The $C^*$-algebras of connected real two-step nilpotent Lie groups

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Abstract

Using the operator valued Fourier transform, the $C^*$-algebras of connected real two-step nilpotent Lie groups are characterized as algebras of operator fields defined over their spectra. In particular, it is shown by explicit computations, that the Fourier transform of such $C^*$-algebras fulfills the norm controlled dual limit property.

1 Introduction

In this article, the structure of the $C^*$-algebras of two-step nilpotent Lie groups will be analyzed. In order to be able to understand these $C^*$-algebras, the Fourier transform is an important tool. The Fourier transform $\mathcal{F}(a) = \hat{a}$ of an element $a$ of a $C^*$-algebra $A$ is defined in the following way: One chooses for every $\gamma$ in $\hat{A}$, the spectrum of $A$, a representation $(\pi_\gamma, \mathcal{H}_\gamma)$ in the equivalence class of $\gamma$ and defines

$$\mathcal{F}(a)(\gamma) := \pi_\gamma(a) \in \mathcal{H}_\gamma \quad \forall \ \gamma \in \hat{A}.$$ 

Then $\mathcal{F}(a)$ is contained in the algebra of all bounded operator fields over $\hat{A}$

$$l^\infty(\hat{A}) = \{ \phi = (\phi(\pi_\gamma) \in B(\mathcal{H}_\gamma))_{\gamma \in \hat{A}} \mid \|\phi\|_\infty := \sup_{\gamma \in \hat{A}} \|\phi(\pi_\gamma)\|_{op} < \infty \}$$

and the mapping

$$\mathcal{F} : A \rightarrow l^\infty(\hat{A}), \ a \mapsto \hat{a}$$

is an isometric $*$-homomorphism.

The structure of the $C^*$-algebras is already known for certain classes of Lie groups: The $C^*$-algebras of the Heisenberg and the thread-like Lie groups have been characterized in [7] and the $C^*$-algebras of the $ax+b$-like groups in [6]. Furthermore, the $C^*$-algebras of the 5-dimensional nilpotent Lie groups have been determined in [11] and H.Regeiba analyzed the $C^*$-algebras of all 6-dimensional nilpotent Lie groups in his doctoral thesis (see [10]). The methods in this paper will partly be similar, but more complex, to the one used for the characterization of the $C^*$-algebra of the Heisenberg Lie group (see [7]), which is also two-step nilpotent and thus serves as an example.

It will be shown that the $C^*$-algebras of two-step nilpotent Lie groups $G$ are characterized by the following conditions. The same conditions hold true for all 5- and 6-dimensional nilpotent Lie groups (see [11]), for the Heisenberg Lie groups and the thread-like Lie groups (see [7]).
1. Stratification of the spectrum:

(a) A finite increasing family \( S_0 \subset S_1 \subset \ldots \subset S_r = \hat{C}^*(G) \cong \hat{G} \) of closed subsets of the spectrum \( \hat{C}^*(G) \cong \hat{G} \) of \( C^*(G) \) or respectively \( G \) will be constructed in such a way that for \( i \in \{1, \ldots, r\} \) the subsets \( \Gamma_i = S_i \setminus S_{i-1} \) are Hausdorff in their relative topologies and such that \( S_0 \) consists of all the characters of \( C^*(G) \) or \( G \), respectively.

(b) For every \( i \in \{0, \ldots, r\} \) a Hilbert space \( H_i \) and for every \( \gamma \in \Gamma_i \) a concrete realization \((\pi_\gamma, H_i)\) of \( \gamma \) on the Hilbert space \( H_i \) will be defined.

2. CCR \( C^* \)-algebra:

It will be shown that \( C^*(G) \) is a separable CCR (or liminal) \( C^* \)-algebra, i.e. a separable \( C^* \)-algebra such that the image of every irreducible representation \((\pi, H)\) of \( C^*(G) \) is contained in the algebra of compact operators \( K(H) \) (which implies that the image equals \( K(H) \)).

3. Changing of layers:

Let \( a \in C^*(G) \).

(a) It will be proved that the mappings \( \gamma \mapsto \mathcal{F}(a)(\gamma) \) are norm continuous on the different sets \( \Gamma_i \).

(b) For any \( i \in \{0, \ldots, r\} \) and for any converging sequence contained in \( \Gamma_i \) with limit set outside \( \Gamma_i \) (thus in \( S_{i-1} \)), there will be constructed a properly converging subsequence \( \mathcal{Y} = (\gamma_k)_{k \in \mathbb{N}} \) (i.e. the subsequences of \( \mathcal{Y} \) have all the same limit set - see Definition 2.3), as well as a constant \( C > 0 \) and for every \( k \in \mathbb{N} \) an involutive linear mapping \( \tilde{\nu}_k = \tilde{\nu}_{\gamma,k} : CB(S_{i-1}) \to B(H_i) \), which is bounded by \( C \| \cdot \|_{S_{i-1}} \), such that

\[
\lim_{k \to \infty} \| \mathcal{F}(a)(\gamma_k) - \tilde{\nu}_k(\mathcal{F}(a)\big|_{S_{i-1}}) \|_{\text{op}} = 0.
\]

Here \( CB(S_{i-1}) \) is the \(*\)-algebra of all the uniformly bounded fields of operators \((\psi(\gamma) \in B(H_j))_{\gamma \in \Gamma_j, j=0, \ldots, i-1}\), which are operator norm continuous on the subsets \( \Gamma_j \) for every \( j \in \{0, \ldots, i-1\} \), provided with the infinity-norm

\[
\| \varphi \|_{S_{i-1}} := \sup_{\gamma \in S_{i-1}} \| \varphi(\gamma) \|_{\text{op}}.
\]

These properties characterize the structure of \( C^*(G) \) (see [11], Theorem 3.5). A \( C^* \)-algebra fulfilling these conditions is called a \( C^* \)-algebra with “norm controlled dual limits”.

The main work of this article consists in the proof of Property 3(b) and in particular in the construction of the mappings \((\tilde{\nu}_k)_k\).

2 Preliminaries

2.1 Two-step nilpotent Lie groups

Let \( \mathfrak{g} \) be a real Lie algebra which is nilpotent of step two. This means that

\[
[\mathfrak{g}, \mathfrak{g}] := \text{span}\{[X,Y] \mid X, Y \in \mathfrak{g}\}
\]

is contained in the center of \( \mathfrak{g} \).

Fix a scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) and take on \( \mathfrak{g} \) the Campbell-Baker-Hausdorff multiplication

\[
u \cdot v = \nu + v + \frac{1}{2} [\nu, v] \quad \forall \, \nu, v \in \mathfrak{g}.
\]
This gives the simply connected connected Lie group \( G = (g, \cdot) \) with Lie algebra \( g \). The exponential mapping \( \exp : g \to G = (g, \cdot) \) is in this case the identity mapping.

The Haar measure of this group is a Lebesgue measure which is denoted by \( dx \).

Then, the \( C^* \)-algebra of \( G \) is defined as the completion of the convolution algebra \( L^1(G, dx) = L^1(G) \) with respect to the \( C^* \)-norm of \( L^1(G, dx) \), i.e.

\[
C^*(G) := \overline{L^1(G, dx)}^{\| \cdot \|_{C^*(G)}} \quad \text{with} \quad \| f \|_{C^*(G)} := \sup_{\pi \in \hat{G}} \| \pi(f) \|_{op}
\]

and a well-known result, that can be found in [3], states that the spectrum of \( C^*(G) \) coincides with the spectrum of \( G \):

\[
\hat{C^*(G)} = \hat{G}.
\]

Now, for a linear functional \( \ell \) of \( g \), consider the skew-bilinear form

\[
B_\ell(X, Y) := \langle \ell, [X, Y] \rangle
\]

on \( g \). Moreover, let

\[
g(\ell) := \{ X \in g | \langle \ell, [X, g] \rangle = \{0\} \}
\]

be the radical of \( B_\ell \) and the stabilizer of the linear functional \( \ell \). Then, as \( g \) is two-step nilpotent, \( [g, g] \subset g(\ell) \) and thus \( g(\ell) \) is an ideal of \( g \).

**Definition 2.1.**
A subalgebra \( p \) of \( g \), that is subordinated to \( \ell \) (i.e. that fulfills \( \langle \ell, [p, p] \rangle = \{0\} \)) and that has the dimension

\[
\dim(p) = \frac{1}{2} (\dim(g) + \dim(g(\ell))),
\]

which means that \( p \) is maximal isotropic for \( B_\ell \), is called a polarization in \( \ell \).

Again since \( g \) is nilpotent of step two, every maximal isotropic subspace \( p \) of \( g \) for \( B_\ell \) containing \( [g, g] \) is a polarization at \( \ell \).

Now, if \( p \subset g \) is any subalgebra of \( g \) which is subordinated to \( \ell \), the linear functional \( \ell \) defines a unitary character \( \chi_\ell \) of \( P := \exp(p) \):

\[
\chi_\ell(x) := e^{-2\pi i \ell(\log(x))} = e^{-2\pi i \ell(x)} \quad \forall \ x \in P.
\]

### 2.2 Induced representations

The induced representation \( \sigma_{\ell, p} = \text{ind}_P^G \chi_\ell \) for a polarization \( p \) in \( \ell \) and \( P := \exp(p) \) can be described in the following way:

Since \( p \) contains \( [g, g] \) and even the center \( z \) of \( g \), one can write \( g = s \oplus p \) and \( p = t \oplus z \) for two subspaces \( t \) and \( s \) of \( g \). The quotient space \( G/P \) is then homeomorphic to \( s \) and the Lebesgue measure \( ds \) on \( s \) defines an invariant Borel measure \( d\dot{g} \) on \( G/P \). The group \( G \) acts by the left translation \( \sigma_{\ell, p} \) on the Hilbert space

\[
L^2(G/P, \chi_\ell) := \left\{ \xi : G \to \mathbb{C} | \xi \text{ measurable, } \xi(gp) = \chi_\ell(p)\xi(g) \forall g \in G \forall p \in P, \right\}
\]

\[
\|\xi\|_2^2 := \int_{G/P} |\xi(g)|^2 \, d\dot{g} < \infty.
\]
Now, if one uses the coordinates $G = \mathfrak{g} \cdot \mathfrak{p}$, one can identify the Hilbert spaces $L^2(G/P, \chi_\ell)$ and $L^2(\mathfrak{g}) = L^2(\mathfrak{g}, d\mathfrak{g})$:

Let $U_\ell : L^2(\mathfrak{g}, d\mathfrak{g}) \to L^2(G/P, \chi_\ell)$ be defined by

$$U_\ell(\varphi)(S \cdot Y) := \chi_\ell(-Y) \varphi(S) \quad \forall \ Y \in \mathfrak{p} \quad \forall \ S \in \mathfrak{g} \quad \forall \ \varphi \in L^2(\mathfrak{g}).$$

Then, $U_\ell$ is a unitary operator and one can transform the representation $\sigma_{\ell,p}$ into a representation $\pi_{\ell,p}$ on the space $L^2(\mathfrak{g})$:

$$\pi_{\ell,p} := U_\ell^* \circ \sigma_{\ell,p} \circ U_\ell.$$  \hspace{1cm} (1)

Furthermore, one can express the representation $\sigma_{\ell,p}$ in the following way:

$$\sigma_{\ell,p}(S \cdot Y) \xi(R) = \xi(Y^{-1} S^{-1} R) = \xi((R - S) \cdot \left(-Y + \frac{1}{2}[R, S] - \frac{1}{2}[R - S, Y]\right)) = e^{2\pi i (\xi(R - S) - \frac{1}{2}[R - S, Y])} \xi(R - S) \quad \forall \ R, S \in \mathfrak{g} \quad \forall \ Y \in \mathfrak{p} \quad \forall \ \xi \in L^2(G/P, \chi_\ell).$$

Hence

$$\pi_{\ell,p}(S \cdot Y) \varphi(R) = e^{2\pi i (\xi(R - S) - \frac{1}{2}[R - S, Y])} \varphi(R - S) \quad \forall \ R, S \in \mathfrak{g} \quad \forall \ Y \in \mathfrak{p} \quad \forall \ \varphi \in L^2(\mathfrak{g}).$$  \hspace{1cm} (2)

### 2.3 Orbit method

By the Kirillov theory (see [2], Section 2.2), for every representation class $\gamma \in \hat{G}$, there exists an element $\ell \in \mathfrak{g}^*$ and a polarization $\mathfrak{p}$ of $\ell$ in $\mathfrak{g}$ such that $\gamma = [\text{ind}^G_{\mathfrak{p}} \chi_\ell]$, where $P := \exp(\mathfrak{p})$.

Moreover, if $\ell, \ell' \in \mathfrak{g}^*$ are located in the same coadjoint orbit $O \in \mathfrak{g}^*/G$ and $\mathfrak{p}$ and $\mathfrak{p}'$ are polarizations in $\ell$ and $\ell'$, respectively, the induced representations $\text{ind}^G_{\mathfrak{p}} \chi_\ell$ and $\text{ind}^G_{\mathfrak{p}'} \chi_{\ell'}$ are equivalent and thus, the Kirillov map which goes from the coadjoint orbit space $\mathfrak{g}^*/G$ to the spectrum $\hat{G}$ of $G$

$$K : \mathfrak{g}^*/G \to \hat{G}, \ Ad^* G \ell \mapsto [\text{ind}^G_{\mathfrak{p}} \chi_\ell]$$

is a homeomorphism (see [1] or [5], Chapter 3). Therefore,

$$\mathfrak{g}^*/G \cong \hat{G}$$

as topological spaces.

For every $\ell \in \mathfrak{g}^*$ and $x \in G = (\mathfrak{g}, \cdot)$

$$\text{Ad}^* (x) \ell = (1_{\mathfrak{g}^*} + \text{ad}^* (x)) \ell \in \ell + \mathfrak{g}(\ell)^\perp.$$

Hence, as $\text{ad}^* (\mathfrak{g}) \ell \in \mathfrak{g}(\ell)^\perp$,

$$O_\ell := \text{Ad}^* (G) \ell = \ell + \mathfrak{g}(\ell)^\perp.$$  \hspace{1cm} (3)

**Definition 2.2.**

Let $T$ be a second countable topological space and suppose that $T$ is not Hausdorff, which means that converging sequences can have many limit points. Denote by $L((t_k)_k)$ the collection of all the limit points of a sequence $(t_k)_k$ in $T$. A sequence $(t_k)_k$ is called properly converging, if $(t_k)_k$ has limit points and if every subsequence of $(t_k)_k$ has the same limit set as $(t_k)_k$.

It is well known that every converging sequence in $T$ admits a properly converging subsequence.
Now, let \((\pi_k)_k \subset \hat{G}\) be a properly converging sequence in \(\hat{G}\) with limit set \(L((\pi_k)_k)\). Let \(O \in \mathfrak{g}^*/G\) be the Kirillov orbit of some \(\pi \in L((\pi_k)_k), O_k\) the Kirillov orbit of \(\pi_k\) for every \(k\) and let \(\ell \in O\). Then there exists for every \(k\) an element \(\ell_k \in O_k\), such that \(\lim_{k \to \infty} \ell_k = \ell\) in \(\mathfrak{g}^*\) (see \([5]\)). One can assume that, passing to a subsequence if necessary, the sequence \((\mathfrak{g}(\ell_k))_k\) converges in the subspace topology to a subalgebra \(u\) of \(\mathfrak{g}(\ell)\) and that there exists a number \(d \in \mathbb{N}\), such that \(\dim(O_k) = d\) for every \(k \in \mathbb{N}\). Then it follows from \([4]\), that

\[
L((O_k)_k) = \lim_{k \to \infty} \ell_k + \mathfrak{g}(\ell_k)^\perp = \ell + u^\perp \subset \mathfrak{g}^*.
\]

(4)

Since \(\mathfrak{g}(\ell_k)\) contains \([\mathfrak{g}, \mathfrak{g}]\) for every \(k\), the subspace \(u\) also contains \([\mathfrak{g}, \mathfrak{g}]\). Hence, the limit set \(L((\pi_k)_k)\) in \(\hat{G}\) of the sequence \((\pi_k)_k\) is the “affine” subset

\[
L((\pi_k)_k) = \{ [\chi_q \otimes \text{ind}_P^G \chi_{\ell}] | q \in u^\perp \}
\]

for a polarization \(p\) in \(\ell\) and \(P := \text{exp}(p)\).

The observations above lead to the following proposition:

**Proposition 2.3.**

There are three different types of possible limit sets of the sequence \((O_k)_k\) of coadjoint orbits:

1. The limit set \(L((O_k)_k)\) is the singleton \(O_{\ell} = \ell + \mathfrak{g}(\ell)^\perp\), i.e. \(u = \mathfrak{g}(\ell)\).

2. The limit set \(L((O_k)_k)\) is the affine subspace \(\ell + u^\perp\) of characters of \(\mathfrak{g}\), i.e. \(\{ \ell, [\mathfrak{g}, \mathfrak{g}] \} = \{ 0 \}\).

