Hermite-Hadamard and Hermite-Hadamard-Fejér type Inequalities for Generalized Fractional Integrals

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Abstract
In this paper we obtain the Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for fractional integrals which generalize the two familiar fractional integrals namely, the Riemann-Liouville and the Hadamard fractional integrals into a single form. We prove that, in most cases, we obtain the Riemann-Liouville and the Hadamard equivalence just by taking limits when a parameter $\rho \to 1$ and $\rho \to 0^+$, respectively.

Keywords: Hermite-Hadamard Inequalities, Hermite-Hadamard-Fejér inequalities, Riemann-Liouville fractional integral, Hadamard fractional integral, Katugampola fractional integral, convexity

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1. Introduction

The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that $f$ is concave if $(-f)$ is convex.

Let $f : I \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},$$

which is known as the Hermite–Hadamard inequality [12]. In [10], Fejér developed the weighted generalization of the Hermite–Hadamard inequality given below.

**Theorem 1.1.** Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality

$$f \left( \frac{a + b}{2} \right) \int_a^b g(x)dx \leq \frac{1}{b - a} \int_a^b f(x)g(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx$$

holds, where $g : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric to $(a + b)/2$.

Since then, many researches generalized and extended the two inequalities [1] and [2]. For related results, for example, see [3, 4, 5, 23, 24, 40, 42] and the references therein. In [37], Sarikaya et al. generalized the Hermite–Hadamard type inequalities via Riemann–Liouville fractional integrals. Then in [13], Işcan extended Sarikaya’s results to Hermite–Hadamard–Fejér type inequalities for fractional integrals. Further results involving the two inequalities in question with applications to fractional integrals can be found, for example, in [6, 14, 37, 38] and the references therein.

In [16], the second author introduces an Erdélyi-Kober type fractional integral operator and uses that integral to define a new fractional derivative in [17], which generalizes the Riemann-Liouville and the Hadamard fractional derivatives to a single form and argued that it is not possible to derive the Hadamard equivalence operators from the corresponding Erdélyi-Kober type operators, thus making the new derivative more appropriate for modeling certain phenomena which undergo bifurcation-like behaviors. For further properties...
of the Erdélyi-Kober operators, the interested reader is referred to, for example, [21, 22, 36]. According to the literature, the newly defined fractional operators are known as the Katugampola fractional integral and derivatives, respectively. For consistency, we use the same name for those operators in question. It can be shown that the derivatives in question satisfy the fractional derivative criteria (test) given in [20, 32]. These operators have applications in fields such as in probability theory [1], theory of inequalities [2, 40, 43], variational principle [2], numerical analysis [3], and Langevin equations [30]. A Caputo-type modification of the operator in question can be found in [4]. The interested reader is referred, for example, to [5, 11, 19, 27–31, 34, 35] for further results on these and similar operators. The Mellin transforms of the generalized fractional integrals and derivatives defined in [16 and 17, respectively, are given in [18]. The same reference also studies a class of sequences that are closely related to the Stirling numbers of the 2nd kind. The \( \rho \)-Laplace and \( \rho \)-Fourier transforms of the Katugampola fractional operators are given in [5].

In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

**Definition 1.2** ([33]). Let \( \alpha > 0 \) with \( n - 1 < \alpha \leq n, \ n \in \mathbb{N}, \) and \( a < x < b \). The left- and right-side Riemann–Liouville fractional integrals of order \( \alpha \) of a function \( f \) are given by

\[
J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) \, dt \quad \text{and} \quad J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t) \, dt
\]

respectively, where \( \Gamma(\cdot) \) is the Euler's gamma function defined by

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt.
\]

**Definition 1.3** ([36]). Let \( \alpha > 0 \) with \( n - 1 < \alpha \leq n, \ n \in \mathbb{N}, \) and \( a < x < b \). The left- and right-side Hadamard fractional integrals of order \( \alpha \) of a function \( f \) are given by

\[
H^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left( \ln \frac{x}{t} \right)^{\alpha-1} \frac{f(t)}{t} \, dt \quad \text{and} \quad H^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \left( \ln \frac{t}{x} \right)^{\alpha-1} \frac{f(t)}{t} \, dt.
\]

In [37], Sarikaya et al. established the Hermite–Hadamard inequalities via the Riemann–Liouville fractional integrals as follows.

**Theorem 1.4.** Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \( [a, b] \), then the following inequalities hold

\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left( J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a) \right) \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

**Theorem 1.5.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \( (a, b) \) with \( a < b \). If \( |f'| \) is a convex function on \( [a, b] \), then the following inequality holds for \( \alpha > 0 \),

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left( J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a) \right) \right| \leq \frac{b-a}{2(\alpha+1)} \left( 1 - \frac{1}{2^\alpha} \right) \left( |f'(a)| + |f'(b)| \right).
\]

Further, in [13], İscan extended these results to Hermite–Hadamard–Fejér type inequalities as follows.

**Theorem 1.6.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function with \( a < b \) and \( f \in L[a, b] \). If \( g : [a, b] \to \mathbb{R} \) is non-negative, integrable and symmetric to \( (a+b)/2 \), then the following inequalities for fractional integrals hold

\[
f \left( \frac{a+b}{2} \right) \left| J^{\alpha}_{a+} g(b) + J^{\alpha}_{b-} g(a) \right| \leq \left| J^{\alpha}_{a+} (gf)(b) + J^{\alpha}_{b-} (gf)(a) \right| \leq \frac{f(a) + f(b)}{2} \left| J^{\alpha}_{a+} g(b) + J^{\alpha}_{b-} g(a) \right|
\]

with \( \alpha > 0 \).
Theorem 1.7. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) and \( f' \in L[a, b] \) with \( a < b \). If \( |f'| \) is convex on \([a, b]\) and \( g : [a, b] \rightarrow \mathbb{R} \) is continuous and symmetric to \((a + b)/2\), then the following inequality holds
\[
\left| \frac{f(a) + f(b)}{2} - \left[ J_{a+}^\alpha g(b) + J_{a-}^\alpha g(a) \right] - \left[ J_{b+}^\alpha (gf)(b) + J_{b-}^\alpha (gf)(a) \right] \right| \leq \frac{(b - a)^{\alpha + 1}}{\Gamma(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \|f'(a)\| + \|f'(b)\|
\]
with \( \alpha > 0 \), where \( \|g\|_{\infty} = \sup_{x \in [a, b]} |g(x)| \).

