Approximating CSPs with Global Cardinality Constraints Using SDP Hierarchies

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Abstract

This work is concerned with approximating constraint satisfaction problems (CSPs) with an additional global cardinality constraints. For example, Max Cut is a boolean CSP where the input is a graph \( G = (V, E) \) and the goal is to find a cut \( S \cup \bar{S} = V \) that maximizes the number of crossing edges, \( |E(S, \bar{S})| \). The Max Bisection problem is a variant of Max Cut with an additional global constraint that each side of the cut has exactly half the vertices, i.e., \( |S| = |V|/2 \). Several other natural optimization problems like Min Bisection and approximating Graph Expansion can be formulated as CSPs with global constraints.

In this work, we formulate a general approach towards approximating CSPs with global constraints using SDP hierarchies. To demonstrate the approach we present the following results:

- Using the Lasserre hierarchy, we present an algorithm that runs in time \( O(n^{\text{poly}(1/\varepsilon)}) \) that given an instance of Max Bisection with value \( 1 - \varepsilon \), finds a bisection with value \( 1 - O(\sqrt{\varepsilon}) \). This approximation is near-optimal (up to constant factors in \( O() \)) under the Unique Games Conjecture.

- By a computer-assisted proof, we show that the same algorithm also achieves a 0.85-approximation for Max Bisection, improving on the previous bound of 0.70 (note that it is Unique Games hard to approximate better than a 0.878 factor). The same algorithm also yields a 0.92-approximation for Max 2-Sat with cardinality constraints.

- For every CSP with a global cardinality constraints, we present a generic conversion from integrality gap instances for the Lasserre hierarchy to a dictatorship test whose soundness is at most integrality gap. Dictatorship testing gadgets are central to hardness results for CSPs, and a generic conversion of the above nature lies at the core of the tight Unique Games based hardness result for CSPs. [Rag08]
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1 Introduction

Constraint Satisfaction Problems (CSP) are a class of fundamental optimization problems that have been extensively studied in approximation algorithms and hardness of approximation. In a constraint satisfaction problem, the input consists of a set of variables taking values over a fixed finite domain (say \{0, 1\}) and a set of \textit{local} constraints on them. The constraints are \textit{local} in that each of them depends on at most \(k\) variables for some fixed constant \(k\). The goal is to find an assignment to the variables that satisfies the maximum number of constraints.

Over the last two decades, there has been much progress in understanding the approximability of CSPs. On the algorithmic front, semidefinite programming (SDP) has been used with great success in approximating several well-known CSPs such as Max Cut [GW95], Max 2-Sat [CMM07] and Max 3-Sat [KZ97]. More recently, these algorithmic results have been unified and generalized to the entire class of constraint satisfaction problems [RS09a]. With the development of PCPs and long code based reductions, tight hardness results matching the SDP based algorithms have been shown for some CSPs such as Max-3-SAT [H01]. In a surprising development under the Unique Games Conjecture, semidefinite programming based algorithms have been shown to be optimal for Max Cut [KKMO07], Max 2-Sat [Aus07] and more generally every constraint satisfaction problem [Rag08].

Unfortunately, neither SDP based algorithms nor the hardness results extend satisfactorily to optimization problems with \textit{non-local} constraints. Part of the reason is that the nice framework of SDP based approximation algorithms and matching hardness results crucially rely on the \textit{locality} of the constraints involved. Perhaps the simplest non-local constraint would be to restrict the cardinality of the assignment, i.e., the number of ones in the assignment. Variants of CSPs with even a single cardinality constraint are not well-understood. Optimization problems of this nature, namely constraint satisfaction problems with global cardinality constraints are the primary focus of this work. Several important problems such as Max Bisection, Min Bisection, Small-Set Expansion can be formulated as CSPs with a single global cardinality constraint.

As an illustrative example, let us consider the Max Bisection problem which is also part of the focus of this work. The Max Bisection problem is a variant of the much well-studied Max Cut problem [GW95, KKMO07]. In the Max Cut problem the goal is to partition the vertices of the input graph into two sets while maximizing the number of crossing edges. The Max Bisection problem includes an additional cardinality constraint that both sides of the partition have exactly half the vertices of the graph. The seemingly mild cardinality constraint appears to change the nature of the problem. While Max Cut admits a factor 0.878 approximation algorithm [GW95], the best known approximation factor for Max Bisection equals 0.7027 [FL06], improving on previous bounds of 0.6514 [FJ97], 0.699 [Ye01], and 0.7016 [HZ02]. These algorithms proceed by rounding the natural semidefinite programming relaxation analogous to the Goemans-Williamson SDP for Max Cut. Guruswami et al. [GMR+11] showed that this natural SDP relaxation has a large integrality gap: the SDP optimum could be 1 whereas every bisection
might only cut less than 0.95 fraction of the edges! In particular, this implies that none of these algorithms guarantee a solution with value close to 1 even if there exists a perfect bisection in the graph. More recently, using a combination of graph-decomposition, brute-force enumeration and SDP rounding, Guruswami et al. [GMR+11] obtained an algorithm that outputs a \(1 - O(e^{1/3} \log(1/e))\) bisection on a graph that has a bisection of value \(1 - \varepsilon\).

A simple approximation preserving reduction from \textsc{Max Cut} shows that \textsc{Max Bisection} is no easier to approximate than \textsc{Max Cut} (the reduction is simply to take two disjoint copies of the \textsc{Max Cut} instance). Therefore, the factor 16/17 NP-hardness [H01, TSSW00] and the factor 0.878 Unique-Games hardness for \textsc{Max Cut} [KKMO07] also applies to the \textsc{Max Bisection} problem. In fact, a stronger hardness result of factor 15/16 was shown in [HK04] assuming \(\text{NP} \not\subseteq \bigcap_{\gamma>0} \text{TIME}(2^{n^\gamma})\). Yet, these hardness results for \textsc{Max Bisection} are far from matching the best known approximation algorithm that only achieves a 0.702 factor.

**SDP Hierarchies.** Almost all known approximation algorithms for constraint satisfaction problems are based on a fairly minimal SDP relaxation of the problem. In fact, there exists a simple semidefinite program with linear number of constraints (see [Rag08, RS09a]) that yields the best known approximation ratio for every CSP. This leaves open the possibility that stronger SDP relaxations such as those obtained using the Lovasz-Schriver, Sherali-Adams and Lasserre SDP hierarchies yield better approximations for CSPs. Unfortunately, there is evidence suggesting that the stronger SDP relaxations yield no better approximation for CSPs than the simple semidefinite program suggested in [Rag08, RS09a]. First, under the Unique Games Conjecture, it is \text{NP}-hard to approximate any CSP to a factor better than that yielded by the simple semidefinite program [Rag08]. Moreover, a few recent works [KS09, Tul09, RS09b] have constructed integrality gap instances for strong SDP relaxations of CSPs, obtained via Sherali-Adams and Lasserre hierarchies. For instance, the integrality gap instances in [KS09, RS09b] demonstrate that up to \((\log \log n)^c\) rounds of the Sherali-Adams SDP hierarchy yields no better approximation to \textsc{Max Cut} than the simple Goemans-Williamson semidefinite program [GW95].

The situation for CSPs with cardinality constraints promises to be different. For the \textsc{Balanced Separator} problem – a CSP with a global cardinality constraint, Arora et al. [ARV04] obtained an improved approximation of \(\sqrt{\log n}\) by appealing to a stronger SDP relaxation with triangle inequalities. In case of \textsc{Max Bisection}, one of the components of the algorithm of [GMR+11] is a brute-force search – a technique that could quite possibly be carried out using SDP hierarchies.

Despite their promise, there are only a handful of applications of SDP hierarchies in to approximation algorithms, most notably to approximating graph expansion [ARV04], graph coloring and hypergraph independent sets. Moreover, there are few general techniques to round solutions to SDP hierarchies, and analyze their integrality gap.

In an exciting development, fairly general techniques to round solutions to SDP hi-
Hierarchies (particularly the Lasserre hierarchy) has emerged in recent works by Barak et al. [BRS11] and Guruswami and Sinop [GS11]. Both these works (concurrently and independently) developed a fairly general approach to round solutions to the Lasserre hierarchy using an appropriate notion of local-global correlations in the SDP solution. As an application of the technique, both the works obtain a subexponential time algorithm for the Unique Games problem using the Lasserre SDP hierarchy. These works also demonstrate several interesting applications of the technique.

Barak et al. [BRS11] obtain an algorithm for arbitrary 2-CSPs with an approximation guarantee depending on the spectrum of the input graph. Specifically, the result implies a quasi-polynomial time approximation scheme for every 2-CSP on low threshold rank graphs, namely graphs with few large eigenvalues.

Guruswami and Sinop [GS11] obtain a general algorithm to optimize quadratic integer programs with positive semidefinite forms and global linear constraints. Several interesting problems including 2-CSPs with global cardinality constraints such as Max Bisection, Min Bisection and Balanced Separator fall into the framework of [GS11]. However, the approximation guarantee of their algorithm depends on the spectrum of the input graph, and is therefore effective only on the special class of low threshold rank graphs.

1.1 Our Results

In this paper, we develop a general approach to approximate CSPs with global cardinality constraints using the Lasserre SDP hierarchy.

We illustrate the approach with an improved approximation algorithm for the Max Bisection and balanced Max 2-SAT problems. For the Max Bisection problem, we show the following result.

**Theorem 1.1.** For every $\delta > 0$, there exists an algorithm for Max Bisection that runs in time $O(n^{\text{poly}(1/\delta)})$ and obtains the following approximation guarantees,

- The output bisection has value at least $0.85 - \delta$ times the optimal max bisection.
- For every $\epsilon > 0$, given an instance $G$ with a bisection of value $1 - \epsilon$, the algorithm outputs a bisection of value at least $1 - O(\sqrt{\epsilon}) - \delta$.

Note that the approximation guarantee of $1 - O(\sqrt{\epsilon})$ on instances with $1 - \epsilon$ is nearly optimal (up to constant factors in the $O()$) under the Unique Games Conjecture. This follows from the corresponding hardness of Max Cut and the reduction from Max Cut to Max Bisection.

