Spin(7)-structures on complex linear bundles and explicit Riemannian metrics with holonomy group SU(4)

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Abstract. A system of differential equations with 5 unknowns is fully investigated; this system is equivalent to the existence of a parallel Spin(7)-structure on a cone over a 3-Sasakian manifold. A continuous one-parameter family of solutions to this system is explicitly constructed; it corresponds to metrics with a special holonomy group, SU(4), which generalize Calabi’s metrics.

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§ 1. Introduction

1.1. The first example of a complete Riemannian metric with holonomy group SU(n) was the Calabi metric, which was described explicitly in terms of algebraic functions in [1]. The Calabi metric is constructed on the total space of an appropriate complex linear bundle on an arbitrary Kähler-Einstein manifold $F$. If we take $F$ to be the complex projective space $\mathbb{C}P^n$, the resulting Calabi metric is asymptotically locally Euclidean (ALE), otherwise it is asymptotically conical (AC). In [1] Calabi also considered hyper-Kähler metrics and constructed a complete Riemannian metric on $T^*\mathbb{C}P^m$ with holonomy group Sp(m) explicitly; this was the first explicit example of a hyper-Kähler metric.

In this paper we make an explicit construction, in algebraic form, of a one-parameter family of complete Riemannian metrics ‘connecting’ these two Calabi metrics in the space of special Kähler metrics in 8-dimensional spaces when $F$ is the complex 3-flag manifold of $\mathbb{C}^3$; we also carry out a full investigation of the existence problem for metrics with holonomy group Spin(7) on an appropriate bundle on $F$. In this case the tangent space of the 4-dimensional quaternionic Kähler manifold $\mathcal{O}$...
associated with $F$ can be ‘split’, which allows us to introduce an additional parameter describing deformations of the metric and to obtain a well-defined system of differential equations.

There is quite a lot of interest in explicit metrics with special holonomy groups (and, in particular, in special Kähler metrics), because only a few examples of this type are known. For instance, Joyce [2], 8.2.5, conjectured that all the other ALE-metrics with holonomy group $SU(n)$ for $n \geq 3$ (apart from the Calabi metric for $F = CP^n$) are ‘transcendental’, that is, they cannot be represented in algebraic form. We stress that the metrics we construct here are asymptotically conical (AC), but not ALE, so our example does not refute Joyce’s conjecture.

1.2. More precisely, let

$$M = SU(3)/U(1)_{1,1,-2}$$

be the Aloff-Wallach space, which carries the structure of a 7-dimensional 3-Sasakian manifold. We consider the Riemannian metric of the following form on $M = M \times \mathbb{R}_+$:

$$dt^2 + A_1(t)^2 \eta_1^2 + A_2(t)^2 \eta_2^2 + A_3(t)^2 \eta_3^2 + B(t)^2 (\eta_4^2 + \eta_5^2) + C(t)^2 (\eta_6^2 + \eta_7^2), \quad (1.1)$$

where $t$ is the variable on $\mathbb{R}_+$ and $\{\eta_i\}$ is an orthonormal coframe on $M$ agreeing with the 3-Sasakian structure (see §2 for details). We resolve the conical singularity of $\tilde{M}$ (for $t = 0$) as follows: on the level $\{t = 0\}$ we contract each circle corresponding to the covector $\eta_i$ to a point. This gives us a manifold whose quotient by $\mathbb{Z}_2$ is diffeomorphic to $H/\mathbb{Z}_2$, the square of the canonical complex linear bundle over the flag space of $\mathbb{C}^3$.

**Theorem 1.** For $0 \leq \alpha < 1$ each Riemannian metric in the family

$$\bar{g}_\alpha = \frac{r^4 (r^2 - \alpha^2)(r^2 + \alpha^2)}{r^8 - 2\alpha^4 (r^4 - 1)} dr^2 + \frac{r^8 - 2\alpha^4 (r^4 - 1)}{r^2 (r^2 - \alpha^2) \eta_1^2} + \frac{r^2 (\eta_2^2 + \eta_3^2)}{r^2 - \alpha^2} (\eta_4^2 + \eta_5^2)$$

$$+ (r^2 + \alpha^2)(\eta_6^2 + \eta_7^2) \quad (1.2)$$

is a complete smooth Riemannian metric on $H/\mathbb{Z}_2$ with holonomy group $SU(4)$. For $\alpha = 0$ the metric (1.2) is isometric to the Calabi metric with holonomy group $SU(4)$; for $\alpha = 1$, (1.2) is isometric to the Calabi metric on $T^*CP^2$ with holonomy group $Sp(2) \subset SU(4)$ (see [1]).

Note that the metrics (1.2) in Theorem 1 corresponding to $\alpha = 0$ and $\alpha = 1$ have a different form from the metrics in [1]; Calabi metrics in this form were investigated in [3] and [4]. The metric (1.2) for $M = SU(3)/U(1)_{1,1,-2}$ was also obtained in [5] as a particular solution of a system of equations for metrics with holonomy group Spin(7).

