ALGEBRAS WITH CONVERGENT STAR PRODUCTS AND THEIR REPRESENTATIONS IN HILBERT SPACES

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Abstract. We study star product algebras of analytic functions for which the power series defining the products converge absolutely. Such algebras arise naturally in deformation quantization theory and in noncommutative quantum field theory. We consider different star products in a unifying way and present results on the structure and basic properties of these algebras, which are useful for applications. Special attention is given to the Hilbert space representation of the algebras and to the exact description of their corresponding operator algebras.

1. Introduction

In this paper, we study star product algebras of analytic functions for which the power series defining the products are absolutely convergent. We discuss the structure and some basic properties of these algebras, consider their representations in Hilbert spaces, and describe precisely the corresponding operator algebras. The question of convergence of star products was previously considered in the context of deformation quantization and specifically for the Weyl-Moyal and Wick products [1]–[4]. It acquires a new significance in noncommutative quantum field theory (NC QFT), because taking the nonlocal nature of the star products into account is important for the physical interpretation of noncommutative models [5]–[8]. As shown in [6], a causality condition for NC QFT can be formulated through the use of a star product algebra associated with the light cone and having the property of absolute convergence, and some simple models satisfy this condition. Since in the literature there is no consensus on the equivalence of the Weyl-Moyal and Wick-Voros formulations of noncommutative field theory [9]–[11], it is useful to define and explore those properties of the convergent star product algebras that are independent of the specific form of the star product.

We consider different quantizations in a unifying way, starting from a general form of star products defined by a constant matrix whose antisymmetric part is the Poisson tensor and whose symmetric part specifies the quantization. In the case of the Weyl-Moyal product the symmetric part is zero. An important role in our approach is played by function spaces that are algebras under each of these products and consist of rapidly decreasing functions. As well known, the usual axiomatic quantum field theory uses the Schwartz space \( S \) of all rapidly decreasing smooth functions, on which quantum fields are defined as operator-valued distributions. The space \( S \) is an algebra under the Weyl-Moyal product but not under the Wick product, for which the symmetric part is pure imaginary diagonal. Moreover, any test function space having the structure

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of an algebra under the Wick product must consist of entire analytic functions. This suggests that any attempt to develop a Wick-Voros formulation of noncommutative QFT should be based on treating it as an essentially nonlocal theory, which can also be confirmed by the simple but instructive example of the normal ordered star product square of a free field, considered for the Weyl-Moyal and Wick products in [8] and [12]. We believe that using appropriate function algebras consisting of rapidly decreasing functions and universal for different star products is very useful in the general theory of quantization on symplectic linear spaces. In particular, their operator representations associated with a representation of the translation group are defined in an obvious way and can be extended by continuity to larger algebras. In [3], a faithful representation of a Wick star product algebra with absolute convergence is defined by means of the Gelfand-Naimark-Segal construction. We construct Hilbert space representations of various star product algebras explicitly and characterize the domain of definition of the corresponding operator algebras.

The paper is organized as follows. In Sec. 2 we introduce the main definitions and notation. In Sec. 3 we consider the spaces $E^\rho$ consisting of entire functions of order $\rho \leq 2$ and of minimal type, and prove that these spaces are topological algebras under any star product $\star_\vartheta$ defined by a constant matrix $\vartheta$. In Sec. 4 we show that the algebra $(E^\rho, \star_\vartheta)$ contains a family of subalgebras associated naturally with cone-shaped regions. If $\rho > 1$, then $(E^\rho, \star_\vartheta)$ has a subalgebra $W^\rho$ consisting of functions rapidly decreasing at real infinity. These are just the functions that can serve as test functions for NC QFT, and we explain the relation of $W^\rho$ with the Gelfand-Shilov spaces $S^\beta$ used in the previous papers [6], [8], and [12]. In Sec. 5 we describe the Fourier transform of the algebra $(W^\rho, \star_\vartheta)$, which is used later in Sec. 6 to construct Hilbert space representations of this algebra. In Sec. 7 we prove that the representations of $(W^\rho, \star_\vartheta)$ can be uniquely extended by continuity to representations of the algebra $(E^\rho, \star_\vartheta)$. We also characterize exactly the domain of definition of the corresponding operator algebra in the space $L^2$ of square integrable functions. Analogous results for the Fock-Bargmann space representation are derived in Sec. 8. The final Sec. 9 contains concluding remarks.

2. STAR PRODUCTS ON SYMPLECTIC LINEAR SPACES

Let $X$ be a real $2n$-dimensional vector space equipped with a symplectic bilinear form $\omega$. Choosing a symplectic basis in $X$, we identify the form $\omega$ with its matrix $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and we denote by $\pi$ the inverse matrix of $\Omega$. Then the Poisson bracket of two functions on $X$ is written as

$$\{f, g\} = \sum_{j,k=1}^{2n} \pi^{jk} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^k}.$$ 

The Weyl-Moyal star product $f \star_\hbar g$ is a noncommutative deformation of the ordinary (pointwise) product of the functions in the direction of the Poisson bracket and is defined by

$$f \star_\hbar g(x) = f(x) e^{\frac{i\hbar}{2} \sum_{j,k} \pi^{jk} \hat{\partial}_j \hat{\partial}_k g(x)}, \quad (1)$$

where the Planck constant $\hbar$ plays the role of a noncommutativity parameter, and where the summation convention for the repeated indices is used. In this paper, we consider
star products of the more general form
\[ (f \star_\vartheta g)(x) = f(x) e^{i\hbar \frac{\partial j}{\vartheta jk} \partial_{x^k}} g(x) = f(x)g(x) + \sum_{m=1}^\infty \frac{(i\hbar)^m}{m!} \left( \vartheta^{jk} \partial_{x^j} \partial_{x^k} \right)^m f(x)g(x') \bigg|_{x=x'} , \] (2)
where \( \vartheta \) is a constant and, in general, complex matrix. By the classic al-quantum correspondence principle, the antisymmetric part of \( \vartheta \) must be equal to \( \frac{1}{2}\pi \). Then, and only then, the product (2) satisfies the limit relation
\[ \lim_{\hbar \to 0} \frac{1}{i\hbar} (f \star_\vartheta g - g \star_\vartheta f) = \{f, g\}. \]
Therefore, in what follows we assume that
\[ \vartheta = Q + \frac{1}{2}\pi, \quad \pi = \Omega^{-1}, \quad Q^{jk} = Q^{kj}. \] (3)

Different choices of the symmetric matrix \( Q \) correspond to different quantizations, i.e., to different rules of association between functions on \( X \) and quantum operators, including the standard, the anti-standard, the normal, and the anti-normal ordering rules. This is demonstrated below by constructing explicitly Hilbert space representations which convert the star products to the operator product. In Secs. 3–5 we write \( \star_\vartheta \) instead of \( \star_\hbar \vartheta \), assuming that the deformation parameter \( \hbar \) is included in the matrix \( \vartheta \).

3. **Convergence of the star products**

We will consider spaces of functions for which the star product (2) is absolutely convergent for any matrix \( \vartheta \). Such algebras were previously studied \([1, 2]\), mostly for the case of Weyl-Moyal product. The generalization to arbitrary star products in not difficult, but we give it for completeness in the form convenient for deriving our main results in Secs. 4–8.

Let \( 0 < \rho < \infty \) and let \( \mathcal{E}^\rho(\mathbb{R}^d) \) denote the space of all smooth functions on \( \mathbb{R}^d \) satisfying the inequalities
\[ |\partial^{\kappa} f(x)| \leq C_L L^{-|\kappa|} (\kappa!)^{1-1/\rho} e^{\frac{|x|}{L^\rho}} , \] (4)
where \( L > 0 \) and can be chosen arbitrarily large, \( C_L \) is a constant depending on \( f \) and \( L \), \( |x| = \max_{1 \leq j \leq d} |x_j| \), and \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{Z}_+^d \). Here and hereafter, we use the standard multi-index notation
\[ \partial^\kappa = \frac{\partial^{|\kappa|}}{\partial x_1^{\kappa_1} \ldots \partial x_d^{\kappa_d}}, \quad |\kappa| = \kappa_1 + \cdots + \kappa_d, \quad \kappa! = \kappa_1! \cdots \kappa_d!. \]

The topology of \( \mathcal{E}^\rho(\mathbb{R}^d) \) is defined by the family of norms
\[ \|f\|_L = \sup_{x, \kappa} L^{\kappa!} |\partial^\kappa f(x)| e^{-|x|/L^\rho} , \quad L > 0. \] (5)
We note that $\kappa!$ in (1), (5) can be replaced with $\kappa^\kappa = \kappa_1^\kappa_1 \ldots \kappa_d^\kappa_d$, because by Stirling’s formula we have
\[
e^{-|\kappa|} \kappa^\kappa \leq \kappa! \leq C_\delta (1 + \delta)^{|\kappa|} e^{-|\kappa|} \kappa^\kappa \quad \text{for any } \delta > 0. \tag{6}\]

However the definition in form (1), (5) is more convenient for what follows.

