SOLITONS OF THE MIDPOINT MAPPING AND AFFINE CURVATURE

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Abstract. For a polygon \( x = (x_j)_{j \in \mathbb{Z}} \in \mathbb{R}^n \) we consider the midpoints polygon \( (M(x))_j = (x_j + x_{j+1})/2 \). We call a polygon a soliton of the midpoints mapping \( M \) if its midpoints polygon is the image of the polygon under an invertible affine map. We show that a large class of these polygons lie on an orbit of a one-parameter subgroup of the affine group acting on \( \mathbb{R}^n \). These smooth curves are also characterized as solutions of the differential equation \( \dot{c}(t) = Bc(t) + d \) for a matrix \( B \) and a vector \( d \). For \( n = 2 \) these curves are curves of constant generalized-affine curvature \( k_{ga} = k_{ga}(B) \) depending on \( B \) parametrized by generalized-affine arc length unless they are parametrizations of a parabola, an ellipse, or a hyperbola.

1. Introduction

We consider an infinite polygon \( (x_j)_{j \in \mathbb{Z}} \) given by its vertices \( x_j \in \mathbb{R}^n \) in an \( n \)-dimensional real vector space \( \mathbb{R}^n \) resp. \( n \)-dimensional affine space \( \mathbb{A}^n \) modelled after \( \mathbb{R}^n \). For a parameter \( \alpha \in (0, 1) \) we introduce the polygon \( M_\alpha(x) \) whose vertices are given by

\[
(M_\alpha(x))_j := (1 - \alpha)x_j + \alpha x_{j+1}.
\]

For \( \alpha = 1/2 \) this defines the midpoints polygon \( M(x) = M_{1/2}(x) \). On the space \( \mathcal{P} = \mathcal{P}(\mathbb{R}^n) \) of polygons in \( \mathbb{R}^n \) this defines a discrete curve shortening process \( M_\alpha : \mathcal{P} \to \mathcal{P} \), already considered by Darboux [4] in the case of a closed resp. periodic polygon. For a discussion of this elementary geometric construction see Berlekamp et al. [1]. The mapping \( M_\alpha \) is invariant under the canonical action of the affine group. The affine group \( \text{Aff}(n) \) in dimension \( n \) is the set of affine maps \( (A, b) : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax + b \). Here \( A \in \text{Gl}(n) \) is an invertible matrix and \( b \in \mathbb{R}^n \) a vector. The translations \( x \mapsto x + b \) determined by a vector \( b \) form a subgroup isomorphic to \( \mathbb{R}^n \). Let \( \alpha \in (0, 1) \). We call a polygon \( x_j \) a soliton for the process \( M_\alpha \) (or affinely invariant under \( M_\alpha \)) if there is an affine
map \((A, b) \in \text{Aff}(n)\) such that
\[
(M_\alpha(x))_j = Ax_j + b
\]
for all \(j \in \mathbb{Z}\). In Theorem 1 we describe these solitons explicitely and discuss under which assumptions they lie on the orbit of a one-parameter subgroup of the affine group acting canonically on \(\mathbb{R}^n\). We call a smooth curve \(c : \mathbb{R} \to \mathbb{R}^n\) a soliton of the mapping \(M_\alpha\) resp. invariant under the mapping \(M_\alpha\) if there is for some \(\epsilon > 0\) a smooth mapping \(s : (-\epsilon, \epsilon) \to (A(s), b(s)) \in \text{Aff}(n)\) such that for all \(s \in (-\epsilon, \epsilon)\) and \(t \in \mathbb{R}\):
\[
\tilde{c}_s(t) := (1 - \alpha)c(t) + \alpha c(t + s) = A(s)c(t) + b(s).
\]
Then for some \(t_0 \in \mathbb{R}\) and \(s \in (-\epsilon, \epsilon)\) the polygon \(x_j = c(js + t_0), j \in \mathbb{Z}\) is a soliton of \(M_\alpha\). The parabola is an example of a soliton of \(M = M_{1/2}\), cf. Figure 1 and Example 1, Case (e). We show in Theorem 2 that the smooth curves invariant under \(M_\alpha\) coincide with the orbits of a one-parameter subgroup of the affine group \(\text{Aff}(n)\) acting canonically on \(\mathbb{R}^n\). For \(n = 2\) we give a characterization of these curves in terms of the general-affine curvature in Section 5.

The authors discussed solitons, i.e. curves affinely invariant under the curve shortening process \(T : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)\) with
\[
(T(x))_j = \frac{1}{4}\{x_{j-1} + 2x_j + x_{j+1}\}
\]
in [9]. The solitons of \(M = M_{1/2}\) form a subclass of the solitons of \(T\), since \((T(x))_j = (M^2(x))_{j-1}\). Instead of the discrete evolution of polygons one can also investigate the evolution of polygons under a linear flow, cf. Viera and Garcia [11] and [9, sec.4] or a non-linear flow, cf. Glickenstein and Liang [5].

