Proof of a conjecture
on the algebraic connectivity of a graph and its complement

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Abstract
For a graph $G$, let $\lambda_2(G)$ denote its second smallest Laplacian eigenvalue. It was conjectured that $\lambda_2(G) + \lambda_2(\overline{G}) \geq 1$, where $\overline{G}$ is the complement of $G$. Here, we prove this conjecture in the general case. Also, we will show that $\max\{\lambda_2(G), \lambda_2(\overline{G})\} \geq 1 - O(n^{-1})$, where $n$ is the number of vertices of $G$.

1 Introduction
Let $G$ be a simple graph with $n$ vertices. The Laplacian of $G$ is defined to be $L := D - A$, where $A$ is the adjacency matrix of $G$ and $D$ is the diagonal matrix of vertex degrees. The eigenvalues of $L$,

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G),$$

expose various properties of $G$. The “algebraic connectivity” of a graph $\lambda(G) := \lambda_2(G)$ is an efficient measure for connectivity of graphs.

The Laplacian spread of a graph $G$ is defined to be $\lambda_n(G) - \lambda_2(G)$. It was conjectured [YL12, ZSH11] that this quantity is at most $n - 1$, or equivalently, $\lambda_2(G) + \lambda_2(\overline{G})$ is at least 1 (since $\lambda_n(G) = n - \lambda_2(\overline{G})$).

Conjecture. For any graph $G$ of order $n \geq 2$, the following holds:

$$\lambda(G) + \lambda(\overline{G}) \geq 1,$$

with equality if and only if $G$ or $\overline{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n - 1$.

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This conjecture was proved for trees \[FXWL08\], unicyclic graphs \[BTF09\], bicyclic graphs \[FLT10\], tricyclic graphs \[CW09\], cactus graphs \[Liu10\], quasi-tree graphs \[XM11\], graphs with diameter not equal to 3 \[ZSH11\], bipartite and \(K_3\)-free graphs \[CD16\]. Also, \[AAMM18\] provided the constant lower bound \(\frac{2}{5}\) for \(\lambda(G) + \lambda(\overline{G})\), by proving that the maximum of \(\lambda(G)\) and \(\lambda(\overline{G})\) is greater than \(\frac{2}{5}\).

Here, we prove the conjecture, in the general case. The main idea of the proof is as follows. Suppose that \(x\) and \(y\), respectively, are eigenvectors of \(L(G)\) and \(L(\overline{G})\), corresponding to the eigenvalues \(\lambda_2(G)\) and \(\lambda_2(\overline{G})\), with \(||x||_2 = ||y||_2 = 1\).

We have
\[
\lambda(G) + \lambda(\overline{G}) = \sum_{\{i,j\} \in E(G)} (x_i - x_j)^2 + \sum_{\{i,j\} \notin E(G)} (y_i - y_j)^2 \\
\geq \sum_{i < j} \min\{(x_i - x_j)^2, (y_i - y_j)^2\}.
\]

So, it is enough to show that
\[
\sum_{i < j} \min\{(x_i - x_j)^2, (y_i - y_j)^2\} \geq 1.
\]

Now, suppose that \(M\) is the maximum of all of numbers \(|x_i - x_j|^2|\) and \(|y_i - y_j|^2|\), for every \(1 \leq i, j \leq n\). A simple algebraic identity (see Lemma 3), shows that \(\sum_{i < j} \min\{|x_i - x_j|^2|, |y_i - y_j|^2|\} \geq \frac{1}{M^2}\). So, if \(M \leq 1\), the proof is complete. It remains to consider the case \(M > 1\). We manage this case with some basic properties of the effective resistances between pairs of vertices of \(G\) and a useful characterization of the algebraic connectivity of graphs due to Fiedler (see Lemma 1).

Moreover, here, we give an asymptotic lower bound for the maximum of \(\lambda(G)\) and \(\lambda(\overline{G})\) by proving the following theorem.

