STATIONARITY AND SELF-SIMILARITY CHARACTERIZATION
OF THE SET-INDEXED FRACTIONAL BROWNIAN MOTION

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Abstract. The set-indexed fractional Brownian motion (sifBm) has been defined by Herbin-Merzbach (2006a) for indices that are subsets of a metric measure space. In this paper, the sifBm is proved to satisfy a strengthened definition of increment stationarity. This new definition for stationarity property allows to get a complete characterization of this process by its fractal properties: The sifBm is the only set-indexed Gaussian process which is self-similar and has stationary increments.

Using the fact that the sifBm is the only set-indexed process whose projection on any increasing path is a one-dimensional fractional Brownian motion, the limitation of its definition for a self-similarity parameter \(0 < H < 1/2\) is studied, as illustrated by some examples. When the indexing collection is totally ordered, the sifBm can be defined for \(0 < H < 1\).

1. Introduction

In [HeMe06a], the set-indexed fractional Brownian motion (sifBm) was defined among processes indexed by a collection of subsets of a measure metric space. The study of its properties showed fractal behaviour such as a kind of increment stationarity and self-similarity. In addition, it is proved that the projection of a sifBm on an increasing path is a one-dimensional fractional Brownian motion. Fine properties of multi-dimensional parameter fractional Brownian motions are studied by several authors (see [Is05, TuXi08, XiZh02] for example).

In this paper, we extend the increment stationarity property defined in [HeMe06a]. Instead of considering a stationarity property on \(\Delta X_C\) (for \(C \in C_0\)) that only involves marginal distributions of the increment process, we consider a property of stationarity of the distribution of the whole process \(\Delta X = \{\Delta X_C; C \in C_0\}\). We obtain a strengthened definition for increment stationarity which is preserved under projections on flows (increasing paths). More precisely, we show that if \(X\) is a set-indexed process satisfying this new property of stationarity, then its projection on any flow is a one-dimensional increment stationary process. For that reason, this new definition can be considered as the most natural one. The set-indexed fractional Brownian motion is proved to satisfy this property.

The new stationarity definition allows us to get the main result of this paper: a complete characterization of the set-indexed fractional Brownian motion as the only set-indexed mean-zero Gaussian process which satisfies the two properties of increment stationarity and self-similarity. This property thus extends the well-known characterization of one-parameter fractional Brownian motion.

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The second point of this paper is the use of flows to understand the limitation of the general sIFBm’s definition for a parameter $H \in (0, 1/2]$, as opposed to one-parameter fractional Brownian motion which is defined for $0 < H < 1$. In the latter case, the behaviour of the process leads to critical values for $H$ (see [Ch01, ChNu05] for instance). Here we observe that the set-indexed fractional Brownian motion can be defined for $0 < H < 1$ when the indexing collection $\mathcal{A}$ is totally ordered. On the contrary, we give new examples of indexing collections $\mathcal{A}$ on which the sIFBm cannot be defined for $H > 1/2$.

The paper is organized as follows: in section 2, indexing collection and the set-indexed fractional Brownian motion are defined, and projections of the sIFBm on flows are studied. Section 3 is devoted to the study of the sIFBm along flows to get a deeper understanding of its properties. Among the results, we get a characterization of the sIFBm by its projection along flows, which constitutes a converse of a result in [HeMe06a]. We prove that a set-indexed process is a set-indexed fractional Brownian motion if and only if its projections on every flows are one-dimensional fractional Brownian motions. This gives a good justification of the definition of the sIFBm and opens the door to a variety of applications. In [HeMe06c], a part of this result was presented and also an integral representation for the sIFBm was given.

In section 4, we use the flows to get a better understanding of the limitation $H \in (0, 1/2]$. This fact was already observed for the fBm indexed by the sphere of $\mathbb{R}^N$ (see [Is05]). The examples given explain why the sIFBm cannot be defined in general for $H > 1/2$. As a byproduct, we prove that the cardinality of a totally ordered indexing collection cannot exceed the continuum.

In section 5, the new strengthened definition for increment stationarity of a set-indexed process is studied.

Then in section 6, we establish the fractal characterization of the set-indexed fractional Brownian motion.

### 2. Projections of the sIFBm on flows

We follow [HeMe06a] for the framework and notation. Our processes are indexed by an indexing collection $\mathcal{A}$ of compact subsets of a locally compact metric space $\mathcal{T}$ equipped with a Radon measure $m$ (denoted $(\mathcal{T}, m)$).

Let $\mathcal{A}(u)$ denotes the class of finite unions from sets belonging to $\mathcal{A}$.

**Definition 2.1 (Indexing collection).** A nonempty class $\mathcal{A}$ of compact, connected subsets of $\mathcal{T}$ is called an indexing collection if it satisfies the following:

(1) $\emptyset \in \mathcal{A}$, and $A^c \neq A$ if $A \notin \{\emptyset, \mathcal{T}\}$. In addition, there exists an increasing sequence $(B_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{A}(u)$ such that $\mathcal{T} = \bigcup_{n \in \mathbb{N}} B_n^c$.

(2) $\mathcal{A}$ is closed under arbitrary intersections and if $A, B \in \mathcal{A}$ are nonempty, then $A \cap B$ is nonempty. If $(A_i)$ is an increasing sequence in $\mathcal{A}$ and if there exists $n \in \mathbb{N}$ s. t. for all $i$, $A_i \subseteq B_n$ then $\bigcup_i A_i \in \mathcal{A}$.

(3) The $\sigma$-algebra generated by $\mathcal{A}$, $\sigma(\mathcal{A}) = \mathcal{B}$, the collection of all Borel sets of $\mathcal{T}$.

(4) Separability from above

There exists an increasing sequence of finite subclasses $\mathcal{A}_n = \{A_{n1}, ..., A_{nk_n}\}$ of $\mathcal{A}$ closed under intersections and satisfying $\emptyset, B_n \in \mathcal{A}_n(u)$ and a sequence of functions $g_n : \mathcal{A} \to \mathcal{A}_n(u) \cup \{\mathcal{T}\}$ such that
The set of all simple (resp. elementary) flows is denoted by \(\mathcal{F} \) (resp. \(\mathcal{E} \)).

- \((a)\) \(g_n\) preserves arbitrary intersections and finite unions.
- \((i.e.)\) \(g_n(\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} g_n(A)\) for any \(A \subseteq \mathcal{A}\), and
- \(\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{m} A_j\), then \(\bigcup_{i=1}^{k} g_n(A_i) = \bigcup_{j=1}^{m} g_n(A_j)\);
- \((b)\) for each \(A \in \mathcal{A}\), \(A \subseteq (g_n(A))^{\circ}\);
- \((c)\) \(g_n(A) \subseteq g_m(A)\) if \(n \geq m\);
- \((d)\) for each \(A \in \mathcal{A}\), \(A = \bigcap_{n} g_n(A)\);
- \((e)\) if \(A, A' \in \mathcal{A}\) then for every \(n\), \(g_n(A) \cap A' \in \mathcal{A}\), and if \(A' \in \mathcal{A}_n\) then
- \(g_n(A) \cap A' \in \mathcal{A}_n\);
- \((f)\) \(g_n(\emptyset) = \emptyset \forall n\).

