PRIMITIVE RECURSIVE BOUNDS FOR THE FINITE VERSION OF GOWERS’ $c_0$ THEOREM

KONSTANTINOS TYROS

Abstract. We provide primitive recursive bounds for the finite version of Gowers’ $c_0$ theorem for both the positive and the general case. We also provide multidimensional versions of these results.

1. Introduction

In 1992 W. T. Gowers (see [3]) obtained a stability result for real valued Lipschitz functions defined on the unit sphere of $c_0$. This result is actually a consequence of a deep infinite dimensional Ramsey type result (see [1, 4, 14] as well as [7] for an elegant proof). Our goal in this paper is to provide primitive recursive bounds for the finite version of this Ramsey type result.

To state our results we need first to introduce some pieces of notation. By $\mathbb{N}$ we denote the set of all non-negative integers. Let $n, k$ be positive integers. By $X_k(n)$ we denote the set of all functions having domain the set $\{0, \ldots, n-1\}$, range the set $\{0, \ldots, k\}$ and achieving the value $k$, i.e.,

$$X_k(n) = \{f : \{0, \ldots, n-1\} \rightarrow \{0, \ldots, k\} \text{ such that } f(i) = k \text{ for some } 0 \leq i < n\}.$$ 

By $X_{[k]}(n)$ we denote the set of all functions having domain the set $\{0, \ldots, n-1\}$ and range the set $\{0, \ldots, k\}$. The set $X_k(n)$ discretizes the positive cone of the unit sphere of $\ell^\infty_n$, while $X_{[k]}(n)$ discretizes the positive cone of the unit ball of $\ell^\infty_n$. The scalar multiplication is captured by the following map. We define $T : X_{[k]}(n) \rightarrow X_{[k]}(n)$ as follows. For every $f \in X_{[k]}(n)$ and $i \in \{0, \ldots, n-1\}$ we set

$$T(f)(i) = \max(0, f(i) - 1).$$

For every $f$ in $X_{[k]}(n)$, by $\text{supp}(f)$, we denote the set of all $i$ in $\{0, \ldots, n-1\}$ such that $f(i) \neq 0$. A sequence $F = (f_i)_{i=0}^{m-1}$ in $X_{[k]}(n)$ is called block of length $m$ if $(\text{supp}(f_i))_{i=0}^{m-1}$ forms a block sequence of nonempty finite subsets of $\mathbb{N}$, that is, $\max \text{supp}(f_i) < \min \text{supp}(f_{i+1})$ for all $0 \leq i < m - 1$. For a block sequence...
\[ F = (f_i)_{i=0}^{m-1} \] in \( X_{[k]}(n) \), we define the positive subspace generated by \( F \) to be

\[
(F)_{k} = \left\{ \sum_{i=1}^{\ell} T^{\varepsilon_i}(f_{j_i}) : \ell \leq m, \ 0 \leq j_1 < \ldots < j_\ell < m, \right. \\
\left. \varepsilon_1, \ldots, \varepsilon_\ell \in \{0, \ldots, k-1\} \text{ and } \min_{1 \leq i \leq \ell} \varepsilon_i = 0 \right\}.
\]

The first result in this paper is the following.

**Theorem 1.** For every triple of positive integers \( k, m, r \), there exists a positive integer \( n_0 \) satisfying the following property. For every integer \( n \geq n_0 \) and every coloring of the set \( X_k(n) \) with \( r \) colors, there exists a block sequence \( F \) in \( X_k(n) \) of length \( m \) such that the set \( (F)_{k} \) is monochromatic. We denote the least \( n_0 \) satisfying the above property by \( G(k, m, r) \).

Moreover, the numbers \( G(k, m, r) \) are upper bounded by a primitive recursive function belonging to the class \( \mathcal{E}_7 \) of Grzegorczyk’s hierarchy.

For a detailed exposition on the Grzegorczyk’s classes we refer the reader to [9]. A similar to Theorem 1 result (not obtaining, however, primitive recursive bounds) have been recently proved in [10]. The proof of Theorem 1 is inspired by Shelah’s proof for the Graham–Rothschild theorem (see [11, Theorem 2.2]), though there are some novel arguments that do not appear in Shelah’s proof. Specifically, given a coloring of the set \( X_k(n) \), the goal is, by passing to a subspace, to canonize this coloring and make it insensitive in the following sense. Every two functions of the form \( f \) and \( f + T^{(k-1)}(f') \) have the same color, where \( f \) and \( f' \) are disjointly supported. In order to achieve this insensitivity of a given coloring, we consider the type of a function, a notion introduced in Section 3, that takes into account the nonzero values of a function with no repetition. Using the finite version of the Milliken–Taylor theorem, we first make the coloring to depend only on the type of a function. An appropriate choice of functions, that eliminates several types, generates a subspace on which the given coloring is insensitive.

Our second result involves functions taking also negative values. To state it, we need to introduce some additional notation. Let \( n, k \) be positive integers. By \( X_{\pm k}(n) \) we denote the set of all functions having domain the set \( \{0, \ldots, n-1\} \), range the set \( \{-k, \ldots, k\} \) and achieving the value \( k \) or \(-k\), i.e.,

\[ X_{\pm k}(n) = \{ f : \{0, \ldots, n-1\} \rightarrow \{-k, \ldots, k\} \text{ such that } |f(i)| = k \text{ for some } 0 \leq i < n \}. \]

By \( X_{\pm k}(n) \) we denote the set of all functions having domain the set \( \{0, \ldots, n-1\} \) and range the set \( \{-k, \ldots, k\} \). Moreover, we extend the map \( T : X_{\pm k}(n) \rightarrow X_{\pm k}(n) \) as follows. We set

\[
T(f)(i) = \begin{cases} 
  f(i) - 1, & f(i) > 0 \\
  0, & f(i) = 0 \\
  f(i) + 1, & f(i) < 0 
\end{cases}
\]
for all \( f \) in \( X_{|±k|}(n) \) and \( i \) in \( \{0, ..., n-1\} \). For a block sequence \( F = (f_i)_{i=0}^{m-1} \) in \( X_{±k}(n) \), i.e., \( (\text{supp}(f_i))_{i=0}^{m-1} \) is a block sequence, we extend the notion of the positive subspace. In particular, we define the subspace generated by \( F = (f_i)_{i=0}^{m-1} \) to be

\[
(F)_{±k} = \left\{ \sum_{i=1}^{\ell} ±T^{ε_i}(f_i) : ℓ \leq m, 0 \leq j_1 < ... < j_\ell < m, \varepsilon_1, ..., \varepsilon_\ell \in \{0, ..., k-1\} \text{ and } \min_{1 \leq i \leq \ell} \varepsilon_i = 0 \right\}.
\]

On \( X_{±k}(n) \) we consider the supremum metric, denoted by \( ρ_∞ \), and defined as usual

\[
ρ_∞(f, g) = \max_{0 \leq i < n} |f(i) - g(i)|
\]

for all \( f, g \) in \( X_{±k}(n) \). Given a finite coloring \( c : X_{±k}(n) \to \{1, ..., r\} \), we say that a subset \( A \) of \( X_{±k}(n) \) is approximately monochromatic if there exists some \( i_0 \) in \( \{1, ..., r\} \) such that for every \( f \) in \( A \) there exists an \( f' \) in \( X_{±k}(n) \) with \( c(f') = i_0 \) and \( ρ_∞(f, f') \leq 1 \).

**Theorem 2.** For every triple of positive integers \( k, m, r \), there exists a positive integer \( n_0 \) satisfying the following property. For every integer \( n \geq n_0 \) and every coloring of the set \( X_{±k}(n) \) with \( r \) colors, there exists a block sequence \( F \) in \( X_{±k}(n) \) of length \( m \) such that the set \( (F)_{±k} \) is approximately monochromatic. We denote the least \( n_0 \) satisfying the above property by \( G_{±}(k, m, r) \).

Moreover, the numbers \( G_{±}(k, m, r) \) are upper bounded by a primitive recursive function belonging to the class \( E^7 \) of Grzegorczyk’s hierarchy.

The reduction of Theorem 2 to Theorem 1 essentially relies on the choice of some appropriate functions in \( X_{±k}(n) \). This choice has been inspired by the approach in [2] that makes use of \( -T \) instead of \( T \) for the proof of the general case.

Finally, we consider multidimensional versions of the Theorems 1 and 2, which are presented in Section 6.

