Finsler gravity action from variational completion

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In the attempts to apply Finsler geometry to construct an extension of general relativity, the question about a suitable generalization of the Einstein equations is still under debate. Since Finsler geometry is based on a scalar function on the tangent bundle, the field equation which determines this function should also be scalar equation on the tangent bundle. In the literature two such equations have been suggested: the one by Rutz and the one by one of the authors. Here we employ the method of canonical variational completion to show that Rutz equation cannot be obtained from a variation of an action and that its variational completion yields the latter field equations. Moreover, to improve the mathematical rigor in the derivation of the Finsler gravity field equation, we formulate the Finsler gravity action on the positive projectivised tangent bundle, which has the advantage of allowing us to apply the classical variational principle, by choosing the domains of integration to be compact and independent of the dynamical variable. In particular in the pseudo-Riemannian case, the vacuum field equation becomes equivalent to the vanishing of the Ricci tensor.

I. INTRODUCTION

A large source of information about the physical properties of spacetime is obtained by observing the motion of point particles. The observed trajectories are identified with the geodesics of the geometry of spacetime and thus by matching the observed curves with the predicted geodesics the viability of a certain geometry can be tested. Conversely, when a spacetime geometry is determined by dynamical physical field equations, its geodesics can be derived and observable effects can be predicted.

General relativity is based on pseudo-Riemannian geometry, i.e. a spacetime manifold equipped with a metric tensor of Lorentzian signature. The metric is determined by the Einstein equations, its geodesics predict the motion of point particles and geodesic deviation is sourced by the curvature of its Levi-Civita connection. On a huge variety of physical scales the predictions made on the basis of general relativity are outstandingly correct and in agreement with observation, however there are the well known shortcomings such as the rotational curves of galaxies and the accelerated expansion of the universe, which led to the introduction of the notions of dark matter and dark energy [1–4]. The most common approach to understand and explain dark matter and dark energy is to postulate the existence of additional particles to the ones in the standard model of particle physics, and the alternative is to look for extensions and modifications in the description of gravity [5]. In this article we follow the latter route and consider Finsler geometry as extended geometry of spacetime, which has been proposed as one possibility to shed light onto the dark universe phenomenology [6–10].

Finsler spacetime geometry is the geometry of a manifold equipped with a so called Finsler function which is a 1-homogeneous function on the tangent bundle of spacetime and defines a length measure for curves. It thus is the most general geometry with a geometric clock in the sense of the clock postulate, namely that the time an observer measures between to events is given by the length of its worldline.

Finsler spacetime geometry, resp. the geodesics of a Finslerian geometry, describe the motion of point particles subject to a dispersion relation, which is non-quadratic in the particle’s four momenta. Such modified dispersion relations (MDRs) appear most naturally in effective field theories in media, such as premetric electrodynamics [11] or wave equations in solids [12]. The former describes among other things the behavior of the electromagnetic field in crystals [13] and the latter for example the propagation of earthquakes waves [14]. Moreover MDRs are used as an effective description of quantum gravity effects [15–18], making spacetime effectively a medium whose origin lies in the four momentum dependent scattering of elementary particles with the graviton. More fundamentally MDRs emerge from non local

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Lorentz invariant field theories [19], like studied in the standard model extension [20], very special and very general relativity [21–23] or again in premetric resp. area metric electrodynamics [24, 25].

The appearance of Finsler geometry in physics raised the question, whether it is possible to find dynamical equations which determine the Finslerian spacetime geometry in the same way as the Einstein equations determine the pseudo-Riemannian geometry of spacetime. Up to now there is no generally accepted equation, but several attempts have been made [9, 26–34]. The difference in the approaches lies in the choice of the fundamental variable, Finsler function or Finsler metric tensor (accordingly in the type of the dynamical equation, scalar or tensorial) and in the way how the equation is obtained, by variation from an action, by formal resemblance to the Einstein equations or from further physical principles.

In this article we argue for physical dynamics for Finsler spacetime geometries which have the following properties:

- the fundamental variable is the geometry defining Finsler function, i.e. the field equation we are looking for is a homogeneous scalar equation on the tangent bundle, which determines a homogeneous function;
- the homogeneity of the Finsler function, which can be understood as equivariance with respect to the common group action of rescaling both on its domain and its codomain, allows taking the quotient with respect to this group action on both sides, which then turns the Finsler function into a mapping from orbits to orbits, which in turn is a section of a bundle over the positively projectivised tangent bundle;
- the field equation is obtained by variational means, starting from a well defined action integral;
- in the case when the Finslerian spacetime geometry is pseudo-Riemannian, the dynamics become equivalent to the Einstein equations

\[
G_{ij} = \frac{8\pi G}{c^4} T_{ij},
\]

where \(G_{ij} = r_{ij} - \frac{1}{2} r g_{ij}\) are the components of the Einstein tensor built from the components of the Ricci tensor \(r_{ij}\) and the Ricci scalar \(r\) of the Levi-Civita connection of the Lorentzian spacetime metric with components \(g_{ij}\).\(^1\)

We prove that the most promising conjectured Finsler spacetime dynamics, the one by Rutz [31] and the one by one of us [32], are actually related in the way that the latter is the variational completion of the former. A similar property can be found in the emergence of the Einstein equations. An early version of the Einstein equations was simply stating that \(r_{ij} \sim T_{ij}\). It has been shown that the left hand side of this equation cannot be obtained by variational calculus, not even in the vacuum case \(T_{ij} = 0\), and its variational completion is given by the Einstein tensor [35]. Hence by the demand of a variational equation for the Finsler function as fundamental variable of a Finslerian spacetime geometry, the simplest self consistent action based field equations are the ones which were derived in [32].

We establish this result in the following way. The variational completion algorithm is based on the notion of the Vainberg-Tonti Lagrangian associated to a given system of partial differential equations. This Lagrangian is determined solely by the PDE system and is the closest Lagrangian which has the PDE system as Euler-Lagrange equations. We find that the Vainberg-Tonti Lagrangian density of Rutz’s equation must be a product between the Lagrangian density of the Einstein–Hilbert action and another quantity, the canonical Finsler curvature scalar.

\[
\pi G
\]

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\[^1\] In order to distinguish Riemannian curvature-related geometric objects from Finslerian ones (as some of them have different definitions), we denote the curvature tensor in a pseudo-Riemannian space by \(r\) (small letter) and the Finslerian curvature-related quantities by capital letters.
The article is structured as follows. We begin by giving the basic definition of Finsler spacetimes and its geometry in Section II. In Section III we review the field equations for a Finsler function that are a scalar equation which determine the Finsler function, these are Rutz’s equation and the one developed by one of us. Afterwards in Section IV the positive projective tangent bundle is introduced as the stage where we formulate the Finsler gravity action. The main result is then presented in Section V where we show that the variational completion of Rutz equation is given by the the one developed by one of then authors. We confirm the field equations by variational calculus in Section VI, before we show their consistency with the Einstein equations in the case that the Finsler spacetime geometry is a pseudo-Riemannian geometry in Section VII. Finally we conclude in Section VIII.

II. FINSLER SPACETIMES

We begin by stating the basic notations and definitions of Finsler geometry we use throughout this article.

A. The Definition

Let $M$ be a connected, oriented, $C^\infty$-smooth manifold of dimension 4 and $(TM, \pi_{TM}, M)$, its tangent bundle. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas on $M$. We denote by $(x^i)_{i=0}^3$ the coordinates of a point $x \in M$ in a local chart $(U_\alpha, \varphi_\alpha)$; denoting, for any vector $\dot{x} \in T_xM$, by $(\dot{x}^i)$ the coordinates in the local natural basis $\{\partial_i = \partial/\partial x^i\}$ of $T_xM$, we obtain, for a point $(x, \dot{x}) \in \pi^{-1}_{TM}(U_\alpha) \subset TM$, the coordinates $(\dot{x}^1, \dot{x}^2, \dot{x}^3, \dot{x}^4)$, then, $\{(\pi^{-1}_{TM}(U_\alpha), d\varphi_\alpha)\}$ is an oriented atlas on TM. We will denote by $\cdot_\beta$ and $\partial_\alpha$ partial differentiation with respect with $x^\beta$ and $x^\alpha$ respectively.

By $F(TM)$, we will mean the set of $C^\infty$-smooth functions on $TM$. For any fibered manifold $\pi : Y \to X$, we will denote by $\Gamma(Y)$, the module of sections of $Y$ and by $\Omega(Y)$, the set of differential forms on $Y$.

A conic subbundle of $TM$ is, [36], a non-empty open submanifold $A \subset TM\backslash\{0\}$, with the following properties:

- $\pi_{TM}(A) = M$;
- conic property: if $(x, \dot{x}) \in A$, then, for any $\lambda > 0$ : $(x, \lambda \dot{x}) \in A$.

The first condition above ensures that $(A, \pi_{TM\mid A}, M)$ has a fibered manifold structure.

