Scalar Perturbations and Stability of a Loop Quantum Corrected Kruskal Black Hole

Ramin G. Daghigh\textsuperscript{1}, Michael D. Green\textsuperscript{2}, Gabor Kunstatter\textsuperscript{3},

\textsuperscript{1} Natural Sciences Department, Metropolitan State University, Saint Paul, Minnesota, USA 55106
\textsuperscript{2} Mathematics and Statistics Department, Metropolitan State University, Saint Paul, Minnesota, USA 55106
\textsuperscript{3} Physics Department, University of Winnipeg, Winnipeg, MB Canada R3B 2E9

Abstract

We investigate the massless scalar field perturbations of a new loop quantum gravity motivated regular black hole proposed by Ashtekar et al. in [Phys.Rev.Lett. 121, 241301 (2018), Phys.Rev.D 98, 126003 (2018)]. The spacetime of this black hole is distinguished by its asymptotic properties: in Schwarzschild coordinates one of the metric functions diverges as \( r \to \infty \) even though the spacetime is asymptotically flat. We show that despite this unusual asymptotic behavior, the quasinormal mode potential is well defined everywhere when Schwarzschild coordinates are used. In addition to calculating the quasinormal mode spectrum and the ringdown waveform of the effective field theory metric, we propose a useful new approximate form of the metric. This new approximation makes the calculations significantly easier and is more suitable for such calculations than the previous approximation used in the literature to probe the global structure of the spacetime. While the calculations are in principle possible using the exact form of the metric, our approximation allows us to produce quasinormal mode frequencies and ringdown waveforms to high accuracy with manageable computation times. Our results indicate that this black hole model is stable against massless scalar field perturbations. We show that, compared to the Schwarzschild black hole, this black hole oscillates with higher frequency and less damping. We also observe a qualitative difference in the power-law tail of the ringdown waveform between this black hole model and the Schwarzschild black hole. This suggests the quantum corrections affect the behavior of the waves at large scales.
1 Introduction

The Nobel prize winning singularity theorem of Penrose [1] proves that general relativity invariably leads to singularity formation inside black hole event horizons, thereby signaling its own demise. It is commonly believed that the singularity will be resolved by the ultimate microscopic theory, presumably a version of quantum gravity, that describes the final stage of collapse. Since we are ostensibly quite far from a complete theory of quantum gravity that can accurately describe such a process, it is useful to utilize toy models in order to study the structure of the presumable non-singular complete spacetime associated with the formation and evaporation of regular black holes (RBHs). Although deviations from the Schwarzschild solution in the spherically symmetric case might normally only be significant near the Planck scale, it may in fact be that new physics enters at a different length scale, one that is accessible to astrophysical observations. This possibility is given more credence by consideration of the so-called information loss paradox [2]. According to Page’s arguments [3] any mechanism that enables information to emerge from an evaporating black hole horizon must become significant about half way through the evaporation process (the Page time) at which point the horizon radius can be very large. It is in fact millions of kilometers for galactic black holes. There are examples in the literature of dynamical theories in which the new length scale associated with singularity resolution is macroscopic. See for example [4] and [5].

Many models of RBH spacetimes have been studied over the years. See some of the Schwarzschild-like RBHs in [6–17]. In many cases, the spacetime metrics are not derived from any underlying microscopic theory and are not closely connected to a potential theory of quantum gravity. Two notable exceptions, in the context of loop quantum gravity (LQG), appear in [18] by Peltola and Kunstatter and more recently in [20,21] by Ashtekar, Olmedo and Singh in which complete regular static black hole spacetimes are derived as solutions to an effective theory motivated by LQG. For brevity we call the former the PK black hole and the latter the AOS black hole. An interesting feature of both these RBHs is that the singularity is in effect avoided by the removal of \( r = 0 \) from the spacetime and its replacement by a minimum area\(^2\) whose value is ultimately determined by the microscopic theory. As in the case of the Schwarzschild region, spatial slices that extend to spatial infinity describe an Einstein-Rosen wormhole with a minimum (throat) radius that changes with Kruskal time. For both the PK and AOS black holes, the throat radius shrinks to its minimal value before re-expanding. In the case of the PK black hole the throat radius re-expands to infinity producing in the future interior a single Kastner-type cosmological spacetime. For the AOS black hole the re-expansion proceeds until the formation of a second horizon whose radius is the same as that of the first horizon. The solution can then be analytically continued to produce a new (time-reversed) asymptotic region. The resulting Penrose diagram consists of an infinite tower of horizons and asymptotic spacetimes.

In a previous paper [22], three of the current authors studied the effects of singularity resolution on the response to perturbations of the PK black hole, including the quasinormal

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1 A RBH spacetime similar to that in [18] was also derived by Modesto [19].
2 LQG naturally leads to such a minimum area element.
mode (QNM) spectrum and the ringdown waveform. Such calculations are potentially relevant to gravitational wave observations as well as to determining the stability \[23\] of the black hole solutions under consideration. In this paper, we focus on scalar or spin-0 perturbations, which may allow us to test the alternative theories of gravity that involve scalar fields using the observational data. See for example \[24\] and \[25\].

The purpose of the present paper is to do a similar study of the regular spacetime of the AOS black hole \[20,21\]. This particular spacetime is noteworthy not only because of its close connection to an underlying microscopic theory, but also because of its interesting analytic structure and global properties. In Schwarzschild-like coordinates, the \(g_{00}\) metric component has a pre-factor of the form \((r/r_H)^\epsilon\), where \(r_H\) is the horizon radius and \(\epsilon \ll 1\) is a dimensionless parameter that derives from the microscopic theory. As a result, the Schwarzschild-like metric does not go to the Minkowski metric at \(r \rightarrow \infty\). As shown in \[26\], the metric is nonetheless asymptotically flat, as can be shown by a time dependent change of coordinates. It is therefore of interest to determine how this structure affects the asymptotic observables such as QNMs and waveforms associated with such macroscopic black holes. We note that perturbations of other RBH models are investigated in, for example, \[27–44\].