(5) Every countable intersection of sets in \(\mathcal{A}(u)\) may be expressed as the closure of a countable union of sets in \(\mathcal{A}\).

(Note: ‘\(\subset\)’ indicates strict inclusion and ‘\(\bigcap\)' and ‘\(\bigcup\)' denote respectively the closure and the interior of a set.)

The set-indexed fractional Brownian motion (sifBm) on \((T, \mathcal{A}, m)\) was defined as the centered Gaussian process \(B^H = \{B^H_U; U \in \mathcal{A}\}\) such that

\[
\forall U, V \in \mathcal{A}; \quad E[B^H_U B^H_V] = \frac{1}{2} \left[ m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H} \right],
\]

where \(0 < H \leq \frac{1}{2}\).

If \(\mathcal{A}\) is provided with a structure of group on \(T\), properties of increment stationarity and self-similarity are studied in [HeMe06a]. In the special case of \(\mathcal{A} = \{[0, t]; t \in \mathbb{R}_+^N\} \cup \{\emptyset\}\), we get a multiparameter process called Multiparameter fractional Brownian motion (MpfBm), whose properties are studied in [HeMe06b].

The notion of flow is the key to reduce the proof of many theorems. It was extensively studied in [Iv03] and [IvMe00].

**Definition 2.2.** An elementary flow is defined to be a continuous increasing function \(f : [a, b] \subset \mathbb{R}_+ \rightarrow \mathcal{A}\), i.e. such that

\[
\forall s, t \in [a, b]; \quad s < t \Rightarrow f(s) \subseteq f(t)
\]

\[
\forall s \in [a, b]; \quad f(s) = \bigcap_{v > s} f(v)
\]

\[
\forall s \in (a, b); \quad f(s) = \bigcup_{u < s} f(u).
\]

A simple flow is a continuous function \(f : [a, b] \rightarrow \mathcal{A}(u)\) such that there exists a finite sequence \((t_0, t_1, \ldots, t_n)\) with \(a = t_0 < t_1 < \cdots < t_n = b\) and elementary flows \(f_i : [t_{i-1}, t_i] \rightarrow \mathcal{A}\) \((i = 1, \ldots, n)\) such that

\[
\forall s \in [t_{i-1}, t_i]; \quad f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j).
\]

The set of all simple (resp. elementary) flows is denoted by \(S(\mathcal{A})\) (resp. \(S^e(\mathcal{A})\)).

In [IvMe00], the projection of a set-indexed process \(X = \{X_U; U \in \mathcal{A}\}\) on any elementary flow \(f\) was considered as the real-parameter process \(X^f = \{X_{f(t)}; t \in [a, b] \subset \mathbb{R}_+\}\).

Here, we define another parametrization of this projection, which allows simpler statements in the sequel.
Definition 2.3. For any set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ on the space $(\mathcal{T}, \mathcal{A}, m)$ and any elementary flow $f : [a, b] \to \mathcal{A}$, we define the $m$-standard projection of $X$ on $f$ as the process

$$X^f,m = \{X^f_m = X_{f \circ \theta^{-1}(t)}; t \in [a, b]\},$$

where $\theta : t \mapsto m[f(t)]$ and $\theta^{-1}$ is its pseudo-inverse function.

The use of this new notation $X^f,m$ avoids any confusion with the projection $X^f$ previously defined.

Notice that since $\theta$ is non-decreasing, the function $\theta^{-1}$ is well-defined and for all $t \in [a, b]$, we have $\theta(\theta^{-1}(t)) = t$.

The following result, proved in [HeMe06a], gives a good justification of the definition of the sifBm.

Proposition 2.4. Let $B^H$ be a sifBm on $(\mathcal{T}, \mathcal{A}, m)$ and $f : [a, b] \to \mathcal{A}$ be an elementary flow. Then the process $(B^H)^f,m = \{B^H_{f \circ \theta^{-1}(t)}; t \in [a, b]\}$, where $\theta : t \mapsto m[f(t)]$, is a one-parameter fractional Brownian motion.

In section 3, we prove the converse to Proposition 2.4. For this purpose, we will use the following lemma proved in [Iv03].

Lemma 2.5. The finite dimensional distributions of an additive $\mathcal{A}$-indexed process $X$ determine and are determined by the finite dimensional distributions of the class $\{X^f, f \in S(\mathcal{A})\}$.

3. Characterization of the sifBm

In the case of $L^2$-monotone outer-continuous set-indexed processes, we prove that the sifBm could be defined as a process whose projections on elementary flows is a one-dimensional fractional Brownian motion.

Recall the following definition (see [IvMe00])

Definition 3.1. A set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ is said $L^2$-monotone outer-continuous if for any decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{A}$,

$$E[|X_{U_n} - X_{\bigcap_{k \in \mathbb{N}} U_k}|^2] \to 0$$

as $n \to \infty$.

Proposition 3.2. The sifBm $B^H$ ($0 < H \leq 1/2$) is $L^2$-monotone outer-continuous.

Proof. Let $(U_n)_{n \in \mathbb{N}}$ be a decreasing sequence in $\mathcal{A}$. As $\bigcap_{k \in \mathbb{N}} U_k \in \mathcal{A}$, by definition of sifBm, we have

$$\forall n \in \mathbb{N}; \quad E\left[|B^H_{U_n} - B^H_{\bigcap_{k \in \mathbb{N}} U_k}|^2\right] = m(U_n \setminus \bigcap_{k \in \mathbb{N}} U_k)^{2H}$$

But, as $(U_n)_{n \in \mathbb{N}}$ is decreasing, by definition of a measure,

$$m(U_n \setminus \bigcap_{k \in \mathbb{N}} U_k) \to 0.$$

Then the result follows. □

The following lemma will be useful for the converse of proposition 2.4, and will be strengthened in section 4 to understand links between structure of $\mathcal{A}$ and flows.
Lemma 3.3. For any $U_1, U_2, \ldots, U_n \in \mathcal{A}$ such that $U_i \subset U_{i+1}$ ($\forall i = 1, \ldots, n-1$), there exist an elementary flow $f : \mathbb{R}_+ \to \mathcal{A}$ and real numbers $0 < t_1 < t_2 < \cdots < t_n$ such that

$$\forall i = 1, \ldots, n; \quad f(t_i) = U_i.$$  

Proof. This result is a particular case of lemma 5.1.7 in [IvMe00] (and lemma 5 in [IV03]). As the sequence $U_1, U_2, \ldots, U_n$ is increasing, $\mathcal{A}' = \{U_1, U_2, \ldots, U_n\}$ constitutes a semilattice of $\mathcal{A}$ with a consistent numbering. The proof of lemma 5.1.7 in [IvMe00] constructs such an elementary flow $f$. Here the increasing property of $(U_i)_{1 \leq i \leq n}$ allows $f$ to take its values in $\mathcal{A} (\subset \mathcal{A}(u))$. □

Theorem 3.4. Let $X = \{X_U; U \in \mathcal{A}\}$ be an $L^2$-monotone outer-continuous set-indexed process.

If the projection $X^f$ of $X$ on any elementary flow $f$, is a time-changed one-parameter fractional Brownian motion of parameter $H \in (0, 1/2]$, then there exists a Borel measure $\nu$ on $\mathcal{T}$ such that $X$ is a set-indexed fractional Brownian motion on $(\mathcal{T}, \mathcal{A}, \nu)$.

