Periodic orbits in the 1:2:3 resonant chain and their impact on the orbital dynamics of the planetary system Kepler-51

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ABSTRACT

Aims. Many exoplanets evolving in (or close to) MMRs and resonant chains have been discovered by space missions. Oftentimes, the published data possess very large uncertainties, due to observational limitations, which deem the system chaotic. In short or large timescales. We propose a study of the dynamics of such systems by exploring particular regions in phase space.

Methods. We exemplify our method by studying the long-term orbital stability of the three-planet system Kepler-51 and either favor or constrain its data. It is a dual process which breaks down in two steps: the computation of the families of periodic orbits in the 1:2:3 resonant chain and the visualization of the phase space through maps of dynamical stability.

Results. We present novel results in the General 4-Body Problem. Stable periodic orbits were found only in the low-eccentricity regime. We demonstrate three possible scenarios safeguarding Kepler-51, each followed by constraints. Firstly, the 2/1 and 3/2 two-body MMRs, in which $e_b < 0.02$, so that such two-body MMRs last for long-time spans. Secondly, the 1:2:3 three-body Laplace-like resonance, in which $e_c < 0.016$ and $e_d < 0.006$ for such a chain to be viable. Thirdly, the combination comprising an 1/1 secondary resonance inside 2/1 MMR for the inner pair of planets and an apsidal difference oscillation for the outer pair of planets in which the observational eccentricities $e_b$ and $e_a$ are favored as long as $e_d \approx 0$.

Conclusions. Aiming to the optimum deduction of the orbital elements, this study showcases the need for dynamical analyses based on periodic orbits being performed in parallel to the fitting processes.

Key words. celestial mechanics – planets and satellites: dynamical evolution and stability – planetary systems – methods: analytical – methods: numerical – chaos

1. Introduction

Kepler, K2 and TESS missions have revealed 3670 planetary systems, 813 of which being multiple-planet⁴(see e.g. Lissauer et al. 2011; Fabrycky et al. 2014; Weiss et al. 2018). Apart from various properties and architectures, such as the number of planets, the planetary period and radii ratios, observed among them, planetary pairs and triplets seem to be evolving near (just outside) or in two-body mean-motion resonances (MMRs) and three-body Laplace-like resonances (resonant chains), respectively. In particular, five out of six planets of TOI-178 are in the 2:4:6:9:12 resonant chain (Lecue et al. 2021), four planets of the HR 8799 are in the 8:4:2:1 chain (Góździewski & Migaszewski 2020), four planets of the system K2-102 evolve near the 12:5:7 resonant chain (Heller et al. 2019), four planets of the system Kepler-223 are in a 8:4:6:8 resonant chain (Miller et al. 2016), while five planets of the K2-138 are in the 3:2 MMR (Christiansen et al. 2018). Other examples include the multiple-planet systems Trappist-1 (Gillon et al. 2017), Kepler-18 (Cochran et al. 2011), Kepler-30 (Panichi et al. 2018), Kepler-47 (Orosz et al. 2019), Kepler-60 (Góździewski et al. 2016) and Kepler-82 (Freudenthal et al. 2019).

Recent studies were devoted to the formation history and dynamical evolution of multiple-planet systems that yield exoplanets locked in or close to a resonant chain (e.g. Morrison et al. 2020; Pichierri & Morbidelli 2020; Lissauer & Gavio (2021) numerically analyzed the long-term orbital stability of three-planet systems when the planetary longitudes and orbital separation vary, while Siegel & Fabrycky (2021) performed a numerical exploration of equilibria dependent on the libration of the three-body resonant angles for chains consisting of first-order MMRs. Furthermore, Tamayo et al. (2021) provided a stability estimator that distinguishes quickly the unstable from the stable planetary configurations among estimated orbital parameters and masses of multiple-planet systems. Charalambous et al. (2018) chose different planetary masses and illustrated maps with stable and unstable domains of three-planet systems at which the mean-motion ratios of the planets were altered, while Petit (2021) analytically approximated first-order resonant chains of three massive planet systems with an averaged Hamiltonian.

Planetary masses and eccentricities can be precisely extracted by the transit-timing variation (TTV) method for exoplanets close to an MMR. However, employing numerical analyses or N-body simulation algorithms, in order to reject orbital parameters that lead to instability events, and which cannot be estimated by observations, can become computationally expensive and time-consuming. Additionally, oftentimes the published observational data possess very large deviations which deem the system chaotic.
stead, spotting out the stable periodic orbits close to the exoplanetary system (for a given MMR or resonant chain, planetary masses and eccentricity values) can immediately unravel the regions where the stability is guaranteed for long-time spans. Hence, the boundaries for all the orbital elements, even the longitudes of pericenter and mean anomalies, can be delineated as the dynamical phase space of resonant exoplanets is crystallized.

Hadjidemetriou & Michalodimitrakis (1981) computed the first stable periodic orbits in the planar General 4-Body Problem (G4BP) (see e.g. Hadjidemetriou 1974, Michalodimitrakis & Grigorets 1989) for the Galilean satellites of Jupiter in the 1:2:4 resonant chain. Following their work, we apply this methodology to the exoplanets. Here, we focus on the system Kepler-51 and aim to provide hints and constraints that favor its survival. We study the dynamics and long-term stability of the system via the families of periodic orbits in the planar G4BP and extend our methodology for pairs of resonant massive exoplanets in mutually inclined orbits (Antoniadou & Voyatzis 2013, 2014) and coplanar orbits (Antoniadou & Voyatzis 2016), to the case of triplets of massive coplanar exoplanets in resonant chains (G4BP).

Our work is organized as follows. In Sect. 2 we provide the key points and tools employed to obtain our dynamical analysis. In Sect. 3 we discuss possible regions of long-term stable evolution in the dynamical vicinity of Kepler-51 based on the periodic orbits in the G4BP and conclude in Sect. 4. In Appendix A we provide the equations of motion for three-planet systems (A.1), define the symmetric periodic orbits (A.3) and elaborate on the continuation method followed for 1:2:3 resonant periodic orbits (A.3). In Appendix B we provide details and examples on the linear stability of the periodic orbits in the G4BP (B.1) and the chaotic indicator used (B.2).

2. Main aspects of the methodology

We considered three massive planets revolving around a star on coplanar orbits. When viewed in an inertial frame of reference, these orbits correspond to Keplerian ellipses with heliocentric osculating elements, i.e. the semimajor axes, $a_i$, the eccentricities, $e_i$, the longitudes of pericenter, $\omega_i$, and the mean anomalies, $M_i$ ($i = 1, 2, 3$), where subscript 1 refers to the inner planet. In computations, we use the normalized set of units where the semimajor axis of the inner planet is equal to 1, $G \times m_{\text{total}} = 1$, and subsequently the period of the inner planet is equal to about $2\pi$.

