On the moments of the gaps between consecutive primes

Marek Wolf

Cardinal Stefan Wyszynski University, Faculty of Mathematics and Natural Sciences. College of Sciences, ul. Wóycickiego 1/3, PL-01-938 Warsaw, Poland, e-mail: m.wolf@uksw.edu.pl

Abstract

We derive heuristically formula for the $k$–moments $M_k(x)$ of the gaps between consecutive primes $< x$ represented directly by $x\pi(x)$ — the number of primes up to: $M_k(x) = \Gamma(k+1)x^k/\pi^{k-1}(x) + O(x)$, We illustrate obtained results by computer data.

Key words: Prime numbers, gaps between primes, moments

Let $p_n$ denotes the $n$-th prime number and $d_n = p_{n+1} - p_n$ denotes the $n$-th gap between consecutive primes. Let us introduce moments of arbitrary order $k$ of gaps between consecutive primes:

$$M_k(x) \equiv \sum_{p_{n+1} \leq x} (p_{n+1} - p_n)^k. \quad (1)$$

The symbol $f(x) \sim g(x)$ means here that $\lim_{x \to \infty} f(x)/g(x) = 1$. Presumably for the first time the second moment of gaps $M_2(x)\sum_{p_n \leq x} (p_{n+1} - p_n)^2$ was considered in 1937 by H. Cramer [3]. Assuming the validity of the Riemann Hypothesis he obtained:

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2 = O(x \log^{3+\epsilon} x) \quad (2)$$

for every $\epsilon > 0$. In 1943 A. Selberg in [11], also assuming the Riemann Hypothesis, has proved:

$$\sum_{p_n < x} \frac{(p_{n+1} - p_n)^2}{p_n} = O(\log^3 x). \quad (3)$$

In [6] D.R. Heath-Brown conjectured that

$$\sum_{p_{n+1} < x} (p_{n+1} - p_n)^2 \sim 2x \log(x). \quad (4)$$
For the history of the problem and review of results see [7]; see also problem A8 in [4]. In [9, p.2056] the Heath-Brown–Oliveira conjecture was formulated:

\[ M_k(x) \sim k! x \log^{k-1}(x). \]  

(5)

In [9] authors made a remark after equation (5) that \( k \geq 1 \), but even for \( k = 0 \) it produces correct answer as \( M_0(x) = \pi(x) - 1 \) (here, as usual, \( \pi(x) = \sum_n \Theta(x - p_n) \) and \( \Theta \) is a unit step function: \( \Theta(x) = 1 \) for \( x > 0 \) and \( \Theta(x) = 0 \) for \( x \leq 0 \)). By the Prime Number Theorem (PNT) the number of prime numbers below \( x \) is very well approximated by the logarithmic integral

\[ \pi(x) \sim \text{Li}(x) \equiv \int_2^x \frac{du}{\ln(u)}. \]

Integration by parts gives the asymptotic expansion which should be cut at the term \( n_0 = \lfloor \ln(x) \rfloor \):

\[ \text{Li}(x) = \frac{x}{\ln(x)} + \frac{x}{\ln^2(x)} + \frac{2! x}{\ln^3(x)} + \frac{3! x}{\ln^4(x)} + \cdots. \]  

(6)

Let \( \tau_d(x) \) denote the number of pairs of consecutive primes smaller than a given bound \( x \) and separated by \( d \):

\[ \tau_d(x) = \#\{p_n, p_{n+1} < x, \text{ with } p_{n+1} - p_n = d\}. \]  

(7)

In [13] (see also [14]) we proposed the following formula expressing function \( \tau_d(x) \) directly by \( \pi(x) \):

\[ \tau_d(x) \sim C_2 \prod_{p|d, p > 2} \frac{p - 1}{p - 2} \frac{\pi^2(x)}{x} \left( 1 - \frac{2\pi(x)}{x} \right)^{\frac{d}{2} - 1} \text{ for } d \geq 6. \]  

(8)

Here

\[ C_2 \equiv 2 \prod_{p > 2} \left( 1 - \frac{1}{(p - 1)^2} \right) = 1.320323631693739\ldots \]

is called the “twins constant”. The pairs of primes separated by \( d = 2 \) (“twins”) and \( d = 4 \) (“cousins”) are special as they always have to be consecutive primes (with the exception of the pair (3,7) containing 5 in the middle)). For \( d = 4 \) we adapt the expression obtained from (8) for \( d = 2 \), which for \( \pi(x) \sim x/\log(x) \) goes into the the conjecture B of G. H. Hardy and J.E. Littlewood [5, eqs. (5.311) and (5.312)]:

\[ \tau_2(x) \left( \approx \tau_4(x) \right) \sim C_2 \frac{\pi^2(x)}{x} \approx C_2 \frac{x}{\ln^2(x)}. \]  

