Improved Vietoris Sine Inequalities for Non-Monotone, Non-Decaying Coefficients

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Abstract

The classical Vietoris sine inequality states that for any non-increasing sequence of positive real numbers \( \{a_k\}_{k=1}^{\infty} \) satisfying
\[
a_{2j-1} \geq \frac{2j}{2j-1} a_{2j} \quad (j = 1, 2, 3, \cdots), \tag{\star}
\]
the following sine polynomials are nonnegative in \([0, \pi]\),
\[
\sum_{k=1}^{n} a_k \sin(kx) \geq 0, \quad x \in [0, \pi], \quad \text{for all } n = 1, 2, 3, \cdots. \tag{\dagger}
\]

Recently, the author has improved this result to include non-monotone sequences.

In this paper, we establish two further extensions. The first states that if \( \{a_k\} \) is a sequence of positive numbers satisfying
\[
a_2 \geq 0.5869890995 \cdots a_3, \quad \text{and} \quad a_{2j} \geq \frac{2j+1}{2j+2} a_{2j+1} \quad (j = 2, 3, \cdots),
\]
then (\star) implies (\dagger). An example is \( \{a_k\} = \{\frac{8}{5}, \frac{4}{3}, 1, \frac{8}{7}, 1, \cdots\} \), with \( a_k = 1 \) for even \( k \geq 4 \) and \( a_k = (k+1)/k \) for odd \( k \geq 3 \).

A second, independent, extension affirms that (\dagger) also holds under (\star) and
\[
a_{2j} \geq \frac{(2j+1)(4j-1)}{2j(4j+3)} a_{2j+1} \quad (j = 1, 2, \cdots).
\]

An example is \( \{3, \frac{4}{2}, \frac{7}{3}, \frac{7}{4}, \frac{11}{5}, \frac{11}{6}, \cdots\} \) where \( a_k = 2 - \frac{(-1)^k}{k} \).

The coefficients in these examples are not monotone and not converging to 0.

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1 Introduction

Excellent surveys on the history and applications of nonnegative trigonometric polynomials can be found, for example, in Alzer, Koumandos and Lamprecht [1], Askey et. al. [4]–[6], Brown [8], and Koumandos [9], and the references therein.

For convenience, we use the acronyms NN to stand for “non-negative”, and PS for P-Sum (a sum with all its partial sums NN). These can be interpreted as an adjective or a noun depending on the context. A sequence of real numbers is denoted by \( \{a_k\}_{k=1}^{\infty} \), or simply, \( \{a_k\} \). A finite \( n \)-tuple of numbers can be interpreted as an infinite sequence by adding 0 to the end. The symbol \( \searrow \) means non-increasing.

Following the convention adopted in [11], we use bold capital letters such as \( \mathbf{F} \) and \( \Phi \) to denote sums of numbers or functions. One of the earliest known PS is

\[
\mathbf{F} = \sum \frac{\sin(kx)}{k},
\]

first conjectured by Ferjér 1910, and confirmed independently by Jackson and Gronwall. Vietoris, in 1958, established a deep result that includes \( \mathbf{F} \).

**Theorem A** (Vietoris [12]). The sum \( \sum a_k \sin(kx) \) is a PS in \([0, \pi]\) (i.e. \((\dagger)\) holds) if \( a_k \searrow 0 \) and

\[
a_{2j-1} \geq \frac{2j}{2j-1} a_{2j}, \quad \text{for} \quad j = 1, 2, \ldots.
\]

**Remark 1.** There is an analogous cosine inequality \( a_1 + \sum a_k \cos(kx) \) is also a PS), but we are only concerned with the sine sum in this paper.

Belov, in 1995, greatly improved Vietoris’ sine inequality, by establishing, under the monotonicity requirement, a necessary and sufficient condition for PS.

**Theorem B** (Belov [7]). Assume \( a_k \searrow 0 \). Then \( \sum a_k \sin(kx) \) is a PS in \([0, \pi]\) iff

\[
\sum_{k=1}^{n} (-1)^{k-1} k a_k \geq 0, \quad \text{for all} \quad n \geq 2.
\]

**Remark 2.** For the cosine analog, condition \((1.3)\) is sufficient but not necessary.

Belov’s Theorem leaves no more room for improvement, unless the \( \searrow \) assumption on \( a_k \) is lifted. In this less restrictive situation, \((1.3)\) is no longer sufficient for PS (it is still necessary). It is not difficult to construct examples of PS sine sums with non-monotone coefficients, as we will see in Section 2. However, no useful general conditions applicable to non-monotone coefficients are known until recently. In [11], the following result was established.

**Theorem C.** Vietoris’ result remains valid when \( \searrow \) (still need \((1.2)\)) is relaxed to

\[
\frac{(2j-1)\sqrt{j} + 1}{2j \sqrt{j}} a_{2j+1} \leq a_{2j}, \quad j = 1, 2, \ldots.
\]
An example is given by the non-monotone sequence of coefficients:

\[ \begin{align*}
1, & \quad \frac{1}{2}, \quad \frac{1}{\sqrt{2}}, \\
& \quad \frac{3}{4\sqrt{2}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{5}{6\sqrt{3}}, \quad \cdots 
\end{align*} \tag{1.4} \]

\[ = 1, \quad 0.5, \quad 0.707 \cdots, \quad 0.530 \cdots, \quad 0.577 \cdots, \quad 0.481 \cdots, \quad \cdots \]

An important tool used in the proof is the well-known Comparison Principle (CP for short). It will continue to play an important role in this paper.