3. The dimension of the orbit \(O_{\ell}\) is strictly greater than 0 and strictly smaller than \(d\). In this case

\[
L((O_k)_k) = \bigcup_{q \in u^\perp} q + O_{\ell}, \quad \text{i.e.} \quad L((\pi_k)_k) = \bigcup_{q \in u^\perp} [\chi_q \otimes \text{ind}_P^G \chi_{\ell}]
\]

for a polarization \(p\) in \(\ell\) and \(P := \text{exp}(p)\).

**2.4 The \(C^*\)-algebra \(C^*(G/U, \chi_{\ell})\)**

Let \(u \subset \mathfrak{g}\) be an ideal of \(\mathfrak{g}\) containing \([\mathfrak{g}, \mathfrak{g}]\), \(U := \text{exp}(u)\) and let \(\ell \in \mathfrak{g}^*\) such that \(\{ \ell, [\mathfrak{g}, \mathfrak{u}] \} = \{ 0 \}\) and such that \(u \subset \mathfrak{g}(\ell)\). Then the character \(\chi_{\ell}\) of the group \(U = \text{exp}(u)\) is \(G\)-invariant. One can thus define the involutive Banach algebra \(L^1(G/U, \chi_{\ell})\) as

\[
L^1(G/U, \chi_{\ell}) := \left\{ f : G \to \mathbb{C} | f \text{ measurable, } f(gu) = \chi_{\ell}(u^{-1})f(g) \forall g \in G \right. \quad \forall u \in U, \left. \|f\|_1 := \int_{G/U} |f(g)| \, d\tilde{g} < \infty \right\}.
\]

The convolution

\[
f * f'(g) := \int_{G/U} f(x)f'(x^{-1}g) \, d\tilde{x} \quad \forall g \in G
\]

and the involution

\[
f^*(g) := \overline{f(g^{-1})} \quad \forall g \in G
\]

are well-defined for \(f, f' \in L^1(G/U, \chi_{\ell})\) and

\[
\|f * f'\|_1 \leq \|f\|_1 \|f'\|_1.
\]
In order to be able to do this, one needs to construct a polarization $\Gamma$. Fix once and for all a Jordan-Hölder basis $\{H_1,\ldots, H_n\}$ of $\mathfrak{g}$, in such a way that $\mathfrak{g}_i := \text{span}\{H_i,\ldots, H_n\}$ for $i \in \{0,\ldots, n\}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is two-step nilpotent, one can first choose a basis $\{H_{\tilde{n}}, \ldots, H_n\}$ of $[\mathfrak{g}, \mathfrak{g}]$ and then add the vectors $H_1, \ldots, H_{\tilde{n}-1}$ to obtain a basis of $\mathfrak{g}$. Let

$$I^{P_{\text{uk}}} = \{i \leq n \mid \mathfrak{g}(\ell) \cap \mathfrak{g}_i = \mathfrak{g}(\ell) \cap \mathfrak{g}_{i+1}\}$$

Moreover, the linear mapping

$$p_{G/U} : \mathcal{L}^1(G) \to \mathcal{L}^1(G/U, \chi_\ell),$$

$$p_{G/U}(F)(g) := \int F(gu)\chi_\ell(u) \, du \quad \forall \, F \in \mathcal{L}^1(G) \quad \forall \, g \in G$$

is a surjective $*$-homomorphism between the algebras $\mathcal{L}^1(G)$ and $\mathcal{L}^1(G/U, \chi_\ell)$.

Let

$$\tilde{G}_{u,\ell} := \{ (\pi, \mathcal{H}_\pi) \in \tilde{G} \mid \pi_{|U} = \chi_\ell|_U I_{\mathcal{H}_\pi} \}.$$

Then $\tilde{G}_{u,\ell}$ is a closed subset of $\tilde{G}$, which can be identified with the spectrum of the algebra $\mathcal{L}^1(G/U, \chi_\ell)$. Indeed it is easy to see that every irreducible unitary representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}}) \in \tilde{G}_{u,\ell}$ defines an irreducible representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ of the algebra $\mathcal{L}^1(G/U, \chi_\ell)$ as follows:

$$\tilde{\pi}(p_{G/U}(F)) := \pi(F) \quad \forall \, F \in \mathcal{L}^1(G).$$

Similarly, if $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ is an irreducible unitary representation of $\mathcal{L}^1(G/U, \chi_\ell)$ then

$$\pi := \tilde{\pi} \circ p_{G/U}$$

defines an element of $\tilde{G}_{u,\ell}$.

Let $\mathfrak{s} \subset \mathfrak{g}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}(\ell) \oplus \mathfrak{s}$. Since $u$ contains $[\mathfrak{g}, \mathfrak{g}]$, it is easy to see that

$$\tilde{G}_{u,\ell} = \{ [\chi_\ell \otimes \pi_\ell] \mid q \in (u + \mathfrak{s})^+ \},$$

leaving $\pi_\ell := \text{ind}_{\mathfrak{p}}^{\tilde{G}}(\chi_\ell)$ for a polarization $\mathfrak{p}$ in $\ell$ and $P := \text{exp}(\mathfrak{p})$.

Denote by $C^\ast(G/U, \chi_\ell)$ the $C^\ast$-algebra of $\mathcal{L}^1(G/U, \chi_\ell)$, whose spectrum can also be identified with $\tilde{G}_{u,\ell}$.

With $\pi_{\ell+q} := \text{ind}_{\mathfrak{p}}^{\tilde{G}}(\chi_{\ell+q})$, the Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}(a)(q) := \pi_{\ell+q}(a) \quad \forall \, q \in (u + \mathfrak{s})^+$$

then maps the $C^\ast$-algebra $C^\ast(G/U, \chi_\ell)$ onto the algebra $C_0((u + \mathfrak{s})^+, \mathcal{K}(\mathcal{H}_{\pi_\ell}))$ of the continuous mappings $\varphi : (u + \mathfrak{s})^+ \to \mathcal{K}(\mathcal{H}_{\pi_\ell})$ vanishing at infinity with values in the algebra of compact operators on the Hilbert space of the representation $\pi_\ell$.

If one restricts $p_{G/U}$ to the Fréchet algebra $\mathcal{S}(G) \subset \mathcal{L}^1(G)$, its image will be the Fréchet algebra

$$\mathcal{S}(G/U, \chi_\ell) = \{ f \in \mathcal{L}^1(G/U, \chi_\ell) \mid f \text{ smooth and for every subspace } \mathfrak{s} \subset \mathfrak{g} \text{ with } \mathfrak{g} = \mathfrak{s}' \oplus u \text{ and for } S' = \text{exp}(\mathfrak{s}') \text{, } f_{|_{S'}} \in \mathcal{S}(S') \}.$$

## 3 Conditions 1, 2 and 3(a)

Now, to start with the proof of the above listed conditions, the families of sets $(S_i)_{i \in \{0,\ldots, r\}}$ and $(\Gamma_i)_{i \in \{0,\ldots, r\}}$ are going to be defined and the Properties 1, 2 and 3(a) are going to be checked.

In order to be able to do this, one needs to construct a polarization $p_{\ell}^\mathfrak{u}$ for $\ell \in \mathfrak{g}^\ast$ as follows:

Fix once and for all a Jordan-Hölder basis $\{H_1, \ldots, H_n\}$ of $\mathfrak{g}$, in such a way that $\mathfrak{g}_i := \text{span}\{H_i,\ldots, H_n\}$ for $i \in \{0,\ldots, n\}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is two-step nilpotent, one can first choose a basis $\{H_{\tilde{n}}, \ldots, H_n\}$ of $[\mathfrak{g}, \mathfrak{g}]$ and then add the vectors $H_1, \ldots, H_{\tilde{n}-1}$ to obtain a basis of $\mathfrak{g}$. Let

$$I^{P_{\text{uk}}} := \{i \leq n \mid \mathfrak{g}(\ell) \cap \mathfrak{g}_i = \mathfrak{g}(\ell) \cap \mathfrak{g}_{i+1}\}$$
be the Pukanszky index set for \( \ell \in \mathfrak{g}^* \). The number of elements \(|I_{\ell}^{Puk}|\) of \( I_{\ell}^{Puk} \) is the dimension of the orbit \( O_\ell \) of \( \ell \).

Moreover, if one denotes by \( g_i(\ell_{[\ell]},.) \) the stabilizer of \( \ell_{[\ell]} \) in \( \mathfrak{g}_i \),

\[
p_Y^V := \sum_{i=1}^n g_i(\ell_{[\ell]})
\]

is the Vergne polarization of \( \ell \) in \( \mathfrak{g} \). Its construction will now be analyzed by a method developed in [2].

Let \( \ell \in \mathfrak{g}^* \). Then choose the greatest index \( j_1(\ell) \in \{1,...,n\} \) such that \( H_{j_1(\ell)} \not\subseteq \mathfrak{g}(\ell) \) and let \( Y_1^{V,\ell} := H_{j_1(\ell)} \). Furthermore, choose the index \( k_1(\ell) \in \{1,...,n\} \) such that \(<\ell,[H_{k_1(\ell)},H_{j_1(\ell)}]> \neq 0 \) and \(<\ell,[H_{i_1(\ell)},H_{j_1(\ell)}]> = 0 \) for all \( i > k_1(\ell) \) and let \( Y_1^{V,\ell} := H_{k_1(\ell)} \).

Next, let \( g_{1,\ell} := \{ U \in \mathfrak{g} | <\ell,[U,Y_1^{V,\ell}]> = 0 \} \). Then \( g_{1,\ell} \) is an ideal in \( \mathfrak{g} \) which does not contain \( X_1^{V,\ell} \), and \( g = \mathbb{R}X_1^{V,\ell} \oplus g_{1,\ell}. \) Now, the Jordan-Hölder basis will be changed, taking out \( H_{k_1(\ell)} \):

Consider the Jordan-Hölder basis \( \{H_1^{1,\ell},...,H_{k_1(\ell)-1,\ell},H_{k_1(\ell)+1,\ell},...,H_n^{1,\ell}\} \) of \( g_{1,\ell} \) with

\[
H_1^{1,\ell} := H_i \quad \forall \ i > k_1(\ell) \quad \text{and} \quad H_i^{1,\ell} := H_i - \frac{<\ell,[H_i,Y_1^{V,\ell}]>Y_1^{V,\ell}}{<\ell,[X_1^{V,\ell},Y_1^{V,\ell}]>} \quad \forall \ i < k_1(\ell).
\]

Then, choose the greatest index \( j_2(\ell) \in \{1,...,k_1(\ell)-1,k_1(\ell)+1,...,n\} \) such that \( H_{j_2(\ell)} \not\subseteq g_{1,\ell} \) and define \( Y_2^{V,\ell} := H_{j_2(\ell)} \). Like above, choose \( k_2(\ell) \in \{1,...,k_1(\ell)-1,k_1(\ell)+1,...,n\} \) such that \(<\ell,[H_{k_2(\ell)},H_{j_2(\ell)}]> \neq 0 \) and that \(<\ell,[H_{k_1(\ell)},H_{j_2(\ell)}]> = 0 \) for all \( i > k_2(\ell) \) and set \( X_2^{V,\ell} := H_{k_2(\ell)} \).

Iterating this procedure, one gets sets \( \{Y_1^{V,\ell},...,Y_d^{V,\ell}\} \) and \( \{X_1^{V,\ell},...,X_d^{V,\ell}\} \) for \( d \in \{0,...,\frac{[\mathfrak{g}]}{2}\} \) with the properties

\[
p_Y^V = \text{span}\{Y_1^{V,\ell},...,Y_d^{V,\ell}\} \oplus g(\ell)
\]

and

\[
<\ell,[X_i^{V,\ell},Y_i^{V,\ell}]> \neq 0, \quad <\ell,[X_i^{V,\ell},Y_j^{V,\ell}]> = 0 \quad \forall \ i \neq j \in \{1,...,d\} \quad \text{and} \quad <\ell,[Y_i^{V,\ell},Y_j^{V,\ell}]> = 0 \quad \forall \ i,j \in \{1,...,d\}.
\]

Now, let

\[
J(\ell) := \{j_1(\ell),...,j_d(\ell)\} \quad \text{and} \quad K(\ell) := \{k_1(\ell),...,k_d(\ell)\}.
\]

Then

\[
I_{\ell}^{Puk} = J(\ell) \cup K(\ell) \quad \text{and} \quad j_1(\ell) > \cdots > j_d(\ell).
\]

It is easy to see that the index sets \( I_{\ell}^{Puk}, J(\ell) \) and \( K(\ell) \) are the same on every coadjoint orbit (see [8]) and can therefore also be denoted by \( I_{O}^{Puk}, J(O) \) and \( K(O) \) if \( \ell \) is located in the coadjoint orbit \( O \).

Now, for the parametrization of \( \mathfrak{g}^*/G \) and thus of \( \hat{G} \) and for the choice of the \( \mathfrak{g} \) in Property 1(b) required concrete realization of a representation, let \( O \in \mathfrak{g}^*/G \). A theorem of L.Pukanszky (see [2]. Part II, Chapter I.3 or [2]). Corollary 1.2.5) states that there exists one unique \( \ell_0 \in O \) such that \( I_{O}(H_i) = 0 \) for every index \( i \in I_{O}^{Puk} \). So, choose this \( \ell_0 \), let \( \Pi_{\ell_0}^{V} := \exp(p_{\ell_0}^V) \) and define the irreducible unitary representation

\[
\sigma_{\ell_0}^{V} := \text{ind}_{\ell_0}^{G} \pi_{\ell_0} \chi_{\ell_0}
\]

associated to the orbit \( O \) and acting on \( L^2(G/P_{\ell_0} \chi_{\ell_0}) \cong L^2(\mathbb{R}^d) \).
Next, one has to construct the demanded sets $\Gamma_i$ for $i \in \{0, \ldots, r\}$:

For this, define for a pair of sets $(J, K)$ such that $J, K \subset \{1, \ldots, n\}$, $|J| = |K|$ and $J \cap K = \emptyset$ the subset $(g^*/G)_{(J, K)}$ of $g^*/G$ by

$$(g^*/G)_{(J, K)} := \{ O \in g^*/G | (J, K) = (J(O), K(O)) \}.$$  

Moreover, let

$$\mathcal{M} := \{ (J, K) | J, K \subset \{1, \ldots, n\}, J \cap K = \emptyset, |J| = |K|, (g^*/G)_{(J, K)} \neq \emptyset \}$$

and

$$(g^*/G)_{2d} := \{ O \in g^*/G | |I_O^{P_u k}| = 2d \}.$$  

Then

$$(g^*/G)_{2d} = \bigcup_{(J, K) \in \mathcal{M}, |J| = |K| = d, J \cap K = \emptyset} (g^*/G)_{(J, K)}$$

and

$$g^*/G = \bigcup_{d \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor \}} (g^*/G)_{2d} = \bigcup_{(J, K) \in \mathcal{M}} (g^*/G)_{(J, K)}.$$  

Now, an order on the set $\mathcal{M}$ shall be introduced.

First, if $|J| = |K| = d$, $|J'| = |K'| = d'$ and $d < d'$, then the pair $(J, K)$ is defined to be smaller than the pair $(J', K')$: $(J, K) < (J', K')$.

If $|J| = |K| = |J'| = |K'| = d$, $J = \{j_1, \ldots, j_d\}$, $J' = \{j'_1, \ldots, j'_d\}$ and $j_1 < j'_1$, the pair $(J, K)$ is again defined to be smaller than $(J', K')$.

Otherwise, if $j_1 = j'_1$, one has to consider $K = \{k_1, \ldots, k_d\}$ and $K' = \{k'_1, \ldots, k'_d\}$ and here again, compare the first elements $k_1$ and $k'_1$: So, if $j_1 = j'_1$ and $k_1 < k'_1$, again $(J, K) < (J', K')$.

But if $k_1 = k'_1$, one compares $j_2$ and $j'_2$ and continues in that way.

If $r + 1 = |\mathcal{M}|$, one can identify the ordered set $\mathcal{M}$ with the interval $\{0, \ldots, r\}$ and assign to each such pair $(J, K) \in \mathcal{M}$ a number $i_{JK} \in \{0, \ldots, r\}$.

Finally, one can therefore define the sets $\Gamma_{i_{JK}}$ and $S_{i_{JK}}$ as

$$\Gamma_{i_{JK}} := \{ [\pi_{\ell_o}^V] | O \in (g^*/G)_{(J, K)} \}$$

and

$$S_{i_{JK}} := \bigcup_{i \in \{0, \ldots, i_{JK}\}} \Gamma_i.$$  

Then obviously, the family $(S_i)_{i \in \{0, \ldots, r\}}$ is an increasing family in $\hat{G}$.

Furthermore, the set $S_i$ is closed for every $i \in \{0, \ldots, r\}$. This can easily be deduced from the definition of the index sets $J(\ell)$ and $K(\ell)$. The indices $j_m(\ell)$ and $k_m(\ell)$ for $m \in \{1, \ldots, d\}$ are chosen in such a way that they are the largest to fulfill a condition of the type $(\ell, [H_m^{-1} \ell, \cdot]) \neq 0$ or $(\ell, [H_m^{-1} \ell, \cdot]) = 0$, respectively.

