Reflection symmetries of isolated self-consistent stellar systems

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ABSTRACT

Isolated, steady-state galaxies correspond to equilibrium solutions of the Poisson–Vlasov system. We show that (i) all galaxies with a distribution function (DF) depending on energy alone \( f(E) \) must be spherically symmetric, and (ii) all axisymmetric galaxies with a DF depending on energy and the angular momentum component parallel to the symmetry axis \( f(E, L_z) \) must also be reflection-symmetric about the plane \( z = 0 \). The former result is known, whilst the latter result is new. These results are subsumed into the Symmetry Theorem, which specifies how the symmetries of the DF in configuration or velocity space can control the planes of reflection symmetries of the ensuing stellar system.

Key words: galaxies: kinematics and dynamics – galaxies: structure.

1 INTRODUCTION

The shapes of isolated, steady-state stellar systems are controlled by gravity. In such systems, the phase-space distribution function (DF) satisfies the collisionless Boltzmann equation (CBE) involving the Newtonian potential, which is also coupled to the density (i.e. an integrated DF) through Poisson’s equation. This imposes severe restrictions on the possible intrinsic shapes of systems. In fact, all known equilibrium models of stellar systems are highly symmetric. Spherically symmetric models were first studied by J. H. Jeans and A. S. Eddington nearly a century ago. Algorithms to find both isotropic and anisotropic DFs for spherical galaxies are now well established (e.g. Eddington 1916; Osipkov 1979; Merritt 1983; Dejonghe 1986; Evans & An 2006). Methods to build axisymmetric models with DFs depending on the two classical integrals (energy \( E \) and angular momentum component parallel to the symmetry axis \( L_z \)) are also known (Lynden-Bell 1962; Hunter & Qian 1993), together with some exact solutions (Toomre 1982; Evans 1993). Both spherically symmetric and axisymmetric models contain an infinite number of reflectional planes of symmetry. There are a very few triaxial models with DFs known (Vandervoort 1980; Hunter de Zeeuw 1992; Sanders & Evans 2015). However, even triaxial models have three reflectional symmetries in the principal planes (the \( D_{2h} \) point group). This led Tremaine (1993) to raise the question as to whether equilibrium models of stellar systems with still fewer symmetries can exist.

For fluid dynamical equilibria, the principal result in this area was established in the early years of the last century (Lichtenstein 1928, see also Grossman 1996 for a simplified treatment). Lichtenstein studied self-gravitating, barotropic fluids and showed that if there is a constant vector field \( \hat{\mathbf{k}} \) such that the velocity field \( \mathbf{v} \) is stratified on the set of planes perpendicular to \( \hat{\mathbf{k}} \) (viz. \( \mathbf{v} \cdot \hat{\mathbf{k}} = 0 \) everywhere), then the figure has a symmetry plane perpendicular to \( \hat{\mathbf{k}} \). For a static fluid, there is a symmetry plane perpendicular to every axis. Hence, all isolated, static, self-gravitating, barotropic fluids must be spherical. The extension of this result to stellar dynamics is given in Binney & Tremaine (2008). These authors pointed out that stellar dynamical models with ergodic DFs \( f(E) \) satisfy the self-same equations – Poisson’s equation, hydrostatic equilibrium and density constant on equipotentials – as barotropic self-gravitating fluids. Thus, any isolated, static, self-gravitating, ergodic stellar system must also be spherically symmetric.

In this paper, we investigate whether one can generalize such arguments to constrain the shape of relaxed stellar systems. In Section 2, we first present the idealized one-dimensional (1D) case, which reduces the fundamental mathematical principle to the level of elementary calculus. In Section 3, we introduce some important mathematical results by Gidas, Ni & Nirenberg (1979, 1981), which essentially develop the 1D analytical idea for higher dimensions. As an illustration, we discuss how to recover the known results on Lichtenstein’s theorem and ergodic systems using these results (cf. Perez & Aly 1996; Ciotti 2001). We then establish new results on the reflection symmetry of systems built by the axisymmetric two-integral DFs, \( f(E, L_z) \) (Section 4), as well as by DFs satisfying certain sets of the symmetry conditions (Section 5).

2 ONE-DIMENSIONAL CASE

Let us consider the 1D stellar system in a steady-state equilibrium with the potential \( \Phi(x) \). The system is described by the DF \( F(x, v) \), which is a solution to the CBE, sometimes also called the Vlasov equation:

\[
\frac{\partial f}{\partial t} + v \frac{\partial F}{\partial x} - \frac{\partial \Phi}{\partial x} \frac{\partial F}{\partial v} = 0,
\]

whose general solution is found via the method of characteristics to be \( F(x, v) = f(v^2/2 + \Phi) \), where \( f(E) \) is an arbitrary non-negative...
function of $E$. In other words, any distribution satisfying the 1D time-independent CBE must be constant on the hypersurfaces of constant energy $E = v^2/2 + \Phi(x)$ (Jeans’ Theorem). The density $\rho$ of the system follows integrating the DF over the momentum (i.e. velocity) space:

$$\rho = \int_{-\infty}^{\infty} dx \frac{F(x, v)}{\sqrt{2\pi \sigma_x} \sigma_v} = \sqrt{2} \int_0^\infty \frac{f(E) dE}{\sqrt{E - \Phi}}$$