The definition of $\mathcal{E}^\rho(\mathbb{R}^d)$ can be reformulated in terms of complex variables. Let $\mathcal{E}^\rho(\mathbb{C}^d)$ be the space of all entire functions of order $\leq \rho$ and minimal type, i.e. satisfying the condition
\[
|f(z)| \leq C_\epsilon e^{\epsilon|z|^\rho},
\]
where $\epsilon > 0$ and can be chosen arbitrarily small, $z = x + iy$, and $|z| = \max_{1 \leq j \leq d} |z_j|$. The natural topology on $\mathcal{E}^\rho(\mathbb{C}^d)$ is defined by the countable system of norms
\[
\|f\|_\epsilon = \sup_z |f(z)| e^{-\epsilon|z|^\rho}, \quad \epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots.
\]
and is hence metrizable. It follows from the elementary properties of analytic functions of several variables that $\mathcal{E}^\rho(\mathbb{C}^d)$ is complete and therefore a Fréchet space.

**Proposition 1.** The space $\mathcal{E}^\rho(\mathbb{C}^d)$ is canonically isomorphic to $\mathcal{E}^\rho(\mathbb{R}^d)$.

**Proof.** The Taylor series expansion of $f \in \mathcal{E}^\rho(\mathbb{R}^d)$ about a point $x \in \mathbb{R}^d$ has an infinite radius of convergence and defines an analytic continuation of $f$ to $\mathbb{C}^d$ which satisfies (7). Indeed, using the first of inequalities (6) and choosing $L' < L$, we obtain
\[
|f(x + iy)| \leq \sum_\kappa \frac{|y^\kappa|}{\kappa!} \|\partial^\kappa f(x)\| \leq \|f\|_L e^{\epsilon|x/L|^\rho} \sum_\kappa \frac{|y^\kappa|}{L^{|\kappa|}(\kappa!)^{1/\rho}} \leq \|f\|_L e^{\epsilon|x/L|^\rho} \sup_\kappa \frac{|y^\kappa| e^{|\kappa|/\rho}}{L^{|\kappa|}(\kappa!)^{1/\rho}} \sum_\kappa \left( \frac{L'}{L} \right)^{|\kappa|} \leq C_{L,L'} \|f\|_L e^{\epsilon|x/L|^\rho + (d/\rho)\epsilon|x/L'|^\rho} \leq C_{L,L'} \|f\|_L e^{(1+d/\rho)\epsilon|x/L'|^\rho}.
\]

Therefore, $f(x + iy) \in \mathcal{E}^\rho(\mathbb{C}^d)$ and $\|f\|_\epsilon \leq C_{\epsilon,L} \|f\|_L$ if $(1 + d/\rho)L^{-\rho} < \epsilon$, and thus the corresponding map from $\mathcal{E}^\rho(\mathbb{R}^d)$ to $\mathcal{E}^\rho(\mathbb{C}^d)$ is continuous.

Conversely, let $f \in \mathcal{E}^\rho(\mathbb{C}^d)$ and let $D_\epsilon = \{ \zeta \in \mathbb{C}^d : |\zeta_j| \leq r_j, j = 1, \ldots, d \}$, where all $r_j$’s are positive. It follows from the Cauchy inequality that
\[
|\partial^\kappa f(z)| \leq \frac{\kappa!}{r^\kappa} \sup_{\zeta \in D_\epsilon} |f(z - \zeta)| \leq \frac{\kappa!}{r^\kappa} \|f\|_L e^{\epsilon|x^2|^{\rho/2} + \epsilon|2r|^\rho}, \tag{10}\]
where $|r|^\rho = \max_j r_j^\rho \leq \sum_j r_j^\rho$. A simple computation gives
\[
\inf_r \frac{e^{\sum_j r_j^\rho}}{r^\kappa} = \frac{(\epsilon \rho e)^{|\kappa|/\rho}}{(\kappa!)^{1/\rho}}. \tag{11}\]
Combining (10) and (11), we obtain
\[
|\partial^\kappa f(z)| \leq \|f\|_L 2^{|\kappa|}(\epsilon \rho e)^{|\kappa|/\rho} \frac{\kappa!}{(\kappa!)^{1/\rho}} e^{\epsilon|x^2|^{\rho/2}}. \tag{12}\]

Using the second of inequalities (6), this can also be written as
\[
|\partial^\kappa f(z)| \leq C_{\epsilon,L} \|f\|_L 2^{|\kappa|}(\epsilon' \rho e)^{|\kappa|/\rho} (\kappa!)^{1-1/\rho} e^{\epsilon|x^2|^{\rho/2}} \quad \text{for any } \epsilon' > \epsilon. \tag{13}\]
We infer that the restriction $f|_{\mathbb{R}^d}$ of the function to the real space belongs to $\mathcal{E}^\rho(\mathbb{R}^d)$ and satisfies
\[
\|f|_{\mathbb{R}^d}\|_L \leq C_{\epsilon,L} \|f\|_\epsilon \quad \text{for } L < \frac{1}{2(1/\rho e)} \min \left(1, \frac{1}{\rho^{1/\rho}} \right),
\]
which completes the proof. \qed
Remark 1. We prefer to say that $\mathcal{E}^\rho(\mathbb{C}^d)$ is isomorphic to $\mathcal{E}^\rho(\mathbb{R}^d)$ instead of saying that these spaces coincide, because the elements of $\mathcal{E}^\rho(\mathbb{R}^d)$ can also be continued as anti-analytic functions.

We now characterize the infinite order differential operators that are endomorphisms of $\mathcal{E}^\rho(\mathbb{R}^d)$.

**Theorem 1.** Let $\rho > 1$ and $\rho' = \rho/(\rho - 1)$. If $G(s) = \sum_k c_k s^k$ is an entire function of order $\leq \rho'$ and finite type, then the differential operator $G(\partial) = \sum_k c_k \partial^k$ maps $\mathcal{E}^\rho(\mathbb{R}^d)$ continuously into itself and the series $\sum_k c_k \partial^k f$ converges absolutely in every norm of this space for any $f \in \mathcal{E}^\rho(\mathbb{R}^d)$. If $0 < \rho' < 1$, then analogous statements hold for each operator of the form $G(\partial)$, where $G(s)$ is an entire function.

**Proof.** In deriving (12), we have already reproduced the well-known estimate of the Taylor coefficients of entire functions with a given order of growth. When applied to a function $G(s)$ of order $\leq \rho'$ and type $< a$, a similar estimate shows that

$$|c_k| = \frac{|\partial^k G(0)|}{k!} \leq C\frac{(a\rho')^{|k|/\rho'}}{(k!)^{1/\rho'}}$$

(14)

where $C$ is a positive constant. If $1/\rho' = 1 - 1/\rho$ and $\varepsilon'' \geq 2\rho\varepsilon$, then it follows from (13) and (11) that

$$\sum_{\kappa} \|c_{\kappa} \partial^\kappa f\|_{\varepsilon''} \equiv \sum_{\kappa} |c_{\kappa}| \sup_z |\partial^\kappa f(z)| e^{-\varepsilon''|z|^\rho} \leq C\varepsilon'' \|f\|_\varepsilon \sum_{\kappa} 2|\kappa|! |\varepsilon'\rho|^{\varepsilon'' |\kappa|/\rho'}$$

where $\varepsilon' > \varepsilon$ and can be taken arbitrarily close to $\varepsilon$. Because $\varepsilon$ can be chosen arbitrarily small, we conclude that the series $\sum_{\kappa} c_{\kappa} \partial^\kappa f$ converges absolutely in $\mathcal{E}^\rho(\mathbb{C}^d) \approx \mathcal{E}^\rho(\mathbb{R}^d)$ and the operator $G(\partial)$ is continuous in the topology of this space. If $\rho \leq 1$, then it suffices to take into account that the Taylor coefficients of any entire function satisfy the inequality $|c_k| \leq C\varepsilon|\kappa|$ with arbitrarily small $\varepsilon > 0$. Combining this inequality with (13), we see that in this case, any operator of the form $G(\partial)$, with $G(s)$ an entire function, maps $\mathcal{E}^\rho(\mathbb{R}^d)$ continuously into itself. The theorem is proved. □

**Proposition 2.** The set of polynomials is dense in $\mathcal{E}^\rho(\mathbb{C}^d)$.

**Proof.** We show that the Taylor series expansion of $f \in \mathcal{E}^\rho(\mathbb{C}^d)$ converges absolutely in $\mathcal{E}^\rho(\mathbb{C}^d)$. Let $\rho' > 2\rho d\varepsilon$. Using the inequality $|d| z|^\rho \geq \sum_j |z_j|^\rho$ together with (11) and (12), we get

$$\sum_{\kappa} \left\| \frac{z^\kappa}{k!} \partial^\kappa f(0) \right\|_{\varepsilon'} \leq \sum_{\kappa} \frac{|\partial^\kappa f(0)|}{k!} \sup_z |z^\kappa| e^{-(\varepsilon'/d) \sum_j |z_j|^\rho} \leq \|f\|_{\varepsilon} \sum_{\kappa} 2|\kappa|! (e\varepsilon'/\rho')^{\kappa / \rho'} \leq C\varepsilon' \|f\|_{\varepsilon}$$

Because the topology of $\mathcal{E}^\rho(\mathbb{C}^d)$ is finer than the topology of simple convergence, we conclude that the Taylor series converges absolutely in the topology of $\mathcal{E}^\rho(\mathbb{C}^d)$ to the function $f$. The proposition is proved. □

**Theorem 2.** If $\rho \leq 2$, then for any matrix $\vartheta$, the space $\mathcal{E}^\rho(\mathbb{R}^d)$ is an associative unital topological algebra under the star product $f \star_{\vartheta} g$. The series defining this product converges absolutely in $\mathcal{E}^\rho(\mathbb{R}^d)$.

