AN ELEMENTARY PROOF OF LELLI CHIESA’S
THEOREM ON CONSTANCY OF SECOND COORDINATE
OF GONALITY SEQUENCE

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Abstract. Let $X$ be a K3 surface and $L$ be an ample line bundle on it. In
this article we will give an alternative and elementary proof of Lelli
Chiesa’s Theorem in the case of $r = 2$. More precisely we will prove that
under certain condition the second co-ordinate of the gonality sequence
is constant along the smooth curves in the linear system $|L|$. Using Lelli
Chiesa’s theorem for $r \geq 3$ we also extend Lelli Chiesa’s Theorem in the
case of $r = 2$ in weaker condition.

1. Introduction

Given a smooth irreducible projective curve $C$ and an integer $r$ one can
associate an integer $d_r$ as the minimal degree of a line bundle with $r + 1$
sections. Thus to each curve one can associate a sequence $(d_1, d_2, ...)$ called
gonality sequence. The first co-ordinate of the gonality sequence is known as
the gonality of $C$. Let $X$ be a smooth projective K3 surface over the field of
complex numbers and $L$ be a line bundle on $X$. Then the natural question
one can ask whether the gonality sequence remains constant as $C$ varies in
$|L|_s$, where $|L|_s = \{ C \in |L| : C$ is smooth $\}$. The answer of the question
is negative. In fact, Donagi and Morrison pointed out the following easy
counter example showing that even the first co-ordinate is not constant.

Example ([3], 2.2). Let $\pi : X \to \mathbb{P}^2$ be a K3 surface obtained as a double
cover of $\mathbb{P}^2$ ramified at a smooth sextic curve. Let $L = \pi^*(\mathbb{O}_{\mathbb{P}^2}(3))$. The
general curve of $|L|$ is a plane sextic and hence they have gonality 5. On the
other hand, $|L|$ contains a subspace of co-dimension 1 consisting of bielliptic
curves which has gonality 4.

However, Ciliberto and Pareschi proved that if $L$ is an ample line bundle
on a K3 surface $X$, such that $X$ and $L$ are not simultaneously as in the
Donagi-Morrison’s example, then gonality remains constant along $|L|_s$ [2].

Naturally one could ask about the behavior of the second co-ordinate.
Note that in the Donagi-Morrison’s example, the second co-ordinate (which
we will call planarity of $C$ and denote by $P(C)$) is constant.

Recently Lelli Chiesa [7] proved that if $C$ is an ample curve in $X$ with
some extra hypothesis and admits a complete $g^r_d$ computing the Clifford

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index of $C$ then every curve in the linear system $|\mathcal{O}_X(C)|$ admits a complete $g^r_d$. However if the Clifford index is bigger than 2, then the extra hypothesis satisfied automatically. Thus if the Clifford index of $C$ is bigger than 2 and $C$ admits a complete $g^2_d$ computing the Clifford index of $C$, then $\mathcal{P}(C)$ is constant as $C$ varies in the linear system $|\mathcal{O}_X(C)|$. The question of constancy of $\mathcal{P}(C)$ still remains open if $C$ does not admit a complete $g^2_d$ computing the Clifford index. For example see section 3.

In this article we will give an independent proof for constancy of $\mathcal{P}(C)$ when $C$ admits a complete $g^2_d$ computing the Clifford index and also few cases when it does not admit a complete $g^2_d$ computing the Clifford index. We prove the following Theorem:

**Theorem 1.1.** Let $X$ be a smooth projective $K3$ surface over the field of complex numbers. Let $L$ be an ample line bundle on $X$ such that there is no bi-elliptic curve in $|L|$. Then every smooth curve in the linear system $|L|$ carry a $g^2_d$, if one of the following holds:

(i) there exist an irreducible smooth curve $C$ in the linear system $|L|$ with a complete $g^r_d$, for some $r$, $2 \leq r \leq 3$ which computes the Clifford index of $C$.