(2)

which depends on the position only through the potential. That is to say, the local density of the 1D steady-state stellar system is constant on locations with an equal value of the potential, and so the density may be considered as a function of the potential, $\rho = \rho(\Phi)$. If the potential $\Phi$ is generated self-consistently by the density field $\rho(\Phi)$, then it must satisfy the 1D Poisson equation $d^2\Phi/dx^2 = 2G\rho(\Phi)$, which results in an autonomous (i.e. not involving the independent variable $x$ explicitly) second-order ordinary differential equation in $\Phi$.

We now show that if there is a critical position in a 1D potential, then the potential and the density must be reflection-symmetric with respect to the critical point. This result has been established before by Schulz et al. (2013), who, however, assumed the existence of a critical point in the potential implicitly (in fact, it is possible to have a solution $\Phi$ that is strictly monotonic everywhere). Here, we rederive the result somewhat more rigorously. The proof may be constructed via solving Poisson’s equation formally for the solution, which is achieved by reducing the degree of the differential equation. In particular, if $N(\Phi)$ is the antiderivative of $4G\rho$ considered as a function of $\Phi$, that is, $dN(\Phi)/d\Phi = 4G\rho$, then Poisson’s equation implies

$$\frac{d}{dx} \left[ \frac{d\Phi}{dx} \right]^2 - N(\Phi) = 2 \left( \frac{d^2\Phi}{dx^2} - 2G\rho \right) \frac{d\Phi}{dx} = 0.$$ 

(3)

In other words, $(\Phi')^2 - N(\Phi)$ is constant for all positions (within a connected interval over which $\Phi$ is finite). Next, suppose that there exists $x_0$ such that $\Phi'(x_0) = 0$. It follows from the constancy of $(\Phi')^2 - N(\Phi)$ that the potential at any location $x$ satisfies

$$\left( \frac{d\Phi}{dx} \right)^2 = N(\Phi) - N(\Phi_0) = 4G \int_{\Phi_0}^\Phi \rho(\Phi) = D(\Phi) \geq 0,$$

(4)

where $\Phi_0 = \Phi(x_0)$, which should be the global minimum of $\Phi$ (i.e. $\Phi \geq \Phi_0$), provided that $\rho \geq 0$ everywhere (conversely, if $\rho \leq 0$ everywhere, then $\Phi \leq \Phi_0$) in order for $D(\Phi)$ to be non-negative. Equation (4) also indicates that $|dx/d\Phi| = D^{-1/2}$, and so it follows (assuming $\rho \geq 0$ and $\Phi \geq \Phi_0$) that

$$|x - x_0| = \int_{\Phi_0}^{\Phi(x)} \frac{d\Phi}{\sqrt{D(\Phi)}}.$$ 

(5)

Unless $\rho(\Phi) = 0$ in an open neighbourhood of $\Phi_0 = \Phi(x_0)$, $D(\Phi)$ is strictly positive and non-decreasing for $\Phi > \Phi_0$. It follows that the right-hand side of equation (5) is also a monotonically increasing function of $\Phi \geq \Phi_0$. Inverting equation (5) for $\Phi(x)$ as a function of $x$ provides the solution to Poisson’s equation with the initial condition that $\Phi'(x_0) = 0$ and $\Phi(x_0) = \Phi_0$. Equation (5) further suggests that $\Phi(x)$, given the initial condition, depends only on the distance $|x - x_0|$ to the critical point $x_0$, which is to say $\Phi(x)$ is symmetric under the reflection about $x = x_0$.

3 THE GIDAS–NI–NIRENBERG THEOREMS

Although the 1D case elucidates the basic principle, applying this analytic result to 3D problems requires considerable mathematical finesse. Instead, we note well-established results that generalize the discussion in the preceding section for multidimensional spaces.

The results by Gidas et al. (1979, 1981) are particularly notable in this respect. Whilst their results are celebrated amongst those who study elliptic partial differential equations, they appear not widely known in the astrophysical community. The most basic result of Gidas et al. (1979, henceforth GNN) is as follows.

**Theorem 1.** (GNN1). If $\psi(x)$ is a positive $C^1$-function on a closed ball of radius $R$ around the origin in $\mathbb{R}^n$ satisfying

$$\nabla^2 \psi + f(\psi) = 0 \quad (||x|| < R),$$

$$\psi(x) = 0 \quad (||x|| = R),$$

where $f(\psi)$ is a $C^1$-function of $\psi$, then $\psi$ is radially symmetric and $\partial \psi/\partial r < 0$ for $0 < r < ||x|| < R$.

Extending to the whole space, we have the following theorem.

