p-ADIC MULTiresolution ANALYSES

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ABSTRACT. We study p-adic multiresolution analyses (MRAs). A complete characterisation of test functions generating a MRA (scaling functions) is given. We prove that only 1-periodic test functions may be taken as orthogonal scaling functions and that all such scaling functions generate Haar MRA. We also suggest a method of constructing sets of wavelet functions and prove that any set of wavelet functions generates a p-adic wavelet frame.

1. Introduction

In the early nineties a general scheme for the construction of wavelets (of real argument) was developed. This scheme is based on the notion of multiresolution analysis (MRA in the sequel) introduced by Y. Meyer and S. Mallat [1], [2] (see also, e.g., [4], [11]). Immediately specialists started to implement new wavelet systems. Nowadays it is difficult to find an engineering area where wavelets are not applied.

In the p-adic setting, the situation is as follows. In 2002 S. V. Kozyrev [3] found a compactly supported p-adic wavelet basis for $L^2(Q_p)$ which is an analog of the Haar basis. It even turned out that these wavelets were eigenfunctions of p-adic pseudo-differential operators [5], J.J. Benedetto and R.L. Benedetto [6], [7], however, discussed if it is possible to construct other p-adic wavelets with the same set of translations which are not a group. In particular, R.L. Benedetto [7, p. 28] had doubts that a MRA-theory could be developed because discrete subgroups do not exist in $Q_p$. Indeed, the latter seems to be an obstacle for the development of a MRA theory. On the other hand, A. Khrennikov and V. Shelkovich [8] conjectured that the equality

$$\varphi(x) = \sum_{r=0}^{p-1} \varphi\left(\frac{1}{p} x - \frac{r}{p}\right), \quad x \in Q_p,$$

may be considered as a refinement equation for the Haar MRA generating Kozyrev’s wavelets. A solution $\varphi$ of this equation (a refinable function) is the characteristic function of the unit disc. We note that equation (1.1) reflects a natural “self-similarity” of the space $Q_p$: the unit disc $B_0(0) = \{x : |x|_p \leq 1\}$ is represented as the union of p mutually disjoint discs $B_{-1}(r) = \{x : |x-r|_p \leq p^{-1}\}, r = 0, \ldots, p-1$. Following this idea, the notion of p-adic MRA was introduced and a general scheme for its construction was described in [9]. Also, using (1.1) as a generating refinement equation, this scheme was realized to construct the 2-adic Haar MRA. In contrast

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to the real setting, the refinable function $\varphi$ generating the Haar MRA is periodic, which implies the existence of infinitely many different orthonormal wavelet bases in the same Haar MRA. One of them coincides with Kozyrev’s wavelet basis. The authors of [10] described a wide class of functions generating a MRA, but all of these functions are 1-periodic. In the present paper we prove that there exist no other orthogonal test scaling functions generating a MRA, except for those described in [9]. Also, the MRAs generated by arbitrary test scaling functions (not necessarily orthogonal) are considered and a criterion for a test function to generate such a MRA is found. The non-group structure of the set of standard translations is compensated by the fact that the sample spaces are invariant with respect to all translations by the elements of $\mathbb{Q}_p$. Finally we develop a method to construct a wavelet frame based on a given MRA.

Here and in what follows, we shall systematically use the notation and the results from [13]. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the sets of positive integers, integers, real numbers, complex numbers, respectively. The field $\mathbb{Q}_p$ of $p$-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic norm $|\cdot|_p$. This $p$-adic norm is defined as follows: $|0|_p = 0$; if $x \neq 0$, $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers $m$, $n$ are not divisible by $p$, then $|x|_p = p^{-\gamma}$. The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. The canonical form of any $p$-adic number $x \neq 0$ is

$$x = p^\gamma \sum_{j=0}^{\infty} x_j p^j$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j \in D_p := \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The fractional part $\{x\}_p$ of the number $x$ equals by definition $p^\gamma \sum_{j=0}^{\gamma-1} x_j p^j$. Thus, $\{x\}_p = 0$ if and only if $\gamma \geq 0$. We also set $\{0\}_p = 0$.

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq p^\gamma\}$ the disc of radius $p^\gamma$ with the center at a point $a \in \mathbb{Q}_p$, $\gamma \in \mathbb{Z}$. Any two balls in $\mathbb{Q}_p$ either are disjoint or one contains the other. We observe that $B_0(0) = \{x \in \mathbb{Q}_p : \{x\}_p = 0\}$.

There exists the Haar measure $dx$ on $\mathbb{Q}_p$ which is positive, invariant under the shifts, i.e., $d(x+a) = dx$, and normalized by $\int_{|x|_p \leq 1} dx = 1$. A complex-valued function $f$ defined on $\mathbb{Q}_p$ is called locally-constant if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that $f(x+y) = f(x)$, $y \in B_{l(x)}(0)$. Denote by $\mathcal{D}$ the linear space of locally-constant compactly supported functions (so-called test functions). The space $\mathcal{D}$ is an analog of the Schwartz space in the real analysis.

The Fourier transform of $\varphi \in \mathcal{D}$ is defined as

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) \varphi(x) \, dx, \quad \xi \in \mathbb{Q}_p,$$

where $\chi_p(\xi) = e^{2\pi i \xi x}_p$ is the additive character for the field $\mathbb{Q}_p$, and $\{\cdot\}_p$ is the fractional part of a $p$-adic number. The Fourier transform is a linear isomorphism taking $\mathcal{D}$ into $\mathcal{D}$. The Fourier transform is extended to $L^2(\mathbb{Q}_p)$ in a standard way and the Plancherel equality holds

$$\int_{\mathbb{Q}_p} f(x) g(x) \, dx = \int_{\mathbb{Q}_p} \hat{f}(\xi) \hat{g}(\xi) \, d\xi, \quad f, g \in L^2(\mathbb{Q}_p).$$
If $f \in L^2(Q_p)$, $0 \neq a \in Q_p$, $b \in Q_p$, then:

$$F[f(a \cdot + b)](\xi) = |a|^{-1}_p \chi_p \left(-\frac{b}{a}\right) \hat{F}[f]\left(\frac{\xi}{a}\right).$$

Besides,

$$F[\Omega(| \cdot |_p)](\xi) = \Omega(|\xi|_p), \quad \xi \in Q_p,$$

where $\Omega$ is the characteristic function of the interval $[0, 1]$.

2. Multiresolution analysis

Let us consider the set

$$I_p = \{a \in Q_p : \{a\}_p = a\}.$$

Since $B_0(0) = \{x \in Q_p : \{x\}_p = 0\}$, we have the following decomposition of $Q_p$ into the union of mutually disjoint discs: $Q_p = \bigcup_{a \in I_p} B_0(a)$. Thus, $I_p$ can be considered as a “natural” set of translations for $Q_p$.

**Definition 2.1.** A collection of closed spaces $V_j \subset L^2(Q_p)$, $j \in \mathbb{Z}$, is called a **multiresolution analysis (MRA)** in $L^2(Q_p)$ if the following axioms hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(Q_p)$;
(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
(d) $f(\cdot) \in V_j \iff f(p^{-j}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) there exists a function $\varphi \in V_0$ such that $V_0 := \operatorname{span}\{\varphi(x-a) : a \in I_p\}$.