We choose the relative frequencies, $\omega_i$ ($i = 1, 2, 3$), of the three planets of period $T_i = \frac{2\pi}{\omega_i}$ ($i = 1, 2, 3$) in the inertial frame, so that they be commensurable:

$$\frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \approx \frac{P}{Q},$$

where $P, Q \in \mathbb{Z}^*$.\footnote{In Appendix A.1 we provide more details about the model set-up and the inertial and rotating frames of reference.}

Then, in the rotating frame of reference we can define the relative period, $T_r$, of a system where planet 2 with period $T_2$ revolves $P$ times about planet 1 with period $T_1$, while planet 3 with period $T_3$ revolves $Q$ times around planet 1, i.e.

$$T_r \approx \frac{T_1}{1 - \frac{P}{T_2}} P \approx \frac{T_1}{1 - \frac{Q}{T_3}} Q. \quad (2)$$

Certain orbits, which fulfill a specific periodicity condition in the rotating frame, are called symmetric periodic orbits (see Appendix A.2 for more details). These periodic orbits correspond to the exact location of the MMR. In this work, we focus exclusively on the symmetric elliptic (resonant) periodic orbits, which have $\omega_i = 0$ or $\pi$ ($i = 1, 2, 3$).

Described within another context, the periodic orbits are the fixed (or periodic) points of a Poincaré map and correspond also to the fixed points (or stationary solutions) of an averaged Hamiltonian, as long as the latter is accurate enough. Following the continuation schemes explained in Appendix A.2 and A.3, these points form the so-called characteristic curves or families of periodic orbits.

The periodic orbits depend on the resonant angles, $\theta_i$ ($i = 1, \ldots, 4$), which, given the 1:2:3 resonant chain studied, take the form

$$\begin{align*}
\theta_1 &= 2\lambda_2 - \lambda_1 - \varpi_1 \\
\theta_2 &= 2\lambda_2 - \lambda_1 - \varpi_2 \\
\theta_3 &= 3\lambda_3 - 2\lambda_2 - \varpi_3 \\
\theta_4 &= 3\lambda_3 - 2\lambda_2 - \varpi_2 \\
\phi_L &= \theta_1 - \theta_2 = 3\lambda_3 - 4\lambda_2 + \lambda_1
\end{align*}$$

with $\lambda_i = \omega_i + M_i$ being the mean longitude, $\Delta\varpi_{21} = \varpi_2 - \varpi_1$ and $\Delta\varpi_{32} = \varpi_3 - \varpi_2$ the apsidal differences and $\phi_L$ the Laplace angle.

If the periodic orbit is symmetric the angles in Eq. 3 are equal to either 0 or $\pi$. In order to distinguish the families of periodic orbits, that belong to different configurations, we present them on the $(e_1 \cos \theta_1, e_2 \cos \theta_2)$ and $(e_2 \cos \theta_2, e_3 \cos \theta_3)$ planes.

The characteristic curves of the families change significantly, when the mass-ratio changes, see e.g. (Antoniadou & Voyatzis) 2013 for the 2/1 MMR and (Antoniadou & Voyatzis 2014) for the 3/2, 5/2, 3/1 and 4/1 MMRs. If the mass values change, but are of the same order and also preserve their ratio, then the main properties of the families (location and stability) do not show significant variations (see e.g. Voyatzis 2008). Therefore, in order to distinguish the families that belong to the same configuration but have been computed for different mass-ratios, we introduce two planetary mass-ratios; one for the innermost and one for the outermost pair, namely $p_i = \frac{m_i}{m_2}$ and $p_0 = \frac{m_0}{m_2}$, respectively.

It is widely known in Hamiltonian systems that stable periodic orbits are surrounded by invariant tori where the motion is regular and quasi-periodic, whereas the neighborhood of unstable periodic orbits can give rise to instability events (either weak chaos or strong chaos with collisions or escapes). In the neighborhood of stable periodic orbits, all the resonant angles librate, while all or some of them exhibit rotation in the vicinity of the unstable ones.

The libration of all the resonant angles $\theta_i$ ($i = 1, \ldots, 4$) signifies a two-body MMR (hereafter denoted by $R_{ij}$), where $\omega_a = \frac{(2\pi)}{a^3} = \frac{(2\pi)}{a^{3/2}} \approx \frac{p_{i+1} - p_i}{p_i}$ and $\omega_b = \frac{(2\pi)}{b^3} = \frac{(2\pi)}{b^{3/2}} \approx \frac{q_{i+1} - q_i}{q_i}$, with $p_i, q_i \in \mathbb{Z}^*$. With $p_i, q_i$ being the order of the MMR, implying that the libration of the Laplace angle, $\phi_L$, is simply

$$\phi_L = \frac{\lambda_3 - 4\lambda_2 + \lambda_1}{3},$$

Asymmetric periodic orbits also exist, having longitudes of pericenter different than 0 or $\pi$, which are not considered herein.
In the following, we also use a notation of where only the apsidal difference librates, may become apsidal difference oscillates and the rest of the angles rotate. At the instance, the combination of a secondary resonance of the outer pair is denoted by \( R_T \), while in the latter case, only the apsidal rates behaviors of the inner and outer pairs of planets. For the system Kepler-51 has been studied regularly (see e.g. Steffen et al. 2013; Hadden & Lithwick 2014; Masuda 2014; Morton et al. 2016; Holczer et al. 2016; Hadden & Lithwick 2017; Berger et al. 2018; Gajdoš et al. 2019; Libby-Roberts et al. 2020; Battley et al. 2021). Its innermost planetary pair evolves close to the 2/1 MMR, according to the period ratio \( T_{2/1} = 1.89 \), while the outermost one is in the 3/2 MMR, as \( T_{3/2} = 1.53 \). In what follows, we present the families of periodic orbits in the 1:2:3 resonant chain and construct DS-maps which unveil the dynamical vicinity of Kepler-51, in order to explore the dynamical mechanisms that may guarantee its orbital stability.

### 3. Dynamical constraints on Kepler-51

We now take into account the planetary eccentricity values provided by \( (1) \) Masuda (2014) and \( (2) \) Libby-Roberts et al. (2020) and \( (3) \) Battley et al. (2021). Table 2: Published data that were utilized for the constraints on Kepler-51.

| Parameter | Kepler-51b | Kepler-51c | Kepler-51d |
|-----------|------------|------------|------------|
| \( m_p (\text{M}_\oplus) \) | 2.1\(^{+1.5}_{-0.8}\) | 4.0 \( \pm 0.4 \) | 7.6 \( \pm 1.1 \) |
| \( T (\text{days}) \) | 45.1540 \( \pm 0.0002 \) | 85.312\(^{+0.003}_{-0.002}\) | 130.194\(^{+0.005}_{-0.002}\) |
| \( a (\text{au}) \) | 0.2514 \( \pm 0.0097 \) | 0.384 \( \pm 0.015 \) | 0.509 \( \pm 0.020 \) |
| \( e \) | 0.04 \( \pm 0.01 \) | 0.014\(^{+0.013}_{-0.009}\) | 0.008\(^{+0.011}_{-0.008}\) |
| \( T_0 (\text{BJD} = 2454833) \) | 881.5977 \( \pm 0.0004 \) | 892.509 \( \pm 0.003 \) | 862.3233 \( \pm 0.0004 \) |

### 3.1. Families of symmetric periodic orbits for Kepler-51

Following the continuation method illustrated in Appendix 3.2, we set \( P = 3 \) and \( Q = 4 \) in Eq. (3.18) which yields for \( k = 4 \)

\[
\left( \begin{array}{ccc}
T_2 & T_3 & T_4 \\
T_2^1 & T_3^1 & T_4^1
\end{array} \right) = \left( \begin{array}{ccc}
2 & 3 & 3 \\
1 & 1 & 2
\end{array} \right)
\]

