Communication and measurement with squeezed states

Horace P. Yuen
Department of Electrical and Computer Engineering,
Department of Physics and Astronomy,
Northwestern University, Evanston, IL 60208

October 25, 2018

Abstract

The principles are elaborated which underlie the applications of general nonclassical states to communication and measurement systems. Relevant classical communication concepts are reviewed. Communication and measurement processes are compared. The possible advantages of nonclassical states in classical information transfer are assessed. The significance of novel quantum amplifiers and duplicators in communication is emphasized. A general approach is developed for determining the ultimate accuracy limit in quantum measurement systems. It is found that bandwidth or mode number is a most important parameter and ultra-high precision measurement is possible in systems with a fixed energy but many modes. The problem of the standard quantum limit in monitoring the position of a free mass is also addressed.

To appear in the book Quantum squeezing, P.D. Drummond and Z. Ficek, Springer-Verlag, to be published.
# Contents

1 Introduction 3

2 Classical communication and measurement 5
   2.1 Classical Information Transmission 5
   2.2 Signal, noise and dimensionality 8
   2.3 Communication versus measurement 12

3 Quantum communication 13
   3.1 Quantum Versus Classical Communication 13
   3.2 Mutual information 15
   3.3 The entropy bound 18
   3.4 Effect of loss 19
   3.5 Quantum amplifiers and duplicators 20

4 Ultimate limit on measurement accuracy 24
   4.1 Measurement System and Ultimate Performance 24
   4.2 Classical rate-distortion limit 26
   4.3 Ultimate quantum measurement system limit 28

5 Position monitoring with contractive states 30

References 33
1 Introduction

In this chapter I will discuss the advantages, in principle, of using nonclassical states [1] in communication and measurement situations involving classical information transfer. Most of the discussions will be concerned with the quantum states of light, in particular, the quadrature squeezed states and number states. Thus, optical terminology will be freely employed even though the principles are generally applicable to fermions also, and gravitational wave detection by a free mass will also be treated. I shall focus on the general theoretical concepts and principles underlying such applications of nonclassical states without extensive mathematical derivations, and also no review of the physics involving these states which are covered elsewhere in this book. I shall mostly avoid precise mathematical definitions and formulations, although the treatment is as precise as most standard treatments in theoretical physics or engineering science.

It is, of course, the defining characteristic of a quadrature squeezed state that the quantum fluctuation in one of its quadrature is reduced below that of a coherent state. Let \( |\alpha\rangle \) be a coherent state (CS) of an optical field mode with photon annihilation operator \( \hat{a} = (\hat{x} + i\hat{y})/2 \) so that
\[
(\Delta x)^2 = (\Delta y)^2 = 1. \tag{1}
\]

In a two-photon coherent state (TCS) \[2\] \( |\mu\nu\alpha\rangle \), which are the pure quadrature squeezed states, one obtains with a proper choice of quadrature \( \hat{x}^\theta \)
\[
(\Delta x^\theta)^2 = (|\mu| - |\nu|)^2, \quad (\Delta \hat{x}^{\theta+\pi/2})^2 = (|\mu| + |\nu|)^2. \tag{2}
\]

Since \( |\mu|^2 - |\nu|^2 = 1 \), \( |\mu\nu\alpha\rangle \) is a minimum uncertainty state on \( \hat{x}^\theta, \hat{x}^{\theta+\pi/2}, \)
\[
(\Delta x^\theta)^2 \left( \Delta \hat{x}^{\theta+\pi/2} \right)^2 = 1. \tag{3}
\]

For simplicity, let \( \mu, \nu, \alpha \) be real and \( |\mu - \nu| < 1 \), thus
\[
\langle \alpha | \hat{x} | \alpha \rangle = 2\alpha = \langle \mu\nu\alpha | \hat{x} | \mu\nu\alpha \rangle \tag{4}
\]
\[
\langle \alpha | (\Delta \hat{x})^2 | \alpha \rangle > \langle \mu\nu\alpha | (\Delta \hat{x})^2 | \mu\nu\alpha \rangle. \tag{5}
\]

This is often taken to mean that in the proper quadrature, a TCS is less noisy than a CS and so is better for communication and measurement. However, \((4)-(5)\) is not a proper justification of such an assertion.

First of all, the states \( |\mu\nu\alpha\rangle \) and \( |\alpha\rangle \), which is \( |\mu\nu\alpha\rangle \) with \( \nu = 0 \), have different energy,
\[
\langle \mu\nu\alpha | a^\dagger a | \mu\nu\alpha \rangle = |\alpha|^2 + |\nu|^2. \tag{6}
\]

It is not a priori clear that if a portion of the energy associated with the mean field \( \alpha \) is moved to increase \( (\Delta y)^2 \) so that \( (\Delta x)^2 \) is less, the overall effect is
beneficial. Assuming that a signal-to-noise ratio (SNR) criterion is appropriate for the present illustration,

\[
\text{SNR} \equiv \frac{\langle x \rangle^2}{(\Delta x)^2}
\]

it was shown [3] that TCS indeed maximizes (7) under the constraint of a fixed energy for an arbitrary state, \(\langle a^\dagger a \rangle \leq S\), with the result

\[
\text{SNR}_{|\mu\nu\alpha\rangle} = 4S(S + 1)
\]

for \(\nu = S/\sqrt{2S + 1}\) as compared to \(\text{SNR}_{|\alpha\rangle} = 4S\). Secondly, in communication with \(|\alpha\rangle\), both quadratures can be used to carry information and thus may yield a higher capacity than the use of \(|\mu\nu\alpha\rangle\) with only one quadrature, which is equivalent to using half of the available bandwidth. It turns out that for the unrestricted capacity [4], and much more so for the binary signaling capacity [5], the use of \(|\mu\nu\alpha\rangle\) does lead to improvement over \(|\alpha\rangle\). The relevant communication concepts and further details are to be discussed in the sequel. The point here is that the advantage of \(|\mu\nu\alpha\rangle\) over \(|\alpha\rangle\) is not as obvious or intuitive as it may first appear. Similarly, while number states \(|n\rangle\) and direct detection produce a noiseless system, it is discrete as compared to the in-principle continuum of states \(|\alpha\rangle\). Again, it is not a priori clear that \(|n\rangle\) would lead to a higher capacity.

The real point involving nonclassical states, I believe, is the following. Historically or typically in physics, one analyzes a given physical phenomenon and sees if it can be useful in application, whereas in engineering one often synthesizes to produce something to perform a certain function efficiently. (This opposition between analysis and synthesis is, of course, neither absolute nor pervasive in physics versus engineering.) For a long time after the laser was invented, the ideal laser state was supposed to be a coherent state, a quantum source one has to live with. Thus, all practical light sources were supposedly characterized by classical states, i.e., pure coherent states or their random superposition. However, states which are not classical, the nonclassical states, are clearly possible to have, at least in principle. In a synthesis or optimization approach, one would want to find out whether such states could lead to a better system for the application under consideration. Thus, the following questions suggest themselves in any given problem situation: What are the appropriate performance criteria and resource constraints? What are the best states or state-measurement combination one should use according to the criteria and the constraints? How much better are they compared to the conventional or standard system? The above discussion surrounding (7) and (8) furnishes an example of answers to such questions. Typically, the answer would involve quantities that are only specified mathematically, such as a TCS. If it seems worthwhile to develop such new systems, further questions on concrete physical realizations would have to be addressed. In these days of “quantum information”, such questions are even more pervasive and important.
Figure 1: Schematic representation of a classical communication system: for $U$ and $V$ capitals denote random quantities, lower case their samples but no such distinction is made for $X^{(in)}$ and $X^{(out)}$.

In the following section, I will review some basic concepts in classical communication, distinguish communication from detection, and discuss how physical measurement fits into both. In Section 3 the issues of quantum communication for classical information transfer will be explained. [Note that “quantum information” is entirely outside the scope of this chapter.] I will discuss the information capacity of nonclassical states, and the apparently only possible useful application of nonclassical states in fiber optic communication, to date — the use of nonclassical amplifiers and duplicators. In Section 4 I will discuss the use of nonclassical states in physical measurement problems, and the communication theoretic limit on the accuracy of measurements. In Section 5 the validity of the standard quantum limit for monitoring free-mass positions is addressed. Throughout I will try to explain the intuitive relevance of the various basic communication parameters, to highlight the main ideas with careful formulation but minimum details, and to dispel a few common misconceptions. Some results are also presented here for the first time.

2 Classical communication and measurement

2.1 Classical Information Transmission

For our purpose, a classical communication system can be schematically represented by Fig. 1. A source generates a classical quantity $u$, which is a member of an alphabet set $U$, $u \in U$, which may be discrete or continuous. Since $u$ is generated probabilistically according to some distribution, the corresponding random variable is denoted by $U$. The transmitter modulates $u$ onto a signal $X^{(in)}(t, u)$, which is a time-varying classical function. The channel, which usually represents all the disturbance in the system from source to destination, yields an output $X^{(out)}(t, u)$ statistically related to the input $X^{(in)}(t, u)$. The receiver processes $X^{(out)}(t, u)$ to produce an estimate $v \in U$ of $u$ to satisfy the performance criteria.

If $U$ is a finite set $\{1, \ldots, M\}$, the criterion of error probability is often employed. If $U$ is continuous, the mean-square error between $U$ and $V$ is often taken as the criterion. In both cases the system is designed, subject to whatever constraints under consideration, to minimize the error or to produce a sufficiently small error. In a communication situation, one has joint design
over the transmitter and receiver whereas in a detection situation, one is concerned only with the receiver design. Thus, in communications one may pick $X^{(in)}(t, u)$ to influence $X^{(out)}(t, u)$, and in detection one is faced with a given statistical description of $X^{(out)}(t, u)$. Clearly, communication is broader than detection. In the communication case it is important to deal explicitly with the time-sequential nature of the source output, with $u$ regarded as a sequence $u_1, u_2, \ldots, u_i, \ldots$ with corresponding $X^{(in)}_i(t, u_i)$ and $X^{(out)}_i(t, u_i)$.

The system constraints in both cases are similar. The physical transmission medium (and often together with the unavoidable disturbance in the receiver structure) specifies the channel representation, the statistical relation between $X^{(in)}(t, u)$ and $X^{(out)}(t, u)$. Constraints on the channel typically include all the physical limitations on the transmitter, the medium, and the receiver. They usually include a power or energy limitation on $X^{(in)}(t, u)$, a total time $T$ and a total bandwidth $W$ available for transmission and reception. In addition to small error, the system objectives include moderate implementation complexity, which is not always easy to quantify, and also large data rate in the case of communication.

