Comments on open string with “massive” boundary term

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Abstract

We discuss possible definition of open string path integral in the presence of additional boundary couplings corresponding to the presence of masses at the ends of the string. These couplings are not conformally invariant implying that as in a non-critical string case one is to integrate over the 1d metric or reparametrizations of the boundary. We compute the partition function on the disc in the presence of an additional constant gauge field background and comment on the structure of the corresponding scattering amplitudes.
1 Introduction

Relativistic string is a remarkably rigid theory: it is hard to modify it while preserving its quantum-mechanical consistency and solvability. One of the key conditions is 2d conformal invariance, the preservation of which imposes constraints on the target space dimension and geometry.

There were several attempts to generalize the open string theory by adding masses at the ends (see, e.g., [1, 2, 3, 4, 5] for some old and recent papers). At the classical level adding masses to the ends of an open string corresponds to considering the action

\[ I = I_0 + I_\partial, \quad I_0 = T \int d^2 \sigma \sqrt{\det h_{ab}}, \quad h_{ab} = \partial_a x^a \partial_b x_n, \quad (1.1) \]

\[ I_\partial = m_1 \int d\tau \sqrt{-\dot{x}^2} \bigg|_{\sigma=0} + m_2 \int d\tau \sqrt{-\dot{x}^2} \bigg|_{\sigma=\pi}. \quad (1.2) \]

While it may be used for developing an effective perturbation theory near a long semiclassical string, this action is not a good starting point for quantization of short fundamental strings being non-linear.

Instead, one may introduce an auxiliary metric \( g_{ab} \) and a boundary metric \( e \) and consider the analog of the Polyakov string path integral [6, 7] with the following action defined, e.g., on a disc

\[ I = I_0 + I_\partial, \quad I_0 = \frac{1}{2} T \int d^2 \sigma \sqrt{g} \partial^a x^a \partial_b x_n, \quad (1.3) \]

\[ I_\partial = \frac{1}{2} \int_0^{2\pi} d\varphi \left( e T_0 + e^{-1} Q_0 \dot{x}^m \dot{x}_m \right). \quad (1.4) \]

Here we use the Euclidean notation, \( T = \frac{1}{2\pi\alpha'} \) is string tension, \( e = e(\varphi) \) is an einbein and \( T_0, Q_0 \) are constant parameters (one of them may be absorbed into \( e \)). Solving for \( e \) classically gives the “mass term” like in (1.2)\(^1\)

\[ \hat{I}_\partial = M \int d\varphi \ |\dot{x}|, \quad M \equiv \sqrt{T_0 Q_0}. \quad (1.5) \]

\( I_\partial \) in (1.3) may be viewed as a special case of the boundary action [8]

\[ I_\partial = \frac{1}{2} \int_0^{2\pi} d\varphi \left[ e \mathcal{T}(x) + e^{-1} Q_{mn}(x) \dot{x}^m \dot{x}^n - i A_m(x) \dot{x}^m + ... \right], \quad (1.6) \]

where \( \mathcal{T} \) may be interpreted as a condensate of an open string tachyon, \( Q_{mn} \) – of a spin 2 massive open string mode, \( A_m \) – of a massless vector field, etc.

In general, for fixed \( e \) the presence of non-trivial \( \mathcal{T}, Q, ... \) couplings breaks scale invariance (beta-functions for \( \mathcal{T}, Q, ... \) will be non-zero, see, e.g., [9]). As a result, if, for example, \( \mathcal{T} = ... \)

\(^1\)Strictly speaking, disc or half-plane with the above mass term at the boundary corresponds to an unphysical choice of \( m_1 = -m_2 = M \) when transforming from the action (1.1) on the strip with two boundaries and two masses at ends. Moreover, in the absence of conformal invariance descriptions using different domains may not be equivalent.
$T_0$, $Q_{mn} = Q_0 \delta_{mn}$ as in (1.3), then $e$ will not decouple, i.e. there will be a scale (and 1d $SL(2)$ conformal) anomaly. This will happen even if the 2d bulk Weyl anomaly cancels out (which, of course, requires $D = 26$ [6]).

