POINCARE SERIES AND INSTABILITY OF EXPONENTIAL MAPS

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Abstract. We relate the properties of the postsingular set for the exponential family to the questions of stability. We calculate the action of the Ruelle operator for the exponential family. We prove that if the asymptotic value is a summable point and its orbit satisfies certain topological conditions, the map is unstable hence there are no Beltrami differentials in the Julia set. Also we show that if the postsingular set is a compact set, then the singular value is summable.

1. Introduction

If f is a transcendental entire map, we denote by $f^n$, $n \in \mathbb{N}$, the n-th iterate of f and write the Fatou set as $F(f) = \{z \in \mathbb{C};$ there is some open set U containing z in which $\{f^n\}$ is a normal family $\}$. The complement of $F(f)$ is called the Julia set $J(f)$. We say that f belongs to the class $S_q$ if the set of singularities of $f^{-1}$ contains at most q points.

Two entire maps g and h are topologically equivalent if there exist homeomorphisms $\varphi, \psi: \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi \circ g = h \circ \psi$. Given a map f, let us denote by $M_f$, the set of all entire maps topologically equivalent to f. It is proved in [5] that $M_f$ has the structure of a $(q+2)$-dimensional complex manifold. The Affine group acts on the space $M_f$ and as it shown in [5] the space $N_f = M_f / \{\text{Affine group}\}$ is a $q$-dimensional complex orbifold.

A measurable field of tangent ellipses of bounded eccentricity determines a complex structure on the sphere. This ellipse field is recorded by a $(-1,1)$-form $\mu(z)dz\wedge d\bar{z}$ with $||\mu||_\infty < 1$, a Beltrami differential. If an entire map f is holomorphic in a complex structure defined by the Beltrami differential $\mu$, then $\mu$ is the invariant Beltrami differential. Since the sphere admits a unique complex structure, there is a homeomorphism $\phi: \mathbb{C} \rightarrow \mathbb{C}$ such that $\mu$ is the pullback of the standard structure and the map $f_\phi = \phi \circ f \circ \phi^{-1}$ is an entire map.

The non existence of an invariant Beltrami differential (invariant line field) on the Julia set is related to the Fatou conjecture, see [9].

Now let us consider the main hero of this paper - Exponential family: $E = \{f_\lambda(z) = \exp(\lambda z), \lambda \in \mathbb{C}^*\}$. Then $N_{f_1} \cong E$, where $f_1 = \exp(z)$. The map $f_{\lambda_0}$ is structurally stable if for any $\lambda$ close enough to $\lambda_0$ there exists a quasiconformal homeomorphism $\phi_\lambda$, such that $f_\lambda = \phi_\lambda \circ f_{\lambda_0} \circ \phi_\lambda^{-1}$.

Due to Mané, P. Sad, D. Sullivan (see [10]) and A. Eremenko, M. Lyubich (see [5]) the following three items are equivalent for $E$:

- Fatou conjecture
- There is no invariant Beltrami differentials supported by the Julia set.
If $J(f_\lambda) = \mathbb{C}$, then $f_\lambda$ is structurally unstable.

In 1985 R. Devaney (see [2]) proves that $\exp(z)$ is structurally unstable, after A. Douady and L. R. Goldberg (see [4]) did show that the maps $\lambda \exp(z), \lambda \geq 1$ are topologically unstable. Zhuan Ye (see [13]) proves that $f_\lambda$ is structurally unstable map if $\lim_{n \to \infty} f_\lambda^n(0) = \infty$.

In this paper we follow the approach of papers [1], [6] and [7]-[8], (case of rational maps) and [3] (case of transcendental entire maps with only algebraic singularities). In the case of Exponential family we have only one asymptotic singularity which is a different situation that in [3].

The stability of a map depends on the behavior of the postsingular set, denoted as $X_\lambda = \{\cup_{n \geq 1} f_\lambda^n(0)\}$.

Let us start with $f_\lambda$ whose Julia set is equal to the plane. Then we have the following simple possibilities:

1. $\lim_{n \to \infty} |(f_\lambda^n)'(0)| = 0$.
2. There exists a subsequence $\{n_i\}$ such that $\lim_{i \to \infty} |(f_\lambda^{n_i})'(0)| = \infty$.
3. There exists a subsequence $\{n_i\}$ such that $\lim_{i \to \infty} |(f_\lambda^{n_i})'(0)| = M < \infty$ and $M \neq 0$.

We believe that the first case contains a contradiction. Since in this situation the forward orbit of 0 must converge to an attractive cycle and hence $0 \notin J(f_\lambda)$. We show this conjecture under very strong additional conditions only as an illustration that this conjecture is not completely false (see theorem 1).

As for the last two cases, the Fatou conjecture claims that $f_\lambda$ is an unstable map.

Define

**Definition 1.** Let $\lambda \in \mathbb{C}^*$, then the Poincaré series for $f_\lambda$ is the following formal series

$$P_\lambda = 1 + \frac{1}{\lambda} \sum_{i=2}^{\infty} \frac{1}{(f_\lambda^{i-2})'(1)}.$$ 

Let

$$S_n = 1 + \frac{1}{\lambda} \sum_{i=2}^{n} \frac{1}{(f_\lambda^{i-2})'(1)}$$

be a particular sums of the Poincaré series $P_\lambda$. Then we have the following theorem

**Theorem 1.**

1. If there exist a sequence $\{n_i\}$ such that $(f_\lambda^{n_i})'(1) \to \infty$ and $\lim_{i \to \infty} \sup|S_{n_i}| > 0$, then $F(f_\lambda) = \emptyset$ and $f_\lambda$ is unstable.
2. If there exist a sequence $\{n_i\}$ such that $(f_\lambda^{n_i})'(1) \cong c$, where $c \neq 0$ is a constant and $\lim_{i \to \infty} \sup|S_{n_i}| = \infty$, then $f_\lambda$ is unstable.
3. Let $\lim_{n \to \infty} (f_\lambda^n)'(1) = 0$, and one of the following conditions holds:

   $$\lim_{n \to \infty} \sup \left| \frac{(f_\lambda^{n+1})'(1)}{(f_\lambda^n)'(1)} \right| < \infty,$$

   or

   $$\lim_{n \to \infty} \inf \left| \frac{(f_\lambda^{n+1})'(1)}{(f_\lambda^n)'(1)} \right| > 0.$$

   Then $F(f_\lambda) \neq \emptyset$.

