The dark sector from interacting canonical and non-canonical scalar fields

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Abstract

In this work general models with interactions between two canonical scalar fields and between one non-canonical (tachyon type) and one canonical scalar field are investigated. The potentials and couplings to the gravity are selected through the Noether symmetry approach. These general models are employed to describe interactions between dark energy and dark matter, with the fields being constrained by the astronomical data. The cosmological solutions of some cases are compared with the observed evolution of the late Universe.

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1. Introduction

It is well known that common matter cannot explain the observed galaxy rotation curves so another type of matter is naturally necessary. Although this is an old problem [1], up till now it is not solved and the most accepted explanation is that there exists a strange matter field which interacts only gravitationally with the known matter—the so-called dark matter [2]. The recent data from the gravitational lensing effect strongly support the existence of dark matter [3, 4].

More recently, the astronomical observations have indicated that the Universe is expanding acceleratedly in late time [5, 6]. But the standard cosmology cannot explain this observed behavior and cosmologists are looking for explanations to the current accelerated period. Until recently the most accepted idea is that there exists an exotic component with negative pressure which causes the accelerated expansion of the Universe—the so-called dark energy. It is generally described by a scalar field [7, 8] and composes the most part of the energy of the Universe at present.

After the discovery of the accelerated expansion of the Universe, several models taking into account dark energy and dark matter—the so-called dark sector—were proposed and the
most of them consider dark energy and dark matter as non-interacting fields. More recently, in the literature models with interaction in the dark sector—an interesting analysis of the viability of such an interaction can be found in [9]—and the effects of a possible dark interaction upon the dynamics of galaxy clusters appear to be in agreement with the observations [10]. In [11] the authors analyze the energy exchange between the dark fields. The works [12–14] propose models which use a priori specified scalar fields for the representation of the dark sector whereas the works [15, 16] suppose certain interactions between the dark fields and represent them by the relations involving their a priori non-specified energy densities which are posteriorly determined. An interacting dark sector non-minimally coupled to the gravity is proposed in [17], and in [18] the author investigates a model of dark energy interacting with neutrinos and dark matter. The growth of structures under the interaction between dark matter and dark energy was investigated in [19]. Also in the matter of scalar fields, the tachyon type scalar field has received considerable attention in cosmology since it can simulate the dark energy with certain success [20–26].

In order to describe the late Universe, we consider in this work a spatially flat, homogeneous and isotropic Universe composed of an interacting dark sector and a common matter field. The dark sector will be investigated from two general models: interacting canonical scalar fields and interacting non-canonical (tachyon type) and canonical scalar fields. The analysis starts from a general action and the potentials and couplings to the gravity are selected from the condition of existence for the Noether symmetry [26–30]. Each set of potentials and couplings satisfying the symmetry condition corresponds to a particular model. The field equations for some particular models resulting from the symmetry are solved and their respective cosmological scenarios are analyzed. The cosmological solutions show that these kinds of models produce decelerated–accelerated regimes from a dynamics with energy exchange among the gravitational field and dark fields. The resulting cosmological scenarios present good agreement with the observational data.

This paper is organized as follows. In the second section the general model of interacting canonical scalar fields is analyzed. In subsection 2.1 the field equations are derived from a point-like Lagrangian. One selects the potentials and couplings by the Noether symmetry approach in subsection 2.2. And in subsection 2.3 the cosmological solutions for the most general cases are obtained. The third section treats the general model of interacting canonical and non-canonical scalar fields. In subsection 3.1 the potentials and couplings are selected by the Noether symmetry. The field equations are derived in section 3.2 (from a point-like Lagrangian) and the resulting equations of energy exchange are obtained. The cosmological solutions for the minimally and non-minimally coupled cases are obtained in subsection 3.3. The conclusions about the results close the paper in the fourth section. In this work we will adopt the signature (+, −, −, −) for the metric and the natural units $8\pi G = c = \hbar = 1$.

2. Interacting canonical scalar fields

2.1. General action and field equations

Let us take a general action for two interacting scalar fields non-minimally coupled to the gravity of the form

$$S = \int d^4x \sqrt{-g} \left\{ \left[ F(\phi) + G(\chi) \right] R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - W(\phi, \chi) \right\} + S_m, $$

(1)

with $S_m = \int d^4x \sqrt{-g} L_m$ being an additional action which represents a common matter field. Here $R$ is the Ricci scalar and $F(\phi), G(\chi)$ denote generic $C^2$ functions which describe
the coupling of the scalar fields to the gravitational field. Furthermore, \( V(\phi) \) is the self-interaction potential of the field \( \phi \) and \( W(\phi,\chi) \) describes the interaction between the fields \( \phi \) and \( \chi \), including the self-interaction of the field \( \chi \). In this action the Einstein coupling is recovered when \( F(\phi) + G(\chi) \to 1/2 \).

By varying the action (1) with respect to the metric tensor \( g_{\mu\nu} \), we obtain the following modified Einstein’s equations:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{T_{\mu\nu}}{2(F + G)},
\]

where \( T_{\mu\nu} = T^m_{\mu\nu} + T^\phi_{\mu\nu} + T^\chi_{\mu\nu} \) denotes the total energy–momentum tensor related to all components of the Universe and the letters \( m, \phi \) and \( \chi \) designate the energy–momentum tensor of the common matter and fields \( \phi \) and \( \chi \), respectively. They are given by

\[
T^m_{\mu\nu} = \frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_m) \delta g^{\mu\nu},
\]

\[
T^\phi_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \left( \frac{1}{2} \partial_\theta \phi \partial_\theta \phi - V \right) g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\theta \nabla_\theta) F, \quad \text{(4)}
\]

\[
T^\chi_{\mu\nu} = \partial_\mu \chi \partial_\nu \chi - \left( \frac{1}{2} \partial_\theta \chi \partial_\theta \chi - W \right) g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\theta \nabla_\theta) G, \quad \text{(5)}
\]

with \( \nabla_\theta \) denoting the covariant derivative.

Let us consider a flat Friedmann–Robertson–Walker (FRW) metric, \( ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2) \)—where \( a(t) \) is the scale factor—and suppose that the scalar fields are homogeneous, \( \phi = \phi(t) \) and \( \chi = \chi(t) \), and that the common matter is a pressureless fluid. Hence, we can write from the action (1) the point-like Lagrangian

\[
\mathcal{L} = 6a \ddot{a} (F + G) + 6a^2 \dot{a} \left( \frac{dF}{d\phi} \dot{\phi} + \frac{dG}{d\chi} \dot{\chi} \right) - a^3 \left( \frac{1}{2} \dot{\phi}^2 - V + \frac{1}{2} \dot{\chi}^2 - W \right) + \rho_m^0, \quad \text{(6)}
\]

where \( \rho_m^0 \) is the energy density of the common matter field at an initial instant and the point represents derivative with respect to time.