**Proof.** According to the definition (2), the product $f \star_{\vartheta} g$ is obtained by applying the operator $\exp(i\vartheta \partial_j \otimes \partial_k)$ to the function $(f \otimes g)(x, x') \in \mathcal{E}^\rho(\mathbb{R}^{2d})$ and then restricting to the diagonal $x = x'$. The entire function $\exp(i\vartheta \partial_j s_j s_k)$ is of order 2 and finite type. From the isomorphism $\mathcal{E}^\rho(\mathbb{R}^d) \approx \mathcal{E}^\rho(\mathbb{C}^d)$, we immediately infer that the restriction map
\[ \mathcal{E}^\rho(\mathbb{R}^d) \rightarrow \mathcal{E}^\rho(\mathbb{R}^d) \] is continuous. Therefore Theorem 1 implies that the series defining \( f \ast_\theta g \) is absolutely convergent. The bilinear map \( (f,g) \rightarrow f \otimes g \) from \( \mathcal{E}^\rho(\mathbb{R}^d) \times \mathcal{E}^\rho(\mathbb{R}^d) \) to \( \mathcal{E}^\rho(\mathbb{R}^d) \) is obviously jointly continuous. Therefore the bilinear map \( (f,g) \rightarrow f \ast_\theta g \) is also jointly continuous. To prove the associativity of this product we use the identity
\[
\partial^\rho_x f(x,x') = (\partial_x + \partial_{x'})^\rho f(x,x') \bigg|_{x=x'},
\]
which holds for any \( f \in \mathcal{E}^\rho(\mathbb{R}^{2d}) \). With the help of (15) we obtain
\[
(f \ast (g \ast h))(x) = e^{i\partial_jk\partial_{x,j}(\partial_x + \partial_{x'})^\rho f(x')}(e^{i\partial_jk\partial_{x,j} g(x')}h(x'')) \bigg|_{x=x'=x''} = \rho \left( V \right)
\]
(16) The right-hand sides of (16) and (17) coincide by Theorem 1 applied to the operator \( \exp(ip\partial_jk\partial_{x,j}(\partial_x + \partial_{x'})^\rho g(x'))\bigg|_{x=x'} \) acting to \( \mathcal{E}^\rho(\mathbb{R}^d) \). Clearly, the function \( f(x) \equiv 1 \) is the unit element of the algebra \( (\mathcal{E}^\rho(\mathbb{R}^d), \ast_\theta) \). The theorem is proved. \( \square \)

4. Subalgebras associated with cone-shaped regions
If \( \rho > 1 \), then the space \( \mathcal{E}^\rho(\mathbb{R}^d) \) has a nontrivial subspace \( \mathcal{W}^\rho(\mathbb{R}^d) \) consisting of functions rapidly decreasing at real infinity. More precisely, \( \mathcal{W}^\rho(\mathbb{R}^d) \) consists of smooth functions on \( \mathbb{R}^d \) satisfying the condition
\[
|\partial^\rho f(x)| \leq C_{N,L}L^{-1}(|x|)(1 + |x|)^{-N} \quad \text{for all } L > 0, \quad N = 0, 1, 2, \ldots
\]
Correspondingly, its topology of is defined by the system of norms
\[
\|f\|_{N,L} = \sup_{x,\kappa} L^{1}(|x|)(1 + |x|)^{N} \|\partial^\rho f(x)\|,
\]
and under this topology \( \mathcal{W}^\rho(\mathbb{R}^d) \) is a Fréchet space. By reasoning similar to the proof of Preposition 1, this space is canonically isomorphic to the space of entire functions satisfying the inequalities
\[
(1 + |x|)^{N} |f(x + \epsilon y)| \leq C_{N,\epsilon} e^{\epsilon |y|\rho}, \quad \epsilon > 0, \quad N = 0, 1, 2, \ldots
\]
and the system of norms (19) is equivalent to the system
\[
\|f\|_{N,\epsilon} = \sup_{x,y} (1 + |x|)^{N} |f(x + \epsilon y)| e^{-\epsilon |y|\rho}.
\]
A similar space can be associated with each open cone \( V \subset \mathbb{R}^d \). We define \( \mathcal{W}^\rho(V) \) to be the space of all entire functions with the finite norms
\[
\|f\|_{V,N,\epsilon} = \sup_{x,y} (1 + |x|)^{N} |f(x + \epsilon y)| e^{-\epsilon \delta_V(x) - \epsilon |y|\rho},
\]
where \( \delta_V(x) = \inf_{\xi \in V} |x - \xi| \) is the distance of \( x \) from \( V \). We note that \( \delta_V(tx) = \delta_V(x) \) for all \( t > 0 \), because cones are invariant under dilations. Clearly, if \( V_1 \subset V_2 \), then \( \mathcal{W}^\rho(V_2) \) is canonically embedded in \( \mathcal{W}^\rho(V_1) \). The space \( \mathcal{W}^\rho(\mathbb{R}^d) \) is embedded in every \( \mathcal{W}^\rho(V) \) and all of these spaces are embedded in \( \mathcal{E}^\rho(\mathbb{R}^d) \). If \( K \) is a closed cone and a functional \( v \in (\mathcal{W}^\rho(\mathbb{R}^d))' \) has a continuous extension to each \( \mathcal{W}^\rho(V) \), where \( V \supset K \setminus \{0\} \), then the cone \( K \) can be thought of as a carrier of \( v \), because this property implies that \( v \) decreases in all directions outside \( K \) more rapidly than any test function.

**Theorem 3.** If \( 1 \leq \rho \leq 2 \), then for any open cone \( V \subset \mathbb{R}^d \), the space \( \mathcal{W}^\rho(V) \) is an associative topological algebra under the star product \( f \ast_\theta g \) and the series defining this product converges absolutely in \( \mathcal{W}^\rho(V) \).
Proof. Let \( f, g \in \slh^\rho(V) \). It is easy to see that \( f \otimes g \in \slh^\rho(V \times V) \) and

\[
\|f \otimes g\|_{V \times V, N, 2k} \leq \|f\|_{V, N, \epsilon} \|g\|_{V, N, \epsilon},
\]

if \( |y| \) in (22) is defined by \( |y| = \max_j |y_j| \). It is clear that if \( h(z, z') \in \slh^\rho(V \times V) \), then the function \( \tilde{h}(z) = h(z, z) \) belongs to \( \slh^\rho(V) \) and \( \|\tilde{h}\|_{V, N, \epsilon} \leq \|h\|_{V \times V, N, \epsilon} \). Therefore, it suffices to prove an analogue of Theorem 1 for \( \slh^\rho(V) \). To this end we derive an estimate similar to (13). Using Cauchy’s inequality and the elementary inequality \((1 + |x|) \leq (1 + |\xi|)(1 + |x - \xi|)\), we obtain

\[
(1 + |x|)^N |\partial^\kappa f(z)| \leq \frac{k!}{r^N} \sup_{\zeta = \xi + i\eta \in D_r} (1 + |x|)^N |f(\zeta - \xi)| \leq \frac{k!}{r^N} \sum_{\zeta = \xi + i\eta \in D_r} (1 + |\zeta - \xi|)^N |f(\zeta)| e^{\epsilon |y|^2} \leq C_{N, \epsilon} \frac{k!}{r^N} \sup_{\zeta = \xi + i\eta \in D_r} e^{\epsilon |y|^2} \leq C_{N, \epsilon} \frac{k!}{r^N} \frac{|y|}{r} \leq C \frac{k!}{r^N} \frac{|y|}{r} \leq C_{N, \epsilon} \frac{k!}{r^N} \frac{|y|}{r},
\]

which holds for any \( \epsilon' > \epsilon \). We now can state that if \( G(s) = \sum_\kappa c_\kappa s^\kappa \) is an entire function of order \( \leq \rho' = \rho/(\rho - 1) \) and finite type, then the differential operator \( \tilde{G}(\partial) = \sum_\kappa c_\kappa \partial^\kappa \) maps \( \slh^\rho(V) \) continuously into itself and the series \( \sum_\kappa c_\kappa \partial^\kappa \) converges absolutely in this space. Indeed, if \( \epsilon' > 2^\rho \epsilon \), then it follows from (13) and (23) that

\[
\sum_\kappa \|c_\kappa \partial^\kappa \|_{V, N, \epsilon'} \equiv \sum_\kappa |c_\kappa| \sup_{z = x + iy} (1 + |x|)^N |\partial^\kappa f(z)| e^{-\epsilon'' |y|^2} \leq C_{\epsilon'} \|f\|_{V, N, \epsilon} \sum_\kappa 2^{N|\kappa|} (2\epsilon' \rho)^{|\kappa|/\rho} (a\rho')^{1/\rho'},
\]

where \( \epsilon' > \epsilon \) and can be taken arbitrarily close to \( \epsilon \). In particular, the operator \( \exp(i\beta j \partial_j \otimes \partial_k) \) is a continuous automorphism of \( \slh^\rho(V \times V) \), and so the theorem is proved. \( \square \)

