FREQUENCY CRITERIA FOR EXPONENTIAL STABILITY

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Abstract. We discuss some frequency domain criteria for the exponential stability of nonlinear feedback systems based on classical dissipativity theory. Applications are given to convergence rates for certain perturbations of the damped harmonic oscillator.

1. Introduction

1.1. Overview. This paper concerns exponential stability for nonlinear feedback systems of Lur’e type. While a more general viewpoint is adopted in §2, the examples we consider are all of the form

\[ \dot{x} = Ax + B\phi(Cx), \]  

(1.1)

where \( A, B, C \) are matrices with values in \( K = \mathbb{C} \) or \( \mathbb{R} \), and \( \phi \) is a continuous function.

In general, an autonomous system \( \dot{x} = f(x) \) with \( f \) continuous is said to be \( r \)-exponentially stable for a given rate \( r \geq 0 \) if \( x = 0 \) is an equilibrium and there is an increasing continuous function \( \kappa \), with \( \kappa(0) = 0 \), such that

\[ |x(t)| \leq \kappa(|x(0)|) \exp(-rt) \]

for each solution \( x \) and every \( t \geq 0 \) for which \( x \) exists (a posteriori these solutions are bounded, hence exist globally).

Criteria for exponential stability of systems (1.1), including frequency domain formulations, can be deduced from standard results in dissipativity theory. A brief but self-contained exposition is presented in §2, complete with simple proofs. Special attention is given to critical cases where the linear block has marginal stability properties and the frequency inequalities are non-strict. We also provide a new version of Popov’s criterion for exponential stability that applies to nonlinearities with uncontrolled growth.

Although frequency criteria for exponential stability have been studied for a long time (see, e.g., [Bli, BLR, HS, AC] for some recent results), it is difficult to find concrete applications in the literature. A class of examples related to the Liénard equation is discussed in §1.2 below.

Although it is not discussed here, exponential stability for discrete analogues of (1.1) has been the subject of intense investigation in recent years because of relationships
with optimization methods for convex functions [LRP, FRMP, LS, TVSL, CHVSL, VSFL, HL].

1.2. Applications. The simplest applications are to second-order equations. As an illustration we consider a class of dissipative Hamiltonian systems on $\mathbb{R}^{2d}$ of the form

$$\dot{x} = (J - S)\nabla H(x),$$

(1.2)

where $J$ is the usual symplectic matrix, and the symmetric part $S \geq 0$ represents resistive elements of the system. Here

$$x = (q, p) \in \mathbb{R}^{2d}, \quad H(x) = \frac{1}{2}|p|^2 + f(q),$$

and $f : \mathbb{R}^d \to \mathbb{R}$ is a given potential. Assume that $S = \text{diag}(\tau I, 2\sigma I)$, where $\sigma > 0$ and $\tau \geq 0$ are constant, so (1.2) becomes

$$\dot{q} = p - \tau \nabla f(q),$$
$$\dot{p} = -2\sigma p - \nabla f(q).$$

(H)

The system (H) with $\tau = 0$ represents a class of Rayleigh–Liénard oscillators with constant damping:

$$\dot{q} = p$$
$$\dot{p} = -2\sigma p - \nabla f(q).$$

(H$_0$)

Because it was investigated in pioneering work of Polyak [Pol] as a way of exploring the minimum of a convex function, (H$_0$) is often referred to in the optimization literature as Polyak’s heavy ball with friction.

The case $\tau \geq 0$ has received less attention. Note that (H) is equivalent to the second-order equation

$$\ddot{q} + (2\sigma + \tau^2 \nabla^2 f(q)) \dot{q} + (1 + 2\sigma \tau \nabla f(q)) = 0.$$

(1.3)

Equations of this form are sometimes called Hessian-damped; although several works have considered (1.3) including other first-order representations, e.g., [AABR, ACPR, APR2, SDJS, ABR, APR1, ACFR], its simple Hamiltonian formulation appears to have been overlooked. Discrete analogues of (H) with $\tau > 0$ play an important role in accelerating the convergence of iterative convex optimization methods [SDJS].

To state the main results, we recall some classes of nonlinearities. Given $m < L \leq \infty$, a continuous function $\phi : \mathbb{K}^n \to \mathbb{K}^n$ is said to belong to the sector $[m, L]$ if

$$\text{Re}\langle my - \phi(y), y - L^{-1}\phi(y) \rangle \leq 0$$

(1.4)

for all $y \in \mathbb{K}^n$, with the obvious interpretation when $L = \infty$. Sector bounded nonlinearities play a central role in the classical passivity-based conditions for Lyapunov stability (e.g., the circle and Popov criteria, cf. §2.5).
Now let $K = \mathbb{R}$, and suppose that $\phi = \nabla f$ for some potential $f : \mathbb{R}^n \to \mathbb{R}$. Even if $\phi$ belongs to a sector $[m, \infty]$, its potential can be far from convex. The condition
\[
f(y) - \langle \nabla f(y), y \rangle + (m/2)|y|^2 \leq f(0) \tag{1.5}
\]
further limits the concavity of $f$. A function whose gradient belongs to $[m, \infty]$ and satisfies (1.5) will be called at most mildly concave (this terminology is not standard).

**Proposition 1.1.** Let $\tau = 0$ and $m > 0$. Suppose that $\sigma, r > 0$ satisfy
\[
r \leq 2\sigma/3, \quad 2r\sigma - r^2 \leq m,
\]
and that at least one of these inequalities is strict. If $\nabla f$ belongs to the sector $[m, \infty]$ and $f$ is at most mildly concave, then $(H_0)$ is $r$-exponentially stable.

The proof of Proposition 1.1 follows from a version of the Popov criterion (Lemma 2.15), which in the time domain is equivalent to the existence of a Lyapunov function
\[
V(x) = \langle Px, x \rangle + f(q) - f(0) \tag{1.6}
\]
satisfying $\dot{V} \leq -2rV$. Actually, an explicit formula for such a $P$ is easily guessed from numerical experiments; one possibility is
\[
P = \begin{bmatrix} r^2 I & rI \\ rI & 1/2 \end{bmatrix},
\]
which has the property that $P + \text{diag}(mI/2, 0)$ is positive definite precisely under the hypotheses of Proposition 1.1. Note that this $P$ depends on the rate $r$, and it is unclear how its form could be reasoned a priori.

When the upper sector bound is unknown, i.e., $\nabla f$ is only known to belong to the sector $[m, \infty]$, the best rate guaranteed by Proposition 1.1 occurs when
\[
r = 2\sigma/3 - \varepsilon, \quad \sigma^2 = 9m/8,
\]
yielding $r = \sqrt{m/2} - \varepsilon$ for any $\varepsilon > 0$. This is in contrast to the linear problem over the same sector, where the best rate is
\[
r = \sqrt{m} - \varepsilon
\]
for any $\varepsilon > 0$, achieved in the critically damped regime $\sigma^2 = m$. Thus the Popov criterion fails to show that the nonlinear problem is stable at the same exponential rate as the linear one.

