Particle Creation in a Universe Filled with Radiation and Dust-Like Matter

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In this article the particle creation process of scalar and spin 1/2 particles in a spatially open cosmological model associated with a universe filled with radiation and dustlike matter is analyzed. The Klein-Gordon and the Dirac equations are solved via separation of variables. After comparing the in and out vacua, we obtain that the number of created particles corresponds to Planckian and Fermi-Dirac distributions for the scalar and Dirac cases respectively.

§ 1. Introduction

During the last two decades there has been increasing interest in the study of quantum effects in cosmological backgrounds. Since the appearance of the pioneer works by Parker1) and Grib et al.2) among others, a great body of papers have been published on the problem of scalar and spin 1/2 particle creation in different cosmological configurations. More recently there have been some attempts to analyze in this framework the problem of particle creation in the vicinity of cosmic strings.3),4)

Since the problem of particle creation in curved space-times is closely related to the definition of particle in a curved background, different approaches and interpretations have appeared during these last two decades, the most popular being the adiabatic and the corpuscle interpretations, and an overview of the most important results and approaches can be found in Refs. 5) and 6). Despite the great effort devoted to a better understanding of quantum phenomena in expanding cosmological universes, the number of articles discussing concrete models is relatively scarce. This is due to the fact that, in most of the cases, the relativistic wave equations in curved backgrounds are very difficult to solve and therefore is necessary to reduce the original problem to a simpler one.

Recently, the exact solution of Klein-Gordon equation in an expanding (asymptotically flat) cosmological universe filled with radiation has been reported,7) the authors obtain a thermal planckian distribution of particles created when they compare the “in” and “out” vacua via the Bogoliubov coefficients. The model presented in Ref. 7) encourages us to analyze more complex configurations.

A good scenario for discussing the problem of scalar and spin 1/2 particle creation in a cosmological background, is an isotropic universe with zero cosmological constant (Λ=0) filled with a mixture of radiation and dust-like matter8),9) in a spatially open Friedmann Robertson-Walker model of the form

\[ ds^2 = -dt^2 + a^2(t)(d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)), \] (1.1)
From (1.1) we obtain
\[ \frac{\dot{e}}{\dot{p} + \epsilon} = -3 \frac{\dot{a}}{a}, \]  
(1.2)
\[ \kappa \epsilon = \frac{3}{a^2} \left[ \frac{(\dot{a})^2}{dt} - 1 \right], \]  
(1.3)
where \( \kappa \) is the constant appearing in Einstein's equations, \( \epsilon \) is the energy and \( p \) is the pressure. Assuming that there is no interaction between radiation and dust, we have \( \epsilon = \epsilon_d + \epsilon_r \) where
\[ \epsilon_d = \frac{A}{a^7}, \quad \epsilon_r = \frac{B}{a^4}, \]  
(1.4)
after introducing the conformal time \( \eta \) related to \( t \) by
\[ \eta = \int \frac{dt}{a}, \]  
(1.5)
and integrating \( a(\eta) \), and imposing as initial condition \( a(\eta) \rightarrow b \) when \( \eta \rightarrow -\infty \), being \( b \) a constant, we obtain the line element
\[ ds^2 = b^2(e^\eta + 1)^2( -d\eta^2 + d\chi^2 + \sinh^2 \chi(d\phi^2 + \sin^2 \theta d\theta^2)). \]  
(1.6)
Among the advantages of the line metric (1.6) we can mention that, this line element is regular for any value of the time parameter \( \eta \). The homogeneous and isotropic spacetime (1.6) evolves in permanent expansion towards and asymptotically flat region at \( \eta \rightarrow +\infty \). This asymptotic behavior resembles the one obtained in the perfect fluid cosmological model. In addition, \( \partial / \partial \eta \) approaches a timelike Killing vector when \( \eta \rightarrow -\infty \). Also, we have that, after separating variables in the Klein-Gordon and Dirac equations, the equations governing the time dependence are solvable in terms of special functions. It is the purpose of the present paper to obtain the quantum spectral distribution of scalar and spin 1/2 particles created in the cosmological model associated with the line element (1.6). The paper is organized as follows: In § 2 we solve the Klein-Gordon equation in the background field (1.6) and, in § 3 we also solve the Dirac equation in this metric. Finally, in § 4, based on the results obtained in §§ 2 and 3, we compute, via the Bogoliubov coefficients, the probability distribution of the number of created particles.

