Replica symmetry breaking in long-range glass models without quenched disorder.

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We discuss mean field theory of glasses without quenched disorder focusing on the justification of the replica approach to thermodynamics. We emphasize the assumptions implicit in this method and discuss how they can be verified. The formalism is applied to the long range Ising model with orthogonal coupling matrix. We find the one step replica-symmetry breaking solution and show that it is stable in the intermediate temperature range that includes the glass state but excludes very low temperatures. At very low temperatures this solution becomes unstable and this approach fails.

The thermodynamics of glasses without quenched disorder is a long standing problem in statistical physics. The interest to this problem was renewed recently when it was understood that powerful methods developed for the glasses with quenched disorder can be often applied to this problem. In both systems the local magnetization in the ground state varies from site to site and different sites are typically non-equivalent. The qualitative reason why glasses without quenched disorder are more difficult to describe theoretically than spin glasses is the following. The mean field theory has to operate with the average magnetization (or its moments), not with quantities which depend on a realization and a particular state. The average quantities appear naturally in spin glasses after averaging over quenched disorder which makes all sites equivalent.

A few methods were suggested to overcome this difficulty for the glasses without quenched disorder. First, a mapping of some glass models to the quenched disordered problems was suggested; this method has an obvious disadvantage that such mapping is difficult to guess. Second, it was noted that a typical dynamics in a glassy system leads not to a ground state but to one of many metastable states providing an effective averaging mechanism which makes all sites equivalent even for glasses without quenched disorder. This method has a disadvantage that dynamical equations are much more difficult to solve than the statical ones. Very recently the cloning method was proposed that is based on the idea that even at low temperatures the system leads not to a ground state but to one of many metastable states providing an effective averaging mechanism which makes all sites equivalent. The main assumptions implicit in this approach are that low lying metastable states are not correlated (otherwise, averaging over them would not remove completely the non-equivalence of different sites) and that configurational entropy associated with these states behaves well as a function of energy at low energies permitting the “right” choice of $m$.

The goal of this paper is to provide an alternative theoretical framework to the cloning method which, albeit somewhat similar in formalism, uses different physical arguments for its justification and allows to check the main assumption of the method mentioned above. We apply this method to the Ising version of the periodic long-range Josephson array model which is a simple example of a glass without quenched disorder and we show that in this model the main assumptions of the method are correct in the intermediate temperature range but become wrong at very low temperatures. The main idea of the approach is that in a system with many low-lying states even a small random field is able to change the energy balance between the states and pull down a different state making it a new ground state of the system. Averaging over this random field is equivalent to the averaging over low lying metastable states. Specifically, in a spin system we add to the physical Hamiltonian a magnetic field part: $\mathcal{H} \rightarrow \mathcal{H} + \sum_i h_i S_i$, with small random $h_i$. The resulting change in the energy of a typical metastable state is of the order of $\sqrt{Nh}$; because this energy interval contains a large amount of metastable states, we expect that a small non-zero field would result in a large rearrangement of their energies but would not change the properties of individual states. Averaging over the random field configurations is performed in the usual way introducing $n$ replicas of the system and taking the limit $n \rightarrow 0$. The assumption of uncorrelated states is equivalent to one step replica symmetry breaking (1RSB) formalism; further, in this case the method is formally equivalent to cloned liquid approach if the size of the blocks in 1RSB is equal to the number of clones (see for the discussion of replica method vs clones for quenched disordered glasses). From the above discussion it is evident that another assumption implicit in this approach is that the energy spacing between low lying states should be much
less than $O(\sqrt{N})$ if it is too big a small magnetic field will not be sufficient to rearrange low lying states, if it is too small, e.g. $dS_{conf}/dF|_{h_i} = \infty$, the effect of random field will be too large and no sensible limit $h_i \to 0$ is possible. The latter situation seems to happen in the periodic long range Josephson array with flux $2\pi$ per strip 2 when all states are exactly degenerate and $S_{conf}(F)$ is very singular at $T = 0$.

