Raising NP Lower Bounds to Parallel NP Lower Bounds

Edith Hemaspaandra†  Lane A. Hemaspaandra‡  Jörg Rothe§

Abstract

A decade ago, a beautiful paper by Wagner [Wag87] developed a “toolkit” that in certain cases allows one to prove problems hard for parallel access to NP. However, the problems his toolkit applies to most directly are not overly natural. During the past year, problems that previously were known only to be NP-hard or coNP-hard have been shown to be hard even for the class of sets solvable via parallel access to NP. Many of these problems are longstanding and extremely natural, such as the Minimum Equivalent Expression problem [GJ79] (which was the original motivation for creating the polynomial hierarchy), the problem of determining the winner in the election system introduced by Lewis Carroll in 1876 [Dod76], and the problem of determining on which inputs heuristic algorithms perform well. In the present article, we survey this recent progress in raising lower bounds.

1 Introduction

Suppose you are given some nice, challenging problem and you are able to prove an NP-hardness lower bound for it. Unfortunately, this lower bound doesn’t match the problem’s known upper bound, which is, say, $P^{NP}$—that is, your problem can be solved by some deterministic polynomial-time Turing machine that is given access to an NP database. Now, when trying to close the gap between the upper and the lower bounds, the first thing you notice is that to solve your problem, you do not need the power of sequential access to the NP database; instead of asking your queries sequentially (which means that the answers to earlier questions may determine later questions—as in, for example, the popular game

---

*Supported in part by grants NSF-INT-9513368/DAAD-315-PRO-fo-ab, NSF-CCR-9322513, and a University of Rochester Bridging Fellowship.
†edith@bamboo.lemoine.edu. Dept. of Mathematics, Le Moyne College, Syracuse, NY 13214, USA.
‡lane@cs.rochester.edu. Dept. of Computer Science, University of Rochester, Rochester, NY 14627, USA.
§rothe@informatik.uni-jena.de. Inst. für Informatik, Friedrich-Schiller-Universität Jena, 07743 Jena, Germany. Work done in part while visiting Le Moyne College.

1We use hardness in the sense in which it is becoming most commonly used, namely, with regard to many-one polynomial-time ($\leq_{m}^{p}$) reductions. That is, a problem $A$ is said to be hard for a complexity class $C$ (for short, $C$-hard) if every problem $B \in C$ polynomial-time many-one reduces to $A$, i.e., there is a polynomial-time computable function $f$ such that, for each $x$, $x \in A \iff f(x) \in B$. If $A$ is hard for $C$ and $A \in C$, then $A$ is said to be $C$-complete.
Twenty Questions or as in binary search), it suffices to ask all queries in parallel (i.e., on a given input $x$, some P machine first computes a list of all questions that it wants answered, then it gives its NP database the entire list at once, and finally, given as the reply a list of yes/no answers corresponding to the listed questions, the P machine correctly determines whether or not $x$ is an instance belonging to the given problem).

The type of restricted access to NP that you have discovered that your problem can be solved with has been studied in the literature. In particular, Papadimitriou and Zachos [PZ83] introduced and discussed the complexity class $P_{\parallel}^\text{NP}$, which by definition contains exactly those problems that can be solved via parallel access to NP. Clearly, $NP \subseteq P_{\parallel}^\text{NP} \subseteq P_{\parallel}$. Hemaspaandra [Hem87] and Kübler, Schöning, and Wagner [KSW87] proved that $P_{\parallel}^\text{NP}$ is also exactly the class of problems that can be solved by $O(\log n)$ sequential Turing queries to NP. Wagner [Wag90] established about half a dozen other characterizations of $P_{\parallel}^\text{NP}$. Furthermore, it is known that if NP contains some problem that is hard for $P_{\parallel}^\text{NP}$, then the polynomial hierarchy collapses to NP. The class $P_{\parallel}^\text{NP}$ is also closely related to whether NP has sparse Turing-hard sets [Kad89], to whether feasible complexity classes can create Kolmogorov-random objects [HW91], and to many other topics (see, e.g., [LS95, Kre88]).

Can you completely close the gap between your problem’s upper and lower bounds by raising its NP-hardness lower bound to a $P_{\parallel}^\text{NP}$-hardness lower bound? Wagner [Wag87] provided a very useful toolkit with which one can seek to establish $P_{\parallel}^\text{NP}$-hardness results. In particular, he showed how to obtain certain “standardized” $P_{\parallel}^\text{NP}$-complete versions associated with some given NP-complete problem such as Clique, and he established a sufficient condition for $P_{\parallel}^\text{NP}$-hardness. Using this approach, he showed, for instance, that the following problem is $P_{\parallel}^\text{NP}$-hard (and, indeed, $P_{\parallel}^\text{NP}$-complete): Given a graph $G$, is the size of $G$’s maximum-sized cliques an odd integer? Unfortunately, such standard versions of $P_{\parallel}^\text{NP}$-complete problems are arguably not too natural (though in fairness we should mention that Wagner’s “equality” and “comparison” problems are potentially more arguably natural). During the past year, Wagner’s toolkit has been used to raise to $P_{\parallel}^\text{NP}$-hardness the lower bounds of a number of clearly natural, longstanding problems for which previously only NP-hardness (or coNP-hardness) lower bounds were known. In this article, we survey this progress.

