More hilltop inflation models

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Abstract. Using analytic expressions, we explore the parameter space for hilltop inflation models with a potential of the form $V_0 \pm m^2 \phi^2 - a\phi^p$. With the positive sign and $p > 2$ this converts the original hybrid inflation model into a hilltop model, allowing the spectral index to agree with the observed value $n = 0.95$. In some cases the observed value is theoretically favored, while in others there is only the generic prediction $|n - 1| \lesssim 1$.

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1. Introduction

The general idea of what has been called [1] hilltop inflation is that cosmological scales leave the horizon while the inflaton is near the top of a hill, with its potential still concave-downward. This allows the initial condition to be set by an era of eternal inflation, whose indefinite duration may remove any concern about the probability of arriving at the hilltop in the first place. Hilltop inflation also ensures that the spectral tilt is negative, though it may not be guaranteed that the amount of tilt is small as is required by observation.

It was noticed earlier [1] that hilltop inflation is more natural than one might think. Starting with any rather flat potential, it is easy to generate a maximum with a reasonably-looking additional term. In this paper we pursue that line of thinking, by considering a potential which covers a range of possibilities, and is yet simple enough to yield analytic formulas for the predictions.

We shall take for granted the basic ideas of slow-roll inflation model-building, as explained for instance in [2]–[5]. Cosmological scales leave the horizon during about ten $e$-folds of inflation. The value $\phi$ of the inflaton field when $N$ $e$-folds of inflation remain is

$$N = M_P^2 \int_{\phi_{\text{end}}}^{\phi(N)} \frac{V}{V'} \, d\phi,$$

where $M_P = (8\pi G)^{-1/2} = 2.2 \times 10^{18}$ GeV. Cosmological scales leave the horizon during about ten $e$-folds of inflation, starting with the largest scale $k = H_0$. (Here $k$ is the
present value of the comoving wavenumber and \( H_0 \) the present Hubble parameter.) For a standard cosmology after inflation, the value of \( N \) when the latter scale leaves the horizon is typically in the range 50–60. In the following we give the predictions for \( N = 60 \).

We assume that the primordial curvature perturbation \( \zeta \) is generated from the vacuum fluctuation of the inflaton, instead of later by some curvaton-type mechanism. The spectrum of the tensor perturbation, as a fraction \( r \) of the observed spectrum of the curvature perturbation, is \( r = 16\epsilon \), where \( \epsilon = (M_P^2/2)(V'/V)^2 \). (Here and in the following all functions of \( \phi \) are to be evaluated when the relevant scale leaves the horizon.) If the potential remains concave-downward for the rest of inflation, this implies \[ r < 0.002 \left( \frac{\Delta \phi}{M_P} \right)^2 \left( \frac{60}{N} \right)^2, \] where \( \Delta \phi \) is the variation of the inflaton field. We will here demand only that the shape of the potential is such as to give \( r \ll 10^{-2} \). This is consistent with present observations.

The cmb anisotropy determines the magnitude of the spectrum of the curvature perturbation \[ P_\zeta = (5 \times 10^{-5})^2, \] with an uncertainty which is negligible in the present context. Invoking the slow-roll prediction for \( P_\zeta \) one finds

\[
 r = 16\epsilon = \left( \frac{V_0^{1/4}}{3.3 \times 10^{16} \text{ GeV}} \right)^4, \tag{3}
\]

We call the second equality the cmb normalization. Our requirement \( r < 0.01 \) corresponds to

\[
 V_0^{1/4} < 1.0 \times 10^{16} \text{ GeV} = 4.2 \times 10^{-3} M_P. \tag{4}
\]

The spectral index of the curvature perturbation is defined by

\[
 n - 1 = \frac{dP_\zeta}{d \ln k} = -\frac{dP_\zeta}{dN}, \tag{5}
\]

where \( N \) is the number of \( e \)-folds remaining after the scale \( k \) leaves the horizon and the final equality assumes slow roll. The slow-roll prediction is

\[
 n = 1 + 2\eta - (3/8)r, \tag{6}
\]

where \( \eta \equiv M_P^2 V''/V \). Assuming \( r \lesssim 10^{-1} \), the cmb anisotropy requires [6]

\[
 n = 0.948^{+0.015}_{-0.018}. \tag{7}
\]

An analysis including other types of observation [7] finds instead \( n = 0.97 \pm 0.01 \). When we invoke the observational value we take the central value of the cmb result.

