A new phenomenological Investigation of KMR and MRW
unintegrated parton distribution functions

M. Modarres†, H.Hosseinkhani†, N.Olanj‡, and M.R.Masouminia†

†Physics Department, University of Tehran, 1439955961, Tehran, Iran.
‡Plasma and Fusion Research School, Nuclear Science and Technology Research Institute, 14395-836, Tehran, Iran.
and
♭Physics Department, Faculty of Science, Bu-Ali Sina University, 65178, Hamedan, Iran.

Abstract

The longitudinal proton structure function, $F_L(x, Q^2)$, from the $k_t$ factorization formalism by using the unintegrated parton distribution functions (UPDF) which are generated through the KMR and MRW procedures. The LO UPDF of the KMR prescription is extracted, by taking into account the PDF of Martin et al, i.e. MSTW2008-LO and MRS T99-NLO and next, the NLO UPDF of the MRW scheme is generated through the set of MSTW2008-NLO PDF as the inputs. The different aspects of $F_L(x, Q^2)$ in the two approaches, as well as its perturbative and non-perturbative parts are calculated. Then the comparison of $F_L(x, Q^2)$ is made with the data given by the ZEUS and H1 collaborations. It is demonstrated that the extracted $F_L(x, Q^2)$ based on the UPDF of two schemes, are consistent to the experimental data, and by a good approximation, they are independent to the input PDF. But the one developed from the KMR prescription, have better agreement to the data with respect to that of MRW. As it has been suggested, by lowering the factorization scale or the Bjorken variable in the related experiments, it may be possible to analyze the present theoretical approaches more accurately.

PACS numbers: 12.38.Bx, 13.85.Qk, 13.60.-r

Keywords: unintegrated parton distribution, DGLAP equation, splitting function

*Corresponding author, Email : mmodares@ut.ac.ir, Tel:+98-21-61118645, Fax:+98-21-88004781
I. INTRODUCTION

In recent years, the extraction of unintegrated parton distribution functions (UPDF) have become very important, since there exists plenty of experimental data on the various events, such as the exclusive and semi-inclusive processes in the high energy collisions in LHC, which indicates the necessity for computation of these \( k_T \)-dependent parton distribution function.

The \( UPDF, f_a(x, k_t^2, \mu^2) \), are the two-scale dependent functions, i.e. \( k_t^2 \) and \( \mu^2 \), which satisfy the Ciafaloni-Catani-Fiorani-Marchesini (CCFM) equations \[1-5\], where \( x, k_t \) and \( \mu \) are the longitudinal momentum fraction (the Bjorken variable), the transverse momentum and the factorization scale, respectively. They are unintegrated over \( k_t \) with respect to the conventional parton distributions (PDF) which satisfy the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations \[6-9\].

But the generation of \( UPDF \) from the CCFM equations is a complicated task. So, in general, the Monte Carlo event generators \[10-17\] are the main users of these equations. Since there is not a complete quark version of the CCFM formalism, the alternative prescriptions are used for producing the quarks and the gluons \( UPDF \). Therefore, to obtain the \( UPDF, \) Kimber, Martin and Ryskin (KMR) \[18, 19\] proposed a different procedure based on the standard DGLAP equations in the leading order (LO) approximation, along with a modification due to the angular ordering condition, which is the key dynamical property of the CCFM formalism. Later on, Martin, Ryskin and Watt (MRW) extended the KMR approach for the next-to-leading order (NLO) approximation \[20-22\], with this aim to improve the exclusive calculations. These two procedures are the modifications to the standard DGLAP evolution equations and can produce the \( UPDF \) by using the PDF as the inputs.

The general behavior and stability of the KMR and MRW prescriptions were investigated in the references \[24, 28\]. Furthermore, to check the reliability of generated \( UPDF \), their relative behaviors were compared and used to calculate the observable, deep inelastic scattering proton structure function \( F_2(x, Q^2) \). Then the predictions of these two methods for the structure functions, \( F_2(x, Q^2) \), were also compared to the electron-proton deep inelastic measurements of NMC \[29\], ZEUS \[30\] and \( H1 + ZEUS \[31\] experimental data. The results were promising \[32\]. It is also concluded that \[32\], while the MRW formalism is
more compliance with the DGLAP evolution equations requisites, but it seems in the KMR case, the angular ordering constraint spreads the UPDF to whole transverse momentum region, and makes the results to sum up the leading DGLAP and Balitski-Fadin-Kuraev-Lipatov (BFKL) logarithms [34–38].