In particular, we will focus on three problems, Carroll Winner, Minimum Equivalent Expression, and the problem of determining for which graphs the Minimum Degree Greedy

---

2For historical sticklers, we mention that this paragraph is slightly ahistorical. In fact, Papadimitriou and Zachos used as their definition what we above have described as the equivalent characterization of the class, and the papers mentioned actually linked their original definition to what is here described as the definition. Just to make things more confusing, the class itself nowadays is most often referred to in the technical literature as $\Theta_p^2$, rather than as $P_{\parallel}^\text{NP}$ or $P_{\parallel}^\text{NP}[O(\log n)]$, as it has become a “named” level of the polynomial hierarchy (see [Wag90]).
Algorithm finds an independent set whose size is within a factor of $r$ (with $r$ fixed) of the size of the maximum independent sets of the graph (formal definitions will be given later). These problems come from three different areas: political science, logic, and graph theory. We now briefly describe these problems and sketch their historical background.

In 1876, Charles L. Dodgson [Dod76] (who is usually referred to today by his pen name, Lewis Carroll) proposed an election system in which the winner is the candidate who with the fewest changes in voters’ preferences becomes a Condorcet winner—a candidate who beats all other candidates in pairwise majority-rule elections. Carroll Winner is the problem of whether a distinguished candidate wins a given election (specified by a list of candidates and a list of voters’ preferences over the candidates). Bartholdi, Tovey, and Trick [BTT89] proved Carroll Winner to be NP-hard. They posed as an open question the issue of whether Carroll Winner is NP-complete. Hemaspaandra, Hemaspaandra, and Rothe [HHR97] recently raised the lower bound of Carroll Winner to $\text{P}^\text{NP}$-hardness, thus pinpointing to the exact computational complexity of checking the winner in Carroll elections (as this lower bound matches the trivial $\text{P}^\text{NP}$ upper bound). This also shows that Carroll Winner cannot be NP-complete unless the polynomial hierarchy collapses, answering the above question raised by Bartholdi, Tovey, and Trick [BTT89]. The claim that this is a natural $\text{P}^\text{NP}$-complete problem is compelling here as elections are exactly the type of setting in which comparisons (seeing who did best) are completely natural. It is also nice that the problem predates by a century the class for which it is complete. Section 2 further discusses the complexity of Carroll’s election system.

Another problem whose lower bound has been recently raised to $\text{P}^\text{NP}$ is Minimum Equivalent Expression (MEE, for short), which asks: Given a boolean formula $\phi$ and a positive integer $k$, is there a boolean expression having at most $k$ occurrences of literals that is equivalent to $\phi$. This is the best-known form of the problem, as this version is presented and discussed in detail in the widely read book of Garey and Johnson [GJ79]. However, the problem has a long history in various related and contrasting versions, and is of immense historical importance in complexity theory. In their seminal paper defining the polynomial hierarchy, Meyer and Stockmeyer [MS72] state explicitly that this is the issue that led to their creation of the polynomial hierarchy:

“We were first led to extend the class NP by considering the language... which denotes the set of well-formed Boolean expressions for which there is no shorter equivalent expression.”

Furthermore, in his journal paper on the polynomial hierarchy, Stockmeyer [Sto77] uses as his motivating problem the language that is the set of pairs $(\phi, k)$ such that $\phi$ is a DNF formula and there is a DNF formula $\phi'$ that is equivalent to $\phi$ and has at most $k$ occurrences of literals.
Returning from our historical digression, we now focus again on \textit{MEE} in the form in which it is stated by Garey and Johnson. Note that \textit{MEE} has a trivial NP\(^{\text{NP}}\) upper bound, yet the best previously known lower bound for \textit{MEE} was only coNP-hardness.\footnote{Garey and Johnson write “NP-hardness,” but by this they mean merely NP-\textit{Turing}-hardness.} Hemaspaandra and Wechsung [HW97] have within the last year raised this lower bound to P\(^{\text{NP}}\)-hardness. Due to space limitations, in this survey we will not further discuss this problem, but instead we point the reader to [HW97].

Section 3 is about the Minimum Degree Greedy Algorithm (MDG, for short), which is a well-known heuristic algorithm for seeking a large independent set in a graph. When does MDG in fact output a maximum independent set—or at least when is MDG’s output \textit{approximately} the size (i.e., within a certain fixed constant factor) of a maximum independent set of the given graph? One line of research tries to identify those graph classes for which MDG has a good approximation ratio (e.g., graphs with bounded degree or bounded average degree [IR94]) or for which MDG actually outputs a maximum independent set (e.g., trees, split graphs, “well-covered” graphs, complete \(k\)-partite graphs, and complements of \(k\)-trees, see [BTY97,TT96]). On the other hand, Bodlaender, Thilikos, and Yamazaki [BTY97] prove that the problem of recognizing (for any fixed rational \(r \geq 1\)) whether, for a given graph \(G\), MDG on input \(G\) outputs a set of size at least \(1/r\) times the size of a maximum independent set of \(G\) is a coNP-hard problem. They also provide a P\(^{\text{NP}}\) upper bound for this recognition problem and leave open the question of whether their coNP-hardness lower bound can be improved to match the upper bound (they actually state their upper bound as a P\(^{\text{NP}}\) upper bound, but it obviously is even a P\(^{\text{NP}}\) upper bound). Hemaspaandra and Rothe [HR97] settle this question by raising this problem’s lower bound to P\(^{\text{NP}}\)-hardness, thus establishing that the problem is complete for P\(^{\text{NP}}\).

