INTEGRATED DENSITY OF STATES FOR RANDOM METRICS ON MANIFOLDS

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Abstract. We study ergodic random Schrödinger operators on a covering manifold, where the randomness enters both via the potential and the metric. We prove measurability of the random operators, almost sure constancy of their spectral properties, the existence of a selfaveraging integrated density of states and a Šubin type trace formula.

1. Introduction

The mathematically rigorous study of random Schrödinger operators commenced in the 70ties. The motivation was to understand the transport properties of random media. Since then a variety of results on the spectral, wave-spreading and conductance properties of random Schrödinger operators have been derived on mathematical grounds. We refer to the textbook accounts [CFKS87, Kir89, CL90, PF92, Sto01] and the references cited therein.

This paper carries over the fundamental properties of random Schrödinger operators to random Laplace-Beltrami operators, i.e. Laplacians with random metrics. Namely, we
(A) discuss a framework for random operators on manifolds with randomness entering both via potential and metrics,
(B) show measurability of the introduced operators, which implies, in particular, almost sure constancy of their spectral features,
(C) prove existence and selfaveraging property of the integrated density of states together with a Šubin type trace formula.

Thereby we extend and apply the earlier [PV02, LPVb]. The main result of this paper is result (C) concerning the integrated density of states (IDS). Physically, the integrated density of states measures the number of electron energy levels per unit volume up to a given energy value. It can be obtained by a macroscopic limit, where ergodicity of the family of operators yields the selfaveraging nature, i.e. the non-randomness, of this quantity. It is sometimes called spectral density function.

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Let us put these results in perspective. Probably the most prominent success of the theory of random Schrödinger operators is the proof of localization. This phenomenon has been explained on physical grounds by Anderson [And58], but only in the late 70ties first rigorous results were established, see the original papers [GMP77, FS83, AM93] or [Sto01] for a monograph exposition.

In [Dav90] Davies studies among others the relation of heat kernels on a manifold associated to different metrics. In this context he raises the question of localization due to random metrics. This should be analogous to the phenomena occurring in quantum wave guides [KS00].

In comparison to localization the study of the integrated density of states undertaken in this paper is a physically more basic and technically less involved question. Still this quantity comprises many important spectral features of the random Schrödinger operator and its understanding can be seen as a first step towards the proof of localization. Namely, the multiscale proof of localization of Fröhlich and Spencer [FS83] derived for (specific models of) random Schrödinger operators in Euclidean space relies on the continuity and asymptotic properties of the integrated density of states. These were first studied by Wegner [Weg81], respectively by Lifshitz [Lif64]. In a forthcoming paper [LPPV], we derive results on the (dis)continuity of the IDS for periodic and random operators on manifolds. There, we furthermore discuss some similarities and differences between random Laplace-Beltrami operators and divergence type operators. For the time being, let us only emphasize that Euclidean random divergence type operators do not cover our models, due to the more general geometry and underlying group structure we consider.

The paper is organized as follows. In the next section we introduce our model and state the main results. In Section 3 we introduce quadratic forms and derive the measurability of the quantities we are considering, thereby giving a precise form to (B) above. Section 4 is devoted to general results on random operators which are proven in an abstract setting in [LPVb]. This presents our treatment of (A) above. A discussion of heat kernels on manifolds is given in Section 5, specializing to the principle of not feeling the boundary in Section 6. We derive uniform bounds for the kernels of the semigroups of a random family of Schrödinger operators acting on a manifold and including singular nonnegative potentials. Using these results, we then prove our main result concerning (C) in Section 7.

2. Model and results

In this section we state the main results about the existence and nonrandomness of the integrated density of states. Beforehand we explain the geometric setting we are working in and the properties of the random Schrödinger operator.
Consider a complete Riemannian manifold $X$ of dimension $n$ with metric $g_0$ and associated volume form $\text{vol}_0$. Let $\Gamma$ be a discrete infinite subgroup of the isometries of $(X, g_0)$, acting cocompactly, freely and properly discontinuously on $X$. Consequently, $M = X/\Gamma$ is a compact Riemannian manifold. Furthermore, let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which $\Gamma$ acts ergodically by measure preserving transformations.

**Definition 2.1.** A family of Riemannian metrics $\{g_\omega\}_{\omega \in \Omega}$ on $X$ with corresponding volume forms $\text{vol}_\omega$ is called a random metric on $(X, g_0)$ if the following properties are satisfied:

(M1) The map $\Omega \times TM \to \mathbb{R}$, $(\omega, v) \mapsto g_\omega(v, v)$ is jointly measurable.

(M2) There is a $C_g \in ]0, \infty[$ such that

$$C_g^{-1}g_0(v, v) \leq g_\omega(v, v) \leq C_gg_0(v, v) \quad \text{for all } v \in TX.$$

(M3) There is a $C_\rho > 0$ such that

$$|\nabla_0 \rho_\omega(x)|_0 \leq C_\rho \quad \text{for all } x \in X,$$

where $\nabla_0$ denotes the gradient w.r.t $g_0$, $\rho_\omega$ is the unique smooth density satisfying $d\text{vol}_0 = \rho_\omega d\text{vol}_\omega$, and $|v|^2_0 = g_0(v, v)$.

(M4) There is a uniform lower bound $K \in \mathbb{R}$ for the Ricci curvatures of all Riemannian manifolds $(X, g_\omega)$.

(M5) The metrics are compatible in the sense that the deck transformations

$$\gamma: (X, g_\omega) \to (X, g_{\gamma^{-1}\omega}), \quad \gamma: x \mapsto \gamma x$$

are isometries.

(M5) implies that, in particular, the induced maps $U_{(\omega, \gamma)}: L^2(X, \text{vol}_{\gamma^{-1}\omega}) \to L^2(X, \text{vol}_{\omega})$, $(U_{(\omega, \gamma)}f)(x) = f(\gamma^{-1}x)$ are unitary operators.

Based on this geometric setting, we consider a family of Schrödinger operators. These operators are defined via quadratic forms, as explained in Section 3.

**Definition 2.2.** Let $\{g_\omega\}$ be a random metric on $(X, g_0)$. For each $\omega \in \Omega$ let $H_\omega = \Delta_\omega + V_\omega$ be a Schrödinger operator defined on the Hilbert space $L^2(X, \text{vol}_\omega)$. $\{H_\omega\}_{\omega \in \Omega}$ is called a random (Schrödinger) operator if it satisfies the following equivariance condition

$$(1) \quad H_\omega = U_{(\omega, \gamma)}H_{\gamma^{-1}\omega}U^*_{(\omega, \gamma)},$$

for all $\gamma \in \Gamma$ and $\omega \in \Omega$, and if the potential $V: \Omega \times X \to \mathbb{R}$ is jointly measurable, nonnegative and $V_\omega = V(\omega, \cdot) \in L^1_{\text{loc}}(X)$, for all $\omega \in \Omega$.

For technical reasons we require that the $\sigma$-algebra $\mathcal{B}_\Omega$ is countably generated. This can always be established by changing to an equivalent version of the defining stochastic processes given by the random potential and the random metric. This has been done for the potential explicitly in Remark 2.8 of [LPVb].
In Section 3 we extend the standard notion of measurability for a family of operators acting on a fixed Hilbert space [KMS82] to operators acting on varying Hilbert spaces. This leads to the fundamental

**Theorem 1.** A random operator \( \{H_\omega\}_{\omega \in \Omega} \) is a measurable family of operators.

From this theorem and the results of [LPVb] we immediately obtain the following result. (Note that \( \sigma_{pp} \) denotes the closure of the set of eigenvalues.)

**Theorem 2.** There exist \( \Omega' \subset \Omega \) of full measure and \( \Sigma, \Sigma_\bullet \subset \mathbb{R} \), such that \( \sigma(H_\omega) = \Sigma, \sigma_\bullet(H_\omega) = \Sigma_\bullet \) for all \( \omega \in \Omega' \) where \( \bullet = \text{disc, ess, ac, sc, pp} \). Moreover, \( \Sigma_{\text{disc}} = \emptyset \).