Recently, Katugampola introduced a new fractional integral that generalizes the Riemann–Liouville and the Hadamard fractional integrals into a single form (see [16][18]). The purpose of this paper is to derive Hermite–Hadamard type and Hermite–Hadamard–Fejér type inequalities using the Katugampola fractional integrals. Since it is a generalization of Hadamard fractional integral, we can also get the inequalities for Hadamard fractional integral in some cases by just taking limits, while we obtain Riemann-Liouville equivalence by taking limits in all the cases.

Definition 1.8 ([17]). Let \([a, b] \subset \mathbb{R}\) be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order \( \alpha > 0 \) of \( f \in X^\rho(a, b) \) are defined by [17],
\[
\rho I_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1} f(t) dt}{(x - t)^{1-\alpha}} \quad \text{and} \quad \rho I_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1} f(t) dt}{(t - x)^{1-\alpha}}
\]
with \( a < x < b \) and \( \rho > 0 \), if the integrals exist.

Theorem 1.9 ([17]). Let \( \alpha > 0 \) and \( \rho > 0 \). Then for \( x > a \),
1. \( \lim_{\rho \to 1} \rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x) \),
2. \( \lim_{\rho \to \rho^+} \rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x) \).

Similar results also hold for right-sided operators.

2. Main Results

First we generalize Sarikaya’s results [37] of the Hermite-Hadamard’s inequalities for the Katugampola fractional integrals.

Theorem 2.1. Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \rightarrow \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in X^\rho(a^\rho, b^\rho) \). If \( f \) is also a convex function on \([a, b]\), then the following inequalities hold:
\[
f \left( \frac{a^\rho + b^\rho}{2} \right) \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} \left[ \rho I_{a+}^\rho f(b^\rho) + \rho I_{b-}^\rho f(a^\rho) \right] \leq \frac{f(a^\rho) + f(b^\rho)}{2}
\]
where the fractional integrals are considered for the function \( f(x^\rho) \) and evaluated at \( a^\rho \) and \( b^\rho \), respectively.

Proof. Let \( t \in [0, 1] \). Consider \( x, y \in [a, b] \), \( a \geq 0 \), defined by \( x^\rho = t^\rho a^\rho + (1 - t^\rho)b^\rho \), \( y^\rho = (1 - t^\rho)a^\rho + t^\rho b^\rho \). Since \( f \) is a convex function on \([a, b]\), we have
\[
f \left( \frac{x^\rho + y^\rho}{2} \right) \leq \frac{f(x^\rho) + f(y^\rho)}{2}
\]
Then we have
\[
2f \left( \frac{a^\rho + b^\rho}{2} \right) \leq f(t^\rho a^\rho + (1 - t^\rho)b^\rho) + f((1 - t^\rho)a^\rho + t^\rho b^\rho)
\]
Multiplying both sides of Eq. (9) by \( t^{\rho-1} \), \( \alpha > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([a, b]\), we obtain
\[
\frac{2}{\alpha^\rho} f \left( \frac{a^\rho + b^\rho}{2} \right) \leq \int_a^b t^{\rho-1} f(t^\rho a^\rho + (1 - t^\rho)b^\rho) \, dt + \int_a^b t^{\rho-1} f((1 - t^\rho)a^\rho + t^\rho b^\rho) \, dt
\]
\[
= \int_b^a \left( \frac{b^\rho - x^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} f(x^\rho) \frac{x^\rho-1}{a^\rho - b^\rho} \, dx + \int_a^b \left( \frac{y^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} f(y^\rho) \frac{y^\rho-1}{b^\rho - a^\rho} \, dy
\]
\[
= \frac{\rho^\alpha \Gamma(\alpha + 1)}{\Gamma(\alpha)} \left[ \rho I_{a+}^\rho f(b^\rho) + \rho I_{b-}^\rho f(a^\rho) \right].
\]
This establishes the first inequality. For the proof of the second inequality in Eq. (8), we first note that for a convex function \( f \), we have
\[
 f(t^\alpha a^\rho + (1 - t^\alpha)b^\rho) \leq t^\alpha f(a^\rho) + (1 - t^\alpha)\ f(b^\rho),
\]
and
\[
 f((1 - t^\alpha)a^\rho + t^\alpha b^\rho) \leq (1 - t^\alpha)\ f(a^\rho) + t^\alpha \ f(b^\rho).
\]
By adding these inequalities, we then have
\[
 f(t^\alpha a^\rho + (1 - t^\alpha)b^\rho) + f((1 - t^\alpha)a^\rho + t^\alpha b^\rho) \leq f(a^\rho) + f(b^\rho).
\]
Multiplying both sides of Eq. (12) by \( t^{\alpha-1} \), \( a > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([a, b]\), we similarly obtain
\[
\frac{\rho^\alpha \Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right] \leq \frac{f(a^\rho) + f(b^\rho)}{\alpha^\rho}.
\]
This completes the proof of the theorem. 

If the function \( f' \) is differentiable, we have the following result.