This theorem states that the time-changes giving to projections the law of a one-parameter fBm, determine a Borel measure $\nu$ such that $X$ is a sifBm on the space $(\mathcal{T}, \mathcal{A}, \nu)$.

A sketch of the proof is given in [HeMe06c]. Here we present a complete proof. In particular, the importance of lemma 3.3 is pointed out.

Proof. Let $f : [a, b] \to \mathcal{A}$ be an elementary flow. As the projected process $X^f$ is a time-changed fBm of parameter $H$, we have

$$\forall s, t \in [a, b]; \quad E \left[ X^f_t - X^f_s \right]^2 = |\theta_f(t) - \theta_f(s)|^{2H}$$  

where $\theta_f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function.

The idea of the proof is the construction of a measure $\nu$ such that for any $f \in S^e(\mathcal{A})$,

$$\forall t \in [a, b]; \quad \theta_f(t) = \nu [f(t)].$$

For all $U \in \mathcal{A}$, let us define

$$F^e_U = \{ f \in S^e(\mathcal{A}) : \exists u_f \in [a, b]; U = f(u_f) \}.$$  

As for all $f$ and $g$ in $F^e_U$, $\theta_f(u_f)^{2H} = \theta_g(u_g)^{2H} = E [X_U]^2$, one can define

$$\psi(U) = \theta_f(u_f) = \left( E [X_U]^2 \right)^{\frac{1}{2H}}.$$  

For all $U$ and $V$ in $\mathcal{A}$ with $U \subset V$, lemma 3.3 implies the existence of an elementary flow $f$ such that

$$\exists u_f, v_f \in [a, b]; \quad u_f \leq v_f; \quad U = f(u_f) \subset f(v_f) = V$$

Then, as the time-change $\theta_f$ is increasing, $\psi$ is non-decreasing in $\mathcal{A}$. 
The definition of $\psi$ on $A$ can be extended to $C = U \setminus \bigcup_{1 \leq i \leq n} U_i$ by the inclusion-exclusion formula

$$\psi(C) = \psi(U) - \sum_{i=1}^{n} \psi(U \cap U_i) + \sum_{i<j} \psi(U \cap (U_i \cap U_j))$$

$$- \cdots + (-1)^n \psi(U \cap \left( \bigcap_{1 \leq i \leq n} U_i \right)).$$

(4)

We denote this class of sets by $C$.

The definition (4) of $\psi$ can be easily extended to the set $C(u)$ of finite unions of elements of $C$ in the same way.

A direct consequence of definition (4) is that, for all $C_1, C_2 \in C$ such that $C = C_1 \cup C_2 \in C$,

$$\psi(C_1 \cup C_2) = \psi(C_1) + \psi(C_2) - \psi(C_1 \cap C_2)$$

(5)

Let us remark that equality (5) holds for any $C_1, C_2 \in C(u)$.

From the pre-measure $\psi$ defined on $C$, the function

$$\nu : E \subset T \mapsto \inf_{C_i \in C} \sum_{i=1}^{\infty} \psi(C_i)$$

defines an outer measure on $T$ (see [Ro70] pp. 9–26). Let us show that $\nu$ defines a Borel measure on the topological space $T$.

Let $\mathcal{M}_\nu$ be the $\sigma$-field of $\nu$-measurable subsets of $T$. It is known that $\nu$ is a measure on $\mathcal{M}_\nu$ (see [Ro70], thm. 3). By definition, any $U \in A$ is $\nu$-measurable if

$$\forall A \subset U, \forall B \subset T \setminus U ; \quad \nu(A \cup B) = \nu(A) + \nu(B)$$

As the inequality $\nu(A \cup B) \leq \nu(A) + \nu(B)$ follows from definition of any outer-measure, it remains to show the converse inequality.

Consider any sequence $(C_i)_{i \in \mathbb{N}}$ in $C$ such that $A \cup B \subset \bigcup_{i} C_i$. The sequence $(C_i)_{i \in \mathbb{N}}$ can be decomposed in the elements $C_i, i \in I$ such that $C_i \cap U = \emptyset$ and the $C_i, i \in J$ such that $C_i \subset U$ (if $C_i \cap U \neq \emptyset$ and $C_i \not\subset U$, cut $C_i = C_i' \cup C_i''$ where $C_i' \subset U$ and $C_i'' \cap U = \emptyset$).

As

$$A \cup B \subset \left[ \bigcup_{i \in I} C_i \right] \cup \left[ \bigcup_{i \in J} C_i \right]$$

we get the implications

$$\forall i \in I ; \ C_i \cap U = \emptyset \quad \Rightarrow \quad A \subset \bigcup_{i \in J} C_i$$

and

$$\forall i \in J ; \ C_i \subset U \quad \Rightarrow \quad B \subset \bigcup_{i \in I} C_i.$$
which leads to $\nu(A \cup B) \geq \nu(A) + \nu(B)$.

We have proved that $\mathcal{A} \subset \mathcal{M}_\nu$. By definition of $\mathcal{A}$, the smallest $\sigma$-field containing $\mathcal{A}$ is the Borel $\sigma$-field $\mathcal{B}$. Therefore, $\mathcal{B} \subset \mathcal{M}_\nu$ and $\nu$ is a measure on $\mathcal{B}$.

The second part of the proof is to show that the measure $\nu$ is an extension of $\psi$, i.e.

$$\forall U \in \mathcal{A}; \quad \nu(U) = \psi(U).$$

(7)

- For any $U \in \mathcal{A}$, by definition of $\nu(U)$,

$$\nu(U) = \inf_{\bigcup C_i \subset U} \sum_{i=1}^{\infty} \psi(C_i) \leq \psi(U)$$

(8)

- To prove the converse inequality, consider $U \in \mathcal{A}$ and a sequence $(C_i)_{i\in\mathbb{N}}$ in $\mathcal{C}$ such that $U \subset \bigcup_i C_i$. For all $n \in \mathbb{N}^*$, we have

$$U \subset \bigcup_{1 \leq i \leq n} C_i \cup \left[ U \setminus \bigcup_{1 \leq i \leq n} C_i \right].$$

Then, (5) implies

$$\psi(U) = \psi\left( \bigcup_{1 \leq i \leq n} C_i \right) + \psi\left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right) \leq \sum_{i=1}^{\infty} \psi(C_i) + \psi\left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right).$$

(9)

Using $L^2$-monotone outer continuity of $X$ and proposition 1.4.8 in [IvMe00], we have

$$\lim_{n \to \infty} \psi\left( U \setminus \bigcup_{1 \leq i \leq n} C_i \right) = 0$$

(10)

Thus, (9) and (10) imply that for all sequence $(C_i)_{i\in\mathbb{N}}$ in $\mathcal{C}$ such that $U \subset \bigcup_i C_i$,

$$\psi(U) \leq \sum_{i=1}^{\infty} \psi(C_i)$$

and then, by definition of $\nu(U)$

$$\psi(U) \leq \nu(U)$$

(11)

Equality (7) follows from (8) and (11).