**Theorem 2.** (GNN4). Let $\psi(x)$ be a positive $C^2$-solution of

$$\nabla^2 \psi + f(\psi) = 0 \quad \text{in } \mathbb{R}^n$$

with a $C^1$-function $f(\psi)$. If $\psi$ admits the asymptotic expansion (with a fixed $m > 0$) up to a translation,

$$\psi = a_0 ||x||^m + \sum_{i,j} a_{ij} x_i x_j + o \left( \frac{1}{||x||^{m+2}} \right), \quad (||x|| \to \infty),$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\psi$ is radially symmetric and $\partial \psi/\partial r < 0$ (where $r$ is the radial coordinate).

This further generalizes to the following theorem.

**Theorem 3.** (GNN4'). If $\psi$ is a positive $C^2$-solution of

$$\nabla^2 \psi + F(x_2, \ldots, x_n; \psi) = 0 \quad \text{in } \mathbb{R}^n$$

with a continuous $\partial F/\partial \psi$, and expressible in the same asymptotic series as the preceding theorem, then $\psi$ is symmetric under $\psi(x_1, x_2, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_n)$ and $\partial \psi/\partial x_1 < 0$ for $x_1 > 0$.

The theorems proved in GNN are of greater generality, although we have here specialized to the specific case of Poisson’s equation. The proofs are examples of the so-called moving plane method, which relies on the maximum principle for the solution to some classes of the elliptic partial differential equations. An accessible introduction for readers interested in the mathematical details of the GNN theorems is provided by the book of Fraenkel (2000). In this paper, we will not attempt to reproduce these proofs, but accept them as established facts to be applied. Nevertheless, we note that the differentiability and the asymptotic behaviour conditions appearing in the statements of the theorems (or the version of theorems with appropriately relaxed conditions) are typically satisfied by potentials due to physical models (characterized by finite spatial extents or a finite total mass with a continuous and bounded force field).

3.1 Lichtenstein’s theorem in fluid mechanics

Partly for pedagogical reasons, here we outline how the GNN theorems lead to Lichtenstein’s theorem in fluid mechanics. Lichtenstein’s original paper (which predates GNN by about a half century) is both lengthy and somewhat inaccessible. Textbooks normally content themselves with either stating the theorem (Tassoul 1978; Binney & Tremaine 2008) or giving a simplified proof for homogeneous fluids (Grossman 1996). The modern proof (see e.g. Lindblom 1992, section 4) employs similar techniques to GNN.
(based on the maximum principle), but still obscures the connection to the more general results.

Henceforth, we restrict ourselves to 3D space. In addition, we also revert to the physics sign convention for the potential: namely, that the acceleration is directed along the downhill direction of the potential, \( \mathbf{g} = -\nabla \Phi \), which results in the Poisson equation of \( \nabla^2 \Phi = 4\pi G \rho \). For a non-negative density \( \rho \geq 0 \), the usual zero-point limit \( \lim_{\rho \to 0} \Phi(r) = 0 \) then implies that the potential is actually negative. Lichtenstein’s theorem is as follows.

**Theorem 4. (Lichtenstein).** A time-independent barotropic fluid solution to the coupled Euler–Poisson equation

\[
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \rho \nabla \Phi = 0; \quad \nabla^2 \Phi = 4\pi G \rho
\]

with a stratified velocity field such that \( \mathbf{v} \cdot \hat{z} = 0 \) (where \( \hat{z} \) is a fixed unit vector) is symmetric with respect to the reflection about a plane perpendicular to \( \hat{z} \). Furthermore, if the fluid is in a static equilibrium, the system must also be spherically symmetric about the centre of mass (and reflection-symmetric with respect to any plane passing the centre).

Although this result predates the GNN theorems, Lichtenstein’s theorem is essentially a corollary. Following the barotropic assumption, it is possible to define ‘specific enthalpy’:

\[
h(p) = \int_0^p \frac{dp}{\rho(p)}, \quad \nabla h = \frac{\nabla p}{\rho}.
\]

Then, the Euler equation is reducible to

\[
(\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla (h + \Phi) = 0,
\]

and the dot product to the fixed vector \( \hat{z} \) in the Cartesian \( z \)-coordinate direction results in \( (\partial/\partial z)(h + \Phi) = 0 \); that is, \( h + \Phi = C(x, y) \) is independent of the coordinate \( z \), where \( C(x, y) \) is an arbitrary function of the coordinate components \( (x, y) \) on the plane perpendicular to \( \hat{z} \). Applying the Laplacian on \( h + \Phi = C \) and using Poisson’s equation then yields

\[
\nabla^2 h + 4\pi G \rho(p) - \nabla^2 C(x, y) = 0.
\]

Since \( \rho \geq 0 \), the enthalpy is a positive increasing function of the pressure. Thus, \( h(p) \) is in principle invertible for the pressure as a function of the enthalpy, \( p = p(h) \), and the barotropic density can also be considered as a function of the enthalpy, \( p(h) \). Then, equation (8) is in the form of \( \nabla^2 h + f_1(x, y; h) = 0 \), and the GNN theorem implies that the solution \( h(r) \) is reflection-symmetric, \( h(x, y, -z) = h(x, y, z) \), with respect to a properly chosen mid-plane. Since the density and the pressure functions are of the enthalpy, they are also symmetric under the same reflection. Alternatively, Poisson’s equation directly indicates \( \nabla^2 \Phi = 4\pi G \rho(p) \), but \( p = p(h) = p(C(x, y) - \Phi) \); that is, \( \nabla^2 \Phi + f_2(x, y; \psi) = 0 \) with \( f_2 = 4\pi G p(C + \psi) \), where \( \psi = -\Phi \). The reflection symmetry of \( \Phi \) is then the result of GNN, whilst those of \( h, p, \rho \) follow \( C = \Phi \).