(ii) $L^2 \geq 8$ and there exists a smooth curve $C \in |L|$ with a complete $g^4_d$, which computes the Clifford index of $C$.

In other words, the second co-ordinate of the gonality sequence of smooth curves is constant along the linear system $|L|$.

**Notation:** We work throughout over the field $\mathbb{C}$ of complex numbers. If $X$ is a smooth, projective variety, we denote by $K_X$ the canonical bundle on $X$. For a coherent sheaf $\mathcal{F}$ on $X$, we denote by $H^i(\mathcal{F})$ the $i$-th cohomology group of $\mathcal{F}$ and by $h^i(\mathcal{F})$ its (complex) dimension. If $V$ is a vector bundle on $X$, we denote by $V^*$ the dual of $V$. For a sub-scheme $Z \subset X$, we denote by $\mathcal{I}_Z$ the ideal sheaf of $Z$. A line bundle of degree $d$ is called a complete $g^r_d$ on a smooth projective curve $C$ if it has exactly $r+1$ sections. We denote by $W^r_d(C)$, the subvariety of Pic$^d(C)$ whose support is the set:

$$\text{Supp}(W^r_d(C)) = \{L \in \text{Pic}^d(C) : h^0(C, L) \geq r + 1\}.$$  

If $r = 0$ we denote $W^0_d(C)$ simply by $W_d(C)$.

2. **Preliminaries**

In this section we recall the basic properties of the bundle $E_{C,A}$ of Lazarsfeld [5] and Tyurin [9], associated to an irreducible smooth curve $C$ in $X$ and a globally generated line bundle $A$ and the basic definitions of Clifford index and Clifford dimension.

Let $X$ be a smooth projective $K3$ surface over the field of complex numbers. Let $C$ be an irreducible smooth curve in $X$ and $A$ be a globally generated line bundle on $C$. Viewing $A$ as a sheaf on $X$, consider the evaluation map

$$H^0(C, A) \otimes \mathcal{O}_X \rightarrow A.$$
Let $F_{C,A}$ be its kernel and $E_{C,A} := F_{C,A}^*$. Then $F_{C,A}$ fits in the following exact sequence on $X$.

$$0 \rightarrow F_{C,A} \rightarrow H^0(C, A) \otimes \mathcal{O}_X \rightarrow A \rightarrow 0.$$ 

It is easy to check that $F_{C,A}$ is locally free. Dualizing the above exact sequence one gets

$$0 \rightarrow H^0(C, A)^* \otimes \mathcal{O}_X \rightarrow E_{C,A} \rightarrow \mathcal{O}_C(C) \otimes A^* \rightarrow 0.$$ 

Then it is easy to check the following properties:

**Lemma 2.1.**
1. Rank of $E_{C,A} = h^0(C, A)$.
2. $\det(E_{C,A}) = \mathcal{O}_X(C)$.
3. $c_2(E_{C,A}) = \text{deg}(A)$.
4. $h^0(X, E_{C,A}^*) = h^1(X, E_{C,A}^*) = 0$.
5. $E_{C,A}$ is generated by its global sections off a finite set.

**2.2. Clifford index.** Let $C$ be a smooth irreducible complex projective curve of genus $g \geq 2$. Recall that the Clifford index of a line bundle $A$ on $C$ is the integer

$$\text{Cliff}(A) = \text{deg}(A) - 2r(A),$$

where $r(A) = h^0(A) - 1$. The Clifford index of $C$ itself is defined to be

$$\text{Cliff}(C) = \min\{\text{Cliff}(A) | h^0(A) \geq 2, h^1(A) \geq 2\}.$$ 

We say that a line bundle $A$ on $C$ contributes to the Clifford index of $C$ if $A$ satisfies the inequalities in the definition of Cliff($C$); it computes the Clifford index of $C$ if in addition Cliff($C$) = Cliff($A$).