**Theorem 2.2.** Let \( f : [a^\rho, b^\rho] \rightarrow \mathbb{R} \) be a differentiable mapping with \( 0 \leq a < b \). If \( f' \) is differentiable on \((a^\rho, b^\rho)\), then the following inequality holds:
\[
\left| \frac{f(a^\rho) + f(b^\rho)}{\alpha^\rho} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right] \right| \leq \frac{(b^\rho - a^\rho)^2}{2(\alpha + 1)(\alpha + 2)} \sup_{t \in [a^\rho, b^\rho]} |f''(t)|. \tag{13}
\]

**Proof.** Using right side of inequality (10) and Eq. (11), we have
\[
\frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right] = \int_0^1 t^{\alpha-1} f(t^\alpha a^\rho + (1 - t^\alpha)b^\rho) \, dt + \int_0^1 t^{\alpha-1} f((1 - t^\alpha)a^\rho + t^\alpha b^\rho) \, dt.
\]

By using integration by parts, we then have
\[
\frac{f(a^\rho) + f(b^\rho)}{\alpha^\rho} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right]
\leq \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^{\alpha-1} \left[ f'((1 - t^\alpha)a^\rho + t^\alpha b^\rho) - f'(t^\alpha a^\rho + (1 - t^\alpha)b^\rho) \right] \, dt. \tag{14}
\]

Using Eq. (14) and applying the mean value theorem for the function \( f' \), we have
\[
\frac{f(a^\rho) + f(b^\rho)}{\alpha^\rho} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right] = \frac{(b^\rho - a^\rho)^2}{\alpha} \int_0^1 t^{\alpha-1} (2t^\rho - 1) f''(\xi(t)) \, dt,
\]
where \( \xi(t) \in (a^\rho, b^\rho) \). This leads us to
\[
\left| \frac{f(a^\rho) + f(b^\rho)}{\alpha^\rho} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \rho^\alpha I^\alpha_{a^\rho} f(b^\rho) + \rho^\alpha I^\alpha_{b^\rho} f(a^\rho) \right] \right|
\leq \frac{(b^\rho - a^\rho)^2}{\alpha} \int_0^1 t^{\alpha-1} |2t^\rho - 1| |f''(\xi(t))| \, dt
\leq \frac{(b^\rho - a^\rho)^2}{\alpha} \sup_{\xi \in [a^\rho, b^\rho]} |f''(\xi)| \left[ \int_0^1 t^{\alpha-1} (2t^\rho - 1) t^{\alpha-1} \, dt + \int_0^1 (2t^\rho - 1) t^{\alpha-1} \, dt \right]
= \frac{(b^\rho - a^\rho)^2}{\alpha \rho(\alpha + 1)(\alpha + 2)} \left( \alpha + 1 \right) \sup_{\xi \in [a^\rho, b^\rho]} |f''(\xi)|.
\]
This gives the desired result.
Theorem 2.3. Let \( f : [a^p, b^p] \to \mathbb{R} \) be a differentiable mapping on \((a^p, b^p)\) with \(0 \leq a < b\). If \(|f'|\) is convex on \([a^p, b^p]\), then the following inequality holds:

\[
\left| \frac{f(a^p) + f(b^p)}{2} - \frac{\alpha^p \Gamma(\alpha + 1)}{2(b^p - a^p)^\alpha} \left[ \rho I_{a^+}^\alpha f(b^p) + \rho I_{b^-}^\alpha f(a^p) \right] \right| \leq \frac{b^p - a^p}{2(\alpha + 1)} \left( \left| f'(a^p) \right| + \left| f'(b^p) \right| \right).
\] (15)

Proof. By using Eq. (14), the triangle inequality and the convexity of \(|f'|\), we get

\[
\left| \frac{f(a^p) + f(b^p)}{\alpha^p} \right| = \frac{\alpha^p \Gamma(\alpha + 1)}{(b^p - a^p)^\alpha} \left( \rho I_{a^+}^\alpha f(b^p) + \rho I_{b^-}^\alpha f(a^p) \right)
\]

\[
\leq \frac{b^p - a^p}{\alpha} \int_0^1 t^\rho(\alpha + 1) \left( f'((1 - t^\rho)a^p + t^\rho b^p) - f'((1 - t^\rho)a^p + (1 - t^\rho)b^p) \right) dt
\]

\[
\leq \frac{b^p - a^p}{\alpha} \int_0^1 t^\rho(\alpha + 1) \left[ f'((1 - t^\rho)a^p + t^\rho b^p) + \left| f'((1 - t^\rho)a^p + (1 - t^\rho)b^p) \right| \right] dt
\]

\[
= \frac{b^p - a^p}{\alpha} \left[ f(a^p) + f(b^p) \right] \int_0^1 t^\rho(\alpha + 1) dt
\]

\[
= \frac{b^p - a^p}{\alpha \rho(\alpha + 1)} \left[ f(a^p) + f(b^p) \right] .
\]

This establishes the result.

Another more strict inequality can be obtain by using the following lemma.

Lemma 2.4. Let \( f : [a^p, b^p] \to \mathbb{R} \) be a differentiable mapping on \((a^p, b^p)\) with \(0 \leq a < b\). Then the following equality holds if the fractional integrals exist:

\[
\left| \frac{f(a^p) + f(b^p)}{2} - \frac{\alpha^p \Gamma(\alpha + 1)}{2(b^p - a^p)^\alpha} \left[ \rho I_{a^+}^\alpha f(b^p) + \rho I_{b^-}^\alpha f(a^p) \right] \right| = \frac{b^p - a^p}{2} \int_0^1 \left( (1 - t^\rho)^\alpha - t^\alpha \right) t^\rho(\alpha + 1) f'(t^\rho a^p + (1 - t^\rho)b^p) dt .
\] (16)

Proof. This can be proved using a similar line of argument as in the proof of Lemma 2 in [37]. To that end, by integration by parts, first note that

\[
\int_0^1 (1 - t^\rho)^\alpha t^\rho(\alpha + 1) f'(t^\rho a^p + (1 - t^\rho)b^p) dt
\]

\[
= \left. \left| \frac{1}{\rho(a^p - b^p)} (1 - t^\rho)^\alpha f(t^\rho a^p + (1 - t^\rho)b^p) \right| \right|_0^1 + \alpha \int_0^1 (1 - t^\rho)^\alpha t^\rho(\alpha + 1) f'(t^\rho a^p + (1 - t^\rho)b^p) dt
\]

\[
= \frac{f(b^p)}{\rho(b^p - a^p)} - \frac{\alpha}{b^p - a^p} \int_b^a \left( x^\rho - a^\rho \right)^{\alpha - 1} \left( \frac{x^\rho}{b^\rho} - \frac{a^\rho}{b^\rho} \right) dx
\]

\[
= \frac{f(b^p)}{\rho(b^p - a^p)} - \frac{\rho(\alpha + 1)}{(b^p - a^p)^{\alpha + 1}} \rho I_{b^-}^\alpha f(x^p) \bigg|_{x = a} .
\]

Similarly, we can also prove that

\[
- \int_0^1 t^\rho(\alpha + 1) t^\rho(\alpha + 1) f'(t^\rho a^p + (1 - t^\rho)b^p) dt = \frac{f(a^p)}{\rho(b^p - a^p)} - \frac{\rho(\alpha + 1)}{(b^p - a^p)^{\alpha + 1}} \rho I_{a^+}^\alpha f(x^p) \bigg|_{x = b} .
\]

These two results lead to the proof of Lemma 2.4.

