Quantum immanants, double Young-Capelli bitableaux
and
Schur shifted symmetric functions

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Abstract

In this paper are introduced two classes of elements in the enveloping algebra $U(gl(n))$: the double Young-Capelli bitableaux $\left[ \begin{array}{c|c} S & T \end{array} \right]$ and the central Schur elements $S_\lambda(n)$, that act in a remarkable way on the highest weight vectors of irreducible Schur modules.

Any element $S_\lambda(n)$ is the sum of all double Young-Capelli bitableaux $\left[ \begin{array}{c|c} S & S \end{array} \right]$, $S$ row (strictly) increasing Young tableaux of shape $\tilde{\lambda}$. The Schur elements $S_\lambda(n)$ are proved to be the preimages - with respect to the Harish-Chandra isomorphism - of the shifted Schur polynomials $s_{\lambda|n}^{\ast} \in \Lambda^\ast(n)$. Hence, the Schur elements are the same as the Okounkov quantum immanants, recently described by the present authors as linear combinations of Capelli immanants. This new presentation of Schur elements/quantum immanants doesn't involve the irreducible characters of symmetric groups. The Capelli elements $H_k(n)$ are column Schur elements and the Nazarov elements $I_k(n)$ are row Schur elements. The duality in $\zeta(n)$ follows from a combinatorial description of the eigenvalues of the $H_k(n)$ on irreducible modules that is dual (in the sense of shapes/partitions) to the combinatorial description of the eigenvalues of the $I_k(n)$.

The passage $n \to \infty$ for the algebras $\zeta(n)$ is obtained both as direct and inverse limit in the category of filtered algebras, via the Olshanski decomposition/projection.

Keyword: Combinatorial representation theory; shifted symmetric functions; superalgebras; central elements in $U(gl(n))$; Capelli identities; superstandard Young tableaux; Schur supermodules.

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7 The algebra $\Lambda^*$ of shifted symmetric functions

1 Introduction

The study of the center $\zeta(n)$ of the enveloping algebra $U(gl(n))$ of the general linear Lie algebra $gl(n, \mathbb{C})$ and the study of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials have noble and rather independent origins and motivations.

The theme of central elements in $U(gl(n))$ is a standard one in the general theory of Lie algebras, see e.g. [28]. It is an old and actual one, since it is an offspring of the celebrated Capelli identity (see, e.g. [19], [22], [33], [30], [54], [62], [64]), relates to its modern generalizations and applications (see, e.g. [3], [39], [10], [16], [17], [19], [50], [58], [61]) as well as to the theory of Yangians (see, e.g. [44], [45], [48]).

The algebra $\Lambda^*(n)$ of shifted symmetric polynomials is a remarkable deformation of the algebra $\Lambda(n)$ of symmetric polynomials and its study fits into the mainstream of generalizations of the classical theory (see, e.g. factorial symmetric functions, [6], [7], [23], [32], [33], [42], [43]).

Since the algebras $\zeta(n)$ and $\Lambda^*(n)$ are related by the Harish-Chandra isomorphism $\chi_n$ (see, e.g. [51]), their investigation can be essentially regarded as a single topic, and this fact gave rise to a fruitful interplay between representation-theoretic methods (e.g., eigenvalues on irreducible representations) and combinatorial techniques (e.g., generating functions).

In this work, we propose a new approach to a systematic study of some of the main features of the theory of the center $\zeta(n)$ of $U(gl(n))$ and of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials that allows the whole theory to be developed, in a transparent and concise way, from a combinatorial representation theoretic point of view, that is entirely in the center $\zeta(n)$.

The paper is organized as follows.

In Section 2 we provide a synthetic presentation of the superalgebraic method of virtual variables for $gl(n)$. For details, we refer the reader to [9], [17] and [18]. This method was developed by the present authors for the general linear Lie superalgebras $gl(m|n)$ (see, e.g. [37], [59]) in the series of notes [8], [9], [11], [12], [13], [14], [16]. The technique of virtual variables is an extension of Capelli’s method of variabili ausiliarie (Capelli [22], see also Weyl [64]).

The superalgebraic method of virtual variables allows us to express remarkable classes of elements in $U(gl(n))$, namely,

- the Capelli bitableaux $[S|T] \in U(gl(n))$
- the Young-Capelli bitableaux $[S \ T] \in U(gl(n))$
- the double Young-Capelli bitableaux $[S \ | \ T] \in U(gl(n))$

as the images - with respect to the $Ad_{gl(n)}$-adjoint equivariant Capelli devirtualization epimorphism - of simple expressions in an enveloping superalgebra $U(gl(m_0|m_1+n))$. 

3
Capelli (determinantal) bitableaux are generalizations of the famous determinantal elements in $U(gl(n))$ introduced by Capelli in 1887 [19] (see, e.g. [18]). Young-Capelli bitableaux were introduced by the present authors several years ago [12], [13], [14] and might be regarded as generalizations of the Capelli determinantal elements in $U(gl(n))$ as well as of the Young symmetrizers of the classical representation theory of symmetric groups (see, e.g. [64]). Double Young-Capelli bitableaux are essentially new and play a crucial role in the present paper.

In plain words, the Young-Capelli bitableau $[S \mid T]$ is obtained by adding a column symmetrization to the Capelli bitableau $[S \mid T]$ (Proposition 2.13) and the double Young-Capelli bitableau $[S \mid T]$ is obtained by adding a further row skew-symmetrization to the Young-Capelli bitableau $[S \mid T]$ (Proposition 2.14).

In Section 3, we regard the supersymmetric superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as $U(gl(m_0|m_1+n))$-module and define the Schur supermodules $\text{Schur}_\lambda(m_0, m_1+n)$ as (irreducible) submodules of $\mathbb{C}[M_{m_0|m_1+n,d}]$. Schur supermodules are isomorphic to the modules constructed by Berele and Regev [4], [5] as tensor modules induced by Young symmetrizers (see, e.g. [64]) when they act by a “signed action” of the symmetric group (see also [38]). The description presented here is simpler than the tensor description, and provides a close connection with the superstraightening theory of Grosshans, Rota and Stein [34]. The classical Schur modules $\text{Schur}_\lambda(n) - gl(n)$-irreducible modules with (nonnegative) integral highest weights - are here regarded as $gl(n)$-submodules of suitable Schur supermodules. The crucial and new result is that double Young-Capelli bitableaux act in a remarkable way on the highest weight vectors of Schur modules (subsection 3.4, Theorem 3.9).

In Section 4, we consider a class of central elements $S_\lambda(n)\leq n$, which arise in a natural way in the context of the virtual method when dealing with symmetry and skew-symmetry in $U(gl(n))$.

These elements are expressed as linear combinations

$$S_\lambda(n) = \frac{1}{H(\lambda)} \sum_S [S \mid S] \in U(gl(n))$$

of double Young-Capelli bitableaux where the sum is extended to all row (strictly) increasing tableaux $S$ of shape $sh(S) = \lambda$.

We call the elements $S_\lambda(n)$ the Schur elements.

The main results are Theorems 4.2 and 4.3 that follow from Theorem 3.9 and provide notable descriptions of the action of the central Schur elements $S_\lambda(n)$ on the highest weight vectors of Schur modules. Theorem 4.2 implies that the set of the elements $S_\lambda(n)$, $\lambda_1 \leq n$ is a basis of the center $\zeta(n)$.

By combining Theorem 4.2 with the Sahi-Okounkov characterization Theorem (57), [51], [49], here quoted as Proposition 4.6, we infer that the Schur elements $S_\lambda(n)$ are the preimages - with respect to the Harish-Chandra isomorphism - of the shifted Schur polynomials $s^{\lambda|n}_\lambda \in \Lambda^*(n)$ [50], [51]. Hence,
the Schur elements are the same as the quantum immanants, first presented by Okounkov as traces of fusion matrices (19, 50) and, recently, described by the present authors as linear combinations (with explicit coefficients) of “diagonal” Capelli immanants, see [17]. Presentation (11) of Schur elements/quantum immanants doesn’t involve the irreducible characters of symmetric groups. Furthermore, it is better suited to the study of the duality in the algebra \( \zeta(n) \) as well as to the study of the limit \( n \to \infty \).

We examine two further classes of central elements, namely, the determinantal Capelli elements \( H_k(n) \), \( k = 1, 2, \ldots, n \) (see, e.g. 19, 20, 21, 9), and the permanental Nazarov elements \( I_k(n) \), \( k \in \mathbb{Z}^+ \) (see, e.g. 47, 48, 61, 63, see also 9), which provide two systems of algebra generators of the center \( \zeta(n) \).

The Capelli elements \( H_k(n) \) are column Schur elements, specifically,

\[
H_k(n) = S_{(1^k)}(n).
\]

The Nazarov elements \( I_k(n) \) are row Schur elements, specifically,

\[
I_k(n) = S_{(k)}(n).
\]

The duality in \( \zeta(n) \) (Theorem 4.27) immediately follows from a combinatorial description of the eigenvalues of the Capelli elements \( H_k(n) \) on irreducible Schur modules (Proposition 4.15) that is dual (in the sense of shapes/partitions) to the combinatorial description of the eigenvalues of the Nazarov elements \( I_k(n) \) (Theorem 1.29).

The passage to the infinite dimensional case \( n \to \infty \) for the algebras \( \zeta(n) \) is rather subtle; the “naive” \( \infty \)-dimensional analogue of the algebras \( U(gl(n)) \), that is the direct limit algebra \( \lim_{\to} U(gl(n)) \) with respect to the “inclusion” monomorphisms, has trivial center. In Section 5 the \( \infty \)-dimensional analogue \( \zeta \) of the algebras \( \zeta(n) \) is obtained as the direct limit algebra \( \lim_{\to} \zeta(n) \) (in the category of filtered algebras) with respect to the family of monomorphisms \( i_{n+1,n} : \zeta(n) \hookrightarrow \zeta(n+1) \), where

\[
i_{n+1,n}(H_k(n)) = H_k(n+1), \quad k = 1, 2, \ldots, n.
\]

An intrinsic/invariant presentation of the monomorphisms \( i_{n+1,n} \) is obtained, in subsection 5.2 via the Olshanski projections \( \mu_{n,n+1} : \zeta(n+1) \to \zeta(n) \) (see also Molev [41]). The Olshanski projections \( \mu_{n,n+1} \) are left inverses of the monomorphisms \( i_{n+1,n} \), and they become two-sided inverses when restricted to the filtration elements \( \zeta(n+1)(m) \) and \( \zeta(n)(m) \), for \( n \) sufficiently large (Propositions 5.2 and Proposition 5.3). The interplay between the monomorphisms \( i_{n+1,n} \) and the projections \( \mu_{n,n+1} \) shows the algebra \( \zeta \) admits a double presentation, both as a direct limit and as an inverse limit.

Since the Olshanski projection \( \mu_{n,n+1} \) maps \( H_k(n+1) \) to \( H_k(n) \), \( I_k(n+1) \) to \( I_k(n) \) and \( S_\lambda(n+1) \) to \( S_\lambda(n) \), then

\[
i_{n+1,n}(H_k(n)) = H_k(n+1), \quad i_{n+1,n}(I_k(n)) = I_k(n+1),
\]
and
\[ i_{n+1,n}(S_\lambda(n)) = S_\lambda(n+1), \]
for \( n \) sufficiently large.

Hence, the direct limits
\[ H_k \overset{\text{def}}{=} \lim_{n \to \infty} H_k(n) \in \zeta \]
and
\[ I_k \overset{\text{def}}{=} \lim_{n \to \infty} I_k(n) \in \zeta \]
and
\[ S_\lambda \overset{\text{def}}{=} \lim_{n \to \infty} S_\lambda(n) \in \zeta \]
can be consistently written as formal series by naturally extending to infinite sums the finite sums (eqs. (3), (30), (33)) that define \( H_k(n), I_k(n) \) and \( S_\lambda(n) \), respectively.

The algebra \( \zeta \) is isomorphic to the algebra \( \Lambda^* \) of shifted symmetric functions (Theorem 7.2). The Olshanski projections are the natural counterpart, in the context of the centers \( \zeta(n) \), of the Okounkov-Olshanski stability principle for the algebras \( \Lambda^*(n) \) of shifted symmetric polynomials \[52\], the isomorphism \( \chi : \zeta \to \Lambda^* \) is indeed the “limit” of the Harish-Chandra isomorphisms \( \chi_n : \zeta(n) \to \Lambda^*(n) \) and it admits a transparent representation-theoretic interpretation (Sections 6 and 7).

2 A glimpse on the superalgebraic method of virtual variables

2.1 The superalgebras \( gl(m_0|m_1 + n) \) and \( C[M_{m_0|m_1+n,d}] \)

2.1.1 The general linear Lie super algebra \( gl(m_0|m_1 + n) \)

Given a vector space \( V_n \) of dimension \( n \), we will regard it as a subspace of a \( \mathbb{Z}_2 \)-graded vector space \( W = W_0 \oplus W_1 \), where

\[ W_0 = V_{m_0}, \quad W_1 = V_{m_1} \oplus V_n. \]
The vector spaces \( V_{m_0} \) and \( V_{m_1} \) (we assume that \( \dim(V_{m_0}) = m_0 \) and \( \dim(V_{m_1}) = m_1 \) are “sufficiently large”) are called the positive virtual (auxiliary) vector space, the negative virtual (auxiliary) vector space, respectively, and \( V_n \) is called the (negative) proper vector space.

The inclusion \( V_n \subset W \) induces a natural embedding of the ordinary general linear Lie algebra \( gl(n) \) of \( V_n \) into the auxiliary general linear Lie superalgebra \( gl(m_0|m_1 + n) \) of \( W = W_0 \oplus W_1 \) (see, e.g. \[37\], \[59\]).

Let \( \mathcal{A}_0 = \{ \alpha_1, \ldots, \alpha_{m_0} \}, \mathcal{A}_1 = \{ \beta_1, \ldots, \beta_{m_1} \}, \mathcal{L} = \{ 1, 2, \ldots, n \} \) denote fixed bases of \( V_{m_0}, V_{m_1} \) and \( V_n \), respectively; therefore \( |\alpha_s| = 0 \in \mathbb{Z}_2 \), and \( |\beta_t| = 1 \in \mathbb{Z}_2 \).

Let
\[ \{ e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L} \}, \quad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2 \]
be the standard \( \mathbb{Z}_2 \)-homogeneous basis of the Lie superalgebra \( gl(m_0|m_1 + n) \) provided by the elementary matrices. The elements \( e_{a,b} \in gl(m_0|m_1 + n) \) are \( \mathbb{Z}_2 \)-homogeneous of \( \mathbb{Z}_2 \)-degree \( |a| + |b| \).

The superbracket of the Lie superalgebra \( gl(m_0|m_1 + n) \) has the following explicit form:

\[
[e_{a,b}, e_{c,d}] = \delta_{bc} e_{a,d} - (-1)^{(|a| + |b|)(|c| + |d|)} \delta_{ad} e_{c,b},
\]

\( a, b, c, d \in A_0 \cup A_1 \cup L \).

In the following, the elements of the sets \( A_0, A_1, L \) will be called positive virtual symbols, negative virtual symbols and negative proper symbols, respectively.