Without assuming mild concavity, we can still show exponential stability provided $\nabla f$ lies in a finite sector. In fact, this applies to rates exceeding $2\sigma/3$ as well. For simplicity we focus only on endpoint values of the damping allowed by the linear problem (namely $2r\sigma - r^2 = m$), since these maximize the overall convergence rate:
Proposition 1.2. Let $\tau = 0$ and $m > 0$. Suppose that $\sigma, r > 0$ satisfy
\[ r < \sigma, \quad 2r\sigma - r^2 = m. \]
If $L = m + 4(\sigma - r)^2$ and $\nabla f$ belongs to the sector $[m, L]$, then $(H_0)$ is $r$-exponentially stable.

If $\tau$ is allowed to be positive, then a quantitative version of the Aizerman conjecture holds for $(H)$: for fixed $m, \sigma > 0$, the optimal rate $r_*$ that holds uniformly for linear $\nabla f$ in the sector $[m, \infty]$ as $\tau$ ranges over $[0, \infty)$ also holds for the nonlinear problem, with the same optimal value of $\tau$.

Proposition 1.3. Given $m, \sigma > 0$, let
\[ r_* = \frac{3\sigma + \sqrt{2m + \sigma^2}}{2}, \quad \tau_* = \frac{\sigma + \sqrt{2m + \sigma^2}}{m}. \]
If $\nabla f$ belongs to the sector $[m, \infty]$, then $(H)$ is $r_*$-exponentially stable.

As mentioned above, this result is sharp in the following sense. If $r > r_*$, then for each $\tau \geq 0$ there exists $k \in [m, \infty)$ such that the conclusion of Proposition 1.3 is false when $f(q) = (k/2)|q|^2$.

Propositions 1.2 and 1.3 do not require $f$ to be at most mildly concave, only that $\nabla f$ belongs to a certain sector. In the time domain, this corresponds to a quadratic Lyapunov function
\[ V(x) = \langle Px, x \rangle. \]
It follows that Propositions 1.2 and 1.3 also apply if $f$ has time dependence, provided the sector bounds are uniform in time. This is contrast to Proposition 1.1, which uses the time invariance of $f$ (and mild concavity) in an essential way.

As far as we are aware, Propositions 1.1 and 1.2 provide the best known global convergence rates for $(H_0)$ under the given hypotheses. For the infinite sector problem, it was previously observed in [Sie] that the Lyapunov function
\[ V(x) = \frac{1}{2}|p + \sigma q|^2 + f(q) - f(0) \]
yields the rate $r = \sqrt{m}/2$ when $\sigma^2 = m$. Proposition 1.3, which quantifies the stabilizing role played $\tau > 0$, appears to be new; weaker results appear in [SDJS, ACFR].

Systems of the form $(H)$ have become popular subjects in the optimization literature vis-à-vis continuous analogues of iterative first-order methods for convex optimization [SBC, WWJ, WRJ, SDJS, SDSJ]. The majority of these works consider only the time domain and construct explicit Lyapunov functions (notable exceptions are [CHVSL, VSFL] in the discrete-time setting).
2. Exponential stability

2.1. Dissipativity. A detailed treatment of dissipativity theory can be found in the foundational papers [Wil3, Wil4], as well as [HM1, HM2]. A more classical approach to stability is given by Popov in the classic textbook [PG]. In this section we provide a brief review of the relevant material.

A continuous dynamical system with values in $\mathbb{K}^d$ is represented by a set of admissible trajectories

$$B \subset L^2_{\text{loc}}(\mathbb{R}; \mathbb{K}^d),$$

called a behavior (following Willems). Dissipativity is defined with respect to a fixed quadratic form $\sigma : \mathbb{K}^d \to \mathbb{R}$, referred to as the supply rate. For $[t_0, t_1] \subset [0, \infty)$ and $w \in B$, the integral

$$E(w, t_0, t_1) = \int_{t_0}^{t_1} \sigma(w(s)) \, ds \tag{2.1}$$

is interpreted as the energy supplied to the system over the time period $[t_0, t_1]$ when the system response is $w$.

Consider a linear time-invariant (LTI) system with state $x \in \mathbb{K}^m$, input $u \in \mathbb{K}^n$, and external variables $w \in \mathbb{K}^d$:

$$\begin{align*}
\dot{x} &= Ax + Bu, \\
n &= C_w x + D_w u. \tag{2.2}
\end{align*}$$

A parallel development for nonlinear systems is given in [Wil3, HM1]. A behavior $B$ is said to be represented by $(A, B, C_w, D_w)$ if for each $w \in B$ there exists $(x, u)$ such that (2.2) holds. In what follows the supply rate $\sigma$ is understood to be fixed.

**Definition 2.1.** If $B$ is represented by $(A, B, C_w, D_w)$, then $V : \mathbb{K}^m \to \mathbb{R}$ is called a storage function if the dissipation inequality

$$V(x(t_1)) - V(x(t_0)) \leq E(w, t_0, t_1) \tag{DIE}$$

holds whenever $[t_0, t_1] \subset [0, \infty)$ and $\dot{x} = Ax + Bu; w = C_w x + D_w u$. If $B$ admits a storage function, then $B$ is said to be cyclodissipative.

In many references, storage functions are required to be nonnegative. There are at least two distinguished candidates for storage functions (when they exist). Define the available storage $V_a$ by

$$V_a(x_0) = \sup\{-E(w, 0, T) : (2.2) \text{ holds, } T \geq 0, x(0) = x_0, x(T) = 0\},$$

which represents the greatest amount of energy that can be extracted in motions driving the system from state $x_0$ to the origin. Similarly, the required supply $V_r$ is defined by

$$V_r(x_0) = \inf\{E(w, 0, T) : (2.2) \text{ holds, } T \geq 0, x(0) = 0, x(T) = x_0\}.$$
If \((A, B)\) is controllable, then \(V_r < \infty\) and \(V_a > -\infty\). The following result gives several characterizations of cyclodissipativity with respect to a given supply rate; the proof follows from straightforward manipulations of the definitions.

**Lemma 2.2** ([Wil3, Wil4]). If \(\mathcal{B}\) is represented by \((A, B, C_w, D_w)\) and \((A, B)\) is controllable, then the following conditions are equivalent:

1. \(\mathcal{B}\) is cyclodissipative,
2. \(V_r > -\infty\),
3. \(V_a < \infty\),
4. \(E(w, 0, T) \geq 0\) whenever \(\dot{x} = Ax + Bu; w = C_w + D_w u\) and \(T \geq 0\) is such that \(x(0) = x(T)\).

If any of these conditions are satisfied, then \(V_r(0) = V_a(0) = 0\), and any storage function normalized by \(V(0) = 0\) satisfies

\[
V_a \leq V \leq V_r.
\]

Because \(\mathcal{B}\) is associated with an LTI system, even more is true: \(V_r(x) = \langle P_- x, x \rangle\) and \(V_a = \langle P_+ x, x \rangle\) for Hermitian matrices \(P_- \leq P_+\), and if \((A, B)\) is controllable, then \(\mathcal{B}\) is cyclodissipative if and only if it admits a quadratic storage function.