The function $\varphi$ from axiom (e) is called *scaling*. One also says that a MRA is generated by its scaling function $\varphi$ (or $\varphi$ generates the MRA). It follows immediately from axioms (d) and (e) that

$$V_j = \operatorname{span}\{\varphi(p^{-j}x-a) : a \in I_p\}, \quad j \in \mathbb{Z}.$$  

An important class of MRAs consists of those generated by so-called **orthogonal scaling functions**. A scaling function $\varphi$ is said to be orthogonal if $\{\varphi(x-a), a \in I_p\}$ is an orthonormal basis for $V_0$. Consider such a MRA. Evidently, the functions $p^{j/2}\varphi(p^{-j}x-a), a \in I_p$, form an orthonormal basis for $V_j$, $j \in \mathbb{Z}$. According to the standard scheme (see, e.g., [11, §1.3]) for the construction of MRA-based wavelets, for each $j$, we define a space $W_j$ (*wavelet space*) as the orthogonal complement of $V_j$ in $V_{j+1}$, i.e., $V_{j+1} = V_j \oplus W_j, j \in \mathbb{Z}$, where $W_j \perp V_j, j \in \mathbb{Z}$. It is not difficult to see that

$$f(\cdot) \in W_j \iff f(p^{-j}\cdot) \in W_{j+1}, \quad \text{for all } j \in \mathbb{Z}$$

and $W_j \perp W_k, j \neq k$. Taking into account axioms (b) and (c), we obtain

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(Q_p) \quad (\text{orthogonal direct sum}).$$

If we now find functions $\psi^{(\nu)} \in W_0, \nu \in A$, such that the functions $\psi^{(\nu)}(x-a), a \in I_p, \nu \in A$, form an orthonormal basis for $W_0$, then, due to (2.2) and (2.3), the system $\{p^{j/2}\psi^{(\nu)}(p^{-j}x-a), a \in I_p, j \in \mathbb{Z}, \nu \in A\}$ is an orthonormal basis for $L^2(Q_p)$. Such a function $\psi^{(\nu)}$ are called a *wavelet function* and the basis is a *wavelet basis*.

Another interesting class of scaling functions consists of functions $\varphi$ for which $\{\varphi(x-a), a \in I_p\}$ is a Riesz system. Probably, adopting the ideas developed for...
the real setting, one can use MRAs generated by such functions for constructing dual biorthogonal wavelet systems. This topic is, however, out of our consideration in the present paper.

In Section 3 we will discuss how to construct a $p$-adic wavelet frame based on an arbitrary MRA generated by a test function.

Let $\varphi$ be an orthogonal scaling function for a MRA $\{V_j\}_{j \in \mathbb{Z}}$. Since the system 
$\{p^{j/2}\varphi(p^{-1}x-a), a \in I_p\}$ is a basis for $V_1$ in this case, it follows from axiom (a) that

$$\varphi(x) = \sum_{a \in I_p} \alpha_a \varphi(p^{-1}x-a), \quad \alpha_a \in \mathbb{C}. \quad (2.4)$$

We see that the function $\varphi$ is a solution of a special kind of functional equation. Such equations are called refinement equations, and their solutions are called refinable functions. It will be shown in Section 3 that any test scaling function (not necessary orthogonal) is refinable.

A natural way for the construction of a MRA (see, e.g., [11, §1.2]) is the following. We start with a refinable function $\varphi$ and define the spaces $V_j$ by (2.1). It is clear that axioms (d) and (e) of Definition 2.1 are fulfilled. Of course, not any such function $\varphi$ provides axiom (a). In the real setting, the relation $V_0 \subset V_1$ holds if and only if the refinable function satisfies a refinement equation. The situation is different in the $p$-adic case. Generally speaking, a refinement equation (2.4) does not imply the including property $V_0 \subset V_1$ because the set of shifts $I_p$ does not form a group. Indeed, we need all the functions $\varphi(-b)$, $b \in I_p$, to belong to the space $V_1$, i.e., the identities $\varphi(x-b) = \sum_{a \in I_p} \alpha_{a,b} \varphi(p^{-1}x-a)$ should be fulfilled for all $b \in I_p$. Since $p^{-1}b + a$ is not in $I_p$ in general, we can not state that $\varphi(-b)$ belongs to $V_1$ for all $b \in I_p$. Nevertheless, we will see below that a wide class of refinable equations provide the including property.

Providing axiom (a) is a key moment for the construction of MRA. Axioms (b) and (c) are fulfilled for a wide class of functions $\varphi$ because of the following statements.

**Theorem 2.2.** If $\varphi \in L^2(\mathbb{Q}_p)$ and $\hat{\varphi}$ is compactly supported, then axiom (c) of Definition 2.1 holds for the spaces $V_j$ defined by (2.1).

**Proof.** Let $\hat{\varphi} \subset B_M(0)$, $M \in \mathbb{Z}$. Assume that a function $f \in L^2(\mathbb{Q}_p)$ belongs to any space $V_j$, $j \in \mathbb{Z}$. Given $j \in \mathbb{N}$ and $\varepsilon > 0$, there exists a function $f_{\varepsilon}(x) := \sum_{a \in I_p} \alpha_a \varphi(p^{j}x-a)$, where the sum is finite, such that $\|f - f_{\varepsilon}\| < \varepsilon$. Using (1.3), it is not difficult to see that supp $\hat{f}_{\varepsilon} \subset$ supp $\hat{\varphi}(p^{-j})$, which yields that $\hat{f}_{\varepsilon}(\xi) = 0$ for any $\xi \notin B_{M-j}(0)$. Due to the Plancherel theorem, it follows that $\hat{f} = 0$ almost everywhere on $B_{M-j}(0)$. Since $j$ is an arbitrary positive integer, $\hat{f}$ is equivalent to zero on $Q_p$. \hfill \Box

Another sufficient condition for axiom (c) was given in [10]:

**Theorem 2.3.** If $\varphi \in L^2(\mathbb{Q}_p)$ and the system $\{\varphi(x-a) : a \in I_p\}$ is orthonormal, then axiom (c) of Definition 2.1 holds for the spaces $V_j$ defined by (2.1).

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1. Usually the terms “refinable function” and “scaling function” are synonyms in the literature, and they are used in both senses: as a solution to the refinable equation and as a function generating MRA. We separate here the meanings of these terms.
Let \( \varphi \in L^2(\mathbb{Q}_p) \), the spaces \( V_j \), \( j \in \mathbb{Z} \), be defined by (2.1), and let \( \varphi(-b) \in \bigcup_{j \in \mathbb{Z}} V_j \) for any \( b \in \mathbb{Q}_p \). Axiom (b) of Definition 2.1 holds for the spaces \( V_j \), \( j \in \mathbb{Z} \), if and only if
\[
\bigcup_{j \in \mathbb{Z}} \text{supp} \hat{\varphi}(p^j \cdot) = \mathbb{Q}_p.
\]

**Remark 2.5.** It is not difficult to see that the assumption \( \varphi(-b) \in \bigcup_{j \in \mathbb{Z}} V_j \) for any \( b \in \mathbb{Q}_p \) is fulfilled whenever \( \varphi \) is a refinable function and \( \hat{\varphi} \subset B_0(0) \). We will see that this assumption is also valid for a wide class of refinable functions \( \varphi \) for which \( \hat{\varphi} \not\subset B_0(0) \).

**Proof.** First of all we show that the space \( \bigcup_{j \in \mathbb{Z}} V_j \) is invariant with respect to all shifts. Let \( f \in \bigcup_{j \in \mathbb{Z}} V_j \), \( b \in \mathbb{Q}_p \). Evidently, \( \varphi(p^{-k} \cdot - t) \in \bigcup_{j \in \mathbb{Z}} V_j \) for any \( t \in \mathbb{Q}_p \) and for any \( k \in \mathbb{Z} \). Since the \( L^2 \)-norm is invariant with respect to the shifts, it follows that \( f(-b) \in \bigcup_{j \in \mathbb{Z}} V_j \). If now \( g \in \bigcup_{j \in \mathbb{Z}} V_j \), then approximating \( g \) by the functions \( f \in \bigcup_{j \in \mathbb{Z}} V_j \), again using the invariance of \( L^2 \)-norm with respect to the shifts, we derive \( g(-b) \in \bigcup_{j \in \mathbb{Z}} V_j \).