Another important observable quantity in this connection is the longitudinal structure function, i.e. $F_L(x, \mu^2)$, which is proportional to the cross section of the longitudinal polarized virtual photon with proton. Particularly at small $x$, it is directly sensitive to the gluon distributions i.e. $g \rightarrow q \bar{q}$ process. Moreover its calculations in this region need the $k_t$ factorization formalism [39–43], which is beyond the standard collinear factorization procedure [44]. Recently, Golec-Biernat and Staśto (GS) have used the $k_t$ and collinear factorizations [39–43] as well as the dipole approach to generate the longitudinal structure function, but by using the DGLAP/BFKL re-summation method, developed by Kwiecinski, Martin and Stasto (KMS) [47], for calculation of the unintegrated gluon density at small $x$. They have parameterized the input non-perturbative gluon distribution such that they could get the best fit to the experimental proton structure function data [47].

On the experimental side, the longitudinal structure function has been measured by both the H1 [48, 49] and ZEUS [50, 51] collaborations at the DESY electron-proton collider HERA. The $Q^2$ ranges have been varied between 12 to 90 and 24 to 110 GeV$^2$ in each experiments, respectively.

As it was pointed out above, similar to our recent publication on $F_2(x, Q^2)$ [32], in the present paper, we intend to calculate $F_L(x, Q^2)$ by working in the the $k_t$-factorization scheme. But rather than KMS re-summation method pointed out above, the KMR and MRW [18–22] formalisms are used to predict the UPDF with the input PDF of the MRST99-NLO [52], MSTW2008-LO [53] and MSTW2008-NLO [53] which covers wide range of $(x, Q^2)$ plane. Then our results can be compared both with the experimental data as well as the theoretical KMS – GS presentation of $F_L(x, Q^2)$. So the paper is organized as follows: In the section II we give a belief review of the KMR and the MRW formalisms [18–22] for extraction of the UPDF form the phenomenological PDF [52, 53]. The formulation of $F_L(x, Q^2)$ based on the $k_t$-factorization scheme is given in the section III. Finally, the section IV is devoted to results, discussions and conclusions.
II. A BRIEF REVIEW OF THE KMR AND THE MRW FORMALISMS

The KMR and MRW ideas for generating the UPDF work as follows: Using the given integrated PDF as the inputs, the KMR and MRW procedures produce the UPDF as their outputs. They are based on the DGLAP equations along with some modifications due to the separation of virtual and real parts of the evolutions, and the choice of the splitting functions at leading order (LO) and the next-to-leading order (NLO) levels, respectively:

(i) In the KMR formalism [18, 19], the UPDF, $f_a(x, k_t^2, \mu^2)$ ($a = q$ and $g$), are defined in terms of the quarks and the gluons PDF, i.e.:

$$f_q(x, k_t^2, \mu^2) = T_q(k_t, \mu) \frac{\alpha_s(k_t^2)}{2\pi} \int_x^{1-\Delta} dz \left[ P_{qq}(z) \frac{x}{z} q \left( \frac{x}{z}, k_t^2 \right) + P_{qg}(z) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) \right],$$

and

$$f_g(x, k_t^2, \mu^2) = T_g(k_t, \mu) \frac{\alpha_s(k_t^2)}{2\pi} \int_x^{1-\Delta} dz \left[ \sum_q P_{gq}(z) \frac{x}{z} q \left( \frac{x}{z}, k_t^2 \right) + P_{gg}(z) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) \right],$$

respectively, where, $P_{aa'}(x)$, are the LO splitting functions, and the survival probability factors, $T_a(k_t, \mu)$, are evaluated from:

$$T_a(k_t, \mu) = \exp \left[ - \int_{k_t^2}^{\mu^2} \frac{\alpha_s(k_t^2)}{2\pi} \frac{dk_t^2}{k_t^2} \sum_{a'} \int_0^{1-\Delta} dz' P_{a'a}(z') \right].$$

The angular ordering condition (AOC) [54, 55], which is a consequence of coherent emission of gluons, on the last step of the evolution process [23], is imposed. The AOC determined the cut off, $\Delta = 1 - z_{max} = \frac{k_t}{\mu + k_t}$, to prevent $z = 1$ singularities in the splitting functions, which arises from the soft gluon emission. As it has been pointed out in the references [18, 19], the KMR approach has several main characteristics. The important one, is the existence of the cut off at the upper limit of the integrals, that makes the distributions to spread smoothly to the region in which $k_t > \mu$ i.e. the characteristic of the small $x$ physics, which is principally governed by the BFKL evolution [34–38]. This feature of the KMR, leads to the UPDF with the behavior very similar to the unified BFKL+DGLAP formalism [18, 19]. The UPDF based on the KMR formalism, have been widely used in the phenomenological calculations which depend on the transverse momentum [50–57].
(ii) In the MRW formalism \[20,22\], the similar separation of real and virtual contributions to the DGLAP evolution is done, but the procedure is performed at the NLO level i.e.,

\[
f_a^{NLO}(x, k_t^2, \mu^2) = \int_x^1 dz T_a(k^2, \mu^2) \frac{\alpha_s(k^2)}{2\pi} \sum_{b=q,g} P_{ab}^{(0+1)}(z) b^{NLO} \left( \frac{x}{z}, k^2 \right) \Theta(\mu^2 - k^2),
\]

where

\[
P_{ab}^{(0+1)}(z) = P_{ab}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{ab}^{(1)}(z), k^2 = \frac{k_t^2}{1-z}.
\]

