NON-SEQUENTIAL DECENTRALIZED STOCHASTIC CONTROL REVISITED: CAUSALITY AND STATIC REDUCIBILITY

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Abstract. In decentralized stochastic control (or stochastic team theory) and game theory, if there is a pre-defined order in a system in which agents act, the system is called sequential, otherwise it is non-sequential. Much of the literature on stochastic control theory, such as studies on the existence analysis, approximation methods, and on dynamic programming or other analytical or learning theoretic methods, have focused on sequential systems. The static reduction method for sequential control problems (via change of measures or other techniques), in particular, has been shown to be very effective in arriving at existence, structural, approximation and learning theoretic results. Many practical systems, however, are non-sequential where the order of agents acting is random, and dependent on the realization of solution paths and prior actions taken. The study of such systems is particularly challenging as tools applicable for sequential models are not directly applicable. In this paper, we will study static reducibility of non-sequential stochastic control systems, including by change of measure methods. We revisit the notion of Causality (a definition due to Witsenhausen and which has been refined by Andersland and Tekentzis), and provide an alternative representation using imaginary agents. Through this representation, we show that Causality is equivalent to Causal Implementability (and Dead-Lock Freeness), thus, generalizing previous results. Via this representation, we show that Causality, under an absolute continuity condition, allows for an equivalent static model whose reduction is policy-independent. This facilitates much of the stochastic analysis available for sequential systems to also be applicable for non-sequential systems. We further show that under more relaxed conditions on the model, such as solvability, such a reduction (when possible) is policy-dependent or includes parameters in the cost of the reduced model, and thus has limited utility. We will also present a further reduction method for partially nested causal non-sequential systems.

1. Introduction. An increasingly important class of optimal stochastic control problems involve setups where a number of decision makers (DMs) / controllers / agents, who have access to different and local information, are present. Such a collection of decision makers who wish to minimize a common cost function and who has an agreement on the system (that is, the probability space on which the system is defined, and the policy and action spaces) is said to be a stochastic team. Such problems are also called decentralized stochastic control problems. For such problems, classical methods involving single-agent stochastic control often are not applicable due to information structure constraints (and notably due to the fact that the information available to agents acting one after another do not necessarily increase, i.e., there is no perfect recall).

Such team problems entail a collection of DMs acting together to optimize a common cost function, but not necessarily sharing all the available information. At each time stage, each DM has only partial access to the global information, which is characterized by the information structure of the problem [41]. If there is a pre-defined order in which the DMs act, then the team is called a sequential team. For sequential teams, if each DM’s information depends only on primitive random variables, the team is static. If at least one DM’s information is affected by an action of another DM, the team is said to be dynamic. Information structures can be further categorized as classical, partially nested (or quasi-classical), and nonclassical. An information structure is classical if the information of decision maker \(i\) (DM\(i\)) includes all of the information available to DM\(k\) for \(k < i\). An information structure is partially nested, if whenever the action of DM\(k\), for some \(k < i\), affects the information of DM\(i\), then the information of DM\(i\) includes the information of DM\(k\). An information structure that is not partially nested is nonclassical.

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Static teams are relatively simpler to study: For teams with finitely many DMs, Marschak \[29\] has studied static teams and Radner \[32\] has established connections between person-by-person (pbp) optimality (i.e. Nash equilibrium), stationarity, and global optimality. Radner’s results were generalized in \[27\] by relaxing optimality conditions. The essence of these results is that in the context of static team problems, convexity of the cost function, subject to minor regularity conditions, suffices for the global optimality of pbp optimal solutions. In the particular case of LQG (Linear Quadratic Gaussian) static teams, this result leads to optimality of linear policies \[32\]. Optimality of linear policies also holds for dynamic LQG teams with partially nested information structures through a transformation of the dynamic team to a static one \[23\]; in \[23\] dynamic LQG teams with partially nested information structures and in \[24\] general dynamic teams with partially nested information structures, satisfying an invertibility assumption, have been shown to be reducible to static team problems, where the aforementioned results for static teams can be applied. Such transformation of dynamic teams to static teams is called static reduction. In the static reduction presented in \[23, 24\], given the policies of the DMs, there is a bijection between observations as a function of precedent actions of DMs and primitive random variables, and observations generated under the transformations, where now they are only functions of primitive random variables. We note that the static reduction procedure in \[23, 24\] depends on the policies that precedent DMs choose.

In \[42\], Witsenhausen introduced a static reduction method for dynamic teams, where observation kernels satisfy an absolute continuity condition. In this static reduction, the probabilistic nature of the problem has been transformed to the cost function by changing the measures of the observations to fixed probability measures. Witsenhausen’s static reduction procedure is independent of the policies that precedent DMs choose, and hence, we refer to this type of static reduction as a policy-independent static reduction \[36\]. The policy-independent static reduction is essentially a change of measure argument, a prominent version of which is known as Girsanov’s transformation \[20, 10\] which has been considered in \[42, Eqn(4.2)\], and later utilized in \[45, p. 114\] and \[45, Section 2.2\] (for discrete-time partially observed stochastic control, similar arguments had been presented, e.g. by Borkar in \[13, 14\]). We refer the reader to \[15\] for relations with the classical continuous-time stochastic control, where the relation with Girsanov’s classical measure transformation \[20, 10\] is recognized. Such reformulation of stochastic dynamic control and filtering problems via a change of measure has proven to be very effective for existence, approximations, filtering, and dynamic programming methods under a variety of contexts. In discrete-time, such a formulation leads to an equivalent system under which measurements are exogenous independent variables. For continuous-time stochastic control, this method has been popular since the analysis in \[10\], to avoid stringent conditions on control policies to satisfy the conditions for existence of strong solutions to controlled stochastic differential equations (see also \[16, 17\]), which allows the control to be a function of an independent Brownian innovations process. This was utilized by Fleming and Pardoux \[18\], who introduced wide-sense admissible control policies for classical partially observed stochastic control problems (POMDPs), where first the measures were reduced to an exogenous Brownian process and a control policy was defined to be wide-sense admissible if the control and (exogenous) measurements were independent from future increments of the measurement process (see also \[11\], where this was utilized further without separating estimation and control in POMDPs). Borkar \[13, 14\] obtained existence results in average cost POMDPs via generalizing Fleming and Pardoux’s notion to discrete-time POMDPs, where, once again, first the measurements were made independent and then the control and measurements form a wide-sense admissible process if they are independent from future measurements. In non-linear filtering theory such methods have found
applications via the Kushner-Kallianpur-Striebel formula [28].

Since Witsenhausen’s paper [42], the static reduction method for sequential control problems has been shown to be very effective in arriving at existence, structural and approximation results. For existence results building on this approach, we refer the reader to [21, 47, 33, 45], for a dynamic programming formulation to [40] for countable spaces and [45] for general spaces, for rigorous approximations with finite models to [35]. Furthermore, it has been shown recently that all sequential decentralized stochastic control problems are nearly static reducible by additive perturbations of measurements of each DM [25].

Despite extensive studies for sequential decentralized stochastic control problems, which also include dynamic programming methods (see e.g. under various formulations [30, 31], [7, 44, 40, 45]), and while ubiquitous in many engineering systems or other application areas such as economics, research on non-sequential stochastic control has been rather limited. One could notice that since the foundational works by Witsenhausen [39] and later by Andersland and Teneketzis [4, 5] and Teneketzis [38], there has been modest progress in this area with the exception of few recent studies such as [26], [22, 1] primarily in the game theory community.

In this paper, we study static reducibility for non-sequential stochastic control problems. During our derivations and results, we also present new interpretations for some of the classical results put forward by Witsenhausen, Andersland, and Teneketzis. It is our hope that research in this field may be revived in the community.

Static reduction also facilitates stochastic optimality analysis via numerical methods and convex relaxations [34] and learning theoretic methods in decentralized stochastic control [6]. Notably, for non-sequential systems it is particularly challenging to apply dynamic programming or reason via backwards induction. One benefit of reduction is that even if the cost function is not written explicitly or known, that there is one such cost function and that the information is static, certifies the approach that one can apply learning algorithms to converge both to equilibria [6] or optimal solutions [43]; for continuous spaces this also applies [2, Section 6.1]. This is a very useful practical implication of static reducibility, in particular for non-sequential teams. In the non-sequential case, static reduction is even more appealing, since it effectively decouples the agents, and the problem can be solved as if all agents act at the same time, independently in an equivalent probability space. This saves the need for a tedious optimality analysis along the realized solution paths, in which the ordering of agents is dependent on the sample path and policies of agents.

With the recognition that the information fields generated by local measurements lead to subtle conditions on solvability and causality, an alternative probabilistic model, based on quantum mechanics, for describing such problems has been proposed by Baras in [9] and [8]. Such quantum team decision theory, for sequential systems however, has also been studied in [34]. Our paper will also have implications for the applicability of such studies for non-sequential models.

Contributions.

(i) We revisit the notion of Causality (a definition due to Witsenhausen and which has been refined by Andersland and Teneketzis), and provide an alternative representation using imaginary agents. Through this representation, we show that Causality is equivalent to Causal Implementability (and Dead-Lock Freeness) in Theorem 2.9 complementing the existing implication relations presented in Theorem 2.8 via generalizing the equivalence result of [4, Theorem 4] for countable spaces to general standard Borel models.

(ii) Via this imaginary agents representation, we show that Causality (C), under an absolute continuity condition, allows for an equivalent static model whose reduction
is policy-independent. This is done in Theorem 3.2 for countable models and in Theorem 3.3 for general standard Borel models. This result facilitates much of the stochastic analysis available for sequential systems to also be applicable for non-sequential systems. In particular, learning algorithms are guaranteed to converge to optimality for finite space models [5, 43] and near optimality for Borel models [2, Section 6.1], even when the exact form of the reduced model is not explicitly written but that such a reduction is available is known.

(iii) We further show that under more relaxed conditions on the model, such as Causal Implementability (CI) and Solvability (SM), such a reduction, when possible at all, is either policy-dependent or includes policies as parameters in the cost of the reduced model, and thus has limited utility, in Theorems 4.1 and 4.2.

(iv) We will also present a further reduction method for partially nested causal non-sequential systems in Theorem 3.4, generalizing the approach presented in [23, 24] for sequential systems.

In Section 2, we outline the general structure of non-sequential decentralized stochastic control problems that will be considered, and we introduce some preliminary definitions pertaining to the study of non-sequential problems, and we will discuss some existing related results for the analysis of such problems, before showing that Causality (Property C) is equivalent to Causal Implementability (Property CI). In Section 3, we will state conditions upon which a non-sequential decentralized stochastic control problem admits a policy-independent static reduction. Also in Section 3 we will present a policy-dependent static reduction procedure for non-sequential decentralized stochastic team problems with partially-nested information structures which also possess Property C (generalizing such reduction methods for sequentially partially nested teams), and also a less practical policy-dependent static reduction procedure for problems possessing weaker properties, such as Property SM or CI, to be defined below. In Section 5 we provide an example to illustrate the main result of the paper, on policy-independent static reduction for

2. Properties of Non-Sequential Decentralized Stochastic Control Problems. Witsenhausen [39] defines several properties for non-sequential systems: At a high-level, Witsenhausen defines causality to be the property that for each \( k \leq N \), the events that a DM acting at time \( k \) can distinguish are events that can be induced by the decisions of the first \( k - 1 \) DMs and \( \omega \). Furthermore, after \( k - 1 \) DMs have acted, the selection of the \( k \)th DM is determined by \( \omega \) and the actions up to time \( k - 1 \). Nonetheless, the DM acting at time \( k \) does not need to know that he/she is \( DM_k \). Witsenhausen has proved that causality implies solvability [39, Theorem 1], which ensures that all control action variables in the system are well-defined random variables (as proper measurable functions), and Andersland and Teneketzis later introduced causal-implementability and deadlock-freeness [4] properties which are more general than causality as they only pertain to the information available to the DMs, and do not impose restrictions on the ordering of DMs. All of these properties are formally defined and studied in this section.