A possible way out is to consider a kind of “light” version of non-critical string theory where the bulk conformal factor is decoupled (in $D = 26$) but one is still to integrate over the 1d metric $e$. This will allow one to absorb the 1d scale anomaly or the corresponding 2d UV divergences into a redefinition of $e$. If the reparametrization invariance is assumed to be preserved by a regularization, we may then fix a 1d reparametrization gauge as $e(\varphi) = L = \text{const}$ and integrate over the remaining constant parameter $L \in (0, \infty)$.

Then the string partition function on a disc will be given by

$$Z = \int_0^{\infty} dL \mu(L) e^{-\pi L T_0} \hat{Z},$$

(1.7)

$$\hat{Z} = \int [dx] \exp \left[ - I_0 - \frac{1}{2} \int_0^{2\pi} d\varphi \left( L^{-1} Q_0 \dot{x}^m \dot{x}^m - i A_m(x) \dot{x}^m + \ldots \right) \right].$$

(1.8)

In the standard critical string theory (where $T_0 = Q_0 = 0$) the integral over $L$ should decouple (and, in fact, it should be removed as a result of dividing over the Weyl gauge group on the disc). One may expect that the measure $\mu(L)$ should be proportional to $L^{k(D-26)}$ and should thus be trivial in the critical dimension. The precise form of $\mu(L)$ remains one of the open questions.

Another issue will be how to define the corresponding vertex operators and thus scattering amplitudes. In general, the corresponding Green’s function will depend on the metric $e$ and thus there will be additional $L$-dependent terms reflecting breaking of 1d scale invariance. For constant gauge field background $F_{mn} = \text{const}$ these are only linear divergences (that can be absorbed into tachyon coupling) but, in general, there will be also non-trivial log divergences. If the condition of decoupling of $e = L$ is not satisfied we cannot use the usual marginality condition to determine the vertex operators. A naive guess is that the vector field vertex operator which does not directly couple to $e$ may still remain the same.

Below in section 2 we shall first consider the disc partition function in the abelian constant gauge field strength background and then in section 3 make comments on the scattering amplitudes.

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2 An example of a quantum-consistent 1d CFT is a string in a constant Maxwell field background with the coupling term $\int d\tau F_{mn} x^m \dot{x}^n$. One may also consider other options like non-local 1d actions that are scale invariant (cf., e.g., [10]).

3 This is, of course, what is done in the familiar proper-time representation of a 1-loop effective action (say, with kinetic operator $\Delta = -\partial^2 + T(x) + \ldots$, i.e. $\Gamma_1 = \frac{1}{2} \log \det \Delta = -\frac{1}{2} \int \frac{dL}{L} \text{tr}(e^{-L \Delta})$).

4 The definition of the measure is subtle as it requires, in particular, a proper choice of the boundary condition for the bulk conformal factor relating it to $e$. The present case is different from the Wilson loop case [11] as the boundary conditions are different – for $M = 0$ we should recover the Neumann boundary condition, not the Dirichlet one.

5 We thank the referee for pointing out some related issues.

6 The condition of marginality of the vector vertex operator will be sensitive to $e$ via modified Green’s function so it is unclear why it should remain to represent a massless particle.
2 Partition function in constant gauge field background

To get an idea of how the integrand of the $L$-integral in (1.7) may look like let us consider the generating functional for scattering of soft photons which in the standard critical string theory is described by the Born-Infeld action. It is given by the disc partition function in constant strength background for the vector field in (1.6), i.e. $A_m = -\frac{1}{2}F_{mn}x^n$. We shall use the discussion in Appendix of [12].

