On static contact of belt and different pulleys

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Abstract. The fitting of a looped belt on two pulleys with different radii is considered. A geometrically nonlinear model with account for tension and transverse shear is applied for modeling the belt. The pulleys are considered rigid bodies, and the belt-pulley contact is assumed frictionless. The problem has an axis of symmetry, therefore the boundary value problem is formulated and solved for a half of the belt. The considered part consists of three segments, two contact segments and a free span segment between them. The introduction of a dimensionless material coordinate at all segments leads to a system of ordinary differential equations of fifteenth order. The nonlinear boundary value problem for this system and boundary conditions is solved numerically with the shooting method and the finite difference method. As a result, the belt shape including the rotation angle, the forces, moments and contact pressure are determined. The contact pressure increases near the end point of contact areas, however no concentrated contact forces occur.

1. Introduction
The first systematic study of belt drives (and elastic creep in it) was reported by Reynolds in [1], where he used the string model. Until recently, one-dimensional models of elastic strings were widely used, cf. [2, 3, 4]. The exact solution for nonlinear steady state equations of extensible string is obtained in [3] by assuming zones of perfect and sliding friction contact between the belt and the pulley. A spatial description of the belt contour motion is suggested in [2]. An idealized point friction model allows extending this result to the transient dynamics, see [4].

However it turned out that the model of extensible string describes just a part of important effects in belt mechanics. Friction forces transmit power between the belt and the pulleys. They are applied on the belt from one side and result not only in tangent forces, but also in distributed moments. The model of extensible string without bending stiffness cannot resist the moment loading. Therefore, we use the rod model (i.e., the one with at least bending stiffness).

The calculation of belt-pulley interaction can be found in works [3, 5, 6, 7, 8, 9]. The study of interaction should be made with account for the transverse shear, because the importance of shear in contact problems of the rod theory is well-known, cf. [10, 11, 12, 13]. A contrary point of view is presented in [14]. The introduction of shear deformation causes the absence of lumped contact forces and promotes better understanding of the contact force distribution. The shear is also required to describe the effect of elastic microslip, see, e.g. [15, 16] for general friction modelling. Particularly the shear is of importance in friction belt drive operation [17, 18, 19].
However, shear is known to be taken into account only together with tension (compression) if the nonlinear theory of rods is applied. The goal of this work is to model the belt as a rod with bending and shear stiffness in the static problem where the belt is set on the pulleys. The system of equations of the corresponding rod theory can be found in [20, 21, 22, 23] (without direct application to belt drive mechanics, though).

The penalty formulation is in common use in numerical modeling of contact problems [24] and includes the choice of penalty coefficient or its distribution (see e.g. [25] for application to contact between rods). This approach is applied for belt drives in [9, 26, 27]. In the present paper we obtain the contact pressure and the stress-strain state in a different manner by describing the form of the belt lying on the rigid pulleys (the constraint formulation or the Hertz–Signorini–Moreau conditions for frictionless contact according to [24]).

We assume that the contact is full in a classical sense, i.e. it is continuous in an interval of the belt which length is to be determined. This assumption will be verified by obtaining the positive contact pressure. In the present study we use computer mathematics to solve difficult boundary value problems for the systems of ordinary differential equations (ODE). The application of standard methods is possible due to introduction of transformed material coordinate.

The presented study is an extension of the work [28]. Here we introduce significant complexity by rejecting the assumption of equal pulleys accepted there.

2. Equations of nonlinear contact problem

Figure 1 displays a general calculation scheme of a rod with transverse shear and tension in the $xy$ plane. The rod is considered a Cosserat line [20, 21, 22, 23]. The position vector in the deformed configuration $r(s)$ is a function of material coordinate $s$, and $\partial(\ldots)/\partial s = (\ldots)'$. The angular orientation of each particle of the rod is given by the orthonormal unit vectors $e_1, e_2$. The angle $\varphi$ denotes the angle between unit vector $e_1$ and Cartesian axis $x$, the latter having the unit vector $i$. For plane deformations the system of equations of nonlinear theory of rods is as follows [20, 21, 22, 23]:

$$Q' + q = 0, \quad M' + k \cdot r' \times Q + m = 0,$$

$$\varphi' - \varphi_0' = AM, \quad r' = P \cdot r_0' + B \cdot Q = e_1(1 + B_1 Q_1) + e_2 B_2 Q_2.$$

(1)

Here we denote: $Q$ is the internal force vector, $M$ is the bending moment, and $q, m$ stand for the external distributed forces and moments, respectively; $k$ is the unit vector of Cartesian axis $z$ perpendicular to the plane of drawing. The index zero indicates the values in an initial state, and $P = e_i e_0 i$ is the rotation tensor ($P \cdot r_0' = e_1$). Elastic characteristics of the rod in plane are determined by three scalar compliances: the bending compliance $A$, the tension compliance $B_1$ and the shear compliance $B_2$. The unit vectors $e_1, e_2$ are directed along the principal axes of the compliance tensor $B$. 

