Lattice QCD and Chiral Lagrangians

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Abstract

After a very brief review of the formalism of lattice gauge theories we show how one can calculate the parameters of the continuum chiral Lagrangians proceeding through the derivation of an effective lattice chiral Lagrangian as an intermediary step. The derivation is done in the strong coupling limit. We also discuss how the derivation could be carried out in the intermediate coupling domain by numerical simulation techniques.

1 Introduction

The lattice regularization of a quantum gauge field theory, coupled with numerical simulation methods, provides a very powerful tool to calculate non-perturbative observables otherwise inaccessible to more conventional analytic techniques. Since the introduction of lattice gauge theory (LGT) simulations in 1979, enormous progress
has been made in the application of these techniques. The proceedings of the international symposia on lattice gauge theories, which have been held yearly since 1983, offer one of the best references for recent advances in the field\textsuperscript{3}. In the course of the years, LGT calculations have increased in scope, scale and degree of sophistication, progressing from the early estimates of the simplest observables to reasonably accurate determinations of several quantities of phenomenological interest. One of the features which characterizes the maturity of LGT simulations is that they have gone way beyond being a (poor) substitute for the laboratory provided nature, good only for a direct approximation of the observables of the continuum theory. Rather, the full power of the numerical laboratory is being more and more exploited. Thus, information about the dynamics of the interactions might be obtained, for instance, from the way in which the finite size of the lattice affects the results of a simulation. While the ultimate goal is, of course, always to use the lattice regularization for deriving the properties of the continuum, an increased interest in the dynamics of the discretized system pays off dividends with the information it indirectly provides about its continuum counterpart.

In this lecture we would like to illustrate how one may derive the parameters of the effective chiral Lagrangians\textsuperscript{4,5,6} used for a phenomenological description of low energy QCD interactions from first principles by lattice techniques. The strategy will be to exploit the fact that the lattice regularized system also exhibits a chiral symmetry in order to perform, directly on the lattice, the same reduction of degrees of freedom which conceptually characterizes the derivation of an effective Lagrangian in the continuum. By taking the limit of low lattice momenta in the lattice chiral Lagrangian one can then calculate the parameters of the continuum chiral Lagrangian itself. The derivation of the lattice chiral Lagrangian will be done here in the strong coupling approximation. The material which we are presenting is taken from research which is currently in progress and a corresponding determination of the lattice chiral Lagrangian for more realistic values of the lattice coupling constant, which must be done numerically, has not been carried out yet. The strong coupling calculation, however, illustrates even better than a numerical simulation the reduction of degrees of freedom that leads to the lattice chiral Lagrangian and which is at the root of the method we propose. A discussion of how the same reduction could be implemented numerically in the intermediate coupling domain will be presented at the end of this paper.

1.1. Lattice QCD

We begin by reviewing in an extremely concise manner the formalism of lattice QCD. By defining a quantum field theory on a lattice one achieves a regularization of the ultraviolet divergences which is non-perturbative and gauge invariant\textsuperscript{1}. In general, Euclidean space-time is discretized by the introduction of a regular lattice of points
and oriented links joining neighboring points. In the vast majority of applications this lattice is a hypercubical lattice, however, for reasons that will become clear later, we will need to work with a different lattice to carry out the strong coupling expansion. Therefore we will leave the detailed geometry of the lattice unspecified for now, but we will assume that the lattice is regular and uniform, with lattice spacing $a$.

We will denote by $x$ the coordinates of the lattice points and by $v$ the fundamental displacement vectors in the lattice. Thus, for clarification, with a hypercubical lattice we would have $x = (i_1 a, i_2 a, i_3 a, i_4 a)$ with integer $i_1, i_2, i_3, i_4$ and $v$ would range over $(a, 0, 0, 0), (0, a, 0, 0), (0, 0, a, 0), (0, 0, 0, a)$. The fundamental variables are the gluon $g$ and the quark fields.

The gauge part of the action is constructed from the plaquette variables, i.e. the transport factors around the elementary closed contours of the lattice. The plaquettes are defined by two displacement vectors, and so we will use the notation $G_{x,v_1v_2}$ to denote a plaquette variable. (For the hypercubical lattice $G_{x,v_1v_2} = G_{x,v_1} G_{x+v_1v_2} G_{x+v_2v_1} G_{x,v_2}^\dagger$). The gauge part of the action is then given by:

$$S_g = \beta K_1 \sum_{x,v_1v_2} \text{Tr} (I - G_{x,v_1v_2}), \quad (1)$$

the coupling parameter $\beta$ being related to the bare coupling constant by $\beta = 6/g_0^2$.

