Some Results on The Incomplete $q$-Gamma Function and Its First Derivative

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Abstract. In this paper, we use neutrix calculus in order to obtain some results on the $q$-analogue of the incomplete gamma function and its first derivative for all real numbers.

1. Introduction and Preliminaries
Quantum calculus, the $q$-analogue of classic calculus, has been used in many research areas such as physics, mathematics, statistic etc. recently [1, 2, 3]. The $q$-analogue $\mathcal{M}_q$ of mathematical object $\mathcal{M}$ has not been given any specific definition, but common admission is that as $q$ tends to 1, the limit of $\mathcal{M}_q$ approaches to $\mathcal{M}$. Therefore many mathematical objects have got more then one $q$-analogue such as exponential function. One of the $q$-analogue of the exponential function $e^x$ is denoted by $E^x_q$ and defined by

$$E^x_q = \sum_{n=0}^{\infty} \frac{q^{n+1} x^n}{[n]!} = (1 - (1 - q)x)_q^\infty$$

where $[x] = \frac{1 - q^x}{1 - q}$; $[n]! = [1][2] \ldots [n - 1][n]$ $n \in \mathbb{Z}^+$ with $[0]! = 1$ and $q$-analogue of $(x - a)^n$ is defined by

$$(x - a)_q^n = \begin{cases} 
1, & n = 0, \\
(x - a)(x - qa) \ldots (x - q^{n-1}a), & n \geq 1.
\end{cases}$$

Let us take a function $f$ on any interval $I$ of real numbers such that if $x \in I$ then $qx \in I$. The $q$-derivative of the function $f$ is defined by

$$D_qf(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{for } x \neq 0 \quad \text{and} \quad (D_qf)(0) = f'(0)$$

provided $f'(0)$ exists. Note that $D_qE^x_q = E^{qx}_q$.

The $q$-integral of a function $f$ from zero to $a$ is defined by

$$\int_0^a f(x)d_qx = a(1 - q) \sum_{n=0}^{\infty} q^n f(q^na)$$

provided the sum converges absolutely and one of the differences between quantum calculus and classic calculus is that the $q$-derivative of the product of two functions is not symmetric. Because
of that, the \( q \)-integration by parts can be given by two ways. First one can be found in [4] and the second one, which will be used in this paper, is given for suitable functions \( f \) and \( g \) as

\[
\int_a^b f(qx) dq g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x) dq f(x).
\]

(1)

The reader may find more information about quantum calculus in [4, 5] and references therein. In our study we also need the definitions of neutrix and neutrix limit given by Van der Corput in [6].

**Definition 1.** (Neutrix) Let \( N' \) be a nonempty set and let \( N \) be a commutative, additive group of functions mapping \( N' \) into a commutative, additive group \( N'' \). The group \( N \) is called neutrix if the function which is identically equal to zero is the only constant function occurring in \( N \). The function which belongs to \( N \) is called “negligible function” in \( N \).

Let \( N \) be a domain lying in a topological space with a limit point \( b \) not belonging to \( N \) and \( N \) be a commutative additive group of functions defined on \( N' \) with the following property:

\[
\forall f \in N, \lim_{x \to b} f(x) = c \text{ (constant)} \text{ for } x \in N' \text{ then } c = 0''.
\]

Then this group \( N \) is a neutrix.

**Definition 2.** (Neutrix Limit) Let \( f \) be a real valued function defined on \( N' \) and suppose that it is possible to find a constant \( c \) such that \( f(x) - c \) is negligible in \( N \). Then \( c \) is called the neutrix limit of \( f(x) \) as \( x \) tends to \( b \) and denoted by

\[
N' \lim_{x \to b} f(x) = c.
\]

In [7], authors defined the incomplete \( q \)-gamma function for \( \alpha > 0 \) by \( q \)-integral as

\[
\gamma_q(\alpha, x) = \int_0^{\alpha} t^{\alpha-1} E_q^{-qt} dt.
\]

