DIRECTIONAL COMPLEXITY AND ENTROPY FOR LIFT MAPPINGS

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Abstract. We introduce and study the notion of a directional complexity and entropy for maps of degree 1 on the circle. For piecewise affine Markov maps we use symbolic dynamics to relate this complexity to the symbolic complexity. We apply a combinatorial machinery to obtain exact formulas for the directional entropy, to find the maximal directional entropy, and to show that it equals the topological entropy of the map.

1. Introduction. There is a well-developed theory of rotation vectors (numbers) and rotation sets (see, for instance, [11] and reference therein). One considers a map $f : M \to M$ generating a dynamical system and an observable $\phi : M \to \mathbb{R}^d$ that classically is a displacement but might be an arbitrary function. The rotation vector of $x$ is the Birkhoff average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x),$$

provided that the limit exists, say, equals $v$. Then we may say that $x$ moves in the direction $v$. A natural question arises: how many points move in the direction $v$ if one measures them in terms of the topological entropy. The authors of [11] have mentioned several attempts to answer the question and have described their own approach. All of them including one of [16] are based on the thermodynamic formalism, in particular, on the variational principle. In our article we use purely topological (metric) approach to describe points moving to the prescribed direction.

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We shall exploit notion of the $\epsilon$-separability introduced by Kolmogorov and Tikhomirov [15] in the context of [4]. A notion of space-time window introduced in [17, 18] for cellular automata and used in [1, 3, 8] for lattice dynamical systems we apply here for maps on $\mathbb{R}^1$ that are lifts for maps of the circle of degree 1. If such a map generates the dynamical system with non-zero topological entropy then, very often, it has a rotation interval different from a single point. It implies the existence of trajectories with different rotation numbers, i.e. with different spatio-temporal features. We suggest here to measure the number of trajectories with a given rotation number using the notion of a directional entropy. Roughly speaking if $X$ is a subset of a the circle such that the trajectories going through $X$ have the rotation number, say, $\alpha$, then the $(\epsilon, n)$-complexity of $X$ behaves asymptotically ($n >> 1$) as $\exp(nH_\alpha)$. We call the number $H_\alpha$ the directional entropy in the direction $\alpha$. The greater $H_\alpha$ the greater the rate of instability manifests by trajectories with the rotation number $\alpha$. But one has to be careful. It can happen (and occurs for mixing systems) that for any fixed rotation number $\alpha$ inside the rotation interval the set of initial points, say $X_\alpha$, corresponding to this rotation number is dense in the circle. So, the topological entropy on $X_\alpha$ coincide with the topological entropy of the whole system. To avoid it we approximate $X_\alpha$ by sets of initial points which trajectories stay in a space-time window, calculate the entropy on this window, and obtain $H_\alpha$ as the limit of these entropies.

In this article we study mainly piecewise affine Markov maps of the circle. For such maps it is possible to replace the calculation of the $(\epsilon, n)$-complexity by that of the symbolic complexity of some subsets of a corresponding topological Markov chain (TMC). The TMC is determined by the Markov partition of the circle and the subsets – by the admissibility condition formulated according to the value of the rotation number. After that the problem becomes purely combinatorial. We use the approach of [20, 21] adjusted for our situation to obtain the explicit formulas for $H_\alpha$. The formulas depend only on the entries of the transition matrix of the TMC and on the weights of the edges of the corresponding oriented graph, where the weights are determined by the Markov partition and the lift map. Moreover, our results on TMC does not depend on the fact that it is originated from a circle map as it explained in Section 8.

The article is organized as follows. In Section 2 we give the definitions of the directional complexity and the directional entropy $H_\alpha$ for a map of the circle. In Section 3 we show that $H_\alpha \neq 0$ only if $\alpha$ belongs to the rotation interval. In Section 4 we define piecewise affine Markov maps and show how to calculate the $(\epsilon, n)$-complexity in terms of symbolic dynamics. Section 5 is devoted to the description of the combinatorial machinery. In Section 6 we describe a specific example where all can be explicitly seen. In Section 7 we construct some invariant probabilistic measures for which measure theoretical entropies coincide with the directional entropies. By using this we show that the topological entropy coincides with a directional entropy for some specific direction. We present a formula for this direction. Section 8 is devoted to the definition of directional entropy for topological Markov chains. Section 9 contains some concluding remarks.

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1 The term directional complexity was used in [12] in another context. In [12] the direction is the physical direction in billiards.
2. Definitions. Let $f : S^1 \to S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$ be a continuous mapping of degree one, i.e. there is a lift mapping $F : \mathbb{R}^1 \to \mathbb{R}^1$ of the form

$$F(x) = x + w + h(x),$$

where $h$ is 1-periodic function such that $\int_0^1 h(x)dx = 0$. Thus, $f(x) = x + w + h(x)$ mod 1.

Let $e = (e_x, e_y)$ be the unit vector in direction $\alpha$, that is $e = \sqrt{1 + \alpha^2 \pi}$. Given $l_1 < l_2$, let

$$W = W(l_1, l_2, \alpha) = \{(x + te_x, te_y) \mid 0 \leq t, l_1 \leq x \leq l_2\}$$

be the “window” in $\mathbb{R} \times \mathbb{R}^+$.\n
Definition 2.1. [4]

1) Two points $x, y \in \mathbb{R}$ are $(\epsilon, W, T)$-separated if $(F^n x, n), (F^n y, n) \in W$ for each $n \leq T$, and there exists $0 \leq n \leq T$ such that $|F^n x - F^n y| \geq \epsilon$.

2) A set $X \subset \mathbb{R}$ is $(\epsilon, W, T)$-separated if any pair $x, y$ in $X$, $x \neq y$, is $(\epsilon, W, T)$-separated.

3) The number

$$C_\epsilon(W, T) = \max\{\text{card } X \mid X \text{ is } (\epsilon, W, T) - \text{separated}\},$$

is called the directional $(\epsilon, W, T)$-complexity (in the direction $e$). Here, $\text{card } X$ is the cardinality (the number of points) of $X$.

4) The number

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{\ln C_\epsilon(W, T)}{T} = \mathcal{H}_\alpha(l_1, l_2),$$

is called the directional entropy in the direction $e$ with respect to the interval $[l_1, l_2]$. The limit

$$\mathcal{H}_\alpha = \lim_{l_1 \to -\infty} \lim_{l_2 \to \infty} \mathcal{H}_\alpha(l_1, l_2)$$

is called the directional entropy in the direction $e$.

5) Given a window $W$, an $(\epsilon, W, T)$-separated set $X$ is optimal if $\text{card } X = C_\alpha(W, T)$.

Remark 1. Roughly speaking, $C_\epsilon$ and $\mathcal{H}_\alpha$ are quantities reflecting the number of orbits “moving” with the velocity $\alpha$ along the circle. Indeed, to be in the window $W$, the point $(F^n x, n)$ must satisfy the inequality

$$l_1 + n\alpha \leq F^n x \leq l_2 + n\alpha,$$

thus the “velocity” $\frac{F^n x}{n}$ is approximately $\alpha$ if $n \gg 1$.