**Theorem 2.3.** (M. Green, R. Lazarsfeld [4]) Let $X$ be a complex projective $K3$ surface, and let $C \subset X$ be a smooth irreducible curve of genus $g \geq 2$. Then

$$\text{Cliff}(C') = \text{Cliff}(C)$$

for every smooth curve $C' \in |C|$. Furthermore, if Cliff($C$) is strictly less than the generic value $\left\lfloor \frac{(g-1)}{2} \right\rfloor$, then there is a line bundle $L$ on $X$ whose restriction to any smooth $C' \in |C|$ computes the Clifford index of $C'$.

Given a curve $C$, we define its Clifford dimension as

$$r = \min\{h^0(A) - 1 | A \text{ computes the Clifford index of } C\}.$$ 

**Proposition 2.4.** (Ciliberto, Pareschi [2]) Let $C$ be a smooth and irreducible curve of genus $g$ sitting on a $K3$ surface $X$ as an ample divisor. Then either $C$ is isomorphic to a smooth plane sextic and $X, \mathcal{O}_X(C)$ are as in Donagi-Morrison’s example or the Clifford dimension of $C$ is 1.
3. An example

In this section we will give an example of a curve $C$ in a K3 surface $X$ such that the Clifford index of $C$ is not computed by a $g_2^d$ but a $g_3^d$. Therefore we can not use Lelli Chiesa’s Theorem to conclude the constancy of the second co-ordinate of the gonality sequence. However we will see that the second co-ordinate remains constant along $\mathcal{O}_X(C)$, which gives an example in support of our Theorem 1.1.

Example: Let $X$ be the K3 surface given by a smooth quartic hypersurface in $\mathbb{P}^3$. Let $C$ be a quadric hypersurface section. In other words, $C$ is a complete intersection of two hypersurfaces of degree 4 and 2 respectively. Clearly $C$ is an ample curve in $X$. Then we have following facts [1, p.199, F-2]:

- $W_1^3(C) = \emptyset$
- $W_4^1(C) \neq \emptyset$
- $W_8^3(C) \neq \emptyset$
- $W_8^3(C) - W_2(C) \subset W_6^1(C)$
- $W_7^2(C) = W_8^3(C) - W_1(C)$.

Thus the Clifford index of $C$ is 2. Since $W_7^2(C) = W_8^3(C) - W_1(C)$ and $W_8^3(C) - W_2(C) \subset W_6^1(C)$, we have $W_7^2 = \emptyset$. Therefore the Clifford index of $C$ can not be computed by a $g_2^d$. On the other hand, since $W_8^3(C)$ is non-empty, the Clifford index is computed by a $g_3^d$. It is clear that $\mathcal{P}(C) = 7$ for all smooth curve $C \in |\mathcal{O}_X(C)|$.

4. Structure of $E_{C,A}$

Let $C$ be a smooth irreducible curve in a K3 surface $X$ and $A$ be a line bundle of minimal degree $d$ with 3 sections. Clearly such a line bundle is globally generated. Let $E_{C,A}$ be the vector bundle constructed as in Section 2. Then by Lemma 2.1 we have,

$$\text{rk}(E_{C,A}) = 3, \det(E_{C,A}) = \mathcal{O}_X(C), c_2(E_{C,A})$$

$$= d, h^0(X, E_{C,A}^*) = h^1(X, E_{C,A}^*) = 0$$

and $E_{C,A}$ is globally generated off a finite set.

The following Proposition is a slight modification of a result of Donagi-Morrison.

**Proposition 4.1.** $E_{C,A}$ is not a simple vector bundle, then we have the following possibilities:

1. There exist a base point free line bundle $N$ and a rank 2 vector bundle $F$, globally generated off a finite set such that $E_{C,A} = F \oplus N$.
2. There exist a base point free line bundle $N$, a rank 2 vector bundle $F$ and a finite set $Z \subset X$ such that $E_{C,A}$ sits in the following exact sequence,

$$0 \to F \to E_{C,A} \to N \otimes \mathcal{I}_Z \to 0$$

and we have $h^0(F) \geq h^0(N) \geq 2$. 
Proof. If $E_{C,A}$ is not simple, then there is an endomorphism $\varphi : E_{C,A} \to E_{C,A}$ which is not of the form $c.Id$ for some scalar $c$, where $Id$ denotes the identity morphism. Let $x \in X$ be a point. Consider an eigen value $c$ of the linear map $\varphi_x : (E_{C,A})_x \to (E_{C,A})_x$. Then the morphism $\psi := \varphi - c(Id)$ is a nonzero morphism, which drops rank everywhere.