The above two theorems and the framework underlying their proofs complete our investigation of (A) and (B) of the introduction.

Next, we introduce the (abstract) density of states for a random operator \( \{H_\omega\} \) as the measure on \( \mathbb{R} \), given by

\[
(2) \quad \rho_H(f) := \frac{\mathbb{E}[\text{tr}(\chi_F f(H_\bullet))]}{\mathbb{E}[\text{vol}_\bullet(F)]}, \quad f \text{ bounded, measurable.}
\]

Here \( F \subset X \) is a precompact \( \Gamma \)-fundamental domain with piecewise smooth boundary, \( \mathbb{E} \) denotes the expectation with respect to \( \mathbb{P} \) and \( \text{tr} = \text{tr}_\omega \) is the trace on the Hilbert space \( L^2(X, \text{vol}_\omega) \), where we suppress the index \( \omega \) in the following. The expression (2) is closely related to a trace \( \tau \) of a von Neumann algebra, as discussed in Section 4 and summarized in

**Theorem 3.** \( \rho_H \) is a spectral measure for the direct integral operator

\[
H := \int_{\Omega} \oplus H_\omega \, d\mathbb{P}(\omega)
\]

and \( \rho_H(f) \) and \( \tau(f(H)) \) coincide for arbitrary bounded measurable \( f \) on \( \mathbb{R} \), up to a fixed constant factor. In particular, the almost sure spectrum \( \Sigma \) coincides with the topological support \( \{\lambda \in \mathbb{R} : \rho([\lambda-\epsilon, \lambda+\epsilon]) > 0 \text{ for all } \epsilon > 0\} \) of \( \rho_H \).

Recall that a measure \( \phi \) on \( \mathbb{R} \) is a spectral measure for selfadjoint operator \( H \) with spectral family \( E_H \) if, for Borel measurable \( B \subset \mathbb{R} \), \( \phi(B) = 0 \iff E_H(B) = 0 \).

To state our main result (see (C) of the introduction), it is indispensable to assume that the underlying discrete group \( \Gamma \) is amenable. We introduce restrictions of operators on \( X \) to open sets \( D \subset X \) with \( \text{vol}_\omega(D) < \infty \). The restriction of \( H_\omega \) to \( D \) with Dirichlet boundary conditions (b.c.) will be denoted by \( H^D_\omega \). The restriction \( H^D_\omega \) is again selfadjoint, bounded below and has purely discrete spectrum. Therefore, we may enumerate its eigenvalues in increasing order, counting multiplicities: \( \lambda_1(H^D_\omega) \leq \lambda_2(H^D_\omega) \leq \ldots \lambda_i(H^D_\omega) \to \infty \). We define the normalized eigenvalue counting function as

\[
(3) \quad N^D_\omega(\lambda) = \frac{\# \{i \mid \lambda_i(H^D_\omega) < \lambda\}}{\text{vol}_\omega(D)}.
\]
$N^D_\omega$ is a distribution function and has countably many discontinuity points.

Amenability of $\Gamma$ guarantees the existence of an exhaustion of $X$ by open sets $\{D^j\}_j$ with very strong additional properties, see Section 7. Such an exhaustion $\{D^j\}_j$ is called an admissible sequence of subsets of $X$. For the associated restricted operators we use the shorthand $H_\omega^j = H_\omega^{D^j}$ and, similarly, $N^j_\omega = N^j_\omega^{D^j}$. Our main result establishes the selfaveraging property of the IDS and expresses it by a Šubin type trace formula [Šub79, Šub82]:

**Theorem 4.** Let $\{D^j\}_j$ be an admissible sequence and $\{H_\omega\}_\omega$ be as above. There exists a set $\Omega'$ of full measure such that

$$\lim_{j \to \infty} N^j_\omega(\lambda) = \rho_H([ - \infty, \lambda[),$$

for every $\omega \in \Omega'$ and every point $\lambda \in \mathbb{R}$ with $\rho_H(\{\lambda\}) = 0$.

The distribution function of $\rho_H$ is denoted by $N_H$ and is called the integrated density of states (IDS) of the random operator $\{H_\omega\}$, i.e.,

$$N_H(\lambda) = \rho_H([ - \infty, \lambda[).$$

Since $N_H$ can be obtained by an exhaustion procedure $D^j \to X$ without integrating over $\Omega$ explicitly, it is called selfaveraging. The proof of Theorem 4 in Section 7 actually also establishes the following result. For a set $D^j \subset X$ in an admissible sequence denote

$$N^j_\omega(\lambda) := \frac{\text{tr} \left( \chi_{D^j} E_\omega(\lambda) \right)}{\text{vol}_\omega(D^j)}.$$  

(4)

Here, the superscript $f$ stands for the fact that this finite volume IDS is defined without the use (i.e. free) of boundary conditions.

**Corollary 2.3.** For almost every $\omega \in \Omega$ the convergence $\lim_{j \to \infty} N^j_\omega(\lambda) = N_H(\lambda)$ holds at every continuity point $\lambda$ of $N_H$.

This means that in the macroscopic limit $D^j \to X$ it is not felt whether the restriction of the operator in space, or the projection on an energy interval took place first.

For simplicity, we have so far assumed the potentials $V$ to be nonnegative. It suffices to assume that the $V_\omega$ are uniformly bounded below by a constant $C$ not depending on $\omega \in \Omega$. Then our results apply to the shifted operator family $\{H_\omega - C\}_{\omega \in \Omega}$. Since $N_H-C(\lambda-C) = N_H(\lambda)$ for all $\lambda \in \mathbb{R}$, and similarly for the normalized eigenvalue counting functions, the results carry over to the original operators.

3. Quadratic forms and measurability

In this section we give a precise definition of the operators we are dealing with and show their measurability.

To introduce our operators, we will use quadratic forms. The relevant theory can be found, e.g., in the first two sections of [Dav89]. It is developed there for $X = \mathbb{R}^n$ but carries directly over to arbitrary manifolds $X$. 
We abbreviate the scalar product in the tangent space by
\[ (v, w)_\omega := g_\omega(x)(v, w) \text{ for all } v, w \in T_x X. \]
For \( D \subset X \) open and each \( \omega \in \Omega \) we define the quadratic forms
\[
\tilde{Q}(\omega^D) : C^\infty_c(D) \times C^\infty_c(D) \to \mathbb{R}, \quad (f, h) \mapsto \int_D \langle \nabla f(x), \nabla h(x) \rangle_\omega d\vol_\omega(x)
\]
and
\[
\tilde{Q}(V^D_\omega) : C^\infty_c(D) \times C^\infty_c(D) \to \mathbb{R}, \quad (f, h) \mapsto \int_D f(x)V_\omega(x)h(x)d\vol_\omega(x).
\]
These forms are closable and their closures \( Q(\omega^D) \) and \( Q(V^D_\omega) \), respectively, give rise to selfadjoint nonnegative operators \( \omega^D \) and \( V_\omega \). Next, consider the form
\[
\tilde{Q}(H^D_\omega) : C^\infty_c(D) \times C^\infty_c(D) \to \mathbb{R}, \quad (f, h) \mapsto Q(\omega^D)(f, h) + Q(V^D_\omega)(f, h).
\]
This form is closable, the closure \( Q(H^D_\omega) \) is the form sum of \( Q(\omega^D) \) and \( Q(V^D_\omega) \), and \( Q(H^D_\omega) \) induces, again, a selfadjoint operator (see [Dav89, Thm. 1.8.1]). Of course, for smooth \( V \) and \( f \in C^\infty_c(D) \) we have \( H^D_\omega f(x) = \omega^D f(x) + V(x)f(x) \). The form \( Q(H^D_\omega) \) is a Dirichlet form (see [Dav89, Thm. 1.3.5]), i.e., it satisfies
\[
\exp(-tH_\omega) : L^\infty(D, \vol_\omega) \to L^\infty(D, \vol_\omega) \text{ is a contraction for every } t > 0
\]
and
\[
\exp(-tH_\omega) : L^2(D, \vol_\omega) \to L^2(D, \vol_\omega) \text{ is positivity preserving for every } t > 0.
\]
Semigroups \( e^{-tH} \) associated to Dirichlet forms are called symmetric Markov semigroups.

