Solutions of Conformal Gravity with Dynamical Mass Generation in the Solar System

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The field equations of Mannheim’s theory of conformal gravity with dynamic mass generation are solved numerically in the interior and exterior regions of a model spherically symmetric sun with matched boundary conditions at the surface. The model consists of a generic fermion field inside the sun, and a scalar Higgs field in both the interior and exterior regions. From the conformal geodesic equations it is shown how an asymptotic gradient in the Higgs field causes an anomalous radial acceleration in qualitative agreement with that observed on the Pioneer 10/11, Galileo, and Ulysses spacecraft. At the same time the standard solar system tests of general relativity are preserved within the limits of observation.

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I. INTRODUCTION

It has become increasingly probable in recent times that general relativity is on its way to being falsified as a classical theory of gravitation. General relativity fails to account for the motions of galactic clusters (1, 2, 3) and rotations of individual galaxies (3, 8, see also [5] and references therein) without the ad hoc assumption of varying amounts and distributions of cold dark matter 4. At the same time the existence of cold dark matter in the universe is becoming more and more in doubt. As determined by the Boomerang Experiment, the small amplitude of the second peak relative to the first in the power spectrum of the cosmic microwave background does not support the cold dark matter hypothesis 5. Even within the solar system general relativity has met with a potential failure. Radio metric data from the Pioneer 10/11, Galileo, and Ulysses spacecraft indicate a constant (with respect to radius from the sun) radial acceleration directed toward the sun of magnitude ∼ \(8.5 \times 10^{-10}\) m s\(^{-2}\), in addition to the expected Newtonian-Einsteinian acceleration that varies as the inverse radius squared 6. Furthermore, Viking Lander ranging data limit any such anomalous radial accelerations on the planets Earth and Mars to less than one hundredth of that detected on the spacecraft, a violation of the equivalence principle. On a more fundamental plane, general relativity is not compatible with dynamical mass generation in that the latter leads to a manifestly traceless energy-momentum tensor as the source for Einstein’s field equation, while the Einstein tensor, the left side of the field equation, is not traceless.

Collectively these problems pose a challenge to general relativity as a classical theory of gravity, and it may be appropriate to seriously consider alternatives. Two such alternatives have appeared in the literature in recent years: (1) Milgrom’s modified Newtonian dynamics (MOND) 8, 9; and (2) the Weyl fourth-order, conformally invariant gravitational theory which has been re-examined recently by Mannheim and Kazanas 10 and extended by Mannheim to include dynamic mass generation 11. Both of these theories have been used to account for galactic rotation curves without dark matter 12, 13. However, only (2) is compatible with dynamical mass generation, in fact requires it, and therefore we believe (2) is the stronger candidate as an alternative to general relativity.

In this paper we investigate conformal gravity with dynamical mass generation in the solar system to see whether the theory can account for the anomalous, constant radial acceleration on the spacecraft while also passing the standard solar system tests, namely light bending at the limb of the sun and perihelion precession. As the field equations of conformal gravity are nonlinear, fourth-order differential equations, exact analytic solutions are not possible, even in a simple solar system model. We first consider approximate solutions and then confirm the general validity of the approximate results with complete numerical solutions of the interior and exterior problems with matched boundary conditions at the limb of the sun. We find that a completely conformal theory, including a scalar Higgs field for dynamical mass generation, does, in fact, predict an anomalous, nearly-constant radial acceleration toward the sun on a test particle such as a spacecraft, while at the same time satisfying the standard solar system tests. Furthermore, we give a qualitative argument why planets should behave differently than spacecraft and not exhibit the same anomalous radial acceleration.

II. CONFORMAL GRAVITY WITH DYNAMIC MASS GENERATION

Conformal gravity is based on the Weyl geometry action,

\[ I_w = -\alpha \int d^4x \sqrt{-g} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \]
where $C_{\mu\nu\lambda\rho}$ is the Weyl conformal tensor, $g \equiv \det (g_{\mu\nu})$, $R_{\mu\nu\lambda\rho}$ is the Riemann tensor, $R_{\mu\nu} \equiv R^{\lambda}_{\mu\lambda\nu}$ is the Ricci tensor, $R \equiv R^\mu_{\mu}$ is the Ricci scalar, and $\alpha$ is a dimensionless constant. The Weyl action is invariant under a conformal transformation, $g_{\mu\nu} (x) \rightarrow [\Omega (x)]^2 g_{\mu\nu} (x)$, which is an arbitrary, continuous deformation of the spacetime manifold. We use a metric signature $(-, +, +, +)$ and the sign conventions of Weinberg [1].