We structure the paper as follows. In Sec. 2 we set up the problem by introducing the QNM wave equation. In Sec. 3 we calculate the QNM complex frequencies of the AOS black hole using the 6\(^{th}\) order Wentzel–Kramers–Brillouin (WKB) method. In Sec. 4 we calculate the QNM frequencies using the improved asymptotic iteration method and compare the results to those in Sec. 3. In Sec. 5 we analyze the QNM spectrum for different values of the parameter \(\epsilon\) and multipole number \(l\). In Sec. 6 we produce and analyze the ringdown waveform for various values of \(\epsilon\) and \(l\). Finally, the summary and conclusion are presented in Sec. 7.

### 2 Wave Equation

A massive scalar field in the background of a black hole spacetime obeys the Klein-Gordon equation

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) - m^2 \Phi = 0,
\]

where \(m\) is the mass of the scalar field, \(g_{\mu\nu}\) is the metric and \(g\) is its determinant. Here we use Planck units where \(c = G = \hbar = 1\).

We apply the separation of variables

\[
\Phi(t, r, \theta, \phi) = Y_l(\theta, \phi) \Psi(t, r)/r,
\]

where \(Y_l(\theta, \phi)\) are spherical harmonics with the multipole number \(l = 0, 1, 2, \ldots\), together with the line element of a completely general, spherically symmetric, static spacetime

\[
ds^2 = -A(r)dt^2 + B(r)^{-1}dr^2 + r^2 d\Omega^2
\]
to obtain the QNM wave equation

\[ \frac{\partial^2 \Psi}{\partial t^2} + \left( -\frac{\partial^2}{\partial r^2} + V(r) \right) \Psi = 0. \]  (4)

In the above equation, \( r_* \) is the tortoise coordinate linked to the radial coordinate according to

\[ dr_* = \frac{dr}{\sqrt{A(r)B(r)}}, \]  (5)

and

\[ V(r) = A(r) \left[ \frac{l(l+1)}{r^2} + m^2 \right] + \frac{1}{2r} \frac{d}{dr} \left[ A(r)B(r) \right] \]  (6)

is the Regge-Wheeler or QNM potential. If we assume the perturbations depend on time as

\[ \Psi(t,r) = e^{-i\omega t} \psi(r), \]  (7)

we obtain the time-independent wave equation

\[ \frac{d^2 \psi}{dr^2_*} + \left[ \omega^2 - V(r) \right] \psi = 0, \]  (8)

where \( \omega \) is the complex QNM frequency to be determined.

The effective field theory metric functions of the static, spherically symmetric line element (3) for the exterior spacetime of the AOS black hole is provided explicitly by Ashtekar and Olmedo in [26]:

\[ A(r) = \left( \frac{r}{r_H} \right)^{2e} \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \left( 2 + \epsilon + \epsilon \left( \frac{r_H}{r} \right)^{1+\epsilon} \right)^2 \left( 2 + \epsilon \right)^2 - \epsilon^2 \left( \frac{r_H}{r} \right)^{1+\epsilon} \]  \[ \frac{16}{1 + \frac{\Lambda^2}{r^4} \left( \frac{r_H}{r} \right)^2} \left( 1 + \epsilon \right)^4 \]  (9)

and

\[ B(r) = \frac{\left( \frac{r}{r_H} \right)^{1+\epsilon} - 1}{\left( \frac{r}{r_H} \right)^{1+\epsilon}} \left( \frac{r}{r_H} \right)^{1+\epsilon} \left( 2 + \epsilon \right)^2 - \epsilon^2 \]  \[ \left( 1 + \frac{\Lambda^2}{r^4} \left( \frac{r_H}{r} \right)^2 \right) \left( \epsilon + \left( \frac{r}{r_H} \right)^{1+\epsilon} \left( 2 + \epsilon \right) \right)^2, \]  (10)

where \( \Lambda = \frac{\delta \tilde{c} L_0 \gamma}{4} \). Here \( \gamma \) is the Barbero-Immirzi parameter of LQG, \( L_0 \) is an infrared regulator introduced to make the phase space description well-defined, and \( \delta \tilde{c} \) is the quantum parameter necessary for the quantum correction in the exterior region of the black hole.

Assuming \( \Lambda \ll \epsilon \ll 1 \), the authors of [26] provide the approximate metric functions

\[ A(r) = \left( \frac{r}{r_H} \right)^{2e} \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \]  (11)

and

\[ B(r) = \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \]  (12)
to probe the global structure of the spacetime outside the horizon. In what follows, we show that the above approximation leads to a noticeably different QNM potential and consequently QNM frequency spectrum. For brevity, in the remainder of this paper, we will refer to the QNM potential (6) with the metric functions (9) and (10) as the AOS potential, \( V_{AOS} \). The QNM potential (6) with the approximate metric functions (11) and (12) will be referred to as the approximate potential, \( V_A \).

We point out that the Taylor series expansion of Eqs. (9) and (11) around \( \epsilon = 0 \) only match in the \( \epsilon^0 \) term. The same is true for Eqs. (10) and (12). To resolve this discrepancy, we introduce an improved approximation where

\[
A(r) = \left( \frac{r}{r_H} \right)^2 \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \frac{1 + \epsilon \left( 1 + \frac{r_H}{r} \right)}{(1 + \epsilon)^3}
\]

and

\[
B(r) = \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \frac{1 + \epsilon}{1 + \epsilon \left( 1 + \frac{r_H}{r} \right)}.
\]

The above functions have the same Taylor series expansion around \( \epsilon = 0 \) as the exact expressions in (9) and (10) up to the order of \( \epsilon \). Note that the functions (13) and (14) are not unique and one can achieve the same Taylor series up to the order of \( \epsilon \) with a variety of functions. However, the multiplication of the two improved approximate metric functions, \( A(r)B(r) \), form a perfect square that leads to a simpler expression for the tortoise coordinate that involves \( \sqrt{A(r)B(r)} \).

For the approximate metric functions (13) and (14), the QNM potential (6) takes the following form

\[
V_{IA}(r) = \left( \frac{r}{r_H} \right)^2 \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right) \frac{1 + \epsilon \left( 1 + \frac{r_H}{r} \right) l(l+1)}{r^2 \left( 1 + \epsilon \right)^2} + \frac{\epsilon + \left( \frac{r_H}{r} \right)^{1+\epsilon}}{r^2 \left( 1 + \epsilon \right)^2} ,
\]

where \( IA \) in the subscript stands for improved approximation. For \( \epsilon = 0 \), the above potential reduces to the Schwarzschild QNM potential for scalar perturbations.

