ON LANDAU-GINZBURG MODELS FOR QUADRICS AND FLAT SECTIONS OF DUBROVIN CONNECTIONS

C. PECH, K. RIETSCH, AND L. WILLIAMS

Abstract. This paper proves a version of mirror symmetry expressing the (small) Dubrovin connection for even-dimensional quadrics in terms of a mirror-dual Landau-Ginzburg model $(\tilde{X}_{\text{can}}, W_q)$. Here $\tilde{X}_{\text{can}}$ is the complement of an anticanonical divisor in a Langlands dual quadric. The superpotential $W_q$ is a regular function on $\tilde{X}_{\text{can}}$ and is written in terms of coordinates which are naturally identified with a cohomology basis of the original quadric. This superpotential is shown to extend the earlier Landau-Ginzburg model of Givental, and to be isomorphic to the Lie-theoretic mirror introduced in [Rie08]. We also introduce a Laurent polynomial superpotential which is the restriction of $W_q$ to a particular torus in $\tilde{X}_{\text{can}}$. Together with results from [PR13] for odd quadrics, we obtain a combinatorial model for the Laurent polynomial superpotential in terms of a quiver, in the vein of those introduced in the 1990’s by Givental for type $A$ full flag varieties. These Laurent polynomial superpotentials form a single series, despite the fact that our mirrors of even quadrics are defined on dual quadrics, while the mirror to an odd quadric is naturally defined on a projective space. Finally, we express flat sections of the (dual) Dubrovin connection in a natural way in terms of oscillating integrals associated to $(\tilde{X}_{\text{can}}, W_q)$ and compute explicitly a particular flat section.

Contents

1. Introduction 1
2. Landau-Ginzburg models for odd quadrics 6
3. Landau-Ginzburg models for even quadrics 11
4. The quiver mirrors $(\tilde{X}_{\text{Lus}}, W_{q,\text{Lus}})$ 24
5. The A-model and B-model connections 26
6. The hypergeometric flat section of a quadric 31
References 36

1. Introduction

Suppose $X$ is a smooth projective complex Fano variety of dimension $N$. Starting from $X$ as the ‘$A$-model’, Dubrovin constructed a flat connection on a trivial bundle with fiber $H^*(X, \mathbb{C})$, using Gromov-Witten invariants of $X$, see Section 5. The ‘$B$-models’ of Fano varieties were first introduced in [Wit97] and [Giv93]. In our setting $X$ will always have Picard rank 1. In this case the base of the trivial bundle on the $A$-side can be taken to be the two-dimensional complex torus $\mathbb{C}^* \times \mathbb{C}^*$ with coordinates $q$ and $\hbar$. The Dubrovin connection is flat and therefore defines a $D$-module $M_A$, where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}](\partial_\hbar, \partial_q)$.
In [Giv96], Givental computed the ‘small $J$-function’ and the ‘quantum differential equation’ of projective hypersurfaces, such as quadrics (see Section 6). He also proved the first mirror theorem in this setting, which states that the coefficients of the $J$-function (and hence the solutions to the quantum differential equation) can be expressed as oscillating integrals. When the cohomology of the hypersurface is generated in degree 2, e.g. for odd-dimensional quadrics, then the coefficients of the $J$-function generate the $A$-model $D$-module $M_A$. For even-dimensional quadrics this is no longer the case.

In this paper, we exploit the fact that quadrics are homogeneous spaces for the special orthogonal group and thus also have mirror LG models defined using Lie theory [Rie08]. We express these Lie theoretic mirrors in certain canonical coordinates and show how to reconstruct in a natural way the entire $D$-module $M_A$ on the mirror side from a Gauss-Manin system $M_B$. In particular, we obtain formulas for flat sections of the Dubrovin connection where the coefficients are oscillating integrals. We also investigate the comparison between various choices of mirrors for quadrics including particularly Givental’s mirror and our canonical LG model.

We begin describing our results by giving an overview of various LG models for quadrics, including the new ones introduced in this paper. We are then able to state our comparison results followed by our versions of the mirror theorem and some applications.

**Acknowledgements.** The authors thank Sasha Givental for useful comments and suggestions, leading to major improvements, particularly in the exposition. We thank Bernard Leclerc for pointing us to the references [GLS08a] and [GLS08b]. The first two authors also thank Yankı Lekili for helpful conversations. The middle author thanks Dale Peterson.

1.1. Overview of LG models for quadrics.

**Givental’s mirror.** Givental’s mirror to the quadric $X = Q_N$ is defined by a smooth affine variety (the Givental mirror manifold)

$$\hat{X}_{q,Giv} = \left\{ (\nu_1, \ldots, \nu_{N+2}) \in (\mathbb{C}^*)^{N+2} \mid \prod_{i=1}^{N+2} \nu_i = q, \nu_{N+1} + \nu_{N+2} = 1 \right\}$$

with superpotential

$$W_{q,Giv}(\nu_1, \ldots, \nu_{N+2}) = \nu_1 + \ldots + \nu_N,$$

and volume form

$$\omega_{q,Giv} = \frac{\Lambda_{i=1}^{N+2} d\log \nu_i}{d(\nu_{N+1} + \nu_{N+2}) \wedge d\log(\prod_{i=1}^{N+2} \nu_i)}.$$

Note that $\hat{X}_{q,Giv}$ is a hypersurface in an $(N + 1)$-dimensional torus and $\omega_{q,Giv}$ is the residue of the standard holomorphic volume form on the torus (compare e.g. [Pha11]). Givental’s mirror theorem expresses the coefficients of the $J$-function of $Q_N$ as oscillating integrals involving $W_{q,Giv}$ and $\omega_{q,Giv}$ over some middle-dimensional cycles in $\hat{X}_{q,Giv}$. 
A canonical mirror. A Laurent polynomial LG model \((\hat{X}_{\text{Prz}}, \mathcal{W}_{q, \text{Prz}})\) for the \(N\)-dimensional quadric \(X = Q_N\),

\[
\hat{X}_{\text{Prz}} = (\mathbb{C}^*)^N, \quad \mathcal{W}_{q, \text{Prz}} = z_1 + z_2 + \ldots + z_{N-1} + \frac{(z_N + q)^2}{z_1 z_2 \cdots z_N},
\]

can be obtained from Givental’s mirror by a change of variables which is essentially the one found in [Prz13, Remark 19], see also [GS13]. We recall the change of variables in Sections 2.2 and 3.7. This LG model is a partial compactification of Givental’s mirror. The torus-invariant volume form on \(\hat{X}_{\text{Prz}}\) restricts to Givental’s volume form \(\omega_{\text{can}}\).

A Lie-theoretic mirror. The smooth quadric \(Q_N\) inside \(\mathbb{P}^{N+1}\) is naturally a homogeneous space for the group \(\text{Spin}_{N+2}(\mathbb{C})\) associated to the defining quadratic form. The mirror construction from [Rie08] applies in this setting. It gives a regular function \(\mathcal{W}_{q, \text{Lie}}\) on an \(N\)-dimensional affine subvariety \(X_{\text{Lie}}\) inside the full flag variety for the Langlands dual group, namely the full flag variety for \(\text{PSO}_{N+1}(\mathbb{C})\) if \(N\) is odd, and for \(\text{PSO}_{N+2}(\mathbb{C})\) otherwise. The precise definition of \((\hat{X}_{\text{Lie}}, \mathcal{W}_{q, \text{Lie}})\) is recalled in Section 3.4.

The affine variety \(X_{\text{Lie}}\) also has a holomorphic volume form \(\omega_{\text{can}}\), which is explicitly described in [Rie08]. Indeed \(X_{\text{Lie}}\) is an affine Richardson variety and it is also a log Calabi-Yau as seen by combining [KLS14, Appendix A] and [KSI14, Section 4.2].

By the main result of [Rie08] there is an isomorphism between the Jacobi ring of \(\mathcal{W}_{q, \text{Lie}}\) and the quantum cohomology ring of \(Q_N\) (with the quantum parameter inverted). This is not true for the mirrors \((\hat{X}_{q, \text{Giv}}, \mathcal{W}_{q, \text{Giv}})\) and \((\hat{X}_{\text{Prz}}, \mathcal{W}_{q, \text{Prz}})\).

A canonical mirror. The canonical mirror of an odd-dimensional quadric \(Q_{2m-1}\) was introduced in [PR13], and is defined on the complement \(X_{\text{can}}\) of an anticanonical divisor in the projective space \(\hat{X} = \mathbb{P}(H^*(Q_{2m-1}, \mathbb{C})^*)\). Suppose \(p_0, \ldots, p_{2m-1}\) are the homogeneous coordinates on \(\hat{X}\) corresponding to the Schubert basis of \(H^*(Q_{2m-1}, \mathbb{C})\). Then \(\mathcal{W}_q : X_{\text{can}} \to \mathbb{C}\) is given by

\[
\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1} p_{2m-1-\ell}}{\delta_\ell} + q \frac{p_1}{p_{2m-1}},
\]

where

\[
\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k} \text{ for } 1 \leq \ell \leq m-1
\]

with \(N = 2m - 1\).

The canonical mirror of an even-dimensional quadric \(Q_{2m-2}\) introduced here is similar in appearance, however the mirror projective space is replaced by a ‘mirror quadric’ \(\hat{X} = \hat{Q}_{2m-2}\). Note first that \(\mathbb{P}(H^*(Q_{2m-2}, \mathbb{C})^*)\) has dimension \(2m -1\) and homogeneous coordinates \(p_0, \ldots, p_{m-1}, p_{2m-1}, \ldots, p_{2m-2}\) corresponding to the Schubert basis of \(H^*(Q_{2m-2}, \mathbb{C})\). The mirror quadric \(\hat{Q}_{2m-2}\) is the quadratic hypersurface inside \(\mathbb{P}(H^*(Q_{2m-2}, \mathbb{C})^*)\) defined by

\[
p_{m-1} p_{2m-1} \cdot p_{m-1} - p_m p_{m-2} + \cdots + (-1)^{m-1} p_{2m-2} p_0 = 0.
\]
The superpotential $W_q$ is defined by the formula
\[
W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + q \frac{p_1}{p_{2m-2}},
\]
which is regular on the complement $\tilde{X}_{\text{can}}$ of an anticanonical divisor in $\tilde{Q}_{2m-2}$. Here $\delta_\ell$ is defined by the formula in equation (5), with $N = 2m - 2$.

Laurent polynomial mirrors with a quiver description. For $X = \tilde{Q}_{2m-1}$ the Laurent polynomial mirror
\[
W_{q,\text{Lus}} = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q \frac{a_1 + b_1}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1},
\]
was introduced in [PR13, Proposition 8]. It was obtained by restricting $W_{q,\text{Lie}}$ to a natural choice of torus $\tilde{X}_{\text{Lus}}$ in $\tilde{X}_{\text{Lie}}$, on which we consider coordinates like the ones used by Lusztig in [Lus94].

For the even quadric $X = \tilde{Q}_{2m-2}$ we define here an analogous Laurent polynomial mirror
\[
W_{q,\text{Lus}} = a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q \frac{a_1 + b_1}{a_1 \cdots a_{m-2} c d b_{m-2} \cdots b_1},
\]
also obtained from a torus $\tilde{X}_{\text{Lus}}$ in $\tilde{X}_{\text{Lie}}$. Note that $(\tilde{X}_{\text{Lus}}, W_{q,\text{Lus}})$ is not isomorphic to the other Laurent polynomial mirror $(\tilde{X}_{\text{Prz}}, W_{q,\text{Prz}})$.

In Section 4 we interpret $(\tilde{X}_{\text{Lus}}, W_{q,\text{Lus}})$ in terms of a quiver, in the spirit of [Giv97, BCFKvS98, BCFKvS00]. The quiver we associate to $Q_N$ looks like an augmentation of a type $D_N$ quiver (see Figure 3). Note that the mirrors for type $A$ homogeneous spaces from [Giv97, EHX97, BCFKvS98, BCFKvS00] also relate to Lusztig coordinates, see [Rie06, Rie08].

1.2. Comparison of the canonical LG model with the other mirrors.

Isomorphism with the Lie-theoretic mirror. It was proved in [PR13] that for the odd-dimensional quadric $Q_{2m-1}$ viewed as a homogeneous space for Spin$_{2m+1}$, there is an isomorphism between the domain $\tilde{X}_{\text{Lie}}$ of $W_{q,\text{Lie}}$ and the domain $\tilde{X}_{\text{can}}$ of the canonical mirror. This isomorphism identifies the superpotentials $W_{q,\text{Lie}}$ and $W_q$.

**Theorem 1.1 (PR13 Theorem 1).** If $X = Q_{2m-1}$ is an odd-dimensional quadric, there is an isomorphism of affine varieties $\tilde{X}_{\text{Lie}} \rightarrow \tilde{X}_{\text{can}}$ such that the following diagram commutes
\[
\begin{array}{ccc}
\tilde{X}_{\text{Lie}} & \sim & \tilde{X}_{\text{can}} \\
\downarrow W_{q,\text{Lie}} & & \downarrow W_q \\
\mathbb{C} & & \mathbb{C}
\end{array}
\]

A key ingredient in the construction of the isomorphism is the geometric Satake correspondence of [Lus83, Gin95, MV07], which identifies the projective space $\tilde{X} = \mathbb{P}(H^*(Q_{2m-1}, \mathbb{C})^*)$ containing $\tilde{X}_{\text{can}}$ as the projectivisation of a representation of PSp$_{2m}(\mathbb{C})$.

In this paper, we prove the same result in the case of even-dimensional quadrics $Q_{2m-2}$ (see Theorem 3.2).
Comparison with the Givental mirror. In Sections 2.2 and 3.7 we relate \((\hat{X}_{\text{can}}, W_q)\) to the Givental mirror \((\hat{X}_{\text{can}, \text{Giv}}, W_{q, \text{Giv}})\). In particular, we prove the following proposition.

**Proposition 1.2.** There is an embedding, \(\hat{X}_{q, \text{Giv}} \hookrightarrow \hat{X}_{\text{can}}\), of the Givental mirror manifold into the canonical mirror such that the volume form \(\omega_{\text{can}}\) on \(\hat{X}_{\text{can}}\) (suitably normalized) pulls back to \(\omega_{q, \text{Giv}}\), and the superpotential \(W_q\) pulls back to \(W_{q, \text{Giv}}\).

An advantage of the mirror \(W_q\) over its predecessor \(W_{q, \text{Giv}}\) is that the former has the expected number of critical points (at fixed generic value of \(q\), namely \(\dim(H^*(Q_N, \mathbb{C}))\)).

**Proposition 1.3.** The superpotential \(W_q : \hat{X}_{\text{can}} \to \mathbb{C}\) for the mirror of \(Q_N\) has \(\dim H^*(Q_N, \mathbb{C})\) many non-degenerate critical points. Precisely two of these in the even \(N\) case, and one of these in the odd \(N\) case, are not contained in the image of the embedding, \(\hat{X}_{q, \text{Giv}} \hookrightarrow \hat{X}_{\text{can}}\), of the Givental mirror manifold.

In the special case of \(Q_4\) this lack of critical points of the classical mirror was already observed in [EHX97]. It was suggested there to solve it using a partial compactification and this was carried out for the first time, albeit in an ad hoc fashion. This was also a motivation for introducing the Lie-theoretic mirrors \((\hat{X}_{\text{Lie}}, W_{q, \text{Lie}})\) in [Rie08]. In the odd quadrics case Proposition 1.2 is proved using a combination of results from [GS13] and [PR13]. In the even quadrics case we prove it in the present paper.

The first part of Proposition 1.3 is an immediate consequence of analogous result for \((\hat{X}_{\text{Lie}}, W_{q, \text{Lie}})\) from [Rie08] together with Theorem 2.1 and Theorem 3.2 respectively. The second part comes from a direct calculation, see Propositions 2.2 and 3.13.

Comparison with \((\hat{X}_{\text{Prz}}, W_{q, \text{Prz}})\). For odd quadrics \(Q_{2m-1}\) it was proved in [PR13] that after a change of variables, \(\hat{X}_{\text{Prz}}\) gets identified with a particular torus inside \(\hat{X}_{\text{can}}\). This embedding identifies the two superpotentials \(W_{q, \text{Prz}}\) and \(W_q\). We recall this result in Section 2.2.

For even quadrics \(Q_{2m-2}\), the situation is more complicated. We consider the complement of a particular hyperplane section in \(\hat{X}_{\text{Prz}}\) for which we construct an embedding into \(\hat{X}_{\text{can}}\) such that \(W_q\) pulls back to \(W_{q, \text{Prz}}\) and show that this embedding cannot be extended. Moreover we observe that the image of the embedding is precisely the embedded Givental mirror manifold inside \(\hat{X}_{\text{can}}\). Therefore the Givental mirror manifold is in a sense the intersection of the mirrors \(\hat{X}_{\text{Prz}}\) and \(\hat{X}_{\text{can}}\). These results are contained in Section 3.7.

Comparison with the quiver mirror. The quiver mirror \((\hat{X}_{\text{Lus}}, W_{q, \text{Lus}})\) is obtained from the Lie-theoretic mirror \((\hat{X}_{\text{Lie}}, W_{q, \text{Lie}})\), and hence from the canonical mirror \((\hat{X}_{\text{can}}, W_q)\), by restricting it to a torus (see Propositions 2.2 and 3.11).