For dynamic mass generation, following Mannheim [11], we choose a conformally invariant matter action in a model consisting of a self-interacting scalar Higgs field $S$ and a generic fermion field $\psi$ with a Yukawa interaction with the scalar field as in the Standard Model of particle physics. The requirement of conformal invariance restricts such a matter action to the form

$$I_m = -\int d^4x \sqrt{-g} \left( \bar{\psi} \gamma^\mu (x) (\partial_\mu + \Gamma_\mu (x)) + ihS \right) \psi + \frac{1}{2} \partial_\mu S \partial^\mu S - \frac{1}{2} \left( \frac{R}{6} \right) S^2 + \lambda S^4 \right),$$  

where $\Gamma_\mu (x)$ is a spin connection, $h$ is a dimensionless Yukawa coupling, and the conformal term in $S$ with $R/6 < 0$ creates a case of spontaneous symmetry breaking and consequent dynamical mass generation.

Setting the variation of the total action, $I = I_w + I_m$, with respect to the metric equal to zero yields the Weyl field equations,

$$W_{\mu\nu} = \frac{1}{4\alpha} T_{\mu\nu},$$  

where the source,

$$T_{\mu\nu} \equiv \delta I_m / \delta g_{\mu\nu} = \bar{\psi} \gamma_\mu (x) \left[ \partial_\nu + \Gamma_\nu (x) \right] \psi + \frac{2}{3} S^\mu S^\nu - \frac{1}{2} g_{\mu\nu} S^\lambda \lambda - \frac{1}{3} S S_\mu^\nu + \frac{1}{3} g_{\mu\nu} S S_\lambda^{\lambda},$$  

is the energy-momentum tensor, which in this theory includes the scalar Higgs field, and the tensor on the left side,

$$W_{\mu\nu} = -\frac{1}{6} g_{\mu\nu} R^{\lambda\lambda} + \frac{2}{3} R_{\mu\nu} + R^{\lambda}_{\mu\lambda\nu} - 2 R^{\lambda\mu\lambda}_{\nu}, -\frac{1}{2} g_{\mu\nu} R^2 + \frac{2}{3} R R_{\mu\nu} - 2 R \Lambda_{\mu\nu} - 2 R_{\lambda\rho} R^{\lambda\rho} - \frac{1}{6} g_{\mu\nu} R^2$$  

was first obtained by Bach [15].

Setting the variation of the action with respect to the two fields to zero yields the two field equations of motion,

$$\left( i\gamma^\mu (x) \left[ \partial_\mu + \Gamma_\mu (x) \right] - hS \right) \psi = 0,$$  

(1)

$$S_{\mu\nu} + \frac{R}{6} S - 4\lambda S^3 = -h \bar{\psi} \psi.$$

Equation (1) is the Dirac equation in curved spacetime with a fermion mass due to spontaneous symmetry breaking given by

$$m = hS.$$

Equation (2) is the Klein-Gordon equation in curved spacetime with a self-interaction, $-4\lambda S^3$, and a fermion source, $-h \bar{\psi} \psi$. The scalar curvature, or more precisely $-R/6$, plays the role of the mass squared of the Higgs boson. In contrast to the Standard Model of particle physics, where the field point of the Higgs vacuum is a universal constant over all spacetime, in the present context the Higgs field is dynamically connected to the metric of spacetime through Eqs. (1), (2), and (3), and is not constant in general. In the present problem the Higgs field varies radially from the sun, and it is the radial gradient of the Higgs field that is responsible for the anomalous radial acceleration.

The traceless property of the energy-momentum tensor with dynamic mass generation is most easily established from Eq. (1) starting in a gauge where the scalar Higgs field takes on a constant, nonzero value $S_0$. In this case, using the field equations of motion, Eqs. (1) and (2), we see that $T_{\mu\nu} = 0$. Since $T_{\mu\nu}$ is a conformal tensor, under conformal transformation, $T_{\mu\nu} = g^{\mu\nu} T_{\mu\nu} \rightarrow \left[ \Omega^2 (x) \right]^{-1} g^{\mu\nu} \Omega^{-2} (x) T_{\mu\nu} = \Omega^{-4} (x) T_{\mu\nu}$. In general the scalar Higgs field transform as $S (x) \rightarrow S (x)/\Omega (x)$. Therefore, in a general gauge in which the Higgs field is not constant, $S (x) = S_0/\Omega (x)$, and the trace of the energy-momentum tensor is still zero since $T_{\mu\nu} = \Omega^4 (x) T_{\mu\nu} = 0$.

