The paper is concerned with the derivation and analysis of nonoverlapping domain decomposition for heterogeneous, anisotropic diffusion problems discretized by the finite element cell-centered (FECC) scheme. Differently from the standard finite element method (FEM), the FECC method involves only cell unknowns and satisfies local conservation of fluxes by using a technique of dual mesh and multipoint flux approximations to construct the discrete gradient operator. Consequently, if the domain is decomposed into nonoverlapping subdomains, the transmission conditions (on the interfaces between subdomains) associated with the FECC scheme are different from those of the standard FEM. However, the substructuring procedure as well as the Neumann-Neumann type preconditioner can be adapted to the domain decomposition-based FECC method naturally. Convergence analysis of a preconditioned iterative algorithm, namely the Dirichlet-Neumann to Neumann-Neumann algorithm, associated with the discrete FECC interface problem is the main focus of this work. Two dimensional numerical results for two subdomains with conforming meshes demonstrate that the preconditioned iterative algorithm converges independently of the mesh size and the coefficient jump.

KEYWORDS:
nonoverlapping domain decomposition, heterogeneous anisotropic coefficients, cell-centered schemes, finite elements, Steklov-Poincaré operator, Neumann-Neumann preconditioner

1 INTRODUCTION

Domain decomposition methods have received great attention from researchers in recent decades due to the strong development of parallel computer architectures and multiprocessor supercomputer designs. The idea is to decompose the domain of calculation into several subdomains, then instead of solving a problem defined on the whole domain, we solve the subproblems defined on the subdomains and couple them through the use of well-chosen transmission conditions on the interfaces between subdomains. This technique is efficient in the sense that it reduces the size of the problem and takes advantage of using parallel computing to solve subdomain problems on different processors in parallel. It is also well-adapted to applications in which the domain of calculation is a union of different subdomains with different physical properties (for instance, the simulation of aircraft, far field simulations of underground nuclear waste disposal, ocean-atmosphere coupling in climate modeling, etc.). There is a large amount of research and numerical algorithms using domain decomposition techniques for different types of linear and nonlinear partial differential equations (see [1,2] and the references therein). Based on physical transmission conditions, a class of nonoverlapping domain decomposition methods is defined by using the Steklov-Poincaré-type operators. These operators
were introduced for stationary problems as natural mathematical tools for analyzing domain decomposition algorithms for both homogeneous and heterogeneous problems. In particular, see for a thorough study of domain decomposition for finite element discretizations of second-order elliptic problems. The convergence of an iterative procedure associated with the discrete counterpart of any Steklov-Poincaré operator (namely, the Schur complement matrix) is accelerated by a use of the Neumann-Neumann preconditioner, a local preconditioner defined by solving Neumann boundary problems in the subdomains. For a decomposition into many subdomains, a technique called balancing domain decomposition was introduced and analyzed in for finite elements, and in for mixed finite elements. Extensions of Steklov-Poincaré operators to parabolic problems were given in for uniform time steps, and in with mixed formulations and with primal formulations for nonconforming time steps in the subdomains.

The FECC method is a numerical scheme which has been recently introduced and analyzed in for two- and three-dimensional diffusion problems. Unlike the standard FEM method which fails to give accurate approximations to problems with discontinuous coefficients, the FECC method can be applied to heterogeneous, anisotropic diffusion problems on general (possibly distorted) meshes. Based on a technique of dual mesh and multipoint flux approximations, the scheme is cell-centered and satisfies local continuity of fluxes. Rigorous convergence analysis is given in and numerical results show that on the same primal mesh, the FECC scheme gives more accurate solutions than those by the FEM, the mixed finite volume method (MFV), the mimetic finite difference method (MFD), the discrete duality finite volume method (DDFV) and the SUSHI method. An extension of the FECC scheme, namely the staggered cell-centered finite element method (SC-FEM), to two- and three-dimensional linear elasticity problems has been studied. The SC-FEM is based on a mixed pressure-displacement formulation and is shown to be stable and convergent with low-order (P0-P1) approximations for the pressure and the displacement.

The aim of this work is to develop and analyze nonoverlapping domain decomposition for the FECC discretization of the diffusion problems with discontinuous, anisotropc coefficients. Due to the specific construction of the FECC discrete gradient operator, the transmission conditions associated with the FECC method are essentially different from those of the FEM. In particular, in addition to the continuity of the nodal unknowns and weak fluxes on the interface, extra transmission conditions representing the continuity of strong fluxes are introduced. These conditions are required to obtain the equivalence between the discrete multidomain problem and the discrete monodomain problem. We generalize the ideas of the discrete Steklov-Poincaré operator to derive a substructuring method associated with the FECC multidomain problem, namely the Dirichlet-Neumann to Neumann-Neumann method (instead of the classical Dirichlet-to-Neumann method). A discrete interface problem is formulated and is solved iteratively. Once the interface unknowns are found, one can easily recover the solution in each subdomain. In addition, we construct a generalization of the Neumann-Neumann preconditioner with weights for this interface problem and perform preconditioned Richardson iteration. Our main result is the proof of the convergence of the preconditioned iterative algorithm for the case of two subdomains. We remark that the interface operator in the proposed method is no longer symmetric as in the case of FEM and the convergence analysis in this situation is distinct from that of FEM. The proof can be easily extended to the case of strip subdomains. For the case of multiple subdomains with cross points, the method can be generalized based on conventional coupling at the cross points together with a coarse problem to remove subdomain singularities (when the preconditioner is performed on a floating subdomain) as well as to enhance the scalability when the number of subdomains increases. However, this subject is beyond the scope of this paper and will be discussed elsewhere.

For an open, bounded domain \( \Omega \) in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega \), we consider the second-order elliptic problem:

\[
-\text{div}(\Lambda(x)\nabla u(x)) = f(x) \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \Lambda : \Omega \rightarrow \mathbb{R}^{2 \times 2} \) is a symmetric, positive definite tensor, and its eigenvalues are bounded in \( \left[ \lambda_1, \lambda_2 \right] \), \( \lambda_1, \lambda_2 > 0 \); the source term \( f \) is a function in \( L^2(\Omega) \). For simplicity, homogeneous Dirichlet boundary conditions are imposed. The analysis given below can be extended to other types of boundary conditions as in Chapter 1, Section 1.4. The weak form of problem (1) is given by:

Find \( u \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} (\Lambda(x)\nabla u(x)) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx, \quad \forall v \in H^1_0(\Omega).
\]

It is well-known (see, for instance, Chapter 1) that under the assumptions made above, problem (2) has a unique solution \( u \) in \( H^1_0(\Omega) \).
In the next section, we introduce the FECC scheme for the discretization of problem (2). In Section 3, we derive the discrete multidomain problem using conforming decomposition into two nonoverlapping subdomains. Section 4 contains our main results which are the formulations of the Dirichlet-Neumann to Neumann-Neumann method and its interface problem, and the convergence proof of the associated iterative algorithm. In Section 5, numerical results confirm theoretical analysis are presented. Finally, detailed calculations of the block stiffness matrices used in the convergence proof are given in Appendix A.

2 | THE FECC FRAMEWORK

In this section, we recall the derivation of the FECC scheme for problem (2) with heterogeneous, anisotropic coefficients: we first describe the construction of the meshes, then define the discrete gradient which satisfies local continuity of fluxes; finally we derive a linear algebraic system associated with (2).

2.1 | The meshes

For completeness, we recall the construction of the two-dimensional meshes in the FECC scheme as presented in\cite{18,19,29} (see \cite{19} Chapter 3 for the scheme in three dimensions). For a polygonal domain $\Omega \subset \mathbb{R}^2$, we consider a triangulation $T_h$ of $\Omega$:

$$\Omega = \bigcup_{K \in T_h} K.$$

We assume that each element $K \in T_h$ is a star-shaped polygon in which we choose a point $C_K \in \text{int}(K)$ and call it the mesh point of $K$. Throughout the paper, we refer to $T_h$ as the primal mesh. Next, we define the dual mesh $T_h^*$ and the dual sub-mesh $T_h^{**}$. For this purpose, we assume that the line joining two mesh points of any two neighboring elements is inside $\Omega$ and it intersects the common edge of the two elements. The latter assumption is necessary to define the scheme for heterogeneous problems (see\cite{19}).

The dual mesh $T_h^*$ is constructed from the primal mesh in a way that each dual control volume of $T_h^*$ corresponds to a vertex of $T_h$. Denote by $\mathcal{N}$ the set of all nodes or vertices of $T_h$:

$$\mathcal{N} := \{ P : P \text{ is a vertex of } T_h \}.$$

For each $P \in \mathcal{N}$, denote by

$$\mathcal{K}_p := \{ K \in T_h : K \text{ shares the vertex } P \},$$

the set of primal elements that have $P$ as their vertex. We consider two cases (see Figure 1):

(a) If $P$ is an interior vertex, we obtain the dual control volume $M_p \in T_h^*$ associated with the vertex $P$ by connecting the mesh points of neighboring elements in $\mathcal{K}_p$.

(b) If $P$ is on the boundary $\partial \Omega$, denote by $\sigma_{p,1}$ and $\sigma_{p,2}$ the two edges on the boundary that have $P$ as their vertex. Let $K_{p,1}$ and $K_{p,2}$ be two (same or different) elements in $\mathcal{K}_p$ such that $\sigma_{p,1} \subset \partial K_{p,1}$ and $\sigma_{p,2} \subset \partial K_{p,2}$. The dual control volume $M_p$ is defined by joining mesh points of neighboring elements in $\mathcal{K}_p$ and the mesh point of $K_{p,1}$ (and $K_{p,2}$) with a chosen interior point (e.g. the midpoint) of $\sigma_{p,1}$ (and $\sigma_{p,2}$ respectively). Note that in this case $M_p$ has $P$ as its vertex as well.

The collection of all $M_p$ defines a dual mesh $T_h^*$ such that

$$\Omega = \bigcup_{p \in \mathcal{N}} M_p.$$
In order to obtain the discrete variational formulation associated with problem (2), we shall define a projection operator \( \Phi \) that projects functions from \( L^2(\Omega) \) to \( \mathbb{P}_0(\mathcal{T}_h^*) \). Finally, we construct the dual sub-mesh \( \mathcal{T}_h^{**} \) as a triangular subgrid of the dual grid: for an element \( M \in \mathcal{T}_h^* \), we construct elements of \( \mathcal{T}_h^{**} \) by connecting \( C_M \) to all vertices of \( \mathcal{T}_h^* \) (see Figure 2):

\[
\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h^{**}} T.
\]

Let \( \mathcal{N}^{**} \) be the set of nodes of elements of \( \mathcal{T}_h^{**} \). We have the following remark.

\textbf{Remark 1.} By construction, we have that

(a) for all interior triangular elements \( T \in \mathcal{T}_h^{**} \) (i.e. \( \partial T \cap \partial \Omega = \emptyset \)), there exits two primal elements \( K \) and \( L \in \mathcal{T}_h \) such that \( T \cap K \neq \emptyset \) and \( T \cap L \neq \emptyset \).

(b) \( \mathcal{N}^{**} \) consists of three sets \( C, C^* \) and \( \mathcal{N}_{d\Omega}^{**} \) containing mesh points of primal elements, mesh points of dual control volumes and points lying on the boundary respectively:

\[
\mathcal{N}^{**} = C \cup C^* \cup \mathcal{N}_{d\Omega}^{**},
\]

where \( C := \{ C_K, \forall K \in \mathcal{T}_h \} \), \( C^* := \{ C_M, \forall M \in \mathcal{T}_h^* \} \) and \( \mathcal{N}_{d\Omega}^{**} := \{ P \in \mathcal{N}^{**} \text{ such that } P \in \partial \Omega \} \).

For each primal element \( K \in \mathcal{T}_h \), we denote by \( \Lambda_K \) the average value of tensor \( \Lambda \) on \( K \):

\[
\Lambda_K = \frac{1}{|K|} \int_K \Lambda(x) dx.
\]

We are interested in the heterogeneous, anisotropic case where \( \Lambda \) is discontinuous across the primal elements, i.e.: \( \Lambda_K \neq \Lambda_L \) for any \( K, L \in \mathcal{T}_h \), \( K \neq L \).

The FECC scheme follows the idea of the standard finite element method applying on the dual sub-mesh and we seek for an approximate solution of problem (2) by finding its values at all nodes \( P \in \mathcal{N}^{**} \). Thus we define by \( X_h \) the set of all vectors \( u_h := (u_P)_{P \in \mathcal{N}^{**}} \), where \( u_P \) is the approximate value of the solution \( u \) at the node \( P \in \mathcal{N}^{**} \):

\[
\mathcal{X}_h = \{ u_h = (u_P)_{P \in \mathcal{N}^{**}}, u_P \in \mathbb{R} \}.
\]

Due to Remark 1(b), we have that

\[
\mathcal{X}_h = (u_P)_{P \in \mathcal{N}^{**}} = (u_{C_K})_{K \in \mathcal{T}_h^*}, (u_{C_M})_{M \in \mathcal{T}_h^{**}}, (u_P)_{P \in \mathcal{N}_{d\Omega}^{**}}.
\]

To simplify the notation, we rewrite (4) as

\[
\mathcal{X}_h = (u_P)_{P \in \mathcal{N}^{**}} = (u_{K})_{K \in \mathcal{T}_h^*}, (u_M)_{M \in \mathcal{T}_h^{**}}, (u_P)_{P \in \mathcal{N}_{d\Omega}^{**}}.
\]

In addition, to handle Dirichlet boundary conditions, we need to define the following subset of \( \mathcal{X}_h \):

\[
\mathcal{X}_h^0 = \{ u_h \in \mathcal{X}_h : u_P = 0, \forall P \in \mathcal{N}_{d\Omega}^{**} \}.
\]

In order to obtain the discrete variational formulation associated with problem (2), we shall define a projection operator \( \Phi(u_h) \) and the discrete gradient \( \nabla_h u_h \) for \( u_h \in \mathcal{X}_h \).
The projection operator and the discrete gradient

2.2 The projection operator and the discrete gradient

The two operators are defined by their restrictions to each element of \( \mathcal{T}_h^{**} \). In particular, the projection operator \( \Phi(u_h) \) is a function in \( L^2(\Omega) \) and it is continuous piecewise linear on each element \( T \in \mathcal{T}_h^{**} \); and the discrete gradient is defined in a way to enforce mass conservation in each element \( T \in \mathcal{T}_h^{**} \) when the coefficient \( \Lambda \) is discontinuous.