Before proceeding with the definitions, we introduce a variation of the intrinsic model presented in [41] and some notation that will be used in the definitions.

2.1. An intrinsic model for non-sequential stochastic control. The general intrinsic model that we will introduce will be a minor modification of Witsenhausen’s sequential model [41]. The model consists of:

1. The number of DMs in the system, \( N \in \mathbb{N} \).
2. A measurable space \((\Omega, \mathcal{F})\) characterizing the underlying event space.
3. \((\mathcal{U}^1, \mathcal{U}^k)\) denotes the standard Borel space from which \( u^k \), the \( k \)th control action is selected. The measurable space containing the collective action \( \mathbf{u} := (u^1, u^2, \cdots, u^N) \)
is denoted by \((\prod_{i=1}^{N} U^i, \mathcal{G}^N, \mathcal{U}^i)\).

4. \((Y^k, \mathcal{Y}^k)\) denotes the standard Borel space from which \(y^k\), the measurement \(DM^k\), takes values from.

5. Measurement functions, \(y^i = \eta^i(\omega, u)\), which output the measurements of each \(DM_i\), for \(i = 1, 2, \ldots, N\). Let \(\mathcal{F}^i\) denote the sigma field induced on \(\Omega \times (\prod_{k=1}^{N} U^k)\) by the measurement function \(\eta^i\).

6. A design constraint which restricts the \(N\)-tuples of control policies to be measurable given the information structure: \(\gamma = \{y^1, y^2, \ldots, y^N\}\), with \(y^k : (Y^k, \mathcal{Y}^k) \rightarrow (U^k, \mathcal{U}^k)\), \(k = 1, 2, \ldots, N\), so that \(u^k = y^k(y^k)\). We let \(\Gamma^k\) denote the set of all such policies for \(DM_k\), and \(\Gamma := \prod_{k=1}^{N} \Gamma^k\) denote the collection of such policies.

7. A probability measure \(P\) on \((\Omega, \mathcal{F})\).

We will also find it useful to refine \((\Omega, \mathcal{F})\) into a collection of signal spaces \((\prod_{i=0}^{N} \Omega_i, \prod_{i=1}^{N} \mathcal{B}_i)\) denoting measurable spaces from which the collection of random variables \(\omega_0, \omega_1, \ldots, \omega_N, \omega_{i0}\) take values from. Here, \((\omega_0, \omega_{i0}) \in \Omega_0\) are the cost and order relevant variables, where \(\omega_0\) is the cost-relevant uncertainty, and \(\omega_{i0}\) is the uncertainty related to the possibly random ordering of DMs. The variables \(\omega_1, \omega_2, \ldots, \omega_N\) represent the noise in the measurements \(y^i, i = 1, 2, \ldots, N\). We note that \((\omega_0, \omega_{i0})\) may not be independent, and thus only their joint probability distribution is referred to.

Given a cost function \(c : \Omega_0 \times \prod_{i=0}^{N} U^i \rightarrow \mathbb{R}^+\), a solution to a stochastic team problem is to identify a policy \(\gamma\) that achieves \(\inf_{\gamma \in \Gamma} J(\gamma)\) with

\[J(\gamma) := E[c(\omega_0, \omega_{i0}, u)].\]

2.2. A generalized classification of information structures for non-sequential systems. Some of the tediousness that comes with studying non-sequential decentralized stochastic control problems is due to the fact that the information structures of decision makers are not as easily characterized as in sequential problems. Since the indexing of DMs does not represent the order in which they act when studying non-sequential problems, it is necessary to adjust the standard classifications from sequential problems, introduced by Witsenhausen, so that they may also apply to non-sequential problems. In the following, and in the rest of the paper, we denote the ordering of DMs for a particular realized solution path as \(s := (s_1, s_2, \ldots, s_N) \in S\), where \(S\) denotes the set of all possible \(N\)-agent orderings, and \(s_k = i\) means that \(DM^i\) is the \(k\)-th DM to act. We offer the definitions of such generalized classifications for non-sequential systems below:

**Definition 2.1. Classifications of Information Structures**

(i) An information structure is classical if the measurement \(y^i\) of the \(i\)-th DM to act contains all of the information available to \(DM^k\) for \(k < i\), for \(i = 1, 2, \ldots, N\), and for all possible orderings.

(ii) An information structure is quasi-classical or partially nested, if whenever \(u^k\), for some \(k < i\) affects \(y^i\) through the measurement function \(\gamma_{sg}\), \(y^i\) contains all of the information available to \(DM^k\). That is \(\sigma(y^k) \subseteq \sigma(y^i)\), for all \(i = 1, 2, \ldots, N\), and for all possible orderings.

(iii) An information structure which is not partially nested is non-classical.

2.3. Causality, Causal Implementability and Solvability. We begin by introducing some notation that will be used in the definitions that follow. Recall that \(\mathcal{F}^k\) denotes the sigma-field generated by the information at \(DM^k\), for \(1 \leq k \leq N\). In particular, let \(y^i = \eta^i(\omega_0, \omega_i, u)\). Let \(S_k\) denote the set of \(k\)-DM orderings of \(k = 1, 2, \ldots, N\) and \(|s|\) denote the range of \(s\), that is the set of indices present in the ordering \(s\), for any \(s \in S_{|s|}\), where \(|s|\) denotes the cardinality of \(|s|\). For any \(s \in S_k, k = 1, 2, \ldots, N\), \(P_k\) is defined as the projection of
coupled. The information field \( u_1 \) as is not truly any one physical DM, but rather a collection of DMs, where only one is active (image of projection). Let \( \Omega \times \prod_{i=1}^{N} U^j \) denote the identity of the DM. The action sets \( \mathcal{F} \) we let \( \mathcal{F} \) the identity of the DM. The action sets \( \mathcal{F} \) we use this definition is that in Witsenhausen’s treatment in \([39]\), the ordering function \( \psi \) is a function from \( \prod_{i=1}^{N} U^j \) to \( S_k \), that is, the ordering function does not take the uncertainty \( \omega \) as a domain variable. The generalization of including the uncertainty \( \omega \) as a domain variable is useful as it allows the first DM to be decided by randomness, rather than having a deterministic starting agent as in \([39]\) Lemma 2].

2.3.1. Causality (C). We begin with causality, which will be of particular importance.

\[ \text{DEFINITION 2.2 (Causality (C)).} \]

An information structure possesses Property C if there exists at least one map \( \psi : \Omega \times \prod_{i=1}^{N} U^j \rightarrow S_N \) such that for any \( s = (s_1, s_2, \ldots, s_k) \in S_k, k = 1, 2, \ldots, N \),

\[ \mathcal{F}^k \cap [T^N_k \psi]^{-1}(s) \subset \mathcal{F}(T^k_{k-1}(s)) \tag{2.1} \]

Alternatively, for \( E \subset \mathcal{F}^k \) the following holds:

\[ E \cap [T^N_k \psi]^{-1}(s) \in \mathcal{F}([T^k_{k-1}(s)]) \tag{2.2} \]

The term \( [T^N_k \psi]^{-1}(s) \) denotes the preimage of the mapping \( T^N_k \psi : \Omega \times \prod_{i=1}^{N} U^j \rightarrow S_k \). Therefore, given the ordering \( s \in S_k \) of the first \( k \) DMs, \( [T^N_k \psi]^{-1}(s) \) will be the set of intrinsic outcomes \((\omega, u) \in \Omega \times \prod_{i=1}^{N} U^j\) that maps, through \( \psi \), to an \( N \)-DM ordering in which the ordering of the first \( k \) agents is \( s \).

An alternative characterization of property C via imaginary DMs acting sequentially: An imaginary sequential model.

We can also present an equivalent definition which will be useful later in our analysis: Property C can be represented in terms of the notion of the join of two sigma fields. The difficulty in presenting this definition will mainly be in the notation, which we propose to be altered in order to view the information structure in terms of imaginary DMs which are indexed by their time of action rather than their identity. In the following, we view \( DM^k \) not as a specific \( DM \), for some realized solution path where \( s_k = i \), but instead we consider \( DM^k \) to be the \( k \)-th DM to act, regardless of the realized solution path. Therefore, \( DM^k \) is not truly any one physical DM, but rather a collection of DMs, where only one is active as \( DM^k \) for any one realized solution path. In this sense, we also change the actions and measurements of each DM to be represented in terms of the order of the actions, rather than the identity of the DM. The action sets \( U^k \) and measurement sets \( \Psi^k \) are therefore seen as coupled. The information field \( \mathcal{J}^k \), of the \( k \)-th DM, is the sigma-field on \( \Omega \times \prod_{i=1}^{N} U^j \) defined by the following:

\[ \mathcal{J}^k = \sigma \left\{ \bigcup_{i=1}^{N} \left[ \mathcal{F}^j \cap [P^N_{T^k_{k-1}}(\omega, u)]^{-1}(P^N_{T^k_{k-1}}(\omega, u)) : \psi(\omega, u)_k = i \right] \right\} \]

Now, in this alternative representation we take the join of the sigma field \( \mathcal{J}^k \) with the sigma field generated by the set \([T^N_k \psi]^{-1}(s)\), for all \( s = (s_1, s_2, \ldots, s_k) \in S_k, k = 1, 2, \ldots, N \).

\[ \text{[We note here that in \([39]\), Witsenhausen defines the projection } \mathcal{P}_j \text{ as being the restriction of } \prod_{i=1}^{N} U^j \text{ to } \prod_{i=1}^{N} U^{j_i}, \text{ whereas we are using the definition which is adopted by Andersland and Tenekezis \([4, 5]\). The reason we use this definition is that in Witsenhausen’s treatment in \([39]\), the ordering function } \psi \text{ is a function from } \prod_{i=1}^{N} U^j \text{ to } S_N, \text{ that is, the ordering function does not take the uncertainty } \omega \text{ as a domain variable. The generalization to include the uncertainty } \omega \text{ as a domain variable is useful as it allows the first DM to be decided by randomness, rather than having a deterministic starting agent as in \([39]\) Lemma 2].} \]
The sigma field generated by the set \( [T_k^N \circ \psi]^{-1}(s) \) refers to the coarsest sigma field over \( \Omega \times (\prod_{i=1}^{N} U_i) \) for which the mapping \( T_k^N \circ \psi \) is measurable. The join of these two sigma fields, denoted by \( \vee \), is the coarsest sigma field over \( \Omega \times \prod_{i=1}^{N} U_i \) containing both. The condition is then that the join of these two fields is a subset of \( \mathcal{F}^*(k-1) \), where \( \mathcal{F}^*(k-1) \) denotes the cylindrical extension of \( \mathcal{F} \times (\prod_{i=1}^{k-1} U_i) \) to \( \Omega \times \prod_{i=1}^{N} U_i \). Note that the difference between \( \mathcal{F}^*(k-1) \) and \( \mathcal{F}(T_k^{k-1}(s)) \) is that \( \mathcal{F}(T_k^{k-1}(s)) \) is defined for any given \( k \in \{1, 2, \ldots, N\} \), while \( \mathcal{F}(T_k^{k-1}(s)) \) is only defined if the order \( s \in S_k \) is defined as well (or \( s \in S_N \) if we apply the map \( T_k^{N}(s) \)). Also, \( \mathcal{F}^*(k-1) \) is the cylindrical extension to \( \Omega \times \prod_{i=1}^{N} U_i \), while \( \mathcal{F}(T_k^{k-1}(s)) \) is the cylindrical extension to \( \Omega \times \prod_{i=1}^{N} U_i \). The condition is therefore:

\[
\mathcal{J}^k \bigvee \sigma\{[T_k^N \circ \psi]^{-1}(s) : s \in S_k\} \subseteq \mathcal{F}^*(k-1) \tag{2.3}
\]

In the imaginary model, with Property C, the measurements \( y^i \) are related to the original measurements \( y^i \) by:

\[
I(y^i) := \sum_{k=1}^{N} y^k 1_{\{y^i(\omega_{s^i_k}, u^{i_1}, \ldots, u^{i_{k-1}}) = k\}} \tag{2.4}
\]

and the actions \( u^i \) are related to \( u^i \) by:

\[
I^u(u^i) := \sum_{k=1}^{N} u^k 1_{\{u^i(\omega_{s^i_k}, u^{i_1}, \ldots, u^{i_{k-1}}) = k\}} \tag{2.5}
\]

Here \( I \) defines an isomorphism between \( (\psi^i, Y^i) \) and \( (\psi^{i_k}, \mathcal{Y}^{i_k}) \), where \( s_k = i \), and likewise so does \( I^u \) between \( (U^i, \mathcal{U}^i) \) and \( (\mathcal{U}^{i_k}, \mathcal{U}^{i_k}) \).