Integrating over the values of the string coordinates at the internal points of the disc we get the following expression for $Z$ in (1.8)

$$
\hat{Z} = c_0 \int d^D x_0 Z , \quad Z = \int [d\xi] e^{-\hat{I}_\partial} , \quad c_0 \sim T^{-D/2} \quad (2.1)
$$

Here $\hat{I}_\partial$ is the effective action at the boundary of the disc and we isolated the constant zero mode in $x^m = x_0^m + \xi^m(\varphi)$, $\xi^m(\varphi) = \frac{i}{\sqrt{\pi}} \sum_{n=1}^{\infty} (a^m \cos n\varphi + b^m \sin n\varphi)$. The scale-invariant non-local operator $G^{-1}$ in (2.2) is the inverse of the restriction of the Neumann function on the disc to its boundary$^7$

$$
G(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos n\varphi_{12} , \quad G^{-1}(\varphi_1, \varphi_2) = \frac{1}{\pi} \sum_{n=1}^{\infty} n \cos n\varphi_{12} , \quad \varphi_{12} = \varphi_1 - \varphi_2 \quad (2.3)
$$

The action thus contains the effectively “first-order” term ($\sim T$) and the second-order term ($\sim \mathcal{M}$) in 1d derivatives and interpolates between the standard massless string theory case $T \neq 0$, $\mathcal{M} = 0$ and the standard particle case $T = 0$, $\mathcal{M} \neq 0$ which appears in the Schwinger computation of $\log \det[-D^2(A)]$ in constant $F_{mn}$ background [13]. The resulting partition function will then interpolate between the Born-Infeld (string) and the Schwinger (particle) expressions.

Putting $F_{mn}$ into the block-diagonal form and concentrating first on a single $(1, 2)$ block we find after integrating over the coordinates $\xi^1, \xi^2$

$$
Z_{12} = Z_{12}(\mathcal{M}) Z_{12}(F, \mathcal{M}) \quad , \quad (2.4)
$$

$$
Z_{12}(\mathcal{M}) \sim \prod_{n=1}^{\infty} \left( Tn + \mathcal{M}n^2 \right)^{-2} \sim \mathcal{M} \left( \prod_{n=1}^{\infty} \left( 1 + \frac{T\mathcal{M}^{-1}}{n} \right) \right)^{-2} \quad , \quad (2.5)
$$

$$
Z_{12}(F, \mathcal{M}) = \prod_{n=1}^{\infty} \left( 1 + \frac{F^2}{(T + \mathcal{M}n)^2} \right)^{-1} , \quad F \equiv F_{12} \quad . \quad (2.6)
$$

$Z_{12}(F, \mathcal{M})$ in (2.6) depends only on the dimensionless ratios $T^{-1}F$ and $T^{-1}\mathcal{M}$. We shall ignore the (power divergent) $F$-independent factor $Z_{12}(\mathcal{M})$ which can be absorbed into the renormalization of tachyon coupling $\mathcal{T}_0$ in (1.7).$^8$

$^7$Note that the $\delta$-function on non-constant $\xi$ is $\delta(\varphi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\varphi$.

$^8$Recall that according to (2.2) we have $\mathcal{M} = L^{-1}Q_0$ so that $\mathcal{M}$ depends on $L$. All linear divergences should be renormalized away; this is also part of subtracting the $SL(2, R)$ Mobius volume in the standard open string theory set-up [14].
When $\mathcal{M} = 0$ the factors in the product in $Z_{12}(F, \mathcal{M})$ are $n$-independent, and using the standard regularization prescription $\prod_{n=1}^{\infty} c = c^{-1/2}$ (with the linear divergence again absorbed into the tachyon coupling) we get the familiar Born-Infeld expression

$$Z_{12}(F, \mathcal{M} = 0) = \sqrt{1 + (T^{-1}F)^2}.$$  \hspace{1cm} (2.7)

For $T = 0$ we get instead the Schwinger expression $Z_{12}(F, \mathcal{M})\mid_{T=0} = \frac{\pi \mathcal{M}^{-1}F}{\sinh(\pi \mathcal{M}^{-1}F)}$.

