HAUSDORFF DIMENSION OF GAUSS–CANTOR SETS AND TWO APPLICATIONS TO CLASSICAL LAGRANGE AND MARKOV SPECTRA

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ABSTRACT. This paper is dedicated to the study of two famous subsets of the real line, namely Lagrange spectrum $L$ and Markov spectrum $M$. Our first result, Theorem 2.1, provides a rigorous estimate on the smallest value $t_1$ such that the portion of the Markov spectrum $(-\infty, t_1) \cap M$ has Hausdorff dimension 1. Our second result, Theorem 3.1, gives a new upper bound on the Hausdorff dimension of the set difference $M \setminus L$. In addition, we also give a plot of the dimension function, which hasn’t appeared previously in the literature to our knowledge.

Our method combines new facts about the structure of the classical spectra together with finer estimates on the Hausdorff dimension of Gauss–Cantor sets of continued fraction expansions whose entries satisfy appropriate restrictions.

1. Introduction

The theory of Diophantine approximations began with the search for rational approximations to the solutions of certain algebraic equations (e.g., $x^2 - 2 = 0$, $x^2 - x - 1 = 0$, etc.) and well-known mathematical constants (e.g., $\pi = 3.14159265 \ldots$). Besides its intrinsic beauty, this topic attracted the attention of several generations of mathematicians thanks to its deep connections with many other areas including Kolmogorov–Arnold–Moser (KAM) theory of quasi-periodic motions for Hamiltonian systems (cf. Siegel and Moser books [30], [24]) and the spectral theory of certain quasi-periodic Schrödinger operators (cf. Avila–Jitomirskaya’s solution [1] to the “ten martini problem”).

The investigation of Diophantine approximations often leads to the study of the smallest values of quadratic forms on lattices (i.e., the so-called geometry of numbers): for instance, if $\alpha \in \mathbb{R}$ and $p/q \in \mathbb{Q}$, then $|\alpha - p/q| = q^{-2}h_{\alpha}(p,q)$ where $h_{\alpha}(p,q) := q^2\alpha - pq \in h_{\alpha}(\mathbb{Z}^2)$. In 1841, Dirichlet used his famous pigeonhole principle to show that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\# \{p/q \in \mathbb{Q} : |q^2\alpha - pq| < 1\} = \infty$, and, subsequently, Hurwitz improved upon Dirichlet’s theorem by proving that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\# \{p/q \in \mathbb{Q} : |q^2\alpha - pq| < 1/\sqrt{5}\} = \infty$, but $\# \{p/q \in \mathbb{Q} : |q^2\left(\frac{1+\sqrt{5}}{2}\right) - pq| < 1/(\sqrt{5}+\varepsilon)\} < \infty$ for all $\varepsilon > 0$. More generally, it was realised that the finite “best constants” of Diophantine approximations for real numbers and real indefinite binary quadratic forms are encoded by the Lagrange spectrum

$$L := \left\{ \limsup_{p,q \to \infty} \frac{1}{|q^2\alpha - pq|} < \infty : \alpha \in \mathbb{R} \right\}$$

and the Markov spectrum

$$M := \left\{ \sup_{(p,q) \in \mathbb{Z}^2, (p,q) \neq (0,0)} \frac{1}{|ap^2 + bpq + cq^2|} < \infty : ax^2 + bxy + cy^2 \text{ real indefinite, } b^2 - 4ac = 1 \right\}.$$ 

In 1879–1880, Markov [18], [19] performed the first systematic study of the Lagrange and Markov spectra: in particular, he showed that

$$L \cap [\sqrt{5}, 3) = M \cap [\sqrt{5}, 3) = \left\{ \sqrt{9 - 4z_n^2} : n \in \mathbb{N} \right\}$$

where $z_n \in \mathbb{N}$ are Markov numbers, i.e., the largest coordinates of a triple $(x_n, y_n, z_n) \in \mathbb{N}^3$ satisfying the Markov–Hurwitz cubic equation

$$x_n^2 + y_n^2 + z_n^2 = 3x_ny_nz_n.$$
Remark 1.1. Zagier [31] showed in 1982 that the set of Markov numbers is sparse (e.g., \#\{z_n \leq x : n \in \mathbb{N}\} = c(\log x)^2 + O(\log x(\log \log x)^2))
Goldman [10] observed that the Markov–Hurwitz cubic surface is a special example of character variety, the Markov numbers are known to describe hyperbolic lengths of closed geodesics on an once-punctured torus [21], and the reduction modulo \( p \) of the Markov–Hurwitz cubic equation leads to an interesting family of graphs [3, 5] which are conjectured by Bourgain–Gamburd–Sarnak to form an expander family.

In 1921 Perron [26] gave a simple (dynamical) characterization of the classical spectra in terms of continued fractions. Following Perron, see also [22], consider the set \((\mathbb{N}^*)^Z\) of bi-infinite sequences of elements of \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\). To any element \(\alpha = (\alpha_n)_{n \in \mathbb{Z}} \in (\mathbb{N}^*)^Z\) and each \(k \in \mathbb{Z}\), we associate a pair of real numbers defined in terms of continued fraction expansions
\[
[\alpha_k; \alpha_{k+1}, \alpha_{k+2}, \ldots] := \alpha_k + \frac{1}{\alpha_{k+1} + \frac{1}{\alpha_{k+2} + \ldots}} \quad \text{and} \quad [0; \alpha_{k-1}, \alpha_{k-2}, \ldots] := \frac{1}{\alpha_{k-1} + \frac{1}{\alpha_{k-2} + \ldots}}
\]
and consider the map \(\lambda_0(\alpha) := [\alpha_0; \alpha_1, \alpha_2, \ldots] + [0; \alpha_{-1}, \alpha_{-2}, \ldots]\). Let us denote by \(\sigma\) the Bernoulli shift on \((\mathbb{N}^*)^Z\) given by \(\sigma((\alpha_n)_{n \in \mathbb{Z}}) = (\alpha_{n+1})_{n \in \mathbb{Z}}\). The Lagrange value of \(\alpha\) is the limit superior of values of \(\lambda_0\) along the \(\sigma\)-orbit of \(\alpha\):
\[
\ell(\alpha) := \limsup_{n \to \infty} \lambda_0(\sigma^n \alpha) = \limsup_{n \to \infty} ([\alpha_n; \alpha_{n+1}, \alpha_{n+2}, \ldots] + [0; \alpha_{n-1}, \alpha_{n-2}, \ldots])
\]
and the Markov value of \(\alpha\) is the supremum of values of \(\lambda_0\) along the \(\sigma\)-orbit of \(\alpha\):
\[
m(\alpha) := \sup_{n \in \mathbb{Z}} \lambda_0(\sigma^n \alpha) = \sup_{n \in \mathbb{Z}} ([\alpha_n; \alpha_{n+1}, \alpha_{n+2}, \ldots] + [0; \alpha_{n-1}, \alpha_{n-2}, \ldots]).
\]
In this setting, Perron established that the Lagrange and Markov spectra are the collections of (finite) Lagrange and Markov values:
\[
L := \{\ell(\alpha) \in \mathbb{R} \mid \alpha \in (\mathbb{N}^*)^Z\} \quad \text{and} \quad M := \{m(\alpha) \in \mathbb{R} \mid \alpha \in (\mathbb{N}^*)^Z\}.
\]

Remark 1.2. The values of \(n\)-ary quadratic forms, \(n \geq 3\), can also be studied using dynamical ideas: for instance, Margulis famously solved Oppenheim’s conjecture using higher rank actions on homogeneous spaces (see [17] for a nice survey). However, we shall not make further comments about this because the techniques in the present paper (inspired from rank one systems such as the Gauss map and the geodesic flow on the modular surface) are fundamentally distinct from the results concerning higher rank systems.

The dynamical result of Perron gives access to many basic properties of the classical spectra (see, e.g., [6]): for example, it is known that \(L \subseteq M\) are closed subsets of the real line such that \(\sqrt{12}, \sqrt{13} \in L\) and
\[
L \cap (\sqrt{12}, \sqrt{13}) = M \cap (\sqrt{12}, \sqrt{13}) = \emptyset.
\]
Moreover, Hall [11] observed in 1947 that certain portions of \(L\) and \(M\) are controlled in terms of the arithmetic sums of Cantor sets of real numbers whose continued fraction expansions satisfy certain restrictions: for instance, if \(E_4 = \{[0; \alpha_1, \alpha_2, \ldots] : \alpha_n \in \{1, 2, 3, 4\} \land n \in \mathbb{N}\}\), then Hall proved that the arithmetic sum
\[
E_4 + E_4 = \{x + y : x, y \in E_4\}
\]
contains an interval of length \(> 1\), and this fact was exploited to show that \([6, \infty) \subseteq L \subseteq M\). Actually, since the classical spectra are closed subsets of the real line, there exists the smallest number \(c_F\) such that \([c_F, +\infty) \cap L = [c_F, +\infty) \cap M = [c_F, +\infty):\) this half-line is usually called Hall’s ray in the literature. As it turns out, the value of \(c_F\) was computed explicitly by Freiman in [8] to be \(c_F = 4.5278 \ldots\) and is called Freiman’s constant.

Remark 1.3. The arithmetic sum \(A + B = \{a + b : a \in A, b \in B\}\) of \(A, B \subseteq \mathbb{R}\) is the image \(A + B = \pi(A \times B)\) of the cartesian product \(A \times B \subseteq \mathbb{R}^2\) under \(\pi : \mathbb{R}^2 \to \mathbb{R}, \pi(x, y) = x + y\). Thus, the results of Hall mentioned above point towards a connection between the classical spectra and the projections of fractal sets which is going to be relevant in the present paper.

1For more details about the standard relationship between continued fractions, Bernoulli shift, and Gauss map, see the book of Einsiedler and Ward [7].

2All numbers are truncated, not rounded.
In contrast to this, the sets $L \cap [3, c]$ and $M \cap [3, c]$ have a complicated and mysterious structure. Nevertheless, some facts have been established. In particular, it was shown by Hall [12] that $M \cap [0, \sqrt{10}]$ has zero Lebesgue measure. A few years later this result was improved by Pavlova and Freiman [9] (cf. [6, Theorem 2, Chapter 6]), when they showed that $M \cap [0, \sqrt{689}/8]$ has zero Lebesgue measure.\(^3\)

More recently, it was shown by the second author in [22] that for any $t > 0$ the sets $(-\infty, t] \cap M$ and $(-\infty, t] \cap L$ have the same Hausdorff dimension:

$$\dim_H((-\infty, t] \cap M) = \dim_H((-\infty, t] \cap L)$$

and, moreover, the function

$$f(t) := \dim_H((-\infty, t] \cap M)$$

is a continuous non-decreasing function on the real line. We now introduce the number which is the subject of our investigations.

**Definition 1.4.**

(1.2) \[ t_1 := \inf \{t \in \mathbb{R} \mid f(t) = 1\}. \]

In view of monotonicity of $f$ the value $t_1$ is usually referred to as the first transition point of the classical Lagrange and Markov spectra. In 1982 Bumby [4] gave a heuristic estimate

(1.3) \[ 3.33437 < t_1 < 3.33440, \]

while the results by Hall [12] and the second author [22] give the best rigorous lower and upper bounds on $t_1$ to date:

(1.4) \[ \sqrt{10} = 3.162277 \ldots < t_1 < \sqrt{12} = 3.464101 \ldots. \]

Our first result, Theorem 2.1 confirms Bumby’s claim and gives a rigorous estimate of $t_1 = 3.334384\ldots$ The proof is built on ideas developed by Bumby and uses a connection between Markov values and Gauss–Cantor sets defined in terms of continued fractions of their elements. The argument is computer—assisted and the result could be refined further with the method we present, subject to more computer time and resources.

Using a similar approach, we can also compute a good approximation to the function $f$ by solving equations $f(x) = k/n$ for $k = 1, 2, \ldots, n - 1$ and $n$ sufficiently large. The plot of the resulting function is shown on the right.

Lima and the second author in [15] recently conjectured\(^4\) that $(t_1, t_1 + \delta) \cap L$ has non-empty interior for all $\delta > 0$. Together with our new result $t_1 = 3.334384\ldots$ this would imply, in particular, that $(3.334384, 3.334385) \cap L$ has non-empty interior and thus would prove an open folklore conjecture that the interior of $(-\infty, \sqrt{10}] \cap L$ is non-empty.

The second part of our paper concerns the set difference of the Markov and Lagrange spectra. It is known that $M \setminus L$ has zero Lebesgue measure. Furthermore, it was proved in [20] and [27] that the Hausdorff dimension of $M \setminus L$ satisfies

$$0.5312 < \dim_H(M \setminus L) < 0.8823.$$ \(^{\text{The proof is also computer—assisted. Following the approach developed by the first two authors [20], we use fine-grained combinatorial analysis of continued fractions to construct a cover $M \setminus L$ by arithmetic sums of Gauss–Cantor sets and the so-called “Cantor sets of the gaps”. We then apply the new method for computing the Hausdorff dimension recently developed by the last two authors [27] to several Gauss–Cantor sets to obtain sharper upper bounds on $\dim_H(M \setminus L)$.}}

\(^{\text{3}}\)Note that $\sqrt{689}/8 = 3.2811\ldots$ and $\sqrt{10} = 3.162277\ldots$

\(^{\text{4}}\)This is motivated by the main results of [15] saying that this conclusion holds in many “similar” examples of dynamical Lagrange and Markov spectra.
We organize this article as follows. In §2, we reduce the problem of computing \( t_1 \) to the problem of constructing two Gauss–Cantor sets \( X \) and \( Y \) such that

\[ \dim_H X < 0.5 < \dim_H Y, \]

and the substrings of \( \alpha \in (\mathbb{N}^*)^\mathbb{Z} \) with \( m(\alpha) \) close to \( t_1 \) are “controlled” by \( X \) and \( Y \). The conditions that \( X, Y \) and \( t_1 \) should jointly satisfy are slightly more subtle and we describe them in detail in §2.1.2. Next §3 is dedicated to the construction and analysis of the arithmetic sums of Gauss–Cantor sets and “Cantor sets of the gaps” which cover \( M \setminus L \), and subsequently allows us to obtain an upper bound on \( \dim_H (M \setminus L) \). The intricate character of the Gauss–Cantor sets involved in estimates in §2 and §3 means that the algorithm for computing the Hausdorff dimension developed in [27] has to be considerably adapted and improved. For completeness, in §4 we explain how the Hausdorff dimension of the complicated Gauss–Cantor sets can be computed and give some details of the numerical implementation; in addition, we also provide pseudocode in the Appendix (with the codes available at https://github.com/Polevita/Gauss_Cantor_sets).

Finally, we discuss in §5 a few lines of future research partly motivated by our main results.

Remark 1.5. On our way to establishing the results mentioned in the previous paragraphs, we encounter some other interesting facts about the structure of the classical spectra. For example, Lemma 3.9 below says that \( 4.5278 \ldots \) is a non-trivial rational point in \( L \) in the sense that it occurs after \( 3 \) and before the beginning \( c = 4.5278 \ldots \) of Hall’s ray. Hence the value \( 4.5 \) is realised as the Markov value of two sequences which arise in our study of \( (4.4984, \sqrt{21}) \cap (M \setminus L) \).

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2. Phase transition in classical spectra

In this section, we give the theoretical basis for the proof of our first main result which provides rigorous bounds on the first transition point \( t_1 \) and construct explicitly the relevant Gauss–Cantor sets. The theoretical background for computer-assisted calculations which are used to obtain estimates on Hausdorff dimension of those Gauss–Cantor sets which are constructed here can be found in §4.

Theorem 2.1. \( t_1 = \inf \{ t \in \mathbb{R} : \dim_H ((-\infty, t] \cap M) = 1 \} = 3.334384 \ldots \), where this value is accurate to the 6 decimal places presented.

2.1. Preliminaries. We begin by describing the basic strategy to deduce bounds on \( t_1 \) which generalizes the approach of Hall. The first Cantor set we introduce is relatively famous and consists of all real numbers whose continued fraction expansion has only digits 1 and 2:

\[ E_2 := \{ a = [0; \alpha_1, \alpha_2, \ldots] \mid \alpha_j \in \{1, 2\}, j \geq 1 \}. \]

Its Hausdorff dimension has been computed to high precision (see, for instance [13], and references therein), and for our purposes it is sufficient to know that

\[ \dim_H E_2 > 0.53128. \]

In what follows, we identify a subset \( A \subseteq E_2 \) with a set of one-sided sequences corresponding to the continued fraction expansions of its elements.
In the sequel we use a simple observation that the constant bi-infinite sequence \( \beta_n \equiv 1, n \in \mathbb{Z} \), has the minimal Markov value among all bi-infinite sequences \( \alpha \in \{1, 2\}^{\mathbb{Z}} \), and, moreover, for any \( \alpha \neq \beta \) we have \( m(\beta) < m(\alpha) \). Straightforward computation gives
\[
m(\beta) = \sqrt{5} \leq m(\alpha) \quad \text{for any } \alpha \in \{1, 2\}^{\mathbb{Z}}.
\]

2.1.1. Approach to lower bound. We fix some threshold \( T \) and attempt to construct a finite set of finite “forbidden” strings \( \beta_{-k} \ldots \beta_1 \beta_0 \beta_1 \ldots \beta_n \) so that all infinite extensions \( \{ \alpha \in \{1, 2\}^{\mathbb{Z}} | \beta_j = \alpha_j, -k \leq j \leq n \} \) of these strings have Markov values \( m(\alpha) \geq T \).

By definition, after excluding from \( E_2 \) all irrational numbers whose continued fraction expansion contains a “forbidden” string, we obtain a Cantor set \( K \subset E_2 \) such that
\[
(2.2) \quad M \cap (\sqrt{5}, T) \subset 2 + K + K.
\]
Recall that \( \dim_H(K + K) \leq \dim_H K + \overline{\text{dim}_B}K \), where \( \overline{\text{dim}_B}K \) denotes the upper box dimension. It is known that \( \overline{\text{dim}_B}K = \dim_H K \) for these types of sets (cf. Chapter 4 of Palis–Takens book [25]) and hence \( \dim_H K < 0.5 \) implies that \( t_1 \geq T \).

2.1.2. Approach to upper bound. Now let \( S \) be the maximal Markov value of strings which do not contain a forbidden string as a substring and let \( K \subset E_2 \) be as above. It was shown in [22, proof of Lemma 3] that
\[
(2.3) \quad \min\{2 \cdot \dim_H K, 1\} \leq \dim_H((\sqrt{5}, S) \cap M),
\]
Therefore we deduce that \( \dim_H K \geq 0.5 \) implies \( t_1 \leq S \).

In order to illustrate this methodology we shall show the double inequality (1.4).

Example 2.2. To establish the lower bound in (1.4), we can use a result by Hall [12] stating that if \( \alpha \in \{1, 2\}^{\mathbb{Z}} \) doesn’t contain the string 121, then \( m(\alpha) < \sqrt{10} \). So we choose
\[
K := \{[0; \alpha_1, \alpha_2, \ldots] | \alpha_j \in \{1, 2\}, (\alpha_j, \alpha_{j+1}, \alpha_{j+2}) \neq (121) \text{ for all } j \geq 1 \},
\]
and apply the algorithm from §4 to show that \( \dim_H K < 0.45 \). Then (2.2) gives
\[
\dim_H \left( \{ \alpha \in \{1, 2\}^{\mathbb{Z}} : \sqrt{5} < m(\alpha) \leq \sqrt{10} \} \right) \leq 2 \dim_H K \leq 0.9
\]
and the lower bound \( t_1 \geq \sqrt{10} \) follows.

To establish the upper bound in (1.4), we recall a result by Perron [26] which states that \( m(\alpha) \leq \sqrt{12} \) if and only if \( \alpha \in \{1, 2\}^{\mathbb{Z}} \). Therefore we may choose the empty set of forbidden strings and \( K = E_2 \). Combining (2.1) with (2.3) we get
\[
\dim_H((\sqrt{5}, \sqrt{12}) \cap M) \geq \min\{2 \cdot \dim_H E_2, 1\} = \min\{2 \cdot 0.54318, 1\} = 1,
\]
and conclude that \( t_1 \leq \sqrt{12} \).

This approach has been used by Bumby to obtain heuristic estimates and we will review it in detail below in §2.2. Since we already know that \( m(\alpha) \leq \sqrt{12} \) if and only if \( \alpha \in \{1, 2\}^{\mathbb{Z}} \), until the end of §2 we study only sequences of 1s and 2s.

2.2. Bumby’s method for building the set of forbidden strings. In this section we explain how to find a suitable set of forbidden strings which can be employed to define a set \( K \) to use in (2.2) or (2.3).

Recall the map \( \lambda \) introduced in §1
\[
\lambda_0 : \mathbb{N}^{\mathbb{Z}} \rightarrow \mathbb{R}, \quad \lambda_0(\alpha) = [\alpha_0; \alpha_1, \alpha_2, \ldots] + [0; \alpha_{-1}, \alpha_{-2}, \ldots].
\]
On the one hand, it is clear from definition that \( m(\alpha) \geq \lambda_0(\alpha) \). On the other hand, it is a well known fact that for any Markov value \( m \in M \), there exists a sequence \( \alpha \) such that \( \lambda_0(\alpha) = m \); see for instance [6, Lemma 6, Chapter 1]. Therefore, one can attempt to construct a suitable set of forbidden strings by studying the function \( \lambda_0 \). This brings us to introducing a function \( J \), which associates to a finite string a closed interval.

Definition 2.3. We denote by \( J(\alpha_{-k:j}) \) the interval given by the convex hull of the set of values \( \lambda_0(\beta) \) for strings \( \beta \in \{1, 2\}^{\mathbb{Z}} \) such that \( \beta_n = \alpha_n \) for all \( -k \leq n \leq j \).
In the sequel we will use the following shorthand notation for certain finite substrings of a string $\alpha$: $\alpha_{-k,j} = \alpha_{-k} \ldots \alpha_{-1} \alpha_0 \alpha_1 \ldots \alpha_j$, where $j, k \geq 0$.

Let us denote by $\overline{\alpha}$ the periodic sequence obtained by infinite repetition of a given finite string $\alpha$. The following technical Lemma allows one to compute the interval $J(\alpha_{-k,j})$ explicitly.

**Lemma 2.4.** For any sequence $\alpha \in \{1, 2\}^\mathbb{Z}$ we have an upper bound

$$\lambda_0(\alpha) \leq \begin{cases} [\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k}] & \text{if } k \text{ and } j \text{ are even,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k-1}] & \text{if } k \text{ is even and } j \text{ is odd,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k-1}] & \text{if } k \text{ is odd and } j \text{ is even,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k}] & \text{if } k \text{ and } j \text{ are odd;} \\
\end{cases}$$

and a lower bound

$$\lambda_0(\alpha) \geq \begin{cases} [\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k}] & \text{if } k \text{ and } j \text{ are even,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k-1}] & \text{if } k \text{ is even and } j \text{ is odd,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k-1}] & \text{if } k \text{ is odd and } j \text{ is even,} \\
[\alpha_0; \alpha_1 \ldots \alpha_j] + [0; \alpha_{-1} \ldots \alpha_{-k}] & \text{if } k \text{ and } j \text{ are odd.} \\
\end{cases}$$

**Proof.** This follows immediately from the fact that $\inf E_2 = [0; 21] = \frac{1}{2}(\sqrt{3} - 1)$ and $\sup E_2 = [0; 12] = \sqrt{3} - 1$, where $21$ and $12$ represent infinite sequences of alternating $1$s and $2$s.

This Lemma also allows us to establish two more properties of the function $J$ which will be useful for our analysis.

1. The function $J$ is invariant under reversal of the string (note that reversal keeps the $0$’th place unchanged).
2. Extensions of a string $\alpha_{-k,j}$ correspond to subintervals of $J(\alpha_{-k,j})$.

$$J(\alpha_{-k,j}) = J(1\alpha_{-k} \ldots \alpha_j) \cup J(2\alpha_{-k} \ldots \alpha_j)$$

$$= J(\alpha_{-k} \ldots \alpha_j) \cup J(\alpha_{-k} \ldots \alpha_j 2);$$

Observe that unions need not to be disjoint.

The second property allows us to organise the intervals obtained from continuations of a given string in a binary tree, so that the union of children is equal to the parent.

Now we can describe a recursive process for the construction of sets of forbidden strings. A basic idea is that we fix a threshold $T$, close to a conjectured lower bound on $t_1$ and look for finite strings $\alpha_{-k,j}$ such that the corresponding intervals $J(\alpha_{-k,j})$ lie to the right of $T$. We call these finite strings “forbidden” and obtain the Cantor set $K$ by removing from $E_2$ all numbers whose continued fraction expansion contains a forbidden substring. If an interval $J(\alpha_{-k,j})$ lies to the left of $T$, we make a record of its right end point as a possible upper bound on $t_1$. If $T \in J(\alpha_{-k,j})$ then we subdivide the interval into two by adding an extra symbol to $\alpha_{-k,j}$ either in the beginning or at the end and study these two new intervals at the next step of the recursive process. When we find a new forbidden substring, we recompute the Hausdorff dimension of the updated set $K$. We may need to lower the original threshold, if $\dim_H K > 0.5$ and the right end points of the intervals which lie to the left of $T_1$ is too large; on the other hand, we may need to increase the original threshold if $\dim_H K < 0.5$ and we look to improve an existing lower bound. We terminate the recursion when we find two sets which are suitable to confirm lower and upper bounds on $t_1$ using (2.2) and (2.3) respectively.

**Remark 2.5.** A similar approach can be used to solve the equation $f(x) = t$ for other values $0 < t < 1$. Namely, Figure 1 was obtained by computing the lower and upper bounds on $x$ with the property that $\dim_H ((-\infty, x) \cap M) = t$ using (2.2) and (2.3) for $t = 3 + t_1 \cdot \frac{1}{10}$, $k = 1, 2, \ldots, 700$.

2.3. **Rigorous verification of the Bumby’s estimate (1.3).** In preparation for our estimate for $t_1$ we will first rigorously confirm bounds close to the heuristic values of Bumby [4]. This analysis will be an integral part of our subsequent improved estimates.

Following Bumby, keeping in mind the heuristic estimate $3.33437 < t_1 < 3.33440$ which we would like to confirm rigorously, let us fix the threshold

$$T_1 = 3.334369.$$
We are ready to start the recursive process of computing the set of forbidden strings. We use the asterisk to mark the zeroth place in the string, i.e. $2^*1$ corresponds to $\alpha_{-1} = 2$, $\alpha_0 = 2$, $\alpha_1 = 1$.