### 2.1.2 The supersymmetric algebra \( \mathbb{C}[M_{m_0|m_1+n,d}] \)

Let

\[
\mathbb{C}[M_{n,d}] = \mathbb{C}[(i[j])_{i=1,...,n,j=1,...,d}
\]

be the polynomial algebra in the (commutative) entries \((i[j])\) of the matrix:

\[
M_{n,d} = [(i[j])_{i=1,...,n,j=1,...,d} = \begin{pmatrix}
(1|1) & \cdots & (1|d) \\
\vdots & & \vdots \\
(n|1) & \cdots & (n|d)
\end{pmatrix}.
\]

We regard the commutative algebra \( \mathbb{C}[M_{n,d}] \) as a subalgebra of the “auxiliary” supersymmetric algebra

\[
\mathbb{C}[M_{m_0|m_1+n,d}]
\]
generated by the \((\mathbb{Z}_2\text{-graded})\) variables

\( (a|j), \quad a \in A_0 \cup A_1 \cup L, \quad j \in \mathcal{P} = \{j = 1, \ldots, d; |j| = 1 \in \mathbb{Z}_2\}, \)

with \(|(a|j)| = |a| + |j| \in \mathbb{Z}_2\), subject to the commutation relations:

\[
(a|h)(b|k) = (-1)^{|(a|h)||b|k)} (b|k)(a|h).
\]

In plain words, \( \mathbb{C}[M_{m_0|m_1+n,d}] \) is the free supersymmetric algebra

\[
\mathbb{C}[(\alpha_s|j), (\beta_i|j), (i|j)]
\]
generated by the \((\mathbb{Z}_2\text{-graded})\) variables \( (\alpha_s|j), (\beta_i|j), (i|j)\), \( j = 1, 2, \ldots, d \), where all the variables commute each other, with the exception of pairs of variables \((\alpha_s|j), (\alpha_t|j)\) that skew-commute:

\[
(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).
\]

In the standard notation of multilinear algebra, we have:

\[
\mathbb{C}[M_{m_0|m_1+n,d}] \cong \Lambda[W_0 \otimes P_d] \otimes \text{Sym}[W_1 \otimes P_d]
\]

\[
= \Lambda[V_{m_0} \otimes P_d] \otimes \text{Sym}[(V_{m_1} \oplus V_n) \otimes P_d]
\]

where \( P_d = (P_d)_1 \) denotes the trivially \( \mathbb{Z}_2 \)-graded vector space with distinguished basis \( \mathcal{P} = \{j = 1, \ldots, d; |j| = 1 \in \mathbb{Z}_2\} \).
2.1.3 Left superderivations and left superpolarizations

A left superderivation \( D^l \) \((\mathbb{Z}_2\text{-homogeneous of degree } |D^l|)\) (see, e.g. [59], [37]) on \( \mathbb{C}[M_{m_0|m_1+n,n}] \) is an element of the superalgebra \( \text{End}_\mathbb{C}[\mathbb{C}[M_{m_0|m_1+n,n}]] \) that satisfies "Leibniz rule"

\[
D^l(p \cdot q) = D^l(p) \cdot q + (-1)^{|D^l||p|} p \cdot D^l(q),
\]

for every \( \mathbb{Z}_2\text{-homogeneous of degree } |p| \) element \( p \in \mathbb{C}[M_{m_0|m_1+n,n}] \).

Given two symbols \( a, b \in A_0 \cup A_1 \cup \mathcal{L} \), the left superpolarization \( D^l_{a,b} \) of \( b \) to \( a \) is the unique left superderivation of \( \mathbb{C}[M_{m_0|m_1+n,n}] \) of \( \mathbb{Z}_2\text{-degree } |D^l_{a,b}| = |a| + |b| \) in \( \mathbb{Z}_2 \) such that

\[
D^l_{a,b}((c|j)) = \delta_{bc} (a|j), \quad c \in A_0 \cup A_1 \cup \mathcal{L}, \quad j = 1, \ldots, n.
\]

Informally, we say that the operator \( D^l_{a,b} \) annihilates the symbol \( b \) and creates the symbol \( a \).

2.1.4 The superalgebra \( \mathbb{C}[M_{m_0|m_1+n,n}] \) as a \( \mathbb{U}(gl(m_0|m_1+n)) \)-module

Since

\[
D^l_{a,b}D^l_{c,d} - (-1)^{(|a|+|b|)(|c|+|d|)} D^l_{c,d}D^l_{a,b} = \delta_{b,c} D^l_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{a,d} D^l_{c,b},
\]

the map

\[
e_{a,b} \mapsto D^l_{a,b}, \quad a, b \in A_0 \cup A_1 \cup \mathcal{L}
\]

is a Lie superalgebra morphism from \( gl(m_0|m_1+n) \) to \( \text{End}_\mathbb{C}[\mathbb{C}[M_{m_0|m_1+n,n}]] \) and, hence, it uniquely defines a representation:

\[
\varrho : \mathbb{U}(gl(m_0|m_1+n)) \to \text{End}_\mathbb{C}[\mathbb{C}[M_{m_0|m_1+n,n}]],
\]

where \( \mathbb{U}(gl(m_0|m_1+n)) \) is the enveloping superalgebra of \( gl(m_0|m_1+n) \).

In the following, we always regard the superalgebra \( \mathbb{C}[M_{m_0|m_1+n,n}] \) as a \( \mathbb{U}(gl(m_0|m_1+n)) \)-supermodule, with respect to the action induced by the representation \( \varrho \):

\[
e_{a,b} \cdot p = D^l_{a,b}(p),
\]

for every \( p \in \mathbb{C}[M_{m_0|m_1+n,n}] \).

We recall that \( \mathbb{U}(gl(m_0|m_1+n)) \)-module \( \mathbb{C}[M_{m_0|m_1+n,n}] \) is a semisimple module, whose simple submodules are - up to isomorphism - Schur supermodules (see, e.g. [11], [12], [8]. For a more traditional presentation, see also [24]).

Clearly, \( \mathbb{U}(gl(0|n)) = \mathbb{U}(gl(n)) \) is a subalgebra of \( \mathbb{U}(gl(m_0|m_1+n)) \) and the subalgebra \( \mathbb{C}[M_{n,d}] \) is a \( \mathbb{U}(gl(n)) \)-submodule of \( \mathbb{C}[M_{m_0|m_1+n,n}] \).
2.1.5 The virtual algebra $\text{Virt}(m_0 + m_1, n)$ and the virtual presentations of elements in $U(gl(n))$

We say that a product

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in U(gl(m_0|n)),$$ 

is an \textit{irregular expression} whenever there exists a right subword

$$e_{a_i b_i} \cdots e_{a_2 b_2} e_{a_1 b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\# \{ j ; b_j = \gamma , j \leq i \} > \# \{ j ; a_j = \gamma , j < i \}. \quad (2)$$

The meaning of an irregular expression in terms of the action of $U(gl(m_0|n))$ by left superpolarization on the algebra $C[M_{m_0|n,d}]$ is that there exists a virtual symbol $\gamma$ and a right subsequence in which the symbol $\gamma$ is \textit{annihilated} more times than it was already \textit{created} and, therefore, the action of an irregular expression on the algebra $C[M_{n,d}]$ is zero.

\textbf{Example 2.1.} Let $\gamma \in A_0 \cup A_1$ and $x_i, x_j \in L$. The product

$$e_{\gamma, x_j} e_{x_i, \gamma} e_{\gamma, x_i}$$

is an irregular expression.

Let $\text{Irr}$ be the \textit{left ideal} of $U(gl(m_0|n))$ generated by the set of irregular expressions.

\textbf{Proposition 2.2.} The superpolarization action of any element of $\text{Irr}$ on the subalgebra $C[M_{n,d}] \subset C[M_{m_0|m_1+n,d}]$ - via the representation $\varphi$ - is identically zero.

\textbf{Proposition 2.3.} The sum $U(gl(0|n)) + \text{Irr}$ is a direct sum of vector subspaces of $U(gl(m_0|n))$.

\textbf{Proposition 2.4.} The direct sum vector subspace $U(gl(0|n)) \oplus \text{Irr}$ is a subalgebra of $U(gl(m_0|n))$.

The subalgebra

$$\text{Virt}(m_0 + m_1, n) = U(gl(0|n)) \oplus \text{Irr} \subset U(gl(m_0|m_1 + n)).$$

is called the \textit{virtual algebra}.

\textbf{Proposition 2.5.} The left ideal $\text{Irr}$ of $U(gl(m_0|n))$ is a two sided ideal of $\text{Virt}(m_0 + m_1, n)$. 


The Capelli devirtualization epimorphism is the surjection

\[ p : \text{Virt}(m_0 + m_1, n) = U(gl(0|n)) \oplus \mathbb{Irr} \rightarrow U(gl(0|n)) = U(gl(n)) \]

with \( \text{Ker}(p) = \mathbb{Irr} \).

Any element in \( M \in \text{Virt}(m_0 + m_1, n) \) defines an element in \( m \in U(gl(n)) \) via the map \( p \) - and \( M \) is called a virtual presentation of \( m \).

Furthermore,

**Proposition 2.6.** The subalgebra \( \mathbb{C}[M_{n,d}] \subseteq \mathbb{C}[M_{m_0|m_1+n,d}] \) is invariant with respect to the action of the subalgebra \( \text{Virt}(m_0 + m_1, n) \).

**Proposition 2.7.** For every element \( m \in U(gl(n)) \), the action of \( m \) on the subalgebra \( \mathbb{C}[M_{n,d}] \) is the same of the action of any of its virtual presentation \( M \in \text{Virt}(m_0 + m_1, n) \). In symbols,

\[ \text{if } p(M) = m \text{ then } m \cdot P = M \cdot P, \text{ for every } P \in \mathbb{C}[M_{n,d}]. \]

Since the map \( p \) a surjection, any element \( m \in U(gl(n)) \) admits several virtual presentations. In the sequel, we even take virtual presentations as the definition of special elements in \( U(gl(n)) \), and this method will turn out to be quite effective.

The superalgebra \( U(gl(m_0|m_1 + n)) \) is a Lie module with respect to the adjoint representation \( Ad_{gl(m_0|m_1+n)} \). Since \( gl(n) = gl(0|n) \) is a Lie subalgebra of \( gl(m_0|m_1 + n) \), then \( U(gl(m_0|m_1 + n)) \) is a \( gl(n) \)-module with respect to the adjoint action \( Ad_{gl(n)} \) of \( gl(n) \). We recall a couple of results from [18].

**Proposition 2.8.** The virtual algebra \( \text{Virt}(m_0 + m_1, n) \) is a submodule of \( U(gl(m_0|m_1 + n)) \) with respect to the adjoint action \( Ad_{gl(n)} \) of \( gl(n) \).

**Proposition 2.9.** The Capelli epimorphism

\[ p : \text{Virt}(m_0 + m_1, n) \rightarrow U(gl(n)) \]

is an \( Ad_{gl(n)} \)-equivariant map.

**Corollary 2.10.** The isomorphism \( p \) maps any \( Ad_{gl(n)} \)-invariant element \( m \in \text{Virt}(m_0 + m_1, n) \) to a central element of \( U(gl(n)) \).

Balanced monomials are elements of the algebra \( U(gl(m_0|m_1 + n)) \) of the form:

- \( e_{i_1, j_1} \cdots e_{i_k, j_k} \gamma_{p_1, j_1} \cdots e_{p_k, j_k} \gamma_{p_k, j_k} \)
- \( e_{i_1, j_1} \cdots e_{i_k, j_k} \theta_{q_1, j_1} \cdots e_{q_k, j_k} \gamma_{p_1, j_1} \cdots e_{p_k, j_k} \gamma_{p_k, j_k} \)
- and so on,

where \( i_1, \ldots, i_k, j_1, \ldots, j_k \in L \), i.e., the \( i_1, \ldots, i_k, j_1, \ldots, j_k \) are \( k \) proper (negative) symbols, and the \( \gamma_{p_1, j_1}, \ldots, \gamma_{p_k, j_k}, \theta_{q_1, j_1}, \ldots, \theta_{q_k, j_k} \) are virtual symbols. In plain words, a balanced monomial is product of two or more factors where the
rightmost one annihilates (by superpolarization) the $k$ proper symbols $j_1, \ldots, j_k$ and creates (by superpolarization) some virtual symbols; the leftmost one annihilates all the virtual symbols and creates the $k$ proper symbols $i_1, \ldots, i_k$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

**Proposition 2.11.** Every balanced monomial belongs to $\text{Virt}(m_0 + m_1, n)$. Hence, the Capelli epimorphism $p$ maps balanced monomials to elements of $U(gl(n))$.

### 2.2 Four special classes of elements in $\text{Virt}(m_0 + m_1, n)$ and their images in $U(gl(n))$

We will introduce four classes of remarkable elements of the enveloping algebra $U(gl(n))$, that we call bitableaux monomials, Capelli bitableaux, Young-Capelli bitableaux and double Young-Capelli bitableaux, respectively.

#### 2.2.1 Partitions and Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p) \vdash h$ be a partition of the positive integer $h \in \mathbb{Z}^+$, where $p = l(\lambda)$ is the length of $\lambda$.

We denote by $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_q)$ the conjugate partition of $\lambda$, that is $$\tilde{\lambda}_i = \#\{j = 1, 2, \ldots, p; \lambda_j \geq i\}, \quad i = 1, 2, \ldots, \lambda_1;$$

clearly, $l(\tilde{\lambda}) = \lambda_1$.

Label the boxes of the Ferrers diagram of the partition $\lambda$ with the numbers $1, 2, \ldots, h$ in the following way:

$$
\begin{array}{cccc}
1 & 2 & \cdots & \lambda_1 \\
\lambda_1 + 1 & \lambda_1 + 2 & \cdots & \lambda_1 + \lambda_2 \\
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & h
\end{array}
$$

A Young tableau $T$ of shape $\lambda$ over the alphabet $A = \{a_1, a_2, \ldots\}$ is a map $T : \{1, 2, \ldots, h\} \to A$; the element $T(i)$ is the symbol in the cell $i$ of the tableau $T$.

The sequences

$$T(1)T(2)\cdots T(\lambda_1),$$

$$T(\lambda_1 + 1)T(\lambda_1 + 2)\cdots T(\lambda_1 + \lambda_2),$$

$$\ldots$$

are called the row words of the Young tableau $T$.

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \ldots, \omega_p)$. Furthermore, the word of the tableau $T$ is the concatenation

$$w(T) = \omega_1 \omega_2 \cdots \omega_p.$$  

(3)
The content of a tableau $T$ is the function $c_T : A \rightarrow \mathbb{N}$, 

$$c_T(a) = \sharp \{ i \in h \mid T(i) = a \}. $$

Set

$$D_\lambda = \begin{pmatrix}
a_1 \ldots \ldots a_{\lambda_1} \\
a_1 \ldots \ldots a_{\lambda_2} \\
\ldots \\
a_1 \ldots \ldots a_{\lambda_p}
\end{pmatrix}. \tag{4}$$

The tableaux of kind $\{4\}$ are called Deruyts tableaux (of shape $\lambda$) in honor of Jacques Deruyts (1862 – 1945), who introduced them in his treatment of semi-invariants/primary covariants of algebraic forms [26] (see also [31] and [8]).

Set

$$C_\lambda = \begin{pmatrix}
a_1 \ldots \ldots a_1 \\
a_2 \ldots \ldots a_2 \\
\ldots \\
a_p \ldots a_p
\end{pmatrix}. \tag{5}$$

Since $C_\lambda$ is the conjugate tableau $C_\lambda = \tilde{D}_\lambda$ of the Deruyts tableau $D_\lambda$ of shape $\tilde{\lambda}$, we refer to the tableaux of kind $\{5\}$ as Coderuyts tableaux (of shape $\lambda$).

Now, assume that the alphabet $A$ is

$$A = A_0 \cup A_1 \cup \mathcal{L}$$

as in Section 2.1.1.

Given a shape/partition $\lambda$, we assume that $|A_0| = m_0 \geq \lambda_1$ and $|A_1| = m_1 \geq \lambda_1$. Let us denote by $\alpha_1, \ldots, \alpha_p \in A_0$, $\beta_1, \ldots, \beta_{\lambda_1} \in A_1$ two arbitrary families of distinct positive and negative virtual symbols, respectively.

Set

$$D_\lambda^* = \begin{pmatrix}
\beta_1 \ldots \ldots \beta_{\lambda_1} \\
\beta_1 \ldots \ldots \beta_{\lambda_2} \\
\ldots \\
\beta_1 \ldots \beta_{\lambda_p}
\end{pmatrix}, \quad C_\lambda^* = \begin{pmatrix}
\alpha_1 \ldots \ldots \alpha_1 \\
\alpha_2 \ldots \ldots \alpha_2 \\
\ldots \\
\alpha_p \ldots \alpha_p
\end{pmatrix}. \tag{6}$$

The tableaux of kind $\{6\}$ are called virtual Deruyts and Coderuyts tableaux of shape $\lambda$, respectively.