We have mentioned the raising to P\(^{\text{NP}}\)-hardness of a number of known NP-hardness/coNP-hardness lower bounds of some quite natural problems. In Section 4, we discuss the question of what raising lower bounds from NP-hardness to hardness for parallel access to NP actually tells us about the computational complexity of a given problem. That is, does P\(^{\text{NP}}\)-hardness give any more insight into the inherent hardness of a problem than an NP-hardness lower bound? We will make the case that the answer to this question depends on which computational model one views as most natural. We discuss this issue for the following computational paradigms: deterministic algorithms, probabilistic algorithms, small circuits, exact counting, unambiguous computation, and approximate computation.
2 Determining the Winner in Lewis Carroll’s Election System

Let us say that an election is specified by a list of candidates and a list of voters’ preferences, where each voter must have strict preferences over the candidates. A candidate \( c \) is a Condorcet winner of a given election if \( c \) defeats (“defeats” here means: is preferred by strictly more than half of the voters) each other candidate in pairwise majority-rule elections. This notion dates to research done in the 1700s by the Marquis de Condorcet ([Con85], see [Bla58]). Not all elections have Condorcet winners. The famous Condorcet Paradox notes that when there are more than two candidates, pairwise majority-rule elections may yield strict cycles in the aggregate preference even if each voter has non-cyclic preferences.

As an example, suppose that three candidates, Clinton (\( C \)), Dole (\( D \)), and Perot (\( P \)), run for president, and there are exactly three voters and they have the following preferences: \( \langle D < C < P \rangle \) (voter 1), \( \langle C < P < D \rangle \) (voter 2), and \( \langle P < D < C \rangle \) (voter 3). Note that in a pairwise majority-rule election \( C \) would beat \( D \), \( P \) would beat \( C \), and \( D \) would beat \( P \)—a strict cycle. That is, though each of the voters is individually rational (i.e., has transitive preferences), the voters’ aggregated preference (“society’s preference”) is irrational!

Lewis Carroll developed an election system that respects the notion of a Condorcet winner, yet that will never create an irrational aggregate preference. Carroll’s voting system ([Dod76], see also [BTT89, HHR97]) works as follows. Each candidate is assigned a score (that we will call his or her Carroll score), namely, the minimum number of sequential exchanges of two adjacent candidates in the voters’ preference orders needed to make the given candidate a Condorcet winner. In particular, any Condorcet winner (and if one exists, he or she is unique) has a Carroll score of 0.

As an example, consider the following electorate: \( \langle D < P < C \rangle \) (voter 1), \( \langle D < C < P \rangle \) (voter 2), \( \langle C < D < P \rangle \) (voter 3), and \( \langle C < P < D \rangle \) (voter 4). Obviously, \( P \)’s Carroll score is 0, since \( P \) is the Condorcet winner. \( C \)’s Carroll score equals 3. It is at most 3, as \( C \) can be made a Condorcet winner by, e.g., exchanging \( C \) upwards in the preferences of voter 2 (once) and voter 4 (twice), which yields the new preferences: \( \langle D < P < C \rangle \), \( \langle D < P < C \rangle \), \( \langle C < D < P \rangle \), and \( \langle P < D < C \rangle \). Also, it is not hard to see that no two exchanges will make \( C \) a Condorcet winner. (By symmetry, \( D \) also has a Carroll score of 3.)

A candidate \( c \) ties-or-defeats a candidate \( d \) if the score of \( d \) is not less than that of \( c \). A candidate \( c \) is said to win a Carroll election if \( c \) ties-or-defeats all other candidates. Of course, due to ties it is possible for two candidates (such as \( C \) and \( D \) in the above example) to tie-or-defeat each other. Thus, some elections may have more than one winner. However, it is important to note that in Carroll elections no strict-preference cycles are possible, since each candidate is assigned an integer score.