We begin with a simple observation that a sequence $\alpha \in \{1, 2\}^\omega$ with $3 < \lambda_0(\alpha) < \sqrt{12}$ satisfies $\alpha_0 = 2$. (Since by Lemma 2.4 we get $J(1^*) = [3/\sqrt{3}, \sqrt{12} - 1]$ and $J(2^*) = [\sqrt{3} + 1, \sqrt{12})$. We may now consider two continuations $2^*2$ and $2^*1$. Applying Lemma 2.4 again we compute

$$1 + \sqrt{3} = [2; 2T_2] + |0; 2T_2| \leq \lambda_0(2^*2) \leq [2; 2T_2] + |0; T_2| = 2 + \frac{2}{\sqrt{3}}.$$ 

We conclude that $J(22^*) = J(2^*2) \subset [1 + \sqrt{3}, 3.155] < T_1$ and proceed to analyse continuations of $x_{-1}x_0x_1 = 12^*1$.

The process of constructing the set of forbidden strings is depicted in Figure 3. We begin at the root marked $2^*$ and follow two edges up adding letters marked by the bar symbol in the beginning, as a prefix, and letters marked by the hat symbol at the end, as a suffix. Thus every vertex corresponds to a finite string and we compute the interval corresponding to this string in order to decide how to proceed further. Starting from the root, after two steps we arrive at a vertex which corresponds to $12^*1$. Taking two steps further we obtain $212^*12$ which is our first excluded string because by Lemma 2.4, the corresponding interval is $J(212^*12) \subset [3.4, \sqrt{12}] > T_1$.

The intervals corresponding to the vertices of the tree which are crucial to our analysis are recorded in Table 1. For completeness in §2.3.4 we give details of our analysis. All intervals are computed using Lemma 2.4.

2.3.1. Exclusion of $21212$. Note that

$$J(112^*11) \subset [3.1547, 3.268], \quad J(112^*12) \subset [3.28, 3.3661], \quad J(212^*12) \subset [3.4, \sqrt{12}].$$

Since $3.268 < T_1 < 3.4$, we exclude $212^*12$ and we analyse the continuation of $212^*11$. For this purpose, we decompose $J(112^*12)$ into $J(1112^*12)$ and $J(2112^*12)$.

Note that

$$J(2112^*12) \subset [3.2802, 3.3193] \quad \text{and} \quad J(1112^*12) \subset [3.3149, 3.3661].$$

Thus, it suffices to study the continuations $x = 112^*12$ (as $T_1 > 3.3193$).

2.3.2. Exclusion of $21112121$ and $21112122$. Let’s consider the decompositions of the intervals $J(1112^*121)$ and $J(1112^*122)$ where the string $21212$ doesn’t appear. For the first interval, it amounts to studying $J(11112^*121)$, $J(21112^*121)$. Note that

$$J(1112^*121) \subset [3.3324, 3.3524] \quad \text{and} \quad J(21112^*121) \subset [3.35, 3.3661].$$

For the second interval, we have

$$J(21112^*1222) \subset [3.337, 3.3419], \quad J(11112^*1222) \subset [3.3189, 3.3282]$$

$$J(21112^*1221) \subset [3.3329, 3.3389], \quad J(11112^*1221) \subset [3.3149, 3.3252].$$

Since $3.3282 < T_1 < 3.337$, we exclude $2112121$ and $21112122$, and we shall consider the decompositions of the intervals $J(11112^*121)$ and $J(21112^*121)$.

2.3.3. Exclusion of $1111121$ and $2111212$. Note that

$$J(11112^*121) \subset [3.3376, 3.3524], \quad J(211112^*121) \subset [3.3324, 3.3456],$$

and

$$J(121112^*121) \subset J(121112^*12) \subset [3.353, 3.3661], \quad J(221112^*121) \subset [3.3329, 3.3356].$$

Because $T_1 < 3.3353$, we exclude $1111121$ and $1211121$, and we consider the decompositions of $J(21112^*121)$ and $J(22112^*1212)$. Actually, given that the string $21212$ is already excluded, our task is to study the decompositions of $J(211112^*1211)$ and $J(221112^*1211)$.

2.3.4. Exclusion of $11112111$. Observe that

$$J(211112^*1211) \subset J(1112^*12111) \subset [3.3351, 3.3588], \quad J(211112^*12112) \subset [3.3324, 3.3348],$$

and

$$J(221112^*1212) \subset [3.33294, 3.3397], \quad J(221112^*12111) \subset [3.341, 3.3356].$$

Because $3.3397 < T_1 < 3.351$, we exclude $11121111$ and we decompose $J(211112^*12112)$ and $J(221112^*12111)$. 
Figure 3. The tree depicting the process of constructing forbidden strings. Each vertex corresponds to a finite string of 1s and 2s, which can be recovered by going down to the root writing the labels along the path marked by bars and going up to the vertex writing the labels marked by hats. The short forbidden words are written explicitly, the longer ones abbreviated as $fw_j$, $j = 9, \ldots, 14$. A dashed edge without a label leading to a forbidden word means that the forbidden word corresponds to a longer interval than the one which corresponds to the vertex it is connected to. The vertices $R_1$, $R_2$, $R_3$ will be used as the roots for new trees which we build in Section 2.4 in order to prove our first main result, Theorem 2.1.
Table 1. Strings and intervals crucial to our analysis. The action column indicates how to proceed with the tree construction further: E — Exclude the string (the case \( J_0 > T_1 = 3.334369 \)), the corresponding vertex is a leaf; A — Abandon the branch (the case \( J_0 < T_1 \)) the corresponding vertex is a leaf; S — Subdivide the interval into two parts (the case \( T_1 \in J_0 \)), the vertex is a branching point. An estimate for \( J(21121112^*21211) \) used the fact that 2112121 is excluded, in addition to Lemma 2.4. There is only one branch out of the 8th vertex because 21212 is already excluded, so the only possible suffix is 1.

2.3.5. Exclusion of 21112121112. Note that \( J(211112^*12112) \) decomposes into \( J(2211112^*12112) \) and

\[
J(2111112^*12112) \subset [3.3324, 3.3339].
\]
Similarly, \( J(221112^*122111) \) decomposes into \( J(221112^*122111) \) and

\[
J(221112^*122111) \subset J(211112^*122112) \subset [3.33, 3.389].
\]

Since \( 3.33 < T_1 < 3.34 \), we exclude \( 21112122112 \), and we decompose \( J(2211112^*1212112) \) and \( J(221112^*122111) \).

2.3.6. **Exclusion of 222111212211**. Note that \( J(2211112^*1212112) \) breaks into \( J(2211112^*121112) \) and

\[
J(2211112^*121112) \subset [3.33, 3.3426].
\]

Analogously, \( J(221112^*122111) \) decomposes into \( J(1221112^*122111) \) and

\[
J(2211112^*122111) \subset J(2221112^*122111) \subset [3.33469, 3.33541].
\]

Given that \( 3.33426 < T_1 < 3.33469 \), we exclude \( 222111212211 \), and we proceed to analyse the decompositions of \( J(2211112^*121112) \) and \( J(1221112^*122111) \).

2.3.7. **Exclusion of 12211112121112**. We break the previous intervals into \( J(12211112^*121112) \), \( J(2211112^*121112) \)
and \( J(1221112^*122111) \), \( J(1221112^*122111) \), \( J(1221112^*122111) \), and we observe that

\[
J(12211112^*121112) \subset [3.33448, 3.33472].
\]

Because \( T_1 < 3.33448 \), we exclude \( 12211112121112 \), and we decompose \( J(2211112^*121112) \), \( J(1221112^*122111) \)
and \( J(1221112^*122111) \).

2.3.8. **Exclusion of two extra strings**. We decompose the interval \( J(2211112^*121112) \) into \( J(2211112^*121112) \) and

\[
J(2211112^*121112) \subset J(2211112^*121121) \subset [3.33, 3.3472].
\]

Similarly, \( J(1221112^*122111) \) subdivides into \( J(21211112^*122111) \) and

\[
J(11221112^*122111) \subset [3.33447, 3.34684].
\]

Analogously, \( J(1221112^*122111) \) breaks into \( J(11221112^*122111) \) and

\[
J(21211112^*122111) \subset [3.3414, 3.3424].
\]

Because \( 3.33424 < T_1 < 3.33441 \), we exclude \( 112211121221111 \) and \( 22111121211121 \), and we analyse \( J(2211112^*1211222) \),
\( J(2121112^*1221111) \) and \( J(11221112^*122111) \).

2.3.9. **Exclusion of three extra strings**. We decompose the interval \( J(12211112^*1221112) \) into

\[
J(11221112^*1221112) \subset [3.3343, 3.34393],
\]

\[
J(211221112^*1221112) \subset J(211221112^*122111) \subset [3.343894, 3.352].
\]

(Here, we estimated the second interval using the fact that \( 21112121 \) is excluded.)

Similarly, we break \( J(21221112^*1221111) \) into

\[
J(21221112^*1221111) \subset [3.3343, 3.34402],
\]

\[
J(21221112^*1221111) \subset J(21112^*1221112) \subset [3.334009, 3.3384].
\]

Finally, we observe that \( J(22111112^*1211122) \) subdivides into \( J(22111112^*1211122) \), \( J(22211112^*1211122) \),
\( J(22211112^*1211122) \) with

\[
J(22211112^*1211122) \subset [3.33435, 3.34375],
\]

\[
J(22211112^*1211222) \subset [3.334371, 3.343876],
\]

and

\[
J(22211112^*1211222) \subset [3.3343899, 3.33441].
\]

Since \( T_1 < 3.3343894 \), we exclude \( 211221112121112 \), \( 211121221112 \) and \( 222111121211222 \).
2.3.10. Upper bound on \( t_1 \) revisited. Using numerical data from the top part of Table 1 we are now in a position to get an upper bound on \( t_1 \) in line with the heuristic estimate of 3.33440 suggested by Bumby. Denote by \( B_1 \) the Cantor set of numbers whose continued fraction expansions in \( \{1, 2\}^\mathbb{N} \) which do not contain the following fourteen strings (nor their transposes) taken from the lines of Table 1 marked for exclusion:

- \( 21212, 21112121, 21112122, 11111212, 12112121, 11212111, 2111212212, \)
- \( 22111212211, 122111212122, 112211212111, 221111212121, \)
- \( 211221112121, 2111212111221, 2221112121222. \)

The algorithm described in Section 4 provides us lower and upper bounds (see Subsection 4.6.1 for numerical data and implementation notes)

\[
0.50001 < \dim B_1 < 0.50005
\]

which confirms Bumby’s heuristics in [4]. Consequently, applying (2.3) we get that \( t_1 \) is bounded from above by the maximum of the right endpoints of the non-excluded intervals that appeared in the process of construction of the set \( B_1 \) (both abandoned and marked for subdivision). This turns out to be the right end point of the interval corresponding to the vertex 17. In particular, we have that

\[
t_1 \leq S_1 = 3.334402.
\]

2.3.11. Lower bound on \( t_1 \) revisited. With a little more work we can get a lower bound on \( t_1 \) which supports Bumby’s lower bound on \( t_1 \) of 3.33437. Continuing to follow Bumby [4], let us further analyse the intervals

\[
J(21221112^*1221111) \text{ and } J(111221112^*1221112),
\]

which correspond to the 16th and 17th vertices of the tree and marked for subdivision in Table 1. Our computations are presented in the bottom part of Table 1. In particular, we see that one can also exclude 111122111212211122 and 22122111212211111 in order to obtain a smaller Cantor set \( B_2 \subsetneq B_1 \). Applying the algorithm for computing Hausdorff dimension described in §4 we obtain estimates on dimension (see §4.6.2 for implementation notes):

\[
0.499975 < \dim_H B_2 < 0.49999
\]

This is quite close to Bumby’s heuristic claim that \( \dim_H(B_2) < 0.499974 \) and we conclude that

\[
t_1 \geq T_1 = 3.334369.
\]

Summing up, we have rigorously confirmed that the heuristic argument by Bumby in favour of looking for \( t_1 \) inside the interval \( (3.33437, 3.33440) \) was correct.

After the above review (and slight improvement) of Bumby’s work [4], we now turn to the proof of our main result Theorem 2.1.

2.4. Proof of Theorem 2.1. Recall that our goal is to show that the first transition point \( t_1 = 3.334384 \ldots \) It is sufficient to prove that

\[
3.3343840 < t_1 < 3.33438495.
\]

For this purpose, let us fix the thresholds

\[
T_2 := 3.334384009 \quad \text{and} \quad S_2 := 3.3343849341.
\]

Our goal now is to modify the Cantor sets \( B_1 \) and \( B_2 \) defined above to obtain two Cantor sets \( X \) and \( Y \) such that the intervals corresponding to forbidden strings used to define \( X \) lie to the right of \( T_2 \) and \( S_2 \) is the right end point of the intervals corresponding to the non-excluded strings which appear in the construction of \( Y \). Furthermore, we also require that the double inequality \( \dim_H X < 0.5 < \dim_H Y \) holds.

In this direction we consider the intervals listed in Table 1 and choose the smallest (by inclusion) intervals which contain both \( T_2 \) and \( S_2 \) in order to subdivide them further and to identify forbidden strings exclusion of which will result in Cantor sets with dimension closer to 0.5 than \( \dim_H B_1 \) and \( \dim_H B_2 \). These turn out to be the intervals corresponding to the vertices \( R_1 \), \( R_2 \), and \( R_3 \). We list the corresponding strings: \( R_1 = 222211112^*12112221 \), \( R_2 = 1111221112^*12211122 \), and \( R_3 = 221221112^*122111111 \). We subdivide each of the intervals \( J(R_1), J(R_2), \) and \( J(R_3) \) following the same process as before, with a separate decision tree in each case.
Table 2. Numerical data for the subdivision of the interval \( J(R_1) = J(22211112^*12112221) \). The corresponding tree is shown in Figure 4. Strings corresponding to the intervals to the right of \( T_2 = 3.334384009 \) marked for exclusion.

| String \( \alpha_{-k,j} \) | Interval \( J_0 \supset J(\alpha_{-k,j}) \) | Action |
|---------------------------|---------------------------------|--------|
| \( R_12 \)               | [3.334371, 3.334381]            | A      |
| \( 2 R_11 \)             | [3.334376, 3.33438141]          | A      |
| \( 1 R_12 \)             | [3.334384049, 3.3343876]        | E      |
| \( 11 R_111 \)           | [3.334381, 3.3343837]           | A      |
| \( 121 R_11 \)           | [3.33438409, 3.3343876]         | E      |
| \( 221 R_1112 \)         | [3.33438368, 3.33438401]        | S      |
| \( 21 R_1111 \)          | [3.33438444, 3.33438551]        | E      |

Figure 4. Continuation of the string \( R_1 = 222111112^*12112221 \).

Table 3. Numerical data for the subdivision of the interval \( J(R_2) = J(1111221112^*12211122) \). The subdivision tree is shown in Figure 5. Strings corresponding to the intervals to the right of \( T_2 = 3.334384009 \) marked for exclusion.

| String \( \alpha_{-k,j} \) | Interval \( J_0 \supset J(\alpha_{-k,j}) \) | Action |
|---------------------------|---------------------------------|--------|
| \( 1 R_2 \)               | [3.334369, 3.33438361]          | A      |
| \( 2 R_21 \)              | [3.33438668, 3.33439261]        | E      |
| \( 2 R_221 \)             | [3.3343815, 3.3343847]          | S      |
| \( 12 R_222 \)            | [3.33438429, 3.3343856]         | E      |

2.4.1. Refinement of \( J(222111112^*12112221) \). The tree depicting continuation of the string \( R_1 = 222111112^*12112221 \) is shown in Figure 4 and the numerical data for the key intervals is given in Table 2 (obtained using Lemma 2.4). Three extra strings are marked for exclusion, namely \( 1 R_112, 121 R_11, \) and \( 21 R_111 \).

2.4.2. Refinement of \( J(1111221112^*12211122) \). The tree depicting continuation of the string \( R_2 = 1111221112^*12211122 \) is shown in Figure 5 and the numerical data for the key intervals is given in Table 3. Based on the threshold \( T_2 \) we exclude \( 2 R_21 = 21111221112^*122111221 \) and \( 12 R_22 = 121111221112^*1221112222 \).

2.4.3. Refinement of \( J(221221112^*12211111) \). The tree depicting continuation of the string \( R_3 = 221221112^*12211111 \) is shown in Figure 6 and the numerical data for the key intervals is shown in Table 4. Based on the threshold \( T_2 \) five additional strings are marked for exclusion: \( R_32, 21 R_33, 1 R_311, 111 R_312, \) and \( 22 R_3112 \).
2.4.4. Lower bound on $t_1$. In order to confirm the lower bound stated in Theorem 2.1, we collect together numerical data from calculations in §§2.4.1–2.4.3. Consider the Cantor set $X \subset E_2$ of numbers which continued fraction expansions do not contain neither any of the following 24 strings nor their transposes:

| String $\alpha_{-k,j}$ | Interval $J_0 \supset J(\alpha_{-k,j})$ | Action |
|-------------------------|--------------------------------------|--------|
| $R_3^2$                 | [3.33439, 3.334402]                 | E      |
| 21$R_3$                 | [3.3343856, 3.334402]               | E      |
| 1$R_3$11                | [3.3343866, 3.3343922]              | E      |
| 211$R_3$12              | [3.334383, 3.33438429]              | S      |
| 111$R_3$121             | [3.33438375, 3.334384636]           | S      |
| 111$R_3$122             | [3.3343846357, 3.3343853]           | E      |
| 12$R_3$1                | [3.334378, 3.33438459]              | S      |
| 22$R_3$12               | [3.334379, 3.3343806]               | S      |
| 22$R_3$111              | [3.3343829, 3.33438403]             | S      |
| 22$R_3$112              | [3.3343847, 3.3343855]              | E      |

The corresponding tree is shown in Figure 6. Strings corresponding to the intervals to the right of $T_2 = 3.334384009$ marked for exclusion.
The 14 words proposed by Bumby, listed in §2.3.10, cf. Table 1: 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112, 2221112122111, 12211112121122, 1122111221111, 2211112121221, 21121112122111, 211112121112, 2222111212112222;
• The 3 words obtained in §2.4.1 as continuations of $R_1$: 12222111121211222112, 1212221111121122111, and 212222111121211221111;
• The 2 words obtained in §2.4.2 as continuations of $R_2$: 211112112121111221, 12111121111221111222;
• The 5 words obtained in §2.4.3 as continuations of $R_3$: 222211121122111112, 2122211112121111112, 12122211111211111112, 221111121211111112, 222211111211111112.

In Subsection 4.6.3 the algorithm described in [27] will be implemented to rigorously establish the bound \( \dim H X < 0.5 - 10^{-9} \). Summing up, we get the desired lower bound

\[
3.334384009 = T_2 \leq t_1.
\]

2.4.5. Upper bound on \( t_1 \). We are now ready to justify the upper bound \( t_1 \leq S_2 = 3.3343849341 \) proposed in Theorem 2.1. Following the method explained in §2.1.2, we need to modify the set \( X \), increasing its dimension, so that the right end point of a non-excluded interval is no smaller than \( S_2 \). Therefore from the intervals marked for exclusion in Tables 2, 3, 4 we choose the shortest ones which contain \( S_2 \). These turn out to be the intervals corresponding to the strings

\[
1 R_1 12 = 1222211112*1211222112, \quad 221 R_1 11 = 221222111112*12112221111,
\]
\[
12 R_2 22 = 1211112121112*121121112, \quad 121 R_1 11 = 12222111112*12112221111.
\]

We proceed to study their subintervals applying Lemma 2.4 while excluding all intervals to the right of the value

\[
T_3 := 3.3343846357 \in (T_2, S_2).
\]

The analysis of the first interval \( J(1R_1 12) \) is relatively simple. More precisely, it breaks into

\[
J(1 R_1 12) \subset [3.334386, 3.334387], \quad J(1 R_1 12) \supset [3.33438473, 3.334385] \supset \supset [3.334384049, 3.33438484] < T_3.
\]

Following the approach explained in the beginning of §2.2, we exclude the string 21112112 corresponding to the first of them, since every element is larger than \( T_3 \).

Similarly, \( J(12 R_2 22) \) breaks into

\[
J(12 R_2 22) \subset [3.33438429, 3.3343849341] < T_3, \quad J(12 R_2 22) \supset [3.3343851, 3.3343856] > T_3.
\]

We exclude the string 21222221 corresponding to the second interval, since it lies to the right of \( T_3 = 3.3343846357 \).

The third interval \( J(221 R_1 111) \) subdivides into

\[
J(221 R_1 111) \subset [3.3343848, 3.334384762] \supset T_3, \quad J(221 R_1 111) \supset [3.33438488, 3.33438551] > T_3.
\]

Finally, \( J(121 R_1 11) \) decomposes as

\[
J(121 R_1 11) \subset [3.3343848, 3.3343856] > T_3, \quad J(121 R_1 11) \supset [3.334384009, 3.33438445] < T_3.
\]

We may now define \( Y \) to be the Cantor set of continued fraction expansions in \( \{1, 2\}^N \) which do not contain the following 25 strings (nor their transposes):

• The 14 words composed by Bumby, listed in §2.2 cf. Table 1: 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112, 2221112122111, 12211112121122, 1122111221111, 22111121211221, 21121112122111, 211112121112, 2222111212112222;
• The 2 words constructed as continuations of \( R_2^2 \): \( 2 R_2 1 = 21111211111221111221 \) (cf. Table 3) and \( 212 R_2 22 = 2121111211112122111112222 \) (see (2.11) above).
• The 5 words obtained in §2.4.3 as continuations of $R_3$: 21221112121111112, 21222111212211111, 111212211121111111, 1112212111221111112, 2222121112211111112 (cf. Table 4 for numerical data on the intervals);
• The 4 words composed as continuations of $R_3$:

\[ 21 R_1 12 = 2122211112121111112, \]
\[ 111 R_1 12 = 1112212111211222112 (by (2.8)), \]
\[ 121 R_1 111 = 1212221112121111112 (by (2.13)), \]
\[ 211 R_1 1112 = 2122211112122111112 (by (2.12)). \]

then the fact that a rigorous estimate in Subsection 4.6.4 gives that $\dim_H Y > 0.5 + 10^{-8}$ allows to conclude that

\[ t_1 \leq S_2 = 3.3343849341, \]

which is the right end point of the non-excluded interval corresponding to $112 R_2 22 = 112111221112^{*} 1211212222$ (see (2.10)).

The inequalities (2.7) and (2.14) complete the proof of Theorem 2.1.

3. Bounds on $\dim_H (M \setminus L)$

In this section we establish our second main result

**Theorem 3.1.** The Hausdorff dimension of the difference of Markov and Lagrange spectra satisfies

\[ 0.537152 < \dim_H (M \setminus L) < 0.796445 \]

3.1. **Lower bounds.** It was shown in [16, §2.5.4] that $\dim_H ((M \setminus L) \cap (3.7, 3.71))$ coincides with the dimension of a certain Gauss–Cantor set $\Omega$ with complicated structure. Implementing the algorithm described in §4, we obtain an estimate $\dim_H \Omega = 0.537152 \ldots$ (see § 4.6.5 for computation notes). A combination of these two results gives the best lower bound on $M \setminus L$ so far:

\[ \dim_H ((M \setminus L) \cap (\sqrt{13}, 3.84)) \geq \dim_H \Omega > 0.537152. \]

3.2. **Upper bounds.** Recall that Freiman and Schecker independently showed circa 1973 that see, e.g. [6]

\[ [\sqrt{21}, +\infty) = L \cap [\sqrt{21}, +\infty) = M \cap [\sqrt{21}, +\infty). \]

More recently, it was shown in [20] and [27] that $\dim_H ((M \setminus L) \cap (\sqrt{5}, \sqrt{13})) < 0.73$. Hence in order to establish an upper bound of 0.796455, it suffices to study $M \setminus L$ within the interval $(\sqrt{13}, \sqrt{21})$.

Let us now set out the strategy which we will employ for the rest of this section. Consider a partition of $(\sqrt{13}, \sqrt{21})$ into several small intervals $(x, y)$ and study the intersections $(M \setminus L) \cap (x, y)$. To find an upper bound for the Hausdorff dimension of $(M \setminus L) \cap (x, y)$, we continue to develop the ideas from [20].

Very roughly speaking, we select two transitive subshifts of finite type $B \subset C \subset (\mathbb{N}^*)^2$ with $m(\alpha) < x$ for all $\alpha \in B$ and any $\beta \in (\mathbb{N}^*)^2$ with $m(\beta) < y$ belongs to $C$. We require that $B$ and $C$ are symmetric in the sense that $K(B) = K^{-}(B)$ and $K(C) = K^{-}(C)$, where $K(A) := \{(0; \alpha_1, \alpha_2, \ldots) \mid (\alpha_n)_{n \in \mathbb{Z}} \in A\}$ and $K^{-}(A) := \{(0; \alpha_1, \alpha_2, \ldots) \mid (\alpha_n)_{n \in \mathbb{Z}} \in A\}$ stand for the unstable and stable Gauss–Cantor sets associated to a given subshift of finite type $A \subset (\mathbb{N}^*)^\mathbb{Z}$.

At this stage, we want to employ a shadowing lemma type argument to get that, up to transposition, any sequence $\zeta$ with $m(\zeta) \in (M \setminus L) \cap (x, y)$ has the property that if $N$ is large, $n \geq N$, $\tau$ is a finite string and $\alpha, \alpha'$ are infinite strings with distinct first elements such that the two sequences $\ldots \zeta_N \ldots \zeta_0 \tau \alpha$ and $\ldots \zeta_N \ldots \zeta_0 \tau \alpha'$ have Markov values in $(M \setminus L) \cap (x, y)$, then the unstable Cantor set $K(B) = \{(0; \theta_1, \theta_2, \ldots) : (\theta_n)_{n \in \mathbb{Z}} \in B\}$ of $B$ doesn’t intersect the interval $[0; \alpha], [0; \alpha']$. In particular, by taking $x \tau = \emptyset$, the allowed continuations of $\zeta$ with $m(\zeta) \in (M \setminus L) \cap (x, y)$ live in a small “Cantor set” $K_{gap}$ in the gaps of $K(B)$, so that $\dim_H ((M \setminus L) \cap (x, y)) \leq \dim_H (K(C)) + \dim_H (K_{gap})$.

As it turns out, the rest of this section relies on the formalisation of the idea of the previous paragraph based on a version of Lemma 6.1 of [20].

**Definition 3.2.** Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let $\alpha \in \Sigma(C)$ be a sequence with $m(\alpha) = \lambda_B(\alpha) = m \in M$. We say that $\alpha$ connects positively to $B$, if for every $k \in \mathbb{N}$ there exist a finite sequence $\tau$ and an infinite sequence $v \in \Sigma^+(B)$ such that for $\tilde{\alpha} := \ldots \alpha_{-2} \alpha_{-1} \alpha_0 \ldots \alpha_k \tau v$ we have

\[ m(\tilde{\alpha}) < m(\alpha) + 2^{-k}. \]

We say that $\alpha$ connects negatively to $B$ if the reversed sequence $\alpha^t$ connects positively to $B$. 


