Lagrangian time-discretization of the Hunter-Saxton equation

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Abstract

We study Lagrangian time-discretizations of the Hunter-Saxton equation. Using the Moser-Veselov approach, we obtain such discretizations defined on the Virasoro group and on the group of orientation-preserving diffeomorphisms of the circle. We conjecture that one of these discretizations is integrable.

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Introduction

In 1991 J. K. Hunter and R. Saxton considered the equation

\[ (u_t + uu_x)_x = \frac{1}{2} u_x^2, \]

which describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. A derivative with respect to \( x \) of the Hunter-Saxton equation

\[ (u_t + uu_x)_{xx} = \left( \frac{1}{2} u_x^2 \right)_x, \quad \text{or simply} \quad u_{txx} = -2u_xu_{xx} - uu_{xxx}, \]

is also often called the Hunter-Saxton equation. It is equation \( \text{[1]} \) that will be central to the present paper.

Let us recall some results concerning the Hunter-Saxton equation. Equation \( \text{[1]} \) was solved in \( \text{[1]} \) using the method of characteristics. A generalization of equation \( \text{[1]} \) was studied by M. V. Pavlov \( \text{[2]} \) and it was also solved. Equation \( \text{[2]} \) was investigated by J. Hunter and Y. Zheng \( \text{[3]} \), and it was proven that \( \text{[2]} \) is a completely integrable, bi-variational, bi-Hamiltonian system.

The Hunter-Saxton equation is related to the Korteweg-de Vries and Camassa-Holm equations in many ways. P. Olver and P. Rosenau described this relation using tri-Hamiltonian structures and scaling arguments. R. Beals, D. H. Sattinger and J. Szmigielski described the...
scattering theory for all three equations in a unified way. B. Khesin and G. Misiołek interpreted these three equations as the Euler equations describing geodesic flows associated to different right-invariant metrics on the Virasoro group, or on an appropriate homogeneous space. Moreover, they proved that these three equations exhaust all generic bi-Hamiltonian systems related to the Virasoro group. The paper was a motivation for the present paper and we will recall some results of in Section 2.

Since the Hunter-Saxton equation is an Euler equation, it is natural to try to discretize it using the Moser-Veselov approach. This approach is the following.

Let \( M \) be a manifold, let \( L \) be a function on \( M \times M \). A discrete Lagrangian system (see the review for references) describes stationary points of a functional \( S = S(X) \) defined on the space of sequences \( X = (x_k), x_k \in M, k \in \mathbb{Z} \), by a formal sum

\[
S(X) = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).
\]

The function \( L \) is called the Lagrangian.

Let us assume that \( M \) is a Lie group \( G \). It was observed by A. P. Veselov and J. Moser that symmetric \( \forall x, y \in M \)

\[
L(x, y) = L(y, x)
\]

and right-invariant (or left-invariant)

\[
L(xg, yg) = L(x, y)
\]

Lagrangians often correspond to integrable systems which are discretizations of the Euler equations corresponding to some right-invariant (or left-invariant) metrics on \( G \). The case of \( SO(n) \) and other classical finite-dimensional groups \( U(n), Sp(n) \) was studied extensively in \cite{9, 10, 11}. The first example of an infinite-dimensional group, namely SDiff\((\mathbb{R}^2)\) was considered in \cite{12, 13}.

We shall study discrete Lagrangian systems on the Virasoro group in order to find discretizations of the Hunter-Saxton equation. Discrete Lagrangian systems on the Virasoro group were studied in \cite{14, 15}. These papers were motivated by the observation by V. Yu. Ovsienko and B. A. Khesin \cite{16} that the Korteweg-de Vries equation is the Euler equation for a right-invariant metric on the Virasoro group. In \cite{14, 15} we investigated discrete Lagrangian systems on the Virasoro group trying to find a time-discretization of the Korteweg-de Vries equation. In \cite{14} we studied a particular class of discrete Lagrangian systems containing an interesting system with remarkable properties. It was noticed however that this system had no visible relation to the Korteweg-de Vries equation. We will show in the present paper that this system is in fact related to the Hunter-Saxton equation. In \cite{15} we considered another particular class of discrete Lagrangian systems which has the Korteweg-de Vries equation as a continuous limit. One of these systems has very interesting properties and we conjectured that this system is integrable, but in fact till now only one integral was found.
In this paper we study a class of discrete Lagrangian systems on the Virasoro group satisfying natural conditions generalizing those considered in [14]. We show that these systems have the Hunter-Saxton equation as their continuous limit. We show that the system considered in [14] which belongs to this class can be considered as a good candidate for a natural integrable Lagrangian time-discretization of the Hunter-Saxton equation.