**Proposition 1.** Does not exist a map $f_\lambda$ such that $\lim_{n \to \infty} |f_\lambda^n(1)| = C > 0$

The next theorems discuss the best conditions on the Poincaré series and on the postsingular set for the map to be unstable.
Definition 2. A point \( a \in \mathbb{C} \) is called "summable" if and only if the series
\[
\sum_{i=0}^{\infty} \frac{1}{(f_\lambda^n)'(a)}
\]
is absolutely convergent. Note that the point \( z = 0 \) is summable if and only if the Poincaré series \( P_\lambda \) is absolutely convergent.

Definition 3. Let \( W \subset E \) be the subset of exponential maps \( f_\lambda \), with summable singular point \( 0 \in J(f_\lambda) \), satisfying one of the following conditions:
1. \( 0 \not\in X_\lambda \),
2. \( X_\lambda \) does not separate the plane,
3. \( m(X_\lambda) = 0 \), where \( m \) is the Lebesgue measure.

Theorem 2. Let \( f_\lambda \in W \). Then \( f_\lambda \) is an unstable map, and hence there is no invariant Beltrami differentials on its Julia set.

Theorem 3. Let \( f_\lambda \in E \), with \( J(f_\lambda) = \mathbb{C} \). Assume \( 0 \not\in X_\lambda \) (i.e. \( 0 \) is non-recurrent), then
1. There exist a subsequence \( n_k \) such that \( (f_\lambda^{n_k})'(1) \to \infty \)
2. If \( X_\lambda \) is bounded, then the singular point \( z = 0 \) is summable for \( f_\lambda \).

In section 2 we discuss and prove Theorem 3 and Proposition 1. In section 3 we consider the basic definitions and properties of the Ruelle operator and the potential of deformations, as a consequence we prove theorem 1. The rest of the paper is devoted to prove Theorem 2.

2. Postsingular set and dynamics

Mañe has a result that establishes expansion properties of rational maps on the compact subsets of their Julia sets, which are far away from the parabolic points and the \( w \)-limit sets of recurrent critical points. Next we will consider this result for our map \( f_\lambda \).

Remark 1. Note that if \( f_\lambda^n(0) \to \infty \) then \( f_\lambda \) is summable. To see this, consider
\[
\left| \frac{1}{(f_\lambda^{n+1})'(a)} \right| / \left| \frac{1}{(f_\lambda^n)'(a)} \right| = \left| \frac{1}{A(f_\lambda^n(a))} \right|
\]
now choose \( a = f_\lambda(1) \) and since the orbit of \( 0 \) tends to \( \infty \) this fraction converges to zero, so the series \( \sum \frac{1}{(f_\lambda^n)'(1)} \) absolutely converges.

2.1. Proof of Theorem 3. The proof of the theorem follows exactly the proof in [12], by Shishikura and Tan Lei. For completeness we will state the lemmas used in the paper above mentioned, restricted to the situation of our case. Hence in order to prove our theorem 3, we will follow their arguments.

Denote by \( d(z, E) \) the Euclidian distance between a point \( z \in \mathbb{C} \) and a closed subset \( E \subset \mathbb{C} \). Let \( d_Y(z, X) \) be the Poincaré distance on a hyperbolic surface \( Y \) between a point \( z \) and a closed subset \( X \subset Y \) and \( \text{diam}_W(W') \) the diameter of \( W' \) with respect of the the Poincaré metric of \( W \).
Lemma 1. ([12] lemma 2.1). For any $0 \leq r \leq 1$, there exist a constant $C(1, r) \geq 0$ such that for any holomorphic proper map $g : V \to \mathbb{D}$ of degree 1, with $V$ simple connected, each component of $g^{-1}(D_r(0))$ has diameter $\leq C(1, r)$ with respect to the Poincaré metric on $V$. Moreover $\lim_{r \to 0} C(1, r) = 0$.

Definition 4. $N_0$: There exist $z_1, \ldots, z_{N_0-1} \in \mathbb{D}$ such that $\left\{ \frac{3}{4} \leq |z| \leq 1 \right\} \subset \bigcup_{i=1}^{N_0-1} D_{\frac{3}{4}}(z_i)$. Let $C_0 = N_0 C(1, \frac{3}{4})$.

The Julia set $J(f_\lambda) = \mathbb{C}$, hence we can choose a periodic point $w$ so that the domain $\Omega = \mathbb{C} \setminus \{\text{forward orbit of the point } w\}$ satisfies: $d_\Omega(0, X_\lambda) \geq 2 C_0$.

Lemma 2. ([12] lemma 2.3) Let $U_0 = D_r(x)$ be a disc centered at $x \in X_\lambda$ with radius $r$ so that $U_0 \subset \Omega$ and $\text{diam}_\Omega(U_0) \leq C_0$, then for every $n \geq 0$ the following is true:

- $\text{deg}(n)$. For every $D_s(z) \subset U_0$ with $0 \leq s \leq d(z, \partial U_0)/2$, and every connected component $V'$ of $f_\lambda^{-n}(D_s(z))$, $V'$ is simply connected and $\text{deg}(f_\lambda^n : V' \to D_s(z)) = 1$;

- $\text{diam}(n)$. For every $D_r(w) \subset U_0$ with $0 \leq r \leq d(w, \partial U_0)/2$ and every connected component of $V$ of $f_\lambda^{-n}(D_r(w))$, $\text{diam}_V \leq C_0$.

