GEOMETRY AND ANALYSIS OF THE YANG–MILLS–HIGGS–DIRAC MODEL

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Abstract. The harmonic sections of the Kaluza–Klein model can be seen as a variant of harmonic maps with additional gauge symmetry. Geometrically, they are realized as sections of a fiber bundle associated to a principal bundle with a connection. In this paper, we investigate geometric and analytic aspects of a model that combines the Kaluza–Klein model with the Yang–Mills action and a Dirac action for twisted spinors. In dimension two we show that weak solutions of the Euler–Lagrange system are smooth. For a sequence of approximate solutions on surfaces with uniformly bounded energies we obtain compactness modulo bubbles, namely, energy identities and the no-neck property hold.

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The action functionals of quantum field theory (QFT) carry a rich and subtle mathematical structure. Investigating that structure is important for physics and mathematics alike. In fact, models from QFT have lead to a host of powerful geometric invariants. In particular, Donaldson could construct powerful invariants for differentiable 4-manifolds from solution spaces of anti-selfdual Yang–Mills connections, and later, Seiberg and Witten derived simpler invariants also from the Yang–Mills functional. The Gromov-Witten invariants are fundamental in symplectic geometry, to name just the most famous and powerful such invariants.

The Yang–Mills functional evaluates the $L^2$-norm of the curvature of a connection on a principal bundle. Such a connection arises as a gauge field in QFT. The first gauge theory was proposed by Hermann Weyl, in order to unify electromagnetism with gravity. The gauge group was the abelian group $U(1)$. While this was not successful as a physical theory, it inspired Yang and Mills to develop gauge theories with non-abelian gauge groups. Yang–Mills–Higgs theory couples the connection from Yang–Mills theory with a section of an associated bundle of the principal bundle, the Higgs field. These theories constitute the basis of the Standard Model of elementary particle physics that unifies the electromagnetic, weak and strong forces. The gauge group here is $SU(3) \times SU(2) \times U(1)$, but mathematically, one can work with any compact linear group. Thus, also grand unified theories with gauge groups like $SU(5)$ have been proposed. The gauge fields, however, constitute only half of the fields of QFT, the bosonic ones. The other fields are the fermionic matter fields. They are mathematically represented by spinors, and the action is of Dirac type. These two types of fields are combined in supersymmetric Yang–Mills theory. The action functional includes commuting gauge fields and anticommuting matter fields, and supersymmetry converts one type of field into the other, while leaving the action invariant. The supersymmetric Yang–Mills action is mathematically very rich. In order to develop tools for its mathematical analysis and to explore its geometric consequences, it has been found expedient to work with simplified versions. For instance, the Seiberg-Witten invariants arise from a reduced version of super Yang–Mills. Perhaps the simplest action functional that still captures the essential mathematical aspects behind super Yang–Mills is the nonlinear supersymmetric sigma model, see for instance [10, Chapter 6]. Here, the gauge connection is replaced by a map into some Riemannian manifold (a sphere in the original model, but mathematically, one can take any Riemannian manifold). The action functional for that map is the Dirichlet action, and its critical points are known as harmonic maps in the mathematical literature. The matter field becomes a spinor field along the map, and the critical points solve a nonlinear Dirac equation. For the details of the algebraic and geometric structure of this action functional, we refer to the systematic investigation [29].

From a semiclassical perspective, one would like to study the critical points of the action functional. They are solutions of certain partial differential equations (PDEs), the Euler–Lagrange equations for the functional. Here, a new mathematical difficulty arises. The fermionic fields are anticommuting, and therefore, they are not amenable to regularity theory for solutions of partial differential equations, because that theory works with analytical inequalities, and these are meaningful only for commuting (real-valued) fields. Therefore,

\footnote{In the mathematical literature, this is usually called an energy instead of an action; in fact, we shall use some energies below for auxiliary purposes in our analysis.}
in [8], a variant of the functional has been constructed that works with commuting fields only. That is, the spinor fields also become commuting fields. This is achieved by changing the Clifford algebra for the representation of the spin group. By that construction, supersymmetry between the fields is lost, but all other symmetries, in particular conformal symmetry, are preserved, and the analytical power of PDE regularity theory is gained. (Note that for the conformal symmetry of the spinor action, we need to perform a suitable rescaling. While this will be important in the analytical part, we ignore it in this introduction.)

The preceding described the simplest theory in that context. The standard model has more fields than the sigma model, and the coupling between those fields is essential. Of particular importance is the Higgs field whose physical role consists in assigning masses to other fields. Therefore, it is natural to develop the geometry of coupled field equations, and from an analytical perspective, at the same time to make all fields commuting. That is what we start in this paper.

More precisely, we develop the geometric construction of a gauged nonlinear sigma model that combines the Dirichlet action for maps and the Dirac action for twisted spinors with the Yang–Mills action for gauge fields. In addition to the conformal invariance of the Dirac-harmonic action, the gauge invariance of the Yang–Mills action will be of fundamental importance.

Yang–Mills theory works most naturally in dimension 4 whereas the sigma model, while being diffeomorphism invariant in any dimension, enjoys conformal symmetry only on 2-dimensional domains. Conformal symmetry, in fact, is the key to the regularity of the solutions of the Euler–Lagrange equations, and in higher dimensions, regularity of the solutions may fail. Also, conformal symmetry naturally connects it to the theory of (super) Riemann surfaces, and this is the key to a deeper mathematical understanding of the action functional. The symmetries of the functional are geometrically induced by (super) diffeomorphisms and (super) Weyl transformations, see [24, 29]. As Atiyah-Bott [2] have demonstrated, Yang–Mills theory also leads to geometric and topological insight in dimension two. Therefore, in this paper, after setting up the general geometric scheme which works in any dimension, we shall develop the regularity theory for the solutions, that is, the critical points of our action functionals, in dimension two. As mentioned, in that dimension, we have conformal invariance, and that is needed for the regularity. Still, the regularity is far from being trivial or easy, but we are able to rely on systematic prior work of ourselves and others. The key ingredient is the blow-up analysis for understanding the formation of singularities.

For the geometric constructions, there also exists some prior work on which we can build. The Yang–Mills–Higgs theory has been analyzed from a mathematical perspective, viewing the Higgs field as a natural generalization of harmonic maps to fiber bundles as C. M. Wood noticed in his work [52, 53] on harmonic sections. David Betounes has clarified that the right geometric setup for Yang–Mills–Higgs theory is given by a Riemannian variant of Kaluza–Klein geometry, see [3, 4, 5]. The Yang–Mills–Higgs functional is also investigated under the name of gauged harmonic maps by Lin–Yang [31]. The Yang–Mills–Higgs theory for symplectic fibrations has a self-duality structure and admits compactifications which leads to new Gromov-Witten invariants, see [37, 38]. The general minimax solutions are studied in [46, 47] and for the boundary value problem see [1].

Supersymmetry requires that both fields, the connection and the Higgs-field, obtain a superpartner and this then results in an additional supersymmetry transformation. For example, [10, Chapter 6] constructs such super Yang–Mills–Higgs models. However, from
The mathematical perspective, analytical properties of its critical points need further study. We have already mentioned the major challenge for the analysis of gauged supersymmetric sigma-models, namely that, in order to allow for supersymmetry, anti-commuting variables are needed, see [22, 29] and references therein. And we have also mentioned that an alternative approach to gauged supersymmetric sigma-models, purely in the realm of standard Riemannian geometry, is possible that uses the method developed for Dirac-harmonic maps in [8], and this requires switching to another Clifford algebra [25], and one can no longer expect supersymmetry in full generality.

The advantage of this approach is that we obtain a model coupling classical gauge theory with the Higgs field and spinors on general fiber bundles with gauge symmetry. Even though we no longer have full supersymmetry, fortunately, the symmetries of the model are still rich and powerful enough to make a detailed regularity theory possible. In fact, without such symmetries, from a pure PDE perspective, we cannot expect regularity of the solutions, because the resulting equations are highly nonlinear. The Noether currents from the symmetries, however, provide us with additional equations that we can exploit.

The equations of motion of our model, which we call the Yang–Mills–Higgs–Dirac model, are a set of coupled partial differential equations on the domain manifold. As explained, we study their analytical properties in the special case when the dimension of the domain is two.

Let us now describe the geometric structure in more detail. Given a G-principal fiber bundle $P$ over the manifold $M$ and a left $G$-manifold $(N,h)$, one can construct the associated fiber bundle $N = P \times_G N$ over $M$. A principal connection $\omega$ on $P$ induces an associated connection on $N$, in particular a splitting $TN = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle in a horizontal and a vertical part. Kaluza–Klein theory constructs a bundle metric $\mathcal{G}$, turning $(N,\mathcal{G})$ into a Riemannian manifold. While the action on the connection is given by the Yang–Mills functional, the Higgs energy of sections $\phi: M \to N$ is not the full Dirichlet energy, since the latter is not compatible with the variation in the space of sections of $N$. Rather, we should restrict it to the vertical part $d^V\phi$ of its differential. For the twisted spinors $\psi \in \Gamma(S \otimes \phi^*\mathcal{V})$, we define a vertical, twisted Dirac-operator. Putting the pieces together, the action of the Yang–Mills–Higgs–Dirac functional is then given by

$$A(\omega, \phi, \psi) = \int_M |F(\omega)|^2 + |d^V\phi|^2 + \langle \psi, D\psi \rangle \, dv_g,$$

where $F(\omega)$ is the curvature of the principal connection $\omega$.

The Euler–Lagrange equations for the action $A$ are given by

$$D^*_\omega F + d\tilde{\mu}_\phi^*(d^V\phi) + Q(\phi, \psi) = 0,$$

$$\tau^V(\phi) - \frac{1}{2} \mathcal{R}^V(\phi, \psi) = 0,$$

$$D\psi = 0.$$

The terms $d\tilde{\mu}_\phi^*(d^V\phi)$ and $Q(\phi, \psi)$ describe the infinitesimal dependence of $|d^V\phi|^2$ and $\langle \psi, D\psi \rangle$ on $\omega$, respectively. The vertical tension field $\tau^V(\phi)$ is a differential operator of order two and $\mathcal{R}^V(\phi, \psi)$ is a contraction of the Riemannian curvature of $\mathcal{G}$. Up to the choice of a gauge, for instance the Coulomb gauge, (1) is locally an elliptic system.

Further analysis of the system (1) depends heavily on the dimension of the domain. While Yang–Mills theory is richest in dimension four, the theory of harmonic maps, that is, the
Higgs-field, meets its singularity already in dimension three. Consequently, we will restrict our attention here to the case of a two-dimensional domain. Since the equation for the connection is subcritical in dimension two, with the help of Rivière’s regularity theory, we can obtain the full regularity of the weak solutions.

**Theorem 1.1** (see Theorem 6.2). Let $(M, g)$ be a closed Riemann surface. Let $(\omega, \phi, \psi)$ be a weak solution of (1). Then there is a gauge transformation $\varphi \in D_2$ such that $(\varphi^* \omega, \varphi(\phi), \varphi(\psi))$ is a smooth triple.

Thereafter we turn to the blow-up analysis for a sequence of approximating solutions. For that purpose we first establish the small energy regularity and a Pohozaev type identity, which is essential to build the energy identities and the no-neck properties. The concentration set is defined as usual, and we show that the connections will not concentrate in dimension less than the critical dimension, hence the connections converge nicely. Along the energy concentration set of the sections, Dirac-harmonic spheres or harmonic spheres can emerge in the limit of rescaling. We summarize this last result in the following theorem.

**Theorem 1.2** (see Theorem 9.1). Let $(\omega_k, \phi_k, \psi_k)$ be a sequence of approximating solutions, i.e., (14) and (15) are satisfied. Assume that they have uniformly bounded energies. Then up to a subsequence they converge weakly to a smooth solution $(\omega_\infty, \phi_\infty, \psi_\infty)$ of (1).

Furthermore, there is a finite set $S_1 = \{x_1, \ldots, x_I\} \subset M$ such that the convergence is strong on any compact subset of $M \setminus S_1$. Moreover, corresponding to each $x_i \in S_1$ there exists a finite collection of Dirac-harmonic spheres $(\sigma_i^l, \xi_i^l)$ from $S^2$ into $N$ for $1 \leq l \leq L_i < \infty$, such that the energy identities hold,

\[
\lim_{k \to \infty} A_{YM}(\omega_k) = A_{YM}(\omega_\infty),
\]

\[
\lim_{k \to \infty} E(\phi_k) = E(\phi_\infty) + \sum_{i=1}^I \sum_{l=1}^{L_i} E(\sigma_i^l),
\]

\[
\lim_{k \to \infty} E(\psi_k) = E(\psi_\infty) + \sum_{i=1}^I \sum_{l=1}^{L_i} E(\xi_i^l),
\]

and the no-neck property holds, i.e., the set $\phi_\infty(M) \cup (\bigcup_{i,l} \sigma_i^l(S^2))$ is connected.

The article is organized as follows. In the first part we give a detailed geometric setup of our gauged nonlinear sigma model on general fiber bundles over Riemannian manifolds. For the convenience of the readers and in order to fix the notation, we recall the Kaluza–Klein geometry and the theory of harmonic sections. Then we construct the combined Yang–Mills–Higgs–Dirac action and discuss possible extensions. We formulate the geometric quantities both globally and locally, the latter for the subsequent local analysis.

In the second part we focus on the case where the domain is a closed Riemann surface. With the help of the tools from Yang–Mills and harmonic map theory, we obtain the regularity of weak solutions and the energy identities and no-neck properties for the approximate sequences.
Part 1. Geometric Construction of the Model in General Dimensions

In this part, we explain the geometric background of Kaluza–Klein geometry and construct the Yang–Mills–Higgs–Dirac action. We compute its energy-momentum tensor and check the gauge invariance of the action functional.

In this part and in contrast to the second part, the dimension of the base manifold can be arbitrary. However, we require the base manifold to be closed. This is mainly a technical assumption to simplify the integration by parts. One could as well work with complete manifolds and consider only integrable geometric quantities which vanish at infinity and appropriate Sobolev spaces.

2. Review of Kaluza–Klein geometry and Harmonic Sections

In this section we mainly introduce notation and review Kaluza–Klein geometry and harmonic sections. Our main references on this topic are [3, 4, 5, 52, 53]. Harmonic sections can be seen as equivariant harmonic maps from a $G$-principal bundle $P \rightarrow M$ to a Riemannian $G$-manifold $N$. Alternatively, and that is our point of view here, as sections of the associated fiber bundle $\mathcal{N} \rightarrow M$ minimizing an action on $M$.

Let $(M,g)$ be an $m$-dimensional oriented closed manifold with a Riemannian metric $g$, and let $G$ be a finite-dimensional compact Lie group with Lie algebra $\mathfrak{g}$. In particular, being compact and finite dimensional, $G$ can be taken as a matrix group. Suppose, $P = P(M,G,\pi,\Psi)$ is a principal $G$-bundle over $M$, where $\pi: P \rightarrow M$ denotes the projection and

$$\Psi: P \times G \rightarrow G, \quad \Psi(p,a) \equiv \Psi_a(p) \equiv \Psi_p(a),$$

denotes a free right $G$-action. Further assume that $(N,h)$ is a left $G$-manifold; that is,

$$\mu: G \times N \rightarrow N, \quad \mu(a,y) \equiv \mu_a(y) \equiv \mu_y(a),$$

is a left action with $\mu(G) \subset \text{Isom}(N,h)$. Then $G$ acts on the product $P \times N$ from the right freely:

$$(\Psi \times \hat{\mu}): (P \times N) \times G \rightarrow P \times N$$

$$((p,y),a) \mapsto (\Psi_a(p),\mu_{a^{-1}}(y)).$$

For further reference we abbreviate $\hat{\mu}_a \equiv \mu_{a^{-1}}$. The orbit space

$$\mathcal{N} := (P \times N)/_G \equiv P \times_G N$$

is a smooth manifold; denote the quotient map by $\iota: P \times N \rightarrow \mathcal{N}$, where $\iota(p,y) = [p,y]$. Note that this is a principal fiber bundle over $\mathcal{N}$ with fiber $G$. As $G$ acts fiberwisely on $P$, there is a unique map $\rho: \mathcal{N} \rightarrow M$ s.t. the following diagram commutes:

$$\begin{array}{ccc}
P \times N & \xrightarrow{pr_1} & (P \times N)/_G = P \times_G N \\
\downarrow{pr_1} & \Downarrow{\exists \rho} & \downarrow{\rho} \\
P & \xrightarrow{\pi} & M
\end{array}$$

It is well-known that $\rho: \mathcal{N} \rightarrow M$ is a fiber bundle with fiber space $N$. The embedding of the fiber is given by the insertion map: For any $x \in M$ and any $p \in P$ with $\pi(p) = x$, the
insertion map

\[ \iota_p : N \to N_x = \rho^{-1}(x), \]

\[ y \mapsto [p, y] \]

is a diffeomorphism. Note that another point \( p' = \Psi_a(p) \) gives rise to another embedding via

\[ \iota_{\Psi_a(p)}(y) = [\Psi_a(p), y] = [p, \mu_a(p)] = \iota_p \circ \mu_a(y); \]

that is, different embeddings differ by some automorphism of \( N \).

Taking the differential \( d\rho \) of the projection \( \rho : \mathcal{N} \to M \) yields a short exact sequence of vector bundles over the fiber bundle \( \mathcal{N} \):

\[ 0 \longrightarrow \mathcal{V} \equiv \ker(d\rho) \xrightarrow{\pi^\mathcal{V}} TN \xrightarrow{\rho^*(TM)} 0 \]

We call \( \pi^\mathcal{V} : \mathcal{V} \to \mathcal{N} \) the vertical bundle over \( \mathcal{N} \), whose fibers are given by the tangent space of \( N \). Indeed, for any \([p, y] \in \mathcal{N}\) and any \( Y \in T_yN \) represented by a curve \( \gamma \) (i.e. \( \gamma(0) = y, \gamma(0) = Y \)), we have

\[ d\iota_p : T_yN \to T_{[p, y]} \mathcal{N}, \]

\[ Y = \gamma(0) \mapsto d\iota_p(Y) = \left. \frac{d}{dt} \right|_{t=0} \iota_p(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} [p, \gamma(t)]. \]

Since \( \rho([p, \gamma(t)]) = \pi(p) \), we see that \( d\iota_p(Y) \in \ker(d\rho)_p = \mathcal{V}_p \). By counting dimensions, we see that any vertical vector is in the image of some \( \iota_p \).