Now comes an important point. Since we are assuming \( r \lesssim 10^{-2} \), the observed value of \( n \) means that we can take the prediction to be simply \( n = 1 + 2\eta \) if it is to fit observation. We conclude that if a slow-roll model of inflation is to generate the observed curvature perturbation with \( r \lesssim 10^{-2} \), its potential must be concave-downward while cosmological scales leave the horizon, with \( \eta \equiv M_P^2 V''/V \simeq -0.02 \).

We should also consider the running \( n' \equiv d n/\ln k = -d n/dN \). The models we consider here give \( n' > 0 \) and this condition should be imposed as a prior when determining the upper bound on \( n' \) allowed by observation. As will be shown elsewhere [8], current observations require in that case something like \( n' < 0.01 \). We impose this constraint where relevant.
2. The potential

We consider a potential of the following form, with $\lambda$ positive.

$$V(\phi) = V_0 \pm \frac{1}{2} m^2 \phi^2 - \lambda \frac{\phi^p}{M_P^{p-4}} + \cdots$$

$$\equiv V_0 \left( 1 + \frac{1}{2} \eta_0 \frac{\phi^2}{M_P^2} \right) - \lambda \frac{\phi^p}{M_P^{p-4}} + \cdots,$$

with

$$\eta_0 = \frac{\pm m^2 M_P^2}{V_0}.$$  \hspace{1cm} (10)

The additional terms are presumed negligible during inflation and we require that $\phi$ rolls towards the origin. There is no need to assume that $p$ is integral if this is just regarded as a parameterization of the potential.

To justify keeping just two terms, especially if $p$ is an integer, we would like $\phi \ll M_P$ after the largest cosmological scale leaves the horizon (which we are taking to correspond to $N = 60$). We will just require that the maximum of $\phi_{\text{end}}$ and $\phi(N = 60)$ is $\lesssim M_P$.

We consider three different regimes of $\eta_0$ and $p$.

1. **Two-term approximation about the hilltop.** The choice $\eta_0 \leq 0$ and $p > 2$ sets $\phi = 0$ at the hilltop, and adds a higher power to the quadratic term. This is sketched in figure 1.

2. **Hilltop mutated/brane hybrid inflation.** The choice $\eta_0 < 0$ and $p < 0$ converts these models to hilltop models, as sketched in figure 2.

3. **Hilltop tree-level hybrid inflation.** The choice $\eta_0 > 0$ and $p > 2$ converts the original hybrid inflation model to a hilltop model, as illustrated in figure 3.
To achieve slow-roll with the potential (9), we need $V \simeq V_0$ giving

$$\frac{V'}{V} = \eta_0 \frac{\phi}{M_P^2} - p\lambda \frac{\phi^{p-1}}{V_0 M_{P}^{p-4}}$$

(11)

$$\frac{V''}{V} = \frac{\eta_0}{M_P^2} - p(p-1)\lambda \frac{\phi^{p-2}}{V_0 M_{P}^{p-4}}.$$  

(12)

Then equation (1) has the analytic solution [9]

$$\left( \frac{\phi}{M_P} \right)^{p-2} = \left( \frac{V_0}{M_P^4} \right) \frac{\eta_0 e^{(p-2)\eta_0 N}}{\eta_0 x + p\lambda \left( e^{(p-2)\eta_0 N} - 1 \right)}$$

(13)

$$x = \left( \frac{V_0}{M_P^4} \right) \left( \frac{M_P}{\phi_{\text{end}}} \right)^{p-2},$$

(14)

leading to the predictions

$$P_{\zeta} = \frac{1}{12\pi^2} \left( \frac{V_0}{M_P^4} \right)^{(p-4)/(p-2)} e^{-2\eta_0 N} \left[ \frac{p\lambda \left( e^{(p-2)\eta_0 N} - 1 \right) + \eta_0 x}{\eta_0 \left( e^{(p-2)(p-2)/2} - \eta_0 \left( e^{(p-2)(p-2)/2} - \eta_0 \phi \right)^2 \right)} \right]$$

(15)
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\[ n - 1 = 2\eta_0 \left[ 1 - \frac{\lambda p(p - 1)e^{(p-2)\eta_0 N}}{\eta_0 x + p\lambda(e^{(p-2)\eta_0 N} - 1)} \right] \]  
(16)

\[ n' = 2\eta_0^2 \lambda p(p - 1)(p - 2) \frac{e^{(p-2)\eta_0 N}(\eta_0 x - p\lambda)}{(\eta_0 x + p\lambda(e^{(p-2)\eta_0 N} - 1))^2} \]  
(17)

In a supergravity theory where \(|\eta_0|\) vanishes in the global supersymmetric limit, the generic expectation is \(|\eta| \sim 1\). For slow-roll inflation per se, all we need is \(|\eta_0| \ll 1\) which does not necessarily imply significant fine-tuning. The problem (usually called the \(\eta\) problem) comes though when one tries to understand why the spectral index given by equation (6) is so close to 1. We will keep this issue in mind and review the final situation in section 7 after considering in turn each of the three cases.