There exist positive, smooth functions \( \rho_\omega \in C^\infty(X) \) such that
\[
\int_X f(x)d\vol_0(x) = \int_X f(x)\rho_\omega(x)d\vol_\omega(x).
\]
More explicitly, \( \rho_\omega(x) \) is given by
\[
\rho_\omega(x) = \left( \det g_0(e^\omega_i, e^\omega_j) \right)^{1/2} = \left( \det g_\omega(e^0_i, e^0_j) \right)^{-1/2},
\]
where \( e^0_1, \ldots, e^0_d \in T_x X \) is any base of \( T_x X \) orthonormal w.r.t. \( g_0 \) and \( e^\omega_1, \ldots, e^\omega_d \in T_x X \) is any base orthonormal w.r.t. \( g_\omega \). Consequently, the operators
\[
S_\omega : L^2(D, \vol_0) \to L^2(D, \vol_\omega), \quad S_\omega(f) = \rho_\omega^{1/2}f
\]
are unitary. The \( L^2 \)-products on \( L^2(D, \vol_0) \) and on \( L^2(D, \vol_\omega) \) are denoted by \( \langle \cdot, \cdot \rangle_0 \) and \( \langle \cdot, \cdot \rangle_\omega \), respectively. The corresponding norms are denoted by \( \| \cdot \|_0 \) and \( \| \cdot \|_\omega \).

It follows from property (M2) of Definition 2.1 that
\[
(5) \quad C_g^{-n/2} \leq \rho_\omega(x) \leq C^g_{m/2} \text{ for all } x \in D, \omega \in \Omega.
\]
Now we introduce the notion of *measurability* of a family of selfadjoint operators, indexed by the elements of $\Omega$. It is a modification of the definition from [KM82] to operators with varying domains of definition.

**Definition 3.1.** A family of selfadjoint operators $\{H_\omega\}_\omega$, where the domain of $H_\omega$ is a dense subspace $D_\omega$ of $L^2(D, \text{vol}_\omega)$, is called a measurable family of operators if

$$\omega \mapsto (f_\omega, F(H_\omega)f_\omega)_\omega$$

is measurable for all bounded, measurable functions $F: \mathbb{R} \to \mathbb{C}$ and all $f: \Omega \times D \to \mathbb{R}$ measurable with $f_\omega \in L^2(D, \text{vol}_\omega)$, $f_\omega(x) = f(\omega, x)$, for every $\omega \in \Omega$.

**Remark 3.2.** In our setting, due to (M2), the above definition can be slightly simplified. Namely, a family of operators $\{H_\omega\}_\omega$ is measurable if and only if

$$\omega \mapsto (f, F(H_\omega)f)_\omega$$

is measurable for all $F: \mathbb{R} \to \mathbb{C}$, $F \in L^\infty$ and all $f \in L^2(D, \text{vol}_\omega)$. (Note that, due to (M2), $L^2(D, \text{vol}_\omega)$ and $L^2(D, \text{vol}_\omega)$ coincide for all $\omega \in \Omega$ as sets, though not in their scalar product.)

To see this, note that (7) implies the same statement for $f(\omega, x)$ replaced by $h(\omega, x) = g(\omega)f(x)$ where $g \in L^2(\Omega)$ and $f \in L^2(D, \text{vol}_\omega)$. Such functions form a total set in $L^2(\Omega \times D, \mathbb{P} \circ \text{vol})$.

Now, consider a measurable $h: \Omega \times D \to \mathbb{R}$ such that $h_\omega := h(\omega, \cdot) \in L^2(D, \text{vol}_\omega)$ for every $\omega \in \Omega$. Then $h^n(\omega, x) := \chi_{h,n}(\omega)h(\omega, x)$ is in $L^2(\Omega \times D, \mathbb{P} \circ \text{vol})$ where $\chi_{h,n}$ denotes the characteristic function of the set $\{\omega | \|h_\omega\|_{L^2(D, \text{vol}_\omega)} \leq n\}$. Since $\chi_{h,n} \to 1$ pointwise on $\Omega$ for $n \to \infty$ we obtain

$$(h^n_\omega, F(H_\omega)h^n_\omega)_\omega \to (h_\omega, F(H_\omega)h_\omega)_\omega$$

which shows that $\{H_\omega\}_\omega$ is a measurable family of operators.

The following proposition (and its proof) is a variant of Proposition 3 in [KM82]. It suits our purposes and shows that our notion of measurability is compatible with theirs: Let $\{A_\omega\}_\omega$ be a family of densely defined nonnegative selfadjoint operators on a fixed Hilbert space $\mathcal{H}$. Denote by $\hat{\Sigma} = \bigcup_{\omega} \sigma(A_\omega)$ the closure of all spectra and by $F_i$ the following classes of functions: $F_1 = \{\chi_{-\infty, \lambda}[\lambda \geq 0]\}$, $F_2 = \{x \mapsto e^{itx} | t \in \mathbb{R}\}$, $F_3 = \{x \mapsto e^{-itx} | t \geq 0\}$, $F_4 = \{x \mapsto (z - x)^{-1} | z \in \mathbb{C} \setminus \hat{\Sigma}\}$, $F_5 = F_4(z_0) = \{x \mapsto (z_0 - x)^{-1}\}$ for a fixed $z_0 \in \mathbb{C} \setminus \hat{\Sigma}$, $F_6 = C_b = \{f: \mathbb{R} \to \mathbb{C} | f \text{ bounded, continuous}\}$, and $F_7 = L^\infty = \{f: \mathbb{R} \to \mathbb{C} | f \text{ bounded, measurable}\}$.

**Proposition 3.3.** The following properties are equivalent:

(F$_i$) $\omega \mapsto (f, F(A_\omega)h)_\mathcal{H}$ is measurable for all $f, h \in \mathcal{H}$ and $F \in F_i$, where $i = 1, \ldots, 7$. 


Proof. For the equivalence of \((F_4)\) and \((F_5)\) we assume \(d(z_0, \tilde{\Sigma}) = \delta\) and that \((z_0 - H_\omega)^{-1}\) is weakly measurable. Using a Neumann series expansion as in [RS80, Theorem VI.5] one infers the weak measurability of \((z - H_\omega)^{-1}\) for all \(z\) with \(d(z, z_0) < \delta\). Iterating this argument, we obtain measurability of \((z - H_\omega)^{-1}\) for all \(z \in \mathbb{C}\setminus \tilde{\Sigma}\).

Now, by the Stone/Weierstrass theorem we obtain the equivalence of \((F_2), (F_3), (F_4), (F_5), (F_6)\).

The equivalence of \((F_1)\) and \((F_7)\) follows by monotone class arguments.

As \((F_7) \Rightarrow (F_6)\) is clear, it only remains to prove \((F_6) \Rightarrow (F_1)\). This is immediate as every characteristic function \(\chi_{[-\infty, \infty]}\) is a pointwise monotone limit of continuous functions.

We prove now that the random operator \(\{H_\omega\}\) introduced in Section 2 is measurable in the sense of Definition 3.1. The first step in the proof is to pull all operators \(\{H_\omega\}\) on the same Hilbert space by the unitary transformation \(S_\omega\) and to show the following comparability property of the associated quadratic forms:

**Proposition 3.4.** Let the selfadjoint operators

\[ A_\omega: (S_\omega)^{-1}D(\Delta_0^D) \subset L^2(D, \text{vol}_0) \rightarrow L^2(D, \text{vol}_0) \]

be defined by \(A_\omega := (S_\omega)^{-1}\Delta_0^D S_\omega\). Let \(Q_0, Q_\omega\) be the quadratic forms associated to the operators \(\Delta_0^D\) and \(A_\omega\). Then there is a constant \(C_A\) such that

\[ C_A^{-1} (Q_0(f, f) + \|f\|_0^2) \leq Q_\omega(f, f) + \|f\|_0^2 \leq C_A (Q_0(f, f) + \|f\|_0^2). \]

for all \(f \in C^\infty_c(D)\) and \(\omega \in \Omega\). Moreover, there exists a dense subspace \(D \subset L^2(D, \text{vol}_0)\) with \(D = D(A_\omega^{1/2}) = D((\Delta_0^D)^{1/2})\) for every \(\omega \in \Omega\) and \(8)\) holds for every \(f \in D\).

