A Fourier transform for sheaves on Lagrangian families of real tori

U. Bruzzo, G. Marelli and F. Pioli

§ Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Beirut 4, 34013 Trieste, Italy
¶ Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

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Abstract. We systematically develop a transform of the Fourier-Mukai type for sheaves on symplectic manifolds $X$ of any dimension fibred in Lagrangian tori. One obtains a bijective correspondence between unitary local systems supported on Lagrangian submanifolds of $X$ and holomorphic vector bundles with compatible unitary connections supported on complex submanifolds of the relative Jacobian of $X$ (suitable conditions being verified on both sides).

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E-Mail addresses: bruzzo@sissa.it, marelli@sissa.it, pioli@sissa.it
1 Introduction

The idea that, in accordance with the Strominger-Yau-Zaslow conjecture [25], a kind of Fourier-Mukai transform should describe transformation properties of D-branes under string-theoretic mirror symmetry dates back to 1996 [9]. The original Fourier-Mukai transform, mapping coherent sheaves on an abelian variety $X$ to coherent sheaves on the dual variety $\hat{X}$, was introduced in [22]. A relative Fourier-Mukai transform for elliptic varieties was developed in [4, 7, 16, 5] and was shown to describe a correct D-brane transformation pattern in the case of K3 surfaces [4, 3]. An analogous result was shown to hold for elliptic Calabi-Yau threefolds in [1].

In the case of Calabi-Yau threefolds which are fibred in (special Lagrangian) real 3-tori, a similar description should be provided by a “real” relative Fourier transform. The presence of singular fibres raises here a big problem because it is not clear how to handle them. As a first step, one may consider the simplified case when there are no singular fibres.

If $X$ is a symplectic family of smooth Lagrangian tori, the dual family $\hat{X}$ has a natural complex structure. Then the relative Fourier transform yields a correspondence between local systems supported on Lagrangian submanifolds of $X$ and holomorphic vector bundles supported on complex submanifolds of $\hat{X}$, where both sets of data satisfy suitable conditions (cf. later in this introduction). Some results along these lines were already contained in [2] but we strengthen and extend them considerably (cf. also [19]). We also carefully spell out the conditions on the submanifold $S$ of $X$ which ensure that the support of the transformed sheaf is a complex submanifold of $\hat{X}$.

The correspondence we get closely resembles Fukaya’s homological mirror symmetry [12]. Comparison with that approach suggests that in order to extend the results presented in this paper to more general Lagrangian submanifolds (e.g., when the Lagrangian submanifold $S$ is ramified over the base of the fibration $X$), or to the situation when $X$ has singular fibres, it is necessary to allow for some kind of quantum corrections. Future extensions of the theory should also allow for the inclusion of a B-field and should investigate the possibility of describing the correspondence between the Floer homology of $X$ and the algebraic cohomology of $\hat{X}$ in terms of the Fourier-Mukai transform studied in this paper. One could also study the relation between the transform presented in this paper and the constructions of Laumon [18] and Rothstein [24]; the local systems we transform...
are $\mathcal{D}$-modules, the same objects considered by these authors. This might also relate to possible applications to generalizations of the Krichever correspondence along the lines of [23].

We describe now the contents of this paper. As a preliminary step, in section 2 we consider the absolute case. In its simplest version one constructs two functors which establish the equivalence between the category $\text{Sky}(T)$ of skyscraper sheaves of finite-dimensional vector spaces on a real torus $T$, and the category $\text{Loc}(\hat{T})$ of unitary local systems (vector bundles with a flat unitary connection) on the dual torus $\hat{T}$. We also study the transformation of a local system supported on an affine subtorus of $T$.

In section 3 we consider the relative case. In 3.1 we set up the general framework. Then, in considering local systems supported on a Lagrangian submanifold of a symplectic torus fibration $X \to B$, we first analyze the two extreme cases, i.e., when the submanifold is a fibre of $X$ or a Lagrangian section (section 3.2). In the first case one gets the usual tautological property of the Fourier-Mukai transforms, while in the second one obtains a bijective correspondence between local systems supported on Lagrangian sections of $X$ and holomorphic bundles with compatible unitary connections, flat along the fibre directions of $\hat{X}$ (and satisfying some further conditions).

The intermediate, non-transversal cases (i.e., when one considers a Lagrangian submanifold $S \subset X$ whose projection onto $B$ has a dimension strictly between 0 and $\text{dim } B$) are more involved, and are analyzed in section 3.3. To get a well-behaved transform one needs to assume that $S$ intersects the fibres $X_b$ of $X$ (here $b \in B$) along subtori $S_b$ of $X_b$, and that the vertical tangent spaces to $S$ undergo parallel displacement under the natural Gauss-Manin connection defined in $TX$. Under this condition the transform of a local system on $S$ is a holomorphic vector bundle supported on a complex submanifold of $\hat{X}$, and again, provided that suitable conditions are satisfied, there is a bijective correspondence between the two sets of data.

These result hold true whatever is the dimension of $X$, and do not require $X$ to be Calabi-Yau (and not even complex). One should note that, when $X$ is a Calabi-Yau manifold, the additional condition on the support $S$ we have previously described is in general quite unrelated to the condition of $S$ being special (in addition to being Lagrangian), and coincides with the latter only when $X$ is complex 3-dimensional, and the projection of the Lagrangian submanifold onto the base is (real) 1-dimensional (this corresponds to a
transformed sheaf which is a line bundle supported on a curve in $\hat{X}$). It is not clear to the authors whether this situation has any implication or motivation in string theory.

In section 4 we draw some conclusions, in particular we comment upon the relation of this construction to Fukaya’s homological mirror symmetry.

Two Appendices contain two proofs that, sketchy as they are, are too lengthy to be included in the main text.

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2 The absolute case

We start this section by offering a description of (smooth) $U(1)$ bundles on real tori in terms of their factor of automorphy which fully parallels the one available (in the holomorphic case) on complex tori (cf. [17]). This description of line bundles will be extensively used in the remainder of the paper. In section 2.2 we describe two complexes which are naturally associated with the Poincaré sheaf. In section 2.3 we introduce and study the functors which establish the equivalence between the category of skyscrapers and that of local systems; the computation of the second functor will require the study of the cohomology of a complex associated with the Poincaré bundle. In section 2.4 we consider local systems supported on subtori.

2.1 Line bundles on real tori

Let $\Lambda$ be a $g$-dimensional lattice in a $g$-dimensional real vector space $V$, and let $T = V/\Lambda$ be the corresponding torus. Let $\text{Pic}(T)$ denote the group of isomorphism classes of $U(1)$ bundles on $T$. The group $\text{Pic}(T)$ is isomorphic to a group $P(\Lambda)$ we may associate with the lattice $\Lambda$ in the following way. As a set, $P(\Lambda)$ is the set of pairs $(A, \chi)$, where $A \in \text{Alt}^2(\Lambda, \mathbb{Z})$ is an alternating two-form on $\Lambda$, and $\chi$ is a semicharacter for $A$, namely, a map $\chi: \Lambda \rightarrow U(1)$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{i\pi A(\lambda,\mu)}$$

for all $\lambda, \mu \in \Lambda$. The group structure is the one given by

$$(A_1, \chi_1) \cdot (A_2, \chi_2) = (A_1 + A_2, \chi_1 \chi_2).$$
The isomorphism $\text{Pic}(T) \simeq P(\Lambda)$ is the Appell-Humbert theorem for real tori. Via the isomorphism $\text{Alt}^2(\Lambda, \mathbb{Z}) \simeq H^2(T, \mathbb{Z})$, the form $A$ is to be identified with the first Chern class. In this way we have an exact sequence

$$0 \to \text{Hom}_{\mathbb{Z}}(\Lambda, U(1)) \to \text{Pic}(T) \xrightarrow{c_1} H^2(T, \mathbb{Z}) \to 0$$

and the kernel $\text{Hom}_{\mathbb{Z}}(\Lambda, U(1))$, whose elements are isomorphism classes of flat $U(1)$ bundles, is isomorphic to the dual torus $\hat{T} = V^\vee / \Lambda^\vee$ (here $V^\vee = \text{Hom}_\mathbb{R}(V, \mathbb{R}); \Lambda^\vee = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$). To every point $y \in \hat{T}$ there corresponds a flat line bundle $L_y$ whose associated pair is

$$A_y = 0, \quad \chi(y) = e^{2\pi i y}.$$

The description of the bundle $L$ by means of the pair $(A, \chi)$ allows one to give an explicit characterization of the global sections of $L$. To this end one introduces the factor of automorphy of the pair $(A, \chi)$, defined as the map

$$a_L: V \times \Lambda \to U(1), \quad a_L(x, \lambda) = \chi(\lambda) e^{i\pi A(x, \lambda)}$$

(here $A$ has been extended to $V \times V$ in the natural way).

**Proposition 2.1.** Let $L$ be a line bundle on $T$, corresponding to the pair $(A, \chi) \in P(\Lambda)$. The global sections of $L$ are in a one-to-one correspondence with the smooth functions $s: V \to \mathbb{C}$ satisfying the automorphy condition

$$s(x + \lambda) = a_L(x, \lambda) s(x)$$

for all $x \in V, \lambda \in \Lambda$.

**Proof.** The proof is a (simplified) replica of the one holding in the case of complex tori [17].

The action of an automorphism of $L$ changes the factor of automorphy; an automorphism of $L$ is induced by a map $\phi: V \to U(1)$, and the new factor of automorphy is

$$a'_L(x, \lambda) = \phi(x + \lambda) a_L(x, \lambda) \phi(x)^{-1}.$$ 

Now we use these tools to describe the Poincaré bundle $P$ on the product $T \times \hat{T}$. The line bundle $P$ is associated with the pair $(A, \chi) \in P(\Lambda \times \Lambda^\vee)$, where

$$A((\lambda_1, \mu_1), (\lambda_2, \mu_2)) = \mu_1(\lambda_2) - \mu_2(\lambda_1), \quad \chi(\lambda, \mu) = e^{i\pi \mu(\lambda)}.$$
The corresponding factor of automorphy is
\[ a_P(x, y, \lambda, \mu) = e^{i\pi[y(\lambda)-\mu(x)-\mu(\lambda)]}. \]

It is convenient to apply the automorphism induced by the map
\[ \phi: V \times V^\vee \to U(1), \quad \phi(x, y) = e^{i\pi y(x)} \]
thus obtaining a new factor of automorphy
\[ a'_P(x, y, \lambda, \mu) = e^{2i\pi y(\lambda)}. \tag{1} \]

This description of the Poincaré bundle shows explicitly that \( P_{|T \times \{ y \}} \simeq \mathcal{L}_y \).