With the help of this lemma, we have the following result.

Theorem 2.5. Let \( f : [a^p, b^p] \to \mathbb{R} \) be a differentiable mapping on \((a^p, b^p)\) with \(0 \leq a < b\). If \(|f'|\) is convex on \([a^p, b^p]\), then the following inequality holds:

\[
\left| \frac{f(a^p) + f(b^p)}{2} - \frac{\alpha^p \Gamma(\alpha + 1)}{2(b^p - a^p)^\alpha} \left[ \rho I_{a^+}^\alpha f(b^p) + \rho I_{b^-}^\alpha f(a^p) \right] \right| \leq \frac{b^p - a^p}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left| f'(a^p) \right| + \left| f'(b^p) \right| .
\] (17)
Proof. Using Lemma 2.4 and the convexity of $|f'|$, we have
\[
\left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha + 1)}{2(\rho^\alpha - a^\rho)^\alpha} \left[ \rho I_{\alpha+}^\rho f(b^\rho) + \rho I_{\alpha-}^\rho f(a^\rho) \right] \right|
\leq \frac{b^\rho - a^\rho}{2} \int_0^1 \left| (1 - t^\rho)^\alpha - t^\alpha \right| \left| f'(t^\rho a^\rho + (1 - t^\rho)b^\rho) \right| \, dt
\leq \frac{b^\rho - a^\rho}{2} \int_0^1 \left| (1 - t^\rho)^\alpha - t^\alpha \right| \left| \rho^\alpha f'(a^\rho) \right| + (1 - t^\rho) \left| f'(b^\rho) \right| \, dt
\leq \frac{b^\rho - a^\rho}{2} \left\{ \int_0^1 \left| (1 - t^\rho)^\alpha - t^\alpha \right| \left| \rho^\alpha |f'(a^\rho)| + (1 - t^\rho) |f'(b^\rho)| \right| \, dt \right\}
+ \int_0^1 \left| (1 - t^\rho)^\alpha - t^\alpha \right| \left| \rho^\alpha |f'(a^\rho)| + (1 - t^\rho) |f'(b^\rho)| \right| \, dt \right\}
= \int_0^1 g(t) \, dt - 2 \int_0^1 \frac{1}{\rho} \left| f'(a^\rho) \right| - \left| f'(b^\rho) \right| \frac{\alpha}{(\alpha + 1)(\alpha + 2)} - 2 \left\{ \left| f'(a^\rho) \right| + \left| f'(b^\rho) \right| \right\} \left( \frac{\rho^\alpha + 2}{\alpha + 1} + \frac{\rho^\alpha + 2}{\alpha + 2} \right)\frac{1}{(\alpha + 1)(\alpha + 2)} - \left| f'(b^\rho) \right| \frac{\alpha}{(\alpha + 1)(\alpha + 2)} - \left| f'(b^\rho) \right| \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right\}
= \frac{b^\rho - a^\rho}{2\rho(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left| f'(a^\rho) \right| + \left| f'(b^\rho) \right| \right|.
\]
This completes the proof of the theorem. □

When $\rho = 1$, Theorem 2.5 will reduce to Theorem 3 of [37]. If 1 is in the domain of $f$ and $f$ is differentiable at 1, then we have the following special case when $\rho \to 0^+$.
\[
\left| f(1) - \frac{\alpha \Gamma(\alpha + 1)}{2(\ln \frac{1}{2})^\alpha} \left[ I_{\alpha+} f(1) + I_{\alpha-} f(1) \right] \right| \leq \frac{1}{(\alpha + 1) \ln \frac{1}{2}} \left( 1 - \frac{1}{2^\alpha} \right) \left| f'(1) \right|.
\]
where $I_{\alpha+} (\cdot)$ and $I_{\alpha-} (\cdot)$ are Hadamard fractional integrals defined in Eq. 3.

3. Further inequalities

In this section, we generalize the results of Jleli et al. [13] further. Let $f : [a, b] \to \mathbb{R}$ be a given function, where $0 < a < b < \infty$. For the rest of the paper, we define $F(x) := f(x) + f(a + b - x)$. Then it is easy to show that if $f(x)$ is convex on $[a, b]$, $F(x)$ is also convex. The function $F$ has several interesting properties, especially,

- $F(x)$ is symmetric to $(a + b)/2$;
- $F(a) = F(b) = f(a) + f(b)$;
- $F\left( \frac{a + b}{2} \right) = 2f\left( \frac{a + b}{2} \right)$.

3.1. Hermite-Hadamard type inequalities

Hermite–Hadamard inequalities can be generalized via Katugampola fractional integrals as follows.

**Theorem 3.1.** If $f$ is a convex function on $[a, b]$ and $f \in L[a, b]$. Then $F(x)$ is also integrable, and the following inequalities hold
\[
F\left( \frac{a + b}{2} \right) \leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\rho^\alpha - a^\rho)^\alpha} \left[ \rho I_{\alpha+} f(b) + \rho I_{\alpha-} f(a) \right] \leq \frac{F(a) + F(b)}{2}
\] (18)
with $\alpha > 0$ and $\rho > 0$. 