Recall that the tangent bundle of the principal bundle \( P \) has an analogously defined vertical bundle \( V_P = \ker d\pi \). In this case the vertical bundle is trivial: \( V_P = P \times \mathfrak{g} \). A principal connection is a \( G \)-equivariant \( \mathfrak{g} \)-valued one form \( \omega \in \Omega^1(P, \mathfrak{g}) \) such that, under the above trivialization, \( \omega((p, a)) = a \in \mathfrak{g} \) for all \((p, a) \in VP = P \times \mathfrak{g} \). The kernel of \( \omega \) is the horizontal distribution \( HP \) which is \( G \)-invariant, and yields \( TP = VP \oplus HP \). Note that at each \( p \in P \), \( HP_p \cong T_{\pi(p)}M \). In other words, \( HP \cong \pi^*TM \). For more details on connections in principal bundles, see [12].

The principal connection \( \omega \) induces a connection \( \sigma \) on the associated bundle \( \mathcal{N} \) by specifying a horizontal distribution \( \mathcal{H} \) complementary to \( \mathcal{V} \) in \( TN \):

\[ \mathcal{H} = d\iota_y((HP)_p). \]

Here, \( \iota_y : P \to \mathcal{N} \) is the map that arises from \( \iota : P \times N \to \mathcal{N} \) by restricting to \( y \in N \) and \( d\iota_y \) its tangent map. The distribution \( \mathcal{H} \) is well-defined, that is, independent of the choice of representative \([p, y] \) because of the equivariance of the connection on \( P \).

By construction we have \( \mathcal{H} = T_xM \) where \( x = \rho([p, y]) = \pi(p) \) and moreover \( \mathcal{H} \cong \rho^*TM \). To make this isomorphism more explicit, let \( \tilde{X} \) be a lift of the vector field \( X \in \Gamma(TM) \) with horizontal part \( \text{hor} \tilde{X} \). The isomorphism

\[ \sigma : \rho^*TM \overset{\cong}{\to} \mathcal{H} \subset TN \]

is then given by

\[ \sigma(\rho^*X)_|[p,y] = d\iota_y(\text{hor} \tilde{X}). \]
Since for all \( y \in \mathcal{N} \) we have \( \rho \circ \iota_y = \pi \) the map \( \sigma \) splits the short exact sequence (2), that is \( d\rho \circ \sigma = \text{Id}_{\rho^*TM}. \) This yields a direct sum decomposition \( T\mathcal{N} = \mathcal{V} \oplus \mathcal{H} \) where the projectors on the horizontal and vertical bundles are given by

\[
\text{hor} = \sigma \circ d\rho, \quad \text{ver} = (1 - \sigma \circ d\rho).
\]

In particular, the map \( \sigma \) defines a connection on \( \mathcal{N}. \)

The embeddings \( \iota_p: \mathcal{N} \to \mathcal{N} \) of fibers at the point \( p \) induce a Riemannian metric \( \bar{h} \) on \( \pi^\mathcal{V} \) in the following way: at \( [p, y] \in \mathcal{N}, \)

\[
\bar{h}|_{[p, y]} := (\iota_p^{-1})^*(h|_y).
\]

The metric \( \bar{h} \) is well-defined since \( \mu(G) \subset \text{Isom}(\mathcal{N}, h) \). Together with the splitting \( T\mathcal{N} = \mathcal{V} \oplus \mathcal{H} \) from the connection we can define the Kaluza–Klein metric

\[
\mathcal{G}(X, Y) = \bar{h}(\text{ver} X, \text{ver} Y) + g_\rho(\text{hor} X, \text{hor} Y)
\]

for \( X, Y \in \Gamma(T\mathcal{N}). \) Here, \( g_\rho \) is the metric on \( \mathcal{H} = \rho^*TM \) obtained via \( \rho \) from the metric \( g \) on \( TM. \) As a Riemannian manifold, \( (\mathcal{N}, \mathcal{G}) \) admits a unique Levi-Civita connection which we denote as \( \nabla. \)

With respect to the Kaluza–Klein metric \( \mathcal{G}, \) the fibration \( \rho: \mathcal{N} \to M \) has totally geodesic fibers. As a consequence, for vector fields \( Y \) and \( Z \) on \( \mathcal{N}, \) the Levi-Civita covariant derivative of the local vertical vector fields \( d\iota_p(Y), d\iota_p(Z) \) on \( \mathcal{N} \) is given by

\[
\nabla_{d\iota_p(Y)} d\iota_p(Z) = d\iota_p(\nabla^\mathcal{h} Y Z).
\]

In other words, for vertical vector fields \( W \) and \( V \) the covariant derivative \( \nabla_W V \) is again vertical. It follows that also for horizontal vector fields \( H \) the field \( \nabla_W H \) is horizontal and \( \nabla_W \text{hor} = \nabla_W \text{ver} = 0. \)

From now on, we will assume the existence of a smooth section \( \phi \in \Gamma(\mathcal{N}). \) In general, there are topological obstructions to the existence of sections of fiber bundles, see [18, §29]. Sections which are at least differentiable once can then be turned into smooth sections by local approximation. The case of dimension two, which we are mainly interested in, is unobstructed if the second homotopy group of the fiber vanishes.

The pullback of (2) along \( \phi \) gives a short exact sequence of vector bundles over \( M: \)

\[
0 \longrightarrow \phi^*\mathcal{V} \longrightarrow \phi^*T\mathcal{N} \overset{d\rho}{\longrightarrow} TM \longrightarrow 0
\]

The horizontal part of the differential \( d\phi \in \Gamma(T^*M \otimes \phi^*T\mathcal{N}) \) is the identity \( 1_{TM} \) and hence has constant length \( \sqrt{m}. \) The vertical part \( d\bar{\phi} \equiv \text{ver} d\phi \in \Gamma(T^*M \otimes \phi^*\mathcal{V}) \) encodes the essential geometric information contained in the gradient of the section. Therefore we consider the effective Dirichlet energy of the section

\[
E(\phi; \sigma) := \int_M |d\bar{\phi}|^2_{g^\mathcal{V} \otimes \mathcal{G}} \text{dvol}_g
\]

where \( g^\mathcal{V} \) denotes the dual metric on the cotangent bundle \( T^*M. \) As the decomposition \( T\mathcal{N} = \mathcal{H} \oplus \mathcal{V} \) is orthogonal with respect to \( \mathcal{G} \) it holds

\[
\int_M |d\phi|^2_{g^\mathcal{V} \otimes \mathcal{G}} \text{dvol}_g = E(\phi; \sigma) + \text{dim}(M) \cdot \text{Vol}(M).
\]
2.1. Harmonic sections. The Dirichlet energy functional (3) can be defined on the space of $W^{1,2}$-sections. Its critical points are known as harmonic sections, see [52, 53]. Let us take a closer look at the variational structure of this functional before combining it with the Yang–Mills action and the Dirac-action in Section 3.

2.1.1. Equations of motion. Let $\phi \in \Gamma(\mathcal{N})$ and take a variation $\phi_t$ of $\phi$ in the space of $W^{1,2}$-sections. Thus the variational field is vertical:

$$V = \frac{d}{dt} \bigg|_{t=0} \phi_t \in \Gamma(\phi^*\mathcal{V}).$$

To obtain the equations of motion we calculate:

$$\frac{d}{dt} E(\phi_t) = \frac{d}{dt} \bigg|_{t=0} \int_M \langle d^V \phi_t(e_\alpha), d^V \phi_t(e_\alpha) \rangle \ d\text{vol}_g$$

$$= 2 \int_M \left\langle \nabla_{\phi_t} \phi_t(e_\alpha), d^V \phi_t(e_\alpha) \right\rangle \ d\text{vol}_g \bigg|_{t=0}$$

$$= 2 \int_M \left\langle \text{ver} \nabla_{\phi_t} \phi_t(e_\alpha), d^V \phi_t(e_\alpha) \right\rangle \ d\text{vol}_g$$

$$= 2 \int_M \left\langle \phi_t(e_\alpha), \nabla^{\phi_t}_{e_\alpha} V, d^V \phi(e_\alpha) \right\rangle \ d\text{vol}_g$$

$$= -2 \int_M \left\langle V, \nabla^{\phi_t}_{e_\alpha} d^V \phi(e_\alpha) + (\text{div}_g e_\alpha) d^V \phi(e_\alpha) \right\rangle \ d\text{vol}_g$$

Since $V$ can be an arbitrary vertical field along the section $\phi$, we conclude that $\phi$ is critical for the action functional $E(\phi)$ if and only if the following equation is satisfied

$$\tau^V(\phi) := \text{ver} \nabla^{\phi_t}_{e_\alpha} d^V \phi(e_\alpha) + (\text{div}_g e_\alpha) d^V \phi(e_\alpha) = 0,$$

where $\text{div}_g e_\alpha \equiv \sum_\beta \langle \nabla_{e_\beta} e_\alpha, e_\beta \rangle$. This tensor $\tau^V(\phi)$ is called the vertical tension field of the section $\phi$, and solutions of $\tau^V(\phi) = 0$ are called harmonic sections. Note that $\tau^V(\phi)$ coincides with the tension field $\tau(\phi)$ if $d\phi$ and $d^V \phi$ coincide. This happens, for example, in the case of the trivial action on $N$ where $\mathcal{N} = M \times N$ and the connection is trivial. Hence harmonic sections generalize harmonic maps to a gauged setting.

In addition, it is shown in [53] that $\phi$ is a harmonic section if and only if its corresponding $G$-equivariant map $\tilde{\phi}: P \to N$ is harmonic with respect to the Kaluza–Klein metric $\mathcal{G}_P$ on $P$. The Kaluza–Klein metric on $P$ is given by

$$\mathcal{G}_P(X, Y) = \langle \text{ver} X, \text{ver} Y \rangle_\mathfrak{g} + g_\pi(\text{hor} X, \text{hor} Y)$$

where $\langle \cdot, \cdot \rangle$ is an ad-invariant scalar product on $VP = P \times \mathfrak{g}$. For more details on $\tilde{\phi}$ see the next section.

Despite the similarities to harmonic maps, the existence results do not immediately transfer to harmonic sections because of the required equivariance properties. For example, constant maps are trivially harmonic maps but do not directly generalize to the bundle case. Rather one would have to consider sections with vanishing vertical differential. However, the existence of such parallel sections might have topological obstructions.

It is shown in [53] that the theory of the heat flow of harmonic maps can be used in certain cases to obtain harmonic sections: If the fiber manifold $(N, h)$ has non-positive curvature and the fiber bundle $\mathcal{N}$ allows for a $C^1$-section, then this section can be deformed via heat...
flow into a harmonic one. The curvature condition excludes singularities of the flow in the fiber manifold and hence guarantees the long time existence of the flow. The limit of the flow is a static solution, that is, a harmonic section.

In addition, when $m = 2$, the model possesses conformal invariance, and one can use the methods in [12, 15] to obtain harmonic sections in a given homotopy class.

2.1.2. Equivariant representatives. Recall that each section $\phi \in \Gamma(N)$ corresponds uniquely to an equivariant map $\tilde{\phi}: P \to N$, such that $\iota \circ \left(\text{Id}_P, \tilde{\phi}\right) = \phi \circ \pi$. Here $G$-equivariance means $\tilde{\phi}(\Psi_a(p)) = \tilde{\mu}_a(\tilde{\phi}(p))$ for all $a \in G$ and $p \in P$. Differentiating the equivariance equation we obtain $G$-equivariance of $d\tilde{\phi}: TP \to TN$: for any $p \in P$ and $Y_p \in T_pP$,

$$d\tilde{\phi}(\Psi_a Y_p) = d\tilde{\mu}_a(d\tilde{\phi}(Y_p)).$$

Here $\Psi'_a$ is the tangent map of the right multiplication $\Psi_a: P \to P$ by $a \in G$.

The differential of $\phi \circ \pi = \iota \left(\text{Id}_P, \tilde{\phi}\right)$ yields

$$(d\phi \circ d\pi) Y_p = d\nu_{\phi(p)} Y + dt_p d\tilde{\phi} Y.$$ 

If we apply this formula to the horizontal part of a lift $\tilde{X} \in \Gamma(TP)$ of a vector field $X \in \Gamma(TM)$, i.e. $d\pi(\tilde{X}) = X$, the first summand is horizontal and the second is vertical:

$$d\phi(X_x) \equiv \underbrace{d\nu_{\phi(p)}(\text{hor} \tilde{X}_p)}_{\text{horizontal}} + \underbrace{dt_p \left(d\tilde{\phi}(p)(\text{hor} \tilde{X}_p)\right)}_{\text{vertical}}.$$

Since $\tilde{X} = \text{hor} \tilde{X} + \text{ver} \tilde{X}$ and $\text{ver} \tilde{X}_p = \Psi'_p(\omega(\tilde{X}_p))$, we have

$$d\nu \phi(X) \equiv dt_p d\tilde{\phi} \left(\tilde{X} - \Psi'_p \omega(\tilde{X})\right) \equiv dt_p \left(d\tilde{\phi}(\tilde{X}) + d\mu_{\tilde{\phi}(p)} \omega(\tilde{X})\right).$$

2.2. Local formulation. For later use we derive the local version of the model. The local representatives of the various geometric quantities will all be induced from a local section $s: U \to \pi^{-1}(U) \subset P$ of the principal bundle $P(M, G)$.

First, this local section $s$ gives rise to a local trivialization of $P$ over the domain of $s$:

$$\chi^P_U: \pi^{-1}(U) \to U \times G$$

$$p \mapsto (\pi(p), \kappa(p))$$

where $\kappa: \pi^{-1}(U) \to G$ is determined by $\Psi_{\kappa(p)}(s(\pi(p))) = p$, i.e. $\kappa(p) = (\Psi_{s(\pi(p))})^{-1}(p)$, known as the structure group mapping. It is characterized by the identity $\kappa(s(x)) = e \in G$, for any $x \in U \subset M$, where $e$ denotes the neutral element of $G$. Then the local form of $\omega$ is given by

$$A = s^* \omega: TU \to \mathfrak{g},$$

that is, $A$ is a $\mathfrak{g}$-valued one-form on $U$. Second, this local section also induces a local trivialization of the associated fiber bundle $N$:

$$\chi^N_U: \rho^{-1}(U) \to U \times N$$

$$[p, y] \mapsto \left(\rho([p, y]) = \pi(p), \mu_{\kappa(p)}(y)\right).$$
This is well-defined since for any \( a \in G \),
\[
\chi^N_U(\Psi_a(p), \mu_{a^{-1}}(y)) = (\pi(\Psi_a(p)), \mu_{\kappa(\Psi_a(p))}(\mu_{a^{-1}}(y)))
\]
\[
= (\pi(p), \mu_{\kappa(p)} \circ \mu_{a^{-1}}(y)) = (\pi(p), \mu_{\kappa(p)}(y)) = \chi^N_U([p, y]).
\]
Given a section \( \phi \in \Gamma(\mathcal{N}) \), its local representative is given by
\[
u := \text{pr}_2 \circ \chi^N_U \circ \phi: U \to \mathcal{N},
\]
\[
u(x) = \text{pr}_2 \circ \chi^N_U(\phi(x)) = \text{pr}_2 \circ \chi^N_U([s(x), \tilde{\phi}(s(x))]) = \mu_{\kappa(s(x))}(\tilde{\phi}(x))
\]
and in this local trivialization the section \( \phi \) has the form \( \chi^N_U \circ \phi(x) = (x, u(x)) \), for any \( x \in U \).
Moreover, in this local trivialization, the tangent bundle of \( \mathcal{N} \) is also locally trivialized:
\[
T\left(\rho^{-1}(U)\right) \overset{d\chi^N_U}{\longrightarrow} TU \times TN,
\]
and the vertical differential of \( \phi \) takes the form
\[
\text{pr}_2 \circ d\chi^N_U(d^V \phi_x(X)) = \text{pr}_2 \circ d\chi^N_U \circ d\mu \left( d\tilde{\phi}_p(\text{hor } \tilde{X}) \right)
\]
(note that \( \text{pr}_2 \circ \chi^N_U \circ \iota_p(f) = \mu_{\kappa(p)}(f) \))
\[
d\mu_{\kappa(p)} \left( d\tilde{\phi}_p(\text{hor } \tilde{X}) \right)
\]
then use the \( G \)-equivariance
\[
d\tilde{\phi}_{s(x)}(d\Psi_{\kappa(p)}^{-1}(\text{hor } \tilde{X}))
\]
\[
d\tilde{\phi}_{s(x)}(\text{hor } \tilde{X})
\]
\( G \)-invariance of horizontal distributions
Here \( \tilde{X} \) is a lifting of \( X \in \Gamma(TU) \) to \( TP \). In particular we could take \( \tilde{X} = s_*X \) and get
\[
\text{pr}_2 \circ d\chi^N_U(d^V \phi_x(X)) = d\tilde{\phi}_{s(x)}(\text{hor } s_*X)
\]
\[
= d\tilde{\phi}_{s(x)}(s_*X) + d\mu_{\tilde{\phi}(s(x))}(\omega(s_*X))
\]
\[
d u(X) + d\mu_{u(x)}(A(X)) \equiv d_A u(X) \in \Gamma(u^*TN).
\]
Furthermore, since \( \iota_p: (N, h) \to (\mathcal{N}_x, \mathcal{G}) = (\rho^{-1}(x), \mathcal{G}) \) is an isometry for \( p = s(x) \in P \), we have, for a local orthonormal frame \( (e_\alpha) \) on \( U \) and writing \( \tilde{e}_\alpha = s_*e_\alpha \),
\[
|d^V \phi|^2(x) = \sum_\alpha |d^V \phi(e_\alpha)|^2_{\mathcal{G}}^2 = \sum_\alpha |d\tilde{\phi}_p(\text{hor } \tilde{e}_\alpha)|^2_h
\]
\[
= \sum_\alpha |d\tilde{\phi}(s_*e_\alpha) + d\mu_{\tilde{\phi}(p)}(\omega(s_*e_\alpha))|^2_h
\]
\[
= \sum_\alpha |d u(e_\alpha) + d\mu_{u(x)}(A(e_\alpha))|^2_h = \sum_\alpha |d_A u(e_\alpha)|^2_h(x).
\]
Therefore, locally we are considering the action
\[
E(u; A) = \int_M |d_A u|_{\mathcal{G} \otimes h}^2 \, d\text{vol}_g = \int_M |d u + d\mu_{u}(A)|^2 \, d\text{vol}_g,
\]
where $g^\gamma$ denotes the induced metric on the cotangent bundle. Locally a variation $(\phi_t)$ of $\phi_0 = \phi$ can be realized $\phi_t(x) = (x, u_t(x)) \in U \times N$ where $u_t : U \to N$ is a family of maps, and the variational field is

$$V = \left. \frac{d}{dt} \right|_{t=0} \phi_t = \left. \frac{d}{dt} \right|_{t=0} (\text{Id}, u_t) = \left( 0, \left. \frac{d}{dt} \right|_{t=0} u_t \right) = (0, W),$$

where $W = \left. \frac{d}{dt} \right|_{t=0} u_t \in \Gamma(u^*TN)$.