Proof. Direct calculation for \(f \in C^\infty_c(D)\) shows

\[ Q_\omega(f, f) = (S_\omega f, \Delta_\omega S_\omega f)_\omega \leq 2 \left( \|\rho_\omega^{1/2} \nabla_\omega f\|_0^2 + \|f \nabla_\omega \rho_\omega^{1/2}\|_0^2 \right). \]

To bound \(\|\rho_\omega^{1/2} \nabla_\omega f\|_\omega\) we consider the \(n \times n\)-matrix \(A = (a_{ij})\) defined by \(e_i^0 = \sum_{j=1}^n a_{ij} e_j^0\). A calculation using (M2) in Definition 2.1 shows \(C_g^{-1} \leq A A^\top \leq C_g\). This implies

\[ C_g^{-1}\|\nabla_\omega f(x)\|_\omega^2 \leq |\nabla_0 f(x)|_0^2 \leq C_g \|\nabla_\omega f(x)\|_\omega^2. \]

and thus

\[ \|\rho_\omega^{1/2} \nabla_\omega f\|_\omega^2 = \int_D |\nabla_\omega f(x)|_\omega^2 d\text{vol}_0(x) \leq C_g \int_D |\nabla_0 f(x)|_0^2 d\text{vol}_0(x) = C_g Q_0(f, f). \]

To estimate \(\|f \nabla_\omega \rho_\omega^{1/2}\|_\omega^2\) we use (5), (10) and (M3) of Definition 2.1 to calculate

\[ \|f \nabla_\omega \rho_\omega^{1/2}\|_\omega^2 \leq C_{1+n/2} \int_D |\nabla_0 \rho_\omega(x)|_0^2 |f(x)|^2 d\text{vol}_\omega(x) \leq C_{1+n} C_\rho^2 \|f\|_0^2. \]
By symmetry, there is also an estimate of the form

\[ C^{-1}_A (Q_0(f,f) + \|f\|_0^2) \leq Q_\omega(f,f) + \|f\|_\omega^2, \]

for all \( f \in C^\infty_c(D) \), and the first statement is proven. The statement follows now from (8), as \( C^\infty_c(D) \) is a core for \( Q_0 \) and \( Q_\omega \).

\[ \square \]

**Proposition 3.5** (see Prop. 1.2.6. in [Sto01]). Let \( Q_\omega, \omega \in \Omega \) and \( Q_0 \) be nonnegative closed quadratic forms with the following properties:

(P1) \( Q_\omega, \omega \in \Omega \) and \( Q_0 \) are defined on the same dense subset \( D \) of a fixed Hilbert space \( \mathcal{H} \).

(P2) There is a fixed constant \( C > 0 \) such that

\[ C^{-1} (Q_0(f,f) + \|f\|_0^2) \leq Q_\omega(f,f) + \|f\|_\omega^2 \leq C (Q_0(f,f) + \|f\|_0^2). \]

(P3) The map \( \omega \mapsto Q_\omega(f,f) \) is measurable, for every \( f \in D \).

Then the family \( \{H_\omega\}_\omega \) of associated selfadjoint operators satisfies the equivalent properties of Proposition 3.3.

The foregoing propositions allow us to show the following:

**Proposition 3.6.** The family \( \{A_\omega\}_\omega \) of Proposition 3.4 is a measurable family of operators.

*Proof.* Since \( C^\infty_c(D) \) is a core for \( Q_\omega \) for all \( \omega \), the closures of this set with respect to one of the equivalent norms in (8) coincide, which shows assumption (P1) of Proposition 3.5. (P2) is just (8) and (P3) is obvious for \( f \in C^\infty_c(D) \). It then follows by approximation for all \( f \in \mathcal{D} \). \[ \square \]

*Proof of Theorem 1.* For \( n \in \mathbb{N} \) and \( \omega \in \Omega \), define bounded functions \( V^n_\omega : X \to \mathbb{R} \) by \( V^n_\omega(x) := \min\{n, V_\omega(x)\} \). Thus, the operator sum \( A^n_\omega := A_\omega + V^n_\omega \) is well defined, where \( A_\omega \) is as in Proposition 3.4 and \( D = X \). Moreover, by [KM82, Prop. 2.4] and Proposition 3.6, the family of operators \( A^n_\omega \) is measurable. In particular, the corresponding semigroups \( \omega \mapsto \exp(-tA^n_\omega) \), \( t > 0 \), are weakly measurable. Now, obviously, the forms of \( A^n_\omega \) converge monotonously towards the form of \( A^\infty_\omega := A_\omega + V_\omega \). By [Kat80, Thms. VIII.3.13a and IX.2.16], this implies that the semigroups of \( A^n_\omega \) converge weakly towards the semigroup \( \omega \mapsto \exp(-tA^\infty_\omega) \) for \( n \to \infty \), and the measurability of the family \( A^\infty_\omega \) follows. Finally, this implies measurability of the family \( H_\omega \), since \( H_\omega = S_\omega A^\infty_\omega S_\omega^{-1} \) and \( S_\omega \) is multiplication with the measurable function \( (x, \omega) \mapsto \rho_\omega(x) \). \[ \square \]

The same arguments show measurability of the restricted operators \( \{H^D_\omega\}_\omega \).

**4. Abstract spectral properties of random operators**

We saw in the last section that a random operator \( \{H_\omega\}_\omega \) is a measurable family of operators. This enables us to make use of the results derived in [LPVb] for random operators in an abstract setting. The following information can be inferred from the cited source.
Definition 4.1. A family \( \{A_\omega\}_{\omega \in \Omega} \) of bounded operators \( A_\omega : L^2(X, \text{vol}_\omega) \to L^2(X, \text{vol}_\omega) \) is called a bounded random operator if it satisfies:

(i) \( \omega \mapsto \langle g_\omega, A_\omega f_\omega \rangle \) is measurable for arbitrary \( f, g \in L^2(\Omega \times X, \mathbb{P} \circ \text{vol}) \).
(ii) There exists a \( C \geq 0 \) with \( \|A_\omega\| \leq C \) for almost all \( \omega \in \Omega \).
(iii) For all \( \omega \in \Omega, \gamma \in \Gamma \) the equivariance condition \( A_\omega = U_{(\omega, \gamma)} A_{\gamma^{-1} \omega} U^*_{(\omega, \gamma)} \) is satisfied.

Two bounded random operators \( \{A_\omega\}_\omega, \{B_\omega\}_\omega \) are called equivalent, \( \{A_\omega\}_\omega \sim \{B_\omega\}_\omega \), if \( A_\omega = B_\omega \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). Each equivalence class of bounded random operators \( \{A_\omega\}_\omega \) gives rise to a bounded operator \( A \) on \( L^2(\Omega \times X, \mathbb{P} \circ \text{vol}) \) by \( (Af)(\omega, x) := A_\omega f_\omega(x) \), see Appendix A in [LPVb]. This allows us to identify the equivalence class of \( \{A_\omega\}_\omega \) with the bounded operator \( A \).