From (3) and (7), the Borel measure $\nu$ defined by (6) satisfies

$$\forall U \in \mathcal{A}; \quad E[|X_U|^2] = \psi(U)^{2H} = \nu(U)^{2H}.$$

Using the Borel measure $\nu$, consider a set-indexed fractional Brownian motion $Y$ on $(T, \mathcal{A}, \nu)$ (which exists as $0 < H \leq 1/2$), defined by

$$\forall U, V \in \mathcal{A}; \quad E[Y_UY_V] = \frac{1}{2} \left[ \nu(U)^{2H} + \nu(V)^{2H} - \nu(U \triangle V)^{2H} \right].$$
According to proposition 6.4 in [HeMe06a], projections of $Y$ on any elementary flow $f : [a, b] \to \mathcal{A}$ is a time-change one-parameter fractional Brownian motion, i.e. such that

$$\forall s, t \in [a, b]; \quad E \left[ Y^f_t - Y^f_s \right]^2 = |\nu[f(t)] - \nu[f(s)]|^{2H} = |\theta_f(t) - \theta_f(s)|^{2H}$$

Then, the projections of the set-indexed processes $X$ and $Y$ on any elementary flow have the same distribution. By additivity, this fact holds also on any simple flow. Thus, lemma 2.5 implies $X$ and $Y$ have the same law. □

Considering only $m$-standard projections on flows, theorem 3.4 gives the following characterization of the sifBm.

**Corollary 3.5.** Let $X = \{X_U; U \in \mathcal{A}\}$ be an $L^2$-monotone outer-continuous set-indexed process. The following two assertions are equivalent:

(i) for any elementary flow $f : [a, b] \to \mathcal{A}$, the $m$-standard projection of $X$ on $f$ is a one-parameter fractional Brownian motion of index $H \in (0, 1/2]$;

(ii) $X$ is a set-indexed fractional Brownian motion of index $H \in (0, 1/2]$ on $(T, \mathcal{A}, m)$.

### 4. Can sifBm be defined for $H > 1/2$?

In [HeMe06a], the set-indexed fractional Brownian motion is defined for a parameter $H \in (0, 1/2]$. As one-dimensional fractional Brownian motion is defined for $H \in (0, 1)$, a natural question arises: Are there conditions on the indexing collection $\mathcal{A}$ such that sifBm on $(T, \mathcal{A}, m)$ can be defined for $H > 1/2$? Projections on flows allow to answer this question.

Let $\Phi^H : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ denote the function

$$\Phi^H : (U, V) \mapsto \frac{1}{2} \left\{ m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H} \right\}.$$ 

The question is: In which cases $\Phi^H$ can be seen as the covariance function of a set-indexed process? In the following, we can see that this question is related to the two different cases either $\mathcal{A}$ is totally ordered or not.

Let us first describe the particular structure of a totally ordered indexing collection.

**Proposition 4.1.** If the indexing collection $\mathcal{A}$ is totally ordered by the inclusion, then there exists a surjective elementary flow $f : \mathbb{R}_+ \to \mathcal{A}$, i.e. such that

$$\forall U \in \mathcal{A}; \quad U \in f(\mathbb{R}_+).$$

**Proof.** By definition of an indexing collection, $\mathcal{A}$ can be discretized by the increasing sequence of finite subclasses $(\mathcal{A}_n)_{n \in \mathbb{N}}$. As subclasses $\mathcal{A}_n$ are finite and totally ordered, lemma 3.3 implies for all $n$, existence of an elementary flow $f_n : \mathbb{R}_+ \to \mathcal{A}$ such that

$$\mathcal{A}_n \subseteq f_n(\mathbb{R}_+).$$

Note that, by construction of flows $(f_n)$ (see [IvMe00], lemma 5.1.7), we have

$$\forall t \in f_n^{-1}(\mathcal{A}_n), \forall m \geq n; \quad f_m(t) = f_n(t). \quad (12)$$

Let us define $\mathcal{I}$, the set of $s \in \mathbb{R}_+$ such that $f_m(s) \in \mathcal{A}_m$ for some $m \in \mathbb{N}$.

From the sequence $(f_n)_{n \in \mathbb{N}}$, we define the function $f : \mathbb{R}_+ \to \mathcal{A}$ in the following way:
• For all \( t \in I \), there exists \( m \in N \) such that \( f_m(t) \in A_m \). By \([12]\), the sequence \((f_n(t))_{n \geq m}\) is constant. We can set \( f(t) = f_m(t) \).

• In the construction of lemma 5.1.6 in \([IvMe00]\), the subset \( I \) is proved to be dense. Let us define for all \( t \notin I \),

\[
f(t) = \bigcap_{s \in I, s > t} f(s).
\]

Let us show that \( f \) satisfies the conclusions of the proposition 4.1.

• \( f \) is non-decreasing: for all \( s, t \in R_+ \) such that \( s < t \), we have clearly

\[
\bigcap_{u \in I, u > s} f(u) \subseteq \bigcap_{u \in I, u > t} f(u)
\]

and then, \( f(s) \subseteq f(t) \).

• \( f \) passes through every elements of \( \bigcup_{n \in N} A_n \): for any \( U \in \bigcup_{n \in N} A_n \), there exist \( m \in N \) and \( t_U \in I \) such that \( U = f_m(t_U) \). Then, by definition of \( f \) on \( I \), we have \( f(t_U) = U \).

• \( f \) is continuous: according to definition 2.2, we must verify that \( f(t) = \bigcap_{s \in I, s > t} f(s) \) and \( f(t) = \bigcup_{s < t} f(s) \). Using density of \( I \), the right-continuity of \( f \) comes directly from its definition

\[
f(t) = \bigcap_{s \in I, s > t} f(s) = \bigcap_{s > t} f(s).
\]

For the second equality, in the proof of lemma 5.1.6 in \([IvMe00]\), it is proved that

\[
\bigcup_{s \in I, s < t} f(s) = \bigcap_{s > t} f(s).
\]

Then the density of \( I \) allows to conclude \( f(t) = \bigcup_{s < t} f(s) \).

• \( f \) passes through all the elements of \( A \): for all \( U \in A \setminus \bigcup_{n \in N} A_n \), its approximations satisfy

\[
\forall n \in N; \quad g_n(U) \in A_n(U) = A_n
\]

because \( A_n \) is totally ordered for all \( n \). Therefore, for all \( n \in N \), there exists \( t_{g_n(U)} \) in \( R_+ \) such that \( g_n(U) = f_n(t_{g_n(U)}) \), and we can write

\[
U = \bigcap_{n \in N} g_n(U) = \bigcap_{n \in N} f_n(t_{g_n(U)}) = \bigcap_{n \in N} f(t_{g_n(U)}).
\]

The sequence \((g_n(U))_{n \in N}\) is non-increasing in \( R_+ \): By definition,

\[
\begin{align*}
g_{n+1}(U) &= f_{n+1}(t_{g_{n+1}(U)}) \\
g_n(U) &= f_n(t_{g_n(U)}) = f_{n+1}(t_{g_n(U)})
\end{align*}
\]

using the construction of \((f_n)_{n \in N}\). Then, as \((g_n(U))_{n \in N}\) is non-increasing and \( f_{n+1} \) is non-decreasing, we get \( t_{g_{n+1}(U)} \leq t_{g_n(U)} \) from \( g_n(U) \subseteq g_{n+1}(U) \).
Consequently, \((t_{g_n(U)})_{n \in \mathbb{N}}\) converges to some \(t_U \in \mathbb{R}_+\). Then, using the continuity of \(f\), we get

\[
\bigcap_{n \in \mathbb{N}} f(t_{g_n(U)}) = f(t_U)
\]

which proves that \(U \in f(\mathbb{R}_+)\).