Since a non-zero scalar multiple of a PS is still a PS, we consider two sequences of coefficients equivalent if they only differ by a non-zero multiple. We say that

\[ \{a_k\} \succeq \{b_k\} \iff \begin{cases} 
(1) \ a_k = 0 \implies b_k = 0, \\
(2) \text{ after skipping those } a_k = 0, \quad \frac{b_k}{a_k} \searrow 0.
\end{cases} \]

This defines a partial ordering among equivalent classes of sequences. With this notation, the CP can be restated as follows.

**Lemma 1.** Let \( \sigma_k(x) \) be a sequence of functions defined on an interval \( I \).

\[ \sum a_k \sigma_k(x) \text{ PS in } I \text{ and } \{a_k\} \succeq \{b_k\} \implies \sum b_k \sigma_k(x) \text{ PS in } I. \]

**Remark 3.** Among all the sequences of coefficients satisfying Vietoris’ conditions, there is a maximal one, namely

\[ \{c_k\} = \left\{ 1, \quad \frac{1}{2}, \quad \frac{1}{\sqrt{2}}, \quad \frac{3}{4\sqrt{2}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{5}{6\sqrt{3}}, \quad \cdots \right\} \tag{1.5} \]

obtained by replacing the inequality sign in (1.2) by equality and letting \( a_{2j} = a_{2j+1} \). The CP reduces the proof of the general Vietoris inequality to just showing that the maximal sum \( V = \sum c_k \sin(kx) \) is PS.

In the same sense, (1.4) is the maximal sequence for Theorem C. On the other hand, there is no maximal sequence for Belov’s result.

In this paper, we present two further improvements of Theorem C. In order to better illustrate some of the main ideas, we first establish, in Section 3, a slightly weaker NN criterion that is associated with the sequence of coefficients

\[ \{\gamma_k\} = \left\{ 2, \quad 1, \quad \frac{4}{3}, \quad 1, \quad \frac{6}{5}, \quad 1, \quad \frac{8}{7}, \quad 1, \quad \cdots \right\}, \quad \gamma_k = \begin{cases} 
\frac{k+1}{k} & \text{k is odd} \\
1 & \text{k is even}
\end{cases}. \quad (1.6) \]

**Lemma 2.** \( \Psi = \sum a_k \sin(kx) \) is a PS in \([0, \pi]\) if (1.2) holds and

\[ a_{2j} \geq \frac{2j+1}{2j+2} a_{2j+1}, \quad \text{for } j = 1, 2, \cdots. \tag{1.7} \]

The maximal sum is given by \( \Phi = \sum \gamma_k \sin(kx) \).
Remark 4. Lemma 2 is already a significant improvement over Theorem A and C. The coefficients $a_k$ that satisfy the hypotheses of these Theorems must decay faster than $1/\sqrt{k}$. The coefficients of $\Phi$, on the other hand, converge to 1.

Lemma 2 can be sharpened in two different ways. Let $\alpha \approx 0.78265213271 \cdots$ be the second largest real root of the polynomial

$$54675a^4 - 2442195a^3 + 2182800a^2 - 115424a - 96429 = 0. \quad (1.8)$$

Theorem 1. $\Psi = \sum a_k \sin(kx)$ is a PS in $[0, \pi]$ if $\{a_k\}$ satisfies (1.2), $a_2 \geq \frac{3\alpha}{4} a_3$, and (1.7) for $j = 2, 3, \cdots$. \hfill (1.9)

The maximal sum is $\Phi_1$ with coefficients $\{2\alpha, \alpha, \gamma_3, \gamma_4, \gamma_5, \cdots\}$.

The value $\alpha$ is best possible; if it is replaced by any smaller positive number, then $\Phi_1(5)$ is not NN.

Remark 5. Note that even though the coefficients of $\Phi_1$ are not monotone, the subsequence of odd-order coefficients is decreasing, while the even-order coefficients are constant. Contrast this with $\Phi_2$ defined below. Its subsequence of even-order coefficients is increasing.

Let

$$\\{\delta_k\} = \left\{3, \frac{3}{2}, \frac{7}{3}, \frac{7}{4}, \frac{11}{5}, \frac{11}{6}, \cdots\right\}, \quad \delta_k = 2 - \frac{(-1)^k}{k}. \quad (1.10)$$

Theorem 2. $\Psi = \sum a_k \sin(kx)$ is a PS in $[0, \pi]$ if $\{a_k\}$ satisfies (1.2) and,

$$a_{2j} \geq \frac{(2j+1)(4j-1)}{2j(4j+3)} a_{2j+1}, \quad \text{for } j = 1, 2, \cdots. \quad (1.11)$$

The maximal sum is $\Phi_2 = \sum \delta_k \sin(kx)$.

Remark 6. Theorems 1 and 2 are independent of each other as their extremal sums are not related to each other by $\succeq$. The same is true for Lemma 2 and Theorem C. On the other hand, each of Theorems 1 and 2 implies both Lemma 2 and Theorem C. Yet, neither extends Belov’s result. It would be ideal if Belov’s result can be combined with Theorems 1 and 2 in a general unified way, but that remains a future goal for now.
By applying the reflection $x \mapsto (\pi - x)$ to $\Phi$ (or $\Phi_1$ and $\Phi_2$), we see that its PS property is equivalent to that of
$$\Theta = \sum (-1)^{k+1} \gamma_k \sin(kx)$$  \hspace{1cm} (1.12)
(and the corresponding $\Theta_i$, $i = 1, 2$).