The connection form of the connection \( \nabla_P \) on the Poincaré bundle is written in the gauge where the factor of automorphy of \( P \) has the form (1) as
\[ A = 2i\pi \sum_{j=1}^{g} y_j dx^j \tag{2} \]
where \((x^1, \ldots, x^g)\) are flat coordinates on \( T \) and \((y_1, \ldots, y_g)\) are dual flat coordinates on \( \hat{T} \).

If we act on \( a_P \) with the automorphism \( \phi(x, y) = e^{-i\pi y(x)} \) we obtain the factor of automorphy \( a'_P(x, y, \lambda, \mu) = e^{-2i\pi \mu(x)} \) which shows that, after the identification \( \hat{T} \simeq T \), the dual bundle \( P^\vee \) is a Poincaré bundle for \( \hat{T} \times T \).

### 2.2 Complexes associated with the Poincaré sheaf

We denote by \( p, \hat{p} \) the projections onto the two factors of \( T \times \hat{T} \). To simplify notation we shall set
\[ \Omega^{m,n}_{m,n} = p^*\Omega^m_T \otimes_{\mathcal{C}^\infty_{T \times \hat{T}}} \hat{p}^*\Omega^n_{\hat{T}}. \]

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1A word about notation: if \( f: X \to Y \) is a differentiable map between two differentiable manifolds, and \( \mathcal{F} \) any sheaf on \( Y \), we shall denote by \( f^{-1}\mathcal{F} \) the sheaf-theoretic inverse image of \( \mathcal{F} \); if \( \mathcal{F} \) is a sheaf of \( \mathcal{C}^\infty_Y \)-modules, we shall denote by \( f^*\mathcal{F} \) its inverse image as a sheaf of modules, i.e.,
\[ f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{C}^\infty_Y} \mathcal{C}^\infty_X. \]
The connection $\nabla_\mathcal{P}$ has a Künneth splitting into two operators

$$
\nabla_1: \mathcal{P} \to \mathcal{P} \otimes \Omega^{1,0}, \quad \nabla_2: \mathcal{P} \to \mathcal{P} \otimes \Omega^{0,1}
$$

both squaring to zero (but their anticommutator is the curvature of $\nabla_\mathcal{P}$). In the gauge of equation (3), the action of $\nabla_1$, $\nabla_2$ on sections is locally written in the form

$$
\nabla_1 s = \sum_{j=1}^{g} \left( \frac{\partial s}{\partial x^j} + 2i\pi y_j s \right) dx^j, \quad \nabla_2 f = \sum_{j=1}^{g} \frac{\partial f}{\partial y_j} dy_j, \quad (3)
$$

where $g$ is the dimension of $T$.

Let $\mathcal{E}$ be a $\mathcal{C}_c^\infty$-module with a flat connection $\nabla$. By pulling the pair $(\mathcal{E}, \nabla)$ back to $T \times \hat{T}$ and coupling it with the pair $(\mathcal{P}, \nabla_1)$ we obtain a complex

$$
0 \to \ker \nabla_1^\mathcal{E} \to p^* \mathcal{E} \otimes \mathcal{P} \xrightarrow{\nabla_1^\mathcal{E}} p^* \mathcal{E} \otimes \mathcal{P} \otimes \Omega^{1,0} \xrightarrow{\nabla_1^\mathcal{E}} p^* \mathcal{E} \otimes \mathcal{P} \otimes \Omega^{2,0} \to \ldots
$$

Since locally the operator $\nabla_1^\mathcal{E}$ coincides with the exterior differential, this sheaf complex is exact, and is a fine resolution of the sheaf $\ker \nabla_1^\mathcal{E}$. Thus we obtain an isomorphism

$$
H^i(T \times U, \ker \nabla_1^\mathcal{E}) \simeq H^i((p^* \mathcal{E} \otimes \mathcal{P} \otimes \Omega^{*,0})(T \times U), \nabla_1^\mathcal{E}), \quad i \geq 0
$$

between the cohomology of the sheaf ker $\nabla_1^\mathcal{E}$ and the cohomology of the complex $(p^* \mathcal{E} \otimes \mathcal{P} \otimes \Omega^{*,0})(T \times U)$ (where $U$ is an open set in $\hat{T}$) acted upon by the differential $\nabla_1^\mathcal{E}$. The sheaf $R^i \hat{p}_* \ker \nabla_1^\mathcal{E}$ associated with the presheaf $U \rightsquigarrow H^i(T \times U, \ker \nabla_1^\mathcal{E})$ is the $i$th higher direct image of the sheaf ker $\nabla_1^\mathcal{E}$ to $\hat{T}$.

The same results hold for the operator $\nabla_2$. The cohomology of the complex $\left( \Gamma(\mathcal{P}^\nabla \otimes \Omega^{0,*}), \nabla_2 \right)$ will be computed in section 2.3.

### 2.3 The equivalence

For every real torus $T$ we consider

(i) the category $\text{Sky}(T)$ of skyscrapers on $T$ of total finite length (i.e., $\dim H^0(T, M) < \infty$ for all $M \in \text{Ob}(\text{Sky}(T))$);

(ii) the category $\text{Loc}(T)$ of unitary local systems on $T$. Objects of this category are pairs $(E, \nabla)$, where $E$ is a smooth complex vector bundle on $T$, and $\nabla$ is a flat unitary
connection. Morphisms in this category are vector bundle morphisms compatible with the connections. The objects in \( \text{Loc}(T) \) can also be regarded as locally free \( \mathbb{C}_T \)-modules of finite rank equipped with a hermitian metric defined up to homothety; under this disguise, a unitary local system will be typically denoted by a gothic letter. We shall freely switch from one picture to the other.

We now define the functor \( F: \text{Sky}(T) \to \text{Loc}(\hat{T}) \). Let \( M \) be a skyscraper of finite length on \( T \) supported on a 0-dimensional subset \( S \), with the \( C_\infty^\infty \)-structure given by evaluation of functions. Denoting by \( P_S \) the restriction of \( P \) to \( S \times \hat{T} \), and by \( p_S, \hat{p}_S \) the projections of \( S \times \hat{T} \) onto its factors, we define \( \hat{E} \) as the sheaf \( \hat{p}_S \ast (p_S \ast M \otimes P_S) \). This is locally free of finite rank, so that it is the sheaf of sections of a vector bundle \( \hat{E} \), with \( \text{rk} \hat{E} = \text{length}(M) \). Moreover, the operator \( \nabla_2 \) naturally extends to \( p_S \ast M \otimes P_S \), and induces an operator \( \hat{\nabla}: \hat{E} \to \hat{E} \otimes \Omega^1_{\hat{T}} \) which is a unitary flat connection. Standard checks show that this procedure does define a functor.

**Example 2.2.** Let \( \mathbb{C}(x) \) denote the one-dimensional skyscraper at \( x \in T \). One has \( F(\mathbb{C}(0)) \simeq \mathbb{C}_{\hat{T}} \). Indeed, in this case we have \( p^*M \otimes P \simeq C_\infty^\infty_{\{0\} \times \hat{T}} \), and, in view of equations \((4), (3)\), the operator \( \nabla_2 \) reduces on this sheaf to the exterior differential along the \( \hat{T} \) direction. As a consequence, \( (\hat{E}, \hat{\nabla}) = (C_\infty^\infty_{\hat{T}}, d) \), and \( F(\mathbb{C}(0)) = \ker d \simeq \mathbb{C}_{\hat{T}} \). △

For every \( x \in T \) let \( t_x \) be the associated translation, \( t_x(x') = x + x' \). Moreover, identify \( \hat{T} \) with \( T \). The following result is easily proved.

**Proposition 2.3.** For every \( x \in T \) and \( M \in \text{Ob}(\text{Sky}(T)) \) there is an isomorphism \( F(t_x^{-1}M) \simeq L_{-x} \otimes F(M) \).

As a consequence, in view of Example 2.2, we have

**Corollary 2.4.** For every \( x \in T \) one has \( F(\mathbb{C}(x)) \simeq L_{-x} \).

This defines the action of the functor \( F \) on the whole category \( \text{Sky}(T) \).

Now we construct the inverse functor. Let \( (\hat{E}, \hat{\nabla}) \) be an object in \( \text{Loc}(\hat{T}) \), and let \( \hat{E} \) be the sheaf of sections of \( \hat{E} \). As we did in the previous section, but reversing the roles of \( T \) and \( \hat{T} \), we consider on the sheaf \( \hat{p}^* \hat{E} \otimes P^\vee \) an operator \( \hat{\nabla}_2^\hat{E} \) obtained by coupling (the pullback of) \( \nabla \) with the operator \( \hat{\nabla}_2 \) (the \( \hat{T} \)-component of the connection of \( P^\vee \)). We shall eventually prove the following.
Proposition 2.5. 1. $R^j p_* \ker \hat{\nabla}_2^\xi = 0$ for $j = 0, \ldots, g - 1$;

2. The sheaf $R^g p_* \ker \hat{\nabla}_2^\xi$ is a skyscraper of finite length.

The functor $\hat{F}$ is defined as $\hat{F}((\hat{E}, \hat{\nabla})) = R^g p_* \ker \hat{\nabla}_2^\xi$.

