On the irreversibility of measurements of correlations

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Abstract. Measurements on a bipartite system $AB$ are classified into ones that are freely implementable with only classical communication between $A$ and $B$ (LOCC measurements), and the others that require consumption of entanglement if they are to be implemented with classical communication. When we notice that measurements on a bipartite system can also be used to create entanglement, we have another natural classification: measurements that are capable of creating entanglement, and the others that have no ability of producing entanglement (separable measurements). Interestingly, there exists a separable measurement that is not an LOCC measurement, namely, a measurement that requires entanglement to implement, but is not capable of creating entanglement at all. Such an example was found for a pair of three-level systems, in conjunction with a discrimination task of an intricate set of eight or nine states. Here we show an example of such a measurement on a pair of two-level systems (qubits), which arises in a discrimination task of just two states that look quite simple and have little intricacy. Such an example suggests that this kind of irreversibility is not only shown by a limited set of cleverly constructed measurements, but is also exhibited by a much larger class of measurements than we had expected.

1. Introduction

When we want to distinguish between classical and quantum correlations, a scenario called LOCC (local operations and classical communication), in which two remote parties Alice and Bob only communicate over a classical communication channel, is often adopted. For example, a bipartite state that cannot be created under the LOCC scenario from scratch is called an entangled state, and is considered to have a genuinely quantum correlation. If we apply this scenario to the measurements on a bipartite system $AB$, we obtain a natural classification according to the feasibility under the LOCC scenario. Let us call a measurement that can be implemented under the LOCC scenario an LOCC measurement, which is in a sense a classical measurement, in contrast to the rest of the measurements that require direct interaction between the two systems $A$ and $B$. When we take into account the fact that one can implement any interaction between $A$ and $B$ as long as one is allowed to consume enough amount of resource of entanglement (an entangled state), we can rephrase the above classification in terms of cost of entanglement under the LOCC scenario: LOCC measurements are the ones with zero cost of entanglement, and the rest of the measurements require a nonzero consumption of entanglement resource to implement.
With the opposite point of view, one may wonder if a similar argument is possible from a measurement on a bipartite system $AB$ shared by Alice and Bob can be used to prepare a correlated state in another bipartite system $A'B'$, in the following way [1, 2]. Alice prepares systems $AA'$ in a maximally entangled state $|\Phi\rangle_{AA'} = d^{-1/2} \sum_{j=0}^{d-1} |j\rangle_A |j\rangle_A'$. Bob also prepares his systems $BB'$ in a maximally entangled state $|\Phi\rangle_{BB'} = d^{-1/2} \sum_{j=0}^{d-1} |j\rangle_B |j\rangle_B'$. At this point, the state of $AA'BB'$, $\hat{\rho}_{AA'BB'} = |\Phi\rangle\langle\Phi|_{AA'} \otimes |\Phi\rangle\langle\Phi|_{BB'}$, has no correlation between Alice and Bob. Now suppose that a measurement with POVM (positive-operator-valued measure) $\{\hat{F}_\mu\}$ is applied on $AB$ and an outcome $\mu$ is obtained. The unnormalized state left in the system $A'B'$ should be $\sigma_{A'B'} = \text{tr}_A(\hat{F}_\mu \otimes \hat{1}_{A'B'})\hat{\rho}_{AA'BB'}$. It is easy to confirm that $d^2 A'\langle j|B'|k|\sigma_{A'B'}|j'|A'|k\rangle_{A'} = A\langle j'|B<k|\hat{F}_\mu|j\rangle_A|k\rangle_B$.