**Definition 2.3.** A behavior \(\mathcal{B}\) represented by \((A, B, C_w, D_w)\) is said to be **dissipative** if it admits a nonnegative storage function.

The following analogue of Lemma 2.2 holds.

**Lemma 2.4** ([Wil3, Wil4]). If \(\mathcal{B}\) is represented by \((A, B, C_w, D_w)\) and \((A, B)\) is controllable, then the following conditions are equivalent:

1. \(\mathcal{B}\) is dissipative,
2. \(V_r \geq 0\),
3. \(E(w, 0, T) \geq 0\) for every \(T \geq 0\) whenever \(\dot{x} = Ax + Bu; w = C_w + D_w u\) and \(x(0) = 0\).

Let \(M = M^* \in \mathbb{K}^{d \times d}\) be such that \(\sigma(w) = \langle M w, w \rangle\). From the differential version of the dissipation inequality (DIE), cyclodissipativity is equivalent to the existence of \(P = P^* \in \mathbb{K}^{m \times m}\) satisfying the linear matrix inequality

\[
\begin{bmatrix}
A^* P + PA & PB \\
B^* P & 0
\end{bmatrix} - \begin{bmatrix}
C_w^* \\
D_w^*
\end{bmatrix} M \begin{bmatrix}
C_w & D_w
\end{bmatrix} \preceq 0.
\]

If \(P\) can be chosen positive semidefinite, then \(\mathcal{B}\) is dissipative. More general linear matrix inequalities are considered in the next section.
2.2. Quadratic storage functions. First we recall necessary and sufficient frequency conditions under which the linear matrix inequality

$$\Lambda(P) = \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \leq 0$$

(LMI)

admits a Hermitian solution $P$. Although (LMI) is algebraically more general than (2.3), it can also be viewed in the behavioral framework as a special case of (2.3) by taking $[C_w \ D_w] = I$. In other words, we consider cyclodissipativity of the input–state behavior, which consists of pairs $w = (x, u)$ such that $\dot{x} = Ax + Bu$. The corresponding supply rate is

$$\sigma(x,u) = \langle Qx, x \rangle + 2 \text{Re} \langle Sx, u \rangle + \langle Ru, u \rangle.$$  \hspace{1cm} (2.4)

When convenient we also write $M$ for the matrix associated with $\sigma$. Introduce the spectral density function

$$\Pi(s) = \begin{bmatrix} (s - A)^{-1}B^* \\ I \end{bmatrix}^* \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} (s - A)^{-1}B \\ I \end{bmatrix},$$

which is defined away from the eigenvalues of $A$. If $P = P^*$ and $s$ is not an eigenvalue of $A$, then

$$\begin{bmatrix} (s - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \begin{bmatrix} (s - A)^{-1}B \\ I \end{bmatrix} = (2 \text{Re}s)((s - A)^{-1}B)^*P(s - A)^{-1}B.$$

(2.5)

If $\Lambda(P) \leq 0$ admits a Hermitian solution, it follows from (2.5) and (LMI) that the frequency condition

$$\Pi(i\omega) \geq 0 \text{ whenever } \omega \in \mathbb{R} \text{ and } \det(i\omega - A) \neq 0$$

(FDI)

holds. Although the following version of the Kalman–Yakubovich–Popov (KYP) lemma is well-known, a short proof is included for the reader’s convenience.

**Lemma 2.5.** If $(A, B)$ is controllable, then $\Lambda(P) \leq 0$ admits a solution $P = P^*$ if and only if (FDI) holds.

**Proof.** For the converse, it suffices to show that $V_\alpha < \infty$, where the available storage is defined with respect to the input–state behavior and the supply rate $\sigma$ is given by (2.4). Let $T \geq 0$. Suppose that $w = (x, u)$ satisfies $\dot{x} = Ax + Bu$ with the initial conditions $x(0) = x_0$ and $x(T) = 0$. Since $(A, B)$ is controllable, there exists $w_1 = (u_1, x_1)$ of compact support with the same properties, such that

$$\text{supp} \ w_1 \subset (-\infty, T], \quad w(t) = w_1(t) \text{ for } t \in [0, T].$$

Moreover, the restriction of $w_1$ to $(-\infty, 0)$ can be chosen to depend only on $x_0$. By Plancherel’s formula, (FDI) implies that $E(w_1, -\infty, T) \geq 0$. In particular,

$$E(w_1, -\infty, 0) \geq -E(w, 0, T),$$
which shows that $V_n(x_0) < \infty$ for all $x_0 \in \mathbb{K}^m$. \hfill \qed

Of course (LMI) is invariant under conjugation by a nonsingular matrix. We make repeated use of two important cases. The first is state feedback; given the sextuple $(A, B, P, Q, S, R)$ and a matrix $N \in \mathbb{K}^{n \times m}$, define $(A_*, B_*, P_*, Q_*, S_*, R_*)$ by

$$A_* = A + BN, \quad B_* = BN, \quad P_* = P,$$

$$Q_* = Q + 2 \text{Re} S^* N + N^* R N, \quad S_* = S + R N, \quad R_* = R$$

(2.6)

Define $\Lambda_*(P_*)$ as in (LMI), replacing $(A, B, P, Q, S, R)$ with $(A_*, B_*, P_*, Q_*, S_*, R_*)$; then

$$\Lambda_*(P_*) = \begin{bmatrix} I & 0 \\ N & I \end{bmatrix} \Lambda(P) \begin{bmatrix} I & 0 \\ N & I \end{bmatrix}.$$ 

Furthermore, if $\Pi_*$ is the corresponding spectral density, then apart from finitely many points,

$$W(s)^* \Pi_*(s) W(s) = \Pi(s), \quad W(s) = I - N(sI - A)^{-1} B.$$ 

In particular, $\Pi(s) \geq 0$ if and only if $\Pi_*(s) \geq 0$, apart from finitely many $s$. It is also obvious that if $(A, B)$ is controllable, then so is $(A_*, B_*)$.

We will also need to consider similarity transformations. Given an invertible $T \in \mathbb{K}^{m \times m}$, define $(A_*, B_*, P_*, Q_*, S_*, R_*)$ by

$$A_* = T^{-1} A T, \quad B_* = T^{-1} B, \quad P_* = T^* P T,$$

$$Q_* = T^* Q T, \quad S_* = S T, \quad R_* = R.$$ 

(2.7)

In that case

$$\Lambda_*(P_*) = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \Lambda(P) \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix},$$

and $\Pi(s) = \Pi_*(s)$.

2.3. Positive semidefinite storage functions. In general it is difficult to characterize the existence of positive semidefinite solutions to $\Lambda(P) \leq 0$. The following notion of minimal stability is due to Yakubovich and Popov.

