\partial \mathbf{d}\varepsilon}{\partial r} \right|^2 d\sigma + \int_{B_r(0)} 2F_{\varepsilon}(\mathbf{d}\varepsilon)dx 
- r \int_{\partial B_r(0)} e^\varepsilon(\mathbf{d}\varepsilon) d\sigma = \int_{B_r(0)} |x| (\frac{\partial \mathbf{d}\varepsilon}{\partial r}, \tau^\varepsilon)dx.$$
Hence
\[
\int_{\partial B_r(0)} e_\varepsilon(\mathbf{d}) d\sigma = \int_{\partial B_r(0)} \left| \frac{\partial \mathbf{d}}{\partial r} \right| d\sigma \\
+ \frac{1}{r} \int_{B_r(0)} 2F_\varepsilon(\mathbf{d}) dx - \frac{1}{r} \int_{B_r(0)} |x| \left( \frac{\partial \mathbf{d}}{\partial r}, \tau^\varepsilon \right) dx.
\]

Integrating from \( r \) to \( R \) yields
\[
\int_{B_R(0) \setminus B_r(0)} e_\varepsilon(\mathbf{d}) dx = \int_{B_R(0) \setminus B_r(0)} \left| \frac{\partial \mathbf{d}}{\partial r} \right|^2 dx \\
+ \frac{1}{r} \int_{B_r(0)} \left( 2F_\varepsilon(\mathbf{d}) - |x| \left( \frac{\partial \mathbf{d}}{\partial r}, \tau^\varepsilon \right) \right) dx d\tau.
\]
(36)

Since \( \Sigma_t = (0,0) \), then there exists \( \gamma > 0 \) such that
\[
e_\varepsilon(\mathbf{d}) dx \Rightarrow \frac{1}{2} |\nabla \mathbf{d}|^2 dx + \gamma \delta_{(0,0)}
\]
as convergence of Radon measure. By sending \( \varepsilon \to 0 \) in (36) we get
\[
\int_{B_R(0) \setminus B_r(0)} \frac{1}{2} |\nabla \mathbf{d}|^2 dx \\
\geq \int_{B_R(0) \setminus B_r(0)} \left| \frac{\partial \mathbf{d}}{\partial r} \right|^2 dx + \int_{r}^{R} \frac{1}{r} \liminf_{\varepsilon \to 0} \int_{B_r(0)} 2F_\varepsilon(\mathbf{d}) dx d\tau \\\n+ \liminf_{\varepsilon \to 0} \int_{r}^{R} -\frac{1}{r} \int_{B_r(0)} |x| \left( \frac{\partial \mathbf{d}}{\partial r}, \tau^\varepsilon \right) dx d\tau.
\]
(37)

Notice that
\[
\left| \int_{r}^{R} -\frac{1}{r} \liminf_{\varepsilon \to 0} \int_{B_r(0)} |x| \left( \frac{\partial \mathbf{d}}{\partial r}, \tau^\varepsilon \right) dx d\tau \right| \\
\leq \limsup_{\varepsilon \to 0} \int_{0}^{R} \left\| \tau^\varepsilon \right\|_{L^2(B_{r}(0))} \left\| \nabla \mathbf{d}^\varepsilon \right\|_{L^2(B_{r}(0))} d\tau \\
= O(R).
\]

As a consequence, we claim that
\[
2F_\varepsilon(\mathbf{d}) \to 0 \text{ in } L^1(B_\delta).
\]
(38)

For, otherwise, then there exists \( \kappa > 0 \) such that
\[
2F_\varepsilon(\mathbf{d}) dx \to \kappa \delta_{(0,0)}.
\]

This implies
\[
\lim_{r \downarrow 0} \int_{r}^{R} \frac{1}{r} \liminf_{\varepsilon \to 0} \int_{B_r(0)} 2F_\varepsilon(\mathbf{d}) dx d\tau = \lim_{r \downarrow 0} \frac{\kappa}{r} d\tau = \infty.
\]
If we choose $X(x) = (x, 0)$ in (35), we obtain that

$$
\frac{1}{2} \int_{B_r(0)} \left( |\partial_x \bar{d}|^2 - |\partial_x \bar{d}|^2 \right) \, dx + \int_{B_r(0)} F_\varepsilon(\bar{d}^\varepsilon) \, dx
$$

$$
= \int_{B_r(0)} x_1 (\partial_x \bar{d}^\varepsilon, \tau^\varepsilon) \, dx + \int_{\partial B_r(0)} \frac{x_1^2}{r} e_\varepsilon(\bar{d}^\varepsilon) \, d\sigma
$$

$$
- \int_{\partial B_r(0)} x_1 (\partial_x \bar{d}^\varepsilon, \frac{\partial \bar{d}^\varepsilon}{\partial r}) \, d\sigma.
$$