This implies that if $\hat{F}_\mu$ is separable (written as a sum of positive product operators like $\sum_j \hat{A}_j \otimes \hat{B}_j$), so is $\sigma_{A'B'}$. If $\hat{F}_\mu$ is not separable, $\sigma_{A'B'}$ is not separable either, namely, $\sigma_{A'B'}$ is an entangled state. Hence we see that any measurement with an inseparable POVM element has an ability to create an entangled state with a nonzero probability. Conversely, if $\hat{F}_\mu$ is separable and the initial state $\hat{\rho}_{AA'BB'}$ is separable, $\sigma_{A'B'}$ is always separable. That is, a measurement with every element of POVM being separable (which we call a separable measurement here) cannot produce entanglement from separable states. We thus obtain another classification of measurement in terms of entanglement-yielding capability. Since entanglement cannot be generated under the LOCC scenario, no cost means no yield, namely, an LOCC measurement is always a separable measurement. But the converse is not necessarily true.

If the converse were also true, namely, if the set of LOCC measurements and the set of separable measurements coincided, then we would have a very simple picture about the classical-quantum boundary of the correlation measurements. But Bennett et al. has found [3] that it is not true, namely, there is a separable measurement (no entanglement-yielding capability) that is not an LOCC measurement (nonzero cost of entanglement), showing interesting irreversibility. Their example shows up in a discrimination task of the following nine orthogonal states $\{|\psi_j\rangle\}_{j=1,...,9}$ of a pair of three-level systems $AB$: $\begin{align*}
|\psi_1\rangle &= |1\rangle_A |1\rangle_B, \quad |\psi_{2,3}\rangle = 2^{-1/2}(|0\rangle_A |0\rangle_B \pm |1\rangle_B), \quad |\psi_{4,5}\rangle = 2^{-1/2}(|1\rangle_A |1\rangle_B \pm |2\rangle_B), \\
|\psi_{6,7}\rangle &= 2^{-1/2}(|1\rangle_A \pm |2\rangle_A)|0\rangle_B, \quad |\psi_{8,9}\rangle = 2^{-1/2}(|0\rangle_A \pm |1\rangle_A)|2\rangle_B,
\end{align*}$

(1)

which are depicted as a set of dominos in Ref. [3]. Since every state is separable, so is the projection operator onto the state. Hence the orthogonal measurement that completely reveals which of the nine states the system $AB$ is in is a separable measurement. Interestingly, they have proved that this measurement cannot be implemented under the LOCC scenario. They have also shown that this measurement cannot be well approximated under the LOCC scenario even when the number of rounds and the amount of the classical communication go to infinity. The latter statement was proved by introducing a measure $P$ for the distinguishing ability of a measurement, based on the mutual information. Let $P_{\text{opt}}^{(\text{sep})}$ be the maximum value of $P$ among all separable measurements, which is trivially given by the orthogonal measurement. Let $P_{\text{opt}}^{(\text{LOCC})}$ be the least upper bound on $P$ for all LOCC measurements. What they proved is the inequality $P_{\text{opt}}^{(\text{sep})} > P_{\text{opt}}^{(\text{LOCC})}$, implying that there is a nonzero gap between the powers of the two classes of measurements. In proving the inequality, they invoked the fact that any LOCC measurement can be implemented as a “continuous” protocol, and noticed that the continuous evolution must pass through a critical moment at which the initial nine states should have evolved to nonorthogonal states.
The above result implies that the optimal separable measurement has a striking irreversibility under LOCC: we must invest entanglement to implement the measurement, whereas we can never use it to create entanglement from scratch. We see interesting parallels between this and a bound entangled state [4]. For the latter, we must invest entanglement to create the state, whereas we can never extract from it a standard form of entanglement like a singlet state of two spin-1/2 particles. In comparison to bound entanglement, our understanding about the irreversibility in the measurements are still rudimentary, including the very reason why the LOCC cannot implement those. For example, we know that the bound entanglement is unique to higher-level systems and does not appear in a qubit-pair system [5], but we know little about the same question for the irreversibility in the measurements.