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Proof. Since \( f(x) \) is a convex function on \([a, b]\), we have for \( x, y \in [a, b] \)
\[
f\left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}.
\]

Set \( x = ta + (1 - t)b \) and \( y = (1 - t)a + tb \), then
\[
2f\left( \frac{a + b}{2} \right) \leq f(ta + (1 - t)b) + f((1 - t)a + tb),
\]
Using the notation of \( F(x) \), we have
\[
F\left( \frac{a + b}{2} \right) \leq F((1 - t)a + tb). \tag{19}
\]

Multiplying both sides of (19) by
\[
\frac{((1-t)a + tb)^{\rho^{-1}}}{[b^\rho - ((1-t)a + tb)^\rho]^{1-\alpha}} \tag{20}
\]
and integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we get
\[
F\left( \frac{a + b}{2} \right) \frac{(b^\rho - a^\rho)^\alpha}{\alpha \rho (b - a)} \leq \int_0^1 \frac{((1-t)a + tb)^{\rho^{-1}}}{[b^\rho - ((1-t)a + tb)^\rho]^{1-\alpha}} F((1-t)a + tb) dt
\]
\[
= \int_a^b \frac{\alpha^{-1} b^\rho}{(b^\rho - a^\rho)^{1-\alpha}} F(a) \frac{b^\rho - a^\rho}{b - a} d\alpha = \frac{\Gamma(\alpha) b^\rho}{b - a} \rho I_{a+}^\alpha F(b)
\]
i.e.
\[
F\left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1) b^\rho}{(b^\rho - a^\rho)^\alpha} I_{a+}^\alpha F(b). \tag{21}
\]

Similarly, multiplying both sides of (19) by
\[
\frac{((1-t)a + tb)^{\rho^{-1}}}{[(1-t)a + tb)^\rho - a^\rho]^{1-\alpha}} \tag{22}
\]
and integrating the resulting inequality over \([0, 1]\), we get
\[
F\left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1) b^\rho}{(b^\rho - a^\rho)^\alpha} I_{b-}^\alpha F(a). \tag{23}
\]

By adding inequalities (21) and (23), we obtain
\[
F\left( \frac{a + b}{2} \right) \leq \frac{\rho \Gamma(\alpha + 1) b^\rho}{2(b^\rho - a^\rho)^\alpha} \left[ \rho I_{a+}^\alpha F(b) + \rho I_{b-}^\alpha F(a) \right].
\]
The first inequality of (18) is proved.

For the second part, since \( f \) is a convex function, then for \( t \in [0, 1] \), we have
\[
f(ta + (1 - t)b) + f((1 - t)a + tb) \leq f(a) + f(b).
\]
Using the notation of \( F(x) \), we then have
\[
F((1 - t)a + tb) \leq \frac{F(a) + F(b)}{2}. \tag{24}
\]

Multiplying both sides of (24) by factor (20) and integrating the resulting inequality over \([0, 1]\) with respect to \( t \), we get
\[
\frac{\Gamma(\alpha) b^\rho}{b - a} \rho I_{a+}^\alpha F(b) \leq \frac{(b^\rho - a^\rho)^\alpha}{\alpha \rho (b - a)} \left[ \rho I_{a+}^\alpha F(b) + \rho I_{b-}^\alpha F(a) \right].
\]
Lemma 3.4. Let 
and the following equalities hold 

The proof is completed.

By adding inequality (25) and (26), we obtain

over \([0, 1]\), we get

Remark 3.2. Theorem [3, 1] is a generalization of Hermite-Hadamard inequality.

1. Letting \(\rho \to 1\) in (18) and noticing that

we immediately get the Riemann-Liouville form of Hermite-Hadamard inequality [1] in Theorem 1.3.

2. If \(f\) is also symmetric to \(\frac{a + b}{2}\), then \(F(x) = f(x) + f(a + b - x) = 2f(x)\), and the inequality (18) becomes

We can get inequality (4) directly by letting \(\rho \to 1\).

On the other hand, letting \(\rho \to 0^+\) in inequality (18), we get the following Hermite-Hadamard inequality for Hadamard fractional integrals.

Corollary 3.3. If \(f\) is a convex function on \([a, b]\) and \(f \in L[a, b]\). Then \(F(x)\) is also convex and \(F \in L[a, b]\), and the following equalities hold

with \(\alpha > 0\) and \(\rho > 0\).

In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.4. Let \(f : [a, b] \to \mathbb{R}\) be a differentiable mapping on \((a, b)\) with \(a < b\). If \(f' \in L[a, b]\), then \(F\) is also differentiable and \(F' \in L[a, b]\), and the following equality holds:

with \(\alpha > 0\) and \(\rho > 0\), where \(K(t) = \{(1 - t)a + bt\}^\rho - a^\rho - [b^\rho - ((1 - t)a + bt)^\rho]\).

Proof. Note that

\[I = \int_0^1 K(t)F'((1 - t)a + bt)dt = \int_0^1 [(1 - t)a + bt]^\rho - a^\rho] F'((1 - t)a + bt)dt - \int_0^1 [b^\rho - ((1 - t)a + bt)^\rho] F'((1 - t)a + bt)dt = I_1 + I_2.\]
Integrating by parts, we get
\[
I_1 = \int_0^1 \left[ (1-t)a + bt \right]^{\rho - \alpha} F'(t) dt = \frac{1}{b-a} \int_a^b [u^{\rho - \alpha}]^\alpha dF(u) \tag{30}
\]
\[
= \left[ (u^{\rho - \alpha}) F(u) \right]_a^b - \frac{\alpha \rho}{b-a} \int_a^b u^{\rho-1} \Gamma(1-\alpha, u) du.
\]
\[
= \frac{(b^{\rho - \alpha})^\alpha}{b-a} F(b) - \frac{\alpha (1+\rho)^\alpha}{b-a} \rho \int_a^b \Gamma(\alpha+1, \rho F(b) - \rho F(a)) \, d\rho.
\]