We consider a triangle \( T = (C_M C_K C_L) \in \mathcal{T}_h^{**} \) where \( K, L \) are two primal elements, \( K, L \in \mathcal{T}_h \), and \( M \) a dual control volume, \( M \in \mathcal{T}_h^* \) (see Figure 3). Denote by \( \sigma \equiv C_M C_K \) the common edge of \( K \) and \( L \) and \( C_\sigma \in \sigma \) the intersecting point between \( C_K C_L \) and \( \sigma \). For any \( u_h \in X_h \), the restriction of \( \Phi(u_h) \) to \( T \), denoted by \( \Phi_T(u_h) \), is a continuous function and it is linear on each of the two sub-triangles \( (C_M C_K C_\sigma) \) and \( (C_M C_L C_\sigma) \).

Let \( u_M^* \), a temporary unknown to be defined later, be an approximation of \( u \) at \( C_\sigma \) seeing from \( M \). In addition, denote by \( n_{C_M C_K}^K \), \( n_{C_K C_L}^K \) and \( n_{C_M C_L}^C \) the outward normal vectors of the triangle \( (C_M C_K C_\sigma) \) such that the lengths of these vectors are equal to the segments \( C_M C_\sigma \), \( C_K C_\sigma \) and \( C_M C_\sigma \) respectively (see Figure 3). We also denote by \( m_{(C_M C_K C_\sigma)} \) the measure of triangle \( (C_M C_K C_\sigma) \). Note that \( n_{C_M C_K}^K + n_{C_K C_L}^K + n_{C_M C_L}^L = 0 \).

For any vector \( u_h \in X_h \), the projection operator \( \Phi(u_h) \) and the discrete gradient \( \nabla_X u_h \) restricted to \( T \) are defined as follows:

(i) On triangle \( (C_M C_K C_\sigma) \), we have

\[
\Phi_T(u_h)|_{(C_M C_K C_\sigma)}(x) = \begin{dcases} 
M & \text{if } x = C_M, \\
K & \text{if } x = C_K, \\
M^* & \text{if } x = C_\sigma.
\end{dcases}
\]

Now using multi-point flux approximations, we define the restriction of \( \nabla_X u_h \) to \( (C_M C_K C_\sigma) \) as

\[
\nabla_X u_h|_{(C_M C_K C_\sigma)} = \frac{-u_M n_{C_M C_\sigma} - u_K n_{C_K C_\sigma} - u_{M^*} n_{C_M C_L}}{2m_{(C_M C_K C_\sigma)}}. \tag{5}
\]

Similarly, the restrictions of \( u_h \) and \( \nabla_X u_h \) to triangle \( (C_M C_L C_\sigma) \) are respectively:

\[
\Phi_T(u_h)|_{(C_M C_L C_\sigma)}(x) = \begin{dcases} 
M & \text{if } x = C_M, \\
L & \text{if } x = C_L, \\
M^* & \text{if } x = C_\sigma.
\end{dcases}
\]

and

\[
\nabla_X u_h|_{(C_M C_L C_\sigma)} = \frac{-u_M n_{C_M C_\sigma} - u_L n_{C_L C_\sigma} - u_{M^*} n_{C_M C_L}}{2m_{(C_M C_L C_\sigma)}}. \tag{6}
\]

(ii) We choose \( \alpha_M^* \) to strongly satisfy the continuity of the flux across \( C_M C_\sigma \):

\[
\Lambda_K \nabla_X u_h|_{(C_M C_K C_\sigma)} \cdot n_{C_M C_\sigma} + \Lambda_L \nabla_X u_h|_{(C_M C_L C_\sigma)} \cdot n_{C_M C_\sigma} = 0. \tag{7}
\]

Substituting (5) and (6) into (7), we obtain

\[
(\beta_{1,M} u_M + \beta_K u_K + \beta_{1,M^*} u_{M^*}) + (\beta_{2,M} u_M + \beta_L u_L + \beta_{2,M^*} u_{M^*}) = 0,
\]

\[
(\beta_{1,L} u_M + \beta_K u_K + \beta_{1,M^*} u_{M^*}) + (\beta_{2,L} u_M + \beta_L u_L + \beta_{2,M^*} u_{M^*}) = 0.
\]

FIGURE 3 Left: An element of the dual sub-grid \( T = (C_M C_K C_L) \); Right: Outward normal vectors of each sub-triangle.
where
\[
\begin{align*}
\beta_{1,M} &= -\frac{(n^K_{cu,c_k})^T\Lambda_k n_{cu,c_k}}{2m(c_u c_k c_c)}, \\
\beta_K &= -\frac{(n^K_{cu,c_c})^T\Lambda_k n^K_{cu,c_k}}{2m(c_u c_k c_c)}, \\
\beta_{1,\sigma} &= -\frac{(n^K_{cu,c_c})^T\Lambda_k n_{cu,c_k}}{2m(c_u c_k c_c)}, \\
\beta_{2,M} &= -\frac{(n^L_{cu,c_k})^T\Lambda_k n_{cu,c_c}}{2m(c_u c_k c_c)}, \\
\beta_L &= -\frac{(n^L_{cu,c_c})^T\Lambda_k n^L_{cu,c_k}}{2m(c_u c_k c_c)}, \\
\beta_{2,\sigma} &= -\frac{(n^L_{cu,c_c})^T\Lambda_k n_{cu,c_k}}{2m(c_u c_k c_c)}.
\end{align*}
\]
Assume that \(\beta_{1,\sigma} + \beta_{2,\sigma} \neq 0\), we deduce from (8) that
\[
u^M_{\sigma} = \tilde{\beta}_K u_K + \tilde{\beta}_L u_L + (\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}) u_M.
\]
where
\[
\tilde{\beta}_K = \frac{-\beta_K}{\beta_{1,\sigma} + \beta_{2,\sigma}}, \quad \tilde{\beta}_L = \frac{-\beta_L}{\beta_{1,\sigma} + \beta_{2,\sigma}}, \quad \tilde{\beta}_{1,M} = \frac{-\beta_{1,M}}{\beta_{1,\sigma} + \beta_{2,\sigma}}, \quad \tilde{\beta}_{2,M} = \frac{-\beta_{2,M}}{\beta_{1,\sigma} + \beta_{2,\sigma}}.
\]

Remark 2. For each internal edge \(\sigma \equiv C_M C_M\) of the primal elements, there are two values of \(u\) at \(C_\sigma\), one seeing from \(M\) \(\left(u^M_{\sigma}\right)\) and another from \(\hat{M}\) \(\left(u^M_{\sigma}\right)\). As for \(u^M_{\sigma}\) in (8), \(u^M_{\sigma}\) can be expressed as a linear combination of \(u_{\hat{M}}, u_K\) and \(u_L\).

For a general mesh and \(\Lambda_K \neq \Lambda_L\), the two different values of \(u\) at \(C_\sigma\) are not equal: \(u^M_{\sigma} \neq u^M_{\hat{M}}\). For homogeneous Dirichlet boundary conditions, \(u^M_{\sigma} = 0\) if \(C_\sigma \in \partial\Omega\).

Substituting (9) into (5) and (6), we conclude that the discrete gradient \(\nabla u_h\) restricted to triangle \(T = (C_M C_K C_L) \in T^*_h\) depends linearly on the three nodal values \(u_M, u_K\) and \(u_L\):
\[
\nabla u_h|_{(C_M C_K C_L)} = \frac{-u_K\hat{n}^K_{(C_M C_K C_L)} - u_L\hat{n}^L_{(C_M C_K C_L)} - u_M\hat{n}^M_{(C_M C_K C_L)}}{2m(c_u c_k c_c)},
\]
\[
\nabla u_h|_{(C_M C_K C_L)} = \frac{-u_K\hat{n}^K_{(C_M C_K C_L)} - u_L\hat{n}^L_{(C_M C_K C_L)} - u_M\hat{n}^M_{(C_M C_K C_L)}}{2m(c_u c_k c_c)},
\]
where
\[
\hat{n}^K_{(C_M C_K C_L)} = n^K_{cu,c_k} + \hat{\beta}_K n_{cu,c_k}, \quad \hat{n}^L_{(C_M C_K C_L)} = n^L_{cu,c_k} + \hat{\beta}_L n_{cu,c_k}, \quad \hat{n}^M_{(C_M C_K C_L)} = n^M_{cu,c_k} + \hat{\beta}_M n_{cu,c_k}.
\]

The discrete variational formulation associated with problem (2) is as follows:
Find \(u_h \in X^0_h\) such that
\[
\int_\Omega (\Lambda(x)\nabla u_h(x)) \cdot \nabla v_h(x) \, dx = \int_\Omega f(x)\Phi(v_h)(x) \, dx, \quad \forall v_h \in X^0_h.
\]

### 2.3 The linear algebraic system

To derive the linear algebraic system associated with (12), for each internal node \(Q \in (N^{**} \setminus N^{**}_{\partial\Omega})\), we choose \(v_h = v^Q_P = (v^Q_P)_{P \in N^{**}} \in X^0_h\) such that
\[
v^Q_P = \begin{cases} 1 & \text{if } P \equiv Q, \\ 0 & \text{if } P \neq Q, \end{cases}
\]
and obtain
\[
\int_\Omega (\Lambda(x)\nabla u_h(x)) \cdot \nabla v^Q_P(x) \, dx = \int_\Omega f(x)\Phi(v^Q_P)(x) \, dx, \quad \forall Q \in (N^{**} \setminus N^{**}_{\partial\Omega}).
\]
in which the discrete gradient depends only on the nodal values \(u_P, P \in N^{**}\) (cf. formula (11)).

To derive a matrix form of (14), we proceed as in pp.12-14 by first choosing \(v_h = v^C_M\) for each \(M \in T^*_h\) in (14) and obtain the linear system:
\[
Du_h|_{r^*_M} + Eu_h|_{r^*_M} = F^*,
\]
where \( u_h|_{r^*} := (u_M)_{M \in T^*_h} \) and \( u_h|_{\Gamma} := (u_K)_{K \in \Gamma_h} \), \( D \) is a symmetric, positive definite, square matrix and \( F^* \) a column matrix depending on \( f \). Next, we take \( v_h = v_h^{CE} \) for each \( K \in \Gamma_h \):

\[
Mu_h|_{r^*} + Nu_h|_{\Gamma} = F,
\]

where \( N \) is a symmetric, square matrix, \( F \) a column matrix depending on \( f \) and \( M \) is the transpose of \( E \). Consequently, we obtain the matrix system associated with (14) as follows:

\[
\begin{pmatrix}
D & E \\
M & N
\end{pmatrix}
\begin{pmatrix}
u_h|_{r^*} \\
u_h|_{\Gamma}
\end{pmatrix} = \begin{pmatrix} F^* \\
F
\end{pmatrix}.
\]

Since inverse matrix of \( D \) exists (see \( \text{[13]} \) p.14), one can compute \( u_h|_{r^*} \) from the first equation of (15):

\[
u_h|_{r^*} = D^{-1}(F^* - Eu_h|_{\Gamma}).
\]

Substituting this into the second equation of (15), we obtain the following linear system involving only primal cell unknowns:

\[
(N - MD^{-1}E)u_h|_{\Gamma} = F - MD^{-1}F^*.
\]

The matrix \( A := N - MD^{-1}E \) is a variant of the stiffness matrix and is symmetric and positive definite on general meshes\( \text{[13]} \).

We also recall Corollary 5.4 in \( \text{[13]} \) that the FECC scheme is convergent, that is to say, \( \Phi(u_h) \) converges to the exact solution \( u_{\text{exact}} \) of problem (2) and \( \nabla_h u_h \) converges to \( \nabla u_{\text{exact}} \) as \( h \) tends to 0, with

\[
h = \sup\{h_T, \text{ the diameter of the triangle } T, \ T \in T_h^{**}\}.
\]

### 3 CONFORMING, NONOVERLAPPING DOMAIN DECOMPOSITION

Using nonoverlapping domain decomposition, we formulate a discrete multidomain problem corresponding to the discrete monodomain problem (14). The formulation is derived for the case of two subdomains for simplicity and can be straightforwardly extended to multiple strip subdomains. We first introduce some notation.

#### 3.1 Notation

We consider a triangulation \( T_h \) of \( \Omega \) and a conforming decomposition of \( \Omega \) into two nonoverlapping subdomains \( \Omega_1 \) and \( \Omega_2 \) (the analysis can be extended to many subdomain case with strip substructures). Denote by \( \Gamma \) the interface between the two subdomains: \( \Gamma = \partial \Omega_1 \cap \partial \Omega_2 \cap \Omega \). Let \( T_{h,i} \), \( i = 1, 2 \), be the triangulation of \( \Omega_i \) such that \( T_{h,i} \) is a subset of \( T_h \). We shall construct the dual mesh \( T_{h,i}^* \) and the dual sub-mesh \( T_{h,i}^{**} \) of the subdomains from those of the monodomain, \( T_h^* \) and \( T_h^{**} \). Since each dual control volume corresponds to a vertex of the primal mesh, we distinguish two cases (see Figure 4):

(a) If the vertex \( P \) of an element of \( T_{h,i} \) does not belong to \( \Gamma \), then its control volume \( M_p \in T_{h,i}^* \) coincides with the control volume \( M_p \in T_h^* \). Hence, the triangular elements of \( T_{h,i}^{**} \) associated with \( M_p \) are those of \( T_h^{**} \) associated with \( M_p \).

(b) Otherwise if \( P \in \Gamma \) (note that \( \Gamma \) now is a part of the boundary of \( \Omega_i \)), its control volume \( M_p \in T_{h,i}^* \) is the intersection of the control volume \( M_p \in T_h^* \) and \( \partial \Omega_i \). The triangular elements of \( T_{h,i}^{**} \) associated with \( M_p \) is then defined as in Subsection 2.1.

The dual sub-meshes \( T_{h,1}^{**} \) and \( T_{h,2}^{**} \) are matching on the interface \( \Gamma \) and they are NOT subsets of \( T_h^{**} \). Denote by \( N_i^{**}, \ i = 1, 2 \), be the set of all vertices of elements of \( T_{h,i}^{**} \) (note that \( N_1^{**} \) and \( N_2^{**} \) are not subsets of \( N^{**} \)). As for the monodomain case (cf. Remark 1(b)), we can decompose \( N_i^{**} \) into three sets \( C_i, C_i^+ \) and \( N_{i,\partial \Omega}^{**} \) as follows:

\[
N_i^{**} = C_i \cup C_i^+ \cup N_{i,\partial \Omega}^{**},
\]

where \( C_i := \{K, \forall K \in T_{h,i}\} \), \( C_i^+ := \{M, \forall M \in T_{h,i}^*\} \) and \( N_{i,\partial \Omega}^{**} := \{P \in N_i^{**} \text{ such that } P \in \partial \Omega_i\} \).