The Euler–Lagrange equations applied to the Lagrangian (6) for \( a, \phi \) and \( \chi \) furnish

\[
2 \dot{H} + 3H^2 = - \frac{p}{2(F + G)}, \quad \text{(7)}
\]

\[
\dot{\phi} + 3H \phi - 6(\dot{H} + 2H^2) \frac{dF}{d\phi} + \frac{dV}{d\phi} + \frac{\partial W}{\partial \phi} = 0, \quad \text{(8)}
\]

\[
\dot{\chi} + 3H \chi - 6(\dot{H} + 2H^2) \frac{dG}{d\chi} + \frac{\partial W}{\partial \chi} = 0, \quad \text{(9)}
\]

respectively. Moreover, by imposing that the energy function associated with the Lagrangian (6) vanishes, one obtains the modified Friedmann equation, i.e.

\[
E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} \ddot{a} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \ddot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\chi}} \ddot{\chi} - \mathcal{L} \equiv 0, \quad \implies \quad H^2 = \frac{\rho}{6(F + G)}, \quad \text{(10)}
\]

In the above equations \( H = \dot{a}/a \) denotes the Hubble parameter. The set (7)–(10) are the field equations, where \( \rho = \rho_m + \rho_\phi + \rho_\chi \) and \( p = p_\phi + p_\chi \), which are given by

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V - 6H \frac{dF}{d\phi}, \quad \text{(11)}
\]
\[ \rho_\chi = \frac{1}{2} \dot{\chi}^2 + W - 6H \frac{dG}{d\chi} \dot{\chi}, \quad (12) \]
\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V + 2 \left( \frac{dF}{d\phi} \dot{\phi} + 2H \frac{dF}{d\phi} \dot{\phi} \right), \quad (13) \]
\[ p_\chi = \frac{1}{2} \dot{\chi}^2 - W + 2 \left( \frac{dG}{d\chi} \dot{\chi} + 2H \frac{dG}{d\chi} \dot{\chi} \right), \quad (14) \]

Now we will calculate the covariant derivative of the total energy–momentum tensor, $\nabla_\mu T^\mu_\nu = \nabla_\mu T^\mu_\phi + \nabla_\mu T^\mu_\chi$, in order to analyze the energy exchange among the fields. Firstly, the computation of the quantity $\dot{\rho} + 3H(\rho + p)$ for the energy densities (11) and (12) and their respective pressures leads to
\[ \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\frac{\partial W}{\partial \phi} \dot{\phi} + \frac{(dF/d\phi) \dot{\phi}}{F + G} \rho, \quad (15) \]
\[ \dot{\rho}_\chi + 3H(\rho_\chi + p_\chi) = \frac{\partial W}{\partial \chi} \dot{\chi} + \frac{(dG/d\chi) \dot{\chi}}{F + G} \rho, \quad (16) \]

where (8), (9) and (10) were used for the simplifications. Equations (15) and (16) are the same as those resulting from the covariant derivative of the energy–momentum tensors for the scalar fields $\phi$ and $\chi$ ($\nabla_\mu T^\mu_\phi$ and $\nabla_\mu T^\mu_\chi$), respectively. Then, remembering that $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, we find that the covariant derivative of the total energy–momentum tensor is
\[ \nabla_\mu T^\mu_\nu = \frac{\rho}{F + G} \left( \frac{dF}{d\phi} \dot{\phi} + \frac{dG}{d\chi} \dot{\chi} \right). \quad (17) \]

By observing (15) and (16), we note that their first terms on the right-hand side represent the energy exchange between the fields $\phi$ and $\chi$ and their second terms on the right-hand side describe the energy exchange among the scalar fields and gravitational field. From (17) one concludes that if $F$ and $G$ are constants, $\nabla_\mu T^\mu_\nu = 0$, meaning that when the coupling is minimal there is no energy exchange among the scalar fields and gravitational field. If this is the case, there exists an energy exchange only between the scalar fields, and consequently the total energy related to the components of the Universe is conserved.

2.2. Couplings and potentials from the Noether symmetry

Starting with a general action, we can restrict the forms of the undefined couplings and potentials through the requirement of mathematical proprieties for the Lagrangian, such as symmetries. The symmetries may generate some formal suggestions for the possible forms of the undefined functions. In this work we will require that the Lagrangian of the general model satisfies the Noether symmetry, which provides a conserved quantity associated with the dynamical system. Interesting results may arise from the Noether symmetry approach, as can be found in [26–30].

A Noether symmetry for a given Lagrangian of the form $L = L(q_i, \dot{q}_i)$ exists if the condition $L_x L = X L = 0$ is satisfied, with $L_x$ designating the Lie derivative with respect to the vector field $X$ defined by
\[ X = \alpha_i \frac{\partial}{\partial q_i} + \frac{d\alpha_i}{dt} \frac{\partial}{\partial \dot{q}_i}, \quad (18) \]
where the \( \alpha_i \)'s are functions of the generalized coordinates \( q_i \). The conserved quantity associated with the Noether symmetry generated by \( X \) is given by

\[
M_0 = \alpha_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.
\]  

The condition of existence for the Noether symmetry \( L_X \mathcal{L} = X \mathcal{L} = 0 \) is applied to the point-like Lagrangian (6), with the vector field \( X \) defined for our problem as follows:

\[
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \chi} + \frac{\partial \alpha}{\partial t} \frac{\partial}{\partial a} + \frac{\partial \beta}{\partial t} \frac{\partial}{\partial \phi} + \frac{\partial \gamma}{\partial t} \frac{\partial}{\partial \chi},
\]  

where \( \alpha, \beta \) and \( \gamma \) are functions of \( (a, \phi, \chi) \). In this case we obtain the following coupled system of differential equations:

\[
(F + G) \left( \alpha + 2a \frac{\partial \alpha}{\partial a} \right) + a \frac{dF}{d\phi} \left( \beta + a \frac{\partial \beta}{\partial a} \right) + a \frac{dG}{d\chi} \left( \gamma + a \frac{\partial \gamma}{\partial a} \right) = 0,
\]

\[
3\alpha - 12 \frac{dF}{d\phi} \frac{\partial \alpha}{\partial \phi} + 2a \frac{\partial \beta}{\partial \phi} = 0,
\]

\[
3\alpha - 12 \frac{dG}{d\chi} \frac{\partial \alpha}{\partial \chi} + 2a \frac{\partial \gamma}{\partial \chi} = 0,
\]

\[
a^2 \frac{d^2 F}{d\phi^2} + \left( 2\alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) \frac{dF}{d\phi} + a \frac{dG}{d\chi} \frac{\partial \gamma}{\partial \phi} + 2 \frac{\partial \alpha}{\partial \phi} (F + G) - \frac{a^2}{6} \frac{\partial \beta}{\partial a} = 0,
\]

\[
a^2 \frac{d^2 G}{d\chi^2} + \left( 2\alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \gamma}{\partial \chi} \right) \frac{dG}{d\chi} + a \frac{\partial \beta}{\partial \phi} \frac{dF}{d\chi} + 2 \frac{\partial \alpha}{\partial \phi} (F + G) - \frac{a^2}{6} \frac{\partial \gamma}{\partial a} = 0,
\]

\[
\frac{\partial \alpha}{\partial \phi} \frac{dG}{d\chi} + \frac{\partial \alpha}{\partial \phi} \frac{dF}{d\chi} - \frac{a}{6} \left( \frac{\partial \beta}{\partial \chi} + \frac{\partial \gamma}{\partial \phi} \right) = 0,
\]

\[
3\alpha (V + W) + a^2 \left( \frac{dV}{d\phi} + \frac{\partial W}{\partial \phi} \right) + a \frac{\partial W}{\partial \chi} = 0.
\]

The solution of the coupled system of differential equations (21)–(27) is not unique and several solutions are found in tables 1 and 2 which contain all sets of functions \( \alpha, \beta, \gamma, F, G, V, W \), where the quantities \( \alpha_0, \beta_0, \gamma_0, F_0, G_0, V_0, W_0 \) are constants and \( K = \beta_0 / \gamma_0 \). We have looked for solutions which always furnish for the function \( W \) an expression different from \((0, \text{constant}, f(\phi), g(\chi))\) in order to guarantee an interaction between the fields \( \phi \) and \( \chi \).