The plaquette variables generalize the field strength tensor $F_{\mu\nu}(x)$ of the continuum theory and the factors in Eq. 1 are arranged so that the lattice action reduces formally to the standard action of the gauge field in the limit of vanishing lattice spacing $a \to 0$ if the gauge variables are identified with continuum transport factors through $G_{x,v} = \exp[i g_0 A_\mu(x) v^\mu]$.

The quark fields are defined over the sites of the lattice and are given by elements of a Grassmann algebra $\psi_x$ (color, spin and flavor indices will frequently be left implicit). The quark matter-field action is given by:

$$S_q = K_2 \sum_x \sum_v (\bar{\psi}_x G_{x,v} \psi_{x+v} - \bar{\psi}_{x+v} G_{x,v}^\dagger \psi_x) + \sum_x \bar{\psi}_x j_x \psi_x. \quad (2)$$

In terms of fundamental variables the (lattice regularized) quantum expectation value of any observable $O(G, \bar{\psi}, \psi)$ is given by

$$<O> = Z^{-1} \int DG_{x,v} D\bar{\psi}_x D\psi_x O(G, \bar{\psi}, \psi) \exp [-S_g(G) - S_q(G, \bar{\psi}, \psi)] \quad (3)$$

with

$$Z = \int DG_{x,v} D\bar{\psi}_x D\psi_x \exp [-S_g(G) - S_q(G, \bar{\psi}, \psi)]. \quad (4)$$

---

$^a$ Gauge fields are customarily denoted by $U_{x,v}$, but eventually, following another standard notation, we will want to reserve the $U$ symbol for the chiral fields. We will therefore denote the lattice gauge fields by $G_{x,v}$.

$^b$ $K_1$ and $K_2$ are suitable normalization constants, dependent on the actual shape of the lattice.
It is to be noticed that, if one considers first a lattice of finite volume \( V \), letting \( V \to \infty \) only at the end of the calculations, all of the integrals in Eqs. 3 and 4 are finite and gauge invariant. Thus the lattice formalism provides a mathematically well defined, non-perturbative and gauge invariant regularization of QCD. Formally, in the limit of vanishing lattice spacing, the above expression for the quantum averages reduces to the continuum path integral, but, of course, the latter is \textit{per se} not well defined. The correct continuum limit is obtained through the process of renormalization, whereby lattice spacing \( a \) and bare coupling constant \( g_0 \) are sent simultaneously to 0 according to a precise functional relationship which, because of the property of QCD of being an asymptotically free theory, can be established by perturbative techniques.

1.2. Effective chiral Lagrangians

The most important property of QCD for low energy phenomenology is that the theory has an approximate \( SU(3) \times SU(3) \) chiral symmetry, which would be exact for \( m_u = m_d = m_s = 0 \), and that this symmetry is spontaneously broken. As a consequence, the spectrum of particle states contains eight pseudoscalar mesons of very low mass, also called pseudo Goldstone bosons, which would be the Goldstone bosons of the spontaneously broken symmetry if the masses of the lightest quarks were indeed equal to zero.

Effective chiral Lagrangians have been considered in other lectures delivered at the ICTP summer workshops and we refer the reader to the proceedings or to the seminal papers of Gasser and Leutwyler for more detailed information. We summarize here only the most basic features of effective chiral Lagrangians and of their use for low energy QCD.

Following Gasser and Leutwyler\(^5,6\), one introduces the generating functional for connected Green’s functions of QCD in the presence of scalar, pseudoscalar, vector and axial vector sources

\[
Z(s, p, v, a) = \int Dq D\bar{q} DGe^{i \int d^4x \mathcal{L}(q, \bar{q}, G; s, p, v, a)},
\]

\[\mathcal{L} = \mathcal{L}^0_{QCD} - \bar{q}(x)[s(x) - i\gamma_5p(x)]q(x) + \bar{q}(x)\gamma^\mu[v_\mu(x) + \gamma_5a_\mu(x)]q(x).\] (6)

In this expression, \( q, \bar{q} \) represent the lightest quark flavor triplet and \( \mathcal{L}^0_{QCD} \) is what remains of the Lagrangian of QCD when the masses of three lightest quarks are set to zero

\[
\mathcal{L}^0_{QCD} = -\frac{1}{4g^2}G_{\mu\nu}(x)G^{\mu\nu}(x) + \bar{q}(x)\gamma^\mu[i\partial_\mu + G_\mu(x)]q(x).
\] (7)