Remark 3.3. Observe that we can replace $2^{-k}$ in (3.1) by any sequence converging to zero, or, in other words, the inequality (3.1) can be replaced by

$$\lim_{k \to \infty} \inf_{\tau \text{ finite word in } C} m(\ldots a_{-2} a_{-1} a_0 \ldots a_k \tau v) = m.$$

The following equivalent definition is slightly more elaborate, but more useful for our purposes.

Definition 3.2.\ Let $\Sigma(B) \subset \Sigma(C)$ be two transitive and symmetric subshifts of finite type. We say that $\alpha \in \Sigma(C)$ connects positively to $B$ if for every $k \in \mathbb{N}$ there exist a finite sequence $\tau$ and a pair of infinite sequences $v_C \in \Sigma^+(C)$ and $v_B \in \Sigma^+(B)$ such that the concatenation $\tilde{\alpha} := v_C^{\tau} \alpha_{-k,k} \tau v_B$ satisfies $m(\tilde{\alpha}) < m(\alpha) + 2^{-k}$.

The advantage of this more complicated alternative definition is that for each $k$ the hypothesis is formulated in terms of the finite subsequence $\alpha_{-k,k}$. Notice that if $\alpha$ does not connect to $B$, then there exists a fixed positive value of $k$ for which the condition above fails. In the sequel, instead of Lemma 6.1 in [20], we shall use the following statement.

Lemma 3.4. Consider two transitive and symmetric subshifts of finite type $\Sigma(B) \subset \Sigma(C)$. Let $x$ be such that $m(\beta) \leq x$ for all $\beta \in \Sigma(B)$. Suppose that a sequence $\gamma \in \Sigma(C)$ satisfies $m(\gamma) = \lambda_0(\gamma) = m > x$ and connects positively and negatively to $B$. Then $m \in L$.

Proof. By Theorem 2 in Chapter 3 of Cusick–Flahive book [6], it is sufficient to show that $m = \lim_{k \to \infty} m(P_k)$ where $P_k$ is a sequence of periodic points in $\Sigma(C)$.

Since $\gamma$ connects positively and negatively to $B$, there exist finite sequences $\tau, \tilde{\tau}$ and infinite sequences $v, \tilde{v} \in \Sigma^+(B)$ such that $m(\ldots \gamma_{-2} \gamma_{-1} \gamma_0 \ldots \gamma_k \tau v) < m + 2^{-k}$ and $m(v^{\tilde{\tau}} \tilde{\gamma}_{-k} \ldots \gamma_0 \gamma_1 \gamma_2 \ldots) < m + 2^{-k}$.

Let $v^k := v_1 \ldots v_k$ and $\tilde{v}^k := \tilde{v}_k \ldots \tilde{v}_1$ be the segments of $v$ and $\tilde{v}$ respectively. By transitivity of $\Sigma(B)$, there exists $\beta \in \Sigma(B)$ which contains non-overlapping occurrences of the strings $v^k$ and $\tilde{v}^k$ in this order. Let us denote by $(v^k \ast \tilde{v}^k)$ a finite substring of $\beta$ which begins with $v^k$ and terminates with $\tilde{v}^k$.

We now want to consider the periodic point $P_k \in \Sigma$ obtained by infinite concatenation of the finite block $\gamma_0 \ldots \gamma_k \tau (v^k \ast \tilde{v}^k) \gamma_{-k} \ldots \gamma_{-1}$.

Recall that for any finite sequence $\xi = \xi_1 \ldots \xi_k$ of positive integers and for any pair of sequences $\alpha', \alpha'' \in (\mathbb{N}^+)^k$ we have $[0; \xi, \alpha'] - [0; \xi, \alpha''] < 2^{-1 - k}$. Therefore for any $j \in \mathbb{Z}$ we get $\lambda_0(\sigma^j(P_k)) \leq m + 2^{-k}$ and $\lambda_0(P_k) > \lambda_0(\gamma) - 2^{-k} = m - 2^{2^{-k}}$.

In particular, $m = \lim_{k \to \infty} m(P_k)$. This completes the argument. \hfill $\square$

Remark 3.5. Assume that $\Sigma(B) \subset \Sigma(C)$ are two transitive symmetric subshifts and let $x$ be such that $m(\beta) \leq x$ for all $\beta \in \Sigma(B)$. Consider a sequence $\gamma \in \Sigma(C)$ with $m(\gamma) = \lambda_0(\gamma) = m > x$. Then for any finite sequence $\tau$ and half-infinite sequence $v \in \Sigma^+(B)$ directly from definition of Lagrange and Markov numbers we get

$$\lim_{j \to +\infty} \sup_{j \to +\infty} \left( \lambda_0(\sigma^j(\ldots \gamma_{-2} \gamma_{-1} \gamma_0 \ldots \gamma_k \tau v)) < m, \right)$$

where $\sigma$ is the Bernoulli shift. Thus, if we want to get that $m(\ldots \gamma_{-2} \gamma_{-1} \gamma_0 \ldots \gamma_k \tau v) < m + 2^{-k}$, then it suffices to check that

$$\lambda_0(\sigma^j(\ldots \gamma_{-2} \gamma_{-1} \gamma_0 \ldots \gamma_k \tau v)) < m + 2^{-k}$$

for finitely many values of $j$, namely, for all $0 \leq j \leq k + |\tau| + l$ where $l$ is sufficiently large (so that $2^{1-l} \leq m - x + 2^{-k}$).

The following elementary fact is quite useful to us.

Lemma 3.6. Let $\Sigma(C)$ be a transitive symmetric subshift. Assume that three half-infinite sequences $\beta_1, \beta_2, \beta_3 \in \Sigma^+(C)$ are such that $[0; \beta_1] < [0; \beta_2] < [0; \beta_3]$. Then for all $\alpha \in \Sigma(C)$ and for all $j \leq n + 1$

$$\lambda_0(\sigma^j(\ldots \alpha \ldots a_0 a_0 \ldots a_n \beta^2)) \leq \max\left( m(\ldots a \ldots a_0 a_0 \ldots a_n \beta_1), m(\ldots a \ldots a_0 a_0 \ldots a_n \beta_2) \right).$$
We will use Lemmas 3.4 and 3.6 and Remark 3.5 in order to estimate Hausdorff dimensions of $(M \setminus L) \cap (x, y)$ in the following way. Recall that [6, Lemma 6, Chapter 1] for any $m \in M$ there exists a sequence $\alpha$ such that $\lambda_0(\alpha) = m(\alpha)$. Therefore to study $(M \setminus L) \cap (\sqrt{13}, \sqrt{21})$ we may consider

$$Y := \{ \alpha \in \{1, 2, 3, 4\}^\mathbb{Z} \mid m(\alpha) = \lambda_0(\alpha) \in M \setminus L \}.$$ 

In order to prove that $\dim_H (M \setminus L) \leq d$, it suffices to consider the cylinder sets $V_n(\alpha) := \{ \tilde{\alpha} \in \{1, 2, 3, 4\}^\mathbb{Z} \mid \tilde{\alpha}_j = \alpha_j, -n \leq j \leq n \}$ and to show that for every $\alpha \in Y$, there is $n \in \mathbb{N}$ such that

$$\dim_H(m(V_n(\alpha) \cap Y)) \leq d.$$

In this direction, we will associate (see Tables 5 and 6) to an interval $(x, y)$ two symmetric transitive subshifts of finite type $\Sigma(B) = \Sigma(B_2) \subset \Sigma(C) = \Sigma(C_y) \subset (\mathbb{N}^*)^\mathbb{Z}$ such that

- $m(\beta) < x$ for all $\beta \in \Sigma(B)$; and
- for all $\gamma \in (\mathbb{N}^*)^\mathbb{Z}$ such that $m(\gamma) < y$ we have $\gamma \in \Sigma(C)$.

If $m(\alpha) = \lambda_0(\alpha) = m \in M \setminus L$, then by Lemma 3.4, $\alpha$ doesn’t connect neither positively nor negatively to $B$. Suppose without loss of generality that it doesn’t connect positively to $B$. Then by Definition 3.2 there exists $k \in \mathbb{N}$ such that, for any $N \geq k + 2$, any finite sequence $\tau$ and infinite sequences $\upsilon_C \in \Sigma^+(C)$ and $\upsilon_B \in \Sigma^+(B)$ the concatenation $\tilde{\alpha} = \upsilon_C^\alpha \cdot \alpha_{-N,N} \cdot \upsilon_B$ satisfies $m(\tilde{\alpha}) \geq m - 2^k \geq m + 2^{-N+2}$.

At this point, we will proceed as follows. In the remainder of this section, for each interval $(x, y)$ introduced below, we will construct a finite collection $X_1, \ldots, X_r$ of finite sets of finite sequences over $\{1, 2, 3, 4\}$ with the following property: if $x < m < m + 2^{-N+2} < y$ and, for some $n \geq N$, a sequence $\upsilon_C^\alpha \alpha_{-N,n} \cdot \upsilon_B$ with continuations $\upsilon^1, \upsilon^2 \in \{1, 2, 3, 4\}^N$ with different subsequent term (of index $n + 1$) leading to Markov values which are smaller than $m + 2^{-N+2}$, then there is $X_\beta$ (depending only on $\alpha_{-N,n}$) such that the initial segment of any $\upsilon \in \{1, 2, 3, 4\}^N$ with the property that $m(\upsilon_C^\alpha \alpha_{-N,n} \cdot \upsilon) < m + 2^{-N+2}$ belongs to $X_\beta$ (these elements tend to live on gaps of $K(B)$).

Notice now that $V_N(\alpha) \cap Y$ is contained in the set of sequences $\beta = \upsilon_C^\alpha \alpha_{-N,N} \cdot \upsilon_B$ such that $m(\beta) < m + 1/2^{N-2}$. Hence, if $s > 0$ is such that, for all $X_1, \ldots, X_s$ and all positive integers $b_1, \ldots, b_s$, we have

$$\sum_{\tau \in X_\beta} |I(b_1, \ldots, b_s, \tau)|^a \leq |I(b_1, \ldots, b_s)|^a,$$

where $I(a_1, \ldots, a_k) = \{ [0; a_1, \ldots, a_k, \rho] : \rho > 1 \}$, then Markov values in $m(V_N(\alpha) \cap Y)$ belong to the arithmetic sum of $K(C)$ with a set $K_{\text{pap}}$ whose Hausdorff dimension is at most $s$, and thus its Hausdorff dimension is at most $d = \dim_H(K(C)) + s$ (by a classical mass transference principle, see e.g. [16, Proposition E.1]).

We can now make a first choice of disjoint subintervals of $(\sqrt{13}, \sqrt{21})$ with their corresponding subshifts $\Sigma(B)$, which we will subdivide further in the next subsections: cf. Table 5 below. Note that these choices of $\Sigma(B)$ are simpler than the original choices in [20] and this is possible because Lemma 3.4 is more flexible than [20, Lemma 6.1].

| $n$ | Interval $R_n$ | $B_n$ | $\mathcal{F}_n$ |
|-----|----------------|-------|----------------|
| 1   | $(\sqrt{13}, 3.92)$ | 1, 2  | $\phi$ |
| 2   | $(3.92, 4.32372)$ | 1, 2, 3 | 13, 31 |
| 3   | $(4.32372, 4.4984)$ | 1, 2, 3 | 131 |
| 4   | $(4.4984, \sqrt{21})$ | 1, 2, 3 | $1313, 3131$ |

Table 5. Subshifts $\Sigma(B_n) = \{ \beta \in B_n^\mathbb{Z} \mid \alpha$ has no substring from $\mathcal{F}_n \}$.

Also, we collect together in Table 6 the subshifts $C_n$ and the rigorous upper bounds on $\dim_H K(C_y)$ (derived from the same method as before, described in §4) we need for the sequel.

We are now ready to proceed to the detailed analysis of the sets $K_{\text{pap}}$ constructed below to analyse different parts of $M \setminus L$. However, for the sake of completeness, let us briefly postpone this to the next subsections while closing the current discussion with an illustration of the method for the region $(M \setminus L) \cap (\sqrt{5}, \sqrt{13})$.

\footnote{In most cases below, $X_j$ is a pair of finite sequences (e.g., $X_j = \{23, 1133\}$ in §3.4), but sometimes we use larger finite sets (e.g., one of the $X_j$ in §3.14 is $X_j = \{34313131, 344434, 213131\}$). In principle, we could explicitly list all $X_j$ appearing below, but, for the sake of simplicity of exposition, we will refrain from doing so: in other words, the relevant sets $X_j$ will always be implicit in our subsequent discussions.}
| $n$ | Interval $S_n$ | $A_n$ | $T_n$ | $\dim_H K(C_n)$ |
|-----|---------------|-------|-------|-----------------|
| 1   | $(\sqrt{5}, 3.042)$ | 1, 2  | 121, 212, 2111222, 2221112 | 0.346453 |
| 2   | $(\sqrt{13}, 3.84)$  | 1, 2, 3 | 13, 31 | 0.537961 |
| 3   | $(3.84, 3.92)$      | 1, 2, 3 | 131, 313, 231, 132, 312, 213 | 0.594179 |
| 4   | $(3.92, 4.01)$      | 1, 2, 3 | 131, 313, 2312, 2132 | 0.643354 |
| 5   | $(4.01, 4.1165)$    | 1, 2, 3 | 131 | 0.666993 |
| 6   | $(4.1165, 4.1673)$  | 1, 2, 3 | 1313, 3131, 1312, 2131, 13111, 11131 | 0.6694154 |
| 7   | $(4.1673, 4.252725)$| 1, 2, 3 | 1313, 3131, 1312, 2131, 131111, 1313112, 211313, 3131113, 3111313 | 0.677846 |
| 8   | $(4.252725, 4.32372)$| 1, 2, 3 | 1313, 3131, 21312 | 0.691289 |
| 9   | $(4.32372, 4.385)$  | 1, 2, 3 | 31313, 21313, 31312, 21312, 113131, 1313111, 3131112, 2111313, 3131113, 3111313 | 0.694718 |
| 10  | $(4.385, \sqrt{20})$ | 1, 2, 3 | 31313, 31312, 21313, 121312, 213121 | 0.697493 |
| 11  | $(\sqrt{20}, 4.4984)$| 1, 2, 3 | 31313 | 0.704213 |
| 12  | $(4.4984, 4.513)$   | 1, 2, 3, 4 | 14, 41, 24, 42, 343, 31313 | 0.704700 |
| 13  | $(4.527, 4.55)$     | 1, 2, 3, 4 | 14, 41, 24, 42, 3433, 3433, 3434, 3434 | 0.708245 |
| 14  | $(4.4984, \sqrt{21})$| 1, 2, 3, 4 | 14, 41, 24, 42 | 0.709394 |

Table 6. Subshifts $\Sigma(C_n) = \{\alpha \in A_n^\mathbb{Z} \mid \alpha \text{ has no substring from } T_n\}$ used in our analysis, and dimension of $K(C_n)$ calculated using the method from §4.

Take $C_1 \subseteq \{1, 2\}^\mathbb{Z}$ where 121, 212, 2111222 and 2221112 are forbidden. Notice that $\lambda_0(12^*1) > 3.15$, in the sense that if $\alpha = (a_n)_{n \in \mathbb{Z}} \in \Sigma(A_1)$ and $(a_1, a_0, a_1) = (1, 2, 1)$ then $\lambda_0(\alpha) > 3.15$. Indeed, in this case, we have $\lambda_0(\alpha) \geq [2; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] + [0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] > 3.15$. We also have $\lambda_0(21^*2) \geq [2; 1, 2, 2, 2] + [0; 2, 2, 2] > 3.06$ and $\lambda_0(22^*1112) \geq [2; 2, 2, 2, 2] + [0; 1, 1, 1, 2, 2, 2] > 3.042$ (and so, by symmetry, $\lambda_0(1112^*22) > 3.042$). These inequalities imply that $M \cap (\sqrt{5}, 3.042] \subseteq 2 + K(C_1) + K(C_1)$, and thus

$$\dim_H((M \setminus L) \cap (\sqrt{5}, 3.042]) \leq \dim_H(M \cap (\sqrt{5}, 3.042]) \leq 2 \dim_H(K(C_1)) < 0.693,$$

since $\dim_H(K(C_1)) < 0.3465$ (as it can be checked with the method from §4).

Now let $(\mu, \nu) = (3.042, \sqrt{13})$. Here we take $C = \{1, 2\}^\mathbb{Z}$ and $B \subseteq \{1, 2\}^\mathbb{Z}$ where 121, 212 and 21112 are forbidden. Note that if $\gamma \in \Sigma(B)$, then $m(\gamma) \leq [2; 2, 2, 2, 2] + [0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] < 3.041$. Given $\gamma \in Y$ with $m = m(\gamma) \in (\mu, \nu)$, we observe that if $N \geq 5$, then $m + 1/2^{N-2} \leq \sqrt{12} + 1/2^3 < \nu$. By the previous discussion (cf. Lemma 3.4), there is an integer $k$ (which we may assume to be at least 3) such that, for any $N \geq k + 2$, any finite sequence $\tau$ and infinite sequences $\gamma \in \Sigma^+(C), \theta \in \Sigma^+(B)$, if $\hat{\alpha} = \gamma^* a_{-N} \ldots a_0 \ldots a_N \tau \theta$, then $m(\hat{\alpha}) \geq m + 1/2^k \geq m + 1/2^{N-2}$.

Suppose that for some $n \geq N$, a sequence $\gamma^* a_{-N} \ldots a_0 \ldots a_n$ has continuations with different subsequent term (of index $n + 1$) whose Markov value are smaller than $m + 1/2^{N-2}$ - in this case this means that this sequence has such a continuation with $a_{n+1} = 1$ and another one with $a_{n+1} = 2$. We claim that these continuations should be of the type $a_n a_{n+1} = a_n 121 a_{n+3}$ and $a_n a_{n+1} = a_n 221 a_{n+3}$ thanks to the presence of the continuations $1121222$ and $221222$. Indeed, we have two cases:

- If $a_n = 1$ can be continued with $a_{n+2} a_{n+3} \neq 12$, since $[0; 1, a_{n+2}, a_{n+3}] > [0; 1, 1, 2]$, it follows from Lemma 3.6 that the Markov values centered at $a_n$, with $k \leq n$ and at $a_{n+1} = 1$ are smaller than $m + 1/2^{N-2}$; by Remark 3.5, it is enough to verify that $[2; 2, 1, 1, 2] + [0; 1, 1, a_n, \ldots] \leq [2; 2, 1, 1, 2] + [0; 1, 1, 1, 2] < 3.021 < m$ in order to conclude that the Markov value of $\gamma^* a_{-N} \ldots a_0 \ldots a_N 1121222$ is smaller than $m + 1/2^{N-2}$ and get the desired contradiction.

- If $a_n = 2$ can be continued with $a_{n+2} a_{n+3} \neq 21$, since $[0; 2, a_{n+2}, a_{n+3}] < [0; 2, 2, 1]$, it follows from Lemma 3.6 that the Markov values centered at $a_n$, with $k \leq n$ and at $a_{n+1} = 2$ are smaller than $m + 1/2^{N-2}$; by Remark 3.5, it is enough to verify that $[2; 1, 1, 2, 2] + [0; 2, a_n, \ldots] \leq [2; 1, 1, 2, 2] + [0; 2, 2, 2] < 3.01 < m$ in order to conclude that the Markov value of $\gamma^* a_{-N} \ldots a_0 \ldots a_N 221222$ is smaller than $m + 1/2^{N-2}$ and derive again a contradiction.
At this point, we recall that it was shown in [20] that, for \( s = 0.174813 \), and all positive integers \( b_1, \ldots, b_n \), we have\(^6\)
\[
|I(b_1, \ldots, b_n, 1, 2)|^s + |I(b_1, \ldots, b_n, 2, 2, 1)|^s \leq |I(b_1, \ldots, b_n)|^s.
\]
Thus, \( \dim_H((M \setminus L) \cap (3.042, \sqrt{13})) \leq 0.174813 + \dim_H(E_2) < 0.174813 + 0.531281 = 0.706094. \)

Hence,
\[
\dim_H((M \setminus L) \cap (-\infty, \sqrt{13})) =
\max\{\dim_H((M \setminus L) \cap (-\infty, 3.042]), \dim_H((M \setminus L) \cap (3.042, \sqrt{13}))\}
\leq \max\{\dim_H(M \cap (-\infty, 3.042]), \dim_H((M \setminus L) \cap (3.042, \sqrt{13}))\}
\leq \{0.693, 0.706094\} = 0.706094.
\]

3.3. \textbf{Improvement of the upper bounds in the region} \((\sqrt{13}, 3.84)\). As specified in Table 6, we choose \( \Sigma(C_2) = \{ \alpha \in \{1, 2, 3\}^2 \mid 13 \text{ and } 31 \text{ are not substrings of } \alpha \} \) and \( \Sigma(B_1) = \{1, 2\}^2 \) to show that if \( \alpha \in Y \) and \( m(\alpha) \in M \setminus L \) then there are two possibilities for the sequences \( \alpha_n = (v_1^1, v_1^{11}, \ldots) \) and \( \beta_n = (v_2^1, v_2^{11}, \ldots) \) with \( v_1^1 \neq v_2^1 \) corresponding Markov values in \((M \setminus L) \cap (\sqrt{13}, 3.84))\):

(A) \( \alpha_n = 3\alpha_{n+1} + \beta_{n+1} \) (i.e., \( v_1^1 = 3, v_2^1 = 2 \))

(B) \( \alpha_n = 2\alpha_{n+1} + \beta_{n+1} \) (i.e., \( v_1^1 = 2, v_2^1 = 1 \))

where \( \alpha_{n+k} := (v_1^{11}, v_1^{111}, \ldots) \) and \( \beta_{n+k} := (v_2^{11}, v_2^{111}, \ldots) \).

Let us first look at (A). It continues with \( \beta_n = 21\beta_{n+2} \) and, in fact, we see that \( 21\beta_{n+2} = 2\beta \) because 2 appears in odd positions, 1 appears in even positions and 13 is forbidden. Thus \([0; \alpha_n], [0; \beta_n] \cap K(B) \neq \emptyset \).

Let us now look at (B). It continues with \( \beta_n = 11\beta_{n+2} \). Since 13 is forbidden, \( 11\beta_{n+2} \to \cdots \to 1121 \in K(B_1). \) Thus \([0; \alpha_n], [0; \beta_n] \cap K(B_1) \neq \emptyset \). Therefore it is not possible to have two different continuations which do not connect to \( B \). Hence \( \dim_H K_{gap} = 0. \)

In particular, \( \dim_H((M \setminus L) \cap (\sqrt{13}, 3.84)) \leq \dim_H(K(C_1)) \leq 0.574. \)

\textbf{Remark 3.7.} This estimate should be compared with the inequality \( \dim((M \setminus L) \cap (\sqrt{13}, 3.84)) \geq \dim(\Omega) > 0.537109 \) from \S 3.1.

3.4. \textbf{Improvement of the upper bounds in the region} \((3.84, 3.92)\). Similarly to [20], we can use \( C_3 \subset \{1, 2, 3\}^2 \) where \( 131, 313, 231, 132 \) are forbidden and a certain block \( B \) to show that the continuations of words with values in \((M \setminus L) \cap (3.84, 3.92)\) are

- 33 and 21
- 23 and 113 or 1121

We affirm that the first Cantor set of gaps is trivial. Indeed, if the continuation 21 is not \( 2\beta \), it must be \( (21)^n \) for some \( n \in \mathbb{N} \), a contradiction because 213 is \textit{forbidden} in this region as
\[
[3, 1, 2, 3, \overline{1}] + [0, 3, 1] > 3.95
\]

Also, a similar argument shows that the option 1121 is trivial. Thus, the Cantor set of the gaps in this region consist of the options 23 and 1133 (since 131 and 132 are forbidden in \( C \)). It follows that the Cantor set of the gaps has dimension \( \dim_H K_{gap} < 0.133 \) because
\[
0.016134^{0.133} + (1/690)^{0.133} < 1.
\]
Since \( \dim_H K(C_3) < 0.5942 \), we deduce that \( \dim_H((M \setminus L) \cap (3.84, 3.92)) < 0.5942 + 0.133 = 0.7272. \)

3.5. \textbf{Refinement of the control in the region} \((3.92, 4.01).\)

\(^6\)Here and in the sequel, we use the well-known formula \( |I(a_1, \ldots, a_k)| = \frac{1}{q_{a_1}q_{a_2} \cdots q_{a_k}} \) where \( q_j \) stands for the denominator of \([0; a_1, \ldots, a_j]. \)
3.5.1. Refinement of the control in the region \((3.92, 3.9623)\). Similarly to [20], we can use \(C \subset \{1, 2, 3\}^\mathbb{Z}\) where 131, 313, 2312, 2132 are forbidden and a certain block \(B\) to show that the continuations of words with values in \((M \setminus L) \cap (3.92, 3.9623)\) are

- 331 and 21
- 23 and 113

Note that in this regime we have
\[
\lambda_0(3^*12) > [3; 1, 2, 3, 1, 1, 1, 3, \overline{1}] + [0; 3, 1, 2, 1, 3, 3, 3, \overline{1}] > 3.96238
\]
so that the strings 312 and 213 are forbidden. Similarly, the strings 3231, 1323, 2231, 1322 are also forbidden.

Thus, the continuations 331 and 21 are not possible in this regime: indeed, given that 213 is forbidden, the smallest continuation of 21 would be \(2\mathbb{T}\), so that we would be able to connect to the block \(B\), a contradiction.

Next, we affirm that the continuation 23 and 113 leads to 231 and 113: otherwise, if 232 or 233 is an allowed continuation, then we could use the largest continuation \(2\mathbb{T}\) to connect to an adequate block \(B\), a contradiction. Since
\[
\left(\frac{|I(a_1, \ldots, a_n, 2, 3, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.153} + \left(\frac{|I(a_1, \ldots, a_n, 1, 1, 3)|}{|I(a_1, \ldots, a_n)|}\right)^{0.153} \\
\leq (0.00254)^{0.153} + (1/63)^{0.153} < 1,
\]
we deduce that \(\dim_H((M \setminus L) \cap (3.92, 3.9623)) < 0.643355 + 0.153 = 0.796355\).