One may observe from table 2 that the general forms of \( W \) provided by the Noether symmetry allow the existence of sums which incorporate terms of the form \( f(\phi) \)—representing an additional term of self-interaction for the field \( \phi \)—namely \( W = f(\phi) + g(\phi, \chi) \). In this case, one must redefine the potentials in the energy–density equations (11) and (12) and
performed. They are summarized in table 3. Due to the interaction between the fields \( \phi \) and \( \chi \) pressure equations (13) and (14) by writing \( W \rightarrow W = W - f(\phi) \) and \( V \rightarrow V = V + f(\phi) \). So we take into account the additional self-interaction term of the field \( \phi \).

Case II in table 1 with \( V = V_0 \phi^2 \) is similar to the model analyzed in [17] and case III in table 2, when \( h = \phi^2 + \chi^2 \), is the model proposed in [12] with \( V = W_0 \phi^4 \) but with an additional self-interaction term of the form \( W_0 \chi^2 \). The models of [12, 17] are the particular cases of the one denoted by I in table 2.

From equation (19) we can write the conserved quantities associated with the cases in tables 1 and 2. They are summarized in table 3.

### 2.3. Solutions of the field equations

Due to the interaction between the fields \( \phi \) and \( \chi \) and the presence of a common matter field in the action (1), the field equations become more complicated than in the case with two non-interacting scalar fields and without a common matter field [28]. Hence, the search for numerical solutions of the system (7)–(10) for the most general cases in tables 1 and 2 will be performed.

Let us transform the derivatives with respect to time in equations (7)–(10) into derivatives with respect to red-shift through the relationships

\[
z = \frac{1}{a} - 1, \quad \frac{d}{dt} = -H(1 + z) \frac{d}{dz},
\]

and divide all the equations by \( \rho_0 \)—the total energy density of the Universe at the present time. Hence, one obtains from equations (7)–(10) the following system of coupled differential equations:

\[
4 \ddot{H} \left(1 + z \right) (F + G) = \ddot{\rho} + \ddot{p},
\]

\[
\ddot{H}^2 (1 + z)^2 \phi'' + \ddot{H} \left( \dot{H}' (1 + z) - 2 \ddot{H} \right) \left[ (1 + z) \phi' + 6 \frac{dF}{d\phi} \right] + \frac{d}{d\phi} \dddot{W} + \ddot{\phi} \frac{dW}{d\phi} = 0.
\]

| Cases | \( M_0 \) |
|-------|----------|
| I–III Tab 1 | \( \frac{1}{2} \sigma_0 a^3 \left[ 2H \left[ 4(F + G) - 3 \left( \frac{\partial W}{\partial \phi} + \frac{\partial W}{\partial \chi} \right) \right] + \left( 4 \frac{dF}{d\phi} + \phi + \left( \frac{\partial W}{\partial \phi} + \chi \right) \dot{\chi} \right) \right] \) |
| I–IV Tab 2 | \( \beta_0 a^3 \left[ 6H \left( \frac{\partial W}{\partial \phi} - \phi \frac{\partial W}{\partial \chi} \right) + \phi \dot{\chi} - \chi \dot{\phi} \right] \) |
| V–VI Tab 2 | \( \gamma_0 a^3 \left[ 6H \left( K \frac{dF}{d\chi} + \frac{\partial W}{\partial \chi} \right) - K \phi - \dot{\chi} \right] \) |

### Table 2. Solutions with \( \alpha = 0 \) and \( (\beta, \gamma) \neq 0 \).

| \( \beta \) | \( \gamma \) | \( F \) | \( G \) | \( V \) | \( W \) |
|-------|-------|-------|-------|-------|-------|
| I \( \beta_0 \chi \) | \( -\beta_0 \phi \) | \( F_0^1 + F_0 \phi^2 \) | \( G_0^1 + G_0 \chi^2 \) | \( \int \left( \frac{\partial W}{\partial \phi} - \frac{\partial W}{\partial \chi} \right) d\phi \) | \( W \) |
| II \( \beta_0 \chi \) | \( -\beta_0 \phi \) | \( F_0^1 + F_0 \phi^2 \) | \( G_0^1 + G_0 \chi^2 \) | \( 0, V_0 \) | \( W(\phi + \chi^2) \) |
| III \( \beta_0 \chi \) | \( -\beta_0 \phi \) | \( F_0 \) | \( G_0 \) | \( \int \left( \frac{\partial W}{\partial \phi} - \frac{\partial W}{\partial \chi} \right) d\phi \) | \( W \) |
| IV \( \beta_0 \chi \) | \( -\beta_0 \phi \) | \( F_0 \) | \( G_0 \) | \( 0, V_0 \) | \( W(\phi + \chi^2) \) |
| V \( \beta_0 \) | \( \gamma_0 \) | \( F_0^1 + F_0 \phi \) | \( G_0^1 - K F_0 \chi \) | \( 0, V_0 \) | \( W(\phi - K \chi) \) |
| VI \( \beta_0 \) | \( \gamma_0 \) | \( F_0 \) | \( G_0 \) | \( 0, V_0 \) | \( W(\phi - K \chi) \) |
\[ \tilde{H}^2 (1 + z)^2 \chi'' + \tilde{H} [\tilde{H}' (1 + z) - 2 \tilde{H}] \left[ (1 + z) \chi' + 6 \frac{dG}{d\chi} \left( \frac{dF}{d\phi} \phi' \right)' \right] + \frac{\partial \tilde{W}}{\partial \chi} = 0, \quad (31) \]

by taking into account equation (10). Above, the prime represents the derivative with respect to \( z \) and the following dimensionless quantities were introduced: \( \tilde{\rho} = \rho / \rho_0 = \rho_m / \rho_0 + \rho_\phi / \rho_0 + \rho_x / \rho_0 = \tilde{\rho}_m + \tilde{\rho}_\phi + \tilde{\rho}_x \), \( \tilde{\rho} = p / \rho_0 = \rho_\phi / \rho_0 + p_x / \rho_0 = \tilde{\rho}_\phi + \tilde{\rho}_x \), \( \tilde{\rho} = H / \sqrt{\rho_0} \), \( \tilde{V} = V / \rho_0 \) and \( \tilde{W} = W / \rho_0 \). Furthermore, the dimensionless energy densities and pressures read