Notably, in the imaginary model, we have that

\[
\mathcal{U}^{i_k} = \bigcup_{i=1}^{N} U^i, \quad \psi^{i_k} = \bigcup_{i=1}^{N} \psi^i
\]

**Remark 2.1.** Note that such an interpretation leads to a sequential model with ‘imaginary’ agents, as there is always the natural ordering.

An important lemma that follows from Property C is the following:

**Lemma 2.3.** \([39\text{ Lemma } 1]\) Suppose that an information structure possesses Property C, then for any \( s = (s_1, s_2, \ldots, s_{||s||}) \in S_{||s||} \), the following holds:

\[
[T_{||s||}^N \circ \psi]^{-1}(s) \in \mathcal{F} (\{T_{||s||}^N \circ \psi\}^{-1}(s)) \tag{2.6}
\]

In terms of sigma fields:

\[
\sigma\{[T_{||s||}^N \circ \psi]^{-1}(s) : s \in S_{||s||}\} \subseteq \mathcal{F}^* (||s|| - 1) \tag{2.7}
\]

This lemma is especially important for a policy-independent static reduction as it essentially states that after the first \( k - 1 \) agents have taken their actions, then the \( k^{th} \) agent to act is thereby determined mathematically. An intuitive example of such an ordering function is one where \( \psi \) is an \( N \)-tuple \( \psi \equiv (\psi^1, \ldots, \psi^N) \), where the index of the \( i^{th} \) DM to act, \( s_i \) is given by \( s_i = \psi(\omega_{s_0}, u^{i_1}, \ldots, u^{i_{k-1}}) \), where \( \omega_{s_0} \) is an exogenous random variable responsible for uncertainty in the ordering of DMs. Put simply, the lemma requires that the ordering of
the system is causal, with the index of the next DM to act decided upon based only on the actions of previous DMs, and exogenous random variables. This condition will be critical in Lemma 3.1 to be presented.

It’s not necessary for the kth DM to know which DMs have acted previously, or even that they are the kth DM, only that the universe (or, umpire, if appropriate in this context) knows.

Due to the following condition being an important classification in non-sequential problems, we call this property random causal sequentiality (RCS), although implicitly this has already been studied by Witsenhausen in [39, Lemma 1]:

**Definition 2.4 (Random Causal Sequentiality (RCS)).** [39, Lemma 1]: An information structure possesses property RCS if for any s = (s1, s2, ..., s|s|) ∈ S|s|, the following holds:

\[ \mathcal{F}(|T||s|^{-1}\omega u)^{-1} \in \mathcal{F}(|T||s|-1(\gamma)) \]

### 2.3.2. Solvability (SM) and Causal-Implementability (CI).

In order for being able to have control actions as well-defined random variables and define the expected cost function under a given policy in the non-sequential setup, the information structure should possess a property called solvability-measurability (SM) [39].

**Definition 2.5 (Solvability-Measurability (SM)).** [39] Property SM holds when, for each γ ∈ Γ and ω ∈ Ω, there exists one and only one u ∈ \( \prod_{i=1}^{N} \mathcal{U}^i \) satisfying the closed loop equations

\[ P_i(u) = \gamma^i(y^i), i = 1, 2, ..., N \]

⇒ u = \( \gamma^i(\eta_i(\omega_0, \omega_i, u^1, u^2, ..., u^N)) \)

Here, P_i is the projection (or restriction) of the product \( \prod_{k=1}^{N} \mathcal{U}^j \) to \( \mathcal{U}^i \). Furthermore, the Property SM requires that the solution map denoted by \( M^T : (\Omega, \mathcal{F}) \rightarrow (\prod_{k=1}^{N} \mathcal{U}^k, \mathcal{B}(\prod_{k=1}^{N} \mathcal{U}^k)) \) is measurable.

Property SM guarantees that the control actions \( u^i, i = 1, 2, ..., N \), are well-defined random variables, given an admissible control policy \( \gamma \equiv (\gamma^1, \gamma^2, ..., \gamma^N) \). Property SM also ensures that the expectation of the cost function is well-defined under a given policy.

A further property in non-sequential stochastic control, called causal-implementability (CI) defined by Andersland and Teneketzis in [4, Definition 2], is necessary and sufficient to ensure that the information available to any DM does not depend on the actions of DMs who act after it [4, Theorem 1]:

**Definition 2.6 (Causal-Implementability (CI)).** [4] An information structure possesses Property CI when there exists at least one map \( \psi : \Omega \times \prod_{i=1}^{N} \mathcal{U}^j \rightarrow S_N \) such that for all k = 1, 2, ..., N, and \( (\omega, u) \in \Omega \times \prod_{i=1}^{N} \mathcal{U}^j \),

\[ \mathcal{F}^k \cap [P_{T_{N-1}(z)}^{-1}(P_{T_{k-1}(z)}(\omega, u))] \subset \emptyset \]

Where \( P_z \) is defined as the projection of \( \Omega \times \prod_{i=1}^{N} \mathcal{U}^j \) onto \( \Omega \times (\prod_{i=1}^{k} \mathcal{U}^i) \), i.e. \( P_z(\omega, u) = (\omega, u^1, u^2, ..., u^k) \), \( P_0(\omega, u) = (\omega) \).

The condition states that for any distinguishable set \( A \in \mathcal{F}^k \), the intersection \( A \cap [P_{T_{N-1}(z)}^{-1}(P_{T_{k-1}(z)}(\omega, u))] \) is either equal to \( [P_{T_{N-1}(z)}^{-1}(P_{T_{k-1}(z)}(\omega, u))] \) or \( \emptyset \).

---

2 We note that this definition of SM is presented differently than in [39], in order to accommodate use of the measurements \( y^i, i = 1, 2, ..., N \). The closed loop equations in [39] are given by: \( P_i(u) = \gamma^i(\omega, u), i = 1, 2, ..., N \).
Once again an equivalent, and perhaps more appropriate, way to state this definition is in terms of sigma fields. In this alternative approach, we simply state that an information structure possesses Property CI if and only if there exists at least one map \( \psi : \Omega \times (\prod_{i=1}^{N} U^{i}) \rightarrow S_{N} \), such that the information field \( J^{k} \) of the \( k \)-th DM contains only events which are distinguishable by chance, \( \omega \), and the actions of previous DMs, \((u^{i})_{i<k}\). Mathematically:

\[
J^{k} \subset \mathcal{F}^{k}(k-1)
\]

(2.9)

Recall that \( \mathcal{F}^{k}(k-1) \) is the cylindrical extension of \( \mathcal{F} \times (\prod_{i=1}^{k-1} U^{i}) \) to \( \Omega \times \prod_{i=1}^{N} U^{i} \).

Following the definition of Property CI, the following property, called deadlock-freeness (DF) (also introduced by \[4\]), is a more easily interpretable and equivalent high-level description of Property CI:

**Definition 2.7. [Deadlock-Freeness (DF)]** \[4\]

An information structure possesses Property DF if for each \( \gamma \in \Gamma \), and for every \( \omega \in \Omega \), there exists an ordering of \( \gamma \)'s \( N \) control laws, say \( \gamma^{1}(\omega), \gamma^{2}(\omega), ..., \gamma^{N}(\omega) \), such that no control action \( u^{i}(\omega), i = 1, 2, ..., N, \) depends on the control actions that follow.

To gain further insight into the definitions above, consider the following two examples from \[38\].

**Example 2.1.** Let \( \Omega = \{ 0, 1 \} \) and

\[
\sigma(I^{1}) = \left\{ \emptyset, \Omega \times \{ 0 \} \times \{ 0 \} \times \{ 0 \}, \{ 0, 1 \} \times \{ 0 \} \times \{ 0 \} \times \{ 0 \} : \omega(1-u^{2})u^{3} = 1 \right\},
\]

\[
\{ (\omega, u^{1}, u^{2}, u^{3}) : \omega(1-u^{2})u^{3} = 0 \},
\]

\[
\sigma(I^{2}) = \left\{ \emptyset, \Omega \times \{ 0 \} \times \{ 0 \} \times \{ 0 \}, \{ 0, 1 \} \times \{ 0 \} \times \{ 0 \} \times \{ 0 \} : \omega(1-u^{3})u^{1} = 1 \right\},
\]

\[
\{ (\omega, u^{1}, u^{2}, u^{3}) : \omega(1-u^{3})u^{1} = 0 \},
\]

\[
\sigma(I^{3}) = \left\{ \emptyset, \Omega \times \{ 0 \} \times \{ 0 \} \times \{ 0 \}, \{ 0, 1 \} \times \{ 0 \} \times \{ 0 \} \times \{ 0 \} : \omega(1-u^{1})u^{2} = 1 \right\},
\]

\[
\{ (\omega, u^{1}, u^{2}, u^{3}) : \omega(1-u^{1})u^{2} = 0 \}.
\]

This system has a deadlock, since no DM can act.

**Example 2.2.** Let \( \Omega = \{ 0, 1 \} \times \{ 0, 1 \} = \emptyset \) and \( \sigma(I^{1}) = 2^{\Omega \times \{ 0, 1 \}}, \sigma(I^{2}) = 2^{\Omega \times \{ 0, 1 \}} \) (where the notation \( 2^{\Omega} \) denotes the power set, that is the collection of all subsets of the set \( \Omega \)).

Consider the following team policy:

\[
\gamma^{1}(\omega, u^{2}) = 0 \times 1_{\{ u^{2}=0 \}} + 1 \times 1_{\{ u^{2}=1 \}}
\]

\[
\gamma^{2}(\omega, u^{1}) = 0 \times 1_{\{ u^{1}=0 \}} + 1 \times 1_{\{ u^{1}=1 \}},
\]

where \( 1_{E} \) denotes the indicator function for event \( E \). For this design, consider the realization \( \omega = 0 \). In this case, \((\omega, u^{1}, u^{2}) = (0, 0, 0)\) as well as \((0, 1, 1)\) are acceptable realizations given

\[3\]In addition to the Properties SM, DF, C, and CI, \[5\] defines analogues of these properties which are policy-dependent in the sense that they apply only for a given policy or a subset of policies but not necessarily for all policies.
the policy stated above. A similar setting occurs for \( \omega = 1 \), since \((1, 0, 0)\) and \((1, 1, 1)\) are acceptable realizations. Hence, for a given cost function \( c \), there does not exist, in general, a well-defined (measurable) cost realization variable \( c(\omega, u^1, u^2) \) under this policy, and the expectation \( E[c(\omega, u^1, u^2)] \) is not well defined given the policy \((\gamma^1, \gamma^2)\). This system is not solvable.

