Phase Transitions in Random Boolean Networks with Different Updating Schemes

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Abstract
In this paper we study the phase transitions of different types of Random Boolean networks. These differ in their updating scheme: synchronous, semi-synchronous, or asynchronous, and deterministic or non-deterministic. It has been shown that the statistical properties of Random Boolean networks change considerably according to the updating scheme. We study with computer simulations sensitivity to initial conditions as a measure of order/chaos. We find that independently of their updating scheme, all network types have very similar phase transitions, namely when the average number of connections of nodes is between one and three. This critical value depends more on the size of the network than on the updating scheme.

Key words: Random Boolean networks, phase transitions, order, chaos, synchronous, asynchronous
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Introduction
Random Boolean networks (RBNs) have been used to model several complex systems. Originally proposed by Kauffman to model genetic regulatory networks [1][2], their study has propagated into different research areas, from mathematics to robotics, from biology to artificial life. It has been noticed that the properties and behaviour of RBNs, such as average number of attractors and shape of attractor basins, change considerably depending on the updating scheme we use [3][4][5]. For this paper, we studied the following question: Does the updating scheme affect the dynamical phase (ordered, chaotic) of the network? In other words, for a given family of RBNs, if we change the updating scheme, could this also change the regime?

In the following section, we review the properties of different types of RBNs
according to their updating scheme: synchronous or asynchronous, deterministic or non-deterministic [4]. In Section 2 we mention previous studies on phase transitions of RBNs. In Section 3 we present experiments we performed for comparing the sensitivity to initial conditions of different types of RBN. The results are discussed in Section 4, for concluding in Section 5.

1 Different Types of Random Boolean Networks

Random Boolean networks consist of \( n \) nodes or units which can take values of zero or one. The actual value of each node is controlled by the values of \( k \) nodes which are “connected” to it, and determined by lookup tables of logic functions. The lookup tables and the connectivity of the network are generated randomly, but remain fixed through the dynamics of the network. RBNs are interesting models because one does not assume the functionality of the network in order to study the generic properties of a type of networks. The parameters which affect the behaviour of networks include number of nodes, number of connections, topology [6][7], and updating scheme [3][4][5].

Classical Random Boolean networks (CRBNs) were proposed by Kauffman to model genetic regulatory networks [1][2]. He used synchronous updating: the values of the nodes at time \( t + 1 \) depend of the values of the nodes at time \( t \). Since they are deterministic, and have a finite number of nodes, sooner or later they reach a state which was visited already. The network has fallen into an attractor. If the attractor consists of a single state, it will be called a point attractor, whereas a cycle attractor consists of several states. CRBNs have been widely studied (e.g. [8][9]).

Asynchronous Random Boolean networks (ARBNs) were developed by Harvey and Bossoaier [3], criticizing the assumption of Kauffman that genetic regulatory networks are synchronous. ARBNs take one node at random each time step and update it. The behaviour of the network changes considerably from CRBNs. The basins of attraction change, and cycle attractors disappear, since these networks are non-deterministic in their updating, and the sequence of transitions will not be repeated. However, DiPaolo was able to use genetic algorithms to find ARBNs with “rhythmic” attractors [10]. Also, it has been shown that asynchronous dynamical systems can reproduce the behaviour of synchronous systems [11][12].

Deterministic Asynchronous Random Boolean networks (DARBNs) [4] were defined criticizing non-determinism in ARBNs, and as an intermediate alternative between CRBNs and ARBNs. Each node can be updated in different time periods, but these are deterministic (although randomly generated).
We need for each node parameters $p$ and $q$ which determine the “period” and the “translation” of the update. So each node is updated when the modulus of time over $p$ equals $q$, $(p > q)$. If more than one node fulfills this condition, the network is updated in a specific sequence for each node, one by one, to simulate the updating of ARBNs. Since the functions, topology, and update parameters are generated randomly, the sequence does not affect the general properties of a family of networks. DARBNs have again cycle attractors, and their properties are similar to the ones of CRBNs. However, as we allow a wider variety of updating periods (increasing $maxP$, the maximum $p$ allowed), the properties are more similar to the ones of ARBNs.

**Generalized Asynchronous Random Boolean networks (GARBNs)** [4] are very similar to ARBNs, only that they choose randomly at each time step which nodes to update, and these are updated synchronously. Therefore, there could be none, one, some, or all of the nodes being updated at a particular time step. We can say that GARBNs are semi-synchronous.

**Deterministic Generalized Asynchronous Random Boolean networks (DGARBNs)** [4], like DARBNs also use parameters $p$ and $q$ to make the dynamics deterministic. The difference with DARBNs is that when several nodes fulfill the updating condition, all them are updated synchronously. Thus, DGARBNs have semi-synchronous but deterministic updating. CRBNs are a particular case of DGARBNs, $(maxP = 1)$. We can describe the set of parameters $p$ and $q$ as the context of the network. We have used DGARBNs as models of contextual discrete dynamical systems with promising results [5]. We have found that changing the context of a particular network can change considerably its attractor basins and number of attractors, without changing the rules or connectivity of the network.