2. Background Material

The proofs of Theorems 1 and 2 make use of the finite version of the Milliken–Taylor theorem [8][12]. To state it we need to introduce some pieces of notation. Let \( m, d \) be positive integers with \( d \leq m \). A finite sequence \( s = (s_i)_{i=0}^{m-1} \) of nonempty finite subsets of \( \mathbb{N} \) is called block if \( \max s_i < \min s_{i+1} \) for all \( 0 \leq i < m-1 \). For a block sequence \( s = (s_i)_{i=0}^{m-1} \) of nonempty finite subsets of \( \mathbb{N} \) we define the set of nonempty unions of \( s \) to be

\[
\text{NU}(s) = \left\{ \bigcup_{i \in t} s_i : t \text{ is a nonempty subset of } \{0, ..., m-1\} \right\}.
\]

We say that a block sequence \( t = (t_i)_{i=0}^{d-1} \) of nonempty finite subsets of \( \mathbb{N} \) is a block subsequence of \( s \) if \( t_i \in \text{NU}(s) \) for all \( 0 \leq i < d \). By \( \text{Block}^d(s) \) we denote the set of all block subsequences of \( s \) of length \( d \). Moreover, for simplicity, by \( \text{Block}^d(m) \), we
denote the set $\text{Block}^d((\{i\})_{i=0}^{n-1})$. The finite version of the Milliken-Taylor theorem is stated as follows.

**Theorem 3.** For every triple $d, m, r$ of positive integers with $d \leq m$, there exists a positive integer $n_0$ with the following property. For every finite block sequence $s$ of nonempty finite subsets of $\mathbb{N}$ of length at least $n_0$ and every coloring of the set $\text{Block}^d(s)$ with $r$ colors, there exists a block subsequence $t$ of $s$ of length $m$ such that the set $\text{Block}^d(t)$ is monochromatic. We denote the least $n_0$ satisfying the above property by $\text{MT}(d, m, r)$.

Moreover, the numbers $\text{MT}(d, m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^6$ of Grzegorczyk’s hierarchy.

Note that the case “$d = 1$” of Theorem 3 is the finite version of Hindman’s theorem [6]. This finite versions follows by the disjoint union theorem [5, 7] and Ramsey’s theorem. The higher dimensional case of Theorem 3 (that is, the case $d \geq 2$), follows from the aforementioned finite version of Hindman’s theorem and a standard iteration (see [2] for further details).

3. Types and Insensitivity

Let us start with some notation. Let $d, n$ be positive integers with $d \leq n$ and $F$ be a block sequence in $X_k(n)$ (resp. in $X_{\pm k}(n)$). We say that a block sequence $G = (g_i)_{i=0}^{d-1}$ in $X_k(n)$ (resp. in $X_{\pm k}(n)$) is a block subsequence of $F$ if $g_i$ belongs to $\langle F \rangle_k$ (resp. $g_i$ belongs to $\langle F \rangle_{\pm k}$) for all $0 \leq i < d$. Moreover, by $\text{Block}^d_k(F)$ (resp. $\text{Block}^d_{\pm k}(F)$) we denote the set of all block subsequences of $F$ of length $d$. For simplicity, by $\text{Block}^d_k(n)$ (resp. $\text{Block}^d_{\pm k}(n)$), we denote the set $\text{Block}^d_k((k \cdot \chi^n_{(i)})_{i=0}^{n-1})$ (resp. $\text{Block}^d_{\pm k}((k \cdot \chi^n_{(i)})_{i=0}^{n-1})$), where by $\chi^n_k$ we denote the characteristic function, defined on $\{0, \ldots, n-1\}$, of a finite nonempty subset $A$ of $\mathbb{N}$ with $\max A < n$.

Finally, for every finite sequence $b$ and every non-negative integer $d$ less that or equal to the length of $b$, by $b|d$, we denote the initial segment of $b$ of length $d$.

The proof of Theorem 3 proceeds by induction on $k$. Central role in the proof of the inductive step possesses the notion of insensitivity, which we are about to define.

**Definition 4.** Let $m, n, k$ be positive integers with $m \leq n$ and $F \in \text{Block}^m_k(n)$. We say that a coloring $c$ of the set $X_k(n)$ is insensitive over $F$ if for every $f, f'$ in $\langle F \rangle_k$ disjoinly supported, we have that $f$ and $f + T^{(k-1)}(f')$ have the same color.

Let us point out that the notion of insensitivity is hereditary. In particular, we have the following easy to observe fact.

**Fact 5.** Let $m, n, k$ be positive integers with $m \leq n$ and $F \in \text{Block}^m_k(n)$. Also let $c$ be a coloring of the set $X_k(n)$ and assume that $c$ is insensitive over $F$. Then for every block subsequence $G$ of $F$, we have that $c$ is insensitive over $G$. 
In order to carry out the inductive step of the proof of Theorem 5, we need to make an arbitrary coloring of the set \( X_k(n) \) insensitive over a long enough block sequence \( F \) in \( X_k(n) \). To this end, we will need the notion of a type.

Let \( n, k \) be positive integers. For every \( d \leq n \), a type of length \( \phi \) over \( X_k(d) \) is a function \( \varphi \) in \( X_k(d) \) such that \( \varphi(i) \neq \varphi(i + 1) \) for all \( 0 \leq i < d - 1 \) and \( \text{supp}(\varphi) = \{0, \ldots, d - 1\} \). By \( |\varphi| \), we denote the length of a type \( \varphi \). Observe that for every function \( f \) in \( X_{\pm k}(n) \) there exist a unique type of some length \( \phi \leq n \), which we denote by \( \text{tp}(f) \), and a unique block sequence of nonempty finite subsets of \( \mathbb{N} \) of length \( \phi \), which we denote by \( \text{bsupp}(f) \), such that \( f = \text{map}(\text{tp}(f), \text{bsupp}(f)) \). Finally, for \( d \leq n \) and \( s = (s_i)_{i=0}^{d-1} \) in \( \text{Block}^d(n) \), we define the spaces generated by \( s \) as \( X_k(s) = \{(k \cdot \chi_{s_i})_{i=0}^{d-1}\}_k \) and \( X_{\pm k}(s) = \{(k \cdot \chi_{s_i})_{i=0}^{d-1}\}_{\pm k} \). By Theorem 3, we obtain a canonicalization of a given coloring with respect to the types.

**Lemma 6.** Let \( m, n, k, r \) be positive integers, with \( n \geq \text{MT}(m, 2m - 1, r^\alpha) \), where \( \alpha = \sum_{d=1}^m d(k-1)^{d-1} \). Then for every coloring of the set \( X_k(n) \) with \( r \) colors, there exists \( s \) in \( \text{Block}^m(n) \) such that every \( f, f' \) in \( X_k(s) \) of the same type have the same color.

**Proof.** First let us observe that for every positive integer \( d \) the number of types over \( k \) of length \( d \) is at most \( d(k-1)^{d-1} \). Thus the number of types over \( k \) of length at most \( m \) is at most \( \alpha \). Let \( c : X_k(n) \to \{1, \ldots, r\} \) be a coloring. Also let \( T \) be the set of all types over \( k \) of length at most \( m \) and \( \mathcal{X} \) the set of all the maps from \( T \) into \( \{1, \ldots, r\} \). As we have already pointed out the set \( T \) has cardinality \( \alpha \) and therefore the set \( \mathcal{X} \) has cardinality \( r^\alpha \). We define a coloring \( \tilde{c} : \text{Block}^m(n) \to \mathcal{X} \) as follows. For every \( t \) in \( \text{Block}^m(n) \), we first define \( q_t \) in \( \mathcal{X} \) as follows. For every type \( \varphi \) in \( T \) we set \( q_t(\varphi) = c(\text{map}(\varphi, t|d)) \), where \( d \) is the length of \( \varphi \). Finally, we set \( \tilde{c}(t) = q_t \) for all \( t \) in \( \text{Block}^m(n) \).

Since \( n \geq \text{MT}(m, 2m - 1, r^\alpha) \), applying Theorem 3, we obtain a block sequence \( s' \in \text{Block}^{2m-1}(n) \) such that the set \( \text{Block}^m(s') \) is \( \tilde{c} \)-monochromatic. That is, there exists \( q \) in \( \mathcal{X} \) such that for every \( t \) in \( \text{Block}^m(s') \) we have that \( q_t = q \). We set \( s = s'|m \) and we observe that \( s \) is as desired. Indeed let \( f, f' \) in \( X_k(s) \) of the same type \( \varphi \). Let \( d \) be the length of \( \varphi \). Clearly \( 1 \leq d \leq m \). Since \( s \) is the initial segment of \( s' \) of length \( m \) and \( s' \) is of length \( 2m - 1 \), we can end-extend both \( \text{bsupp}(f) \) and \( \text{bsupp}(f') \) into \( t \) and \( t' \) respectively elements of \( \text{Block}^m(s') \). Clearly, \( f = \text{map}(\varphi, t|d) \) and \( f' = \text{map}(\varphi, t'|d) \). Thus \( c(f) = q_t(\varphi) = q(\varphi) = q_{t'}(\varphi) = c(f') \). The proof is complete. \( \Box \)
The above lemma is the main tool to obtain the color insensitivity by passing to a subspace. In particular, we have the following.