As a first step we compute the action of $\hat{F}$ on the trivial line bundle, i.e., we take $\hat{E} = C_\infty^\mathbb{C}$ and $\hat{\nabla} = d$. Thus we want to compute the sheaves $R^j p_* \ker \hat{\nabla}_2^\xi$. To this end we shall study the presheaves

$$U \rightsquigarrow H^j(U \times \hat{T}, \ker \hat{\nabla}_2) \simeq H^j\left(\left(P^\mathbb{V} \otimes \Omega^{0,\bullet}\right)(U \times \hat{T}), \hat{\nabla}_2^\xi\right)$$

whose associated sheaves are exactly the sheaves we are interested in.

As a first result we have

**Proposition 2.6.** $H^0(U \times \hat{T}, \ker \hat{\nabla}_2) = 0$ for all open sets $U \subset T$, so that $p_* \ker \hat{\nabla}_2 = 0$.

**Proof.** An element of $H^0(U \times \hat{T}, \ker \hat{\nabla}_2)$ restricted to $\{x\} \times \hat{T}$, with $x \in U$, yields a global section of $L_x$, which is zero unless $x = 0$. By a density argument we get the result.

To compute the higher-order direct images we first consider the case $g = 1$.

**Proposition 2.7.** If $g = 1$, then $R^1 p_* \ker \hat{\nabla}_2 \simeq \mathbb{C}(0)$.

**Proof.** We compute the cohomology groups $H^1(U \times \hat{T}, \ker \hat{\nabla}_2) \simeq H^1((\mathbb{P}^\mathbb{V} \otimes \Omega^{0,\bullet})(U \times \hat{T}), \hat{\nabla}_2^\xi)$. We represent $T$ as $\mathbb{R} = \mathbb{R}/\mathbb{Z} \lambda$, with $\lambda \in \mathbb{R}^*$, and $\hat{T} = \mathbb{R}/\mathbb{Z} \mu$, with $\mu = 1/\lambda$. Let $W$ be the inverse image of $U$ in $\mathbb{R}$.

We work now in a gauge where the factor of automorphy of $\mathbb{P}^\mathbb{V}$ is $e^{2i\pi \mu(x)}$, and the operator $\hat{\nabla}_2$ is the $\hat{T}$-part of the exterior differential. An element in $((\mathbb{P}^\mathbb{V} \otimes \Omega^{0,1})(U \times \hat{T}), \ker \hat{\nabla}_2)$ may be written as $\tau = t(x, y) dy$, where $t$ is a function on $W \times V^\mathbb{V}$ satisfying the automorphy condition

$$t(x, y + \mu) = t(x, y) e^{2i\pi \mu(x)}.$$

If $\tau$ is a coboundary, $\tau = \hat{\nabla}_2 s$, one has

$$s(x, y) = \int_0^y t(x, u) du + c(x).$$

The function $s$ must satisfy the automorphy condition, which amounts to the following condition on $c$:

$$c(x)(1 - e^{2i\pi \mu(x)}) = -\int_0^\mu t(x, u) du. \quad (4)$$
If $0 \notin U$ this condition may be solved for $c$, so that $H^1(U \times \hat{T}, \ker \hat{\nabla}_2) = 0$. Thus $R^1p_* \ker \hat{\nabla}_2$ is a skyscraper supported at $0 \in T$.

If $0 \in U$, the condition (1) may be solved if and only if

$$\int_0^\mu t(0, u) \, du = 0$$

so that $H^1(U \times \hat{T}, \ker \hat{\nabla}_2) \simeq \mathbb{C}$. This proves the claim.

We move to the higher-dimensional case by means of a Künneth-type argument.

**Proposition 2.8.** If $\dim T = g$ we have

1. $R^j\pi_* \ker \hat{\nabla}_2 = 0$ for $j = 0, \ldots, g - 1$;
2. $R^g\pi_* \ker \hat{\nabla}_2 \simeq \mathbb{C}(0)$.

**Proof.** A choice of flat coordinates $(x^1, \ldots, x^g)$ on $T$ fixes an isomorphism $T \simeq S^1 \times \ldots S^1$. The Poincaré sheaf $\mathcal{P}$ on $T \times \hat{T}$ is the product of the Poincaré sheaves $\mathcal{P}_i$ on the $i$ factors of $T \times \hat{T}$, as one can check for instance by describing the Poincaré bundles by their factors of automorphy. Let $U \subset T$ be of the form $U = U_1 \times \ldots \times U_g$ where each $U_i$ lies in a factor of $\hat{T}$. If $g = 2$, a word-by-word translation of the Künneth theorem for de Rham cohomology (cf. e.g. [1]) gives a decomposition

$$H^j(U \times \hat{T}, \ker \hat{\nabla}_2) = \bigoplus_{m+n=j} H^m(U_1 \times S^1, \ker \hat{\nabla}_2^1) \otimes H^n(U_2 \times S^1, \ker \hat{\nabla}_2^2)$$

whence we have, by Proposition 2.6,

$$H^j(U \times \hat{T}, \ker \hat{\nabla}_2) = 0 \quad \text{for} \quad j = 0, 1, \quad H^2(U \times \hat{T}, \ker \hat{\nabla}_2) \simeq \mathbb{C}.$$

Induction on $g$ then yields, for every $g$,

$$H^j(U \times \hat{T}, \ker \hat{\nabla}_2) = 0 \quad \text{for} \quad j = 0, \ldots, g - 1, \quad H^g(U \times \hat{T}, \ker \hat{\nabla}_2) \simeq \mathbb{C}.$$

This proves both claims.

So we have also obtained

$$H^j(T \times \hat{T}, \ker \hat{\nabla}_2) = \begin{cases} 0 & \text{for} \quad j = 0, \ldots, g - 1, \\ \mathbb{C} & \text{for} \quad j = g. \end{cases}$$

Let $\mathcal{L}_x$ be the local system corresponding to the line bundle $\mathcal{L}_x$ with its flat connection. In analogy with Proposition 2.3, we have
Proposition 2.9. \( \hat{F}(\mathcal{L}_x \otimes_{C_{\hat{T}}} \mathcal{S}) \simeq t_x^{-1}\hat{F}(\mathcal{S}) \) for every \( x \in T \) and every local system \( \mathcal{S} \) on \( \hat{T} \).

Corollary 2.10. \( \hat{F}(\mathcal{L}_x) \simeq \mathbb{C}(x) \) for every \( x \in T \).

Remark 2.11. Since any flat vector bundle on a torus is a direct sum of flat line bundles (i.e., every local system on \( \hat{T} \) is a direct sum of local systems of the type \( \mathcal{L}_x \)), Corollary 2.10 completely describes the action of the functor \( \hat{F} \).

\[ \triangle \]

Corollaries 2.4 and 2.10 and Remark 2.11 eventually prove

Theorem 2.12. The functors \( \mathcal{F}, \hat{F} \) are inverse to each other, and establish an equivalence between the categories \( \text{Sky}(T) \) and \( \text{Loc}(\hat{T}) \).

Again, any question related to the behaviour of morphisms under the functor \( \hat{F} \) is simply a matter of routine checks.

2.4 Subtori

We consider now the transformation of \( U(1) \) local systems supported on affine subtori of the \( g \)-dimensional torus \( T = V/\Lambda \).

Definition 2.13. A subtorus of \( T \) is a subset \( S \subset T \) of the form \( S = W/W \cap \Lambda \), where \( W \) is \( k \)-dimensional linear subspace of \( V \) such that the lattice \( W \cap \Lambda \) has rank \( k \). An affine subtorus is a subset of the form \( S + x \) for an element \( x \in T \).

Let \( \mathcal{L} \equiv (\mathcal{L}, \nabla) \) be a \( U(1) \) local system supported on a \( k \)-dimensional affine subtorus \( S \) of \( T \). By coupling the pullback of \( \nabla \) with the connection of \( \mathcal{P}_S \) and projecting on the \( T \)-differentials one obtains a complex

\[
0 \to \ker \nabla^{\mathcal{L}}_1 \to p^*_S \mathcal{L} \otimes \mathcal{P}_S \xrightarrow{\nabla^{\mathcal{L}}} p^*_S \mathcal{L} \otimes \mathcal{P}_S \otimes \Omega^{1,0} \xrightarrow{\nabla^{\mathcal{L}}} p^*_S \mathcal{L} \otimes \mathcal{P}_S \otimes \Omega^{2,0} \to \ldots
\]

Proposition 2.14. 1. \( R^j \hat{p}_S^* \ker \nabla^{\mathcal{L}}_1 = 0 \) for all \( j \neq k \);

2. \( R^k \hat{p}_S^* \ker \nabla^{\mathcal{L}}_1 \) is supported on a \((g-k)\)-dimensional affine subtorus \( \hat{S} \) of \( \hat{T} \), which is normal to \( S \).
3. If $L$ is trivial, then $\hat{S}$ goes through the origin of $\hat{T}$, otherwise it is an affine subtorus translated by the element of $T$ corresponding to $L^*$. 

4. The sheaf $R^k\hat{p}_S\ast \ker \nabla_L$ on $\hat{S}$ is a $U(1)$ bundle, and has a compatible flat connection which makes it into a $U(1)$ local system $\hat{L}$.