\[ \tilde{\rho}_m = \tilde{\rho}_m^0 (1 + z)^3, \quad (32) \]
\[ \tilde{\rho}_\phi = \frac{\tilde{H}^2 (1 + z)^2 \phi^2}{2} + \tilde{V} + 6 \tilde{H}^2 (1 + z) \frac{dF}{d\phi} \phi', \quad (33) \]
\[ \tilde{\rho}_x = \frac{\tilde{H}^2 (1 + z)^2 \chi^2}{2} + \tilde{W} + 6 \tilde{H}^2 (1 + z) \frac{dG}{d\chi} \chi', \quad (34) \]
\[ \tilde{p}_\phi = \frac{\tilde{H}^2 (1 + z)^2 \phi^2}{2} - \tilde{V} + 2 \tilde{H} (1 + z) \left\{ \tilde{H} (1 + z) \left( \frac{d^2 F}{d\phi^2} \phi'^2 + \frac{dF}{d\phi} \phi'' \right) \right\} + \tilde{H} (1 + z) - \tilde{H} \frac{dF}{d\phi} \phi', \quad (35) \]
\[ \tilde{p}_x = \frac{\tilde{H}^2 (1 + z)^2 \chi^2}{2} - \tilde{W} + 2 \tilde{H} (1 + z) \left\{ \tilde{H} (1 + z) \left( \frac{d^2 G}{d\chi^2} \chi'^2 + \frac{dG}{d\chi} \chi'' \right) \right\} + \tilde{H} (1 + z) - \tilde{H} \frac{dG}{d\chi} \chi', \quad (36) \]

Our aim is to use the solutions given in tables 1 and 2 in order to describe the dark sector as an interacting structure. Then one considers that the fields \( \phi \) and \( \chi \) correspond to the dark energy and dark matter fields, respectively. This choice requires the following features for each field: (i) the field \( \phi \) must have a negative pressure in the late time and its energy density composes the most part of the total energy density of the Universe at the present time; (ii) the field \( \chi \) has a small positive pressure in comparison to the pressure modulus of the dark energy and its energy density still represents a considerable fraction of the total energy density of the Universe at the present time.

To satisfy the above requirements we will use the initial conditions for the system (29)–(31) which match the astronomical data. At \( z = 0 \) one introduces the quantities \( \tilde{\rho}_m (0) = \rho_m^0 / \rho_0 = \Omega_m^0 \), \( \tilde{\rho}_\phi (0) = \rho_\phi^0 / \rho_0 = \Omega_\phi^0 \) and \( \tilde{\rho}_x (0) = \rho_x^0 / \rho_0 = \Omega_x^0 \), where \( \Omega_i^0 \) denotes the value of the density parameter of each component at the present time whereas \( \Omega_0 = \Omega_m^0 + \Omega_\phi^0 + \Omega_x^0 \) refers to the total density parameter. The values of the density parameters adopted here are \( \Omega_0^0 = 0.05 \), \( \Omega_\phi^0 = 0.72 \) and \( \Omega_x^0 = 0.23 \) (see e.g. [31]). Further, in agreement with requirement (i) one has that \( \phi(0)^2 \ll 1 \), which means that the field \( \phi \) varies very slowly in the late time, i.e. \( \phi'(0) = 0 \), with \( \epsilon \) very small. From (33) and this last condition it follows that \( \tilde{V} (0) \approx \Omega_\phi^0 \) and one may obtain the initial condition for \( \phi \). Once \( \phi(0) \) is fixed, one may determine the initial condition for \( \chi \) from the necessity that the coupling has the present value 1/2, i.e. \( F(0) + G(0) = 1/2 \). Satisfying requirement (ii) by the condition \( \tilde{\rho}_x (0) = \Omega_x^0 \), and since one knows \( \chi (0) \), from (34) we may determine the initial condition for \( \chi \). The part of requirement (ii) that is related to the value of \( p_x \) can be satisfied through the adjustments of the constants that appear in the functions of the couplings and potentials. From (10) we have for the Hubble parameter the initial condition \( \tilde{H} (0) = \sqrt{\Omega_0^0/6[ F(0) + G(0)]} = 1/\sqrt{3} \). Summing up one has

(a) \( \tilde{H} (0) = \frac{1}{\sqrt{3}} \);
(b) $\tilde{V}(0) = G^0_0$ determines $\phi(0)$, $\phi'(0)^2 \ll 1$;
(c) $G(0) = \frac{1}{2} - F(0)$ determines $\chi(0)$;
(d) $\chi'(0)^2 + 6\tilde{W}(0) + 12\chi'(0)\frac{dW}{d\chi}|_{z_0} = 6G^2_0$ determines $\chi'(0)$.

For case I in table 1 we take

$$V = V_0\phi^2, \quad f\left(\frac{\chi}{\phi}\right) = \frac{\chi}{\phi}, \quad \text{which implies} \quad W = W_0\phi\chi.$$  \hspace{1cm} (37)

By using the above equations, the initial conditions are

$$\phi(0) = \sqrt{\frac{0.72}{V_0}}, \quad \chi(0) = \sqrt{\frac{1/2 - F_0\phi(0)^2}{G_0}},$$
$$\chi'(0) = \sqrt{6\left[0.23 + 24G^2_0\chi(0)^2 - \tilde{W}_0\phi(0)\chi(0)\right] - 12G_0\chi(0)},$$

where

$$F_0 \leq \frac{1}{2\phi(0)^2} \quad \text{and} \quad \tilde{W}_0 \leq \frac{0.23/\chi(0) + 24G^2_0\chi(0)}{\phi(0)}.$$  \hspace{1cm} (38)

For this case we have adopted the following values for the fixed constants in the numerical computations: $F_0 = -0.002366$, $G_0 = 0.04651$, $V_0 = 0.010101$ and $W_0 = 0.01856$.

Now, for case I in table 2, by considering $F^0_0 = G^1_0 = 0$ without loss of generality, one takes

$$h(\phi^2 + \chi^2) = \phi^2 + \chi^2 \quad \text{which implies} \quad W = W_0(\chi^4 + 2\phi^2\chi^2), \quad V = W_0\phi^4,$$

with $W_0$ from table 2 replaced by $4W_0$.

For this case one has the initial conditions

$$\phi(0) = \left(\frac{0.72}{W_0}\right)^{1/4}, \quad \chi(0) = \sqrt{\frac{1/2 - F_0\phi(0)^2}{G_0}},$$
$$\chi'(0) = \sqrt{6\left[0.23 + (24G^2_0 - \tilde{W}_0)\chi(0)^4 - 2\tilde{W}_0\phi(0)^2\chi(0)^2\right] - 12G_0\chi(0)},$$

where

$$F_0 \leq \frac{1}{2\phi(0)^2} \quad \text{and} \quad \tilde{W}_0 \leq \frac{0.23 + 24G^2_0\chi(0)^4}{\chi(0)^4 + 2\phi(0)^2\chi(0)^2}.$$  \hspace{1cm} (39)

In this case we have taken for the fixed constants the values $F_0 = 0.2064$, $G_0 = 0.03333$, and $W_0 = 0.1250$.