3. Rotation intervals and directional entropy. The ratio $\frac{F^n x}{n}$ is not only the velocity but also is related to the rotation number of the orbit going through the point $x$.

Definition 3.1. [19],[13]. The set

$$\bigcup_{x \in [0, 1]} \lim_{n \to \infty} \frac{F^n x}{n} = I,$$

i. e., the set of all points of accumulation for all initial points $x \in [0, 1]$ (the upper topological limit), is called the rotation interval of $f$.\n
It is known ([13],[19],[7]) that the rotation interval is a closed interval and for every \( \mu \in I \) there is \( x \in [0,1] \) such that \( \lim_{n \to \infty} \frac{F_n^\mu x}{n} = \mu \).

**Lemma 3.2.** The entropy \( H_\alpha = 0 \) if \( \alpha \notin I \).

**Proof.** Denote by \( a \) (\( b \)) the left (right) endpoint of the segment \( I \). It is known (see [6]) that there are functions \( F_1, F_2 : \mathbb{R} \to \mathbb{R} \) such that:

i) \( F_i \) are weakly monotone, i.e. the inequality \( x < y \) implies \( F_i(x) \leq F_i(y) \), \( i = 1, 2 \);

ii) there exist limits

\[
\lim_{n \to \infty} \frac{F_1^n(x)}{n} = a, \quad \lim_{n \to \infty} \frac{F_2^n(x)}{n} = b
\]

for any \( x \in \mathbb{R} \);

iii) for any \( x \in \mathbb{R} \) one has \( F_1(x) \leq F(x) \leq F_2(x) \).

The properties i) and ii) imply that

\[
F_1^n(x) \leq F^n(x) \leq F_2(x)
\]

(3)

for every \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

Assume now that \( H_\alpha > 0 \) and \( \alpha > b \). It means that there exists \( \epsilon > 0 \) and \( l_1 < l_2 \) such that \( H_\alpha(l_1, l_2) > b + \alpha \). Therefore there exists \( x \in \mathbb{R} \) such that the inequalities (2) hold for each \( n \in \mathbb{N} \). The inequalities (2) and (3) imply that

\[
F_2^n(x) > l_1 + na \geq l_1 + n(b + \epsilon)
\]

or

\[
\frac{F_2^n(x)}{n} \geq \frac{l_1}{n} + (b + \epsilon).
\]

Taking the limit as \( n \to \infty \) we obtain a contradiction. In the same way we prove that \( H_\alpha \) cannot be positive if \( \alpha < a \).

\[\square\]

**4. Piecewise affine Markov maps.** In this section we consider arbitrary piecewise affine Markov maps on the circle. For that, we represent \( S^1 \) as \( \mathbb{R} / \mathbb{Z} \) or as the interval [0, 1] with the identified endpoints. Let \( \mathcal{D} = \{d_0 = 0 < d_1 < \cdots < d_p = 1\} \), \( i = 0, \ldots, p-1 \), be an ordered collection of points on \( S^1 \). We introduce the following class of maps \( f : S^1 \to S^1 \):

i) \( f \) is a continuous map of degree 1,

ii) \( f(\mathcal{D}) \subset \mathcal{D} \),

iii) \( f \) is an affine map on each interval \([d_i, d_{i+1}]\): \( f(x) = a_i x + b_i \), \( i = 0, \ldots, p-1 \), \( a_i \neq 0 \); so, in particular \( f \) is one-to-one on \([d_j, d_{j+1}]\)

iv) \( |f'(x)| > 1, x \notin \mathcal{D} \), or \( |a_i| > 1 \), \( i = 0, \ldots, p-1 \).

Remark that the condition (ii) says that the points \( \mathcal{D} \) determine a Markov partition for \( f \) on \( S^1 \), and the condition (iv) claims that \( f \) is expanding on each element of this partition. Let us emphasize that this class of maps is interesting and large enough: first of all, Markov maps are dense in the space of expanding maps endowed with the topology of uniform convergence, and second, any Markov expanding map is semi-conjugated to a piecewise affine Markov map (is conjugated in the transitive case), see, for instance, [6].

Given \( f \) of this class, let us choose the lifting map \( F : \mathbb{R} \to \mathbb{R} \) such that \( F(0) \in [0,1], F(1) \in [1,2] \). Since \( f \) is of degree 1, such a lift always exists.

Let \( \xi_i = [d_i, d_{i+1}] \) be the \( i \)-th element of the Markov partition \( \xi, i = 0, \ldots, p-1 \). Without loss of generality one may assume that \( \text{diam} F(\xi_i) < 1, i = 0, \ldots, p-1 \). If
it is not so, one may consider the dynamical refinement \( \xi^{(n)} = \xi \cap f^{-1} \xi \cap \ldots \cap f^{-n+1} \xi \).

Because of the condition (iv), the diameter of an element \( \xi^{(n)} \) goes to 0 as \( n \to \infty \), so one may find out \( n_0 \), such that \( \text{diam} F(\xi^{(n)}) < 1 \) for every element \( \xi^{(n)} \) \( \in \xi^{n_0} \) and treat \( \xi^{(n)} \) as the original partition \( \xi \). Because of that, one may see that, first, if \( f(\text{int} \xi_i) \cap \text{int} \xi_j \neq 0 \) then \( f(\text{int} \xi_i) \supset \text{int} \xi_j \) (\( \text{int} \xi_i = (d_i, d_{i+1}) \), the open interval), and, second, for \( x \in \xi_j \) the set \( f^{-1} x \cap \xi_i \) consists of exactly one point if \( f(\text{int} \xi_i) \cap \xi_j \neq 0 \), and \( f^{-1} x \cap \xi_i = \emptyset \) if \( f(\text{int} \xi_i) \cap \xi_j = \emptyset \).

As usual, we identify the elements \( \xi_i \) with the symbols \( i \), consider the \( p \times p \)-matrix \( A = (a_{ij}) \), \( a_{ij} = 1 \) iff \( f(\text{int} \xi_i) \cap \text{int} \xi_j \neq 0 \), and introduce the one-sided topological Markov chain \((\Omega_A, \sigma)\) where \( \Omega_A = (\omega_0, \omega_1, \ldots) \mid \omega_k \in \{0, 1, \ldots, p-1\} \), \( \omega_k \) can follow \( \omega_{k-1} \) iff \( a_{\omega_{k-1} \omega_k} = 1 \), \( k = 1, \ldots \). We endow \( \Omega_A \) with the distance

\[
d(\omega, \omega') = \sum_{k=0}^{\infty} \frac{|\omega_k - \omega'_k|}{p^k},
\]

so, the shift map \( \sigma : \Omega_A \to \Omega_A \), \((\sigma \omega)_k = \omega_{k+1}, k \in \mathbb{Z}_+ \), will be continuous. The coding map \( \chi : \Omega_A \to \mathbb{S}' \) is well-defined in such a way that, for \( \omega = (\omega_0, \omega_1, \ldots) \in \Omega_A \)

\[
\chi(\omega) = \bigcap_{n=1}^{\infty} \Delta_{\omega_0 \ldots \omega_{n-1}}
\]

where \( \Delta_{\omega_0 \ldots \omega_{n-1}} = \xi_{\omega_0} \cap f^{-1} \xi_{\omega_1} \cap \ldots \cap f^{-n+1} \xi_{\omega_{n-1}} \). Since, for \( \omega \in \Omega_A \),

\[
\text{diam} \bigcap_{n=1}^{\infty} \Delta_{\omega_0 \ldots \omega_{n-1}} \prod_{k=0}^{n-1} |\omega_k^{-1}| \to 0 \text{ as } n \to \infty,
\]

then \( \chi(\omega) \) consists of the only one point.

