PARAMETER DEPENDENCE OF SOLUTIONS OF THE CAUCHY-RIEMANN EQUATION ON SPACES OF WEIGHTED SMOOTH FUNCTIONS

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Abstract. We study the inhomogeneous Cauchy-Riemann equation on spaces $\mathcal{E}V(\Omega, E)$ of weighted $C^\infty$-smooth $E$-valued functions on an open set $\Omega \subset \mathbb{R}^2$ whose growth on strips along the real axis is determined by a family of continuous weights $V$ where $E$ is a locally convex Hausdorff space over $\mathbb{C}$. We derive sufficient conditions on the weights $V$ such that the kernel $\ker \partial$ of the Cauchy-Riemann operator $\partial$ in $\mathcal{E}V(\Omega) \coloneqq \mathcal{E}V(\Omega, \mathbb{C})$ has the property $(\Omega)$ of Vogt. Then we use previous results and conditions on the surjectivity of the Cauchy-Riemann operator $\partial : \mathcal{E}V(\Omega) \to \mathcal{E}V(\Omega)$ and the splitting theory of Vogt for Fréchet spaces and of Bonet and Domański for (PLS)-spaces to deduce the surjectivity of the Cauchy-Riemann operator on the space $\mathcal{E}V(\Omega, E)$ if $E \coloneqq F_b'$ where $F$ is a Fréchet space satisfying the condition $(DN)$ or if $E$ is an ultrabornological (PLS)-space having the property $(P\Lambda)$. As a consequence, for every family of right-hand sides $(f_\lambda)_{\lambda \in U}$ in $\mathcal{E}V(\Omega)$ which depends smoothly, holomorphically or distributionally on a parameter $\lambda$ there is a family $(u_\lambda)_{\lambda \in U}$ in $\mathcal{E}V(\Omega)$ with the same kind of parameter dependence which solves the Cauchy-Riemann equation $\partial u_\lambda = f_\lambda$ for all $\lambda \in U$.

1. Introduction

Let $E$ be a linear space of functions on a set $U$ and $P(\partial) : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ be a linear partial differential operator with constant coefficients which acts continuously on a locally convex Hausdorff space of (generalized) differentiable scalar-valued functions $\mathcal{F}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$. We call the elements of $U$ parameters and say that a family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depends on a parameter w.r.t. $E$ if the map $\lambda \mapsto f_\lambda(x)$ is an element of $E$ for every $x \in \Omega$. The question of parameter dependence is whether for every family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depending on a parameter $\lambda$ there is a family $(u_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ with the same kind of parameter dependence which solves the partial differential equation

$$P(\partial) u_\lambda = f_\lambda, \quad \lambda \in U.$$

In particular, it is the question of $C^k$-smooth (holomorphic, distributional, etc.) parameter dependence if $E$ is the space $C^k(U)$ of $k$-times continuously partially differentiable functions on an open set $U \subset \mathbb{R}^d$ (the space $O(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}$, the space of distributions $\mathcal{D}(U)'$ on an open set $U \subset \mathbb{R}^d$, etc.).

The question of parameter dependence has been subject of extensive research varying in the choice of the spaces $E$, $\mathcal{F}(\Omega)$ and the properties of the partial differential operator $P(\partial)$, e.g. being (hypo)elliptic, parabolic or hyperbolic. Even partial differential differential operators $P_\lambda(\partial)$ where the coefficients also depend...
$C^b([0,1])$-smoothly $\mathbb{R}_x^2$ and $C^\infty$-smoothly $[31,52]$, holomorphically $[63,64,81]$ or differentiable resp. real analytic $[20]$ on the parameter $\lambda$ were considered. The case that the coefficients of the partial differential differential operator $P(x,\partial)$ are non-constant functions in $x \in \Omega$ was treated for $\mathcal{F}(\Omega) = \mathcal{S}'(\mathbb{R}^n)$, the space of real analytic functions on $\mathbb{R}^n$, as well $[3,4]$.

The answer to the question of $C^b$-smooth (holomorphic, distributional, etc.) parameter dependence is obviously affirmative if $P(\partial)$ has a linear continuous right inverse. The problem to determine those $P(\partial)$ which have such a right inverse was posed by Schwartz in the early 1950s (see $[31]$, p. 680). In the case that $\mathcal{F}(\Omega)$ is the space of $C^\infty$-smooth functions or distributions on an open set $\Omega \subset \mathbb{R}^n$ the problem was solved in $[63,66,67]$ and in the case of ultradifferentiable functions or ultradistributions in $[60]$ by means of Phragmén-Lindelöf type conditions. The case that $\mathcal{F}(\Omega)$ is a space of weighted $C^\infty$-smooth functions on $\Omega = \mathbb{R}^n$ or its dual was handled in $[48,51]$, even for some $P(x,\partial)$ with smooth coefficients, the case of tempered distributions in $[49]$ and of Fourier (ultra-)hyperfunctions in $[57,58]$. For Hörmander’s spaces $B^\infty_p(\Omega)$ as $\mathcal{F}(\Omega)$ the problem was studied in $[31]$. The same problem for differential systems on distributions was considered in $[28]$ and on ultradifferentiable functions or ultradistributions in $[34]$.

The conditions of Phragmén-Lindelöf type were analysed in $[11,12,62,68,70,71]$ for spaces of $C^\infty$-smooth functions or distributions, in $[10,72]$ for spaces of real analytic or ultradifferentiable functions of Roumieu type and in $[14,15,16]$ for ultradifferentiable functions or ultradistributions of Beurling type.

The necessary condition of surjectivity of the partial differential operator $P(\partial)$ was studied in many papers, e.g. in $[1,32,57,61,89]$ on $C^\infty$-smooth functions and distributions, in $[13,30,54,55,56]$ on real analytic functions, in $[9,21]$ on Gevrey classes, in $[17,19,52,53,73]$ on ultradifferentiable functions of Roumieu type, in $[31]$ on ultradistributions of Beurling type, in $[4,18]$ on ultradifferentiable functions and ultradistributions and in $[60]$ on the multiplier space $\mathcal{O}_M$.

However, if $P(\partial):C^\infty(\Omega) \rightarrow C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is elliptic, then $P(\partial)$ has a linear right inverse (by means of a Hame basis of $C^\infty(\Omega)$) and it has a continuous right inverse due to Michael’s selection theorem $[74]$, Theorem 3.2”, p. 367 and $[40]$, Satz 9.28, p. 217, but $P(\partial)$ has no linear continuous right inverse if $n \geq 2$ by a result of Grothendieck $[83]$, Theorem C.1, p. 109. Nevertheless, the question of parameter dependence w.r.t. $E$ has a positive answer for several locally convex Hausdorff spaces $E$ due to tensor product techniques. In this case the question of parameter dependence obviously has a positive answer if the topology of $E$ is stronger than the topology of pointwise convergence on $U$ and

$$P(\partial)^E:C^\infty(\Omega,E) \rightarrow C^\infty(\Omega,E)$$

is surjective where $C^\infty(\Omega,E)$ is the space of $C^\infty$-smooth $E$-valued functions on $\Omega$ and $P(\partial)^E$ the version of $P(\partial)$ for $E$-valued functions. If $E$ is complete, we have the topological isomorphy $C^\infty(\Omega,E) \cong C^\infty(\Omega)\varepsilon E$ where the latter space is Schwartz’ $\varepsilon$-product. By Grothendieck’s classical theory of tensor products $[33]$ the $\varepsilon$-product is topologically isomorphic to the completion of the projective tensor product $C^\infty(\Omega)\mathcal{B}_\varepsilon E$, implying $C^\infty(\Omega,E) \cong C^\infty(\Omega)\mathcal{B}_\varepsilon E$, since $C^\infty(\Omega)$ with its usual topology is a nuclear space. From this tensor product representation and the surjectivity of the elliptic operator $P(\partial)$ on the Fréchet space $C^\infty(\Omega)$ follows the surjectivity of $P(\partial)^E$ by $[40]$, Satz 10.24, p. 255 if $E$ is a Fréchet space. Hence the answer to the question of $C^b$-smooth or holomorphic parameter dependence is affirmative but the case of distributional parameter dependence is not covered as $\mathcal{D}(U)^*\varepsilon$ with the strong dual topology is not a Fréchet space. However, the surjectivity result for $P(\partial)^E$ can even be extended beyond the class of Fréchet spaces $E$.
due to the splitting theory of Vogt for Fréchet spaces \([56, 57]\) and of Bonet and Domaniş for (PLS)-spaces \([3, 7]\). Namely, we have that \(P(\partial)^E, n \geq 2\), is surjective if \(E := F'_\kappa\) where \(F\) is a Fréchet space satisfying the condition \((DN)\) by \([56\text{, Theorem}\ 2.6, \ p.\ 174]\) or if \(E\) is an ultrabornological (PLS)-space having the property \((PA)\) by \([22\text{, Corollary}\ 3.9, \ p.\ 1112]\) since \(\ker P(\partial)\) has the property \((\Omega)\) by \([56\text{, Proposition}\ 2.5 (b), \ p.\ 173]\). The latter result covers the case of distributional parameter dependence.

In general, Grothendieck’s classical theory of tensor products can be applied if \(P(\partial)\) is surjective and \(\mathcal{F}(\Omega)\) is a nuclear Fréchet space. If in addition \(\ker P(\partial)\) has the property \((\Omega)\), the splitting theory of Vogt for Fréchet spaces and of Bonet and Domański for (PLS)-spaces can be used. In the case that \(\mathcal{F}(\Omega)\) is not a Fréchet space the question of surjectivity of \(P(\partial)^E\) can still be handled. For (PLS)-spaces \(\mathcal{F}(\Omega)\), e.g. (ultra-)distributions, one can apply the splitting theory of Bonet and Domański for (PLS)-spaces, and for (PLH)-spaces \(\mathcal{F}(\Omega)\), e.g. \(\mathcal{D}_{\text{loc}}\) and \(\mathcal{B}^{\text{loc}}_{\alpha}(\Omega)\) which are non-(PLS)-spaces, the splitting theory of Dierolf and Sieg for (PLH)-spaces \([22, 23]\) is available. For applications we refer the reader to the already mentioned papers \([5, 7, 22, 23, 86, 87]\) as well as \([6, 25, 26]\) where \(\mathcal{F}(\Omega)\) is the space of ultradistributions of Beurling type or of ultradifferentiable functions of Roumieu type and \(E\), amongst others, the space of real analytic functions and to \([11]\) where \(\mathcal{F}(\Omega)\) is the space of \(C^\infty\)-smooth functions or distributions.

Notably, the preceding results imply that the inhomogeneous Cauchy-Riemann equation with a right-hand side \(f \in E(\Omega, E) := C^\infty(\Omega, E)\), where \(\Omega \subset \mathbb{R}^2\) is open and \(E\) a locally convex Hausdorff space over \(\mathbb{C}\) whose topology is induced by a system of seminorms \((p_\alpha)_{\alpha \in \mathfrak{A}}\), given by

\[
\bar{\partial}^E u := (1/2)(\bar{\partial}_1^E + i \partial_2^E)u = f
\]

has a solution \(u \in \mathcal{E}(\Omega, E)\) if \(E\) is a Fréchet space or \(E := F'_\kappa\) where \(F\) is a Fréchet space satisfying the condition \((DN)\) or if \(E\) is an ultrabornological (PLS)-space having the property \((PA)\). Among these spaces \(E\) are several spaces of distributions like \(\mathcal{D}(U)'\), the space of tempered distributions, the space of ultradistributions of Beurling type etc. In the present paper we study this problem under the constraint that the right-hand side \(f\) fulfills additional growth conditions given by an increasing family of positive continuous functions \(V := (\nu_\alpha)_{\alpha \in \mathfrak{A}}\) on an increasing sequence of open subsets \((\Omega_n)_{n \in \mathbb{N}}\) of \(\Omega\) with \(\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n\), namely,

\[
|f|_{n,m,\alpha} := \sup_{x \in \Omega_n} p_\alpha((\partial^\beta)^E f(x)) |\nu_\alpha(x)| < \infty
\]

for every \(n \in \mathbb{N}, m \in \mathbb{N}_0\) and \(\alpha \in \mathfrak{A}\). Let us call the space of such functions \(\mathcal{E}(\Omega, E)\). Our interest is in conditions on \(V\) and \((\Omega_n)_{n \in \mathbb{N}}\) such that there is a solution \(u \in \mathcal{E}(\Omega, E)\) of \([11]\), i.e. we search for conditions that guarantee the surjectivity of

\[
\bar{\partial}^E : \mathcal{E}(\Omega, E) \to \mathcal{E}(\Omega, E).
\]

From the previous considerations for the Cauchy-Riemann operator on the space of non-weighted \(C^\infty\)-smooth functions our task is evident and a part of it is already done. The spaces \(\mathcal{E}(\Omega) := \mathcal{E}(\Omega, \mathbb{C})\) are Fréchet spaces by \([11\text{, 3.4 Proposition, p.}\ 6]\), in \([11\text{, 3.1 Theorem, p.}\ 12]\) we derived conditions on the family of weights \(V\) and the sequence of sets \((\Omega_n)_{n \in \mathbb{N}}\) such that \(\mathcal{E}(\Omega)\) becomes a nuclear space and in \([11\text{, 4.8 Theorem, p.}\ 20]\) such that \(\bar{\partial}\) is surjective on \(\mathcal{E}(\Omega)\). Furthermore, we obtained the topological isomophism \(\mathcal{E}(\Omega, E) \cong \mathcal{E}(\Omega)(\varepsilon E)\) for complete \(E\) in \([11\text{, 5.10 Example}\ c, \ p.\ 24]\). Therefore we already have a solution in the case that \(E\) is Fréchet space at hand (see \([11\text{, 4.9 Corollary, p.}\ 21]\)). What remains to be done is to characterise conditions on the kernel \(\ker \bar{\partial}\) in \(\mathcal{E}(\Omega)\) to have the property
(Ω) which allow us to extend the surjectivity result beyond the class of Fréchet spaces E. Concerning the sequence (Ωn)n∈ℕ, we concentrate on the case that it is a sequence of strips along the real axis, i.e., Ωn := \{z ∈ ℂ | |\text{Im}(z)| < n\}. The case that this sequence has holes along the real axis is treated in [43].