3. Two-term approximation

The two-term approximation may apply in a wide variety of cases, and has a long history. In using our parameterization we choose the origin \(\phi = 0\) to be a maximum of \(V\) (at least in the regime \(\phi > 0\)). The necessary condition \(V'(0) = 0\) might be ensured by a symmetry (with the origin the fixed point of the transformation) but that is not essential. The maximum of the potential might occur because the potential is periodic (corresponding to \(\phi\) being a pseudo Nambu–Goldstone boson (PNG)) or else through the interplay of two terms as in our cases (2) and (3).

We will focus on the non-hybrid case, where \(V(\phi)\) descends smoothly to a minimum with \(V = 0\). Then we might be dealing with modular inflation corresponding to \(\Delta \phi \sim M_P\). (`Moduli’ allowing this kind of inflation may be expected in string theory as discussed, for instance, in [11].) Alternatively we might be dealing with a small-field model, corresponding to \(\Delta \phi\) some orders of magnitude below \(M_P\). If the origin is a fixed point of symmetries involving \(\phi\) we deal with what one might call new inflation. (The original new inflation model [12] (see also [15]) corresponds more or less to \(p = 4\) and \(\eta_0 = 0\) which we consider below.)

This case has been investigated in [9, 10], where the formulas equations (13)–(16) were first given. Less complete investigations were made earlier, keeping just the \(\phi^p\) term [2, 4, 5], both terms with \(p = 4\) [2, 28, 29], and both terms with generic \(p\) [30]. In the following we present a further investigation of this interesting case.

3.1. Spectral index

The potential is concave-downward throughout inflation. As a parameterization over a limited range it makes sense to allow \(p\) to be non-integral, with \(p \gtrsim 3\).

Since \(V'/V\) is increasing during inflation, one expects the value of \(\phi(N)\) given by equation (1) to be insensitive to \(\phi_{\text{end}}\), at least in some part of the parameter space. That would correspond to \(x = 0\) being a good approximation. We will not make that approximation, but instead take \(\phi_{\text{end}}\) (the point at which slow-roll fails) to be the point

\[ \phi_{\text{end}} \]
Figure 4. $p \geq 3$, contours of $n$ in the $\log(-\eta_0)-p$ plane.

where $\eta(N = 0) = -1$ which corresponds to $n(N = 0) = -1$ or

$$\frac{V_0}{M_P^4} = \frac{p(p-1)\lambda}{1-\eta_0} \left( \frac{\phi_{\text{end}}}{M_P} \right)^{p-2}. \quad (18)$$

Since slow-roll requires $|\eta_0| \ll 1$, our requirement $\phi_{\text{end}} \lesssim M_P$ becomes

$$V_0/M_P^4 \lesssim \lambda. \quad (19)$$

Using equation (18) we find that the spectral index depends only on $p$ and $\eta_0$. For $\eta_0 = 0$ we recover the known result [2]

$$n = 1 - 2 \left( \frac{p-1}{p-2} \right) \frac{1}{N}. \quad (20)$$

This is automatically within the observed range.

For $\lambda = 0$ we recover the known result

$$n - 1 = 2\eta_0. \quad (21)$$

In that case, the requirement $\phi_{\text{end}} < M_P$ requires $|\eta_0| \gtrsim 1$ corresponding to $n - 1 \simeq -1$. The value of $|\eta_0|$ is in mild conflict with the slow-roll requirement, and of course the value of $n$ is far below the observed one.

In figure 4 we plot contours of constant $n$ in the $p-\eta_0$ plane. The nearly horizontal lines correspond to the limit (20). The nearly vertical lines correspond to the limit (21) being practically attained at $\phi(N = 60)$, the term of $V$ proportional to $\phi^p$ ensuring that slow-roll ends at $\phi \lesssim M_P$.

### 3.2. Modular inflation

We now look at the cmb normalization, beginning with modular inflation. Here the minimum of $V$ is expected to be of the order of $M_P$, corresponding roughly to $\lambda = V_0/M_P^4$. We will impose that equality, leaving only the parameters $\eta_0$ and $p$.