We now provide the details of our formalism and its application to the simplest model of a glass without disorder. Our model consists of two sets of Ising spins (which we shall refer as “upper” and “lower” in the following) interacting via

$$H = -\frac{1}{2} \sum_{m,n} S_{im} J_{mn}^\alpha S_{jn}^\alpha,$$

Here the spin $S_{im}$ has a site index $(m = 1 \ldots N)$ and a components index $i = 1, 2$ corresponding to the upper and lower spins and matrix $J$ is

$$J_{mn} = \begin{pmatrix} 0 & J_{mn}^\alpha \\ J_{mn}^\alpha & 0 \end{pmatrix},$$

with $J_{mn} = (J_0 \sqrt{2/N}) \cos(\frac{2\pi m}{N}(m - 1/2)(n - 1/2))$. For $\alpha = 1/2$ we obtain the orthogonal limit $\sum_n J_{mn} J_{nk} = J_0^2 \delta_{mk}$, in what follows we shall focus on this case. This Ising spin model is similar to the XY spin model of long ranged Josephson array [2] and to the Bernasconi model [8]. As well as in these models its lowest states correspond to “pseudorandom” sequences with flat Fourier transform. So, we expect that this model also displays glassy properties, in particular that it has extensive configurational entropy at low temperatures. Further, one expects that in a model with long range interaction the barriers separating metastable states become infinite in the thermodynamic limit. We have verified numerically that the configurational entropy in this model is indeed extensive and its dependence on energy is similar to the one obtained for other infinite range glasses (see Fig. 1). Note, however, the important difference between this model and the XY spin model of [8]: in the orthogonal limit the ground state of Ising model does not become extensively degenerate (i.e. degeneracy stays finite as $N \to \infty$, see Fig. 1) whereas in the XY spin model the ground state becomes extensively degenerate in the unitary limit making it very complicated [2].

Taking the Gaussian distribution for the random magnetic field $\langle h_i h_j \rangle = 2 h_0^2 \delta_{ij}$, we get the replica Hamiltonian

$$H_s = \sum_\alpha \mathcal{H}(S_\alpha) + h_0^2 \sum_{\alpha,\beta} S_{im}^\alpha S_{im}^\beta,$$

where the replica indexes $\alpha, \beta$ run from 1 to $n$ and the limit $n \to 0$ should be taken. The glass transition corresponds to an appearance of a non replica-symmetric solution of the Hamiltonian (3) in the limit $h_0 \to 0$.

In the large $N$ limit a long range model containing $N$ sites can be reduced to an effective single-site model with a free energy density $F$

$$-\beta F = \frac{1}{2} \text{Tr} \gamma(B) + \frac{1}{2} \sum_j S_j^\alpha B_{\alpha\beta} S_j^\beta,$$

where $S_j^\alpha$ is Ising spin field retaining only replica and component index dependence, $B$ is an order parameter matrix in the replica space. The function $\gamma(B)$ can be determined from the condition that all single site correlation functions of the model (4) coincide with the correlation functions of the original model (3). Instead of comparing the spin correlation functions of these two models it is easier to decouple Ising spins by auxiliary field $\psi$, sum over Ising spins and compare the correlation functions of conjugate field $\psi$ in the two new models

$$\beta H_{\psi} = \frac{T}{2} \sum_{m,n,\alpha} \psi_{im}^\alpha (\hat{J}^{-1})_{mn}^\alpha \psi_{jn}^\alpha - \sum_{m,\alpha,j} V(\psi_{jm}^\alpha), \quad (5)$$

$$\beta F_{\psi} = -\frac{1}{2} \left[ \text{Tr} \gamma(B) - \sum_{\alpha,\beta,j} \psi_{ij}^\alpha B_{\alpha\beta}^{-1} \psi_{ij}^\beta \right] - \sum_{\alpha,j} V(\psi_{ij}^\alpha). \quad (6)$$