The above result was obtained when we were making a systematic investigation of metrics of the form (1.1) with holonomy group Spin(7) by a method developed in [6] and then used in [7] and [8]: a metric (1.1) is constructed for an arbitrary 7-dimensional 3-Sasakian manifold $M$ and carries a natural Spin(7)-structure. The condition that this structure be parallel reduces to the following nonlinear system
of ordinary differential equations:

\[
A_1' = \frac{(A_2 - A_3)^2 - A_1^2}{A_2 A_3} + \frac{A_1^2 (B^2 + C^2)}{B^2 C^2},
\]

\[
A_2' = \frac{A_1^2 - A_2^2 + A_3^2}{A_1 A_3} - \frac{B^2 + C^2 - 2A_2^2}{BC},
\]

\[
A_3' = \frac{A_1^2 + A_2^2 - A_3^2}{A_1 A_2} - \frac{B^2 + C^2 - 2A_3^2}{BC},
\]

\[
B' = -\frac{CA_1 + BA_2 + BA_3}{BC} - \frac{(C^2 - B^2)(A_2 + A_3)}{2A_2 A_3 C},
\]

\[
C' = -\frac{BA_1 + CA_2 + CA_3}{BC} - \frac{(B^2 - C^2)(A_2 + A_3)}{2A_2 A_3 B}.
\]

Note that for \( B = C \) the system (1.3) was investigated fully in [6] and [8]. To obtain a smooth metric (1.1) we must resolve the conical singularity of \( \overline{M} \) using one of two methods, which gives a space \( \mathcal{M}_1 \) or \( \mathcal{M}_2 \). We shall describe this scheme in § 2. Then the family of metrics (1.2) on \( \mathcal{M}_2/\mathbb{Z}_2 \) is obtained by integrating the system (1.3) for \( A_2 = -A_3 \) (this is the subject of § 3).

1.3. In § 4 we prove the following result, which completes our analysis of system (1.2) in the case of the space \( \mathcal{M}_2 \).

**Theorem 2.** Let \( M \) be a 3-Sasakian 7-manifold. Let \( p = 2 \) or \( p = 4 \), depending on whether the general leaf of the 3-Sasakian foliation on \( M \) is \( \text{SO}(3) \) or \( \text{SU}(2) \). Then the orbifold \( \mathcal{M}_2/\mathbb{Z}_p \) carries the following complete regular Riemannian metrics \( \overline{g} \) of the form (1.2), with holonomy group \( H \subset \text{Spin}(7) \):

1) if \( A_1(0) = 0, -A_2(0) = A_3(0) > 0 \) and \( 2A_2^2(0) = B^2(0) + C^2(0) \), then the metric \( \overline{g} \) in (1.1) has holonomy group \( \text{SU}(4) \subset \text{Spin}(7) \) and is homothetical to some metric in (1.2);

2) if \( A_1(0) = 0 \) and \( -A_2(0) = A_3(0) < B(0) = C(0) \), then there exists a regular ALC-metric \( \overline{g} \) of the form (1.1) with holonomy group \( \text{Spin}(7) \); this was found in [6]. At infinity such metrics converge to the product of a cone over the twistor space \( \mathcal{Z} \) and the circle \( S^1 \).

Moreover, each complete regular metric of the form (1.1) on \( \mathcal{M}_2/\mathbb{Z}_p \), which has the \( \text{Spin}(7) \)-structure mentioned above and a holonomy group \( H \subset \text{Spin}(7) \) is isometric to one of the above metrics.

§ 2. The description of a \( \text{Spin}(7) \)-structure on a cone over a 3-Sasakian manifold

2.1. In this section we describe briefly how to construct the spaces on which we find metrics with holonomy \( \text{Spin}(7) \). In our notation and definitions relating to a 3-Sasakian manifold we follow [6]. For more detail we direct the reader to [9].

By a cone \( \overline{M} \) over a smooth closed Riemannian manifold \( (M, g) \) we mean a Riemannian manifold \( (\mathbb{R}_+ \times M, dt^2 + t^2 g), t \in \mathbb{R}_+ = (0, \infty) \). The manifold \( M \) is said to be 3-Sasakian if the metric on \( \overline{M} \) is hyper-Kähler, that is, its holonomy group lies in \( \text{Sp}(\mathbb{m}+1)/4 \). Then there exist three parallel complex structures \( J^1, J^2 \) and \( J^3 \) on \( \overline{M} \), which satisfy \( J^j J^i = -\delta^{ij} + \varepsilon_{ijk} J^k \). Identifying \( M \) with the ‘base’
The fields $\xi^i$ are called \textit{characteristic fields of the 3-Sasakian manifold} $M$, and the dual 1-forms $\eta_i$ are called \textit{characteristic forms}. We can show that the $\xi^i$ form the Lie algebra $\mathfrak{su}(2)$ with respect to the Lie bracket of vector fields, so we have a fibration $\pi: M \to \mathcal{O}$ with general fibre $\text{SU}(2) = S^3$ (or $\text{SO}(3) = \mathbb{R}P^3$) over some 4-dimensional quaternionic Kähler orbifold $\mathcal{O}$. Let $\mathcal{H}$ be the bundle of horizontal vectors (with respect to $\pi$) on $M$.