Our approach is robust in that it also yields similar approximation guarantees to the more general $\alpha$-Max Cut problem where the goal is to find a cut with exactly $\alpha$-fraction of vertices on one side of the cut. More generally, the algorithm also generalizes to a weighted version of Max Bisection, where the vertices have weights and the cut has...
approximately half the weight on each side. \footnote{Note that in the weighted case, finding any exact bisection is at least as hard as subset-sum problem.}

The same algorithm also yields an approximation to the complementary problem of \textsc{Min Bisection}. Formally, we obtain the following approximation algorithm for \textsc{Min Bisection} and \textsc{$\alpha$-Balanced Separator}.

\textbf{Theorem 1.2.} For every $\delta > 0$, there exists an algorithm running in time $O(n^{O(\text{poly}(1/\delta))})$, which given a graph with a bisection (\textsc{$\alpha$-balanced separator}) cutting $\epsilon$-fraction of the edges, finds a bisection (\textsc{$\alpha$-balanced separator}) cutting at most $O(\sqrt{\epsilon}) + \delta$-fraction of edges.

Towards showing a matching hardness results for CSPs with cardinality constraints, we construct a dictatorship test for these problems. Dictatorship testing gadgets lie at the heart of all optimal hardness of approximation results for CSPs (both \textsc{NP}-hardness and unique games based hardness results). In fact, using techniques from the work of Khot et al. [KKMO07], any dictatorship test for a CSP yields a corresponding unique games based hardness result. More generally, a large fraction of hardness of approximation results (not necessarily CSPs) have an underlying dictatorship testing gadget.

Building on earlier works, Raghavendra [Rag08] exhibited a generic reduction that starts with an arbitrary integrality gap instance for certain SDP relaxation of a CSP to a dictatorship test for the same CSP. In turn, this implied optimal hardness results matching the integrality gap of the SDP under the unique games conjecture. Using techniques from [Rag08], we exhibit a generic reduction from integrality gap instances to the Lasserre SDP relaxation of a CSP with cardinality constraints, to a dictatorship test for the same. While the reduction applies in general for every CSP with cardinality constraints, for the sake of exposition, we present the special case of \textsc{Max Bisection}. For \textsc{Max Bisection}, we show the following.

\textbf{Theorem 1.3. (Informal Statement)} For every $\epsilon, \delta > 0$, given an integrality gap instance for poly($1/\epsilon$)-round Lasserre SDP for \textsc{Max Bisection}, with SDP value $c$ and optimum integral value $s$, there exists a dictatorship test for \textsc{Max Bisection} with completeness $c - O(\epsilon + \delta)$ and soundness $s + O(\epsilon + \delta)$.

The formal statement of the result and its proof is presented in Section 6. Unfortunately, this dictatorship test does not yet translate in to a corresponding hardness result for \textsc{Max Bisection}. First, observe that the framework of Khot et al. [KKMO07] to show unique games based hardness results does not apply to \textsc{Max Bisection} due to the global constraint on the instance. This is the same reason why the unique games conjecture is not known to imply hardness results for \textsc{Balanced Separator}. The reason being that the hard instances of these problems are required to have certain global structure (such as expansion in case of \textsc{Balanced Separator}). In case of \textsc{Max Bisection}, a hard instance must not decompose in to sets of small size ($en$ vertices), else the global balance condition can be easily satisfied by appropriately flipping the cut in each set independently. Gadget reductions from a unique games instance preserve the global properties.
of the unique games instance such as lack of expansion. Therefore, showing hardness for \textit{Balanced Separator} or \textit{Max Bisection} problems require a stronger assumption such as unique games with expansion or the Small Set expansion hypothesis \cite{RS10}.

2 Overview of Techniques

In this section, we outline the our approach to approximating the \textit{Max Bisection} problem. The techniques are fairly general and can be applied to other CSPs with global cardinality constraints.

**Global Correlation.** For the sake of exposition, let us recall the Goemans and Williamson algorithm for \textit{Max Cut}. Given a graph $G = (V, E)$, the Goemans-Williamson SDP relaxation for \textit{Max Cut} assigns a unit vector $v_i$ for every vertex $i \in V$, so as to maximize the average squared length $E_{i,j \in E} ||v_i - v_j||^2$ of the edges. Formally, the SDP relaxation is given by,

$$\text{maximize } \mathbb{E}_{i,j \in E} ||v_i - v_j||^2 \text{ subject to } ||v_i||_2^2 = 1 \forall i \in V$$

The rounding scheme picks a random halfspace passing through the origin and outputs the partition of the vertices induced by the halfspace. The value of the cut returned is guaranteed to be within a 0.878-factor of the SDP value.

The same algorithm would be an approximation for \textit{Max Bisection} if the cut returned by the algorithm was near-balanced, i.e., $|S| \approx |V|/2$. Indeed, the expected number of vertices on either side of the partition is $|V|/2$, since each vertex $i \in V$ falls on a given side of a random halfspace with probability $\frac{1}{2}$.

If the balance of the partition returned is concentrated around its expectation then the Goemans and Williamson algorithm would yield a 0.878-approximation for \textit{Max Bisection}. However, the balance of the partition need not be concentrated, simply because the values taken by vertices could be highly correlated with each other!

**SDP Relaxation.** To exploit the correlations between the vertices we use a $k$-round Lasserre SDP \cite{Las01} of \textit{Max Bisection} for a sufficiently large constant $k$. On a high level, the solutions to a Lasserre’s SDP hierarchy are vectors that \textit{locally behave} like a distribution over integral solutions. The $k$-round Lasserre SDP has the following properties similar to a true distribution over integral solutions.

- **Marginal Distributions** For any subset $S$ of vertices with $|S| \leq k$, the SDP will yield a distribution $\mu_S$ on partial assignments to the vertices $\{\{-1, 1\}^S\}$. The marginals of $\mu_S$, $\mu_T$ for a pair of subsets $S$ and $T$ are consistent on their intersection $S \cap T$. 
– Conditioning. Analogous to a true distribution over integral solutions, for any subset \( S \subseteq V \) with \(|S| \leq k\) and a partial assignment \( \alpha \in \{-1, 1\}^S \), the SDP solution can be conditioned on the event that \( S \) is assigned \( \alpha \).

A detailed description of the Lasserre’s SDP hierarchy for \textsc{Max Bisection} and other CSPs will be given in Section 3.

**Measuring Correlations.** In this work, we will use mutual information as a measure of correlation between two random variables. We refer the reader to Section 3 for the definitions of Shannon entropy and mutual information. The correlation between vertices \( i \) and \( j \) is given by

\[ I_{\mu_{i,j}}(X_i; X_j) = H(X_i) - H(X_i | X_j), \]

where the random variables \( X_i, X_j \) are sampled using the local distribution \( \mu_{i,j} \) associated with the Lasserre SDP. An SDP solution will be termed \( \alpha\)-independent if the average mutual information between random pairs of vertices is at most \( \alpha \), i.e., \( \mathbb{E}_{i,j \in V}[I(X_i; X_j)] \leq \alpha \).

For most natural rounding schemes such as the halfspace-rounding, the variance of the balance of the cut returned is directly related to the average correlation between random pairs of vertices in the graph. In other words, if the rounding scheme is applied to an \( \alpha \)-independent SDP solution then the variance of the balance of the cut is at most \( \text{poly}(\alpha) \).

**Obtaining Uncorrelated SDP Solutions.** Intuitively, if it is the case that globally all the vertices are highly correlated, then conditioning on the value of a vertex should reveal information about the remaining vertices, therefore reducing the total entropy of all the vertices.

Formally, let us suppose the \( k \)-round Lasserre SDP solution is not \( \alpha \)-independent, i.e., \( \mathbb{E}_{i,j \in V}[I(X_i; X_j)] > \alpha \). Let us pick a vertex \( i \in V \) at random, sample its value \( b \in \{-1, 1\} \) and condition the SDP solution to the event \( X_i = b \). This conditioning reduces the average entropy of the vertices \( \mathbb{E}_{j \in V}[H(X_j)] \) by at least \( \alpha \) in expectation. If the conditioned SDP solution is \( \alpha \)-independent we are done, else we repeat the process.

The initial average entropy \( \mathbb{E}_{j \in V}[H(X_j)] \) is at most 1, and the quantity always remains non-negative. Therefore, within \( \frac{\alpha}{\alpha} \) conditionings, the SDP solution will be \( \alpha \)-independent. Starting with a \( k \)-round Lasserre SDP solution, this process produces a \( k-t \) round \( \alpha \)-independent Lasserre SDP solution for some \( t > \frac{1}{\alpha} \).

**Rounding Uncorrelated SDP Solutions.** Given an \( \alpha \)-independent SDP solution, for many natural rounding schemes the balance of the output cut is concentrated around its expectation. Hence it suffices to construct rounding schemes that output a balanced cut in expectation. We exhibit a simple rounding scheme that preserves the bias of each vertex individually, thereby preserving the global balance property. The details of the rounding algorithm will be described in Section 5.
3 Preliminaries

Constraint Satisfaction Problem with Global Cardinality Constraints. In this section we formally define CSPs with global constraints.

Definition 3.1 (Constraint Satisfaction Problems with Global Cardinality Constraints). A constraint satisfaction problem with global cardinality constraints is specified by $\Lambda = ([q], \mathbb{P}, k, c)$ where $[q] = \{0, \ldots, q - 1\}$ is a finite domain, $\mathbb{P} = \{P : [q] \rightarrow [0, 1] | t \leq k\}$ is a set of payoff functions. The maximum number of inputs to a payoff function is denoted by $k$. The map $c : [q] \rightarrow [0, 1]$ is the cardinality function which satisfies $\sum_i c_i = 1$. For any $0 \leq i \leq q - 1$, the solution should contain $c_i$ fraction of the variables with value $i$.

Remark 3.2. Although some problems (e.g., Balanced Separator) do not fix the cardinalities to be some specific quantities, they can be easily reduced to the above case.

Definition 3.3. An instance $\Phi$ of constraint satisfaction problems with global cardinality constraints $\Lambda = ([q], \mathbb{P}, k, c)$ is given by $\Phi = (V, \mathbb{P}_V, W)$ where

- $V = \{x_1, \ldots, x_n\}$: variables taking values over $[q]$
- $\mathbb{P}_V$ consists of the payoffs applied to subsets $S$ of size at most $k$
- Nonnegative weights $W = \{w_S\}$ satisfying $\sum_{|S| \leq k} w_S = 1$. Thus we may interpret $W$ as a probability distribution on the subsets. By $S \sim W$, we denote a set $S$ chosen according to the probability distribution $W$
- An assignment should satisfy that the number of variables with value $i$ is $c_in$ (we may assume this is an integer).