\( (4) \)

i.e. the resonant chain of the 2/1 and 3/2 MMRs is established for the innermost and outermost pairs of Kepler-51.
Fig. 1: Groups of families, $S_i$ ($i = 1...6$), of planar symmetric periodic orbits in the 1:2:3 resonant chain with stable segments close to the dynamical vicinity of Kepler-51 presented on the ($e_1,e_2$) and ($e_2,e_3$) planes. Each of these six groups consists of families of the same configuration (shown in Table 3), but has different planetary mass-ratios, i.e. $\rho_i = \frac{m_c}{m_b}$ and $\rho_o = \frac{m_d}{m_c}$ (shown in Table 1 and labeled from 1 to 4 on each panel). Label 1 corresponds to Masuda (2014), label 2 to Libby-Roberts et al. (2020), label 3 to default and label 4 to highmass priors considered by Battley et al. (2021). The stable (unstable) periodic orbits are depicted by blue (red) solid curves.

We observe that the segments of stable periodic orbits do not alter in extent as the considered planetary mass-ratios vary, even though the reflection of the four different sets of planetary masses (shown in Table 1) on the the mass-ratios seems quite important and significantly ranging, e.g. from 1.2 (Libby-Roberts et al. 2020) to 2.57 (Battley et al. 2021). When the dynamics is being studied, apart from the masses themselves, one major factor that affects it is the order of the mass values. More precisely, the divergence of the families and their stability segments as the mass-ratios vary are affected differently for giant Jovian planets and terrestrial ones; and all studies performed for the super-puff planets of Kepler-51 attribute masses of the same order ($10^{-6} - 10^{-5}$).

Since the change on the dynamics is not significant among the various planetary masses considered in Table 1 in what follows, we choose to focus on the study of Masuda (2014). Our choice is also justified by both Battley et al. (2021) and Libby-Roberts et al. (2020) who conclude that their mass values are less constrained and not improved when compared to Masuda (2014).

In the top panels of Fig. 2 six families of planar periodic orbits, $S_i$ ($i = 1...6$), in the 1:2:3 resonant chain are presented on the signed-eccentricity planes. In the bottom panels, we provide all (nine) families up to high-eccentricity values, even though only low-eccentricity stable periodic or-
Fig. 2: Families, $S_i$ ($i = 1...9$), of planar symmetric periodic orbits in the 1:2:3 resonant chain close to the dynamical vicinity of Kepler-51 (top panels) and until their termination points (bottom panels) presented on the $(e_1 \cos \theta_1, e_2 \cos \theta_2)$ and $(e_2 \cos \theta_2, e_3 \cos \theta_3)$ planes. The angles for the different configurations of the families are shown in Table 3. Kepler-51 is identified in the top panels by the cyan ‘+’ symbol, while the errors provided by Masuda (2014) are delineated by magenta dashed lines.

More precisely, solely the families $S_1$, $S_4$, $S_5$ and $S_6$ possess stable (blue colored) periodic orbits. These are found in the configurations $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, \pi, 0, \pi)$ and $(0, \pi, \pi, \pi)$ for the family $S_1$, $(\pi, 0, 0, 0)$ for the family $S_4$, $(\pi, \pi, 0, \pi)$ for the family $S_5$ and $(0, \pi, \pi, 0)$ for the family $S_6$. 

bits were found. The families were computed for the planetary masses of Kepler-51, shown in Table 2. In Table 3, we provide the longitudes of pericenter, mean anomalies and resonant angles along each family together with the eccentricity values at which a transition of the configuration takes place. Given the observational values of the eccentricities (see Table 2), depicted by a cyan ‘+’ symbol, and their deviations, demonstrated by magenta dashed lines, we focus on the low-eccentricity regime (top panels).
Discerning distinct regions of regular and chaotic motion in phase space can sometimes be demanding for Hamiltonian systems of 5 degrees of freedom (planar 3HB). The use of chaotic indicators can assist in deciphering such domains. However, all indicators have their pros and cons (see e.g. Maffione et al. 2011, for a comparison between the Lyapunov Indicator (LI), the Mean Exponential Growth factor of Nearby Orbits (MEGNO), the Smaller Alignment Indicator (SALI), the FLI and the Relative Lyapunov Indicator (RLI)). In this paper, we use a version of the FLI (called DFLI; see Appendix B.2 for its definition). Concerning dynamical systems like the planetary ones, the DFLI was established as efficient and reliable by Voyatzis (2008).

With regard to the computation of the DS-maps, we constructed $100 \times 100$ grid planes and focused on the families $S_1$, $S_5$ and $S_6$, i.e. we chose specific orbital elements from the symmetric periodic orbits that belong to these families (their angles are reported in Table 3), since their stable segments are closer to the observational eccentricity values of Kepler-51. We remind the reader that the chosen families of periodic orbits have been computed for the masses of Kepler-51 (identified by label 1 (namely Masuda 2014)) in Fig. 1 and shown in full extent in Fig. 2) and for the 1:2:3 resonant chain. We note also that the values of the semimajor axes vary slightly along the families, but the resonance remains almost constant (see Appendix A.3 for the differences between these elliptic, $S_i$ families and the circular family along which the resonance varies). More precisely, on each grid plane we vary a pair of the orbital elements, while keeping the rest of the orbital elements and masses of the periodic orbit fixed. In the following, we mention the orbital elements of the selected periodic orbits used for the construction of each DS-map and justify their selection.

In order to classify the orbits, we chose a maximum integration time for the computation of the DFLI equal to $t_{\text{max}} = 3\text{Myr}$, which corresponds approximately to 24 million orbits of the innermost planet $b$ or 8.5 million orbits of the outermost planet $d$ and was deemed appropriate, in order to distinguish chaos from order reliably for the particular application. We used the Bulirsch–Stoer integrator with tolerance $10^{-14}$. For small values of DFLI (dark-colored domains), regular evolution of the planets is expected. In our study, we stop the integrations either when DFLI($t$) > 30 or when $t_{\text{max}}$ is reached.

### Notes
Transitions to different configurations as the families, $S_i$ ($i = 1..9$), of symmetric periodic orbits in the 1:2:3 resonant chain (presented in Fig. 2) evolve from the origin of the axes (circular family) up to high eccentricity values. The transitions along the families $S_7$ and $S_8$ begin from the configuration where $e_1 > 0$ and are monitored as they evolve thereafter. The eccentricity values $e_i^*$ ($i = 1,2,3$) represent the periodic orbits where a change of a configuration takes place along the family.