2.2.2 Bitableaux monomials in $U(gl(m_0 + m_1, n))$

Let $S$ and $T$ be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $A_0 \cup A_1 \cup \mathcal{L}$:

$$S = \begin{pmatrix}
z_{i_1} \ldots \ldots z_{i_{\lambda_1}} \\
z_{j_1} \ldots \ldots z_{j_{\lambda_2}} \\
\ldots \\
z_{s_1} \ldots z_{s_{\lambda_p}}
\end{pmatrix}, \quad T = \begin{pmatrix}
z_{h_1} \ldots \ldots z_{h_{\lambda_1}} \\
z_{k_1} \ldots \ldots z_{k_{\lambda_2}} \\
\ldots \\
z_{t_1} \ldots z_{t_{\lambda_p}}
\end{pmatrix}. \tag{7}$$
To the pair \((S, T)\), we associate the bitableau monomial:

\[
e_{S,T} = e_{z_{1,1},z_{1,1}} \cdots e_{z_{\lambda_1,1},z_{\lambda_1,1}} e_{z_{1,1},z_{1,1}} \cdots e_{z_{\lambda_2,1},z_{\lambda_2,1}} \cdots e_{z_{1,1},z_{1,1}} \cdots e_{z_{\lambda_p,1},z_{\lambda_p}}(8)
\]

in \(U(gl(m_0|m_1 + n))\).

### 2.2.3 Capelli bitableaux in \(U(gl(n))\)

Given a pair of Young tableaux \(S, T\) of the same shape \(\lambda\) on the proper alphabet \(L\), consider the elements

\[
e_{S,C^*} e_{C^*,T} \in U(gl(m_0|m_1 + n)).
\]

Since these elements are balanced monomials in \(U(gl(m_0|m_1 + n))\), then they belong to the virtual subalgebra Virt\((m_0 + m_1, n)\).

Hence, we can consider their images in \(U(gl(n))\) with respect to the Capelli epimorphism \(p\).

We set

\[
p(e_{S,C^*} e_{C^*,T}) = [S|T] \in U(gl(n)), (9)
\]

and call the element \([S|T]\) a Capelli bitableau.

The elements defined in (9) do not depend on the choice of the virtual Corderuyts tableau \(C^*_\lambda\).

From [18], we recall that the Capelli bitableaux \([S|T] \in U(gl(n))\) are the preimages of the determinantal bitableaux \((S|T)\) in the polynomial algebra \(\mathbb{C}[M_{n,n}]\) (see section 2.3 below) with respect to the Koszul equivariant isomorphism

\[
K : U(gl(n)) \to \mathbb{C}[M_{n,n}] \cong \text{Sym}(gl(n)).
\]

Hence, Capelli bitableaux \([S|T]\) admit explicit Laplace expansions: let \(S\) and \(T\) be the Young tableaux

\[
S = \begin{pmatrix}
i_{p_1} & \cdots & i_{p_{\lambda_1}} \\
i_{q_1} & \cdots & i_{q_{\lambda_2}} \\
\vdots \\
i_{r_1} & \cdots & i_{r_{\lambda_m}}
\end{pmatrix}, \quad T = \begin{pmatrix}
j_{s_1} & \cdots & j_{s_{\lambda_1}} \\
j_{t_1} & \cdots & j_{t_{\lambda_2}} \\
\vdots \\
j_{v_1} & \cdots & j_{v_{\lambda_m}}
\end{pmatrix}.
\]

**Proposition 2.12.** ([18], Corollary 8.3) We have

\[
[S|T] = \sum_{\sigma_1,\ldots,\sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{pmatrix}
i_{p_{\sigma_1(1)}} & \cdots & j_{s_1} \\
i_{p_{\sigma_1(\lambda_1)}} & \cdots & j_{s_{\lambda_1}} \\
\vdots & \cdots & \vdots \\
i_{r_{\sigma_m(1)}} & \cdots & j_{v_1} \\
i_{r_{\sigma_m(\lambda_m)}} & \cdots & j_{v_{\lambda_m}}
\end{pmatrix}
\]
Given a pair of Young tableaux $S, T$ of the same shape $\lambda$ on the proper alphabet $\mathcal{L}$, consider the elements

$$e_{S,C_\lambda^*} e_{C_\lambda^* D_\lambda^*} e_{D_\lambda^* T} \in U(gl(m_0|m_1 + n)).$$

Since these elements are balanced monomials in $U(gl(m_0|m_1 + n))$, then they belong to the virtual subalgebra $Virt(m_0 + m_1, n)$.

Hence, we can consider their images in $U(gl(n))$ with respect to the Capelli epimorphism $p$.

We set

$$p\left(e_{S,C_\lambda^*} e_{C_\lambda^* D_\lambda^*} e_{D_\lambda^* T}\right) = [S \overline{T}] \in U(gl(n)). \quad (10)$$

and call the element $[S \overline{T}]$ a Young-Capelli bitableau.

The elements defined in $(10)$ do not depend on the choice of the virtual Deruyts and Coderuyts tableaux $D_\lambda^*$ and $C_\lambda^*$.

In plain words, the Young-Capelli bitableau $[S \overline{T}]$ is obtained from the Capelli bitableau $[S|T]$ by adding a column symmetrization on the right Young tableau $T$. Indeed, we have

**Proposition 2.13.** Any Young-Capelli bitableau equals the sum of Capelli bitableaux:

$$[S \overline{T}] = \sum [S|T],$$

where the sum is extended to all Young tableaux $T$ obtained from $T$ by permutations of the elements of each column.
2.2.5 Double Young-Capelli bitableaux in $U(gl(n))$

Given a pair of Young tableaux $S, T$ of the same shape $\lambda$ on the proper alphabet $L$ consider the elements

$$e_{S,C}^\lambda \cdot e_{C,D}^\lambda \cdot e_{D,C}^\lambda \cdot e_{C,T}^\lambda \in U(gl(m_0|m_1+n)).$$

Since these elements are balanced monomials in $U(gl(m_0|m_1+n))$, then they belong to the virtual subalgebra $\text{Virt}(m_0+m_1,n)$.

Hence, we can consider their images in $U(gl(n))$ with respect to the Capelli epimorphism $p$.

We set

$$p\left( e_{S,C}^\lambda \cdot e_{C,D}^\lambda \cdot e_{D,C}^\lambda \cdot e_{C,T}^\lambda \right) = [S|T] \in U(gl(n)).$$ (11)

and call the element $[S|T]$ a double Young-Capelli bitableau.

The elements defined in (11) do not depend on the choice of the virtual Deruys and Coderuys tableaux $D^\lambda$ and $C^\lambda$.

In plain words, the double Young-Capelli bitableau $[S|T]$ is obtained from the Young-Capelli bitableau $[S|T]$ by adding a further row skew symmetrization. Indeed, we have

**Proposition 2.14.** Any double Young-Capelli bitableau equals the sum of Young-Capelli bitableaux:

$$[S|T] = (-1)^{|T|} \sum_{\sigma} (-1)^{|\sigma|} [S|T^\sigma],$$

where the sum is extended to all Young tableaux $T^\sigma$ obtained from $T$ by permutations of the elements of each row, and $(-1)^{|\sigma|}$ is the product of the signatures of row permutations.

2.3 Bitableaux in $\mathbb{C}[M_{m_0|m_1+n,d}]$ and the standard monomial theory

2.3.1 Biproducts in $\mathbb{C}[M_{m_0|m_1+n,d}]$

Embed the algebra

$$\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}[(\alpha_s|j),(\beta_t|j),(i|j)]$$

into the (supersymmetric) algebra $\mathbb{C}[(\alpha_s|j),(\beta_t|j),(i|j), (\gamma|j)]$ generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j),(\beta_t|j),(i|j), (\gamma|j)$, $j = 1,2,\ldots,d$, where

$$|(\gamma|j)| = 1 \in \mathbb{Z}_2 \text{ for every } j = 1,2,\ldots,d,$$

and denote by $D_{z_i,\gamma}^t$ the superpolarization of $\gamma$ to $z_i$. 

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Let $\omega = z_1 z_2 \cdots z_p$ be a word on $A_0 \cup A_{upL}$, and $\varpi = j_1 j_2 \cdots j_q$ a word on the alphabet $P = \{1, 2, \ldots, d\}$. The biproduct
\[(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_1 j_2 \cdots j_q)\]
is the element
\[D^l_{z_1, \gamma} D^l_{z_2, \gamma} \cdots D^l_{z_p, \gamma} \left((\gamma | j_1) (\gamma | j_2) \cdots (\gamma | j_q)\right) \in \mathbb{C}[M_{m_0| m_1 + n_0}]\]
if $p = q$ and is set to be zero otherwise.

**Claim 2.15.** The biproduct $(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_1 j_2 \cdots j_q)$ is supersymmetric in the $z$’s and skew-symmetric in the $j$’s. In symbols
1. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_1 j_2 \cdots j_{i+1}) = (-1)^{|z_i|} (z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_1 j_2 \cdots j_i)$
2. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_1 j_2 \cdots j_{i+1}) = (z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_1 j_{i+1} \cdots j_i \cdots j_q)$.

**Proposition 2.16. (Laplace expansions)** We have
1. $(\omega_1 \omega_2 | \varpi) = \Sigma(\varpi) (-1)^{|\varpi(1)|} (\omega_1 | \varpi(1)) (\omega_2 | \varpi(2))$.
2. $(\omega | \omega_1 \omega_2) = \Sigma(\omega) (-1)^{|\omega(2)|} (\omega(1) | \omega_1) (\omega(2) | \omega_2)$

where
\[\triangle(\varpi) = \Sigma(\varpi) \varpi(1) \otimes \varpi(2), \quad \triangle(\omega) = \Sigma(\omega) \omega(1) \otimes \omega(2)\]
denote the coproducts in the Sweedler notation (see, e.g. [1]) of the elements $\varpi$ and $\omega$ in the supersymmetric Hopf algebra of $W$ (see, e.g. [3]) and in the free exterior Hopf algebra generated by $j = 1, 2, \ldots, d$, respectively.

**Example 2.17.** Let $\omega = \alpha_1 \alpha_2 3$, $\varpi = 123$, where $|(\alpha_1 | j)| = |(\alpha_2 | j)| = 1$, $j = 1, 2, 3$ and $|(3 | j)| = 0$, $j = 1, 2, 3$. Then
\[(\omega|\varpi) = D^l_{\alpha_1, \gamma} D^l_{\alpha_2, \gamma} D^l_{3, \gamma} \left((\gamma | 1)(\gamma | 2)(\gamma | 3)\right)\]
\[= D_{\alpha_1, \gamma} D_{\alpha_2, \gamma} \left((3|1)(\gamma | 2)(\gamma | 3) - (\gamma | 1)(3|2)(\gamma | 3) + (\gamma | 1)(\gamma | 2)(3|3)\right)\]
\[= D_{\alpha_1, \gamma} \left((3|1)(\alpha_2 | 2)(\gamma | 3) + (3|1)(\gamma | 2)(\alpha_2 | 3) - (\alpha_2 | 1)(3|2)(\gamma | 3)\right)\]
\[= \left(3|1)(\alpha_2 | 2)(\gamma | 3) + (3|1)(\alpha_1 | 2)(\alpha_2 | 3) - (\alpha_2 | 1)(3|2)(\alpha_2 | 3)\right)\]
\[= (\alpha_1 | 3|2)(\alpha_2 | 3) + (\alpha_1 | 1)(\alpha_2 | 2)(\alpha_2 | 3) + (\alpha_1 | 1)(\alpha_2 | 2)(3|3).\]

From Proposition 2.14.1, by setting $\omega_1 = 12$, $\omega_2 = 3$, it follows
\[(\omega|\varpi) = (\alpha_1 \alpha_2 | 12)(3|3) + (\alpha_1 | 3|2)(\alpha_2 | 3) + (\alpha_2 | 3|2)(\alpha_1 | 3).\]

From Proposition 2.16.2, by setting $\omega_1 = \alpha_1 \alpha_2$, $\omega_2 = 3$, it follows
\[(\omega|\varpi) = (\alpha_1 \alpha_2 | 12)(3|3) - (\alpha_1 \alpha_2 | 13)(3|2) + (\alpha_1 \alpha_2 | 23)(3|1).\]
2.3.2 Biproducts in $\mathbb{C}[M_{n,d}]$

Let $\omega = i_1i_2\cdots i_p$, $\varpi = j_1j_1\cdots j_p$ be words on the negative alphabet $\mathcal{L} = \{1,2,\ldots,n\}$ and on the negative alphabet $\mathcal{P} = \{1,2,\ldots,d\}$.

From Proposition 2.16, we infer

Corollary 2.18. The biproduct of the two words $\omega$ and $\varpi$

$$(\omega|\varpi) = (i_1i_2\cdots i_p|j_1j_2\cdots j_p)$$

(12)

is the signed minor:

$$(\omega|\varpi) = (-1)^{|\omega|} \det(i_r,j_s)_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,d}].$$

2.3.3 Biproducts and polarization operators

Following the notation introduced in the previous sections, let

$$Super[W] = Sym[W_0] \otimes \Lambda[W_1]$$

denote the (super)symmetric algebra of the space

$$W = W_0 \oplus W_1$$

(see, e.g. [59]).

By multilinearity, the algebra $Super[W]$ is the same as the superalgebra $Super[A_0 \cup A_1 \cup \mathcal{L}]$ generated by the "variables"

$$\alpha_1,\ldots,\alpha_{m_0} \in A_0, \quad \beta_1,\ldots,\beta_{m_1} \in A_1, \quad x_1,\ldots,x_n \in \mathcal{L},$$

modulo the congruences

$$zz' = (-1)^{|z||z'|}z'z, \quad z,z' \in A_0 \cup A_1 \cup \mathcal{L}.$$

Let $d^l_{z,z'}$ denote the (left)polarization operator of $z'$ to $z$ on

$$Super[W] = Super[A_0 \cup A_1 \cup \mathcal{L}],$$

that is the unique superderivation of $\mathbb{Z}_2$-degree

$$|z| + |z'| \in \mathbb{Z}_2$$

such that

$$d^l_{z,z'}(z'') = \delta_{z',z''} \cdot z,$$

for every $z,z',z'' \in A_0 \cup A_1 \cup \mathcal{L}$.

Clearly, the map

$$e_{z,z'} \rightarrow d^l_{z,z'}$$

is a Lie superalgebra map and, therefore, induces a structure of

$$gl(m_0|m_1+n) - \text{module}$$

on $Super[A_0 \cup A_1 \cup \mathcal{L}] = Super[W]$.  

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Proposition 2.19. Let \( \varpi = j_{t_1} j_{t_2} \cdots j_{t_q} \) be a word on \( P = \{1, 2, \ldots, d\} \). The map
\[
\Phi_{\varpi} : \omega \mapsto (\omega \mid \varpi),
\]
\( \omega \) any word on \( A_0 \cup A_1 \cup L \), uniquely defines \( gl(m_0|m_1 + n) \)-equivariant linear operator
\[
\Phi_{\varpi} : Super[A_0 \cup A_1 \cup L] \rightarrow \mathbb{C}[M_{m_0|m_1+n}],
\]
that is
\[
\Phi_{\varpi}(e_{z,z'} \cdot \omega) = \Phi_{\varpi}(d^{l}_{z,z'}(\omega)) = D^{l}_{z,z'}((\omega \mid \varpi)) = e_{z,z'} \cdot (\omega \mid \varpi),
\]
for every \( z, z' \in A_0 \cup A_1 \cup L \).