**Definition 1** By a Finsler spacetime, we understand a triple $(M, A, L)$, where $A$ is a conic subbundle of $TM$ such that $TM\backslash A$ has zero measure and $L : TM \to \mathbb{R}$ has the following properties:

1. $L$ is continuous on $TM$ and smooth on $A$;
2. $L$ is positively 2-homogeneous with respect to its $\dot{x}$ argument, i.e. $\forall \lambda > 0 : L(x, \lambda \dot{x}) = \lambda^2 L(x, \dot{x})$;
3. at any $(x, \dot{x}) \in A$ and in any local chart around $(x, \dot{x})$, the Hessian (called L-metric)
   \[g^L_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = \frac{1}{2} L_{ij}\]
   is nondegenerate;
4. at any $x \in M$, the set $\Omega_x = \{\dot{x} \in A \cap T_xM \mid L(x, \dot{x}) = \epsilon, \ g(x, \dot{x}) \text{ has signature } (\epsilon, -\epsilon, -\epsilon, -\epsilon)\}$ has at least one non-empty closed connected component $S_x$.

The set $A$ where the matrix $(g^L_{ij})$ exists and is invertible will be called the set of admissible vectors and we denote by $A_0 \subset A$ all admissible vectors which are non-lightlike, i.e. $L(x, \dot{x}) \neq 0 \forall (x, \dot{x}) \in A_0$. The existence of a set $S_x$ ensures the existence of unit timelike vectors, see [32, Sec. V] for an illustrative example. The Finsler function $F$, usually employed in standard textbooks about Finsler geometry [30, 37], is defined as $F = \sqrt{L}$ and the length measure for curves $\gamma : [a, b] \to M$ on $M$ is given by

\[\ell[\gamma; a, b] = \int_a^b F(\gamma, \dot{\gamma}) \, d\tau.\]

Observe that Definition 1 refines the classical one by Beem [38] and differs from the ones used in [39], [32] and in [36], since it does neither demand smoothness of $L$ on the boundary of $A$ or on the non-trivial null directions $\dot{x}$ for which $L(x, \dot{x}) = 0$ nor that there exists a certain power of $L$ which is smooth on the slit tangent bundle.
$TM \setminus (x, 0)$. In particular it includes Randers spacetimes $L = (\sqrt{g_{ab} \dot{x}^a \dot{x}^b} + A_a \dot{x}^a)^2$ and polynomial Finsler spacetimes for which the 2-homogeneous $L$ is obtained as appropriate root of the norm of a higher order polynomial in $\dot{x}$ like $L = \epsilon(G_{a_1 \ldots a_n}(x) \dot{x}^{a_1} \ldots \dot{x}^{a_n})^p$, $\epsilon = \text{sign}(G_{a_1 \ldots a_n}(x) \dot{x}^{a_1} \ldots \dot{x}^{a_n})$. From the viewpoint of physics the former describes the motion of a charged particle in an electromagnetic potential and, for $n = 4$ for example, the latter describes the propagation of light in premetric electrodynamics [11, 24, 30, 41].

### B. The Geometry

The geometry of Finsler spacetimes is constructed from objects obtained from derivatives acting on $L$. All details on geometry based on non-linear connections and of Finsler spacetimes can be found in the books [30, 37, 42]. Here we recall the notions we need throughout this article.

In any local chart of $TM$ the first derivative of $L$ w.r.t. $\dot{x}$ defines the momenta, or lower index velocities,

$$p_{(x, \dot{x})} = \dot{x}_i dx^i, \quad \dot{x}_i = \frac{1}{2} L_{,i},$$

the second derivative of defines $L$ the $L$-metric and its inverse

$$g^L_{(x, \dot{x})} = g^{ij}_L(x, \dot{x}) dx^i \otimes dx^j, \quad g^{ij}_L = \frac{1}{2} L^{,ij},$$

and the third derivative the so called Cartan tensor

$$C_{(x, \dot{x})} = C_{ijk}(x, \dot{x}) dx^i \otimes dx^j \otimes dx^k, \quad C_{ijk} = \frac{1}{2} g^{L,ijk} = \frac{1}{4} L^{,ijk}.$$  

By the homogeneity of $L$ the following equalities hold in every local coordinate chart

$$L(x, \dot{x}) = g^L_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j, \quad L_i = 2 \dot{x}_i = 2 g^L_{ij}(x, \dot{x}) \dot{x}^j, \quad \dot{x}_{ij} = g^{L,ij}_L(x, \dot{x}), \quad \dot{x}^i C_{ijk}(x, \dot{x}) = 0.$$  

A canonical invariant volume form can be defined on $\mathcal{A}$ by using the determinant of the $L$-metric as

$$dV = |\det(g^L)| dx^0 \land \ldots \land dx^3 \land d\dot{x}^0 \land \ldots \land d\dot{x}^3.$$  

The fundamental ingredient of the geometry of a Finsler spacetime is the geodesic spray from which one obtains the canonical non-linear connection, defining parallel transport. The geodesic equation of (3) in arclength parametrization can be written as

$$\ddot{\gamma}^i + 2G^i_{\gamma \dot{\gamma}} = 0,$$

where the geodesic spray coefficients are given by

$$2G^i = \frac{1}{2} g^{L,ij}(\dot{x}^k L_{,k,j} - L_{,j}),$$

and exist on the fibered manifold $\mathcal{A} \subset TM$. They define the coefficients $G^i_{\gamma j}$ of the canonical Cartan non-linear connection, which will be understood as defining a splitting of the tangent bundle $(T \mathcal{A}, \pi \mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ into a vertical subbundle $V \mathcal{A} = \ker(\pi \mathcal{A})$ and a horizontal subbundle $\mathcal{H} \mathcal{A}$ such that $T \mathcal{A} = \mathcal{H} \mathcal{A} \oplus V \mathcal{A}$. The local adapted basis will be denoted by $(\delta_i, \partial_i)$, where $\delta_i = \partial_i - G^i_{\gamma j} \partial_j$ and $\partial_i = \partial_i$. The connection coefficients are defined as

$$G^i_{\gamma j} = G^i_{\gamma j}.$$  

Besides the fundamental non-linear connection it is possible to define several linear connections on Finsler spacetimes. We do not regard these linear connections as fundamental but rather as tools to define tensorial quantities. For the purposes of this article we will use the so called Chern-Rund linear connection $D$ on $TM$ restricted to $\mathcal{A}$. It is locally defined by

$$D_{\delta_i} \delta_j = \Gamma_{\gamma kj} \delta_i, \quad D_{\delta_i} \partial_j = \Gamma_{\gamma kj} \partial_i, \quad D_{\delta_i} \delta_j = D_{\partial_i} \partial_j = 0,$$

where $\Gamma_{\gamma kj} := \frac{1}{2} g^{L,ij}(\delta_k g^{L}_{ij} + \delta_j g^{L}_{ik} - \delta_i g^{L}_{kj})$. We denote by $|D|$ $D$-covariant differentiation with respect to $\delta_i$. The difference between the derivative of the non-linear connection coefficients $G^i_{\gamma j k}$ and the Chern-Rund connection coefficients $\Gamma^i_{\gamma j k}$ defines the Landsberg tensor $P = P^i_{\gamma j k} \partial_i \otimes dx^j \otimes d\dot{x}^k$, with

$$P^{a}_{\gamma j k} = G^{a}_{\gamma j k} - \Gamma^{a}_{\gamma j k}, \quad \dot{x}^a P^{a}_{\gamma j k}(x, \dot{x}) = 0.$$
The geometric objects introduced so far satisfy some important identities regarding their differentiation w.r.t. the Chern Rund connection and the operator $\nabla = \dot{x}^i\partial_i$ (e.q. [37]):

$$\delta_i L = L_{|i} = 0, \quad \nabla L = 0 \quad (14)$$
$$\dot{x}^i|_j = 0, \quad \nabla \dot{x}^i = 0 \quad (15)$$
$$g^i_{|jk} = 0, \quad \nabla g^i_{jk} = 0 \quad (16)$$
$$\nabla C^i_{jk} = P^i_{jk}. \quad (17)$$

To understand the motivation of the Finsler gravity equation suggested by Rutz it is necessary to recall that the geodesic deviation equation on Finsler spacetimes takes the following form. Let $\gamma$ be a Finsler geodesic and $\dot{\gamma} = (\gamma, \dot{\gamma})$ be its lift to the tangent bundle with tangent $\dot{\gamma} = \dot{\gamma}^i\delta_i$. Moreover let $V$ be a deviation vector field on spacetime with canonical horizontal lift $\dot{V} = V^i\delta_i$. Then the geodesic deviation equation is

$$(\nabla \nabla V)_{(\gamma, \dot{\gamma})} = R(\dot{\gamma}, V). \quad (18)$$

The curvature tensor $R = R^i_j dx^j \otimes \delta_i$, which sources the geodesic deviation is derived from the curvature of the non-linear connection as

$$R^i_j = R^i_{jk}\dot{x}^k, \quad R^i_{jk} = [\delta_j, \delta_k]^i = \delta_k G^i_{jk} - \delta_j G^i_{kc} \quad (19)$$

The canonical Finsler Ricci scalar $R$ is given by its trace

$$R = R^i_i = R^i_{ik}\dot{x}^k. \quad (20)$$

It is important to observe that the curvature tensors appearing here are defined solely in terms of the canonical Cartan non-linear connection, as the operator $\nabla : \Gamma(HTM) \to \Gamma(HTM)$ only requires this structure on spacetime. The Finsler linear connections, which one may define, are not entering here.