It is shown in [7] that the Hunter-Saxton equation can also be considered as an Euler equation on the homogeneous space Diff⁺(S¹)/Rot(S¹) of the group of orientation-preserving diffeomorphisms of the circle modulo the group of rotations of the circle. We will also consider possible discretizations of the Hunter-Saxton equation on the group Diff⁺(S¹). However, our primary interest will be the Virasoro group since it plays a more fundamental role for the Hunter-Saxton equation. Moreover, we will see that in some sense discrete Lagrangian systems on Diff⁺(S¹) are “reductions” of discrete Lagrangian systems on the Virasoro group.

The plan of the paper is as follows. In Section 1 we recall necessary facts about the Virasoro group and the Virasoro algebra. In Section 2 we recall the results of paper [7] describing links between the Hunter-Saxton equation and the Virasoro group. In Section 3 we study discrete Lagrangian systems on the Virasoro group. In Section 4 we recall the results of [14] and show that the system studied in [14] can be considered as a good candidate for a natural integrable Lagrangian time-discretization of the Hunter-Saxton equation. In the final Section 5 we will study discrete Lagrangian systems on the group Diff⁺(S¹) and their relation to the Hunter-Saxton equation. We will see that the “reduction” to Diff⁺(S¹) of the system studied in Section 4 is particularly simple.

1 The Virasoro group and the Virasoro algebra

Let Diff⁺(S¹) be the group of diffeomorphisms of S¹ preserving the orientation. We shall represent an element of Diff⁺(S¹) as a diffeomorphism f : R → R such that

1. f ∈ C∞(R),
2. f'(x) > 0,
3. f(x + 2π) = f(x) + 2π.

Such a representation is not unique. Indeed, the functions f + 2πk, k ∈ Z represent one element of Diff⁺(S¹).

There exists a non-trivial central extension of Diff⁺(S¹) which is unique up to an isomorphism.

This extension is called the Virasoro group (or the Bott-Virasoro group) and is denoted by Vir. Elements of Vir are pairs (f, F), where f ∈ Diff⁺(S¹), F ∈ R. The product of two elements is defined with the help of the Bott cocycle as

\[(f, F) \circ (g, G) = (f \circ g, F + G + \int_0^{2\pi} \log(f \circ g)' d \log g').\]
The unit element of Vir is \((id, 0)\). The inverse element of \((f, F)\) is \((f^{-1}, -F)\).

The Virasoro algebra vir is a Lie algebra corresponding to the Virasoro group. It is the central extension of the algebra \(vect(S^1)\) of vector fields on the circle \(S^1\)

\[
\text{vir} = vect(S^1) \oplus \mathbb{R}.
\]

We represent an element of \(vect(S^1)\) as \(v(x)\partial_x\), where \(v\) is a 2\(\pi\)-periodic function. Thus an element of the Virasoro algebra is a pair \((v(x)\partial_x, a)\).

The algebra commutator in vir is defined with the help of the Gelfand-Fuchs cocycle as

\[
[(v(x)\partial_x, a), [w(x)\partial_x, b]] = \left((-vw_x + v_x w)\partial_x, \frac{2\pi}{0} \int v_{xxxx} w dx\right).
\]

2 The Hunter-Saxton equation and the Virasoro group

In this section we briefly recall some results of 
ocusing a link between the Hunter-Saxton equation and the Virasoro group.

Let \(G\) be a Lie group, \(\mathfrak{g}\) its Lie algebra. A right-invariant metric on \(G\) is completely defined by its restriction to \(\mathfrak{g}\)

\[
E(v) = \frac{1}{2}\langle v, Av \rangle, \quad v \in \mathfrak{g},
\]

where \(A\) is a linear map \(A : \mathfrak{g} \rightarrow \mathfrak{g}^*\) called the inertia operator.

To describe a geodesic \(g(t)\) on \(G\), we transport its velocity vector to the identity by the right translation

\[
v(t) = R_{g^{-1}(t)} \frac{d}{dt} g(t).
\]

Since \(v(t)\) is an element of \(\mathfrak{g}\), we can consider \(m = Av\) which is an element of the dual space \(\mathfrak{g}^*\). Then \(m\) satisfies the Euler equation given by the following explicit formula:

\[
\frac{dm}{dt} = -ad^*_{A^{-1}m} m.
\]

This is a standard result which can be found in [3]. If we start with a left-invariant metric, the sign in (3) is reversed.