For any $k \in (1, \infty)$, define
$$\phi_k(x) = \sin((k - 1)x) + \frac{k - 1}{k} \sin(kx),$$  \hspace{1cm} (1.13)
$$\theta_k(x) = \sin((k - 1)x) - \frac{k - 1}{k} \sin(kx).$$  \hspace{1cm} (1.14)

The partial sums $\Phi(n)$ and $\Theta(n)$ have the representations
$$\Phi(n) = 2 \phi_2(x) + \frac{4}{3} \phi_4(x) + \cdots + \frac{2n}{2n - 1} \phi_{2n}(x) + \left[ \frac{(2n + 2) \sin(nx)}{2n + 1} \right],$$  \hspace{1cm} (1.15)
$$\Theta(n) = 2 \theta_2(x) + \frac{4}{3} \theta_4(x) + \cdots + \frac{2n}{2n - 1} \theta_{2n}(x) + \left[ \frac{(2n + 2) \sin(nx)}{2n + 1} \right],$$  \hspace{1cm} (1.16)
where $\bar{n}$ denotes the largest integer less than or equal to $n/2$, and the notation $[\cdot]$ means that the term is present only if $n$ is an odd integer.

**Remark 7.** An alternative way to see that Theorem 1 implies Lemma 2 is to note that $\Phi = 2(1 - \alpha)\phi_2 + \Phi_1$.

The first term on the righthand side is NN and the second term is a PS. Likewise,
$$\Phi = F + \Phi_2,$$  \hspace{1cm} (1.17)
where $F$ is the Ferjér-Jackson-Gronwall PS, shows that Theorem 2 implies Lemma 2.

The following well-known identities will be used in subsequent proofs.

$$\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin((2n - 1)x) = \frac{\cos(\frac{x}{4}) - \cos(\frac{(2n+1)x}{4})}{\sin(\frac{x}{4})},$$  \hspace{1cm} (1.18)
$$\sin(x) + \sin(2x) + \sin(3x) + \cdots + \sin(nx) = \frac{\cos(\frac{x}{2}) - \cos(\frac{2nx}{2})}{\sin(\frac{x}{2})}.$$  \hspace{1cm} (1.19)
$$\cos(x) + \cos(3x) + \cos(5x) + \cdots + \cos((2n - 1)x) = \frac{\sin(2nx)}{2\sin(x)}. \hspace{1cm} (1.20)$$
$$\cos(x) - \cos(2x) + \cos(3x) - \cdots + (-1)^n \cos(nx) = \frac{1}{2} + (-1)^{n} \frac{\cos((2n+1)x/2)}{2\cos(x/2)}. \hspace{1cm} (1.21)$$

The rest of the paper is organized as follows. In Section 2 we give some examples of PS sine sums with non-monotone coefficients that can be easily constructed using known results. These examples should be contrasted with those covered by Theorems 1 and 2. The proofs of Lemma 2 and Theorems 1 and 2 are given in Sections 3, 4, and 5, respectively. Section 6 presents some further examples and remarks.
2  Trivial Examples of PS with Non-Monotone Coefficients

**Example 1.** Assume $b_k \searrow 0$. Then $B = \sum b_k \sin((2k-1)x)$ is a PS in $[0, \pi]$.

Consider
\[ C(n) = \sin(x) + \sin(3x) + \cdots + \sin((2n-1)x). \] (2.1)
From (1.18), we see that
\[ 2\cos(x) C(n) = 1 - \cos(2nx) \geq 0. \] (2.2)
Hence, $C$ is PS. It follows from the CP that $B$ is also PS.

Even though the sequence $\{b_k\}$ is decreasing, from the point of view of the full sine sum, the coefficient sequence is actually $\{b_1, 0, b_2, 0, b_3, 0, \cdots\}$, which is not monotone.

**Example 2.** Let $B$ and $C$ be as in Example 1 and $V$ be the Vietoris sum as in Remark 3.
\[ C + V = 2 \sin(x) + \frac{1}{2} \sin(2x) + \frac{3}{2} \sin(3x) + \frac{3}{8} \sin(4x) + \cdots, \] (2.3)
is a PS with non-monotone coefficients. More generally, $\beta B + V$ is a PS for any $\beta > 0$.

**Example 3.** By applying the reflection $x \mapsto \pi - x$ to $V$, we see that
\[ V_2 = \sum (-1)^{k+1} c_k \sin(kx) \] (2.4)
is a PS in $[0, \pi]$, so is $2V + V_2$ with coefficients
\[ 3, \frac{1}{2}, \frac{3}{2}, \frac{3}{8}, \frac{9}{8}, \frac{5}{16}, \ldots \] (2.5)

**Example 4.** It is easy to construct specific sine polynomials with a finite number of terms and non-monotone coefficients that are PS in $[0, \pi]$. For example
\[ 2\sin(x) + \sin(2x) + \left(1 + \frac{\sqrt{3}}{2}\right) \sin(3x) \] (2.6)
and
\[ 3\sin(x) + \sin(2x) + \left(\frac{3}{2} + \sqrt{2}\right) \sin(3x) \] (2.7)
are both PS in $[0, \pi]$ with non-monotone coefficients. We refer the readers to [10] for a discussion of how these and similar polynomials can be constructed.
It is also easy to prove that for any positive integer \( m \),
\[
\sin(x) + \frac{\sin(mx)}{m}
\]
is a PS in \([0, \pi]\) with non-monotone coefficients.

If one insists on constructing examples with an infinite number of terms, simply add an appropriate multiple of one of these to \( V \).

We consider all such examples trivial because they are easy corollaries of Vietoris’ result and other known examples.

3 Proof of Lemma 2

Lemma 2 is obviously true for \( n = 1, 2 \) and 3. Hence, we assume \( n \geq 4 \) in the following.