### 2.4. Equivalence of Property C and Property CI (and Property DF)

We first present the following, due to Witsenhausen, and Andersland and Tenekezis:

**Theorem 2.8.** [39, Theorem 1] [4, Theorems 1 and 2] The following relationships hold:

\[
C \Rightarrow CI \iff DF \Rightarrow SM
\]

We note that both of the relations in \( C \Rightarrow CI \iff DF \Rightarrow SM \) follow essentially directly from the (alternative and equivalent) \( \sigma \)-field characterizations of these definitions. The relations presented above were first established in [4], although Witsenhausen proved in [39, Theorem 1] that \( C \Rightarrow SM \), which is proved indirectly by the fact that \( C \Rightarrow CI \Rightarrow SM \).

Property CI, in view of the definitions and discussions above, states that for any possible ordering the information available to each DM at the time they take their action cannot depend on the action of any DM that has not yet acted. Note that this definition does not impose any restrictions on the information available to select the ordering of DMs, unlike in Property C, which requires that the ordering function has nested information fields. The only difference between Property C and Property CI is indeed that under C, \( \psi \) is causal whereas this is not the case under CI. We note that Property SM does not imply Property C. That Property C and CI are near equivalent under countability conditions has been reported in [4, Theorem 4].

The following result shows that Causality and Properties CI and DF are essentially equivalent.

**Theorem 2.9.** For the case with standard Borel measurement and action spaces, Property CI implies Property C, in the sense that for every non-sequential system with Property CI one can construct an equivalent non-sequential system with Property C. Therefore, (together with Theorem 2.8) we have

\[
C \iff CI \iff DF \Rightarrow SM
\]

**Proof.** We have that Property C implies Property CI by Theorem 2.8. We prove the other direction. We will first obtain a policy independence property. Let us consider first \( k = 1 \).

\[
P^f(dy^k | \omega_0, \omega_{s_0}) = \sum_{s \in S_1} P^f(dy^k, s | \omega_0, \omega_{s_0}) = \sum_{s \in S_1} P^f(dy^k | s, \omega_0, \omega_{s_0}) P^f(s | \omega_0, \omega_{s_0}) = \sum_{s \in S_1} P(dy^k | s, \omega_0, \omega_{s_0}) P(s | \omega_0, \omega_{s_0})(2.10)
\]

Here (2.10) builds on Property CI in view of (2.8). (2.11) builds on the property that the future policy \( \gamma^{k+1}, \ldots, N \) does not impact the action at time \( k = 1 \) by the CI property as explicitly stated.
in the equivalent Definition 2.7, and since \( P(dy^{k} | s_{1}, o_{0}, o_{s_{0}}) \) is independent of future policies, it must be that \( P(s_{1} | o_{0}, o_{s_{0}}) \) is essentially independent of future policies, in the following sense: If we define an equivalence class \( s \equiv s' \) (with \( B_{s} = \{ s' \in S_{k} : s' \equiv s \} \)) defined with

\[
P(dy^{1} | s_{1}, o_{0}, o_{s_{0}}) = P(dy'^{1} | s_{1}, o_{0}, o_{s_{0}}),
\]

then the expressions \( \hat{p}^{1..N}_{k} (B_{s}|o_{0}, o_{s_{0}}) \) and \( \hat{p}^{1..N}_{k} (B_{s}|o_{0}, o_{s_{0}}) \) must be equal for any future policies \( \hat{p}^{k..N}_{k} \) or \( \hat{p}^{k..N}_{k} \). In the absence of such a property, different future policies can be constructed to lead to different conditional probabilities on the equivalence classes \( B_{s} \), which will in turn affect the actions taken at time \( k \), and therefore Definition 2.7 will not apply. Thus, we can construct (by assigning positive measures to all realizations within an equivalence class) probability measures \( P(s_{1} | o_{0}, o_{s_{0}}) \) which is policy independent and which satisfies (2.11).

Now that \( P(s_{1} | o_{0}, o_{s_{0}}) \) is policy independent, we move to \( k = 2 \) and beyond, by constructing the probability measures inductively. For \( k = 2 \), once again,

\[
P^{c}(dy^{k} | o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1}) = \sum_{s \in S_{k}} P^{c}(dy^{k}, s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1})
\]

\[
= \sum_{s \in S_{k}} P^{c}(dy^{k}) \left( s, o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1} \right)
\]

\[
\times P^{c}(s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1})
\]

\[
= \sum_{s \in S_{k}} P^{c}(dy^{k}) \left( s, o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1} \right)
\]

\[
\times P^{c}(s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1})
\]

\[
= \sum_{s \in S_{k}} P^{c}(dy^{k}) \left( s, o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1} \right)
\]

\[
\times P^{c}(s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1})
\]

\[
= \sum_{s \in S_{k}} P^{c}(dy^{k}) \left( s, o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1} \right)
\]

\[
\times P^{c}(s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1})
\]

(2.12)

Once again (2.12) builds on Property CI. Given (2.12), as above, (2.13) builds on the property that the future policy \( \hat{p}^{k..N}_{k} \) does not impact the action at time \( k = 2 \) by the CI property as explicitly stated in the equivalent Definition 2.7 and since \( P(dy^{k} | s_{1}, o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1}) \) is independent of future policies, it must be that \( P^{c}(dy^{k-1} | s|o_{0}, o_{s_{0}}, y^{1}, u^{1}, \ldots, y^{k-1}, u^{k-1}) \) is essentially independent of future policies, in the following sense: If we define again an equivalence class \( s \equiv s' \) (with \( B_{s} = \{ s' \in S_{k} : s' \equiv s \} \) and compatible with the construction
With \( u^{i_1}, \ldots, u^{i_{k-1}} \) for measurable functions admit stochastic realizations (see Lemma 1.2 in [19], or Lemma 3.1 of [12]) so that we can express probability measures can construct (by assigning positive measures to all realizations within an equivalence class) will in turn affect the actions taken at time \( t_k \). Since, once \( y^{i_1} \) is independent of future policies. Since, once \( y^{i_1} \) and \( u^{i_1} \) is determined, we conclude that \( s_2 \) is also only dependent on the past (in a policy-independent fashion).

The reasoning applies also for \( k = 3, \ldots, N \). Now, the stochastic kernels (starting with \( k = 1 \),

\[
P \left( dy^k \mid s, \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}} \right) = P \left( dy^k \mid s', \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}} \right),
\]

then for any given \( y^{i_1}, \ldots, y^{i_{k-1}} \), the expressions \( P y^{i_1, \ldots, y^{i_{k-1}}, \tilde{y}^{k-1}, \tilde{y}^{k-1}N} (B_s \mid \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \)

\( P y^{i_1, \ldots, y^{i_{k-1}}, \tilde{y}^{k-1}, \tilde{y}^{k-1}N} (B_s \mid \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \) must be equal for any future policies \( \tilde{y}^{k-1}, N \) or \( \tilde{y}^{k-1}, N \). In the absence of such a property, different future policies can be constructed to lead to different conditional probabilities on the equivalence classes \( B_s \), which will in turn affect the actions taken at time \( k \), and therefore Definition 2.7 will not apply. We can construct (by assigning positive measures to all realizations within an equivalence class) probability measures \( P(y^{i_0}, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \).

Since

\[
P y^{i_1}(s_2, s_1 \mid \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1})
\]

is independent of future policies, we have that

\[
P y^{i_1}(s_2 \mid \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1})
\]

is independent of future policies. Since, once \( y^{i_1} \) and \( u^{i_1} \) is specified, the effect of \( y^{i_1} \) is determined, we conclude that \( s_2 \) is also only dependent on the past (in a policy-independent fashion).

The reasoning applies also for \( k = 3, \ldots, N \). Now, the stochastic kernels (starting with \( k = 1 \),

\[
P \left( dy^k \mid s, \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}} \right) = P \left( s \mid \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}} \right)
\]

admit stochastic realizations (see Lemma 1.2 in [19], or Lemma 3.1 of [12]) so that we can express

\[
y^k = \eta_k(s, \omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}, \omega_{s_k}^f)
\]

\[
s = \kappa_k(\omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}, \omega_{s_k}^o)
\]

for measurable functions \( \eta_k, \kappa_k \) and \([0, 1] \)-valued independent noise variables \( \omega_{s_0}^f, \omega_{s_0}^o \). Substituting \( s \) in the above, we arrive at

\[
y^k = \eta_k'(\omega_0, \omega_{s_0}, y^{i_1}, u^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}, \omega_{s_k}^o)
\]

With \( \kappa_k, \eta_k' \) as constructed, the system satisfies Property C and induces (given a fixed policy) the same probability measure on the measurement and action sequences and expected cost as the original model; and this is an equivalent model which satisfies Property C.

\[ \diamond \]

3. Static Reduction of Non-Sequential Decentralized Stochastic Control.
3.1. Policy-Independent Static Reduction under Causality: Case with Countable Measurement Spaces. We now consider the case in which a non-sequential stochastic control problem admits a policy-independent static reduction. It turns out that if a non-sequential decentralized stochastic control problem possesses Property C, then a static reduction is possible in which the conditional expectation of the equivalent static problem will be policy-independent. Here we use the imaginary sequential model, and so we refer to the measurements \( y^i \) and actions \( u^i \) for \( i = 1, \ldots, N \). This is stated formally in this section.

First, under Property C, we establish the following (policy-independence) lemma for the measurement kernels in the imaginary sequential model.

**Lemma 3.1.** Under any team policy \( \gamma \), under Property C

\[
P^T(d y^k | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}}) = P(d y^k | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

(3.1)

That is, the conditional probabilities defining measurement kernels are policy independent.

**Proof.** We have that, for all \( 1 \leq k \leq N \)

\[
P^T(d y^k | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}}) = \sum_{s \in S_k} P(d y^k, s | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

\[
= \sum_{s \in S_k} P \left( d y^k | s, \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}} \right) \times P(s | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

\( = \sum_{s \in S_k} P \left( d y^k | s, \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}} \right) \times P(s | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

(3.2)

\[
\times P(s | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

(3.3)

\( = P(d y^k | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})
\]

(3.4)

that is, the conditional probability is policy independent. Equations (3.2) and (3.3) critically follow from Property C, via (2.3); as the past control and measurements uniquely determine both the ordering as well as the conditional probability.

**Theorem 3.2.** Suppose that an N-DM non-sequential stochastic control problem possesses Property C for all admissible policies \( \gamma \in \Gamma \), and that the set \( \var{v} \) is countable for \( i = 1, 2, \ldots, N \), then the problem admits a policy-independent static reduction.

**Proof.**

**Step 1.** Let us first write:

\[
P^T(d \omega_0, d \omega_s_0, d y, d u) = P^T(d \omega_0, d \omega_s_0) P^T(d y, d u | \omega_0, \omega_s_0)
\]

where:

\[
P^T(d y, d u | \omega_0, \omega_s_0)
\]

\[
= P^T(d y^{s_1} | \omega_0, \omega_s_0) P^T(d u^{s_1} | y^{s_1})
\]

\[
\times P^T(d y^{s_2} | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}) P^T(d u^{s_2} | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, y^{s_2})
\]

\[
\cdots \times P^T(d y^{s_N} | \omega_0, \omega_s_0, y^{s_1}, u^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}})
\]

(3.5)
\[
\times P^T(du^N | y^N, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{N-1}}, u^{i_{N-1}}) \tag{3.5}
\]

**Step 2.** We have that, for all \(1 \leq k \leq N\)
\[
P^T(dy^k | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) = P(dy^k | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}})
\]
that is, the conditional probability is policy independent. This due to Lemma 3.1.