The Bach tensor, the left side of the field equations, Eq. (3), is also traceless, as can be seen from Eq. (2) which gives

$$W_{\mu\nu} = -\frac{2}{3} R^{\lambda\lambda} + \frac{2}{3} R_{\mu\nu} + R^{\lambda}_{\mu\lambda\nu} - 2 R^{\lambda\mu\lambda}_{\nu} + \frac{2}{3} R^2 - 2 R \Lambda_{\mu\nu} - 2 R_{\lambda\rho} R^{\lambda\rho} - \frac{2}{3} R^2$$  

$$= 0,$$

where we have used the covariant divergence of the contracted Bianchi identity, $(R^{\lambda\lambda} - g^{\lambda\lambda} R/2)_{;\lambda} = 0$, to establish the cancellation of the third and fourth terms. Thus dynamical mass generation is compatible with conformal gravity but not with general relativity since the Einstein tensor is not traceless in general: $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -R \neq 0$.

In general relativity the predictive power of the theory lies in the postulate that a test particle of mass $m$ follows a world line on the spacetime manifold that minimizes the action

$$I = -mc \int d\tau,$$  

(9)
where \( d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \) is the line element. With the mass \( m \) of the test particle constant, the path is a geodesic. However, the action of Eq. (3) is not suitable for a conformal theory because it is not invariant under conformal transformation. In fact, conformal gravity has been criticized by Perlick and Xu [7] as being non-predictive except for null geodesics because the action of Eq. (3) is arbitrarily variable under conformal transformation. What these authors have failed to recognize is that a conformal theory must be completely conformal. The action of Eq. (3) must be replaced by a conformally invariant action, and the way to make this change is apparent from Eqs. (3) and (4). The mass of the test particle in a conformal theory must no longer be a formally invariant action, and the way to make this change for a conformal theory because it is not invariant under conformal transformation. In fact, conformal gravity for a conformal theory because it is not invariant under a conformal transformation. However, the action of Eq. (9) is not suitable for a conformal theory constant, the path is a geodesic. However, the action of Eq. (9) is not suitable for a conformal theory because it is not invariant under conformal transformation. Thus we see that a conformal theory can pro-

In their original paper on conformal gravity Mannheim and Kazanas [10] obtained an analytic solution of the homogeneous equations, 
\[
W_{\mu\nu} = 0 ,
\]
for a static, spherically-symmetric metric corresponding to the line element,
\[
ds^2 = -b(r) c^2 dt^2 + \frac{1}{b(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .
\]
It is given by
\[
b(r) = 1 - \frac{\beta (2 - 3\beta \gamma)}{r} - 3\beta \gamma + \gamma r - kr^2 ,
\]
where \( \beta, \gamma, \) and \( \kappa \) are integration constants.

As Mannheim and Kazanas have shown, the above simple form of the metric suffices in a conformal theory. Starting with the general line element in terms of a radial coordinate \( \rho \),
\[
ds^2 = -B(\rho) c^2 dt^2 + A(\rho) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]
one makes a change of radial variable \( \rho = p(r) \) and rewrites the line element as
\[
ds^2 = \left[ \frac{p(r)}{r} \right]^2 \left[ -b(r) c^2 dt^2 + a(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] ,
\]
where \( b(r) \equiv r^2 B(p(r)) / |p(r)|^2 \) and \( a(r) \equiv r^2 A(p(r)) (dp/dr)^2 / |p(r)|^2 \). Then, if one chooses the function \( p(r) \) to satisfy
\[
\frac{1}{p(r)} = - \int \frac{dr}{r^2 \sqrt{A(p(r)) B(p(r))}} ,
\]
the line element becomes
\[
ds^2 = \left[ \frac{p(r)}{r} \right]^2 \left[ -b(r) c^2 dt^2 + \frac{1}{b(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] ,
\]
which is related to the simple line element of Eq. (13) by a conformal transformation with \( \Omega (r) = r/p(r) \).

Mannheim and Kazanas [10] interpret Eq. (14) as an exterior vacuum solution with a Schwarzschild-like metric in a background de Sitter spacetime, where the integration constant \( \gamma \) measures the departure from the Schwarzschild metric. They have argued that with \( \gamma \ll 1 \) the metric of Eq. (14) enjoys all of the successes of general relativity in the solar system, while on longer interstellar scales in a galaxy the linear term \( \gamma r \) can provide the increasing gravitational potential with radius.
required to supplement the decreasing Newtonian potential in order to account for the plateau characteristics of galactic rotation curves \[13\]. They have also argued that the integration constant \( \kappa \) is so small that the de Sitter term is negligible except on cosmological distance scales. Perlick and Xu \[17\] correctly have insisted that the values of the integration constants, \( \beta, \gamma, \) and \( \kappa \), cannot be chosen at will, but must be determined by matching boundary conditions with an interior solution at the inner boundary with the source and at infinity.