Lemma 3. For all \( k > 1 \),
\[
\theta_k(x) \geq 0 \quad \text{for} \ x \in \left[0, \frac{\sigma}{k}\right],
\]
where \( \sigma \approx 4.493409458 \) is the first positive zero of the function
\[
f(z) = \sin(z) - z \cos(z).
\]

Proof. Let \( \mu = 1 - \frac{1}{k} \in (0, 1) \) and \( y = kx \). Then, from the definition (1.13),
\[
\frac{\theta_k(x)}{\mu} = \frac{\sin(\mu y)}{\mu} - \sin(y).
\]

\[
\frac{\partial}{\partial \mu} \left( \frac{\theta_k(y)}{\mu} \right) = -\frac{\sin(\mu y) - \mu y \cos(\mu y)}{\mu^2} = -\frac{f(\mu y)}{\mu^2}.
\]

For \( x \in [0, \sigma/k] \), \( \mu y \in [0, \sigma] \). Since \( f(z) \) is positive in \((0, \sigma)\), the righthand side of (3.4) is negative, implying that \( \theta_k(y)/\mu \) is a decreasing function of \( \mu \). Hence,
\[
\frac{\theta_k(x)}{\mu} \geq \lim_{k \to \infty} \frac{\theta_k(x)}{\mu} = 0.
\]

Lemma 4. For any integer \( n > 0 \),
\[
\Phi(n) \geq 0 \quad \text{in} \quad \left[0, \frac{\pi}{n}\right] \cup \left[\pi - \frac{\pi}{n}, \pi\right].
\]
Proof. In $[0, \pi/n]$, every term in $\Phi(n)$ is NN and so is their sum.

The assertion $\Phi(n) \geq 0$ in $[\pi - \pi/n, \pi]$ is equivalent to $\Theta$ being NN in $[0, \pi/n]$. We make use of the representation (1.16) of $\Theta(n)$. If $n$ is even, $\Theta$ is a sum of positive multiples of $\theta_{2j}(x)$, for $j = 1, \cdots, \tilde{n}$. By Lemma 3, each of these is NN in $[0, \sigma/2\tilde{n}] \supset [0, \pi/n]$. Hence, their sums is NN in $[0, \pi/n]$ and the conclusion still holds. 

In view of Lemma 4 to complete the proof of Lemma 2, it remains to show that $\Phi(n)$ is NN in $I_n = [\pi/n, \pi - \pi/n]$ for all $n$.

Let $m = n$ if $n$ is odd, and $n - 1$ otherwise. It is the largest odd integer $\leq n$. Then $\Phi(n) = S_1(n) + T(m)$, where

$S(n) = \sin(x) + \sin(2x) + \cdots + \sin(nx)$

and

$T(m) = \sin(x) + \frac{\sin(3x)}{3} + \cdots + \frac{\sin(mx)}{m}$. 

Identity (1.19) gives a lower bound for $S(n)$.

$S(n) \geq \frac{\cos(x/2) - 1}{2\sin(x/2)}$

$= \frac{-\tan(x/4)}{2}$

$\geq -\frac{1}{2}$. 

The proof of Lemma 2 is thus complete if we can show that

$T(m) \geq \frac{1}{2}, \quad x \in I_n, \quad n \geq 4$. 

(3.11)

When $n$ is even, $n$ and $n - 1$ use the same $T(m)$, but $I_{n-1} \subset I_n$. Hence, if (3.11) can be proved for $n$, then it will also hold for $n - 1$. In other words, we only have to establish (3.11) for even $n$, in which case $m = n - 1$. Note that $T(m)$ is an even function about $x = \pi/2$. Thus, it suffices to show (3.11) for odd $m$ and $x \in J_n = [\pi/n, \pi/2]$.

An alternative representation for $T(m)$ can be given using (1.20).

$T(m) = f_n(x) := \int_0^x (\cos(s) + \cos(3s) + \cdots + \cos((n - 1)s)) \, ds$

$= \int_0^x \frac{\sin(ns)}{2\sin(s)} \, ds$. 

(3.12)

For convenience, we revert back to using $n = m - 1$ instead of $m$. Besides being easier to estimate, another advantage of the alternative representation is that the definition of $f_n(x)$
can be extended to all real $n \in (0, \infty)$. Even though we only need (3.11) for even integers $n$, we are going to prove the stronger inequality

$$ f_n(x) \geq \frac{1}{2}, \quad x \in J_n, \quad n \geq 4. \quad (3.13) $$

The graph of one of these functions, $f_{23}(x)$, is depicted in Figure 1.

Since $f_n'(x) = \sin(nx)/\sin(x)$, the critical points of $f_n(x)$ in $J_n$ are $\pi/n, 2\pi/n, 3\pi/n, \cdots$. The first is the left endpoint of $J_n$ and is a local maximum, so are all other odd-order points. The even-order points $x_2 = 2\pi/n, x_4 = 4\pi/n, \cdots$ are local minima. A lower bound for $f_n(x)$ in $J_n$ is, therefore,

$$ \min_{x \in J_n} f_n(x) = \min\{ f_n(x_2), f_n(x_4), \cdots \}. \quad (3.14) $$

Integration by parts gives

$$ f_n(x_{2k+2}) - f_n(x_{2k}) = \int_{x_{2k}}^{x_{2k+2}} \frac{\sin(ns)}{2\sin(s)} \, ds $$

$$ = \int_{x_{2k}}^{x_{2k+2}} \frac{(1 - \cos(ns)) \cos(s)}{2n \sin^2(s)} \, ds $$

$$ > 0. \quad (3.15) $$

Hence, $f_n(x_2) < f_n(x_4) < f_n(x_6) < \cdots$ and it follows from (3.14) that

$$ f_n(x) \geq f_n(x_2). \quad (3.16) $$

Now (3.13) follows from the next Lemma and the proof of Lemma 2 is complete.
Lemma 5. The sequence \( f_n(x_2), n = 4, 5, \cdots \) is increasing.

\[
\frac{2}{3} = f_4(x_2) < f_5(x_2) < \ldots < f_n(x_2) < f_{n+1}(x_2) < \ldots \tag{3.17}
\]

Proof. That \( f_4(x_2) = 2/3 \) can be verified directly. In fact, each \( f_n(x_2) \) can be computed exactly using Maple.