**Step 3.** We have that, for all \(1 \leq k \leq N\)
\[
P^T(du^k | y^k, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}})
\]
\[
= \sum_{s_k} P^T(du^k | s_k, y^k, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) P^T(s_k | y^k, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}})
\]
\[
= \sum_{s_k} P^T(du^k | s_k, y^k) P(s_k | y^k, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \tag{3.6}
\]
\[
= \sum_{s_k} P^T(dy^k | y^k) P(s_k | y^k, o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \tag{3.7}
\]
\[
= \sum_{s_k} P^T(dy^k | y^k) P(s_k | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{k-1}}, u^{i_{k-1}}) \tag{3.8}
\]

Here, (3.6) follows also from Property C; note that \(s_k\) is determined given the conditioned information. (3.7) follows from the structure given in (3.10) since given the policy and \(s_k\) the measurement determines the action: With Property C, the measurements \(y^{i_1}\) are related to the original measurements \(y^i\) by:
\[
I(y^i) = \sum_{k=1}^{N} y^k 1_{\{y^k(o_0, o_{s_0}, \ldots, o^{i_{k-1}}) = k\}} \tag{3.9}
\]
and the actions \(u^{i_1}\) are related to \(u^i\) by:
\[
I^u(u^{i_1}) = \sum_{k=1}^{N} u^k 1_{\{u^k(o_0, o_{s_0}, \ldots, o^{i_{k-1}}) = k\}} \tag{3.10}
\]

Here \(I\) defines an isomorphism between \((\mathcal{Y}^i, \mathcal{Y}^i)\) and \((\mathcal{Y}^{i_1}, \mathcal{Y}^{i_1})\), where \(s_k = i\), and likewise so does \(I^u\) between \((\mathcal{U}^i, \mathcal{U}^i)\) and \((\mathcal{U}^{i_1}, \mathcal{U}^{i_1})\).

Equation (3.8) also follows from Property C.

**Step 4.** Thus, we write (3.5) as
\[
P^T(dy, du | o_0, o_{s_0})
\]
\[
= P(dy^1 | o_0, o_{s_0}) P^{i_1}(du^1 | o_0, o_{s_0}, y^{i_1})
\]
\[
\times P(dy^2 | o_0, o_{s_0}, y^{i_1}, u^{i_1}) P^{i_2}(du^2 | o_0, o_{s_0}, y^{i_1}, u^{i_1}, y^{i_2})
\]
\[
\times \cdots \times P(dy^N | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{N-1}}, u^{i_{N-1}}) P^{i_N}(du^N | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{N-1}}, u^{i_{N-1}})
\]
\[
\times P^T(dy^N, du^N | o_0, o_{s_0}, y^{i_1}, \ldots, y^{i_{N-1}}, u^{i_{N-1}}) \tag{3.11}
\]
We then have that, in view of Step 3,

\[
P^T(dy, du|\omega_0, \omega_{n_0}) = P(dy^1|\omega_0, \omega_{s_0}) \left( \sum_{s_1} P^r_{s_1}(du^{s_1}|y^{s_1})P(s_1|\omega_0, \omega_{s_0}) \right) \times P(dy^2|\omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}) \left( \sum_{s_2} P^r_{s_2}(du^{s_2}|y^{s_2})P(s_2|\omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}) \right) \times \cdots \times P(dy^N|\omega_0, \omega_{s_0}, y^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}}) \left( \sum_{s_N} P^r_{s_N}(du^{s_N}|y^{s_N})P(s_N|\omega_0, \omega_{s_0}, y^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}}) \right) = c(\omega_0, \omega_{s_0}, s_{[1,N]}, u^{s_{1:s_N}})
\]

(3.12)

The stochastic kernels \(P(dy^i|\omega_0, \omega_{s_0}, y^{s_1}, \ldots, y^{s_{i-1}}, u^{s_1}, \ldots, u^{s_{i-1}})\) are policy-independent, as we are conditioning on the actions of all previous DMs as well as the information used to select those actions, thus the policy that maps the past DMs’ measurements to their actions is irrelevant.

Now, when the measurement space is countable, the following absolute continuity condition always holds [42], for some \(Q_{s_i}\):

\[
P(y^{s_i} \in A|\omega_0, y^{s_1}, \ldots, y^{s_{i-1}}, u^{s_1}, \ldots, u^{s_{i-1}}) = \int_A f_{s_i}(y^{s_i}, \omega_0, y^{s_1}, \ldots, y^{s_{i-1}}, u^{s_1}, \ldots, u^{s_{i-1}})Q_{s_i}(dy^{s_i})
\]

(3.13)

and thus we can write the total expected cost function as:

\[
J(y) = \int P(d\omega_0, d\omega_{s_0}) \times f_{s_1}(y^{s_1}, \omega_{s_0}, \omega_0)Q_{s_1}(dy^{s_1}) \left( \sum_{s_1} P^r_{s_1}(du^{s_1}|y^{s_1})P(s_1|\omega_0, \omega_{s_0}) \right) \times f_{s_2}(y^{s_2}, \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1})Q_{s_2}(dy^{s_2}) \left( \sum_{s_2} P^r_{s_2}(du^{s_2}|y^{s_2})P(s_2|\omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}) \right) \times \cdots \times \sum_{s_N} P^r_{s_N}(du^{s_N}|y^{s_N})P(s_N|\omega_0, \omega_{s_0}, y^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}})Q_{s_N}(dy^{s_N})
\]

(3.15)

where

\[
c(\omega_0, \omega_{s_0}, s_{[1,N]}, u^{s_{1:s_N}}) = c(\omega_0, \omega_{s_0}, u_1, \ldots, u_N) = c(\omega_0, \sigma^{-1}_s(u^{s_{1:s_N}}))
\]

with the permutation (given the ordering \(s \in S_N\)): \(\sigma_s: [u^1, \ldots, u^N] \mapsto [u^{s_1}, \ldots, u^{s_N}]\) determining the agent ordering.

---

4As Witsenhausen notes while studying [42 Eqn (4.2)], when the measurement variables take values from countable set, a reference measure can be constructed (e.g., \(Q_s(z) = \sum_{i=1}^{2^{-N}} 1_{z=m_i} \)) where \(V^i = \{m_i, i \in N\}\) so that the absolute continuity condition always holds.
We also note that under Causality, the conditional probabilities \( P(s_k | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}}) \) are in fact indicator functions. Therefore, we can take out the summation over \( s_k, k = 1, \ldots, N \) and simplify the expression further. Thus, we have an equivalent static problem with independent measurements (generated under measures \( Q_{s_1}(d y^{s_1}), \ldots, Q_{s_N}(d y^{s_N}) \)), and the formulation is policy-independent. In particular, we have

\[
J(\gamma) = \int P(d \omega_0, d \omega_{s_0}) \left( Q_{s_1}(d y^{s_1}) P^{s_1}(du^{s_1} | y^{s_1}) Q_{s_2}(d y^{s_2}) P^{s_2}(du^{s_2} | y^{s_2}) \cdots Q_{s_N}(d y^{s_N}) P^{s_N}(du^{s_N} | y^{s_N}) \right)
\]

\[
\times f_{s_1}(y^{s_1}, \omega_{s_0}, \omega_0) \left( \sum_{s_1} P(s_1 | \omega_0, \omega_{s_0}) \right)
\]

\[
\times f_{s_2}(y^{s_2}, \omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}) \left( \sum_{s_2} P(s_2 | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}) \right)
\]

\[
\times f_{s_N}(y^{s_N}, \omega_{s_0}, \omega_0, y^{s_1}, \ldots, y^{s_{N-1}}, u^{s_1}, \ldots, u^{s_{N-1}}) \left( \sum_{s_N} P(s_N | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}}) \right)
\]

\[
\times \tilde{c}(\omega_0, \omega_{s_0}, s_{[1,N]}, u^{[s_1:s_N]})
\]

or

\[
J(\gamma) = \int P(d \omega_0, d \omega_{s_0}) \left( Q_{s_1}(d y^{s_1}) P^{s_1}(du^{s_1} | y^{s_1}) Q_{s_2}(d y^{s_2}) P^{s_2}(du^{s_2} | y^{s_2}) \cdots Q_{s_N}(d y^{s_N}) P^{s_N}(du^{s_N} | y^{s_N}) \right)
\]

\[
\times c(\omega_{s_0}, \omega_0, y^{[s_1:s_N]}, u^{[s_1:s_N]})
\]

with the static-reduced cost \( c \) being:

\[
c(\omega_{s_0}, \omega_0, y^{[s_1:s_N]}, u^{[s_1:s_N]})
\]

\[
= f_{s_1}(y^{s_1}, \omega_{s_0}, \omega_0) \left( \sum_{s_1} P(s_1 | \omega_0, \omega_{s_0}) \right)
\]

\[
\times f_{s_2}(y^{s_2}, \omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}) \left( \sum_{s_2} P(s_2 | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}) \right)
\]

\[
\times f_{s_N}(y^{s_N}, \omega_{s_0}, \omega_0, y^{s_1}, \ldots, y^{s_{N-1}}, u^{s_1}, \ldots, u^{s_{N-1}}) \left( \sum_{s_N} P(s_N | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{N-1}}, u^{s_{N-1}}) \right)
\]

\[
\times \tilde{c}(\omega_0, \omega_{s_0}, s_{[1,N]}, u^{[s_1:s_N]})
\]

Step 5.

We may note again that the conditional probabilities

\[
P(s_k = i | \omega_0, \omega_{s_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}}) = I_{\psi(\omega_0, \omega_{s_0}, y^{s_1}, \omega_{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, \omega_{s_{k-1}}, u^{s_{k-1}})}(i)
\]

are indicator functions.

Accordingly, we can further simplify (3.16) with
\[ J(\gamma) = \int P(\omega_0, \omega_N) \sum_{s \in S_N} \left( Q_{s_1}(d\psi^1) P_{s_2}(dy^{s_2}) P_{s_2}(du^{s_2}) \cdots Q_{s_N}(dy^{s_N}) P_{s_N}(du^{s_N}) \right) \]
\[ \times f_{s_1}(y^{s_1}, \omega_0, \omega_0) \left( \sum_{s_1} 1_{\{\psi(\omega_0, \omega_0) = s_1\}} \right) \]
\[ \times f_{s_2}(y^{s_2}, \omega_0, \omega_0, y^{s_1}, u^{s_1}) \left( \sum_{s_2} 1_{\{\psi(\omega_0, \omega_0, y^{s_1}, u^{s_1}) = s_2\}} \right) \]
\[ \times f_{s_N}(y^{s_N}, \omega_N, y^{s_1}, \cdots, y^{s_{N-1}}, u^{s_1}, \cdots, u^{s_{N-1}}) \]
\[ \left( 1_{\{\psi(\omega_0, \omega_0, y^{s_1}, \cdots, y^{s_{N-1}}, u^{s_1}, \cdots, u^{s_{N-1}}) = s_N\}} \right) \]
\[ \times \tilde{c}(\omega_0, \omega_N, s_{[1,N]}, u^{[s_1,s_N]}), \] (3.19)

In (3.19) by applying another permutation map, we can write the simpler expression (as far as the original measurement spaces are involved)

\[ J(\gamma) = \int P(\omega_0, \omega_N) \sum_{s \in S_N} 1_{\{\sigma_s([1,\cdots,N]) = [s_1, \cdots, s_N]\}} \left( \tilde{Q}_1(dy^1) \tilde{Q}_2(dy^2) \tilde{P}_{s_2}(du^2) \cdots \tilde{Q}_N(dy^N) \tilde{P}_{s_N}(du^N) \right) \]
\[ \times \left( 1_{\{\psi(\omega_0, \omega_N) = s_{[1,N]}\}} 1_{\{\psi(\omega_0, \omega_0, y^{s_1}, u^{s_1}) = s_2\}} \cdots 1_{\{\psi(\omega_0, \omega_0, y^{s_1}, \cdots, y^{s_{N-1}}, u^{s_1}, \cdots, u^{s_{N-1}}) = s_N\}} \right) \]
\[ \times \tilde{c}(\omega_0, \omega_N, s_{[1,N]}, u^{[s_1,s_N]}), \] (3.20)

where for any \( m = 1, \cdots, N, \tilde{Q}_m(dy^m) = Q_{s_N, y^m}(dy^m) \) is the restriction of \( Q_{s_N} \) on \( y^m \).