Spherical symmetry is an immediate corollary to the reflection symmetry, since \( \mathbf{v} \cdot \hat{z} = 0 \) for any \( \hat{z} \) in the static system. However, the barotropic assumption in a static equilibrium is actually redundant as the barotropy automatically follows the static Euler equation; that is, \( \nabla \times (\rho^{-1} \nabla p + \nabla \Phi) = -\rho^{-1} (\nabla \rho \times \nabla p) = 0 \Rightarrow \nabla \rho \times \nabla p \) (or \( \nabla \rho = 0 \)). In fact, \( \nabla \times (\nabla p + \rho \nabla \Phi) = \nabla \rho \times \nabla \Phi = 0 \) by itself implies \( \nabla \rho \parallel \nabla \rho \) and \( \rho = \rho(\Phi) \); that is, the isolated fluid system in a self-gravitating static equilibrium must be spherically symmetric, thanks to GNN.

**3.2 Ergodic distribution functions in stellar dynamics**

The GNN theorems also generalize the symmetry theorem of the 1D system proved in Section 2. In the 1D case, the general solution to the time-independent CBE is an arbitrary function of the specific energy. The Jeans theorem generalizes this for a 3D system; that is, if \((I_1, I_2, I_3)\) is the set of three independent isolating integrals of motion (admitted by the one-particle Hamiltonian), any DF (composed of \( N \) identical particles governed by the same one-particle Hamiltonian) in equilibrium must be constant over the joint level surfaces of \((I_1, I_2, I_3)\) in phase space. This is usually stated as the DF is a function of the integrals, \( F(r, \mathbf{v}) = f(I_1, I_2, I_3) \).

Since any time-independent Hamiltonian is itself an integral of motion, the simplest DF in equilibrium is of the form \( f(\mathcal{H}) \). The Hamiltonian of a free particle in a fixed potential \( \Phi \) is \( \mathcal{H} = ||\mathbf{v}||^2/2 + \Phi(r) \). As the Hamiltonian is isotropic (depending only on the magnitude \( v = ||\mathbf{v}|| \)), the local density resulting from the DF of \( f(\mathcal{H}) \) is

\[
\rho = 4\pi \int_0^\infty dv^3 F(r, \mathbf{v}) = 2^{5/2} \pi \int_0^\infty d\mathcal{H} \sqrt{\mathcal{H} - \Phi} f(\mathcal{H}), \tag{9}
\]

which again depends on the position only through the potential, \( \rho = \rho(\Phi) \). Therefore, following GNN, we come to the following conclusion.

**Theorem 5.** The stellar dynamical system specified by an ergodic DF, \( F(r, \mathbf{v}) = f(||\mathbf{v}||^2/2 + \Phi) \), must be spherically symmetric about the centre of mass, provided that the total mass of the system is finite and the potential \( \Phi \) has been generated self-consistently without any external potential.

Binney & Tremaine (2008, box 4.1) established this result based on Lichtenstein’s theorem, after showing that an ergodic DF results in the equation of hydrostatic equilibrium. However, the GNN theorems directly imply the sphericity of an ergodic stellar dynamical system, as has already been noted in astrophysical literatures (e.g. Perez & Aly 1996; Ciotti 2001; Rein & Guo 2003).

**4 AXISYMMETRIC STELLAR SYSTEMS**

We now apply the results of GNN to axisymmetric stellar systems to obtain new results. Note that the stress tensor in such systems is anisotropic, and there is no possibility of recourse to fluid mechanics and Lichtenstein’s theorem, as the scalar pressure is not defined. The situation is rectified, thanks to the results of GNN, which are still applicable.

If the potential is axisymmetric as in \( \Phi(R, z) \), where \((x, y, z)\) are the rectangular coordinates and \( R^2 = x^2 + y^2 \), then the axial component of the angular momentum, \( L_z = L \cdot \hat{z} = xv_z - yv_x \) (where \( L = r \times \mathbf{v} \) is the specific angular momentum and \( \hat{z} \) is the unit vector in the Cartesian \( z \)-direction), is also an integral of motion, and so the two-integral DF \( F(E, L_z) \) satisfies the CBE. Although, in most studies, an axisymmetric two-integral system is usually assumed to be symmetric about the reflection with respect to the mid-plane, the assumption appears not to have been explicitly proven previously.