In general, for $\mathcal{M} \neq 0$, $Z_{12}(F, \mathcal{M})$ in (2.6) may be expressed in terms of $\Gamma$-functions as

$$Z_{12}(F, \mathcal{M}) = \prod_{p=1}^{D/2} \left[ \frac{\Gamma\left(\frac{T+\mathcal{M}+iF_p}{\mathcal{M}}\right) \Gamma\left(\frac{T+\mathcal{M}-iF_p}{\mathcal{M}}\right)}{\left[\Gamma\left(\frac{T+\mathcal{M}}{\mathcal{M}}\right)\right]^2} \right] \left[\Gamma\left(\frac{T+\mathcal{M}}{\mathcal{M}}\right)\right]^{2}.$$

(2.8)

The general expression for the partition function is the product of factors for each eigenvalue $F_p$ of the field strength $F_{mn}$

$$Z(F, \mathcal{M}) = \prod_{p=1}^{D/2} \left[ \frac{\Gamma\left(\frac{T+\mathcal{M}+iF_p}{\mathcal{M}}\right) \Gamma\left(\frac{T+\mathcal{M}-iF_p}{\mathcal{M}}\right)}{\left[\Gamma\left(\frac{T+\mathcal{M}}{\mathcal{M}}\right)\right]^2} \right] \left[\Gamma\left(\frac{T+\mathcal{M}}{\mathcal{M}}\right)\right]^{2}.$$

(2.9)

Once again, in the “point-particle” limit $T \to 0$ we get the standard Schwinger expression [13]

$$Z(F, \mathcal{M})\mid_{T \to 0} = \prod_{p=1}^{D/2} \frac{\pi \mathcal{M}^{-1}F_p}{\sinh(\pi \mathcal{M}^{-1}F_p)}.$$  \hspace{1cm} (2.10)

Taking the limit $\mathcal{M} \to 0$ and using the Stirling formula $\Gamma(z \to \infty) = \sqrt{2\pi z} \frac{z^z e^{-z}}{[1 + O(1)]}$ we find from (2.9)

$$Z\mid_{\mathcal{M} \to 0} = \prod_{p=1}^{D/2} \sqrt{1 + (T^{-1}F_p)^2} \left[1 + (T^{-1}F_p)^2\right]^{-\mathcal{M}^{-1}T} \left(\frac{1 + iT^{-1}F_p}{1 - iT^{-1}F_p}\right)^{\mathcal{M}^{-1}F_p} \left[1 + O(\mathcal{M})\right].$$  \hspace{1cm} (2.11)

The mass parameter $\mathcal{M} = L^{-1}Q_0$ thus plays here the role of a UV cutoff with linear divergences proportional to $L$. Eq. (2.11) then reduces to the standard Born-Infeld expression times an extra divergent factor

$$Z\mid_{\mathcal{M} \to 0} = \sqrt{\det(\delta_{mn} + T^{-1}F_{mn})} \ e^{\mathcal{M}^{-1}f(F)},$$

(2.12)

that can be “renormalized” away by absorbing it into the $\mathcal{T}_0$ term in (1.7) that also scales linearly with $L$.

Finally, it remains to substitute the expression for $\hat{Z}$ in (2.1),(2.9) into (1.7) and integrate over $L$. Let us assume for simplicity that we have just one magnetic field component $F_{12} = F$ and plug (2.8) into the integral over $L$ in (1.7)

$$Z = c_0 V_D (1 + \bar{F}^2) \int_{0}^{\infty} dL \ \mu(L) \ e^{-\pi T^{-1}L^2} \ H(\bar{L}, \bar{F}) ,$$

(2.13)

$$H(\bar{L}, \bar{F}) \equiv \frac{\Gamma(\bar{L} + i\bar{F}) \ \Gamma(\bar{L} - i\bar{F})}{[\Gamma(\bar{L})]^2} , \quad \bar{L} \equiv TQ_0^{-1}L , \quad \bar{F} \equiv T^{-1}F .$$  \hspace{1cm} (2.14)
Here $V_D$ is the volume factor in (2.1) and we used the definition $M^2 = T_0 Q_0$ in (1.5). The standard massless string theory limit corresponds to $Q_0 \to 0$, $M^2 \to 0$ or $\bar{L} \to \infty$ for fixed $T$. This is also the limit $\mathcal{M} \to \infty$ in (2.11). If the measure $\mu(L) \sim L^\gamma$ with $\gamma > 0$ then the resulting integral over $L$ is regular and gives a finite expression for the partition function as a function of the tension $T$, magnetic field $F$ and the mass parameter $M$.

