ASYMPTOTIC BEHAVIORS AND STOCHASTIC TRAVELING WAVES IN STOCHASTIC FISHER-KPP EQUATIONS

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Abstract. Fisher-KPP equations are an important class of mathematical models with practical background. Previous studies analyzed the asymptotic behaviors of the front and back of the wavefront and proved the existence of stochastic traveling waves, by imposing decrease constraints on the growth function. For the Fisher-KPP equation with a stochastically fluctuated growth rate, we find that if the decrease restrictions are removed, the same results still hold. Moreover, we show that with increasing the noise intensity, the original equation with Fisher-KPP nonlinearity evolves into first the one with degenerated Fisher-KPP nonlinearity and then the one with Nagumo nonlinearity. For the Fisher-KPP equation subjected to the environmental noise, the established asymptotic behavior of the front of the wavefront still holds even if the decrease constraint on the growth function is ruled out. If this constraint is removed, however, the established asymptotic behavior of the back of the wavefront will no longer hold, implying that the decrease constraint on the growth function is a sufficient and necessary condition to ensure the asymptotic behavior of the back of the wavefront. In both cases of noise, the systems can allow stochastic traveling waves.

1. Introduction. In the natural world, the propagation and diffusion of energy/substance usually take wave forms, e.g., the ripple formed by dropping a stone on the lake surface, the wave shape by shaking a rope suspended in the air, the transformation of crystalline state in physics, and reaction processes in chemistry. In these transition processes of energy/substance, some characteristics subjected to sufficiently low frictional damping are reflected in limited propagation speeds and amplitudes as well as almost consistent shapes in propagation.

As a powerful tool to reveal transition processes of energy/substance, reaction-diffusion equations have received extensive attention, mainly because of their advantages in modeling and interpreting many natural and social phenomena [8, 10, 20].

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The reaction-diffusion equations can allow a special category of solutions, named by traveling waves. Main characteristics of these waves are that their shape is kept almost constant as the time goes on. Traveling wave solutions have good properties such as limited propagation speeds and amplitudes, and shape invariance in the propagation and diffusion of energy/substance.

The most representative model in the family of reaction-diffusion equations is the Fisher-KPP equation, written in form

$$u_t = u_{xx} + ru(u - K), \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.1)$$

where parameter $r$ represents the growth rate and parameter $K$ represents the habitat capacity. This model was initially used to describe the propagation of protogene in one-dimensional habitat (see [9]). It has been proved that Eq.(1.1) has a family of nonnegative traveling wave solutions of the form $u(t, x) = u(x - \gamma t)$ if speed $\gamma$ satisfies the constraint of $\gamma \geq 2\sqrt{rK}$ (see [1]). Subsequently, the study of traveling waves was extended to a wider class of Fisher-KPP equations driven by problems in physics [7, 34], chemistry [17, 22], ecology [3, 6] and epidemic disease [4, 39]. Moreover, traveling waves can be used to describe natural phenomena like phase transition in physics, concentration variation in chemistry, species invasion in biology and pathophoresis in epidemic disease.

On the other hand, stochastic perturbations are ubiquitous in the real world. Based on the background of particle branching in a particle system approximation, some authors [5, 23, 24, 25, 26, 37] introduced and analyzed the following two kinds of stochastic Fisher-KPP equations

$$u_t = u_{xx} + u(\theta - u) + \sqrt{u}\xi_{t,x}, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.2)$$

and

$$u_t = u_{xx} + u(1 - u) + \epsilon\sqrt{u(1 - u)}\xi_{t,x}, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.3)$$

where $\xi_{t,x}$ represents the time-space white noise and $\epsilon$ the noise intensity. Tribe [37] showed that the solution of Eq.(1.2), $u(t, x)$, with a shift of front maker $\gamma(t)$, i.e., $u(t, x + \gamma(t))$, is a stationary process with respect to time, indicating the existence of stochastic traveling waves in Eq.(1.2); Mueller and Sowers [23] also obtained similar results. Subsequently, Conlon and Doering [5] gave estimates on the wave speeds of stochastic traveling waves in Eq.(1.2) by using the ergodic theorem, where wave speed is defined by the limit $\lim_{t \to \infty} \gamma(t)/t$. Mueller, et al. [25, 26] used some skillful techniques to improve results in ref. [5], and derived better estimates.

In addition, based on the background of species diffusion or pathophoresis, some authors [13, 14, 29, 30] introduced and analyzed the Fisher-KPP equation subjected to the environmental noise

$$du = u_{xx}dt + c(u)udt + kudW_t, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.4)$$

where $k$ represents the noise intensity, $W_t$ is a standard Brownian motion, and growth function $c(u)$ is supposed to be decreasing as $u \in [0, \infty)$ and $c(K) = 0$ for some $K > 0$ with $K$ representing the environmental capacity. It is not difficult to see that the environmental noise $udW_t$ originates from randomizing parameter $K$ if we write $c(u) = K - u$. Øksendal et al. [29, 30] analyzed the asymptotic behaviors of the solution to Eq.(1.4) starting at the Heaviside function $H$ defined by $H(x) = 0$ for $x \geq 0$ and $H(x) = 1$ for $x < 0$. Following the study of Øksendal, et al., Huang and Liu [13], and Huang, et al. [14] proved the existence of stochastic traveling waves in Eq.(1.4) using different methods. Subsequently, some stochastic
Fisher-KPP equations driven by double noise were also introduced and analyzed. Wang, et al. [40] considered Eq. (1.4) with a stochastic convection, that is,
\[ du = u_{xx}dt + c(u)udt + k_1udu_1 + k_2u_xdW_2, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.5) \]
where the Brownian motions \( W_1 \) and \( W_2 \) are correlated, i.e., \( dW_1 \cdot dW_2 = \rho dt \), \(-1 \leq \rho \leq 1\) holds. They also analyzed the asymptotic behaviors of solutions starting at the Heaviside function, which are followed by the existence of stochastic traveling waves. Huang and Liu [15] studied the following stochastic Fisher-KPP equation driven simultaneously by both environmental noise and growth rate noise
\[ du = u_{xx}dt + ru(K - u)dt + k_1udu_1 + k_2u(K - u)dW_2, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.6) \]
where the Brownian motions were supposed to be independent of each other, i.e., \( \rho = 0 \). They also analyzed the asymptotic behaviors of solutions but did not prove the existence of stochastic traveling waves. In addition, bifurcations of asymptotic behaviors induced by noise intensities \( k_1, k_2 \) were addressed in Refs. [40] and [15].

Finally, some authors paid attention to stochastic traveling waves generated by reaction diffusion equations in random media. Shen [31] gave a theoretical framework for proving the existence of stochastic traveling waves, which was subsequently adopted by the authors of Refs. [31, 27, 28, 32, 33] to deal with problems of traveling waves in reaction-diffusion equations with Fisher-KPP nonlinearity, Nagumo nonlinearity and ignition nonlinearity, in random media. Other authors focused on the stability and robustness of the deterministic traveling waves perturbed by a small noise as well as the convergence rate of the perturbed traveling waves to the original traveling waves [36, 11, 12]. Literatures on study of traveling waves in stochastic neural field equations can be referred to Refs. [2, 16, 19, 20] wherein the noise effects on the deterministic traveling waves were addressed from two aspects: the difference of solution and the deviation of wave front induced by the noise.

Our study is motivated by the existing works on the models (1.4)-(1.6) (see Refs. [29, 30, 13, 14, 40, 15]). We devote to studying the following stochastic Fisher-KPP equation
\[ du = u_{xx}dt + ru(K - u)dt + ku(K - u)dW_t, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.7) \]
where \( W_t \) is the standard Brownian motion and \( k \) represents the noise intensity. Note that in Eq. (1.5), both the environmental noise \( udu_t \) and the convection noise \( u_xdW_t \) are linear. However, appropriate transformations make the former be transferred to a stochastic coefficient of the model while the latter to the special variable of a solution. Both the environmental noise and the convection noise cannot produce any additional nonlinear effect on the growth function. Also note that the noise term \( u(K - u)dW_t \) with volatility \( u(K - u) \) being a quadratic polynomial, whose stochastic effect is practically to perturb the growth rate (i.e., randomizing the parameter, \( r \)), can produce some additional nonlinear effects on the growth function, or equivalently, the growth rate noise can enhance the nonlinearity of the reaction function. In this case, however, characteristics of the original Fisher-KPP equation would be lost, leading to some difficulties in establishing a comprehensive dynamics of solutions.