2. THE AFFINE GROUP AND SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

The affine group \(\text{Aff}(n)\) is a semidirect product of the general linear group \(\text{Gl}(n)\) and the group \(\mathbb{R}^n\) of translations. There is a linear representation
\[
(A, b) \in \text{Aff}(n) \to \left( \begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array} \right) \in \text{Gl}(n+1),
\]
of the affine group in the general linear group \(\text{Gl}(n+1)\), cf. [8, Sec.5.1]. We use the following identification
\[
\left( \begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array} \right) \left( \begin{array}{c} x \\ 1 \end{array} \right) = \left( \frac{Ax + b}{1} \right).
\]
Hence we can identify the image of a vector $x \in \mathbb{R}^n$ under the affine map $x \mapsto Ax + b$ with the image $\left(\frac{Ax + b}{1}\right)$ of the extended vector $\left(\frac{x}{1}\right)$. Using this identification we can write down the solution of an inhomogeneous system of linear differential equations with constant coefficients using the power series $F_B(t)$ which we introduce now:

**Proposition 1.** For a real $(n,n)$-matrix $B \in M_\mathbb{R}(n)$ we denote by $F_B(t) \in M_\mathbb{R}(n)$ the following power series:

\[
F_B(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} B^{k-1}.
\]

(a) We obtain for its derivative:

\[
\frac{d}{dt} F_B(t) = \exp(Bt) = BF_B(t) + 1.
\]

The function $F_B(t)$ satisfies the following functional equation:

\[
F_B(t + s) = F_B(s) + \exp(Bs)F_B(t),
\]

resp. for $j \in \mathbb{Z}, j \geq 1$:

\[
F_B(j) = \{1 + \exp(B) + \exp(2B) + \ldots + \exp((j - 1)B)\} F_B(1) = (\exp(B) - 1)^{-1} (\exp(jB) - 1) F_B(1).
\]
(b) The solution \( c(t) \) of the inhomogeneous system of linear differential equations

\[
  \dot{c}(t) = Bc(t) + d
\]

with constant coefficients (i.e. \( B \in M_\mathbb{R}(n,n) \), \( d \in \mathbb{R}^n \)) and with initial condition \( v = c(0) \) is given by:

\[
  c(t) = v + F_B(t)(Bv + d) = \exp(Bt)(v) + F_B(t)(d).
\]

Proof. (a) Equation (6) follows immediately from Equation (5). Then we compute

\[
  \frac{d}{dt}(F_B(t + s) - \exp(Bs)F_B(t)) = \exp(B(t + s)) - \exp(Bs)\exp(Bt) = 0.
\]

Since \( F_B(0) = 0 \) Equation (7) follows. And this implies Equation (8).

(b) We can write the solution of the differential equation (8)

\[
  \frac{d}{dt}\begin{pmatrix} c(t) \\ 1 \end{pmatrix} = \begin{pmatrix} B & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c(t) \\ 1 \end{pmatrix}
\]

as follows:

\[
  \begin{pmatrix} c(t) \\ 1 \end{pmatrix} = \exp\left(\begin{pmatrix} B & d \\ 0 & 0 \end{pmatrix}t\right) \begin{pmatrix} v \\ 1 \end{pmatrix} = \\
  \begin{pmatrix} \exp(Bt) & F_B(t)(d) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} \exp(Bt)(v) + F_B(t)(d) \\ 1 \end{pmatrix}
\]

which is Equation (9). One could also differentiate Equation (9) and use Equation (6). □

Remark 1. Equation (2) shows that \( c(t) \) is the orbit

\[
  t \in \mathbb{R} \mapsto c(t) = \exp\left(\begin{pmatrix} B \\ 0 \end{pmatrix}t\right) \begin{pmatrix} v \\ 1 \end{pmatrix} \in \mathbb{R}^n.
\]

of the one-parameter subgroup

\[
  t \in \mathbb{R} \mapsto \exp\left(\begin{pmatrix} B \\ 0 \end{pmatrix}t\right) \in \text{Aff}(n)
\]

of the affine group \( \text{Aff}(n) \) acting canonically on \( \mathbb{R}^n \).

3. Polygons invariant under \( M_\alpha \)

Theorem 1. Let \((A,b) : x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^n \) be an affine map and \( v \in \mathbb{R}^n \). Assume that for \( \alpha \in (0,1) \) the value \( 1 - \alpha \) is not an eigenvalue of \( A \), i.e. the matrix \( A_\alpha := \alpha^{-1}(A + (\alpha - 1)\mathbb{1}) \) is invertible. Then the following statements hold:
(a) There is a unique polygon \( x \in \mathcal{P}(\mathbb{R}^n) \) with \( x_0 = v \) which is a soliton for \( M_\alpha \) resp. affinely invariant under the mapping \( M_\alpha \) with respect to the affine map \((A, b)\), cf. Equation (1). If \( b_\alpha = \alpha^{-1}b \), then for \( j > 0 \): 

\[
x_j = A_j^\alpha(v) + A_j^{-1}(b_\alpha) + \ldots + A_\alpha(b_\alpha) + b_\alpha \\
= v + (A_j^\alpha - 1) \left( v + (A_\alpha - 1)^{-1}(b_\alpha) \right).
\]

(11)

and for \( j < 0 \):

\[
x_j = A_j^\alpha(v) - A_j^{-1}(b_\alpha) + \ldots + A_{\alpha}^{-1}(b_\alpha) \\
= v + (A_j^\alpha - 1) \left( v - (A_{\alpha}^{-1} - 1)^{-1}(A_{\alpha}^{-1}(b_\alpha)) \right).
\]

(12)

(b) If \( A_\alpha = \exp(B_\alpha) \) for a \((n, n)\)-matrix \( B_\alpha \) and if \( b_\alpha = F_{B_\alpha}(1)(d_\alpha) \) for a vector \( d_\alpha \in \mathbb{R}^n \) then the polygon \( x_j \) lies on the smooth curve 

\[
c(t) = v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha)
\]

i.e. \( x_j = c(j) \) for all \( j \in \mathbb{Z} \).