where $V(\psi) = \ln 2 \cosh(\psi)$. For both models one can construct a formal perturbation theory in the interaction $\ln 2 \cosh(\psi_{ij}^\alpha)$ and verify that these expansions coincide. We begin with the model (5). Inspecting the terms of the perturbation theory for the correlator $G_{ij}^\alpha = \langle \psi_{im}^\alpha \psi_{jn}^\beta \rangle$ one verifies that in the leading order in $1/N$ it is given by $G = [T \hat{J}^{-1} - \Sigma]^{-1}$ with the self energy $\Sigma$ which is diagonal in the site index: $\Sigma = A \delta_{mn} \delta_{ij}$. This approach is similar to a locator expansion [4] but in our case the locator matrix $A$ might be non-trivial in the replica space. Using the orthogonality of $\hat{J}$ we get that the single site correlation function $G_{ij} = G_{ij}^{\alpha\beta}$ (that we need to establish the correspondence between the models) becomes

$$G_{ij} = \left[ T \hat{J}^{-1} - \Sigma \right]^{-1}.$$
\[ \mathcal{G} = [-A + (j_0^2 A)^{-1}]^{-1} \]  

where \( j_0 = \beta J \).

Now we turn to the model \( \mathcal{G} \). Here the self energy is diagonal in the site index by construction, further, the interaction part of this model is the same as for model \( \mathcal{F} \); assuming that their single site correlation functions coincide we conclude that their single site self-energies are equal as well. Thus, the spin correlator obtained for this model is \( \mathcal{G} = [B^{-1} - A]^{-1} \), comparing this expression with \( \mathcal{F} \) we conclude that \( B = j_0^2 A \).

The correlator of the dual field \( \psi \) can be related to the correlator of original spins: consider a Gaussian functional \( \mathcal{F} = \int d\psi \exp(-\beta J_0^{-1} \psi^2/2 + S\psi) \) and use it to express \( \mathcal{G} \) via correlator \( D_{\alpha,\beta} \equiv \langle S^\alpha S^\beta \rangle \), we get: \( \mathcal{G} = B + BDB \). Solving this equation for the spin correlator \( D \) and using \( \mathcal{F} \) and the relation \( B = j_0^2 A \) we obtain

\[ D = B[j_0^2 - B^2]^{-1}. \]

Finally, the saddle point condition for the free energy \( \mathcal{H} \)

\[ 2D = -\gamma'(B) \]

therefore integrating Eq.\( \mathcal{F} \) we find

\[ \gamma(B) = \ln(1 - j_0^2 B^2), \]

Note that the free energy \( \mathcal{H} \) coincides with the free energy of the model considered in Ref. \( \mathcal{F} \) although their properties at finite \( N \) are markedly different. Furthermore, this free energy is the same as obtained by family Hamiltonian approach \( \mathcal{G} \).

**Paramagnetic state.** In this state we take the replica symmetric ansatz \( B_{\alpha,\beta} = \mu \delta_{\alpha,\beta} \) and free energy \( \mathcal{F} \) becomes

\[ \mathcal{F} = \ln \left( j_0^2 - \ln(j_0^2 - \mu^2) \right) / 2 - \mu - 2 \ln 2. \]  

Variation with respect to \( \mu \) gives \( \mu = [\sqrt{1 + 4j_0^2} - 1] / 2 \). Usual thermodynamic relations between energy and entropy give \( E = -T \mu, S = \ln[4\sqrt{\pi}] / j_0 \).

One can see that the entropy of the normal solution becomes negative at \( T < T_K = J_0 / (4\sqrt{15}) \approx 0.064550 J_0 \) which is the Kauzmann temperature for this model \( \mathcal{F} \); one expects that the glass transition takes place at some temperature, \( T_c \), above \( T_K \).