Proof. The proof of this Proposition is given in Appendix A. \hfill \Box

Let us describe the content of this Proposition in local coordinates; while this is just simple linear algebra, the explicit equations we are going to write will help to understand the more complicated relative situation. Let $(y^1, \ldots, y^g)$ be flat coordinates in $T$, $(w_1, \ldots, w_g)$ the corresponding dual flat coordinates in the dual torus $\hat{T}$, and write the equation for the affine subtorus $S$ in the form

$$\sum_{j=1}^{g} a^i_j y^j + \chi^i = 0, \quad i = 1, \ldots, g - k.$$ 

The equations $\sum_{j=1}^{g} a^i_j y^j = 0$ describe a corresponding “linear subtorus” $S_0$; the equations of the dual torus $S_0^\ast$ may be written implicitly as

$$\sum_{j,\ell=1}^{g} a^i_j g^{i\ell} w_\ell = 0, \quad i = 1, \ldots, g - k,$$

where the constant functions $g^{i\ell}$ are the components of the natural flat metric on $\hat{T}$, or explicitly as

$$w_\ell = \sum_{m=1}^{k} \tilde{a}_\ell^m \xi_m, \quad \ell = 1, \ldots, g$$

(5)

for a suitable $k \times (g - k)$ matrix $\tilde{a}$. The specification of the local system $\mathcal{L}$ corresponds to a choice of the parameters $(\xi_1, \ldots, \xi_k)$ in equation (5). The support $\hat{S}$ of the transformed local system is given by equations

$$\sum_{j=1}^{g} \gamma^j_m w_j + \xi_m = 0, \quad m = 1, \ldots, k,$$

where $\gamma^j_m$ is a matrix satisfying $\sum_{j=1}^{g} \gamma^j_i a^i_j = 0$. The local system $\hat{\mathcal{L}}$ is given by the point in $\hat{S}_0^\ast$ whose coordinates are the numbers $\chi^j$.

The pair $(\hat{S}, \hat{\mathcal{L}})$ is the Fourier-Mukai transform of the pair $(S, \mathcal{L})$. Of course we may perform the same transformation from $\hat{T}$ to $T$ (in addition to the obvious replacements, one twists by $\mathcal{P}^\vee$ instead of $\mathcal{P}$), and we have:
Proposition 2.15. The Fourier-Mukai transform of $(\hat{S}, \hat{L})$ is naturally isomorphic to the pair $(S, L)$.

Let $\text{Loc}_k(T)$ be the category of $U(1)$ local systems supported on affine subtori of $T$ of dimension $k$. Objects of this category are triples $(S, L, \nabla)$ (where $S$ is an affine subtorus in $T$, $L$ a line bundle on $S$, and $\nabla$ a flat unitary connection on $L$) modulo isomorphisms, i.e., modulo vector bundle isomorphisms which commute with the actions of the connections (the two line bundles having the same support). The space of morphisms between two objects $(S_1, L_1, \nabla_1)$ and $(S_2, L_2, \nabla_2)$ of $\text{Loc}_k(T)$ is defined by taking into account that the intersection $S = S_1 \cap S_2$ is a (possibly empty) finite collection of (possibly zero-dimensional) affine tori $R_i$, and one sets

$$\text{Mor}((S_1, L_1, \nabla_1), (S_2, L_2, \nabla_2)) = \bigoplus_i \text{Mor}_\nabla((R_i, L_1, \nabla_1), (R_i, L_2, \nabla_2)),$$

where $\text{Mor}_\nabla((R_i, L_1, \nabla_1), (R_i, L_2, \nabla_2))$ is the set of morphisms between $L_1|_{R_i}$ and $L_2|_{R_i}$ compatible with the connections $\nabla_1$ and $\nabla_2$. It is easy to check that the Fourier-Mukai transform yields an equivalence of categories

$$\text{Loc}_k(T) \simeq \text{Loc}_{g-k}(\hat{T}).$$

3 Relative theory

3.1 The geometric setting

Let $(X, \omega)$ be a connected symplectic manifold admitting a map $f: X \to B$ whose fibres are $g$-dimensional smooth Lagrangian tori. We assume that $f$ admits a Lagrangian section $\sigma: B \to X$; according to [1], this makes $X$ isomorphic, as a symplectic manifold fibred in Lagrangian submanifolds, to a quotient bundle $T^*B/\Lambda$, where $\Lambda$ is a Lagrangian covering of $B$. The symplectic form $\omega$ provides an isomorphism $\text{Vert} T X \simeq f^*T^*B$. We also have an identification $TB \simeq R^1 f_*\mathbb{R} \otimes \mathbb{C}_B$, and this endows $TB$ with a flat, torsion-free connection $\nabla_{GM}$ — the Gauss-Manin connection of the local system $R^1 f_*\mathbb{R}$. The holonomy of this connection coincides with the monodromy of the covering $\Lambda$ (indeed, the horizontal tangent spaces may be identified with the first homology groups of the fibres with real coefficients).

Let $\hat{X} = R^1 f_*\mathbb{R}/R^1 f_*\mathbb{Z}$ be the dual family, with projection $\hat{f}: \hat{X} \to B$. Dualizing the isomorphism $\text{Vert} T X \simeq f^*T^*B$ we get a new isomorphism $\text{Vert} T \hat{X} \simeq \hat{f}^*TB$; combining
this with the splitting of the Atiyah sequence

\[ 0 \to \text{Vert} T\hat{X} \to T\hat{X} \to \hat{f}^*TB \to 0 \]

provided by the Gauss-Manin connection (which can be regarded as a connection on \( T\hat{X} \)), one has a splitting

\[ T\hat{X} \simeq \hat{f}^*TB \oplus \hat{f}^*TB. \]

By letting \( J(\alpha, \beta) = (-\beta, \alpha) \) this induces a complex structure on \( \hat{X} \), such that the holomorphic tangent bundle to \( \hat{X} \) is isomorphic, as a smooth bundle, to \( \hat{f}^*TB \otimes \mathbb{C} \).

We shall systematically use on \( X \) local action-angle coordinates \((x^1, \ldots, x^g, y_1, \ldots, y_g)\): thus the \( x \)'s are local coordinates on \( B \), and for fixed values of the \( x \)'s, the \( y \)'s are flat coordinates on the corresponding torus, dual to an integral homology basis. On \( \hat{X} \) we consider local coordinates \((x^1, \ldots, x^g, w^1, \ldots, w^g)\) such that the \( w \)'s are dual coordinates to the \( y \)'s. Local holomorphic coordinates on \( \hat{X} \) are given by \( z^j = x^j + iw^j \).

In this relative context it is natural to consider the fibre product \( Z = X \times_B \hat{X} \) of the fibrations \( X \) and \( \hat{X} \). We shall denote by \( p, \hat{p} \) the projections of \( Z \) onto its factors. On \( Z \) there is a Poincaré bundle \( \mathcal{P} \) which may be described in an intrinsic way, however, it is enough to say that \( \mathcal{P} \) a line bundle on \( X \times_B \hat{X} \) equipped with a \( U(1) \) connection \( \nabla_\mathcal{P} \) whose connection form may be written in a suitable gauge as

\[ A = 2i \pi \sum_{j=1}^g w^j dy_j. \]

Moreover, \( \mathcal{P} \) has the property that for every \( \xi \in \hat{X} \), \( \mathcal{P}|_{\hat{p}^{-1}(\xi)} \) is isomorphic to \( \mathcal{L}_\xi \) (the line bundle parametrized by \( \xi \)) as a \( U(1) \) bundle.

If \( S \) is a closed submanifold of \( X \), we define \( Z_S = S \times_B \hat{X} \), with projections \( p_S, \hat{p}_S \) onto the two factors, and denote \( \mathcal{P}_S = \mathcal{P}|_{Z_S} \). We shall assume that \( \hat{X}^S = \hat{p}(Z_S) \) is a closed submanifold of \( \hat{X} \), and that \( \hat{p}_S: Z_S \to \hat{X}^S \) is a submersion (in the remainder of this section we shall tacitly understand that these conditions are satisfied, while in the subsequent sections they will hold as a consequence of other assumptions). We consider the exact sequence

\[ 0 \to \hat{p}_S^*\Omega^1_{\hat{X}^S} \to \Omega^1_{Z_S} \overset{r}{\to} \Omega^1_{Z_S/\hat{X}^S} \to 0 \]

which defines the sheaf \( \Omega^1_{Z_S/\hat{X}^S} \) of \( \hat{p}_S \)-relative differentials. The Gauss-Manin connection \( \nabla_{GM} \) provides a splitting of this exact sequence. Analogously, if \( \hat{S} \) is a closed submanifold
of $\hat{X}$, we have a split exact sequence

$$0 \to p^*_S \Omega^1_{X^S} \to \Omega^1_{Z^S} \overset{r}{\to} \Omega^1_{Z^S/X^S} \to 0$$

(7)

which defines the sheaf $\Omega^1_{Z^S/X^S}$ of $p^*_S$-relative differentials. For every sheaf $\mathcal{E}$ of $\mathcal{C}_S$-modules endowed with a flat connection $\nabla$, one defines the following differential operators:

(i) the operator

$$\nabla^\mathcal{E}: p^*_S \mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^\bullet \to p^*_S \mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^{\bullet+1},$$

obtained by coupling the pullback of the connection $\nabla$ with the connection of the Poincaré sheaf;

(ii) the operators $\nabla^\mathcal{E}_r, \hat{\nabla}^\mathcal{E}_r$ obtained by composing $\nabla^\mathcal{E}$ with the projections $r, \hat{r}$ onto the relative differentials. One has $(\nabla^\mathcal{E})^2 = (\hat{\nabla}^\mathcal{E})^2 = 0$.

We shall consider the higher direct images $R^i \hat{p}_{S,*} \ker \nabla^\mathcal{E}_r$, which are the cohomology sheaves of the complex

$$\hat{p}_{S,*}(p^*_S \mathcal{E} \otimes \mathcal{P}_S) \xrightarrow{\nabla^\mathcal{E}_r} \hat{p}_{S,*}(p^*_S \mathcal{E} \otimes \Omega^1_{Z^S/X^S}) \xrightarrow{\nabla^\mathcal{E}_r} \hat{p}_{S,*}(p^*_S \mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^2_{Z^S/X^S}) \to \cdots$$

(8)

As in the usual theory of the Fourier-Mukai transform, it is convenient to introduce a WIT notion.\footnote{Let us recall that “WIT” stands for “weak index theorem.”}

**Definition 3.1.** The pair $(\mathcal{E}, \nabla)$ is said to be WIT$_k$ if $R^i \hat{p}_{S,*} \ker \nabla^\mathcal{E}_r = 0$ for $i \neq k$.

