Three Body Bound State in Non-Commutative Space

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Abstract

The Bethe-Salpeter equation in non-commutative QED (NCQED) is considered for three-body bound state. We study the non-relativistic limit of this equation in the instantaneous approximation and derive the corresponding Schrödinger equation in non-commutative space. It is shown that the experimental data for Helium atom puts an upper bound on the magnitude of the parameter of non-commutativity, $\theta \sim 10^{-9}\lambda_e^2$.

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Non-commutative spaces [1] and their phenomenological aspects [2-10] have been recently considered by many authors. Among the others P. M. Ho and H. C. Kao in their paper [2] has claimed that there is not any non-commutative correction to the Hydrogen atom spectrum at the tree level owing to the opposite non-commutativity of the particles with opposite charges. Following these arguments, to detect the effect of non-commutativity at the tree level one should consider the bound states including three or more particles. A question here is the accuracy of experimental data. Fortunately atomic transitions $2^3P_1 \to 2^3P_0$ and $2^3P_1 \to 2^3P_2$ in Helium atom is recently measured [11] (and theoretically calculated [12]) with an accuracy about 1 kHz as precise as hyper-fine splitting in positronium. Due to long life time of these transitions, 98 ns, Helium atom is considered as one of the best labs to test QED. Therefore, it seems reasonable to study the effect of non-commutativity on the spectrum of Helium atom. For this purpose we calculate the corresponding corrections on the spectrum of the three body bound state up to the order $\theta \alpha^4$ by using the Bethe-Salpeter equation (BS) in the framework of NCQED.

First we extend the non-relativistic results of reference [2], regarding the lack of corrections to the spectrum of Hydrogen on noncommutative spaces, to the full non-relativistic regime using the BS equation [13]. The BS equation for the bound state of a fermion-(anti)fermion system has the form (Fig. 1):

$$
\Psi(p; P) = S\left(\frac{P}{2} + p\right)S\left(\frac{P}{2} - p\right) \int \frac{d^4k}{(2\pi)^4} I(k, p) \Psi(p + k; P),
$$

(1)

where $P$ is the momentum of the centre of mass, $\Psi(p; P)$ is the BS amplitude for the bound state and $S(p)$ is the fermion field propagator. $I(k, p)$ is the kernel of the interaction which is the sum of all possible irreducible graphs. The effects of non-commutativity are introduced in QED by modifying the vertices and including 3-photon and 4-photon interactions. Therefore, in non-commutative QED (NCQED) the kernel of Eq.(1) can be easily constructed by using the Feynman rules of the theory which are completely given in references [9, 14]. For instance, for the Hydrogen atom the tree level calculations correspond to the ladder approximation for the kernel with free particle propagators which is given in Fig. 2. At this level one has

$$
I^l_0(k, p) = e^{\frac{1}{\theta} [p^0 (\theta^1 + \theta^2) k]} I^l(k, p),
$$

(2)

where the superscript $l$ stands for ladder and $I^l(k, p)$ is the interaction kernel in the commutative space in the ladder approximation. The parameter of non-commutativity $\theta$, is an antisymmetric real valued tensor defined as follows:

$$
\theta^{\mu\nu} = -i [x^\mu, x^\nu],
$$

(3)
Figure 1: Bethe-Salpeter equation for two particles bound state.

Figure 2: Bethe-Salpeter equation in the ladder approximation. The coupling constant
\[ g_i = i q_i \gamma_\mu e^{-\frac{i}{2} p_i \theta_i \cdot (p - k)} \],
where \( \theta_i \), \( i = 1, 2 \) are the non-commutativity parameters corresponding to the particles with charges \( q_i \).

Of course \( \theta_0 \) is assumed to be vanishing since \( \theta_0 \neq 0 \) leads to some problems with the concepts of causality and the unitarity of field theories \([15]\). The superscripts introduced in Eq.(2) for \( \theta \) refer to the possibility that in an effective NCQED that considers the interactions between particles with internal structures, different non-commutativity parameters should be assigned to different particles. Furthermore, one can easily see that when \( \theta_1 = -\theta_2 \), there is not any non-commutative effect in tree level (see Eq.(2)).