The differential of the group action $\mu : G \times N \to N$ is given by $d\mu : TG \times TN \to TN$ over $\mu$. If we restrict it to the identity of $G$, we obtain a bundle map $g_N \oplus TN \to TN$ over $N$, still denoted by $d\mu$, where $g_N$ denotes the trivial bundle with fiber $g$ over $N$. Let now $a$ be a section of the trivial bundle $g_N$, and $W$ a section of $TN$. Then $d\mu(a, W) = d\mu(a, 0) + W$ because $\mu$ is the identity when restricted to $e \in G$. We will sometimes abbreviate $d\mu(a, 0)$ as $d\mu(a)$ for simplicity, which can also be viewed as a partial tangent map (with fixed $y \in N$). For later convenience we write

$$\partial_1 \partial_2 \mu(a, W) \equiv \nabla^N_W d\mu(a, 0) - d\mu(\nabla^N_W a, 0) \in \Gamma(TN).$$

where $\nabla^N_W a$ is the trivial covariant derivative on the trivial bundle $g$. With respect to a basis $\epsilon_i$ of $g$ and $a = a^i \epsilon_i$ we have $\nabla^N_W a = W(a^i) \epsilon_i$. Notice that $\partial_1 \partial_2 \mu(a, W)$ is bilinear in $a$ and $W$ and can be seen as the off-diagonal part of the Hessian of $\mu$.

By assuming that the variations are compactly supported inside $U$ such that integration by parts without boundary term is allowed, we can calculate the variation formula of the energy functional (with respect to a fixed connection $A = s^\ast \omega$) as follows

$$\left. \frac{d}{dt} \right|_{t=0} \int_M E(u_t; A) = \left. \frac{d}{dt} \right|_{t=0} \int_M |du_t + d\mu_{u_t}(A)|^2 \text{dvol}_g$$

$$= \left. \frac{d}{dt} \right|_{t=0} \int_M \delta^{\alpha\beta} h_{u_t}(du_{\alpha}(e_\alpha) + (d\mu)_{u_t}(A(e_\alpha), 0), du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0)) \text{dvol}_g$$

$$= 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( \nabla^TN_{\partial_t} (du_{\alpha}(e_\alpha) + (d\mu)_{u_t}(A(e_\alpha), 0), du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0)) \text{dvol}_gight)$$

$$= 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( \nabla^W_{e_\alpha} W, du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0)) \text{dvol}_g$$

$$+ 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( \nabla^TN_{\partial_t} (d\mu)_{u_t}(A(e_\alpha), 0), du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0)) \text{dvol}_g$$

$$= -2 \int_M \delta^{\alpha\beta} h_{u_t} \left( W, \nabla^N_{e_\alpha} (du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0))) \text{dvol}_g$$

$$- 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( W, (\text{div}_{g} e_\alpha) (du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0))) \text{dvol}_g$$

$$+ 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( (\partial_1 \partial_2 \mu)(A(e_\alpha), W), du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0) \right) \text{dvol}_g$$

$$= -2 \int_M h_{u_t} \left( W, (\tau(u) + \delta^{\alpha\beta} (\partial_1 \partial_2 \mu)_u(A(e_\beta), du_{\alpha})) + d\mu_u((\text{div}_{g} A), 0) \right) \text{dvol}_g$$

$$+ 2 \int_M \delta^{\alpha\beta} h_{u_t} \left( (\partial_1 \partial_2 \mu)(A(e_\alpha), W), du_{\beta}(e_\beta) + (d\mu)_{u_t}(A(e_\beta), 0) \right) \text{dvol}_g.$$
Claim 1. For any $a \in \Gamma(\mathfrak{g}_N)$ and any $Y, Z \in \Gamma(TN)$,
\begin{equation}
\langle \partial_1 \partial_2 \mu(a, Y), Z \rangle_h + \langle Y, \partial_1 \partial_2 \mu(a, Z) \rangle_h = 0
\end{equation}

Argument for Claim 1. Let $\{e_\alpha\}$ be an orthonormal basis for $\mathfrak{g}$. Then the field $d\mu(e_\alpha, 0)$ is Killing on $N$. Hence, for any $Y, Z \in \Gamma(TN)$, we have
\[ \langle \nabla_Y d\mu(e_\alpha, 0), Z \rangle + \langle Y, \nabla_Z d\mu(e_\alpha, 0) \rangle = 0. \]
Therefore, write $a = a^\alpha e_\alpha$, then
\[ \langle \partial_1 \partial_2 \mu(a, Y), Z \rangle = a^\alpha \langle \partial_1 \partial_2 \mu(e_\alpha, Y), Z \rangle = a^\alpha \langle \nabla_Y d\mu(e_\alpha, 0), Z \rangle = -a^\alpha \langle Y, \nabla_Z d\mu(e_\alpha, 0) \rangle = -\langle Y, \partial_1 \partial_2 \mu(a, Z) \rangle, \]
as claimed. \qed

Applying this observations to the integrand above, we obtain
\begin{align*}
\langle \partial_1 \partial_2 \mu(A(e_\alpha), W), du(e_\alpha) \rangle &= -\langle W, \partial_1 \partial_2 \mu(A(e_\alpha), du(e_\alpha)) \rangle, \\
\langle \partial_1 \partial_2 \mu(A(e_\alpha), W), d\mu_u(A(e_\alpha)) \rangle &= -\langle W, \partial_1 \partial_2 \mu(A(e_\alpha), d\mu_u(A(e_\alpha))) \rangle.
\end{align*}
Hence, we are finally led to
\[
\frac{d}{dt} \bigg|_{t=0} E(u_t; A) = -2 \int_M \langle W, \tau(u) + \partial_1 \partial_2 \mu(A(e_\alpha), du(e_\alpha)) + d\mu_u(\text{div}(A)) \rangle \text{dvol}_g \\
- 2 \int_M \langle W, \partial_1 \partial_2 \mu(A(e_\alpha), du(e_\alpha)) + \partial_1 \partial_2 \mu(A(e_\alpha), d\mu_u(A(e_\alpha))) \rangle \text{dvol}_g.
\]

Thus the Euler–Lagrange equations for the energy functional in terms of the local representative $u$ reads
\[ \tau(u) + 2\partial_1 \partial_2 \mu(A(e_\alpha), du(e_\alpha)) + d\mu_u(\text{div}(A)) + \partial_1 \partial_2 \mu(A(e_\alpha), d\mu_u(A(e_\alpha))) = 0. \]
This is the local form of $\tau^V(\phi) = 0$.

2.3. Diffeomorphism invariance and conformal invariance. By construction, the functional is diffeomorphism invariant: for a diffeomorphism $f \in \text{Diff}(M)$, it holds that
\[ E(\phi; \omega, g) = E(f^*\phi; f^*\omega, f^*g). \]
Notice that the section $\phi \in \Gamma(\mathcal{N})$ is pulled back to a section $f^*\phi$ of the fiber bundle $f^*\mathcal{N}$, which is associated to the principal $G$-bundle $f^*P \to M$, and the connection $\omega$ is also pulled back to a connection $f^*\omega$ on $f^*P$ whose local representative is given by the local form $f^*A$. The diffeomorphism invariance formula can then be verified by change of variables.

In the special case where the base manifold is a surface, this energy is invariant under rescaling of the metric $g$ on $M$ by a positive smooth function $\lambda \in C^\infty(M)$:
\[ E(\phi; \omega, g) = E(\phi; \omega, \lambda^2 g). \]
As a consequence the action $E(\phi; \omega, g)$ is invariant also under conformal diffeomorphisms.
3. Coupling with Yang–Mills and Dirac

In this section we construct the Yang–Mills–Higgs–Dirac action which combines the Dirichlet action for sections with the Yang–Mills action on the connection $\omega$ and the Dirac action for a twisted spinorial field. In physics, the coupling between the section $\phi$ with the twisted spinor is motivated by supersymmetry, see [10]. Mathematically, one might say that we extend Dirac-harmonic maps, see [8], to a gauged setting. Previous works in this direction include [46] for the analysis of the Yang–Mills–Higgs action and [17] for the regularity of the Dirac equation.

3.1. The Yang–Mills–Higgs part. Recall that the connection $\omega$ is an ad-equivariant, $g$-valued one-form on $P$. Its curvature $\hat{F} = D_\omega(\omega)$ is a horizontal equivariant $g$-valued two-form, satisfying

$$\hat{F} = d\omega + \frac{1}{2}[\omega, \omega], \quad D_\omega \hat{F} = 0.$$  

Recall that horizontal, ad-equivariant $k$-forms on $P$ can be reduced to $\text{Ad}(P)$-valued $k$-forms on the base manifold $M$, where $\text{Ad}(P)$ is the adjoint bundle induced by the adjoint action of $G$ on $g$ i.e. $\text{Ad}(P) = P \times_{\text{Ad}} g$. Equipping the compact Lie group $G$ with a bi-invariant Riemannian structure $\langle \cdot, \cdot \rangle$, and hence $g$ with an $\text{Ad}$-invariant inner product $\langle \cdot, \cdot \rangle_g$, we get a Riemannian structure on $\text{Ad}(P)$, still denoted by $\langle \cdot, \cdot \rangle$ for simplicity.

In particular, the curvature can be identified with a section

$$F = (e^\alpha \wedge e^\beta) \otimes F_{\alpha \beta} \in \Gamma((T^*M \wedge T^*M) \otimes_M \text{Ad}(P)) \equiv \Omega^2(\text{Ad}(P))$$

with norm $|F(x)|^2 = \sum_{\alpha, \beta} \langle F_{\alpha \beta}(x), F_{\alpha \beta}(x) \rangle$, where $(e^\alpha)$ is a local orthonormal coframe on $(M, g)$. The Yang–Mills functional is

$$A_{YM}(\omega) = \int_M |F|^2 \text{dvol}_g.$$  

The variation formula for $A_{YM}(\omega)$ is standard, see, for example, [42]. Let $\tilde{\zeta}$ be an arbitrary ad-equivariant horizontal one-form on $P$ with values in $g$, and consider the family of connections $\omega_t = \omega + t\tilde{\zeta}$. The corresponding curvatures are $\hat{F}_t$ which are identified with $F_t \in \Omega^2(\text{Ad}(P))$. Identify also the variational field $\tilde{\zeta}$ with a section $\zeta \in \Omega^1(\text{Ad}(P))$. Then

$$\frac{d}{dt} \big|_{t=0} \int_M |F_t|^2 \text{dvol}_g = 2 \int_M \langle D_\omega \zeta, F \rangle = 2 \int_M \langle \zeta, D_\omega^* F \rangle \text{dvol}_g,$$

where $D_\omega^*: \Omega^2(\text{Ad}(P)) \to \Omega^1(\text{Ad}(P))$ is the adjoint of $D_\omega$ on $\Omega^1(\text{Ad}(P))$ with respect to the global $L^2$ inner product.

Using the local section $s$ as before and writing $A = s^*\omega$, the local representative of the curvature is given by $F_A = s^*\hat{F}$ which satisfies

$$F_A = dA + \frac{1}{2}[A, A], \quad D_A F_A = dF_A + [A, F_A] = 0.$$  

Its codifferential is

$$D_A^* F_A = d_A^*(dA + \frac{1}{2}[A, A]) = d^* dA + \frac{1}{2} d^* [A, A] + A^* dA + \frac{1}{2} A^* [A, A]$$

$$= d^* dA + \frac{1}{2} d^* [A, A] - A_\perp dA - \frac{1}{2} A_\perp [A, A],$$
where the last step holds because $A^*$ acts on forms as contraction by $-A$.

Note that the "energy" $E(\phi; \omega)$ also depends on the gauge potential $\omega$. Actually, from the construction we see that both the Kaluza–Klein metric $\g$ on $N$ and the vertical differential $d^V \phi$ depend on the induced connection $\sigma$ and hence on the principal connection $\omega$. However, by the computation in the previous section we see that for a local orthonormal frame $(e_\alpha)$ which lifts to $(\tilde{e}_\alpha)$ on $P$,

$$|d^V \phi|_{\g}^2(x) = \g_{\phi(x)}(\text{ver}(d\phi(e_\alpha(x))), \text{ver}(d\phi(e_\alpha(x))))$$

$$= \g_{[p, \tilde{\phi}(p)]}\left(\frac{\partial}{\partial s} \tilde{\phi}(\text{hor} \tilde{e}_\alpha(p)), \frac{\partial}{\partial s} \tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))\right)$$

$$= g_{\phi(p)}\left(\frac{d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p)), d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))}{2}\right)$$

$$= |d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))|_h^2,$$

where $\pi(p) = x$. That is,

$$\text{hor} \tilde{e}_\alpha(p) = \tilde{e}_\alpha - \Psi'_p(\omega(\tilde{e}_\alpha(p)))$$

is the only part depending on the connection $\omega$. Moreover,

$$\frac{d}{dt} \left|_{t=0} \frac{d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))}{d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))}\right|_{h_{\tilde{\phi}(p)}} = 2 \left\langle d\tilde{\phi}(\Psi'_p(\zeta(e_\alpha(x))), d\tilde{\phi}(\text{hor} \tilde{e}_\alpha(p))) \right\rangle_{\g_{\phi(x)}}$$

$$= 2 \left\langle d_{t,p} d\mu_{\phi(x)}(\zeta(e_\alpha(x))), d_{t,p} d\phi(\text{hor} \tilde{e}_\alpha(p))) \right\rangle_{\g_{\phi(x)}},$$

where $d\mu_{\phi(x)}(\zeta(e_\alpha))$ is defined in the following way. For a point $z = [p, y] \in N$ consider the map

(6) \(d\tilde{\mu}_z: \text{Ad}(P) \rightarrow \mathcal{V}_z,\)

\[ [p, \zeta] \mapsto [p, d\mu_y(\zeta)], \]

where $d\mu_y$ is the differential of the evaluation map $\mu_y: G \rightarrow N$ and $\zeta \in \g$ is a Lie algebra element. This is well-defined: for any $a \in G$, $[p, y] = [\Psi_a(p), \mu_{a^{-1}}(y)]$ and $[p, \zeta] = [\Psi_a(p), Ad_{a^{-1}}(\zeta)]$, we have

$$[\Psi_a(p), d\mu_{\mu_{a^{-1}}(y)} Ad_{a^{-1}}(\zeta)] = [\Psi_a(p), d\mu_{a^{-1}} d\mu_y(\zeta)] = [p, d\mu_y(\zeta)],$$

where we have used, for a curve $c(t)$ representing $\zeta \in \g$, that

$$d\mu_{\mu_{a^{-1}}(y)} Ad_{a^{-1}}(\zeta) = \frac{d}{dt} \bigg|_{t=0} \mu(a^{-1} c(t) a, \mu_{a^{-1}}(y)) = d\mu_{a^{-1}} d\mu_y(\zeta).$$

Denote by $\text{Ad}(P) \times_M N$ the fiber product (the pull-back in the sense of category theory) of $\text{Ad}(P)$ and $N$ over $M$. We have a well-defined map

$$d\tilde{\mu}: \text{Ad}(P) \times_M N \rightarrow \mathcal{V}.$$
Therefore we have
\[
\frac{d}{dt} \bigg|_{t=0} \int_M |d^\nabla\phi|^2_g(x) \, dv g = 2 \int_M \langle d\bar{\mu}_\phi(\zeta(e_\alpha)), d^\nabla\phi(\epsilon_\alpha) \rangle \, dv g \\
= 2 \int_M \langle \zeta, d\bar{\mu}_\phi(d^\nabla\phi) \rangle \, dv g,
\]
where \( d\bar{\mu}_\phi \) denotes the formal \( L^2 \)-adjoint of \( d\mu_\phi \) in \( \text{Hom}(\text{Ad}(P), \phi^*\mathcal{V}) \).

In the local formulation, this is much easier. Recall \( d_A u = du + d\mu_u(A) \) and that \( E(u; A) \) is given by (3) in which the metric does not depend on the connections. Variation with respect to the family \( (\omega_t = \omega + t\zeta) \) gives
\[
\frac{d}{dt} \bigg|_{t=0} E(u; A_t) = 2 \int_M \langle d\mu_u(\zeta), d_A u \rangle \, dv g \equiv 2 \int_M \langle \zeta, d\mu_u^*(d_A u) \rangle \, dv g.
\]

3.2. The Dirac action. From now on we assume that the base manifold \((M, g)\) is spin and fix a spin structure. Let \( S \to M \) be a spinor bundle associated with the Clifford map \( \gamma: TM \to \text{End}(S) \) satisfying the Clifford relation
\[
\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X, Y), \quad \forall X, Y \in \Gamma(TM).
\]
The Levi-Civita connection of \((M, g)\) can be lifted to a connection on the spin principal bundle and thus induces a spin connection on \( S \). We denote the corresponding covariant derivative by \( \nabla^s \). The spin Dirac operator \( \hat{\phi} s = \gamma(e_\alpha)\nabla^s_{e_\alpha} s \) is a first-order self-adjoint elliptic differential operator on \( S \). Without loss of generality we assume that the spinor bundle \( S \) is always equipped with a \(
abla\text{Spin}(m)\) invariant metric \( g_s \). For more about spin geometry we refer to [30, 21, 14].

It is important to note that the self-adjointness of \( \hat{\phi} \) depends crucially on the minus sign in the Clifford relation, compare the discussion in [25]. Without this minus sign, the Dirac-operator would be anti-selfadjoint and the Dirac-term in the action below would vanish. In contrast, in the physics literature, it is customary to use the Clifford relation without minus sign and to obtain a selfadjoint Dirac-operator via anti-commuting variables. The idea that anti-commuting variables can be avoided by using the minus sign in the Clifford relation goes back to [8].

Given a \( C^1 \) section \( \phi \in \Gamma(\mathcal{N}) \), we consider twisted spinors field along \( \phi \), that is, sections \( \psi \in \Gamma(S \otimes \phi^*\mathcal{V}) \). We still denote by \( \gamma: TM \to \text{End}(S \otimes \phi^*\mathcal{V}) \) the Clifford map that arises from the Clifford map on \( S \) acting on the first factor. The covariant derivative \( \nabla \) on \( TN \) can be restricted to a covariant derivative on \( \mathcal{V} \) by setting \( \nabla^\mathcal{V} = \text{ver} \nabla \).

Thus \((S \otimes \phi^*\mathcal{V}, \nabla^{S \otimes \phi^*\mathcal{V}}, \gamma, g_s \otimes \phi^*\hat{h})\) is a Dirac bundle in the sense of [30]. The corresponding twisted Dirac-operator \( \hat{D}\psi = \gamma(e_\alpha)\nabla^{S \otimes \phi^*\mathcal{V}}_{e_\alpha}{\psi} \) is again an essentially self-adjoint first-order differential operator.