In addition, the sets $\Gamma_i$ are Hausdorff. For this, let $i = i_{JK}$ for $(J, K) \in \mathcal{M}$ and $(O_k)_k$ in $(g^*/G)_{(J, K)}$ a sequence of orbits such that the sequence $([\pi_{\ell_o}^V])_k$ converges in $\Gamma_i$, i.e. $(O_k)_k$ converges in $(g^*/G)_{(J, K)}$ and thus has a limit point $O$ in $(g^*/G)_{(J, K)}$. If now $O_k \ni \ell_k \rightarrow \ell \in O$, then by $[4]$, it follows that the limit $u$ of the sequence $(g(\ell_k))_k$ is equal to $g(\ell)$. Therefore, the sequence $(O_k)_k$ and thus also the sequence $([\pi_{\ell_o}^V])_k$ have unique limits and hence $\Gamma_i$ is Hausdorff.

Moreover, one can still observe that for $d = 0$ the choice $J = K = \emptyset$ represents the only possibility to get $|J| = |K| = d$. So, the pair $(\emptyset, \emptyset)$ is the first element in the above defined order and therefore corresponds to 0. Thus

$$\Gamma_0 = \{ [\pi_{\ell_o}^V] | I_O^{P_u k} = \emptyset \},$$

which is equivalent to the fact that $g(\ell_O) = 0$ which again is equivalent to the fact that every $\pi_{\ell_o}^V \in \Gamma_0$ is a character. Hence, $S_0 = \Gamma_0$ is the set of all characters on $g$, as demanded.
Next, let
\[ g \in \text{a compact operator}. \]
\[ G \]
\[ L \]
\[ \text{Lie group} \]
\[ s_{tO} = \text{span}\{X^V_{1tO}, \ldots, X^V_{d_tO}\}, \]
\[ L^2(\mathbb{R}^d) \]
\[ \text{as in } [1] \]
\[ \text{one can suppose that the representation } \pi^V_{tO} \text{ acts on the Hilbert space} \]
\[ L^2(\mathbb{R}^d) \]
\[ \text{for every } O \in (\mathfrak{g}^* / G)_{2d}. \]

Hence, the first condition is fulfilled. For the proof of the Properties 2 and 3(a), a proposition will be shown:

**Proposition 3.1.**

For every \( a \in C^*(G) \) and every \( (J, K) \in \mathcal{M} \) with \( |J| = |K| = d \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\} \), the mapping

\[ \Gamma_{J,K} \rightarrow L^2(\mathbb{R}^d), \; \gamma \mapsto F(a)(\gamma) \]

is norm continuous and the operator \( F(a)(\gamma) \) is compact for all \( \gamma \in \Gamma_{J,K} \).

**Proof:**

The compactness follows directly from a general theorem which can be found in [2] (Chapter 4.2) or [9] (Part II, Chapter II.5) and states that the \( C^* \)-algebra \( C^*(G) \) of every connected nilpotent Lie group \( G \) fulfills the CCR condition, i.e. the image of every irreducible representation of \( C^*(G) \) is a compact operator.

Next, let \( d \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\} \) and \( (J, K) \in \mathcal{M} \) such that \( |J| = |K| = d \).

First, one has to observe that the polarization \( p^V_\ell \) is continuous in \( \ell \) on the set \( \{ \ell_O \mid O' \in (\mathfrak{g}^* / G)_{(J,K)} \} \).

Now, let \( (O_k)_k \) be a sequence in \( (\mathfrak{g}^* / G)_{(J,K)} \) and \( O \in (\mathfrak{g}^* / G)_{(J,K)} \) such that \( \pi^V_{tO_k} \xrightarrow{k \to \infty} \pi^V_{tO} \)

and let \( a \in C^*(G) \). Then \( \ell_{O_k} \xrightarrow{k \to \infty} \ell_O \) and by the observation above, the associated sequence of polarizations \( (p^V_{\ell_{O_k}})_k \) converges to the polarization \( p^V_{\ell_O} \). By Theorem 2.3 in [11], thus \( \pi^V_{tO_k}(a) \xrightarrow{k \to \infty} \pi^V_{tO}(a) \) in the operator norm.

Since \( C^*(G) \) is obviously separable, this proposition proves the desired Properties 2 and 3(a) and hence, it remains to show Property 3(b):

### 4 Condition 3(b)

#### 4.1 Introduction to the setting

For simplicity, in the following, the representations will be identified with their equivalence classes.

Let \( d \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\} \) and \( (J, K) \in \mathcal{M} \) with \( |J| = |K| = d \). Furthermore, fix \( i = i_{J,K} \in \{0, \ldots, r\} \).

Let \( (\pi^V_k)_k = (\pi^V_{tO_k})_k \) be a sequence in \( \Gamma \), whose limit set is located outside \( \Gamma_i \). Since every converging sequence has a properly converging subsequence, it will be assumed that \( (\pi^V_k)_k \) is properly converging and the transition to a subsequence will be omitted.

The corresponding sequence of coadjoint orbits \( (O_k)_k \) is contained in \( (\mathfrak{g}^* / G)_{(J,K)} \) and in particular every \( O_k \) has the same dimension \( 2d \). Moreover, it converges properly to a set of orbits \( L((O_k)_k) \).

In addition, since \( S_i \) is closed, the limit set \( L((\pi^V_k)_k) \) of the sequence \( (\pi^V_k)_k \) is contained in \( S_{i-1} \) and therefore for every element \( O \in L((O_k)_k) \) there exists a pair \( (J_O, K_O) < (J, K) \) such that \( \pi^V_{tO} \in \Gamma_{J_O,K_O} \) or equivalently, \( O \in (\mathfrak{g}^* / G)_{(J_O,K_O)} \).
4.2 Changing the Jordan-Hölder basis.

Let \( \tilde{\ell} \in \tilde{O} \subset L((O_k)_k) \). Then, there exists a sequence \( (\tilde{\ell}_k)_k \) in \( O_k \) such that \( \tilde{\ell} = \lim_{k \to \infty} \tilde{\ell}_k \).

Since one is interested in the orbits \( O_k = \tilde{\ell}_k + g(\tilde{\ell}_k)^{\perp} \), one can change the sequence \( (\tilde{\ell}_k)_k \) to a sequence \( (\ell_k)_k \) by letting \( \ell_k(A) = 0 \) for every \( A \in g(\tilde{\ell}_k)^{\perp} = g(\ell_k)^{\perp} \).

Thus, one obtains another converging sequence \( (\ell_k)_k \) in \( (O_k)_k \) whose limit \( \ell \) is located in an orbit \( O \subset L((O_k)_k) \).

As above, one can suppose that the subalgebras \( g(\ell_k)_k \) converge to a subalgebra \( u \), whose corresponding Lie group \( \exp(u) \) is denoted by \( U \). These subalgebras \( g(\ell_k)_k \) can be written as

\[
g(\ell_k) = [g, g] \oplus s_k,
\]

where \( s_k \subset [g, g]^{\perp} \). In addition, let \( n_{k,0} \) be the kernel of \( \ell_k|_{[g, g]} \) and \( s_{k,0} \) the kernel of \( \ell_k|_{[g, g]} \) for all \( k \in \mathbb{N} \). One can assume that \( s_{k,0} \neq s_k \) and choose \( T_k \in s_k \) orthogonal to \( s_{k,0} \) of length 1. The case \( s_{k,0} = s_k \) for \( k \in \mathbb{N} \), being easier, will be omitted.

Similarly, choose \( Z_k \in [g, g] \) orthogonal to \( n_{k,0} \) of length 1. One sees that such a \( Z_k \) must exist: If \( \ell_k|_{[g, g]} = 0 \) for \( k \in \mathbb{N} \), then \( \pi_{T_{O_k}}^V \) is a character and thus contained in \( S_0 = \Gamma_0 \). But \( S_0 = \Gamma_0 \) is closed and thus \( (\pi_{T_{O_k}}^V)_k \) cannot have a limit set outside \( \Gamma_0 \).

Furthermore, let \( r_k = g(\ell_k)^{\perp} \subset g \).

One can assume that, passing to a subsequence if necessary, \( \lim_{k \to \infty} Z_k =: Z \), \( \lim_{k \to \infty} T_k =: T \) and \( \lim_{k \to \infty} r_k =: r \) exist.

Now, new polarizations \( p_k \) in \( \ell_k \) are needed:

The restriction to \( r_k \) of the skew-form \( B_k := B_{\ell_k} \) defined in Chapter 2 is non-degenerate on \( r_k \) and there exists an invertible endomorphism \( S_k \) of \( r_k \) such that

\[
\langle x, S_k(x') \rangle = B_k(x, x') \quad \forall \ x, x' \in r_k.
\]

Then \( S_k \) is skew-symmetric, i.e. \( S_k^t = -S_k \), and with the help of Lemma 6.1 one can decompose \( r_k \) into an orthogonal direct sum

\[
r_k = \sum_{j=1}^d V_j^k
\]

of two-dimensional \( S_k \)-invariant subspaces. Choose an orthonormal basis \( \{X_j^k, Y_j^k\} \) of \( V_j^k \). Then,

\[
[X_i^k, X_j^k] \in n_{k,0} \quad \forall \ i, j \in \{1, ..., d\},
\]

\[
[Y_i^k, Y_j^k] \in n_{k,0} \quad \forall \ i, j \in \{1, ..., d\}
\]

and

\[
[X_i^k, Y_j^k] = \delta_{i,j} c_j^k Z_k \mod n_{k,0} \quad \forall \ i, j \in \{1, ..., d\},
\]

where \( 0 \neq c_j^k \in \mathbb{R} \) and \( \sup_{k \in \mathbb{N}} c_j^k < \infty \) for every \( j \in \{1, ..., d\} \).

Again, by passing to a subsequence if necessary, the sequence \( (c_j^k)_k \) converges for every \( j \in \{1, ..., d\} \) to some \( c_j \).

Since \( X_j^k, Y_j^k \in r_k \) and \( \ell_k(A) = 0 \) for every \( A \in r_k \), \( \ell_k(X_j^k) = \ell_k(Y_j^k) = 0 \) for all \( j \in \{1, ..., d\} \).

Furthermore, one can suppose that the sequences \( (X_j^k)_k, (Y_j^k)_k \) converge in \( g \) to vectors \( X_j, Y_j \) which form a basis modulo \( u \) in \( g \).

It follows that

\[
\langle \ell_k, [X_j^k, Y_j^k] \rangle = c_j^k \lambda_k, \text{ where } \lambda_k = \langle \ell_k, Z_k \rangle \xrightarrow{k \to \infty} \langle \ell, Z \rangle =: \lambda.
\]

As \( Z_k \) was chosen orthogonal to \( n_{k,0} \), \( \lambda_k \neq 0 \) for every \( k \).
Now, let

\[ p_k := \text{span}\{Y^k_1, \ldots, Y^k_d, g(\ell_k)\} \]

and \( P_k := \exp(p_k) \). Then \( p_k \) is a polaratization at \( \ell_k \). Furthermore, define the representation \( \pi_k \) as

\[ \pi_k := \text{ind}_{P_k}^G \chi_{\ell_k}. \]

Then, since \( \pi_k \), as well as \( \pi_k^\ell \), are induced representations of polarizations and of the characters \( \chi_{\ell_k} \) and \( \chi_{\ell_k} \), whereat \( \ell_k \) and \( \ell_k \) lie in the same coadjoint orbit \( O_k \), the two representations are equivalent, as observed in Chapter 4.

Let \( a_k := n_{k,0} + s_{k,0} \). Then \( a_k \) is an ideal of \( g \) on which \( \ell_k \) is 0. Therefore, the normal subgroup \( \exp(a_k) \) is contained in the kernel of the representation \( \pi_k \). Moreover, let \( a := \lim_{k \to \infty} a_k \).

In addition, let \( p \in \mathbb{N}, \tilde{p} \in \{1, \ldots, p\} \) and let \( \{A^k_1, \ldots, A^k_\tilde{p}\} \) denote an orthonormal basis of \( n_{k,0} \), the part of \( a_k \) which lies inside \([g, g]\), and \( \{A^k_\tilde{p}+1, \ldots, A^k_p\} \) an orthonormal basis of \( a_{k,0} \), the part of \( a_k \) outside \([g, g]\). Then \( \{A^k_1, \ldots, A^k_p\} \) is an orthonormal basis of \( a_k \) and as above, one can assume that \( \lim_{k \to \infty} A^k_j = A_j \) exists for all \( j \in \{1, \ldots, p\} \).

Now, for every \( k \in \mathbb{N} \) one can take as an orthonormal basis for \( g \) the set of vectors

\[ \{X^k_1, \ldots, X^k_d, Y^k_1, \ldots, Y^k_d, T_k, Z_k, A^k_1, \ldots, A^k_p\} \]

as well as the set

\[ \{X_1, \ldots, X_d, Y_1, \ldots, Y_d, T, Z, A_1, \ldots, A_p\}. \]

This gives the following Lie brackets:

\[
\begin{align*}
[X^k_i, Y^k_j] &= \delta_{i,j} c^k_j Z_k \mod a_k, \\
[X^k_i, X^k_j] &= 0 \mod a_k \quad \text{and} \\
[Y^k_i, Y^k_j] &= 0 \mod a_k.
\end{align*}
\]

The vectors \( Z_k \) and \( T_k \) are central modulo \( a_k \).

Before starting the analysis of \( (\pi_k)_{k \in \mathbb{N}} \), some notations have to be introduced:

### 4.3 Definitions

Choose for \( j \in \{1, \ldots, d\} \) the Schwartz functions \( \eta_j \in \mathcal{S}(\mathbb{R}) \) such that \( \|\eta_j\|_{L^2(\mathbb{R})} = 1 \) and \( \|\eta_j\|_{L^\infty(\mathbb{R})} \leq 1 \).

Furthermore, for \( x_1, \ldots, x_d, y_1, \ldots, y_d, t, z, a_1, \ldots, a_p \in \mathbb{R} \), write

\[
\begin{align*}
(x)_k := (x_1, \ldots, x_d)_k := \sum_{j=1}^d x_j X^k_j, \\
(y)_k := (y_1, \ldots, y_d)_k := \sum_{j=1}^d y_j Y^k_j, \\
(t)_k := tT_k, \\
(z)_k := zZ_k, \\
(\tilde{a})_k := (a_1, \ldots, \tilde{a})_k := \sum_{j=1}^p a_j A^k_j, \\
(\tilde{a})_k := (a_{\tilde{p}+1}, \ldots, a_p)_k := \sum_{j=\tilde{p}+1}^p a_j A^k_j \quad \text{and} \\
(a)_k := (\tilde{a}, \tilde{a})_k := (a_1, \ldots, a_p)_k = \sum_{j=1}^p a_j A^k_j,
\end{align*}
\]

whereat \( (\cdot, \cdot, \ldots, \cdot)_k \) is defined to be the \( d, \tilde{p}, (p - \tilde{p}) \)- or the \( p \)-tuple with respect to the bases \( \{X^k_1, \ldots, X^k_d\}, \{Y^k_1, \ldots, Y^k_d\}, \{A^k_1, \ldots, A^k_p\}, \{A^k_{\tilde{p}+1}, \ldots, A^k_p\} \) and \( \{A^k_1, \ldots, A^k_p\} \), respectively, and let

\[
\begin{align*}
(g)_k := (x_1, \ldots, x_d, y_1, \ldots, y_d, t, z, a_1, \ldots, a_p)_k := ((x)_k, (y)_k, (t)_k, (z)_k, (\tilde{a})_k, (\tilde{a})_k) \\
&= ((x)_k, (h)_k) \\
&= \sum_{j=1}^d x_j X^k_j + \sum_{j=1}^d y_j Y^k_j + tT_k + zZ_k + \sum_{j=1}^p a_j A^k_j,
\end{align*}
\]

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where \((h)_k\) is in the polarization \(p_k\) and the \((2d + 2 + p)\)-tuple \((\cdot, \ldots, \cdot)_k\) is regarded with respect to the basis \(\{X_1^k, \ldots, X_d^k, Y_1^k, \ldots, Y_d^k, Tk, Z_k, A_1^k, \ldots, A_\tilde{p}^k\}\).

Moreover, define the limits

\[
(x)_\infty := (x_1, \ldots, x_d)_\infty := \lim_{k \to \infty} (x)_k = \sum_{j=1}^d x_j X_j, \quad (y)_\infty := (y_1, \ldots, y_d)_\infty := \lim_{k \to \infty} (y)_k = \sum_{j=1}^d y_j Y_j,
\]

\[
(t)_\infty := \lim_{k \to \infty} (t)_k = tT, \quad (z)_\infty := \lim_{k \to \infty} (z)_k = zZ, \quad (\hat{a})_\infty := (a_1, \ldots, a_\tilde{p})_\infty := \lim_{k \to \infty} (\hat{a})_k = \sum_{j=1}^\tilde{p} a_j A_j,
\]

\[
(a)_\infty := (\hat{a}, \tilde{a})_\infty = (a_1, \ldots, a_p)_\infty := \lim_{k \to \infty} (a)_k = \sum_{j=1}^p a_j A_j \quad \text{and}
\]

\[
(g)_\infty := (x, y, t, z, \hat{a}, \tilde{a})_\infty := \lim_{k \to \infty} (g)_k = \sum_{j=1}^d x_j X_j + \sum_{j=1}^d y_j Y_j + tT + zZ + \sum_{j=1}^p a_j A_j.
\]

Now, the representations \((\pi_k)_{k \in \mathbb{N}}\) can be computed:

### 4.4 Formula for \(\pi_k\)

Let \(f \in L^1(G)\).