The study of Ref. [15] indicated that to obtain the asymptotic behaviors of the front and back of the wavefront, the intensity of the growth rate noise has to be subjected to some constraint with a upper threshold, which implies that the reaction function has strong Fisher-KPP nonlinearity, and the growth function depending on \( u \) is decreasing and positive as \( u \in (0, K) \). With this constraint, Eq. (1.6) can
generate an approximate traveling wave profile. In this paper, we improve the threshold of the noise intensity to a larger one, that is, the constraint on the noise intensity can be relaxed. Moreover, the reaction function is extended to general Fisher-KPP nonlinearity, i.e., the growth function only needs to be positive as \( u \in [0, K] \) and the monotonicity condition is unnecessary. This improvement can also hold for more general reaction functions with which asymptotic behaviors and stochastic traveling waves are still obtained.

Before presenting the main results of the current paper, we restate the definition of a stochastic traveling wave.

**Definition 1.1.** A stochastic travelling wave is a solution \( u(t) \) with values in the state space and for which the centered process \( \tilde{u}(t) := u(t, \cdot + \gamma(t)) \) is a stationary process with respect to time, where \( \gamma(t) \) is some appropriate wavefront marker. The law of a stochastic traveling wave is the law of \( \tilde{u}(0) \) on the state space. If the centered process forms a Markov process. Then an equivalent definition is that the law of a stochastic travelling wave is an invariant measure for the centered process.

The main result of the paper is summarized in the following theorem, whose proof is given in Section 3.

**Theorem 1.2.** Let \( u \) be the solution of Eq.(1.7) starting at the Heaviside function \( H \). Suppose that the noise intensity \( k \) satisfies \( k^2 < 2r/K \). Then, the equation has the following asymptotic behaviors

\[
\lim_{t \to \infty} \sup_{x \geq (\gamma + h)t} u(t, x) = 0, \quad \lim_{t \to \infty} \inf_{x \leq (\gamma - h)t} u(t, x) = K
\]

almost surely, for any \( h > 0 \), where the wave speed \( \gamma \) is given by

\[
\gamma = \sqrt{2K(2r - k^2K)}.
\]

In addition, there exist stochastic traveling waves in Eq.(1.7).

We point out that if the decrease constraint on the growth function \( c(u) \) in Eq.(1.4) is removed, then the asymptotic behavior of the front of the wavefront established in [29, 30] can be recovered following our argument, but the asymptotic behavior of the back of the wavefront established in [29, 30] no longer holds. A possible reason is because of the uniform boundedness of solutions. The solutions to Eq.(1.7) have a uniform upper boundary if they are restricted in the range of interval \([0, K]\), which is contrary to the case of Eq.(1.4) where the deterministic equilibrium \( u \equiv K \) is destroyed by the noise and the uniform upper boundary becomes stochastic. The detailed explanation is given in Section 4.

This paper is organized as follows. In Section 2, we derive an implicit stochastic Feynman-Kac formula for Eq.(1.7) through the Wong-Zakai approximation that is a main analytical tool to prove Theorem 1.2. Section 3 presents a proof of Theorem 1.2. In Section 4, we analyze the asymptotic behaviors of solutions of Eq.(1.4) with a growth function that has no monotonic decrease constraint. And in Section 5, we conclude the paper with remarks on our future works.

2. Preliminaries. The aim of this section is to derive the Feynman-Kac formula for Eq.(1.7), which is taken as a fundamental analytical tool in obtaining the asymptotic behaviors given in Theorem 1.2. We first present a proposition that summarizes the properties of solutions to Eq.(1.7), which will be used later.
Proposition 1. Let \( W_t \) be a \( \mathcal{F}_t \) Brownian motion defined on a filtered space \((\Omega, \mathcal{F}_t, \mathbf{F}, \mathbb{P})\), where \( \mathcal{F}_0 \) contains the \( \mathbb{P} \) null sets. Given any \( u_0 : \mathbb{R} \to [0, K] \) which is \( \mathcal{B}(\mathbb{R}) \times \mathcal{F}_0 \) measurable, there exists a progressively measurable solution \( u(t, x) \), \( t \geq 0, x \in \mathbb{R} \) to Eq.(1.7), driven by \( W_t \) and initial condition \( u_0 \). The paths of \( u \) lie almost surely in \( C([0, \infty), L_{loc}^1(\mathbb{R})) \) and solutions are pathwise unique in this space. If \( P_\varphi \) is the law of the solution starting at \( \varphi \) on the space \( C([0, \infty), L_{loc}^1(\mathbb{R})) \), then the family \( (P_\varphi, \varphi \in L_{loc}^1(\mathbb{R})) \) forms a strong Markov family. The associated Markov semigroup is Feller. Moreover, there exists a regular version of any solution, where the paths \( u(t, x) \), \( t \geq 0, x \in \mathbb{R} \) lie almost surely in \( C^{0,2}([0, \infty) \times \mathbb{R}) \). The following additional properties hold for the regular versions:

(i) Two solutions \( u, \hat{u} \) with initial conditions \( u_0(x) \leq \hat{u}_0(x) \) for all \( x \in \mathbb{R} \) almost surely, remain coupled, namely that \( u(t, x) \leq \hat{u}(t, x) \) for all \( t \geq 0, x \in \mathbb{R} \) almost surely.

(ii) If \( u_0 \in S \) almost surely, then \( u(t) \in S \) for all \( t \geq 0 \) almost surely. Moreover, \( u(t, x) > 0 \) and \( u_\varepsilon(t, x) < 0 \) for all \( t > 0, x \in \mathbb{R} \) almost surely, where the state space is defined as

\[ S = \{ \varphi : \mathbb{R} \to [0, K] \text{ decreasing and right continuous, } \varphi(-\infty) = K, \varphi(\infty) = 0 \}. \]

The results of this proposition are standard, so we omit its proof here. One route to reach the strict positivity in part (ii) is to follow the argument in Ref.[35]. If the corresponding method is applied to the equation

\[ dv = v_{xx} dt + r(K - 2u)vdW_t + k(K - 2u)vvdW_t \]

for the derivative \( v = u_\varepsilon \) over any time interval \([t_0, \infty)\), then we can show the strict negativity of function \( v(t, x) \) for all \( t > 0, x \in \mathbb{R} \) almost surely.

2.1. Wong-Zakai approximation. To derive an implicit stochastic Feynman-Kac formula for Eq.(1.7), we need to make use of the Wong-Zakai approximation. Define a piecewise linear approximation to the Brownian motion \( W_t \) by

\[ W^\varepsilon_t = W_{k\varepsilon} + \frac{1}{\varepsilon}(W_{(k+1)\varepsilon} - W_{k\varepsilon})(t - k\varepsilon), \]

for \( t \in [k\varepsilon, (k + 1)\varepsilon] \), \( k = 0, 1, \cdots, N \), where \( \varepsilon > 0 \). Then the equation

\[ (u_\varepsilon)_t = (u_\varepsilon)_{xx} + r(u_\varepsilon(K - u_\varepsilon)) - \frac{k^2}{2} u_\varepsilon(K - u_\varepsilon)(K - 2u_\varepsilon) + ku_\varepsilon(K - u_\varepsilon)W^\varepsilon_t \quad (2.1) \]

can be solved over each interval \([k\varepsilon, (k + 1)\varepsilon]\) path by path, given an appropriate initial condition. The following lemma suggests that the solution of Eq.(2.1) converges to that of Eq.(1.7) as \( \varepsilon \) tends to zero.

Lemma 2.1. Let \( u \) and \( u_\varepsilon \) be solutions of Eq.(1.7) and Eq.(2.1) with the initial conditions \( u(0) \) and \( u_\varepsilon(0) \in S \) satisfying \( u(0) = u_\varepsilon(0) \) almost surely. Then the following result

\[ u_\varepsilon(t, x) \xrightarrow{L^2} u(t, x) \text{ as } \varepsilon \to 0 \]

holds for \( t \geq 0, x \in \mathbb{R} \).

The proof of this lemma is tedious and will be given as an appendix (see Appendix A). In addition, to obtain the derivation of the Feynman-Kac formula for Eq.(1.7), we need another approximation, whose proof can be found in Appendix B.
Lemma 2.2. Let \( u \) and \( u_\epsilon \) be solutions of Eq. (1.7) and Eq. (2.1) with the initial conditions \( u(0) \) and \( u_\epsilon(0) \) \( \in \mathcal{S} \) satisfying \( u(0) = u_\epsilon(0) \) almost surely. Then we have
\[
\int_0^t u_\epsilon(s, x) \dot{W}_\epsilon t ds \overset{L^2}{\to} \int_0^t u(s, x) dW_t + \frac{k}{2} \int_0^t u(s, x)(K - u(s, x)) ds,
\]
as \( \epsilon \to 0 \) for \( t \geq 0, \ x \in \mathbb{R} \).