The Dirac action of interest has the form
\[
A_D(\psi; \omega, \phi) = \int_M \langle \psi, \hat{D}\psi \rangle_{g_s \otimes \phi^*\hat{h}} \, dv g.
\]
3.2.1. Equations of motion. The derivation of the equations of motion of \( A_D(\psi; \omega, \phi) \) is mostly straightforward. Note that the spinor field \( \psi \) depends on the section \( \phi \) and hence \( \phi \) and \( \psi \) cannot be varied independently. Therefore, we use the same method as in [25]. Thus let \( (\phi_t, \psi_t) \) be a variation family of \( (\phi = \phi_0, \psi = \psi_0) \) for \( t \) in a neighborhood of 0. Then

\[
\frac{d}{dt} \int_M \left\langle \psi_t, D\phi_t \psi_t \right\rangle \, dvol_g = \int_M \nabla^{S+\phi \gamma \nu} \left\langle \psi_t, D\phi_t \psi_t \right\rangle \, dvol_g = \int_M \left\langle \nabla^{S+\phi \gamma \nu} \psi_t, D\phi_t \psi_t \right\rangle \, dvol_g + \int_M \left\langle \psi_t, \nabla^{S+\phi \gamma \nu} D\phi_t \psi_t \right\rangle \, dvol_g
\]

\[
= \int_M \left\langle \nabla^{S+\phi \gamma \nu} \psi_t, D\phi_t \psi_t \right\rangle \, dvol_g + \int_M \left\langle \psi_t, \nabla^{S+\phi \gamma \nu} \tilde{D} \phi_t \psi_t \right\rangle \, dvol_g
\]

\[
+ \int_M \left\langle \psi_t, \gamma(e_\alpha) R^{S+\phi \gamma \nu}(\partial_t, e_\alpha) \psi_t \right\rangle \, dvol_g.
\]

Since the Dirac operator is self-adjoint and the spinor bundle does not change with \( t \), we have

\[
\frac{d}{dt} \bigg|_{t=0} \int_M \left\langle \psi_t, D\phi_t \psi_t \right\rangle \, dvol_g = 2 \int_M \left\langle \tilde{D} \psi, \nabla^{S+\phi \gamma \nu} \psi_t \right\rangle \bigg|_{t=0} \, dvol_g + \int_M \left\langle \psi, \gamma(e_\alpha) R^{S+\phi \gamma \nu}(\partial_t, e_\alpha) \psi \right\rangle \bigg|_{t=0} \, dvol_g.
\]

The curvature term arises from the permutation of the Dirac operator and the covariant derivative, compare the corresponding calculation in [3]. The curvature term is tensorial in \( \phi_* (\partial_t) \) and thus we define \( \mathcal{R}^V(\phi, \psi) \in \Gamma((\phi^* T \mathcal{N})^*) \) by

\[
\left\langle \psi, \gamma(e_\alpha) R^{S+\phi \gamma \nu}(\partial_t, e_\alpha) \psi \bigg|_{t=0} \right\rangle \equiv \left\langle \phi_* (\partial_t), \mathcal{R}^V(\phi, \psi) \right\rangle.
\]

Therefore the variation formula with respect to \( (\phi, \psi) \) is

\[
\frac{d}{dt} \bigg|_{t=0} A_D(\phi_t, \psi_t; \omega) = \int_M 2 \left\langle \tilde{D} \psi, \nabla^{S+\phi \gamma \nu} \psi_t \right\rangle_{t=0} \bigg| + \left\langle \mathcal{R}^V(\phi, \psi), \phi_* (\partial_t) \right\rangle_{t=0} \, dvol_g.
\]

3.2.2. Local description. We will now derive local expressions for the twisted Dirac operator in suitable local normal coordinates. Let \( x_0 \) be a point in \( M \), \( y_0 \) a point in \( N \) and \( p_0 \) a point in \( P \) above \( x_0 \). Let \( (x^\alpha)_{\alpha=1, \ldots, m} \) be normal coordinates with respect to \( g \) in an open neighborhood \( U \) of \( x_0 \) in \( M \).

Assume that there is a local section \( s: U \rightarrow P \) giving a local trivialization of \( P \) as \( U \times G \). In this trivialization the connection \( \omega \) is given by \( A = s^* \omega \). We can choose the section \( s \) such that \( A(x_0) = 0 \). Indeed, if \( A(x_0) \neq 0 \), let \( \tilde{s} = \Psi_{\varphi(x)}(s(x)) \) where \( \varphi: U \rightarrow G \) such that \( \varphi(x_0) = e \in G \) and \( d\varphi(x_0) = -A(x_0) \). Then \( \tilde{A} = \tilde{s}^* \omega \) is given by \( \tilde{A} = ad_{\varphi^{-1}}(A) + \varphi^{-1} d\varphi \), and

\[
\tilde{A}(x_0) = ad_{\varphi}(A)(x_0) + d\varphi(x_0) = A(x_0) - A(x_0) = 0.
\]

In the following we will always assume a trivialization such that \( A(x_0) = 0 \). Let \( z^\nu \) be local coordinates on \( G \) around \( e \). We denote the lift of \( x^\alpha \) and \( z^\nu \) to the product \( U \times G \) by \( (\tilde{x}^\alpha, \tilde{z}^\nu) \).

Let now \( (y^i)_{i=1, \ldots, n} \) be normal coordinates around \( y_0 \) in \( N \) with respect to \( h \). The fiber bundle \( \mathcal{N} \) is locally around \( [p_0, y_0] \) a fiber product and we denote the lift of the coordinates \( x^\alpha \) and \( y^i \) to this product by \( (\tilde{x}^\alpha, \tilde{y}^i) \) gives a local coordinate system. By construction, the vector field \( \partial_{\tilde{y}^i} \) is vertical, but the vector fields \( \partial_{\tilde{x}^\alpha} \) are not necessarily horizontal.
Indeed, noting that \( \bar{x}^\alpha \circ \iota_y = \tilde{x}^\alpha \) as local functions on \( P \), we have

\[
d\iota_y \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right) = \frac{\partial}{\partial \bar{x}^\alpha},
\]
and hence

\[
\text{hor} \left( \frac{\partial}{\partial \bar{x}^\alpha} \right) = d\iota_y \left( \text{hor} \frac{\partial}{\partial \tilde{x}^\alpha} \right) = d\iota_y \left( \frac{\partial}{\partial \tilde{x}^\alpha} - \Psi'_p \omega \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right) \right) = \frac{\partial}{\partial \bar{x}^\alpha} + dt_p \circ d\mu_y \circ \omega \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right).
\]

Thus the vertical part is

\[
\text{ver} \left( \frac{\partial}{\partial \bar{x}^\alpha} \right) = - dt_p \circ d\mu_y \circ \omega \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right) = - d\bar{\mu}_z \circ A \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right)
\]

where \( A = s^* \omega \) is the local representative of the connection and \( d\bar{\mu}_z \) is given in (3).

Recall the Kaluza–Klein metric on \( N \):

\[
G(X,Y) = \bar{h} \left( \text{ver} X, \text{ver} Y \right) + g_{\rho} \left( \text{hor} X, \text{hor} Y \right)
\]

In terms of the local coordinates introduced above, we have

\[
\begin{align*}
G_{\alpha\beta} &= G \left( \frac{\partial}{\partial \bar{x}^\alpha}, \frac{\partial}{\partial \bar{x}^\beta} \right) = g \left( \frac{\partial}{\partial \tilde{x}^\alpha}, \frac{\partial}{\partial \tilde{x}^\beta} \right) + \bar{h} \left( \text{ver} \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right), \text{ver} \left( \frac{\partial}{\partial \tilde{x}^\beta} \right) \right) \\
&= g_{\alpha\beta} + \bar{h} \left( d\bar{\mu}_z A \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right), d\bar{\mu}_z A \left( \frac{\partial}{\partial \tilde{x}^\beta} \right) \right) = g_{\alpha\beta} + \bar{h}_{\alpha\beta}, \\
G_{ij} &= G \left( \frac{\partial}{\partial \bar{y}^i}, \frac{\partial}{\partial \bar{y}^j} \right) = h_{ij}, \\
G_{\alpha i} &= G \left( \frac{\partial}{\partial \bar{x}^\alpha}, \frac{\partial}{\partial \bar{y}^i} \right) = \bar{h} \left( \text{ver} \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right), \frac{\partial}{\partial \bar{y}^i} \right) = - \bar{h} \left( d\bar{\mu}_z A \left( \frac{\partial}{\partial \tilde{x}^\alpha} \right), \frac{\partial}{\partial \bar{y}^i} \right).
\end{align*}
\]

Notice that we use Greek indices for coordinates of the base and Latin fiber indices.

At the point \([p_0, y_0]\), by construction of the normal coordinates we have

\[
(7) \quad G_{\alpha\beta} = \delta_{\alpha\beta}, \quad G_{ij} = \delta_{ij}, \quad G_{\alpha i} = 0,
\]

and all the Christoffel symbols of \( \nabla \) vanish at this given point.

Any spinor field \( \psi \) along the section \( \phi \) can be expressed as

\[
\psi = \psi^i \otimes \phi^* \left( \frac{\partial}{\partial \bar{y}^i} \right).
\]

The vertical connection acts on such twisted spinors in the following way: for any \( X \in \Gamma(TM) \),

\[
\nabla_X^{S \otimes \phi^* V} \psi = \nabla^S_X \psi^i \otimes \phi^* \left( \frac{\partial}{\partial \bar{y}^i} \right) + \psi^i \otimes \nabla^\phi \phi^* \left( \frac{\partial}{\partial \bar{y}^i} \right),
\]

where

\[
\nabla^\phi \phi^* \left( \frac{\partial}{\partial \bar{y}^i} \right) = \text{ver} \phi^* \left( \nabla_{d\phi(X)} \frac{\partial}{\partial \bar{y}^i} \right).
\]
Writing $d\phi(X) = X(\phi^\beta)\partial_{x^\beta} + X(\phi^j)\partial_{y^j}$ and noting that the fibers are totally geodesic, we have

$$\nabla_X^{\phi^*} \phi^* \left( \frac{\partial}{\partial y^i} \right) = \text{ver} \left( X(\phi^\beta)\nabla_{\partial_{x^\beta}}\partial_{y^i} + X(\phi^j)\nabla_{\partial_{y^j}}\partial_{y^i} \right)$$

$$= \text{ver} \left( X(\phi^\beta)\Gamma^\eta_\beta_i \frac{\partial}{\partial x^\eta} + X(\phi^\beta)\Gamma^k_\beta_i \frac{\partial}{\partial y^k} + X(\phi^j)\Gamma^k_j \frac{\partial}{\partial y^k} \right)$$

$$= -X(\phi^\beta)\Gamma^\eta_\beta_i d\tilde{\mu}(x)A \left( \frac{\partial}{\partial x^\eta} \right) + X(\phi^\beta)\Gamma^k_\beta_i \frac{\partial}{\partial y^k} + X(\phi^j)\Gamma^k_j \frac{\partial}{\partial y^k}. $$

The associated Dirac operator $\slashed{D}$ on $S \otimes \phi^*\mathcal{V}$ is defined in the canonical way: taking a local orthonormal basis $(e_\alpha)$ on $M$, for any spinor $\psi$ along the section $\phi$,

$$\slashed{D}\psi = \gamma(e_\alpha)\nabla_{e_\alpha}^{S\otimes \phi^*\mathcal{V}} \psi = \partial\psi^i \otimes \phi^* \left( \frac{\partial}{\partial y^i} \right) + \gamma(e_\alpha)\psi^i \otimes \text{ver} \phi^* \left( \nabla_{d\phi(e_\alpha)}\frac{\partial}{\partial y^i} \right).$$

### 3.2.3. Dependence on the gauge potential.

Finally we need to consider the variation with respect to the gauge potential $\omega$. As before we consider $\omega_t = \omega + t\zeta$ with $\zeta$ a horizontal one-form on $P$. Note that the Kaluza–Klein metric $\mathcal{G}$ depends on $\omega_t$ via $\sigma$, while the vertical metric $\bar{h}$ does not. Hence the Dirac operator $\slashed{D}_t \equiv \slashed{D}_{\omega_t}$ depends also on $t$ via the Levi-Civita connection, and we have

$$\frac{d}{dt} \slashed{D}_t \psi = \frac{d}{dt} \left( \partial\psi^i \otimes \phi^* \left( \frac{\partial}{\partial y^i} \right) + \gamma(e_\alpha)\psi^i \otimes \nabla_{e_\alpha}^{t\phi^*\mathcal{V}} \phi^* \left( \frac{\partial}{\partial y^i} \right) \right)$$

$$= \gamma(e_\alpha)\psi^i \otimes \frac{d}{dt} \phi^* \left( \nabla_{\phi^*}(e_\alpha) \frac{\partial}{\partial y^i} \right).$$

Thus, the problem is reduced to analyzing the dependence of $\nabla^\mathcal{V}$ on the connection $\omega$. As $\nabla^\mathcal{V}$ is the vertical part of the Levi-Civita connection $\nabla$ of $\mathcal{G}$ we need to understand their dependence on $\omega$. Here

$$\mathcal{G}_t(X,Y) = g_\rho(\text{hor}_t X, \text{hor}_t Y) + \bar{h}(\text{ver}_t X, \text{ver}_t Y)$$

depends on $\omega$ via the horizontal and vertical projectors.

The coordinates $(\bar{x}^\alpha, \bar{y}^i)$ of $\mathcal{N}$ are normal coordinates with respect to the metric $\mathcal{G} = \mathcal{G}_0$ at the point $(x_0, y_0)$ by construction. For a general $t \neq 0$, the local vectors $\{\partial/\partial \bar{y}^i\}_{1 \leq i \leq n}$ stay orthonormal and vertical, but the vectors $\{\partial/\partial \bar{x}^\alpha\}$ are in general neither horizontal nor orthonormal. Hence,

$$\frac{d}{dt} \mathcal{G}_{ij} \bigg|_{t=0} = \frac{d}{dt} \bigg|_{t=0} \bar{h}_{ij} = 0,$$

$$\frac{d}{dt} \mathcal{G}_{\alpha\beta} = \bar{h} \left( \frac{d}{dt} \bigg|_{t=0} \text{ver}_t \left( \frac{\partial}{\partial \bar{x}^\alpha} \right), \text{ver}_t \left( \frac{\partial}{\partial \bar{y}^\beta} \right) \right) + \bar{h} \left( \text{ver}_t \left( \frac{\partial}{\partial \bar{x}^\alpha} \right), \frac{d}{dt} \bigg|_{t=0} \text{ver}_t \left( \frac{\partial}{\partial \bar{y}^\beta} \right) \right),$$

$$\frac{d}{dt} \mathcal{G}_{\alpha i} = \bar{h} \left( \frac{d}{dt} \bigg|_{t=0} \text{ver}_t \left( \frac{\partial}{\partial \bar{x}^\alpha} \right), \text{ver}_t \left( \frac{\partial}{\partial y^i} \right) \right).$$
Therefore we get
\[
\frac{d}{dt}\left|_{t=0} \text{ver}_t \left( \frac{\partial}{\partial x^\alpha} \right) = \frac{d}{dt}\left|_{t=0} \right. - dt_p \, d\mu_y \omega_t \left( \frac{\partial}{\partial x^\alpha} \right) \\
\quad = - dt_p \, d\mu_y \zeta \left( \frac{\partial}{\partial x^\alpha} \right) = - d\bar{\mu}_z \circ \zeta \left( \frac{\partial}{\partial x^\alpha} \right).
\]

Therefore we get
\[
\frac{d}{dt}\left|_{t=0} \mathcal{G}_{\alpha \beta} = h \left( d_t \, d\mu_y \zeta \left( \frac{\partial}{\partial x^\alpha} \right), d_t \, d\mu_y A \left( \frac{\partial}{\partial x^\beta} \right) \right) \\
\quad + h \left( d_t \, d\mu_y A \left( \frac{\partial}{\partial x^\alpha} \right), d_t \, d\mu_y \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right) \\
\quad = h \left( d\bar{\mu}_z \zeta \left( \frac{\partial}{\partial x^\alpha} \right), d\bar{\mu}_z A \left( \frac{\partial}{\partial x^\beta} \right) \right) + h \left( d\bar{\mu}_z A \left( \frac{\partial}{\partial x^\alpha} \right), d\bar{\mu}_z \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right) \\
\quad = (\bar{\mu}_z \hat{h}) \left( \zeta \left( \frac{\partial}{\partial x^\alpha} \right), A \left( \frac{\partial}{\partial x^\beta} \right) \right) + (\bar{\mu}_z \hat{h}) \left( A \left( \frac{\partial}{\partial x^\alpha} \right), \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right),
\]

\[
\frac{d}{dt}\left|_{t=0} \mathcal{G}_{a i} = - \hat{h} \left( d\bar{\mu}_z \zeta \left( \frac{\partial}{\partial x^\alpha} \right), \frac{\partial}{\partial y^i} \right).
\]

Now we continue to compute (9). The points under consideration are \( \phi(x) = [p, y] \in \mathcal{N} \) and \( y = \hat{\phi}(p) \in \mathcal{N} \). Write
\[
\phi^* (e_\alpha) = \phi_\alpha^2 \frac{\partial}{\partial x^\beta} + \phi_\alpha^j \frac{\partial}{\partial y^j};
\]
and denote the Christoffel symbols of \( \mathcal{G}(t) \) by \( \Gamma(t) \), and \( A_t = A + t\zeta \), then
\[
\text{ver}_t \nabla_{\phi(e_\alpha)} \frac{\partial}{\partial y^i} = \phi_\alpha^2 \Gamma^\eta_{\beta j} (t) d\bar{\mu} \phi(x) A_t \left( \frac{\partial}{\partial x^\eta} \right) + \phi_\alpha^2 \Gamma^k_{\beta j} (t) \frac{\partial}{\partial y^k} + \phi_\alpha^j \Gamma^k_{\beta j} (t) \frac{\partial}{\partial y^k}.
\]
Note that \( \frac{d}{dt}\left|_{t=0} \right. \Gamma^k_{\beta j} = 0 \) since the vertical part stays unchanged when perturbing the connection. We have to compute the \( t \)-derivatives of the Christoffel symbols \( \Gamma^k_{\beta j} \). Since we have take normal coordinates around \( \phi(x_0) \in \mathcal{N} \) such that (11) holds there, we have, at the point \( \phi(x_0) \),
\[
\frac{d}{dt}\left|_{t=0} \right. \Gamma^k_{\beta i} = \frac{1}{2} \frac{d}{dt}\left|_{t=0} \right. \left\{ \mathcal{G}^{kn} \left( \frac{\partial \mathcal{G}_{ni}}{\partial x^\beta} + \frac{\partial \mathcal{G}_{nj}}{\partial y^i} - \frac{\partial \mathcal{G}_{ni}}{\partial y^i} \right) + \mathcal{G}^{kl} \left( \frac{\partial \mathcal{G}_{li}}{\partial x^\beta} + \frac{\partial \mathcal{G}_{lj}}{\partial y^k} - \frac{\partial \mathcal{G}_{lj}}{\partial y^k} \right) \right\} \\
\quad = \frac{1}{2} \mathcal{G}^{kl} \left( \frac{\partial}{\partial x^\beta} \frac{d}{dt}\left|_{t=0} \right. \mathcal{G}_{li} + \frac{\partial}{\partial y^k} \frac{d}{dt}\left|_{t=0} \right. \mathcal{G}_{lj} - \frac{\partial}{\partial y^k} \frac{d}{dt}\left|_{t=0} \right. \mathcal{G}_{lj} \right) \\
\quad = \frac{1}{2} \left[ \frac{\partial}{\partial y^k} \frac{d}{dt}\left|_{t=0} \right. \mathcal{G}_{k \beta} - \frac{\partial}{\partial y^k} \frac{d}{dt}\left|_{t=0} \right. \mathcal{G}_{\beta i} \right] \\
\quad = \frac{1}{2} \left[ \frac{\partial}{\partial y^k} \hat{h} \left( d\bar{\mu} \phi(x) \left( \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right), \frac{\partial}{\partial y^k} \right) - \frac{\partial}{\partial y^k} \hat{h} \left( d\bar{\mu} \phi(x) \left( \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right), \frac{\partial}{\partial y^k} \right) \right] \\
\quad = \frac{1}{2} \left[ \hat{h} \left( \partial_1 \partial_2 \bar{\mu} \phi(x) \left( \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right), \frac{\partial}{\partial y^k} \right) - \hat{h} \left( \partial_1 \partial_2 \bar{\mu} \phi(x) \left( \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right), \frac{\partial}{\partial y^k} \right) \right] \\
\quad = \hat{h} \left( \partial_1 \partial_2 \bar{\mu} \phi(x) \left( \zeta \left( \frac{\partial}{\partial x^\beta} \right) \right), \frac{\partial}{\partial y^k} \right),
\]
where in the last step we used (5) in Claim 1.