Now we begin to prove the theorem 3. If only $\infty$ is a point of accumulation of $\{U_n f_\lambda(0)\}$, then by the remark 1 above the point $z = 0$ is a summable and hence $\lim_{n \to \infty} |(f_\lambda^n)'(0)| = \infty$.

Now let $y \in X_\lambda$ be another point of accumulation of the orbit of $z = 0$. Let $n_i$ be any subsequence such that $y = \lim_{i \to \infty} f_\lambda^{n_i}(1)$. Then we claim:

Claim $\lim_{i \to \infty} |(f_\lambda^{n_i})'(1)| = \infty$.

To prove the claim we repeat the arguments of Shishikura and Tan Lei. Assume there exist a number $M < \infty$ and a sequence of natural numbers $\{n_j\} \subset \{n_i\}$ such that $|(f_\lambda^{n_j})'(1)| \leq M$. Then by the lemma 2 there exist an integer $N$ and a number $r$ such that components $W_j \subset f_\lambda^{-n_j}(D_r(y))$ containing the point $z = 1$ are simply connected and the respective restriction maps $f_\lambda^{n_j} : W_j \to D_r(y)$ are univalent for all $j \geq N$. Now let $B \subset \Omega$ be the hyperbolic ball of the radius $C_0$ centered at the point $z = 1$, then $B$ is a precompact subset of $\Omega$ and hence has a bounded Euclidian diameter in $\mathbb{C}$. Besides, again by the lemma 2, the set $\{ \cup_j W_j \} \subset B$. Let $g_j : D \to W_j$ be the inverse maps, then it is a normal family. Hence after passing to a subsequence we can assume that $g_j$ converge. Let $g_\infty$ be a limit map, then $g_\infty \neq \text{const}$ since the derivatives are $\geq \frac{1}{M}$ by hypothesis. Then there is a neighborhood $U_0$ of $z = 1$ such that $U_0 \subset g_j(D)$ for large $j$. Then $f_\lambda^{n_j}$ is a normal in $U_0$, but there are many periodic expansive points in $U_0 \subset J(f_\lambda)$ and the derivative diverges. Which is a contradiction. The claim and the first part of the theorem are done.

Finally for the proof of the second part, we again repeat arguments of Shishikura and Tan Lei in [12]. So assume that $f_\lambda$ is not expansive on $X_\lambda$, i.e. there are $n_k \to \infty$, $x_k \in X_\lambda$, such that $|(f_\lambda^{n_k})'(x_k)| \leq 1$. Now using the compactness of $X_\lambda$ and the arguments above, we obtain a contradiction. Expansivity immediately implies summability of the point $z = 1$ and completes the theorem.

2.2. Proof of proposition 1.
Proof. We have \( \lim_{n \to \infty} \frac{1}{f_\lambda^n(z)} = 1 \). Since \( \lim_{n \to \infty} \frac{f_\lambda^n(z)}{f_\lambda(z)} = 1 \), then \( |f_\lambda^n(z)| \) is near \( \frac{1}{n} \) for all large values of \( n \).

This implies that \( X_\lambda \) is bounded, hence compact and 0 is non-recurrent, by Theorem 3, \( f_\lambda \) is summable. That is a contradiction with the hypothesis. □

3. Ruelle Operator: Definitions and Properties

For any \( \lambda \in \mathbb{C}^* \) we define the following operators (compare with [7], [8], [9]).

**Definition 5.**

- **Ruelle operator (or push-forward operator)**
  
  \[
  R_\lambda^*(\varphi)(z) := \sum_{\xi_i} \varphi(\xi_i)\xi_i^2 = \frac{1}{\lambda z^2} \sum_{\xi_i} \varphi(\xi_i),
  \]

  where the summation is taken over all branches \( \xi_i \) of \( f_\lambda^{-1} \).

- **Modulus of the Ruelle operator** \( |R_\lambda^*(\varphi)| = \frac{1}{|\lambda z^2|} \sum_{\xi_i} \varphi(\xi_i) \).

- **Beltrami operator** \( B_\lambda(\varphi) = \varphi(f_\lambda)\frac{\partial f_\lambda}{\partial \bar{z}} \).

Then we have the following simple lemma.

**Lemma 3.** For all \( \lambda \);

1. \( R_\lambda^* : L_1(\mathbb{C}) \to L_1(\mathbb{C}) \) and \( \| R_\lambda^* \|_{L_1} \leq 1 \).
2. \( |R_\lambda^*| : L_1(\mathbb{C}) \to L_1(\mathbb{C}) \), \( \| |R_\lambda^*| \|_{L_1} \leq 1 \) and the fixed points of \( |R_\lambda^*| \) define a finite, complex-valued, invariant, and absolutely continuous measures on \( \mathbb{C} \).
3. \( B_\lambda : L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C}) \), is the dual operator to \( R_\lambda^* \), and \( \| B_\lambda \|_{L_\infty} = 1 \).

**Proof.** Immediately follows from the definitions. □

3.1. Potential of Deformations. The open unit ball \( B \) of the space \( \text{Fix}(B_\lambda) \subset L_\infty(\mathbb{C}) \) of fixed points of \( B_\lambda \) is called the space of invariant Beltrami differentials for \( f_\lambda \) and describes all quasiconformal deformations of \( f_\lambda \).