In the case of positronium, there is also an annihilation diagram in the lowest order. In such a diagram, each vertex contains a factor \( e^{-\frac{i}{2} p \theta \cdot (-p)} = 1 \), thus there is not any non-commutativity correction at this level due to the annihilation graph.

The BS formalism for three body bound state is very complicated. But here, since we are interested in lowest order diagrams, the kernel of the BS equation reduces to three copies of two body kernel. At the tree level, the kernel is,

\[ I \Psi = (I_{\text{ope}} + I_{\text{ann}}) \Psi, \tag{4} \]

where \( \text{ope} \) stands for one photon exchange and \( \text{ann} \) for one photon annihilation interaction. Of course, there is no annihilation diagram in the case of three body bound states like the Helium atom. The \( I_{\text{ope}} \) contains three terms, shown in Fig. 3, as follows:

\[ I_{\text{ope}} = I_1 + I_2 + I_3, \tag{5} \]

where for example

\[ I_1 = q_2 q_3 \left( \frac{\gamma_2 \gamma_3}{K^2} - \left( \frac{\gamma_2 \gamma_3 k_0^2}{k^2 K^2} - \frac{\gamma_2 \cdot \gamma_3}{k^2} \right) \right). \tag{6} \]
Figure 3: One photon exchange diagrams. The coupling constant \( g_i = i q_i \gamma \mu e^{-i \mathbf{p}_i \cdot \mathbf{p}'_i} \), where \( \theta \)'s are the non-commutativity parameters corresponding to the particles with charge \( q_i \).

To obtain the above relation, we have used Feynman gauge and separated the instantaneous interaction from retarded one. In the present work, we do not consider the retarded interaction and the annihilation diagram since they result in higher order corrections in comparison with the instantaneous interaction. It is straightforward to show that considering only the instantaneous interaction, in non-relativistic approximation, the BS equation with the kernel (5), leads to the Schrödinger equation for three body bound state. To this end we consider the BS equation as follows:

\[
\Psi(p_1, p_2; P) = S_1 S_2 S_3 \int \frac{d^4 p_1'}{(2\pi)^4} \frac{d^4 p_2'}{(2\pi)^4} I(p_1, p_2, p_1', p_2'; P) \Psi(p_1', p_2'; P), \tag{7}
\]

where \( P \) is the momentum of the centre of mass. The propagators at lowest order are,

\[
S_i = \frac{2m_i}{p_i^2 - m_i^2 + i \epsilon}, \quad i = 1, 2, 3, \tag{8}
\]

in which \( p_3 = P - p_1 - p_2 \). The instantaneous interaction is

\[
I_{\text{inst}} = \sum_{i<j} \frac{q_i q_j}{|\mathbf{K}_{ij}|^2}, \tag{9}
\]

where \( q_i, i = 1, 2, 3 \), stand for the charges of the particles and \( \mathbf{K}_{ij} \) is the three-momentum of the exchanged photon between the \( i \)th and \( j \)th lines. Defining

\[
\phi(p_1, p_2) \equiv \int dp_1^0 dp_2^0 \Psi(p_1, p_2; P), \tag{10}
\]

Eq.(7) can be written as follows,

\[
\phi(p_1, p_2) = Z(p_1, p_2)(V\phi)(p_1, p_2), \tag{11}
\]

where

\[
Z(p_1, p_2) = \int dp_1^0 dp_2^0 S_1 S_2 S_3, \tag{12}
\]
and

\[ (V\phi)(p_1, p_2) = \int \frac{d^3p_1'}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} I_{\text{inst}}\phi(p_1', p_2'). \]  

(13)

To obtain \( Z(p_1, p_2) \), we use the definition

\[ \omega_i \equiv \sqrt{p_i^2 + m_i^2}, \quad i = 1, 2, 3, \]  

(14)

and rewrite Eq.(12) as follows,

\[
\frac{Z(p_1, p_2)}{8m_1m_2m_3} = \int \frac{1}{(x^2 - \omega_1^2 + i\epsilon)(y^2 - \omega_2^2 + i\epsilon)((P^0 - x - y)^2 - \omega_3^2 + i\epsilon)} \frac{1}{4\omega_1\omega_2((P^0 - \omega_1 - \omega_2)^2 - \omega_3^2)} + \frac{1}{4\omega_2\omega_3((P^0 - \omega_2 - \omega_3)^2 - \omega_1^2)} + \frac{1}{4\omega_1\omega_3((P^0 - \omega_1 + \omega_3)^2 - \omega_2^2)}. \]  