Let us now compute \(ad^*\) for the Virasoro algebra. Let a space

\[
\{(u(dx)^2, a) | u \in C^\infty, u(x) = u(x + 2\pi), a \in \mathbb{R}\}
\]

be the dual space \(\text{vir}^*\) to the Virasoro algebra with the natural pairing given by

\[
\langle (u(dx)^2, a), (w\partial_x, c) \rangle = \frac{2\pi}{0} uv dx + ac.
\]
Thus, $ad^*$ is defined by the formula
\[
\langle ad^*_{(v\partial_x, b)}(u(dx)^2), a \rangle = \langle (u(dx)^2), a \rangle [((v\partial_x, b), (w\partial_x, c))].
\]

Using the definition of the commutator in $\text{vir}$ and integration by parts we obtain that the right-hand-side is equal to
\[
\int_0^{2\pi} w(2uv_x + u_x v - av_{xxx}) \, dx.
\]
Thus we obtain
\[
ad^*_{(v\partial_x, b)}(u(dx)^2, a) = ((2uv_x + u_x v - av_{xxx})(dx)^2, 0).
\]

Now let us consider an inertia operator $A : \text{vir} \to \text{vir}^*$, given by
\[
A(v\partial_x, b) = ((-\Lambda v)(dx)^2, b),
\]
where $\Lambda = -\partial_x^2$. The corresponding scalar product on $\mathfrak{g}$ is given by the formula
\[
E((v\partial_x, b)) = \int_0^{2\pi} vv_x dx + b^2.
\]
Using integration by parts, we obtain
\[
E((v\partial_x, b)) = \int_0^{2\pi} (v_x)^2 dx + b^2. \quad (4)
\]

In the Euler equation (3) we have $A^{-1}m$, but our inertia operator $A$ is degenerate on $\text{vir}$ since $\Lambda$ has a non-trivial kernel consisting of constant vector fields on $S^1$. It is necessary to consider the quotient space $\text{Vir}/\text{Rot}(S^1)$ of the Virasoro group by the subgroup of rotations of the circle. This is not difficult since it is sufficient to consider $m$ only from the image of the inertia operator, see [7] for details.

We obtain the Euler equation (3) on $\text{vir}^*$
\[
\frac{d}{dt}(u(dx)^2, a) = -ad^*_{A^{-1}(u(dx)^2), a}(u(dx)^2, a) =
\]
\[
= -(2u\Lambda^{-1} u_x + u_x \Lambda^{-1} u - a\Lambda^{-1} u_{xxx})(dx)^2, 0).
\]
If we put $v = \Lambda^{-1} u$ we obtain two equations:
\[
a_t = 0
\]
(so $a$ is a constant) and
\[
v_{txx} = -2v_x v_{xx} - v_{xxx} - av_{xxx}.
\]
(5)

After the change of variables $u(x, t) = v(x, t) + a$ we obtain the Hunter-Saxton equation (2).

It has also been shown [7] that the Hunter-Saxton equation can be considered as an Euler equation on the homogeneous space $\text{Diff}_+(S^1)/\text{Rot}(S^1)$.
of the group of orientation-preserving diffeomorphisms of the circle modulo the group of rotations of the circle. The construction is quite analogous to the construction on the Virasoro group. It is necessary to consider an inertia operator $A : \text{vect}(S^1) \rightarrow \text{vect}(S^1)^*$ given by
\[ A(v\partial_x) = (-\Lambda v)(dx)^2. \]
This inertia operator will give an equation
\[ v_{txx} = -2v_x v_{xx} - vv_{xxx}, \]
i.e. the Hunter-Saxton equation. This coincides with equation (5) for $a = 0$. We will see that in the discrete case the situation is analogous.

## 3 Discrete Lagrangian systems on the Virasoro group

Let $M$ be a manifold, let $L$ be a function on $M \times M$. A discrete Lagrangian system describes stationary points of a functional $S = S(X)$ defined on the space of sequences $X = (x_k), x_k \in M, k \in \mathbb{Z}$, by a formal sum
\[ S(X) = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}). \]

The function $L$ is called the Lagrangian.