By (1) and the last section, the resolvents, spectral projections and the semigroup associated to \( \{H_\omega\}_\omega \) are all bounded random operators. Theorem 3.1 in [LPVb] states that the set of bounded random operators forms a von Neumann algebra \( \mathcal{N} \). Choose a measurable \( u : \Omega \times X \to \mathbb{R}^+ \) with \( \sum_{\gamma \in \Gamma} u_{\gamma^{-1} \omega}(\gamma^{-1} x) \equiv 1 \) on \( \Omega \times X \) and define the mapping

\[
\tau(A) := \mathbb{E}[\text{tr}(u \cdot A_\bullet)]
\]

on the set of non-negative operators in \( \mathcal{N} \). This \( \tau \) is independent of \( u \) (chosen as above) and defines a trace on \( \mathcal{N} \) of type \( \Pi_\infty \), which is closely related to the IDS. Namely, the spectral projections \( \{E_\omega(\lambda)\}_\omega \) onto the interval \( ]-\infty, \lambda[ \) of a random operator \( \{H_\omega\}_\omega \) form a bounded random operator. Thus it is an element of \( \mathcal{N} \) and agrees with the spectral projection of \( H := \int_{\Omega} H_\omega \, d\mathbb{P}(\omega) \) onto \( ]-\infty, \lambda[ \). Hence \( \tau(E(\lambda)) \) is well defined and the choice \( u_\omega(x) = \chi_F(x) \) yields the identity \( \tau(E(\lambda)) = \mathbb{E}(\text{vol} \cdot \mathcal{F}) N_H(\lambda) \), where \( \mathcal{F} \) is a fundamental domain as discussed after (2).

Now, Theorems 2 and 3 follow from Sections 4 and 5 of [LPVb].

5. Heat kernels

In this section we investigate existence and properties of the kernels of the semigroups \( \exp(-tH_\omega) \) and \( \exp(-tH_\omega^D) \). It will be of particular importance to us to keep track of the dependence of the estimates both on the potential and the metric, since they vary with the random parameter \( \omega \in \Omega \).

We start with the kernels of the Laplacians \( \Delta_\omega \). Sobolev embedding theorems and spectral calculus directly show that

\[
\exp(-t\Delta_\omega) : L^2(X, \text{vol}_\omega) \to L^\infty(X, \text{vol}_\omega)
\]

is bounded for every \( t > 0 \).

Thus, \( \exp(-t\Delta_\omega) \) is ultracontractive, and, by [Dav89, Lemma 2.1.2], this implies that \( \exp(-t\Delta_\omega) \) has a kernel \( k_{\Delta_\omega} \) with

\[
0 \leq k_{\Delta_\omega}(t, x, y) \leq \|\exp(-t\Delta_\omega)\|_{1, \infty} =: C_t^\omega, \quad \text{for almost all } x, y \in X,
\]

and agrees with the spectral projection of

\[
\mathcal{N} \to \text{tr}(\omega \cdot h_\bullet)
\]
where \( \|A\|_{1,\infty} \) denotes the norm of \( A: L^1 \to L^\infty \). By the Trotter product formula we see that, for \( f \geq 0, f \in L^1(\text{vol}_\omega) \),
\[
0 \leq \exp(-tH_\omega)f(x) \leq \exp(-t\Delta_\omega)f(x) \leq C_t^\omega \|f\|_{L^1}
\]
for almost every \( x \in X \). Thus, \( \exp(-tH_\omega): L^1(\text{vol}_\omega) \to L^\infty(\text{vol}_\omega) \) is also bounded by \( C_t^\omega \) and we have
\[
0 \leq k_\omega(t, x, y) \leq C_t^\omega \text{ for almost every } x, y \in X.
\]
To obtain a better estimate, we show that the \( L^2 \)-kernel of the heat semigroup coincides with the fundamental solution of the heat equation as defined, e.g., in [Dod83] or [Cha84] in the case of the pure Laplacian. This allows us to apply estimates of [LY86] for the fundamental solution. Uniqueness of the fundamental solution and its agreement with the \( L^2 \)-kernel are well-known (see, e.g., [Dod83]). For completeness reasons, we give a short alternative functional analytic proof of this agreement based on a theorem of [Dav89] in the more general case with a smooth potential \( W \).

**Theorem 5.** Let \( \{H_\omega\}_\omega = \Delta_\omega + W_\omega \) be a random operator with smooth potential \( W_\omega \in C^\infty(X) \). Then, the kernel \( k_\omega \) has a nonnegative representative in \( C^\infty([0, \infty) \times X \times X) \). Moreover, we have:

(HE) \( k_\omega \) is a solution of the heat equation: \( (\frac{d}{dt} + \Delta_\omega + W_\omega)k_\omega(t, x, y) = 0 \), where \( \Delta_\omega \) denotes the \( \omega \) operator acting on the variable \( y \).

(W) \( k_\omega(t, x, \cdot) \) converges weakly to the point mass in \( x \):
\[
\int k_\omega(t, x, y)f(y)dy \to f(x), \text{ as } t \to 0, \text{ for every bounded continuous } f \text{ on } X \text{ and every } x \in X.
\]

**Proof.** Mimicking the proof of [Dav89, Thm. 5.2.1], we infer that \( k_\omega \) has a representative in \( C^\infty([0, \infty) \times X \times X) \). Since \( \exp(-tH_\omega) \) is positivity preserving, we conclude that \( k_\omega \geq 0 \). Now, direct calculations show (HE). To show (W), we recall that \( \exp(-tH_\omega) \) is a contraction. Thus, by standard measure theory (see, e.g., [Bau92, Satz 30.8]) it suffices to consider only \( f \in C_c(X) \). We can even further restrict the set of functions to \( f \in C^\infty_c(X) \), since \( C^\infty_c(X) \) is dense in \( C_c(X) \) with respect to the sup-norm. By elliptic regularity and Sobolev embeddings, there exist \( a, b > 0 \) and \( j \in \mathbb{N} \) with
\[
\|\exp(-tH_\omega)f - f\|_\infty \leq a\|H_\omega^j(\exp(-tH_\omega)f - f)\|_2 + b\|\exp(-tH_\omega)f - f\|_2
\]
for every \( t \geq 0 \). By spectral calculus, the right hand side tends to zero as \( t \to 0 \), and the theorem is proven.

To formulate the results of Li and Yau [LY86] which we will be using we denote by \( d_\omega: X \times X \to [0, \infty] \) the Riemannian distance function on \( X \) with respect to \( g_\omega \), and similarly by \( d_0 \) the one with respect to the metric \( g_0 \).

**Proposition 5.1.** For every \( t > 0 \) there exist constants \( C_t > 0, \alpha_t > 0 \) with
\[
(13) \quad k_{\Delta_\omega}(t, x, y) \leq C_t \exp\left(-\alpha_t d_\omega^2(x, y)\right)
\]
for all \( \omega \in \Omega \). In particular, the following holds:

(i) \( C_t^\omega \leq C_t \) for every \( \omega \in \Omega \), where \( C_t^\omega \) was defined in (11).
(ii) For all $a > 0$, there exists a $B_{t,a} < \infty$ such that the estimate

$$\int_X k_{\Delta, a}(t, x, y) \, d\text{vol}_\omega(y) \leq B_{t,a}$$

holds uniformly in $x \in X$ and $\omega \in \Omega$. We set $B_t := B_{t,1}$.

Proof. Using [LY86, Cor. 3.1], property (M2) of Definition 2.1 and (5), we obtain the estimate (13) for the fundamental solution of the heat equation without potential. Note that the uniform lower bound $K > -\infty$ for the Ricci curvatures of $(X, g_\omega)$ enters into the constant $C_t$. By Theorem 5 the fundamental solution agrees with the $L^2$-kernel of the semigroup $\exp(-t \Delta_\omega)$. Given this estimate, (i) and (ii) are easy consequences. Note for (ii) that the volume of metric balls of radius $r$ can be estimated (uniformly in $\omega$) from above by $C_1 \exp(C_2 r)$ with fixed constants $C_1, C_2 > 0$, by property (M2), (5), and the Bishop volume comparison theorem (cf. [Cha93, Thm. 3.9]).

Corollary 5.2. For arbitrary open $D$ and $t > 0$ the following holds

$$0 \leq k^{D, \omega}(t, x, y) \leq k_\omega(t, x, y) \leq C_t \exp(-\alpha t d_\omega^2(x, y)),$$

for almost all $x, y \in D$. The constants are as in the previous proposition. In particular, the integral estimate in 5.1(ii) holds also for the perturbed operator, for almost all $x \in X$.

