Riemannian Stochastic Hybrid Gradient Algorithm for Nonconvex Optimization

Jiabao Yang

Abstract

In recent years, Riemannian stochastic gradient descent (R-SGD), Riemannian stochastic variance reduction (R-SVRG) and Riemannian stochastic recursive gradient (R-SRG) have attracted considerable attention on Riemannian optimization. Under normal circumstances, it is impossible to analyze the convergence of R-SRG algorithm alone. The main reason is that the conditional expectation of the descending direction is a biased estimation. However, in this paper, we consider linear combination of three descent directions on Riemannian manifolds as the new descent direction (i.e., R-SRG, R-SVRG and R-SGD) and the parameters are time-varying. At first, we propose a Riemannian stochastic hybrid gradient (R-SHG) algorithm with adaptive parameters. The algorithm gets a global convergence analysis with a decaying step size. For the case of step-size is fixed, we consider two cases with the inner loop fixed and time-varying. Meanwhile, we quantitatively research the convergence speed of the algorithm. Since the global convergence of the R-SHG algorithm with adaptive parameters requires higher functional differentiability, we propose a R-SHG algorithm with time-varying parameters. And we obtain similar conclusions under weaker conditions.

1 Introduction

Consider the following finite-sum optimization problems definition on a smooth Riemannian manifold $\mathcal{M}$

$$
\min_{\omega \in \mathcal{M}} f(\omega) := \frac{1}{n} \sum_{i=1}^{n} f_i(\omega) \quad (P)
$$

where the function $f_i : \mathcal{M} \to \mathbb{R}, i = \{1, 2, ..., n\}$.

Problem (P) has many applications; including principal component analysis [1, 2], low-rank matrix completion [3, 4], Riemannian centroid computation [7], independent component analysis [8], dictionary learning [9, 10] and so on.

Since some constrained optimization problems in Euclidean space can be converted to unconstrained problems on manifolds, it is interested in solving problem (P) over the Riemannian manifold space via Riemannian gradient methods. A common idea is that the negative of Riemannian gradient direction is used as the descent direction, that is calculate the Riemannian full gradient of function $f$: $\text{grad} f(\omega) = \frac{1}{n} \sum_{i=1}^{n} \text{grad} f_i(\omega)$, where the $\text{grad} f_i(\omega)$ denotes Riemannian gradient of the $i$th. If $n$ is large, the cost of computing and operating is expensive.

In Euclidean space, a popular choice to solve problem (P) is stochastic gradient descent (SGD) algorithm. Some scholars have achieved better results by improving robustness [22], adapting learning [21] etc. Inspired by the SGD algorithm in Euclidean space, other scholars have proposed the R-SGD algorithm on Riemannian manifold. Bonnabel proposed a
R-SGD algorithm to extend SGD algorithm from Euclidean space to Riemannian manifold. However, we should point out that a popular choice is random selection of partial function gradients without taking the gradients of all functions. But R-SGD algorithm needs exponential mapping and parallel translation operation in each iteration. If these computational costs are lower than the computational Riemannian gradient, we can ignore them. Similar to the SGD algorithm in Euclidean space, when we use a large step size in R-SGD algorithm, the loss of training will decrease rapidly at first, but it may have a great influence around the solution. On the contrary, in order to obtain convergence, we require a large number of iterations when using smaller steps. Therefore, R-SGD algorithm can start with a large step size and gradually reduces the step size to avoid these problems. But due to the attenuation of step-size sequence, the convergence of R-SGD algorithm is slow.

In recent years, the technique of stochastic variance reduction have attracted considerable attention for minimizing the average of finite-number of loss functions. In Euclidean space, scholars prove that the method of variance reduction can accelerate SGD algorithm convergence [27]. The main idea is that by periodically calculating the gradient to correct the deviation of stochastic gradient, and the gradient variance decreases with the progress of training. This leads to linear convergence.

Because of this, the paper [11] proposed a R-SVRG algorithm. Inspired by the variance reduction of non-convex optimization, [1] has analyzed the R-SVRG algorithm of geodesic strongly convex function through a new theoretical analysis and explained the nonlinear (curve) geometric shape of the Riemannian manifold. This produce a linear convergence rate. The works are parallel with paper [12]. The main idea is that by periodically calculating the gradient to correct the deviation of stochastic gradient, and the gradient variance decreases with the progress of training. This leads to linear convergence.

Because of this, the paper [11] proposed a R-SVRG algorithm. Inspired by the variance reduction of non-convex optimization, [1] has analyzed the R-SVRG algorithm of geodesic strongly convex function through a new theoretical analysis and explained the nonlinear (curve) geometric shape of the Riemannian manifold. This produce a linear convergence rate. The works are parallel with paper [12]. From the idea of paper [12], paper [1] proves the global convergence of the algorithm under retraction mapping and vector transport. But [11] is carried out under exponential mapping and parallel translation. It should emphasize that the local convergence rate is analyzed in [1]. If the function $f$ is assumed to have global strong convexity in the search space, the global iterative complexity can be obtained.

Since R-SVRG algorithm uses double loop iteration, we need to add the condition that $\omega_0$ is transported to $\omega_t$, the vector transport of R-SVRG algorithm between the iterations of two distant points is required in the calculation. Its cost and difficulty will be improved. Therefore, a R-SRG algorithm independent of two distant points is proposed in [13], to avoid the calculation of contraction inverse and makes the calculation efficiency higher. The advantage of R-SRG algorithm over R-SVRG algorithm is more notable in the Riemannian than Euclidean case [13]. In addition, from [23, 24], Riemannian stochastic recursive momentum (R-SRM) algorithm is proposed in [14]. The author considers the linear combination of R-SGD and R-SVRG, and obtained the R-SRM algorithm (the linear combination coefficient and step size of the algorithm are time-varying), and assumes that the optimization function is an unbiased estimation. It is proved that the expectation converges at the convergence rate of $O\left(\frac{1}{T^2}\right)$.

Because the calculation of exponential mapping and parallel translation are expensive, therefore, in this paper, we consider the situations with retraction mapping and vector transport. Inspired by [14], we consider the linear combination of three descent directions on Riemannian manifolds as the new descent direction (i.e., R-SRG, R-SVRG and R-SGD) and propose the two algorithms. And the linear combination of the parameters in the algorithms are time-varying. Compared to the existing works, the key contributions of our paper are listed as follows

1) Commonly, the global convergence of R-SRG algorithm can not analyze alone as [1], the main reason is the conditional expectation of the descent direction is biased. In contrast, [1] is unbiased. Therefore, we propose a R-SHG algorithm with adaptive parameters. In this way, the conditional expectation of the descent direction after the combination is still a biased estimation. For the case of reduced step size, by adapting the parameters of the R-SHG algorithm, we can get the global convergence. If special parameters are chosen, our
results can be degenerated into [1]. Moreover, for the case of fixed step size, we quantitatively research the convergence rate of the algorithm.

2) Research [14] considers the linear combination of R-SRG and R-SGD. Our second algorithm (i.e., R-SHG algorithm with time-varying parameters) can obtain a faster convergence rate than them. If we consider the problem of expectation (online) minimization algorithm (i.e., R-SHG algorithm with time-varying parameters) can obtain a faster convergence rate under fixed step size. These convergence conditions are weakly than the R-SHG algorithm with adaptive parameters.

3) Usually, choosing time-varying step size may accelerate the convergence of the algorithm. In Riemannian manifold, the main consideration is to improve convergence speed by using time-varying step size. However, our results imply that the convergence rate can also be accelerated under the condition of fixed step size by changing the parameters.

4) Convergence analysis(convergence rate) is complex in the algorithm, which is in itself a challenging problem. But our algorithm can do convergence analysis under time-varying step size and fixed step size. We use retraction mapping and vector transport, which is more general in Riemannian manifold than exponential mapping and parallel transport.

5) For the three special situations (i.e., the descent direction only use the R-SRG and R-SVRG term, the retraction mapping and vector transport is taken as exponential mapping and parallel transport), the retraction mapping and vector transport is taken as exponential mapping and parallel translation operations, and function $f$ is $\tau$-gradient dominated), we give the better conclusions.

The rest of the paper is organized as follows. Section 2 describes the Riemannian preliminaries and assumptions. Section 3 and 4 introduce the algorithm and prove the proposed algorithms’ global convergence and local convergence rate. We also consider the convergence in several special cases. In Section 5, the conclusions and future research topics are given.

Notation and symbols: $|A|$: the cardinality of set $A$; $a_n = O(b_n)$: $\limsup_{n \to \infty} \frac{a_n}{b_n} < \infty$

2 Preliminaries and Assumption

2.1 Preliminaries

A manifold whose tangent spaces are endowed with a smoothly varying inner product is called a Riemannian manifold. The smoothly varying inner product is called the Riemannian metric [19]. The inner product $g_x: T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$. For the convenience of the following, we let $g_x(\eta_x, \zeta_x) = \langle \eta_x, \zeta_x \rangle_x = \langle \eta_x, \zeta_x \rangle$, for any $\eta_x, \zeta_x \in T_x\mathcal{M}$. Let $\|\eta_x\| = \sqrt{\langle \eta_x, \eta_x \rangle}$. $\mathbb{E}[\mathcal{F}_t]$ denotes the conditional expectation with respect to the random variable $I^t$, where $\mathcal{F}_t = \sigma\{I_0^t, I_1^t, I_2^t, ..., I_0^t, I_1^t, I_2^t, ..., I_{t-1}^t\}$ is the $\sigma$-algebra and $I_0^t, s \in 1, 2, ..., S$ is equal to complete set. $\text{grad} f_{I_t}(\omega) = \frac{1}{|I_t|} \sum_{i \in I_t} \text{grad} f_i(\omega)$, where $I_t^* \subset \{1, 2, ..., n\}$ is an index set with cardinality $|I_t^*|$. The gradient $\text{grad} f(\omega)$ is defined as the unique element of $T_\omega\mathcal{M}$ that satisfies

$$Df(\omega)[\xi_\omega] = \langle \text{grad} f(\omega), \xi_\omega \rangle \quad \xi_\omega \in T_\omega\mathcal{M}$$

where $Df(\omega): T_\omega\mathcal{M} \to \mathbb{R}$ is the derivative of $f$ at $\omega$. The exponential map $\text{Exp}_x: T_x\mathcal{M} \to \mathcal{M}$ maps a tangent vector $\eta_x \in T_x\mathcal{M}$ along the geodesic leading to $y = \text{Exp}_x(\eta_x) \in \mathcal{M}$ such that $\gamma(0) = x, \gamma(1) = y, \gamma'(0) = \eta_x$. And the distant between $x$ and $y$ denotes $\text{dist}(x, y) = \|\eta_x\|$. For $\delta > 0$, we denote $\mathbb{B}_x(0, \delta) = \{y \in \mathcal{M} | \text{dist}(x, y) \leq 1\}$. In this paper, our analysis focus on retraction mapping and vector transport. The definition of a retraction is as follows [19].

**Definition 1.** $R: T\mathcal{M} \to \mathcal{M}$ is called a retraction on $\mathcal{M}$ if the restriction $R_\omega: T_\omega\mathcal{M} \to \mathcal{M}$ to $T_\omega\mathcal{M}$ for all $\omega \in \mathcal{M}$ satisfies:
1. $R_\omega(0_\omega) = \omega$, where $0_\omega$ is the zero vector in $T_\omega M$
2. $D R_\omega(0_\omega)[\xi] = \xi$, for all $\xi \in T_\omega M$

Let $\Gamma^y_x$ be the parallel translation operator by the exponential mapping linking $x$ and $y$. However, parallel translation sometimes computationally expensive, so we consider using vector transport replacing parallel translation.

**Definition 2.** A vector transport on a manifold $M$ is a smooth mapping

$$T_M \otimes T_M \rightarrow T_M : (\eta_\omega, \xi_\omega) \mapsto T_{\eta_\omega}(\xi_\omega) \in T_M$$

satisfying the following diagram properties for all $x \in M$

1. $T_{\eta_\omega}(\xi) = \xi$, where $\xi \in T_\omega M, \omega \in M$
2. $T_{\eta_\omega}(a \xi_\omega + b \theta_\omega) = a T_{\eta_\omega}(\xi_\omega) + b T_{\eta_\omega}(\theta_\omega)$, where $a, b \in \mathbb{R}, \eta_\omega, \xi_\omega, \theta_\omega \in T_x M$

For the convenience of the following, we let $T_{\eta_\omega}^{R_\omega}(\eta_\omega)(\xi_\omega) = T_{\eta_\omega}(\xi_\omega)$, for any $\eta_\omega, \xi_\omega \in T_\omega(M)$. Here, we further introduce the concept of $\tau-$gradient dominated function \[25\] [26] which will also be used in this paper.

**Definition 3.** We say function $f : M \rightarrow \mathbb{R}$ is $\tau-$gradient dominated, if for any $\omega \in M$, we have $f(\omega) - f(\omega^*) \leq \tau \| \text{grad} f(\omega) \|^2$, where $\omega^*$ is a global minimizer of $f$.

### 2.2 Assumption

In this article, we will use following assumptions.

**Assumption 1.a** Function $f$ and its component functions $f_i, i = 1, 2...n$ are continuously differentiable.

**Assumption 1.b** Function $f$ is thrice continuously differentiable, and its component functions $f_i, i = 1, 2...n$ are twice continuously differentiable.

**Assumption 2** Iterate sequences produced by algorithms stay continuously in a compact neighbourhood $\Omega$, where the $\Omega$ is a neighbourhood around $\omega^*$. Additionally, $\Omega$ is a $\rho$-totally retractive neighborhood of $\omega^*$ where retraction $R$ is a diffeomorphism. And for all $t \geq 0, s \geq 1, \tau \in [0, 1], R_{\omega^*}(-\tau \alpha^*_t V^*_t) \in \Omega$.

The $\rho$-totally neighborhood $\Omega$ of $\omega^*$ is a set such that for all $\omega \in \Omega$, $\Omega \subset R_{\omega}(0_\omega, \rho)$, and $R_{\omega}(\cdot)$ is a diffeomorphism on $R_{\omega}(0_\omega, \rho)$. Assumption 1.b and 2 are basic for standard analysis.

**Assumption 3** The sequence $\{\alpha^*_t\}$ of step sizes satisfies $\sum_{s=1}^{\infty} \sum_{t=0}^{m-1} \alpha^*_t = \infty$ and $\sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\alpha^*_t)^2 < \infty$.

The conditions of assumption 3 are satisfied, for example, $\alpha^*_t = \frac{1}{t+s+1}$.

**Assumption 4** The vector transport $T$ is continuous and isometric on $M$, i.e., for any $\omega \in M, \eta, \xi, \zeta \in T_\omega M$, satisfies $\langle T_\eta \xi, T_\eta \zeta \rangle = \langle \xi, \zeta \rangle$.

Similar to [13] [19], we also can construct an isometric vector transport such that assumption 4 holds.