(15)

Simplifying the above result one finds that

\[
Z(p_1, p_2) = \frac{2m_1m_2m_3}{\omega_1\omega_2\omega_3} \left( \frac{\omega_2 + \omega_3}{(P^0 - \omega_1)^2 - (\omega_2 + \omega_3)^2} + \frac{\omega_1 + \omega_2}{(P^0 + \omega_3)^2 - (\omega_1 + \omega_2)^2} \right). \]  

(16)

Using Eq.(16) in Eq.(11), we obtain

\[
V\phi(p_1, p_2) = \frac{m_1m_2m_3}{\omega_1\omega_2\omega_3} \left( \frac{\omega_2 + \omega_3}{\frac{\omega_2}{2} + \omega_2 + \omega_3} + \frac{\omega_1 + \omega_2}{\frac{\omega_1}{2} + \omega_3 + \omega_1 + \omega_2} \right) V \frac{\omega_1}{2 + \omega_3} \right) (V\phi)(p_1, p_2), \]  

(17)

where \( V \equiv (P^0 - \omega_1 - \omega_2 - \omega_3) \). Assuming \( V \ll \omega_i, i = 1, 2, 3 \) we have

\[
V\phi(p_1, p_2) = \frac{m_1m_2m_3}{\omega_1\omega_2\omega_3} (V\phi)(p_1, p_2), \]  

(18)

Therefore, in configuration space the above equation in non-relativistic limit i.e. \( \omega_i \approx m_i \), leads to :

\[
H(x_1, x_2, x_3) \psi(x_1, x_2, x_3; t) = E\psi(x_1, x_2, x_3; t), \]  

(19)

in which \( E = P^0 - m_1 - m_2 - m_3 \), \( x_i \) is the position of the \( i \)-th particle, and

\[
H = -\frac{\nabla_{x_1}^2}{2m_1} - \frac{\nabla_{x_2}^2}{2m_2} - \frac{\nabla_{x_3}^2}{2m_3} + V(x_1, x_2, x_3), \]  

(20)

where

\[
V(x_1, x_2, x_3) = \frac{q_1q_2}{|x_1 - x_2|} + \frac{q_2q_3}{|x_2 - x_3|} + \frac{q_1q_3}{|x_1 - x_3|}. \]  

(21)

Now we are ready to examine the BS equation in the non-commutative space. Here, the vertices in kernel \( I_{\text{ope}} \), given in Eq.(5), is multiplied by a momentum dependent phase factor similar to Eq.(2), e.g. :

\[
I_1 \rightarrow I_1^{NC} = e^{-\frac{i\theta_{\mu\nu}p_{\mu}p_{\nu}}{2}} e^{-\frac{i\theta_{\mu\nu\rho}p_{\mu}p_{\nu}p_{\rho}}{6}} I_1. \]  

(22)
It can be easily verified that the lowest order corrections due to the \( \theta \)-dependent terms are of order \( \theta^4 \) and correspond to the instantaneous interaction. Since it is assumed that \( \theta_0 = 0 \), in the above procedure nothing changes but the term \((V \phi)(p_1, p_2)\) becomes

\[
(V \phi)(p_1, p_2, \theta) = \int \frac{d^3p'_1}{(2\pi)^3} \frac{d^3p'_2}{(2\pi)^3} (I^{NC})_{inst}(p'_1, p'_2),
\]

where

\[
(I^{NC})_{inst} = \sum_{i=1}^{3}(I^{NC}_i)_{inst},
\]

and for example

\[
(I^{NC}_1)_{inst} = \frac{e^{-\frac{i}{2} \theta_{23}^2 p'_1 p'_2} e^{-\frac{i}{2} \theta_{31}^2 p'_3 p'_3}}{|K_{23}|^2}.
\]

From Eq.(23), one can easily verify that the Schrödinger equation in non-commutative space can be modified as follows,

\[
i\frac{\partial}{\partial t}\psi(x_1, x_2, x; t) = H(x_1, x_2, x) \ast \psi(x_1, x_2, x; t),
\]

which is the familiar Schrödinger equation in NCQM. The lowest order \( \theta \)-dependent term in NCQED corresponds to the instantaneous interaction and is of the order \( \theta^4 \) (see Eq.(25)). Therefore, to obtain the lowest order terms in \( \theta \) in NCQED, it is sufficient to calculate the corresponding terms from Eq.(26) which is the non-relativistic limit of the BS equation in non-commutative space.