(39)

Since $e_\varepsilon(\bar{d}^\varepsilon) \, dx \rightharpoonup \frac{1}{2} |\nabla d|^2 \, dx$ in $B_{2r} \setminus B_r$ for $r > 0$, it is easy to see

$$
\int_{\partial B_r(0)} x_1 (\partial_x \bar{d}^\varepsilon, \frac{\partial \bar{d}^\varepsilon}{\partial r}) \, d\sigma \rightarrow \int_{\partial B_r(0)} x_1 (\partial_x d, \frac{\partial d}{\partial r}) \, d\sigma,
$$

and by (38),

$$
\int_{B_r(0)} F_\varepsilon(\bar{d}^\varepsilon) \, dx \rightarrow 0.
$$

With the fact that

$$
\left| \int_{B_r} x_1 (\partial_x \bar{d}^\varepsilon, \tau^\varepsilon) \, dx \right| = O(r),
$$

by sending $\varepsilon \rightarrow 0$ in (39) we obtain

$$
\frac{1}{2} \int_{B_r} \left( |\partial_x x|^2 - |\partial_x (x_1)|^2 \right) \, dx + \alpha = O(r)
$$

which implies $\alpha = 0$ after sending $r \rightarrow 0$.

Similarly, if we choose $X(x) = (0, x_1)$ in (35), by performing the same argument we will arrive at

$$
\frac{1}{2} \int_{B_r} \langle \partial_x x, \partial_x y \rangle \, dx + \beta = O(r).
$$

Hence $\beta = 0$. This implies almost surely convergence of Ericksen stress tensor field (25). From (20) and (21) we can conclude that for any $1 < p < \infty$, it holds

$$
E' \sup_{0 \leq t \leq T} \left[ \|u^\varepsilon(t)\|^2 + \|\nabla \bar{d}^\varepsilon(t)\|^2 + \int_{D \times \{t\}} F_\varepsilon(\bar{d}^\varepsilon) \, dx \right]^p
$$

$$
+ E' \left[ \int_0^T (\|\nabla u^\varepsilon\|^2 + \|\Delta \bar{d}^\varepsilon - f_\varepsilon(\bar{d}^\varepsilon)\|^2) \, dt \right]^p
$$

$$
= E \left[ \|u^\varepsilon(T)\|^2 + \|\nabla \bar{d}^\varepsilon(T)\|^2 + \int_{D \times \{T\}} F_\varepsilon(\bar{d}^\varepsilon) \, dx \right]^p
$$

$$
+ E \left[ \int_0^T (\|\nabla u^\varepsilon\|^2 + \|\Delta \bar{d}^\varepsilon - f_\varepsilon(\bar{d}^\varepsilon)\|^2) \, dt \right]^p
$$

$$
\leq C.
$$

Thus we have for any $\xi \in L^2(\Omega'; J)$, it holds

$$
\lim_{\varepsilon \rightarrow 0} E' \left[ \int_D \langle M_{u^\varepsilon}(t), \xi \rangle \, dx \right]
$$

(41)
we get
\begin{align*}
\int_{D} (\langle \nabla f, \nabla \xi \rangle - \langle f \otimes f + \nabla f \otimes \nabla f - \frac{1}{2} |\nabla f|_{2}^{2}, \nabla \xi \rangle) dx ds
\end{align*}
\begin{align*}
&= \mathbb{E}' \left[ \int_{D} (u(t) - u_{0}, \xi) dx \right]
\end{align*}
\begin{align*}
&= \mathbb{E}' \left[ \int_{D} (u(t) - u_{0}, \xi) dx \right] + \int_{0}^{t} \int_{D} (\langle \nabla u, \nabla \xi \rangle - \langle u \otimes u + \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|_{2}^{2}, \nabla \xi \rangle) dx ds
\end{align*}
\begin{align*}
= \mathbb{E}' \left[ \int_{D} (M_{u}(t), \xi) dx \right].
\end{align*}