**Figure 1.** Nonlinear elastic rod with account for shear and tension. Deformed configuration, unit vectors $e_1, e_2$, tangent vector $r'$, and angle $\varphi$. 

Algorithms and results for the contact of belt and pulleys without shear are presented in [5, 6]. In the contact segment we introduce the spatial coordinate $s$ which is the arc coordinate in the actual configuration. The peculiarity of the considered problem is due to the fact that the function $r(s) = R(\sigma)$ is unknown because the arc coordinate $\sigma$ is not yet determined.

We use the obvious geometric relations and the expression of $r'$ from (1):

$$e_1 = i \cos \varphi + j \sin \varphi, \quad e_2 = -i \sin \varphi + j \cos \varphi;$$
$$r'(s)ds = R(\sigma)d\sigma, \quad |\dot{R}| = 1,$$
$$\sigma' = |r'| \equiv D = \sqrt{(1 + B_1 Q_1)^2 + (B_2 Q_2)^2}, \quad s = D^{-1}. \quad (2)$$

The last equations determine the relation of material coordinate $s$ with arc coordinate $\sigma$ (a dot indicates the derivative with respect to $\sigma$).

Consider the full contact with the pulley. We write the tangent unit vector and integrate it

$$\dot{R} = i \sin \kappa \sigma + j \cos \kappa \sigma, \quad r' = D \dot{R},$$
$$R = \kappa^{-1} [i (1 - \cos \kappa \sigma) + j \sin \kappa \sigma]. \quad (3)$$

Here $\kappa$ is the curvature of the pulley. We also assume $R(0) = 0$ and note that

$$r' \cdot e_1 = 1 + B_1 Q_1 = D \sin(\varphi + \kappa \sigma),$$
$$r' \cdot e_2 = B_2 Q_2 = D \cos(\varphi + \kappa \sigma). \quad (4)$$

Hence it is possible to express the transverse force $Q_2$ as a function of arc coordinate, angle and axial force in the following form:

$$Q_2(\sigma, \varphi, Q_1) = B_2^{-1}(1 + B_1 Q_1) \cot \alpha, \quad \alpha(\sigma, \varphi) \equiv \varphi + \kappa \sigma. \quad (5)$$

As a result we have four unknown functions of $s$ which are $\sigma, \varphi, M, Q_1$.

We assume no tangential contact between the belt and the pulley. Therefore the vectors $q, r'$ are orthogonal:

$$Q' \cdot r' = 0; \quad e_1' = \varphi' e_2, \quad e_2' = -\varphi' e_1;$$
$$(Q_1' - \varphi' Q_2)(1 + B_1 Q_1) + (Q_2' + \varphi' Q_1) B_2 Q_2 = 0. \quad (6)$$

In the last equation we express the derivatives using (1), (2), and $Q_2 = Q_2(\sigma, \varphi, Q_1)$ as follows:

$$\varphi' = \varphi'' + AM,$$
$$Q_2' = \frac{\partial Q_2}{\partial Q_1} Q_1' + \frac{\partial Q_2}{\partial \varphi} D + \frac{\partial Q_2}{\partial \varphi'} \varphi' = \frac{B_1}{B_2} Q_1' \cot \alpha - \frac{1 + B_1 Q_1}{B_2 \sin^2 \alpha} (\varphi' + \kappa \sigma'). \quad (7)$$

The combination of (6) and (7) provides the differential equation for $Q_1$ not stated here because it is cumbersome. Then we substitute $r'$ from (1) into the balance of moment equation and obtain

$$M' = ((B_2 - B_1) Q_1 - 1) Q_2. \quad (8)$$

The expressions $\sigma', \varphi', M', Q_1'$ form a fourth order ODE system:

$$Y' = F(s, Y), \quad Y \equiv (\sigma \quad \varphi \quad M \quad Q_1)^T. \quad (9)$$