For \( X \subset L^2(\mathbb{Q}_p) \), set \( \hat{X} = \{ \hat{f} : f \in X \} \). By the Wiener theorem for \( L^2 \) (see, e.g., [11]; all the arguments of the proof given there may be repeated word for word with replacing \( \mathbb{R} \) by \( \mathbb{Q}_p \)), a closed subspace \( X \) of the space \( L^2(\mathbb{Q}_p) \) is invariant with respect to the shifts if and only if \( \hat{X} = L^2(\Omega) \) for some set \( \Omega \subset \mathbb{Q}_p \). If now \( X = \bigcup_{j \in \mathbb{Z}} V_j \), then \( \hat{X} = L^2(\Omega) \). Thus \( X = L^2(\mathbb{Q}_p) \) if and only if \( \Omega = \mathbb{Q}_p \). Set \( \varphi_j = \varphi(p^{-j} \cdot) \), \( \Omega_0 = \bigcup_{j \in \mathbb{Z}} \text{supp} \hat{\varphi}_j \) and prove that \( \Omega = \Omega_0 \). Since \( \varphi_j \in V_j \), \( j \in \mathbb{Z} \), we have \( \text{supp} \hat{\varphi}_j \subset \Omega \), and hence \( \Omega_0 \subset \Omega \). Now assume that \( \Omega \setminus \Omega_0 \) contains a set of positive measure \( \Omega_1 \). Let \( f \in V_j \). Given \( \epsilon > 0 \), there exists a function \( f_\epsilon(x) := \sum_{a \in I_p} \alpha_a \varphi(p^j x - a) \), where the sum is finite, such that \( \| f - f_\epsilon \| < \epsilon \).

Using (1.3), we see that \( \text{supp} \hat{f_\epsilon} \subset \text{supp} \hat{\varphi}(p^{-j} \cdot) \), which yields that \( \hat{f_\epsilon}(\xi) = 0 \) for any \( \xi \notin \Omega_1 \). Due to the Plancherel theorem, it follows that \( \hat{f} = 0 \) almost everywhere on \( \Omega_1 \). Hence the same is true for any \( f \in \bigcup_{j \in \mathbb{Z}} V_j \). Passing to the limit we deduce that that the Fourier transform of any \( f \in X \) is equal to zero almost everywhere on \( \Omega_1 \), i.e., \( L^2(\Omega) = L^2(\Omega_0) \). It remains to note that \( \text{supp} \hat{\varphi}_j = \text{supp} \hat{\varphi}(p^j \cdot) \)

A real analog of Theorem 2.4 was proved by C. de Boor, R. DeVore and A. Ron in [14].

### 3. Refinable functions

We are going to study \( p \)-adic refinable functions \( \varphi \). Let us restrict ourselves to the consideration of \( \varphi \in D \). Evidently, each \( \varphi \in D \) is a \( p^M \)-periodic function for some \( M \in \mathbb{Z} \). Denote by \( D_N^M \) the set of all \( p^M \)-periodic functions supported on \( B_N(0) \). Taking the Fourier transform of the equality \( \varphi(x - p^M) = \varphi(x) \), we obtain \( \chi_p(p^M \xi) \hat{\varphi}(\xi) = \hat{\varphi}(\xi) \), which holds for all \( \xi \) if and only if \( \text{supp} \hat{\varphi} \subset B_M(0) \). Thus, the set \( D_N^M \) consists of all locally constant functions \( \varphi \) such that \( \text{supp} \varphi \subset B_N(0) \), \( \text{supp} \hat{\varphi} \subset B_M(0) \).

**Proposition 3.1.** Let \( \varphi, \psi \in L^2(\mathbb{Q}_p) \), \( \text{supp} \varphi, \text{supp} \psi \subset B_N(0) \), \( N \geq 0 \), and let \( b \in I_p \), \( |b| \leq p^N \). If
\[
\psi(-b) \in \text{span} \{ \varphi(p^{-j} x - a) : a \in I_p \}
\]

\( j = 0 \). Taking the Fourier transform of the equality \( \varphi(x - p^M) = \varphi(x) \), we obtain \( \chi_p(p^M \xi) \hat{\varphi}(\xi) = \hat{\varphi}(\xi) \), which holds for all \( \xi \) if and only if \( \text{supp} \hat{\varphi} \subset B_M(0) \). Thus, the set \( D_N^M \) consists of all locally constant functions \( \varphi \) such that \( \text{supp} \varphi \subset B_N(0) \), \( \text{supp} \hat{\varphi} \subset B_M(0) \).
then
\begin{equation}
\psi(x-b) = \sum_{k=0}^{p^{N+1}-1} h_{k,b} \varphi \left( \frac{x-k}{p^{N+1}} \right) \quad \forall x \in Q_p.
\end{equation}

Proof. Given \( \epsilon > 0 \), there exist functions
\[ f_\epsilon(x) := \sum_{a \in I_p} \alpha_a \varphi(p^j x - a), \quad g_\epsilon(x) := \sum_{a \in I_p} \alpha_a \varphi(p^j x - a), \]
where the sums are finite, such that \( \| \psi(\cdot - b) - f_\epsilon - g_\epsilon \| < \epsilon \). If \( x \in B_N(0) \), \( |a|_p > p^{N+1} \), then \( |p^{-1} x - a|_p > p^{N+1} \) and hence \( \varphi(p^{-1} x - a) = 0 \). So, \( g_\epsilon(x) = 0 \) whenever \( x \in B_N(0) \). If \( x \notin B_N(0) \), then \( \varphi(x - b) = 0 \) and \( \varphi(p^{-1} x - a) = 0 \) for all \( a \in I_p \), \( |a|_p \leq p^{N+1} \). So, \( \varphi(\cdot - b) - f_\epsilon(x) = 0 \) whenever \( x \notin B_N(0) \). It follows that
\[ \| \psi(\cdot - b) - f_\epsilon \|^2 = \int_{B_N(0)} |\psi(x-b) - f_\epsilon|^2 dx = \int_{B_N(0)} |\psi(x-b) - f_\epsilon - g_\epsilon| dx \leq \epsilon^2. \]

Hence
\[ \psi(\cdot - b) \in \text{span} \{ \varphi(p^{-1} x - a), \ a \in I_p, \ |a|_p \leq p^{N+1} \}, \]
which implies (3.2).

\[ \square \]

Corollary 3.2. If \( \varphi \in L^2(Q_p) \) is a refinable function and \( \text{supp} \varphi \subset B_N(0), \ N \geq 0 \), then its refinement equation is
\begin{equation}
\varphi(x) = \sum_{k=0}^{p^{N+1}-1} h_k \varphi \left( \frac{x-k}{p^{N+1}} \right) \quad \forall x \in Q_p.
\end{equation}

The proof immediately follows from Proposition 3.1.

Corollary 3.3. Let \( \varphi \in L^2(Q_p) \) be a scaling function of a MRA. If \( \text{supp} \varphi \subset B_N(0), \ N \geq 0 \), then \( \varphi \) is a refinable function satisfying (3.3).

The proof follows by combining axiom (a) of Definition 2.1 with Proposition 3.1.

Taking the Fourier transform of (3.3) and using (1.3), we can rewrite the refinable equation in the form
\begin{equation}
\tilde{\varphi}(\xi) = m_0 \left( \frac{\xi}{p^N} \right) \tilde{\varphi}(p\xi),
\end{equation}
where
\begin{equation}
m_0(\xi) = \frac{1}{p} \sum_{k=0}^{p^{N+1}-1} h_k \chi_p(k\xi)
\end{equation}
is a trigonometric polynomial. It is clear that \( m_0(0) = 1 \) whenever \( \tilde{\varphi}(0) \neq 0 \).

Proposition 3.4. If \( \varphi \in L^2(Q_p) \) is a solution of refinable equation (3.3), \( \tilde{\varphi}(0) \neq 0 \), \( \tilde{\varphi}(\xi) \) is continuous at the point 0, then
\begin{equation}
\tilde{\varphi}(\xi) = \tilde{\varphi}(0) \prod_{j=0}^{\infty} m_0 \left( \frac{\xi}{p^{N-j}} \right).\]
**Corollary 3.5.** If $\varphi \in \mathcal{D}_N^M$ is a refinable function, $N \geq 0$, and $\hat{\varphi}(0) \neq 0$, then (3.6) holds.

This statement follows immediately from Corollary 3.3 and Proposition 3.4.

**Lemma 3.6.** Let $\hat{\varphi}(\xi) = C \prod_{j=0}^{\infty} m_0 \left( \frac{\xi}{p^{N-j}} \right)$, where $m_0$ is a trigonometric polynomial with $m_0(0) = 1$ and $C \in \mathbb{R}$. If $\text{supp} \hat{\varphi} \subset B_M(0)$, then there exist at most $\frac{\deg m_0}{p}$ integers $n$ such that $0 \leq n < p^{M+N}$ and $\hat{\varphi} \left( \frac{n}{p^M} \right) \neq 0$.