### 1.3. The mirror theorem for \(A\)-model and \(B\)-model \(D\)-modules.

Recall that the Dubrovin connection for \(Q_N\) gives rise to a module \(M_A\) over the ring of differential operators \(D = \mathbb{C}[[\hbar^{\pm 1}, q^{\pm 1}]](\partial_\hbar, \partial_q)\), see (18). On the \(B\)-side we obtain a \(D\)-module \(M_B\) by considering a Gauss-Manin system associated to the mirror \((\hat{X}_{\text{can}}, W_q)\), see Definition 5.1. For odd-dimensional quadrics it is already known that there is an isomorphism between \(M_A\) and \(M_B\). This follows from
The isomorphism takes a particularly natural form in the canonical coordinates, as recalled in Theorem 5.2.

For even dimensional quadrics we construct in Section 5 an explicit isomorphism from the $A$-model $D$-module $M_A$ to a natural submodule of the $B$-model $D$-module $M_B$, see Theorem 5.3. We conjecture that this submodule is in fact all of $M_B$, so that $M_A$ and $M_B$ are isomorphic. Here our canonical mirror $(\check{X}_{\text{can}}, W_q)$ takes place on a dual quadric. We note that there is a non-trivial cluster algebra structure on the coordinate ring of $\check{X}_{\text{can}}$, which plays an important role in our proof of the isomorphism.

1.4. Applications.

In Section 6 we turn to the problem of constructing flat sections $S: \mathbb{C}^*_\hbar \times \mathbb{C}^*_q \to H^*(X, \mathbb{C})$ for a dual version of the Dubrovin connection of the quadric $Q_N$, using the $B$-model. Namely we are interested in solutions to the partial differential equation

\begin{align}
\frac{1}{\hbar} \frac{\partial S}{\partial q} &= \frac{1}{2\pi i} \sigma_1 * q S, \\
\frac{1}{\hbar} \frac{\partial S}{\partial \hbar} &= -\frac{1}{2\pi i} c_1(TX) * q S - \text{Gr}(S).
\end{align}

First we observe that one can write coefficients of flat sections from the $B$-model as oscillating integrals using $(\check{X}_{\text{can}}, W_q)$. This goes as follows. Consider any critical point $p$ of $W_q$. By a procedure outlined by Givental in the setting of full flag varieties in [Giv97, Section 2], there should be an associated non-compact, middle-dimensional cycle $\Gamma_p$ in $\check{X}_{\text{can}}$ for which $\Re\left(\frac{1}{\hbar} W_q \right) \to -\infty$ rapidly in any unbounded direction of $\Gamma_p$ (here we suppress the dependence on $\hbar$ and $q$ in the notation for simplicity). Then, as in [MR13, Section 4.2], the integrals $\int_{\Gamma_p} e^{\frac{1}{\hbar} W_q p_i \omega_{\text{can}}}$ locally determine coefficients of a section $S_{\Gamma_p}$. This section is given by the formula

\[ S_{\Gamma_p} = \frac{1}{(2\pi i)^N} \sum_{i=0}^{N} \left( \int_{\Gamma_p} e^{\frac{1}{\hbar} W_q p_i \omega_{\text{can}}} \right) \sigma_{N-i} \]

in the odd quadric case, and by a similar formula in the even quadric case. The local section $S_{\Gamma_p}$ is a solution to (8) as a consequence of Theorems 5.2 and 5.3. With an appropriate partial compactification these cycles should have an interpretation in terms of Lefschetz thimbles, compare [Sei08].

If we replace the cycle $\Gamma_p$ with a compact torus $(S^1)^N$ we obtain a global holomorphic flat section of the dual Dubrovin connection whose coefficients are given by residue integrals. In Section 6 we construct this solution explicitly by expanding it as a power series, using the quiver mirror $(\check{X}_{\text{lus}}, W_{q, \text{lus}})$ to express the integrals in coordinates. Moreover we verify the resulting formula in a different way on the $A$-side.

2. Landau-Ginzburg models for odd quadrics

The quadrics are cominuscule homogeneous spaces (for the Spin groups). Therefore, in addition to the Givental approach [Giv98] for constructing LG models, there is another LG model for each quadric on an affine variety (generally larger than a torus), which was defined by the second-named author using a Lie-theoretic construction [Rie08]. Namely for any projective homogeneous space $X = G/P$ of a simple complex algebraic group, [Rie08] constructed a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. We call...
it the Lie-theoretic LG model. It was shown in [Rie08] that this LG model recovers
the Peterson variety presentation [Pet97] of the quantum cohomology of \( X = G/P \).
It therefore defines an LG model whose Jacobi ring has the correct dimension. In
this section we will rewrite the Lie-theoretic LG model in terms of natural projective
coordinates on \( \mathbb{P}(H^*(Q_N, \mathbb{C}))^* \). We call the resulting LG model the canonical
LG model of \( Q_N \).

Note that for odd-dimensional quadrics \( Q_{2m-1} \) a recent paper [GS13] of Gor-
bounov and Smirnov constructed directly a partial compactification of the Givental
mirrors, without making use of [Rie08].

2.1. The canonical LG model for \( Q_{2m-1} \). LG models for odd-dimensional
quadrics with the expected number of critical points have been constructed in
[Rie08] (where they appear as a special case), [GS13], and finally [PR13]. Here we
recall the main results from the paper [PR13], which contains the formulation for
[Rie08] (where they appear as a special case), [GS13], and finally [PR13]. Here we
call the resulting LG model the canonical LG model of \( Q_N \).

In this section our A-model variety \( X = X_N = X_{2m-1} \) is the quadric \( Q_N =
Q_{2m-1} \). Recall that an odd-dimensional quadric has one-dimensional cohomology
groups in even degrees spanned by Schubert classes \( \sigma_i \in H^{2i}(Q_{2m-1}, \mathbb{C}) \) for \( 0 \leq
i \leq 2m - 1 \), and no other cohomology. To construct its canonical mirror first
consider the projective space \( \hat{X} = \hat{X}_{2m-1} = \mathbb{P}^{2m-1} \) with homogeneous coordinates
\( (p_0 : p_1 : \cdots : p_{2m-1}) \) in one-to-one correspondence with these Schubert classes \( \sigma_i \).

Inside \( \hat{X} \) we have the open affine subvariety \( \hat{X}_{\text{can}} \subset \mathbb{P}^{2m-1} \) defined by:

\[
\hat{X}_{\text{can}} = \hat{X}_{2m-1} := \hat{X} \setminus D,
\]

where \( D := D_0 + D_1 + \cdots + D_{m-1} + D_m \), the divisors \( D_i \) being given by
\[
D_0 := \{ p_0 = 0 \},
\]
\[
D_\ell := \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-1-\ell+k} = 0 \right\} \quad \text{for } 1 \leq \ell \leq m - 1,
\]
\[
D_m := \{ p_{2m-1} = 0 \}.
\]

The divisor \( D \) is an anticanonical divisor. Indeed, the index of \( \hat{X} = \mathbb{P}^{2m-1} \) is \( 2m \). As
a result, there is a unique up to scalar \( (2m-1) \)-form \( \omega_{\text{can}} \), which is regular on \( \hat{X}_{\text{can}} \)
and has logarithmic poles on \( D \). For all \( 1 \leq j \leq m - 1 \), take \( r_j \in \{ p_j, p_{2m-1-j} \} \).
Setting \( p_0 = 1 \), the restriction of \( \omega_{\text{can}} \) to the torus \( \{ r_j \neq 0 \mid 1 \leq j \leq m - 1 \} \)
inside \( \hat{X}_{\text{can}} \) is given by

\[
\omega_{\text{can}} = \frac{\bigwedge_{1 \leq j \leq m-1} dr_j \wedge \bigwedge_{1 \leq \ell \leq m-1} d\delta_\ell \wedge dp_{2m-1}}{\delta_1 \cdots \delta_{m-1} p_{2m-1}}.
\]

We have:

**Theorem 2.1** ([PR13, Theorem 1]). The Lie-theoretic LG model \( W_{q, \text{Lie}} : \hat{X}_{\text{Lie}} \to
\mathbb{C} \) from [Rie08] for \( X = Q_{2m-1} \) is isomorphic to the canonical LG model \( W_q :
\hat{X}_{2m-1} \to \mathbb{C} \) defined by

\[
W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1} p_{2m-1-\ell}}{\delta_\ell} + q \frac{p_1}{p_{2m-1}},
\]

where \( \delta_\ell \) is given by \([\delta_\ell]\) with \( N = 2m - 1 \).

We also have another expression for the superpotential:
Proposition 2.2 ([PR13], Proposition 8]). For \( X = Q_{2m-1} \) and \( \mathcal{W}_q \) as above, there is a torus \( \tilde{X}_{\text{Lus}} := (\mathbb{C}^*)^{2m-1} \) \( \hookrightarrow \tilde{X}_{\text{can}} \) to which \( \mathcal{W}_q \) pulls back giving the Laurent polynomial expression

\[
\mathcal{W}_{q,\text{Lus}} = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q \frac{a_1 + b_1}{a_1 \cdots a_{m-1} c b_{m-1} \cdots b_1}.
\]

We call the Laurent polynomial LG model \( (\tilde{X}_{\text{Lus}}, \mathcal{W}_q) \) from Proposition 2.2 the quiver mirror. The reason for this denomination will be made clear in Section 4.

2.2. Comparison with the Givental and Laurent polynomial mirrors for odd quadrics. Let us recall the Laurent polynomial LG model of \( Q_{2m-1} \) from Equation (4)

\[
\mathcal{W}_{q,\text{Prz}} = z_1 + \cdots + z_{2m-2} + \frac{(z_{2m-1} + q)^2}{z_1 z_2 \cdots z_{2m-1}},
\]

defined over the torus

\[ \tilde{X}_{\text{Prz}} := \{(z_1, \ldots, z_{2m-1}) \mid z_i \neq 0 \ \forall \ i\}, \]

and the Givental LG model from Equation (2)

\[
\mathcal{W}_{q,\text{Giv}} = \nu_1 + \cdots + \nu_{2m-1},
\]

defined over the affine variety

\[ \tilde{X}_{q,\text{Giv}} = \left\{ (\nu_1, \ldots, \nu_{2m+1}) \mid \nu_i \neq 0 \ \forall \ i, \prod_{i=1}^{2m+1} \nu_i = q, \ \nu_{2m} + \nu_{2m+1} = 1 \right\}. \]

These two LG models are related by a birational change of coordinates analogous to that of [Prz13], Remark 19], namely

\[
\begin{align*}
    z_i &= \begin{cases} 
        \frac{\nu_{i+1}}{q^i} & \text{for } 1 \leq i \leq 2m-2; \\
        q & \text{for } i = 2m-1;
    \end{cases} \\
    \nu_i &= \begin{cases} 
        \frac{(z_{2m-1} + q)^2}{z_1 z_2 \cdots z_{2m-1}} & \text{for } i = 1; \\
        \frac{z_{i-1}}{z_{2m-1}} & \text{for } 2 \leq i \leq 2m-1; \\
        \frac{z_{2m-1} + q}{q} & \text{for } i = 2m; \\
        \frac{z_{2m-1} + q}{q} & \text{for } i = 2m + 1.
    \end{cases}
\end{align*}
\]

This change of variables defines an isomorphism

\[ \tilde{X}_{q,\text{Prz}} \setminus \{z_{2m-1} + q = 0\} \cong \tilde{X}_{q,\text{Giv}} \]

which identifies the superpotentials \( \mathcal{W}_{q,\text{Prz}} \) and \( \mathcal{W}_{q,\text{Giv}} \).

Let us now compare these two LG models with ours. Consider the change of coordinates

\[
\begin{align*}
    z_i &= \begin{cases} 
        \frac{\nu_{i+1}}{q^i} & \text{for } 1 \leq i \leq m-1; \\
        \frac{\nu_{i+1}}{q} \delta_{2m-3-i} & \text{for } m \leq i \leq 2m-3; \\
        \frac{\nu_{i+1}}{q} \delta_{2m-2-i} & \text{for } 2m-2-i \geq 2m-3; \\
        \frac{\nu_{i+1}}{q} \delta_{2m-1-i} & \text{for } i = 2m-2; \\
        \frac{\nu_{i+1}}{q} \delta_{m-1} & \text{for } i = 2m-1;
    \end{cases}
\end{align*}
\]
It is well-defined on the cluster torus \( \{ p_i \neq 0 \mid \forall 1 \leq i \leq m - 1 \} \) inside \( \tilde{X}_{\text{can}} \). Moreover, an easy calculation shows that it transforms the canonical LG model \( \mathbb{P} \) into the Laurent polynomial LG model \( \mathbb{P} \) for odd quadrics.

Indeed, using this change of variables we see that \( z_1 \ldots z_{2m-1} \) maps to

\[
\frac{p_{m-1}}{p_0} \frac{p_{m-2} \delta_0}{\delta_{m-2} p_1} q \frac{p_{m-2}}{p_{2m-1}} \left( \frac{q^{m-1} p_m}{\delta_{m-1}} \right)^2 = q^2 \left( \frac{p_{m-1}}{\delta_{m-1}} \right)^2.
\]

Moreover \( (z_{2m-1}+q)^2 \) maps to \( \left( \frac{2 m_{-1} p_m}{\delta_{m-1}} \right)^2 \) since \( \delta_{m-1} + \delta_{m-2} = p_{m-1} p_m \). It follows that \( (z_{2m-1}+q)^2 \) maps to \( \frac{2 m_{-1} p_m}{\delta_{m-1}} \). We also see that for \( 2 \leq j \leq m-1 \), \( z_j + z_{2m-1-j} \) maps to \( \frac{p_j p_{2m-j}}{\delta_j \delta_{m-j}} \) since \( \delta_j + \delta_{m-j} = p_j p_{2m-j} \). Hence via this change of variables, the Laurent polynomial superpotential \( W_{q,Prz} \) maps to

\[
\frac{p_1}{p_0} + \sum_{j=2}^{m-1} \frac{p_j p_{2m-j}}{\delta_j \delta_{m-j}} + q \frac{p_1}{p_{2m-1}} + \frac{p_m^2}{\delta_{m-1}}.
\]

which is precisely the expression of \( W_q \).

Note that this change of coordinates between \( (\tilde{X}_{Prz}, W_{q,Prz}) \) and \( (\tilde{X}_{\text{can}}, W_q) \) may also be obtained by combining the isomorphism between \( \mathbb{P} \) and the Gorbounov-Smirnov mirror from [PR13 Section 6], with the comparison between the Gorbounov-Smirnov mirror and the Laurent polynomial mirror (there called the Hori-Vafa mirror) in [GS13].

Combining both changes of coordinates, we obtain an embedding of the Givental mirror variety \( \tilde{X}_{q,Giv} \hookrightarrow \tilde{X}_{\text{can}} \), corresponding to the change of coordinates

\[
\mu_i = \begin{cases} 
\frac{p_2}{\delta_{m-1}} & \text{for } i = 1; \\
\frac{p_{i-1}}{p_1} & \text{for } 2 \leq i \leq m; \\
\left( \frac{p_{m-1} \delta_{2m-1-j}}{p_{2m-1} \delta_{m-1}} \right) & \text{for } m + 1 \leq i \leq 2m - 2; \\
\frac{p_{2m-1} \delta_{m-2}}{p_m - 1} & \text{for } i = 2m - 1; \\
\frac{p_{m-1} \delta_{m-2}}{p_m - 1} & \text{for } i = 2m; \\
\frac{\delta_{m-2}}{p_m - 1} & \text{for } i = 2m + 1.
\end{cases}
\]

The embedding identifies \( \tilde{X}_{q,Giv} \) with the intersection of cluster tori \( \{ p_i \neq 0 \mid \forall 1 \leq i \leq m \} \) in \( \tilde{X} \), the superpotential \( W_{q,Giv} \) with \( W_q \), and the form \( \omega_{q,Giv} \) with \( \omega_{\text{can}} \). This proves Proposition 1.2 from the introduction in the case of odd quadrics.

2.3. The critical points of the canonical mirror. Since the canonical mirror \( (\tilde{X}_{\text{can}}, W_q) \) is isomorphic to the Lie-theoretic mirror \( (\tilde{X}_{\text{Lie}}, W_q, \omega_{\text{Lie}}) \), it follows from [Ric08] that \( W_q \) has the ‘correct’ number of critical points on \( \tilde{X}_{\text{can}} \), that is, \( \dim H^*(Q_{2m-1}, \mathbb{C}) = 2m \). Here we give explicit expression for the critical points, and compare with the critical points of the classical mirrors \( (\tilde{X}_{q,Giv}, W_{q,Giv}) \) and \( (\tilde{X}_{Prz}, W_{q,Prz}) \).

**Proposition 2.3.** The critical points of the superpotential \( W_q \) on \( \tilde{X}_{\text{can}} \) are given by

\[
p_j = \begin{cases} 
\zeta^j & \text{if } 1 \leq j \leq m - 1; \\
\frac{1}{\zeta} \zeta^j & \text{if } m \leq j \leq 2m - 2; \\
q & \text{if } j = 2m - 1,
\end{cases}
\]
where \( \zeta \) is a primitive \((2m - 1)\)-st root of \(4q\). The associated critical value is \((2m - 1)\zeta\). Moreover there is an extra critical point given by \(p_1 = \cdots = p_{2m-2} = 0\), \(p_{2m-1} = -q\) with corresponding critical value 0. This critical point does not belong to \(\hat{X}_{Prz}, \hat{X}_{Giv}\) or \(\hat{X}_{Las}\).