In the equations (4) and (5) the \(P^{(0)}\) and the \(P^{(1)}\) denote the LO and the NLO contributions of the splitting functions, respectively. It is obvious from equation (4) that in the MRW formalism, the UPDF are defined such that to ensure \(k^2 < \mu^2\). Also, the survival probability factor, \(T_a(k^2, \mu^2)\), are obtained as follows:

\[
T_a(k^2, \mu^2) = \exp \left( -\int_{k^2}^{\mu^2} \frac{\alpha_s(k^2)}{2\pi} \frac{dk^2}{k^2} \sum_{b=q,g} \int_0^1 d\zeta \zeta P_{ba}^{(0+1)}(\zeta) \right),
\]

where \(P_{ab}^{(i)}\) (which is singular in the \(z \to 1\)) is given in the reference \[68\]. MRW have demonstrated that the sufficient accuracy can be obtained by keeping only the LO splitting functions together with the NLO integrated parton densities. So, by considering angular ordering, we can use \(P^{(0)}\) instead of \(P^{(0+1)}\). As it is mentioned above unlike the KMR formalism, where the angular ordering is imposed to the all of terms of the equations (1) and (2), in the MRW formalism, the angular ordering is imposed to the terms in which the splitting functions are singular, i.e. the terms that include \(P_{qq}\) and \(P_{gg}\).

III. THE FORMULATION OF \(F_L(x, Q^2)\) IN THE \(k_t\)-FACTORIZATION APPROACH

The \(k_t\)-factorization approach has been discussed in the several works i.e. references \[3, 39, 42, 69, 70\]. In the following equation \[45, 71, 73\], the different terms i.e the perturbative and the non-perturbative contributions to the \(F_L(x, Q^2)\) has been broken into the sum of gluons from the quark-box (the first term i.e. the \(k_t\) factorization part), see figure 1 \[22\], quarks (the second term) and the non-perturbative gluon (the third term) Parts:

\[
F_L(x, Q^2) = \left[ \frac{Q^4}{\pi} \sum_q e_q^2 \int \frac{dk_t^2}{k_t^4} \Theta(k^2 - k_t^2) \int_0^1 d\beta \int d^2 \kappa \alpha_s(\mu^2) \beta^2 (1 - \beta)^2 \left( \frac{1}{D_1} - \frac{1}{D_2} \right)^2 \right. \times
\]
\begin{equation}
\begin{aligned}
  f_g \left( \frac{x}{z}, k_t^2, \mu^2 \right) \Theta(1 - \frac{x}{z}) + 
  & \left[ \frac{4}{3} \int_x^1 \frac{dy}{y} \frac{\alpha_s(Q^2)}{\pi} \left( \frac{x}{y} \right)^2 F_2(y, Q^2) \right] + 
  \frac{\alpha_s(Q^2)}{\pi} \left[ \sum_q e_q^2 \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 y g(y, k_0^2) \right],
\end{aligned}
\end{equation}

where the second term is (see [74, 75]):

\begin{equation}
\sum_q e_q^2 \frac{\alpha_s(Q^2)}{\pi} \frac{4}{3} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 [q_1(y, Q^2) + \bar{q}_1(y, Q^2)].
\end{equation}

In the above equation, in which the graphical representations of $k_t$ and $\kappa_t$ have been introduced in the figure 1, the variable $\beta$ is defined as the light-cone fraction of the photon momentum carried by the internal quark [70]. Also, the denominator factors are:

\begin{align}
D_1 &= \kappa_t^2 + \beta(1 - \beta)Q^2 + m_q^2, \\
D_2 &= (\kappa_t - k_t)^2 + \beta(1 - \beta)Q^2 + m_q^2.
\end{align}

Then by defining $\kappa'_t = \kappa_t - (1 - \beta)k_t$, the variable $y$ takes the following form:

\begin{equation}
y = x(1 + \frac{\kappa_t^2 + m_q^2}{\beta(1 - \beta)Q^2}),
\end{equation}

and

\begin{equation}
\frac{1}{z} = 1 + \frac{\kappa_t^2 + m_q^2}{(1 - \beta)Q^2} + \frac{k_t^2 + \kappa^2 - 2k_t.k_t + m_q^2}{\beta Q^2}.
\end{equation}