Now we pass to the convergence of $\bar{f}$. We claim that up to a subsequence,
\begin{align*}
\Delta \bar{f} - f_{0}(\bar{f}) \rightarrow \Delta d + |\nabla d|^{2} d \quad \text{in} \quad L^{2}(\Omega' \times [0, T] \times D).
\end{align*}
From (16) we can assume that there exists $g \in L^{2}(\Omega' \times [0, T] \times D)$ such that
\begin{align*}
\Delta \bar{f} - f_{0}(\bar{f}) \rightarrow g \quad \text{in} \quad L^{2}(\Omega' \times [0, T] \times D).
\end{align*}
First we claim that
\begin{align*}
g \perp d \quad \text{for almost all} \quad (\omega', t, x) \in \Omega' \times [0, T] \times D.
\end{align*}
In fact, for any test function $\phi = \phi(\omega', x)$, if we apply the Itô formula to
\begin{align*}
\Psi(\bar{f}) = \int_{D} \frac{|\bar{f}|^{2}}{2} \phi dx,
\end{align*}
it hold that (see Appendix A)
\begin{align*}
\mathbb{E}' \left[ \int_{D} \frac{|\bar{f}|^{2}(t)}{2} \phi dx \right] - \mathbb{E}' \left[ \int_{D} \frac{|\bar{f}|^{2}(t-\delta)}{2} \phi dx \right] = -\mathbb{E}' \left[ \int_{t-\delta}^{t} \int_{D} \phi \nabla \bar{f} \cdot \nabla |\bar{f}|^{2} dx ds \right] + \mathbb{E}' \left[ \int_{t-\delta}^{t} \int_{D} (\Delta \bar{f} - f_{0}(\bar{f}), \bar{f}) \phi dx ds \right].
\end{align*}
Now we pass $\varepsilon$ to 0, using the fact that $|d| = 1$ for almost all $(\omega', t, x) \in \Omega' \times [0, T] \times D$ we get
\begin{align*}
\mathbb{E}' \left[ \int_{t-\delta}^{t} \int_{D} (g, d) \phi dx ds \right] = 0.
\end{align*}
Since $\phi$ and $\delta$ can be arbitrary, $(g, d) = 0$ for almost all $(\omega', t, x) \in \Omega' \times [0, T] \times D$. Hence (43) holds. By taking the cross product of (42) with $\bar{f} \phi$ we get
\begin{align*}
0 &\quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}' \left[ \int_{0}^{T} \int_{D} (\Delta \bar{f} - f_{0}(\bar{f}) - g) \times \bar{f}, \phi) dx dt \right]
\end{align*}
\begin{align*}
= \lim_{\varepsilon \rightarrow 0} \mathbb{E}' \left[ \int_{0}^{T} \int_{D} (\nabla \bar{f}) \times \bar{f}, \phi) dx dt - \int_{0}^{T} \int_{D} (g \times \bar{f}, \phi) dx dt \right]
\end{align*}
\begin{align*}
= \lim_{\varepsilon \rightarrow 0} \mathbb{E}' \left[ - \int_{0}^{T} \int_{D} (\nabla \bar{f}) \times \bar{f}, \phi) dx dt - \int_{0}^{T} \int_{D} (g \times \bar{f}, \phi) dx dt \right]
\end{align*}
\begin{align*}
= \mathbb{E}' \left[ - \int_{0}^{T} \int_{D} (\nabla d \times \bar{f}, \phi) dx dt - \int_{0}^{T} \int_{D} (g \times d, \phi) dx dt \right].
\end{align*}
This implies \((g - \Delta d) \times d = 0\) and hence there exists \(\lambda = \lambda(\omega', t, x) : \Omega' \times [0, T] \times D \to \mathbb{R}\) such that

\[
g - \Delta d = \lambda d.
\]

From (43) and \(\langle \Delta d, d \rangle = -|\nabla d|^2 d\) we get

\[
\lambda = (g - \Delta d, d) = |\nabla d|^2.
\]

Thus (42) holds. By (22), (40) and (42), we have for any \(\zeta \in L^2(\Omega'; H^1(D, \mathbb{R}^3))\), it holds

\[
\lim_{\varepsilon \to 0} \mathbb{E}' \left[ \int_{D} \langle M_{\varepsilon}^T(t), \zeta \rangle dx \right]
= \lim_{\varepsilon \to 0} \mathbb{E}' \left[ \int_{D} \langle \tilde{d}^T(t) - d_0, \zeta \rangle dx \right]
- \int_{0}^{t} \int_{D} \left( \langle \tilde{u}^T \otimes \tilde{d}^T, \nabla \zeta \rangle - (\Delta \tilde{d}^T - f_{\varepsilon}(\tilde{d}^T), \zeta) \right) dx ds
- \lim_{\varepsilon \to 0} \mathbb{E}' \left[ \frac{1}{2} \int_{0}^{t} \int_{D} \langle \tilde{d}^T \times h \times h, \zeta \rangle dx ds \right]
= \mathbb{E}' \left[ \int_{0}^{t} \langle d(t) - d_0, \zeta \rangle dx \right]
- \int_{0}^{t} \int_{D} \left( \langle u \otimes d, \nabla \zeta \rangle - (\Delta d + |\nabla d|^2 d, \zeta) \right) dx ds
- \mathbb{E}' \left[ \frac{1}{2} \int_{0}^{t} \int_{D} \langle d \times h \times h, \zeta \rangle dx ds \right]
= \mathbb{E}' \left[ \int_{D} \langle M_d(t), \zeta \rangle dx \right].
\]

Taking the limit \(\varepsilon \to 0\) in (30) and applying the lower semicontinuity yields (5).