Bartholdi, Tovey, and Trick [BTT89] provided a lower bound—NP-hardness—on the
computational complexity of determining the election winner(s) in Carroll’s system, a problem we will call Carroll Winner. Formally, Carroll Winner is the problem of determining, given a set $C$ of candidates, a distinguished member $c$ of $C$, and a list $V$ of the voters’ preference orders on $C$, whether $c$ ties-or-defeats all other candidates in the election. Bartholdi, Tovey, and Trick leave open the problem of what the exact computational complexity of Carroll Winner is. Hemaspaandra, Hemaspaandra, and Rothe\footnote{HHR97} solve this question by proving Carroll Winner to be P\textsuperscript{NP-}\textsuperscript{complete}. The upper bound can easily be seen as follows. First note that for a given candidate $c$ and a given integer $k$, it can be decided in NP whether $c$’s score is at most $k$\footnote{BTT89}. Also, given an election as input, the number of candidates and the highest possible score for each candidate are both polynomially bounded in the input length. Hence, we can in parallel ask to an NP set all plausible Carroll scores for each of the given candidates and thus can compute the exact Carroll score for each candidate. After having done so, it is easy to decide whether or not the designated candidate $c$ ties-or-defeats all other candidates in the election. This easy argument establishes the upper bound.

Unfortunately, the proof of the lower bound is too long and complicated to be included here (interested readers are referred to \footnote{HHR97}). A short, handwaving outline of the argument’s flavor follows.

The general proof strategy is as follows: In order to prove that Carroll Winner is hard (P\textsuperscript{NP-}\textsuperscript{hard}), Hemaspaandra, Hemaspaandra, and Rothe build a broad set of polynomial-time algorithms for manipulating (in the common English-language sense, not the term-of-art political science sense) Carroll elections. In particular, they build algorithms that manipulate the parity of elections, that allow groups of elections to be “summed” in such a way that the score of the sum equals the sum of the scores, and that allow elections to be merged in such a way as to preserve information yet avoid interference. Using these algorithms, Wagner’s toolkit, and going through some intermediate problems and arguments, the P\textsuperscript{NP-} lower bound is established.

3 Greed is Hard to Understand: Approximation and Maximum Independent Sets

The Minimum Degree Greedy Algorithm (MDG) chooses some vertex of minimum degree from the input graph, adds this vertex to its output set, deletes all neighbors of this vertex, and repeats this procedure with the accordingly updated graph until an empty graph is left. How well does MDG approximate a maximum independent set of the given graph (i.e., a maximum-sized subset $S$ of the vertices of the graph such that for no pair of vertices in $S$ is there an edge connecting them)? We denote the size of the maximum independent sets of graph $G$ by $\alpha(G)$. Let mdg($G$) denote the maximum size of the output set of MDG on
input $G$, where the maximum is taken over all the possible choices MDG has among the vertices of minimum degree when there is more than one such vertex.

Bodlaender, Thilikos, and Yamazaki [BTY97] define (for any fixed rational $r \geq 1$) the class of graphs for which MDG, taking a best possible sequence of choices, approximates the size of a maximum independent set within a constant factor of $r$, i.e., for which $\alpha(G)/r \leq \text{mdg}(G)$. They denote this recognition problem by $\mathcal{S}_r$ and prove that for any rational $r \geq 1$, $\mathcal{S}_r$ is coNP-hard and is contained in $P_{||}^{NP}$. They leave open the question of whether the lower and/or the upper bound can be improved. For the special case of $r = 1$, they show that $\mathcal{S}_1$ even is DP-hard, where DP [PY84] denotes the class of sets that can be represented as the difference of two NP sets (DP clearly contains both NP and coNP and is equal to neither NP nor coNP unless the polynomial hierarchy collapses [Kad88]), again leaving open the issue of whether this lower bound can be raised.

Hemaspaandra and Rothe [HR97] settle these questions by establishing the exact computational complexity of recognizing all these graph classes: For each rational $r \geq 1$, $\mathcal{S}_r$ is $P_{||}^{NP}$-complete. The upper bound is implicit in the proof of [BTY97, Lemma 6]. Namely, to determine whether a given graph with $n$ vertices does not belong to $\mathcal{S}_r$, it is sufficient to ask whether for some $k$, $1 \leq k \leq n$, it holds that (i) $\alpha(G) \geq k$, and (ii) for no possible sequence of choices does MDG on input $G$ output a set of size at least $k/r$, i.e., $\text{mdg}(G) < k/r$. Note that all these queries are NP-type or coNP-type queries, and can be resolved via parallel access to the NP-complete set SAT (using the standard complementation-of-the-answer trick for the coNP-type questions). This algorithm places the complement of $\mathcal{S}_r$, and thus $\mathcal{S}_r$ itself, into $P_{||}^{NP}$.

For the lower bound, we will here describe only one special case: the proof that $\mathcal{S}_1$ in fact is $P_{||}^{NP}$-hard, which improves upon the previously known DP lower bound. This special case of the reduction is short enough to fit into a footnote. Interested readers are referred to [HR97] for the more complicated proof of the general case, i.e., the proof for arbitrary rationals $r \geq 1$.