For \( \mu \in B \) and for any \( t \) with \( |t| < \frac{1}{|\mu|} \), the element \( \mu_t = t\mu \in B \). Let us denote by \( h_t \) their corresponding quasiconformal maps; then we have the following functional equation as explained in [7], [9]:

\[
F_\mu(f_\lambda(z)) - f_\lambda(z)F_\mu(z) = G_\mu(z)
\]

where \( h_t \circ f_\lambda \circ h_t^{-1} = f_{\lambda(t)} \in M_f \) and \( G_\mu(z) = \frac{\partial f_\lambda(z)}{\partial \bar{z}}(z)|_{t=0} = z \exp(\lambda z)\lambda'(t)|_{t=0} \).

The function

\[
F_\mu(a) = \frac{\partial h_t}{\partial t}|_{t=0} = -\frac{a(a-1)}{\pi} \int_{\mathbb{C}} \frac{\mu(z)}{z(z-1)(z-a)}
\]

is called the potential of the qc-deformations generated by \( \mu \) and \( \overline{\partial}F_\mu = \mu \) in the sense of distributions, see [11].

**Lemma 4.** If \( F(f_\lambda) = \emptyset \), then \( G_\mu = 0 \) if and only if \( \mu = 0 \).

**Proof.** If \( G_\mu = 0 \), then \( F_\mu(f_\lambda(z)) = f_\lambda(z)F_\mu(z) \). Hence \( F_\mu = 0 \) on the set of repelling periodic points and hence \( F_\mu = 0 \) on the Julia set. Then \( \mu = \overline{\partial}F_\mu = 0 \). The lemma is finished. □
Then by an inductive argument we have that

\[ F_\mu(f^n_\lambda(a)) = f^n_\lambda(a) \left( F_\mu(a) + \frac{c}{\lambda} \sum_{i=1}^{n} \frac{1}{(f^{n-2}_\lambda)'(a)} \right). \]

from above \( G_\mu(a) = \frac{a f_\lambda(t)c}{\lambda} \), where the constant \( c = \lambda'(t) \) and by the lemma 4 above \( c \neq 0. \)

\[ F_\mu(f^n_\lambda(a)) = f^n_\lambda(a) \left( F_\mu(a) + \frac{ac}{\lambda^2} \sum_{i=1}^{n} \frac{1}{(f^{n-2}_\lambda)'(a)} \right) \]

3.2. Proof of Theorem 1. Firstly we show (3). Such that \( \frac{|(f^{n+1}_\lambda)'(0)|}{|(f^n_\lambda)'(0)|} = |\lambda f^{n+1}_\lambda(0)| \), then assumption either

\[ \lim_{n \to \infty} \sup \frac{|(f^{n+1}_\lambda)'(1)|}{|(f^n_\lambda)'(1)|} \leq C < \infty \]

or

\[ \lim_{n \to \infty} \inf \frac{|(f^{n+1}_\lambda)'(1)|}{|(f^n_\lambda)'(1)|} > 0. \]

implies either \( X_\lambda \) is a compact subset of the plane or \( 0 \notin X_\lambda \), respectively. Assume \( F(f_\lambda) = \emptyset \), then an application of the theorem 3 implies a contradiction with \( \lim_{n \to \infty} |(f^n_\lambda)'(0)| = 0 \). Hence we are done.

Now we show (1) and (2). Assume \( f_\lambda \) is stable.

From the equation (1) above, we have that

\[ \frac{F_\mu(f^n_\lambda(a))}{(f^n_\lambda)'(a)} = F_\mu(a) + \frac{ac}{\lambda^2} \sum_{i=1}^{n} \frac{1}{(f^{n-2}_\lambda)'(a)} \]

From [11] we have the following inequality

\[ |F_\mu(a)| \leq M |a| \log |a|, \]

where \( M \) is a constant depending only on \( \mu \). Applying this estimate above we obtain:

\[ \frac{|F_\mu(f^n_\lambda(a))|}{|(f^n_\lambda)'(a)|} \leq M |f^n_\lambda(a)| \log |f^n_\lambda(a)|. \]

Easy calculation shows \( \log |f^n_\lambda(a)| \) and \( (f^n_\lambda)'(a) = \lambda^2 f^n_\lambda(a) f^{n-1}_\lambda(a)(f^{n-2}_\lambda)'(a). \) Hence

\[ \frac{|F_\mu(f^n_\lambda(a))|}{|(f^n_\lambda)'(a)|} \leq \frac{M}{\lambda^4}. \]

Now let \( n_j \) be the sequence from the assumptions of theorem 1 items (1)-(2) and the point \( a = 1 \). Since \( F_\mu(1) = 0 \), then from the equation 2 we obtain the following equation:

\[ \frac{F_\mu(f^{n_j+2}_\lambda(1))}{(f^{n_j+2}_\lambda)'(1)} = \frac{c}{\lambda} + \frac{c}{\lambda^2} \sum_{i=2}^{n_j+2} \frac{1}{(f^{i-2}_\lambda)'(1)} = \frac{c}{\lambda} \cdot S_{n_j}. \]

Then this equation produces a contradiction in the both cases with the hypothesis over \( S_{n_j} \), so \( f_\lambda \) is unstable.
4. Calculation of the Ruelle Operator

In this section we calculate the action of the Ruelle operator on the family of rational functions $\gamma_a(z) = \frac{a(z-1)}{z^2-1(z-a)}$, such that $a \neq 0, 1$. Let us recall that any rational integrable differential is a linear combination of such $\gamma_a(z)$.

Let $S = \mathbb{C}\setminus\{0,1\}$ be the trice punctured sphere.