With a slight abuse of notation, we will write (13) in the form
\[
D^{l}_{z,z'}((\omega \mid \varpi)) = (d^{l}_{z,z'}(\omega) \mid \varpi).
\]

2.3.4 Bitableaux in \( \mathbb{C}[M_{m_0|m_1+n}] \)

Let \( S = (\omega_1, \omega_2, \ldots, \omega_p) \) and \( T = (\varpi_1, \varpi_2, \ldots, \varpi_p) \) be Young tableaux on \( A_0 \cup A_1 \cup L \) and \( P = \{1, 2, \ldots, d\} \) of shapes \( \lambda \) and \( \mu \), respectively.

If \( \lambda = \mu \), the Young bitableau \((S|T)\) is the element of \( \mathbb{C}[M_{m_0|m_1+n}] \) defined as follows:
\[
(S|T) = \begin{pmatrix}
\omega_1 & \varpi_1 \\
\omega_2 & \varpi_2 \\
\vdots & \vdots \\
\omega_p & \varpi_p 
\end{pmatrix}
= \pm (\omega_1 | \varpi_1)(\omega_2 | \varpi_2) \cdots (\omega_p | \varpi_p),
\]
where
\[
\pm = (-1)^{\omega_1 | \varpi_1 + \omega_2 | \varpi_2 + \cdots + \omega_p | \varpi_p}.
\]
If \( \lambda \neq \mu \), the Young bitableau \((S|T)\) is set to be zero.

2.3.5 Bitableaux and polarization operators

By naturally extending the slight abuse of notation (13), the action of any polarization on bitableaux can be explicitly described:

Proposition 2.20. Let \( z, z' \in A_0 \cup A_1 \cup L \), and let \( S = (\omega_1, \ldots, \omega_p) \), \( T = (\varpi_1, \ldots, \varpi_p) \).
(ω₁, ..., ωₚ). We have the following identity:

\[ e_{z,z'} \cdot (S | T) = D_{z,z'}^l \left( \begin{array}{c|c} \omega_1 & \omega_1 \\ \omega_2 & \omega_2 \\ \vdots & \vdots \\ \omega_p & \omega_p \end{array} \right) \]

\[ = \sum_{s=1}^{p} (-1)^{(|z|+|z'|)\epsilon_s} \left( \begin{array}{c|c} \omega_1 & \omega_1 \\ \omega_2 & \omega_2 \\ \vdots & \vdots \\ d_{z,z'}^l(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \omega_p \end{array} \right), \]

where \( \epsilon_1 = 1, \) \( \epsilon_s = |\omega_1| + \cdots + |\omega_{s-1}|, \) \( s = 2, \ldots, p. \)

**Example 2.21.** Let \( \alpha_i \in A_0, 1, 2, 3, 4 \in L, |D_{\alpha_i,2}| = 1. \) Then

\[ e_{\alpha_i,2} \cdot \left( \begin{array}{c|c} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 4 & 2 & 1 \end{array} \right) = D_{\alpha_i,2}^l \left( \begin{array}{c|c} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 4 & 3 & 1 \end{array} \right) = \]

\[ = \left( \begin{array}{c|c} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 4 & 2 & 1 \end{array} \right) - \left( \begin{array}{c|c} 1 & 2 & 3 \\ \alpha_i & 2 & 3 \\ 4 & 3 & 1 \end{array} \right) + \left( \begin{array}{c|c} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 4 & 3 & 1 \end{array} \right). \]

**2.3.6 The straightening algorithm and the standard basis theorem for \( \mathbb{C}[M_{m_0}|m_1+n,d] \)**

Consider the set of all bitableaux \((S | T) \in \mathbb{C}[M_{m_0}|m_1+n,d], \) where \( sh(S) = sh(T) \vdash_{l} h, h \) a given positive integer. In the following, let denote by \( \leq \) the partial order on this set defined by the following two steps:

1. \((S | T) < (S' | T')\) whenever \( sh(S) <_{l} sh(S'), \)

2. \((S | T) < (S' | T')\) whenever \( sh(S) = sh(S'), w(S) >_{l} w(S'), w(T) >_{l} w(T'), \)

where the shapes and the row-words are compared in the lexicographic order.

The next results are superalgebraic versions of classical, well-known results for the symmetric algebra \( \mathbb{C}[M_{n,d}] \) (39, 27, 25, for the general theory of standard monomials see, e.g. 54, Chapt. 13) and of their skew-symmetric analogues (29, 2).

**Theorem 2.22.** (*The straightening algorithm*) [34]
Let \((P|Q) \in \mathbb{C}[M_{m_0|m_1+n,d}]\). Then \((P|Q)\) can be written as a linear combination, with rational coefficients,

\[
(P|Q) = \sum_{S,T} c_{S,T} (S|T),
\]

of standard bitableaux \((S|T)\), where \((S|T) \geq (P|Q)\) and \(c_S = c_P, c_T = c_Q\).

Since standard bitableaux are linearly independent in \(\mathbb{C}[M_{m_0|m_1+n,d}]\), the expansion \((15)\) is unique.

Following [4], [5], [11], a partition \(\lambda\) satisfies the \((m_0, m_1+n)\)-hook condition (in symbols, \(\lambda \in H(m_0, m_1+n)\) if and only if \(\lambda_{m_0+1} \leq m_1 + n\). We have:

**Lemma 2.23.** There exists a standard tableau on \(A_0 \cup A_1 \cup L\) of shape \(\lambda\) if and only if \(\lambda \in H(m_0, m_1 + n)\).

Given a positive integer \(h \in \mathbb{Z}^+\), let \(C_h[M_{m_0|m_1+n,d}]\) denote the \(h\)-th homogeneous component of \(\mathbb{C}[M_{m_0|m_1+n,d}]\).

From Theorem 2.22, it follows

**Corollary 2.24.** (The Standard basis theorem for \(C_h[M_{m_0|m_1+n,d}]\), [34])

The following set is a basis of \(C_h[M_{m_0|m_1+n,d}]\):

\[
\{(S|T) \text{ standard}; \ sh(S) = sh(T) = \lambda \vdash h, \lambda \in H(m_0, m_1 + n), \lambda_1 \leq d \}.
\]

### 3 The Schur (covariant) modules and supermodules

#### 3.1 The Schur \(U(gl(m_0|m_1+n))\)-supermodules as submodules of \(\mathbb{C}[M_{m_0|m_1+n,d}]\)

Given \(\lambda \in H(m_0, m_1 + n)\), the Schur supermodule \(Schur_\lambda(m_0, m_1 + n)\) is the subspace of \(\mathbb{C}[M_{m_0|m_1+n,d}]\), \(d \geq \lambda_1\), spanned by the set of all bitableaux \((S|D^\lambda_P)\) of shape \(\lambda\), where \(S\) is a Young tableau on the alphabet \(A_0 \cup A_1 \cup L\), and \(D^\lambda_P\) is the Deruyts tableau on \(P = \{1, 2, \ldots, d\}\)

\[
D^\lambda_P = \begin{pmatrix}
1 & 2 & 3 & \ldots & \lambda_1 \\
1 & 2 & 3 & \ldots & \lambda_2 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 2 & 3 & \ldots & \lambda_p
\end{pmatrix}, \quad p = l(\lambda).
\]

From Corollary [2.24] and the straightening algorithm (see, e.g. [3]), it follows

**Proposition 3.1.** The set

\[
\{(S|D^\lambda_P); \ S \text{ superstandard} \}
\]

is a \(\mathbb{C}\)-linear basis of \(Schur_\lambda(m_0, m_1 + n)\).
Furthermore, we recall Proposition 3.2. (\cite{III, p}.) The submodule $\text{Schur}_\lambda(m_0, m_1 + n)$ is an irreducible $U(gl(m_0|m_1 + n))$-submodule of $\mathbb{C}[M_{m_0|n_1 + n,d}]$, with highest weight $(\lambda_1, \ldots, \lambda_{m_0}; \tilde{\lambda}_1 - m_0, \tilde{\lambda}_2 - m_0, \ldots)$.

### 3.2 The classical Schur $U(gl(n))$-modules

Given $\lambda$ such that $\lambda_1 \leq n$, the Schur module $\text{Schur}_\lambda(n)$ is the subspace of $\mathbb{C}[M_{n,d}]$, $d \geq \lambda_1$, spanned by the set of all bitableaux $(X|D_\lambda)$ of shape $\lambda$ and $X$ is a Young tableau on the alphabet $L$.

**Proposition 3.3.** The set

$$\left\{ (X|D_\lambda^p); X \text{ standard } \right\}$$

is a $\mathbb{C}$-linear basis of $\text{Schur}_\lambda(n)$.

Furthermore, $\text{Schur}_\lambda(n)$ is an irreducible $U(gl(n))$-submodule of $\mathbb{C}[M_{n,d}]$, with highest weight $\tilde{\lambda}$.

Let

$$D_\lambda = \begin{pmatrix}
1 & 2 & 3 & \cdots & \lambda_1 \\
1 & 2 & 3 & \lambda_2 \\
\cdots \\
1 & 2 & 3 & \cdots & \lambda_p
\end{pmatrix}$$

denote the (proper) Deruyts tableau on the alphabet $L = \{1, 2, \ldots, n\}$. The bitableau

$$v_{\tilde{\lambda}} = (D_\lambda|D_\lambda^p)$$

is the “canonical” highest weight vector of the irreducible $gl(n)$-module $\text{Schur}_\lambda(n)$ with highest weight $\tilde{\lambda}$.

### 3.3 The classical Schur modules as $gl(n)$-submodules of Schur supermodules

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition such that $\lambda_1 \leq n$.

Consider a Schur supermodule

$$\text{Schur}_\lambda(m_0, m_1 + n)$$

(clearly, $\lambda \in H(m_0, m_1 + n)$, for every $m_0, m_1$).

The Schur module $\text{Schur}_\lambda(n)$ can be regarded as a $U(gl(n))$-submodule of the $U(gl(m_0|m_1 + n))$-supermodule $\text{Schur}_\lambda(m_0, m_1 + n)$.

Let $p$ be the Capelli epimorphism

$$p : \text{Virt}(m_0 + m_1,n) \rightarrow U(gl(n)), \quad \text{Ker}(p) = \text{Irr}.$$
Proposition 3.4. The Schur module Schur_λ(n) is invariant (as a subspace of Schur_λ(m_0, m_1 + n)) with respect to the action of the subalgebra $\text{Virt}(m_0 + m_1, n) \subset U(gl(m_0|m_1 + n))$.

From Proposition 2.7 we infer

Proposition 3.5. For every element $M \in \text{Virt}(m_0 + m_1, n)$, the action of $M$ on the Schur module Schur_λ(n) is the same of the action of its image $p(M) = m \in U(gl(n))$.

3.4 The action of double Young-Capelli bitableaux on highest weight vectors of Schur modules

Let’s start with some lemmas.

In the following, given partitions $\lambda$, $\mu$ and their conjugates $\tilde{\lambda}$ and $\tilde{\mu}$, we assume that $m_0 \geq \lambda_1, \mu_1, \quad m_1, d \geq \tilde{\lambda}_1, \tilde{\mu}_1$.

Let $v_\mu = (D_\tilde{\mu}|D_\tilde{\mu}^P)$ be the “canonical” highest weight vector of weight $\mu$ of the irreducible $gl(n)$-module Schur_\tilde{\mu}.

Lemma 3.6. We have

If $|\tilde{\mu}| < |\tilde{\lambda}|$, then $e_{C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P) = 0, \quad \forall S$ (16)

If $|\tilde{\mu}| = |\tilde{\lambda}|$, $\tilde{\mu} \neq \tilde{\lambda}$, then $e_{D_\lambda^*C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P) = 0, \quad \forall S$. (17)

The assertions of eqs. (16), (17) are special cases of standard elementary facts of the method of virtual variables (see, e.g. [8]).

Lemma 3.7. If $\tilde{\lambda} \not\subseteq \tilde{\mu}$, then

$e_{D_\lambda^*C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P) = 0, \quad \forall S$.

Proof. Assume that $|\tilde{\mu}| \geq |\tilde{\lambda}|$ to avoid trivial cases (by eq. (16)). The action $e_{C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P)$ produces a linear combination of bitableaux $(T|D_\tilde{\mu}^P) \in \text{Schur}_\tilde{\mu}(m_0, m_1 + n)$, where each tableau $T$ contains exactly $\tilde{\lambda}_i$ occurrences of the positive virtual symbols $\alpha_i \in A_0$. By straightening each of them (see, e.g. [8]), the element $e_{C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P)$ is uniquely expressed as a linear combination of (super)standard tableaux

$e_{C_\lambda^*S} \cdot (D_\tilde{\mu}|D_\tilde{\mu}^P) = \sum_i (S_i|D_\tilde{\mu}^P) \in \text{Schur}_\tilde{\mu}(m_0, m_1 + n), \quad (18)$

where in each $S_i$ the positive virtual symbols $\alpha_i \in A_0$ occupies a subshape $\tilde{\lambda}' \subseteq \tilde{\mu}$ such that $\tilde{\lambda}' \triangleright \lambda$. If $\tilde{\lambda} \not\subseteq \tilde{\mu}$, any element $(S_i|D_\tilde{\mu}^P)$ in the canonical form (18) is such that $\tilde{\lambda}' \triangleright \tilde{\lambda}, \tilde{\lambda}' \neq \tilde{\lambda}$. Then $e_{D_\lambda^*C_\lambda^*S} \cdot (S_i|D_\tilde{\mu}^P) = 0$, by skew-symmetry, and the assertion follows. ☐
We recall that, given a shape/partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p) \), the hook length \( H(x) \) of a box \( x \) in the Ferrers diagram \( F_\lambda \) of the shape \( \lambda \) is the number of boxes that are in the same row to the right of it plus those boxes in the same column below it, plus one (for the box itself). The hook number the shape \( \lambda \) is the product \( H(\lambda) = \prod_{x \in F_\lambda} H(x) \). Furthermore, we write \( \lambda! \) for the product \( \lambda_1! \lambda_2! \cdots \lambda_p! \).

