A protocol to estimate the average fidelity of the bipartite system

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In the field of quantum process tomography, the average fidelity quantifies how well the quantum channel preserves quantum information. In present work, we shall develop a protocol to estimate the average fidelity for the bipartite system. We show that the average fidelity should be known if the three measurable quantities, the average survive probability of the product state and the average survive probability of each subsystem, have been decided. Our protocol can be also applied to decide the selected element of the quantum process matrix.

PACS numbers: 03.67.Lx

I. INTRODUCTION

The characterization of the evolution of a quantum system is one of the main tasks to accomplish to achieve quantum information processing. A general class of methods, which have been developed in quantum information theory to accomplish this task is known as quantum process tomography (QPT) - for a review of quantum tomography, see Refs. [1–3]. The various protocols of getting the complete information about the quantum process can be divided into two classes: The standard quantum process tomography (SQPT) [1,4,5], with the central idea of preparing a set of linearly independent inputs and measuring the outputs via the quantum state tomography (QST), works without requiring any additionally quantum resources. Another is the so-called ancilla-assisted quantum process tomography (AAQPT) with the approach of encoding all information about the transformation into a single bipartite system-ancilla quantum state [6-8].

As a known fact, the complete characterization of a unknown quantum channel is a non-scalable task: For the $N d$-level system, there are about $d^{4N}$ elements to be decided. Naturally, one may ask: Can the number of experiment be scalable if partial information about the quantum process is to be characterized? The average fidelity is an important quantity in QPT. It quantifies how well the quantum map preserves quantum information [9,10]. Recently, it has been shown that it is possible to estimate the average fidelity via a technique known as twirling [9-15]. The Haar-twirl channel can be produced by the so-called Haar twirling procedure. It consists of applying a unitary, which is randomly chosen with the Haar measure, before the process to be characterized, followed by the inverse of the same unitary. The average fidelity can be estimated by preparing an arbitrary pure state for the input of the Haar-twirl channel and then measuring its survive probability (the overlap between the input and output). Another twirling procedure, where the unitary is sampled uniformly from the Clifford group, has been proposed in [12]. The resulted Clifford-twirl channel was shown to be equivalent with the Haar-twirl one. Still, the exact measurement of the average fidelity with the Clifford twirling protocol, which involves finite but exponentially large resources, is a non-scalable task. However, to experimentally characterize the fidelity of a quantum process on $n$ qubits for a desired accuracy, an efficient protocol has been constructed with quantum circuits of size $O(n)$ without requiring any ancilla qubits [12].

Besides the twirling protocol, another important ancilla-less way for deciding the average fidelity has also been developed. The average fidelity should be known if the survive probability of each state, which belongs to the state 2-design, has been decided [16-18]. Furthermore, it was found that the state 2-design protocol can be also adapted to estimate an arbitrary element of the quantum process matrix. For the $D$-dimensional system, the state 2-design usually has more than $D^2$ elements in it. Therefore, an efficient method to estimate a selected element, where the error scales as $\sqrt{1/M}$ with $M$ the number for the repetitions of the experiment, has also been proposed in [16-18].

The concept of the average fidelity can be generalized to the gate fidelity, a quantity characterizing how well the quantum map approximate a quantum gate [9]. It is demonstrated that twirling experiments previously used to characterize the average fidelity of quantum memories efficiently can be easily adapted to estimate the average fidelity of the experimental implementation of important quantum computation processes, such as unitaries in the Clifford group, in a practical and efficient manner[19].

In present work, we shall develop a protocol to estimate the average fidelity of the quantum channel for a bipartite system. Our work is motivated by such an interesting case: In the Bell-type experiment, two spin-$s$ particles are initially prepared in an arbitrary state from a quantum source, then each particle is sent to Alice and Bob, the two users who are space separated, respectively. In general, we suppose that there exists a quantum map which relates the initial state (for the two particles in the source) to the final state (for the two particles in the users’ hand). Now, the average fidelity is still an important quantity to characterize the quantum process where the state should be kept unchanged. For such a case, the average fidelity is hard to be measured in a directly way: By its definition, one should measure the survive probability of an arbitrary state. However, the problem appears when the two particles are prepared in
an entangled state. As a solution for it, we introduce
three directly measurable quantities, the average survive
probability of the product state and the average survive
probability of each subsystem, for the bipartite system
and give a formula to estimate the average fidelity with
the introduced quantities. Furthermore, we show that
our protocol can be also applied to decide an arbitrary
selected element of the quantum process matrix.

The rest content of present work can be divided into
following parts. In Sec. II we shall give a brief review
of the known ancilla-less methods used to estimate the
average fidelity of the quantum channel. Especially, we
introduce the convenient tool where a bounded matrix is
related to a vector in the enlarged Hilbert space. As an
application of it, we show that the average fidelity can
be calculated as the expectation of the quantum pro-
cess super operator with the separable Werner state. In
Section III we shall firstly define the three quantities,
the average survive probability of the product state and
the average survive probability of each subsystem, for
the bipartite system and then design several protocols to
measure them. A formula, where the average fidelity is
related to the three average survive probabilities, should
be constructed there. In Sec. IV we define the quantum
process matrix and its elements in an explicit way. By
following the idea presented in [18], the protocol used to
measure the average fidelity is adapted to decide an ar-
bitrary element of the quantum process matrix. In Sec.
V we shall develop an efficient protocol to measure the
average fidelity. Finally, we end our work with a short
discussion.

II. MEASURING THE AVERAGE FIDELITY
WITH THE PROTOCOL OF TWIRLING

In this section, we shall firstly give a brief review of
the known ancilla-less methods applied to measure the
average fidelity of the quantum channel. Let \{\{i\}\} \text{\text{\text{\text{\text{\text{}\text{\text{}\text{\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{}\text{'
protocol can be also applied to get the average fidelity. Formally, we express it as

\[ f^{\text{avg}}_c(\varepsilon) = \frac{1}{K} \sum_{i=1}^{K} \text{Tr}[\Psi_0 \Psi_i^\dagger \varepsilon(C_i \Psi_0 C_i^\dagger) C_i]. \quad (3) \]

Recently, it has been found that a state 2-design can be also applied to measure the average fidelity. The state 2-design, \( \{\Psi_x\}_{x=1}^{N} \), is a set of states satisfying the constraint that

\[ \frac{1}{N} \sum_{x=1}^{N} \Psi_x \otimes \Psi_x = \frac{1}{D(D+1)} \sum_{i,j=1}^{D} |jk\rangle \langle jk| + |kj\rangle \langle kj|. \quad (4) \]

Strictly to say, this definition is suitable for the case where all the states \( \Psi_x \) are equal weighted. For a more general definition of the state 2-design, please see [20].