Here we give a few examples of CSPs with global cardinality constraints.

Definition 3.4 (Max(Min) Bisection). Given a (weighted) graph $G = (V, E)$ with $|V|$ even, the goal is to partition the vertices into two equal pieces such that the number (total weights) of edges that cross the cut is maximized (minimized).

More generally, in an $\alpha$-Max Cut problem, the goal is to find a partition having $\alpha n$ vertices on one side, while cutting the maximum number of edges. Furthermore, one could allow weights on the vertices of the graph, and look for cuts with exactly $\alpha$-fraction of the weight on one side. Most of our techniques generalize to this setting.

Throughout this work, we will have a weighted graph $G$ with weights $W$ on the vertices. The weights on the vertices are assumed to form a probability distribution. Hence the notation $i \sim W$ refers to a random vertex sampled from the distribution $W$.

Definition 3.5 (Edge Expansion). Given a graph (w.l.o.g. we may assume it is an unweighted regular graph) $G = (V, E)$, and $\delta \in (0, 1/2)$, the goal is to find a set $S \subseteq V$ such that $|S| = \delta|V|$ and the edge expansion of $S$: $\Phi(S) = \frac{E(S, \bar{S})}{|S|}$ is minimized.
Information Theoretic Notions.

**Definition 3.6.** Let \( X \) be a random variable taking values over \([q]\). The *entropy* of \( X \) is defined as

\[
H(X) \overset{\text{def}}{=} -\sum_{i \in [q]} \mathbb{P}(X = i) \log \mathbb{P}(X = i)
\]

**Definition 3.7.** Let \( X \) and \( Y \) be two jointly distributed variables taking values over \([q]\). The *mutual information* of \( X \) and \( Y \) is defined as

\[
I(X; Y) \overset{\text{def}}{=} \sum_{i,j \in [q]} \mathbb{P}(X = i, Y = j) \log \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(X = i) \mathbb{P}(Y = j)}
\]

**Definition 3.8.** Let \( X \) and \( Y \) be two jointly distributed variables taking values over \([q]\). The *conditional entropy* of \( X \) conditioned on \( Y \) is defined as

\[
H(X|Y) = \mathbb{E}_{i \in [q]} [H(X|Y = i)]
\]

We also give two well-known theorems in information theory below.

**Theorem 3.9.** Let \( X \) and \( Y \) be two jointly distributed variables taking value on \([q]\), then

\[
I(X; Y) = H(X) - H(X|Y)
\]

**Theorem 3.10.** (Data Processing Inequality) Let \( X, Y, Z, W \) be random variables such that \( H(X|W) = 0 \) and \( H(Y|Z) = 0 \), i.e., \( X \) is fully determined by \( W \) and \( Y \) is fully determined by \( Z \), then

\[
I(X; Y) \leq I(W; Z)
\]

**Lasserre SDP Hierarchy for Globally Constrained CSPs.** Let \( \Lambda = ([q], \mathbb{P}, k, c) \) be a CSP with global constraints and \( \Phi = (V, \mathbb{P}_V, W) \) be an instance of \( \Lambda \) on variables \( X = \{x_1, ..., x_n\} \). A solution to the \( k \)-round Lasserre SDP consists of vectors \( v_{S,\alpha} \) for all vertex sets \( S \subseteq V \) with \(|S| \leq k \) and local assignments \( \alpha \in [q]^S \). Also for each subset \( S \subseteq V \) with \(|S| \leq k \), there is a distribution \( \mu_S \) on \([q]^S\). For two subsets \( S, T \) such that \(|S|, |T| \leq k \), we require that the corresponding distributions \( \mu_S \) and \( \mu_T \) are consistent when restricted to \( S \cap T \). A Lasserre solution is feasible if for any \(|S \cup T| \leq k, \alpha \in [q]^S, \beta \in [q]^T\), we have

\[
\langle v_{S,\alpha}, v_T, \beta \rangle = \mathbb{P}_{\mu_S \cap T} \{X_S = \alpha, X_T = \beta\}
\]

The SDP also has a vector \( l \) that denotes the constant 1. The global cardinality constraints can be written in terms of the marginals of each variable. Specifically, for every \( S \) with \(|S| \leq k - 1 \) and \( \alpha \in [q]^S \), we have

\[
\mathbb{E}_{j \in W} \mathbb{P}_{\mu_S}(x_j = i|X_S = \alpha) = c_i
\]

The objective of the SDP is to maximize

\[
\mathbb{E}_{S \in W} \left( \sum_{\beta \in [q]^S} P_S(\beta(S)) \mathbb{P}_{\mu_S}(S, \beta) \right)
\]
While the complete description of the Lasserre SDP hierarchy is somewhat complicated, there are few properties of the hierarchy that we need. The most important property is the existence of consistent local marginal distributions \( \{ \mu_S \} \subseteq V | |S| \leq k \) whose first two moments match the inner products of the vectors. We stress that even though the local distributions are consistent, there might not exist a global distribution that agrees with all of them. The second property of the \( k \)-round Lasserre SDP solution is that although the variables are not jointly distributed, one can still condition on the assignment to any given variable to obtain a solution to the \( k - 1 \) round Lasserre’s SDP that corresponds to the conditioned distribution.

4 Globally Uncorrelated SDP Solutions

As remarked earlier, it is easy to round SDP solutions to a CSP with cardinality constraint if the variables behave like independent random variables. In this section, we show a very simple procedure that starts with a solution to the \((k + l)\)-round Lasserre SDP and produces a solution to the \( l \)-round Lasserre SDP with the additional property that globally the variables are somewhat "uncorrelated". To this end, we define the notion of \( \alpha \)-independence for SDP solutions below. We remark that all the definitions and results in this section can be applied to all CSPs.

Definition 4.1. Given a solution to the \( k \)-round Lasserre SDP relaxation, it is said to be \( \alpha \)-independent if 
\[
\mathbb{E}_{i, j \sim W}[\mathbb{I}_{\mu_{i,j}}(X_i; X_j)] \leq \alpha
\]
where \( \mu_{i,j} \) is the local distribution associated with the pair of vertices \( \{i, j\} \).

Remark 4.2. We stress again that the variables in the SDP solution are not jointly distributed. However, the notion is still well-defined here because of the locality of mutual information: it only depends on the joint distribution of two variables, which is guaranteed to exist by the SDP. Also, \( \mu_{i,j} \) in the expression can be replaced with \( \mu_S \) for arbitrary \( S \) with \( i, j \in S \) and \(|S| \leq k \) because of the consistency of local distributions.

The notion of \( \alpha \)-independence of random variables using mutual information, easily translates into more familiar notion of statistical distance. Specifically, we have the following relation.

Fact 4.3. Let \( X \) and \( Y \) be two jointly distributed random variables on \([q]\) then,
\[
I(X; Y) \geq \frac{1}{2 \ln 2} \sum_{i, j \in [q]} (\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i) \mathbb{P}(Y = j))^2,
\]
in particular for all \( i, j \in [q] \)
\[
|\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i) \mathbb{P}(Y = j)| \leq \sqrt{2I(X; Y)}
\]
As a consequence, if \( X \) and \( Y \) are two random variables defined on \([-1, 1] \), \( \text{Cov}(X, Y) \leq O(\sqrt{I(X; Y)}) \)
For the sake of completeness, we include the proof of this observation in Appendix B. Now we describe the procedure of getting an $\alpha$-independent $l$-rounds Lasserre’s solution. A similar argument was concurrently discovered in [BRS11]. Here we reproduce the argument in information theoretic terms, while [BRS11] present the argument in terms of covariance. The information theoretic argument is somewhat robust and cleaner in that it is independent of the sample space involved.

**Algorithm 4.4.** Input: A feasible solution to the $(k+l)$ round Lasserre SDP relaxation as described in Section 3 for $k = 1/\sqrt{\alpha}$.

**Output:** An $\alpha$-independent solution to the $l$ round Lasserre SDP relaxation.

Sample indices $i_1, \ldots, i_k \subseteq V$ independently according to $W$. Set $t = 1$.

Until the SDP solution is $\alpha$-independent repeat

- Sample the variable $X_{i_t}$ from its marginal distribution after the first $t - 1$ fixings, and condition the SDP solution on the outcome.

- $t = t + 1$.

The following lemma shows that there exists $t$ such that the resulting solution is $\alpha$-independent after $t$-conditionings with high probability.

**Lemma 4.5.** There exists $t \leq k$ such that $E_{i_1, \ldots, i_k \sim W} E_{i, j \sim W} [I(X_i, X_j | X_{i_1}, \ldots, X_{i,t-1})] \leq \frac{\log q \cdot t}{k-1}$

**Proof.** By linearity of expectation, we have that for any $t \leq k - 2$

$$E_{i_1, i_2 \sim W} [H(X_i | X_{i_1}, \ldots, X_{i_k})] = E_{i_1, i_2 \sim W} [H(X_i | X_{i_1}, \ldots, X_{i_{k-1}})] - E_{i_1, i_2 \sim W} [I(X_i, X_i | X_{i_1}, \ldots, X_{i_{k-1}})]$$

adding the equalities from $t = 1$ to $t = k - 2$, we get

$$E_{i \sim W} [H(X_i)] - E_{i_1, \ldots, i_{k-2} \sim W} [H(X_i | X_{i_1}, \ldots, X_{i_{k-2}})] = \sum_{1 \leq k \leq k-1} E_{i_1, \ldots, i_{k-1} \sim W} [I(X_i, X_i | X_{i_1}, \ldots, X_{i_{k-1}})]$$

The lemma follows from the fact that for each $i$, $H(X_i) \leq \log q$. $\square$

**Theorem 4.6.** For every $\alpha > 0$ and positive integer $\ell$, there exists an algorithm running in time $O(n^{poly(1/\alpha) + \ell} )$ that finds an $\alpha$-independent solution to the $\ell$-round Lasserre SDP, with an SDP objective value of at least $\text{OPT} - \alpha$, where $\text{OPT}$ denotes the optimum value of the $\ell$-round Lasserre SDP relaxation.

