Some Applications of a Simple Stationary Line Element for the Schwarzschild Geometry

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ABSTRACT

Guided by a Hamiltonian treatment of spherically symmetric geometry, we are led to a remarkably simple – stationary, but not static – form for the line element of Schwarzschild (and Reissner-Nordstrom) geometry. The line element continues smoothly through the horizon; by exploiting this feature we are able to give a very simple and physically transparent derivation of the Hawking radiance. We construct the complete Penrose diagram by enforcing time-reversal symmetry. Finally we outline how an improved treatment of the radiance, including effects of self-gravitation, can be obtained.
Schwarzschild found his remarkable exact solution for the geometry outside a star in general relativity quite soon after Einstein derived the field equations. Further study of this geometry over the course of several decades revealed a series of surprises: the existence and physical relevance of pure vacuum “black hole” solutions; the incompleteness of the space-time covered by the original Schwarzschild coordinates, and the highly non-trivial global structure of its completion; and the dynamic nature of the physics in this geometry despite its static mathematical form, revealed perhaps most dramatically by the Hawking radiance [1]. Discussions of this material can now be found in advanced textbooks [2], but they are hardly limpid.

In the course of investigating an improvement to the standard calculation of this radiance to take into account its self-gravity, as we shall sketch below, we came upon a remarkably simple form for the line element of Schwarzschild (and Reissner-Nordstrom) geometry. This line element has an interesting history [3], but as far as we know it has never been discussed from a modern point of view. We have found that several of the more subtle features of the geometry become especially easy to see when this line element is used.

1. Hamiltonian Form, and Derivation

The general spherically symmetric line element can be written in the form

\[ ds^2 = -(N^t)^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \]  

In this expression it is to be understood that \( L, R, N^t \) and \( N^r \) are functions of \( r \) and \( t \). When one inserts this form into the Einstein-Hilbert action, one finds that no time derivatives of the lapse \( N^t \) or of the shift \( N^r \) occur, so that variation of these quantities generates constraints. These are [4,5]

\[ \mathcal{H}_t^G = L \pi_L^2 / 2R^2 - \pi_L \pi_R / R + \left( \frac{RR'}{L} \right)' - \left( \frac{R'^2}{2L} \right) - L/2 = 0 \]  

(1.2)
\[ \mathcal{H}_r^G = R' \pi_R - L \pi'_L = 0 \]  

(1.3)

where \( t \) represents \( \frac{d}{dt} \) and \( \cdot \) represents \( \frac{d}{dr} \), and of course \( \pi_L = (N^r R^r - RR^r)/N^t \), \( \pi_R = (N^r LR^r)/N^t - (LR^r)/N^t \) are the canonical conjugates of \( L, R \) respectively.

Furthermore the action is invariant under reparametrization, so that one should put extra restrictions on \( L \) and \( R \) (fix a gauge) in order to eliminate spurious degrees of freedom. The form of the solutions obtained, though not their physical content, will depend on one’s choice for these restrictions.

Our choice is simply \( L = 1, R = r \). With this choice the equations simplify drastically, and one easily solves to find \( \pi_L = \sqrt{2Mr} \), \( \pi_R = \sqrt{\frac{M}{2r}} \) and then \( N^t = \pm 1, N^r = \pm \sqrt{\frac{2M}{r}} \). Thus for the line element we have

\[ ds^2 = -dt^2 + (dr \pm \sqrt{\frac{2M}{r}} dt)^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(1.4)

\( M \), which appears as an integration constant, of course is to be interpreted as the mass of the black hole described by this line element.

For the Reissner-Nordstrom geometry, the same gauge choice leads to a metric of the same form, with the only change that \( 2M \rightarrow 2M - Q^2/r \).

These line elements are stationary – that is, invariant under translation of \( t \), but not static – that is, invariant under reversal of the sign of \( t \). Indeed reversal of this sign interchanges the \( \pm \) in (1.4), a feature we will interpret further below. Another peculiar feature is that each constant time slice \( dt = 0 \) is simply flat Euclidean space!
2. Global Structure

Now let us discuss the global properties of our coordinate system. Perhaps the clearest approach to such questions is via consideration of the properties of light rays. Taking for definiteness the upper sign in (1.4), and without any essential loss of generality restricting to the case $d\theta = d\phi = 0$ appropriate to the $\theta = \pi/2$ sections, we find that $ds^2 = 0$ when

$$\frac{dr}{dt} = -\sqrt{\frac{2M}{r}} \mp 1 .$$

(2.1)

For the class of light rays governed by the upper sign, we can cover the entire range $0 < r < \infty$ as $t$ varies. In particular one meets no obstruction, nor any special structure, at the horizon $r = 2M$. For the class of light rays governed by the lower sign there is structure at $r = 2M$. When $r > 2M$ one has a positive slope for $\frac{dr}{dt}$, and $r$ ranges over $2M < r < \infty$. When $r < 2M$ one has a negative slope for $\frac{dr}{dt}$, and $r$ ranges over $0 < r < 2M$. When $r = 2M$ it does not vary with $t$. From these properties, one infers that our light rays cover regions I and II in the Penrose diagram, as displayed in Figure 1. Let us emphasize that the properties of the Penrose diagram can be inferred from the properties of the light rays, although we will not belabor that point here.