In figure 1 the density parameters of the common matter, dark energy and dark matter for cases 1 and 2 in the left and right frames, respectively, are represented. From this figure we can observe the evident difference of the increase of the density parameter of the quintessence with the red-shift and the corresponding decrease of the density parameter of the dark matter for the two cases. Since the gravitational coupling has quadratic forms in both cases, the different red-shift evolutions of the density parameters are determined uniquely by the interaction and self-interaction potentials of the fields, which are responsible for the energy transfer between the scalar fields and among the scalar fields and gravitational field, as can be observed from (15) and (16). The energy transfer among the fields (scalar field–scalar field and gravitational field–scalar field) have a definitive role in the variety of behaviors which can be produced by models with scalar fields, as can be seen from these two cases in figure 1. This is definitely verified when one observes from figure 1 that the common matter field, which is not coupled to the other fields, has a red-shift evolution of its density parameter quite identical in the two cases, meaning that it is just submitted to the dilution caused by the expansion of the
Figure 1. Density parameters of the common matter and scalar fields as the functions of the red-shift $z$. Left frame: case 1; right frame: case 2.

Figure 2. Ratio of the pressure and energy density of the scalar fields as the functions of the red-shift $z$. Left frame: case 1; right frame: case 2.

The ratio of the pressure and energy density of the scalar fields $\omega_\phi = p_\phi/\rho_\phi$ and $\omega_\chi = p_\chi/\rho_\chi$ are plotted in figure 2, where the left frame corresponds to case 1 and the right frame to case 2. These figures show that the pressure relative to the energy density of the dark matter field is small in comparison to the one (in modulus) of the dark energy for both cases. But the dark matter pressure has a significant role in the determination of the epoch where the transition of a decelerated expansion to an accelerated expansion of the Universe occurs, since its participation in the total composition of the Universe is significant. Observe that in the present day $\omega_\phi \rightarrow -1$, which corresponds to a cosmological constant.

In the left frame of figure 3 the effective coupling $F+G$ and in the right frame the deceleration parameter $q = \frac{\dot{a}}{a} + \frac{3}{2} \frac{\dot{\rho}}{\rho}$, for cases 1 and 2, are plotted. We can infer from figure 3 that the variation of the effective gravitational coupling presents a small value around its present value $F(0) + G(0) = 1/2$. This variation is about 10%, meaning that the effective gravitational 'constant' varies approximately 10% in the considered interval. The right frame of figure 3 shows that there exists a small difference between the red-shifts of the transition from a decelerated regime to an accelerated regime for the two cases. For cases 1 and 2 the
values of the red-shift transitions are \( z_T = 0.43 \) and \( z_T = 0.52 \) whereas the present values of the deceleration parameter read \( q(0) = -0.53 \) and \( q(0) = -0.52 \), respectively. These results are in good agreement with the observational data, namely \( z_T = 0.74 \pm 0.18 \) (from [32]) and \( q(0) = -0.46 \pm 0.13 \) (from [33]).

3. Interacting canonical and non-canonical scalar fields

3.1. General action and Noether symmetry

Now let us take an action where one scalar field is non-canonical and represented by \( \varphi \), being a tachyon type field, and the other is a canonical scalar field represented by \( \phi \):

\[
S = \int d^4x \sqrt{-g} \left\{ \left[ F(\varphi) + G(\phi) \right] R - V(\varphi) \sqrt{1 - \partial_\mu \varphi \partial^\mu \varphi} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - W(\varphi, \phi) \right\} + S_m, \tag{40}
\]

where \( F(\varphi) \) and \( G(\phi) \) represent generic \( C^2 \) functions which describe the coupling of the scalar fields to the gravity, \( V(\varphi) \) is the self-interaction potential of the field \( \varphi \) and \( W(\varphi, \phi) \) describes the self-interaction of the field \( \phi \) and the interaction between the fields \( \varphi \) and \( \phi \). As before, when \( F(\varphi) + G(\phi) \to \frac{1}{2} \) we recover the Einstein coupling.

From the variation of the action (40) with respect to \( g_{\mu \nu} \) one obtains the modified Einstein’s equations with the same form of (2):

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = - \frac{T_{\mu \nu}}{2(F + G)}, \tag{41}
\]

being \( T_{\mu \nu} = T_{\mu \nu}^m + T_{\mu \nu}^\varphi + T_{\mu \nu}^\phi \) defined as follows:

\[
T_{\mu \nu}^m = \frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} L_m \right)}{\delta g^{\mu \nu}}, \tag{42}
\]

\[
T_{\mu \nu}^\varphi = V \left( \frac{\partial_\mu \varphi \partial_\nu \varphi}{\sqrt{1 - \partial_\mu \varphi \partial^\mu \varphi}} + g_{\mu \nu} \sqrt{1 - \partial_\mu \varphi \partial^\mu \varphi} \right) + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \nabla^\alpha \nabla^\alpha) F, \tag{43}
\]

\[
T_{\mu \nu}^\phi = \partial_\mu \phi \partial_\nu \phi - \left( \frac{1}{2} \partial_\mu \phi \partial^\beta \phi - W \right) g_{\mu \nu} + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \nabla^\alpha \nabla^\alpha) G. \tag{44}
\]
By considering again a flat FRW metric and the scalar fields homogeneous, \( \varphi = \varphi(t) \) and \( \phi = \phi(t) \), with the common matter being a pressureless fluid, the point-like Lagrangian which follows from the action (40) reads

\[
\mathcal{L} = 6a^2 \dot{a}^2 (F + G) + 6a^2 \dot{a} \left( \frac{dF}{d\varphi} \dot{\varphi} + \frac{dG}{d\phi} \dot{\phi} \right) + a^3 \sqrt{1 - \dot{\varphi}^2} - a^3 \left( \frac{1}{2} \dot{\varphi}^2 - W \right) + \rho_m^0.
\] (45)

From the condition of the existence for the Noether symmetry \( L_x \mathcal{L} = X \mathcal{L} = 0 \) applied to the point-like Lagrangian (45), with the vector field \( X \) now defined as

\[
X = \alpha \frac{\partial}{\partial \alpha} + \beta \frac{\partial}{\partial \beta} + \gamma \frac{\partial}{\partial \gamma} + \delta \frac{\partial}{\partial \delta} + \partial \frac{\partial}{\partial \partial \phi},
\] (46)

where \( \alpha, \beta, \gamma \) and \( \delta \) are functions of \( (a, \varphi, \phi) \), we obtain the following system of partial differential equations:

\begin{align*}
(F + G) \left( \alpha + 2a \frac{\partial \alpha}{\partial a} \right) + a \frac{dF}{d\varphi} \left( \beta + a \frac{\partial \beta}{\partial a} \right) + a \frac{dG}{d\phi} \left( \gamma + a \frac{\partial \gamma}{\partial a} \right) &= 0, \\
\frac{\partial \alpha}{\partial \varphi} &= 0, \\
\frac{\partial \beta}{\partial \varphi} &= 0, \\
\frac{\partial \gamma}{\partial \varphi} &= 0,
\end{align*}
(47)

\begin{align*}
\frac{a\beta}{\partial \varphi} dF &= 0, \\
\frac{a\gamma}{\partial \varphi} dG &= 0,
\end{align*}
(48)

\begin{align*}
3a - 12 \frac{dG}{d\phi} \frac{\partial \alpha}{\partial \phi} + 2a \frac{d\gamma}{\partial \phi} &= 0, \\
\frac{a^2 \gamma}{\partial \phi} &= 0, \\
\frac{a^2 \gamma}{\partial \phi} &= 0,
\end{align*}
(49)

\begin{align*}
\frac{a\beta}{\partial \varphi} dF &= 0, \\
\frac{a\gamma}{\partial \varphi} dG &= 0,
\end{align*}
(50)

\begin{align*}
\frac{\partial \alpha}{\partial \phi} + \frac{\partial \alpha}{\partial \delta} dF &= 0, \\
\frac{\partial \gamma}{\partial \phi} &= 0,
\end{align*}
(51)

\begin{align*}
3a V + a\beta \frac{dV}{d\varphi} &= 0, \\
3a W + a\beta \frac{dW}{d\varphi} + a\gamma \frac{dW}{d\phi} &= 0.
\end{align*}
(52)