3 Scattering amplitudes

Next, we may look at the generalization of the standard vector scattering amplitudes to the case of non-zero mass parameter $M$ in (1.4),(1.5). To compute the scattering amplitudes we need to specify (i) vertex operators, (ii) the modified ($L$-dependent) boundary Green’s function, and (iii) integrate over $L$ as in (1.7).

Let us first comment on the Green’s function. To find (2.3) we followed [8] and started with the Neumann function on the disc. This does not restrict the boundary value of the string coordinate and just amounts to integrating out its values in internal points of the disc. For example, if one has boundary coupling to an external vector, it classically modifies the boundary conditions and that leads to an $F$-dependent Green’s function [15]. However, the same result is obtained by restricting the Neumann function to the boundary and then considering the purely boundary theory as in (2.2). Explicitly, if $z = re^{i\phi}$ ($0 < r < 1$) is a coordinate on a disc then

$$N(z, z') = -\frac{1}{2\pi} \left( \log |z - z'| + a \log |z - z'|^{-1} \right), \quad G(\phi, \phi') \equiv N(e^{i\phi}, e^{i\phi'}), \quad (3.1)$$

where $a = 1$ corresponds to the Neumann function and $a = -1$ to the Dirichlet function. Then the boundary value $G(\phi, \phi')$ is (for $a = 1$)

$$G(\phi, \phi') \equiv G(\phi - \phi') = -\frac{1}{2\pi} \log \left[ 2 - 2 \cos(\phi - \phi') \right] = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos n(\phi - \phi'), \quad (3.2)$$

$$G^{-1}(\phi) = -\frac{d^2}{d\phi^2} G(\phi) = -\frac{1}{4\pi \sin^2 \frac{\phi - \phi'}{2}} = \frac{1}{\pi} \sum_{n=1}^{\infty} n \cos n(\phi - \phi'), \quad (3.3)$$

where we used that $\log(1 + b^2 - 2b \cos \alpha) = -2 \sum_{n=1}^{\infty} \frac{b^n}{n} \cos n\alpha$.

Integrating over the string fluctuations inside the disk we get the boundary action [8] in (2.2), i.e.\(^9\)

$$\hat{I}_0 = \frac{1}{2} T \int_0^{2\pi} d\phi_1 d\phi_2 \xi^m(\phi_1) G^{-1}(\phi_1) \xi^m(\phi_2) + \frac{1}{2} \mathcal{M} \int_0^{2\pi} d\phi \, \xi^m \dot{\xi}^m. \quad (3.4)$$

\(^9\)Here we may not distinguish between $\bar{x}(\phi)$ and its non-constant part $\xi(\phi)$ as the constant $x_0$ drops out under the integral.
Here $G$ where according to (3.8) we get
\[ \hat{I}_0 = \frac{1}{2} T \int_0^{2\pi} d\varphi_1 d\varphi_2 \dot{\xi}^m(\varphi_1) \hat{G}(\varphi_{12}) \dot{\xi}^m(\varphi_2) = \frac{1}{2} T \int_0^{2\pi} d\varphi_1 d\varphi_2 \xi^m(\varphi_1) G(\varphi_{12}) \xi^m(\varphi_2), \] (3.5)
\[ \hat{G}(\varphi) \equiv G(\varphi) + \varkappa \delta(\varphi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} + \varkappa \cos n\varphi, \quad \hat{G}(\varphi) = -\frac{d^2}{d\varphi^2} G(\varphi), \] (3.6)
\[ \varkappa \equiv T^{-1} \mathcal{M} = LT^{-1} Q_0. \] (3.7)

Here $G$ is the effective boundary Green’s function corresponding to the action (2.2) containing the “mass term” $\mathcal{M} \hat{G}^2$

\[ G(\varphi, \varphi') \equiv G(\varphi - \varphi') = \frac{1}{\pi} \sum_{n=1}^{\infty} g_n \cos n(\varphi - \varphi'), \quad g_n = \frac{1}{n + \varkappa n^2}. \] (3.8)

$n$ in $g_n^{-1}$ comes from $G^{-1}$ in (2.3) and $n^2$ from second-derivative term in (2.2). It is the determinant of $G$ that appeared in (2.5). The same expression for the Green’s function should be found if one first modifies the classical boundary conditions due to the presence of the boundary mass term $\mathcal{M} \hat{G}^2$.