Finally we determine the contact pressure $p$:

$$Dp = q \cdot k \times r' = k \cdot Q' \times r' = (Q_1' - \varphi' Q_2)B_2 Q_2 - (Q_2' + \varphi' Q_1)(1 + B_1 Q_1). \quad (10)$$

The contact pressure must be non-negative. Unlike the case without shear, in the considered problem we expect no lumped contact reactions. The similar case of the classical contact between an initially straight nonlinear shearable rod and a rigid straight obstacle is discussed e.g. in [11]. Also the distributed contact moment which plays the key role in the elastic microslip (see [18]) in loaded drive is zero because of the frictionless contact.
3. Contact of belt with pulleys

The scheme of fitting the belt on the pulleys is shown in Fig. 2. In the initial state the belt is the circle of radius $a_0$. Hence the initial angle and its derivative are $\varphi_0 = \pi/2 - a_0^{-1} s$, $\varphi_0' = -a_0^{-1}$. The pulleys have radii $a_1$ and $a_2$. The centre distance (i.e. the distance between the pulley centres) is $2a_0 - a_1 - a_2 + \delta$, where the pulley displacement $\delta$ increases from zero under force $P$ that takes the pulleys apart.

Because of the problem symmetry it is sufficient to consider only the upper half of the belt $0 \leq s \leq L = \pi a_0$. In the segment $0 \leq s \leq s_1$ the full contact is assumed, however the coordinate $s_1$ is unknown. The formulation for this segment is presented above. For the segment of contact with another pulley we have $s_2 \leq s \leq L$, $s_2$ is also unknown. We will use upper indices: $(...)^{(1)}$ for the values in the left contact segment and $(...)^{(3)}$ for the values in the right contact segment. We assign the spatial coordinate origin by the equation $\sigma^{(3)} = 0$, inside the segment $\sigma^{(3)}$ is negative. Similarly to (3) we write

$$\dot{R} = -i \sin \kappa_2 \sigma^{(3)} + j \cos \kappa_2 \sigma^{(3)}, \quad \dot{r} = D^{(3)} \dot{R};$$

$$\dot{R} = i \left[ 2R + \delta - \kappa_2^{-1} \left( 1 - \cos \kappa_2 \sigma^{(3)} \right) \right] - j \kappa_2^{-1} \sin \kappa_2 \sigma^{(3)}. \quad (11)$$

We make use of the formula (11) as the boundary condition below. Also we transform the equation (4) to the form:

$$\dot{r} \cdot e_1 = 1 + B_1 Q_1^{(3)} = -D^{(3)} \sin \left( \varphi^{(3)} + \kappa_2 \sigma^{(3)} \right)$$

$$\dot{r} \cdot e_2 = B_2 Q_2^{(3)} = -D^{(3)} \cos \left( \varphi^{(3)} + \kappa_2 \sigma^{(3)} \right). \quad (12)$$

This gives four equations for the segment similar to (9).

Let us turn to the free segment $s_1 < s < s_2$. As follows from the force balance equation in (1), the force $Q$ is constant. Using the symmetry we can determine this force:

$$Q = \frac{P}{2} i + Q_y j. \quad (13)$$

The value $P$ is prescribed, however the dependence $\delta(P)$ should be determined. $Q_y$ is the unknown constant. Due to the obvious absence of lumped contact reaction at point $s_1$ (and at $s_2$ also) we have the continuity of $Q$:

$$Q_1 = \frac{P}{2} \cos \varphi + Q_y \sin \varphi,$$

$$Q_2 = -\frac{P}{2} \sin \varphi + Q_y \cos \varphi \quad (14)$$
(it also holds everywhere in the free segment). Because the force is constant, we are able to
integrate the equation of moments (1):

\[ M = \frac{P}{2} y - Q_y x + M_s. \]  

(15)

We use the continuity of moment at points \( s_1, s_2 \) in order to determine the constants \( M_s, Q_y \).

Then we derive the following equations from (1):

\[ \varphi' = -a_0^{-1} + AM, \quad x' = (1 + B_1Q_1) \cos \varphi - B_2Q_2 \sin \varphi, \quad y' = (1 + B_1Q_1) \sin \varphi + B_2Q_2 \cos \varphi. \]  

(16)

The functions \( \varphi, x, y \) are continuous at points \( s_1, s_2 \).