**Proof.** Pick $k = \left\lfloor \frac{\log q}{\alpha} \right\rfloor$. Solve the $k + \ell$ round Lasserre SDP solution, and use it as input to the conditioning algorithm described earlier. Notice that the algorithm respects the marginal distributions provided by the SDP while sampling the values to variables. Therefore, the expected objective value of the SDP solution after conditioning is exactly equal to the SDP objective value before conditioning. Also notice that the SDP value is at most 1. Therefore, the probability of the SDP value dropping by at least $\alpha$ due to conditioning is at most $1/(1 + \alpha)$. 

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Also, by Lemma 4.5 and Markov Inequality, the probability of the algorithm failing to find a \( \sqrt{\log q / k} \)-independent solution is at most \( \sqrt{\log q / k} \). Therefore, by union bound, there exists a fixing such that the SDP value is maintained up to \( \alpha \), and the solution after conditioning is \( \alpha \)-independent. Moreover, this particular fixing can be found using brute-force search.

\[ \square \]

5 Rounding Scheme for Max Bisection

In this section, we present and analyze a natural rounding scheme for Max Bisection. Given an globally uncorrelated SDP solution to a 2-round Lasserre SDP relaxation of Max Bisection, the rounding scheme will output a cut with the approximation guarantees outlined in Theorem 1.1. The same rounding scheme also yields a 0.92-approximation algorithm for arbitrary globally constrained Max 2-SAT problem.

Constructing Goemans-Williamson type SDP solution. In the 2-round Lasserre SDP for Max Bisection, there are two orthogonal vectors \( v_0 \) and \( v_1 \) for each variable \( x_i \). This can be used to obtain a solution to the Goemans-Williamson SDP solution by simply defining \( v_i \overset{\text{def}}{=} v_{i0} - v_{i1} \). The following proposition is an easy consequence.

**Proposition 5.1.** Let \( v_i = v_{i0} - v_{i1} = (2p_i - 1)I + w_i \) where \( p_i = P(x_i = 0) \). Then, for each edge \( e = (i, j) \in E \), \( P_{\mu_e}(x_i \neq x_j) = \|v_i - v_j\|^2 / 4 \).

**Proof.**

\[ \|v_i - v_j\|^2 = 2 - 2\langle v_{i0} - v_{i1}, v_{j0} - v_{j1} \rangle = 2 - 2(P_{\mu_e}(x_i = x_j) - P_{\mu_e}(x_i \neq x_j)) = 4P_{\mu_e}(x_i \neq x_j) \]

\[ \square \]

Let \( w_i \) be the component of \( v_i \) orthogonal to the \( I \) vector, i.e., \( w_i \overset{\text{def}}{=} (v_i - \langle v_i, I \rangle I) \).

Using \( v_{i0} + v_{i1} = I \) and \( \langle v_{i0}, v_{i1} \rangle = 0 \), we get \( v_{i0} = \langle v_{i0}, I \rangle I + w_i / 2 \) and \( v_{i1} = \langle v_{i1}, I \rangle I - w_i / 2 \).

We remark that \( w_i \) is the crucial component that captures the correlation between \( x_i \) and other variables. To formalize this, we show the following lemma.

**Lemma 5.2.** Let \( v_i \) and \( v_j \) be the unit vectors constructed above, \( w_i \) and \( w_j \) be the components of \( v_i \) and \( v_j \) that orthogonal to \( I \). Then \( |\langle w_i, w_j \rangle| \leq 4 \sqrt{2I(x_i, x_j)} \)

**Proof.** Let \( p_i \overset{\text{def}}{=} P(x_i = 0) = \langle v_{i0}, I \rangle \) and \( p_j \overset{\text{def}}{=} P(x_j = 0) = \langle v_{j0}, I \rangle \). Notice that

\[
|P(x_i = 0, x_j = 0) - P(x_i = 0)P(x_j = 0)| = ||\langle p_i I + w_i / 2, p_j I + w_j / 2 \rangle - p_i p_j || = |\langle w_i, w_j \rangle| / 4
\]

By applying Fact 4.3, we get \( |\langle w_i, w_j \rangle| \leq 4 \sqrt{2I(x_i, x_j)} \)

\[ \square \]
Henceforth we will switch from the alphabet \{0, 1\} to \{-1, 1\}. After this transformation, we can interpret the inner product \(\mu_i = \langle v_i, I \rangle = p_i - (1 - p_i)\) as the bias of vertex \(i\).

### 5.1 Rounding Scheme

Roughly speaking, the algorithm applies a hyperplane rounding on the vectors \(w_i = v_i - \langle v_i, I \rangle I\) associated with the vertices \(i \in V\). However, for each vertex \(i \in V\), the algorithm shifts the hyperplane according to the bias of that vertex.

**Algorithm 5.3.** Given: A set of unit vectors \(\{v_1, \ldots, v_n\}\) where \(v_i = \mu_i I + w_i\), where \(w_i\) is the component of \(v_i\) orthogonal to \(I\).

Pick a random Gaussian vector \(g\) orthogonal to \(I\) with coordinates distributed as \(N(0, 1)\). For every \(i\),

1. Project \(g\) on the direction of \(w_i\), i.e., \(\xi_i = \langle g, \tilde{w_i} \rangle\), where \(\tilde{w_i} = \frac{w_i}{\sqrt{1 - \mu_i^2}}\) is the normalized vector or \(w_i\). Note that \(\xi_i\) is also a standard Gaussian variable.
2. Pick threshold \(t_i\) as follows:
   \[
   t_i = \Phi^{-1}(\mu_i/2 + 1/2)
   \]
3. If \(\xi_i \leq t_i\), set \(x_i = 1\), otherwise set \(x_i = -1\).

Notice that, the threshold \(t_i\) is chosen so that individually the bias of \(x_i\) is exactly \(\mu_i\). Therefore, the expected balance of the rounded solution matches the intended value. The analysis of the rounding algorithm consists of two parts: first we show that the cut returned by the rounding algorithm has high expected value, then we show the that the balance of the cut is concentrated around its expectation.

### 5.2 Analysis of the Cut Value

Analyzing the cut value of the rounding scheme is fairly standard albeit a bit technical. The analysis is local as in the case of other algorithms for CSPs, and reduces to bounding the probability that a given edge is cut. The probability that a given edge \(u, v\) is cut corresponds to a probability of an event related to two correlated Gaussians.

By using numerical techniques, we were able to show that the cut value is at least 0.85 times the SDP optimum. Analytically, we show the following asymptotic relation.

**Lemma 5.4.** Let \(u = \mu_1 I + w_1, v = \mu_2 I + w_2\) be two unit vectors satisfying \(||u - v||^2/4 \leq \varepsilon\), then the probability of them being separated by Algorithm 5.3 is at most \(O(\sqrt{\varepsilon})\).

The proof of this lemma is fairly technical and is deferred to Appendix A.

---

2The mapping is given by 0 \(\rightarrow\) 1 and 1 \(\rightarrow\) −1
5.3 Analysis of the Balance

In this section we show that the balance of the rounded solution will be highly concentrated. We prove this fact by bounding the variance of the balance. Specifically, we show that if the SDP solution is $\alpha$-independent, then the variance of the balance can be bounded above by a function of $\alpha$.

The proof in this section is information theoretical – although this approach gives sub-optimal bound, but the proof itself is very simple and clean.

Lemma 5.5. Let $v_i = \mu_i I + w_i$ and $v_j = \mu_j I + w_j$ be two vectors in the SDP solution that satisfy $|\langle w_i, w_j \rangle| \leq \zeta$. Let $y_i$ and $y_j$ be the rounded solution of $v_i$ and $v_j$, then

$$I(y_i; y_j) \leq O\left( \frac{\zeta^2}{3} \right)$$

Proof. Since

$$|\langle w_i, w_j \rangle| = \sqrt{1 - \mu_i^2} \sqrt{1 - \mu_j^2} |\langle \bar{w}_i, \bar{w}_j \rangle| \leq \zeta$$

It implies that one of the three quantities in the equation above is at most $\zeta^2/3$. If it is the case that $\sqrt{1 - \mu_i^2} \leq \zeta^2/3$ or $\sqrt{1 - \mu_j^2} \leq \zeta^2/3$ (w.l.o.g we can assume it’s the first case), then we have

$$\min(|1 - \mu_i|, |1 + \mu_i|) \leq O(\zeta^{2/3})$$

We may assume $\mu_i > 0$, therefore $1 - \mu_i \leq O(\zeta^{2/3})$. Notice that our rounding scheme preserves the bias individually, which implies $y_i$ is a highly biased binary variable, hence

$$I(y_i; y_j) \leq H(y_i) = O(- (1 - \mu_i) \log(1 - \mu_i)) \leq O(\zeta^{1/3})$$

Now let’s assume it’s the case that $|\langle \bar{w}_i, \bar{w}_j \rangle| \leq \zeta^{1/3}$. Let $g_1 = g \cdot \bar{w}_1$ and $g_2 = g \cdot \bar{w}_2$ as described in the rounding scheme, and $\rho = \langle \bar{w}_i, \bar{w}_j \rangle$. Hence $g_1$ and $g_2$ are two jointly distributed standard Gaussian variables with covariance matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

The mutual information of $g_1$ and $g_2$ is

$$I(g_1, g_2) = -\frac{1}{2} \log(\det \Sigma) \leq O(- \log(1 - \zeta^{2/3})) \leq O(\zeta^{1/3})$$

Notice that $y_i$ is fully dependent on $g_i$, therefore by the data processing inequality (Theorem 3.10), we have $I(y_i; y_j) \leq I(g_1, g_2) \leq O(\zeta^{1/3})$.