But there is an even more serious criticism of the vacuum solution of Mannheim and Kazanas, and that is it is not a vacuum solution at all. The vacuum in conformal gravity is not \( T_{\mu \nu} = 0 \) because even outside the source the Higgs field must be nonzero. Otherwise a massive test particle would not be possible. Even in a gauge where the Higgs field is constant, outside the source the final term in Eq. (4) contributes to a nonzero energy-momentum tensor. The next to last term also contributes for all metrics in which the Ricci tensor is nonzero, including the Schwarzschild metric satisfies \( R_{\mu \nu} = 0 \), and therefore it also satisfies \( W_{\mu \nu} = 0 \). So the Schwarzschild metric is a first approximation to the conformal metric with \( T_{\mu \nu} \approx 0 \). As we see from Eq. (4), the energy momentum tensor will be small outside the sun if the radial gradient of the Higgs field \( S \) and the self coupling constant \( \lambda \) are both small with respect to unity. We will see later from our numerical calculations that both of these assumptions are valid in the asymptotic region. Note that for the Schwarzschild metric the next to last term in \( T_{\mu \nu} \) is identically zero since the scalar curvature \( R = R^\mu_{\mu \rho} \mu = 0 \). But realistically the metric must be near to Schwarzschild, but not exactly Schwarzschild because the mass squared of the Higgs particle is \( R/6 \).

Assuming spherical symmetry such that \( S = S(r) \) and for a line element of the form of Eq. (3) with

\[
b(r) = 1 - \frac{2m}{r},
\]

where \( m \equiv GM/c^2 \) is the geometric radius of the sun, the conformal geodesic equation, Eq. (11), evaluated for \( \rho = 0, 1, 2, 3 \) is given respectively by \( (c = 1) \)

\[
\frac{d}{d\tau} bS \frac{dt}{d\tau} = 0,
\]

\[
d^2r \over d\tau^2 + \frac{1}{2} \frac{db}{dr} \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{2b} \frac{db}{dr} \left( \frac{dr}{d\tau} \right)^2 - br \left( \frac{d\theta}{d\tau} \right)^2 - b\sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{S} \frac{dS}{dr} \left[ b + \left( \frac{dr}{d\tau} \right)^2 \right] = 0,
\]

\[
d^2\theta \over d\tau^2 + \frac{2}{r} \frac{d\tau}{d\tau} - \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{S} \frac{dS}{dr} \frac{d\tau}{d\tau} = 0,
\]

\[
d^2\phi \over d\tau^2 + 2 \cot \theta \frac{d\tau}{d\tau} \frac{d\phi}{d\tau} + \frac{2}{r} \frac{d\tau}{d\tau} \frac{d\phi}{d\tau} - \frac{1}{S} \frac{dS}{dr} \frac{d\tau}{d\tau} = 0.
\]

For spherical symmetry, we lose no generality by restricting motion to the \( \theta = \pi/2 \) plane, in which case Eq. (23) is trivially satisfied. Equations (20) and (22) may be integrated once \( (l \) and \( k \) are constants of integration),

\[
\frac{dt}{d\tau} = \frac{l}{bS},
\]

\[
r^2 \frac{d\phi}{d\tau} = \frac{k}{S},
\]

and the result substituted into Eq. (21) giving

\[
\frac{d^2r}{d\tau^2} + \frac{1}{2b} \frac{db}{dr} \left[ \frac{l^2}{S^2} - \left( \frac{dr}{d\tau} \right)^2 \right] + \frac{1}{S} \frac{dS}{dr} \left[ b + \left( \frac{dr}{d\tau} \right)^2 \right] - \frac{k^2b}{r^3S^2} = 0.
\]
From the line element with \( d\theta = 0 \) and \( l/bS \) substituted for \( dr/d\tau \) according to Eq. (24), we write

\[
1 = \frac{l^2}{bS^2} - \frac{1}{b} \left( \frac{dr}{d\tau} \right)^2 - \frac{k^2}{r^2 S^2} .
\]  

(27)

Solving Eq. (27) for \( (dr/d\tau)^2 \) and substituting into Eq. (24) gives

\[
d^2r \over dr^2 + \frac{1}{2} \left( 1 + \frac{k^2}{r^2 S^2} \right) + \frac{1}{S^2} \frac{dS}{dr} \left( l^2 - k^2 b \right) - \frac{k^2}{r^2 S^2} = 0 ,
\]