**Assumption 5** $\Gamma^y_x$ is the parallel transport operator from $y$ to $x$, there exists a constant $M > 0$, for any $x, y = R_x(\xi) \in \Omega$, satisfying $\frac{1}{n} \sum_{i=0}^{n-1} \| \text{grad} f_i(x) - \Gamma^y_x \text{grad} f_i(y) \|^2 \leq M^2 \| \xi \|^2$.

**Assumption 6** [15] Difference between vector transport $T_x^y$ and parallel transport $\Gamma^y_x$ associated with the same retraction $R$ is bounded. There exists a constant $\theta > 0$, for all
exists a constant $L > R - SHG$ algorithm with adaptive parameters

Algorithm 1

1: Input: step size $\alpha_i$, frequency $m > 0$, $0 < \mu < 1$, parameter $\psi_i$, $\phi_i$
2: Initialize: $\omega_0$
3: for for $s=1,2,\ldots,S$ do
4: Caclulate the full Riemannian gradient $\nabla f(\hat{\omega}^{s-1})$
5: Store $\omega_0 = \hat{\omega}^{s-1}$, $V_0 = \nabla f(\hat{\omega}_0)$, $\omega_1 = R_{\omega_0}(-\alpha_0 V_0$)
6: for $t=1,2,\ldots,m-1$ do
7: Choose $I_t \in \{1, 2, \ldots, n\}$ uniformly at random
8: Caclulate the value of $(\nabla_{\omega_{t-1}}(V_{t-1}^s - \nabla f(\omega_{t-1}^s)), \nabla f(\omega_t^s))$
9: if $(\nabla_{\omega_{t-1}}(V_{t-1}^s - \nabla f(\omega_{t-1}^s)), \nabla f(\omega_t^s)) \neq 0$ then
10: $\tilde{\psi}_t^s = \min\{\psi_t^s, \frac{\mu\|\nabla f(\omega_t^s)\|^2}{\|\nabla_{\omega_{t-1}}(V_{t-1}^s - \nabla f(\omega_{t-1}^s)), \nabla f(\omega_t^s))\|}\}$
11: else if
12: then $\tilde{\psi}_t^s = \psi_t^s$
13: end if
14: $V_t^s = \phi_t^s \left(\nabla f_{I_t}(\omega_t^s) - T^s_{\omega_t^s} \left(\nabla f_{I_t}(\omega_t^s) - \nabla f(\omega_t^s)\right)\right) + \tilde{\psi}_t^s \left(\nabla f_{I_t}(\omega_t^s) - T^s_{\omega_t^s} \left(\nabla f_{I_t}(\omega_{t-1}^s) - V_{t-1}^s\right)\right) + (1 - \phi_t^s - \tilde{\psi}_t^s) \nabla f_{I_t}(\omega_t^s)$

15: $\omega_{t+1} = R_{\omega_t^s}(-\alpha_t V_t^s)$
16: end for
17: $\hat{\omega}^s = \omega_m$
18: end for
19: We choose: option 1: $\omega_a = \hat{\omega}^S$
20: We choose: option 2: $\omega_a$ uniformly randomly from $\{\omega_t^s\}_{t=0}^{S-1}$
21: Output: $\omega_a$

3 Riemannian Stochastic Hybrid Gradient Algorithm with Adaptive Parameters

In this section, firstly, we present the R-SHG algorithm with adaptive parameters. For the case of reduced step size, we qualitatively analyze the convergence of the algorithm. For the case of fixed step size, we quantitatively research the convergence of the algorithm. Throughout this section, we consider two special cases. Now, we propose the first algorithm.
In the algorithm, we require that the parameters $\psi^*_t$, $\phi^*_t$ must be satisfies $0 \leq \psi^*_t + \phi^*_t \leq 1$ and $\psi^*_t, \phi^*_t \geq 0$. In this paper, we always suppose the step size and the parameters are positive. By the definition of $\psi^*_t$, then we can get following inequalities.

**Lemma 1.** Let $\tilde{\psi}^*_t$ be defined by the algorithm 1, then

$$- \mu \| \nabla f(\omega^*_t) \|^2 \leq \tilde{\psi}^*_t \langle \mathcal{T}^t_{\omega^*_t} (V^s_{t-1} - \nabla f(\omega^*_t-1)), \nabla f(\omega^*_t) \rangle \leq \mu_1^* \| \nabla f(\omega^*_t) \|^2$$  \hspace{1cm} (3)

Especially, if $\psi^*_t \equiv 0$, then $\tilde{\psi}^*_t \equiv 0$, the inequalities also hold for any $0 < \mu < 1$.

This result is important in the convergence of the algorithm. Normally, we can not determine the positive or negative number of $\langle \mathcal{T}^t_{\omega^*_t} (V^s_{t-1} - \nabla f(\omega^*_t-1)), \nabla f(\omega^*_t) \rangle$. Therefore, we adjust $\tilde{\psi}^*_t$ to make $\tilde{\psi}^*_t \langle \mathcal{T}^t_{\omega^*_t} (V^s_{t-1} - \nabla f(\omega^*_t-1)), \nabla f(\omega^*_t) \rangle$ is bounded and sufficiently small. The proof of lemma [1] is given in Appendix [A].

### 3.1 Step size is reduced

We first prove the following lemma of the estimator $V^s_t$. This lemma plays an important role in this section.

**Lemma 2.** Let $V^s_t$ be defined by algorithm 1, then,

$$E[V^s_t|F^s_t] = \nabla f(\omega^*_t) + \tilde{\psi}^*_t \mathcal{T}^t_{\omega^*_t} (V^s_{t-1} - \nabla f(\omega^*_t-1))$$  \hspace{1cm} (4)

Furthermore, if $\psi^*_t \neq 0$, then $V^s_t$ is a biased estimate.

**Proof.** Suppose for any $\omega \in \Omega$ and $\omega$ is $F_t^s$ measurable. By the definition of $I^s_t$ and $F^s_t$, we get

$$E[\nabla f(I^s_{t})|F^s_t] = \frac{1}{b} \sum_{i=1}^{n} \nabla f_i(\omega) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\omega) = \nabla f(\omega)$$  \hspace{1cm} (5)

Since $V^s_{t-1}$ is $F^s_t$ measurable and $\mathcal{T}^t_{\omega^*_t} = \mathcal{T}^t_{\omega^*_t-V^s_{t-1}}$, we obtain that $\mathcal{T}^t_{\omega^*_t}$ is also $F^s_t$ measurable. This together with (5) gives

$$E[V^s_t|F^s_t]$$

$$= \mathbb{E}[\phi^*_t(\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t}) - \nabla f(\omega^*_t)))]$$

$$+ \tilde{\psi}^*_t (\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t-1}) - V^s_{t-1}))$$

$$+ (1 - \phi^*_t - \tilde{\psi}^*_t) \nabla f(I^s_{t} - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t-1} - V^s_{t-1}))|F^s_t]$$

$$= \phi^*_t \mathbb{E}[\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t}) - \nabla f(\omega^*_t))|F^s_t]$$

$$+ \tilde{\psi}^*_t \mathbb{E}[\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t-1}) - V^s_{t-1})|F^s_t]$$

$$+ (1 - \phi^*_t - \tilde{\psi}^*_t) \mathbb{E}[\nabla f(I^s_{t} - \mathcal{T}^t_{\omega^*_t} (\nabla f(I^s_{t-1} - V^s_{t-1}))|F^s_t]$$

$$= \phi^*_t \mathbb{E}[\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\mathbb{E}[\nabla f(I^s_{t})|F^s_t] - \nabla f(\omega^*_t))|F^s_t]$$

$$+ \tilde{\psi}^*_t \mathbb{E}[\nabla f(I^s_{t}) - \mathcal{T}^t_{\omega^*_t} (\mathbb{E}[\nabla f(I^s_{t-1})|F^s_t] - V^s_{t-1})|F^s_t]$$

$$+ (1 - \phi^*_t - \tilde{\psi}^*_t) \mathbb{E}[\nabla f(I^s_{t} - \mathcal{T}^t_{\omega^*_t} (\mathbb{E}[\nabla f(I^s_{t-1})|F^s_t] - V^s_{t-1}))|F^s_t]$$

$$= \nabla f(\omega^*_t) + \tilde{\psi}^*_t \mathcal{T}^t_{\omega^*_t} (V^s_{t-1} - \nabla f(\omega^*_t-1))$$  \hspace{1cm} (6)

\[\square\]

Now, we give the mean-square convergence of the proposed algorithm under the assumptions.
Theorem 1. Suppose the assumption 1.b and assumption 2-4 hold. The sequences \{\omega^*_i\} produced by algorithm 1, then \{\mathbb{E}[\|\text{grad}f(\omega^*_i)\|^2]\} \to 0

To prove theorem 1 we need the following two lemmas. These two lemmas are very useful in stochastic algorithms.

Lemma 3 ([17]). Let \{x(k), \mathcal{F}(k)\}, \{\alpha(k), \mathcal{F}(k)\}, \{\beta(k), \mathcal{F}(k)\} and \{\gamma(k), \mathcal{F}(k)\} be nonnegative adaptive sequences satisfying

\[ \mathbb{E}[x(k + 1)|\mathcal{F}(k)] \leq (1 + \alpha(k))x(k) - \beta(k) + \gamma(k), k \geq a.s. \]

if \(\sum_{k=0}^{\infty} (\alpha(k) + \gamma(k)) < \infty \) a.s., then \(x(k)\) converges to a finite random variable a.s. and \(\sum_{k=0}^{\infty} \beta(k) < \infty \) a.s.

Lemma 4 ([18]). Let \{x(k)\} be a nonnegative stochastic process with bounded positive variations, i.e., \(\sum_{k=0}^{\infty} \mathbb{E}[|x(k + 1) - x(k)|\mathcal{F}(k)] \leq \infty\), where \(x^+ = \max\{0, x\}\), Then \(x(k)\) is a quasi martingale, i.e.,

\[ \sum_{k=0}^{\infty} \mathbb{E}[|x(k + 1) - x(k)|\mathcal{F}(k)] < \infty \text{ a.s.} \quad \text{\textbf{x(k) converges a.s}} \]

Proof of Theorem 1

Proof. Since \(\Omega\) is compact, all continuous functions on \(\Omega\) can be bounded. Hence, there exists a positive constant \(N\), for all \(\omega \in \Omega\), such that \(\|\text{grad}f(\omega)\| \leq N\) and \(\|\text{grad}f_i(\omega)\| \leq N, i = 1, 2, \ldots, n\). We reindex the sequence \({\alpha^*_t}\), \({\omega^*_t}\) as \({x_0, x_1, \ldots, x_{m-1}, x_0, x_1, \ldots, x_{m-1}, \ldots}\). From assumption 3, there exists \(s_0\) such that for \(s \geq s_0\), we have \(\alpha^*_t \leq 1\), for any \(t \in \{0, 1, \ldots, m-1\}\). Next we will use the mathematical induction proof that \(s \geq s_0\), \(\|V^*_t\| \leq 3N\). If \(t = 0\), \(\|V^*_0\| = \|\text{grad}f(\omega^*_0)\| \leq N \leq 3N\), the conclusion is hold.

Suppose that the conclusion is hold for any \(t - 1\), according to (2) and assumption 4, we get

\[ \|V^*_t\| \leq \|\phi^*_t(\text{grad}f_t(\omega^*_t^*) - \mathcal{T}_{\omega^*_t^*}^* (\text{grad}f_t(\omega^*_0^*) - \text{grad}f(\omega^*_0^*)))\| \]

\[ + \|\tilde{\psi}_t^* (\text{grad}f_t(\omega^*_t^*) - \mathcal{T}_{\omega^*_0^*}^* (\text{grad}f_t(\omega^*_0^*) - V^*_0^*))\| \]

\[ + \|[(1 - \phi^*_t - \tilde{\psi}_t^*)\text{grad}f_t(\omega^*_0^*)]\| \]

\[ \leq \phi^*_t \|[\text{grad}f_t(\omega^*_t^*)] + \|[(\text{grad}f_t(\omega^*_0^*)) + \|\text{grad}f(\omega^*_0^*)]\| \]

\[ + \|[(1 - \phi^*_t - \tilde{\psi}_t^*)\text{grad}f_t(\omega^*_0^*)]\| \]

\[ \leq 3N \]

Then \(s \geq s_0\), we have \(\|V^*_0\| \leq 3N\). Denote \(\Omega' = [0, 1] \times \{(\omega, V)|\omega \in \Omega, V \in \mathbb{B}(0,3N)\}\). Defining \(h(\tau, \omega, V) : \Omega' \to \mathcal{R}, h(\tau, \omega, V) := (f \circ R_\omega)(-\tau V)\), from assumption 1.b and \(\Omega'\) is a compact, there exists a constant \(N' > 0\) such that \(\frac{\partial^2}{\partial \tau^2} h(\tau, \omega, V) \leq N'.\) By the Taylor expansion

\[ f(\omega^*_{t+1}) - f(\omega^*_t) = f \circ R_{\omega^*_t}(-\alpha^*_t V^*_t) - f \circ R_{\omega^*_t}(0) \]

\[ = h(1, \omega^*_t, \alpha^*_t V^*_t) - h(0, \omega^*_t, V^*_t) \]

\[ = \frac{\partial}{\partial \tau} h(\tau, \omega^*_t, \alpha^*_t V^*_t)|_{\tau=0} \alpha^*_t + \int_0^1 (1 - \tau) \frac{\partial^2}{\partial \tau^2} h(\tau, \omega^*_t, \alpha^*_t V^*_t) d\tau \]

\[ = \frac{\partial}{\partial \tau} h(\tau, \omega^*_t, \alpha^*_t V^*_t)|_{\tau=0} \alpha^*_t + \int_0^1 (1 - \tau) \frac{\partial^2}{\partial \tau^2} h(\alpha^*_t \tau, \omega^*_t, V^*_t) d\tau \]
\[ \leq -\alpha_t^s \langle V_t^s, \nabla f(\omega_t^s) \rangle + \frac{N'}{2} (\alpha_t^s)^2 \] (8)

Since \( \omega_t^s \) is measurable in \( \mathcal{F}_t^s \), which together with lemma 1 and lemma 2 lead to
\[
\begin{align*}
E[\langle V_t^s, \nabla f(\omega_t^s) \rangle | \mathcal{F}_t^s] &= \| \nabla f(\omega_t^s) \|^2 + \tilde{\psi}_t^s (V_{t-1}^s - \nabla f(\omega_t^s)) \cdot \nabla f(\omega_t^s) \\
&\geq (1 - \mu) \| \nabla f(\omega_t^s) \|^2
\end{align*}
\] (9)

and
\[
\begin{align*}
E[\langle V_t^s, \nabla f(\omega_t^s) \rangle | \mathcal{F}_t^s] &= \| \nabla f(\omega_t^s) \|^2 + \tilde{\psi}_t^s (V_{t-1}^s - \nabla f(\omega_t^s)) \cdot \nabla f(\omega_t^s) \\
&\leq (1 + \mu) \| \nabla f(\omega_t^s) \|^2
\end{align*}
\] (10)