Proof. (a) By Equation (1) we have

\[(1 - \alpha)x_j + \alpha x_{j+1} = Ax_j + b\]

for all \( j \in \mathbb{Z} \). Hence the polygon is given by \( x_0 = v \) and the recursion formulae 

\[x_{j+1} = A_\alpha(x_j) + b_\alpha; \quad x_j = A_\alpha^{-1}(x_{j+1} - b_\alpha)\]

for all \( j \in \mathbb{Z} \). Then Equation (11) and Equation (12) follow.

(b) For \( A_\alpha = \exp(B_\alpha), b_\alpha = F_{B_\alpha}(d_\alpha) \) we obtain from Equation (6) for all \( j \in \mathbb{Z} : A_\alpha - 1 = B_\alpha F_{B_\alpha}(1) \) and \( A_j^\alpha - 1 = B_\alpha F_{B_\alpha}(j) \). Hence for \( j > 0 \):

\[
x_j = v + (A_j^\alpha - 1) \left( v + (A_\alpha - 1)^{-1}(b_\alpha) \right) \\
= v + B_\alpha F_{B_\alpha}(j) \left( v + (B_\alpha F_{B_\alpha}(1))^{-1}(b_\alpha) \right) \\
= v + F_{B_\alpha}(j)(B_\alpha v + d_\alpha) = c(j).
\]

The Functional Equation (7) for \( F_B(t) \) implies \( 0 = F_B(0) = F_B(-1 + 1) = F_B(-1) + \exp(-B)F_B(1) \), hence

\[F_B(-1) = -\exp(-B)F_B(1); \quad F_B(-1)^{-1} = -\exp(B)F_B(1)^{-1}.
\]
Note that the matrices $B, F_B(t), F_B(t)^{-1}$ commute. With this identity we obtain for $j < 0$:

$$x_j = v + (A_j^t - 1) \left( v - (A_\alpha^{-1} - 1)^{-1} (A_\alpha^{-1} b_\alpha) \right)$$

$$= v + B_\alpha F_{B_\alpha}(j) \left( v - (B_\alpha F_{B_\alpha}(-1))^{-1} \exp(-B_\alpha)(b_\alpha) \right)$$

$$= v + B_\alpha F_{B_\alpha}(j) \left( v - F_{B_\alpha}(-1)^{-1} B_\alpha^{-1} \exp(-B_\alpha)F_{B_\alpha}(1)(d_\alpha) \right)$$

$$= v + F_{B_\alpha}(j) (B_\alpha v + d_\alpha) = c(j).$$

□

**Remark 2.**

(a) Using the identification Equation 14 we can write

$$\begin{align*}
\frac{x_{j+1}}{1} &= \begin{pmatrix} A_\alpha & b_\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_j \\ 1 \end{pmatrix} : \begin{pmatrix} x_j \\ 1 \end{pmatrix} = \begin{pmatrix} A_\alpha & b_\alpha \\ 0 & 1 \end{pmatrix}^j \begin{pmatrix} v \\ 1 \end{pmatrix}
\end{align*}$$

for all $j \in \mathbb{Z}$. 

(b) If $A_\alpha = \exp(B_\alpha)$ for a $(n, n)$-matrix $B_\alpha$ and if $b_\alpha = F_{B_\alpha}(1) (d_\alpha)$ for a vector $d_\alpha \in \mathbb{R}^n$ then we obtain from Equation 10:

$$\begin{align*}
\begin{pmatrix} c(t) \\ 1 \end{pmatrix} &= \exp \left( \begin{pmatrix} B_\alpha & d_\alpha \\ 0 & 0 \end{pmatrix} t \right) \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} \exp(B_\alpha t) & F_{B_\alpha}(t)(d_\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \exp(B_\alpha t)(v) + F_{B_\alpha}(t)(d_\alpha) \\ 1 \end{pmatrix} = \begin{pmatrix} v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha) \\ 1 \end{pmatrix}
\end{align*}$$

Hence $t \in \mathbb{R} \mapsto c(t) \in \mathbb{R}^n$ is the orbit of a one-parameter subgroup of the affine group applied to the vector $v$. 

### 4. Smooth curves invariant under $M_\alpha$

For a smooth curve $c : \mathbb{R} \to \mathbb{R}^n$ and a parameter $\alpha \in (0, 1)$ we define the one-parameter family $\tilde{c}_s : \mathbb{R} \to \mathbb{R}^n$, $s \in \mathbb{R}$ by Equation 2. And we call a smooth curve $c : \mathbb{R} \to \mathbb{R}^n$ a **soliton** of the mapping $M_\alpha$ (resp. affinely invariant under $M_\alpha$) if there is $\epsilon > 0$ and a smooth map $\in (-\epsilon, \epsilon) \to (A, b) \in \text{Aff}(n)$ such that

$$\tilde{c}_s(t) = (1 - \alpha)c(t) + \alpha c(t + s) = A(s)(c(t)) + b(s).$$

Then we obtain as an analogue of [9 Thm.1]:

**Theorem 2.** Let $c : \mathbb{R} \to \mathbb{R}^n$ be a soliton of the mapping $M_\alpha$ satisfying Equation 14. Assume in addition that for some $t_0 \in \mathbb{R}$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent.