(3)

In this paper, we let \( N \) be the neutrix having domain the open interval \( N' = \{ \varepsilon : 0 < \varepsilon < \infty \} \) and range \( N'' \) as the real numbers with negligible functions being finite linear sums of the functions

\[
\varepsilon^\lambda \ln^{r-1} \varepsilon, \ \ln^r \varepsilon, \ [\varepsilon]^\lambda \ (\lambda < 0, \ r = 1, 2, \ldots)
\]

and all being functions \( f(\varepsilon) \) which converges to zero in the usual sense as \( \varepsilon \) tends to zero. In [8], the author used the neutrix limit in order to define the incomplete \( q \)-gamma function with its derivatives as

\[
\gamma_q^{(r)}(\alpha, x) = N' \lim_{\varepsilon \to 0} \int_\varepsilon^\alpha t^{\alpha-1} \ln^r t E_q^{-qt} dt
\]

(4)

for all real values of \( \alpha \) and \( x > 0 \) and showed that

\[
\gamma_q(-n, x) = \int_1^x t^{-n-1} E_q^{-qt} dt + \int_0^1 t^{-n-1} \left[ E_q^{-qt} - \sum_{i=0}^n \frac{(-1)^i q^{i(i+1)}}{[i]!} t^i \right] dt + \sum_{i=0}^{n-1} \frac{(-1)^i q^{i(i+1)}}{[i-n][i]!}
\]

(5)

for \( n \in \mathbb{Z}^+, \ x > 0 \).

2. Main Results

In this section, we aim to give some equalities for the incomplete \( q \)-gamma function with its first derivative for all real values of \( \alpha \). First we show the following result.
for $n=1, 2, \ldots$

$$\gamma_q(-n, x) = \frac{1}{[-n]} \gamma_q(-n + 1, x) + \frac{x^{-n} E_q^{-x}}{[-n]} - \frac{(-1)^n q^{n(-n+1)}}{[-n][n!]}.$$  \hfill (7)

**Proof.** The proof of equation (6) is straightforward from equation (1). Now for equation (7), we have

$$\gamma_q(-n, x) = \frac{1}{[-n]} \left\{ \int_1^x t^{-n} E_q^{-qt} d_q t + \int_0^1 t^{-n} \left[ E_q^{-qt} - \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i]!} t^i \right] d_q t \right\}$$

$$+ \frac{x^{-n} E_q^{-x}}{[-n]} - \frac{E_q^{-1}}{[-n]} + \frac{E_q^{-1}}{[-n]} - \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i]!} - \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i][i-n]}$$

by using $q$-integration by parts in equation (5). By calculating the last two terms, we get desired result. \hfill \Box

The recurrence formula of incomplete $q$-gamma function for negative integers is also obtained in \cite{8} as

$$\gamma_q(-n, x) + \frac{q^n}{[n]} \gamma_q(-n + 1, qx) = \frac{(-1)^n q^{n(-n+1)}}{[n][n!]} - \frac{q^n}{[n]} x^{-n} E_q^{-qx}$$ \hfill (8)

by $q$-integration by parts in \cite{4}. Remark that, the classical version is given by

$$\gamma(-n, x) + \frac{1}{n} \gamma(-n + 1, x) = \frac{(-1)^n}{nn!} = \frac{1}{n} e^{-x} x^{-n}$$ \hfill (9)

in \cite{9}. Nevertheless to say, both of the equations (7) and (8) tend to (9) as $q \to 1$ and the equations (6) and (7) tends to the result in \cite{10} as $x \to \infty$.