We consider the following 2-forms on $M$:

$$\omega_i = d\eta_i + \sum_{j,k} \varepsilon_{ijk} \eta_j \wedge \eta_k, \quad i = 1, 2, 3.$$ 

We see immediately (see \cite{[6]}) that the $\omega_i$ span a subspace of $\Lambda^2 \mathcal{H}^*$, so we can pick an orthonormal system of 1-forms $\eta_4, \eta_5, \eta_6, \eta_7$ in $\mathcal{H}$ such that

$$\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7), \quad \omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5),$$

$$\omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6).$$

Consider the standard Euclidean space $\mathbb{R}^8$ with coordinates $x^0, \ldots, x^7$. Setting $e^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$, we define the following self-dual 4-form on $\mathbb{R}^8$:

$$\Phi_0 = \varepsilon^{0123} + \varepsilon^{04567} + \varepsilon^{0145} - \varepsilon^{02467} - \varepsilon^{0167} + \varepsilon^{02367} + \varepsilon^{0246} + \varepsilon^{1346} - \varepsilon^{0275} + \varepsilon^{1357} + \varepsilon^{0347} - \varepsilon^{1247} - \varepsilon^{0356} + \varepsilon^{1256}.$$ 

We know that the group of linear transformations of $\mathbb{R}^8$ which preserve $\Phi_0$ is isomorphic to $\text{Spin}(7)$, and this group $\text{Spin}(7)$ also preserves the orientation and the metric $g_0 = \sum_{i=0}^{7} (e^i)^2$. Let $N$ be an oriented Riemannian 8-manifold. We say that a differential form $\Phi \in \Lambda^4 N$ defines a $\text{Spin}(7)$-structure on $N$ if there exists an orientation-preserving isometry $\varphi_p: T_pN \to \mathbb{R}^8$ in a neighbourhood of each point $p \in N$ such that $\varphi^* p \Phi_0 = \Phi|_p$. If $\Phi$ is a parallel form, then the holonomy group of the Riemannian manifold $N$ reduces to the subgroup $\text{Spin}(7) \subset \text{SO}(8)$, that is, $\text{Hol}(N) \subset \text{Spin}(7)$. It is well-known that $\Phi$ is parallel if and only if it is closed:

$$\nabla \Phi = 0 \iff d\Phi = 0$$

(see \cite{[10]}). We shall construct a $\text{Spin}(7)$-structure on $\overline{M}$. We take the following form for $\Phi$:

$$\Phi = \varepsilon^{0123} + C^2 B^2 \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 + \frac{B^2 + C^2}{4} (\varepsilon^{01} - \varepsilon^{023}) \wedge \omega_1$$

$$+ \frac{B^2 - C^2}{4} (\varepsilon^{01} - \varepsilon^{023}) \wedge \omega + \frac{BC}{2} (\varepsilon^{02} - \varepsilon^{31}) \wedge \omega_2 + \frac{BC}{2} (\varepsilon^{03} - \varepsilon^{12}) \wedge \omega_3,$$

where

$$e^0 = dt, \quad e^i = A_i \eta_i, \quad i = 1, 2, 3, \quad e^j = B \eta_j, \quad j = 4, 5,$$

$$e^k = C \eta_k, \quad k = 6, 7,$$

and $A_1(t), A_2(t), A_3(t), B(t)$ and $C(t)$ are some smooth functions. It is easy to see that $\Phi$ corresponds to a Riemannian metric of the form (1.1) on $\overline{M}$. 
2.2. We shall assume that the quaternionic Kähler orbifold $\mathcal{O}$ is Kähler, so we can pick a basis $\eta_i$, $i = 4, 5, 6, 7$, such that the form $\omega = 2(\eta_4 \wedge \eta_5 + \eta_6 \wedge \eta_7)$ defines a Kähler structure on $\mathcal{O}$ and, in particular, is closed. This assumption closes the exterior algebra of forms under consideration and allows us to derive a well-defined system of equations with respect to the functions $A, B, C$. Note that if $\mathcal{O}$ is not assumed to be Kähler, then generally speaking we must assume that $B = C$ to close the algebra of forms.

**Lemma 1.** The fact that the form $\Phi$ is parallel is equivalent to the following system of ordinary differential equations:

\[
\begin{align*}
A_1' &= \frac{(A_2 - A_3)^2 - A_1^2}{A_2 A_3} + \frac{A_2^2(B^2 + C^2)}{B^2 C^2}, \\
A_2' &= \frac{A_1^2 - A_2^2 + A_3^2}{A_1 A_3} - \frac{B^2 + C^2 - 2A_2^2}{BC}, \\
A_3' &= \frac{A_1^2 + A_2^2 - A_3^2}{A_1 A_2} - \frac{B^2 + C^2 - 2A_3^2}{BC}, \\
B' &= -\frac{CA_1 + BA_2 + BA_3}{BC} - \frac{(C^2 - B^2)(A_2 + A_3)}{2A_2 A_3 C}, \\
C' &= -\frac{BA_1 + CA_2 + CA_3}{BC} - \frac{(B^2 - C^2)(A_2 + A_3)}{2A_2 A_3 B}.
\end{align*}
\]  