**Theorem.** For all simple graphs \(G\) with \(n\) vertices,
\[
\max\{\lambda(G), \lambda(\overline{G})\} \geq 1 - O(n^{-\frac{1}{4}}).
\]

Also, for each \(n \geq 4\), there is a graph \(G\) which has \(n\) vertices and the maximum of \(\lambda(G)\) and \(\lambda(\overline{G})\) is less than 1. So, if \(c_n\) denote the minimum of \(\max\{\lambda(G), \lambda(\overline{G})\}\) over all graphs \(G\) with \(n\) vertices, then \(c_n = 1 - o(1)\).

The organization of the remaining of the paper is as follows. In Section 2, we gives some preliminaries and notations, which is necessary. In Section 3, we present two main results of the paper.

## 2 Preliminaries and notations

Throughout this paper \(G\) is a simple graph, with \(n \geq 2\) vertices \(V(G) := \{v_1, \ldots, v_n\}\) and edges \(E(G)\). The notation \(\{i, j\} \in E(G)\), for \(i, j \in \{1, \ldots, n\}\),
means that \( v_i \) and \( v_j \) are adjacent in \( G \). Also, \( A(G) \) denotes the adjacency matrix of \( G \). \( D(G) \) is the diagonal matrix of vertex degree. We denote the Laplacian matrix of \( G \) by \( L(G) := D(G) - A(G) \) and its eigenvalues by \( \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G) \leq n \) and the algebraic connectivity of \( G \) by \( \lambda(G) := \lambda_2(G) \). \( \overline{G} \) and \( \text{diam}(G) \), respectively, denote the complement of \( G \) and the diameter of \( G \). If \( v \) is a vertex of \( G \), then \( N_G(v) \) denotes the set of vertices which are adjacent to \( v \) in \( G \).

Recall that for each \( 1 \leq i < n \), we have

\[
\lambda_{i+1}(G) = n - \lambda_{n+1-i}(\overline{G}).
\]

The effective resistance between two vertices \( v_r \) and \( v_s \) in a graph \( G \) is denoted by \( R_{G}^{r,s} \) and defined by:

\[
\frac{1}{R_{G}^{r,s}} := \min \sum_{\{i,j\} \in E(G)} (x_i - x_j)^2,
\]

where the minimum runs over all \( x \in \mathbb{R}^n \), with \( x_r - x_s = 1 \). The effective resistance obeys the basic rules of the total resistance for parallel and series electrical circuits. Also, the effective resistance (as is trivial by the definition) does not increase by adding edges.

We conclude this section with a useful lemma, due to Fiedler.

**Lemma 1** ([Fie75]). \( \lambda(G) \) is the largest real number for which the following inequality holds for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \):

\[
\sum_{\{i,j\} \in E(G)} (x_i - x_j)^2 \geq \frac{\lambda(G)}{n} \sum_{i<j} (x_i - x_j)^2.
\]

### 3 Main results

#### 3.1 Sum of the algebraic connectivity of a graph and its complement

**Theorem 1.** For any graph \( G \) of order \( n \geq 2 \), the following holds:

\[
\lambda(G) + \lambda(\overline{G}) \geq 1,
\]

with equality if and only if \( G \) or \( \overline{G} \) is isomorphic to the join of an isolated vertex and a disconnected graph of order \( n - 1 \).

Note that for any connected graph \( G \), \( \lambda(G) \) is a positive number. So, when \( G \) is a disconnected graph, Theorem 1 can be reformulated as the following lemma:

**Lemma 2.** If \( G \) is a disconnected graph, then \( \lambda(\overline{G}) \geq 1 \), with equality if and only if \( \overline{G} \) is the join of an isolated vertex and a disconnected graph.