Proposition 2.

$$R^*_\lambda(\gamma_a(z)) = \frac{1}{(f_\lambda)'(a)} \gamma_{f_\lambda(a)}(z) - \frac{a}{(f_\lambda)'(1)} \gamma_{f_\lambda(1)}(z).$$

Proof. Let $h_a(z) = R^*_\lambda(\gamma_a(z)) - \frac{1}{(f_\lambda)'(a)} \gamma_{f_\lambda(a)}(z) + \frac{a}{(f_\lambda)'(1)} \gamma_{f_\lambda(1)}(z)$ be a function. Our aim is to show that $h_a(z)$ defines a holomorphic integrable function on the surface $S$, hence $h_a(z) = 0$ and we are done. By the lemma 3 the function $h_a(z)$ is integrable over the plane. Therefore it is enough to show that $h_a(z)$ is holomorphic on $S$.

Let $\varphi \in C^\infty(S)$ be any differentiable function with compact support in $S$. Then

$$\int_C \varphi h_a(z) = \int_C B_\lambda(\varphi) \gamma_a(z) - \frac{1}{(f_\lambda)'(a)} \int_C \varphi \gamma_{f_\lambda(a)}(z) + \frac{a}{(f_\lambda)'(1)} \int_C \varphi \gamma_{f_\lambda(1)}(z) =$$

$$= \int_C \varphi(f_\lambda) \frac{(f_\lambda)^\prime}{(f_\lambda)^\prime} \gamma_a(z) - \frac{1}{(f_\lambda)'(a)} \int_C \varphi \gamma_{f_\lambda(a)}(z) + \frac{a}{(f_\lambda)'(1)} \int_C \varphi \gamma_{f_\lambda(1)}(z) = (*)$$

On the other hand

$$\int_C \varphi(f_\lambda) \frac{(f_\lambda)^\prime}{(f_\lambda)^\prime} \gamma_a(z) = (a-1) \int_C \frac{1}{z(z-1)(z-a)(f_\lambda)^\prime} \int_C (\varphi \circ f_\lambda) \frac{1}{z(f_\lambda)^\prime} \frac{z}{f_\lambda)^\prime} =$$

$$= (a-1) \int_C \frac{1}{z(z-1)(f_\lambda)^\prime} - a \int_C \frac{1}{z(z-a)(f_\lambda)^\prime}.$$ 

Such that always $\varphi(0) = 0$, $\varphi(1) = 0$. Then

$$(a-1) \int_C \frac{1}{z(z-1)(f_\lambda)^\prime} = \frac{a-1}{f_\lambda(0)^\prime} \varphi(f_\lambda(0)) = 0$$

$$a \int_C \frac{1}{z(z-1)(f_\lambda)^\prime} = \frac{a}{f_\lambda(1)^\prime} \varphi(f_\lambda(1))$$

$$\int_C \frac{1}{z(z-a)(f_\lambda)^\prime} = \frac{1}{f_\lambda(a)^\prime} \varphi(f_\lambda(a))$$

the same decompositions show

$$\frac{1}{(f_\lambda)'(a)} \int_C \varphi \gamma_{f_\lambda(a)}(z) = \frac{1}{(f_\lambda)'(a)} \varphi(f_\lambda(a))$$

$$\frac{a}{(f_\lambda)'(1)} \int_C \varphi \gamma_{f_\lambda(1)}(z) = \frac{a}{(f_\lambda)'(1)} \varphi(f_\lambda(1))$$

and as a result we obtain

$$(*) = 0.$$

By the Weyl’s Lemma $h_a(z)$ is a holomorphic function on $S$. Hence we are done. □
Corollary 1. If $F(f_{\lambda}) = \emptyset$ and $\mu \neq 0 \in B$, then

$$G_{\mu}(a) = \frac{af_{\lambda}'(a)}{f_{\lambda}'(1)}F_{\mu}(f_{\lambda}(1))$$

Proof. Let $\mu \neq 0 \in B$ be invariant Beltrami differential for $f_{\lambda}$, then by the proposition 3 we have

$$\pi F_{\mu}(a) = \int \int \gamma_{a}(z)\mu = \int \int \mathcal{R}_{\lambda}^{*}(\gamma_{a}(z))\mu = \frac{1}{f_{\lambda}'(1)}(-\pi)F_{\mu}(f_{\lambda}(a)) - \frac{a}{f_{\lambda}'(1)}(-\pi)F_{\mu}(f_{\lambda}(1)).$$

Hence

$$F_{\mu}(a) = \frac{1}{f_{\lambda}'(a)}F_{\mu}(f_{\lambda}(a)) - \frac{a}{f_{\lambda}'(1)}F_{\mu}(f_{\lambda}(1)),$$

and

$$G_{\mu}(a) = F_{\mu}(f_{\lambda}(a)) - f_{\lambda}'(a)F_{\mu}(a) = \frac{af_{\lambda}'(a)}{f_{\lambda}'(1)}F_{\mu}(f_{\lambda}(1)).$$

Now, by the linearity of the Ruelle operator together with an easy induction argument, for any $n \geq 0$ we have

$$(*) \quad (\mathcal{R}_{\lambda})^{n}(\gamma_{a}(z)) = \frac{1}{(f_{\lambda}')^{n}(a)}\gamma f_{\lambda}(a)(z) - \frac{f_{\lambda}^{n-1}(1)}{(f_{\lambda}')^{n-1}(a)f_{\lambda}'(1)}\gamma f_{\lambda}(1)(z) - \frac{f_{\lambda}^{n-2}(1)}{(f_{\lambda}')^{n-2}(a)f_{\lambda}'(1)}\mathcal{R}_{\lambda}(\gamma f_{\lambda}(1)(z)) - \cdots - \frac{a}{f_{\lambda}'(1)}(\mathcal{R}_{\lambda})^{n-1}(\gamma f_{\lambda}(1)(z)).$$

Define the following series

$$B(a) = \frac{1}{f_{\lambda}'(1)}\sum_{j=1}^{\infty} \frac{f_{\lambda}^{j-1}(a)}{(f_{\lambda}')^{j-1}(a)}.$$

5. Proof of the Theorem 2

Assume $f_{\lambda}$ is a stable map, then the summability of the singular value implies $F(f_{\lambda}) = \emptyset$.