As in the reference [47], the scale $\mu$ which controls the *unintegrated* gluon and the QCD coupling constant $\alpha_s$, is chosen as follows:

\begin{equation}
\mu^2 = k_t^2 + \kappa_t^2 + m_q^2.
\end{equation}

One should note that the coefficients used for quark and non-perturbative gluon contributions depend on the transverse momentum. As it has been briefly explained before, the main prescription for $F_L$ consists of three terms; the first term is the $k_t$ factorization which explains the contribution of the UPDF into the $F_L$. This term is derived with the use of pure gluon contribution. However, it only counts the gluon contributions coming from the perturbative region, i.e. for $k_t > 1$ GeV, and does not have anything to do with the non-perturbative contributions. In the reference [74], it has been shown that a proper
non-perturbative term can be derived from the $k_t$ factorization term, compacting the $k_t$ dependence and the integration with the use of a variable-change, i.e. $y$, that carries the $k_t$ dependence. Nevertheless, there is a calculable quark contribution in the longitudinal structure function of the proton, which comes from the collinear factorization, i.e. the second term of the equation \ref{eq:7}.

For the charm quark, $m$ is taken to be $m_c = 1.4 GeV$, and $u$, $d$ and $s$ quarks masses are neglected. We also use the same approximation to save the computation time \ref{19}, the one we did for the calculation of $F_2(x, Q^2)$ \ref{32} i.e the representative ”average” value for $\phi$, $\langle \phi \rangle = \frac{\pi}{4}$ for perturbative gluon contribution. This approximation has been checked in the reference \ref{19} (page 83). The rest of $\phi$ angular integration can be performed analytically by using a series of integral identities given in the reference \ref{76}. We will also verify this approximation in the next section. The unintegrated gluon distributions are not defined for $k_t$ and $\kappa_t < k_0$, i.e. the non-perturbative region. So, according to the reference \ref{71}, $k_0$ is chosen to be about one GeV which is around the charm mass in the present calculation, as it should be. On the other hand, one expects that the discrepancy between the $k_t$-factorization calculation and the experimental data can be eliminated by using the PDF, which have been fitted to the same data for $F_2(x, Q^2)$ \ref{77} with respect to the re-summation method of KMS \ref{47}.

IV. RESULTS, DISCUSSIONS AND CONCLUSIONS

In the figure 2, the longitudinal proton structure functions in the frameworks of KMR (left panels) and MRW (right panels) formalisms, by using the MRST99 \ref{52} and the MSTW2008 – NLO \ref{53} PDF inputs, versus $x$, for $Q^2=2, 4, 6, 12$ and 15 GeV$^2$ are plotted, respectively. Their total $F_L(x, Q^2)$ and the contributions from $k_t$ factorization scheme, the quarks and the no-perturbative parts (see the equation \ref{7}) are presented with different curve styles. The behavior of $F_L(x, Q^2)$ mostly comes from the $k_t$ factorization contribution especially as the $Q^2$ is increased and it is more sizable in case of MRW approach. By rising up the $Q^2$ values the contribution of the $k_t$-factorization becomes dominant. Another point is the decrease of non-perturbative parts at small $x$, in the case of MRW scheme. As we discussed in our pervious works, this is expected. Since the KMR constraint spreads the UPDF to the whole transverse momentum region \ref{32} and it sums up the both leading DGLAP
and BFKL logarithms contributions. The general behavior of two schemes in the figure 2 shows some differences also at lower $Q^2$ scales, while the values and behaviors of quarks and $k_t$-factorization portions in both formalisms are almost similar, the non-perturbative contributions have more different values and behavior in the $x \simeq 0.01$. The later point plays the main role in the discrepancies of the total $F_L(x,Q^2)$ at lower $Q^2$. On the other hand the non-perturbative contribution in each case remains almost fix through the variation of $Q^2$. These effects have root in the parent PDF sets at non-perturbative boundary which is very sensitive to the discipline and procedure of the PDF generating group. This figure can also be compared with the figure 2 of GS [45] at $Q^2=2,4$ and 6 GeV$^2$. There are general agreements between our approaches and those of GS, which have used the DGLAP/BFKL re-summation method, developed by Kwiecinski, Martin and Stasto (KMS) [47], for calculation of the unintegrated gluon density at small $x$. This agreement is more visible at larger $Q^2$ and in the KMR approach, which is expected. However our longitudinal proton structure function results go smoothly to zero with respect to those GS as $x$ becomes larger. The reason comes from both our input PDF, which is valid for the whole $(x,Q^2)$ plane, and the calculation of $UPDF$ which are calculated by using the KMR and MRW approaches, which are full fill the DGLAP requirements.