With \(p_k := \langle \ell_k, T_k \rangle\), \(c^k := (c_1^k, \ldots, c_d^k)\) and \((s)_k := (s_1, \ldots, s_d)_k = \sum_{j=1}^d s_j X_j^k\) for \(s_1, \ldots, s_d \in \mathbb{R}\), where again \((\cdot, \ldots, \cdot)_k\) is the \(d\)-tuple with respect to the basis \(\{X_1^k, \ldots, X_d^k\}\), as in (2), the representation \(\pi_k\) acts on \(L^2(G/P_k, \chi_{\ell_k})\) in the following way:

\[
\pi_k((g)_k)\xi((s)_k) = \xi((g)_k^{-1} \cdot (s)_k) = e^{2\pi i \langle (f_k, -y, -t, -z, -\hat{a}, -\tilde{a})_k + ((s)_k + \frac{1}{2}(x)_k - \frac{1}{2}((s)_k)_k)\rangle} \xi((s - x)_k) = e^{2\pi i \langle -t\rho_k - z\lambda_k + \sum_{j=1}^d \lambda_j c_j (s_j - \frac{1}{2}(x)_k)_j\rangle} \xi((s - x)_k) = e^{2\pi i \langle -t\rho_k - z\lambda_k + c^k(s) - \frac{1}{2}(x)_k(s)_k\rangle} \xi((s - x)_k),
\]

since \(\ell_k(Y_j^k) = 0\) for all \(j \in \{1, \ldots, d\}\).

Now, identify \(G\) with \(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{p-\tilde{p}} \cong \mathbb{R}^{2d+2+p}\), let \(\xi \in L^2(\mathbb{R}^d)\) and \(s \in \mathbb{R}^d\). Moreover, identify \(\pi_k\) with a representation acting on \(L^2(\mathbb{R}^d)\) which will also be called \(\pi_k\). To stress the dependence on \(k\) of the above fixed function \(f \in L^1(G)\), denote by \(f_k \in L^1(\mathbb{R}^{2d+2+p})\) the function \(f\) applied to an element in the \(k\)-basis:

\[
f_k(g):= f((g)_k).
\]
Moreover, the two representations $\chi_\pi$ since

$$f_k(g)\pi_k(g)\xi(s)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\beta} f_k(x, y, t, z, \tilde{a}, \tilde{a}) e^{2\pi i (-t \rho_k - z \lambda_k + \lambda_k \xi(s - \frac{1}{2}x))} \xi(s - x) \, d(x, y, t, z, \tilde{a}, \tilde{a})$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^\beta} f_k(s - x, y, t, z, \tilde{a}, \tilde{a}) e^{2\pi i (-t \rho_k - z \lambda_k + \lambda_k \xi(s + x))} \xi(x) \, d(x, y, t, z, \tilde{a}, \tilde{a})$$

$$= \int_{\mathbb{R}^d} F^{2,3,4,5,6}_k (s - x, -\frac{\lambda_k \xi(s + x) - \rho_k}{2}, \rho_k, \lambda_k, 0, 0) \xi(x) \, dx,$$

where $F^{2,3,4,5,6}_k$ denotes the Fourier transform in the 2nd, 3rd, 4th, 5th and 6th variable.

**4.5 First case**

First consider the case that $L((O_k)_k)$ consists of one single limit point $O$. In this case, for every $k$,

$$2d = \dim(O_k) = \dim(O).$$

Thus, the regarded situation occurs if and only if $\lambda \neq 0$ and $c_j \neq 0$ for every $j \in \{1, \ldots, d\}$.

Consider again the above chosen sequence $(\ell_k)_k$ which converges to $\ell \in O$. As the dimensions of the orbits $O_k$ and $O$ are the same, there exists a subsequence of $(\ell_k)_k$ (which will also be denoted by $(\ell_k)_k$ for simplicity) such that $p := \lim_{k \to \infty} p_{\ell_k}$ is a polarization for $\ell$, but not necessarily the Vergne polarization. Moreover, define $P := exp(p) = \lim_{k \to \infty} P_{\ell_k}$ and let

$$\pi := \text{ind}^{G}_{P} \chi_{\ell}.$$

Now, if one identifies the Hilbert spaces $H_{\pi}^{\pi}$ and $H_{\pi}$ of $\pi^{\pi}_{\ell_k} = \text{ind}^{G}_{F_{\ell_k}} \chi_{\ell_k}$ and $\pi$ with $L^2(\mathbb{R}^d)$, from [I], Theorem 2.3, one can conclude that

$$\|\pi^{\pi}_{\ell_k}(a) - \pi(a)\|_{op} = \|\text{ind}^{G}_{F_{\ell_k}} \chi_{\ell_k}(a) - \text{ind}^{G}_{P} \chi_{\ell}(a)\|_{op} \to 0 \quad \forall \ a \in C^*(G).$$

Since $\pi$ and $\pi^{\pi}_{\ell_k} = \text{ind}^{G}_{F_{\ell_k}} \chi_{\ell}$ are both induced representations of polarizations and of the same character $\chi_{\ell}$, they are equivalent and hence, there exists a unitary intertwining operator

$$F : \mathcal{H}_{\pi^{\pi}} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi} \cong L^2(\mathbb{R}^d)$$

such that $F \circ \pi^{\pi}_{\ell_k} = \pi(a) \circ F \quad \forall \ a \in C^*(G)$.

Moreover, the two representations $\pi^{\pi}_{\ell_k} = \pi^{\pi}_{\ell_{O_k}} = \text{ind}^{G}_{F_{\ell_{O_k}}} \chi_{\ell_{O_k}}$ and $\pi^{\pi}_{\ell_k} = \text{ind}^{G}_{F_{\ell_k}} \chi_{\ell_k}$ are equivalent for every $k \in \mathbb{N}$ because $\ell_{O_k}$ and $\ell_k$ are located in the same coadjoint orbit $O_k$ and $p_{\ell_{O_k}}^{\pi}$ and $p_{\ell_k}^{\pi}$ are polarizations. Thus there exist further unitary intertwining operators

$$F_{\ell_k} : \mathcal{H}_{\pi^{\pi}} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi^{\pi}_{\ell_k}} \cong L^2(\mathbb{R}^d)$$

with $F_{\ell_k} \circ \pi^{\pi}_{\ell_k}(a) = \pi^{\pi}_{\ell_k}(a) \circ F_{\ell_k} \quad \forall \ a \in C^*(G)$.

Now, define the required operators $\tilde{\nu}_k$ as

$$\tilde{\nu}_k(\varphi) := F_{\ell_k}^{*} \circ F \circ \varphi(\pi^{\pi}_{\ell_k}) \circ F^{*} \circ F_{\ell_k} \quad \forall \ \varphi \in CB(S_{i-1}),$$

where $CB(S_{i-1})$ denotes the space of continuous bounded functions on the boundary of $S_{i-1}$.
which makes sense since $\pi^Y_k$ is a limit point of the sequence $(\pi^Y_k)_k$ and hence contained in $S_{i-1}$, as seen in Chapter 4.4.

As $\varphi(\pi^Y_k) \in B(L^2(\mathbb{R}^d))$ and $F$ and $F_k$ are intertwining operators and thus bounded, the image of $\tilde{\nu}_k$ is contained in $B(L^2(\mathbb{R}^d))$, as requested.

Next, it needs to be shown that $\tilde{\nu}_k$ is bounded: By the definition of $\|\cdot\|_{S_{i-1}}$, one has for every $\varphi \in CB(S_{i-1})$

$$\|\tilde{\nu}_k(\varphi)\|_{op} = \|F_k^* \circ F \circ \varphi(\pi^Y_k) \circ F^* \circ F_k\|_{op} \leq \|\varphi(\pi^Y_k)\|_{op} \leq \|\varphi\|_{S_{i-1}}.$$  

In addition, one can easily observe that $\tilde{\nu}_k$ is involutive: For every $\varphi \in CB(S_{i-1})$

$$\tilde{\nu}_k(\varphi)^* = (F_k^* \circ F \circ \varphi(\pi^Y_k) \circ F^* \circ F_k)^* = F_k^* \circ F \circ \varphi*(\pi^Y_k) \circ F^* \circ F_k = \tilde{\nu}_k(\varphi^*).$$

Now, the last thing to check is the required convergence of Condition 3(b): For every $a \in C^*(G)$

$$\|\pi^Y_k(a) - \tilde{\nu}_k(F(a))\|_{op} = \|\pi^Y_k(a) - F_k^* \circ F \circ \pi^Y_k(a) \circ F^* \circ F_k\|_{op} = \|F_k^* \circ \pi^Y_k(a) \circ F_k - F_k^* \circ \pi(a) \circ F_k\|_{op} = \|F_k^* \circ (\pi^Y_k -\pi)(a) \circ F_k\|_{op} \leq \|\pi^Y_k(a) - \pi(a)\|_{op} \xrightarrow{k \to \infty} 0.$$  

Therefore, the representations $(\pi^Y_k)_k$ and the constructed $(\tilde{\nu}_k)_k$ fulfill Condition 3(b) and thus, in this case, the claim is shown.

### 4.6 Second case

In the second case the situation that $\lambda = 0$ or $c_j = 0$ for every $j \in \{1, \ldots, d\}$ must be considered.

In this case,

$$\langle \ell_k, [X_j^k, Y_j^k] \rangle = c_j^k \lambda \xrightarrow{k \to \infty} c_j \lambda = 0 \quad \forall \ j \in \{1, \ldots, d\},$$

while $c_j^k \lambda_k \neq 0$ for every $k$ and every $j \in \{1, \ldots, d\}$.

Then $\ell|_{\{0\}} = 0$ and so every limit orbit $O$ in the set $L((O_k)_k)$ has the dimension 0.

As in Calculation [2] in Chapter 4.3 identify $G$ again with $\mathbb{R}^{2d+2+p}$. From now on, this identification will be used most of the time. Only in some cases where one applies $\ell_k$ or $\ell$ and thus it is important to know whether one is using the basis depending on $k$ or the limit basis, the calculation will be done in the above defined bases $(\cdot)_k$ or $(\cdot)_\infty$.

Now, adapt the methods developed in [7] to this given situation.

Let $s = (s_1, \ldots, s_d)$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ and define

$$\eta_{k,\alpha,\beta}(s) = \eta_{k,\alpha,\beta}(s_1, \cdots, s_d) := \varepsilon^{2\pi i os} \prod_{j=1}^d [\lambda_k c_j^k]^{*} \eta_j \left( \left[ \lambda_k c_j^k \right] \frac{1}{2} \left( s_j + \frac{\beta_j}{\lambda_k c_j^k} \right) \right).$$

Moreover, let $c_{\alpha,\beta}^k$ be the coefficient function defined by

$$c_{\alpha,\beta}^k(g) := \langle \pi_k(g) \eta_{k,\alpha,\beta}, \eta_{k,\alpha,\beta} \rangle \quad \forall \ \ g \in G \cong \mathbb{R}^{2d+2+p}$$

and $\ell_{\alpha,\beta}$ the linear functional

$$\ell_{\alpha,\beta}(g) = \ell_{\alpha,\beta}(x, y, t, z, a) := \alpha x + \beta y \quad \forall \ g = (x, y, t, z, a) \in G \cong \mathbb{R}^{2d+2+p}.$$  

Then, as in [7], one can show by similar computations that the functions $c_{\alpha,\beta}^k$ converge uniformly on compacta to the character $\chi_{\ell + \ell_{\alpha,\beta}}.$
4.6.1 Definition of the \( \nu_k \)'s

For \( 0 \neq \tilde{\eta} \in L^2(G/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^d) \) let

\[
P_{\tilde{\eta}} : L^2(\mathbb{R}^d) \to \mathbb{C} \tilde{\eta}, \quad \xi \mapsto \tilde{\eta}(\xi, \tilde{\eta}).
\]

Then \( P_{\tilde{\eta}} \) is the orthogonal projection onto the space \( \mathbb{C} \tilde{\eta} \).

Let \( h \in C^*(G/U, \chi_\ell) \). Again, identify \( G/U \) with \( \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d} \) and as already introduced in Chapter 4.4 in order to show the dependence on \( k \), here the utilization of the limit basis will be expressed by an index \( \infty \) if necessary:

\[
h_\infty(x, y) := h((x, y)_\infty).
\]

Now, \( \hat{h}_\infty \) can be seen as a function in \( C_0(\ell + u^2) \cong C_0(\mathbb{R}^{2d}) \) and, using this identification, define the linear operator

\[
\nu_k(h) := \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) P_{\eta_k, \ell, \tilde{\sigma}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|}.
\]

Then, the following proposition holds:

**Proposition 4.1.**

1. For every \( k \in \mathbb{N} \) and \( h \in S(G/U, \chi_\ell) \) the integral defining \( \nu_k(h) \) converges in the operator norm.
2. The operator \( \nu_k(h) \) is compact and \( \|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U, \chi_\ell)} \).
3. \( \nu_k \) is involutive, i.e. \( \nu_k(h)^* = \nu_k(h^*) \) for every \( h \in C^*(G/U, \chi_\ell) \).

**Proof:**

1. Let \( h \in S(G/U, \chi_\ell) \cong S(\mathbb{R}^{2d}) \). Since

\[
\|P_{\eta_k, \ell, \tilde{\sigma}}\|_{op} = \|\eta_k, \ell, \tilde{\sigma}\|_{\frac{1}{2}}^2 = 1,
\]

one can estimate the operator norm of \( \nu_k(h) \) as follows:

\[
\|\nu_k(h)\|_{op} = \left\| \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) P_{\eta_k, \ell, \tilde{\sigma}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} \right\|_{op}
\]

\[
\leq \int_{\mathbb{R}^{2d}} \left| \hat{h}_\infty(\tilde{x}, \tilde{y}) \right| \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} = \frac{\|\hat{h}\|_{L^1(\mathbb{R}^{2d})}}{\prod_{j=1}^d |\lambda_k c_j^k|}.
\]

Therefore, the convergence of the integral \( \nu_k(h) \) in the operator norm is shown for \( h \in S(\mathbb{R}^{2d}) \cong S(G/U, \chi_\ell) \).

2. First, let \( h \in S(G/U, \chi_\ell) \cong S(\mathbb{R}^{2d}) \).

Define for \( s = (s_1, \ldots, s_d) \in \mathbb{R}^d \)

\[
\eta_{k, \beta}(s) := \prod_{j=1}^d |\lambda_k c_j^k|^{\frac{1}{2}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}} (s_j + \frac{\beta_j}{\lambda_k c_j^k}) \right).
\]

Then

\[
\eta_{k, \alpha, \beta}(s) = e^{2\pi i s \cdot \eta_{k, \beta}(s)}
\]
and thus one has for \( \xi \in \mathcal{S}(\mathbb{R}^d) \) and \( s \in \mathbb{R}^d \)

\[
\nu_k(h)\xi(s)
= \int_{\mathbb{R}^d} \hat{h}_{\infty}(\tilde{x}, \tilde{y}) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^{d} |\lambda_k c_j|^2} \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s}(s) \left( \int_{\mathbb{R}^d} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^{d} |\lambda_k c_j|^2} \right) \eta_{\tilde{x}, \tilde{y}, s}(s) d\tilde{x} d\tilde{y}.
\]

Hence, as the kernel function

\[
h_{k}(s, r) := \int_{\mathbb{R}^d} \hat{h}_{\infty}(s-r, \tilde{y}) \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s}(s) \frac{d\tilde{y}}{\prod_{j=1}^{d} |\lambda_k c_j|^2}
\]

of \( \nu_k(h) \) is in \( \mathcal{S}(\mathbb{R}^{2d}) \), \( \nu_k(h) \) is a compact operator.