(9)
We will assume that for sufficiently regular functions \( f(n) \) the following formula holds:

\[
\sum_{k=1}^{\infty} \prod_{p|k, p>2} \frac{p-1}{p-2} f(k) = \frac{1}{\prod_{p>2}(1 - \frac{1}{(p-1)^2})} \sum_{k=1}^{\infty} f(k) \quad (10)
\]

In other words we will replace the product over \( p|d \) in (8) by its mean value as E. Bombieri and H. Davenport [1] have proved that the number \( 1/\prod_{p>2}(1 - \frac{1}{(p-1)^2})C_2/2 \)

is the arithmetical average of the product \( \prod_{p|k, p>2} \frac{p-1}{p-2} \):

\[
\sum_{k=1}^{n} \prod_{p|k, p>2} \frac{p-1}{p-2} = \frac{n}{\prod_{p>2}(1 - \frac{1}{(p-1)^2})} + O(\log^2(n)). \quad (11)
\]

Later H.L. Montgomery [8, eq.(17.11)] has improved the error term to \( O(\log(n)) \).

### TABLE I

The ratios of the sums of squares of gaps between consecutive primes for \( x = 2^{24}, \ldots, x = 4 \times 10^{18} \) and closed formulas for \( M_2(x) \) given by eq.(5), eq.(12) and eq.(13) respectively presented up to 4 figures.

| \( x \) | \( M_2(x)/M_2^{(1)}(x) \) | \( M_2(x)/M_2^{(2)}(x) \) | \( M_2(x)/M_2^{(3)}(x) \) |
|---------|----------------|----------------|----------------|
| \( 2^{24} = 1.6777 \ldots \times 10^7 \) | 0.7971 | 0.9104 | 0.8519 |
| \( 2^{26} = 6.7109 \ldots \times 10^7 \) | 0.8102 | 0.9151 | 0.8611 |
| \( 2^{28} = 2.6844 \ldots \times 10^8 \) | 0.8221 | 0.9198 | 0.8696 |
| \( 2^{30} = 1.0737 \ldots \times 10^9 \) | 0.8323 | 0.9237 | 0.8769 |
| \( 2^{32} = 4.2950 \ldots \times 10^9 \) | 0.8414 | 0.9272 | 0.8833 |
| \( 2^{34} = 1.7180 \ldots \times 10^{10} \) | 0.8495 | 0.9303 | 0.8890 |
| \( 2^{36} = 6.8719 \ldots \times 10^{10} \) | 0.8567 | 0.9332 | 0.8942 |
| \( 2^{38} = 2.7488 \ldots \times 10^{11} \) | 0.8632 | 0.9358 | 0.8988 |
| \( 2^{40} = 1.0995 \ldots \times 10^{12} \) | 0.8692 | 0.9382 | 0.9031 |
| \( 2^{42} = 4.3980 \ldots \times 10^{12} \) | 0.8746 | 0.9404 | 0.9069 |
| \( 2^{44} = 1.7592 \ldots \times 10^{13} \) | 0.8796 | 0.9425 | 0.9105 |
| \( 2^{46} = 7.0369 \ldots \times 10^{13} \) | 0.8841 | 0.9444 | 0.9138 |
| \( 2^{48} = 2.8147 \ldots \times 10^{14} \) | 0.8883 | 0.9462 | 0.9168 |
| \( 1.61 \times 10^{18} \) | 0.9087 | 0.9549 | 0.9315 |
| \( 4 \times 10^{18} \) | 0.9104 | 0.9556 | 0.9327 |

We will use the notation \( \widetilde{M}_k^{(i)}(x) \) for the \( i \)-th analytical formula for \( M_k(x) \). The superscript \( i = 1 \) will refer to the conjecture [5]: \( \widetilde{M}_k^{(1)}(x) = k!x \log^{k-1}(x) \) and expressions for \( i = 2 \) and \( i = 3 \) we will derive below. For second moments using the differentiated geometrical series we obtain (we have extended the summation over \( d = 2n \) up to infinity and used (10), then the dependence on \( c_2 \) drops out)

\[
M_2(x) = \sum_{p_n < x} (p_n - p_{n-1})^2 = \sum_{d=2,4,6,\ldots} d^2 \tau_d(x) \approx \frac{8\pi^2(x)}{(x - 2\pi(x))} \sum_{n=1}^{\infty} n^2 \left(1 - \frac{2\pi(x)}{x}\right)^n
\]

3
\[
\frac{2x^2}{\pi(x)} \left( 1 - \frac{\pi(x)}{x} \right) \equiv \tilde{M}_2^{(2)}(x). \tag{12}
\]

For large \(x\) skipping in the big bracket above term \(\pi(x)/x \sim 1/\log(x)\) we obtain

\[
M_2(x) \sim \frac{2x^2}{\pi(x)} \equiv \tilde{M}_2^{(2)}(x) \tag{13}
\]

what for \(\pi(x) \sim x/\log(x)\) gives exactly \(4\).