We denote by \( N_1^{**} \) the set of vertices of elements of \( T_{h,1}^{**} \), \( i = 1, 2 \), that belong to \( \Gamma \) and by \( E_1^{**} \) the set of edges of elements of \( T_{h,1}^{**} \) that lie on \( \Gamma \). Due to the decomposition, in addition to the nodes of the primal mesh \( T_h \) lying on \( \Gamma \) (magenta circled points in Figure 4), \( N_1^{**} \) also consists of extra points (magenta squared points in Figure 4) resulting from case (b). We then write

\[
N_1^{**} = (N_{1,\Gamma_\circ}^{**} \cup N_{1,\Gamma\cap}^{**}) \subseteq N_{i,\partial \Omega}^{**}, \ i = 1, 2.
\]
Note that the points in $\mathcal{N}^{**}_{1,\Omega}$ play the same role as $C_\sigma$ in the construction of the discrete gradient (Subsection 2.2) and $\mathcal{N}^{**}_{1,\varnothing} \cap \mathcal{N}^{**} = \emptyset$. We also denote by $C^i_\Gamma$ ($i = 1, 2$) the sets of mesh points of primal elements of $\Omega_i$ that have edges lying on $\Gamma$:

$$
C^i_\Gamma := \{ C_K, K \in T_{h,i}, \partial K \cap \Gamma \neq \emptyset \}, \ i = 1, 2. \tag{16}
$$

To derive a multidomain problem associated with (14), we need to introduce the space $G_h := \{ \Phi(v_h)|_{\Gamma_i} \mid \forall v_h \in X_h^0 \} \subset L^2(\Gamma)$, consisting of discontinuous, piecewise linear functions on $\Gamma$ (actually, a function in $G_h$ is continuous linear on each $e \in E^{**}_{1,\Sigma}$, cf. Subsection 2.2). The vector set associated with $G_h$, denoted by $G_{\mathcal{h}}$, is defined by:

$$
G_{\mathcal{h}} := \left\{ u_\Gamma = \left( (u_p)_{p \in \mathcal{N}^{**}_{1,\Sigma}}, \left( u^M_\sigma, u^M_\sigma \right)_{C_\sigma \in \mathcal{N}^{**}_{1,\varnothing} \cap \mathcal{N}^{**}_1} \right), \text{ for any } u_h \in X_h^0 \right\}. \tag{17}
$$

Recall that due to the construction of the discrete gradient, there are two different unknowns at $C_\sigma \in \mathcal{N}^{**}_{1,\varnothing}$: $u^M_\sigma \neq u^M_\sigma$, where $M$ and $\tilde{M}$ are dual control volumes whose mesh points $C_M$ and $C_{\tilde{M}}$ belong to $\mathcal{N}^{**}_{1,\varnothing}$ (see Remark 2). We shall make use of such notation for the rest of the paper. We also introduce the following sets:

$$
X_{h,i} := \left\{ u_{h,i} = (u_{i,p})_{p \in \mathcal{N}^{**}_{1,i}}, u_{i,p} \in \mathbb{R} \right\},
$$

$$
X^0_{h,i} := \left\{ u_{h,i} \in X_{h,i} : u_{i,p} = 0, \forall P \in \left( \mathcal{N}^{**}_{1,i} \setminus \mathcal{N}^{**}_{1,\varnothing} \right) \right\}.
$$

Finally, we define the projection $\Phi_i$, the discrete gradient $\nabla_{\Lambda_i}$ and the right-hand side data $f_i$ for $i = 1, 2$, as the restrictions of $\Phi$, $\nabla_{\Lambda}$ and $f$ to $\Omega_i$ respectively.

### 3.2 A discrete multidomain problem

With the above notation, the discrete multidomain problem equivalent to the monodomain problem (14) consists of:

1. Solving in the subdomains the following problems:

   Find $u_{h,i} \in X^0_{h,i} \cap \mathcal{N}^{**}_{1,i}$ such that

   $$
   \int_{\Omega_i} \left( \Lambda_i \nabla_{\Lambda_i} u_{h,i} \right) \cdot \nabla_{\Lambda_i} v^Q_{h,i} \, dx - \int_{\Gamma_i} \left( \Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i \right) \Phi_i \left( v^Q_{h,i} \right) \, d\gamma = \int_{\Omega_i} f_i \Phi_i \left( v^Q_{h,i} \right) \, dx, \quad \forall Q \in \mathcal{N}^{**}_{1,i} \setminus \mathcal{N}^{**}_{1,\varnothing}, \tag{17}
   $$

   for $i = 1, 2$, where $v^Q_{h,i} = \left( v^Q_{i,p} \right)_{p \in \mathcal{N}^{**}_{1,i}} \in X^0_{h,i} \cap \mathcal{N}^{**}_{1,i}$, for $Q \in \mathcal{N}^{**}_{1,i}$ and $i = 1, 2$, is defined as

   $$
u^Q_{i,p} = \begin{cases} 1 & \text{if } P \equiv Q, \\ 0 & \text{if } P \not\equiv Q, P \not\in \mathcal{N}^{**}_{1,\varnothing}. \end{cases} \tag{18}
   $$

2. Together with three transmission conditions on $\Gamma$ expressing respectively...
Recall that the test vector \( \Phi_1(u_{h,1}) \rvert_\Gamma = \Phi_2(u_{h,2}) \rvert_\Gamma \).

or equivalently,

\[
\Phi_1(u_{h,1}) = \Phi_2(u_{h,2}),
\]

where \( u_{i,\Gamma} = \left( \begin{pmatrix} u_{i,p} \\ u_{i,s} \end{pmatrix} p \in N_{i,\Gamma}^{s*} \right) \) \( \in \mathcal{G}_h \), \( i = 1, 2 \).

The second transmission condition \((21)\) results from the construction of the discrete gradient in the FECC scheme and is used to determine the values of the solution at points \( C_{\phi_i,\Gamma} \in \mathcal{N}_{i,\Gamma}^{s*} \).

We now examine in more detail the third transmission condition \((22)\) in the context of the FECC scheme. One easily sees that

\[
\forall Q \in \mathcal{N}_{i,\Gamma}^{s*} \subset \mathcal{N}_{i,\Gamma}^{s*} : \quad \Phi \left( u^0 \right) \rvert_\Gamma \in \mathcal{G}_h \quad \text{and} \quad \Phi \left( u^0 \right) \rvert_\Gamma \neq 0.
\]

Recall that the test vector \( u^0 \in \mathcal{X}_h \) is defined as in \((19)\). Furthermore, because of the construction of the discrete gradient, we also have

\[
\forall Q \in \mathcal{C}_i^{\Gamma} : \quad \Phi \left( u^0 \right) \rvert_\Gamma \in \mathcal{G}_h \quad \text{and} \quad \Phi \left( u^0 \right) \rvert_\Gamma \neq 0,
\]

where the set \( \mathcal{C}_i^{\Gamma} \) is defined in \((16)\). Thus, the condition of flux continuity \((22)\) can be replaced by

\[
\int \left( \sum_{i=1}^{2} \Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i \right) \Phi \left( u^0 \right) \rvert_\Gamma d\gamma = 0, \quad \forall Q \in \mathcal{N}_{i,\Gamma}^{s*}, (24a)
\]

\[
\int \left( \Lambda_1 \nabla_{\Lambda,1} u_{h,1} \cdot n_1 \right) \Phi \left( u^0 \right) \rvert_\Gamma d\gamma = \int \left( \Lambda_2 \nabla_{\Lambda,2} u_{h,2} \cdot n_1 \right) \Phi \left( u^0 \right) \rvert_\Gamma d\gamma, \quad \forall Q \in \mathcal{C}_1^{\Gamma}, (24b)
\]

\[
\int \left( \Lambda_2 \nabla_{\Lambda,2} u_{h,2} \cdot n_2 \right) \Phi \left( u^0 \right) \rvert_\Gamma d\gamma = \int \left( \Lambda_1 \nabla_{\Lambda,1} u_{h,1} \cdot n_2 \right) \Phi \left( u^0 \right) \rvert_\Gamma d\gamma, \quad \forall Q \in \mathcal{C}_2^{\Gamma}. (24c)
\]

Here the projection \( \Phi \) is global (i.e., defined on the whole domain \( \Omega \)). In the following we will transform these equations into a form such that local projections \( \Phi_i, \ r = 1, 2, \) are used.

(a) For \( Q \in \mathcal{N}_{i,\Gamma}^{s*} \), by the definition of the projection \( \Phi \), we have that

\[
\Phi \left( u^0 \right) \rvert_\Gamma = \Phi_1 \left( u_{h,1} \right) \rvert_\Gamma = \Phi_2 \left( u_{h,2} \right) \rvert_\Gamma.
\]

(b) For fixed \( i \in \{1, 2\} \) and \( Q_i \in \mathcal{C}_i^{\Gamma} \) (thus \( Q_i \) is an interior node in \( \Omega_i \)),

\[
\Phi \left( u^0 \right) \rvert_\Gamma = \Phi_i \left( u_{h,i} \right) \rvert_\Gamma.
\]

Denote by \( j = (3 - i) \), we define an extension operator \( E_{h,j} \) from \( \mathcal{G}_h \) to \( X_{h,j} \) as follows: for \( \nu_{\Gamma} = \left( \nu_{P} \right) p \in \mathcal{N}_{\Gamma}^{s*} \in \mathcal{G}_h \), let

\[
E_{h,j}(\nu_{\Gamma}) = \nu_{h,j} := \begin{cases} \nu_{P} & \text{if } P \in \mathcal{N}_{\Gamma}^{s*} \\ 0 & \text{if } P \in \left( \mathcal{N}_{\Gamma}^{s*} \setminus \mathcal{N}_{\Gamma}^{s*} \right) \end{cases} \in X_{h,j},
\]

i.e. \( E_{h,j}(\nu_{\Gamma}) \) equals \( \nu_{\Gamma} \) at the nodes on the interface and vanishes at the internal nodes in \( \Omega_j \). Then

\[
\Phi \left( u^0 \right) \rvert_\Gamma = \Phi_j \left( E_{h,j}(u^0_{h,j}) \right) \rvert_\Gamma,
\]

where \( u^0_{h,i} := \left( \nu_{i,P} \right) p \in \mathcal{N}_{\Gamma}^{s*} \in \mathcal{G}_h \).
Using the relations established in (25) and (27) we can rewrite the transmission condition (24) as follows:

$$
\int_{\Gamma} \sum_{i=1}^{2} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \Phi_i \left( v_{h,i}^Q \right) \, d\gamma = 0, \quad \forall Q \in \mathcal{N}_{i,\partial}^{**}, \quad (28a)
$$

$$
\int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,1} \cdot n_1) \Phi_1 \left( v_{h,1}^Q \right) \, d\gamma = \int_{\Gamma} (\Lambda_2 \nabla_{\Lambda_2} u_{h,2} \cdot n_1) \Phi_1 \left( E_{h,2} v_{h,2}^Q \right) \, d\gamma, \quad \forall Q \in \mathcal{C}_{1}^F, \quad (28b)
$$

$$
\int_{\Gamma} (\Lambda_2 \nabla_{\Lambda_2} u_{h,2} \cdot n_2) \Phi_2 \left( v_{h,2}^Q \right) \, d\gamma = \int_{\Gamma} (\Lambda_1 \nabla_{\Lambda_1} u_{h,1} \cdot n_2) \Phi_2 \left( E_{h,1} v_{h,1}^Q \right) \, d\gamma, \quad \forall Q \in \mathcal{C}_{2}^F. \quad (28c)
$$

The last two equations are used as Neumann data on boundary $\Gamma$, which is needed to solve the subdomain problems. This is the essential difference between the FECC-based multidomain formulation and the standard domain decomposition formulations (see Chapters 1 and 2) and it is due to the discrete gradient constructed in the FECC scheme.

### 4 A DIRICHLET-NEUMANN TO NEUMANN-NEUMANN METHOD

From the multidomain problem (17) with the transmission conditions (20)-(21) and (28) and by using substructuring techniques, we derive an interface problem that can be solved iteratively. In particular, we imposed Dirichlet boundary conditions (cf. Equation (20)) and Neumann boundary conditions (cf. Equations (28b) and (28c)) on the interface to solve the subdomain problems, then we enforce the remaining transmission conditions (Equations (21) and (28)) to obtain the interface problem. We first introduce some notation and several operators. Let $\mathbf{y}_h$ be a vector in $G_h$ and $\mathbf{\eta}_{h,i}$ be a vector in $\mathbb{R}^{\text{card}(\mathcal{C}_i)}$ representing respectively the Dirichlet and Neumann boundary data on the interface for the problem in $\Omega_i$, $i = 1, 2$. We also denote by $\left( \mathbf{\eta}_{h,i} \right)_Q$ the component of $\mathbf{\eta}_{h,i}$ associated with the node $Q \in \mathcal{C}_i^\Gamma$.