**Proof.** Setting \(p_0 = 1\) we get the following relations at a critical point of \(W_q\):

\[
\frac{\partial W_q}{\partial p_1} = 1 + \left( \sum_{\ell=1}^{m-1} \frac{(-1)^{\ell + 1} p_{\ell + 1} p_{2m-1-\ell}}{\delta_\ell^2} \right) p_{2m-2} + \frac{q}{p_{2m-1}} = 0
\]

(12)

\[
\frac{\partial W_q}{\partial p_j} = \frac{p_{2m-j}}{\delta_{j-1}} + \left( \sum_{\ell=j}^{m-1} \frac{(-1)^{\ell + 1 - j} p_{\ell + 1} p_{2m-1-\ell}}{\delta_\ell^2} \right) p_{2m-1-j} = 0 \quad (2 \leq j \leq m - 1)
\]

(13)

\[
\frac{\partial W_q}{\partial p_m} = \frac{p_m (p_m - 2 \delta_{m-2})}{\delta_{m-1}^2} = 0
\]

(14)

\[
\frac{\partial W_q}{\partial p_j} = -\frac{p_{2m-j} \delta_{2m-2-j}}{\delta_{2m-1-j}^2} + \left( \sum_{\ell=2m-j}^{m-1} \frac{(-1)^{\ell + j - 2m} p_{\ell + 1} p_{2m-1-\ell}}{\delta_\ell^2} \right) p_{2m-1-j} = 0 \quad (m + 1 \leq j \leq 2m - 2)
\]

(15)

\[
\frac{\partial W_q}{\partial p_{2m-1}} = \sum_{\ell=1}^{m-1} \frac{(-1)^{\ell - 1} p_{\ell + 1} p_{2m-1-\ell}}{\delta_\ell^2} - q \frac{p_1}{p_{2m-1}^2} = 0.
\]

(16)

From Equation (14) it follows that we have two possibilities, i.e. \(p_m = 0\), or \(p_m - 1 p_m = 2 \delta_{m-2}\). If \(p_m = 0\), using (13) for \(j = m - 1, m - 2, \ldots, 2\) shows that \(p_m = p_{m+1} = p_{2m-2} = 0\). Then (12) implies \(p_{2m-1} = -q\). Using (15) for \(j = m + 1, m + 2, \ldots, 2m - 2\) shows that \(p_{m-1} = p_{m-2} = \cdots = p_2 = 0\). Finally (16) implies \(p_1 = 0\). At the corresponding critical point \((0, \ldots, 0, -q)\), the value of \(W_q\) (the critical value) is clearly 0.

Let us now assume \(p_m \neq 0\) and \(p_m - 1 p_m = 2 \delta_{m-2}\), so that \(\delta_{m-1} = \delta_{m-2}\). Combining Equations (13) for \(j = m - 1\) and (15) for \(j = m + 1\), we obtain \(p_{m-2} p_{m+1} = 2 \delta_{m-3}\), hence \(\delta_{m-2} = \delta_{m-3}\). Iteratively, we obtain

\[
\delta_{m-1} = \delta_{m-2} = \cdots = \delta_0;
\]

(17)

\[
p_{j} p_{2m-1-j} = 2 \delta_{j-1} \quad \forall 1 \leq j \leq m - 1.
\]

(18)

Combining Equations (12) and (16) with the identity (17), we get that

\[
p_{2m-1} = q,
\]

hence all the \(\delta_j\) are equal to \(q\).

Now Equations (13) for \(j = m - 1\) and (17) imply \(p_{m-1} p_{m+1} = 2 \delta_{m-1}^2\). Then Equation (13) for \(j = m - 2\) and (17) imply that \(p_{m-2} p_{m+2} = 2(p_{m-1} p_{m+1} - \delta_{m-1}^2)\). Inductively for \(j = m - 3, \ldots, 2\) we obtain

\[
\sum_{\ell=j}^{m} (-1)^{\ell - j} p_{\ell} p_{2m-\ell} = \delta_{m-1}^2
\]

(20)
for all $2 \leq j \leq m$. Then (12) implies that $p_m^2 = qp_1$.

Finally, (14) and (20) imply $p_j = \frac{1}{2} p_1^j$ for $m \leq j \leq 2m - 2$, while (13) and (20) imply $p_j = p_1^j$ for $1 \leq j \leq m - 1$. Then $p_m = \frac{1}{2} p_1^m$ together with $p_m^2 = qp_1$ implies that $p_1^{2m - 1} = 4q$, which concludes the proof.

3. LANDAU-GINZBURG MODELS FOR EVEN QUADRICS

We view the quadric $X = X_{2m-2} := Q_{2m-2}$ of dimension $2m - 2$ as a homogeneous space for the Spin group Spin$_{2m}(C)$. In this section we will introduce a canonical LG model for $X_{2m-2}$ which will be defined on an open subvariety of a dual quadric $\mathbb{H}_{2m-2} = P \setminus \text{PSO}_{2m}(C)$, see Section 3.2. Note that the projective special orthogonal group PSO$_{2m}(C)$ is the Langlands dual group to Spin$_{2m}(C)$, and both groups have the same Dynkin diagram, namely the Dynkin diagram of type $D_m$.

The main result of this section, Proposition 3.3, shows that the new LG-model is isomorphic to one defined earlier [Rie08] on a Richardson variety $X_{\text{Lie}}$ inside the full flag variety of PSO$_{2m}(C)$.

Note that in the following we will denote the group PSO$_{2m}(C)$ by $G$, since this is the group we will primarily be working with. Then the $A$-model symmetry group is $G^\vee = \text{Spin}_{2m}(C)$, and we have $X_{2m-2} = G^\vee / P^\vee$, where $P^\vee$ is the parabolic subgroup associated to the first node of the Dynkin diagram of type $D_m$.

![Dynkin diagram for $D_m$]

3.1. Notations and definitions. Let $V = C^{2m}$ with fixed quadratic form

$$Q = \begin{pmatrix} & & & & \emptyset & \\
& & & & & -1 \\
& & & & & & -1 \\
& & & & & & & -1 \\
& & & & & & & & -1 \\
\end{pmatrix}.$$  

In other words $Q(v_i, v_j) = (-1)^{\max(i,j)} \delta_{i+j, 2m+1}$ where $\{v_i\}$ is the standard basis of $C^{2m}$. For $G = \text{PSO}(V, Q) = \text{PSO}(V)$ we fix Chevalley generators $(e_i)_{1 \leq i \leq m}$ and $(f_i)_{1 \leq i \leq m}$. To be explicit we embed $\mathfrak{so}(V, Q)$ into $\mathfrak{gl}(V)$ and set

$$e_i = \begin{cases} E_{i,i+1} + E_{2m-i,2m-i+1} & \text{if } 1 \leq i \leq m - 1, \\
E_{m-1,m+1} + E_{m,m+2} & \text{if } i = m, \\
\end{cases}$$

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \ldots, m$. Here $E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group PSO$(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in PSO$(V)$, respectively. Here $U_+$ and $U_-$ are the unipotent radicals of $B_+$ and $B_-$, respectively, and $T$ is the maximal torus of PSO$(V)$, consisting of diagonal matrices $(d_{ij})$ with non-zero entries $d_{i,i} = d_{2m-i+1,2m-i+1}^{-1}$. We let $X(T) = \text{Hom}(T, C^*)$, $R \subset X(T)$ the set of roots, and $R^+$ the positive roots. We denote the set of simple roots by $\Pi = \{\alpha_i \mid 1 \leq i \leq m\} \subset R^+ \subset R \subset X(T)$, and the set of fundamental weights (which is the dual basis in $X(T)$) by $\{\omega_i \mid 1 \leq i \leq m\} \subset X(T) \otimes \mathbb{R}.$
The parabolic subgroup $P$ of $\text{PSO}(V)$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the $e_i$ together with $f_2, \ldots, f_m$, leaving out $f_1$. Let $x_i(a) := \exp(a e_i)$ and $y_i(a) := \exp(a f_i)$. The Weyl group $W$ of $\text{PSO}(V)$ is generated by simple reflections $s_i$ for which we choose representatives

$$s_i = y_i(-1)x_i(1)y_i(-1).$$

We let $W_P$ denote the parabolic subgroup of the Weyl group $W$, namely $W_P = \langle s_2, \ldots, s_m \rangle$. The length of a Weyl group element $w$ is denoted by $\ell(w)$. The longest element in $W_P$ is denoted by $w_P$. We also let $w_0$ be the longest element in $W$. Next $W^P$ is defined to be the set of minimal length coset representatives for $W/W_P$. The minimal length coset representative for $w_0$ is denoted by $w^P$.

We introduce the following notation for the elements of $W^P$. Namely, $W^P = \{e, w_1, \ldots, w_{m-1}, w_m, w_{m+1}, \ldots w_{2m-2}\}$, where

$$w_k = \begin{cases} s_k s_{k-1} \cdots s_1 & \text{if } 1 \leq k \leq m-2, \\ s_{m-1} s_{m-2} \cdots s_1 & \text{if } k = m-1, \\ s_m s_{m-1} s_{m-2} \cdots s_1 & \text{if } k = m, \\ s_{2m-1-k} \cdots s_{m-2} s_m s_{m-1} s_{m-2} \cdots s_1 & \text{if } m+1 \leq k \leq 2m-2. \end{cases}$$

and $w_{m-1} = s_m s_{m-2} \cdots s_1$.

For any $w \in W$ let $\hat{w}$ denote the representative of $w$ in $G$ obtained by setting $\hat{w} = s_{i_1} \cdots s_{i_r}$, where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression and $\hat{w}$ is as in (21). Each $\hat{w}_k \in \text{PSO}(V)$ can be represented by a matrix $[w_k] \in \text{SO}(V)$ such that

$$[w_k] \cdot v_{2m} = \begin{cases} v_{2m-k} & 1 \leq k < m-1, \\ v_{2m-k-1} & m-1 \leq k \leq 2m-2, \end{cases}$$

and $[w_{m-1}] \cdot v_{2m} = v_m$ and $[w_{m-1}] \cdot v_{2m} = v_{m+1}$.

### 3.2. The dual quadric and its Plücker coordinates

Consider the homogeneous space $\tilde{X}_{2m-2} = P/\text{PSO}(V)$. It is canonically identified with the isotropic Grassmannian of lines in $V^*$, when this Grassmannian is viewed as a homogeneous space via the action of $\text{PSO}(V)$ from the right. Moreover the isotropic Grassmannian of lines is also a $(2m-2)$-dimensional quadric $Q_{2m-2}$, now in $\mathbb{P}(V^*)$. So in this case, the varieties $X$ and $\tilde{X}$ are (non-canonically) isomorphic. The reason for this isomorphism of varieties is that the group $G^\vee$ is of simply-laced type. However Lie-theoretically we still think of $X_{2m-2}$ and $\tilde{X}_{2m-2}$ as being very different homogeneous spaces, with $X_{2m-2} = \text{Spin}_{2m}(\mathbb{C})/P^\vee$ and $\tilde{X}_{2m-2} = P/\text{PSO}_{2m}(\mathbb{C})$.

**Definition 3.1** (Plücker coordinates). The Plücker coordinates for $\tilde{X}_{2m-2} = P/\text{PSO}(V)$ are the homogeneous coordinates coming from the embedding of $\tilde{X}_{2m-2}$ into $\mathbb{P}(V^*)$ as the (right) $G$-orbit of the line $C v_{2m}^*$:

$$\tilde{X}_{2m-2} = P/\text{PSO}(V) \rightarrow \mathbb{P}(V^*): P g \mapsto (C v_{2m}^*) \cdot g.$$ 

We think of the Plücker coordinates as corresponding to the elements of $W^P$. Let $v_{\omega_i}$ (respectively $v_{\omega_i}^+$) denote lowest and highest weight vectors in the highest weight representation $V_{\omega_i}$. Then the Plücker coordinates may be defined by:

$$p_0(g) = \langle v_{2m}^* \cdot [g], v_{2m} \rangle,$$

$$p_k(g) = \langle v_{2m}^* \cdot [g], [w_k] \cdot v_{2m} \rangle \text{ for } 1 \leq k \leq 2m-2,$$

$$p_{m-1}'(g) = \langle v_{2m}^* \cdot [g], [w_{m-1}'] \cdot v_{2m} \rangle.$$
where \([g] \in \text{SO}(V)\) is any fixed matrix representing \(g \in \text{PSO}(V)\). The homogeneous coordinates of \(P_g\) are then given by

\[(p_0(g) : \ldots : p_{m-2}(g) : p_{m-1}(g) : p_{m-1}'(g) : p_m(g) : \ldots : p_{2m-2}(g))\].

These are simply the bottom row entries of \([g]\) read from right to left, keeping in mind \([22]\).

We may now write down the equation of the quadric \(\mathbb{X}_{2m-2}\) in terms of Plücker coordinates:

\[(23)\quad p_{m-1}'p_{m-1} - p_{m-2}p_m + p_{m-3}p_{m+1} - \cdots + (-1)^{m-1}p_0p_{2m-2} = 0.\]

We note that as in the case of the odd quadric these Plücker coordinates are to be thought of as \(B\)-model incarnations of the Schubert classes of \(Q_{2m-2}\). Namely, recall that \(H^*(Q_{2m-2}, \mathbb{C})\) has a Schubert basis \(\{\sigma_w\}\) indexed by \(W^P\). We will use the notation \(\sigma_i = \sigma_{w_i}, \sigma_{i-1}' = \sigma_{w_i'},\) and \(\sigma_0 = \sigma_e\), where the \(w_i\) are defined in Section 3.1. As a special case of the geometric Satake correspondence \([\text{Lus}83, \text{Gin}95, \text{MV}07]\) we have that the (defining) projective representation \(V\) of \(\text{PSO}_{2m}(V)\) is identified with the cohomology of \(Q_{2m-2}\):

\[V = H^*(Q_{2m-2}, \mathbb{C}),\]

and the standard basis \(v_i\) agrees with the Schubert basis via \(v_{2m} = \sigma_0\) and

\[(24)\quad [w_1] \cdot v_{2m} = \sigma_1, \quad [w_{m-1}'] \cdot v_{2m} = \sigma_{m-1}'.\]

The Schubert classes \(\sigma_w\) are in this way naturally identified with the Plücker coordinates.

3.3. The superpotential for \(Q_{2m-2}\) on a dual quadric. In this section we state our theorem describing a superpotential for \(Q_{2m-2}\) in terms of Plücker coordinates on the dual quadric \(\mathbb{X}_{2m-2} = \mathbb{Q}_{2m-2}\). Consider

\[(25)\quad \mathbb{X}_{\text{can}} = \mathbb{X}_{2m-2} := \mathbb{X} \setminus D,\]

where \(D := D_0 + D_1 + \ldots + D_{m-2} + D_{m-1} + D_{m-1}'\), the \(D_i\) being given by

\[D_0 := \{p_0 = 0\},\]

\[D_\ell := \left\{ \sum_{k=0}^\ell (-1)^k p_{\ell-k}p_{2m-2-\ell+k} = 0 \right\} \text{ for } 1 \leq \ell \leq m - 3,\]

\[D_{m-2} := \{p_{2m-2} = 0\},\]

\[D_{m-1} := \{p_{m-1} = 0\},\]

\[D_{m-1}' := \{p_{m-1}' = 0\}.\]

The divisor \(D\) is an anticanonical divisor in \(\mathbb{X}\) (see \([\text{KLS}14\] Lemma 5.4). For simplicity, we will define

\[(26)\quad \delta_\ell = \sum_{k=0}^\ell (-1)^k p_{\ell-k}p_1p_{N-\ell+k} \text{ for } 1 \leq \ell \leq m - 3.\]

(For even quadrics, \(N = 2m - 2\).)

As in the odd case, we have a unique up to scalar \((2m-2)\)-form \(\omega_{\text{can}}\) which is regular on \(\mathbb{X}_{\text{can}}\) and has logarithmic poles along \(D\). For all \(1 \leq j \leq m - 2,\)
take \( r_j \in \{p_j, p_{2m-2-j}\} \). Setting \( p_0 = 1 \), the restriction of \( \omega_{\text{can}} \) to the torus \( \{r_j \neq 0 \mid 1 \leq j \leq m-2\} \) inside \( \tilde{X}_{\text{can}} \) is given by

\[
\omega_{\text{can}} = \frac{\prod_{1 \leq j \leq m-2} r_j \wedge \prod_{1 \leq \ell \leq m-3} \delta_\ell \wedge \delta_1 \cdots \delta_{m-3} p_{2m-2} p_{m-1} p_{m-1}}{\delta_1 \cdots \delta_{m-3} p_{2m-2} p_{m-1} p_{m-1}}.
\]

Our first result is the following theorem.

**Theorem 3.2.** The Lie-theoretic LG model \((\tilde{X}_{\text{Lie}}, \mathcal{W}_q, \text{Lie})\) for \( Q_{2m-2} = \text{Spin}_{2m}/P^\vee \) from [Rie08] is isomorphic to the canonical LG model \((\tilde{X}_{2m-2}, \mathcal{W}_q)\), where \( \mathcal{W}_q : \tilde{X}_{2m-2} \to \mathbb{C} \) is defined by

\[
\mathcal{W}_q = p_1 + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell} + p_m}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p_{m-1}'} + q \frac{p_1}{p_{m-2}}.
\]

This isomorphism is defined in Section 3.4. Before we begin the proof we need to recall the definition of the Lie-theoretic LG model from [Rie08].