3.5.2. Refinement of the control in the region \((3.9623, 3.9845)\). Similarly to [20], we can use \(C \subset \{1, 2, 3\}^\mathbb{Z}\) where 131, 313, 2312, 2132 are forbidden and a certain block \(B\) to show that the continuations of words with values in \((M \setminus L) \cap (3.9623, 3.9845)\) are

- 331 and 21
- 23 and 113

Since the strings 3231 and 1323 are forbidden in this regime (as \(\lambda_0(323^*1) > 3.99\)), the same argument of the previous subsection says that 23 and 113 actually must be 231 and 113 where
\[
\left(\frac{|I(a_1, \ldots, a_n, 2, 3, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.153} + \left(\frac{|I(a_1, \ldots, a_n, 1, 1, 3)|}{|I(a_1, \ldots, a_n)|}\right)^{0.153} \\
\leq (0.00254)^{0.153} + (1/63)^{0.153} < 1,
\]
Thus, it suffices to analyse the case \(a_n = 331\alpha_{n+3} + \beta_n = 21\beta_{n+2}\). For this sake, note that, in the current region, the strings 1213 and 3121 are forbidden because \(\lambda_0(3^*121) > [3; 1, 2, 1, 1, \overline{3}] + [0; 3, 1, 2, 1, 3, \overline{1}] > 3.9866\) (as 313 is forbidden). Also, the strings 23122 and 33122 are forbidden because \(\lambda_0(333^*12) > \lambda_0(233^*12) > [3; 1, 2, \overline{3}, \overline{1}, \overline{3}] + [0; 3, 2, 3, 2, \overline{3}, \overline{1}] > 3.98459\) (as 3231 is forbidden).

We claim that the \(n\)th digit \(a_n\) (before \(\alpha_n\) and \(\beta_n\)) is 2 or 3: otherwise, we would have a continuation \(1\beta_n = 121\beta_{n+2}\) connecting to the block \(B\) (as the smallest continuation would be \(2\mathbb{T}\)). In view of the fact that \(a_n \in \{2, 3\}\), we have that \(a_n\alpha_n = a_n331\alpha_{n+3} = a_n3311\alpha_{n+5}\) (as 313, 2312 and 33312 are forbidden) and, \textit{a fortiori}, \(a_n\alpha_n = a_n33111\alpha_{n+5}\) thanks to the presence of the continuation \(33\mathbb{T}\). Indeed, if \(a_n3311\) can be continued with \(r > 1\), since \([0; 3, 3, 1, 1, r] < [0; 3, 3, 1, 1, 1] < [0; 2, 1]\), by Remark 3.5, it follows that the Markov values centered at \(a_k\), with \(k \leq n\) and at \(a_{n+1} = 3\) are smaller than \(m\), and it is enough to verify that \([3; 1] + [0; 3, a_n, \ldots] \leq [3; T] + [0; 3, 3, T] < 3.93 < m\). We will use implicitly this kind of argument in several forthcoming cases. Since
\[
\left(\frac{|I(a_1, \ldots, a_n, 3, 1, 1, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.15} + \left(\frac{|I(a_1, \ldots, a_n, 2, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.15} \\
\leq (2/3619)^{0.15} + (0.0718)^{0.15} < 1,
\]
we derive that \(\dim_H((M \setminus L) \cap (3.9623, 3.9845)) < 0.643355 + 0.153 = 0.796355\).
3.5.3. *Refinement of the control in the region* (3.9845, 4.01). Similarly to [20], we can use \( C \subset \{1, 2, 3\}^\mathbb{Z} \) where 131, 313, 2312, 2132 are forbidden and a certain block \( B \) to show that the continuations of words with values in \((M \setminus L) \cap (3.9845, 4.01)\) are

- \( \alpha_n = 331 \alpha_{n+3} \) and \( \beta_n = 21 \beta_{n+2} \);
- \( \alpha_n = 23 \alpha_{n+2} \) and \( \beta_n = 113 \beta_{n+3} \).

Let us analyse the first possibility depending on the \( n \)th digit \( a_n \) appearing before \( 331 \alpha_{n+3} \) and \( 21 \beta_{n+2} \):

- if \( a_n = 1 \), then \( \alpha_n = 3312 \alpha_{n+4} \) thanks to the presence of the continuation 3312\( \mathbb{I} \) (which is valid as \( \lambda_0(133^*12) < 3.984 \));
- if \( a_n \in \{2, 3\} \), then \( \beta_n = 213 \beta_{n+3} \) thanks to the continuation 21331\( \mathbb{I} \).

Similarly, we can decompose the second possibility into two subcases depending on the digit appearing before \( 23 \alpha_{n+2} \) and \( 113 \beta_{n+3} \):

- if \( a_n = 1 \), then \( \alpha_n = 231 \alpha_{n+3} \) in view of 231\( \mathbb{I} \);
- if \( a_n \in \{2, 3\} \), then \( \beta_n = 1132 \beta_{n+4} \) in view of 1132\( \mathbb{I} \).

Since

\[
\left( \frac{|I(a_1, \ldots, a_n, 3, 3, 1, 2)|}{|I(a_1, \ldots, a_n)|} \right)^{0.152} + \left( \frac{|I(a_1, \ldots, a_n, 2, 1)|}{|I(a_1, \ldots, a_n)|} \right)^{0.152} \leq (1/1504)^{0.152} + (0.0718)^{0.152} < 1,
\]

\[
\left( \frac{|I(a_1, \ldots, a_n, 3, 3, 1)|}{|I(a_1, \ldots, a_n)|} \right)^{0.133} + \left( \frac{|I(a_1, \ldots, a_n, 2, 1, 3)|}{|I(a_1, \ldots, a_n)|} \right)^{0.133} \leq (1/255)^{0.133} + (0.0071)^{0.133} < 1,
\]

\[
\left( \frac{|I(a_1, \ldots, a_n, 2, 3, 1)|}{|I(a_1, \ldots, a_n)|} \right)^{0.153} + \left( \frac{|I(a_1, \ldots, a_n, 1, 1, 3)|}{|I(a_1, \ldots, a_n)|} \right)^{0.153} \leq (0.0071)^{0.153} + (1/63)^{0.153} < 1,
\]

\[
\left( \frac{|I(a_1, \ldots, a_n, 2, 3)|}{|I(a_1, \ldots, a_n)|} \right)^{0.14} + \left( \frac{|I(a_1, \ldots, a_n, 1, 3, 2)|}{|I(a_1, \ldots, a_n)|} \right)^{0.14} \leq (0.0162)^{0.14} + (1/368)^{0.14} < 1,
\]

we derive that \( \dim_H ((M \setminus L) \cap (3.9845, 4.01)) < 0.643355 + 0.153 = 0.796355 \).

3.6. *Refinement of the control in the region* (4.01, 4.1165).

3.6.1. *Refinement of the control in the region* (4.01, 4.054). Similarly to [20], we can use \( C = \{1, 2, 3\}^\mathbb{Z} \) and a certain block \( B \) to show that the continuations of words with values in \((M \setminus L) \cap (4.01, \sqrt{20})\) are

- \( \alpha_n = 331 \alpha_{n+3} \) and \( \beta_n = 213 \beta_{n+2} \);
- \( \alpha_n = 23 \alpha_{n+2} \) and \( \beta_n = 113 \beta_{n+3} \).

Since \( \lambda_0(13^*1) > 4.1165 \), the string 131 is forbidden in our current regime. Also, \( \lambda_0(213^*2) \geq |3; 2, 13, 3, 2, 3| + |0, 1, 2, 3, 1, 3, 2| > 4.054 \) when \( 13^*1 \) is forbidden. Hence, the first transition extends as 331\( \alpha_{n+3} \) and 2133\( \beta_{n+4} \).

Next, since 131 is forbidden and \( \lambda_0(113^*2\mathbb{I}) < 4.0078 \), the second transition above is actually \( \alpha_n = 23 \alpha_{n+2} \) and \( \beta_n = 1132 \beta_{n+4} \). This transition extends in two possible ways:

- if the digit appearing before is \( a_n \in \{1, 2\} \), we have \( \lambda_0(a_n 23^*1\mathbb{I}) < 4.0014 \) and, hence, the transition becomes \( \alpha_n = 231 \alpha_{n+3} \) and \( \beta_n = 1132 \beta_{n+4} \);
- if the digit appearing before is \( a_n = 3 \), we have \( \lambda_0(a_n 113^*2\mathbb{I}) < 4.0026 \) and, thus, the transition becomes \( \alpha_n = 23 \alpha_{n+2} \) and \( \beta_n = 11323 \beta_{n+5} \).
Now, we recall that \( \frac{|I(a_1, \ldots, a_n, 3, 1, 3, 1)|}{|I(a_1, \ldots, a_n, 3, 1)|} \leq \frac{1}{255} \), \( \frac{|I(a_1, \ldots, a_n, 2, 1, 3, 1)|}{|I(a_1, \ldots, a_n)|} \leq 0.000641 \), \( \frac{|I(a_1, \ldots, a_n)|}{|I(a_1, \ldots, a_n, 1, 1, 3, 2)|} \leq 0.0071 \), and \( \frac{|I(a_1, \ldots, a_n, 1, 1, 3, 2)|}{|I(a_1, \ldots, a_n)|} \leq 0.000688 \). Therefore, \( \max\left\{ \left( \frac{1}{255} \right)^{0.11} + (0.000641)^{0.11}, \left( \frac{1}{3905} \right)^{0.11} + (0.0071)^{0.11}, \left( \frac{1}{3905} \right)^{0.11} + (0.00126)^{0.11} \right\} < 1 \).

Therefore, \( \operatorname{dim}_H((M \setminus L) \cap (4.01, 4.054)) < 0.667 + 0.129 = 0.796 \) (because the Cantor set of continued fraction expansions in \( \{1, 2, 3\}^\infty \) which avoid dimension < 0.667).

3.6.2. Refinement of the control in the region (4.054, 4.06326). The string 131 is still forbidden in our current regime and the same argument of the previous subsection can be employed to treat the second transition. Thus, it remains only to analyse the first transition \( \alpha_n = 331\alpha_{n+3} \) and \( \beta_n = 213\beta_{n+3} \).

If the digit appearing before the first transition is \( a_n = 1 \), we get a valid continuation \( \lambda_0(a_n, 33^*1331\overline{2}) < 4.0468 \), so that the first transition becomes \( \alpha_n = 331\alpha_{n+4} \) and \( \beta_n = 213\beta_{n+4} \).

If the digit appearing before the first transition is \( a_n = 2 \), we claim that \( a_n213 \) is forbidden: indeed, \( \lambda_0(213^*2b_m) > 4.1 \) when \( b_m \in \{2, 3\} \), \( \lambda_0(213^*211) > 4.072 \), \( \lambda_0(a_n213^*212) > 4.067 \), \( \lambda_0(a_n213^*2133) > 4.06352 \), and \( \lambda_0(a_n213^*21321) > [3; 2, 1, 3, 1, 2, 3, \overline{1}] + [0; 1, 2, 1, 3, 2, 3] > 4.06326 \) (here we used that 213211 and 131 are forbidden), so that continuations of \( a_n213 \) are large. Thus, the first transition becomes \( \alpha_n = 331\alpha_{n+3} \) and \( \beta_n = 213\beta_{n+4} \).

If the digit appearing before the first transition is \( a_n = 3 \), we have that \( a_n3313 \) is also forbidden (as \( \lambda_0(a_n, 33^*13) > 4.0679 \) and \( a_n33127 \) is a valid continuation (as \( \lambda_0(a_n, 33^*127) < 4.03845 \)), so that the first transition becomes \( \alpha_n = 331\alpha_{n+3} \) and \( \beta_n = 213\beta_{n+3} \). Actually, the fact that we are looking at Markov values below 4.0679 makes that the same argument above can still be employed to treat the first transition in the case of the digits appearing before are \( a_n \in \{1, 3\} \).

Finally, if the digit appearing before is \( a_n = 2 \), then \( a_n3313\overline{2} \) is a valid continuation because the fact that 131 and 2132\( \overline{m} \), \( b_m \in \{2, 3\} \) are forbidden says that \( \lambda_0(a_n, 33^*13\overline{2}) \leq [3; 1, 3, 3, 1, 2, 1] + [0; 3, 2, 1, 3, 2, 1, 3, 2, 3] < 4.063251 \).

In particular, the first transition becomes \( 331\alpha_{n+4} \) and \( 213\beta_{n+3} \) and we derive that \( \operatorname{dim}_H((M \setminus L) \cap (4.06326, 4.0679)) < 0.667 + 0.129 = 0.796 \).

3.6.4. Refinement of the control in the region (4.0679, 4.1). Since 131 is forbidden here, it suffices to analyse the first transition \( \alpha_n = 331\alpha_{n+3} \) and \( \beta_n = 213\beta_{n+3} \). Moreover, the same argument above can still be employed to treat the first transition in the case of the digit appearing before is \( a_n = 1 \). Furthermore, 2132\( \overline{m} \), \( b_m \in \{2, 3\} \) is also forbidden here, so that the same argument above also treats the case of the first transition when the digit appearing before is \( a_n = 2 \). Finally, if the digit appearing before is \( a_n = 3 \), we see that the first transition becomes \( 331\alpha_{n+3} \) and \( 213\beta_{n+3} \) as \( a_n213\overline{2} \) is a valid continuation (since \( \lambda_0(a_n, 213^*27) < 4.063582 \)). Given that \( \frac{|I(a_1, \ldots, a_n, 3, 1, 3, 1)|}{|I(a_1, \ldots, a_n)|} \leq \frac{1}{255} \), \( \frac{|I(a_1, \ldots, a_n, 2, 1, 3, 1)|}{|I(a_1, \ldots, a_n)|} \leq 0.0071 \), and \( \frac{|I(a_1, \ldots, a_n)|}{|I(a_1, \ldots, a_n, 2, 1, 3, 2)|} \leq 0.000688 \), and \( \max\left\{ \left( \frac{1}{255} \right)^{0.11} + (0.000641)^{0.11}, \left( \frac{1}{3905} \right)^{0.11} + (0.0071)^{0.11}, \left( \frac{1}{3905} \right)^{0.11} + (0.00126)^{0.11} \right\} < 1 \), we obtain that \( \operatorname{dim}_H((M \setminus L) \cap (4.06326, 4.0679)) < 0.667 + 0.129 = 0.796 \).

3.6.5. Refinement of the control in the region (4.1, 4.1165). Using for the last time that 131 is forbidden, we will again concentrate only on the first transition \( \alpha_n = 331\alpha_{n+3} \) and \( \beta_n = 213\beta_{n+3} \). Here, we observe that \( 3313\overline{2} \) is a valid continuation (as \( \lambda_0(33^*13\overline{2}) < 4.0721 \), so the first transition becomes \( 331\alpha_{n+4} \) and \( 213\beta_{n+3} \) and we get \( \operatorname{dim}_H((M \setminus L) \cap (4.1, 4.1165)) < 0.667 + 0.129 = 0.796 \).

3.7. Refinement of the control in the region (4.1165, 4.1167).
3.7.1. Refinement of the control in the region (4.1165, 4.1271). Recall that in the region \((M \setminus L) \cap (4.01, \sqrt{20})\) our task is to analyse the transitions

- \(\alpha_n = 3310\alpha_{n+3} + \beta_n = 213\beta_{n+3};\)
- \(\alpha_n = 23\alpha_{n+2} + \beta_n = 113\beta_{n+3};\)

We begin by observing that \(\lambda_0(313^1) > 4.2372, \lambda_0(213^1) > 4.2527275\) (when 3131 is forbidden), and the dimension of \(\mathcal{C} = 1, 2, 3\) with 3131, 2131, 13111 and their transposes forbidden is \(< 0.66942\). Recalling that 331333 is a valid continuation (as \(\lambda_0(333^113\overline{2}) < 0.0721\), the first transition always becomes 3313\(\alpha_{n+4}\) and 213\(\beta_{n+3}\) in the region between 4.1165 and 4.2527275. Since \(\frac{|I(a_1, a_2, a_3, a_4)|}{|I(a_3, a_4)|} \leq 1, \frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} \leq 0.0071, (1/2592)^{0.111} + (0.0071)^{0.111} < 1, \) and 0.66942 + 0.11 = 0.78042, the first transition is completely treated in the region between 4.1165 and 4.2527275.

Let us now focus on the second transition. Since 23T is a valid continuation (as \(\lambda_0(23^1T) < 0.06\), the second transition becomes 231\(\alpha_{n+3}\) and 113\(\beta_{n+3}\).

If the digit appearing before is \(a_n \in \{1, 2\}, \) since \(\lambda_0(a_{n+1}13^1) > 4.134215\) and 1133\(\overline{3}\) is a valid continuation (as \(\lambda_0(113^133) < 0.079\)), the second transition becomes 231\(\alpha_{n+3}\) and 113\(\beta_{n+3}\). Given that \(\frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} \leq 0.0071, (1/3905)^{0.11} + (0.0071)^{0.11} < 1, \) and 0.66942 + 0.11 = 0.77942, we are done.

If the digit appearing before is \(a_n = 3\), we observe that \(\lambda_0(a_{n+1}23^1b_m) > 4.1271\) for \(b_m \in \{2, 3\}\) and \(\lambda_0(23^1113) < 4.081,\) so that the second transition becomes 2311\(\alpha_{n+6}\) and 113\(\beta_{n+3}\). Given that \(\frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} \leq 0.0001, \frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} < 1/63, (0.0001)^{0.11} + (1/63)^{0.11} < 1, \) and 0.66942 + 0.11 = 0.77942, we are done. In summary, we showed that dim\((M \setminus L) \cap (4.1165, 4.1271)\) < 0.78042.

3.7.2. Refinement of the control in the region (4.1271, 4.12733). In view of the arguments of the previous subsection, our task is reduced to discuss the second transition 231\(\alpha_{n+3}\) and 113\(\beta_{n+3}\) when the digit appearing before is \(a_n = 3\).

Note that \(\lambda_0(a_{n+1}23^13) = \lambda_0(323^13) > 4.199.\) Moreover, we claim that all continuations of \(a_{n+1}2312\) are large. Indeed, \(\lambda_0(a_{n+1}23^112b_m) > 4.1358\) for \(b_m \in \{1, 2\}, \lambda_0(a_{n+1}23^112c_m) > 4.1296\) for \(c_m \in \{2, 3\}, \lambda_0(a_{n+1}23^112d_m) > 4.1275\) for \(d_m \in \{1, 2\}.\) Since \(\lambda_0(a_{n+1}23^11233) > 4.12733, \lambda_0(23^1132) > 4.1288, \) and 3131 is forbidden, we conclude that \(a_{n+1}2312\) has no short continuation. In view of the valid continuation 2311\(\overline{1}33\) (with \(\lambda_0(23^1113) < 4.081),\) we see that the second transition becomes 2311\(\alpha_{n+5}\) and 113\(\beta_{n+3}\). Hence, we can apply again the argument from the previous subsection to derive that dim\((M \setminus L) \cap (4.1271, 4.12733)\) < 0.78042.

3.7.3. Refinement of the control in the region (4.12733, 4.12762). In view of the arguments of the previous subsection, our task is again reduced to discuss the second transition 231\(\alpha_{n+3}\) and 113\(\beta_{n+3}\) when the digit appearing before is \(a_n = 3\).

If the digit appearing before \(a_n = a_{n-1} \in \{1, 2\},\) we have \(\lambda_0(a_{n-1}13^133) < 4.1264,\) so that the second transition becomes 231\(\alpha_{n+3}\) and 113\(\beta_{n+3}\) (since 1313, 1312, 13111 and 13112 are forbidden for any sequence with Markov value < 4.134215). Given that \(\frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} \leq 0.0071, \frac{|I(a_1, a_2, a_3)|}{|I(a_3)|} \leq 0.000241, (0.0071)^{0.11} + (0.00241)^{0.11} < 1, \) and 0.66942 + 0.11 = 0.77942, we are done.

If the digit appearing before \(a_n = a_{n-1} = 3,\) we have three possibilities. If the digit before \(a_{n-1} = a_{n-2} = 3,\) we have \(\lambda_0(a_{n-2}a_{n-1}13^133) < 4.12729969\) and, hence, the argument of the previous paragraph can be repeated. If \(a_{n-2} = 2,\) we recall that \(\lambda_0(323^13) > 4.199, \lambda_0(2332^112) > 4.1277\) (when 3131 is forbidden) and \(\lambda_0(2311\overline{1}33) < 4.081\) to get that the second transition becomes 2311\(\alpha_{n+6}\) and 113\(\beta_{n+3}\). Finally, if \(a_{n-2} = 1,\) then we have three subcases: if \(a_{n-3} \in \{2, 3\},\) we note that \(\lambda_0(a_{n-3}a_{n-2}a_{n-1}13^1) \geq \lambda_0(2133113131) \geq \lambda_0(21331131133) \geq 4.1277\) (when 1313, 1312, 13111 and 13112 are forbidden) and \(\lambda_0(113^1\overline{2}35) < 4.079,\) so that the second transition becomes 231\(\alpha_{n+3}\) and 113\(\beta_{n+5}\); if \(a_{n-3} = 1\) and \(a_{n-4} = 2,\) we get \(\lambda_0(a_{n-4}a_{n-3}a_{n-2}a_{n-1}13^1) \geq \lambda_0(211331133113) \geq 4.12762\) (because \(\lambda_0(23^1132) > 4.1288),\) so that the second transition becomes 2311\(\alpha_{n+6}\) and 113\(\beta_{n+3}\). In any event, we conclude that dim\((M \setminus L) \cap (4.12733, 4.12762)\) < 0.78042.
3.7.4. Refinement of the control in the region (4.12762, 4.134215). In view of the arguments of the previous subsection, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n = 3$, $a_{n-1} = 3$ and $a_{n-2} \in \{1,2\}$.

If $a_{n-2} = 2$, we have $\lambda_0(233113^{*}T\overline{T}T) < 4.12751$, and 1313, 1312, 13111 and 13112 are forbidden on any sequence with Markov value $< 4.134215$, so that the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+3}$ and we are done.

If $a_{n-2} = 1$ and $a_{n-3} \in \{2,3\}$, we recall that $\lambda_0(323^{*}13) > 4.199$, $\lambda_0(233^{*}12b_n) > 4.1358$ for $b_n \in \{1,2\}$ and $\lambda_0(a_n^{-3}2a_n^{-2}a_n^{-1}a_n23^{*}123133\overline{T}) < 4.127471$, so that the second transition becomes $23123\alpha_{n+5}$ and $113\beta_{n+3}$. Given that $\frac{|I(a_1, ..., a_3, 1, 2, \overline{3})|}{|I(a_1, ..., a_3)|} \leq 0.000111$, $\frac{|I(a_1, ..., a_3, 1, 3)|}{|I(a_1, ..., a_3)|} \leq 1/63$, $0.000111 \cdot 0.111 + (1/63)0.111 < 1$, and 0.66942 + 0.111 = 0.78042, we are done in this case. If $a_{n-2} = 1 = a_{n-3}$ and $a_{n-4} \in \{1,2\}$, we get $\lambda_0(a_n^{-4}a_{n-4}^{-3}a_{n-2}^{-2}a_{n-1}^{-1}a_n23^{*}1231332113) < 4.12761982$ and the second transition still is $23123\alpha_{n+5}$ and $113\beta_{n+3}$, and we are done. If $a_{n-2} = 1 = a_{n-3}$ and $a_{n-4} = 3$, we have $\lambda_0(a_n^{-4}a_{n-4}^{-3}a_{n-2}^{-2}a_{n-1}^{-1}a_n113^{*}113113\overline{T}T) < 4.127618$, so that the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+6}$, and we are done.

In any case, we get that $\dim((M \setminus L) \cap (4.127672, 4.134215)) < 0.78042$.

3.7.5. Refinement of the control in the region (4.134215, 4.137519). In view of the arguments of the previous subsections, our task is to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n = 2$ and $a_n = 3$. For later reference, we remark that the Cantor set $C = 1, 2, 3$ where 1313, 1312, 13111 and their transposes are forbidden has dimension $< 0.6964155$.

If $a_n = 2$, we have two possibilities. If the digit appearing before appearing $a_{n-2} = 2$ is $a_{n-1} \in \{2,3\}$, then $\lambda_0(232^{*}131^{*}1) > 4.143241$ and $\lambda_0(113^{*}23^{*}) < 4.079$, so that the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+3}$ and we are done. If the digit before $a_{n-2} = 2$ is $a_{n-1} = 1$, we have $\lambda_0(a_{n-1}23^{*}13) > 4.1837$, $\lambda_0(a_{n-1}a_{n-2}23^{*}121) > 4.137519$ (as 1313, 1312 and 13111 are forbidden), and $\lambda_0(1223^{*}122^{*}) < 4.127$, so that the second transition becomes $231223\alpha_{n+5}$ and $113\beta_{n+3}$. Given that $\frac{|I(a_1, ..., a_2, 3, 1, 2, \overline{3})|}{|I(a_1, ..., a_2)|} \leq 0.000021$, $\frac{|I(a_1, ..., a_2, 1, 1, 3)|}{|I(a_1, ..., a_2)|} \leq 1/63$, $(0.000021)^{0.111 + (1/63)0.111} < 1$, and 0.67 + 0.115 = 0.785, we are done.

If $a_n = 3$, since 1313, 1312 are forbidden and $\lambda_0(3113^{*}23^{*}) < 4.128$, the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+3}$. Since $\frac{|I(a_1, ..., a_2, 3, 1, 1)|}{|I(a_1, ..., a_2)|} \leq 0.000742603$, $\frac{|I(a_1, ..., a_2, 1, 1, 3)|}{|I(a_1, ..., a_2)|} \leq 1/400$, $0.000742603^{0.1270292} + (1/400)^{0.1270292} < 1$, and 0.6694155 + 0.1270292 = 0.7964447 < 0.796445, we are done. In any event, we get that $\dim((M \setminus L) \cap (4.134215, 4.137519)) < 0.796445$.

3.7.6. Refinement of the control in the region (4.137519, 4.1407). In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digits appearing before are $a_n = 2$ and $a_{n-1} = 1$.

If the digit before $a_{n-1} = a_{n-2} \in \{2,3\}$, we have $\lambda_0(1223^{*}13) > 4.1837$ and $\lambda_0(a_{n-2}a_{n-1}23^{*}121) > 4.1409$ (thanks to the fact that 1313, 1312, 13111 are forbidden), so that we are back to the situation in the previous subsection.

If the digit before $a_{n-1} = a_{n-2} = 1$, we have $\lambda_0(a_{n-2}a_{n-1}113^{*}1) > 4.1407$ (as 1313, 1312, 13111 are forbidden) and $\lambda_0(2113^{*}23^{*}) < 4.027$, so that the second transition becomes $231\alpha_{n+3}$ and $113\beta_{n+5}$ and we are done.

In summary, we get that $\dim((M \setminus L) \cap (4.137519, 4.1407)) < 0.796445$.

3.7.7. Refinement of the control in the region (4.1407, 4.1673). In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$ when the digit appearing before is $a_n = 2$ (since 11313 is forbidden because $\lambda_0(11313^{*}1) > 4.1673$).

If the digit before $a_n$ is $a_{n-1} \in \{2,3\}$, we have $\lambda_0(a_{n-1}a_n23^{*}12T) < 4.1387$ and $\lambda_0(a_{n-1}a_n23^{*}13) > 4.175$, so that the second transition becomes $23121\alpha_{n+5}$ and $113\beta_{n+3}$. Since $\frac{|I(a_1, ..., a_2, 3, 1, 2, \overline{3})|}{|I(a_1, ..., a_2)|} \leq 0.00051$, $\frac{|I(a_1, ..., a_2, 1, 1, 3)|}{|I(a_1, ..., a_2)|} \leq 1/63$, $(0.00051)^{0.122} + (1/63)^{0.122} < 1$, and 0.67 + 0.122 = 0.792, we are done.