Now we want to state a condition for the sheaves $R^i \hat{p}_{S,*} \ker \nabla^\mathcal{E}_r$ to admit a connection induced, so to say, by the part of the operator $\nabla^\mathcal{E}$ complementary to $\nabla^\mathcal{E}_r$. The splitting of the exact sequence (8) provided by the Gauss-Manin connection $\nabla_{GM}$ allows one to make a splitting

$$\nabla^\mathcal{E} = \nabla^\mathcal{E}_r + \hat{\nabla}^\mathcal{E}.$$

The $\hat{\nabla}^\mathcal{E}$ operator induces connections on the higher direct images $R^i \hat{p}_{S,*} \ker \nabla^\mathcal{E}_r$ provided it anticommutes with the operator $\nabla^\mathcal{E}_r$. The anticommutator $\nabla^\mathcal{E}_r \circ \hat{\nabla}^\mathcal{E} + \hat{\nabla}^\mathcal{E} \circ \nabla^\mathcal{E}_r$ may be regarded as an operator $p^*_S \mathcal{E} \otimes \mathcal{P}_S \to p^*_S \mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^2_{Z^S}$ and as such it coincides with the restriction to $Z^S$ of $1 \otimes \mathbb{F}$, where $\mathbb{F}$ is the curvature of the connection $\nabla_{\mathcal{P}}$ of the Poincaré bundle. As a consequence, we have:
Proposition 3.2. Assume that the sheaf $\mathcal{E}$ is supported on a closed submanifold $S \subset X$, the sheaf $R^j\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r$ is supported on a closed submanifold $\hat{S} \subset \hat{X}$, and the curvature operator $\hat{\nabla}^\mathcal{E}$ vanishes on $S \times_B \hat{S} \subset Z$. Then the operator $\hat{\nabla}^\mathcal{E}$ induces a connection on the sheaf $R^j\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r$.

Eventually, we may introduce the Fourier-Mukai transform we shall study in the remainder of this paper.

Definition 3.3. If the pair $(\mathcal{E}, \nabla)$ is WIT, and satisfies the condition in Proposition 3.2, the pair $(\hat{\mathcal{E}}, \hat{\nabla})$, where $\hat{\mathcal{E}} = R^k\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r$ and $\hat{\nabla}$ is the connection induced as in Proposition 3.2, is called the Fourier-Mukai transform of $(\mathcal{E}, \nabla)$.

We end this section with an easy lemma which is useful when checking if the WIT property holds for some sheaf and connection.

Lemma 3.4. Let $(\mathcal{E}, \nabla)$ be a local system supported on a closed submanifold $S$ of $X$ which intersects every fibre $X_b$ along a closed submanifold $S_b$. For every $j = 1, \ldots, g$ there is a canonical isomorphism

$$
(R^j\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r)|_{\hat{X}_b} \sim R^j\hat{p}_{S_b,*}\ker\nabla^\mathcal{E}_b,
$$

where $b = \hat{p}(\xi)$, $\hat{p}_b: X_b \times \hat{X}_b \to \hat{X}_b$ is the projection onto $\hat{X}_b$ and $\mathcal{E}_b$ is the restriction of $\mathcal{E}$ to $S_b$.

Proof. The restriction $(R^j\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r)|_{\hat{X}_b}$ is defined as $\check{j}_b^{-1}R^j\hat{p}_{S,*}\ker\nabla^\mathcal{E}_r \otimes \check{j}_b^{-1}C^\infty_X C^\infty_{\hat{X}_b}$ (here $j_b: S_b \to X$ and $\check{j}_b: \hat{X}_b \to \hat{X}$ are the natural inclusions). The result is proved by applying the topological base change [13] to the diagram

$$
\begin{array}{ccc}
S_b \times \hat{X}_b & \xrightarrow{j_b \times j_b} & S \times_B \hat{X} \\
\downarrow \hat{p}_b & & \downarrow \hat{p} \\
\hat{X}_b & \xrightarrow{j_b} & \hat{X}
\end{array}
$$

3.2 Fibres and Lagrangian sections

In studying the transformation of local systems supported by Lagrangian submanifolds we start by considering the case where the submanifold is either a fibre or a Lagrangian section.
The first case is the simplest to deal with. It is enough to consider the case rank $\mathfrak{L} = 1$, since the higher rank case reduces immediately to this. The isomorphism class of the local system $\mathfrak{L}^*$ singles out a point in $\hat{X}$, which we denote by $[\mathfrak{L}^*]$. Since $X_b \times_B \hat{X} \simeq X_b \times \hat{X}_b$, we obtain the usual “tautological” property of the Fourier-Mukai transform.

**Proposition 3.5.** The pair $(\mathcal{L}, \nabla) \equiv \mathfrak{L}$ is WIT$_g$, and the sheaf $\hat{\mathcal{L}} = R^g \hat{p}_* \ker \nabla^\mathcal{L}$ is isomorphic to the skyscraper $\mathbb{C}([\mathfrak{L}^*])$.

Now we construct a transform for $U(1)$ local systems supported on Lagrangian sections of $X \to B$. This will generalize the tautological correspondence that in the absolute case holds between skyscrapers of length one on a torus and $U(1)$ local systems on the dual torus. The transform will produce holomorphic line bundles on $\hat{X}$ with compatible $U(1)$ connections which satisfy some further conditions.

Let $S \subset X$ be the image of a Lagrangian section of $X \to B$, and $\mathfrak{L} \equiv (\mathcal{L}, \nabla)$ a $U(1)$ local system on $S$.

**Proposition 3.6.** 1. The pair $(\mathcal{L}, \nabla)$ is WIT$_0$;

2. $\hat{\mathcal{L}} = \hat{p}_{S*} \ker \nabla^\mathfrak{L}$ is a rank-one locally free $\mathcal{C}_X^\infty$-module.

**Proof.** Both claims follows from Lemma 3.4 and the absolute case.  

Since $\mathbb{F}_{|S \times_B X} = 0$ the conditions of Proposition 3.2 are met, so that $\hat{\mathcal{L}}$ carries a $U(1)$ connection $\hat{\nabla}$. Let us express this connection in (action-angle) coordinates. We write the local equations of $S$ as $y_j = \epsilon_j(x)$; as $S$ is Lagrangian, one has $\frac{\partial \epsilon_j}{\partial x^m} = \frac{\partial \epsilon_m}{\partial x^j}$. Moreover, the $x$’s can be thought of as local coordinates on $S$. If the connection form of $\nabla$ is $A = i \sum_{j=1}^g A_j(x) \, dx^j$, with $\frac{\partial A_j}{\partial x^j} = \frac{\partial A_i}{\partial x^j}$, then $\hat{\nabla}$ may be represented by the connection form

$$\hat{A} = i \sum_{j=1}^g A_j(x) \, dx^j - 2i\pi \sum_{j=1}^g \epsilon_j(x) \, dw^j.$$ 

In these coordinates the components of the connection form $\hat{A}$ do not depend on the $w$’s. Moreover, both the horizontal and vertical part (with respect to the splitting given by the Gauss-Manin connection) are flat, and in particular, the restriction of $\hat{\nabla}$ to any fibre $\hat{X}_b$ of $\hat{X} \to B$ is flat.

**Remark 3.7.** The independence of the components $\hat{A}$ on the $w$’s can be stated invariantly in a variety of ways. For instance, one can use the fact that the zero-section of $\hat{X}$ makes
the latter into a (trivial) principal $T^g$-bundle over $B$; then, $\hat{\nabla}$ commutes with the action of $T^g$ on $\hat{X}$. 

The Hodge components of curvature form $\hat{F}$ of this connection may be written — recalling that in the complex structure we have given to $\hat{X}$ the coordinates $z^j = x^j + iw^j$ are complex holomorphic — as

$$\hat{F}^{2,0} = \frac{\pi}{2} \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} dz^k \wedge d\bar{z}^j$$

$$\hat{F}^{0,2} = -\frac{\pi}{2} \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} d\bar{z}^k \wedge dz^j$$

$$\hat{F}^{1,1} = \frac{\pi}{2} \sum_{k,j} \left( \frac{\partial \epsilon_k}{\partial x^j} + \frac{\partial \epsilon_j}{\partial x^k} \right) dz^k \wedge d\bar{z}^j.$$

Since $S$ is Lagrangian we have $\hat{F}^{0,2} = \hat{F}^{2,0} = 0$, so that $\hat{L}$ may be given a holomorphic structure compatible with the connection $\hat{\nabla}$. Moreover, we have

$$\hat{F}^{1,1} = \pi \sum_{k,j} \frac{\partial \epsilon_k}{\partial x^j} dz^k \wedge d\bar{z}^j.$$

**Definition 3.8.** The Fourier transform of $(S, \mathcal{L})$ is the pair $(\hat{L}, \hat{\nabla})$.

### 3.3 The non-transversal case

The results in section 3.2 can be generalized to local systems supported on Lagrangian submanifolds of $X$ other than sections. This allows us to enlarge the “dual” category on which the inverse Fourier-Mukai transform (defined in section 3.4) acts, to a category of sheaves with connection (satisfying some suitable conditions) supported by complex submanifolds of $\hat{X}$. Such sheaves arise naturally in Fukaya’s treatment of mirror symmetry [12]. We shall be interested in transforming local systems supported on a submanifold $S$ of $X$ such that:

1. **(C1)** $S$ is a Lagrangian subvariety of $X$;
2. **(C2)** the intersection $S_b = S \cap X_b$ of $S$ with a fibre of $X$, when nonempty, is a (possibly affine) subtorus $S_b$ of $X_b$ whose dimension does not depend on $b$.  