(2.1)

**Proof.** Using the relations imposed on the exterior algebra of forms in [6],

\[
d e^0 = 0,
\]

\[
d e^i = A_i' e^0 \wedge e^i + A_i \omega_i - \frac{2A_i}{A_{i+1} A_{i+2}} e^{i+1} \wedge e^{i+2}, \quad i = 1, 2, 3 \, (\text{mod} \, 3),
\]

\[
d \omega_i = \frac{2}{A_{i+2}} \omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}} e^{i+1} \wedge \omega_{i+2}, \quad i = 1, 2, 3 \, (\text{mod} \, 3),
\]

and also the relations $d\omega = 0$ and $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3$, after straightforward calculations we obtain

\[
d \Phi = \left[\frac{B^2 - C^2}{2A_2 A_3} A_1 - \frac{BB' - CC'}{2} - \frac{B^2 - C^2}{4A_2} A_2' - \frac{B^2 - C^2}{4A_3} A_3'\right] e^{023} \wedge \omega
\]

\[
+ \left[-A_1 - \frac{BC}{A_2} - \frac{BC}{A_3} + \frac{B^2 + C^2}{2A_2 A_3} A_1 - \frac{BB' + CC'}{2} - \frac{B^2 + C^2}{4A_2} A_2' - \frac{B^2 + C^2}{4A_3} A_3'\right] e^{023} \wedge \omega_1
\]

\[
+ \left[A_2 + \frac{BC}{A_1} - \frac{BCA_2}{A_1 A_3} + \frac{B^2 + C^2}{2A_3} + \frac{B'C + BC'}{2} + \frac{BCA_1}{2A_1} + \frac{BCA_2}{2A_3}\right] e^{013} \wedge \omega_2
\]

\[
- \left[A_3 + \frac{BC}{A_1} - \frac{BCA_3}{A_1 A_2} + \frac{B^2 + C^2}{2A_2} + \frac{B'C + BC'}{2} + \frac{BCA_1}{2A_1} + \frac{BCA_2}{2A_2}\right] e^{012} \wedge \omega_3
\]

\[
- \frac{1}{4} \left[2BCA_2 + BCA_3 + (B^2 + C^2)A_1 + C^2 BB' + B^2 CC'\right] e^0 \wedge \omega_1 \wedge \omega_1.
\]

Solving a system of 5 linear equations with respect to the ‘unknowns’ $A_1', A_2', A_3', B'$ and $C'$ we obtain (2.1). The proof is complete.
For $B = C$ we obtain the following system, which was investigated in [6]:

\[
\begin{align*}
A'_1 &= \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2A_3}, \\
A'_2 &= \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1A_3}, \\
A'_3 &= \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1A_2}, \\
B' &= -\frac{A_1 + A_2 + A_3}{B}.
\end{align*}
\]

(2.2)

To get a smooth Riemannian metric on a manifold (orbifold) we must prescribe boundary conditions for (2.1). In [6] the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ corresponding to the two different methods of resolving the conical singularity of $\mathcal{M}$ were described. Below we describe the space $\mathcal{M}_2$, on which we shall seek a metric with holonomy group $H \subset \text{Spin}(7)$.

Let $S \simeq S^1$ be the subgroup of $\text{SU}(2)$ (or $\text{SO}(3)$) integrating one of the Killing fields, for instance $\xi^1$. Then we have a principal bundle $\pi' : M \to \mathcal{S}$ with structure group $S$, where $\mathcal{S} = M/S$ is the twistor space. We consider the natural action of $S$ on $\mathbb{R}^2 = \mathbb{C}$: $e^{i\varphi} \in S : z \to ze^{i\varphi}$ and associate the fibred space $\mathcal{M}_2$ with fibre $\mathbb{C}$ on which we have the above action with $\pi'$. Thus the orbifold $\mathcal{S}$ is embedded in $\mathcal{M}_2$ as the zero section, and $\mathcal{M}_2 \setminus \mathcal{S}$ is foliated by ‘spherical’ sections diffeomorphic to $M$ and contracting to the zero section $\mathcal{S}$ as $t \to 0$.

Now let $p \in \mathbb{N}$ and $\mathbb{Z}_p \subset S$. The group $\mathbb{Z}_p$ acts by isometries on $\mathcal{M}_2$, so we have a well-defined orbifold $\mathcal{M}_2/\mathbb{Z}_p$, which is a manifold if and only if $\mathcal{M}_2$ is a manifold. It is easy to see that $\mathcal{M}_2/\mathbb{Z}_p$ is a bundle with fibre $\mathbb{C}$, which is associated with the principal bundle $\pi' : M \to \mathcal{S}$ by means of the action $e^{i\varphi} \in S : z \to ze^{i\varphi}$. Note that if $M$ is a regular 3-Sasakian manifold (that is, the foliation by 3-dimensional 3-Sasakian leaves is regular), then all the fibres of $\pi$ are isometric to $S^3 = \text{SU}(2)$ or $\text{SO}(3)$ and the orbifolds $\mathcal{O}$, $\mathcal{S}$ and $\mathcal{M}_2$ are smooth manifolds. We know that this is possible only when $M$ is isometric to $S^7$, $\mathbb{R}P^7$ or $N_{1,1} = \text{SU}(3)/T_{1,1}$ (see [9]). However, among these examples only the Aloff-Wallach space $N_{1,1}$ has a Kähler base, so we can only obtain new metrics on a smooth manifold in that case.