The concept of data rate or information rate is fundamental in communication. It is usually measured in bits per second, or bits per use which is immediately converted to bits per second when multiplied by uses per second. For a data source generating one of $M$ equiprobable messages per $T$ seconds, the data rate $R$ is defined to be

$$R = (\log_2 M)/T.$$  \hfill (9)

This definition explicitly indicates that it is the number of message possibilities that characterizes the rate of a source. It immediately shows why one can have more than one bit per photon. Indeed, one can have an infinite number of bits per photon if that photon can fall into, say, one of an infinite number of different time slots. For a general statistical source, the Shannon entropy $H$ for the source is used, in bits per use of the source or bits per source symbol. A full description of communication, information, and detection theory can be found in [6]-[9]. In the present treatment, only some significant relevant points would be highlighted.

The concept of data rate (9) already forces upon us a fundamental discrete view of nature in any realistic physical process. If one can assess a true continuum, or indeed a true discrete infinity (in communications the word “discrete” often means discrete and finite), one would be able to get infinite data rate, e.g., when one can distinguish the real numbers between 0 and 1 with infinite precision. In reality, a continuum can support only a finite number of bits either from unavoidable disturbance or from the laws of quantum physics. A discussion of certain points relating to this finite/infinite dichotomy can be found in [10]. Here I would like to emphasize that communication is inherently a finite (discrete, digital) process. Any continuous quantity would finally appear in some discrete fashion in actual utilization.

Not surprisingly, the desirable goals of large data rate and small error probability are in conflict with each other. It is easy to see from the law of large
number that if one slows down the data rate, say by repeatedly sending the same message, one can decrease the error probability, indeed to zero asymptotically but with the data rate also going to zero. What is not obvious, but given by Shannon’s channel coding theorem, is that for a fixed channel representation, there is a nonzero rate called channel capacity below which one can transmit with arbitrarily small error probability by using increasingly long codes. A (channel) code is a signaling scheme in which all the signaling symbols in a sequence over many uses are processed simultaneously, which clearly makes the implementation more complex. However, long sequences have the statistical regularity given by the probabilistic description similar to the law of large number, which, e.g., implies that in a long sequence of fair coin tosses there is roughly one half heads, compared to nothing that can be said which is applicable to a single or a few tosses. It is this statistical regularity in long sequences that leads to the possibility of vanishingly small error probability with a nonzero rate as given by Shannon’s theorem.

The maximum such nonzero rate under whatever constraints and specifications on a channel is called the capacity of that constrained or specific channel, and is equal to the mutual information between the channel input and output. Referring to Fig. 1, we will later discuss the time-varying signal aspect but for the moment consider just channel input $X^{(in)}$ and output $X^{(out)}$ from alphabets X and Y with the channel specified statistically by the conditional probability $p(X^{(out)}|X^{(in)})$, $X^{(out)} \in Y$, $X^{(in)} \in X$, interpreted as a probability density or probability mass according to whether the alphabet is continuous or discrete. With an input probability $p(X^{(in)})$, the joint probability $p(X^{(out)},X^{(in)}) = p(X^{(out)}|X^{(in)}) p(X^{(in)})$ completely specifies the channel action and the mutual information $I(X;Y)$ is defined by, in the continuous case

$$I(X;Y) = \int p(X^{(in)},X^{(out)}) \log \frac{p(X^{(in)}|X^{(out)})}{p(X^{(in))}} dX^{(in)} dX^{(out)}, \tag{10}$$

and similarly, in the discrete case,

$$I(X;Y) = \sum_{X^{(in)},X^{(out)}} p(X^{(in)},X^{(out)}) \log \frac{p(X^{(in)}|X^{(out)})}{p(X^{(in))}}. \tag{11}$$

The Shannon entropy $H(U)$ of a single random variable $U$ can be defined as average self information, or

$$H(U) \equiv - \int p(u) \log p(u) du, \tag{12}$$

$$H(U) \equiv - \sum_u p(u) \log p(u) \tag{13}$$

in the continuous and discrete case. Note that (13) is always nonnegative while (12) can be negative. Shannon’s source-channel coding theorem and its converse [7,9,12] state that successive independent samples of a discrete $U$ can be transmitted over a memoryless channel $p(X^{(out)}|X^{(in)})$ with arbitrarily small (but not
exactly zero) error probability between $U$ and $V$ (see Fig. [1]) if $H(U) < I(X;Y)$, and the block error $P_e \to 1$ for $H(U) > I(X;Y)$. The case $H(U) = I(X;Y)$ forms a boundary with $P_e$ bounded away from zero in general unless the channel is noiseless. It is important to observe the conceptual distinction between a source output $U$ and a channel input $X$, even though they may happen to be the same physical quantity. Note also that a continuous alphabet channel in reality still has a finite capacity and so can reliably transmit only a discrete quantity. If $U$ is a continuous random variable, some performance criterion such as mean-square error would need to be adopted which cannot be made vanishingly small. The extent to which it can be minimized is dealt with in rate-distortion theory [7-13] discussed in Section 4. It is important to note that for a noisy channel, the use of long codes to obtain a reliable system with high rate significantly increases the system complexity, especially in the decoding operation.

The name “capacity” is usually applied to the $I(X;Y)$ maximized with respect to $p(X^{(in)})$ under whatever constraints, but it is also used to refer to whatever maximum $I(X;Y)$ obtained by different restrictions on the utilization of a given channel, e.g., under discretization (usually called quantization in the communication and signal processing literature) of the input and output of a continuous channel. The point is that with various special restrictions including a fixed $p(X^{(in)})$, a given channel would give rise to many other channels, each with its own “capacity.” Even more proliferation occurs in the quantum case. It is essential to understand the exact conditions under which a so-called “capacity” is obtained, for it is often not a really meaningful capacity in the sense of ultimate capability limit on the transmission medium or system.

2.2 Signal, noise and dimensionality

I will now try to describe qualitatively the effect of noise on data rate, finally leading to the famous Shannon capacity formula for an additive white Gaussian noise (AWGN) channel which is directly applicable to squeezed states. Let $P$ and $N$ be the total average signal and noise power of an AWGN channel represented by

$$X^{(out)}(t) = X^{(in)}(t) + n(t),$$

where $n(t)$ is the white noise. Let $W$ be the available bandwidth, i.e., the duration in frequency occupied by the signals $X^{(in)}(t)$. Then the optimizing input signals for capacity is a white Gaussian process with resulting

$$C = W \log(1 + \frac{P}{N}).$$

(15)

In terms of the noise spectral density $N_0$, one has the famous formula

$$C = W \log(1 + \frac{P}{N_0 W}).$$

(16)

Equation (15) can be derived from the mutual information expression (10), as given by Shannon [11] and later more rigorously in [7]. The intuitive reason why
(15) takes the form it does, according to such a derivation, would then have to be traced through the reason why (10) or (11) provides a general capacity for information transmission. In the discrete case (11), this can be gleaned from Shannon’s original proof in [11], and the continuous case may be viewed as a discrete limit as developed in [7]. However, a direct approach can be given for a Gaussian channel, also provided by Shannon [13], which explains the nature of various relevant quantities quite succinctly.

Consider the transmission and reception of a single continuous real variable in noise

\[ X^{(out)} = X^{(in)} + n. \]  

(17)

If \( X^{(in)} \) is restricted to an interval of length \( L \), an infinite number of bits per channel use is obtained in the absence of noise, \( n = 0 \), for any \( L > 0 \). If the noise \( n \) always has value in the interval \([−∆/2, ∆/2]\), the number of bits per use is reduced to the finite

\[ (L + ∆)/∆ \]

including edge effects. If \( X \) and \( n \) are independent continuous random variables with variances \( P \) and \( N \), or standard deviations \( \sqrt{P} \) and \( \sqrt{N} \), a crude estimate patterning after (18) would suggest that the number of amplitudes that can be well distinguished, or equivalently the number of bits per use, is

\[ \sim k\sqrt{(P + N)/N}, \]  

(19)

where \( k \) is a constant in the neighborhood of unity depending on how “well distinguished” is to be interpreted. We may recall earlier in this section it was mentioned that in a long sequence of independent trials, statistical regularity appears and provides deterministic features to the sequence. This kind of effect would indeed turn the approximate relation (19) into an exact one similar to (18). In the case of time-varying signals, this comes about in the long signal duration \( T \) limit as follows.

First of all, the collection of time functions of “approximate time duration” \( T \) and “approximate bandwidth” \( W \) span a linear space of dimension

\[ D \sim 2TW \]  

(20)

according to the Dimensionality Theorem [6], an improved version of the sampling theorem [13]. The word “approximate” above is necessary because no time function with a Fourier transform can be both strictly time-limited and strictly band-limited, but the exact definitions of “approximate” do not alter the final result [14]. The Dimensionality Theorem (20), as I discussed elsewhere [15], has momentous consequences in the description of nature. Here, it cuts down, even in the absence of noise, an otherwise infinite dimensional space to a finite dimension in a realistic system where both \( T \) and \( W \) have to be finite. Thus, a signal or time function can be viewed geometrically as a point in a finite dimensional Hilbert space. (The linear space is readily given an inner product via \( \int_T X_1^{(in)}(t)X_2^{(in)}(t)dt \).)
Figure 2: Geometric representation of the receiver: four possible signals 1 to 4 in a 2-dimensional space with corresponding minimum distance decision regions I to IV formed by the dotted lines. The additive white Gaussian noise vector $n_1$ added to signal 1 would be decoded correctly, while the noise vector $n_2$ pushes signal 1 to the decision region IV for signal 4, and would be decoded incorrectly.

In this geometric representation, the effect of an additive noise is to add a noise vector to the signal vector. The effect of a power constraint $P$ on $X^{(m)}(t)$ is to have it lie inside a $D$-dimensional sphere of radius $\sqrt{P}$ in the whole space $R^D$, the Euclidean space of dimension $D$. For white Gaussian processes, the coefficients of its expansion in any orthonormal basis are independent Gaussian random variables, with variances all given by the same quantity, the average power of the process [6-8,13]. The receiver looks at the received point $A$ in the signal space, and picks the nearest signal point in Euclidean distance to $A$ for minimizing the error probability assuming equiprobable messages. The situation is illustrated in Fig. 2.