Now it will be shown that

\[
\|\nu_k(h)\|_{op} \leq \|\hat{h}\|_{\infty}.
\]

For \( \xi \in \mathcal{S}(\mathbb{R}^d) \) one has similar as in \([7]\)

\[
\|\nu_k(h)\xi\|_2^2
= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{h}_{\infty}^2(s-r, \tilde{y}) \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s}(s) \frac{d\tilde{y}}{\prod_{j=1}^{d} |\lambda_k c_j|^2} dr ds \right|^2 ds
= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{h}_{\infty}^2(s-r, \tilde{y}) * (\xi \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s})(s) \frac{d\tilde{y}}{\prod_{j=1}^{d} |\lambda_k c_j|^2} \right|^2 ds
\]

Cauchy–Schwarz,

\[
\leq \prod_{j=1}^{d} |\lambda_k c_j|^2 \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{h}_{\infty}^2(s-r, \tilde{y}) * (\xi \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s})(s) \right|^2 ds \frac{d\tilde{y}}{\prod_{j=1}^{d} |\lambda_k c_j|^2} \]

Plancherel

\[
\leq \|\hat{h}\|_{\infty}^2 \prod_{j=1}^{d} \frac{1}{|\lambda_k c_j|^2} \int_{\mathbb{R}^d} \|\xi \eta_{\mathcal{R}, \tilde{x}, \tilde{y}, s}\|_2^2 d\tilde{y}
= \prod_{j=1}^{d} \frac{1}{|\lambda_k c_j|^2} \int_{\mathbb{R}^d} \|\xi\|_2^2.
\]
Thus, since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$,
\[
\|\nu_k(h)\|_{op} = \sup_{\xi \in L^2(\mathbb{R}^d), \|\xi\|_2 = 1} \|\nu_k(h)(\xi)\|_2 \leq \|\hat{h}\|_\infty
\]
for $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_\ell)$. Therefore, with the density of $\mathcal{S}(G/U, \chi_\ell)$ in $C^*(G/U, \chi_\ell)$, one gets the compactness of the operator $\nu_k(h)$ for $h \in C^*(G/U, \chi_\ell)$, as well as the desired inequality
\[
\|\nu_k(h)\|_{op} \leq \|\hat{h}\|_{C^*(G/U, \chi_\ell)}.
\]

3. The proof of the involutivity of $\nu_k$ is straightforward.

This proposition firstly shows that the image of the operator $\nu_k$ is located in $\mathcal{B}(L^2(\mathbb{R}^d)) = \mathcal{B}(\mathcal{H}_i)$ as required in Condition 3(b). Secondly, the proposition gives the boundedness and the involutivity of the linear mappings $\nu_k$ for every $k \in \mathbb{N}$. For the analysis of the sequence $(\pi_k)_k$, it remains to show the convergence condition.

### 4.6.2 Theorem - Second Case

**Theorem 4.2.**

Define as in Subsection 2.4
\[
p_{G/U} : L^1(G) \to L^1(G/U, \chi_\ell),
\]
\[
p_{G/U}(f)(\tilde{g}) := \int_U f(\tilde{g}u)\chi_\ell(u) \, du \quad \forall \tilde{g} \in G \quad \forall f \in L^1(G)
\]
and canonically extend $p_{G/U}$ to a mapping going from $C^*(G)$ to $C^*(G/U, \chi_\ell)$. Furthermore let $a \in C^*(G)$. Then
\[
\lim_{k \to \infty} \| \pi_k(a) - \nu_k(p_{G/U}(a)) \|_{op} = 0.
\]

**Proof:**

For $u = (t, z, \hat{a}, \hat{a})_\infty \in U = \text{span}\{T, Z, A_1, ..., A_{\hat{p}}, A_{\hat{p}+1}, ..., A_p\}$
\[
\chi_\ell(u) = e^{-2\pi i (t, z, \hat{a}, \hat{a})_\infty} = e^{-2\pi i (t \hat{p} + z \lambda)}
\]
and therefore, identifying $U$ again with $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\hat{p}} \times \mathbb{R}^{p - \hat{p}}$ and $L^1(G/U, \chi_\ell)$ with $L^1(\mathbb{R}^{2d})$, for $f \in L^1(G) \cong L^1(\mathbb{R}^{2d+2+p})$ and $\tilde{g} = (\hat{x}, \hat{y}, 0, 0, 0, 0) \in \mathbb{R}^{2d}$ one has
\[
\left(p_{G/U}(f)\right)_\infty(\tilde{g}) = \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\hat{p}} \times \mathbb{R}^{p - \hat{p}}} f_\infty(\hat{x}, \hat{y}, \tilde{\xi}, \tilde{\zeta}, \hat{\alpha}, \hat{\alpha}) e^{-2\pi i (\hat{t} \hat{p} + \hat{z} \lambda)} \, d(0, 0, \tilde{\xi}, \tilde{\zeta}, \hat{\alpha}, \hat{\alpha})
\]
\[
= \int_{\mathbb{R}^{3,4,5,6}} f_\infty(\hat{x}, \hat{y}, \rho, \lambda, 0, 0),
\]
whereat $f_\infty(\hat{x}, \hat{y}, \rho, \lambda, 0, 0) = f((\hat{x}, \hat{y}, \rho, \lambda, 0, 0)_\infty)$.

Now, let $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$ such that its Fourier transform in $[\mathfrak{g}, \mathfrak{g}]$ has a compact support on $G \cong \mathbb{R}^{2d+2+p}$. If one then writes the elements $g$ of $G$ as $g = (x, y, t, z, \hat{a}, \hat{a})$ like above, whereat
\[
x \in \text{span}\{X_1, ..., X_d\}, \quad y \in \text{span}\{Y_1, ..., Y_d\}, \quad t \in \text{span}\{T\},
\]
\[
z \in \text{span}\{Z\}, \quad \hat{a} \in \text{span}\{A_1, ..., A_{\hat{p}}\}, \quad \hat{a} \in \text{span}\{A_{\hat{p}+1}, ..., A_p\}
\]
or, respectively

\[ x \in \text{span}\{X^k_1 \times \ldots \times \text{span}\{X^k_d\}, \quad y \in \text{span}\{Y^k_1 \times \ldots \times \text{span}\{Y^k_d\}, \quad t \in \text{span}\{T_k\}, \]

\[ z \in \text{span}\{Z_k\}, \quad \tilde{a} \in \text{span}\{A^k_1 \times \ldots \times \text{span}\{A^k_d\}, \quad \tilde{a} \in \text{span}\{A^k_{p+1} \times \ldots \times \text{span}\{A^k_p\}, \]

this means that the partial Fourier transform \( \hat{f}_{\mathbb{J}} \) has a compact support in \( G \), since \( [\mathfrak{g}, \mathfrak{g}] = \text{span}\{Z_k, A^k_1, \ldots, A^k_p\} = \text{span}\{Z, A_1, \ldots, A_p\} \).

Moreover, let \( s \in \mathbb{R}^d \) and define

\[ \eta_{k,0}(s) := \prod_{j=1}^{d} |\lambda_k c_j^k| \hat{\xi}_j \left( \frac{1}{|\lambda_k c_j^k|} \xi_j(s) \right). \]

(Compare the definition of \( \eta_{k,\beta} \) in the last proof.)

If \( \xi \in \mathcal{S}(\mathbb{R}^d) \), one has

\[
\int_{\mathbb{R}^d} \hat{f}_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \xi(r) \, dr
\]

\[
= \int_{\mathbb{R}^d} \frac{1}{|\lambda_k c_j^k|} \Xi_j \left( \frac{1}{|\lambda_k c_j^k|} \xi_j(s) \right) \hat{\xi}_j \hat{\eta}_{k,0}(\tilde{y}) \eta_{k,0}(\tilde{y}) \, d\tilde{y} dr,
\]

The just obtained integrals are now divided into five parts. To do so, new functions \( q_k, u_k, v_k, n_k \) and \( w_k \) are defined:

\[ q_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \hat{\eta}_{k,0}(\tilde{y} + r - s) \left( f_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \right) dr,
\]

\[ u_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \hat{\eta}_{k,0}(\tilde{y} + r - s) \left( f_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \right) dr,
\]

\[ v_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \hat{\eta}_{k,0}(\tilde{y} + r - s) \left( f_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \right) dr,
\]

\[ n_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \hat{\eta}_{k,0}(\tilde{y} + r - s) \left( f_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \right) dr,
\]

\[ w_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \hat{\eta}_{k,0}(\tilde{y} + r - s) \left( f_{\mathbb{J}}(s, r, -\lambda_k c_k^k) \right) dr,
\]
\[ u_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \overline{\pi_{k,0}(\tilde{y} + r - s)} \left( \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda, 0, 0) - \tilde{f}_k^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) \right) dr, \]

\[ n_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \overline{\pi_{k,0}(\tilde{y} + r - s)} \left( \tilde{f}_k^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) - \tilde{f}_k^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) \right) dr \]

and

\[ w_k(s) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi(r) \eta_{k,0}(\tilde{y}) \left( \overline{\eta_{k,0}(\tilde{y})} - \pi_{k,0}(\tilde{y} + r - s) \right) \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0) d\tilde{y} dr. \]

Then,

\[ (\pi_k(f) - \nu_k(p_{G/U}(f))) \xi(s) = \int_{\mathbb{R}^d} q_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) \tilde{y} d\tilde{y} + \int_{\mathbb{R}^d} u_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) \tilde{y} d\tilde{y} + \int_{\mathbb{R}^d} v_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) \tilde{y} d\tilde{y} + \int_{\mathbb{R}^d} n_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) \tilde{y} d\tilde{y} + w_k(s). \]

In order to show that

\[ \| \pi_k(f) - \nu_k(p_{G/U}(f)) \|_{op} \xrightarrow{k \to \infty} 0, \]

it suffices to prove that there are \( \kappa_k, \gamma_k, \delta_k, \omega_k \) and \( \epsilon_k \) which are going to 0 for \( k \to \infty \), such that

\[ \|q_k\|_2 \leq \kappa_k \|\xi\|_2; \quad \|u_k\|_2 \leq \gamma_k \|\xi\|_2; \quad \|v_k\|_2 \leq \delta_k \|\xi\|_2; \quad \|n_k\|_2 \leq \omega_k \|\xi\|_2; \quad \text{and} \quad \|w_k\|_2 \leq \epsilon_k \|\xi\|_2. \]

First, regard the last factor of the function \( q_k \):

\[ \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0) - \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda_k, 0, 0) \]

\[ = (\rho_k - \rho) \int_0^1 \partial_t \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho + t(\rho_k - \rho), \lambda_k, 0, 0) dt. \]

Thus, since \( f \) is a Schwartz function, one can find a constant \( C_1 > 0 \) (depending on \( f \)), such that

\[ \left| \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0) - \tilde{f}_k^{2,3,4,5,6}(s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda_k, 0, 0) \right| \leq |\rho_k - \rho| \frac{C_1}{(1 + \|s - r\|)^{2d}}. \]
Hence, one gets the following estimation for $q_k$:

$$
\|q_k\|_2^2 = \int_{\mathbb{R}^d} |q_k(s, \bar{y})|^2 \, d(s, \bar{y})
$$

$$
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla \eta_k,0(\bar{y} + r - s)||\rho_k - \rho| \frac{C_1}{(1 + \|s - r\|)^2d} \, dr \right)^2 \, d(s, \bar{y})
$$

$$
\leq C_2^2 |\rho_k - \rho|^2 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla \eta_k,0(\bar{y} + r - s)| \frac{\xi(r)}{(1 + \|s - r\|)^d} \, dr \right)^2 \, d(s, \bar{y})
$$

$$
= C_4' |\rho_k - \rho|^2 \int_{\mathbb{R}^d} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^2d} |\eta_k,0(\bar{y} + r - s)|^2 \, d(r, s, \bar{y})
$$

$$
\|q_{k,0}\|_2 = 1 \leq C_4'' |\rho_k - \rho|^2 \|\xi\|_2^2,
$$

where $C_4' > 0$ and $C_4'' > 0$ are matching constants depending on $f$. Thus, for $\kappa_k := \sqrt{C_4''} |\rho_k - \rho|$, $\kappa_k \xrightarrow{k \to \infty} 0$, since $\rho_k \xrightarrow{k \to \infty} \rho$, and

$$
\|q_k\|_2 \leq \kappa_k \|\xi\|_2.
$$

As $\lambda_k \xrightarrow{k \to \infty} \lambda$, the estimation for the function $u_k$ can be done analogously.

Now, regard $v_k$. Like for $q_k$ and $u_k$, one has

$$
\int_0^1 \partial_t \int_{\mathbb{R}^d} |\nabla \eta_k,0(\bar{y} + r - s)| \cdot \left| c_k \cdot \frac{1}{2}(r - s) - (r - s + \bar{y}) \right| \, dt,
$$

where $\cdot$ is the scalar product, and hence there exists again an on $f$ depending constant $C_3$ such that

$$
|f_k,2,3,4,5,6(s - r, -\frac{\lambda_k c_k}{2}(s + r), \rho, \lambda, 0, 0) - f_k,2,3,4,5,6(s - r, \lambda_k c_k(\bar{y} - s), \rho, \lambda, 0, 0)|
$$

$$
\leq |\lambda_k| \left( \|c_k(r - s)\| + \|c_k(r - s + \bar{y})\| \right) \frac{C_3}{(1 + \|s - r\|)^{2d + 1}}.
$$

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Therefore, defining $\tilde{\eta}_j(t) := \| t \| \eta_j(t)$, one gets a similar estimation for $v_k$:

$$\| v_k \|^2 \leq \int_{\mathbb{R}^d} |v_k(s, \tilde{y})|^2 \, d(s, \tilde{y})$$

$$\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\xi(r)\eta_{k,0}(\tilde{y} + r - s)| |\lambda_k| \left( \| e^k(r - s) \| + \| e^k(r - s + \tilde{y}) \| \right) \right) \frac{C_3}{\left(1 + \|s - r\|\right)^{2d+1}} \, dr \, d(s, \tilde{y})$$

$$\leq 2C_3' \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{|\xi(r)|^2}{\left(1 + \|s - r\|\right)^{2d+2}} \left| \eta_{k,0}(\tilde{y} + r - s) \right|^2 |\lambda_k|^2 \| e^k(r - s + \tilde{y}) \|^2 \, d(r, s, \tilde{y}) \right)$$

$$+ 2C_3' \int_{\mathbb{R}^d} \frac{|\xi(r)|^2}{\left(1 + \|s - r\|\right)^{2d+2}} \left| \eta_{k,0}(\tilde{y} + r - s) \right|^2 |\lambda_k|^2 \| e^k(r - s) \|^2 \, d(r, s, \tilde{y})$$

$$\leq 2C_3' \| \lambda_k e^k \|^2 \int_{\mathbb{R}^d} \frac{|\xi(r)|^2}{\left(1 + \|s - r\|\right)^{2d+2}} \left| \eta_{k,0}(\tilde{y} + r - s) \right|^2 \, d(r, s, \tilde{y})$$

$$\leq 2C_3' \| \lambda_k e^k \|^2 \left( \int_{\mathbb{R}^d} \frac{|\xi(r)|^2}{\left(1 + \|s - r\|\right)^{2d+2}} \, d(r, s) \right)$$

$$\leq 2C_3' \| \lambda_k e^k \|^2 \left( \prod_{j=1}^d \| \lambda_k e_j^k \|^2 \| |\tilde{\eta}_j(r)\| \|^2 \, d(r, s, \tilde{y}) \right)$$

$$\leq 2C_3' \| \lambda_k e^k \|^2 \left( \prod_{j=1}^d \| \tilde{\eta}_j \|^2 + \| \lambda_k e^k \|^2 \| \xi \|^2 \right)$$

with constants $C_3' > 0$ and $C_5'' > 0$, again depending on $f$.

Now, since $\lambda_k e^k \rightharpoonup_k 0$, $\delta_k := \left( C_3' \left( \prod_{j=1}^d \| \tilde{\eta}_j \|^2 + \| \lambda_k e^k \|^2 \right)^2 \right)^{1/2}$ fulfills $\delta_k \rightharpoonup_k 0$ and

$$\| v_k \|_2 \leq \delta_k \| \xi \|_2.$$

For the estimation of $\eta_k$, the fact that the Fourier transform in $[g, \tilde{g}]$, i.e. in the 4th and 5th
variable, $\hat{f}^{4.5} =: \hat{f}$ has a compact support will be needed. Therefore, let the support of $f$ be located in the compact set

$$K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{p-\rho}$$

and let $K := K_2 \times K_3 \times K_6 \subset \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{p-\rho}$.