2.3. The following lemma presents conditions on the functions $A_i$, $B$ and $C$ which ensure that a solution of system (2.1) defines a smooth metric (1.1) on $\mathcal{M}_2$.

**Lemma 2.** Let $(A_1(t), A_2(t), A_3(t), B(t), C(t))$ be a $C^\infty$-smooth solution of (2.1), $t \in [0, \infty)$. Let $p = 4$ or $p = 2$ depending on whether the general fibre in $M$ is isometric to $\text{Sp}(1)$ or $\text{SO}(3)$. The metric (1.1) extends to a smooth metric on $\mathcal{M}_2/\mathbb{Z}_p$ if and only if the following conditions are satisfied:

1. $A_1(0) = 0$, $|A'_1(0)| = 4$;
2. $A_2(0) = -A_3(0) \neq 0$, $A'_2(0) = A'_3(0)$;
3. $B(0) \neq 0$, $B'(0) = 0$;
4. $C(0) \neq 0$, $C'(0) = 0$;
5. the functions $A_1$, $A_2$, $A_3$, $B$, $C$ have constant sign on the interval $(0, \infty)$.

Lemma 2 was proved in [6] for $B = C$. The proof can be carried over to the general case with no modifications apart from the following observation: in the construction of $\mathcal{M}_2$ in [6] it is not important how we choose the field $\xi^i$ along which the circle $S$ is ‘collapsed’, because the system (2.2) has extra symmetries. However, a simple analysis of (2.1) shows that we must take $\xi^1$ as a generator for $S$, so that only the function $A_1$ can vanish at the initial moment of time.
§ 3. Constructing explicit solutions on $\mathcal{M}_2$

3.1. In (2.1) we make the substitution $A_2 = -A_3$. Then adding together the second and third equations we obtain $B^2 + C^2 = 2A_2^2$, and subtracting the fifth equation form the fourth we conclude that $(B^2 - C^2)' = 0$. Thus we may assume that $B^2 = A_2^2 + \alpha^2$ and $C^2 = A_2^2 - \alpha^2$ for some nonnegative constant $\alpha$, and (2.1) reduces to the system

$$A_1' = -4 + \frac{A_1^2}{A_2^2} + 2\frac{A_1^2 A_2^2}{A_2^2 - \alpha^2}, \quad (A_2^2)' = -A_1.'$$

This is easy to integrate. Namely, we introduce a new variable $\rho$ by setting $d\rho = -2A_1 dt$. By shifting in $\rho$ we can always ensure that $A_2^2 = \rho$. Setting $A_1^2 = F$ we obtain

$$\frac{dF}{d\rho} + FG = 4,$$

where

$$G(\rho) = \frac{1}{\rho} + \frac{1}{\rho - \alpha^2} + \frac{1}{\rho + \alpha^2}.$$ 

This system is solved in the standard way (by introducing an integrating factor). Setting $r^2 = \rho$ we obtain

$$F = \frac{r^8 - 2\alpha^4 r^4 + \beta}{r^2 (r^4 - \alpha^4)};$$

where $\beta$ is the integration constant. So the metric (1.1) takes the following form:

$$\bar{g} = \frac{r^4 (r^2 - \alpha^2) (r^2 + \alpha^2)}{r^8 - 2\alpha^4 r^4 + \beta} dr^2 + \frac{r^8 - 2\alpha^4 r^4 + \beta}{r^2 (r^2 - \alpha^2) (r^2 + \alpha^2)} (\eta_1^2 + r^2 (\eta_2^2 + \eta_3^2)) + (r^2 + \alpha^2) (\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2) (\eta_6^2 + \eta_7^2).$$

To have a regular metric on $\mathcal{M}_2/\mathbb{Z}_p$ we need the polynomial $r^8 - 2\alpha^4 r^4 + \beta$ to have real roots and its largest root $r_0$ to be greater than $\alpha$. In this case the metric will be defined for $r \geq r_0$. Obviously, taking a metric homothetical to the original one, we can normalize the largest root by the condition $r_0 = 1$. Thus we can readily calculate that $0 \leq \alpha < 1$ and $\beta = 2\alpha^4 - 1$. Thus the metric (1.1) takes the following form:

$$\bar{g}_\alpha = \frac{r^4 (r^2 - \alpha^2) (r^2 + \alpha^2)}{r^8 - 2\alpha^4 (r^4 - 1) - 1} dr^2 + \frac{r^8 - 2\alpha^4 (r^4 - 1) - 1}{r^2 (r^2 - \alpha^2) (r^2 + \alpha^2)} (\eta_1^2 + r^2 (\eta_2^2 + \eta_3^2)) + (r^2 + \alpha^2) (\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2) (\eta_6^2 + \eta_7^2),$$

where $0 \leq \alpha < 1$ and $r \geq 1$. An immediate verification of the assumptions of Lemma 2 demonstrates that for $r \geq 1$, (3.1) represents a family of smooth metrics on $\mathcal{M}/\mathbb{Z}_p$ for $0 \leq \alpha < 1$, and $\bar{g}_0$ coincides with the Calabi metric with holonomy group SU(4) constructed in [1].