**Lemma 7.** Let $m, n, k, r$ be positive integers, with

$$n \geq MT(m(2k-1), 2m(2k-1) - 1, r^\alpha),$$

where $\alpha = \sum_{d=1}^{m} d(k-1)^{d-1}$. Then for every coloring $c$ of the set $X_k(n)$ with $r$ colors, there exists $F$ in $\text{Block}^m_k(n)$ such that $c$ is insensitive over $F$.

**Proof.** Let $c$ be a coloring of the set $X_k(n)$ with $r$ colors and $p = m(2k-1)$. Since $n \geq MT(m(2k-1), 2m(2k-1) - 1, r^\alpha)$, applying Lemma 6, we obtain a block sequence $s = (s_i)_{i=0}^{p-1} \in \text{Block}^p(n)$ such that every $f, f'$ in $X_k(s)$ of the same type have the same color. The desired block sequence $F = (f_i)_{i=0}^{m-1}$ is defined as follows. For every $i = 0, ..., m-1$, we set

$$f_i = \sum_{q=-(k-1)}^{k-1} (k - |q|) \cdot \chi^{n}_{s_{j_i+q}},$$

where $j_i = i(2k-1) + k - 1$. It is easy to see that $F$ belongs to $\text{Block}^m_k(n)$ and $\langle F \rangle_k$ is a subset of $X_k(n)$. Thus every $f, f'$ in $\langle F \rangle_k$ of the same type have the same color. Moreover, for every $i \in \{0, ..., m-1\}$ the function $f_i$ has a “pyramid” shape and, in particular, for every $\varepsilon \in \{0, ..., k-1\}$ we have that

$$T^{(c)}(f_i)(\max \text{ supp}(T^{(\varepsilon)}(f_i))) = T^{(\varepsilon)}(f_i)(\min \text{ supp}(T^{(\varepsilon)}(f_i))) = 1.$$}

This easily yields that for every $f, f'$ disjointly supported in $\langle F \rangle_k$ the functions $f$ and $f + T^{(k-1)}(f')$ are of the same type, since $T^{(k-1)}(f')$ is of the form $\chi^s_n$ for some $s$ in $\text{NU}(s)$ disjoint to the support of $f$, and therefore of the same color. That is, the color $c$ is insensitive over $F$. \qed

4. **Proof of Theorem 1**

For the proof of Theorem 1 we need the following notation. For $m, n$ positive integers with $m \leq n$ and $F = (f_i)_{i=0}^{m-1}$ block sequence in $X_k(n)$, we set

$$\langle F \rangle_k = \left\{ \sum_{i=1}^{\ell} T^{(\varepsilon)}(f_{j_i}) : \ell \leq m, 0 \leq j_1 < \ldots < j_{\ell} < m, \varepsilon_1, \ldots, \varepsilon_{\ell} \in \{0, \ldots, k-1\} \right\}.$$\)

**Proof of Theorem 1.** We proceed by induction on $k$. The case $k = 1$ of the theorem follows by Theorem 4 for “$d = 1$”. In particular, it is easy to observe that

(1) \[ G(1, m, r) = MT(1, m, r) \]

for every choice of positive integers $m$ and $r$. Assume that for some integer $k \geq 2$, the theorem holds for $k - 1$. We will establish the validity of the theorem for $k$, by showing that

(2) \[ G(k, m, r) \leq MT(G(k-1, m, r)(2k-1), 2G(k-1, m, r)(2k-1) - 1, r^\alpha), \]

...
where $\alpha = \sum_{d=1}^{G(k-1,m,r)} d(k-1)^{d-1}$, for every choice of positive integers $m$ and $r$. Indeed, let $m, r$ be positive integers and set $M = G(k-1,m,r)$. Also let $n$ be a positive integer with

$$n \geq MT(M(2k-1), 2M(2k-1) - 1, r^\alpha),$$

where $\alpha = \sum_{d=1}^{M} d(k-1)^{d-1}$, and $c$ a coloring of the set $X_k(n)$ with $r$ colors. By 3 and Lemma 7 applied for “$m = M$”, we obtain $F' = (f')_{i=0}^{M-1}$ in Block$_k^M(n)$ such that the coloring $c$ is insensitive over $F'$. We define a map $Q : X_{[k-1]}(M) \to (F')_{[k]}$ by setting

$$Q(g) = \sum_{i \in \mathrm{supp}(g)} T^{(k-1-g(i))}(f'_i)$$

for all $g \in X_{[k-1]}(M)$. Let us isolate the following easy to observe properties of the map $Q$.

(a) If $g \in X_{k-1}(M)$, then $Q(g) \in (F')_{k}$.

(b) If $(g_i)_{i=0}^{d-1}$ is a block sequence in $X_{k-1}(M)$ then $(Q(g_i))_{i=0}^{d-1}$ is also a block sequence in $(F')_{k}$.

(c) For every $g \in X_{[k-1]}(M)$, we have $Q(T(g)) = T(Q(g)) + \sum_{i \in s} T^{(k-1)}(f'_i)$, where $s = \mathrm{supp}(g) \setminus \mathrm{supp}(T(g))$.

Hence, if $(g_i)_{i=0}^{d-1}$ is a block sequence in $X_{k-1}(M)$ and $\varepsilon_0, ..., \varepsilon_{d-1} \in \{0, ..., k - 2\}$ with $\min_{0 \leq i < d} \varepsilon_i = 0$, then

$$Q \left( \sum_{i=0}^{d-1} T^{(\varepsilon_i)} g_i \right) = \sum_{i=0}^{d-1} T^{(\varepsilon_i)} (Q(g_i)) + \sum_{i \in s} T^{(k-1)}(f'_i),$$

where $s = \cup_{i=0}^{d-1} \mathrm{supp}(g_i) \setminus \mathrm{supp}(T^{(\varepsilon_i)}(g_i))$, and therefore, by the insensitivity of the coloring $c$ over $F'$, we have that

$$c \left( Q \left( \sum_{i=0}^{d-1} T^{(\varepsilon_i)} g_i \right) \right) = c \left( \sum_{i=0}^{d-1} T^{(\varepsilon_i)} (Q(g_i)) \right).$$

We define a coloring $\tilde{c}$ of the set $X_{k-1}(M)$ by setting $\tilde{c}(g) = c(Q(g))$ for all $g \in X_{k-1}(M)$. Let us point out that the coloring $\tilde{c}$ is well defined due to property (a) above. By the choice of $M$ and the inductive assumption, we obtain a block sequence $G = (g_i)_{i=0}^{m-1}$ in $X_{k-1}(M)$ such that the set $(G)_{k-1}$ is monochromatic with respect to $\tilde{c}$. Therefore, setting $F = (f_i)_{i=0}^{m-1} = (Q(g_i))_{i=0}^{m-1}$, by the definition of $\tilde{c}$ and 4, we have that the set

$$\left\{ \sum_{i=1}^{\ell} T^{\varepsilon_i} (f_{j_i}) : \ell \leq m, 0 \leq j_1 < ... < j_\ell < m, \varepsilon_1, ..., \varepsilon_\ell \in \{0, ..., k - 2\} \text{ and } \min_{1 \leq i \leq \ell} \varepsilon_i = 0 \right\}$$

is monochromatic with respect to $c$. Since $c$ is insensitive over $F'$ and $F$ is a block subsequence of $F'$, by Fact 5 we get that that $c$ is insensitive over $F$. Hence, by 4,
we get that \((\mathbf{F})_k\) is monochromatic with respect to \(c\) and the proof of the inductive step is complete.

Finally, by (11), (2) and the fact that the numbers \(\text{MT}(d, m, r)\) are upper bounded by a function belonging to the class \(\mathcal{E}^6\) of Grzegorczyk’s hierarchy we have that the numbers \(G(k, m, r)\) are upper bounded by a function belonging to the class \(\mathcal{E}^7\). □

5. Proof of Theorem 2

First let us extend the notion of the support. Let \(m, n, k\) be positive integers with \(m \leq n\) and \(s\) in Block\(^m(n)\). We define the support of a function \(f \in X_{\pm k}(s)\) with respect to \(s\) as

\[
\text{supp}_s(f) = \text{supp}(g),
\]

where \(g\) is the unique function in \(X_{\pm k}(m)\) such that \(f = \text{map}(g, s)\). For \(f, f'\) in \(X_{\pm k}(s)\), we will say that the pair \((f, f')\) is of \(s\)-displacement at most one if

\[
\min \text{supp}_s(f) \leq \min \text{supp}_s(f') \leq \min \text{supp}_s(f) + 1 \quad \text{and} \quad \max \text{supp}_s(f) \leq \max \text{supp}_s(f') \leq \max \text{supp}_s(f) + 1
\]

Moreover, for \(\mathbf{F} = (f_i)_{i=0}^{d-1}\) and \(\mathbf{F}' = (f'_i)_{i=0}^{d-1}\) block sequences in \(X_{\pm k}(s)\) of some length \(d \leq m\), we will say that the pair \((\mathbf{F}, \mathbf{F}')\) is of \(s\)-displacement at most one if the pair \((f_i, f'_i)\) is of \(s\)-displacement at most one for all \(0 \leq i < d\). Finally, we will say that a block sequence \(\mathbf{F} = (f_i)_{i=0}^{d-1}\) in \(X_{\pm k}(s)\) of some length \(d \leq m\) is \(s\)-skipped block if \(\max \text{supp}_s(f_i) + 1 \leq \min \text{supp}_s(f_{i+1})\) for all \(0 \leq i < d - 1\). Under this terminology, we have the following immediate fact.