### 3.4. The Lie-theoretic LG model for \( Q_{2m-2} \)

Following [Rie08] consider the (open) Richardson variety \( \tilde{X}_{\text{Lie}} := \tilde{R}_{wp,w_0} \subset G/B_- \), namely

\[ \tilde{X}_{\text{Lie}} := \tilde{R}_{wp,w_0} = (B_+ \tilde{w}_p B_- \cap B_- \tilde{w}_p B_-)/B_- . \]

This Richardson variety \( \tilde{X}_{\text{Lie}} \) is irreducible of dimension \( 2m-2 \), and its closure is the Schubert variety \( B_+ \tilde{w}_p B_- / B_- \). Let \( T^{W_P} \) be the \( W_P \)-fixed part of the maximal torus \( T \). Note that since we are in the setting of Section 3.4, we have that \( T^{W_P} \cong \mathbb{C}^* \) with isomorphism given by \( \alpha_1 \). The inverse isomorphism is \( \omega_1' : C^* \to T^{W_P} \). We fix a \( d \in T^{W_P} \). Then one can define

\[
Z_d := B_- \tilde{w}_0 \cap U_+ \tilde{w}_p U_- \subset G,
\]

and the map

\[
\pi_R : Z_d \to \tilde{X}_{\text{Lie}} : g \mapsto gB_-.
\]

is an isomorphism from \( Z_d \) to the open Richardson variety [Rie08] Section 4.1].

Let \( q \) be the non-vanishing coordinate on the 1-dimensional torus \( T^{W_P} \) given by \( \alpha_1 : T^{W_P} \to \mathbb{C}^* \). The mirror LG model is a regular function on \( \tilde{X}_{\text{Lie}} \) depending also on \( q \), and hence a regular function on \( \tilde{X}_{\text{Lie}} \times T^{W_P} \). It is defined as follows [Rie08]:

\[
\mathcal{F} : (u_1 \tilde{w}_p B_-, d) \mapsto g = u_1 \tilde{w}_p B_- \in Z_d \mapsto \sum c^*_i(u_1) + \sum f^*_i(\tilde{u}_2),
\]

where \( u_1 \in U_+, \tilde{u}_2 \in U_- \), and where \( \tilde{u}_2 \) is determined by \( u_1 \) and the property that \( u_1 \tilde{w}_p \tilde{u}_2 \in Z_d \).

The corresponding map from \( \tilde{X}_{\text{Lie}} \), when the coordinate \( q \) is fixed, is denoted

\[ \mathcal{W}_{q, \text{Lie}} : \tilde{X}_{\text{Lie}} \to \mathbb{C} : u_1 \tilde{w}_p B_- \mapsto \mathcal{F}(u_1 \tilde{w}_p B_-, \omega_1'(q)). \]

**Remark 1.** Note that if \( g = u_1 \tilde{w}_p \tilde{u}_2 \in Z_d \), then we have a simple identity concerning the Plücker coordinates:

\[
(p_0(g) : \ldots : p_{2m-2}(g)) = (p_0(\tilde{u}_2) : \ldots : p_{2m-2}(\tilde{u}_2)).
\]

The remainder of Section 3 will be devoted to proving Theorem 3.2, which now says that there is an isomorphism \( \tilde{X}_{2m-2} \cong \tilde{X}_{\text{Lie}} \) under which \( \mathcal{W}_q \) is identified with \( \mathcal{W}_{q, \text{Lie}} \).
3.5. **Isomorphism between $\tilde{X}_{\text{can}}$ and $\tilde{X}_{\text{Lie}}$.** To prove Theorem 3.2 the first step is to construct an isomorphism between $\tilde{X}_{2m-2}$ and the open Richardson variety $\tilde{X}_{\text{Lie}}$. We define the following maps:

$$\tilde{X} = P\backslash G \xrightarrow{\pi_L} Z_d = B_-\bar{\omega}_0 \cap U_+d\bar{w}^pU_-, \quad \tilde{X}_{\text{Lie}},$$

\[ Pg \leftrightarrow g, \quad g \mapsto gB_- \]
given by taking left and right cosets, respectively. Note that $g$ is equal to $b_-\bar{\omega}_0$ in our previous notation and factorizes (a priori non-uniquely) as

$$g = u_1d\bar{w}^p\bar{u}_2.$$  

Moreover $\pi_R$ is an isomorphism, so we have $\pi := \pi_L \circ \pi_R^{-1}: \tilde{X}_{\text{Lie}} \to \tilde{X}_{2m-2}$. Our next goal is to prove:

**Proposition 3.3.** $\pi_L$ defines an isomorphism from $Z_d$ to $\tilde{X}_{2m-2}$. As a consequence, $\pi$ defines an isomorphism from $\tilde{X}_{\text{Lie}}$ to $\tilde{X}_{2m-2}$.

Our proof uses a presentation of the coordinate ring of the unipotent cell

$$(32) \quad U^p := U_- \cap B_+ (\bar{w}^p)^{-1} B_+$$
due to [GLS11]. The strategy of the proof of Proposition 3.3 is as follows.

- The first step is to show that the natural map $\pi_L: Z_d \to \tilde{X}$ factorizes as $\phi \circ \theta$ where $\phi: U^p \to \tilde{X}$ with $\phi(\bar{u}) = P\bar{u}$ and $\theta: Z_d \to U^p$ is an isomorphism which will be constructed in Lemma 3.4.

- We then use the presentation of the coordinate ring of $U^p$ to show that the image of the map $\phi$ lands in $\tilde{X}_{2m-2}$ and not just $\tilde{X}_{2m-2}$. That is, the Plücker coordinates $p_0, p_{2m-2}, p_{m-1}, p'_m, p'_{m-1}$ and the functions $\delta_\ell$ (defined in (3)) do not vanish. Finally, we show that $\phi$ is an isomorphism from $U^p$ to $\tilde{X}_{2m-2}$. The main step is to find a pre-image for each of the functions generating $\mathbb{C}[U^p]$.

**Lemma 3.4.** There exists an isomorphism $\theta: Z_d \to U^p$ such that for $b\bar{w}0 \in Z_d$, \n
$$(33) \quad Pb\bar{w}0 = P\bar{u}_2,$$

where $\bar{u}_2 := \theta(b\bar{w}0)$.

To prove Lemma 3.4 we use an isomorphism introduced by Berenstein and Zelevinsky in [BZ97] (and joint with Fomin in type $A$ [BFZ96]) which is sometimes called the BZ twist (or BFZ twist).

**Theorem 3.5.** [BZ97] Theorem 1.2] Let $y \in U_- \cap B_+ \bar{w}^{-1} B_+$. There exists a unique $x \in U_+ \cap B_- \bar{w} B_-$ such that $U_+ \cap B_- \bar{w}y = \{x\}$. The resulting map $\tilde{\eta}_w : U_- \cap B_+ \bar{w}^{-1} B_+ \to U_+ \cap B_- \bar{w} B_-$ sending $y$ to $x$ is an isomorphism. In particular we have an inverse isomorphism

$$\varepsilon_w : U_+ \cap B_- \bar{w} B_- \to U_- \cap B_+ \bar{w}^{-1} B_+.$$  

**Remark 2.** We note that the original twist map of Berenstein and Zelevinsky is an automorphism $\eta_w : U_+ \cap B_- \bar{w} B_- \to U_+ \cap B_- \bar{w} B_-$. Our map $\tilde{\eta}_w$ is related to $\eta_w$ by

$$\tilde{\eta}_w(y) = \eta_w(y^T),$$

where $y^T$ denotes the transpose of $y$. We have

$$\tilde{\eta}_w(y) = x \iff B_-wy = B_-x.$$
Here we may write $B_-\dot{w}$ for $B_-\dot{w}_0$, as the coset doesn’t depend on the representative of $w$.

**Proof of Lemma 3.4** The idea is to consider the two birational maps

$$\Psi_1: U_+^P \to P\backslash G, \quad \bar{u}_2 \mapsto P\bar{u}_2,$$

$$\pi_L: Z_d \to P\backslash G, \quad b_-\dot{w}_0 = u_1\dot{w}_0 P \bar{u}_2 \mapsto Pb_-\dot{w}_0,$$

and to show that the composition

$$(34) \quad \theta := \Psi_1^{-1} \circ \pi_L: Z_d \to U_-^P,$$

is an isomorphism. We construct a commutative triangle of maps as follows.

$$\begin{array}{ccc}
Z_d & \xrightarrow{\mu} & U_-\dot{w}_0 \cap B_+\dot{w}_0 U_- \\
\downarrow{\xi} & & \downarrow{\theta} \\
U_-^P & \xrightarrow{\xi} & U_-^P
\end{array}$$

Here $\mu: Z_d \to U_-\dot{w}_0 \cap B_+\dot{w}_0 U_-$ is an isomorphism defined by $b_-\dot{w}_0 \mapsto [b_-]_0^{-1} b_-\dot{w}_0$, where $[b_-]_0$ is the torus part of $b_-$. The inverse isomorphism $\mu^{-1}$ is given by $b_+\dot{w}_0 U_- \mapsto d[b_+]_0^{-1} b_+\dot{w}_0 U_-$. Note that clearly $Pz = P\mu(z)$ for all $z \in Z_d$.

We now define a composition $\xi$ of isomorphisms as follows,

$$U_-\dot{w}_0 \cap B_+\dot{w}_0 U_- \xrightarrow{\ell_{\dot{w}_0}^{-1}} U_+ \cap B_-w^P B_- \xrightarrow{\varepsilon_{w^P}} U_- \cap B_+(w^P)^{-1}B_+,$$

where $\ell_{\dot{w}_0}^{-1}$ is the left multiplication by $\dot{w}_0^{-1}$ map. Hence we obtain an isomorphism

$$\xi: U_-\dot{w}_0 \cap B_+\dot{w}_0 U_- \xrightarrow{\xi} U_-^P.$$

Suppose $u_-\dot{w}_0 \in U_-\dot{w}_0 \cap B_+\dot{w}_0 U_-$. To prove the identity (34) it remains to check that $P u_-\dot{w}_0 = P\bar{u}_2$ where $\bar{u}_2 = \xi(u_-\dot{w}_0)$. This follows from the defining property of $\varepsilon_{w^P}$. Namely if $u_-\dot{w}_0 \in U^P\dot{w}_0$ then if $y = \varepsilon_{w^P}(\dot{w}_0^{-1} w\dot{w}_0)$, we have

$$B_-\dot{w}_0^{-1} u_-\dot{w}_0 = B_- w^P \bar{u}_2 = B_- w_0 w^P \bar{u}_2.$$

Therefore $B_+u_-\dot{w}_0 = B_+wP\bar{u}_2$. \hfill $\square$

For the second step of the proof of Proposition 3.3 we use a result of GLS11 to describe the coordinate ring of the unipotent cell $U_+^P$. In Lemma 3.10 we then explicitly relate the coordinates on $U_+^P$ to the coordinates on $\bar{X}_{2m-2}$, which are the Plücker coordinates from Definition 3.1. In this way we show that the map

$$\phi: U_+^P \to \bar{X}, \quad \bar{u}_2 \mapsto P\bar{u}_2$$

restricts to an isomorphism onto its image, and that this image is $\bar{X}_{2m-2}$.

We must first define the generalized minors involved in the presentation due to GLS11. Let $G^{sc}$ be the simply-connected covering group of $G = \mathrm{PSO}(V)$, with Borel subgroup $B^{sc}$ and unipotent radical $U^{sc}$ projecting to $B_-$ and $U_-$ in $G$. Here $G^{sc} = \mathrm{Spin}(V)$. Since $U^{sc} \cong U_-$ via this projection, we may use representations of $G^{sc}$ to define generalized minors of elements of $U_-$. For $u \in U_-$ we denote by $u^{sc}$ its lift to $U^{sc}$, and similarly for elements of $U_+$.

Let $w \in W$ have reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. Write

$$\bar{s}_j = g^{sc}_j x^{sc}_j (-1)g^{sc}_j (1)$$
and \( \bar{w} = s_i, s_{i_2} \ldots s_{i_r} \).

**Definition 3.6.** Let \( w \in W \) and \( \omega_j \) be a fundamental weight of \( G^{sc} \). Let \( V_{\omega_j} \) be the irreducible representation of \( G^{sc} \) with highest weight \( \omega_j \) and \( v_{\omega_j}^+ \) be a fixed highest weight vector. Define for any \( a \in U_- : \)

\[
\Delta_{\omega_j, w, \omega_j}(u) = \langle u^{sc} \cdot v_{\omega_j}^+, \bar{w} \cdot v_{\omega_j}^+ \rangle.
\]

Here \( \langle u^{sc} \cdot v_{\omega_j}^+, \bar{w} \cdot v_{\omega_j}^+ \rangle \) denotes the highest weight coefficient of \( \bar{w}^{-1}u^{sc} \cdot v_{\omega_j}^+ \) in terms of the weight space decomposition.

Note that the smallest representative \( w^P \) in \( W \) of \( [w_0] \in W/W_P \) has the following reduced expression:

\[(35) \quad w^P = s_1 \ldots s_{m-2m-1} s_{m-2} \ldots s_1.\]

Here we state the result from [GLS11] applied to our particular setting.

**Theorem 3.7 ([GLS11] Section 8).** Consider the reduced expression \( s_{i_1} \ldots s_{i_{2m-2}} = s_1 \ldots s_{m-2m-1} s_{m-2} \ldots s_1 \) for \( (\bar{w}^P)^{-1} \) coming from \( (35) \). The coordinate ring of the unipotent cell \( U_P^P := U_- \cap B_+(\bar{w}^P)^{-1} B_+ \) inside \( PSO_{2m} \) is

\[
\mathbb{C}[U_P^P] = \mathbb{C}[\Delta_{\omega_j, P \otimes \omega_j}^{-1}]\text{ for } j < m \]

where

- \( 1 \leq r \leq 2m - 2; m - 1 \leq s \leq 2m - 2; \)
- \((\bar{w}^P)^{-1} := s_{i_1} \ldots s_{i_r} \).

If \( j \leq m \) then \( \Delta_{\omega_j, w, \omega_j}(u) \) is a minor in the usual sense for the unique matrix \( u^{SO_{2m}} \) in \( U_-^{SO_{2m}} \) representing \( u \). We denote the minor of \( u^{SO_{2m}} \) with row set \( \{i_1, \ldots, i_p\} \) and column set \( \{j_1, \ldots, j_p\} \) by \( D_{j_1, \ldots, j_p}(u) \). We now reformulate Theorem 3.7 as follows.

**Corollary 3.8.** The coordinate ring \( \mathbb{C}[U_P^P] \) is generated by the minors

\[
D_{1,2,\ldots,r+1}^{2m-1-s,m+1}, \quad 1 \leq r \leq m - 2;
\]

\[
D_{1,2,\ldots,2m-1-s,m}^{2m-1-s,m+1}, \quad m + 1 \leq s \leq 2m - 3, \text{ and } D_1^{2m};
\]

the functions

\[
\Delta_{\omega, m, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m-1} - \epsilon_m)+m} \text{ and } \Delta_{\omega, m-1, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m})},
\]

which are Pfaffians; the inverses of minors

\[
(D_{1,2,\ldots,2m-1-s}^{2m-1-s,m})^{-1}, \quad m + 1 \leq s \leq 2m - 3, \text{ and } (D_1^{2m})^{-1};
\]

and the inverses of Pfaffians

\[
\Delta_{\omega, m, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m-1} - \epsilon_m)} \text{ and } \Delta_{\omega, m-1, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m})}.
\]

To relate the minors and Pfaffians of Corollary 3.8 to the Plücker coordinates we will need to use a specific factorisation of generic elements of \( U_P^P \). By an application of Bruhat’s lemma [Lm94], a generic element in \( U_P^P \) can be assumed to have a particular factorisation:

\[(36) \quad \bar{a}_2 = y_1(a_1) \ldots y_{m-2}(a_{m-2}) y_m(d) y_{m-1}(c) y_{m-2}(b_{m-2}) \ldots y_1(b_1), \]

where \( a_1, c, d, b_j \neq 0 \).
We have the following standard expression for the $p_k$ on factorized elements, which is a simple consequence of their definition.

**Lemma 3.9.** Fix $0 \leq k \leq 2m - 2$ an integer. Then if $\bar{u}_2$ is of the form \([36]\), we have

$$p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1 \ldots a_{k-1}(a_k + b_k) & \text{if } 1 \leq k \leq m - 2, \\ a_1 \ldots a_{m-2}c & \text{if } k = m - 1, \\ a_1 \ldots a_{m-2}cd & \text{if } k = m, \\ a_1 \ldots a_{m-2}cdm_2 - b_{2m-1-k} & \text{otherwise.} \end{cases}$$

and

$$p_{m-1}^\prime(\bar{u}_2) = a_1 \ldots a_{m-2}d.$$ 

We can now prove the lemma we need.

**Lemma 3.10.** We have the following equalities of generalised minors and Plücker coordinates evaluated on $\bar{u}_2 \in U^p$:

(37) \[ D_{2m}^1(\bar{u}_2) = p_{2m-2}(\bar{u}_2). \]

(38) \[ \Delta_{\omega_m \cdots \omega_1}^{\frac{1}{2}[-\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m]}(\bar{u}_2) = p_{m-1}(\bar{u}_2), \]

(39) \[ \Delta_{\omega_m \cdots \omega_1}^{\frac{1}{2}[-\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m - 1 - \epsilon_m]}(\bar{u}_2) = p_{m-1}^\prime(\bar{u}_2), \]

(40) \[ D_{2m}^{2m-1-s,m+1}(\bar{u}_2) = \delta_{s-m}(\bar{u}_2), \text{ for } m + 1 \leq s \leq 2m - 3, \]

where we recall that $\delta_{s-m} = \sum_{k=s}^{m} (-1)^{s-k} p_{k-m} p_{3m-2k}$.