Similarly,
\[
I_2 = -\int_0^1 \left[ (1-t)a + bt \right]^{\rho - \alpha} F'(t) dt \tag{31}
\]
\[
= \frac{(b^{\rho - \alpha})^\alpha}{b-a} F(b) - \frac{\alpha (1+\rho)^\alpha}{b-a} \rho \int_a^b \Gamma(\alpha+1, \rho F(b) + \rho F(a)) \, d\rho.
\]

By adding (30) and (31), we get
\[
I = \frac{(b^{\rho - \alpha})^\alpha}{b-a} [F(a) + F(b)] - \frac{\alpha (1+\rho)^\alpha}{b-a} \left[ \rho \int_a^b \Gamma(\alpha+1, \rho F(b) - \rho F(a)) \, d\rho + \rho \int_a^b \Gamma(\alpha+1, \rho F(b) + \rho F(a)) \, d\rho \right].
\]

Then, multiplying both sides by \( \frac{b-a}{2(\beta - \alpha)} \) we obtain equality (29). \( \square \)

We are now ready to prove the following Hermite–Hadamard type inequality.

**Theorem 3.5.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \) and \( f' \in L[a, b] \). Then \( F \) is also differentiable and \( F' \in L[a, b] \). If \( |f'| \) is convex on \([a, b]\), then the following inequality holds:
\[
\left| \frac{F(a) + F(b)}{2} - \frac{\rho \Gamma(\alpha+1)}{2(\beta - \alpha)^\alpha} \left[ \rho \int_a^b F(b) + \rho \int_a^b F(a) \right] \right| \leq \frac{b-a}{2(\beta - \alpha)^\alpha} \int_0^1 |K(t)| dt \left( |f'(a)| + |f'(b)| \right) \tag{32}
\]

with \( \alpha > 0 \) and \( \rho > 0 \), where \( K(t) = \left| \left(1-t\right)a + bt \right|^{\rho - \alpha} - \left| (1-t)a + bt \right|^{\rho - \alpha} \).

**Proof.** Notice that \( f'(x) = f'(x) - f'(a + b - x) \). By the convexity of \( |f'| \), we have
\[
|f'((1-t)a + bt)| = |f'((1-t)a + bt) - f'(a + (1-t)b)| \leq (1-t)|f'(a)| + t|f'(b)| + t|f'(a)| + (1-t)|f'(b)| \tag{33}
\]
\[
= |f'(a)| + |f'(b)|.
\]

By inequalities (20) and (34), we get
\[
\left| \frac{F(a) + F(b)}{2} - \frac{\rho \Gamma(\alpha+1)}{2(\beta - \alpha)^\alpha} \left[ \rho \int_a^b F(b) + \rho \int_a^b F(a) \right] \right| \leq \frac{b-a}{2(\beta - \alpha)^\alpha} \int_0^1 |K(t)| dt \left( |f'(a)| + |f'(b)| \right) \leq \frac{b-a}{2(\beta - \alpha)^\alpha} \int_0^1 |K(t)| dt \left( |f'(a)| + |f'(b)| \right).
\]

**Remark 3.6.** In Theorem 3.5, by letting \( \rho \to 1 \), inequality (29) becomes inequality (15) of Theorem 1.6. As
\[
\lim_{\rho \to 1} \int_0^1 |K(t)| dt = \int_0^1 t^\alpha - (1-t)^\alpha dt \tag{34}
\]
\[
= \int_0^1 t^\alpha \left( 1 - \frac{1}{2^\alpha} \right) dt = 2(b-a)^{\alpha/2} \cdot \left( 1 - \frac{1}{2^\alpha} \right).
\]
On the other hand, by letting $\rho \to 0^+$ in equality (29), we get the following result for Hadamard fractional integrals.

**Corollary 3.7.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$ and $f' \in L[a, b]$. Then $F$ is also differentiable and $F' \in L[a, b]$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{F(a) + F(b)}{2} - \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\ln_\rho^\alpha)^{\alpha}} \left[ H^\alpha_{a+}F(b) + H^\alpha_{b-}F(a) \right] \right| \leq b - a \int_0^1 |K(t)|dt \left( |f'(a)| + |f'(b)| \right) \tag{34}$$

with $\alpha > 0$, where $K(t) = \left[ ((1 - t)a + bt)^\rho a - a^\rho \right]^\alpha - \left[ b^\rho - ((1 - t)a + bt)^\rho \right]^\alpha$.

### 3.2. Hermite–Hadamard–Fejér type inequalities

The Hermite–Hadamard–Fejér inequalities can also be generalized via Katugampola fractional integrals as follows.

**Theorem 3.8.** Let $f : [a, b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a, b]$. Then $F(x)$ is also convex and $F \in L[a, b]$. If $g : [a, b] \to \mathbb{R}$ is nonnegative and integrable, then the following inequalities hold:

$$\left\{ \begin{array}{ll}
\frac{a + b}{2} \left[ \rho I^\alpha_{a+}g(b) + \rho I^\alpha_{b-}g(a) \right] \leq \left[ \rho I^\alpha_{a+}(gF)(b) + \rho I^\alpha_{b-}(gF)(a) \right] \leq \frac{F(a) + F(b)}{2} \left[ \rho I^\alpha_{a+}g(b) + \rho I^\alpha_{b-}g(a) \right] \\
\end{array} \right. \tag{35}$$

with $\alpha > 0$ and $\rho > 0$.