**Theorem 6.** Consider a stellar dynamical system in an axisymmetric (about the \( z \)-axis) potential \( \Phi \). Suppose the system is specified by the two-integral DF, \( F(r, \mathbf{v}) = f(E, L_z) \), where \( E \) is the specific energy and \( L_z \) is the \( z \)-component of the specific angular momentum. If the total mass of the system is finite, and the potential \( \Phi \) has been generated self-consistently without any external potential, the system must be symmetric with respect to the reflection about the
plane passing through the centre of mass and perpendicular to the z-axis.

In terms of the velocity component projected on to the orthonormal frame for the cylindrical coordinates (R, ϕ, z), that is, (vR, vϕ, vz) = (R, Rϕ, z), we find E = (vR2 + vϕ2 + vz2)/2 + Φ and Lz = R2ϕ = R0Rϕ. Then, the density due to the DF f(E, Lz) is obtained by the integral (here vR2 = vR2 + vz2)

\[ \rho = 2\pi \int_{v_R \geq 0} F(r, v) \, dv_R \, dv_\phi \, dv_z = \frac{2\pi}{R} \int_{L_z \geq 2R^2(E-\Phi)} dE \, dL_z \, f(E, L_z), \]

(10)

which is axisymmetric (i.e. independent of the azimuth ϕ). Equation (10) indicates that the z-dependence of the density is only through the potential Φ, that is, \( \rho = \rho(R, \Phi) \). Then, Poisson’s equation for the self-consistent system leads to the partial differential equation on \( \psi = -\Phi \) as in \( \nabla^2 \psi + 4\pi G \rho(\sqrt{x^2 + y^2}, -\psi) = 0 \), which is the form considered in the theorem of GNN\(^4\). Therefore, the potential \( \Phi \) is reflection-symmetric with respect to a plane perpendicular to the z-axis (which may be considered as the \( z = 0 \) mid-plane without loss of generality), and thus the conclusion of Theorem 6 holds. In particular, the symmetry of the density immediately follows the symmetry of the potential, \( \rho(R, -z) = \rho(R, \Phi(R, z)) \) (and so the centre of mass consequently lies on the mid-plane), whilst the symmetry of the velocity distribution is a simple consequence of the symmetry of both \( E \) and \( L_z \). So if the two-integral DF exists, it is reflection-symmetric.

5 SYMMETRIC VELOCITY DISTRIBUTIONS

Let us rearrange the CBE in rectangular coordinates as

\[ \frac{\partial F}{\partial x} + v_x \frac{\partial F}{\partial y} - \frac{\partial \Phi}{\partial x} \frac{\partial F}{\partial v_x} - \frac{\partial F}{\partial y} \frac{\partial F}{\partial v_y} = \frac{\partial F}{\partial z} \frac{\partial F}{\partial v_z}, \]

and consider the DF in equilibrium with both left- and right-hand sides vanishing separately. The general solution for the vanishing right-hand side is found to be \( F = f(v_x^2/2 + \Phi; x, y, v_y, v_z) \) via the method of characteristics (e.g. Garabedian 1998) – here \( v_x^2/2 + \Phi = H - (v_y^2 + v_z^2)/2 \) and so the DF is also of the form \( F = f(H; x, y; v_x = p_x, v_y, v_z) \) albeit with a different function \( f \). The density resulting from this DF is found from the integral (where \( E_z = v_z^2/2 + \Phi \))

\[ \rho = \sqrt{\frac{2}{\pi}} \int_{E_z \geq 0} dE_z \, dv_x \, dv_y \, dv_z \, \frac{f(E_z; x, y; v_x, v_y)}{\sqrt{E_z - \Phi}}, \]

(12)

whose \( z \)-dependence is only through \( \Phi \), namely, \( \rho = \rho(\Phi; x, y) \). This is again the form of the source term for Poisson’s equation assumed for the GNN theorems, and so the potential \( \Phi \) and the density \( \rho \) are reflection-symmetric with respect to a plane perpendicular to the \( z \)-axis. Since the DF is also an even function of \( v_x \), the velocity distribution is invariant under the same reflection too.

How can the DF satisfy the Cartesian \( z \) part of the CBE separately from the \( (x, y) \) part? Let us observe that, with \( F = f(v_x^2/2 + \Phi; x, y, v_y, v_z) \), equation (11) divides the even and odd parts on \( v_x \) into two opposite sides. In general, an arbitrary DF may be decomposed into \( F(x, y, z; v_x, v_y, v_z) = F_+ + F_\mp \), where \( 2F_\pm = F(x, y, z; v_x, v_y, v_z) \pm F(x, y, z; v_x, v_y, -v_z) \). The CBE for the even-odd decomposed DF is then itself reassembled separately for the even and odd parts, which results in

\[ \frac{\partial \Phi}{\partial z} \frac{\partial F_\pm}{\partial v_z} - \frac{\partial F_\pm}{\partial z} \frac{\partial \Phi}{\partial v_z} = \left( v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \frac{\partial \Phi}{\partial v_x} \frac{\partial \Phi}{\partial y} \right) F_\pm. \]

(13)

Hence, if \( F_\pm = 0 \), then \( F = f(v_x^2/2 + \Phi; x, y, v_y, v_z) \).