**Proof.** The identity \([37]\) follows immediately from the definition of the Plücker coordinates. For the identity \([38]\), write

$$\Delta_{\omega_m \cdots \omega_1}^{\frac{1}{2}[-\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m]}(\bar{u}_2) = (D_{1,\ldots,m-1,m+1}^{2m}(\bar{u}_2))^\frac{1}{2}.$$ 

Note that in the definition of $\Delta_{\omega_j \cdots \omega_1}$, we have chosen the representative $\bar{w}$ in such a way that evaluated on a factorized $\bar{u}_2$ the generalized minors will be nonnegative for any positive choice of the coordinates $a_i, b_i, c, d$ (i.e. on ‘totally positive’ $\bar{u}_2$).

This determines the choice of square root. Then developing $D_{1,\ldots,m-1,m+1}^{2m}(\bar{u}_2)$ with respect to the last column, we get

$$D_{1,\ldots,m-1,m+1}^{2m}(\bar{u}_2) = D_{1,\ldots,m-1}^{2m}(\bar{u}_2) D_{m+1}^{2m}(\bar{u}_2) = p_{m-1}(\bar{u}_2) D_{1,\ldots,m-1}^{2m}(\bar{u}_2)$$

using the definition of $p_{m-1}(\bar{u}_2)$. Finally, since the matrix is $\bar{u}_2$ orthogonal:

$$D_{1,\ldots,m}^{2m}(\bar{u}_2) = D_{1,\ldots,m+1}^{2m}(\bar{u}_2).$$

Developing again with respect to the last column, we obtain

$$D_{1,\ldots,m+1}^{2m}(\bar{u}_2) = D_{1,\ldots,m}^{2m}(\bar{u}_2) D_{m+1}^{2m}(\bar{u}_2) = p_{m-1}(\bar{u}_2),$$

using the definition of $p_{m-1}(\bar{u}_2)$ and the fact that $\bar{u}_2$ is lower unipotent. The identity \([38]\) then follows. The proof of the identity \([39]\) is similar.

Let us now prove the identity \([40]\). Developing $D_{1,2,\ldots,2m-1-s}^{2m-1-s,m+1}(\bar{u}_2)$ with respect to the $(2m - 1 - s)$-th column, we see that it is equal to

$$D_{m+1}^{2m-1-s}(\bar{u}_2) D_{1,2,\ldots,2m-2-s}^{2m-1-s}(\bar{u}_2) - D_{1,2,\ldots,2m-1-s}^{2m-2-s,m+1}(\bar{u}_2).$$

Since $\bar{u}_2$ is orthogonal for $Q$, we have

$$D_{1,2,\ldots,2m-1-s}^{2m-1-s}(\bar{u}_2) = D_{1,\ldots,s+1,2m}^{1,s+1,2m}(\bar{u}_2),$$
and since \( \bar{u}_2 \) is in \( U_+ \),
\[
D_{1,\ldots,s+2}^{1,\ldots,s+1,2m}(\bar{u}_2) = D_{s+2}^{2m}(\bar{u}_2) = p_{2m-2-s}(\bar{u}_2).
\]
Finally
\[
D_{1,2,\ldots,2m-1-s}^{2,\ldots,2m-1-s,m+1}(\bar{u}_2) = D_{2m-1-s}^{m+1}(\bar{u}_2)p_{2m-2-s}(\bar{u}_2) - D_{1,\ldots,2m-2-s}^{2,\ldots,2m-1-s,m+1}(\bar{u}_2),
\]
hence
\[
D_{1,2,\ldots,2m-1-s}^{2,\ldots,2m-1-s,m+1}(\bar{u}_2) = \sum_{k=s}^{2m-2} (-1)^{s-k} D_{2m-1-s}^{m+1}(\bar{u}_2)p_{2m-2-s}(\bar{u}_2).
\]
We also have
\[
D_{2m-1-s}^{m+1}(\bar{u}_2) = db_{2m-2}\ldots b_{2m-1-s} \quad \text{for} \quad m+1 \leq s \leq 2m-2.
\]
Indeed, if \( \bar{u}_2 \in U_+ \), then by Corollary 3.3 the minors \( D_{1,2,\ldots,2m-1-s}^{2,\ldots,2m-1-s,m+1}(\bar{u}_2) \) and \( D_{1}^{2m}(\bar{u}_2) \) and the Pfaffians \( \Delta_{\omega_m,\chi_1,\ldots,\chi_{m-1}-\epsilon_m}(\bar{u}_2) \) and \( \Delta_{\omega_m,\chi_{-1,\ldots,\chi_{m-1}}}(\bar{u}_2) \) do not vanish. Since we have proved in Lemma 3.10 that those correspond precisely to the divisors involved in defining \( \bar{X}_{2m-2} \), it follows that \( \bar{P}\bar{u}_2 \in \bar{X}_{2m-2} \). We may now prove that \( \phi \) is an isomorphism between \( U_+ \) and \( \bar{X}_{2m-2} \).

Injectivity of the pullback map \( \phi^* : \mathbb{C}[\bar{X}_{2m-2}] \to \mathbb{C}[U_+] \) is a simple consequence of the fact that the map \( U_+ \to \bar{X}_{2m-2} \) is dominant. We now prove that \( \phi^* \) is surjective by observing that each of the functions generating \( \mathbb{C}[U_+] \) (as in Corollary 3.3) has a preimage.

We have already seen that the inverses of minors and Pfaffians correspond to the inverses of denominators of \( W_q \). Let us now consider the minors \( D_{1,2,\ldots,r}^{2,\ldots,r+1} \) for \( 1 \leq r \leq m-2 \) and \( D_{1,2,\ldots,2m-1-s}^{2,\ldots,2m-1-s,m+1} \) for \( m+1 \leq s \leq 2m-3 \). In Lemma 3.10 we proved that
\[
D_{1,2,\ldots,2m-1-s}^{2,\ldots,2m-1-s,m+1} = \phi^*(\delta_{s-m})
\]
and
\[
D_{1,2,\ldots,r}^{2,\ldots,r+1} = D_{1,\ldots,r}^{2m-1-r} = D_{2m-r}^{2m} = \phi^*(p_r).
\]
Finally, \( D_{1}^{2m} = \phi^*(p_{2m-2}) \), and the Pfaffians
\[
\Delta_{\omega_m,\chi_{-1,\ldots,\chi_{m-1}}}(\bar{u}_2) \quad \text{and} \quad \Delta_{\omega_m,\chi_{-1,\ldots,\chi_{m-1}}}(\bar{u}_2)
\]
are pullbacks of the Plücker coordinates \( p'_{m-1} \) and \( p_{m-1} \), by Lemma 3.10 This concludes the proof.
3.6. **Comparison of the superpotentials.** In this section we will prove Theorem 3.2. We saw in the previous section that $\pi = \pi_L \circ \pi_R^\sim : \tilde{X}_{\text{Lie}} \to \tilde{X}_{2m-2}$ is an isomorphism. Note that we have a commutative diagram

$$
\begin{array}{ccc}
Z_d & \xrightarrow{\pi_R} & \tilde{X}_{\text{Lie}} \\
\sim & \downarrow & \\
\mathbb{C} & \rightarrow & \tilde{W}_{q,\text{Lie}}
\end{array}
$$

Therefore

$$(\pi^{-1})^*(\tilde{W}_{q,\text{Lie}}) = (\pi_L^{-1})^*(F_q).$$

This gives a regular function on $\tilde{X}_{2m-2}$ which we denote by $\tilde{W}_q$. The statement of Theorem 3.2 says that $\tilde{W}_q$ and $W_q$ agree. We will prove this by expressing both functions in terms of coordinates introduced earlier. Namely we consider the set of factorized elements $F\bar{u}_2$ with $\bar{u}_2$ as in (30) with nonzero coordinates $a, b, c, d$ as defining an open dense subvariety inside $\tilde{X}_{2m-2}$ which is isomorphic to a torus. We call this subvariety $X_{2m-2}$. To finish the proof we will show that the restrictions of $\tilde{W}_q$ and of $W_q$ to $X_{2m-2}$ agree. This will additionally give an interesting Laurent polynomial formula for the superpotential, which we will use in Section 6 to describe a flat section of the Dubrovin connection.

**Proposition 3.11.** $\tilde{W}_q$ and $W_q$ restricted to a particular torus $\tilde{X}_{\text{Lus}}$ inside $X_{2m-2}$ have the following Laurent polynomial expression

$$W_{q,\text{Lus}} = a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q \frac{a_1 + b_1}{a_1 \cdots a_{m-2} c d b_{m-2} \cdots b_1}.$$ 

We call ($\tilde{X}_{\text{Lus}}, \tilde{W}_{q,\text{Lus}}$) the quiver mirror. To prove Proposition 3.11 we will need the following:

**Lemma 3.12.** If $u_1 \in U_+, \bar{u}_2 \in U_-, u_1 d\tilde{w}_p \bar{u}_2 \in Z_d$, and $\bar{u}_2$ can be written as in (30), then we have the following identities:

(41) 
$$f_i^*(\bar{u}_2) = \begin{cases} 
a_i + b_i & \text{if } 1 \leq i \leq m - 2, \\
c & \text{if } i = m - 1, \\
d & \text{if } i = m. 
\end{cases}$$

(42) 
$$e_i^*(u_1) = \begin{cases} 
0 & \text{if } 2 \leq i \leq m, \\
\frac{a_i + b_i}{q \prod_{s=1}^{m-2} a_s c d b_{m-1} \cdots b_1} & \text{if } i = 1. 
\end{cases}$$

**Proof.** Equation (41) is obtained immediately from the definition of $\bar{u}_2$. For Equation (42), notice that

$$e_i^*(u_1) = \frac{\langle u_1^{-1}, e_i : v_{\bar{u}_2} \rangle}{\langle u_1^{-1}, v_{\bar{u}_2} \rangle} = \frac{\langle d\tilde{w}_p \bar{u}_2, v_{\bar{u}_2} \rangle}{\langle d\tilde{w}_p \bar{u}_2, v_{\bar{u}_2} \rangle}.$$ 

Assume $2 \leq i \leq m$. Then $e_i^*(u_1) = 0$ if and only if $\langle \bar{u}_2 : v_{\bar{u}_2}^{-1}, \bar{u}_2^{-1} e_i : v_{\bar{u}_2}^{-1} \rangle = 0$. Now the vector $w_p^{-1} e_i : v_{\bar{u}_2}^{-1}$ is in the $\mu$-weight space of the $i$-th fundamental representation,
where $\mu = w_p^{-1} s_i ( - \omega_i )$. Moreover, $\bar{u}_2 \in B_+ ( \bar{w}_p^{-1} )$, hence $\bar{u}_2 \cdot v_{w_1}^+$ can have non-zero components only down to the weight space of weight $( w_p^{-1} ) ( \omega_i ) = w_p^{-1} ( - \omega_i )$. Since $l( w_p^{-1} s_i ) > l( w_p^{-1} )$ for $2 \leq i \leq m$, this is higher than $\mu$, which proves that $e_1^* ( u_1 ) = 0$.

Now assume $i = 1$. We have

$$e_1^* ( u_1 ) = \frac{ (dw_p \bar{u}_2 \cdot v_{w_1}^+, e_1 \cdot v_{w_1}^- ) }{ (dw_p \bar{u}_2 \cdot v_{w_1}^+, v_{w_1}^- ) } = ( \omega_1 + \alpha_1 - \omega_1 ) ( d ) \frac{ \langle \bar{u}_2 \cdot v_{w_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{w_1}^- \rangle }{ \langle \bar{u}_2 \cdot v_{w_1}^+, \bar{w}_p v_{w_1}^- \rangle } = q \frac{ \langle \bar{u}_2 \cdot v_{w_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{w_1}^- \rangle }{ \langle \bar{u}_2 \cdot v_{w_1}^+, v_{w_1}^- \rangle }.$$

First look at the denominator. The only way to go from the highest weight vector $v_{w_1}^+$ of the first fundamental representation to the lowest weight vector $v_{w_1}^-$ is to apply $g \in B_+ w B_+$ for $w \geq ( w_p )^{-1}$. Since $\bar{u}_2 \in B_+ ( \bar{w}_p^{-1} ) B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v_{w_1}^-$ appropriately, we get

$$\langle \bar{u}_2 \cdot v_{w_1}^+, v_{w_1}^- \rangle = a_1 \ldots a_{m-1} c d b_{m-1} \ldots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2 \cdot v_{w_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{w_1}^- \rangle$. The vector $\bar{w}_p^{-1} e_1 \cdot v_{w_1}^-$ has weight

$$\mu' = \bar{w}_p^{-1} s_1 ( - \omega_1 ) = \bar{w}_p^{-1} ( - \epsilon_2 ) = \epsilon_2.$$

Write $\bar{w}_p^{-1} s_1$ as a prefix $w' = s_1 s_2 \ldots s_{m-2} s_{m-1} s_{m-2} \ldots s_2$ of $( w_p )^{-1}$. We have $w' s_1 = ( w_p )^{-1}$, hence the way from $v_{w_1}^+$ to $w' \cdot v_{w_1}^-$ is through $s_1$. From the factorization of $\bar{u}_2$ in (36), it follows that $\langle \bar{u}_2 \cdot v_{w_1}^+, \bar{w}_p^{-1} e_1 \cdot v_{w_1}^- \rangle = a_1 + b_1$. \hfill \square

**Proof of Proposition 5.11** Using the expression (31) of the superpotential from [Ric08], we immediately deduce expression for $\tilde{W}_q$ as a Laurent polynomial from Lemma 5.12. \hfill \square

Next, using Lemma 5.9 and Proposition 5.11, we express $\tilde{W}_q$ in terms of Plücker coordinates and deduce the theorem.

**Proof of Theorem 5.2.** From Lemma 5.9, it follows that for $\bar{u}_2$ as in (36)

$$p_{\ell+1} ( \bar{u}_2 ) p_{2m-2-\ell} ( \bar{u}_2 ) = ( a_{\ell+1} + b_{\ell+1} ) ( a_1 \ldots a_{\ell} )^2 a_{\ell+1} \ldots a_{m-2} c d b_{m-2} \ldots b_{\ell+1}$$

for $0 \leq \ell \leq m-3$. We also get that $p_k ( \bar{u}_2 ) p_{2m-2-k} ( \bar{u}_2 )$ is equal to

$$\begin{cases} a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_1 & \text{if } k = 0; \\ (a_1 + b_1) a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_2 & \text{if } k = 1; \\ (a_k + b_k) a_1 \ldots a_{k-1} a_k \ldots a_{m-2} c d b_{m-2} \ldots b_{k+1} & \text{if } 2 \leq k \leq m-3. \end{cases}$$

Using (43), we find that most terms in $\delta_\ell ( \bar{u}_2 ) = \sum_{k=0}^{\ell} ( -1 )^k p_{\ell-k} ( \bar{u}_2 ) p_{2m-2+k-\ell} ( \bar{u}_2 )$ cancel, and

$$\delta_\ell ( \bar{u}_2 ) = ( a_1 \ldots a_{\ell} )^2 a_{\ell+1} \ldots a_{m-2} c d b_{m-2} \ldots b_{\ell+1}.$$

This proves that

$$\frac{ p_{\ell+1} p_{2m-2-\ell} ( \bar{u}_2 ) }{ \delta_\ell } = a_{\ell+1} + b_{\ell+1}$$
for \(0 \leq \ell \leq m - 3\). Moreover:
\[
\frac{p_m}{p_{m-1}}(\bar{u}_2) = \frac{a_1 \ldots a_{m-2} c d}{a_1 \ldots a_{m-2} c} = d,
\]
and
\[
\frac{p_m}{p_{m-1}}(\bar{u}_2) = \frac{a_1 \ldots a_{m-2} c d}{a_1 \ldots a_{m-2} d} = c.
\]
For the first and last terms, we obtain
\[
\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1,
\]
and
\[
\frac{p_1}{p_{2m-2}}(\bar{u}_2) = \frac{a_1 \ldots a_{m-2} c b_{m-1} \ldots b_1}{a_1 \ldots a_{m-1} c d b_{m-1} \ldots b_1}
\]
as easy consequences of Lemma 3.9. Using Proposition 3.11, this proves that \(\tilde{W}_q\) coincides with the definition of \(W_q\) from Equation (28):
\[
W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p_{m-1}'} + q \frac{p_1}{p_{2m-2}}. \quad \square
\]

3.7. Comparison with the Givental and Laurent polynomial mirrors for even quadrics. Let us recall the Laurent polynomial LG model of \(Q_{2m-2}\) from Equation (4)
\[
W_{q, Prz} = z_1 + \cdots + z_{2m-3} + \frac{(z_{2m-2} + q)^2}{z_1 z_2 \ldots z_{2m-2}}
\]
defined over the torus
\[
\tilde{X}_{Prz} := \{(z_1, \ldots, z_{2m-2}) | z_i \neq 0 \; \forall \; i\},
\]
and the Givental LG model from Equation (2)
\[
W_{q, Giv} = \nu_1 + \cdots + \nu_{2m-2},
\]
defined over the affine variety
\[
\tilde{X}_{q, Giv} = \{(\nu_1, \ldots, \nu_{2m}) | \nu_i \neq 0 \; \forall \; i, \; \prod_{i=1}^{2m} \nu_i = q, \; \nu_{2m-1} + \nu_{2m} = 1\}.
\]
These two LG models are related by a birational change of coordinates analogous to that of [Prz13, Rmk. 19], namely
\[
\tilde{z}_i = \begin{cases} 
\nu_{i+1} & \text{for } 1 \leq i \leq 2m-3; \\
q^{\nu_{2m-2}} & \text{for } i = 2m-2;
\end{cases}
\]
and conversely
\[
\nu_i = \begin{cases} 
\frac{(z_{2m-2} + q)^2}{z_1 \ldots z_{2m-2}} & \text{for } i = 1; \\
z_{i-1} & \text{for } 2 \leq i \leq 2m-2; \\
z_{2m-2} + q & \text{for } i = 2m-1; \\
z_{2m-2} + q & \text{for } i = 2m.
\end{cases}
\]
This change of variables defines an isomorphism
\[
\tilde{X}_{Prz} \setminus \{z_{2m-2} + q = 0\} \cong \tilde{X}_{q, Giv}
\]
which identifies the superpotentials \(W_{q, Prz}\) and \(W_{q, Giv}\).
Let us now compare these two LG models with ours. Consider the change of coordinates

\[
z_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \leq i \leq m - 2; \\ \frac{p_{2m-3-i}p_{2m-5-i}}{p_{2m-4-i}p_{2m-6-i}} & \text{for } m - 1 \leq i \leq 2m - 5; \\ \frac{p_{m}}{p_{m-1}} & \text{for } i = 2m - 4; \\ \frac{p_{m-1}p_{m-5}}{\delta_{m-3}} & \text{for } i = 2m - 3; \\ \frac{p_{m-1}}{\delta_{m-2}} & \text{for } i = 2m - 2. \end{cases}
\]