As $\phi$ is not far below $M_P$ we should view equation (9) just as an approximation, valid hopefully for some $p \gtrsim 3$. In figure 6 we plot the cmb-normalized $V_0^{1/4}$ against $n$, for a few values of $\eta_0$ and the range $3 < p < 100$. This plot shows if we demand a high inflation scale in a modular inflation, then $n$ in a modular inflation model cannot be far below 1.
3.3. New inflation

For new inflation corresponding to $\phi_{\text{end}} \ll M_P$, it is reasonable to suppose that the term $\propto \phi^p$ is the leading next term in a power-series expansion, further terms being suppressed by the small value of $\phi$. Then $p$ will be an integer bigger than 2. The integer can be bigger than 3 because in new inflation the origin is supposed to be a fixed point of symmetries.

3.3.1. Case $p = 4$. Let us suppose that odd $p$ are forbidden by a symmetry $\phi \to -\phi$, making the leading term $p = 4$. This gives the original new inflation model [12], except that the slight running of $\lambda$ with $\phi$ invoked there is absent. A famously small value of $\lambda$ is required. The original version of the model, where $\lambda$ is a gauge coupling, was therefore rejected but it was soon pointed out [13] that $\lambda$ could instead be a Yukawa coupling making the small value perhaps acceptable. More recently it has been noticed [14] that the predicted spectral index is compatible with observation.

In the version of the last paragraph this model is not supersymmetric so that there is no natural expectation that $|\eta_0|$ will be significant, but still we are free to consider that case. Moreover, this potential with significant $|\eta_0|$ has been motivated in other ways, both non-supersymmetric [28] and supersymmetric [29]. The second case, however, is very fine-tuned [2]. It therefore appears that the case $p = 4$ is best realized without supersymmetry, especially if $|\eta_0|$ is included.

After imposing the cmb normalization there is one parameter which we take to be $\eta_0$. In figure 7 we show $n$ and $\log \lambda$ as functions of $\eta_0$. We see in figure 7 that increasing $|\eta_0|$ decreases both $\lambda$ and $n$, allowing the latter to be well below the observed value.

3.3.2. Case $p = 6$. The need for the quartic term to be very suppressed is a generic feature of inflation models after the cmb normalization is imposed [2]. Supposing it to be negligible and still taking $V$ to be even, we expect the leading term to be $p = 6$.

In figure 8, we show contours of $\log(V_0^{1/4}/M_P)$ in the $\eta_0 - \log \lambda$ plane, assuming that $x$ is given by equation (18). We also show the lines $\phi_{\text{end}} = 0.1M_P$ and $\phi_{\text{end}} = 1.0M_P$. In figures 9 and 10 we show $n$ against $\eta_0$. From figure 10, we can see the spectral index can be far below the observed value.

To suppress the quartic term we would like to impose supersymmetry. With a minimal Kahler potential we arrive at the $\eta_0 = 0$ case with a superpotential of the form [30]

$$W = V_0^{1/2} \Psi \left[ 1 - \left( \frac{\Phi}{v} \right)^{p/2} \right] + \cdots,$$

where the extra terms ensure $\Psi = 0$ without affecting the potential in the direction of the inflaton $\phi \equiv |\Phi|$.

4. Hilltop mutated hybrid inflation

Now we come to the second case, characterized by $\eta_0 < 0$ and $p < 0$. The prediction for $n$ is given in figure 5. Regarding the limit of large $p$ and small $|\eta_0|$, the remarks made for the previous case apply.

This case corresponds to what is called mutated hybrid inflation [22]–[24]. It differs from ordinary hybrid inflation in the following respect. Ordinary hybrid inflation assumes
Figure 5. $p < 0$, contours of $n$ in the $\log(-\eta_0)$-$p$ plane.

Figure 6. Model 1, $p = 3-100$, different values of $\eta_0$ in the $V_0^{1/4}$-$n$ plane. We use $\lambda = V_0/M_P^4$ which corresponds to modular inflation.

Figure 7. Model 1, $p = 4$, $n$ and $\log \lambda$ versus $\eta_0$. 
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Figure 8. Model 1, $p = 6$, contours of $\log(V_{1/4}^{1/4}/M_P)$ in the $\eta_0-\log \lambda$ plane. We also show the lines $\phi_{\text{end}} = 0.1 M_P$ and $\phi_{\text{end}} = 1.0 M_P$.

Figure 9. Model 1, $p = 6$, $n$ versus $\eta_0$.