Let us briefly outline the content of our paper. In Section 2 we summarise the necessary definitions and preliminaries which are needed in the subsequent sections. The kernel ker ∂ is a projective limit and in Section 3 we prove that it is weakly reduced under suitable assumptions on V and (Ωn)n∈ℕ (see Corollary 3.4). The weak reducibility is used in Section 4 to obtain property (Ω) for the kernel in the case that (Ωn)n∈ℕ is a sequence of strips along the real axis (see Theorem 4.3 Corollary 4.5). In our final Section 5 we use the preceding conditions on the weights V to deduce the surjectivity of the Cauchy-Riemann operator on EV(Ω, E) for E := F′b where F is a Fréchet space satisfying the condition (DN) or an ultrabornological (PLS)-space E having the property (PA) (see Theorem 5.4). In particular, we apply our results in the case that (Ωn)n∈ℕ is a sequence of strips along the real axis (see Corollary 5.6) and for example νn(z) := exp(a_n|\text{Re}(z)|γ) for some 0 < γ ≤ 1 and a_n > 0 (see Corollary 5.7).

2. Notation and Preliminaries

The notation and preliminaries are essentially the same as in [43, 46, Section 2]. We define the distance of two subsets M0, M1 ⊂ ℝ2 w.r.t. a norm ‖⋅‖ on ℝ2 via

\[ d_{11}(M_0, M_1) := \begin{cases} \inf_{x \in M_0, y \in M_1} |x - y| & , M_0, M_1 \neq \emptyset, \\ \infty & , M_0 = \emptyset \text{ or } M_1 = \emptyset. \end{cases} \]

Moreover, we denote by ‖⋅‖∞ the sup-norm, by ‖⋅‖ the Euclidean norm on ℝ2, by \( B_r(x) := \{w ∈ ℝ^2 | |w - x| < r\} \) the Euclidean ball around \( x ∈ ℝ^2 \) with radius \( r > 0 \) and identify ℝ2 and ℂ as (normed) vector spaces. We denote the complement of a subset M ⊂ ℝ2 by \( M^C := ℝ^2 \setminus M \), the closure of M by \( \overline{M} \) and the boundary of M by \( \partial M \). For a function \( f : M → ℂ \) and \( K ⊂ M \) we denote by \( f|_K \) the restriction of \( f \) to \( K \) and by

\[ \|f\|_K := \sup_{x ∈ K}|f(x)| \]

the sup-norm on \( K \). By \( L^1(Ω) \) we denote the space of (equivalence classes of) \( ℂ \)-valued Lebesgue integrable functions on a measurable set \( Ω ⊂ ℝ^2 \) and by \( L^q(Ω) \), \( q ∈ ℕ \), the space of functions \( f \) such that \( f^q ∈ L^1(Ω) \).

By \( E \) we always denote a non-trivial locally convex Hausdorff space over the field \( ℂ \) equipped with a directed fundamental system of seminorms \( (p_α)_{α ∈ Γ} \). If \( E = ℂ \), then we set \( (p_α)_{α ∈ Γ} := \{\|\cdot\|\} \). Further, we denote by \( L(F, E) \) the space of continuous linear maps from a locally convex Hausdorff space \( F \) to \( E \) and sometimes write \( (T, f) := T(f), f ∈ F, \) for \( T ∈ L(F, E) \). If \( E = ℂ \), we write \( F^′ := L(F, ℂ) \) for the dual space of \( F \). If \( F \) and \( E \) are (linearly topologically) isomorphic, we write \( F ∼ E \). We denote by \( L_1(F, E) \) the space \( L(F, E) \) equipped with the locally convex topology of uniform convergence on the finite subsets of \( F \) if \( t = σ \), on the precompact subsets of \( F \) if \( t = γ \), on the absolutely convex, compact subsets of \( F \) if \( t = κ \) and on the bounded subsets of \( F \) if \( t = b \).

The so-called \( ε \)-product of Schwartz is defined by

\[ F ∈ E := L_ε(F′_κ, E) \]

where \( L(F′_κ, E) \) is equipped with the topology of uniform convergence on equicontinuous subsets of \( F′_κ \). This definition of the \( ε \)-product coincides with the original one by Schwartz [75, Chap. 1, §1, Définition, p. 18].
We recall the following well-known definitions concerning continuous partial differentiability of vector-valued functions (c.f. [44, p. 4]). A function \( f: \Omega \rightarrow E \) on an open set \( \Omega \subset \mathbb{R}^2 \) to \( E \) is called continuously partially differentiable (\( f \in C^1 \)) if for the \( n \)-th unit vector \( e_n \in \mathbb{R}^2 \) the limit
\[
(\partial^n f)(x) := \lim_{h \to 0, h \neq 0} \frac{f(x + he_n) - f(x)}{h}
\]
even in \( E \) for every \( x \in \Omega \) and \((\partial^n f)(x)\) is continuous on \( \Omega \) \((\partial^n f) \in C^0\) for every \( n \in \{1, 2\} \). For \( k \in \mathbb{N} \) a function \( f \) is said to be \( k \)-times continuously partially differentiable (\( f \in C^k \)) if \( f \) is \( C^1 \) and all its first partial derivatives are \( C^{k-1} \). A function \( f \) is called infinitely continuously partially differentiable (\( f \in C^\infty \)) if \( f \in C^k \) for every \( k \in \mathbb{N} \). The linear space of all functions \( f: \Omega \rightarrow E \) which are \( C^\infty \) is denoted by \( C^\infty(\Omega, E) \). Let \( f \in C^\infty(\Omega, E) \). For \( \beta = (\beta_n) \in \mathbb{N}_0^2 \) we set \((\partial^{\beta_n}) f := f \) if \( \beta_n = 0 \), and
\[
(\partial^{\beta_n}) f := (\partial^{\beta_n}) \ldots (\partial^{\beta_n}) f
\]
if \( \beta_n \neq 0 \) as well as
\[
(\partial^{\beta}) f := (\partial^{\beta_1}) (\partial^{\beta_2}) f.
\]
Due to the vector-valued version of Schwarz’ theorem \((\partial^{\beta}) f \) is independent of the order of the partial derivatives on the right-hand side, we call \( |\beta| := \beta_1 + \beta_2 \) the order of differentiation and write \( \partial^{\beta} f := (\partial^{\beta}) f \).

A function \( f: \Omega \rightarrow E \) on an open set \( \Omega \subset \mathbb{C} \) to \( E \) is called holomorphic if the limit
\[
(\frac{\partial}{\partial z}) f(z_0) := \lim_{h \to 0, h \neq 0} \frac{f(z_0 + h) - f(z_0)}{h}
\]
even in \( E \) for every \( z_0 \in \Omega \) and the space of such functions is denoted by \( \mathcal{O}(\Omega, E) \). The exact definition of the spaces from the introduction is as follows.

2.1. Definition ([44, 3.1 Definition, p. 5]). Let \( \Omega \subset \mathbb{R}^2 \) be open and \((\Omega_n)_{n \in \mathbb{N}}\) a family of non-empty open sets such that \( \Omega_n \subset \Omega_{n+1} \) and \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \). Let \( \mathcal{V} := (\nu_n)_{n \in \mathbb{N}} \) be a countable family of positive continuous functions \( \nu_n: \Omega \rightarrow (0, \infty) \) such that \( \nu_n \leq \nu_{n+1} \) for all \( n \in \mathbb{N} \). We call \( \mathcal{V} \) a directed family of continuous weights on \( \Omega \) and set for \( n \in \mathbb{N} \)

a) \[
E\nu_n(\Omega_n, E) := \{ f \in C^\infty(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A}, m \in \mathbb{N}_0^2 : |f|_{n, m, \alpha} < \infty \}
\]
and
\[
E\mathcal{V}(\Omega, E) := \{ f \in C^\infty(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in E\nu_n(\Omega_n, E) \}
\]
where
\[
|f|_{n, m, \alpha} := \sup_{x \in \Omega_n} \nu_n(\alpha) |(\partial^{\beta}) f(x)|_{\mathbb{N}_0^2, |\beta| \leq m}
\]
b) \[
E\nu_{n, \mathcal{V}}(\Omega_n, E) := \{ f \in E\nu_n(\Omega_n, E) \mid f \in \ker \partial_e^E \}
\]
and
\[
E\mathcal{V}_{\mathcal{V}}(\Omega, E) := \{ f \in E\mathcal{V}(\Omega, E) \mid f \in \ker \partial_e^E \}.
\]
c) \[
O\nu_n(\Omega_n, E) := \{ f \in O(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A} : |f|_{n, \alpha} < \infty \}
\]
and
\[
O\mathcal{V}(\Omega, E) := \{ f \in O(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in O\nu_n(\Omega_n, E) \}.
\]
where
\[ |f|_{n, \alpha} := \sup_{x \in \Omega_n} (f(x)) \nu_n(x). \]

The subscript \( \alpha \) in the notation of the seminorms is omitted in the \( \mathbb{C} \)-valued case.

The letter \( E \) is omitted in the case \( E = \mathbb{C} \) as well, e.g., we write \( \mathcal{E}_n(\Omega_n) := \mathcal{E}_n(\Omega, \mathbb{C}) \land \mathcal{E}_n(\Omega) := \mathcal{E}(\Omega, \mathbb{C}). \)

The spaces \( \mathcal{F}_n(\Omega, E), \mathcal{F} = \mathcal{E}, \mathcal{O}, \) are projective limits, namely, we have
\[ \mathcal{F}_n(\Omega, E) \cong \lim_{\rightarrow/\leftarrow} \mathcal{F}_n(\Omega, E), \] where the spectral maps are given by the restrictions
\[ \pi_{k,n} : \mathcal{F}_n(\Omega_k, E) \rightarrow \mathcal{F}_n(\Omega_n, E), f \mapsto f|_{\Omega_n}, k \geq n. \]

3. Weak reducibility of \( \mathcal{O}_n(\Omega) \)

The goal of this section is to show that the projective limit \( \mathcal{O}_n(\Omega) \) is weakly reduced under suitable assumptions, i.e., for every \( n \in \mathbb{N} \) there is \( m \in \mathbb{N} \) such that \( \mathcal{O}_n(\Omega) \) is dense in \( \mathcal{O}_m(\Omega_m) \) w.r.t. the topology of \( \mathcal{O}_n(\Omega_n) \). First, we show that \( \mathcal{O}_n(\Omega) \land \mathcal{O}_n(\Omega) \) coincide topologically under mild assumptions on weights \( V \) and the sequence of sets \( \{\Omega_n\} \). Then we use a similar result for \( \mathcal{E}_n(\Omega) \) which was obtained in [46] to prove the weak reducibility of \( \mathcal{O}_n(\Omega) \). For corresponding results in the case that \( \Omega_n = \Omega \) for all \( n \in \mathbb{N} \) see [29] Theorem 3, p. 56], [50], 13 Lemma, p. 418] and [77], Theorem 1, p. 145.