**Glass state.** At the glass transition temperature \( T_c \) the replica symmetry is broken, we assume that it is described by one step replica symmetry breaking (1RSB) and then verify that it is indeed a stable solution below \( T_c \). The 1RSB ansatz is \( B_{\alpha,\beta} = \mu \delta_{\alpha,\beta} + \eta R_{\alpha,\beta} \), where the matrix \( R \) is a block-diagonal matrix consisting of \( m \times m \) blocks with all elements equal 1, we get the free energy functional

\[ \beta \mathcal{F} = \ln j_0^2 - (1 - 1/m) \ln (j_0^2 - \mu^2) / 2 \]

\[ -2 \ln 2 - (\ln X) / 2m - \mu - 2 f(\eta, m) / m, \]  

where \( X = j_0^2 - (\mu + \eta m)^2 \) and the function \( f \) is

\[ f(\eta, m) = \ln \left( \int P_m(z) dz \right), P_m(z) = e^{-z^2 / 2} \cosh^{m}(z \sqrt{\eta}). \]  

Taking the derivatives of \( \mathcal{F} \) with respect to \( \mu, \eta, m \) we get

\[ \left( \frac{1}{m} - 1 \right) \frac{\mu}{j_0^2 - \mu^2} - \frac{\eta m + \mu}{X m} + 1 = 0, \]

\[ - (\eta m + \mu) / X + q(m - 1) + 1 = 0, \]

\[ \frac{1}{2m^2} \log [(j_0^2 - \mu^2) / X] + \frac{2}{m} \frac{\partial}{\partial \eta} \mathcal{F}(\eta, m) \]

\[ - \eta (\mu + \eta m) / m X - 2 f(\eta, m) / m^2 = 0, \]  

where \( q = \int \tanh^2(z) P_n(z) dz / \int P_n(z) dz \) is the spin overlap of different replicas belonging to the same block \( D_{\alpha,\beta} = (1 - q) \delta_{\alpha,\beta} + q R_{\alpha,\beta} \), that coincides with Edwards-Anderson (EA) order parameter. Eqs.\( \mathcal{F} \) can be solved with respect to \( m, \eta \) giving

\[ \mu = \eta \frac{1 + (1 - q + qm)\eta m \eta / (1 - q - 2\eta (1 - q + qm)) }{q/(1-q) - 2\eta (1 - q + qm)} \]

and \( j_0^2 = \mu^2 + \mu^2/(1 - q) \). For a given \( m \) we can solve Eq.\( \mathcal{F} \) numerically with respect to \( \eta \) and get all quantities as functions of \( m \). The resulting dependence of \( m(T) \) for \( J_0 = 1 \) is shown on Fig. 2. In the limit \( n \to 0 \), the values of \( m \) should lie within the interval \((0,1) \) and \( m = 1 \) defines the thermodynamic critical temperature \( T_c \approx 0.064593 \), it is larger than \( T_K \) as expected. The value of the EA order parameter \( q \) at the glass transition is very close to 1, \( 1 - q = 0.00017116 \), so in this sense the phase transition is strongly first order but (similar to p-spin model), the energy and entropy do not change discontinuously at the transition. The numerical solution shows that when the temperature decreases, the entropy of the glass state monotonically decreases and eventually becomes negative below \( T_K \approx 2.8 \times 10^{-4} \). The explanation of such unphysical behavior is that 1RSB ansatz, in fact, becomes unstable in this low temperature regime.

**Stability of the thermodynamical solution.** In order to analyze stability of 1RSB ansatz we expand the Eq.\( \mathcal{F} \) to the second order in fluctuation of the order parameter \( \delta B \) and consider different families of fluctuation matrices \( \delta B \). This calculation is very similar to the analysis of the stability of paramagnetic solution and Parisi solution in SK model \( \mathcal{F} \), so we only sketch it here, for details see Appendix. We find that the most dangerous direction in the fluctuation space corresponds to the "replicon" modes \( \mathcal{F} \) that are fluctuations within diagonal blocks.
solution is unstable at $T < T_c$. By some temperature range below that the main assumptions of this approach are correct in clones \[6\.\] The stability of the 1RSB solution indicates a dynamical approach with a free energy functional that is obtained in the cloned liquid. Corresponding to 1RSB ansatz is equivalent to the free energy characterized by $\delta B$ satisfying the conditions $\delta B R_{\alpha,\beta} = 0$, $\delta B_{\alpha,\alpha} = 0$. The eigenvalue corresponding to these modes is

$$\Lambda = 2(1-q)/\mu + 2(1-q)^2 - 2(r-q^2)$$

where $r = \int \tanh^4(z) P_m(z) dz / \int P_m(z) dz$. Numerical solution shows that $\Lambda_1$ is positive at temperatures $T > T_{uns} \approx 6.1 \times 10^{-3}$ but changes sign at $T_{uns}$, thus 1RSB solution is unstable at $T < T_{uns}$.