□

Theorem 5.6. Given an $\alpha$-independent solution to 2-rounds Lasserre’s SDP hierarchy. Let $\{y_i\}$ be the rounded solution after applying Algorithm 5.3. Define $S = \mathbb{E}_{i \sim W} y_i$, then

$$\text{Var}(S) \leq O(\alpha^{1/12})$$

Proof:

$$\text{Var}(S) = \mathbb{E}_{i,j \sim W} [\text{Cov}(y_i, y_j)]$$
\[ \mathbb{E}_{i,j \sim W} \left[ O \left( \sqrt{I(y_i; y_j)} \right) \right] \quad \text{(by Fact 4.3)} \]
\[ \mathbb{E}_{i,j \sim W} \left[ O \left( \sqrt{|w_i, w_j|^{1/3}} \right) \right] \quad \text{(by Lemma 5.5)} \]
\[ \mathbb{E}_{i,j \sim W} \left[ O \left( \sqrt{I(x_i; x_j)^{1/6}} \right) \right] \quad \text{(by Lemma 5.2)} \]
\[ \leq O \left( \mathbb{E}_{i,j \sim W} [I(x_i; x_j)]^{1/12} \right) \quad \text{(by concavity of the function } x^{1/12}) \]
\[ \leq O(\alpha^{1/12}) \]

\[ \square \]

**Corollary 5.7.** Given an \( \alpha \)-independent solution to 2-rounds Lasserre’s SDP hierarchy \( v_i = \mu_i + w_i \). The rounding algorithm will find an \( O(\alpha^{1/24}) \)-balanced (that is, the balance of the cut differs from the expected value by at most \( O(\alpha^{1/24}) \) fraction of the total weights) with probability at least \( 1 - O(\alpha^{1/24}) \).

### 5.4 Wrapping Up

Here we present the proofs of the main theorems of this work.

**Proof of Theorem 1.2.** Suppose we’re given a M\( \text{in} \) B\( \text{isection} \) instance \( G = (V, E) \) with value at most \( \varepsilon \) and constant \( \delta > 0 \). By setting \( \alpha = \delta^{24} \) and applying Theorem 4.6, we will get an \( \alpha \)-independent solution with value at most \( \varepsilon + \alpha \). By Lemma 5.4 and the concavity of the function \( \sqrt{x} \), the expected size of the cut returned by Algorithm 5.3 is at most \( O(\sqrt{\varepsilon + \alpha}) = O(\sqrt{\varepsilon} + \sqrt{\alpha}) \). Therefore, with constant probability (say \( 1/2 \)), the cut returned by the rounding algorithm has size at most \( O(\sqrt{\varepsilon} + \sqrt{\alpha}) \). Also, by Corollary 5.7, the cut will be \( O(\alpha) \)-balanced with probability at least \( 1 - O(\delta) \). Therefore, by union bound, the algorithm will return an \( O(\alpha) \)-balanced cut with value at most \( O(\sqrt{\varepsilon} + \sqrt{\alpha}) \) with constant probability. Notice that this probability can be amplified to \( 1 - \varepsilon \) by running the algorithm \( O(\log(1/\varepsilon)) \) times. Given such a cut, we can simply move \( O(\alpha) \) fraction of the vertices with least degree from the larger side to the smaller side to get an exact bisection – this process will increase the value of the cut by at most \( O(\delta) \). Therefore, in this case, we get a bisection of value at most \( O(\sqrt{\varepsilon} + \sqrt{\alpha} + \delta) = O(\sqrt{\varepsilon} + \delta) \). Hence, the expected value of the bisection returned by the rounding algorithm is at most \( (1 - \varepsilon)O(\sqrt{\varepsilon} + \delta) + \varepsilon = O(\sqrt{\varepsilon} + \delta) \).

**Proof of Theorem 1.1.** The proof is similar in the case of M\( \text{ax} \) B\( \text{isection} \). The only difference is that we have to use the fact that the rounding scheme is balanced, i.e., \( \mathbb{P}(F(v) \neq F(-v)) = 1 \). Hence, by Lemma 5.4, for any edge \( (u, v) \) with value \( 1 - \varepsilon \) in the SDP solution, the algorithm separates them with probability at least \( 1 - O(\sqrt{\varepsilon}) \). The rest of the proof is identical.

Using a computer-assisted proof, we can show that the approximation ratio of this algorithm for M\( \text{ax} \) B\( \text{isection} \) is between 0.85 and 0.86. Thus further narrowing down the
6 Dictatorship Tests from Globally Uncorrelated SDP Solutions

A dictatorship test $\text{DICT}$ for the Max Bisection problem consists of a graph on the set of vertices $\{\pm 1\}^R$. By convention, the graph $\text{DICT}$ is a weighted graph where the edge weights form a probability distribution (sum up to 1). We will write $(z, z') \in \text{DICT}$ to denote an edge sampled from the graph $\text{DICT}$ (here $z, z' \in \{\pm 1\}^R$).

A cut of the $\text{DICT}$ graph can be thought of as a boolean function $F : \{\pm 1\}^R \to \{\pm 1\}$. The value of a cut $F$ given by $\text{DICT}(F) = \frac{1}{2} \mathbb{E}_{(z, z') \in \text{DICT}} \left[ 1 - F(z)F(z') \right]$, is the probability that $z, z'$ are on different sides of the cut. It is also useful to define $\text{DICT}(F)$ for non-boolean functions $F : \{\pm 1\}^R \to [-1, 1]$ that take values in the interval $[-1, 1]$. To this end, we will interpret a value $F(z) \in [-1, 1]$ as a random variable that takes $\{\pm 1\}$ values. Specifically, we think of a number $a \in [-1, 1]$ as the following random variable

$$a = \begin{cases} -1 & \text{with probability } \frac{1-a}{2} \\ 1 & \text{with probability } \frac{1+a}{2} \end{cases}$$

(6.1)

With this interpretation, the natural definition of $\text{DICT}(F)$ for such a function is as follows:

$$\text{DICT}(F) = \frac{1}{2} \mathbb{E}_{(z, z') \in \text{DICT}} \left[ 1 - F(z)F(z') \right].$$

Indeed, the above expression is equal to the expected value of the cut obtained by randomly rounding the values of the function $F : \{\pm 1\}^R \to [-1, 1]$ to $\{\pm 1\}$ as described in Equation (6.1).

We will construct a dictatorship test for the weighted version of Max Bisection. In particular, each vertex $x \in \{\pm 1\}^R$ of $\text{DICT}$ is associated a weight $W(x)$, and the weights $W$ form a probability distribution over $\{\pm 1\}^R$ (sum up to 1). The balance condition on the cut can now be expressed as $\mathbb{E}_{z \sim W}[F(z)] = 0$.

The dictatorship test $\text{DICT}$ can be easily transformed in to a dictatorship test $\text{DICT}'$ for unweighted Max Bisection. The idea is to replace each vertex $x \in \{\pm 1\}^R$ with a cluster $V_x$ of $\lfloor W(x) \cdot M \rfloor$ vertices for some large integer $M$. For every edge $(x, y)$ in $\text{DICT}$, connect every pair of vertices in the corresponding clusters $V_x, V_y$ with edge of the same weight. Given any bisection $F' : \text{DICT}' \to \{\pm 1\}$ of the graph $\text{DICT}'$ with value
c, define \( \mathcal{F}(z) = \mathbb{E}_{v \in V} \mathcal{F}'(v) \). By slightly correcting the balance of \( \mathcal{F} \), it is easy to obtain a bisection \( \mathcal{F} : [\pm 1]^R \rightarrow [-1, 1] \) satisfying

\[
\text{DICT}(\mathcal{F}) \geq c - o_M(1) \quad \mathbb{E}_z \mathcal{F}(z) = 0.
\]

Conversely, given a bisection \( \mathcal{F} : [\pm 1]^R \rightarrow [-1, 1] \) of \( \text{DICT} \), assign \((1 + \mathcal{F}(z))/2\) fraction of vertices of \( V_z \) to be 1 and the rest to \(-1\). The resulting partition of \( \text{DICT}' \) is very close to balanced (up to rounding errors), and can be modified in to a bisection with value \( \text{DICT}(\mathcal{F}) - o_M(1) \).

The dictator cuts are given by the functions \( \mathcal{F}(z) = z(\ell) \) for some \( \ell \in [R] \). The dictatorship test graph is so constructed that each dictator cut will yield a bisection and the completeness of the test \( \text{DICT} \) is the minimum value of a dictator cut, i.e.,

\[
\text{Completeness}(\text{DICT}) = \min_{\ell \in [R]} \text{DICT}(z^{(\ell)})
\]

The soundness of the dictatorship test is the value of bisections of \( \text{DICT} \) that are far from every dictator. We will formalize the notion of being far from every dictator using the notion of influences.

**Influences and Noise Operators.** To this end, we recall the definitions of influences and noise operators. Let \( \Omega = ([\pm 1], \mu) \) denote the probability space with atoms \([\pm 1]\) and a distribution \( \mu \) on them. Then, the influences and noise operators for functions over the product space \( \Omega^R \) are defined as follows.

**Definition 6.1** (Influences). The influence of the \( \ell \)th coordinate on a function \( \mathcal{F} : [\pm 1]^R \rightarrow \mathbb{R} \) under a distribution \( \mu \) over \([\pm 1]\) is given by

\[
\text{Inf}_\ell(\mathcal{F}) = \mathbb{E}_{x \leftarrow \Omega} \left[ \mathbb{V}_{x^{(\ell)}}[\mathcal{F}(x)] \right] = \sum_{S \ni \ell} \mathcal{F}_S^2.
\]

**Definition 6.2.** For \( 0 \leq \varepsilon \leq 1 \), define the operator \( T_{1-\varepsilon} \) on \( L_2(\Omega^R) \) as,

\[
T_{1-\varepsilon} \mathcal{F}(z) = \mathbb{E}[\mathcal{F}(\tilde{z}) | z]
\]

where each coordinate \( \tilde{z}^{(i)} \) of \( \tilde{z} \) is equal to \( z^{(i)} \) with probability \( 1-\varepsilon \) and a random element from \( \Omega \) with probability \( \varepsilon \).

**Invariance Principle.** The following invariance principle is an immediate consequence of Theorem 3.6 in the work of Isaksson and Mossel [IM09].

**Theorem 6.3.** (Invariance Principle [IM09]) Let \( \Omega \) be a finite probability space with the least non-zero probability of an atom at least \( \alpha \leq 1/2 \). Let \( \mathcal{L} = \{\ell_1, \ell_2\} \) be an ensemble of random variables over \( \Omega \). Let \( \mathcal{G} = \{g_1, g_2\} \) be an ensemble of Gaussian random variables satisfying the following conditions:

\[
\mathbb{E}[\ell_i] = \mathbb{E}[g_i] \quad \mathbb{E}[\ell_i^2] = \mathbb{E}[g_i^2] \quad \mathbb{E}[\ell_i \ell_j] = \mathbb{E}[g_i g_j] \quad \forall i, j \in \{1, 2\}
\]
Let $K = \log(1/\alpha)$. Let $F$ denote a multilinear polynomial and let $H = (T_{1-\varepsilon} F)$. Let the variance of $H$, $\var[H]$ be bounded by 1 and all the influences are smaller than $\tau$, i.e., $\text{Inf}_i(H) \leq \tau$ for all $i$.