§ 2. Solution of the Klein-Gordon equation

The covariant generalization of the Klein-Gordon equation reads:
\[ (g^{ab}\nabla_a \nabla_b - \xi R - m^2)\Phi = 0, \]  
(2.1)
where \( R \) is the scalar curvature
\[ R = \frac{6}{b^2(e^\eta + 1)^3} \]  
(2.2)
and \( \xi \) is a dimensionless parameter which for a conformally coupling takes the value
Introducing a solution of the form
\[ \Phi = T(\eta)X(\chi)Y(\theta, \phi) \]  
and substituting (2·3) into (2·1), we find that, after separating variables, Eq. (2·1) reduces to the following set of ordinary differential equations:

\[ \left( \frac{d^2}{d\eta^2} + \cot \delta \frac{d}{d\eta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y = l(l+1)Y, \]  
\[ \left( \frac{a^2}{\eta^2} + 2 \frac{d}{d\eta} + \frac{d}{d\eta} + ma^2 + \sigma^2 - 1 \right) T = 0, \]  
\[ \left( \frac{d^2}{d\chi^2} + 2 \coth \chi \frac{d}{d\chi} + \frac{l(l+1)}{\sinh^2 \chi} + \sigma^2 \right) X = 0, \]

where \( l \) and \( \sigma \) are constants of separation, with \( l \geq 0 \) and \( \sigma^2 > 1 \).

After introducing the auxiliary function \( Z(\eta) \),
\[ T(\eta) = \frac{Z(\eta)}{a(\eta)}, \]  
Eq. (2·5) takes the form
\[ \left( \frac{d^2}{d\eta^2} + ma^2(\eta) + \sigma^2 - 1 \right) Z = 0. \]

Substituting the expansion factor \( a(\eta) = b(e^\eta + 1) \) into (2·8), we obtain the equation,
\[ \left( \frac{d^2}{d\eta^2} + m^2 b^2 e^{2\eta} + 2m^2 b^2 + b^2 m^2 + \sigma^2 - 1 \right) Z = 0, \]
whose solution can be expressed in terms of confluent hypergeometric functions \( M(a, b, z) \) and \( U(a, b, z) \) as follows:
\[ Z(\eta) = e^{\sigma c} \exp(-imb\nu) \left( c_1 M \left( \frac{1}{2} + c + ibm, 2c + 1, 2imb\nu \right) + c_2 U \left( \frac{1}{2} + c + ibm, 2c + 1, 2imb\nu \right) \right), \]

where \( c_1 \) and \( c_2 \) are arbitrary constants and \( c \) is given by the expression
\[ c = -i(\sigma^2 + m^2 b^2 - 1)^{1/2}. \]

The solutions of (2·4) are given in terms of the harmonic polynomials: \( Y = Y_{lm}(\theta, \phi) \), also we have that the solution of Eq. (2·6) reads
\[ X = \frac{1}{\sqrt{\sinh \chi}} P_{\nu}^{l-1/2}(\cosh \chi), \]
where \( P_{\nu}^{l}(\cosh \xi) \) are the torus functions and \( \nu^2 = \sigma^2 - 1. \)
§ 3. Solution of the Dirac equation

The covariant generalization of the Dirac equation in curved space-time reads

\[ \{ \tilde{\gamma}^\mu (\partial_\mu - \Gamma_\mu) + m \} \Psi = 0, \]  
(3·1)

where \( \tilde{\gamma}^\mu \) are the curved gamma matrices satisfying the relation,

\[ \{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \} = 2g^{\mu\nu} \]  
(3·2)

and \( \Gamma_\mu \) are the spin connections. Choosing to work in the diagonal tetrad gauge for \( \tilde{\gamma}^\mu \),

\[ \tilde{\gamma}^0 = a^{-1} \gamma^0, \quad \tilde{\gamma}^1 = a^{-1} \gamma^1, \quad \tilde{\gamma}^2 = a^{-1} (\sinh \chi)^{-1} \gamma^2, \]  
(3·3)

\[ \tilde{\gamma}^3 = a^{-1} (\sinh \chi)^{-1} (\sin \theta)^{-1} \gamma^3, \]  
(3·4)

where \( \gamma^\mu \) are the standard Dirac flat matrices, we have that the Dirac equation (3·1) takes the form

\[ \left\{ \frac{1}{a} \gamma^0 \partial_\eta + \frac{1}{a} \gamma^1 \partial_\chi + \frac{1}{\sinh \chi} \gamma^1 \partial_\theta + \frac{1}{\sinh \chi \sin \theta} \gamma^3 \partial_\varphi + m \right\} \Psi = 0, \]  
(3·5)

where we have introduced the spinor \( \Psi \),

\[ \Psi = a^{-2} (\sinh \chi)^{-1} (\sin \theta)^{-1/2} \phi. \]  
(3·6)