2.2. Feynman-Kac formula. The Feynman-Kac formula for Eq. (2.1) can be written as
\[
u_\epsilon(t, x) = \hat{E} \left[ u_\epsilon(0, x + \sqrt{2} \hat{W}_t) \exp \left( \int_0^t f_0(u_\epsilon(t-s, x + \sqrt{2} \hat{W}_s)) ds \right) \times \exp \left( k \int_0^t (K - u_\epsilon(s, x + \sqrt{2} \hat{W}_s)) \dot{\hat{W}}_s ds \right) \right],
\]
where
\[f_0(u) = r(K - u) - \frac{k^2}{2}(K - u)(K - 2u),\]
and \( \hat{W} \) is a Brownian motion defined on an auxiliary probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) and \( \hat{E} \) is the expectation with respect to \( \hat{\mathbb{P}} \). The auxiliary probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) is complete independent of the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\), which implies that the Brownian motion \( \hat{W}_\epsilon \) is also independent of the Brownian motion \( W_t \). \( \dot{\hat{W}}_t \) is the time derivative of the approximated Brownian motion \( \hat{W}_\epsilon \).
Thus by Lemma 2.1 and 2.2, taking the limit as \( \epsilon \to 0 \) in (2.2), we have
\[
u(t, x) = \hat{E} \left[ u(0, x + \sqrt{2} \hat{W}_t) \exp \left( \int_0^t f(u(t-s, x + \sqrt{2} \hat{W}_s)) ds \right) \times \exp \left( k \int_0^t (K - u(s, x + \sqrt{2} \hat{W}_s)) \dot{\hat{W}}_s ds \right) \right],
\]
where
\[f(u) = (K - u) \left( r - \frac{k^2 K}{2} + \frac{k^2}{2} u \right).
\]

3. Proof of Theorem 1.2. In this section, we present the proof of Theorem 1.2. The asymptotic behaviors of a solution involve two parts: the front and the back of the wavefront, which will be discussed in the following subsections respectively. First of all, we give an estimate for the stochastic integral in the Feynman-Kac formula, i.e., Eq. (2.3), which will be frequently used in the following argument. For any \( x \in \mathbb{R} \), the stochastic integral is a time-changed Brownian motion, i.e.
\[
\int_0^t (K - u(s, x)) dW_s = B \left( \int_0^t (K - u(s, x))^2 ds \right),
\]
where \( B(r) \) is a Brownian motion with time \( r \). Then for any \( \epsilon > 0 \), by the fact of \( u \leq K \) as well as the Doob’s inequality, we have
\[
\mathbb{P} \left( \sup_{x \in \mathbb{R}} \int_0^t (K - u(s, x)) dW_s \bigg| > \epsilon t \right)
= \mathbb{P} \left( \sup_{x \in \mathbb{R}} \left| B \left( \int_0^t (K - u(s, x))^2 ds \right) \right| > \epsilon t \right)
\leq \mathbb{P} \left( \sup_{0 \leq s \leq t} |B(s)| > \epsilon t \right).
which suggests that there exists $\Omega^\epsilon_t \subset \Omega$ depending on $t$ and $\epsilon$, satisfying $\mathbb{P}(\Omega^\epsilon_t) \geq 1 - e^{-c^2t/2}$, such that

$$\sup_{x \in \mathbb{R}} \left| \int_0^t (K - u(s, x, \omega))dW_s(\omega) \right| \leq \epsilon t. \tag{3.1}$$

for any $\omega \in \Omega^\epsilon_t$.

### 3.1. Asymptotic behavior of the front of the wavefront.

The asymptotic behavior of a solution to Eq.(1.7) in the front of the wavefront is summarized in the following lemma.

**Lemma 3.1.** Let $u$ be the solution of Eq.(1.7) with the Heaviside function $H$ as the initial condition. Suppose that the noise intensity satisfies $k^2 < \frac{2}{K}$. Then there holds

$$\sup_{x \geq (\gamma + h)t} u(t, x) \to 0 \text{ as } t \to \infty$$

almost surely for any $h > 0$, where $\gamma$ is given in (1.8).

**Proof.** For any $\epsilon > 0$, take the subset $\Omega^\epsilon_t \subset \Omega$ satisfying $\mathbb{P}(\Omega^\epsilon_t) \geq 1 - e^{-\epsilon^2t/2}$ such that (3.1) holds. The proof is divided into two steps. In the first step, suppose that the noise intensity satisfies $k^2 \leq \frac{r}{K}$. Then the function $f(u)$ given in (2.4) is decreasing as $u \geq 0$. For $\omega \in \Omega^\epsilon_t$, using the Feynman-Kac formula, i.e. (2.3), and by the Doob’s inequality, we have

$$u(t, x, \omega) \leq \mathbb{E}[H(x + \sqrt{2W_t})] \exp(f(0)t + \epsilon kt)$$

$$\leq \mathbb{P}(x + \sqrt{2W_t} < 0) \exp(f(0)t + \epsilon kt)$$

$$\leq \exp\left(-\frac{x^2}{4t} + f(0)t + \epsilon kt\right). \tag{3.2}$$

For any $h > 0$, take $\epsilon$ sufficiently small such that $\epsilon k < \gamma h/2 + h^2/2$. Thus if $x \geq (\gamma + h)t$, we have

$$u(t, x, \omega) \leq \exp\left(-\frac{(\gamma + h)^2t}{4} + f(0)t + \epsilon t\right)$$

$$= \exp\left(-\frac{\gamma ht}{2} - \frac{h^2t}{4} + \epsilon kt\right)$$

$$\to 0 \text{ as } t \to \infty.$$ 

In the second step, suppose that the noise intensity satisfies $r/K < k^2 < 2r/K$. It is easy to note that the function $f(u)$ is increasing as $0 \leq u \leq K/2 - r/k^2$. For $\epsilon \leq K/2 - r/k^2$, define $\alpha(t) = \sup_{0 \leq s \leq t} \beta(t - s)$, where $\beta(t - s)$ satisfies

$$\beta(t - s) = \inf\{x : u(t - s, x + \sqrt{2W_s}) \leq \epsilon\}.$$ 

Then for $x \geq \alpha(t)$, it holds that $u(t - s, x + \sqrt{2W_s}) \leq \epsilon$ for $0 \leq s \leq t$. Similar to (3.2), we have

$$u(t, x, \omega) \leq \mathbb{E}[H(x + \sqrt{2W_t})] \exp(f(\epsilon)t + \epsilon kt)$$

$$= \mathbb{P}(x + \sqrt{2W_t} < 0) \exp(f(\epsilon)t + \epsilon kt)$$

$$\leq \exp\left(-\frac{x^2}{4t} + f(\epsilon)t + \epsilon kt\right).$$
For any $h > 0$, take $\epsilon$ so small that $\epsilon(f'(0) + k) < \gamma h/2 + h^2/2$. Then if $x \geq (\gamma + h)t \vee \alpha(t)$, we have

$$u(t, x, \omega) \leq \exp \left( -\frac{(\gamma + h)^2}{4} f(0)t + \epsilon f'(0)t + \epsilon kt \right) = \exp \left( -\frac{\gamma ht}{2} - \frac{h^2t}{4} \epsilon f'(0)t + \epsilon kt \right) \to 0 \text{ as } t \to \infty. \quad (3.3)$$

We claim that for sufficiently large $t$, $(\gamma + h)t \geq \alpha(t)$ holds almost surely. Indeed, by (3.3), it is not difficult to see that there exists $T > 0$ such that for $t \geq T$, it holds that

$$u(t, (\gamma + h)t, \omega) \leq \epsilon = u(t, \beta(t), \omega),$$

which suggests that for $t \geq T$, $\beta(t) \leq (\gamma + h)t$ as $u(t, x)$ is decreasing with respect to $x$. This implies that $\lim_{t \to \infty} \beta(t)/t \leq \gamma + h$ almost surely. According to the definition of $\alpha(t)$, we know that

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} \leq \lim_{t \to \infty} \frac{\beta(t)}{t} \leq \gamma + h$$

holds almost surely. \hfill \Box

3.2. **Asymptotic behavior of the back of the wavefront.** To obtain the asymptotic behavior of a solution to Eq.(1.7) in the back of the wavefront, we first present a lemma.

**Lemma 3.2.** Let $u$ be the solution of Eq.(1.7) with the Heaviside function $H$ as the initial condition. Suppose that the noise intensity satisfies $k^2 < 2r/K$. Then for any $h > 0$, there holds

$$\lim_{t \to \infty} \frac{1}{t} \log u(t, \gamma t) \geq -h \quad (3.4)$$

almost surely, where $\gamma$ is given in (1.8).