![Graph](image_url)

FIG. 2. Main plot: Dependence of the configuration entropy on the temperature for the marginally stable solution. Insert: size of 1RSB block, $m$, for thermodynamic (solid line) and marginally stable (dashed line) solutions. The value of $T$ at which $m = 1$ gives thermodynamical (dynamical) critical temperature.

**Marginal solution.** One expects that in a glass a typical dynamical process leads to a most abundant state which is, therefore, marginally stable. We note that although plausible, this assumption might be violated if the attraction basins of the low lying states are much larger than those of the marginally stable ones \[13\.\] Assuming that it is not the case, a dynamical freezing leads to the states with $\Lambda = 0$ instead of the states with the minimal free energy characterized by $dF/dm = 0$. Thus, to get the properties of the states selected in a "dynamical" process we replace Eq. 15 by $\Lambda = 0$. The resulting dependence of the temperature on the size of 1RSB block $m$ is shown in Fig. 2. The value $m = 1$ defines the "dynamical" critical temperature $T_g \approx 0.13363$. The free energy functional corresponding to 1RSB ansatz is equal to the free energy functional that is obtained in the cloned liquid approach with $m$ being equal to the number of the clones \[1\.\] The stability of the 1RSB solution indicates that the main assumptions of this approach are correct in some temperature range below $T_g$ and, therefore, in this temperature range the configurational entropy is given by $S_{conf} = m^2 2\pi$. The dependence $S_{conf}$ on temperature is shown in Fig. 2. Decreasing the temperature, it first increases, goes through the maximum at $T_m$, and eventually becomes negative at the temperature $T_{uns}$, at which the thermodynamical solution becomes unstable.

It is not clear however that the 1RSB solution is a correct solution in the whole temperature range $T_{uns} < T < T_g$, on the contrary, it is quite likely that another solution is preferred by the system below some $T_c' < T_g$. We have only indirect arguments for this: first, it seems unphysical that $S_{conf}$ decreases with temperature decrease, usually at lower temperatures additional states appear. Second, the obtained $S_{conf}$ does not match the results of numerics if one believes that this solution remains correct at $T < T_m$. Finally, note the analogy with higher temperatures: paramagnetic solution is always stable but is eventually replaced by 1RSB solution.

In conclusion we have justified the application of replica method to some systems without quenched disorder and discussed situations in which cases it fails. We identify two dangers: correlations between metastable states close to the ground state and too large degeneracy of the ground state. We apply the formalism to the periodic Ising spin model with orthogonal coupling matrix and find that it gives the same free energy as fiduciary Hamiltonian approach \[0\.\] Further we show that it works in the intermediate temperature range but fails at low temperatures when metastable states become correlated. Two questions remain open: whether generalization of this method to continuous symmetry breaking would allow one to study the models with correlated metastable states and what to do if the ground state of the model is highly degenerate as it is, e.g., in the case of unitary coupling matrix.

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I. APPENDIX. STABILITY ANALYSIS.