To finish the construction, we need to show that for every \(t \geq 0\) and \(\varepsilon > 0\), let

\[
\hat{F}_{\varepsilon}^t := \sigma \left( \sigma \left( \langle u \varepsilon(s), d^T(s), \tilde{W}_1(s), \tilde{W}_2(s) \rangle s \leq t \right) \cup \mathcal{N} \right),
\]

\[
F_{\varepsilon}^t := \sigma \left( \sigma \left( \langle u(s), d(s), W_1(s), W_2(s) \rangle s \leq t \right) \cup \mathcal{N} \right).
\]

Since \(L(u \varepsilon, d^T, \tilde{W}_1, \tilde{W}_2) = L(u, d^T, W_1, W_2)\), \((\tilde{W}_1, \tilde{W}_2)\) form a sequence of cylindrical Wiener processes. Moreover, for \(0 \leq s < t \leq T\) the increments \((\tilde{W}_1(s) - \tilde{W}_1(t))(\tilde{W}_2(s) - \tilde{W}_2(t))\) are independent of \(\hat{F}_{\varepsilon}^t\) for \(r \in [0, s]\). Let \(k \in \mathbb{N}\) and \(s_0 = 0 < s_1 < \cdots < s_k \leq T\) be a partition of \([0, T]\). By the characterization of \(K_2\)-valued \(K_1\)-cylindrical Wiener process [6, Remark 2.8], for each \(\xi \in K_2^1\) we have

\[
\mathbb{E}' \left[ e^{i \sum_{j=1}^{k} \langle \xi, \tilde{W}_1(s_j) - \tilde{W}_1(s_{j-1}) \rangle_{K_2^1, K_2^2}} \right]
= \mathbb{E} \left[ e^{i \sum_{j=1}^{k} \langle \xi, W_1(s_j) - W_1(s_{j-1}) \rangle_{K_2^1, K_2^2}} \right]
= e^{-\frac{1}{2} \sum_{j=1}^{k} (s_j - s_{j-1}) \langle \xi, \xi \rangle_{K_2^1}}.
\]
Thanks to (22) and the Lebesgue Dominated Convergence Theorem, we have
\[
\lim_{\varepsilon \to 0} \mathbb{E}' \left[ e^{i \sum_{j=1}^{b} (\xi, \mathcal{W}_{1}(s_j) - \mathcal{W}_{1}(s_{j-1})) K_{2}^* K_{2}} \right] = \mathbb{E}' \left[ e^{i \sum_{j=1}^{b} (\xi, \mathcal{W}_{1}(s_j) - \mathcal{W}_{1}(s_{j-1})) K_{2}^* K_{2}} \right] = e^{-\frac{1}{2} \sum_{j=1}^{b} (s_j - s_{j-1}) |\xi|^{2}}.
\]
Hence the finite dimensional distribution of \(W_1'\) is Gaussian. The same argument also works for \(W_2'\). Next we want to show that \((W_1'(t) - W_1'(s), W_2'(t) - W_2'(s))\), \(0 \leq s < t \leq T\) is independent of \(\mathcal{F}_{r}^{\varepsilon}\) for \(r \in [0, s]\). Consider \(\{\phi_{j}\}_{j=1}^{r} \in C_b(W^{-2,p}(D) \times L^{\frac{2}{3}}(D)), \{\psi_{j}\}_{j=1}^{r} \in C_b(K_2 \times \mathbb{R})\), let \(0 \leq t_1 < \cdots < r_k \leq s < t \leq T\), \(\psi \in C_b(K_2), \zeta \in C_b(\mathbb{R})\).