4.1. Estimates from above. We introduce an oriented graph \( \Gamma_A \) having \( p \) vertices such that there exists an edge starting at the vertex \( i \) and ending at \( j \) iff \( a_{ij} = 1 \). By \( L^*_\Gamma \), we denote all \( \Gamma \)-admissible finite words (paths: \((\omega_0, \omega_1, \ldots, \omega_{n-1}) \in L^*_\Gamma \) iff \((\omega_j, \omega_j)\) is a \( \Gamma \)-edge for all \( j = 1, \ldots, n-1 \). As the graph \( \Gamma \) is normally fixed we sometimes omit the subscript \( \Gamma \). We relate a weight \( k_{ij} \in \mathbb{Z} \) to every edge \((i, j)\) of the graph \( \Gamma_A \) as follows: \( k_{ij} = s \) iff \( F(\xi_i) \supset \xi_j + s \) where \( \xi_j + s = \{x + s \mid x \in \xi_j\} \). Since \( F \) is continuous, the collection \( \{k_{ij} \mid a_{ij} = 1\} = \{s_0, s_0+1, \ldots, s_0+\rho\}, s_0 \leq 0, -s_0, \rho \in \mathbb{N} \). Now we want to estimate \( C_L(W, T) \) through the cardinality of different sets of words generated by \( \Gamma_A \). Let us start with some notation and definitions. For a finite word \( w = w_0 \ldots w_{n-1} \in L^*_\Gamma \) we denote:

- \(|w| = n\), the length of the sequence,
- \(w[i : j] = w_i w_{i+1} \ldots w_j; w[: j] = w_0 \ldots w_j\),
- \(v(w) = \sum_{i=1}^{n-1} k_{w[i-1, i]}\), the weight of \( w \),
- \(L^n = \{w \in L^*_\Gamma \mid |w| = n\}\), the collection of all admissible words of length \( n \),
- \(L^m_n = \{w \in L^n \mid v(w) = m\}\), the collections of admissible \( n \)-words of the weight \( m \).
- For any \( w \in L^n \) let \(|w| \subseteq \Omega_A \) be the corresponding cylinder, i.e. \(|w| = \{\omega \in \Omega_A \mid \omega[: n-1] = w\}\).

**Lemma 4.1.** Given \( w \in L^n \), for any \( x \in \chi(|w|) = \Delta_{\omega_0 \ldots \omega_{n-1}} \) one has

\[
m \leq F^{n-1} x \leq m + 1,
\]

where \( m = v(w) \).
Lemma 4.3. An estimate from below.

Proof. In fact, the statement directly follows from the definition of $k_{ij}$. Indeed, if $0 \leq x \leq 1$ then $Fx \in [k_{w_0w_1}, k_{w_0w_1} + 1]$ and so on.

Proposition 1. If, for $x \in [0,1]$, the inequality (4) is satisfied then $x \in \chi([w])$, $w \in L_{m-1}^* \cup L_m^* \cup L_{m+1}^*$.

Proof. Since the images of the cylinders $\{\chi([w]) \mid w \in L^n\}$ form a partition of the interval $[0,1]$ then $x \in \chi([w])$, $w = w_0 \ldots w_{n-1}$. Let $q = \sum_{j=0}^{n-1} k_{w_jw_{j+1}}$. If $q > m + 1$ (q < m - 1) then, because of Lemma 4.1, $F_{n-1}x \geq m + 1$ ($F_{n-1}x \leq q + 1 < m$), the contradiction with (4).

For $\alpha \in \mathbb{R}_+$, $r, n \in \mathbb{N}$, let $B_{n,\alpha, r} = \{w \in L^n \mid \forall j = 1, \ldots, n - 1 \alpha_j - r \leq v(w[: j]) \leq \alpha_j + r\}$. The following proposition is an easy implication of the definition of $B_{n,\alpha, r}$.

Proposition 2. Let $|w| = n$. Then $w \in B_{n,\alpha, r}$ if and only if for any $j = 1, \ldots, n - 1$ one has $w[: j] \in \bigcup_{m = \alpha_j - r}^{\alpha_j + r} L_{m}^{j+1}$.

We want to estimate $C_{\epsilon}(W(\alpha, [-r, r]), n)$ using the cardinalities of the sets $B_{n,\alpha, r+1}$.

Lemma 4.2. The following estimate holds

$$C_{\epsilon}(W(\alpha, [-r, r]), n) \leq \left[\frac{1}{\epsilon}\right]|B_{n,\alpha, r+1}|$$ (5)

Proof. Let $P$ be a an $(\epsilon, W, n)$-separated optimal set. By definition, if $x \in P$ then $(t - 1)\alpha - r \leq F_{n-1}x \leq (t - 1)\alpha + r$ for $t = 1, \ldots, n$. Now, $x \in \chi([w])$ where $w = w_0 \ldots w_{n-1}$. Because of Proposition 1,

$$w \in \bigcup_{m = \alpha_j - r}^{\alpha_j + r} L_{m}^{j+1}.$$ So, by Proposition 2, $x \in \Delta_w$ with $w \in B_{n,\alpha, r+1}$. Since $F_{n-1}$ is one-to-one on $\Delta_{w_0 \ldots w_{n-1}}$ and $|F_{t}^t(y)| > 1$ then $|F_{t}^t x - F_{t-1}^t y| \geq \epsilon$ for some $t < n$ and $x, y \in \Delta_{w_0 \ldots w_{n-1}}$, implies $|F_{n-1}^n x - F_{n-1}^n y| \geq \epsilon$.

Since $F_{n-1}^n \Delta_{w_0 \ldots w_{n-1}}$ is an interval of length less than 1, the number of points of $P$ inside $\Delta_{w_0 \ldots w_{n-1}}$ does not exceed $\left[\frac{1}{\epsilon}\right]$. Thus

$$|P| = C_{\epsilon}(W) \leq \left[\frac{1}{\epsilon}\right]|B_{n,\alpha, r+1}|.$$
Fix a maximal $S$ satisfying this property. One can check that $x \in \Delta_w$ and $y \in \Delta_v$ are $(\epsilon, W, km)$-separated for $w, v \in S$ and $w \neq v$. So, $C_{\epsilon_m}(W(\alpha, [-r, r]), km) \geq |S|$. We only need to estimate $|S|$. For $w \in B_{km, \alpha, r}$ let

$$U(w) \{ v \in B_{km, \alpha, r} \mid \forall 0 \leq j < k \text{ the intervals } \Delta_v[(jm:j+1)m] \text{ and } \Delta_w[(jm:j+1)m] \text{ are equal or successive} \}$$

Observe that $|U(w)| \leq 3k$ and $B_{km, \alpha, r} = \bigcup_{w \in S} U(w)$ due to the maximality of $S$. The estimate follows.