We thus get

\[
\frac{d}{dt} \left| t = 0 \right| \nabla_{\phi(x)} \frac{\partial}{\partial y^j} = \phi^\beta_\alpha \hbar \left( \partial_1 \partial_2 \tilde{\mu}_\phi(x) (\zeta(\frac{\partial}{\partial x^\alpha}), \frac{\partial}{\partial y^k}) \right) \frac{\partial}{\partial y^k}
\]

\[
= \partial_1 \partial_2 \tilde{\mu}_\phi(x) \left( \zeta(\frac{\partial}{\partial x^\alpha}), \frac{\partial}{\partial y^i} \right)
\]

\[
\equiv \left\langle \zeta, e_\alpha \otimes \partial_1 \partial_2 \tilde{\mu}_\phi(x) \left( \frac{\partial}{\partial y^j} \right) \right\rangle.
\]

It follows that

\[
\frac{d}{dt} \left| t = 0 \right| \mathcal{D}_t \psi = \gamma(e_\alpha) \psi^j \otimes \partial_1 \partial_2 \tilde{\mu}_\phi(x) \left( \zeta(e_\alpha), \frac{\partial}{\partial y^i} \right),
\]

and

\[
\frac{d}{dt} \left| t = 0 \right| \langle \psi, \mathcal{D}_t \psi \rangle = \langle \psi^j, \gamma(e_\alpha) \psi^j \rangle \cdot \hbar \left( \partial_1 \partial_2 \tilde{\mu}_\phi(x) (\zeta(e_\alpha), \frac{\partial}{\partial y^i}), \frac{\partial}{\partial y^i} \right) \equiv \langle \zeta, Q(\phi, \psi) \rangle.
\]

Note that both factors in the middle are antisymmetric in \(i\) and \(j\).

3.3. The coupled action. In the remainder of this article we will be concerned with the model which is given by the action

\[
A(\omega, \phi, \psi) = A_{YM}(\omega) + E(\phi; \omega) + A_D(\psi; \omega) = \int_M |F|^2 + |dV\phi|^2 + \langle \psi, \mathcal{D}\psi \rangle \, dvol_g.
\]

This Yang–Mills–Higgs–Dirac action might also be considered as a gauged version of Dirac-harmonic maps. We have already seen that the total variation formula has the form

\[
\delta A = \int_M \langle \zeta, D^*_\omega F \rangle - 2 \langle \tau^V(\phi), \delta \phi \rangle + \langle \zeta, d\bar{\mu}_\phi^*(dV\phi) \rangle + 2 \langle \mathcal{D}_t \psi, \delta \psi \rangle + \langle \mathcal{R}^V(\phi, \psi), \delta \phi \rangle + \langle \zeta, Q(\phi, \psi) \rangle \, dvol_g,
\]

where \(\tilde{\zeta} = \delta \omega\) as before. The goal is to study the critical points of this coupled action functional, that is, the solution of the following Euler–Lagrange equations:

\[
(10) \quad D^*_\omega F + d\bar{\mu}_\phi^*(dV\phi) + Q(\phi, \psi) = 0,
\]

\[
\tau^V(\phi) - \frac{1}{2} \mathcal{R}^V(\phi, \psi) = 0,
\]

\[
\mathcal{D}_t \psi = 0.
\]

This is a coupled system, with the equation for \(\phi\) and \(\psi\) being elliptic. The equation for \(\omega\) is actually also (locally) elliptic, up to the choice of a local gauge, which we can fix as a Coulomb gauge, as explained later. In the remainder of this article we will study symmetry properties of the action \(A\) as well as regularity of its critical points, and their blow-up behaviour. However, we cannot include the existence part in this article.

While we focus for simplicity on the Yang–Mills–Higgs–Dirac action, several extensions have been considered in the literature:
• Instead of the Dirac action $A_D$ one might consider a Dirac-action with mass-term given by

$$\int_M \langle \psi, \hat{D} \psi \rangle - \lambda |\psi|^2 \, dvol_g,$$

see, for example, [36]. Here the parameter $\lambda \geq 0$ is interpreted as the mass of the spinors in physics. In this case, the Dirac equation is $\hat{D} \psi = \lambda \psi$. However, the mass term behaves badly under scaling (see Lemma 5.1) and is dropped in our analysis.

• In addition to the Yang–Mills–Higgs-Dirac action one might consider a curvature term for the twisted spinor $\psi = \psi^i \otimes \phi^* \partial_{y^i}$:

$$\frac{1}{6} \int_M g_s(\psi^i, \psi^k)g_s(\psi^j, \psi^l) \mathcal{G}(R^N(\partial_{y^i}, \partial_{y^j})\partial_{y^k}, \partial_{y^l}) \, dvol_g.$$

The derivation of the additional terms in the equations of motion is straightforward, compare also [6].

• Often for applications in physics, an additional potential term is needed. The functional takes the form

$$A_V(\omega, \phi, \psi) = \int_M |F(\omega)|^2 + |d^\omega \phi|^2 + V(\phi) + \langle \psi, \hat{D} \psi \rangle \, dvol_g,$$

where $V: \mathcal{N} \to \mathbb{R}$ stands for a $G$-invariant function, known as a potential. For example, when the fiber is linear a polynomial potential is usually used and when the fiber is symplectic, the momentum map is used in [37, 38, 46, 47]. We do not include this potential term since it does not affect our analysis too much in dimension two, as long as the integrability of the potential is guaranteed and certain abstract growth conditions are posed. Most of the results can be directly extended to the potential case. More generally the potential term could also depend on the spinorial field, and it is then helpful to obtain minimax solutions, see [18, 19, 54]. We shall discuss this potential a bit more in the concluding part.

• Instead of the Levi-Civita connection $\nabla$ on $\mathcal{N}$ one might consider more general metric connection, allowing for torsion, compare also [7].

• In [10] Chapter 6, a fully supersymmetric variant of the Yang–Mills–Higgs-Dirac action is given, which has motivated our study here. The fully supersymmetric theory requires an additional twisted spinor $\lambda \in S^* \otimes \text{ad } P$ as a superpartner of the connection. The action for $\lambda$ is also the Dirac-action together with lower order terms coupling to $\phi$ and $\psi$. It is straightforward to calculate the necessary terms in the equations of motion. In case the equation for the additional spinorial field is subcritical, the analysis could be carried out by extending the methods here. Notice, however, that we cannot expect full supersymmetry in our model, even when extended by $\lambda$. The reason is that supersymmetry requires anti-commuting variables which we are avoiding for the sake of analysis.

4. Energy-Momentum tensor

The energy-momentum tensor $T$ is the variation of the action $A$ with respect to the metric $g$. This tensor is interesting as it is a conserved quantity for the diffeomorphism invariance by Noether’s theorem. For a detailed explanation in a similar model, see [28].
Let \((g(t))\) be a family of Riemannian metrics on the surface \(M\), with variational field

\[
\frac{d}{dt} \bigg|_{t=0} g(t) = k \in \Gamma(\text{Sym}^2 T^*M).
\]

There is a family of self-adjoint endomorphisms \((H_t) \subset \text{End}(TM)\) such that \(g(t)(\cdot, \cdot) = g(H_t(\cdot), \cdot)\), and set \(b_t = H_t^{-1/2} \in \text{Aut}(TM)\). Then \(b_t: (TM, g) \to (TM, g(t))\) is an isometry of Riemannian vector bundles. Let \((e_\alpha)\) be a local oriented \(g\)-orthonormal frame, then \(\{E_\alpha(t) = b_t(e_\alpha)\}\) is a local oriented \(g(t)\)-orthonormal frame, and

\[
\frac{d}{dt} \bigg|_{t=0} E_\alpha(t) = \frac{d}{dt} \bigg|_{t=0} b_t(e_\alpha) = -\frac{1}{2} Ke_\alpha = -\frac{1}{2} K^{\beta}_\alpha e_\beta.
\]

Here \(K\) is the endomorphism associated to \(k\), which is also the \(t\)-derivative of \(H_t\) at \(t = 0\).

For the Dirac action part, since the fiber bundle metric \(\mathcal{G}\) is quite involved, we calculate as follows. First, note that the spinor bundle \(S_g\) depends on the choice of the Riemannian metric \(g\) on \(M\), and so does the spinor bundle metric \(g_s\). Thus, let \(S_{g_t}\) be the corresponding spinor bundle with induced metric \(g_s(t)\), and we set \(\beta_t: S_g \to S_{g_t}\) to be the isometry induced from \(b_t\) above. Then, for a given spinor \(\psi \in \Gamma(S_g \otimes \phi^*\mathcal{V})\), we should consider the isometric version

\[
\psi_t = (\beta_t \otimes 1)\psi = \beta_t(\psi^i) \otimes \phi^* \left( \frac{\partial}{\partial y^i} \right) \in \Gamma(S_{g_s(t)} \otimes \phi^*\mathcal{V}).
\]

Then with respect to \(g(t)\) and \(g_s(t)\), the Dirac operator acts as

\[
\hat{D}_{g(t)} \psi_t = \hat{D}_{g_s(t)} (\beta_t \psi^i) \otimes \phi^* \left( \frac{\partial}{\partial y^i} \right) + \gamma_t(E_\alpha(t)) \beta_t \psi^i \otimes \text{ver} \phi^* \left( \nabla_{d\phi(E_\alpha(t))} \frac{\partial}{\partial y^i} \right).
\]

Here \(\gamma_t\) denotes the Clifford map with respect to \(g(t)\) and \(\mathcal{G}(t)\) denotes the corresponding fiber bundle metric.

In terms of the frame \((E_\alpha(t))\), the action with respect to \(g(t)\) is given by

\[
\mathcal{A}(\omega, \phi, \psi_t; g(t)) = \int_M |F|_{g(t)}^2 + |d^\omega \phi|_{g(t)}^2 + \langle \hat{D}_{g(t)} \psi_t, \psi_t \rangle \ d\text{vol}_{g(t)}
\]

\[
= \int_M \sum_{\alpha, \beta} |F(E_\alpha, E_\beta)|_{g}^2 + \sum_{\alpha} \langle d^\omega \phi(E_\alpha), d^\omega \phi(E_\alpha) \rangle_{g} + \langle \hat{D}_{g(t)} \psi_t, \psi_t \rangle \ d\text{vol}_{g(t)}.
\]

Now, the curvature part is as usual,

\[
\frac{d}{dt} \bigg|_{t=0} \int_M |F|_{g(t)}^2 \ d\text{vol}_{g(t)} = \frac{d}{dt} \bigg|_{t=0} \int_M \sum_{\alpha, \beta} |F(E_\alpha, E_\beta)|_{g}^2 \ d\text{vol}_{g(t)}
\]

\[
= \int_M \sum_{\alpha, \beta} 2 \left( F(e_\alpha, e_\beta), F(-\frac{1}{2} K(e_\alpha), E_\beta) + F(\alpha, -\frac{1}{2} K(e_\beta)) \right)_{g} + \frac{1}{2} |F|_{g(t)}^2 \text{Tr}_g(K) \ d\text{vol}_{g(t)}
\]

\[
= -\int_M k_{\alpha \beta} \left( \sum_\eta 2 \langle F_{\alpha \eta}, F_{\beta \eta} \rangle_{g} - \frac{1}{2} |F|_{g_{\alpha \beta}}^2 \right) \ d\text{vol}_{g}.
\]
For the vertical energy of the section, the situation is not much different from the harmonic map case:

\[
\frac{d}{dt}igg|_{t=0} \int_M \langle d^\nu \phi(E_\alpha), d^\nu \phi(E_\beta) \rangle_\alpha d\text{vol}_{g(t)}
= - \int_M k_{\alpha \beta} \left( \langle d^\nu \phi(e_\alpha), d^\nu \phi(e_\beta) \rangle - \frac{1}{2} |d^\nu \phi|^2 g_{\alpha \beta} \right) d\text{vol}_g.
\]

For the Dirac action part, note that

\[
\text{For the other summand, applying (8) to the metric}
\]

\[
\text{coordinates and at the point under consideration, we have}
\]

\[
\text{Using the skew-adjointness of the Clifford multiplication, we have}
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= - \int_M k_{\alpha \beta} \left( \langle d^\nu \phi(e_\alpha), d^\nu \phi(e_\beta) \rangle - \frac{1}{2} |d^\nu \phi|^2 g_{\alpha \beta} \right) d\text{vol}_g.
\]
Note that the connection is fixed and hence the vertical-horizontal decomposition is unaffected by the variation of the metric, hence at the center of the normal coordinates,
\[
\frac{d}{dt}\bigg|_{t=0} \Gamma^k_{\beta i} = \frac{1}{2} \frac{d}{dt}\bigg|_{t=0} \left\{ g^{k\eta} \left( \frac{\partial G^\eta_i}{\partial \bar{x}^\beta} + \frac{\partial G^\eta_i}{\partial \bar{y}^\beta} - \frac{\partial G^\eta_i}{\partial \bar{x}^\eta} \right) + g^{kl} \left( \frac{\partial G^l_i}{\partial \bar{x}^\beta} + \frac{\partial G^l_i}{\partial \bar{y}^\beta} - \frac{\partial G^l_i}{\partial \bar{x}^l} \right) \right\} = 0.
\]
Therefore,
\[
\frac{d}{dt}\bigg|_{t=0} \left\langle \mathcal{D}_{g(t)} \psi, \psi \right\rangle \, dvol_{g(t)} = - \int_M K^\alpha_\alpha \left( \frac{1}{4} \left( \gamma(e_\alpha) \nabla_{e_\beta}^{\otimes \phi^*} \psi + \gamma(e_\beta) \nabla_{e_\alpha}^{\otimes \phi^*} \psi, \psi \right) - \frac{1}{2} \left( \mathcal{D}_\psi, \psi \right) g_{\alpha \beta} \right) \, dvol_g,
\]
where we used the symmetrization that emerged from the above calculation since \( k \) is itself symmetric. Consequently we get
\[
\frac{d}{dt}\bigg|_{t=0} \mathcal{A}(\omega, \phi, \psi; g(t)) = - \int_M \langle T, k \rangle \, dvol_g,
\]
where the inner product under the integral is the induced metric on the symmetric two-forms and \( T = T_{\alpha \beta} e^\alpha \otimes e^\beta \) is the energy-momentum tensor of \( \mathcal{A} \) given by
\[
T_{\alpha \beta} = T_{\beta \alpha} = 2 \langle F_{\alpha \eta}, F_{\beta \eta} \rangle - \frac{1}{2} |F|^2 g_{\alpha \beta} + \langle d^\nu \phi(e_\alpha), d^\nu \phi(e_\beta) \rangle - \frac{1}{2} |d^\nu \phi|^2 g_{\alpha \beta} + \frac{1}{4} \left( \gamma(e_\alpha) \nabla_{e_\beta}^{\otimes \phi^*} \psi + \gamma(e_\beta) \nabla_{e_\alpha}^{\otimes \phi^*} \psi, \psi \right) - \frac{1}{2} \left( \mathcal{D}_\psi, \psi \right) g_{\alpha \beta}.
\]
The trace of \( T \) is
\[
\text{Tr}(T) = (2 - \frac{m}{2}) |F|^2 + (1 - \frac{m}{2}) |d^\nu \phi|^2 + \frac{1 - m}{2} \left( \mathcal{D}_\psi, \psi \right).
\]
Note that when \( m = 4 \) the curvature would not appear in the trace, and when \( m = 2 \) the Dirichlet energy would not appear. Obviously, in general \( \text{Tr}_g T \) is not zero which reflects the failure of the conformal symmetry. The scaling behaviour of the spinor terms could, however, be fixed by applying an additional scaling, compare [28].

By construction, the functional \( \mathcal{A} \) is invariant under the transformations induced by diffeomorphisms. Hence Noether’s theorem implies that the energy-momentum tensor is divergence-free, if the system of equations (10) is satisfied. That is, for all \( \alpha \),
\[
\sum_\beta \nabla_\beta T_{\alpha \beta} = 0.
\]
Noether’s theorem holds if the solutions are at least twice differentiable. However, weak solutions with Sobolev regularity satisfying the divergence-free formula have additional properties.