3.1. Condition [46, 3.3 Condition, p. 7]. Let \( V := (\nu_n)_{n \in \mathbb{N}} \) be a directed family of continuous weights on an open set \( \Omega \subset \mathbb{R}^2 \) and \( (\Omega_n)_{n \in \mathbb{N}} \) a family of non-empty open sets such that \( \Omega_n \subset \Omega_{n+1} \land \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \). For every \( k \in \mathbb{N} \) let there be \( \rho_k \in \mathbb{R} \) such that \( 0 < \rho_k < d^{1/\infty}(\{x\}, \partial \Omega_{k+1}) \) for all \( x \in \Omega_k \) and let there be \( q \in \mathbb{N} \) such that for any \( n \in \mathbb{N} \) there is \( \psi_n \in L^q(\Omega_k), \psi_n > 0, \land J_i(n) \geq n \land C_i(n) > 0 \) such that for any \( x \in \Omega_k \):

\[
(\omega.1) \quad \sup_{|\xi| \leq \rho_k} \nu_n(x + \xi) \leq C_1(n) \inf_{|\xi| \leq \rho_k} \nu_n(x + \xi) \\
(\omega.2) \quad \nu_n(x) = C_2(n) \psi_n(x) \nu_{J_i(n)}(x).
\]

3.2. Example [44, 3.7 Example, p. 9]. Let \( \Omega \subset \mathbb{R}^2 \) be open and \( (\Omega_n)_{n \in \mathbb{N}} \) a family of non-empty open sets such that

(i) \( \Omega_n := \mathbb{R}^2 \) for every \( n \in \mathbb{N} \).

(ii) \( \Omega_n \subset \Omega_{n+1} \land d^{1/\infty}(\Omega_n, \partial \Omega_{n+1}) > 0 \) for every \( n \in \mathbb{N} \).

(iii) \( \Omega_n := \{x = (x_i) \in \Omega \mid i \in I : |x_i| < n + N \land d^{1/\infty}(\{x\}, \partial \Omega) > 1/(n + N)\} \)
where \( I \subset [1, 2] \land \partial \Omega \neq \emptyset \) and \( N \in \mathbb{N}_0 \) is big enough.

(iv) \( \Omega_n := \{x = (x_i) \in \Omega \mid i \in I : |x_i| < n\} \) where \( I \subset [1, 2] \land \Omega := \mathbb{R}^2 \).

(v) \( \Omega_n := K_n \) where \( K_n \subset K_{n+1}, K_n \neq \emptyset \), is a compact exhaustion of \( \Omega \).

Let \( (a_n)_{n \in \mathbb{N}} \) be strictly increasing such that \( a_n > 0 \) for all \( n \in \mathbb{N} \) or \( a_n < 0 \) for all \( n \in \mathbb{N} \). The family \( \mathcal{V} := (\nu_n)_{n \in \mathbb{N}} \) of positive continuous functions on \( \Omega \) given by
\[ \nu_n(x) := \Omega \rightarrow (0, \infty), \nu_n(x) := e^{a_n \rho(x)}, \]
with some function \( \rho : \Omega \rightarrow [0, \infty) \) fulfills \( \nu_n \leq \nu_{n+1} \) for all \( n \in \mathbb{N} \) and Condition 3.1 for every \( q \in \mathbb{N} \) with \( \psi_n(x) := (1 + |x|^2)^{-1}, x \in \mathbb{R}^2 \), for every \( n \in \mathbb{N} \)

a) there is some \( 0 < \gamma \leq 1 \) such that \( \rho(x) := |(x_1, x_2)|^{\gamma}/\gamma \land (x_1, x_2) \in \Omega \), where \( I_0 := [1, 2] \land I \) with \( \gamma \notin \{1, 2\} \) and \( (\Omega_n)_{n \in \mathbb{N}} \) from (iii) or (iv).

b) \( \lim_{n \rightarrow \infty} a_n = \infty \) or \( \lim_{n \rightarrow \infty} a_n = 0 \) and there is some \( m \in \mathbb{N}, m \leq 5 \) such that
\[ \rho(x) = |x|^m, x \in \Omega, \] with \( (\Omega_n)_{n \in \mathbb{N}} \) from (i) or (ii).

c) \( a_n = n/2 \) for all \( n \in \mathbb{N} \) and \( \mu(x) = \ln(1 + |x|^2), x \in \mathbb{R}^2 \), with \( (\Omega_n)_{n \in \mathbb{N}} \) from (i).

d) \( \mu(x) = 0, x \in \Omega, \) with \( (\Omega_n)_{n \in \mathbb{N}} \) from (v).
In this section we only need property \((\omega. 1)\).

3.3. **Proposition.** Let \(V := (\nu_n)_{n \in \mathbb{N}}\) be a directed family of continuous weights on an open set \(\Omega \subset \mathbb{R}^2\) and \((\Omega_n)_{n \in \mathbb{N}}\) a family of non-empty open sets such that \(\Omega_n \subset \Omega_{n+1}\) and \(\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n\). If \((\omega. 1)\) is fulfilled, then

a) for every \(n \in \mathbb{N}\) and \(m \in \mathbb{N}_0\) there is \(C > 0\) such that

\[
|f|_{n,m} \leq C |f|_{2J_1(n)}, \quad f \in \mathcal{O}_{\nu_{2J_1(n)}}(\Omega_{2J_1(n)}).
\]

b) \(\mathcal{E}_{V,\Omega}(\Omega) = \mathcal{O}(\Omega)\) as Fréchet spaces.

**Proof.** a) Let \(n \in \mathbb{N}\) and \(m \in \mathbb{N}_0\). First, we note that \(\Omega_{n+1} \subset \Omega_{2J_1(n)}\) and \(\partial^\beta f(x) = \frac{i^{|eta|} f^{(|\beta|)}(x)}{\beta!}, \quad x \in \Omega_{2J_1(n)},\) holds for all \(\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2\) and \(f \in \mathcal{O}_{\nu_{2J_1(n)}}(\Omega_{2J_1(n)})\) where \(f^{(|\beta|)}\) is the \(|\beta|\)th complex derivative of \(f\). Then we obtain via \((\omega. 1)\) and Cauchy's inequality

\[
|f|_{n,m} = \sup_{x \in \Omega_n} |\partial^\beta f(x)| \nu_n(x) \leq \sup_{x \in \Omega_n} \frac{\|\partial^\beta f(x)\|}{\rho_n} \max_{|\zeta - x| \leq \rho_n} |f(\zeta)| \nu_n(x)
\]

\[
\leq C_1 \sup_{x \in \Omega_n} \frac{\|\partial^\beta f(x)\|}{\rho_n} \max_{|\zeta - x| \leq \rho_n} |f(\zeta)| \nu_n(x) \leq C_1 \sup_{x \in \Omega_n} \frac{\|\partial^\beta f(x)\|}{\rho_n} |f|_{2J_1(n)}.
\]

b) The space \(\mathcal{E}_{V,\Omega}(\Omega)\) is a Fréchet space since it is a closed subspace of the Fréchet space \(\mathcal{E}(\Omega)\) by [44, 3.4 Proposition, p. 6]. From part a) and \(|f|_n = |f|_{n,0}\) for all \(n \in \mathbb{N}\) and \(f \in \mathcal{E}_{V,\Omega}(\Omega)\) follows the statement.

If in addition \((\omega. 2)\) is fulfilled, then the space \(\mathcal{E}_{V,\Omega}(\Omega)\) is nuclear and thus its subspace \(\mathcal{O}(\Omega)\) as well which we need in our last section. The following conditions guarantee a kind of weak reducibility of the projective limit \(\mathcal{E}_{V,\Omega}(\Omega)\).

3.4. **Condition (46, 4.2 Condition, p. 10).** Let \(V := (\nu_n)_{n \in \mathbb{N}}\) be a directed family of continuous weights on an open set \(\Omega \subset \mathbb{R}^2\) and \((\Omega_n)_{n \in \mathbb{N}}\) a family of non-empty open sets such that \(\Omega_n \subset \mathbb{R}^2\), \(\Omega_n \subset \Omega_{n+1}\) for all \(n \in \mathbb{N}\), \(d_{n,k} := \max_{n \in \mathbb{N}} \Omega_n \setminus \Omega_k > 0\) for all \(n, k \in \mathbb{N}, \ k > n\), and \(\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n\).

a) For every \(n \in \mathbb{N}\) let there be \(g_n \in \mathcal{O}(\mathbb{C})\) with \(g_n(0) = 1\) and \(I_f(n) > n\) such that

i) for every \(\varepsilon > 0\) there is a compact set \(K \subset \Omega_n\) with \(\nu_n(x) \leq \varepsilon \nu_{I_f(n)}(x)\) for all \(x \in \Omega_n \setminus K\).

ii) there is an open set \(X_{I_f(n)} \subset \mathbb{R}^2 \setminus \overline{I_f(n)}\) such that there are \(R_n, r_n \in \mathbb{R}\) with \(0 < 2R_n < d_{1}(X_{I_f(n)}, \Omega_{I_f(n)}) = d(X_{I_f(n)}, \Omega_{I_f(n)})\) and \(R_n < r_n < d_{X_{I_f(n)} - R_n}\) as well as \(A_2(n, z): X_{I_f(n)} + B_{R_n}(0) \to (0, \infty), A_2(n, z) X_{I_f(n)}\) locally bounded, satisfying

\[
\max\{|g_n(\zeta)| \nu_{I_f(n)}(\zeta) | \zeta \in \mathbb{R}^2, |\zeta - (z - x)| = r_n\} \leq A_2(x, n, n)
\]

for all \(z \in \Omega_{I_f(n)}\) and \(x \in X_{I_f(n)} + B_{R_n}(0)\).

iii) for every compact set \(K \subset \mathbb{R}^2\) there is \(A_3(n, K) > 0\) with

\[
\int_K \frac{|g_n(x - y)| \nu_n(x)}{|x - y|} dy \leq A_3(n, K), \quad x \in \Omega_n.
\]

b) Let a)i) be fulfilled. For every \(n \in \mathbb{N}\) let there be \(I_f(n) > n\) and \(A_4(n) > 0\) such that

\[
\int_{\Omega_{I_f(n)}} \frac{|g_{I_f(n)}(x - y)| \nu_k(y)}{|x - y| \nu_k(y)} dy \leq A_4(n), \quad x \in \Omega_k.
\]
for \((k,p) = (I_4(n), n)\) and \((k,p) = (I_{14}(n), I_{14}(n))\) where \(I_{14}(n) := I_1(I_4(n))\).

c. Let a)(i)-(ii) and b) be fulfilled. For every \(n \in \mathbb{N}\), every closed subset \(M \subset \overline{\Omega}_n\) and every component \(N\) of \(M^c\) we have

\[ N \cap \overline{\Omega}_n^C \neq \emptyset \Rightarrow N \cap X_{I_{14}(n)} \neq \emptyset \]

where \(I_{214}(n) := I_2(I_{14}(n))\).

We will see that \(\Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\} \) and \(\nu_n(z) := \exp(a_n|\text{Re}(z)|^\gamma)\) for some \(0 < \gamma \leq 1\) and \(a_n > 0\) or \(a_n \xrightarrow{\infty} \) fulfill the conditions above with \(g_n(z) := \exp(-z^2)\).

3.5. Theorem (\[12\], 4.3 Theorem, p. 10]). Let \(n \in \mathbb{N}\). If Condition 3.4 is fulfilled, then \(\pi_{I_{214}(n), \nu_{I_{214}(n)}}(\Omega_{I_{214}(n)})\) is dense in \(\pi_{I_{14}(n), \nu_{I_{14}(n)}}(\Omega_{I_{14}(n)})\) w.r.t. \(|\cdot|_n\) where \(J_{I_{14}(n)} := J_1(I_{14}(n))\) and \(\pi_n : \Omega_{\nu_{I_{14}(n)}} \to \Omega_{\nu_{I_{14}(n)}}, \pi_n(f) := f|_{\Omega_n}\).