Applying the algebraic method of separation of variables it is possible to write Eq. (3·5) as a sum of two first order differential operators \( \tilde{K}_1, \tilde{K}_2 \) satisfying the relation

\[ [\tilde{K}_1, \tilde{K}_2] = 0, \quad (\tilde{K}_1 + \tilde{K}_2) \phi = 0, \]  
(3·7)

\[ \tilde{K}_1 \phi = \lambda \phi = - \tilde{K}_2 \phi, \]  
(3·8)

where

\[ \Psi = \gamma^1 \gamma^2 \gamma^3 \phi, \]  
(3·9)

\[ \tilde{K}_1 = (\gamma^0 \partial_\eta + am) \gamma^1 \gamma^2 \gamma^3, \]  
(3·10)

\[ \tilde{K}_2 = \left( \gamma^1 \partial_\chi + \frac{1}{\sinh \chi} \gamma^2 \partial_\theta + \frac{1}{\sinh \chi \sin \theta} \gamma^3 \partial_\varphi \right) \gamma^1 \gamma^2 \gamma^3. \]  
(3·11)

In this way, we have separated the time variable \( \eta \) from the spatial \( \chi, \theta \) and \( \varphi \) variables. The problem arises when we try to reduce (3·11) to the form (3·8). It is not difficult to see that there are no separating matrices allowing that step. In order to go further we write equation \( (\tilde{K}_2 + \lambda) \phi = 0 \) as follows:

\[ (\tilde{K}_3 + \tilde{K}_4 \gamma^1 \gamma^2) \phi = 0, \]  
(3·12)

where \( \tilde{K}_3 \) and \( \tilde{K}_4 \) are two commuting differential operators given by the expressions...
After introducing the auxiliary spinor $\Sigma$

$$\Phi = (\bar{K}_4 + i\gamma^2 \overline{K}_3) \Sigma,$$  
where

$$\bar{N}_2 = \frac{(\sinh x)^2}{\sinh^2 x} (\coth x - \gamma^2 \gamma^2 \coth x + \gamma^2),$$  
$$\bar{N}_0 = (\partial_x^2 + i\gamma^2 \gamma^3 k(x)^{-2} \cos \vartheta - k(x)^{-2}),$$

where $k(x)$ is the eigenvalue of the operator $-i\partial_x$.

Choosing to work in the following representation of the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},$$

we have that the spinor $\Sigma$ takes the form

$$\Sigma = \begin{pmatrix} p(x)A(\vartheta) \\ q(x)B(\vartheta) \\ r(x)C(\vartheta) \\ s(x)D(\vartheta) \end{pmatrix},$$

where the functions $p(x)$, $q(x)$, $r(x)$ and $s(x)$ satisfy the differential equation

$$(\sinh x(d_x^2 + i\lambda))(\sinh x(d_x^2 + i\lambda))X(x) + k^2 X(x) = 0,$$

where the upper signs correspond to $p(x)$, $r(x)$, and the lower signs are related to $q(x)$ and $s(x)$.

The functions $A(\vartheta)$, $B(\vartheta)$, $C(\vartheta)$, $D(\vartheta)$ are solutions of the equation

$$(d_\vartheta + k(x)^{-1})(d_\vartheta + k(x)^{-1})Y(\vartheta) - k^2 Y(\vartheta) = 0,$$

where the upper signs correspond to $A(\vartheta)$ and $C(\vartheta)$, and the lower ones correspond to $B(\vartheta)$ and $D(\vartheta)$. After substituting the expression (3-20) into (3-14) we arrive at
\[
\Phi = ik \begin{pmatrix} p(\chi)B(\theta)(1+i)\alpha \\ q(\chi)A(\theta)(1-i)\alpha \\ r(\chi)D(\theta)(1+i)\beta \\ s(\chi)C(\theta)(1-i)\beta \end{pmatrix} e^{ikr},
\]

(3.23)

where \( \alpha \) and \( \beta \) are functions depending on the variable \( \eta \). The functions \( A, B, C, D, p, q, r, s \) are related by the following systems of coupled partial equations:

\[
\begin{aligned}
(d_s - \frac{k_p}{\sin \vartheta}) A &= -k B, \\
(d_s + \frac{k_p}{\sin \vartheta}) B &= -k D,
\end{aligned}
\]