**Proof.** For any $h > 0$, take $\epsilon$ sufficiently small such that $(3 + k)\epsilon \leq h$. Then there exists $\Omega' \subset \Omega$ satisfying $P(\Omega') \geq 1 - e^{-\epsilon^2t/2}$, such that (3.1) holds for $\omega \in \Omega'$. To prove (3.4), it is sufficient to show that there exists a $T > 0$, such that for any $t \geq T$ and $\omega \in \Omega'$, the inequality

$$u(t, \gamma t, \omega) \geq e^{-ht}$$

holds. To make a contradiction, we suppose that for any $T > 0$, there exist $t' \geq T$ and $\omega' \in \Omega'$, such that

$$u(t', \gamma t', \omega') \leq e^{-ht'}.$$

If the noise intensity $k$ satisfies $k^2 \leq r/K$, then

$$f(0) \geq f(u(t', \gamma t', \omega')) \geq f(e^{-ht'}) \to f(0)$$

as $T \to \infty$. If the noise intensity $k$ satisfies $r/K < k^2 < 2r/K$, then we can take a sufficiently large $T$ such that $e^{-ht'} \leq K/2 - r/k^2$ and

$$f(0) \leq f(u(t', \gamma t', \omega')) \leq f(e^{-ht'}) \to f(0)$$

as $T \to \infty$. This implies that

$$\lim_{T \to \infty} \frac{1}{t' \int_0^{t'} f(u(s, \gamma t' + \sqrt{2}W_{t'-s}, \omega')) ds} = f(0)$$

as $T \to \infty$. 


where we have used the fact $\lim_{t \to \infty} \sup_{0 \leq s \leq t} |\bar{W}_s|/t = 0$ almost surely. Taking a sufficiently large $T$ such that

$$\int_0^{t'} f(u(s, \gamma t' + \sqrt{2W_{t'-s}}, \omega')) ds \geq f(0)t' - \epsilon t'$$

holds as $t' \geq T$. By the Feynman-Kac formula detailed by (2.3), we have

$$u(t', \gamma t', \omega') \geq \mathbb{E}\left[ H(\gamma t' + \sqrt{2W_{t'}}) \exp\left( \int_0^{t'} f(u(s, \gamma t' + \sqrt{2W_{t'-s}}, \omega')) ds \right) e^{-\epsilon t'} \right]$$

\begin{align*}
&\geq \mathbb{E}[H(\gamma t' + \sqrt{2W_{t'}})]e^{f(0)t' - \epsilon t' - \epsilon t'}
&= \mathbb{P}(\sqrt{2W_{t'}} < -\gamma t')e^{f(0)t' - \epsilon t' - \epsilon t'},
\end{align*}

where

$$\mathbb{P}(\sqrt{2W_{t'}} < -\gamma t') = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2W_{t'}}}^\infty \exp\left( -\frac{x^2}{2} \right) dx.$$  

Again take a sufficiently large $T$ such that $\gamma \geq \sqrt{\pi}(\gamma^2 + 4\epsilon)e^{-t'}$ and $1 \geq \sqrt{s}e^{-s}$ holds for $s \geq t'$. Then we have

$$\mathbb{P}(\sqrt{2W_{t'}} < -\gamma t') = -\frac{1}{\sqrt{2\pi}} \int_{\sqrt{2W_{t'}}}^\infty \frac{d}{ds} \left( \int_s^\infty \exp\left( -\frac{x^2}{2} \right) dx \right) ds$$

\begin{align*}
&= -\frac{\gamma}{4\sqrt{\pi}} \int_{\sqrt{2W_{t'}}}^\infty \frac{1}{\sqrt{s}} \exp\left( -\frac{\gamma^2 s}{4} \right) ds
&\geq -\frac{\gamma}{4\sqrt{\pi}} \int_{\sqrt{2W_{t'}}}^\infty \exp\left( -\frac{\gamma^2 s - \epsilon s}{4} \right) ds
&= \frac{\gamma}{\sqrt{\pi}(\gamma^2 + 4\epsilon)} \exp\left( -\frac{\gamma^2 t' - \epsilon t'}{4} \right)
&\geq \exp\left( -\frac{\gamma^2 t'}{4} - 2\epsilon t' \right).
\end{align*}

The combination of the above analysis implies

$$u(t', \gamma t', \omega') \geq \exp\left( -\frac{\gamma^2 t'}{4} + f(0)t' - 3\epsilon t' - \epsilon kt' \right) \geq e^{-3\epsilon t' - \epsilon kt'} \geq e^{-ht'}.$$

which contradicts to the hypothesis. \hfill \Box

Then, we use the following lemma to summarize the asymptotic behavior of solution to Eq. (1.7) in the back of the wavefront.

**Lemma 3.3.** Let $u$ be the solution of Eq. (1.7) with the Heaviside function $H$ as the initial condition. Suppose that the noise intensity satisfies $k^2 < 2\epsilon/K$. Then there holds

$$\inf_{x \leq (\gamma - h)t} u(t, x) \to K \text{ as } t \to \infty \quad (3.5)$$

almost surely for any $h > 0$, where $\gamma$ is given in (1.8).

**Proof.** For any $\epsilon > 0$, there exists $\Omega^\epsilon \subset \Omega$ satisfying $\mathbb{P}(\Omega^\epsilon) \geq 1 - e^{-\epsilon^2 t/2}$, such that (3.1) holds for $\omega \in \Omega^\epsilon$. To prove (3.5), it is sufficient to show that for any $\lambda > 0$, $\omega \in \Omega^\epsilon$, there exists a $T > 0$, such that

$$\inf_{x \leq (\gamma - h)t} u(t, x, \omega) \geq (1 - \lambda)K. \quad (3.6)$$
holds for \( t \geq T \). To obtain a contradiction, we suppose that (3.6) is false, i.e., for any \( T > 0 \), there exists \( t' > T \), \( \omega' \in \Omega_0 \), and \( x' \leq (\gamma - h)t' \), such that
\[
u(t', x', \omega') < (1 - \lambda)K.
\]

Since \( u(t, x) \) is decreasing with respect to \( x \), then we have
\[
u(t', (\gamma - h)t', \omega') \leq u(t', x', \omega') < (1 - \lambda)K. \quad (3.7)
\]

Denote by
\[
\theta(r) = (t' - r, (\gamma - h)t' + \sqrt{2}W_r), \quad r \leq t',
\]
and
\[
D = \{(s, x) : s \geq 0, x \leq \gamma s, u(s, x) \leq (1 - \lambda/2)K\}.
\]

Then, (3.7) suggests
\[
u(t', (\gamma - h)t', \omega') \leq (1 - \lambda/2)K,
\]
which implies \( \theta(0) \in D \).

Define a stopping time as
\[
\tau = \inf\{r \geq 0 : \theta(r) \notin D\} \land t'.
\]

The process \( \theta(r) \) may exit from the boundary \( x = \gamma t \), which is denoted by \( \partial D_1 \), or from the remaining part of the boundary, denoted by \( \partial D_2 \). Take a \( h \) satisfying \( \gamma h < h \), and define
\[
\hat{\Omega}_1 = \{\hat{\omega} \in \hat{\Omega} : \theta(\tau) \in \partial D_1, \tau \notin [ht', t'/2]\},
\]
\[
\hat{\Omega}_2 = \{\hat{\omega} \in \hat{\Omega} : \theta(\tau) \in \partial D_1, \tau \in [ht', t'/2]\},
\]
and
\[
\hat{\Omega}_3 = \{\hat{\omega} \in \hat{\Omega} : \theta(\tau) \in \partial D_2\}.
\]