The aim of this paper is to provide a discrimination problem exhibiting such a phenomenon, which is much simpler in many ways: the system is a qubit pair, and the number of states to be distinguished is just two. The two states are chosen to be

\[
\hat{\rho}_0 = |00\rangle\langle 00|, \quad \hat{\rho}_1 = (1/2)(|++\rangle\langle++| + |-\rangle\langle-|),
\]

where \(|\pm\rangle \equiv 2^{-1/2}(|0\rangle \pm |1\rangle)\). The two states may look rather mundane and with little contraption, but we can still prove \(P_{\text{opt}}^{(\text{sep})} > P_{\text{opt}}^{(\text{LOCC})}\) for a suitably defined measure \(P\). The key element in the proof is the observation that refinement process of any LOCC measurement must proceed alternately between the two systems. This should be contrasted to the property used in the proof of [3], that the refinement process can always be regarded as a continuous one.

This paper is organized as follows. In Sec. 2, we give the precise description of the discrimination task and define \(P\). In Sec. 3, we determine \(P_{\text{opt}}^{(\text{sep})}\) and characterize the separable measurements achieving the optimality. In Sec. 4, we pick up several properties shared by the LOCC measurements, which are used in the proof in Sec. 5 that the optimal separable measurements are never implemented under the LOCC scenario. Section 6 briefly discusses the nonzero gap between \(P_{\text{opt}}^{(\text{sep})}\) and \(P_{\text{opt}}^{(\text{LOCC})}\). Section 7 summarizes this paper.

2. Discrimination task

The problem we consider here is the unambiguous discrimination [6, 7, 8] between the two separable states \(\hat{\rho}_0\) and \(\hat{\rho}_1\) of a two-qubit system defined in Eqs. (2) and (3). The two qubits are secretly prepared in one of the two states before one qubit is sent to Alice and the other one to Bob. They should determine the identity of the prepared state by a quantum measurement. We allow them to declare that the measurement has been a failure, but otherwise they must identify the state with no errors. Such a measurement is generally described by a POVM \(\{\hat{F}_m\}_{m=0,1,2}\) satisfying

\[
\text{tr}(\hat{F}_0\hat{\rho}_1) = \text{tr}(\hat{F}_1\hat{\rho}_0) = 0,
\]

where the outcome \(m = 2\) corresponds to the failure and \(m = 0, 1\) to the identification of state \(\hat{\rho}_m\). For \(j = 0, 1\), let \(\gamma_j \equiv \text{tr}(\hat{F}_j\hat{\rho}_j)\) be the probability of success when the prepared state was \(\hat{\rho}_j\). Here we allow the possibility of the weights of the two states being asymmetric [9]. That is to say, what we seek is the supremum of \(\gamma_1\) over all the allowed measurements with a fixed value of \(\gamma_0\), which we will denote by \(P_{\text{opt}}(\gamma_0)\). A related problem is the optimization of the averaged success rate \(\gamma_{\text{ave}} \equiv \eta_0\gamma_0 + \eta_1\gamma_1\) when \(\hat{\rho}_j\) is prepared with probability \(\eta_j\). The optimized value \(Q_{\text{opt}}\) can be straightforwardly calculated once we learn the function \(P_{\text{opt}}(\gamma_0)\), namely, \(Q_{\text{opt}} = \max_{\gamma_0}[\eta_0\gamma_0 + \eta_1P_{\text{opt}}(\gamma_0)]\).

If we allow Alice and Bob arbitrary global quantum operations, the optimal probability \(\gamma_1 = P_{\text{opt}}^{(\text{sep})}(\gamma_0)\) is known [10] to be achieved by the following protocol. First, they globally
measure whether their bit values coincide, namely, project the state to the subspace spanned by \{\{00\}, \{11\}\} and to the one by \{\{01\}, \{10\}\}. If the prepared state was \(\hat{\rho}_1\), the latter case occurs with probability 1/2 and then the measurement successfully identifies it. Otherwise, they are left with the discrimination between \(|\psi_0\rangle \equiv |00\rangle\) and \(|\psi_1\rangle \equiv 2^{-1/2}(|00\rangle + |11\rangle)\) with \(s \equiv |\langle \psi_0 | \psi_1 \rangle|^2 = 1/2\). This is a well-known problem and \(|\psi_0\rangle\) can be identified with probability \(\gamma_0(1/2)\) while \(|\psi_1\rangle\) can be identified with \(1 - s/(1 - \gamma_0)\) \([11]\). The whole protocol achieves \(\gamma_1\) equal to

\[
P_{\text{opt}}(\gamma_0) = 1 - [4(1 - \gamma_0)]^{-1}
\]

for \(0 \leq \gamma_0 \leq 1/2\).