Then taking mathematical expectations on both sides of (8), and substituting (9) back to (8) gives
\[ E[f(\omega_{t+1}^s) \mathcal{F}_t^s] \leq f(\omega_t^s) - (1 - \mu) \alpha_t^s \| \nabla f(\omega_t^s) \|^2 + \frac{N'}{2} (\alpha_t^s)^2 \] (11)

which yields
\[ (1 - \mu) \alpha_t^s E[\| \nabla f(\omega_t^s) \|^2] \leq E[f(\omega_t^s) - f(\omega_{t+1}^s)] + \frac{N'}{2} (\alpha_t^s)^2 \] (12)

Summing this result over \( t = 0, \ldots, m - 1 \) and \( s = 1, \ldots, S \) gives
\[ \sum_{s=s_0}^{S} \sum_{t=0}^{m-1} (1 - \mu) \alpha_t^s E[\| \nabla f(\omega_t^s) \|^2] \leq E[f(\omega_0^s) - f(\omega_m^S)] + \sum_{s=s_0}^{S} \sum_{t=0}^{m-1} \frac{N'}{2} (\alpha_t^s)^2 \] (13)

Since \( f \) is continuous on \( \Omega \), there exists a constant \( C \geq 0 \), for any \( \omega \in \Omega, -C \leq f(\omega) \leq C \), which gives \( f(\omega_0^s) - f(\omega_m^S) \leq 2C \). Let \( S \to \infty \), the above inequality gives
\[ \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (1 - \mu) \alpha_t^s E[\| \nabla f(\omega_t^s) \|^2] \leq 2C + \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} \frac{N'}{2} (\alpha_t^s)^2 \] (14)

From assumption 3, \( 2C + \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} \frac{N'}{2} (\alpha_t^s)^2 = 2C + \frac{N'}{2} \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (\alpha_t^s)^2 < \infty \), such that
\[ \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (1 - \mu) \alpha_t^s E[\| \nabla f(\omega_t^s) \|^2] < \infty \] (15)

Since \( 0 < \mu < 1 \), we get \( \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} \alpha_t^s E[\| \nabla f(\omega_t^s) \|^2] < \infty \), which together with \( \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (\alpha_t^s)^2 = \infty \) implies that \( \liminf_{s \to \infty} E[\| \nabla f(\omega_t^s) \|^2] = 0 \). Next, we will prove \( E[\| \nabla f(\omega_t^s) \|^2] \) converge to a real number. By assumption 3 and the properties of continuous functions on compact sets. Bounding the largest eigenvalue of the Hessian of \( \| \nabla f(\omega_t^s) \|^2 \) from above by \( \gamma_1 \) along the curve defined by the retraction \( R \) linking \( \omega_{t+1}^s \) and \( \omega_t^s \). A lower bound of the minimum eigenvalue of the Hessian of \( f \) is \( \gamma_2 \). Let \( \lambda_t^s \) be the eigenvalue of the Hessian of \( f \) about \( V_t^s \). By Taylor expansion, combining the above inequalities (9) and (10), we have
\[
\begin{align*}
E[\| \nabla f(\omega_{t+1}^s) \|^2 - \| \nabla f(\omega_t^s) \|^2 | \mathcal{F}_t^s] &\leq E[-2\alpha_t^s \langle \nabla f(\omega_t^s), \nabla f(\omega_t^s) \rangle V_t^s] + (\alpha_t^s)^2 \| V_t^s \|^2 \gamma_1 | \mathcal{F}_t^s|
\end{align*}
\]
From assumption 3 and (15), we know that a better conclusion can be obtained after the function \( f \) produced by algorithm 1. No matter what choose, we have \( \lim_{k \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \), for all \( k \geq 1 \). If we choose option 2, according to theorem 1, for all \( k \geq 1 \), \( \lim_{N \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \), if we choose option 1, \( \lim_{k \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \), \( \lim_{k \to \infty} a(k) = a \).

Taking mathematical expectations on both sides of (16) yields

\[
E[\|\text{grad}(\omega^k)\|^2] \leq E[\|\text{grad}(\omega^k)\|^2 + 2\alpha_k^2|\gamma_2|/(1 + \mu)E\|\text{grad}(\omega^k)\|^2 + (\alpha_k^2)^2(3N)^2\gamma_1
\]

From assumption 3 and (15), we know \( \lim_{k \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \), we can obtain the convergence of the output of algorithm 1. Before this, we need to prove a lemma.

**Lemma 5.** Let \( a(k) \) be a real number sequence and satisfies \( \lim_{k \to \infty} a(k) = a \), then \( \lim_{k \to \infty} \frac{a(1) + a(2) + \cdots + a(k)}{k} = a \).

**Proof.** For any \( \varepsilon > 0 \), there exists a positive constant \( N_1 \), such that \( |a(k) - a| < \varepsilon, k > N_1 \). Denote \( M = \max\{|a(1) - a|, \cdots, |a(N_1) - a|\} \), we get

\[
\left| \frac{a(1) + a(2) + \cdots + a(k)}{k} - a \right| = \left| \frac{a(1) - a + a(2) - a + \cdots + a(k) - a}{k} \right| \\
\leq \frac{1}{k} \left( |a(1) - a| + |a(2) - a| + \cdots + |a(k) - a| \right) \\
\leq \frac{M}{k} + \frac{k - N_1}{k} \varepsilon < \frac{MN_1}{k} + \varepsilon
\]

Note that \( \lim_{k \to \infty} \frac{MN_1}{k} = 0 \), for the above \( \varepsilon \), there exists a constant \( N_2 \), such that \( \frac{MN_1}{k} < \varepsilon, k > N_2 \). Denote \( N = \max\{N_1, N_2\} \), \( \frac{a(1) + a(2) + \cdots + a(k)}{k} - a < 2\varepsilon, k > N \).

**Corollary 1.** Suppose the assumption 1.b and assumption 2-4 hold. The sequences \( \{\omega^k\} \) produced by algorithm 1. No matter what choose, we have \( E[\|\text{grad}(\omega^k)\|^2] \to 0 \).

**Proof.** If we choose option 1, \( \lim_{k \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \) implies that \( \lim_{k \to \infty} E[\|\text{grad}(\omega^k)\|^2] = 0 \). If we choose option 2, according to theorem 1 for all \( t \), we have \( \lim_{S \to \infty} E[\|\text{grad}(\omega^S)\|^2] = 0 \), using lemma 5 then \( \lim_{S \to \infty} E[\|\text{grad}(\omega^S)\|^2] = \lim_{S \to \infty} \frac{1}{m^2} \sum_{s=1}^{m} \sum_{t=0}^{m-1} E[\|\text{grad}(\omega^s)\|^2] = 0 \).

The above theorem has no special requirements for function \( f \). The following theorem introduces that a better conclusion can be obtained after the function \( f \) satisfies other properties. That is, if the function \( f \) satisfies \( f \geq 0 \), we can get an almost sure convergence.

**Theorem 2.** Suppose the assumption 1.b and assumption 2-4 hold. The sequences \( \{\omega^k\} \) produced by algorithm 1. If \( f \geq 0 \), then \( \{\text{grad}(\omega^k)\} \to 0 \) a.s. and \( \{f(\omega^k)\} \) converges to a finite random variable a.s.

**Remark 1.** If \( \psi^k \equiv 0 \) is hold, we can be degenerated into the situation in [7].
Proof. From the inequality (11), we have

$$
E[f(\omega_{t+1}^s) | \mathcal{F}_t^s] \leq f(\omega_t^s) - (1 - \mu)\alpha_t^s \|\nabla f(\omega_t^s)\|^2 + \frac{N'}{2} (\alpha_t^s)^2
$$

(18)

We can get from assumption 3, that is \( \sum_{s=0}^{\infty} \sum_{t=0}^{m-1} \frac{N'}{2} (\alpha_t^s)^2 = \frac{N'}{2} \sum_{s=0}^{\infty} \sum_{t=0}^{m-1} (\alpha_t^s)^2 < \infty \). By condition \( f \geq 0 \), using lemma [3] then \( \{f(\omega_t^s)\} \) converges to a finite random variable a.s., and

$$
\sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (1 - \mu)\alpha_t^s \|\nabla f(\omega_t^s)\|^2 < \infty \text{ a.s.}
$$

(19)

Moreover, since \( 0 < \mu < 1 \), we obtain

$$
\sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} \alpha_t^s \|\nabla f(\omega_t^s)\|^2 < \infty \text{ a.s}
$$

(20)

which gives together with \( \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} \alpha_t^s = \infty \) leads to \( \lim_{s \to \infty} \|\nabla f(\omega_t^s)\|^2 = 0 \text{ a.s.} \). Next, we will prove that \( \|\nabla f(\omega_t^s)\|^2 \) converges to a finite random variable a.s. According to the inequalities (16)

$$
E[||\nabla f(\omega_{t+1}^s)\|^2 - ||\nabla f(\omega_t^s)\|^2|\mathcal{F}_t^s] \]

$$
= \max\{0, E[||\nabla f(\omega_{t+1}^s)\|^2 - ||\nabla f(\omega_t^s)\|^2|\mathcal{F}_t^s]\}

$$
\leq \max\{0, 2\alpha_t^s\gamma_2(1 + \mu)||\nabla f(\omega_t^s)||^2 + (\alpha_t^s)^2(3N)^2\gamma_1\}

= 2\alpha_t^s\gamma_2(1 + \mu)||\nabla f(\omega_t^s)||^2 + (\alpha_t^s)^2(3N)^2\gamma_1

(21)

this together with assumption 3 and (20), we get \( \sum_{s=s_0}^{\infty} \sum_{t=0}^{m-1} (2\alpha_t^s\gamma_2(1 + \mu)||\nabla f(\omega_t^s)||^2 + (\alpha_t^s)^2(3N)^2\gamma_1) < \infty \). From lemma [3] \( ||\nabla f(\omega_t^s)||^2 \) is a quasi martingale, i.e., \( ||\nabla f(\omega_t^s)||^2 \) converges to a finite random variable a.s. Combining \( \lim_{s \to \infty} \|\nabla f(\omega_t^s)\|^2 = 0 \text{ a.s.} \) gives the desired result \( \lim_{s \to \infty} \|\nabla f(\omega_t^s)\|^2 = 0 \text{ a.s.} \).

\[\Box\]

### 3.2 Step size is fixed

Theorem [1] and theorem [2] qualitatively research the convergence of R-SHG with adaptive parameters when the step size is reduced. For the case of fixed step size, theorem [1] and theorem [2] will not be satisfied, and the reason is that the conditions of assumption 3 will not be satisfied. Furthermore, we can use the weaker differentiability of the function \( f \), and quantitatively research the convergence speed of the algorithm. Before that, we will prove the following two lemmas. At the rest of this article, we suppose that the \( N \) is defined by theorem [1] i.e., for any \( \omega \in \Omega \), \( ||\nabla f_\omega(\omega)|| \leq N \).

**Lemma 6.** Suppose assumption 1.a, assumption 5 and assumption 6 hold, for any \( \omega_1, \omega_2 \in \Omega \) are \( \mathcal{F}_t^s \) measurable, \( \omega_2 = R_{\omega_1}(\xi_\omega) \), such that

$$
E[||\nabla f(\omega_2) - T_{\omega_1} \nabla f(\omega_1)||^2|\mathcal{F}_t^s] \leq 2(M^2 + \theta^2 N^2)||\xi_\omega||^2
$$

(22)

**Lemma 7.** Suppose assumption 1.a and assumption 4-7 hold, we have

$$
E[||V_t^s - \nabla f(\omega_t^s)||^2|\mathcal{F}_t^s] \leq 4(M^2 + \theta^2 N^2)(\phi_t^s)^2||\xi_{\omega_t^s}||^2 + 4N^2(1 - \phi_t^s)^2 + (\psi_t^s)^2 + (\psi_t^s)^2||V_t^{s-1} - \nabla f(\omega_{t-1}^s)||^2
$$

(23)
The proofs of the above two lemmas are shown in Appendix A. Lemma 7 gives the mean-square bound between $V_t^*$ and $\nabla f(\omega^*_t)$. The idea of lemma 7 is important and there are many similar proofs in this paper. Now, we will give the third theorem that research the convergence speed of algorithm 1 when step size and inner loop parameters are fixed. But the convergence of the algorithm is local.

**Theorem 3.** Suppose assumption 1.a, assumption 2 and assumption 4-9 hold. The sequences \{\omega_t^*\} produced by algorithm 1 with option 2. Let $\alpha_t^* \equiv C_\alpha$, $\phi_t^* = \phi^*$ and $\psi_t^* = \psi^*$, satisfying $C_\alpha \leq \frac{2}{L + \sqrt{L^2 + 4\nu}}$, where $\nu = 4m^3(M^2 + \theta^2 N^2)C_1^2C_2^2$, and $\sum_{s=1}^{\infty}((1 - \phi^*)^2 + (\psi^*)^2) < \infty$, then

\[
\mathbb{E}[\|\nabla f(\omega_s^*)\|^2] \leq \frac{2}{mSC_\alpha}(f(\omega_0^*) - f(\omega^*)) + \frac{4mN^2}{S} \sum_{s=1}^{\infty}((1 - \phi^*)^2 + (\psi^*)^2)
= C(\frac{1}{S})
\]

(24)

**Remark 2.** It is easily verified that $\phi^*$ and $\psi^*$ satisfy the condition of theorem 3 if $\phi^* = 1 - \frac{1}{(1 + t)}$, $\psi^* = \frac{1}{t}$, $\nu = \frac{1}{t}$. It is also hold $\phi^* = 1$, i.e., $\psi^* = 0$. To compare with [1], we also give the convergence rate analysis under fixed step size.