\( s(x), p(x), v_\mu(x) \) and \( a_\mu(x) \) are Hermitian matrices in color space. The effective chiral Lagrangian is introduced to provide a description of phenomena involving small external momenta. Such phenomena will lead to the excitation of the pseudo Goldstone
bosons only, and correspondingly the chiral Lagrangian will be formulated in terms of an octet of pseudoscalar fields \( \pi^a(x) \), collected in a unitary 3 \( \times \) 3 matrix:

\[
U(x) = \exp \left[ \frac{i\pi^a(x)\lambda^a}{F_0} \right], \quad U(x)U(x)^\dagger = 1. \tag{8}
\]

\( \lambda^a, a = 1, \ldots, 8 \) are Hermitian generators of \( SU(3) \), with the normalization

\[
\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}, \tag{9}
\]

\( F_0 \) is the pion decay constant in the chiral limit:

\[
< 0 \mid q(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}q(x) \mid \pi^b > = if_\pi\delta^{ab},
\]

\[
f_\pi = F_0(1 + O(m_q)) \approx 93.3 \text{MeV}. \tag{10}
\]

The chiral Lagrangian itself takes the form of a perturbative expansion\[^1\] in the external momenta \( p \):

\[
Z(s, p, v, a) = \int DU e^{i\int d^4x L_2(U; s, p, v, a) + L_4(U; s, p, v, a) + \ldots}, \tag{11}
\]

Here \( L_2 \) is a Lagrangian involving terms of order \( p^2 \):

\[
L_2 = \frac{F_0^2}{4} \text{Tr}\{[D_\mu U(x)][D^\mu U(x)]^\dagger + \chi(x)U(x)^\dagger + \chi(x)^\dagger U(x)\}. \tag{12}
\]

In this expression,

\[
D_\mu U(x) = \partial_\mu U(x) - ir_\mu(x)U(x) + iU(x)r_\mu(x)
\]

is a flavor covariant derivative and matrix \( \chi(x) \) collects scalar and pseudoscalar sources:

\[
\chi(x) = 2B_0[s(x) + ip(x)]. \tag{14}
\]

The constant \( B_0 \) is related to the quark condensate in the chiral limit:

\[
< 0 \mid \bar{u}u \mid 0 > = < 0 \mid \bar{d}d \mid 0 > = < 0 \mid \bar{s}s \mid 0 > = -F_0^2B_0(1 + O(m_q)). \tag{15}
\]

Similarly, \( L_4 \) is the most general Lagrangian of order \( p^4 \):

\[
L_4 = L_1[\{[D_\mu U]^\dagger(D^\mu U)\}]^2 + L_2[\{[D_\mu U]^\dagger(D^\mu U)\} \text{Tr}\{[D^\mu U]^\dagger(D^\mu U)\}
+ L_3 \text{Tr}\{[D_\mu U]^\dagger(D^\mu U)(D_\nu U)^\dagger(D^\nu U)\}] + L_4 \text{Tr}\{[D_\mu U]^\dagger(D^\mu U)\} \text{Tr}(\chi^\dagger U + U^\dagger \chi)
+ L_5 \text{Tr}\{[D_\mu U]^\dagger(D^\mu U)(\chi^\dagger U + U^\dagger \chi)\} + L_6 \text{Tr}(\chi^\dagger U + U^\dagger \chi)^2
+ L_7 \text{Tr}(\chi^\dagger U - U^\dagger \chi)^2 + L_8 \text{Tr}(\chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi)
+ iL_9 \text{Tr}[F_\mu^R(D^\nu U)(D^\nu U)^\dagger + F_\mu^L(D^\nu U)^\dagger(D^\nu U)] + L_{10} \text{Tr}(U^\dagger F_\mu^R U F^L_{\mu\nu})
+ H_1 \text{Tr}(F_\mu^R F^R_{\mu\nu} + F_\mu^L F^L_{\mu\nu}) + H_2 \text{Tr}(\chi^\dagger \chi). \tag{16}
\]

\[^1\]It can be shown that in this expansion vector and axial vector sources are of order \( p \), whereas scalar and pseudoscalar sources are of order \( p^2 \).
Here $F_{\mu\nu}^R, F_{\mu\nu}^L$ are field strengths for right and left-handed fields:

$$
F_{\mu\nu}^R = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu],
$$

$$
F_{\mu\nu}^L = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu].
$$

(17)

As we see from the above, at leading order, chiral symmetry restricts the number of terms in the effective chiral Lagrangian to only two, so we need to know only two constants, $F_0$ and $B_0$, in order to describe all low energy phenomena in QCD. At next to leading order, we need ten more constants, $L_1 - L_{10}$. (Constants $H_1$ and $H_2$ are of no physical significance.) Typically, these coupling constants are fixed by comparison of the predictions that follow from the Lagrangian with a subset of the experimental data. One can then derive further predictions that can be tested against experiment or used as input in the study of other interactions (e.g. one might need to calculate strong corrections to electroweak matrix elements).