Our longitudinal proton structure function results for larger values of $Q^2$, with the different input PDF i.e. MERST99 [52], MSTW2008 – LO (using KMR formalism) and MSTW2008 – NLO [53] (using MRW formalism) are given the figures 3, 4 and 5, respectively. Again the total $F_L(x,Q^2)$ and the contributions from $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles. The results are mostly decreasing function $x$, for the various values of $Q^2$. There are sizable differences between the MERST99 and MSTW2008 – LO. On the other hand, as one should expect, for large value of $Q^2$ the results of the KMR and MRW behave more similarly. As we pointed out before, again the $k_t$ factorization contributions are dominant. The increase in the values of $F_L(x,Q^2)$ in the figure 4 is due to the increase of the input PDF at LO approximation. The reason that the results of $F_L(x,Q^2)$ approach to same values as $x$ and $Q^2$ increases, which is a heritage of the parent DGLAP evolution.

In order to analyze the above $Q^2$ dependent more clearly, in the figure 6, the longitudinal proton structure functions are plotted against $Q^2$ for two different values of $x = 0.001$ and 0.0001. Note, that for large $Q^2$, especially the MRW approach, needs large computation
time. So we have stopped at $Q^2 = 100 \text{ GeV}^2$ for this procedure. There are sizable differences between the two approaches and results coming from the two different input PDFs. But this should not be very important regarding the experimental data, that we will discuss later on.

In the figure 7, a comparison is made between the three different, $F_L(x, Q^2)$ results, namely KMR procedure with MERST99 and MSTW2008 − LO inputs and MRW scheme with MSTW2008 − NLO inputs. Especially there are large differences between KMR and MRW approaches at large $Q^2$. The above results can be directly compared that of GS [45] (see their figure 3). Very similar behavior is observed especially between the $k_t$ factorization approaches.

In the figures 8, 9 and 10, we present our results in the range of energy available in the H1 and ZEUS data [31], respectively. Note that for $Q^2 \geq 80 \text{ GeV}^2$, because of large computation time, we have only given four points (filled squares) for the MRW case. Very good agreements is observed between our result and those of experimental data at different $Q^2$ and x values. It seems with present existed data the UPDF of gluons generated with different input PDF and constraints procedures, one can reasonably explain the H1 and the ZEUS experimental data. It looks that even at low energies and small x values (see the figure 8); we find good agreement between our calculation and available data. However, as we mentioned before and it has been stated by several authors, the $F_L$ is mainly driven through the gluons distributions, especially at low values of x. The fact that $F_2$ is not accurately fit the data (see our previous work [32]), but we get good agreement between the $F_L$ calculations and H1 and ZEUS data, could be caused of the quark-quark contributions which has more contribution to $F_2$. Since $F_L$ is more sensitive to the gluons UPDF with respect to $F_2$. So one can conclude that present calculation can confirm that the KMR and MRW procedures (for generating the gluon UPDF) and the $k_t$-factorization scheme can reproduce reasonable $F_2$ (considering our previous work [32]) and present $F_L$. On the other hand, as we stated previously:(1) Present results also shows good agreement with the theoretical calculations of GS, which have used more complicated approach such as KMS. (2) It is interesting that the KMR and MRW UPDF can generate reasonable $F_L$ without using any free parameter in the $(x, Q^2)$-plane even at low $Q^2$ (regarding figure 8), especially the UPDF generated for gluons.

Finally, the verification of the fact that the $\phi$ integration of perturbative gluon contribution can be averaged by setting $<\phi> = \pi/4$, which was discussed in the end of previous
section, is presented in the figure 11, for four values of $Q^2 = 3.5, 12, 60$ and 110 Gev$^2$ by using the $KMR$ formalism and the $MRST99$. It is clearly seen that the above approximation does work properly and one can save much computation time.