Remark 2. The family of subalgebras \( \slh^2(V) \) of the maximal algebra \( \mathcal{E}^2(\mathbb{R}^d) \) with the absolute convergence property plays a special role. In [6] and [12], we used the notation \( \mathcal{I}^{1/2} \) instead of \( \slh^2 \), because this space can be regarded as the projective limit of the Gelfand-Shilov spaces \( S^{1/2,b} \) as \( b \to 0 \). The notation \( \slh^2 \) is here more convenient and agrees with the notation in [15], where the isomorphism between \( S^{\beta,b} \) and \( W^{1/(1-\beta),b} \) is established for \( \beta < 1 \). The spaces \( \slh^\rho = \bigcap_{b \to 0} W^{\rho,b} \) should be distinguished from the spaces \( W^\rho = \bigcup_{b \to \infty} W^{\rho,b} \) considered by Gelfand and Shilov [15]. The space \( \mathcal{I}^{1/2} = \slh^2 \) was proposed in [6] as a universal test function space for noncommutative quantum field theory. It was argued there that the spaces \( \mathcal{I}^{1/2}(V) \) associated with cones can be used to formulate a generalized causality condition and shown that some simple noncommutative models satisfy this condition.

5. DEFORMED CONVOLUTION PRODUCT

We now turn to describing the Fourier transform of the algebra \( (\slh^\rho, \star_\rho) \). As in the case [13] of the spaces \( W^\rho \), the behavior of the Fourier transforms of the elements of \( \slh^\rho \) is characterized by an indicator function which is the Young dual of \( y^\rho \).
Let, as before, \( \rho > 1 \) and let \( \rho' \) be the dual exponent of \( \rho \), i.e., \( 1/\rho' + 1/\rho = 1 \). We define \( \mathcal{W}_\rho'(\mathbb{R}^d) \) to be the space of all smooth functions on \( \mathbb{R}^d \) with the finite norms

\[
\|g\|_{N,a} = \max_{|\kappa| \leq N} \sup_{\sigma} |\partial^\kappa g(\sigma)| e^{a|\sigma|^{\rho'}}, \quad a > 0, \ N = 0, 1, 2, \ldots .
\]

(24)

In what follows we use the notation \( \sigma \cdot x = \sum_j \sigma_j x^j, \sigma \cdot \vartheta t = \sum_j \sigma_j \vartheta^j t^j \). The Fourier transform is defined by \( \hat{f}(\sigma) = (2\pi)^{-d/2} \int e^{-i\sigma x} f(x) dx \).

**Theorem 4.** If \( \rho' \geq 2 \), then for any complex matrix \( \vartheta \), the space \( \mathcal{W}_\rho'(\mathbb{R}^d) \) is an associative topological algebra under the operation \( g_1, g_2 \to g_1 \circ_\vartheta g_2 \), where

\[
(g_1 \circ_\vartheta g_2)(\sigma) = \frac{1}{(2\pi)^{d/2}} \int g_1(\sigma - \tau) g_2(\tau) e^{-i(\sigma - \tau) \cdot \vartheta \tau} d\tau = \frac{1}{(2\pi)^{d/2}} \int g_1(\tau) g_2(\sigma - \tau) e^{-i\sigma \cdot \vartheta(\sigma - \tau)} d\tau.
\]

(25)

The Fourier transformation is an isomorphism of \( (\mathcal{W}_\rho'(\mathbb{R}^d), \star_{\vartheta}) \) onto \( (\mathcal{W}_{\rho'}(\mathbb{R}^d), \circ_{\vartheta} ) \), where \( \rho' = \rho(\rho - 1) \).

**Proof.** We note that the bilinear operation defined by (25) is a noncommutative deformation of the ordinary convolution (up to a factor \( (2\pi)^{-d/2} \)). In the case when \( \vartheta \) is a real skewsymmetric matrix, it is called the twisted convolution, and this case corresponds to the Weyl-Moyal star product. For the properties of the twisted convolution product and its relation to the Weyl-Heisenberg group, we refer the reader to [16] and [17]. Let us now show that the Laplace transformation

\[
g \to f(x + iy) = (2\pi)^{-d/2} \int e^{i\sigma \cdot (x + iy)} g(\sigma) d\sigma
\]

(26)

maps \( \mathcal{W}_\rho'(\mathbb{R}^d) \) onto \( \mathcal{W}_\rho'(\mathbb{R}^d) \). Under the condition \( \rho' \geq 2 \), the integral in (26) is well defined and analytic in \( \mathbb{C}^d \), and we have the chain of inequalities

\[
(1 + |x|^N |f(x + iy)| \leq C_N \max_{|\kappa| \leq N} |x^\kappa f(x + iy)| \leq C_N' \max_{|\kappa| \leq N} \int |e^{i\sigma \cdot (x + iy)} \partial^\kappa g(\sigma)| d\sigma
\]

\[
\leq C'_N \|g\|_{N,(d+1)a'} \int e^{-\sigma \cdot y - (d+1)a' |\sigma|^\rho'} d\sigma \leq C_{N,a'} \|g\|_{N,(d+1)a'} \sup_{|\sigma|} e^{-\sigma \cdot y - a' \sum_j |\sigma_j|^\rho'}.
\]

A standard calculation gives

\[
\sup_{\sigma} \left(-\sigma \cdot y - a' \sum_j |\sigma_j|^\rho'\right) = a \sum_j |y_j|^\rho \leq ad |y|^\rho,
\]

where

\[
(a' \rho')(a \rho') = 1.
\]

(27)

We infer that \( f \in \mathcal{W}_\rho'(\mathbb{R}^d) \) and \( \|f\|_{N,\varepsilon} \leq C_{N,a'} \|g\|_{N,(d+1)a'} \) for \( \varepsilon \geq ad \). Conversely, let \( f \) belong to \( \mathcal{W}_\rho'(\mathbb{R}^d) \). Then the integral \( (2\pi)^{-d/2} \int e^{-i\sigma \cdot (x + iy)} f(x + iy) dx \) is independent of \( y \). Denoting it by \( g(\sigma) \) and assuming that \( |\kappa| \leq N \), we have

\[
|\partial^\kappa g(\sigma)| \leq (2\pi)^{-d/2} \inf_y \int e^{\sigma \cdot y} |(x + iy)^\kappa f(x + iy)| dx
\]

\[
\leq (2\pi)^{-d/2} \inf_y \int e^{\sigma \cdot y} (1 + |y|^N) (1 + |x|^N) |f(x + iy)| dx
\]

\[
\leq C_{N,a} \|f\|_{N + d + 1, a} \inf_y e^{\sigma \cdot y + a \sum_j |y_j|^\rho} \leq C_{N,a} \|f\|_{N + d + 1, e^{-a'|\sigma|^\rho'}},
\]
where \( a > \epsilon \) can be taken arbitrarily close to \( \epsilon \), and \( \alpha' \) and \( \rho' \) are defined by (27). We see that \( g \in \mathcal{W}_\rho'(\mathbb{R}^d) \) and the map \( f \rightarrow g = \hat{f} \) from \( \mathcal{W}_\rho'(\mathbb{R}^d) \) to \( \mathcal{W}_\rho'(\mathbb{R}^d) \) is continuous.