\[
\mathbb{E}' \left[ \prod_{j=1}^{k} \phi_{j}(\mathcal{W}_{1}^{\varepsilon}(r_j), \mathcal{W}_{1}^{\varepsilon}(r_j)) \prod_{j=1}^{k} \psi_{j}(\mathcal{W}_{2}^{\varepsilon}(r_j), \mathcal{W}_{2}^{\varepsilon}(r_j)) \right]
\times \psi(\mathcal{W}_{1}^{\varepsilon}(t) - \mathcal{W}_{1}^{\varepsilon}(s)) \zeta(\mathcal{W}_{2}^{\varepsilon}(t) - \mathcal{W}_{2}^{\varepsilon}(s))
= \mathbb{E}' \left[ \prod_{j=1}^{k} \phi_{j}(\mathcal{W}_{1}^{\varepsilon}(r_j), \mathcal{W}_{1}^{\varepsilon}(r_j)) \prod_{j=1}^{k} \psi_{j}(\mathcal{W}_{2}^{\varepsilon}(r_j), \mathcal{W}_{2}^{\varepsilon}(r_j)) \right]
\times \mathbb{E}' \left[ \psi(\mathcal{W}_{1}^{\varepsilon}(t) - \mathcal{W}_{1}^{\varepsilon}(s)) \right] \mathbb{E}' \left[ \zeta(\mathcal{W}_{2}^{\varepsilon}(t) - \mathcal{W}_{2}^{\varepsilon}(s)) \right].
\]

Again by the Lebesgue Dominated Convergence theorem, if we send \(\varepsilon \to 0\) in (48) we can see (48) also holds for \((u, d, W_1'; W_2')\) in the limit. Furthermore, it is easy to show that \(W_1'\) is independent of \(W_2'\).

For any \(\delta > 0\), let \(\eta_{\delta}\) be a standard mollifier with support in \((0, t)\). Define
\[
S^{\delta}(u(s)) = \int_{-\infty}^{\infty} \eta_{\delta}(s - r) S(u(r)) dr.
\]
Let \(M_{u}^{\delta}\) and \(M_{\mathcal{W}}^{\delta}\) be respectively defined by
\[
M_{u}^{\delta}(t) = \int_{0}^{t} S^{\delta}(u(s)) d\mathcal{W}_{1}(s),
\]
\[
M_{\mathcal{W}}^{\delta}(t) = \int_{0}^{t} S^{\delta}(u(s)) dW_{1}(s).
\]
By the property of mollifiers, we can get for any \(v \in H\)
\[
\lim_{\delta \to 0} \mathbb{E}' \int_{0}^{t} \|S^{\delta}(v(s)) - S(v(s))\|_{L_{2}(K_{1}, H)}^{2} ds = 0.
\]
Hence, for any \(t \in (0, T)\), we have the following uniform approximation
\[
\lim_{\delta \to 0} \sup_{0 < \varepsilon < \delta} \mathbb{E}' \left\| M_{u}^{\delta}(t) - \int_{0}^{t} S(u) d\mathcal{W}_{1}(s) \right\|^{2} = 0,
\]
and
\[
\lim_{\delta \to 0} \mathbb{E}' \left\| M_{\mathcal{W}}^{\delta}(t) - \int_{0}^{t} S(u) dW_{1}(s) \right\|^{2} = 0.
\]
Next, we need to show that for any \(\delta > 0\)
\[
\lim_{\varepsilon \to 0} \mathbb{E}' \left\| M_{\mathcal{W}}^{\delta}(t) - M_{u}^{\delta}(t) \right\|^{2} = 0.
\]
By the Itô isometry, we have that
\[
M^\delta_w(t) - M^\delta_u(t) = \sum_{i=1}^\infty \int_0^t S^\delta(\underbar{e}(s)) (e_i) d\overline{B}^i_s(s) - \sum_{i=1}^\infty \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s).
\]
By Young’s convolution inequality, we have that
\[
\left\| \sum_{i=1}^\infty \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s) \right\|_2^2 \leq C \sum_{i=1}^\infty \int_0^t \left\| S^\delta(\overline{u}(s)) (e_i) \right\|_2^2 ds \leq C.
\]
Thus, for any \( \gamma > 0 \), there exists an \( N \in \mathbb{N}_+ \) such that
\[
\sum_{i=N+1}^\infty \mathbb{E}' \int_0^t \left\| S^\delta(\overline{u}(s)) (e_i) \right\|^2 ds < \gamma.
\]
Since
\[
\lim_{\varepsilon \to 0} \mathbb{E}' \int_0^t \left\| S^\delta(\overline{u}(s)) - S^\delta(\underbar{e}(s)) \right\|_2^2 \leq 0,
\]
there exists an \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \),
\[
\sum_{i=N+1}^\infty \mathbb{E}' \int_0^t \left\| S^\delta(\underbar{e}(s)) (e_i) \right\|^2 ds < 2\gamma.
\]
Now we split \( M^\delta_w(t) - M^\delta_u(t) \) into three parts
\[
M^\delta_w(t) - M^\delta_u(t) = \sum_{i=1}^N \left( \int_0^t S^\delta(\underbar{e}(s)) (e_i) d\overline{B}^i_s(s) - \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s) \right)
\]
\[
+ \sum_{i=N+1}^\infty \int_0^t S^\delta(\underbar{e}(s)) (e_i) d\overline{B}^i_s(s)
\]
\[
+ \sum_{i=N+1}^\infty \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s) := J^\delta_{\varepsilon,1}(t) + J^\delta_{\varepsilon,2}(t) + J^\delta_{\varepsilon,3}(t).
\]
By the Itô isometry, we have that
\[
\mathbb{E}'\|J^\delta_{\varepsilon,2}\|^2 = \sum_{i=N+1}^\infty \mathbb{E}' \int_0^t \left\| S^\delta(\overline{u}(s)) (e_i) \right\|^2 ds < 2\gamma,
\]
\[
\mathbb{E}'\|J^\delta_{\varepsilon,3}\|^2 = \sum_{i=N+1}^\infty \mathbb{E}' \int_0^t \left\| S^\delta(\overline{u}(s)) (e_i) \right\|^2 ds < \gamma.
\]
For \( J^\delta_{\varepsilon,1}(t) \), we write
\[
J^\delta_{\varepsilon,1}(t) = \sum_{i=1}^N \left( \int_0^t S^\delta(\underbar{e}(s)) (e_i) d\overline{B}^i_s(s) - \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s) \right)
\]
\[
+ \sum_{i=1}^N \left( \int_0^t S^\delta(\underbar{e}(s)) (e_i) dB^i_s(s) - \int_0^t S^\delta(\overline{u}(s)) (e_i) dB^i_s(s) \right)
\]
\[
:= J^\delta_{\varepsilon,1} + J^\delta_{\varepsilon,2}.
\]
For $I_{\epsilon,1}^\delta(t)$, by integration by parts we obtain that