3. Optimal separable measurements

In this section, we determine the optimal probability \(\gamma_1 = P_{\text{opt}}^{(\text{sep})}(\gamma_0)\) among separable measurements. It is convenient to use the high symmetry of states \(\hat{\rho}_0\) and \(\hat{\rho}_1\), which are written in the matrix representation as

\[
\hat{\rho}_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\rho}_1 = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
\]

These states are invariant under any of the following maps — Exchange of the two qubits, given by the map \(S^{(1)}\): \(|ij\rangle\langle kl| \rightarrow |ji\rangle\langlelk|\); simultaneous phase flip, given by \(S^{(2)}\): \(|ij\rangle\langle kl| \rightarrow (-1)^{i+j+k+l}|ij\rangle\langle kl|\); (global) transpose, given by \(S^{(3)}\): \(|ij\rangle\langle kl| \rightarrow |kl\rangle\langle ij|\); and partial transpose, given by \(S^{(4)}\): \(|ij\rangle\langle kl| \rightarrow |il\rangle\langle kj|\). As a result, if a separable measurement \(\{\hat{F}_m\}\) achieves success probabilities \((\gamma_0, \gamma_1)\), all the POVMs generated by applying the above maps are physically valid (note the separability of \(\hat{F}_m\) for \(S^{(4)}\) \([12]\)) and give the same probabilities \((\gamma_0, \gamma_1)\).

Then, the symmetrizing measurement, which executes one of those \(2^4\) measurements randomly, also achieves the same \((\gamma_0, \gamma_1)\). Let us define the symmetrizing map via the composition \(S = \prod_{n=1}^{4} 2^{-4}(I + S^{(n)})\), where \(I\) is the identity map. The POVM of the symmetrized measurement is given by \(\{S(\hat{F}_m)\}\). Any symmetrized self-adjoint operator has a simple form with four real parameters in the matrix representation as

\[
\begin{pmatrix}
a & 0 & 0 & \mu \\
0 & c & \mu & 0 \\
0 & \mu & c & 0 \\
\mu & 0 & 0 & b
\end{pmatrix}.
\]

It is positive (semidefinite) if and only if

\[
a \geq 0, \quad ab \geq \mu^2, \quad c \geq |\mu|.
\]

The conditions \(\text{tr}[\hat{\rho}_0 S(\hat{F}_0)] = \gamma_0\) and \(\text{tr}[\hat{\rho}_1 S(\hat{F}_0)] = 0\) require that

\[
S(\hat{F}_0) = \gamma_0 \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}.
\]
Finally, they determine the final outcome process as a trajectory in a $y$ plane. This can be done by specifying real functionals $\hat{F}$, and by assigning the point $(x(A), y(B))$ on the $xy$ plane for a product operator $\hat{A} \otimes \hat{B}$. Then, a refinement process of the POVM elements, $\hat{G}_i$, requires $c \leq 1 - 2\gamma_0$ and $b \leq (1 - 2\gamma_0)/(1 - \gamma_0)$. The optimal value of $\gamma_1 = \text{tr}[\hat{\rho}_1 S(\hat{F}_1)] = (b + 2c)/4$ is thus given by

$$P_{\text{opt}}^{(\text{sep})}(\gamma_0) = 1 - \gamma_0 - [4(1 - \gamma_0)]^{-1}$$