(3.24)

\[
\sinh \chi (d_x + il) P = k q, \\
\sinh \chi (d_x - il) q = -k r,
\]

(3.25)

where \( \epsilon \) and \( \omega \) are constants. From (3.23) and (3.10) we can put \( p(\chi) = r(\chi), q(\chi) = s(\chi), A(\theta) = C(\theta) \) and \( B(\theta) = D(\theta) \). The functions \( \alpha \) and \( \beta \) satisfy the system:

\[
(\partial_\eta + ima) \beta = -\lambda \alpha, \\
(\partial_\eta - ima) \alpha = \lambda \beta
\]

(3.26)

from which we obtain:

\[
(d_\eta^2 + m^2 b^2 e^{2\vartheta} + e^\vartheta (2m^2 b^2 - imb) + m^2 b^2 + \lambda^2) \alpha = 0,
\]

(3.27)

\[
(d_\eta^2 + m^2 b^2 e^{2\vartheta} + e^\vartheta (2m^2 b^2 + imb) + m^2 b^2 + \lambda^2) \beta = 0,
\]

(3.28)

whose solution can be written in terms of confluent hypergeometric functions as follows:

\[
\begin{aligned}
\alpha(\eta) &= e^{\gamma \vartheta} e^{-imb\eta} \left( -c_0 \frac{(c+imb)}{\lambda} M(c+imb+1, 2c+1, 2imb\eta) \\
&+ c_1 \frac{(c^2 + m^2 b^2)}{\lambda} U(c+imb+1, 2c+1, 2imb\eta) \right),
\end{aligned}
\]

(3.29)

\[
\beta(\eta) = e^{\gamma \vartheta} e^{-imb\eta} (c_0 M(c+imb, 2c+1, 2imb\eta) + c_1 U(c+imb, 2c+1, 2imb\eta)),
\]

(3.30)

where \( c \) is given by the expression:

\[
c = -i(\lambda^2 + m^2 b^2)^{1/2}.
\]

(3.31)

The solution of the coupled system of Eq. (3.24) can be expressed in terms of Jacobi polynomials as follows:

\[
A(\theta) = \sin^k \varphi(\theta) \cos(\theta/2) P_n^{(k \varphi^{1/2}, k \varphi^{-1/2})} (\cos \theta),
\]

(3.32)

\[
B(\theta) = \sin^k \varphi(\theta) \sin(\theta/2) P_n^{(k \varphi^{-1/2}, k \varphi^{1/2})} (\cos \theta),
\]

(3.33)

where \( n \) is given by the expression

\[
n = |k| - |k_r| - \frac{1}{2}.
\]

(3.34)
The system (3·25) can be rewritten in the following matrix form

\[
\left( \mathbf{I} \partial_x + i \lambda \sigma^a - \frac{ik}{\sinh \chi} \sigma^a \right) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}
\]

(3·35)

after applying the matrix transformation \( S \)

\[
S^{-1} \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} (1 + i \sigma^1)
\]

(3·36)

we reduce (3·25) to the form

\[
\left( \frac{d_x}{\sinh \chi} - \frac{ik}{\sinh \chi} \right) Q = \lambda P, \quad \left( d_x + \frac{ik}{\sinh \chi} \right) P = -\lambda Q
\]

(3·37)

with

\[
P + iQ = p, \quad Q + iP = q .
\]

(3·38)

The solution of the system (3·37) can be expressed in terms of Gauss hypergeometric functions,

\[
P(x) = \sinh ik(x) \cosh (x/2)
\]

(3·39)

\[
\times F \left( \frac{1}{2} + i \lambda + ik, \frac{1}{2} - i \lambda + ik, 1 - \cosh x/2 \right),
\]

\[
Q(x) = \left( \frac{1}{2} + ik \right) \sinh ik(x) \sinh (x/2)
\]

(3·40)

\[
\times F \left( \frac{1}{2} + i \lambda + ik, \frac{1}{2} - i \lambda + ik, 1 - \cosh x/2 \right).
\]