If the digit before $a_n$ is $a_{n-1} = 1$, we have two possibilities. If the digit before $a_{n-1} = a_{n-2} = 1$, then $\lambda_0(1223^{*}13) > 4.1837$ and $\lambda_0(a_{n-2}a_{n-1}a_n23^{*}1213313T) < 4.13997$ (as 1313, 1312, 13111 are forbidden), so that the second transition becomes $23121\alpha_{n+5}$ and $113\beta_{n+3}$ and we are back to the situation of the previous paragraph. If the digit before $a_{n-1}$
is $a_{n-2} \in \{2,3\}$, then the facts that $1313, 1312$ are forbidden, and $\lambda_0(a_{n-2}a_{n-1}a_n113^*\overline{T}3) < 4.13984$ imply that the second transition is $231\alpha_{n+3}$ and $11311\beta_{n+5}$ and we are done.

In summary, we get that $\dim((M \setminus L) \cap (4.1407, 4.1673)) < 0.796445$.

3.8. **Refinement of the control in the region** $(4.1673, 4.2527275)$. In view of the arguments of the previous subsections, our task is reduced to discuss the second transition $231\alpha_{n+3}$ and $113\beta_{n+3}$. We shall describe the possible extensions of this transition in terms of the digits appearing before and/or the Markov values of the words.

If the digit appearing before is $a_n = 1$, the second transition becomes $23132\alpha_{n+5}$ and $113\beta_{n+3}$ because $3131$ is forbidden and $\lambda_0(a_n23132\overline{T}3) < 4.1619$.

If the digit appearing before is $a_n = 2$, the facts that $3131$ and $2131$ are forbidden, $\lambda_0(a_n23^*13) > 4.1991, \lambda_0(a_n23^*\overline{T}3) < 4.149$, and $\lambda_0(a_n113^*\overline{T}3) < 4.128$ can be used to say that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1673, 4.199)$. Furthermore, the fact that $\lambda_0(a_n113^*111\overline{T}2) < 4.1785$ allows to conclude that the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.199, 4.2527275)$.

If the digit appearing before is $a_n = 2$, the facts that $3131$ and $2131$ are forbidden, $\lambda_0(a_n23^*13) > 4.175, \lambda_0(a_n23^*\overline{T}3) < 4.132$, and $\lambda_0(a_n113^*\overline{T}3) < 4.1521$ can be used to say that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1673, 4.175)$. Moreover, the fact that $\lambda_0(a_n113^*111) > 4.1857$ and $\lambda_0(a_n113^*111\overline{T}) < 4.16781$ allows to conclude that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.175, 4.1857)$. Hence, it remains to analyse the region $(4.1857, 4.2527275)$ when $a_n = 2$.

If the digit appearing before is $a_n = 2$ is $a_{n-1} \in \{2,3\}$, the facts that $3131$ and $2131$ are forbidden, $\lambda_0(a_{n-2}223^*13) < 4.18261$, and $\lambda_0(a_n113^*\overline{T}3) < 4.1521$ imply that the second transition becomes $2313\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1587, 4.2527275)$. Thus, it suffices to treat the case $a_n = 2$ and $a_{n-1} = 1$ in the region $(4.1857, 4.2527275)$.

If the digit appearing before is $a_n = 2$ is $a_{n-1} = 12$ is $a_{n-2} = 1$, the facts that $3131$ and $2131$ are forbidden, $\lambda_0(a_n2113^*111) > 4.189$, and $\lambda_0(a_n113^*111\overline{T}) < 4.16781$ imply that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.189)$. Also, the second transition becomes $2313\alpha_{n+4}$ and $113111\beta_{n+5}$ in the region $(4.189, 4.2527275)$ because $\lambda_0(a_n21223^*135) < 4.1881$.

If the digit appearing before is $a_{n-1} = 12$ is $a_{n-2} = 3$, the facts that $3131$ and $2131$ are forbidden, $\lambda_0(a_{n-2}2223^*13) > 4.1889, \lambda_0(a_n23^*\overline{T}3) < 4.132$ and $\lambda_0(a_n113^*\overline{T}3) < 4.1521$ imply that the second transition becomes $2312\alpha_{n+4}$ and $11311\beta_{n+5}$ in the region $(4.1889, 4.2527275)$. Also, the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.1889, 4.2527275)$ because $\lambda_0(a_n2113^*1111\overline{T}2) < 4.1865$.

At this point, it remains only to investigate the region $(4.1857, 4.2527275)$ when the digit appearing before $a_{n-1} a_n = 12$ is $a_{n-2} = 2$. For this sake, we shall distinguish three subcases.

3.8.1. **The subcase** $a_{n-3} = 3$ and $a_{n-2}a_{n-1}a_n = 212$. Since $3131$ and $2131$ are forbidden, $\lambda_0(a_{n-3}212113^*111) > [3; 11111\overline{T}3] + [0; 12122a_{n-3}3T] > 4.1876$. Because $\lambda_0(a_n113^*111\overline{T}) < 4.16781$, we conclude that the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.1876)$. Moreover, $\lambda_0(a_{n-3}22132^*133132\overline{T}) < 4.1874$ and $\lambda_0(a_n113^*\overline{T}3) < 4.1521$, so that the second transition becomes $2313\alpha_{n+4}$ and $113111\beta_{n+5}$ in the region $(4.1876, 4.2527275)$.

3.8.2. **The subcase** $a_{n-3} = 1$ and $a_{n-2}a_{n-1}a_n = 212$. Since $3131$ and $2131$ are forbidden, $\lambda_0(a_{n-3}221223^*13) > 4.1878$. Because $\lambda_0(a_n23^*\overline{T}3) < 4.132$ and $\lambda_0(a_n113^*\overline{T}3) < 4.1521$, we conclude that the second transition becomes $2312\alpha_{n+4}$ and $113111\beta_{n+5}$ in the region $(4.1857, 4.1878)$. Also, $\lambda_0(a_{n-3}2212113^*111111\overline{T}32) < 4.1873$, so that the second transition becomes $231\alpha_{n+3}$ and $113111\beta_{n+6}$ in the region $(4.1878, 4.2527275)$.

3.8.3. **The subcase** $a_{n-3} = 2$ and $a_{n-2}a_{n-1}a_n = 212$. If the digit appearing before is $a_{n-3} = 1$, we have $\lambda_0(a_{n-4}2212213^*111) > [3; 111131123\overline{T}3T] + [0; 12122113] > 4.187546$ in the region $(4.1857, 4.199)$ because the strings $3131, 2131, 113111$ are forbidden. Thus, the second transition becomes $231\alpha_{n+3}$ and $113112\beta_{n+6}$ in the region $(4.1857, 4.187546)$ since $\lambda_0(a_n113^*112\overline{T}) < 4.16781$. Moreover, $\lambda_0(a_n221223^*133132\overline{T}) < 4.187543$, so that the second transition becomes $2313\alpha_{n+4}$ and $113111\beta_{n+4}$ in the region $(4.187546, 4.2527275)$. 
In summary, we showed that the possibilities for the second transition in the region (4.1673, 4.2527275) are $2313a_{n+5} - 113\beta_{n+3}$, $2312a_{n+4} - 11311\beta_{n+5}$, $2310a_{n+3} - 11311\beta_{n+6}$, $231\alpha_{n+4} - 113112\beta_{n+6}$ and $2313\alpha_{n+4} - 11311\beta_{n+5}$. By combining this information with the facts that the Cantor set $C = 1, 2, 3$ with 3131 and 2131 forbidden has dimension $< 0.67785$, and $0.67785 + 0.118 = 0.79585$, we derive that $\dim((M \setminus L) \cap (4.1673, 4.2527275)) < 0.79585$.

3.9. Refinement of the control in the region (4.2527275, 4.32372). Recall that in the region $(4.01, \sqrt{20})$, we have to investigate the transitions:

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Since $3131$ is forbidden here, $\lambda_0(33*1323) < 4.081$, $\lambda_0(3313**23) < 4.21$, $\lambda_0(23***2) < 4$, the first transition becomes $331323a_{n+6}$ and $213\beta_{n+3}$.

Similarly, since $3131$ is forbidden, $\lambda_0(23*13212) < 4.2171$, $\lambda_0(113*1131) < 4.2022$ and $\lambda_0(113113*1) < 4.18$, so the second transition becomes $231323a_{n+5}$ and $1131\beta_{n+4}$.

Given that $|\frac{|I(a_1, \ldots, a_n, 2.3.1.3.2.3)}{|I(a_1, \ldots, a_n)|}| < \frac{1}{160678}$, $|\frac{|I(a_1, \ldots, a_n, 2.3.1.3.2)}{|I(a_1, \ldots, a_n)|}| < 0.0071$, $|\frac{|I(a_1, \ldots, a_n, 2.3.1.3.2)}{|I(a_1, \ldots, a_n)|}| < 0.00012$, $|\frac{|I(a_1, \ldots, a_n, 1.1.1, 3.1)}{|I(a_1, \ldots, a_n)|}| < \frac{1}{1007}$, and

\[
(\frac{1}{1007})^{0.09} + (0.0071)^{0.09} < 1, 
(0.00012)^{0.1021} + (\frac{1}{1007})^{0.1021} < 1,
\]
we conclude that $\dim((M \setminus L) \cap (4.2527275, 4.32372)) < 0.6913 + 0.1021 = 0.7934$ thanks to the fact that $C = 1, 2, 3$ with 1313, 3131 and 2132 forbidden has dimension $< 0.6913$.

3.10. Refinement of the control in the region (4.32372, 4.385). Similarly to [20], we can use $C = \{1, 2, 3\}^Z$ and a block $B$ to show that the continuations of words with values in $(M \setminus L) \cap (4.01, \sqrt{20})$ are

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

For later use, we note that $31313, 21313, 31312, 21312, 1113131, 1313111, 3131112, 2111313, 3131113, 3111313$ are forbidden here, and the corresponding Cantor set has dimension $< 0.6948$.

Let us discuss the first transition. The continuation $231113231$ is valid in the region (4.3, 4.385), so that the first transition becomes $3313a_{n+4}$ and $2131\beta_{n+5}$.

Let us now investigate the second transition. The validity of the continuations $23133$ and $113113\beta_{n+4}$ say that the second transition becomes $2313a_{n+4}$ and $113\beta_{n+4}$.

In the region (4.3353, 4.385), the second transition is $231311a_{n+4}$ and $1131\beta_{n+3}$ thanks to the valid continuation $231311311113231$ and the fact that $3131$ and $31312$ are forbidden.

Next, we observe that $313112, 211313, 313111$ and $111313$ are forbidden in the region (4.32372, 4.3353). We will analyse the second transition in this region depending on the digits appearing before it.

If the digit before is $a_n = 3$, the second transition becomes $2313a_{n+4}$ and $11313\beta_{n+5}$ in the region (4.332, 4.3353) because the continuation $a_n, 1131313132$ is valid, and it becomes $2313231a_{n+8}$ and $11311\beta_{n+4}$ in the region (4.3272, 4.332) because the string $23131$ is forbidden and the continuation $a_n, 2313231$ is valid.

If the digit before is $a_n = 2$, the second transition becomes $2313a_{n+4}$ and $113121\beta_{n+5}$ in the region (4.3272, 4.3353), because the continuation $a_n, 1131211323$ is valid and $211313$ is forbidden.

If the digits before are $a_n-1 a_n = 11$, the second transition is $2313a_{n+4}$ and $11312\beta_{n+5}$ in the region (4.3272, 4.3353) because $11313$ is forbidden and $a_{n-1} a_n, 113123132$ is valid.

If the digits before are $a_n-1 a_n = 21$, the second transition is $2313a_{n+4}$ and $11312\beta_{n+5}$ in the region (4.3272, 4.3353) because $11313$ is forbidden and $a_{n-1} a_n, 113123132$ is valid, and it becomes $2313231a_{n+8}$ and $11312\beta_{n+4}$ in the region (4.3272, 4.329) since $23131$ is forbidden and $2313231$ is valid.

If the digits before are $a_n-1 a_n = 31$, the second transition is $23131a_{n+5}$ and $1131\beta_{n+4}$ in the region (4.332, 4.3353), because $a_{n-1} a_n, 2313111311132$ is valid, and it becomes $2313a_{n+4}$ and $1131113\beta_{n+7}$ in the region (4.3272, 4.332) because $a_{n-1} a_n, 113112$ and $a_{n-1} a_n, 113113$ are forbidden and $a_{n-1} a_n, 1131113132$ is valid.
In summary, we established that the possible transitions are $3313\alpha_{n+4} - 2131\beta_{n+5}$, $231311\alpha_{n+6} - 1131\beta_{n+4}$, $2313\alpha_{n+4} - 11313\beta_{n+5}$, $231233\alpha_{n+8} - 1131\beta_{n+4}$, $2313\alpha_{n+4} - 11312\beta_{n+5}$, $2313\alpha_{n+4} - 1131113\beta_{n+7}$. Since $0.6948 + 0.10155 = 0.79635$, we conclude that $\dim((M \setminus L) \cap (4.32372, 4.385)) < 0.79635$.

3.11. **Refinement of the control in the region** (4.385, 4.41). Recall that in the region between 4.01 and $\sqrt{20}$ our goal is to study the transitions:

- $\alpha_n = 331\alpha_{n+3}$ and $\beta_n = 213\beta_{n+3}$;
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 113\beta_{n+3}$.

Let us discuss the second transition. It extends as $2313111\alpha_{n+7}$ and $1131\beta_{n+4}$ in this region because the strings $31313$, $31312$ and $21313$ are forbidden and the continuations $2313111\beta_{n+2}$ and $1131\beta_{n+4}$ are valid.

We observe that the Cantor set $C = 1, 2, 3$ with $31313$, $31312$, $21313$, $121312$ and $213121$ forbidden (as they are in this region) has Hausdorff dimension $< 0.6975$.

The first transition becomes $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in this region due to the valid continuations $3313111\beta_{n+2}$ and $2131\beta_{n+4}$ and to the fact that $331312$ and $31313$ are forbidden.

In summary, we showed that the possible transitions in our region are $3313111\alpha_{n+7} - 2131\beta_{n+4}$, and $2313111\alpha_{n+7} - 1131\beta_{n+4}$.

Since $0.6975 + 0.0963 = 0.7938$, we conclude that $\dim((M \setminus L) \cap (4.385, 4.41)) < 0.7938$.

3.12. **Refinement of the control in the region** (4.41, $\sqrt{20}$). The second transition extends as $23131\alpha_{n+5}$ and $11313\beta_{n+5}$ in this region due to the valid continuations $23131\beta_{n+2}$ and $11313\beta_{n+2}$.

Let us now discuss the first transition. Since $33131\beta_{n+2}$ and $2131\beta_{n+4}$ are valid continuations when the Markov value is $> 4.332$, the first transition extends $33131\alpha_{n+5}$ and $2131\beta_{n+4}$ in our region.

In the region $(4.46151, \sqrt{20})$, we have a valid continuation $331312\beta_{n+2}$, so that the first transition becomes $331312\alpha_{n+6}$ and $2131\beta_{n+4}$ (as $31313$ is forbidden). Thus, it remains only to treat the region $(4.41, 4.46151)$.

If the digits appearing before the first transition are $a_{n-1}a_n = 13$, the first transition becomes:

- $33131\alpha_{n+5}$ and $2131\beta_{n+5}$ in the region $(4.4608, 4.46151)$ thanks to the valid continuation $23131\beta_{n+2}$;
- $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in the region $(4.41, 4.461)$ due to the valid continuation $3313111\beta_{n+2}$ and the fact that $a_{n-1}\alpha_n31312$ and $31313$ are forbidden.

If the digits appearing before the first transition are $a_{n-1}a_n \neq 13$, the first transition becomes:

- $33131\alpha_{n+5}$ and $21312\beta_{n+5}$ in the region $(4.456, 4.46151)$ thanks to the fact that $a_{n-1}\alpha_n21313$ is forbidden and the validity of the continuation $213121\beta_{n+2}$;
- $3313111\alpha_{n+7}$ and $2131\beta_{n+4}$ in the region $(4.41, 4.459)$ due to the valid continuation $3313111\beta_{n+2}$ and the fact that $a_{n-1}\alpha_n21313$ and $31313$ are forbidden.

In summary, we showed that the possible transitions in our region are $331312\alpha_{n+6} - 2131\beta_{n+4}$, $33131\alpha_{n+5} - 21312\beta_{n+6}$, $3313111\alpha_{n+7} - 2131\beta_{n+4}$, $33131\alpha_{n+5} - 21313\beta_{n+5}$, and $23131\alpha_{n+5} - 11313\beta_{n+5}$. Since $0.7057 + 0.0903 = 0.796$, we conclude that $\dim((M \setminus L) \cap (4.41, \sqrt{20})) < 0.796$.

3.13. **Refinement of the control in the region** ($\sqrt{20}, 4.4984$). Similarly to [20], we can use $C \subset \{1, 2, 3, 4\}^\mathbb{Z}$ where 14, 41, 24, 42 are forbidden and a certain block $B$ to show that the continuations of words with values in $(M \setminus L) \cap (\sqrt{20}, 4.4984)$ are

- $\alpha_n = 4\alpha_{n+1}$ and $\beta_n = 33131\beta_{n+4}$, or
- $\alpha_n \in \{33131\alpha_{n+5}, 34\alpha_{n+3}\}$ and $\beta_n = 2131\beta_{n+4}$, or
- $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}$.
Note that a sequence containing the strings 343 or 31313 has Markov value $> 4.52$. In particular, we can refine $C$ into $C = 1, 2, 3, 4$ where $14, 41, 24, 42, 343, 31313$ are forbidden. Note that $\dim(C) < 0.705$.

If the sequence $\theta$ in $C$ contains 4 and it is not $\mathcal{T}$ (whose Markov value is $\sqrt{20}$), then it contains 43 and, a fortiori, $\theta = \ldots 443 \ldots$.

Suppose that $\lambda_0(\ldots 44^*3\ldots) \geq \lambda_0(\ldots 4^*43\ldots)$, i.e.,

$$4 + \alpha + \frac{1}{4+\beta} \geq 4 + \beta + \frac{1}{4 + \alpha}$$

where $\alpha = [0; 3, \ldots]$. This would imply that $\alpha \geq \beta$, so that

$$\lambda_0(\ldots 44^*3\ldots) \geq 4 + \alpha + \frac{1}{4 + \alpha}$$

for $\alpha = [0; 3, \ldots]$. Because the minimal value of $\alpha$ extracted from a sequence $\theta \in C$ is

$$\alpha \geq [0; 3, 1, 3, 1, 2, 1],$$

we would have that

$$\lambda_0(\ldots 44^*3\ldots) \geq [4; 3, 1, 3, 1, 2, 1] + [0; 4, 3, 1, 3, 1, 2, 1] > 4.4984.$$ 

Therefore, we can assume that 4 doesn’t appear in sequences $\theta$ producing Markov values in the interval $(\sqrt{20}, 4.4984)$. In particular, the continuations of words with values in $(M \setminus L) \cap (\sqrt{20}, 4.4984)$ are actually

(i) $\alpha_n = 33131\alpha_{n+5}$ and $\beta_n = 2131\beta_{n+4},$ or

(ii) $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}.$

We affirm that (i) has $\alpha_n = 331312\alpha_{n+6}$: indeed, this happens because of the presence of the continuation $331312\mathcal{T}$ (which is valid as $\lambda_0(\ldots 331312\mathcal{T}) \leq 4.463 < \sqrt{20}$). Similarly, we affirm that (ii) has $\alpha_n = 23131\alpha_{n+5}$ and $\beta_n = 1131\beta_{n+6}$: in fact, this happens because of the presence of the continuations $231\mathcal{T}$ and $113\mathcal{T}$ (which are valid as $\lambda_0(\ldots 231\mathcal{T}) < 4.394 < \sqrt{20}$ and $\lambda_0(\ldots 113\mathcal{T}) < 4.42521 < \sqrt{20}$).

Since

$$\left( \frac{[I(a_1, \ldots, a_n, 3, 3, 1, 3, 1, 2)]}{[I(a_1, \ldots, a_n)]} \right)^{0.09} + \left( \frac{[I(a_1, \ldots, a_n, 2, 1, 3, 1)]}{[I(a_1, \ldots, a_n)]} \right)^{0.09}$$

$$= \left( \frac{r + 1}{(53r + 173)(72r + 235)} \right)^{0.09} + \left( \frac{r + 1}{(5r + 14)(9r + 25)} \right)^{0.09}$$

$$\leq \left( \frac{1}{34691} \right)^{0.09} + \left( \frac{0.003106}{0.09} \right) < 0.985 < 1$$

and

$$\left( \frac{[I(a_1, \ldots, a_n, 2, 3, 1, 3, 1)]}{[I(a_1, \ldots, a_n)]} \right)^{0.09} + \left( \frac{[I(a_1, \ldots, a_n, 1, 1, 3, 1, 3, 1)]}{[I(a_1, \ldots, a_n)]} \right)^{0.09}$$

$$= \left( \frac{r + 1}{(19r + 43)(34r + 77)} \right)^{0.09} + \left( \frac{r + 1}{(24r + 43)(43r + 77)} \right)^{0.09}$$

$$\leq \left( \frac{0.00331}{0.09} \right) + \left( \frac{1}{33111} \right)^{0.09} < 0.966 < 1$$

for $0 < r < 1$, we derive that $\dim((M \setminus L) \cap (\sqrt{20}, 4.4984)) < 0.705 + 0.09 = 0.795$.

**Remark 3.8.** Even though this fact will not be used here, we note that 4.4984 is somewhat close to the point $\alpha + \frac{1}{\alpha} = 4.49846195 \ldots$, where

$$\alpha = [4; 313121311312313121112331312112],$$

which is the smallest element of the Lagrange spectrum accumulated by Lagrange values of sequences containing the letter 4 infinitely often: cf. [28].
3.14. **Refinement of the control in the region** $(4.4984, \sqrt{2}t)$. Similarly to [20], we can use $C \subset \{1, 2, 3, 4\}^2$ where 14, 41, 24, 42 are forbidden and a certain block $B$ to show that the continuations of words $\gamma$ with values in $(M \setminus L)$ and $(4.4984, \sqrt{2}t)$ are

1. $\alpha_n = 4\alpha_{n+1}$ and $\beta_n = 3131\beta_{n+4}$, or
2. $\alpha_n \in \{33314\alpha_{n+5}, 34\alpha_{n+3}\}$ and $\beta_n = 2131\beta_{n+4}$, or
3. $\alpha_n = 2\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}$.

In the sequel, we shall significantly refine the analysis of these continuations.

3.14.1. **The case** $(1)$ of $\alpha_n = 4\alpha_{n+1}$ and $\beta_n = 3131\beta_{n+4}$. We affirm that $\alpha_n = 44\alpha_{n+2}$ in this situation. In fact, given the nature of $C$, our task is to rule out the other possibility that $\alpha_n = 43\alpha_{n+2}$. In this direction, the following lemma (obtained from a direct calculation) will be helpful:

**Lemma 3.9.** $[4; 3] + [0; 3, 1, 3, 1, 2] = \frac{\partial}{\partial} = [3; 1, 3, 4] + [0; 1, 2, 1, 3]$. In particular,

$$
9/2 = m(\overline{4}3\overline{1}3\overline{1}4) = \lim_{n \to \infty} m(\overline{4}4\overline{3}(1312)^n1313) \in L \cap \mathbb{Q}.
$$

If $\gamma$ has allowed continuations $\alpha_n = 43\alpha_{n+2}$ and $\beta_n = 3131\beta_{n+4}$, then its Markov value is $< 9/2$. Otherwise, its Markov value $m(\gamma)$ would be $> 9/2$ (as $9/2 \in L$ and $\gamma$ is assumed to give rise to an element of $M \setminus L$), and the previous lemma would permit us to connect $\gamma$ with an adequate block $B$ via $4\alpha_{n}3132$, a contradiction.

Now, if $\gamma$ has allowed continuations $\alpha_n = 43\alpha_{n+2}$ and $\beta_n = 3131\beta_{n+4}$, and its Markov value is $< 9/2$, then let us write $\gamma = \theta\alpha_{n+1} \ldots$. select $\beta \in [\theta, \theta^4]$ with $[\beta^3] > [4; 4]$ and let us consider the Markov value $m(3\beta) < 9/2$ of an allowed continuation $3\beta$. If $[0; 3, \beta] \geq [0; 3, 1, 3, 1, 2]$, then $m(3\beta) \geq [\beta^3] + [0; 3, \beta^3] > [4] + [0; 3, 1, 3, 1, 2] = 9/2$, a contradiction. If $[0; 3, \beta^3] < [0; 3, 1, 3, 1, 2]$, then $\rho = 13\mu$ with $[\rho] > [0; 1, 2, 1, 3]$ and $\mu(3\beta) < 9/2$ would force $[3; \mu] + [0; 1, 3, \beta^3] < 9/2$, so that

$$
9/2 > m(3\beta) > [\beta^3] + [0; 3, 1, 3, \mu] > [\beta^3] + [0; 3, 1, 9/2 - [0; 1, 3, \beta^3]].
$$

This is a contradiction because the right-hand side is an increasing function of $[\beta^3] > [\overline{3}]$ whose value at $[\beta^3] = [\overline{3}]$ is $9/2$ after the previous lemma.

In summary, we proved that, in any scenario, the case $(1)$ is actually

(1') $\alpha_n = 44\alpha_{n+2}$ and $\beta_n = 3131\beta_{n+4}$.

In what follows, we shall analyse the natural subdivision $(1')$ into two scenarios:

(1i) $\alpha_n = 444\alpha_{n+3}$ is an allowed continuation,

(1ii) $\alpha_n = 44\alpha_{n+2}$ and $\beta_n = 3131\beta_{n+4}$.

3.14.2. **The subcase** $(1i)$. We affirm that $\beta_n = 313123\beta_{n+7}$ in this situation. In fact, let us begin by noticing that the Markov value of $\gamma = \rho\alpha_{n+1} \ldots$ with allowed continuations of type $(1i)$ is $< 4.513$: otherwise, we would be able to connect to an adequate block $B$ by continuing with $\alpha_n' = 443\mathbf{1}$ (since $[4; 3, \overline{1}] + [0; 4, \rho^3] \leq [4; 3, \overline{1}] + [0; 4, 4, 4] < 4.513$). Next, we observe that the strings 31313 and 343 are forbidden for any sequence with Markov value $< 4.513$.