(28)

and finally for radial motion \( (k = 0) \),

\[
d^2r \over dr^2 + \frac{l^2}{2} \frac{dS}{dr} = 0 .
\]

(29)

As the radiometric measurements of the accelerations of the spacecraft in the solar system are based on coordinate rather than proper time, we convert Eq. (29) to coordinate time using \( dt/d\tau = l/bS \) and restore \( c \) by letting \( t \to ct \),

\[
d^2r \over dt^2 - \left( \frac{1}{b} \frac{db}{dr} + \frac{1}{S} \frac{dS}{dr} \right) \left( \frac{dr}{dt} \right)^2 + \frac{b^2 c^2 S^2}{2l^2} \frac{db}{dr} + \frac{b^2 c^2}{S} \frac{dS}{dr} = 0 .
\]

(30)

It is the final term in Eq. (30) that has the potential of accounting for a constant radial acceleration for large radius. In this limit \( b = 1 - 2m/r \to 1 \), so we set

\[
c^2 \frac{d\ln S}{dr} = a(r) ,
\]

(31)

where \( a(r) \) is an anomalous acceleration. The next to final term in Eq. (30) accounts for the Newtonian acceleration.

\[
\frac{b^2 c^2 S^2}{2l^2} \frac{db}{dr} = \frac{1}{l^2} \left( 1 - \frac{2m}{r} \right)^2 S^2 \frac{GM}{r^2} \to \frac{S^2 \frac{GM}{l^2}}{r^2} .
\]

(32)

since for \( r = 40 \ \text{AU}, 2m/r = 2.0 \times 10^{-8} \ll 1 \). Turning our attention to the second term in Eq. (30), we see from Eqs. (13) and (21) that

\[
\left( \frac{1}{b} \frac{db}{dr} + \frac{1}{S} \frac{dS}{dr} \right) \left( \frac{dr}{dt} \right)^2 = \left[ \frac{2m}{r} - \frac{a(r)}{c^2} \right] \left( \frac{dr}{dt} \right)^2 \simeq \left( \frac{2GM}{r^2} + a(r) \frac{1}{c^2} \right) \left( \frac{dr}{dt} \right)^2 ,
\]

(33)

which is negligible for nonrelativistic speeds.

Based upon the above stated assumptions and approximations, the conformal geodesic equation, for nonrelativistic radial motion with a Schwarzschild metric, reduces asymptotically to

\[
d^2r \over dt^2 = - \frac{S^2 \frac{GM}{l^2}}{r^2} - a(r) .
\]

(34)

The requirement, based upon the radiometric data from the spacecraft, that \( a(r) = a_0 \) be asymptotically constant determines the asymptotic form that the scalar Higgs field must take through Eq. (31):

\[
d\ln S = \frac{a_0}{c^2} dr , \quad \ln S = \frac{a_0}{c^2} r + \ln S_0 , \quad S = S_0 \exp \left( \frac{a_0}{c^2} r \right) .
\]

(35)

For \( a_0 = 8.5 \times 10^{-10} \ \text{m s}^{-2} \) and \( r = 40 \ \text{AU}, a_0 r/c^2 = 5.6 \times 10^{-14} \ll 1 \) and the required asymptotic form is well approximated by

\[
S = S_0 \left( 1 + \frac{a_0}{c^2} r \right) .
\]

(36)

Then the Newtonian term in Eq. (34) becomes asymptotically

\[
- \frac{S^2 \frac{GM}{l^2}}{r^2} = - \left( \frac{a_0}{c^2} r \right)^2 \frac{GM}{r^2} \approx - \frac{GM}{r^2} .
\]

(37)

if we set the two integration constants equal, \( l = S_0 \), and neglect \( a_0 r/c^2 \) with respect to 1. Note from Eq. (24) that \( l \) sets the scale of proper time, and we are free to measure proper time in any units we wish. Finally, if the Higgs field takes on the asymptotic form of Eq. (34), then the conformal geodesic equation becomes in the asymptotic region

\[
d^2r \over dt^2 = - \frac{GM}{r^2} - a_0 .
\]

(38)