**Proof.** We know that the eigenvalues of the Laplacian of \( G \) are the union of the eigenvalues of its connected components where each of which has less than \( n \)
vertices. So, \( \lambda(G) = n - \lambda_n(G) \geq 1 \), with equality if and only if \( G \) is the disjoint union of an isolated vertex and a connected graph \( H \) with \( n - 1 \) vertices, where \( \lambda_{n-1}(H) = n - 1 \). But, note that \( \lambda_{n-1}(H) = n - 1 - \lambda_2(H) \). So, we have \( \lambda_{n-1}(H) = n - 1 \), if and only if \( \overline{H} \) is disconnected. Therefore, we have equality \( \lambda(G) = 1 \), if and only if \( \overline{G} \) is the join of an isolated vertex and a disconnected graph \( \overline{H} \).

\[\Box\]

**Lemma 3.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two vectors in \( \mathbb{R}^n \) with \( \sum_i y_i = \sum_i x_i = 0 \). Then

\[\sum_{i<j}(x_i - x_j)^2(y_i - y_j)^2 \geq \|x\|^2\|y\|^2.\]

**Proof.** Denote \( A = \sum_{i<j}(x_i - x_j)^2(y_i - y_j)^2 \). We have

\[2A = \sum_{i,j}(x_i^2 + x_j^2 - 2x_i x_j)(y_i^2 + y_j^2 - 2y_i y_j)\]

\[= 2\sum_i x_i^2 y_i^2 + 2\sum_i x_i^2 \sum_i y_i^2 + 4\sum_i x_i y_i^2 \geq 2\|x\|^2\|y\|^2.\]

\[\Box\]

**Lemma 4.** Suppose that \( G \) is a connected graph with \( n \geq 2 \) vertices \( \{v_1, \ldots, v_n\} \). Let \( x = (x_1, \ldots, x_n) \) be an eigenvalue of \( L(G) \) corresponding to \( \lambda_2(G) \). If

\[x_1 = \max_i x_i, \quad x_2 = \min_i x_i,\]

and the distance between \( v_1 \) and \( v_2 \) is at most equal to \( 2 \), then \( \lambda(G) \geq 1 \). In particular, for any graph with diameter less than \( 3 \), we have \( \lambda(G) \geq 1 \).

**Proof.** Since \( x \) is orthogonal to the constant-entries vector \( (1, \ldots, 1) \), \( \sum_i x_i = 0 \). So, \( x_2 < 0 < x_1 \). Now, from \( L(G)x = \lambda x \), we get \( \lambda x_1 = \sum_{(i,j) \in E(G)}(x_i - x_j) \). But, for each \( i \), \( x_i - x_j \geq 0 \). Thus, for every \( v_i \sim v_1 \), we have \( x_1 - x_i \leq \lambda x_1 \), or, equivalently, \( x_i \geq (1 - \lambda)x_1 \). Similarly, for every \( v_j \sim v_2 \), we have \( x_j \leq (1 - \lambda)x_2 \). Therefore, if \( \lambda < 1 \), for every \( v_i \sim v_1 \) and \( v_j \sim v_2 \) we have \( x_j < 0 < x_i \). So \( v_1 \) and \( v_2 \) are not adjacent and have no common neighbor. Thus, the distance between \( v_1 \) and \( v_2 \) is greater than \( 2 \).

\[\Box\]

**Proof of Theorem 1.** According to Lemma\(^2\) we can suppose that both \( G \) and \( \overline{G} \) are connected. Also, we consider \( n > 2 \) (the case \( n = 2 \) is obvious). Denote \( \lambda_2(G) \) and \( \lambda_2(\overline{G}) \), respectively by \( \lambda \) and \( \overline{\lambda} \). Let \( x = (x_1, \ldots, x_n) \) be a normal eigenvector of \( L(G) \) corresponding to \( \lambda_2(G) \) and \( y = (y_1, \ldots, y_n) \) be a normal eigenvector of \( L(\overline{G}) \) corresponding to \( \lambda_2(\overline{G}) \). Note that \( y \) is also an eigenvector of \( L(\overline{G}) \) corresponding to \( \lambda_2(\overline{G}) \). So \( x \) and \( y \) are two orthonormal vectors in \( \mathbb{R}^n \) which are orthogonal to \( e = (1, \ldots, 1) \) (the eigenvector of \( \lambda_1(G) = 1 \)).
0). Now, without loss of generality, we can suppose that \( \max_{i<j} |x_i - x_j| \geq \max_{i<j} |y_i - y_j| \) and

\[
x_1 = \max_{1 \leq i \leq n} x_i, \quad x_2 = \min_{1 \leq i \leq n} x_i.
\]