In the similar manner for third moment we obtain using \(8\) the expression:

\[
M_3(x) \sim \tilde{M}_3^{(2)}(x) \equiv \frac{6x^3}{\pi^2(x)} \left( 1 - 2 \frac{\pi(x)}{x} + \frac{2 \pi^2(x)}{3x^2} \right). \tag{14}
\]

Putting here \(\pi(x) \sim x/\log(x)\) in the limit of large \(x\) we obtain

\[
M_3(x) \sim 6x \log^2(x),
\]

i.e. \(4\) for \(k = 3\).

For fourth moment similarly we obtain:

\[
M_4(x) \approx \tilde{M}_4^{(2)}(x) \equiv 24 \frac{x^4}{\pi^3(x)} \left( 1 - 3 \frac{\pi(x)}{x} + \frac{7 \pi^2(x)}{3x^2} - \frac{1 \pi^3(x)}{3x^3} \right). \tag{15}
\]

and for large \(x\) it goes to \(4!x \log^3(x)\).

**TABLE II** The ratios of the sums of cubes of gaps between consecutive primes for \(x = 2^{24} \ldots, x = 4 \ldots \times 10^{18}\) and closed formulas for \(M_3(x)\) given by eq.(5) for \(k = 3\), eq.(14) and eq.(19) for \(k = 3\) presented up to 4 figures.

| \(x\)            | \(M_3(x)/M_3^{(1)}(x)\) | \(M_3(x)/M_3^{(2)}(x)\) | \(M_3(x)/M_3^{(3)}(x)\) |
|------------------|--------------------------|--------------------------|--------------------------|
| \(2^{24}\) = 1.6777 \ldots \times 10^7 | 0.6104                  | 0.7975                   | 0.6972                   |
| \(2^{26}\) = 6.7109 \ldots \times 10^7 | 0.6331                  | 0.8087                   | 0.7152                   |
| \(2^{28}\) = 2.6844 \ldots \times 10^8 | 0.6540                  | 0.8195                   | 0.7318                   |
| \(2^{30}\) = 1.0737 \ldots \times 10^9 | 0.6722                  | 0.8287                   | 0.7461                   |
| \(2^{32}\) = 4.2950 \ldots \times 10^9 | 0.6885                  | 0.8367                   | 0.7588                   |
| \(2^{34}\) = 1.7180 \ldots \times 10^{10} | 0.7030                  | 0.8438                   | 0.7700                   |
| \(2^{36}\) = 6.8719 \ldots \times 10^{10} | 0.7162                  | 0.8504                   | 0.7803                   |
| \(2^{38}\) = 2.7488 \ldots \times 10^{11} | 0.7283                  | 0.8564                   | 0.7896                   |
| \(2^{40}\) = 1.0995 \ldots \times 10^{12} | 0.7393                  | 0.8619                   | 0.7981                   |
| \(2^{42}\) = 4.3980 \ldots \times 10^{12} | 0.7495                  | 0.8670                   | 0.8059                   |
| \(2^{44}\) = 1.7592 \ldots \times 10^{13} | 0.7588                  | 0.8716                   | 0.8131                   |
| \(2^{46}\) = 7.0369 \ldots \times 10^{13} | 0.7674                  | 0.8759                   | 0.8198                   |
| \(2^{48}\) = 2.8147 \ldots \times 10^{14} | 0.7754                  | 0.8800                   | 0.8259                   |
| \(1.61 \times 10^{18}\) | 0.8147                  | 0.8997                   | 0.8561                   |
| \(4 \times 10^{18}\) | 0.8180                  | 0.9014                   | 0.8586                   |
The ratios of the sums of fourth powers of gaps between consecutive primes for $x = 2^{24} \ldots, x = 4 \ldots < 10^{18}$ and closed formulas for $M_k(x)$ given by eq. [5] for $k = 4$, eq. [15] and eq. [19] for $k = 4$ presented up to 4 figures.