### Table 3: Configurations of families of symmetric periodic orbits in the 1:2:3 resonant chain.

| Family | $\varpi_1$ | $\varpi_2$ | $\varpi_3$ | $M_1$ | $M_2$ | $M_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\phi_L$ | $e_1^*$ | $e_2^*$ | $e_3^*$ |
|--------|------------|------------|------------|------|------|------|------------|------------|------------|--------|-------|-------|-------|
| $S_1$  | 0          | $\pi$      | 0          | 0    | $\pi$  | 0    | 0          | 0          | 0          | 0      | 0.064725 | 0.0026077 | 0.0   |
|        | 0          | $\pi$      | 0          | 0    | $\pi$  | 0    | 0          | 0          | 0          | 0      | 0.0991335 | 0.0     | 0.0006738 |
|        | 0          | 0          | 0          | 0    | 0      | 0    | 0          | 0          | 0          | 0      | 0.2391378 | 0.0     | 0.0581071 |
|        | $\pi$      | $\pi$      | $\pi$      | $\pi$ | 0      | 0    | $\pi$      | $\pi$      | $\pi$      | 0      | 0          | 0.0     | 0.2666913 |
| $S_2$  | 0          | 0          | $\pi$      | 0    | 0      | 0    | 0          | 0          | 0          | 0      | 0.3948695 | 0.3015711 | 0.0   |
|        | 0          | 0          | 0          | 0    | 0      | 0    | 0          | 0          | 0          | 0      | 0          | 0.0     | 0       |
| $S_3$  | $\pi$      | 0          | $\pi$      | 0    | 0      | 0    | $\pi$      | 0          | 0          | 0      | 0          | 0       | 0       |
| $S_4$  | $\pi$      | 0          | 0          | 0    | 0      | 0    | 0          | 0          | 0          | 0      | 0          | 0       | 0       |
| $S_5$  | $\pi$      | 0          | $\pi$      | 0    | 0      | 0    | $\pi$      | 0          | 0          | 0      | 0          | 0       | 0       |
| $S_6$  | 0          | $\pi$      | 0          | 0    | 0      | 0    | $\pi$      | $\pi$      | 0          | 0      | 0          | 0       | 0       |
| $S_7$  | 0          | 0          | $\pi$      | 0    | 0      | 0    | 0          | 0          | 0          | 0      | 0.0       | 0.0682943 | 0.2815644 |
|        | $\pi$      | 0          | 0          | 0    | 0      | 0    | $\pi$      | 0          | 0          | 0      | 0.1066329 | 0.0     | 0.343046 |
|        | $\pi$      | $\pi$      | $\pi$      | $\pi$ | 0      | 0    | $\pi$      | $\pi$      | $\pi$      | 0      | 0          | 0       | 0       |
| $S_8$  | 0          | 0          | $\pi$      | 0    | 0      | 0    | $\pi$      | 0          | 0          | 0      | 0.0       | 0.1101254 | 0.488658 |
|        | 0          | $\pi$      | 0          | 0    | $\pi$  | 0    | 0          | 0          | 0          | 0      | 0.1764644 | 0.0     | 0.5815609 |
| $S_9$  | 0          | 0          | $\pi$      | 0    | 0      | 0    | $\pi$      | 0          | 0          | 0      | 0          | 0       | 0       |

### 3.2. DS-maps and constraints for Kepler-51

Discerning distinct regions of regular and chaotic motion in phase space can sometimes be demanding for Hamiltonian systems of 5 degrees of freedom (planar 3HB). The use of chaotic indicators can assist in deciphering such domains. However, all indicators have their pros and cons (see e.g. Maffione et al. 2011, for a comparison between the Lyapunov Indicator (LI), the Mean Exponential Growth factor of Nearby Orbits (MEGNO), the Smaller Alignment Index (SALI), the FLI and the Relative Lyapunov Indicator (RLI)). In this paper, we use a version of the FLI (called DFLI; see Appendix B.2 for its definition). Concerning dynamical systems like the planetary ones, the DFLI was established as efficient and reliable by Voyatzis (2008).

With regard to the computation of the DS-maps, we constructed $100 \times 100$ grid planes and focused on the families $S_1$, $S_5$ and $S_6$, i.e. we chose specific orbital elements from the symmetric periodic orbits that belong to these families (their angles are reported in Table 3), since their stable segments are closer to the observational eccentricity values of Kepler-51. We remind the reader that the chosen families of periodic orbits have been computed for the masses of Kepler-51 (identified by label 1 (namely Masuda 2014)) in Fig. 1 and shown in full extent in Fig. 2) and for the 1:2:3 resonant chain. We note also that the values of the semimajor axes vary slightly along the families, but the resonance remains almost constant (see Appendix A.3 for the differences between these elliptic, $S_i$ families and the circular family along which the resonance varies). More precisely, on each grid plane we vary a pair of the orbital elements, while keeping the rest of the orbital elements and masses of the periodic orbit fixed. In the following, we mention the orbital elements of the selected periodic orbits used for the construction of each DS-map and justify their selection.

In order to classify the orbits, we chose a maximum integration time for the computation of the DFLI equal to $t_{\text{max}} = 3\text{Myr}$, which corresponds approximately to 24 million orbits of the innermost planet $b$ or 8.5 million orbits of the outermost planet $d$ and was deemed appropriate, in order to distinguish chaos from order reliably for the particular application. We used the Bulirsch–Stoer integrator with tolerance $10^{-14}$. For small values of DFLI (dark-colored domains), regular evolution of the planets is expected. In our study, we stop the integrations either when DFLI($t$) > 30 or when $t_{\text{max}}$ is reached.

Article number, page 6 of 18
In Fig. 3 we present DS-maps on the \((e_1, e_2)\) and \((e_2, e_3)\) planes, where the family \(S_6\) is projected, and the \((\Delta \varpi_{21}, M_{21})\) and \((\Delta \varpi_{32}, M_{32})\) planes. Kepler-51 is identified by the magenta ‘+’ symbol, while the errors are delineated by magenta dashed lines. The color bar illustrates the DFLI values; dark colors correspond to regular evolution, while pale ones signify chaoticity. \(R_L\) indicates that both pairs of planets are locked in a two-body MMR (2/1 and 3/2 MMR), while \(R_{S,L}\) indicates an 1/1 secondary resonance inside the 2/1 MMR for the inner pair and a locking in the 3/2 MMR for the outer pair of planets.

In Fig. 4 we present DS-maps by varying the eccentricities on the planes \((e_1, e_2)\) and \((e_2, e_3)\) (top panels) and \((\Delta \varpi_{21}, M_{21})\) and \((\Delta \varpi_{32}, M_{32})\) (bottom panels). The rest of the orbital elements, which remain fixed for each of the initial conditions on the DS-maps, are selected from a planar stable (blue) periodic orbit of family \(S_5\) being closer to the observational values \(e_c\) and \(e_d\) (errors counted in). Its orbital elements are: \(a_1 = 1.04268089, a_2 = 1.65662643, a_3 = 2.17192389, e_1 = 0.0075847, e_2 = 0.0150072, e_3 = 0.0080335, \varpi_1 = 0^\circ, \varpi_2 = 180^\circ, \varpi_3 = 0^\circ, M_1 = 0^\circ, M_2 = 0^\circ\) and \(M_3 = 180^\circ\), with configuration \((\theta_1, \theta_2, \theta_3, \theta_4) = (0, \pi, \pi, 0)\).
Fig. 4: DS-maps on the eccentricity planes \((e_1,e_2)\) and \((e_2,e_3)\) where the family \(S_5\) is projected. \(R_L\) denotes a two-body MMR (2/1 and 3/2 MMR for the inner and outer pairs of planets), \(R_T\) indicates a three-body Laplace-like resonance (1:2:3 resonant chain), while \(R_{S,A}\) indicates an 1/1 secondary resonance inside the 2/1 MMR for the inner pair and an apsidal difference oscillation/rotation observed for the outer pair of planets. Colors and lines as in Fig. 3.