\(\square\)

In [IvMe00], proposition 1.3.5 shows that by definition, an indexing collection is not allowed to be too big. More precisely, the cardinality of \(A\) cannot exceed the cardinality of \(\mathcal{P}(\mathbb{R})\), the set of subsets of \(\mathbb{R}\). In the particular case of a totally ordered indexing collection, this upper bound for size of \(A\) can be sharpened. From surjectivity of the flow \(f\) in proposition 4.1, we can state

Corollary 4.2. If the indexing collection \(A\) is totally ordered by the inclusion, then its cardinality cannot exceed cardinality of \(\mathbb{R}\).

From this study of the particular case of a totally ordered collection \(A\), we can prove the existence the set-indexed fractional Brownian motion for a parameter \(H \in (0, 1)\), as in one-parameter case.

Theorem 4.3. If the indexing collection \(A\) is totally ordered by the inclusion, then the set-indexed fractional Brownian motion on \((T, A, m)\) can be defined for \(0 < H < 1\).

Proof. According to proposition 4.1 as \(A\) is totally ordered, there exists an elementary flow \(f : \mathbb{R}_+ \to A\) passing through every \(U \in A\), i.e. such that

\[
\forall U \in A;\quad U \in f(\mathbb{R}_+).
\]

Then, the existence of the sifBm is equivalent to the existence of its projection on the flow \(f\). For any \(H \in (0, 1)\), let us consider a one-parameter fractional Brownian motion \(B^H = \{B^H_t; t \in \mathbb{R}_+\}\) of self-similarity parameter \(H\).

The set-indexed process \(X = \{X_U = B^H_{\theta \circ f^{-1}(U)}; U \in A\}\), where \(\theta : t \mapsto m[f(t)]\), is a mean-zero Gaussian process such that for all \(U, V \in A\),

\[
E[X_U X_V] = \frac{1}{2} \left[ (\theta \circ f^{-1}(U))^{2H} + (\theta \circ f^{-1}(V))^{2H} - (\theta \circ f^{-1}(U) - \theta \circ f^{-1}(V))^{2H} \right]
\]

\[
= \frac{1}{2} \left[ m(U)^{2H} + m(V)^{2H} - |m(U) - m(V)|^{2H} \right].
\]

As \(A\) is totally ordered, we have either \(U \subseteq V\) or \(V \subseteq U\), and then

\[
|m(U) - m(V)| = m(U \triangle V).
\]

Thus, the covariance structure of \(X\) is given by

\[
\forall U, V \in A;\quad E[X_U X_V] = \frac{1}{2} \left[ m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H} \right]
\]

and it follows that \(X\) is a sifBm of parameter \(H\) on \((T, A, m)\). \(\square\)
The cross terms are measure in $\mathbb{R} = \mathbb{A} \cup \{0\}$. Let us consider the four points of $(2$ $H > 1/2$ the function $\Phi^H$ is not positive definite. In fact, for $H > 1/2$ the function $\Phi^H$ is not positive definite. Let us consider the indexing collection constituted by rectangles of $\mathbb{R}^2$, $\mathcal{A} = \{[0,t]; t \in \mathbb{R}^2_+ \} \cup \{\emptyset\}$. $\mathcal{A}$ is not totally ordered, and then, theorem 4.3 cannot be applied. In fact, for $H > 1/2$ the function $\Phi^H$ is not positive definite. The following example shows that even in the simple case of rectangles in $\mathbb{R}^2$, the sifBm may not be defined for $H > 1/2$.

**Example 4.4.** Let us consider the indexing collection constituted by rectangles of $\mathbb{R}^2$, $\mathcal{A} = \{[0,t]; t \in \mathbb{R}^2_+ \} \cup \{\emptyset\}$. $\mathcal{A}$ is not totally ordered, and then, theorem 4.3 cannot be applied. In fact, for $H > 1/2$ the function $\Phi^H$ is not positive definite. Let us consider the four points of $\mathbb{R}^2$, defined by their coordinates $t_1 = (1, 1), t_2 = (2, 1), t_3 = (1, 2)$ and $t_4 = (2, 2)$. The points $t_i$ ($1 \leq i \leq 4$) define four elements $U_i = [0,t_i]$ of $\mathcal{A}$. We compute $\Phi^H(U_i, U_j)$ for all $1 \leq i, j \leq 4$ ($m$ is the Lebesgue measure in $\mathbb{R}^2$). The diagonal terms are

$$
\Phi^H(U_1, U_1) = m(U_1)^{2H} = 1; \quad \Phi^H(U_2, U_2) = m(U_2)^{2H} = 2^{2H};
$$

$$
\Phi^H(U_3, U_3) = m(U_3)^{2H} = 2^{2H}; \quad \Phi^H(U_4, U_4) = m(U_4)^{2H} = 4^{2H}.
$$

The cross terms are

$$
\Phi^H(U_1, U_2) = \frac{1}{2} \left[ m(U_1)^{2H} + m(U_2)^{2H} - m(U_2 \setminus U_1)^{2H} \right]
$$

$$
= \frac{1}{2} \left[ 1 + 2^{2H} - 1 \right] = 2^{2H-1},
$$

$$
\Phi^H(U_1, U_3) = 2^{2H-1};
$$

$$
\Phi^H(U_1, U_4) = \frac{1}{2} \left[ 1 + 4^{2H} - 3^{2H} \right];
$$

and

$$
\Phi^H(U_2, U_3) = \frac{1}{2} \left[ 2^{2H} + 2^{2H} - 2^{2H} \right] = 2^{2H-1};
$$

$$
\Phi^H(U_2, U_4) = \frac{1}{2} \left[ 2^{2H} + 4^{2H} - 2^{2H} \right] = 2^{4H-1};
$$

$$
\Phi^H(U_3, U_4) = 2^{4H-1}.
$$

By computation, the matrix

$$
\begin{pmatrix}
1 & 2^{2H-1} & 2^{2H-1} & \frac{1 + 2^{4H-3^{2H}}}{2} \\
2^{2H-1} & 2^{2H} & 2^{2H-1} & 2^{4H-1} \\
2^{2H-1} & 2^{2H-1} & 2^{2H} & 2^{4H-1} \\
\frac{1 + 2^{4H-3^{2H}}}{2} & 2^{4H-1} & 2^{4H-1} & 2^{4H}
\end{pmatrix}
$$

is not positive definite for $H = 3/4$ (although it is for $H = 1/2$). Therefore $\Phi^{3/4}$ is not positive definite and consequently, the sifBm cannot be defined on $(\mathbb{R}^2_+, \mathcal{A}, m)$ for $H = 3/4$.

The following example shows that sifBm’s definition can be used to obtain an extension of fractional Brownian motion indexed by a differential manifold. In that case, the choice of the indexing collection on the manifold is crucial and can lead to different processes, whose definitions are limited to $0 < H \leq 1/2$ or not.