---

\footnote{We reduce the problem $\text{MIS}_\alpha$ to $\mathcal{S}_1$, where $\text{MIS}_\alpha$ is defined to be the set of all pairs $(G, H)$ of graphs $G$ and $H$ such that $\alpha(G) = \alpha(H)$. By Wagner’s work [Wag87] (see also [HR97]), $\text{MIS}_\alpha$ is $P_{||}^{NP}$-complete. So let a pair of graphs, $(G, H)$, be given. It is easy to see that without loss of generality we can assume that $G$ and $H$ have the same number, $k$, of edges. Now we use the construction of [BTY97, Theorem 4] to transform $G$ and $H$ into two new graphs $G'$ and $H'$ such that: (i) $G'$ and $H'$ both are in $\mathcal{S}_1$, (ii) $\alpha(G') = \alpha(G) + k$, and (iii) $\alpha(H') = \alpha(H) + k$. Let $\ell$ be some integer larger than the larger of the number of vertices in $G'$ and the number of vertices in $H'$. The actual construction is as follows. Take two copies of $G'$, two copies of $H'$, and two copies of a set consisting of $\ell$ isolated new vertices, and connect these six subgraphs as shown in Figure 2, where a “×” between two subgraphs denotes their Cartesian product, i.e., any two vertices $u$ and $v$ (where $u$ is from the one subgraph and $v$ is from the other) are joined by an edge. Call the resulting graph $\widehat{G}$. Since $G' \in \mathcal{S}_1$ and $H' \in \mathcal{S}_1$, we have $\text{mdg}(\widehat{G}) = \alpha(G') + \alpha(H') + \ell = \alpha(G) + \alpha(H) + 2k + \ell$. On the other hand, $\alpha(\widehat{G}) = 2 \cdot \max\{\alpha(G'), \alpha(H')\} + \ell = 2 \cdot \max\{\alpha(G), \alpha(H)\} + 2k + \ell$. Thus, $\widehat{G} \in \mathcal{S}_1$ if and only if $\alpha(G) = \alpha(H)$.}
4 Final Remarks: Lower Bounds and Differing Forms of Computation

In this final section, we will be dealing with the question: What impact does raising some problem \textit{FOO}'s lower bound from NP-hardness to $P^{NP}$-hardness actually have regarding the problem's potential solvability via other models of computation (as captured by their respective complexity classes). By the definition of hardness, this can be formally phrased as follows: \textit{If $C$ is some complexity class, is it currently known to hold that NP $\subseteq C$ if and only if $P^{NP} \subseteq C$?} An affirmative answer means that the raised lower bound is worthless with regard to $C$. A negative answer, however, implies that the raised lower bound may have some value in the model captured by $C$ (on the other hand, it might just be the case that NP $\subseteq C$ $\iff$ $P^{NP} \subseteq C$ truly holds and researchers have simply to date failed to establish that it holds). That is, if the implication NP $\subseteq C$ $\Rightarrow$ $P^{NP} \subseteq C$ is not currently known to hold, then in light of the problem \textit{FOO}'s new $P^{NP}$ lower bound, placing \textit{FOO} into $C$ gives us a potentially stronger conclusion than what was previously known: $P^{NP} \subseteq C$ instead of merely NP $\subseteq C$. In other words, we may take “\textit{FOO} is $P^{NP}$-hard” to be potentially stronger evidence that \textit{FOO} cannot be solved by the computational power modeled by $C$ (i.e., that \textit{FOO} $\notin C$) than was available from the fact that “\textit{FOO} is NP-hard.” (The converse implication, $P^{NP} \subseteq C$ $\Rightarrow$ NP $\subseteq C$, is trivial and so needs no discussion.)

We mention, however, that independent of the “connections to other computational models” issues discussed in this section, raising from NP-hardness to $P^{NP}$-hardness the
lower bounds of such natural and longstanding problems as those we have discussed is a genuine improvement in terms of placement within the polynomial hierarchy (unless the polynomial hierarchy itself collapses).

In light of the above discussion, we now discuss for various computational models (as captured by their complexity classes \(\mathcal{C}\)) whether: \(\text{NP} \subseteq \mathcal{C} \iff \text{P}^{\text{NP}} \subseteq \mathcal{C}\).

**Deterministic Polynomial Time (P).** The class P is so low in power that from its viewpoint, it does not matter at all whether a certain problem has a \(\text{P}^{\text{NP}}\) lower bound or just an NP lower bound, since clearly \(\text{NP} = \text{P}\) if and only if \(\text{P}^{\text{NP}} = \text{P}\). Of course, this fact has been well-known since the seminal paper of Meyer and Stockmeyer [MS72], where \(\text{P} = \text{NP} \iff \text{P} = \text{PH}\) is explicitly noted.

**Probabilistic Polynomial Time with unbounded and bounded two-sided error (PP and BPP).** PP [Sim75,Gil77] (respectively, BPP [Gil77]) is defined to be the class of languages \(L\) for which there exists a probabilistic polynomial-time Turing machine \(M\) such that, for all inputs \(x\), if \(x \in L\) then \(M\) accepts its input \(x\) with probability \(\geq 3/4\), and if \(x \notin L\) then \(M\) accepts its input \(x\) with probability \(< 1/2\) (respectively, \(\leq 1/4\)). Clearly, BPP \(\subseteq\) PP and \(\text{NP} \cup \text{coNP} \subseteq \text{P}^{\text{NP}}\). It is known that \(\text{P}^{\text{NP}} \subseteq \text{PP}\) [BHW91]. This latter inclusion immediately implies that \(\text{NP} \subseteq \text{PP}\) if and only if \(\text{P}^{\text{NP}} \subseteq \text{PP}\), since both are outright true. BPP and NP probably are incomparable. However, since BPP is closed under Turing reductions, we easily have: \(\text{NP} \subseteq \text{BPP}\) if and only if \(\text{P}^{\text{NP}} \subseteq \text{BPP}\). Thus, raising a set’s lower bound from NP-hardness to \(\text{P}^{\text{NP}}\)-hardness does not in and of itself give one any heightened level of evidence that the problem is not in BPP or is not in PP.