**Proof.** Using assumption 8, we get

\[
f(\omega_{t+1}^*) - f(\omega_t^*)
\leq -C_\alpha \langle \nabla f(\omega_t^*), V_t^* \rangle + \frac{LC_2^2}{2} \|V_t^*\|^2
= -\frac{C_\alpha}{2} \|V_t^*\|^2 - \frac{C_\alpha}{2} \|\nabla f(\omega_t^*)\|^2 + \frac{C_\alpha}{2} \|V_t^* - \nabla f(\omega_t^*)\|^2 + \frac{LC_2^2}{2} \|V_t^*\|^2
= -\frac{C_\alpha}{2} \|\nabla f(\omega_t^*)\|^2 + \frac{C_\alpha}{2} \|V_t^* - \nabla f(\omega_t^*)\|^2 + (\frac{LC_2^2}{2} - \frac{C_\alpha}{2}) \|V_t^*\|^2
\]

(25)

Taking the mathematical expectations on both sides of (25), thus

\[
\mathbb{E}[\|\nabla f(\omega_t^*)\|^2]
\leq \frac{2}{C_\alpha} \mathbb{E}[f(\omega_t^*) - f(\omega_{t+1}^*)] + \mathbb{E}[\|V_t^* - \nabla f(\omega_t^*)\|^2] + (LC_\alpha - 1)\mathbb{E}[\|V_t^*\|^2]
\]

(26)

According to assumption 9, we obtain

\[
\|\nabla f(\omega_t^*)\|^2 = \|R_{-1}^{-1}(\omega_t^*)\|^2
\leq C_1^2 \sum_{i=0}^{\infty} \|V_t^*\|^2
\]

(27)

Let the parameters, step size and (27) back to (23), we have

\[
\mathbb{E}[\|V_t^* - \nabla f(\omega_t^*)\|^2]\bigg| F_t^*
\leq 4(M^2 + \theta^2 N^2)(\phi^*)^2 C_1^2 C_2^2 C_2^2 t \sum_{i=0}^{t-1} \|V_t^*\|^2 + 4N^2((1 - \phi^*)^2 + (\psi^*)^2)
\]

\[
+ 4(1 - \phi^*)^2 \|V_{t-1}^* - \nabla f(\omega_{t-1}^*)\|^2
\leq 4(M^2 + \theta^2 N^2) C_1^2 C_2^2 C_2^2 m \sum_{i=0}^{t-1} \|V_t^*\|^2 + 4N^2((1 - \phi^*)^2 + (\psi^*)^2)
\]

(28)
+\|V^*_{s-1} - \text{grad}f(\omega^*_{t-1})\|^2
\leq 4(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2m^2 \sum_{t=0}^{m-1} \|V^*_t\|^2 + 4tN^2((1 - \phi^*)^2 + (\psi^*)^2)
+\|V^*_0 - \text{grad}f(\omega^*_0)\|^2
\leq 4(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2m^2 \sum_{t=0}^{m-1} \|V^*_t\|^2 + 4mN^2((1 - \phi^*)^2 + (\psi^*)^2) \tag{28}

The second inequality is due to $0 \leq \phi^* \leq 1$, $0 \leq \psi^* \leq 1$ and $t \leq m$. The last inequality is bases on the fact that $V^*_0 = \text{grad}f(\omega^*_0)$. Note that $\mathbb{E}[\mathbb{E}[\mathcal{F}^*_t]] = \mathbb{E}[\cdot]$, taking the mathematical expectations with respect to (28), we get

\[
\sum_{t=0}^{m-1} \sum_{s=1}^{S} \mathbb{E}[\|V^*_t - \text{grad}f(\omega^*_t)\|^2] 
\leq \sum_{t=0}^{m-1} \sum_{s=1}^{S} \left( 4(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2m^2 \sum_{s=1}^{S} \mathbb{E}[\|V^*_t\|^2] + 4mN^2((1 - \phi^*)^2 + (\psi^*)^2) \right)
\leq 4(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2m^2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V^*_t\|^2] + 4mN^2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} ((1 - \phi^*)^2 + (\psi^*)^2) \tag{29}

Summing the result of (26) gives

\[
\sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad}f(\omega^*_t)\|^2] 
\leq \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V^*_t - \text{grad}f(\omega^*_t)\|^2] + (LC_\alpha - 1) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V^*_t\|^2] 
+ \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega^*_m)]
\leq \left( 4m^3(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2 + LC_\alpha - 1 \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V^*_t\|^2]
+ 4m^2N^2 \sum_{s=1}^{\infty} \sum_{t=0}^{\infty} ((1 - \phi^*)^2 + (\psi^*)^2) + \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega^*_m)]
\leq \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega^*_m)] + 4m^2N^2 \sum_{s=1}^{\infty} \sum_{t=0}^{\infty} ((1 - \phi^*)^2 + (\psi^*)^2)
\leq \frac{2}{C_\alpha} (f(\omega^0) - f(\omega^*)) + 4m^2N^2 \sum_{s=1}^{\infty} \sum_{t=0}^{\infty} ((1 - \phi^*)^2 + (\psi^*)^2) \tag{30}

The third inequality applies $4m^3(M^2 + \theta^2N^2)C_1^2C_2^2C_3^2 + LC_\alpha - 1 \leq 0$, if $C_\alpha \leq \frac{2}{L + \sqrt{L + 4\theta}}$. The last inequality follows from $f(\omega^*_m) > f(\omega^*)$. Hence, we have

\[
\mathbb{E}[\|\text{grad}f(\omega^*_m)\|^2] = \frac{1}{mS} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad}f(\omega^*_t)\|^2]
\]
\[ \leq \frac{2}{mSC_\alpha}(f(\tilde{\omega}) - f(\omega^*)) + \frac{4mN^2}{S} \sum_{s=1}^{\infty}((1 - \phi^s)^2 + (\psi^s)^2) \]
\[ = O\left(\frac{1}{S}\right) \] (31)

### 3.3 Special case 1

Suppose the descent direction only use R-SVRG and R-SRG term, i.e.,

\[ V_t^s = \tilde{\psi}_t^s \left( \text{grad}f_{t^s}(\omega_t^s) - \mathcal{T}_{\omega_0^s}^\alpha \left( \text{grad}f_{t^s}(\omega_0^s) - \text{grad}f(\omega^*) \right) \right) \]
\[ + \tilde{\psi}_t^s \left( \text{grad}f_{t^s}(\omega_t^s) - \mathcal{T}_{\omega_{t-1}^s}^\alpha \left( \text{grad}f_{t^s}(\omega_{t-1}^s) - V_{t-1}^s \right) \right) \] (32)

where \( \tilde{\phi}_t^s = 1 - \tilde{\psi}_t^s \). For the step size is reduced, lemma 1 and lemma 2 are satisfied. Therefore, theorem 1 and theorem 2 are still hold. Compared to theorem 3, we can get a similar conclusion under weaker conditions; that is, it is not necessary to fix the inner loop parameters. Before this, let us give a lemma.

**Lemma 8.** Suppose assumption 1.a and assumption 4-7 hold, let the descent direction be \( V_t^s \), then

\[ \mathbb{E}[\| V_t^s - \text{grad}f(\omega_t^s) \|^2 | F_t^s] \leq 4(M^2 + \theta^2 N^2)(\tilde{\psi}_t^s)^2 \| \tilde{\psi}_t^s \|^2 + 8N^2(\tilde{\psi}_t^s)^2 \]
\[ + (\tilde{\psi}_t^s)^2 \| V_{t-1}^s - \text{grad}f(\omega_{t-1}^s) \|^2 \] (33)

The proof of the lemma is in Appendix A.

**Theorem 4.** Suppose assumption 1.a, assumption 2 and assumption 4-9 hold. The sequences \( \{\omega_t^s\} \) produced by algorithm 1 with option 2 and the descent direction is \( V_t^s \). Let \( \alpha_t^s \equiv C_\alpha, \) satisfying \( C_\alpha \leq \frac{1}{L + \sqrt{L^2 + 4\nu}} \), where \( \nu = 4m^2(M^2 + \theta^2 N^2)C_1^2C_2^2 \), and \( \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 < \infty \), such that

\[ \mathbb{E}[\| \text{grad}f(\omega_t^s) \|^2] \leq \frac{2}{mSC_\alpha} \mathbb{E}[f(\tilde{\omega}) - f(\omega^*)] + \frac{8N^2}{S} \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 \]
\[ = O\left(\frac{1}{S}\right) \] (34)

**Proof.** Let the parameters, step size and \( (27) \) back to (33) gives

\[ \mathbb{E}[\| V_t^s - \text{grad}f(\omega_t^s) \|^2 | F_t^s] \]
\[ \leq 4(M^2 + \theta^2 N^2)(\tilde{\psi}_t^s)^2 C_1^2 C_2^2 C_\alpha \sum_{i=0}^{t-1} \| V_{i}^s \|^2 + 8N^2(\tilde{\psi}_t^s)^2 \]
\[ + (\tilde{\psi}_t^s)^2 \| V_{t-1}^s - \text{grad}f(\omega_{t-1}^s) \|^2 \]
\[ \leq 4(M^2 + \theta^2 N^2)C_1^2 C_2^2 C_\alpha \sum_{t=0}^{m-1} \| V_{t}^s \|^2 + 8N^2(\tilde{\psi}_t^s)^2 \]
\[ + \| V_{t-1}^s - \text{grad}f(\omega_{t-1}^s) \|^2 \]
\[ \leq 4(M^2 + \theta^2 N^2)C_1^2 C_2^2 C_\alpha m \sum_{t=0}^{m-1} \| V_{t}^s \|^2 + 8mN^2(\tilde{\psi}_t^s)^2 \]
\[ + \| V_{0}^s - \text{grad}f(\omega_0^s) \|^2 \]

13
\[ = 4(M^2 + \theta^2 N^2) C_1^2 C_2^2 C_3^2 m^2 \sum_{t=0}^{m-1} \| V_t \|^2 + 8mN^2(\psi_t)^2 \] (35)

The above inequality applies \(0 \leq \phi_t \leq 1, \psi_t \leq \psi_t^*\) and \(t \leq m\). Similar to the proof of (25)-(30) in theorem 3, we get

\[
\begin{align*}
& \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\| \text{grad} f(\omega_t^a) \|^2] \\
\leq & \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\| V_t^s - \text{grad} f(\omega_t^s) \|^2] + (LC_\alpha - 1) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\| V_t^s \|^2] \\
& + \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega_m^s)] \\
\leq & \left(4m^3(M^2 + \theta^2 N^2) C_1^2 C_2^2 C_3^2 + LC_\alpha - 1\right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\| V_t^s \|^2] \\
& + 8mN^2 \sum_{s=1}^{\infty} (\psi_t^s)^2 + \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega_m^s)] \\
\leq & \frac{2}{C_\alpha} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega^s)] + 8mN^2 \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 \\
\leq & \frac{2}{C_\alpha} (f(\tilde{\omega}^0) - f(\omega^s)) + 8mN^2 \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 \\
\leq & \frac{2}{C_\alpha} (f(\tilde{\omega}^0) - f(\omega^s)) + 8mN^2 \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 \\
& \leq O\left(\frac{1}{S}\right) \\
\end{align*}
\] (36)

Hence, we obtain

\[
\mathbb{E}[\| \text{grad} f(\omega_a) \|^2] = \frac{1}{mS} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\| \text{grad} f(\omega_t^s) \|^2] \\
\leq \frac{2}{mSC_\alpha} (f(\tilde{\omega}^0) - f(\omega^s)) + \frac{8N^2}{S} \sum_{s=1}^{\infty} \sum_{t=0}^{m-1} (\psi_t^s)^2 \\
= O\left(\frac{1}{S}\right) \\
\] (37)

3.4 Special case 2

The previous two subsections present a local convergence rate analysis of the algorithm with retraction mapping and vector transport. In this subsection, we consider a special case of the result in the previous subsection, where exponential mapping and parallel translation are chosen as retraction and vector transport. The previous theorems still hold when the retraction mapping is taken as exponential mapping and the vector transport is taken as parallel transport. For theorem 3, if the exponential mapping and parallel transport are used, then the convergence is global convergence. For this special case, we give only a sketch of the proofs and the result as the following corollary.

**Corollary 2.** Suppose the conditions in theorem 3 are hold and consider algorithm 1 with \(R = \text{Exp} \) and \(T = \Gamma\). Let \(C_\alpha \leq \frac{2}{L+\sqrt{L^2+4\nu}}\) and \(\sum_{s=1}^{\infty} ((1-\phi^s)^2 + (\psi^s)^2)^2 < \infty\) where \(\nu = 4m^3M^2\),
such that

\[ \mathbb{E}[\|\operatorname{grad}f(\omega_s)\|^2] \leq \frac{2}{mSC_\alpha} (f(\omega^0) - f(\omega^*)) + \frac{4mN^2}{S} \sum_{s=1}^{\infty} ((1 - \varphi_s)^2 + (\psi_s)^2) \]

\[ = O\left(\frac{1}{S}\right) \quad (38) \]

**Proof.** If the retraction mapping is taken as exponential mapping and the vector transport is taken as parallel transport, then the inequality (23) in lemma 7 becoming

\[ \mathbb{E}[\|V^*_t - \operatorname{grad}f(\omega^*_t)\|^2 | F^*_t] \leq 4M^2(\varphi^*_t)^2 \|\xi^*_0\|^2 + 4N^2((1 - \varphi^*_t)^2 + (\psi^*_t)^2) \|V^*_{t-1} - \operatorname{grad}f(\omega^*_{t-1})\|^2 \]

And the inequality (27) in theorem 3 becoming

\[ \|\xi^*_0\|^2 \leq C_2^2 \sum_{i=0}^{t-1} \|V^*_{i}\|^2 \]

\[ \quad (39) \]

Other proofs are the same as theorem 3. \qed

**Remark 3.** These are equivalent to \( \theta = 0, C_1 = C_2 = 1 \). But the convergence rate of corollary is global.

### 4 Riemannian Stochastic Hybrid Gradient Algorithm with time-varying Parameters

In this section, we will prove the convergence rate under both step size are reduced and fixed. When the step size is reduced, if we choose \( \omega_s = \tilde{\omega}_s \), it is difficult to analyse the convergence of \( \omega_s \). It is different from algorithm 1, and we only consider option 2 of R-SHG algorithm with time-varying parameters to analyse the convergence. We can quantitatively research the convergence of \( \omega_s \). Of course, the advantage is that we only need to use assumption 1.a. Here we will propose the second algorithm in this paper.

**Algorithm 2** R-SHG algorithm with time-varying parameters

1: **Input:** step size \( \alpha^s \), frequency \( m > 0 \), the positive parameters \( \psi^s, \phi^s \).
2: **Initialize:** \( \tilde{\omega}^0 \).
3: for \( s=1,2,...,S \) do
4: Caculate the full Riemannian gradient \( \operatorname{grad}f(\tilde{\omega}^{s-1}) \)
5: Store \( \omega_0^s = \tilde{\omega}^{s-1}, V_0^s = \operatorname{grad}f(\omega_0^s), \omega_s^1 = R_{\omega_0^s}(-\alpha_0^sV_0^s) \)
6: for \( t=1,2,...,m-1 \) do
7: Choose \( I_t^s \in \{1,2,...,n\} \) uniformly at random
8: Caculate the descent direction
\[ V^*_s = \phi^t_\tilde{\omega}_s \left( \operatorname{grad}f_{I_t^s}(\omega^s_t) - T_{\omega_0^s}^{\omega_{I_t^s}}(\operatorname{grad}f_{I_t^s}(\omega_0^s) - \operatorname{grad}f(\omega_0^s)) \right) \]
\[ + \psi^t_\tilde{\omega}_s \left( \operatorname{grad}f_{I_t^s}(\omega^s_t) - T_{\omega_{I_t}^{s-1}}^{\omega^s_t}(\operatorname{grad}f_{I_t^s}(\omega_{I_t}^{s-1}) - V^*_{I_t}^{s-1}) \right) \]
\[ + (1 - \phi^t_\tilde{\omega}_s - \psi^t_\tilde{\omega}_s)\operatorname{grad}f_{I_t^s}(\omega^s_t) \]
9: \( \omega^s_{t+1} = R_{\omega_t^s}(-\alpha^s_tV^*_t) \)
10: end for
11: \( \tilde{\omega}^* = \omega^s_m \)
12: end for
Remark 4. If \( \psi^* = 0 \), then the two algorithms are equivalent. Therefore, the following results are also a supplement to the literature [11] if \( \psi^* = 0 \).