While Eq. (35) gives the required asymptotic form of the Higgs field in order for the conformal theory to account for the anomalous radial acceleration, it remains to be determined whether the conformal field equations have such an asymptotic solution. The answer to this question requires the full interior and exterior solutions with matched boundary conditions at the limb of the sun. But before turning our attention to a full solution of the field equations, it is instructive to consider an approximate analytic solution of Eq. (9), the equation of motion of the Higgs field, in the exterior region (\( \psi = 0 \)), assuming the metric is exactly Schwarzschild (\( R = 0 \)) and neglecting the self-interaction term (\( \lambda = 0 \)). In this case and with spherical symmetry Eq. (7) becomes

\[
S'' + \frac{S'}{r} = \frac{2m}{r} \left( \frac{2m}{r} + 2 \right) = 0 .
\]

(39)

and this differential equation has a solution,

\[
S = C_1 \ln \left( 1 - 2m \right) + C_2 .
\]

(40)

Although there are two constants of integration in the solution of the second-order differential equation, physically only their ratio is important, as can be seen from
the conformal geodesic equation, Eq. (41), in which for
the present static case with spherical symmetry the Higgs
dependent factor in Eq. (20) is
$$\frac{1}{S} \frac{dS}{dr} = \left( \frac{2m}{r^2} \right) \left( 1 - \frac{2m}{r} \right)^{-\frac{3}{2}} \ln \left( 1 - \frac{2m}{r} \right) \left[ 1 + \frac{C_2}{C_1} \right]^{-1}.$$

(41)

Thus we can choose either \(C_1\) or \(C_2\) arbitrarily and it is
convenient to evaluate \(C_2\) by setting \(S = S_0 = 1\) m\(^{-1}\) at
\(r = R\) at the limb of the sun. Then we have
$$C_2 = 1 - C_1 \ln \left( 1 - \frac{2m}{R} \right).$$

(42)

Ideally we should evaluate \(C_1\) by matching the slope,
$$\frac{dS}{dr} = \frac{2mC_1}{r^2 - 2mr},$$

(43)

with the slope of an interior solution at the limb of the sun. But since at the present stage of the exposition we
do not have an interior solution, we instead evaluate \(C_1\)
by the observational requirement of an anomalous radial
acceleration of \(a_0 = -8.5 \times 10^{-10}\) ms\(^{-2}\) at a radius of
\(r = R_{40} = 40\) AU, giving
$$C_1 = S_0 \left[ \ln \left( 1 - \frac{2m}{R_{40}} \right) - \ln \left( 1 - \frac{2m}{R} \right) \right] - \frac{2mc^2}{a_0 R_{40}^2} \left( \frac{1}{2} - \frac{2m}{R_{40}} \right)^{-1}.$$

(44)

Evaluating Eqs. (43) and (44), we obtain \(C_1 = 1.20 \times 10^{-4}\) m\(^{-1}\) and \(C_2 = 1 + 5.14 \times 10^{-10} \approx 1\) m\(^{-1}\).

With these constants, the Higgs field \(S\) is found to have a
value that is nearly constant for \(r > 0.1\) AU, with a slope
that is positive and very small with respect to unity (see Fig. 4, noting that only the first half AU is shown for clarity, which establishes the asymptotic trend). This
approximate solution of the scalar field equation of motion
shows that the features the scalar field can be expected to
have are intuitively reasonable. Although the slope is not constant as would be required by observation, the
inclusion of a nearly, but not exactly, Schwarzschild metric
and a small but nonzero self coupling is enough to reduce the variation with radius significantly, as we shall see.
The approximations used (zero Higgs mass and self
coupling), although quite sweeping, are not so extreme as to take the analytic solution of the reduced equation
far from the solutions of the full equation.

IV. SOLAR SYSTEM TESTS

Two further questions need to be considered, and these
are the effect of the scalar field on the deflection of light
and perihelion precession, the standard solar system tests
of any gravitational theory. These questions can be an-
swered by the orbit equation (\(u \equiv 1/r\) and \(h\) is the an-
gular momentum per unit mass),
$$\frac{d^2 u}{dt^2} = -\frac{db u^2}{2} - bu - \frac{1}{h^2} \left( \frac{S^2}{2} \frac{db}{du} + b S \frac{dS}{du} \right),$$

(45)

which is derived from the conformal geodesic equation,
Eq. (11), for the case of the conformal line element given by
Eq. (13). Regarding the deflection of light, we note that
the angular momentum per unit mass of a photon is
infinite, and therefore the final term in Eq. (45) in-
volving the scalar field is zero. This result is compatible
with the fact that the photon, being massless, does not
couple directly with the Higgs field. Thus there is no
direct effect on the deflection of light. Indirectly the pre-
diction of the conformal theory will differ from that of
general relativity by the degree that \(b (r)\) differs from the
Schwarzschild metric. We will see in the numerical solu-
tion that this difference is minimal. On the question of perihelion precession, while the final term in Eq. (45)
is nonzero we note that it does not contain the inverse
radius, \(u\). In the standard iterative solution of Eq. (45)
assuming small eccentricity, the zeroth order approxima-
tion, \(u = u_0 (1 + e \cos \phi)\), is substituted into the right-
hand side of the equation, resulting in a secular term,
\((3m^3/h^4) e \phi \sin \phi\), that grows with the angle traversed.
Again, the presence of the Higgs field does not contribute
directly to the secular term and consequently to perihel-
ion precession. Only through the deviation of \(b (r)\) from
Schwarzschild will the prediction of the conformal theory
differ from general relativity.