Proof. We omit the restriction maps in our proof. Due to Proposition 3.3 a) the restrictions to \(\Omega_{I_{14}(n)}\) of functions from \(\Omega_{I_{214}(n)}\) are elements of \(\nu_{I_{14}(n)}(\Omega_{I_{214}(n)})\). Let \(\varepsilon > 0\) and \(f_0 \in \Omega_{I_{214}(n)}(\Omega_{I_{214}(n)})\). For every \(j \in \mathbb{N}\) there exists

\[(i) f_j \in \nu_{I_{214}(n+j-1)}(\Omega_{I_{214}(n+j-1)})\]

\[(ii) f_j|_{\Omega_{I_{14}(n+j)}} \in \nu_{I_{14}(n+j)}(\Omega_{I_{14}(n+j)}) \subset \nu_{I_{14}(n+j)}(\Omega_{I_{14}(n+j)})\]

such that

\[|f_j - f_{j-1}|_{n+j-1} < \varepsilon \]

by Theorem 3.5 and the condition \(I_{214}(k) \geq I_{14}(k + 1)\) for all \(k \in \mathbb{N}\). Therefore we obtain for every \(k \in \mathbb{N}\)

\[|f_k - f_0|_{n} = \sum_{j=1}^{k} |f_j - f_{j-1}|_{n} \leq \sum_{j=1}^{k} |f_j - f_{j-1}|_{n+j-1} \leq \sum_{j=1}^{k} |f_j - f_{j-1}|_{n+j-1} \leq \frac{\varepsilon}{2} (1 - \frac{1}{2k}) < \frac{\varepsilon}{2} \]

Now, let \(\varepsilon_0 > 0\) and \(l \in \mathbb{N}\). We choose \(l_0 \in \mathbb{N}, l_0 \geq l\), such that \(\frac{\varepsilon}{2l_0} < \varepsilon_0\). Similarly, we get for all \(p \geq k \geq l_0\)

\[|f_p - f_k|_{l} \leq |f_p - f_{k-l}|_{l} \leq \sum_{j=k-l+1}^{p} |f_j - f_{j-1}|_{l} \leq \sum_{j=k-l+1}^{p} \frac{\varepsilon}{2l_0} = \frac{\varepsilon}{2} \left( \frac{1}{2l_0} - \frac{1}{2p} \right) \]

\[< \frac{\varepsilon}{2^{l_0+1}} \leq \frac{\varepsilon}{2^{l_0+1}} < \varepsilon_0.\]

Hence, \((f_k)_{k \geq l_0}\) is a Cauchy sequence in the Banach space \(\nu_{I_{14}(n+m_0)}(\Omega_{I_{14}(n+m_0)})\) for every \(m_0 \in \mathbb{N}\) and thus has a limit \(F_{m_0} \in \nu_{I_{14}(n+m_0)}(\Omega_{I_{14}(n+m_0)})\). These limits
coincide on their common domain because for every \( n_1, n_2 \in \mathbb{N} \) with \( I_{14}(n + n_1) \leq I_{14}(n + n_2) \) and \( \varepsilon_1 > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( k \geq N \)
\[
|F_{n_1} - F_{n_2}|_{I_{14}(n + n_1)} \leq |F_{n_1} - f_k|_{I_{14}(n + n_1)} + |F_{n_2} - f_k|_{I_{14}(n + n_1)} \\
\leq |F_{n_1} - f_k|_{I_{14}(n + n_1)} + |F_{n_2} - f_k|_{I_{14}(n + n_2)} < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.
\]
We deduce that the glued limit function \( f \) given by \( f := F_{n_0} \) on \( \Omega_{I_{14}(n + n_0)} \) for all \( n_0 \in \mathbb{N}_0 \) is well-defined and we have \( f \in \bigcap_{n_0 \in \mathbb{N}_0} \mathcal{O} \mathcal{V}_{I_{14}(n + n_0)}(\Omega_{I_{14}(n + n_0)}) = \mathcal{O} \mathcal{V}(\Omega) \) since \( I_{14}(n + n_0) \geq n + n_0 \). By the definition of \( f \) there exists \( N \in \mathbb{N} \) such that for every \( k \geq N \)
\[
|f - f_0|_n \leq |f - f_k|_n + |f_k - f_0|_n \quad n \leq I_{14}(n + 0) \leq \frac{\varepsilon}{2} + |f_k - f_0|_n \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
which proves our statement. \( \square \)

4. \( \mathcal{O} \mathcal{V} \)-spaces on strips

Using Corollary 3.16 and a decomposition theorem of Langenbruch, we prove that the space \( \mathcal{O} \mathcal{V}(\Omega) \) where the \( \Omega_n \) are strips along the real axis satifies the property \( (\Omega) \) of Vogt for suitable weights \( \mathcal{V} \). Let us recall that a Fréchet space \( F \) with an increasing fundamental system of seminorms \((\|\cdot\|_k)_{k \in \mathbb{N}}\) satisfies \( (\Omega) \) if
\[
\forall \ p \in \mathbb{N} \exists \ q \in \mathbb{N} \forall \ k \in \mathbb{N} \exists \ n \in \mathbb{N}, \ C > 0 \forall \ r > 0 : U_q \subset C r^n U_k + \frac{1}{r} U_p \quad \text{(7)}
\]
where \( U_k := \{ x \in F \mid \|x\|_k \leq 1 \} \) (see [72], Chap. 29, Definition, p. 367). The weights we want to consider are generated by a function \( \mu \) with the following properties.

4.1. Definition (strong) weight generator. A continuous function \( \mu : \mathbb{C} \to [0, \infty) \)
is called a weight generator if \( \mu(z) = \mu(\text{sign}(\text{Re}(z))) \) for all \( z \in \mathbb{C} \), the restriction \( \mu_{|[0,\infty)} \)
is strictly increasing,
\[
\lim_{x \to \infty} \frac{\ln(1 + |x|)}{\mu(x)} = 0
\]
and
\[
\exists \Gamma > 1, C > 0 \forall \ x \in [0, \infty) : \mu(x + 1) \leq \Gamma \mu(x) + C.
\]
If \( \mu \) is a weight generator which fulfills the stronger condition
\[
\exists \Gamma > 1 \forall \ n \in \mathbb{N} \exists \ C > 0 \forall \ x \in [0, \infty) : \mu(x + n) \leq \Gamma \mu(x) + C,
\]
then \( \mu \) is called a strong weight generator.

Weight generators are introduced in [59], Definition 2.1, p. 225) and strong weight generators in [59], Definition 2.2.2, p. 43 where they are simply called weight functions resp. strong weight functions. For a weight generator \( \mu \) we define the space
\[
H_\varepsilon(S_t) := \{ f \in \mathcal{O}(S_t) \mid \|f\|_{\tau, t} := \sup_{z \in S_t} |f(z)| e^{\tau \varepsilon} < \infty \}
\]
for \( t > 0 \) and \( \tau \in \mathbb{R} \) with the strip \( S_t := \{ z \in \mathbb{C} \mid |\text{Im}(z)| < t \} \).

4.2. Theorem ([59], Theorem 2.2, p. 225]). Let \( \mu \) be a weight generator. There are \( \overline{\tau}, K_1, K_2 > 0 \) such that for any \( \tau_0, \tau < \tau_2 \) there is \( C_0 = C_0(\text{sign}(\tau)) \) such that for any \( 0 < \tau_0 < \tau_2 < \overline{\tau} \) with
\[
t_0 \leq \min \left[ K_1, K_2 \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} \right]
\]
\(^1\) A superfluous constant depending on \( \text{sign}(\tau_0) \) is omitted.
there is $C_1 \geq 1$ such that for any $r \geq 0$ and any $f \in H_r(S_1)$ with $\|f\|_{r,2} \leq 1$ the following holds: there are $f_2 \in \mathcal{O}(S_{t_2})$ and $f_0 \in \mathcal{O}(S_{t_0})$ such that $f = f_0 + f_2$ on $S_{t_0}$ and

$$\|f_0\|_{C_0\tau_0, t_0} \leq C_1 e^{-Gr}$$

and

$$\|f_2\|_{\tau_2, t_2} \leq e^{r}$$

where

$$G := K_1 \min \left[ 1, \frac{t - t_0}{2t}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} \right].$$

To apply this theorem, we have to know the constants involved. In the following the notation of $[59]$ is used and it is referred to the corresponding positions resp. conditions for these constants. We have

$$T := \frac{1}{4 \ln(\Gamma)}$$

by $[59]$, Lemma 2.4, (2.15), p. 228 with $\Gamma$ from Definition 4.1 such that $\Gamma \geq e^{1/4}$. The choice $\Gamma \geq e^{1/4}$ comes from wanting $\hat{t} \leq 1$ in $[59]$ Lemma 2.4, p. 228. By $[59]$, Corollary 2.6, p. 230-231] we have

$$C_0 := \begin{cases} 4\Gamma B_3 = \frac{64 \cosh(1)}{\cos(1/2)} \Gamma^2 > 1 & , \tau < 0, \\ 4t B_2 = \frac{1}{\sinh(1)} \frac{\cosh(1)}{1/2} \Gamma^2 < 1 & , \tau \geq 0, \end{cases}$$

where $B_3 := \frac{16 \cosh(1/2) \Gamma}{\cos(1/2)}$ by $[59]$, Lemma 2.4, p. 228-229]. To get the constants $K_1$ and $K_2$, we have to analyze the conditions for $t_0$ in the proof of $[59]$ Theorem 2.2, p. 225]. By the assumptions on $\tau_0$, $\tau$ and $\tau_2$ and the choice of $C_0$ we obtain

$$\tau_2 - C_0\tau_0 > \tau_2 - C_0\tau \geq \tau_2 - \tau > 0 \quad \text{(8)}$$

and

$$\tau - C_0\tau_0 > \tau - C_0\tau = \tau(1 - C_0) > 0. \quad \text{(9)}$$

By choosing $D > 0$ in the proof of $[59]$ Theorem 2.2, (2.22), p. 232-233] as $D := \min(\frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau_0)2t_0})$, the estimate

$$D = \frac{\tau - C_0\tau_0}{(\tau_2 - C_0\tau_0)2t_0} \geq \min \left( \frac{1}{2t_0}, \frac{1}{2t}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} \right) \geq \min \left( \frac{1}{2t_0}, \frac{1}{2t}, \frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} \right) \geq \frac{1}{2t_0}$$

holds where $\Gamma_0 = \max(\Gamma, \bar{\Gamma})$ with $\bar{\ Gamma}, \bar{\Gamma} > 1$ from the proof. With $\theta \geq \frac{2\pi}{\ln(\Gamma)}$ (p. 232) we get on p. 233, below (2.24), due to the condition $t_0 \leq T_0 := \min(\frac{\theta}{2}, \frac{3t_0}{3a^2 B_1 t_0})$,

$$\min \left( \frac{\theta}{2}, D, 1 \right) \geq \min \left( \frac{1}{2}, \frac{1}{2t_0}, \frac{1}{2t_0} \right) \geq \frac{1}{2t_0} \left( \tau - C_0\tau_0 \right) \left( \tau_2 - C_0\tau_0 \right) =: G \quad \text{(10)}$$

where $a = \ln(\Gamma)$ (in the middle of p. 231) and $B_1 := 2\cosh(1)$ by the proof of $[59]$ Lemma 2.3, p. 226-227]. The assumptions $2t_0 < t$ and $t_0 < K_1$ in Theorem
Theorem. 4.3. By the choice of \(a\) for all \(n\)\(\), we need a linear transformation between strips to get the desired decomposition on the desired strip, desired in the spirit of Corollary 3.6. We choose use the theorem above, we need a linear transformation between strips to get the desired decomposition on the desired strip, desired in the spirit of Corollary 3.6.

Proof. Let \(p \in \mathbb{N}\). As \((a_n)_{n \in \mathbb{N}}\) is strictly increasing and \(\lim_{n \to \infty} a_n = 0\) or \(\lim_{n \to \infty} a_n = \infty\), we may choose \(q \in \mathbb{N}\) such that \(a_{2J_1I_{14}(p)}/C_0 < a_q\) and \(4J_1I_{14}(p) < q\). To use the theorem above, we need a linear transformation between strips to get the decomposition on the desired strip, desired in the spirit of Corollary 3.6. We choose \(\Gamma \geq e^{1/3}\) and \(T \in \mathbb{R}\) such that
\[
0 < T < \frac{1}{4\max(q + 1, 2J_1I_{14}(k))\ln(\Gamma)}
\]
which also fulfills
\[
T \leq \frac{1}{2J_1I_{14}(p)} \min\left(\frac{1}{2\Gamma_0}, \frac{1}{2\cosh(1)\ln(\Gamma)}, \frac{1}{2\sqrt{\cosh(1)\Gamma_0 \ln(\Gamma)}} \right).
\]
(10)