Then the curve $c$ is the unique solution of the differential equation

$$\dot{c}(t) = Bc(t) + d$$
for $B = \alpha^{-1}A'(0), d = \alpha^{-1}b'(0)$ with initial condition $v = c(0)$.
And $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), b(s) = \alpha F_B(s)(d)$.

Hence the curve $c(t)$ is the orbit of a one-parameter subgroup

$$t \in \mathbb{R} \mapsto B(t) := \exp\left(\begin{pmatrix} B & d \\ 0 & 0 \end{pmatrix} t\right) = (\exp(Bt), F_B(t)(d)) \in \text{Aff}(n)$$

of the affine group, i.e.

$$c(t) = B(t)\begin{pmatrix} v \\ 1 \end{pmatrix} = v + F_B(t)(Bv + d),$$

cf. Remark 4.

**Remark 3.** For an affine map $(A, b) \in \text{Gl}(n), b \in \mathbb{R}^n$ the linear isomorphism $A$ is called the **linear part**. For $n = 2$ we discuss the possible normal forms of $A \in \text{Gl}(2)$ resp. the normal forms of the one-parameter subgroup $\exp(tB)$ and of the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs)$ introduced in Theorem 2. This will be used in Section 5.

1. $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ for $\lambda, \mu \in \mathbb{R} - \{0\}$, i.e. $A$ is diagonalizable (over $\mathbb{R}$), then $A$ is called **scaling**, for $\lambda = \mu$ it is called **homothety**. For an endomorphism $B$ which is diagonalizable over $\mathbb{R}$ the one-parameter subgroup $B(t) = \exp(Bt)$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs)$ consists of scalings.

2. $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for $a, b \in \mathbb{R}, b \neq 0$, i.e. $A$ has no real eigenvalues. Then $A$ is called a **similarity**, i.e. a composition of a **rotation** and a **homothety**. For an endomorphism $B$ with no real eigenvalues the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), s \neq 0$ consist of affine mappings without real eigenvalues, i.e. compositions of non-trivial rotations and homotheties.

3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is called **shear transformation**. Hence the matrix $A$ has only one eigenvalue 1 and is not diagonalizable. If $B$ is of the form $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$ i.e. $B$ is nilpotent, then the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), s \neq 0$ consist of shear transformations.

4. $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \in \mathbb{R} - \{0, 1\}$. Then $A$ is invertible with only one eigenvalue $\lambda \neq 1$ and not diagonalizable. This linear map is a composition of a homothey and a shear transformation. The one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well
as the one-parameter family $A(s) = (1 - \alpha)\mathbb{I} + \alpha \exp( Bs ), s \neq 0$ consist of linear mappings with only one eigenvalue different from 1 which are not diagonalizable. Hence they are compositions of non-trivial homotheties and shear transformations, too.

We use the following convention: For a one-parameter family $s \mapsto c_s$ of curves or a one-parameter family $s \mapsto A(s), s \mapsto b(s)$ of affine maps we denote the differentiation with respect to the parameter $s$ by $'$. On the other hand we use for the differentiation with respect to the curve parameter $t$ of the curves $t \mapsto c(t), t \mapsto c_s(t)$ the notation $\dot{c}, \dot{c}_s$.

Proof. The proof is similar to the Proof of Theorem [9, Thm.1]: Let

$$c_s(t) = A(s)c(t) + b(s) = (1 - \alpha)c(t) + \alpha c(t + s). \tag{15}$$

For $s = 0$ we obtain $c(t) = c_0(t) = A(0)c(t) + b(0)$ for all $t \in \mathbb{R}$, resp. $(A(0) - \mathbb{I})(c(t)) = -b(0)$ for all $t$. We conclude that

$$(A(0) - \mathbb{I}) (c^{(k)}(t)) = 0 \tag{16}$$

for all $k \geq 1$. Since for some $t_0$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent by assumption we conclude from Equation (16): $A(0) = \mathbb{I}, b(0) = 0$. Equation (15) implies for $k \geq 1$:

$$A(s)c^{(k)}(t) = (1 - \alpha)c^{(k)}(t) + \alpha c^{(k)}(t + s)$$

and hence

$$A'(s)c^{(k)}(t) = \alpha c^{(k+1)}(t + s).$$

We conclude from Equation (15):

$$\frac{\partial c_s(t)}{\partial s} = A'(s)c(t) + b'(s) = \frac{\partial c_s(t)}{\partial t} - (1 - \alpha) \dot{c}(t) = (A(s) - (1 - \alpha)\mathbb{I}) \dot{c}(t).$$

Since $A(0) = \mathbb{I}$ the endomorphisms $A(s) + (\alpha - 1)\mathbb{I}$ are isomorphisms for all $s \in (0, \epsilon)$ for a sufficiently small $\epsilon > 0$. Hence we obtain for $s \in (0, \epsilon)$:

$$\dot{c}(t) = (A(s) + (\alpha - 1)\mathbb{I})^{-1} A'(s)c(t) + (A(s) + (\alpha - 1)\mathbb{I})^{-1} b'(s). \tag{17}$$