In conclusion, the longitudinal proton structure functions, $F_L(x, Q^2)$, were calculated based on the $k_t$ factorization formalism, by using the $UPDF$ which are generated through the $KMR$ and $MRW$ procedures. The $LO UPDF$ of the $KMR$ prescription is extracted, by taking into account the $PDF$ of $MSTW2008-LO$ and $MRST99-NLO$ and also, the $NLO UPDF$ of the $MRW$ scheme is generated through the set of $MSTW2008-NLO PDF$ as the inputs. The different aspects of the $F_L(x, Q^2)$ in the two approaches, as well as its perturbative and non-perturbative parts were calculated and discussed. It was shown that our approaches are in agreement with those given $GS$. Then the comparison of $F_L(x, Q^2)$ was made with the data given by the $ZEUS$ and $H1$ collaborations at $HERA$. It was demonstrated that the extracted longitudinal proton structure functions based on the $UPDF$ of above two schemes, were consistent with the experimental data, and by a good approximation, they are independent to the input $PDF$. But as it was pointed out in our previous work [32], the one developed from the $KMR$ prescription, have better agreement to the data with respect to that of $MRW$. Although the $MRW$ formalism is in more compliance with the $DGLAP$ evolution equations requisites, but it seems in the $KMR$ case, the angular ordering constraint spreads the $UPDF$ to whole transverse momentum region, and makes the results to sum up the leading $DGLAP$ and $BFKL$ logarithms. At first, it seems that there should be a theoretical support for applying the angular ordering condition only to the diagonal splitting functions, in accordance with reference [22]. But as it has been mentioned in the references [32, 33], this phenomenological modifications of the $KMR$ approach (including the application of the $AOC$ to all splitting functions) works as an ”effective model” that spreads the $UPDF$ to the $k_t > \mu$ (a characteristic of low x physics) which enables it to represent a good level of agreement with the data. Beside this in our new work [33], in which we have calculated the $F_L$ in the dipole approximation according to the $LO$ prescription of reference [22], it is shown that there is not much difference if one applies the $AOC$ to the all splitting functions i.e. to use the $KMR UPDF$ instead of using $LO$ prescription of reference [22]. On the other hand, in this paper we have focused on comparison of the $LO$ and the $NLO$ calculation of $F_L$ and since the calculations are very time consuming we restricted the results to the $LO - KMR$ and $NLO - MRW$. 
As it has been suggested in the reference \cite{45}, by lowering the factorization scale or the Bjorken variable in the experimental measurements, it may be possible to analyze the present theoretical approaches more accurately.

Acknowledgments

$MM$ would like to acknowledge the Research Council of University of Tehran and Institute for Research and Planning in Higher Education for the grants provided for him.