Changing the updating scheme of a RBN changes its behaviour considerably. Even among deterministic RBNs, the attractor basins shift for different updating schemes. Different types of RBNs have different statistical properties, such as average number of attractors and percentage of states in attractors. However, we have realized that point attractors are the same for all types of RBNs [4], since no matter which node is updated, the state will not change. Thus, the updating does not affect the point attractors. Also all the RBNs have very similar behaviour for the anomalous case $k = 0$, because there are no actual dynamics: all initial states tend to a single point attractor.

We have shown [5] that there is a smooth transition of the statistical properties of RBNs from deterministic to asynchronous non-deterministic, passing through asynchronous deterministic as we allow a wider variety of their updating periods (i.e. increase $maxP$).
2 Phase Transitions in RBNs

Random Boolean networks can have different dynamic regimes. On one hand, we have dynamics which stabilize quickly, and are robust to perturbations, which are called ordered. On the other hand, we have a chaotic regime, where networks take time to reach an attractor (this can be infinite in practice), and are very sensitive to initial conditions. Kauffman conjectured that biological networks should be “at the edge of chaos”, with enough flexibility to explore new functionalities, but enough robustness to keep the present ones.

Classical RBNs with uniform connectivity have been found to have a phase transition from ordered to chaotic for a critical value for $k = 2$ [13]. Bias in the functions (not an equal opportunity for zeros and ones) and topology of the networks have been shown to change the critical value [14][7]

Mesot and Teuscher [15] studied critical values for GARBNs inspired in the annealed approximation method [13], but found no critical value, in the sense that GARBNs amplify small perturbations and reduce big ones. The results we obtained measuring sensitivity to initial conditions question their findings.

3 Experiments

We carried out experiments in an open source software laboratory, RBNLab, which we developed for studying different properties of Random Boolean networks. It is available to the public (Java source code included) [16].

For our experiments, we used “standard” RBNs: homogeneous probability in the boolean functions (no bias for more zeroes or ones) and uniform topology (all nodes have exactly $k$ connections).

There are several features which can be used to measure chaos, and which can yield different results. We decided to take sensitivity to initial conditions as a measure for chaos, which is similar to the study of minimal perturbations or damages [14]. We first created randomly an initial state A, and flip one node to have another initial state B. We run each initial state in the network for ten thousand time steps, obtaining states $A'$ and $B'$. Then we compare the normalized Hamming distance (1) of the final states with the one of the initial states to obtain a parameter $\delta$ (2).
\[ H(A, B) = \frac{1}{n} \sum_{i} |a_i - b_i| \] (1)

\[ \delta = H_{t \to \infty} - H_{t=0} \] (2)

If \( \delta \) is negative, it means that the Hamming distance was reduced. Since the initial distance is minimal \( \left( \frac{1}{n} \right) \), a negative \( \delta \) indicates that both initial states tend to the same attractor. This implies that the network is stable, or on an ordered phase. A positive \( \delta \) indicates that the dynamics for very similar initial states diverge. This is a common characteristic of chaos in dynamical systems.

Since the initial states are chosen randomly, the comparison we make is equivalent to see B as a perturbed version of A, and observe if the perturbation affects the dynamics.

To compare the regimes of different types of RBN, we created \( NN \) number of networks (200 unless specified), and evaluated for each \( NS \) number of states (200 unless specified) for all five types of RBN (using \( maxP = 7 \) for the deterministic asynchronous RBNs), for different network sizes \( (n) \) and connectivity density \( (k) \).

We can observe the averages of \( \delta \) for networks with \( n = 5 \) in Figure 1. The error bars indicate the standard deviations. We can see that all networks have an average phase transition from ordered to chaotic for values of \( k \) between two and three (although the standard deviations indicate us that there can very well be chaotic networks for \( k = 2 \) and ordered for \( k > 2 \)). It is curious to note that for this network size, the “most ordered” updating schemes of the ordered phase are also the “most chaotic” of the chaotic phase, and vice versa.

Independently of network size, all RBNs will have \( \delta = -\frac{1}{n} \) for \( k = 0 \), since all states tend to a single point attractor.

In Figure 2, we can observe that for \( n = 10 \) all network types except CRBN are already on the chaotic regime for \( k = 2 \), although still close to the “edge of chaos”.

This is similar for \( n = 15 \), although CRBN has average \( \delta \) values very close to zero, as we can see in Figure 3.
As we increase $n$, the point where $\delta$ values of CRBNs cross to be higher than the others shifts towards larger values of $k$. As shown in Figure 4, for $n = 20$ and large values of $k$, DARBNs turn out to be “less chaotic” than the asynchronous non-deterministic networks. For large values of $k$ and $n$, all networks tend asymptotically to an average maximum value of $\delta$, where almost all close states tend to different attractors, independently on the updating scheme.

A similar pattern follows as we increase the network size, as seen in Figures
Fig. 3. Sensitivity to initial conditions for networks with $n = 15$

Fig. 4. Sensitivity to initial conditions for networks with $n = 20$

5, 6, 7, and 8. What changes is that for very large networks, CRBNs seem to be always “less chaotic” than others, even in the chaotic phase.