for $0 \leq \gamma_0 \leq 1/2$. This is achieved if and only if $S(\hat{F}_m) = \hat{F}_m^{(\text{sep})}$ with

$$\hat{F}_0^{(\text{sep})} = 2\gamma_0 (\hat{P}_+ \otimes \hat{P}_- + \hat{P}_- \otimes \hat{P}_+)$$

$$\hat{F}_1^{(\text{sep})} = (1 - 2\gamma_0) (\hat{P}_0 \otimes \hat{P}_1 + \hat{P}_0 \otimes \hat{P}_1)$$

$$+ (1 - \xi_0) \hat{P}_1 \otimes \hat{P}_1$$

$$\hat{F}_2^{(\text{sep})} = [(1 + \xi_0)/2](\hat{P}_{\gamma+} \otimes \hat{P}_{\gamma+} + \hat{P}_{\gamma-} \otimes \hat{P}_{\gamma-})$$

where $\xi_0 \equiv \gamma_0/(1 - \gamma_0)$, $|\gamma\pm\rangle \equiv \sqrt{1 - \gamma_0}|0\rangle \pm \sqrt{\gamma_0}|1\rangle$, and we denoted $\hat{P}_X \equiv |X\rangle\langle X|$. 

4. Properties of LOCC measurements

In this section, we emphasize two properties that are common to all LOCC measurements. These come from the fact that any protocol implementing an LOCC measurement under the LOCC scenario can be regarded as alternating processes of refining POVM elements.

A general measurement protocol under the LOCC scenario, which produces outcome $m = 0, 1, 2$, is always rephrased into the following form. Alice first obtains an outcome $i_1$ and tells Bob. (If Bob is the first one to communicate, simply define $i_1$ to be a dummy with no information.) If they stopped the protocol here, it would be regarded as a measurement with POVM $\{\hat{G}_{i_1}\}_{i_1}$ with $\hat{G}_{i_1} = \hat{A}_{i_1} \otimes \hat{1}$ and $\sum_{i_1} \hat{G}_{i_1} = \hat{1} \otimes \hat{1}$. Next, Bob chooses a measurement depending on $i_1$, and obtains an outcome $i_2$ and tells Alice. The POVM is refined to $\{\hat{G}_{i_1i_2}\}_{i_1,i_2}$ with $\hat{G}_{i_1i_2} = \hat{A}_{i_1} \otimes \hat{B}_{i_1i_2}$ at this point, where $\sum_{i_2} \hat{G}_{i_1i_2} = \hat{G}_{i_1}$. Repeating such a process $n$ rounds, they carry out a measurement with POVM $\{\hat{G}_{i_1\ldots i_n}\}_{i_1\ldots i_n}$, with $\hat{G}_{i_1\ldots i_n} = \hat{A}_{i_1i_2\ldots i_n} \otimes \hat{B}_{i_1i_2\ldots i_n}$.

Finally, they determine the final outcome $m = 0, 1, 2$ according to a rule $m = \Omega(i_1, \ldots, i_n)$.

In order to have a better understanding about this refinement process, let us depict the process as a trajectory in a $xy$ plane. This can be done by specifying real functionals $x(A)$ and $y(B)$, and by assigning the point $(x(A), y(B))$ on the $xy$ plane for a product operator $\hat{A} \otimes \hat{B}$. Then, a refinement process of the POVM elements,

$$\hat{1} \otimes \hat{1} \rightarrow \hat{G}_{i_1} \rightarrow \hat{G}_{i_1i_2} \rightarrow \hat{G}_{i_1i_2i_3} \rightarrow \cdots,$$
is represented by a trajectory

\[(x(\hat{1}), y(\hat{1})) \rightarrow (x(\hat{A}_{i_1}), y(\hat{1})) \rightarrow (x(\hat{A}_{i_1}), y(\hat{B}_{i_1i_2})) \rightarrow (x(\hat{A}_{i_1i_2i_3}), y(\hat{B}_{i_1i_2})) \rightarrow \cdots. \tag{17}\]

We notice that either \(x\) or \(y\), and not both, changes in a single step. Hence the trajectory on the \(xy\) plane is a zigzag line as in Fig. 1. This property, reflecting the fact that Alice and Bob can refine the measurement operator only alternately to arrive at the final outcome, is crucial in showing that the optimal separable measurement is not feasible.