Now, let us study the possible extensions of $\beta_n = 3131\beta_{n+4}$. We have that $\gamma = \rho\alpha_{n+1} \ldots$ where $\rho$ ends with 3 or 4 (because $\alpha_n = 4 \ldots$ is allowed and 14, 24 are forbidden strings). If $\rho$ ends with 4, we observe that the estimate

$$
[3; 1, 3, \rho] + [0; 1, 2, 1, 3, 4] < [3; 1, 3, 4; 4, 3, 4] + [0; 1, 2, 1, 3, 1, 3] < 4.49838
$$

would allow to connect $\gamma$ to an adequate block $B$ unless $\beta_n = 3131213\beta_{n+7}$. Similarly, if $\rho$ ends with 3, say $[\rho^3] = 3+x$ with $0 < x < [0; 1, 3, 1, 2, 1, 3]$, then the continuation $\alpha_n = 44\alpha_{n+3}$ would lead to an estimate

$$
m(\rho44\alpha_{n+3}) \geq [4; 3 + x] + [0; 4, 4, \alpha_{n+3}] \geq [4; 3 + x] + [0; 4, 4, 4, 3, 4] \\
\geq [3; 1, 3, 3 + x] + [0; 1, 2, 1, 3, 3, 1, 3] + 0.000076
$$

allowing to connect $\gamma$ to an adequate block $B$ unless $\beta_n = 3131213\beta_{n+7}$.

In other terms, we showed that $(1i)$ actually is

(1i') $\alpha_n \in \{444\alpha_{n+3}, 44\alpha_{n+2}\}$ and $\beta_n = 3131213\beta_{n+4}$.
Here, note that the relevant Cantor set $C = 1, 2, 3, 4$ with where 14, 41, 24, 42, 343, 31313 are forbidden has dim$(C) < 0.705$, and

$$\frac{|I(a_1, \ldots, a_n, 4, 4)|^{0.086} + |I(a_1, \ldots, a_n, 3, 1, 3, 1, 2, 1, 3)|^{0.086}}{|I(a_1, \ldots, a_n)|^{0.086}} \leq \left( \frac{r + 1}{(4r + 17)(5r + 21)} \right)^{0.086} + \left( \frac{r + 1}{(71r + 269)(90r + 341)} \right)^{0.086} \leq \left( \frac{1}{273} \right)^{0.086} + \left( \frac{1}{73270} \right)^{0.086} < 1.$$  

3.14.3. The subcase $(1ii)$. If the Markov value of $\gamma$ is $m(\gamma) \leq 4.5274$, then the string 31313 is forbidden. In particular, $\beta_n = 313121\beta_{n+6}$ (by comparison with 31312T). Here,

$$\frac{|I(a_1, \ldots, a_n, 4, 4, 3)|^{0.087} + |I(a_1, \ldots, a_n, 3, 1, 3, 1, 2, 1)|^{0.087}}{|I(a_1, \ldots, a_n)|^{0.087}} \leq \left( \frac{r + 1}{(13r + 55)(17r + 72)} \right)^{0.087} + \left( \frac{r + 1}{(14r + 53)(19r + 72)} \right)^{0.087} \leq \left( \frac{1}{3026} \right)^{0.087} + \left( \frac{2}{6097} \right)^{0.087} < 1.$$  

If the Markov value of $\gamma$ is $m(\gamma) > 4.5274$, then we affirm that it can not continued as $\alpha_n = 4431\alpha_{n+4}$; otherwise, we would have a continuation 44323T connecting to an adequate block $B$, a contradiction. This leaves us with two possibilities:

$(1ii') m(\gamma) \leq 4.53422$, so $\alpha_n = 4430\alpha_{n+3}$ cannot extend as 4433 nor 4434, and thus $\alpha_n = 4430\alpha_{n+3}$ extends only as $\alpha_n = 4432\alpha_{n+4}$;

$(1ii'') m(\gamma) > 4.53422.$

In the subcase $(1ii')$, we observe that

$$\frac{|I(a_1, \ldots, a_n, 4, 4, 3, 2)|^{0.0881} + |I(a_1, \ldots, a_n, 3, 1, 3, 1)|^{0.0881}}{|I(a_1, \ldots, a_n)|^{0.0881}} \leq \left( \frac{r + 1}{(30r + 127)(43r + 182)} \right)^{0.0881} + \left( \frac{r + 1}{(5r + 19)(9r + 34)} \right)^{0.0881} \leq \left( \frac{2}{35325} \right)^{0.0881} + \left( \frac{1}{516} \right)^{0.0881} < 1.$$  

In this case we will use the fact that the Cantor set $C = 1, 2, 3, 4$ where 41, 42, 434, 433 and their transposes are forbidden has dimension $< 0.7081$. Notice that $0.7081 + 0.0881 = 0.7962$.

In the subcase $(1ii'')$, we note that $\beta_n = 31313\beta_{n+5}$; otherwise, a continuation 31313432 would allow to connect to an adequate block $B$, a contradiction. Here, we observe for later use that

$$\frac{|I(a_1, \ldots, a_n, 4, 4, 3)|^{0.084} + |I(a_1, \ldots, a_n, 3, 1, 3, 1, 3)|^{0.084}}{|I(a_1, \ldots, a_n)|^{0.084}} \leq \left( \frac{r + 1}{(13r + 55)(17r + 72)} \right)^{0.084} + \left( \frac{r + 1}{(19r + 72)(24r + 91)} \right)^{0.084} \leq \left( \frac{1}{3026} \right)^{0.084} + \left( \frac{2}{10465} \right)^{0.084} < 1.$$
3.14.4. The case (3) of $\alpha_n = 23\alpha_{n+2}$ and $\beta_n = 1131\beta_{n+4}$. Analogously to the analysis of this situation in the region $(\sqrt{20}, 4.4984)$, we have that $\alpha_n = 23131\alpha_{n+5}$ and $\beta_n = 113131\beta_{n+6}$ together with the estimate

$$
\left(\frac{|I(a_1, \ldots, a_n, 2, 3, 1, 3, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.0857} + \left(\frac{|I(a_1, \ldots, a_n, 1, 1, 3, 1, 3, 1)|}{|I(a_1, \ldots, a_n)|}\right)^{0.0857} = \left(\frac{(r+1)}{(24r+43)(43r+77)}\right)^{0.0857} \leq (0.0031)^{0.0857} + (1/3311)^{0.0857} < 0.9997 < 1.
$$

3.14.5. The case (2) of $\alpha_n \in \{33131\alpha_{n+5}, 34\alpha_{n+3}\}$ and $\beta_n = 2131\beta_{n+4}$. Suppose that both continuations $33131\alpha_{n+5}$ and $34\alpha_{n+3}$ are allowed. In this context, any extension of $34\alpha_{n+3}$ which does not increase Markov values would be valid. Among them, we see from the discussion in the beginning of subsection about the region $(\sqrt{20}, 4.4984)$ that such a minimal extension has the form $\rho_{3343}\rho$. Thus, $\rho$ and $\rho^f$ cannot connect on an adequate block $B$ and, hence, we could use Proposition 7.8 in [20] to get that the set of Markov values associated to such $\gamma = \rho\alpha_{n+1} \ldots$ has Hausdorff dimension $< 2 - 0.713 < 0.35$.

Therefore, there is no loss of generality in assuming that only one of the continuations $33131\alpha_{n+5}$ and $34\alpha_{n+3}$ is allowed.

If the continuation $34\alpha_{n+3}$ is not allowed, we have two possibilities:

- if $m(\gamma) < 4.52$, then the strings 31313 and 343 are forbidden and we get $\alpha_n = 33132\alpha_{n+6}$ (by comparison with $33131\alpha_{n+6}$) and $\beta_n = 2131\beta_{n+4}$;
- if $m(\gamma) \geq 4.52$, then we get $\alpha_n = 33131\alpha_{n+5}$ and $\beta_n = 213131\beta_{n+6}$ (thanks to the continuation $213131\beta_{n+6}$ with $\lambda_0(\ldots 213^{*13}) < 4.5197$).

In the first case, since $r+1 \leq 34691$, and

$$
\left(\frac{1}{34691}\right)^{0.09} + (0.0031)^{0.09} < 0.9851,
$$

recalling that, if $C = 1, 2, 3, 4$ where $14, 41, 24, 42, 343, 31313$ are forbidden, then $\dim(C) < 0.705$ we will get the upper estimate $0.705 + 0.09 = 0.795$. In the second case, since $r+1 \leq (24r+67)(43r+120) \leq 0.000136$, and

$$
\left(\frac{2}{11745}\right)^{0.08} + (0.000136)^{0.08} < 0.991,
$$

it remains only to treat the possibility of $\alpha_n = 34\alpha_{n+3}$ and $\beta_n = 2131\beta_{n+4}$ being the unique allowed continuation.

In the case of $\alpha_n = 34\alpha_{n+3}$ and $\beta_n = 2131\beta_{n+4}$, if $m(\gamma) < 4.527$, then the strings 31313 and 343 are forbidden and $\alpha_n = 3444\alpha_{n+4}$. If both continuations $\alpha_n = 3444\alpha_{n+5}$ and $\alpha_n = 3444\alpha_{n+5}$ are allowed, then any continuation $3444 \ldots$ which does not increase Markov values would be allowed and the same analysis of the first paragraph of this subsection (considering sequences of the type $\gamma 344443\gamma^f$) implies that the corresponding set of Markov values $m(\gamma) < 4.527$ has Hausdorff dimension $< 0.35$. In other words, there is no loss of generality in assuming that only one of the continuations $\alpha_n = 3443\alpha_{n+5}$ or $\alpha_n = 3444\alpha_{n+5}$ when $m(\gamma) < 4.527$. Since $\frac{|I(a_1, \ldots, a_n, 3, 4, 4, 4)|}{|I(a_1, \ldots, a_n)|} \leq \frac{2}{71065}$, $\frac{|I(a_1, \ldots, a_n, 3, 4, 4, 3)|}{|I(a_1, \ldots, a_n)|} \leq 0.00311$ and

$$
\left(\frac{2}{71065}\right)^{0.09} + (0.00311)^{0.09} < 0.985 < 1,
$$

our task is reduced to discuss the case of $\alpha_n = 34\alpha_{n+3}, \beta_n = 2131\beta_{n+4}$, and $m(\gamma) \geq 4.527$. In this regime, the continuation $21313\beta_{n+7}$ is allowed, so that $\beta_n = 213131\beta_{n+7}$. If $4.527 \leq m(\gamma) \leq 4.55$, the strings 3433, 3434 and
2131313 are forbidden (as \(\lambda_0(34^*33) > 4.56593\) and \(\lambda_0(2131^*13) > 4.55065\)) so that \(\beta_n = 21313121\beta_{n+9}\) (thanks to the continuation 2131312T) in this context. Since
\[
\left(\frac{I(a_1, \ldots, a_n, 3, 4)}{I(a_1, \ldots, a_n)}\right)^{0.08745} + \left(\frac{I(a_1, \ldots, a_n, 2, 1, 3, 1, 3, 1, 2, 1)}{I(a_1, \ldots, a_n)}\right)^{0.08745} \\
\left(\frac{2}{357}\right)^{0.08745} + (9.71 \times 10^{-6})^{0.08745} < 0.9999 < 1,
\]
In this case we will use the fact that the Cantor set \(C = 1, 2, 3, 4\) where 41, 42, 3433, 3434 and their transposes are forbidden has dimension \(< 0.7083\). Notice that \(0.7083 + 0.08745 = 0.79575\).

It remains to treat the case \(\alpha_n = 34\alpha_n\beta_3, \beta_n = 213131\beta_{n+7}, \text{ and } m(\gamma) > 4.55\). If \(\alpha_n\) can not be extended as both 343 or 344, we can use the estimates \(\frac{[I(a_1, \ldots, a_n, 3, 4)]}{[I(a_1, \ldots, a_n)]]} \leq \frac{2}{4835349}\), \(\frac{[I(a_1, \ldots, a_n, 3, 4, 4, 4, 4, 3, 3)]}{[I(a_1, \ldots, a_n)]} \leq \frac{1}{11142680}\), and \((\frac{2}{4835349})^{0.0857} + (\frac{1}{11142680})^{0.0857} + (0.000136)^{0.0857} < 0.999 < 1\), the analysis of the case \(2\) is now complete.

3.14.6. End of the study of the region \((4.4984, \sqrt{2T})\). Our discussion of the cases \(1\), \(2\) and \(3\) above shows that \(\dim((M \setminus L) \cap (4.4984, \sqrt{2T})) < \max\{0.7094 + 0.0857, 0.7962\} = 0.7962\).

4. Algorithm for computing the Hausdorff dimension

It remains to get rigorous bounds for \(\dim_H B_1, \dim_H B_2, \dim_H X, \dim_H Y\) and \(\dim_H \Omega\) claimed in the previous sections. Our approach uses connection between the Hausdorff dimension of a limit set and the eigenvalue of the transfer operator. It was applied, for instance, in [13] to estimate \(\dim_H E_2\). The algorithm used in [13] is the so-called periodic points method and requires, in particular, accurate computation of all periodic points up to a large period \(n\) in order to get accurate estimate on dimension. The number of periodic points grows exponentially with \(n\), which makes this method non-applicable to systems with large number of maps.

In [27] a new approach has been developed for approximation of the eigenvalue of the transfer operator using an approximation of the corresponding eigenfunction. It is based on the Chebyshev spectral collocation method. However, the complex character of the Gauss–Cantor sets \(B_1, B_2\) and others means that the associated space of functions will have a very large dimension about \(3 \times 10^7\), which makes an accurate computation of the eigenfunction impossible at first sight. Nevertheless it turns out that in the cases we are interested in the eigenfunction can be approximated by a polynomial which lies in a subspace of dimension less than 4000! This allows us to approximate it sufficiently accurately.

In this section we would like to explain how to adapt the method developed in [27] to the present setting, to make the computation practicable.

In Appendix we give pseudocode for the computations described below. The master program is given in Algorithm 1. It splits into two parts. The first is combinatorial, where the problem of computing the dimension of a subset of \(E_2\) is turned into a problem of computing the dimension of the limit set of a certain iterated function scheme; following description in §4.1–§4.3 below; this part is covered by Algorithm 2. The second part deals with the computation of the Hausdorff dimension and uses approach via transfer operators inspired by thermodynamic formalism theory. This is covered by the main program given in Algorithm 3, it has several subroutines given in Algorithms 4–8.
4.1. **The setting.** We begin with a general setting.

**Definition 4.1.** Let $A = \{1, 2\}$ be an alphabet. Let $\bar{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ be the vector of lengths. Consider a set of forbidden words

$$F := F(\bar{r}) = \left\{ d_1^{(1)} d_2^{(1)} \ldots d_1^{(r_1)} \in A^{r_1}, d_1^{(2)} d_2^{(2)} \ldots d_2^{(r_2)} \in A^{r_2}, \ldots, d_1^{(k)} d_2^{(k)} \ldots d_k^{(k)} \in A^{r_k} \right\}.$$ 

A set $X_F \subset [0, 1]$ is defined by continued fraction expansions of its elements with extra Markov conditions:

$$X_F := \left\{ [0; a_1, a_2, \ldots] \mid a_n \in \{1, 2\}, \text{ such that for all } j \geq 1 \right.$$ 

$$a_j a_{j+1} \ldots a_{j+r_1} \neq d_1^{(1)} d_2^{(1)} \ldots d_1^{(r_1)}$$

$$a_j a_{j+1} \ldots a_{j+r_2} \neq d_1^{(2)} d_2^{(2)} \ldots d_2^{(r_2)}$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_j a_{j+1} \ldots a_{j+r_k} \neq d_1^{(k)} d_2^{(k)} \ldots d_k^{(k)} \right\} \subseteq E_2.$$

We next want to introduce a Markov iterated function scheme of uniformly contracting maps whose limit set is $X_F$. We take two maps $T_1, T_2 : [0, 1] \to [0, 1]$ defined by $T_1(x) = \frac{x}{2^r + 2}$ and $T_2(x) = \frac{1}{2^r + 2}$ and consider all possible compositions of length $n := \max_{1 \leq i \leq k} r_i - 1$, i.e. we consider a collection of maps

$$\mathcal{T}_n = \{ T_{a_1} \circ \ldots \circ T_{a_n} \mid a_n = a_1 \ldots a_n \in \{1, 2\}^n \}.$$ 

The Markov condition can be written as a $2^n \times 2^n$-matrix $M = M(a^n, b^n)$ where

$$M(a^n, b^n) = \begin{cases} 0, & \text{if concatenation } a_i^j b_k \text{ contains } d_1^{(i)} \ldots d_i^{(i)} \text{ as a subword for some } 1 \leq i \leq k; \\ 1, & \text{otherwise.} \end{cases}$$

It is a simple observation that the limit set of $(\mathcal{T}_n, M)$ is equal to $X_F$.

4.2. **The transfer operator.** In order to compute the Hausdorff dimension of the limit set of $(\mathcal{T}_n, M)$ we follow a general approach which dates back to Bowen and Ruelle [29]. More precisely, we use the connection between Hausdorff dimension of the limit set and the spectral radius of a transfer operator.

The transfer operator associated to a Markov iterated function scheme is a linear operator acting on the space of Hölder-continuous functions $C^\alpha([1, \ldots, 2^n] \times [0, 1])$, where $\{1, \ldots, 2^n\} \times [0, 1]$ represents a disjoint union of $2^n$ copies of $[0, 1]$ ([27], Section 2.4). It is defined by

$$\mathcal{L}_t : (f_{a_1}, \ldots, f_{a_n}) \mapsto (F_{b_1}^t, \ldots, F_{b_n}^t),$$

where

$$F_{b_i}^t(x) = \sum_{j=1}^{2^n} M(a^j_{b_i}, b^k_{b_i}) \cdot f_{a^j_{b_i}}(T_{a^j_{b_i}}(x)) \cdot |T_{a^j_{b_i}}(x)|^t.$$ 

(4.1)

Our method is based on the following result (originally due to Ruelle, generalizing a more specific result of Bowen for limit sets of Fuchsian-Schottky groups):

**Proposition 4.2** (after [29]). Assume that the maximal positive eigenvalue of $\mathcal{L}_t$ is equal to 1. Then $\dim_H X_F = t$.

In order to obtain lower and upper bounds on the maximal eigenvalue of $\mathcal{L}_t$ we use min-max inequalities as described in ([27], Section 3.1) which has the dual advantages of being easy to implement and also leading to rigorous results. More precisely, our numerical estimates are based on the following, realised in practice using Algorithms 6, 7 and 8.

**Lemma 4.3** ([27]). Assume that there exist two positive functions

$$\bar{f} = (f_{a_1}, \ldots, f_{a_2^n}), \quad \bar{g} = (g_{a_1}, \ldots, g_{a_2^n}) \in C^\alpha(\{1, \ldots, 2^n\} \times [0, 1])$$

such that for $\mathcal{L}_{t_0} \bar{f} = (F_{a_1}^t, \ldots, F_{a_2^n}^t)$ and $\mathcal{L}_{t_1} \bar{g} = (G_{a_1}^t, \ldots, G_{a_2^n}^t)$ we have

$$\min \inf_j \frac{F_{a_j}^t(x)}{f_{a_j}^t(x)} > 1 \quad \text{and} \quad \max \sup_j \frac{G_{a_j}^t(x)}{g_{a_j}^t(x)} < 1$$

(4.2)

Then $t_0 \leq \dim_H X_F \leq t_1$. 
We attempt to construct good choices of functions $f_{q_j}$ and $g_{q_j}$ for $1 \leq j \leq 2^n$ as positive polynomials of a relatively small degree using the collocation method. We fix a small natural $m$ and define $m$ Chebyshev nodes by

$$x_k := \frac{1}{2} \left( 1 + \cos \left( \frac{\pi (2k-1)}{2m} \right) \right) \in [0,1] \text{ for } k = 1, \ldots, m.$$  

The Lagrange interpolation polynomials are defined by $p_i(x) := \prod_{k=1}^{m} \frac{x-x_k}{x_i-x_k}$. These are the unique polynomials of minimal degree with the property that $p_i(x_k) = \delta_{ik}^2$. We then consider the subspace of $C^\alpha([1, \ldots, 2^n] \times [0,1])$ spanned by $2^n$ copies of the space $\langle p_k \rangle_{k=1}^m$:

$$\Pi(n, m) := \langle \{1, \ldots, 2^n \} \times \langle p_1, \ldots, p_m \rangle \rangle \subset C^\alpha([1, \ldots, 2^n] \times [0,1]).$$

Then the components of any $\bar{q} = (q_1, \ldots, q_{2^n}) \in \Pi(n, m)$ are uniquely defined by their values at the Chebyshev nodes:

$$q_j(x) = \sum_{i=1}^m q_j(x_i)p_i(x) \in \mathbb{R}[x]; \quad j = 1, \ldots, 2^n. \quad (4.3)$$

In particular, the formula (4.3) defines a bijection $I : \mathbb{R}^{2^n m} \to \Pi(n, m)$. We introduce a projection operator $P : C^\alpha([1, \ldots, 2^n] \times [0,1]) \to \mathbb{R}^{2^n m}$ given by

$$P(f_1, \ldots, f_{2^n}) \mapsto (f_1(x_1), \ldots, f_1(x_m), f_2(x_1), \ldots, f_2(x_m), \ldots, f_{2^n}(x_1), \ldots, f_{2^n}(x_m)) \in \mathbb{R}^{2^n m}.$$  

We may now consider a finite rank linear operator $B^t : \mathbb{R}^{2^n m} \to \mathbb{R}^{2^n m}$ defined by

$$B^t \bar{v} = P \mathcal{L} t \bar{v} \quad (4.4)$$

and construct the test functions $\tilde{f}$ and $\tilde{g}$ in (4.2) from the eigenvectors $\nu_t$ and $\nu_{t_1}$ corresponding to the leading eigenvalues of $B_{t_0}$ and $B^t_{t_1}$ respectively using the formulae $\tilde{f} = I \nu_t$ and $\tilde{g} = I \nu_{t_1}$. The pseudocode is given in Algorithms 4 and 5.

**Remark 4.4.** This approach appears to be relatively straightforward to implement numerically compared to other methods. The bisection method can be used to get a refined estimate.

Nevertheless, practical implementation is challenging for large values of $n$. The first complication here is the computation of the matrix $M$ that gives the Markov condition, since at first sight it requires analysing of $2^{2n}$ words of length $2n$ searching for forbidden substrings, and the resulting matrix of the size $2^{2n}$ would take about 2GB of computer memory\footnote{A very optimistic estimate is that one needs at least $2^{2n}$ bits and for $n = 17$ we get $2^{34}$ bits, which is $2^{31}$ bytes, exactly 2GB.} to store for a modest value $n = 17$ and for larger values $n > 19$ the resulting Markov matrix wouldn’t fit into RAM memory of a personal computer.

Furthermore, the matrix $B^t$ is even larger and requires much more space as it is not a binary matrix and its values need to be computed with higher accuracy. Typically we would like to work with 128 bits precision, so for modest values of $n = 17$ and $m = 6$ it would require 1512GB just to store.

A final complication is that the computation of the eigenvector of a huge matrix with high accuracy is also very time-consuming in practice. The best method here would be the power method, which has complexity of the matrix multiplication. The latter depends on the realisation, but is no less than $O(n^{2.5})$.

In the remainder of the section we explain how to refine the basic algorithm to make it more practical.

### 4.3. Simplifying the computation of the Markov matrix $M$.

The next statement gives the basis for our approach for making the computation possible.

**Proposition 4.5.** Assume that the columns $j_1$ and $j_2$ of the Markov matrix $M$ are identical, i.e. for all $1 \leq k \leq 2^n$ we have that $M(a_k, b_{1k}) \equiv M(a_k, b_{2k})$. Then any eigenvector $\tilde{f}$ of $B^t$ lies in the subspace of $\Pi(n, m)$ for which $f_{q_1} = f_{q_2}$.

We postpone the proof of this Proposition until Section 4.4. Fortunately, it turns out that for the sets of forbidden words we need to deal with, the Markov matrix has a very small number of pairwise different columns compared to its size.

**Example 4.6.** For the specific sets which we study in this paper, we have the following.

1. In the case of the set $B_3$ which appears in Section §2.3.11 the Markov matrix has 41186 columns, of which only 138 are pairwise distinct.
2. In the case of the set $B_2$ which appears in Section §2.3.10 the Markov matrix has 79034 columns of which only 184 are pairwise distinct.
(3) In the case of the set $X$ which is used in Section §2.4.4 to obtain a lower bound on the transition value $t_1$, the Markov matrix has $3940388$ columns of which only $429$ are pairwise distinct.

(4) In the case of the set $Y$ which is used in Section §2.4.5 to obtain the upper bound on the transition value $t_1$, the Markov matrix has $3940438$ columns of which only $434$ are pairwise distinct.

(5) In the case of the set $\Omega$ defined in [20] the Markov matrix has $45059$ columns, of which only $114$ are pairwise distinct.

Proposition 4.10 below gives an upper bound on the number of pairwise different columns in the transition matrix in terms of forbidden words.

Therefore instead of computing (and storing) the entire Markov matrix $M$ it is sufficient to identify and to compute only unique columns, and to keep a record of the indices of columns which are identical. This is a significant saving in memory already, but there is room for even more.

**Remark 4.7.** We note that if the rows $i_1$ and $i_2$ and the columns $j_1$ and $j_2$ of the Markov matrix agree, i.e. $M(a_{i_1}^j, b_{n_i}^i) = M(a_{i_2}^j, b_{n_i}^i)$ and $M(a_n^k, b_{n_i}^j) = M(a_n^k, b_{n_i}^j)$ for all $1 \leq k \leq 2^n$, then $M(a_{i_1}^j, b_{n_i}^i) = M(a_{i_2}^j, b_{n_i}^i) = M(a_{i_2}^j, b_{n_i}^i) = M(a_{i_2}^j, b_{n_i}^i)$.

This simple observation allows us to reduce significantly the memory needed to store the Markov matrix $M$. In particular, in our considerations the Markov matrix $M$ can be replaced by a smaller reduced Markov matrix $\hat{M}$ as follows.

**Step 1.** Identify the words $\hat{a}_i^j, j = 1, \ldots, K$ such that the rows $M(\hat{a}_i^j, \cdot)$ are pairwise different and define the map $R$ which associates a row $j$ with a row $R(j)$ from the set of unique rows.

**Step 2.** Identify the words $\hat{b}_i^j, j = 1, \ldots, K$ such that the columns $M(\cdot, \hat{b}_i^j)$ are pairwise different and define the map $C$ which associates a column $j$ with a column $C(j)$ from the set of unique columns.

**Step 3.** Compute the reduced Markov matrix $\hat{M} = \hat{M}(\hat{a}_i^j, \hat{b}_i^j)$ of the size $K \times K$.

It is clear that the huge Markov matrix $M$ can be easily recovered from the reduced matrix $\hat{M}$ using the correspondence maps $R$ and $C$, since $M(\hat{a}_i^j, \hat{b}_i^j) = \hat{M}(\hat{a}_i^{R(j)}, \hat{b}_i^{C(k)})$, $1 \leq k \leq 2^n$. Therefore, the main step in computing the Markov matrix $M$ is the computation of the sets of words which give unique columns and rows of $M$ together with the maps $R$ and $C$.