To compare, in Figure 1, we plot all the three QNM potentials, i.e. \( V_{AOS} \), \( V_A \) and \( V_{IA} \), together with the Schwarzschild QNM potential. It is noteworthy that despite the divergence of the metric function \( A(r) \) as \( r \to \infty \), which has been argued to be unsettling in [45, 46], the QNM potential of this black hole model is well behaved as \( r \to \infty \) as can be seen in Figure 1. The two QNM potentials \( V_{AOS} \) and \( V_{IA} \) are almost identical, but \( V_A \) is significantly different than all the other cases in Figure 1.

The tortoise coordinate for this spacetime can be derived by combining Eqs. (5), (13) and (14):

\[
*\! = \int \frac{(1 + \epsilon) dr}{\left( \frac{r}{r_H} \right)^2 \left( 1 - \left( \frac{r_H}{r} \right)^{1+\epsilon} \right)} = -\frac{(1 + \epsilon)}{2} \frac{r^2}{r_H} 2F_1 \left( 1, \frac{2}{1 + \epsilon}; \frac{2}{1 + \epsilon}; \left( \frac{r}{r_H} \right)^{1+\epsilon} \right) + C
\]

where \( 2F_1(a, b; c; z) \) is the Hypergeometric function and \( C \) is the constant of integration, which should be chosen such that the tortoise coordinate has no imaginary component.
In the rest of this paper, we also choose $C$ so that the peak of the QNM potential in the tortoise coordinate is centered at $r_* = 0$. Note that when $\epsilon = 0$, the relationship between the tortoise coordinate and the radial coordinate reduces to the Schwarzschild case where $r_* = r + r_H \ln(r - r_H) + C$.

With our choice of time-dependence \[ (7) \], the boundary conditions at the event horizon and infinity are, respectively,

$$\begin{align*}
\psi(x) &\rightarrow e^{-i\omega r_*} \quad \text{as} \quad r_* \rightarrow -\infty \quad (r \rightarrow r_H), \\
\psi(x) &\rightarrow e^{i\omega r_*} \quad \text{as} \quad r_* \rightarrow \infty \quad (r \rightarrow \infty).
\end{align*} \tag{17}$$

For the purpose of brevity, in the rest of this paper, we choose units in which $r_H = 1$. That means $r, r_*$, and $t$ are expressed in units of $r_H$. The QNM frequency $\omega$ is in units of $r_H^{-1}$ and the units for the QNM potential $V(r)$ are $r_H^{-2}$.

## 3 The WKB Method

To calculate the QNMs of the AOS black hole, we use the WKB method. This method was originally applied to the problem of black hole perturbations by Schutz and Will in \[47\]. The formula for the 3rd order WKB method is derived in \[48\] and it is extended to the 6th order by Konoplya in \[49\].

To determine the QNM spectrum using the WKB method, one needs to solve the following equation \[48\]

$$\frac{i[\omega^2 - V(r_*)]_{r_*}}{\sqrt{2V''(r_*)}} - \sum_{j=2}^{N} \Lambda_j(n) = n + \frac{1}{2}, \tag{18}$$

where $r_*$ is the location of the maximum of the QNM potential $V(r_*)$ in the tortoise coordinate. Prime indicates differentiation with respect to $r_*$, and $\Lambda_j(n)$ are the WKB
correction terms. \( N \) indicates the order of the the WKB method. \( \Lambda_{2,3} \) are given in [48] and \( \Lambda_{4,5,6} \) can be found in [49].

In Tables I and II, for comparison, we provide the QNM complex frequencies for the multipole numbers \( l = 0 \) and \( l = 3 \) for the three different potentials \( V_{AOS}, V_A \) and \( V_{IA} \). The value of \( \epsilon \) in Table I is 0.1 and in Table II is 0.01. For \( \epsilon = 0.1 \), the QNMs of the potential with the improved approximation, \( V_{IA} \), matches the AOS potential, \( V_{AOS} \), up to three significant figures. The accuracy further improves for \( \epsilon = 0.01 \) where the QNMs of the two potentials (\( V_{IA} \) and \( V_{AOS} \)) match up to five significant figures. This shows that the improved approximation produces accurate results for small \( \epsilon \). On the other hand, the approximate potential \( V_A \) matches only up to one significant figure for \( \epsilon = 0.1 \) and up to two significant figures for \( \epsilon = 0.01 \).

The AOS metric functions [9] and [10] are far more complicated than the metric functions [13] and [14] in the improved approximation. The calculations are analytically more complex and numerically more intense for the AOS metric functions. Since the difference in the results is negligible for \( \epsilon \lesssim 0.1 \), as is evident in Tables I and II, it makes more sense to use the improved approximation. In the rest of this paper, we will focus mainly on the improved approximation.

| \( n, l \) | \( V_{AOS} \) | \( V_A \) | \( V_{IA} \) |
|---|---|---|---|
| 0,0 | \( 0.235193 - 0.196029i \) | \( 0.258287 - 0.215511i \) | \( 0.234806 - 0.195919i \) |
| 1,0 | \( 0.183430 - 0.679378i \) | \( 0.201299 - 0.747111i \) | \( 0.182999 - 0.679192i \) |
| 0,3 | \( 1.37350 - 0.191270i \) | \( 1.46611 - 0.208058i \) | \( 1.37062 - 0.191030i \) |
| 1,3 | \( 1.34342 - 0.579258i \) | \( 1.43196 - 0.630346i \) | \( 1.34052 - 0.578534i \) |
| 2,3 | \( 1.28746 - 0.982962i \) | \( 1.36835 - 1.07050i \) | \( 1.28450 - 0.981761i \) |
| 3,3 | \( 1.21360 - 1.41032i \) | \( 1.28428 - 1.53766i \) | \( 1.21057 - 1.40866i \) |
| 4,3 | \( 1.13183 - 1.86567i \) | \( 1.19091 - 2.03699i \) | \( 1.12868 - 1.86357i \) |
| 5,3 | \( 1.05210 - 2.35036i \) | \( 1.09928 - 2.57050i \) | \( 1.04880 - 2.34787i \) |