It is interesting to note that the standard deviation decreases as we increase the size of the networks. This is good, since larger networks need more computational resources to be analysed. We should also note that always the highest variance is around $k = 3$. This indicates a wider diversity in the types of networks. This is a desirable property in evolving networks. It should be mentioned that also at $k=3$ point attractors are “harder to find” (they have the
Fig. 5. Sensitivity to initial conditions for networks with \( n = 25 \)

Fig. 6. Sensitivity to initial conditions for networks with \( n = 50 \). NN=100, NS=100 most skewed distribution among all possible networks: few networks with lots of point attractors, many with none) [3].

We can see that for very large networks \( n > 100 \), even CRBNs with \( k = 2 \) fall into the chaotic regime.

The detailed data and graphics obtained for these experiments can be found online [17].
Fig. 7. Sensitivity to initial conditions for networks with \( n = 100 \). NN=50, NS=50

Fig. 8. Sensitivity to initial conditions for networks with \( n = 200 \). NN=20, NS=20

4 Discussion

Even when there are differences in the average values of \( \delta \) for different types of RBN, all of them follow a similar path, crossing from ordered to chaotic for values \( 1 < k < 3 \). The precise value of the transition, in practice, does not depend only on the updating scheme, but also on the size of the network. However, we should note that our results show only averages, and that it is
always possible to find ordered RBNs of an arbitrary size and connectivity\(^1\). What our results show is that networks with complex behaviour are easier to find for values \(1 < k < 3\), independently of the updating scheme. This is consistent with previous results for CRBNs. It is interesting that in practice smaller networks have transitions at values of \(k\) closer to three, while larger networks have values closer to one.

Mesot and Teuscher [15] did not find any phase transition for GARBNs. They used a different approach, namely the annealed approximation method[13]. In their simulations, they studied networks of \(n = 200\) and \(k >= 1\). As we can see in Figure 8, the behaviour for \(k = 1\) is very similar than the one for \(k = 0\). It seems that the explanation for this is that very large networks with only one connection are “frozen” easily, because with an uniform topological distribution, it is hard for perturbations to propagate. On the other hand, if we have two connections per node, distant nodes can affect each other indirectly. This explains why larger networks tend to be less stable, even with the same number of connections. For \(n = 200\), the behaviour of the networks is on average either static \((k <= 1)\) or chaotic \((k >= 2)\), but independently of the updating scheme. It seems that the edge of chaos “shrinks” as we increase the size of networks. In other words, complex behaviour will still be found somewhere near a certain critical value, only that on average it will be harder to find particular networks which exhibit complex behaviour, i.e. for larger networks there is a lower percentage of complex networks in all possible networks.

It is also worth noting that for large networks, asynchronous updating schemes, deterministic or not, tend to be very similar in terms of \(\delta\). It seems that synchronicity plays a more important role in phase transitions than the determinism of the updates. This is opposite to the network properties, where the determinism makes a greater difference than the synchronicity [4], mainly because cycle attractors are lost in non-deterministic updating schemes. Deterministic asynchronous RBNs have the advantage that they are easier to study than the non-deterministic ones, but we have seen that they have similar phase transitions. This is yet another reason for preferring GARBNs for modelling complex networks[5], although specific considerations can suggest another network type.

GARBNs could also be preferred because they can “process more information” than other types of networks of the same size and connectivity, since they can “throw information into their context”[5]. This is because any deterministic asynchronous RBN can be mapped into a redundant CRBN, incorporating the information of the updating periods into the network [4]. But this takes

\(^1\) An example can be a network which states represent numbers in base two. If each state goes to the state representing the next ascending number, and the last \((2^n)\) to itself, we have a very ordered network independently of \(n\).
the RBN into a family of networks of the chaotic phase, since we add more connections. With the results presented in this paper, we can see that if we vary the updating periods of a RBN, the phase would still be similar. This allows the same network with the same rules and topology to have different functionality in different situations or contexts. Therefore, more information can be processed by DGARBNs by complexifying its updating periods, but without moving the dynamics into the chaotic phase. This is an evolutionary advantage, and we should expect natural systems to exploit their context or environment in order to process more information without increasing their own information processing capacity. An example of this are humans using their environment (books, computers, culture) to enhance their cognitive abilities[18].

The role of the topology on the properties and behaviour of RBNs should be studied for different types of RBNs, especially for scale-free topologies, following the work of Aldana [7].

5 Conclusions

We have tested experimentally the sensitivity to initial conditions of different types of Random Boolean networks. We found no significant difference between the different updating schemes and the phase transition. All RBNs have a phase transition from ordered to chaotic for values $1 < k < 3$, independently of their updating scheme. In practice, the precise critical value of $k$ depends more on the size of the network than on the updating scheme. This is interesting because the statistical properties, such as average number of attractors and average attractor length, do change with the updating scheme. It implies that the phase of a RBN (ordered, critical, chaotic) does not depend really on the updating scheme, but on the network size, connectivity (and topology), and functionality of the network.

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