While a systematic way of constructing vertex operators in the “non-critical” massive case remains to be found we may be guided by the fact that at least the vector field couples to the open string ends in a (classically) scale invariant way. Let us thus assume that we may start with the standard vertex operator corresponding to the vector coupling in (1.7), i.e.

\[ V(\zeta, p) = \int d\varphi \zeta_m(p) \dot{\xi}^m(\varphi) e^{ipm\xi^m(\varphi)}. \] (3.9)

The generating functional for correlators of unintegrated vertex operators (computed now with the Green’s function in (3.8)) may be written as (cf. (1.7),(2.1),(2.2))

\[ Z(\zeta, p) = \int_0^{\infty} d\mu(L) e^{-\pi L \hat{T}_0} W, \] (3.10)
\[ W = \int [d\xi] \exp \left[ \int_0^{2\pi} d\varphi \left( -\frac{1}{2} T \xi^m G^{-1} \xi^m + \hat{\zeta}_m \dot{\xi}^m + i\hat{p}_m \xi^m \right) \right]. \] (3.11)

Here $\dot{\zeta}_m = \sum_{k=1}^{N} \zeta_m^{(k)} (p_k) \delta(\varphi - \varphi_k), \quad \hat{p}_m = \sum_{k=1}^{N} p_m^{(k)} \delta(\varphi - \varphi_k)$ and to find the expression for the $N$-point scattering amplitude one needs to take the relevant multi-linear term in $\zeta^{(k)}$.

The integral over the constant zero mode $x_0^m$ in (1.7) gives, as usual, the total momentum conservation delta-function.

Doing the Gaussian integral over $\xi^m$ gives

\[ W = \exp \left[ \frac{1}{2} T^{-1} \sum_{k,k'=1}^{N} \left( -\hat{G}_{kk'} \zeta^{(k)} \cdot \zeta^{(k')} + 2i \hat{G}_{kk'} \zeta^{(k)} \cdot p^{(k')} - G_{kk'} p^{(k')} \cdot p^{(k')} \right) \right] \] (3.12)

where according to (3.8) we get $G_{kk'} = \frac{1}{\pi} \sum_{n=1}^{\infty} g_n \cos n(\varphi_k - \varphi_{k'}), \quad g_n = \frac{1}{n + \varkappa n^2}$, and thus

\[ \hat{G}_{kk'} = -\frac{1}{\pi} \sum_{n=1}^{\infty} n g_n \sin n(\varphi_k - \varphi_{k'}), \quad \hat{G}_{kk'} = -\frac{1}{\pi} \sum_{n=1}^{\infty} n^2 g_n \cos n(\varphi_k - \varphi_{k'}). \] (3.13)
This leads to the standard integrands for the vector amplitudes on the disc when \( \mathcal{M} = 0 \), i.e. when \( g_n = \frac{1}{n} \). Under the integral over \( L \) one should still have \( SL(2, R) \) Mobius symmetry so one may use it to fix 3 points. One may also choose not to fix the Mobius symmetry explicitly – as the disc Mobius group volume is only power divergent [14] this divergence should also be possible to absorb into a renormalization of the tachyon coupling \( T_0 \).