Also we want to note that by mathematical induction one can find that

$$\gamma_q(-n, x) = \frac{(-1)^n q^{n(n+1)}}{[n]!} [\varphi_q(n) + \gamma(0, x)] + \mu_q(n) E_q^{-x}.$$ \hfill (10)

for $n=1, 2, \ldots$ where the functions are defined by

$$\mu_q(n) = \sum_{i=0}^{n-1} (-1)^i \eta_q(i) x^{-n+i}, \quad \varphi_q(n) = \sum_{i=1}^{n} \frac{1}{[n]}, \quad \eta_q(i) = \prod_{j=0}^{i} \frac{q^{n-j}}{[n-j]}$$

respectively. The equation (10) approaches to the result in \cite{11} as $x \to \infty, q \to 1$.

Before we give the result for the first derivative of incomplete $q$-gamma function at negative integers we need the following properties.

**Lemma 1.**

$$\gamma_q(\alpha, x) = \int_1^x t^{\alpha-1} E_q^{-qt} d_q t + \int_0^1 t^{\alpha-1} \left[ E_q^{-qt} - \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i]!} t^i \right] d_q t + \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i][i+\alpha][i+\alpha]} t^i$$ \hfill (11)

and

$$\gamma_q'(\alpha, x) = \int_1^x t^{\alpha-1} \ln t E_q^{-qt} d_q t + \int_0^1 t^{\alpha-1} \ln t \left[ E_q^{-qt} - \sum_{i=0}^{n-1} \frac{(-1)^i q^{(i+1)x}}{[i]!} t^i \right] d_q t \int_0^1 t^{\alpha-1} \ln t$$
We also have taking neutrix limit on both side and using the equation (5) we get (11) that
\[
\gamma_q(-n, x) = \int_1^x t^{-n-1} \ln t E_q^{-qt} dt + \int_0^1 t^{-n-1} \ln t \left[ E_q^{-qt} - \sum_{i=0}^{\alpha} \left( -1 \right)^i q \frac{t^{i+1}}{i!} \right] dt \\
+ \sum_{i=0}^{n-1} \frac{(-1)^i q^{i+1}}{i!} \left[ \ln q^{-1} \frac{1}{[\alpha + i]} + \ln q^{-1} \frac{1}{(q-1)[\alpha + i]^2} \right]
\]
for \(-n < \alpha < -n + 1, n = 1, 2, \ldots, x > 0\) and
\[
\gamma_q(-n, x) = \int_1^x t^{-n-1} \ln t E_q^{-qt} dt + \int_0^1 t^{-n-1} \ln t \left[ E_q^{-qt} - \sum_{i=0}^{\alpha} \left( -1 \right)^i q \frac{t^{i+1}}{i!} \right] dt \\
+ \sum_{i=0}^{n-1} \frac{(-1)^i q^{i+1}}{i!} \left[ \ln q^{-1} \frac{1}{[\alpha + i]} + \ln q^{-1} \frac{1}{(q-1)[\alpha + i]^2} \right]
\]
for \(n = 1, 2, \ldots\).

**Proof.** For \(\varepsilon > 0\), we have
\[
\int_\varepsilon^x t^{\alpha-1} E_q^{-qt} dt = \int_1^x t^{\alpha-1} E_q^{-qt} dt + \int_1^\varepsilon t^{\alpha-1} \left[ E_q^{-qt} - \sum_{i=0}^{\alpha} \left( -1 \right)^i q \frac{t^{i+1}}{i!} \right] dt \\
+ \sum_{i=0}^{n-1} \frac{(-1)^i q^{i+1}}{i!} \left[ \ln q^{-1} \frac{1}{[\alpha + i]} + \ln q^{-1} \frac{1}{(q-1)[\alpha + i]^2} \right].
\]
Since the last term is a negligible function, we get equation (11). The proofs of the equation (12) and (13) can be obtained similarly. \(\square\)

We can now prove the following theorem.

**Theorem 2.**
\[
\gamma_q^{(r)}(\alpha, x) = N - \lim_{\varepsilon \to 0} \gamma_q^{(r)}(\alpha + \varepsilon, x)
\]
for all real values of \(\alpha\) and \(r = 0, 1\).