| $x$ | $M_4(x)/M_4^{(1)}(x)$ | $M_4(x)/M_4^{(2)}(x)$ | $M_4(x)/M_4^{(3)}(x)$ |
|-----|------------------------|------------------------|------------------------|
| $2^{24} = 1.6777 \ldots \times 10^6$ | 0.4586 | 0.6854 | 0.5598 |
| $2^{26} = 6.7109 \ldots \times 10^6$ | 0.4862 | 0.7024 | 0.5838 |
| $2^{28} = 2.6844 \ldots \times 10^8$ | 0.5123 | 0.7190 | 0.6063 |
| $2^{34} = 1.0737 \ldots \times 10^9$ | 0.5354 | 0.7332 | 0.6261 |
| $2^{32} = 4.2950 \ldots \times 10^9$ | 0.5560 | 0.7453 | 0.6433 |
| $2^{24} = 1.7180 \ldots \times 10^{10}$ | 0.5746 | 0.7560 | 0.6587 |
| $2^{36} = 6.8719 \ldots \times 10^{10}$ | 0.5919 | 0.7661 | 0.6731 |
| $2^{38} = 2.7488 \ldots \times 10^{11}$ | 0.6078 | 0.7753 | 0.6861 |
| $2^{40} = 1.0995 \ldots \times 10^{12}$ | 0.6225 | 0.7837 | 0.6982 |
| $2^{12} = 4.3980 \ldots \times 10^{12}$ | 0.6360 | 0.7915 | 0.7093 |
| $2^{14} = 1.7592 \ldots \times 10^{13}$ | 0.6486 | 0.7987 | 0.7195 |
| $2^{46} = 7.0369 \ldots \times 10^{14}$ | 0.6603 | 0.8054 | 0.7290 |
| $2^{48} = 2.8147 \ldots \times 10^{14}$ | 0.6712 | 0.8116 | 0.7379 |
| $1.61 \times 10^{18}$ | 0.7256 | 0.8422 | 0.7816 |
| $4 \times 10^{18}$ | 0.7303 | 0.8448 | 0.7853 |

We stop with these particular moments and we will derive the formula for moments of general order $k$. From the formula (8) we obtain:

$$M_k(x) = \sum_{n<x} (p_n - p_{n-1})^k = \sum_{d=2,4,6,\ldots} d^k \tau_d(x) \sim 2 \pi^2(x) x - 2 \pi(x) \sum_{n=1}^{\infty} (2n)^k \left(1 - \frac{2\pi(x)}{x}\right)^n$$

(16)

To proceed further we need formula for the $k$-times differentiated geometrical series:

$$\sum_{n=1}^{\infty} n^k q^n = (\frac{qd}{d^k}) \left(1 - q\right) = \frac{1}{(1-q)^{k+1}} \sum_{i=0}^{k-1} \left(\begin{array}{c} k \\ i \end{array}\right) q^{k-i}$$

(17)

where $|q| < 1$ and $\left(\begin{array}{c} n \\ i \end{array}\right)$ are Eulerian numbers (should not be confused with Euler numbers $E_n$), see [10] p.54 and eq. (7) in entry Eulerian numbers in [12]. In our case $q = 1 - 2\pi(x)/x$ and for large $x$ we have $q \rightarrow 1$ hence in nominator we obtain $k!$ because the Eulerian numbers satisfy the identity

$$\sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array}\right) = k!$$

(18)

see [2, eq.(1.8)] and entry Eulerian numbers in [12]. The denominator is $(2\pi(x)/x)^{k+1}$ and the power $2^{k+1}$ cancels out. Finally we obtain

$$M_k(x) \equiv \sum_{n<x} (p_n - p_{n-1})^k \sim k! \frac{x^k}{\pi^{k-1}(x)} \equiv \tilde{M}_k^{(3)}(x)$$

(19)
and for $\pi(x) \sim x/\log(x)$ it goes into (5). For $k = 1$ from above equation we obtain $M_1(x) = x$ and for $k = 0$ we obtain $M_0(x) = \pi(x)$ as it should be.

During over a seven months long run of the computer program we have collected the values of $\tau_d(x)$ up to $x = 2^{48} \approx 2.8147 \times 10^{14}$. The data representing the function $\tau_d(x)$ were stored at values of $x$ forming the geometrical progression with the ratio 2, i.e. at $x = 2^{15}, 2^{16}, \ldots, 2^{47}, 2^{48}$. Such a choice of the intermediate thresholds as powers of 2 was determined by the employed computer program in which the primes were coded as bits. The data is available for downloading from [http://pracownicy.uksw.edu.pl/mwolf/gaps.zip](http://pracownicy.uksw.edu.pl/mwolf/gaps.zip). At the Tomás Oliveira e Silva web site [http://sweet.ua.pt/tos/gaps.html](http://sweet.ua.pt/tos/gaps.html) we have found values of $\tau_d(x)$ for $x = 1.61 \times 10^{18}$ and $x = 4 \times 10^{18}$. In the tables I, II and III we present comparison of the actual values of $M_k(x)$ calculated from these computer data $k = 2, 3, 4$ and the prediction given by formulas for $\tilde{M}_k^{(i)}(x)$ for $i = 1, 2, 3$ and the set of values of $x$. As the rule the best approximations are given by (13), (14) and (15), next by (19) and the least accurate are values predicted by (5).