In case the Finsler Lagrangian takes the form $L = g_{ij}(x)\dot{x}^i\dot{x}^j$, where $g_{ij}(x)$ are the components of a Lorentzian metric, the geometry of a Finsler spacetime $(M, A, L)$ becomes essentially the geometry of the pseudo Riemannian spacetime manifold $(M, g)$. The $L$-metric becomes the Lorentzian metric, the Cartan tensor vanishes, the non-linear connection coefficients and the non-linear curvature tensor become the Christoffel symbols and the Riemann curvature tensor of the Levi-Civita connection of $g$, up to a contraction with a velocity $\dot{x}$. Observe that the Finsler Ricci scalar becomes $R(x, \dot{x}) = r_{jk}(x)\dot{x}^j\dot{x}^k$ and is not equal to the Riemannian Ricci scalar $r = r_{ij}g^{ij}$ in this case.

When we construct an action for Finsler gravity in section VI we will work on the positive projective tangent bundle with 0-homogeneous objects. On $A_0$ we can introduce the 0-homogenized Ricci scalar$^2$

$$R_0 = \frac{1}{L} R, \quad (21)$$

which will be the key ingredient to the Lagrangian density defining the gravity action. Additionally we need a canonical invariant 0-homogeneous volume form on $A_0$, which is given by

$$dV_0 = \frac{1}{L^2} \sqrt{\det(g^L)}|dx^0 \wedge \ldots \wedge dx^3 \wedge d\dot{x}^0 \wedge \ldots \wedge d\dot{x}^3. \quad (22)$$

This volume form is indeed 0-homogeneous w.r.t. $\dot{x}$, which can be seen from the fact that

$$\mathcal{L}_C dV_0 = 0, \quad (23)$$

where the Liouville vector field $C = \dot{x}^i\partial_i$ is the generator of the homotheties $(x, \dot{x}) \mapsto (x, \lambda \dot{x})$. During the derivation of the Finsler gravity field equation the following divergence formula for horizontal and vertical vector fields, $X = X^i\delta_i$ and $Y = Y^i\delta_i$, on $A_0$, turn out to be very useful

$$\text{div}(X)dV_0 = \mathcal{L}_X (dV_0) \Leftrightarrow \text{div}(X) = (X^|i_P^i X^i), \quad \text{with} \quad P_i = P^j_{ji}, \quad (24)$$
$$\text{div}(Y)dV_0 = \mathcal{L}_Y (dV_0) \Leftrightarrow \text{div}(Y) = (Y^|i + 2C^i Y^k - \frac{4}{L} Y^j \dot{x}_j), \quad \text{with} \quad C^i = C^j_{ji}, \quad (25)$$

which imply for any $f : A_0 \to \mathcal{R}$,

$$\text{div}(f \dot{x}^i \delta_i) = \nabla f. \quad (26)$$

$^2 R_0$ is commonly denoted by $\text{Ric}$ in the literature. We choose the subscript 0 here to indicate its connection to the set $A_0$. 

III. FINSLER SPACETIME DYNAMICS

The demand that Finsler spacetime dynamics shall use the Finsler function as fundamental variable selects among the conjectured Finsler spacetime dynamics to the ones suggested in [31, 32, 43]. The first field equation which took the Finsler function as fundamental variable and was itself a scalar equation on the tangent bundle on the manifold was obtained by Rutz [31]. It was argued that from the geodesic deviation equation one finds the relevant curvature structure of spacetime which causes tidal forces between neighboring trajectories, and that its trace is a suitable approach as gravitational vacuum field equation. The same argument was applied in the pseudo-Riemannian case by Pirani to obtain the Einstein vacuum field equations [44]. Rutz’s equation simply states that the canonical non-linear Finsler curvature scalar (20) vanishes

\[ R = 0. \] (27)

It measures the trace of the geodesic deviation operator (18), understood as function of the Finsler Lagrangian \( L \) and its derivatives.

Action based Finsler field equations using the Finsler function as fundamental variable have been obtained by calculus of variation in [32] and [43] independently, in the first case Finsler spacetimes, in the later case for positive definite Finsler spaces. The action employed is

\[ S[L] = \int_{\Sigma \subset TM} \text{vol}(\Sigma) R|_\Sigma, \] (28)

where \( \Sigma = \{(x, \dot{x}) \in TM| F(x, \dot{x}) = 1\} \) is the unit tangent bundle and \( \text{vol}(\Sigma) \) the volume form on \( \Sigma \) defined from the Finsler metric. Variation with respect to \( L \) yields the Finsler spacetime vacuum dynamics\(^3\)

\[ 2R - \frac{L}{3}g^{ij}R_{ij} + \frac{2L}{3}g^{ij}(\nabla P_i)_j + P_{ij} - P_iP_j = 0. \] (29)

In case of a pseudo-Riemannian Finsler Lagrangian determined by a Lorentzian metric \( g \) Rutz’s equation as well as the action based Finsler spacetime dynamics are equivalent to the Einstein vacuum equations \( r_{ab} = 0 \). For the action based spacetime dynamics it is possible to add a matter field action so that the resulting gravitational dynamics reduce to the Einstein equations (1) [32].

We will see in section V in detail that Rutz’s equation has the disadvantage that it can not be obtained as an Euler-Lagrange equation. However, applying the variational completion algorithm developed in [35] to (27) yields the field equations (29). The analogue statement holds for the field equations \( r_{ij} = 0 \) and \( \tilde{G}_{ij} = 0 \) in general relativity. The former have the disadvantage that they can not be obtained as Euler-Lagrange equations, while the latter are the result of the variational completion algorithm applied to the former. Thus only the latter can be completed consistently to non-vacuum dynamics.

IV. THE STAGE FOR A FINSLER GRAVITY ACTION

As we have seen in the previous sections, all geometric objects in Finsler geometry possess homogeneity properties with respect to their dependence on the directions \( \dot{x} \). This means that they are equivariant under the action of a Lie Group, which makes it more appropriate to describe them on a bundle that takes this equivariance into account. In the previous approaches to action based Finsler gravity equations [32, 43] this fact was taken care of by constructing an action on the unit tangent bundle \( \Sigma = \{(x, \dot{x}) \in TM| F(x, \dot{x}) = 1\} \). However this construction has the drawback, that the domain of integration depends on the dynamical variable one is interested in, and, in the case of a Lorentzian signature of the Finsler metric, is non-compact.

To avoid this complication we construct the action integral for Finsler gravity on the projective tangent bundle \( PTM^+ \) in section VI. The advantage is that \( PTM^+ \) can be defined without any further structure on \( TM \), an so is in particular independent of the Finsler function. Here we introduce the projective tangent bundle and how one can understand the Finsler function as section of an associated vector bundle over \( PTM^+ \). This construction allows us a mathematically rigorous formulation of the Finsler gravity action as well as a technically precise derivation of the Euler-Lagrange equations from the action.

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3 Observe that in [43], the Landsberg tensor, called \( S^i_{jk} \) there, is defined with a different sign, \( S^i_{jk} = -P^i_{jk} \).
A. The positive projective tangent bundle

The positive projective tangent bundle $PTM^+$ can be constructed from the slit tangent bundle $\tilde{T}M := TM \setminus \{0\}$ in the following way: it consists of rays $\{(x, \lambda \hat{x}) \mid \lambda > 0\}$ and is defined by the equivalence relation

$$ (x, \hat{x}) \sim (x, u) \iff u = \lambda \hat{x} \text{ for some } \lambda > 0 $$

between elements $(x, \hat{x})$ and $(x, u)$ in $\tilde{T}M$. To be precise

$$ PTM^+ := \{[(x, \hat{x})_\sim] \mid (x, \hat{x}) \in T\tilde{M}^\circ\}. $$

For a 4-dimensional base manifold $M$ the positive projective tangent bundle is itself a 7-dimensional manifold, where the manifold structure by an atlas $\{(U_i^+, \varphi_i^+), (U_i^-, \varphi_i^-)\}$, where, e.g., $U_i^+ = \{(x^0, \ldots, x^3, \hat{x}^0, \ldots, \hat{x}^i, \ldots, \hat{x}^3) \mid x^i > 0\}$, $U_i^- = \{(x^0, \ldots, x^3, \hat{x}^0, \ldots, \hat{x}^i, \ldots, \hat{x}^3) \mid x^i < 0\}$ and we will denote collectively $\varphi_i^+$ and $\varphi_i^-$ as $(x^j, u^\alpha)$, $\alpha = 1, 2, 3$:

$$ (x^i, u^\alpha) = \left(x^0, \ldots, x^3, \frac{\hat{x}^0}{x^i}, \ldots, \frac{\hat{x}^{i-1}}{x^i}, \frac{\hat{x}^{i+1}}{x^i}, \ldots, \frac{\hat{x}^3}{x^i}\right). $$

The manifold $PTM^+$ is compact and orientable. This can be seen easily, e.g., as $PTM^+$ is diffeomorphic to the unit sphere bundle of an arbitrarily chosen (positive definite) Riemannian metric on $M$.