$$I_{\epsilon,1}^\delta(t) = \sum_{i=1}^{N} \left( \int_{0}^{t} [\eta'_i * S(\mathbf{u}(s))](e_i)B_i'(s)ds - \int_{0}^{t} [\eta'_i * S(\mathbf{u}(s))](e_i)B_i'(s)ds \right)$$

$$= - \sum_{i=1}^{N} \left( \int_{0}^{t} [\eta'_i * S(\mathbf{u}(s))](e_i)|\mathbf{B}_i'(s)| ds - B_i'(s)ds \right).$$

From the Burkholder–Davis–Gundy inequality, we get for any $p > 1$, any $i = 1, 2, \ldots, N$,

$$\sup_{\epsilon > 0} \mathbb{E}' \sup_{s \in [0,T]} \left( |\mathbf{B}_i'(s)|^p + |B_i'(s)|^p \right) \leq CT^\frac{p}{2}. \quad (52)$$

Hence, by the uniform integrability (52) and the almost surely convergence (22) we have that for $i = 1, 2, \ldots, N$,

$$\lim_{\epsilon \to 0} \mathbb{E}' \int_{0}^{t} |\mathbf{B}_i'(s) - B_i'(s)|^p ds = 0.$$

This implies

$$\mathbb{E}' \|I_{\epsilon,1}^\delta(t)\|^2 = \mathbb{E}' \left\| \sum_{i=1}^{N} \int_{0}^{t} [\eta'_i * S(\mathbf{u}(s))](e_i)(\mathbf{B}_i'(s) - B_i'(s))ds \right\|^2$$

$$\leq N \sum_{i=1}^{N} \mathbb{E}' \left\| \int_{0}^{t} [\eta'_i * S(\mathbf{u}(s))](e_i)(\mathbf{B}_i'(s) - B_i'(s))ds \right\|^2$$

$$\leq N \sum_{i=1}^{N} \mathbb{E}' \left[ \int_{0}^{t} \|\eta'_i * S(\mathbf{u}(s))\|(e_i)\|\mathbf{B}_i'(s) - B_i'(s)\|ds \right]^2$$

$$\leq N \sum_{i=1}^{N} \mathbb{E}' \left[ \int_{0}^{t} \|\eta'_i * S(\mathbf{u}(s))\|(e_i)\|\mathbf{B}_i'(s) - B_i'(s)\|^2ds \right]$$

$$\leq C\frac{N}{\delta^2} \sum_{i=1}^{N} \mathbb{E}' \left[ \int_{0}^{t} \|S(\mathbf{u}(s))\|_{L^2(K_1, H)}^2ds \int_{0}^{t} |\mathbf{B}_i'(s) - B_i'(s)|^2 ds \right]$$