V. NUMERICAL SOLUTION

Finally we present a numerical solution of the Weyl
field equations and the scalar field equation of motion
that has been carried out over both the interior and ex-
terior domains, with boundary condition matching at the
limb of the source, a prototypical sun which is presumed
static and spherically symmetric. The exterior domain
of the numerical solution has been extended in fine reso-
nution to a radius of 60 AU, and in coarser resolution to
further radius. The simple solar model used to perform
these solutions was a polytrope of order three, and the
equations were solved for the metric coefficient \(b\) and the
scalar field \(S\), with the metric coefficient \(a\) removed by
coordinate and conformal transformations. The fermion
field was handled by following the averaging procedure
used in [13] and solving in terms of generic fermion num-
ber density and pressure. The main methods of solution
were Runge-Kutta integrations and Newton-Raphson re-
laxations.

Boundary conditions at the limb of the source were
provided by requiring compatibility with observation.
For the metric field, the Schwarzschild solution is known

[54x308]value that is nearly constant for
[54x308]r>
[54x319]With these constants, the Higgs field
[54x-1072]as to take the analytic solution of the reduced equation
coupling), although quite sweeping, are not so extreme
see. The approximations used (zero Higgs mass and self
reduce the variation with radius significantly, as we shall
[54x-844]inclusion of a nearly, but not exactly, Schwarzschild met-
[54x-731]ric and a small but nonzero self coupling is enough to
have are intuitively reasonable. Although the slope
shows that the features the scalar field can be expected
to have are intuitively reasonable. Although the slope
is not constant as would be required by observation, the
inclusion of a nearly, but not exactly, Schwarzschild metric
and a small but nonzero self coupling is enough to reduce the variation with radius significantly, as we shall see.
The approximations used (zero Higgs mass and self
coupling), although quite sweeping, are not so extreme as to take the analytic solution of the reduced equation
far from the solutions of the full equation.

IV. SOLAR SYSTEM TESTS

Two further questions need to be considered, and these
are the effect of the scalar field on the deflection of light
and perihelion precession, the standard solar system tests
of any gravitational theory. These questions can be an-
swered by the orbit equation (\(u \equiv 1/r\) and \(h\) is the an-
gular momentum per unit mass),
$$\frac{d^2 u}{dt^2} = -\frac{db u^2}{2} - bu - \frac{1}{h^2} \left( \frac{S^2}{2} \frac{db}{du} + b S \frac{dS}{du} \right),$$

(45)

which is derived from the conformal geodesic equation,
Eq. (11), for the case of the conformal line element given by
Eq. (13). Regarding the deflection of light, we note that
the angular momentum per unit mass of a photon is
infinite, and therefore the final term in Eq. (45) in-
volving the scalar field is zero. This result is compatible
with the fact that the photon, being massless, does not
couple directly with the Higgs field. Thus there is no
direct effect on the deflection of light. Indirectly the pre-
diction of the conformal theory will differ from that of
general relativity by the degree that \(b (r)\) differs from the
Schwarzschild metric. We will see in the numerical solu-
tion that this difference is minimal. On the question of perihelion precession, while the final term in Eq. (45)
is nonzero we note that it does not contain the inverse
radius, \(u\). In the standard iterative solution of Eq. (45)
assuming small eccentricity, the zeroth order approxima-
tion, \(u = u_0 (1 + e \cos \phi)\), is substituted into the right-
hand side of the equation, resulting in a secular term,
\((3m^3/h^4) e \phi \sin \phi\), that grows with the angle traversed.
Again, the presence of the Higgs field does not contribute
directly to the secular term and consequently to perihel-
ion precession. Only through the deviation of \(b (r)\) from
Schwarzschild will the prediction of the conformal theory
differ from general relativity.