**Example 4.5.** Suppose we aim to extend fractional Brownian motion for indices in the unit circle $\mathbb{S}_1$ in $\mathbb{R}^2$. Let us fix a point $O \in \mathbb{S}_1$ and define $\mathcal{A}$ as the collection
\{0\hat{M}; M \in S_1\} \cup \{\emptyset\}, where \(0\hat{M}\) denotes the positive oriented arc from \(O\) to \(M\). \(\mathcal{A}\) is clearly an indexing collection which is totally ordered. Then, theorem 4.3 implies the existence of a sifBm on \((S_1, \mathcal{A}, m)\) for a parameter \(H \in (0, 1)\), where \(m\) denotes the Lebesgue measure on \(S_1\). It is defined as the mean-zero Gaussian process \(X = \{X_M; M \in S_1\}\) such that

\[
\forall M, M' \in S_1; \quad E[X_M - X_{M'}]^2 = m(0\hat{M} \triangle 0\hat{M'})^{2H} = m(0\hat{M})^{2H} = m(0\hat{M'})^{2H}.
\]

Another choice of indexing collection is \(\mathcal{A}' = \{0\hat{M}; M \in S_1\} \cup \{\emptyset\}\), where \(0\hat{M}\) denotes the smallest arc of circle from \(O\) to \(M\). As \(\mathcal{A}'\) is not totally ordered, there is a priori a limitation of sifBm’s definition on \((S_1, \mathcal{A}', m)\) for a parameter \(H \in (0, 1/2)\).

Another point of view is followed in Istas’ extension of fractional Brownian motion indexed by points on the unit circle, considered as a metric space (and not as a measure space). In \[Is05\], the periodical fractional Brownian motion (PFBM) is defined as the mean-zero Gaussian process \(X = \{X_M; M \in S_1\}\) such that \(X_O = 0\) (for some \(O \in S_1\)) almost surely and

\[
\forall M, M' \in S_1; \quad E[|X_M - X_{M'}|^2] = [d(M, M')]^{2H}
\]

where \(d(M, M')\) is the distance between \(M\) and \(M'\) on the circle \(S_1\).

This process is different from the two previous definitions based on set-indexed fractional Brownian motion, in the sense that the covariance function cannot be expressed in terms of some measure on \(S_1\).

However, if we only consider the positive half-circle \(\frac{1}{2}S_1\) starting from \(O \in S_1\), then

\[
\forall M, M' \in \frac{1}{2}S_1; \quad m(0\hat{M} \triangle 0\hat{M'}) = m(0\hat{M})^{2H} = m(0\hat{M'})^{2H} = [d(M, M')]^{2H}
\]

and the three covariance functions are identical. Therefore the three different processes defined on \(S_1\) coincide on \(\frac{1}{2}S_1\). In that sense, Istas’ PFBM on the half-circle can be seen as a particular case of the sifBm. Consequently, fractal properties such as stationarity and self-similarity are satisfied by this process on \(\frac{1}{2}S_1\) (cf. section 3) but stationarity cannot hold on the whole unit circle.

Moreover, as seen later, the characterization by fractal properties leads naturally to our first definition (cf. section 6).

5. Fractal properties

5.1. Increment stationarity. The increments of a set-indexed process are defined from the collection of subsets \(\mathcal{C}\).

For all \(C = U \setminus \bigcup_{1 \leq i \leq n} U_i\), we define the increment of the process \(X\) on \(C\) by

\[
\Delta X_C = X_U - \sum_{i=1}^n X_{U \cap U_i} + \sum_{i<j} X_{U \cap (U_i \cap U_j)} - \cdots + (-1)^n X_{U \cap (\bigcap_{1 \leq i \leq n} U_i)}.
\]

(13)

This increment is always well defined for the sifBm \(B^H\) since for all \(U, V \in \mathcal{A}\) such that \(U \cup V \in \mathcal{A}\), we have \(E[|X_U + X_V - X_{U \cap V} - X_{U \cup V}|^2] = 0\).
In [HeMe06a], we defined a stationarity property for a set-indexed process \( X = \{X_U; U \in \mathcal{A}\} \) on \((T, \mathcal{A}, m)\) by

\[
\forall C, C' \in \mathcal{C}_0; \quad m(C) = m(C') \Rightarrow \Delta X_C \overset{d}{=} \Delta X_{C'}
\]

(14)

where \( \mathcal{C}_0 \) denotes the set of elements \( U \setminus V \) with \( U, V \in \mathcal{A} \). Property (14) is called \( \mathcal{C}_0 \)-stationarity.

As \( \mathcal{C}_0 \)-stationarity only concerns marginal distributions of the increment process \( \Delta X \), but not distribution of the process, the property is weaker than the classical increment stationarity property for one-parameter processes.

In that view, it seems judicious to strengthen the increment stationarity definition in such a way that projections of a increment stationary set-indexed process on any flow give increment stationary one-parameter processes.

**Definition 5.1.** A set-indexed process \( X = \{X_U; U \in \mathcal{A}\} \) is said to have \( m \)-stationary \( \mathcal{C}_0 \)-increments if for any integer \( n \), for all \( V \in \mathcal{A} \) and for all increasing sequences \((U_i)_{1 \leq i \leq n}\) and \((A_i)_{1 \leq i \leq n}\) in \( \mathcal{A} \),

\[
\forall i, \quad m(U_i \setminus V) = m(A_i) \quad \Rightarrow \quad (\Delta X_{U_1 \setminus V}, \ldots, \Delta X_{U_n \setminus V}) \overset{d}{=} (\Delta X_{A_1}, \ldots, \Delta X_{A_n}).
\]

**Proposition 5.2.** The sifBm has \( m \)-stationary \( \mathcal{C}_0 \)-increments.

**Proof.** Let \( X = \{X_U; U \in \mathcal{A}\} \) be a sifBm. For any integer \( n \), let us consider \( V \in \mathcal{A} \) and increasing sequences \((U_i)_{1 \leq i \leq n}\) and \((A_i)_{1 \leq i \leq n}\) in \( \mathcal{A} \) such that \( m(U_i \setminus V) = m(A_i) \) (\( \forall 1 \leq i \leq n \)). Without loss of generality, we can assume \( V \subseteq U_i \) (\( \forall i \)). Let us compute for all \( \lambda_1, \ldots, \lambda_n \) in \( \mathbb{R} \),

\[
E \left[ \lambda_1 \Delta X_{U_1 \setminus V} + \cdots + \lambda_n \Delta X_{U_n \setminus V} \right]^2 = \sum_{i,j} \lambda_i \lambda_j E \left[ \Delta X_{U_i \setminus V} \Delta X_{U_j \setminus V} \right]
\]

\[
= \sum_{i,j} \lambda_i \lambda_j E \left[ (X_{U_i} - X_V)(X_{U_j} - X_V) \right]
\]

\[
= \sum_{i,j} \lambda_i \lambda_j \left( E \left[ X_{U_i} X_{U_j} \right] - E \left[ X_{U_i} X_V \right] - E \left[ X_V X_{U_j} \right] + E \left[ X_V \right]^2 \right)
\]

\[
= \sum_{i,j} \lambda_i \lambda_j \left( m(U_i \Delta V)^{2H} + m(U_i \Delta V)^{2H} - m(U_i \Delta U_j)^{2H} \right).
\]