The change of variable, \( s = t/n \) gives

\[
f_n(x_2) = \int_0^{2\pi/n} \frac{\sin(ns)}{2\sin(s)} ds = \int_0^{2\pi} \frac{\sin(t)}{2n\sin(t/n)} dt = \int_0^\pi k_n(t) \sin(t) ds, \tag{3.18}\]

where

\[
k_n(t) = \frac{1}{2n\sin(t/n)}. \tag{3.19}\]

Thus,

\[
f_{n+1}(x_2) - f_n(x_2) = \int_0^{2\pi} (k_{n+1}(t) - k_n(t)) \sin(t) dt. \tag{3.20}\]

In the next Lemma, we show that

\[
h_n(t) = k_n(t) - k_{n+1}(t) \tag{3.21}\]

is a positive increasing function of \( t \in [0, 2\pi] \). Anticipating this fact, we see that

\[
f_{n+1}(x_2) - f_n(x_2) = \int_0^\pi |\sin(t)| h_n(t) dt - \int_0^{2\pi} \sin(t) h_n(t) dt \]

\[
> h_n(\pi) \int_0^\pi |\sin(t)| dt - h_n(\pi) \int_0^{2\pi} \sin(t) dt
\]

\[
= 0. \tag{3.22}\]

as desired. ■

Lemma 6. For \( n \geq 4 \), \( h_n(t) \) is a positive increasing function of \( t \) in \([0, 2\pi]\).

Proof. The NN of \( h_n(t) \) follows from the fact that, for fixed \( t \), \( k_n(t) \) is a decreasing function of \( n \), which is equivalent to the fact that \( n\sin(t/n) \) is an increasing function of \( n \).

The increasing property of \( h_n(t) \) is true if we can prove that

\[
\frac{\partial^2}{\partial n \partial t} k_n(t) \leq 0. \tag{3.23}\]

Direct computation gives the numerator of \(-\frac{\partial^2}{\partial n \partial t} k_n(t)\) as the function

\[
\xi(t) = 2\cos^2(t/n) + t \sin^2(t/n) - 2n \cos(t/n) \sin(t/n). \tag{3.24}\]
For convenience, we have suppressed the dependence of $\xi(t)$ on $n$. Now it suffices to show that $\xi(t) \geq 0$ for $t \in [0, 2\pi]$. Since $\xi(0) = 0$, if we can show that $\xi'(t) \geq 0$, the proof is complete.

$$\xi'(t) = \sin\left(\frac{t}{n}\right) \left[3 \sin\left(\frac{t}{n}\right) - \frac{2t}{n} \cos\left(\frac{t}{n}\right)\right] = \sin\left(\frac{t}{n}\right) \xi_2(t). \quad (3.25)$$

It now suffices to show that $\xi_2(t)$ is NN. The desired conclusion follows from the facts $\xi_2(0) = 0$, and

$$\xi_2'(t) = \frac{1}{n} \cos\left(\frac{t}{n}\right) + \frac{2t}{n^2} \sin\left(\frac{t}{n}\right) \geq 0. \quad (3.26)$$

4 Proof of Theorem 1

We first take care of $n > 20$. The partial sums $\Phi_1(n)$ can be represented as

$$\Phi_1(n) = \Phi(n) - \lambda \phi_2(x)$$

$$= S(n) + f_n(x) - \lambda \phi_2(x), \quad (4.1)$$

where $\lambda = 2 - 2\alpha \approx 0.434695735$. In view of (3.9) and Lemma 5, we get, for all $n > 20$, $x \in [0, \pi]$,

$$\Phi_1(n) \geq F(x) := -\frac{\tan(x/4)}{2} + f_{20}(x_2) - \frac{4347}{10000} \phi_2(x). \quad (4.2)$$

Maple gives

$$f_{20}(x_2) = \frac{2}{15} + \frac{1580}{4641} \cos\left(\frac{\pi}{5}\right) + \frac{1820}{1881} \cos\left(\frac{2\pi}{5}\right) > \frac{73542}{103909}. \quad (4.3)$$

It follows from (4.2) and (4.3) that

$$\Phi_1(n) \geq F_1(x) := \frac{73542}{103909} - \frac{\tan(x/4)}{2} - \frac{4347}{10000} \left(\sin(x) + \frac{\sin(2x)}{2}\right). \quad (4.4)$$

Let $T = \tan(x/4)$. Since $x \in [0, \pi]$, we have $T \in [0, 1]$. Then

$$F_1(x) = \frac{73542}{103909} - \frac{T}{2} - \frac{4347}{1250} \frac{T(1 - T^2)^3}{(1 + T^2)^4}$$

$$= \frac{P(T)}{(1 + T^2)^4},$$

where

$$P(T) = -45963750 T^9 + 91927500 T^8 + 267837423 T^7 + 367710000 T^6 - 1630859769 T^5 + 551565000 T^4 + 1171222269 T^3 + 367710000 T^2 - 497656173 T + 91927500.$$
The classical Sturm Theorem, provides a way to find the number of real roots of an algebraic polynomial with real coefficients within any given subinterval of the real line. It can be used (see [10] and the discussion below) to show that $P(T) > 0$ in $[0, 1]$. With this fact, we conclude that $\Phi_1(n) > 0$ for $x \in [0, \pi], n > 20$.

For $n \leq 20$, the above argument does not work because when $f_{20}(x_2)$ is replaced by any $f_n(x_2)$ with $n < 20$, the resulting $F(x)$ is no longer NN in $[0, \pi]$. Our verification of Theorem 1 for $n \leq 20$, relies on a brute-force technique based on the Sturm Theorem. The method is explained in great details in [10]. See also [2] which discusses its use in the study of Rogosinski-Szegö-type inequalities [3].