| \( n, l \) | \( V_{AOS} \) | \( V_A \) | \( V_{IA} \) |
|---|---|---|---|
| 0,0 | \( 0.222271 - 0.201083i \) | \( 0.22488 - 0.203094i \) | \( 0.222265 - 0.201083i \) |
| 1,0 | \( 0.178608 - 0.688112i \) | \( 0.180386 - 0.695005i \) | \( 0.178600 - 0.688124i \) |
| 0,3 | \( 1.35286 - 0.192846i \) | \( 1.36199 - 0.194566i \) | \( 1.35283 - 0.192843i \) |
| 1,3 | \( 1.32341 - 0.584098i \) | \( 1.33214 - 0.589335i \) | \( 1.32338 - 0.584099i \) |
| 2,3 | \( 1.26910 - 0.991208i \) | \( 1.27708 - 1.00019i \) | \( 1.26906 - 0.991195i \) |
| 3,3 | \( 1.19847 - 1.42165i \) | \( 1.20547 - 1.43471i \) | \( 1.19843 - 1.42163i \) |
| 4,3 | \( 1.12197 - 1.87886i \) | \( 1.12790 - 1.89639i \) | \( 1.12193 - 1.87883i \) |
| 5,3 | \( 1.05000 - 2.36289i \) | \( 1.05490 - 2.38535i \) | \( 1.04996 - 2.36286i \) |

In Table III, we provide the QNM complex frequencies for different values of the overtone number \( n \) and the multipole number \( l \) for the potential \( V_{IA} \). As indicated in [49], the WKB method works more accurately for lower values of \( n \) and higher values of \( l \). For example, for \( l = 0, 1, 2 \) we only find less than six reliable roots, while for \( l = 6 \) we find twelve reliable roots.

\(^3\Lambda_2 \) in [48] is missing a factor of \( i \) in the numerator.
of the WKB method. For example, in Figure 2 we see the real and imaginary parts of \( \omega \) for different values of \( \epsilon \) using 6\textsuperscript{th} order WKB method.

The reliability of the roots is determined by comparing the results for different orders of the WKB method. For example, in Figure 2 we see the real and imaginary parts of \( \omega \) converge as we increase the order of the WKB method for \( l = 0 \) and \( n = 1 \). That is not the case for \( l = 0 \) and \( n = 2 \). Convergence for higher values of \( n \) improves with higher...
values of \( l \). We illustrate this in Figure 2 where we plot the real and imaginary parts of \( \omega \) for \( l = 2 \) and \( n = 2 \) to be compared with the case of \( l = 0 \) and \( n = 2 \).

We also look at convergence plots, similar to Figure 2, to determine if the accuracy of the WKB method depends on the value of the parameter \( \epsilon \). We find no correlation.

4  The Improved Asymptotic Iteration Method

As mentioned earlier, the WKB method becomes less accurate for lower values of \( l \) and higher values of \( n \). Since the accuracy of the WKB method is in question, it is useful to compare the results to those found using other methods.

In this section, we use the improved asymptotic iteration method (AIM) described in [51] to determine QNM frequencies for the potential \( V_{IA} \) provided in Eq. (15). We briefly explain the AIM below and show how to adapt the method to our black hole model.

The AIM is useful for studying linear second order differential equations of the form

\[
\chi'' = \lambda_0(x) \chi' + s_0(x) \chi. \tag{19}
\]

For such equations, the higher derivatives of \( \chi \) can be expressed in terms of \( \chi' \) and \( \chi \) as

\[
\chi^{(n+1)} = \lambda_{n-1}(x) \chi' + s_{n-1}(x) \chi, \tag{20}
\]

where

\[
\begin{align*}
\lambda_n(x) &= \lambda_{n-1}(x) + s_{n-1}(x) + \lambda_0(x) \lambda_{n-1}(x) \\
s_n(x) &= s'_{n-1}(x) + s_0(x) \lambda_{n-1}(x). \tag{21}
\end{align*}
\]

One then can expand \( \lambda_n \) and \( s_n \) in a Taylor series around some point \( x_0 \):

\[
\begin{align*}
\lambda_n(x) &= \sum_{i=0}^{\infty} c_n^i (x - x_0)^i \\
s_n(x) &= \sum_{i=0}^{\infty} d_n^i (x - x_0)^i. \tag{22}
\end{align*}
\]

The recurrence relations for \( \lambda_n \) and \( s_n \) can now be written in terms of the coefficients \( c_n \) and \( d_n \) as follows:

\[
\begin{align*}
c_n^i &= (i + 1) c_{n-1}^{i+1} + d_{n-1}^i + \sum_{k=0}^{i} c_0^k c_{n-1}^{i-k} \\
d_n^i &= (i + 1) c_{n-1}^{i+1} + \sum_{k=0}^{i} d_0^k c_{n-1}^{i-k}. \tag{23}
\end{align*}
\]

We then make the assumption that, for large \( n \),

\[
\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)}. \tag{25}
\]
After combining Eqs. (22)-(25), one obtains an equation in terms of the Taylor series coefficients:

\[ d_n^0 c_n^{0} - d_{n-1}^0 c_n^0 = 0. \]  

(26)

The QNM frequency spectrum can be determined by solving the above equation.

To implement AIM, first we find the asymptotic behavior of the solution to the wave equation (8) for \( V(r) = V_{IA}(r) \) at the boundaries:

\[ \psi \xrightarrow{r \to 1} (r^{1+\epsilon} - 1)^{-i\omega} \text{ and } \psi \xrightarrow{r \to \infty} r^{i\omega(1-\epsilon)}e^{i\omega \frac{1+\epsilon}{1+\epsilon} r^{1-\epsilon}}. \]  

(27)

We then scale out the asymptotic behavior using

\[ \psi(r) = \left( \frac{r^{1+\epsilon} - 1}{r^{1+\epsilon}} \right)^{-i\omega} r^{i\omega(1-\epsilon)}e^{i\omega \frac{1+\epsilon}{1+\epsilon} (\frac{r}{r^{1+\epsilon}} - 1)}(r) \chi(r). \]  

(28)

The AIM requires a finite domain so we introduce the change of variable

\[ \xi = 1 - \frac{r^{1+\epsilon}}{1}, \]  

(29)

which transforms the domain \([1, \infty)\) to \([0, 1]\). Combining Eqs. (8), (28) and (29) leads to a differential equation for \( \chi(\xi) \) in the form of Eq. (19) where

\[ \lambda_0 = -\frac{1}{(1-\xi)} \left( \frac{2i\omega}{(1-\xi)^{1+\epsilon}} + 2i\omega \frac{\epsilon(1-\xi)}{1-\epsilon} + \frac{2i\omega}{\xi} - \frac{3 + \epsilon - 4i\omega}{1+\epsilon} \right) \]  