Let
\[
\tau_0 := \frac{a_{2J_1I_{14}(p)}}{C_0}, \quad \tau := a_q, \quad \tau_2 := \max(a_{q+1}, a_{2J_1I_{14}(k)}) \quad t_0 := 2J_1I_{14}(p)T, \quad t := qT, \quad t_2 := \max(q + 1, 2J_1I_{14}(k))T.
\]
By the choice of \(q\) we have
\[
\tau_0 = \frac{a_{2J_1I_{14}(p)}}{C_0} < \tau < \max(a_{q+1}, a_{2J_1I_{14}(k)}) = \tau_2.
\]
By the choice of \(q\) and (10) we get
\[
0 < 2t_0 = 4J_1I_{14}(p)T < qT = t < \max(q + 1, 2J_1I_{14}(k))T = t_2 < \frac{1}{4\ln(\Gamma)} = \tilde{t}.
\]
Further, we deduce from (11) that
\[
t_0 = 2J_1I_{14}(p)T \leq \min[\frac{\tau}{\tau - C_0\gamma_0}, \frac{\tau}{\tau - C_0\gamma_0}].
\]
Let \(r \geq 0\) and \(f \in \mathcal{O}\mathcal{V}(\mathbb{C})\) such that \(|f|_q = \|f\|_{a_q} \leq 1\). We set \(\bar{f}: S_qT \to \mathbb{C}, \bar{f}(z) := f(z/T)\), and define
\[
H_T(S_q) := \{g \in \mathcal{O}(S_q) \mid \|g\|_{x,T} := \sup_{z \in S_q} |g(z)|e^{r\bar{f}(z)} < \infty\}
\]
where \( \tilde{\mu} := \mu(fT) \). We note that for \( \tilde{\mu} := [1/T] \), where \( [\cdot] \) is the ceiling function, there is \( C > 0 \) such that for all \( x \geq 0 \)

\[
\tilde{\mu}(x + 1) = \mu\left(\frac{x + 1}{T}\right) \leq \mu\left(\frac{x}{T} + \frac{1}{T}\right) = \mu\left(\frac{x}{T}\right) + C = \Gamma\tilde{\mu}(x) + C,
\]

because \( \mu \) is a strong weight generator. We conclude that \( \tilde{\mu} \) is also a weight generator with the same \( \Gamma \) as \( \mu \) which is independent of \( T \). Moreover, from

\[
\|\hat{f}\|_{s,t} = \sup_{z \in S_{r,s}} |\hat{f}(z)| e^{\alpha_{s,t}(z)} = \sup_{z \in S_{r,s}} |\hat{f}(z)| e^{\alpha_{s,t}(z)} = |f|_{q} \leq 1
\]

follows by Theorem 4.2 that there are \( \hat{f}_j \in O(S_{t_j}), j \in \{0, 2\} \), such that

\[
\hat{f}(z) = \hat{f}_{0}(z) + \hat{f}_{2}(z), \quad z \in S_{r_{0}}\tag{12}
\]

and

\[
C_{1} e^{-Gr} \geq \|\hat{f}_{0}\|_{C_{0}, r_{0}} = \sup_{z \in S_{r_{0}}} |\hat{f}_{0}(z)| e^{C_{0} n \tilde{\mu}(z)} = \sup_{z \in S_{r_{0}}/T} |\hat{f}_{0}(z)| e^{C_{0} n \tilde{\mu}(Tz)} = \sup_{z \in S_{r_{0}}/T} |\hat{f}_{0}(z)| e^{C_{0} n \tilde{\mu}(Tz)}\tag{13}
\]

where \( f_0 \in O(S_{2J_{0}T}) \), as well as

\[
|f_{2}(z)| e^{\tau_{2} n \tilde{\mu}(z)} = \sup_{z \in S_{r_{2}}} |f_{2}(z)| e^{\tau_{2} n \tilde{\mu}(z)} = \sup_{z \in S_{r_{2}}/T} |f_{2}(z)| e^{\tau_{2} n \tilde{\mu}(Tz)} \geq \sup_{z \in S_{r_{2}}/T} |f_{2}(z)| e^{\alpha_{2J_{1}T_{t}T}(z)} = |f_{2}|_{2J_{1}T_{t}(k)}\tag{14}
\]

where \( f_{2} \in O(S_{r_{2}}/T) \) and the inclusion is justified by the identity theorem. Furthermore, for \( z \in S_{r_{0}}/T = S_{2J_{1}T_{t}(p)} \) the equation

\[
f(z) = \hat{f}(Tz) = \hat{f}_{0}(Tz) + \hat{f}_{2}(Tz) = f_{0}(z) + f_{2}(z)\tag{15}
\]

holds, thus \( f = f_{0} + f_{2} \) on \( S_{2J_{1}T_{t}(p)} \). By virtue of Corollary 3.6 the following is valid:

\[
\forall \varepsilon > 0 \exists \hat{f}_0, \hat{f}_2 \in \mathcal{O}(C) : (i) \ |\hat{f}_0 - f_0|_p < \varepsilon \quad \text{and} \quad (ii) \ |\hat{f}_2 - f_2|_k < \varepsilon.\tag{15}
\]

Now, we have to consider two cases. Let \( \varepsilon := C_{1} e^{-Gr} \). For \( k \leq p \) we get via (15) (i) \( f = \hat{f}_0 + (f_2 + \hat{f}_0) \) on \( S_{2J_{1}T_{t}(p)} \), so

\[
f_2 + f_0 - \hat{f}_0 = f - \hat{f}_0 = \hat{T}_2 \quad \text{on} \quad S_{2J_{1}T_{t}(p)}\tag{16}
\]

where the function \( \hat{T}_2 \in \mathcal{O}(C) \) and thus is a holomorphic extension of the left-hand side on \( C \). Hence we clearly have \( f = \hat{f}_0 + \hat{T}_2 \) and

\[
|\hat{f}_0|_p \leq |\hat{f}_0 - f_0|_p + |f_0|_p \leq \varepsilon + |f_0|_p \leq \varepsilon + |f_0|_{2J_{1}T_{t}(p)} \leq 2C_1 e^{-Gr} = C_2 e^{-Gr}\tag{17}
\]

as well as

\[
|\hat{T}_2|_k \leq |\hat{T}_2 - f_2|_k + |f_2|_k \leq |\hat{f}_0 - f_0|_p + |f_2|_{2J_{1}T_{t}(k)} \leq \varepsilon + |f_2|_{2J_{1}T_{t}(k)} \leq C_1 e^{-Gr} + \varepsilon \leq C_1 e^{-Gr} + e^{-Gr} \leq C_3 e^{r}.\tag{18}
\]

Analogously, for \( k > p \) we obtain via (15) (ii)

\[
f = \hat{f}_2 + (f_0 + f_2 - \hat{f}_0) \quad \text{on} \quad S_{2J_{1}T_{t}(p)}
\]

so

\[
f_0 + f_2 - \hat{f}_2 = f - \hat{f}_2 = \hat{T}_0 \quad \text{on} \quad S_{2J_{1}T_{t}(p)}\tag{19}
\]
where the function \( \mathcal{F}_0 \in \mathcal{O}\mathcal{V}(\mathbb{C}) \) and thus is a holomorphic extension of the left-hand side on \( \mathbb{C} \). Hence we clearly have \( f = \mathcal{F}_0 + \mathcal{F}_2 \) and

\[
\left| \mathcal{F}_0 \right| = |f - \mathcal{F}_2| = |f_0 + f_2 - \mathcal{F}_2| \leq |f_2 - \mathcal{F}_2| + |f_0| \leq |f_2 - \mathcal{F}_2| + |f_0|_p \leq \varepsilon + |f_0|_{2,t_1,t_4(p)} \leq 2C_1 e^{-Gr} = C_2 e^{-Gr}
\]

for all \( k \geq p \) and

\[
\left| \mathcal{F}_2 \right| = |\mathcal{F}_2| \leq \varepsilon + |f_2|_{2,t_1,t_4(k)} \leq C_1 e^{-Gr} + \varepsilon \leq C_3 e^{-\bar{r}}.
\]

Next, we set \( \bar{r} = [1/G] \) and \( C := C_3 e^{ln(C_2)/G} \). Let \( \bar{r} \geq 1 \). For \( \bar{r} \geq 1 \) there is \( r \geq 0 \) such that

\[
\bar{r} = e^{Gr - ln(C_2)} = \frac{e^{Gr}}{C_2}
\]

and we have by (17) and (18) for \( k \leq p \)

\[
|\mathcal{F}_0| \leq C_2 e^{-Gr} = \frac{1}{\bar{r}}, \quad |\mathcal{F}_2| \leq C_3 e^{-r} = C_3 e^{\frac{1}{\bar{r}} (Gr - ln(C_2))} = C e^{\frac{1}{\bar{r}} p} \leq C p^n,
\]

as well as by (20) and (21) for \( k > p \)

\[
|\mathcal{F}_0| \leq \frac{1}{\bar{r}}, \quad |\mathcal{F}_2| \leq C\bar{r}^n.
\]

For \( 0 < \bar{r} < 1 \) we have, since \( q \geq p \),

\[
|f|_p \leq |f|_q \leq 1 < \frac{1}{\bar{r}}.
\]

Thus our statement is proved. \( \square \)

Let us remark that the choice of the sequence \( (a_n)_{n \in \mathbb{N}} \) in the preceding theorem does not really matter.

4.4. Remark. Let \( \mu : \mathbb{C} \to [0, \infty) \) be continuous, \( (a_n)_{n \in \mathbb{N}} \) strictly increasing, \( a_n < 0 \) for all \( n \in \mathbb{N} \) or \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), \( \lim_{n \to \infty} a_n = 0 \) or \( \lim_{n \to \infty} a_n = \infty \), \( \mathcal{V} := (\exp(a_n\mu))_{n \in \mathbb{N}} \) and \( \Omega_n := S_n \) for all \( n \in \mathbb{N} \). Set \( \mathcal{V}_\mu := (\exp((-1/n)\mu))_{n \in \mathbb{N}} \) and \( \Omega_\mu := (\exp(n\mu))_{n \in \mathbb{N}} \). Then

\[\mathcal{O}\mathcal{V}(\mathbb{C}) \cong \mathcal{O}\mathcal{V}_\mu(\mathbb{C}), \quad \text{if} \ a_n < 0, \quad \text{and} \quad \mathcal{O}\mathcal{V}(\mathbb{C}) \cong \mathcal{O}\mathcal{V}_\mu(\mathbb{C}), \quad \text{if} \ a_n \geq 0,\]

which is easily seen. Thus one may choose the most suitable sequence \( (a_n)_{n \in \mathbb{N}} \) for one’s purpose without changing the space.

4.5. Corollary. Let \( (a_n)_{n \in \mathbb{N}} \) be strictly increasing, \( a_n < 0 \) for all \( n \in \mathbb{N} \) or \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), \( \lim_{n \to \infty} a_n = 0 \) or \( \lim_{n \to \infty} a_n = \infty \), \( \mathcal{V} := (\exp(a_n\mu))_{n \in \mathbb{N}} \) and \( \Omega_n := S_n \) for all \( n \in \mathbb{N} \) where

\[\mu : \mathbb{C} \to [0, \infty), \quad \mu(z) := \|\text{Re}(z)\|^\gamma, \quad \text{for some} \ 0 < \gamma \leq 1. \quad \text{Then} \ \mathcal{O}\mathcal{V}(\mathbb{C}) \ \text{satisfies} \ (\Omega).\]

Proof. We only need to check that the conditions of Theorem 4.3 are fulfilled. Obviously, \( \mu(z) = \mu(|\text{Re}(z)|) \) for all \( z \in \mathbb{C} \), \( \mu \) is strictly increasing on \([0, \infty)\) and \( \lim_{x \to \infty, x \in \mathbb{R}} \frac{\ln(1+|z|)}{\mu(x)} = 0 \). The observation

\[\mu(x + n) - \mu(x) = |x + n|^\gamma - |x|^\gamma \leq |x + n - n|^\gamma = n^\gamma, \quad n \in \mathbb{N}, \ x \in [0, \infty),\]

implies that \( \mu \) is a strong weight generator with any \( \Gamma > 1 \) and \( C := n^\gamma \) by Definition 4.1. In addition, condition (\( \omega, 1 \)) is fulfilled by Example 4.2.2). Let us turn to Condition 4.3. If \( a_n < 0 \) for all \( n \in \mathbb{N} \), then Condition 4.4 is fulfilled by 4.10 Example a), p. 22 where we used \( \tilde{\mu}(z) := |z|^\gamma \) instead of \( \mu \); which does not make a difference since

\[|\text{Re}(z)|^\gamma \leq |z|^\gamma \leq |\text{Re}(z)|^\gamma + n^\gamma, \quad z \in \Omega_n = S_n.\]
If $a_n \geq 0$ for all $n \in \mathbb{N}$, we only have to modify [46, 4.10 Example a], p. 22] a bit. We choose $I_j(n) := 2n$ for $j \in \{1, 2, 4\}$ and define the open set $X_{I_j(n)} := S_{4n}^2$. Then we have

$$I_{214}(n) = 8n \geq 4n + 4 = I_{14}(n + 1), \quad n \in \mathbb{N}.$$ 

Furthermore, we have $d_{n,k} = |n - k|$ for all $n, k \in \mathbb{N}$.

Condition 3.4 a)(i) and c): Verbatim as in [46, 4.10 Example a], p. 22].

Condition 3.4 a)(ii): We have $d_{X, I_2} = 2n$. We choose $g_n : \mathbb{C} \to \mathbb{C}$, $g_n(z) := \exp(-z^2)$, as well as $r_n := 1/(4n)$ and $R_n := 1/(6n)$ for $n \in \mathbb{N}$. Let $z = z_1 + iz_2 \in \Omega_{I_2(n)} = S_{2n}$ and $x \in X_{I_2(n)} + B_{R_n}(0)$. For $\zeta = \zeta_1 + i\zeta_2 \in \mathbb{C}$ with $|\zeta - (z - x)| = r_n$ we have

$$|g_n(\zeta)|e^{a_{2n}|\zeta|^2} = e^{-\text{Re}(\zeta^2)} e^{a_{2n} |\text{Re}(\zeta)|} \leq e^{-\zeta_1^2 + \zeta_2^2} e^{a_{2n}(1 + |z_1|)}$$

$$\leq e^{(r_n |z_1|^2 + |z_2|^2) + a_{2n}(1 + r_n |x_1|)} e^{-|\zeta_1|^2 + a_{2n}|\zeta_1|}$$

$$\leq e^{(r_n + 2n |z_1|^2 + |z_2|^2) + a_{2n}(1 + r_n |x_1|)} \sup_{t \in \mathbb{R}} e^{-t^2 + a_{2n} t}$$

$$= e^{(r_n + 2n |z_1|^2 + |z_2|^2) + a_{2n}(1 + r_n |x_1|) + a_{2n}^2 / 4} = A_2(x, n)$$

and observe that $A_2(\cdot, n)$ is continuous and thus locally bounded on $X_{I_2(n)}$.