In a nutshell, given any specific sine polynomial, we can expand it into a product of $\sin(x)$ and an algebraic polynomial $p(Y)$ of the variable $Y = \cos(x) \in [-1, 1]$. The Sturm Theorem can then be invoked to check if $p(Y)$ is NN or not. This procedure works with one polynomial at a time. It is, therefore, not adequate to prove general results like Theorem 1, which involves an infinite number of polynomials. Nevertheless, we can comfortably use this technique to deal with the first 20 of such polynomials.

The procedure we implemented in Maple, however, has one limitation. It works only when the coefficients of the sine polynomial are given rational numbers. For this reason, it cannot be directly applied to the sine polynomials of Theorem 1 because they involve the irrational number $\alpha$. The procedure is modified as follows. For $n \leq 20$, except $n = 5$, we replace $\alpha$ by the slightly smaller rational number $\alpha = 171/100 < \alpha$. The corresponding partial sums $\Phi_1(n)$ is shown to be NN using the Maple procedure. It then follows from the CP that $\Phi_1(n)$ is also NN.

With $\Phi_1(5)$, the above approach encounters a different problem. No matter what $\alpha < \alpha$ is chosen, $\Phi_1$ is not NN. In fact, $\alpha$ has been chosen to be critical in some sense, namely,

$$\alpha = \inf \{a \mid p_a(Y) \geq 0 \text{ in } [0, \pi] \}.$$  

Here $p_a(Y)$ is the algebraic polynomial

$$p_a(Y) = 144Y^4 + 60Y^3 - 68Y^2 + (15a - 30)Y + (15a - 1). \quad (4.5)$$

associated with the sine polynomial

$$2a \sin(x) + a \sin(2x) + \sum_{k=3}^{5} \gamma_k \sin(kx) \geq 0. \quad (4.6)$$

For large $a$, for example $a = 2$, $p_a(Y)$ is NN in $[-1, 1]$; its graph lies above and away from the $Y$-axis. On the other hand, when $a = 0$, the graph crossed the $Y$-axis. As $a$ increases from 0, the graph of $p_a(Y)$ rises monotonically. By continuity, there is a value of $a = \alpha$ when the graph is about to leave the $Y$-axis; it is tangent to the $Y$-axis at one or more points. To determine $\alpha$, note that each point of tangency corresponds to a double root of $p_a(Y) = 0$. 

12
A necessary condition for having a double root is the vanishing of the discriminant. With
the help of Maple, the discriminant, after deleting a numerical factor, is found to be 1.8.
Numerical computation yields four real roots of 1.8: -0.17, 0.30, 0.78, and 43.76. Hence,
\( \alpha \) is the second largest root.

5 Proof of Theorem 2

As in the proof of Theorem 1, we can use the Sturm procedure to confirm Theorem 2 for
small \( n \), more specifically, we have done that for \( n \leq 20 \). Hence, we assume \( n > 20 \) in the
rest of this section.

The partial sums of \( \Phi_2 \) have the representation

\[
\Phi_2(n) = 2S(n) + U(n),
\]

where \( S \) is given by (3.7) and

\[
U(n) = \sin(x) - \frac{\sin(2x)}{2} + \cdots - \frac{(-1)^n \sin(nx)}{n} \]

\[
= \int_0^x \left( \cos(s) - \cos(2s) + \cdots - (-1)^n \cos(ns) \right) ds
\]

\[
= \frac{x}{2} + (-1)^n \int_0^x \frac{\cos \left( \frac{(2n+1)s}{2} \right)}{2 \cos \left( \frac{s}{2} \right)} ds.
\]

We have used (1.21) to derive the last equality. By Lemma 4, we only have to show that
\( \Phi_2(n) \geq 0 \) in \( I_n = [\pi/n, \pi - \pi/n] \). Using (3.9), (5.1) and (5.2), we see that

\[
\Phi_2(n) \geq -\tan \left( \frac{x}{4} \right) + \frac{x}{2} - h_n(x),
\]

where

\[
h_n(x) = (-1)^{n+1} \int_0^x \frac{\cos \left( \frac{(2n+1)s}{2} \right)}{2 \cos \left( \frac{s}{2} \right)} ds.
\]

Since \( \tan(x/4) \leq 0.32x \) for \( x \in [0, \pi] \), (5.3) leads to

\[
\Phi_2(n) \geq 0.18 x - h_n(x).
\]

Hence, Theorem 2 is proved if we can show that

\[
h_n(x) \leq 0.18 x, \quad x \in I_n, \quad n \geq 21.
\]

With change of variables, \( s \mapsto 2t, x \mapsto 2y \) and \( 2n + 1 \mapsto \hat{m} \), (5.6) becomes

\[
g_{\hat{m}}(y) \leq 0.18 y, \quad y \in I_{\hat{m}}, \quad \hat{m} = 43, 45, 47, \cdots ,
\]

where

\[
g_{\hat{m}}(y) = (-1)^{(\hat{m}+1)/2} \int_0^y \frac{\cos(\hat{m}t)}{2 \cos(t)} dt
\]

and \( I_{\hat{m}} = [\pi/(\hat{m} - 1), \pi/2 - \pi/(\hat{m} - 1)] \). In fact, we claim that (5.7) holds in the bigger
interval \( J_m = [\pi/\hat{m}, \pi/2] \).
The wavy curve in Figure 2 depicts the graph of $g_{23}(y)$ and the dashed line is the graph of $0.18y$. It is clear from the figure that, in this case, (5.7) fails for small positive $y$. When $\hat{m} = 1 \pmod{4}$, however, $g_{\hat{m}}(y)$ is negative for $y \in [0, \pi/\hat{m}]$ and it can be shown that (5.7) holds in the whole interval $[0, \pi/2]$.