Let $U_i^r$, $i = 1, 2$, be the solution operator that associates with the Dirichlet boundary data, Neumann boundary data and the right hand side $(\mathbf{y}_h, \mathbf{\eta}_{h,i}, f_i) \in G_h \times \mathbb{R}^{\text{card}(\mathcal{C}_i)} \times L^2(\Omega_i)$, the solution $u_{h,i} \in X_{h,i}^{0,\partial\Omega_i \cap \partial\Omega}$ of the subdomain problem:

$$
\int_{\Omega_i} (\Lambda_i \nabla_{\Lambda_i} u_{h,i}) \cdot \nabla_{\Lambda_i} v_{h,i}^Q \, dx - \int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \Phi_i \left( v_{h,i}^Q \right) \, d\gamma = \int_{\Omega_i} f_i \Phi_i \left( v_{h,i}^Q \right) \, dx, \quad \forall Q \in \mathcal{N}_{i,\partial}^{**} \setminus \mathcal{N}_{i,\partial\Omega_i},
$$

$$
\int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \Phi_i \left( v_{h,i}^Q \right) \, d\gamma = \left( \mathbf{y}_h \right)_Q, \quad \forall Q \in \mathcal{C}_i^\Gamma, \quad (29)
$$

The operator $U_i^r : (\mathbf{y}_h, \mathbf{\eta}_{h,i}, f_i) \mapsto u_{h,i} = U_i^r(\mathbf{y}_h, \mathbf{\eta}_{h,i}, f_i)$ is well-defined as the solution to (29) exists uniquely. We also make use of the following interface operators:

$$
P_{j}^{str} : u_{h,i} \mapsto \left( \int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \, d\gamma \right)_{Q \in \mathcal{N}_{i,\partial}^{**}}, \quad i = 1, 2,
$$

$$
P_{j}^{w} : u_{h,i} \mapsto \left( \int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \Phi_i \left( v_{h,i}^Q \right) \, d\gamma \right)_{Q \in \mathcal{N}_{i,\partial}^{**}}, \quad i = 1, 2,
$$

$$
P_{j}^{c} : u_{h,i} \mapsto \left( \int_{\Gamma} (\Lambda_i \nabla_{\Lambda_i} u_{h,i} \cdot n_i) \Phi_j \left( E_{h,j} v_{h,j}^Q \right) \, d\gamma \right)_{Q \in \mathcal{C}_j^F}, \quad i = 1, 2, \ j = (3 - i).
The transmission conditions (28) lead to the following interface problem

\[
\begin{align*}
\sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, f_i) &= 0 \\
\sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, f_i) &= 0 \\
\eta_{h,1} + F^c_1 \circ \mathcal{U}_2 (\gamma_h, \eta_{h,1}, f_2) &= 0 \\
\eta_{h,2} + F^c_2 \circ \mathcal{U}_1 (\gamma_h, \eta_{h,2}, f_1) &= 0,
\end{align*}
\]

where

\[
S \begin{pmatrix} \gamma_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, 0) \\ \sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, 0) \\ \eta_{h,1} + F^c_1 \circ \mathcal{U}_2 (\gamma_h, \eta_{h,1}, 0) \\ \eta_{h,2} + F^c_2 \circ \mathcal{U}_1 (\gamma_h, \eta_{h,2}, 0) \end{pmatrix}, \quad \text{and} \quad \chi = \begin{pmatrix} -\sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (0, 0, f_i) \\ -\sum_{i=1}^{2} F^w_i \circ \mathcal{U}_i (0, 0, f_i) \\ -F^c_1 \circ \mathcal{U}_2 (0, 0, f_2) \\ -F^c_2 \circ \mathcal{U}_1 (0, 0, f_1) \end{pmatrix}.
\]

Here we have used the fact that the problem is linear and write the interface problem in a way such that the left-hand side depends on the interface unknowns \( \gamma_h, \eta_{h,1} \) and \( \eta_{h,2} \). The operator \( S \) is called a discrete Dirichlet-Neumann to Neumann-Neumann operator and the number of equations and unknowns in (30) is equal to \( |N^{\ast} | + 2 |N^{\ast} | + |C^1 | + |C^2 | \). Problem (30) can be solved iteratively using a Richardson procedure or a Krylov subspace iteration method (e.g. GMRES). To speed up the convergence, we shall derive a preconditioner for (30). We write

\[
S = S_1 + S_2,
\]

where

\[
S_i \begin{pmatrix} \gamma_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} = \begin{pmatrix} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, 0) \\ F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,i}, 0) \\ \eta_{h,1} + F^c_1 \circ \mathcal{U}_2 (\gamma_h, \eta_{h,1}, 0) \\ \eta_{h,2} + F^c_2 \circ \mathcal{U}_1 (\gamma_h, \eta_{h,2}, 0) \end{pmatrix}, \quad \text{and} \quad S_2 \begin{pmatrix} \gamma_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} := \begin{pmatrix} F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,2}, 0) \\ F^w_i \circ \mathcal{U}_i (\gamma_h, \eta_{h,2}, 0) \\ \eta_{h,1} + F^c_1 \circ \mathcal{U}_2 (\gamma_h, \eta_{h,2}, 0) \\ \eta_{h,2} + F^c_2 \circ \mathcal{U}_1 (\gamma_h, \eta_{h,2}, 0) \end{pmatrix}.
\]

Following the idea of the Neumann-Neumann preconditioner \( M \), define by

\[
M = \sigma_1 S_1^{-1} + \sigma_2 S_2^{-1},
\]

a preconditioner of (30) where \( S_i^{-1} \) is the inverse of \( S_i \) and \( \sigma_i \geq 0 \) is the weight, \( i = 1, 2 \). In order to derive an explicit formula for \( S_i^{-1} \), we define the solution operator \( \mathcal{U}_i : (\xi^w_1, \xi^w_2, \eta_{h,i}) \mapsto \mathcal{U}_i (\xi^w_1, \xi^w_2, \eta_{h,i}) \in X_{h,i}^{0, \Omega, \omega_{1,2}} \), where \( \mathcal{U}_{h,i} \) is the solution to the Neumann problem

\[
\begin{align*}
\int\limits_{\Omega_h} (\Lambda_i \nabla_{A_i} \tilde{u}_{h,i} \cdot \nabla_{A_i} \psi^0_{h,i}) \, dx - \int\limits_{\Gamma} (\Lambda_i \nabla_{A_i} \tilde{u}_{h,i} \cdot \mathbf{n}) \cdot \Phi_i (\psi^0_{h,i}) \, d\gamma &= 0, & \forall Q \in \left( N^{\ast} \setminus N^{\ast,1} \right) \cup N^{\ast} \setminus N^{\ast,1}, \\
\int\limits_{\Gamma} (\Lambda_i \nabla_{A_i} \tilde{u}_{h,i} \cdot \mathbf{n}) \cdot \Phi_i (\psi^0_{h,i}) \, d\gamma &= (\xi^w_i)_{\varepsilon}, & \forall \varepsilon \in E_i^{\ast}, \\
\int\limits_{\Gamma} (\Lambda_i \nabla_{A_i} \tilde{u}_{h,i} \cdot \mathbf{n}) \cdot \Phi_i (\psi^0_{h,i}) \, d\gamma &= (\xi^w_i)_Q, & \forall Q \in N_i^{\ast}, \\
\int\limits_{\Gamma} (\Lambda_i \nabla_{A_i} \tilde{u}_{h,i} \cdot \mathbf{n}) \cdot \Phi_i (\psi^0_{h,i}) \, d\gamma &= \frac{(\eta_{h,i})_Q}{2}, & \forall Q \in C_i^{1}.
\end{align*}
\]

We also use the trace operator

\[
T : \tilde{u}_{h,i} = (\tilde{u}_{i,p})_{p \in N_i^{\ast}} \mapsto \tilde{u}_{i,1} = \left( \tilde{u}_{i,p} \right)_{p \in N_i^{\ast}, \sigma} \left( \tilde{u}_{i,1, \sigma} \right)_{C_{1,2,\sigma} \in C_{1,2,\sigma}} \in G_h.
\]
Finally, the formulations of $S_1^{-1}$ and $S_2^{-1}$ are given by:

\[
S_1^{-1} \begin{pmatrix} \xi^{\text{str}}_h \\ \xi^w_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} := \begin{pmatrix} \mathcal{T} \circ \mathcal{U}_1(\xi^{\text{str}}_h, \eta^w_{h,1}) \\ \tilde{\eta}_{h,1} + \frac{1}{2} \mathcal{U}_1 \circ \mathcal{U}_1(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,1}) \\ \tilde{\eta}_{h,2} + \frac{1}{2} \mathcal{U}_2 \circ \mathcal{U}_1(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,2}) \end{pmatrix}, \quad \text{and} \quad S_2^{-1} \begin{pmatrix} \xi^{\text{str}}_h \\ \xi^w_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} := \begin{pmatrix} \mathcal{T} \circ \mathcal{U}_2(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,2}) \\ \tilde{\eta}_{h,1} + \frac{1}{2} \mathcal{U}_2 \circ \mathcal{U}_1(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,2}) \end{pmatrix}.
\]

Depending on the choice of the weights, one may obtain different preconditioners. For discontinuous coefficients, one may choose the weights in order to obtain convergence independent of the jump of the coefficients. This will be addressed in Section 3 for numerical experiments. However, for the analysis, we perform a Richardson procedure for the interface problem (30) with $S_2^{-1}$ as a preconditioner and obtain the following algorithm. Of course one can also use the preconditioner $S_1^{-1}$, which results in another algorithm with the same analysis as the one presented below.

### 4.1 Dirichlet-Neumann to Neumann-Neumann (DN-NN) algorithm

For given initial guess $\gamma^0_h \in G_h \subset \mathbb{R}^{\text{card}(\mathcal{N}_1) + \text{card}(\mathcal{N}_\Gamma)}$, $\gamma^0_{h,1} \in \mathbb{R}^{\text{card}(\mathcal{N}_1)}$ and $\gamma^0_{h,2} \in \mathbb{R}^{\text{card}(\mathcal{N}_\Gamma)}$, solve for each $n = 1, 2, \ldots$, the following problems:

Find $u^{n+1}_{h,1} \in \chi^{0,\text{dir}}_{0,\Omega} \cap \Omega_1$ such that

\[
\int_{\Omega_1} \left( \Lambda_1 \nabla \cdot \Lambda_1 u^{n+1}_{h,1} \right) \cdot \nabla \cdot \nu^{Q}_{h,1} d\Omega - \int_{\Gamma} \left( \Lambda_1 \nabla \cdot \Lambda_1 u^{n+1}_{h,1} \cdot \mathbf{n}_1 \right) \nu^{Q}_{h,1} d\gamma = \int_{\Gamma} f_1 \nu^{Q}_{h,1} d\gamma , \quad \forall Q \in \mathcal{N}_1 \setminus \mathcal{N}_1^{\text{dir}},
\]

and

\[
S_2^{-1} \begin{pmatrix} \xi^{\text{str}}_h \\ \xi^w_h \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} := \begin{pmatrix} \mathcal{T} \circ \mathcal{U}_2(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,2}) \\ \tilde{\eta}_{h,1} + \frac{1}{2} \mathcal{U}_2 \circ \mathcal{U}_1(\xi^{\text{str}}_h, \xi^w_h, \tilde{\eta}_{h,2}) \end{pmatrix}.
\]

\[u^{n+1}_{h,1} = \gamma^n_{h,1}, \quad n = 1, 2, \ldots \]

Then find $u^{n+1}_{h,2} \in \chi^{0,\text{dir}}_{0,\Omega} \cap \Omega_2$ such that:

\[
\int_{\Omega_2} \left( \Lambda_2 \nabla \cdot \Lambda_2 u^{n+1}_{h,2} \right) \cdot \nabla \cdot \nu^{Q}_{h,2} d\Omega - \int_{\Gamma} \left( \Lambda_2 \nabla \cdot \Lambda_2 u^{n+1}_{h,2} \cdot \mathbf{n}_2 \right) \nu^{Q}_{h,2} d\gamma = \int_{\Gamma} f_2 \nu^{Q}_{h,2} d\gamma , \quad \forall Q \in \left( \mathcal{N}_2^{\text{dir}} \setminus \mathcal{N}_2^{\text{dir}} \right) \cup \mathcal{N}_\Gamma^{\text{dir}},
\]

\[
\frac{1}{2} \int_{\Gamma} \left( \Lambda_2 \nabla \cdot \Lambda_2 u^{n+1}_{h,2} \cdot \mathbf{n}_2 \right) \nu^{Q}_{h,2} d\gamma = \int_{\Gamma} \left( \Lambda_1 \nabla \cdot \Lambda_1 u^{n+1}_{h,1} \cdot \mathbf{n}_2 \right) \nu^{Q}_{h,1} d\gamma , \quad \forall Q \in \mathcal{N}_\Gamma^{\text{dir}},
\]

\[
\frac{1}{2} \int_{\Gamma} \left( \Lambda_2 \nabla \cdot \Lambda_2 u^{n+1}_{h,2} \cdot \mathbf{n}_2 \right) \nu^{Q}_{h,2} d\gamma = \int_{\Gamma} \left( \Lambda_1 \nabla \cdot \Lambda_1 u^{n+1}_{h,1} \cdot \mathbf{n}_2 \right) \nu^{Q}_{h,1} d\gamma , \quad \forall Q \in \mathcal{N}_\Gamma^{\text{dir}},
\]

\[u^{n+1}_{h,2} = \gamma^n_{h,2}, \quad n = 1, 2, \ldots \]

with

\[
\begin{pmatrix} \gamma^n_{h,1} \\ \eta^n_{h,1} \\ \eta^n_{h,2} \end{pmatrix} := (1 - \theta) \begin{pmatrix} \gamma_{h,1} \\ \eta_{h,1} \\ \eta_{h,2} \end{pmatrix} + \theta \begin{pmatrix} \eta^n_{h,1} \\ \eta^n_{h,2} \end{pmatrix},
\]

\[\begin{pmatrix} \gamma^n_{h,1} \\ \eta^n_{h,1} \\ \eta^n_{h,2} \end{pmatrix}.
\]
where \( \theta \) is a positive acceleration parameter and

\[
\eta_{h,1}^{n+1} - \eta_{h,1}^n = \frac{1}{2} \left( \int \gamma \left( \Lambda_1 \nabla \phi_{1,2} u_{h,2}^{n+1} \cdot n_1 \right) \Phi_2 (E_{h,2} \psi_{1,2}^n) d\gamma \right)_{Q \in C_1^0} \\
\eta_{h,2}^{n+1} - \eta_{h,2}^n = \frac{1}{2} \left( \int \gamma \left( \Lambda_1 \nabla \phi_{1,1} u_{h,1}^{n+1} \cdot n_2 \right) \Phi_1 (E_{h,1} \psi_{1,1}^n) d\gamma \right)_{Q \in C_1^0}.
\]

\subsection{Convergence analysis}

The convergence of the DN-NN algorithm is guaranteed by the following theorem:

\textbf{Theorem 1.} For any given initial guess

\[
\begin{pmatrix}
\gamma_h^n \eta_{h,1}^n \\
\eta_{h,1}^n \eta_{h,2}^n
\end{pmatrix} \in \mathbb{R}^{\text{card}(\mathcal{N}_r^\gamma)} \times \mathbb{R}^{\text{card}(\mathcal{N}_r^\gamma)} ,
\]

the sequence of iterates \((
u^n_h, \eta_{h,1}^n, \eta_{h,2}^n) \in X_{h,1}\) converges to the solution \(u_h = (u_h |_{\Omega_1}, u_h |_{\Omega_2})\) of the monodomain problem \((12)\) in the following sense:

\[
\sum_{i=1}^{2} \sum_{p \in \mathcal{N}_r^\gamma} (u_{i,p} - u_{h,p})^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

The proof of Theorem 1 is based on the following lemma which says that the iterative DN-NN solution converges to the monodomain solution if the iterative interface unknowns converge to the interface counterpart of the monodomain solution.