**Definition 2.6.** $(A, B)$ is said to be minimally stable with respect to the supply rate (2.4) if for each $x_0 \in \mathbb{K}^m$ there exists $w = (x, u)$ with $\dot{x} = Ax + Bu$, such that

$$x(0) = x_0, \quad E(w, 0, t) \leq 0 \text{ for all } t \geq 0, \quad x(t) \to 0 \text{ as } t \to \infty.$$ 

If $(A, B)$ is minimally stable, then every continuous storage function normalized by $V(0) = 0$ is nonnegative: minimal stability implies that

$$V(x(t)) \leq V(x_0),$$

at which point it suffices to let $t \to \infty$ to deduce that $V(x_0) \geq 0$. In particular, every Hermitian solution of $\Lambda(P) \leq 0$ is positive semidefinite.
The simplest way to establish minimal stability is via feedback stabilization. If \( A \) is Hurwitz and \( Q \leq 0 \), then \((A, B)\) is minimally stable: simply take \( u = 0 \) and let \( x \) solve the asymptotically stable system \( \dot{x} = Ax \) with \( x(0) = x_0 \). Equivalently, this can be seen from the Lyapunov inequality
\[
\text{Re} A^*P \leq Q
\]
implied by the upper left block of (LMI). More generally the following holds.

**Lemma 2.7.** If there exists \( K \) such that \( A + BK \) is Hurwitz and
\[
Q + 2 \text{Re} S^*K + K^*RK \leq 0,
\]
then \((A, B)\) is minimally stable.

**Proof.** Apply (2.6) with \( N = K \). \(\Box\)

If \( Q \leq 0 \), then (2.8) holds for \( K = -\delta S \) provided \( \delta \in [0, 2/\|R\|] \). Thus, one way to verify the hypotheses of Lemma 2.7 is to show that \( A - \delta BS \) is Hurwitz for \( \delta \) in this range.

Another path towards positivity involves strengthened frequency domain conditions. From (2.5), if \( \Lambda(P) \leq 0 \) admits a solution \( P \geq 0 \), then
\[
\Pi(s) \geq 0 \text{ whenever } \text{Re}(s) \geq 0 \text{ and } \det(s - A) \neq 0.
\]
(FDI+) This is of course a stronger condition than (FDI) in general. It was claimed in [Wil1] that the converse also holds, but Willems later provided a counterexample [Wil2]. In the latter reference it was pointed out that sufficiency does hold if \( M \) has at most \( n \) positive eigenvalues, where recall the dimension \( n \) is determined by \( R \in \mathbb{K}^{n \times n} \). In [Wil2] this is phrased (equivalently) in terms of the existence of a decomposition
\[
\sigma(x, u) = |C_1x + D_1u|^2 - |C_2x + D_2u|^2,
\]
where \( D_1 \) is square. A complete proof is somewhat involved [RV] (see also [Moy, WT]); for simplicity we only consider the special case where (2.8) holds (certainly implying that \( M \) has at most \( n \) positive eigenvalues), which was studied previously by Moylan [Moy].

**Lemma 2.8.** If \((A, B)\) is controllable and there exists \( K \) such that
\[
Q + 2 \text{Re} S^*K + K^*RK \leq 0,
\]
then \( \Lambda(P) \leq 0 \) admits a solution \( P \geq 0 \) if and only if (FDI+) holds.

**Proof.** We show that (FDI+) implies there is a solution \( P \geq 0 \) to \( \Lambda(P) \leq 0 \). By a preliminary feedback (2.6) with \( N = K \), we assume that \( Q \leq 0 \). First, suppose that
$R > 0$. With the feedback $N = -R^{-1}S$, we obtain the new problem

$$\begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} = \begin{bmatrix} Q - SR^{-1}S & 0 \\ 0 & R \end{bmatrix}.$$  

If we factor $Q - SR^{-1}S = -C^*C$ for some $C$, then $\Pi_*(s) = -H_*(s)H_*(s) + R$, where $H_*(s) = C(sI - A_*)^{-1}B_*$. If $C = 0$, then $P = 0$ solves $\Lambda_*(P) \leq 0$. Otherwise, by the discussion surrounding (2.7), we can assume that $A$ has a Kalman decomposition

$$A_* = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B_* = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix},$$  

(2.9)

where $(A_{11}, C_1)$ is observable and $(A_{11}, B_1)$ is controllable. Now (FDI+) implies that $H(s)$ has no poles in the closed right half-plane; since the poles of $H(s)$ are the eigenvalues of $A_{11}$, we conclude that $A_{11}$ is Hurwitz. Since (FDI) holds, there is a Hermitian solution

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

to $\Lambda_*(P) \leq 0$, where the block decomposition is the same as in (2.9). Thus

$$2 \text{Re} A_{11}^*P_{11} \leq -C_1^*C_1,$$

so $P_{11} \geq 0$ (and in fact $P_{11} > 0$ by observability — cf. Lemma 2.10 below). If we now set $P_0 = \text{diag}(P_{11}, 0)$, then $P_0 \geq 0$ and $\Lambda_*(P_0) \leq 0$ as well.

Even if $R$ is not invertible, the argument above applies if $R$ is replaced by $R + \varepsilon I$ for any $\varepsilon > 0$ (this should be done after the initial feedback, i.e., we are assuming that $Q \leq 0$). This implies the dissipation inequality

$$\int_0^T \sigma(x(s), u(s)) + \varepsilon |u(s)|^2 ds \geq 0$$

for every $T \geq 0$, whenever $\dot{x} = Ax + Bu$ and $x(0) = 0$. Let $\varepsilon \to 0$ and apply Lemma 2.4 to finish the proof. \qed

The relationship is between Lemmas 2.7 and 2.8 may not be clear at first glance. To complete our discussion, we show that they are equivalent when $(A, S)$ is observable. For simplicity we assume that $Q \leq 0$.

**Lemma 2.9.** Suppose that $(A, B)$ is controllable, $(A, S)$ is observable, and $Q \leq 0$. If (FDI) holds, then the following are equivalent.

(1) $(A, B)$ is minimally stable,

(2) Every Hermitian solution of $\Lambda(P) \leq 0$ satisfies $P \geq 0$,

(3) (FDI+) holds,

(4) There exists $\delta \in (0, 2/\|R\|)$ such that $A - \delta BS$ is Hurwitz.
Proof. Clearly (1) ⇒ (2) ⇒ (3) and (4) ⇒ (1), so it remains to show that (3) ⇒ (4). If
\[ G(s) = S(s-A)^{-1}B \text{ and } G_*(s) = S(s-A+\delta BS)^{-1}B, \]
then one has the usual formula
\[ G_*(s) = (I+\delta G(s))^{-1}G(s). \]
Since \((A-\delta BS, B)\) is controllable and \((A-\delta BS, S)\) is observable, the eigenvalues of
\( A-\delta BS \) and the poles of \( G_* \) coincide. If \( \delta > 0 \), then \( G_* \) has a pole at \( s \) if and only if
\( s \) is not a pole of \( G \) and \( \det(I+\delta G(s)) = 0 \). But if \( \Re s \geq 0 \) and \( s \) is not a pole of \( G \), then
\[ 2 \Re G(s) + R \geq \Pi(s) \geq 0 \]
since \( Q \leq 0 \). Thus \( \det(I+\delta G(s)) \neq 0 \) provided \( \delta \in (0, 2/\|R\|) \), and hence \( A-\delta BS \) is Hurwitz.