The other property of the LOCC measurement we will use in the proof is the common property of any sequence of measurements. Consider a real linear functional \(f(\hat{G})\). The fact that the \(j\)-th measurement refines the POVM element \(\hat{G}_{i_1\cdots i_{j-1}}\) as \(\hat{G}_{i_1\cdots i_{j-1}} = \sum_{i_j} \hat{G}_{i_1\cdots i_{j-1}i_j}\) leads to \(f(\hat{G}_{i_1\cdots i_{j-1}}) = \sum_{i_j} f(\hat{G}_{i_1\cdots i_{j-1}i_j})\). Hence if \(f(\hat{G}_{i_1\cdots i_{j-1}}) > 0\), there exists a value of \(i_j\) satisfying \(f(\hat{G}_{i_1\cdots i_{j-1}i_j}) > 0\). That is to say, once the refinement process yields an element with positive \(f\), that property is succeeded along at least one branch of the refining tree. This leads to the following implication:

\[f(\hat{G}_{i_1\cdots i_k}) > 0 \rightarrow \exists \{j_{k+1} \cdots j_n\} \text{ s.t. } f(\hat{G}_{i_1\cdots i_kj_{k+1} \cdots j_n}) > 0, \tag{18}\]

which means that a property in the middle of the trajectory is always possessed by at least one final outcome.

5. The optimal separable measurements are not LOCC measurements

In this section, we will show that the optimal separable measurement \(\{\hat{F}_m^{(sep)}\}\) for \(0 < \gamma_0 < 1/2\) cannot be implemented under LOCC. Let us introduce a “weight” of a POVM element \(\hat{G}\) by \(w(\hat{G}) \equiv \langle 00|\hat{G}|00\rangle = \text{tr}(\hat{\rho}_0\hat{G})\). As long as \(w > 0\), we can write down a zigzag trajectory by defining

\[x(\hat{A}) \equiv \text{Re} A_{01}/A_{00}, \quad y(\hat{B}) \equiv \text{Re} B_{01}/B_{00}. \tag{19}\]
where $A_{ik} \equiv \langle j|\hat{A}|k \rangle$ and $B_{jk} \equiv \langle j|\hat{B}|k \rangle$. The starting point of the trajectory, for $\hat{I} \otimes \hat{I}$, is $(x(1), y(1)) = (0, 0)$. The parameters $w, x, y$ take distinct values for $m = 0, 1$. If $m = \Omega(i_1, \ldots, i_n) = 0$ and hence $\text{tr}(\hat{G}_{i_1 \ldots i_n}) = 0$, then $\hat{G}_{i_1 \ldots i_n}$ should be proportional to $\hat{P}_+ \otimes \hat{P}_-$ or to $\hat{P}_- \otimes \hat{P}_+$, namely,

$$
(x, y) = (1, -1) \text{ or } (-1, 1) \text{ for } m = 0.
$$

If $m = \Omega(i_1, \ldots, i_n) = 1$, $\text{tr}(\hat{G}_{i_1 \ldots i_n}) = 0$ means that

$$
w(\hat{G}_{i_1 \ldots i_n}) = 0 \text{ for } m = 1.
$$

In order to achieve $\gamma_1 = P_{\text{opt}}^{(\text{sep})}(\gamma_0)$, the outcomes with $m = \Omega(i_1, \ldots, i_n) = 2$ must also have distinct values for $(x, y)$, namely,

$$
(x, y) = (\sqrt{\xi_0}, \sqrt{\xi_0}) \text{ or } (-\sqrt{\xi_0}, -\sqrt{\xi_0}) \text{ for } m = 2.
$$