By formula (2.3) and using the strong Markov property, we have
\[
u(t', (\gamma - h)t', \omega') = \mathbb{E}\left[\nu(\theta(\tau), \omega') \exp\left(\int_0^\tau f(\nu(u(s), \omega'))ds\right)\right.
\times \exp\left(k\int_0^\tau (K - \nu(u\theta(t' - s), \omega'))dW_s\right)\biggr]\biggr.
\[
= 3\sum_{i=1}^3 \nu_i(t', (\gamma - h)t', \omega'),
\]
where for \( i = 1, 2, 3 \), and
\[
u_i(t', (\gamma - h)t', \omega') = \mathbb{E}\left[\chi_{\Omega_i} \nu(\theta(\tau), \omega') \exp\left(\int_0^\tau f(\nu(u(s), \omega'))ds\right)\right.
\times \exp\left(k\int_0^\tau (K - \nu(u\theta(t' - s), \omega'))dW_s\right)\biggr]\biggr]
\[
= \mathbb{E}\left[\chi_{\Omega_2} \nu(\theta(\tau), \omega') \exp\left(f\left(\left(1 - \frac{\lambda}{2}\right)K\right)\tau - \epsilon k\tau\right)\biggr]\biggr]
\[
\geq \mathbb{P}(\Omega_2) \exp\left(f\left(\left(1 - \frac{\lambda}{2}\right)K\right)\tau - \epsilon k\tau - \epsilon\right).
\]
Indeed, we can take a sufficiently small $\epsilon$ such that $\epsilon(k + 1) < f((1 - \lambda/2)K)$, and a sufficiently large $T$ such that $\exp((1 - \lambda/2)K - \epsilon k T) > K$. This implies
\[ u_2(t', (\gamma - h)t', \omega') \geq \tilde{P}(\hat{\Omega}_2)K. \] (3.8)
For $u_3(t', (\gamma - h)t', \omega')$, we have
\[ u_3(t', (\gamma - h)t', \omega') \geq \tilde{P}[\chi_{\Omega_3}](1 - \frac{\lambda}{2})K \exp \left( f\left( \left(1 - \frac{\lambda}{2}\right)K \right) \tau - \epsilon k \tau \right). \]
According to the above setting of $\epsilon$ and $T$, we can further have
\[ u_3(t', (\gamma - h)t', \omega') \geq \tilde{P}(\hat{\Omega}_3)\left(1 - \frac{\lambda}{2}\right)K. \] (3.9)
On the other hand, in the region $\hat{\Omega}_1$, if $\tau < \hat{h}t'$, then $\theta(r)$ will meet the line $x = \gamma s$ at a time $\tau < \hat{h}t'$. This suggests
\[ (\gamma - h)t' + \sqrt{2\hat{W}_\tau} = \gamma(t' - \tau) \geq \gamma(1 - \hat{h})t', \]
which is equivalent to $\sqrt{2\hat{W}_\tau} \geq (h - \gamma\hat{h})t'$. Then by the Doob’s inequality, we have
\[ \tilde{P}(\tau < \hat{h}t', (\gamma - h)t' + \sqrt{2\hat{W}_\tau} = \gamma(t' - \tau)) \leq \tilde{P}(\sqrt{2\hat{W}_\tau} \geq (h - \gamma\hat{h})t') \leq e^{-\frac{(h - \gamma\hat{h})^2t'}{4}}. \]
Similarly, if $\tau > t'/2$, then $\theta(r)$ will meet the line $x = \gamma s$ at a time $\tau > t'/2$. This suggests
\[ (\gamma - h)t' + \sqrt{2\hat{W}_\tau} = \gamma(t' - \tau) \leq \frac{\gamma t'}{2}, \]
which is equivalent to $\sqrt{2\hat{W}_\tau} \leq -\gamma t'/2 + \hat{h}t'$. Again by the Doob’s inequality, we have
\[ \tilde{P}(\tau > t'/2, (\gamma - h)t' + \sqrt{2\hat{W}_\tau} = \gamma(t' - \tau)) \leq \tilde{P}(\sqrt{2\hat{W}_\tau} \leq -\gamma t'/2 + \hat{h}t') \leq e^{-\frac{(\gamma - \gamma\hat{h})^2t'}{4}}. \]
Now, we take a sufficiently large $T$ such that
\[ e^{-\frac{(h - \gamma\hat{h})^2t'}{4}} + e^{-\frac{(\gamma - \gamma\hat{h})^2t'}{4}} \leq \lambda, \]
which implies that $\tilde{P}(\hat{\Omega}_1) \leq \lambda$. Then the combination of (3.8) and (3.9) yields
\[ u(t', (\gamma - h)t', \omega') = \sum_{i=1}^{3} u_i(t', (\gamma - h)t', \omega') \geq \left(1 - \frac{\lambda}{2}\right)K(1 - \tilde{P}(\hat{\Omega}_1)) \geq (1 - \lambda)K, \]
which contradicts to (3.7). Thus, (3.6) holds.

3.3. Existence of stochastic traveling waves. The existence of stochastic traveling waves can be guaranteed following the theoretical framework in Ref.[38]. Here we gives a proof for this existence. For a fixed $a \in (0, K)$, define a wavefront marker as
\[ \gamma(t) = \inf\{x : u(t, x) < a\}. \]
In addition, define a centered solution as
\[ \tilde{u}(t, x) = u(t, x + \gamma(t)). \]
and the space as
\[ \mathcal{S}_c = \{\varphi : \mathbb{R} \to [0, K] \text{ decreasing and right continuous}\}. \]
Then $S_c$ is compact under the $L_{1,\text{loc}}$ topology. The centered solution $\tilde{u}(t)$, which is considered as random variable taking values in $S_c$, converges in distribution to a limit law $\nu_c \in \mathcal{M}(S_c)$, as $t \to \infty$, where $\mathcal{M}(S_c)$ is the space of Borel probability measures on $S_c$ with the topology of weak convergence (Proposition 10 in Ref. [38]).

According to the asymptotic behaviors in the front and back of the wavefront, specified in Lemma 3.1 and Lemma 3.3, we can rule out the possibility that the wavefront gets wider and wider and that the limit $\nu_c$ is concentrated on the flat profile. Let $\nu$ be the restriction of $\nu_c$ to $S$. The fact of $\mathcal{L}(\tilde{u}(t)) \to \nu$ in $\mathcal{M}(S_c)$ implies that $\mathcal{L}(\tilde{u}(t)) \to \nu$ in $\mathcal{M}(S)$. By the Dynkin criterion, which gives a simple transition kernel condition, i.e., a function of a Markov process is still Markovian, we know that the centered solution $\tilde{u}(t)$ is a Markov process. Then we let $P_t(\varphi, d\psi)$ be the Markov transition kernel for a solution of Eq. (1.7) on $S$ and $\tilde{P}_t(\varphi, d\psi)$ be the Markov transition kernels for the centered solution on $S_0$, where

$$S_0 = \{ \varphi \in S : \inf \{ x : \varphi(x) < a \} = 0 \}.$$ 

Denote by $P_t$ ($\tilde{P}_t$) by the associative semigroup generated by these kernels acting on measurable function $F : S \to \mathbb{R}$ ($F : S_0 \to \mathbb{R}$) and $P^*_t$ ($\tilde{P}^*_t$) by the dual semigroups acting on $\mathcal{M}(S)$ ($\mathcal{M}(S_0)$). The centered law $\tilde{\nu}$, which is the image of $\nu$ under the centering map, changes only on $S_0$. Take $F : S \to \mathbb{R}$, which is bounded, continuous and translationally invariant. Then the Feller property and translational invariance of the solution imply that $P_tF$ remains bounded, continuous and translationally invariant. Let $F_0$ be the restriction of $F$ to $S_0$. The translational invariance of $F$ implies $\tilde{P}_tF_0(\varphi) = P_tF(\varphi)$. Then we have

$$\int_{S_0} F_0d(\tilde{P}^*_s\tilde{\nu}) = \int_{S_0} \tilde{P}_sF_0d\tilde{\nu}$$

$$= \int_S P_sFd\nu \text{ (by translation invariance of } P_tF)$$

$$= \lim_{t \to \infty} \int_S P_sFd\mathcal{L}(\tilde{u}(t)) \text{ (by the convergence to } \nu)$$

$$= \lim_{t \to \infty} \int_S P_sFd\mathcal{L}(u(t)) \text{ (by translation invariance of } P_sF)$$

$$= \lim_{t \to \infty} \int_S P_sFd\mathcal{L}(u(t + s)) \text{ (by the Markov property of } u)$$

$$= \lim_{t \to \infty} \int_S P_sFd\mathcal{L}(\tilde{u}(t + s)) \text{ (by translation invariance of } F)$$

$$= \int_S Fd\nu$$

$$= \int_{S_0} F_0d\tilde{\nu}.$$ 

This equality can be extended to the case of all bounded measurable and translationally invariant functions $F : S \to \mathbb{R}$. Then taking $F(\varphi) = \chi(\tilde{\varphi} \in A)$ for some measurable set $A \subseteq S$, we have

$$\tilde{P}^*_t\tilde{\nu}(A \cap S_0) = \int_{S_0} \chi(\tilde{\varphi} \in A)d\tilde{P}^*_s\tilde{\nu} = \int_{S_0} \chi(\tilde{\varphi} \in A)d\tilde{\nu} = \tilde{\nu}(A \cap S_0),$$

which yields $\tilde{P}^*_t\tilde{\nu} = \tilde{\nu}$, showing that $\tilde{\nu}$ is the law of a stochastic traveling wave. It is not difficult to show that $\nu$ changes also only $S_0$ and $\tilde{\nu} = \nu$. 