**Proof.** First of all we note that $\hat{\varphi}$ is a $p^N$-periodic function satisfying (3.4). Denote by $O_p$ the set of positive integers not divisible by $p$. Since $\text{supp} \hat{\varphi} \subset B_M(0)$, we have $\hat{\varphi} \left( \frac{k}{p^M} \right) = 0$ for all $k \in O_p$. By the definition of $\hat{\varphi}$ the equality $\hat{\varphi} \left( \frac{k}{p^M} \right) = 0$ holds if and only if there exists $\nu = 1 - N, \ldots, M+1$ such that $m_0 \left( \frac{k}{p^M} \right) = 0$. Set

$$\sigma_\nu := \left\{ l \in O_p : l < p^{N+\nu}, m_0 \left( \frac{l}{p^{N+\nu}} \right) = 0, m_0 \left( \frac{l}{p^{N+\nu}} \right) \neq 0 \forall \mu = 1 - N, \ldots, \nu - 1 \right\},$$

$$v_\nu := \# \sigma_\nu.$$ Evidently, $\sigma_\nu \subset O'_p$ for all $\nu$, where $O'_p = \{ k \in O_p : k < p^{M+N+1} \}$, and $\sigma_\nu \cap \sigma_{\nu'} = \emptyset$ whenever $\nu' \neq \nu$. If $\hat{\varphi} \left( \frac{k}{p^M} \right) = 0$ for some $k \in O_p$, then there exist a unique $\nu = 1 - N, \ldots, M+1$ and a unique $l \in \sigma_\nu$ such that $k \equiv l \pmod{p^{N+\nu}}$. Moreover, for any $l \in \sigma_\nu$ there are exactly $p^{M-\nu+1}$ integers $k \in O'_p$ (including $l$) satisfying the above comparison. It follows that

$$\sum_{\nu=1-N}^{M+1} p^{M-\nu+1} v_\nu = 2 O'_p = p^{M+N} (p-1). \quad (3.7)$$

Now if $l \in \sigma_\nu$, $\nu \leq M$, then $\hat{\varphi} \left( \frac{p^\gamma k}{p^M} \right) = 0$ for all $\gamma = 0, 1, \ldots, M-\nu$, $k = l + rp^{N+\nu}$, $r = 0, 1, \ldots, p^{M-\nu-\gamma}-1$, i.e., each $l \in \sigma_\nu$ generates at least $1 + p + \cdots + p^{M-\nu}$ distinct positive integers $n < p^{M+N}$ for which $\hat{\varphi} \left( \frac{n}{p^M} \right) = 0$. Hence

$$v := \# \left\{ n : n = 0, 1, \ldots, p^{M+N} - 1, \hat{\varphi} \left( \frac{n}{p^M} \right) = 0 \right\} \geq$$

$$\sum_{\nu=1-N}^{M} (1 + p + \cdots + p^{M-\nu}) v_\nu = \frac{1}{p-1} \sum_{\nu=1-N}^{M} (p^{M-\nu+1} - 1) v_\nu =$$

$$\frac{1}{p-1} \sum_{\nu=1-N}^{M+1} (p^{M-\nu+1} - 1) v_\nu.$$
Since $\sum_{\nu=1-N}^{M+1} v_\nu \leq \deg m_0$, by using (3.7), we obtain

$$v \geq \frac{1}{p-1} \left( \sum_{\nu=1-N}^{M+1} p^{M-\nu+1} v_\nu - \deg m_0 \right) \geq p^{M+N} - \frac{\deg m_0}{p-1}. \tag{3.10}$$

For each $\varphi \in \mathcal{D}_N^M$, $M, N \geq 0$, we assign the set

$$L_\varphi = \left\{ l = 0, 1, \ldots, p^{M+N} - 1 : \hat{\varphi} \left( \frac{l}{p^M} \right) \neq 0 \right\}. \tag{3.11}$$

**Theorem 3.7.** Let $\varphi \in \mathcal{D}_N^M$, $M, N \geq 0$ and $\hat{\varphi}(0) \neq 0$. If

$$\varphi(\cdot - b) \in \text{span} \{ \varphi(p^{-1}x - a), a \in I_p \} \tag{3.12}$$

for all $b \in I_p$, $|b|_p \leq p^N$, then $\sharp L_\varphi \leq p^N$.

**Proof.** Let $b \in I_p$, $|b|_p \leq p^N$. Because of Proposition 3.1, we can rewrite (3.12) in the form

$$\varphi(x - b) = \sum_{k=0}^{p^{N+1}-1} h_{k,k} \hat{\varphi} \left( \frac{x - k}{p^{N+1}} \right) \quad \forall x \in \mathbb{Q}_p. \tag{3.13}$$

Taking the Fourier transform, we obtain

$$\hat{\varphi}(\xi) \chi_p(b\xi) = m_b \left( \frac{\xi}{p^N} \right) \hat{\varphi}(p\xi), \quad \forall \xi \in \mathbb{Q}_p, \tag{3.14}$$

where $m_b$ is a trigonometric polynomial, $\deg m_b < p^{N+1}$. Combining (3.15) for $b = 0$ with (3.15) for arbitrary $b$, we obtain

$$\hat{\varphi}(p\xi) \left( m_0 \left( \frac{\xi}{p^N} \right) \chi_p(b\xi) - m_b \left( \frac{\xi}{p^N} \right) \right) = 0 \quad \forall \xi \in \mathbb{Q}_p, \tag{3.16}$$

which is equivalent to

$$F(\xi) := \hat{\varphi} \left( p^{N+1}\xi \right) (m_0(\xi) \chi_p(p^N b\xi) - m_b(\xi)) = 0 \quad \forall \xi \in \mathbb{Q}_p. \tag{3.17}$$

Since $\text{supp} F \subset B_{M+N+1}(0)$ and $F$ is a 1-periodic function, (3.16) holds if and only if $\hat{\varphi} \left( \frac{l}{p^{M+N+1}} \right) = 0$, $l = 0, 1, \ldots, p^{M+N+1} - 1$.

First suppose that $\deg m_0 \geq p^N(p-1), i.e.,$

$$m_0(\xi) = \sum_{k=0}^{K} h_k \chi_p(k\xi), \quad h_K \neq 0, \tag{3.18}$$

where $K = K_N p^N + K_{N-1} p^{N-1} + \cdots + K_0$, $K_j \in D_p$, $j = 0, 1, \ldots, N$, $K_N = p - 1$ (indeed, if $K_N < p - 1$, then $\deg m_0 = K \leq (p-2)p^N + (p-1)(1+p+\cdots+p^{N-1}) = p^{N+1} - p^N - 1 < p^N(p-1)$). Set $b := p - p^{-N} K$. It is not difficult to see that $b \in I_p$, $|b|_p \leq p^N$ and $K + b p^N = p^{N+1}$. We see that the degree of the polynomial

$$t(\xi) := m_0(\xi) \chi_p(p^N b\xi) - m_b(\xi), \tag{3.19}$$

is exactly $p^{N+1}$, and hence there exist at most $p^{N+1}$ integers $l$ such that $0 \leq l < p^{M+N+1}$, $t \left( \frac{l}{p^{M+N+1}} \right) = 0$. Thus,

$$\sharp \left\{ l : l = 0, 1, \ldots, p^{M+N+1} - 1, \hat{\varphi} \left( \frac{l}{p^M} \right) = 0 \right\} \geq p^{M+N+1} - p^{N+1}. \tag{3.20}$$
Taking into account that \( \hat{\varphi} \) is a \( p^N \)-periodic function, we obtain
\[
(3.12) \quad \sharp \left\{ l : l = 0, 1, \ldots, p^{M+N} - 1, \hat{\varphi} \left( \frac{l}{p^M} \right) = 0 \right\} \geq p^{M+N} - p^N.
\]
It remains to note that (3.12) is also fulfilled whenever \( \deg m_0 < p^N(p-1) \) because of Lemma 3.6 and Corollary 3.5. \( \square \)

**Theorem 3.8.** Let \( \varphi \in D_N^M, M, N \geq 0, \sharp L_\varphi \leq p^N \), then
\[
(3.13) \quad \varphi(x - b) = \sum_{a \in I_p} \alpha_{a,b} \varphi(x - a) \quad \forall b \in \mathbb{Q}_p,
\]
where the sum is finite.