Let us note that an alternative representation for the path integral appeared in the context of computing the Wilson loop (WL) expectation value (see [11]) with fixed target-space contour: there the integral over 1d metric \( e \) is equivalent to integrating over reparametrizations \( s(t) \) of the boundary \( (ds = e(t)dt) \). It is useful to map the unit disk \((r, \varphi)\) onto the upper half-plane \( z = i \frac{1}{1 - r e^{i \varphi}} \); then the boundary at \( r = 1 \) is mapped onto the real axis \( -\infty < t < +\infty \) by \( t(\varphi) = -\cot \frac{\varphi}{2} \). The boundary restriction of the Green’s function in (2.3),(3.2) mapped to the half-plane is

\[
G(t) = -\frac{1}{\pi} \log |t| , \quad G^{-1}(t) = -\frac{1}{\pi t^2} .
\]

Ignoring the boundary mass term and introducing the integral over reparametrizations one gets (see [16, 17, 18])

\[
A[\vec{x}] = \int [ds(t)] e^{-i}, \quad \hat{I} = -\frac{1}{2\pi} T \int dt_1 dt_2 \dot{x}(t_1) \log |s(t_1) - s(t_2)| \dot{x}(t_2) = \frac{1}{2\pi} T \int ds_1 ds_2 \frac{[\dot{x}(t_1) - \dot{x}(t_2)]^2}{(s_1 - s_2)^2} \]

(3.16)

where the integrals go over the real line (cf. (3.5)) and \( \vec{x} \) is the counterpart of \( \xi^n \) in (3.4). The analog of the mass term is \( I_0 = \frac{1}{2} \mathcal{M} \int dt \dot{x}(s(t)) \dot{x}(s(t)) \). If we are interested, for example, in tachyon scattering amplitudes we may just insert momentum-dependent factors and integrate over all boundary functions \( \vec{x}(t) \). Alternatively, one may do Fourier transform of the WL expectation value and then pick up a step-function - like contour for \( \vec{p}(t) \) [18]. This may be viewed as a particular off-shell prescription for tachyon scattering amplitudes.

To appreciate the technical difficulty between the \( \mathcal{M} = 0 \) and \( \mathcal{M} \neq 0 \) cases it is useful to find the explicit form of the counterpart of the Green’s function (3.8) in the half-plane parametrization. We may write \( G \) in (3.8) as

\[
G(\varphi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n + \varphi n^2} = G(\varphi; 0) - G(\varphi; \varphi^{-1}) , \quad G(\varphi; b) \equiv \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n + b} .
\]

(3.17)

Let us set \( w = e^{i\varphi}, \ w' = e^{i\varphi'} \) so that \( G(\varphi - \varphi'; b) \) may be written as

\[
G(w, w'; b) \equiv \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n + b} (\frac{w}{w'})^n + \text{c.c.} \equiv \frac{1}{2\pi} \Phi(\frac{w}{w'}, 1, b) - b^{-1} + \text{c.c.} = \frac{1}{2\pi} \sum_{r=0}^{\infty} (-b)^r \text{Li}_{r+1}(\frac{w}{w'}) + \text{c.c.} .
\]

(3.18)
Here \( \Phi(u, r, b) \equiv \sum_{k=0}^{\infty} \frac{u^k}{(k+b)'} \) is the Lerch transcendent generalizing the Hurwitz \( \zeta \)-function, i.e.

\[
\Phi(u, 1, b) - b^{-1} = \sum_{n=1}^{\infty} \frac{u^n}{n + b} = \sum_{r=0}^{\infty} (-b)^r \text{Li}_{r+1}(u),
\]

(3.19)

where \( \text{Li}_s(u) = \sum_{k=1}^{\infty} \frac{u^k}{k^s} \) is the polylogarithm function. Note that \( \sum_{n=1}^{\infty} \frac{u^n}{n+b} = z\Phi(u, 1, 1+b) \).