$$\leq C\frac{N}{\delta^2} \sum_{i=1}^{N} \mathbb{E}' \left[ \int_{0}^{t} (1 + \|\mathbf{u}(s)\|^2)ds \int_{0}^{t} |\mathbf{B}_i'(s) - B_i'(s)|^2 ds \right]$$

$$\leq \frac{CNT^\frac{1}{2}}{\delta^2} \left( \mathbb{E}' \sup_{0 \leq s \leq t} (1 + \|\mathbf{u}(s)\|)^{\frac{1}{4}} \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_{0}^{t} |\mathbf{B}_i'(s) - B_i'(s)|^4 ds \right)^{\frac{1}{2}}$$

$$\leq \frac{CNT^\frac{1}{2}}{\delta^2} \sum_{i=1}^{N} \left( \mathbb{E}' \int_{0}^{t} |\mathbf{B}_i'(s) - B_i'(s)|^4 ds \right)^{\frac{1}{2}} \to 0,$$

as $\epsilon \to 0$. Using a similar argument, we can show that

$$\lim_{\epsilon \to 0} \mathbb{E}' \|I_{\epsilon,2}^\delta(t)\|^2 = 0.$$

Since $\gamma$ can be arbitrarily small, we get

$$\lim_{\epsilon \to 0} \mathbb{E}' \left[ \|J_{\epsilon,1}^\delta\|^2 + \|J_{\epsilon,2}^\delta(t)\|^2 + \|J_{\epsilon,3}^\delta(t)\|^2 \right] = 0, \ \forall t \in (0, T],$$

where $J_{\epsilon,1}^\delta$, $J_{\epsilon,2}^\delta(t)$, and $J_{\epsilon,3}^\delta(t)$ represent the terms derived from the original equations after linearization.
This implies (51). Then we can conclude from (49), (50) and (51) that for every \( t \in (0, T] \),
\[
\lim_{\varepsilon \to 0} \mathbb{E}' \left\| \int_0^t S(u^\varepsilon(s))dW^\varepsilon_1(s) - \int_0^t S(u(s))dW_1(s) \right\|^2 = 0.
\]
Similarly, we can show
\[
\lim_{\varepsilon \to 0} \mathbb{E}' \left\| \int_0^t (d^\varepsilon \times h)dW^\varepsilon_2(s) - \int_0^t (d \times h)dW'_2(s) \right\|^2 = 0.
\]
Hence, the convergence of martingale terms (46) and (47) holds. Putting (41), (45), (46) and (47) together completes the proof.

Appendix A. Itô’s formulas for functionals of \( d \). Consider the functional
\[
\Psi(d^\varepsilon) := \int_D \frac{|d^\varepsilon|^2}{2} \phi dx.
\]
It is easy to obtain the first and second Fréchet derivatives of \( \Psi(d^\varepsilon) \)
\[
\Psi'(d^\varepsilon)[g] = \int_D (d^\varepsilon, g) \phi dx,
\]
\[
\Psi''(d^\varepsilon)[g, g] = \int_D (g, g) \phi dx.
\]
Applying the Itô formula to \( \Psi(d^\varepsilon) \) gives
\[
d\Psi(d^\varepsilon) = \Psi'(d^\varepsilon)[dd^\varepsilon] + \frac{1}{2} \Psi''(d^\varepsilon)[dd^\varepsilon, dd^\varepsilon].
\]
Since,
\[
(dd^\varepsilon = (-u^\varepsilon \cdot \nabla d^\varepsilon + \Delta d^\varepsilon - f^\varepsilon(d^\varepsilon) + \frac{1}{2}(d^\varepsilon \times h) \times h) dt + (d^\varepsilon \times h) dW_2,
\]
we then obtain that for \( 0 < \delta < t \),
\[
\Psi(d^\varepsilon)(t) - \Psi(d^\varepsilon)(t - \delta) = \int_{t-\delta}^t \left( \Psi'(d^\varepsilon)[j(s)] + \frac{1}{2} \Psi''(d^\varepsilon)[k(s), k(s)] \right) ds
\]
\[
+ \int_{t-\delta}^t \Psi'(d^\varepsilon)[k(s)]dW_2(s)
\]
\[
= \int_0^t \int_D \langle -u^\varepsilon \cdot \nabla d^\varepsilon, d^\varepsilon \rangle \phi dxdxds + \int_0^t \int_D \langle \Delta d^\varepsilon - f^\varepsilon(d^\varepsilon), d^\varepsilon \rangle \phi dxdxs
\]
\[
+ \frac{1}{2} \int_{t-\delta}^t \int_D \langle (d^\varepsilon \times h) \times h, d^\varepsilon \rangle \phi dxdxds + \frac{1}{2} \int_{t-\delta}^t \int_D |d^\varepsilon \times h|^2 \phi dxdxs
\]
\[
+ \int_{t-\delta}^t \int_D \langle d^\varepsilon \times h, d^\varepsilon \rangle \phi dxdW_2(s)
\]
\[
= \int_{t-\delta}^t \frac{\partial}{\partial t} \int_D \frac{|d^\varepsilon|^2}{2} \phi dxdxds + \int_{t-\delta}^t \int_D \langle \Delta d^\varepsilon - f^\varepsilon(d^\varepsilon), d^\varepsilon \rangle \phi dxdxs,
\]
where we use the fact the vector triple product
\[
\langle (d^\varepsilon \times h) \times h, d^\varepsilon \rangle = -|d^\varepsilon \times h|^2.
\]
and