Now we define the center of mass coordinates as

\[
\mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2 + m_3 \mathbf{x}_3}{m_1 + m_2 + m_3},
\]

\[
\mathbf{r}_j = \mathbf{x}_j - \mathbf{x}_3, \quad j = 1, 2.
\]

Assume the non-commutativity algebra,

\[
[x_j, x_j] = i\delta_{ij}c_j \theta,
\]

where for simplicity we have assumed that \( \theta^i = c_i \theta \). This algebra leads to the following algebra for the center of mass coordinates:

\[
[r_1, r_2] = ic_3 \theta, \\
[r_j, R] = i\eta_j \theta, \\
[r_j, r_j] = i\kappa_j \theta, \quad j = 1, 2, \\
[R, R] = i\beta \theta.
\]
where,
\[ \eta_j := \frac{m_j c_j - m_3 c_3}{m_1 + m_2 + m_3}, \]
\[ \kappa_j := i(c_j + c_3), \quad j = 1, 2, \]
\[ \beta := \frac{m_1 c_1 + m_2 c_2 + m_3 c_3}{(m_1 + m_2 + m_3)^2}, \]

Considering Eq.(29) and the translation
\[ r_j \rightarrow r_j + \frac{1}{2} \eta_j \theta. K, \]
in which \( K \) is the wave vector of the center of mass [2], one can show that
\[ V \ast \psi = q_1 q_3 V \left( r_1 - \frac{i}{2} \kappa_1 \theta. \nabla_1 - \frac{i}{2} c_3 \theta. \nabla_2 \right) + 1 \leftrightarrow 2 \]
\[ + q_1 q_2 V \left( r_1 - r_2 - \frac{i}{2} \theta. (c_1 \nabla_1 - c_2 \nabla_2) \right) \psi, \]
where \( V(x) = \frac{1}{|x|}. \) Expanding the right hand side of Eq.(32) in terms of \( \theta \), one finds that Eq.(26) can be written as
\[ E\psi = \left( H + H_{NC}^I \right) \psi, \]
where, the operator \( H \) is the Hamiltonian of the three body bound state in the commu-
tative space and
\[ H_{NC}^I = -\frac{q_1 q_3}{2} \theta. \left( \frac{r_1 \times (\kappa_1 p_1 + c_3 p_2)}{|r_1|^3} \right) + 1 \leftrightarrow 2 + \frac{q_1 q_2}{2} \theta. \left( \frac{(r_1 - r_2) \times (c_1 p_1 - c_2 p_2)}{|r_1 - r_2|^3} \right). \]

To obtain Eq.(34) we have used the relation
\[ u. \theta. v = \Theta. (u \times v), \]
for arbitrary vectors \( u \) and \( v \) in which \( \Theta \) is a vector valued parameter defined as follows [3]:
\[ \Theta := (\theta_{23}, \theta_{31}, \theta_{12}). \]
For example for the bound state of two electrons \((c_i = 1, i = 1, 2)\), and a nucleus \((i = 3)\), e.g. the Helium atom, \( H_{NC}^I \) takes the following form,
\[ H_{NC}^I = \frac{Z e^2}{2} \kappa \theta. \left( \frac{\ell_1}{|r_1|^3} + \frac{\ell_2}{|r_2|^3} \right) + \frac{e^2}{2} \theta. \ell \left( \frac{\ell}{|r|^3} + \frac{c_3 \theta}{|r|^3} \right) + \frac{Z e^2}{2} c_3 \theta. \left( \frac{r_1 \times p_2}{|r_1|^3} + \frac{r_2 \times p_1}{|r_2|^3} \right), \]
where \( r = r_1 - r_2, \ell = r \times (p_1 - p_2) \) and \( \ell_i = r_i \times p_i, i = 1, 2 \). One should note that these terms result in energy shifts of order \( \alpha^4 \) (tree level corrections) and they should be added to the terms with the same order of \( \alpha \) in \( H \). For example, the second term of Eq.(37) changes the spin-other-orbit interaction term [16] to
\[ u_{s-o-o}^{nc} = -\frac{\alpha}{2m^2} \left[ (\sigma_1 + m^2 \theta) \cdot \left( \frac{r \times p_2}{|r|^3} \right) + 1 \leftrightarrow 2 \right]. \]
It is interesting to note that if one assign a non-commutativity parameter with opposite sign relative to that of electron to the nucleus, i.e. $c_3 = -1$ \[2\], then only the first term in Eq.(37) vanishes and that equation reduces to