**Lemma 3.8.** (Regonati’s Hook Lemma, [55]) Let \( H(\lambda) = H(\tilde{\lambda}) \) denotes the hook number of the shape/partition \( \lambda \). We have

\[
e^{C_{\tilde{\lambda}}^* D_{\lambda}} \cdot v_{\lambda} = e^{C_{\tilde{\lambda}}^* D_{\lambda}} \cdot (D_{\lambda}^P | D_{\lambda}^P) = (-1)^{H(\tilde{\lambda})} \frac{H(\tilde{\lambda})}{\lambda!} (C_{\lambda}^* | D_{\lambda}^P) \tag{19}
\]

and

\[
e^{C_{\tilde{\lambda}}^* D_{\lambda}} \cdot (D_{\lambda}^P | D_{\lambda}^P) = (-1)^{H(\tilde{\lambda})} \frac{H(\tilde{\lambda})}{\lambda!} (C_{\lambda}^* | D_{\lambda}^P). \tag{20}
\]

Furthermore

\[
e^{D_{\lambda}^* C_{\lambda}} \cdot (C_{\lambda}^* | D_{\lambda}^P) = \tilde{\lambda}! (D_{\lambda}^* | D_{\lambda}^P). \tag{21}
\]

**Theorem 3.9.** We have:

1. If \( sh(S) = sh(S') = \tilde{\lambda} \), \( |\mu| < |\tilde{\lambda}| \), then

\[
[[ S' \mid S ]] \cdot v_{\mu} = 0,
\]

2. If \( sh(S) = sh(S') = \tilde{\lambda} \), \( |\mu| = |\tilde{\lambda}| \), \( \tilde{\mu} \neq \tilde{\lambda} \), then

\[
[[ S' \mid S ]] \cdot v_{\mu} = 0,
\]

3. If \( \tilde{\mu} = \tilde{\lambda} \), then

\[
[[ D_{\lambda} \mid D_{\lambda} ]] \cdot v_{\lambda} = H(\tilde{\lambda})^2 v_{\lambda},
\]

4. If \( sh(S) = sh(S') = \tilde{\lambda} \), \( \tilde{\lambda} \not\in \tilde{\mu} \), then

\[
[[ S' \mid S ]] \cdot v_{\mu} = 0.
\]

**Proof.** Since

\[
[[ S' \mid S ]] = p \left( e^{s_{S'}} C_{\lambda}^* \cdot e^{C_{\lambda}^* D_{\lambda}} \cdot e^{D_{\lambda}^* C_{\lambda}} \cdot e^{C_{\lambda}^* S} \right) \in U(gl(n)),
\]

then

\[
[[ S' \mid S ]] \cdot v_{\mu} = e^{s_{S'}} C_{\lambda}^* \cdot e^{C_{\lambda}^* D_{\lambda}} \cdot e^{D_{\lambda}^* C_{\lambda}} \cdot e^{C_{\lambda}^* S} \cdot v_{\mu},
\]

from Proposition 2.7.
Hence, item 1) follows from eq. (16), item 2) follows from eq. (17) and item 4) follows from Lemma 5.6. Then, we prove item 3). We have

\[
\left[ \begin{array}{c|c} D_\lambda & D_\bar{\lambda} \\ \end{array} \right] \cdot v_\lambda = e_{D_\lambda \cdot C_\lambda^*} e_{C_\lambda^* \cdot D_\lambda} \cdot (D_\lambda | D_\bar{\lambda}).
\]

From eq. (20), this equals

\[
\frac{1}{\lambda!} (-1)^{\ell(\lambda)} H(\lambda) e_{D_\lambda \cdot C_\lambda^*} e_{C_\lambda^* \cdot D_\lambda} \cdot (C_{\lambda^*}^* | D_\bar{\lambda});
\]

from eq. (22) this equals

\[
(-1)^{\ell(\lambda)} H(\bar{\lambda}) e_{D_\lambda \cdot C_\lambda^*} e_{C_\lambda^* \cdot D_\lambda} \cdot (D_\lambda | D_\bar{\lambda});
\]

from eq. (21) this equals

\[
= (-1)^{\ell(\lambda)} H(\bar{\lambda}) \frac{1}{\lambda!} (-1)^{\ell(\bar{\lambda})} e_{D_\lambda \cdot C_\lambda^*} \cdot (C_{\lambda^*}^* | D_\bar{\lambda})
\]

\[
= H(\bar{\lambda})^2 (D_\lambda | D_\bar{\lambda}) = H(\bar{\lambda})^2 v_\lambda.
\]

\[\square\]

4 The center \(\zeta(n)\) of \(U(gl(n))\)

4.1 The Schur elements \(S_\lambda(n) \in \zeta(n)\)

Let \(\mu\) be a partition, let \(\bar{\mu}\) be its conjugate partition. Assume \(\mu_1 \leq n\), and \(m_1 \geq \mu_1\), \(m_0 \geq \bar{\mu}_1\); hence, the virtual Deruyts tableau \(D_\mu^\ast\) and the virtual Coderuys tableau \(C_\mu^\ast\) can be constructed. Let \(S_1, S_2\) be tableaux on the proper alphabet \(L = \{1, 2, \ldots, n\}\) of shape \(\mu\). We notice that any element

\[e_{S_1 \cdot C_\mu^\ast} \cdot e_{C_\mu^\ast \cdot D_\mu^\ast} \cdot e_{D_\mu^\ast \cdot C_\mu^\ast} \cdot e_{C_\mu^\ast \cdot S_2} \in \text{Virt}(m_0 + m_1, n),\]

is skew-symmetric in the rows of \(S_1\) and \(S_2\), respectively.

Given a partition \(\lambda\), assume \(\bar{\lambda}_1 \leq n\), \(m_1 \geq \bar{\lambda}_1\), \(m_0 \geq \lambda_1\). We set

\[
S_\lambda(n) = \frac{1}{H(\lambda)} \sum_{S} \varphi(e_{S_1 \cdot C_\lambda^*} \cdot e_{C_\lambda^* \cdot D_\lambda} \cdot e_{D_\lambda \cdot C_\lambda^*} \cdot e_{C_\lambda^* \cdot S}) \tag{23}
\]

\[
= \frac{1}{H(\bar{\lambda})} \sum_{S} [\begin{array}{c|c} S & S \end{array}] \in U(gl(n)), \tag{24}
\]

where the sum is extended to all row (strictly) increasing tableaux \(S\) of shape \(\lambda\) on the proper alphabet \(L = \{1, 2, \ldots, n\}\). Notice that \(H(\lambda) = H(\bar{\lambda})\).

By convention, if \(\lambda\) is the empty partition, we set \(S_\emptyset(n) = 1 \in \zeta(n)\).

The element \(S_\lambda(n) \in U(gl(n))\) is called the Schur element of weight \(\lambda\) (and shape \(\bar{\lambda}\)) in dimension \(n\).
**Theorem 4.1.** The Schur elements $S_{\lambda}(n)$ are central in $U(gl(n))$.

*Proof.* Consider the element

$$
\sum_{S} e_{S,C_1^\lambda} \cdot e_{C_1^\lambda,D_1^\lambda} \cdot e_{D_1^\lambda,C_1^\lambda} \cdot e_{C_1^\lambda} \cdot e_{S} \in \text{Virt}(m_0 + m_1, n),
$$

where the sum is extended to all row (strictly) increasing tableaux $S$ on the proper alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$.

Since the adjoint representation acts by derivation, we have

$$
ad(e_{ij}) \left( \sum_{S} e_{S,C_1^\lambda} \cdot e_{C_1^\lambda,D_1^\lambda} \cdot e_{D_1^\lambda,C_1^\lambda} \cdot e_{C_1^\lambda} \cdot e_{S} \right) = 0,
$$

for every $e_{ij} \in gl(n)$. Hence, the assertion follows from Corollary 2.10. \qed

Let $\zeta(n)^{(m)}$ denote the $m$-th filtration element of $\zeta(n)$ with respect to the filtration induced by the standard filtration of $U(gl(n))$. Clearly,

$$
S_{\lambda}(n) \in \zeta(n)^{(m)}, \quad (25)
$$

for every $m \geq |\lambda|$.

**Theorem 4.2.** *(Triangularity/orthogonality of the actions on highest weight vectors)* We have:

*If* $|\mu| < |\lambda|$, *then* $S_{\lambda}(n) \cdot v_{\mu} = 0$,  \quad (26)

*If* $|\mu| = |\lambda|$, *then* $S_{\lambda}(n) \cdot v_{\mu} = \delta_{\lambda,\mu} H(\lambda) v_\lambda$.  \quad (27)

*Proof.* The first assertion is an immediate consequence of Theorem 3.9 item 1). The fact that, if $|\mu| = |\lambda|$, $\mu \neq \lambda$, then $S_{\lambda}(n) \cdot v_{\mu} = 0$, is an immediate consequence of Theorem 3.9 item 2).

We examine the case $\lambda = \mu$. The value

$$
S_{\lambda}(n) \cdot v_\lambda \overset{\text{def}}{=} \frac{1}{H(\lambda)} \sum_{S} p(e_{S,C_1^\lambda} \cdot e_{C_1^\lambda,D_1^\lambda} \cdot e_{D_1^\lambda,C_1^\lambda} \cdot e_{C_1^\lambda} \cdot e_{S}) \cdot (D_\lambda|D_\lambda^P)
$$

equals

$$
\frac{1}{H(\lambda)} \sum_{S} e_{S,C_1^\lambda} \cdot e_{C_1^\lambda,D_1^\lambda} \cdot e_{D_1^\lambda,C_1^\lambda} \cdot e_{C_1^\lambda} \cdot e_{S} \cdot (D_\lambda|D_\lambda^P),
$$

by Proposition 2.7. Clearly, this reduces to

$$
\frac{1}{H(\lambda)} e_{D_\lambda,C_1^\lambda} \cdot e_{C_1^\lambda,D_1^\lambda} \cdot e_{D_1^\lambda,C_1^\lambda} \cdot e_{C_1^\lambda} \cdot D_\lambda \cdot (D_\lambda|D_\lambda^P).
$$

This value equals

$$
\frac{1}{H(\lambda)} [D_\lambda \mid D_\lambda] \cdot v_\lambda = H(\lambda) v_\lambda,
$$

by item 3) of Theorem 3.9. \qed 25
Theorem 4.3. *(Vanishing theorem)* If $\lambda \nsubseteq \mu$, then $S_\lambda(n) \cdot v_\mu = 0$.

*Proof.* It is an immediate consequence of item 4) of Theorem 3.9. \qed

Theorem 4.4. For every $m \in \mathbb{Z}^+$, the set

$$\{ S_\lambda(n); \lambda_1 \leq n, |\lambda| \leq m \}$$

is a linear basis of $\zeta(n)^{(m)}$.

The set

$$\{ S_\lambda(n); \lambda_1 \leq n \}$$

is a linear basis of the center $\zeta(n)$.

### 4.2 The Sahi-Okounkov Characterization Theorem

We reword Theorem 4.2 in terms of the Harish-Chandra isomorphism

$$\chi_n : \zeta(n) \longrightarrow \Lambda^*(n),$$

where $\Lambda^*(n)$ denotes the algebra of shifted symmetric polynomials in $n$ variables (see Section 6 below).

**Proposition 4.5.** Given $\lambda$, $\lambda_1 \leq n$ and $\mu$, $\mu_1 \leq n$, we have:

- If $|\mu| < |\lambda|$, then $\chi_n(S_\lambda(n))(\mu) = 0$,
- If $|\mu| = |\lambda|$, then $\chi_n(S_\lambda(n))(\mu) = \delta_{\lambda,\mu} H(\lambda)$.

We recall the Sahi-Okounkov Characterization Theorem for the Schur shifted symmetric polynomials

$s^*_{\lambda|n}$, $\lambda_1 \leq n$

(Theorem 1 of [57] and Theorem 3.3 of [51], see also [49]).

**Proposition 4.6.** Given $\lambda$, $\lambda_1 \leq n$, the polynomial $s^*_{\lambda|n}$ is the unique element of $\Lambda^*(n)$ such that $\deg s^*_{\lambda|n} \leq |\lambda|$ and

$$s^*_{\lambda|n}(\mu) = \delta_{\lambda,\mu} H(\lambda),$$

for all partitions $\mu$ such that $|\mu| \leq |\lambda|$ and $\mu_1 \leq n$.

From Propositions 4.5 and 4.6 we obtain

**Corollary 4.7.** Given $\lambda$, $\lambda_1 \leq n$, we have

$$\chi_n(S_\lambda(n)) = s^*_{\lambda|n}.$$
It follows that the Schur elements $S_\lambda(n)$ are the same as the Okounkov quantum immanant associated to $\lambda$ (see [49, see also [50] and [48]).

We recall that the Schur element/Okounkov quantum immanant $S_\lambda(n)$ also admits a representation as linear combination of Capelli immanants (see [17, section 5])

$$Cimm_\mu[i_1j_2 \cdots i_hz_1j_2 \cdots j_h] = \sum_{\sigma \in S_h} \chi^\mu(\sigma) \begin{bmatrix} i_{\sigma(1)} & j_1 \\
 i_{\sigma(2)} & j_2 \\
 \vdots & \vdots \\
 i_{\sigma(h)} & j_h \end{bmatrix},$$

where $\chi^\mu$ denotes the irreducible character associated to the irreducible representation of shape $\mu$ of the symmetric group $S_h$ and

$$\begin{bmatrix} i_{\sigma(1)} & j_1 \\
 i_{\sigma(2)} & j_2 \\
 \vdots & \vdots \\
 i_{\sigma(h)} & j_h \end{bmatrix},$$

are column Capelli bitableaux (see, e.g. [17, [18]):

**Proposition 4.8.** ([17, Theorem 6.2 ) Given $\lambda, \lambda_1 \leq n$, we have

$$S_\lambda(n) = (-1)^{\binom{n}{2}} \sum_{h_1 + \cdots + h_n = h} \frac{1}{h_1! \cdots h_n!} Cimm_\lambda[1^{h_1} \cdots n^{h_n}; 1^{h_1} \cdots n^{h_n}].$$

Hence, we have the remarkable identity:

**Corollary 4.9.**

$$S_\lambda(n) = \frac{1}{H(\lambda)} \sum_{S} \left| \begin{array}{c}
S \\
S \end{array} \right| \left| \begin{array}{c}
S \\
S \end{array} \right|,$$

$$=( -1)^{\binom{n}{2}} \sum_{h_1 + \cdots + h_n = h} \frac{1}{h_1! \cdots h_n!} Cimm_\lambda[1^{h_1} \cdots n^{h_n}; 1^{h_1} \cdots n^{h_n}],$$

where the sum of double Young-Capelli bitableaux in eq. (28) is extended to all row (strictly) increasing tableaux $S$ of shape $\lambda$ on the proper alphabet $L = \{1, 2, \ldots, n\}$ and eq. (29) is a sum of diagonal Capelli immanants of shape $\lambda$.

Corollary 4.9 was announced, without proof, in our recent paper [17].
4.3 The determinantal Capelli generators $H_k(n)$

Let $(k)$ be the row shape of length $k$, $\alpha \in \mathcal{A}_0$ a positive virtual symbol. The element

$$[i_1i_2\cdots i_k|j_1j_2\cdots j_k] = p(e_{i_1,\alpha}e_{i_2,\alpha} \cdots e_{i_k,\alpha}e_{\alpha,j_1}e_{\alpha,j_2} \cdots e_{\alpha,j_k})$$

is the Capelli bitableau

$$p(e_S c_{(\alpha)}^* e_{C_{(\alpha)}^*}, T) \in \mathbf{U}(\mathfrak{gl}(n)),$$

where $S = (i_1i_2\cdots i_k)$ and $T = (j_1j_2\cdots j_k)$. Clearly, the elements $[i_1i_2\cdots i_k|j_1j_2\cdots j_k]$ are skew-symmetric both in the left and the right sequences. In particular,

$$[i_k\cdots i_2i_1|i_1i_2\cdots i_k] = (-1)^{\binom{k}{2}}[i_1i_2\cdots i_k|i_1i_2\cdots i_k].$$

In the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$, given any integer $k = 1, 2, \ldots, n$, consider the Capelli elements

$$H_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k\cdots i_2i_1|i_1i_2\cdots i_k]. \quad (30)$$

We recall that the Capelli elements admit a classical presentation as a column determinant [19].

Proposition 4.10. For every $k = 1, 2, \ldots, n$, we have:

$$H_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k\cdots i_2i_1|i_1i_2\cdots i_k]$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \text{cdet} \begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \cdots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \cdots & e_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ e_{i_k,i_1} & e_{i_k,i_2} & \cdots & e_{i_k,i_k} \end{pmatrix}.$$

Proof. See [18], Proposition 8.6 (see also [41]). \[\square\]

Proposition 4.11. Let $(1)^k$ be the column shape of depth $k$. Then,

$$H_k(n) = S_{(1^k)}(n) \in \zeta(n).$$

Proof. We have

$$S_{(1^k)}(n) = \frac{1}{H((k))} \sum_S p(e_S c_{(\alpha)}^* e_{C_{(\alpha)}^*} d_{(\alpha)}^* e_{D_{(\alpha)}^*}, S)$$

$\footnote{The column determinant of a matrix $A = [a_{ij}]$ with noncommutative entries is, by definition, \text{cdet}(A) = \sum_{\sigma} (-1)^{\sigma} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}.$}
where the sum is extended to all strictly increasing row tableaux $S$ of shape $(k)$ and $H((k)) = k!$. 

Notice that

$$p(e_{S,C^*_k} \cdot e_{C^*_k} e_{D^*_k} c_{(k)} \cdot e_{C^*_k} s)$$

equals

$$(-1)^{\frac{k(k-1)}{2}} p(e_{S,C^*_k} e_{C^*_k} c_{(k)} e_{C^*_k} S),$$

that, in turn, equals

$$(-1)^{\frac{k(k-1)}{2}} k! p(e_{S,C^*_k} e_{C^*_k} e_{C^*_k} e_{C^*_k} e_{C^*_k} e_{C^*_k} \cdot e_{a,i_k} \cdot e_{a,i_k} \cdot e_{a,i_k} \cdot e_{a,i_k}).$$

Hence,

$$S_{(1^k)}(n) = (-1)^{\frac{k(k-1)}{2}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} p(e_{i_1,a} e_{i_2,a} \cdots e_{i_k,a} e_{a,i_1} e_{a,i_2} \cdots e_{a,i_k})$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} p(e_{i_1,a} \cdots e_{i_2,a} e_{i_1,a} e_{i_2,a} e_{a,i_1} e_{a,i_2} \cdots e_{a,i_k})$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = H_k(n).$$

\[ \square \]

**Corollary 4.12.** The Capelli elements $H_k(n)$ are central. Furthermore,

$$H_k(n) \in \zeta(n)^{(m)},$$

for every $m \geq k$.