A coupling between the inflaton field $\phi$ and the waterfall field $\chi$ which is some function of $\phi$ times $\chi^2$. This fixed $\chi$ at the origin during slow-roll inflation, which ends when $\phi$ passes through some critical value which destabilizes $\chi$. In mutated hybrid inflation, the coupling involves a higher power than $\chi^2$. As a result, the waterfall field is not fixed, but instead adjusts to minimize the potential as the inflaton field slowly rolls. The inflationary potential is $V(\phi, \chi(\phi))$ and inflation ends when slow-roll fails.

Until now, it has usually been assumed that $V$ with $\chi = 0$ is absolutely flat. The most general potential that has been considered [24] is then

$$ V = V_0 - A \chi^s + B \chi^q \phi^r. \quad (23) $$

When the potential is minimized by $\chi(\phi)$ at fixed $\phi$, the potential becomes

$$ V(\phi, \chi(\phi)) = V_0 - \left[ A \left( \frac{Bq}{As} \right)^{s/(s-q)} - B \left( \frac{Bq}{As} \right)^{q/(s-q)} \right] \phi^{-rs/(q-s)}. \quad (24) $$

This is of the form of equation (9) with $\eta_0 = 0$ and $p = -(sr)/(q-s)$. If $\phi = 0$ is a fixed point of the symmetries, it is quite reasonable to add a term $\pm \frac{1}{2} m^2 \phi^2$ to the
Figure 10. Model 1, $p = 6$, $n$ versus $\eta_0$. This plot shows we can have a large range of $n$. The limit of $n$ when $\eta_0 \to 0$ is $n = 0.959$.

above potential, to arrive at equation (9). More generally though, a linear term would be allowed in the potential and so it would be more reasonable to add a term $\pm \frac{1}{2} m^2 (\phi - \phi_0)^2$ with $\phi_0$ a parameter.

The latter is the case for the original mutated hybrid inflation model [22]. It is not covered by our parameterization, and we focus instead on what was called smooth hybrid inflation [23]. There the origin is a fixed point, with

$$V = \left( V_0^{1/2} - \frac{\chi^4}{16M_P^2} \right)^2 + \frac{\chi^6 \phi^2}{16M_P^4}. \quad (25)$$

During inflation we can drop the $\chi^8$ term to get equation (23) with $s = 4$, $q = 6$ and $r = 2$. This leads to equation (9) with $p = -4$ and

$$\lambda = (2/27)(V_0^{1/4}/M_P)^6, \quad (26)$$

and of course $\eta_0 = 0$. We can easily compare this with [23]. First, we use equation (11) and take the limit $\eta_0 \to 0$ for $p = -4$. We obtain $n = 1 - 5/180 \simeq 0.97$, which is exactly the same as [23]. Second, we can see from (10) and using the relation $V_0 = m^2 M_P^2/\eta_0$ we obtain $V_0/M_P^4 \simeq 1.12 \times 10^{-8} \lambda^{1/4}$. Using the above relation $\lambda = (2/27)(V_0^{1/4}/M_P)^6$ we can solve for $(V_0^{1/4}/M_P) \simeq 5.08 \times 10^{-4}$.

The inclusion of a nonzero $\eta_0$ for smooth hybrid inflation has been considered in [25] and we further investigate it now. In figure 12 we see that $|\eta_0|$ cannot be very big. From figures 11 and 13 we see that this requires a high inflation scale and a spectral index around the observed value.

The case $p = -4$ has also been derived in a colliding brane scenario [26]. Our potential does not apply in that case, because the origin will not be a fixed point of symmetries so that a linear term in $\phi$ is allowed which would go beyond our parameterization. (Also, non-canonical normalization is allowed in this case, though there is a significant regime of parameter space where the normalization is canonical.)

We also considered mutated hybrid inflation with $p = -2$, assuming that the origin is a fixed point of the symmetries. We plot the spectral index $n$ against $\eta_0$ in figures 15
Figure 11. $p = -4$, $\lambda = (2/27)(V_0^{1/4}/M_P)^6$, $V_0^{1/4}/M_P$ versus $\eta_0$.

Figure 12. Model 2, $p = -4$, We show the lines $V' = 0$, $\phi = 1.0 M_P$, and $\phi = 0.1 M_P$ in the $\eta_0$–log $\lambda$ plane. Below the line $V' = 0$, the inflaton rolls to the right. The allowed region in our model is the left lower corner in the plot. $\lambda_{\text{smooth}} = 1.27 \times 10^{-21}$ corresponds to smooth hybrid inflation.