**Proof.** First we assume that \( b \in Q_p, |b|_p \leq p^N \), and prove that
\[
(3.14) \quad \varphi(x - b) = \sum_{k=0}^{p^{N-1}} \alpha_{k,b} \varphi \left( x - \frac{k}{p^N} \right) \quad \forall x \in \mathbb{Q}_p.
\]
Taking the Fourier transform, we reduce (3.14) to
\[
(3.15) \quad \hat{\varphi} \left( \chi_p(b) \right) = m_b \left( \frac{\xi}{p^N} \right) \hat{\varphi}(\xi), \quad \forall \xi \in \mathbb{Q}_p,
\]
where \( m_b \) is a trigonometric polynomial, \( \deg m_b < p^N \), which is equivalent to
\[
(3.16) \quad f(\xi) := \hat{\varphi} \left( p^N \xi \right) \left( \chi_p \left( p^N b \xi \right) - m_b(\xi) \right) = 0 \quad \forall \xi \in \mathbb{Q}_p.
\]
Since \( \text{supp} f \subset B_{M+N}(0) \) and \( f \) is a 1-periodic function, (3.16) is equivalent to
\[
f \left( \frac{l}{p^{M+N}} \right) = 0, \forall l = 0, 1, \ldots, p^{M+N} - 1,
\]
which holds if and only if
\[
(3.17) \quad m_b \left( \frac{l}{p^{M+N}} \right) = \chi_p \left( \frac{bl}{p^M} \right), \quad \forall l \in L_\varphi.
\]
Hence we can find \( m_b \) by solving the linear system (3.17) with respect to the unknown coefficients of \( m_b \). So, we proved (3.15), and hence (3.14).

Next let \( b \in Q_p, |b|_p = p^{N+1} \), i.e., \( b = \frac{b_{N+1}}{p^{N+1}} + b', b_{N+1} \in D_p, b_{N+1} \neq 0, |b'|_p \leq p^N \). Using (3.14) with \( b = b' \), we have
\[
\varphi(x - b) = \sum_{k=0}^{p^N-1} \alpha_{k,b'} \varphi \left( x - \frac{k}{p^N} - \frac{b_{N+1}}{p^{N+1}} \right) = \sum_{k=0}^{p^N-1} \alpha_{k,b'} \varphi \left( x - \frac{pk + b_{N+1}}{p^{N+1}} \right).
\]
Taking into account that \( pk + b_{N+1} \leq p(p^N - 1) + (p - 1) = p^{N+1} - 1 \), we derive
\[
\varphi(x - b) = \sum_{k=0}^{p^{N+1}-1} \alpha_{k,b'} \varphi \left( x - \frac{k}{p^{N+1}} \right) \quad \forall x \in \mathbb{Q}_p.
\]
Similarly, we can prove by induction on \( n \) that
\[
\varphi(x - b) = \sum_{k=0}^{p^{N+n} - 1} \alpha_{k,b} \varphi \left( x - \frac{k}{p^{N+n}} \right) \quad \forall x \in \mathbb{Q}_p,
\]
whenever \( b \in \mathbb{Q}_p, |b|_p = p^{N+n} \).

As a consequence we have the following statements.

**Corollary 3.9.** Let \( \varphi \in D^M_N \) be a refinable function, \( M, N \geq 0, L_\varphi \leq p^N \), and let the spaces \( V_j \) be defined by (2.1). Then axiom (a) of Definition 2.1. holds.

**Corollary 3.10.** If a test function \( \varphi \) with \( \hat{\varphi}(0) \neq 0 \) generates a MRA, then the corresponding spaces \( V_j, j \in \mathbb{Z} \), are invariant with respect to all translations.

**Theorem 3.11.** A function \( \varphi \in D^M_N, M, N \geq 0 \), with \( \hat{\varphi}(0) \neq 0 \) generates a MRA if and only if

1. \( \varphi \) is refinable;
2. there exist at most \( p^N \) integers \( l \) such that \( 0 \leq l < p^{M+N} \) and \( \hat{\varphi}\left(\frac{l}{p^M}\right) \neq 0 \).

**Proof.** If \( \varphi \) is a scaling function of a MRA, then (1) follows from Corollary 3.3, and (2) follows from (1) and Theorem 3.7.

Now let conditions (1), (2) be fulfilled. Define the spaces \( V_j, j \in \mathbb{Z} \), by (2.1). Axioms (d) and (e), evidently, hold. Axiom (a) follows from Corollary 3.9. Axiom (b) follows from Theorems 3.8 and 2.4. Axiom (c) follows from Theorems 2.2. □

**Example 3.12.** Let \( p = 2, N = 2, M = 1 \varphi \) be defined by (3.6), where \( \hat{\varphi}(0) \neq 0 \), \( m_0 \) is given by (3.5), \( m_0(1/4) = m_0(3/8) = m_0(7/16) = m_0(15/16) = 1 \) and \( m_0(0) = 0 \). It is not difficult to see that \( \text{supp} \hat{\varphi} \subset B_1(0) \), \( \text{supp} \hat{\varphi} \not\subset B_0(0) \) and \( \hat{\varphi}\left(\frac{1}{2}\right) = \hat{\varphi}\left(\frac{3}{4}\right) = \hat{\varphi}\left(\frac{5}{8}\right) = \hat{\varphi}(1) = 0 \), i.e., all the assumptions of Theorem 3.11 are fulfilled.

## 4. Orthogonal Scaling Functions

Now we are going to describe all orthogonal scaling functions \( \varphi \in D^M_N \).

**Theorem 4.1.** Let \( \varphi \in D^M_N, M, N \geq 0 \). If \( \{\varphi(x-a) : a \in I_p\} \) is an orthonormal system, then

\[
\sum_{l=0}^{p^{M+N}-1} \left| \hat{\varphi}\left(\frac{l}{p^M}\right) \right|^2 \chi_p\left(\frac{lk}{p^{M+N}}\right) = p^N \delta_{k0}, \quad k = 0, 1, \ldots, p^N - 1.
\]

**Proof.** Let \( a \in I_p \). Due to the orthonormality of \( \{\varphi(x-a) : a \in I_p\} \), using the Plancherel theorem, we have

\[
\delta_{a0} = \langle \varphi(\cdot), \varphi(\cdot-a) \rangle = \int_{\mathbb{Q}_p} \varphi(x)\varphi(x-a) \, dx = \int_{B_M(0)} |\hat{\varphi}(\xi)|^2 \chi_p(a\xi) \, d\xi.
\]
Let $\xi \in B_M(0)$. There exists a unique $l = 0, 1, \ldots, p^{M+N} - 1$ such that $\xi \in B_{-N}(b_l)$, $b_l = \frac{\xi}{p^N}$. It follows that

$$
\int_{B_M(0)} |\widehat{\varphi}(\xi)|^2 \chi_p(a\xi) \, d\xi = \sum_{k=0}^{p^{M+N} - 1} \int_{|\xi - b_l|_p \leq p^{-N}} |\widehat{\varphi}(\xi)|^2 \chi_p(a\xi) \, d\xi
= \sum_{l=0}^{p^{M+N} - 1} |\widehat{\varphi}(b_l)|^2 \int_{|\xi - b_l|_p \leq p^{-N}} \chi_p(a\xi) \, d\xi = \frac{1}{p^n} \Omega(|p^N a|_p) \sum_{l=0}^{p^{M+N} - 1} |\widehat{\varphi}(b_l)|^2 \chi_p(ab_l).
$$

To prove (4.1) it only remains to note that $\Omega(|p^N a|_p) = 0$ whenever $a \in I_p$, $p^N a \neq 0, 1, \ldots, p^N - 1$.