The Fourier transformation converts the operator \( \exp(i\sigma_j \partial_j \otimes \partial_k) \) into the multiplication by the function \( \exp\{-i\sigma_j g^{jk} \tau_k\} \) which is obviously a pointwise multiplier of \( \mathcal{W}_\rho'(\mathbb{R}^{2d}) \) for \( \rho' \geq 2 \). Let \( h(\sigma, \tau) \) be an arbitrary function in \( \mathcal{W}_\rho'(\mathbb{R}^{2d}) \). The formula

\[
\frac{1}{(2\pi)^{d/2}} \int \left\{ \frac{1}{(2\pi)^{d/2}} \int h(\sigma - \tau, \tau) d\tau \right\} e^{i\sigma \cdot x} d\sigma = \frac{1}{(2\pi)^d} \int h(\sigma, \tau) e^{i\sigma \cdot x + i\tau \cdot x'} d\sigma d\tau \bigg|_{x=x'}
\]

shows that the operation of restriction to the diagonal is converted by the Fourier transformation into the operation

\[
\mathcal{W}_\rho'(\mathbb{R}^{2d}) \rightarrow \mathcal{W}_\rho'(\mathbb{R}^d): h(\sigma, \tau) \rightarrow \frac{1}{(2\pi)^{d/2}} \int h(\sigma - \tau, \tau) d\tau.
\]

Consequently,

\[
\hat{f_1} \star_{\theta} \hat{f_2} = \hat{f_1} \circ_{\theta} \hat{f_2}, \tag{28}
\]

which completes the proof. \( \square \)

6. OPERATOR REPRESENTATIONS OF STAR PRODUCTS

Let \( T \) be a unitary projective representation of the translation group of a symplectic space \((X, \omega)\) and let \( \exp\{1/\Omega \omega\} \) be its multiplier, so that we have the relation

\[
T_x T_{x'} = e^{i\frac{1}{2\pi} \omega(x,x')} T_{x+x'} \tag{29}
\]

We let \( H \) denote the Hilbert space on which this representation acts. The Weyl quantization associates Hilbert space operators with functions on \( X \approx \mathbb{R}^{2n} \) in the following way

\[
f \mapsto \hat{f} = \frac{1}{(2\pi)^n} \int \hat{f}(s) T_{hs} ds, \tag{30}
\]

where \( \hat{f} \) is the symplectic Fourier transform of \( f \), defined by

\[
\hat{f}(s) = \frac{1}{(2\pi)^n} \int f(x) e^{i\omega(x,s)} dx. \tag{31}
\]

Letting, as before, \( \Omega \) denote the matrix of the symplectic form \( \omega \), we have \( \omega(x,s) = x \cdot \Omega s = x^j \Omega_{jk} s^k \). Then \( \hat{f}(s) = \hat{f}(-\Omega s) \), and after the change of variables \( \sigma = -\Omega s \), the formula (28) can be written as

\[
\hat{f_1} \star_{\theta} \hat{f_2} = \hat{f_1} \circ_{\tilde{\theta}} \hat{f_2}, \quad \text{where} \quad \tilde{\theta} = \Omega^t \theta \Omega = -\Omega \theta \Omega. \tag{32}
\]

The integral in (30) is well defined if \( \hat{f} \) is integrable, and the Weyl transform takes the star product (11) into the operator product.

Let now the matrix \( \theta \) be of the form given by (11), and consequently,

\[
\tilde{\theta} = Q - \frac{1}{2} \Omega, \quad \text{where} \quad Q = -\Omega Q \Omega. \tag{33}
\]

Then the Weyl correspondence (30) should be replaced by

\[
f \mapsto \hat{f} = \frac{1}{(2\pi)^n} \int \hat{f}(s) e^{i\frac{h}{\hbar} Q^t \theta T_{hs} ds}. \tag{34}
\]

Theorem 5. If \( 1 \leq \rho \leq 2 \), then for any \( \theta \) of the form (11), the map (34) is a continuous homomorphism of the algebra \((\mathcal{W}_\rho'(\mathbb{R}^{2n}), \star_{\theta})\) into the algebra \( \mathcal{B}(H) \) of bounded operators on the Hilbert space \( H \).
Proof. It follows from Theorem 4 and from the invariance of \( \mathcal{W}^{\rho}(\mathbb{R}^{2n}) \) under linear change of variables that the symplectic Fourier transformation is a topological and algebraic isomorphism of \( \mathcal{W}^{\rho}(\mathbb{R}^{2n}) \) onto \( \mathcal{W}^{\rho'}(\mathbb{R}^{2n}) \). Because \( \rho' \geq 2 \), we have

\[
\|f\| \leq \frac{1}{(2\pi)^{n}} \int |\hat{f}(s)| e^{\frac{ih}{2s} |Q| |s|^2} ds \leq C_a \|\hat{f}\|_{0,a},
\]

where \( a > (h/2)|\tilde{Q}| \). It remains to show that the map

\[
\mathcal{W}^{\rho'}(\mathbb{R}^{2n}) \longrightarrow \mathcal{B}(H): \hat{f} \longmapsto f = \frac{1}{(2\pi)^{n}} \int \hat{f}(s) e^{\frac{ih}{2s} \tilde{Q}s} T_{hs} ds
\]

is an algebra homomorphism. Using (29), (33), the symmetry of the matrix \( \tilde{Q} \), and the Fubini theorem, we obtain

\[
f = \frac{1}{(2\pi)^{n}} \int (\tilde{f}_1 \otimes_{h^0} f_2)(s) e^{\frac{ih}{2s} \tilde{Q}s} T_{hs} ds = \frac{1}{(2\pi)^{2n}} \int \int \tilde{f}_1(t) \tilde{f}_2(s-t) e^{-ih t \cdot \tilde{Q}(s-t)} + \frac{ih}{2s} \tilde{Q}s T_{hs} dtds = \frac{1}{(2\pi)^{2n}} \int \int \tilde{f}_1(t) \tilde{f}_2(t') e^{-ih t \cdot \tilde{Q} + \frac{ih}{2(t+t')} \tilde{Q}(t+t')} T_{h(t+t')} dt dt' = \frac{1}{(2\pi)^{2n}} \int \int \tilde{f}_1(t) \tilde{f}_2(t') e^{\frac{ih}{2(t+t')} \tilde{Q} + \frac{ih}{2(t+t')} \tilde{Q}' T_{ht + T_{ht}} dt dt' = f_1 f_2.
\]

The theorem is proved. \( \square \)

In quantum mechanics on phase space, the following notation is usually used: \( x = (p_1, \ldots, p_n; q^1, \ldots, q^n) \), where \( q^i \) are the coordinate variables and \( p_j \) are their conjugate momentums. The standard projective representation of the phase space translations in the Hilbert space \( L^2(\mathbb{R}^n) \) is written as

\[
T_{p,q} = e^{\frac{i}{\hbar}(pq - qp)},
\]

(35)

where \( q^i \) is the operator of multiplication by the \( j \)th coordinate function and \( \frac{d}{dp_j} \) is the operator of differentiation with respect to the same coordinate. Let \( s = (u, v) \) and \( \tilde{Q} = \frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \). Then \( s \cdot \tilde{Q}s = u \cdot v \) and the correspondence (33) takes the form

\[
f \longmapsto \hat{f} = \frac{1}{(2\pi)^{n}} \int \int \hat{f}(u,v) e^{\frac{i}{\hbar} u \cdot v + i(uq - vp)} dudv = \frac{1}{(2\pi)^{n}} \int \int \hat{f}(u,v) e^{iuq} e^{-ivp} dudv,
\]

(36)

which defines the so-called \( qp \)-quantization, i.e., the standard ordering of the operators. We let \( j_{st} \) denote the map defined by (36). In the theory of pseudodifferential operators, it is usually called the Kohn-Nirenberg correspondence. Under this correspondence the monomial \( p^\kappa q^\lambda \) transforms into the operator \( q^\lambda p^\kappa \). Formally, this follows from the relations

\[
p^\kappa q^\lambda = (2\pi)^n (-i)^{\kappa |\lambda|} \partial^\kappa \delta(v) \partial^\lambda \delta(u), \quad \partial^\kappa_q \partial^\lambda_p e^{iuq} e^{-ivp} = (-i)^{\kappa |\lambda|} q^\lambda e^{iuq} p^\kappa e^{-ivp},
\]

but a rigorous proof is based on the extension of map (36) to tempered distributions [18]. An alternative proof, based on extending it to the space \( \mathcal{E}^2 \), is given in Sec. 7. The star product corresponding to the standard ordering is written as

\[
(f \ast_{st} g)(p, q) = f(p, q) e^{-ih \frac{\partial}{\partial q_j}} \partial^\lambda_q g(p, q).
\]

(37)
Let us point out some other important cases. The matrix $\tilde{Q} = -\frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ defines the correspondence

$$f \mapsto \tilde{f} = \frac{1}{(2\pi)^n} \iint \tilde{f}(u,v)e^{-iu\rho}e^{iv\rho}du dv,$$

under which the monomial $p^n q^l$ transforms to $p^n q^l$, and so we have the anti-standard ordering. Now, let $\tilde{Q} = -\frac{i}{2} \begin{pmatrix} \varsigma^{-2}I_n & 0 \\ 0 & \varsigma^2I_n \end{pmatrix}$, where the dimensional parameter $\varsigma$ is such that $\varsigma q$ and $\varsigma^{-1}p$ are of the same dimension. In this case, it is convenient to use the holomorphic variables

$$z^j = \frac{1}{\sqrt{2}}(\varsigma q^j + i\varsigma^{-1}p_j), \quad \bar{z}^j = \frac{1}{\sqrt{2}}(\varsigma q^j - i\varsigma^{-1}p_j). \quad (38)$$

By [1], the elements of $E^\rho(\mathbb{R}^{2n})$ and $\mathcal{F}^\rho(\mathbb{R}^{2n})$ can be regarded as entire functions of these variables, considered as independent variables. The restriction to the real space $\mathbb{R}^{2n}$ amounts then to identifying $\bar{z}$ with the complex conjugate of $z$. With these variables, the formula (34) becomes

$$\tilde{f}(w,\bar{w}) = \frac{1}{(2\pi i)^n} \iint f(z,\bar{z}) e^{z \bar{w} - \bar{z} w} dz \wedge d\bar{z},$$

where $w = (\varsigma u - i\varsigma^{-1}v)/\sqrt{2}$. As usual, we introduce the annihilation and creation operators

$$a^j = \frac{1}{\sqrt{2}}(\varsigma q^j + i\varsigma^{-1}p_j), \quad a^j = \frac{1}{\sqrt{2}}(\varsigma q^j - i\varsigma^{-1}p_j).$$