Let $F := \ker(\psi), N' = im(\psi)$. If $E_{C,A}$ is decomposable then we are in situation (1). Let us assume $E_{C,A}$ is indecomposable. If the rank of the endomorphism $\psi$ is 2, then one can easily see that the rank of $\psi^2$ is 1. Thus with out loss of generality we can assume that $rk(F) = 2$ and we have a short exact sequence of the form,

$$0 \rightarrow F \rightarrow E_{C,A} \rightarrow N' \rightarrow 0.$$ 

Since $X$ is a surface, any reflexive sheaf over $X$ is locally free. Thus $F$ is locally free.

Note that $N := N''$ is a line bundle and $N' = N \otimes \mathcal{I}_Z$, for some finite set $Z \subset X$. Thus we have a sequence

$$0 \rightarrow F \rightarrow E_{C,A} \rightarrow N \otimes \mathcal{I}_Z \rightarrow 0.$$ 

Since $E_{C,A}$ is globally generated off a finite set, $N$ is also globally generated off a finite set. Since a line bundle on a K3 surface has no base points outside its fixed component [Corollary 3.2, [8]], it is globally generated. Moreover, since $h^0(X, (E_{C,A})^*) = 0$, $N$ is non-trivial. Thus $h^0(N) \geq 2$. If $\psi^2 \neq 0$ then the sequence splits and again we are in the situation (1). If $\psi^2 = 0$, then $h^0(F \otimes N^*) > 0$. Therefore, we have $h^0(F) \geq h^0(N) \geq 2$. □

**Remark 4.2.** Note that if we are in the second case, then $h^0(N^* \otimes F) \neq 0$. Thus $F$ and hence $E_{C,A}$ contains a line subbundle $M$ which admits at least 2 sections. Thus $E_{C,A}$ fit in the following exact sequence,

$$0 \rightarrow M \rightarrow E_{C,A} \rightarrow F \rightarrow 0,$$

where $F$ is a torsion free sheaf of rank two generated by its global sections off a finite set and we have the following exact sequence

$$0 \rightarrow F \rightarrow F^{**} \rightarrow S \rightarrow 0,$$

where $F^{**}$ is the double dual of $F$, $S$ is a coherent sheaf of finite length, in particular supported on a zero-dimensional subscheme $Z$. Also note that $c_2(F) = c_2(F^{**}) + |Z|$, where $|Z|$ denotes the length of $Z$.

**Lemma 4.3.** If $E$ is a globally generated vector bundle off a finite set and $c_1(E)^2 > 0$, then $c_2(E) \geq 0$.

**Proof.** If $E$ is a globally generated vector bundle off a finite set then for a general subspace $V \subset H^0(E)$ of dimension $rk(E)$, we have the following exact sequence [2, See P.18 ]

$$0 \rightarrow V \otimes \mathcal{O}_X \rightarrow E \rightarrow B \rightarrow 0$$
where $B$ is a line bundle on a smooth curve $C \subset X$. Dualizing the exact sequence we have,

$$0 \to E^* \to V^* \otimes O_X \to A \to 0$$

where $A = K_C \otimes B^*$. If $\deg(A) < 0$, then degree of $B \geq 2g' - 1$, where $g'$ is the genus of $C$ and hence $h^0(B) \geq g'$. Thus $h^0(E) \geq g' + 2$. On the other hand since $c_1(E)^2 > 0$, we have $h^0(E) \leq h^0(c_1(E)) = g' + 1$ [Proposition 1.5, [4]]. Thus $c_2(E) = \deg(A) \geq 0$. □

5. Trigonal curve in K3 surface

In this section we will prove an interesting property of a trigonal curve in a K3 surface.