If one chooses instead the lower sign in (1.4), and performs a similar analysis, one finds that regions I and II' are covered. Patching these together with the sectors found previously, one still does not have a complete space-time. However our line element is not yet exhausted. For in drawing Figure 1 we have implicitly assumed that $t$ increases along light rays which point up (“towards the future”). Logically, and to maintain symmetry, one should consider also the opposite case, that the coordinate $t$ increases towards the past. By doing this, one generates coordinate systems covering regions $I'$ and $II'$ respectively $I'$ and $II'$, for the upper and lower signs in (1.4). Thus the complete Penrose diagram is covered with patches each governed by a stationary – but not static – metric, and with non-trivial regions of overlap.
In the Reissner-Nordstrom case the generalization of (1.4) has a coordinate singularity at \( r = Q^2/2M \). However this singularity is inside both horizons, and does not pose a serious obstruction to a global analysis. One obtains the complete Penrose diagram also in this case by iterating constructions similar to those just sketched.

The usual Schwarzschild line element appears to be time reversal symmetric, but when the global structure of the space-time it defines is taken into account one sees that this appearance is misleading. The fully extended light-rays in Figure 1 go from empty space to a singularity as \( t \) advances (they pass from region I into region II), which is definitely distinguishable from the reverse process. There is a symmetry which relates these to the corresponding rays going from region II' to region I', however it involves not merely changing the sign of \( t \) in the Schwarzschild metric, but rather going to a completely disjoint region of the space-time. This actual symmetry of the space-time is if anything more obvious in our construction than in the standard one. Thus by taking the line-element in region I stationary rather than static we have lost some false symmetry while making the true symmetry – and its necessary connection with the existence of region I' (constructed, as we have seen, by simultaneously reversing the sign of \( t \) and interchanging the future with the past) – more obvious.

3. Boundary Conditions and Radiance

Taking with the upper sign in (1.4) and the normal time-orientation, we have a coordinate system that goes through the horizon smoothly and contains future infinity. We can use it to discuss the problem of defining boundary conditions on the quantum fields, such that a freely falling observer will see no singular behavior when passing through the horizon. This presumably corresponds to the physical situation for the geometry defined by collapse, since there is nothing singular or special in the local geometry at the horizon – and indeed, strictly speaking the position of the horizon actually depends on future events!
Thus we seek to construct a vacuum state which has a non-singular stress-energy as measured by freely falling observers. For concreteness, let us consider how we should construct the vacuum state for a massless scalar field $\phi$. By the equivalence principle, and standard quantum field theory in flat space, we should leave all the positive-frequency modes empty – where positive frequency is defined in a coordinate system that is locally flat. For the metric we are considering, it is convenient to work along a curve $dr + \sqrt{2M/r} \, dt = 0$; then the condition is simply positive frequency with respect to $t$ near this curve.

Now at spatial infinity (more accurately: conformal infinity $\mathcal{I}^+$) the vacuum state is defined locally by the requirement that modes having positive frequency with respect to the variable $u = t_s - r_*$ are unoccupied, where $t_s$ is Schwarzschild time and $r_* = r + 2M \ln(r - 2M)$ is the tortoise coordinate. We wish to find the relationship between this requirement and the preceding one.

The relationship between $t$ and $t_s$ is

$$t = t_s + 2\sqrt{2Mr} + 2M \ln \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}$$

so that

$$u = t_s - r_* = t - 2\sqrt{2Mr} - r - 4M \ln(\sqrt{r} - \sqrt{2M}),$$

and thus one finds that along a curve with $dr + \sqrt{2M/r} \, dt = 0$,

$$\frac{du}{dt} = 2 + \sqrt{\frac{2M}{r}} + \frac{2M}{r - \sqrt{2Mr}}.$$  \hspace{1cm} (3.3)

Because the last term on the right hand side is singular, the two definitions of positive frequency – with respect to $u$ or to $t$ – do not coincide. To remove the singularity, note that along any of the curves of interest $e^{-u/4M}$ has a simple zero at $r = 2M$, but is otherwise positive. Clearly then demanding positive frequency
with respect to $t$ along such curves requires positive frequency not with respect to $u$ but rather with respect to

\[ U = -e^{-u/4M}. \] (3.4)

In this way we have arrived at the famous Unruh boundary conditions [6]. From these boundary conditions, one readily derives the Hawking radianc.