Let us assume that $M$ is a Lie group $G$. Our goal is to investigate potential candidates for a time-discretization of the Euler equation. Since the Euler equation describes geodesics on a Lie group equipped with right-invariant metric, it is natural to require that the function $L$ is right-invariant
\[ \forall g \in G \quad L(xg, yg) = L(x, y). \]

Let $H(x) = L(x, e)$, where $e$ is the identity element of $G$. A right-symmetric Lagrangian $L(x, y)$ is completely determined by $H$. Indeed,
\[ L(x, y) = L(xy^{-1}, e) = H(xy^{-1}). \]

We shall consider a class of functions of the type
\[ H((f, F)) = F^2 + \int_0^{2\pi} U(f'(x)) \, dx, \]
where $f$ is a diffeomorphism, $F \in \mathbb{R}$. It is a quite natural class since the scalar product in vir defining the Hunter-Saxton equation (6) depends only on a derivative of a vector field.

Let us consider a functional
\[ S = \sum_{k \in \mathbb{Z}} L((f_k, F_k), (f_{k+1}, F_{k+1})), \]
where $\{(f_k, F_k)\}$ is a sequence of points on Vir and $L$ is defined using the function $H$ (6) as described above:
\[ L((f_k, F_k), (f_{k+1}, F_{k+1})) = H((f_k, F_k) \circ (f_{k+1}, F_{k+1})^{-1}). \]
Theorem 1 The discrete Euler-Lagrange equations \( \frac{dS}{df_k,F_k} = 0 \) are the following:

\[
- \Omega_k + \Omega_{k+1} = 0, \tag{7}
\]

\[
-2\Omega_k (\log(\omega_k'))'' + U''(\omega_k')\omega_k'' + 2\Omega_{k+1} (\log((\omega_{k+1}')'))'' + U''(\omega_{k+1}')\omega_{k+1}'' = 0, \tag{8}
\]

where \((\omega_k, \Omega_k)\) and \((\omega_{k+1}, \Omega_{k+1})\) are discrete analogues of angular velocities,

\[
(\omega_l, \Omega_l) = (f_{l-1}, F_{l-1}) \circ (f_l, F_l)^{-1}, \quad l \in \mathbb{Z}.
\]

Proof. It sufficient to consider a variation of the form

\[
(id + \varepsilon v, \varepsilon A) \circ (f_k, F_k)
\]

and to find a Taylor series of the variation of

\[
L((f_{k-1}, F_{k-1}), (f_k, F_k)) + L((f_k, F_k), (f_{k+1}, F_{k+1}))
\]

up to order \(O(\varepsilon^2)\). We obtain

\[
L((f_{k-1}, F_{k-1}), (id + \varepsilon v, \varepsilon A) \circ (f_k, F_k)) +
\]

\[
+L((id + \varepsilon v, \varepsilon A) \circ (f_k, F_k), (f_{k+1}, F_{k+1})) =
\]

\[
= L((f_{k-1}, F_{k-1}), (f_k, F_k)) + L((f_k, F_k), (f_{k+1}, F_{k+1})) +
\]

\[
+\varepsilon A(-2\Omega_k + 2\Omega_{k+1}) + \varepsilon \int_0^{2\pi} -2\Omega_k (\log(\omega_k'))'' + U''(\omega_k')\omega_k''
\]

\[
+2\Omega_{k+1} (\log((\omega_{k+1}')'))'' + U''(\omega_{k+1}')\omega_{k+1}'' +
\]

\[
+2\Omega_{k+1} \left( C_{k+1}(\omega_{k+1}')'' + U''(\omega_{k+1}')\omega_{k+1}'' \right) \right) v \, dx + O(\varepsilon^2) = 0.
\]

Since \( A \) is an arbitrary constant and \( v \) is an arbitrary periodic function, we obtain our formulae for the discrete Euler-Lagrange equation \( \Box \).  

The first equation \( \Box \) is quite simple: it says that \( \Omega_{k+1} = \Omega_k \), so \( \Omega_k \) is an integral of our discrete Lagrangian system. The second equation is quite complicated. It is necessary to remark that this is not a differential-difference equation. Indeed, it includes \( \omega_{k+1}^{-1} \), which is a diffeomorphism inverse to \( \omega_{k+1} \). So, it is a complicated relation between \( \omega_k \) and \( \omega_{k+1} \), and it is better to view it as a correspondence (i.e. multivalued mapping)

\[
\omega_k \mapsto \omega_{k+1}
\]

from \( \text{Diff}_+(S^1) \) to \( \text{Diff}_+(S^1) \).