Note that since \( \sum_i x_i = 0 \) and \( x \) is nonzero, we have \( x_2 < 0 < x_1 \).

**Step 1 (the case \( x_1 - x_2 < 1 \)).** We have

\[
\lambda + \bar{\lambda} = \sum_{(i,j) \in E(G)} (x_i - x_j)^2 + \sum_{(i,j) \notin E(G)} (y_i - y_j)^2 \\
\geq \sum_{i < j} \min \{(x_i - x_j)^2, (y_i - y_j)^2\} \\
\geq \frac{\sum_{i<j} (x_i - x_j)^2(y_i - y_j)^2}{\max_{i<j} \max \{(x_i - x_j)^2, (y_i - y_j)^2\}} \\
\geq \frac{1}{(x_1 - x_2)^2}. \quad \text{(by Lemma 3)}.
\]

Thus, in the case \( x_1 - x_2 < 1 \), we have \( \lambda + \bar{\lambda} > 1 \) and the proof is complete. So, we can suppose that \( x_1 - x_2 \geq 1 \).

**Step 2 (the case \( x_1 - x_2 \geq 1 \)).** Since \( \lambda \) and \( \bar{\lambda} \) are positive, if one of them is greater than or equal to 1 then \( \lambda + \bar{\lambda} > 1 \). So, we can suppose that \( \lambda, \bar{\lambda} < 1 \). Let \( d \) be the distance between \( v_1 \) and \( v_2 \) in \( G \).

**Step 2.1 \( (d \leq 2 \text{ or } d > 3) \).** If \( d \leq 2 \) then, by Lemma 4, we have \( \lambda \geq 1 \). On the other hand, if \( d > 3 \) then \( \text{diam}(G) > 3 \). So the diameter of \( G \) is at most equal to 2. Thus, by Lemma 3 \( \bar{\lambda} > 1 \). Therefore, we can suppose that \( d = 3 \).

**Step 2.2 \( (d = 3) \).** Now, suppose that \( s \geq 1 \) is the maximum number of disjoint paths with length 3 between two vertices \( v_1, v_2 \) in \( G \).

Let \( S \) be the union of the vertices of \( s \) disjoint paths between \( v_1 \) and \( v_2 \) with length 3. Denote \( S_1 := S \cap N_G(v_2) \) and \( S_2 := S \cap N_G(v_1) \). Note that because \( v_1 \) and \( v_2 \) have no common neighbor in \( G \), the vertices of \( S_1 \) are not adjacent to \( v_1 \) and the vertices of \( S_2 \) are not adjacent to \( v_2 \). Also, \( |S_1| = |S_2| = s \) and \( S = S_1 \cup S_2 \cup \{v_1, v_2\} \). Therefore, if we denote \( A := N_G(v_1) \setminus S, B := N_G(v_2) \setminus S, \) and \( C := V(G) \setminus (A \cup B \cup S) \) then \( \{A, B, C, S_1, S_2, \{v_1, v_2\}\} \) is a partition of the vertices of \( G \). Furthermore we can suppose that \( a \leq b \).
Suppose that $l$ is the maximum of $|N_G(v) \cap A|$ for all $v \in S_1$ and $|N_G(u) \cap B|$ for all $u \in S_2$. So $G$ contains a subgraph as illustrated in Figure 1. We have

$$\lambda \geq \frac{(x_1 - x_2)^2}{R_{1,2}} \geq \frac{1}{3} + \cdots + \frac{1}{3} \underbrace{1 + \frac{2}{l+1}}_{s-1 \text{ times}} = \frac{s-1}{3} + \frac{l+1}{l+3},$$

(1)

which is a lower bound for $\lambda$ in terms of $s$ and $l$ (The terms $\frac{1}{3}$ correspond to other $s-1$ paths of length 3 between $v_1$ and $v_2$ in $G$).