Let $\mu \neq 0$ be an invariant Beltrami differential. Then the formula $(*)$ above, the invariance of $\mu$, and the definition of the potential $F_{\mu}$ give the following

$$(**) \quad F_{\mu}(a) = \frac{1}{f_{\lambda}'(a)}F_{\mu}(f_{\lambda}'(a)) - B_{n}(a)F_{\mu}(f_{\lambda}(1)),$$

where $B_{n}(a)$ is the $n$th partial sum of the series $B(a)$ above.

Let $a$ be a summable point, then the series $B(a)$ is absolutely convergent and by the arguments of the theorem 1, item (1), the expression $\frac{1}{f_{\lambda}'(a)}F_{\mu}(f_{\lambda}'(a)) \to 0$ as $n \to \infty$.

Then passing to the limit in the formula $(**)$ above we have:

$$F_{\mu}(a) = -B(a)F_{\mu}(f_{\lambda}(1))$$

Now set $a = f_{\lambda}(1)$, then:
and we have two possibilities:

1) \( F_\mu(f_\lambda(1)) = 0 \)

Then by the Corollary 1, \( G_\mu = 0 \) and by the lemma 4, \( \mu = 0 \) which contradicts the assumption above.

2) \( B(f_\lambda(1)) = -1 \).

Now we finish the theorem 2 in 3 steps. Let \( \varphi \) be the following series

\[
\varphi(z) := \sum_{n \geq 0} \frac{1}{(f_\lambda^n)'(f_\lambda(1))} \gamma f_\lambda^n(f_\lambda(1))(z),
\]

then summability of the point \( z = 0 \) implies \( \varphi \in L_1(\mathbb{C}) \).

In the first step we show that under assumption 2) above, the function \( |\varphi| \) presents a density of a finite, invariant measure which is absolutely continuous with respect to Lebesgue measure on the plane.

**Lemma 5.** Under assumption (2) above we have:

\[
R^*_\lambda(|\varphi|) = |\varphi|.
\]

**Proof.** For any \( n \geq 0 \), by the formula (**) we have the following expression

\[
R^*_\lambda\left( \frac{1}{(f_\lambda^n)'(f_\lambda(1))} \gamma f_\lambda^n(f_\lambda(1))(z) \right) = \frac{1}{(f_\lambda^{n+1})'(f_\lambda(1))} \gamma f_\lambda^{n+1}(f_\lambda(1))(z) - \frac{1}{f_\lambda'(1)} \gamma f_\lambda(1)(z) \frac{f_\lambda^n(f_\lambda(1))}{(f_\lambda)'(f_\lambda(1))}.
\]

Then summation over all \( n \geq 0 \) gives

\[
R^*_\lambda(\varphi) = R^*_\lambda\left( \sum_{n \geq 0} \frac{\gamma f_\lambda^n(f_\lambda(1))(z)}{(f_\lambda^n)'(f_\lambda(1))} \right) = \sum_{n \geq 0} \frac{\gamma f_\lambda^{n+1}(f_\lambda(1))(z)}{(f_\lambda^{n+1})'(f_\lambda(1))} - \frac{1}{f_\lambda'(1)} \gamma f_\lambda(1)(z) \sum_{n \geq 0} \frac{f_\lambda^n(f_\lambda(1))}{(f_\lambda)'(f_\lambda(1))} = \varphi(z) - \gamma f_\lambda(1)(z) - \gamma f_\lambda(1)(z) [B(f_\lambda(1))] = \varphi(z)
\]

by hypothesis.

\[\square\]

**Lemma 6.** In assumption of the lemma 5 above the function \( |\varphi| \) is a fixed point for the modulus of the Ruelle operator,

\[
|R^*_\lambda(|\varphi|)| = |\varphi|.
\]

**Proof.** We recall that by definition, for every function \( \varphi \)

\[
|R^*_\lambda(|\varphi|)| = \sum_{\zeta_i} |\varphi(\zeta_i)| |\zeta_i|^2
\]

where summation is over all branches \( \zeta_i \) of inverses of \( f_\lambda(z) = e^{\lambda z} \).

By assumption
Now define for each index \( i \), \( \alpha_i = \varphi(\zeta_i)(\zeta'_i)^2 \), \( \beta_i = \sum_{j \neq i} \varphi(\zeta_j)(\zeta'_j)^2 = \varphi - \alpha_i \).

With this notations we have

\[
\| \varphi \| = \| R_{\lambda}^*(\varphi) \| = \iint_{\mathcal{C}} \left| \sum_{i} \varphi(\zeta_i)(\zeta'_i)^2 \right|.
\]

Hence all inequalities above are really equalities, then for each index \( i \) we have

\[
\iint_{\mathcal{C}} |\alpha_i + \beta_i| = \iint_{\mathcal{C}} |\alpha_i| + \iint_{\mathcal{C}} |\beta_i|
\]

which implies that \( |\alpha_i + \beta_i| = |\alpha_i| + |\beta_i| \) almost everywhere with respect to Lebesgue measure. Then for each index \( i \)

\[
|\alpha_i + \beta_i| = |\alpha_i + \sum_{j \neq i} \varphi(\zeta_j)(\zeta'_j)^2| = |\alpha_i| + \sum_{j \neq i} |\alpha_j|,
\]

and by the induction we obtain

\[
|\sum_i \alpha_i| = \sum_i |\alpha_i|.
\]

That implies that

\[
\| \varphi \| = \sum_i |\alpha_i| = \sum_i |\alpha_i| = \sum_i |\varphi(\zeta_i)||\zeta'_i|^2 = |R_{\lambda}^*(\| \varphi \|)|.
\]

□

By the lemma 3 the measure \( \sigma(A) = \iint_A |\varphi(z)| \) is a non-negative invariant absolutely continuous probability measure, where \( A \subset \mathcal{C} \) is a measurable set. We have complete the first step.