Furthermore, since a Fourier transform is independent of the choice of the basis $[\mathfrak{g}, \mathfrak{g}] = \text{span} \{ Z_k, A_{k, p} \}_{k=1}^4$ expressed in the $k$-basis and its value expressed in the limit basis are the same:

$$f(\cdot, \cdot, (z, \bar{u}), \cdot) = f(\cdot, \cdot, (z, \bar{u})_\infty, \cdot).$$

So, in the course of this proof, the limit basis will be chosen for the representation of the 4th and 5th position of an element $g$. Then

$$\hat{f}_k^{2,3,4,5,6}(s - r, \lambda_k e^k (\bar{y} - s), \rho, \lambda, 0, 0) - \hat{f}_\infty^{2,3,4,5,6}(s - r, \lambda_k e^k (\bar{y} - s), \rho, \lambda, 0, 0)$$

$$= \int_{\mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}^{p-\rho}} \left( \hat{f}_k(s - r, y, t, \lambda, 0, \bar{u}) - \hat{f}_\infty(s - r, y, t, \lambda, 0, \bar{u}) \right) e^{-2\pi i \rho (\bar{y} - s) y + \rho t} d(y, t, \bar{u})$$

$$= \int_{\mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}^{p-\rho}} \left( \hat{f}((s - r, y, t), (\lambda, 0)_\infty, (\bar{u})_k) - \hat{f}((s - r, y, t), (\lambda, 0)_\infty, (\bar{u})_\infty) \right) e^{-2\pi i \rho (\bar{y} - s) y + \rho t} d(y, t, \bar{u})$$

Furthermore,

$$\hat{f}((s - r, y, t), (\lambda, 0)_\infty, (\bar{u})_k) - \hat{f}((s - r, y, t), (\lambda, 0)_\infty, (\bar{u})_\infty)$$

$$= \hat{f} \left( \sum_{i=1}^d (s_i - r_i) X_i^k + \sum_{i=1}^d y_i Y_i^k + t T_k + \lambda Z + \sum_{i=p+1}^p a_i A_i^k \right)$$

$$- \hat{f} \left( \sum_{i=1}^d (s_i - r_i) X_i + \sum_{i=1}^d y_i Y_i + t T + \lambda Z + \sum_{i=p+1}^p a_i A_i \right)$$

$$= \left( \sum_{i=1}^d (s_i - r_i)(X_i^k - X_i) + \sum_{i=1}^d y_i (Y_i^k - Y_i) + t(T_k - T) + \sum_{i=p+1}^p a_i (A_i^k - A_i) \right)$$

$$\cdot \int_0^1 \partial \hat{f} \left( \sum_{i=1}^d (s_i - r_i)(X_i^k - X_i) + \sum_{i=1}^d y_i (Y_i^k - Y_i) + t(T_k - T) + \sum_{i=p+1}^p a_i (A_i^k - A_i) \right) d\bar{t}.$$
Now, with the help of the two calculations above, \( \| n_k \|_2^2 \) can be estimated:

\[
\| n_k \|_2^2 = \int_{\mathbb{R}^d} |n_k(s, \tilde{y})|^2 d(s, \tilde{y})
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \xi(r) \nabla_{k,0}(\tilde{y} + r - s) \left( f_k^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) - f_k^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) \right) dr \right)^2 d(s, \tilde{y})
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| \xi(r) \nabla_{k,0}(\tilde{y} + r - s) \right| \left( \| s - r \| \omega_k^1 + \| y \| \omega_k^2 + \| t \| \omega_k^3 + \| \tilde{y} \| \omega_k^4 \right) \right)^2 d(s, \tilde{y})
\]

The Cauchy-Schwarz inequality gives:

\[
C_4 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + \| s - r \|)^{2d+1}} \left| e^{-2\pi i (\lambda_k c^k(\tilde{y} - s) y + \rho t)} \right| d(y, t, \tilde{a}) \right)^2 d(s, \tilde{y})
\]

where \( C_4 > 0 \) and \( C_4' > 0 \) depending on \( f \) and \( \omega_k^{k-\infty} \) for \( i \in \{5, ..., 7\} \). Thus, \( \omega_k := C_4' \omega_k^5 \) fulfills \( \omega_k^{k-\infty} \) and

\[
\| n_k \|_2 \leq \omega_k \| \xi \|_2.
\]

Last, it still remains to examine \( u_k \):

\[
\nabla_{k,0}(\tilde{y}) - \nabla_{k,0}(\tilde{y} + r - s)
= \sum_{j=1}^d (r_j - s_j) \int_0^1 \partial_j \nabla_{k,0}(\tilde{y} + t(r - s)) dt
= \sum_{j=1}^d (r_j - s_j) \int_0^1 \left( \prod_{i=1}^d |\lambda_k c_i^k|^j \nabla_{j}(\nabla_{i} \nabla_{j}(\tilde{y} + t(r - s))) \right) dt.
\]

Thus, since \( f \) and the functions \( \eta_{j} \) are Schwartz functions, one can find an on
\((\eta_j)_{j \in \{1, \ldots, d\}}\) depending constant \(C_5\) such that
\[
\left| \left( \pi_{k,0}(\tilde{y}) - \pi_{k,0}(y + r - s) \right) \right| F^2 3.4.5.6 \left( s - r, -\frac{\lambda_k e^k}{2} (s + r), \rho_k, \lambda_k, 0, 0 \right) \leq \|r - s\| \left( \sum_{j=1}^{d} \prod_{l=1}^{d} |\lambda_k c_l^k| \right) \frac{C_5}{(1 + \|r - s\|)^{2d+1}}.
\]
Now, one has the following estimation for \(\|w_k\|_2\), which is again similar to the above ones:
\[
\|w_k\|_2^2 = \int_{\mathbb{R}^d} |w_k(s)|^2 ds \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi(r)||\eta_{k,0}(\tilde{y})| \|r - s\| \left( \sum_{j=1}^{d} \prod_{l=1}^{d} |\lambda_k c_l^k| \right) \right) \frac{C_5}{(1 + \|r - s\|)^{2d+1}} \|r - s\|^{2d+1} d(r, \tilde{y}, s)
\]
where the constants \(C_5'^{'} > 0\) and \(C_5'^{''} > 0\) depend on \((\eta_j)_{j \in \{1, \ldots, d\}}\). Therefore, for \(\epsilon_k := \left( C_5'^{'} \left( \sum_{j=1}^{d} \prod_{l=1}^{d} |\lambda_k c_l^k| \right) \right)^{1/2}\), the desired properties \(\epsilon_k \xrightarrow{\|\xi\|_2} 0\) and
\[
\|w_k\|_2 \leq \epsilon_k \|\xi\|_2
\]
are fulfilled.

Thus, for those \(f \in S(\mathbb{R}^{2d+2+p}) \cong S(G)\) whose Fourier transform in \([\mathfrak{g}, \mathfrak{g}]\) has a compact support,
\[
\|\pi_k(f) - \nu_k(p_{G/U}(f))\|_{op} = \sup_{\xi \in L^2(\mathbb{R}^d)} \|\left( \pi_k(f) - \nu_k(p_{G/U}(f)) \right)(\xi)\|_{2 \rightarrow \infty} = 0.
\]
Because of the density in \(L^1(G)\) and thus in \(C^*(G)\) of the set of Schwartz functions \(f \in S(G)\) whose partial Fourier transform has a compact support, the claim is true for general \(a \in C^*(G)\).

### 4.6.3 Transition to \((\pi_k^V)^k\)

As for every \(k \in \mathbb{N}\) the two representations \(\pi_k^V\) and \(\pi_k^V\) are equivalent, there exist unitary intertwining operators
\[
F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \rightarrow \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{with} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall \ a \in C^*(G).
\]
Futhermore, since the limit set \(L((\pi_k^V)_k)\) of the sequence \((\pi_k^V)_k\) is contained in \(S_{\ell-1}\), as discussed in Section 4.1 identifying \(\tilde{G}\) with the set of coadjoint orbits \(\mathfrak{g}^*/G\), one can restrict an operator field \(\varphi \in CB(S_{\ell-1})\) to \(L((\mathcal{O}_k)_k) = \ell + u^{\dagger}\) and obtains an element in \(CB(\ell + u^{\dagger})\). Thus, as
\[
\{F(a)|L((\mathcal{O}_k)_k)| a \in C^*(G)\} = C_0(L((\mathcal{O}_k)_k)) = C_0(\ell + u^{\dagger}),
\]

one can define the *-isomorphism
\[ \tau : C_0(\mathbb{R}^2) \cong C_0(\ell + u^1) \rightarrow C^*(G/U, \chi_L) \cong C^*(\mathbb{R}^2), \quad F(a)|_{L((O_k)_{k})} \mapsto p_{G/U}(a). \]

Now, define \( \tilde{\nu}_k \) as
\[ \tilde{\nu}_k(\varphi) = F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_{k})}) \circ F_k \quad \forall \varphi \in CB(S_{-1}). \]

Since the image of \( \nu_k \) is in \( B(L^2(\mathbb{R}^d)) \) and \( F_k \) is an intertwining operator and thus bounded, the image of \( \tilde{\nu}_k \) is contained in \( B(L^2(\mathbb{R}^d)) \) as well.

Moreover, the operator \( \tilde{\nu}_k \) is bounded: From the boundedness of \( \nu_k \) (Propostition 4.1) and using that \( \tau \) is an isomorphism, one gets for every \( \varphi \in CB(S_{-1}) \)
\[ \| \tilde{\nu}_k(\varphi) \|_{op} = \| F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_{k})}) \circ F_k \|_{op} \]
\[ \leq \| (\nu_k \circ \tau)(\varphi|_{L((O_k)_{k})}) \|_{op} \]
\[ \leq \| \tau(\varphi|_{L((O_k)_{k})}) \|_{C^*(\mathbb{R}^2)} \]
\[ \leq \| (\varphi|_{L((O_k)_{k})})\|_{\infty} \leq \| \varphi \|_{S_{-1}}. \]

The involutivity of \( \tilde{\nu}_k \) follows from the involutivity of \( \tau \) and \( \nu_k \) (Proposition 4.1).

Finally, the demanded convergence of Condition 3(b) can also be shown: With the above stated equivalence of the representations \( \pi_k \) and \( \pi_k \), one gets
\[ \| \pi_k^\dagger (a) - \tilde{\nu}_k(F(a)|_{S_{-1}}) \|_{op} = \| F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ (\nu_k \circ \tau)(F(a)|_{L((O_k)_{k})}) \circ F_k \|_{op} \]
\[ = \| F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ \nu_k(p_{G/U}(a)) \circ F_k \|_{op} \]
\[ = \| F_k^* \circ (\pi_k(a) - \nu_k(p_{G/U}(a))) \circ F_k \|_{op} \]
\[ \leq \| \nu_k(p_{G/U}(a)) - \pi_k(a) \|_{op} \xrightarrow{k \rightarrow \infty} 0. \]

Therefore, the representations \( (\pi_k)_{k} \) fulfill Property 3(b) and the conditions of the theorem are thus proved.

### 4.7 Third case

In the third and last case \( \lambda \neq 0 \) and there exists \( 1 \leq m < d \) such that \( c_j \neq 0 \) for every \( j \in \{1, \ldots, m\} \) and \( c_j = 0 \) for every \( j \in \{m+1, \ldots, d\} \).

This means that
\[ \langle \ell_k, [X_j^k, Y_j^k] \rangle = c_j^k \lambda_k \xrightarrow{k \rightarrow \infty} c_j \lambda = 0 \quad \Leftrightarrow \quad j \in \{m+1, \ldots, d\}. \]

In this case \( p := \text{span}\{X_{m+1}, \ldots, X_d, Y_1, \ldots, Y_d, T, Z, A_1, \ldots, A_p\} \) is a polarization for \( \ell \).

Moreover, for \( \tilde{p}_k := \text{span}\{X_{m+1}^k, \ldots, X_d^k, Y_1^k, \ldots, Y_d^k, T_k, Z_k, A_1^k, \ldots, A_p^k\} \), one has \( \tilde{p}_k \xrightarrow{k \rightarrow \infty} p \).

Let \( P := \exp(p) \) and \( \tilde{P}_k := \exp(\tilde{p}_k) \).

#### 4.7.1 Convergence of \( (\pi_k)_{k} \) in \( \tilde{G} \)

Let
\[ (x)_\infty = (\tilde{x}, \tilde{x})_\infty \quad \text{with} \quad (\tilde{x})_\infty := (x_1, \ldots, x_m)_\infty \quad \text{and} \quad (\tilde{x})_\infty := (x_{m+1}, \ldots, x_d)_\infty \]
and analogously
\[ (y)_\infty = (\tilde{y}, \tilde{y})_\infty \quad \text{with} \quad (\tilde{y})_\infty := (y_1, \ldots, y_m)_\infty \quad \text{and} \quad (\tilde{y})_\infty := (y_{m+1}, \ldots, y_d)_\infty. \]

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Moreover, as in Chapter 13 above, let

\[(a)_{\infty} = (\dot{a}_1, \ldots, \dot{a}_d)_{\infty} \text{ with } (\ddot{a})_{\infty} = (a_1, \ldots, a_p)_{\infty}\]

and let

\[(g)_{\infty} = (\dot{x}, \dot{y}, \dot{y}, t, z, \dot{a})_{\infty} = (x, y, t, z, \dot{a})_{\infty} = (x, h)_{\infty}.\]

Now, let \(\vec{a} := (\alpha_{m+1}, \ldots, \alpha_d) \in \mathbb{R}^{d-m}\) and \(\vec{\beta} := (\beta_{m+1}, \ldots, \beta_d) \in \mathbb{R}^{d-m}\), consider \(\vec{a}\) and \(\vec{\beta}\) as elements of \(\mathbb{R}^d\) identifying them with \((0, \ldots, 0, \alpha_{m+1}, \ldots, \alpha_d)\) and \((0, \ldots, 0, \beta_{m+1}, \ldots, \beta_d)\), respectively and let

\[\vec{\pi} := \vec{\pi}_{\vec{a}, \vec{\beta}} := \text{ind}_{\mathbb{P}}^{\mathbb{Q}} \chi_{\ell + \ell, \vec{a}, \vec{\beta}}.\]

Then, for a function \(\dot{\xi}\) in the representation space \(\mathcal{H}_{\vec{\pi}} = L^2(G/P, \chi_{\ell, \vec{a}, \vec{\beta}})\) of \(\vec{\pi}\) and for \(\gamma_1, \ldots, \gamma_m \in \mathbb{R}\), \((\dot{\gamma})_{\infty} = (\gamma_1, \ldots, \gamma_m)_{\infty} \in \text{span}\{X_1\} \times \ldots \times \text{span}\{X_m\}\) and \(\dot{c} = (c_1, \ldots, c_m)\), letting \(\rho := (\ell, T)\) one has similarly as in [5]:

\[\vec{\pi}((g)_{\infty})\dot{\xi}((\dot{\gamma})_{\infty}) = e^{2\pi i \ell g \gamma_1 + \ell \gamma_2 + \ldots + \ell \gamma_m} e^{-2\pi i \dot{a}(\dot{\gamma})_{\infty} + \dot{\beta}(\dot{\gamma})_{\infty}} \vec{\xi}((\dot{\gamma} - \dot{\gamma})_{\infty}),\]

since \(\ell(Y_j') = \ell(X_j) = 0\) for all \(j \in \{1, \ldots, d\}\).

From now on, most of the time, \(G\) will be identified with \(\mathbb{R}^{d+2+p}\).

Define for \(\vec{s} = (s_{m+1}, \ldots, s_d) \in \text{span}\{X_{m+1}\} \times \ldots \times \text{span}\{X_d\} \cong \mathbb{R}^{d-m}\)

\[\vec{\eta}_{k, \vec{a}, \vec{\beta}}(\vec{s}) := \prod_{j=m+1}^d |\lambda_k c_j^k|^\frac{1}{2} |\pi_k c_j^k|^\frac{1}{2} (s_j + \dot{\beta}_j c_j^k)\]

and furthermore for \(\vec{\xi} \in \mathcal{H}_{\vec{\pi}} = L^2(G/P, \chi_{\ell, \vec{a}, \vec{\beta}}) \cong L^2(\mathbb{R}^m)\) and \(s = (\dot{s}, \vec{s})\) in \((\text{span}\{X_1\} \times \ldots \times \text{span}\{X_m\}) \times (\text{span}\{X_{m+1}\} \times \ldots \times \text{span}\{X_d\}) \cong \mathbb{R}^m \times \mathbb{R}^{d-m}\)

\[\xi_k(s) := \vec{\xi}(\vec{s})\vec{\eta}_{k, \vec{a}, \vec{\beta}}(\vec{s}).\]

Then, as above in the second case, the coefficient functions \(c_{k, \vec{a}, \vec{\beta}}^k\) defined by

\[c_{k, \vec{a}, \vec{\beta}}^k(g) := \langle \pi_k(g)\xi_k, \xi_k \rangle \quad \forall \ g \in G \cong \mathbb{R}^{d+2+p}\]

converge uniformly on compacta to \(\eta_{k, \vec{a}, \vec{\beta}}\) which in turn is defined by

\[c_{k, \vec{a}, \vec{\beta}}(g) := \langle \vec{\pi}_{k, \vec{a}, \vec{\beta}}(g)\check{\xi}, \check{\xi} \rangle \quad \forall \ g \in G \cong \mathbb{R}^{d+2+p}.\]

4.7.2 Definition of the \(\nu_k\)’s

For \(0 \neq \eta' \in L^2(\tilde{P}_k/P_k, \chi_{\ell, \vec{a}}) \cong L^2(\mathbb{R}^{d-m})\) let

\[P_{\eta'} : L^2(\mathbb{R}^{d-m}) \to \mathbb{C}\eta', \ \xi \mapsto \eta'(\xi, \eta').\]

Then \(P_{\eta'}\) is the orthogonal projection onto the space \(\mathbb{C}\eta'.\)

Define now for \(k \in \mathbb{N}\) and \(h \in C^\ast(G/U, \chi)\) the linear operator

\[\nu_k(h) := \int_{\mathbb{R}^{d-m}} \pi_{\ell + (\check{x}, \check{y})}(h) \otimes P_{\eta_k, \vec{x}, \vec{\beta}} \frac{d(\vec{x}, \vec{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|^\frac{1}{2}}.\]
whereat $\pi_{t+(\hat{x},\hat{y})}$ is defined as $\text{ind}f_2^{\ell}\chi_{t+(\hat{x},\hat{y})}$ for an element $\ell + (\hat{x},\hat{y})$ located in 
$\ell + ((\text{span}\{X_{m+1}\} \times \ldots \times \text{span}\{X_d\}) \times (\text{span}\{Y_{m+1}\} \times \ldots \times \text{span}\{Y_d\}))^* \cong \ell + \mathbb{R}^{2(d-m)}$.