Then the representation operator (35) can be written as

$$T_z = e^{\pi(z a^j - z a)},$$

In terms of the new variables, the quadratic form $s \cdot \tilde{Q} s$ becomes $-iw \cdot \bar{w}$, and the correspondence (34) takes the form

$$f \mapsto \tilde{f} = \frac{1}{(2\pi i)^n} \iint \tilde{f}(w,\bar{w}) e^{w a^j} e^{-\bar{w} a} dw \wedge d\bar{w}. \quad (39)$$

Therefore, in this case we have the Wick quantization, or in other words, the normal ordering, under which the monomial $\bar{z}^n z^l$ corresponds to the operator $(a^j)^n a^l$. The corresponding star product is

$$(f \ast_{\text{Wick}} g)(z,\bar{z}) = f(z,\bar{z}) e^{\frac{\pi}{\varsigma} z a^j - \frac{\pi}{\varsigma} \bar{z} a^j} g(z,\bar{z}).$$

By changing the sign of $\tilde{Q}$, we obtain the quadratic form $iw \cdot \bar{w}$, which defines the anti-Wick quantization and the anti-normal ordering $a^l(a^j)^n$.

7. Extensions of representations

We now show that the map (34) can be naturally extended to functions in $E^\rho(\mathbb{R}^{2n})$, where $\rho \leq 2$. For clarity we first consider the representation in the space of square integrable functions $L^2(\mathbb{R}^n)$. In doing so we use another function space whose elements decrease at infinity faster than the elements of $E^\rho$ increase.

Let $\gamma > 1$ and $\rho > 1$. The space $W^\rho_\gamma(\mathbb{R}^n)$ consists of all entire analytic functions satisfying the condition

$$|\psi(q + iy)| \leq C e^{-a|q|^\rho + b|y|^\gamma} \quad (40)$$

where $a$, $b$, and $C$ are positive constants depending on $\psi$. As before, we really deal with the restrictions of these functions to $\mathbb{R}^n$, using the analytic continuation as an auxiliary
tool. The space $W_{\rho}^{\gamma}$ coincides with the Gelfand-Shilov space $S_{1/\rho}^{1-1/\gamma}$ and is nontrivial if and only if $\gamma \geq \rho$. Its natural topology is the inductive limit topology with respect to the family of Banach spaces $W_{\rho,a}^{1,b}$ equipped with the norms

$$
\|\psi\|_{a,b} = \sup_{q,y} |\psi(q + iy)| e^{a|q|^{\rho} - b|y|^\gamma}.
$$

(41)

Using reasoning similar to that in the proof of Theorem 4, it is easy to show that $W_{\rho}^{\gamma}$ is invariant under the Fourier transformation if and only if $1/\gamma + 1/\rho = 1$. It is important that $W_{\rho}^{\gamma}$ is dense in $L^2$. Indeed, if $f \in L^2$, $\varphi \in W_{\rho}^{\gamma}$ and $\int \varphi(q) dq = 1$, then the sequence $f_{\nu}(q) = \nu^n \int f(q - \xi) \varphi(\nu \xi) d\xi$ belongs to $W_{\rho}^{\gamma}$ and converges to $f$ in the norm of $L^2$ as $\nu \to \infty$.

We let $\mathcal{A}_{W_{\rho}^{\gamma}}(L^2)$ denote the algebra of operators which are defined on the subspace $W_{\rho}^{\gamma}$ of the Hilbert space $L^2$ and map $W_{\rho}^{\gamma}$ continuously into itself, and we endow this algebra with the topology of uniform convergence on bounded subsets of $W_{\rho}^{\gamma}$.

**Theorem 6.** If $1 < \rho \leq 2$, then the map (34) defined in Theorem 5 as an algebra homomorphism from $\mathcal{W}^\rho(\mathbb{R}^{2n})$ to $\mathcal{B}(L^2)$ extends uniquely to a continuous homomorphism of the algebra $(\mathcal{E}^\rho(\mathbb{R}^{2n}), \ast_{\rho})$ to the algebra $\mathcal{A}_{W_{\rho'}^{\gamma}}(L^2)$, where $\rho' = \rho/(\rho - 1)$.

**Proof.** Let $f(p,q) \in \mathcal{E}^\rho(\mathbb{R}^{2n})$. This function can be written as

$$
f(p,q) = \sum_{\kappa,\lambda} c_{\kappa,\lambda} p^{\kappa} q^{\lambda},
$$

(42)

where by the estimate (12) the coefficients $c_{\kappa,\lambda} = \frac{1}{\kappa! \lambda!} \partial^\kappa_p \partial^\lambda_q f(0,0)$ satisfy

$$
|c_{\kappa,\lambda}| \leq \|f\| e^{(2^\rho e^\rho e)^{(|\kappa| + |\lambda|)/\rho}} (\kappa^\rho \lambda^\rho)^{1/\rho}.
$$

(43)

We first extend the transformation (36) that defines the standard ordering. If such a continuous extension to $\mathcal{E}^\rho$ exists, then it is unique because $\mathcal{W}^\rho$ is dense in $\mathcal{E}^\rho$ by 2. Indeed, every polynomial $P(z)$ can be approximated in the topology of $\mathcal{E}^\rho$ by a sequence of functions of the form $f_{\nu}(z) = P(z) f_0(z/\nu)$, where $f_0 \in \mathcal{W}^\rho$ is such that $f_0(0) = 1$. Let $\psi \in W_{\rho}^{\gamma}(\mathbb{R}^n)$. We define an operator $\hat{f}$ by

$$
(\hat{f}\psi)(q) = \sum_{\kappa,\lambda} c_{\kappa,\lambda} (-i\hbar)^{|\kappa|} q^{\kappa} \partial^\kappa \psi(q)
$$

(44)

and show that the series on the right-hand side of (44) converges to an element of $W_{\rho}^{\rho'}$, and $\hat{f}$ is hence well defined as an operator from $W_{\rho}^{\rho'}$ into itself. To this end, we estimate the derivatives of $\psi$ in the complex domain, using the Cauchy inequality again. Let $z = q + iy$. Taking into account that $\rho' \geq \rho$, we get

$$
|\partial^\kappa \psi(z)| \leq \frac{\kappa!}{\kappa^\rho} \sup_{\xi,\eta \in D_r} |\psi'(z - \xi)| \leq \frac{\kappa!}{\kappa^\rho} \|\psi\|_{a,b} e^{-a|q/2|^{\rho'} + b|2y|^\rho'} \sup_{\xi,\eta \in D_r} e^{a|\xi|^{2\rho'} + b|2\eta|^{\rho'}} \leq \frac{\kappa!}{\kappa^\rho} \|\psi\|_{a,b} e^{-a|q/2|^{\rho'} + b|2y|^\rho'} e^{(a + \rho' b) \sum_j r_j^{\rho'}}.
$$

Using (11) with $\rho'$ instead of $\rho$, we obtain

$$
|\partial^\kappa \psi(z)| \leq C \|\psi\|_{a,b} (\kappa/\rho')^{1/\rho'} e^{-a|q/2|^{\rho'} + b|2y|^\rho'},
$$

(45)