Thus, in time $T$ there are $2TW$ independent Gaussian amplitudes from (20), and from (19) the total number of well distinguished signals is

$$M = \left[ k \sqrt{\frac{P + N}{N}} \right]^{2TW}.$$ (21)
The number of bits per second is, from (9),
\[ \log_2 \frac{M}{T} = W \log_2 k^2 \frac{P + N}{N}. \quad (22) \]

The capacity formula (15) for AWGN channel follows from (22) with \( k = 1 \).
It comes about more precisely as follows. As a result of the statistical regularity mentioned above, for large \( T \) the signals \( X^{(in)}(t) \) must almost all lie on a sphere of radius \( \sqrt{2TW} \), signal plus noise \( X^{(out)}(t) \) on a sphere of radius \( \sqrt{2TW(P + N)} \), with noise \( n(t) \) on a sphere of radius \( \sqrt{2TWN} \) centered at the original signal point. Note that the Euclidean structure of the signal space, which is absent in the general discrete case, is crucial here — the different coordinate values of, say, \( n(t) \) in the \( D \)-dimensional space are Gaussian distributed yielding the given average value \( DN \) for the norm \( \int_T n^2(t) dt \) of the \( n(t) \) vector, so that \( n(t) \) is on a sphere of radius \( \sqrt{DN} \) around the source signal point. For arbitrarily small error probability, one would want the noise spheres around different signal points to overlap arbitrarily little. A “sphere-packing” argument (see (35) below also) then readily establishes the converse to the coding theorem for (15), namely that it is impossible to transmit with arbitrarily small error probability at rates above \( C \). For the positive statement that it is indeed possible to so transmit at rates below \( C \), a “random coding” argument is required which in fact establishes the following amazing result: if the signals are selected at random, with probability one the resulting error probability is arbitrarily small. The dichotomy at \( C \), for all rates \( R < C \) the block error \( P_e \to 0 \) for almost all long codes \( (n \to \infty) \) while for rates above \( C \), \( P_e \to 1 \) for almost all long codes, is exactly like a phase transition. In practice, it turns out that long codes or signal sets that have enough structure to be readily described, encoded and decoded, do not approach capacity although the situation seems to be changing very recently. For more details of the above description see [6] and [13].

Besides communication of information, the problem of estimating a continuous entity is also of prime concern in this chapter. Consider a Gaussian random variable \( U \) with zero mean (or normalize it away) and variance \( \sigma^2 \), which is received in the form \( Au \) in Gaussian noise with a possible gain or loss \( A \), i.e.,
\[ X^{(out)} = Au + n. \quad (23) \]

This may arise in linear modulation, or in the estimation of \( U \) in any experiment. (The word “detection” is commonly reserved for the “estimation” of a discrete \( U \).) If the mean-square error \( \bar{\epsilon}^2 \) between the estimate \( V = \hat{u}(X^{(out)}) \) and \( U \) is to be minimized, the best estimate is given by [6,8] the conditional mean \( E[U|X^{(out)}] \),
\[ \hat{U}_{MMSE}(X^{(out)}) = \frac{X^{(out)}}{1 + N/\sigma^2 A^2}, \quad (24) \]

where \( N \) is the noise variance, with resulting
\[ \bar{\epsilon}^2 = \frac{\sigma^2}{1 + \sigma^2 A^2/N} = \sigma^2 (1 + \frac{S}{N})^{-1} \quad (25) \]
in terms of the a priori variance \( \sigma^2 \) and a signal-to-noise ratio \( S/N \).

In the physics literature on quantum information and its applications, the criterion of mutual information is often used in place of detection or estimation error in situations (such as cryptographic eavesdropping) for which no coding is possible. Depending on the problem, the best possible outcome of such use would be a bound rather than the desired performance criterion.

### 2.3 Communication versus measurement

The estimation/detection problem clearly parallels the problem of producing an estimate of a desired quantity \( u \) from the measured data \( X^{(out)} \) in a physical experiment. More generally, in an estimation problem one is given a fixed statistical specification \( X^{(out)}(t, u) \) and forms in general a nonlinear estimate \( \hat{u}(X^{(out)}) \) so that a cost function \( C(\hat{U}, U) \) is minimized. For mean-square error,

\[
C(\hat{U}, U) = \mathbb{E}\epsilon^2 = \mathbb{E}[|U - \hat{U}|^2],
\]

where the expectation \( \mathbb{E} \) is taken over all the random quantities involved. In a communication situation, the channel is a statistical transformation \( F \) on the input \( X^{(in)}(t, u) \)

\[
X^{(out)}(t, u) = F[X^{(in)}(t, u)],
\]

which reads, for an additive noise channel,

\[
X^{(out)}(t, u) = X^{(in)}(t, u) + n(t)
\]

for which one can control \( X^{(in)}(t, u) \) subject to the system constraints. In contrast to the estimation case, a direct optimization approach to a communication problem with joint transmitter/receiver optimization has never been developed in a useful way. Instead of asking for the optimum system for a fixed time duration \( T \), \( T \) itself is floated as a design parameter in the development of channel encoding-decoding design subject to Shannon’s coding theorems.

It can be seen that a physical measurement is generally not just an estimation problem, because \( X^{(out)}(t, u) \) can be influenced to some extent through the choice of the physical measurement process, although perhaps not as much as controlling \( X^{(in)}(t, u) \) in (27). In particular, it is a major part of the measurement system design to find an appropriate physical variable \( X^{(in)} \) to couple to the desired information parameter \( u \) to form \( X^{(in)}(t, u) \) for information extraction after the corruption of \( X^{(in)}(t, u) \) to \( X^{(out)}(t, u) \) by the “channel” is taken into account. However, there is usually no question of data rate in a measurement. Thus, physical measurements, which are of prime concern to us, are described somewhere between communication and detection/estimation. This situation already obtains in classical measurements, and in cases of fixed quantum states and quantum measurements. It becomes more so in quantum communications where the quantum states and quantum measurements can be freely chosen. As developed in Section 4, the feasibility of choosing quantum states moves a physical measurement problem away from being a pure estimation problem to becoming more like a communication problem, although it never fully becomes a standard communication problem.
3 Quantum communication

3.1 Quantum Versus Classical Communication

By “quantum communication” we mean more than the study of quantum effects in communication systems involving classical information transfer. Specifically, in quantum communications we are concerned with the system performance under a variety of different quantum measurements and quantum states. Referring to Fig. 3, the statistical specification of the channel plus transmitter, e.g., is given by a conditional probability \( p(X^{\text{out}}|u) \). The classical variable \( X^{\text{out}} \) may well be of quantum origin, say it is the eigenvalue of a quantum observable obtained in a measurement. However, as far as the analysis of this system is concerned, the fact that \( p(X^{\text{out}}|u) \) arises from quantum mechanics makes no difference, and it would proceed just like a classical communication system. If this \( p(X^{\text{out}}|u) \) arises from a quantum state \( \rho(u) \) and a quantum measurement of a selfadjoint \( X^{\text{out}} \) with eigenstates \( |X^{\text{out}}\rangle \), \( p(X^{\text{out}}|u) = \langle X^{\text{out}}|\rho(u)|X^{\text{out}}\rangle \), one may well ask whether other possible choices of \( \rho(u) \) and observable with resulting different \( p(X^{\text{out}}|u) \) may lead to better performance. These additional freedoms of quantum measurement and quantum state selection are absent in a classical communication system. They constitute the new content of quantum communication.

A general quantum communication system is depicted schematically in Fig. 3. The channel input and output signals \( \hat{X}^{(\text{in})}(t) \) and \( \hat{X}^{(\text{out})}(t) \) are now field operators in quantum states \( \rho(u) \) and \( \rho_R(u) \).

Generally, as indicated by the Dimensionality Theorem (20), a finite number of modes each with two degrees of freedom, such as an optical mode with two quadratures, would suffice so that the density operators \( \rho \) and \( \rho_R \) are well-defined. A specific classical statistical characterization of the system would result upon a choice of quantum measurement at the receiver. The most general characterization of a quantum measurement is the so-called “completely positive operation measure” with the corresponding measurement statistics given by a positive operator-valued measure (POM) [16]. Let \( \hat{O}(X^{\text{out}}) \) be a POM on
\[ \hat{X}^{(out)}(t), \text{ the channel output, thus} \]
\[ \sum_{X^{(out)}} \hat{O}(X^{(out)}) = I \quad \text{or} \quad \int \hat{O}(X^{(out)}) dX^{(out)} = I \quad (29) \]

and each \( \hat{O}(X^{(out)}) \) is a nonnegative selfadjoint operator, with \( \hat{O}(X^{(out)}) = |X^{(out)}\rangle \langle X^{(out)}| \) for orthogonal \( |X^{(out)}\rangle \) in the case of a selfadjoint observable. The statistics are given by
\[ p(X^{(out)}|u) = \text{tr} \rho_R(u) \hat{O}(X^{(out)}). \quad (30) \]

The output state \( \rho_R(u) \) is determined by the channel action on the input state \( \rho(u) \). The additional quantum “freedoms” in quantum communication consist in the selection of \( \hat{O}(X^{(out)}) \) and \( \rho(u) \). Note that it may be more convenient, as in the case of frequency modulation, to enter the information variable \( u \) in \( \hat{X}^{(in)}(t) \) in parallel with the classical case, and specify the transmitter in terms of \( \hat{X}^{(in)}(t, u) \) and \( \rho \) rather than \( \hat{X}^{(in)}(t) \) and \( \rho(u) \). In this formulation, the quantum state \( \rho \) and the classical modulation process are decoupled. However, if the information \( u \) enters through the quadratures, it would be necessary to use \( \rho(u) \), for which the quantum state selection and modulation selection are tied together.

Historically, the serious study of optical communication began immediately after the laser was experimentally realized, for which quantum effects are clearly important as \( h\omega/k \sim 10^4 K \). While the evaluation of system performance went on for coherent states and the three standard measurements: direct, homodyne, and heterodyne detections, quantum communication theory in our sense was also developed. Forney [17] and Gordon [18] proposed the entropy bound for information transfer with a fixed set of states valid for arbitrary measurement, to be discussed in section 3.3. Helstrom [19] studied the quantum measurement optimization problems in the spirit of classical detection/estimation theory, which were further developed by Holevo [20] and Yuen [21]. Each of the above three standard measurements corresponds, respectively, to the quantum measurement of photon number, single field quadrature, and joint quadratures described by a POM but not a selfadjoint operator [22]. In such work, which actually has many applications in physics [23] but will not be further discussed in this chapter, the states are fixed and the quantum measurement is selected so that the resulting classical statistical system leads to the best possible performance compared to other measurements. The issue of quantum channel representation was treated [4,24,25] and the possibility of receiver state control is suggested [2,4,25,26,27]. The general problem of transmitter quantum state selection was considered by Yuen [3,4], leading to the development of TCS as indicated in Section 1. The question of optimal state influence on channel capacity was also implicit in connection with the application of the entropy bound, indicating that number states and photon counting are best for free boson fields [4,28]. For recent advances in determining the capacities and error exponents of various quantum channels by Holevo and the Hirota group, see [29-31]. For other advances including work on quantum tomography by the D’Ariano group, see [32-34]. For applications of squeezed states to quantum cryptography, see [35, 36].
3.2 Mutual information

The capacities, or mutual informations maximized over the input distributions, for various boson channels are discussed extensively in [37]. Here I would like to focus on five capacities for the narrowband free boson channel under an average power constraint: number state and photon counting, TCS and homodyning, coherent state and the three standard measurements. Hopefully, it would become clear within this and the next subsection that they are the most important cases capturing the essence of the situation.