4. **Asymptotic behaviors of Eq.(1.4).** In this section, we remove the decrease constraint on the growth function \( c(u) \) in Eq.(1.4) and analyze asymptotic behaviors. The existence and uniqueness of solution to Eq.(1.4) can be referred to Ref.[29]. Moreover, properties similar to (i) and (ii) in Proposition 1 hold also for Eq.(1.4) if the state space \( S \) is replaced with \( C^{0,2}([0, \infty) \times \mathbb{R}) \). The Feynman-Kac formula for Eq.(1.4) can be written as (see Ref.[40])

\[
  u(t, x) = \mathbb{E}\left[u(0, x + \sqrt{2\hat{W}_t}) \exp\left(\int_0^t c(u(t-s, x + \sqrt{2\hat{W}_s}))ds - \frac{k^2t}{2} + kW_t\right)\right], \tag{4.1}
\]

The noise intensity \( k \) is supposed to satisfy \( k^2 < 2c(0) \) (see Refs.[13, 14, 29, 30]). The following lemma summarizes the asymptotic behavior of the front of the wavefront.

**Proposition 2.** Let \( u \) be a solution of Eq.(1.4) with the Heaviside function \( H \) as the initial condition. Suppose that the noise intensity \( k \) satisfies \( k^2 < 2c(0) \). Then there holds

\[
  \sup_{x \geq (\gamma + h)t} u(t, x) \to 0 \text{ as } t \to \infty
\]

almost surely for any \( h > 0 \), where \( \gamma \) is given by

\[
  \gamma = \sqrt{2(c(0) - k^2)}. \tag{4.2}
\]

**Proof.** Since the growth function \( c(u) \) is continuous, i.e., for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
  |c(u) - c(0)| \leq \epsilon \text{ for } 0 \leq u \leq \delta.
\]

Define \( \alpha(t) = \sup_{0 \leq s \leq t} \beta(t-s) \), where \( \beta(t-s) \) satisfies

\[
  \beta(t-s) = \inf\{x : u(t-s, x + \sqrt{2\hat{W}_s}) \leq \delta\}.
\]

Then for \( x \geq \alpha(t) \), we have \( u(t-s, x + \sqrt{2\hat{W}_s}) \leq \delta \) and

\[
  c(u(t-s, x + \sqrt{2\hat{W}_s})) \leq c(0) + \epsilon.
\]

By the Doob’s inequality, there exists \( \Omega'_t \subseteq \Omega \) satisfying \( \mathbb{P}(\Omega'_t) \geq 1 - e^{-c^2/2} \) such that \( \hat{W}_t(\omega) \leq \epsilon t \) holds for \( \omega \in \Omega'_t \). By the Feynman-Kac formula given by (4.1), we have

\[
  u(t, x) \leq \mathbb{E}[H(x + \sqrt{2\hat{W}_t})] \exp\left(c(0)t - \frac{k^2t}{2} + k\epsilon t + \epsilon t\right)
\]

\[
  = \mathbb{P}(x + \sqrt{2\hat{W}_t} < 0) \exp\left(c(0)t - \frac{k^2t}{2} + k\epsilon t + \epsilon t\right)
\]

\[
  \leq \exp\left(-\frac{x^2}{4t} + c(0)t - \frac{k^2t}{2} + k\epsilon t + \epsilon t\right)
\]

for \( \omega \in \Omega'_t \). For any \( h \), take a sufficiently small \( \epsilon \) such that \( c(k+1) < \gamma h/2 + h^2/4 \). Then if \( x \geq (\gamma + h)t \cap \alpha(t) \), then we have

\[
  u(t, x) \leq \exp\left(-\frac{\gamma h}{2} t - \frac{h^2}{4} t + c\epsilon t + \epsilon t\right) \to 0 \text{ as } t \to \infty. \tag{4.3}
\]

Similarly to the proof of Lemma 3.1, we claim that for sufficiently large \( t \), there holds \( (\gamma + h)t \geq \alpha(t) \) almost surely. Indeed, by (4.3), it is not difficult to see that there exists \( T > 0 \) such that for \( t \geq T \), there holds

\[
  u(t, (\gamma + h)t, \omega) \leq \delta = u(\beta(t), \omega),
\]
which suggests that $\beta(t) \leq (\gamma + h)t$ holds for $t \geq T$. This implies that $\lim_{t \to \infty} \beta(t)/t \leq \gamma + h$ almost surely. According to the definition of $\alpha(t)$, there holds

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} \leq \lim_{t \to \infty} \frac{\beta(t)}{t} \leq \gamma + h$$

almost surely.

The asymptotic behavior of the back of the wavefront for Eq.(1.4) with the decreasing growth function is established in Refs.[29, 30], that is, almost surely.

$$\sup_{x \geq (\gamma - h)t} |u(t, x) - Z(t)| \to 0 \text{ as } t \to \infty$$

(4.4)

almost surely for any $h > 0$, where $\gamma$ is given by (4.2) and $Z(t)$ is a stationary solution of the corresponding stochastic differential equation of the form

$$dZ = c(Z)Zdt + kZdW_t.$$  

(4.5)

However, we find that if the decrease constraint on the growth function $c(u)$ is removed, then the asymptotic behavior no longer holds. We can show that if the asymptotic behavior given by (4.4) holds, the growth function $c(u)$ has to be decreasing. We can claim that the decrease constraint on the growth function is a sufficient and necessary condition to ensure the asymptotic behavior of the back of the wavefront. Indeed, define $v$ as $v = u/Z$, where $u$ is the solution of Eq.(1.4). Then $v$ satisfies

$$v_t = v_{xx} + (c(Z(t)v) - c(Z(t)))v = v_{xx} - c'(Z_v(t))Z(t)(1 - v)v,$$

where the process $Z_v(t)$ satisfies $Z(t)v \leq Z_v(t) \leq Z(t)$. Moreover, $0 \leq v \leq 1$ holds if the initial data satisfies $0 \leq v(0) \leq 1$. Then the asymptotic behavior given by (4.4) is equivalent to

$$\inf_{x \geq (\gamma - h)t} v(t, x) \to 1 \text{ as } t \to \infty$$

almost surely for any $h > 0$, which suggests that for some constant $C > 0$ taken as the initial data of $v$, $v(t)$ is spatially homogeneous and converges to 1 almost surely. Meanwhile, $v(t)$ can be expressed as

$$v(t) = \frac{C \exp(-\int_0^t c'(Z_v(s))Z(s)ds)}{1 + C \exp(-\int_0^t c'(Z_v(s))Z(s)ds)}.$$

Thus the fact that $v(t) \to 1$ almost surely as $t \to \infty$ implies that $-\int_0^t c'(Z_v(s))Z(s)ds \to \infty$ almost surely as $t \to \infty$, which further implies that at least $-c'(Z(t)) > 0$ for sufficiently large $t$. On the other hand, suppose that the growth function $c(u)$ satisfies the constraint given in Ref.[40], i.e., there exist constants $a$, $b$ and $q > 0$ such that $c(u) \leq a - bu^q$, which changes the form of the constraint on $c(u)$ given in Refs.[29, 30]. Then according to the theory of random dynamic system, we know that the stably stationary solution $Z(t)$ considered on $(0, \infty)$ is ergodic. Therefore, we conclude that the growth function $c(u)$ should satisfy $c'(u) \leq 0$, i.e., $c(u)$ should be decreasing as $u \in (0, \infty)$.

5. Conclusion. In this paper, we have studied asymptotic behaviors and stochastic traveling waves in the system of the stochastic Fisher-KPP equation perturbed by growth rate noise, i.e., Eq.(1.7). The structural characteristics of this equation is that has a nonlinear noise term, $u(K - u)dW_t$, which is different from the stochastic Fisher-KPP equation perturbed by environmental noise $udW_t$ that is linear. The nonlinear noise term $u(K - u)dW_t$ with the volatility of quadratic
Appendix A. Proof of Lemma 2.2. For $s \in I_k(t) = [k \epsilon \land t, (k + 1)\epsilon \land t]$, define $s_-(\epsilon) = k \epsilon$ and $s_+ (\epsilon) = (k + 1)\epsilon$. For any $u : [0, \infty) \times \mathbb{R} \to [0, K]$, define $u^-$ by $u^-(t, x) = u(t^-(\epsilon), x)$. Define auxiliary equations as follows:

\begin{align}
\dot{d}u &= \bar{u}_{xx} dt + ru^- (K - u^-) dt + Ku^- (K - u^-) dW_t, \quad (A.1) \\
\dot{d}u &= \hat{u}_{xx} dt + ru^- (K - u^-) dt + Ku^- (K - u^-) dW_t, \quad (A.2) \\
\dot{d}u &= \hat{u}_{xx} dt + rKu^- (1 - u^-) dt + Ku^- (1 - u^-) dW_t. \quad (A.3)
\end{align}

Suppose that Eqs. (1.7), (2.1), (A.1), (A.2) and (A.3) start at the same initial conditions in the state space $S$. The existence and uniqueness of solution hold for Eqs. (A.1)-(A.3). By the triangle inequality, we can write

$$|u - u^-|^2 \leq C(|u - \bar{u}|^2 + |\bar{u} - \hat{u}|^2 + |\hat{u} - \hat{u}|^2 + |\hat{u} - u^-|^2).$$