Fig. 5: DS-maps on the eccentricity planes \((e_1,e_2)\) and \((e_2,e_3)\), where the family \(S_1\) is projected. \(R_{S,A}\) indicates an 1/1 secondary resonance inside the 2/1 MMR for the inner pair and an apsidal difference oscillation/rotation observed for the outer pair of planets. Colors and lines as in Fig. 3.

\[
\begin{align*}
  a_3 &= 2.05151434, \quad e_1 = 0.0025747, \quad e_2 = 0.0050208, \quad e_3 = 0.0030763, \\
  \varpi_1 &= 180^\circ, \quad \varpi_2 = 180^\circ, \quad \varpi_3 = 0^\circ, \quad M_1 = 180^\circ, \\
  M_2 &= 180^\circ \text{ and } M_3 = 0^\circ, \quad \text{which correspond to the configuration } (\theta_1, \theta_2, \theta_3, \theta_4) = (\pi, \pi, 0, \pi).
\end{align*}
\]
Fig. 6: DS-maps that showcase the extent of the 2/1 (left panels) and the 3/2 (right panels) MMRs in relation to each eccentricity value. The distinct ‘$R_L$-regions’ appear at the values $\frac{a_2}{a_1} \approx 1.58$ and $\frac{a_3}{a_2} \approx 1.31$. The magenta lines showcase the nominal values of each MMR. The system Kepler-51 (magenta dashed lines enclosing the ‘+’ symbol) is found at $\frac{a_c}{a_b} \approx 1.527$ and $\frac{a_d}{a_c} \approx 1.325$, where an apsidal resonance takes place. Colors and lines as in Fig. 3.

Likewise, in Fig. 5 we present DS-maps by varying the eccentricities on the planes $(e_1,e_2)$ and $(e_2,e_3)$. The rest of the orbital elements, which remain fixed for each of the initial conditions on the DS-maps, are selected from a planar stable periodic orbit of family $S_6$, where $e_1 \approx e_3$. Its orbital elements are: $a_1 = 1.02641224$, $a_2 = 1.62959097$, $a_3 = 2.13398349$, $e_1 = 0.0400329$, $e_2 = 0.0030374$, $e_3 = 0.0000763$, $\varpi_1 = 0^\circ$, $\varpi_2 = 180^\circ$, $\varpi_3 = 0^\circ$, $M_1 = 0^\circ$, $M_2 = 180^\circ$ and $M_3 = 0^\circ$, with configuration $(\theta_1,\theta_2,\theta_3,\theta_4) = (0, \pi, 0, \pi)$.

In Fig. 6, we provide the DS-maps that showcase the extent of each resonance (the 2/1 MMR in the left panels and the 3/2 MMR in the right ones) in relation to each planetary eccentricity value. We were guided by the same periodic orbit of the family $S_6$ that was used in Fig. 3 in the configuration $(\theta_1,\theta_2,\theta_3,\theta_4) = (0, \pi, 0, \pi)$. The observational values of the semimajor axes and eccentricities are denoted by the magenta ‘+’ symbol.

Fig. 7: Evolution of the orbital elements and resonant angles along an orbit initiated by the non-resonant low-eccentricity orbits showcasing an apsidal resonance found in Fig. 6.
Fig. 8: Evolution of the orbital elements and resonant angles along orbits initiated by the ‘RL-region’ (left panel) and the ‘RS,L-region’ (right panel) dynamically unveiled by family S₆ shown in Fig. 3.

3.2.1. Apsidal resonance for non-resonant low-eccentricity orbits

At the low-eccentricity non-resonant orbits presented in Fig. 6, i.e. away from the distinct ‘RL-regions’ found around the values $a_2 \approx 1.58$ and $a_3 \approx 1.31$, an apsidal resonance is found at the position of Kepler-51. In such an evolution shown in Fig. 7, all the resonant angles rotate, but the apsidal differences librate about $\pi$, given the configuration $(\theta_1, \theta_2, \theta_3, \theta_4)=(0, \pi, \pi, 0)$ of the periodic orbit from the family $S_6$ used.
3.2.2. Two-body MMRs, $R_L$

The dynamical mechanism of two independent two-body MMRs, namely the 2/1 for the innermost planets and the 3/2 for the outermost ones, was encountered in the neighborhood of families $S_6$ and $S_5$ ($R_L$ symbols on the DS-maps in Figs. 3 and 4). The evolutions of the orbital elements and the resonant angles from the initial conditions taken from these two $R_L$-regions are demonstrated in the left panels of Figs. 8 and 9. In the former case, the resonant angles, $(\theta_1, \theta_2, \theta_3, \theta_4)$, librate about $(0, \pi, \pi, 0)$, while in the latter, the resonant angles librate about $(\pi, \pi, 0, \pi)$.

Regarding the family $S_5$, this mechanism is stronger the closer we get to the family, as the libration widths of the resonant angles are very small. Therefore, we may conjecture that the two-body MMRs cannot act as a safeguard...
for the regular evolution of Kepler-51 when found in its dynamical vicinity.

In contrary, the family $S_6$ may provide a region that could host Kepler-51 for long-time spans should the pairs be in two-body MMRs. More precisely, the dark-colored regular domain on the $(e_2, e_3)$ plane of Fig. 3 almost engulfs the observational eccentricities, $e_c$ and $e_d$, along with their deviations. Given this overlap, a constraint should be imposed on the eccentricity $e_b$, since the ‘$R_L$-region’ takes place when $e_1 < 0.02$, which is by far lower than the lowest deviation boundary (magenta dashed line) on the $(e_1, e_2)$ plane of Fig. 3. Additionally, guided by the same stable periodic orbit, the boundaries for the mean anomalies and apsidal differences can be deduced by the bottom panels of Fig. 3.

Fig. 10: Evolution of the orbital elements and resonant angles along orbits initiated by the ‘$RS_A$-regions’ dynamically unveiled by families $S_5$ (left panel) and $S_1$ (right panel) shown in Figs. 4 and 5 respectively.
3.2.3. Three-body Laplace-like resonance, $R_T$

The dynamical mechanism of a three-body Laplace-like resonance, namely the 1:2:3 resonant chain, was encountered in the neighborhood of family $S_3$ ($R_T$ symbols in the left panel of Fig. 4). The evolution of the orbital elements and the resonant angles, with initial conditions taken from the ‘$R_T$-region’ within the magenta dashed lines, is shown in the right panel of Fig. 4, where the resonant angles rotate, but the Laplace angle librates about $0°$.

We may cautiously impose further constraints on the observational eccentricities for the 1:2:3 resonant chain to be long-term viable, by taking into account the regular domains (‘$R_T$-regions’) and the overlapping deviations (magenta dashed lines). More particularly, $e_c < 0.016$ (deduced from the regular ‘$R_T$-region’ of left panel of Fig. 4), while $e_d < 0.006$ (which is the highest value of $e_3$ in the stable segment of the family $S_6$).