Finally, from (3·6), (3·9) and (3·23) we find that the solution of the Dirac equation (3·1) reads

\[
\Psi = a^{-2} (\sinh \chi)^{-1} (\sin \theta)^{-1/2} \left( \begin{pmatrix} p(x) B(\theta)(1 + i) \beta(\eta) \\ q(x) A(\theta)(1 - i) \alpha(\eta) \end{pmatrix} + i \lambda \sigma^a \begin{pmatrix} p(x) B(\theta)(1 + i) \beta(\eta) \\ q(x) A(\theta)(1 - i) \alpha(\eta) \end{pmatrix} + \lambda^2 \sinh \chi \right) e^{ik\varphi} .
\]

(3·41)

Here some comments are in order. The election of a diagonal tetrad for separating variables and solving the Dirac equation in the background field (1·6) was done for the sake of simplicity. If we want to reobtain an angular dependence of the spinor (3·41) in terms of spherical harmonics, it is enough to achieve the transformation relating the diagonal tetrad gauge with the Cartesian one. Obviously, this transformation does not affect the time dependence of the spinor solution.

§ 4. Particle creation

In this section we proceed to analyze the phenomena of scalar and Dirac particle
creation in the cosmological background associated with the metric (1·6). The model in question begins at $\eta = -\infty$ and evolves to $\eta \to +\infty$. Among the particularities of our model we have to mention that, like Ref. 7) the universe (1·6) is asymptotically flat, but does not present a period of contraction during its evolution toward the future.

In order to establish the asymptotic behavior of the solutions, at $\eta = -\infty$ and at $\eta = +\infty$, of the corresponding wave equations, we write the Hamilton-Jacobi equation

$$g^{ab}S_aS_b + m^2 = 0$$

(4·1)

in the background field (1·6). Then we obtain

$$-(S,\tau)^2 + (S,\chi)^2 + \frac{1}{\sinh^2 \chi} \left( (S,\phi)^2 + \frac{1}{\sin^2 \phi} (S,\varphi)^2 \right) + m^2 a^2 = 0.$$  

(4·2)

We can separate in Eq. (4·2) the time dependence from the spatial one as follows:

$$S = T(\eta) + F(\chi, \vartheta, \varphi),$$

(4·3)

and then, we obtain

$$(T,\tau)^2 = m^2 a^2 + \Lambda^2, \quad T = \pm \int(m^2 a^2 + \Lambda^2)^{1/2} d\eta,$$

(4·4)

where $\Lambda$ is a constant of separation. Then, the quasiclassical asymptotes of the solutions are

$$\Psi \to F(\chi, \vartheta, \varphi) \exp \left( \pm i \int \left( m^2 a^2 + \Lambda^2 \right)^{1/2} d\eta \right).$$

(4·5)

The asymptotic behavior of the function $T(\eta)$ is

$$\lim_{\eta \to -\infty} T = \pm (m^2 b^2 + \Lambda^2)^{1/2} \eta, \quad \lim_{\eta \to +\infty} T = \pm m b e^\eta.$$  

(4·6)

The relation between the conformal time $\eta$ and the cosmic time $t$ can be obtained from (1·5) giving as result,

$$b(e^\eta + \eta) = t.$$  

(4·7)

Note that the function $b(e^\eta + \eta)$ monotonically increases, and $t \to \pm \infty$ when $\eta \to \pm \infty$, and, when $\eta \to +\infty$ the metric (1·6) becomes a Milne universe.5)

We have that the solutions of the Klein-Gordon equation with the asymptotic behavior (4·5) can be written as

$$Z_{-\infty}(\eta) = (2c)^{-1/2} e^{\gamma \eta} \exp(-imb^\eta) M\left( \frac{1}{2} + c - ibm, 2c+1, 2ibm \right)$$

(4·8)

for $\eta \to -\infty$, and

$$Z_{+\infty}(\eta) = e^{inc\tau} e^{\gamma \eta} \exp(-imb^\eta) U\left( \frac{1}{2} + c - ibm, 2c+1, 2ibm \right)$$

(4·9)

for $\eta \to +\infty$, where, in order to analyze the form (4·8) takes at infinity, we have used the formula:11)
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\[ U(a, \gamma, z) \rightarrow z^{-a} \left(1 + O\left(\frac{1}{z}\right)\right), \quad (4 \cdot 10) \]

and the relation (4 \cdot 7).