Of course, it would be very important to be able to calculate the values of the coupling constants appearing in the chiral Lagrangian from first principles, directly from the fundamental QCD Lagrangian. This requires that one carries out explicitly an integration over the high energy degrees of freedom of QCD, trading off quark and gluon fields for the fields of the pseudo Goldstone bosons. In the phenomenological applications one assumes the resulting form of the Lagrangian, but, as we have just stated above, one derives its parameters from comparison with experiment. What we will show in the remainder of this lecture is that the integration over quark and gluon fields can actually be performed explicitly in the lattice formulation in the large $N$, strong coupling limit and that it leads to a lattice chiral Lagrangian from which the coupling constants of Eq. 16 can be derived. Finally we will discuss how it may be possible to use numerical simulations to go beyond the strong coupling approximation and derive the chiral Lagrangian for more physical values of the lattice coupling constant $g_0$.

## 2 Lattice chiral Lagrangian in the strong coupling approximation

In order to derive a lattice chiral Lagrangian one must integrate out the high energy degrees of freedom, i.e. the quantum fluctuations of the gauge and quark fields, re-expressing the generating functional in terms of pseudoscalar fields conjugate to the external sources. Thus one effectively performs a bosonization of the original theory. For general values of the bare lattice coupling constant this can only be done numerically, but in the strong coupling and in the large $N$ (number of colors) limits, the bosonization can be done analytically. The techniques for deriving an effective Lagrangian in the strong coupling limit were laid down in pioneering papers by Kluberg-Stern, Morel, Napoly and Petersson$^7$ and Kawamoto and Smit$^8$. We add, however, to the work of these authors an ingredient that plays a crucial role for the
possibility of obtaining the terms of order $p^4$ in the chiral Lagrangian. As in most lattice calculations, Refs. 7 and 8 deal with a hypercubical lattice. While such a lattice obviously does not have the symmetry of the continuum, as is well known all tensors of rank up to two symmetric under the group of lattice transformations are also symmetric under the full group of 4-dimensional rotations of Euclidean space-time. However, this property does not carry through to tensors of higher order, so that, although one could expand the chiral Lagrangian derived on a hypercubical lattice to terms of $4^{th}$ order in lattice momentum, one would obtain terms that cannot be identified with continuum counterparts.

Inspired by the use of lattices of higher symmetry in the theory of lattice gases, we will formulate the theory and perform the strong coupling expansion on a body-centered hypercubical (BCH) lattice\textsuperscript{d}. The BCH lattice can be obtained from a hypercubical (HC) lattice in two equivalent ways. One can define it as a HC lattice with all the sites of odd (or even) parity removed. Equivalently one can take as sites of the BCH lattice all the sites of a HC lattice together with all the centers of its elementary cells (hence the name BCH). In the BCH lattice every site has 24 nearest neighbors. As fundamental displacement vectors $v$ in the positive direction we will take the following:

$$v_{ij}^\alpha = \frac{\vec{e}_i + \alpha \vec{e}_j}{\sqrt{2}}, \{1 \leq i < j \leq 4, \alpha = \pm 1\}. \quad (18)$$

The BCH lattice has the largest symmetry group of all four dimensional lattices, with 1152 elements (for comparison, the symmetry group of the HC lattice has 384 elements). As a consequence of this larger symmetry group, all tensors of rank up to 4 with the symmetry of the BCH lattice are also invariant under the full 4-dimensional rotation group. This property will be very important for our derivation.