**Theorem 5.1.** Let $C$ be a trigonal curve of genus $g \geq 5$ in a K3 surface $X$. Then the following holds: There exists an irreducible curve $\Delta$ such that $p_a(\Delta) = 1$ and $\Delta.C = 3$.

**Proof.** Since $C$ is a trigonal curve, its Clifford index is 1. Note that if a $g^r_d$ computes the Clifford index of $C$, then $d = 2r + 1$. On the other hand, for a trigonal curve if $d_r$ is the minimal degree of a line bundle with at least $r + 1$ sections, then we have (see [6, Remark 4.5(b)])

$$d_r = 3r, 1 \leq r \leq \left\lfloor \frac{g - 1}{3} \right\rfloor,$$

$$= r + g - 1 - \frac{g - r - 1}{2}, \frac{g - 1}{3} \leq r \leq g - 1$$

$$= r + g, r \geq g$$

Therefore, if a line bundle of degree $2r + 1$ has at least $r + 1$ sections, then $2r + 1 = d_r$. Now from the expression of $d_r$ in 1, one can conclude that the possibilities are $r = 1$ and $r = g - 2$. In other words, the Clifford index of a trigonal curve can be computed only by a pencil $L$ and $K \otimes L^*$.

On the other hand, there exist a line bundle $M$ on $X$ such that $M|_C$ computes the Clifford index [4]. Therefore, $h^0(C, M|_C) = 2$ or $h^0(O_X(C) \otimes M^*)|_C) = 2$. Without loss of generality we assume $h^0(C, M|_C) = 2$. Since $K_C \otimes (M|_C)^*$ also computes the Clifford index, once can see that $h^0(M \otimes O(-C)) = 0$ and hence $h^0(X, M) = 2$ and $\deg(M|_C) = M.C = 3$. Therefore a general curve $\Delta$ in $|M|$ is irreducible and has arithmetic genus 1. Also we have $\Delta.C = 3$, which conclude the Theorem. □

6. Main theorem

In this section we prove the main theorem. If $X$ and $L$ are as in the Donagi-Morrison’s example, then we have seen that the planarity remains constant along the smooth curves in $|L|$. Let assume $X$ and $L$ are not as in the Donagi-Morrison’s example. Let $C$ be an irreducible smooth curve $C$ in the linear system $|L|$ with a complete $g^r_d$, where $2 \leq r \leq 4$, which computes the Clifford index of $C$. It is known that
the gonality is constant along the smooth curves in the linear system $|L|$ \cite{2}. Let $d$ be the gonality. Also we have the Clifford dimension of every curve in the linear system $|L|$ is 1 \cite{2}. Thus the Clifford index of every curve is $d - 2$.

**Proof.** of main Theorem:

**Case I: $r=2$**

Let $C \in |L|$ be a smooth curve. If $C$ is hyperelliptic then the Theorem holds trivially. We assume $C$ is not hyperelliptic. Let $A$ be a complete $g^2_2$, on $C$, computing the Clifford index. Therefore, the degree $d'$ of $A$ is $d + 2$ and such a line bundle is necessarily globally generated.

Note that $d + 2$ is the minimal degree of a line bundle with at least 3 sections.

If the vector bundle $E_{C,A}$ is simple, then we have

$$h^0(E_{C,A} \otimes E_{C,A}^*) = 1.$$  

Thus

$$\chi(E_{C,A} \otimes E_{C,A}^*) = 2 - h^1(E_{C,A} \otimes E_{C,A}^*).$$

On the other hand, by Riemann-Roch, we have

$$\chi(E_{C,A} \otimes E_{C,A}^*) = \frac{c_1(E_{C,A} \otimes E_{C,A}^*)^2}{2} - c_2(E_{C,A} \otimes E_{C,A}^*) + \text{rk}(E_{C,A} \otimes E_{C,A}^*) \chi(\mathcal{O}_X).$$