Let us now find a continuous limit of our equations \( \Box \). Our definition of a continuous limit is the following. Firstly, we suppose that the angular velocity is of the form

\[
(\omega_l, \Omega_l) = (id + \varepsilon v_l(x), \varepsilon A_l),
\]

i.e. the angular velocity is the identity element of Vir up to \( O(\varepsilon) \). Secondly, we suppose that

\[
v_k(x) = v(x, t), \quad A_k = A(t), \quad v_{k+1}(x) = v(x, t + \varepsilon), \quad A_{k+1} = A(t + \varepsilon).
\]

(9)
This is a quite natural discretization of time. We substitute these formulae in the Euler-Lagrange equations (7,8), find a Taylor series, and finally we define a continuous limit as an equation arising in this Taylor series as the term of lowest order with respect to ε.

**Theorem 2** Let $U''(1) \neq 0$. Then the continuous limit of the Euler-Lagrange equations (7,8) is the following:

$$\Omega = \text{const},$$

$$v_{txx} = 2v_xv_{xx} + vv_{xxx} - \frac{4\Omega}{U''(1)}v_{xxx}.$$  \hspace{1cm} (11)

**Proof.** We substitute the formulae (9) in the equations (7,8) and we use a Taylor series. From the equation (7) we obtain

$$\varepsilon^2 \frac{d}{dt} A(t) + O(\varepsilon^3) = 0,$$

thus $A(t)$ is a constant. Let us denote it by $\Omega$. This gives us equation (10). From equation (8) we obtain

$$\varepsilon^2 (-4\Omega v_{xxx}(x,t) + 2U''(1)v_x(x,t)v_{xx}(x,t) + U''(1)vv_{xxx} -$$

$$- U''(1)v_{xx}(x,t)) + O(\varepsilon^3) = 0.$$

Since $U''(1) \neq 0$, this gives us equation (11). \hspace{1cm} ☐

If we do a simple change of time $t \mapsto -t$ and change our constant $a = \frac{4\Omega}{U''(1)}$, we obtain the Hunter-Saxton equation exactly in the same form as this equation arises as the Euler equation (5):

$$v_{txx} = -2v_xv_{xx} - vv_{xxx} - av_{xxx}.$$

As we see, a wide class of discrete Lagrangian systems has the Hunter-Saxton equation as a continuous limit. We have a natural question: how to find a “correct” discretization? We will discuss it in the next section.

4 A candidate for a natural integrable Lagrangian time-discretization of the Hunter-Saxton equation

Since the Hunter-Saxton equation is integrable, it is natural to suppose that a “correct” discretization is integrable in some sense.

To find a function $U$ giving an integrable discretization we can use the observation by A. P. Veselov and J. Moser [11, 12] mentioned in the Introduction. Namely that known integrable Lagrangian discretizations correspond to right-invariant (or left-invariant) symmetric Lagrangians:

$$L(x, y) = L(y, x).$$

Thus, let us also assume that our Lagrangian is symmetric. The Lagrangian is given by the function $H$:

$$L(x, y) = L(xy^{-1}, e) = H(xy^{-1}).$$
It is easy to see that symmetric Lagrangians correspond to inverse-invariant function $H$:

$$H(x^{-1}) = H(x).$$

Let us consider $H$ of the form (6). Which functions $U$ give us an inverse-invariant $H$? This question was studied in [14] and a sufficient condition was found:

**Lemma 1** [14] If a function $U$ satisfy the condition

$$xU \left( \frac{1}{x} \right) = U(x),$$

then a function $H : \text{Vir} \rightarrow \mathbb{R}$ defined by formula (6) is inverse-invariant:

$$H((f, F)^{-1}) = H((f, F)).$$

A second derivative of the identity (12) is

$$U'' \left( \frac{1}{x} \right) \frac{1}{x^3} = U''(x).$$

Using this identity we can rewrite equation (8) as

$$\left[ -2\Omega_k \left( \log((\omega_k)'') \right)' - U'' \left( \frac{1}{(\omega_k)''} \right) + 2\Omega_{k+1} \left( \log((\omega_{k+1}^{-1})'') \right)' - U'' \left( \frac{1}{(\omega_{k+1}^{-1})''} \right) \right]' = 0.$$ (13)

This is very symmetric expression and it is a complete derivative with respect to $x$. We note that the Hunter-Saxton equation arising from the Euler equation (5) is also a complete derivative with respect to $x$.