Figure 13. Model 2, $p = -4$, $n$ versus $\eta_0$. 

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Figure 14. Model 2, \( p = -4 \), contours of \( \log(V_0^{1/4}/M_P) \) in the \( \eta_0 - \log \lambda \) plane. We require \( \phi < M_P \).

Figure 15. Model 2, \( p = -2 \), \( n \) versus \( \eta_0 \).

and 16 for \( p = -2 \). These plots show that basically we can have a large range of spectral index \( n \) by introducing nonzero \( \eta_0 \) in our models. We also plot contours of \( \log(V_0^{1/4}/M_P) \) in the \( \eta_0 - \log \lambda \) plane for the cases \( p = -4 \) and \( p = -2 \) in figures 14 and 17.

5. \( F \)- and \( D \)-term inflation

Now we consider the standard \( F \)- and \( D \)-term inflation models [16, 17]. Here, sticking to the simplest versions of those models, \( \lambda \phi^p/M_P^{p-4} \) is replaced by \( V_0(g^2/4\pi^2) \log(\phi/Q) \) with \( Q \) a constant of the order of \( \phi \). Since we are assuming \( V \simeq V_0 \), other, this is equivalent to taking the limit \( p \to 0 \) with \( \lambda p \) fixed at the value

\[
\lambda_p = -\frac{V_0 g^2}{M_P^4 4\pi^2}.
\]

This case has been analyzed for both positive [18]–[20] and [1, 21] negative \( \eta_0 \), but the latter choice leading to hilltop inflation is more interesting and makes it easier for the models to agree with observation and we analyze it further now.
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Figure 16. Model 2, $p = -2$, $n$ versus $\eta_0$. This plot shows we can have a large range of $n$. The limit of $n$ when $\eta_0 \to 0$ is $n = 0.975$.

Figure 17. Model 2, $p = -2$, contours of $\log(V_0^{1/4}/M_P)$ in $\eta_0$–log $\lambda$ plane. We require $\phi < M_P$.

We find
\[
\phi(N = 60) = \frac{g}{2\pi} \left[ -\eta_0/\left(1 + \eta_0\right) - \left(1 - e^{-120 \eta_0}\right)\right]^{1/2}
\tag{28}
\]
which is independent of $V_0$. This is shown in figure 18. We see that the requirement $\phi \ll M_P$ requires $g \ll 1$.

The spectral index is
\[
n = 1 + 2\eta_0 \left[ 1 - \frac{e^{-120 \eta_0}}{1 - e^{-120 \eta_0} + \eta_0/(1 + \eta_0)} \right], \tag{29}
\]
shown in figure 19.

The cmb normalization is
\[
\frac{V_0^{1/4}}{\sqrt{g}} = \left( \frac{2.5 \times 10^{-5} \sqrt{12} (-\eta_0)^{1/2} (1 + \eta_0)/(1 + \eta_0))}{e^{-60 \eta_0} \left[ -\eta_0/(1 + \eta_0) - \left(1 - e^{-120 \eta_0}\right)\right]^{1/2}} \right)^{1/2}. \tag{30}
\]
Figure 18. This plot shows contours of $\phi$ in the $\eta_0 - g$ plane.

Figure 19. $p = 0$, $n$ versus $\eta_0$. The upper bound from cosmic strings applies to the $D$-term case [21].

This is plotted in figure 20. In both the $F$- and $D$-term models, $V_0^{1/4}/\sqrt{g}$ is the vev of the waterfall field. In the $F$-term case one supposes that the waterfall field is a GUT Higgs field making its vev of the order of the GUT scale $10^{16}$ GeV, and a similar value is expected in the $D$-term case with a high string scale. Imposing that restriction, we see from figure 19 that $n$ cannot be far below the observed value.

6. Hilltop tree-level hybrid inflation

6.1. Generic case

The third case has never been considered before. Here we are seeing whether the original hybrid inflation model [27], which gives $n \geq 1$ in contradiction with observations, can be saved by the addition of the $\phi^p$ term, as in figure 3.

In this case we keep $x$ as a parameter of the model. The value of $x$ depends on $\phi_{\text{end}} = \phi_c$. If the coupling of the inflaton to the waterfall field is $\lambda_{\phi \chi} \chi^2 \phi^q / M_p^{q-2}$, then
Figure 20. $p = 0$, $(V_0/g^2)^{1/4}$ versus $\eta_0$ in Planck units $M_P = 1$. We also show the requirement as in equation (4) in the dashed line which corresponds to an upper bound for $g = 1$.