Proof. Using the Feynman-Kac formula, we obtain

$$0 \leq \int_X k_\omega(t, x, y) f(y) \, dy = \mathbb{E}_x(\exp(-\int_0^t V_\omega(X_s) \, ds) f(X_t)) \leq \mathbb{E}_x(f(X_t)) \leq \int_X k_{\Delta, \omega}(t, x, y) f(y) \, dy,$$

for all nonnegative $f \in L^2(X, \text{vol}_\omega)$. Proposition 5.1 implies

$$0 \leq k_\omega(t, x, y) \leq k_{\Delta, \omega}(t, x, y) \leq C_t \exp(-\alpha t d_\omega^2(x, y)),$$

for almost all $x, y \in X$. The inequality for the Dirichlet operator follows by so called domain monotonicity, see e.g. [Dav89, Thm. 2.1.6]. The proof of this theorem carries directly over from $\mathbb{R}^n$ to manifolds.
Finally, note that, for $\text{vol}_\omega(D) < \infty$, the estimate $0 \leq k^D_\omega(t, x, y) \leq C_t$ implies that $\exp(-tH^D_\omega)$ is a Hilbert-Schmidt operator and thus $H^D_\omega$ has purely discrete spectrum by the spectral mapping theorem.

6. THE PRINCIPLE OF NOT FEELING THE BOUNDARY

In this section we show that the semigroups associated to our random operators satisfy a principle of not feeling the boundary.

Let $D$ be a open set on the manifold $X$. One expects the difference between the Dirichlet heat kernel $k^D(t, x, y)$ and $k^H(t, x, y)$ to be small as long as $t > 0$ is small and $x$ and $y$ stays away from the boundary of $D$. This phenomenon is called principle of not feeling the boundary. To treat it rigorously, we introduce the notion of thickened boundary. For $h > 0$, let $\partial_h D := \{x \in X | d_0(x, \partial D) \leq h\}$ and $D_h$ be the interior of the set $D \setminus \partial_h D$.

Note that (M2) of Definition 2.1 implies the following inequalities

$$C^{-1}_g d_0(x, y) \leq d_\omega(x, y) \leq C_g d_0(x, y).$$

The main result of this section is the following:

**Theorem 6.** For all $t, \epsilon > 0$, there exists an $h = h(t, \epsilon) > 0$ such that for every open set $D \subset X$ and all $\omega \in \Omega$, we have

$$0 \leq k_\omega(t, x, y) - k^D_\omega(t, x, y) \leq \epsilon,$$

for almost all $x, y \in D_h$.

The Proof of the Theorem follows from the next two propositions. More precisely, in view of the next proposition, it is enough to prove the theorem for vanishing potential. This, however, is accomplished in Proposition 6.3. Let $\tau^D_x$ denote the first exit time from $D$ for Brownian motion starting in $x$.

**Proposition 6.1.** The following statements are equivalent.

(i) For all $t, \epsilon > 0$, there exists an $h = h(t, \epsilon) > 0$ such that for every open set $D \subset X$ and all $\omega \in \Omega$, we have

$$0 \leq k_\omega(t, x, y) - k^D_\omega(t, x, y) \leq \epsilon,$$

for almost all $x, y \in D_h$.

(ii) For all $t, \epsilon > 0$, there exists an $h = h(t, \epsilon) > 0$ such that for every open set $D \subset X$ and all $\omega \in \Omega$

$$E_x(\chi_{D_h}(X_t)\chi_{\tau^D_x < t}) \leq \epsilon,$$

for almost every $x \in D_h$.

(iii) For every random operator $\{H_\omega\}$ and for all $t, \delta > 0$, there exists an $r = r(t, \epsilon, H) > 0$ such that for every open set $D \subset X$ and all $\omega \in \Omega$, we have

$$0 \leq k_\omega(t, x, y) - k^D_\omega(t, x, y) \leq \delta,$$

for almost all $x, y \in D_r$. 
Proof. (i)⇒(ii). By the Feynman-Kac formula for the unperturbed $\Delta_\omega$-operator, we have

$$E_x(\chi_{D_h}(X_t) \chi_{\{r_D < t\}}) = \int [k_{\Delta_\omega}(t, x, y) - k_{\Delta_D}(t, x, y)] \chi_{D_h}(y) \, dy$$

which can be bounded using the Hölder inequality and domain monotonicity by

$$\text{ess sup}_{y \in D_h} |k_{\Delta_\omega}(t, x, y) - k_{\Delta_D}(t, x, y)| \frac{1}{2} \int |(k_{\Delta_\omega}(t, x, y) - k_{\Delta_D}(t, x, y)) \chi_{D_h}(y)| \, dy$$

$$\leq \text{ess sup}_{y \in D_h} |k_{\Delta_\omega}(t, x, y) - k_{\Delta_D}(t, x, y)| \frac{1}{2} \int k_{\Delta_\omega}(t, x, y) \frac{1}{2} \chi_{D_h}(y) \, dy$$

The first term can be seen to be small by (i) for almost every $x \in D_h$, and the second term is bounded by $B_{t,1/2}$, due to Proposition 5.1 (ii).

(ii)⇒(iii). We have to show that

$$\chi_{D_h}(\exp(-tH_\omega) - \exp(-tH_\omega^D)) \chi_{D_h} : L^1(D_h, \text{vol}_\omega) \to L^\infty(D_h, \text{vol}_\omega)$$

is arbitrarily small for $h$ large enough (independently of $\omega$ and $D$). Let $R := \exp(-\frac{t}{2}H_\omega) - \exp(-\frac{t}{2}H_\omega^D)$. Note that

$$\|e^{-\frac{t}{2}H_\omega^D}\|_{1 \to 2} \leq \sqrt{B_{t/2,2}},$$

by Corollary 5.2. Thus, since

$$\chi_{D_h}(e^{-tH_\omega} - e^{-tH_\omega^D}) \chi_{D_h} = \chi_{D_h} e^{-\frac{t}{2}H_\omega} R \chi_{D_h} + \chi_{D_h} R e^{-\frac{t}{2}H_\omega^D} \chi_{D_h}$$

and by duality

$$\|R \chi_{D_h}\|_{1 \to 2} = \|\chi_{D_h} R\|_{2 \to \infty}$$

it suffices to show that

$$\chi_{D_h}(\exp(-tH_\omega) - \exp(-tH_\omega^D)) \chi_{D_h} : L^2(D_h, \text{vol}_\omega) \to L^\infty(D_h, \text{vol}_\omega)$$

is arbitrarily small for $h$ large enough (independently of $\omega$ and $D$). Using the Feynman-Kac Formula for the perturbed operator, we obtain

$$(\exp(-tH_\omega) - \exp(-tH_\omega^D)) f(x)$$

$$= E_x(\exp(-\int_0^t V_\omega(X_s) \, dX_t) \chi_{D_h}(X_t) f(X_t) \chi_{\{r_D^* < t\}})$$

$$\leq E_x(f(X_t))^2 \frac{1}{2} E_x(\chi_{D_h}(X_t) \chi_{\{r_D^* < t\}})^2$$

$$\leq (\int_X k_\omega(t, x, y)|f(y)|^2 \, dy)^{1/2} E_x(\chi_{D_h}(X_t) \chi_{\{r_D^* < t\}})^{1/2}$$

$$\leq C t^{1/2} \|f\|_2 E_x(\chi_{D_h}(X_t) \chi_{\{r_D^* < t\}})^{1/2}.$$

The proof is finished by invoking (ii).