It follows from Lemma 1 that the holonomy group Hol($\bar{g}_\alpha$) of the metric (3.1) lies in Spin(7). Now consider the 2-form

$$\bar{\Omega}_1 = -e^0 \wedge e^1 + e^2 \wedge e^3 + e^4 \wedge e^5 - e^6 \wedge e^7.$$ 

Obviously, it is compatible with the metric (1.1). A direct calculation shows that it is closed precisely when $A_2 = -A_3$, so that it is the Kähler form of the metric (3.1). Thus Hol($\bar{g}_\alpha$) $\subset$ SU(4).
3.2. Now we make a detailed investigation of the case when the metrics $\bar{g}_\alpha$ are defined on a smooth manifold, that is, when $M = N_{1,1}$. We start by introducing our notation for the following subgroups of SU(3):

$$S_{1,1} = \{ \text{diag}(z, z, z^2) \mid z \in S^1 \subset \mathbb{C} \}, \quad K_1 = \left\{ \begin{pmatrix} \text{det}(A) & 0 \\ 0 & A \end{pmatrix} \mid A \in U(2) \right\}, \quad K_3 = \left\{ \begin{pmatrix} A & 0 \\ 0 & \text{det}(A) \end{pmatrix} \mid A \in U(2) \right\}.$$ 

Consider the 3-dimensional complex space $\mathbb{C}^3$ and the unit sphere $S^5 \subset \mathbb{C}^3$ in it. Assume that the unit circle $S^1$ acts diagonally on $\mathbb{C}^3$ and associated spaces. We use square brackets for equivalence classes defined by such an action: $[u, v], [u]$ etc.

Let

$$\bar{E} = \{ (u_1, u_2) \mid |u_1| = 1, \langle u_1, u_2 \rangle_{\mathbb{C}} = 0 \} \subset S^5 \times \mathbb{C}^3.$$ 

Consider the diagonal action of the circle $S^1$ on the space $\bar{E}$ and the projection $\bar{\pi}_1: (u_1, u_2) \mapsto u_1$ of $\bar{E}$ onto $S^5$, which is a fibration with fibre $\mathbb{C}^2$. The space of the spherical subbundle of $\bar{\pi}_1$ is

$$\bar{E}^1 = \{ (u_1, u_2) \in E \mid |u_1| = |u_2| = 1, \langle u_1, u_2 \rangle_{\mathbb{C}} = 0 \};$$

it is diffeomorphic to the group SU(3). The bundle $\bar{\pi}_1$ gives rise (by means of the action of $S^1$) to the vector bundle $\pi_1: E = \bar{E}/S^1 \to \mathbb{C}P^2$ with fibre $\mathbb{C}^2$ and spherical subbundle $E^1 = \bar{E}^1/S^1 = SU(3)/S_{1,1} = N_{1,1} \to \mathbb{C}P^2 = SU(3)/K_1$. It is easy to see that $\pi_1$ can be identified with the cotangent bundle $T^*\mathbb{C}P^2 \to \mathbb{C}P^2$.

In a similar way we consider the space

$$\tilde{H} = \{ (u_1, u_2, [u_3]) \mid |u_1| = |u_3| = 1, \langle u_i, u_j \rangle_{\mathbb{C}} = 0, i, j = 1, 2, 3 \} \subset S^5 \times \mathbb{C}^3 \times \mathbb{C}P^2$$

and the projection $\tilde{\pi}_2: (u_1, u_2, [u_3]) \mapsto (u_1, [u_3])$ of the space $\tilde{H}$ onto

$$\bar{F} = \{ (u_1, [u_3]) \mid |u_1| = |u_3| = 1, \langle u_1, u_3 \rangle_{\mathbb{C}} = 0 \},$$

with fibre $\mathbb{C}$. The total space of the spherical subbundle of $\tilde{\pi}_2$ coincides with

$$\tilde{H}^1 = \{ (u_1, u_2, [u_3]) \mid \langle u_i, u_j \rangle_{\mathbb{C}} = 0, |u_i| = 1, i, j = 1, 2, 3 \}$$

and can be identified with $SU(3) = \bar{E}^1$ in the obvious way. Through the same action of $S^1$ the bundle $\tilde{\pi}_2$ gives rise to a bundle $\pi_2: H = \tilde{H}/S^1 \to F = \bar{F}/S^1$ with fibre $\mathbb{C}$, whose spherical subbundle coincides with the map $E^1 = \bar{E}^1 = N_{1,1} \to SU(3)/T$. The base of $\pi_2$ is the complex flag manifold $F = SU(3)/T$, which can be represented as follows:

$$F = \{ ([u_1], [u_3]) \mid u_i \in \mathbb{C}^3, |u_i| = 1, \langle u_1, u_3 \rangle_{\mathbb{C}} = 0, i = 1, 3 \}.$$ 

**Definition 1.** We call the complex linear bundle $\pi_2: H \to F$ the canonical bundle on the complex flag manifold $F$ of $\mathbb{C}^3$.