For the case \( N = D(D+1) \), the state 2-design is known to be a complete set of mutually unbiased bases (MUBs) [21-22]; That is a set of \( D+1 \) bases for \( H_D \) with a constant overlap of 1/\( D \) between elements of different bases,

\[ |\langle \psi^j_x | \psi^j'_x \rangle|^2 = \frac{1}{D} \delta_{mm'}, \quad j = j', \quad j \neq j'. \quad (5) \]

If \( N = D^2 \), the state 2-design is unique: It’s just the symmetric information complete (SIC) set \( \{\psi^x_{\text{SIC}}\}_{x=1}^{D^2} \) introduced in [23]. The normalized states \( |\psi^x_{\text{SIC}}\rangle \) have the property that

\[ |\langle \psi^x_{\text{SIC}} | \psi^x_{\text{SIC}} \rangle|^2 = \frac{1 + D\delta_{xy}}{1 + D}. \quad (6) \]

The way of applying the state 2-design for estimate the average fidelity is shown in FIG. 1d: Preparing a state \( |\psi_x\rangle \), which belongs to a given set of a state 2-design, for the quantum channel \( \varepsilon \), after the evolution, one measure the expectation value of the projective operator \( \Psi_x \) with the output \( \varepsilon(\Psi_x) \). The average fidelity should be known by repeating this process with a number of \( N \) different inputs,

\[ f^\text{avg}_d(\varepsilon) = \frac{1}{N} \sum_{x=1}^{N} \text{Tr}[\Psi_x \varepsilon(\Psi_x)]. \quad (7) \]

In the above argument, we have assumed that all the quantities \( f^\text{avg}_k(\varepsilon) \), which are measured with the protocols depicted from FIG. 1a to FIG. 1d, should equal the average fidelity defined in (1),

\[ f^\text{avg}_k(\varepsilon) \equiv f^\text{avg}_c(\varepsilon), \quad k = a, b, c, d \quad (8) \]

(Here, we suppose \( f^\text{avg}_a(\varepsilon) = f^\text{avg}_c(\varepsilon) \)). Although this equivalence has been verified in previous works, for the convenience of reading, we would still like to give a self-contained proof for it. To complete this task, we shall at first introduce the convenient tool where a bounded matrix in \( H_D \) is related to a vector in the enlarged Hilbert space \( H_D^{\otimes 2} \). Let \( A \) to be a bounded matrix in the \( D \)-dimensional Hilbert space \( H_D \), with \( A_{ij} = \langle i|A|j \rangle \) the matrix elements for it, an isomorphism between \( A \) and a \( D^2 \)-dimensional vector \( |A\rangle \) is defined as

\[ |A\rangle = \sqrt{D} A \otimes I_D |S_+\rangle = \sum_{i,j=1}^{D} A_{ij} |ij\rangle, \quad (9) \]

in which \( |S_+\rangle \) is the maximally entangled state for \( H_D^{\otimes 2} \),

\[ |S_+\rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D} |kk\rangle \text{ with } |ij\rangle = |i\rangle \otimes |j\rangle \text{. This isomorphism offers a one-to-one mapping between the matrix and its vector form. Suppose that } A, B, \text{ and } \rho \text{ are three arbitrary bounded matrices in } H_D, \text{ there should be} \]

\[ \text{Tr}[A \otimes B] = \text{Tr}[A^\dagger B], \quad |A \rangle \langle B| = A \otimes B^\dagger \rho \rangle \langle \rho|, \quad (10) \]

with \( B^\dagger \) denoting the transpose of \( B \). Especially, if \( A \) takes the form \( \rho = |\psi\rangle \langle \phi| \), its corresponding vector should be

\[ |||\langle \phi|\rangle|| = ||\psi\rangle \otimes |\phi^\ast\rangle|. \quad (11) \]

With the isomorphism in (9) and its properties in (10-11), recalling \( \varepsilon(\rho) = \sum_n A_n \rho A_n^\dagger \), we are able to express the average quantities measured in FIG. 1 in the way like

\[ f^\text{avg}_k(\varepsilon) = \text{Tr}[\hat{F}_k \sum_n A_n \otimes A_n^\dagger]], \quad k = a, b, c, d \quad (12) \]

where \( \hat{F}_k \), the super operators in \( H_D^{\otimes 2} \), are defined as

\[ \hat{F}_a = \int d\mu_{H}(\Psi) |\langle \Psi|\rangle \langle \Psi|\rangle, \quad (13) \]

\[ \hat{F}_b = \int d\mu_{H}(U) |U \otimes U^\dagger|\langle \Psi_0|\langle \Psi_0| \langle U \otimes U^*|\rangle, \quad (14) \]

\[ \hat{F}_c = \int \frac{1}{K} \sum_{j=1}^{K} C_j \otimes C_j^\dagger |\Psi_0\rangle \langle \Psi_0| \langle C_j \otimes C_j^\dagger|, \quad (15) \]

\[ \hat{F}_d = \int \frac{1}{N} \sum_{x=1}^{N} |\Psi_x\rangle \langle \Psi_x| \langle \Psi_x|, \quad (16) \]

Now, if a separable Werner state \( \rho^\text{sep}_W \) for \( H_D^{\otimes 2} \) is introduced as

\[ \rho^\text{sep}_W = \frac{1}{D(D+1)} (I_D \otimes I_D + D |S_+\rangle \langle S_+|) \]

\[ = \frac{1}{D(D+1)} (I_D \otimes I_D + |I_D\rangle \langle I_D|), \quad (17) \]

one may conclude that all the super operators \( \hat{F}_k \) are equivalent since that

\[ \hat{F}_k = \rho^\text{sep}_W. \quad (18) \]

A simple reasoning, where the above conclusion can be achieved at, is like this: At first, we take it for granted that \( \hat{F}_a = \hat{F}_b \) since that the arbitrary state \( |\psi\rangle \) in FIG. 1a is related to the arbitrary unitary transformation \( U \).
in FIG. 1b via the simple relation, $|\psi\rangle = U|\psi_0\rangle$. At the same time, the relation $F_b = F_b^\dagger$ should also hold because that both the Haar twirling and the Clifford twirling will result the same channel, $\varepsilon^{CT} = \varepsilon^H$. The proof for $F_b = \rho_W^{sep}$ is given in Appendix. Recall that $\Psi_x$ is a Hermitian operator, $\Psi_x^* = \Psi_x^\dagger$. By performing the partial transposition on both sides of (4), the definition of the state 2-design may have an equivalent version:

$$
\frac{1}{N} \sum_{x=1}^N \Psi_x \otimes \Psi_x^* = \rho_W^{sep}.
$$

Noting that $\Psi_x \otimes \Psi_x^*$ are product states, this is the reason why we call the Werner state in (17) the separable one. From (11), there should be $|\Psi_x\rangle\langle\Psi_x| = \Psi_x \otimes \Psi_x^*$. Therefore, the relation, $F_d = \rho_W^{sep}$, can be easily verified.