If $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with Lipschitz constant $C_0$ (with respect to the $L_2$ norm) then

$$\left| \mathbb{E}\left[\Psi(H(L^R))\right] - \mathbb{E}\left[\Psi(H(G^R))\right]\right| \leq C \cdot C_0 \cdot \tau^{e^{18K}} = o(1)$$

for some constant $C$.

**Construction.** Let $G = (V, E)$ be an arbitrary instance of $\text{Max Bisection}$. Let $V = \{v_{i,0}, v_{i,1}\}_{i \in V}$ denote a globally uncorrelated feasible SDP solution for two rounds of the Lasserre hierarchy. Specifically, for every pair of vertices $i, j \in V$, there exists a distribution $\mu_{ij}$ over $\{\pm 1\}$ assignments that match the SDP inner products. In other words, there exists $\{\pm 1\}$ valued random variables $z_i, z_j$ such that

$$\langle v_i, v_j \rangle = \mathbb{E}[z_i \cdot z_j].$$

Furthermore, the correlation between random pair of vertices is at most $\delta$, i.e.,

$$\mathbb{E}_{i, j \in V}[I(z_i, z_j)] \leq \delta.$$

Starting from $G = (V, E)$ along with the SDP solution $V$ and a parameter $\varepsilon$ we construct a dictatorship test $\text{DICT}_V^\varepsilon$. The dictatorship test gadget is exactly the same as the construction by Raghavendra [Rag08] for the $\text{Max Cut}$ problem. For the sake of completeness, we include the details below.
\textbf{DICT}_\epsilon^V (\textsc{Max Bisection}) The set of vertices of DICT\_\epsilon^V consists of the R-dimensional hypercube \{\pm 1\}^R. The distribution of edges in DICT\_\epsilon^V is the one induced by the following sampling procedure:

- Sample an edge \(e = (v_i, v_j) \in E\) in the graph \(G\).
- Sample \(R\) times independently from the distribution \(\mu_e\) to obtain \(z_i^R = (z_i^{(1)}, \ldots, z_i^{(R)})\) and \(z_j^R = (z_j^{(1)}, \ldots, z_j^{(R)})\), both in \{\pm 1\}^R.
- Perturb each coordinate of \(z_i^R\) and \(z_j^R\) independently with probability \(\epsilon\) to obtain \(\tilde{z}_i^R, \tilde{z}_j^R\) respectively. Formally, for each \(\ell \in [R]\),
  \[
  \tilde{z}_i^{(\ell)} = \begin{cases} 
  z_i^{(\ell)} & \text{with probability } 1 - \epsilon \\
  \text{random sample from distribution } \mu_i & \text{with probability } \epsilon 
  \end{cases}
  \]
- Output the edge \((\tilde{z}_i^R, \tilde{z}_j^R)\).

The weights on the vertices of DICT\_\epsilon^V is given by

\[
W(x) = \mathbb{E}_{\mu_i \in \mu_i^R} \left[ \mathbb{P} \left[ z = x \right] \right].
\]

We will show the following theorem about the completeness and soundness of the dictatorship test.

\textbf{Theorem 6.4.} There exist absolute constants \(C, K\) such that for all \(\epsilon, \tau \in [0, 1]\) there exists \(\delta\) such that following holds. Given a graph \(G\) and a \(\delta\)-independent SDP solution \(V = \{v_i, 0, v_i, 1 \mid i \in V\}\) for the two round Lasserre SDP for \textsc{Max Bisection}, the dictatorship test DICT\_\epsilon^V is such that

- The dictator cuts are bisections with value within \(2\epsilon\) of the SDP value, i.e.,
  \(\text{Completeness}(\text{DICT}_\epsilon^V) \geq \text{val}(V) - 2\epsilon\)
- If \(\mathcal{F} : \{\pm 1\}^R \rightarrow [-1, 1]\) is a bisection of DICT\_\epsilon^V (\(\mathbb{E}_{x \sim W} [\mathcal{F}(x)] = 0\)) and all its influences are at most \(\tau\), i.e.,
  \[
  \text{Inf}_i^{\mu_i}(\mathcal{F}) \leq \tau \quad \forall i \in V, \ell \in [R],
  \]
  then,
  \[
  \text{DICT}_\epsilon^V(\mathcal{F}) \leq \text{opt}(G) + C\tau^K \epsilon.
  \]

\textbf{Proof.} The analysis of the dictatorship test is along the lines of the corresponding proof for \textsc{Max Cut} in [Rag08].

\textbf{Completeness.} First, the dictatorship test gadget is exactly the same as that constructed for \textsc{Max Cut} in [Rag08]. Therefore from [Rag08], the fraction of edges cut
by the dictators is at least \( \text{val}(V) - 2\varepsilon \). To finish the proof of completeness, we need to show that the dictator cuts are indeed balanced. However, this is an easy calculation since the balance of the \( j \)-th dictator cut is given by,

\[
\mathbb{E}_{x \in W}[x^{(j)}] = \mathbb{E}_{i \in V, x \in \mu^R_i}[x^{(j)}] = \mathbb{E}_{i \in V} \mathbb{E}[a] = 0,
\]

where the last equality uses the fact that the SDP solution satisfies the balance condition.

**Soundness.** Let \( \mathcal{F} : \{\pm 1\}^R \to [-1, 1] \) be a balanced cut all of whose influences are at most \( \tau \). As in [Rag08], we will use the function \( \mathcal{F} \) to round the SDP solution \( V \). The rounding algorithm is exactly the same as the one in [Rag08]. For the sake of completeness, we reproduce the rounding scheme below.

**Round\( \mathcal{F} \) Scheme**

**Truncation Function.** Let \( f_{[-1,1]} : \mathbb{R} \to [-1, 1] \) be a Lipschitz-continuous function such that for all \( x \in [-1, 1] \), \( f_{[-1,1]}(x) = x \). Let \( C_0 \) denote the Lipschitz constant of the function \( f_{[-1,1]} \).

**Bias.** For each vertex \( i \in V \), let the bias of vertex \( i \) be \( \theta_i = \langle v_{i,0}, I \rangle \) and let \( w_i = v_{i,0} - \langle v_{i,0}, I \rangle v_{i,0} \) be the component of \( v_{i,0} \) orthogonal to the vector \( I \).

**Scheme.** Sample \( R \) vectors \( \zeta^{(1)}, \ldots, \zeta^{(R)} \) with each coordinate being i.i.d normal random variable.

For each \( i \in V \) do

- For all \( 1 \leq j \leq R \), compute the projection \( g^{(j)}_i \) of the vector \( w_i \) as follows:

\[
g^{(j)}_i = \theta_i + \langle w_i, \zeta^{(j)} \rangle
\]

and let \( g_i = (g^{(1)}_i, \ldots, g^{(R)}_i) \).

- Let \( F_i \) denote the multilinear polynomial corresponding to the function \( \mathcal{F} \) under the distribution \( \mu^R_i \) and let \( H_i = T_{1-\varepsilon} F_i \). Evaluate \( H_i \) with \( g^{(j)}_i \) as inputs to obtain \( p_i \), i.e., \( p_i = H_i(g^{(1)}_i, \ldots, g^{(R)}_i) \).

- Round \( p_i \) to \( p^*_i \in [-1, 1] \) by using the Lipschitz-continuous truncation function \( f_{[-1,1]} : \mathbb{R} \to [-1, 1] \).

\[
p^*_i = f_{[-1,1]}(p_i).
\]

- Assign the vertex \( i \) to be 1 with probability \( (1 + p^*_i)/2 \) and \(-1\) with the remaining probability.

Let \( \text{Round}_{\mathcal{F}}(V) \) denote the expected value of the cut returned by the rounding scheme \( \text{Round}_{\mathcal{F}} \) on the SDP solution \( V \) for the MAX BISECTION instance \( G \).
Again, by appealing to the soundness analysis in [Rag08], we conclude that the fraction of edges cut by the resulting partition is lower bounded by

$$\text{Round}_\mathcal{F}(V) \geq \text{DICT}_V^e(\mathcal{F}) - C'\tau^{Ke}.$$  

for an absolute constant $C'$. To finish the proof, we need to argue that if the SDP solution $V$ is $\delta$-independent, then the resulting partition is close to balanced with high probability.

First, note that the expected balance of the cut is given by,

$$\mathbb{E} \left[ \mathbb{E}[p_i^*] \right] = \mathbb{E} \left[ \mathbb{E}[f_{[-1,1]}(H_i(g_i))] \right].$$

Fix a vertex $i \in V$. By construction, the random variables $z_i^{(f)} \sim \mu_i$ and $q_i^{(f)}$ have matching moments up to order two for each $\ell \in [R]$. Therefore, by applying the invariance principle of Isaksson and Mossel [IM09] with the smooth function $f_{[-1,1]}$ and the multilinear polynomial $F_i$ yields the following inequality,

$$\mathbb{E} \left[ f_{[-1,1]}(H_i(g_i)) \right] \leq \mathbb{E} \left[ f_{[-1,1]}(H_i(z_i^R)) \right] + C\tau^{Ke}.$$

Since the cut $\mathcal{F}$ is balanced we can write,

$$\mathbb{E} \mathbb{E} \left[ f_{[-1,1]}(H_i(z_i^R)) \right] = \mathbb{E} \mathbb{E} \left[ H_i(z_i^R) \right] = \mathbb{E} \mathbb{E} \left[ F_i(z_i^R) \right] = \mathbb{E} \mathbb{E} \left[ \mathcal{F}(z_i^R) \right] = 0.$$

In the previous calculation, the first equality uses the fact that $f_{[-1,1]}(x) = x$ for $x \in [-1,1]$ while the second equality uses the fact that $\mathbb{E}_z[T_{-x}H_i(z)] = \mathbb{E}_z[F_i(z)]$. Therefore, we get the following bound on the expected value of the balance of the cut, $\mathbb{E}_\xi \left[ f_{[-1,1]}(H_i(g_i)) \right] \leq C\tau^{Ke}.$

Finally, we will show that the balance of the cut is concentrated around its expectation. To this end, we first show the following continuity of the rounding algorithm.