**Fact 8.** Let \(d, m, n, k\) be positive integers with \(d \leq m \leq n\) and \(s \in \text{Block}^m(n)\). Also let \(\mathbf{F} = (f_i)_{i=0}^{d-1}\) be an \(s\)-skipped block sequence in \(X_{\pm k}(s)\) and \(f'_0, ..., f'_{d-1}\) elements of \(X_{\pm k}(s)\) such that the pair \((f_i, f'_i)\) is of \(s\)-displacement at most one for all \(0 \leq i < d\). Then the sequence \((f'_i)_{i=0}^{d-1}\) is a block sequence in \(X_{\pm k}(s)\) and the pair \((\sum_{i=0}^{d-1} f_i, \sum_{i=0}^{d-1} f'_i)\) is of \(s\)-displacement at most one.

Moreover, if in addition we have \(\rho_{\infty}(f_i, f'_i) \leq 1\) for all \(i = 0, ..., d - 1\), then we have that \(\rho_{\infty}(\sum_{i=0}^{d-1} f_i, \sum_{i=0}^{d-1} f'_i) \leq 1\).

The following functions posses central role in the proof of Theorem 2, and they are inspired by the approach in [7]. Again let us fix a triple of positive integers \(m, n, k\) with \(m \leq n\) and \(s\) in \(\text{Block}^m(n)\). For every \(\delta \in \{1, ..., k\}\) and \(k - 1 \leq \ell \leq m - k\), we define the function

\[
q(\delta, \ell, s) = \sum_{j=-\lfloor \delta \rfloor}^{\delta-1} (-1)^j (\delta - |j|) \chi^n_{\alpha_{s+j}}
\]

while for every non-positive integer \(\delta\) by \(q(\delta, \ell, s)\) we denote the constant zero function. The basic properties of the functions \(q(\delta, \ell, s)\) are summarized by the following easy to prove lemma.
Lemma 10. Let $\ell, m, n, \delta, k$ be positive integers with $m \leq n$, $k - 1 \leq \ell \leq m - k$ and $1 \leq \delta \leq k$. Also let $s \in \text{Block}^m(n)$. Then we have the following.

(i) $T(q(\delta, \ell, s)) = q(\delta - 1, \ell, s)$.
(ii) If $\ell < m - k$, then setting $f = -q(\delta, \ell, s)$ and $f' = q(\delta, \ell + 1, s)$ we have that
(a) the pair $(f, f')$ is of $s$-displacement at most one and
(b) $\rho_\infty(f, f') = 1$.

Let us also extend the notion of the positive subspace for block sequences in $X_{\pm k}(n)$. For $m, n, k$ positive integers with $m \leq n$ and $F = (f_i)_{i=0}^{m-1}$ in $\text{Block}^m\pm k(n)$ we set

$$(F)_k = \left\{ \sum_{i=1}^{\ell} T^{\varepsilon_i}(f_{j_i}) : \ell \leq m, \ 0 \leq j_1 < \ldots < j_\ell < m, \right. \varepsilon_1, \ldots, \varepsilon_\ell \in \{0, \ldots, k - 1\} \text{ and } \left. \min_{1 \leq i \leq \ell} \varepsilon_i = 0 \right\}.$$

The proof of Theorem 2 makes use of the following lemma, which gathers the properties of the functions $q(\delta, \ell, s)$ that we shall need.

Lemma 9. Let $\ell, m, n, \delta, k$ be positive integers with $2km \leq n$ and $s \in \text{Block}^{2km}(n)$. We set $G = (g_i)_{i=0}^{m-1} = (q(k, \ell, s))_{i=0}^{m-1}$ where $\ell_i = 2ki + k - 1$ for all $0 \leq i < m$. Let $F$ be a block sequence in $\langle G \rangle_k$. Then for every $f \in \langle F \rangle_{\pm k}$ there exist $f' \in X_{\pm k}(s)$ and $f'' \in \langle F \rangle_k$ such that

(i) the functions $f', f''$ are of the same type,
(ii) $\rho_\infty(f, f') \leq 1$,
(iii) the pair $(f, f')$ is of $s$-displacement at most one and
(iv) $\text{supp}(f) = \text{supp}(f'')$.

Proof. Let $d$ be the length of $F$ and $F = (f_j)_{j=0}^{d-1}$. Then for every $j = 0, \ldots, d - 1$ there exist a subset $s_j$ of $\{0, \ldots, m - 1\}$ and a family $(\varepsilon_i)_{i \in s_j}$ of elements from the set $\{0, \ldots, k - 1\}$ such that

$$(6) \quad \min_{i \in s_j} \varepsilon_i = 0$$

and $f_j = \sum_{i \in s_j} T(\varepsilon_i)(g_i)$. By the definition of $G$ and part (i) of Lemma 9 we get that

$$(7) \quad f_j = \sum_{i \in s_j} q(k - \varepsilon_i, \ell, s)$$

for all $j = 0, \ldots, d - 1$.

Let $f \in \langle F \rangle_{\pm k}$. Then there exist a subset $\tilde{s}$ of $\{0, \ldots, d - 1\}$, a family $(a_j)_{j \in \tilde{s}}$ of elements from $\{-1, 1\}$ and a family $(\tilde{\varepsilon}_j)_{j \in \tilde{s}}$ of elements from $\{0, \ldots, k - 1\}$ such that

$$(8) \quad \min_{j \in \tilde{s}} \tilde{\varepsilon}_j = 0$$
and

\[ f = \sum_{j \in \mathcal{E}} a_j T(\tilde{\epsilon}_j)(f_j). \]  

Since \( F \) is a block subsequence of \( G \), we have that \((s_j)_{j=0}^{p-1}\) is a block sequence of nonempty finite subsets of \( N \) and, in particular, consists of pairwise disjoint sets. Thus, setting \( p \) to be the cardinality of the set \( \cup_{j \in \mathcal{E}} s_j \) and \( \cup_{j \in \mathcal{E}} s_j = \{i_0 < \ldots < i_{p-1}\} \), we have that for every \( x = 0, \ldots, p-1 \) there exists unique \( j_x \) in \( \mathcal{E} \) such that \( i_x \in s_{j_x} \). By equation (9) we have that

\[ f = \sum_{j \in \mathcal{E}} a_j T(\tilde{\epsilon}_j)(f_j) \supset \sum_{j \in \mathcal{E}} a_j T(\tilde{\epsilon}_j) \left( \sum_{i \in s_{j_x}} q(k - \varepsilon_i, \ell_i, s) \right) \]

\[ = \sum_{j \in \mathcal{E}} a_j \sum_{i \in s_{j_x}} T(\tilde{\epsilon}_j) \left( q(k - \varepsilon_i, \ell_i, s) \right) \quad \text{(Part (i) of Lemma 9)} \]

\[ = \sum_{j \in \mathcal{E}} \sum_{i \in s_{j_x}} a_j q(k - \varepsilon_i - \tilde{\epsilon}_{j_x}, \ell_i, s) = \sum_{x=0}^{p-1} a_x q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell_{i_x}, s). \]

For every \( x = 0, \ldots, p-1 \) we set \( \ell'_{i_x} = \ell_{i_x} \) if \( a_{j_x} = 1 \) and \( \ell'_{j_x} = \ell_{j_x} + 1 \) if \( a_{j_x} = -1 \). The sequence \((q(k, \ell_i, s))_{i=0}^{m-1}\), by its definition, is \( s \)-skipped block and therefore the sequence \((q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell_{i_x}, s))_{x=0}^{p-1}\) is \( s \)-skipped block. By part (ii) of Lemma 9 for every \( x = 0, \ldots, p-1 \) we have that the pair

\[ (q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell_{i_x}, s), q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s)) \]

is of \( s \)-displacement at most one and

\[ \rho_\infty(q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell_{i_x}, s), q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s)) \leq 1. \]

Thus, by Fact 8 we have that \((q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s))_{x=0}^{p-1}\) is a block sequence and setting

\[ f' = \sum_{x=0}^{p-1} q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s) \]

we have that \( \rho_\infty(f, f') \leq 1 \) and the pair \((f, f')\) is of \( s \)-displacement at most one, i.e., parts (ii) and (iii) of the lemma are satisfied. Moreover, again by the fact that the sequence \((q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s))_{x=0}^{p-1}\) is block and by part (i) of Lemma 9 we have that

\[ f' = \sum_{x=0}^{p-1} q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s) = \sum_{x=0}^{p-1} T(\varepsilon_{i_x} + \tilde{\varepsilon}_{j_x}) q(k, \ell'_{i_x}, s). \]

Hence, by equations (6) and (8), we have that \( f' \in X_{\pm k}(s) \).