Thus the canonical bundle on $F$ and the cotangent bundle of $\mathbb{C}P^2$ have the same total space $N_{1,1}$ of the spherical subbundle, which can be fibred in two different ways. It is known that $M = N_{1,1}$ carries the structure of a 3-Sasakian manifold,
whose twistor bundle coincides with $\pi_2: N_{1,1} \rightarrow F = \mathcal{Z}$ and whose 3-Sasakian foliation is given by the projection $\pi'_2: N_{1,1} \rightarrow \text{SU}(3)/K = \mathbb{CP} = \mathcal{O}$ with fibre $\text{SO}(3)$ (see [9]). Obviously, in this case $\mathcal{M}_2$ coincides with the space $H$ of the fibration $\pi_2$ on the complex flag manifold $F$ which was considered above. For $0 \leq \alpha < 1$ the metric (3.1) we have constructed is a smooth metric on $H/\mathbb{Z}_2$, the space of the complex linear bundle $\pi_2 \otimes \pi_2$. For $\alpha = 1$, (3.1) reduces to a metric on $E = T^* \mathbb{CP}^2$ coinciding with the Calabi metric (see [1]).

3.3. The proof of Theorem 1. Here we finish the proof and present a fuller statement of Theorem 1 given in the introduction.

**Theorem 3.** For $M = N_{1,1}$ the Riemannian metrics $\overline{g}_\alpha$ constructed explicitly in (3.1) are pairwise nonhomothetical smooth complete metrics. They have the following properties:

1) for $0 \leq \alpha < 1$, $\overline{g}_\alpha$ is a smooth metric on the space $H/\mathbb{Z}_2$ of the tensor square of the canonical bundle $\pi_2: H \rightarrow F$ on the complex flag manifold $F$ of $\mathbb{C}^3$ and has the holonomy group $\text{SU}(4)$; $\overline{g}_0$ coincides with the Calabi metric (see [1]);

2) the metric $\overline{g}_1$ has holonomy $\text{Sp}(2) \subset \text{SU}(4)$ and coincides with Calabi’s hyper-Kähler metric (see [1]) on $T^* \mathbb{CP}^2$.

**Proof.** To see that $\overline{g}_1$ is hyper-Kähler, it is sufficient to consider an additional pair of Kähler forms, which together with $\overline{\Omega}_1$ form a hyper-Kähler structure:

$$
\overline{\Omega}_2 = e^0 \wedge e^2 + e^1 \wedge e^3 - e^4 \wedge e^6 + e^7 \wedge e^5 = e^0 \wedge e^2 + e^1 \wedge e^3 - \frac{BC}{2} \omega_2,
$$

$$
\overline{\Omega}_3 = e^0 \wedge e^3 + e^1 \wedge e^2 - e^4 \wedge e^7 + e^5 \wedge e^6 = -e^0 \wedge e^3 + e^1 \wedge e^2 - \frac{BC}{2} \omega_3.
$$

Direct calculation shows that the forms $\overline{\Omega}_2$ and $\overline{\Omega}_3$ are closed precisely for $\alpha = 1$, which reduces the holonomy group to $\text{Sp}(2) \subset \text{SU}(4)$ in the case of the metric $\overline{g}_1$.

To complete the proof it remains to show that $\overline{g}_\alpha$ is not hyper-Kähler for $0 \leq \alpha < 1$. In fact, if

$$
\text{Hol}(\overline{g}_\alpha) = \text{Hol}(\mathcal{M}_2/\mathbb{Z}_2) \subset \text{Sp}(2), \quad 0 \leq \alpha < 1,
$$

then the limiting metric has the same property: $\text{Hol}(\overline{M}/\mathbb{Z}_2) \subset \text{Sp}(2)$. However, it is clear that after taking the quotient of the cone $\overline{M}$ by $\mathbb{Z}_2$, the generator of $\mathbb{Z}_2$ must be added to the holonomy group of $\overline{M}$. This generator corresponds to the transformation $\mathbb{H}^2 \rightarrow \mathbb{H}^2: (q_1, q_2) \mapsto (q_1', q_2')$, where $q_l = u_l + v_l j$ and $q_l' = u_l + v_l j$, $u_l, v_l \in \mathbb{C}$, $l = 1, 2$. It is clear that although this transformation belongs to $\text{SU}(4)$, it stays outside $\text{Sp}(2)$. Hence $\text{Hol}(\overline{M}/\mathbb{Z}_2)$ does not lie in $\text{Sp}(2)$ and $\text{Hol}(\overline{g}_\alpha) = \text{SU}(4)$, which completes the proof.