**Randomized Polynomial Time with one-sided and zero-sided error (R and ZPP).** R [Gil77] is defined to be the class of languages \(L\) for which there exists a probabilistic polynomial-time Turing machine \(M\) such that, for all inputs \(x\), if \(x \in L\) then \(M\) accepts its input \(x\) with probability at least \(1/2\), and if \(x \notin L\) then \(M\) accepts its input \(x\) with probability 0. Clearly, \(\text{P} \subseteq \text{R} \subseteq \text{NP}\) and \(\text{R} \subseteq \text{BPP}\). Like NP, the class R is not known to be closed under Turing reductions or even under complementation—R and coR perhaps differ. ZPP [Gil77] equals the class of languages that can be solved in expected polynomial time, and it is known that ZPP = R \(\cap\) coR. Since ZPP (like BPP) is closed under Turing reductions, the above comment about BPP applies analogously to ZPP: \(\text{NP} = \text{ZPP}\) if and only if \(\text{P}^{\text{NP}} = \text{ZPP}\). Thus, raising a set’s lower bound from NP-hardness to \(\text{P}^{\text{NP}}\)-hardness does not in and of itself give one any heightened level of evidence that the problem is not in ZPP.

In contrast, it is not known whether \(\text{NP} = \text{R}\) implies \(\text{P}^{\text{NP}} = \text{R}\). (The best result known in this direction is that \(\text{NP} = \text{R}\) implies \(\text{P}^{\text{NP}} \subseteq \text{BPP}\), due to the fact that BPP \(\subseteq \text{BPP}\) [ZH86].)
is closed under Turing reductions and \( R \subseteq \text{BPP} \). Thus, raising a set’s lower bound from NP-hardness to \( \text{P}_{||} \)-hardness potentially provides a heightened level of evidence that the problem is not in \( R \). (Of course, if one believes as an article of faith that \( R \neq \text{NP} \) then this heightening claim is not applicable, as in that case one cannot believe that even one NP-hard set might be in \( R \).)

**Exact Counting (\( C_{\text{P}} \)).** \( C_{\text{P}} \) \[^{[Sim75,Wag86]}\] is the class of sets \( L \) such that there is a polynomial-time function \( f \) and a nondeterministic polynomial-time Turing machine \( N \) such that, for each \( x, x \in L \) if and only if \( N \) on input \( x \) has exactly \( f(x) \) accepting paths. It is well-known that \( \text{coNP} \subseteq C_{\text{P}} \subseteq \text{PP} \) \[^{[Sim75,Wag86]}\]. Like \( R \), \( C_{\text{P}} \) is neither known to be closed under Turing reductions nor known to be closed under complementation. We claim that \( \text{NP} \subseteq C_{\text{P}} \Rightarrow \text{P}_{||} \subseteq C_{\text{P}} \) nonetheless holds. We will prove this using other closure properties that \( C_{\text{P}} \) is known to possess.

**Theorem 4.1** \( \text{NP} \subseteq C_{\text{P}} \iff \text{P}_{||} \subseteq C_{\text{P}} \).

**Proof:** Assuming \( \text{NP} \subseteq C_{\text{P}} \) and recalling \( \text{coNP} \subseteq C_{\text{P}} \), we have that DP (see Section \[^{[3]}\]) is contained in \( C_{\text{P}} \), since each DP set is the intersection of an NP set and a coNP set, and Gundermann, Nasser, and Wechsung \[^{[GNW90]}\] have shown that \( C_{\text{P}} \) is closed under intersection. Since \( C_{\text{P}} \) is known \[^{[GNW90]}\], see the discussion in \[^{[Rot93]}\] and \[^{[BCO93]}\] to also be closed under disjunctive truth-table reductions \[^{[LLS75]}\] and since the disjunctive truth-table closure of DP is equal to \( \text{P}_{||} \) it follows that \( \text{P}_{||} \subseteq C_{\text{P}} \). Regarding the claim we just made that \( \{ L \mid (\exists A \in \text{DP})[L \leq_{\text{dtt}}^P A] \} = \text{P}_{||} \), or equivalently, using “R” notation,

\[
R_{\text{dtt}}(\text{DP}) = \text{P}_{||},
\]

we claim that this is implicit and immediate from the fact that a certain set known as \( \text{PARITY}_{\omega}^{\text{SAT}} \) that Buss and Hay \[^{[BH91]}\] proved \( \text{P}_{||} \)-complete is clearly in \( R_{\text{dtt}}(\text{DP}) \). \( \blacksquare \)