Differentiating with respect to $s$:

$$\left((A(s) + (\alpha - 1)\mathbb{I})^{-1} A'(s)\right)'(c(t)) + \left((A(s) + (\alpha - 1)\mathbb{I})^{-1} b'(s)\right)' = 0$$
and differentiating with respect to $t$:

$$\left((A(s) + (\alpha - 1)\mathbb{I})^{-1} A'(s)\right)' c^{(k)}(t) = 0; \ k = 1, 2, \ldots, n.$$ 

By assumption the vectors $c(t_0), c(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent. Therefore we obtain $\left((A(s) + (\alpha - 1)\mathbb{I})^{-1} A'(s)\right)' = 0$. Let $B = \alpha^{-1}A'(0), d = \alpha^{-1}b'(0)$. Then we conclude

$$A'(s) = (A(s) + (\alpha - 1)\mathbb{I}) B; \ b'(s) = (A(s) + (\alpha - 1)\mathbb{I}) (d),$$

We obtain from Equation [17]:

$$\dot{c}(t) = Bc(t) + d.$$ 

Equation [18] with $A(0) = \mathbb{I}$ implies $A(s) = (1 - \alpha)\mathbb{I} + \alpha \exp Bs$. And we obtain $b'(s) = \alpha \exp Bs(d) = \alpha F_B(s)(d)$. Hence $b(s) = \alpha F_B(s)(d)$ since $b(0) = 0$. \hfill \Box

As a consequence we obtain the following

**Theorem 3.** For a $(n,n)$-matrix $B$ and a vector $d$ any solution of the inhomogeneous linear differential equation $\dot{c}(t) = Bc(t) + d$ with constant coefficients is a soliton of the mapping $M_\alpha$. These solitons are orbits of a one-parameter subgroup of the affine group, i.e. they are of the form given in Equation [9].

**Proof.** Any solution of the equation $\dot{c}(t) = Bc(t) + d$ has the form

$$c(t) = v + F_B(t)(Bv + d)$$

with $v = c(0)$, cf. Proposition [1]. Then with $A(s) = (1 - \alpha)\mathbb{I} + \alpha \exp Bs$ and $b(s) = \alpha F_B(s)(d)$ we conclude from Equation [6] and Equation [7]:

$$\ddot{c}_s(t) = (1 - \alpha)c(t) + \alpha c(t + s)$$

$$= v + (1 - \alpha)F_B(t)(Bv + d) + \alpha F_B(t + s)(Bv + d)$$

$$= v + (1 - \alpha)(c(t) - v) + \alpha (F_B(s) + \exp Bs F_B(t))(Bv + d)$$

$$= (1 - \alpha)c(t) + \alpha \exp Bs(v) + \alpha F_B(s)(d) + \alpha \exp Bs(c(t) - v)$$

$$= ((1 - \alpha)\mathbb{I} + \alpha \exp Bs) c(t) + \alpha F_B(s)(d)$$

$$= A(s)c(t) + b(s).$$

Hence $c$ is a soliton of the mapping $M_\alpha$, cf. Equation [14]. \hfill \Box

The curves $c$ invariant under the process $T$ considered in [9] define a class of curves containing the orbits of the one-parameter subgroups of the affine group. They are solutions of
the second order differential equation $\ddot{c} = Bc + d$ which on the other hand can be reduced to a system of first order differential equations, cf. [9, Rem.1].

5. Curves with constant affine curvature

The orbits of one-parameter subgroups of the affine group $\text{Aff}(2)$ acting on $\mathbb{R}^2$ can also be characterized as curves of constant general-affine curvature parametrized proportional to general-affine arc length unless they are parametrizations of a parabola, an ellipse or a hyperbola. This will be discussed in this section. The one-parameter subgroups are determined by an endomorphism $B$ and a vector $d$. We describe in Proposition 2 how the general-affine curvature can be expressed in terms of the matrix $B$.

For certain subgroups of the affine group $\text{Aff}(2)$ one can introduce a corresponding curvature and arc length. One should be aware that sometimes in the literature the curvature related to the equi-affine subgroup $S\text{Aff}(2)$ generated by the special linear group $\text{SL}(2)$ of linear maps of determinant one and the translations is also called affine curvature. We distinguish in the following between the equi-affine curvature $k_{ea}$ and the general-affine curvature $k_{ga}$ as well as between the equi-affine length parameter $s_{ea}$ and the general-affine length parameter $s_{ga}$.

We recall the definition of the equi-affine and general-affine curvature of a smooth plane curve $c : I \rightarrow \mathbb{R}^2$ with $\det(\dot{c}(t) \ddot{c}(t)) = |\dot{c}(t) \ddot{c}(t)| \neq 0$ for all $t \in I$.