Assumptions on \((U_i)\) and \((A_i)\) imply \( m(U_i \Delta V) = m(U_i \setminus V) = m(A_i) \), and as \((U_i)\) is increasing, \( m(U_i \Delta U_j) = |m(U_i \setminus V) - m(U_j \setminus V)| = |m(A_i) - m(A_j)| \). Then, for all \( \lambda_1, \ldots, \lambda_n \) in \( \mathbb{R} \),

\[
E \left[ \lambda_1 \Delta X_{U_1 \setminus V} + \cdots + \lambda_n \Delta X_{U_n \setminus V} \right]^2 = E \left[ \lambda_1 \Delta X_{A_1} + \cdots + \lambda_n \Delta X_{A_n} \right]^2
\]

and, as the process \( \Delta X \) is centered Gaussian, the result follows. \( \square \)

**Example 5.3.** Following the notation of example 4.3, the sifBm defined on \((S_1, \mathcal{A}, m)\), which provides an extension of fractional Brownian motion indexed by points of the unit circle of \( \mathbb{R}^2 \), has \( m \)-stationary \( \mathcal{C}_0 \)-increments. By definition, \( \mathcal{C}_0 \) consists of all elements \( \widehat{MM'} \) where \( M, M' \in S_1 \). Then this stationarity property states that the law of the process \( \Delta X \) is invariant by translations along \( S_1 \).
The following proposition shows that definition 5.1 provides a natural extension of stationarity property for one-parameter processes. Then, it justifies this new definition for stationarity of set-indexed processes.

**Proposition 5.4.** A set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ has $m$-stationary $C_0$-increments if and only if the $m$-standard projection of $X$ on any elementary flow $f : \mathbb{R}_+ \rightarrow \mathcal{A}$ has stationary increments, i.e.,

$$\left\{X_{t+h}^{f,m} - X_h^{f,m}; \ t \in \mathbb{R}_+ \right\} \overset{(d)}{=} \left\{X_t^{f,m} - X_0^{f,m}; \ t \in \mathbb{R}_+ \right\}$$

where $X_t^{f,m} = \{X_{f \circ \theta^{-1}(i)}; \ t \in \mathbb{R}_+\}$ and $\theta : t \mapsto m(f(t))$.

**Proof.** We prove the first implication. Assume that $X$ has $m$-stationary $C_0$-increments and that $f$ is an elementary flow.

For all $t_1 < t_2 < \cdots < t_n$ and $h$ in $\mathbb{R}_+$, consider for $1 \leq i \leq n$, $U_i = f \circ \theta^{-1}(t_i + h)$, $V = f \circ \theta^{-1}(h)$ and

$$C_i = U_i \setminus V = f \circ \theta^{-1}(t_i + h) \setminus f \circ \theta^{-1}(h) \quad \text{and} \quad A_i = f \circ \theta^{-1}(t_i).$$

As $(U_i)_{1 \leq i \leq n}$ and $(A_i)_{1 \leq i \leq n}$ are increasing and $V \subset U_i \ (\forall i)$, we have

$$\Delta X_{U_i \setminus V} = X_{U_i} - X_V$$

$$= X_{f \circ \theta^{-1}(t_i + h)} - X_{f \circ \theta^{-1}(h)}$$

$$= X_{t_i + h}^{f,m} - X_h^{f,m}$$

and

$$m(U_i \setminus V) = m(f \circ \theta^{-1}(t_i + h)) - m(f \circ \theta^{-1}(h))$$

$$= \theta \circ \theta^{-1}(t_i + h) - \theta \circ \theta^{-1}(h)$$

$$= t_i$$

$$= m(A_i).$$

Then, $m$-stationarity of the set-indexed process $X$ implies

$$(\Delta X_{U_1 \setminus V}, \ldots, \Delta X_{U_n \setminus V}) \overset{(d)}{=} (X_{A_1}, \ldots, X_{A_n})$$

which gives

$$(X_{t_1 + h}^{f,m} - X_h^{f,m}, \ldots, X_{t_n + h}^{f,m} - X_h^{f,m}) \overset{(d)}{=} (X_{t_1}^{f,m}, \ldots, X_{t_n}^{f,m})$$

and the increment stationarity of the $m$-standard projection of $X$ on $f$.

In the same way, using lemma 3.3 to define a flow passing through every $U_i \ (1 \leq i \leq n)$, we prove the converse. □

### 5.2. Self-similarity

In [HeMe06a], we defined the self-similarity property of a set-indexed process with respect to action of a group $G$ satisfying the following assumptions.

We suppose that $\mathcal{A}$ is provided with the operation of a non trivial group $G$ that can be extended satisfying

$$\forall U, V \in \mathcal{A}, \forall g \in G; \ g.(U \cup V) = g.U \cup g.V$$

$$g.(U \setminus V) = g.U \setminus g.V$$

(15)
and assume there exists a surjective function \( \mu : G \rightarrow \mathbb{R}^* \)
\[
\forall U \in \mathcal{A}, \forall g \in G; \quad m(gU) = \mu(g)m(U). \tag{16}
\]

A set-indexed process \( X = \{X_U; \ U \in \mathcal{A}\} \) is said to be self-similar of index \( H \), if there exists a group \( G \) which operates on \( \mathcal{A} \), and satisfies (15) and (16), such that for all \( g \in G \),
\[
\{X_{gU}; \ U \in \mathcal{A}\} \stackrel{(d)}{=} \{\mu(g)^H X_U; \ U \in \mathcal{A}\} \tag{17}
\]

**Remark 5.5.** In the case of \( \mathcal{A} = \{[0, u]; \ u \in \mathbb{R}^N_+ \} \cup \{\emptyset\} \), the operation of \( G = \mathbb{R}^+ \) defined by
\[
\mathbb{R}^+ \times \mathcal{A} \rightarrow \mathcal{A}
\]

\[
(a, [0, u]) \mapsto [0, au]
\]

satisfies assumptions (15) and (16).

On the contrary to stationarity property, self-similarity of a set-indexed process does not imply self-similarity of projections on flows in a natural way. This is essentially due to the fact that there is no connection between zooming in \( \mathcal{A} \) (through operation of \( G \)) and zooming along a flow (through multiplication by \( \mathbb{R}^+ \)). For instance, in the frame of remark 5.5, if \([0, u] \in \mathcal{A}\) belongs to an elementary flow \( f \), i.e. if there exists \( t \in \mathbb{R}^+ \) such that \( f(t) = [0, u] \), the element \([0, au] (a > 0)\) of \( \mathcal{A} \) does not belong necessarily to \( f \).

However, under some additional assumptions either about flows or about the set-indexed process, standard projections can inherit the self-similarity property.

**Proposition 5.6.** Let \( X = \{X_U; \ U \in \mathcal{A}\} \) be a set-indexed process on \((T, \mathcal{A}, m)\) which satisfies the two following properties:

1. self-similarity of index \( H \) (with respect to operation of a group \( G \) satisfying assumptions (15) and (16)),
2. \( m \)-stationarity of \( C_0 \)-increments.