The solution of the system (47)–(54) is not unique and the solutions that we found are given in table 4 containing the sets of \( a, \beta, \gamma, F, G, V, W \), where \( a_0, \beta_0, \gamma_0, F_0, F_1, G_0, G_1 \) and \( V_0 \) are constants and \( \mu = 3a_0/\beta_0 \) and \( K = \beta_0/\gamma_0 \). Here we also looked for solutions which present potentials of the form \( W \neq 0 \) (constant, \( f(\varphi), g(\phi) \)) in order to provide an interaction between the fields \( \varphi \) and \( \phi \). And from (19) we have the respective conserved quantities, which are given in table 5.

It is interesting to observe that solution 1 generalizes the model analyzed in [26].

### 3.2. Field equations and energy exchange

From the Euler–Lagrange equations applied to the Lagrangian (45) for \( a, \varphi \) and \( \phi \), respectively, one has

\[
2(F + G)(2H + 3H^2) - V\sqrt{1 - \dot{\varphi}^2} + \frac{1}{2} \dot{\varphi}^2 - W \\
+ 2 \left( \frac{dF}{d\varphi} \dot{\varphi} + 2H \frac{dF}{d\varphi} \dot{\varphi} + \frac{d^2F}{d\varphi^2} \dot{\varphi}^2 \right) + 2 \left( \frac{dG}{d\phi} \dot{\phi} + 2H \frac{dG}{d\phi} \dot{\phi} + \frac{d^2G}{d\phi^2} \dot{\phi}^2 \right) = 0.
\] (55)
and (62), one has 

\[ \alpha \beta \gamma F G V W \]

IV–V associated with the Lagrangian (45) is null, it follows that 

I–III Cases

Class. Quantum Grav. 27 (2010) 175006 R C de Souza and G M Kremer

Table 4. Solutions.

| Cases | \( \alpha \beta \gamma F G V W \) |
|-------|------------------|
| I     | \( \alpha_0 \alpha \beta_0 -3 \alpha_0 \phi/2 F_0 e^{-i\phi} G_0 \phi^2 V_0 e^{-i\phi} f(\phi e^{i\phi}) e^{-i\phi} \) |
| II    | \( \alpha_0 \alpha \beta_0 -3 \alpha_0 \phi/2 F_0 e^{-i\phi} 0 V_0 e^{-i\phi} f(\phi e^{i\phi}) e^{-i\phi} \) |
| III   | \( \alpha_0 \alpha \beta_0 -3 \alpha_0 \phi/2 0 G_0 \phi^2 V_0 e^{-i\phi} f(\phi e^{i\phi}) e^{-i\phi} \) |
| IV    | \( \beta_0 \gamma_0 F_0 + F_0 \phi G_0 - K F_0 \phi V_0 W(\psi - K \phi) \) |
| V     | \( \beta_0 \gamma_0 F_0 G_0 V_0 W(\psi - K \phi) \) |

Table 5. Conserved quantities.

| Cases | \( M_0 \) |
|-------|-----------|
| I–III | \( \frac{1}{2} \alpha_0 \alpha \frac{1}{V} \left[ 2H \left( 4(F + G) + 3 \left( \frac{1}{2F} - \phi \frac{dG}{d\phi} \right) \right) + \left( \frac{4DF}{d\phi} - \frac{2V}{\sqrt{1 - \psi^2}} \right) \phi + \left( \frac{4dG}{d\phi} + \phi \right) \phi \right] \) |
| IV–V  | \( \gamma_0 \alpha \frac{1}{2} \left[ 6H \left( K \frac{dF}{d\phi} + \frac{dG}{d\phi} \right) - \frac{K \psi}{\sqrt{1 - \psi^2}} \right] \) |

\[ \frac{\dot{\psi}}{1 - \psi^2} + 3H \psi + \frac{1}{V} \frac{dV}{d\phi} - 6(\dot{H} + 2H^2) \frac{dG}{d\phi} V = 0. \] (56)

\[ \ddot{\psi} + 3H \dot{\psi} - 6(\dot{H} + 2H^2) \frac{dG}{d\phi} + \frac{\partial W}{\partial \phi} = 0. \] (57)

As in the previous section, by imposing that the energy function \( E_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \psi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \) associated with the Lagrangian (45) is null, it follows that

\[ 6(F + G)H^2 - \frac{\rho_m}{a^3} - \frac{V}{\sqrt{1 - \psi^2}} + 6H \frac{dF}{d\psi} \psi - \frac{1}{2} \dot{\phi}^2 - W + 6H \frac{dG}{d\phi} \frac{\dot{\phi}}{\psi} = 0. \] (58)

From equations (55)–(58) one defines \( \rho = \rho_m + \rho_\phi + \rho_\psi \) and \( p = p_\psi + p_\phi \), with their forms given by

\[ \rho_\phi = \frac{V}{\sqrt{1 - \psi^2}} - 6H \frac{dF}{d\psi} \psi, \] (59)

\[ \rho_\psi = \frac{1}{2} \dot{\phi}^2 + W - 6H \frac{dG}{d\phi}, \] (60)

\[ p_\psi = -V \sqrt{1 - \psi^2} + 2 \left( \frac{dF}{d\psi} \psi + 2H \frac{dF}{d\psi} \psi + \frac{d^2F}{d\psi^2} \psi \right), \] (61)

\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - W + 2 \left( \frac{dG}{d\psi} \dot{\phi} + 2H \frac{dG}{d\phi} \dot{\phi} + \frac{d^2G}{d\phi^2} \dot{\phi} \right)^2, \] (62)

in agreement with the definitions of the energy–momentum tensors (43) and (44).

By using the definitions of the energy densities (59) and (60) and their respective pressures (61) and (62), one has

\[ \dot{\rho}_\psi + 3H(\rho_\phi + p_\phi) = -\frac{\partial W}{\partial \psi} \psi + \frac{(dF/d\psi) \psi}{F + G} \rho, \] (63)
as in section 2, we have the covariant derivative of the total energy–momentum tensor

\[
\nabla_{\mu} T^{\mu\nu} = \frac{\rho}{F + G} \left( \frac{dF}{d\phi} \psi + \frac{dG}{d\phi} \phi \right)
\]

which has the same form as of (17).

From these results, we will consider the interacting dark sector model as before: one takes the field \( \phi \) to represent the dark energy and the field \( \psi \) to represent the dark matter. And following the astronomical constrains: (i) the field \( \phi \) composes the major part of the total energy density and has an expressive negative pressure in the late time; (ii) the field \( \psi \) has a small positive pressure and its energy density represents a considerable fraction of the total energy density in the present time.

### 3.3. Cosmological solutions

Having in view the difficulties of integration, we will search for numerical solutions for the system (55)–(58). In order to analyze the cosmological scenarios that these models can describe, the solutions for some cases in table 4 will be considered.