By eventually changing the orientation of the curve we can assume $|\dot{c}(t) \ddot{c}(t)| > 0$ for all $t \in I$. A reference is the book by P. and A. Schirokow [10], §10 or the recent article by Kobayashi and Sasaki [7]. Then $s_{ea}(t) := \int |\dot{c}(t) \ddot{c}(t)|^{1/3} \, dt$ is called equi-affine arc length. We denote by $t = t(s_{ea})$ the inverse function, then $\tilde{c}(s_{ea}) = c(t(s_{ea}))$ is the parametrization by equi-affine arc length. Then $\tilde{c}'''(s_{ea}), \tilde{c}'(s_{ea})$ are linearly dependent and the equi-affine curvature $k_{ea}(s)$ is defined by

$$
\tilde{c}'''(s_{ea}) = -k_{ea}(s_{ea}) \tilde{c}'(s_{ea})
$$

resp.

$$
k_{ea}(s) = |\tilde{c}''(s_{ea}) \tilde{c}'''(s_{ea})|.
$$

Assume that $c = c(s_{ea}), s_{ea} \in I$ is a smooth curve parametrized by equi-affine arc length for which the sign $\epsilon = \text{sign}(k_{ea}(s)) \in \{0, \pm 1\}$ of the equi-affine curvature is constant. If $\epsilon = 0$ then the curve is up to an affine transformation a parabola $(t, t^2)$. Now assume $\epsilon \neq 0$
and let $K_{ea} = |k_{ea}| = \epsilon k_{ea}$. Then the general-affine arc length $s_{ga} = s_{ga}(s_{ea})$ is defined by
\begin{equation}
    s_{ga} = \int \sqrt{K_{ea}(s_{ea})} \, ds_{ea}.
\end{equation}

We call a curve $c = c(t)$ parametrized proportional to general-affine arc length if $t = \lambda_1 s_{ga} + \lambda_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq 0$. The general-affine curvature $k_{ga} = k_{ga}(s)$ is defined by
\begin{equation}
    k_{ga}(s) = K_{ea}'(s)K_{ea}(s)^{-3/2} = -2 \left(K_{ea}^{-1/2}(s)\right)'.
\end{equation}

If the general-affine curvature $k_{ga}$ (up to sign) and the sign $\epsilon$ is given with respect to the equi-affine arc length parametrization, then the equi-affine curvature $k_{ea} = k_{ea}(s_{ea})$ is determined up to a constant by Equation (20). Hence the curve is determined up to an affine transformation. The invariant $k_{ga}$ already occurs in Blaschke’s book [2, §10, p.24]. Curves of constant general-affine curvature are orbits of a one-parameter subgroup of the affine group. These curves already were discussed by Klein and Lie [6] under the name $W$-curves.

**Proposition 2.** For a non-zero matrix $B \in M_\mathbb{R}(2,2)$ and vectors $d, v \in \mathbb{R}^2$ where $Bv + d$ is not an eigenvector of $B$ let $c : \mathbb{R} \rightarrow \mathbb{R}^2$ be the solution of the differential equation $c'(t) = Bc(t) + d; c(0) = v$, i.e. $c(t) = v + F_B(t)(Bv + d) = \exp(tB)(v) + F_B(t)(d)$. We assume that $\beta = |c'(0) c''(0)|^{1/3} = |Bv + d \ B(Bv + d)|^{1/3} > 0$. Define
\begin{equation}
    k = k(B) = -2 + 9 \det(B)/\text{tr}^2(B) ; \ K = K(B) = |k(B)|^{-1/2}.
\end{equation}

(a) If $\text{tr}(B) = 0$ then the curve is parametrized proportional to equi-affine arc length and the equi-affine curvature is constant $k_{ea} = \det(B)/\beta^2$ and $\epsilon = \text{sign}(\det(B))$, the curve is a parabola, if $\epsilon = 0$, an ellipse, if $\epsilon > 0$, or a hyperbola, if $\epsilon < 0$, cf. Remark 4.

(b) If $\text{tr}(B) \neq 0$ then we can choose a parametrization by equi-affine arc length $s_{ea}$ such that the equi-affine curvature $k_{ea}$ is given by:
\begin{equation}
    k_{ea}(s_{ea}) = k(B)s_{ea}^{-2}.
\end{equation}

If $k(B) = 0$ the curve has vanishing equi-affine curvature and is a parametrization of a parabola, cf. the Remark 4. If $k(B) \neq 0$ then the general-affine curvature is defined and constant:
\begin{equation}
    k_{ga}(s_{ea}) = -2K(B).
\end{equation}
Up to an additive constant the general-affine arc length parameter $s_{ga}$ is given by:

$$s_{ga} = -\frac{\text{tr}B}{3K(B)} \cdot t.$$ 

Hence the curve $c(t)$ is parametrized proportional to general-affine arc length.

**Remark 4.** It is well-known that the curves of constant equi-affine curvature are parabola, hyperbola or ellipses, cf. [2, §7]. For $k_{ea} = 0$ we obtain a parabola: $c(t) = c(0) + c'(0)s_{ea} + c''(0)s_{ea}^2/2$, for $k_{ea} > 0$ the ellipse $c(s_{ea}) = (a \cos(\sqrt{k_{ea}}s_{ea}), b \sin(\sqrt{k_{ea}}s_{ea}))$ with $k_{ea} = (ab)^{-2/3}$ and for $k_{ea} < 0$ the hyperbola $c(s_{ea}) = (a \cosh(\sqrt{-k_{ea}}s_{ea}), b \sinh(\sqrt{-k_{ea}}s_{ea}))$ with $k_{ea} = -(ab)^{-2/3}$. Here $a, b > 0$.