The map from the disc to half-plane is

\[
w = \frac{z-i}{z+i}
\]

(3.20)

with \( |w| = 1 \) if \( z \) is real. The massless boundary Green’s function expressed in terms of real \( z, z' \) or \( |w| = |w'| = 1 \) is

\[
G(w, w'; 0) = -\frac{1}{\pi} \log |1 - \frac{w}{w'}| = -\frac{1}{\pi} \log |w - w'| = -\frac{1}{\pi} \log \frac{2|z-z'|}{|z+i||z'+i|} \rightarrow -\frac{1}{\pi} \log |z - z'|,
\]

(3.21)

where we used the freedom in the Green’s function \( G \rightarrow G + f(z) + \bar{f}(z') \) to get the standard expression (cf. (3.14)) at the boundary of half-plane. Then \( G(z, z') \) can be found directly from (3.18),(3.20). For example, the small \( \varkappa \) expansion of \( G \) in (3.17) is

\[
G = -\frac{1}{\pi} \log |1 - \frac{w}{w'}| + \frac{1}{2\pi} \left[ \varkappa \left( \frac{w}{w - w'} + \frac{\bar{w} \bar{w}'}{(w - w')^2} \right) + \bar{\varkappa} \left( \frac{ww'}{(w - w')^2} + \frac{\bar{w} \bar{w}'}{(\bar{w} - \bar{w}')^2} \right) \right] + O(\varkappa^3),
\]

(3.22)

where \( w \) is to be replaced by (3.20). As a result,

\[
G = -\frac{1}{\pi} \log |z - z'| + \frac{1}{2\pi} \varkappa - \frac{1}{4\pi} \bar{\varkappa} \frac{(1 + z^2)(1 + z'^2)}{(z - z')^2} + O(\varkappa^3),
\]

(3.23)

where \( z, z' \) are real. The Green’s function (3.17) will depend on \( e = L \) not only at the coinciding points (as is the case also for \( \varkappa = 0 \) if one uses a covariant cutoff) but also explicitly via \( M \) or \( \varkappa \) and this makes the integral over \( e \) non-trivial. This is a reflection of the explicit breaking of the conformal invariance by the mass term at the boundary.

The constant \( \varkappa \) term in (3.23) will not contribute to on-shell amplitudes so the first non-trivial correction will be at order \( \varkappa^2 \). Note, however, that significance of this expansion is unclear as we are effectively to integrate over \( \varkappa \) which depends on \( L \) according to (2.2).\(^\text{10}\)

### 4 Remarks

In the standard Polyakov path integral approach the integral over the conformal factor \( \rho \) decouples for \( D = 26 \) only in the vacuum partition function; if one considers correlators of vertex operators then the condition of decoupling of \( \rho \) leads to the mass shell restriction on external momenta [19]. In the “massive” case discussed above the path integral over the boundary value

\(^\text{10}\)Formally, the expansion in small \( \varkappa \) can be done directly in the path integral and will correspond to the insertion of \( \int dt \dot{x}^2 \) operators in addition to the tachyon or vector vertex operators: this will be an amplitude with an extra “off-shell” spin 2 open string mode vertex operators at zero momentum.
of the conformal factor or the 1d metric will no longer be trivial (i.e. will not just give, e.g.,
the delta-function of the on-shell tachyon scattering condition \( \alpha' p_i^2 = 1 \)).

One may study the off-shell amplitudes and hope that looking at their consistency condi-
tions may help determine the modified mass shell restrictions on the external momenta. This
requires extracting the dependence on the 1d metric from the explicit factors in the mass term
as well as the anomalous contributions coming from the use of a covariant cutoff in Green’s
functions at coinciding points (giving terms proportional to \( p_i^2 \)). An open problem is to find
an approximation (e.g., small or large mass expansion) that may make the study of scattering
amplitudes tractable.

For example, ref. [4] considered a modification of the Veneziano amplitude due to massive
(\( \sim m \)) string ends assuming that \( \frac{1}{\sqrt{\alpha'}} \ll m \ll s, t \) where \( s, t \) are kinematic variables. There just
the leading semiclassical approximation was used assuming that the amplitude is still given by
the expectation value of product of \( e^{i \vec{p}_i \cdot \vec{x}(t_i)} \) insertions (i.e. ignoring the issue that the structure
of vertex operators can no longer be fixed using conformal invariance condition). It would
be interesting to perform a similar computation in the setting described above where one is
supposed to integrate over the 1d metric.

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