\textbf{Lemma 1.} Assume that there exists a positive parameter \(\theta^*\) such that for all \(\theta \in (0, \theta^*)\), the sequence \((\gamma_h^n, \eta_{h,1}^n, \eta_{h,2}^n)\) converges in \(\mathbb{R}^{\text{card}(\mathcal{N}_r^\gamma)} \times \mathbb{R}^{\text{card}(\mathcal{N}_r^\gamma)} \times \mathbb{R}^{\text{card}(\mathcal{N}_r^\gamma)}\) in the Euclidean norm \(||\cdot||\) as \(n\) tends to infinity. Then (36) holds.

\textbf{Proof.} We first rewrite the linear system \((15)\) of the FECC scheme on the monodomain case as follows:

\[
\begin{pmatrix}
D_{11}^{(1)} & 0 & 0 & E_{11}^{(1)} & 0 \\
0 & D_{22}^{(2)} & 0 & 0 & E_{22}^{(2)} \\
0 & 0 & D_{33}^{(3)} & 0 & E_{33}^{(3)} \\
M_{11} & 0 & M_{13} & N_{11}^{(1)} & N_{12}^{(1)} \\
0 & M_{22} & M_{23} & N_{21}^{(2)} & N_{22}^{(2)}
\end{pmatrix}
\begin{pmatrix}
u_{i,1,1}^{n,\text{int}} \nu_{i,2,1}^{n,\text{int}} \nu_{i,1,2}^{n,\text{int}} \nu_{i,2,2}^{n,\text{int}} \nu_{i,1,3}^{n,\text{int}} \nu_{i,2,3}^{n,\text{int}} \nu_{i,1,4}^{n,\text{int}} \nu_{i,2,4}^{n,\text{int}}
\end{pmatrix}
= \begin{pmatrix}
F_{11}^{(1)} \\
F_{12}^{(2)} \\
F_{13}^{(3)} \\
F_{14}^{(1)} \\
F_{22}^{(2)} \\
F_{23}^{(3)} \\
F_{24}^{(1)}
\end{pmatrix},
\]

where the vectors \(\nu_{i,1,1}^{n,\text{int}} = (u_{i,M} |_{T_{i,1}^r \setminus \mathcal{N}_r^\gamma}, u_{i,M} |_{T_{i,2}^r \setminus \mathcal{N}_r^\gamma}, u_{i,M} |_{T_{i,3}^r \setminus \mathcal{N}_r^\gamma}, u_{i,M} |_{T_{i,4}^r \setminus \mathcal{N}_r^\gamma}, u_{i,M} |_{K \in T_{i,1}^r}, u_{i,M} |_{K \in T_{i,2}^r}, u_{i,M} |_{K \in T_{i,3}^r}, u_{i,M} |_{K \in T_{i,4}^r})\). The matrices \(D, E, M\) and \(N\) are block matrices and \(D\) is invertible:

\[
D = \begin{pmatrix}
D_{11}^{(1)} & 0 & 0 \\
0 & D_{22}^{(2)} & 0 \\
0 & 0 & D_{33}^{(3)}
\end{pmatrix}, \quad E = \begin{pmatrix}
E_{11}^{(1)} & 0 \\
0 & E_{22}^{(2)} \\
0 & E_{33}^{(3)}
\end{pmatrix}, \quad M = \begin{pmatrix}
M_{11} & 0 & M_{13} \\
0 & M_{22} & M_{23}
\end{pmatrix}, \quad N = \begin{pmatrix}
N_{11}^{(1)} & N_{12}^{(1)} \\
N_{21}^{(2)} & N_{22}^{(2)}
\end{pmatrix}.
\]

Next, from (35), we have

\[
\gamma_{h}^{n+1} = (1 - \theta) \gamma_{h}^{n} + \theta \nu_{2,1}^{n+1}.
\]

or

\[
\gamma_{h}^{n+1} - \gamma_{h}^{n} = \theta \left( \nu_{2,1}^{n+1} - \gamma_{h}^{n} \right) = \theta \left( \nu_{2,1}^{n+1} - \nu_{2,1}^{n+1} \right). \tag{38}
\]

Similarly, from (35), (33c) and (34d), we deduced that

\[
\eta_{h,1}^{n+1} - \eta_{h,1}^{n} = \theta \left( \int \gamma \left( \Lambda_1 \nabla \phi_{1,1} u_{h,1}^{n+1} \cdot n_1 \right) \Phi_1 (E_{h,1} \psi_{1,1}^n) d\gamma - \int \gamma \left( \Lambda_1 \nabla \phi_{1,1} u_{h,1}^{n+1} \cdot n_1 \right) \Phi_2 (E_{h,2} \psi_{1,1}^n) d\gamma \right)_{Q \in C_1^0}, \tag{39}
\]

and

\[
\eta_{h,2}^{n+1} - \eta_{h,2}^{n} = \theta \left( \int \gamma \left( \Lambda_1 \nabla \phi_{1,1} u_{h,2}^{n+1} \cdot n_2 \right) \Phi_1 (E_{h,1} \psi_{1,2}^n) d\gamma - \int \gamma \left( \Lambda_2 \nabla \phi_{1,2} u_{h,2}^{n+1} \cdot n_2 \right) \Phi_2 (E_{h,2} \psi_{1,2}^n) d\gamma \right)_{Q \in C_1^0}. \tag{40}
\]
By performing elementary calculations as shown in Appendix A and using (38)-(40), we obtain the linear algebraic system associated with the DN-NN algorithm (33) and (34) as follows:

\[
\begin{pmatrix}
D^{(1)}_{11} & 0 & 0 & E^{(1)}_{11}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}^{n+1}_{h,1/head} & \mathbf{u}^{n+1}_{h,2/head} & \mathbf{u}^{n+1}_{h,1/root} & \mathbf{u}^{n+1}_{h,2/root}
\end{pmatrix}
\begin{pmatrix}
0
0
\end{pmatrix}
\begin{pmatrix}
R_{1}^{DN}(\gamma^{n+1}_{h} - \gamma^{n}_{h})
R_{2}^{DN}(\gamma^{n+1}_{h} - \gamma^{n}_{h}) + \mathbf{G}_{1}^{DN}(\eta^{n+1}_{h,1} - \eta^{n}_{h,1}),
R_{2}^{DN}(\gamma^{n+1}_{h} - \gamma^{n}_{h}) + \mathbf{G}_{2}^{DN}(\eta^{n+1}_{h,2} - \eta^{n}_{h,2})
\end{pmatrix}
\begin{pmatrix}
\mathbf{F}^{*}_{1}
\mathbf{F}^{*}_{2}
\end{pmatrix},
\]

(41)

where \(\mathbf{u}^{n+1}_{h,1/head} = (\mathbf{u}^{n+1}_{1,M/p} p \in \mathcal{N}_{h,1,\text{root}}^{*}), \mathbf{u}^{n+1}_{h,1/root} = (\mathbf{u}^{n+1}_{1,K/K} \in \mathcal{N}_{h,1,\text{root}}^{*}), \mathbf{u}^{n+1}_{h,2/head} = (\mathbf{u}^{n+1}_{1,M/K} \in \mathcal{N}_{h,2,\text{root}}^{*})\), and \(\mathbf{u}^{n+1}_{h,2/root} = (\mathbf{u}^{n+1}_{2,K/K} \in \mathcal{N}_{h,2,\text{root}}^{*})\). Explicit forms of the \(\mathbf{R}\)'s and \(\mathbf{G}\)'s matrices are given in Appendix A.

As \((\gamma^{n}_{h}, \eta^{n}_{h,1}, \eta^{n}_{h,2})\) converges, we have that \((\gamma^{n+1}_{h} - \gamma^{n}_{h}), (\eta^{n+1}_{h,1} - \eta^{n}_{h,1})\) and \((\eta^{n+1}_{h,2} - \eta^{n}_{h,2})\) tend to 0 when \(n\) tends to infinity. By comparing the two systems (37) and (41) and due to the uniqueness of the solution of (37), convergence of the multidomain solution to the monodomain solution then follows.

**Proof. (of Theorem 1)** In view of Lemma 1, we first show that the sequence \(\gamma^{n+1}_{h} = \mathbf{u}^{n+1}_{1,\Gamma}\) converges in \(\Gamma_{h}\). Toward this end, from the system (41) we find an explicit formula for

\[
\mathbf{u}^{n+1}_{1,\Gamma} = \left(\mathbf{u}_{1,p}^{n+1} p \in \mathcal{N}_{h,1,\text{root}}^{*}, \mathbf{u}_{1,K}^{n+1} K \in \mathcal{N}_{h,1,\text{root}}^{*}\right) := \left(\mathbf{u}_{h,1,1,\text{root}}, \mathbf{u}_{h,1,1,\text{root}}\right).
\]

Using the third row of (41), we compute \(\mathbf{u}^{n+1}_{h,1,\text{root}}\) as follows:

\[
\mathbf{u}^{n+1}_{h,1,\text{root}} = (\mathbf{D}_{33}^{T})^{-1} \left[\mathbf{E}_{31}^{(1)} - \mathbf{E}_{32}^{(2)} \mathbf{D}_{33}^{T}\right] \left(\mathbf{F}^{*}_{1} - \mathbf{R}_{1}^{DN} \mathbf{u}^{n+1}_{2,\Gamma,1,\text{root}} - \mathbf{R}_{2}^{DN} \mathbf{u}^{n+1}_{2,\Gamma,1,\text{root}}\right) \mathbf{F}_{2}^{-1}.
\]

(42)

We also deduce from (41) that

\[
\left(\begin{array}{c}
\mathbf{u}^{n+1}_{h,1/head,\text{root}}
\mathbf{u}^{n+1}_{h,2/head,\text{root}}
\mathbf{u}^{n+1}_{h,1/root,\text{root}}
\end{array}\right) = \mathbf{D}^{-1} \left[\begin{array}{c}
\mathbf{F}^{*}_{1}
\mathbf{F}^{*}_{2}
\mathbf{F}_{2}^{-1}
\end{array}\right] - \mathbf{E} \left(\begin{array}{c}
\mathbf{u}^{n+1}_{h,1/head,\text{root}}
\mathbf{u}^{n+1}_{h,2/head,\text{root}}
\mathbf{u}^{n+1}_{h,1/root,\text{root}}
\end{array}\right) - \left(\begin{array}{c}
0
0
\mathbf{R}_{1}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right)
\mathbf{R}_{2}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right)
\end{array}\right).\]

(43)

and

\[
\mathbf{M} \left(\begin{array}{c}
\mathbf{u}^{n+1}_{h,1/head,\text{root}}
\mathbf{u}^{n+1}_{h,2/head,\text{root}}
\mathbf{u}^{n+1}_{h,1/root,\text{root}}
\end{array}\right) + \mathbf{N} \left(\begin{array}{c}
\mathbf{u}^{n+1}_{h,1/head,\text{root}}
\mathbf{u}^{n+1}_{h,2/head,\text{root}}
\mathbf{u}^{n+1}_{h,1/root,\text{root}}
\end{array}\right) = \left(\begin{array}{c}
\mathbf{F}^{*}_{1}
\mathbf{F}^{*}_{2}
\mathbf{F}_{2}^{-1}
\end{array}\right) - \left(\begin{array}{c}
\mathbf{R}_{1}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right)
\mathbf{R}_{2}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right)
\end{array}\right).
\]

(44)

Substituting (43) into (44), we obtain

\[
\begin{pmatrix}
\mathbf{N} - \mathbf{M} \mathbf{D}^{-1} \mathbf{E}
\end{pmatrix} \left(\begin{array}{c}
\mathbf{u}^{n+1}_{h,1/head,\text{root}}
\mathbf{u}^{n+1}_{h,2/head,\text{root}}
\mathbf{u}^{n+1}_{h,1/root,\text{root}}
\end{array}\right) = \left(\begin{array}{c}
\mathbf{F}^{*}_{1}
\mathbf{F}^{*}_{2}
\mathbf{F}_{2}^{-1}
\end{array}\right) - \mathbf{R}_{1}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right) - \mathbf{R}_{2}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right).
\]

(45)

Matrix \(\mathbf{A}\) is symmetric and positive definite (cf. Section 2), thus we can first compute \(\mathbf{u}^{n+1}_{h,1/head,\text{root}}\) explicitly by inverting \(\mathbf{A}\), then substitute the result into (42) and obtain:

\[
\mathbf{u}^{n+1}_{h,1/head,\text{root}} = \mathbf{F}^{*}_{1} - \mathbf{R}_{1}^{DN} \left(\mathbf{u}^{n+1}_{h,2/head} - \mathbf{u}^{n+1}_{h,2/root}\right),
\]

where

\[
\mathbf{F}^{*}_{1} = (\mathbf{D}_{33}^{T})^{-1} \left(\mathbf{E}_{31}^{(1)} \mathbf{E}_{32}^{(2)} \mathbf{A}^{-1} \mathbf{M}_{11} \left[D_{11}^{(1)} F_{1}^{*} + \mathbf{M}_{13} \left[D_{33}^{(1)} F_{2}^{*} + \mathbf{M}_{23} \left[D_{33}^{(2)} F_{2}^{*} \right]^{-1} \right]^{-1} \right]^{-1} \right) + (\mathbf{D}_{33}^{T})^{-1} \mathbf{F}_{2}^{*} - (\mathbf{D}_{33}^{T})^{-1} \left(\mathbf{E}_{31}^{(1)} \mathbf{E}_{32}^{(2)} \mathbf{A}^{-1} \mathbf{F}_{1}^{*}\right),
\]

(46)
and

\[ R_{N_{10}^*} = R_T + (D_{33}^{(i)})^{-1} \left( E_{31}^{(i)} E_{32}^{(i)} A^{-1} \begin{pmatrix} M_{13} \left[ D_{33}^{(i)} \right]^{-1} R_T - R_1 \end{pmatrix} \right). \]

We aim to derive a formula for \( u_{h1}^{n+1} \mid_{N_{10}^*} \) similar to (46). Using the condition of strong flux continuity (34b) for \( e = [C_0, C_M] \in E_T^{\ast +} \) (see Figure 4), we have that

\[
\left( \beta_K u_{1,K}^{n+1} + \beta_1 u_{1,1}^{n+1} + \beta_{2,M} u_{2,M}^{n+1} \right) + \left( \beta_{2,L} u_{2,L}^{n+1} + \beta_{2,2} u_{2,2}^{n+1} + \beta_{2,3} u_{2,3}^{n+1} \right) = 0,
\]

where the coefficients are given in (8). We rewrite (47) equivalently as

\[
- (\beta_{1,1} + \beta_{2,2}) u_{1,1}^{n+1} = \beta_{2,L} u_{2,L}^{n+1} + (\beta_{1,1} + \beta_{2,2}) u_{1,1}^{n+1} + u_{2,2}^{n+1} - u_{1,1}^{n+1}.
\]

Since \( \beta_{1,1} + \beta_{2,2} \neq 0 \), we can compute \( u_{1,1}^{n+1} \) using the values of \( u_h \mid_{r_1}, u_h \mid_{r_2}, u_h \mid_{N_{10}^*} \) and \( (u_{2,2} - u_{1,1}) \). In other words, we can rewrite (48) in the following matrix form:

\[
u_{h1}^{n+1} \mid_{N_{10}^*} = L u_{h1}^{n+1} \mid_{N_{10}^*} + K \left( u_{h1}^{n+1} \mid_{r_1}, u_{h2}^{n+1} \mid_{r_2} \right) + H \left( u_{h2}^{n+1} \mid_{r_2} - u_{1,1}^{n+1} \right).
\]

Substituting (45) and (46) into (49), we obtain

\[
u_{h1}^{n+1} \mid_{N_{10}^*} = F_{N_{10}^*} - R_{N_{10}^*} \left( u_{2,2}^{n+1} - u_{1,1}^{n+1} \right).
\]

We skip the detailed calculations of matrices \( L, K, H \) and \( F_{N_{10}^*}, R_{N_{10}^*} \) for the sake of simplicity. Using (50) together with (46), we find that

\[
\gamma_{h}^{n} = u_{h1}^{n+1} = F - R \left( u_{2,2}^{n+1} - u_{1,1}^{n+1} \right),
\]

with \( F = \begin{pmatrix} F_{N_{10}^*} \ 0 \\ R_{N_{10}^*} \ 0 \end{pmatrix} \), \( R = \begin{pmatrix} R_{N_{10}^*} \ 0 \\ 0 \ R_{N_{10}^*} \end{pmatrix} \). Inserting (51) into (38), we have

\[
\frac{1}{\beta} R e_{h1}^{n+1} = F + \left( \frac{1}{\beta} R - I \right) \gamma_{h}.
\]

where \( I \) is the identity matrix.