**Lemma 4.2.** Let $c_0, \ldots, c_{n-1}$ be mutually distinct elements of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Suppose that there exist nonzero reals $x_j$, $j = 0, 1, \ldots, n-1$, such that

$$
\sum_{j=0}^{n-1} c_j^k x_j = \delta_{k0}, \quad k = 0, 1, \ldots, n-1.
$$

Then $x_j = 1/n$ for all $j$, and up to reordering

$$
c_j = c_0 e^{2\pi j/n}, \quad j = 0, 1, \ldots, n-1.
$$

**Proof.** In accordance with Cramer’s rule we have $x_j = \frac{\Delta_j}{\Delta}$, $0 \leq j \leq n-1$, where $\Delta = V(c)$ is the Vandermonde determinant corresponding to $c = (c_0, \ldots, c_{N-1})$, and $\Delta_j$ is obtained from $\Delta$ by replacing the $j$-th column with the transpose of the row $(1,0,\ldots,0)$. A straightforward computation shows that

$$
\Delta_j = (-1)^j V(c^{(j)}) \prod_{k \neq j} c_k,
$$

where $c^{(j)}$ is obtained from $c$ by removing the $j$-th coordinate. Thus,

$$
x_j = (-1)^j V(c^{(j)}) \prod_{k \neq j} c_k = (-1)^j \prod_{k \neq j} c_k \prod_{k \neq j} (c_k - c_j) \prod_{k > l} (c_k - c_l)
= \prod_{k \neq j} \frac{c_k}{c_k - c_j} = \prod_{k \neq j} \frac{1}{1 - c_k^{-1} c_j}.
$$

Next, for any $\alpha \in \mathbb{R}$, we have

$$
1 - e^{i\alpha} = 2 \sin \frac{\alpha}{2} \left( \sin \frac{\alpha}{2} - i \cos \frac{\alpha}{2} \right) = 2 \sin \frac{\alpha}{2} e^{i\left( \frac{\alpha}{2} - \frac{\pi}{2} \right)}.
$$

Let us define $\alpha_j$, $j = 0, 1, \ldots, n-1$, by $c_j = e^{i\alpha_j}$. Then from the above arguments and (4.4) it follows that

$$
x_j = \prod_{k \neq j} \frac{1}{1 - c_k^{-1} c_j} = e^{i\gamma} \sum_{k \neq j} \left( 2 \sin \frac{\alpha_k - \alpha_j}{2} \right)^{-1},
$$
By Lemma 4.2, \( \theta = \frac{n}{2} \alpha_j + \pi \) for all \( j \). Thus, up to reordering \( \alpha_j = \alpha_0 + \frac{2\pi}{n} \), which implies (4.3), and consequently that \( x_j = 1/n \) for all \( j \).

**Theorem 4.3.** Let \( \varphi \in \mathcal{D}_N^M \) be an orthogonal scaling function and \( \hat{\varphi}(0) \neq 0 \). Then \( \text{supp} \hat{\varphi} \subset B_0(0) \).

**Proof.** Without loss of generality, we can assume that \( M, N \geq 0 \). Combining Theorems 3.11 and 4.1, we have

\[
\gamma = \sum_{k \neq j} \frac{\alpha_k - \alpha_j + \pi}{2} = \theta - \frac{n}{2} \alpha_j, \quad \theta = \frac{1}{2} \left( (n - 1)\pi + \sum_{k=0}^{n-1} \alpha_k \right)
\]

By the lemma’s hypothesis \( x_j \in \mathbb{R} \), whence \( \gamma \equiv 0 \) (mod \( \pi \)) and consequently \( n\alpha_j \equiv 2\theta \) (mod \( 2\pi \)). Thus, it follows from Theorem 3.11 that \( \hat{\varphi}(0) = 0 \), which implies (4.3), and consequently that \( x_j = 1/n \) for all \( j \).

So any test function \( \varphi \) generating a MRA belongs to the class \( \mathcal{D}_N^M \). All such functions were described in [10]. The following theorem summarizes these results.

**Theorem 4.4.** Let \( \hat{\varphi} \) be defined by (3.5), where \( m_0 \) is the trigonometric polynomial (3.6) with \( m_0(0) = 1 \). If \( m_0 \left( \frac{l}{p^M} \right) = 0 \) for all \( k = 1, \ldots, p^{N+1} - 1 \) not divisible by \( p \), then \( \varphi \in \mathcal{D}_N^M \). If, furthermore, \( m_0 \left( \frac{k}{p^M} \right) \neq 0 \) for all \( k = 1, \ldots, p^{N+1} - 1 \) divisible by \( p \), then \( \{ \varphi(x - a) : a \in I_p \} \) is an orthonormal system. Conversely, if \( \text{supp} \hat{\varphi} \subset B_0(0) \) and the system \( \{ \varphi(x - a) : a \in I_p \} \) is orthonormal, then \( m_0 \left( \frac{k}{p^M} \right) = 0 \) whenever \( \frac{k}{p^M} \) is not divisible by \( p \), \( m_0 \left( \frac{k}{p^M} \right) = 1 \) whenever \( \frac{k}{p^M} \) is divisible by \( p \), \( k = 1, 2, \ldots, p^{N+1} - 1 \), and \( |\hat{\varphi}(x)| = 1 \) for any \( x \in B_0(0) \).

**Theorem 4.5.** There exists a unique MRA generated by an orthogonal scaling test function.

**Proof.** Let a MRA \( \{ V_j \}_{j \in \mathbb{Z}} \) be generated by a test scaling function \( \varphi \) such that the system \( \{ \varphi(x - a) : a \in I_p \} \) is orthonormal. We prove that this MRA coincides with the Haar MRA \( \{ V_j^H \}_{j \in \mathbb{Z}} \) generated by the scaling function \( \varphi^H = \Omega(\cdot \theta) \). Evidently, it suffices to check that \( V_0 = V_0^H \). Let \( f \in V_0 \). It follows from Theorem 4.3 that \( \text{supp} \hat{f} \subset B_0(0) \). Hence \( \text{supp} \hat{f} \subset B_0(0) \), i.e. \( \hat{f} \in L^2(B_0(0)) \). It is well known that each continuous character of the additive group of the ring \( \mathbb{Z}_p \) is of the form \( \chi_p(a\xi), a \in I_p \). Since this group is compact, the Peter-Weyl theorem the set of all these characters is an orthonormal basis for \( L^2(B_0(0)) = L^2(\mathbb{Z}_p) \) (see, e.g. [16]). Thus we have \( \hat{f}(\xi) = \varphi^H(\xi) \sum_{a \in I_p} \alpha_a \chi_p(a\xi), \sum_{a \in I_p} |\alpha_a|^2 < \infty \). Taking the Fourier transform and using (1.3) and (1.4), we obtain \( f(x) = \sum_{a \in I_p} \varphi^H(x - a) \). So \( V_0 \subset V_0^H \). To prove the inclusion \( V_0^H \subset V_0 \) we will check that \( \varphi^H(x - b) \in V_0 \) for any \( b \in I_p \). By Theorem 4.4, the function \( (\hat{\varphi})^{-1} \) is bounded on \( B_0(0) \). This yields that \( (\hat{\varphi})^{-1} \in L^2(B_0(0)) \). Hence, \( \chi_p(b\xi)(\hat{\varphi}(\xi))^{-1} = \varphi^H(\xi) \sum_{a \in I_p} \beta_a \chi_p(a\xi) \),
\[ \sum_{a \in I_p} |\beta_a|^2 < \infty, \] which may be rewritten as \( \chi_p(b \xi) \varphi^H(\xi) = \tilde{\varphi}(\xi) \sum_{a \in I_p} \beta_a \chi_p(a \xi). \) Taking the Fourier transform and using again (1.3), (1.4), we obtain
\[ \varphi^H(x - b) = \sum_{a \in I_p} \beta_a \varphi(x - a). \]

5. Construction of wavelet frames

**Definition 5.1.** Let \( H \) be a Hilbert space. A system \( \{f_n\}_{n=1}^\infty \subset H \) is said to be a frame if there exist positive constants \( A, B \) (frame boundaries) such that
\[ A\|f\|^2 \leq \sum_{n=1}^\infty |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \quad \forall f \in H. \]

We are interested in the construction of \( p \)-adic wavelet frames, i.e., frames in \( L^2(\mathbb{Q}_p) \) consisting of functions \( p^{j/2} \psi^{(\nu)}(p^{-j}x - a), a \in I_p, \nu = 1, \ldots, r \).