The positive projective tangent bundle is defined without any reference to additional geometric structure on $\tilde{T}M$. However, if the slit tangent bundle is equipped with a Finsler function with Finsler metric of positive definite signature, that consists of rays collectively $\varphi_i$ for $i = 0, \ldots, 3$. For Finsler spacetimes, and general Finsler function with Finsler metrics of indefinite signature, such a global diffeomorphism does not exist. What however does exist, as soon as one has a 1-homogeneous function on $\tilde{T}M$ is available, is a diffeomorphism between certain compact subsets $D^+$ of $PTM^+$ and specific compact subsets $D$ of the unit tangent bundle $\Sigma \subset \tilde{T}M$. We will now construct this diffeomorphism and relate the integration over $D^+ \subset PTM^+$ to the integration over $D \subset \tilde{T}M$. This will considerably simplify the work in local coordinates with an action for Finsler gravity on $PTM^+$.

The first step towards this goal is to observe that $\tilde{T}M$ with the action of the multiplicative group $R_+^*$

$$ \cdot : R^* \times T\tilde{M} \rightarrow T\tilde{M}, \quad (x, \hat{x}) \cdot \lambda = (x, \lambda \hat{x}), $$

is a principal bundle over $PTM^+$. This can be seen easily, as $R_+^*$ acts freely and transitively on the fibers of $\tilde{T}M$ relative to the projection:

$$ \pi^+: \tilde{T}M \rightarrow PTM^+, \quad (x, \hat{x}) \mapsto [x, \hat{x}]. $$

Conventionally, we will write this action as a right action - though this is not relevant, as the operation that defines it is commutative. The fibers $R_+^*$ of $(\tilde{T}M, \pi^+, PTM^+, R_+^*)$ are 1-dimensional Lie groups, generated by the Liouville vector field $C = \hat{x} \partial_\hat{x}$. That is, $C$ is a $\pi^+$-vertical vector field on $T\tilde{M}$.

Second, the projection $\pi^+$ allows us to treat differential forms on $PTM^+$ as certain particular differential forms on $T\tilde{M}$ via pullback. Let $\rho^+$ be a differential form on $PTM^+$, then $\rho := (\pi^+)^*\rho^+$ is a basic form with respect to $\pi^+$, i.e. it satisfies:

- equivariance with respect to the action of the Lie group $(R_+^*, \cdot)$, or, in other words, they are 0-homogeneous in their dependence on $\hat{x}$
  $$ \mathcal{L}_C \rho = 0, $$

- horizontality with respect to the projection $\pi^+$, which means that contracted with $C$ they satisfy
  $$ I_C \rho = 0. $$
Exterior differentiation of forms $\rho^+ \in \Omega(PTM^+)$ can be carried out identically to exterior differentiation of the corresponding form $\rho \in \Omega(TM)$ as, $d \circ (\pi^+)^* = (\pi^+)^* \circ d$.

The third step is to realize that $\pi^+$ is a diffeomorphism between a compact subset $D$ of the level hypersurface of a 1-homogeneous smooth map $\varphi$ on $\hat{T}M$

$$\varphi^{-1}(1) = \{(x, \dot{x}) \in \hat{T}M | \varphi(x, \dot{x}) = 1\}$$

and its image $\pi^+(D) \subset PTM^+$. This is easy to see. Injectivity is immediate. Moreover, considering $\hat{T}M$ in local coordinates of the form $(x^i, \varphi, \varphi u^\alpha)$, where $u^\alpha = \frac{\dot{x}^\alpha}{\varphi}$, $\alpha = 1, 3$, are 0-homogeneous functions, the Jacobian determinant

$$\det(\frac{\partial(x^i, \dot{x}^i)}{\partial(x^i, u^\alpha, \varphi)}) = \varphi^3$$

is nonzero on $D$. Consequently we can state

**Lemma 2** Pick any smooth function $\varphi : \hat{T}M \to \mathbb{R}$, that is 1-homogeneous in $\dot{x}$. For any connected, compact subset $D^+ \subset PTM^+$ such that $\varphi((\pi^+)^{-1}(D^+)) \neq 0$ and any 7-form $\rho^+$ on $PTM^+$:

$$\int_{D^+} \rho^+ = \int_D \rho,$$

where $\rho = (\pi^+)^* \rho^+$ is a 0-homogeneous differential form on $\hat{T}M$ and $D = (\pi^+)^{-1}(D^+) \subset \varphi^{-1}(1)$ is a compact, connected subset.

Hence we related integrals on $PTM^+$ to integrals on $\hat{T}M$.

Observe that vector fields $X^+$ on $D^+ \subset PTM^+$ can be identified with zero homogeneous vector fields $X$ on $D = (\pi^+)^{-1}(D^+)$, i.e., vector fields satisfying $[X, C] = 0$, via the tangent map $d(\pi^+)^{-1}$.

Let us next apply this construction to Finsler spacetimes.

### B. Finsler spacetime action integrals on $PTM^+$

On a Finsler spacetime $(M, A, L)$ integrals on $PTM^+$ can be understood as follows. Choose as 1-homogeneous function $\varphi = F = \sqrt{|L|}$, which is smooth on the connected components of $A_0$. The hypersurface $\varphi^{-1}(1) = \Sigma \cap A_0$ is the intersection of the unit tangent bundle with the set of non-lightlike admissible directions. Thus, according to our finding in Lemma 2, any integral over a compact domain $D$ in $\Sigma \cap A_0$ (where the domain of integration depends on the Finsler function) can be equated to an integral over a compact domain $D^+ \subset PTM^+$, which is defined independently of the Finsler function.

Any 7-form $\rho^+$ can be decomposed into a product of the canonical volume form $dV_0^+$ on $PTM^+$ and a function. The volume form can be obtained via the canonical Hilbert form. Consider the set

$$A_0^+ = \pi^+(A_0) \subset PTM^+.$$ 

The functions $l_i = \partial_i F$ are well defined on $A_0$ and, by their 0-homogeneity with respect to $\dot{x}$, also on $A_0^+$. This implies that the Hilbert form

$$\omega = l_i dx^i,$$

is a well defined coordinate invariant 1-form on $A_0^+$ and

$$\omega \wedge (d\omega) \wedge (d\omega) \wedge (d\omega) \neq 0$$

Observe that a more general statement is true: $\pi^+$ is a diffeomorphism between $Q = \varphi^{-1}(1) \cap \hat{T}M$ and $\pi^+(Q)$. However for our applications the restriction to compact sets $D$ and $D^+$ suffices.
is a well defined 7-form on $\mathcal{A}_0^+$ [45]. Hence, a coordinate invariant, well-defined volume form on $\mathcal{A}_0^+$ is given by

$$
\begin{align*}
    dV_0^+ &= \frac{\epsilon}{3} \omega \wedge (d\omega) \wedge (d\omega),
\end{align*}
$$

where $\epsilon = \text{sign} (\det g^L)$. The pullback of $dV_0^+$ by $\pi^+$ yields a 7-form on $\mathcal{A}_0$, which can be expressed in terms of the volume form $\omega$ on $\mathcal{A}$, see (8), respectively $\div(\mathcal{I}_c \omega) = \frac{1}{L^2} \mathcal{I}_c dV$.

Note that for 0-homogeneous vector fields $X$ on $T_v \Omega$, their divergence with respect to the volume form $dV_0$ satisfies

$$
\div(X) \mathcal{I}_c dV_0 = d(\mathcal{I}_c \mathcal{I}_c dV_0) .
$$

Hence, on Finsler spacetimes, integrals on compact domains $D^+ \subset PTM^+$ can be written as integrals on $D \subset (\Sigma \cap \mathcal{A}_0)$

$$
\int_{D^+} f dV_0^+ = \int_D f \mathcal{I}_c dV_0 ,
$$

where $f$ is a 0-homogeneous function on $T_v \Omega$. For us $f$ will be the Lagrangian which we will obtain from variational completion of Rutz’s equation in section $\textbf{V}$. An important Lemma, inspired by a similar statement on Finsler spaces found in [43], which allows us to evaluate and manipulate the action integral later is

**Lemma 3** Let $(M, \mathcal{A}, L)$ be a general Finsler spacetime. Fix an arbitrary compact domain $D^+ \subset PTM^+$, such that $D^+ = \pi^+(D)$, with $D \subset (\Sigma \cap \mathcal{A}_0)$. Then for any smooth function $f : PTM^+ \rightarrow \mathbb{R}$ and in any local chart

$$
\begin{align*}
    g^{Lij}(Lf)_{i,j} &= 8f + \div(Lg^{Lij} f_j \dot{\gamma}_i) ,
\end{align*}
$$

which implies

$$
\begin{align*}
    \int_{D^+} g^{Lij}(Lf)_{i,j} dV_0^+ &= 8 \int_{D^+} f dV_0^+ + \int_{\partial D^+} Lg^{Lij} f_i n_j d\sigma^+ ,
\end{align*}
$$

where $n$ is the oriented surface normal to $\partial D^+$ and $d\sigma^+ = \mathcal{I}_c dV_0^+$. The proof the Lemma can be found in the Appendix B. For example, applied to $f = \dot{x}^m \dot{x}^n L^{-1}$, it shows that the contraction with the Finsler metric is the same as the contraction with directions $\dot{x}$ up to a boundary term

$$
\begin{align*}
    \int_{D^+} g^{Lij}(\dot{x}^m \dot{x}^n)_{i,j} dV_0^+ = 2 \int_{D^+} \dot{x}^m \dot{x}^n L^{-1} dV_0^+ + \int_{\partial D^+} Lg^{Lij}(\dot{x}^m \dot{x}^n/L) n_j d\sigma^+ .
\end{align*}
$$