Our idea is to show that each term on the right hand side is either of order $\epsilon$ or an integral copy of the left hand side, and to use a Gronwall argument to prove the left hand side convergences to zero. This can be done by a string of lemmas. In the following argument, constant $C$ may change line on line, but its specific value makes no difference.
Lemma A.1. Let $u_\epsilon$ be a solution of Eq. (2.1). Then there holds
\[
\mathbb{E}[|u_\epsilon(t, x) - u_\epsilon(t^-, x)|^2] \leq C \left( \epsilon \ln^2 \epsilon + \epsilon + \left( \frac{\epsilon}{t^-} \right)^2 \right),
\]
Proof. We can write
\[
|u_\epsilon(t, x) - u_\epsilon(t^-, x)|^2 \leq C(I_1 + I_2 + I_3 + I_4 + I_5),
\]
where
\[
I_1 = \left| \int_{\mathbb{R}} u_\epsilon(0, y) (G_t(y) - G_t^-(y)) dy \right|^2,
\]
\[
I_2 = \left| \int_0^{t^-} \int_{\mathbb{R}} f(u_\epsilon(s, y)) (G_{t-s}(x-y) - G_{t^-}(x-y)) dy ds \right|^2,
\]
\[
I_3 = \left| \int_0^{t^-} \int_{\mathbb{R}} u_\epsilon(K - u_\epsilon) (G_{t-s}(x-y) - G_{t^-}(x-y)) dy dW_t \right|^2,
\]
\[
I_4 = \left| \int_{t^-}^t \int_{\mathbb{R}} f(u_\epsilon(s, y)) G_{t-s}(x-y) dy ds \right|^2,
\]
\[
I_5 = \left| \int_{t^-}^t \int_{\mathbb{R}} u_\epsilon(K - u_\epsilon) G_{t-s}(x-y) dy dW_t \right|^2,
\]
in which $G$ is a Green function and
\[
f(u) = ru(K - u) - \frac{t^2}{2} u(K - u)(K - 2u).
\]
We will take each of these terms to obtain the following estimate
\[
\int_{\mathbb{R}} |G_{t-s}(y) - G_{t^-}(y)| dy \leq C \left| \frac{t - t^-}{t^- - s} \right| \wedge 1
\]
for all $0 \leq s < t^- \leq t < \infty$. For $I_1$, it is easy to get
\[
I_1 \leq \left| \int_{\mathbb{R}} |G_{t-s}(y) - G_{t^-}(y)| dy \right| \leq C \left( \frac{\epsilon}{t} \right)^2.
\]
For $I_2$, we have
\[
I_2^{1/2} \leq C \int_0^{t^-} \int_{\mathbb{R}} |G_{t-s}(x-y) - G_{t^-}(x-y)| dy ds\]
\[
\leq C \int_0^{t^-} \frac{\epsilon}{t^- - s} ds + C \int_{t^-}^{t^- - \epsilon} ds\]
\[
\leq C \epsilon (- \ln \epsilon + \ln t^- + 1)\]
\[
\leq - C \epsilon \ln \epsilon.
\]
For $I_3$, we have

$$
E[I_3^2] \leq C \mathbb{E} \left[ \int_0^t |\bar{W}^*_t| \int_\mathbb{R} |G_{t-s}(x-y) - G_{t-s}(x-y)| dy ds \right]^2 \\
\leq C \int_0^t \int_0^t \mathbb{E} \left[ |\bar{W}^*_t||\bar{W}^*_t| \right] \left( \frac{\epsilon}{|t-s|} \right) \wedge 1 \cdot \left( \frac{\epsilon}{|t-r|} \right) \wedge 1 ds dr \\
\leq C \int_0^t \int_0^t \frac{1}{\epsilon} \left( \frac{\epsilon}{|t-s|} \right) \wedge 1 \cdot \left( \frac{\epsilon}{|t-r|} \right) \wedge 1 ds dr \\
\leq C \left( \int_0^{t-\epsilon} \frac{\epsilon}{|t-s|} ds + \int_{t-\epsilon}^t \frac{\epsilon}{ds} ds \right)^2 \\
\leq C \frac{1}{\epsilon} (-\epsilon \ln \epsilon + et^- + \epsilon)^2 \\
\leq C \epsilon \ln^2 \epsilon.
$$

It is easy to show that the remaining terms are of order $\epsilon$. \hfill \Box

**Lemma A.2.** Let $u$, $\bar{u}$ be the solutions of Eqs. (1.7) and (A.1) respectively with the same initial conditions. Then there holds

$$
E[|u(t, x) - \bar{u}(t, x)|^2] \leq C \sqrt{\epsilon}.
$$

**Proof.** The Green function’s representation gives

$$
|u(t, x) - \bar{u}(t, x)|^2 \leq C (I_1 + I_2),
$$

where

$$
I_1 = \left| \int_0^t \int_\mathbb{R} G_{t-s}(x-y)(u(K-u) - u^-(K-u^-)) dy ds \right|^2,
$$

$$
I_2 = \left| \int_0^t \int_\mathbb{R} G_{t-s}(x-y)(u(K-u) - u^-(K-u^-)) dy dW_t \right|^2.
$$

By Lemma A.1, there holds

$$
E[|u(K-u) - u^-(K-u^-)|^2] \leq C \left( \epsilon + \left( \frac{\epsilon}{s^-} \right)^2 \right), \quad (A.4)
$$

which implies by the Cauchy-Schwarz’s inequality that

$$
E[I_1] \leq C \mathbb{E} \left[ \int_0^t \left( \int_\mathbb{R} G_{t-s}(x-y)(u(K-u) - u^-(K-u^-)) dy \right)^2 ds \right] \\
\leq C \left( \int_0^t \left( \int_\mathbb{R} G_{t-s}(x-y)(u(K-u) - u^-(K-u^-)) dy \right)^2 ds \right) \\
\leq C \int_0^t \epsilon^2 ds \\
\leq C \epsilon \sqrt{\epsilon},
$$

(A.5)
For $I_2$, we have
\[
E[I_2] = \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)(u(K-u)-u^-(K-u^-))dy \right] \, ds \\
\leq \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)|u(K-u)-u^-(K-u^-)|^2 dyds \right] \\
\leq C \int_0^t \left( \epsilon + \left( \frac{s}{\epsilon} \right)^2 \right) ds \\
\leq C \sqrt{\epsilon}.
\]

Lemma A.3. Let $u_\epsilon$, $\tilde{u}$ be the solutions of Eqs. (2.1) and (A.3) respectively with the same initial conditions. Then there holds
\[
\mathbb{E}[|u_\epsilon(t, x) - \tilde{u}(t, x)|^2] \leq C \sqrt{\epsilon}.
\]

Proof. Again by the Green function’s representation, we can write
\[
|u_\epsilon(t, x) - \tilde{u}(t, x)|^2 \leq C(I_1 + I_2),
\]
where
\[
I_1 = \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)(f(u_\epsilon) - f(u^-))dyds \right|^2,
\]
\[
I_2 = \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)(u_\epsilon(K-u_\epsilon) - u^-(K-u^-))dydW_t^\epsilon \right|^2 \\
- \frac{k}{2} \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u^-\epsilon(K-u^-)(K-2u^-)dyds \right|^2.
\]

For $I_1$, along similar lines to the above estimate given in (A.5), it is not difficult to see that $\mathbb{E}[I_1]$ is of order $\sqrt{\epsilon}$. Now we focus on $I_2$, which is a key estimate in our approximation. We make use of the Taylor’s Theorem of the second order remainder. For $s \in I_{\epsilon}(t)$ and some $\eta$s lying between $s$ and $s^-$, we can write
\[
u_\epsilon(s)(K-u_\epsilon(s)) = u_\epsilon(s^-)(K-u_\epsilon(s^-)) + h_1(u_\epsilon(s^-))(s-s^-) + h_2(u_\epsilon(\eta_s))(s-s^-)^2,
\]
where
\[
h_1(u_\epsilon) = (K-2u_\epsilon)f(u_\epsilon) + k(K-2u_\epsilon)u_\epsilon(K-u_\epsilon)W_t^\epsilon
\]
and
\[
h_2(u_\epsilon) = q_1(u_\epsilon) + q_2(u_\epsilon)W_t^\epsilon + q_3(u_\epsilon)(W_t^\epsilon)^2,
\]
in which $q_1$, $q_2$, $q_3$ are polynomials of $u_\epsilon$, and their concrete expressions are complex but they make no difference in the estimate. Therefore, we omit the expressions here. Then we can write $I_2$ as
\[
I_2 \leq C(I_{21} + I_{22} + I_{23}),
\]
where
\[
I_{21} = \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)(K-2u_\epsilon^-)f(u^-)(s-s^-)dydW_t^\epsilon \right|^2,
\]
\[
I_{22} = \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)(K-2u^-\epsilon)(s-s^-)dy\left((s-s^-)(W_t^\epsilon)^2 - \frac{1}{2}\right)ds \right|^2,
\]
\[
I_{23} = \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)h_2(u_\epsilon(\eta_s))(s-s^-)^2dydW_t^\epsilon \right|^2.
\]
For $I_{21}$, we have