Then, the \( m \)-standard projection of \( X \) on any elementary flow \( f \) is self-similar of index \( H \), i.e.
\[
\forall a \in \mathbb{R}^+; \quad \left\{ X_{at}^{f,m}; \ t \in \mathbb{R}^+ \right\} \stackrel{(d)}{=} \left\{ a^H X_t^{f,m}; \ t \in \mathbb{R}^+ \right\}
\]

where \( X_t^{f,m} = X_{f \circ \theta^{-1}(t)} \) and \( \theta : t \mapsto m[f(t)] \).

**Proof.** Let \( f \) be any elementary flow, \( a \in \mathbb{R}^+ \) and \( t_1 < t_2 < \cdots < t_n \) a sequence of elements of \( \mathbb{R}^+ \). For all \( i = 1, \ldots, n \), consider \( U_i = f \circ \theta^{-1}(t_i) \).
As \( \mu \) is a surjective function, there exists \( g \in G \) such that \( a = \mu(g) \).
As
\[
\forall i = 1, \ldots, n; \quad m(f \circ \theta^{-1}(at_i)) = \theta \circ \theta^{-1}(at_i) = at_i
\]
\[
= \mu(g)m(U_i) = m(gU_i),
\]
by \( m \)-stationarity, we have
\[
(X_{gU_1}, \ldots, X_{gU_n}) \stackrel{(d)}{=} (X_{at_1}^{f,m}, \ldots, X_{at_n}^{f,m}) \tag{18}
\]
and by self-similarity,
\[
(X_{gU_1}, \ldots, X_{gU_n}) \stackrel{(d)}{=} (\mu(g)^H X_{U_1}, \ldots, \mu(g)^H X_{U_n}). \tag{19}
\]
The result follows from (18) and (19). □

In the previous proof, the stationarity allows to guarantee for any flow \( f \) and \( U \in \mathcal{A} \) the existence of \( g \in G \) such that \( g.U \) belongs to \( f \), up to equality with respect to the law of \( X \). In that context, the \( m \)-stationarity definition allowing deformation of objects in \( \mathcal{A} \) is the key of its special importance.

**Remark 5.7.** The particular case of set-indexed fractional Brownian motion, which satisfies both properties (1) and (2) of proposition 5.6, shows that projections of sifBm on any elementary flow is a self-similar one-parameter process. Of course, this fact is already known as \( m \)-standard projections of sifBm are fBm.

**6. Characterization of the sifBm by stationarity and self-similarity**

Real-parameter fractional Brownian motion is well known as the only Gaussian process satisfying the two properties of self-similarity and increment stationarity.

In [HeMe06a], we proved that a set-indexed process \( X = \{ X_U; U \in \mathcal{A} \} \) satisfying the two following properties:

(i) \( C_0 \)-increment stationarity (property (14))
(ii) self-similarity of index \( H \in (0, 1/2) \)

must verify, for all \( U \) and \( V \) in \( \mathcal{A} \) such that \( U \subset V \)

\[
E [X_U X_V] = K \left[ m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H} \right]
\]

where \( K \in \mathbb{R}_+ \).

Characterizing covariance between only comparable elements of \( \mathcal{A} \), the two fractal properties (i) and (ii) only provide a pseudo-characterization of the sifBm.

Here, we prove that using the new definition 5.1 of stationarity for a set-indexed process, we get a complete characterization of the sifBm. As we see in the proof, the statement which only consider distributional properties of set-indexed processes, relies on the characterization of the sifBm by its projections on flows (see theorem 3.4).  

**Theorem 6.1.** The sifBm \( B^H \) on \((\mathcal{T}, \mathcal{A}, m)\) is the only \( L^2 \)-monotone outer-continuous Gaussian set-indexed process, which is self-similar of index \( H \in (0, 1/2) \) and has \( m \)-stationary \( C_0 \)-increments.

**Proof.** From [HeMe06a] and proposition 5.2, we know that the sifBm is Gaussian, self-similar and has \( m \)-stationary \( C_0 \)-increments.

Conversely, consider a Gaussian set-indexed process \( X \), which is self-similar and has \( m \)-stationary \( C_0 \)-increments. For any elementary flow \( f : [a, b] \subset \mathbb{R}_+ \rightarrow \mathcal{A} \), propositions 5.4 and 5.6 imply that the standard projection of \( X \) on \( f \) satisfies

- \( X^{f,m} \) is Gaussian,
- \( X^{f,m} \) is self-similar of index \( H \),
- \( X^{f,m} \) has stationary increments.

Therefore, by the well-known characterization of one-parameter fBm, \( X^{f,m} \) is a fractional Brownian motion, and then

\[
\forall t \in [a, b]; \quad E \left[ X_{f \circ \theta^{-1}(t)} \right]^2 = t^{2H}
\]

where \( \theta : t \mapsto m[f(t)] \).
Then, theorem 3.4 states the existence of a Borel measure $\nu$ on $\mathcal{T}$ such that $X$ is a sifBm on $(\mathcal{T}, \mathcal{A}, \nu)$. Consequently, the centered Gaussian process $X$ is defined by
\[
\forall U, V \in \mathcal{A}; \quad E [X_U - X_V]^2 = \nu(U \triangle V)^{2H},
\]
and particularly
\[
\forall U \in \mathcal{A}; \quad E [X_U]^2 = \nu(U)^{2H}.
\]
Then, according to proposition 2.4, for any elementary flow $f : [a, b] \to \mathcal{A}$, the process $\{X_{f_\psi^{-1}(t)}; t \in [a, b]\}$ with $\psi : t \mapsto \nu[f(t)]$ is a one-parameter fractional Brownian motion. This leads to
\[
\forall t \in [a, b]; \quad E [X_{f_\psi^{-1}(t)}]^2 = t^{2H}. \tag{21}
\]
From (20) and (21), we get for any elementary flow $f : [a, b] \to \mathcal{A}$
\[
\forall t \in [a, b]; \quad E [X_{f(t)}]^2 = m[f(t)]^{2H} = \nu[f(t)]^{2H}.
\]
Considering a flow passing through any given $U \in \mathcal{A}$, this implies
\[
\forall U \in \mathcal{A}; \quad m(U) = \nu(U)
\]
and consequently, the set-indexed process $X$ is a sifBm on $(\mathcal{T}, \mathcal{A}, m)$.

\[\square\]

Example 6.2. Let us come back to example 4.5. With the same notation, the sifBm on $(S_1, \mathcal{A}, m)$ is the only mean-zero Gaussian process $X$ indexed by $S_1$ such that the two following conditions are satisfied:

(i) the law of the increment process $\Delta X$ is invariant against translations along the circle;
(ii) $X$ is self-similar of parameter $H$, i.e.
\[
\forall a > 0; \quad \{X_{a.M}; M \in S_1\} \overset{(d)}{=} \{a^H.X_M; M \in S_1\},
\]
where $a.M$ denotes the point $M'$ of $S_1$ defined by $m(\text{c}^0 M') = a.m(\text{c}^0 M)$.

Remark 6.3. In the view of theorem 6.1, it is natural to wonder about existence of Gaussian set-indexed processes which are self-similar of index $H \in (1/2, 1)$ and have $m$-stationary $C_0$-increments. According to theorem 3.4, this question is related to the existence of set-indexed processes whose standard projections on any flow are fBm of parameter $H \in (1/2, 1)$. As we saw in section 4, the answer depends on the structure of the indexing collection $\mathcal{A}$.

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