4.3.1. An upper bound for the number of unique rows and columns.

**Definition 4.8.** We call the word $w' = w_n \ldots w_1$ the semordnilap or reverse of the word $w = w_1 \ldots w_n$.

It is easy to see that if the set of forbidden words includes every word together with its reverse, then the number of pairwise different columns in the Markov matrix is equal to the number of pairwise different rows. In particular, the structure of the set of forbidden words we consider implies that the reduced Markov matrix is a square matrix.

We need the following notation for the sequel.

**Definition 4.9.** For any $1 \leq k \leq n - 1$ we call a subword $w_k \ldots w_1$ a suffix of the word $w_n \ldots w_1$ and a subword $w_n \ldots w_{k+1}$ a prefix of the word $w_n \ldots w_1$.

We now want to give an absolute upper bound on the number of unique columns or, equivalently, rows, of the reduced Markov matrix $\hat{M}$ in terms of forbidden words.

**Proposition 4.10.** Assume that there are $k$ forbidden words which have $P$ different suffixes in total. Then the number of pairwise distinct rows in the Markov matrix is no more than $P + 1$. Similarly, the number of pairwise distinct suffixes gives an upper bound on the number of pairwise distinct columns.

The first step in the computation, as outlined in Algorithm 2, is to identify all the words which contain a forbidden word as a subword, since all of them give the zero row (or column) in the transition matrix. After removing them from our consideration, we obtain the set of allowed words.

$$A := \{ w_n = w_1 \ldots w_n \in \mathbb{A}^n, w_j \ldots w_{j+r}, \notin F, \text{ for all } 1 \leq j \leq n - r_i, 1 \leq i \leq k \}.$$
Once the set $A$ is computed, it is of course possible to study all concatenations of allowed words and to identify those which give the unique rows and columns to the transition matrix. However, this would require $O \left( \left( \sum_{i=1}^{k} r_i \right) \cdot \# A \right)$ operations, which is prohibitively time-consuming, since $\# A$ is typically very large; more precisely, in the examples we consider we have $\# A \approx 2^{n-1}$. In the next subsection we give a faster algorithm, which requires only $O \left( \# A \left( \sum_{i=1}^{k} r_i \right) \right) + O \left( \left( \sum_{i=1}^{k} r_i \right)^4 \right)$ operations.

We denote by $P_F$ the set of prefixes of forbidden words and we denote by $S_F$ the set of suffixes of forbidden words. Observe that entries of a column which corresponds to a word are determined by suffixes of forbidden words it starts with.

Proof. (of Proposition 4.10). We may consider a mapping $g: A \to P_F$ that associates to every allowed word $w$ the longest prefix from $P_F$ which is a suffix of $w$. In other words $g(w) = \overline{w}$, if $w = w'\overline{w}$, where $\overline{w} \in P_F$ and $w'$ is the shortest word with this property. Evidently, the function $g$ takes at most $P + 1$ different values.

We claim that if $g(w_1) = g(w_2)$, then the rows corresponding to the words $w_1$ and $w_2$ in the transition matrix are identical. Indeed, assume for a contradiction that the rows are different. In other words, there exist a word $u \in A$ such that concatenation $w_1 u$ contains a forbidden subword $f w \in F$, and concatenation $w_2 u$ doesn’t contain any words from $F$. Since $w_1, u \in A$, we deduce that $w_1$ contains a non-empty prefix of $f w$ as a suffix and $u$ contains a non-empty suffix of $f w$ as a prefix:

$$w_1 = w'_1 f w, \quad u = f w' u, \quad f w = f w' f w,$$

where $w'_1 \neq \emptyset$ and $w' \neq \emptyset$ are some words. Therefore we may write $g(w_1) = g(w_2) = g(w)$. Since by assumption $g(w_1) = g(w_2)$, we see that $w_2 = w'_2 w'' f w$. Hence concatenation $w_2 u$ contains the word $f w$ and we get a contradiction. \hfill \square

Evidently, the total number of suffixes (or prefixes) is bounded by the sum of the lengths of all forbidden words: $\sum_{j=1}^{k} r_j$.

4.3.2. Computation of the reduced Markov matrix $\widehat{M}$. This can be realised by a number of technical steps (see Algorithm 2 for pseudocode). Let us denote by $|w|$ the length of the word $w$.

1. Compute the sets $P_F = \{ \overline{w} \text{ is a prefix of } w \mid w \in F \}$ and $S_F = \{ \widehat{w} \text{ is a suffix of } w \mid w \in F \}$.
2. For every word $w \in A$ we compute:
   a. The set of suffixes of forbidden words which are prefixes of $w$: $SF_w = \{ \overline{w} \in S_F \mid \overline{w} \text{ is a prefix of } w \}$; and
   b. The set of prefixes of forbidden words which are suffixes of $w$: $PF_w = \{ \widehat{w} \in P_F \mid \widehat{w} \text{ is a suffix of } w \}$.
3. We say that two words $w_1, w_2 \in A$ are “suffix–equivalent” if $SF_{w_1} = SF_{w_2}$ and we say that $w_1, w_2 \in A$ are “prefix–equivalent” if $PF_{w_1} = PF_{w_2}$. Thus we split the set of allowed words in equivalence classes by suffixes $A/\sim_S$ and prefixes $A/\sim_P$. It turns out that there is relatively small number of equivalence classes compared to the number of allowed words.

   In the next steps we explain that in order to decide whether two words are compatible it is sufficient to work with their equivalence classes.

4. The following encoding is handy to study compatibility of words based on equivalence classes. First, we fix enumeration of the set of forbidden words $F = \{ d_1, \ldots, d_k \}$. To every suffix $\overline{d} \in S_F$ of a forbidden word we associate a set of pairs $\{ (j, |\overline{d}|) \mid \overline{d} \text{ is a prefix of } d_j, d_j \in F \}$. To every prefix $\widehat{d} \in P_F$ we associate a set pairs $\{ (j, |d_j| - |\widehat{d}|) \mid \widehat{d} \text{ is a prefix of } d_j, d_j \in F \}$.

   Note that concatenation of a prefix and a suffix is a forbidden word, if their encodings are the same.

5. For any allowed word $w \in A$ we apply the encoding described above to the equivalence classes $A/\sim_S$ and $A/\sim_P$.

6. It is clear that the concatenation of the words $w_1$ and $w_2$ doesn’t have a forbidden subword, if and only if the corresponding equivalence classes $AP_{w_1}$ and $AS_{w_2}$ do not have any common pairs after encoding. Therefore, instead of computing the Markov matrix for the set of allowed words it is sufficient to compute the compatibility matrix for the equivalence classes.

7. We identify unique rows and columns in the compatibility matrix for the equivalence classes and choose representatives from each class to obtain words which give unique rows and columns in the reduced matrix $\widehat{M}$.

The main advantage of this approach is that in order to compute the equivalence classes $A/\sim_S$ and $A/\sim_P$ it is sufficient to parse the huge set of allowed words only once. The number of operations on subsequent steps is $O \left( \left( \sum_{i=1}^{k} r_i \right)^3 \right)$. 

4.4. Computation of the test functions. In order to construct the test functions to use in Lemma 4.3 we need to compute the eigenvector of the matrix $B^t$ defined by (4.4). By straightforward computation we can obtain the explicit form of $B^t$. Indeed for any $v \in \mathbb{R}^{2^m}$ we have

\[
Iv = (q_1(x), \ldots, q_{2^n}(x)), \quad q_j(x) = \sum_{l=1}^{m} v_{(j-1)m+l} \cdot p_l(x), \quad \text{for } 1 \leq l \leq 2^n.
\]

Therefore using (4.1) we get

\[
\mathcal{L}_t Iv = (Q_1, \ldots, Q_{2^n}) \text{ where for all } 1 \leq k \leq 2^n \text{ we have}
\]

\[
Q_k(x) = \sum_{j=1}^{2^n} M(a_{j,k}^l, b_{j,k}^l) \cdot q_j(T_{2^n}^l(x)) \cdot |T_{2^n}^l(x)|^t
\]

\[
= \sum_{j=1}^{2^n} M(a_{j,k}^l, b_{j,k}^l) \cdot |T_{2^n}^l(x)|^t \cdot \left( \sum_{l=1}^{m} v_{(j-1)m+l} \cdot p_l(T_{2^n}^l(x)) \right).
\]

Hence the components of $u^t = (u_1^t, \ldots, u_{2^n}^t)$ are given by

\[
u_{(k-1)m+l} = Q_k(x) = \sum_{j=1}^{2^n} M(a_{j,k}^l, b_{j,k}^l) \cdot |T_{2^n}^l(x)|^t \cdot p_l(T_{2^n}^l(x)) \cdot v_{(j-1)m+l}.
\]

Introducing $2^n$ small $m \times m$ matrices

\[
B^t(i,l) := |T_{2^n}^l(x)|^t \cdot p_l(T_{2^n}^l(x))
\]

we get

\[
B^t = \left( \begin{array}{cccc}
M(a_{1,1}^1, b_{1,1}^1) \cdot B^{1,t} & M(a_{1,2}^1, b_{1,2}^1) \cdot B^{2,t} & \cdots & M(a_{1,n}^1, b_{1,n}^1) \cdot B^{n,t} \\
M(a_{2,1}^1, b_{2,1}^1) \cdot B^{1,t} & M(a_{2,2}^1, b_{2,2}^1) \cdot B^{2,t} & \cdots & M(a_{2,n}^1, b_{2,n}^1) \cdot B^{n,t} \\
\vdots & \vdots & \ddots & \vdots \\
M(a_{n,1}^1, b_{n,1}^1) \cdot B^{1,t} & M(a_{n,2}^1, b_{n,2}^1) \cdot B^{2,t} & \cdots & M(a_{n,n}^1, b_{n,n}^1) \cdot B^{n,t}
\end{array} \right).
\]

We are now ready to prove Proposition 4.5.

**Proof.** (of Proposition 4.5). Since by assumption $M(a_{j,k}^l, b_{j,k}^l) = M(a_{j,k}^l, b_{j,k}^l)$ for all $1 \leq k \leq 2^n$, using representation (4.7) of the matrix $B^t$ we conclude that

\[
B^t((j_1 - 1)m + l, k) = B^t((j_2 - 1)m + l, k)
\]

for all $1 \leq l \leq m$ and $1 \leq k \leq 2^n$. Therefore for any $v \in \mathbb{R}^{2^m}$ and $u = B^tv$ we have

\[
u_{(j_1 - 1)m + l} = \sum_{k=1}^{2^n} B^t((j_1 - 1)m + l, k) \cdot v_k = \sum_{k=1}^{2^n} B^t((j_2 - 1)m + l, k) \cdot v_k = u_{(j_2 - 1)m + l}.
\]

The result follows from (4.5) applied to $u$.}

We proceed to computing the leading eigenvector of $B^t$. By Proposition 4.5 it belongs to the subspace of dimension $\mathbb{R}^{Km}$ where $K$ is the number of pairwise different columns of the Markov matrix $M$. We have already mentioned that it is not possible to work with the matrix $B^t$ itself. The next Lemma allows us to replace the matrix $B^t$ with a smaller reduced matrix $\tilde{B}^t$ in our considerations.

**Definition 4.11.** Assume that the Markov matrix $M$ has $K$ pairwise different columns. Let $j_1, \ldots, j_K$ be the indices of the unique columns of $M$ and let $i_1, \ldots, i_K$ be the indices of the unique rows of $M$. Let $R$ be the correspondence map as constructed in Step 1. We define the reduced matrix $\tilde{B}^t$ by

\[
\tilde{B}^t((k-1)m + i, (l-1)m + j) = \sum_{x: R(x) = i, 1 \leq s < 2^n} M(i_k, s) B^{s,t}(i, j), \quad 1 \leq l \leq K.
\]

**Remark 4.12.** Note that in order to compute the matrix $\tilde{B}^t$ there is no need to store the matrix $B^t$. It is sufficient to add elements of $B^t$ to the corresponding elements of $\tilde{B}^t$ as we compute them.

**Lemma 4.13.** Let $\tilde{v}$ be the eigenvector of $\tilde{B}^t$. Then the eigenvector of $B^t$ can be computed using the formula $v_{(j-1)m+l} := \tilde{v}_{(R(j-1)m+l)}$ for $1 \leq l \leq m$ and $1 \leq j \leq 2^n$. 

Proof. Let \( P : \mathbb{R}^{2^n m} \to \mathbb{R}^{K m} \) be the orthogonal projector onto the subspace defined by the system of equations \( v_{i_k} = v_s \) where \( R(s) = i_k \) for all \( 1 \leq s \leq 2^n \) and \( 1 \leq k \leq K \). Then \( \tilde{B}^t = PB^tP^* \), where \( P^* \) stands for the transposed matrix \( P \).

Therefore, in order to recover the eigenvector of \( B^t \) and to compute the test functions, it is sufficient to compute the eigenvector of a much smaller reduced matrix \( \tilde{B}^t \), defined above. The latter can be realised using simple iterations method (see Algorithms 4 and 5 for pseudocode).

4.5. Verification of the min-max inequalities. Finally, to verify the conditions of Lemma 4.3 numerically, we follow the same method as proposed in [27].

First, we compute the coefficients of the polynomials \( p_1, \ldots, p_K \) from the eigenvector of \( \tilde{B}^t \).

The transfer operator \( \mathcal{L}_t \) given by (4.1) can be written using the reduced Markov matrix \( \tilde{M} \) and the correspondence map \( R \). More precisely, let \( j_1, \ldots, j_K \) be the indices of the unique columns in matrix \( M \). Let \( p_1, \ldots, p_K \) be the polynomials constructed from the eigenvector of the \( \tilde{B}^t \). Then the transfer operator takes the form \( \mathcal{L}_t : (p_1, \ldots, p_K) \mapsto (Q_1^t, \ldots, Q_K^t) \) where

\[
Q_k^t(x) = \sum_{i=1}^{2^n} \tilde{M}(a_n^{R(i)}, b_n^{j_k}) \cdot p_{R(i)}(T_{a_n^{i}}(x)) \cdot |T_{a_n^{i}}(x)|^t.
\]

In order to obtain upper and a lower bounds on \( \frac{Q_k^t}{p_k} \), we take a partition of the interval \([0,1]\) into 256 equal intervals. Then we evaluate \( \frac{Q_k}{p_k} \) at the centre and to compute \( \sup \frac{d}{dx} \frac{Q_k(x)}{p_k(x)} \) on each interval using Taylor series expansion. The latter is realised using arbitrary precision ball arithmetic [14]. Pseudocode is given in Algorithms 7 and 8.

4.6. Computational aspects. Here we give some numerical data. The decimal numbers which we give in this section are truncated, not rounded.

4.6.1. Set \( B_1 \). We apply the method described above to estimate the dimension of the set \( B_1 \). In this case, the set of forbidden words constitutes of 27 words of length from 5 to 17.

\[
F = \{21212, 21112121, 12111212, 21112122, 111112121, 111121211, \\
21121221, 2211121211, 1221111212112, 1122111212111, \\
2221111212122, 212211121212112, 21111212112212, 2222111212112222,
\]

and their reverses \}

Therefore we will consider the words of length 16 and we begin by computing the set of allowed words \( A_F \), which do not contain a forbidden word as a subword. The computation leaves us with 41186 allowed words (down from \( 2^{16} = 65536 \)). We also compute the coefficients of the Möbius maps corresponding to allowed compositions \( T_{j_1} \circ \ldots \circ T_{j_{16}} \).

Then we employ the algorithm described in §4.3.2 to identify the words which give unique columns and rows to the Markov matrix together with correspondence maps \( R \) and \( C \). It turns out that there are 138 such words. Finally, we calculate the Markov matrix itself. The computations we have done so far take less than a minute.

Afterwards, we choose \( m = 8, t_0 = 0.5 \) and compute the reduced matrix \( \tilde{B}^t \) using the formulae (4.6) and (4.8) and find its eigenvector using the power method. We work 128 bit for precision and the eigenvector is computed with an error of \( 10^{-26} \). We recover 138 polynomials from the eigenvector applying the formula (4.3). Then we take a uniform partition of the interval \([0,1]\) into 256 intervals and estimate the ratios \( \frac{F_{j_0}}{F_j} \) on each of the intervals using ball-precision arithmetic. To obtain accurate bounds on the numerator of the derivative \( \frac{F_{j_0}}{F_j} \), we compute the first 4 of its derivatives. We omit the eigenvector here, but we note that for \( t_0 = 0.5 \) the leading eigenvalue of \( \tilde{B}^{t_0} \) is 1.0004258... > 1 and the ratios satisfy

\[
1.000425 < \frac{F_{j_0}^{t_0}}{F_j} < 1.000426 \text{ for } j = 1, \ldots, 138.
\]
We then test another two values to get a more accurate estimate. For \( t_1 = 0.50001 \) we have that the leading eigenvalue of \( \hat{B}^{t_1} \) is \( 1.0002239 \ldots > 1 \) and the ratios can be bounded as

\[
1.000223 < \frac{F_{t_1}^j}{f_j} < 1.000225 \text{ for } j = 1, \ldots, 138.
\]

For \( t_2 = 0.50005 \) we get that the leading eigenvalue of \( \hat{B}^{t_2} \) is \( 0.99941699 \ldots < 1 \) and the ratios can be bounded as

\[
0.999416 < \frac{F_{t_2}^j}{f_j} < 0.999418 \text{ for } j = 1, \ldots, 138.
\]

It takes about 20 minutes to complete the estimates for a single value of \( t \) using 8 threads running in parallel. \(^8\)

4.6.2. Set \( B_2 \). In this case we have 33 forbidden words of length from 5 to 18. Thus we consider the words of length 17 and after removing those which contain a forbidden word as a subword, obtain 79034 allowed words. Among those we identify 184 words which give unique columns to the Markov matrix \( M \) and (another) 184 words which give unique rows. Using the same parameters \( m = 8 \) and \( t_0 = 0.5 \) we compute the matrix \( \hat{B}^{t_0} \) and its eigenvector with an error of \( 10^{-40} \). The leading eigenvalue is \( 0.9996 \ldots \) and after another 40 minutes we have lower and upper bounds on the ratios

\[
0.999606 < \frac{F_{t_0}^j}{f_j} < 0.999607 \text{ for } j = 1, \ldots, 184.
\]

Therefore we deduce that \( \dim_H B_2 < 0.5 \). In order to obtain more refined estimates we consider another two values \( t_1 = 0.499975 \) and \( t_2 = 0.499995 \). It turns out that the largest eigenvalue or \( \hat{B}^{t_1} \) is \( 1.0001426 \ldots > 1 \) and we have the following bounds for the ratios

\[
1.000141 < \frac{F_{t_1}^j}{f_j} < 1.000143 \text{ for } j = 1, \ldots, 184.
\]

The largest eigenvalue of \( \hat{B}^{t_2} \) is \( 0.99971391 \ldots < 1 \) and the bounds for the ratios are

\[
0.999713 < \frac{F_{t_2}^j}{f_j} < 0.999714 \text{ for } j = 1, \ldots, 184.
\]

Therefore we conclude that \( 0.499975 < \dim_H B_2 < 0.499995 \). It takes about 90 minutes to obtain estimates on \( \frac{F_{t_1}^j}{f_j} \) for a single value of \( t \). The time is evidently affected by the number of functions.

4.6.3. Set \( X \). The set \( X \) is specified by exclusion of 46 words of length from 5 to 24. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 23. After removing all compositions which correspond to forbidden words, we are left with 3940388 maps, which is slightly less than a half of \( 2^{23} \). The algorithm also identifies 429 unique columns and rows in the Markov matrix \( M \); some of them are repeated as many as 141030 times. These computations take about 5 minutes.

We then choose \( m = 8 \) and \( t_0 = 0.5 \) as initial dimension guess and work with precision of 190 bits. It takes about 2.5 hours to compute the eigenvector of the reduced matrix \( \hat{B}^{t_0} \) of dimension 429 \cdot 8 \) with an error of \( 10^{-40} \) and to obtain coefficients of 429 polynomials of degree 7. The corresponding eigenvalue is \( 0.999973 \ldots < 1 \).

Most of the time is then taken by calculation of the images of these polynomials under the map \( \mathcal{L} \), as it involves taking compositions with all 3940388 maps. We use a partition of the interval \([0, 1]\) into 1024 intervals. The computation takes around 15 days with 8 threads running in parallel. Finally, we obtain

\[
0.9999732 < \frac{F_{t_0}^j}{f_j} < 0.9999738 \text{ for } j = 1, \ldots, 429.
\]

which allows us to conclude that \( \dim_H X < t_0 = 0.5 \).

\(^8\) Computations were done using 4 Core 8 Threads Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz.
4.6.4. Set $Y$. The set $Y$ is specified by exclusion of 48 words of length from 5 to 24. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 23. After removing all compositions which correspond to forbidden words, we are left with 3940438 maps, which is slightly less than a half of $2^{23}$. The algorithm also identifies 434 unique columns and rows in the Markov matrix $M$; some of them are repeated as many as 176015 times. These computations take about 5 minutes. We then choose $m = 8$ and $t_0 = 0.5$ as initial dimension guess and work with precision of 190 bits. It takes about 2.5 hours to compute the eigenvector of the reduced matrix $\tilde{B}^{t_0}$ of dimension $434 \cdot 8$ with an error of $10^{-40}$ and to obtain coefficients of 434 polynomials of degree 7. The corresponding eigenvalue is $1.0000162 \ldots > 1$.

Most of the time is then taken by calculation of the images of these polynomials under the map $\mathcal{L}_{t_0}$ as it involves taking compositions with all 3940438 maps. This time in attempt to make the computation faster we use a uniform partition of the interval $[0,1]$ into 256 intervals. The computation takes around 3 days with 8 threads running in parallel. Finally, we obtain

$$1.0000160 < \frac{F_{t_0}}{f_j} < 1.0000166 \quad \text{for} \quad j = 1, \ldots, 434.$$ 

which allows us to conclude that $\dim_H Y > t_0 = 0.5$.

4.6.5. Set $\Omega$. The set $\Omega$ is specified by exclusion of 26 words of length from 5 to 15. To compute its Hausdorff dimension we consider an iterated function scheme of compositions of length 14. After removing all compositions which correspond to forbidden words, we are left with 45059 maps, which is ten times less than $3^{14}$. The algorithm also identifies 114 unique columns and rows in the Markov matrix $M$; some of them are repeated as many as $3745$ times, but some occur only once. These computations take less than a minute. We then choose $m = 8$ and $t_0 = 0.5$ as initial dimension guess and work with precision of 190 bits. It less than a minute to compute the eigenvector of the reduced matrix $\tilde{B}^{t_0}$ of dimension $114 \cdot 8$ with an error of $10^{-40}$ and to obtain coefficients of 114 polynomials of degree 7. The corresponding eigenvalue is $1.956 \ldots > 1$.

The subsequent estimates of the ratios $\frac{F_{t_0}}{f_j}$ using partition of the interval $[0, 1]$ into 256 intervals take about 15 minutes and give

$$1.956990 < \frac{F_{t_0}}{f_j} < 1.9569915 \quad \text{for} \quad j = 1, \ldots, 114.$$ 

We therefore conclude that $\dim_H \Omega > 0.5$ and apply bisection method to get a better estimate. Taking the value $t_1 = 0.537152$ we get the leading eigenvalue $1.000031 \ldots > 1$ and

$$1.0000315 < \frac{F_{t_1}}{f_j} < 1.0000320 \quad \text{for} \quad j = 1, \ldots, 114.$$ 

and for $t_2 = 0.537155$ we get the leading eigenvalue of $\tilde{B}^{t_2}$ to be $0.999977 < 1$ and

$$0.999977 < \frac{F_{t_2}}{f_j} < 0.999979 \quad \text{for} \quad j = 1, \ldots, 114.$$ 

We therefore conclude that $0.537152 < \dim_H \Omega < 0.537155$.

This information is summarized in Table 7 below.

5. SOME COMMENTS AND OPEN QUESTIONS

Closing this article, let us briefly mention some problems left open by the present paper.

5.1. Modulus of continuity of the dimension function. The function

$$f(t) = \dim_H ((-\infty, t] \cap M)$$

is not Hölder continuous at 3 (cf. [22, p.147]), but an estimate on its modulus of continuity at any $t \in [3, c]$ was given in [22]. In particular, it is not clear what should be expected about its local Hölder continuity properties at non-isolated points of $L$. 

| Set | $A$ | $#F$ | $n$ | $#A_F$ | $K$ | $t$ | $\lambda_{max}$ | $r_1$ | $r_2$ | time |
|-----|-----|-----|-----|------|----|-----|---------|------|------|------|
| $B_1$ | $\{1,2\}$ | 27 | 16 | 41186 | 138 | 0.5 | 1.000425 | 1.000424 | 1.000426 | 15mins |
|       |     |     |     |      |     | 0.50001 | 1.000223 | 1.000222 | 1.000225 |   |
|       |     |     |     |      |     | 0.50005 | 0.999416 | 0.499415 | 0.499418 |   |
| $B_2$ | $\{1,2\}$ | 33 | 17 | 79034 | 184 | 0.5 | 0.999606 | 0.999600 | 0.999607 | 90 mins |
|       |     |     |     |      |     | 0.499975 | 1.000142 | 1.000141 | 1.000143 |   |
|       |     |     |     |      |     | 0.499995 | 0.999713 | 0.999712 | 0.999714 |   |
| $X$  | $\{1,2\}$ | 46 | 23 | 3940388 | 429 | 0.5 | 0.999973 | 0.999972 | 0.999974 | 4 days |
|       |     |     |     |      |     | 0.5 | 1.000016 | 1.000015 | 1.000017 |   |
| $Y$  | $\{1,2\}$ | 48 | 23 | 3940438 | 434 | 0.5 | 1.000016 | 1.000015 | 1.000017 |   |
|       |     |     |     |      |     | 0.537152 | 1.000031 | 1.000030 | 1.000032 |   |
|       |     |     |     |      |     | 0.537155 | 0.999977 | 0.999976 | 0.999979 |   |
| $\Omega$ | $\{1,2,3\}$ | 26 | 14 | 45059 | 114 | 0.5 | 1.956 | 1.955 | 1.957 | 12mins |
|       |     |     |     |      |     | 0.537152 | 1.000031 | 1.000030 | 1.000032 |   |
|       |     |     |     |      |     | 0.537155 | 0.999977 | 0.999976 | 0.999979 |   |

Table 7. Numerical output of the algorithm for computing dimension of the sets $B_1$, $B_2$, $X$, $Y$, $\Omega$.