Next, we give a lower bound for $\bar{\lambda}$. Note that $\bar{G}$ contains the edges of the graph illustrated in Figure 2. Also, according to the definition of $l$, each vertex of $S_1$ and of $S_2$, respectively are adjacent to at most $l$ vertices of $A$ and $B$. Now, we define the subgraphs $H_1, H_2$ and $H_3$ of $\bar{G}$ as follows:

1. $H_1$ is the union of the edges which join $v_1$ to the vertices of $\{v_2\} \cup B \cup C \cup S_1$, 

Figure 1: The subgraph of $G$ in the proof of Theorem 1.

Figure 2: The subgraph of $\bar{G}$ in the proof of Theorem 1.
2. \( H_2 \) is the union of the edges which join \( v_2 \) to the vertices of \( A \cup C \cup S_2 \),

3. \( H_3 \) is the union of the edges which join \( v_2 \) to the vertices of \( \{ v_1 \} \cup A \cup S_2 \).

Next, note that each of \( H_1, H_2, \) and \( H_3 \) is a star graph. Thus, \( \lambda(H_i) = 1 \), for \( i = 1, 2, 3 \). So, by Lemma 1, one can easily verify that

\[
\begin{align*}
(c + b + s + 2) & \sum_{(i,j) \in E(H_1)} (y_i - y_j)^2 + \\
(c + a + s + 1) & \sum_{(i,j) \in E(H_2)} (y_i - y_j)^2 + \\
(a + s + 2) & \sum_{(i,j) \in E(H_3)} (y_i - y_j)^2 \geq \sum_{(i,j) \in X} (y_i - y_j)^2,
\end{align*}
\]

where \( X \) contains all pairs \((i,j)\), for \( 1 \leq i, j \leq n \), expect the elements of \( A \times B \), \( S_1 \times S_2 \), \( S_1 \times A \), and \( S_2 \times B \) and their inverses.

Since the resistance of a path with length 3 is equal to 3, we have

\[
\sum_{v_i \in S_1, v_j \in S_2} (y_i - y_j)^2 \leq 3 \sum_{v_i \in S_1} ((y_i - y_1)^2 + (y_1 - y_2)^2 + (y_2 - y_j)^2)
\]

\[
= 3s \sum_{v_i \in S_1} (y_i - y_1)^2 + 3s \sum_{v_j \in S_2} (y_1 - y_j)^2 + 3s^2(y_1 - y_2)^2. \tag{3}
\]

Also, if we define \( A_i := A \setminus N_G(v_i) \), for each \( v_i \in S_1 \), then \( |A_i| \leq l \), for each \( v_i \in S_i \), and

\[
\sum_{v_i \in S_1} \sum_{v_j \in A_i} (y_i - y_j)^2 \leq 3 \sum_{v_i \in S_1} \sum_{v_j \in A_i} ((y_i - y_1)^2 + (y_1 - y_2)^2 + (y_2 - y_j)^2)
\]

\[
\leq 3sl\left( \sum_{v_i \in S_1} (y_i - y_1)^2 + \sum_{v_j \in A} (y_2 - y_j)^2 + (y_1 - y_2)^2 \right). \tag{4}
\]

Similarly, if we define \( B_i := B \setminus N_G(v_i) \), for each \( v_i \in S_2 \), we have

\[
\sum_{v_i \in S_2} \sum_{v_j \in B_i} (y_i - y_j)^2 \leq 3sl\left( \sum_{v_i \in S_2} (y_i - y_2)^2 + \sum_{v_j \in B} (y_1 - y_j)^2 + (y_1 - y_2)^2 \right). \tag{5}
\]