Let us firstly transform the derivatives with respect to time in the system (55)–(58) into derivatives with respect to red-shift. In addition, by substituting \( H^2 \) from equation (58) into equation (55), we obtain the following final system of coupled differential equations to solve:

\[
4\dddot{H} - (1 + z) (F + G) = \ddot{\rho} + \ddot{\rho}_m,
\]

\[
\frac{\dddot{H}^2 (1 + z)^2 \dddot{\psi}'}{1 - \dddot{H}^2 (1 + z)^2 \dddot{\psi}'^2} + \frac{1}{\dddot{H} (1 + z)} \frac{d\dddot{V}}{d\dddot{\psi}'} - 3 \dddot{H} (1 + z) \dddot{\psi}' = 0, \tag{66}
\]

\[
\dddot{H}^2 (1 + z)^2 \dddot{\psi}' + \dddot{H} [\dddot{H}' (1 + z) - 2\dddot{H}] \left( (1 + z) \dddot{\psi}' + 6 \dddot{G} \right) \frac{d\dddot{W}}{d\dddot{\phi}} = 0, \tag{67}
\]

with the line representing derivative with respect to \( z \), where \( \dddot{H} = H / \sqrt{\rho_0}, \dddot{\psi} = \sqrt{\rho_0} \psi, \dddot{V} = V / \rho_0, \dddot{W} = W / \rho_0, \dddot{\rho} = \rho / \rho_0 = \rho_m / \rho_0 + \rho_\phi / \rho_0 + \rho_\psi / \rho_0 = \dddot{\rho}_m + \dddot{\rho}_\phi + \dddot{\rho}_\psi \) and \( \dddot{\rho}_m = \rho / \rho_0 = \rho_m / \rho_0 + \rho_\phi / \rho_0 + \rho_\psi / \rho_0 = \dddot{\rho}_m + \dddot{\rho}_\phi + \dddot{\rho}_\psi \), which are dimensionless quantities. The energy densities and pressures are now given by

\[
\dddot{\rho}_m = \dddot{\rho}_m (1 + z)^3, \tag{69}
\]

\[
\dddot{\rho}_\psi = \frac{\dddot{V}}{\sqrt{1 - \dddot{H}^2 (1 + z)^2 \dddot{\psi}'^2}} + 6 \dddot{H}^2 (1 + z) \frac{dF}{d\psi'}, \tag{70}
\]

\[
\dddot{\rho}_\phi = \frac{\dddot{W}}{2} + \dddot{W} + 6 \dddot{H}^2 (1 + z) \frac{dG}{d\phi'}, \tag{71}
\]

\[
\dddot{\rho}_\psi = 2 \dddot{H} (1 + z) \left[ \dddot{H} (1 + z) \left( \frac{d^2F}{d\psi'^2} + \frac{dF}{d\psi'} \frac{d\dddot{V}}{d\dddot{\psi}'} \right) + \dddot{H}' (1 + z) - \dddot{H} \frac{dF}{d\psi'} \frac{d\dddot{V}}{d\dddot{\psi}'} \right]
\]

\[
- \dddot{V} \sqrt{1 - \dddot{H}^2 (1 + z)^2 \dddot{\psi}'^2}, \tag{72}
\]
where \( \bar{\phi} = 2\bar{H}(1 + z) \left\{ \bar{H}(1 + z) \left( \frac{d^2 G}{d\phi^2} \phi'' + \frac{dG}{d\phi} \phi' \right) + \left[ \bar{H}'(1 + z) - \bar{H} \right] \frac{dG}{d\phi} \phi' \right\} + \frac{\bar{H}^2(1 + z)^2 \phi'^2}{2} - \bar{W}. \) (73)

Requirements (i) and (ii) for the fields \( \psi \) and \( \phi \) will be satisfied by using the initial conditions for the system (66)–(68) determined from the astronomical data, as is done in the canonical–canonical model. From requirement I we have that \( \bar{\psi}'(0)^2 \ll 1 \), that is, the field \( \psi \) is varying very slowly in the late time (the same consideration as in section 2). This last condition and (70) imply in the relation that \( \bar{V}(0) \sim \Omega_0^0 \). Remembering that the gravitational coupling must have the value 1/2 at present, \( F(0) + \tilde{G}(0) = 1/2 \), equation (58) furnishes the initial condition \( \bar{H}(0) = \sqrt{\Omega_0^0/6 \left[ F(0) + \tilde{G}(0) \right]} = 1/\sqrt{3} \) to the Hubble parameter, just as it was in the first case. All these relations will be employed to perform comparisons to the observational data.

From now on, we will analyze cases I and V from table 4, which represent interacting models non-minimally and minimally coupled to the gravity, respectively.

For case I, we have chosen

\[
f(\phi, e^{2\varphi}) = e^{-\varphi} \frac{\phi}{\bar{\phi}}, \quad \text{which implies} \quad W = W_0 e^{-\mu \varphi} \phi.
\] (74)

For these functions, one may determine that the initial conditions are given by

\[
\bar{\psi}(0) = \frac{\ln[V_0/0.72]}{\bar{\mu}}, \quad \phi(0) = \sqrt{1/2 - F_0 e^{-\bar{\psi}(0)}} G_0, \\
\phi'(0) = \sqrt{6 \left[ 0.23 + 24G_0^2\phi(0)^2 - \bar{W}_0 \phi(0)^{-1} e^{-\bar{\mu}\varphi} \right] - 12G_0\phi(0)},
\]

where \( \bar{\mu} = \mu / \sqrt{\bar{\psi}_0} \) and

\[
F_0 \leq \frac{e^{-\bar{\psi}(0)}}{2}, \quad \bar{W}_0 \leq \left[ 0.23 + 24G_0^2\phi(0)^2 \right] \phi(0) e^{\frac{\bar{\mu}}{4}}.
\] (75)

For the derivative of the field \( \varphi \) at \( z = 0 \) we have chosen \( \bar{\varphi}'(0) = 10^{-6} \) and the following values have been adopted for the fixed constants: \( F_0 = -8.5 \times 10^{-3}; G_0 = 6.9 \times 10^{-3}; V_0 = 2.2 \times 10^{-3}; \bar{W}_0 = 1.9 \times 10^{-3}. \) Two values for the coefficient in the exponential term were adopted, namely, \( \bar{\mu}_1 = 10^{-3} \) and \( \bar{\mu}_2 = 10^{-2}. \)

For case V we have considered that the pressure of the dark matter vanishes at \( z = 0 \) and that the interaction potential of the scalar fields is given by

\[
W(\varphi - \kappa \phi) = W_0 e^{-\kappa(\varphi - \phi)}.
\]

From the subtraction and sum of equations (71) and (73) one obtains the initial conditions for \( \phi(0) \) and \( \phi'(0) \), respectively,

\[
\phi(0) = \frac{\bar{\varphi}(0) + \ln(0.115/\bar{W}_0)}{\tilde{\xi} / \bar{\kappa}}, \quad \phi'(0) = \sqrt{0.69},
\]

where \( \tilde{\xi} = \xi / \sqrt{\bar{\psi}_0} \) and \( \bar{\kappa} = \sqrt{\bar{\mu}\bar{\psi}_0}. \) The initial conditions for \( \bar{\varphi}(0) \) and \( \bar{\varphi}'(0) \) are free and were chosen as \( \bar{\varphi}(0) = 1.0 \) and \( \bar{\varphi}'(0) = 10^{-2} \). For the fixed constants the following values were adopted: \( F_0 + G_0 = 1/2; V_0 = 0.72; \bar{W}_0 = 10^{-2}; \bar{\kappa} = 0.5405 \). As in the previous case, two values for the coefficient in the exponential term were adopted, namely \( \bar{\mu}_1 = 4.9 \) and \( \bar{\mu}_2 = 4.45. \)
Figure 4. Left frame: the ratio of the pressure and energy density of the scalar fields for case I, represented by the straight line for $\tilde{\mu}_1 = 10^{-3}$ and by the dashed line for $\tilde{\mu}_2 = 10^{-2}$. Right frame: the ratio of the pressure and energy density of the scalar fields for case V, represented by the straight line for $\tilde{\xi}_1 = 4.900$ and by the dashed line for $\tilde{\xi}_2 = 4.450$.