The condition (12) is very restrictive, however there are many functions satisfying (12). If we consider other cases studied, there are no indications how to find an appropriate $U$. In the case $G = SO(n)$ [9, 10, 11] the final form of a Lagrangian is obtained by imposing the condition that the Lagrangian $L(x, y)$ should be bilinear in $x, y$. It is not clear what is the analog of this condition in our case. In the case $G = \text{SDiff}(\mathbb{R}^2)$ [12, 13] the Lagrangian is chosen as a natural generalization of the $SL(2) = Sp(2)$ case. It leads to an integrable system, but there is no evidence that it is the unique Lagrangian with this property.

All we can do is to guess which function $U$ satisfying the condition (12) gives us an integrable system. The answer to this question is still unknown, but we have a good candidate. If we look at the equation (13), we can see that if we have $\omega_k$, we must solve a differential equation to find $\omega_{k+1}$ and then use the inversion operation in the Virasoro group to find $\omega_{k+1}$. It is natural to suppose that an integrable system corresponds to an integrable differential equation for $\omega_{k+1}^{-1}$. At least, analogous situation was observed in the case of $\text{SDiff}(\mathbb{R}^2)$ [12]. In [14] we showed that a function $U(x) = \sqrt{x}$ gives us a system such that if we have $\omega_k$, we can find $\omega_{k+1}$ by solving a linear first-order differential equation and using the inversion operation in Vir. In the rest of this section we follow [14].
Let us consider the equation (13) with $U(x) = \sqrt{x}$:

$$-rac{2\Omega_k}{k} \left( \log((\omega_k'))' \right) - \frac{1}{2} \sqrt{(\omega_k')'} + 2\Omega_{k+1} \left( \log((\omega_{k+1}^{-1})')' \right) - \frac{1}{2} \sqrt{(\omega_{k+1}^{-1})'} = 0.$$  (14)

Let us integrate this equation once and put

$$\Phi = \frac{1}{\sqrt{(\omega_k')'}}, \quad \Psi = \frac{1}{\sqrt{(\omega_{k+1}^{-1})'}}.$$

We obtain the equation

$$8\Omega \left( \frac{\Phi'}{\Phi} + \frac{\Psi'}{\Psi} \right) + \frac{\Phi}{\Phi} + \frac{1}{\Psi} + C = 0,$$

where $C$ is a constant of integration and $\Omega = \Omega_k = \Omega_{k+1}$ (remember that $\Omega_k$ is an integral). This equation is equivalent to the equation

$$\Psi' + \Psi \left( \frac{C}{8\Omega} + \frac{1}{8\Omega \Phi} - \frac{\Phi'}{\Phi} \right) + \frac{1}{8\Omega} = 0.$$

This is a linear first-order differential equation for $\Psi$ with periodic coefficients depending on $\Phi$. For generic $\Phi$ it has only one solution, so $\Psi$ is determined by $\Phi$ up to a constant $C$. Reconstructing $\omega_{k+1}$ from $\Psi$ we obtain another constant, so we have a following result: $\omega_{k+1}$ is obtained from $\omega_k$ by a two-parametric correspondence. To find $\omega_{k+1}$ starting from $\omega_k$, we must solve a first-order linear differential equation to find $\omega_{k+1}$, and then reconstruct $\omega_{k+1}$ from $\omega_{k+1}$ by inversion. As mentioned before, the analogous situation was observed in the case of $SDiff(R^2)$ [12] which is integrable [13]. For this reason we consider system (14) to be a good candidate for an integrable Lagrangian discretization of the Hunter-Saxton equation.

5 Discrete Lagrangian systems on the group of orientation-preserving diffeomorphisms of the circle

As explained in Section 2 we can obtain the Hunter-Saxton equation using not only the Virasoro group, but also the group $Diff_+(S^1)$, and the construction is quite analogous. Similarly, we can consider discrete Lagrangian systems on the group $Diff_+(S^1)$ and obtain results analogous to the results for the Virasoro group. In this Section we formulate these results, proofs are omitted since they are analogous to the proofs of the last two sections.

We shall consider a class of functions $H : Diff_+(S^1) \rightarrow \mathbb{R}$ of the type

$$H(f) = \int_0^{2\pi} U(f'(x)) \, dx,$$  (15)
where \( f \) is a diffeomorphism.