**Remark 3.1.** Note that when the cost function \( c \) is permutation invariant, with
\[ c(\omega_0, \omega_N, u_1, \cdots, u_N) = c(\omega_0, \sigma^{-1}_2(u^{[s_1,s_N]})) \]
the reduction simplifies, as we will see in the example in Section 5.

### 3.2. Policy-Independent Static Reduction under Causality: Case with Standard Borel Spaces
The results in Theorem 3.2 are stated for the case in which the set \( \Psi^i \) is countable, for which the continuity condition (3.13) always holds. The results can be generalized to the standard Borel setup. Suppose that the joint probability measure on \((\omega_0, \omega_N, y^{s_1}, \cdots, y^{s_N}, u^{s_1}, \cdots, u^{s_N})\) is denoted by \( P \), and the joint probability measure on \( \omega_0 \) and \( \omega_0 \) is denoted by \( \mathbb{P}^{0} \). Then if for any Borel set \( A \), the absolute continuity condition (3.13) holds, then under any admissible policy \( \gamma = (\psi^1, \psi^2, \cdots, \psi^N) \), there exists a joint reference probability measure \( Q \) on \((\omega_0, \omega_N, y^{s_1}, \cdots, y^{s_N}, u^{s_1}, \cdots, u^{s_N})\) such that the probability measure \( P \) is absolutely continuous with respect to \( Q \). Thus for every Borel set \( A \) in \((\Omega_0, \Omega_N, \prod_{i=1}^{N}(\Psi^i \times \Psi^i))\), we have:

\[ P(A) = \int_A \frac{dP}{dQ}(\omega_0, \omega_N, dy^{s_1}, \cdots, dy^{s_N}, du^{s_1}, \cdots, du^{s_N}), \]

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where:
\[
Q(d\omega_0, d\omega_{s_0}, d\gamma^1, \ldots, d\gamma^N, d\upsilon^{i_1}, \ldots, d\upsilon^{i_N}) = \mathbb{P}^0(d\omega_0, d\omega_{s_0}) \prod_{i=1}^{N} Q_{s_i}(d\gamma^i) 1_{\{\gamma^i(\gamma^j) \in du^j\}},
\]
leads to a Radon-Nikodym derivative, which is policy-independent:
\[
\frac{dP}{dQ}(\omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^N, \upsilon^{i_1}, \ldots, \upsilon^{i_N}) = \prod_{i=1}^{N} f_{s_i}(\gamma^i, \omega_0, \omega_{s_0}, \gamma^i, \ldots, \gamma^{i-1}, \upsilon^{i_1}, \ldots, \upsilon^{i-1}).
\]

The new cost function would then be defined as:
\[
c(\omega_{s_0}, \omega_0), y^{[s_1-s_N]}, u^{[s_1-s_N]})) = f_{s_1}(\gamma^1, \omega_0, \omega_{s_0}) \left( \sum_{s_1} P(s_1 | \omega_0, \omega_{s_0}) \right)
\times f_{s_2}(\gamma^2, \omega_0, \omega_{s_0}, \gamma^1, \upsilon^{i_1}) \left( \sum_{s_2} P(s_2 | \omega_0, \omega_{s_0}, \gamma^1, \upsilon^{i_1}) \right)
\times f_{s_N}(\gamma^N, \omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^{N-1}, \upsilon^{i_1}, \ldots, \upsilon^{i_{N-1}})
\left( \sum_{s_N} P(s_N | \omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^{N-1}, \upsilon^{i_1}, \ldots, \upsilon^{i_{N-1}}) \right)
\times c(\omega_0, \omega_{s_0}, s_1[N], u^{[s_1-s_N]}),
\]

**Theorem 3.3.** Suppose that an N-DM non-sequential stochastic control problem possesses Property C (for all admissible policies $\gamma \in \Gamma$), and that there exists a reference probability measure $Q$ on $(\omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^N, \upsilon^{i_1}, \ldots, \upsilon^{i_N})$ such that the joint probability measure $P$ on $(\omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^N, \upsilon^{i_1}, \ldots, \upsilon^{i_N})$ is absolutely continuous with respect to $Q$ ($P \ll Q$) where $Q$ is given in (3.21). Then the dynamic non-sequential problem admits a policy-independent static reduction.

**Proof.** The proof is the same as for Theorem 3.2 except the relation (3.13) now holds almost-surely for sets $A$ in the standard Borel case under the assumption that $P \ll Q$.

We note that a sufficient condition for the absolute continuity condition under Condition C is that, for every realization $\omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^{i-1}, \upsilon^{i_1}, \ldots, \upsilon^{i_{i-1}},$
\[
P(\gamma^i_1 \in A | \omega_0, \omega_{s_0}, \gamma^1, \ldots, \gamma^{i-1}, \upsilon^{i_1}, \ldots, \upsilon^{i_{i-1}}) \ll \sum_{i=1}^{N} 1_{[s_1=m]} \psi^i(\cdot),
\]
where $\psi^i \in \mathcal{P}(\mathbb{V}^i)$ is a reference probability measure. In this case, we can take
\[
Q_{s_i}(d\gamma^i) = \sum_{i=1}^{N} \psi^i(d\gamma^i \cap \mathbb{V}^i) \in \mathcal{P}(\cup_{i=1}^{N} \mathbb{V}^i).
\]
In particular, for every $m = 1, \ldots, N$, $Q_{s_m}(d\gamma^m)$ can be the same reference measure. As in Step 5 in the proof of Theorem 3.2 the reduction can be further simplified.
3.3. Reduction of Partially-Nested Non-Sequential Teams Satisfying Property C.

We now consider the case in which a non-sequential problem admits a policy-dependent static reduction of the form discussed in [36, 37]. First, we recall that policy-dependent static reduction, as categorized by [36, 37], has its limitations due to the lack of preservation of person-by-person optimality and stationarity (i.e., policies that satisfy variational optimality inequalities). Nonetheless, for global optimality, isomorphism holds in that a policy which is globally optimal in the dynamic setup is still so under a static reduction. The policy-dependent static reduction in this case is different from that presented earlier, as it does not involve a change of measure, and the cost function in the static problem remains unaltered.

This policy-dependent static reduction applies to stochastic team problems with classical and quasi-classical information structures, which also possess Property C. Consider a non-sequential problem with quasi-classical information structure, where the observation of the \( k \)th DM to act, is given by:

\[
y_{s_k}^D = \{y_{1:s_k}^D, \hat{y}_{s_k}^D : = h_{s_k}(g_{s_k}(\omega), u_{1:s_k}^D)\}
\]  

(3.23)

Here \( y_{s_k}^D \) is the observation used to select the action \( u_{s_k}^D \) of the \( k \)th DM, where we have used the superscript \( D \), since these observations are still dynamic. \( y_{1:s_k}^D \) represents the information available to each DM that acts prior to \( DM^k \), whose action affects the measurement of \( DM^k \). \( \omega := (\omega_0, \omega_h, \omega_1, ..., \omega_N) \) represents the uncertainty in the system, and \( h_{s_k}, g_{s_k} \) are measurable functions. The importance of Property C here is that \( y_{s_k}^D, y_{s_{k+1}}^D, ..., y_{s_k}^D \) are fixed, given \( \omega, u_{s_1}^D, u_{s_2}^D, ..., u_{s_{k-1}}^D \). We present the following theorem:

**Theorem 3.4.** Consider a non-sequential stochastic team problem with a quasi-classical information structure, where the measurements can be written in the form (3.23). Suppose also that the information structure possesses Property C, then if we assume that the function \( h_{s_k}(\cdot, u_{1:s_k}^D) \) is invertible for fixed \( u_{1:s_k}^D \), then the problem admits a policy-dependent static reduction in which the cost function remains unaltered.

**Proof.**

First, assuming that \( h_{s_k}(\cdot, u_{1:s_k}^D) \) is invertible, let us define the static measurements as:

\[
y_{s_k}^S = \{y_{1:s_k}^S, \hat{y}_{s_k}^S := g_{s_k}(\omega)\}
\]

Here \( \hat{y}_{s_k}^S := g_{s_k}(\omega) = h^{-1}_{s_k}(\hat{y}_{s_k}^D y_{1:s_k}^D(y_{s_k}^D)) \), and:

\[
u_{s_k}^S = y_{s_k}^S(y_{s_k}^S)
\]

(3.24)

The assumption that the function \( h_{s_k}(\cdot, u_{1:s_k}^D) \) is invertible for fixed \( u_{1:s_k}^D \), for all \( k = 1, 2, ..., N \) guarantees the existence of a bijection between \( y_{s_k}^S \) and \( y_{s_k}^D \). Furthermore, we have that for any admissible policy \( \gamma_{s_k}^D \), an admissible policy \( \gamma_{s_k}^S \) can be constructed such that

\[
u_{s_k}^S = y_{s_k}^S(y_{s_k}^D) = y_{s_k}^S(\gamma_{s_k}^S), P - a.s.
\]

(3.25)

The construction is as follows. Suppose that \( y_{s_k}^D \) and \( y_{s_k}^S \) are related by the bijection \( F_{s_k}^D : y_{s_k}^D \to y_{s_k}^S \), where \( F_{s_k}^S \) is policy-dependent. Then, for any \( \gamma^D \in \Gamma^D \), an admissible policy \( \gamma^S \in \Gamma^S \) can be constructed as \( \gamma_{s_k}^S = \gamma_{s_k}^D \circ F_{s_k}^S \), and for any \( \gamma^S \in \Gamma^S \), an admissible \( \gamma^D \in \Gamma^D \).
can be constructed as $y^D_{\kappa_k} = y^S_{\kappa_k} \circ F^{\kappa_k} \downarrow_{\kappa_k}$.

**Remark 3.2.** The policies used in this static reduction are required to be deterministic, as the DMs must be able to compute the actions of previous DMs based upon their observations. As noted in [36], this does not pose any issues for optimality in the team setup, as globally optimal policies can always be chosen among those that are deterministic [47] Theorems 2.3 and 2.5.

The policy-dependent static reduction for partially-nested non-sequential teams is almost the same as for sequential teams, with the key difference being that Property C must hold so that the invertibility condition may hold. It would be just as correct to state the theorem only with the condition that the function $h_{i_k}(\cdot, u^D_{\kappa_k})$ is invertible for fixed $u_{i_k}$, as it would then just be implied that Property C holds, although it seems appropriate to include Property C due to its relevance in the rest of the paper.

4. Beyond Causality: Two Further Policy-Dependent Equivalent Formulations under Property SM where Policy Appears as a Cost Parameter.