The use of the BCH lattice entails another advantage in the strong coupling limit. It is of rather technical nature and so we will mention it only briefly. The definition of lattice fermions encounters some notorious problems. A straightforward discretization of the Dirac operator leads to the introduction of additional poles in the propagators of the fermions at the corners of the Brillouin zone (species doubling). Although these extra fermionic modes would in general be coupled through the gluonic quantum fluctuations, on a hypercubical lattice there is a fourfold degeneracy of totally decoupled fermionic degrees of freedom. This is akin to starting with one flavor of fermions, but discovering that the theory actually contains four decoupled flavors. As a consequence, the spontaneous breaking of chiral symmetry gives origin to 16 pseudo Goldstone bosons, even though one started with a single flavor. On a BCH lattice, because of the higher coordination number of its vertices, such a separation of the fermionic fields into four decoupled components (which is not demanded by

\textsuperscript{d} This lattice, also known as $F_4$ lattice, has been considered in prior unrelated LGT investigations, see for instance Refs. 10-13. However it is the motivation for its use in the theory of lattice gases that comes closer to the one underlying our own work.
any general theorem on lattice fermions) does not take place and one maintains the relationship between number of explicit flavor indices and number of pseudo Goldstone bosons proper of the continuum theory.

Our goal is to express the generating functional of Eq. 5, formulated over the BCH lattice (cf. Eqs. 1-4), in terms of lattice bosonic variables $U_x = \exp \left[ i \pi_x \lambda^\alpha / F_0 \right]$. We follow the work of Refs. 7, 8, adapting it to the fact that we are on the BCH lattice. Also, for the sake of brevity, we will not include into our equations explicit reference to the external sources $v$ and $a$ coupled to the vector and axial vector currents. These can be introduced via suitable chiral transport factors defined over the links of the lattice. We will include the results for the relevant corresponding coefficients in $\mathcal{L}_4$ only in the final table.

In the strong coupling expansion, the gauge term (Eq. [1]) is suppressed by $1 / g^2$. Therefore, we can neglect it to the leading order in the strong coupling expansion, and the action reduces to Eq. [2]. Now the gauge variables appear only in the first two terms, moreover they appear only linearly. This means that the product of integrals over gauge variables factors into a product of one-link integrals:

$$Z = \int D\psi D\bar{\psi} e^{-\sum_x \bar{\psi}_x j_x \psi_x} \prod_{x,v} \int dG_{x,v} e^{-W_{x,v}}.$$  \hspace{1cm} (19)

Here we have defined:

$$W_{x,v} = \frac{1}{12} (\bar{\psi}_x \gamma^\mu G_{x,v} \psi_{x+v} - \bar{\psi}_{x+v} \gamma^\mu G_{x,v}^T \psi_x).$$  \hspace{1cm} (20)

One link integrals of this type have been calculated for the case when the gauge field couples to a bosonic source in Ref. 14. In Ref. 7, it was shown that this result applies equally to an interaction of gauge fields with fermionic fields. The interested reader should consult Ref. 7 for the details. Here we just give the final result for the generating functional to the leading order in the strong coupling, large-N limit:

$$Z = \int D\psi D\bar{\psi} \exp \left\{ N \sum_{x,v} \text{Tr}[\lambda(x, v)] - \sum_x \bar{\psi}_x j_x \psi_x \right\}. \hspace{1cm} (21)$$

In this equation, $\lambda(x, v)$ is a matrix in Dirac-flavor space, defined by its elements:

$$\lambda(x, v)_{\alpha\beta} = \frac{1}{9N^2} \sum_{i,j=1}^{4n} \sum_{\delta,\gamma,\epsilon=1}^{4n} \left[ \bar{\psi}_{i}^\alpha (x) \psi_{j}^\delta (x) (\bar{\psi})^\gamma (x + v) \psi_{j}^\epsilon (x + v) (\bar{\psi})_{\beta} \right]. \hspace{1cm} (22)$$

It has been argued in Ref. 11 that the use of the BCH lattice actually worsens the problem of lattice fermions as one tries to recover the continuum limit. This is irrelevant for our purposes, because our motivation for its use comes solely from the need of obtaining better symmetry properties in the strong coupling limit. We would not advocate using the BCH lattice in numerical simulations done in the scaling regime, where there is good evidence for the recovery of rotational symmetry with the ordinary HC lattice.

On this lattice the values of constants in Eqs. [3] and [4] are: $K_1 = K_2 = 1/12$.

Greek letters will denote indices in the direct product of Dirac and flavor spaces, and Latin letters the color indices. In this section we will take the number of flavors to be $n$ and will specialize to $n = 3$ only at the end.
$F(\lambda)$ is the following matrix function of $\lambda$:

$$F(\lambda) = 1 - \sqrt{1 - \lambda} + \ln \frac{1 + \sqrt{1 - \lambda}}{2},$$

which should be truncated for finite $N$ as a power series in Grassmann variables. Therefore, Eq. (21) is valid only in the large-$N$ limit.