**Theorem 4.4.** Let

$$e_{\alpha,r} = \log \lim_{n \to \infty} \sqrt[n]{|B_{n,\alpha,r}|}.$$

Then the entropy

$$H_{\alpha} = \lim_{r \to \infty} e_{\alpha,r}. \quad (6)$$

**Proof.** Let

$$\lim_{n \to \infty} \frac{\ln C_{\epsilon}(W(\alpha, [-r, r]), n)}{n} = H_{\alpha}(\epsilon, r).$$

Lemma 4.2 and Lemma 4.3 together say that

$$3^{-k}|B_{km, \alpha, r}| \leq C_{\epsilon}(W(\alpha, [-r, r]), km) \leq \left[ \frac{1}{\epsilon^m} \right] |B_{n,\alpha}, r+1|,$$

for $\epsilon \leq \epsilon_m$. Taking $\ln(\sqrt[k]{3})$ from all parts of the above inequality and directing $k \to \infty$ one gets

$$-\frac{1}{m} \ln(3) + e_{\alpha,r} \leq H_{\alpha}(\epsilon, r) \leq e_{\alpha, r+1}.$$  

The smaller $\epsilon$ is the larger $m$ can be taken ($\epsilon \leq \epsilon_m \to 0$ when $m \to \infty$). So,  

$$e_{\alpha,r} \leq \lim_{\epsilon \to 0} H_{\alpha}(\epsilon, r) \leq e_{\alpha, r+1}.$$  

Finally we obtain the formula (6) \[ \square \]

**Remark 2.** We believe that formula (6) can be obtained by using the technique developed by M. Misiurewicz (see, for instance [6]). But, since we deal generally with non-invariant sets, this technique should be adjusted to the “non-invariant situation”. So, we decided to make a direct proof here.

5. **Combinatorial part.** Let

$$e_{\alpha} = \log \lim_{n \to \infty} \sqrt[n]{|L_{[\alpha]}^n|}.$$  

The aim of this subsection is to show that (under some conditions)

$$e_{\alpha} = \lim_{r \to \infty} e_{\alpha,r} = H_{\alpha}$$

and to explain how to calculate $e_{\alpha}$.  

Let $D \subset L^*$ be finite subset. Let the matrix $M(D) \in \text{Mat}_{p \times p}(\mathbb{N})$ be such that $M(D)_{ij}$ is the number of words in $D$ starting from $i$ and ending by $j$. Given  

$X, Y \subseteq L^*$ and $B \in \text{Mat}_{p \times p}(\{0, 1\})$ let $X \times Y \{ uw \mid u = u_1 \ldots u_n \in X, v_1 \ldots v_m \in Y \ B(u, v_1) = 1 \}$. The following proposition is a direct corollary of the above definitions.

**Proposition 3.**

$$M(X \times Y) = M(X)BM(Y)$$
Recall that \( L^n \) is the set of admissible words related to matrix \( A \). It is known that \( M(L^n) = A^{n-1} \), see, for instance, [2]. Let us represent the matrix \( A \) in the form
\[
A = \sum_{s \in S} A_s
\]
according to weight of the edges of \( \Gamma \). Precisely, \( A_s \in \text{Mat}(\{0,1\}) \), \( A_s(i,j) = 1 \) if and only if \( k_{ij} = s \). Here the set \( S \) is the set of all possible weights.

**Proposition 4.**

i) \( M(L_0) = E \) and \( M(L_1^m) = 0 \) if \( m \neq 0 \).

ii) For \( n \in \mathbb{Z}_+ \) the following equality holds
\[
M(L_{m}^{n+1}) = \sum_{s \in S} M(L_{m-s}^{n})A_s,
\]

*Proof.* By definition \( L^1 = \{0, \ldots, p-1\} \). Any word of length \( n+1 \) has a form \( wj \), where \( w \) is a word of length \( n \) and \( j \in \{1, \ldots, p\} \) and \( v(wj) = v(w) + v(w_{n-1}j) \). So, one has
\[
L_{m}^{n+1} = \bigcup_{s \in S} L_{m-s}^{n} \times \{0, \ldots, p-1\}.
\]
So, Proposition 3 implies the statement. \( \square \)

The following proposition is a consequence of definition of \( B_{n,\alpha,r} \) and \( L_m^n \).

**Proposition 5.**

\[
B_{n,\alpha,r} \subset \bigcup_{m=\lceil (n-1)\alpha \rceil - r}^{\lfloor (n-1)\alpha \rfloor + r} L_m^n,
\]

(7)

For \( B_j \in \text{Mat}_{p \times p}(\{0,1\}) \) we use below the notation \( (X_1 \times X_2 \times Y)Y = X_1 \times Y \cup X_2 \times Y \).

**Proposition 6.** Fix \( t \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). Let \( r \in \mathbb{N} \) be large enough \( (r > (t-1) \cdot (\max\{|s-\alpha| \mid s \in S\}) \), \( m_j = \lfloor t\alpha \rfloor - \lfloor (j-1)\alpha \rfloor \). Then for any \( c \in \mathbb{N} \) one has:
\[
(\bigcup_{s \in S} L_{m_1-s}^t A_s)(\bigcup_{s \in S} L_{m_2-s}^t A_s)\ldots(L_{m_c}^t A_s) \subset B_{ct,\alpha,r}.
\]

(8)

Moreover, \( m_j = \lfloor t\alpha \rfloor \) or \( m_j = \lfloor t\alpha \rfloor + 1 \) for \( j = 0, 1, \ldots c \).

*Proof.* The words of the set \( B_{ct,\alpha,r} \) are the words such that the weights of their initial subwords are in the \([-r,r]\)-strip with slope \( \alpha \). In the words from l.h.s. of the equation (8) we fix the weights of the initial subwords with the length being multiple of \( t \). Because \( r \) is large enough the weights have no chance to leave the \([-r,r]\)-strip. Now we make the corresponding calculations. Let \( w \) be in l.h.s. of the inclusion. It means that \( v[w[1:t]) = m_1 \), \( v[w[t:2t]) = m_2 \). Generally, \( v[w[(j-1)t:jt]) = m_j \) for \( j = 1, \ldots, c-1 \), and \( v[w[(c-1)t:ct-1]) = m_c \). So, \( v(w[jt]) = m_1 + m_2 + \ldots + m_j = \lfloor j\alpha \rfloor \). Now, \( |v[w[ shame zone]j+1] - \alpha(jt+k)| < 1 + \max|x_s| - \alpha|k| \). Here \( k \leq t-1 \), so \( w \in B_{ct,\alpha,r} \). \( \square \)