V. NUMERICAL SOLUTION

Finally we present a numerical solution of the Weyl
field equations and the scalar field equation of motion
that has been carried out over both the interior and ex-
terior domains, with boundary condition matching at the
limb of the source, a prototypical sun which is presumed
static and spherically symmetric. The exterior domain
of the numerical solution has been extended in fine reso-
nution to a radius of 60 AU, and in coarser resolution to
further radius. The simple solar model used to perform
these solutions was a polytrope of order three, and the
equations were solved for the metric coefficient \(b\) and the
scalar field \(S\), with the metric coefficient \(a\) removed by
coordinate and conformal transformations. The fermion
field was handled by following the averaging procedure
used in [13] and solving in terms of generic fermion num-
ber density and pressure. The main methods of solution
were Runge-Kutta integrations and Newton-Raphson re-
laxations.

Boundary conditions at the limb of the source were
provided by requiring compatibility with observation.
For the metric field, the Schwarzschild solution is known
to be very accurate in the inner solar system for light deflection and precession observations, so the metric field was required to match well with the Schwarzschild solution at the limb of the sun. The scalar field has its value normalised to one at this point. The limb gradient of the scalar field can be limited in magnitude by compatibility with existing observations, or a more specific value can be produced by reference to the anomalous motion of the Pioneer spacecraft. If the observed acceleration of these spacecraft is accurate (and a transponder signal should be the most sensitive test we have of these effects to date), the scalar field gradient corresponding to the observed acceleration at 40 AU would be $S' \sim 10^{-24}$ m$^{-2}$. The exterior numerical solution gives a corresponding limb value of $S' \sim 10^{-19}$ m$^{-2}$.

With these boundary conditions in hand, a consistent interior/exterior numerical solution was produced (Figures 2, 3). The interior metric field merges well into an exterior solution of the Schwarzschild form. The scalar field increases across the interior domain before smoothly assuming a very slight gradient as required in the exterior region. Newton-Raphson iterations on the exterior solution near the source show that the metric field produced is in agreement with Schwarzschild to one part in $10^{45}$, even in the presence of a scalar field. From our arguments given above, we would conclude that theory passes the standard solar system tests to this accuracy.

The far-field ($30 – 60$ AU) behaviour of the fields is also satisfactory. The metric field continues to agree with Schwarzschild to good order, up to the accuracy limits of the numerical methods. The scalar field assumes a very slight and asymptotically decreasing gradient which varies very slowly across the exterior domain. The results are very similar to the restricted analytic solution described above, but with the inclusion of the metric field in the mass term resulting in a slower variation of the scalar field gradient. It is expected that with improving accuracy of the far-field methods that the scalar field will vary more slowly again.

The values that the constants of the theory assumed with these boundary conditions were $\alpha \sim 3.5 \times 10^{15}$ and $\lambda \leq 10^{-45}$. The value of $\alpha$ is set through the boundary conditions, and that of $\lambda$ by the stability of the solution at large radius, larger values producing runaway behaviour in the fields at large radius due to the $\lambda S^3$ term in the equations.

VI. PLANETARY MOTION

The form we have found for the scalar field shows a radially and rapidly decreasing gradient in the vicinity of the source. This feature could have relevance as to why anomalous accelerations are not observed for planetary bodies, at least to the accuracy of ranging experiments to date ([1]; see also comments in [18]). While it is not yet known how the scalar field sourced by more than one body would be configured, we expect that a secondary body, such as a planet, would superpose a similar characteristic well upon the background field produced by the primary. Such a superposition may well result in an average of the scalar field gradient over the volume of the planet that is less than the single-source field gradient. Thus the constituent particles of the planet could, on average, experience less of an anomalous radial acceleration towards the sun than they would if the rest of the planet were not present. In this respect a planet is unlike a spacecraft, which would not appreciably affect the scalar field in its own vicinity.

VII. CONCLUSION

In this paper we have investigated the theory of conformal gravity with dynamical mass generation, as originally presented by Mannheim [1]. The theory has been presented here in a logically consistent format that clarifies some areas in the construction of the theory that have been confused in the past. In particular it has been emphasized here that since conformal gravity not only is compatible with dynamical mass generation, but also requires it to be predictive, the scalar Higgs field is an integral part of the theory and cannot be ignored. From this framework, analytic and numerical results have been extracted. The import of these results is that it appears that conformal gravity with dynamical mass generation can reproduce the gravitational effects within the solar system, including the possible Pioneer spacecraft accelerations, while still satisfying the standard solar system tests. This successful outcome was made possible through the inclusion of the scalar field required for dynamical mass generation. Solutions of the full system of equations have not been obtained before.

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FIG. 1. Exterior Scalar Field
FIG. 2. Scalar Field - Boundary Matching
FIG. 3. Metric Coefficient - Boundary Matching