Now $c_2(E_{C,A} \otimes E_{C,A}^*) = 6c_2(E_{C,A}) - 2c_1(E_{C,A})^2$. Thus we have,

$$\chi(E_{C,A} \otimes E_{C,A}^*) = 18 - 6c_2(E_{C,A}) + 2c_1(E_{C,A})^2 = 18 - 6(d + 2) + 2(2g - 2) = 2 - 2\rho(g, 2, d + 2).$$

Comparing (2) and (3) we have, $\rho(g, 2, d + 2) \geq 0$. Thus $W^2_{d+2}(C)$ is non-empty for every smooth curve $C$ in $|L|$. Hence the second co-ordinate of the gonality sequence is constant.

Let assume $E_{C,A}$ is not simple. Then by Remark 4.2, we have an exact sequence of the form

$$0 \to M \to E_{C,A} \to F \to 0,$$

where $F$ is a rank 2 torsion free sheaf, generated by its global sections off a finite set and $M$ is line bundle, with at least two sections and $F$ fits in the following exact sequence,

$$0 \to F \to F^{**} \to \mathcal{O}_Z \to S \to 0,$$

where $S$ is a coherent sheaf of finite length, in particular supported on a zero-dimensional subscheme $Z$.

Let $N := c_1(F)$. Note that

$$c_2(E_{C,A}) = d + 2 = MN + |Z| + c_2(F^{**}).$$

Since $F$ is globally generated by its section of a finite set, $F^{**}$ is also globally generated off a finite set. Also note that as $F^{**}$ is globally generated by it’s sections off a finite set, $N$ is globally generated off a finite set and since on a K3-surface a line bundle can have no fixed point outside fixed components,
$N$ is globally generated.
Also by Lemma 4.3, $c_2(F^{**}) \geq 0$.
Claim: $h^1(N) \leq 1$.
Since $N$ is base point free, $h^1(N) \neq 0$ implies that $N = O(k\Gamma)$ [Proposition 2.6 [8]], where $\Gamma$ is an elliptic curve and $k$ is an integer $\geq 2$. Also we have $h^1(N) = k - 1$ and $h^0(N) = k + 1$. Thus if $h^1(N) > 1$, then $k \geq 3$. Since $c_2(F) \geq 0$, we have $C.2\Gamma < M.N \leq d + 2$. But $O_C(2\Gamma)$ has 3 sections, which is a contradiction to the minimality of $d + 2$.

In the case when $h^1(N) = 1$ we have, $N = O(2\Gamma)$. If $|Z| + c_2(F^{**}) > 0$, then $\deg(N|_C) < d + 2$ and $h^0(N|_C) = 3$. Thus we get a contradiction.
If $|Z| + c_2(F) = 0$, then $N|_C$ has 3 sections and degree of $N|_C = d + 2$ for all $C \in |L|$, which proves our theorem.

Let us assume $h^1(N) = 0$.
If $h^0(N) = 2$, then $N = O_X(E)$, where $E$ is a smooth elliptic curve.
On the other hand, since $F^{**}$ is globally generated off a finite set, by [4, Proposition 1.5], $F^{**} = O_X(\Delta) \oplus O_X(\Delta)$, where $\Delta$ is a smooth irreducible curve on $X$ which moves in a base-point free pencil. Thus $N = O_X(2\Delta)$, a contradiction.

Let $h^0(N) \geq 3$.