$$
H_I^{NC} = \frac{e^2}{2} \frac{\Theta \ell}{|r|^3} + \frac{Ze^2}{2} c_3 \Theta \left( \frac{r_1 \times p_2}{|r_1|^3} + \frac{r_2 \times p_1}{|r_2|^3} \right). \tag{39}
$$

Consequently, also in this case, there exist three level corrections for Helium atom.

To calculate the energy shift resulted by $H_I^{NC}$ given in Eq.(37), we consider the state $(1s)(np)$ and assume the electron labeled $i = 1$ as inner electron and that labeled $i = 2$ as outer one. Since the inner electron is in the ground state and the outer electron in an excited state, we can take an unsymmetrized wave function as $u_0(r_1)u_n(r_2)$ in which $u_i$, are approximated by Hydrogen-like wave functions. It is known that the error committed in leaving out symmetrization is of the order of the ratio of the radii of the orbits \[17\]. Consequently the third term (and similar expressions in the second term) in Eq.(37) has no contribution in the energy shift as a consequence of their odd parity. Therefore,

$$
\delta E = \langle 0n | H_I^{NC} | 0n \rangle \\
\approx \frac{e^2}{2} (\kappa Z - 1) \Theta \langle 0n | \frac{\ell_2}{|r_2|^3} | 0n \rangle \\
= f(n, l) \frac{\Theta |\alpha^4}{\lambda_e^2} m, \quad m = -\ell_2, \cdots, \ell_2, \tag{40}
$$

where

$$
f(n, l) = \frac{(\kappa Z - 1)(Z - 1)^3}{n^3(2\ell_2 + 1)(\ell_2 + 1)\ell_2}, \tag{41}
$$

and $\lambda_e$ is the Compton wave length of the electron. In the last equality of Eq.(40) we have assumed $\Theta = |\Theta| \hat{z}_2$. The above calculations are reasonable at least if the outer electron is in a highly excited state. Of course these terms can be calculated more accurately using the method of reference \[18\]. Eq.(40) gives a Zeeman effect caused by non-commutativity. Since the reported uncertainties on the experimental values of atomic levels $2P$ in Helium atom are about 1 kHz, from Eq.(40) one can see that such an uncertainty gives an upper bound $|\Theta|/\lambda_e^2 \sim 10^{-9}$ or $|\Theta| \sim 10^{-33}m^2$.

**Summary**

Although in first sight it seems that Helium atom is not suitable to be considered as a test on non-commutativity but we found that the upper bound on $|\Theta|$ obtained in such a system is even more accurate than that expected for Hydrogen atom and positronium.
For example it is shown that there is not any correction at the order $\alpha^4$ for the hyper fine splitting of positronium and the correction to the energy shift are of order $\frac{|\Theta|}{\lambda^2} \alpha^6$ which gives an upper bound $\frac{|\Theta|}{\lambda^2} \sim \frac{10^{-9}}{\alpha^2} \sim 10^{-5}$ for 1 kHz uncertainty in experiment [4]. The atomic transition $(2^3P_1) \rightarrow (2^3P_2)$ in Helium atom is measured with an accuracy 1kHz [11]. Eq.(40) shows that the states $(2^3P_1)$ and $(2^3P_2)$ are split themselves and the energy intervals are of order $\frac{|\Theta|}{\lambda^2} \alpha^4$. Thus the upper bound on $|\Theta|$ obtained here is $\frac{|\Theta|}{\lambda^2} \sim 10^{-9}$.

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