Let $\mathcal{L} \equiv (\mathcal{L}, \nabla)$ be a $U(1)$ local system on $S$. Its Fourier-Mukai transform at the sheaf level is

$$\hat{\mathcal{L}} = R^{m}\hat{p}_{S,*}\ker \nabla_{r}^{\mathcal{L}}$$

where $m$ is the dimension of the tori $S_{b}$. Indeed, one has:

**Proposition 3.9.** If the conditions $C1$ and $C2$ are satisfied, the sheaf $\mathcal{L}$ is WIT$_{m}$.

**Proof.** It follows from Lemma 3.4 and Proposition 2.14.

Lemma 3.4 and Proposition 2.14 also imply that after restriction to its support, $\hat{\mathcal{L}}$ is a line bundle. We shall now show that, under some suitable conditions on the support $S$, the transform $\hat{\mathcal{L}}$ is supported on a complex submanifold $\hat{S}$ of the dual family $\hat{X}$. More precisely, we assume:

(C3) the vertical tangent spaces of the family of subtori $\{S_{b}\}_{b \in f(S)}$ are parallelly transported by the Gauss-Manin connection $\nabla_{GM}$ regarded as a connection in $TX$.

This requirement can be translated into a more explicit form in terms of the action-angle coordinates $(x, y)$ we have previously introduced, in that it amounts to the condition that the family of subtori $\{S_{b}\}$ can be written as

$$\sum_{j=1}^{g} a_{i}^{j} y_{j} + \chi_{i} = 0, \quad i = 1, \ldots, g - m$$

with the matrix $a_{i}^{j}$ constant and the $\chi_{i}$’s local functions on $B$.

**Lemma 3.10.** Conditions $C1$, $C2$ and $C3$ imply that $f(S)$ is a submanifold of $B$ of dimension $k = g - m$, and that it can be parametrized by the first $k$ action coordinates $x^{j}$.

**Proof.** The first claim follows from the fact that the horizontal part of the tangent space to $S$ has constant dimension; the second from the Lagrangian condition which implies that the local equations of $f(S)$ in $B$ are linear in the action coordinates.

**Proposition 3.11.** Let $(S, \mathcal{L}, \nabla)$ be a local system supported on a Lagrangian submanifold $S$ fulfilling the conditions $C1$ and $C2$. The condition $C3$ is satisfied if and only if the support $\hat{S}$ of the transform $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$. 19
Proof. A proof is given in Appendix B. □

Remark 3.12. In our setting there is no constraint on the dimension of $X$, the latter space is assumed to be just symplectic, and we consider local systems supported on Lagrangian submanifolds of $X$. On the other hand, string-theoretic mirror symmetry assumes, on physical grounds, that $X$ is a (usually 3-dimensional) Calabi-Yau manifold, and one considers special Lagrangian supports\[.\] In this case, the condition that $S$ is special Lagrangian implies, for $k = 1$, that the coefficients $a_i^j$ are constant, so that this is a particular case within our treatment. On the contrary, for $k = 2$ the speciality property seems to be unrelated to the conditions that ensure the support $\hat{S}$ to be complex holomorphic. △

Proposition 3.13. Under the conditions of Proposition [3.11], the operator $\hat{\nabla}_L$ (cf. section 3.1) induces on $\hat{\mathcal{L}}$ a $U(1)$ connection.

Proof. This will use the proof of Proposition [3.11] given in Appendix B. We know that $\hat{\nabla}_L$ induces a connection on the Fourier-Mukai transform if the curvature $\mathbb{F}$ of the Poincaré bundle on $Z = X \times_B \hat{X}$ vanishes on $S \times_B \hat{S}$, where $S$ and $\hat{S}$ are the supports of $\mathcal{L}$ and $\hat{\mathcal{L}}$, respectively. In view of the form of $\mathbb{F}$, this condition is met if for each $b \in B$ the intersections of $S$ and $\hat{S}$ with the fibres $X_b$, $\hat{X}_b$ yield subtori of $X_b$, $\hat{X}_b$ that are normal to each other. But looking at the equations of the supports, (10) and (13), and comparing with the absolute case (Proposition 2.14), we see that this condition is fulfilled. □

We shall now prove that $\hat{\mathcal{L}}$, as a line bundle on $\hat{S}$, has a holomorphic structure. Let $\hat{\nabla}$ be the connection induced on $\hat{\mathcal{L}}$.

Proposition 3.14. If the support $\hat{S}$ of the transformed sheaf $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$, then $\hat{\nabla}$ induces a holomorphic structure on $\hat{\mathcal{L}}$.

Proof. The connection 1-form of the connection $\nabla$ can be written in an appropriate gauge as

$$A = i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k) \, dx^j + 2i \pi \sum_{\ell=1}^{g-k} \xi^\ell \, dy^\ell,$$

\[3\] Let us recall that a special Lagrangian submanifold of a Calabi-Yau $n$-fold $X$ is an oriented real $n$-dimensional submanifold $Y$ which is Lagrangian w.r.t. the Kähler form of $X$, and such that one can choose a global trivialization of the canonical bundle of $X$ whose imaginary part vanishes on $Y$. For more details cf. [15].
with the quantities $\xi^\ell$ constant. From the proof of Proposition 2.14 given in the Appendix A we know that the transformed connection $\hat{\nabla}$ is given in local action-angle coordinates by the 1-form $\hat{A}$

$$\hat{A} = -2i\pi \sum_{\ell=g-k+1}^{g} \chi_{\ell}(x^1, \ldots, x^k) \, dw^\ell + i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k) \, dx^j.$$ 

Rewriting this in terms of $w^1, \ldots, w^k$ we obtain

$$\hat{A} = -2i\pi \sum_{\ell=g-k+1}^{g} \sum_{j=1}^{k} \chi_{\ell}(x^1, \ldots, x^k) \, \hat{\gamma}_j^\ell \, dw^j + i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k) \, dx^j$$

where the coefficients $\hat{\gamma}_j^\ell$ are constant. Since $d(\sum_j \alpha_j \, dx^j) = 0$ because of the flatness of $\nabla$, it follows that the curvature of $\hat{\nabla}$ is given by

$$\hat{F} = -2i\pi \sum_{\ell=g-k+1}^{g} \sum_{j,m=1}^{k} \frac{\partial \chi_{\ell}}{\partial x^j} \hat{\gamma}_m^\ell \, dx^j \wedge dw^m.$$ 

Since $\hat{\gamma}_m^{g-k+j} = \frac{\partial \zeta^{g-k+j}}{\partial x^m}$, where the functions $\zeta^{g+j}$ are those of the equations (10), the condition $\hat{F}^{0,2} = 0$ can be written as

$$\sum_{\ell=g-k+1}^{g} \left[ \frac{\partial \chi_{\ell}}{\partial x^j} \frac{\partial \chi_{\ell}}{\partial x^m} - \frac{\partial \chi_{\ell}}{\partial x^m} \frac{\partial \chi_{\ell}}{\partial x^j} \right] = 0, \quad 1 \leq j < m \leq k.$$ 

But this is the system of equations (14), therefore when $S$ is Lagrangian, this condition is automatically satisfied.

Remark 3.15. (The higher rank case.) So far we have for simplicity considered only the transformation of local systems of rank one. However the higher rank case, under the same conditions, can be treated along the same lines, obtaining on the $\hat{X}$ side holomorphic vector bundles of the corresponding rank supported on complex submanifolds of $\hat{X}$.

3.4 Invertibility

In this section we shall prove that the Fourier-Mukai transform we have defined inverts. However we shall only discuss the inverse transform of rank 1 sheaves. The higher rank

\footnote{Also in this case we find that the transformed connection $\hat{\nabla}$ satisfies the condition of Remark 3.7.}
case requires to consider Lagrangian submanifolds of $X$ which ramify over $B$, and this will be done in a future paper.

We shall therefore consider a holomorphic line bundle $\hat{L}$ supported on a $k$-dimensional complex submanifold $\hat{S}$ of $\hat{X}$, equipped with a compatible $U(1)$ connection $\hat{\nabla}$. Moreover, we shall assume that:

(D1) $\hat{S}$ intersects the fibres of $\hat{X}$ along affine subtori of complex dimension $k$;

(D2) the horizontal part of the connection $\hat{\nabla}$ is flat (horizontality is given by the Gauss-Manin connection);

(D3) the connection $\hat{\nabla}$ is invariant under the action of $T^g$ on $\hat{X}$ (cf. Remark 3.7).

These conditions allow us to write the local connection form of $\hat{\nabla}$ as

$$\hat{A} = i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k) \, dx^j + 2i\pi \sum_{j=1}^{k} \beta_j(x^1, \ldots, x^k) \, dw^j,$$

where the functions $\alpha_j$ satisfy (as a consequence of D2) the closure condition $\frac{\partial \alpha_j}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}$. This shows that the restriction of $\hat{\nabla}$ to any fiber $\hat{X}_b$ of $\hat{X} \to B$ yields a flat connection on $\hat{L}|_{\hat{X}_b}$.

Let $\hat{p}_S, \hat{p}_{\hat{S}}$ the canonical projections of $X \times_B \hat{S}$ onto its factors. We consider the operator

$$\nabla^{\hat{\nabla}}_{\hat{r}} = \hat{r} \circ (\hat{p}_S^{\ast} \nabla^{\hat{\nabla}} \otimes 1 + 1 \otimes \nabla^{\hat{\nabla}}_{\hat{S}})$$

and in terms of it we define a Fourier-Mukai transform from sheaves on $\hat{X}$ to sheaves on $X$ (notice that we twist with the dual Poincaré bundle $P^\vee$).

**Proposition 3.16.** $L$ is WIT$_k$, and $L = R^k p_{\hat{S},s} \ker \nabla^{\hat{\nabla}}_{\hat{r}}$ is supported on a Lagrangian submanifold $S$ of $X$ such that every intersection $S_b = S \cap X_b$ is an affine subtorus of $X_b$ of dimension $g - k$ (when nonempty). Moreover the family of subtori $S_b$ is parallelly transported by the Gauss-Manin connection $\nabla_{GM}$. Finally, a flat connection $\nabla$ is naturally induced on $L$.