3.2.4. Combination of secondary resonance, two-body MMR and apsidal difference oscillation, $R_{S,L}$ and $R_{S,A}$

The dynamical mechanism of an 1/1 secondary resonance inside the 2/1 MMR for the innermost planets and a locking in the 3/2 MMR for the outermost ones, was encountered in the neighborhood of family $S_0$ ($R_{S,L}$ symbols in the top left panel of Fig. 4). The evolution of the orbital elements and the resonant angles, with initial conditions taken from the ‘$R_{S,L}$-region’ within the magenta dashed lines, is shown in the right panel of Fig. 8 where a libration is observed for $\theta_1$ about $0°$, $\theta_2$ about $180°$, $\theta_4$ about $0°$ and $\Delta\pi_{32}$ about $180°$ and a rotation for the rest of angles.

The regular domain exhibiting this mechanism is particularly thin and the region delineated by the deviations of observational eccentricities (magenta dashed lines) is populated mainly by chaotic orbits. Hence, we do not put any constraints on Kepler-51 stemming from the ‘$R_{S,L}$-region’ dynamically unveiled by the family $S_0$, since such an evolution may not be probable.

The dynamical mechanism of the 1/1 secondary resonance inside the 2/1 MMR for the innermost planets and an apsidal difference oscillation/rotation observed for the outer pair planets, was encountered in the neighborhood of the families $S_0$ and $S_1$ ($R_{S,A}$ symbols in Figs 4 and 5). The evolution of the orbital elements and the resonant angles, with initial conditions taken from the ‘$R_{S,A}$-region’ within the magenta dashed lines of Figs. 4 and 5 is shown in the left and right panels of Fig. 10 respectively. In the ‘$R_{S,A}$-region’ originating from the family $S_0$, $\theta_1$ librates about $180°$ and $\Delta\pi_{32}$ oscillates about $180°$, following the configuration of the chosen periodic orbit which was used for the DS-map construction. As for the ‘$R_{S,A}$-region’ established by the periodic orbit of the family $S_1$, $\theta_1$ librates about $0°$, while $\Delta\pi_{32}$ alternates between large amplitude oscillations about $0°$ and circulations, showing the chaotic nature of this motion.

Regarding the ‘$R_{S,A}$-region’ in the DS-map of Fig. 4, we observe that $e_c$ and $e_d$ along with their deviations fall mainly on the unstable (red colored) periodic orbits. Therefore, the long-term stability may be possible but not probable within these very small regular domains.

Concerning the ‘$R_{S,A}$-region’ in the DS-maps of Fig. 5, we observe that $e_b$ and $e_c$ along with their deviations fall entirely on the regular domain. As a result, we performed a search for possible boundaries for the mean anomalies and apsidal differences guided by the same stable periodic orbit used for the DS-maps on the eccentricity planes. We found that essentially all values in the domain $[0, 180]$ yield a regular $R_{S,A}$ mechanism. However, we remind that the above hold for eccentricity values $e_d \approx 0$ (see family $S_1$).

4. Conclusions

Motivated by the increasing number of exoplanets evolving in (or close to) MMRs and resonant chains, we analyzed the long-term orbital stability of the system Kepler-51. We presented novel results regarding the 1:2:3 resonant symmetric periodic orbits of the G4BP, which was used as a model to provide hints on the dynamics of the three planets.

For the planetary masses of Kepler-51, only four families were found to possess stable segments in the low-eccentricity dynamical neighborhood of the system. These families are in the symmetric configurations $(\theta_1, \theta_2, \theta_3, \theta_4) = (0, \pi, 0, \pi)$ and $(0, \pi, \pi, \pi)$ (family $S_1$), $(\pi, 0, 0, 0)$ (family $S_4$), $(\pi, \pi, 0, \pi)$ (family $S_5$) and $(0, 0, \pi, 0)$ (family $S_6$).

Guided by the stable symmetric periodic orbits of these families, we computed DS-maps, which basically unraveled the three main dynamical mechanisms which secure the long-term orbital stability of the system Kepler-51. More precisely, the 1/1 and 3/2 two-body MMRs (denoted by $R_L$), the 1:2:3 three-body Laplace-like resonance (denoted by $R_T$) and a combination of mechanisms for the inner and outer pairs separately, i.e. the 1/1 secondary resonance inside the 2/1 MMR for the inner pair with either a 3/2 MMR or an apsidal difference oscillation for the outer pair (denoted by $R_{S,L}$ and $R_{S,A}$, respectively).

Based on the regular domains demonstrated in the DS-maps, we put possible constraints on the observational eccentricities, mean anomalies and apsidal differences. For the first scenario in the ‘$R_L$-region’, we concluded that $e_d < 0.02$ and presented possible islands of stability for the angles, so that such two-body MMRs last long-time spans. For the second scenario in the ‘$R_T$-region’, we found that $e_c < 0.016$ and $e_d < 0.006$ for such a chain to be viable. For the third scenario, we deduced that an evolution comprising an 1/1 secondary resonance with a 3/2 MMR (‘$R_{S,L}$-region’) may be possible but not probable. Additionally, we showed that an evolution in an ‘$R_{S,A}$-region’, although of chaotic nature, fits very well with the observational eccentricities $e_b$ and $e_c$ (and almost any value for the mean anomalies and the apsidal differences) as long as $e_d \approx 0$.

With regard to two-planet systems, many fitting methods have been developed and dynamical analyses have been performed for giant planets locked in MMRs in tandem with migration simulations (see e.g. Hadden & Payne 2020). An efficient fitting of the observational data for systems of three planets in (or near) a resonance is beyond any question. Aiming to the optimum deduction of the orbital elements, this study exemplified the need for dynamical analyses based on periodic orbits being performed in parallel to the data fitting methods.

Asymmetric configurations in resonant chains may also exist, but indications of such stability domains in three-planet systems were found in moderate eccentricity values when Jovian masses were assumed for the three planets (Voyatzis 2016).
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Article number, page 14 of 18
Appendix A: Hadjidemetriou’s approach of the planar 4-Body Problems

A.1. Equations of Motion

Let us first consider a system of 4 bodies, \( p_i \) (\( i = 0, \ldots, 3 \)), with masses \( m_i \) (\( i = 0, \ldots, 3 \)), respectively, which move under their mutual gravitational attraction in the inertial frame \( XGY \), where \( G \) is their center of mass and \( \mathbf{R}_i \) the position vector with respect to \( G \). Then, we define a rotating frame of reference, \( xOy \), whose origin, \( O \), is the center of mass of \( p_0 \) and \( p_1 \) which move on the \( Ox \)-axis (positive direction from \( p_0 \) to \( p_1 \)). The position vector for each body in this frame with respect to \( O \) is \( \mathbf{r}_i \), while \( x_1, x_2, x_3, y_2, y_3 \) are the relative coordinates, \( \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_2, \dot{y}_3 \) are the relative velocities, \( \theta \) is the angle between the \( Ox \) and \( GX \) axes and \( \dot{\theta} \) the angular velocity. The motion takes place on a plane, while the planes \( XGY \) and \( xOy \) coincide. In this work, subscript 0 refers to the Star, while the total mass of the system, \( m = \sum_{i=0}^{3} m_i \), and the gravitational constant, \( G \), are normalized to unity, so that \( m_0 = 1 - m_1 - m_2 - m_3 \) and \( G = 1 \), respectively.