Theorem 7. The finite-mass self-consistent stellar dynamic system specified by the steady-state DF such that \( F(x, y, z; v_x, v_y, v_z) \) must possess a plane of reflection symmetry perpendicular to the \( z \)-axis – i.e. there exists \( z_0 \) such that \( \Phi(x, y, z) = \Phi(x, y, z) \) and \( \rho(x, y, 2z_0 - z) = \rho(x, y, z) \). The DF is also expressible as \( F = f(v_x^2/2 + \Phi; x, y; v_y, v_z) \) and so the whole system is also invariant under the reflection about the \( z = z_0 \) plane (i.e. \( z \rightarrow 2z_0 - z \) and \( v_z \rightarrow -v_z \)).

This actually supersedes Theorem 6, for \( f(E, L_z) \) is invariant under \( v_z \rightarrow -v_z \), given that \( L_z = v_y - y_0 \) is independent of \( v_z \) and \( E \). Thus, \( (v_z^2 + v_y^2)/2 + \Phi \) is an even function of \( v_z \).

In fact, similar even–odd splits of the CBE can be applied for the DFs with alternative sets of symmetries. In particular, the most general result is given by the following theorem.

Theorem 8. (The Symmetry Theorem). The same conclusion as Theorem 7 holds if the DF is subject to any one of the alternative symmetries:

(i) \( F(x, y, z; v_x, v_y, v_z) = F(x, y, z; v_x, v_y, v_z) \),
(ii) \( F(x, y, z; v_x, v_y, v_z) = F(x, y, z; v_x, v_y, v_z) \),
(iii) \( F(x, y, z; v_x, v_y, v_z) = F(x, y, z; v_x, v_y, v_z) \),
(iv) \( F(x, y, z; v_x, v_y, v_z) = F(x, y, z; v_x, v_y, v_z) \),

where \( x_0, y_0 \) and \( z_0 \) are fixed constants.

Here, the even–odd splits based on the last two symmetry assumptions actually require the accompanying symmetry of the potential so \( \Phi(x, y, z) = \Phi(x, y, z) \) or \( \Phi(2x_0 - x, 2y_0 - y, z) = \Phi(x, y, z) \). This, however, is a natural consequence of the self-consistency condition (once the symmetry of the density is established following the integration of the DF over the momentum space) and redundant.

Condition 3 appears to be the same as the conclusion, but the conclusion is actually more restrictive. In fact, condition 3 does not describe the proper reflection symmetry of the DF with respect to the \( z = z_0 \) plane since it does not involve the transformation of the velocity field (i.e. the true reflection symmetry follows from invariance under \( z \rightarrow 2z_0 - z \) and \( v_z \rightarrow -v_z \)). On the other hand, the conclusion of the theorem implies the reflection symmetry of the density, potential and DF, plus the DF being an even function of \( v_z \).

Similarly, Condition 4 is not in fact the true antipodal symmetry about the axis defined by \( x = x_0 \) and \( y = y_0 \) (i.e. the invariance under the 180°-rotation around the same axis). Rather, the condition indicates that the density and the potential is antipodally symmetric, whilst the velocity distributions in the axially antipodal points are invariant under the rigid translation (but not necessarily under 180°-rotation). Consequently, neither the DF with rectangular reflection symmetry nor the one with axial rotational symmetry satisfies Condition 4, unless some additional conditions are imposed on the behaviour of the velocity distributions. One such condition for the axially symmetric DF would be the isotropy within the \( v_x \)-plane as in \( F = f(R, v_y; v_x^2 + v_z^2, v_z) \), whereas the DF given by \( F = f(x^2, y^2, z; v_x^2, v_y^2, v_z) \) is symmetric under the individual reflections, \( x \rightarrow -x \) and \( y \rightarrow -y \), and also satisfies Condition 4.
5.1 Properties of systems satisfying the symmetry theorem

Since the DF in the form of \( F = f(v_x^2 + \Phi; x, y; v_x, v_y) \) is an even function of \( v_x \), any velocity moment with an odd power to \( v_x \) for this system vanishes. According to An & Evans (2016, corollary 8), this implies that the potential is separable like \( \Phi(x, y, z) = \Phi_1(x, y) + \Phi_2(z) \), unless \( ((v_x^2) - (v_y^2))((v_x^2) - (v_y^2)) = (v_x, v_y)^2 \). However, such separable potentials give rise to self-consistent density profiles like \( \rho = \rho_1(x, y) + \rho_2(z) \), which cannot be of a finite total mass (except for \( \rho = 0 \)).