For the monodomain problem (37), performing similar calculations as for (46) and (50), we find that

\[
u_{h1}^{n} = \left( u_{h} \mid_{N_{10}^*} \ 0 \right) = \begin{pmatrix} F_{N_{10}^*} \ 0 \\ R_{N_{10}^*} \ 0 \end{pmatrix} = F.
\]

Setting \( e_{h}^{n} = \gamma_{h}^{n} - u_{h1}^{n} \), we deduce from (52) and (53) that

\[
\frac{1}{\beta} R e_{h1}^{n+1} = \left( I - \frac{1}{\beta} R \right) e_{h}^{n}.
\]

By Theorem 3.2.1 in Chapter 3, for sufficiently large \( |\beta| \) such that \( \frac{|R|}{|\beta|} < 1 \), the matrix \( \left( I - \frac{1}{\beta} R \right) \) is invertible and its inverse is equal to \( \sum_{k=0}^{\infty} \frac{1}{\beta k} R^{k} \). As a results, we can deduce from (54) that

\[
\left[ \frac{1}{\beta} \left( I - \frac{1}{\beta} R \right)^{-1} \right] e_{h1}^{n+1} = \left( -\sum_{k=0}^{\infty} \frac{1}{\beta k+1} R^{k+1} \right) e_{h}^{n+1} = e_{h}^{n},
\]

or

\[
\left( -\sum_{k=0}^{\infty} \frac{1}{\beta k+1} R^{k+1} \right) e_{h1}^{n+1} = e_{h}^{n}.
\]
On the other hand, we have
\[- \sum_{k=0}^{\infty} \frac{1}{\theta^{k+1}} R^{k+1} = I - \sum_{k=0}^{\infty} \frac{1}{\theta^k} R^k = I - S(R, \theta),\]
where \( S(R, \theta) = \sum_{k=0}^{\infty} \frac{1}{\theta^k} R^k \) converges due to Lemma 3.2.3 in Chapter 3. If we choose a sufficiently large value \(|\theta|\) such that 
\[\|S(R, \theta)\| < 1,\]
then there exists the inverse of \((I - S(R, \theta))^{-1}\). As a consequence, from (55) we deduce that
\[\|e^n_h\| = \|[(I - S(R, \theta))^{-1}]^n e^0_h\| \leq \|I - S(R, \theta)\|^n \|e^0_h\| = \|\sum_{k=0}^{\infty} S^k(R, \theta)\|^n \rightarrow 0,\]
for sufficiently large \(|\theta|\), where \(\|I - S(R, \theta)\|\) \(\leq \sum_{k=0}^{\infty} \|S(R, \theta)\|^k < 1\) according to Theorem 3.2.1 Chapter 3. By the definition of \(e^n_h\), it follows that \(e^n_h\) converges to \(u_{h,1}\) as \(n\) tends to infinity. We further discuss a corollary of this convergence, which will be useful later. From (57), we find that \(u_h\) satisfies the following systems:
\[A \begin{pmatrix} u_h & r_{h,1} \\ u_h & r_{h,2} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - MD^{-1} \begin{pmatrix} F_1^+ \\ F_2^+ \end{pmatrix}.\]
Subtracting (57) from (45), we obtain
\[A \begin{pmatrix} u_h & r_{h,1} \\ u_h & r_{h,2} \end{pmatrix} = \begin{pmatrix} R_1 \begin{pmatrix} u_{h,1}^{\ell+1} - u_{h,1}^{\ell+1} \\ u_{h,2}^{\ell+1} - u_{h,2}^{\ell+1} \end{pmatrix} \\ R_2 \begin{pmatrix} u_{h,1}^{\ell+1} - u_{h,1}^{\ell+1} \\ u_{h,2}^{\ell+1} - u_{h,2}^{\ell+1} \end{pmatrix} \end{pmatrix} - MD^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} R_3 \begin{pmatrix} u_{h,1}^{\ell+1} - u_{h,1}^{\ell+1} \\ u_{h,2}^{\ell+1} - u_{h,2}^{\ell+1} \end{pmatrix} \]
Due to (55) and the fact that \(A\) is symmetric, positive definite, we can rewrite the above equation equivalently as
\[\begin{pmatrix} u_h & r_{h,1} \\ u_h & r_{h,2} \end{pmatrix} = \begin{pmatrix} R_1 \begin{pmatrix} \gamma^{\ell+1} - \gamma^{\ell} \\ \gamma^{\ell+1} - \gamma^{\ell} \end{pmatrix} \\ R_2 \begin{pmatrix} \gamma^{\ell+1} - \gamma^{\ell} \\ \gamma^{\ell+1} - \gamma^{\ell} \end{pmatrix} \end{pmatrix} - \frac{1}{\theta} A^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} R_3 \begin{pmatrix} \gamma^{\ell+1} - \gamma^{\ell} \\ \gamma^{\ell+1} - \gamma^{\ell} \end{pmatrix}.\]
Since \((\gamma^{\ell+1} - \gamma^{\ell}) \to 0\), we deduce that
\[\begin{pmatrix} u_h & r_{h,1} \\ u_h & r_{h,2} \end{pmatrix} \to 0\]
as \(n \to 0\).

To prove that \(\left(\eta^{\ell+1}_{h,1} - \eta^{n}_{h,1}\right)\) tends to 0, we evaluate (59) at \(Q \equiv K \in C_1\) (see Figure 4). Using (33a) and (A10), we obtain:
\[\frac{1}{\theta} (\eta^{\ell+1}_{h,1} - \eta^{n}_{h,1}) \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \int_{Q} \left(\begin{array}{c} A_1 \nabla_A u_{h,1}^{\ell+1} - \nabla_A u_{h,1} \quad A_1 \nabla_A u_{h,1}^{\ell+1} \end{array}\right) \nabla_A E_{h,2} v_{h,1}^{\ell+1} \quad dx \]
\[+ \int_{Q} \left(\begin{array}{c} A_1 \nabla_A u_{h,1}^{\ell+1} - \nabla_A E_{h,2} v_{h,1}^{\ell+1} \\ A_1 \nabla_A u_{h,1}^{\ell+1} - \nabla_A E_{h,2} v_{h,1}^{\ell+1} \end{array}\right) \nabla_A E_{h,2} v_{h,1}^{\ell+1} \quad dx \]
\[+ \int_{Q} \left(\begin{array}{c} A_1 \nabla_A u_{h,1}^{\ell+1} - \nabla_A v_{h,1}^{\ell+1} \\ A_1 \nabla_A u_{h,1}^{\ell+1} - \nabla_A v_{h,1}^{\ell+1} \end{array}\right) \nabla_A E_{h,2} v_{h,1}^{\ell+1} \quad dx \]
\[\left(\begin{array}{c} f_1 \Phi_1 \left(v_{h,1}^{\ell+1}\right) \quad f_2 \Phi_2 \left(E_{h,2} v_{h,1}^{\ell+1}\right) \end{array}\right) \quad dx.\]
Subtracting (59) from (A11), we obtain
\[\frac{1}{\theta} (\eta^{\ell+1}_{h,1} - \eta^{n}_{h,1}) \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \int_{Q} \left(\begin{array}{c} A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \quad A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \end{array}\right) \nabla_A v_{h,1}^{\ell+1} \quad dx \]
\[+ \int_{Q} \left(\begin{array}{c} A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \quad A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \end{array}\right) \nabla_A E_{h,2} v_{h,1}^{\ell+1} \quad dx \]
\[+ \int_{Q} \left(\begin{array}{c} A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \quad A_1 \nabla_A \left(u_{h,1}^{\ell+1} - u_h\right) \end{array}\right) \nabla_A E_{h,2} v_{h,1}^{\ell+1} \quad dx \]
\[\left(\begin{array}{c} f_1 \Phi_1 \left(v_{h,1}^{\ell+1}\right) \quad f_2 \Phi_2 \left(E_{h,2} v_{h,1}^{\ell+1}\right) \end{array}\right) \quad dx.\]
Next, we shall derive upper bounds for the above integrals. Using the discrete gradient formulas \(^5\) and the definition of test vectors \(^{13}\), we have:

\[
\int_{(C_u C_k C_v)} \left[ \Lambda_1 \nabla \left( u_{h,1}^{n+1} - u_h \right) \cdot \nabla \Lambda_{\frac{1}{2}} u_{h,1}^K \right] d\mathbf{x} \leq \frac{\left| \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{u_{h,1}^K}^C \right| + \left| \tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right|}{4m(C_u C_k C_v)} \leq \frac{\left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)}.
\]

\[
+ \left| \left( u_{h,1}^{n+1} - u_{h} \right) \right| \frac{\left| \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right| + \left| \tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right|}{4m(C_u C_k C_v)}.
\]

\[
+ \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \frac{\left| \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right| + \left| \tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right|}{4m(C_u C_k C_v)}. \tag{61}
\]

We need to introduce some notation: let \( h_{(C_u C_k C_v)} \) be the diameter of the triangle \((C_u C_k C_v) \) is \( T_{1,h}^{\ast \ast} \), \( d(C_u C_k C_v) \) the Euclidean distance between the mesh point of \((C_u C_k C_v) \) and its edge \( e \in \{C_k C_v, C_m C_v, C_k C_m \} \). We have the following estimate:

\[
\left| \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right| \leq \frac{\tilde{\lambda}_1}{2} \frac{|C_k C_v||C_u C_v| \cos \left( \angle(C_m C_v, C_k, C_u) \right)}{2d(C_u C_k C_v) + 2d(C_k C_m C_v) + d(C_u C_m C_v)} \leq \frac{\tilde{\lambda}_1}{2} \left[ \frac{\max \left\{ \frac{h_{(C_k C_m C_v)}}{d(C_u C_k C_v)}, \frac{h_{(C_k C_m C_v)}}{d(C_k C_m C_v)}, \frac{h_{(C_u C_k C_v)}}{d(C_u C_k C_v)} \right\}}{^2} \right]
\]. \tag{62}

We make the following assumption on the regularity of two sub-dual meshes \( T_{1,h}^{\ast \ast}, T_{2,h}^{\ast \ast} \); assume that there exists \( \theta > 0 \) such that \( \theta_1, \theta_2 \) are less than \( \theta \), with

\[
\theta_1 = \max_{T, \in T_{1,h}^{\ast \ast}} \frac{h_{T}}{d_{T,\sigma}}, \quad \theta_2 = \max_{T, \in T_{2,h}^{\ast \ast}} \frac{h_{T}}{d_{T,\sigma}}.
\]

This together with \(62\) yields \( \left| \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u \right| \leq \frac{\tilde{\lambda}_1}{6} \theta^2 \). A similar argument can be applied for other terms in \(61\). As a result, we obtain:

\[
\left( u_{h,1}^{n+1} - u_h \right) \left( u_{h}^{n+1} - u_{\sigma} \right) \leq \frac{\tilde{\lambda}_1}{6} \theta^2.
\]

Besides, we clearly see that

\[
\left( u_{h,1}^{n+1} - u_h \right) \left( u_{h}^{n+1} - u_{\sigma} \right) \leq \left| \left( u_{h,1}^{n+1} - u_h \right) \right| \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \leq \left| \left( u_{h,1}^{n+1} - u_h \right) \right| \leq \left| \left( u_{h,1}^{n+1} - u_h \right) \right| \leq \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \leq \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right|.
\]

Moreover, \( \tilde{\beta}_K \) is bounded according to Lemmas 5.1 and 5.2 in \( \text{[11]} \). This together with \(58\), \(61\), \(63\) and \(64\) implies that

\[
\int_{(C_u C_k C_v)} \left[ \Lambda_1 \nabla \left( u_{h,1}^{n+1} - u_h \right) \right] \cdot \nabla \Lambda_{\frac{1}{2}} u_{h,1}^K \] \left. \right|_{C_u C_k C_v} d\mathbf{x} \to 0 \quad \text{as } n \to \infty.
\]

We next evaluate

\[
\int_{(C_u C_k C_v)} \left[ \Lambda_2 \nabla \left( u_{h,2}^{n+1} - u_h \right) \right] \cdot \nabla \Lambda_{\frac{1}{2}} u_{h,2}^K \] \left. \right|_{C_u C_k C_v} d\mathbf{x} \leq \left| \left( u_{h,2}^{n+1} - u_h \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)} + \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)}.
\]

\[
\right. \right|_{C_u C_k C_v} d\mathbf{x} \leq \left| \left( u_{h,2}^{n+1} - u_h \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)} + \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)}.
\]

\[
\right. \right|_{C_u C_k C_v} d\mathbf{x} \leq \left| \left( u_{h,2}^{n+1} - u_h \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)} + \left| \left( u_{h}^{n+1} - u_{\sigma} \right) \right| \frac{\tilde{\beta}_K \left( \Lambda, \mathbf{n}_{C_k} C_v \right) \cdot \mathbf{n}_{C_v} C_u}{4m(C_u C_k C_v)}.
\]
Using (38), we find that

\[ \mathbf{u}_{2,1}^{n+1} = \frac{1}{J} (\mathbf{y}^{n+1} - \mathbf{y}^n) + \mathbf{e}_h^n. \]

This implies that

\[ \left| \mathbf{u}_{2,1}^{n+1} - \mathbf{u}_M \right| + \left| \mathbf{u}_{2,\sigma}^{n+1} - \mathbf{u}_\sigma^M \right| \leq \left\| \mathbf{u}_{2,1}^{n+1} - \mathbf{u}_h \right\| \leq \left\| \frac{1}{J} (\mathbf{y}^{n+1} - \mathbf{y}^n) \right\| + \left\| \mathbf{e}_h^n \right\|. \]

In addition, as \( L \) is an interior point in \( \Omega_2 \), we have

\[ \left| \mathbf{u}_{2,1}^{n+1} - \mathbf{u}_L \right| \leq \left\| \mathbf{u}_h \right\|_{T_{h,2}} - \left\| \mathbf{u}_h^{n+1} \right\|_{T_{h,2}}. \]

It follows that

\[ \int_{(C_{h,1}^2 C_{h,2}^2)} \Lambda_2 \mathbf{v}_{h,2}^{n+1} \cdot \nabla_2 \mathbf{u}_1^{n+1} \, d\mathbf{x} \to 0 \quad \text{as } n \to \infty. \]

Similarly to (65) and (66), we can prove that the other integrals in the right hand side (60) tend to 0 as \( n \) tends to infinity. Hence \( (\eta_h^{n+1} - \eta_h^n) \to 0 \) as \( n \to \infty \). In a same manner, one can show that \( (\eta_h^{n+1} - \eta_h^n) \) converges to 0 as \( n \) approaches infinity.