Figure 5. Left frame: the deceleration parameter for case I, represented by the straight line for $\tilde{\mu}_1 = 10^{-3}$ and by the dashed line for $\tilde{\mu}_2 = 10^{-2}$. Right frame: the deceleration parameter for case V, represented by the straight line for $\tilde{\xi}_1 = 4.900$ and by the dashed line for $\tilde{\xi}_2 = 4.450$.

In figure 4 the ratios of the pressure and energy density of the scalar fields, $\omega_\phi = p_\phi/\rho_\phi$ and $\omega_\psi = p_\psi/\rho_\psi$, where the left and right frames represent cases I and V, respectively, are plotted. From this figure one can infer that when the parameter $\tilde{\mu}$ is varied from $10^{-3}$ to $10^{-2}$ (case I) the ratio $\omega_\phi$ changes its red-shift evolution drastically. This behavior can be understood by observing that—according to (63)—this ratio is related to the direct exchange of energy between the field $\psi$ and the gravitational field. Then the behavior of the dark energy changes from a cosmological constant type $\omega_\phi \approx -1$ for $\tilde{\mu}_1 = 10^{-3}$ to a variable $\omega_\phi$ for $\tilde{\mu}_2 = 10^{-2}$ as a consequence of the modification in the direct energy exchange with the gravitational field. However, the ratio $\omega_\psi$ has a smooth variation when the values of the coefficient in the exponential term are changed. For case V, when the parameter $\tilde{\xi}$ is varied, one notes by observing the behavior of the ratio $\omega_\psi$ that the dark matter suffers a significant influence, while the dark energy always has an approximated cosmological constant-type behavior, since a very small variation of the ratio $\omega_\psi$ occurs. Observe that in case V there is no direct energy exchange with the gravitational field due to the conditions $\{F, G\}$ = constant.
The deceleration parameter $q = 1/2 + 3p/2\rho$ is represented in figure 5, for cases I (left frame) and V (right frame). The left frame of this figure shows us that the deceleration parameter exhibits a small modification when one varies the value of the coefficient $\tilde{\mu}$ in the exponential term. However, one may infer that when $\tilde{\mu}_2 = 10^{-2}$ and $\omega_\phi$ goes asymptotically to $-1$ in the late time, the red-shift transition from a decelerated regime to an accelerated regime is smaller than that for $\tilde{\mu}_1 = 10^{-3}$. Indeed, in this last situation, $\omega_\phi \approx -1$ in the whole evolution of the density parameter, which implies an earlier transition of the regime. For case V, one may observe that for $\xi_2 = 4.45$ the red-shift transition is smaller than that for $\xi_1 = 4.9$. This can be understood by looking at figure 4 again, where one can infer that $\omega_\phi$ is larger for $\xi_2 = 4.45$ than for $\xi_1 = 4.9$, meaning that in this situation the dark matter has a larger relative pressure, which contributes to retard the transition of the regime. The values of the red-shift transition for case I are $z_T = 0.45 (\tilde{\mu}_1 = 10^{-3})$ and $z_T = 0.40 (\tilde{\mu}_2 = 10^{-2})$, whereas those for case V are $z_T = 0.46 (\xi_1 = 4.9)$ and $z_T = 0.40 (\xi_2 = 4.45)$. Furthermore, the values of the deceleration parameter $q_0$ at $z = 0$ for case I are $q_0 = -0.55 (\tilde{\mu}_1 = 10^{-3})$ and $q_0 = -0.51 (\tilde{\mu}_2 = 10^{-2})$, while those for case V are $q_0 = -0.57 (\xi_1 = 4.9)$ and $q_0 = -0.51 (\xi_2 = 4.45)$. In order to perform comparisons to the observational data, the recent observed values are $z_T = 0.74 \pm 0.18$ (from [32]) for the red-shift transition and $q_0 = -0.46 \pm 0.13$ (from [33]) for the deceleration parameter at $z = 0$. Hence, one may conclude that there exists a good agreement of the results with the observational data.

The density parameters of the common matter, dark energy and dark matter fields are represented in figure 6 for cases I (left frame) and V (right frame). From the left frame one observes that for $\tilde{\mu}_2$ the density parameters of the dark energy and dark matter become equal earlier than for $\tilde{\mu}_1$, but its red-shift transition occurs at a smaller red-shift than that for $\tilde{\mu}_1$. This shows that the change of the behavior of $\omega_\phi$ caused by the change of the energy exchange with the gravitational field is really responsible for a smaller red-shift transition in case I. On the other hand, by observing the right frame, one notes that for $\xi_2$ the density parameters of the dark energy and dark matter become equals earlier than for $\xi_2$. This reinforces the delay of the red-shift transition caused by a larger relative pressure of the dark matter for $\xi_2$.

The effective coupling $F+G$ for case I is plotted in figure 7. One observes from this figure that the effective gravitational coupling has a small variation in comparison to its present value $F(0) + G(0) = 1/2$. There is a variation of less than 10% around the value 1/2, and consequently the effective gravitational ‘constant’ has its value changed about 10% in the considered interval. This result is similar to that of the canonical–canonical case.
4. Conclusions

By applying the Noether symmetry approach we have restricted the possible functions of the undefined couplings and potentials of the general models to families of functions. Using this tool we could analyze the cosmological solutions of some particular interacting dark sector models, which correspond to the potentials and couplings satisfying the symmetry condition for the general actions. Some of the resulting models from the symmetry condition generalize certain interacting dark sector models that have appeared in the literature. The energy exchange which occurs among the fields (scalar field–scalar field and gravitational field–scalar fields) has strong influence on the behaviors described by models with scalar fields. An important verification is that non-minimal couplings can have a very significant influence on the evolution of the energy densities and pressures of the components of the Universe. The results for both general models (canonical–canonical and non-canonical–canonical) showed distinct ways for the energy density and pressure evolutions in similar regimes of expansion of the Universe. Further, both can reproduce a decelerated–accelerated regime, describing the recent transition from a decelerated to an accelerated expansion in agreement with the observational data. Canonical–non-canonical models can reproduce behaviors very similar to that of the cosmological constant model to the late Universe (with respect to the ratio of the pressure and energy density of the dark energy field), but with the additional advantage of presenting more possible ways for the energy density evolution of the matter fields.

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