For the free electromagnetic field at optical frequencies, all the current or foreseeable future systems are narrowband, i.e., the available bandwidth is only a small fraction of the center frequency. Due to various facts of nature, it would be extremely difficult and inefficient to utilize photons at higher frequencies, say X-rays, in a communication situation. Thus, there is no practical significance in studying wideband photonic channels. The constraint of average power can be separated into two parts: average with respect to the statistics of the information variable $U$ and average with respect to the quantum nature of the state $\rho$. In either case a peak power (or energy or power spectral density) constraint can also be applied. In the case of classical signals the peak power constraint is indeed quite meaningful and realistic, but is often hard to handle mathematically and usually avoided. In the case of quantum states, a peak energy constraint would cut off the Hilbert space of states $\mathcal{H}$ at a maximum number state eigenvalue $n_m$ so that $\mathcal{H}$ becomes finite-dimensional. This, however, is unrealistic or at least hard to handle in so far as one considers a coherent state $|\alpha\rangle$, which has components in all $|n\rangle$, to be realizable. Some discussion on this issue is given in [10], although in its full scope it is a complicated and profound issue. Here I would advocate, if only on the ground of mathematical convenience, that energy constraint is to be applied to the quantum state average $\text{tr}\rho a^\dagger a$, and not to yield an $n_m$.

Let $P = h f_0 WS$ be the available signal power of a narrowband channel of center frequency $f_0$ and photon numbers $S$ per mode. The photon number capacity is [28,37,38]

$$C_{op} = W[(S + 1) \log(S + 1) - S \log S].$$

(31)

For TCS with homodyne detection [4,37],

$$C_{TCS} = W \log(1 + 2S).$$

(32)

If both quadratures of the TCS are utilized under the same power constraint with optimized TCS-heterodyne [22] or joint quadrature measurement, it can be shown from the Kuhn-Tucker optimality conditions of nonlinear programming that as $S$ is increased from 1 the optimum capacity is indeed achieved through utilization of only one quadrature. The coherent state heterodyne and homodyne capacities are

$$C_{het} = W \log(1 + S),$$

(33)

$$C_{hom} = \frac{W}{2} \log(1 + 4S).$$

(34)
Figure 4: Comparison of capacities in bits per second for $f_0 \sim 5.7 \times 10^{14}$ Hz and $W \sim 10^{14}$Hz.

Equations (32)-(34) are easily derived from (15) because the corresponding channels are AWGN ones — the fluctuations in a TCS or a coherent state, which is merely a classical amplitude superposed on vacuum, are behaving as independent additive Gaussian noises, and are white noises under the narrowband assumption. The coherent-state photon counting capacity $C_{ph}$ does not have a simple closed form but is readily computed numerically [39]. These five capacities are compared numerically in Fig. 4 reproduced from [4], for a fairly wide bandwidth. It may be observed that $C_{TCS}$ is always larger than $C_{het}$ and $C_{hom}$, and is also larger than $C_{ph}$ in the case of more than a fraction of a photon per mode.

However, the difference between $C_{TCS}$ and $C_{het}, C_{hom}$ is not big. Indeed, $C_{TCS}$ is less than twice $C_{hom}$ although the SNR of TCS is the square of that for coherent states. Because the data rate for a mode goes as $\log(1+\text{SNR})$ from
(15), the square in SNR becomes less than a multiplicative factor of 2. The difference between \( C_{\text{TCs}} \) and \( C_{\text{het}} \) is even less, (32) is equivalent to doubling the signal power in \( C_{\text{het}} \) with the same bandwidth. The underlying reason can be understood as follows. In the geometric representation of signal and noise sketched in section 2.2, it can be seen that the effect of noise is to move a given signal point away from its position. If the noise is big enough, it would move it closer to another signal point B as compared to the original point A, and the optimum receiver would decide it is this other signal B that was transmitted, hence making an error, as illustrated in Fig. 2. Thus, a good system would have the signal points as far apart as possible from the viewpoint of errors, and have as many signal points as possible from the viewpoint of data rate, two conflicting goals. For a fixed dimension \( D \sim 2WT \), a larger power \( P \) yields a larger sphere and the same number of \( M \) signal points can be placed further apart inside the sphere, leading to a smaller error for a fixed noise power \( N \). Increasing \( W \), however, is more beneficial than increasing \( P \), thus \( C_{\text{het}} > C_{\text{hom}} \) as \( W \) increases, even though \( C_{\text{hom}} \) has a bigger SNR. To see the role of \( W \) versus \( P \), recall the discussion around (21) and (32) that one wants the noise spheres around different signal points to be almost nonoverlapping to yield small error probability. As a result of this “sphere packing,” the number of well distinguished signals is roughly the ratio of the signal plus noise volume to the noise volume. The volume \( V_{D}(r) \) of a \( D \)-dimensional sphere of radius \( r \) is \( B_{D} r^{D} \) for a \( D \)-dependent constant \( B_{D} \), which implies

\[
V_{D}(\sqrt{D(P+N)}/V_{D}(\sqrt{DN}) = (1 + \frac{P}{N})^{D} \tag{35}
\]

since the radii of the signal plus noise and noise spheres is \( \sqrt{2TW(P+N)} \) and \( \sqrt{2TWN} \) respectively. The quantity \( \delta \) grows exponentially in \( D \) or \( W \) but only to a fixed power in \( P \). This more important role of \( W \) versus \( P \) clearly manifests in (15) and (16).

Having understood why the apparent large gain in SNR given by (3) for TCS leads only to a small gain in capacity, the question becomes whether TCS would be significant in improving optical communications compared to coherent states. This rest of this section (3) is devoted to a detailed examination of this issue. We may first observe that complicated coding, especially the decoding process, is required to approach capacity given by any of (32)-(34). If one looks at the error behavior of information transfer under specific simple signaling scheme, e.g., the antipodal signals discussed in [5], the full SNR square advantage may appear. That is, more restricted “capacities” than (32)-(34) may show a large advantage with TCS. In Section 3.3 we will see that the number state capacity \( C_{\text{op}} \), which is so close to \( C_{\text{TCs}} \), is actually the optimum rate for any states and measurements subject to the average power constraint. This capacity \( C_{\text{op}} \) can be obtained \textit{without} the need for complicated decoding because the ideal number state channel is noiseless — there is no need to use long sequences to yield statistical regularity. Thus, the use of number states can be considered as an \textit{alternative to channel coding}. Number states, as intensity squeezed states, have a lot of similarity to TCS in regard to their physical generation and propagation.
characteristics. Unfortunately, the use of such nonclassical states as information sources would not be advisable in practical communication systems. In addition to various problems of a more practical nature, such as phase coherence for TCS and good detectors for number states, the inevitable presence of significant loss would wipe out the advantage of nonclassical states. This issue will be treated in section 3.4 after the following discussion of the entropy bound that established the optimality of $C_{op}$ given by (31).

### 3.3 The entropy bound

Given a fixed set of density operators $\rho_\lambda$ dependent on a discrete or continuous random variable $\Lambda$ with probability (density) $p(\lambda)$, define

$$\mathcal{P} \equiv \sum_\lambda p(\lambda) \rho_\lambda \quad \text{or} \quad \int p(\lambda) \rho_\lambda d\lambda. \quad (36)$$

Let $S(\rho) \equiv -tr \rho \log \rho$ be the Von Neumann entropy of $\rho$, and let $\hat{O}(X^{(out)})$ be the POM giving the measurement probability. Then the mutual information between $\lambda$ and $X^{(out)}$ is bounded by

$$I(\Lambda; Y) \leq S(\mathcal{P}) - S(\rho_\lambda) \quad (37)$$

$$S(\rho_\lambda) = \sum_\lambda p(\lambda) S(\rho_\lambda) \quad \text{or} \quad \int p(\lambda) S(\rho_\lambda) d\lambda. \quad (38)$$

This entropy bound (37), first given by Forney [17] and Gordon [18], was proved for finite discrete $\Lambda$ and finite dimensional Hilbert space $\mathcal{H}$ by Zador [40] and independently by Holevo [41], and general $\Lambda$ and infinite dimensional $\mathcal{H}$ by Ozawa [38]. The long complicated history of this bound is outlined in [10]. Recently, the inequality in (37) is shown to be achievable if the measurement can be made over a long sequence of states instead of just symbol by symbol [42,43]. However, while this may be considered to establish the capacity of a quantum channel defined by the mapping $\lambda \mapsto \rho_\lambda$, such a specification of a quantum channel is neither general nor practical. The main reason is that there is no way to tell whether the particular map $\lambda \mapsto \rho_\lambda$ is optimum under the constraint of the problem. As we have emphasized in section 3.1, a general quantum communication problem involves both the choice of states and measurements. Under an average energy constraint for a single mode,

$$\sum_\lambda p(\lambda) tr \rho_\lambda a^\dagger a \leq S \quad (39)$$

one cannot tell a priori what the optimal $\lambda \mapsto \rho_\lambda$ should be, even if one assumes all $\rho_\lambda$ are coherent states.

On the other hand, the bound (37) in its full generality readily shows [38] that for a single boson mode under (38), the maximum $I(\Lambda; Y)$ is achieved by taking $\lambda$ and $X^{(out)}$ as a nonegative integer, with number states $\rho_{\lambda=n} = \ldots$
and the $I$ value given by (31) for $W = 1$. The wideband capacity can be similarly derived [38]. Note that apart from showing that more general processing such as feedback would not increase $I$, this joint optimization over state/measurement and modulation (the map $\lambda \mapsto \rho_\lambda$) demodulation (the map $X^{(\text{out})} \mapsto \lambda$) does establish that (31) is the ultimate quantum limit on the possible rate of information transfer for a boson mode of average energy $S$.

### 3.4 Effect of loss

The effect of linear loss on a mode can be represented as a transformation on the modal photon annihilation operator [2,4,44-46],

$$b = \eta^{1/2} a + (1 - \eta)^{1/2} d,$$

where $\eta$ is the transmittance and the $d$-mode is in vacuum. Note that the effect of quantum efficiency on a detector can be so represented as well. It follows immediately from (40) that the resulting quadrature fluctuation in $b$ has a floor level $(1 - \eta)/4$, which is essentially the coherent state noise level for $\eta \ll 1$.

Similarly for a number state, the $b$-mode photon number fluctuation

$$\langle \Delta N_b^2 \rangle = \eta^2 \langle \Delta N_a^2 \rangle + \eta(1 - \eta) \langle N_a \rangle$$

contains a partition noise equal to the mean $\langle N_b \rangle = \eta \langle N_a \rangle$ for $\eta \ll 1$, washing away the sub-Poissonian character of the $a$-mode. Generally, one can readily show from (40) that the state $\rho_b$ is very close to a coherent state of mean $\eta^{1/2} \langle a \rangle$ for large loss, thus any nonclassical state becomes essentially classical.

The implication of this fact on the utility of nonclassical states is profound, especially in engineering applications where significant loss is usually present, e.g., in fiber optic communications. Unless a special environment is created [4] to compensate for the squeezing or nonclassical effect in the presence of loss, there is no way to keep a nonclassical state at the reception end. While this is possible in principle, it seems that is not worth the trouble. Even in scientific experiments or in the process of nonclassical state generation, loss places a severe limit on the amount of squeezing obtainable. The sensitivity of nonclassical states to loss and interference would place strenuous requirements on all the system components, making any such system extremely difficult to implement. To me, a similar kind of argument leads to a similar implication in the field of quantum information.