The shape of $g_{\hat{m}}(y)$ is strikingly similar to that of $f_{\hat{m}}(x)$ in Figure 1. Indeed, by using the reflection mapping $y = \pi/2 - x$, one can show that $g_{\hat{m}}(y) = f_{\hat{m}}(\pi/2 - y) - f_{\hat{m}}(\pi/2)$.

For instance, the critical points of $g_{\hat{m}}(y)$ are given by the sequence

$$y_{(\hat{m} - 1)/2} = \frac{\pi}{2\hat{m}} < y_{(\hat{m} - 3)/2} = \frac{3\pi}{2\hat{m}} < \cdots < y_1 = \frac{(\hat{m} - 2)\pi}{2\hat{m}}.$$ 

Note that we have numbered the critical points $y_k$ from right to left. The first one, $y_1$, is always a local maximum and then they alternate as local minimum and maximum. The last one, $y_{(\hat{m} - 1)/2}$ is is a minimum or maximum depending on whether $(\hat{m} + 1)/2$ is odd or even. The sequence of local maximum (minimum) values $g_{\hat{m}}(y_i)$ is decreasing (increasing) as $i$ increases. The global maximum of $g_{\hat{m}}(y)$ is attained at $y_1$.

**Lemma 7.** For all odd integers $\hat{m} \geq 43$,

$$g_{\hat{m}}(y) \leq 0.22, \quad y \in [0, \pi/2].$$

**Proof.** Let us estimate

$$g_{\hat{m}}(y_1) - g_{\hat{m}}(y_2) = \int_{y_2}^{y_1} \frac{|\cos(\hat{m}t)|}{2\cos(t)} dt = \int_{\pi}^{2\pi} \frac{\sin(s)}{2\hat{m} \sin(s/\hat{m})} ds.$$
For fixed \( s \in [\pi, 2\pi] \), \( \hat{m} \sin(s/\hat{m}) \) is an increasing function of \( \hat{m} \geq 43 \). As a result, \( g_{\hat{m}}(y_1) - g_{\hat{m}}(y_2) \) is a decreasing function of \( \hat{m} \). In particular,

\[
g_{\hat{m}}(y_1) - g_{\hat{m}}(y_2) \leq g_{43}(y_1) - g_{43}(y_2) = 0.21731814075 \ldots.
\]

Here we have abused the notation: \( y_1 \) and \( y_2 \) on the lefthand side of the inequality are different from those on the other side. Since \( g_{\hat{m}}(y_2) < 0 \), the desired conclusion follows. \( \blacksquare \)

Obviously, Lemma 7 implies that (5.7) holds on \([11/9, \pi/2]\). It remains to show (5.7) on \([\pi/\hat{m}, 11/9]\). Our next Lemma shows that in this subinterval, (5.9) can be greatly improved.

**Lemma 8.** For all odd integers \( \hat{m} \geq 43 \),

\[
g_{\hat{m}}(y) \leq 0.06, \quad y \in [0, 11/9].
\]

**Proof.** For \( \hat{m} = 43 \), the first (counting from \( y_1 \)) local maximum that falls within the subinterval \([0, 11/9]\) is \( y_5 \), and we compute

\[
g_{43}(y_5) - g_{43}(y_6) = 0.05955292306 \ldots.
\]

Using the same arguments as in the proof of Lemma 7, we see that \( g_{\hat{m}}(y_5) - g_{\hat{m}}(y_6) \) is a decreasing function of \( \hat{m} \). Hence,

\[
g_{\hat{m}}(y_5) - g_{\hat{m}}(y_6) < 0.05955292306 \ldots
\]

and the desired conclusion follows. \( \blacksquare \)

Lemma 8 implies that (5.7) holds on \([1/3, 11/9]\). It remains to show (5.7) on \([\pi/\hat{m}, 1/3]\). We use a different method to estimate \( g_{\hat{m}}(y) \) in this interval. For \( t \in [0, 1/3] \),

\[
1 = \frac{1}{\cos(t)} \leq \frac{1}{\cos(1/3)} < 1.06.
\]

It follows that

\[
\frac{\cos(\hat{m}t)}{2\cos(t)} \leq \frac{1}{2} \cos(\hat{m}t) + 0.03
\]

and

\[
- \frac{\cos(\hat{m}t)}{\cos(t)} \leq - \frac{1}{2} \cos(\hat{m}t) + 0.03.
\]

We consider two cases. When \( (\hat{m} + 1)/2 \) is even, then from (5.8) and (5.15), we obtain

\[
g_{\hat{m}}(y) \leq \frac{\sin(\hat{m}y)}{2\hat{m}} + 0.03y.
\]

It is not difficult to see that this implies (5.7) in \([\pi/\hat{m}, 1/3]\).

In the complementary case, when \( (\hat{m} + 1)/2 \) is odd, we use (5.8) and (5.16) to obtain

\[
g_{\hat{m}}(y) \leq - \frac{\sin(\hat{m}y)}{2\hat{m}} + 0.03y.
\]

This implies (5.7) in \([0, 1/3] \supset [\pi/\hat{m}, 1/3]\), and completes the proof of Theorem 2.
6 Further Examples and Remarks

Example 5. Theorem 1 can be applied to show that the sum
\[
\frac{\phi_2(x)}{\sqrt{2}} + \frac{\phi_4(x)}{\sqrt{3}} + \frac{\phi_6(x)}{\sqrt{4}} + \cdots + \frac{\phi_{2n}(x)}{\sqrt{n+1}} + \left[ \frac{\sin(nx)}{\sqrt{n+2}} \right]
\]
is a PS. It is not covered by Theorem 2. More generally, Theorem 1 implies that
\[
\frac{\phi_2(x)}{\sqrt{\beta + 1}} + \frac{\phi_4(x)}{\sqrt{\beta + 2}} + \frac{\phi_6(x)}{\sqrt{\beta + 3}} + \cdots + \frac{\phi_{2n}(x)}{\sqrt{\beta + n}} + \left[ \frac{\sin(nx)}{\sqrt{\beta + n + 1}} \right]
\]
is PS for \( \beta \geq \frac{8 - 9\alpha^2}{9\alpha^2 - 4} \approx 1.64393 \). Numerical experiments suggest that the sum is PS for \( \beta > 1.76923 \).