The last technical construction to write down the Finsler gravity action in section $\textbf{VI}$, is to understand our dynamical variable $L$ as a section of a fibered manifold [46]. It turns out that the most natural such choice is an associated bundle to the principal bundle $PTM^+$. We have already discussed that $(\mathcal{O}, \pi^+, PTM^+)$ is a principal bundle. By the definition of the following equivalence relation on $T_v \times \mathbb{R}$

$$
(x, \dot{x}, z) \sim (x, \lambda \dot{x}, \lambda^2 z)
$$

for all $\lambda > 0$ we can construct the associated bundle $(Y, \pi_Y, PTM^+)$, with

$$
Y := (\mathcal{O} \times \mathbb{R}^*_+) \sim, \quad \pi_Y([x, \dot{x}, z]) = [x, \dot{x}]
$$

and fiber $\mathbb{R}^*_+$. Coordinates in a fibered chart on this manifold then are $(x, \dot{x}, \mu^\alpha, z)$. It is now easy to see that there is a one-to-one correspondence between 2-homogeneous maps $L : T_v \rightarrow \mathbb{R}$ and sections $\gamma$ of $Y$

$$
L \mapsto \gamma : PTM^+ \rightarrow Y , \gamma ([x, \dot{x}]) = [x, \dot{x}, L(x, \dot{x})] .
$$

This can be checked as follows. The mapping is well-defined, as, for any $(x, \lambda \dot{x}) \in [x, \dot{x}]$, we have $[x, \lambda \dot{x}, L(x, \lambda \dot{x})] = [x, \dot{x}, L(x, \dot{x})]$ by virtue of (49). Its injectivity and surjectivity are immediate.

This completes the discussion of the technical ingredients to apply the variational completion algorithm to Rutz’s equation.
V. RUTZ’S EQUATION AND ITS VAINBERG-TONTI LAGRANGIAN ON $PT^M^+$

Canonical variational completion [35] is a powerful algorithm to assess whether a certain set of field equations can be obtained by variation of an action functional or not. In case it can be obtained by variation, the algorithm determines the action, and, in the contrary case, the algorithm determines a closest completion of the equations to make them variational. Before we apply the technique to Rutz’s equation (27), we recall its main steps.

Consider a set of $m$ partial differential equations (PDEs) of order $r$ in the independent variables $x^A \in \mathbb{R}^N$ (regarded as coordinates in a local chart $U$ on some manifold $X$) and the dependent variables $y^\mu = y^\mu(x^A)$

$$\varepsilon_\sigma(x^A, y^\mu, y^\mu_A, \ldots, y^\mu A_1 \ldots A_r) = 0,$$

where $A = 1, \ldots, N$ and $\mu, \nu = 1, \ldots, m$. The subscripts on $y^\mu$ denote partial differentiation, i.e. $y^\mu_A = \partial_x^A y^\mu$ and so on, as usual for jet bundle coordinates. Note that the number $m$ of equations coincides with the number of dependent variables.

From equations (52), we can build, on a given coordinate chart, the so-called Vainberg-Tonti Lagrangian density

$$\mathcal{L} = y^\sigma \int_0^1 \varepsilon_\sigma(x^A, ty^\mu, ty^\mu_A, \ldots, ty^\mu A_1 \ldots A_r) dt .$$

The Vainberg-Tonti Lagrangian density $\mathcal{L}$ is the "closest" Lagrangian density to our PDE system, in the sense that, if equations (52) are locally variational, i.e., if they can be locally written as the Euler-Lagrange equations attached to some Lagrangian density, then, this Lagrangian density is, up to a total derivative term, $\mathcal{L}$.

The quantities which measure the departure of the original PDE system of interest of being variational are the components of the so called Helmholtz form

$$H_\sigma := E_\sigma - \varepsilon_\sigma ,$$

where

$$E_\sigma = \frac{\partial \mathcal{L}}{\partial y^\sigma} - \frac{d}{du^\nu} \frac{\partial \mathcal{L}}{\partial y^\sigma_i} + \ldots + (-1)^r \frac{d}{du^{\nu r}} \frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 \ldots i_r}}$$

are the Euler-Lagrange expressions of $\mathcal{L}$. The following result is the key to examine if the original PDEs we started with were variational, see [35]:

**Proposition 4** The PDE system (52) is locally variational if and only if, in any local chart the Helmholtz conditions

$$H_\sigma = 0 , \quad \sigma = 1, \ldots, m ,$$

hold.

The canonical variational completion, see again [35], of the PDE system $\varepsilon_\sigma = 0$ are the PDEs

$$E_\sigma = 0 .$$

The term canonical comes from the fact that, adding to the left hand sides of equations (57) any locally variational term will still result in a variational PDE system. But (57) are the closest variational equations to the initial ones, as indicated by (56). In particular, they are completely determined by the functions $\varepsilon_\sigma$ alone.

To illustrate the framework notice that a typical example of variational completion is the derivation of the completion of the Einstein vacuum equations. On a Lorentzian manifold $(M, g)$, the canonical variational completion of the equations $r_{ij} = 0$ are the full equations $r_{ij} - \frac{1}{4} r g_{ij} = 0$ [35].

Finally let us apply the canonical variational completion to Rutz’s equation (27). The setup is

$$X = D^+ \subset PT^M^+ , \quad x^A = (x^i, \dot{x}^i) \text{ or } x^A = (x^i, u^\alpha) , \quad y = y^1 = L \text{ (i.e. } \mu = 1) ,$$

where the first coordinates mentioned are homogeneous coordinates [45], while the second are the ones introduced in (32).

In order to get a correct scalar density, let us multiply Rutz’s equation by $|\det g^L|$. In addition we multiply it by $L$ to an arbitrary power $\alpha$, in order to be able to adjust the homogeneity of the desired Lagrange density later. Thus, Rutz’s equation becomes:

$$\varepsilon := L^\alpha R |\det g^L| = 0 .$$
Taking into account the local expression of $R = R_{i,j}^l \dot{x}^l$, we see that $\varepsilon = \varepsilon(\dot{x}^i, L, L_{,i}, L_{,i}, \ldots, L_{,i,j,k,l})$ depends on $L$ and its partial derivatives up to order four. For each coordinate neighborhood on $D^\alpha$, we find the local Vainberg-Tonti Lagrangian as,

$$\mathcal{L} = \mathcal{L} = L \int_0^1 \varepsilon(x^i, \dot{x}^i, tL, tL_{,i}, tL_{,i}, \ldots, tL_{,i,j,k,l}) \, dt. \quad (60)$$

To evaluate the integral in the Lagrangian we note that, with respect to the fiber homotheties $L \mapsto \tilde{L} := tL$, the metric tensor components $g_{ij}^L$ and the inverse metric transform as

$$\tilde{g}_{ij} = t g_{ij}^L, \quad \tilde{g}^{Lij} = t^{-1} g^{Lij}. \quad (61)$$

The geodesic spray (10) behaves thus as

$$\tilde{G}^i(\tilde{L}, \tilde{L}_{,j}, \tilde{L}_{,i,j}) = G^i(L, L_{,j}, L_{,i,j}, L_{,i,j}), \quad (62)$$

which implies the same behaviour for the curvature and most importantly for the Finsler Ricci scalar

$$\tilde{R}^i_{jk} = R^i_{jk}, \quad \tilde{R} = R. \quad (63)$$

The last missing ingredient in the Vainberg-Tonti Lagrangian is the volume form factor which, by the fact that we are considering a four dimensional manifold $M$, transforms as

$$|\det \tilde{g}^L| = t^{|L|} |\det g^L|. \quad (64)$$

Employing the scaling behaviours just discussed we find the desired Lagrangian

$$\mathcal{L} = L^{\alpha+1} R |\det g^L| \int_0^1 t^{\alpha+4} dt = \frac{1}{\alpha + 5} L^{\alpha+1} R |\det g^L|. \quad (65)$$

In order to define a proper Lagrangian on $PTM^+$, we must construct a 4-form of the type $\rho = f \varepsilon dV_0$, with a zero homogeneous $f$, as discussed below (45). To achieve this the above expression for $\mathcal{L}$ must be $-4$-homogeneous and so $\alpha$ must be chosen to be $-4$, since $R$ is 2-homogeneous and $\det g^L$ is 0-homogeneous.

Thus we conclude that the Lagrange density which yields the variationally completed field equations to Rutz’s equation is

$$\mathcal{L} = L^{-3} R |\det g^L|. \quad (66)$$

This Lagrange density coincides with the ones suggested in [32] and [43] (for positive definite Finsler spaces), here derived by the means of variational completion.

Following the canonical variational completion algorithm we found that if Rutz’s equation (27) is variational then the Lagrangian from which it shall be obtained by variation is given by (66). What we will find in the next section is that the Euler-Lagrange equation of (66) does not coincide with Rutz’s equation, so Rutz’s equation can not be variational but must be variationally completed by the terms we will find next.