$$
E[I_{21}] = E \left[ \sum_{k=0}^{N} \int_{I_k(t)} \int_{\mathbb{R}} G_{t-s}(x-y) (K - 2u_\epsilon) f(u_\epsilon)(s-s^-) d\hat{W}_t^\epsilon ds \right]^2
$$

$$
\leq C \sum_{k=0}^{N} E \left[ \int_{I_k(t)} \int_{\mathbb{R}} G_{t-s}(x-y) f(u_\epsilon) d\hat{W}_t^\epsilon (s-s^-) ds \right]^2
\leq C \sum_{k=0}^{N} E \left[ \int_{I_k(t)} \left| \hat{W}_t^\epsilon \right| (s-s^-) ds \right]^2
\leq C \sum_{k=0}^{N} \int_{I_k(t)} (s-s^-)^2 ds
\leq C^2,
$$

where $N = \lfloor T/\epsilon \rfloor + 1$. Denote by $q(u_\epsilon)$ a polynomial of $u_\epsilon$. Then for $m \leq 3$, by the Cauchy-Schwarz’s inequality, we can have

$$
E \left[ \left| \int_{0}^{t} \int_{\mathbb{R}} G_{t-s}(x-y) q(u_\epsilon(\eta_s))(s-s^-)^2 d\hat{W}_t^\epsilon \right| \right]^2
\leq C E \left[ \sum_{k=0}^{N} \int_{I_k(t)} \int_{\mathbb{R}} G_{t-s}(x-y) q(u_\epsilon(\eta_s)) |dy(s-s^-)^2| \left| \hat{W}_t^\epsilon \right|^m ds \right]^2
\leq C \sum_{k=0}^{N} \left| \hat{W}_t^\epsilon \right|^{2m} \sum_{k=0}^{N} \int_{I_k(t)} (s-s^-)^2 ds \right]^2
\leq C \sum_{k=0}^{N} \left| \hat{W}_t^\epsilon \right|^{2m}
\leq C \epsilon^{-m-1}
\leq C \epsilon^{-m+4},
$$

which implies that $I_{23}$ is of order $\epsilon$. For $I_{22}$, note that

$$
E \left[ \int_{I_k(t)} (s-s^-)(\hat{W}_t^\epsilon)^2 - \frac{1}{2} ds \Bigg| \mathcal{F}_{k\epsilon} \right] = 0,
$$

from which together with the Cauchy-Schwarz’s inequality, we have

$$
E[I_{22}] = E \left[ \sum_{k=0}^{N} \int_{I_k(t)} \int_{\mathbb{R}} G_{t-s}(x-y) g(u_\epsilon) \left( (s-s^-)(\hat{W}_t^\epsilon)^2 - \frac{1}{2} \right) ds \right]^2
= \sum_{k=0}^{N} E \left[ \int_{I_k(t)} \int_{\mathbb{R}} G_{t-s}(x-y) g(u_\epsilon) \left( (s-s^-)(\hat{W}_t^\epsilon)^2 - \frac{1}{2} \right) ds \right]^2
\leq C \sum_{k=0}^{N} \int_{I_k(t)} (s-s^-)(\hat{W}_t^\epsilon)^2 - \frac{1}{2} ds \right]^2
$$
Finally, it is easy to get
\[
E[|\bar{u}(t, x) - \tilde{u}(t, x)|^2] \leq C \epsilon.
\]
where \( g(u) = (K - 2u)u(K - u) \). Combining the estimates for \( I_{21}, I_{22}, I_{23} \) yields that \( I_2 \) is of order \( \epsilon \).

**Lemma A.4.** Let \( \bar{u}, \tilde{u} \) be the solutions of Eqs. (A.2) and (A.3) respectively with the same initial conditions. Then there holds
\[
E[|\bar{u}(t, x) - \tilde{u}(t, x)|^2] \leq C \epsilon.
\]

**Proof.** We write
\[
|\bar{u}(t, x) - \tilde{u}(t, x)|^2 \leq C \left| \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)u_e^-(K - u_e^-)dy(dW_t - dW_t^e) \right|^2.
\]
Since \( \int_{I_k(t)} (dW_t - dW_t^e) = 0 \), we have
\[
E\left[ \left| \int_0^t \int_{\mathbb{R}} (G_{t-s}(x-y) - G_{t-s}^e(x-y))u_e^-(K - u_e^-)dydW_t \right|^2 \right] 
\leq C E\left[ \left| \int_0^t \int_{\mathbb{R}} (G_{t-s}^e(x-y) - G_{t-s}^e(x-y))u_e^-(K - u_e^-)dydW_t \right|^2 \right] 
\]
\[
+ C E\left[ \left| \int_0^t \int_{\mathbb{R}} (G_{t-s}(x-y) - G_{t-s}^e(x-y))u_e^-(K - u_e^-)dydW_t^e \right|^2 \right].
\]
Finally, it is easy to get
\[
E[|\bar{u}(t, x) - \tilde{u}(t, x)|^2] \leq C \int_0^t \sup_{y \in \mathbb{R}} E[|u(s^-, y) - u_e(s^-, y)|^2] ds.
\]
Combing the estimates in Lemma A.1-A.4 gives
\[
\sup_{x \in \mathbb{R}} \mathbb{E}[|u(t, x) - u_\epsilon(t, x)|^2] \leq C \int_0^t \sup_{y \in \mathbb{R}} \mathbb{E}[|u(s^-, y) - u_\epsilon(s^-, y)|^2] ds + C \sqrt{\epsilon},
\]
which implies
\[
R_\epsilon(t) \leq C \int_0^t R_\epsilon(s) ds + C \sqrt{\epsilon},
\]
where
\[
R_\epsilon(t) = \sup_{s \leq t} \sup_{x \in \mathbb{R}} \mathbb{E}[|u(s, x) - u_\epsilon(s, x)|^2].
\]
Then by making use of Gronwall’s inequality, we can conclude that \(R_\epsilon(t) \to 0\) as \(\epsilon \to 0\).

Appendix B. Proof of Lemma 2.3. We can write
\[
\left| \int_0^t u_\epsilon \bar{W}_t^\epsilon ds - \int_0^t u dW_t - \frac{k}{2} \int_0^t u(K - u) ds \right|^2 \leq C(I_1 + I_2 + I_3 + I_4 + I_5 + I_6),
\]
where
\[
I_1 = \left| \int_0^t u_\epsilon \bar{W}_t^\epsilon ds - \int_0^t u_\epsilon \bar{W}_t^\epsilon ds - \frac{k}{2} \int_0^t u_\epsilon (K - u_\epsilon) ds \right|^2,
\]
\[
I_2 = \left| \int_0^t u_\epsilon (K - u_\epsilon) ds - \int_0^t u_\epsilon (K - u_\epsilon) ds \right|^2,
\]
\[
I_3 = \left| \int_0^t u_\epsilon (K - u_\epsilon) ds - \int_0^t u(K - u) ds \right|^2,
\]
\[
I_4 = \left| \int_0^t u_\epsilon \bar{W}_t^\epsilon ds - \int_0^t u \bar{W}_t^\epsilon ds \right|^2,
\]
\[
I_5 = \left| \int_0^t u \bar{W}_t^\epsilon ds - \int_0^t u dW_t \right|^2,
\]
\[
I_6 = \left| \int_0^t u dW_t - \int_0^t u dW_t \right|^2.
\]
For \(I_1\), we have
\[
I_1 \leq C(I_{11} + I_{12} + I_{13}),
\]
where
\[
I_{11} = \left| \int_0^t f(u_-)(s - s^-) \bar{W}_t^\epsilon ds \right|^2,
\]
\[
I_{12} = \left| \int_0^t u_\epsilon (K - u_\epsilon) \left( (s - s^-)(\bar{W}_t^\epsilon)^2 - \frac{1}{2} \right) ds \right|^2,
\]
\[
I_{13} = \left| \int_0^t \frac{1}{2} \partial_t^2 u_\epsilon(\eta_\epsilon)(s - s^-)^2 \bar{W}_t^\epsilon ds \right|^2.
\]
Following the similar argument to Lemma A.3, we can show that \(I_1\) is of order \(\epsilon\).

Then by Lemma 2.2 and Lemma A.1, it is easy to show that the remaining terms are all of order \(\sqrt{\epsilon}\).
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