Note that, for the constant \( c \) given by (2 \cdot 10), \( T_{-\infty} \) and \( T_{\infty} \) can be associated with a positive frequency mode at \( \eta \rightarrow -\infty \) and \( \eta \rightarrow \infty \) respectively. The next step is to compute the Bogoliubov coefficients relating the two different vacua at \( \eta = -\infty \) and \( \eta \rightarrow \infty \). In the present case it is possible to obtain \( \beta \) without performing the integration. Indeed, using the recurrence formula between the confluent hypergeometric functions \( U(a, \gamma, z) \) and \( M(a, \gamma, z) \),

\[ M(a, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-a)} e^{i\alpha a} U(a, \gamma, z) + \frac{\Gamma(\gamma)}{\Gamma(a)} e^{i\alpha(\gamma-a)} e^{\alpha z} U(\gamma-a, \gamma, -z) \quad (4 \cdot 11) \]

substituting \( a = (1/2) + c + ib \), \( \gamma = 2c + 1 \) and \( z = 2i\text{me}^\eta \) into (4 \cdot 10) and taking into account the relation

\[ e^{\alpha z} U(\gamma-a, \gamma, -z) = e^{\alpha z} z^{1-\gamma} (-1)^{1-\gamma} U(1-a, 2-\gamma, -z), \quad (4 \cdot 12) \]

we arrive at

\[ Z_{-\infty}(\eta) = \frac{\Gamma(2c+1)}{(2c)^{1/2} \Gamma\left(\frac{1}{2} + c - ib\right)} e^{i\alpha(a-c)} e^{-\frac{1}{2}mb} Z_m(\eta) \]

\[ + \frac{\Gamma(2c+1)}{(2c)^{1/2} \Gamma\left(\frac{1}{2} + c - ib\right)} e^{-i\alpha(2c+1)} e^{-\frac{1}{2}mb} (2c)^{-2c} Z^*(\eta). \quad (4 \cdot 13) \]

Then, using the relations for the \( \Gamma \) function \(^{12}\)

\[ 2^{2z-1} \Gamma(z) \Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi} \Gamma(2z), \quad (4 \cdot 14) \]

\[ |\Gamma(iy)|^2 = \frac{\pi}{y \sinh(\pi y)}, \quad (4 \cdot 15) \]

and considering the limit case when \( b \) is small, we arrive at

\[ |eta|^2 = \frac{1}{e^{2\nu} - 1}, \quad (4 \cdot 16) \]

where \( \nu = (\alpha^2 - 1)^{1/2} \).

Analogously, we have that the Bogoliubov coefficients for the Dirac case can be obtained by means of the functions \( a(\eta) \) and \( \beta(\eta) \) determining the time dependence of the spinor (3 \cdot 14). Then, considering the corresponding positive frequency solutions at the origin and at infinity,

\[ \beta_{-\infty}(\eta) = e^{\nu c} e^{-imb\eta} M(c + imb, 2c + 1, 2imbe^\eta), \quad (4 \cdot 17) \]

\[ \beta_{\infty}(\eta) = e^{i\alpha c} e^{\nu c} e^{-imb\eta} U(c + imb, 2c + 1, 2imbe^\eta), \quad (4 \cdot 18) \]

where the constant \( c \) is given by (3 \cdot 31). Taking into account the relation (4 \cdot 10) and (4 \cdot 11), we obtain
\[
\beta_{-\omega}(\eta) = \frac{\Gamma(2c+1)}{\Gamma(c+1+imb)} e^{i\pi(c+imb)} e^{-i\pi c/2} \beta_{\omega}(\eta)
\]
\[
+ \frac{\Gamma(2c+1)}{\Gamma(c-imb)} e^{-i\pi(c+imb+1)} e^{i\pi c/2} (2\text{mb})^{-2c} \beta_{\omega}(\eta),
\]
then, for small values of \(b\), using the relation for the gamma function \(\Gamma(z)\),
\[
|\Gamma(iy + \frac{1}{2})|^2 = \frac{\pi}{\cosh(\pi y)}
\]
and the expression (4·14) we arrive at
\[
|\beta|^2 = \frac{1}{2} \frac{1}{e^{2\pi \lambda} + 1},
\]
where \(\lambda\) in (4·21) is the separation constant appearing in (3·8).

§ 5. Concluding remarks

In this paper we have analyzed the process of creation of scalar and Dirac particles in an isotropic and homogeneous cosmological universe filled with non-interacting radiation and dust-like matter. It is worth noticing that, also in this model we have obtained that the average number of particles created satisfies a thermal distribution law. This result shows that the equation of state does not determine the quantum spectrum but the time behavior of the line element associated with the cosmological universe. From a mathematical point of view, this paper shows the capabilities of the algebraic method of separation of variables and how it works when first order separation is not possible. We hope that the results presented in this article will be of help in understanding quantum phenomena in cosmological backgrounds.

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