## VI. FINSLER GRAVITY ACTION AND ITS FIRST VARIATION

The last step in the variational completion algorithm is to check whether the seed equation (52) can be obtained by variational calculus from the action defined by its Vainberg-Tonti Lagrangian (60). If so, the seed equation itself is variational, if not we find the closest variational completion of the seed equation.

The classical variational principle, [46], requires the existence of a fibered manifold $(Y, \pi, X)$, dim $X = n$, dim $Y = m + n$. The manifold $Y$ is called the configuration manifold and $X$, the base (typically - but not necessarily - spacetime) manifold. Sections $\Gamma \in \Gamma(Y)$ will be interpreted as fields. Deformations of a field (section) are given by 1-parameter groups of fibered automorphisms, generated by projectable vector fields on $Y$.

In this setting, a Lagrangian of order $r$ is regarded as a horizontal differential form on the jet bundle $J^r Y$. Denoting by $(x^A, y^a, y^a_A, \ldots, y^a_{A_1 \cdots A_n})$ the fibered coordinates on $J^r Y$, a Lagrangian is locally expressed as $\lambda = \mathcal{L} d^m x^A$, where the Lagrangian density $\mathcal{L} = \mathcal{L}(x^A, y^a, y^a_A, \ldots, y^a_{A_1 \cdots A_n})$ is a real-valued function on some open subset of $J^r Y$. 
A. The Finsler gravity action on $PTM^+$

Consider an arbitrary admissible compact domain $D^+ = \pi^+(D) \subset PTM^+$, where $D \subset (\Sigma \cap A_0)$: As $D^+$ contains no lightlike directions, $L$ has a constant sign on $D^+$ and we can assume with no loss of generality that $L > 0$ on $D^+$. Moreover consider the Vainberg-Tonti Lagrangian (66) we constructed in the previous section. Then, the action associated to the Lagrangian $\mathcal{L}$ and to the compact domain $D^+$ is the mapping $S_D : \Gamma(Y) \to \mathbb{R}$, $\gamma \mapsto S_D(\gamma)$ (recall the definition of $Y$ in (50)), given by

$$S_{D^+}(\gamma) = \int_{D^+} J^4 \gamma^* \lambda,$$

(67)

where $J^4 \gamma^* \lambda$ is the pull-back of $\lambda$ to $PTM^+$, along the jet prolongation $J^4 \gamma$ of $\gamma$ to the jet bundle $J^4 \mathcal{Y}$.

In local homogeneous coordinates on $PTM^+$, $\gamma$ can be expressed as $\gamma(x, \dot{x}) = (x, \dot{x}, L(x, \dot{x}))$ and the action becomes

$$S_{D^+}(\gamma) = \int_{D^+} R_0 dV^+_0 = \int_{D^+} R_0 i_{\bar{x}} dV_0 = \int_{D^+} R \frac{i}{L^8} \det g^L | i_{\bar{x}}(d\bar{x}^0 \land \ldots \land d\bar{x}^3 \land d\dot{x}^0 \land \ldots \land d\dot{x}^3),$$

(68)

where we used the results from Lemma 2 to rewrite the integral over $D^+ \subset PTM^+$ into an integral over $D \subset (\Sigma \cap A_0)$.

Using Lemma 3, an equivalent form of the action is given by

$$\int_{D^+} R_0 dV^+_0 = \frac{1}{8} \int_{D^+} g^{Lij} R_{ij} \ dV^+_0 - \frac{1}{8} \int_{\partial D^+} g^{Lij} R_{0;ij} \ d\sigma^+, \quad (70)$$

which will be used in section VII to relate the Finsler gravity action to the Einstein–Hilbert action.

B. Derivation of the field equations

Take an arbitrary vertical vector field $\Xi := 2v \frac{\partial}{\partial z}$ on $Y$, where $v = v(x^i, \dot{x}^i, z)$, has its support strictly contained in $D^+$ and denote by $\{\Phi_t\}$ the 1-parameter group of fibered automorphisms of $Y$ generated by $\Xi$. The deformed sections:

$$\bar{\gamma} := \Phi_t(\gamma)$$

automatically correspond to equivariant (i.e, 2-homogeneous) functions $\bar{L}$ on non-lightlike, admissible domains of $TM$. This also implies the fact that the functions $v$ have to be 2-homogeneous in $\dot{x}$, $v$ and its partial derivatives will vanish on the boundary $\partial D^+$. We notice the relation:

$$\delta L := \frac{dL}{dt}|_{t=0} = 2v. \quad (71)$$

Moreover, for small enough $t$, the signature of the corresponding Hessian remains the same, i.e., $\bar{L}$ is a spacetime Finsler function.

The first variation of the action $S_{D^+}(\gamma)$ is:

$$\delta S_{D^+}(\gamma) := \frac{d}{dt}|_{t=0}(S_{D^+}(\bar{\gamma})). \quad (72)$$

Critical points, or extremals, $[\bar{\gamma}]$, are defined by the condition that, for any admissible, non-lightlike compact domain $D^+ \subset PTM^+$ and any $v$ with support contained in $D^+$, $\delta S_{D^+}(\gamma) = 0$.

The expression $\delta S_{D^+}(\gamma)$ will be split into three integrals

$$\delta S_{D^+}(\gamma) = (I_1 + I_2 + I_3), \quad (73)$$

where

$$I_1 = - \int_{D^+} \frac{3}{L^2} \frac{dL}{dt}|_{t=0} R dV^+_0, \quad (74)$$

$$I_2 = \int_{D^+} \frac{1}{L} \frac{dR}{dt}|_{t=0} dV^+_0, \quad (75)$$

$$I_3 = \int_{D^+} R \frac{1}{\det g^L} \frac{d(\det g^L)}{dt}|_{t=0} dV^+_0. \quad (76)$$
The first integral is easily evaluated to
\[ I_1 = -\int_{D^+} \frac{6}{L^2} R v \, dV_0^+ \, . \] (77)

The other two integrals involve rather lengthy calculation which we will display in detail in Appendix A and yield
\[ I_2 = -\int_{D^+} \frac{2}{L} R^{Lij} (P_{lij} - P_i P_j + (\nabla P_i)_j) v \, dV_0^+ \, , \] (78)
\[ I_3 = \int_{D^+} \frac{1}{L} R^{Lij} R_{i,j} v \, dV_0^+ \, . \] (79)

Thus finally we obtain that the extremal points of the Finsler gravity vacuum action (68), formulated in the positive projective tangent bundle, must satisfy
\[ \delta S_{D^+}(\gamma) = \int_{D^+} \left[ \frac{L}{2} R^{Lij} R_{i,j} - 3R - L R^{Lij} (P_{lij} - P_i P_j + (\nabla P_i)_j) \right] \frac{2v}{L} dV_0^+ = 0 \, , \] (80)

which leads us to formulate

**Theorem 5** Critical points of the Finsler gravity action (68) formulated on the positive projective tangent bundle \( PTM^+ \) are given by the equation:
\[ \frac{1}{2} g^{Lij} R_{i,j} - 3R - \frac{g^{Lij}}{L} (P_{lij} - P_i P_j + (\nabla P_i)_j) = 0 \, . \] (81)

This equation is identical to the one found in [32] on the unit tangent bundle and in [43] for positive definite Finsler spaces. The important new ingredients here are that the integration domains on \( PTM^+ \) are compact and do not depend on the Finsler Lagrangian as well as that the Lagrange density used in the action (68) was obtained by variational completion in the previous section V.

**VII. THE PSEUDO-RIEMANNIAN CASE**

Before concluding, we exemplify our findings for pseudo-Riemannian Finsler geometries \( L(x, \dot{x}) = g_{ij}(x) \dot{x}^i \dot{x}^j \), which are defined by a metric \( g = g_{ij}(x) dx^i \otimes dx^j \) with Lorentzian signature.

For such Finsler Lagrangians the Landsberg tensor vanishes \( P_i = 0 \), the components of the Finsler metric become identical to the components of the Lorentzian metric \( g^L_{ij}(x, \dot{x}) = g_{ij}(x) \), the Finsler Ricci scalar is the contracted Ricci tensor \( R = -r_{ij}(x) \dot{x}^i \dot{x}^j \) and thus, multiplying by \( L \), the Finsler gravity equation becomes
\[ 0 = 3R - \frac{L}{2} g^{Lij} R_{i,j} = -3r_{ij}(x) \dot{x}^i \dot{x}^j + (g_{ij}(x) \dot{x}^i \dot{x}^j) r(x) \, , \] (82)

where \( r(x) = r_{ij}(x) g^{ij}(x) \) is the Riemannian Ricci scalar. Taking a second derivative with respect to \( \dot{x} \) the equations turn out to be equivalent to
\[ 3r_{ij} - g_{ij} r = 0 \, , \] (83)

which implies \( r = 0 \) and hence we find equivalence to the Einstein vacuum equations \( r_{ij} = 0 \). Notice that (83) is not \( 2r_{ij} - g_{ij} r = 0 \), which one may have expected, however, for the vacuum equations this does not matter, since we end up with \( r_{ij} = 0 \) in both cases. One might wonder if this factor of three is contradicting the Bianchi identities. In an upcoming work, where we will discuss the invariance of the Finsler gravity action under manifold induced diffeomorphism, we will show that this is not the case, but quite the opposite is true. The factor is necessary for consistency with the Bianchi identities.