(30)

and

\[ s_0 = \frac{(1+\epsilon)^2 - \xi(1+\epsilon) + (1+\epsilon + \epsilon(1-\xi))}{(1-\xi)^{1+\epsilon}} l(l+1) - \frac{\omega^2}{\xi(1-\xi)^{1+\epsilon}} \]

\[ + \frac{i\omega(1-\xi)^{2+\epsilon}}{(1-\xi)^{2+\epsilon}} \left\{ \frac{4(1-\epsilon)^2}{(1-\xi)^{2+\epsilon}} + (1-\epsilon)^4 (1-\xi)^{1+\epsilon} + \frac{\epsilon(1-\epsilon^2)(2+\epsilon - 4i\omega)}{(1-\xi)^{1+\epsilon}} \right\} \]

\[ - (1-\epsilon^2) \left[ 5 + \epsilon(-2 + \epsilon + 4i\omega) - 4i\omega \right] - \frac{2i\omega}{(1-\xi)^{3+\epsilon}} \frac{(1-\epsilon^2)(1+\epsilon)}{(1-\xi)^{3+\epsilon}} - \frac{2i\omega}{(1-\xi)^{1+\epsilon}} - \frac{4i\omega}{(1-\xi)^{1+\epsilon}} \frac{(1-\epsilon^2)(1+\epsilon)}{(1-\xi)^{1+\epsilon}} \]

\[ + \frac{2i\omega}{(1-\xi)^{2+\epsilon}} - \frac{2i\omega}{(1-\xi)^{3+\epsilon}} - \frac{1}{(1-\xi)^{2+\epsilon}} - \frac{1}{(1-\xi)^{3+\epsilon}} \]

\[ - \frac{i\omega}{(1-\xi)^{2+\epsilon}} + \frac{2i\omega}{(1-\xi)^{2+\epsilon}} - \frac{\epsilon(1-\epsilon^2)(3+\epsilon - 2i\omega(3-\epsilon))}{(1-\xi)^{2+\epsilon}} \]

\[ + \frac{\epsilon(1-\epsilon^2)(1-2i\omega(1-\epsilon))}{(1-\xi)^{3+\epsilon}} + \frac{1}{(1-\xi)^{3+\epsilon}} - \frac{1}{(1-\xi)^{2+\epsilon}} - \frac{1}{(1-\xi)^{3+\epsilon}} \]

\[ - \frac{(1-\epsilon^2 - \frac{1}{(1-\xi)^{2+\epsilon}} - \frac{6 + \epsilon(14-9\epsilon + 3\epsilon^2)}{(1-\xi)^{2+\epsilon}})}{(1-\xi)^{2+\epsilon}} \].  

(31)
We now expand $\lambda_0$ and $s_0$ in a Taylor series around a point $\xi_0$. After selecting an appropriate depth [$n$ in Eq. (26)], we substitute the coefficients into (26) to obtain an equation in $\omega$. A root finder is then used to find the QNMs. Although the choice of $\xi_0$ should not make a difference, in practice there are small variations when changing $\xi_0$. We find setting $\xi_0$ to the $\xi$-value corresponding to the maximum of the potential produces the best results.

We present the results for $l = 0, 1, 2, 6$ for different values of the parameter $\epsilon$ in Table IV. These results are in good agreement with the WKB results presented in Table III. The roots produced by the AIM become more accurate as the depth ($n$ in Eq. (26)) increases. The roots included in Table IV are calculated using a depth of 110. However, to determine which of the roots ($\omega_n$) found at this depth are most reliable, we compare them with roots calculated at a depth of 100 ($\omega'_n$) and throw out those where $|\omega_n - \omega'_n| < 0.01$. For higher values of $l$ and lower values of $\epsilon$, the AIM is able to find more roots for this particular black hole.

| $n, l$ | $\epsilon = 0$ | $\epsilon = 0.01$ | $\epsilon = 0.1$ |
|-------|----------------|-----------------|----------------|
| 0.0   | 0.22091$ - 0.20979i$ | 0.22212$ - 0.20937i$ | 0.23333$ - 0.20557i$ |
| 0.1   | 0.58587$ - 0.19532i$ | 0.58708$ - 0.19514i$ | 0.59773$ - 0.19313i$ |
| 1.1   | 0.52989$ - 0.61254i$ | 0.52992$ - 0.61196i$ | 0.53867$ - 0.60543i$ |
| 1.2   | 0.45908$ - 1.08030i$ | 0.45913$ - 1.0817i$ | 0.4754$ - 1.0874i$ |
| 0.2   | 0.96729$ - 0.19352i$ | 0.96890$ - 0.19336i$ | 0.98272$ - 0.19150i$ |
| 1.2   | 0.92770$ - 0.59121i$ | 0.92922$ - 0.59070i$ | 0.94205$ - 0.58496i$ |
| 2.2   | 0.86109$ - 1.0171i$ | 0.86235$ - 1.0163i$ | 0.87256$ - 1.0067i$ |
| 3.2   | 0.78773$ - 1.4762i$ | 0.78806$ - 1.4754i$ | 0.80673$ - 1.4740i$ |
| 0.6   | 2.5038$ - 0.19261i$ | 2.5074$ - 0.19246i$ | 2.5383$ - 0.19067i$ |
| 1.6   | 2.4875$ - 0.57947i$ | 2.4911$ - 0.57900i$ | 2.5217$ - 0.57360i$ |
| 2.6   | 2.4557$ - 0.97120i$ | 2.4593$ - 0.97041i$ | 2.4891$ - 0.96130i$ |
| 3.6   | 2.4098$ - 1.3708i$ | 2.4133$ - 1.3697i$ | 2.4418$ - 1.3568i$ |
| 4.6   | 2.3521$ - 1.7810i$ | 2.3554$ - 1.7795i$ | 2.3820$ - 1.7628i$ |
| 5.6   | 2.2854$ - 2.2035i$ | 2.2884$ - 2.2018i$ | 2.3124$ - 2.1815i$ |
| 6.6   | 2.2130$ - 2.6396i$ | 2.2157$ - 2.6375i$ | 2.2365$ - 2.6142i$ |
| 7.6   | 2.1383$ - 3.0889i$ | 2.1405$ - 3.0867i$ | 2.1620$ - 3.0745i$ |
| 8.6   | 2.0641$ - 3.5506i$ | 2.0683$ - 3.5500i$ | 2.0895$ - 3.5378i$ |

5 An Analysis of the QNM Spectrum

In Figure 3, we plot the QNM spectrum for $\epsilon = 0.1$ for different values of $l$. As $l$ increases, the real part of the $n^{th}$ QNM frequency ($\omega_R$) increases, by a nearly fixed amount, and there is a small decrease in the magnitude of the imaginary part ($|\omega_I|$).