**Proof.** Since $f$ is convex on $[a, b]$, for all $t \in [0, 1]$, we have

$$2f\left( \frac{a + b}{2} \right) \leq f(ta + (1 - t)b) + f((1 - t)a + tb),$$

That is

$$F\left( \frac{a + b}{2} \right) \leq F((1 - t)a + tb). \tag{36}$$

Multiplying both sides of (36) by

$$\frac{((1 - t)a + tb)^{\rho - 1}}{[b^\rho - ((1 - t)a + tb)^\rho]^1 - \alpha} g((1 - t)a + tb) \tag{37}$$

and integrating the resulting inequality with respect to $t$ over $[0, 1]$, we get

$$\frac{\rho^\alpha \Gamma(\alpha) \rho I^\alpha_{a+}g(b)F\left( \frac{a + b}{2} \right)}{b - a} \leq \int_0^1 \frac{((1 - t)a + tb)^{\rho - 1}}{[b^\rho - ((1 - t)a + tb)^\rho]^1 - \alpha} g((1 - t)a + tb)F((1 - t)a + tb)dt$$

$$= \frac{\Gamma(\alpha) \rho^\alpha \rho I^\alpha_{a+}(gF)(b)}{b - a},$$

i.e.

$$F\left( \frac{a + b}{2} \right) \rho I^\alpha_{a+}g(b) \leq \rho I^\alpha_{a+}(gF)(b). \tag{38}$$

Similarly, we have

$$F\left( \frac{a + b}{2} \right) \rho I^\alpha_{b-}g(a) \leq \rho I^\alpha_{b-}(gF)(a). \tag{39}$$

By adding inequalities (38) and (39), we obtain

$$F\left( \frac{a + b}{2} \right) \left[ \rho I^\alpha_{a+}g(b) + \rho I^\alpha_{b-}g(a) \right] \leq \left[ \rho I^\alpha_{a+}(gF)(b) + \rho I^\alpha_{b-}(gF)(a) \right]$$

The first inequality of (35) is proved.

For the second inequality, since $f$ is a convex function, then for all $t \in [0, 1]$, we have

$$f(ta + (1 - t)b) + f((1 - t)a + tb) \leq f(a) + f(b),$$

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which can be rewritten as

\[ F((1 - t)a + tb) \leq \frac{F(a) + F(b)}{2}. \]  (40)

Multiplying both sides of (40) by factor \( \frac{1}{2} \) and integrating over \([0, 1]\) with respect to \( t \), we get

\[
\frac{\Gamma(\alpha)^{\rho-1}}{b - a} \int_a^b (\frac{1}{g^\rho} - \frac{1}{f^\rho})^{\frac{1}{\alpha}-1} F(\rho I_{a+}^\alpha g(b)) \leq \frac{1}{2} \frac{\Gamma(\alpha)^{\rho-1}}{b - a} \int_a^b \rho I_{a+}^\alpha g(b) F(a) + F(b)
\]

i.e.

\[
\rho I_{a+}^\alpha (gF)(b) \leq \frac{F(a) + F(b)}{2} \rho I_{a+}^\alpha g(b)
\]  (41)

Similarly, we have

\[
\rho I_{b-}^\alpha (gF)(a) \leq \frac{F(a) + F(b)}{2} \rho I_{b-}^\alpha g(a)
\]  (42)

Adding inequality (41) and (42), we obtain

\[
\left[ \rho I_{a+}^\alpha (gF)(b) + \rho I_{b-}^\alpha (gF)(a) \right] \leq \frac{F(a) + F(b)}{2} \left[ \rho I_{a+}^\alpha g(b) + \rho I_{b-}^\alpha g(a) \right].
\]

The proof is completed. \( \square \)

**Remark 3.9.** Theorem 3.8 is a generalization of Hermite–Hadamard–Fejér inequalities [13].

1. If \( f \) is symmetric to \( \frac{a + b}{2} \), then \( F(x) = f(x) + f(a + b - x) = 2f(x) \), inequality (35) becomes

\[
f(\frac{a + b}{2}) \left[ \rho I_{a+}^\alpha g(b) + \rho I_{b-}^\alpha g(a) \right] \leq \left[ \rho I_{a+}^\alpha (gF)(b) + \rho I_{b-}^\alpha (gF)(a) \right] \leq \frac{f(a) + f(b)}{2} \left[ \rho I_{a+}^\alpha g(b) + \rho I_{b-}^\alpha g(a) \right] \]  (43)

with \( \alpha > 0 \) and \( \rho > 0 \).

2. If we take \( g(x) = 1 \) in inequality (35), then it becomes inequality (18) of Theorem 3.4.

3. If \( g(x) \) is symmetric to \( (a + b)/2 \), then letting \( \rho \to 1 \), inequality (35) becomes inequality (10) of Theorem 1.6.

Since

\[
\lim_{\rho \to 1} \rho I_{a+}^\alpha (gF)(b) = \lim_{\rho \to 1} \frac{1}{\Gamma(\alpha)} \int_a^b (\frac{1}{b^\rho} - \frac{1}{f^\rho})^{\frac{1}{\alpha}-1} g(t) dt = \frac{1}{\Gamma(\alpha)} \int_a^b (b - t)^{\alpha-1} g(t) dt + \frac{1}{\Gamma(\alpha)} \int_a^b (b - t)^{\alpha-1} g(t) f(a + b - t) dt = J_{a+}^\alpha (gF)(b) + J_{b-}^\alpha (gF)(a)
\]

and similarly

\[
\lim_{\rho \to 1} \rho I_{b-}^\alpha (gF)(a) = J_{a+}^\alpha (gF)(a) + J_{b-}^\alpha (gF)(b).
\]

To prove the inequality in Theorem 3.11 we need the following lemma.

**Lemma 3.10.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( 0 \leq a < b \) and \( f' \in L[a, b] \). Then \( F(x) \) is also differentiable and \( F' \in L[a, b] \). If \( g : [a, b] \to \mathbb{R} \) is integrable, then the following equality holds:

\[
\frac{F(a) + F(b)}{2} \left[ \rho I_{a+}^\alpha g(b) + \rho I_{b-}^\alpha g(a) \right] - \left[ \rho I_{a+}^\alpha (gF)(b) + \rho I_{b-}^\alpha (gF)(a) \right] = \frac{\rho^{1-\alpha}}{2\Gamma(\alpha)} \int_a^b \left[ \int_a^s G(s) g(s) ds - \int_s^b G(s) g(s) ds \right] F'(t) dt
\]  (44)

with \( \alpha > 0 \) and \( \rho > 0 \), where

\[ G(s) = \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} + \frac{s^{\rho-1}}{(s^\rho - a^\rho)^{1-\alpha}}. \]
Integrating by parts, we get

\[ I = \int_a^b \left[ \int_t^b G(s)g(s)ds \right] F'(t)dt \]