### 4.1 Step size is reduced

Before the theorem, we need to introduce the following two lemmas. Here, we first present a lemma that bounds the estimation error of the estimator.

**Lemma 9.** Suppose assumption 1.a and assumption 4-7 hold, then

\[
\mathbb{E}[\|V_t^s - \nabla f(\omega_t^s)\|^2 | \mathcal{F}_t] 
\leq 6(\phi_t^s)^2(M^2 + \theta^2 N^2)\|\xi_{\omega_t^s}\|^2 + 6(\psi_t^s)^2(M^2 + \theta^2 N^2)\|\xi_{\omega_t^s}\|^2 
+ 12(1 - \phi_t^s - \psi_t^s)^2 N^2 + (\psi_t^s)^2\|V_{t-1}^s - \nabla f(\omega_{t-1}^s)\|^2 
\tag{41}
\]

Now, we introduce another lemma. The bound produced by the lemma is very important in the proof of the theorem.

**Lemma 10.** Suppose assumption 1.a, assumption 2 and assumption 4-8 hold. The sequences \( \{\omega_t^s\} \) produced by algorithm 2. Let \( \gamma > 1, \kappa = \inf \{x \in \mathbb{R}^+ | x \geq \gamma\} \), \( \alpha_t^s = (t + s + \kappa + 2)^{-P}C_\alpha \) and \( \psi_t = 1 - (t + s + \kappa + 1)^{-Q}C_\psi \), satisfying \( \max \{\frac{Q-1}{\gamma - 1}, 2\} \leq P < Q, 0 < P, Q < 1 \). And \( C_\psi \geq P + C_\alpha^2 \cdot 6\beta(M^2 + \theta^2 N^2) \), where \( \beta > 2 \), for any \( 2 \leq t \leq m - 1 \) such that

\[
\frac{\mathbb{E}[\|V_t^s - \nabla f(\omega_t^s)\|^2] - \mathbb{E}[\|V_{t-1}^s - \nabla f(\omega_{t-1}^s)\|^2]}{\alpha_{t-1}^s} 
\leq \frac{6(M^2 + \theta^2 N^2)(\phi_t^s)^2 \rho^2 + 12(M^2 + \theta^2 N^2)\alpha_{t-1}^s \mathbb{E}[\|\nabla f(\omega_{t-1}^s)\|^2]}{\alpha_{t-1}^s} 
+ \frac{6(2 - \beta)(M^2 + \theta^2 N^2)\alpha_{t-1}^s \mathbb{E}[\|V_{t-1}^s - \nabla f(\omega_{t-1}^s)\|^2]}{\alpha_{t-1}^s} 
+ \frac{12(1 - \psi_t^s)^2 N^2}{\alpha_{t-1}^s} 
\tag{42}
\]

The proofs of lemma 9 and 10 see Appendix A. The following theorem will introduce the main result about the reduced step size.

**Theorem 5.** Suppose the conditions in lemma 10 are hold, if \( C_\alpha \leq \min \{\frac{1}{P}, \sqrt{\frac{(1-P)}{6\beta(M^2 + \theta^2 N^2)}}\} \), where \( \beta > 4 \), let \( \phi_t^s = (t + s + \kappa + 1)^{-R}C_\phi \), \( C_\phi \leq C_\psi \) and \( R \geq Q \), then

\[
\mathbb{E}[\|\nabla f(\omega_\alpha)\|^2] 
\leq \frac{2(\beta-2)}{\beta-4} \left[ \mathbb{E}[f(\tilde{\omega}_0) - f(\omega^*)] + \sum_{s=1}^{S} \sum_{t=0}^{S-1} \left( \frac{\rho^2 C_s^2}{2(\beta-2)C_\alpha} + \frac{12\eta N^2 C_s^2}{C_\alpha} \right)(t + s + \kappa + 2)^{(P-2)Q} \right] 
\leq mSC_\alpha(m + S + \kappa + 1)^{-P} 
\leq O\left(\frac{1}{\sqrt{Q(P)}}\right) 
\tag{43}
\]

Furthermore, the fastest convergence rate of \( \mathbb{E}[\|\nabla f(\omega_\alpha)\|^2] \) at least can get \( O(\frac{1}{S^{t+\epsilon}}) \rightarrow O(\frac{1}{t}) \)

**Remark 5.** The inequalities in theorem can get some results and guarantee some conditions hold. \( C_\alpha \leq \sqrt{\frac{(1-P)}{6\beta(M^2 + \theta^2 N^2)}} \) can ensure a positive constant \( C_\psi \) exists, such that \( C_\psi \in \{P + C_\alpha^2 \cdot 6\beta(M^2 + \theta^2 N^2), 1\} \), and \( C_\alpha < \frac{1}{P} \) implies \( \phi_t^s < \frac{1}{P} \). Moreover, the conditions \( C_\phi \leq C_\psi, R \geq Q \) can make \( \psi_t^s + \phi_t^s \leq 1 \) hold. Since assumption 9 is not used, the convergence is global.
Proof. By assumption 8 and $C_\alpha \leq \frac{1}{2}$, we have

$$f(\omega_{t+1}^*)$$

$$\leq f(\omega_t^*) - \langle \alpha_t^* V_t^*, \nabla f(\omega_t^*) \rangle + \frac{(\alpha_t^*)^2 L}{2} \Vert V_t^* \Vert^2$$

$$= f(\omega_t^*) - \frac{\alpha_t^*}{2} \Vert \nabla f(\omega_t^*) \Vert^2 - \frac{\alpha_t^*}{2} \Vert V_t^* \Vert^2 + \frac{\alpha_t^*}{2} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2 + \frac{(\alpha_t^*)^2 L}{2} \Vert V_t^* \Vert^2$$

$$\leq f(\omega_t^*) - \frac{\alpha_t^*}{2} \Vert \nabla f(\omega_t^*) \Vert^2 + \frac{\alpha_t^*}{2} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2$$

(44)

According to lemma 10 for all $1 \leq t \leq m - 2$, we obtain

$$\frac{1}{12(\beta - 2)(M^2 + \theta^2 N^2)} \left( \frac{\mathbb{E} \Vert V_{t+1}^* - \nabla f(\omega_{t+1}^*) \Vert^2}{\alpha_t^*} - \frac{\mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2}{\alpha_{t-1}^*} \right)$$

$$\leq \frac{1}{12(\beta - 2)(M^2 + \theta^2 N^2)} \left( 6(M^2 + \theta^2 N^2) \frac{(\phi_{t+1}^*)^2}{\alpha_t^*} \rho^2 + 12(M^2 + \theta^2 N^2) \alpha_t^* \mathbb{E} \Vert \nabla f(\omega_t^*) \Vert^2 \right) + \frac{12(1 - \psi_{t+1}^*)^2 N^2}{\alpha_t^*}$$

$$+ 6(2 - \beta)(M^2 + \theta^2 N^2) \alpha_t^* \mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2 \right) \right)$$

$$= \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^*)^2}{\alpha_t^* N^2} + \frac{\alpha_t^*}{2} \frac{(1 - \psi_{t+1}^*)^2}{\alpha_{t-1}^*} \mathbb{E} \Vert \nabla f(\omega_t^*) \Vert^2 - \frac{\alpha_t^*}{2} \mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2 \right)$$

(45)

Taking the mathematical expectation about (44) and combining (45), for any $1 \leq t \leq m - 2$, we get

$$\frac{1}{12(\beta - 2)(M^2 + \theta^2 N^2)} \left( \frac{\mathbb{E} \Vert V_{t+1}^* - \nabla f(\omega_{t+1}^*) \Vert^2}{\alpha_t^*} - \frac{\mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2}{\alpha_{t-1}^*} \right)$$

$$+ \mathbb{E} \left[ \frac{f(\omega_{t+1}^*) - f(\omega_t^*)}{\rho^2} \cdot \frac{(\phi_{t+1}^*)^2}{\alpha_t^* N^2} + \frac{\alpha_t^*}{2} \frac{(1 - \psi_{t+1}^*)^2}{\alpha_{t-1}^*} \mathbb{E} \Vert \nabla f(\omega_t^*) \Vert^2 - \frac{\alpha_t^*}{2} \mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2 \right]$$

$$\leq \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^*)^2}{\alpha_t^* N^2} + \frac{\alpha_t^*}{2} \frac{(1 - \psi_{t+1}^*)^2}{\alpha_{t-1}^*} \mathbb{E} \Vert \nabla f(\omega_t^*) \Vert^2$$

(46)

We can definite $V_m^*$ to make above formula is hold for $1 \leq t \leq m - 1$ (The $V_m^*$ is exists, for example $V_m^*$ give by algorithm 2). Let $\eta = \frac{1}{12(\beta - 2)(M^2 + \theta^2 N^2)}$ and for any $1 \leq t \leq m - 1$, we have

$$\mathbb{E} [f(\omega_{t+1}^*) - f(\omega_t^*)]$$

$$\leq \eta \left( \frac{\mathbb{E} \Vert V_t^* - \nabla f(\omega_t^*) \Vert^2}{\alpha_{t-1}^*} - \frac{\mathbb{E} \Vert V_{t+1}^* - \nabla f(\omega_{t+1}^*) \Vert^2}{\alpha_t^*} \right)$$

$$+ \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^*)^2}{\alpha_t^*} + \frac{(4 - \beta)\alpha_t^*}{2(\beta - 2)} \mathbb{E} \Vert \nabla f(\omega_t^*) \Vert^2 + 12\eta N^2 \cdot \frac{(1 - \psi_{t+1}^*)^2}{\alpha_t^*}$$

(47)
Summing this result over \(1 \leq t \leq m - 1\) gives

\[
\begin{aligned}
\mathbb{E}[f(\omega_m^n) - f(\omega_0^n)] &\leq \eta \left( \frac{\mathbb{E}[\|V_t^n - \text{grad}f(\omega_t^n)\|^2]}{\alpha_t^n} - \frac{\mathbb{E}[\|V_m^n - \text{grad}f(\omega_m^n)\|^2]}{\alpha_{m-1}^n} \right) \\
&\quad + \sum_{t=1}^{m-1} \left( \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^n)^2}{\alpha_t^n} + \frac{(4 - \beta)\alpha_t^n}{\alpha_{m-1}^n} \right) \mathbb{E}[\|\text{grad}f(\omega_t^n)\|^2] + 12\eta N^2 \cdot \frac{(1 - \psi_{t+1}^n)^2}{\alpha_t^n}
\end{aligned}
\] (48)

Since \((78\), \((44)\) and \((1 - \phi_t^n - \psi_t^n)^2 \leq (1 - \phi_t^n)^2\) yields

\[
\begin{aligned}
\mathbb{E}[f(\omega_0^n) - f(\omega_0^n)] &\leq \eta \left( \frac{\mathbb{E}[\|V_t^n - \text{grad}f(\omega_t^n)\|^2]}{\alpha_t^n} \right) \\
&\quad + \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^n)^2}{\alpha_t^n} + \frac{(4 - \beta)\alpha_t^n}{\alpha_{m-1}^n} \mathbb{E}[\|\text{grad}f(\omega_0^n)\|^2] + 12\eta N^2 \cdot \frac{(1 - \psi_{t+1}^n)^2}{\alpha_t^n}
\end{aligned}
\] (49)

Combining the above two inequalities, we get

\[
\begin{aligned}
\mathbb{E}[f(\omega_m^n) - f(\omega_0^n)] &\leq \mathbb{E}[f(\omega_0^n) - f(\omega_0^n)] + \mathbb{E}[f(\omega_m^n) - f(\omega_0^n)] \\
&\leq \mathbb{E}[f(\omega_t^n) - f(\omega_0^n)] + \eta \left( \frac{\mathbb{E}[\|V_t^n - \text{grad}f(\omega_t^n)\|^2]}{\alpha_t^n} - \frac{\mathbb{E}[\|V_m^n - \text{grad}f(\omega_m^n)\|^2]}{\alpha_{m-1}^n} \right) \\
&\quad + \sum_{t=1}^{m-1} \left( \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^n)^2}{\alpha_t^n} + \frac{(4 - \beta)\alpha_t^n}{\alpha_{m-1}^n} \mathbb{E}[\|\text{grad}f(\omega_t^n)\|^2] + 12\eta N^2 \cdot \frac{(1 - \psi_{t+1}^n)^2}{\alpha_t^n} \right)
\end{aligned}
\]

The second inequality uses the fact \(\mathbb{E}[\|V_m^n - \text{grad}f(\omega_m^n)\|^2] \geq 0\). The last inequality is due to the reduced step size, i.e., \(\alpha_t^n > \alpha_{m-1}^n\). Since \(\beta > 4\) such that \(\frac{(4 - \beta)}{2(\beta - 2)} < 0\), we have

\[
\begin{aligned}
\alpha_{m-1}^n \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad}f(\omega_t^n)\|^2] &\leq \frac{2(\beta - 2)}{\beta - 4} \left( \mathbb{E}[f(\omega_0^n) - f(\omega_m^n)] + \sum_{t=0}^{m-1} \left( \frac{\rho^2}{2(\beta - 2)} \cdot \frac{(\phi_{t+1}^n)^2}{\alpha_t^n} + 12\eta N^2 \cdot \frac{(1 - \psi_{t+1}^n)^2}{\alpha_t^n} \right) \right)
\end{aligned}
\] (51)

Using \(\alpha_t^n > \alpha_{m-1}^n\) again, then

\[
\mathbb{E}[\|\text{grad}f(\omega_t^n)\|^2] = \frac{1}{mS} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad}f(\omega_t^n)\|^2]
\]

18
Furthermore, by conditions max
\[ E \phi \leq E \leq Q \]
\[ x < \text{to proof the theorem, we will elaborate in the remark (6).} \]

In this subsection, we analyze the case of fixed step size. The reason why we use lemma 9

4.2 Step size is fixed

Since \( \sum_{s=1}^{S-1} \sum_{t=0}^{m-1} (t + s + \kappa + 2)^x \leq \frac{m \ln(m + S + \kappa + 1)}{\kappa+1} + m \int_{\kappa+1}^{m+S+\kappa+1} \kappa+1 \]

Thus
\[ \sum_{s=1}^{S} \sum_{t=0}^{m-1} (t + s + \kappa + 2)^x \leq \left\{ \begin{array}{ll} m \ln(m + S + \kappa + 1), & x = -1 \\ \frac{m}{\kappa+1} (m + S + \kappa + 1)^{x+1}, & -1 \leq x < 0 \end{array} \right. \]

Since \( R \geq Q \), we obtain that \( \frac{(\phi_{\kappa+1}^*)^2}{\alpha_{\kappa+1}^2} = (t + s + \kappa + 2)^{P-2Q} \). Given by \[ (52) \text{ and } (54) \], we have
\[ \mathbb{E}[\|\text{grad} f(\omega_n)\|^2] \leq \frac{2(\beta-2)}{\beta-4} \left( \mathbb{E}[f(\omega_n)] - f(\omega^*) \right) + \sum_{s=1}^{S} \sum_{t=0}^{m-1} \left( \frac{\beta^2 C_\alpha^2}{2(\beta-2)C_n} + \frac{12\eta N^2 C_\alpha^2}{C_n} \right) (t + s + \kappa + 2)^{P-2Q} \]

\[ = O\left( \frac{1}{S^{2(Q-P)}} \right) \]

Furthermore, by conditions max\( \{ \frac{2Q-1}{\gamma+1}, \frac{Q}{2} \} \leq P \leq Q, 0 < P, Q < 1 \), and using linear programming knowledge, we know \( 2(Q - P) \) get maximum at \( Q = \frac{2}{\gamma+1} \) and \( P = \frac{1}{\gamma+1} \). The maximum value is \( [2(Q - P)]_{\max} = \frac{2}{\gamma+1} \). By the condition \( \gamma > 1 \), the fastest convergence rate of \( \mathbb{E}[\|\text{grad} f(\omega_n)\|^2] \) at least can get \( O\left( \frac{1}{S^{2(Q-P)}} \right) \to O\left( \frac{1}{S} \right) \)

Andi Han et al.\[ 14 \] consider the problem of expectation (online) minimization over Riemannian manifold \( \mathcal{M} \). They assumption the stochastic gradient is an unbiased estimation, i.e., \( E_{\omega} \text{grad} f(x, \omega) = \text{grad} F(x) \). And they get a convergence rate of \( O\left( \frac{1}{T^2} \right) \) for the case of the reduced step size. In our paper, we do not need this assumption (similar to lemma 2) and we can get faster convergence. If \( \phi_i^* = 1 \) and consider the problem is online, our results can be degenerated into \[ 14 \] and more faster.