Figure 21. Model 3, $p = 4$, log $x$ and log $\lambda$ versus $\eta_0$.

\[ \phi_c^2 = \lambda_{\phi\chi}^{-1} m^2 \chi M_P^2 \] (where $m_\chi$ is the tachyonic mass of the waterfall field). This gives

\[ \eta_0 x = \left( \frac{m_\chi}{M_P} \right)^2 \left( \frac{\lambda_{\phi\chi} M_P^2}{m^2_\chi} \right)^{(p-2)/q}. \] (31)

The plausible range of $x$ is clearly very large.

To obtain a general idea about the allowed parameter space, we shall fix $n = 0.95$ (the central observed value). Consider first the case $p = 4$. In figure 21 we show log $x$ and log $\lambda$ as a function of $\eta_0$. At $\eta_0 = 0.0125$, $\lambda = x = 0$, which is also the line $\phi = \phi_{\text{end}}$ above which we have $\phi > \phi_{\text{end}}$. The curve of $\lambda$ against $\eta_0$ also represents the contour of $n = 0.95$ in the $\lambda-\eta_0$ plane. We can generalize this for $n = 0.9, 0.95, 0.98$ which we plot.
Figure 22. Model 3, $p = 4$, contours of $n$ in the log $\lambda$–$\eta_0$ plane with the constraint $n = 0.9$: $0.025 < \eta_0 < 0.0578$, $n = 0.95$: $0.0125 < \eta_0 < 0.0588$, $n = 0.98$: $0.005 < \eta_0 < 0.0593$. The upper bound is provided by $|n'| < 0.01$ and the lower bound is provided by $\phi_{\text{end}} < \phi$.

Figure 23. Model 3, $p = 6$, contours of $\log(V_0^{1/4}/M_P) = -2, -4, -6, -8, -10$ and $\log x = -5, 0, 5, 10, 15$. The dashed line (dotted line) denotes $\phi = 0.1M_P$ ($1M_P$). The top horizontal line ($\eta_0 = 0.033$) represents the upper bound on $\eta_0$ by a condition $|n'| < 0.01$. The bottom horizontal line ($\eta_0 = 0.06$) represents the lower bound by a requirement $\phi_{\text{end}} < \phi$.

in figure 22. In figure 22, we also show the upper bound for $\eta_0$, which is provided by $n' < 0.01$.

Now consider $p = 6$. In figure 23 we show in the $\eta_0$–log $\lambda$ plane contours of $\log x$ and contours of $\log(V_0^{1/4}/M_P)$. We also show the lines $\phi(N) = M_P$ and $\phi(N) = 0.1M_P$. The trapezium-like area (on the rhs of $\phi = 0.1$ (which means $\phi < 0.1$)) and $0.006 < \eta_0 < 0.033$) represents the allowed region for the inflation model. The upper horizontal line $\eta_0 = 0.033$ represents $n' = 0.01$, underneath which is the requirement $|n'| < 0.01$. The lower horizontal line is $\phi(N) = \phi_{\text{end}}$, above which is the requirement $\phi(N) > \phi_{\text{end}}$ corresponding
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Figure 24. Model 3, \( p = 4 \), \( P_{\zeta}^{1/2}(N = 0) \) versus \( n' \). The upper bound \( n' < 0.01 \) corresponds to \( P_{\zeta}^{1/2}(N = 0) < 6 \times 10^{-4} \).

to motion towards \( \phi = 0 \). Using equations (10) and (11), we can see at \( \eta_0 = 0.006, x = 6\lambda \) which makes \( m = 0 \) therefore \( V_0 = 0 \) and \( \phi = \phi_{\text{end}} = 0 \) for all finite \( \lambda \). That is why the plot of \( V_0 \) and \( \phi \) behaves oddly there.

In figure 21, we see that \( \lambda \) has to be very small, in accordance with the known generic result. On the assumption that it will actually be negligible after suppression by (say) a supersymmetry mechanism, we choose instead \( p = 6 \), to obtain the plot shown in figure 23.

The term in the potential proportional to \( \phi^p \) may be regarded as parameterizing physics beyond the ultraviolet cutoff. Taking the cutoff to be \( M_P \), one might generically expect \( \lambda \sim 1 \) in \( \lambda \phi^p / M_P^{p-4} \). With a lower cutoff \( M \) one might expect \( \lambda' \sim (M/M_P)^{p-4} \) in \( \lambda' \phi^p / M^{p-4} \) (equivalent to \( \lambda \sim 1 \) with the replacement \( M_P \rightarrow M \)).