### III. ESTIMATING THE AVERAGE FIDELITY OF THE BIPARTITE SYSTEM

In this section, we shall develop a protocol to estimate the fidelity of a bipartite system $H = H_A^2 \otimes H_B^2$. For this $D^2$-dimensional Hilbert system, we use $\Lambda$ to describe a trace preserving quantum map:

$$
\Lambda(\rho) = \sum_m E_m \rho(E_m)^\dagger, \sum_m (E_m)^\dagger E_m = 1_{D^2}.
$$

For an arbitrary $\Lambda$, we can introduce the following three average quantities,

$$
\bar{f}_{AB}(\Lambda) = \int d\mu_H(\Psi)d\mu_H(\Phi) Tr[\Psi \otimes \Phi \Lambda(\Psi \otimes \Phi)],
$$

$$
\bar{f}_{A}(\Lambda) = \int d\mu_H(\Psi)d\mu_H(\Phi) Tr[\Psi \otimes I_D \Lambda(\Psi \otimes \Phi)],
$$

$$
\bar{f}_{B}(\Lambda) = \int d\mu_H(\Psi)d\mu_H(\Phi) Tr[I_D \otimes \Phi \Lambda(\Psi \otimes \Phi)],
$$

where $\Psi \otimes \Phi$ denotes an arbitrary product state in $H_A^2 \otimes H_B^2$ while $d\mu_H(\Psi)$ and $d\mu_H(\Phi)$ are the Haar measures of the states on $H_B$. In present work, $\bar{f}_{AB}(\Lambda)$ is referred to as the average survival probability of the product states (for channel $\Lambda$), $\bar{f}_{A}(\Lambda)$ and $\bar{f}_{B}(\Lambda)$ is the average survival probability for subsystem system $H_A^2$ and $H_B^2$, respectively. An experimental protocol of measuring the above three quantities is depicted in FIG. 2a: Preparing an arbitrary product state $|\psi\rangle \otimes |\phi\rangle$ as the input for the quantum channel $\Lambda$, after the evolution, one may simultaneously measure the three expectations with the output $\Lambda(\Psi \otimes \Phi)$, $\bar{f}_{AB} = Tr[\Psi \otimes \Phi \Lambda(\Psi \otimes \Phi)]$, $\bar{f}_{A} = Tr[\Psi \otimes I_D \Lambda(\Psi \otimes \Phi)]$, and $\bar{f}_{B} = Tr[I_D \otimes \Phi \Lambda(\Psi \otimes \Phi)]$. By sampling $\Psi$ and $\Phi$ randomly with the Haar measure, the average survival probability in (19-21) should be known.

The average survive probabilities are defined in an integral version. To carry out the integrations, we shall also apply the isomorphism in (9). To let it has a form suitable for the bounded operators in the $D^2$-dimensional Hilbert space case, we introduce the following definition: Letting $|\Omega\rangle$ be a maximally entangled states in $H_H^4$, $|\Omega\rangle = \sum_{i,j=1}^D |ij\rangle$, with $|ij\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle$, a vector $|\Gamma\rangle$ in $H_H^4$ is related to the bounded operator $\Gamma$ in $H_H^{2^2}$, with its matrix elements to be $\Gamma_{ij,kl} \equiv (ij|\Gamma|kl)$, via the isomorphism,

$$
|\Gamma\rangle = D \cdot \Gamma \otimes I_D^{\otimes 2} |\Omega\rangle = \sum_{i,j,k,l=1}^D \Gamma_{ij,kl} |ij\rangle |kl\rangle.
$$

Suppose that $\Gamma$, $\Delta$, and $\Sigma$ are three arbitrary bounded matrices in $H_H^{2^2}$, there should be

$$
Tr[\Gamma^\dagger \Delta] = (\Gamma^\dagger \Delta), Tr[\Sigma \Delta] = \Gamma \otimes \Delta^T \Sigma).
$$

If $\Gamma = |\Psi\rangle\langle\Phi|$, there exists such a relation,

$$
||\Psi\rangle\langle\Phi|| = |\langle\Psi| \otimes |\Phi^*\rangle|.
$$

As an application of the above isomorphism, we find that the average survive probabilities in (19-21) can be rewritten as

$$
\bar{f}_{AB}(\Lambda) = Tr[\hat{F}_{AB} \lambda],
$$

$$
\bar{f}_{A}(\Lambda) = Tr[\hat{F}_{A} \lambda],
$$

$$
\bar{f}_{B}(\Lambda) = Tr[\hat{F}_{B} \lambda],
$$

where the four super operators in $H_D^{\otimes 4}$ are defined as

$$
\hat{F}_{AB} = \int d\mu_H(\Psi)d\mu_H(\Phi)|\Psi \otimes \Phi\rangle \langle \Psi \otimes \Phi|,
$$

$$
\hat{F}_{A} = \int d\mu_H(\Psi)d\mu_H(\Phi)|\Psi \otimes \Phi\rangle \langle \Psi \otimes I_D|,
$$

$$
\hat{F}_{B} = \int d\mu_H(\Psi)d\mu_H(\Phi)|\Psi \otimes \Phi\rangle \langle I_D \otimes \Phi|,
$$

$$
\lambda = \sum_m E_m \otimes E_m^*.
$$
Here, it should emphasize that $\lambda$ is a physical meaningful super operator: Suppose $\rho$ to be an arbitrary input for the quantum channel $\Lambda$ and $\Lambda(\rho) = \sum_m E_m \rho E_m^\dagger$ to be the corresponding output. Applying the result in (23), we see that the two vectors, $|\Lambda(\rho)\rangle$ and $|\rho\rangle$, are simply related by $\lambda$:

$$|\Lambda(\rho)\rangle = \sum_m E_m \rho E_m^\dagger \equiv \lambda|\rho\rangle.$$  

To carry out the integrations above, we introduce a special unitary transformation $\beta$ in $H_D^{\otimes 4}$,

$$\beta = \sum_{i,j,k,l=1}^D |ijkl\rangle \langle ikjl|.$$  

One may check that $\beta$ is also a Hermitian operator,

$$\beta = \beta^\dagger = \beta^{-1}.$$  

It has a nice property that

$$\beta |\Psi \otimes \Phi\rangle = |\Psi\rangle \otimes |\Phi\rangle.$$  

With the above property of $\beta$, we can reexpress $\bar{F}_{AB}$ as

$$\bar{F}_{AB} = \beta \langle \int d\mu_H(\Psi)|\Psi\rangle \langle\langle \Psi| \otimes \langle \int d\mu_H(\Phi)|\Phi\rangle \langle\langle \Phi|\rangle\beta.$$  

Recalling our results in (13-18), we have

$$\bar{F}_{AB} = \beta (\rho_{\text{sep}}^{AB} \otimes \rho_{\text{sep}}^{AB}) \beta.$$  

Different protocols of measuring the $\bar{F}_{AB}(\Lambda)$ are depicted in FIG.2. We call the one in FIG. 2b as the product Haar twirling procedure: Let $|\psi_0\rangle$ and $|\Phi_0\rangle$ the fixed state for the subsystem $H^A_D$ and $H^B_D$, respectively. The two arbitrary states in (19), $\Psi$ and $\Phi$, may be related to the arbitrary unitary operation $U$ and $V$

$$\Psi = U|\psi_0\rangle, \Phi = V|\Phi_0\rangle, \quad U,V \in U(D),$$  

respectively.