**Unambiguous Polynomial Time (UP).** \( \text{UP} \) \[^{[Val76]}\] is the class of those NP sets that are accepted via some NP machine that, on each input, has at most one accepting path. Clearly, \( P \subseteq \text{UP} \subseteq \text{NP} \). Like \( R \) and \( C_{\text{P}} \), UP is not known to be closed under Turing reductions (or even under complementation). Unlike \( C_{\text{P}} \), however, UP (though clearly closed under intersection) is not known to possess any other useful closure properties that might be exploited instead. Thus, it is an open question whether \( \text{NP} = \text{UP} \) implies \( \text{P}_{||} = \text{UP} \). This leaves open the possibility that raising NP lower bounds to \( \text{P}_{||} \)-hardness is an improvement in terms of giving evidence that a given problem is not in UP. (Of course, if one believes as an article of faith that \( \text{UP} \neq \text{NP} \) then this heightening claim is not applicable, as in that case one cannot believe that even one NP-hard set might be in UP.)

**Small circuits (P/poly) and Approximation Models (P-close and APT).** P/poly denotes the class of all sets that can be decided by polynomial-size circuits. By
a result of Meyer (reported in [BH77]), a set \( A \) is in \( P/\text{poly} \) if and only if \( A \in P^S \) for some sparse set \( S \). Thus, \( P/\text{poly} \) is clearly closed under Turing reductions, which gives: \( NP \subseteq P/\text{poly} \) if and only if \( P^{NP} \| \subseteq P/\text{poly} \).

A set \( A \) is P-close [Sch86] if there is a P set \( B \) such that the symmetric difference of \( A \) and \( B \) is a sparse set. That is, each P-close set in a sense is “approximated” by a P set. However, Schöning [Sch86] proved that if every NP set is P-close, then \( NP = P \), which in turn implies that every \( P^{NP} \| \) set is P-close. Thus, we have: \( NP \subseteq P \)-close if and only if \( P^{NP} \| \subseteq P \)-close. An interesting special case of P-closeness is provided by the class APT (almost polynomial time) [MP79]. APT is the class of sets having deterministic algorithms that run in polynomial time for all inputs except those in a sparse set. Since every APT set is P-close, the above result of Schöning easily gives: \( NP \subseteq APT \) if and only if \( P^{NP} \| \subseteq APT \).

Stepping back to summarize the contents of this section and this article: A number of recent results improve from NP-hardness (or coNP-hardness) to \( P^{NP} \| \)-hardness the lower bounds for natural problems whose computational complexities have long been open issues. In fact, two of these natural problems are complete for \( P^{NP} \| \), lending credibility to the naturalness of the class \( P^{NP} \| \). However, as a cautionary note, we pointed out that many other computational modes are so sharply orthogonal to complexity as measured in the polynomial hierarchy that raising lower bounds in the polynomial hierarchy does not speak directly to raising complexity in these other computational modes.

Acknowledgments: We thank Gerd Wechsung for pointing out the DP claim in the proof of Theorem 4.1.

References

[BCO93] R. Beigel, R. Chang, and M. Ogiwara. A relationship between difference hierarchies and relativized polynomial hierarchies. *Mathematical Systems Theory*, 26:293–310, 1993.

[BH77] L. Berman and J. Hartmanis. On isomorphisms and density of NP and other complete sets. *SIAM Journal on Computing*, 6(2):305–322, 1977.

[BH91] S. Buss and L. Hay. On truth-table reducibility to SAT. *Information and Computation*, 91(1):86–102, 1991.

[BHW91] R. Beigel, L. Hemachandra, and G. Wechsung. Probabilistic polynomial time is closed under parity reductions. *Information Processing Letters*, 37(2):91–94, 1991.

---

\(^6\) A set \( S \) is said to be sparse if there is a polynomial \( p \) such that, for all lengths \( n \), the number of elements of \( S \) up to length \( n \) is bounded by \( p(n) \).
Bla58] D. Black. Theory of Committees and Elections. Cambridge University Press, 1958.

BTT89] J. Bartholdi III, C. Tovey, and M. Trick. Voting schemes for which it can be difficult to tell who won the election. Social Choice and Welfare, 6:157–165, 1989.

BTY97] H. Bodlaender, D. Thilikos, and K. Yamazaki. It is hard to know when greedy is good for finding independent sets. Information Processing Letters, 61:101–106, 1997.

Con85] M. J. A. N. de Caritat, Marquis de Condorcet. Essai sur l’Application de L’Analyse à la Probabilité des Décisions Rendues à la Pluraliste des Voix. 1785. Facsimile reprint of original published in Paris, 1972, by the Imprimerie Royale.

Dod76] C. Dodgson. A method of taking votes on more than two issues, 1876. Pamphlet printed by the Clarendon Press, Oxford, and headed “not yet published” (see the discussions in [MU93,Bla58], both of which reprint this paper).

Gil77] J. Gill. Computational complexity of probabilistic Turing machines. SIAM Journal on Computing, 6(4):675–695, 1977.