where $b_1 = (a + \rho' b)\rho' e$. (45)
It follows from (13) and (45) that
\[
\sum_{\kappa,\lambda} |c_{\kappa,\lambda} h^{[\kappa]} q^\lambda \partial^\kappa \psi(z)| \leq C \|f\|_\epsilon \|\psi\|_{a,b} e^{-a|q/2|^a+b|2y|^b} \sum_\kappa \sum_\lambda (2^\rho \epsilon \rho e \epsilon^\rho \rho_1^{\kappa/\rho} \rho_1^{\kappa/\rho} \rho_1^\rho) \sum_\lambda |q^\lambda (2^\rho \epsilon \rho e \epsilon^\rho \rho_1^{\kappa/\rho} \rho_1^{\kappa/\rho} \rho_1^\rho)|
\]
where $\rho_1^{\kappa/\rho} \rho_1^{\kappa/\rho} \rho_1^\rho = \kappa$. Using (6), we see that the sum over $\kappa$ does not exceed $\sum_\kappa (\epsilon \rho_1 h^\rho)^{\kappa/\rho}$ and is dominated by a constant if $\epsilon < 1/(\epsilon \rho_1 h^\rho)$. The sum over $\lambda$ is estimated similar to the sum in (3) and does not exceed $C e^{\epsilon \rho |2q\rho|}$, where $\epsilon' > \epsilon$ and can be taken arbitrarily close to $\epsilon$. We infer that
\[
\sum_{\kappa,\lambda} |c_{\kappa,\lambda} h^{[\kappa]} q^\lambda \partial^\kappa \psi(z)| \leq C \epsilon \|f\|_\epsilon \|\psi\|_{a,b} e^{-a|q/2|^a+b|2y|^b} \]
if $\epsilon$ is sufficiently small compared to $a$ and $b$. Therefore, $f \psi \in W^\rho_\rho (\mathbb{R}^n)$. The operator $\mathfrak{f}$ maps $W^\rho_\rho$ into itself continuously, because the right-hand side of (46) contains the factor $\|\psi\|_{a,b}$. Moreover, the Kohn-Nirenberg map $j_{st} : f \rightarrow \mathfrak{f}$ is continuous from $\mathcal{E}^\rho$ to $\mathcal{A}_{W^\rho_\rho} (L^2)$, because the right-hand side of (46) contains $\|f\|_\epsilon$. We now show that if $f \in \mathcal{W}^\rho (\mathbb{R}^{2n})$, then the definition of $\mathfrak{f}$ by (14) is equivalent to the above definition (36). To accomplish this, we consider the matrix element $\langle \varphi, \mathfrak{f} \psi \rangle$, where $\varphi, \psi \in W^\rho_\rho (\mathbb{R}^n)$ and the angle brackets denote the inner product of $L^2(\mathbb{R}^n)$. The function
\[
\chi(u, v) = \langle \varphi, e^{iuq} e^{-ivp} \psi \rangle = \int \hat{\varphi}(q) e^{iuq} \hat{\psi}(q - hv) dq
\]
belongs to $W^\rho_\rho (\mathbb{R}^{2n})$, because $\hat{\varphi} \otimes \hat{\psi} \in W^\rho_\rho (\mathbb{R}^{2n})$ and this space is invariant under linear changes of variables and under the partial Fourier transform. From (36), we have
\[
\langle \varphi, \mathfrak{f} \psi \rangle = \frac{1}{(2\pi)^n} \int \int \hat{f}(u, v) \chi(u, v) dudv = \frac{1}{(2\pi)^n} \int \int f(p, q) \hat{\chi}(-p, -q) dpdq = \frac{1}{(2\pi)^n} \sum_{\kappa,\lambda} c_{\kappa,\lambda} \int p^\kappa q^\lambda \hat{\chi}(-p, -q) dpdq,
\]
where the order of summation and integration can be interchanged because of absolute convergence. Taking the (inverse) symplectic Fourier transform, we obtain
\[
\langle \varphi, \mathfrak{f} \psi \rangle = \sum_{\kappa,\lambda} c_{\kappa,\lambda} (-i)^{[\kappa]} \hat{c}\partial^\kappa \hat{\delta} \delta, \chi \rangle = \sum_{\kappa,\lambda} c_{\kappa,\lambda} (-i\hbar)^{[\kappa]} \int \hat{\varphi}(q) q^\lambda \partial^\kappa \psi(q) dq.
\]
Thus, the definitions (36) and (14) are indeed consistent. The star product (37) is continuous in $\mathcal{E}^\rho$ by Theorem 2 and the operator product is separately continuous in $\mathcal{A}_{W^\rho_\rho} (L^2)$ by the definition of the topology of bounded convergence. Because $\mathcal{W}^\rho$ is dense in $\mathcal{E}^\rho$, we conclude that the constructed extension of correspondence (36) is an algebra homomorphism from $(\mathcal{E}^\rho, \ast h_0)$ to $\mathcal{A}_{W^\rho_\rho} (L^2)$. We let $Q_{st}$ denote the matrix $-\frac{1}{2} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ defining the map $j_{st}$ and let $j_Q$ denote the symbol-operator correspondence of general form (31). If $f \in \mathcal{W}^\rho$, then for any $Q$, $\tilde{f}(s)e^{sQ}f$ is the symplectic Fourier transform of $e^{-Q_i^k \partial_i \partial_k} f$. Hence we have the relation
\[
j_Q(f) = j_{st} \left( e^{\frac{i\hbar}{2} (Q_{st}^{jk} - Q_{st}^{kj}) \partial_j \partial_k} f \right),
\]
which holds for all \( f \in \mathcal{W}^\rho \). We can now extend \( j_Q \) to a map from \( \mathcal{E}^\rho \) into \( A_{W^\rho}(L^2) \) by replacing \( j_{st} \) in (47) with its extension constructed above. The map defined in this way is continuous, because the differential operator on the right-hand side of (47) is a continuous automorphism of \( \mathcal{E}^\rho \) by Theorem 1. This completes the proof.

8. THE FOCK-BARGMANN SPACE REPRESENTATION

The Wick correspondence is usually realized using a representation in the Fock-Bargmann space \( \mathcal{F}^2 \), which consists of antiholomorphic functions on \( \mathbb{C}^n \) satisfying the condition

\[
\int |\varphi(\bar{z})|^2 e^{-\frac{1}{\hbar}||z||^2} d\mu(z) < \infty,
\]

where \( d\mu(z) = (2\pi i)^{-n}dz \wedge \bar{z} = \pi^{-n}d(\text{Re } z)d(\text{Im } z) \) and \( ||z||^2 = z \cdot \bar{z} = \sum_j |z_j|^2 \). It is a Hilbert space with inner product given by

\[
\langle \varphi, \psi \rangle = \frac{1}{\hbar^n} \int \overline{\varphi(\bar{z})} \psi(z) e^{-\frac{1}{\hbar}||z||^2} d\mu(z).
\]

The representation (35) on the space \( L^2 \) can be transferred to a representation on \( \mathcal{F}^2 \) by means of the Bargmann transformation

\[
\psi(q) \longrightarrow (B\psi)(\bar{z}) \overset{\text{def}}{=} (\pi\hbar)^{-n/4} e^{\frac{1}{2\hbar}z\bar{z}2} \int e^{\frac{1}{\hbar}(\sqrt{2}z-q)^2} \psi(q)dq,
\]

which is a unitary isomorphism from \( L^2 \) onto \( \mathcal{F}^2 \) (see [18], [19]). When realized on \( \mathcal{F}^2 \), the creation operator is the multiplication by \( \bar{z} \) and the annihilation operator is \( \hbar \partial_z \). Accordingly, the representation of the phase-space translations is realized by the operators

\[
B T_w B^{-1} = T_w = e^{\frac{1}{\hbar}(\bar{z}w-w\bar{z})} = e^{-\frac{1}{2\hbar}w\bar{w}} e^{\frac{1}{\hbar}w\bar{z}} e^{-\hbar \partial_z},
\]

and the Wick correspondence takes the form

\[
f \longrightarrow \hat{f} = \int \bar{f}(w, \bar{w}) e^{w\bar{z}} e^{-\hbar \partial_z} d\mu(w),
\]

Let \( 1 < \rho \leq 2 \) and \( A > 0 \). We define \( E^{2,A}_\rho \) to be the space of all entire antiholomorphic functions satisfying the condition

\[
|\varphi(\bar{z})| \leq Ce^{A ||z||^2-r||z||^\rho},
\]

where \( C \) and \( r \) are constants depending on \( \varphi \), and we give \( E^{2,A}_\rho \) the topology of the inductive limit of Banach spaces with the norms

\[
||\varphi||_r = \sup_z |\varphi(\bar{z})| e^{-A ||z||^2+r||z||^\rho}.
\]

Theorem 7. Suppose \( 1 < \rho \leq 2 \) and \( \rho'/\rho = (\rho - 1) / (\rho - 1) \). Then the Bargmann transformation defined by (45) maps the space \( W_{\rho'}^{\rho,a} \) isomorphically onto \( E^{2,A}_\rho \), where \( A = 1/(2\hbar) \).

Proof. The transformation (45) is the composition of the following three operations: (1) convolution by the function \( e^{-\frac{1}{2\hbar}z\bar{z}2} \), (2) dilation by a factor of \( \sqrt{2} \), and (3) multiplication by \( (\pi\hbar)^{-n/4} e^{\frac{1}{2\hbar}z\bar{z}2} \). The first operation is equivalent to the multiplication of \( \hat{\psi}(\sigma) \) by \( ce^{-\frac{1}{2}\sigma^2} \), where \( \sigma^2 = \sum_j \sigma_j^2 \) and the precise value of the constant \( c \) is of no importance for the proof. Let \( \hat{\psi} \in W_{\rho',\rho,a}(\mathbb{R}^n) \) and \( \hat{\psi}_1(\sigma) = \hat{\psi}(\sigma)e^{-\frac{1}{2\hbar}\sigma^2} \). Then we have

\[
||\hat{\psi}_1(\sigma + i\tau)|| \leq ||\psi||_{a,b}e^{-\frac{1}{2\hbar}(\sigma^2 - \tau^2) - a|\sigma|^\rho + b|\tau|^\rho}.
\]
Taking the inverse Laplace transform, setting \( z = x + iy \), and using the Cauchy-Poincaré theorem, we get