**Proof.** Since \(\gamma_q^{(r)}(\alpha, x)\) is continuous function for \(\alpha \neq 0, -1, -2, \ldots\), the result follows immediately for \(\alpha \neq 0, -1, -2, \ldots\). If \(0 < \varepsilon < 1\) and \(r = 0\), we have from the equation (11) that
\[
\gamma_q(-n + \varepsilon, x) = \int_1^x t^{-n-1} E_q^{-qt} dt + \int_0^1 t^{-n-1} \left[ E_q^{-qt} - \sum_{i=0}^{\alpha} \left( -1 \right)^i q \frac{t^{i+1}}{i!} \right] dt \\
+ \sum_{i=0}^{n-1} \frac{(-1)^i q^{i+1}}{i!} \left[ \ln q^{-1} \frac{1}{[\alpha + i]} + \ln q^{-1} \frac{1}{(q-1)[\alpha + i]^2} \right].
\]
The last term on the right side of the equation is a negligible function. Because of that, by taking neutrix limit on both side and using the equation (5) we get
\[
N - \lim_{\varepsilon \to 0} \gamma_q(-n + \varepsilon, x) = \gamma_q(-n, x).
\]
We also have
\[
\gamma_q(-n - \varepsilon, x) = \int_1^x t^{-n-1} E_q^{-qt} dt + \int_0^1 t^{-n-1} \left[ E_q^{-qt} - \sum_{i=0}^{\alpha} \left( -1 \right)^i q \frac{t^{i+1}}{i!} \right] dt \\
+ \sum_{i=0}^{n-1} \frac{(-1)^i q^{i+1}}{i!} \left[ \ln q^{-1} \frac{1}{[\alpha + i]} + \ln q^{-1} \frac{1}{(q-1)[\alpha + i]^2} \right].
\]
Now taking neutrix limits of both sides, using the equation (5) and the fact that \([-\varepsilon] = -q^\varepsilon[\varepsilon]\), we get
\[
N - \lim_{\varepsilon \to 0} \gamma_q(-n - \varepsilon, x) = \gamma_q(-n, x).
\]
The case \(r = 1\) can be proved similarly. \(\square\)
Theorem 3. \[ \gamma_q(\alpha + 1, x) = N - \lim_{\varepsilon \to 0} \left( [\alpha + \varepsilon] \gamma_q(\alpha + \varepsilon, x) - x^{\alpha + \varepsilon} E_q^{-x} \right) \]
for all real values of \( \alpha \).

Proof. The result follows because of the continuity of \( \gamma_q(\alpha, x) \) for \( \alpha \neq -1, -2, \ldots \). By using equation (14) we get
\[ \gamma_q(-n + \varepsilon + 1, x) = [-n + \varepsilon] \gamma_q(-n + \varepsilon, x) - x^{-n + \varepsilon} E_q^{-x}. \]
for \( 0 < |\varepsilon| < 1 \) and \( n = 1, 2, \ldots \).
Then using theorem 2 we obtain
\[ \gamma_q(-n + 1, x) = N - \lim_{\varepsilon \to 0} \gamma_q(-n + \varepsilon + 1, x) = N - \lim_{\varepsilon \to 0} \left( [-n + \varepsilon] \gamma_q(-n + \varepsilon, x) - x^{-n + \varepsilon} E_q^{-x} \right). \]

One can find that the result goes to the one in [13] as \( x \to \infty \).

In the next theorem we will show that the first derivative of incomplete \( q \)-gamma function at negative integers can be presented by itself.