There exist other frequency conditions under which \( \Lambda(P) \leq 0 \) admits positive semi-
definite solutions [Mol, TR, WT], but these are not discussed here.

2.4. Positive definite storage functions. In general, observability conditions are
needed to conclude that a Hermitian solution of \( \Lambda(P) \leq 0 \) is nonsingular (and hence
\( P > 0 \) if \( P \geq 0 \)). The following fact about the Lyapunov equation is well known —
see, e.g., [Che] for a proof.

**Lemma 2.10.** Let \((A, S)\) be observable. If \( P = P^* \) satisfies
\[ \Re A^* P \leq -S^* S, \tag{2.10} \]
then \( \det P \neq 0 \) and \( A \) has no imaginary eigenvalues. Furthermore the inertia of \( P \) is
equal to the inertia of \(-A\).

**Lemma 2.11.** Suppose that \( P = P^* \) satisfies \( \Lambda(P) \leq 0 \). If \((A, S)\) is observable and
there exists a matrix \( K \) such that
\[ Q + 2 \Re S^* KS + (KS)^* R(KS) \leq 0, \quad \det(I + RK) \neq 0, \]
then \( \det P \neq 0 \).

Proof. First assume that \( Q \leq 0 \). Given \( \delta > 0 \), apply (2.6) with \( N = -\delta S \). From the
upper left block of the resulting inequality \( \Lambda_*(P) \leq 0 \),
\[ 2 \Re(A-\delta BS)^* P \leq -S^* (2\delta I - \delta^2 R)S. \]
Take any \( \delta \in (0, 2/\|R\|) \). Since \((A-\delta BS, S)\) is also observable, an application of
Lemma 2.10 shows that \( \det P \neq 0 \). In general, first apply the feedback (2.6) with
\( N = KS \). The previous result applies provided the pair \((A+BKS, (I+RK)S)\) is
observable, which is true if \( I + RK \) is invertible. \( \square \)
There is one notable case in which definiteness can be established without observability, but this requires a brief detour. Given a solution \( P = P^* \) of \( \Lambda(P) \leq 0 \), factor

\[
\begin{bmatrix}
A^*P + PA & PB \\
B^*P & 0
\end{bmatrix}
- \begin{bmatrix}
Q & S^* \\
S & R
\end{bmatrix}
= - \begin{bmatrix}
L^* \\
W^*
\end{bmatrix}
\begin{bmatrix}
L \\
W
\end{bmatrix},
\]  
(2.11)

where \( \begin{bmatrix} L & W \end{bmatrix} \) has full row rank. Recall the notation \( P_+ \) and \( P_- \) for the Hermitian matrices corresponding to the required supply and available storage, respectively (when they exist).

**Lemma 2.12.** Let \((A, B)\) be controllable, and suppose that \( \Lambda(P) \leq 0 \) admits a Hermitian solution. If \( L\pm, W\pm \) are such that the factorization (2.11) holds for \( P\pm \), then

\[ \dim \ker \begin{bmatrix} A - sI & B \\
L_\pm & W_\pm
\end{bmatrix}^* = 0 \text{ whenever } \pm \Re s < 0. \]

**Proof.** The following elementary argument is adapted from [Meg] (when \( R \) is nonsingular this follows from classical facts about the algebraic Riccati equation [Wil1]). We only consider \( P_+ \), with the \( P_- \) case being similar. Suppose on the contrary that there exists \( \lambda \) such that \( \Re \lambda < 0 \) and

\[ \begin{bmatrix} v \\ z \end{bmatrix}^* \begin{bmatrix} A - \lambda I & B \\
L_+ & W_+
\end{bmatrix} = 0, \quad \begin{bmatrix} v \\ z \end{bmatrix} \neq 0. \]  
(2.12)

Note that \( v \neq 0 \), otherwise \( z = 0 \) as well since \( \begin{bmatrix} L_+ & W_+ \end{bmatrix} \) has full row rank. Also \( z \neq 0 \) since \((A, B)\) is controllable. If \( \dot{x} = Ax + Bu \), then from (2.12) the function \( e^{-\lambda T} \langle x, v \rangle \) satisfies

\[ e^{-\lambda T} \langle x(T), v \rangle - \langle x(0), v \rangle = - \int_0^T e^{-\lambda s} \langle L_+ x(s) + W_+ u(s), z \rangle \, ds. \]

Since \( z \neq 0 \) and \( \Re \lambda < 0 \), there exists \( c > 0 \) such that if \( x(0) = 0 \) and \( x(T) = x_0 \), then

\[ c |\langle x_0, v \rangle|^2 \leq \int_0^T |L_+ x(t) + W_+ u(t)|^2 \, dt. \]

Crucially, this \( c \) is independent of \( T \) as well. From the dissipation inequality,

\[ V_t(x_0) + \int_0^T |L_+ x(t) + W_+ u(t)|^2 \, dt \leq E(w, 0, T). \]

Now choose \( x_0 \neq 0 \) such that \( \langle x_0, v \rangle \neq 0 \). Taking the infimum over all \( T \) and \((u, x)\) with \( x(0) = 0 \) and \( x(T) = 0 \) contradicts the definition of \( V_t \). \( \square \)

If \( R > 0 \), then \( W_\pm \) is invertible; by Schur complements, Lemma 2.12 implies that the spectrum of \( A - BW_\pm^{-1}L_\pm \) is contained in \( \{ \pm \Re s \geq 0 \} \). The next argument may have originated with Popov.
Lemma 2.13. Let \((A, B)\) be controllable and minimally stable with respect to the supply rate (2.4). If \(R > 0\), then \(\Lambda(P) \leq 0\) admits a solution \(P > 0\) if and only if \((\text{FDI})\) holds.

Proof. By controllability, minimal stability, and \((\text{FDI})\), the maximal solution \(P_+\) of \(\Lambda(P) \leq 0\) exists and is positive semidefinite. To see that \(\det P_+ \neq 0\), suppose to the contrary that \(P_+x_0 = 0\). Let \(w = (x, u)\) be a trajectory from the definition of minimal stability, namely \(\dot{x} = Ax + Bu\) with

\[
x(0) = x_0, \quad E(w, 0, t) \leq 0 \text{ for all } t \geq 0, \quad x(t) \to 0 \text{ as } t \to \infty.
\]

Since \(P_+ \geq 0\), the dissipation inequality implies that \(L_+x + W_+u = 0\) almost everywhere. Thus \(u = -W_+^{-1}L_+x\) (redefining \(u\) on a set of measure zero if necessary), so

\[
\dot{x} = (A - BW_+^{-1}L_+)x.
\]

But the spectrum of \(A - BW_+^{-1}L_+\) precludes \(x(t) \to 0\) unless \(x_0 = 0\). \(\square\)

2.5. Stability criteria. In this section we record two frequency conditions for exponential stability which are analogues of the classical circle and Popov criteria. Although these can be stated in terms of interconnection of dissipative systems, we take a more direct approach closer to Popov’s notion of hyperstability [PG]. Consider an LTI system

\[
\dot{z} = Az + Bu, \\
y = Cz + Du,
\]

and a given supply rate \(\sigma(y, u)\). Assume (2.5) admits a positive definite storage function \(V(z) = \langle Pz, z \rangle\), so that

\[
V(z(t)) \leq V(z(0)) + E(y, u, 0, t).
\]

Assume that the admissible inputs \(u\) are restricted so that \(E(y, u, 0, t) \leq 0\) for all \(t \geq 0\). The latter condition can be phrased more elaborately in terms of the modern theory of integral quadratic constraints or Safonov’s topological separation property [MR, SA, CS], but we do not pursue this here. Consequently, there exists \(c > 0\) such that

\[
|z(t)|^2 \leq c|z(0)|^2.
\]

A version of this discussion for exponential stability follows by conjugation with an exponential weight (see below).