In Figure 4, we compare the QNM spectrum of the AOS black hole for $\epsilon = 0.1$ with the Schwarzschild spectrum ($\epsilon = 0$) for $l = 3$ and $l = 6$. In both graphs, we include the roots generated by the AIM and WKB method. The roots from both methods match well. For $\epsilon = 0.1$, the real part of the low damping QNMs is larger than the Schwarzschild modes, but $\omega_R$ becomes smaller than the Schwarzschild case for higher overtone QNMs. This is different than the behavior of the PK black hole QNM spectrum [22], which stays
parallel to the Schwarzschild spectrum. The damping rate ($|\omega_I|$) of the AOS black hole QNM spectrum is slightly lower than the Schwarzschild spectrum.

Figure 3: Scalar QNM spectrum for $\epsilon = 0.1$ for different values of $l$. The roots are generated using the WKB method.

Figure 4: Scalar QNM spectrum for $l = 3$, on the left, and $l = 6$, on the right, for $\epsilon = 0, 0.1$. Both graphs include the roots generated by the AIM and WKB method.
6 Ringdown Waveform

To generate the ringdown waveform, we numerically solve the time-dependent wave equation (4) using the initial data

$$\Psi(\tau_*, 0) = A \exp \left( -\frac{(\tau_* - \bar{\tau}_*)^2}{2\sigma^2} \right), \quad \partial_t \Psi|_{t=0} = -\partial_{\tau_*} \Psi(\tau_*, 0),$$ (32)

where we use $\sigma = 1$, $\bar{\tau}_* = -40$, and $A = 20$. We choose the observer to be located at $\tau_* = 90$. To carry out the calculations, we use the built-in *Mathematica* commands for solving partial differential equations.

In Figure 5, we compare the shape of the Schwarzschild potential ($\epsilon = \Lambda = 0$) in tortoise coordinate for $l = 0$ and $l = 2$ with the shape of $V_{AOS}$, $V_{IA}$ and $V_A$ for $\epsilon = 0.1$ and $\Lambda = 0$. As one can see, the height of the Schwarzschild potential is the lowest of all. The two QNM potentials $V_{AOS}$ and $V_{IA}$ are almost identical, but $V_A$ is significantly taller than all the other cases.

![Figure 5: QNM potential versus tortoise coordinate for $\epsilon = 0.1$, $\Lambda = 0$, $l = 0$ (left) and $l = 2$ (right). For comparison, we include $V_{AOS}$ in dashed black, $V_A$ in green, $V_{IA}$ in red. We also include the Schwarzschild QNM potential ($\epsilon = \Lambda = 0$) in blue. $V_{AOS}$ and $V_{IA}$ show almost perfect agreement.](image)

In Figure 6, we plot the ringdown waveform $\Psi$ and $\ln |\Psi|$, as a function of time, for $l = 0$ for all the three potentials $V_{AOS}$, $V_{IA}$, and $V_A$. We also include the Schwarzschild ringdown waveform for comparison. $V_{AOS}$ and $V_{IA}$ produce almost identical waveforms, but the waveform produced by $V_A$ is visibly different. The oscillation periods are easier to see in the log plot, where it is clear that the oscillation frequency is the lowest for the Schwarzschild case and highest for $V_A$. Another interesting feature observed in Figure 6 is the behavior of the power-law tail of the ringdown waveform, where the Schwarzschild case is different from all the other cases. This does not occur when $l \neq 0$ or for other RBHs, for which the waveform asymptotes to Schwarzschild. The studies of the Schwarzschild spacetime show a link between the power-law tail and the scattering waves at large radius \[52\]. Therefore, it is reasonable to infer that the difference in the power-law tail is due to the asymptotic behavior of the QNM potential of the Schwarzschild black hole, which drops as $1/r^3$ for $l = 0$, while $V_{AOS}$, $V_{IA}$ and $V_A$ all drop as $\approx 1/r^2$. 

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Figure 6: $\Psi$ (left) and $\ln |\Psi|$ (right) as a function of time for $l = 0$. In both graphs, we include the cases $\epsilon = 0$ (Schwarzschild) in dashed blue and $\epsilon = 0.1$ in dashed black for $V_{AOS}$, solid red for $V_{IA}$, and dotted green for $V_A$. $\Lambda$ is taken to be zero. The ringdown waveform for $V_{AOS}$ and $V_{IA}$ show almost perfect agreement.

Figure 7: $\Psi$ (left) and $\ln |\Psi|$ (right) as a function of time for $l = 1$. In both graphs, we include the cases $\epsilon = 0$ (Schwarzschild) in dashed blue and $\epsilon = 0.1$ in solid red for $V_{IA}$.

Figure 8: $\Psi$ (left) and $\ln |\Psi|$ (right) as a function of time for $l = 2$. In both graphs, we include the cases $\epsilon = 0$ (Schwarzschild) in dashed blue and $\epsilon = 0.1$ in solid red for $V_{IA}$.

In Figures 6 and 7 we plot the ringdown waveform $\Psi$ and $\ln |\Psi|$, as a function of time,
for $l = 1$ and $l = 2$ for the Schwarzschild potential and $V_{IA}$. In the log plot, it is clear that the oscillation frequency is higher for the AOS black hole.

To further check for the consistency between the numerically generated ringdown waveforms and QNM data provided in Tables III and IV, we use the Prony method \cite{53} to extract the first ($n = 0$) QNM frequency from the waveforms generated for $\epsilon = 0.1$ shown in Figures 9, 7, and 8. In the case of $l = 0$, we find $0.23334 - 0.20535i$ for $V_{IA}$ and $0.23373 - 0.20545i$ for $V_{AOS}$. For $l = 1$ and $l = 2$, the results are $0.59776 - 0.19300i$ and $0.98265 - 0.19145i$ respectively. These are all in good agreement with the data presented in Tables III and IV.