4.2 Step size is fixed

In this subsection, we analyze the case of fixed step size. The reason why we use lemma 9 to proof the theorem, we will elaborate in the remark (6).

Theorem 6. Suppose assumption 1.a, assumption 2 and assumption 4-9 hold. The sequences \( \{\omega_i^*\} \) produced by algorithm 2 with \( \alpha_i^* \equiv C_\alpha \) and \( \psi_i^* = \psi^*, \phi_i^* = \phi^* \), satisfying
\[ C_\alpha \leq \frac{2}{L + \sqrt{L^2 + 4\nu}}, \quad \nu = 6m^2(M^2 + \theta^2 N^2)(C_1^2 C_2^2 m^2 + 1) \]
Remark 6. There exists parameters $\phi^*, \psi^*$ satisfy the conditions. For example $\phi^* + \psi^* = 1 - \frac{1}{s+1}$. We can analyze the convergence similar to theorem 3. The result is similar, but the condition is $\sum_{s=1}^{\infty} (1 - \phi^*)^2 + (\psi^*)^2 < \infty$. However, if $\phi^* = \psi^* = \frac{1}{2} (1 - \frac{1}{s+1})$, we can easily find that the parameters satisfy $\sum_{s=1}^{\infty} (1 - \phi^*)^2 + (\psi^*)^2 < \infty$ and not satisfy $\sum_{s=1}^{\infty} (1 - \phi^*)^2 + (\psi^*)^2 < \infty$. Therefore, we will use lemma 9. The advantage is that we use weaker conditions to obtain similar conclusions.

Proof. According to lemma 9 taking the parameters, step size and $[27]$ into $[41]$, we have

$$E[\|\nabla f(\omega_s)\|^2] \leq 12(1 - \phi^* - \psi^*)^2 N^2 + (\psi^*)^2 E[\|V_{s-1}^* - \nabla f(\omega_{s-1})\|^2]$$

$$+ 6C_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m \sum_{i=0}^{m-1} E[\|V_i^*\|^2] + (\psi^*)^2 \sum_{i=0}^{t-1} E[\|V_i^*\|^2] \right)$$

$$\leq 12t(1 - \phi^* - \psi^*)^2 N^2 + E[\|V_0^* - \nabla f(\omega_0^*)\|^2]$$

$$+ 6C_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m \sum_{i=0}^{m-1} E[\|V_i^*\|^2] + (\psi^*)^2 \sum_{i=0}^{t-1} E[\|V_i^*\|^2] \right)$$

$$\leq 12m(1 - \phi^* - \psi^*)^2 N^2 + E[\|V_0^* - \nabla f(\omega_0^*)\|^2]$$

$$+ 6C_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m^2 \sum_{i=0}^{m-1} E[\|V_i^*\|^2] + (\psi^*)^2 \sum_{i=0}^{m-1} E[\|V_i^*\|^2] \right)$$

$$= 6C_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m^2 + (\psi^*)^2 \sum_{i=0}^{m-1} E[\|V_i^*\|^2] \right)$$

$$+ 12m(1 - \phi^* - \psi^*)^2 N^2$$

(57)

The above inequality applies $V_0^* = \nabla f(\omega_0^*)$ and $t \leq m$. Summing this result over $t = 0, ..., m - 1$ and $s = 1, ..., S$ gives

$$\sum_{s=1}^{S} \sum_{t=0}^{m-1} E[\|V_t^* - \nabla f(\omega_t^*)\|^2]$$

$$\leq \sum_{s=1}^{S} \left[ 6mC_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m^2 + (\psi^*)^2 \sum_{i=0}^{m-1} E[\|V_i^*\|^2] \right) \right]$$

$$+ 12N^2 m(1 - \phi^* - \psi^*)^2$$

$$= \sum_{s=1}^{S} \left[ 6mC_2^2 (M^2 + \theta^2 N^2) \left(C_1^2 C_2^2 (\phi^*)^2 m^2 + (\psi^*)^2 \sum_{i=0}^{m-1} E[\|V_i^*\|^2] \right) \right]$$

$$+ 12N^2 m(1 - \phi^* - \psi^*)^2$$

$$= O\left(\frac{1}{S}\right)$$
The third inequality is based on the fact that

$$\sum_{s=1}^{S} \left[ 6mC_{\alpha}^2 (M^2 + \theta^2 N^2) \left( C_{1}^2 C_{2}^2 m^2 + 1 \right) \sum_{t=0}^{m-1} \mathbb{E}[\|V_t^s\|^2 + 12N^2 m^2 (1 - \phi^s - \psi^s)^2] \right]$$

$$\leq 6mC_{\alpha}^2 (M^2 + \theta^2 N^2) \left( C_{1}^2 C_{2}^2 m^2 + 1 \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V_t^s\|^2]$$

$$+ 12N^2 m^2 (1 - \phi^s - \psi^s)^2 \sum_{s=1}^{\infty} (1 - \phi^s - \psi^s)^2 + (58)$$

By the condition in assumption 8, summing with (26), we can easily verify that

$$\sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad} f(\omega_t^s)\|^2]$$

$$\leq \frac{2}{C_{\alpha}} \mathbb{E}[f(\tilde{\omega}) - f(\omega_{m}^s)] + \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V_t^s - \text{grad} f(\omega_t^s)\|^2] + (LC_{\alpha} - 1) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V_t^s\|^2]$$

$$\leq \frac{2}{C_{\alpha}} \mathbb{E}[f(\tilde{\omega}) - f(\omega_{m}^s)] + 12N^2 m^2 \sum_{s=1}^{\infty} (1 - \phi^s - \psi^s)^2$$

$$+ \left[ 6mC_{\alpha}^2 (M^2 + \theta^2 N^2) \left( C_{1}^2 C_{2}^2 m^2 + 1 \right) + LC_{\alpha} - 1 \right] \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|V_t^s\|^2]$$

$$\leq \frac{2}{C_{\alpha}} \mathbb{E}[f(\tilde{\omega}) - f(\omega_{m}^s)] + 12N^2 m^2 \sum_{s=1}^{\infty} (1 - \phi^s - \psi^s)^2$$

$$\leq \frac{2}{C_{\alpha}} (f(\tilde{\omega}) - f(\omega^*)) + 12N^2 m^2 \sum_{s=1}^{\infty} (1 - \phi^s - \psi^s)^2 + (59)$$

The third inequality is based on the fact that

$$6mC_{\alpha}^2 (M^2 + \theta^2 N^2) \left( C_{1}^2 C_{2}^2 m^2 + 1 \right) + LC_{\alpha} - 1 \leq 0,$$

if $C_{\alpha} \leq \frac{-L + L\sqrt{4v + 4v}}{2v} = \frac{2}{L + \sqrt{L^2 + 4v}}$. Thus

$$\mathbb{E}[\|\text{grad} f(\omega_s)\|^2] = \frac{1}{mS} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|\text{grad} f(\omega_t^s)\|^2]$$

$$\leq \frac{2}{mC_{\alpha}S} (f(\tilde{\omega}) - f(\omega^*)) + \frac{12mN^2}{S} \sum_{s=1}^{\infty} (1 - \phi^s - \psi^s)^2$$

$$= O\left(\frac{1}{S}\right)$$

(60)

4.3 Special case

We now turn to a special case of problem (P) with $\tau$-gradient dominated function. As an important class of non-convex function, we can establish linear convergence for this non-convex functions. Here, we only consider this special case, and other special cases are similar to above section, so we will not consider in this subsection.

**Theorem 7.** Suppose the conditions in theorem 6 are hold. If $\psi^s + \phi^s = 1$, $S = \lceil \frac{2\tau\gamma}{mC_{\alpha}} \rceil$, $\gamma > 1$ and $\tilde{\omega}^{k+1} = \text{Alg2}(\tilde{\omega}^k, \tilde{\omega}^0, m, S, \psi^s, \phi^s)$, $0 \leq k \leq K - 1$, the function $f$ is a $\tau$-gradient dominated functions. Then

$$\mathbb{E}[\|\text{grad} f(\tilde{\omega}^K)\|^2] \leq \gamma^{-K} \mathbb{E}[\|\text{grad} f(\tilde{\omega}^0)\|^2]$$

$$\mathbb{E}[f(\tilde{\omega}^K) - f(\omega^*)] \leq \gamma^{-K} \mathbb{E}[f(\tilde{\omega}^0) - f(\omega^*)]$$

(61)
Furthermore, we obtain \( \lim_{K \to \infty} \mathbb{E}[\|\nabla f(\hat{\omega}^K)\|] = 0 \) and \( \lim_{K \to \infty} \mathbb{E}[f(\hat{\omega}^K)] = f(\omega^*) \).

**Proof.** By the condition \( S = \left[ \frac{2T \gamma}{mC_\alpha} \right] \), we have \( \frac{2T}{mC_\alpha S} \leq \frac{1}{\gamma} \). From theorem \( \square \) for any \( 1 \leq k \leq K - 1 \), we have

\[
\begin{align*}
\mathbb{E}[\|\nabla f(\hat{\omega}^{k+1})\|^2] & \leq \frac{2}{mC_\alpha S} \mathbb{E}[f(\hat{\omega}^k) - f(\omega^*)] \\
& \leq \frac{2}{mC_\alpha S} \mathbb{E}[\|\nabla f(\hat{\omega}^k)\|^2] \leq \frac{1}{\gamma} \mathbb{E}[\|\nabla f(\hat{\omega}^k)\|^2] \\
& \leq \frac{1}{\gamma} \cdot \frac{2}{mC_\alpha S} \mathbb{E}[(f(\hat{\omega}^{k-1}) - f(\omega^*))] \\
& \leq \frac{1}{\gamma} \cdot \frac{2}{mC_\alpha S} \mathbb{E}[(f(\hat{\omega}^{k-2}) - f(\omega^*))] \\
& \leq \cdots \leq \frac{1}{\gamma^k} \mathbb{E}[(f(\hat{\omega}^0) - f(\omega^*))]
\end{align*}
\]

The second inequality is due to \( f \) is a \( \tau \)-gradient dominated functions. Iterate on both sides of the inequality, we get

\[
\begin{align*}
\mathbb{E}[\|\nabla f(\hat{\omega}^{k+1})\|^2] & \leq \frac{1}{\gamma} \mathbb{E}[\|\nabla f(\hat{\omega}^k)\|^2] \leq \cdots \leq \frac{1}{\gamma^k} \mathbb{E}[\|\nabla f(\hat{\omega}^0)\|^2] \\
\mathbb{E}[(f(\hat{\omega}^k) - f(\omega^*))] & \leq \frac{1}{\gamma} \mathbb{E}[(f(\hat{\omega}^{k-1}) - f(\omega^*))] \leq \cdots \leq \frac{1}{\gamma^k} \mathbb{E}[(f(\hat{\omega}^0) - f(\omega^*))]
\end{align*}
\]

5 Conclusions

This paper proposes R-SHG algorithm with adaptive parameters and time-varying parameters by the linear combination of R-SRG, R-SVRG and R-SGD. We have studied the finite-sum optimization problems on a smooth Riemannian manifold \( \mathcal{M} \). Two R-SHG algorithms with two different step sizes have been considered. Compared to the existing literature, our model is more widely applicable in the sense that 1) we do not need the descent direction to be an unbiased estimate; 2) our analysis focuses on retraction mapping and vector transport, do not need exponential mapping or vector transport. At the algorithm of R-SHG with adaptive parameters and time-varying parameters, we get global convergence when the step size is reduced and quantitatively research the convergence when the step size is fixed. For some special cases, we give better results. In this paper, there is no special requirement for function \( f \). Next, we will research whether the function satisfying certain conditions can have better properties and consider adaptive batch size gradient of a reference point.