4.1. A Static Reduction with Policy as a Parameter in the Cost Function. We now consider the case in which a non-sequential decentralized stochastic control problem admits a static reduction that is policy-dependent under more relaxed conditions. Compared with Theorem 4.2, countability is not imposed. We present the following theorem:

**Theorem 4.1.** Suppose that an $N$-DM non-sequential stochastic control problem possesses Property SM for all admissible policies $\gamma \in \Gamma$, and that there exists a reference probability measure $Q^\gamma$ on $(\omega_0, \omega_{s_0}, y^1, \ldots, y^N, u^1, \ldots, u^N)$, for each $\gamma \in \Gamma$, such that the joint probability measure $P$ on $(\omega_0, \omega_{s_0}, y^1, \ldots, y^N, u^1, \ldots, u^N)$ is absolutely continuous with respect to $Q^\gamma$ ($P \ll Q^\gamma$), then the problem admits a policy-dependent static reduction.

**Proof.** Let us first write

$$P(d\omega_0, d\omega_{s_0}, dy, du) = P(d\omega_0, d\omega_{s_0})P(dy, du|\omega_0, \omega_{s_0})$$

Now:

$$P(dy, du|\omega_0, \omega_{s_0}) = P(dy^1|\omega_0, \omega_{s_0})P(du^1|y^1)P(dy^2|\omega_0, \omega_{s_0}, y^1, u^1)P(du^2|y^2)\ldots$$

where the kernels $P(dy^i|\omega_0, \omega_{s_0}, y^i, \ldots, y^{i-1}, u^1, \ldots, u^{i-1})$ are well-defined by the stochastic mapping:

$$y^i = \eta_i(\omega_0, \omega_{s_0}, y^1, u^1, \ldots, u^N)$$

Here $\omega_i, i = 1, 2, \ldots, N$ are independent [0,1] random noise variables, and the control actions $u^k, k = 1, 2, \ldots, N$, are well-defined random variables given a control policy $\gamma \equiv (y^1, y^2, \ldots, y^N)$ and $\omega \in \Omega$, as guaranteed by the assumption that Property SM holds. Note that in this reduction we use the normal indices $y^i, u^i, i = 1, 2, \ldots, N$, since there is no ordering function that is guaranteed under SM, and so we formulate the reduction under some arbitrary ordering. The probability measure on the actions $u^1, u^2, \ldots, u^N$ depends, however, on the policy used, which is why this will be a policy-dependent static reduction. Now, let’s write:

$$P(d\omega_0, d\omega_{s_0}, dy, du) = P(d\omega_0, d\omega_{s_0}) \prod_{i=1}^N \int_{y^i \in du^i} P(dy^i|\omega_0, \omega_{s_0}, y^1, \ldots, y^{i-1}, u^1, \ldots, u^{i-1}),$$

...
under the assumption that the policy $\gamma$ is deterministic. Now, given the assumption that there exists $Q'$ such that $P \ll Q'$, the following absolute continuity condition holds for each $\gamma \in \Gamma$:

$$P(y^i \in A|\omega_0, \omega_{s_0}, y^1, ..., y^{i-1}, u^1, ..., u^{i-1}) = \int_A f^T_i(y^i, \omega_0, \omega_{s_0}, y^1, ..., y^{i-1}, u^1, ..., u^{i-1})Q'_i(dy^i),$$

Where we have written $f^T_i, Q'_i$ to emphasize the policy-dependence of these terms.

In the standard Borel setup, the change of measure will involve a Radon-Nikodym derivative. The expected total cost can then be written as:

$$J(\gamma) = \int P(d\omega_0, d\omega_{s_0}) \prod_{i=1}^N (f^T_i(y^i, \omega_0, \omega_{s_0}, y^1, ..., y^{i-1}, u^1, ..., u^{i-1})Q'_i(dy^i)c(\omega_0, \omega_{s_0}, u))$$

So now if we define the new (policy-dependent) cost function, $c^T_\gamma$, as:

$$c^T_\gamma(\omega_0, \omega_{s_0}, y, u) = \prod_{i=1}^N (f^T_i(y^i, \omega_0, \omega_{s_0}, y^1, ..., y^{i-1}, u^1, ..., u^{i-1}))c(\omega_0, \omega_{s_0}, u),$$

then we have an equivalent static problem with independent measurements. Now, let us confirm that the new static problem is equivalent to the original problem. First, the policy correspondence is simply $\tilde{\gamma} = \gamma$. Now, the isomorphism condition is also trivially satisfied:

1. $N = \tilde{N}$
2. $(\mathcal{Y}, \mathcal{Y}) = (\hat{\mathcal{Y}}, \hat{\mathcal{Y}})$.
3. $(\mathcal{U}^\gamma, \mathcal{U}^\gamma) = (\hat{\mathcal{U}}, \hat{\mathcal{U}})$.

Now, in order to confirm that the value condition is satisfied, let’s write:

$$J(\tilde{\gamma}) = E_\tilde{\gamma}[c^T_\gamma(\omega_0, \omega_{s_0}, y, u)]$$

$$= \int P(d\omega_0, d\omega_{s_0}) \prod_{i=1}^N f^T_i(y^i, \omega_0, \omega_{s_0}, y^1, ..., y^{i-1}, u^1, ..., u^{i-1})Q'_i(dy^i)c(\omega_0, \omega_{s_0}, u)$$

and by the absolute continuity condition, we can write:

$$J(\tilde{\gamma}) = \int P(d\omega_0, d\omega_{s_0}, dy^1, ..., dy^N, du^1, ..., du^N)c(\omega_0, \omega_{s_0}, u) = E_\gamma[c(\omega_0, \omega_{s_0}, u)] = J(\gamma),$$

which satisfies the value condition.

This policy-dependent static reduction has many differences with both the policy-independent static reduction, and the policy-dependent static reduction presented previously under Property C. First of all, the cost function in this reduction is altered unlike the policy-dependent reduction presented previously, and furthermore, the cost function takes the policy as a parameter. Also, the policy-dependence in this case is far less practical than the one presented previously, as the policies are used in order to generate probabilities on the actions of future DMs, which can be an exhaustive search, especially when the space is uncountable.

The above analysis leads to a policy independent static reduction under SM for countable spaces, as we make more explicit in the following.

**Theorem 4.2.** Under any team policy $\gamma$, under either Property SM (and therefore also under Property CI), which implies SM, when the measurement and action spaces are countable a static reduction exists, where policy appears in the reduced cost as a parameter.
Proof. Case 1: CI
We have that, for all $1 \leq k \leq N$

$$P'(dy^k | \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$= \sum_{s \in S_k} P'(dy^k, s | \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$= \sum_{s \in S_k} P'(dy^k | s, \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times P'(s | \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}}))$$

$$= \sum_{s \in S_k} P(dy^k | s, \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times P(s, y' | \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$= \sum_{s \in S_k} P(dy^k | s, \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times P(s | y', \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times P(y' | \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$= \sum_{s \in S_k} P(dy^k | s, \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times P(s | y', \omega_0, \omega_{y_0}, y^{s_1}, u^{s_1}, \ldots, y^{s_{k-1}}, u^{s_{k-1}})$$

$$\times 1_{|y'|=y}$$

(4.1)

(4.2)

Case 2: SM In the SM case, (4.1) will also be policy dependent.

In either case, accordingly, absolute continuity with respect to a reference measure holds (since the measurement space is countable), and a static reduction is possible in view of (3.13) and the rest of the proof of Theorem 3.2. Equation (4.1) builds on the CI-property.

4.2. Andersland’s Method for Sequentializing Non-Sequential Problems. One further method for decoupling non-sequential problems was proposed by Andersland in [3], where again policies appear as cost parameters. The method and results are summarized briefly in this section.

First let us relax what it means for a stochastic control problem to be sequential, under a more relaxed interpretation as defined in [3].

Definition 4.3. [3] A Problem is sequential when its information structure satisfies Property CI for some constant order function $\psi$.

[3] converts a non-sequential problem into a sequential one by the following steps [3]:

1. Introduce an additional $DM$, with action $\hat{\alpha} := (\hat{\alpha}^1, \ldots, \hat{\alpha}^N)$, that effectively simulates actions $\alpha^1, \ldots, \alpha^N$ via a $T/\hat{T}$-measurable function of $\omega$, say $\theta$ (ie. $\hat{\alpha}_\omega^k = \theta(\omega)$).

2. Select the action $\hat{\alpha}^k, k = 1, 2, \ldots, N$ as a function $\gamma^k \in \Gamma^k$ of $\omega$ and the simulation $\hat{\alpha}_\omega^k$ rather than $\omega$ and the actual actions $\alpha_\omega^k$ (ie. $\hat{\theta}_{\omega}^k = \gamma(\omega, \hat{\alpha}_\omega^k)$).
Now the problem is sequential as the actions of each $DM$ depends only on the exogenous random variable $\omega$, and the simulation $\hat{u}_\omega^\alpha$. The system’s cost function, or in Andersland’s treatment in [3], the payoff function now depends on the simulation. Specifically, when the simulation is not correct (ie. $\hat{u}_\omega^\alpha \neq \gamma(\omega, \hat{u}_\omega^\alpha)$), the decoupled designs expected cost does not have to equal the actual expected cost. Andersland tackles this by setting the payoff to zero (or the cost infinitely high) when the simulation is not correct. Andersland gave his formal definition of the decoupled problem as follows [3]:

1. Let the new information structure be given by $(\Omega, \mathcal{B}, (\bar{\mathcal{U}}^k, \mathcal{U}^k), \mathcal{J}^k : 1 \leq k \leq N + 1)$, where

   $$(\bar{\mathcal{U}}^k, \mathcal{U}^k) = \begin{cases} 
   (U, \mathcal{U}), & k = 1 \\
   (U^{k-1}, \mathcal{U}^{k-1}), & k = 2, 3, ..., N + 1
   \end{cases}$$

   and

   $$\mathcal{J}^k = \begin{cases} 
   \mathcal{B} \otimes \{\emptyset, U\} \otimes \{\emptyset, U\}, & k = 1 \\
   \mathcal{J}^{k-1} \otimes \{\emptyset, U\}, & k = 2, 3, ..., N + 1
   \end{cases}$$

2. Let $\hat{\Gamma} := \prod_{i=1}^{N+1} \hat{\Gamma}^i$, where $\hat{\Gamma}^k, k = 1, 2, ..., N + 1$, denotes the set of all $\hat{\mathcal{J}}^k / \hat{\mathcal{U}}^k$-measurable functions.

Given this decoupled information structure and decoupled design constraint set $\hat{\Gamma}$, paired with a real, upper bounded $\mathcal{B} \otimes \mathcal{U}$-measurable payoff function $V$, the problem is as follows: Identify a design $(\theta, \gamma) \in \hat{\Gamma}$ that achieves

$$\sup_{(\theta, \gamma) \in \hat{\Gamma}} E_{\omega}[V(\omega, u^\theta, \gamma) \mathbb{1}_{\{\hat{u} = u\}}(\hat{u}_\omega^\alpha, u^\theta_\omega, \gamma)]$$

exactly, or within $\epsilon > 0$. This sequential problem is denoted by (SE-Q), where the original problem is denoted by (Q). Andersland was able to prove the following two theorems based on this discussion:

**THEOREM 4.4.** [3, Theorem 1] Problem (SE-Q) is a sequential, $(N + 1)$-action problem of the form $(P)$ (given in detail in [3]) when, for all $k = 1, 2, ..., N$, $\mathcal{U}^k$ is countably generable.

**THEOREM 4.5.** [3, Theorem 2] Given that, for all $k = 1, 2, ..., N$, $\mathcal{U}^k$ is countably generable, and given that the information structure possesses Property CI, when the payoff of $(\theta, \gamma) \in \hat{\Gamma}$ is within $\epsilon > 0$ of optimal for problem (SE-Q), the payoff of $\gamma \in \Gamma$ is within $\epsilon$ of optimal for problem (Q).