Our general scheme of construction looks as follows. Let \( \{V_j\}_{j \in \mathbb{Z}} \) be a MRA. As above, we define the wavelet space \( W_j, j \in \mathbb{Z}, \) as the orthogonal complement of \( V_j \) in \( V_{j+1}, \) i.e., \( V_{j+1} = V_j \ominus W_j. \) It is not difficult to see that \( f \in W_j \) if and only if \( f(p^{j/2}) \in W_0, \) and \( W_j \perp W_k \) whenever \( j \neq k. \) If now there exist functions \( \psi^{(\nu)} \in L^2(\mathbb{Q}_p), \nu = 1, \ldots, r, \) \( (a \text{ set of wavelet functions}) \) such that
\[ W_0 = \text{span} \{ \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, a \in I_p \}, \]
then we have a wavelet system
\[ \{p^{j/2} \psi^{(\nu)}(p^{-j}x - a), \nu = 1, \ldots, r, a \in I_p, j \in \mathbb{Z}\}. \]

It will be proved that such a system is a frame in \( L^2(\mathbb{Q}_p) \) whenever \( \psi^{(\nu)} \) are compactly supported functions.

**Theorem 5.2.** Let \( \psi^{(\nu)}, \nu = 1, \ldots, r, \) be a set of compactly supported wavelet functions for a MRA \( \{V_j\}_{j \in \mathbb{Z}}. \) Then the system (5.2) is a frame in \( L^2(\mathbb{Q}_p). \)

**Proof.** First we will prove that the system \( \{\psi^{(\nu)}(\cdot - a), \nu = 1, \ldots, r, a \in I_p\} \) is a frame in the wavelet space \( W_0. \) Let \( \text{supp} \psi^{(\nu)} \subset B_N(0), \nu = 1, \ldots, r, N \geq 0. \) Set \( a^{n,l} = \frac{l}{p^{n-l}}, l \in L(n), \) where \( L(n) \) is the set of integers \( l, 0 \leq l < p^n, \) which are not divisible by \( p, \)
\[ W_0^n = \text{span} \{ \psi^{(\nu)}(x - a) : \nu = 1, \ldots, r, a \in I_p \cap B_N(0) \}, \]
\[ W_0^{n,l} = \text{span} \{ \psi^{(\nu)}(x - a) : \nu = 1, \ldots, r, a \in I_p \cap B_N(a^{n,l}) \}, \quad n \in \mathbb{N}, \quad l \in L(n). \]

Since the disks \( B_N(0), B_N(a^{n,l}) \) are mutually disjoint and the union of them is \( \mathbb{Q}_p, \) each function \( f \in W_0 \) may be represented in the form
\[ f = f^0 + \sum_{n=1}^\infty \sum_{l \in L(n)} f^{n,l}, \quad f^0 = f \big|_{B_N(0)}, \quad f^{n,l} = f \big|_{B_N(a^{n,l})}. \]

Due to (5.1), given \( \epsilon > 0, \) there exists a sum \( \sum_{a \in I_p} \sum_{\nu = 1}^r \alpha_\nu \psi^{(\nu)}(x - a) =: f_\epsilon(x), \) such that \( \|f - f_\epsilon\| < \epsilon. \) If \( x \in B_N(0), \) then \( f_\epsilon(x) = \sum_{a \in I_p \cap B_N(0)} \sum_{\nu = 1}^r \alpha_\nu \psi^{(\nu)}(x - a) =: f_\epsilon^0(x). \)

Since \( \text{supp} f^0 \subset B_N(0), \) \( \text{supp} f^0_\epsilon \subset B_N(0), \) we have
\[ \|f - f_\epsilon\|^2 \geq \int_{B_N(0)} |f - f_\epsilon|^2 = \int_{B_N(0)} |f_\epsilon^0 - f^0_\epsilon|^2 = \|f^0 - f^0_\epsilon\|^2. \]
Hence, \( f^0 \in W_0^0 \). Similarly, \( f^{n,l} \in W_0^{n,l} \). It is not difficult to see that the spaces \( W_0^0, W_0^{n,l} \) are mutually orthogonal. Thus we proved that

\[
W_0 = W_0^0 \oplus \left( \bigoplus_{n=1}^{\infty} \bigoplus_{i \in L(n)} W_0^{n,i} \right).
\]

Since \( W_0^0 \) is a finite dimensional space and \( \{\psi^{(\nu)}(\cdot - a), \, \nu = 1, \ldots, r, \, a \in I_p \cap B_N(0)\} \) is a representing system for \( W_0^0 \), this system is a frame. Hence there exist positive constants \( A, B \) such that

\[
A\|f^0\|^2 \leq \sum_{a \in I_p \cap B_N(0)} \sum_{\nu=1}^{r} |\langle f^0, \psi^{(\nu)}(\cdot - a) \rangle|^2 \leq B\|f^0\|^2 \quad \forall f^0 \in W_0^0.
\]

If \( f^{n,l} \in W_0^{n,l} \), we have

\[
\sum_{a \in I_p \cap B_N(a^{n,l})} \sum_{\nu=1}^{r} |\langle f^{n,l}, \psi^{(\nu)}(\cdot - a) \rangle|^2 = \sum_{a \in I_p \cap B_N(0)} \sum_{\nu=1}^{r} |\langle f^{n,l}, \psi^{(\nu)}(\cdot - a^{n,l} - a) \rangle|^2 = \sum_{a \in I_p \cap B_N(0)} \sum_{\nu=1}^{r} |\langle f^{n,l}(\cdot + a^{n,l}), \psi^{(\nu)}(\cdot - a) \rangle|^2.
\]

Since \( f^{n,l}(\cdot + a^{n,l}) \in W_0^0 \), it follows from (5.4) that

\[
A\|f^{n,l}\|^2 \leq \sum_{a \in I_p \cap B_N(n,l)} \sum_{\nu=1}^{r} |\langle f^{n,l}, \psi^{(\nu)}(\cdot - a) \rangle|^2 \leq B\|f^{n,l}\|^2 \quad \forall f^{n,l} \in W_0^{n,l}.
\]

Taking into account (5.3), we derive

\[
A\|f\|^2 \leq \sum_{a \in I_p \nu=1} \sum_{\nu=1}^{r} |\langle f, \psi^{(\nu)}(\cdot - a) \rangle|^2 \leq B\|f\|^2 \quad \forall f \in W_0.
\]

So, we proved that the system \( \{\psi^{(\nu)}(x - a), \, \nu = 1, \ldots, r, \, a \in I_p\} \) is a frame in \( W_0 \). Evidently, the system \( \{p^{1/2} \psi^{(\nu)}(px - a), \, \nu = 1, \ldots, r, \, a \in I_p\} \) is a frame in \( W_j \) with the same frame boundaries for any \( j \in \mathbb{Z} \). Since \( \bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{Q}_p) \), it follows that the union of these frames is a frame in \( L^2(\mathbb{Q}_p) \).