The average survive probability of the product states may be expressed as $\bar{F}_{AB}(\Lambda) = \int d\mu_H(U) \int d\mu_H(V) \text{Tr}[\Psi_0 \otimes \Phi_0 (U \otimes V) \Lambda (U \otimes V)(\Psi_0 \otimes \Phi_0)]$. In experiment, we first prepare $|\psi_0\rangle \otimes |\Phi_0\rangle$ as the fixed input, then apply the operation $U \otimes V$ before the map $\Lambda$ and an operation $(U \otimes V)^\dagger$ after. Finally, we measure the survive probability of $|\psi_0\rangle \otimes |\Phi_0\rangle$ with the output. By sampling $U$ and $V$ randomly with the Haar measure of $U(D)$, we shall get the quantity $\bar{F}_{AB}(\Lambda)$ defined above.

With the isomorphism in (22), one may express $\bar{F}_{AB}(\Lambda)$ as

$$\bar{F}_{AB}(\Lambda) = \text{Tr}[\hat{F}_b \otimes \hat{F}_b \lambda],$$  

where the super operator $\hat{F}_{AB}$ has the form

$$\hat{F}_{AB} = \beta (\hat{F}_b \otimes \hat{F}_b) \beta$$  

with $\hat{F}_b$ defined in (14). Applying the result in (18), we find $\bar{F}_{AB}(\Lambda)$ equals the quantity $\bar{F}_{AB}(\Lambda)$. Therefore, the product Haar twirling procedure in FIG. 2b represents a possible way of getting $\bar{F}_{AB}(\Lambda)$.

The way of applying the product Clifford twirling procedure to measure the average survive probability of the product states is shown in FIG. 2c. Its experimental data can be collected as $\bar{F}_{AB}(\Lambda) = \frac{1}{N^2} \sum_{x,y=1}^N \text{Tr}[\Psi_x \otimes \Phi_y (C_i \otimes C_j)(\Psi_x \otimes \Phi_y)]$, where $\{C_i\}_{i=1}^K$ is the Clifford group of $H_D$. Via the similar argument above, we have $\bar{F}_{AB}(\Lambda) = \text{Tr}[\hat{F}_{AB} \lambda]$ with the super operator $\hat{F}_{AB}$ to be $\hat{F}_{AB} = \beta (\hat{F}_c \otimes \hat{F}_c) \beta$. Obviously, $\bar{F}_{AB}(\Lambda)$ measured with the product Clifford twirling protocol equals $\bar{F}_{AB}(\Lambda)$ in (19) since that $\bar{F}_{AB} = \bar{F}_{AB}$.

Let $\{\Psi_x\}_{x=1}^N$ and $\{\Phi_y\}_{y=1}^N$ to be the state 2-designs defined in (4), the quantity $\bar{F}_{AB}(\Lambda)$, $\bar{F}_{AB}(\Lambda) = \frac{1}{N^2} \sum_{x,y=1}^N \text{Tr}[\Psi_x \otimes \Phi_y (A(\Psi_x \otimes \Phi_y))\Lambda (A(\Psi_x \otimes \Phi_y))]$, can be directly measured through the method in FIG. 2d. Joining the result $\bar{F}_{AB}(\Lambda) = \text{Tr}[\hat{F}_{AB} \lambda]$ with the product state $\Lambda$ in (18), we can get the formula $\bar{F}_{AB}(\Lambda) = \bar{F}_{AB}(\Lambda)$.

After giving a detail discussion about how to measure $\bar{F}_{AB}(\Lambda)$ in experiment, we shall focus on $\bar{F}_{A}(\Lambda)$ and $\bar{F}_{B}(\Lambda)$, the average survive probabilities for the subsystems, defined in (20) and (21), respectively. Let’s consider $\bar{F}_{A}(\Lambda)$ at first. Using Shur’s lemma,

$$\langle \int d\mu_H(\Phi)\Phi = \frac{1}{D^2} I_D,$$

we can simplify the expression of $\bar{F}_{A}$ in (29) as

$$\bar{F}_{A} = \frac{1}{D^2} \int d\mu_H(\Psi)|\Psi \otimes I_D\rangle \langle\langle \Psi| \otimes I_D|\Psi \otimes I_D\rangle.$$  

With the unitary $\beta$ in (32), it has an equivalent form

$$\bar{F}_{A} = \beta \langle\langle \int d\mu_H(\Psi)|\Psi \otimes I_D\rangle \langle\langle \Psi| \otimes I_D|\Psi \otimes I_D\rangle\beta.$$  

Recalling our definition of the super operator $\hat{F}_{a}$ in (13) and the result in (18), we can get the formula

$$\bar{F}_{A} = \beta (\rho_{\text{sep}}^{AB} \otimes \frac{I_D}{D}) \langle\langle I_D|\rangle\langle I_D|\rangle \beta.$$  

Via a similar argument, there should be

$$\bar{F}_{B} = \beta (\frac{I_D}{D}) \langle\langle I_D|\rangle\langle I_D|\rangle \beta.$$  

With the result in (18), where the different ways to expand the separable Werner are given, we are able to prove that the average survive probability for a selected subsystem can be measured by the various protocols in FIG. 2. For example, when the survive probability of subsystem $H^A_D$ is measured with the product state 2-design protocol depicted in FIG. 2d, the experimental data can be organized in the way like

$$\bar{F}_{A}(\Lambda) = \frac{1}{N^2} \sum_{x,y=1}^N \text{Tr}[\Psi_x \otimes \Phi_y (\Lambda(\Psi_x \otimes \Phi_y))].$$  


Formally, it can be transferred into $\tilde{f}_A^d(\Lambda) = \text{Tr}[\tilde{F}_A^d(\Lambda)]$ with the super operator $\tilde{F}_A^d$ to be

$$\tilde{F}_A^d = \frac{1}{N^2} \sum_{x,y=1}^{N} |\Psi_x \otimes \Phi_y)(\Psi_x \otimes I_A|.$$  

As it is shown in [20], the state 2-design has the property that

$$\frac{1}{N} \sum_{y=1}^{N} \Phi_y = \frac{1}{D} I_D.$$ 

With this property in hand, we rewrite $\tilde{F}_A^d$ as

$$\tilde{F}_A^d = \beta \frac{1}{N} \sum_{x=1}^{N} \langle\langle \Psi_x|\langle\langle (I_D) \otimes \frac{|I_D|}{D})|\Psi_x\rangle\rangle \beta.$$ 

Jointing it with (16) and (18), we find $\tilde{F}_A^d = \tilde{F}_A$. Therefore, $\tilde{f}_A^d(\Lambda)$ can be measured in the way presented in (36).