It’s worth noting that in Andersland’s method for decoupling non-sequential problems the reduced cost is policy-dependent, due to the term $\mathbb{1}_{\{\hat{u} = u\}}(\hat{u}_\omega^\alpha, u^\theta_\omega, \gamma)$, as it is required that $\hat{u}_\omega^\alpha = \gamma(\omega, \hat{u}_\omega^\alpha)$.

**5. Example.** We present in this section an example to illustrate the non-sequential policy-independent static reduction process. Consider the non-sequential control problem defined as follows:

1. $N = 3$
2. $\Omega = \{0, 1\}$
3. $\mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \{0, 1\}$
4. 
\[ J^1 = \emptyset, \{(\omega, u) : \omega, u_0 + u_0 = 0\}, \{(\omega, u) : \omega, u_0 + u_0 = 1\}, \{(\omega, u) : \omega, u_0 u_1 u_2^2 = 1\}, \{(\omega, u) : \omega, u_0 u_1 u_2^2 = 0\}, \Omega \times U \} \]
\[ J^2 = \emptyset, \{(\omega, u) : \omega, u_0 = 1\}, \{(\omega, u) : \omega, u_0 = 0\}, \{(\omega, u) : \omega, u_0 + u_1 u_2 = 0\}, \{(\omega, u) : \omega, u_0 + u_1 u_2^3 \neq 0\}, \Omega \times U \} \]
\[ J^3 = \emptyset, \{(\omega, u) : \omega, u_1 = 1\}, \{(\omega, u) : \omega, u_1 = 0\}, \{(\omega, u) : u^1 + u^2 = 0\}, \{(\omega, u) : u^1 + u^2 \neq 0\}, \Omega \times U \} \]

5. \( \psi^1 = \psi^2 = \psi^3 = \{0, 1/2, 1\} \)

6. 
\[ y^1 = \eta_1(\omega_0, \omega_{s_1}, u^2, u_3^2) = \begin{cases} 1, & \omega_{s_0} + \omega_0 = 0 \\ 0, & \omega_{s_0} \omega_1 u_2^2 = 1 \\ 1/2, & \text{otherwise} \end{cases} \]

\[ y^2 = \eta_2(\omega_0, \omega_{s_0}, \omega_1, u^1, u_3^2) = \begin{cases} 1, & \omega_{s_0} \omega_0 = 1 \\ 0, & \omega_{s_0} + u_1 u_3^2 = 0 \\ 1/2, & \text{otherwise} \end{cases} \]

\[ y^3 = \eta_3(\omega_0, \omega_{s_0}, \omega_3, u^1, u_2^2) = \begin{cases} 1, & \omega_3 u_1 = 1 \\ 0, & u_1 + u_2 = 0 \\ 1/2, & \text{otherwise} \end{cases} \]

7. \( \psi := (\psi_1, \psi_2, \psi_3) \), such that:

\[ \psi_1(\omega_{s_0}) = \begin{cases} 1, & \omega_{s_0} = 0 \\ 2, & \omega_{s_0} = 1 \end{cases} \]

\[ \psi_2(\omega_{s_0}, u^1) = \begin{cases} 1, & \omega_{s_0} = 1 \\ 3, & \omega_{s_0} = 0, u^1 = 1 \\ 2, & \text{otherwise} \end{cases} \]

\[ \psi_3(\omega_{s_0}, u^1, u^2) = \begin{cases} 2, & \omega_{s_0} = 0, u^1 = 1 \\ 3, & \text{otherwise} \end{cases} \]

Clearly, the above problem possesses Property C, as the ordering function \( \psi \) is causal, and no matter which ordering is taken, the measurement variables depend only on the actions of DMs who have acted previously. Additionally, we can write the information field \( J^{s_1} \) of the first DM to act, as:

\[ J^{s_1} = \emptyset, \{(\omega, u) : \omega_0 = 1\}, \{(\omega, u) : \omega_0 = 0\}, \{(\omega, u) : \omega_{s_0} \omega_0 = 1\}, \{(\omega_{s_0} \omega_0 = 0\}, \Omega \times U \} \]

Similarly, \( J^{s_2} \) can be written as:
\[ J^{s_2} = \{ (\omega, \mathbf{u}) : \omega_{s_0} \omega_1 u^{s_1} = 1 \}, \{ (\omega, \mathbf{u}) : \omega_{s_0} \omega_1 u^{s_1} = 0 \}, \{ (\omega, \mathbf{u}) : \omega_{s_2} u^{s_2} = 1 \}, \{ (\omega, \mathbf{u}) : \omega_{s_2} u^{s_2} = 0 \}, \Omega \times U \]  

and finally:

\[ J^{s_3} = \{ (\omega, \mathbf{u}) : \omega_{s_0} + \omega_1 u^{s_1} = 0 \}, \{ (\omega, \mathbf{u}) : \omega_{s_0} + u^{s_1} u^{s_2} \neq 0 \}, \{ (\omega, \mathbf{u}) : u^{s_1} + u^{s_2} = 0 \}, \{ (\omega, \mathbf{u}) : u^{s_1} + u^{s_2} \neq 0 \}, \] \[ \{ (\omega, \mathbf{u}) : \omega_2 (u^{s_1} 1_{\omega_{s_0}=0} + u^{s_2} 1_{\omega_{s_0}=1}) = 1 \}, \{ (\omega, \mathbf{u}) : \omega_2 (u^{s_1} 1_{\omega_{s_0}=0} + u^{s_2} 1_{\omega_{s_0}=1}) = 0 \}, \Omega \times U \]  

Now, define the cost function as: \( c(\omega_0, \mathbf{u}) = \omega_0 + u^1 + u^2 + u^3 \). Then, let us write:

\[
P(y^{s_1} \in A|\omega_{s_0}, \omega_0) = \int_A f_{x_1}(y^{s_1}, \omega_{s_0}, \omega_0) Q_{x_1}(dy^{s_1})
\]

\[
= \sum_{y^{s_1} \in A} \frac{P(y^{s_1}, \omega_{s_0}, \omega_0)}{P(\omega_{s_0}, \omega_0)}
\]

\[
= \int_A \frac{P(y^{s_1}|\omega_{s_0}, \omega_0)P(\omega_{s_0}, \omega_0)}{P(\omega_{s_0}, \omega_0)} \mu^{s_1}(dy^{s_1})
\]

Where \( \mu^{s_1} \) is the uniform distribution on \( Y^{s_1} \), and \( P(y^{s_1} \in A|\omega_{s_0}, \omega_0) \) is a well defined stochastic kernel representing either the stochastic mapping \( g_1 \) or \( g_2 \). For example, if \( \omega_{s_0} = 1 \) and \( \omega_0 = 1 \), then \( P(y^{s_1} = 1|\omega_{s_0}, \omega_0) = P(y^2 = 1|\omega_{s_0}, \omega_0) = 1 \). That is, the mapping is deterministic since \( \omega_2 \) is irrelevant in this case. More interestingly, in the case that \( \omega_{s_0} = 1 \) and \( u^2 = 1 \), we have that:

\[
P(y^{s_2} = 0|\omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}) = P(y^1 = 0|\omega_{s_0}, \omega_0, y^2, u^2) = P(\omega_1 = 1)
\]

Now, we can construct the change in measure formula similarly for \( y^{s_2} \) and \( y^{s_3} \):

\[
P(y^{s_2} \in A|\omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}) = \int_A \frac{P(y^{s_2}, \omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}, u^{s_2})}{P(\omega_{s_0}, \omega_0, y^{s_1}, u^{s_1}) \mu^{s_2}(y^{s_2})} \mu^{s_2}(dy^{s_2})
\]

And:

\[
P(y^{s_3} \in A|\omega_{s_0}, \omega_0, y^{s_1}, y^{s_2}, u^{s_1}, u^{s_2}) = \int_A \frac{P(y^{s_3}, \omega_{s_0}, \omega_0, y^{s_1}, y^{s_2}, u^{s_1}, u^{s_2})}{P(\omega_{s_0}, \omega_0, y^{s_1}, y^{s_2}, u^{s_1}, u^{s_2}) \mu^{s_3}(y^{s_3})} \mu^{s_3}(dy^{s_3})
\]

So now the total cost can be written as:

\[
J(\gamma) = \int P(d\omega_0, d\omega_{s_0}) \frac{P(y^{s_1}, \omega_{s_0}, \omega_0)}{P(\omega_{s_0}, \omega_0)} \mu^{s_1}(dy^{s_1}) \frac{P(y^{s_2}, \omega_{s_0}, \omega_0, y^{s_1}, u^{s_2})}{P(\omega_{s_0}, \omega_0, y^{s_1}, u^{s_1})} \mu^{s_2}(dy^{s_2})
\]
Now, the new cost function, we can write:

\[
\sum P(y^3, \omega_{y^0}, \omega_0, y^1, y^2, u^1, u^2) \times \frac{P(y^3, \omega_{y^0}, \omega_0, y^1, y^2, u^1, u^2)}{P(\omega_{y^0}, \omega_0, y^1) \mu^1(y^1) \mu^2(y^2) \mu^3(y^3)(\omega_0 + u^1 + u^2 + u^3)}
\]

Where \(u' = u^1 \{\omega_1(\omega_0) = i\} + u^2 \{\omega_2(\omega_0, u^1) = i\} + u^3 \{\omega(\omega_0, u^1, u^2) = i\}\), and since each \(u'\) appears in the cost function with equal contribution (and the cost is permutation invariant), we can write:

\[
J^*(y^*) = \int P(\omega_0, y^0, y) \frac{P(y^1, \omega_{y_0}, \omega_0)}{P(\omega_{y_0}, \omega_0, y^1) \mu^1(y^1)} \frac{P(y^2, \omega_{y_0}, \omega_0, y^1, u^2)}{P(\omega_{y_0}, \omega_0, y^1) \mu^2(y^2)} \frac{P(y^3, \omega_{y_0}, \omega_0, y^1, y^2, u^1, u^2)}{P(\omega_{y_0}, \omega_0, y^1, y^2) \mu^3(y^3)} \mu^3(\omega_0 + u^1 + u^2 + u^3)
\]

Now, the new cost function, \(C_x\) is defined as:

\[
C_x(\omega_0, \omega_{y_0}, y, u) = \frac{P(y^1, \omega_{y_0}, \omega_0)}{P(\omega_{y_0}, \omega_0, y^1) \mu^1(y^1)} \frac{P(y^2, \omega_{y_0}, \omega_0, y^1, u^2)}{P(\omega_{y_0}, \omega_0, y^1) \mu^2(y^2)} \frac{P(y^3, \omega_{y_0}, \omega_0, y^1, y^2, u^1, u^2)}{P(\omega_{y_0}, \omega_0, y^1, y^2) \mu^3(y^3)} (\omega_0 + u^1 + u^2 + u^3)
\]

and the new joint measure on \((\omega_0, \omega_{y_0}, y)\) is \(P(\omega_0, \omega_{y_0}) \mu^1(y) \mu^2(y) \mu^3(y)\).

6. Conclusion. We revisited the notion of Causality, provided an alternative representation using imaginary agents, and showed that Causality is equivalent to Causal Implementability (and Dead-Lock Freeness) for standard Borel models. Via this representation, we showed that Causality, under an absolute continuity condition, allows for an equivalent static model whose reduction is policy-independent. We further showed that under more relaxed conditions on the model, such as solvability, such a reduction (when possible) is policy-dependent or includes policies as parameters in the cost of the reduced model, and thus has limited utility. We also presented a further reduction method for partially nested causal non-sequential systems.

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