Next we proceed to convert the fermionic path integral into one over bosonic variables as in Refs. 7, 8. In Eq. (21), $N \text{Tr}[F(\lambda)]$ is the interaction term. It is important to notice that it depends only on color singlet combinations $\bar{\psi}_x \gamma^\alpha \psi_x$ at different points on the lattice. Since these appear also in the source term, $- \sum_x \bar{\psi}_x j_x \psi_x$, we can rewrite the full generating functional as:

$$Z = e^{N \sum_{x,v} \text{Tr}[F(\lambda(x,v))]} \int D\psi D\bar{\psi} e^{-\sum_x \bar{\psi}_x j_x \psi_x},$$

with:

$$\lambda(x, v)_{\alpha\beta} = \frac{1}{9N^2} \sum_{\delta, \gamma, \epsilon=1}^{4n} \left[ \frac{\partial}{\partial j_{x+\delta}} (\bar{\phi}_\gamma \phi_\delta) \frac{\partial}{\partial j_{x+v}} (\bar{\phi}_\epsilon \phi_\beta) \right].$$

Now we have to perform the integration over fermionic variables. This integral splits into a product of one-site integrals:

$$Z_0 \equiv \int D\psi D\bar{\psi} e^{-S} = \prod_x \int d\psi_x d\bar{\psi}_x e^{-\bar{\psi}_x j_x \psi_x} = \prod_x z_0(j_x),$$

$$z_0(j_x) \equiv \int d\psi_x d\bar{\psi}_x e^{-\bar{\psi}_x j_x \psi_x}.$$

There are two methods of integration over the fermion fields. The first method is based on the use of Laplace transforms. We will follow the method of Ref. 8, where it was shown that $z_0(j_x)$ can be written in the following form:

$$z_0(j_x) = \int d\mathcal{M}_x e^{N\text{Tr}(j_x \mathcal{M}_x) - N\text{Tr}(\ln \mathcal{M}_x) + \text{const}}.$$

Here $\mathcal{M}_x$ is a unitary bosonic matrix in Dirac-flavor space, the trace is over Dirac-flavor indices and the constant is irrelevant so it will be omitted hereafter.

Using Eqs. (24) and (25) we obtain an expression for the full generating functional:

$$Z = \int D\mathcal{M} e^{N\sum_{x,v} \text{Tr}[F(\lambda(x,v))] + \sum_x \text{Tr}(j_x \mathcal{M}_x) - \sum_x \text{Tr}(\ln \mathcal{M}_x)} \equiv \int D\mathcal{M} e^{S(\mathcal{M})},$$

where $\lambda$ is now a bosonic matrix:

$$\lambda(x, v) = \frac{1}{9} \mathcal{M}_x \gamma^\alpha \mathcal{M}_x \gamma^\alpha \mathcal{M}_x.$$  

We look for a translationally invariant saddle point of the action in (29) of the form:

$$\mathcal{M}_{x,v}^0 = v,$$
\( v \) being proportional to the unit matrix in Dirac-flavor space. From this we obtain the effective potential:

\[
V_{\text{eff}} = -\frac{S(M_{x,v}^{0})}{\text{volume}}.
\]  

(32)

The saddle point for the massless case is given by:

\[
v_{(a\bar{m} = 0)} = \sqrt{\frac{23}{4}} \approx 1.20.
\]

We will parameterize the field \( M_{x} \) in a nonlinear way:

\[
M_{x} = v \exp \left[ i \frac{\pi_{x} \gamma_{5}}{F_{0}} \right].
\]  

(33)

Here \( S, \pi, V_{\nu}, A^{\nu} \) and \( T^{\nu\rho} \) are sixteen Hermitian matrices in the flavor space. By finding the propagators of these fields, one can show that only \( \pi \) are Goldstone bosons.

In principle, we would need to integrate out the scalar, vector, axial vector and tensor fields in order to obtain an effective action for pseudoscalar mesons. In first approximation we will neglect these contributions\(^{10}\) and concentrate only on the direct interactions among Goldstone bosons\(^{11}\):

\[
M_{x} = v \exp \left( i \frac{\pi_{x} \gamma_{5}}{F_{0}} \right).
\]  

(34)

We will expand the action in Eq. 29 up to the fourth order in Taylor series around the vacuum. This will be sufficient to extract coefficients of effective chiral Lagrangian to \( O(p^{4}) \) since the deviation from the vacuum \( \lambda_{x,v} - \lambda_{0} \) is of \( O(p) \).