Given the close value of $C_{\text{TCS}}$ to $C_{\text{het}}$ in (32)-(33) in the absence of loss, it should be clear that there is hardly any advantage left in the presence of loss. While the ultimate quantum capacity $C_\ell$ of a lossy channel is not known, an upper bound on $C_\ell$ can be easily derived. Under an average energy constraint $S$ and loss $\eta$, equation (32) for $C_{\text{op}}$ with $S$ replaced by $\eta S$ would provide a bound on $C_\ell$. The gap between $C_{\text{op}}$ and $C_{\text{het}}$ with $\eta S$ is the largest gain, probably not actually achievable, that one can possibly obtain with nonclassical states in a lossy channel. The smallness of this gap, as seen from Fig. 4, shows that there is little significance in pursuing quantum communications with nonclassical sources in practice, a conclusion I drew over twenty years ago.
3.5 Quantum amplifiers and duplicators

Not all is lost, however. As to be presently explained, the use of novel quantum amplifiers and related devices on coherent state sources can lead to a number of significant communication applications not possible with the usual phase-insensitive linear amplifier (PIA). A characteristic feature of these novel devices is that their outputs are often nonclassical states for coherent state inputs, even though it is not the nonclassical nature of these states that is relevant in the application.

Corresponding to the three standard quantum measurements are three quantum amplifiers, the photon number amplifier (PNA), the phase-sensitive linear amplifier (PSA), and the PIA. If $b$ and $a$ are the output and input modal photon annihilation operator of the amplifier, these three amplifiers can be represented as [45-48], with a power gain $G > 1$,

\[
\text{PIA} \quad b = G^{1/2}a + (G - 1)^{1/2}v^\dagger, \quad [v, v^\dagger] = 1 \tag{42}
\]

\[
\text{PSA} \quad b_1 = G^{1/2}a_1, \quad b_2 = G^{-1/2}a_2 \tag{43}
\]

\[
\text{PNA} \quad b^\dagger b = Ga^\dagger a, \quad G \text{ integer} \tag{44}
\]

A fourth quantum phase amplifier [49,50] is

\[
\text{QPA} \quad e_+ = e_+^G, \quad e_+ = (a^\dagger a + 1)^{1/2}a^\dagger, \quad \text{integer} \tag{45}
\]

which is related to the ideal phase measurement [19,21,51] described by a POM involving the Susskind-Glogower states and corresponding phase-coherent states [51].

| DETECTION | AMPLIFIER | STATES | DUPLICATORS |
|-----------|-----------|--------|-------------|
| heterodyne| PIA       | CS     | BQD         |
| homodyne  | PSA       | TCS    | SQD         |
| direct    | PNA       | NS     | PND         |
| phase(ideal) | QPA   | PCS    | QPD         |

(The column on states merely emphasizes that the nature of these states would be preserved only by the corresponding amplifier, not that the amplifier is noiseless only for those states.)
In (42) and (44), the photon operator $b$ has to be defined on two modes. For a fuller discussion of these amplifiers, see [48] and [49] which also contains an extensive treatment of duplicators to be discussed later in this section. The main point about (42)-(44) is that the amplifier output of each is, for the corresponding measurement, a perfect “noiseless” scaled (amplified) version of the input for arbitrary input state, i.e., they are noiseless amplifiers for the corresponding detection scheme. Thus, the often found statement that quantum amplifier necessarily introduces noise, say in the sense of having a noise figure $F \equiv \text{SNR}_a/\text{SNR}_b > 1$, is wrong. As summarized in Table 1, if the proper amplifier matching the measurement is used there is no additional noise ideally, similar to the classical case. All the noise then arises inherently from the quantum nature of the input. (This is also true in both balanced and unbalanced homodyne/heterodyne detection for which the effective amplifier, the local oscillator, introduces no noise in the high gain limit. See [52]. It is a pervasive misconception that the noise in homodyne/heterodyne detection is local-oscillator shot noise.) Without going into a detailed exposition, this is actually clear intuitively from the basic principles of quantum physics. When you fix a measurement, the situation is classical for any given state as discussed in Section 3.1 on quantum vs. classical communication, in the sense that a fixed probabilistic description is obtained. The situation is a little more subtle in the case of POM rather than selfadjoint operator, but can be understood by analyzing the POM as commuting selfadjoint operators measurement on an extended space which can always be done [20].

The generation mechanism of PSA is identical to quadrature squeezing, which, being piecewise linear, is not exactly a nonlinear effect. On the other hand, PNA, QPA and the duplicators involve truly nonlinear quantum effects [47-50] which would not be discussed here. None of these new quantum devices except PSA has been successfully demonstrated experimentally in a useful manner.

At this point, I would like to address a confusing point about the capability of
amplifiers. It is often stated that an amplifier at the receiver could improve the receiver performance. The optimum receiver performance is determined by the specification \( \hat{X}^{(out)}(t), \rho(u) \) in Fig. 3. Nothing, and certainly no amplifier, can ever improve that as a matter of tautology. What can be improved is a specific receiver structure that does not lead to the optimum performance. In such a case, the use of an amplifier or some other device may improve the suboptimum receiver performance. This point is related to, but different from, the so-called data processing theorem [7] in information theory which shows that no processing can increase the information transfer over a channel.

The above amplifiers can be used as pre-amplifiers to suppress subsequent receiver noise in the corresponding detections, in either engineering or scientific applications. They can also be used to advantage [53] in the attempt to create a transparent optical local area network. For such a purpose, however, the duplicators [46-48,54] would be perfect. A photon number duplicator (PND) is a device with one input \( a \) in state \( \rho_a \) and two outputs \( b, c \) such that each of the output photon counting statistics is the same as that of the input

\[
\langle n|\rho_a|n \rangle = \langle n|\rho_b|n \rangle = \langle n|\rho_c|n \rangle.
\]

Typically, the output photon counts for the \( b \) and \( c \) modes are perfectly correlated, thus PND also provides a perfect realization of a photon number quantum nondemolition measurement (QND) with only a finite energy [47]. Single and double quadrature duplicators can be similarly described.

The amplifiers can be used as line amplifiers in long distance optical fiber communications. For example, the use of PSA not only improves the SNR by a factor of 2 for coherent state sources in a long amplifier chain, it also significantly reduces the Gordon-Haus soliton timing error [55]. Considerable experimental progress [56] has been made on such possible application, but the required phase coherence renders it impractical. For on-off signals, the use of PNA leads to the following error probability

\[
P_e = \frac{1}{2} \exp\{-S[1 - f_n(G)]\},
\]

where the functions \( f_n(G) \) obey the recurrence relation

\[
f_{n+1}(G) = (1 - G^{-1})^G [1 + (G - 1)^{-1} f_n(G)]^G
\]

with \( f_0(G) = 0 \). Equations (47)-(48) apply to a chain of \( n \) amplifiers of gain \( G \) and loss \( G^{-1} \) between two adjacent amplifiers, assuming direct detection. In Fig. 3, this error exponent \( 1 - f_n(G) \) is compared with that of the PIA line, \( \frac{1}{4n} \), obtained under the Gaussian approximation for direct detection.

As can be seen in the figure, even more improvement, in fact the optimum improvement, is obtained with the use of a photon on-off amplifier [57] (POA) tailored for the situation. In the state description, a POA acts on two modes but for the input mode it reads

\[
\begin{align*}
|0\rangle & \quad \mapsto \quad |0\rangle \\
\text{POA} & \quad |1\rangle \quad \mapsto \quad |\alpha\rangle
\end{align*}
\]
Figure 5: Comparison of error exponents $S^{-1}\ln 2P_e$ as a function of stages $n$ — the PIA line exponent is independent of $S$ and $G$, the PNA exponent is independent of $S$ and the POA exponent is independent of $G$.

$$|n\rangle \mapsto |\alpha\rangle \mapsto$$

where $|\alpha\rangle, |0\rangle$ are the two on-off coherent states, $S = |\alpha|^2$. The resulting error probability is

$$P_e = \frac{1}{2} \left[ 1 - (1 - e^{-S})^n \right],$$

which is the same as that obtained by a repeater, i.e., by direct detection and retransmission at each of the $n$ stages. In general, it is possible to write down a perfect quantum amplifier for any given signaling and detection scheme which performs as well as a repeater, although the actual installation of POA or any such amplifiers in a long line would entail the loss of flexibility, as compared to PNA, for adapting to other signaling schemes.

Quantum amplifiers are also useful in quantum cryptography [48]. A major problem of the quantum cryptographic schemes is that they cannot be amplified to compensate for the loss without disrupting the operation of the scheme. In [58] a new quantum cryptographic scheme is introduced that allows amplification, which greatly extends the distance over a fiber for which the scheme works.
4 Ultimate limit on measurement accuracy

4.1 Measurement System and Ultimate Performance

In this section, the question is addressed on the ultimate, quantum as well as classical, limit on the measurement accuracy obtainable with various measurement systems. The optimum performance ideally achievable with a measurement system is of course an important piece of design information, but more importantly I would like to assess the potential of such systems, and ways to realize them in principle, in order to explore the feasibility of developing ultra-high precision measurement systems important in many applications, especially scientific ones. My approach [58] is based on the communication characterization of measurement discussed so far, especially in section 2.3, while adopting quantum and classical communication theory to provide the answers. Since the correspondence between communication and measurement is not exactly isomorphic, we will find that it is possible to obtain limits on the measurement accuracy, but not always possible to be assured that those limits are attainable. Indeed, even if the correspondence is perfect, there are still additional questions, such as what systems are actually available, that would resist a complete mathematical characterization in the foreseeable future. Nevertheless, as to be discussed presently, some of the results obtained are somewhat surprising, and also promising. In the next section 2.2 the rate distortion limit in classical communication theory will be explained, and in section 4.3 the corresponding quantum limits will be presented. Here I would like to first highlight the results and their implications.

The final error in a measurement system may depend, even in principle excluding nonideal environmental perturbations, on more than a single source or variety. For example, in the detection of very weak gravitational radiation by a Michelson interferometer, the radiation pressure error needs to be added to the photon detection error to form the total error. The application of squeezed states in this situation is treated elsewhere in this book and would not be discussed. Here, the general theory would be illustrated only with a measurement medium or channel that can be characterized as a free boson field, so that the results in section 3 may be utilized. The general approach, however, is applicable to any specific measurement system.