With the separable Werner state defined in (17), we shall get a relation, $\tilde{I}_D^\text{sep} = D(D+1)\tilde{\rho}_W^\text{sep} - |I_D|/(I_D)$, from it, we can expand the identity operator $\tilde{I}_D^\text{sep}$ as

$$\tilde{I}_D^\text{sep} = D^2(D+1)^2\tilde{\rho}_W^\text{sep} \otimes \rho_W^\text{sep} + |I_D|/(I_D) \otimes |I_D|/(I_D) - D(D+1)[\tilde{\rho}_W^\text{sep} \otimes |I_D|]/(I_D) + |I_D|/(I_D) \otimes \rho_W^\text{sep}.$$ 

Note that $\tilde{I}_D^\text{sep}$ is invariant under the transformation of $\beta$, $\tilde{I}_D^\text{sep} = \beta \tilde{I}_D^\text{sep} \beta$. Recalling our definitions of the super operators, $\tilde{F}_{AB}$ in (33), $\tilde{F}_A$ in (34), and $\tilde{F}_B$ in (35), we shall find that the identity operator, $\tilde{I}_D^\text{sep}$, can also be expanded as

$$\tilde{I}_D^\text{sep} = D^2(D+1)^2[\tilde{F}_{AB} - \frac{\tilde{F}_A + \tilde{F}_B}{D+1}] + |I_D^2|/(I_D^2).$$  

Putting (40) into it, we shall arrive at one of the main results of present work,

$$f^{\text{ave}}(\Lambda) = \text{Tr}[\tilde{\rho}_W^\text{sep} \lambda].$$  

By jointing it with (39) and (41), the well-known relation, for the case where the process super operator $\Lambda$ is defined for the $D^2$-dimensional system, is recovered here

$$f^{\text{ave}}(\Lambda) = \frac{D^2 f^{\text{ent}}(\Lambda) + 1}{D^2 + 1}.$$  

Putting (40) into it, we shall arrive at one of the main results of present work

$$f^{\text{ave}}(\Lambda) = \frac{1}{D^2 + 1}[2 + (D+1)^2 \tilde{f}_{AB} - (D+1)[\tilde{f}_A + \tilde{f}_B]],$$  

where $|\Omega\rangle$, as it has been introduced in (22), is the maximally entangled state of $H \otimes H^{\text{anc}}$ with $H^{\text{anc}}$ to be a $D^2$-dimensional ancilla system. With the super operator $\lambda$ in (31), one may express the entanglement fidelity with a more compact form:

$$f^{\text{ent}}(\Lambda) = \frac{1}{D^4} \text{Tr}[\lambda].$$  

Jointing it with our expanding of the identity operator in (37) and the equations (25-27), we find that the entanglement fidelity should be known if the three average survive probabilities, $\tilde{f}_{AB}(\Lambda)$, $\tilde{f}_A(\Lambda)$ and $\tilde{f}_B(\Lambda)$, have been decided,

$$f^{\text{ent}}(\Lambda) = \frac{1}{D^2} [1 + (D+1)^2 \tilde{f}_{AB} - (D+1)[\tilde{f}_A + \tilde{f}_B]].$$  

For simplicity, we have omitted the symbol (\Lambda) in the expression for each average survive probability.) In the derivation of it, we have used the formula

$$(\bar{I}_{D}^{\text{sep}}| \lambda = \sum_m (E_m^t E_m | = (I_{D}^{\text{sep}})|,$$  

which can be viewed as a vector-form expression of trace-preserving condition of $\Lambda$.

Now, we are able to show that the average fidelity can also be estimated with the three quantities, $\tilde{f}_{AB}(\Lambda)$, $\tilde{f}_A(\Lambda)$ and $\tilde{f}_B(\Lambda)$. As a well-known fact [9,10], there exists a beautiful formula where the average fidelity is simply related to the entanglement fidelity. Instead of directly citing this formula, we shall give a simple reasoning to recover it. We use $\tilde{\rho}_W^\text{sep}$ as the generalization of the separable Werner state (for the $D^4$-dimensional system) in (17) for the $D^4$-dimensional system,

$$\tilde{\rho}_W^\text{sep} = \frac{1}{D^2(D^2 + 1)}(\bar{I}_{D}^{\text{sep}} + (\bar{I}_{D}^{\text{sep}})^2).$$  

Correspondingly, the results in (12) and (18) can also be generalized into the following form

$$f^{\text{ave}}(\Lambda) = \text{Tr}[\tilde{\rho}_W^\text{sep} \lambda].$$  

By jointing it with (39) and (41), the well-known relation, for the case where the process super operator $\Lambda$ is defined for the $D^2$-dimensional system, is recovered here

$$f^{\text{ave}}(\Lambda) = \frac{D^2 f^{\text{ent}}(\Lambda) + 1}{D^2 + 1}. $$  

Putting (40) into it, we shall arrive at one of the main results of present work

$$f^{\text{ave}}(\Lambda) = \frac{1}{D^2 + 1}[2 + (D+1)^2 \tilde{f}_{AB} - (D+1)[\tilde{f}_A + \tilde{f}_B]],$$  

where we show that the average fidelity $f^{\text{ave}}(\Lambda)$ can be estimated with the average survival probabilities, $\tilde{f}_{AB}(\Lambda)$, $\tilde{f}_A(\Lambda)$ and $\tilde{f}_B(\Lambda)$.

IV. MEASURING THE SELECTED ELEMENT OF THE PROCESS MATRIX

The proposal, which suggests that the protocol designed for measuring the average fidelity can be also applied to decide an arbitrary element of the quantum process matrix, originated from the work in [16]. In the first version of it, the off-diagonal elements should be measured by introducing ancilla system. Recently, an improved scheme, where all the elements can be estimated without introducing any ancilla system, has been presented in [18]. In the argument below, following the idea presented in [18], we shall develop a method where an arbitrary element of the process matrix can be decided with the average survive probabilities.
At beginning, let’s introduce the definition of the process matrix. For a reason which will be clear later, we suppose \( \Lambda \) to be an arbitrary quantum map for the joint system \( H = H^A \otimes H^B \),

\[
\Lambda(\rho) = \sum_n \tilde{E}_n \rho \tilde{E}_n^\dagger, \quad \sum_n \tilde{E}_n^\dagger \tilde{E}_n = I^2_D. \tag{45}
\]