\[ \langle d^\varepsilon \times h, d^\varepsilon \rangle = 0. \] (54)

Recall the energy functional

\[ \Phi_\varepsilon(d^\varepsilon) = \frac{1}{2}||\nabla d^\varepsilon||^2 + \int_D F_\varepsilon(d^\varepsilon) dx. \]

The first and second Fréchet derivatives of \( \Phi_\varepsilon \) are given by

\[ \Phi'_\varepsilon(d^\varepsilon)[g] = \int_D (\langle \nabla d^\varepsilon, \nabla g \rangle + \langle f_\varepsilon(d^\varepsilon), g \rangle) dx \]
\[ = \int_D (-\Delta d^\varepsilon + \frac{|d^\varepsilon|^2 - 1}{\varepsilon^2} d^\varepsilon, g) dx, \]
\[ \Phi''_\varepsilon(d^\varepsilon)[g, g] = \int_D \left( \langle \nabla g, \nabla g \rangle + \frac{|d^\varepsilon|^2 - 1}{\varepsilon^2} |g|^2 + \frac{2}{\varepsilon^2} \langle d^\varepsilon, g \rangle^2 \right) dx \]

for every \( g \in H^1(D; \mathbb{R}^3) \). Then, the Itô formula for \( \Phi_\varepsilon(d^\varepsilon) \) reads

\[ d\Phi_\varepsilon(d^\varepsilon) = \Phi'_\varepsilon(d^\varepsilon)[dd^\varepsilon] + \frac{1}{2} \Phi''_\varepsilon(d^\varepsilon)[dd^\varepsilon, dd^\varepsilon]. \]

From the identity (53) and (54) we obtain

\[ \Phi_\varepsilon(d^\varepsilon)(t) - \Phi_\varepsilon(d^\varepsilon) \]
\[ = \int_0^t \left( \Phi'_\varepsilon(d^\varepsilon)[k(s)] + \frac{1}{2} \Phi''_\varepsilon(d^\varepsilon)[k(s), k(s)] \right) ds + \int_0^t \Phi'_\varepsilon(d^\varepsilon)[k(s)] dW_2(s) \]
\[ = \int_0^t \int_D (u^\varepsilon \cdot \nabla d^\varepsilon, \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)) dx ds - \int_0^t \int_D |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2 dx ds \]
\[ + \frac{1}{2} \int_0^t \int_D \langle -\Delta d^\varepsilon + \frac{|d^\varepsilon|^2 - 1}{\varepsilon^2} d^\varepsilon, (d^\varepsilon \times h) \times h \rangle dx ds \]
\[ + \frac{1}{2} \int_0^t \int_D \left( |\nabla (d^\varepsilon \times h)|^2 + \frac{|d^\varepsilon|^2 - 1}{\varepsilon^2} |d^\varepsilon \times h|^2 \right) dx ds \]
\[ + \frac{1}{2} \int_0^t \int_D \langle -\Delta d^\varepsilon + f_\varepsilon(d^\varepsilon), d^\varepsilon \times h \rangle dx dW_2(s) \]
\[ = \int_0^t \int_D (u^\varepsilon \cdot \nabla d^\varepsilon, \Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)) dx ds - \int_0^t \int_D |\Delta d^\varepsilon - f_\varepsilon(d^\varepsilon)|^2 dx ds \]
\[ + \frac{1}{2} \int_0^t \int_D \langle -\Delta d^\varepsilon + f_\varepsilon(d^\varepsilon), d^\varepsilon \times h \rangle dx dW_2(s) \]
\[ + \frac{1}{2} \int_0^t \int_D \langle -\Delta d^\varepsilon + f_\varepsilon(d^\varepsilon), d^\varepsilon \times h \rangle dx .

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