We recall the following fundamental result, indeed proved by Capelli in two papers ([20], [21]) with deceiving titles.

**Theorem 4.13.** (Capelli, 1893)

The set

$$H_1(n), H_2(n), \ldots, H_n(n)$$

is a set of algebraically independent generators of the center $\zeta(n)$ of $U(gl(n))$.

As usual in the theory of symmetric functions, given a shape

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_p), \ \lambda_1 \leq n,$$

we set

$$H_\lambda(n) = H_{\lambda_1}(n) H_{\lambda_2}(n) \cdots H_{\lambda_p}(n).$$

By convention, if $\lambda$ is the empty partition, we set $H_\emptyset(n) = 1 \in \zeta(n)$.

From Theorem [1.13] one infers
Corollary 4.14. The set
\[ \{ H_\lambda(n); \lambda_1 \leq n, |\lambda| \leq m \} \]
is a linear basis of \( \zeta(n)^{(m)} \).

We recall that \( v_\mu = (D_\mu|D_\mu^P) \) denotes the “canonical” highest weight vector of the Schur module \( \text{Schur}_\mu(n), \mu_1 \leq n \), which is indeed of weight \( \mu \) (Subsection 3.2).

Furthermore, we will write \( H_k(n) \cdot v_\mu \) to mean the action of the central element \( H_k(n) \) on \( v_\mu \).

For every \( k = 1, 2, \ldots, n \), let
\[ e_k^*(\mu) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (\mu_{i_1} + k - 1)(\mu_{i_2} + k - 2) \cdots (\mu_{i_k}). \tag{31} \]

Remark 4.15. In formula (31), the sum can be regarded as extended over all Ferrers subdiagrams obtained from the Ferrers diagram of the partition \( \mu \) by selecting \( k \) columns \( i_1 < i_2 < \cdots < i_k \), and each summand
\[ (\mu_{i_1} + k - 1)(\mu_{i_2} + k - 2) \cdots (\mu_{i_k}) \]
is the product of the hook length \( H(x) \) of the boxes of the first row of each Ferrers subdiagram.

The classical determinatal presentation (Proposition 4.10) of the \( H_k(n) \)'s implies the following result.

Proposition 4.16. We have
\[ H_k(n) \cdot v_\mu = e_k^*(\mu) \cdot v_\mu, \quad e_k^*(\mu) \in \mathbb{Z}. \]

Corollary 4.17. If \( \mu_1 < k \), then
\[ H_k(n) \cdot v_\mu = 0, \]

The “virtual definition” (30) of the \( H_k(n) \)'s leads to a further combinatorial description of the integer eigenvalues \( e_k^*(\mu) \), which will turn out to be crucial in the section on duality.

Proposition 4.18. We have
\[ e_k^*(\mu) = \sum hstrip_\mu(k)!, \]
where the sum is extended to all “horizontal strips”\(^2\) of length \( k \) in the Ferrers diagram of the partition \( \mu \), and the symbol \( hstrip_\mu(k)! \) denotes the products of the factorials of the cardinality of each “horizontal component”\(^3\) of the horizontal strip.

---

\(^2\) In this work, we use the expression horizontal strip in a generalized sense. To wit, a horizontal strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same column.

\(^3\) For each each generalized horizontal strip, a horizontal component is the set all cells on the same row.
Proof. Let
\[
H_k(n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]
\]
\[
= \sum_{1 \leq i_1 < \ldots < i_k \leq n} p(e_{i_k, \alpha} \cdots e_{i_2, \alpha} e_{i_1, \alpha} e_{\alpha, i_1} e_{\alpha, i_2} \cdots e_{\alpha, i_k}).
\]

Let \(v_{\bar{\mu}} = (D_\mu | D_\mu^P)\) be the canonical highest weight vector of the irreducible \(gl(n)\)-module \(\text{Schur}_\mu(n)\) (of weight \(\bar{\mu}\)).

Recall that
\[
p(e_{i_k, \alpha} \cdots e_{i_2, \alpha} e_{i_1, \alpha} e_{\alpha, i_1} e_{\alpha, i_2} \cdots e_{\alpha, i_k}) \cdot (D_\mu | D_\mu^P) =
\]
\[
e_{i_k, \alpha} \cdots e_{i_2, \alpha} e_{i_1, \alpha} e_{\alpha, i_1} e_{\alpha, i_2} \cdots e_{\alpha, i_k} \cdot (D_\mu | D_\mu^P),
\]
by Proposition 2.7.

The action of each summand of the “virtualizing part”
\[
e_{\alpha, i_1} e_{\alpha, i_2} \cdots e_{\alpha, i_k}
\]
distributes the \(k\) occurrences of \(\alpha\) in all horizontal strips of length \(k\) (with column positions \(i_1, i_2, \ldots, i_k\)) in the Ferrers diagram of the partition \(\mu\), with signs according to Remark 2.20 - since \(|e_{\alpha, i_h}| = 1\). By applying the “devirtualizing part”
\[
e_{i_k, \alpha} \cdots e_{i_2, \alpha} e_{i_1, \alpha}
\]
it is easy to see that, for each horizontal strip, we obtain a sum of tableaux that:

- to be non zero, have the occurrences of \(\alpha\) - in any horizontal component of the strip - replaced by a permutation of the elements that have been previously polarized into \(\alpha\),

- have a sign that is easily seen to be the product of the signs of the permutations of the elements in each horizontal component.

By reordering each horizontal component, all the signs cancel out. Therefore, we get the “canonical” highest weight vector \(v_{\bar{\mu}} = (D_\mu | D_\mu^P)\) with a positive integer coefficient that is the product of the factorials of the lengths of the horizontal components.

### 4.4 The permanental Nazarov generators \(I_k(n)\)

In this section we provide the virtual form of the set of the preimages in \(\zeta(n)\) - with respect to the Harish-Chandra isomorphism - of the sequence of shifted complete symmetric polynomials \(h_k^*(x_1, x_2, \ldots, x_n)\), \(k \in \mathbb{Z}^+\) (see [51] and [44], Theorem 4.9).

The central elements \(I_k(n)\), \(k \in \mathbb{Z}^+\), coincide (see [9]) with the “permanental generators” of \(\zeta(n)\) originally discovered and studied - through the machinery of Yangians - by Nazarov [47] and later described by Umeda [63] as sums of
column permanents\footnote{The column permanent of a matrix $A = [a_{ij}]$ with noncommutative entries is, by definition, $\text{cper}(A) = \sum_\sigma a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.} in $\mathbf{U}(gl(n))$ (see e.g. Example 4.20 below, see also \cite{[46, 48], and Turnbull [60]).}

The element
\[
[n^{h_1} \cdots 2^{h_2} 1^{h_1} 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* = p(e_{n,\beta}^{h_n} \cdots e_{2,\beta}^{h_2} e_{1,\beta}^{h_1} e_{1,\beta,2} e_{\beta,1}^{h_1} e_{\beta,2} e_{\beta,n}^{h_n}),
\]
where $\beta \in A_1$ denotes any negative virtual symbol, is symmetric both in the left and the right sequences. In particular,
\[
[n^{h_1} \cdots 2^{h_2} 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* = [1^{h_1} 2^{h_2} \cdots n^{h_n}]^*.
\]

Remark 4.19. Let $k = h_1 + h_2 + \cdots + h_n$, and let $(1)^k$ be the column shape of depth $k$. Since
\[
e_{1,\beta}^{h_1} e_{2,\beta}^{h_2} e_{1,\beta,2} e_{\beta,1}^{h_1} e_{\beta,2} e_{\beta,n}^{h_n} \in e_{1,\beta}^{h_1} e_{2,\beta}^{h_2} e_{1,\beta,2} e_{\beta,1}^{h_1} e_{\beta,2} e_{\beta,n}^{h_n},
\]
the element \footnote{The column permanent of a matrix $A = [a_{ij}]$ with noncommutative entries is, by definition, $\text{cper}(A) = \sum_\sigma a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.} \eqref{eq:32} equals the bitableau monomial (see formula \eqref{eq:31})
\[
T, T_{(1)^k} \cdot e_{(1)^k} \cdot e_{(1)^k} \cdot T \in \mathbf{U}(gl(m_0|m_1 + n)),
\]
where $T$ is the column tableau of shape $(1)^k$ with $T = (1^{h_1} 2^{h_2} \cdots n^{h_n})$, $sh(T) = (k)$.

In the enveloping algebra $\mathbf{U}(gl(n))$, given any positive integer $k \in \mathbb{Z}^+$, consider the Nazarov elements
\[
I_k(n) = \sum_{(h_1, h_2, \ldots, h_n)} (h_1! h_2! \cdots h_n!)^{-1} [n^{h_1} \cdots 2^{h_2} 1^{h_1} 1^{h_1} 2^{h_2} \cdots n^{h_n}]^*,
\]
where the sum is extended to all $n$-tuples $(h_1, h_2, \ldots, h_n)$ such that $h_1 + h_2 + \cdots + h_n = k$. Clearly, formula \eqref{eq:33} can be rewritten as
\[
\sum_{\overline{i} = (1 \leq i_1 \leq \cdots \leq i_k \leq n)} (h_1(i_1) \cdots h_n(i))!^{-1} p(e_{i_k,\beta} \cdots e_{i_1,\beta,j_1} \cdots e_{j,ik})
\]
where, given a non decreasing $k$-tuple $\overline{i} = (1 \leq i_1 \leq \cdots \leq i_k \leq n)$, we set
\[
h_j(i) = \sharp \{i_s = j; s = 1, \ldots, k\}, \quad j = 1, 2, \ldots, n.
\]

In “nonvirtual form”, the summands
\[
[n^{h_1} \cdots 2^{h_2} 1^{h_1} 1^{h_1} 2^{h_2} \cdots n^{h_n}]^*
\]
can be written as column permanent in the algebra $\mathbf{U}(gl(n))$ (see, e.g. \cite{[63]}).
Example 4.20.

\[ I_3(3) = \frac{1}{3!} [111][111]^* + \frac{1}{2!} [121][112]^* + \frac{1}{2!} [311][113]^* + \frac{1}{2!} [221][122]^* + [321][123]^* + \frac{1}{2!} [331][333]^* + \frac{1}{2!} [222][222]^* + \frac{1}{2!} [322][223]^* + \frac{1}{2!} [332][233]^* + \frac{1}{3!} [333][333]^* = \]

\[ = \frac{1}{3!} \text{cper} \begin{pmatrix} e_{1,1} - 2 & e_{1,1} & e_{1,1} \\ e_{1,1} - 2 & e_{1,1} - 1 & e_{1,1} \\ e_{1,1} - 2 & e_{1,1} - 1 & e_{1,1} \end{pmatrix} + \frac{1}{2!} \text{cper} \begin{pmatrix} e_{1,1} - 2 & e_{1,1} - 1 & e_{1,2} \\ e_{1,1} - 2 & e_{1,1} - 1 & e_{1,2} \\ e_{2,1} & e_{2,1} & e_{2,1} \end{pmatrix} + 
\]

\[ + \frac{1}{2!} \text{cper} \begin{pmatrix} e_{1,1} - 2 & e_{1,1} - 1 & e_{1,3} \\ e_{1,1} - 2 & e_{1,1} - 1 & e_{1,3} \\ e_{3,1} & e_{3,1} & e_{3,3} \end{pmatrix} + \frac{1}{2!} \text{cper} \begin{pmatrix} e_{1,1} - 2 & e_{1,2} & e_{1,3} \\ e_{1,1} - 2 & e_{1,3} & e_{1,3} \\ e_{2,1} & e_{2,2} - 1 & e_{2,2} \end{pmatrix} + 
\]

\[ + \text{cper} \begin{pmatrix} e_{1,2} - 2 & e_{2,1} & e_{2,2} \end{pmatrix} + \frac{1}{2!} \text{cper} \begin{pmatrix} e_{2,2} - 2 & e_{2,2} - 1 & e_{2,2} \\ e_{2,2} - 2 & e_{2,2} - 1 & e_{2,2} \\ e_{3,2} & e_{3,2} & e_{3,3} \end{pmatrix} + \]

\[ + \frac{1}{2!} \text{cper} \begin{pmatrix} e_{2,2} - 2 & e_{2,3} & e_{2,3} \\ e_{3,2} & e_{3,3} - 1 & e_{3,3} \\ e_{3,2} & e_{3,3} - 1 & e_{3,3} \end{pmatrix} + \frac{1}{3!} \text{cper} \begin{pmatrix} e_{3,3} - 2 & e_{3,3} - 1 & e_{3,3} \\ e_{3,3} - 2 & e_{3,3} - 1 & e_{3,3} \\ e_{3,3} - 2 & e_{3,3} - 1 & e_{3,3} \end{pmatrix}. \]

\[ \square \]

Proposition 4.21. Let \( (k) \) be the row shape of length \( k \). Then,

\[ I_k(n) = S_{(k)}(n). \]

Proof. By formula (33), we have

\[ S_{(k)}(n) = \frac{1}{H((1)^k)} \sum_{S} p(e_{S,C^*_{(1k)}} e_{D^*_{(1k)}} e_{D^*_{(1k)}} e_{C^*_{(1k)}} e_{C^*_{(1k)}} s), \]

where the sum is extended to all column tableaux \( S \) of shape (1\(^k\)) and \( H((1)^k) = k! \).

Since \( S \) is a column tableaux of shape (1\(^k\)) and the column tableau \( C^*_{(1k)} \) is

\[ C^*_{(1k)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix}, \]

where the \( \alpha_i \)'s are distinct positive virtual symbols, then each summand

\[ p(e_{S,C^*_{(1k)}} e_{D^*_{(1k)}} e_{D^*_{(1k)}} e_{C^*_{(1k)}} e_{C^*_{(1k)}} s) \]

33
equals
\[ p(e_{S,D}^* e_{D}^*, S). \]
Hence,
\[ S_{(k)}(n) = \frac{1}{k!} \sum_S p(e_{S,D}^* e_{D}^*, S) \]
\[ = \frac{1}{k!} \sum_{(h_1, \ldots, h_n)} \sum_T p(e_{T,D}^* e_{D}^*, T), \]
where the outer sum is extended over all indexes \( h_1 + \cdots + h_n = k \) and inner sum is extended over all column tableaux \( T \) with \( h_1 \) occurrences of 1, \( h_2 \) occurrences of 2, \ldots, \( h_n \) occurrences of \( n \). Moreover, since each element \( e_{T,D}^* \) and \( e_{D}^* \) are commutative, then the inner sum
\[ \sum_T e_{T,D}^* e_{D}^*, T \]
equals
\[ \left( h_1, h_2, \ldots, h_n \right) \]
\[ e_{1,\beta_1}^{h_1} \cdots e_{n,\beta_1}^{h_1} \cdots e_{n,\beta_n}^{h_n} \]
where there are \( h_1 \) occurrences of 1, \( h_2 \) occurrences of 2, \ldots, \( h_n \) occurrences of \( n \). Hence, from Remark 4.19 we infer
\[ S_{(k)}(n) = \frac{1}{k!} \sum_{(h_1, \ldots, h_n)} \sum_T p(e_{T,D}^* e_{D}^*, T) \]
\[ = \frac{1}{k!} \sum_{(h_1, \ldots, h_n)} \left( h_1, h_2, \ldots, h_n \right) \]
\[ \frac{p(e_{1,\beta_1}^{h_1} \cdots e_{n,\beta_1}^{h_1} \cdots e_{n,\beta_n}^{h_n})}{h_1! h_2! \cdots h_n!} \]
\[ = \sum_{(h_1, h_2, \ldots, h_n)} \frac{1}{h_1! h_2! \cdots h_n!} [n^{h_n} 2^{h_2} 1^{h_1}] = I_k(n). \]

\[ \square \]

**Corollary 4.22.** The Nazarov elements \( I_k(n) \) are central. Furthermore,
\[ I_k(n) \in \zeta(n)^{(m)}, \]
for every \( m \geq k \).