It is well-defined on the following intersection \( \tilde{T} \) of two cluster tori

\[
\tilde{T} := \{ x \in \tilde{X}_{\text{can}} \mid p_i(x) \neq 0 \text{ for all } 0 \leq i \leq m - 2 \text{ and } p_m(x) \neq 0 \}.
\]

The inverse change of coordinates is given by

\[
p_i = \begin{cases} z_1 \ldots z_i & \text{for } 1 \leq i \leq m - 2; \\ qz_1 \ldots z_{m-2} \frac{z_{2m-3}}{z_{2m-2}} & \text{for } i = m - 1; \\ qz_1 \ldots z_{m-2} \frac{z_{2m-4}z_{2m-3}}{z_{2m-2}^2} & \text{for } i = m; \\ qz_1 \ldots z_{i-2} \left( 1 + \frac{z_{2m-1-i}}{z_{i-2}} \right) \frac{z_{2m-6+i}}{z_{2m-2}^2q} & \text{for } m + 1 \leq i \leq 2m - 3; \\ qz_1 \ldots z_{2m-3} \frac{z_{i-1}}{z_{2m-2}^2} & \text{for } i = 2m - 2. \end{cases}
\]

and \( p'_m = qz_1 \ldots z_{m-2} \frac{z_{2m-3}}{z_{2m-2}} \). Moreover, we have

\[
\delta_j = \begin{cases} \frac{z_{2m-j+1}}{z_{2m-3} \ldots z_{m-5}} & \text{for } 1 \leq j \leq m - 3; \\ \frac{q}{z_{2m-2} \ldots z_{m-5}} & \text{for } i = 2m - 2. \end{cases}
\]

We see that the inverse change of coordinates is well-defined over \( \tilde{X}_{\text{Prz}} \setminus \{ z_{2m-2} + q = 0 \} \), which is isomorphic to the Givental mirror manifold \( \tilde{X}_{q,Giv} \). Hence we obtain an isomorphism

\[
\tilde{X}_{\text{can}} \cong \tilde{T} \cong \tilde{X}_{\text{Prz}} \setminus \{ z_{2m-2} + q = 0 \} \cong \tilde{X}_{q,Giv} \subset \tilde{X}_{\text{Prz}}
\]

which identifies the (restrictions of) the superpotentials \( W_q \) and \( W_{q,\text{Prz}} \). It also identifies the form \( \omega_{q,Giv} \) with \( \omega_{can} \). This proves Proposition 1.2 from the introduction in the case of even quadrics.

### 3.8. The critical points of the canonical mirror

Since the canonical mirror \( (\tilde{X}_{\text{can}}, W_q) \) is isomorphic to the Lie-theoretic mirror \( (\tilde{X}_{Lie}, W_{q,Lie}) \), it follows from [Ric08] that \( W_q \) has the `correct’ number of critical points on \( \tilde{X}_{\text{can}} \), that is, \( \dim H^*(Q_{2m-2}, \mathbb{C}) = 2m \). Here we give explicit expression for the critical points, and compare with the critical points of the classical mirrors \( (\tilde{X}_{q,Giv}, W_{q,Giv}) \) and \( (\tilde{X}_{\text{Prz}}, W_{q,\text{Prz}}) \).

**Proposition 3.13.** The critical points of the superpotential \( W_q \) on \( \tilde{X}_{\text{can}} \) are given by

\[
p_j = \begin{cases} \zeta^j & \text{if } 1 \leq j \leq m - 2; \\ \frac{1}{2} \zeta^j & \text{if } m - 1 \leq j \leq 2m - 3; \\ q & \text{if } j = 2m - 2, \end{cases}
\]

and \( p'_m = \frac{1}{2} \zeta^{m-1} \), where \( \zeta \) is a primitive \((2m - 2)\)-st root of \( 4q \). The associated critical value is \((2m - 2)c \). Moreover there are two extra critical points given by
\[ p_1 = \cdots = p_{m-2} = p_m = p_{2m-3} = 0, \quad p_{m-1} = -p'_{m-1} = \pm \sqrt{q}, \quad p_{2m-2} = -q \quad \text{with corresponding critical value 0.} \] These two critical points do not belong to \( \mathcal{X}_{Prz}, \mathcal{X}_{q,Giv} \) or \( \mathcal{X}_{Lus} \).

**Proof.** The proof is very similar to that of Proposition 2.3 and we don’t repeat it here. \( \square \)

4. **The quiver mirrors \( (\mathcal{X}_{Lus}, \mathcal{W}_{q,Lus}) \)**

In this section we will explain how our quiver superpotential \( \mathcal{W}_{q,Lus} \) for \( Q_N \) can be read off from a certain quiver, justifying its name. This is analogous to the type \( A \) complete flag variety case [Giv97] and partial flag variety case [BCFKvS98, BCFKvS00], where one can also read off Laurent polynomial superpotentials from quivers.

We begin by explaining the [BCFKvS98] formula for the Grassmannian \( Gr_2(4) \). Note that since \( Gr_2(4) \) is defined by a single (quadratic) Plücker relation, it is isomorphic to the quadric \( Q_4 \).

For \( Gr_2(4) \) the quiver from [BCFKvS98] is shown in Figure 1. The Laurent polynomial superpotential can be read off easily. There are two versions. In the left hand picture the coordinates \( t_{ij} \) of the torus \( (\mathbb{C}^*)^4 \) are in bijection with vertices of the quiver. To each arrow we associate a Laurent monomial by taking the coordinate at the head of the arrow divided by the coordinate at the tail. The Laurent polynomial corresponding to the quiver is the sum of all of the Laurent monomials associated to the arrows.

![Figure 1. The quiver for \( Gr_2(4) \) and two choices of coordinates.](image)

The labels \( m_i \) of the arrows in the right hand version are another natural choice of coordinates on the same torus. Indeed these are coordinates related to factorizations into one-parameter subgroups of Lie-theoretic mirrors used in [Lus94], compare [MR13]. We suppose the remaining arrows are labelled in such a way that the square commutes and any path leading from 1 to \( q \) has labels whose product equals \( q \). These are Laurent monomials in the variables \( m_i \) (depending on \( q \)). Then the Laurent polynomial superpotential is obtained in [BCFKvS98] as the sum of the labels of all of the arrows of the quiver. In the case of \( Gr_2(4) \) it is

\[
\begin{align*}
m_1 + m_2 + m_3 + m_4 + m_1 m_2 + q \frac{1}{m_1 m_2 m_3}.
\end{align*}
\]

Since \( Gr_2(4) \) is isomorphic to \( Q_4 \), this suggests it should be related to the superpotential \( (\mathcal{X}_{Lus}, \mathcal{W}_{q,Lus}) \) from [7] for \( Q_4 \),

\[
\begin{align*}
a_1 + c + d + b_1 + q \frac{a_1 + b_1}{a_1 b_1 cd}.
\end{align*}
\]
There is indeed a toric change of coordinates turning Equation (44) into Equation (45):

\[ m_1 \mapsto \frac{q}{a_1cd}; \quad m_2 \mapsto a_1; \quad m_3 \mapsto c; \quad m_4 \mapsto b_1. \]

Note that the torus of the other Laurent polynomial mirror \((\hat{X}_{Prz}, W_{q,Prz})\) for \(Q_4\) is a different one, as seen in Section 3.7.

The superpotential (45) also comes from a quiver, see Figure 2. This generalises to all quadrics \(Q_N\). Indeed our Laurent polynomial superpotentials (6) and (7) for \(Q_N\) can be described using quivers as in Figure 3. The factorisation of \(\bar{u}_2\) from (36) can also be naturally read off the quiver (compare with \([MR13\text{ Section 5.3}])

Let the \(N-2\) vertical arrows on the left-hand edge be labelled from top to bottom by \(a_2, a_3, \ldots, a_{m-1}, c, b_{m-1}, \ldots, b_2\) for odd quadrics \(Q_{2m-1}\), and by \(a_2, a_3, \ldots, a_{m-2}, d, c, b_{m-2}, \ldots, b_2\) for even quadrics \(Q_{2m-2}\). The diagonal arrow with the same tail
as \( b_1 \) is labelled by \( a_1 \). The arrows below are not labelled. The labelled arrows can be organized into ‘levels’ starting with \( a_1, b_1 \) at the bottom level. The levels are associated to the one-parameter subgroups \( y_i \) (of \( \text{PSO}_{2m} \) for \( X = Q_{2m-2} \), respectively of \( \text{PSp}_{2m} \) for \( X = Q_{2m-1} \)) as shown in the \( Q_5 \) and \( Q_6 \) examples. Reading off column by column from right to left and from top to bottom we recover the factorization (36).

**Remark 3.** It is interesting to note that our quivers (restricted to the vertices which are not labelled by \( q \)) are orientations of type \( D \) Dynkin diagrams with a special vertex added at either end. So we have three ways to associate a Dynkin diagram to a quadric: the type of its symmetry group, the type of the cluster algebra associated to the coordinate ring of its mirror, and the type of the quiver defining its superpotential. See Table 1.

| Quadric | Symmetry group | Cluster type of mirror | Superpotential Quiver |
|---------|---------------|------------------------|----------------------|
| \( Q_3 \) | \( B_2 \) | \( A_1 \) | \( D_3 \) |
| \( Q_4 \) | \( D_3 \) | \( A_1 \) | \( D_4 \) |
| \( Q_5 \) | \( B_3 \) | \( A_1^2 \) | \( D_5 \) |
| \( Q_6 \) | \( D_4 \) | \( A_1^3 \) | \( D_6 \) |
| \( Q_7 \) | \( B_4 \) | \( A_1^4 \) | \( D_7 \) |

Table 1. Dynkin diagrams associated to quadrics

5. **The A-model and B-model connections**

Our expression for the canonical LG model \( W_q \) in terms of homogeneous coordinates coming from \( \tilde{X}_{\text{can}} \subset \mathbb{P}(H^*(X, \mathbb{C}))^* \) makes it possible to compare in a very natural way the (small) Dubrovin connection on the \( A \) side and the Gauss-Manin connection on the \( B \) side. We recall first the relevant definitions on the \( A \) side.

Let \( X = Q_N \). Consider \( H^*(X, \mathbb{C}[\hbar, q]) \) as a space of sections on a trivial bundle with fiber \( H^*(X, \mathbb{C}) \), over the base \( \mathbb{C} \hbar \times \mathbb{C} q \), where the \( \hbar \) and \( q \) are the coordinates. Let \( \text{Gr} \) be the operator on sections defined on the fibres as the ‘grading operator’ \( H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}) \) which multiplies \( \sigma \in H^{2k}(X, \mathbb{C}) \) by \( k \). We define the Dubrovin connection by

\[
A^\nabla_{\hbar q} S := \frac{\partial S}{\partial \hbar} + \frac{1}{\hbar} \sigma_1 \ast_q S, \tag{46}
\]

\[
A^\nabla_{\hbar h} S := \hbar \frac{\partial S}{\partial \hbar} - \frac{1}{\hbar} \sigma_1 (TX) \ast_q S + \text{Gr}(S), \tag{47}
\]

following the conventions of Iritani [Iri09], where \( \ast_q \) denotes the quantum cup product in the quantum cohomology, and \( S \) may be any meromorphic or formal section of the above vector bundle. The above defines a meromorphic connection which is flat, see also [Dub96, Giv96, CK99]. It therefore turns \( H^*(X, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]) \) into
a $D$-module for $\mathbb{C}[h^{\pm 1}, q^{\pm 1}](\partial_h, \partial_q)$, which we will call the $A$-model $D$-module and denote by $M_A$. Explicitly

$$M_A := H^*(X, \mathbb{C}[h^{\pm 1}, q^{\pm 1}]),$$

with $\partial_h \sigma := A\nabla_{\partial_h} \sigma$ and $\partial_q \sigma := A\nabla_{\partial_q} \sigma$.

This is the $D$-module we consider on the $A$-model side.

We now define the $D$-module $M_B$. Let $\Omega^k(X_{\text{can}})$ denote the space of all algebraic $k$-forms on $X_{\text{can}}$.

**Definition 5.1.** Define the $\mathbb{C}[h, q]$-module

$$G^{W_q}_0 := \Omega^0(X_{\text{can}})[h, q]/(hd + dW_q \wedge -)\Omega^{n-1}(X_{\text{can}})[h, q].$$

It has a meromorphic (Gauss-Manin) connection given by

$$B\nabla_{q\partial_q} \alpha = q \frac{\partial}{\partial q} [\alpha] + \frac{1}{\hbar} \left[ q \frac{\partial W_q}{\partial q} \right],$$

(49)

$$B\nabla_{h\partial_h} \alpha = \hbar \frac{\partial}{\partial h} [\alpha] - \frac{1}{\hbar} [W_q \alpha].$$

Let $M_B = G^{W_q}_0 \otimes_{\mathbb{C}[h, q]} \mathbb{C}[h^{\pm 1}, q^{\pm 1}]$. We view $M_B$ as a $\mathbb{C}[h^{\pm 1}, q^{\pm 1}](\partial_h, \partial_q)$-module with $\partial_q$ acting by $B\nabla_{\partial_q}$ and $\partial_h$ acting by $B\nabla_{\partial_h}$.

On the $A$-model side a special role is played by the element $1 \in M_A$ corresponding to the identity in $H^*(X, \mathbb{C})$. For the $B$-model there is also a distinguished element. Recall that $X_{\text{can}}$ is the complement of an anticanonical divisor in $X$. Therefore we saw that there is an up to scalar unique non-vanishing logarithmic $N$-form on $X_{\text{can}}$ which we called $\omega_{\text{can}}$ (see Equations [10] and [27]). This is the same form as the one appearing in [GHK11] Lemma 5.14, and it also agrees with the one from [Rie08] after the isomorphism of $X_{\text{can}}$ with $X_{\text{Lie}}$. It determines an element $[\omega_{\text{can}}]$ in $M_B$.

5.1. **The case of odd-dimensional quadrics.** For odd-dimensional quadrics we recall the isomorphism between the $D$-modules on the two sides, proved using results from [GS13].

**Theorem 5.2 ([PR13 Corollary 13]).** For $X = Q_{2m-1}$ with its mirror LG-model $(\tilde{X}_{2m-1}, W_q)$ from Theorem [27], the map

$$M_A \quad \rightarrow \quad M_B,$$

$$\sigma_i \quad \mapsto \quad [p_i \omega_{\text{can}}]$$

defines an isomorphism of $D$-modules.

5.2. **The case of even-dimensional quadrics.** For even quadrics $Q_{2m-2}$ we prove the following.

**Theorem 5.3.** For $X = Q_{2m-2}$ and the canonical mirror $(\tilde{X}_{2m-2}, W_q)$, see [28], the map

$$\Psi : \quad M_A \quad \rightarrow \quad M_B,$$

$$\sigma_i \quad \mapsto \quad [p_i \omega_{\text{can}}],$$

$$\sigma_{m-1}' \quad \mapsto \quad [p_{m-1}' \omega_{\text{can}}],$$

defines an injective homomorphism of $D$-modules. In particular, the $\mathbb{C}[h^{\pm 1}, q^{\pm 1}]$-submodule of $M_B$ generated by the classes $[p_i \omega_{\text{can}}]$ and $[p_{m-1}' \omega_{\text{can}}]$ is a submodule also for $D = \mathbb{C}[h^{\pm 1}, q^{\pm 1}](\partial_h, \partial_q)$. 

Remark 4. In the odd quadrics case, [GS13] (with Némethi and Sabbah) prove an additional property, cohomological tameness, for the superpotential, which implies that the dimension of $M_R$ agrees with the number of critical points of $W_q$. It is an interesting question whether this proof could be adapted to give a proof of cohomological tameness in the even case. Since by Proposition 1.3 the number of critical points of $W_q$ agrees with the dimension of $H^*(X, \mathbb{C})$ this would imply that the injective homomorphism in Theorem 5.3 is an isomorphism.

To prove Theorem 5.3 we consider a cluster algebra structure on our mirror $\tilde{X}_{2m-2}$. Cluster algebras were introduced by Fomin and Zelevinsky in the seminal paper [FZ02], which was the first of the series [FZ03, BFZ05, FZ07].

The coordinate ring $\mathbb{C}[\tilde{X}_{2m-2}]$ has a cluster algebra structure of type $A_{2m-2}^1$ which is described in detail in [GLS08b, GLS08a, GLS08c] and which we review here. Note that the coordinates $\{y_1, y_2, \ldots, y_{2m}\}$ in [GLS08b, GLS08a, GLS08c] correspond to our coordinates $\{p_0, p_1, \ldots, p_m-2, p_m-1, p_{m-1}, p_m, \ldots, p_{2m-2}\}$ here, while the coordinates $\{p_i\}$ in [GLS08b, GLS08a, GLS08c] correspond to our coordinates $\{\delta_i\}$.