4. Radiance: Dynamical Treatment

In the previous section, we have discussed the implementation of physical boundary conditions on external quantum fields, regarding the background geometry as fixed and given. For many reasons, one would like to go beyond this treatment, and to consider the geometry also as a quantum variable. In its full generality this problem seems out of reach at present. However there is a situation in which the standard treatment breaks down for a simple concrete physical reason, which plausibly can be repaired by a relatively modest expansion of the formalism.

We are referring to the problem of radiance of a near-extremal Reissner-Nordstrom hole [7]. As one approaches extremality ($Q \to M$), the formal temperature $T$ of the Hawking radiation approaches zero, in such a way that radiation of a single typical quantum of energy $T$ changes the temperature by an amount large compared to its value. In such circumstances, the approximation of treating the geometry as fixed is obviously inadequate, since it matters very much in the formulae whether $M$ is the mass before or after the radiation of a quantum. This ambiguity will be removed if one expands the formalism to include the effects of the gravitational self-interaction of the radiation in a consistent way.

With this motivation, and also simply to have a tractable but non-trivial problem in quantum geometry, we consider the following truncation of the full dynamical problem. We model the s-wave dynamics of an emitted particle by quantizing a spherically symmetric self-gravitating shell around the black hole. The classical action for this system is the sum of the action for the shell itself, and the action of
the gravitational field it produces. To quantize, we have found it most convenient to use a Hamiltonian path integral description, for which the action takes the form

$$S = \int dt \, p_r \dot{r} + \int dt \, dr \left[ \pi_L \dot{L} + \pi_R \dot{R} - N^t (\mathcal{H}_t^G + \mathcal{H}_t^M) - N^r (\mathcal{H}_r^G + \mathcal{H}_r^M) \right] - \int dt \, M$$

(4.1)

where

$$\mathcal{H}_t^M = \sqrt{\frac{p_r^2}{L^2} + m_o^2 \delta(r - \hat{r})},$$

$$\mathcal{H}_r^M = -p_r \delta(r - \hat{r}),$$

(4.2)

and $\mathcal{H}_t^G$, $\mathcal{H}_r^G$ are given by (1.2) and (1.3). In the above, $\hat{r}$ is the coordinate of the shell, $m_o$ its rest mass, and $M$ is the ADM mass of the system. Once again, varying with respect to $N^t$ and $N^r$ yields constraints, but now the constraints depend on the shell variables as well. To obtain an effective action depending only on the position and momentum of the shell, we can solve the constraints to find expressions for the metric variables in terms of the shell variables, and then insert these relations back into the action. Quantization of this action leads to a wave equation modified to include self-interaction, as will be detailed in a forthcoming publication.

The physical rationale for considering the above model is that for a large $(M \gg M_{PL})$ near extremal hole the only important fluctuations in the geometry come from the occasional emission of a quantum. It then seems reasonable to treat the collapsing matter that formed the hole classically. Furthermore, since the hole is cold the radiation is sporadic, so the approximation of including self-interaction while ignoring interparticle interactions would appear to be a good one. In this way we arrive at a picture reminiscent of the path integral treatment of pair production at strong coupling in a weak electric field [8], where an effective action is obtained by including the electric field of the produced pairs.

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In the Hamiltonian formalism spacetime is foliated by a series of spacelike surfaces parametrized by a time variable. For our purposes, it is crucial that the
surfaces pass smoothly through the horizon and are labelled by coordinates which are free of singularities. In principle the familiar Kruskal coordinates could be used, as they satisfy these requirements, but their rather complicated form leads to a cumbersome expression for the effective action. The advantage of the $L = 1, R = r$ gauge in this context is the simplification that is brought about by the vanishing of the terms $\pi_R \dot{L} + \pi_R \dot{R}$. In addition, even after quantization the coordinate $r$ retains a clear physical meaning, as it can be inferred by measuring the area of the constant $r, t$ 2-sphere. In this gauge, then, the model is simple enough, both conceptually and mathematically, to allow one to obtain concrete results, yet it plausibly contains the necessary ingredients to provide for a realistic treatment of the radiation from a near extremal hole.
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FIGURE CAPTIONS

1) Penrose diagram for the Schwarzschild geometry. As described in the text, $r$ and $t$ in one coordinate patch, for the upper sign in the line element, cover regions I and II. As is clear from the diagram, the ingoing light rays are captured in their entirety (into the singularity), whereas the outgoing light rays cannot be traced back past the horizon.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9406042v2