**Proof.** Following Proposition 1 we obtain as solution of the differential equation: $c(t) = v + F_B(t)(Bv + d)$, hence for the derivatives: $c^{(k)}(t) = B^{k-1}\exp(tB)(Bv + d)$. Then:

$$|\dot{c}(t) \ddot{c}(t)| = \exp(Bt)||bv + d| B(Bv + d)| = \exp(\text{tr}(B)t)|Bv + d| B(Bv + d)|.$$

Let $\beta = (|Bv + d| B(Bv + d)|)^{1/3}$ and $\tau = \text{tr}(B)$. Then

$$|\dot{c}(t) \ddot{c}(t)| = \beta^3 \exp(\tau t).$$

(a) If $\tau = 0$ then $s_{ea} = t\beta$, i.e. the curve is parametrized proportional to equi-affine arc length and

$$\dot{c}(s_{ea}) = c(t(s_{ea})) = c(s_{ea}/\beta) = v + F_B(s_{ea}/\beta)(Bv + d).$$

Then

$$\ddot{c}(s_{ea}) = \beta^{-1}\exp(Bs_{ea}/\beta)(Bv + d)$$

$$\dddot{c}(s_{ea}) = \beta^{-3}B^2 \exp(Bs_{ea}/\beta)(Bv + d) = -\det(B)\beta^{-2}\dot{c}(s_{ea}).$$

Here we use that by Cayley-Hamilton $B^2 - \tau B = B^2 = -\det(B) \cdot 1$. Hence we obtain $k_{ea}(s) = \det(B)/\beta^2$ and $\epsilon = \text{sign}(\det(B))$. Then the claim follows from Remark 4.

(b) Assume $\tau \neq 0$. Then the equi-affine arc length $s_{ea} = s_{ea}(t)$ is given by

$$s_{ea}(t) = \beta \int \exp(\tau t/3) \, dt = \frac{3\beta}{\tau} \exp(\tau t/3).$$

Hence the equi-affine arc length parametrization of $c$ is given by

$$\dot{c}(s_{ea}) = v + F_B\left(\frac{3}{\tau} \ln \left(\frac{\tau}{3\beta s_{ea}}\right)\right)(Bv + d).$$
Then we can express the derivatives:

\[
\begin{align*}
\tilde{c}'(s_{ea}) &= \frac{3}{\tau} \frac{1}{s_{ea}} \exp \left( \frac{3}{\tau} B \ln \left( \frac{\tau}{3\beta s_{ea}} \right) \right) (Bv + d) \\
\tilde{c}''(s_{ea}) &= \left( \frac{3}{\tau} B - 1 \right) \frac{1}{s_{ea}} \tilde{c}'(s_{ea}) \\
\tilde{c}'''(s_{ea}) &= \left( \frac{3}{\tau} B - 1 \right) \left( \frac{3}{\tau} B - 2 \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea}) \\
&= -\left( \frac{9 \det(B)}{\tau^2} - 2 \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea}) .
\end{align*}
\]

Here we used that by Cayley-Hamilton \( B^2 - \tau B = -\det B \cdot 1 \). Hence we obtain for the equi-affine curvature

\[
\kappa_{ea}(s_{ea}) = \frac{k(B)}{s_{ea}^2} .
\]

Then \( \epsilon = \text{sign} k(B) \) and for \( k(B) \neq 0 \) we obtain from Equation (20) and Equation (25):

\[
\kappa_{ga}(s_{ea}) = -\frac{2}{K(B)}
\]

And for the general-affine arc length we obtain

\[
s_{ga} = \ln(|s_{ea}|)/K(B)
\]

resp. up to an additive constant:

\[
s_{ga} = \frac{\tau t}{3K(B)}
\]

using Equation (24).

The parametrization by general-affine arc length is given by

\[
c^*(s_{ga}) = v + F_B (3K(B)s_{ga}/\tau) (Bv + d) .
\]

Example 1. Depending on the real Jordan normal forms of the endomorphism \( B \) we investigate the solitons \( c(t) \), their special and general affine curvature. The normal forms of the corresponding one-parameter subgroup \( B(t) = \exp(Bt) \) as well of the one-parameter family \( A(s) = (1 - \alpha) \mathbb{1} + \exp(Bs) \) follow from Remark 3. Since \( c(\mu t) = \exp(\mu Bt) \) the multiplication of \( B \) with a non-zero real \( \mu \) corresponds to a linear reparametrization of the curve. If \( B \) has a non-zero real eigenvalue we can assume without loss of generality that it is 1 and in the case of a non-real eigenvalue we can assume that it has modulus 1.
(a) Let \( B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \), \( d = (0, 0) \), \( c(0) = (1, 1) \) and \( \lambda \neq 0, 1 \). Then \( \beta = (\lambda(1 - \lambda))^{1/3} \neq 0 \), \( \text{tr} B = 1 + \lambda \) and \( c(t) = (\exp(t), \exp(\lambda t)) \). Up to parameterization we have \( c(u) = (u, u^\lambda) \).

If \( \lambda = -1 \) then \( c \) is a parameterization of a hyperbola, \( \text{tr} B = 0 \) and \( k_{ga} = -2^{-2/3} \), cf. Remark 4.