**Proof.** The WIT condition follows immediately from Lemma 3.4. To show the remaining part of the claim we write local equations for $\hat{S}$ as

$$\begin{cases} x^{k+j} = \zeta^{k+j}(x^1, \ldots, x^k), & j = 1, \ldots, g - k \\ w^{k+j} = \sum_{i=1}^{k} P_{i}^{k+j}(x^1, \ldots, x^k) w^i + Q^{k+j}(x^1, \ldots, x^k), & j = 1, \ldots, g - k. \end{cases}$$
Performing a fibrewise transform we obtain the following equations for the support $S$ of the transform $\mathcal{L}$:

$$y_l + \sum_{m=k+1}^{g} P_l^m(x^1, \ldots, x^k) y_m + \beta_l(x^1, \ldots, x^k) = 0$$

where $l = 1, \ldots, k$. It remains to show that $S$ is Lagrangian and that the family $\{S_b\}_{b \in \hat{f}(S)}$ is parallely transported by the Gauss-Manin connection. The latter point follows from the complex structure of $\hat{S}$ (cf. Proposition 3.11): the Cauchy-Riemann equations for $\hat{S}$ imply that the coefficients $P^{k+j}_l$ and $Q^{k+j}_m$ are constant. As far as the Lagrangian property of $S$ is concerned, the holomorphicity of $\hat{S}$ and $\hat{\mathcal{L}}$ imply the equations (11) in the proof of Proposition 3.11 (in Appendix B). Therefore $S$ is Lagrangian. Observe that the transformed connection $\nabla$ has a 1-form given by

$$A = i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k) dx^j - 2i \pi \sum_{m=k+1}^{g} Q^m dy_m,$$

whence we can immediately deduce its flatness.

4 Conclusions

We summarize here the main results of this paper. We have shown that a suitably defined Fourier-Mukai transform $\mathcal{F}$ maps a $U(1)$ local system supported on a Lagrangian subvariety $S$ of $X$ satisfying the conditions C1, C2 and C3 (cf. section 3.3) into a holomorphic line bundle $\hat{\mathcal{L}}$ supported a complex subvariety $\hat{S}$ of $\hat{X}$; moreover $\hat{\mathcal{L}}$ is endowed with a $U(1)$ connection such that conditions D1, D2, D3 (section 3.4) are satisfied.

Conversely, if we start with a holomorphic line bundle supported on a complex subvariety $\hat{S}$ of $\hat{X}$ equipped with a $U(1)$ connection $\hat{\nabla}$ such that conditions D1, D2, D3 are satisfied, we define a dual Fourier-Mukai transform $\hat{\mathcal{F}}$ that maps such objects into a $U(1)$ local system supported on a Lagrangian subvariety $S$ such that conditions C1, C2 and C3 are fulfilled. The explicit forms of the two transforms we have written in sections 3.2 and 3.3 show that the transforms are one the inverse of the other. This parallels the classical result in [22] and generalizes the one in [4], whose authors consider the case where $X$ and $\hat{X}$ are $S^1$-fibrations over $S^1$ ($\hat{X}$ is actually an elliptic curve) and $\mathcal{L}$ is a local system on an affine line $S \subset X$. Observe that in this case the conditions C1, C2, C3 and D1, D2, D3 are trivially satisfied.
Finally, we would like to comment upon the relation of the construction we have described in this paper with Fukaya’s homological mirror symmetry. First we notice that, in the absence of the B-field and with no singular fibres, our “mirror manifold” \( \hat{X} \) coincides with Fukaya’s, also taking into account its complex structure. Let \( S \) be a Lagrangian submanifold of \( X \), and \( \beta = (\mathcal{L}, \nabla) \) a local system on it. Fukaya proposes to construct on \( \hat{X} \) a coherent sheaf whose fibre at a point \( (b, \alpha) \in \hat{X} \) (where \( \alpha = (L_{\alpha}, \nabla_{\alpha}) \) is a local system on the fibre \( X_b \)) is given by the Floer homology

\[
HF^*((X_b, \alpha), (S, \beta)).
\]

This homology may be proved \([11]\) to be isomorphic to

\[
H^{*-\eta(X_b, S)}(S \cap X_b, \mathcal{H}om_{\nabla}(\mathcal{L}_{\alpha}, \mathcal{L})),
\]

where \( \eta(X_b, S) \) is a Maslov index, and \( \mathcal{H}om_{\nabla}(\mathcal{L}_{\alpha}, \mathcal{L}) \) is the sheaf of \( \nabla \)-compatible morphisms between \( \mathcal{L}_{\alpha} \) and \( \mathcal{L} \). It is not difficult to show that only one of these cohomology groups does not vanish (in the correct degree), and that it is isomorphic, up to a dual, to the fibre of our transform \( \hat{\mathcal{L}} \). However, the concrete construction done in \([12]\) is not in terms of Floer homology, but it is an \emph{ad hoc} one, which may be compared with ours when \( X = T^{2g}, B = T^g \) and \( S \) is a Lagrangian embedding of \( T^g \). In this case the vector bundle constructed on \( \hat{X} \) coincides with ours.

It should be noted that our construction provides on the “mirror side” \( \hat{X} \) more data, in that we obtain on \( \hat{\mathcal{L}} \) a connection. It is interesting to note that this connection is not invariant under Hamiltonian diffeomorphisms of \( X \), while the remaining geometric data on \( \hat{X} \) are.

\section*{Appendix A}

We provide here a sketch of the proof of Proposition 2.14. It involves a number of computations but it is conceptually very easy, being a generalization of the proof of Proposition 2.7. For further details we refer to \([20]\).

We first consider the case when \( S \) is a 1-dimensional affine subtorus of \( T \). The direct image \( R^i\hat{p}_{S*}\ker \nabla^\xi_1 \) is by definition the sheaf associated to the presheaf \( \hat{U} \sim H^i(S \times \hat{U}, \ker \nabla^\xi_1) \approx H^i \left( \Omega^{*,0}(p_S^*L \otimes \mathcal{P}_S)(S \times \hat{U}), \nabla_1^\xi \right) \). When \( i = 0 \), take an element \( s \) in \( H^0(S \times \hat{U}, 0 \subset \ker \nabla_1^\xi) \).
\( \hat{U}, \ker \nabla_1^\xi \) and consider its restriction to \( S \times \{ y \} \), with \( y \in \hat{U} \), which is a global section of \( \mathcal{L} \otimes \mathcal{P}_{|S \times \{ y \}} \). This means that \( \mathcal{P}_{|S \times \{ y \}} \) is non trivial for every \( y \) in the complement of \( \ker \psi \), which is a dense subset of \( \hat{T} \) (here \( \psi \) is the natural map \( \psi : \hat{\mathcal{T}} \to \text{Hom}(\pi_1(S), U(1)) \)). The same holds for \( \mathcal{L} \otimes \mathcal{P}_{|S \times \{ y \}} \). Since \( \nabla_1^\xi s_{|S \times \{ y \}} = 0 \) for every \( y \in \hat{T} \), the restriction of \( s \) to \( S \times \{ y \} \) vanishes for \( y \) in a dense subset of \( \hat{T} \), so that \( s \) vanishes everywhere.

When \( i = 1 \) we need to write the equation of \( S \) explicitly. For simplicity we only give some details in the case \( \dim T = 2 \). Let \((y^1, y^2)\) be flat coordinates on \( T \) and \((w_1, w_2)\) flat dual coordinates on \( \hat{T} \). We pick a gauge where the Poincaré bundle has an automorphy factor

\[
a_p(y^1, y^2, w_1, w_2, \lambda^1, \lambda^2, \mu_1, \mu_2) = e^{2\pi i (\lambda^1 w_1 + \lambda^2 w_2)}.
\]

The equation of \( S \) in the universal cover of \( T \) is given by an affine line \( y^2 = ay^1 + b \). Let \( A = \dot{y}_1 dy^1 \) be the connection form of the local system \((\mathcal{L}, \nabla^\xi)\) on \( S \). We need to compute \( H^1(S \times \hat{U}, \ker \nabla_1^\xi) \). So take an element \( \tau \in (\Omega^{1,0}(\rho_S^* \mathcal{L} \otimes \mathcal{P}_S)(S \times \hat{U}), \nabla_1^\xi) \). Observe that \( \tau \) is closed with respect to \( \nabla_1^\xi \) because \( \dim S = 1 \). If we let

\[
\tau = \phi(\xi, w_1, w_2) d\xi
\]

where \( \xi \) is the natural coordinate on \( S \), the automorphy condition satisfied by \( \tau \) can be expressed in the form

\[
\phi(\xi + \sqrt{p^2 + q^2}, w_1, w_2) = e^{p(w_1 + \bar{w}_1) + q w_2} \phi(\xi, w_1, w_2)
\]

having set \( a = q/p \) with \( q, p \) coprime.

Suppose that \( \tau \) is exact so that we can write \( \tau = \nabla_1^\xi s \) where \( s \in C^\infty(S \times \hat{U}, \ker \nabla_1^\xi) \). Then \( s \) can be written in the form

\[
s(\xi, w_1, w_2) = \int_0^\xi \phi(u, w_1, w_2) du + c(w_1, w_2)
\]

but this is well defined if and only if the automorphy condition is satisfied, and one can easily check that this amounts to

\[
c(w_1, w_2)(1 - e^{2\pi i (p(w_1 + \bar{w}_1) + qw_2)}) = -\int_0^{\sqrt{p^2 + q^2}} \phi(u, w_1, w_2) du.
\]

This equation may be solved for \( c \) in the complement of the set \( \hat{S} \) defined by

\[
w_2 = -\frac{1}{a} w_1 - \frac{\bar{w}_1}{a};
\]
thus, arguing as in Proposition 2.7, we obtain that the support of $R^1\hat{p}_{S,*}\ker\nabla_1^L$ is exactly $\hat{S}$.