Let us consider a functional
\[
S = \sum_{k \in \mathbb{Z}} L(f_k, f_{k+1}),
\]
where \( \{f_k\} \) is a sequence of points on \( \text{Diff}_+(S^1) \) and \( L \) is defined using the function \( H \):
\[
L(f_k, f_{k+1}) = H(f_k \circ f_{k+1}^{-1}).
\]

**Theorem 3** The discrete Euler-Lagrange equation \( \frac{dS}{df_k} = 0 \) is the following:
\[
U''(\omega_k)\omega_k'' + U''\left(\frac{1}{(\omega_{k+1})'}\right)\left(\frac{(\omega_{k+1}^{-1})''}{((\omega_{k+1})')^2}\right) = 0,
\]
where \( \omega_k \) and \( \omega_{k+1} \) are discrete analogues of angular velocities,
\[
\omega_l = f_l \circ f_{l-1}^{-1}, \quad l \in \mathbb{Z}.
\]

This is exactly equation (8) with \( \Omega_k = \Omega_{k+1} = 0 \). It is similar to the relation between the Euler equations on \( \text{Vir}/\text{Rot}(S^1) \) and \( \text{Diff}_+(S^1)/\text{Rot}(S^1) \), see the end of Section 3. Thus, Lagrangian discrete systems on \( \text{Diff}_+(S^1) \) are in this sense the “reductions” of Lagrangian discrete systems on the Virasoro group. Using this remark, it is easy to find a continuous limit of equation (16).

**Theorem 4** Let \( U''(1) \neq 0 \). Then the continuous limit of the Euler-Lagrange equation (16) is the following:
\[
v_{txx} = 2v_x v_{xx} + v v_{xxx}.
\]

**Lemma 2** If a function \( U \) satisfies the condition (12) then a function \( H : \text{Diff}_+(S^1) \rightarrow \mathbb{R} \) defined by formula (15) is inverse-invariant:
\[
H(f^{-1}) = H(f).
\]

If the function \( U \) satisfies the condition (12), we can rewrite equation (14) as
\[
\left[U'(\frac{1}{(\omega_k)'} + U'(\frac{1}{(\omega_{k+1}^{-1})'}))\right]' = 0.
\]

As in the previous Section let us consider \( U(x) = \sqrt{x} \). Equation (17) becomes particularly simple
\[
\left[\sqrt{(\omega_k)'} + \sqrt{(\omega_{k+1}^{-1})'}\right]' = 0.
\]

To find \( \omega_{k+1} \) starting from \( \omega_k \), we can rewrite the previous equation as
\[
(\omega_{k+1}^{-1})' = (C - \sqrt{(\omega_k)'}^2),
\]
where \( C \) is an arbitrary constant. We see that given \( \omega_k \), we can find \( \omega_{k+1} \) using integration and the inversion operation in the group \( \text{Diff}_+(S^1) \). We see that system (18) has the same property as the equation (14) and can be also considered as a good candidate for an integrable Lagrangian discretization of the Hunter-Saxton equation.
6 Conclusions

We studied discrete Lagrangian systems on the Virasoro group and the group of orientation-preserving diffeomorphisms of the circle. It is shown that under some very natural assumptions, these systems have the Hunter-Saxton equation as their continuous limit. In particular we studied a special discrete Lagrangian system on the Virasoro group which has the following properties:

1. Its continuous limit is the Hunter-Saxton equation.
2. It can be solved by solving a first-order linear differential equation and using the inversion operation in the Virasoro group.
3. The Lagrangian is symmetric and right-invariant.

The “reduction” of this system to $\text{Diff}_+(S^1)$ has properties 1 and 3, but 2 is replaced but simpler property: it can be solved by integration and the inversion operation in the group $\text{Diff}_+(S^1)$.

In the case of $\text{SDiff}(\mathbb{R}^2)$ studied in [12, 13] the Lagrangian discrete system is integrable (in the sense that the dynamics are linearized). We conjecture that the same is true for two systems studied in this paper since their properties are analogous to the properties of the system on $\text{SDiff}(\mathbb{R}^2)$.

The Lagrangian discrete system on $\text{SDiff}(\mathbb{R}^2)$ can be interpreted as a chain of Bäcklund transformations for the (integrable) Monge-Ampère equation [12]. This is nowadays a standard way to think about integrable discretizations. It would be very interesting to prove similar theorems for the systems discussed in this paper.

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