Let \( Y = \mathbb{C} - X_{\lambda} \) be the complement to the postsingular set \( X_{\lambda} \). In the second step we show that \( \varphi = 0 \) identically on \( Y \).

In the notation of the lemmas above we have:

**Lemma 7.** If \( \alpha_j \neq 0 \) identically on \( Y \), then the function \( k_j = \frac{\beta_j}{\alpha_j} \) is a non-negative constant on any component of \( Y \).

**Proof.** We have \( |1 + \frac{\beta_j}{\alpha_j}| = 1 + \frac{\beta_j}{\alpha_j} \), then if \( \frac{\beta_j}{\alpha_j} = \gamma_1^j + i\gamma_2^j \) we have

\[
\left(1+(\gamma_1^j)^2\right)^2 + (\gamma_2^j)^2 = \left(1+\sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}\right)^2 = \left(1+\sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}\right)^2 = 1+(\gamma_1^j)^2 + (\gamma_2^j)^2 + 2\sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}.
\]

Hence \( \gamma_2^j = 0 \) and \( \frac{\beta_j}{\alpha_j} = \gamma_1^j \) is a real-valued function but \( \frac{\beta_j}{\alpha_j} \) is meromorphic function. So \( \gamma_1^j = k_j \) is constant on every connected component of \( Y \) and the condition \( |1 + k_j| = 1 + |k_j| \) shows \( k_j \geq 0 \).
Definition 6. A measurable set $A \in \hat{\mathbb{C}}$ is called back wandering if and only if $m(f^{-n}(A) \cap f^{-k}(A)) = 0$, for $k \neq n$.

Corollary 2. If $\varphi \neq 0$ on $Y$, then (i) $m(X_\lambda) = 0$, where $m$ is the Lebesgue measure and (ii) $\frac{\mu}{|\varphi|}$ defines an invariant Beltrami differential.

Proof. (i) If $m(X_\lambda) > 0$, then $m(f_\lambda^{-1}(X_0)) = 0$ so $m(f_\lambda^{-1}(X_0) - X_0) > 0$ since $f_\lambda^{-1}(X_0) \neq X_0$, $X_\lambda \notin \mathbb{C}$, denote by $Z_1 = f_\lambda^{-1}(X_\lambda) - X_\lambda$. Then $Z_1$ is back wandering thus $\varphi = 0$, on the orbit of $Z_1$, which is dense in $J(f_\lambda)$, hence $\varphi = 0$ in $Y$. Therefore, $m(X_\lambda) = 0$.

(ii) By notations and the proofs of Lemmas 5 and 6 we have $k_i(x) = \frac{\partial x}{\alpha_i} = \frac{\varphi}{\alpha_i} - 1$ so $\varphi(x) = (1 + k_i(x))\alpha_i = (1 + k_i(x))(\varphi(\zeta_i(x))(\zeta_i')^2(x)$. Hence,

$$\frac{\varphi(x)}{||\varphi(x)||} = \frac{(1 + k_i(x))\varphi(\zeta_i(x))(\zeta_i')^2(x)}{(1 + k_i(x))|\varphi(\zeta_i(x)||(|\zeta_i')^2(x))},$$

and so for any branch $\zeta_i$ we have

$$\mu = \frac{\varphi(\zeta_i)}{|\varphi(\zeta_i)|} = \frac{\varphi(\zeta_i)\zeta_i'}{|\varphi(\zeta_i)|} = \mu(\zeta_i)\frac{\zeta_i'}{\zeta_i}$$

as result $\mu = \frac{\varphi}{|\varphi|}$ is an invariant line field. Thus the corollary is proved.

Now we prove the main result of the second step.

Proposition 3. If $\varphi \neq 0$ on $Y$, then $f_\lambda$ is unstable.

Proof. Let us show first that $X_\lambda = \bigcup f_\lambda^i(1)$. We will use a McMullen argument as in [2]. By Corollary 2, $\mu = \frac{\varphi}{|\varphi|}$ is an invariant Beltrami differential. That implies that $\varphi$ is dual to $\mu$ and $\varphi$ is defined by $\mu$ up to a constant. We will construct a meromorphic function $\psi$, dual to $\mu$ and such that $\psi$ has finite number of poles on each disc $D_R$ of radius $R$ centered at 0.

For that suppose that for $z \in \mathbb{C}$ there exists a branch $g$ of a suitable $f_\lambda^n$, such that $g(U_z) \subset Y$, where $U_z$ is a neighborhood of $z$. Then define $\psi(\zeta) = \varphi(g(\zeta))(g')^2(\zeta)$, for all $\zeta \in U_z$. Note that $\psi(\eta)$ is dual to $\mu$ and has no poles in $U_z$. If there is no such branch $g$, then $\zeta$ is in the postsingular set, and there is a branched covering $F$ from a neighborhood of $\zeta$ to $U_z$, then define $\psi(\zeta) = F^*(\varphi)$, with $F^*$ the Ruelle operator of $F$. The map $\psi$ is a meromorphic function dual to $\mu$ in $U_z$ and has finite number of poles.

By considering $R \to \infty$ we construct a meromorphic function $\psi$ which is dual to $\mu$. The poles of $\psi$ forms a discrete set accumulating to $z = \infty$. Since $\varphi$ is a dual to $\mu$, then $\varphi = C \cdot \psi$, where $C$ is a constant. Hence $X_\lambda = \bigcup f_\lambda^i(1)$ is a discrete closed set accumulating to $z = \infty$ and $Y$ is connected.