Condition 3.4 a)(iii): Let $K \subset \mathbb{C}$ be compact and $x = x_1 + ix_2 \in \Omega_n$. Then there is $b > 0$ such that $|y| \leq b$ for all $y = y_1 + iy_2 \in K$ and from polar coordinates and Fubini’s theorem follows that

$$\int_{K} \frac{|g_n(x - y)|}{|x - y|} dy$$

$$\leq \sup_{w \in K} e^{a_{2n} |\text{Re}(w)|} \int_{K} \frac{e^{-\text{Re}((x-y)^2)/|x-y|}}{|x-y|} e^{-a_{2n}|y|} dy$$

$$\leq C_1 \left( \int_{\mathbb{C}_1(x)} \frac{e^{-\text{Re}((x-y)^2)/|x-y|}}{|x-y|} e^{-a_{2n}|y|} dy + \int_{\mathbb{C}_2(x)} \frac{e^{-\text{Re}((x-y)^2)/|x-y|}}{|x-y|} e^{-a_{2n}|y|} dy \right)$$

$$\leq C_1 \left( \int_{0}^{2\pi} \frac{1}{r} e^{-r^2 \cos(2\varphi)} e^{-a_{2n}|x_1 + r \cos(\varphi)|} r dr d\varphi + \int_{\mathbb{C}_2(x)} e^{-\text{Re}((x-y)^2)/|x-y|} e^{-a_{2n}|y|} dy \right)$$

$$\leq C_1 \left( 2\pi e^{1+o_{2n}} e^{-a_{2n}|x_1|} + \int_{\mathbb{R}} e^{(x_2-y_2)^2} dy_2 \int_{\mathbb{R}} e^{-(x_1-y_1)^2 + a_{2n} |x_2-y_2|} dy_1 e^{-a_{2n}|x_1|} \right)$$

$$\leq C_1 \left( 2\pi e^{1+o_{2n}} + 2be^{(x_2+0)^2} \int_{\mathbb{R}} e^{-y_2^2*a_{2n}|y_1|} dy_1 e^{-a_{2n}|x_1|} \right)$$

$$= C_1 \left( 2\pi e^{1+o_{2n}} + 2be^{(x_2+0)^2} e^{a_{2n}^2/4} \int_{\mathbb{R}} e^{-(|y| - a_{2n}/2)^2} dy_1 e^{-a_{2n}|x_1|} \right)$$

$$= C_1 \left( 2\pi e^{1+o_{2n}} + 4be^{(x_2+0)^2} e^{a_{2n}^2/4} \int_{\mathbb{R}} e^{-y_1^2} dy_1 e^{-a_{2n}|x_1|} \right)$$

$$\leq C_1 \left( 2\pi e^{1+o_{2n}} + 4\sqrt{\pi} be^{(n+b)^2 + a_{2n}^2/4} e^{-a_{2n}|x_1|} \right).$$

We conclude that Condition 3.4 a)(iii) holds since

$$e^{-a_{2n}|x_1| e^{o_{a_{2n}}|\text{Re}(x)|}} \leq e^{a_{n}-a_{2n}) |x_1| + a_n} \leq e^{a_n}.$$
Condition \((\text{[5.4] } \text{b})\): Let \(p, k \in \mathbb{N}\) with \(p \leq k\). For all \(x = x_1 + i x_2 \in \Omega_p\) and \(y = y_1 + i y_2 \in \Omega_{1_4(n)}\) we note that
\[
a p | \Re(x)| ^ \gamma - a k | \Re(y)| ^ \gamma \leq a k | x_1 - y_1 | ^ \gamma \leq a k (1 + | x_1 - y_1 |)
\]
because \((a_n)_{n \in \mathbb{N}}\) is non-negative and increasing and \(0 < \gamma \leq 1\). Like before we deduce that
\[
\int_{\Omega_{1_4(n)}} \left| \frac{y_n(x - y) | \nu_p(x) |^{y} | \nu_k(y) |}{| x - y |} \right| dy
\]
\[
= \int_{\Omega_{2n}} e^{- \Re((x-y)^2)} e^{a p | \Re(x)| ^ \gamma - a k | \Re(y)| ^ \gamma} | | \Re(x) - \Re(y)| ^ \gamma \right| dy
\leq \frac{2 \pi}{2n} \int_{0}^{1} \frac{e^{- x^2 \cos(2x)}}{r} e^{a \Re(x)^ \gamma} rdrd\varphi + \int_{\Omega_{2n} \setminus \Omega_{1}} e^{- \Re((x-y)^2)} e^{a \Re(x) - \Re(y)| ^ \gamma} dy
\leq 2 \pi e^{1+ak} + 2n \int_{R} e^{(x^2 - y)^2} dy_2 \int_{\Omega_{2n}} e^{- (x_1 - y_1)^2 + a_k | x_1 - y_1 |} dy_1
\leq 2 \pi e^{1+ak} + 8 \pi e^{a_k + (|x^2| + 2n)^2 + a_l^2} / 4
\leq 2 \pi e^{1+ak} + 8 \pi e^{a_k + (|x^2| + 2n)^2 + a_l^2} / 4
\]
for \((k, p) = (I_4(n), n)\) and \((k, p) = (I_{14}(n), I_{14}(n))\) as \((a_n)_{n \in \mathbb{N}}\) is non-negative and increasing.

\[\square\]

5. Surjectivity of the Cauchy-Riemann operator

In our last section we prove our main result on the surjectivity of the Cauchy-Riemann operator on \(\mathcal{EV}(\mathbb{C}, E)\) where \(\Omega_n := \{ z \in \mathbb{C} \mid | \Im(z) | < n \}\) for all \(n \in \mathbb{N}\). We recall the corresponding result for \(E = \mathbb{C}\) which we will need. It is a consequence of the approximation Theorem \([\text{[31] } \text{Theorem 4.4.2, p. 94}]\) in combination with Hörmander’s solution of the \(\bar{\mathcal{D}}\)-problem in weighted \(L^2\)-spaces \([\text{[38] } \text{Theorem 4.4.2, p. 94}]\) and the Mittag-Leffler procedure.

5.1. **Theorem** \([\text{[16] } \text{4.8 Theorem, p. 20}]\). Let Condition \([\text{[3.7] } \text{with } \psi_n(z) := (1 + | z^2 |)^2, z \in \Omega, and Condition \([\text{[5.4] } \text{with } I_{211}(n) \geq I_{11}(n + 1) \text{ be fulfilled and } - \ln \nu_n \text{ be subharmonic on } \Omega \text{ for every } n \in \mathbb{N}. \text{ Then}\)

\[\bar{\mathcal{D}} \mathcal{EV}(\Omega) \to \mathcal{EV}(\Omega)\]

is surjective.

An application of this theorem yields the following corollary.

5.2. **Corollary** \([\text{[16] } \text{4.10 Example a, p. 22}]\). Let \((a_n)_{n \in \mathbb{N}}\) be strictly increasing, \(a_n < 0\) for all \(n \in \mathbb{N}\), \(\mathcal{V} := \langle \exp(a_n \mu) \rangle_{n \in \mathbb{N}}\) and \(\Omega_n := \{ z \in \mathbb{C} \mid | \Im(z) | < n \}\) for all \(n \in \mathbb{N}\) where

\[\mu : \mathbb{C} \to [0, \infty), \mu(z) := | \Re(z) | ^ \gamma,\]

for some \(0 < \gamma \leq 1\). Then

\[\bar{\mathcal{D}} \mathcal{EV}(\mathbb{C}) \to \mathcal{EV}(\mathbb{C})\]

is surjective.

The restriction to negative \(a_n\) comes from the condition that \(- \ln \nu_n\) should be subharmonic. We note that the \(E\)-valued versions of Theorem \(\text{[5.1] and Corollary [5.2] where } E \text{ is a Fréchet space over } \mathbb{C} \text{ hold as well by the classical theory of tensor products for nuclear Fréchet spaces (see [16] 4.9 Corollary, p. 21]). Since we will use the } \varepsilon\text{-product } \mathcal{EV}(\Omega) \varepsilon E \text{ to enlarge our collection of locally convex Hausdorff}
space $E$ for which $\overline{\partial}^E$ is surjective, we remark the following (cf. [42, 5.23 Lemma, p. 92]).

5.3. Proposition.  
(a) Let $X$ be a semi-reflexive locally convex Hausdorff space and $Y$ a Fréchet space. Then $L_b(X'_b, Y'_b) \cong L_b(Y, (X'_b)_0)$ via taking adjoints.

(b) Let $X$ be a Montel space and $E$ a locally convex Hausdorff space. Then $L_b(X'_b, E) \cong X \in E$ where the topological isomorphism is the identity map.

Proof.  
(a) We consider the map

$$^t(\cdot): L_b(X'_b, Y'_b) \rightarrow L_b(Y, (X'_b)_0), \ u \mapsto ^t u,$$

defined by $^t u(y)(x') := u(x')(y)$ for $y \in Y$ and $x' \in X'$. First, we prove that $^t(\cdot)$ is well-defined. Let $u \in L(X'_b, Y'_b)$ and $y \in Y$. Since $u \in L(X'_b, Y'_b)$ and $\{y\}$ is bounded in $Y$, there are a bounded set $B \subset X$ and $C > 0$ such that

$$\sup_{x' \in B} |^t u(y)(x')| = \sup_{x' \in B} |u(x')(y)| \leq C \sup_{x' \in B} |x'(x)|$$

for all $x' \in X'$ implying $^t u(y) \in (X'_b)'$.

Let us denote by $\langle \| \cdot \|_{Y, n} \rangle_{n\in\mathbb{N}}$ the (directed) system of seminorms generating the metrisable locally convex topology of $Y$. The canonical embedding $J: Y \rightarrow (Y'_b)'$ is a topological isomorphism between $Y$ and $J(Y)$ by [42, Corollary 25.10, p. 298] because $Y$ is a Fréchet space. For a bounded set $M \subset X'_b$ we note that

$$\sup_{x' \in M} \sup_{y \in N} |u(x')(y)| \leq C \sup_{x' \in M} \sup_{x' \in B} |x'(x)| < \infty,$$

where the last estimate follows from the boundedness of $M \subset X'_b$. Hence $u(M)$ is bounded in $Y'_b$. By the remark about the canonical embedding there are $n \in \mathbb{N}$ and $C_0 > 0$ such that

$$\sup_{x' \in M} |^t u(y)(x')| = \sup_{y \in u(M)} |(J(y), y')| \leq C_0 \|y\|_{Y, n},$$

so $^t u \in L(Y, (X'_b)_0)$ and the map $^t(\cdot)$ is well-defined.

Let us turn to injectivity. Let $u, v \in L(X'_b, Y'_b)$ with $^t u = ^t v$. This is equivalent to

$$u(x')(y) = ^t u(y)(x') = ^t v(y)(x') = v(x')(y)$$

for all $y \in Y$ and $x' \in X'$. This implies $u(x') = v(x')$ for all $x' \in X'$, hence $u = v$.

Next, we turn to surjectivity. We consider the map

$$^t(\cdot): L_b(Y, (X'_b)_0) \rightarrow L_b(X'_b, Y'_b), \ u \mapsto ^t u,$$

defined by $^t u(x')(y) := u(y)(x')$ for $x' \in X'$ and $y \in Y$. We show that this map is well-defined. Let $u \in L_b(Y, (X'_b)_0)$ and $x' \in X'$. Since $u \in L_b(Y, (X'_b)_0)$ and $\{x'\}$ is bounded in $X'$, there are $n \in \mathbb{N}$ and $C > 0$ such that

$$|^t u(y)(x')| = |u(y)(x')| \leq C \|y\|_{Y, n}$$

for all $y \in Y$ yielding to $^t u(x') \in Y'$. Let $B \subset Y$ be bounded. The semi-reflexivity of $X$ implies that for every $u(y)$, $y \in B$, there is a unique $x_{u(y)} \in X$ such that $u(y)(x') = x'_{u(y)}(y)$ for all $x' \in X'$. Then we get

$$\sup_{y \in B} |^t u(y)(x')| = \sup_{y \in B} |u(y)(x')| = \sup_{y \in B} |x'_{u(y)}(y)|.$$
We claim that \( D := \{ x_{u(y)} \mid y \in B \} \) is a bounded set in \( X \). Let \( N \subset X' \) be finite. Then the set \( M := \{ x'(x') \mid x' \in N \} \subset Y' \) is finite. We have
\[
\sup \sup_{y \in B} |x'(x_u(y))| = \sup \sup_{y \in B} |x'(x'_u(y))| = \sup \sup_{y \in M} |y(y)| < \infty
\]
where the last estimate follows from the fact that the bounded set \( B \) is weakly bounded. Thus \( D \) is weakly bounded and by [72 Mackey's theorem 23.15, p. 268] bounded in \( X \). Therefore, it follows from
\[
\sup \sup_{y \in B} |x'(x'_u(y))| = \sup \sup_{y \in B} |x'(x_u(y))| = \sup \sup_{x \in D} |x'(x)|
\]
for all \( x' \in X' \) that \( ^t u \in L(X'_b, Y'_b) \) which means that \( ^t (\cdot) \) is well-defined. Let \( u \in L(Y, (X'_b)^t) \). Then we have \( ^t u \in L_b(X'_b, Y'_b) \). In addition, for all \( y \in Y \) and all \( x' \in X' \)
\[
^t (^t u)(y)(x') = ^t u(x')(y) = u(y)(x')
\]
is valid and so \( ^t (^t u)(y) = u(y) \) for all \( y \in Y \) proving the surjectivity.