Letting \( \{\Gamma_\mu\}_{\mu=1}^{D^4} \) to be an orthogonal operator basis for \( H \),

\[
\text{Tr}[\Gamma_\mu^\dagger \Gamma_\nu] = D^2 \delta_{\mu\nu}, \tag{46}
\]

one may rewrite (45) as

\[
\Lambda(\rho) = \sum_{\mu, \nu=1}^{D^4} \Gamma_\mu \rho \Gamma_\nu^\dagger \chi_{\mu,\nu}(\Lambda), \tag{47}
\]

where the coefficients \( \chi_{\mu,\nu}(\Lambda) \),

\[
\chi_{\mu,\nu}(\Lambda) = \frac{1}{D^4} \sum_n \text{Tr}[\Gamma_\mu^\dagger \tilde{E}_n](\text{Tr}[\Gamma_\nu \tilde{E}_n])^*, \tag{48}
\]

are the entries of a \( D^4 \times D^4 \) process matrix \( \chi(\Lambda) \) which is Hermitian by definition. Here, it should be noted that the factor \( D^{-4} \) comes from the fact that the basis operators in (46) are not normalized. As it has been done in previous works, we suppose that \( \Gamma_\mu \) are Hermitian and unitary operators,

\[
\Gamma_\mu = \Gamma_\mu^\dagger = \Gamma_\mu^{-1}. \tag{49}
\]

Furthermore, we rewrite (48) with a convenient form

\[
\chi_{\mu,\nu}(\Lambda) = \frac{1}{D^4} \text{Tr}[(\sum_n \tilde{E}_n \otimes \tilde{E}_n^\dagger)(\Gamma_\mu \otimes \Gamma_\nu)]. \tag{50}
\]

From it, we see that the diagonal matrix elements should take the form,

\[
\chi_{\mu,\mu}(\Lambda) = \frac{1}{D^4} \text{Tr}[(\sum_n \tilde{E}_n \otimes \tilde{E}_n^\dagger)(\Gamma_\mu \otimes \Gamma_\mu)]. \tag{51}
\]

Now, let’s assume that the quantum map \( \Lambda \) discussed in above section is related to \( \Lambda \) via the simple form,

\[
\Lambda = \Lambda \circ \Gamma_\mu, \tag{52}
\]

where the process super operator \( \Lambda \circ \Gamma_\mu \) is defined as \( \Lambda \circ \Gamma_\mu(\rho) = \sum_n \tilde{E}_n (\Gamma_\mu \rho \Gamma_\mu^\dagger) \tilde{E}_n^\dagger \). We call the so-defined \( \Lambda \) the modified map (of \( \Lambda \)). Based on this assumption, we find that the super operator \( \lambda \) in (31) should be

\[
\lambda = (\sum_n \tilde{E}_n \otimes \tilde{E}_n^\dagger)(\Gamma_\mu \otimes \Gamma_\mu^\dagger). \tag{53}
\]

Recalling the formula for the calculation of the entanglement fidelity in (39) and the one for the diagonal matrix elements in (50), we shall find an interesting result,

\[
\chi_{\mu,\mu}(\Lambda) = f_{\text{ent}}(\Lambda \circ \Gamma_\mu) \equiv f_{\text{ent}}(\Lambda). \tag{54}
\]

In other words, one may transfer the task of measuring the diagonal matrix element, which is defined for the quantum map \( \Lambda \), into the one of deciding the entanglement fidelity of the modified map \( \Lambda = \Lambda \circ \Gamma_\mu \). Based on this principle, the diagonal matrix element \( \chi_{\mu,\mu}(\Lambda) \) can be estimated with \( f_{AB}(\tilde{\Lambda} \circ \Gamma_\mu) \), \( f_A(\Lambda \circ \Gamma_\mu) \) and \( f_B(\Lambda \circ \Gamma_\mu) \), which are the average survive probabilities of the modified quantum map \( \Lambda \circ \Gamma_\mu \),

\[
\chi_{\mu,\mu}(\Lambda) = \frac{1}{D^2} \{1 + (D + 1)^2 \bar{f}_{AB}(\tilde{\Lambda} \circ \Gamma_\mu) - (D + 1)(\bar{f}_A(\Lambda \circ \Gamma_\mu) + \bar{f}_B(\Lambda \circ \Gamma_\mu))\}. \tag{55}
\]

In FIG. 3, we give a protocol to measure the diagonal elements of the process matrix: Suppose that \( \Psi_x \) and \( \Phi_y \) are the elements of the state 2-design in (4). Preparing \( |\psi_x\rangle \otimes |\phi_y\rangle \) as the input, perform an operation \( \Gamma_\mu \) before the quantum \( \Lambda \), then measure the expectations \( f_{AB}(\tilde{\Lambda} \circ \Gamma_\mu) = \text{Tr}[\Psi_x \otimes \Phi_y \Lambda \circ \Gamma_\mu(\Psi_x \otimes \Phi_y)] \), \( f_A(\Lambda \circ \Gamma_\mu) = \text{Tr}[\Psi_x \otimes I_D \Lambda \circ \Gamma_\mu(\Psi_x \otimes \Phi_y)] \) and \( f_B(\Lambda \circ \Gamma_\mu) = \text{Tr}[I_D \otimes \Phi_y \Lambda \circ \Gamma_\mu(\Psi_x \otimes \Phi_y)] \). By repeating this process with a number of \( N^2 \) different inputs, one may get the averaged survive probabilities which can be used to estimate the diagonal element of the process matrix according to (55).

As it has been shown in [18], the way to decide the off-diagonal matrix elements is different form the one to get the diagonal matrix elements. Let’s organize the proposal developed there in the way like: For the two known operators, \( \Gamma_\mu \) and \( \Gamma_\nu \), \( \{\mu \neq \nu\} \), define the following four operators,

\[
\Gamma_{\pm} = \Gamma_\mu \pm i \Gamma_\nu, \tag{56}
\]

From it, one may easily verify that

\[
\Gamma_\mu \otimes \Gamma_\nu = \frac{1}{2} \{\Gamma_+ \otimes \Gamma_+ - \Gamma_- \otimes \Gamma_- + i(\Gamma_+ \otimes \Gamma_- - \Gamma_- \otimes \Gamma_+)\},
\]

\[
\Gamma_\mu \otimes \Gamma_\nu = \frac{1}{2} \{\Gamma_+ \otimes \Gamma_+ - \Gamma_- \otimes \Gamma_- + i(\Gamma_+ \otimes \Gamma_- - \Gamma_- \otimes \Gamma_+)\},
\]

Jointing this result with (49), we observe that the off-diagonal elements of the process matrix can be estimated by the way

\[
\chi_{\mu,\nu}(\Lambda) = \frac{1}{2} [\omega_+ - \omega_- - i(\tilde{\omega}_+ - \tilde{\omega}_-)], \tag{57}
\]

\[
\chi_{\nu,\mu}(\Lambda) = \frac{1}{2} [\omega_+ - \omega_- + i(\tilde{\omega}_+ - \tilde{\omega}_-)]. \tag{58}
\]
in which the four quantities, $\omega_{\pm}$ and $\bar{\omega}_{\pm}$, are defined as

$$\omega_{\pm} = \frac{1}{D^4} \text{Tr}[(\sum_n \tilde{E}_n \otimes \tilde{E}_n)(\Gamma_{\pm} \otimes \Gamma_{\pm}^\dagger)] \quad (54)$$

$$\bar{\omega}_{\pm} = \frac{1}{D^4} \text{Tr}[(\sum_n \tilde{E}_n \otimes \tilde{E}_n)(\tilde{\Gamma}_{\pm} \otimes \tilde{\Gamma}_{\pm}^\dagger)] \quad (55)$$