\[
|\psi_1(\bar{z})| \leq (2\pi)^{-n/2} \|\hat{\psi}\|_{a,b} \inf_{\tau} \left| \int e^{i(x - iy)(\sigma + i\tau)} \psi_1(\sigma + i\tau) \, d\sigma \right| \leq C_{a'} \|\hat{\psi}\|_{a,b} \inf_{\tau} \left( -x\tau + \frac{\hbar}{2} \tilde{\tau}^2 + \tilde{b}|\tau|^p \right) \leq e^{-\frac{\hbar}{2} \tilde{\tau}^2 - a'|\sigma|^p},
\]

where \( a' < a/n^p/2 \). Because \( \rho' \geq 2 \), we have \( \tau^2 \leq n + \sum_j |\tau_j|^{p'} \), and the infimum over \( \tau \) can hence be estimated in a manner similar to that used in proving Theorem 4. Setting \( B' = b + h/2 \), we obtain

\[
\inf_{\tau} \left( -x\tau + \frac{\hbar}{2} \tilde{\tau}^2 + \tilde{b}|\tau|^p \right) \leq \frac{n h}{2} + \inf_{\tau} \left( -x\tau + B' \sum_j |\tau_j|^p \right) \leq \frac{n h}{2} - B|x|^p,
\]

where \((B' \rho')(B \rho)' = 1\). We now show that there exists a positive constant \( \epsilon \) such that

\[
y\sigma - \frac{\hbar}{2} \sigma^2 - a''|\sigma|^p \leq \frac{1}{2h} y^2 - \epsilon \|y\| \quad \text{for} \quad \|\sigma\| \geq \sqrt{n}/2.
\]

After a suitable rescaling of the variables, (52) takes the form

\[
(y - \sigma)^2 \geq \epsilon' \|y\|^2 - a'' \|\sigma\|^p,
\]

where \( \epsilon' = (2\hbar)^{p/2} \epsilon, a'' = (2/\hbar)^{p/2} a' \), and \( \|\sigma\| \geq 1 \). Let us consider the ray \( y = t \sigma \), \( t \geq 0 \) in \( \mathbb{R}^{2n} \). Because \( \rho \leq 2 \) and \( \|\sigma\| \geq 1 \), the inequality (53) holds on this ray if

\[
(t - 1)^2 \geq \epsilon' t^2 - a''.
\]

This condition in turn is satisfied for all \( t \) if \((t - 1)^2 \geq \epsilon' t^2 - a''\), or equivalently, \( \epsilon' \leq a''/(1 + a'') \). We note that for all \( \sigma \) in the ball \( \|\sigma\| \leq 1 \), we have \((y - \sigma)^2 \geq \epsilon' \|y\|^2 - C_{t'} \) with some constant \( C_{t'} > 0 \). Thus, there exists a sufficiently small positive number \( r(a, b, h) \) such that

\[
|\psi_1(x - iy)| \leq C_{a'} \|\hat{\psi}\|_{a,b} e^{\frac{1}{2h} y^2 - r \|x\|^2} \|y\|^p.
\]

Therefore, the function \( \varphi(z) = e^{\frac{1}{2h} z^2} \psi_1(\sqrt{2} z) \) satisfies (50) with \( A = 1/(2h) \) and with a preexponential factor proportional to \( \|\hat{\psi}\|_{a,b} \). We conclude that the Bargmann operator \( \mathcal{B} \) maps \( W^\rho \) continuously to \( \tilde{E}^{2,1/2h}_\rho \).

Conversely, let \( \varphi \in \tilde{E}^{2,1/2h}_\rho \) and \( \|\varphi\|_r < \infty \). Then the function \( \varphi_1(z) = e^{-\frac{1}{2h} z^2} \varphi(\bar{z}/\sqrt{2}) \) satisfies

\[
|\varphi_1(z)| \leq \|\varphi\|_r e^{\frac{1}{2h} y^2 - r' \|x\|^2} \|y\|^p,
\]

where \( r' = r/2^{1+p/2} \). Hence its Fourier-Laplace transform satisfies the estimate

\[
|\hat{\varphi}_1(\sigma + i\tau)| \leq (2\pi)^{-n/2} \inf_{y} \left| \int e^{-i(x - iy)(\sigma + i\tau)} \varphi_1(x - iy) \, dx \right| \leq C_b \|\varphi\|_r e^{-\frac{1}{2h} y^2 - r' \|y\|^2} \|y\|^p \sup_x e^{-\|x\|^2 - |\tau| - b \|x\|^2} [y]^p,
\]

with \( b < r' \). Substituting \( y = h \sigma \), we obtain

\[
\inf_{y} \left( -y\sigma + \frac{1}{2h} y^2 - r' \|y\|^p \right) \leq -\frac{h}{2} \sigma^2 - r' h^p \|\sigma\|^p.
\]
Taking the supremum over $x$ gives the conjugate convex function $b'\|\tau\|^{p'}$. Therefore, it follows from (55) that
\[
\left| e^{\frac{b}{2}(\sigma+i\tau)^2} \hat{\varphi}_1(\sigma + i\tau) \right| \leq C \|\varphi\| e^{-a|\sigma|^{p'}+b'|\tau|^{p'}} ,
\]
where $a = \rho'\hbar^2$ and $b' = \rho'\hbar^{1/2}$. We conclude that $\mathcal{B}^{-1}\varphi \in W^\rho$ and the inverse Bargmann transformation is continuous from $\mathcal{F}^{2,1/2}_\rho$ to $W^\rho$. The theorem is proved. 

**Corollary 1.** For any $\rho$ satisfying $1 < \rho \leq 2$, the Wick correspondence (49) has a unique extension to a continuous algebra homomorphism from $(\mathcal{E}^\rho(\mathbb{R}^n), \star_{\text{Wick}})$ to $\mathcal{A}_{E^\rho, A}(\mathcal{F}^2)$, where $A = 1/(2\hbar)$.

### 9. Concluding remarks

In this paper, we consider only analytic symbols of operators, but the above method of extending the star products and their representations by continuity, starting from a suitable auxiliary space like $\mathcal{W}^\rho$ or $W^\rho$, has a wider field of application. In particular, this approach allows us to extend the Weyl symbol calculus beyond the traditional framework of tempered distributions, see [20], [21]. Such a generalization is also desirable for the anti-Wick correspondence because not all bounded operators have anti-Wick symbols among the usual functions [18].

If $\rho > 2$, then the space $W^\rho$ is not an algebra under the Wick star product. However, it is a topological algebra with respect to the Weyl-Moyal product $\star_{\hbar}$. It follows from a theorem of [17] that the elements of $\mathcal{E}^\rho$ are two-sided multipliers of the algebra $(S^1_{1/\rho}, \star_{\hbar})$ which coincides with the algebra $(W^\rho, \star_{\hbar})$ for $\rho > 1$. This implies that for all $\mathcal{E}^\rho$ and $\varphi \in W^\rho$, the products $f \star_{\hbar} \varphi$ and $\varphi \star_{\hbar} f$ belong to $W^\rho$ and are continuous in $f$ and $\varphi$. Furthermore, the following associativity relations hold:
\[
(f \star_{\hbar} \varphi) \star_{\hbar} \psi = f \star_{\hbar} (\varphi \star_{\hbar} \psi), \quad (\varphi \star_{\hbar} f) \star_{\hbar} \psi = \varphi \star_{\hbar} (f \star_{\hbar} \psi), \quad (\varphi \star_{\hbar} \psi) \star_{\hbar} f = \varphi \star_{\hbar} (\varphi \star_{\hbar} f) .
\]

As shown in [21], the Weyl correspondence defines an isomorphism between the left multiplier algebra of $(W^\rho, \star_{\hbar})$ and the operator algebra $\mathcal{A}_{W^\rho}(L^2)$. Therefore, for the case of Weyl-Moyal product, Theorem [3] can be derived as a consequence of these results. The direct proof given here is more illuminating and applies to any star product of the form [3].

Finally, we note that along with the spaces $\mathcal{E}^\rho$, it is natural to consider the spaces of entire functions of the same order and finite type, i.e., satisfying the condition
\[
|f(z)| \leq C e^{a|z|^\rho} ,
\]
where $a$ and $C$ are constants depending on $f$. We let $E^\rho$ denote this space and endow it with the inductive limit topology defined by the family of Banach spaces $E^{\rho,a}$ with the norms $\|f\|_a = \sup_x |f(z)| e^{-a|z|^\rho}$. It is easy to see that an analogue of Theorem [2] holds for $E^\rho$, but only if the strong inequality $\rho < 2$ holds. It is essential that if $1/\rho' + 1/\rho = 1$ and $\rho \leq 2$, then the series defining the star product $f \star_{\ast} g$ converges absolutely in $E^\rho$ for all $f \in E^\rho$ and $g \in E^\rho$. Moreover, it can be proved that the algebra $(\mathcal{E}^\rho, \star_{\ast})$ acts continuously on the space $E^\rho$ and on its subspace $W^\rho$ by the left and right $\ast_{\ast}$-multiplication. In other words, the function spaces $E^\rho$ and $W^\rho$ (as well as the analogous spaces associated with cone-shaped regions) are topological bimodules over the algebra $(\mathcal{E}^\rho, \star_{\ast})$ for any $\ast$. 


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