The next characterization of the eigenvalues \( h_k^* (\tilde{\mu}) \) of the elements \( I_k(n) \), in combination with characterization of the eigenvalues \( e_k^* (\tilde{\mu}) \) of the elements \( H_k(n) \) (see Proposition 4.15), will play a crucial role in our treatment of duality in the center \( \zeta(n) \) (see Section 4.5 below).
Theorem 4.23. We have:
\[ I_k(n) \cdot v_{\tilde{\mu}} = h^*_k(\tilde{\mu}) \cdot v_{\tilde{\mu}}, \quad h^*_k(\tilde{\mu}) \in \mathbb{N} \]
with
\[ h^*_k(\tilde{\mu}) = \sum \text{vstrip}_\mu(p)(k)!, \]
where the sum is extended to all "vertical strips" of length \( k \) in the Ferrers diagram of the partition \( \mu \), and the symbol \( \text{vstrip}_\mu(p)(k)! \) denotes the product of the factorials of the cardinality of each vertical component of the vertical strip.

Proof. The action of the “virtualizing part”
\[ e^{h_1^\beta} e^{h_2^\beta} \cdots e^{h_n^\beta} \]
of each summand in expression \( [33] \) distributes \( k \) occurrences of the virtual variable \( \beta \) in the Ferrers diagram of the shape \( \mu \), with \( h_1 \) occurrences in column 1, \( h_2 \) occurrences in column 2, and so on. Since \( |\beta| = 1 \), in order to get a non zero result, these \( \beta \)'s must appear in different rows - by skew-symmetry - and, therefore, they form a vertical strip. Clearly, this configuration is created \( h_1! h_2! \cdots h_n! \) times. Again by skew-symmetry, the action of the “devirtualizing part”
\[ e^{h_n^\beta} \cdots e^{h_2^\beta} e^{h_1^\beta} \]
gives a non zero result if and only if the \( \beta \)'s in column 1 are replaced by 1, the \( \beta \)'s in column 2 are replaced by 2, and so on. Therefore we obtain again the highest weight vector \( v_{\tilde{\mu}} = (D_{\tilde{\mu}}|D_{\tilde{\mu}}^\beta) \) with multiplicity \( h_1! h_2! \cdots h_n! \). Note that, since \( |e_{\beta,p}| = |e_{p,\beta}| = 0 \), for every \( p = 1, 2, \ldots, n \), no signs are involved in the proof.

Corollary 4.24. If \( \tilde{\mu}_1 < k \), then
\[ I_k(n) \cdot v_{\tilde{\mu}} = 0. \]

The eigenvalue \( h^*_k(\tilde{\mu}) \) admits a further description that relates it to complete homogeneous shifted symmetric polynomials.

Theorem 4.25. We have:
\[ h^*_k(\tilde{\mu}) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} (\tilde{\mu}_{i_1} - k + 1)(\tilde{\mu}_{i_2} - k + 2) \cdots (\tilde{\mu}_{i_k - 1} - 1)\tilde{\mu}_{i_k} \quad (35) \]

Proof. The action of the “virtualizing part”
\[ e^{h_{i_1}^\beta} e^{h_{i_2}^\beta} \cdots e^{h_{i_k}^\beta}, \quad \hat{i} = (i_1 \leq i_2 \leq \cdots \leq i_k), \]
In this work, we use the expression vertical strip in a generalized sense. To wit, a vertical strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same row.
of each summand in expression (34) of $I_k(n)$, distributes one occurrence of the virtual variable $\beta$ in the Ferrers diagram of the shape $\mu$, in column $i_k$, ..., $i_2$, $i_1$. Since $|\beta| = 1$, in order to get a non zero result, these $\beta$’s must be distributed into different rows, by skew-symmetry. Clearly, this procedure can be done into

$$(\tilde{\mu}_{i_1} - k + 1)(\tilde{\mu}_{i_2} - k + 2)\cdots(\tilde{\mu}_{i_k} - 1)\tilde{\mu}_{i_k}$$

ways. Again by skew-symmetry, the action of the “devirtualizing part”

$$e_{i_k,\beta}e_{i_{k-1},\beta}\cdots e_{i_1,\beta}$$

gives a non zero result of and only if the $\beta$ in column $i_1$ is replaced by $i_1$, the $\beta$ in column $i_2$ is replaced by $i_2$, and so on. Therefore we obtain again the highest weight vector $v_{\tilde{\mu}} = (D_{\mu} | D_{\mu}^P)$ with multiplicity $h_1(\tilde{\mu})! \cdots h_n(\tilde{\mu})!$.

4.5 Duality in $\zeta(n)$

Let

$$\mathcal{W}_n : \zeta(n) \to \zeta(n)$$

be the algebra automorphism defined by setting

$$\mathcal{W}_n \left( H_k(n) \right) = I_k(n), \quad k = 1, 2, \ldots, n.$$  

Clearly, Proposition 4.18 and Theorem 4.23 imply the following result.

**Proposition 4.26.** If $\mu_1, \tilde{\mu}_1 \leq n$, then

$$e_k^* (\tilde{\mu}) = h_k^* (\mu),$$ \hspace{1cm} (36)

that is, the eigenvalue of $H_k(n)$ on the Schur module of shape $\mu$ equals the eigenvalue of $I_k(n)$ on the Schur module of shape $\tilde{\mu}$.

Notice that the following **Duality Theorem** is an immediate consequence of the preceding Proposition.

**Theorem 4.27.** Let $\mu$ be such that $\mu_1, \tilde{\mu}_1 \leq n$. For every $\varrho \in \zeta(n)$ the eigenvalue of $\varrho$ on the $gl(n)$-irreducible module $\text{Schur}_\mu(n)$ (with highest weight $\mu$) equals eigenvalue of $\mathcal{W}_n \left( \varrho \right)$ on the $gl(n)$-irreducible module $\text{Schur}_\mu(n)$ (with highest weight $\tilde{\mu}$).

The preceding result, in combination with the characterization results of subsection 4.2 implies

**Corollary 4.28.** Let $\tilde{\lambda}_1, \lambda_1 \leq n$. Then

$$\mathcal{W}_n \left( S_\lambda(n) \right) = S_\lambda(n).$$
Proof. We have:

If $|\mu| < |\lambda|$, then $S_{\lambda}(n) \cdot v_\mu = 0$,
If $|\mu| = |\lambda|$, then $S_{\lambda}(n) \cdot v_\mu = \delta_{\lambda,\mu} H(\lambda) \cdot v_\mu$.

On the other hand, from Theorem 4.27, it follows:

If $|\tilde{\mu}| < |\tilde{\lambda}|$, then $W_n(S_{\lambda}(n)) \cdot v_{\tilde{\mu}} = 0$,
If $|\tilde{\mu}| = |\tilde{\lambda}|$, then $W_n(S_{\lambda}(n)) \cdot v_{\tilde{\mu}} = \delta_{\lambda,\mu} H(\lambda) \cdot v_{\tilde{\mu}}$.

Since $\delta_{\lambda,\mu} H(\lambda) = \delta_{\lambda,\mu} H(\lambda)$, the assertion follows from Propositions 4.5 and 4.6.

Since $I_k(n) = S_k(n)$ and $H_k(n) = S_{(1^k)}(n)$, then

$W_n(I_k(n)) = H_k(n), \quad k = 1, 2, \ldots, n.$

by Corollary 4.28.

Corollary 4.29. The algebra automorphism $W_n$ is an involution.

5 The limit $n \to \infty$ for $\zeta(n)$: the algebra $\zeta$

5.1 The monomorphisms $i_{n+1,n}$ and the epimorphisms $\pi_{n,n+1}$

Given $n \in \mathbb{Z}^+$, let $H_k(n), \quad k = 1, \ldots, n$

be the Capelli free generators of the center $\zeta(n)$ of the enveloping algebra $U(gl(n))$, for every $n \in \mathbb{Z}^+$.

For every $n \in \mathbb{Z}^+$, let

$i_{n+1,n} : \zeta(n) \to \zeta(n+1)$

be the algebra monomorphism

$i_{n+1,n} : H_k(n) \to H_k(n+1), \quad k = 1, 2, \ldots, n.$

Given $m \in \mathbb{Z}^+$, let $\zeta(n)^{(m)}$ denote the $m$-th filtration element of $\zeta(n)$ (with respect to the filtration induced by the standard filtration of $U(n)$). Clearly, the monomorphisms $i_{n+1,n}$ are morphisms in the category of filtered algebras, that is

$i_{n+1,n} \left[ \zeta(n)^{(m)} \right] \subseteq \zeta(n+1)^{(m)}$

We consider the direct limit (in the category of filtered algebras):

$$\lim_{\to} \zeta(n) = \zeta.$$ (37)
The algebra $\zeta$ inherits a structure of filtered algebra, where

$$\zeta^{(m)} = \lim_{n \to \infty} \zeta^{(m)}(n).$$

On the other hand, given $n \in \mathbb{Z}^+$, we may consider the algebra epimorphism

$$\pi_{n,n+1} : \zeta(n+1) \to \zeta(n),$$

such that

$$\pi_{n,n+1}(H_k(n+1)) = H_k(n) \quad k = 1, 2, \ldots, n,$$

$$\pi_{n,n+1}(H_{n+1}(n+1)) = 0.$$

The following Propositions are fairly obvious from the definitions.

**Proposition 5.1.** We have

1. $\ker(\pi_{n,n+1}) = \left( H_{n+1}(n+1) \right)$, the bilateral ideal of $\zeta(n+1)$ generated by the element $H_{n+1}(n+1)$.
2. The epimorphism $\pi_{n,n+1}$ is the (filtered) left inverse of the monomorphism $i_{n+1,n}$. In symbols,

$$\pi_{n,n+1} \circ i_{n+1,n} = \text{Id}_{\zeta(n)}.$$

**Proposition 5.2.** If $n \geq m$, then the restriction $\pi_{n,n+1}^{(m)}$ of $\pi_{n,n+1}$ to $\zeta(n+1)$ and the restriction $i_{n+1,n}^{(m)}$ of $i_{n+1,n}$ to $\zeta(n)$ are the inverse of each other.

The crucial point is that the projections $\pi_{n,n+1}$ admit an intrinsic/invariant presentation that is founded upon the Olshanski decomposition.

### 5.2 The Olshanski decomposition/projection

We recall a special case of an essential costruction due to Olshanski [52], [53]. For the sake of simplicity, we follow Molev ([44], pp. 928 ff.).

Let $U(gl(n+1))^0$ be the centralizer in $U(gl(n+1))$ of the element $e_{n+1,n+1}$ of the standard basis of $gl(n+1)$, regarded as an element of $U(gl(n+1))$.

Let $I(n+1)$ be the left ideal of $U(gl(n+1))$ generated by the elements

$$e_{i,n+1}, \quad i = 1, 2, \ldots, n+1.$$

Let $I(n+1)^0$ be the intersection

$$I(n+1)^0 = I(n+1) \cap U(gl(n+1))^0. \quad (38)$$

We recall that $I(n+1)^0$ is a bilateral ideal of $U(gl(n+1))^0$, and the following direct sum decomposition hold

$$U(gl(n+1))^0 = U(gl(n)) \oplus I(n+1)^0. \quad (39)$$
Therefore, the Olshanski map
\[ \mathcal{M}_{n+1} : U((gl(n+1))^0) \to U(gl(n)) \]
that maps any element in the direct summand \( U(gl(n)) \) to itself and any element in the direct summand \( I(n+1)^0 \) to zero is a well-defined algebra epimorphism.

Since \( \zeta(n+1) \) is a subalgebra of \( U((gl(n+1))^0) \), the direct sum decomposition \( (39) \) induces a direct sum decomposition of any element in \( \zeta(n+1) \) and the \( \mathcal{M}_{n+1} \) map defines, by restriction, an algebra epimorphism \( \mu_{n,n+1} : \zeta(n+1) \to \zeta(n) \).

In plain words, any element \( \varrho \in \zeta(n+1) \) admits a unique decomposition
\[ \varrho = \varrho' + \varrho^0, \quad \varrho' \in \zeta(n), \quad \varrho^0 \in I(n+1)^0. \]  
(40)

We call the decomposition (40) the Olshanski decomposition of the element \( \varrho \in \zeta(n+1) \).

In this notation, the projection \( \mu_{n+1,n} : \zeta(n+1) \to \zeta(n) \),
\[ \mu_{n+1,n}(\varrho) = \varrho', \quad \varrho \in \zeta(n+1) \]
is defined.

**Proposition 5.3.** We have

1. if \( k \leq n \), then
   \[ H_k(n+1) = H_k(n) + H_k(n+1)^0; \]
   where
   \[ H_k(n+1)^0 = H_k(n+1) - H_k(n) \in I(n+1)^0, \]
   and
   \[ H_k(n) \in \zeta(n); \]

2. \( \mathbf{H}_{n+1}(n+1) = \mathbf{H}_{n+1}(n+1)^0. \)

**Example 5.4.** We have:
\[ \mathbf{H}_2(4) = [21|12] + [31|13] + [41|14] + [32|23] + [42|24] + [43|34] \]
\[ = \mathbf{H}_2(3) + \mathbf{H}_2(4)^0, \]
where
\[ \mathbf{H}_2(3) = [21|12] + [31|13] + [32|23] \in \zeta(3), \]
and
\[ \mathbf{H}_2(4)^0 = [41|14] + [42|24] + [43|34] \in I(4)^0. \]
Corollary 5.5. We have

1. if \( k \leq n \), then \( \mu_{n,n+1}(H_k(n+1)) = H_k(n) \),
2. \( \mu_{n,n+1}(H_{n+1}(n+1)) = 0 \).

**Proposition 5.6.** The map \( \mu_{n,n+1} \) is the same as the map \( \pi_{n,n+1} \).

**Proof.** The family
\[
\{H_1(n+1), H_2(n+1), \ldots, H_n(n+1), H_{n+1}(n+1)\}
\]
is a system of algebraically independent generators of the algebra \( \zeta(n+1) \). From Proposition 5.5, we obtain:

- if \( k \leq n \), then
  \[\mu_{n,n+1}(H_k(n+1)) = H_k(n) = \pi_{n,n+1}(H_k(n+1));\]
- \( \mu_{n,n+1}(H_{n+1}(n+1)) = 0 = \pi_{n,n+1}(H_{n+1}(n+1)). \]

In the following, we refer to the projections

\[ \mu_{n,n+1} = \pi_{n,n+1} \]
as the *Capelli-Olshanski projections.*

From Proposition 5.2, the algebra \( \zeta \) (direct limit) is the same as the *inverse limit in the category of filtered algebras*

**Proposition 5.7.** We have

\[ \zeta = \lim_{\leftarrow} \zeta(n) \]

with respect to the system of Capelli-Olshanski projections.