The last step is to prove the continuity of \( ^t (\cdot) \) and its inverse. Let \( M \subset Y \) and \( B \subset X' \) be bounded sets. Then
\[
\sup \sup_{y \in M} |u(y)(x')| = \sup \sup_{y \in B} |u(x')(y)| = \sup \sup_{y \in M} |u(x')(y)|
\]
holds for all \( u \in L(X'_b, Y'_b) \). Therefore, \( ^t (\cdot) \) and its inverse are continuous.

b) Let \( T \in L(X'_b, E) \). For \( \alpha \in \mathfrak{A} \) there are a bounded set \( B \subset X \) and \( C > 0 \) such that
\[
p_{\alpha}(T(x')) \leq C \sup_{x \in B} |x'(x)| \leq C \sup_{x \in B} |x'(x)|
\]
for every \( x' \in X' \). The set \( \overline{\text{conv}}(B) \) is absolutely convex and compact by [39, 6.2.1 Proposition, p. 103] and [39, 6.7.1 Proposition, p. 112] since \( B \) is bounded in the Montel space \( X \). Hence we gain \( T \in L(X'_a, E) \).

Let \( M \subset X' \) be equicontinuous. Due to [39, 8.5.1 Theorem (a), p. 156] \( M \) is bounded in \( X'_b \). Therefore,
\[
id: L_b(X'_b, E) \rightarrow L_c(X''_b, E) = X \subset E
\]
is continuous.

Let \( T \in L(X'_b, E) \). For \( \alpha \in \mathfrak{A} \) there are an absolutely convex compact set \( B \subset X \) and \( C > 0 \) such that
\[
p_{\alpha}(T(x')) \leq C \sup_{x \in B} |x'(x)|
\]
for every \( x' \in X' \). Since the compact set \( B \) is bounded, we get \( T \in L(X'_b, E) \).

Let \( M \) be a bounded set in \( X'_b \). Then \( M \) is equicontinuous by virtue of [84 Theorem 33.2, p. 349], as \( X \), being a Montel space, is barrelled by [72 Remark 24.24 (a), p. 286]. Thus
\[
id: L_c(X'_b, E) \rightarrow L_b(X'_b, E)
\]
is continuous.

Now, we use the results obtained so far and splitting theory to obtain our main theorem on the surjectivity of the Cauchy-Riemann operator on the space \( E V(\Omega, E) \). We recall that a Fréchet space \( (F, (\| \cdot \|_k)_{k \in \mathbb{N}}) \) satisfies \( (DN) \) by [72 Chap. 29, Definition, p. 359] if
\[
\exists \rho \in \mathbb{N} \forall k \in \mathbb{N} \exists n \in \mathbb{N}, C > 0 \forall x \in F : \| x \|_k^\rho \leq C \| x \|_k \| x \|_n .
\]
A (PLS)-space is a projective limit \( X = \lim X_N \), where the inductive limits \( X_N = \lim_{n \to 0} X_{n+N} \) are (DFS)-spaces (which are also called (LS)-spaces), and it satisfies (PA) if
\[
\forall N \exists M \forall K \exists n \forall m \forall \eta > 0 \exists k, C, r_0 > 0 \forall r > r_0 \forall x' \in X'_N:
\]
\[
\left| x' \circ i_{N,M} \right|_{M,m} \leq C(r^n) \left| x' \circ i_N \right|_{K,k} + \frac{1}{r} \left| x' \circ i_N \right|_{N,n}
\]
where \( \left\| \cdot \right\| \) denotes the norm of \( \left\| \cdot \right\| \) (see [3], Section 4, Eq. (24), p. 577).

5.4. Theorem. Let Condition \( \text{S7} \) with \( \psi_n(z) = (1 + |z|^2)^{-1}, z \in \Omega \), and Condition \( \text{S2} \) with \( I_{214}(n) \geq I_{14}(n + 1) \) be fulfilled and \(-\ln \nu_n \) be subharmonic on \( \Omega \) for every \( n \in \mathbb{N} \). If \( \mathcal{O} \mathcal{V} \mathcal{O} \) satisfies property \((\Omega)\) and

a) \( E := F_2 \) where \( F \) is a Fréchet space over \( \mathbb{C} \) satisfying \((\text{DN})\), or
b) \( E \) is an ultrabornological (PLS)-space over \( \mathbb{C} \) satisfying \((\text{PA})\),

then
\[
\varphi^\text{Fr} : \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega, E) \to \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega, E)
\]
is surjective.

Proof. Throughout this proof we use the notation \( X'' := (X'_1)' \) for a locally convex Hausdorff space \( X \). In both cases, \( a) \) and \( b) \), the space \( E \) is a complete locally convex Hausdorff space. The space \( \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} \) is a Fréchet space by [42, 3.4 Proposition, p. 6] and \( \mathcal{O} \mathcal{V} \mathcal{O} \) as well since it is a closed subspace by Proposition 3.3 b). Both spaces are also nuclear and thus reflexive by [43, 3.1 Theorem, p. 12], [45, 2.7 Remark, p. 5] and [42, 2.3 Remark b), p. 3] because \((\omega.1)\) and \((\omega.2)\) from Condition 3.4 are fulfilled. As a consequence the map
\[
S : \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni E \to \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega, E), \quad u \mapsto [z \mapsto u(\delta_z)]
\]
is a topological isomorphism by [43, 5.10 Example c), p. 24] where \( \delta_z \) is the point-evaluation at \( z \in \Omega \). We denote by \( J : E \to E'' \) the canonical injection in the algebraic dual \( E'' \) of the topological dual \( E' \) and for \( f \in \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega, E) \) we set
\[
R_f : \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \to E'', \quad y \mapsto \left( e' \mapsto y(e' \circ f) \right).
\]
Then the map \( f \mapsto J^{-1} \circ R_f \) is the inverse of \( S \) by [42, 3.14 Theorem, p. 9]. The sequence
\[
0 \to \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni 0,
\]
where \( i \) means the inclusion, is an exact sequence of Fréchet spaces by Theorem 5.1 and hence topologically exact as well. Let us denote by \( J_0 : \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \to \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \) and \( J_1 : \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \to \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \) the canonical embeddings which are topological isomorphisms since \( \mathcal{O} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \) and \( \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \) are reflexive. Then the exactness of (22) implies that
\[
0 \to \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \ni \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \ni 0,
\]
where \( i_0 := J_0 \circ i \circ J_0^{-1} \) and \( i_1 := J_1 \circ i \circ J_1^{-1} \), is an exact topological sequence.

Topological as the (strong) bidual of a Fréchet space is again a Fréchet space by [72, Corollary 25.10, p. 298].

a) Let \( E := F_2 \) where \( F \) is a Fréchet space with \((\text{DN})\). Then \( \text{Ext}^1(F, \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' = 0 \) by [47, 5.1 Theorem, p. 186] since \( \mathcal{O} \mathcal{V} \mathcal{O} (\Omega) \) satisfies \((\Omega)\) and therefore \( \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \) as well. Combined with the exactness of (22) this implies that the sequence
\[
0 \to L(F, \mathcal{O} \mathcal{V} \mathcal{V} \mathcal{O} (\Omega)'' \ni \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \ni L(F, \mathcal{E} \mathcal{V} \mathcal{O} \mathcal{V} \mathcal{O} (\Omega)'' \ni 0
\]
is exact by [76, Proposition 2.1, p. 13-14] where \( i^*_0(B) := i_0 \circ B \) and \( \overline{\mathfrak{J}}_1(D) := \overline{\mathfrak{J}} \circ D \) for \( B \in \mathcal{L}(F, \mathcal{O}(\Omega)^\ast) \) and \( D \in \mathcal{L}(F, \mathcal{E}(\Omega)^\ast) \). In particular, we obtain that
\[
\overline{\mathfrak{J}}_1^\ast : \mathcal{L}(F, \mathcal{E}(\Omega)^\ast) \to \mathcal{L}(F, \mathcal{E}(\Omega)^\ast)
\]
is surjective. Via \( E = F' \) and Proposition \([5,3] \) \((X = \mathcal{E}(\Omega) \text{ and } Y = F)\) we have the topological isomorphism
\[
\psi := S \circ i^\ast : \mathcal{L}(F, \mathcal{E}(\Omega)^\ast) \to \mathcal{E}(\Omega, E), \psi(u) = (S \circ i^\ast(u))(u) = \left[ z \mapsto \iota^* u(\delta_z) \right],
\]
and the inverse
\[
\psi^{-1}(f) = (S \circ i^\ast)^{-1}(f) = \left( i^\ast \circ S^{-1} \right)(f) = \left( J^{-1} \circ R_j^\ast \right), \quad f \in \mathcal{E}(\Omega, E).
\]

Let \( g \in \mathcal{E}(\Omega, E) \). Then \( \psi^{-1}(g) \in \mathcal{L}(F, \mathcal{E}(\Omega)^\ast) \) and by the surjectivity of \([21]\) there is \( u \in \mathcal{L}(F, \mathcal{E}(\Omega)^\ast) \) such that \( \overline{\mathfrak{J}}_1 u = \psi^{-1}(g) \). So we get \( \psi(u) \in \mathcal{E}(\Omega, E) \).

Next, we show that \( \overline{D}^E \psi(u) = g \) is valid. Let \( x \in F \), \( z \in \Omega \) and \( h \in \mathbb{R}, \ h \neq 0 \), and \( e_k \) denote the \( k \)th unit vector in \( \mathbb{R}^2 \). From
\[
\left( \delta_z + h e_k - \delta_z \right)(f) = \frac{f(z + h e_k) - f(z)}{h} \underset{h \to 0}{\to} \partial^e \left( f(z) \right),
\]
for every \( f \in \mathcal{E}(\Omega) \) follows that \( \delta_z + h e_k - \delta_z \) converges to \( \delta_z \circ \partial^e \in \mathcal{E}(\Omega)^\ast \). Since the nuclear Fréchet space \( \mathcal{E}(\Omega) \) is in particular a Montel space, we deduce that \( \delta_z + h e_k - \delta_z \) converges to \( \delta_z \circ \partial^e \) in \( \mathcal{E}(\Omega)^\ast \) by the Banach-Steinhaus theorem. Let \( B \subset F \) be bounded. As \( \iota^* u \in \mathcal{L}(\mathcal{E}(\Omega)^\ast, F' \) there are a bounded set \( B_0 \subset \mathcal{E}(\Omega) \) and \( C > 0 \) such that
\[
\sup_{x \in B} \left| \iota^* u(\delta_z + h e_k) - \iota^* u(\delta_z) \right| = \sup_{x \in B} \left| \iota^* u(\delta_z + h e_k - \delta_z) \right| \leq C \sup_{x \in B} \left| \delta_z \circ \partial^e \left( f(z) \right) \right| \to 0
\]
yielding to \( \overline{D}^E \psi(u)(z) = \iota^* u(\delta_z \circ \overline{D}) \). This implies \( \overline{D}^E \psi(u)(z) = \iota^* u(\delta_z \circ \overline{D}) \).