For two matrices \( M, N \) of the same size over \( \mathbb{Z} \) we write \( M \preceq N \) if \( M_{ij} \leq N_{ij} \) for all admissible indexes. The equations (7) (8) imply the following inequalities for \( M \)-matrices:
\[
(\sum_{s \in S} M(L_{m_1-s}^t)A_s)(\sum_{s \in S} M(L_{m_2-s}^t)A_s)\ldots(M(L_{m_c}^t)A_s) \leq
\]
Applying Proposition 4 to this inequality we obtain

**Proposition 7.**

\[
M(L_{m_1}^{t+1})M(L_{m_2}^{t+1}) \ldots M(L_{m_{n-1}}^{t+1})M(L_{m_n}^{t}) \leq M(B_{ct,\alpha,r}) \leq \sum_{m=\lfloor (ct-1)\alpha \rfloor - r}^{\lfloor (ct-1)\alpha \rfloor + r} M(L_m^t),
\]

where \( m_j = \lfloor j\alpha \rfloor - \lfloor (j-1)\alpha \rfloor \). Moreover, \( m_j = \lfloor t\alpha \rfloor \) or \( m_j = \lfloor t\alpha \rfloor + 1 \).

For a positive sequence \( a_n \) we call \( \lim_{n \to \infty} \sqrt[n]{a_n} \) the exponent of \( a_n \) (if exists). The relation between exponents of \( D_n \) and \( M(D_n) \) is clear:

\[
\lim \sqrt[n]{|D_n|} = \max_{ij} \{ \lim \sqrt[n]{|m_{ij}(n)|} \},
\]

where \( m_{ij} \) are matrix entries of \( M(D_n) \). Using this fact and estimates of Proposition 7 one gets \( e_{\alpha,r} \leq \lim sup_{\epsilon \to 0} \{ \epsilon \beta \mid \beta \in [\alpha - \epsilon, \alpha + \epsilon] \} \). So, the following lemma holds.

**Lemma 5.1.** If \( e_{\alpha} \) depends continuously on \( \alpha \), then \( e_{\alpha,r} \leq e_{\alpha} \).

The estimates from below may be more tricky to obtain. We overcome this difficulty by imposing a rather general sufficient condition.

**Lemma 5.2.** Let \( M(L_{[a\alpha]}^n) \) have a diagonal entry with exponent \( e_{\alpha} \) then \( \lim_{r \to \infty} e_{\alpha,r} \geq e_{\alpha} \).

**Proof.** Let \( M(L_{[a\alpha]}^n)_{ij} \) be a diagonal entry with exponent \( e_{\alpha} \). Let

\[
d(t) = \min \{ (M_{[a\alpha]}^{t+a})_{ij} \mid a, b = 0, 1 \}.
\]

Then Proposition 7 implies the inequality

\[
d(t)^e \leq M_{jj}(B_{ct,\alpha,r}).
\]

Applying \( \sqrt[n]{ \cdot } \) and allowing \( c \to \infty \) one gets

\[
\sqrt[n]{d(t)} \leq \lim_{r \to \infty} e_{\alpha,r},
\]

But \( \sqrt[n]{d(t)} \to e_{\alpha} \) by our assumptions. \( \square \)

In the next subsection we explain how to calculate \( M(L_{[a\alpha]}^n) \).

### 5.1. Generating function.

Let \( S = \{ s_0, s_0 + 1, \ldots, s_0 + \rho \} \). We define the matrix generating function for \( M(L_m^n) \) as

\[
G(x,y) = \sum_{n=1}^{\infty} \sum_{m=n-1}^{(n-1)(s_0+\rho)} M(L_m^n)x^{n-1}y^{m-(n-1)s_0}.
\]

We chose this type of generating function to avoid negative powers and to keep track of the number of total transitions.

**Lemma 5.3.** \( G(x,y) = \left( E - x(A_{s_0} + yA_{s_0+1} + \ldots + y^iA_{s_0+i} + \ldots + y^\rho A_{s_0+\rho}) \right)^{-1} \)
Proof. Taking into account the formula \((E - X)^{-1} = E + X + X^2 \ldots\) it suffices to show that
\[
(A_{s_0} + yA_{s_0+1} + \ldots + y^iA_{s_0+i} + \ldots + y^nA_{s_0+n})^n = \sum_{m=n_{s_0}}^{n_{s_0}+\rho} M(L_m^n)y^{m-n_{s_0}}.
\]
We prove it by induction on \(n\). For \(n = 0\) the equality holds by the statement \(i\) of Proposition 4. Supposing the equality for \(n - 1\) we obtain
\[
(A_{s_0} + yA_{s_0+1} + \ldots + y^iA_{s_0+i} + \ldots + y^nA_{s_0+n})^n
\]
\[
\left(\sum_{m=(n-1)s_0}^{(n-1)s_0} (A_{s_0} + yA_{s_0+1} + \ldots + y^iA_{s_0+i} + \ldots + y^nA_{s_0+n})\right) \left(\sum_{m=n_{s_0}}^{n_{s_0}+(n-1)\rho} M(L_m^n)y^{m-n_{s_0}}\right)
\]
\[
\sum_{m=n_{s_0}}^{n_{s_0}+n_{s_0}+\rho} \left(\sum_{j=0}^{\rho} M(L_{m-s_0-j}^n)A_{s_0+j}\right)y^{m-n_{s_0}}.
\]
In the last equality we use the simple fact that \(L_m^n = \emptyset\) for \(m < (n-1)s_0\) and \(m > (n-1)(s_0 + \rho)\). The induction step follows because of Proposition 4.

Let \(H(x, y) = \det(E - x \sum_{j=0}^{\rho} y^jA_{s_0+j})\). It follows from the formula of an inverse matrix that \(HG\) is a polynomial matrix. In order to calculate the asymptotics we need to study the zeros of \(H\), particularly, we need the so called minimal solutions, see [20, 21, 22].

**Definition 5.4.** Let \(f(x, y)\) be a \(C\)-polynomial. Consider the equation
\[
f(x, y) = 0 \tag{10}
\]
A solution \((x_0, y_0) \in C^2\) of (10) is said to be minimal if equation (10) has no solution \((x, y)\) satisfying \(|x| < |x_0|\) and \(|y| < |y_0|\). A solution \((x_0, y_0) \in C^2\) of the equation (10) is said to be strictly minimal if the inequalities \(|x| \leq |x_0|\) and \(|y| \leq |y_0|\) for any solution \((x, y)\) imply \(x = x_0, y = y_0\).

The following proposition describes the minimal solutions for
\[
H(x, y) = 0 \tag{11}
\]

**Proposition 8.** Let \(A\) be a primitive matrix. Let \((x_0, y_0) \in C^2, y_0 \neq 0\) be a minimal solution of the equation (11). Then the maximal (by the absolute value) eigenvalue of the matrix \(A(x_0, y_0) = x_0 \sum_{j} y_j^jA_{s_0+j}\) is 1. Moreover, if rank of \((A(1, e^{i\phi})) > 1\) for all \(\phi \in \mathbb{R}\) then \((x_0, y_0) \in \mathbb{R}^2\) and \((x_0, y_0)\) is strictly minimal.