Now we discuss how to construct a desirable set of wavelet functions \( \psi^{(\nu)} \), \( \nu = 1, \ldots, r \). Let a MRA \( \{V_j\}_{j \in \mathbb{Z}} \) is generated by a scaling function \( \varphi \in \mathcal{D}'_N \), \( \hat{\varphi}(0) \neq 0 \). First of all we should provide \( \psi^{(\nu)} \in V_1 \). Let us look for \( \psi^{(\nu)} \) in the form

\[
\psi^{(\nu)}(x) = \sum_{k=0}^{p^{N+1}-1} g^{(\nu)}_k \varphi \left( \frac{x}{p} - \frac{k}{p^{N+1}} \right).
\]

Taking the Fourier transform and using (1.3), we have

\[
\hat{\psi}^{(\nu)}(\xi) = n^{(\nu)}_0 \left( \frac{\xi}{p^{N+1}} \right) \hat{\varphi}(p\xi),
\]

where \( n^{(\nu)}_0 \) is a trigonometric polynomial (wavelet mask) given by

\[
n^{(\nu)}_0(\xi) = \frac{1}{p} \sum_{k=0}^{p^{N+1}-1} g^{(\nu)}_k \chi_p(k\xi)
\]
Evidently,  \( \psi^{(\nu)} \in D_N^{M+1} \). By Theorem 3.7, there exist at least \( p^M + N - p^N \) integers \( l \) such that \( 0 \leq l < p^{M+N} \), \( \tilde{\varphi} \left( \frac{l}{p^N} \right) = 0 \). Choose \( u_0^{(\nu)} \) satisfying the following property: if \( l \in L_\varphi \), i.e. \( \tilde{\varphi} \left( \frac{l}{p^N} \right) \neq 0 \) for some \( l = 0, 1, \ldots, p^{M+N} - 1 \), then \( u_0^{(\nu)} \left( \frac{l}{p^N} \right) = 0 \). This yields that \( \tilde{\psi^{(\nu)}} \left( \frac{l}{p^N} \right) = 0 \) whenever \( 0 \leq l < p^{M+N} \), \( \tilde{\varphi} \left( \frac{l}{p^N} \right) \neq 0 \).

Let \( a, b \in I_p \). Using the Plancherel theorem and the arguments of Theorem 4.1, we have

\[
\langle \varphi(x - a), \psi^{(\nu)}(x - b) \rangle = \int_{\mathbb{Q}_p} \varphi(x - a) \overline{\psi^{(\nu)}(x - b)} \, dx = \\
\int_{B_M(0)} \tilde{\varphi}(\xi) \overline{\tilde{\psi^{(\nu)}}(\xi)} \chi_p((b - a)\xi) \, d\xi = \\
\sum_{l=0}^{p^{M+N}-1} \int_{|\xi| \leq p^{-N}} |\xi|^{-1} \tilde{\varphi}(\frac{l}{p^N}) \overline{\tilde{\psi^{(\nu)}}(\frac{l}{p^N})} \chi_p((b - a)\xi) \, d\xi = \\
\sum_{l=0}^{p^{M+N}-1} \tilde{\varphi} \left( \frac{l}{p^N} \right) \overline{\tilde{\psi^{(\nu)}} \left( \frac{l}{p^N} \right) \chi_p((b - a)\xi) \, d\xi = 0.}
\]

It follows that \( \overline{\text{span} \{ \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, a \in I_p \}} \subset V_0 \). On the other hand, due to Theorem 3.8, we have \( \overline{\text{span} \{ \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, a \in I_p \}} \subset V_1 \). Hence,

\[
\text{span} \{ \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, a \in I_p \} \subset W_0.
\]

It is clear from the proof of Theorem 3.8 that

\[
\varphi \left( x - \frac{l}{p^N} \right) = \sum_{k=0}^{p^{N+1}-1} h_{kl} \varphi \left( \frac{x - k}{p^{N+1}} \right), \quad l = 0, \ldots, p^N - 1,
\]

\[
\psi^{(\nu)} \left( x - \frac{l}{p^N} \right) = \sum_{k=0}^{p^{N+1}-1} g_{kl}^{(\nu)} \varphi \left( \frac{x - k}{p^{N+1}} \right), \quad l = 0, \ldots, p^N - 1, \quad \nu = 1, \ldots, r.
\]

If the functions in the right hand side can be expressed as linear combinations of the functions in the left hand side of (5.6), (5.7), i.e.

\[
\text{span} \left\{ \varphi \left( \frac{x}{p} - a \right), \ a \in I_p \cap B_{N+1}(0), \right\} \subset \text{span} \left\{ \varphi(x - a), \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, \ a \in I_p \cap B_N(0) \right\},
\]

then \( W_0 \subset \text{span} \{ \psi^{(\nu)}(x - a), \nu = 1, \ldots, r, a \in I_p \} \). Taking into account (5.5), we deduce that \( \psi^{(\nu)}, \nu = 1, \ldots, r, \) is a set of wavelet functions.
Inclusion (5.8) is, evidently, fulfilled whenever the linear system

\[ \sum_{k=0}^{p^{N+1}-1} h_{kl} x_k = 0, \quad l = 0, \ldots, p^N - 1, \]

\[ \sum_{k=0}^{p^{N+1}-1} g_{kl}^{(\nu)} x_k, \quad l = 0, \ldots, p^N - 1, \quad \nu = 1, \ldots, r \]

has no non-trivial solutions. In particular, in the case \( r = p - 1 \), the system has no non-trivial solutions if and only if the determinant is not equal to zero. It is not quite clear how to construct functions \( \psi \) providing (5.8) for arbitrary \( \varphi \), but we will show how to succeed in the case \( \deg m_0 \leq (p - 1)p^N \). Such a mask with \( p = 2 \) was presented in Example 3.12.

Assume that \( \deg m_0 \leq (p - 1)p^N \). In this case

\[ \varphi(x) = \sum_{k=0}^{(p-1)p^N} h_k \varphi \left( \frac{x}{p} - \frac{k}{p^{N+1}} \right). \]

Define the wavelet masks \( \eta_0^{(\nu)} \), \( \nu = 1, \ldots, p - 1 \), by

\[ n_0^{(\nu)}(\xi) = \chi_p \left((\nu - 1)p^N \xi \right) (\chi_p(\xi) - 1)^{p^N - L} L_{\varphi} \prod_{l \in L_{\varphi}} \left( \chi_p(\xi) - \chi_p \left( \frac{l}{p^{M+N}} \right) \right) = \frac{1}{p} \sum_{k=(\nu-1)p^N}^{\nu p^N} g_{k-(\nu-1)p^N} \chi_p(k\xi), \quad \nu = 1, \ldots, p - 1, \]

(recall that \( ^* L_{\varphi} \leq p^N \) because of Theorem 3.7). So, system (5.6), (5.7) looks as follows:

\[ \varphi \left( x - \frac{l}{p^N} \right) = \sum_{k=0}^{(p-1)p^N+l} h_{k-l} \varphi \left( \frac{x}{p} - \frac{k}{p^{N+1}} \right), \quad l = 0, \ldots, p^N - 1, \]

\[ \psi^{(\nu)} \left( x - \frac{l}{p^N} \right) = \sum_{k=(\nu-1)p^N+l}^{\nu p^N+l} g_{k-(\nu-1)p^N-l} \psi \left( \frac{x}{p} - \frac{k}{p^{N+1}} \right), \quad \nu = 1, \ldots, p - 1, \quad l = 0, \ldots, p^N - 1. \]

The determinant of the system equals to

\[
\begin{array}{cccccccc}
0 & h_0 & h_1 & \ldots & h_{p^N-1} & h_{p^N} & \ldots & h_{(\nu-1)p^N} & 0 & \ldots & 0 \\
0 & h_0 & h_{p^N-2} & \ldots & h_{p^N-1} & h_{(\nu-1)p^N-1} & \ldots & h_{(\nu-1)p^N} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & h_0 & h_1 & \ldots & h_{(\nu-2)p^N+1} & h_{(\nu-2)p^N+2} & \ldots & h_{(\nu-1)p^N} \\
g_{0} & g_1 & \ldots & g_{p^N-1} & g_{p^N} & 0 & \ldots & 0 & \ldots & 0 \\
g_0 & g_0 & \ldots & g_{p^N-2} & g_{p^N-1} & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & g_1 & g_2 & \ldots & g_{p^N} & \end{array}
\]

This determinant is so called resultant. The resultant is not equal to zero if and only if the algebraic polynomials with the coefficients \( g_0, g_1, \ldots, g_{p^N} \) and \( h_0, h_1, \ldots, h_{p^N} \) respectively do not have joint zeros (see, e.g., [15]). But this holds because the trigonometric polynomials \( m_0 \) and \( n_0^{(1)} \) do not have joint zeros by construction.
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