5. Local geometry and gauge transformations

It is a fundamental result of gauge theory that the Yang–Mills action is gauge invariant, that is invariant under vertical automorphisms of the principal bundle. In this section we will show that also the Yang–Mills–Higgs–Dirac action is gauge invariant. Furthermore, we recall some analytical properties of gauge transformations.
The space of connections $\mathfrak A$ on $P(M,G)$ is an affine space modeled by $\Omega^1_{\text{hor},G}(P,G) \cong \Omega^1(\text{Ad}(P))$. More precisely, any other connection $\tilde{\omega}$ can be written as $\tilde{\omega} = \omega + \alpha$ for $\alpha \in \Omega^1_{\text{hor},G}(P,G)$.

A gauge transformation is a vertical automorphism $\varphi \in \text{Aut}_M(P)$ of the bundle $P$. Equivalently, it is an equivariant mapping from $P$ to $G$. The group of local gauge transformations will be denoted by $\mathcal D$. The gauge invariance of the curvature term is standard: For a gauge transformation $\varphi \in \mathcal D$,

$$\varphi^*\omega = \text{Ad}_{\varphi^{-1}}(\omega), \quad \varphi^*(\tilde{F}) = \text{Ad}_{\varphi^{-1}}(\tilde{F}).$$

Hence $|F|^2$ is gauge-invariant.

Now consider the associated fiber bundle $\mathcal N$. A gauge transformation acts on a section $\phi \in \Gamma(\mathcal N)$ by

$$\varphi(\phi)(x) = [\Psi_p(\varphi(\pi(p))), \tilde{\phi}(p)] = [\pi_p, \mu_{\varphi(\pi(p))}^{-1}(\tilde{\phi})(p)] \equiv [\pi_p, (\varphi^*\tilde{\phi})(p)].$$

Moreover, the connection $\varphi^*(\omega)$ induces a connection $\varphi^*\sigma$ on $T\mathcal N$, given in the following way:

$$\varphi^*(X_{\pi(p)}) = dt_p \left( \text{hor}_{\varphi^*\omega}(\tilde{X}_p) \right) \in T[p, y]_p N.$$

Thus the transformed vertical differential at $x = \pi(p)$ is

$$\text{d}^V\varphi(\phi)(X_x) = dt_p \left( \text{d}(\varphi^*\tilde{\phi})_p(\tilde{X}_p) + \text{d}(\varphi^*\tilde{\phi})(p)\varphi^*\omega(\tilde{X}_p) \right)$$

$$= dt_p \left( \text{d}\mu_{\varphi(x)^{-1}} \text{d}\tilde{\phi}_p(\tilde{X}_p) + \text{d}\mu_{\varphi(x)^{-1}} \text{d}\tilde{\phi}(p)\omega(\tilde{X}_p) \right).$$

Since $dt_p$ and $\text{d}\mu_{\varphi(x)^{-1}}$ both are isometries, we see that $|\text{d}^V\phi|^2(x) = |\text{d}^V\varphi(\phi)|^2(x)$ and hence the energy term of the section is gauge invariant.

The argument for the Dirac action part works similarly, as long as one notes that the Lie group $G$ has no action on the pure spinor bundle $S$ while it acts on $(N,h)$ via isometries. A local argument was suggested already in [17]. Thus, our Yang–Mills–Higgs–Dirac action is gauge invariant. Hence it is a well-defined extension of the classical gauge theory that includes spinorial fields.

Let’s write them in local representations. That is, we express it in a passive manner, via the change of the local sections or the local trivializations. Recall that, to carry out the local computations, we have to use a local section $s: U \to P$. It induces a local trivialization $\chi^P_U: \pi^{-1}(U) \to U \times G$ as well as $\chi^N_U: \rho^{-1}(U) \to U \times N$. Then a section $\phi: M \to \mathcal N$ can be locally expressed as $(\text{Id}_U, u)$ where $u: U \to N$ is a local representative, and the vertical differential locally reads

$$\text{d}^V\phi(x) = du(x) + d\sigma_{u(x)}(A) \in T_x^*M \otimes V_{\phi(x)} \cong T_x^*M \otimes T_{u(x)}N.$$

A local gauge transformation can be expressed by a map $\varphi: U \to G$. The action of $\varphi$ on $A$ is given by\footnote{It is this formula which guarantees that it is possible to make the local connection form $A$ vanish at a given point. We have used this in the previous sections.}

$$\varphi^*(A) = \varphi^{-1}d\varphi + \varphi^{-1}A\varphi.$$
Indeed, let $X \in \Gamma(U, TM)$, and temporarily replacing $d_A$ by $d^A$, then we have

\[
(\varphi^* d^A)_x u_x = \varphi^{-1} d^A_X (\sigma(\varphi(x), u(x))) = \varphi^{-1} (d\varphi(x)(du(X)) + ds_{\varphi u(x)}(d\varphi(X)) + d\sigma_{\varphi u(x)}(A(X))) = du(X) + d\sigma_u(x) (\varphi^{-1} d\varphi(X) + \varphi^{-1} A(X) d\sigma_{\varphi(x)}) = du(X) + d\sigma_u(x) ((\varphi^* A)(X)).
\]

Actually, all these computations are essentially carried out in the adjoint bundle $Ad(P)$, thus the form of the action on the local representative $A$ is the same as the classical one.

Next, some local analysis preliminaries. For convenience we take $U$ to be the unit disk $B_1(0)$ with Euclidean metric and assume the bundles are trivialized there. For $r > 0$, denote the dilation

\[ \theta_r: B_1(0) \to B_r(0), \quad x \mapsto r x. \]

With respect to the Euclidean metrics on both sides, we see that $\theta_r^* g_0 = r^2 g_0$.

**Lemma 5.1.** Consider the trivial bundle $P_r = B_r(0) \times G \to B_r(0)$ with connection form $A$ and let $u: B_r(0) \to N$ be a section of the (associated) fiber bundle. Let $A_r(x) \equiv r A(rx)$ be the connection form on $B_1(0)$ for the pullback bundle $\theta_r^*(P_r)$, while $u_r(x) := \theta_r^* u(x) = u(rx)$ and $\psi_r(x) = \frac{m-1}{r} \psi(rx) \in \Gamma(S \otimes u_r^* T N \to B_r(0))$. Then

\[
\int_{B_1(0)} |F(A_r)|^2 dx = r^{4-m} \int_{B_r(0)} |F(A)|^2 dx,
\]

\[
\int_{B_1(0)} |d_{A_r}(\theta_r^* u)|^2 dx = r^{2-m} \int_{B_r(0)} |d_A u|^2 dx,
\]

\[
\int_{B_1(0)} \langle D^u \psi_r, \psi_r \rangle dx = \int_{B_r(0)} \langle D \psi, \psi \rangle dx,
\]

\[
\int_{B_1(0)} |\psi_r|^2 dx = \frac{1}{r} \int_{B_r(0)} |\psi|^2 dx.
\]

The proof is standard and omitted. This tells us that, for $r \in (0, 1)$, the Dirac term stays rescaling invariant if an additional scaling is taken into account, and the $L^2$ norm of the spinor field behaves abnormally (for this reason in our analysis we usually turn the mass term off, namely setting $\lambda = 0$); meanwhile

- $m = 2$: the Yang–Mills term shrinks, and the Higgs term is scaling invariant;
- $m = 3$: the Yang–Mills term shrinks, and the Higgs term expands;
- $m = 4$: the Yang–Mills term is scaling invariant while the Higgs term expands;
- $m \geq 5$: both terms expand.

From this we see that the dimension two is already critical for the action, due to the presence of the Dirichlet type Higgs potential and the nonlinearity of the fibers. This is in great contrast to the Yang–Mills–Higgs theory where the associated bundles are linear. In the second part of the article we will focus on the lowest critical case, that is, dimension two.

The Euler–Lagrange equation for the connection fails to be elliptic in general. Thanks to a result by K. Uhlenbeck, we can choose a nice Coulomb gauge to make it elliptic.

Before stating this result, we need some preliminaries. Let $A^{k,p}$ be the space of $W^{k,p}$ connections, and $G^{k+1,p}$ the space of $W^{k+1,p}$ gauges. Then we know that $G^{k+1,p}$ acts on $A^{k,p}$.
Proposition 5.2 ([50] Lemma 1.2). Let \((k + 1)p > m = \dim M\). Then
1. The gauge group \(\mathcal{G}^{k+1,p}\) is a smooth Lie group.
2. The induced map
\[
\mathcal{G}^{k+1,p} \times \mathfrak{A}^{k,p} \to \mathfrak{A}^{k,p}, \quad (\varphi, \omega) \mapsto \varphi^* \omega
\]
is smooth.
3. If \(\omega\) and \(\varphi^* \omega\) both are in \(\mathfrak{A}^{k,p}\), then the gauge transformation \(\varphi\) has regularity \(W^{k+1,p}\), i.e. \(\varphi \in \mathcal{G}^{k+1,p}\).

Theorem 5.3 ([50] Theorem 2.1, [51] Theorem 6.1]). Let \(p \in (\frac{m}{2}, m]\) and \(G\) be compact. Consider a connection \(\omega\) on the bundle \(B_1(0) \times G\) with local representative \(\tilde{A}\). Then there exist \(\kappa = \kappa(m) > 0\) and \(c = c(m) > 0\) such that if \(\|F(\tilde{A})\|_{L^m(B_1)} \leq \kappa\), then \(\tilde{A}\) is gauge equivalent to a local connection form \(A\) such that
1. \(d^* A = 0\);
2. \((x \cdot A) = 0\) on \(\partial B_1(0)\);
3. \(\|A\|_{W^{1,m/2}} \leq c(m)\|F(\tilde{A})\|_{L^{m/2}}\);
4. \(\|A\|_{W^{1,p}} \leq c(m)\|F(\tilde{A})\|_{L^p}\).

The gauge transformation in the above theorem is usually referred to as a Coulomb gauge. We remark that in [50] the theorem was stated with \(p \in (\frac{m}{2}, m]\), while it actually works for \(p \geq \frac{m}{2}\), see [51] Chapter 6.

Part 2. Analysis of the Model in Dimension two

In this part we obtain regularity of weak solutions of the Euler–Lagrange equations of the Yang–Mills–Higgs–Dirac model and analyze the compactness of the solution space for the case of two-dimensional domain. The reason for the restriction of domain dimension is that in this case although the model is not conformally invariant, it stays bounded during rescaling by a small \(0 < r < 1\).

6. Regularity of weak solutions

In the case of \(m = \dim M = 2\), the Yang–Mills–Higgs–Dirac action is naturally defined on the space
\[
\text{Dom}(\mathcal{A}) := \left\{ (\omega, \phi, \psi) \mid \omega \in \mathfrak{A}^{1,2}, \phi \in W^{1,2}(\Gamma(N)), \psi \in W^{1,\frac{2m}{m+1}}(\Gamma(S \otimes \phi^* V)) \right\}.
\]

Definition 6.1. A triple \((\omega, \phi, \psi) \in \text{Dom}(\mathcal{A})\) is called a weak solution of the system (10) if it satisfies the system (10) in the sense of distributions. More precisely, for any smooth triple \((\zeta, V, \eta)\) with \(\zeta \in \Gamma(\text{Ad}(P))\), \(V \in \Gamma(\phi^* V)\), and \(\eta \in \Gamma(S \otimes \phi^* V)\), it holds that
\[
\int_M \langle D\zeta, F(\omega) \rangle + 2 \langle \nabla^\phi V, d^\phi \phi(e_\alpha) \rangle + \langle d\bar{\mu}(\zeta), d^\phi \phi \rangle \, d\text{vol}_g
\]
\[
+ \int_M 2 \langle \psi, d^\phi \eta \rangle + \langle \psi, \gamma(e_\alpha) R^\phi V(V, d^\phi \phi(e_\alpha)) \psi \rangle \, d\text{vol}_g
\]
\[
+ \int_M \langle \psi \tilde{\phi} \rangle \gamma(e_\alpha) \psi \rangle \, \partial_1 \partial_2 \bar{\mu} \left( \zeta(e_\alpha), \partial_{\psi^i} \right) \, d\text{vol}_g = 0.
\]
The aim of this section is to prove the following

**Theorem 6.2.** Let \((M, g)\) be a closed Riemann surface. Let \((\omega, \phi, \psi)\) be a weak solution as above. Then there is a gauge transformation \(\varphi \in \mathcal{D}^{2,2}\) such that \((\varphi^* \omega, \varphi(\phi), \varphi(\psi))\) is a smooth triple.

The strategy is similar to that for harmonic maps, but in addition, we need to glue the local gauges together to get a good Coulomb global gauge. Note that \(m = 2\) is a subcritical dimension for the Yang–Mills part, thus we can easily improve the regularity for the connection, at least locally.

**Proof of Theorem 6.2.**

**Step 1.** We first deal with the local regularity.

Let us take a local geodesic ball, say \(B_1(0)\) since by rescaling we could always assume it is a unit ball, on which the fiber bundle is trivialized: \(\mathcal{N}|_{B_1(0)} \cong B_1(0) \times N\). We further embed \(N\) isometrically into some Euclidean space \(\mathbb{R}^K\), with second fundamental form \(\mathbb{I}\). Let \(A\) be the local representative of \(\omega\), and \(u\) the local representative of \(\phi\). In terms of such local data, \(d^V \phi\) is represented by

\[
d_A \phi = du + d\mu_u(A) \in \Gamma(T^*M \otimes u^*TN \to B_1(0)).
\]

The spinor along the section \(\phi\) is now locally a spinor along the map \(u : B_1(0) \to N\), and with respect to a local (normal) coordinate system \((y^i)\) the spinorial field takes the form

\[
\psi = \psi^i \otimes u^*(\partial_{y^i}) \in \Gamma(S \otimes u^*TN).
\]

A basis for \(\mathfrak{g}\) is denoted by \((\epsilon_a), 1 \leq a \leq \dim G\), with dual basis \(\epsilon^a\). Then \((A, u, \psi)\) satisfies the equations in (10) weakly on \(B_1(0)\):

\[
\begin{align*}
d^* dA &= -\frac{1}{2} d^*[A, A] + A_\perp dA + \frac{1}{2} A_\perp [A, A] - (d\mu_u)^i (du + d\mu_u(A)) \\
&\quad - \langle \psi^j, \gamma(\epsilon_a) \psi^i \rangle (\partial_i \partial_j \mu(e_a, \partial_{y^i}), \partial_{y^j}) \epsilon^a \otimes e_a, \\
\Delta u &= \text{Tr} \mathbb{I}(u)(du, du) - 2 \text{Tr} \partial_i \partial_j \mu(A, du) - d\mu_u(\text{div} A) - \text{Tr} \partial_i \partial_j \mu_u(A, d\mu_u) \\
&\quad + \frac{1}{2} \langle \psi^j, \gamma(\epsilon_a) \psi^i \rangle (\partial_i, \gamma(\partial_{y^k}, du(e_a) + d\mu_u(A(e_a)))) \partial_{y^k} \rangle h^{kl}(u) u^*(\partial_{y^l}), \\
\partial \psi^i &= \left\{ -\Gamma_{\alpha k}^\gamma(u)(d\mu_u A(\partial_{e^\gamma}))^i + \Gamma_{\alpha k}^i(u) + \Gamma_{\beta j}^i(u) w^{j, \alpha} \right\} \gamma(\epsilon_a) \psi^k
\end{align*}
\]

Thanks to Theorem 5.3, by applying a Coulomb gauge if necessary, we may assume from the beginning that the local trivialization is chosen such that \(d^* A = 0\). Therefore, the left hand side of the equation for \(A\) can be rewritten as

\[
d^* dA + d d^* A = -\Delta A
\]

and the system now is elliptic of mixed order on coupled fields. One key observation is that after such a local gauge transformation, the equation for \(A\) becomes *elliptic* and *subcritical*, which allows us to improve the regularity of weak solutions. We sketch it here for completeness.

From the equation for \(A\) and the regularity assumptions on the weak solutions, by Sobolev embedding we see that

\[
\Delta A \in L^p(B_1)
\]
for any $1 \leq p < 2$. This implies that $A \in W_{loc}^{2p}(B_1(0))$ for any $p \in [1,2)$. In particular, $A \in W_{loc}^{1,q}(B_1(0))$ for any $q \in [1, +\infty)$.

Then we turn to the spinor field $\psi$. Applying [25, Lemma 6.1] to this equation we get that $\psi \in L_{loc}^p(B_1(0))$ for any $p \in [1, +\infty)$. Then the regularity theory for the Dirac operator $\mathcal{D}$ implies $\psi \in W_{loc}^{1,2}(B_1(0))$.

Finally we turn to the equation for $u$. It is well-known that the equation can be rewritten in the form

$$-\Delta u = \Omega \cdot \nabla u + f$$

where $f = f(A, \psi, u) \in L_{loc}^p(B_1(0))$ for any $p \in [1,2)$. Thanks to the regularity theory developed in [39, 41, 40, 44], we conclude that $u \in W_{loc}^{2p}(B_1(0))$ for any $p \in [1,2)$.

Now the situation is subcritical for all the fields, and a bootstrap argument then implies that they are actually in $C^\infty(B_{1/2}(0))$.

Step 2. Gluing the local Coulomb gauge to obtain global smoothness.

Now we suppose that there is a finite open cover $\{U_\alpha\}_{1 \leq \alpha \leq l}$ such that each $U_\alpha$ is a geodesic ball, and on each $U_\alpha$ there exists a Coulomb gauge $\varphi_\alpha$ such that the triple $(\varphi_\alpha^* \omega, \phi, \psi)$ is smooth on $U_\alpha$.

Now, on $U_\alpha \cap U_\beta$, the two connection $\varphi_\alpha^* \omega$ and $\varphi_\beta^* \omega$ are both smooth. Therefore by Proposition 5.2 the gauge $\varphi_\alpha^{-1} \circ \varphi_\beta$ is smooth. Moreover, by precomposing with a smooth gauge if necessary, we may assume that both $\varphi_\alpha$ and $\varphi_\beta$ are close to $e \in G$, hence we could glue them together to obtain a gauge $\varphi_{\alpha \beta}$ on $U_\alpha \cup U_\beta$ such that $(\varphi_{\alpha \beta}^* \omega, \phi, \psi)$ is smooth throughout $U_\alpha \cup U_\beta$. The detailed constructions can be found, for example, in [50] or [46]. Since there are only finitely many open sets in the cover, we obtain a global gauge $\varphi \in \mathcal{G}^{2,2}$ such that $(\varphi^* \omega, \varphi(\phi), \varphi(\psi))$ is smooth.

We remark that, since the term “Coulomb gauge” usually means that the transformed local representatives are locally co-closed, and we do not claim that the gauge in the above theorem is Coulomb.