First we record a version of the circle criterion that applies to systems of the form \(\dot{x} = Ax + B\phi(Cx)\); the proof is entirely standard. Let

\[
H(s) = C(s - A)^{-1}B
\]

denote the transfer matrix of the associated linear block.
Lemma 2.14. Let $r \geq 0$, and let $\phi : \mathbb{K}^n \to \mathbb{K}^n$ belong to the sector $[0, L]$, where $0 < L \leq \infty$. Assume that $(A, B)$ is controllable, $(A, C)$ is observable, and $A + kBC + rI$ is Hurwitz for some $k \in [0, L]$. If the frequency condition
\[
\Re H(i\omega - r) \leq 1/L
\]
holds whenever $\omega \in \mathbb{R}$ and $\det(A - (i\omega - r)I) \neq 0$, then the system $\dot{x} = Ax + B\phi(Cx)$ is $r$-exponentially stable.

Proof. If $\dot{x} = Ax + B\phi(Cx)$ and $r \geq 0$, then the function $z = e^{rt}x$ satisfies the LTI system
\[
\dot{z} = (A + rI)z + Bu,
y = Cz,
\]
where the inputs are subject to the feedback law $u = e^{rt}\phi(e^{-rt}y)$. Define the supply rate $\sigma(y, u) = 2L^{-1}|u|^2 - 2\Re\langle u, y \rangle$, in which case
\[
\begin{bmatrix}
Q & S^* \\
S & R
\end{bmatrix} = \begin{bmatrix}
0 & -C^* \\
-C & 2/L
\end{bmatrix}.
\]
The corresponding spectral density function is $\Pi(s) = 2/L - 2\Re H(s - r)$.

According to Lemma 2.7, the frequency condition (2.13) and the Hurwitz assumption on $A + kBC + rI$ implies that $A(P) \leq 0$ admits a solution $P \geq 0$. Since $(A, C)$ is observable, Lemma 2.11 implies that $P > 0$. Furthermore, by the sector condition,
\[
u = e^{rt}\phi(e^{-rt}y) \implies E(u, y, 0, t) \leq 0
\]
for all $t \geq 0$. By the discussion preceding the lemma, there exists $c > 0$ such that $e^{rt}|x(t)| \leq c|x(0)|$ whenever $\dot{x} = Ax + B\phi(Cx)$. $\square$

Next, we give a version of the Popov criterion for systems $\dot{x} = Ax + B\nabla f(Cx)$, which appears to be new. A similar result was given in [Bli], but the latter only applies if $\nabla f$ belongs to a finite sector. In contrast, the notion of at most mild concavity (cf. (1.5)) allows us to treat the infinite sector case, and is more useful for the applications in §3. We state the result of [Bli] in Lemma 2.16 below so that it can be compared with Lemma 2.15.

Lemma 2.15. Let $r \geq 0$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\phi = \nabla f$ belongs to the sector $[0, L]$, where $0 < L \leq \infty$. If $r > 0$, assume in addition that $f$ is at most mildly concave. Define the multiplier
\[
M(s) = \mu_0 + \mu_1(s + 2r),
\]
where $\mu_0, \mu_1 \geq 0$ satisfy $\mu_0 + \mu_1 > 0$ and
\[
M(s) \neq 0 \text{ whenever } \det(A - sI) = 0.
\]

(2.15)
Assume that $(A, B)$ is controllable, $(A, C)$ is observable, and $A + kBC + rI$ is Hurwitz for some $k \in [0, L]$. If the frequency condition

\[
\text{Re } M(i\omega - r)H(i\omega - r) \leq \mu_0/L \tag{2.16}
\]

holds whenever $\omega \in \mathbb{R}$ and $\det(A - (i\omega - r)I) \neq 0$, then $\dot{x} = Ax + B\nabla f(Cx)$ is $r$-exponentially stable.

**Proof.** The proof is similar to that of Lemma 2.14. This time augment the LTI system (2.14) with an additional output

\[
y' = C(A + 2rI)z + CBu. \tag{2.17}
\]

We can assume that $\mu_1 = 1$ and $\mu_0 = \mu > 0$, otherwise we are in the setting of Lemma 2.14. Define the supply rate

\[
\sigma(y, y', u) = \mu \left(2L^{-1}|u|^2 - 2\text{Re}\langle u, y\rangle\right) - 2\text{Re}\langle u, y'\rangle.
\]

(the real parts are superfluous since the signals can all be taken real-valued.) First we show dissipativity, observing that

\[
\begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} = \begin{bmatrix} 0 & -(A + 2rI + \mu I)^*C^* \\ -C(A + 2rI + \mu I) & 2/L - 2\text{Re}(CB) \end{bmatrix}.
\]

Since $(A, B)$ is controllable, the frequency condition (2.16) implies the existence of a Hermitian solution to $\Lambda(P) \leq 0$.

The existence of a solution $P \geq 0$ does not follow directly from Lemma 2.7. Instead, we present an argument from [Bar]. Given $\delta > 0$, apply (2.6) with $N = \delta C$. Thus

\[
Q_\ast = -2\delta \text{Re}(A + 2rI + \mu I)^*C^*C + 2\delta^2C^*(L^{-1} - \text{Re}(CB))C.
\]

From the upper left block of $\Lambda_\ast(P) \leq 0$, we obtain $2\text{Re}(A + \delta BC + rI)^*P - Q_\ast \leq 0$, or equivalently

\[
\text{Re}(A + \delta BC + rI)^*(P + \delta C^*C) \leq -\delta(r + \mu - \mu\delta L^{-1})C^*C.
\]

Since $(A + \delta BC + rI, C)$ is observable, by Lemma 2.10 we see that $A + \delta BC + rI$ has no imaginary eigenvalues whenever $\delta \in (0, L)$. On the other hand, $A + kBC + rI$ is Hurwitz by assumption for some $k \in (0, L)$. Since the boundary of Hurwitz matrices consists of matrices with at least one imaginary eigenvalue, it follows that $A + \delta BC + rI$ must be Hurwitz for all $\delta \in (0, k]$. By the Lyapunov theorem,

\[
P + \delta C^*C > 0
\]

for all $\delta \in (0, k]$. Let $\delta \to 0$ to conclude that $P \geq 0$.