Consider the problem of estimating a parameter \( U \) with Gaussian density \( p_G(u) \) of zero mean and variance \( \sigma^2 \) via a single mode optical field of average energy \( S \). While the optimization of (4) yields TCS as the solution, two choices have already been fixed in advance: the parameter \( u \) is to be modulated into the mean \( \alpha_1 \) of the state, and homodyning or measurement of \( \alpha_1 \) is to be performed. If one relaxes these two conditions in accordance with the general quantum communication approach of section 3.1, one may pick a state \( \rho(u) \) subject to

\[
\int du p_G(u) \text{tr} \rho(u) a^\dagger a \leq S \tag{51}
\]
and a general measurement represented by the POM $\hat{O}(y)$, so that the mean-square error $\epsilon^2$ between $y$ and $u$ is minimized. It is not clear at all that the combination of linear modulation, TCS and homodyne is the optimum solution or yields a near optimum performance to this problem. As the following development shows, a lower bound for the root-mean-square error $\delta u \equiv (\epsilon^2)^{1/2}$ under (51) can be derived

$$\delta u \geq \sigma \frac{S^S}{(S + 1)^{S+1}} \sim \frac{\sigma}{\epsilon S}, \quad S \gg 1,$$

which is very close to the TCS linear modulation performance,

$$\delta u_{TCS} = \frac{\sigma}{2S + 1} \sim \frac{\sigma}{\epsilon S}, \quad S \gg 1.$$  \hspace{1cm} (53)

Partly because it is not even clear whether the lower bound (52) can indeed be achieved, one would ordinarily be quite satisfied with the difference between $1/2$ and $e^{-1}$ and stop looking for another system unless the TCS system is not practical for whatever reason. One may say the linear TCS system is essentially optimum. The corresponding coherent state performance is $\delta u' \sim \sigma/\sqrt{S}$.

For a uniformly distributed phase parameter $\phi \in [-\pi, \pi)$, the corresponding lower bound for the root-mean-square error is

$$\delta \phi = \lambda \frac{S^S}{(S + 1)^{S+1}} \sim \frac{\lambda}{\epsilon S}, \quad S \gg 1,$$

where $\lambda \sim 1.35$. This single-mode $1/S$ behavior, improved over the $1/\sqrt{S}$ dependence for coherent state, has been obtained previously for two different concrete systems utilizing TCS [60,61] and number states [62]. This should not be surprising given the closeness between the number state and TCS capacities, (31)-(32).

In the case of a narrowband field with $m = D/2 = WT$ modes but the same total energy or photon number $S$, the lower bound for the measurement of a Gaussian $U$ is

$$\delta u \geq \sigma \frac{S^S m^m}{(S + m)^{S+m}} \sim (1 + \frac{m}{S})^{-S} e^{-S}, \quad m \gg 1.$$ \hspace{1cm} (55)

For the uniform phase $\phi$, $\delta \phi$ is given by (54) with $\sigma$ replaced by $\lambda$ as in (52) and (54). Note that (54) goes to zero as $D \to \infty$. This is because infinite capacity is obtained when the narrowband expression is extrapolated to infinite bandwidth, which is not physically meaningful as the $hf$ dependence becomes important [21,37]. The capacity is finite when such dependency is taken into account [28,37-38].

The result (52) or (54) is rather remarkable. For the same total energy $S$ spread over a large number of modes, the performance can be improved from $1/S$ to $e^{-S}$ assuming the bound can be approached. As a communication limit
for which one can control the modulation, this is certainly the case. Intuitively, it can be traced to the relative importance of $D$ or $W$ over $P$ or $S$ as discussed in sections 2.1 and 3.2. As the quantum state affects only the SNR and not $D$, one may expect such behavior for coherent states also, which is indeed born out to be the case as developed in section 4.3. Even in the measurement situation, the improvement of a fixed energy spread over many modes is a real one, to be demonstrated for a concrete frequency modulation scheme also developed in section 4.3.

(An aside between multimode and single mode results. It is sometimes said that two-mode squeezing is basically different from single mode squeezing because two modes are needed. However, by a simple modal transformation equivalent to removing cross terms in the multimode hamiltonian, two-mode squeezing can be reduced to single mode squeezing, that is, in the multimode situation one picks the right mode that yields squeezing. The general theory is given in [63,64], which indeed was what led to the prediction of squeezing in degenerate four-wave mixing [65].)

### 4.2 Classical rate-distortion limit

The rate-distortion function $R(d)$ of a random variable $U$ was introduced by Shannon [11], with by now a very extensive literature. Here we consider just continuous $U$ with density function $p(u)$ although discrete $U$ works the same. For a distortion measure $d(u,v)$ between $u$ and $v$, such as $|u-v|^2$ or $|u-v|$, the average distortion is

$$E[d(U,V)] \equiv \int d(u,v)p(u)p(v|u)dudv. \quad (57)$$

The rate distortion function $R(d)$ of $u$ is defined to be the minimum mutual information

$$R(d) \equiv \min_{E[d(U,V)] \leq d} I(U;V) \quad (58)$$

over all possible choices of $p(v|u)$ subject to the constraint that $E[d(U,V)]$ is less than or equal to a given level $d \geq 0$. One may think of $V$ as a data-compressed version of $U$ — $V$ represents $U$ with an average distortion $d$, thus it takes less bits to represent $V$ than $U$ for $d > 0$. Shannon’s source coding theorem with a fidelity criterion and its converse [7,9,12] state that a source variable $U$ can be asymptotically represented with an average distortion $d$ if and only if at least $R(d)$ bits per source symbol is provided. Similar to channel coding, long sequence encoding and decoding that ensures statistical regularity are in general required to achieve such minimum in the asymptotic limit. Nevertheless, roughly speaking $R(d)$ is the minimum number of information bits per symbol required to represent a source with an average distortion $d$ per symbol.

The channel capacity $C$ may be written as a function $C(\beta)$ where $\beta$ denote the resource parameters available, including power and bandwidth, as well as other characteristics of the channel such as noise power. Referring to Fig. 1, the
question arises on the minimum distortion $d$ one can obtain for transmitting a source variable $U$ over a channel with capacity $C(\beta)$. The answer is provided by Shannon’s joint source-channel coding theorem [7,9,12] in the so-called rate distortion limit or rate distortion bound. Recall that roughly speaking, the channel coding theorem says that $C(\beta)$ is the maximum number of information bits one can transfer error-free over a channel with parameters $\beta$. By combining the source and channel coding theorem, one has $C(\beta) \geq R(d)$ so that, since $R$ is a monotone decreasing function of $d$,

$$d \geq R^{-1}(\beta).$$

Intuitively, this works as a lower bound as a consequence of the converse to the coding theorem because otherwise one can transmit more than the capacity rate or compress smaller than the source rate. The positive coding theorem assures that the bound may be approached arbitrarily closely in a communications situation.

Even though the rate distortion bound (59) is generally achieved only with source and channel coding, it is occasionally achieved without any coding or nonlinear modulation. Consider the transmission of a zero-mean Gaussian $U$ of variance $\sigma^2$ under with the mean-square error criterion, $d(u, v) = |u - v|^2$, over an additive Gaussian noise channel

$$X^{(out)} = X^{(in)} + n$$

with noise variance $N$. Then [7,9,11-13] for $U$

$$R_u(d) = \begin{cases} \frac{1}{2} \log(\sigma^2/d), & 0 \leq d \leq \sigma^2 \\ 0, & d \geq \sigma^2 \end{cases}$$

and

$$C(S) = \frac{1}{2} \log(1 + \frac{S}{N})$$

under $E[X^2] \leq S$. The rate distortion limit (59) becomes

$$d \geq \sigma^2(1 + \frac{S}{N})^{-1}.$$ 

If one sends $U$ as $X$ over the channel as in (23) so that $S = A^2\sigma^2$, and use the estimate (24), the resulting mean-square error (23) is exactly the lower limit (63). This shows that for this problem of transmitting a Gaussian parameter matched, in per use or per symbol to a Gaussian noise channel, even in the full generality of Fig. 1 there is nothing that can do better than linear modulation-demodulation! Indeed, no way other than through $R(d)$ has ever been successfully employed to show the optimality of linear modulation in this problem.
The rate distortion function of the uniform phase variable $\phi$ is difficult to evaluate exactly. However, the Shannon upper and lower bounds [7, 11] on $R(d)$ for $\phi$ differs only by about 0.3 bit per symbol. Thus the upper bound

$$R_\phi(d) \leq \log \frac{1.35}{\sqrt{d}}$$

would be used for $R_\phi(d)$.

There are two complications in the application of the rate distortion bound to measurement problems. The first arises from the fact that in a measurement one has little or no room for source coding as the parameter $U$ is usually out of one’s control before modulating onto the physical channel input variable $X^{(in)}$. Thus, while (61) remains a limit, in general there may be no way to approach it. One may try to replace $R(d)$ of (61) by some realistic $R(d)$ obtained with whatever one can do to $U$, but contrary to what is stated in ref [56], it is not clear how such $R(d)$ may be evaluated. On the other hand, from experience the $R(d)$ for a Gaussian parameter with different encoding criteria vary little, and so the exact form of $R(d)$ is not expected to make any major difference in the final result (59).

The second problem is, I believe, more serious and closer to the heart of the matter. It arises because no channel coding may be employed in a typical measurement situation. One can similarly try to replace $C(\beta)$ by a mutual information $J(\beta)$ incorporating the realistic limitations and freedom, which again seems hard to do. This is an essential connection, however, because the modulation of $U$ into $X$ represents how one physically couples $U$ into the measurement medium in the measurement system. It makes a difference whether the optical field couples to $U$ via an interferometer configuration or a source impressing configuration, and e.g., whether the frequency or the amplitude is modulated by $U$. In any case, if such a meaningful $J(\beta)$ can be obtained, then a measurement rate distortion limit can be obtained from $J(\beta) \geq R(d)$ in the form similar to (59)

$$d \geq R^{-1}J(\beta).$$

4.3 Ultimate quantum measurement system limit

By combining the above rate distortion theory with classical capacities replaced by quantum capacities, one obtains quantum rate distortion limits for a general quantum system of Fig. 3, taking into account all the freedom of classical modulation-demodulation and quantum measurement as well as state selections. Note that the uncertainty principles are far from sufficient to determine such ultimate limits. In the original form they are merely restrictions on quantum states, and even in their extended form [20] they do not account for the many freedom represented in Fig. 3. A few more remarks on this may be found in [10] and [59].

From (61) and the single mode version of (32), one finds (52) using the optimum number state capacity. With $C_{TCS}$ of (32), one finds the same $\delta u_{TCS}$
as (53) for linear modulation without coding! This is exactly the situation around (25) and (63) pointed out above. For $C_{\text{het}}$ of (30), one obtains

$$
\delta u^{\text{CS}} = \frac{\sigma}{S+1}, \quad (66)
$$

which may be compared to

$$
\delta u' \sim \frac{\sigma}{\sqrt{S}} \quad (67)
$$

obtained in coherent state systems without coding or nonlinear modulation. The reason why coding or nonlinear modulation is necessary in the coherent state case is that bandwidth expansion, two quadratures in a coherent state versus the single real parameter to be estimated, has to be utilized. Thus, apart from a gain on $\delta u$ by a factor of 2, the use of TCS for measurement is essentially the same as coding on a coherent state system as far as the performance goes, a rather unexpected result.