On the other hand, for all pairs \((i,j)\) such that \( v_i \in S_1 \) and \( v_j \in A \setminus A_i \), or \( v_i \in S_2 \) and \( v_j \in B \setminus B_i \), or \( v_i \in A \) and \( v_j \in B \), we have the trivial inequality

\[
(y_i - y_j)^2 \leq (y_i - y_j)^2. \tag{6}
\]

Now, note that for all of the terms \((y_i - y_j)^2\) which appear in the greater hand side of the inequalities \((2), (3), (4), (5), \) and \( (6) \), we have \( \{i,j\} \in E(G) \), and
on the other hand, for each $i < j$, one of $(y_i - y_j)^2$ or $(y_j - y_i)^2$ appears in the smaller hand side of one of these inequalities. Since $H_1$ and $H_2$ have no common edges, $c + b + s + 2 \geq c + a + s + 1$, and the terms $(y_i - y_j)^2$ which appear in (6) does not appear in other inequalities, by summing up both sides of these inequalities, we will get

$$\sum_{i<j}(y_i - y_j)^2 \leq ((c + b + s + 2) + (a + s + 2) + 3s^2 + 3sl + 3l) \sum_{(i,j) \in E(G)} (y_i - y_j)^2$$

$$= (n + 2 + 3s^2 + 6sl) \sum_{(i,j) \in E(G)} (y_i - y_j)^2.$$

Therefore, since $L(G)y = \lambda y$ and $\sum_{(i,j) \in E(G)} (y_i - y_j)^2 = \frac{\lambda}{n} \sum_{i<j} (y_i - y_j)^2$, we have (7)

$$\lambda \geq \frac{n}{n + 2 + 3s^2 + 6sl}.$$

A lower bound for $\lambda + \overline{\lambda}$. Therefore, by inequalities (1) and (7),

$$\lambda + \overline{\lambda} \geq \frac{n}{n + 2 + 3s^2 + 6sl} + \frac{s - 1}{3} + \frac{l + 1}{l + 3}.$$

Now by inequality (1), for $s \geq 3$, $\lambda \geq 1$. So we can suppose that $s = 1$ or 2.

The case $s = 1$. In this case,

$$\lambda + \overline{\lambda} \geq \frac{l + 1}{l + 3} + \frac{n}{n + 5 + 6l}.$$

Thus, for each $n \geq 12$,

$$\lambda + \overline{\lambda} \geq 1 - \frac{2}{l + 3} + \frac{12}{17 + 6l} > 1.$$

The case $s = 2$. Similarly, in this case,

$$\lambda + \overline{\lambda} \geq \frac{1}{3} + \frac{l + 1}{l + 3} + \frac{n}{n + 14 + 12l}.$$

Note that, when $l \geq 3$, $\lambda \geq \frac{1}{3} + \frac{l + 1}{l + 3} \geq 1$. So, $\lambda + \overline{\lambda} > 1$. Also, for each of cases $l = 0, 1, 2$, when $n \geq 10$,

- $l = 0$ : $\lambda + \overline{\lambda} \geq \frac{2}{3} + \frac{10}{24} > 1$,
- $l = 1$ : $\lambda + \overline{\lambda} \geq \frac{1}{3} + \frac{1}{2} + \frac{10}{36} > 1$,
- $l = 2$ : $\lambda + \overline{\lambda} \geq \frac{1}{3} + \frac{3}{5} + \frac{10}{48} > 1$.

On the other hand, one can numerically verify the theorem in the cases $n \leq 11$. Thus, the theorem holds for each $n \geq 2$. □
3.2 Maximum of the algebraic connectivity of a graph and its complement

As a by-product of the proof of Theorem 1, we prove the following theorem.