### 5.3 Main results

From Theorem 4.13 and Proposition 5.5, we infer

**Proposition 5.8.** We have

1. \( I_k(n+1) = I_k(n) + I_k(n+1)^0 \), where

\[ I_k(n+1)^0 = I_k(n+1) - I_k(n) \in I_k(n+1)^0, \]

and

\[ I_k(n) \in \zeta(n). \]

Then

\[ \pi_{n,n+1}(I_k(n+1)) = I_k(n). \quad (41) \]
2. $S_\lambda(n+1) = S_\lambda(n) + S_\lambda(n+1)^0$, where

$$S_\lambda(n+1)^0 = S_\lambda(n+1) - S_\lambda(n) \in I_k(n+1)^0,$$

and $S_\lambda(n) \in \zeta(n)$. Then

$$\pi_{n,n+1}(S_\lambda(n+1)) = S_\lambda(n). \quad (42)$$

By combining the preceding Proposition with Proposition 5.2, we get

**Theorem 5.9.** We have:

1. Given a positive integer $k$, if $n \geq k$ then

   $$i_{n+1,n}(I_k(n)) = I_k(n+1);$$

2. Given a partition $\lambda$, if $n \geq |\lambda|$ then

   $$i_{n+1,n}(S_\lambda(n)) = S_\lambda(n+1).$$

Passing to the direct limit $\varinjlim \zeta(n) = \zeta$, we set:

1. $H_k \overset{def}{=} \varinjlim H_k(n) \in \zeta.$

2. $I_k \overset{def}{=} \varinjlim I_k(n) \in \zeta.$

3. $S_\lambda \overset{def}{=} \varinjlim S_\lambda(n) \in \zeta.$

From the definition of the monomorphisms $i_{n+1,n}$ and Theorem 5.9, the elements $H_k, I_k, S_\lambda \in \zeta$ can be consistently written as *formal infinite sums*.

**Proposition 5.10.** We have

1. $H_k = \sum_{i_1 < \cdots < i_k} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k],$

   where the sum is extended to all increasing $k$-tuples $i_1 < i_2 < \cdots < i_k$ in $\mathbb{Z}^+.$

2. $I_k = \sum_{j_1 < j_2 < \cdots < j_p} \left( i_{j_1} ! i_{j_2} ! \cdots i_{j_p} ! \right)^{-1} \left[ j_{p^1} \cdots j_{p^2} \cdots j_{1^1} i_{j_1} i_{j_2} \cdots j_{1^p} \cdots j_{p^p} \right],$

   where the sum is extended to all $p$-tuples $j_1 < j_2 < \cdots < j_p$ in $\mathbb{Z}^+$, and to all the $p$-tuples of exponents $(i_{j_1}, i_{j_2}, \cdots, i_{j_p})$ such that

   $$i_{j_1} + i_{j_2} + \cdots + i_{j_p} = k.$$
3. \[
S_\lambda(n) = \frac{1}{H(\lambda)} \sum_S \delta(S, \bar{\lambda}),
\]

where the sum is extended to all row-increasing tableaux \( S \) of shape \( \bar{\lambda} \) on the alphabet \( \mathbb{Z}^+ \).

From Proposition 5.7 it follows

**Corollary 5.11.** We have:

1. \( \lim \leftarrow H_k(n) = H_k \in \zeta \),
2. \( \lim \leftarrow I_k(n) = I_k \in \zeta \),
3. \( \lim \leftarrow S_\lambda(n) = S_\lambda \in \zeta \).

Due the fact that the algebra \( \zeta \) is defined as a direct limit, we infer:

**Theorem 5.12.**

1. The set \( \{ H_k; k \in \mathbb{Z}^+ \} \)
   is a system of free algebraic generators of \( \zeta \).
2. The set \( \{ I_k; k \in \mathbb{Z}^+ \} \)
   is a system of free algebraic generators of \( \zeta \).
3. The set \( \{ S_\lambda; \lambda \text{ any partition} \} \)
   is a linear basis of \( \zeta \).

**5.4 Duality in \( \zeta \)**

Let \( W : \zeta \rightarrow \zeta \)
denote the automorphism such that

\( W(H_k) = I_k, \) for every \( k \in \mathbb{Z}^+ \).

Since \( \lim \chi(n) = \zeta \), Corollary 4.28 implies

**Theorem 5.13.**

1. For every partition \( \lambda \),
   \( W(S_\lambda) = S_{\bar{\lambda}} \).
2. In particular,
   \( W(I_k) = H_k, \) for every \( k \in \mathbb{Z}^+ \);
   then, the automorphisms \( W \) is an involution.
In this subsection we follow Okounkov and Olshanski [51].

The algebra $\Lambda^*(n)$ of shifted symmetric polynomials is the algebra of polynomials $p(x_1, x_2, \ldots, x_n)$ that satisfy the shifted symmetry condition:

$$p(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = p(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n),$$

for $i = 1, 2, \ldots, n - 1$.

The Harish-Chandra isomorphism $\chi_n$ is the algebra isomorphism $\chi_n : \zeta(n) \to \Lambda^*(n)$, $A \mapsto \chi_n(A)$, where $\chi_n(A)$ is the shifted symmetric polynomial such that, for every highest weight module $V_\mu$, the evaluation $\chi_n(A)(\mu_1, \mu_2, \ldots, \mu_n)$ equals the eigenvalue of $A \in \zeta(n)$ in $V_\mu$ (see, e.g. [51]).

From Corollary 4.16, it follows

**Proposition 6.1.**

$$\chi_n(H_k(n)) = e_k^*(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (x_{i_1} + k - 1)(x_{i_2} + k - 2)\cdots(x_{i_k})$$

for every $k = 1, 2, \ldots, n$.

Clearly, $\chi_n(H_0(n)) = e_0^*(x_1, x_2, \ldots, x_n) = 1$.

The polynomials $e_k^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n)$ are the elementary shifted symmetric polynomials.

From Theorem 4.23, it follows

**Proposition 6.2.**

$$\chi_n(I_k(n)) = h_k^*(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 < \cdots < i_k \leq n} (x_{i_1} - k + 1)(x_{i_2} - k + 2)\cdots(x_{i_k})$$

for every $k = 1, 2, \ldots, n$.

Clearly, $\chi_n(I_0(n)) = h_0^*(x_1, x_2, \ldots, x_n) = 1$.

The polynomials $h_k^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n)$ are the complete shifted symmetric polynomials.

Recall that, given a variable $z$ and a natural integer $p$, the symbol $(z)_p$ denotes the falling factorial polynomial:

$$(z)_p = z(z - 1)\cdots(z - p + 1), \quad p \geq 1, \quad (z)_0 = 1.$$

Let $\mu$ be a partition, $\tilde{\mu}_1 \leq n$.  


Following [51], consider the polynomial

\[ s_\lambda^*(x_1, \ldots, x_n) = \frac{\det(x_i + n - i)_{\lambda_i + n - j}}{\det(x_i + n - i)_{n - j}} \]

\[ = \sum_{T \in \text{RSSYT}^*(\lambda)} (x_{T(s)} - c(s)), \quad (44) \]

where \( \text{RSSYT}(\mu) \) denotes the set of all reverse semistandard Young tableaux \( T \) of shape \( \lambda \) over the set \( \{1, 2, \ldots, n\} \). \( T(s) \) denotes the symbol of in the cell \( s \) of the Ferrers diagram of \( \mu \) and \( c(s) = j - i \) is the content of the cell \( s \) in position \( (i, j) \).

The polynomials \( s_\mu^*(x_1, \ldots, x_n) \in \Lambda^*(n) \) are the shifted Schur polynomials.

From the Characterization Theorem for the Schur elements \( S_\lambda(n) \in \zeta(n) \) (see subsection 4.2) and the Characterization Theorem for the shifted Schur polynomials [51], we have:

**Theorem 6.3.** For every \( \lambda, \tilde{\lambda}_1 \leq n \),

\[ \chi_n(S_\lambda(n)) = s_\lambda^*(x_1, \ldots, x_n). \]

From Theorem 4.13 and Proposition 6.1, it follows

**Proposition 6.4.**

1. The set
   \[ \{ e_k^*(x_1, x_2, \ldots, x_n); \ k = 1, 2, \ldots, n \} \]
   is a set of free algebra generators of the polynomial algebra \( \Lambda^*(n) \).

2. The set
   \[ \{ h_k^*(x_1, x_2, \ldots, x_n); \ k = 1, 2, \ldots, n \} \]
   is a set of free algebra generators of the polynomial algebra \( \Lambda^*(n) \).

3. The set
   \[ \{ s_\lambda^*(x_1, \ldots, x_n); \ \tilde{\lambda}_1 \leq n \} \]
   is a linear basis of the polynomial algebra \( \Lambda^*(n) \).

**7 The algebra \( \Lambda^* \) of shifted symmetric functions**

Let

\[ \mathbf{i}_{n+1,n}^* : \Lambda^*(n) \rightarrow \Lambda^*(n + 1) \]

---

*6 A Young tableau whose entries belong to \( \{1, \ldots, n\} \) and weakly decrease from left to right along each row and strictly decrease down each column.*
be the algebra monomorphism such that

\[ i_{n+1,n}^* (e_k^*(x_1, x_2, \ldots, x_n)) = e_k^*(x_1, x_2, \ldots, x_n, x_{n+1}), \]

for \( k = 1, 2, \ldots, n \).

Given \( m \in \mathbb{Z}^+ \), let \( \Lambda^*(n)^{(m)} \) denote the \( m \)-th filtration element of \( \Lambda^*(n) \) (with respect to the filtration induced by the standard filtration of the algebra of polynomials in the variables \( x_1, x_2, \ldots, x_n \)).

Clearly, the monomorphisms \( i_{n+1,n}^* \) are morphisms in the category of filtered algebras, that is

\[ i_{n+1,n}^* \left[ \Lambda^*(n)^{(m)} \right] \subseteq \Lambda^*(n+1)^{(m)}. \]

The algebra of shifted symmetric functions \( \Lambda^* \) is the direct limit (in the category of filtered algebras):

\[ \Lambda^* = \lim_{\rightarrow} \Lambda^*(n). \]

The algebra \( \Lambda^* \) inherits a structure of filtered algebra, where

\[ \Lambda^*(m) = \lim_{\rightarrow} \Lambda^*(n)^{(m)}. \]

Let

\[ \pi_{n,n+1}^* : \Lambda^*(n+1) \rightarrow \Lambda^*(n) \]

be the algebra epimorphism such that

\[ \pi_{n,n+1}^* (e_k^*(x_1, x_2, \ldots, x_n, x_{n+1})) = e_k^*(x_1, x_2, \ldots, x_n), \]

for \( k = 1, 2, \ldots, n \), and

\[ \pi_{n,n+1}^* (e_{n+1}^*(x_1, x_2, \ldots, x_n, x_{n+1})) = 0. \]

Clearly

\[ \pi_{n,n+1}^* (f^*(x_1, x_2, \ldots, x_n, x_{n+1})) = f^*(x_1, x_2, \ldots, x_n, 0), \]

for every \( f^*(x_1, x_2, \ldots, x_n, x_{n+1}) \in \Lambda^*(n+1) \).

As for the centers \( \zeta(n+1) \) and \( \zeta(n) \), the following Remarks and Proposition on \( \Lambda^*(n+1) \) and \( \Lambda^*(n) \) are obvious from the definitions.

**Proposition 7.1.** We have:

1. \( \text{Ker}(\pi_{n,n+1}^*) \) is the bilateral ideal

   \[ (e_{n+1}^*(x_1, x_2, \ldots, x_n, x_{n+1})) \]

   of \( \Lambda^*(n+1) \) generated by the element \( e_{n+1}^*(x_1, x_2, \ldots, x_n, x_{n+1}) \).

2. The projection \( \pi_{n,n+1}^* \) is the left inverse of the monomorphism \( i_{n+1,n}^* \). In symbols,

   \[ \pi_{n,n+1}^* \circ i_{n+1,n}^* = \text{Id}_{\Lambda^*(n)}. \]
3. If \( m \leq n \), then the restriction \( \pi_{n,n+1}^* \) of \( \pi_{n,n+1} \) to \( \Lambda^*(n+1)^{(m)} \) and the restriction \( \iota_{n+1,n}^* \) of \( \iota_{n+1,n} \) to \( \Lambda^*(n)^{(m)} \) are the inverse of each other.

From Proposition 7.1, the algebra \( \Lambda^* \) (direct limit) is the same as the inverse limit in the category of filtered algebras

\[
\Lambda^* = \lim_{\leftarrow} \Lambda^*(n)
\]

with respect to the system of the projections \( \pi_{n,n+1}^* \), and therefore, the algebra \( \Lambda^* \) is the algebra of shifted symmetric functions of [51].

Consider the commutative diagram:

\[
\begin{array}{ccc}
\zeta^{(m)}(n) & \xleftarrow{\pi_{n,n+1}} & \zeta^{(m)}(n+1) \\
\downarrow{\chi_n} & & \downarrow{\chi_{n+1}} \\
\Lambda^*(m)(n) & \xleftarrow{\pi_{n,n+1}^*} & \Lambda^*(m)(n+1) \\
\end{array}
\]

\[ \text{(46)} \]

**Theorem 7.2.** If \( m \leq n \), the pairs of horizontal arrows in the commutative diagram (46) denote mutually inverse isomorphisms.

Passing to the direct limit, we get the isomorphism of filtered algebras:

\[
\chi : \zeta \to \Lambda^*.
\]

Given \( \varphi = \lim \varphi(n) \in \zeta^{(m)} \) and every partition \( \mu \), if

\[
n \geq \max\{m, l(\bar{\mu}) = \mu_1\},
\]

then

\[
\chi_n(\varphi(n))(\bar{\mu}) = \chi_{n+1}(\iota_{n+1,n}^*(m)(\varphi(n)))(\bar{\mu}) = \chi_{n+1}(\varphi(n+1))(\bar{\mu}).
\]

Therefore, the sequence

\[
\left( \chi_n((\varphi(n)))(\bar{\mu}) \right)_{n \in \mathbb{N}^+}
\]

is definitively constant and the eigenvalue

\[
\chi(\varphi)(\bar{\mu}) = \chi_n(\varphi(n))(\bar{\mu}),
\]

\( n \) sufficiently large, is well-defined.
Corollary 7.3.

1. For every \( k \in \mathbb{Z}^+ \),
\[
\chi(H_k) = e^*_k \in \Lambda^*,
\]
where
\[
e^*_k = \sum_{i_1 < i_2 < \cdots < i_k} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,
\]
e\( _k \) the \( k \)-th elementary shifted symmetric function;

2. For every \( k \in \mathbb{Z}^+ \),
\[
\chi(I_k) = h^*_k \in \Lambda^*,
\]
where
\[
h^*_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,
\]
h\( _k \) the \( k \)-th complete shifted symmetric function.

Since \( \chi_n(S_\lambda(n)) = s^*_\lambda(n) \), from Proposition 5.8 item 2) and Theorem 5.9 item 2) we have

Corollary 7.4.

1. (stability property [51]) \( \pi^*_{n,n+1}(s^*_\lambda(n+1)) = s^*_\lambda(n), \quad n \in \mathbb{Z}^+ \).
2. If \( n \geq |\lambda| \), then
\[
\iota^*_{n+1,n}(s^*_\lambda(n)) = s^*_\lambda(n+1).
\]
The shifted symmetric Schur function \( s^*_\lambda \) is the (direct/inverse) limit
\[
s^*_\lambda = \lim(s^*_\lambda(n)) = \lim(s^*_\lambda(n)).
\]
Then

Corollary 7.5. For every \( \lambda \), we have

1. \( \chi(S_\lambda) = s^*_\lambda \).

2. \( s^*_\lambda = \sum_{T \in \text{RSSYT}(\lambda)} (x_T - c(s)), \)
where RSSYT(\( \lambda \)) is the set of all reverse semistandard Young tableaux \( T \) of shape \( \lambda \) over the set \( \mathbb{Z}^+ \).
Let \( W : \zeta \to \zeta \) denote the automorphism such that
\[
W(H_k) = I_k, \quad \text{for every } k \in \mathbb{Z}^+,
\]
and let
\[
w : \Lambda^* \to \Lambda^*
\]
denote the automorphism such that
\[
w(e_k^*) = h_k^*, \quad \text{for every } k \in \mathbb{Z}^+.
\]
Clearly,
\[
\chi \circ W = w \circ \chi.
\]

**Corollary 7.6.**

1. For every partition \( \lambda \),
\[
w(s_\lambda^*) = s_\lambda^* 
\]

2. In particular,
\[
w(h_k^*) = e_k^*, \quad \text{for every } k \in \mathbb{Z}^+;
\]
then, the automorphism \( w \) is an involution.

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