To solve the formulated problem as a single boundary value problem, we introduce the new
nondimensional coordinate \( \xi \):

\[ s \leq s_1 : \quad s = \xi s_1 \Rightarrow \frac{d}{d\xi} = s_1 \frac{d}{ds}; \]

\[ s_1 \leq s \leq s_2 : \quad s = s_1 + \xi(s_2 - s_1) \Rightarrow \frac{d}{d\xi} = (s_2 - s_1) \frac{d}{ds}; \]

\[ s_2 \leq s \leq L : \quad s = L - \xi(L - s_2) \Rightarrow \frac{d}{d\xi} = (s_2 - L) \frac{d}{ds}. \]  

(17)

We combine the equations at both segments into a single system and distinguish the values by
the indices \((...)^{(1)}, (...)^{(2)}, (...)^{(3)}\). We note that this transformation extends the formulation for
the problem of laying process of underwater pipeline \([11]\) where contact of an initially straight
beam with a flat rigid surface is considered. Thus we obtain the boundary value problem for
the fifteenth order ODE system:

\[ Z' = G(\xi, Z), \quad Z(\xi) \equiv \left( \sigma^{(1)} \varphi^{(1)} M^{(1)} Q_1^{(1)} \sigma^{(3)} \varphi^{(3)} M^{(3)} Q_1^{(3)} \varphi^{(2)} x y M_s s_1 s_2 Q_y \right)^T. \]  

(18)

Here a prime indicates the derivative with respect to the new coordinate \( \xi \) (and the functions
written in (18) are the functions of \( \xi \)). We must not forget the relations between the derivatives
(17). For the first eight unknown variables we have already derived the equations of type (9).

The remaining seven equations are equations (16) and conditions for the unknown constants:
\( M_s', s_1', s_2', Q_y' = 0 \).

Now, we consider the boundary conditions for the system (18). We note that the right end
\( \xi = 1 \) corresponds to the unknown boundary of the contact segment \( s = s_1 \) for the first segment
and \( s = s_2 \) for the second and third segments whereas the left end \( \xi = 0 \) corresponds to the start
point \( s = 0 \), to the unknown boundary \( s = s_1 \), and to the end point \( s = L \) for three segments
sequentially. For the points with material coordinate \( s = 0 \) and \( s = s_2 \) the conditions are

\[ \xi = 0 : \quad \sigma^{(1)} = 0, \quad \sigma^{(3)} = 0, \quad \varphi^{(1)} = \frac{\pi}{2}, \quad \varphi^{(3)} = -\frac{\pi}{2}. \]

\[ \xi = 1 : \quad \varphi^{(2)} = \varphi^{(3)}, \quad Q_1^{(3)} = \frac{P}{2} \cos \varphi^{(2)} + Q_y \sin \varphi^{(2)}, \quad -a_2 \sin \kappa \sigma^{(3)} = y; \]

\[ Q_2 \left( \sigma^{(3)}, \varphi^{(3)} Q_1^{(3)} \right) = \frac{P}{2} \sin \varphi^{(2)} + Q_y \cos \varphi^{(2)}, \quad M^{(3)} = \frac{P}{2} y - Q_y x + M_s. \]  

(19)

And for \( s = s_1 \) the conditions are

\[ s = s_1 : \quad \varphi^{(1)} \bigg|_{\xi=1} = \varphi^{(2)} \bigg|_{\xi=0}, \quad M^{(1)} \bigg|_{\xi=1} = \left( \frac{P}{2} y - Q_y x + M_s \right) \bigg|_{\xi=0}. \]  

5
\[
\begin{align*}
a_1 \left( 1 - \cos \kappa_1 \sigma^{(1)} \right) \bigg|_{\xi = 1} &= x \bigg|_{\xi = 0}, \quad a_1 \sin \kappa_1 \sigma^{(1)} \bigg|_{\xi = 1} = y \bigg|_{\xi = 0}, \\
Q_1^{(1)} \bigg|_{\xi = 1} &= \left( \frac{P}{2} \cos \varphi^{(2)} + Q_y \sin \varphi^{(2)} \right) \bigg|_{\xi = 0}, \\
Q_2 \left( \varphi^{(1)}, \sigma^{(1)}, Q_1^{(1)} \right) \bigg|_{\xi = 1} &= \left( -\frac{P}{2} \sin \varphi^{(2)} + Q_y \cos \varphi^{(2)} \right) \bigg|_{\xi = 0}.
\end{align*}
\]

(20)

4. Numerical results

The formulated nonlinear boundary value problem of fifteenth order (18), (19), (20) is solved by the shooting method in Wolfram Mathematica using the standard built-in function \texttt{NDSolve}. Also we use the finite difference method for this system following the scheme described in [11]. The results of both methods agree.