5 | NUMERICAL EXPERIMENTS

We consider two subdomains decomposed from a unit square \( \Omega = [0, 1]^2 \). The coefficients are constant in the subdomains and discontinuous across the interface. Both isotropic and anisotropic diffusion tensors are studied and two different mesh types are considered (see Figure 5) - Type 1 with triangular primal elements and Type 2 with rectangular primal elements. The associated dual sub-mesh for each type is shown in the same figure. In Table 1, the number of elements of the primal mesh \( h \) and the number of nodes of the dual sub-mesh \( h^{**} \) for different mesh size are presented. Recall that the FECC scheme has the same accuracy as the standard finite element method on the dual sub-mesh, however the computational cost is much lower since only primal cell unknowns are involved in the linear algebraic system (the scheme is cell-centered). Furthermore, one can easily see that for the same mesh size \( h \), the number of elements of mesh type 1 is many more than that of mesh type 2.

We perform GMRES on the interface problem (30) with or without using the generalized Neumann-Neumann (GNN) preconditioner (31). To handle discontinuous coefficients, we use the following formula to calculate the weights (32):

\[ \sigma_i = \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^2, \quad i = 1, 2, \]
TABLE 1 Numbers of elements of the primal mesh $\mathcal{T}_h$ and numbers of nodes of the dual sub-mesh $\mathcal{T}^{**}_h$ for different mesh sizes.

| Mesh size $h$ of $\mathcal{T}_h$ | #elements of $\mathcal{T}_h$ | #nodes of $\mathcal{T}^{**}_h$ |
|---------------------------------|-----------------------------|-------------------------------|
| $h_1 = \frac{1}{8}$            | 224                         | 385                          |
| $h_2 = \frac{1}{16}$           | 896                         | 1441                         |
| $h_3 = \frac{1}{32}$           | 3584                        | 5569                         |
| $h_4 = \frac{1}{64}$           | 14336                       | 21889                        |
| $h_5 = \frac{1}{128}$          | 57344                       | 86785                        |
| $h_6 = \frac{1}{256}$          | 229376                      | 345601                       |

TABLE 2 Number of iterations required to reach an error reduction $10^{-6}$ for mesh type 1 (Test case 1).

| Diffusion ratio $r$ | Method       | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ |
|---------------------|-------------|-------|-------|-------|-------|-------|-------|
| 10                  | No preconditioner | 26    | 39    | 51    | 64    | 78    | 93    |
|                     | GNN preconditioner | 12    | 12    | 13    | 14    | 14    | 14    |
| 100                 | No preconditioner  | 27    | 41    | 56    | 75    | 107   | 143   |
|                     | GNN preconditioner | 9     | 9     | 9     | 9     | 9     | 9     |
| 1000                | No preconditioner  | 27    | 45    | 64    | 86    | 116   | 156   |
|                     | GNN preconditioner | 7     | 7     | 7     | 8     | 8     | 8     |

TABLE 3 Number of iterations required to reach an error reduction $10^{-6}$ for mesh type 2 (Test case 1).

| Diffusion ratio $r$ | Method       | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ |
|---------------------|-------------|-------|-------|-------|-------|-------|-------|
| 10                  | No preconditioner | 14    | 24    | 34    | 43    | 53    | 63    |
|                     | GNN preconditioner | 8     | 9     | 9     | 9     | 9     | 10    |
| 100                 | No preconditioner  | 15    | 23    | 36    | 48    | 65    | 89    |
|                     | GNN preconditioner | 7     | 7     | 7     | 7     | 8     | 8     |
| 1000                | No preconditioner  | 16    | 23    | 39    | 56    | 76    | 104   |
|                     | GNN preconditioner | 5     | 6     | 6     | 7     | 7     | 7     |

where $\lambda_i$ is the maximum eigenvalue of matrix $\Lambda_i$, $i=1, 2$. We first study the error equation with isotropic diffusion tensors in Subsection 5.1 for different jumps in coefficients between the two subdomains. Then we consider a problem with a known analytical solution and with anisotropic, discontinuous diffusion tensor in Subsection 5.2.

5.1 Test case 1: with isotropic, discontinuous diffusion tensors

The diffusion matrix is defined as $\Lambda_i = \lambda_i \mathbf{I}$, $i=1, 2$, where $\mathbf{I}$ is the 2D identity tensor. We fix $\lambda_1 = 1$ and vary $\lambda_2 \in \{10, 100, 1000\}$. Denote by $r := \lambda_2/\lambda_1$ the diffusion ratio. We solve the error equation (with a zero solution) and start with a random initial guess. We calculate the normalized error at each iteration in $L^2(\Omega)$ norm, $(\|\varepsilon^n\|/\|\varepsilon^0\|)$. Tables 2 and 3 show the numbers of iterations required to reach an error reduction $10^{-6}$ for mesh type 1 and type 2 respectively. The results are for different diffusion ratios and different mesh sizes. We first see that the algorithm converges with or without preconditioner, however, the convergence is very slow and very sensitive to the diffusion ratio as well as the mesh size if no preconditioner is used. The GNN preconditioner works well in the sense that the number of iterations is small and its convergence is almost independent of the mesh size. We notice that when the ratio is higher, the preconditioned algorithm converges faster while the no preconditioner algorithm converges slower. Similar convergence behavior is observed for both mesh type 1 and mesh type 2, except that the convergence when mesh type 2 is used is faster when mesh type 1 is used. This is because for the same mesh size, the number of nodes involved in mesh type 2 is smaller that that of mesh type 1 (see Table 1).
TABLE 4 Number of iterations required to reach a relative error of $10^{-6}$ (Test case 2).

| Mesh type | Method           | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ |
|-----------|------------------|-------|-------|-------|-------|-------|-------|
| Type 1    | No preconditioner| 21    | 25    | 31    | 35    | 36    | 33    |
|           | GNN preconditioner| 7     | 7     | 8     | 8     | 8     |       |
| Type 2    | No preconditioner| 7     | 12    | 17    | 21    | 25    | 31    |
|           | GNN preconditioner| 7     | 6     | 6     | 6     | 6     | 5     |

5.2 | Test case 2: with an anisotropic, discontinuous diffusion tensor

We solve the diffusion problem with an exact solution given by

$$ u_{\text{exact}} = \begin{cases} 
\cos(\pi x) \sin(\pi y) & \text{if } x \leq 0.5, \\
10^{-2} \cos(\pi x) \sin(\pi y) & \text{if } x > 0.5 \end{cases} \quad \text{with} \quad \Lambda = \begin{bmatrix} I & \text{if } x \leq 0.5, \\
10^2 & 0 \\
0 & 0.01 \end{bmatrix} \text{if } x > 0.5. $$

For this case, we compute the relative error between the multidomain solution and the monodomain solution in $L^2(\Omega)$-norm and stop the iteration when the error is smaller than $10^{-6}$. In Table 4, we show the number of iterations for mesh type 1 and mesh type 2. We see that even for anisotropic, discontinuous coefficients, the GNN preconditioner is very efficient and requires only a few number of iterations for convergence no matter how small the mesh size is. Clearly, the no preconditioner algorithm converges but very slow compared to the preconditioned algorithm.

6 | CONCLUSIONS

We have formulated the Dirichlet-Neumann to Neumann-Neumann (DN-NN) method for anisotropic, heterogeneous diffusion problems discretized by the FECC scheme. A discrete interface problem is derived and a generalized Neumann-Neumann (GNN) preconditioner with weights is introduced to accelerate the convergence of the iterative algorithm associated with such a interface problem. The convergence of the iterative solution (obtained by the Dirichlet-Neumann to Neumann-Neumann algorithm) to the monodomain solution is rigorously proved. Numerical results confirm our theory and show that the GNN preconditioner efficiently handles the discontinuous coefficients for both isotropic and anisotropic tensors and its convergence is almost independent of the mesh size. Work underway addresses the Optimized Schwarz method\(^ {23}\) for FECC-based discretizations in which Robin transmission conditions, instead of classical transmission conditions, are considered with some parameters that can be optimized to enhance the convergence of the associated iterative algorithm. In addition, extensive numerical results shall be carried out to compare the performance of the DN-NN and the Optimized Schwarz methods with FECC discretization on more realistic test cases with multiple subdomains possibly having cross points.

ACKNOWLEDGEMENTS

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02 – 2017.06. The support is gratefully acknowledged.

References

1. Quarteroni A, Valli A. *Domain decomposition methods for partial differential equations*. Oxford New York: Clarendon Press; 1999.

2. Toselli A, Widlund O. Domain decomposition methods—algorithms and theory. In: Springer Series in Computational Mathematics, vol. 34: Springer-Verlag 2005.
3. Mathew T. Domain Decomposition Methods for the Numerical Solution of Partial Differential Equations. In: Lecture Notes in Computational Science and Engineering, vol. 61: Springer 2008.

4. Agoshkov VI. Poincaré-Steklov’s operators and domain decomposition methods in finite-dimensional spaces. In: First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris). Philadelphia, PA: SIAM 1988 (pp. 73-112).

5. Widlund OB. Iterative substructuring methods: algorithms and theory for elliptic problems in the plane. In: First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris). SIAM 1988 (pp. 113-128).

6. Bjørstad PE, Brækhus J, Hvidsten A. Parallel substructuring algorithms in structural analysis, direct and iterative methods. In: Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow). SIAM 1991 (pp. 321-340).

7. Quarteroni A, Valli A. Theory and application of Steklov-Poincaré operators for boundary-value problems: the heterogeneous operator case. In: Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow). SIAM 1991 (pp. 58-81).

8. Pasciak JE. Domain decomposition preconditioners for elliptic problems in two and three dimensions: first approach. In: First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987). SIAM 1988 (pp. 62-72).

9. Bourgat JF, Glowinski R, Le Tallec P, Vidrascu M. Variational formulation and algorithm for trace operator in domain decomposition calculations. In: Domain decomposition methods (Los Angeles, CA), SIAM, 1989 (pp. 3-16).

10. De Roeck YH, Le Tallec P. Analysis and test of a local domain-decomposition preconditioner. In: Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow, 1990), SIAM, 1991 (pp. 112–128).

11. Mandel J. Balancing domain decomposition. Comm. Numer. Methods Engrg.. 1993;9(3):233–241.

12. Mandel J, Brezina M. Balancing domain decomposition for problems with large jumps in coefficients. Math. Comp.. 1996;65(216):1387–1401.

13. Cowsar LC, Mandel J, Wheeler MF. Balancing domain decomposition for mixed finite elements. Math. Comp.. 1995;64(211):989–1015.

14. Dryja M. Substructuring methods for parabolic problems. In: Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Moscow, 1990), SIAM, 1991 (pp. 264–271).

15. Gastaldi L. A domain decomposition for the transport equation. In: Domain decomposition methods in science and engineering (Como), Amer. Math. Soc. 1994 (pp. 97–102).

16. Hoang TTP, Jaffré J, Japhet C, Kern M, Roberts JE. Space-Time Domain Decomposition Methods for Diffusion Problems in Mixed Formulations. SIAM J. Numer. Anal.. 2013;51(6):3532–3559.

17. Gander MJ, Kwok F, Mandal BC. Dirichlet-Neumann and Neumann-Neumann Waveform Relaxation Algorithms for Parabolic Problems. Electron. T. Numer. Ana.. 2016;45:424–456.

18. Le Potier C, Ong TH. A cell-centered scheme for heterogeneous anisotropic diffusion problems on general meshes. International Journal on Finite Volumes. 2012;8:1-40.

19. Ong TH. Cell-centered scheme for heterogeneous anisotropic diffusion problems on general meshes. PhD thesis, Université Paris-Est; 2012.