**Proof.** Clearly, \(H(x_0, y_0) = 0\) iff 1 is an eigenvalue of \(A(x_0, y_0)\). If \(\lambda\) is an eigenvalue of \(A(x_0, y_0)\) with \(|\lambda| > 1\) then \(H(x_0, \lambda, y_0) = 0\), a contradiction with the minimality of \((x_0, y_0)\).

For a vectors \(u, v \in \mathbb{R}^p\) we write \(u \geq v\) if \(u_i \geq v_i\) for all \(i = 1, \ldots, p\). We write \(u > v\) if \(u \geq v\) and \(u \neq v\). Let \((x_0, y_0) \notin \mathbb{R}^2\) and \(A(x_0, y_0)\xi = \xi\) for \(\xi \in \mathbb{C}_{p}\). Define \(v \in \mathbb{R}^p\) as \(v_i = |\xi_i|\). Observe that \(A(|x_0|, |y_0|)v \geq v\). If \(A(|x_0|, |y_0|)v > v\) then the maximal real eigenvalue of \(A(|x_0|, |y_0|)\) is greater than 1 by Proposition 9 (see below).
and \((x_0, y_0)\) is not minimal, a contradiction. Assume now that \(A([x_0], [y_0])v = v\) and \(A(x_0, y_0) = \{a_{jk}\}\). It follows that \(\arg(a_{jk} \xi_k) = \arg(\xi_j)\), or, the same, \(\arg(a_{jk}) = \arg(\xi_j) - \arg(\xi_k)\). In our situation it means that \(A(1, e^{i\phi})_{jk} = e^{i(\phi_0 + \phi_j - \phi_k)}\), where \(\phi_0 = \arg(y_0)\) and \(\phi_j = \arg(\xi_j)\). So, \(\text{rank}(A(1, e^{i\phi})) = 1\), a contradiction. \(\square\)

**Proposition 9.** Let \(b(A)\) be the greatest real eigenvalue of a matrix \(A\). Let \(A\) be primitive and \(Av > v\) for some \(v > 0\). Then \(b(A) > 1\).

**Proof.** There exists \(n\) such that all entries of \(A^n\) are positive. Observe that if \(u > v\) then \((A^n)_{ii} > (A^n)_{ij}\), for all \(i = 1, \ldots, p\). Observe also that \(A^n v > v\). Thus, there exists \(\beta > 1\) such that \(A^{2n} v > \beta A^n v\). Inductively, \(A^{kn} v > \beta^{k-1} A^n v\). Recall that \(b(A) = \lim_{n \to \infty} \sqrt[n]{\|A^n\|}\). So, \(b(A) \geq \sqrt[n]{\beta} > 1\). \(\square\)

### 5.2. Asymptotics for 2-variable generating functions.

In this section we suppose that \(A\) is primitive and the rank condition of Proposition 8 is satisfied. All entries of \(G(x, y)\) have the form \(f(x, y) / H(x, y)\), where \(f\) is a polynomial. We are interesting in asymptotics of \(a_{n, \lfloor \alpha n \rfloor}\) where \(a_{n,m}\) are the coefficients of the expansion

\[
\frac{f(x, y)}{H(x, y)} = \sum a_{n,m} x^n y^m
\]

We estimate \(a_{n,m}\) using the Wilson-Pemantle technique [20, 21]. The asymptotics depend on minimal points. Under the conditions of Proposition 8 all minimal points are strictly minimal and we may adapt Theorem 3.1 of [20] (see also [22, 21]) as follows

**Theorem 5.5.** Let \((x_0, y_0) \in \mathbb{R}^+_2\) be the unique (in \(\mathbb{R}^+_2\)) solution of

\[
\begin{aligned}
H & = 0 \\
\alpha x \partial_x H & = y \partial_y H,
\end{aligned}
\tag{12}
\]

such that 1 is a maximal eigenvalue of \(A(x_0, y_0)\). Then \((x_0, y_0)\) is a strictly minimal solution of the equation (11) and the following asymptotics takes place:

\[
a_{n, \lfloor \alpha n \rfloor} \sim \frac{f(x_0, y_0)}{\sqrt{2\pi}} x_0^{-n} y_0^{-\alpha n} \sqrt{\frac{-x \partial_x H(x_0, y_0)}{nQ(x_0, y_0)}},
\]

where \(Q(x, y) = -x H_y(y H_y)^2 - y H_y(x H_x)^2 - y^2 x^2 (H_y)^2 H_{xx} + (H_x)^2 H_{yy} - 2 H_x H_y H_{xy}\). Particularly, it implies that

\[
\lim_{n \to \infty} \frac{\ln(a_{n, \lfloor \alpha n \rfloor})}{n} = -\ln(x_0) - \alpha \ln(y_0),
\]

if \(f(x_0, y_0) \neq 0\) and \(Q(x_0, y_0) \neq 0\).

In the following, we assume, without loss of generality, that \(s_0 = 0\). (If not, one should make a change \(\alpha \to \alpha - s_0\). Theorem 5.5 with Lemma 5.1 and Lemma 5.2 imply

**Theorem 5.6.** Let \((x_0, y_0) \in \mathbb{R}^+_2\) solution of the system (12). Let the polynomial matrix \(HG\) have a non-zero diagonal entry evaluated at \((x_0, y_0)\) and \(Q(x_0, y_0) \neq 0\). Then \(\mathcal{H}_\alpha = -\ln(x_0) - \alpha \ln(y_0)\).
6. Example. In this section we consider an example that, in fact, contains all main features of systems on the circle possessing a Markov partition.

Consider the map $f$ for which

$$F(x) = \begin{cases} 
\frac{1}{3} + 2x, & 0 \leq x \leq \frac{1}{3}, \\
\frac{2}{3} - x, & \frac{1}{3} \leq x \leq \frac{2}{3}, \\
-\frac{2}{3} + 2x, & \frac{2}{3} \leq x \leq 1.
\end{cases}$$

The map $f$ has the Markov partition $\xi$ of 3 intervals: $\xi_1 = [0, \frac{1}{3}]$, $\xi_2 = [\frac{1}{3}, \frac{2}{3}]$, $\xi_3 = [\frac{2}{3}, 1]$ (see Fig. 1), and the corresponding topological Markov chain is determined by the transition matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, corresponding to the graph $G$ (see Fig. 2).