So for all \( x \in F \) and \( z \in \Omega \) we have
\[
\overline{D}^E \psi(u)(z)(x) = \iota^* u(\delta_z \circ \overline{D})(x) = u(x)(\delta_z \circ \overline{D}) = \{ \delta_z \circ \overline{D}, J^{-1}(u(x)) \}
\]
\[
= \{ \delta_z, \mathfrak{J}_1^{-1}(u(x)) \} = \{ \mathfrak{J}_1 \circ \overline{D}, J^{-1}(u(x)) \}
\]
\[
= \{ \mathfrak{J}_1 \circ \overline{D}, J^{-1}(u(x)) \} = \mathfrak{J}_1^{-1}(\mathfrak{J}(u(x)))(\delta_z) = \mathfrak{J}_1^{-1}(\mathfrak{J}(u(x)))(\delta_z)
\]
\[
= (\mathfrak{J}_1 \circ \mathfrak{J}^{-1})(\delta_z)(x) = \mathfrak{J}_1^{-1}(\mathfrak{J}(u(x)))(\delta_z) = \overline{D}(\delta_z)(x) = g(z)(x).
\]
Thus \( \overline{D}^E \psi(u)(z) = g(z) \) for every \( z \in \Omega \) which proves the surjectivity.

b) Let \( E \) be an ultrabornological (PLS)-space satisfying \((\text{PA})\). Since the nuclear Fréchet space \( \mathcal{O}(\Omega) \) is also a Schwartz space, its strong dual \( \mathcal{O}(\Omega)^\ast \) is a (DFS)-space. By \([\text{5}], \text{Theorem } 4.1, \text{p. } 577\) we obtain \( \text{Ext}_{\text{PLS}}(\mathcal{O}(\Omega)^\ast, E) = 0 \) as the bidual \( \mathcal{O}(\Omega)^{\ast\ast} \) satisfies \((\Omega) \), \( E \) is a (PLS)-space satisfying \((\text{PA})\) and condition \((c)\) in the theorem is fulfilled because \( \mathcal{O}(\Omega)^\ast \) is the strong dual of a nuclear Fréchet space. Moreover, we have \( \text{Proj}^1 E = 0 \) due to \([\text{5}], \text{Corollary } 3.3.10, \text{p. } 46\) because \( E \) is an ultrabornological (PLS)-space. Then the exactness of the sequence \([\text{23}], \text{p. } 567\) and \([\text{5}], \text{Lemma } 3.3, \text{p. } 567\) (in the lemma the same condition \((c)\) as in \([\text{5}], \text{Theorem } 4.1, \text{p. } 577\) is fulfilled and we choose \( H = \mathcal{O}(\Omega)^{\ast\ast} \) and \( F = G \in \mathcal{E}(\Omega)^{\ast\ast} \), imply that the sequence
\[
0 \to L(E, \mathcal{O}(\Omega)^{\ast\ast}) \xrightarrow{i_0} L(E, \mathcal{E}(\Omega)^{\ast\ast}) \xrightarrow{\overline{J}_1} L(E, \mathcal{E}(\Omega)^{\ast\ast}) \xrightarrow{0} 0
\]
is exact. The maps \( i_0^\ast \) and \( \overline{J}_1 \) are defined like in part \( a \). Especially, we get that
\[
\overline{J}_1 : L(E, \mathcal{E}(\Omega)^{\ast\ast}) \to L(E, \mathcal{E}(\Omega)^{\ast\ast})
\]
\[(25)\]
is surjective.

By [27, Remark 4.4, p. 1114] we have \( L_0(\mathcal{E}\mathcal{V}(\Omega)_b', E'') \cong L_0(E'_b, \mathcal{E}\mathcal{V}(\Omega)'') \) via taking adjoints since \( \mathcal{E}\mathcal{V}(\Omega) \), being a Fréchet-Schwartz space, is a (PLS)-space and hence its strong dual an (LFS)-space, which is regular by [88, Corollary 6.7, 10, \( \Leftrightarrow 11 \), p. 114], and \( E \) is an ultrabornological (PLS)-space, in particular, reflexive by [24, Theorem 3.2, p. 58]. In addition, the map

\[
T: L_0(\mathcal{E}\mathcal{V}(\Omega)_b', E'') \to L_0(\mathcal{E}\mathcal{V}(\Omega)_b', E),
\]

defined by \( T(u)(y) := \mathcal{J}^{-1}(u(y)) \) for \( u \in L(\mathcal{E}\mathcal{V}(\Omega)_b', E'') \) and \( y \in \mathcal{E}\mathcal{V}(\Omega)' \), is a topological isomorphism because \( E \) is reflexive. Due to Proposition 5.3(b) we obtain the topological isomorphism

\[
\psi := S \circ \mathcal{J}^{-1} \circ \psi'(\cdot) : L_0(E'_b, \mathcal{E}\mathcal{V}(\Omega)'' \to \mathcal{E}\mathcal{V}(\Omega), E),
\]

with the inverse given by

\[
\psi^{-1}(f) = (S \circ \mathcal{J}^{-1} \circ \psi'(\cdot))^{-1}(f) = \psi'(f \circ S^{-1}) = \psi'(\mathcal{J} \circ \mathcal{J}^{-1} \circ \mathcal{R}''_f) = \psi'(\mathcal{R}''_f)
\]

for \( f \in \mathcal{E}\mathcal{V}(\Omega, E) \).

Let \( g \in \mathcal{E}\mathcal{V}(\Omega, E) \). Then \( \psi^{-1}(g) \in L(E'_b, \mathcal{E}\mathcal{V}(\Omega)'') \) and by the surjectivity of \( \mathcal{J} \) there exists \( u \in L(E'_b, \mathcal{E}\mathcal{V}(\Omega)'' \) such that \( \mathcal{J} u = \psi^{-1}(g) \). So we have \( \psi(u) \in \mathcal{E}\mathcal{V}(\Omega, E) \). The last step is to show that \( \mathcal{J} \circ \psi(u) = g \). Like in part a) we gain for every \( z \in \Omega \)

\[
\mathcal{J} \circ \psi(u)(z) = \mathcal{J}^{-1}(\psi(u)(\delta_z))
\]

and for every \( x \in E' \)

\[
\psi(u)(\delta_z)(\bar{\delta}_z(x)) = u(x)(\delta_z \circ \bar{\delta}_z) = (\mathcal{J} \circ \psi(u))(x)(\delta_z) = \psi^{-1}(g)(x)(\delta_z) = \psi'(\mathcal{R}''_f)(x)(\delta_z)
\]

Thus we have \( \psi(u)(\delta_z) = \mathcal{J} \circ \psi(u)(\delta_z) \) and therefore \( \mathcal{J} \circ \psi(u)(\delta_z) = \mathcal{J} \circ \psi(u)(\delta_z) \) for all \( z \in \Omega \).

Due to [54, 1.4 Lemma, p. 110] and [4, Proposition 4.2, p. 577] we have the following relation between the cases a) and b) in Theorem 5.4.

5.5. Remark. Let \( F \) be a Fréchet-Schwartz space. Then \( F \) satisfies (DN) if and only if the (DFS)-space \( E := F_b \) satisfies (PA).

Thus case a) is included in case b) if \( F \) is a Fréchet-Schwartz space. Therefore a) is only interesting for Fréchet spaces \( F \) which are not Schwartz spaces.

5.6. Corollary. Let \( \mu \) be a subharmonic strong weight generator, \( (a_n)_{n \in \mathbb{N}} \) strictly increasing, \( a_n < 0 \) for all \( n \in \mathbb{N} \), \( \lim_{n \to \infty} a_n = 0 \) and \( \mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}. \) Let Condition 3.1 with \( \psi(z) := (1 + |z|^2)^{-2}, \) \( z \in \mathbb{C} \), and Condition 3.2 with \( I_{14}(n) \geq I_{214}(n) + 1 \) for all \( n \in \mathbb{N} \) be fulfilled. If

a) \( E := F_b' \) where \( F \) is a Fréchet space over \( \mathbb{C} \) satisfying (DN), or
b) \( E \) is an ultrabornological (PLS)-space over \( \mathbb{C} \) satisfying (PA),

then

\[
\mathcal{J} : \mathcal{E}\mathcal{V}(\mathbb{C}, E) \to \mathcal{E}\mathcal{V}(\mathbb{C}, E)
\]

is surjective.

Proof. The assertion is a direct consequence of Theorem 5.4 and Theorem 4.3. \( \square \)

Corollary 5.6 generalises a part of [42, 5.24 Theorem, p. 95] \( (K = \emptyset) \) which is the case \( \gamma = 1 \) of the next corollary.
5.7. Corollary. Let \((a_n)_{n\in\mathbb{N}}\) be strictly increasing, \(a_n < 0\) for all \(n \in \mathbb{N}\), \(\lim_{n \to \infty} a_n = 0\), \(\mathcal{V} := (\exp(a_n \mu))_{n \in \mathbb{N}}\) and \(\Omega_n := \{ z \in \mathbb{C} \mid |\text{Im}(z)| < n \}\) for all \(n \in \mathbb{N}\) where 
\(\mu : \mathbb{C} \to [0, \infty), \mu(z) := |\text{Re}(z)|^\gamma\),
for some \(0 < \gamma \leq 1\). If
\begin{itemize}
  \item a) \(E := F'_b\) where \(F\) is a Fréchet space over \(\mathbb{C}\) satisfying (DN), or
  \item b) \(E\) is an ultrabornological (PLS)-space over \(\mathbb{C}\) satisfying (PA),
\end{itemize}
then 
\[ \overline{\mathcal{D}}^E : \mathcal{V}(\mathbb{C}, E) \to \mathcal{V}(\mathbb{C}, E) \]
is surjective.

Proof. Follows from Corollary 5.6 and Corollary 5.5 \(\square\)

To close this section we provide some examples of ultrabornological (PLS)-spaces satisfying (PA) and spaces of the form \(E := F'_b\) where \(F\) is a Fréchet space satisfying (DN).

5.8. Example. a) The following spaces are ultrabornological (PLS)-spaces with property (PA) and also strong duals of a Fréchet space satisfying (DN):
\begin{itemize}
  \item the strong dual of a power series space of infinite type \(\Lambda^\infty(\alpha)'_b\),
  \item the strong dual of any space of holomorphic functions \(\mathcal{O}(U)'_b\) where \(U\) is a Stein manifold with the strong Liouville property (for instance, for \(U = \mathbb{C}^d\)),
  \item the space of germs of holomorphic functions \(\mathcal{O}(K)\) where \(K\) is a completely pluripolar compact subset of a Stein manifold (for instance \(K\) consists of one point),
  \item the space of tempered distributions \(\mathcal{S}(\mathbb{R}^d)'_b\) and the space of Fourier ultrahyperfunctions \(\mathcal{P}'_*\) (with the strong topology),
  \item the weighted distribution spaces \((K\{pM\})'_b\) of Gelfand and Shilov if the weight \(M\) satisfies
    \[ \sup_{|x| \leq 1} M(x + y) \leq C \inf_{|x| \leq 1} M(x + y), \quad x \in \mathbb{R}^d, \]
    \(\mathcal{D}(K)'_b\) for any compact set \(K \subset \mathbb{R}^d\) with non-empty interior,
    \(\mathcal{C}^\infty(U)'_b\) for any non-empty open bounded set \(U \subset \mathbb{R}^d\) with \(C^1\)-boundary.
\end{itemize}
b) The following spaces are ultrabornological (PLS)-spaces with property (PA):
\begin{itemize}
  \item an arbitrary Fréchet-Schwartz space,
  \item a (PLS)-type power series space \(\Lambda^\infty(\alpha, \beta)'\) whenever \(s = \infty\) or \(\Lambda^\infty(\alpha, \beta)\) is a Fréchet space,
  \item the spaces of distributions \(\mathcal{D}(U)'_b\) and ultradistributions of Beurling type \(\mathcal{D}(\omega)(U)'_b\) for any open set \(U \subset \mathbb{R}^d\),
  \item the kernel of any linear partial differential operator with constant coefficients in \(\mathcal{D}(U)'_b\) or in \(\mathcal{D}(\omega)(U)'_b\) when \(U \subset \mathbb{R}^d\) is open and convex,
  \item the space \(L_b(X, Y)\) where \(X\) has (DN), \(Y\) has (Ω) and both are nuclear Fréchet spaces. In particular, \(L_b(\Lambda^\infty(\alpha, \Lambda^\infty(\beta))\) if both spaces are nuclear.
\end{itemize}
c) The following spaces are strong duals of a Fréchet space satisfying (DN):
\begin{itemize}
  \item the strong dual \(F'_b\) of any Banach space \(F\),
  \item the strong dual \(\Lambda^2(A)'_b\) of the Köthe space \(\Lambda^2(A)\) with a Köthe matrix \(A = (a_{j,k})_{j,k \in \mathbb{N}_0}\) satisfying
    \[ \exists \ p \in \mathbb{N}_0 \forall \ k \in \mathbb{N}_0 \exists \ n \in \mathbb{N}_0, C > 0 \colon a_{j,k}^{q} \leq C a_{j,p} a_{j,n}. \]
\end{itemize}

Proof. The statement for the spaces in a) and b) follows from [27] Corollary 4.8, p. 1116, [72] Proposition 31.12, p. 401, [72] Proposition 31.16, p. 402 and Remark 5.5 The first part of statement c) is obvious since Banach spaces clearly satisfy the
property \((DN)\). The second part on the Köthe space \(\lambda^2(A)\) follows from [41, Satz 12.11 a), p. 305]. □

We note that the cases that \(E\) is a Fréchet-Schwartz space or that \(E = \Lambda_{r,s}(\alpha,\beta)\) is a Fréchet space or that \(E = F'_b\) where \(F\) is a Banach space are already contained in the case that \(E\) is a Fréchet space (see [42, 4.9 Corollary, p. 21]).

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