Given the above we have

\[
\mathbf{r} = \frac{1}{m_0 + m_1} \sum_{i=0}^{3} m_i \mathbf{R}_i \tag{A.1}
\]

with \( \mathbf{r}_i = \overrightarrow{OG} \) and

\[
x_0 = -qx_1 \tag{A.2}
\]

where \( q = m_1/m_0 \). Equivalently, we can define the position of \( p_0 \) and \( p_1 \) as

\[
x_0 = -(1 - \mu_0_1)r \\
x_1 = \mu_0_1r \tag{A.3}
\]

where \( \mu_0_1 = m_0/(m_0 + m_1) \) and \( r = x_1 - x_0 \).

The Lagrangian of such a system in the General 4-Body Problem (G4BP) is

\[
L = \frac{1}{2}m_0 \left[ \frac{m_0}{m_1} + m_0 \right] \dot{x}_1^2 + \frac{1}{2}m_2(1 - m_2)(\dot{x}_2^2 + \dot{y}_2^2) + \\
\frac{1}{2}m_3(1 - m_3)(\dot{x}_3^2 + \dot{y}_3^2) - m_2m_3(\dot{x}_2\dot{x}_3 + \dot{y}_2\dot{y}_3) + \\
\frac{1}{2}m_0m_1 \frac{m_0}{m_1 + m_0} \dot{x}_1^2 + m_2(1 - m_2)(\dot{x}_2^2 + \dot{y}_2^2) + \\
m_3(1 - m_3)(\dot{x}_3^2 + \dot{y}_3^2) - 2m_2m_3(\dot{x}_2\dot{x}_3 + \dot{y}_2\dot{y}_3) + \\
\dot{\theta}m_2(1 - m_2)(\dot{y}_2x_2 - \dot{x}_2y_2) + \\
m_3(1 - m_3)(\dot{y}_3x_3 - \dot{x}_3y_3) + \\
m_2m_3(\dot{x}_2y_2 + \dot{x}_3y_3 - \dot{y}_2x_2 - \dot{y}_3x_3) \right] - V \tag{A.4}
\]

where \( V = \sum_{i \neq j} \sum_{k} \frac{m_i m_j}{r_{ij}} \) with \( r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \) (\( i, j = 0, \ldots, 3 \)).

It is evident from Eq. A.4 that \( \dot{\theta} \) is an ignorable variable \( (\frac{\partial L}{\partial \dot{\theta}} = 0) \) and hence, the angular momentum integral, \( P_{th} = \frac{\partial L}{\partial \theta} \), exists:

\[
P_{th} = \dot{\theta}(m_1\mu_0_1\dot{r}^2 + m_2(1 - m_2)(\dot{x}_2^2 + \dot{y}_2^2) + \\
m_3(1 - m_3)(\dot{x}_3^2 + \dot{y}_3^2) - 2m_2m_3(\dot{x}_2\dot{x}_3 + \dot{y}_2\dot{y}_3) + \\
m_2(1 - m_2)(\dot{y}_2x_2 - \dot{x}_2y_2) + \\
m_3(1 - m_3)(\dot{y}_3x_3 - \dot{x}_3y_3) + \\
m_2m_3(\dot{x}_2y_2 + \dot{x}_3y_3 - \dot{y}_2x_2 - \dot{y}_3x_3)) \tag{A.5}
\]

Therefore, we have a system of 5 degrees of freedom in the rotating frame with the following equations of motion:

\[
\ddot{x}_1 = \frac{-m_0 \mu_0_1 r^2}{r_{12}^3} \tag{A.5a}
\]

\[
\ddot{x}_2 = \frac{-m_0 \mu_0_1 r^2}{r_{12}^3} + m_2 \left( \ddot{x}_2 - \dot{\theta}^2 x_2 + \dot{\theta}^2 y_2 + B_2 + P_2 \right) \tag{A.5b}
\]

\[
\ddot{y}_2 = -\dot{\theta} x_2 - 2\ddot{x}_2 + \dot{\theta}^2 y_2 + C_2 + Q_2 \tag{A.5c}
\]

\[
\ddot{x}_3 = \frac{-m_0 \mu_0_1 r^2}{r_{13}^3} - m_3 \left( \ddot{x}_3 - \dot{\theta}^2 x_3 + \dot{\theta}^2 y_3 + B_3 + P_3 \right) \tag{A.5d}
\]

\[
\ddot{y}_3 = -\dot{\theta} x_3 - 2\ddot{x}_3 + \dot{\theta}^2 y_3 + C_3 + Q_3 \tag{A.5e}
\]

where

\[
\begin{align*}
A &= -\frac{m_1 + m_0}{r^2} + \\
B_2 &= -\frac{(1 - \mu_0_1)x_2 - \mu_0_1 r^2}{r_{12}} - m_2 \left( \ddot{x}_2 - \dot{\theta}^2 x_2 + \dot{\theta}^2 y_2 + B_2 + P_2 \right) \tag{A.7a}
\end{align*}
\]

while the quantity \( \dot{\theta} \) is found by differentiating Eq. A.5 with respect to time.

For our numerical computations in the G4BP, we set \( \dot{\theta}(0) = 1 \) and arbitrarily choose \( \theta(0) = 0 \) without loss of generality.

In the Restricted 4BP (R4BP), we consider the motion of the massless planet of \( p_1 \) (\( m_3 = 0 \)), which does not affect the motion of the other three main bodies (see e.g. Hadjidemetriou 1980). By taking the limit \( m_3 \to 0 \) in the equations of motion (Eq. A.6), Eqs. A.6d and A.6e uncouple from Eqs. A.6a-A.6c, which now constitute the equations of motion of the 3-body problem, and so we obtain the equations of motion in the R4BP.
A.2. Symmetric periodicity conditions and continuation methods

An orbit $\mathbf{X}(t) = (x_1(t), x_2(t), x_3(t), y_2(t), y_3(t), \dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t), \dot{y}_2(t), \dot{y}_3(t))$ is periodic of period $T$ in the G4BP if it satisfies the periodic conditions:

\[
\begin{align*}
\dot{x}_1(T) &= \dot{x}_1(0) = 0, \\
\dot{x}_2(T) &= \dot{x}_2(0), \\
\dot{x}_3(T) &= \dot{x}_3(0), \\
\dot{y}_2(T) &= \dot{y}_2(0), \\
\dot{y}_3(T) &= \dot{y}_3(0).
\end{align*}
\tag{A.8}
\]

A periodic orbit is symmetric with respect to the $Oz$-axis of the rotating frame if it remains invariant under the fundamental symmetry $\Sigma$ (see e.g. Henon 1977):

\[
\Sigma : (t, x, y) \rightarrow (-t, x, -y),
\tag{A.9}
\]

that the system in Eq. A.3 follows. Following Hadjidemetriou & Michalodimitrakis [1981], we consider initial conditions on a Poincaré surface of section $y_2 = 0$. A symmetric periodic orbit crosses perpendicularly the section twice in one period and therefore the conditions

\[
y_3(0) = y_3(t^*) = 0, \quad \dot{x}_3(0) = \dot{x}_3(t^*) = 0,
\tag{A.10}
\]

where $t^*$ is the time of the first section cross, are sufficient for the periodicity of the orbit with period $T = 2t^*$. The rest non-zero initial conditions form the 5-dimensional space

\[
\Pi_5 = \{(x_1(0), x_2(0), x_3(0), y_2(0), y_3(0))\}
\tag{A.11}
\]

with $\dot{x}_1(0) = \dot{x}_3(0) = 0 = \dot{x}_2(0) = y_2(0) = y_3(0) = 0$.