In the finite difference method we divide the segments into \( N \) uniform intervals with the step \( \varepsilon = 1/N \) (with respect to the transformed coordinate \( \xi \)). To approximate the system of equations we use implicit symmetric one-step difference scheme that has second order of accuracy [29, 30, 31]:

\[
\frac{Z_{i+1} - Z_i}{\varepsilon} = \frac{(G_i + G_{i+1})}{2}, \quad (i = 0, \ldots, N - 1).
\]

As a result of transformations we obtain the system of \( 11N \) nonlinear algebraic equations. Total number of unknowns consists of the values of eleven desired functions at the nodes \( 11(N + 1) \) and four additional constants \( M_s, s_1, s_2, Q_y \). The system of equations is complemented with the fifteen boundary conditions (in our case the Dirichlet boundary conditions) projected into mesh. Furthermore we apply the Newton method using built-in function \texttt{FindRoot} (Mathematica). We use our solution of the problem with equal pulleys [28] as an initial guess in order to begin the computations performed with the numerical continuation method.

The form of the belt before and after the deformation is shown in Fig. 3. The parameters are: the Young modulus is \( E = 10^9 \), the Poisson coefficient is \( \nu = 0.5 \), the sides of the square cross-section are \( h = 10^{-2} \), the initial radius of the belt is \( a_0 = 0.25 \), the pulley radii are \( a_1 = 0.15, \ a_2 = 0.1 \), the current force is \( P = 200 \) (all values are in SI units). The results of computations are: the end coordinate of the first contact segment \( s_1 = 0.163 \), the second contact segment has the length \( L - s_2 = 0.0585 \), the displacement of the pulley \( \delta = 0.132 \). The central angles of the contact zones are introduced in Fig. 2, their values are \( \beta_1 = 1.088, \ \beta_2 = 0.586 \) in the example considered.

Then we determine the contact pressure using the formula (10). We take into account equation (6) and rewrite it as follows:

\[
p = D^{-1} \left( Q_1' - \varphi'Q_2 \right) \left[ B_2Q_2 + \left( 1 + B_1Q_1 \right)^2 \right].
\]

(21)
The result of calculation is shown in Fig. 4. The pressure is positive. We see the concentration of pressure at the boundary of the contact segment which is characteristic for the problems with shear [11]. For the unshearable models we have the constant pressure distribution with the lumped contact reaction at the contact boundary [5, 6]. In the present case there is no lumped reaction force at the boundary. The right contact segment is smaller, therefore the pressure distribution is smoother in it. The maximal values of pressure are $p^{(1)} = 10574$, $p^{(2)} = 14879$.

Finally we demonstrate the loading diagrams. The load $P$ increases from zero while other parameters are the same as above. We solve a number of static problems with the numerical continuation method to show the dependences. The calculations were performed until the contact segments reduce itself to a point.

The dependence of force $P(\delta)$ on the pulley displacement is depicted in Fig. 5. It is nonlinear from the beginning, its behaviour is similar to the one obtained for problem with equal pulleys.

The dependence $\beta(\delta)$ of the angle of contact zone on displacement of pulley is shown in Fig. 6. The angle of the left contact zone is wider than the angle of the right contact zone as expected.

5. Conclusion
The main results of the present work are listed below:

- a system of ODE for the belt as plane nonlinear elastic rod with account for tension and...
transverse shear is derived;
• the equations for the free and contact segments which have qualitatively different
formulation are combined in a single system of ODE. This transformation is done in a
novel way by introducing the new coordinate;
• the formulated boundary value problem is suitable for numerical solving by the shooting
method and finite difference method;
• computer mathematics is utilized for determining all the variables including the coordinates
of boundaries of contact areas and contact pressure;
• the effect of different pulley radii on the belt-pulley interaction problem is studied in detail.

These results will be used in further work including the study of belt-pulley interaction with
friction and the dynamical modeling, both stationary and transient.

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