**Theorem 2.** For all graphs $G$ with $n$ vertices,

$$\max\{\lambda(G), \lambda(\overline{G})\} \geq 1 - O(n^{-\frac{3}{2}}).$$

**Proof.** It is sufficient to prove the theorem for a connected graph $G$ with $n$ vertices, for sufficiently large integers $n$. With the notations and assumptions at the beginning of the proof of Theorem 1, we know

$$\lambda + \overline{\lambda} \geq \frac{1}{(x_1 - x_2)^2}. $$

So, if $(x_1 - x_2)^2 \leq \frac{1}{2}$, then $\lambda + \overline{\lambda} \geq 2$ and the maximum of $\lambda$ and $\overline{\lambda}$ is at least $1$. Thus we can suppose that $(x_1 - x_2)^2 \geq \frac{1}{2}$. Now, similar to Step 2 of the proof of Theorem 1, we can suppose that both $\lambda$ and $\overline{\lambda}$ are smaller than 1. This implies that the distance between $v_1$ and $v_2$ in $G$ is equal to 3. Let $s \geq 1$ be the maximum number of disjoint paths with length 3 between two vertices $v_1$ and $v_2$ in $G$. Therefore,

$$1 > \lambda \geq \frac{(x_1 - x_2)^2}{R_{k,2}^G} \geq \frac{1}{2} \times \frac{s}{3} = \frac{s}{6}.$$ 

So $s \leq 5$. Now, by (7), with the notations which was defined in Step 2.2 of the proof of Theorem 1, we have

$$\overline{\lambda} \geq \frac{n}{n + 2 + 3s^2 + 6sl} \geq \frac{n}{n + 80 + 30l}.$$ 

On the other hand, note that according to the definition of $l$, $G$ contains a subgraph which was illustrated in Figure 1. Without loss of generality, we can assume that the left vertex in Figure 1 is $v_1$ and the right vertex is $v_2$. Now we have

$$\lambda x_1 = \sum_{(i,j) \in E(G)} (x_i - x_j) \geq (x_1 - x_k).$$

So $x_k \geq (1 - \lambda)x_1$ and $x_k - x_2 \geq (1 - \lambda)(x_1 - x_2)$. Therefore,

$$1 > \lambda = \sum_{(i,j) \in E(G)} (x_i - x_j)^2 \geq \frac{(x_k - x_2)^2}{R_{k,2}^G} \geq (1 - \lambda)^2 \times \frac{1}{2} \times \frac{l + 1}{2}.$$ 

Thus, $(1 - \lambda)^2 \leq \frac{4}{l + 1}$ and $\lambda \geq 1 - \frac{2}{\sqrt{l + 1}}$. Now, we consider two cases:

1. $n < l^2$. Thus $n^3 < l^2$ and

$$\lambda \geq 1 - \frac{2}{\sqrt{l}} \geq 1 - \frac{2}{n^3}.$$
2. \( n \geq l^2 \). So \( l \leq n^{\frac{2}{3}}, \frac{l}{n} \leq n^{-\frac{4}{3}}, \) and

\[
\lambda \geq 1 - \frac{80 + 30l}{n + 80 + 30l} \geq 1 - \frac{80}{n} \geq 1 - \frac{110}{n^\frac{4}{3}}.
\]

Therefore, in all cases we have \( \max\{\lambda, \overline{\lambda}\} \geq 1 - O(n^{-\frac{4}{3}}). \)

**Remark.** For each \( n \geq 4 \), define a graph \( G \) with vertices \( \{v_1, \ldots, v_n\} \) such that the induced subgraph on \( \{v_3, \ldots v_n\} \) is a complete graph, \( v_1 \) is only adjacent to \( v_3 \) and \( v_2 \) is only adjacent to \( v_4 \). One can observe that

\[
\lambda_2(G) = \lambda_2(\overline{G}) = \frac{n - \sqrt{n^2 - 4n + 8}}{2} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right) < 1.
\]

Therefore, for each \( n \geq 4 \), there exist a graph \( G \) which has \( n \) vertices and the maximum of \( \lambda_2(G) \) and \( \lambda_2(\overline{G}) \) is less than 1.

**References**

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