Consequently, the DFs of Theorem 7 or 8 must also be constrained so that \( ((v_x^2) - (v_y^2))((v_x^2) - (v_y^2)) = (v_x, v_y)^2 \). Since \( (v_x, v_y) = (v_x, v_y) = 0 \), the characteristic polynomial of the matrix resulting from the stress tensor is \( ((v_x^2) - \lambda)((v_y^2) - \lambda)(v_x^2 - \lambda) - (v_x, v_y)^2) = 0 \); that is, \( \lambda = (v_x^2) \) is one of the eigenvalues, whereas the constraints \( (v_x^2) - (v_y^2) = (v_x^2) = (v_x, v_y)^2 \) are equivalent to \( v_x^2 \) being one of the two remaining eigenvalues. In other words, the velocity ellipsoids must be spheroidal with its unequal axis aligned within the \( x-y \) plane or spherical everywhere. The two-integral distribution \( F = f(E, L_z) \) is an example of a DF satisfying such a constraint.

There are also restrictions on the potential. In particular, the vanishing left-hand side of equation (11) results in the additional partial differential equation

\[
v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} = \frac{\partial \Phi}{\partial x} \frac{\partial F}{\partial v_x} + \frac{\partial \Phi}{\partial y} \frac{\partial F}{\partial v_y}.
\]

(14)

Next, the \( z \)-derivative of this with the substitution \( v_z(\partial F/\partial z) = (\partial \Phi/\partial z)(\partial F/\partial v_z) \) results in

\[
\frac{\partial^2 \Phi}{\partial z \partial v_z} \left( v_x \frac{\partial F}{\partial v_x} - v_y \frac{\partial F}{\partial v_y} - \frac{\partial \Phi}{\partial v_z} \right) = \frac{\partial^2 \Phi}{\partial v_y \partial v_z} \left( v_x \frac{\partial F}{\partial v_x} - v_y \frac{\partial F}{\partial v_y} - \frac{\partial \Phi}{\partial v_z} \right).
\]

(15)

However, the DF satisfying \( v_z(\partial F/\partial z) = (\partial \Phi/\partial z)(\partial F/\partial v_z) \) is also expressible as \( F = f(H; x, y; v_x, v_y) \), and the partial derivatives of \( F \) with respect to the phase-space coordinate \( (x, y, z; v_x, v_y) \) are related to the partial derivatives of \( f \) via

\[
\frac{\partial F}{\partial x} = \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial H} + \frac{\partial f}{\partial \Phi}, \quad \frac{\partial F}{\partial y} = \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial H} + \frac{\partial f}{\partial \Phi},
\]

\[
\frac{\partial F}{\partial z} = \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial H} + \frac{\partial f}{\partial \Phi}.
\]

(16)

Thus, equation (15) reduces to a differential equation on \( f \):

\[
\frac{\partial^2 \Phi}{\partial x \partial v_x} + \frac{\partial^2 \Phi}{\partial y \partial v_y} = 0.
\]

(17)

Here, the second-order derivatives on \( \Phi \) cannot be identically zero.\(^1\)

Then, the general solution of equation (17) for \( f \) at a fixed \( (H; x, y) \) follows the method of characteristics; that is, \( f = \tilde{f}(H; x, y; av_x + bv_y) \), where \( a \) and \( b \) should be functions of only \( (x, y) \) that satisfies

\[
a(x, y) \frac{\partial^2 \Phi}{\partial x \partial v_x} + b(x, y) \frac{\partial^2 \Phi}{\partial y \partial v_y} = 0.
\]

(18)

This is possible only if \( [\partial^2 \Phi/(\partial x \partial z)][\partial^2 \Phi/(\partial y \partial z)] \) is independent of \( z \). Equivalently, there exist functions \( a, b, c \) and \( d \) of \( (x, y) \) and

\[
G(x, y, z) \text{ such that the potential is restricted to be}
\]

\[
\frac{\partial \Phi}{\partial x} = c(x, y) + b(x, y)G(x, y, z),
\]

\[
\frac{\partial \Phi}{\partial y} = d(x, y) - a(x, y)G(x, y, z).
\]

(19)

In general, if equations (18) and (19) hold, we can introduce an orthonormal frame that is locally rotated by \( \psi \) relative to the rigid Cartesian frame such that the velocity components projected on to the frame are \( (\partial v_x + b v_y, \partial v_y - a v_x, v_z) \), where \( a = a/(a^2 + b^2)^{1/2} = \cos \psi \) and \( b = b/(a^2 + b^2)^{1/2} = \sin \psi \). This frame then corresponds to the set of eigenvectors for the stress tensor resulting from the DF of \( \tilde{f} = f(H; x, y; av_x + bv_y) \) in particular, \( (\partial v_x + b v_y)(\partial v_y - a v_x) = 0 \). Furthermore, the resulting velocity dispersions are also constrained such that \( (\partial v_x + b v_y)^2 \) is.