![Figure 1. The graph of $F$ and the Markov partition.](image)

![Figure 2. The oriented graph $G$ for the map $F$ and the partition $\xi$.](image)

One can see that the transition $(3, 1)$ corresponds to the change of the integer part of $F$. So, we represent the transition matrix $A = A_0 + A_1$ where $A_0$ corresponds to all transitions without (31) and $A_1$ corresponds to (31):

$$A_0 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
Figure 3. The graph of $h(\alpha)$

We calculate the generating function $G(x, y) = (E - xA_0 - xyA_1)^{-1} = \frac{1}{-x^3y - x^2y - x + 1} \begin{pmatrix} -x + 1 & -x^2 + x & x^2 + x \\ x^2y & -x^2y - x + 1 & x \\ xy & x^2y & 1 \end{pmatrix}$

Now we can find the asymptotics using Theorem 5.5. Let $H = -x^3y - x^2y - x + 1$
We have to find positive solutions of the system

\[
\begin{cases}
H = 0 \\
\alpha xH_x = yH_y 
\end{cases}
\]

Using SAGE (see [24]) we have found:

\[x = \frac{\alpha \pm \sqrt{5\alpha^2 - 4\alpha + 1}}{2\alpha - 1}.
\]

In this example $\alpha$ is a fraction of (31)-transition ($A_1$-transition). If $\alpha > 1/2$ then 2 consecutive $A_1$ transitions should appear. But there is no word with consecutive (31)-transition. So, we have to consider the interval $0 < \alpha \leq 1/2$ only. The positive branch for $0 < \alpha < 1/2$ is

\[x = \frac{\alpha - \sqrt{5\alpha^2 - 4\alpha + 1}}{2\alpha - 1}.
\]

Equation $H = 0$ implies

\[y = \frac{1 - x}{x^3 + x^2}.
\]

The dependence of the entropy on $\alpha$ is given by the formula $h = -\ln(x) - \alpha \ln(y)$ shown on the figure 3. One can see that our case satisfies Theorem 5.6, so, $H(\alpha) = h(\alpha)$.

7. Measures and entropy.

7.1. Construction of the measure. Recall, that under the conditions of Theorem 5.5 the matrix $A(x_0, y_0)$ has 1 as the greatest simple eigenvalue. Let $l$ be a row-vector ($r$ be a column-vector) such that $lA(x_0, y_0) = l (A(x_0, y_0)r = r)$. By the Perron-Frobenius theorem $l$ and $r$ are positive. Normalize $l$ and $r$ such that $lr = 1$. Let $A(x_0, y_0) = \{a_{jk}\}$. Define (see [14]) the matrix $\Pi = \Pi(x_0, y_0)$ as $\Pi_{jk} = \frac{a_{jk}r_j}{r_j}$. Let $q_j = l_j r_j$ and $q = q_1, q_2, \ldots, q_p$. Observe that $\Pi$ is a stochastic matrix and $q$ is
its left 1-eigenvector. The measure $\mu_{II}$ of the cylinder $[w_1, w_2, w_3, ..., w_n]$ is defined as

$$\mu_{II}([w_1, w_2, w_3, ..., w_n]) = q_{w_1} \Pi_{w_1} \Pi_{w_2} \Pi_{w_3} \cdots \Pi_{w_{n-1}} w_n.$$ 

The entropy of the subshift with respect to $\mu_{II}$ can be calculated by the formula

$$h(\mu_{II}) = -\sum_{jk} q_{j} \Pi_{jk} \ln(\Pi_{jk}),$$ \hspace{1cm} (13)

see [14].

7.2. $h(\mu_{II}) = H_\alpha$. We are going to show that $h(\mu_{II}) = \ln(x_0) + \alpha \ln(y_0)$. In our situation the equation (13) can be rewritten as

$$-h(\mu_{II}) = \sum_{ik} l_i a_{ik} r_k \ln(\frac{a_{ik} r_k}{r_i}) = \sum_{ik} l_i a_{ik} r_k \ln(a_{ik}) +$$

$$\sum_{ik} l_i a_{ik} r_k \ln(r_k) - \sum_{ik} l_i a_{ik} r_k \ln(r_i).$$

Observe that the last line of the equation is 0. (Indeed, evaluating the first sum over $i$ and the second one over $k$ and taking into account that $l(r)$ is a left (right) 1-eigenvector of $A$ we obtain that $\sum_k l_k r_k \ln(r_k) - \sum_i l_i r_i \ln(r_i) = 0$.) Let $A_j = \{(i, k) | (A_{s_0+j})_{ik} = 1\}$. Now we can write:

$$-h(\mu_{II}) = \sum_j \sum_{(i, k) \in A_j} r_i r_k x_0 y_0^1 \ln(x_0 y_0^1)$$

$$\ln(x_0) \sum_j \sum_{(i, k) \in A_j} r_i r_k x_0 y_0^1 + \ln(y_0) \sum_j \sum_{(i, k) \in A_j} r_i r_k x_0 y_0^1$$

$$\ln(x_0)(lA(x_0, y_0)r) + \ln(y_0)(l\tilde{A}(x_0, y_0)r) \ln(x_0) + \ln(y_0)(l\tilde{A}(x_0, y_0)r).$$

where $\tilde{A}(x_0, y_0) = y_0 A_y(x_0, y_0) = \sum_j j x_0 y_0^1 A_{s_0+j}$. So, in order to prove the equality

$H_\alpha = h(\mu_{II})$ we should show that $(l\tilde{A}(x_0, y_0)r) = \alpha$, of course, under the condition that $lA(x_0, y_0) = l$, $A(x_0, y_0)r = r$, $lr = 1$, $(x_0, y_0)$ is the solution of the system (12) satisfying the condition of Theorem 5.5.

To this end we need the following result (recall that $H(x, y) = \det(E - A(x, y))$).

**Proposition 10.** Let $B \in \text{Mat}(\mathbb{C})$, $\det(B) = 0$ and 0 be a simple spectral point of $B$. Let $l$ be a vector-row and $r$ be a vector-column such that $lB = 0$, $Br = 0$, and $lr = 1$. Let $\beta = \lambda_1 \lambda_2 \cdots \lambda_{p-1}$ be the product of all non-zero eigenvalues of $B$ (counted with multiplicity). Then the Frechet derivative $D\text{det}(B)$ of $\text{det}(B)$ (applied to an arbitrary matrix $X$) is equal to

$$D(\text{det}(B))(X) = \beta(lXR)$$

**Proof.** The multilinearity of $\text{det}()$ implies that

$$\text{det}(B + \epsilon X) = \epsilon \sum_{ij} B_{ij} X_{ij} + O(\epsilon^2),$$