Thus, for $L^2(\mathbb{R}^d) \ni \xi = \sum_{i=1}^{\infty} \xi_i \otimes \bar{\xi}_i$ with $\xi_i \in L^2(\mathbb{R}^m)$ and $\bar{\xi}_i \in L^2(\mathbb{R}^{d-m})$ for all $i \in \mathbb{N}$, one has

$$
\nu_k(h)(\xi) := \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{t+(\hat{x},\hat{y})}(h)(\xi_i) \otimes P_{\eta_{k,x,y}}(\bar{\xi}_i) \frac{d(\bar{x},\hat{y})}{\prod_{j=m+1}^{d} |\lambda_k c_j^k|}.
$$

**Proposition 4.3.**

1. For every $k \in \mathbb{N}$ and $h \in S(G/U, \chi_{\ell})$ the integral defining $\nu_k(h)$ converges in the operator norm.

2. The operator $\nu_k(h)$ is compact and $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U, \chi_{\ell})}$.

3. $\nu_k$ is involutive.

**Proof:**

Let $\mathcal{K} = \mathcal{K}(L^2(\mathbb{R}^m))$ be the $C^*$-algebra of the compact operators on the Hilbert space $L^2(\mathbb{R}^m)$ and $C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$ the $C^*$-algebra of all continuous mappings from $\mathbb{R}^{2(d-m)}$ into $\mathcal{K}$ vanishing at infinity.

Define for $\varphi \in C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$ and $k \in \mathbb{N}$ the linear operator

$$
\mu_k(\varphi) := \int_{\mathbb{R}^{2(d-m)}} \varphi(\bar{x},\hat{y}) \otimes P_{\eta_{k,x,y}} \frac{d(\bar{x},\hat{y})}{\prod_{j=m+1}^{d} |\lambda_k c_j^k|}
$$

on $L^2(\mathbb{R}^d)$. Then, as $\mathcal{F}(h) \in C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$ for $h \in C^*(G/U, \chi_{\ell})$,

$$
\nu_k(h) = \mu_k(\mathcal{F}(h)).
$$

1. Since $\mathcal{F}(h) \in S(\mathbb{R}^{2(d-m)}, \mathcal{K})$ for $h \in S(G/U, \chi_{\ell})$ and since

$$
\|\mu_k(\varphi)\|_{op} \leq \int_{\mathbb{R}^{2(d-m)}} \|\varphi(\bar{x},\hat{y})\|_{op} \frac{d(\bar{x},\hat{y})}{\prod_{j=m+1}^{d} |\lambda_k c_j^k|}
$$

for every $\varphi \in S(\mathbb{R}^{2(d-m)}, \mathcal{K})$ and $k \in \mathbb{N}$, the first assertion follows immediately.

2. As $p_{G/U}$ is surjective from the space $S(G)$ onto the space $S(G/U, \chi_{\ell})$, for every $h \in S(G/U, \chi_{\ell}) \cong S(\mathbb{R}^{2d})$ there exists a function $f \in S(G) \cong S(\mathbb{R}^{2d+2+\rho})$ such that $h = p_{G/U}(f)$ and, as shown in the second case, for $\tilde{g} = (\hat{x},\bar{x},\tilde{y},\bar{y},0,0,0,0) \in G/U \cong \mathbb{R}^{2d}$ one has

$$
h_\infty(\tilde{g}) = \int_{\mathbb{R}^{5,6,7,8}} (\hat{x},\bar{x},\tilde{y},\bar{y},\rho,\lambda,0,0),
$$

where again $h_\infty = h((\cdot)_\infty)$ and $f_\infty = f((\cdot)_\infty)$.

Now, let $s_1, \ldots, s_m \in \mathbb{R}$ and $\hat{\nu} = (s_1, \ldots, s_m)_\infty = \sum_{j=1}^{m} s_j X_j$ and moreover, let

$$
(g)_\infty = (\hat{x},\bar{x},\tilde{y},\bar{y},\hat{t},\bar{z},\tilde{a},\bar{a})_\infty = (x,y,t,z,a)_\infty \text{ with } \hat{x},\bar{x},\tilde{y},\bar{y},\hat{a},\bar{a} \text{ and } \hat{\nu} \text{ as above.}
$$

Then, one gets for $\xi_i \in L^2(\mathbb{R}^m)$

$$
\pi_{t+(\hat{x},\hat{y})}(g)_\infty \xi_i(\hat{\nu}_i)_\infty = e^{2\pi i(-tp-z\lambda+\hat{\nu}((\hat{\nu})_\infty-\frac{1}{2}(\hat{\nu})_\infty)-(\hat{\nu}(\hat{\nu})_\infty-\hat{\nu}(\hat{\nu})_\infty))} \xi_i((\hat{\nu}-\hat{x})_\infty).
$$
Using the Equality (7) and identifying $G$ with $\mathbb{R}^{2d+2+p}$, one gets for a function $h = p_{G/U}(f) \in S(G/U, \chi f) \cong S(\mathbb{R}^{2d})$ with $f \in S(G) \cong S(\mathbb{R}^{2d+2+p})$

$$\pi_{\ell+\hat{x},\hat{y}}(h)\hat{\xi}_i(\hat{s}) = \int_{\mathbb{R}^{2d}} \left( p_{G/U}(f) \right)_\infty(\hat{g})\pi_{\ell+\hat{x},\hat{y}}(\hat{g})\hat{\xi}_i(\hat{s}) \, d\hat{g}$$

$$= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d+2+p-(p-p)}} f_\infty(\hat{g}u)\chi_\ell(u) \, du \, \pi_{\ell+\hat{x},\hat{y}}(\hat{g})\hat{\xi}_i(\hat{s}) \, d\hat{g}$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2d+p}} f_\infty(x, \bar{x}, \bar{y}) e^{2\pi i\lambda c((s-\hat{s})y)-(\bar{x},\bar{y})} e^{-2\pi i(t+p+\lambda\hat{s})}\hat{\xi}_i(\hat{s} - \hat{x}) \, d(\hat{x}, \bar{y})$$

$$= \int_{\mathbb{R}^d} \hat{h}_\infty^{2,4,5,6,7,8}(x, \bar{x}, \bar{y}) e^{2\pi i\lambda c((s-\hat{s})y)}\hat{\xi}_i(\hat{s} - \hat{x}) \, d(\hat{x}, \bar{y})$$

$$= \int_{\mathbb{R}^m} \hat{h}_\infty^{2,4,8}(x, \bar{x}, \bar{y}) e^{2\pi i\lambda c((s-\hat{s})y)}\hat{\xi}_i(\hat{s} - \hat{x}) \, d\hat{x}.$$  \hspace{1cm} (10)

Regard now the second factor $P_{\eta_k,\zeta,\beta}$ of the tensor product:

As in the second case above, define

$$\tilde{\eta}_{k,\beta}(\hat{s}) := \prod_{j=m+1}^d \left| \lambda_k c^j \right|^{\frac{1}{2}} \eta_j \left( \left| \lambda_k c^j \right|^{\frac{1}{2}} \left( s_j + \frac{\beta_j}{\lambda_k c^j} \right) \right).$$

Then

$$\tilde{\eta}_{k,\alpha,\beta}(\hat{s}) = e^{2\pi i\hat{x}\hat{s}}\tilde{\eta}_{k,\beta}(\hat{s})$$

and therefore with $\tilde{\xi}_i \in L^2(\mathbb{R}^{d-m})$

$$P_{\eta_k,\zeta,\beta}(\tilde{\xi}_i)(\hat{s}) = \left\langle \tilde{\xi}_i, \tilde{\eta}_{k,\zeta,\beta} \right\rangle \tilde{\eta}_{k,\zeta,\beta}(\hat{s})$$

$$= \left( \int_{\mathbb{R}^{d-m}} \tilde{\xi}_i(\tilde{r})\tilde{\eta}_{k,\zeta,\beta}(\tilde{r}) \, d\tilde{r} \right) \tilde{\eta}_{k,\zeta,\beta}(\hat{s})$$

$$= \left( \int_{\mathbb{R}^{d-m}} \tilde{\xi}_i(\tilde{r}) e^{-2\pi i\hat{x}\tilde{r}} \tilde{\eta}_{k,\zeta,\beta}(\tilde{r}) \, d\tilde{r} \right) e^{2\pi i\hat{x}\hat{s}}\tilde{\eta}_{k,\zeta,\beta}(\hat{s})$$

$$= \int_{\mathbb{R}^{d-m}} \tilde{\xi}_i(\tilde{r}) e^{2\pi i\hat{x}^{-1}\tilde{r}} \tilde{\eta}_{k,\zeta,\beta}(\tilde{r}) \, d\tilde{r} \tilde{\eta}_{k,\zeta,\beta}(\hat{s}).$$

Joining together the calculation above and the one for the first factor of the tensor product (11), one gets for $S(\mathbb{R}^d)$ \exists $\xi = \sum_{i=1}^\infty \tilde{\xi}_i \otimes \tilde{\xi}_i$ with $\tilde{\xi}_i \in S(\mathbb{R}^m)$ and $\tilde{\xi}_i \in S(\mathbb{R}^{d-m})$ for all $i \in \mathbb{N},$
Therefore, the kernel function
\[ h \in \mathcal{S}(\mathbb{R}^{2d}) \text{ and } s = (\tilde{s}, \bar{s}) \in \mathbb{R}^{d} \]
\[ \nu_{h}(h)(\xi)(s) \]
\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{i+}(\tilde{x}, \bar{y})(h)(\xi_{i})(\tilde{s}) \cdot P_{\eta_{k,i},(\xi_{i})}(\bar{s}) \frac{d(\tilde{x}, \bar{y})}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \]
\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \left( \int_{\mathbb{R}^{m}} h_{-2,3,4}^{\infty}(\tilde{x}, \bar{y}, \lambda \xi_{i}(\tilde{\tilde{s}} - \bar{s})) \frac{d(\tilde{x}, \bar{y})}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \right) \]
\[ \cdot \left( \int_{\mathbb{R}^{m}} \xi_{i}(\bar{r}) e^{2\pi i (\tilde{s} - \bar{s})} \eta_{k,i}(\bar{r}) \frac{d\bar{r}}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \right) \]
\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \int_{\mathbb{R}^{2(d-m)}} \int_{\mathbb{R}^{m}} h_{-2,3,4}^{\infty}(\tilde{x}, \bar{s} - \bar{r}, \lambda \xi_{i}(\tilde{\tilde{s}} - \bar{s})) \frac{d\bar{s}}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \]
\[ = \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \int_{\mathbb{R}^{2(d-m)}} \int_{\mathbb{R}^{m}} h_{-2,3,4}^{\infty}(\tilde{\tilde{s}} - \tilde{\tilde{\tilde{s}}}, \bar{s} - \bar{r}, \lambda \xi_{i}(\tilde{\tilde{s}} - \bar{s})) \frac{d\bar{s}}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \]
\[ \xi(\tilde{x}, \bar{r}) \ d(\tilde{x}, \bar{r}). \quad (11) \]

Therefore, the kernel function
\[ h_{K}(\tilde{s}, \bar{s}, (\tilde{x}, \bar{r})) := \int_{\mathbb{R}^{2(d-m)}} h_{-2,3,4}^{\infty}(\tilde{s} - \tilde{x}, \bar{s} - \bar{r}, \lambda \xi_{i}(\tilde{\tilde{s}} - \bar{s})) \eta_{k,i}(\bar{r}) \frac{d\bar{s}}{\prod_{j=m+1}^{d} |\lambda_{k}c_{j}|} \]

of \( \nu_{k}(h) \) is contained in \( \mathcal{S}(\mathbb{R}^{2d}) \) and thus \( \nu_{k}(h) \) is a compact operator for \( h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_{\ell}) \) and with the density of \( \mathcal{S}(G/U, \chi_{\ell}) \) in \( C^{*}(G/U, \chi_{\ell}) \), it is compact for every \( h \in C^{*}(G/U, \chi_{\ell}) \).

Now, it is shown that for every \( \varphi \in C_{0}(\mathbb{R}^{2(d-m)}, \mathcal{K}) \)
\[ ||\mu_{k}(\varphi)||_{\text{op}} \leq ||\varphi||_{\infty} := \sup_{(\tilde{s}, \bar{s}) \in \mathbb{R}^{2(d-m)}} ||\varphi(\tilde{x}, \bar{y})||_{\text{op}}. \]
For this, for any \( \psi \in L^2(\mathbb{R}^d) \), define

\[
  f_{\psi,k}(x,y)(s) := \int_{\mathbb{R}^{d-m}} \psi(s, \tilde{s}) \eta_{h,k,x,y}(\tilde{s}) \, d\tilde{s} \quad \forall \ (x,y) \in \mathbb{R}^{2(d-m)} \quad \forall \ s \in \mathbb{R}^m.
\]

Then, as

\[
  \|\psi\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(x,y)\|_{L^2(\mathbb{R}^m)}^2 \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|},
\]

one gets the identity

\[
  \|\psi\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(x,y)\|_{L^2(\mathbb{R}^m)}^2 \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|}.
\]  

Now, for \( \xi, \psi \in L^2(\mathbb{R}^d) \)

\[
  \left| \langle \mu_k(\varphi)\xi, \psi \rangle_{L^2(\mathbb{R}^d)} \right| \leq \int_{\mathbb{R}^{2(d-m)}} \left| \langle \varphi(x,y) \otimes \eta_{h,k,x,y} \rangle_{L^2(\mathbb{R}^m)} \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right|
\]

\[
  \leq \left( \int_{\mathbb{R}^{2(d-m)}} \|\varphi(x,y) f_{\xi,k}(x,y)\|_{L^2(\mathbb{R}^m)}^2 \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(x,y)\|_{L^2(\mathbb{R}^m)}^2 \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^d)}
\]

\[
  \leq \sup_{(x,y) \in \mathbb{R}^{2(d-m)}} |\varphi(x,y)|_{op} \left( \int_{\mathbb{R}^{2(d-m)}} \|f_{\xi,k}(x,y)\|_{L^2(\mathbb{R}^m)}^2 \frac{d(x,y)}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^d)}
\]

\[
  \leq \|\varphi\|_{\infty} \|\xi\|_{L^2(\mathbb{R}^d)} \|\psi\|_{L^2(\mathbb{R}^d)}.
\]

Hence, for every \( h \in C^*(G/U, \chi_h) \),

\[
  \|\nu_k(h)\|_{op} = \|h_k(F(h))\|_{op} \leq \|F(h)\|_{\infty} = \|h\|_{C^*(G/U, \chi_k)}.
\]

3. To show that \( \nu_k \) is involutive is as straightforward as in the second case.
The demanded convergence of Condition 3(b) remains to be shown:

4.7.3 Theorem - Third Case

Theorem 4.4.

For \( a \in C^\infty (G) \)
\[
\lim_{k \to \infty} \| \pi_k(a) - \nu_k(p_{G/U}(a)) \|_{op} = 0.
\]

Proof:

Let \( f \in S(G) \cong S(\mathbb{R}^{2d+2+p}) \) such that its Fourier transform in \([3,9]\) has a compact support on \( G \cong \mathbb{R}^{2d+2+p} \). In the setting of this third case, this means that \( f^{6,7} \) has a compact support in \( G \) (see Theorem 4.2).

Now, identify \( G \) with \( \mathbb{R}^{2d+2+p} \) again, let \( \xi \in L^2(\mathbb{R}^d) \) and \( s = (s_1, ..., s_d) = (\dot{s}, \ddot{s}) \) be located in \( \mathbb{R}^m \times \mathbb{R}^{d-m} \cong \mathbb{R}^d \) and define
\[
\tilde{\eta}_{k,0}(\tilde{s}) := \prod_{j=m+1}^{d} \left| \lambda_k c_j^k \right|^\frac{1}{2} \eta_j \left( \left| \lambda_k c_j^k \right|^\frac{1}{2} (s_j) \right).
\]

Moreover, let \( \dot{c} = (c_1, ..., c_m), \quad \ddot{c} = (c_{m+1}, ..., c_d) = (0, ..., 0), \quad \dot{c}^k = (c_1^k, ..., c_m^k) \) and \( \ddot{c}^k = (c_{m+1}^k, ..., c_d^k) \).