7 Summary and Conclusion

We studied massless scalar field perturbations in the background of the exterior of the AOS black hole. The spacetime of this black hole is rather unique in the sense that, in Schwarzschild coordinates, one of the metric functions diverges as $r \to \infty$ even though it was shown in [26] that the spacetime is asymptotically flat. We showed that despite this unusual asymptotic behavior, the QNM potential $V_{AOS}$ is well defined everywhere when Schwarzschild coordinates are used.

In addition to calculating the QNM spectrum and the ringdown waveform of the effective field theory metric, we proposed a useful new approximate form of the metric. This new approximation makes the calculations significantly easier and is more suitable for such calculations than the previous approximation used in [26] to probe the global structure of the spacetime. While the calculations are in principle possible using the exact form of the metric, our approximation allows us to produce QNM frequencies and ringdown waveforms to high accuracy with manageable computation times. In the the case of AIM, our approximation ($V_{IA}$) led to highly complicated expressions. The AIM may be intractable using the exact potential $V_{AOS}$.

We calculated the QNMs of the AOS black hole using the 6th order WKB method and the AIM. Both methods gave consistent results. The 6th order WKB method was applied to $V_{AOS}$, $V_A$, and $V_{IA}$. The QNM spectra of $V_{AOS}$ and $V_{IA}$ are almost identical for $\epsilon \lesssim 0.1$ and they both differ form the QNM spectrum produced by $V_A$. The AIM was applied to $V_{IA}$ and consistent results were found. We also examined the ringdown waveform of this black hole for all three cases ($V_{AOS}$, $V_A$ and $V_{IA}$) and compared all our results with the Schwarzschild case. Once again, The ringdown waveform produced by $V_{AOS}$ and $V_{IA}$ are almost identical and they differ from $V_A$ and the Schwarzschild case. As a consistency check, we calculated the least damped QNM ($n = 0$) of the ringdown waveforms using the Prony method and we found consistent results.

The QNM frequencies of the AOS black hole follow closely the QNM spectrum of a Schwarzschild black hole. We found no modes with positive damping, which indicates such a RBH is stable against massless scalar perturbations. We showed that an increase in the magnitude of $\epsilon$ increases the height of the QNM potential and gives oscillations with higher frequency and less damping.
An interesting aspect of the AOS black hole is the asymptotic behavior of corrections to the QNM potential, which drop off as $\approx 1/r^2$. When $l = 0$, the Schwarzschild QNM potential drops as $1/r^3$, so the $1/r^2$ correction dominates the potential for large $r$. For non-zero $l$, the correction is of the same order as the leading $r \to \infty$ term. This is in contrast to other regular asymptotically Schwarzschild black holes for which the quantum correction becomes negligible compared to the classical terms at large $r$. It seems reasonable to draw the conclusion that this $1/r^2$ correction is responsible for the qualitative difference observed in the power-law tail in Figure 6 between AOS and Schwarzschild waveforms at large times. This conclusion is supported by the studies of the Schwarzschild spacetime, which show that the power-law tail is caused by the scattering waves off the potential at large radius [52]. This suggests the quantum corrections affect the behavior of the waves at large scales.