This is because we see from Eq. (15) that the range of $\mathcal{S}(\hat{F}_2) = \hat{F}_2^{(\text{sep})}$ is spanned by $\{|\gamma+\rangle, |\gamma-\rangle\}$ and includes no other product states since $\gamma_0 > 0$. Hence $\hat{G}_{i_1 \ldots i_n}$ should be proportional to $\hat{P}_{\gamma_0} \otimes \hat{P}_{\gamma_0}$. (Otherwise, the range of $\hat{F}_2$ would be strictly larger than that of $\hat{F}_2^{(\text{sep})}$, implying $\mathcal{S}(\hat{F}_2) \neq \hat{S}_{\text{opt}}^{(\text{sep})}$.) This allows us to find a pair of linear functionals $f_{\pm}$ that are nonpositive for any final outcome $\hat{G}_{i_1 \ldots i_n}$. These are defined by

$$
f_{\pm}(\hat{F}) = F_{0011} + F_{0110} + F_{1001} + F_{1100} - 4\xi_0 F_{0000}
\pm(1 + \xi_0)(F_{0010} + F_{1000} - F_{0001} - F_{0100}),
$$

where $F_{ijkl} \equiv \langle ij|\hat{F}|kl \rangle$. For a product operator $\hat{G} = \hat{A} \otimes \hat{B}$ with $x(\hat{A}) = x$ and $y(\hat{B}) = y$, we have

$$
f_{\pm}(\hat{G}) = w(\hat{G})[4(xy - \xi_0) \pm 2(1 + \xi_0)(x - y)]
= w(\hat{G})[\chi_{\pm}(x, y) + (1 - \xi_0)^2],
$$

where

$$
\chi_{\pm}(x, y) \equiv [2x \mp (1 + \xi_0)] [2y \pm (1 + \xi_0)].
$$

Comparing Eqs. (20)–(22) and (24), we conclude that $f_{\pm}(\hat{G}_{i_1 \ldots i_n})$ should be either 0 or $-8(1 + \xi_0)w$ in order to achieve $\gamma_1 = P_{\text{opt}}^{(\text{sep})}(\gamma_0)$. As we will see, this is impossible with LOCC.

Since $\gamma_0 > 0$, there is an outcome with $m = 0$ satisfying $w(\hat{G}_{i_1 \ldots i_n}) > 0$. From Eq. (20), we may assume that $(x, y) = (1, -1)$ for this outcome (since the case for $(-1, 1)$ is similar). Let us define regions $R_{\pm} \equiv \{(x, y)|\chi_{\pm}(x, y) \geq 0\}$, which are shown by shaded areas in Fig. 1. It is obvious that the zigzag path from the origin must land on $R_+$ at least once in order to reach $(1, -1)$. Let $\hat{G}_{i_1 \ldots i_m}$ be the point on $R_+$. From Eq. (25), we see $f_{\pm}(\hat{G}_{i_1 \ldots i_m}) \geq w(\hat{G}_{i_1 \ldots i_m})(1 - \xi_0)^2 \geq w(\hat{G}_{i_1 \ldots i_m})(1 - \xi_0)^2 > 0$ since $\gamma_0 < 1/2$. As we have seen in the previous section, this implies that there exists at least one final outcome satisfying $f_{\pm}(\hat{G}_{i_1 \ldots i_m j_{m+1} \ldots j_n}) > 0$. Hence we conclude that no finite-round LOCC protocols achieve $\gamma_1 = P_{\text{opt}}^{(\text{sep})}(\gamma_0)$.
Figure 2. Bounds on success probabilities \((\gamma_0, \gamma_1)\). The ordinate is not \(\gamma_1\) but is offset by \(P^{(\text{orth})}(\gamma_0) = 3(1 - 2\gamma_0)/4\), in order to make the tiny gap visible. \(\gamma_1 = P^{(\text{orth})}(\gamma_0)\) is achieved by mixing two orthogonal measurements, one on basis \(\{|0\rangle, |1\rangle\}\) for each party, and the other one on basis \(\{|+, -\rangle\}\). Hence the graph can be understood as representing how much improvement over that trivial strategy is possible. The boundary curve \(\gamma_1 = P^{(\text{LOCC})}_{\text{opt}}(\gamma_0)\) should lie in the shaded region.