From (45) and (46), we get

\[ \rho \text{ and similarly} \]

\[ \text{Proof.} \text{ Note that} \]

\[ \rho > \alpha \]

Then the following inequality holds:

\[ \text{With Lemma 3.10, we have the following Hermite–Hadamard–Fejér type inequality.} \]

\[ I = \Gamma(\alpha)\rho^{\alpha-1}[\rho I_{a^+}^\alpha g(b) + \rho I_{b^-}^\alpha g(a)]F(a) - \Gamma(\alpha)\rho^{\alpha-1}[\rho I_{a^+}^\alpha (gF)(b) + \rho I_{b^-}^\alpha (gF)(a)]. \]

From (45) and (46), we get

\[ I = \Gamma(\alpha)\rho^{\alpha-1}[\rho I_{a^+}^\alpha g(b) + \rho I_{b^-}^\alpha g(a)](F(a) + F(b)) - 2\Gamma(\alpha)\rho^{\alpha-1}[\rho I_{a^+}^\alpha (gF)(b) + \rho I_{b^-}^\alpha (gF)(a)]. \]

Then, multiplying both sides by \( \frac{t-a}{2\Gamma(\alpha)} \), we get the conclusion. \( \square \)

With Lemma 3.10 we have the following Hermite–Hadamard–Fejér type inequality.

**Theorem 3.11.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) and \( f' \in L[a, b] \) with \( 0 \leq a < b \). Then \( F(x) \) is also differentiable and \( F'' \in L[a, b] \). If \( |f'| \) is convex on \([a, b]\) and \( g : [a, b] \to \mathbb{R} \) is continuous, then the following inequality holds:

\[ \left| \frac{F(a) + F(b)}{2} - \frac{\rho I_{a^+}^\alpha g(b) + \rho I_{b^-}^\alpha g(a)}{2} - \frac{\rho I_{a^+}^\alpha (gF)(b) + \rho I_{b^-}^\alpha (gF)(a)}{2} \right| \]

\[ \leq \frac{(b-a)|g|_{\infty}}{\rho^2\Gamma(\alpha + 1)} \left( |f'(a)| + |f'(b)| \right) \int_0^1 |K(t)|dt \]

with \( \alpha > 0 \) and \( \rho > 0 \), where \( |g|_{\infty} = \sup_{t \in [a, b]} |g(x)| \), and

\[ K(t) = \left( (1-t)a + bt \right)^\rho - a^\rho - \left( 1-t \right)a + bt b^\rho \]

as defined in Lemma 3.4.

**Proof.** Notice that \( F'(t) = f'(t) - f'(a + b - t) \), and by the convexity of \( f'(t) \), we have

\[ |F'(t)| = |f'(t) - f'(a + b - t)| \leq |f'(t)| - |f'(a + b - t)| \]

\[ = |f'(b) - a + t - a b - a| + |f'(t) - a + t - a b - a| \]

\[ \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| + \frac{b-t}{b-a} |f'(a)| + \frac{b-t}{b-a} |f'(b)| \]

\[ = |f'(a)| + |f'(b)|. \]
Also
\[
\int_a^t G(s)ds - \int_t^b G(s)ds
= \int_a^t \left( \frac{(b^\rho - s^\rho)^{1-\alpha}}{(s^\rho - a^\rho)^{1-\alpha}} + \frac{s^\rho-1}{(s^\rho - a^\rho)^{1-\alpha}} \right) ds - \int_t^b \left( \frac{(b^\rho - s^\rho)^{1-\alpha}}{(s^\rho - a^\rho)^{1-\alpha}} + \frac{s^\rho-1}{(s^\rho - a^\rho)^{1-\alpha}} \right) ds
= \left[ \frac{(b^\rho - s^\rho)^{1-\alpha}}{\alpha \rho} + \frac{(s^\rho - a^\rho)^{1-\alpha} b}{\alpha \rho} \right]_a^b - \left[ \frac{(b^\rho - s^\rho)^{1-\alpha}}{\alpha \rho} + \frac{(s^\rho - a^\rho)^{1-\alpha} b}{\alpha \rho} \right]_t
= \frac{2}{\alpha \rho} [(t^\rho - a^\rho)^{1-\alpha} - (b^\rho - t^\rho)^{1-\alpha}].
\]

Hence by Lemma 3.10
\[
\frac{F(a) + F(b)}{2} \left[ F_a^\rho g(b) + F_b^\rho g(a) - \frac{\rho}{\Gamma(a+1)} \int_a^b G(s)ds - \int_t^b G(s)ds \right] F'(t)dt \\
\leq \frac{\rho^{1-\alpha} \|g\|_\infty}{2 \Gamma(a)} \int_a^b \left| G(s)ds - \int_t^b G(s)ds \right| dt \left( \|f'(a)\| + \|f'(b)\| \right)
\leq \frac{\|g\|_\infty}{\rho^{1-\alpha}\Gamma(a+1)} \int_a^b \left| (t^\rho - a^\rho)^{1-\alpha} - (b^\rho - t^\rho)^{1-\alpha} \right| dt \left( \|f'(a)\| + \|f'(b)\| \right)
= \frac{(b - a)\|g\|_\infty}{\rho^{1-\alpha}\Gamma(a+1)} \int_a^b \left| K(t) \right| dt \left( \|f'(a)\| + \|f'(b)\| \right)
\]
where \( K(t) = \left| ((1-t)a + bt)^\rho - a^\rho \right| - \left| b^\rho - ((1-t)a + bt)^\rho \right| \).

\[ \square \]

**Remark 3.12.** In Theorem 3.11

1. if we take \( g(x) = 1 \) in inequality (47), then it becomes inequality (32) in Theorem 3.5
2. if, in addition, \( g(x) \) is symmetric to \( (a + b)/2 \), letting \( \rho \to 1 \), inequality (47) becomes inequality (7) of Theorem 1.7.

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