7. SMALL ENERGY REGULARITY

This is a preparation for the blow-up analysis in the following section. Recall that the small energy regularity contains the key estimates for establishing the energy identities for harmonic maps, see e.g. [13].

As the Dirac action part may be negative, which makes the action functional non-coercive, we have to use here another energy of the spinorial field to control the spinorial term in our functional. More precisely, we introduce the following energies for the three fields in our model: for an open subset $U \subset M$,

$$A_{YM}(\omega; U) = \int_U |F(\omega)|^2 \, d\text{vol}_g,$$

$$E(\phi; U) = \int_U |d^\wedge \phi|^2 \, d\text{vol}_g,$$

$$E(\psi; U) = \int_U |\psi|^4 \, d\text{vol}_g.$$
When $U = M$, we will omit the domain if there is no confusion. The basic principle is that, if these “energies” are small enough on $U$, then the fields are as regular as one expects, with uniform estimates of their higher derivatives by these energies. Due to the conformal invariance/covariance in dimension two, it is reasonable to have the smallness assumptions on small domains. Thus we can restrict the model to a small disk where the bundles are trivialized. For simplicity of notation we may assume that the local metric is Euclidean.

Let $B$ be a Euclidean disk and consider the trivialized bundle $P = B \times G$ with connection $\omega$. The associated bundle is $N = B \times N$, and the section is locally given by a map $u : B \to N$. The induced covariant derivative is as before given by

$$d_A u = du + d\sigma_u(A).$$

As a local map, the Dirichlet energy of $u$ is

$$E(u; B) = \int_B |du|^2 \, dx.$$

By (11), and up to a gauge if necessary, we have

$$\| du \|_{L^2(B)} - \| d_A u \|_{L^2(B)} \leq C \| A \|_{L^2(B)} \leq CA_Y M(A; B).$$

Thus, locally, we may not distinguish the classical Dirichlet energy of $u$ with its vertical energy as a local section.

To be slightly more general, and also for later convenience, let us consider the approximating (local) system:

$$\Delta u = \text{Tr} \Pi(u)(du, du) - 2 \text{Tr} \partial_1 \partial_2 \mu(A, du) - 2 \text{Tr} \partial_1 \partial_2 \mu_u(A, du) - d\mu_u(\text{div} A) - \text{div} A,$$

and

$$\Delta u = \text{Tr} \Pi(u)(du, du) - d\mu(A),$$

with $\chi_1, \chi_2, \chi_3$ being vector valued error terms such that

$$\| \chi_1^2 \|_{L^2(B)} + \| \chi_2^2 \|_{L^2(B)} + \| \chi_3^2 \|_{L^2(B)} \leq C < +\infty.$$

**Proposition 7.1.** Let $(A, u, \psi)$ be a $C^2$ triple on $B$ satisfying the system (12). There exists $\varepsilon_0 > 0$ such that if

$$A_Y M(A; B) \leq \varepsilon_0,$$

then for any open disk $B' \in B$, there exists $C = C(B, B') > 0$ such that

$$\| A \|_{W^{2,2}(B')} \leq C \left( A_Y M(A; B) + E(u; B) + E(\psi; B) \right) + C\| \chi_1 \|_{L^2(B)}^2.$$
Let \( \eta \in C_0^\infty(B) \) be a local cutoff function with \( \eta \equiv 1 \) on \( U_2 \). Note that
\[
d^* (\eta[A,A]) = \eta d^*[A,A] - d\eta A[A,A].
\]

The localized equation for \( A \) then reads
\[
\Delta(\eta A) = (\Delta \eta)A + 2\nabla \eta \cdot \nabla A + \eta(\Delta A)
\]
\[
= (\Delta \eta)A + 2\nabla \eta \cdot \nabla A + \eta \left( \frac{1}{2} d^*[A,A] - A\omega\eta \right) + (d\mu_a^t)(du + d\mu_a(A))
\]
\[
+ \eta \left( \langle \psi^2, \gamma(e_a) \psi^t \rangle \langle \partial_1 \partial_2 \mu(e_a, \partial_y) \rangle e^a \otimes e_a \right) - \eta \chi_1
\]
\[
= (\Delta \eta)A + 2\nabla \eta \cdot \nabla A + \frac{1}{2} d^*[\eta A,A] - \frac{1}{2} d\eta A[A,A] - \eta A\omega[\eta A,A]
\]
\[
+ (d\mu_a^t)(d(\eta u) - (d\mu_a)^t(u \, d\eta) + (d\mu_a)^t(d\mu_a)(\eta A))
\]
\[
+ \langle \eta \psi^2, \gamma(e_a) \psi \rangle \langle \partial_1 \partial_2 \mu(e_a, \partial_y) \rangle e^a \otimes e_a - \eta \chi_1.
\]

Since \( \text{supp}(\eta) \subseteq B \), hence by Sobolev embedding, for any \( p < \infty \),
\[
|A|_{\text{supp} \eta} ||_{L^p} \leq C |A| ||_{W^{1,2}(B)}.
\]

Then we can estimate,
\[
||\Delta(\eta A)||_{L^2(B)} \leq C(\eta) ||A||_{L^2(B)}^2 + C(\eta) ||dA||_{L^2(B)}
\]
\[
+ C||\nabla(\eta A)||_{L^4(B)} ||A||_{L^4(\text{supp} \eta)} + C||\eta A||_{L^\infty(B)} ||\nabla A||_{L^2(\text{supp} \eta)}
\]
\[
+ C(\eta) ||A||_{L^4(b(B))}^2 + ||A||_{L^4(\text{supp} \eta)} ||A||_{L^4(\text{supp} \eta)}^2
\]
\[
+ C(\mu, N) \left( ||d(\eta u)||_{L^2(B)} + C(\eta) ||u||_{L^2(B)} + ||A||_{L^2(B)} \right)
\]
\[
+ C(\mu) ||\psi||_{L^4(\text{supp} \eta)}^2 + ||\chi_1||_{L^2(B)}
\]

By Sobolev embedding and the smallness assumptions on the energies, we can get
\[
||A||_{W^{2,2}(U_2)} \leq C ||\eta A||_{W^{2,2}(B)} \leq C \left( ||dA||_{L^2(B)} + ||\nabla u||_{L^2(B)} + ||\psi||_{L^4(B)}^2 + ||\chi_1||_{L^2(B)} \right),
\]
where \( C = C(\mu, N, \eta) > 0 \). Since \( N \) and \( \mu \) are fixed, hence universal, and since the dependence on \( \eta \) is actually a dependence on the relative position of \( B = U_1 \) and \( U_2 \supset B' \), we have \( C = C(B, B') \).

In the same way one can obtain the small energy regularity for the other two fields. The derivation is standard but more tedious. We omit the details; one could refer to e.g. \[27\].

**Proposition 7.2.** Let \( (A, u, \psi) \) be a \( C^2 \) triple on \( B \) satisfying the system (12). There exists \( \varepsilon_0 > 0 \) such that if
\[
\max\{A_{YM}(A;B), E(u;B), E(\psi;B)\} \leq \varepsilon_0,
\]
then for any open disk \( B' \subseteq B \), there exists \( C = C(B, B') > 0 \) such that
\[
||u - \bar{u}||_{W^{2,2}(B')} + ||\psi||_{W^{1,4}(B')}^2 + ||A||_{W^{2,2}(B')} \leq C \left( A_{YM}(A;B) + E(u;B) + E(\psi;B) \right)
\]
\[
+ C \left( ||\chi_1||_{L^2(B)}^2 + ||\chi_2||_{L^2(B)}^2 + ||\chi_3||_{L^4(B)}^4 \right).
\]

Here \( \bar{u} \) is the mean value of \( u \) over \( B \).

By the Sobolev embedding \( W^{2,2}(\mathbb{R}^2) \subset C^\beta(\mathbb{R}^2) \), we get the following control on the oscillation of the section \( u \).
Corollary 7.3. Under the assumption of Proposition 7.2, we have

$$\text{Osc}_B u \leq C \left( A_{YM}(A; B) + E(u; B) + E(\psi; B) \right) + C \left( \|\chi_1\|^2_{L^2(B)} + \|\chi_2\|^2_{L^2(B)} + \|\chi_3\|^2_{L^2(B)} \right).$$

If we can control the higher order derivatives of the error terms, then we can also control the higher order derivatives of the three fields under consideration. In that case in the interior of the disk $B$ the solutions with small energies are smoothly bounded.

8. A Pohozaev type identity

As remarked, this model does not possess conformal symmetry. Consequently, we fail to come up with a holomorphic current from the possible solutions. But the equation for the section still allow us to make up a Pohozaev type identity with controllable error terms. Though it is complicated and no explicit geometric meaning is known, it is still useful in the analysis, especially when one considers the compactness properties and performs the local blow-ups.

Proposition 8.1. Let $(A, u, \psi)$ be a solution of (12) on a small Euclidean disk $B_\delta(0) \subset \mathbb{R}^2$. Then, for any $r \in (0, \delta)$ the following identity holds

$$r \int_{\partial B_r(0)} \left| \frac{\partial u}{\partial r} \right|^2 - \left| \frac{1}{r} \frac{\partial u}{\partial \theta} \right|^2 \, ds$$

$$= r \int_{\partial B_r(0)} \langle \psi, \gamma(r^{-1} \partial_\theta) \nabla^0_r \psi \rangle \, ds - \int_{B_r(0)} \langle \psi, \mathcal{D}^0 \psi \rangle \, dx - 2 \int_{B_r(0)} \langle r \nabla^0_\partial \psi, \mathcal{D}^0 \psi \rangle \, dx$$

$$+ \int_{B_r(0)} \langle \psi, \gamma(e_\alpha) R^\alpha \mathcal{D}^0 (\vec{x}, d\mu_u(A(e_\alpha))) \psi \rangle \, dx + 2 \int_{B_r(0)} \langle \Upsilon + \chi_2, r \frac{\partial u}{\partial r} \rangle \, dx,$$

where $\mathcal{D}^0$ is the Dirac operator w.r.t. $\nabla^S \mathcal{D}^0 \equiv \nabla^0$ in the local trivialization and

$$\Upsilon \equiv -2 \text{Tr} \partial_i \partial_j \mu(A, du) - d\mu_u(\text{div} A) - \text{Tr} \partial_i \partial_j \mu_u(A, d\mu(A)).$$

We remark that locally we can use the connection $\nabla^S \mathcal{D}^0 \mathcal{D}^0 \equiv \nabla^0$ in the computation. It is more convenient since later we will view $A$ as a perturbation.

Proof. This follows from testing the equation

$$\Delta u = \text{Tr} \mathbb{I}(u)(du, du) + \Upsilon + \chi_2$$

$$+ \frac{1}{2} \langle \psi^i, \gamma(e_\alpha) \psi^j \rangle \langle \partial_{g^i}, R (\partial_{g^j}, du(e_\alpha) + d\mu_u(A(e_\alpha))) \partial_{g^i} \rangle h^{kl}(u) u^*(\partial_{g^l})$$

against the vector-valued function $\vec{x} \cdot \nabla u = x^\beta \partial_\beta u = r \frac{\partial u}{\partial r}$, where $\vec{x} = x^\beta e_\beta \in \mathbb{R}^2$. It is well-known that the left-hand-side gives

$$\int_{B_r(0)} (\vec{x} \cdot \nabla u) \Delta u \, dx = \frac{1}{2} r \int_{\partial B_r(0)} \left| \frac{\partial u}{\partial r} \right|^2 - \frac{1}{r} \frac{\partial u}{\partial \theta} \right|^2 \, ds.$$
On the right-hand-side, since the second fundamental form part is in the normal direction and hence perpendicular to \(\vec{x} \cdot \nabla u\), it suffices to deal with the spinorial part. Actually

\[
\int_{B_r(0)} \langle \psi^i, \gamma(e_\alpha) \psi^j \rangle \langle \partial_{y^i}, R (\partial_{y^k}, du(e_\alpha)) \partial_{y^j} \rangle (\vec{x} \cdot \nabla u^k) \, dx
\]

\[
= \int_{B_r(0)} \langle \psi, \gamma(e_\alpha) R^{\alpha \beta} T_N (\vec{x}, e_\alpha) \psi \rangle \, dx = \int_{B_r(0)} \langle \psi, \gamma(e_\alpha) R^{S \otimes u} T_N (\vec{x}, e_\alpha) \psi \rangle \, dx
\]

\[
= \int_{B_r(0)} \langle \psi, \nabla^0_\alpha \psi - \nabla^0_\alpha \nabla^0_\beta \psi - \nabla^0_\alpha [\vec{x}, e_\alpha] \psi \rangle \, dx
\]

\[
= \int_{B_r(0)} \nabla^0_\alpha \left( x^2 \langle \psi, \nabla^0_\alpha \psi \rangle \right) - \langle \nabla^0_\alpha (x^2 \psi), \nabla^0_\alpha \psi \rangle + \langle \psi, \nabla^0_\alpha \psi \rangle \, dx
\]

\[
= \int_{\partial B_r(0)} \langle \psi, \nabla^0_\alpha \psi \rangle \vec{x} \cdot \nu + \langle \gamma(e_\alpha) \psi, \nabla^0_\alpha \psi \rangle e_\alpha \cdot \nu \, ds
\]

\[
+ \int_{B_r(0)} - \langle \psi, \nabla^0_\alpha \psi \rangle - \langle x^2 \nabla^0_\alpha \psi, \nabla^0_\alpha \psi \rangle - \langle \nabla^0_\alpha \psi, \nabla^0_\alpha \psi \rangle \, dx
\]

\[
r \int_{\partial B_r(x_0)} \langle \psi, \nabla^0_\alpha \psi \rangle + \langle \gamma(\nu) \psi, \nabla^0_\alpha \psi \rangle \, ds
\]

\[
+ \int_{B_r(0)} - \langle \psi, \nabla^0_\alpha \psi \rangle - 2r \langle \nabla^0_\alpha \psi, \nabla^0_\alpha \psi \rangle \, dx
\]

\[
r \int_{\partial B_r(x_0)} \langle \psi, \gamma(r^{-1} \partial_\nu) \nabla^0_{r^{-1} \partial_\nu} \psi \rangle \, ds
\]

\[
+ \int_{B_r(0)} - \langle \psi, \nabla^0_\alpha \psi \rangle - 2r \langle \nabla^0_\alpha \psi, \nabla^0_\alpha \psi \rangle \, dx
\]

where in the last step we have used \(\nu = \partial_r\) on \(\partial B_r(0)\) and

\[
\langle \psi, \nabla^0_\alpha \psi \rangle + \langle \gamma(\nu) \psi, \nabla^0_\alpha \psi \rangle = \langle \psi, \gamma(\partial_r) \nabla_\alpha \psi + \gamma(r^{-1} \partial_\nu) \nabla^0_{r^{-1} \partial_\nu} \psi \rangle - \langle \psi, \gamma(\partial_r) \nabla^0_\alpha \psi \rangle
\]

\[
= \langle \psi, \gamma(r^{-1} \partial_\nu) \nabla^0_{r^{-1} \partial_\nu} \psi \rangle.
\]

The desired formula follows. \(\square\)

9. Blow-up Analysis

In this section we establish a compactness result for the solution space. More precisely we consider a sequence of approximating solutions \((\omega_k, \phi_k, \psi_k)\) of the Euler–Lagrange system and show that they converges up to bubbles. The motivation for this section is to check Palais-Smale (PS) properties for the action functional. Unfortunately our functional does not satisfy a (PS) condition, for at least two reasons. One is the non-definiteness of the Dirac operator and the other is the possibility of bubbles. With the energy of the spinor fields defined as before, we can avoid the first reason and get the bubble convergence, as in the following.
Let \((\omega_k, \phi_k, \psi_k)\) be a sequence in the space \(\mathfrak{A}^{1,2} \times W^{1,2}(\Gamma(N)) \times W^{1,4}(\Gamma(S \otimes \phi^* V))\) which satisfies the Euler–Lagrange system (10) with small errors:

\[
\begin{align*}
D^*_\omega_k F(\omega_k) + d\mu^*_\phi_k (d^V \phi_k) + \mathcal{Q}(\phi_k, \psi_k) &= a_k, \\
\tau^V(\phi_k) - \frac{1}{2} \mathcal{R}^V(\phi_k, \psi_k) &= b_k, \\
\mathcal{D}\psi_k &= c_k,
\end{align*}
\]

where \(a_k \in L^2(\Gamma(\text{Ad}(P))), b_k \in L^2(\Gamma(\phi_n^* V)), \) and \(c_k \in L^4(\Gamma(S \otimes \phi_n^* V))\), which converges to zero with respect to the corresponding norms:

\[
\max(||a_k||_{L^2}, ||b_k||_{L^2}, ||c_k||_{L^4}) \to 0 \quad \text{as} \quad k \to \infty.
\]

**Theorem 9.1.** Let \((\omega_k, \phi_k, \psi_k)\) be a sequence of approximating solutions, i.e. solutions of (14) and (15). Assume that they have uniformly bounded energies. Then up to a subsequence they converge weakly to a smooth solution \((\omega_\infty, \phi_\infty, \psi_\infty)\) of (10).

Furthermore, there is a finite set \(S_1 = \{x_1, \ldots, x_l\} \subset M\) such that the convergence is strong on any compact subset of \(M \setminus S_1\). Moreover, corresponding to each \(x_i \in S_1\) there exists a finite collection of Dirac-harmonic spheres \((\sigma^i_1, \xi^i_1)\) from \(S^2\) into \(N\) for \(1 \leq l \leq L_i < \infty\), such that the energy identities and the no-neck property hold,

\[
\lim_{k \to \infty} A_{YM}(\omega_k) = A_{YM}(\omega_\infty),
\]

\[
\lim_{k \to \infty} E(\phi_k) = E(\phi_\infty) + \sum_{i=1}^I \sum_{l=1}^{L_i} E(\sigma^i_l),
\]

\[
\lim_{k \to \infty} E(\psi_k) = E(\psi_\infty) + \sum_{i=1}^I \sum_{l=1}^{L_i} E(\xi^i_l),
\]

and the set \(\phi_\infty(M) \cup (\bigcup_{l} \sigma^i_l(S^2))\) is connected.

Before the proof let us make some comments on this result. On one side, as we are in a subcritical situation for the connections, in the limit of the blow-up procedure the connection will not appear, and we are back into a conformally invariant setting and the limits are solutions on spheres. On the other hand, to obtain the desired statement, we need to control the connections, which is not trivial since there is no good holomorphic current available, as conformal symmetry does not hold.