Let’s consider $\omega_+$ at first. Following the argument from (50) to (51), we can also interpret $\omega_+$ as the entanglement fidelity of the non-physical map $\tilde{\Lambda} \circ \Gamma_+$

$$\omega_+ = f_{\text{ent}}(\Lambda \circ \Gamma_+)$$

with the process super operator $\Lambda \circ \Gamma_+$ defined as $\Lambda \circ \Gamma_+(\rho) = \sum_n \tilde{E}_n(\Gamma_+ \rho \Gamma_+^\dagger)\tilde{E}_n^\dagger$. With a simple calculation, which is similar with the one for deriving (40), we shall get

$$\omega_+ = \frac{1}{D^4} \{2 + (D + 1)^2 \tilde{f}_{AB}(\tilde{\Lambda} \circ \Gamma_+) - (D + 1)[\tilde{f}_{A}(\Lambda \circ \Gamma_+) + \tilde{f}_{B}(\Lambda \circ \Gamma_+)]. \quad (56)$$

In the derivation of it, we have used the relation, $\text{Tr}[|D^{\otimes 2}\rangle\langle D^{\otimes 2}|(\sum_n \tilde{E}_n \otimes \tilde{E}_n)(\Gamma_{\pm} \otimes \Gamma_{\pm}^\dagger)] = 2D^2$. Now, we shall encounter the problem of measuring the average survive probabilities for a non-physical map $\Lambda \circ \Gamma_+$. A solution for it, as it has been suggested in [18], is to prepare a set of states $|\psi_{xy}\rangle\rangle\langle\langle |\phi_{xy}\rangle\rangle\rangle$ where $\{|\psi_{xy}\rangle\rangle\langle\langle |\phi_{xy}\rangle\rangle\rangle\}$ are the inputs for the quantum channel $\tilde{\Lambda}$, then decide the average survived probabilities, which are needed in (56), through the way in below:

$$\tilde{f}_{AB}(\tilde{\Lambda} \circ \Gamma_+) = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(\Psi_i \otimes \Psi_j)\Lambda(\Psi_{xy})],$$

$$\tilde{f}_{A}(\Lambda \circ \Gamma_+) = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(\Psi_i \otimes I_D)\Lambda(\Psi_{xy})],$$

$$\tilde{f}_{B}(\Lambda \circ \Gamma_+) = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(I_D \otimes \Psi_j)\Lambda(\Psi_{xy})].$$

The product state 2-design protocol designed to get the above average quantities is depicted in FIG. 4. The above method, which is developed for measuring $\omega_+$, can be easily generalized for the cases where the rest of the quantities defined in (54-55) should be measured.

V. APPROXIMATED MEASUREMENT OF THE AVERAGE SURVIVE PROBABILITIES

In section III, we have introduced three types average survive probabilities for a bipartite system. Several ways of measuring these quantities are depicted in FIG. 2. It should be noted that these measurements are non-scalable: Taking the product state 2-design protocol, which is given in FIG. 2d, for an example, there are about $N^2(N \geq D^2)$ runs of experiment to be performed. A solution for this problem is to find an efficient approximated way where the error introduced by the approximation should depend on the number of operations irrespective of the dimension of the joint system.

In this section, we shall develop such an efficient way to measure the average survive probabilities. In what follows we restrict ourself to the case where a set of product states, $\{|\Psi_i \otimes \Psi_j\rangle\rangle\langle\langle |\Psi_i \otimes \Psi_j\rangle\rangle\rangle$ should be prepared as the inputs of the quantum channel $\Lambda$ while the average survive probabilities defined in (19-21) are approximated with the way like

$$\bar{f}_{AB}^{\text{appr}} = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(\Psi_i \otimes \Psi_j)\Lambda(\Psi_{xy})],$$

$$\bar{f}_{A}^{\text{appr}} = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(\Psi_i \otimes I_D)\Lambda(\Psi_{xy})],$$

$$\bar{f}_{B}^{\text{appr}} = \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}[(I_D \otimes \Psi_j)\Lambda(\Psi_{xy})].$$

These average quantities can be also measured with the twirling procedures by letting $\Psi_j = U_j \Psi_0 U_j^\dagger$ with $\Psi_0$ and $U_j$ to be a fixed state and the arbitrary unitary transformations in $H_D$, respectively. By following the steps for getting the super operators in (33-35), the above equations can be also expressed

$$\bar{F}_a^{\text{appr}} = \text{Tr}[\bar{F}_a^{\text{appr}}], \quad \alpha = A, B, AB. \quad (58)$$

in which the super operators $\bar{F}_a^{\text{appr}}$ are defined as

$$\bar{F}_A^{\text{appr}} = \beta(\frac{1}{M^2} \sum_{i,j=1}^M \langle \Psi_i | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | \rangle \beta), \quad (59)$$

$$\bar{F}_B^{\text{appr}} = \beta(\frac{1}{M^2} \sum_{i,j=1}^M \langle I_D | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | | \langle I_D | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | \rangle \beta), \quad (60)$$

$$\bar{F}_{AB}^{\text{appr}} = \beta(\frac{1}{M^2} \sum_{i,j=1}^M \langle \Psi_i | \langle I_D | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | | \langle \Psi_i \otimes | \Psi_j \rangle \langle \Psi_j | \rangle \beta). \quad (61)$$

In above sections, we have shown that the average fidelity can expressed as the expectation of the super operator $\Lambda$, which is defined in (31), with the separable
Werner state $\tilde{\rho}_{\text{W}}$, $f_{\text{avg}}(\Lambda) = \text{Tr}[\tilde{\rho}_{\text{W}}]\Lambda$. With the expanding formula of the identity operation in (37), one may rewrite $\tilde{\rho}_{\text{W}}$ as

$$\tilde{\rho}_{\text{W}} = \frac{2}{D^2(D^2 + 1)}|\Delta_{D}^{\otimes 2}|(1_{D}^{\otimes 2}) + \frac{1}{D^2 + 1}\Delta,$$  

(62)

in which the operator $\Delta$ is defined as

$$\Delta = \frac{1}{D^2}(|\Delta_{D}^{\otimes 4} - |\Delta_{D}^{\otimes 2}|^2|\Delta_{D}^{\otimes 2}|) = (D + 1)^2\tilde{F}_{AB} - (D + 1)(\tilde{F}_{A} + \tilde{F}_{B}).$$  

(63)

Now, the average fidelity is approximated in the way like

$$f_{\text{avg}}(\Lambda) = \text{Tr}[\tilde{\rho}_{\text{appr}}]\Lambda$$

with $\tilde{\rho}_{\text{appr}}$ the approximated separable Werner state,

$$\tilde{\rho}_{\text{appr}} = \frac{2}{D^2(D^2 + 1)}|\Delta_{D}^{\otimes 2}|^2 + \frac{1}{D^2 + 1}\Delta_{\text{appr}},$$  