We set

\[ f'' = \sum_{x=0}^{p-1} q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell_{i_x}, s). \]

By (10) and the definition of \( f'' \), we have that part (iv) of the lemma is satisfied. Recalling that the sequence \((q(k - \varepsilon_{i_x} - \tilde{\varepsilon}_{j_x}, \ell'_{i_x}, s))_{x=0}^{p-1}\) is block, we observe that
Let \( f, f' \) be of the same type and therefore part (i) of the lemma is satisfied. Finally, we have that
\[
f'' = \sum_{x=0}^{p-1} q(k - \varepsilon x - \tilde{\varepsilon}_j, \ell_i, s) = \sum_{j \in \mathbb{S}} \sum_{i \in s_j} q(k - \varepsilon_i - \tilde{\varepsilon}_j, \ell_i, s)
\]

(11)
\[
= \sum_{j \in \mathbb{S}} \sum_{i \in s_j} T(\tilde{\varepsilon}_j)(q(k - \varepsilon_i, \ell_i, s)) = \sum_{j \in \mathbb{S}} T(\tilde{\varepsilon}_j)(f_j),
\]

where the third equality holds by part (i) of Lemma 9. By (8) and (11), we have that \( f'' \in \langle F \rangle_k \). The proof is complete. \( \square \)

Finally, identical arguments to the ones used in the proof of Lemma 6 yield the following.

Lemma 11. Let \( m, n, k, r \) be positive integers, with \( n \geq MT(m, 2m-1, r^d) \), where \( \beta = \sum_{d=1}^{m} 2d(2k - 1)^{d-1} \). Then for every coloring of the set \( X_{\pm k}(n) \) with \( r \) colors, there exists \( s \) in Block\(^m\)(n) such that every \( f, f' \) in \( X_{\pm k}(s) \) of the same type have the same color.

Actually, we prove the following slightly stronger version of Theorem 2. For its proof, we will need some notation. For \( m, n, k \) positive integers with \( m \leq n \) and \( F = (f_i)_{i=0}^{m-1} \) in Block\(^m\)(n) we set
\[
\langle F \rangle_k^{[]} = \left\{ \sum_{i=1}^{\ell} T^{\varepsilon_i}(f_{j_i}) : \ell \leq m, 0 \leq j_1 < \ldots < j_\ell < m, \varepsilon_1, \ldots, \varepsilon_\ell \in \{0, \ldots, k-1\} \right\}.
\]

Theorem 12. Let \( m, n, k, r \) be positive integers with \( n \geq MT(2kM, 4kM-1, r^d) \), where \( M = G(k, m, r) \) and \( \beta = \sum_{d=1}^{2kM} 2d(2k - 1)^{d-1} \). Then for every coloring \( c : X_{\pm k}(n) \to \{1, \ldots, r\} \) there exist \( s \) in Block\(^{2kM}\)(n) and \( F \) in Block\(^m\)(s) such that

(i) \( F \) is \( s \)-skipped block and

(ii) there exists \( i_0 \) in \( \{1, \ldots, r\} \) such that for every \( f \in \langle F \rangle_{\pm k} \) there exists \( f' \in X_{\pm k}(s) \) such that \( c(f') = i_0, \rho_{\infty}(f, f') \leq 1 \) and the pair \( (f, f') \) is of \( s \)-displacement at most one.

In particular, we have that \( \langle F \rangle_{\pm k} \) is approximately monochromatic.

Proof. Let \( c : X_{\pm k}(n) \to \{1, \ldots, r\} \) be a coloring. Since \( n \geq MT(2kM, 4kM-1, r^d) \), by Lemma 11 applied for “\( m = 2kM \)”, we have that there exists an \( s \) in Block\(^{2kM}\)(n) such that every \( f, f' \) in \( X_{\pm k}(s) \) of the same type have the same color. We set \( G = (g_i)_{i=0}^{M-1} = (q(k, \ell_i, s))_{i=0}^{M-1} \) where \( \ell_i = 2ki + k - 1 \) for all \( 0 \leq i < M \) as in Lemma 10. Observe that \( G \) is \( s \)-skipped block.
We define a map $Q : X_{|k|}(M) \to \langle G \rangle_{|k|}$ by setting

$$Q(g) = \sum_{i=0}^{M-1} T^{(k-b(i))}(g_i)$$

for all $g$ in $X_{|k|}(M)$. It is easy to observe that $Q$ is 1-1 and onto, while $Q$ restricted to $X_k(M)$ is 1-1 and onto $\langle G \rangle_k$. Moreover, the image of every block sequence in $X_k(M)$ is a block sequence in $\langle G \rangle_k$ and the pre-image of every block sequence in $\langle G \rangle_k$ is a block sequence in $X_k(M)$. Finally, for every $g$ in $X_{|k|}(M)$ we have that $Q(T(g)) = T(Q(g))$. Under these remarks, for every block sequence $H = (h_i)_{i=0}^{d-1}$ in $X_k(M)$, setting $F = (Q(h_i))_{i=0}^{d-1}$, we have that

$$Q[\langle H \rangle_k] = \langle F \rangle_k.$$ 

We define a coloring $\tilde{c} : X_k(M) \to \{1, ..., r\}$ by setting $\tilde{c}(g) = c(Q(g))$ for all $g \in X_k(M)$. By the definition of $M$ and applying Theorem 1, we obtain a block sequence $H = (h_i)_{i=0}^{m-1}$ in $X_k(M)$ of length $m$ such that the set $\langle H \rangle_k$ is $\tilde{c}$-monochromatic.

We set $F = (Q(h_i))_{i=0}^{m-1}$. Then $F$ is a block subsequence of $G$ of length $m$. Moreover, since $G$ is $s$-skipped block, we have that $F$ is also $s$-skipped block, that is, part (i) of the theorem is satisfied. By the definition of $\tilde{c}$, the fact that $\langle H \rangle_k$ is $\tilde{c}$-monochromatic and Theorem 12, we have that the set $\langle F \rangle_k$ is $c$-monochromatic. Let $i_0$ in $\{1, ..., r\}$ such that $c(f) = i_0$ for all $f$ in $\langle F \rangle_k$.

In order to check the validity of part (ii) of the theorem, we fix $f \in \langle F \rangle_{\pm k}$. By the choice of $G$, the fact that $F$ is a block subsequence of $G$ and Lemma 10, we have that there exist $f' \in X_{\pm k}(s)$ and $f'' \in \langle F \rangle_k$ such that

(i) the functions $f'$, $f''$ are of the same type,
(ii) $p_{\infty}(f, f') \leq 1$ and
(iii) the pair $(f, f')$ is of $s$-displacement at most one.

By (i) and the choice of $s$, we have $c(f') = c(f'') = i_0$. The proof is complete. $\square$

Clearly, Theorem 12 yields Theorem 2 and, in particular, for every choice of positive integers $m, k, r$ we have that

$$G_{\pm}(k, m, r) \leq MT(2kM, 4kM - 1, r^\beta),$$

where $M = G(k, m, r)$ and $\beta = \sum_{d=1}^{2kM} 2d(2k-1)^{d-1}$. Therefore, since the numbers $MT(d, m, r)$ are upper bounded by a function belonging to the class $\mathcal{E}^6$ of Grzegorczyk’s hierarchy and the numbers $G(k, m, r)$ are upper bounded by a function belonging to the class $\mathcal{E}^7$, we have that the numbers $G_{\pm}(k, m, r)$ are upper bounded by a function belonging to the class $\mathcal{E}^7$.

6. Multidimensional Versions

In this section we provide multidimensional versions of the Theorems 1 and 2. A standard iteration of Theorem 1 yields the following.
Theorem 13. For every quadrat of positive integers \(k, d, m, r\) with \(d \leq m\), there exists a positive integer \(n_0\) satisfying the following property. For every integer \(n \geq n_0\) and every coloring of the set \(\text{Block}_{\pm k}^d(n)\) with \(r\) colors, there exists a block sequence \(F\) in \(X_k(n)\) of length \(m\) such that the set \(\text{Block}_{\pm k}^d(F)\) is monochromatic. We denote the least \(n_0\) satisfying the above property by \(MG(k, d, m, r)\).

Moreover, the numbers \(MG(k, d, m, r)\) are upper bounded by a primitive recursive function belonging to the class \(\mathcal{E}^9\) of Grzegorczyk’s hierarchy.