9.1. **Proof of Theorem 9.1.** Given the sequence as above, let us define the following concentration sets,

\[
S_1 := \bigcap_{r > 0} \left\{ x \in M \mid \liminf_{k \to \infty} \int_{B_r(x)} |d^V \phi_k|^2 \, dv_{g} \geq \varepsilon_0 \right\},
\]

\[
S_2 := \bigcap_{r > 0} \left\{ x \in M \mid \liminf_{k \to \infty} \int_{B_r(x)} |\psi_k|^4 \, dv_{g} \geq \varepsilon_0 \right\},
\]

\[
S_3 := \bigcap_{r > 0} \left\{ x \in M \mid \liminf_{k \to \infty} \int_{B_r(x)} |F(\omega_k)|^2 \, dv_{g} \geq \varepsilon_0 \right\}.
\]
As we assumed that the sequence has uniformly bounded energies, each of the above concentration sets consists of at most finitely many points (possibly none).

**Lemma 9.2.** \( \mathcal{S}_3 = \emptyset \).

**Proof of Lemma 9.2.** Suppose the contrary and let \( x \in \mathcal{S}_3 \). Passing to a subsequence we may assume that

\[
\lim_{r \searrow 0} \lim_{k \to \infty} \int_{B_r(x)} |F(\omega_k)|^2 \, d\text{vol}_g = \alpha(x) \geq \varepsilon_0.
\]

Choose \( 0 < r << 1 \) so small that \( 2r^2 \alpha(x) < \varepsilon_0 \) and

\[
\int_{B_r(x)} |F(\omega_k)|^2 \, d\text{vol}_g \leq 2\alpha(x).
\]

Then, by rescaling via the map \( \theta_r : B_1(0) \to B_r(x) \) as in Lemma 5.1 we see that on \( B_1(0) \) the rescaled connections \( (\omega_k)_r \) satisfy

\[
\int_{B_r(x)} |F((\omega_k)_r)|^2 \, dx < \varepsilon_0.
\]

Then the estimate (13) implies that, up to subsequences, \( (\omega_k)_r \) converges strongly on \( B_1(0) \) in \( W^{1,2} \), say to \( \omega_\infty \in W^{2,2}(B_1(0)) \). Scaling it back, we see that \( \omega_k \) converges strongly in \( W^{1,2} \) to \( (\omega_\infty)_1 \) on \( B_r(x) \), hence

\[
\lim_{r \searrow 0} \lim_{k \to \infty} \int_{B_r(x)} |F(\omega_k)|^2 \, d\text{vol}_g = \lim_{r \searrow 0} \int_{B_r(0)} |F((\omega_\infty)_1)|^2 \, d\text{vol}_g = 0,
\]

which contradicts the concentration inequality (10).

From Lemma 9.2 we see that the concentration set \( \mathcal{S}_1 \) for the sections can be equivalently characterized by

\[
\mathcal{S}_1 = \left\{ x \in M \mid \liminf_{k \to \infty} \int_{B_r(x)} |du_k|^2 \, d\text{vol} \geq \varepsilon_0 \right\},
\]

since \( u \) has bounded values and the \( A_k \) part does not concentrate.

**Lemma 9.3.** \( \mathcal{S}_2 \subset \mathcal{S}_1 \).

**Proof of Lemma 9.3.** Consider the equation for spinors

\[
\mathcal{D} \psi_k^i = \left\{ \Gamma_{\alpha l}^i(u_k)(d\mu_{\alpha k} A_k(\partial_x))^i - \Gamma_{\alpha l}^i(u_k) - \Gamma_{j l}^i(u_k)(u_k)_{\alpha}^i \right\} \gamma(e_{\alpha}) \psi_k^i + \chi_{3k},
\]

where \( c_k \) is locally represented by \( \chi_{3k} \). Taking a cutoff function \( \eta \) as before, we can localize the above equation as

\[
\mathcal{D}(\eta \psi_k^i) = \left\{ \Gamma_{\alpha l}^i(u_k)(d\mu_{\alpha k} (\eta A_k(\partial_x))_{\alpha}^i) - \eta \Gamma_{\alpha l}^i(u_k) - \Gamma_{j l}^i(u_k)(\eta u_k)_{\alpha}^i + \Gamma_{j l}^i(u_k) u_k^l \nabla_{\alpha} \eta \right\} \gamma(e_{\alpha}) \psi_k^i + \chi_{3k}.
\]

Then for any \( \frac{1}{3} < q < 2, \)

\[
\| \mathcal{D}(\eta \psi_k) \|_{L^q(B_r(x))} \leq C \left( \| A_k \|_{L^2(B_r(x))} + \| du \|_{L^2(B_r(x))} \right) \| \eta \psi \|_{L^{\frac{2q}{2q-2}}(B_r(x))} + \| \eta \psi_k \|_{L^q(B_r(x))} + \| \chi_{3k} \|_{L^q(B_r(x))}.
\]
If there was a point \( x \in S_1 \setminus S_2 \), then by taking \( r \) small, we may assume that
\[
2C \left( \|A_k\|_{L^2(B_r(x))} + \|du\|_{L^2(B_r(x))} \right) \|\eta \psi\|_{L^{\frac{2r}{r-2}}(B_r(x))} < C_q^{-1}
\]
with \( C_q \) being the Sobolev constant such that
\[
\|\eta \psi_k\|_{W^{1,q}(B_r(x))} \leq C_q \|\hat{\phi}(\eta \psi_k)\|_{L^q(B_r(x))}.
\]
Then, shrinking \( r \) a little, we could control the \( W^{1,q} \) norm of \( \psi_k \) uniformly
\[
\|\psi_k\|_{W^{1,q}(B_r(x))} \leq C \left( \|\psi_k\|_{L^q(B_r(x))} + \|\chi_k\|_{L^q(B_r(x))} \right).
\]
Since the Sobolev embedding \( W^{1,q} \hookrightarrow L^4 \) is compact in dimension two, it follows that, up to subsequences, \((\psi_k)\) converges strongly in \( L^4(B_r(x)) \) for \( r \) small. This contradicts the concentration phenomenon. \( \square \)

**Corollary 9.4.** On \( M \setminus S_1 \), up to subsequences, the sequence \((\omega_k, \phi_k, \psi_k)\) converges strongly.

**Proof of Corollary 9.4** This follows from the small energy regularity, Theorem 7.2 \( \square \)

Now the uniform bound on energies implies the weak convergence (up to a subsequence) and the small energy regularity implies the strong convergence away from \( S_1 \). It remains to analyze the convergence near the finite set \( S_1 \equiv \{x_1, x_2, \ldots, x_I\} \). Note that, the weak limit \((\omega_\infty, \phi_\infty, \psi_\infty)\), being itself a weak solution, is smooth by Theorem 6.2

Choose \( \delta_i > 0 \) small, \( 1 \leq i \leq I \), such that the balls \( B_{\delta_i}(x_i) \) are disjoint. By passing to a subsequence we may assume that
\[
\lim_{\delta_i \searrow 0} \lim_{k \to \infty} \int_{B_{\delta_i}(x_i)} |d^\nabla \phi_k|^2 \, d\text{vol}_g = \alpha(x_i) \geq \varepsilon_0.
\]

**Lemma 9.5.** For each \( i \in \{1, \ldots, I\} \), there exist a number \( L_i \in \mathbb{N} \) and maps \( v_i^l : S^2 \to N \), spinors \( \xi_i^l \in \Gamma(S(S^2) \otimes (v_i^l)^*TN \to S^2) \), for \( i \leq l \leq L_i \), such that \((v_i^l, \xi_i^l)\) are Dirac-harmonic spheres, and the following energy identities hold
\[
\lim_{\delta_i \searrow 0} \lim_{k \to \infty} E(\phi_k; B_{\delta_i}(x_i)) = \sum_{l=1}^{L_i} E(v_i^l, S^2),
\]
\[
\lim_{\delta_i \searrow 0} \lim_{k \to \infty} E(\psi_k; B_{\delta_i}(x_i)) = \sum_{l=1}^{L_i} E(\xi_i^l, S^2).
\]
Moreover, the image \( \phi_\infty(M) \cup \{v_i^l(x) : x \in S^2, 1 \leq l \leq L_i, 1 \leq i \leq I\} \) is a connected set.

The Dirac-harmonic spheres are called the bubbles of the blow-up procedure. It follows immediately that the energy identities and the no-neck property hold in this situation.

As this kind of blow-up procedure is purely local, we can restrict ourselves to a sufficiently small disk \( B_{\delta_i}(x_i) \) on which we fix trivializations of the bundles. For simplicity of notation, we will assume that the Riemannian metric \( g \) on such a disk is Euclidean, while in the general case the metric may differ by a small term if we employ geodesic normal coordinates. It then suffices to consider the following.

**Theorem 9.6.** Let \( A_k \in \mathfrak{A}^{1,2}, u_k \in W^{1,2}(B_{\delta}(0), N \subset \mathbb{R}^K) \), and \( \psi_k \in W^{1,4}(B_{\delta}(0), S \otimes u_k^*\mathbb{R}^K) \) be a sequence of solutions on the disk \( B_{\delta}(0) \) of the system (12), with uniformly bounded...
Then there exists a positive integer $I \in \mathbb{N}$ such that for each $1 \leq i \leq I$, there exist a sequence of small numbers $\lambda_k^i \searrow 0$ such that

1. for any $i \neq j$,

$$\frac{\lambda_k^i}{\lambda_k^j} + \frac{\lambda_k^j}{\lambda_k^i} + \frac{|x_k^i - x_k^j|}{\lambda_k^i + \lambda_k^j} = \infty;$$

2. for each $i$, the rescaled sequence

$$\hat{A}_k^i(x) := \lambda_k^i A_k(x_k^i + \lambda_k^i x), \quad \hat{u}_k^i(x) := u(x_k^i + \lambda_k^i x), \quad \hat{\psi}_k^i(x) := \sqrt{\lambda_k^i} \psi_k(x_k^i + \lambda_k^i x),$$

converges to $(0, \sigma^i, \xi^i)$ in $W^{1,2}_{\text{loc}} \times W^{1,2}_{\text{loc}} \times W^{1,\frac{4}{3}}_{\text{loc}}(\mathbb{R}^2)$, where $(\sigma^i, \xi^i)$ extends to a Dirac-harmonic sphere; moreover, if $0 \in S_2$, then $\xi^i \equiv 0$ and $\sigma^i$ defines a harmonic sphere;

3. the energy identities hold:

$$\lim_{k \to \infty} A_{YM}(A_k; B_\delta(0)) = A_{YM}(A_\infty; B_\delta(0)),$$

$$\lim_{k \to \infty} E(u_k; B_\delta(0)) = E(u_\infty; B_\delta(0)) + \sum_{i=1}^I E(\sigma^i; S^2),$$

$$\lim_{k \to \infty} E(\psi_k; B_\delta(0)) = E(\psi_\infty; B_\delta(0)) + \sum_{i=1}^I E(\xi^i; S^2);$$

4. there is no neck between bubbles, i.e. the set $u_\infty(B_\delta(0)) \cup (\cup_{1 \leq i \leq I} \sigma^i(S^2))$ is connected.

**Proof of Theorem 9.6.** By passing to a subsequence if necessary, we assume that $A_k$ converges to $A_\infty$ in $W^{1,2}$ strongly, and $(u_k, \psi_k)$ converge to $(u_\infty, \psi_\infty)$ weakly in $W^{1,2} \times W^{1,\frac{4}{3}}_{\text{loc}}(B_\delta(0))$ and strongly in $W^{1,2}_{\text{loc}} \times W^{1,\frac{4}{3}}_{\text{loc}}(B_\delta(0) \setminus \{0\})$, with

$$\lim_{k \to \infty} \int_{B_r(0)} |du_k|^2 \, dx \geq \varepsilon_0.$$
For each $k$, we choose $\lambda_k > 0$ such that
\[
\sup_{x \in B_k(0)} E(u_k, \psi_k; B_{\lambda_k}(x)) = \frac{\varepsilon_0}{4}
\]
and then choose $x_k \in B_\delta(0)$ such that
\[
E(u_k, \psi_k; B_{\lambda_k}(x_k)) = \sup_{x \in B_k(0)} E(u_k, \psi_k; B_{\lambda_k}(x)) = \frac{\varepsilon_0}{4}.
\]
By our assumption that the sequence converges strongly away from the origin, we conclude that $|x_k| \to 0$ and $\lambda_k \searrow 0$. The rescaled sequences are
\[
\hat{A}_k(x) := \lambda_k A_k(x + \lambda_k x), \quad \hat{u}_k(x) := u(x + \lambda_k x), \quad \hat{\psi}_k(x) := \sqrt{\lambda_k} \psi(x + \lambda_k x),
\]
which are defined on the ball $B_{\delta/2\lambda_k}(0) \not
\supset \mathbb{R}^2$ as $k \to \infty$. From Lemma 5.1 for an arbitrary $R > 1$,
\[
A_{YM}(\hat{A}_k, B_R(0)) = (\lambda_k)^2 A_{YM}(A_k; B_{\lambda_k R}(x_k)) \leq (\lambda_k)^2 \Lambda \to 0.
\]
It follows that, up to Coulomb gauges, $\hat{A}_k \to 0$ in $W^{1,p}(B_R(x))$ for any $B_R(0) \subset \mathbb{R}^2$ and any $1 < p < \infty$. Meanwhile $(\hat{u}_k, \hat{\psi}_k)$ satisfies the system
\[
(17) \Delta \hat{u}_k = \text{Tr} \hat{\Pi}(\hat{u}_k)(d\hat{u}_k, d\hat{u}_k) + \frac{1}{2} \left\langle \hat{\psi}_k^i, \gamma(e_\alpha) \hat{\psi}_k^j \right\rangle \left\langle \partial_{g^i}, R \left( \partial_{g^j}, d\hat{u}_k(e_\alpha) \right) \partial_{g^j} \right\rangle h^{q_l}(\hat{u}_k)^* \hat{u}_k(\partial_{g^i})
\]
\[-2 \text{Tr} \partial_1 \partial_2 \mu_0(\hat{A}_k, d\hat{u}_k) - d\mu_0(\text{div} \hat{A}_k) - \text{Tr} \partial_1 \partial_2 \mu_0(\hat{A}_k, d\mu_0(\hat{A}_k))
\]
\[+ \frac{1}{2} \left\langle \hat{\psi}_k^i, \gamma(e_\alpha) \hat{\psi}_k^j \right\rangle \left\langle \partial_{g^i}, R \left( \partial_{g^j}, d\mu_0(\hat{A}_k(e_\alpha)) \right) \partial_{g^j} \right\rangle h^{q_l}(\hat{u}_k)^* \hat{u}_k(\partial_{g^i}) + \hat{\chi}_2,
\]
\[\hat{\phi} \hat{\psi}_k^i = \Gamma^i_{jl}(\hat{u}_k)(\hat{u}_k)^j \gamma(e_\alpha) \hat{\psi}_k^l
\]
\[+ \left\{ -\Gamma^q_{\alpha l}(\hat{u}_k)(d\mu_0(\hat{A}_k(\partial_{g^q})^i) + \Gamma^i_{\alpha l}(\hat{u}_k) \right\} \gamma(e_\alpha) \hat{\psi}_k^j + \hat{\chi}_3,
\]
and their energies are bounded on both sides:
\[
E(\hat{u}_k, \hat{\psi}_k; B_1(0)) = E(u_k, \psi_k; B_{\lambda_k}(x_k)) = \frac{\varepsilon_0}{4},
\]
\[
E(\hat{u}_k, \hat{\psi}_k; B_R(0)) = E(u_k, \psi_k; B_{\lambda_k R}(x_k)) \leq \Lambda < \infty.
\]
The system (17) can be seen as an approximate Dirac-harmonic system, see [26], with the error terms for the sections in $L^2$ and for the spinors in $L^4$, and the error terms go to zero uniformly. Moreover, they scale in the right way. Then we can use the conclusion there directly in our situation and the convergence, energy identities and no-neck statement follows.

To see the other statements in Theorem 9.6: the first item is hidden in the reduction process on the number of bubbles, and it says that when blowing up, the rescaling parameter should separate the concentration points; details can be found in e.g. [33]; for the rest, note that if $0 \notin \mathcal{S}_2$, then the spinor fields will not blow up there and in the limit the $\xi^i$’s are vanishing, hence the bubbles are only $\sigma^i$’s, which are obviously harmonic spheres. This finishes the proof.

By patching up the above local blow-up analysis, we obtain Theorem 9.1.
9.2. Concluding Remarks.

Remark 1. As in \cite{26}, the proof of the blow-up actually can give
\[
\lim_{k \to \infty} \int_{B_3(0)} |\nabla \psi_k|^\frac{4}{3} \, dx = \int_{B_3(0)} |\nabla \psi_\infty|^\frac{4}{3} \, dx + \sum_{i=1}^{I} \int_{S^2} |\nabla \xi_i|^\frac{4}{3} \, dvol.
\]
Therefore we can get the global convergence of the action: in the notation of Theorem 9.1, denoting \( \omega_i^l \equiv 0 \), so for each \( i \) and \( l \), the bundles are all trivial, and
\[
\lim_{k \to \infty} A(\omega_k, \phi_k, \psi_k) = A(\omega_\infty, \phi_\infty, \psi_\infty) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} A(\omega_i^l, \sigma_i^l, \xi_i^l).
\]
As a corollary, if the fiber manifold \((N, G)\) does not admit Dirac-harmonic spheres, then an approximating sequence with uniformly bounded energies must sub-converge to a smooth solution.

Remark 2. Consider the functional with a potential
\[
A_V(\omega, \phi, \psi) = \int_M |F(\omega)|^2 + |d^V \phi|^2 + \langle \psi, D\psi \rangle + V(\omega, \phi, \psi) \, dvol_g,
\]
where \( V: S \times N \to \mathbb{R} \) is \( G \)-equivariant in the second variable. Moreover, the partial derivatives satisfy
\[
|V_\omega| \leq C(|d_\omega \phi||\psi|^\frac{2}{3} + |\psi|^s), \quad |V_\phi| \leq C(|d_\omega \phi||\psi|^\frac{2}{3} + |\psi|^s), \quad |V_\psi| \leq C(|d_\omega \phi||\psi|^\frac{2}{m} + |\psi|^{s-1})
\]
with \( s < \frac{2m}{m-1} \) such that the perturbations caused by the potential are subcritical. The Euler–Lagrange system for this functional \( A_V \) is
\[
D_\omega^* F + d\bar{\mu}^*_\phi (d^V \phi) + Q(\phi, \psi) + V_\omega = 0, \\
\tau^V(\phi) - \frac{1}{2} R^V(\phi, \psi) + V_\phi = 0, \\
D\psi + V_\psi = 0.
\]
With a similar argument one can show that the weak solutions in dimension two are regular, and under the assumption of uniformly bounded energies, the energy identities still persist. The difficult part here is to choose a right potential, which both allows mathematical analysis and makes physical or geometric sense.

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