Although observability of $(A, C)$ is at our disposal, it is not actually needed to deduce the existence of a solution $P \geq 0$ from (FDI) via the argument above. Indeed, if $C = 0$
then \( P = 0 \) is a solution to \( \Lambda(P) \leq 0 \); if \( C \neq 0 \), then by a Kalman decomposition we can assume that

\[
A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}
\]

with \((A_{11}, C_1)\) observable. Since \( Q = 0 \), there exists a block diagonal solution to \( \Lambda(P) \leq 0 \) of the form \( P = \text{diag}(P_{11}, 0) \) — simply take any Hermitian solution and extract its upper left entry (cf. Lemma 2.8). The previous argument then shows that \( P_{11} \geq 0 \), and hence \( P \geq 0 \).

Observability of \((A, C)\) is only used to conclude that \( \det P \neq 0 \). This follows from Lemma 2.11 and the fact that (2.15) implies observability of \((A, S)\). Finally, since we can assume \( f \geq 0 \) by replacing \( f \) with \( f - f(0) \), it is easy to see that

\[
u = e^{rt} \nabla f(e^{-rt} y) \implies E(y, y', u, 0, t) \leq f(y(0))
\]

for all \( t \geq 0 \) by differentiating \( e^{2rt} f(e^{-rt} y) \). When \( r > 0 \), this uses the at most mild concavity of \( f \). We deduce the existence of \( c > 0 \) such that \( e^{2rt} |x(t)|^2 \leq c (|x(0)|^2 + f(Cx(0))) \)

whenever \( \dot{x} = Ax + B \nabla f(Cx) \), which implies \( r \)-exponential stability. \( \square \)

The observability hypotheses of Lemmas 2.14 and 2.15 can weakened in various ways, but we do not pursue this here. Finally, we state a different version of the Popov criterion adapted from [Bli].

**Lemma 2.16 ([Bli]).** Let \( r \geq 0 \), and let \( f : \mathbb{R}^n \to \mathbb{R} \) be such that \( \phi = \nabla f \) belongs to the sector \([0, L]\), where \( 0 < L < \infty \). Define the multiplier

\[
M(s) = \mu_0 + \mu_1 s,
\]

where \( \mu_0, \mu_1 \geq 0 \) satisfy \( \mu_0 + \mu_1 > 0 \) and

\[M(s) \neq 0 \text{ whenever } \det(A - sI) = 0.\]

Assume that \((A, B)\) is controllable, \((A, C)\) is observable, and \( A + kBC + rI \) is Hurwitz for some \( k \in [0, L] \). If the frequency condition

\[
rL \mu_1 H(i \omega - r)^*H(i \omega - r) + \text{Re} \, M(i \omega - r)H(i \omega - r) \leq \mu_0/L
\]

holds whenever \( \omega \in \mathbb{R} \) and \( \det(A - (i \omega - r)I) \neq 0 \), then \( \dot{x} = Ax + B \nabla f(Cx) \) is \( r \)-exponentially stable.

**Proof.** The proof is similar to Lemma 2.15, but this time (2.17) should be replaced by \( y' = CAz + CBu \), and the supply rate is

\[
\mu \left( 2L^{-1} |u|^2 - 2 \text{Re}(u, y) \right) - 2rL |y|^2 - 2 \text{Re}(u, y').
\]
In showing that \( u = e^{rt} \nabla f(e^{-rt}y) \implies E(y, y', u, 0, t) \leq f(y(0)) \), we can now bound \( f(y) \leq (L/2)|y|^2 \) by the sector condition instead of using at most mild concavity. \( \square \)

3. Dissipative Hamiltonian systems

3.1. Preliminaries. Let \( x = (q, p) \) be coordinates on the phase space \( \mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d \), where \( d \geq 1 \). Given a \( C^1 \) function \( f : \mathbb{R}^d \to \mathbb{R} \), define the Hamiltonian

\[
H(x) = \frac{1}{2}|p|^2 + f(q) - f(0).
\]

By replacing \( f \) with \( f - f(0) \) we will assume that \( f(0) = 0 \). Assume \( \nabla f \) belongs to a sector \([m, L] \) for some \( 0 < m \leq L \leq \infty \) (see §1.2). Consider the system

\[
\dot{x} = (J - S) \nabla H(x),
\]

where \( J \) is the standard symplectic matrix, and \( S \in \mathbb{R}^{2d \times 2d} \) is positive semidefinite. Write

\[
S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}
\]

where \( S_{11}, S_{22} \) are positive semidefinite and \( S_{12}^* = S_{21} \). Rewrite (3.1) as a feedback system

\[
\dot{x} = Ax + Bu,
\]

\[
u = \nabla g(Cx),
\]

where \( g = f - (m/2)|q|^2 \) and the matrices \( A, B, C \) are given by

\[
A = \begin{bmatrix} -mS_{11} & I - S_{12} \\ -m(I + S_{21}) & -S_{22} \end{bmatrix}, \quad B = \begin{bmatrix} -S_{11} \\ -I - S_{21} \end{bmatrix}, \quad C = [I \ 0].
\]

Next, assume that \( S_{12} = 0 = S_{21} \). In that case it is easy to see \((A, B)\) is controllable and \((A, C)\) is observable. Finally, assume that \( S_{11}, S_{22} \) are multiples of the identity

\[
S_{11} = \tau I, \quad S_{22} = 2\sigma I,
\]

where \( \sigma > 0 \) and \( \tau \geq 0 \). From the resulting block structure, we can assume that \( d = 1 \) as far as verifying the hypotheses of Lemma 2.14 or 2.15 goes. Thus

\[
H(s) = \frac{1 + 2\sigma \tau + \tau s}{\det(s - A)} = \frac{-(1 + 2\sigma \tau + \tau s)}{s^2 + (2\sigma + \tau m)s + m(1 + 2\sigma \tau)},
\]

where \( H(s) = C(sI - A)^{-1}B \) is the (scalar) transfer function.
3.2. The linear problem. In order for the system (3.1) to be \( r \)-exponentially stable when \( \nabla f \) belongs to a sector \([m, L]\), it is necessary for the matrix

\[
A + (k - m)BC + rI = \begin{bmatrix}
r - k\tau & 1 \\
-k & r - 2\sigma
\end{bmatrix}
\]

(3.4)
to be marginally stable for every \( k \in [m, L] \). The characteristic polynomial of (3.4) is

\[
\chi(s) = s^2 + (2\sigma - 2r + k\tau)s + k + r^2 - 2r\sigma + (2\sigma - r)k\tau.
\]

(3.5)
The roots of \( \chi \) must lie in the closed left half-plane. Equivalently, as a function of \( s \), the linear and constant terms of \( \chi \) must be nonnegative. Furthermore, any imaginary roots must be simple. First we record the observations needed for Proposition 1.1.

Lemma 3.1. If \( \tau = 0 \), then (3.4) is marginally stable for every \( k \geq m \) if and only if \( r \leq \sigma \) and \( m \geq 2r\sigma - r^2 \), and at least one of these inequalities is strict. Furthermore, the following hold.