20. Aavatsmark I, Barkve T, Bøe O, Mannseth T. Discretization on Unstructured Grids for Inhomogeneous, Anisotropic Media. Part I: Derivation of the Methods. SIAM Journal on Scientific Computing. 1998;19(5):1700-1716.
APPENDIX

A COMPUTATION OF MATRICES ASSOCIATED WITH THE DIRICHLET-NEUMANN TO NEUMANN-NEUMANN ALGORITHM

In this section, we present detailed calculations of the matrices in (41). The first two rows of (41) correspond to a choice of test vectors \( v_{Q_1}^{h_1} \) with \( Q_1 \in T_{h_1}^* \setminus \mathcal{N}_{\Gamma_0}^{**} \) in Equation (33a) and \( v_{Q_2}^{h_2} \) with \( Q_2 \in T_{h_2}^* \setminus \mathcal{N}_{\Gamma_0}^{**} \) in Equation (34a), respectively. The third row of (41) is obtained by choosing test vectors \( v_{C_M}^{h_2} \) with \( C_M \in \mathcal{N}_{\Gamma_0}^{**} \) in Equation (34a):

\[
\int_{\Omega_2} \left( \nabla_{\Lambda_2} u^{n+1} \right) \cdot \nabla_{\Lambda_2} v_{h_2}^{C_M} \, dx - \int_{\Gamma} \left( \nabla_{\Lambda_2} u^{n+1} \cdot n_2 \right) \Phi_2 \left( v_{h_2}^{C_M} \right) \, dy = \int_{\Omega_2} f \, \Phi_2 \left( v_{h_2}^{C_M} \right) \, dx, \quad \forall C_M \in \mathcal{N}_{\Gamma_0}^{**},
\]  

(A1)
Using the Neumann condition (34c) and by Green’s formula, we have
\[
\int_{\Gamma} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \cdot n_2 \right) \Phi_2 \left( v^M_{h,2} \right) \, d\gamma = - \int_{\Gamma} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \cdot n_1 \right) \Phi_1 \left( v^C_{h,1} \right) \, d\gamma \\
= - \int_{\Omega_1} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \right) \cdot \nabla_{\Lambda,1} v^C_{h,1} \, dx + \int_{\Omega_2} f_1 \Phi_1 \left( v^C_{h,1} \right) \, dx.
\]

Substituting this into (A1), we find that
\[
\int_{\Omega_1} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \right) \cdot \nabla_{\Lambda,1} v^C_{h,1} \, dx + \int_{\Omega_2} \left( \Lambda_2 \nabla_{\Lambda,2} u^{n+1}_{h,2} \right) \cdot \nabla_{\Lambda,2} v^C_{h,2} \, dx = \int_{\Omega_1} f_1 \Phi_1 \left( v^C_{h,1} \right) \, dx + \int_{\Omega_2} f_2 \Phi_2 \left( v^C_{h,2} \right) \, dx,
\]
(A2)

\[\forall C_M \in \mathcal{N}^{**}_{1,\sigma} \square\]

With the notation in Figure A1, we rewrite (A2) as follows

**FIGURE A1** The triangular elements \((C_M C_N C_\sigma)\), \((C_K C_M C_N)\), \((C_K C_M C_\sigma)\), \((C_K C_M C_\omega)\), \((C_L C_M C_\sigma)\), \((C_L C_M C_\omega)\), \((C_L C_\sigma C_\omega)\), \((C_\sigma C_\omega C_\sigma)\) of \(\mathcal{T}^{**}_{h,1}\) and \((C_M C_P C_\sigma)\), \((C_L C_M C_\sigma)\), \((C_L C_\sigma C_\omega)\), \((C_\sigma C_\omega C_\sigma)\) of \(\mathcal{T}^{**}_{h,2}\).

\[
\int_{\Omega_1 \setminus \{ (C_M C_N C_\sigma) \cup (C_M C_K C_\sigma) \}} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \right) \cdot \nabla_{\Lambda,1} v^C_{h,1} \, dx + \int_{(C_M C_K C_\sigma)} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \right) \cdot \nabla_{\Lambda,1} v^C_{h,1} \, dx \\
+ \int_{(C_M C_N C_\sigma)} \left( \Lambda_2 \nabla_{\Lambda,2} u^{n+1}_{h,2} \right) \cdot \nabla_{\Lambda,2} v^C_{h,2} \, dx + \int_{(C_M C_K C_\sigma)} \left( \Lambda_2 \nabla_{\Lambda,2} u^{n+1}_{h,2} \right) \cdot \nabla_{\Lambda,2} v^C_{h,2} \, dx = \int_{\Omega_1} f_1 \left( v^C_{h,1} \right) \, dx + \int_{\Omega_2} f_2 \left( v^C_{h,2} \right) \, dx.
\]
(A3)

We first evaluate the integral on triangle \((C_M C_K C_\sigma)\). According to the construction of the discrete gradient (5), we have
\[
\nabla_{\Lambda,1} v^C_{h,2} \bigg|_{(C_M C_K C_\sigma)} = \frac{-n_{c_\sigma c_\sigma} - \left( \tilde{\beta}_{1,M} + \tilde{\beta}_{2,M} \right) n_{c_M c_K}}{2m_{(C_M C_K C_\sigma)}},
\]
(A4)

where the coefficients \(\tilde{\beta}_{1,M}\) and \(\tilde{\beta}_{2,M}\) are given in (10). Using this we have
\[
\int_{(C_M C_K C_\sigma)} \left( \Lambda_1 \nabla_{\Lambda,1} u^{n+1}_{h,1} \right) \cdot \nabla_{\Lambda,1} v^C_{h,1} \, dx \\
= \int_{(C_M C_K C_\sigma)} \Lambda_1 \frac{-u^{n+1}_{M,n} n_{c_K c_\sigma} - u^{n+1}_{K,n} n_{c_M c_\sigma} - u^{M,n+1}_{1,\sigma} n_{c_M c_K}}{2m_{(C_M C_K C_\sigma)}} - \frac{n_{c_K c_\sigma} - \left( \tilde{\beta}_{1,M} + \tilde{\beta}_{2,M} \right) n_{c_M c_K}}{2m_{(C_M C_K C_\sigma)}} \cdot \frac{-n_{c_K c_\sigma} - \left( \tilde{\beta}_{1,M} + \tilde{\beta}_{2,M} \right) n_{c_M c_K}}{2m_{(C_M C_K C_\sigma)}}.
\]
or simply,
\[
\int_{(C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^C d\mathbf{x} = \chi_{1,M} u_{1,M}^{p+1} + \chi_{1,K} u_{1,K}^{p+1} + \chi_{1,\sigma} u_{1,\sigma}^{M,p+1},
\]
where
\[
\chi_{1,M} = \frac{1}{4m(C_u C_k C_s)} \left( n_{C_k C_s} + (\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}) n_{C_k u} \right) \Lambda_1 n_{C_k C_s},
\]
\[
\chi_{1,K} = \frac{1}{4m(C_u C_k C_s)} \left( n_{C_k C_s} + (\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}) n_{C_k u} \right) \Lambda_1 n_{C_k C_s},
\]
\[
\chi_{1,\sigma} = \frac{1}{4m(C_u C_k C_s)} \left( n_{C_k C_s} + (\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}) n_{C_k u} \right) \Lambda_1 n_{C_k C_s}.
\]
On the other hand, using strong flux continuity (34b) on the edge \( e = C_M C_s \in \mathcal{E}_1^{**} \), we have
\[
\int_{C_u C_s} \Lambda_1 \nabla \mathbf{u}_{h,2}^{p+1} \cdot n_z \, d\gamma = \int_{C_u C_s} \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \cdot n_z \, d\gamma,
\]
or equivalently as in (8):
\[
\left( \beta_{1,M} u_{1,M}^{p+1} + \beta_{1,K} u_{1,K}^{p+1} + \beta_{1,\sigma} u_{1,\sigma}^{M,p+1} \right) + \left( \beta_{2,M} u_{2,M}^{p+1} + \beta_{2,K} u_{2,K}^{p+1} + \beta_{2,\sigma} u_{2,\sigma}^{M,p+1} \right) = 0.
\]
We can rewrite this equation as follows
\[
(\beta_{1,\sigma} + \beta_{2,\sigma}) u_{1,\sigma}^{M,p+1} = -\beta_{K} u_{1,K}^{p+1} - \beta_{L} u_{2,L}^{p+1} - (\beta_{1,\sigma} + \beta_{2,\sigma}) u_{1,\sigma}^{p+1} - \beta_{2,\sigma} \left( u_{2,\sigma}^{p+1} - u_{1,\sigma}^{p+1} \right).
\]
Thus
\[
u_{1,\sigma}^{M,p+1} = \frac{\beta_{K} u_{1,K}^{p+1} + \beta_{L} u_{2,L}^{p+1}}{\beta_{1,\sigma} + \beta_{2,\sigma}} + \left( \begin{array}{c} \beta_{1,\sigma} + \beta_{2,\sigma} \\ \beta_{1,\sigma} + \beta_{2,\sigma} \\ \beta_{1,\sigma} + \beta_{2,\sigma} \end{array} \right) \left( u_{1,\sigma}^{p+1} - u_{1,\sigma}^{p+1} \right) - \frac{\beta_{1,\sigma} + \beta_{2,\sigma}}{\beta_{1,\sigma} + \beta_{2,\sigma}} \left( u_{2,\sigma}^{p+1} - u_{1,\sigma}^{p+1} \right).
\]

where the coefficients \( \tilde{\beta} \)'s are defined in (10). Substituting (A6) into (A5) we obtain
\[
\int_{(C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^C d\mathbf{x} = \chi_{1,M} + \chi_{1,K} \frac{\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}}{\beta_{1,\sigma} + \beta_{2,\sigma}} u_{1,K}^{p+1} + \chi_{1,\sigma} \frac{\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}}{\beta_{1,\sigma} + \beta_{2,\sigma}} u_{1,\sigma}^{p+1} - \chi_{1,\sigma} \frac{\beta_{2,\sigma}}{\beta_{1,\sigma} + \beta_{2,\sigma}} u_{2,\sigma}^{p+1} + \chi_{1,K} \frac{\beta_{2,\sigma}}{\beta_{1,\sigma} + \beta_{2,\sigma}} u_{2,K}^{p+1} + \chi_{1,\sigma} \beta_{1,\sigma} \beta_{2,\sigma} u_{2,L}^{p+1}.
\]

For the discrete monodomain solution, using (A1) and (A4) we can verify that the coefficients are similar to those in (A7), except for the two last terms which are the coefficients of matrix \( \mathbf{R}_2^{DN} \) in (41):
\[
\int_{(C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^C d\mathbf{x} = \left( \chi_{1,K} + \chi_{1,\sigma} \frac{\tilde{\beta}_{1,M} + \tilde{\beta}_{2,M}}{\beta_{1,\sigma} + \beta_{2,\sigma}} \right) u_{1,K}^{p+1} + \chi_{1,\sigma} \beta_{1,\sigma} \beta_{2,\sigma} u_{2,L}^{p+1}.
\]
Similar calculations can be done for the remaining integrals in (A3).

Now to derive an explicit form of \( \mathbf{R}_1^{DN} \) and \( \mathbf{G}_1^{DN} \), we choose test vectors \( \mathbf{u}_{h,1}^K \) with \( K \in C_1 \) in Equation (33a) (similar calculations can be done to find \( \mathbf{R}_2^{DN} \) by choosing \( \mathbf{u}_{h,2}^L \) with \( L \in C_2 \) in Equation (34a)). We have
\[
\int_{\Omega_1 \setminus (C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^K d\mathbf{x} + \int_{(C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^K d\mathbf{x} + \int_{(C_M C_k C_s)} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \right) \cdot \nabla \mathbf{u}_{h,1}^K d\mathbf{x}
\]
\[
- \int_{C_u C_s} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \cdot n_1 \right) \Phi_1 (\mathbf{u}_{h,1}^K) d\gamma - \int_{C_u C_s} \left( \Lambda_1 \nabla \mathbf{u}_{h,1}^{p+1} \cdot n_1 \right) \Phi_1 (\mathbf{u}_{h,1}^K) d\gamma = \int f (\Phi_1 (\mathbf{u}_{h,1}^K)) d\mathbf{x},
\]
For the interface integral, we write

\[
\int_{\Gamma} \left( \Lambda_1 \nabla u_{h,1}^{n+1} \cdot n_1 \right) \Phi_1 (v^K_{h,1}) d\gamma = \int_{\Gamma} \left( \Lambda_2 \nabla u_{h,2}^{n+1} \cdot n_1 \right) \Phi_2 \left( E_{h,2} v^K_{1,1} \right) d\gamma + \int_{\Gamma} \left( \Lambda_2 \nabla u_{h,2}^{n+1} \cdot n_1 \right) \Phi_2 \left( E_{h,2} u^K_{1,1} \right) d\gamma
\]

(A9)

Recall that the extension operator \( E_{h,2} \) is defined in (26). By Green's formula, we have

\[
\int_{\Gamma} \left( \Lambda_2 \nabla u_{h,2}^{n+1} \cdot n_1 \right) \Phi_2 \left( E_{h,2} v^K_{1,1} \right) d\gamma = -\int_{\Gamma} \left( \Lambda_2 \nabla u_{h,2}^{n+1} \right) \cdot \nabla E_{h,2} v^K_{1,1} d\gamma + \int_{\Gamma} f_2 \Phi_2 \left( E_{h,2} v^K_{1,1} \right) d\gamma
\]

(A10)

Substituting this into (A9) we find that the interface integral can be expressed as a linear combinations of \( u_{h,1}^{n+1}, u_{h,2}^{n+1}, u_{h,2}^{n+1} - u_{h,1}^{n+1} \) and \( \int_{\Gamma} \left( \Lambda_2 \nabla u_{h,1}^{n+1} \cdot n_1 \right) \Phi_1 (v^K_{1,1}) d\gamma - \int_{\Gamma} \left( \Lambda_2 \nabla u_{h,2}^{n+1} \cdot n_1 \right) \Phi_2 \left( E_{h,2} u^K_{1,1} \right) d\gamma \).

This also holds for all the terms on the left hand side of (A8), which defines the coefficients of matrices \( R_{1}^{DN} \) and \( G_{1}^{DN} \). In addition, from (A9), we see that \( G_{1}^{DN} \) is a sparse matrix with 1 and \(-1\) entries.

For the monodomain problem, choosing the test vector \( v^K_{h} \) with \( K \in C_1 \) in (14) lead to:

\[
\int_{\Omega_1} \left( \Lambda_1 \nabla u_{h} \cdot \nabla v_{h} \right) dx + \int_{\Gamma} \left( \Lambda_1 \nabla u_{h} \cdot \nabla v_{h} \right) d\gamma + \int_{\Omega_1} \left( \Lambda_2 \nabla u_{h} \cdot \nabla v_{h} \right) dx + \int_{\Gamma} \left( \Lambda_2 \nabla u_{h} \cdot \nabla v_{h} \right) d\gamma = \int f_1 \Phi (v^K_{h}) dx + \int_{\Omega_1} f_2 \Phi (v^K_{h}) dx,
\]

(A11)

in which the left hand side is a linear combination of \( u_{h,1}^{n+1}, u_{h,2}^{n+1}, u_{h,2}^{n+1} - u_{h,1}^{n+1} \) and the associated coefficients are similar to those in (A8) (by using (A9) and (A10), and following similar calculations in (A5)-(A7)).