**Lemma 6.5.** For each $i \in V$ and any vector $w_i'$ satisfying $\|w_i'\|_2 = \|w_i\|_2$, if $p_i'$ denotes the output of the rounding scheme $\text{Round}_\mathcal{F}$ with $w_i'$ instead of $w_i$ then,

$$\| \mathbb{E}[\xi](p_i' - p_i)^2 \| \leq C(R)\|w_i - w_i'\|_2^2,$$

for some function of $R$ ($C(R) = 2^{2R}$ suffices).

**Proof.** Let $g_i' = (g_i'^{(1)}, \ldots, g_i'^{(R)})$ denote the projections of the vector $w_i'$ along the directions $z^{(1)}, \ldots, z^{(R)}$. The output of the rounding scheme on $w_i'$ is given by $p_i' = f_{[-1,1]}(H_i(g_i'))$. Recall that the output of the rounding scheme is given by $p_i^* = f_{[-1,1]}(H_i(g_i))$.

The result is a consequence of the fact that the function $f_{[-1,1]} \circ H_i$ is Lipschitz continuous. Since the variance of $\mathcal{F}(z_i^R)$ is at most 1, the sum of squares of coefficients
of $H_i$ is at most 1. Therefore, all the $2^R$ coefficients of $H_i$ are bounded by 1 in absolute value.

The proof is a simple hybrid argument, where we replace $q_j^{(i)}$ by $q_j^{(i')}$ one by one. The details of the proof are deferred to the full version.

\[\square\]

**Lemma 6.6.** For every $i, j$,

\[|E[p_i^*p_j^*] - E[p_i^*] E[p_j^*]| \leq C(R)(w_i, w_j)\]  

for some function $C(R)$ of $R$ ($C(R) = 1002^R$ suffices).

\[Proof.\] Set $w'_j = w_j - \langle w_i, w_j \rangle \frac{w_i}{\| w_i \|} + \langle w_i, w_j \rangle \bar{u}$ for a unit vector $\bar{u}$ orthogonal to $w_i$ and $w_j$. Note that $w'_j$ is orthogonal to $w_i$ and satisfies $\| w_j - w'_j \| \leq 4\| (w_i, w_j) \|$. Let $p'_j$ denote the output of the rounding with $w'_j$ instead of $w_j$. Since $w'_j$ is orthogonal to $w_i$ all their projections are independent random variables, which implies that,

\[E[p'_j] = E[p_j] E[p_i^*].\]

Moreover, by Lemma 6.5 we have,

\[E[(p'_j - p_j)^2] \leq C(R)\| w_j - w'_j \|^2 \leq C(R) \cdot 16\| (w_i, w_j) \|^2.\]

Combining these inequalities and using Cauchy-Schwartz, we finish the proof as follows,

\[|E[p_i^*p_j^*] - E[p_i^*] E[p_j^*]| \leq |E[p_i^*(p_i^* - p_j')]| + |E[p_i^*] E[p'_j - p_j^*]|\]

\[\leq 2 \left( E[(p'_j - p_j)^2] \right)^{\frac{1}{2}} \left( E[(p_j^*)^2] \right)^{\frac{1}{2}}\]

\[\leq 8C(R)(w_i, w_j).\]

\[\square\]

To finish the proof, now we bound the variance of the balance of the cut returned using Lemma 6.6. The variance of the balance of the cut returned is given by,

\[E(\sum_i (E[p_i^*])^2 - (E \sum_i p_j^*)^2) = E \sum_{i,j} \left| E[p_i^*p_j^*] - E[p_i^*] E[p_j^*] \right| \leq C(R) E(\| (w_i, w_j) \|)\]

For a $\delta$-independent SDP solution, the above quantity is at most $C(R) \text{poly}(\delta)$. This gives the desired result.  

\[\square\]
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A Analysis of Cut Value

We analyze the rounding algorithm in an indirect way – first we show that under certain conditions, Algorithm 5.3 returns a better cut compared to Goemans-Williamson algorithm (in expectation). Then we use an union-bound type argument to give the proof for general cases.

First, we present a bound on the tail of the standard gaussian distribution.

Lemma A.1. For $t \geq 0$,

$$\Phi^c(t) = 1 - \Phi(t) \leq \frac{\sqrt{2/\pi} e^{-t^2/2}}{t + \sqrt{t^2 + 8/\pi}}$$

Proof. We apply the following bound on the error function given in [Kom55]

$$e^x^2 \int_x^\infty e^{-y^2} dy \leq \frac{1}{x + \sqrt{x^2 + 4/\pi}}$$

by replacing $x$ with $\sqrt{\frac{2}{\pi}}$, we get the desired bound.

From now on, let $\mu_0 = \sqrt{1 - 4/\pi^2} \approx 0.7712$ and $t_0 = \Phi^{-1}(\mu_0/2 + 1/2) \approx 1.2034$.

Lemma A.2. Let $g(t) = e^t^2/2(1 - \mu^2(t))$, where $\mu(t) = 2\Phi(t) - 1$. $g(t)$ is decreasing when $t \geq t_0$.

Proof. By simple calculation, we get

$$g'(t) = 4\left(te^{t^2/2}(1 - \Phi(t))\Phi(t) + \frac{1}{\sqrt{2\pi}}(1 - 2\Phi(t))\right)$$

we want to show

$$te^{t^2/2}(1 - \Phi(t))\Phi(t) + \frac{1}{\sqrt{2\pi}}(1 - 2\Phi(t)) < 0$$

by applying Lemma A.1, we only need to show

$$te^{t^2/2} \frac{\sqrt{2/\pi} e^{-t^2/2}}{t + \sqrt{t^2 + 8/\pi}}\Phi(t) + \frac{1}{\sqrt{2\pi}}(1 - 2\Phi(t)) < 0$$

by simplification, we get

$$2\Phi(t) - 1 > \frac{t}{\sqrt{t^2 + 8/\pi}}$$

By applying the lemma again and further simplification, we get

$$e^t - t^2 > \frac{8}{\pi}$$
This can easily be verified for $t = t_0$. Also LHS is increasing when $t \geq t_0$, therefore the lemma follows. □

**Lemma A.3.** Let $f_1(x)$ and $f_2(x)$ be twice differentiable decreasing functions defined on $[0, \infty)$ satisfying the following conditions

1. $f_1(0) = f_2(0)$
2. $\lim_{x \to \infty} f_1(x) = \lim_{x \to \infty} f_2(x)$
3. $\lim_{x \to 0} \frac{f'_1(x)}{f'_2(x)} > 1$
4. $\frac{f'_1(x)}{f'_2(x)} = 1$ has only one solution

then

$$f_1(x) \leq f_2(x), \quad \forall x \geq 0$$

**Proof.** For the sake of contradiction we assume there exists $x_0$ such that $f_1(x_0) > f_2(x_0)$. By the mean value theorem, there exists $x_1 < x_0$ such that $f'_1(x_1) > f'_2(x_1)$, which means $\frac{f'_1(x_1)}{f'_2(x_1)} < 1$ (since both $f'_1$ and $f'_2$ are negative). By the fourth assumption, for any $x > x_0 > x_1$, $f'_1(x) > f'_2(x)$, therefore $f_1(x) - f_2(x) \geq f_1(x_0) - f_2(x_0) > 0$, contradicting the second assumption. □

Now we show the key lemma in this section.

**Lemma A.4.** Let $u = \mu I + w_1$ and $v = \mu I + w_2$ be two unit vectors with the same projection on the direction of $I$. Also we assume that $\langle \bar{w}_1, \bar{w}_2 \rangle = 1 - \rho \geq 0$, where $\bar{w}_1$ and $\bar{w}_2$ are the normalized vectors of $w_1$ and $w_2$. Then the probability that these two vectors are separated by a random hyperplane is at least the probability that these two vectors are cut by Algorithm 5.3.

**Proof.** First notice that since $u$ and $v$ have the same bias $\mu$, they will be assigned the same threshold $t = \Phi^{-1}(2\mu - 1)$ in Algorithm 5.3.

Henceforth, we fix $\langle \bar{w}_1, \bar{w}_2 \rangle = 1 - \rho \geq 0$, and express the probabilities as a function of $\mu$ and $t$. We stress that $\mu$ and $t$ are fully dependent on each other, therefore the functions are only single variable functions. We use both $\mu$ and $t$ (and other notations that are about to be introduced) in the expression only for simplicity.

Let $\varepsilon = (1 - \mu^2)\rho$, which characterizes $\langle u, v \rangle$ as a function of $\mu$, i.e.,

$$\langle u, v \rangle = \langle \mu I + \sqrt{1 - \mu^2} \bar{w}_1, (\mu I + \sqrt{1 - \mu^2} \bar{w}_2) \rangle = 1 - \varepsilon$$

Let $H(t)$ be the probability of the two vectors being separated by a random hyperplane. It is well-known that [GW95]

$$H(t) = \arccos(u \cdot v)/\pi = \arccos(1 - \varepsilon)/\pi$$

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For Algorithm 5.3, notice that \( \tilde{\omega}_1 \cdot g \) and \( \tilde{\omega}_2 \cdot g \) are two jointly distributed standard Gaussian variables with covariance matrix \( \Sigma = \begin{pmatrix} 1 & 1 - \rho \\ 1 - \rho & 1 \end{pmatrix} \). Thus the probability of \( u \) and \( v \) being separated by Algorithm 5.3 is

\[
B(t) = 2 \int_{-\infty}^t \int_t^\infty \frac{1}{2\pi|\Sigma|^{1/2}} e^{-(x_1, x_2)' \Sigma^{-1} (x_1, x_2)} dx_1 dx_2
\]

It’s easy to see that when \( \mu = t = 0 \), these two rounding schemes are equivalent, thus \( B(0) = H(0) \). Also \( \lim_{t \to \infty} B(t) = \lim_{t \to \infty} H(t) = 0 \). The derivatives of \( H(t) \) and \( B(t) \) are as follows:

\[
H'(t) = -\frac{2\sqrt{2\rho}}{\pi^{3/2}} \frac{\Phi(t)e^{-t^2/2}}{\sqrt{2E - \varepsilon^2}}
\]

and

\[
B'(t) = -\frac{2}{\pi} \frac{\Phi(at)e^{-t^2/2}}{\sqrt{2 \rho - \rho^2}}
\]

where \( a = \frac{\rho}{\sqrt{2\rho - \rho^2}} \leq 1 \) when \( \rho \leq 1 \), and \( \bar{\Phi}(t) \) is defined as

\[
\bar{\Phi}(t) = \Phi(t) - \Phi(-t)
\]

Let \( f(t) = \frac{B(t)}{H(t)} \). Notice that \( f(0) = \pi/2 \geq 1 \), thus by Lemma A.3, we only have to show that \( f(t) = 1 \) has only one solution. Moreover, it suffices to show that \( f'(t) < 0 \) when \( f(t) \leq 1 \).