For example, with an axisymmetric potential \( \Phi = \Phi(R, z) \), where \( R = x^2 + y^2 \), we find that \( \partial \Phi/\partial x = (x/R)(\partial \Phi/\partial R) \) and \( \partial \Phi/\partial y = (y/R)(\partial \Phi/\partial R) \). Then \( a = -1 \) \( (x, y, 0, 0) \), and \( G = R^{-1}(\partial \Phi/\partial R) \), whereas \( [\partial^2 \Phi/(\partial x \partial z)][\partial^2 \Phi/(\partial y \partial z)] = x/y \). In fact, the general solution to equation (17) in this case is then \( f = f(H, L_z; x, y) \), where \( L_z = x \nu_x - y \nu_y \); and the CBE now reduces to

\[
v_z(\partial f/\partial x) = v_z(\partial f/\partial y) = 0,
\]

with general solution is \( f = f(x \nu_x - y \nu_y) \) at a fixed \( (H, L_z) \). In other words, the only possible DFs in equilibrium with an axisymmetric but non-separable – i.e. \( \Phi^2/(\partial R \partial z) \) – potential subject to one of the conditions in Symmetry Theorem 8 are the two-integral DFs of \( F = f(E, L_z) \) (including the ergodic DF as a special case). As is well known, the resulting velocity ellipsoids of such DFs are all axially aligned \( (\nu_x \nu_y = 0) \) with \( (v_x^2) = (v_y^2) \).

6 CONCLUSIONS

There is an ample body of work on the reflection symmetries of isolated, self-gravitating, equilibrium fluid systems – more colloquially, stars. It is already known that all isolated equilibrium stellar models have spatial symmetries (Lindblom 1992, and references therein). In Newtonian gravity, all non-rotating spherical models must be spherically symmetric. Also, all equilibrium stars must have a reflection symmetry perpendicular to the rotation axis of the star. Some of these results have even been extended to relativistic stars (see e.g. Lindblom & Masood-Ul-Alam 1994).

By contrast, the analogous problem in stellar dynamics has received scant attention. Isolated, self-gravitating, equilibrium stellar systems (or more colloquially, galaxies) must satisfy the Poisson and collisionless Boltzmann equations (also known as the Poisson-Vlasov system). This imposes strict requirements on the properties of the solutions, and hence on the shapes of stellar dynamical equilibria. It has long been known that isolated systems with ergodic DFs \( f(E) \) must be spherically symmetric. This was strongly hinted by early works such as Eddington (1915), whilst Binney & Tremaine (2008) show how it can be deduced from the famous theorem of Lichtenstein (1928) in fluid mechanics. Its relationship to the symmetry theorems of elliptic differential equations (Gidas et al. 1979, 1981) has been also noted by Perez & Aly (1996) and Ciotti (2001).

Here, we have used the Gidas et al. (1979) theorems to derive a number of new results on axisymmetric systems (Theorems 6–8). Specifically, we have shown that axisymmetric two-integral DFs \( f(E, L_z) \) must give rise to stellar systems with a plane of reflectional symmetry perpendicular to the symmetry axis (which can be taken without loss of generality as the plane \( z = 0 \)). Although this is often assumed, it does not appear to have been proved previously. We
have also stated a new theorem – the Symmetry Theorem – which gives a set of sufficient conditions on the DF to ensure that the model has an underlying plane of reflectional symmetry. Strictly speaking, our theorems presuppose the existence of a solution to the Poisson and collisionless Boltzmann equations, but if such a solution exists, it must have the specified symmetries.

There are further outstanding open problems in this area, of which we highlight two. First, although the Symmetry Theorem provides sufficient conditions for the existence of a plane of reflectional symmetry, it does not provide necessary conditions. This is illustrated by the DFs of spheroidal St¨ackel models (e.g. Dejonghe & de Zeeuw 1988). What are the necessary and sufficient conditions for the existence of a plane of reflectional symmetry? Secondly, as pointed out by Tremaine (1993), all known isolated, static, stellar dynamical equilibria are highly symmetric – either spherical, axisymmetric or triaxial. They have at least three planes of reflectional symmetry. Do static, isolated, stellar dynamical equilibria with fewer symmetries exist? We suspect that the answer to this question is negative, but a solid proof is lacking.

Of course, the problem of the shapes of equilibrium models is not just of academic interest. Many galaxies are clearly not in equilibrium, being still shaped by violent merging and accretion events or ongoing star formation. However, there are isolated, steady-state galaxies known, as well as galaxies in the voids of large-scale structure (e.g. Sulentic et al. 2006). Their equilibrium shapes will ultimately be controlled by the balance between gravity and motion (or more accurately, gradient of the momentum flux), rather than the effects of the environment. It is to these lonely souls that our work is directly applicable.

ACKNOWLEDGEMENTS

This paper originated from the discussion during the first author’s (JA’s) visit to the Institute of Astronomy (Cambridge), which was supported in part by the Science and Technology Facilities Council (STFC, UK)’s Consolidated Grant award to the University of Cambridge. Work by JA is supported by the Chinese Academy of Sciences (CAS) Fellowships for Young International Scientists (Grant No. 2009Y2AJ7) and also by grants from the National Science Foundation of China (NSFC), including ‘LAMOST and Galactic Dynamics (Grant No. 11390372)’. JLS thanks the STFC for financial support.

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