Since $h^0(M) \geq 2$ and $h^0(M) \leq h^0(M|_C), M|_C$ contributes in the Clifford index. Since $K_C = O_C(C)$, we have $K_C \otimes M|_C = N|_C$. From the exact sequence,
\[
0 \to O(N - C) \to N \to N|_C \to 0
\]
we have $h^0(N|_C) = h^0(N) + h^1(M)$. Also by Riemann-Roch, we have $h^0(N) = \frac{N^2}{2} + 2$. Thus
\begin{equation}
(6)
\text{Cliff}(M|_C) = \text{Cliff}(K_C \otimes M|_C) = \text{Cliff}(N|_C) = N.C - 2(h^0(N) + h^1(M)) + 2
\end{equation}
\[
= N.C - N^2 - 4 - 2h^1(M) + 2
\]
\[
= M.N - 2h^1(M) - 2
\]
\[
= d + 2 - |Z| - c_2(F^{**}) - 2h^1(M) - 2.
\]
But $d - 2 = \text{Cliff}(C) \leq \text{Cliff}(M|_C)$, thus we have
\[
d - 2 \leq d - |Z| - c_2(F^{**}) - 2h^1(M)
\]
or
\[
|Z| + c_2(F^{**}) + 2h^1(M) \leq 2.
\]
In particular $c_2(F^{**}) \leq 2$. Since $F^{**}$ is globally generated off a finite set, for a general two dimensional subspace $V$ of $H^0(F^{**})$, we have
\begin{equation}
(7)
0 \to V \otimes O_X \to F^{**} \to B \to 0
\end{equation}
where $B$ is a line bundle on a smooth curve $D \in |N|$.
Dualizing the above exact sequence we get,
\[
0 \to F^{***} \to V^* \otimes O_X \to B' \to 0
\]
where $B^* = \mathcal{O}_D(D) \otimes B^*$. Now from the long exact sequence of (4), we have $h^0(F^*) = h^2(F) = 0$. Thus we have $h^0(B^*) \geq 2$. Also we have $c_2(F^{**}) = \deg(B^*)$. But $c_2(F^{**}) = \deg(B^*)$ and $B^*$ has at least 2 sections. Therefore the curve $D$ is hyperelliptic. If $D$ has genus 2 then deg$(\mathcal{O}(D)|_C) = D.C = D^2 + M.N = 2 + M.N$. Since $c_2(F^{**}) = 2, |Z| = 0$, then from 5 it follows that $M.N = 2$, which implies $D.C = d + 2$. Therefore $\mathcal{O}(D)|_C$ will give a complete $g_{d+2}^2$ for all $C \in |L|$. If $D$ has genus bigger than 2, then the following two cases can occur [[8, Theorem 5.2] : 
(i) There exists an irreducible elliptic curve $\Delta$ such that $\Delta.D = 2$.
(ii) There exists an irreducible hyperelliptic curve $B$ of genus 2 such that $D \sim 2B$.

In case (i), we can further assume genus of $D$ is bigger than 3, thus we can decompose $D$ as $\Delta + D'$, with $D'.\Delta = 2$. Now $(D - 2\Delta)^2 = D^2 - 8$. Thus if $D - 2\Delta$ is not effective, then $D' = 6$ and hence $D'' = 2$. Therefore the restriction of $\mathcal{O}(D')$ on each curve in $|L|$ will give a complete $g_{d+2}^2$. If $D - 2\Delta$ is effective then we can decompose $D$ as $D'' + 2\Delta$ and $L = \mathcal{O}(2\Delta + D'') \otimes M$. It is easy to see that $(D'' + c_1(M))^2 > 0$. Thus $D''.c_1(M) > 2$, [[8, Lemma 3].

On the other hand,

\begin{equation}
\text{deg}(\mathcal{O}(2\Delta)|_C) = 4 + 2\Delta.c_1(M) \leq M.N + c_2(F^{**}) + |Z| = d + 2
\end{equation}

Therefore, $\mathcal{O}(2\Delta)|_C$ will give a $g_{d+2}^2$ for all $C \in |L|$ or deg$(\mathcal{O}(2\Delta)|_C < d + 2$, a contradiction.

In case (ii), Considering the line bundle $\mathcal{O}(B)$. Note that since, $C$ is neither hyper-elliptic nor bi-elliptic, by Mumford’s Theorem for $g_d^2$ [1], $W_{d+2}(C)$ is non-empty, if and only if, $d + 2 - 6 \geq 0$ that is $d + 2 \geq 6$, i.e., $d \geq 4$.