The metric \(\rho_\infty\) defined on \(X_{\pm k}(n)\) naturally induces a metric on \(\text{Block}_{\pm k}^d(n)\), which we denote, abusing notation, by \(\rho_\infty\). In particular, for every \(F = (f_i)_{i=0}^{d-1}\) and \(G = (g_i)_{i=0}^{d-1}\) in \(\text{Block}_{\pm k}^d(n)\), we define the distance between \(F\) and \(G\) as

\[
\rho_\infty(F, G) = \max_{0 \leq i < d} \rho_\infty(f_i, g_i).
\]

Finally, given a finite coloring \(c : \text{Block}_{\pm k}^d(n) \rightarrow \{1, \ldots, r\}\), we say that a subset \(A\) of \(\text{Block}_{\pm k}^d(n)\) is approximately monochromatic if there exists some \(i_0 \in \{1, \ldots, r\}\) such that for every \(F\) in \(A\) there exists \(F'\) in \(\text{Block}_{\pm k}^d(n)\) with \(c(F') = i_0\) and \(\rho_\infty(F, F') \leq 1\). We have the following multidimensional version of Theorem 2.

Theorem 14. For every quadrat of positive integers \(k, d, m, r\) with \(d \leq m\), there exists a positive integer \(n_0\) satisfying the following property. For every integer \(n \geq n_0\) and every coloring of the set \(\text{Block}_{\pm k}^d(n)\) with \(r\) colors, there exists a block sequence \(F\) in \(X_{\pm k}(n)\) of length \(m\) such that the set \(\text{Block}_{\pm k}^d(F)\) is approximately monochromatic. We denote the least \(n_0\) satisfying the above property by \(MG_{\pm}(k, d, m, r)\).

Moreover, the numbers \(MG_{\pm}(k, d, m, r)\) are upper bounded by a primitive recursive function belonging to the class \(\mathcal{E}^9\) of Grzegorczyk’s hierarchy.

6.1. Proof of Theorem[13] The method for obtaining Theorem[13] from Theorem 1 is quit standard. However, we include it for seasons of completeness. For every two finite sequences \(F\) and \(G\) by \(F^ \bowtie G\) be denote the concatenation of \(F\) and \(G\).

Lemma 15. Let \(d, \ell_1, N, N', n, k, r\) be positive integers such that \(d \leq \ell_1\), \(N = G(k, N', r^{(k+1)\ell_1})\) and \(\ell_1 + N \leq n\). Also let \(G\) and \(F\) be block sequences in \(X_k(n)\) such that

1. \(G\) is of length at most \(\ell_1\),
2. \(F\) is of length \(N\) and
3. \(G^ \bowtie F\) is a block sequence.

Finally, let \(c : \text{Block}_{k}^{d+1}(n) \rightarrow \{1, \ldots, r\}\) be a coloring. Then there exists \(F'\) in \(\text{Block}_{N'}^d(F)\) such that for every \(H\) in \(\text{Block}_{k}^{d}(G)\) and every \(f, f'\) in \(\langle F' \rangle_k\) we have that \(c(H^ \bowtie (f)) = c(H^ \bowtie (f'))\).

Proof. Let \(\mathcal{X}\) be the set of all functions from \(\text{Block}_{k}^{d}(G)\) into \(\{1, \ldots, r\}\). It is easy to observe that the set \(\text{Block}_{k}^{d}(G)\) is of cardinality at most \((k+1)^{\ell_1}\) and therefore the set \(\mathcal{X}\) is of cardinality at most \(r^{(k+1)^{\ell_1}}\). We define a coloring \(\tilde{c} : \langle F \rangle_k \rightarrow \mathcal{X}\) by
we obtain $F, f, f''$ of Lemma 15. Indeed, assume that for some $p < \ell$ it holds that there exists $G$ in $\text{Block}_k^d(N)$ such that for every $H$ in $\text{Block}_k^d(G)$ we have that $c(H^\sim(f)) = \tilde{c}(f)(H) = \tilde{c}(f')(H) = c(H^\sim(f'))$. The proof is complete. □

The next lemma follows by an iterated use of the above lemma. We will need the following invariants. In particular, we define a function $h : \mathbb{N}^5 \to \mathbb{N}$ as follows. For every positive integers $d, \ell, r, k$ we inductively define

$$h(d, \ell, r, k, 0) = 0,$$

$$h(d, \ell, r, k, x + 1) = G(k, h(d, \ell, r, k, x) + 1, r^{(k+1)d}).$$

Since the numbers $G(k, m, r)$ are upper bounded by a function belonging to the class $\mathcal{E}^7$ of Grzegorczyk’s hierarchy, we have that the function $h$ is upper bounded by a function belonging to the class $\mathcal{E}^8$.

**Lemma 16.** Let $d, \ell, n, k, r$ be positive integers such that $d < \ell$ and $n \geq d + h(d, \ell, r, k, \ell - d)$. Also let $c : \text{Block}_{k+1}^d(n) \to \{1, \ldots, r\}$ be a coloring. Then there exists $G$ in $\text{Block}_k^d(n)$ such that for every $J, J'$ in $\text{Block}_k^d(G)$ with $J[d = J'[d$, we have that $c(J) = c(J')$.

**Proof.** For every $p = 0, \ldots, \ell - d$ we set $N_p = h(d, \ell, r, k, \ell - d - p)$. Then by (14) we have that

$$N_{p-1} = G(k, N_p + 1, r^{(k+1)d})$$

for every $p \in \{1, \ldots, \ell - d\}$. We also set $G_0 = (k\chi^d_{i=0})_{i=0}^{d-1}$ and $F_0 = (k\chi^d_{i=d-1})_{i=0}^{N_0}$. We inductively construct two sequences $(F_p)_{p=0}^{\ell-d}$ and $(G_p)_{p=0}^{\ell-d}$ satisfying the following for every $p = 0, \ldots, \ell - d$.

- (C1) $G_p$ belongs to $\text{Block}_k^{d+p}(n)$ and $F_p = (f^p_p)_{i=0}^{N_p}$ belongs to $\text{Block}_k^{N_{p+1}}(n)$.
- (C2) If $p < \ell + d$, then $G_p F_p$ is a block sequence, where $F_p = (f^p_{i+1})_{i=0}^{N_p - 1}$.
- (C3) If $p \geq 1$, then $F_p$ belongs to $\text{Block}_k^{N_{p+1}}(F_{p-1})$.
- (C4) If $p \geq 1$, then $G_p = G_{p-1}(f^p)$.
- (C5) If $p \geq 1$, then for every $H$ in $\text{Block}_k^d(G_{p-1})$ and $f, f'$ in $(F_p)_k$, we have that $c(H^\sim(f)) = c(H^\sim(f'))$.

Observe that for $p = 0$ conditions C1 and C2 are satisfied, while C3-C5 are meaningless. The inductive step of the construction is a straightforward application of Lemma 15. Indeed, assume that for some $p$ in $\{1, \ldots, \ell - d\}$ the sequences $(G_i)_{i=0}^{p-1}$ and $(F_i)_{i=0}^{p-1}$ have been properly chosen. By (14), applying Lemma 15 for “$\ell_1 = \ell \geq d + p - 1$”, “$N = N_{p-1}$”, “$N' = N_p + 1$”, “$G = G_{p-1}$” and “$F = F_{p-1}$”, we obtain $F_p \in \text{Block}_k^{N_{p+1}}(F_{p-1})$ such that for every $H$ in $\text{Block}_k^d(G_{p-1})$ and every $f, f'$ in $(F_p)_k$ we have that $c(H^\sim(f)) = c(H^\sim(f'))$. Clearly conditions C1, C3
and C5 are satisfied. Setting $G_n = G_{p-1}(f_0^n)$ the proof of the inductive step is complete. We set $G = G_{\ell-d}$. It is easy to check that $G$ is as desired.

**Proof of Theorem 13.** We proceed by induction on $d$. The base case $d = 1$ is established by Theorem 1. In particular, we have

$$MG(k, 1, m, r) = G(k, m, r)$$

for every choice of positive integers $k, m, r$. Assume that the theorem holds for some positive integer $d$ and let $m, k, r$ be positive integers. We will prove the inductive step by showing that

$$MG(k, d + 1, m, r) \leq d + h(d, M + 1, r, k, M - d + 1),$$

where $M = MG(k, d, m - 1, r)$ and $h$ is as defined in (13). Indeed, let $n$ be a positive integer with

$$n \geq d + h(d, M + 1, r, k, M - d + 1).$$

Also let $c : \text{Block}_{k+1}^d(n) \to \{1, \ldots, r\}$ be a coloring. By (17) and Lemma 16 for $"\ell = M + 1"$, we have that there exists $G = (g_i)_{i=0}^M$ in $\text{Block}_{k+1}^{M+1}(n)$ such that for every $J, J'$ in $\text{Block}_{k}^d(G)$ with $|J| = |J'|$, we have that $c(J) = c(J')$. We set $G^* = (g_i)_{i=0}^{M-1}$. We define a coloring $\tilde{c} : \text{Block}_{k}^d(G^*) \to \{1, \ldots, r\}$ by setting

$$\tilde{c}(\mathbf{H}) = c(\mathbf{H}^\gamma(g_M)).$$

By the inductive assumption and the definition of $M$ we obtain $F^*$ in $\text{Block}_{k}^{m-1}(G^*)$ such that the set $\text{Block}_{k}^d(F^*)$ is $\tilde{c}$-monochromatic. We set $F = F^\gamma(g_M)$. It is easy to see that $F$ is as desired.