By the Corollary 2 the functions $k_i$ are globally defined constants on $Y$. Moreover by the argument of the lemma 7 $\varphi(x) = (1 + k_i)(\varphi(\zeta_i(x))(\zeta_i')^2(x)$ for any $x \in \mathbb{C}$, thus $k_i = k_j$ for any $i, j$.

So we have $\sum_i \frac{\varphi(x)}{1+k_i} = \sum_i \varphi(\zeta_i(x))(\zeta_i')^2(x) = \varphi(x)$, since the first term of the equation is infinite, this can be only iff $\varphi = 0$.

□
Now to obtain a contradiction, in the step 3 we show that if \( f_\lambda \in W \) is a structurally stable, then \( \varphi \neq 0 \) identically on \( Y \).

The following proposition is proved in [8].

Proposition 4. Let \( a_i \in \mathbb{C}, a_i \neq a_j \), for \( i \neq j \) be points such that \( Z = \bigcup_i a_i \subset \mathbb{C} \) is a compact set. Let \( b_i \neq 0 \) be complex numbers such that the series \( \sum b_i \) is absolutely convergent. Then the function \( I(z) = \sum_i \frac{1}{z-a_i} \neq 0 \) identically on \( Y = \mathbb{C} \setminus Z \) in any of the following cases

1. the set \( Z \) has zero Lebesgue measure.
2. if diameters of components of \( \mathbb{C} \setminus Z \) uniformly bounded below from zero.
3. If \( O_j \) denote the components of \( Y \), then \( Z \in \cup_j \partial O_j \).

Proposition 5. Let \( f_\lambda \) be the exponential map and \( 0 \) a summable point. Then \( \varphi(z) \neq 0 \) identically on \( Y \) in any of the following cases

1. if \( 0 \notin X_\lambda \),
2. if diameters of components of \( Y \) is uniformly bounded below from zero,
3. if \( m(X_\lambda) = 0 \), where \( m \) is the Lebesgue measure on \( \mathbb{C} \).

Proof. Let us prove (1). Denote \( d_\lambda = f_\lambda(1) \).

Assume now that the set \( X_\lambda \) is bounded. Then by Proposition 4 we have that \( \varphi(z) = \frac{C_1}{z} + \frac{C_2}{(f_\lambda)'(d_\lambda)} + \sum_i \frac{1}{(f_\lambda)'(d_\lambda)(z-f_\lambda(d_\lambda))} = I(z) \neq 0 \). The other cases follows directly from Proposition 4 also.

Now let \( X_\lambda \) be unbounded. Let \( y \in \mathbb{C} \) be a point such that the point \( 1 - y \in Y \), then the map \( g(z) = \frac{ze^{\lambda z} - \lambda ze^{-\lambda z}}{e^{\lambda z} - e^{-\lambda z}} \) maps \( X_\lambda \) into \( \mathbb{C} \).

Let us consider the function \( G(z) = \frac{1}{y} \sum_i \frac{(f_\lambda'(d_\lambda))^{-1}}{(f_\lambda'(d_\lambda))^2} \) and \( C_2 = \sum_i \frac{f_\lambda'(d_\lambda)}{(f_\lambda'(d_\lambda))^2} \), then by proposition 4 \( G(z) \neq 0 \) identically on \( g(Y) \).

Now we Claim that \( G(g(z))g'(z) = \phi(z) \).

Proof of the claim. Let us define \( C_1 = \sum_i \frac{(f_\lambda'(d_\lambda))^{-1}}{(f_\lambda'+(d_\lambda))^2} \) and \( C_2 = \sum_i \frac{f_\lambda'(d_\lambda)}{(f_\lambda'(d_\lambda))^2} \), then we have

\[
\frac{C_1}{g(z)} = \frac{C_1(z+y-1)}{yz} \quad \text{and} \quad \frac{C_2}{g(z) - 1} = \frac{C_2(z+y-1)}{(y-1)(z-1)}
\]

and for any \( n \)

\[
\frac{1}{g(z) - g(f_\lambda^n(d_\lambda))} = \frac{(z+y-1)(f_\lambda^n(d_\lambda) + y - 1)}{y(z-f_\lambda^n(d_\lambda))} = \frac{1}{y(y-1)} \left( \frac{(z+y-1)^2}{z-f_\lambda^n(d_\lambda)} + 1 - y - z \right),
\]

then

\[
G(g(z)) = \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)} + \sum_i \frac{1}{(f_\lambda'(d_\lambda))g(z) - g(f_\lambda(d_\lambda))} = \frac{1}{y(y-1)} \left( 1 - y - z \right) \sum_i \frac{1}{(f_\lambda'(d_\lambda))^2} + (z+y-1)^2 \sum_i \frac{1}{(f_\lambda'(d_\lambda))(z-f_\lambda'(d_\lambda))} + \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)} = *
\]
and

\[ * = \frac{1}{g'(z)} \left( \phi(z) + \sum \frac{f_i'(d_\lambda)}{(f_i')'(d_\lambda)} \right) - \sum \frac{f_i'(d_\lambda)}{z - 1} + \sum \frac{1}{1 - y - z} + \right. \\
\left. + \frac{1}{g'(z)} \left( \frac{C_1(y - 1)}{z(z + y - 1)} - \frac{C_2y}{(z - 1)(z + y - 1)} \right) = \right. \\
\left. = \frac{\phi(z)}{g'(z)}. \right. \\

Hence \( \phi(z) = 0 \) identically on \( Y \) if and only if \( G(z) = 0 \) identically on \( g(Y) \). So by proposition 4 we complete the proof of this proposition.

Step 3 and the theorem 2 are finished.

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