1. If \( r < \sigma \) and \( m > 2r\sigma - r^2 \), then \( A + rI \) is Hurwitz.
2. If \( r < \sigma \) and \( m = 2r\sigma - r^2 \), then \( A + rI \) has distinct eigenvalues \( 0 \) and \( 2(r - \sigma) \), and \( A + \delta BC + rI \) is Hurwitz for any \( \delta > 0 \).

Proof. This follows directly from (3.5).

If \( r = \sigma \) and \( m > 2r\sigma - r^2 \), then \( A + rI \) has distinct eigenvalues \( \pm i(m - \sigma^2)^{1/2} \), but \( A + \delta BC + rI \) is not Hurwitz for any \( \delta \geq 0 \).

The corresponding facts needed for Proposition 1.3 are as follows.

Lemma 3.2. If \( \tau > 0 \) and (3.4) is marginally stable for each \( k \geq m \), then

\[
r \leq \min(2\sigma + \tau^{-1}, \sigma + m\tau/2), \quad m\tau(2\sigma + \tau^{-1} - r) \geq 2r\sigma - r^2.
\]

(3.6)
If \( r = 2\sigma + \tau^{-1} = \sigma + \tau m/2 \), then \( A + rI \) has distinct imaginary eigenvalues, and \( A + \delta BC + rI \) is Hurwitz for any \( \delta > 0 \).

Proof. The constraints (3.6) come from the roots of (3.5) for \( k = m \) and in the limit as \( k \to \infty \). If \( r = \sigma + \tau m/2 \), then the linear term in \( \chi \) vanishes for \( k = m \); if additionally \( r = 2\sigma + \tau^{-1} \), then the constant term is \( r^2 - 2r\sigma \) for \( k = m \), which is strictly positive since \( r^2 = (2\sigma + \tau^{-1})r > 2\sigma r \). The Hurwitz statement follows because both the linear and constant terms are strictly increasing with \( k \) when \( \tau > 0 \).

Thus the optimal pair \((r_*, \tau_*)\) for the linear problem over the sector \([m, \infty] \) is obtained by solving \( r = 2\sigma + \tau^{-1} = \sigma + \tau m/2 \). The resulting values are

\[
r_* = \frac{3\sigma + \sqrt{2m + \sigma^2}}{2}, \quad \tau_* = \frac{\sigma + \sqrt{2m + \sigma^2}}{m},
\]

(3.7)
as in the statement of Proposition 1.3.
3.3. The infinite sector case. Define the multiplier \( M(s) = \mu_0 + \mu_1(s + 2r) \), as in Lemma 2.15. From (3.3), in the infinite sector case the frequency condition (2.16) is equivalent to

\[
a_2\lambda^2 + a_1\lambda + a_0 \geq 0 \quad \text{for all } \lambda \geq 0,
\]

where we have written \( \lambda = \omega^2 \), and the coefficients are given by

\[
a_0 = (\mu_0 + \mu_1 r)(1 + 2\sigma \tau - r\tau)(m + (r - 2\sigma)(r - \tau m)),
\]
\[
a_1 = \mu_1 \left(2\sigma(1 + 2\sigma \tau) - r \left(3 + 4\sigma \tau - m\tau^2\right)\right) - \mu_0(1 + \tau(r - m\tau))
\]
\[
a_2 = \mu_1 \tau.
\]

The inequality (3.8) holds if and only if either \( a_0, a_1, a_2 \geq 0 \), or

\[
a_0 > 0, \quad a_2 > 0, \quad 4a_2a_0 - a_1^2 \geq 0.
\]

It is now easy to prove Propositions 1.1 and 1.3 using Lemma 2.15.

**Proof of Proposition 1.1.** Throughout, assume that \( r > 0 \) and \( \tau = 0 \). Thus \( a_2 = 0 \), so (3.8) is equivalent to \( a_0, a_1 \geq 0 \). Now \( a_1 = -\mu_0 + \mu_1(2\sigma - 3r) \), so we choose

\[
\mu_1 = 1, \quad \mu = \mu_0 \geq 0.
\]

Now we show that Lemma 2.15 can be applied (for a suitable choice of \( \mu \geq 0 \)) whenever \( m \leq 2r\sigma - r^2 \) and \( r \leq 2\sigma/3 \), where at least one of these inequalities is strict. First note that \( a_0 \geq 0 \) since \( m \geq 2r\sigma - r^2 \). Also \( a_1 \geq 0 \), is equivalent to

\[
\mu \leq 2\sigma - 3r.
\]

By Lemma 3.1, if \( m \geq 2r\sigma - r^2 \) and \( r \leq 2\sigma/3 \), then \( A + \delta BC + rI \) is Hurwitz for some \( \delta \geq 0 \). To apply Lemma 2.15, it remains to verify (2.15). Equivalently, we show that \( \mu \geq 0 \) can be chosen so that \( -2r - \mu \) is not an eigenvalue of \( A \) and (3.9) holds. Consider the following two possibilities:

1. If \( r < 2\sigma/3 \), then either \( -2r \) is not an eigenvalue, in which case we take \( \mu = 0 \), or otherwise \( -2r - \mu \) is not an eigenvalue for arbitrarily small \( \mu > 0 \) satisfying (3.9).

2. If \( r = 2\sigma/3 \), then we must take \( \mu = 0 \), but recall that \( m > 2r\sigma - r^2 \). If \( -2r \) was an eigenvalue, then \( m = 4r\sigma - 4r^2 \) according to (3.5). When \( r = 2\sigma/3 \) this implies that \( m = 8\sigma^2/9 \), but this contradicts \( m > 2r\sigma - r^2 = 8\sigma^2/9 \).

The proof is now complete by Lemma 2.15.

**Proof of Proposition 1.3.** Let \( \tau = \tau_* \) and \( r = r_* \), where these quantities are defined in (3.7). Choosing \( \mu_1 = 0 \) renders \( a_0 = a_1 = a_2 = 0 \), and hence the frequency condition (3.8) holds. Furthermore, \( A + \delta BC + r_* \) is Hurwitz according to Lemma 3.2. Since \( \mu_1 = 0 \), Lemma 2.14 applies.
3.4. **The finite sector case.** Next, we address Proposition 1.2 using Lemma 2.14; it can be shown that neither Lemma 2.15 nor Lemma 2.16 yield a sharper result. Since \( \tau = 0 \), the frequency condition (2.13) is equivalent to

\[
\lambda^2 + b_1 \lambda + b_0 \geq 0 \text{ for all } \lambda \geq 0,
\]

where again \( \lambda = \omega^2 \), and the coefficients are given by

\[
b_1 = 4\sigma^2 - 2(2r\sigma - r)^2 - m - L,
b_0 = (m + r(r - 2\sigma))(L + r(r - 2\sigma)).
\]

**Proof of Proposition 1.2.** Note that \( b_0 = 0 \) since we are taking \( m = 2r\sigma - r^2 \). The condition \( b_1 \geq 0 \) is then equivalent to

\[
L - m \leq 4(r - \sigma)^2.
\]

By Lemma 3.1 we certainly have that \( A + \delta BC + rI \) is Hurwitz for \( \delta \in (0, L] \), so Lemma 2.14 applies.

\[\square\]

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