Consider the following initial quiver:

Here the initial cluster variables correspond to the vertices in the top row of the quiver, while the frozen variables (or coefficients) correspond to the vertices in the bottom row. Recall that the $p_i$ are Plücker coordinates, and the $\delta_i$ are defined as in (26). We see from this description that the coordinate ring of $\tilde{X}_{2m-2}$ has a cluster structure of type $A_{2m-2}^1$. In particular, it is of finite type, and there are $2m-2$ different clusters, consisting of

- the cluster variables $r_1, \ldots, r_{m-2}$, where $r_i \in \{p_i, p_{2m-2-i}\}$;
- the frozen variables (or coefficients) $\delta_1, \ldots, \delta_{m-3}$, $p_0$, $p_{m-1}$, $p_{m-1}'$, and $p_{2m-2}$.

The exchange relations are

$$ p_i p_{2m-2-i} = \begin{cases} 
 p_0 p_{2m-2} + \delta_1 & \text{for } i = 1; \\
 \delta_{i-1} + \delta_i & \text{for } 1 \leq i \leq m - 3; \\
 \delta_{m-3} + p_{m-1} p_{m-1}' & \text{for } i = m - 2.
\end{cases} $$

(51)

Note that the exchange relation for $i = m - 2$ is a Plücker relation: it is the equation of the dual quadric $28$.

Remark 5. In the case of $\tilde{X}_{2m-3}^2$ the isomorphism with the Richardson variety combined with [GLS11] also gives a cluster algebra structure of type $A_{m-2}^1$, with a similar quiver to the one shown on page 28 but where the frozen vertices labelled $p_{m-1}$ and $p_{m-1}'$ are identified.

Proof of Theorem 5.3 For the injectivity of $\Psi$ we refer to [MRT13, Section 5]. It remains to prove that $\Psi$ preserves the $D$-module structure. We use a change of
coordinates to reduce the problem to checking only the action of \(q\partial_q\). Namely, this follows by replacing \((p_i, q, h)\) with \((\tilde{p}_i, q, h)\), where
\[
\tilde{p}_i = h^{-i}p_i, \quad p'_{m-1} = h^{1-m}p_{m-1}, \quad q = h^{-N}q, \quad h = h,
\]
and observing that written in these coordinates the Gauss-Manin system for \(\frac{1}{h}W_q\) no longer involves the \(h\).

Now we check that the map \(\Psi\) preserves the action of \(q\partial_q\). We consider the following identities in \(QH^*(Q_{2m-2}, \mathbb{C})\), which are a special case of results in \([FW04]\):

\[
\sigma_1 \star_q \sigma_i = \begin{cases} 
\sigma_{i+1} & \text{for } 0 \leq i \leq m - 3 \text{ or } m - 1 \leq i \leq 2m - 4; \\
\sigma_{m-1} + \sigma'_{m-1} & \text{for } i = m - 2; \\
\sigma_{2m-2} + q\sigma_0 & \text{for } i = 2m - 3; \\
q\sigma_1 & \text{for } i = 2m - 2,
\end{cases}
\]

\[(52) \quad \sigma_1 \star_q \sigma_{m-1} = \sigma_m.
\]

We need to prove that there are similar identities on the \(B\) side:

\[
\frac{\partial W_q}{\partial q} p_i \omega_{\text{can}} = \begin{cases} 
[p_{i+1}\omega_{\text{can}}] & \text{for } 0 \leq i \leq m - 3 \text{ or } m - 1 \leq i \leq 2m - 4; \\
[(p_{m-1} + p'_{m-1})\omega_{\text{can}}] & \text{for } i = m - 2; \\
[(p_{2m-2} + q)\omega_{\text{can}}] & \text{for } i = 2m - 3; \\
[q\omega_{\text{can}}] & \text{for } i = 2m - 2,
\end{cases}
\]

\[
\frac{\partial W_q}{\partial q} p'_{m-1} \omega_{\text{can}} = [p_m \omega_{\text{can}}],
\]

(53)

where \(\omega_{\text{can}}\) is the canonical \((2m - 2)\)-form on \(\tilde{X}_{\text{can}}\).

The proof of these identities in \(M_B\) proceeds by constructing closed \((2m - 3)\)-forms \(\nu_i\) and \(\nu'_{m-1}\) such that the relation corresponding to \(p_i\) will follow from the fact that
\[
[dW_q \wedge \nu_i] = [(hd + dW_q \wedge -)\nu_i] = 0
\]
and similarly for \(p'_{m-1}\). (The first equality above comes from the fact that \(\nu_i\) is closed, and the second comes from the definition of \(M_B\).)

Concretely, we will pick a cluster \(\mathcal{C}\) containing a particular Plücker coordinate, say \(p_i\), and use the following Ansatz for constructing \(\nu_i\). We define a vector field
\[
\xi_i = \frac{\sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c \partial_c}{\sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c}
\]
and define an associated \((2m - 3)\)-form by insertion \(\nu_i = \iota_{\xi_i} \omega_{\text{can}}\), and analogously for \(\nu'_{m-1} = \iota_{\xi'_{m-1}} \omega_{\text{can}}\). Here the \(m_c\)'s are constants and \(\iota\) is the interior product.

To see that these \((2m - 3)\)-forms are closed, write \(\omega_{\text{can}} = \bigwedge_{p \in \mathcal{C}} \frac{dp}{p}\). For \(c \in \mathcal{C}\), we have \(\iota_{c\partial_c} \omega_{\text{can}} = \bigwedge_{p \in \mathcal{C} \setminus \{c\}} \frac{dp}{p}\), and so \(\nu_i\) is a \(\mathcal{C}\)-linear combination of terms of the form \(p_i \bigwedge_{p \in \mathcal{C} \setminus \{c\}} \frac{dp}{p}\) for \(c \neq p_i\). Such a term is closed, because \(p_i\) lies in \(\mathcal{C} \setminus \{c\}\).
Using the fact that $d\mathcal{W}_q \wedge \omega_{\text{can}} = 0$, we get $d\mathcal{W}_q \wedge \nu_i = \pm d\mathcal{W}_q(\xi_i)\omega_{\text{can}}$. It follows that

$$d\mathcal{W}_q \wedge \nu_i = p_i \left( \sum_{c \in C \setminus \{p_i\}} m_c \frac{\partial \mathcal{W}_q}{\partial c} \right) \omega_{\text{can}}.$$ 

Therefore e.g. in order to prove that $q \frac{\partial \mathcal{W}_q}{\partial q} p_i - p_i + 1 \omega_{\text{can}} = 0$, we will show that $q \frac{\partial \mathcal{W}_q}{\partial q} p_i - p_i + 1$ has the form $p_i \left( \sum_{c \in C \setminus \{p_i\}} m_c \frac{\partial \mathcal{W}_q}{\partial c} \right)$, for some choice of coefficients $m_c$.

To prove these identities, we will work with two clusters:

- the initial cluster $C_1 = \{p_1, \ldots, p_{m-2}, \delta_1, \ldots, \delta_{m-3}, p_0, p_{m-1}, p_{m-1}', p_{2m-2}\};$
- the cluster $C_2 = \{p_{2m-3}, \ldots, p_m, \delta_1, \ldots, \delta_{m-3}, p_0, p_{m-1}, p_{m-1}', p_{2m-2}\}$.

Let us first start with $C_1$ and express $\mathcal{W}_q$ in terms of it using the exchange relations \[^{[51]}\]. To simply our calculations, we set $p_0 = 1$, and let $\delta_0$ denote $p_0 p_{2m-2} = p_{2m-2}$.

$$\mathcal{W}_q = p_1 + \sum_{i=1}^{m-3} \left( \frac{p_{i+1} \delta_{i-1}}{p_i \delta_i} + \frac{p_{i+1}}{p_i} \right) + \frac{\delta_{m-3}}{p_{m-2} p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}'} + \frac{p_{m-1}'}{p_{m-2}} + \frac{p_{m-1}}{p_{m-2}} + \delta_0.$$

The partial derivatives of $\mathcal{W}_q$ are:

$$q \frac{\partial \mathcal{W}_q}{\partial q} = q \frac{p_1}{\delta_0},$$

$$p_1 \frac{\partial \mathcal{W}_q}{\partial p_1} = p_1 - \frac{p_2 \delta_0}{p_1 \delta_1} - \frac{p_2}{p_1} + \frac{p_1}{\delta_0},$$

$$p_i \frac{\partial \mathcal{W}_q}{\partial p_i} = \frac{p_i \delta_{i-1}}{p_{i-1} \delta_i} + \frac{p_i}{p_{i-1}} - \frac{p_{i+1} \delta_{i-1}}{p_{i} \delta_i} - \frac{p_{i+1}}{p_i} \text{ for } 2 \leq i \leq m-3,$$

$$p_{m-2} \frac{\partial \mathcal{W}_q}{\partial p_{m-2}} = \frac{p_{m-2} \delta_{m-4}}{p_{m-3} \delta_{m-3}} + \frac{p_{m-2}}{p_{m-3}} - \frac{\delta_{m-3}}{p_{m-2} p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}'} + \frac{p_{m-1}}{p_{m-2}} - \frac{p_{m-1}'}{p_{m-2}}.$$

$$\delta_0 \frac{\partial \mathcal{W}_q}{\partial \delta_0} = \frac{p_2 \delta_0}{p_1 \delta_1} - \frac{p_1}{\delta_0},$$

$$\delta_i \frac{\partial \mathcal{W}_q}{\partial \delta_i} = \frac{p_{i+1} \delta_{i-1}}{p_i \delta_i} + \frac{p_{i+2} \delta_i}{p_{i+1} \delta_{i+1}} \text{ for } 1 \leq i \leq m-4,$$

$$\delta_{m-3} \frac{\partial \mathcal{W}_q}{\partial \delta_{m-3}} = \frac{p_{m-2} \delta_{m-4}}{p_{m-3} \delta_{m-3}} + \frac{\delta_{m-3}}{p_{m-2} p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}'},$$

$$p_{m-1} \frac{\partial \mathcal{W}_q}{\partial p_{m-1}} = -\frac{\delta_{m-3}}{p_{m-2} p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}'} + \frac{p_{m-1}}{p_{m-2}} \text{ and }$$

$$p_{m-1}' \frac{\partial \mathcal{W}_q}{\partial p_{m-1}'} = -\frac{\delta_{m-3}}{p_{m-2} p_{m-1}'} - \frac{\delta_{m-3}}{p_{m-2}'} + \frac{p_{m-1}'}{p_{m-2}}.$$


Hence

\[ \frac{q}{\partial q} W_a p_i - p_{i+1} = -p_i \left( \sum_{j=i+1}^{m-1} p_j \frac{\partial W_a}{\partial p_j} + p_{m-1}' \frac{\partial W_a}{\partial p_{m-1}} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_a}{\partial \delta_j} + \sum_{j=i}^{m-3} \delta_j \frac{\partial W_a}{\partial \delta_j} \right) \]

for \( 0 \leq i \leq m - 3 \), and

\[ \frac{q}{\partial q} W_{m-2} - (p_{m-1} + p_{m-1}') = -p_{m-2} \left( p_{m-1} \frac{\partial W_a}{\partial p_{m-1}} + p_{m-1}' \frac{\partial W_a}{\partial p_{m-1}} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_a}{\partial \delta_j} \right). \]

Since the right-hand sides of the above equations have the form \( \sum_{c \in C \setminus \{ p_i \}} m_c c_c \partial W_a \),

this proves identity (54) for \( 0 \leq i \leq m - 2 \).

To prove the remaining identities, we use the cluster \( C_2 \). In this cluster chart, \( W_a \) takes the following form:

\[ W_a = \frac{\delta_0}{p_{2m-3}} + \frac{\delta_1}{p_{2m-3}} + \sum_{\ell=1}^{m-4} \left( \frac{p_{2m-2-\ell}}{p_{2m-3-\ell}} + \frac{p_{2m-2-\ell} \delta_{\ell+1}}{p_{2m-3-\ell} \delta_\ell} \right) + \frac{p_m}{p_{m-1}} + \frac{p_{m+1}}{p_m} + \frac{p_{m-1} p_{m+1}}{p_m \delta_{m-3}^3} + \frac{q}{p_{2m-3}} + \frac{q \delta_1}{p_{2m-3} \delta_0}. \]

Working out the partial derivatives of \( W_a \) as before, we get

\[ q \frac{\partial W_a}{\partial q} p_{m-1} - p_m = p_{m-1} \left( p_{m-1}' \frac{\partial W_a}{\partial p_{m-1}} + \sum_{j=m}^{2m-3} p_j \frac{\partial W_a}{\partial p_j} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_a}{\partial \delta_j} \right) \]

(56)

\[ q \frac{\partial W_a}{\partial q} p_{m-1} - p_m = p_{m-1} \left( p_{m-1}' \frac{\partial W_a}{\partial p_{m-1}} + \sum_{j=m}^{2m-3} p_j \frac{\partial W_a}{\partial p_j} + \sum_{j=0}^{m-3} \delta_j \frac{\partial W_a}{\partial \delta_j} \right) \]

(57)

\[ q \frac{\partial W_a}{\partial q} p_{i} - p_{i+1} = p_i \left( \sum_{j=i+1}^{2m-3} p_j \frac{\partial W_a}{\partial p_j} - \sum_{j=0}^{m-3-i} \delta_j \frac{\partial W_a}{\partial \delta_j} \right) \]

(58)

\( \text{for } m \leq i \leq 2m - 4, \)

Recall that \( \delta_0 \) is \( p_{2m-2} \). The final two relations are

\[ q \frac{\partial W_a}{\partial q} p_{2m-3} - (p_{2m-2} + q) = -p_{2m-3} \delta_0 \frac{\partial W_a}{\partial \delta_0} \]

and

\[ q \frac{\partial W_a}{\partial q} p_{2m-2} - qp_1 = 0 \]

(59)

(60)

This gives us the identities (54) for \( m - 1 \leq i \leq 2m - 2, \) as well as (55). \( \square \)

6. THE HYPERGEOMETRIC FLAT SECTION OF A QUADRIC

Givental in [Giv96] constructed flat sections of a dual version of the Dubrovin connection (see Equations (61) and (62) below) in terms of Gromov-Witten invariants. In this section we directly and explicitly compute all the components of a distinguished flat section and the resulting invariants, in two different ways. The first component we consider is also a particular component of Givental’s \( J \)-function.
6.1. The dual Dubrovin connection and the $J$-function. We begin by defining Givental’s $J$-function and what we call the ‘quantum differential operators’. Consider the dual connection to $A\nabla$ with respect to the pairing
\[
\langle \sigma, \tau \rangle = (2\pi \hbar)^N \int_X \sigma \cup \tau.
\]
Here $\sigma \cup \tau$ is the usual cup product of $\sigma$ and $\tau$, which we will subsequently also denote by $\sigma \tau$. Explicitly, the dual connection is given by the formulas:
\[
A\nabla_{q\hbar}^\vee S := q \frac{\partial S}{\partial q} - \frac{1}{\hbar} \sigma_1 \ast q S,
\]
\[
A\nabla_{h\hbar}^\vee S := h \frac{\partial S}{\partial h} + \frac{1}{\hbar} c_1(TX) \ast q S + \text{Gr}(S),
\]
compare [Iri09, Definition 3.1]. For the purposes of the $J$-function we ignore the $A\nabla_{h\hbar}^\vee$ part of the covariant derivative and consider $A\nabla_{q\hbar}^\vee$ as a family of connections (in the parameter $\hbar$). Formal flat sections indexed by the cohomology basis were written down by Givental [Giv96] in terms of descendent Gromov-Witten invariants. We denote these sections by $S_0, \ldots, S_{2m-1}$ in the case of $Q_{2m-1}$, and by $S_0, \ldots, S_{m-1}, S_{m-1}', S_m, \ldots, S_{2m-2}$ for $Q_{2m-2}$, in keeping with the notation from [21] for Schubert classes. See [CK99] (10.14) for a precise definition of the sections $S_i$.

**Definition 6.1.** We define Givental’s $J$-function in our setting as
\[
J = (2\pi \hbar)^N \sum \langle S_j, \sigma_0 \rangle \sigma_{PD(j)},
\]
where the sum is over all the Schubert classes, including $\sigma_{m-1}'$ in the even case, and where $\sigma_{PD(j)}$ stands for the Poincaré dual cohomology class to $\sigma_j$.

In the case of a quadric (or, indeed, of any projective Fano complete intersection), the $J$-function is computed explicitly in [Giv96, Theorem 9.1] from the $J$-function of projective space. Namely
\[
J^{Q_N} = e^{\frac{\ln(q)}{\hbar}} \sum_{d \geq 0} \prod_{j=1}^{2d} (2\sigma_1 + j\hbar)^N q^d.
\]
We consider a family of differential operators which annihilate the $J$-function:

**Definition 6.2** ([CK99, Definition 10.3.2]). The differential operators $P$ which are formal power series in $\hbar q \partial_q, q, \hbar$ and which annihilate the coefficients of Givental’s $J$-function are called quantum differential operators.