For the uniform phase parameter, one finds (54) from (31) and (64), and similarly

$$
\delta \phi^{\text{TCS}} \sim \frac{1}{2S}, \quad S \gg 1 \quad (68)
$$

$$
\delta \phi^{\text{CS}} \sim \frac{1}{S}, \quad S \gg 1 \quad (69)
$$

Again, the $1/S$ behavior can be obtained without coding on TCS or number state systems, while coherent state systems without coding yields

$$
\delta \phi' \sim 1/\sqrt{S}. \quad (70)
$$

In the multimode narrowband situation, one similarly obtains (53) for the optimum case, with $m = D/2 = WT$, and

$$
\delta u^{\text{TCS}} = \sigma(1 + \frac{2S}{m})^{-m} \quad (71)
$$

$$
\delta u^{\text{CS}} = \sigma(1 + \frac{S}{m})^{-m} \quad (72)
$$

Similarly results for $\phi$ can be written down with $\sigma$ replaced by $\lambda$ as in (52) and (54). From (71) and (72),

$$
\delta u^{\text{TCS}} \to \sigma e^{-2S}, \quad \delta u^{\text{CS}} \to \sigma e^{-S}, \quad D \to \infty \quad (73)
$$

an exponential decrease in $S$ versus $1/S$ in the single mode case. Even though the narrowband assumption is violated in $D \to \infty$, the exponential improvement is real from (53) or (71) to (72) as $D$ can be very large at optical frequencies.

To show that multimode system is indeed better for measurement for the same total energy, consider the following pulse frequency modulation scheme

$$
X^{(in)}(t, \phi) = \sqrt{\frac{2S}{T}} \sin(w_0 + \beta \phi)t, \quad 0 \leq t \leq T, \quad (74)
$$
where $\beta$ is a known fixed constant and $S$ the total energy in the signal. Classically, it is known [8] that in the presence of additive white Gaussian noise, the use of (74) and corresponding nonlinear demodulation lead to a decrease of root-mean-square error $\delta \phi$ by a factor $\sim \frac{1}{m}$ compared to the linear modulation case, when a threshold constraint involving $S$, $T$, $\beta$ and the noise variance $N_0$ is satisfied which occurs for sufficiently large $D$ or $S$. If (74) is used in either a coherent state-heterodyne or TCS-homodyne systems, one would obtain

$$\delta \phi_{CS}^{TCS} \sim \frac{1}{mS}, \quad \delta \phi^{CS} \sim \frac{1}{m\sqrt{S}}$$

compared to (68) and (70). While showing the importance of bandwidth, the net gain $1/m$ is in itself already significant as $D$ is large.

5 Position monitoring with contractive states

As a final application of squeezed states, we discuss the problem of repeatedly measuring the position of a free mass for which the state after each measurement is important as it determines the state at the next measurement instant. This feature makes the problem, relevant to gravitational-wave interferometers treated elsewhere in this book, quite different from the other ones we have discussed so far in this chapter, for which all the information can be extracted from the system by one measurement. There is still considerable confusion in the literature on the validity of the so-called “standard quantum limit” (SQL) on how small the position fluctuation $\langle \Delta \hat{X}^2(t) \rangle$ can be obtained in a sequence of position measurements, although the issues in principle have been cleared up entirely over ten years ago. Perhaps this is partly because some of the following clarification never appeared in print.

The SQL states that [66,67] if a position measurement is made at $t = 0$, the fluctuation at $t > 0$ is at least

$$\langle \Delta \hat{X}^2(t) \rangle_{SQL} = \frac{\hbar t}{m},$$

where $m$ is the mass of a fermion. The derivation of (74), however, was incorrectly taken to be universally valid as a consequence of the Uncertainty Principle, and it was concluded that the free mass position is not a “QND observable” — namely, that the disturbance to the system from the first position measurement demolishes the possibility of an accurate second measurement after an interval of free evolution. To delineate how a position monitoring scheme works, consider the monitoring of weak classical forces $f_1(t), f_2(t)$ coupled linearly to a free mass with position $\hat{X}$ and momentum $\hat{P}$,

$$H_I = f_1(t) \hat{X} + f_2(t) \hat{P}.$$  \hspace{1cm} (77)

In the Heisenberg picture,

$$\hat{X}(t) = \hat{X}(0) + \hat{P}(0) t/m + \int_0^t f_2(t') dt' - \int_0^t dt' \int_0^{t'} d\tau f_1(\tau)/m,$$ \hspace{1cm} (78)
\[ \dot{\hat{P}}(t) = \hat{P}(0) - \int_0^t f_1(t') dt'. \]  

Typically, \( f_2 = 0 \) and \( \hat{X} \) is more readily measurable than \( \hat{P} \) in practice. From (78), information on \( f_1(t) \) can be obtained by measurements on \( \hat{X}(t) \) at different times.

If a position measurement at \( t = 0 \) is made in the sense of Pauli’s first-kind measurement [68], the position eigenstates \( |X\rangle \) is to be used to compute the measurement probability \( p(X) \) and the state of the mass after the measurement with a reading \( X' \) is \( |X'\rangle \). Thus, first-kind measurement of a selfadjoint operator is one for which the Von Neumann projection postulate applies. From (79), the position fluctuation \( \langle \Delta \hat{X}^2(t) \rangle \) is often concluded to be infinite, because the “back-action” causes \( \langle \Delta^2 \hat{P}(0) \rangle = \infty \) with \( \langle \Delta \hat{X}^2(0) \rangle = 0 \) from the Uncertainty Principle. Since \( \langle \hat{P}^2(0) \rangle = \langle \Delta^2 \hat{P}(0) \rangle + \langle \hat{P}(0) \rangle^2 \), an infinite average energy is obtained for the mass in a position eigenstate \( |X\rangle \), thus one can actually only make “approximate” position measurements which are generally described by POM as far as the measurement statistics goes, with \( \langle \Delta \hat{X}^2(0) \rangle > 0 \). In any event, it was concluded that whatever position measurement is used the Uncertainty Principle implies the SQL (76).

In [69], it was pointed out that this conclusion is not valid from (78) when \( \langle \Delta \hat{X}(0) \Delta \hat{P}(0)+(\Delta \hat{P}(0)\Delta \hat{X}(0)) \rangle \) is negative. It was also pointed out that \( \langle \Delta \hat{X}^2(t) \rangle \) can be arranged to be as small as desired at any \( t > 0 \) if the state after measurement is left in a “contractive state” \( |\mu\nu\omega\rangle \), which is a TCS \( |\mu\nu\rangle \) with the frequency \( \omega \) put back explicitly and the parameters \( \mu, \nu, \omega \) chosen appropriately so that the “generalized minimum uncertainty wave packet” \( \langle X|\mu\nu\omega\rangle \) contracts rather than spreads in \( t \) up to a desired measurement time. It was observed that measurements of the second kind, in particular a class of measurements formally described by Gordon and Lonisell, may be used to beat the SQL. Specifically, the measurement described by [68,69]

\[ |\mu\nu\omega\rangle\langle \mu'\nu'\alpha\omega| \]  

would work, where \( |\mu'\nu'\alpha\omega\rangle \) is used to compute the measurement probability with reading \( \alpha = \alpha_1 + i\alpha_2 \),

\[ \alpha_1 = x(m\omega/2\hbar)^{1/2}, \quad \alpha_2 = p/(2hm\omega)^{1/2}, \]  

which may be considered a joint approximate measurement of \( \hat{X} \) and \( \hat{P} \) similar to TCS-heterodyne [22], and \( |\mu\nu\omega\rangle \) is the state after measurement of reading \( \alpha \) arranged to be a contractive state for the next measurement. The position measurement would be sharp if \( \langle \Delta \hat{X}^2 \rangle \sim |\mu' - \nu'|^2 \to 0 \), while \( \mu, \nu, \omega \) are chosen so that the mass state has a sharply defined position at the next measurement instant.

Two criticisms were made on the success of this approach to beat the SQL. First, it was pointed out that it was not clear a measurement described as in (80) is realizable in principle. A quantum measurement realization can be described by the coupling of a “probe” to the system with commuting selfadjoint operators being measured on the probe, and with all the quantities computed by the
usual rules of quantum mechanics (without the need for the projection postulate as emphasized by Ozawa [16].) While two realizations were produced [71], they were criticized on the ground that the probe-system interaction hamiltonians $H_I$ are time-dependent and so are equivalent to “state preparation.” While these realizations are actually quite different from the state preparations that were discussed and are in fact full-fledged quantum measurement realizations in accordance with standard quantum measurement theory, the situation is resolved beyond dispute when a time-independent $H_I$ was found [72] for realizing (80).

More significantly, Ozawa [16,73] has obtained a complete characterization of quantum measurement including the state after measurement in the concept of a completely positive operation measure mentioned in Section 3.1, and he showed that any Gordon-Lonisell measurement representation, in which a complete but not necessarily orthogonal set of states is used to yield the measurement statistics and the state after measurement depends only the measured value, is indeed realizable.

To discuss the second criticism, one needs to examine more closely how the measurement scheme based on (80) actually works. Let $\alpha'$ be the reading at $t = 0$ so that the state at $t = 0+$ is $|\mu \omega \nu \alpha'\rangle$. After another time $t$, the free mass is in state $|\mu t \nu t \alpha' \rangle$ with $|\mu - \nu| \rightarrow 0$. From (78)-(79) with $f_2 = 0$, the value $\alpha'$ is given by

$$\alpha_{1}' = \alpha_1' + \alpha_2' t/m - \int_0^t dt' \int_0^{t'} d\tau f_1(\tau)/m$$

and

$$\alpha_{2}' = \alpha_2' - \int_0^t f_1(\tau) d\tau.$$  

Equations (82) and (83) provide the average of the reading $\alpha''$ at $t$, which can be represented by

$$\alpha_{1}'' = \alpha_1' + \alpha_2' t/m - \int_0^t dt' \int_0^{t'} d\tau f_1(\tau)/m + n_1,$$

$$\alpha_{2}'' = \alpha_2' + n_2,$$

where the fluctuation of $n_1$ is vanishingly small from $|\mu - \nu| \rightarrow 0$ while the noise $n_2$ is big. From (84), one may use the $\alpha_1''$ reading to estimate $f_1$ after it is subtracted from the value of $\alpha_1' + \alpha_2' t/m$ known at time $t$. The reading $\alpha_2''$ is also taken so that it could be used for the subtraction at the next measurement, although it is not used for estimating $f_1$ as it is noisy and helps little. It is clear that this scheme beats the SQL to any arbitrary level in a sequence of measurements.