In order to find a point in $\Pi_5$ that corresponds to a periodic orbit, we keep $x_1(0)$ fixed and we determine computationally the rest four conditions so that they satisfy Eq. A.10. Then, by varying $x_1(0)$ we compute a set of periodic orbits (a family) that forms a characteristic curve in $\Pi_5$. These characteristic curves are presented as projections in planes of initial conditions of the rotating frame or, after conversion, in planes of orbital elements.

The above continuation method can take place also by starting with zero planetary masses and then increasing them. However, it is proved that such a continuation does not hold when the period of the periodic orbit, $T$, is a multiple of $2\pi$ or the mean-motion resonance is of first-order. In particular, the continuation is not applied when $T = 2\pi k$, i.e. given the Eq. 2 at the mean-motion resonances

\[
\frac{T_1}{T_2} = \frac{k - P}{k}, \quad \frac{T_1}{T_3} = \frac{k - Q}{k}, \quad k \in \mathbb{Z}^*.
\tag{A.12}
\]

The proof of the continuation methods for the $N$-body problems, along with the above mentioned exceptions, can be found in Hadjidemetriou [1976, 1977].

A.3. Continuation of periodic orbits in the 1:2:3 resonant chain

In our study, we started from the degenerate (unperturbed) case, where all bodies move on circular Keplerian orbits and all masses are equal to zero. In this case, the resonant periodic orbits in the 1:2:3 resonant chain have a period $T = 12\pi$ (see Eq. 2).

In Fig. A.1 we show the gaps that are formed at the first-order resonances when $m_i \neq 0$ ($i = 1, 2, 3$), between the bifurcation points (blue circles) of the circular family, along which the resonance varies, in the unperturbed case (found at $e_1 = 0$) and some of the families in the G4BP (found at $e_1 \cos \theta_1 \neq 0$). In other words, when we switched on the masses and performed a continuation with respect to the mass until the observational mass value for each planet of Kepler-51 was reached, the periodic orbits in the G4BP were not connected smoothly with the ones in the unperturbed case, since the continuation with respect to the mass cannot be applied (see Eq. A.12). Then, we followed the continuation method which alters the $x_2$ variable of the system by keeping these mass values fixed. The families in the G4BP are deflected from the resonant periodic orbit in the unperturbed case, while the resonance along them remains almost constant. This way, we obtained the families of symmetric periodic orbits in the 1:2:3 resonant chain up to high eccentricity values illustrated in Fig. 2. In Poincaré’s terminology, these are the periodic orbits of second kind.\footnote{The periodic orbits of first kind are the ones of planetary type (non-zero masses) describing nearly circular motion.}

Fig. A.1: The circular family for the unperturbed case at $e_1 = 0$ and examples of families in the G4BP deflected from the resonant periodic orbits found by Eq. A.12 at $\left(\frac{T_1}{T}, \frac{T_2}{T}, \frac{T_3}{T}\right) = (\frac{5}{7}, \frac{4}{7}, \frac{3}{7}, \frac{2}{7})$ for $k = 3$, $(\frac{7}{4}, \frac{5}{4}, \frac{3}{4}, \frac{2}{4})$ for $k = 4$, and $(\frac{7}{4}, \frac{5}{4}, \frac{3}{4}, \frac{2}{4})$ for $k = 5$ (blue circles) computed for $m_1 = m_2 = m_3 = 10^{-6}$. 
ther at close encounters between at least one pair of planets or when the continuation method stalls, as the convergence to the periodicity conditions becomes very slow at very high eccentricity values.

We note that the bifurcation points of the circular family can generate different families of the same resonant chain, which differ in the phases of planetary configurations. These configurations correspond to the initial location at $t = 0$ of the 3 bodies, namely $p_1$, $p_2$ and $p_3$, on the x-axis and are reflected on the values of the longitudes of pericenter and the mean anomalies of the periodic orbits in the G4BP.

Appendix B: Orbital Stability

B.1. Linear stability of periodic orbits

Let us denote with the vector $x = (x_1, ..., x_{10})$ the set of 10 variables of the system $\{x_1, x_2, x_3, y_2, y_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_2, \dot{y}_3\}$. Then, the solution $x(t; x_0)$ will correspond to the initial conditions $x_0 = x(0)$. The variational equations of the system in Eq. A.6 and their solutions are written as

$$\dot{\eta} = J(t) \eta \Rightarrow \eta = \Delta(t) \eta_0$$

(B.1)

where $J$ is the Jacobian matrix of the system and $\Delta(t)$ the fundamental matrix of solutions (called also matrix or state transition matrix).

If the solution $x(t; x_0)$ corresponds to a periodic orbit of period $T$, $\Delta(T)$ is the monodromy matrix. If and only if all the eigenvalues (shown with red dots in Fig. A.2) of $\Delta(T)$ lie on the complex unit circle, the periodic orbit is classified as linearly stable and $\eta(t)$ remains bounded.

We remark that the eigenvalues are in conjugate pairs, as $\Delta(T)$ is symplectic. Moreover, due to the existence of the energy integral, one pair of eigenvalues (denoted here by $\lambda_1$ and $\lambda_2$) is always equal to unity. Some possible configurations of the other four pairs of eigenvalues and different types of instability are shown in Fig. A.2.

B.2. Chaotic Indicator

Telling whether the eigenvalues lie on the unit circle or not can sometimes become ambiguous, due to the limited accuracy. Therefore, apart from the linear stability along a family of periodic orbits, we also compute a chaotic indicator, and in particular a simple detrended Fast Lyapunov Indicator (DFLI, see e.g. [Voyatzis 2008]) defined as

$$DFLI(t) = \log \left( \frac{1}{t} \|\xi(t)\| \right),$$

(B.2)

where $\xi$ is the deviation vector computed after numerical integration of the variational equations. In Fig. B.1, we illustrate the DFLI’s behavior for the system Kepler-51. For a stable periodic orbit its evolution remains almost constant over time and takes small values, whereas it increases exponentially when the orbit is unstable.

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Fig. A.2: Examples of distribution of eigenvalues (red dots) with respect to the complex unit circle with a magnification on the right of each panel for particular orbits from the families of the 1:2:3 resonant chain. (a) Triple instability with three pairs of real eigenvalues (family $S_1$). (b) Double instability with two pairs of real eigenvalues (family $S_3$). (c) u-complex instability with one pair of real and two pairs of complex eigenvalues outside the unit circle (family $S_5$). (d) Complex instability with two pairs of complex eigenvalues outside the unit circle, while the rest remain on it (family $S_5$). (e) Linear stability with all pairs on the unit circle (family $S_1$).
Fig. B.1: Evolution of DFLI for an unstable (red) and a linearly stable (blue) periodic orbit with eigenvalues shown in panel (a) and (e) in Fig. A.2, respectively.