6. Existence of a nonzero gap

Since the reasoning in the last section still allows for the existence of LOCC protocols achieving \(\gamma_1\) arbitrarily close to \(P^{(\text{sep})}_{\text{opt}}(\gamma_0)\), we need a more quantitative argument to show that there is a nonzero gap \(P^{(\text{sep})}_{\text{opt}}(\gamma_0) - P^{(\text{LOCC})}_{\text{opt}}(\gamma_0) > 0\). As seen in the last section, outcomes satisfying \(f_{\pm}(G_{i_1\ldots i_n}) > 0\) are inevitable in an LOCC measurement. Therefore, if we can quantify how much \(\gamma_1\) decreases from the optimal value \(\gamma_1 = P^{(\text{sep})}_{\text{opt}}(\gamma_0)\) as a result of the existence of such an outcome, we may obtain an upper bound on \(P^{(\text{LOCC})}_{\text{opt}}(\gamma_0)\). This was carried out in Ref. [13], and it was shown that

\[
P^{(\text{LOCC})}_{\text{opt}}(\gamma_0) \leq u(\gamma_0) \equiv P^{(\text{sep})}_{\text{opt}}(\gamma_0) - \frac{(1 - 2\gamma_0)^2\gamma_0}{4(1 - \gamma_0)^2[1 + 4(1 - \gamma_0)^2]},
\]

which proves that there is a nonzero gap when \(0 < \gamma_0 < 1/2\), as seen in Fig. 2.

We also give lower bounds on \(P^{(\text{LOCC})}_{\text{opt}}(\gamma_0)\) by finding specific LOCC protocols. If Alice first measures on \(\{|+, -\rangle\}\), Bob’s task is either the discrimination of \(\{|0\rangle, |+\rangle\}\) or that of \(\{|0\rangle, |-\rangle\}\), and hence they can achieve \(\gamma_1 = l_1(\gamma_0) \equiv 1 - [2(1 - \gamma_0)]^{-1}\). If Alice and Bob both measure on \(\{|0\rangle, |1\rangle\}\), they achieve \((\gamma_0, \gamma_1) = (0, 3/4)\). Mixture of the \((\gamma_0, \gamma_1) = (0, 3/4)\) protocol and the \((\gamma_0, \gamma_1) = (\sqrt{2} - 1, l_1(\sqrt{2} - 1))\) protocol gives another linear lower bound \(l_2(\gamma_0)\). These are shown in Fig. 2.

7. Summary

We have found a simple problem of state discrimination with \(P^{(\text{sep})}_{\text{opt}} > P^{(\text{LOCC})}_{\text{opt}}\). This means that, even for a qubit pair, there exists a measurement that requires entanglement to execute, but never produces entanglement from scratch. The measurement is still useful, having a definite
advantage over any LOCC measurement. The pair of states in our example are very simple — in fact, they are one of the simplest in a sense, considering that \( P_{\text{opt}}^{(glo)} = P_{\text{opt}}^{(sep)} = P_{\text{opt}}^{(LOCC)} \) has already been proved for discrimination of any two pure product states [14]. This suggests that the existence of the gap may not be rare but could show up in many other problems, all the more reason to investigate this striking irreversibility in more detail. Duan and coauthors [15] have also raised an interesting example suggesting the wideness of the gap. The qubit pair is a convenient playground for such investigation, because operations are parametrized by fewer parameters and entanglement is understood more quantitatively. We hope that the present example would serve as a driving force toward a better understanding about the subtle boundary between quantum and classical interactions.

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