Finally, since the numbers $G(k, m, r)$ are upper bounded by a function belonging to the class $E^7$ of Grzegorczyk’s hierarchy and $h$ is upper bounded by a function belonging to the class $E^8$, by (15) and (16), we have that the numbers $MG(k, d, m, r)$ are upper bounded by a function belonging to the class $E^9$. □

6.2. **Proof of Theorem 14.** The reduction of Theorem 14 to Theorem 13 is similar to the one of Theorem 2 to Theorem 1. We will need the analogues of Lemmas 6 and 10 for block sequences. To this end, we define the type of a block sequence $F = (f_i)_{i=0}^{m-1}$ in $X_{\pm k}^d(n)$ by setting

$$\text{tp}(F) = (\text{tp}(f_i))_{i=0}^{m-1}.$$ 

Moreover for every $k, d, m, n$ positive integers, with $d \leq m \leq n$, and $s$ in $\text{Block}^m(n)$, by $\text{Block}_{\pm k}^d(s)$ we denote the set of all block sequences in $X_{\pm k}(s)$ of length $d$.

**Lemma 17.** Let $d, m, n, k, r$ be positive integers, with $d \leq m$ and

$$n \geq \text{MT}(m, 2m - 1, r^\gamma),$$

where $\gamma = (m\choose d)^2d^d(2k-1)^{d(\gamma-1)}$. Then for every coloring of the set $\text{Block}_{\pm k}^d(n)$ with $r$ colors, there exists $s$ in $\text{Block}^m(n)$ such that every $F, F'$ in $\text{Block}_{\pm k}^d(s)$ of the same type have the same color.
Proof. We set $T$ to be the set of sequences $\varpi = (\varphi_i)_{i=0}^{d-1}$ of length $d$, such that $\varphi_i$ is a type over $\pm k$ for all $0 \leq i < d$ and $\sum_{i=0}^{d-1} |\varphi_i| \leq m$. Observing that the cardinality of the set of all types over $\pm k$ of some length $\ell \leq m$ is upper bounded by $2\ell(2k-1)^{\ell-1} \leq 2m(2k-1)^{m-1}$, it is easy to check that the cardinality of $T$ is at most $\gamma$. Therefore, setting $X$ to be the set of all maps from $T$ into $\{1,\ldots, r\}$, we have that the cardinality of $X$ is at most $r^\gamma$.

Let $c : \text{Block}^d_{\pm k}(n) \to \{1,\ldots, r\}$ be a coloring. Next we define a new coloring $\tilde{c} : \text{Block}^m(n) \to X$ as follows. For every $\varpi = (\varphi_i)_{i=0}^{d-1}$ in $T$ we define $t_0^\varpi = 0$ and $t_i^\varpi = \sum_{j=0}^{i-1} |\varphi_j|$ for all $1 \leq i < d$. Moreover, for every $\varpi = (\varphi_i)_{i=0}^{d-1}$ in $T$ and every $t = (t_i)_{i=0}^{m-1}$ in $\text{Block}^m(n)$ we define $\text{bl}(\varpi, t)$ in $\text{Block}^d_{\pm k}(t)$ by the rule

$$\text{bl}(\varpi, t) = \left(\text{map}(\varphi_i, (t_i)_{j=0}^{||\varphi_i|-1})\right)_{i=0}^{d-1}.$$ 

Clearly, for every choice of $\varpi$ and $t$ as above, we have that $\text{bl}(\varpi, t)$ is of type $\varpi$. Finally, for every $t$ in $\text{Block}^m(n)$ we define an element $q_t$ of $X$ by setting for every $\varpi$ in $T$

$$q_t(\varpi) = c(\text{bl}(\varpi, t)).$$

We define $\tilde{c}$ by setting $\tilde{c}(t) = q_t$ for all $t$ in $\text{Block}^m(n)$.

Since $n \geq MT(m, 2m - 1, r^\gamma)$, applying Theorem 3, we obtain a block sequence $s' \in \text{Block}^{2m-1}(n)$ such that the set $\text{Block}^m(s')$ is $\tilde{c}$-monochromatic. That is, there exists $q$ in $X$ such that for every $t$ in $\text{Block}^m(s')$ we have that $\tilde{c}(t) = q_t = q$. We set $s = s'|m$ and we observe that $s$ is as desired. Indeed, let $F = (f_i)_{i=0}^{d-1}$ and $F' = (f'_i)_{i=0}^{d-1}$ in $\text{Block}^d_{\pm k}(s)$ of the same type $\varpi = (\varphi_i)_{i=0}^{d-1}$. Also let $\ell = \sum_{i=0}^{d-1} |\varphi_i|$. Clearly $d \leq \ell \leq m$. Also let

$$t_1 = \text{bsupp}(f_1) \cup \cdots \cup \text{bsupp}(f_{d-1}) \text{ and } t'_1 = \text{bsupp}(f'_1) \cup \cdots \cup \text{bsupp}(f'_{d-1}).$$

Clearly, both $t_1$ and $t'_1$ are of length $\ell$. Since $s$ is the initial segment of $s'$ of length $m$ and $s'$ is of length $2m - 1$, we can end-extend both $t_1$ and $t'_1$ into $t$ and $t'$ respectively elements of $\text{Block}^m(s')$. Since $F = \text{bl}(\varpi, t)$ and $F' = \text{bl}(\varpi, t')$, we get that $c(F) = q_t(\varpi) = q(\varpi) = q_t(\varpi) = c(F')$. The proof is complete. \hfill $\Box$

By Lemma 10 we have the following consequence. To state it we need a slight modification of the existing terminology. For $k,d,n$ positive integers with $d \leq n$ and $F$ a block sequence in $X_{\pm k}(n)$ of length at least $d$, we denote by $\text{Block}^d_k(F)$ the set of all block sequences in $(F)_k$ of length $d$.

**Corollary 18.** Let $d, m, n, k$ be positive integers, with $d \leq m$ and $2km \leq n$, and $s \in \text{Block}^{2km}(n)$. We set $G = (g_i)_{i=0}^{m-1} = (q(k, \ell, i, s))_{i=0}^{m-1}$ where $\ell_i = 2ki + k - 1$ for all $0 \leq i < m$. Let $F$ be a block sequence in $(G)_k$. Then for every $H = (h_i)_{i=0}^{d-1}$ in $\text{Block}^d_{\pm k}(F)$ there exist $H'$ in $\text{Block}^d_{\pm k}(s)$ and $H'' = (h''_i)_{i=0}^{d-1}$ in $\text{Block}^d_k(F)$ such that

(i) the sequences $H', H''$ are of the same type,

(ii) $\rho_\infty(H, H') \leq 1$, 
(iii) the pair \((H, H')\) is of \(s\)-displacement at most one and

(iv) \(\text{supp}(h_i) = \text{supp}(h''_i)\), for all \(0 \leq i < d\).

Actually, we prove the following slightly stronger version of Theorem 19.

**Theorem 19.** Let \(d, m, n, k, r\) be positive integers with \(d \leq m\) and

\[
(18) \quad n \geq MT(2kM, 4kM - 1, r^{\gamma}),
\]

where \(M = MG(k, d, m, r)\) and \(\gamma = \binom{M}{d}2^d M^d (2k-1)^{d(M-1)}\). Then for every coloring \(c : \text{Block}^d_{\pm k}(n) \to \{1, \ldots, r\}\) there exist \(s \in \text{Block}^2_{kM}(n)\) and \(F \in \text{Block}^m_{\pm k}(s)\) such that

(i) \(F\) is \(s\)-skipped block and

(ii) there exists \(i_0 \in \{1, \ldots, r\}\) such that for every \(H \in \text{Block}^d_{\pm k}(F)\) there exists \(H' \in \text{Block}^d_{\pm k}(s)\) such that \(c(H') = i_0, \rho_\infty(H, H') \leq 1\) and the pair \((H, H')\) is of \(s\)-displacement at most one.

In particular, we have that \(\text{Block}^d_{\pm k}(F)\) is approximately monochromatic.