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References

[1] R. Penrose, Gravitational Collapse and Space-Time Singularities, Phys. Rev. Lett. 14 57 (1965).
[2] See the review by D. Harlow, Jerusalem Lectures on Black Holes and Quantum Information, Rev. Mod. Phys. 88 015002 (2016).
[3] D. Page, Information in Black Hole Radiation, Phys. Rev. Lett. 71 3743 (1993).
[4] K.A. Bronnikov, V.N. Melnikov and H. Dehnen, On a general class of brane-world black holes, Phys. Rev. D68 024025 (2003).
[5] M. Okounkova, L.C. Stein, M.A. Scheel, and S.A. Teukolsky, Numerical binary black hole collisions in dynamical Chern-Simons gravity, Phys. Rev. D100 104026 (2019).
[6] J.M. Bardeen, Non-singular general-relativistic gravitational collapse, in Proceedings of the International Conference GR5, Tbilisi, USSR (Tbilisi University Press, Tbilisi, 1968), p. 174.
[7] E. Poisson and W. Israel, Structure of the Black Hole Nucleus, Class. Quant. Grav. 5 201-205 (1988).
[8] I. Dymnikova, Vacuum nonsingular black hole, Gen. Rel. Grav. 24 235-242 (1992).
[9] C. Barrabes and V.P. Frolov, How many new worlds are inside a black hole?, Phys. Rev. D53 3215 (1996).
[10] M. Mars, M.M. Martín-Prats, J.M.M. Senovilla, Models of regular Schwarzschild black holes satisfying weak energy conditions, Class. Quant. Grav. 13 L51 (1996).
[11] A. Cabo and E. Ayon-Beato, About black holes without trapping interior,
[12] A. Bogojevic and D. Stojkovic, A Nonsingular black hole, Phys. Rev. D61 084011 (2000).
[13] R. Casadio, A. Fabbri and L. Mazzacurati, New black holes in the brane-world?, Phys. Rev. D65 084040 (2002).
[14] S.A. Hayward, Formation and evaporation of regular black holes, Phys. Rev. Lett. 96 031103 (2006).
[15] K.A. Bronnikov and J.C. Fabris, Regular Phantom Black Holes, Phys. Rev. Lett. 96 251101 (2006).
[16] A. Simpson and M. Visser, Black-bounce to traversable wormhole, JCAP 02 042 (2019).
[17] D. Glavan and C. Lin, Einstein-Gauss-Bonnet gravity in 4-dimensional space-time, Phys. Rev. Lett. 124 081301 (2020).
[18] A. Peltola and G. Kunstatter, Complete, Single-Horizon Quantum Corrected Black Hole Spacetime, Phys. Rev. D79 061501 (2009); Effective Polymer Dynamics of D-Dimensional Black Hole Interiors, Phys. Rev. D80 044031 (2009).
[19] L. Modesto, Loop Quantum Black Hole, Class. Quant. Grav. 23 5587 (2006); Black Hole Interior from Loop Quantum Gravity, Adv. High Energy Phys. 2008, 459290 (2008); Space-Time Structure of Loop Quantum Black Hole, Int. J. Theor. Phys. 49 1649 (2010).
[20] A. Ashtekar, J. Olmedo, and P. Singh, Quantum Transfiguration of Kruskal Black Holes, Phys. Rev. Lett. 121 241301 (2018).
[21] A. Ashtekar, J. Olmedo, and P. Singh, Quantum Extension of the Kruskal Space-time, Phys. Rev. D98 126003 (2018).
[22] R.G. Daghigh, M.D. Green, J.C. Morey, and G. Kunstatter, Scalar Perturbations of a Single-Horizon Regular Black Hole, Phys. Rev. D102 104040 (2020).
[23] T. Regge and J.A. Wheeler, Stability of a Schwarzschild Singularity, Phys. Rev. 108 1063 (1957).
[24] Y. Hagihara, N. Era, D. Iikawa, N. Takeda, and H. Asada, Condition for directly testing scalar modes of gravitational waves by four detectors, Phys. Rev. D101 041501 (2020).
[25] C. Dalang, P. Fleury, and L. Lombriser, Scalar and tensor gravitational waves, arXiv:2009.11827.
[26] A. Ashtekar and J. Olmedo, Properties of a recent quantum extension of the Kruskal geometry, Int.J.Mod.Phys.D 29 (2020) 10, 2050076; arXiv:2005.02309.
[27] J.H. Chen and Y.J. Wang, “Complex frequencies of a massless scalar field in loop quantum black hole spacetime,” Chin. Phys. B20 030401 (2011).
[28] S. Fernando and J. Correa, Quasinormal Modes of Bardeen Black Hole: Scalar Perturbations, Phys. Rev. D86 064039 (2012).
[29] K.A. Bronnikov, R.A. Konoplya, A. Zhidenko, Instabilities of wormholes and regular black holes supported by a phantom scalar field, Phys. Rev. D86 024028 (2012).
[30] A. Flachi, J. P. S. Lemos, Quasinormal modes of regular black holes, Phys. Rev. D87 024034 (2013).
[31] K. Lin, J. Li, S.Z. Yang, Quasinormal modes of Hayward regular black hole, Int. J. Theor. Phys. 52 3771-3778 (2013).
[32] J. Li, M. Hong and K. Lin, Dirac quasinormal modes in spherically symmetric regular black holes, Phys. Rev. D88 064001 (2013).
[33] S. Fernando, T. Clark, Black holes in massive gravity: quasi-normal modes of scalar perturbations, Gen. Relativ. Gravit. 46 1834 (2014).
[34] C.F.B. Macedo, L.C.B. Crispino, Absorption of planar massless scalar waves by Bardeen regular black holes, Phys. Rev. D90 064001 (2014).
[35] J. Li, K. Lin, N. Yang, Nonlinear electromagnetic quasinormal modes and Hawking radiation of a regular black hole with magnetic charge, Eur. Phys. J. C75 131 (2015).
[36] B. Toshmatov, A. Abdujabbarov, Z. Stuchlík and B. Ahmedov, Quasinormal modes of test fields around regular black holes, Phys. Rev. D91 083008 (2015).
[37] V. Santos, R.V. Maluf, C.A.S. Almeida, Quasinormal frequencies of self-dual black holes, Phys. Rev. D93 084047 (2016).
[38] J. Li, K. Lin, H. Wen, W.-L. Qian, Gravitational Quasinormal Modes of Regular Phantom Black Hole, Advances in High Energy Phys. 2017 5234214 (2017).
[39] B. Toshmatov, Z. Stuchlík, J. Schee and B. Ahmedov, Electromagnetic perturbations of black holes in general relativity coupled to nonlinear electrodynamics, Phys. Rev. D97 084058 (2018).
[40] B. Toshmatov, Z. Stuchlík and B. Ahmedov, Electromagnetic perturbations of black holes in general relativity coupled to nonlinear electrodynamics: Polar perturbations, Phys. Rev. D98 085021 (2018).
[41] B. Toshmatov, Z. Stuchlík, B. Ahmedov, D. Malafarina, Relaxations of perturbations of spacetimes in general relativity coupled to nonlinear electrodynamics, Phys. Rev. D99 064043 (2018).
[42] H. Chakrabarty, A. A. Abdujabbarov and C. Bambi, Scalar perturbations and quasi-normal modes of a non-linear magnetic-charged black hole surrounded by quintessence, Eur. Phys. J. C79 179 (2019).
[43] M.B. Cruz, C.A.S. Silva, F.A. Brito, Gravitational axial perturbations and quasinormal modes of loop quantum black holes, Eur. Phys. J. C79 157 (2019).
[44] Liu, Tao Zhu, Qiang Wu, Kimet Jusufi, Mubasher Jamil, Mustapha Azreg-Aïnou, Anzhong Wang, Shadow and Quasinormal Modes of a Rotating Loop Quantum Black Hole, Phys. Rev. D101 084001 (2020).
[45] M. Bouhmadi-López, S. Brahma, C.-Y. Chen, P. Chen, D. Yeom, Asymptotic non-flatness of an effective black hole model based on loop quantum gravity, Phys.Dark Univ. 30 (2020) 100701
[46] V. Faraoni, A. Giusti, Unsettling physics in the quantum-corrected Schwarzschild black hole, Symmetry 12 (2020) 1264, arXiv:2006.12577
[47] B.F. Schutz and C.M. Will, Black hole normal modes - A semianalytic approach, Astrophys.
J. 291 L33 (1985).

[48] S. Iyer and C.M. Will, Black-hole normal modes: A WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering, Phys. Rev. D35 3621 (1987).

[49] R.A. Konoplya, Quasinormal behavior of the D-dimensional Schwarzshild black hole and higher order WKB approach, Phys. Rev. D68 024018 (2003).

[50] These results are generated using Mathematica code provided by Roman Konoplya.

[51] H.T. Cho, A.S. Cornell, J. Doukas, T.R. Huang, W. Naylor, A New Approach to Black Hole Quasinormal Modes: A Review of the Asymptotic Iteration Method, Adv. Math. Phys. 2012 281705 (2012)

[52] R.H. Price, Nonspherical Perturbations of Relativistic Gravitational Collapse. I. Scalar and Gravitational Perturbations, Phys. Rev. D5 2419 (1972).

[53] G. de Prony, Journal de l’École Polytechnique 1(2), 24 (1795); S. L. Marple, Digital spectral analysis with applications, (Prentice-Hall, New Jersey, 1987).