Note that $M.N = d$ and $B.C = B.(M + N) = B(2B + M) = 2B^2 + \frac{M.N}{2} \leq d + 2$. Thus either $\mathcal{O}_X(B)|_C$ is a complete $g_d^2$ for all $C \in |L|$ or we will get a contradiction.

**Case II: r = 3**

Let $A$ be a line bundle of degree $d'$ computing the Clifford index of $C$ with $h^0(A) = 4$. We can assume there is no curve in $|L|$ with a line bundle with 3 sections, computing the Clifford index. Since $d'$ computes the Clifford index of $C$ and the Clifford index of $C$ is $d - 2$, one has $d' = d + 4$. In this case every curve in the linear system $|L|$ admits a complete $g_{d+4}^3$ [7, Theorem 4.1]. For a general point $x \in C$, $A \otimes \mathcal{O}_C(-x)$ admits 3 sections. Thus $W_{d+3}^2$ is non-empty. If $W_{d+2}^2 \neq \emptyset$, then one can get a line bundle computing the Clifford index of $C$ with 3 sections, a contradiction. Thus $W_{d+2}^2 = \emptyset$. This is true for every smooth irreducible curve in $|L|$. Thus the planarity of every curve in the linear system $|L|$ is $d + 3$.

**Case III: r = 4**

Again let $A$ be a line bundle of degree $d'$ computing the Clifford index of $C$ with $h^0(A) = 5$. In this case $d' = d + 6$ and as previous case by [7, Theorem 4.1], every curve in the linear system $|L|$ admits a complete $g_{d+6}^4$. 

Now for general two points \( x, y \in C \), \( A \otimes \mathcal{O}_C(-x - y) \) admits 3 sections. Thus \( W^2_{d+4}(C) \) is non-empty for every smooth irreducible curve \( C \in |L| \). If \( W^2_{d+3}(C) = \emptyset \) for all \( C \in |L| \), then planarity of every curve is \( d + 4 \) and we are done.

Let \( C \in |L| \) such that \( W^2_{d+3}(C) \neq \emptyset \) and let \( A \in W^2_{d+3}(C) \).

Let \( E_{C,A}, F^{**}, M, N, Z, D \) are as in Case I. Then from 6, we have

\[
\begin{align*}
\d - 2 \leq M.N - 2h^1(M) - 2 &= d + 3 - |Z| - c_2(F^{**}) - 2h^1(M) - 2 \\
\text{Or} \quad c_2(F^{**}) + |Z| + 2h^1(M) &\leq 3
\end{align*}
\]

If \( c_2(F^{**}) \leq 2 \) then we can conclude the Theorem as Case I. Let \( c_2(F^{**}) = 3 \). Then the degree of the line bundle \( B \) on \( D \) in 7 is 3 and admits 2 sections. Hence \( D \) is a trigonal curve. Therefore by Theorem 5.1, there exist an elliptic curve \( \Delta \) such that \( \Delta.D = 3 \) and \( D \) can be decomposed as \( D' + \Delta \). If \( D' \) contains any \(-2\)-curve \( \Gamma \), then \( \Gamma \) will be a fixed component of \( N \), a contradiction, since \( N \) is base point free. Thus we can assume \( D'^2 \geq 0 \).

If \( 0 \leq D'^2 \leq 2 \), then by similar analysis as in Case \( r = 2 \), we can conclude the Theorem. Thus we can assume that \( D'^2 \geq 4 \), that is, \( D'^2 \geq 10 \). If \( D^2 \geq 12 \), then \( D \) can be decomposed as \( 2\Delta + D' \) and we are done as earlier.

Let \( D^2 = 10 \). Then \( D'^2 = 4 \). Therefore, \( D' \) is either hyperelliptic or trigonal. Thus we have a decomposition of \( D \) as \( 2\Delta + D' \), which conclude the Theorem.

\[ \square \]

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