Notice that when \( f(t) \leq 1 \), we have

\[
\frac{\sqrt{2E - \varepsilon^2} \Phi(at)}{\sqrt{2\rho - \rho^2} a \Phi(t)} \leq \frac{2}{\pi} \quad (\text{By convexity of } \Phi, \frac{\Phi(at)}{a \Phi(t)} \geq 1 \text{ when } a \leq 1)
\]

\[
\Rightarrow \frac{2E - \varepsilon^2}{2\rho - \rho^2} \leq \frac{4}{\pi^2}
\]

\[
\Rightarrow \frac{\varepsilon}{\rho^2 - \rho^2} \leq \frac{4}{\pi^2}
\]

\[
\Rightarrow (1 - \mu^2)^2 \frac{1 - \rho^2}{\rho^2} \leq \frac{4}{\pi^2} \quad \left( \frac{\varepsilon}{\rho} = 1 - \mu^2 \right)
\]

\[
\Rightarrow \mu \geq \sqrt{1 - 4/\pi^2} = \mu_0 \quad \left( \frac{2 - \rho}{2 - \varepsilon} \leq 1 \right)
\]

\[
\Rightarrow t \geq t_0
\]

By calculation, one can show that

\[
f'(t) = \frac{\sqrt{2\pi e^{-t^2/2}}}{\Phi(t)} \sqrt{2E - \varepsilon^2} \left( \frac{1 - \varepsilon}{2E - \varepsilon^2} (2\mu \rho) \bar{\Phi}(at) + e^{(1-a^2)t^2/2} a - \frac{\Phi(at)}{\Phi(t)} \right)
\]

Now we show \( f'(t) < 0 \) when \( t \geq t_0 \). In order to show this, one only needs to show that

\[
\frac{1 - \varepsilon}{2E - \varepsilon^2} (2\mu \rho) \bar{\Phi}(at) + \frac{\Phi(at)}{\Phi(t)} > e^{(1-a^2)t^2/2} a
\]

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By substituting $\varepsilon = (1 - \mu^2)\rho$ and simplification, we get
\[
\frac{\Phi'(at)}{\Phi(t)} = \frac{1}{1 - \mu^2} \left( \frac{1 - \varepsilon}{2} - 2\mu^2 + 1 - \mu^2 \right) \geq e^{(1-x^2)/2}.
\]

Since $\frac{\Phi'(at)}{\Phi(t)} \geq 1$ when $a \leq 1$ and $e^{(1-x^2)/2} \leq e^2/2$, it suffices to show
\[
\left( 2\mu^2 \frac{1 - \varepsilon}{2 - \varepsilon} + 1 - \mu^2 \right) \geq e^2/2(1 - \mu^2)
\]
holds when $t \geq t_0$.

By Lemma A.2, we know that RHS is decreasing when $t \geq t_0$. Now we show LHS is increasing when $\mu \geq \mu_0$. It can be shown that the derivative of LHS is
\[
2\mu \rho (1 - \mu^2) \mu^2 - (2\mu^3) (2 - \varepsilon) \geq -\mu (2 - 4\mu^2)(2 - \varepsilon) \geq 0
\]
when $\mu \geq \mu_0$.

Now we only have to verify the inequality when $t = t_0$, and that can be done numerically. The calculation shows that LHS($t_0$) $\approx 0.8489$ while RHS($t_0$) $\approx 0.836$.

\[
\square
\]

Finally, we show Lemma 5.4.

**Lemma A.5.** (Restatement of Lemma 5.4) Let $u = \mu_1 I + w_1, v = \mu_2 I + w_2$ be two unit vectors satisfying $\|u - v\|^2/4 \leq \varepsilon$, then the probability of them being separated by Algorithm 5.3 is at most $O(\sqrt{\varepsilon})$.

**Proof:** (Proof of Lemma 5.4)

First we prove the case when $\mu_1 = \mu_2 = \mu$. Notice that when $\langle w_1, w_2 \rangle > 0$, the lemma follows from Lemma A.4 and the fact that Goemans-Williamson algorithm will separate $u$ and $v$ with probability $O(\sqrt{\varepsilon})[GW95]$.

If $\langle w_1, w_2 \rangle < 0$, then $\|u - v\|^2/4 = \|w_1 - w_2\|^2/4 \geq (\|w_1\|^2 + \|w_2\|^2)/4 = (1 - \mu^2)/2$. Hence $|\mu| > 1 - O(\sqrt{\varepsilon})$. By union bound, the probability of the algorithm separating $u$ and $v$ is at most $O(\sqrt{\varepsilon})$.

Now we consider the case when $\mu_1 \neq \mu_2$, w.l.o.g. we may assume $|\mu_1| > |\mu_2|$. We construct an auxiliary vector $v'$ as follow: $v' = \mu_1 I + \sqrt{1 - \mu_1^2} w_2$. It’s easy to see that $\|u - v'\| \leq \|u - v\|$. Let $F$ denote the rounding function, we analyze the probability of $u$ and $v$ being separated as follows:

\[
\mathbb{P}(F(u) \neq F(v)) = \mathbb{P}(F(u) \neq F(v'), F(v') = F(v)) + \mathbb{P}(F(u) = F(v'), F(v') \neq F(v)) \\
\leq \mathbb{P}(F(u) \neq F(v')) + \mathbb{P}(F(v') \neq F(v))
\]
Since $\|u - v\| \leq \|u - v'\|$ and $\langle u, I \rangle = \langle v', I \rangle = \mu_1$, by the first part of the proof $\mathbb{P}(F(u) \neq F(v')) \leq O(\sqrt{\varepsilon})$. Also,

$$\mathbb{P}(F(v') \neq F(v)) \leq \frac{|\mu_1 - \mu_2|}{2} \leq \frac{\|u - v\|}{2} \leq O(\sqrt{\varepsilon}).$$

Therefore the lemma follows. □

**B Mutual Information, Statistical Distance and Independence**

Intuitively, when two random variables have low mutual information, they should be close to being independent. In this section we formalize this intuition by giving an explicit bound on the statistical distance between the joint distribution and the independent distribution. We stress that all the results here are sufficient for our use in this work, but we believe the parameters could be further optimized.

We start by defining a few notions that measures the correlation of two random variables.

**Definition B.1.** Let $\Omega$ be a finite sample space, $P$ and $Q$ be two probability distributions on $\Omega$. The square Hellinger distance of $P$ and $Q$ is defined as

$$H^2(P, Q) = \frac{1}{2} \sum_{x \in \Omega} (\sqrt{P(x)} - \sqrt{Q(x)})^2$$

**Definition B.2.** Let $\Omega$ be a finite sample space, $P$ and $Q$ be two probability distributions on $\Omega$. The Kullback-Leibler divergence of $P$ and $Q$ is defined as

$$D_{KL}(P \parallel Q) = \sum_{x \in \Omega} P(x) \log \frac{P(x)}{Q(x)}$$

Now we give a few facts regarding mutual information, Hellinger distance and Kullback-Leibler divergence without proving them.

**Fact B.3.** Let $X$ and $Y$ be two jointly distributed random variables taking value in $[q]$, then

$$I(X; Y) = D_{KL}(p(x, y) \parallel p(x) \times p(y)).$$

where $p(x, y)$ is the joint distribution of $X$ and $Y$ on $[q]^2$ and $p(x) \times p(y)$ is the product distribution of the marginal distributions of $X$ and $Y$.

**Fact B.4.** Let $\Omega$ be a finite sample space, $P$ and $Q$ be two probability distribution on $\Omega$, then

$$D_{KL}(Q \parallel P) \geq \frac{2}{\ln 2} H^2(P, Q)$$

Combining the facts mentioned above, we get the following relation between mutual information and statistical distance.
Fact B.5. (Restatement of Fact 4.3) Let $X$ and $Y$ be two jointly distributed random variables on $[q]$ then,

$$I(X; Y) \geq \frac{1}{2 \ln 2} \sum_{i,j \in [q]} (\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i)\mathbb{P}(Y = j))^2,$$

in particular for all $i, j \in [q]$

$$|\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i)\mathbb{P}(Y = j)| \leq \sqrt{2I(X; Y)}$$

As a consequence, if $X$ and $Y$ are two random variables defined on $\{-1, 1\}$, $\text{Cov}(X, Y) \leq O(\sqrt{I(X; Y)})$

Proof.

$$I(X; Y) = D_{KL}(p(x, y)\|p(x) \times p(y))$$

$$\geq \frac{2}{\ln 2} H^2(p(x, y), p(x) \times p(y))$$

$$= \frac{2}{\ln 2} \sum_{i,j \in [q]} \left( \sqrt{\mathbb{P}(X = i, Y = j)} - \sqrt{\mathbb{P}(X = i)\mathbb{P}(Y = j)} \right)^2$$

$$= \frac{2}{\ln 2} \sum_{i,j \in [q]} \left( \frac{\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i)\mathbb{P}(Y = j)}{\sqrt{\mathbb{P}(X = i, Y = j)} + \sqrt{\mathbb{P}(X = i)\mathbb{P}(Y = j)}} \right)^2$$

$$\geq \frac{1}{2 \ln 2} \sum_{i,j \in [q]} (\mathbb{P}(X = i, Y = j) - \mathbb{P}(X = i)\mathbb{P}(Y = j))^2$$

Upper bounding $\ln 2$ by 1, finishes the proof. \qed