In [74], a “predictive sense” of the SQL was proposed to suggest that the SQL was not beaten in that sense. This predictive sense can be described by the stipulation that prior to any measurement, $\alpha_1'$ and $\alpha_2'$ in (84) and (83) are unknown and random, thus $\alpha_1''$ is also more random than $n_1$ and indeed obeys the SQL. But since we know we will have the reading value $\alpha'$ available at $t$ which would be subtracted from (84), the reading $\alpha''$ at $t$, we can indeed predict...
we will get $\langle \Delta \hat{X}^2(0) \rangle$, $\langle \Delta \hat{X}^2(t) \rangle$, and so on, arbitrarily small. Thus, the SQL is beaten by (80) in the predictive sense. Further elucidation of this point and discussion on the working of this scheme (80)-(85) was provided in [75].

Actually, this issue would not even arise if the measurement $|\mu\alpha\omega\rangle \langle \mu'\nu'\alpha\omega|$ is employed instead of (80), for which the state after measurement always has $\langle \hat{X} \rangle = \langle \hat{P} \rangle = 0$. This measurement is a special degenerate case of Gordon-Lonisell measurement, and thus realizable by Ozawa’s theorem. Indeed, an explicit Hamiltonian realization can be developed for (80) [76,77].

Since the positions of a free mass can be repeatedly measured accurately, it is not appropriate to say that $\hat{X}$ is not a QND observable. The term QND measurement is often used just to refer to a first-kind measurement, which is an acceptable terminology. What has never been demonstrated is that there is, in principle, any observable which is not a QND observable in the generic sense. In fact, it should be clear from the development in this section, and it can indeed be readily shown in principle, that any observable can be repeatedly measured arbitrarily accurately in the absence of particular constraints. The key point is that, as in (80), the state used to compute the measurement probability and the state after measurement need not be the same.

References

(All unpublished manuscripts by this author are available upon request.)

1. L. Mandel and E. Wolf, *Coherence and Quantum Optics*, Cambridge University Press, 1996.

2. H. P. Yuen, Phys. Rev. A 13, 2226 (1976). The notation $|\beta\rangle_g$ in this reference is equivalent to $|\mu\nu\alpha\rangle$ in this paper with $\beta = \mu\alpha + \nu\alpha^*$.

3. H. P. Yuen, Phys. Lett. A 56, 105 (1976).

4. H. P. Yuen, “Generalized Coherent States and Optical Communications,” in Proc. 1975 Conf. Information Sciences and System, Johns Hopkins Univ., pp. 171-177, 1975.

5. J. H. Shapiro, H. P. Yuen and J. A. Machado Mata, IEEE Trans. Inform. Theory, IT-25, 179 (1979).

6. J. M. Wozencroft and I. M. Jacobs, *Principles of Communication Engineering*, Wiley, New York, 1965.

7. R. G. Gallager, *Information Theory and Reliable Communication*, Wiley, New York, 1968.
8. H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I*, Wiley, New York, 1968.

9. T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, 1991.

10. H. P. Yuen, “Quantum Information Theory, the Entropy Bound, and Mathematical Rigor in Physics,” in *Quantum Communication, Computing, and Measurement*, ed. by Hirota, etc., Plenum, New York, pp. 17-23, 1997.

11. C. E. Shannon, Bell Sys. Tech. J. **27**, pp. 379-423, 623-656 (1948).

12. C. E. Shannon, IRE National Convention Record, Part 4, pp. 142-163 (1959).

13. C. E. Shannon, Proc. IRE **37**, 10 (1949).

14. D. Slepian, Proc. IEEE **64**, 292 (1976).

15. H. P. Yuen, “A New Approach to Quantum Computation,” in *Quantum Communications, Computations, and Measurements*, ed. by P. Kumar, H. Hirota, and M. d’Ariano, Plenum, 1999.

16. M. Ozawa, “Realization of Measurement and the Standard Quantum Limit,” in *Squeezed and Nonclassical Light*, ed. by P. Tombesi and E. R. Pike, Plenum, 263-286, 1989.

17. G. D. Forney, Jr., “The Concept of State and Entropy in Quantum Mechanics,” S. M. thesis, Dept. of Electrical Engineering, 1963.

18. J. P. Gordon, “Noise at Optical Frequency and Information Theory,” in *Quantum Electronics and Coherent Light*, Proceedings of the International School of Physics “Enrico Fermi,” XXXI, ed. P. A. Miles, Academic Press, 156-1964.

19. C. W. Helstrom, *Quantum Detection and Estimation Theory*, Academic press, 1976.

20. A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, 1982.

21. H. P. Yuen and M. Lax, IEEE Trans. Inform. Thoery **19**, 740 (1973); H. P. Yuen, R. S. Kennedy, and M. Lax, IEEE Trans. Inform. Theory **21**, 125 (1975).

22. H. P. Yuen and J. H. Shapiro, IEEE Trans. Inform. Theory, IT-**26**, 78-92 (1980).

23. S. L. Braunstein, C. M. Caves, and G. J. Milburn, Amer. Phys. (N.Y.) **247**, 135 (1996).
24. H. Takahasi, “Information Theory of Quantum Mechanical Channels,” in *Advances in Communication Systems*, ed. by A. Balakrishnan, vol. 1, Academic Press, pp. 227-310, 1965.

25. H. P. Yuen and J. H. Shapiro, IEEE Trans. Inform. Theory *24*, 657 (1978).

26. O. Hirota and S. Ikehara, Trans. IECE of Japan, *EGI*, 273 (1978).

27. J. H. Shapiro, Opt. Lett. *5*, 351 (1986).

28. J. I. Bowen, IEEE Trans. Inform. Theory *13*, 230 (1967).

29. A. S. Holevo, M. Sohma, and O. Hirota, Phys. Rev. A, *59*, 1820 (1999).

30. M. Sohma and O. Hirota, Phys. Rev. A, *62*, 052312 (2000).

31. A. S. Holevo, M. Sohma, and O. Hirota, Rept. Math. Phys. *46*, 343 (2000).

32. M. Hall, Phys. Rev. A *55*, 100 (1997).

33. G. M. D’Ariano, C. Macchiavello, and M. F. Sacchi, Phys. Lett. A *248*, 103 (1998).

34. G. M. D’Ariano and P. Lo Presti, Phys. Rev. Lett. *86*, 4195 (2001).

35. H. P. Yuen, “High-rate Strong-signal Quantum Cryptography,” in *Proceedings of the 1995 Conference on Squeezed States and Uncertainty Relations*, NASA Conference publication 3322, pp. 363-368, 1996.

36. D. Gottesman and J. Preskill, Phys. Rev. A. *63*, 22309 (2001).

37. C. M. Caves and P. D. Drummond, Rev. Mod. Phys. *88*, 481 (1994).

38. P. Yuen and M. Ozawa, Phys. Rev. Lett. *70*, 363 (1993).

39. J. P. Gordon, Proc. IRE *50*, 1898 (1962).

40. P. L. Zador, Bell Telephone Laboratories Technical Memorandum, MM65-1359-4, Murray Hill, NJ, 1965.

41. A. S. Holevo, Probl. Inf. Transm. *9*, 177 (1973).

42. P. Hausladen, R. Josza, B. Schumacher, M. Westmoreland and W. Wooters, Phys. Rev. A *54*, 1869 (1996).

43. A. S. Holevo, “The Capacity of Quantum Channel with General Signal States,” preprint.

44. H. P. Yuen and J. H. Shapiro, IEEE Trans. Inform. Theory *24*, 657 (1978).

45. H. P. Yuen, “Quantum Communication, Quantum Measurement, TCS and QND,” in *Quantum Optics, Experimental Gravity, and Measurement Theory*, ed. P. Meystre and M. O. Scully, Plenum, pp. 249-268, 1983.
46. H. P. Yuen, “Nonclassical Light,” in *Photons and Quantum Fluctuations*, ed. E. R. Pike and H. Walther, Adam Hilger, pp. 1-9, 1988.

47. H. P. Yuen, “Optical Communication with Novel Quantum Devices,” in *Quantum Aspects of Optical Communications*, Lecture Notes in Physics 378, ed. C. Benjakallab, O. Hirota, and S. Reynard, Springer, pp. 333-341, 1991.

48. H. P. Yuen, *Quantum Semiclass. Opt.* 8, 939 (1996).

49. G. M. D'Ariano, *Int. J. Mod. Phys. B* 6, 1291 (1992).

50. G. M. D'Ariano, C. Macchiavello, N. Sterpi and H. P. Yuen, Phys. Rev. A 54, 4712 (1996).

51. J. H. Shapiro and S. R. Shepard, Phys. Rev. A 43, 3795 (1991).

52. H. P. Yuen and V. Chan, Opt. Lett. 8, 177 (1983).

53. H. P. Yuen, Opt. Lett. 12, 789 (1987).

54. H. P. Yuen, “Photon Number Duplication and Quantum Nondemolition Measurements,” unpublished manuscript, 1991; presented at the Oct. 91 Optical Society of America meeting.

55. H. P. Yuen, Opt. Lett. 17, 73 (1992).

56. G. D. Bartolini, D. K. Serkland, P. Kumar and W. L. Kath, IEEE Photon. Technol. Lett. 9, 1020 (1997).

57. A. Mecozzi, P. Kumar and H. P. Yuen, unpublished manuscript, 1998.

58. H. P. Yuen, “Quantum versus Classical Noise Cryptography,” in the book referred to in [15].

59. H. P. Yuen, “The Ultimate Quantum Limits on the Accuracy of Measurements,” in *Proceedings of the Workshop on Squeezed States and Uncertainty Relations*, NASA Conference Publication 3135, pp. 13-22, 1991.

60. C. M. Caves, Phys. Rev. D 23, 1693 (1981).

61. R. S. Bondurant and J. H. Shapiro, Phys. Rev. D 30, 2548 (1984).

62. H. P. Yuen, Phys. Rev. Lett. 56, 2176 (1986).

63. H. P. Yuen, “Generalized Coherent States of the Radiation Field,” unpublished manuscript, 1975.

64. H. P. Yuen, Nuclear Phys. B 6, 309 (1989).

65. H. P. Yuen and J. H. Shapiro, Opt. Lett. 4, 334 (1979).

66. V. B. Braginskii and Yu I. Vorontsov, Sov. Phys. Usp. 17, 644 (1975).
67. C. M. Caves, etc., Rev. Mod. Phys. 52, 341 (1980).
68. W. Pauli, Handbuch der Physik, vol. 5, Springer, 1958.
69. H. P. Yuen, Phys. Rev. Lett. 51, 719 (1983).
70. H. P. Yuen, Phys. Rev. Lett. 52, 1730 (1984).
71. H. P. Yuen, “Violation of the Standard Quantum Limit by Realizable Quantum Measurements,” unpublished manuscript, 1985.
72. M. Ozawa, Phys. Rev. Lett. 60, 385 (1988).
73. M. Ozawa, J. Math. Phys. 25, 79 (1984).
74. R. Lynch, Phys. Rev. Lett. 54, 1599 (1985).
75. H. P. Yuen, “General Quantum Measurements and the Standard Quantum Limit,” unpublished manuscript, 1985.
76. M. Ozawa, Phys. Rev. A 41, 1735 (1990).
77. M. Ozawa, private communications, 1998.