A Proofs of lemmas in section 3 and section 4

**Proof of Lemma** \( \square \)

**Proof.** The inequalities are discussed in two cases

If \( \langle T_{\omega_{t-1}}^{\omega_t^*} (V_{t-1}^* - \nabla f(\omega_{t-1}^*)), \nabla f(\omega_t^*) \rangle > 0 \), then

\[
\begin{align*}
0 & \geq \psi_t^* \langle T_{\hat{\omega}_{t-1}}^{\omega_t^*} (V_{t-1}^* - \nabla f(\omega_{t-1}^*)), \nabla f(\omega_t^*) \rangle \\
& = \min \left\{ \psi_t^*, \frac{\mu \|\nabla f(\omega_t^*)\|^2}{\|\langle T_{\hat{\omega}_{t-1}}^{\omega_t^*} (V_{t-1}^* - \nabla f(\omega_{t-1}^*)), \nabla f(\omega_t^*) \rangle\|} \right\} \\
& \times \langle T_{\hat{\omega}_{t-1}}^{\omega_t^*} (V_{t-1}^* - \nabla f(\omega_{t-1}^*)), \nabla f(\omega_t^*) \rangle \\
& \leq \mu \|\nabla f(\omega_t^*)\|^2
\end{align*}
\]
If \((T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1}))), \text{grad} f(\omega^s_t)) < 0\), then
\[
0 \leq \hat{\psi}_t \langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle \\
= \min \{ \hat{\psi}_t, \frac{\mu \| \text{grad} f(\omega^s_t) \|^2}{\langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle} \} \\
\times \langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle \\
= -\min \{ \psi_t, \frac{\mu \| \text{grad} f(\omega^s_t) \|^2}{\langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle} \} \\
\times -\left( \langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle \right) \\
= \max \{ \psi_t \times \langle T^\omega_{t-1} (V^s_{t-1} - \text{grad} f(\omega^s_{t-1})), \text{grad} f(\omega^s_t) \rangle, -\mu \| \text{grad} f(\omega^s_t) \|^2 \} \\
\geq -\mu \| \text{grad} f(\omega^s_t) \|^2 
\] (65)

\[
\square
\]

**Proof of Lemma [6]**

**Proof.**

\[
\mathbb{E}[\| \text{grad} f_{I_t}(\omega_2) - T^\omega_{t^2} \text{grad} f_{I_t}(\omega_1) \|^2 | F_t] \\
= \mathbb{E}[\| \frac{1}{b} \sum_{i \in I_t} \text{grad} f_i(\omega_2) - T^\omega_{t^2} \text{grad} f_i(\omega_1) \|^2 | F_t] \\
\leq \frac{1}{b} \sum_{i \in I_t} \mathbb{E}[\| \text{grad} f_i(\omega_2) - T^\omega_{t^2} \text{grad} f_i(\omega_1) \|^2 | F_t] \\
= \frac{1}{n} \sum_{i = 1}^n \mathbb{E}[\| \text{grad} f_i(\omega_2) - T^\omega_{t^2} \text{grad} f_i(\omega_1) \|^2 | F_t] \\
\leq \frac{2}{n} \sum_{i = 1}^n (\| \text{grad} f_i(\omega_2) - \Gamma^\omega_{t^2} \text{grad} f_i(\omega_1) \|^2 + \| \Gamma^\omega_{t^2} \text{grad} f_i(\omega_1) - T^\omega_{t^2} \text{grad} f_i(\omega_1) \|^2) \\
\leq \frac{2}{n} \sum_{i = 1}^n (M^2 \| \xi \|^2 + \theta^2 \| \text{grad} f_i(\omega_1) \|^2 \| \xi \|^2) \\
\leq \frac{2}{n} \sum_{i = 1}^n (M^2 + \theta^2 N^2) \| \xi^\omega_{t^2} \|^2 \\
= 2(M^2 + \theta^2 N^2) \| \xi^\omega_{t^2} \|^2 
\] (66)

\[
\square
\]

**Proof of Lemma [7]**

**Proof.** By the definition of \(V^s_t\), we have
\[
\mathbb{E}[\| V^s_t - \text{grad} f(\omega^s_t) \|^2 | F_t] \\
= \mathbb{E}[\| \phi^s_t \left( \text{grad} f_{I_t}(\omega^s_t) - T^\omega_{t^0} (\text{grad} f_{I_t}(\omega^s_0) - \text{grad} f(\omega^s_0)) \right) \\
+ \hat{\psi}_t (\text{grad} f_{I_t}(\omega^s_t) - T^\omega_{t-1} (\text{grad} f_{I_t}(\omega^s_{t-1}) - V^s_{t-1})) \]
The first inequality follows from

\[ \psi \cdot \nabla \phi - \psi \cdot \nabla \tilde{\phi} \]

For the second item at the right side of (67), similar to the proof of (66), it is easy to find that

\[ \left(1 - \phi \right) \nabla \phi - \psi \cdot \nabla \tilde{\phi} \]
By the definition of \( \phi_t^* \), we have
\[
-((1 - \phi_t^*) \nabla f(\omega_t^*) - \tilde{\phi}_t^* \nabla f(\omega_{t-1}^*))^2 [\mathcal{F}_t]
\leq \mathbb{E}[||((1 - \phi_t^*) \nabla f(\omega_t^*) - \tilde{\phi}_t^* \nabla f(\omega_{t-1}^*))||^2 [\mathcal{F}_t]]
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} ((1 - \phi_t^*) \nabla f_i(\omega_t^*) - \tilde{\phi}_t^* \nabla f_i(\omega_{t-1}^*))^2
\]
\[
\leq \frac{2}{n} \sum_{i=1}^{n} ((1 - \phi_t^*) \nabla f_i(\omega_t^*))^2 + ||\tilde{\phi}_t^* \nabla f_i(\omega_{t-1}^*)||^2
\]
\[
\leq \frac{2}{n} \sum_{i=1}^{n} ((1 - \phi_t^*)^2 N^2 + (\tilde{\phi}_t^*)^2 N^2)
\]
\[
= 2N^2((1 - \phi_t^*)^2 + (\tilde{\phi}_t^*)^2)
\]
\[
\leq 2N^2((1 - \phi_t^*)^2 + (\psi_t^*)^2)
\]

(69)

For the third item at the right side of (67), using assumption 4, we obtain
\[
\mathbb{E}[||\tilde{\phi}_t^* \nabla f(\omega_t^*) - \nabla f(\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
= \tilde{\phi}_t^* \mathbb{E}[||\nabla f(\omega_t^*) - \nabla f(\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
= \tilde{\phi}_t^* \mathbb{E}[||\nabla f(\omega_t^*) - \nabla f(\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
\leq \tilde{\phi}_t^* ||\nabla f(\omega_t^*) - \nabla f(\omega_{t-1}^*)||^2 [\mathcal{F}_t]
\]

(70)

Combining the inequalities (67)-(70), we get
\[
\mathbb{E}[||V_t^* - \nabla f(\omega_t^*)||^2 [\mathcal{F}_t]]
\]
\[
\leq 4(M^2 + \theta^2 N^2)(\phi_t^*)^2 ||\nabla f(\omega_t^*)||^2 + 4N^2((1 - \phi_t^*)^2 + (\psi_t^*)^2)
\]
\[
+ (\psi_t^*)^2 ||\nabla f(\omega_{t-1}^*)||^2 [\mathcal{F}_t]
\]

(71)

Proof of Lemma 8

Proof. By the definition of \( V_t^* \), we get
\[
\mathbb{E}[||V_t^* - \nabla f(\omega_t^*)||^2 [\mathcal{F}_t]]
\]
\[
= \mathbb{E}[||\tilde{\phi}_t^* \nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*) - \tilde{\phi}_t^* \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
+ \tilde{\phi}_t^* \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
+ \tilde{\phi}_t^* \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
= \mathbb{E}[||\tilde{\phi}_t^* \nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
+ \tilde{\phi}_t^* \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
+ \tilde{\phi}_t^* \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
\leq 2(\phi_t^*)^2 \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]
\[
\leq 2(\phi_t^*)^2 \mathbb{E}[||\nabla f_t^* (\omega_t^*) - \nabla f_t^* (\omega_{t-1}^*)||^2 [\mathcal{F}_t]]
\]

(72)

\[\square\]
\[ +2(\psi_t^*)^2\mathbb{E}[||\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*)) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] + \mathbb{E}[||\nabla \mathcal{T}_{\omega_{t-1}^*} V_{t-1}^* - \psi_t^* \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \]

The fourth inequality is due to \(\mathbb{E}[(1 - \psi_t)\nabla f_I^t(\omega_t^*) - \phi_t^* \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - \nabla f(\omega_t^*)) - (1 - \psi_t)\nabla f(\omega_t^*)+ \bar{\psi}_t (\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*))|\mathcal{F}_t^* = 0 \) and \(\mathcal{T}_{\omega_{t-1}^*} V_{t-1}^* - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)\) is measurable in \(\mathcal{F}_t^*\). For the second item at the right side of (72). Similarly the proof of (66) yields

\[ \mathbb{E}[||\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*)) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \leq \mathbb{E}[||\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} ||\nabla f_i(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_i(\omega_{t-1}^*)||^2 \]

\[ \leq \frac{2}{n} \sum_{i=1}^{n} (||\nabla f_i(\omega_t^*)||^2 + ||\mathcal{T}_{\omega_{t-1}^*} \nabla f_i(\omega_{t-1}^*)||^2) \]

\[ \leq \frac{2}{n} \sum_{i=1}^{n} (||\nabla f_i(\omega_t^*)||^2 + ||\nabla f_i(\omega_{t-1}^*)||^2) \]

\[ \leq \frac{2}{n} \sum_{i=1}^{n} (N^2 + N^2) \]

\[ = 4N^2 \] (73)

The first inequality holds due to \(\mathbb{E}[||x - \mathbb{E}[x]||^2|\mathcal{F}_t^*] \leq \mathbb{E}[||x||^2|\mathcal{F}_t^*\) Combining the inequalities (68), (70) and (72)-(73) gives

\[ \mathbb{E}[||\nabla \mathcal{T}_{\omega_{t-1}^*} V_{t-1}^* - \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \leq 4(M^2 + \theta^2 N^2)(\bar{\psi}_t)^2||\mathcal{T}_{\omega_{t-1}^*}||^2 + 8N^2(\bar{\psi}_t)^2 \]

\[ \leq 4(M^2 + \theta^2 N^2)(\bar{\psi}_t)^2||\mathcal{T}_{\omega_{t-1}^*}||^2 + 8N^2(\bar{\psi}_t)^2 \]

\[ \leq 4N^2 \] (74)

**Proof of Lemma 9**

Proof. By the definition, we have

\[ \mathbb{E}[||V_t^* - \nabla f(\omega_t^*)||^2|\mathcal{F}_t^*] \]

\[ = \mathbb{E}[||\phi_t^* (\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*)) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)) + \psi_t^* (-\mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - V_{t-1}^*) + (1 - \phi_t^* - \psi_t^*) \nabla f_I^t(\omega_{t-1}^*) - \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \]

\[ = \mathbb{E}[||\phi_t^* (\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*)) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)) + \psi_t^* (-\mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - V_{t-1}^*) + (1 - \phi_t^* - \psi_t^*) \nabla f_I^t(\omega_{t-1}^*) - \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \]

\[ = \mathbb{E}[||\phi_t^* (\nabla f_I^t(\omega_t^*) - \mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - (\nabla f(\omega_t^*)) - \mathcal{T}_{\omega_{t-1}^*} \nabla f(\omega_{t-1}^*)) + \psi_t^* (-\mathcal{T}_{\omega_{t-1}^*} \nabla f_I^t(\omega_{t-1}^*) - V_{t-1}^*) + (1 - \phi_t^* - \psi_t^*) \nabla f_I^t(\omega_{t-1}^*) - \nabla f(\omega_{t-1}^*)||^2|\mathcal{F}_t^*] \]

\[ \boxed{26} \]
Proof. Proof of Lemma 10

\[ F(\leq + (1 - \phi^*_t - \psi^*_t)(\text{grad } f_I (\omega^*_t) - \text{grad } f(\omega^*_t))) \]
\[ + \psi^*_t T_{\omega^*_t-1}^t V^*_t - \psi^*_t T_{\omega^*_t-1}^t \text{grad } f(\omega^*_t-1)|^2|F^*_t] \]
\[ = \mathbb{E}[||\phi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t (\text{grad } f_I (\omega^*_0) - \text{grad } f(\omega^*_0))) - \text{grad } f(\omega^*_t))|| \]
\[ + \psi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1))) \]
\[ + (1 - \phi^*_t - \psi^*_t)(\text{grad } f_I (\omega^*_t) - \text{grad } f(\omega^*_t))) \]
\[ + ||\psi^*_t T_{\omega^*_t-1}^t V^*_t - \psi^*_t T_{\omega^*_t-1}^t \text{grad } f(\omega^*_t-1)||^2 \]
\[ + (\phi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t (\text{grad } f_I (\omega^*_0) - \text{grad } f(\omega^*_0))) - \text{grad } f(\omega^*_t)) \]
\[ + \psi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1))) \]
\[ + (1 - \phi^*_t - \psi^*_t)(\text{grad } f_I (\omega^*_t) - \text{grad } f(\omega^*_t))) \]
\[ + ||\psi^*_t T_{\omega^*_t-1}^t V^*_t - \psi^*_t T_{\omega^*_t-1}^t \text{grad } f(\omega^*_t-1)||^2|F^*_t] \]
\[ \leq 3(\phi^*_t)^2 \mathbb{E}[||\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t (\text{grad } f_I (\omega^*_0) - \text{grad } f(\omega^*_t)) - \text{grad } f(\omega^*_t))||^2|F^*_t] \]
\[ + 3(\psi^*_t)^2 \mathbb{E}[||\text{grad } f_I (\omega^*_t) - T_{\omega^*_t-1}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1)))||^2|F^*_t] \]
\[ + (\phi^*_t + \psi^*_t)^2 \mathbb{E}[||T_{\omega^*_t-1}^t V^*_t - \psi^*_t T_{\omega^*_t-1}^t \text{grad } f(\omega^*_t-1)||^2|F^*_t] \]
\[ (75) \]

The fifth equality is based on \( \mathbb{E}[\phi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t (\text{grad } f_I (\omega^*_0) - \text{grad } f(\omega^*_0))) - \text{grad } f(\omega^*_t)) + \psi^*_t (\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1))) + (1 - \phi^*_t - \psi^*_t)(\text{grad } f_I (\omega^*_t) - \text{grad } f(\omega^*_t))) \)
\( \mathbb{E}[||\text{grad } f_I (\omega^*_t) - T_{\omega^*_t}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1)))||^2|F^*_t] = 0 \) and \( T_{\omega^*_t-1}^t V^*_t - \psi^*_t T_{\omega^*_t-1}^t \text{grad } f(\omega^*_t-1)||^2|F^*_t] \)
\( \) is measurable in \( F^*_t \). Now, we consider the each term on the right side of (75). By lemma \( \Box \) we get
\[ \mathbb{E}[||\text{grad } f_I (\omega^*_t) T_{\omega^*_t-1}^t \text{grad } f_I (\omega^*_t-1)) (\text{grad } f(\omega^*_t) - T_{\omega^*_t}^t \text{grad } f(\omega^*_t-1)))||^2|F^*_t] \]
\[ \leq 2(M^2 + \theta^2 N^2)||\xi_{\omega^*_t}^2||^2 \]
\[ (76) \]

Combining the inequalities (68, 70, 73) and (76), we have
\[ \mathbb{E}[||V^*_t - \text{grad } f(\omega^*_t)||^2|F^*_t] \]
\[ \leq 6(\phi^*_t)^2 (M^2 + \theta^2 N^2)||\xi_{\omega^*_t}^2||^2 + 6(\psi^*_t)^2 (M^2 + \theta^2 N^2)||\xi_{\omega^*_t}^2||^2 \]
\[ + 12(1 - \phi^*_t - \psi^*_t)^2 N^2 + (\psi^*_t)^2||V^*_t-1 - \text{grad } f(\omega^*_t-1)||^2 \]
\[ (77) \]