**Proof.** Let \(c : \text{Block}^d_{\pm k}(n) \to \{1, \ldots, r\}\) be a coloring. By (13) and Lemma 17 applied for \(m = 2kM\), there exists an \(s\) in \(\text{Block}^2_{kM}(n)\) such that every \(H'\) in \(\text{Block}^d_{\pm k}(s)\) of the same type have the same color. We define \(G = (g_i)_{i=0}^{M-1} = (q(k, \ell_i, s))_{i=0}^{M-1}\), where \(\ell_i = 2ki + k - 1\) for all \(0 \leq i < M\), as in Corollary 18. We define a map \(Q : X_M(M) \to \langle G \rangle_M\), as in the proof of Theorem 2 by setting

\[
Q(g) = \sum_{i=0}^{M-1} T^{(k-h(i))}(g_i)
\]

for all \(g \in X_M(M)\). Moreover, we define a map \(Q_d : \text{Block}^d_{k}(M) \to \text{Block}^d_{k}(G)\) setting

\[
Q_d(h') = (Q(h'_i))_{i=0}^{d-1}
\]

for every \(H' = (h'_i)_{i=0}^{d-1}\) in \(\text{Block}^d_{k}(M)\). It is easy to observe that \(Q_d\) is 1-1 and onto. Moreover, for every \(F' = (f'_i)_{i=0}^{\ell-1}\) block sequence in \(X_k(M)\), setting \(F = (Q(f'_i))_{i=0}^{\ell-1}\), we have that the restriction of \(Q_d\) on \(\text{Block}^d_{k}(F')\) is 1-1 and onto \(\text{Block}^d_{k}(F)\). Finally, we define a coloring \(\tilde{c} : \text{Block}^d_{k}(M) \to \{1, \ldots, r\}\), setting \(\tilde{c}(H') = c(Q(H'))\) for all \(H'\) in \(\text{Block}^d_{k}(M)\).

By the choice of \(M\) and Theorem 13 applied for the coloring \(\tilde{c}\), we have that there exists a block sequence \(F' = (f'_i)_{i=0}^{m-1}\) in \(X_k(n)\) of length \(m\) such that the set \(\text{Block}^d_{k}(F')\) is monochromatic with respect to \(\tilde{c}\). We set \(F = (Q(f'_i))_{i=0}^{m-1}\). By the definition of the coloring \(\tilde{c}\), we have that \(\text{Block}^d_{k}(F)\) is monochromatic with respect to \(c\). That is, there exists \(j_0 \in \{1, \ldots, r\}\) such that \(c(H) = j_0\) for all \(H\) in \(\text{Block}^d_{k}(F)\).

Let us observe that \(F\) is as desired. Indeed, observe that \(G\) is an \(s\)-skipped block sequence and therefore, since \(F\) is a block subsequence of \(G\), we have that \(F\) is an \(s\)-skipped block sequence too, that is, part (i) of the theorem is satisfied. Let \(H\) in \(\text{Block}^d_{\pm k}(F)\). By the definition of \(G\), the fact that \(F\) is a block sequence in \(\langle G \rangle_k\)
and Corollary [15] we have that there exist \( H' \) in \( \text{Block}_d^{\pm k}(s) \) and \( H'' \) in \( \text{Block}_d^k(F) \) such that

(i) the sequences \( H', H'' \) are of the same type,
(ii) \( \rho_\infty(H, H') \leq 1 \) and
(iii) the pair \( (H, H') \) is of \( s \)-displacement at most one.

Since \( H', H'' \) both belong to \( \text{Block}_d^{\pm k}(s) \), by the choice of \( s \) and by (i) above, we have that

\[
(19) \quad c(H') = c(H'').
\]

Since \( H'' \) belongs to \( \text{Block}_d^k(F) \), we have that

\[
(20) \quad c(H'') = j_0.
\]

By (19) and (20), we have that \( c(H') = j_0 \). By (ii) and (iii) above, the proof is complete. \( \square \)

Clearly, Theorem [19] yields Theorem [14] and in particular, for every choice of positive integers \( d, m, k, r \) with \( d \leq m \) we have that

\[
\text{MG}_\pm(k, d, m, r) \leq \text{MT}(2kM, 4kM - 1, r^\gamma),
\]

where \( M = \text{MG}(k, d, m, r) \) and \( \gamma = \left(\frac{M}{d}\right) 2^d M^d (2k - 1)^{d(M-1)} \). Therefore, since the numbers \( \text{MG}(k, d, m, r) \) are upper bounded by a function belonging to the class \( E^9 \) of Grzegorczyk’s hierarchy and the numbers \( \text{MT}(d, m, r) \) are upper bounded by a function belonging to the class \( E^6 \), we have that the numbers \( \text{MG}_\pm(k, d, m, r) \) are upper bounded by a function belonging to the class \( E^9 \).

7. Concluding Remarks

It is easy to observe that \( X_{\pm k}(n) \) does not admit the Ramsey property, just by considering a coloring depending on the sign of the function at the minimum of its support. It is natural then to ask whether \( X_{\pm k}(n) \) admits a Ramsey degree. The answer turns out to be negative. In particular, the following holds.

**Proposition 20.** For every pair of positive integers \( K, n \) with \( n \geq 2K \) there exists a coloring \( c : X_{\pm 1}(n) \to \{1, ..., K\} \) such that for every block sequence \( F \) in \( X_{\pm k}(n) \) of length \( 2K \), we have that \( \langle F \rangle_{\pm k} \) realizes all the colors.

**Proof.** Let \( K, n \) be positive integers with \( 2K \leq n \). We define a coloring \( c : X_{\pm 1}(n) \to \{1, ..., K\} \) by the rule

\[
c(f) = (|\text{tp}(f)| \mod K) + 1.
\]

The color essentially counts the “jumps” between the two non-zero values of the function \( f \). Let \( F = (f_i)_{i=0}^{2K-1} \) be a block sequence in \( X_{\pm k}(n) \). For every \( i = 0, ..., K - 1 \), we set

\[
g_i = f_{2i}(\min \text{supp}(f_{2i})) \cdot f_{2i} + f_{2i+1}(\max \text{supp}(f_{2i+1})) \cdot f_{2i+1}.
\]
Let us observe that \((g_i)_{i=0}^{K-1}\) forms a block sequence of length \(K\) and \(g_i(\min \text{supp}(g_i)) = g_i(\max \text{supp}(g_i)) = 1\), for all \(0 \leq i < K\). We set
\[
h_0 = \sum_{j=0}^{K-1} g_j\text{ and } h_i = \sum_{j=0}^{i-1} (-1)^j g_j + (-1)^i \sum_{j=i}^{K-1} g_j
\]
for all \(1 \leq i < K\). By this choice we have that \(h_i \in (F)_{ \pm k}\) and
\[
|\text{tp}(h_i)| = |\text{tp}(h_0)| + i
\]
for all \(0 \leq i < K\). By (21) we have that the set \(\{h_i : 0 \leq i < K\}\) itself realizes all the colors. The proof is complete. \(\Box\)

REFERENCES

[1] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Vol. 1, Amer. Math. Soc. Colloquium Publications 48, American Mathematical Society, Providence, RI, (2000).
[2] P. Dodos and V. Kanellopoulos, Topics in Ramsey Theory, preprint.
[3] W. T. Gowers, Lipschitz functions on classical spaces, Europ. J. Combinatorics, 13, 141–151, (1992).
[4] W. T. Gowers, Ramsey methods in Banach spaces, Handbook of the Geometry of Banach Spaces, Vol. 2, pp. 1071–1097, W. B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam (2003).
[5] R. L. Graham and B. L. Rothschild, Ramsey's theorem for \(n\)-parameter sets, Trans. Amer. Math. Soc. 159 (1971), 257–292.
[6] N. Hindman, Finite sums from sequences within cells of a partition of \(\mathbb{N}\), J. Comb. Theory, Ser. A 17 (1974), 1–11.
[7] V. Kanellopoulos, A proof of W. T. Gowers’ \(c_0\) theorem, Proc. Amer. Math. Soc. 132 (2004), no. 11, 3231–3242 (electronic).
[8] K. Milliken, Ramsey’s theorem with sums or unions, J. Comb. Theory, Ser. A 18 (1975), 276–290.
[9] H. E. Rose, Subrecursion: functions and hierarchies, Oxford Logic Guide 9, Oxford Univ. Press, Oxford, 1984.
[10] D. Ojeda-Aristizabal, Finite forms of Gowers’ Theorem on the oscillation stability of \(c_0\), available at arxiv.org/abs/1312.4639.
[11] S. Shelah, Primitive recursive bounds for van der Waerden numbers, J. Amer. Math. Soc. 1 (1988), 683–697.
[12] A. D. Taylor, A canonical partition relation for the finite subsets of \(\omega\), J. Comb. Theory, Ser. A 21 (1976), 137–146.
[13] A. D. Taylor, Bounds for the disjoint unions theorem, J. Comb. Theory, Ser. A 30 (1981), 339–344.
[14] S. Todorcevic, Introduction to Ramsey Spaces, Annals of Mathematics Studies, No. 174, Princeton University Press 2010.

Department of Mathematics, University of Toronto, Toronto, Canada M5S 2E4
E-mail address: ktyros@math.toronto.edu