(64)

$$\Delta_{\text{appr}} = (D + 1)^2F_{AB} - (D + 1)(\tilde{F}_{A} + \tilde{F}_{B}).$$  

(65)

Considering the fact that the super operator $\lambda$, which is decided by the quantum map $\Lambda$, is usually unknown, for simplicity, we supposed that the accuracy of the approximation should depend on the difference between the separable Werner state and its approximated form. As it has been done in [25], we introduce the Hilbert-Schmidt norm of their difference,

$$||\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}}|| := \text{Tr}[(\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}})(\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}})^\dagger],$$  

(66)

to characterize how well the separable Werner state is approximated. With the definitions from (62) to (64), we simplify the above equation into the form

$$||\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}}|| = \frac{1}{(D^2 + 1)^2}||\Delta - \Delta_{\text{appr}}||.$$  

(67)

In order to perform the above calculation in an easy way, we shall introduce the operators $R_i$,

$$R_i = (D + 1)\Psi_i - I_D,$$  

(68)

and reexpress the operator $\Delta_{\text{appr}}$ in (65) with a more compact form

$$\Delta_{\text{appr}} = \frac{1}{M^2}\sum_{i,j=1}^{M}(|\Psi_i \otimes \Psi_j)(R_i \otimes R_j - I_D^{\otimes 2}|).$$  

(69)

With above formula, one may easily verify that

$$\text{Tr}[\Delta(\Delta)^\dagger] = \text{Tr}[\Delta(\Delta)_{\text{appr}}]\Delta = \text{Tr}[\Delta_{\text{appr}}(\Delta)^\dagger] = \frac{2(D^2 - 1)}{D^2},$$

Therefore, the Hilbert-Schmidt norm is

$$||\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}}|| = \frac{1}{(D^2 + 1)^2}(||\Delta_{\text{appr}}|| - \frac{2(D^2 - 1)}{D^2})$$  

(70)

where the quantity $||\Delta_{\text{appr}}||$ should be decided by our actual choice of the states $\Psi_i$.

Now, we assume that the states $\Psi_i$ are chosen from the symmetric information complete set (SIC) introduced in (6), $\Psi_i \neq \Psi_j$ for $i \neq j$. Based on this assumption, we find that

$$||\Delta_{\text{appr}}|| = \frac{D^4(D + 2)^2 + (D^2 - 2)D^2}{M^2(1 + D)^2} - \frac{4D}{M(1 + D)} + \frac{D - 1}{D + 1}.$$  

(71)

Putting it back into (70), we can get the upper bound of the Hilbert-Schmidt norm,

$$||\tilde{\rho}_{\text{W}} - \tilde{\rho}_{\text{appr}}|| < \frac{1}{M^2} \frac{(1 + D)^2}{D^2}, \quad 1 < M < D^2.$$  

(72)

For the case $D \geq 4$, there is $\frac{(1 + D)^2}{D^2} < 2$. Therefore, we conclude that the separable Werner state is approximated in an efficient way: The upper bound of the error introduced by the approximation process scales better than $1/M^2$ with the number of repetitions $M^2$ of the experiment.

**VI. DISCUSSION**

In previous works, the state 2-design has also been defined in the way like:

$$\frac{1}{N} \sum_{x=1}^{N} \text{Tr}[\Psi_x A \Psi_x B] = \frac{\text{Tr}[A]\text{Tr}[B]}{D(D + 1)}.$$  

(73)

One may check that the definition in (4) is consisted with it. Applying the isomorphism introduced in (9), we rewrite the left side of the above equation as

$$\frac{1}{N} \sum_{x=1}^{N} \text{Tr}[\Psi_x A \Psi_x B] = \text{Tr}[(\frac{1}{N} \sum_{x=1}^{N} \Psi_x \otimes \Psi_x^*)(A \otimes B^T)].$$

As it has been shown in section II, from the definition of the state 2-design in (4), we can get an extremely simple relation between the state 2-design and the separable Werner state, $\sum_{x=1}^{N} \Psi_x \otimes \Psi_x^* = \tilde{\rho}_{\text{W}}$. With the results, $\text{Tr}[B] = \text{Tr}[B^T]$ and $(I_D^{\otimes 2}|A^T \otimes B^T||I_D^{\otimes 2}) = \text{Tr}[AB]$, we have

$$\text{Tr}[\tilde{\rho}_{\text{W}}^\dagger(A \otimes B^*)] = \frac{\text{Tr}[A]\text{Tr}[B] + \text{Tr}[AB]}{D(D + 1)}.$$  

From it, one may conclude that the definition in (4) is consisted with the one in (73).

Finally, let’s end our work with a short conclusion. For the bipartite system, we have introduced three directly measurable quantities, the average survive probability of the product states and the survive probability for each subsystem, and developed several protocols to measure
them. These average quantities can be applied to estimate the average fidelity of the quantum channel and the selected element of the quantum process matrix.

Appendix. In Section II, we have introduced a super operator $\hat{F}_b$, $\hat{F}_b = \int d\mu_H(U)U \otimes U^* |\Psi_0\rangle\langle |\Psi_0\rangle |\Psi_0\rangle$ (17), and stated that it should equal with the separable Werner state defined in (17). Although this statement can be viewed as a known result in other works, (for example, the one in [10]), for the completeness of present work, we shall still give a compact proof for it. At first, applying the isomorphism in (22), we reexpress $\hat{F}_b$ in the vector form as $|\hat{F}_b\rangle = T(|\Psi_0\rangle) \otimes |\Psi_0\rangle$ with the super operator $T$ to be

$$T = \int d\mu_H(U)U \otimes U^* \otimes U^* U.$$

In stead of directly carrying out the integration, we shall cite a result given by Scott in [26]:

$$\int d\mu_H(U)U \otimes U \otimes U^\dagger \otimes U^\dagger = {1 \over D^2-1}(P_{3412} + P_{4321}) - {1 \over D(D^2-1)}(P_{4312} + P_{3421}),$$

with $P_{abcd}$ defined as the permutation operator (For its definition in detail, please see the original work in [26].) With the above calculation in hands, we find the expression of the $T$ should be

$$T = \frac{1}{D^2-1}[|1_D^2\rangle\langle 1_D^2| - |1_D^2\rangle\langle 1_D^2|] + D^2|S_+\rangle(S_+ - |S_+\rangle)\langle S_+|),$$

in which we use $S_+$ to denote the projective operator, $S_+ = |S_+\rangle\langle S_+|$ with $|S_+\rangle$ the maximally entangled state defined in (9). Performing the operation $T$ on an arbitrary product state $|\Psi_0\rangle \otimes |\Psi_0\rangle$, we shall get

$$T(|\Psi_0\rangle \otimes |\Psi_0\rangle) = |\rho_{\text{sep}}^W\rangle,$$

the vector form of the desired result $\hat{F}_b = \rho_{\text{sep}}^W$. 

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