Proof of Lemma 10

Proof. From assumption 2, if \( ||\xi_{\omega^*_0}^2|| > \rho \), then \( \omega^*_t = R_{\omega^*_0}(\xi_{\omega^*_0}^2) \notin \Omega \). This is contradicted with assumption 2 \( \omega^*_t \in \Omega \). Hence, \( ||\xi_{\omega^*_0}^2||^2 \leq \rho^2 \), substituting this result into lemma \( \Box \) and taking the mathematical expectation, we have
\[ \mathbb{E}[||V^*_t - \text{grad } f(\omega^*_t)||^2] \]

27
\[
\begin{align*}
\leq & \ 6(M^2 + \theta^2 N^2)((\phi^*_t)^2 \rho^2 + (\psi^*_t)^2 (\alpha^*_t)^2 \mathbb{E} \|V^*_t\|^2) \\
& + 12(1 - \phi^*_t - \psi^*_t)^2 N^2 + (\psi^*_t)^2 E(\|\nabla f(\omega^*_t)\|^2) \\
\leq & \ 6(M^2 + \theta^2 N^2)(\phi^*_t)^2 \rho^2 + 12(M^2 + \theta^2 N^2)(\psi^*_t)^2 (\alpha^*_t)^2 \mathbb{E} \|\nabla f(\omega^*_t)\|^2) \\
& + 12(1 - \phi^*_t - \psi^*_t)^2 N^2 + (\psi^*_t)^2 \mathbb{E} \|V^*_t\|^2. \\
\end{align*}
\] (78)

The last equality is due to \((a + b)^2 \leq 2a^2 + 2b^2\). For any \(2 \leq t \leq m - 1\), we have
\[
\frac{\|V^*_t\|^2}{\alpha^*_t - 1} \leq \frac{6(M^2 + \theta^2 N^2)((\phi^*_t)^2 \rho^2 + 12(M^2 + \theta^2 N^2)(\psi^*_t)^2 (\alpha^*_t)^2 \mathbb{E} \|\nabla f(\omega^*_t)\|^2)}{\alpha^*_t - 2} \\
+ \left[ - \frac{1}{\alpha^*_t - 2} \right. + \frac{1}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1] (\psi^*_t)^2 \mathbb{E} \|V^*_t\|^2. \\
\leq \frac{6(M^2 + \theta^2 N^2)(\phi^*_t)^2 \rho^2 + 12(M^2 + \theta^2 N^2)(\psi^*_t)^2 (\alpha^*_t)^2 \mathbb{E} \|\nabla f(\omega^*_t)\|^2)}{\alpha^*_t - 1} \\
+ \left[ - \frac{1}{\alpha^*_t - 2} \right. + \frac{\psi^*_t}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1] \mathbb{E} \|V^*_t\|^2. \\
\leq \frac{12(1 - \phi^*_t - \psi^*_t)^2 N^2}{\alpha^*_t - 1} + (\psi^*_t)^2 \mathbb{E} \|V^*_t\|^2. \\
\] (79)

The second inequality is due to \((\psi^*_t)^2 \leq 1\) and \((\psi^*_t)^2 \geq \psi^*_t\). Now, we consider the third term on the right side of (79)
\[
\begin{align*}
- \frac{1}{\alpha^*_t - 2} + \frac{\psi^*_t}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1 &= \frac{1}{\alpha^*_t - 1} - \frac{1}{\alpha^*_t - 2} - \frac{1}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1. \\
\end{align*}
\] (80)

Giving by conditions \(\alpha^*_t = (t + s + \kappa + 2)^{-P} C_\alpha\) and \(\psi^*_t = 1 - (t + s + \kappa + 1)^{-Q} C_\psi\), then
\[
\frac{1 - \psi^*_t}{\alpha^*_t - 1} = \frac{C_\psi}{C_\alpha} \cdot (t + s + \kappa + 1)^{-P} C_\psi,\text{ and } \frac{1}{\alpha^*_t - 2} = \frac{C_\psi}{C_\alpha} \cdot (1 + s + \kappa + 1)^{-P} C_\psi. \\
\]

Let \(g(x) = (x + \kappa)^P, 0 < P < 1\), it is easily verified that \(g''(x) = P(P - 1)(x + \kappa)^{P-2}\), and \(g''(x) < 0, x > 0\). Thus \(g(x + 1) \leq g(x) + g'(x), \text{ i.e., } (t + s + \kappa + 1)^P - (t + s + \kappa)^P \leq P(t + s + \kappa)^{P-1} \). Substituting this result into (80), then we have
\[
\begin{align*}
\frac{1}{\alpha^*_t - 1} - \frac{1}{\alpha^*_t - 2} - \frac{1 - \psi^*_t}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1 \\
& \leq \frac{P}{C_\alpha} (t + s + \kappa)^P - 1 - \frac{C_\psi}{C_\alpha} \cdot (t + s + \kappa + 1)^P - Q + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1. \\
\end{align*}
\] (81)

By conditions \(P \geq \frac{Q - 1}{\gamma - 1}, \frac{t - P}{Q - P} \geq \gamma > 1\) and the definition of \(\kappa\). If \(x \geq \kappa\), it is known that \(0 < x + 1 \leq x^\gamma \leq x^{1+\kappa} \). Note that \(0 < Q - P < Q < 1\), the function \(y = x^{Q - P}\) is a monotonically increasing in \(x \geq 1\). We can easily verify that \(0 < (x + 1)^{Q - P} \leq (x^{1+\kappa})^{Q - P} = x^{(1+\kappa)(Q - P)} \). Therefore, \(\frac{1}{(x + 1)^{Q - P}} \leq (x^{1+\kappa})^{Q - P} \), i.e., \(x^{P - 1} \leq (t + s + \kappa + 1)^P - Q, x \geq \kappa\), implies that \((t + s + \kappa)^P - 1 \leq (t + s + \kappa + 1)^P - Q\). This together with (81) leads to
\[
\begin{align*}
\frac{1}{\alpha^*_t - 1} - \frac{1}{\alpha^*_t - 2} - \frac{1 - \psi^*_t}{\alpha^*_t - 1} + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1 \\
& \leq \frac{P}{C_\alpha} \cdot (t + s + \kappa + 1)^P - Q + 12(M^2 + \theta^2 N^2)\alpha^*_t - 1. \\
\end{align*}
\]
\[
\alpha_{t-1}^{s} = 6(M^2 + \theta^2N^2)C_\alpha(-\beta(t + s + \kappa + 1)^{P-Q} + 2(t + s + \kappa + 1)^{-P}) \tag{82}
\]

Noting that \(0 < a < 1, y = a^x\) is a monotonically decreasing function in \(x > 0\), hence, \(y = \left(\frac{1}{t+s+\kappa+1}\right)^x\) is a monotonically decreasing function in \(x > 0\). By condition \(\frac{Q}{2} \leq P \leq Q\), we get \(\left(\frac{1}{t+s+\kappa+1}\right)^{Q-P} \geq \left(\frac{1}{t+s+\kappa+1}\right)^P\), i.e., \((t+s+\kappa+1)^{P-Q} \geq (t+s+\kappa+1)^{-P}\). Substituting this result back to (82), we have

\[
\begin{align*}
\frac{1}{\alpha_{t-1}^{s}} - \frac{1}{\alpha_{t-2}^{s}} - \frac{1 - \psi_t^s}{\alpha_{t-1}^{s}} + \frac{4(M^2 + \theta^2N^2)}{b} \alpha_{t-1}^{s} \\
\leq 6(M^2 + \theta^2N^2)C_\alpha(-\beta(t + s + \kappa + 1)^{P-Q} + 2(t + s + \kappa + 1)^{-P}) \\
\leq 6(M^2 + \theta^2N^2)C_\alpha(-\beta(t + s + \kappa + 1)^{-P} + 2(t + s + \kappa + 1)^{-P}) \\
= 6(2-\beta)(M^2 + \theta^2N^2)\alpha_{t-1}^{s} \\ \\ 
\end{align*}
\tag{83}
\]

Combining the inequalities (79) and (83), we obtain

\[
\begin{align*}
\mathbb{E}[\|V_t^s - \text{grad} f(\omega_t^s)\|^2] - \mathbb{E}[\|V_{t-1}^s - \text{grad} f(\omega_{t-1}^s)\|^2] \\
\leq \frac{6(M^2 + \theta^2N^2)}{\alpha_{t-1}^{s}}(\psi_t^s)^2\rho^2 + 12(M^2 + \theta^2N^2)(\psi_t^s)^2\alpha_{t-1}^{s}\mathbb{E}[\|\text{grad} f(\omega_{t-1}^s)\|^2] \\
+ \left[ - \frac{1}{\alpha_{t-2}^{s}} + \frac{\psi_t^s}{\alpha_{t-1}^{s}} + 12(M^2 + \theta^2N^2)\alpha_{t-1}^{s}\right]\mathbb{E}[\|V_{t-1}^s - \text{grad} f(\omega_{t-1}^s)\|^2] \\
+ \frac{12(1 - \psi_t^s - \psi_t^{s-1})N^2}{\alpha_{t-1}^{s}} \\
\leq \frac{6(M^2 + \theta^2N^2)}{\alpha_{t-1}^{s}}(\psi_t^s)^2\rho^2 + 12(M^2 + \theta^2N^2)\alpha_{t-1}^{s}\mathbb{E}[\|\text{grad} f(\omega_{t-1}^s)\|^2] \\
+ 6(2-\beta)(M^2 + \theta^2N^2)\alpha_{t-1}^{s}\mathbb{E}[\|V_{t-1}^s - \text{grad} f(\omega_{t-1}^s)\|^2] \\
+ \frac{12(1 - \psi_t^s)^2N^2}{\alpha_{t-1}^{s}} \\ \\ \end{align*}
\tag{84}
\]

\[
\square
\]

References

[1] Sato, H., Kasai, H., and Mishra, B.: Riemannian stochastic variance reduced gradient algorithm with retraction and vector transport, SIAM Journal on Optimization, vol. 29, no. 2, pp. 1444-1472, 2019.

[2] Balzano, L., Nowak, R., and Recht, B.: Online identification and tracking of subspaces from highly incomplete information, In Allerton, pp. 704-711, 2010.

[3] Boumal, N., and Absil, P.-a. RTRMC: A Riemannian trust-region method for low-rank matrix completion, in Advances in Neural Information Processing Systems, 2011, pp. 406-414.

[4] Kasai, H., and Mishra, B.: Low-rank tensor completion: a Riemannian manifold preconditioning approach, In ICML, 2016.

[5] Boumal, N., Mishra, B., Absil, P.-A., and Sepulchre, R.: Manopt: a Matlab toolbox for optimization on manifolds, JMLR, 15(1):1455-1459, 2014.

[6] Meyer, G., Bournabel, S., and Sepulchre, R.: Linear regression under fixed-rank constraints: A Riemannian approach, In ICML, 2011.
[7] Yuan X., Huang W., Absil, P.-A., and Gallivan, K. A.: A Riemannian limited-memory BFGS algorithm for computing the matrix geometric mean, Procedia Computer Science, vol. 80, pp. 2147-2157, 2016.

[8] Theis, F. J., Cason, T. P., and Absil, P.-A.: Soft dimension reduction for ica by joint diagonalization on the Stiefel manifold, in International Conference on Independent Component Analysis and Signal Separation. Springer, 2009, pp. 354-361.

[9] Cherian, A., and Sra, S.: Riemannian dictionary learning and sparse coding for positive definite matrices, IEEE trans. on Neural Networks and Learning Systems, 28(12):2859-2871, 2017.

[10] Sun, J., Qu, Q., and Wright, J.: Complete dictionary recovery over the sphere ii: Recovery by Riemannian trust-region method, IEEE Trans. on Information Theory, 63(2):885-914, 2017.

[11] Zhang, H., Reddi, S. J., and Sra, S.: Riemannian svrg: Fast stochastic optimization on Riemannian manifolds, in Advances in Neural Information Processing Systems, 2016, pp. 4592-4600.

[12] Bonnabel, S.: Stochastic gradient descent on Riemannian manifolds, IEEE Trans. Automat. Control, 58 (2013), pp. 2217-2229.

[13] Sato, H., Kasai, H., and Mishra, B.: Riemannian stochastic variance reduced gradient, arXiv preprint: arXiv:1702.05594, 2017.

[14] Andi H., and Junbin, G.: Riemannian stochastic recursive momentum method for non-convex optimization, arxiv preprint: arXiv:2008.04555, 2020.

[15] Huang, W., Gallivan, K. A., and Absil, P.-A.: A broyden class of quasi-newton methods for Riemannian optimization, SIAM Journal on Optimization, vol. 25, no. 3, pp. 1660-1685, 2015.

[16] Huang, W., Absil, P.-A., and Gallivan, K. A.: A Riemannian symmetric rank-one trust-region method, Math. Program., 150 (2015), pp. 179-216.

[17] Robbins, H., and Siegmund, D.: A convergence theorem for non negative almost supermartingales and some applications, in Selected Papers, T. L. Lai, and D. Siegmund, Eds. New York, NY, USA: Springer-Verlag,1985

[18] Fisk, D. L.: Quasi-martingales, Trans. Amer. Math. Soc., 120 (1965), pp. 369-389.

[19] Absil, P.-A., Mahony, R., Sepulchre, R.: Optimization Algorithms on Matrix Manifolds, Princeton University Press, Princeton, NJ (2008)

[20] Robbins, H., and Monro, S.: A stochastic approximation method, 1951, The Annals of Mathematical Statistics 400-407.

[21] Kingma, D. P., and Ba, J. Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980, 2014.

[22] Staib, M., Reddi, S. J., Kale, S., Kumar, S., and Sra, S.: Escaping saddle points with adaptive gradient methods. arXiv preprint arXiv:1901.09149, 2019.

[23] Cutkosky, A., and Orabona, F.: Momentum-based variance reduction in nonconvex sgd. In Advances in Neural Information Processing Systems, 15236-15245, 2019.
[24] Tran-Dinh, Q., Pham, N. H., Phan, D. T., and Nguyen, L. M.: Hybrid stochastic gradient descent algorithms for stochastic nonconvex optimization. arXiv preprint arXiv:1905.05920, 2019.

[25] Polyak, B.: Gradient methods for the minimisation of functionals, USSR Computational Mathematics and Mathematical Physics, vol. 3, no. 4, pp. 864–878, 1963.

[26] Nesterov, Y., and Polyak, B.: Cubic regularization of Newton method and its global performance, Mathematical Programming, vol. 108, no. 1, pp. 177–205, 2006.

[27] Johnson, R., and Zhang, T.: Accelerating stochastic gradient descent using predictive variance reduction, in Adv. Neural Inf. Process. Syst. 26, Curran Associates, Red Hook, NY, 2013, pp. 315–323.