Theorem 4. For \( n \in \mathbb{Z}^+ \), we have
\[ \gamma'_q(-n, x) = \frac{x^{-n} \ln(q^{-1} x) E_q^{-x}}{[-n]} + \frac{\ln q^{-1}}{(q - 1)[-n]} \gamma_q(-n, x) + \frac{\ln q^{-1}}{[-n]} \gamma_q(-n + 1, x) + \frac{1}{[-n]} \gamma'_q(-n + 1, x). \]

Proof. By using \( q \)-integration by parts on equation (13) we get

Then we add missing series parts of the definition of incomplete \( q \)-gamma function and its first derivative into the corresponding parentheses, we obtain
\[ \gamma'_q(-n, x) = \frac{x^{-n} \ln q^{-1} x E_q^{-x}}{[-n]} + \frac{\ln q^{-1}}{(q - 1)[-n]} \gamma_q(-n, x) + \frac{\ln q^{-1}}{[-n]} \gamma_q(-n + 1, x) + \frac{1}{[-n]} \gamma'_q(-n + 1, x) \]
\[ + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-1} \frac{(-1)^i q^{i(i+1)}}{[i]!} \ln q^{-1} + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-1} \frac{(-1)^i q^{i(i+1)}}{[i]!} \gamma_q(-n + 1, x) \]
\[ + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-2} \frac{(-1)^i q^{i(i+1)}}{[i]! [i]!} + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-2} \frac{(-1)^i q^{i(i+1)}}{[i]! [i]! [i]!} \gamma_q(-n + 1, x) \]
\[ + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-2} \frac{(-1)^i q^{i(i+1)}}{[i]! [i]! [i]! [i]!} + \frac{\ln q}{(q - 1)[-n]} \sum_{i=0}^{n-2} \frac{(-1)^i q^{i(i+1)}}{[i]! [i]! [i]! [i]!} \gamma'_q(-n + 1, x). \]
The sums of the first, fourth and sixth series and also the remaining series’ sums are equal to zero. Hence it completes the proof. □

This result also tends to the one in [10] as \( x \to \infty \).

Differentiating equation (14), we obtain

\[
\gamma_q'(\alpha + 1, x) = \frac{q^\alpha \ln q}{q - 1} \gamma_q(\alpha, x) + [\alpha] \gamma_q'(\alpha, x) - x^\alpha \ln x E_q^{-x} 
\]  

(16)

for \( \alpha \neq 0, -1, -2, \ldots \).

In the following theorem we use neutrix limits in order to generalize equation (16) for all real numbers.

**Theorem 5.**

\[
\gamma_q(\alpha + 1, x) = N - \lim_{\varepsilon \to 0} \left[ \frac{q^{\alpha + \varepsilon} \ln q}{q - 1} \gamma_q(\alpha + \varepsilon, x) + [\alpha + \varepsilon] \gamma_q'(\alpha + \varepsilon, x) - x^{\alpha + \varepsilon} \ln x E_q^{-x} \right] 
\]

for all real values of \( \alpha \).

**Proof.** The result can easily be obtained because of the continuity of \( \gamma_q'(\alpha, x) \) for \( \alpha \neq 0, -1, -2, \ldots \). Equation (15) also satisfies for all real values of \( \alpha \). By rewriting this equation for \( n \in \mathbb{Z}^+ \) and \( 0 < \varepsilon < 1 \) we have

\[
\gamma_q'(n + \varepsilon + 1, x) = [-n + \varepsilon] \gamma_q'(-n + \varepsilon, x) + \frac{\ln q}{q - 1} \gamma_q(-n + \varepsilon, x) + \ln q \gamma_q(-n + \varepsilon + 1, x) - x^{-n+\varepsilon} E_q^{-x} \ln(q^{-1}x) 
\]

\[
+ x^{-n+\varepsilon} E_q^{-x} \ln q - x^{-n+\varepsilon} E_q^{-x} \ln x 
\]

\[
= [-n + \varepsilon] \gamma_q'(-n + \varepsilon, x) + \frac{\ln q}{q - 1} (1 + q^{-n+\varepsilon} - 1) \gamma_q(-n + \varepsilon, x) - x^{-n+\varepsilon} E_q^{-x} \ln x 
\]

Then by taking the neutrix limits of the both sides and theorem 2 the proof is completed. □

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