(iii)⇒(i). This is immediate by choosing $V_\omega \equiv 0$. □
The following lemma is an adaptation of Proposition 1.1 in [Tay96, Chp. 6]. It is useful in our proof of “not feeling the boundary”:

**Lemma 6.2** (Maximum principle for heat equation with nonnegative potential). Let \( D \subset X \) be open with compact closure, \( V \geq 0 \), and \( u \in C([0, T[ X D] \cap C^2([0, T[ X D) \) be a solution of the heat equation \( \frac{\partial}{\partial t} u + (\Delta + V) u = 0 \) on \([0, T[ X D\) with nonnegative supremum \( s = \sup\{u(t, x) \mid (t, x) \in [0, T[ X D\). Then,

\[
s = \max \left\{ \max_{x \in D} u(0, x), \sup_{(0, T[ \times \partial D} u(t, x) \right\}.
\]

**Proof.** It suffices to prove that, for any \( c \geq 0 \), the assumption

\[
u < c \quad \text{on } ((0, T[ X D)) \cup ((0, T[ X \partial D)\]

implies \( u \leq c \) on \([0, T[ X D\). To this aim we introduce the auxiliary function \( u_\delta(t, x) = u(t, x) - \delta t, \delta > 0 \), and show that (15) implies \( u_\delta(t_0, x_0) < c \) on \([0, T[ X D\).

Assume that the conclusion is wrong. Then there exists \((t_0, x_0) \in [0, T[ X D\) such that \( u_\delta(t_0, x_0) \geq c \). By continuity the function \( f(t) := \max_{x \in D} u_\delta(t, x) \) is well defined and \( t_1 := \min_{t \geq 0} \{ t \mid f(t) = c \} \) exists. By (15), we have \( 0 < t_1 \leq t_0 \), and there exists an \( x_1 \in D \) such that \( u_\delta(t_1, x_1) = c \).

On the one hand we have \( \frac{\partial}{\partial t}(t_1, x_1) \geq 0 \) and, on the other, since \( u_\delta(t_1, \cdot) \) has a global maximum at \( x_1 \): \( (\Delta_x u_\delta)(t_1, x_1) \geq 0 \). Evaluating at \((t_1, x_1)\) yields the desired contradiction:

\[
0 \leq \frac{\partial u_\delta}{\partial t} = \frac{\partial u}{\partial t} - \delta = -\Delta u - Vu - \delta \leq -\delta < 0
\]

We now prove the principle of not feeling the boundary for the free Laplacian using an idea of H. Weyl (cf. [Dod81, Lemma 3.5] for a Euclidean version).

**Proposition 6.3.** For any fixed \( t, \epsilon > 0 \), there exists an \( h = h(t, \epsilon) > 0 \) such that for every open set \( D \subset X \) and all \( \omega \in \Omega \)

\[
0 \leq k_{\Delta_\omega}(t, x, y) - k_{\Delta_\omega}(t, x, y) \leq \epsilon,
\]

for all \( x \in D, y \in D_h \).

**Proof.** We prove that the proposition is true for any \( h > 0 \) satisfying

\[
C_1 \exp(-\alpha t C_2^{-2}(h/2)^2) \leq \epsilon.
\]

Let \( \omega \in \Omega \) be fixed, and \( f_\delta \in C_0^\infty(B_\delta(y)) \), with \( 0 < \delta < h/2 \), be a nonnegative approximation of the \( \delta \)-distribution at \( y \in D_h \). Here, \( B_\delta(y) \) denotes the open \( \delta \)-ball around \( y \) with radius \( \delta \). Denote by \( k(t, x, y) = k_{\Delta_\omega}(t, x, y) \) the heat kernel of the semigroup \( e^{-t\Delta_\omega} \) and set

\[
u_1(t, x) := \int_X k(t, x, z) f_\delta(z) dvol_\omega(z) = \int_D k(t, x, z) f_\delta(z) dvol_\omega(z).
\]
Moreover, let 
\[ k_D(t, x, y) = k_{\Delta_D}(t, x, y) \]
be the heat kernel of the semigroup \( e^{-t\Delta_D} \) on \( D \) with Dirichlet data on the boundary \( \partial D \), and set
\[ u_2(t, x) := \int_D k_D(t, x, z) f_\delta(z) d\text{vol}_\omega(z). \]
The difference \( u_1(t, x) - u_2(t, x) \) solves the differential equation \( (\partial_t + \Delta_\omega) u = 0 \) and satisfies the initial condition \( u_1(0, x) - u_2(0, x) = f_\delta(x) - f_\delta(x) = 0 \) for all \( x \in D \). Now, by domain monotonicity we know \( k(t, x, z) - k_D(t, x, z) \geq 0 \), thus
\[ u_1(t, x) - u_2(t, x) = \int_D (k(t, x, z) - k_D(t, x, z)) f_\delta(z) d\text{vol}_\omega(z) \geq 0 \]
for all \( t > 0 \) and \( x \in D \). The application of the maximum principle yields
\[ u_1(t, x) - u_2(t, x) \leq \max_{[0, t] \times \partial D} \{ u_1(s, w) - u_2(s, w) \}. \]
The expression on the right hand side can be further estimated as:
\[ u_1(s, w) - u_2(s, w) \leq \int_D k(s, w, z) f_\delta(z) d\text{vol}_\omega(z) = \int_{D_{h/2}} k(s, w, z) f_\delta(z) d\text{vol}_\omega(z). \]
Since \( w \in \partial D \) and \( z \in D_{h/2} \), we conclude with (13) in Proposition 5.1:
\[ \int_{D_{h/2}} k(s, w, z) f_\delta(z) d\text{vol}_\omega(z) \leq C_t \exp(-\alpha t C^{-2}(h/2)^2) \leq \epsilon. \]
Taking the limit \( \delta \to 0 \), proves the proposition. \( \square \)

7. CONSTRUCTION OF THE IDS BY AN EXHAUSTION PROCEDURE

Using the strategy of [PV02], we show that the IDS, defined in (2), coincides with the limit of an exhaustion procedure, for almost all \( \omega \in \Omega \). This proves the selfaveraging property of the IDS stated in Theorem 4.

We first introduce the notion of an admissible sequence of subsets of \( X \). As explained in [AS93, Section 3], let \( F \subset X \) be a polyhedral fundamental domain of the group \( \Gamma \). Any finite subset \( I \subset \Gamma \) defines a corresponding set
\[ \phi(I) := \text{int} \left( \bigcup_{\gamma \in I} \gamma F \right) \subset X. \]

Now, admissible sequences are defined via tempered Følner sequences:

**Definition 7.1.** (a) A sequence \( \{ I_j \}_j \) of finite subsets in \( \Gamma \) is called a Følner sequence if \( \lim_{j \to \infty} \frac{|I_j \Delta I_j \gamma|}{|I_j|} = 0 \) for all \( \gamma \in \Gamma \).

(b) A Følner sequence \( \{ I_j \}_j \) is called a tempered Følner sequence if it is monotonously increasing and satisfies \( \sup_{j \in \mathbb{N}} \frac{|I_{j+1} \setminus I_j|}{|I_j|} < \infty \).

(c) A sequence \( \{ D^j \}_j \) of subsets of \( X \) is called admissible if there exists a tempered Følner sequence \( \{ I_j \}_j \) in \( \Gamma \) with \( D^j = \phi(I_j), j \in \mathbb{N} \).
By Lemma 2.4 in [PV02], an admissible sequence satisfies the isoperimetric property

\[(17) \lim_{j \to \infty} \frac{\text{vol}_0(\partial_d D^j)}{\text{vol}_0(D^j)} = 0, \text{ for all } d > 0.\]

Existence of a Følner sequence is a geometrical description of amenability of the group \(\Gamma\). The notion of “tempered Følner sequence” is due to A. Shulman [Shu88] and used by Lindenstrauss in the proof of the following pointwise ergodic theorem [Lin01].

**Theorem 7.** (a) Every Følner sequence has a tempered subsequence. In particular, every amenable group admits a tempered Følner sequence.

(b) Let \(\Gamma\) be an amenable discrete group and \((\Omega, A, \mathbb{P})\) be a probability space. Assume that \(\Gamma\) acts ergodically on \(\Omega\) by measure preserving transformations \(\{T_\gamma\}_\gamma\). Let \(\{I_j\}_j\) be a tempered Følner sequence. Then we have, for every \(f \in L^1(\Omega)\)

\[(18) \lim_{j \to \infty} \frac{1}{|I_j|} \sum_{\gamma \in I_j^{-1}} f(T_\gamma \omega) = E(f)\]

in almost-sure and \(L^1\)-topology.