To compute the sheaf $R^1\hat{p}_{S,*}\ker\nabla_1^L$ we note that the map

$$\varpi: \Omega^{1,0}(p^*_S\mathcal{L} \otimes \mathcal{P})(S \times \hat{U}) \rightarrow \mathcal{C}^\infty(\hat{S} \cap \hat{U})$$

$$\tau \mapsto -\int_0^{\sqrt{p^2+q^2}} \phi(u, w_1, w_2) \, du$$

is surjective: if $f \in \mathcal{C}^\infty(\hat{S} \cap \hat{U})$ and $s$ is a section of the Poincaré bundle over $S \times \hat{S}$, then the 1-form $\tau = \phi \, d\xi$ defined by

$$\phi(\xi, w_1, w_2) = \beta s(\xi, w_1) \, f(w_2),$$

with $1/\beta = -\int_0^{\sqrt{p^2+q^2}} s(u, 0) \, du$, satisfies $\varpi(\tau) = f$ and the correct automorphy condition. Thus, $H^1(S \times \hat{U}, \ker\nabla_1^L) = \mathcal{C}^\infty(\hat{S} \cap \hat{U})$.

The transformed sheaf is endowed with a flat connection induced by $\nabla_2^L$, the $(0, 1)$ part of the connection $p^*_S\nabla^L \otimes 1 + 1 \otimes \nabla_{p_S}$, because the anticommutator between $\nabla_1^L$ and $\nabla_2^L$ equals the curvature of the Poincaré bundle, which restricts to zero on $S \times \hat{S}$ (cf. Proposition 2.2).

Of course, $R^i\hat{p}_{S,*}\ker\nabla_1^L = 0$ for $i > 1$ because $S$ is 1-dimensional. This proof is extended to the case $\dim S > 1$ by using a Künneth formula.

### Appendix B

Here we prove Proposition 3.11. For notational convenience we suppose that $k \leq g/2$; the complementary case $k > g/2$ can be treated similarly. In the action-angle coordinates $x, y$ we can write the local equations for $S$ as

$$\left\{ \begin{array}{ll}
y_{g-k+j} = \eta_{g-k+j}(x^1, \ldots, x^k, y_1, \ldots, y_{g-k}), & j = 1, \ldots, k \\
x^{k+i} = \zeta^{k+i}(x^1, \ldots, x^k), & i = 1, \ldots, g-k \end{array} \right. \quad (10)$$
Since $S$ is Lagrangian one has
\begin{equation}
\left\{ \begin{aligned}
\delta_j^m + \sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \eta^\ell}{\partial y_m} &= 0, & j, m = 1, \ldots, k \\
\frac{\partial \zeta^{g+i}}{\partial x^m} + \sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \eta^\ell}{\partial y_{g+i}} &= 0, & i = 1, \ldots, g - 2k; m = 1, \ldots, k; \\
\sum_{\ell=g-k+1}^{g} \left[ \frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \eta^\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \eta^\ell}{\partial x^j} \right] &= 0, & 1 \leq j < m \leq k.
\end{aligned} \right. \tag{11}
\end{equation}

The equations of the subtori $S_b$ can be written in a linear form
\begin{equation}
y_{g-k+j} = \sum_{m=1}^{g-k} a^m_{g-k+j}(x^1, \ldots, x^k) y_m + \chi_{g-k+j}(x^1, \ldots, x^k), \quad j = 1, \ldots, k. \tag{12}
\end{equation}

To find the equations of $\tilde{S}$ we shall perform a fibrewise transform and use the Künneth formula as in Proposition 2.8. First we split every subtorus $S_b$ as a product of 1-dimensional tori $r_i(b)$ which have linear equations given by
\begin{equation}
\left\{ \begin{aligned}
y_l &= 0, & l = 1, \ldots, g - k, \ell \neq i; \\
y_{g-k+j} &= a^i_{g-k+j}(x^1, \ldots, x^k) y_l + \chi_{g-k+j}(x^1, \ldots, x^k), & j = 1, \ldots, k.
\end{aligned} \right.
\end{equation}
Observe that we can also split the local system $L$ on $S_b$ as a box product of local systems $L_i(b)$ on $r_i(b)$ where $i = 1, \ldots, g - k$. Transforming the local system $L_i(b)$ on $r_i(b)$ we get the following equation for the support of $L_i(b)$ (see Appendix A):
\begin{equation}
w^i + \sum_{\ell=g-k+1}^{g} \gamma^i_\ell(x^1, \ldots, x^k) w^\ell + \xi^i
\end{equation}
where the constant term $\xi^i$ describes the automorphy of $L_i$ (here $i$ is fixed), and the matrix $\gamma^i_\ell$ satisfies the condition $\sum_{j=1}^{g} \gamma^i_\ell a^j_\ell = 0$. Then $\tilde{S}$ is the intersection of the supports $\tilde{r}_i$, so that its equations are of the form
\begin{equation}
w^{k+i} = \sum_{j=1}^{k} \gamma^{k+i}_j(x^1, \ldots, x^k) w^j + \zeta^{k+i}(x^1, \ldots, x^k), \quad i = 1, \ldots, g - k \tag{13}
\end{equation}

\begin{equation}
\text{together with the second set of equations (10). Here we have solved with respect to} \ w^1, \ldots, w^k. \text{ These equations may be used to replace the functions $\eta$ in (10), thus getting}
\end{equation}
\begin{equation}
\left\{ \begin{aligned}
\delta_j^m + \sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^j} a^m_\ell &= 0, & j, m = 1, \ldots, k \\
\frac{\partial \zeta^{g+i}}{\partial x^m} + \sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^m} a^{g+i}_\ell &= 0, & i = 1, \ldots, g - 2k; m = 1, \ldots, k.
\end{aligned} \right. \tag{14}
\end{equation}
\[
\sum_{\ell=g-k+1}^{g} \left[ \frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \chi^\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \chi^\ell}{\partial x^i} \right] = 0, \quad 1 \leq j < m \leq k. \tag{15}
\]

The solution of (14) is
\[
\frac{\partial \zeta^{k+i}}{\partial x^j} = \tilde{\gamma}_j^{k+i}, \quad j = 1, \ldots, k, \quad i = 1, \ldots, g - k. \tag{16}
\]

If the submanifold \( S \) is Lagrangian, the conditions (16) admit solutions in \( \zeta \). We must check that the support \( \hat{S} \) is holomorphic, i.e., the equations that define it fulfil the Cauchy-Riemann conditions. The latters are satisfied if and only if the coefficients \( \tilde{\gamma}_j^{k+i} \) do not depend on the \( x \)'s, but this is true if and only if the coefficients \( \gamma_j^{k+i} \) are in turn independent of the \( x \)'s. As a result, we have proved that when \( S \) is Lagrangian, the tangent spaces to the \( S_b \)'s are parallelly transported by \( \nabla_{GM} \) if and only if \( \hat{S} \) is holomorphic.

One may note that the coefficients \( \chi^j \) play no role in the specification of the complex structure of \( \hat{S} \). Moreover, let us remark that equation (13) shows that the intersections of the support \( \hat{S} \) with the fibres \( \hat{X}_b \) are affine subtori.

References

[1] B. Andreas, G. Curio, D. Hernández Ruipérez, S.-T. Yau, Fourier-Mukai transform and mirror symmetry for D-branes on elliptic Calabi-Yau, \texttt{math.AG/0012196}.

[2] D. Arinkin, A. Polishchuk, Fukaya category and Fourier transform, \texttt{math.AG/9811023}.

[3] P. Aspinwall, R. Donagi, The heterotic string, the tangent bundle, and derived categories, Adv. Theor. Math. Phys. 2 (1998), 1041–1074.

[4] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, J.M. Muñoz Porras, Mirror symmetry on K3 surfaces via Fourier-Mukai transform, Commun. Math. Phys. 195 (1998), 79–93.

[5] ———, Relatively stable bundles on elliptic surfaces, Preprint (1999).

[6] R. Bott, L.W. Tu, Differential forms in algebraic topology, Springer-Verlag, New York-Berlin (1982).

[7] T. Bridgeland, Fourier-Mukai transforms for elliptic surfaces, J. reine angew. Math. 498 (1998), 115–133.
[8] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes Math. 163, Springer-Verlag, Berlin-New York (1970).

[9] M.R. Douglas, G. Moore, *D-branes, quivers, and ALE instantons*, hep-th/9603167.

[10] J.J. Duistermaat, *On global action-angle coordinates*, Commun. Pure Appl. Math. 33 (1980), 687-706.

[11] K. Fukaya, *Floer homology and mirror symmetry I*, Preprint (1999). Available from the web page http://www.kusm.kyoto-u.ac.jp/~fukaya/mirror1.pdf

[12] ———, *Mirror symmetry of Abelian varieties and multi-theta functions*, Preprint (2000). Available from the web page http://www.kusm.kyoto-u.ac.jp/~fukaya/abelrev.pdf

[13] S.I. Gel'fand, Yu.I. Manin, *Methods of homological algebras*, Springer-Verlag, Berlin 1991.

[14] M. Gross, *Topological mirror symmetry*, math.AG/9909015.

[15] R. Harvey, H.B. Lawson Jr., *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[16] D. Hernández Ruipérez, J.M. Muñoz Porras, *Structure of the moduli space of stable sheaves on elliptic surfaces*, math.AG/9809019.

[17] H. Lange, Ch. Birkenhake, *Complex abelian varieties*, Springer-Verlag, Berlin (1992).

[18] G. Laumon, *Transformation de Fourier généralisée*, alg-geom/9603004.

[19] N.C. Leung, S.-T. Yau, E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, math.DG/0005113.

[20] G. Marelli, PhD thesis, International School for Advanced Studies, Trieste (2001).

[21] S. Mukai, *Semi-homogeneous vector bundles on an abelian variety*, J. Math. Kyoto Univ. 18 (1978), 239–272.

[22] ———, *Duality between D(X) and D(ŠX) with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153–175.

[23] A. Nakayashiki, *Structure of Baker-Akhiezer modules of principally polarized Abelian varieties, commuting partial differential operators and associated integrable systems*, Duke Math. J. 62 (1991), 315–358.
[24] M.J. Rothstein, *Sheaves with connection on abelian varieties*, Duke Math. J. **84** (1996), 565–598.

[25] A. Strominger, S.-T.Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nucl. Phys. **B479** (1996), 243–259.