Here we provide the details of the stability analysis of 1RSB solution. We start by expanding the Eq. (I) to the second order in fluctuation of the order parameter \( \delta B \):

\[
\delta F = \frac{1}{4} \delta B_{\alpha,\beta} M_{\alpha,\beta,\gamma,\delta} \delta B_{\gamma,\delta},
\]

where the coefficients \( a, b, c \) come from the expansion of Tr \( \gamma(B) \)/2

\[
a = 2q^2 - 2[Y \eta/\mu + q] [Y \eta/(\mu + \eta m) - q],
\]

\[
b = q[\mu^{-1} + 4(1 - q)] + Y \eta [(1 - q) \eta m - 1]/[\mu^2 + \eta \mu m],
\]

\[
c = 2(1 - q)[2(1 - q) + \mu^{-1}],
\]

with \( Y = 1 - q + q m \). The function \( \Gamma \) is the four spin replica correlation function \( \Gamma_{\alpha,\beta,\gamma,\delta} = \langle S_a S_{\beta} S_\gamma S_d \rangle \), defined with respect to the Hamiltonian (I). When at least two indexes, e.g. \( \alpha \) and \( \beta \), are equal this function is

\[
\Gamma_{\alpha,\alpha,\gamma,\delta} = (1 - q) \delta_{\gamma,\delta} + q R_{\gamma,\delta}.
\]

If \( \alpha, \beta, \gamma, \delta \) are all different, the function \( \Gamma \) is not zero only if either \( \alpha, \beta, \gamma, \delta \) belong to the same block, or two of indexes (e.g. \( \alpha, \beta \)) belong to one block and the other two (\( \gamma, \delta \)) to another one. In the first case \( \Gamma_{\alpha,\beta,\gamma,\delta} = r \), where \( r \) is

\[
r = \int P_m(z) \tanh^4(\zeta) dz / \int P_m(z) dz.
\]

In the second case, since \( \alpha, \beta \) and \( \gamma, \delta \) belong to different blocks \( \Gamma_{\alpha,\beta,\gamma,\delta} = q^2 \).

The eigenvalue equation for the matrix \( M \) is

\[
M_{\alpha,\beta,\gamma,\delta} \delta B_{\gamma,\delta} = \Lambda \delta B_{\alpha,\beta}.
\]

The eigenvalue equation can be simplified using that \( \delta B \) is a symmetric matrix. The resulting equation is

\[
[b - 2q(1 - q)](\delta B R + R \delta B)_{\alpha,\beta} + (a - 2q^2)(R \delta B R)_{\alpha,\beta} - 2(4q - 3r - 1) \delta_{\alpha,\beta} \delta B_{\alpha,\alpha} + (3q - r) R_{\alpha,\gamma} \delta B_{\gamma,\delta} R_{\beta,\delta} + 2(2 - q - r) \left[ 2 \delta_{\alpha,\beta}(R \delta B)_{\alpha,\alpha} + R_{\alpha,\beta} \delta B_{\alpha,\alpha} + R_{\beta,\alpha} \delta B_{\beta,\beta} \right] +\]

\[
+ (q^2 - r) \left[ \delta_{\alpha,\beta} R_{\beta,\gamma} \delta B_{\gamma,\gamma} + 2 R_{\alpha,\beta} \delta B_{\alpha,\delta} - 2 R_{\alpha,\beta}(\delta B R)_{\alpha,\alpha} - 2 R_{\alpha,\beta}(\delta B R)_{\beta,\beta} - \delta_{\alpha,\beta}(R \delta B R)_{\alpha,\alpha} - R_{\alpha,\gamma} \delta B_{\gamma,\gamma} R_{\gamma,\beta} \right] = [\Lambda + 2(1 - q)^2 - c] \delta B_{\alpha,\beta}.
\]

This equation has a block-diagonal structure, therefore one can divide the fluctuation matrix \( \delta B \) into blocks \( \delta B \) of \( m \times m \) size and consider the fluctuations within each block independently. Moreover, all eigenvalue equations corresponding to diagonal (off-diagonal) blocks are equivalent. Therefore we are left with two cases: (I) fluctuations within an off-diagonal block and (II) fluctuations within a diagonal block.