Time refers to the time needed to compute the lower and the upper bounds on the ratios $r_1 < \frac{F_t}{f_j} < r_2$ for a single value of $t$.

5.2. **Interior of the intermediate of the classical spectra.** The folklore conjecture that $(3, \sqrt{12}) \cap L$ has non-empty interior is natural because Perron showed that $[\sqrt{5}, \sqrt{12}] \cap M = \{m(\alpha) : \{1,2\}^2\}$, so that $[\sqrt{5}, \sqrt{12}] \cap M$ is closely related to the arithmetic sum $E_2 + E_2$ where $E_2 = \{[0; a_1, a_2, \ldots] : a_n \in \{1,2\} \forall n \in \mathbb{N}\}$. In particular, $[\sqrt{5}, \sqrt{12}] \cap M$ should have non-empty interior because Marstrand’s projection theorem and Moreira–Yoccoz’s work (cf. [23]) say that $E_2 + E_2$ is expected to contain intervals as $E_2$ is a “nonlinear” Cantor set with Hausdorff dimension $> 0.5$. Nonetheless, $E_2 + E_2$ probably does not contain large intervals (as the Hausdorff dimension of $E_2$ is very close to 0.5), and, hence, it is not easy to convert this intuition into a concrete result. On the other hand, the first two authors and L. Jeffreys hope to use these ideas to establish in a future work that $L \cap [3, c]$ has non-empty interior (where $c$ is Freiman’s constant).

**Remark 5.1.** In a related direction, let us recall that Berstein [2] conjectured in 1973 that $[4.1, 4.52] \subset L$.

5.3. **The Hausdorff dimension of $M \setminus L$.** Despite our efforts to establish Theorem 3.1, we could not compute the first digit of $\dim_H(M \setminus L)$. Here, the answer to this problem (a digit in $\{5, 6, 7\}$) seems to depend on better bounds on $\sup(M \setminus L)$: for instance, if one can find new elements $M \setminus L$ “sufficiently close” to Freiman’s constant $c = 4.5278 \ldots$ (say, nearby 4.5251), then it is likely that the first digit of $\dim_H(M \setminus L)$ is 7.
Algorithm 1 Computing the first $K$ digits of the dimension of the Gauss–Cantor set specified by forbidden words.

**Input:** Alphabet $\mathcal{A} \subset \mathbb{N}$, finite list of forbidden words $\mathcal{F} := \{f_{w_j}\}_{j=1}^k \subset \mathcal{A}^n$, the desired accuracy $K \in \mathbb{N}$ digits, lower bound $T_0$ (optional, default value $T_0 = 0$), upper bound $T_1$ (optional, default value $T_1 = 1$)

1. **Program** `GAUSSCANTORSETS`($\mathcal{A}$, $\mathcal{F}$, $K$, $T_0 = 0$, $T_1 = 1$)

   // Global variables specifying the IFS:
   2. `sm`: int[$n$,4]
      /* Coefficients of the maps of the IFS written in rows
         $x \mapsto ax + b/cx + d$. */
   3. `rc`: int[$u$,maxmul]
      /* Indices of the identical rows in the Markov matrix, written in rows. */
   4. `cc`: int[$n$]
      /* For the column $k$ of the Markov matrix, $cc[k]$ is the smallest
       index in its column class. */
   5. `rmm`: bool[$u$,u]
      /* Reduced Markov matrix. */

   // Compute all of them:
   6. $(sm, rmm, rc, cc) \leftarrow$ `COMBINATORICSETUP($\mathcal{A}$, $\mathcal{F}$)`

   // Now compute the dimension:
   7. $\text{dim} \leftarrow$ `COMPUTEDIMENSION($K$, $T_0$, $T_1$)`
   8. return $\text{dim}$
   9. end program

Algorithm 2 Computing the Hausdorff dimension of the Gauss–Cantor set. Combinatorics setup: matrices that define the IFS and Markov condition.

**Input:** Alphabet $\mathcal{A} \subset \mathbb{N}$, finite list of forbidden words $\mathcal{F} := \{f_{w_j}\}_{j=1}^k \subset \mathcal{A}^n$.

1. **Function** `COMBINATORICSETUP($\mathcal{A}$, $\mathcal{F}$)`

   2. $n \leftarrow \max|f_{w_j}| - 1$.
   3. $PF \leftarrow$ the prefix tree of forbidden words
   4. $SF \leftarrow$ the suffix tree of forbidden words
   5. $k \leftarrow 0$
   6. $ws$: int[$2^n$,n] $\leftarrow$ all words in alphabet $\mathcal{A}$ of length $n$.
   7. for $w \in ws$ do
      8. if $w$ doesn’t contain a path in $SF$ from root to leaf then
         9. $k \leftarrow k + 1$
      10. $aw[k,:] \leftarrow w$
      11. $sm[k,:] \leftarrow$ matrix of $T_{w_1} \circ T_{w_2} \ldots T_{w_n}$
      12. end if
      13. end for
      14. for $w \in aw$ do
         15. Walking the tree $SF$, label by $w$ the nodes with the property:
            the path to the root is a prefix of $w$, but paths to the children are not prefixes of $w$
         16. Walking the tree $PF$, label by $w$ the nodes with the property:
            the path to the root is a suffix of $w$, but paths to the children are not suffixes of $w$.
      17. end for
   18. AP, AS $\leftarrow$ Walk the trees $PF$ and $SF$, read equivalence classes from labels and apply encoding to elements
      of equivalence classes as described in §4.3.2. Order the encoding pairs lexicographically.
   19. $N \leftarrow$ total number of pairs $(j, k)$ in encoding of equivalence classes
   20. $N_p \leftarrow$ total number of classes $AP$
   21. $N_s \leftarrow$ total number of classes $AS$
matrix is given by

\[ rmm \]

transfer matrix as specified by

\[ \text{linear fractional transformations given by} \]

\[ sm \]

Computing the first

\[ \text{Algorithm 3} \]

\[ \text{Algorithm 2} \]

\[ if \]

\[ 30 \]

\[ (31 \]

\[ 29 \]

\[ end while \]

\[ it \]

\[ 28 \]

\[ M \]

\[ 27 \]

\[ rmm \]

\[ 26 \]

\[ M \]

\[ 25 \]

\[ rmm \]

\[ 24 \]

\[ m \]

\[ 23 \]

\[ if \]

\[ end if \]

\[ m \]

\[ 22 \]

\[ nuls \]

\[ 21 \]

\[ if \]

\[ end if \]

\[ m \]

\[ 20 \]

\[ \text{Algorithm 3} \]

\[ \text{Computing the first} \]

\[ K \]

\[ \text{digits of the Hausdorff dimension of the limit set of the iterated function scheme of linear fractional transformations given by} \]

\[ sm. \]

\[ \text{The maps are divided in classes in two ways: according to the rows of transfer matrix as specified by} \]

\[ rc \]

\[ \text{and according to the columns of transfer matrix as specified by} \]

\[ cc. \]

\[ \text{The reduced Markov matrix is given by} \]

\[ rmm. \]

\[ function \]

\[ \text{COMPUTEDIMENSION}(K, T_0, T_1) \]

\[ // \text{Initialisation:} \]

\[ prec \leftarrow 64 \]

\[ /* \text{Initial precision, balancing the speed and accuracy. */} \]

\[ \beta \leftarrow \frac{T_0 + T_1}{2} \]

\[ /* \text{Dimension guess */} \]

\[ it \leftarrow 0 \]

\[ /* \text{Number of attempts to compute approximation to eigenfunctions */} \]

\[ m \leftarrow 6 \]

\[ /* \text{Number of Chebyshev nodes */} \]

\[ \varepsilon \leftarrow 10^{-K-1} \]

\[ /* \text{"Number of digits desired" \rightarrow accuracy */} \]

\[ \lnp \leftarrow -8 \]

\[ /* \text{Logarithm of the length of intervals in the partition used to obtain lower and upper bounds on functions using ball arithmetic */} \]

\[ // \text{Local Variables:} \]

\[ f: \text{array of} \]

\[ u \]

\[ \lambda: \text{eigenvalue of the matrix} \]

\[ B^\beta \]

\[ \text{approximating} \]

\[ \mathcal{L}_\beta. \]

\[ while \]

\[ (|T_1 - T_0| > \varepsilon \text{ and} \]

\[ it < 8 \text{ and} \]

\[ \lnp > -20) \text{ do} \]

\[ nuls \leftarrow \text{Zeros of the Chebyshev polynomial} \]

\[ T_m \]

\[ \text{with precision} \]

\[ prec \]

\[ /* \text{sw is the success switch */} \]

\[ (f, \lambda, \lnp, \text{sw}) \leftarrow \text{LEADINGVECTOR}(\beta, m, \text{prec}) \]

\[ /* \text{sw is the success switch */} \]

\[ while \]

\[ (\text{sw} \neq \text{OK} \text{ and} \]

\[ prec < 2048 \text{ and} \]

\[ m \cdot u < 3500) \text{ do} \]

\[ \text{if} \]

\[ \text{sw} = \text{precision_low} \text{ then} /* \text{Power method failed — increase precision */} \]

\[ \text{prec} \leftarrow \text{prec} + 128 \]

\[ \text{nuls} \leftarrow \text{Zeros of the Chebyshev polynomial} \]

\[ T_m \]

\[ \text{with precision} \]

\[ prec \]

\[ end if \]

\[ \text{if} \]

\[ \text{sw} = \text{precision_low_and_degree_low} \text{ then} /* \text{Non-positive polynomial, increase degree and precision */} \]

\[ m \leftarrow m + 2 \]

\[ /* \text{Non-positive polynomial, increase degree and precision */} \]

\[ \text{prec} \leftarrow \text{prec} + 128 \]

\[ \text{nuls} \leftarrow \text{Zeros of the Chebyshev polynomial} \]

\[ T_m \]

\[ \text{with precision} \]

\[ prec \]

\[ end if \]

\[ \text{if} \]

\[ \text{sw} = \text{degree_low} \text{ then} /* \text{To save time increase the degree now for better approximation otherwise} \]

\[ R \]

\[ \text{ATBOUNDS} \]

\[ \text{will give} \]

\[ b_1 < 1 < b_2 \text{ */} \]

\[ m \leftarrow m + 2 \]

\[ /* \text{To save time increase the degree now for better approximation otherwise} \]

\[ \text{RATBOUNDS} \]

\[ \text{will give} \]

\[ b_1 < \frac{C_{\beta j}}{f_j} < b_2, j = 1, \ldots, u. */} \]

\[ \text{nuls} \leftarrow \text{Zeros of the Chebyshev polynomial} \]

\[ T_m \]

\[ \text{with precision} \]

\[ prec \]

\[ end if \]

\[ \text{(f, \lambda, \lnp, \text{sw})} \leftarrow \text{LEADINGVECTOR}(\beta, m, \text{nuls, \lnp, \text{prec})} \]

\[ it \leftarrow it + 1 \]

\[ end while \]

\[ \text{if} \]

\[ \text{sw} = \text{OK} \text{ then} /* f is an array of} \]

\[ u \]

\[ \text{positive polynomials of degree} \]

\[ m */} \]

\[ (b_1, b_2) \leftarrow \text{RATBOUNDS}(f, \beta, \text{nuls, m, \lnp, \text{prec})} \]

\[ /* \text{Lower and upper bounds} b_1 < \frac{C_{\beta j}}{f_j} < b_2, j = 1, \ldots, u. */}
Algorithm 3 Continued

32 if $b_1 > 1$ then /* dim $\in (\beta, T_1)$ */
33 \[ T_0 \leftarrow \beta \]
34 \[ \beta \leftarrow \frac{T_0 + T_1}{2} \]
35 else if $b_2 < 1$ then /* dim $\in (T_0, \beta)$ */
36 \[ T_1 \leftarrow \beta \]
37 \[ \beta \leftarrow \frac{T_0 + T_1}{2} \]
38 else /* The assumption of min-max Lemma doesn’t hold for $f$ */
39 \[ \ln p \leftarrow \ln p - 1 \] /* Reduce the size of partition intervals */
40 if $m \cdot u < 3500$ then /* Increase the degree of approximating polynomials */
41 \[ m \leftarrow m + 2 \]
42 end if
43 it $\leftarrow$ it + 1
44 end if
45 end while
46 if $|T_1 - T_0| < \varepsilon$ then
47 return $\frac{T_1 + T_0}{2}$
48 else /* Cannot achieve the desired accuracy */
49 return $-1$
50 end if
51 end function
Algorithm 4 Computation of the polynomials, approximating the eigenfunctions

1 function LEADINGVECTOR(sm, rmm, rc, cc, β, m, nuls, prec, lnp)
   // Local variables:
2   B: arb[m · u, m · u] the matrix $B^β$, approximating the transfer operator;
3   sw: success switch.
4   ℓp, j = 1, . . . , m: Lagrange polynomials associated to nuls.
5   $B \leftarrow$ APPROXMATRIX(β, m, nuls, prec)
6   $v \leftarrow 1$
   // Compute the leading eigenvector and its eigenvalue; 250 iterations to kick off
7   ($v, λ, err$) $\leftarrow$ POWERMETHOD(B, v, 250, prec)
   // The degree of approximating polynomials and the allowed approximation error depend on $|λ - 1|$; the choice
   // of constants is based on heuristic experiments
8   $d_{\text{min}} \leftarrow \max(6, -1.25 \log |λ - 1|)$
9   $δ \leftarrow \exp(- \max(16, 12 \log |λ - 1|))$
10  if $d_{\text{min}} < m$ then
11     sw = degree_low goto exit /* Increase the degree for better approximation. */
12  end if
13  it $\leftarrow 0$
14  while $(err > δ$ and $it < 10)$ do
15     it $\leftarrow it + 1$ /* Attempting more iterations */
16     $δ_0 \leftarrow err$
17     ($v, λ, err$) $\leftarrow$ POWERMETHOD(B, v, 50, prec)
18     if $err > δ_0$ then /* Attempt failed: */
19         sw = precision_low goto exit /* error increased — increase precision */
20     end if
21     if $err > δ$ then /* Power method failed — increase precision */
22         sw = precision_low goto exit
23     else /* We have good approximation to the true eigenvector */
24         $f_j(x) \leftarrow \sum_{k=1}^{m} v[(j-1)m+k] \cdot ℓp_k(x), j = 1, . . . , u$
25         if ALLPOSITIVE(f, lnp, prec) then /* NB: lnp might have changed */
26             sw = OK goto exit /* Non-positive polynomial, increase degree and precision */
27         else
28             sw = precision_low_and_degree_low goto exit
29         end if
30     end if
31 end while
32 if $err > δ$ then /* We have good approximation to the true eigenvector */
33     sw = precision_low goto exit
34 else
35     $f_j(x) \leftarrow \sum_{k=1}^{m} v[(j-1)m+k] \cdot ℓp_k(x), j = 1, . . . , u$
36     if ALLPOSITIVE(f, lnp, prec) then /* NB: lnp might have changed */
37         sw = OK goto exit /* Non-positive polynomial, increase degree and precision */
38     else
39         sw = precision_low_and_degree_low goto exit
40     end if
41 end if
42 exit return ($f, λ, lnp, sw$
43 end function

34 function POWERMETHOD(B, v, n, prec)
35 for j $\leftarrow 1$ to n do /* All computations are done with precision prec */
36     $v \leftarrow \|Bv\|_1$ /* Here we use ℓ_1-norm $\|v\|_1 = \sum |v[j]|$ */
37 end for /* $v$ is approximation to the eigenvector */
38 $λ \leftarrow \|Bv\|_1 / \|v\|_1$ /* The corresponding eigenvalue */
39 return ($v, λ, \|Bv - λv\|_1$) /* Here we use sup norm */
40 end function
Algorithm 5 Calculation of the matrix $B^3$: $\text{arb}[m \cdot u, m \cdot u]$ approximating the operator $L_\beta$

1 function APPROXMATRIX($\beta, m, \text{nuls}, \text{prec}$)
   // rc: int[$u, \text{maxmul}$] consists of indices of the identical rows in the Markov matrix, written in rows.
   // $F_{rc[j,k]}$ is a map with coefficients from the row $sm[rc[j,k],\cdot]$. 
   // $rc[j,k]$ is a map with coefficients from the row $sm[rc[j,k],\cdot]$.
2 $B \leftarrow 0$
3 for $k \leftarrow 1$ to $u$ do /* For each class of rows */
4     for $k_0 \leftarrow 1$ to $rc[k,1]$ do /* For each word in the class $k$ */
5         temp $\leftarrow 0$
6         /* A matrix $\text{arb}[m, m]$. */
7         for $k_1 \leftarrow 1$ to $m$ do /* For each word in the class $k$ */
8             $(z_1, z'_1) \leftarrow \left(F_{rc[k,k_0]}(\text{nuls}[k_1]), F'_{rc[k,k_0]}(\text{nuls}[k_1])\right)$ /* $F_{rc[k,k_0]}$ and its derivative at every node */
9             temp[$k_1, k_2$] $\leftarrow (-1)^{k_2} |z_1'|^\beta \cdot T_m(2z_1 - 1) \cdot \frac{\sqrt{\text{nuls}[k_2] - (\text{nuls}[k_2])^2}}{m(z_1 - \text{nuls}[k_2])}$ /* Contribution to the matrix $B$ from $F_{rc[k,k_0]}$ */
10            /* $T_m$ is the Chebyshev polynomial of the first kind */
11         end for
12     end for
13     for $j \leftarrow 1$ to $u$ do /* If transition between a word of class $j$ and a word of class $cc[k,1]$ is allowed, add temp to a block of $B$, that corresponds to this transition. */
14         if $\text{rmm}[j, cc[k,1]]$ then
15             $B[(j-1)m+1:jm, (k-1)m+1:km] \leftarrow B[(j-1)m+1:jm, (k-1)m+1:km] + temp$
16         end if
17     end for
18 end function

19 return $B$
Algorithm 6 Computation of the lower and upper bounds on $\frac{[c, f_j]}{f_j}$, $j = 1, \ldots, u$. 

1 function RATBOUNDS($f$, $\beta$, $nuls$, $deg$, $lnp$, $prec$) 

   // Local Variables: 
   $p$, $cp$ : (arb[2$-lnp$],arf[2$-lnp$]) partition intervals and their centres; 
   $n$ : floor($\frac{\text{deg} + 1}{2}$) the number of the derivatives we calculate to get an upper bound using the Taylor series; 
   $tops$ : arb[1$u$, 2$-lnp$, $n + 1$] is an array of $u$ matrices. For each $j$ the first column of $tops[j; ;]$ contains the values $[\mathcal{L}_\beta f_j](cp)$. The second column contains the numerator of $\left(\frac{(\mathcal{L}_\beta f_j)'}{f_j}\right)(cp)$, i.e. $g_j(cp) := \left((\mathcal{L}_\beta f_j)' f_j - (\mathcal{L}_\beta f_j) f_j'\right)(cp)$. The next $n - 2$ columns contain derivatives $g_j^{(k)}(cp)$, $k = 1, \ldots, n - 2$. The last column contains the $n - 1$th derivative on the interval $g_j^{(n-1)}(p)$.

   $pow \leftarrow [-2\beta - d, -1 : -2\beta - d - (n - 1)]$

   $tops \leftarrow \text{NBN}(f, p, cp, pow, deg, lnp)$ /* Calculate $g_j^{(k)}$, $j = 0, \ldots, n - 1$ symbolically and evaluate $g_j^{(k)}(cp)$, $k = 0, \ldots, n - 2$ and $g_j^{(n-1)}(p)$ */

   $(b_1, b_2) \leftarrow (100, -100)$ /* Lower and upper bounds */

   for $j \leftarrow 1$ to $u$ do 

      for $k \leftarrow 1$ to $2$$^{-lnp}$ do 

         $y_1 \leftarrow tops[j,k,1] / f_j(cp[k])$ /* $(\mathcal{L}_\beta f_j)(cp[k]) = y_1$ */

         $ny_1' \leftarrow \text{UPPERBOUND}(tops[j,k,2 ; n + 1], 2$ $^-lnp-1, n)$ /* $|((\mathcal{L}_\beta f_j)' f_j - (\mathcal{L}_\beta f_j) f_j') (p[k])| < ny_1'$ */

         $\delta \leftarrow \frac{ny_1'}{(x_1[1])}$ $2$ $^-lnp-1$

         if $1 \in (y_1 - \delta, y_1 + \delta)$ then /* hypothesis of min-max Lemma failed */

            $(b_1, b_2) \leftarrow (y_1 - \delta, y_1 + \delta)$ goto exit

         end if

         if $y_1 + \delta > b_2$ then /* $y_1 + \delta$ is the new upper bound */

            $b_2 \leftarrow y_1 + \delta$

         end if

         if $y_1 - \delta < b_1$ then /* $y_1 - \delta$ is the new lower bound */

            $b_1 \leftarrow y_1 - \delta$

         end if

         if $1 \in (b_1, b_2)$ then /* hypothesis of min-max Lemma failed */

            goto exit

         end if

      end for

   end for

   exit return $(b_1, b_2)$

28 end function

   // Compute lower an upper bound for the function $f$ on the interval $(c - r, c + r)$ from the values $fn = [f(c), f'(c), f''(c), \ldots, f^{(n-2)}(c), f^{(n-1)}(c - r, c + r)]$

function UPPERBOUND($fn$, $r$, $n$) 

   for $k \leftarrow 0$ to $n - 2$ do 

      $x \leftarrow \max |fn[n - k]|$

      $fn[n - k - 1] \leftarrow (fn[n - k - 1] - r \cdot x, fn[n - k - 1] + r \cdot x)$

   end for

   return $\max |fn[1]|$

35 end function
Algorithm 7 Evaluation of \((\mathcal{L}_\beta f)_j^j f_j - (\mathcal{L}_\beta f)_{j-1} f_{j-1}\) and its derivatives on intervals of partition \((p, cp)\).

1 function NBN\((f, p, cp, pow, deg, ln)\)
   \(//\) pow = \((-2\beta - d, -2\beta - d - 1, \ldots, -2\beta - d - (n - 1))\)
   \(//\) Local Variables:
2   \(rp: int[n]; rp[j] \leftarrow rc[k, 2]\), where \(k = k(j)\) is such that \(j \in rc[k, 1]\).
3   \(yp: arb[2^{-lnp}]\)
4   \(ycp: arb[2^{-lnp}, n]\)
5   \(g_0(\cdot), g_1(\cdot):\) polynomials of degree \(deg\)
6   \(h(\cdot):\) polynomial of degree 2 \(deg\)
7   \(lin(\cdot):\) linear function (polynomial of degree 1)

8 \(\text{tops} \leftarrow 0\)
9 \(\text{for } k_0 \leftarrow 1 \text{ to } N \text{ do}\)
10 \(\quad (g_0, \text{lin}) \leftarrow \text{SAN}(f_{k_0}, sm[k_0, :], deg)\)
11 \(\quad yp[k_2] \leftarrow (\text{lin}(p[k_2]))^{\text{pow}[n]}, k_2 = 1, \ldots, 2^{-lnp}\)
12 \(\quad ycp[k_2, k_1] \leftarrow (\text{lin}(cp[k_2]))^{\text{pow}[k_1]}, k_2 = 1, \ldots, 2^{-lnp}, k_1 = 1, \ldots, n - 1\)
13 \(\quad \text{tops}[k_3, k_2, 1] \leftarrow \text{rmm}(k_3, cc[k_0]) \cdot g_0(cp[k_2]) \cdot ycp[k_2, 1], k_2 = 1 \ldots 2^{-lnp}, k_3 = 1 \ldots u\)
14 \(\quad g_1 \leftarrow g' \cdot \text{lin} - (2\beta + d) \cdot \text{sm}[k_0, 3] \cdot g_0\)
15 \(\quad \text{for } k_3 \leftarrow 1 \text{ to } u \text{ do}\)
16 \(\quad \quad \text{if } \text{rmm}[k_3, cc[k_0]] \text{ then}\)
17 \(\quad \quad \quad h \leftarrow g_1 \cdot f_{k_3} - f'_{k_3} \cdot \text{lin} \cdot g_0\)
18 \(\quad \quad \quad \text{for } k_1 \leftarrow 2 \text{ to } n - 1 \text{ do}\)
19 \(\quad \quad \quad \quad \quad \text{for } k_2 \leftarrow 1 \text{ to } 2^{-lnp} \text{ do}\)
20 \(\quad \quad \quad \quad \quad \quad \text{tops}[k_3, k_2, k_1] \leftarrow \text{tops}[k_3, k_2, k_1] + h(cp[k_2]) \cdot ycp[k_2, k_1]\)
21 \(\quad \quad \quad \quad \end{for}\)
22 \(\quad \quad \quad \text{h} \leftarrow \text{sm}[k_0, 3] \cdot \text{pow}[k_1] \cdot h + h' \cdot \text{lin}\)
23 \(\quad \quad \end{for}\)
24 \(\quad \text{for } k_2 \leftarrow 1 \text{ to } 2^{-lnp} \text{ do}\)
25 \(\quad \quad \quad \text{tops}[k_3, k_2, k_1] \leftarrow \text{tops}[k_3, k_2, k_1] + h(p[k_2]) \cdot yp[k_2]\)
26 \(\quad \quad \end{for}\)
27 \(\quad \end{if}\)
28 \(\quad \text{end for}\)
29 \(\text{end for}\)
30 \(\text{return tops}\)
31 end function
32 function SAN\((f, a, b, c, d, deg)\)
\(/*\) Returns the numerator of the function \(f(f(ax + b)/(cx + d))\). */
33 \(\text{q}[\text{deg} + 1] \leftarrow \text{Coefficients of } f\)
34 \(\text{lin}(x) \leftarrow cx + d\)
35 \(g_0(x) \leftarrow \sum_{j=0}^{\text{deg}} \text{q}[j] \cdot (ax + b)^j \cdot (cx + d)^{\text{deg} - j}\)
36 \(\text{return } (g, \text{lin})\)
37 end function
Algorithm 8 Checking that approximating polynomials $f_j, j = 1, \ldots, u$ are all positive.

1 procedure ALLPOSITIVE($f, lnp, prec$)
2 for $j \leftarrow 1$ to $u$ do
3 if $f_j(0) < 0$ then
4 $f_j \leftarrow -f_j$
5 end if
6 end for
7 $sw \leftarrow$ YES
8 $lnp \leftarrow lnp + 1$
9 while $sw = YES$ and $lnp > -20$ do
10 $lnp \leftarrow lnp - 1$
11 $p \leftarrow$ Partition of the interval $[0, 1]$ into $2^{-lnp}$ intervals.
12 $sw \leftarrow$ NO
13 for $j \leftarrow 1$ to $u$ do
14 for $k \leftarrow 1$ to $2^{-lnp}$ do
15 $sw \leftarrow 0 \in f_j(p[k])$? /* Relying on the ball arithmetic from the arb library */
16 end for
17 end for
18 end while
19 return $sw, lnp$
20 end procedure
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