Example 6. Theorem 1 implies that
\[
\phi_2(x) + \frac{\phi_4(x)}{2^\gamma} + \cdots + \frac{\phi_{2n}(x)}{n^\gamma} + \left[ \frac{\sin(nx)}{(n+1)^\gamma} \right]
\]
is PS for \( \gamma \geq 0.26 \). Theorem 2 performs worse in this case, giving only \( \gamma \geq 0.36258 \). Numerical experiments suggest that the sum may be a PS for \( 0.24 \leq \gamma < 0.26 \), but not for \( \gamma = 0.23 \). In the latter case, all partial sums except the sixth are NN in \([0, \pi]\).

These two examples indicate that Theorem 1 and 2 are not best possible.

Remark 8. In Theorems A, C, 1 and 2, the extremal sums are characterized by their respective subsequences of odd-order coefficients, namely
\[
\{c_{2j-1}\} = \left\{ 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \cdots \right\},
\]
\[
\left\{ 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots \right\},
\]
\[
\{2\alpha, \gamma_{2j+1}\} = \left\{ 2\alpha, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \cdots \right\},
\]
and
\[
\{\delta_{2j-1}\} = \left\{ 3, \frac{7}{3}, \frac{11}{5}, \frac{15}{7}, \cdots \right\}.
\]
The relative strength of the various results can be determined by comparing these sequences according to the CP. For instance, sequence 1 \( \succeq \) sequence 2, while each of sequences 3 and 4 is \( \succeq \) sequences 1 and 2. To look for an improvement of Theorems 1 and 2, one searches find a sequence \( \succeq \) sequence 3 or 4 that yields a PS. Note that \( \{1, 1, \cdots\} \succeq \) sequence 3 and 4, but its associated sine sum is not a PS. In other words, \( \{1, \cdots\} \) is a strict upper bound of all possible improvements of Vietoris’ sine result.
Remark 9. Theorem 1 relaxes the first condition, in (1.2), of the Vietoris result. It is natural to ask whether the second condition in (1.2) can also be relaxed by replacing some of the factors $\rho_j = \frac{2j}{2j-1}$ with larger constants. The following observation concerning Belov’s necessary condition (1.3) leads to the answer no.

**Lemma 9.** (i) A necessary condition for any sine polynomial $\sum_{k=1}^{n} a_k \sin(a_k x)$ to be NN in some neighborhood $[\pi - \epsilon, \pi]$, $0 < \epsilon < \pi$ is

$$\sum_{k=1}^{n} (-1)^{k-1} k a_k \geq 0. \quad (6.1)$$

(ii) A necessary condition for $\sum_{k=1}^{n} a_k \sin(a_k x)$ to be NN in some neighborhood $[0, \epsilon]$, $0 < \epsilon < \pi$ is

$$\sum_{k=1}^{n} k a_k \geq 0 \quad (6.2)$$

**Proof.** Let us prove (i). By assumption

$$0 \leq \sum_{k=1}^{n} \frac{a_k \sin(kx)}{\pi - x} \quad (6.3)$$

for all $x \in [\pi - \epsilon, \pi]$ By taking the limit as $x \to \pi$, we get (using, for example, L’Hôpital’s rule)

$$0 \leq \lim_{x \to \pi} \sum_{k=1}^{n} \frac{a_k \sin(kx)}{\pi - x} = \sum_{k=1}^{n} (-1)^{k+1} a_k. \quad (6.4)$$

The proof of (ii) is similar. □

Remark 10. In the hypotheses of the Lemma, $a_k$ are not required to be of the same sign or monotone. Also note that unlike in the Belov condition, we are assuming in the hypothesis only that the sine polynomial itself (not any of its proper partial sums) is NN, and only one inequality (6.1) is required to hold (not for all $n$).

Remark 11. As Belov already pointed out, his condition (1.3) is no longer sufficient without the additional monotonicity requirement on the coefficients. We give an example related to our sum $\Phi$. It is easy to verify that the polynomial

$$2 \sin(x) + \sin(2x) + \frac{4}{3} \sin(3x) + \sin(4x) + \frac{6}{5} \sin(5x) + \frac{6}{8} \sin(8x)$$

is not NN in $[0, \pi]$, although it satisfies (1.3). This polynomial is constructed by taking $\Phi(5)$, the first five terms of $\Phi$, skipping the terms involving $\sin(6x)$ and $\sin(7x)$ and add the next term with a suitable coefficient to satisfy (1.3). The same is true for the polynomial...
constructed using $\Phi(5)$ and $\sin(10x)$. However, we notice that, after that, all polynomials of the form

$$\Phi(5) + \frac{6}{n} \sin(nx), \quad n = 12, 14, 16, \cdots$$

are all PS.

**Remark 12.** Another natural question to ask is whether our Theorem 1 has a cosine counterpart, namely, whether $\sum \gamma_k \cos(kx)$ is a PS, if $\gamma_0 = \gamma_1$ and $\gamma_k$ is given by (1.6) for $k = 1, 2, \cdots$. The answer is also no. For $x = \pi$, the cosine series becomes

$$\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots$$

and every partial sum with an even number of terms is negative, because $\gamma_2 < \gamma_3$, $\gamma_4 < \gamma_5$, etc. A similar observation applies to the analogous sum $\sum \delta_k \cos(kx)$.

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