We begin our analysis of eigenvalues with the case (I) for which the eigenvalue equation is reduced to

\[
(b - 2q(1 - q))(\delta B \mathcal{E} + R \delta B)_{\alpha,\beta} + (a - 2q^2)(\mathcal{E} \delta B \mathcal{E})_{\alpha,\beta} = \Lambda' \delta B_{\alpha,\beta},
\]

where \( \Lambda' = \Lambda - c + 2(1 - q)^2 \) and \( \mathcal{E} \) is the \( m \times m \) matrix with all elements equal 1. The first eigenvalue corresponds to \( \delta B \) satisfying \( (\delta B \mathcal{E})_{\alpha,\beta} = 0 \) and it is

\[
\Lambda^{(1)} = c - 2(1 - q)^2.
\]

The second eigenvalue is
\[ \Lambda_2^{(1)} = c + mb - 2Y(1 - q) \]  
\[(27) \]
and it corresponds to \( \delta B \) satisfying \((\delta B \mathcal{E})_{\alpha,\beta} \neq 0, \ (\mathcal{E} \delta B \mathcal{E})_{\alpha,\beta} = 0 \). The last eigenvalue of type I corresponds to \( \delta B = \mathcal{E} \) and it is

\[ \Lambda_3^{(1)} = c + mG - 2Y^2, \]  
\[(28) \]

where \( G = ma + 2b \).

The eigenvalues of type II satisfy the equation

\[
\mathcal{E}_{\alpha,\beta} \left[ 2(1-q)\delta B_{\alpha,\alpha} + 2(1-q)\delta B_{\beta,\beta} + 2(1-q)^2 \right] \left[ \delta B \mathcal{E} \right]_{\alpha,\alpha} + \left( r - q^2 \right) \left( \delta B \mathcal{E} \right)_{\alpha,\alpha}
\]

\[
- \delta_{\alpha,\beta} \left[ 2(4q-3r-1)\delta B_{\alpha,\alpha} + 4(r-q)\left( \mathcal{E} \delta B \right)_{\alpha,\alpha} + (r-q^2) \left( \delta B \mathcal{E} \right)_{\alpha,\alpha} \right]
\]

\[
- 2q(1-q) + b \right) \left( \delta B \mathcal{E} + \mathcal{E} \delta B \right)_{\alpha,\beta} + \left( a - r + q^2 \right) \left( \mathcal{E} \delta B \mathcal{E} \right)_{\alpha,\beta} = \left[ \Lambda' + 2(r - q^2) \right] \delta B_{\alpha,\beta}
\]\n\[(29) \]

The first eigenvalue of this type corresponds to \((\delta B \mathcal{E})_{\alpha,\beta} = 0\) and it is

\[ \Lambda_1^{(2)} = c - 2(1-q)^2 - 2(r - q^2). \]  
\[(30) \]

All other eigenvalues of type II correspond to \( \delta B \) of the form \( \delta B_{\alpha,\beta} = x_{\alpha} + y_{\beta} \mathcal{E}_{\alpha,\beta} + y_{\alpha} \mathcal{E}_{\alpha,\beta} \), where \( x_{\alpha}, y_{\alpha} \) are such that either \( \sum_{\alpha} x_{\alpha} = \sum_{\alpha} y_{\alpha} = 0 \) or \( x_{\alpha} = x, y_{\alpha} = y \). The corresponding eigenvalues are:

\[ \Lambda_2^{(2)} = c + \left[ mb + 2E \pm \sqrt{(bm + 2E)^2 - 16Eb} \right]/2, \]

\[ \Lambda_4^{(2)} = c + \left[ mG + F \pm \sqrt{(mG + F)^2 - 4GF} \right]/2, \]

where \( E = 4q - 3r - 1 + m(r - q) \) and \( F = 4\partial^2/\partial \eta^2 f(\eta, m)/(1 - m)m \). The numerical solution shows that for the thermodynamical solution all eigenvalues are positive for temperature higher than \( T_{unw} \). At temperature less than \( T_{unw} \) the eigenvalue \( \Lambda_1^{(2)} \) corresponding to the “replicon” mode becomes negative. The marginal (“dynamical”) solution is defined by \( \Lambda_1^{(2)} = 0 \), all the other eigenvalues for this solution are strictly positive in the region where it exists. Thus the replicon mode is always the most relevant fluctuation.