The action in Eq. 29 can be written as:

\[
S = N \sum_{x,v} \sum_{n=1}^{4} \frac{1}{n!} \left( \frac{\partial^{n} F_{0}}{\partial \lambda_{n}} \right)_{\lambda_{0}} \text{Tr}[\{(\lambda_{x,v} - \lambda_{0})^{n}\}] + Nv \sum_{x} \text{Tr}(j_{x}M_{x}).
\]  

(35)

We will evaluate the Dirac trace in a basis where \( \gamma_{5} = \text{diag}(1,1,-1,-1) \). By using the relation

\[
\not{\phi}M_{x+v} = M_{x+v} \not{\phi},
\]  

(36)

which follows from the definition of \( M \), we can take the Dirac traces to obtain:

\[
\text{Tr}[\{(\lambda_{x,v} - \lambda_{0})^{n}\}] = 2\lambda_{0}^{n} \text{Tr}_{f}[\{(U_{x+v}^{\dagger}U_{x} - 1)^{n}\} + \text{h.c.}].
\]  

(37)

Here we have defined new variables which are matrices in flavor space only:

\[
U_{x} \equiv \exp \left( i \frac{\pi_{x}}{F_{0}} \right).
\]  

(38)

\(^{10}\) They are suppressed by the masses of relevant resonances.

\(^{11}\) In this paper we will not consider the effects of the U(1) anomaly in any detail, so we will restrict our attention to the traceless part of \( \pi_{x} \).
\( \text{Tr}_f \) indicates a trace over flavor indices, and from now on we will omit the subscript \( f \). Evaluating the second term in Eq. [35] is straightforward and one obtains the expression for the action in terms of new variables:

\[
S = 2N \sum_{x,v} \sum_{n=1}^{4} \frac{\lambda^n_0}{n!} \left( \frac{\partial^n F}{\partial \lambda^n} \right)_{\lambda_0} \text{Tr}[(U_{x+v}^x U_x - 1)^n + \text{h.c.}] \\
+ 2N v \sum_x \text{Tr}(\bar{m} U_x + \text{h.c.}).
\] (39)

From now on we will denote by a subscript \( v \) any variable which under the lattice symmetry transforms in the same way as the link \( v_{ij}^\alpha \) on which it is defined. For example:

\[ a_v \equiv a_i + \alpha a_j \sqrt{2}, a_v^2 \equiv (a_v)^2, \quad \text{etc.} \] (40)

Partial derivatives in the direction of the link \( v_{ij}^\alpha \) are defined similarly:

\[ \partial_v a \equiv \frac{\partial_i a + \alpha \partial_j a}{\sqrt{2}}, \partial_v^2 a \equiv \partial_v(\partial_v a), \quad \text{etc.} \] (41)

These derivatives\(^{[3]}\) appear in the Taylor series expansion of \( U_{x+v} \) in powers of \( a \):

\[ U_{x+v} = U + a(\partial_v U) + \frac{a^2}{2} (\partial_v^2 U) + \ldots, \] (42)

One can now substitute the previous Taylor series expansions into Eq. [39]. Using the following results for summation of tensors of rank two and four (which are easy to prove from definitions in Eq. [40]) over the twelve links in the positive direction:

\[
\sum_v a_v b_v = 3 \sum_\mu a_\mu b^\mu, \\
\sum_v a_v b_v c_v d_v = \frac{1}{2} \sum_{\mu,\nu} (a_\mu b^\mu a_\nu b^\nu + a_\mu b_\nu a^\mu b^\nu + a_\mu b_\nu a^\nu b^\mu),
\] (43)

one obtains a manifestly Lorentz invariant expressions for the action. We will not give the details\(^{[4]}\) of this rather straightforward but tedious calculation and just give the final result in the following table.

\(^{[3]}\)Here we are using the convention that, when there is no subscript \( x \), a variable (or its derivative), is evaluated at \( x \).