As in the second case, the expression \( (\pi_k(f) - \nu_k(p_{G/U}(f))) \) is now going to be regarded, composed into several parts and then estimated: For this, Equation (6) from Chapter 4.4 will be used again but its notation needs to be adapted:

\[
\pi_k(f)(s) = \int_{\mathbb{R}^d} f^{2,3,4,5,6}(s - r, \frac{\lambda_k c^k}{2} (s + r), \rho_k, \lambda_k, 0, 0) \xi(r) \, dr
\]
\[
= \int_{\mathbb{R}^d} f^{3,4,5,6,7,8}(\dddot{s} - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \dot{c}^k}{2} (\dot{s} + \dot{r}) - \frac{\lambda_k \ddot{c}^k}{2} (\dddot{s} + \dddot{r}), \rho_k, \lambda_k, 0, 0) \xi(\ddot{r}, \ddot{r}) \, d(\ddot{r}, \ddot{r}).
\]
Using the above equation, $\Pi$ and the fact that $p_{G/U}(f) = \hat{f}^{5,6,7,8}(\cdot, \cdot, \cdot, \cdot, \rho, \lambda, 0, 0)$, one gets

\[
\left( \pi_k(f) - v_k(p_{G/U}(f)) \right) \xi(s) \]

\[
\equiv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho_k, \lambda_k, 0, 0 \right) \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{G/U}(f) 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, \frac{\lambda}{2} \tilde{c} (-\tilde{r} - \tilde{s}), \tilde{y} \right) \tilde{\eta}_{k,0}(\tilde{r}) \tilde{\eta}_{k,0}(\tilde{s}) \ d\tilde{y} \frac{d\tilde{y}}{\prod_{j=m+1}^{d} |\lambda_k c_j^l|} \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
\|\tilde{\eta}_{k,0}\|_2 = 1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho_k, \lambda_k, 0, 0 \right) \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, \frac{\lambda}{2} \tilde{c} (-\tilde{r} - \tilde{s}), \tilde{y}, \rho, \lambda, 0, 0 \right) \tilde{\eta}_{k,0}(\tilde{r}) \tilde{\eta}_{k,0}(\tilde{s}) \ d\tilde{y} \frac{d\tilde{y}}{\prod_{j=m+1}^{d} |\lambda_k c_j^l|} \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho_k, \lambda_k, 0, 0 \right) \tilde{\eta}_{k,0}(\tilde{r}) \tilde{\eta}_{k,0}(\tilde{s}) \ d\tilde{y} \frac{d\tilde{y}}{\prod_{j=m+1}^{d} |\lambda_k c_j^l|} \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda}{2} \tilde{c} (s + \tilde{r}), \lambda_k \bar{c}^k (\tilde{y} - \tilde{s}), \rho, \lambda, 0, 0 \right) \tilde{\eta}_{k,0}(\tilde{r} + \tilde{s}) \tilde{\eta}_{k,0}(\tilde{y}) \ d\tilde{y} \frac{d\tilde{y}}{\prod_{j=m+1}^{d} |\lambda_k c_j^l|} \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r}).
\]

Similar as for the second case, functions $q_k, u_k, o_k, n_k$ and $w_k$ are going to be defined in order to divide the above integrals into six parts:

\[
q_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(\tilde{r}, \tilde{r}) \tilde{\eta}_{k,0}(\tilde{y} + \tilde{r} - \tilde{s})
\]

\[
\left( \int_{k} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho_k, \lambda_k, 0, 0 \right) \right) d(\tilde{r}, \tilde{r})
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho, \lambda, 0, 0 \right) \tilde{\eta}_{k,0}(\tilde{r}) \tilde{\eta}_{k,0}(\tilde{s}) \ d\tilde{y} \frac{d\tilde{y}}{\prod_{j=m+1}^{d} |\lambda_k c_j^l|} \xi(\tilde{r}, \tilde{r}) \ d(\tilde{r}, \tilde{r})
\]

\[
u_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(\tilde{r}, \tilde{r}) \tilde{\eta}_{k,0}(\tilde{y} + \tilde{r} - \tilde{s})
\]

\[
\left( \int_{k} 3.4.5.6.7.8 \left( s - \tilde{r}, \tilde{s} - \tilde{r}, -\frac{\lambda_k \bar{c}^k}{2} (s + \tilde{r}), -\frac{\lambda_k \bar{c}^k}{2} (\tilde{s} + \tilde{r}), \rho, \lambda, 0, 0 \right) \right) d(\tilde{r}, \tilde{r})
\]

\[
32
\]
\[ v_k(s, \bar{y}) := \int_{\mathbb{R}^d} \xi(\hat{r}, \hat{\bar{r}}) \overline{\eta}_{k,0}(\bar{y} + \hat{\bar{r}} - \hat{s}) \]
\[ \quad \times \left( \int_k^{3.4.5.6.7.8} \left( \frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), -\frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), \rho, \lambda, 0, 0 \right) \right) d(\hat{r}, \hat{\bar{r}}), \]
\[ o_k(s, \bar{y}) := \int_{\mathbb{R}^d} \xi(\hat{r}, \hat{\bar{r}}) \overline{\eta}_{k,0}(\bar{y} + \hat{\bar{r}} - \hat{s}) \]
\[ \quad \times \left( \int_k^{3.4.5.6.7.8} \left( \frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), \lambda_k c_k^2 (\bar{y} - \bar{s}), \rho, \lambda, 0, 0 \right) \right) d(\hat{r}, \hat{\bar{r}}), \]
\[ n_k(s, \bar{y}) := \int_{\mathbb{R}^d} \xi(\hat{r}, \hat{\bar{r}}) \overline{\eta}_{k,0}(\bar{y} + \hat{\bar{r}} - \hat{s}) \]
\[ \quad \times \left( \int_k^{3.4.5.6.7.8} \left( \frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), \lambda_k c_k^2 (\bar{y} - \bar{s}), \rho, \lambda, 0, 0 \right) \right) d(\hat{r}, \hat{\bar{r}}) \]
and
\[ w_k(s) := \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^d} \xi(\hat{r}, \hat{\bar{r}}) \overline{\eta}_{k,0}(\bar{y}) \left( \overline{\eta}_{k,0}(\bar{y}) - \overline{\eta}_{k,0}(\bar{y} + \hat{\bar{r}} - \hat{s}) \right) \]
\[ \quad \times \int_k^{3.4.5.6.7.8} \left( \frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), -\frac{\lambda_k c_k^2}{2} (\hat{s} + \hat{r}), \rho, \lambda, 0, 0 \right) d(\hat{r}, \hat{\bar{r}}) d\bar{y}. \]

Then,
\[ (\pi_k(f) - \nu_k(p_{CG/\mathcal{U}}(f))) \xi(s) = \int_{\mathbb{R}^{d-m}} q_k(s, \bar{y}) \overline{\eta}_{k,0}(\bar{y}) d\bar{y} + \int_{\mathbb{R}^{d-m}} w_k(s, \bar{y}) \overline{\eta}_{k,0}(\bar{y}) d\bar{y} \]
\[ + \int_{\mathbb{R}^{d-m}} v_k(s, \bar{y}) \overline{\eta}_{k,0}(\bar{y}) d\bar{y} + \int_{\mathbb{R}^{d-m}} o_k(s, \bar{y}) \overline{\eta}_{k,0}(\bar{y}) d\bar{y} \]
\[ + \int_{\mathbb{R}^{d-m}} n_k(s, \bar{y}) \overline{\eta}_{k,0}(\bar{y}) d\bar{y} + w_k(s). \]

As in the second case, to show that
\[ \| \pi_k(f) - \nu_k(p_{CG/\mathcal{U}}(f)) \|_{op} \xrightarrow{k \to \infty} 0, \]
one has to prove that there are \( \kappa_k, \gamma_k, \delta_k, \tau_k, \omega_k \) and \( \epsilon_k \) which are going to 0 for \( k \to \infty \), such that
\[ \| q_k \|_2 \leq \kappa_k \| \xi \|_2, \quad \| u_k \|_2 \leq \gamma_k \| \xi \|_2, \quad \| v_k \|_2 \leq \delta_k \| \xi \|_2, \quad \| o_k \|_2 \leq \tau_k \| \xi \|_2, \]
\[ \| n_k \|_2 \leq \omega_k \| \xi \|_2 \quad \text{and} \quad \| w_k \|_2 \leq \epsilon_k \| \xi \|_2. \]

The estimation of the functions \( q_k, u_k, v_k, o_k \) and \( w_k \) is very similar to their estimation in the second case and will thus be skipped. So, it just remains the estimation of \( o_k \):
For this, first regard the last factor of the function $o_k$:

$$
\int_0^{\lambda_{\tilde{k}}^{3,4,5,6,7,8}} \left( \frac{\lambda_{\tilde{k}}^{3,4,5,6,7,8}}{2} (\dot{s} + \ddot{\bar{r}}) - \frac{\lambda_{\tilde{k}}^{3,4,5,6,7,8}}{2} (\dot{s} + \ddot{\bar{r}}), \lambda_{\tilde{k}}^{3,4,5,6,7,8} (\ddot{\bar{y}} - \ddot{\bar{s}}), \rho, \lambda, 0, 0 \right) + \frac{1}{2} (\dot{\lambda} - \lambda_{\tilde{k}}^{3,4,5,6,7,8}) (\dot{\bar{s}} + \ddot{\bar{r}})
$$

Thus, for those $x \in S$ such that $\lambda \in \mathbb{R}^{2d}$ has a compact support, we get

$$
\|o_k\|_2^n \leq C_1 \frac{C_2^n \|\dot{\lambda} - \lambda_{\tilde{k}}^{3,4,5,6,7,8}\|^2}{(1 + \|\dot{\bar{s}} + \ddot{\bar{r}}\|)^{2d+1} (1 + \|\dot{s} - \ddot{s}\|)^{2d}} d(\bar{s}, \bar{y}, \bar{\bar{y}})
$$

Hence, one gets

$$
\|o_k\|_2^n = \int_{\mathbb{R}^{2d}} |o_k(s, \bar{y})|^2 d(s, \bar{y})
$$

Cauchy-Schwarz

$$
\leq \int_{\mathbb{R}^{2d}} \xi(\bar{s}, \bar{y}) \frac{1}{(1 + \|\dot{s} + \ddot{\bar{r}}\|)^{2d+1} (1 + \|\dot{s} - \ddot{s}\|)^{2d}} d(\bar{s}, \bar{y}, \bar{\bar{y}})
$$

with matching constants $C_1 > 0$ and $C_2 > 0$, depending on $f$. Hence, $\tau_k := \sqrt{C_1^n} \|\dot{\lambda} - \lambda_{\tilde{k}}^{3,4,5,6,7,8}\|$ fulfills $\lim_{k \to \infty} \tau_k = 0$ and

$$
\|o_k\|_2^n = \tau_k \xi
$$

Thus, for those $f \in S(\mathbb{R}^{2d+2})$ whose Fourier transform in $[\xi, \bar{\xi}]$ has a compact support, we get

$$
\|\pi_k(f) - \nu_k(p_G(U(f)))\|_{op} = \sup_{\xi \in L^2(\mathbb{R}^n) \|\|\xi\|_2 = 1} \left\|\pi_k(f) - \nu_k(p_G(U(f))(\xi)\right\|_{1}^{k \to \infty} \to 0.
$$
As in the second case, because of the density of the set of Schwartz functions whose partial Fourier transform has a compact support, the claim follows for all \( a \in C^*(G) \).

Now, the assertions for the sequence \( (\pi_k^V)_k \) can be deduced:

### 4.7.4 Transition to \( (\pi_k^V)_k \)

Again, because of the equivalence of the representations \( \pi_k \) and \( \pi_k^V \) for every \( k \in \mathbb{N} \), there exist unitary intertwining operators

\[
F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{with} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall \ a \in C^*(G).
\]

With the injective \(*\)-homomorphism

\[
\tau : C_0(\mathbb{R}^{2(d-m)}, K) \to C^*(G/U, \chi_\ell), \quad F(a)|_{L((O_k)_a)} \mapsto p_{G/U}(a)
\]

define

\[
\tilde{\nu}_k(\varphi) := F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_a)}) \circ F_k \quad \forall \ \varphi \in CB(S_{i-1}).
\]

Then, like in the second case, \( \tilde{\nu}_k \) complies with the demanded requirements and thus, the original representations \( (\pi_k^V)_k \) fulfill Property 3(b).

Finally, one obtains the following result:

**Theorem 4.5** (Main result).

The \( C^*\)-algebra \( C^*(G) \) of a connected real two-step nilpotent Lie group is isomorphic (under the Fourier transform) to the set of all operator fields \( \varphi \) defined over \( \hat{G} \) such that

1. \( \varphi(\gamma) \in K(\mathcal{H}_i) \) for every \( i \in \{1, ..., r\} \) and every \( \gamma \in \Gamma_i \).
2. \( \varphi \in l^\infty(\hat{G}) \).
3. The mappings \( \gamma \mapsto \varphi(\gamma) \) are norm continuous on the different sets \( \Gamma_i \).
4. For any sequence \( (\gamma_k)_{k \in \mathbb{N}} \subseteq \hat{G} \) going to infinity \( \lim_{k \to \infty} \| \varphi(\gamma_k) \|_{\text{op}} = 0 \).
5. For \( i \in \{1, ..., r\} \) and any properly converging sequence \( \overline{\gamma} = (\gamma_k) \subseteq \Gamma_i \) whose limit set \( L(\overline{\gamma}) \) is contained in \( S_{i-1} \) (taking a subsequence if necessary) and for the mappings \( \tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma}, k} : CB(S_{i-1}) \to B(\mathcal{H}_i) \) constructed in the preceding sections, one has

\[
\lim_{k \to \infty} \| \varphi(\gamma_k) - \tilde{\nu}_k(\varphi|_{S_{i-1}}) \|_{\text{op}} = 0.
\]

### 5 Example: The free two-step nilpotent Lie groups of 3 and 4 generators

In the case of the free two-step nilpotent Lie groups of \( n = 3 \) and \( n = 4 \) generators, the stabilizer of a linear functional \( \ell \), the in Section 3 constructed polarization \( p_i^V \), the coadjoint orbits, as well as the sets \( S_i \) and \( \Gamma_i \) can easily be calculated.

For \( n = 3 \), there are coadjoint orbits of the dimensions 0 and 2 and for \( n = 4 \), the dimensions 0, 2 and 4 appear.

For the free two-step nilpotent Lie groups of 3 generators, the third case regarded in the proof above does not appear: For this, one has to find a sequence of orbits \( (O_k)_k \) whose limit set \( L((O_k)_k) \) consists of orbits of the dimension strictly greater than 0 but strictly smaller than \( \dim(O_k) \). But as for \( n = 3 \) only orbits of the dimensions 0 and 2 appear, such a sequence \( (O_k)_k \) does not exist. However, for the free two-step nilpotent Lie groups of 4 generators, this discussed third case exists.
For both $n = 3$ and $n = 4$, one can also see that the situation occurs where the polarizations $p^V_{t'}$ are discontinuous in $t$ on the set $\{tO' \mid O' \in (g^*/G)_{(J,K)}^{2d}\}$. This shows the necessity of regarding the sets $\{tO' \mid O' \in (g^*/G)_{(J,K)}\}$ instead.

Some calculations for the example of the free two-step nilpotent Lie groups of 3 and 4 generators can be found in the doctoral thesis of R.Lahiani (see [4]).

### 6 Appendix

**Lemma 6.1.**

Let $V$ be a finite-dimensional euclidean vector space and $S$ an invertible, skew-symmetric endomorphism. Then $V$ can be decomposed into an orthogonal direct sum of two-dimensional $S$-invariant subspaces.

**Proof:**

$S$ extends to a complex endomorphism $S_C$ on the complexification $V^C$ of $V$, which has purely imaginary eigenvalues.

If $i\lambda \in i\mathbb{R}$ is an eigenvalue, then also $-i\lambda$ is a spectral element. Denote by $E_{i\lambda}$ the corresponding eigenspace. These eigenspaces are orthogonal to each other with respect to the Hilbert space structure of $V^C$ coming from the euclidean scalar product $\langle \cdot, \cdot \rangle$ on $V$.

Let for $i\lambda$ in the spectrum of $S_C$

$$V^\lambda : = (E_{i\lambda} + E_{-i\lambda}) \cap V. $$

If $\lambda \neq 0$, $\text{dim}(V^\lambda)$ is even and $V^\lambda$ is $S$-invariant and orthogonal to $V^{\lambda'}$, whenever $|\lambda| \neq |\lambda'|$:

Indeed, one then has for $x \in V^\lambda, x' \in V^{\lambda'}$ that

$$x + iy \in E_{i\lambda} \quad \text{and} \quad x - iy \in E_{-i\lambda} \quad \text{for some} \quad y \in V \quad \text{as well as} \quad x' + iy' \in E_{i\lambda'} \quad \text{and} \quad x' - iy' \in E_{-i\lambda'} \quad \text{for some} \quad y' \in V.$$

Therefore,

$$\langle x + iy, x' + iy' \rangle = 0 \quad \text{and} \quad \langle x - iy, x' + iy' \rangle = 0.$$

Thus, one has

$$\langle x, x' + iy' \rangle = 0 \quad \text{and hence} \quad \langle x, x' \rangle = 0.$$

Suppose that $\text{dim}(V^\lambda) > 2$, choose a vector $x \in V^\lambda$ of length 1 and let $y = S(x)$. Since $S_C^2 = -\lambda^2 \text{Id}$, both on $E_{i\lambda}$ and on $E_{-i\lambda}$,

$$S(y) = S^2(x) = -\lambda^2 x.$$

This shows that $W^\lambda_1 : = \text{span}\{x, y\}$ is an $S$-invariant subspace of $V^\lambda$. If $V^\lambda_1$ denotes the orthogonal complement of $W^\lambda_1$ in $V^\lambda$, then $V^\lambda_1$ is $S$-invariant, since $S' = -S$.

In this way one can find a decomposition of $V^\lambda$ into an orthogonal direct sum of two-dimensional $S$-invariant subspaces $W^\lambda_j$ and by summing up over the eigenvalues, one obtains the required decomposition of $V$.

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