GJ79] M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, 1979.

GNW90] T. Gundermann, N. Nasser, and G. Wechsung. A survey on counting classes. In Proceedings of the 5th Structure in Complexity Theory Conference, pages 140–153. IEEE Computer Society Press, July 1990.

Hem87] L. Hemachandra. The strong exponential hierarchy collapses. In Proceedings of the 19th ACM Symposium on Theory of Computing, pages 110–122, May 1987.

HHR97] E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Exact analysis of Dodgson elections: Lewis Carroll’s 1876 voting system is complete for parallel access to NP. In Proceedings of the 24th International Colloquium on Automata, Languages, and Programming. Springer-Verlag Lecture Notes in Computer Science, 1997. To appear. A full version of this paper, including all proofs, can be found at http://www.cs.rochester.edu/trs as UR-CS-TR-96-640.

HR94] M. Halldorsson and J. Radhakrishnan. Greed is good: Approximating independent sets in sparse and bounded-degree graphs. In Proceedings of the 26th ACM Symposium on Theory of Computing, pages 439–448. ACM Press, 1994.
[HR97] E. Hemaspaandra and J. Rothe. Recognizing when greedy can approximate maximum independent sets is complete for parallel access to NP. Technical Report Math/Inf/97/14, Institut für Informatik, Friedrich-Schiller-Universität–Jena, Jena, Germany, 1997.

[HW91] L. Hemachandra and G. Wechsung. Kolmogorov characterizations of complexity classes. *Theoretical Computer Science*, 83:313–322, 1991.

[HW97] E. Hemaspaandra and G. Wechsung. The minimization problem for boolean formulas. Technical Report Math/Inf/97/13, Institut für Informatik, Friedrich-Schiller-Universität–Jena, Jena, Germany, 1997.

[Kad88] J. Kadin. The polynomial time hierarchy collapses if the boolean hierarchy collapses. *SIAM Journal on Computing*, 17(6):1263–1282, 1988. Erratum appears in the same journal, 20(2):404.

[Kad89] J. Kadin. $P^{NP[\log n]}$ and sparse Turing-complete sets for NP. *Journal of Computer and System Sciences*, 39(3):282–298, 1989.

[Kre88] M. Krentel. The complexity of optimization problems. *Journal of Computer and System Sciences*, 36:490–509, 1988.

[KSW87] J. Köbler, U. Schöning, and K. Wagner. The difference and truth-table hierarchies for NP. *RAIRO Theoretical Informatics and Applications*, 21:419–435, 1987.

[LLS75] R. Ladner, N. Lynch, and A. Selman. A comparison of polynomial time reducibilities. *Theoretical Computer Science*, 1(2):103–124, 1975.

[LS95] T. Long and M. Sheu. A refinement of the low and high hierarchies. *Mathematical Systems Theory*, 28:299–327, 1995.

[MP79] A. Meyer and M. Paterson. With what frequency are apparently intractable problems difficult? Technical Report MIT/LCS/TM-126, MIT Laboratory for Computer Science, Cambridge, MA, 1979.

[MS72] A. Meyer and L. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential space. In *Proceedings of the 13th IEEE Symposium on Switching and Automata Theory*, pages 125–129, 1972.

[MU95] I. McLean and A. Urken. *Classics of Social Choice*. University of Michigan Press, 1995.

[PY84] C. Papadimitriou and M. Yannakakis. The complexity of facets (and some facets of complexity). *Journal of Computer and System Sciences*, 28(2):244–259, 1984.
C. Papadimitriou and S. Zachos. Two remarks on the power of counting. In Proceedings 6th GI Conference on Theoretical Computer Science, pages 269–276. Springer-Verlag Lecture Notes in Computer Science #145, 1983.

J. Rothe. Some closure properties of GAP-definable classes. Technical Report TR 6-93, Institut für Informatik, Friedrich-Schiller-Universität–Jena, Jena, Germany, 1993.

U. Schöning. Complete sets and closeness to complexity classes. Mathematical Systems Theory, 19(1):29–42, 1986.

J. Simon. On Some Central Problems in Computational Complexity. PhD thesis, Cornell University, Ithaca, N.Y., January 1975. Available as Cornell Department of Computer Science Technical Report TR75-224.

L. Stockmeyer. The polynomial-time hierarchy. Theoretical Computer Science, 3:1–22, 1977.

D. Tankus and M. Tarsi. Well-covered claw-free graphs. Journal of Combinatorial Theory, 66:293–302, 1996.

L. Valiant. The relative complexity of checking and evaluating. Information Processing Letters, 5:20–23, 1976.

K. Wagner. The complexity of combinatorial problems with succinct input representations. Acta Informatica, 23:325–356, 1986.

K. Wagner. More complicated questions about maxima and minima, and some closures of NP. Theoretical Computer Science, 51(1–2):53–80, 1987.

K. Wagner. Bounded query classes. SIAM Journal on Computing, 19(5):833–846, 1990.

S. Zachos and H. Heller. A decisive characterization of BPP. Information and Control, 69:125–135, 1986.