Within the context of supersymmetry, a simple realization of hybrid inflation has been termed [31] supernatural inflation. Here, the superpotential provides only the coupling between the waterfall field and the inflaton. All other terms in the potential, including \( V_0 \), come from soft supersymmetry breaking. As in the generic case, our addition of the term proportional to \( \phi^p \) can rescue the model by allowing it to give a spectral index in agreement with observation.

6.2. Black hole formation

As sketched in figure 3, the potential starts out concave-downward as required by observation, but then turns up. As a result it can be much flatter at the end of inflation when cosmological scales leave the horizon. This allows \( P_\zeta \) at the end of inflation to be much bigger than the observed value and the question arises whether it can be of the order of \( 10^{-2} \) or so, leading to the production of black holes.

After fixing \( n = 0.95 \) and apply the cmb normalization, for both the cases \( p = 4 \) and 6, we can express \( P_{\zeta}^{1/2}(N = 0) \) and \( n' \) as a function of \( \eta_0 \). We show the plots of \( P_{\zeta}^{1/2}(N = 0) \) versus \( n' \) for the cases \( p = 4 \) and 6 in figures 24 and 25, respectively. It is seen that black hole formation would require running \( n' \sim 0.1 \), far in excess of what is allowed by observation.
It is clear that any potential allowing black hole formation will have a shape like the one in figure 3. In a companion paper [8] it is shown that suitable potentials definitely exist. A well-motivated example that seems viable at present is the running-mass model [32]. With the parameters used for figure 2 of [33], black hole formation is possible with $n' = 0.009$. By altering the gauge group it should be possible to achieve black hole formation with a lower $n'$.

7. Conclusion

We have extended the investigation of [1], demonstrating that hilltop inflation is an absolutely generic possibility for both hybrid and non-hybrid inflation models. This is a welcome development, in that it allows a whole range of models to fit observation while being at the same time well motivated from the particle physics viewpoint. The development is disturbing though, in that it muddies the clean classification of the models presented in, for instance, [2] which could formerly be made on the basis of the sign and likely value of the spectral index. However, discrimination between the models will still be possible if the running can be measured with an accuracy $\Delta n' \sim 10^{-3}$ (i.e. an order of magnitude better than the bound [8] provided by present data).

We have explored the parameter space of each model after imposing the cmb normalization $P^{1/2}_\zeta = 5 \times 10^{-5}$ on the spectrum. Regarding the spectral index, we have exhibited the effect of imposing the observed value $n = 0.95$, but we have also asked what range would have been allowed theoretically. The latter question is reasonable, because in contrast with the normalization of the spectrum there does not seem to be any anthropic constraint on the spectral index. Therefore, one would like a value in the right ballpark to be an automatic consequence of imposing the cmb normalization.

This desirable state of affairs seems to be achieved for the case of smooth hybrid inflation and $F$-term inflation, when we require that these models be part of a GUT theory. It also seems to be the case for $D$-term inflation and modular inflation, if we require a high inflation scale corresponding to a high string scale.
In other cases, including new inflation, the spectral index might have taken any value in the range \(0 \leq 1 - n \leq 1\) demanded by slow-roll. Tree-level hybrid inflation even allows the whole range \(-1 \leq 1 - n \leq 1\). It was pointed out some time ago [35] that the same is true for \(A\)-term inflation [34], where one deals with a potential that is well approximated by

\[
V(\phi) = V'\phi + \frac{1}{6}V'''\phi^3.
\]

(The same kind of potential has been found recently [36] in the context of colliding brane inflation.) With this potential, \(V\) has a maximum in about half of the parameter space, giving a hilltop model. Therefore, if hilltop inflation is favored on the ground that eternal inflation can provide the initial condition these models automatically give \(n < 1\) but they do not automatically place \(n\) close to 1.

There is a proposal [11] even in these cases, for understanding why \(n\) is so close to 1. This is to demand as much slow-roll inflation as possible, so that the inflated volume created by slow-roll is as large as possible. Indeed, this demand will drive \(n\) as close to 1 as is allowed by the parameter space. For new inflation and modular inflation we have argued that the demand will make \(1 - n\) of the order of a few divided by \(N\), placing it in the right ballpark. For tree-level hybrid inflation and the potential (32) the demand will instead drive \(n\) to be indistinguishable from 1. In any case, it is not clear to us why one should want to maximize the amount of slow-roll inflation, if there has already been a much larger amount of eternal inflation.

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