For positive definite Finsler spaces and an action integral on the unit tangent bundle \( \Sigma \) it has been shown, see [43], that
\[ \int_{\Sigma} R_0 \, i_{\Sigma} dV_0 = \text{const.} \int_{M} d^4 x \sqrt{|\det g|} \, r \, . \] (84)

The proof involves integration over the whole fibers of the unit tangent bundle, which are the unit norm Finsler spheres in this case. In the Lorentzian signature case this is not possible since the fibers of \( \Sigma \) are no longer compact.
and $R_0$ as well as the volume form $i_C dV_0$ are not well defined where $L = 0$, i.e. along lightlike directions. Therefore, any extension to the Lorentzian signature case has to be done with maximum care.

In the following we will prove that a local analogue of the equivalence statement of the positive definite case can be established for a metric $g$ with Lorentzian signature.

We begin by noticing

**Proposition 6** Let $(M, A, L)$ be a Finsler spacetime defined by a Lorentzian metric $g$ such that $L = g_{ij}(x) \dot{x}^i \dot{x}^j$. The Finsler gravity Lagrangian $R_0$ is proportional to the Ricci scalar of $g$, up to a divergence

$$R_0 = - \frac{1}{4} r(x) + \frac{1}{8} \text{div}(Lg^{ij} f_j \partial_i),$$  

(85)

This implies that the Lagrangians $R_0$ and $r(x)$ are dynamically equivalent, i.e. have the same Euler-Lagrange equations.

This can easily be proven by using (46) with $f = R_0$.

An identity, resp. proportionality, of the Einstein–Hilbert action and the Finsler gravity action, as in the Riemannian case, is only true modulo boundary terms. More precisely consider small admissible, non-lightlike compact domains $D^+ \subset PTM^+$, i.e. $D^+ = \pi^+(D)$, where $D \subset (\Sigma \cap A_0) = D_M \times D_1$ for $D_M = \pi(D) \subset M$. This holds for $D$ being contained in a single vector chart on $TM$. Then the Finsler gravity action (68) differs from the usual Einstein–Hilbert action $\int_{D_M} d^4x \sqrt{|\det g|} r$ by a boundary term

$$\int_{D^+} R_0 \ dV_0^+ = - \frac{\text{vol}(D_1)}{4} \int_{D_M} \sqrt{|\det g(x)|} \ r(x) \ d^4x + \text{boundary term}.$$  

(86)

To see this note that the first integral on the right hand side of the action (68) built from (85) can be split in the following way

$$- \frac{1}{4} \int_D r(x) \ i_C dV_0 = - \frac{1}{4} \int_{D_M} [r(x) |\det g(x)| \int_{D_1} i_C (dx^0 \wedge \ldots \wedge dx^3 \wedge dz^0 \wedge \ldots \wedge dz^3)],$$  

(87)

since $r(x)$ is independent of $\dot{x}$ and $L = 1$ on $D$. To evaluate the integration over $D_1$ fix $x \in M$ and introduce a $g$-orthonormal basis on $T_x M$ as $\dot{x}^i = a_i^\mu z^\mu$ (the greek alphabet letters indicate the ON basis tensor components). In this new basis $dz^0 \wedge \ldots \wedge dz^3 = \det(a_i^\mu) dz^0 \wedge \ldots \wedge dz^3$ as well as $\eta_{\mu\nu} = g_{ij} a_i^\mu a_j^\nu$ and $\det(a_i^\nu) = \sqrt{|\det(g_{ij})|}^{-1}$.

Hence

$$- \frac{1}{4} \int_D r(x) \ i_C dV_0 = - \frac{1}{4} \int_{D_M} [r(x) \sqrt{|\det g(x)|} \ \text{vol}(D_1) \ d^4x],$$  

(88)

where

$$\text{vol}(D_1) = \int_{D_1} i_C (dz^0 \wedge \ldots \wedge dz^3),$$  

(89)

is the volume of the compact set $D_1$ seen as subset of the level set $|\eta_{\mu\nu} z^\mu z^\nu| = 1$. In case the compact set $D_1$ does not depend on $x \in D_M$ we can pull the volume factor out of the integral. The second term in the action (68) built from (85) is a divergence and hence gives the boundary term.

**VIII. CONCLUSION**

To enrich and focus the discussion about a proper viable Finsler gravity generalization of the Einstein equations we presented strong arguments which identify the equation (81) as the simplest, mathematically consistent action based gravitational field equation which determines the Finsler function of a Finsler spacetime.

Starting from the physical argument that gravity causes the tidal forces in geodesic deviation, Rutz identified the Cartan non-linear curvature tensor as the relevant mathematical object which encodes the gravitational interaction on the basis of Finsler geometry. Here we proved, by using canonical variational completion, that the field equation suggested by her in [31] can not be obtained by the means of calculus of variations, and that the field equations suggested in [32] are the variational completion of the former ones. In order to rigorously apply this formalism, we first showed that it is equivalent to formulate the Finsler gravity action on parts of the positive projective tangent bundle, which is defined independently of the Finsler structures in consideration, and the corresponding parts on the
unit tangent bundle defined by the Finsler function. Further, we showed that for pseudo-Riemannian metrics, the gravity action conjectured in [32] and found by variational completion here becomes the Einstein–Hilbert action up to a boundary term. This result may be understood as an additional argument supporting the action we have found here.

Our results open the possibility for different further directions of research. One question that naturally arises from our findings is the validity of the Bianchi identities, which are commonly connected with the invariance of the action under diffeomorphisms of the spacetime manifold, and which we plan to investigate in more detail in future work. Other important tasks are to analyze the properties of the Finsler gravity equation further and to construct a consistent coupling of Finsler gravity to matter, expressed either in terms of certain tensor or spinor fields, or as a kinetic fluid. Finally, To understand the predictivity of a Finsler gravity theory it is necessary to understand the initial value problem of the Finsler gravity equation. The correct initial value formulation for field equation on the projective tangent bundle must be constructed and the Finsler gravity equation shall be cast into an initial value problem of the Finsler gravity equation.

Regarding the matter coupling, one possibility is reconsider the tangent bundle matter actions suggested in [32] and reformulate them on the projective tangent bundle, in the same fashion as we did with the gravitational action. A new approach to matter couplings, which shall be investigated in the future, is offered by Lemma 3. It allows us to rewrite contractions with a metric into an integral over contractions with velocities and hence, to rewrite kinetic terms in the usual matter field actions on spacetime into matter field actions on the projective tangent bundle in a canonical way. The most promising outlook for a coupling between matter and a Finslerian tangent bundle geometry is the direct coupling of a kinetic gas to the geometry of spacetime. A kinetic gas is directly described on the tangent bundle in terms of one-particle distribution functions \[ \text{I} \], which can naturally be described in the Finsler language [48]. In its standard formulation the gas back reacts to gravity via averaging, since its energy momentum-tensor, which couples to gravity via the Einstein equations, is obtained by averaging over the velocities of the constituents of the gas. The reformulation of the kinetic gas on the positive projective tangent bundle allows us to directly couple it to the Finsler gravity and omitting the averaging procedure. This offers the perspective of a more precise description of gravitating kinetic gases and its applications to cosmology for possible insight to dark energy as averaging effect.

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Appendix A: Evaluation of the integrals \( I_2 \) and \( I_3 \)

In section VI we encountered the two integrals

\[
I_2 = \int_D \frac{1}{L} \frac{dR}{dt} \bigg|_{t=0} \text{i}_C dV_0 \quad \text{and} \quad I_3 = \int_D \frac{R}{|\det g^{L}|} \frac{d|\det g^{L}|}{dt} \bigg|_{t=0} \text{i}_C dV_0, \tag{A1}
\]

which we will evaluate here in detail.

1. The integral \( I_2 \)

The first step is to investigate the variation of the geodesic spray coefficients (10), since they are the building blocks of the curvature scalar (20). Denoting the derivatives of the variation by \( v_i = v_{,i} \) and \( v_{ij} = v_{,ij} \) we have

\[
\bar{L}_i \approx L_i + 2tv_i, \quad \bar{g}_{ij} \approx g_{ij} + tv_{ij}, \quad \bar{g}^{ij} \approx g^{ij} + v^{ij}, \tag{A2}
\]

where \( v^{ij} = g^{Lmi}g^{Lnj}g^L_{mn} \) and the symbol \( \approx \) means equality modulo higher than linear order in \( t \). As a consequence the variation of the geodesic spray coefficients becomes

\[
2\bar{G}^i = \frac{1}{2} \bar{g}^{Lij}(x^k L_{j,k} - \bar{L}_{,j}) \approx 2G^i + t_y \bar{g}^{Lij}(x^k v_{j,k} - v_{,j} - 2\bar{G}^k v_{jk}) \tag{A3}
\]
Since \( \dot{x} \)-differentiation preserves the tensor character, \( v_i \) are covector components and it makes sense to speak about covariant derivatives thereof with respect to the Chern connection \( (12) \): 
\[
 v_{ij} = \dot{v}_{ij} - G^k_{ij} v_{jk} - \Gamma^k_{ij} v_k.
\]
Contracting the last index with \( \dot{x}^j \) and taking into account the identities \( \Gamma^i_{ij} \dot{x}^j = G^i_j \) and \( G^k_{ij} \dot{x}^j = 2 \dot{G}^k \) yields
\[
v_{ij} \dot{x}^j = v_{ij} \dot{x}^j + 2 \dot{G}^k v_{ik} + G^k_i v_k, \quad \text{and} \quad v_i = v_i + G^k_i v_k. \quad (A4)
\]
Substituting these equalities into \( (A3) \) yields
\[
2 \dot{G}^i 
\geq 2 \dot{G}^i + 2 t A^i, \quad \text{with} \quad A^i = \frac{1}{2} g^{lij} (\nabla v_j - v_{ij}), \quad (A5)
\]
which agrees with the expression found in \([43]\).