\begin{thebibliography}{99}
\itemsep=0pt
\item[1] M. Ciafaloni, Nucl.Phys.B, \textbf{296} (1988) 49.
\item[2] S. Catani, F. Fiorani, and G. Marchesini, Phys.Lett.B, \textbf{234} (1990) 339.
\item[3] S. Catani, F. Fiorani, and G. Marchesini, Nucl.Phys.B, \textbf{336} (1990) 18.
\item[4] G. Marchesini, Proceedings of the Workshop QCD at 200 TeV Erice, Italy, edited by L. Cifarelli and Yu.L. Dokshitzer, Plenum, New York (1992) 183.
\item[5] G. Marchesini, Nucl.Phys.B, \textbf{445} (1995) 49.
\item[6] V.N. Gribov and L.N. Lipatov, Yad. Fiz., \textbf{15} (1972) 781.
\item[7] L.N. Lipatov, Sov.J.Nucl.Phys., \textbf{20} (1975) 94.
\item[8] G. Altarelli and G. Parisi, Nucl.Phys.B, \textbf{126} (1977) 298.
\item[9] Y.L. Dokshitzer, Sov.Phys.JETP, \textbf{46} (1977) 641.
\item[10] H. Kharraziha and L. L"onnblad, JHEP, \textbf{03} (1998) 006.
\item[11] G. Marchesini and B. Webber, Nucl.Phys.B, \textbf{349} (1991) 617.
\item[12] G. Marchesini and B. Webber, Nucl.Phys.B, \textbf{386} (1992) 215.
\item[13] H. Jung, Nucl.Phys.B, \textbf{79} (1999) 429.
\item[14] H. Jung and G.P. Salam, Eur.Phys.J.C, \textbf{19} (2001) 351.
\item[15] H Jung, J.Phys.G: Nucl.Part.Phys, \textbf{28} (2002) 971.
\item[16] H. Jung, et al., Eur.Phys.J.C, \textbf{70} (2010) 1237.
\item[17] H. Jung, M. Kraemer, A.V. Lipatov and N.P. Zotov, JHEP, \textbf{01} (2011) 085.
\item[18] M.A. Kimber, A.D. Martin and M.G. Ryskin, Phys.Rev.D, \textbf{63} (2001) 114027.
\item[19] M.A. Kimber, Unintegrated Parton Distributions, Ph.D. Thesis, University of Durham, U.K. (2001).
\end{thebibliography}
[20] A.D. Martin, M.G. Ryskin, G. Watt, Eur.Phys.J.C, 66 (2010)163.
[21] G. Watt, Parton Distributions, Ph.D. Thesis, University of Durham, U.K. (2004).
[22] G. Watt, A.D. Martin, and M.G. Ryskin, Eur.Phys.J.C, 31 (2003) 73.
[23] G. Watt, A.D. Martin, and M.G. Ryskin, Phys.Rev.D, 70 (2004) 014012.
[24] M. Modarres, H. Hosseinkhani, N. Olanj, Nucl.Phys.A, 902 (2013) 21.
[25] M. Modarres, H. Hosseinkhani, Few-Body Syst., 47 (2010) 237.
[26] M. Modarres, H. Hosseinkhani, Nucl.Phys.A, 815 (2009) 40.
[27] H. Hosseinkhani, M. Modarres, Phys.Lett.B, 694 (2011) 355.
[28] H. Hosseinkhani, M. Modarres, Phys.Lett.B, 708 (2012) 75.
[29] NMC: Arneodo et al., Nucl.Phys.B, 483 (1997) 3.
[30] ZEUS: Derrick et al., Zeit.Phys.C, 72 (1996) 399.
[31] H1 and ZEUS collaborations, JHEP 01 (2010) 109.
[32] M. Modarres, H. Hosseinkhani and N. Olanj, Phys.Rev.D, 89 (2014) 034015.
[33] M. Modarres, M.R. Masouminia, H. Hosseinkhani and N. Olanj, (2015) submitted for publication.
[34] V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys.Lett.B, 60 (1975) 50.
[35] L.N. Lipatov, Sov.J.Nucl.Phys., 23 (1976) 642.
[36] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov.Phys.JETP, 44 (1976) 45.
[37] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov.Phys.JETP, 45 (1977) 199.
[38] Ya.Ya. Balitsky and L.N. Lipatov, Sov.J.Nucl.Phys., 28 (1978) 822.
[39] S. Catani, M. Ciafaloni and F. Hautmann, Phys.Lett.B, 242 (1990) 97.
[40] S. Catani, M. Ciafaloni and F. Hautmann, Nucl.Phys.B, 366 (1991) 135.
[41] J.C. Collins and R.K. Ellis, Nucl.Phys.B, 360 (1991) 3.
[42] E.M. Levin, M.G. Ryskin, Yu.M. Shabelski and A.G. Shuvaev, Sov.J.Nucl.Phys., 54 (1991) 867.
[43] J. Kwiecinski, A.D. Martin and A.M. Stasto, Acta Physica Polonica B, 28 (1997) 2577.
[44] R.G. Robersts, “ The Structure of the proton”, Cambridge University Press (1993).
[45] K. Golec-Biernat and A.M. Stasto, Phys.Rev.D, 80 (2009) 014006.
[46] A.M. Stasto, Phys.Lett.B, 679 (2009) 288.
[47] J. Kwiecinski, A.D. Martin and A.M. Stasto, Phys.Rev.D, 56 (1997) 3991.
[48] F.D. Aaron et al (H1 Collaboration), Phys.Lett.B, 665 (2008) 139.
[49] V. Andreev et al (H1 Collaboration), Eur.Phys.J.C, 74 (2014) 2814.
[50] ZEUS collaboration, Desy Report No. Desy-09-045, (2009).
[51] ZEUS collaboration, Phys.Rev.D 90 072002 (2014) 072002.
[52] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, Eur.Phys.J.C, 14 (2000) 133.
[53] A.D. Martin, W.J. Stirling, R.S. Thorne and G. Watt, Eur.Phys.J.C, 63 (2009) 189.
[54] G. Marchesini and B.R. Webber, Nucl.Phys.B, 310 (1988) 461.
[55] Yu.L. Dokshitzer, V.A. Khoze, S.I. Troyan and A.H. Mueller, Rev.Mod.Phys., 60 (1988) 373.
[56] V.A. Saleev, Phys.Rev.D, 80 (2009) 114016.
[57] A.V. Lipatov and N.P. Zotov, Phys.Rev.D, 81 (2010) 094027.
[58] S.P. Baranov, A.V. Lipatov and N.P. Zotov, Phys.Rev.D, 81 (2010) 094034.
[59] B.A. Kniehl, V.A. Saleev and A.V. Shipilova, Phys.Rev.D, 81 (2010) 094010.
[60] H. Jung, M. Kraemer, A.V. Lipatov and N.P. Zotov, JHEP, 01 (2011) 085.
[61] S.P. Baranov, A.V. Lipatov and N.P. Zotov, Eur.Phys.J.C, 71 (2011) 1631.
[62] A.V. Lipatov, M.A. Malyshev and N.P. Zotov, Phys.Lett.B, 699 (2011) 93.
[63] H. Jung, M. Kraemer, A.V. Lipatov and N.P. Zotov, arXiv:1105.5071 [hep-ph] (2011).
[64] H. Jung, M. Kraemer, A.V. Lipatov and N.P. Zotov, Phys.Rev.D, 85, 034035 (2012).
[65] A.V. Lipatov and N.P. Zotov, Phys.Lett.B, 704 (2011) 189.
[66] B.A. Kniehl, V.A. Saleev, A.V. Shipilova and E.V. Yatsenko, arXiv:1107.1462 [hep-ph] (2011).
[67] H. Jung, M. Kraemer, A.V. Lipatov and N.P. Zotov, Proceedings of 19th International Workshop On Deep-Inelastic Scattering And Related Subjects (DIS 2011), arXiv:1107.4328 [hep-ph] (2011).
[68] W. Furmanski, R. Petronzio, Phys.Lett.B, 97 (1980) 437.
[69] S. Catani and F. Hautmann, Nucl.Phys.B, 427 (1994) 745.
[70] M. Ciafaloni, Phys.lett.B, 356 (1995) 74.
[71] A.J. Askew, J. Kwiecinski, A.D. Martin and P.J. Sutton, Phys.Rev.D, 47 (1993) 3775.
[72] A.J. Askew, J. Kwiecinski, A.D. Martin and P.J. Sutton, Phys.Rev.D, 49 (1994) 4402.
[73] A.J. Askew, Small x Physics, thesis presented for the degree of Doctor of Philosophy, University of Durham, (1995).
[74] A.M. Stasto, Acta.Phys.Polo.B, 27 (1996) 1353.
[75] A.M. Stasto, QCD Analysis of Deep Inelastic Lepton-Hadron Scattering in the Region of Small Values of the Bjorken Parameter, thesis presented for the degree of Doctor of Philosophy,
University of Durham, (1999).