where $\tilde{B} = \{\tilde{B}_{ij}\}$ is the matrix of the cofactors of $B$. Because of the equalities $BB^T = B^T B = \text{det}(B)E = 0$, the columns (rows) of $\tilde{B}$ are proportional to $l(r)$. Thus, $\tilde{B}_{ij} = \gamma_l r_j$ for some $\gamma$. Observe that $\gamma = \text{trace}(\tilde{B})$. Let $D = \text{diag}(-1, 1, -1, 1 \ldots, (-1)^p)$. The matrix $D^{-1} \tilde{B} D$ is the matrix of the minors of $B$. By a theorem due to Kronecker (see [10]) the eigenvalues of $D^{-1} \tilde{B} D$ (as well as of
we have reduced the calculation of the directional entropy for Markov
implies that
\[ \alpha \] because of cancellation we do not need normalization here. Now let
\( H_{\lambda} \) be a right eigenvalue of \( \tilde{A} \) of the maximal entropy. Observe that
Lemma 7.1. \( h(\mu_{\Pi}) = \mathcal{H}_\alpha \), where \( \Pi = \Pi(x_0, y_0) \) and \( (x_0, y_0) \) is the minimal solution of the system (12).
Remark 3. The direct computation shows that
\[ \int_{\Omega_A} v(w): 1) \] \( \mu_{\Pi} \) is probably implies that the support of \( \mu_{\Pi} \) consist of initial words with rotation number \( \alpha \). Moreover, the measure \( \mu_{\Pi} \) is the measure of maximal entropy among measures \( \nu \) such that
\[ \int_{\Omega_A} v(w): 1) \] \( \nu \) is defined in Section 4.1. With shift invariance of \( \mu_{\Pi} \) it probably
\[ \Pi = \Pi(x_0, y_0) \] \( (x_0, y_0) \) is the minimal solution of the system (12).
7.3. When \( \mathcal{H}_\alpha = h_{top} \). Lemma 7.1 implies that \( \mathcal{H}_\alpha = h_{top} \) if \( \mu_{\Pi} \) is the measure of the maximal entropy. Observe that \( A(x, 1) = xA \). So, our construction of \( \mu_{\Pi} \) in the case of \( y_0 = 1 \), in fact, coincides with the construction of the measure of maximal entropy in [14]. Substituting \( y_0 = 1 \) to the system (12), we can find \( \alpha \) and \( x_0 \). It is clear that, in fact, \( x_0 = e^{-h_{top}} \), the inverse value of the greatest eigenvalue of \( A \) since \( A(x_0, 1) = x_0A \). We can formulate the procedure of finding the angle, corresponding the topological entropy in the form of the following
Theorem 7.2. Let \( \lambda \) be the greatest eigenvalue of \( A \); \( l \) (r) be its left (right) \( \lambda \)-eigenvector. Let
\[ \alpha = \frac{lA(1, 1)r}{lA(1, 1)r} + s_0. \]
Then \( \mathcal{H}_\alpha = h_{top} \).
In our example. \( A = A_0 + A_1 \), where
\[ A_0 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]
Denote by \( \lambda \) the maximal eigenvalue of \( A \). Let \( l \) be a left \( \lambda \)-eigenvector of \( A \) and \( r \) be a right \( \lambda \)-eigenvector of \( A \). Calculations show that \( \lambda \approx 1.839 \),
\[ l \approx (1, 0.5436890126920763, 1.839286755214161) \]
\[ r \approx (1, 0.647798871261043, 1.191487883953119) \]
(because of cancellation we do not need normalization here). Now let \( \alpha_{\max} \) be such that \( H_{\alpha_{\max}} = h_{top} \). We can calculate:
\[ \alpha_{\max} = \frac{rA} {lA} \approx 0.2821918053244515. \]
8. Directional complexity and entropy for topological Markov chains. In Section 5 we have reduced the calculation of the directional entropy for Markov maps of the circle to the calculation of some quantities related to the corresponding symbolic systems. It was pointed out by our referee that we have defined, in a hidden way, the directional complexity and entropy for topological Markov chains. We make it explicit in this section. The notion of rotation sets for topological Markov chains
was introduced in [23] following general approach of [11]. In our notations it can
be described as follows. We consider a topological Markov chain \((\Omega_A, \sigma)\) for which
the edges \((i, j)\) are endowed with integer weights \(k_{i,j}\). We introduce a function \(\phi : \Omega_A \to \mathbb{Z}\) as follows: given \(\omega = (\omega_0, \omega_1, \ldots) \in \Omega_A\) let \(\phi(\omega) = k_{\omega_0, \omega_1}\). Then the rotation set \(J\) of \(\omega\) is
\[
J(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\sigma^i \omega)\left\lfloor \frac{\omega_0, \omega_1, \ldots, \omega_{n-1}}{v} \right\rfloor,
\]
where \(v\) is the weight, defined in subsection 4.1 and \(\mu\) is the upper topological limit.
The rotation set of the system \((\Omega_A, \sigma)\) is, by definition, \(\bigcup_{\omega \in \Omega_A} J(\omega)\). The results of
[23] imply that, under some conditions, the rotation set is a closed interval. Now,
there are points \(\omega \in \Omega_A\) for which \(\alpha = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\sigma^i \omega)\) exists, and for a given \(\alpha\)
we may define directional complexity and entropy.

**Definition 8.1.**
1. The number \(C_{n,\alpha,r} = |B_{n,\alpha,r}|\) is called the \(\alpha\)-directional \(r\)-
complexity.
2. The number \(e_{\alpha,r} = \ln \lim_{n \to \infty} \sqrt[n]{C_{n,\alpha,r}}\)
is called the directional \(r\)-entropy.
3. The number \(e_{\alpha} = \lim_{r \to \infty} e_{\alpha,r}\) is called the \(\alpha\)-directional entropy.

Let us remind that \(B_{n,\alpha,r} = \{w \in L^n \mid \forall j = 1, \ldots, n-1 \alpha j - r \leq v(w[j:j]) \leq \alpha j + r\}\), i.e. we admit only those \(n\)-cylinders for which the weight of a \(j\)-subcylinder can differ from \(\alpha j\) no more than by \(\pm r\). It is the direct analogy with the definition of the “window-separated points”. There is another way to define the directional entropy which was suggested in Section 5.

**Definition 8.2.** The upper topological entropy of the system \((\Omega_A, \sigma)\) is \(\hat{e}_\alpha = \ln \lim_{n \to \infty} \sqrt[n]{L^n_{\alpha n}}\).

**Theorem 8.3.** If \(\hat{e}_\alpha\) is a continuous at \(\alpha\) and \(M(L^n_{\alpha n})\) have a diagonal entry with
exponent \(\hat{e}_\alpha\), then \(e_\alpha = \hat{e}_\alpha\).

**Remark 4.** The method of calculating of \(\hat{e}_\alpha\) described in Section 5 works in this
more general situation.

**Remark 5.** Theorem 8.3 leaves the possibility that \(e_\alpha \neq \hat{e}_\alpha\). The open question is
it really may happens.

9. **Concluding remarks.** Following ideas of Milnor [17, 18] and also [4, 1, 3, 8]
we have introduced and studied the directional complexity and entropy for dynamical
systems generated by degree one maps of the circle. In particular, we have
considered the maps that admit a Markov partition and have positive topological
entropy. For them we have reduced the calculation of the \((\epsilon, n)\)-complexity on a set
of initial points having a prescribed rotation number to that of symbolic complexity
of admissible cylinders of a topological Markov chain (TMC). The admissibility of
the cylinders is constructively determined by the rotation number. To calculate the
symbolic complexity we have used a combinatorial machinery developed in [20, 21]
adjusted to our situation. As a result we have obtained exact formulas for the
directional entropy corresponding to every rotation number. Using these formulas we
have shown that the directional entropy coincides with the measure-theoretic entropy related to a Markov measure (different for different direction). In particular, we have proved that the measure of maximal entropy determines the direction in which the directional entropy equals the topological entropy of the original dynamical system and, also, we have found an exact formula for this direction.

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