The second step is the variation of the Finsler Ricci scalar. By definition, we find the variation of the non-linear connection coefficients \( (11) \) to be
\[
\tilde{G}^i_{ij} \dot{t} = \tilde{G}^i_{ij} + t A^i, \quad \text{where} \quad A^i = A^i_{ij}. \quad (A6)
\]
Further, using the Landsberg tensor \( (13) \) \( G^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \) we may write the variation of the non-linear curvature tensor \( (19) \) as
\[
\dot{R}^i_{jk} \dot{t} = \dot{R}^i_{jk} + t (A^i_{jk} - A^i_{kj}) + t (A^i_{j k l} \dot{t} - A^i_{k l} P_{jl}). \quad (A7)
\]
Contracting this equation with \( \dot{x}^k \) and taking into account \( P_{j k} \dot{x}^k = 0 \) as well as \( A^i_{k} \dot{x}^k = 2 A^i \) we get
\[
\dot{R}^i_{jk} \dot{t} = \dot{R}^i_{jk} + t (\nabla A^i_{j} - 2 A^i_{j i} - 2 A^i P_{jl}), \quad (A8)
\]
which finally leads us to \( \dot{R} \simeq R + t (\nabla A^i_{j} - 2 A^i_{j i} - 2 A^i P_{jl}) \).

The third and final step is the isolation of the variation in the integral. To do so we substitute our findings
\[
I_2 = \int_D \frac{1}{L} (\nabla A^i_{j} - 2 A^i_{j i} - 2 A^i P_{jl}) v_i dV_0. \quad (A9)
\]
Observe that the first term gives a boundary term which we can neglect. This is so since \( \nabla L = 0 \) and thus \( \nabla (\dot{G}^i_{ij}) \) is a divergence of a \( 0 \)-homogeneous vector field according to \( (26) \), which is equal to a total derivative according to \( (44) \). It remains to write \( L^{-1} A^i_{j} = \text{div}(L^{-1} A^i \delta_i) + L^{-1} A^i P_{i} \), see \( (24) \), to find
\[
I_2 = - \int_D \frac{1}{L} \frac{1}{L} A^i P_{i} v_i dV_0. \quad (A10)
\]
Using the definition of \( A^i \) we can expand the integrand as
\[
-4 L^{-1} A^i P_{i} = 2 \left[ (L^{-1} v P^i)_{i} - L^{-1} v A^i_{j} + \nabla (L^{-1} v P^i) + L^{-1} v P^i \right], \quad (A11)
\]
and observe that \( \nabla (L^{-1} v P^i) \) is a total divergence of a \( 0 \)-homogeneous vector field again and that \( (L^{-1} v P^i)_{i} = \text{div}(L^{-1} v P^i \delta_i) + L^{-1} v P^i P_i \), which implies
\[
-4 L^{-1} A^i P_{i} = \text{div}(\ldots) + 2 L^{-1} (P^i P_i - P^i_{j i}) v + 2 L^{-1} v_m \nabla P^m. \quad (A12)
\]
Hence, the last we need to investigate is \( 2 L^{-1} v_m \nabla P^m \). By the Leibniz rule we have
\[
L^{-1} v_m \nabla P^m = (L^{-1} v \nabla P^i)_{i} - L^{-1} v \nabla (L^{-1} \nabla P^i)_{i}. \quad (A13)
\]
The second term on the right hand side vanishes since \( \dot{x}_i \nabla P^i = \nabla (\dot{x} \dot{P}_i) = 0 \). The first term can be written into a divergence of a vertical vector field according to \( (25) \) as \( (L^{-1} v \nabla P^i)_{i} = \text{div}(L^{-1} v \nabla P^i \delta_i) - 2 C_i L^{-1} v \nabla P^i + 0 \). Summing up yields
\[
L^{-1} v_m \nabla P^m = \text{div}(\ldots) - v L^{-1} (2 C_i L^{-1} \nabla P^i + (\nabla P^i)_{i}). \quad (A14)
\]
It is now straightforward to see that \( 2 C_i L^{-1} \nabla P^i + (\nabla P^i)_{i} = g^{lij} (\nabla P^j)_{j} \) and so altogether
\[
-4 L^{-1} A^i P_{i} = \text{div}(\ldots) + 2 L^{-1} (P^i P_i - P^i_{j i}) - g^{lij} (\nabla P^j)_{j} v. \quad (A15)
\]
Finally, the integral \( I_2 \) becomes, neglecting the boundary terms,
\[
I_2 = \int_D \frac{2}{L} (P^i P_i - P^i_{j i}) - g^{lij} (\nabla P^j)_{j} v i_c dV_0, \quad (A16)
\]
which is the desired expression \( (78) \).
2. The integral $I_3$

For the integral $I_3$ observe that

$$|\det \dot{g}^L| \overset{t}{\approx} |\det g^L| + tg^{Lij}v_{ij}| \det g^L|,$$

where we used the derivative formula for the determinant

$$\frac{d}{dt} \det g^L = \det \dot{g}^L \dot{g}^{Lij} \frac{d}{dt} \dot{g}_{ij}.$$

The integral thus becomes

$$I_3 = \int_D \frac{R}{L} g^{Lij} v_{ij} i_C dV_0$$

The Leibniz rule, together with (25) imply

$$L^{-1} R g^{Lij} v_{ij} = \text{div}(L^{-1} R g^{Lij} v_{ij} \hat{\partial}_i) - 2L^{-1} R C^i v_i + 8L^{-2} R v - (L^{-1} R g^{Lij}), v_{ij}$$

(A20)

$$= \text{div}(L^{-1} R g^{Lij} v_{ij} \hat{\partial}_i) + 12L^{-2} R v - L^{-1} R g^{Lij}$$

(A21)

$$= \text{div}(L^{-1} R g^{Lij} v_{ij} \hat{\partial}_i) - \text{div}(L^{-1} R g^{Lij} v_{ij} \hat{\partial}_i) + L^{-1} v g^{Lij} R_{ij}.$$ (A22)

Hence the integral turns out to be, neglecting again the boundary terms,

$$I_3 = \int_D \frac{1}{L} g^{Lij} R_{ij} v_i i_C dV_0,$$ (A23)

which again is the result presented in (79).

Appendix B: Proof of Lemma 3

In Lemma 3 we displayed a useful formula to understand the Finsler gravity action in the case of pseudo-Riemannian geometry. Here we provide the proof of the formula

$$\int_{D^+} g^{Lij}(L f)_{ij} dV_0^+ = 8 \int_{D^+} f dV_0^+ + \int_{D^+} L g^{Lij} f_i n_j d\sigma^+.$$ (B1)

We have $g^{Lij}(L f)_{ij} = g^{Lij}(L_{ij} f + L f_i f_j + L f_i f_j + L f_{ij})$. Taking into account the 0-homogeneity of $f$ and $L_{ij} = 2g_{ij}$ and $g^{Lij} g^L_{ij} = 4$ we find $g^{Lij}(L f)_{ij} = 8f + L g^{Lij} f_{ij}$.

The second term can be further manipulated employing equation (25) and the 0-homogeneity of $f$,

$$L g^{Lij} f_{ij} = (L g^{Lij} f_{ij})_i - (L g^{Lij})_i f_{ij} = \text{div}(L g^{Lij} f_{ij} \hat{\partial}_i) - 2C_i L g^{Lij} f_{ij} + \frac{4}{L^2} L g^{Lij} f_{ij} - (L g^{Lij})_i f_{ij}$$

(B2)

$$= \text{div}(L g^{Lij} f_{ij} \hat{\partial}_i),$$ (B3)

and thus we can equate

$$g^{Lij}(L f)_{ij} = 8f + \text{div}(L g^{Lij} f_{ij} \hat{\partial}_i)$$ (B4)

which makes the integral

$$\int_{D^+} g^{Lij}(L f)_{ij} dV_0^+ = 8 \int_{D^+} f dV_0^+ + \int_{D^+} \text{div}(L g^{Lij} f_{ij} \hat{\partial}_i) dV_0^+.$$ (B5)

The last step in the proof is to apply Stokes’ theorem and to realize that $L = 1$ on $D^+$ and $\partial D^+$,

$$\int_{D^+} \text{div}(L g^{Lij} f_{ij} \hat{\partial}_i) dV_0^+ = \int_{\partial D^+} L g^{Lij} f_{ij} n_j d\sigma^+.$$ (B6)

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