If \( \lambda \neq -1 \) we obtain for the equi-affine curvature with respect to an equi-affine parameterization \( s_{ea} \) from Equation (22):

\[
k_{ea}(s_{ea}) = \left( \frac{\det B}{\text{tr}^2 B} - 2 \right) \frac{1}{s_{ea}^2} = -\frac{(\lambda - 2)(2\lambda - 1)}{(\lambda + 1)^2} \frac{1}{s_{ea}^2}.
\]

For \( \lambda = 1/2, 2 \) we obtain a parameterization of a parabola with vanishing equi-affine curvature, cf. Remark 4. Now we assume \( \lambda \neq 1/2, 2 \). Hence \( \epsilon = 1 \) if and only if \( 1/2 < \lambda < 2 \). The affine curvature \( k_{ga} \) is constant:

\[
k_{ga} = -2 \frac{|\lambda + 1|}{\sqrt{|(\lambda - 2)(2\lambda - 1)|}},
\]

cf. [7, Ex.2.14]. We have \( \epsilon = 1 \) if and only if \( 1/2 < \lambda < 2 \), then \( k_{ga} \in (-\infty, -4) \). And \( \epsilon = -1 \) if and only if \( \lambda < 1/2, \lambda \neq 0 \) or \( \lambda > 2 \), then \( k_{ga} \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, 0) \).

Hence in this case the corresponding one-parameter subgroup

\[
B(t) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(\lambda t) \end{pmatrix}
\]

as well as the one-parameter family

\[
A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & 0 \\ 0 & 1 - \alpha + \alpha \exp(\lambda s) \end{pmatrix}
\]

consist of scalings.

(b) \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( d = (1, 0) \). Then the solution of the Equation \( \dot{c}(t) = Bc(t) + d \) with \( c(0) = (0, 1) \) is of the form \( c(t) = (t, \exp(t)) \). Then we obtain \( \epsilon = -1 \) and \( k_{ga} = -\sqrt{2} \). The corresponding one-parameter subgroup \( B(t) \) as well as the one-parameter family \( A(s) \) consist of scalings, the affine transformation \( (A(s), b(s)) \) is given by \( (A(s), b(s)) = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 - \alpha + \alpha \exp(s) \end{pmatrix}, \alpha \begin{pmatrix} s \\ 0 \end{pmatrix} \right) \), i.e. a composition of scalings and translations.

(c) If \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( d = (0, 0) \), \( c(0) = (1, 1) \) then \( c(t) = ((t + 1)\exp(t), \exp(t)) \) i.e. up to an affine transformation and a reparametrization the curve is of the form \( c(u) = (u, u \ln(u)) \). Then \( \epsilon = 1 \) and \( k_{ga} = -4 \). The corresponding one-parameter
Figure 2. The soliton $c(t) = ((t + 1) \exp(t), \exp(t))$

with the family $c_s(t) = A(s)c(t)$.

subgroup $B(t)$ as well as the one-parameter family $A(s)$ consist of compositions of a homothety and a shear transformations;

$$B(t) = \exp(t) \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & \alpha s \exp(s) \\ 0 & 1 - \alpha + \alpha \exp(s) \end{pmatrix},$$

cf. Figure 2.

(d) If $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b \neq 0, a^2 + b^2 = 1, d = 0, c(0) = (1,0)$ then $c(t) = \exp(at) (\cos(bt), \sin(bt))$. For $a = 0$, this is a circle with $k_{ea} = 1$. Now we assume $a \neq 0$: Then $\epsilon = \text{sign}(9 \det(B) - 2tr^2(B)) = \text{sign}(a^2 + 9b^2) = 1$ and we obtain for the general-affine curvature $k_{ga} = -4|a|/\sqrt{a^2 + 9b^2} = -4|a|/\sqrt{9 - 8a^2}$, i.e. $k_{ga} \in (-4,0)$. The corresponding one-parameter subgroup $B(t)$ as well as the one-parameter family $A(s)$ consist of similarities.

(e) If $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ one can choose $c(0) = (0,0), d = (0,1)$ and obtain $c(t) = (t^2/2, t)$, i.e. a parabola. In this case the one parameter subgroup $B(t) = \exp(tB)$ consists of shear transformations. The one-parameter family $(A(s), b(s)) = \left( \begin{pmatrix} 1 & \alpha s \\ 0 & 1 \end{pmatrix}, \alpha \begin{pmatrix} s^2/2 \\ s \end{pmatrix} \right)$ consists of a composition of a shear transformation and a translation, cf. Figure 1.

For the affine curve shortening flow the parabola is a translational soliton. Therefore it is also called the affine analogue of the grim reaper, cf. [3, p.192]. For the
curve shortening process $T$ defined by Equation (3) the parabola is also a translational soliton, cf. [9, Sec.5, Case (5)].

Note that the parabola occurs twice, in Case (a) it occurs with the parametrization $c(t) = (\exp(t), \exp(2t))$, in Case (e) it occurs with a parametrization proportional to equi-affine arc length. Summarizing we obtain from Theorem \textsuperscript{2} and Theorem \textsuperscript{3} together with Proposition \textsuperscript{2} resp. Example \textsuperscript{1} the following

**Theorem 4.** Let $c : \mathbb{R} \rightarrow \mathbb{R}^2$ be a smooth curve for which $\dot{c}(0), \ddot{c}(0)$ are linearly independent. Then $c$ is a soliton of the mappings $M_\alpha, \alpha \in (0, 1)$, in particular of the midpoints mapping $M = M_{1/2}$, if it is a curve of constant equi-affine curvature parametrized proportional to equi-affine arc length, or a parabola with the parametrization $c(t) = (\exp(t), \exp(2t))$ up to an affine transformation, or if it is a curve of constant general-affine curvature parametrized proportional to general-affine arc length.

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