We can try to determine the form of error terms in the formulas (12), (13), (14) and (19). In figure 1 we present plots of the differences of experimental values of moments $M_k(x)$ calculated from the real computer data and appropriate formulas for $\tilde{M}_k^{(i)}(x)$. All these plots suggest that the error term is given by $A_k x^\alpha$, where $\alpha$ is very close to 1 and the prefactors $A_k$ increases rapidly with the order $k$ of moments. Because all approximate expressions $\tilde{M}_k^{(i)}(x)$ give values larger than experimental values of moments we write:

$$M_k(x) = \tilde{M}_k^{(i)}(x) - A_k^{(i)} x. \quad (20)$$

In the Table IV we present a sample of coefficients $A_k^{(i)}$ calculated from the above equation for $x = 4 \times 10^{18}$, as then the exponent in power of $x$ is closest to 1. Thus, generalizing to non–integer $k$, we formulate the Conjecture:

$$M_k(x) = \frac{\Gamma(k+1)x^k}{\pi^{k-1}(x)} + O(x). \quad (21)$$

| $k$  | $A_k^{(1)}$ | $A_k^{(2)}$ | $A_k^{(3)}$ |
|-----|------------|------------|------------|
| 2   | 7.674      | 3.624      | 5.624      |
| 3   | 2003.517   | 985.198    | 1482.890   |
| 4   | 508697.096 | 252978.305 | 376492.431 |

In paper [9] on p. 2057 the authors consider corrections to (5) given by the series in powers of $1/\log(x)$:

$$M_k(x) \approx k! \log^{k-1}(x) \sum_{n=0}^{N} \frac{d_{kn}}{\log^n(x)}. \quad (22)$$
In this paper the table of values of $d_{kn}$ obtained from the least-squares fitting to data for $10^{10} < x < 4 \times 10^{18}$ is given for $N = 2$ for $k = 2, 3, 4$. We have checked that increasing the order $N$ completely changes the values of coefficients $d_{kn}$, except $d_{k0} \approx 1$, thus they depend on the order $N$. To explain this we notice that the fitting was done in the very short interval $(1/\log(4 \times 10^{18}), 1/\log(10^{10}) = (0.0233 \ldots, 0.031 \ldots)$. On such a narrow interval each smooth function by the Taylor expansion is a linear function in the first approximation plus a part of parabola plus a cubic term etc. In the Taylor expansion of $f(x)$ around point $x = a$ coefficients are $f^{(n)}(a)/n!$ and they do not change with increasing the number of terms. However in [9] $d_{kn}$ were determined from the least-squares method.

The correct expansion in powers of $1/\log(x)$ of formulas for moments we obtain using the asymptotic series for the logarithmic integral in (6) and putting it into ours expressions for moments involving the prime counting function $\pi(x)$. In this manner we obtain from (13) for second moment:

$$\tilde{M}_2^{(2)} = 2x \log(x) \left(1 - \frac{2}{\log(x)} - \frac{2}{\log^2(x)} - \frac{3}{\log^3(x)} + \frac{17}{\log^4(x)} + \ldots\right)$$

and for third moment:

$$\tilde{M}_3^{(2)} = 3x \log^2(x) \left(1 - \frac{4}{\log(x)} + \frac{5}{3 \log^2(x)} - \frac{2}{\log^3(x)} + \frac{47}{\log^4(x)} + \ldots\right).$$

In general from our conjecture (19) we get

$$\tilde{M}_k^{(3)} = k! x \log^{k-1}(x) \left(1 + \frac{1 - k}{\log(x)} + \frac{4 - 5k + k^2}{2 \log^2(x)} + \frac{6 - \frac{47k}{6} + 2k^2 - \frac{k^3}{6}}{\log^3(x)} + \ldots\right).$$

For the coefficients $d_{k0}$ in [9] the values very close to 1 were obtained and indeed from above expansions we have that they are always 1.

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Figure 1: The plot of differences between experimental values of moments $k(x)$ and calculated from $\tilde{M}_k^{(1)}$ and $\tilde{M}_k^{(2)}$ for $k = 2, 3, 4$. 