\(^{[4]}\)The interested reader should consult our longer paper, currently in preparation, where we give all the details as well as a discussion on how one includes currents.
Table 1: Comparison of the strong coupling expansion with experimental results from Refs. 6, 18. The left column contains our results from the strong coupling expansion. The right column contains the experimental values $L_i^r(m_\eta)$. The numbers are in units of $10^{-3}$.

|   | theory | experiment |
|---|--------|------------|
| $L_1$ | $0.5 (F_0 a)^2$ | $0.9 \pm 0.3$ |
| $L_2$ | $1.1 (F_0 a)^2$ | $1.7 \pm 0.7$ |
| $L_3$ | $10.4 (F_0 a)^2$ | $-4.4 \pm 2.5$ |
| $L_4$ | $0 (F_0 a)^2$ | $0 \pm 0.5$ |
| $L_5$ | $0 (F_0 a)^2$ | $2.2 \pm 0.5$ |
| $L_6$ | $0 (F_0 a)^2$ | $0 \pm 0.3$ |
| $L_7$ | $0.9 (F_0 a)^2$ | $-0.4 \pm 0.15$ |
| $L_8$ | $2.6 (F_0 a)^2$ | $1.1 \pm 0.3$ |
| $L_9$ | $-6.7 (F_0 a)^2$ | $7.4 \pm 0.2$ |
| $L_{10}$ | $-6.7 (F_0 a)^2$ | $-5.7 \pm 0.3$ |

3 Discussion

We have chosen to express the entries in Table 1 in terms of a common factor $(F_0 a)^2$ because of a point we will be making below, but the value of this quantity is also determined by the strong coupling expansion, from the $O(p^2)$ terms in the lattice chiral Lagrangian, and is given by $2.1N$. With 3 colors the entries in the table come out about one order of magnitude higher than the values derived from experiment. We take this to be an indication that the strong coupling limit produces too tight a binding between quark and antiquark, which in turn leads to an unacceptably high value when $F_0$ is expressed in terms of $a^{-1}$ (or, equivalently, to too large a value for $a$). If we allowed ourselves to set $a^{-1} \approx F_0$, which would be a more reasonable scale for a strong coupling calculation, then the theoretical predictions for several of the coefficients would compare rather well with the experimental results. Of course we should not make too much of this agreement, because we expect the values of the coefficients to depend on detailed dynamical features of the interactions which may not be well reproduced by the strong coupling approximation. What is much more important is the demonstration, through the strong coupling expansion, of the main point we wanted to make, namely that one can formulate a chiral effective Lagrangian on the lattice and that this can be the vehicle for the derivation of the parameters of the continuum chiral Lagrangian.

Another point that should be mentioned is that the expansion for small lattice momenta which we have used to derive the coefficients corresponds to using the lattice chiral Lagrangian in the tree approximation. We have calculated a few of the corrections that would be induced by one loop diagrams and have found these to be
small. However, the same factor of \((F_0a)^2\) which we discussed above, enters also as a coefficient in the denominator of the loop diagrams, so that the statement that the loop corrections are small should be taken with caution. They would become larger if one could assign to \(F_0a\) the smaller value which we advocated above. However, even if the loop corrections turned out to be large in a more realistic calculation, this would only represent a technical problem and not a conceptual difficulty for the whole approach. The very important fact is that, because we are working with a regularized theory, all physical quantities are well defined and finite. Moreover, since the lattice system also exhibits spontaneous breaking of chiral symmetry and Goldstone bosons in the limit of vanishing quark mass, it will always be possible to perform an expansion for small lattice momenta. The coefficients of such expansion are in any case well defined quantities. Thus the question is whether they can be calculated by a perturbative expansion of the chiral effective theory, which would be more convenient, or whether, failing such possibility, one will have to resort to numerical techniques. Nevertheless, even in the latter case, one would still be dealing with a bosonic system, and would therefore avoid the need of simulating the quantum fluctuations of fermionic variables, which represents today the major difficulty one faces in the implementation of QCD numerical simulations.

Finally, in order to obtain reliable values for the coefficients one should perform the calculation in the range of values of the lattice coupling constant \((\beta = 6/g_0^2 \approx 5.7 - 6, \text{including quark degrees of freedom in the simulation})\) where one witnesses the onset of scaling to the continuum. Such a calculation can at present only be done by numerical techniques. The way we envisage it could be carried out would be by assuming a sufficiently large set of couplings for the lattice effective theory and fixing them through the matching of an overcomplete set of expectation values. This is very similar to procedures commonly used in numerical studies of renormalization group transformations. The crucial point is that, because the lattice effective theory already accounts for the long range excitations of QCD, the matching should only require a reasonably small lattice size, of the order of the inverse \(\rho\) mass. The two theories (the one derived from the original QCD Lagrangian and the effective theory) should produce exactly the same values for the observables on any lattice size, because they are mathematically equivalent. Working with a small lattice, we are confident, one would be able to fix the parameters of the lattice effective theory with a good degree of accuracy and would thus establish a solid base for the subsequent determination of the parameters of the continuum chiral effective Lagrangian.

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