[76] I.S. Gradshteyn and I.M. Ryzhik, "Table of integrals, Series, and Products, corrected and enlarged edition", Academic Press (1980).

[77] M.A. Kimber, J. Kwiecinski, A.D. Martin and A.M. Stasto, Phys.Rev.D, 62 (2000) 094006.

FIG. 1: The quarks-box and exchanged diagrams in the photon-gluon fusion process discussed in the $k_t$ factorization formula in the text.

FIG. 2: The longitudinal proton structure functions in the frameworks of $KMR$ (left panels, using the MRST99 $PDF$ data as inputs) and $MRW$ (right panels, using the $MSTW2008-NLO$ data as inputs) $UPDF$, versus $x$, for $Q^2=2, 4, 6, 12$ and $15 \text{ GeV}^2$. Their total value and the contributions of $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles.

FIG. 3: The longitudinal proton structure functions in the frameworks of $KMR$ by using the $MRST99 PDF$ data versus $x$, for $Q^2=12, 15, 20, 25, 35, 45, 60, 80, 90$ and $110 \text{ GeV}^2$. Their total value and the contributions of $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles.
FIG. 4: The longitudinal proton structure functions in the frameworks of KMR by using the MSTW2008 – LO PDF data versus x, for $Q^2=12, 15, 20, 25, 35, 45, 60, 80, 90$ and 110 GeV$^2$. Their total value and the contributions of $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles.

FIG. 5: The longitudinal proton structure functions in the frameworks of MRW and by using the MSTW2008 – NLO PDF data versus x, for $Q^2=12, 15, 20, 25, 35$ and 45 GeV$^2$. Their total value and the contributions of $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles.

FIG. 6: The longitudinal proton structure functions in the frameworks of KMR and MRW by using the MRST99, MSTW2008 – LO and MSTW2008 – NLO PDF data versus $Q^2$ (GeV$^2$), for fix x=0.001 and 0.0001. Their total values and the contributions of $k_t$ factorization scheme, the quarks and the no-perturbative parts are presented with different curve styles.

FIG. 7: The comparison of total longitudinal proton structure functions in the frameworks of KMR and MRW by using the MRST99, MSTW2008 – LO and MSTW2008 – NLO PDF data versus $Q^2$ (GeV$^2$), for the fix x=0.001 and 0.0001.

FIG. 8: The comparison of total longitudinal proton structure functions, in the frameworks of KMR and MRW by using the MRST99, MSTW2008 – LO and MSTW2008 – NLO PDF data versus x at $Q^2=2, 2.5, 3.5, 5, 6.5, 8.5$ and 9 GeV$^2$, with the corresponding ZEUS and H1 data (filled-triangles and bold points), respectively.

FIG. 9: The comparison of total longitudinal proton structure functions, in the frameworks of KMR and MRW by using the MRST99, MSTW2008 – LO and MSTW2008 – NLO PDF data versus x at $Q^2=12, 15, 20, 25, 35, 45, 60$ and 90 GeV$^2$, with the corresponding H1 data (bold points).

FIG. 10: The comparison of total longitudinal proton structure functions, in the frameworks of KMR and MRW by using the MRST99, MSTW2008 – LO and MSTW2008 – NLO PDF data versus x at $Q^2=24, 32, 45, 80$ and 110 GeV$^2$, with the corresponding ZEUS and H1 data (filled-triangle and bold points), respectively.
FIG. 11: The comparison of perturbative gluon contribution to $F_L$ by performing the $\phi$ integration (exact) and the approximated one with $\phi = \pi/4$, in the frameworks of $KMR$ by using the $MRST99$ versus $x$ at $Q^2=3.5, 12, 60$ and $110$ GeV$^2$. 