The results of the last two sections are used to prove the following heat kernel lemma:

**Lemma 7.2.** Let \(\{D^j\}, j \in \mathbb{N},\) be an admissible sequence and let \(\{H^j_\omega\}_\omega\) be a random operator. Then the following holds.

(a) \(\sup_{\omega \in \Omega} \text{vol}^{-1}_\omega(D^j) \left| \text{tr}(\chi_{D^j} \exp(-tH^j_\omega)) - \text{tr}(\exp(-tH^j_\omega)) \right| \to 0, \quad n \to \infty.\)

(b) There exists a constant \(C > 0\) with \(\text{tr}(\chi e^{-tH_\omega}) \leq C\) for all \(\omega \in \Omega.\)

(c) The map \(\omega \mapsto \text{tr}(\chi e^{-tH_\omega})\) is measurable.

**Proof.** (a) By \(\exp(-tH) = \exp(-\frac{t}{2}H) \exp(\frac{t}{2}H)\) for arbitrary \(H \geq 0\) and standard calculations for integral kernels, we have

\[(19) \quad \text{tr}(\chi_{D^j} e^{-tH_\omega}) = \int_{D^j} \int_{D^j} k_\omega(t/2, x, y)^2 d\text{vol}_\omega(x)d\text{vol}_\omega(y)\]

and

\[(20) \quad \text{tr}(e^{-tH^j_\omega}) = \int_{D^j} \int_{D^j} k_{H^j_\omega}(t/2, x, y)^2 d\text{vol}_\omega(x)d\text{vol}_\omega(y).\]

We express the difference of (19) and (20) using \(k_\omega^2 - k_{H^j_\omega}^2 = [k_\omega - k_{H^j_\omega}] [k_\omega + k_{H^j_\omega}]\) and use the following decomposition of the integration domain:

\[
\int_{D^j} \int_{D^j} [k_\omega(t/2, x, y) - k_{H^j_\omega}(t/2, x, y)] [k_\omega(t/2, x, y) + k_{H^j_\omega}(t/2, x, y)] d\text{vol}_\omega(x, y)
= \int_{\partial_h D^j} \int_{\partial_h D^j} \ldots + \int_{D^j} \int_{D^j} \ldots + \int_{\partial_h D^j} \int_{\partial_h D^j} \ldots + \int_{D^j} \int_{D^j} \ldots
\]
Each of the first three terms can be bounded by \(2C_{t/2}B_{t/2}\text{vol}_\omega(\partial_RD^j)\) by inferring the following consequences of Section 5:

\[
0 \leq k_{H^2}(t/2, x, y) \leq k_\omega(t/2, x, y) \leq C_{t/2} \quad \text{and} \quad \int k_\omega(t/2, x, y) \text{dvol}_\omega(y) \leq B_{t/2}
\]

for almost every \(x, y \in X\). As for the last term, we fix \(\epsilon > 0\) and choose \(h = h(t/2, \epsilon)\) according to Theorem 6 and obtain the bound \(2\epsilon B_{t/2}\text{vol}_\omega(D^j)\). By (5), we conclude that the sequence \(D^j\) satisfies the isoperimetric property

\[
\lim_{j \to \infty} \frac{\text{vol}_\omega(\partial_d D^j)}{\text{vol}_\omega(D^j)} = 0, \quad \text{for all } d > 0,
\]

for the metric \(g_\omega\), as well. This shows part (a).

Now, (b) follows from (5), Proposition 5.1 (ii) and the analog of (19) for \(\chi_\mathcal{F}\), while (c) follows from measurability of \(\omega \mapsto e^{-tH_\omega}\), after choosing a suitable orthonormal basis according to Appendix A of [LPVb]. \(\square\)

Finally, we present the proof of our main result:

**Proof of Theorem 4.** A criterion of Pastur and Šubin [Pas71, Šub79] establishes the convergence of the normalized eigenvalue counting functions \(N^j_\omega\) to a selfaveraging limit, if their Laplace transforms \(L^j_\omega(\cdot)\) converge. To apply this criterion (cf. [PV02, LPVb], as well) we note that the random operator \(\{H_\omega\}_\omega\) is non-negative and by 7.2 (b) the \(L^j_\omega(t)\) are bounded by a constant depending only on \(t\). So it remains to show:

\[
\lim_{j \to \infty} L^j_\omega(t) := \lim_{j \to \infty} \int \text{e}^{-t\lambda} \text{d}N^j_\omega(\lambda) = \int \text{e}^{-t\lambda} \text{d}N_H(\lambda)
\]

for all \(t > 0\), in \(L^1\) and \(\mathbb{P}\)-almost sure-sense. This is done by applying Lindenstrauss’ ergodic theorem (Theorem 7). We introduce the equivalence relation \(a_j \xrightarrow{j \to \infty} b_j\) for two arbitrary sequences \(a_j(\omega), b_j(\omega), j \in \mathbb{N}\), satisfying \(a_j - b_j \to 0\), as \(j \to \infty\), in \(L^1\) and \(\mathbb{P}\)-almost surely. By definition we have

\[
L^j_\omega(t) = \text{vol}_\omega(D^j)^{-1} \text{tr}(\text{e}^{-tH_\omega^j}).
\]

Using the previous lemma, equivariance and Theorem 7, we derive

\[
|I_j|^{-1} \text{tr}(\text{e}^{-tH_\omega^j}) \overset{j \to \infty}{\sim} |I_j|^{-1} \text{tr}(\chi_\mathcal{F}_j e^{-tH_\omega}) = |I_j|^{-1} \sum_{\gamma \in I_j} \text{tr}(\chi_\mathcal{F}\gamma e^{-tH_\omega})
\]

\[
= |I_j|^{-1} \sum_{\gamma \in I_j^{-1}} \text{tr}(\chi_\mathcal{F}\gamma e^{-tH_\omega}) \overset{j \to \infty}{\sim} \mathbb{E}(\text{tr}(\chi_\mathcal{F} e^{-tH_\omega})).
\]

Similarly, we infer

\[
|I_j|^{-1} \text{vol}_\omega(D^j) = |I_j|^{-1} \sum_{\gamma \in I_j} \text{vol}_\omega(\gamma \mathcal{F}) = |I_j|^{-1} \sum_{\gamma \in I_j^{-1}} \text{vol}_\omega(\mathcal{F}) \overset{j \to \infty}{\sim} \mathbb{E}\{\text{vol}_\omega(\mathcal{F})\}.
\]

Putting this together, and noting that, by (5),

\[
C_{\text{g}}^{-n/2} \text{vol}_0(\mathcal{F}) \leq |I|^{-1} \text{vol}_\omega(\phi(I)) \leq C_{\text{g}}^{-n/2} \text{vol}_0(\mathcal{F}),
\]

for all \(n \geq 1\), we can conclude that

\[
\lim_{j \to \infty} \frac{\text{vol}_\omega(D_j)}{\text{vol}_\omega(\mathcal{F})} = \frac{\text{vol}_0(\mathcal{F})}{\text{vol}_0(\mathcal{F})} = 1
\]

as \(j \to \infty\), in \(\mathbb{P}\)-almost sure-sense. This completes the proof of Theorem 4.

**Remark.** Theorem 4 extends to \(\omega\)-almost sure-sense the results of [LPVb], and [Pas71, Šub79] to a selfaveraging limit.
for all finite sets $I \subset \Gamma$, we obtain

$$L^j_\omega(t) = \text{vol}_\omega(D^j)^{-1} \frac{\text{tr}(e^{-tH^j_\omega})}{|I_j|^{-1} \text{vol}_\omega(D^j)} \sim \frac{\mathbb{E}\{\text{tr}(\chi_F e^{-tH^\star})\}}{\mathbb{E}\{\text{vol}_\star(F)\}}.$$ 

By (2),

$$\frac{\mathbb{E}\{\text{tr}(\chi_F e^{-tH^\star})\}}{\mathbb{E}\{\text{vol}_\star(F)\}} = \int_{\mathbb{R}} e^{-t\lambda} d\mu_H(\lambda).$$

This finishes the proof. □

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