6.2. The hypergeometric term of the $J$-function. Among Givental’s flat sections $S_i$, the flat section $S_N$ corresponding to the class of a point has the property that all its coefficients are power series in $q = h^{-N} q$. Moreover, a special role is played by the coefficient $(2\pi \hbar)^N \langle S_N, \sigma_0 \rangle$, also appearing as the coefficient of the fundamental class in the definition of $J$-function. We define it as in [BCFKvS98, Definition 5.1.1]:

**Definition 6.3.** The hypergeometric series $A_X$ of $X$ is the unique power series of the form $A_X(q) = 1 + \sum_{k=1}^{\infty} a_k q^k$, for which $P(q \partial_q, q, 1)A_X = 0$ for all quantum differential operators $P(lq \partial_q, q, \hbar)$ specialized to $\hbar = 1$. We denote the hypergeometric series $A_{Q_N}$ of the quadric $Q_N$ by $A_N$. 
The hypergeometric series $A_N$ of the quadric $Q_N$ may be obtained by setting $\hbar$ to 1 in $(2\pi i\hbar)^N \langle S_N, \sigma_0 \rangle$. Alternatively we have $\langle S_N, \sigma_0 \rangle = A_N(h^{-N}q)$.

We recall the geometric interpretation of the coefficients of $A_X$ below. The flat sections $S_i$ and in particular the $J$-function encode certain descendent Gromov-Witten invariants. Let

\[ J^{Q_N} = (2\pi i\hbar)^N \sum_{i=1}^N J_i^{Q_N} \sigma_{PD(i)}, \]

then in fact $A_N(h^{-N}q) = \langle S_N, \sigma_0 \rangle = J^{Q_N}$ and we have

\[ A_N(h^{-N}q) = J^{Q_N} = 1 + \sum_{k=1}^\infty q^k I_k \left( \frac{\sigma_N e^{\frac{\ln(q)\gamma_1}{\hbar - \psi}}}{\hbar - \psi}, \sigma_0 \right) \]

\[ = 1 + \sum_{k=1}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\ln(q)\gamma_1}{\hbar} \right)^j \frac{1}{j!} \left( \frac{\psi}{\hbar} \right)^i, \sigma_0 \right). \]

The cup-product $\sigma_N \cup \left( \frac{\ln(q)\gamma_1}{\hbar} \right)^j$ is nonzero if and only if $j = 0$. Therefore we have

\[ J^{Q_N} = 1 + \sum_{k=1}^\infty \sum_{j=0}^\infty \sum_{i=0}^\infty \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\psi}{\hbar} \right)^i, \sigma_0 \right). \]

Now the dimension of the moduli space of stable maps $\overline{M}_{0,2}(Q_N, k)$ is equal to $(k+1)N - 1$, hence

\[ J^{Q_N} = 1 + \sum_{k=1}^\infty \frac{q^k}{\hbar} I_k \left( \sigma_N \left( \frac{\psi}{\hbar} \right)^{kN-1}, \sigma_0 \right). \]

Next we use the fundamental class axiom to get

\[ J^{Q_N} = 1 + \sum_{k=1}^\infty \left( \frac{q}{\hbar^{N+2}} \right)^k I_k \left( \sigma_N \psi^{kN-2} \right). \]

If we set $\hbar = 1$ in $J^{Q_N}$, this gives exactly the hypergeometric series of the quadric, since $J^{Q_N}_N = A_N(h^{-N}q)$. Hence we obtain the following geometric interpretation of the coefficient $a_k$ of $q^k$ in $A_N(q)$:

\[ a_k = I_k \left( \sigma_N \psi^{kN-2} \right). \]

6.3. The hypergeometric flat section of the dual Dubrovin connection. In this Section, as an illustration of the mirror theorem, we compute explicitly the coefficients of the hypergeometric flat section $S_N$ of the Dubrovin connection for $Q_N$, once using the $A$-model and once using the $B$-model. The main result of the computations is the following.
Theorem 6.4. The hypergeometric flat section $S_N$ of the dual Dubrovin connection for $Q_N$ is given by the expansion

$$S_N = \frac{1}{(2\pi i)^N} \sum_{\ell=0}^{N} \langle S_N, \sigma_{\ell} \rangle \sigma_{PD(\ell)},$$

where $\sum'$ means that we add an extra summand $\langle S_N, \sigma_{m-1} \rangle \sigma_{m-1}$ when $N = 2m-2$. The coefficients are given by the following formulas:

$$\langle S_N, \sigma_{\ell} \rangle = \begin{cases} \sum_{k \geq 0} \frac{k^\ell}{k!(k+1)!} \cdot \binom{2k}{k} \cdot q^k & \text{if } 0 \leq \ell \leq \lfloor \frac{N-1}{2} \rfloor, \\ \sum_{k \geq 0} \frac{2^{\ell-k} k^\ell (k+1)!}{2^{\ell-k} (k+1)!} \cdot \binom{2k}{k} \cdot q^k & \text{if } \lfloor \frac{N+1}{2} \rfloor \leq \ell \leq N-1, \\ \sum_{k \geq 0} \frac{2^{\ell-k} k^\ell (k+1)!}{2^{\ell-k} (k+1)!} \cdot \binom{2k-2}{k-1} \cdot q^k & \text{if } \ell = N. \end{cases}$$

Moreover, when $N = 2m - 2$ is even, we have

$$\langle S_N, \sigma_{m-1} \rangle = \sum_{k \geq 0} \frac{k^{m-1}}{2^{kN+1-m(k+1)!}} \cdot \binom{2k}{k} \cdot q^k.$$

The $\ell = 0$ special case of Theorem 6.4 gives the following.

Corollary 6.5. The hypergeometric series of the quadric $Q_N$ is

$$(66) \quad A_N(q) = 1 + \sum_{k \geq 1} \frac{1}{(k!)^N} \binom{2k}{k} q^k.$$

The Gromov-Witten invariant $I_k(\sigma_N \psi^{Nk-2})$ is given by

$$(67) \quad I_k(\sigma_N \psi^{Nk-2}) = \frac{1}{(k!)^N} \binom{2k}{k}.$$

This corollary is easily verified in the A-model. The formula (67) follows from equations (66) and (65), while the second formula, (66), follows easily from the formula (55) for the $J$-function of $Q_N$. In the odd quads case the $D$-module is cyclic and hence the constant term determines all of the other terms of the flat section. However for even quadrics this is not the case. We now give a direct A-model proof of Theorem 6.4 which works in the even and odd case alike.

A-model proof. Our A-model proof works by recovering Theorem 6.4 from the recurrence relations of Kontsevich-Manin for Gromov-Witten invariants [KM98]. Define

$$(68) \quad \beta_{\ell,k} = I_k(\psi^{Nk-1-\ell} \sigma_N, \sigma_{\ell}).$$

Let us first assume that $N = 2m - 1$. Using the divisor axiom and topological recursion, we get:

$$k \beta_{\ell,k} = I_k(\psi^{Nk-1-\ell} \sigma_N, \sigma_{\ell}, \sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m-1, N-1, N\}, \\ 2\beta_{m,k} & \text{if } \ell = m-1, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N-1, \\ \beta_{1,k-1} & \text{if } \ell = N. \end{cases}$$

A straightforward computation then gives

$$\frac{\beta_{\ell,k+1}}{\beta_{\ell,k}} = \begin{cases} \frac{2(2k+1)}{k(k+1)^N+1} & \text{if } 0 \leq \ell \leq N-1, \\ \frac{2(2k+1)}{k(k-1)^N} & \text{if } \ell = N, \end{cases}$$
and $\beta_{1,1} = 2$, which yields Theorem 6.4.

Similarly, in the case where $N = 2m - 2$:

$$k\beta_{\ell,k} = I_k^0(\psi N^{k-1-\ell})(N,\sigma,\sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m - 2, N - 1, N\}, \\ \beta_{m-1,k} & \text{if } \ell = m - 2, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N - 1, \\ \beta_{1,k-1} & \text{if } \ell = N, \end{cases}$$

and

$$k\beta_{m-1,k} = \beta_{m,k}.$$ 

Theorem 6.4 is then easily checked.

$B$-model proof. We consider the distinguished flat section of the Dubrovin connection whose coefficients are expressed in terms of the $B$-model as residue integrals, see Section 6.4 and compare with [MR13, Theorem 4.2]. Explicitly, we let $\Gamma$ be a compact cycle inside $\mathcal{X}_{\text{can}}$ such that $\int_{\Gamma_0}\omega_{\text{can}} = 1$. Then the integral formula

$$S_{\Gamma_0}(h,q) := \frac{1}{(2\pi i)^N} \sum \int_{\Gamma_0} e^{\frac{1}{\hbar}W_0(h)} \sigma_{N-i}$$

defines a flat section of the Dubrovin connection in the $N = 2m - 1$ case, and with $\int_{\Gamma_0} e^{\frac{1}{\hbar}W_0(h)} \sigma_{m-1}$ replaced by $\int_{\Gamma_0} e^{\frac{1}{\hbar}W_0(h)} \sigma_{m-1} + \int_{\Gamma_0} e^{\frac{1}{\hbar}W_0(h)} \sigma'_{m-1}$ in the $N = 2m - 2$ case.

We will prove the formula in Theorem 6.4 in one representative case, but omit the other cases, which are extremely similar.

Let us consider the case that $N = 2m - 2$, and $m \leq \ell \leq 2m - 3$. In this case recall that $p_\ell = a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_{2m-1-\ell}$, and recall from (7) that the superpotential $W_q$ equals

$$a_1 + \ldots + a_{m-2} + c + d + b_{m-2} + \ldots + b_1 + \frac{q}{a_2 \ldots a_{m-2} c d b_{m-2} \ldots b_1} + \frac{q}{a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_2}$$

in terms of the usual coordinates on $\mathcal{X}_{\text{Lus}}$ viewed as a torus chart in $\mathcal{X}_{\text{can}}$.

To compute the constant term of $p_\ell \exp(\frac{1}{\hbar}W_q)$, we consider

$$p_\ell \left( 1 + \frac{q}{\hbar^1} W_q + \frac{q}{\hbar^2} W_q^2 + \frac{q}{\hbar^3} W_q^3 + \ldots \right),$$

and we pick out from each $p_\ell \frac{W_q^i}{\hbar^i}$ every term which has the form $\lambda q^i$ where $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$. Here we just need to look at each $\frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}}$ for $k = 1, 2, \ldots$, because the expansion of $p_\ell \frac{W_q^i}{\hbar^i}$ for $i$ not of the form $kN - \ell$ will contain no terms of the form $\lambda q^i$ for $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$.

Now let us analyze $p_\ell \frac{W_q^{kN-\ell}}{\hbar^{kN-\ell}}$ for $N = 2m - 2$. A (Laurent) monomial in the expansion of $p_\ell W_q^{kN-\ell}$ is obtained by choosing one term in each of the $k(2m - 2) - \ell$ factors. Some of the monomials in the expansion will be pure in the variable $q$ alone – in which case they will equal $q^k$. We need to show that the number of such monomials divided by $(k(2m - 2) - \ell)!$ equals $\frac{1}{2}(k^2)k^\ell/(k!)^{k(2m - 2)}$. To count the number of such monomials, we need to pick one term in each of the $k(2m - 2) - \ell$ factors so that we:

- choose $i$ terms which are $\frac{q}{a_2 \ldots a_{m-2} c d b_{m-2} \ldots b_1}$ for some $0 \leq i \leq k$;
choose \( k - i \) terms which are \( a_1 \ldots a_{m - 2} a_2 \); 
• choose \( k - 1 \) terms which are \( c \); 
• choose \( k - 1 \) terms which are \( d \); 
• choose \( i \) terms which are \( b_1 \); 
• choose \( k - i - 1 \) terms which are \( a_1 \); 
• for each \( j \) such that \( 2 \leq j \leq m - 2 \), choose \( k - 1 \) terms which are \( a_j \); 
• for each \( j \) such that \( 2 \leq j \leq 2m - 2 - \ell \), choose \( k \) terms which are \( b_j \). 

The number of ways to do this is the sum of multinomial coefficients

\[
\sum_{i=0}^{k} \binom{k(2m - 2) - \ell}{i, i, k - i - 1, k \ldots k, k - 1 \ldots k - 1},
\]

where the number of \( k \)'s in the string \( k \ldots k \) above is \( 2m - 2 - \ell - 1 \), and the number of \( k - 1 \)'s in the string \( k - 1 \ldots k - 1 \) above is \( \ell - 1 \). When we simplify (70) and divide by \( (k(2m - 2) - \ell)! \), we obtain \( \frac{1}{2} (2^k) k^\ell / (k!)^{k(2m-2)} \), as desired. \( \square \)

REFERENCES

[BCFKvS98] Victor V. Batyrev, Ionuţ Ciocan-Fontanine, Bumsig Kim, and Duco van Straten, Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians, Nuclear Phys. B 514 (1998), no. 3, 640–666. MR 1619529 (99m:14074)
[BCFKvS00] , Mirror symmetry and toric degenerations of partial flag manifolds, Acta Math. 184 (2000), no. 1, 1–39. MR 1756568 (2001f:14077)
[BFZ96] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, Parametrizations of canonical bases and totally positive matrices, Adv. Math. 122 (1996), no. 1, 49–149. MR 1405449 (98j:17008)
[BFZ05] , Cluster algebras. III. Upper bounds and double Bruhat cells, Duke Math. J. 126 (2005), no. 1, 1–52. MR 2110627 (2005i:16065)
[BZ97] Arkady Berenstein and Andrei Zelevinsky, Total positivity in Schubert varieties, Comment. Math. Helv. 72 (1997), no. 1, 128–166. MR 1456321 (99g:14064)
[CK09] David A. Cox and Sheldon Katz, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999. MR 1677117 (2000d:14048)
[Dub96] Boris Dubrovin, Geometry of 2D topological field theories, Integrable Systems and Quantum Groups 1620 (1996), 120 – 348.
[EHX97] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong, Gravitational quantum cohomology, Internat. J. Modern Phys. A 12 (1997), no. 9, 1743–1782.
[FW04] W. Fulton and C. Woodward, On the quantum product of Schubert classes, J. Algebraic Geom. 13 (2004), no. 4, 641–661.
[FZ02] Sergey Fomin and Andrei Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529 (electronic). MR 1887642 (2003f:16050)
[FZ03] , Cluster algebras. II. Finite type classification, Invent. Math. 154 (2003), no. 1, 63–121. MR 2004457 (2004m:17011)
[FZ07] , Cluster algebras. IV. Coefficients, Compos. Math. 143 (2007), no. 1, 112–164. MR 2295199 (2008d:16049)
[GHK11] Mark Gross, Paul Hacking, and Sean Keel, Mirror symmetry for log Calabi-Yau surfaces I, 2011.
[Gin95] V. Ginzburg, Perverse sheaves on a Loop group and Langlands’ duality, arXiv:9511007, 1995.
[Giv95] Alexander B. Givental, Homological geometry and mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 472–480. MR 1403947 (97j:58013)
[Giv96] , Equivariant Gromov-Witten invariants, IMRN 13 (1996), 613–663.
[Giv97] Givental, A. stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, Topics in singularity theory, Amer. Math. Soc. Transl. Ser. 2, vol. 180, Amer. Math. Soc., Providence, RI, 1997, pp. 103–115. MR 1767115 (2001d:14063)

[GLS08a] Christof Geiß, Bernard Leclerc, and Jan Schröer, Partial flag varieties and preprojective algebras, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 3, 825–876. MR 2427512 (2009f:14104)

[GLS08b] Preprojective algebras and cluster algebras, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 253–283. MR 2484728 (2009m:16024)

[GLS11] Kac-Moody groups and cluster algebras, Adv. Math. (2011), no. 687, 329–433.

[GS13] Vassily Gorbounov and Maxim Smirnov, Some remarks on Landau-Ginzburg potentials for odd-dimensional quadrics, 2013.

[Iri09] Hiroshi Iritani, An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, Adv. Math. 222 (2009), no. 3, 1016–1079.

[KLS14] Allen Knutson, Thomas Lam, and David E Speyer, Projections of Richardson varieties, Journal für die reine und angewandte Mathematik (Crelles Journal) (2014), no. 687, 133–157.

[KM98] M. Kontsevich and Yu. Manin, Relations between the correlators of the topological sigma-model coupled to gravity, Comm. Math. Phys. 196 (1998), no. 2, 385–398. MR 1645019 (99k:14040)

[KS14] Shrawan Kumar and Karl Schwede, Richardson varieties have Kawamata log terminal singularities, Int. Math. Res. Not. IMRN (2014), no. 3, 842–864. MR 3163569

[Lus83] George Lusztig, Singularity of character formulas, and a \( q \)-analog of weight multiplicities, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229.

[Lus94] G. Lusztig, Total positivity in reductive groups, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568. MR 1327548 (96m:20071)

[MKS07] I. Mirković and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[Pet97] D. Peterson, Quantum cohomology of \( G/P \), Lecture Course, MIT, Spring Term, 1997.

[Pha11] Frédéric Pham, Singularities of integrals, Universitext, Springer, London; EDP Sciences, Les Ulis, 2011, Homology, hyperfunctions and microlocal analysis, With a foreword by Jacques Bros, Translated from the 2005 French original, With supplementary references by Claude Sabbah. MR 2798679 (2012b:58031)

[PR13] C. Pech and K. Rietsch, A comparison of Landau-Ginzburg models for odd-dimensional quadrics, arXiv:1306.4016, 2013.

[Rie06] Konstanze Rietsch, A mirror construction for the totally nonnegative part of the Peterson variety, Nagoya Math. J. 183 (2006), 105–142. MR 2253887 (2007i:14055)

[Rie08] A mirror symmetric construction of \( qH^*_P(G/P) \), Adv. Math. 217 (2008), no. 6, 2401–2442.

[Sei08] Paul Seidel, Fukaya categories and Picard-Lefschetz theory, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2441780 (2009m:53143)

[Wit97] Edward Witten, Phases of \( N = 2 \) theories in two dimensions, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 143–211. MR 1416338