§ 4. Analysing the general problem of the existence of solutions on $\mathcal{M}_2$

4.1. Recall that a metric (1.1) is called **locally conical** if the functions $(A_i, B, C)$ are linear in $t$. If moreover, none of $(A_i, B, C)$ is a constant function, the metric (1.1) is called **conical**. If there exists a (locally) conical metric defined by functions $(\overline{A}_i, \overline{B}, \overline{C})$ such that

$$
\lim_{t \rightarrow \infty} \left| 1 - \frac{A_i(t)}{\overline{A}_i(t)} \right| = 0, \quad \lim_{t \rightarrow \infty} \left| 1 - \frac{B(t)}{\overline{B}(t)} \right| = 0, \quad \lim_{t \rightarrow \infty} \left| 1 - \frac{C(t)}{\overline{C}(t)} \right| = 0,
$$
then (1.1) is called an *asymptotically (locally) conical metric* (which we abbreviate to AC- or ALC-metric).

This section is mainly devoted to the proof of Theorem 2.

The central idea of the proof is to use the fact that system (2.1) has a homogeneous right-hand side and to come over to a dynamical system on the sphere $S^4 \subset \mathbb{R}^5$. Consider a vector $R(t) = (A_1(t), A_2(t), A_3(t), B(t), C(t)) \in \mathbb{R}^5$ and the map $V: \mathbb{R}^5 \to \mathbb{R}^5$ defined by the right-hand side of (2.1) (strictly speaking, $V$ is only partially defined, for $A_i, B, C \neq 0$). Thus we can write system (2.1) in the following form:

$$\frac{dR}{dt} = V(R).$$

Now we consider the substitution $R(t) = f(t)S(t)$, where $f(t) = |R(t)|$ and

$$S(t) = (\alpha_1(t), \ldots, \alpha_5(t)) \in S^4 = \left\{ (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mid \sum_{i=1}^{5} \alpha_i^2 = 1 \right\}.$$  

Since $V(fR) = V(R)$, the original system splits into its tangential and radial parts:

$$\frac{dS}{du} = V(S) - \langle V(S), S \rangle S = W(S), \quad (4.1)$$

$$\frac{1}{f} \frac{df}{du} = \langle V(S), S \rangle, \quad dt = f du. \quad (4.2)$$

We see that to solve (2.1) it is sufficient to solve the autonomous system (4.1) on $S^4$, after which we can find a solution to (2.1) by simply integrating equations (4.2).

The remaining part of the proof of Theorem 2 is structured as follows. First we find all the stationary and conditionally stationary points of system (4.1) (Lemmas 4 and 5); they determine the asymptotic behaviour of the corresponding metrics (Lemma 6). Next we describe the initial points $S_0$ corresponding to the necessary conditions for the smoothness of the metric in Lemma 2; we prove that there is a unique trajectory of system (4.1) going out of any such point (Lemma 7). After that it remains to understand the limiting behaviour of these trajectories. To do this we define invariant domains $\Pi$ and $\Gamma$ of system (4.1) and establish some differential relations, which hold along trajectories of the system and are useful for what follows (Lemma 8); these relations demonstrate that certain specially selected functions are monotonic along trajectories, so that their asymptotic behaviour can be described precisely (Proposition 1).

4.2. Symmetries, stationary and conditionally stationary points of system (4.1). The following lemma is obvious.

**Lemma 3.** System (4.1) has the discrete symmetry group $G = D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the following transformations:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, -\alpha_2, -\alpha_3, \alpha_5, -\alpha_4),$$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, \alpha_3, \alpha_2, \alpha_4, \alpha_5),$$

$$(\alpha_1(u), \ldots, \alpha_5(u)) \mapsto (-\alpha_1(-u), \ldots, -\alpha_5(-u),$$

where $D_4$ is the dihedral group.
Recall that a point $S$ is said to be stationary for (4.1) if $W(S) = 0$. Obviously, a vector field $W$ is defined precisely at the points $S \in S^4$ at which the vector field $V$ is defined. We say that a point $S \in S^4$ at which $W$ is not defined is conditionally stationary if there exists a real analytic curve $\gamma(u): (-\varepsilon, \varepsilon) \to S^4$ on the sphere such that $\gamma(0) = S$ and $\lim_{u \to 0} W(\gamma(u)) = 0$.

**Lemma 4.** All the stationary points of (4.1) can be obtained from the points

$$
\left(\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{\sqrt{5}}{\sqrt{13}}, \frac{\sqrt{5}}{\sqrt{13}}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)
$$

by the action of $G$.

**Proof.** Let $T$ denote the expression $\langle V(S), S \rangle$ in (4.1). We are looking for stationary solutions of the equation $W(S) = 0$, that is, for $S = (\alpha_1, \ldots, \alpha_5)$ such that no $\alpha_i$ vanishes.

We start with the case $\alpha_4 = \pm \alpha_5$. Note that system (2.1) is invariant under the substitution $$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (\alpha_1, -\alpha_2, -\alpha_3, \alpha_4, -\alpha_5),$$
so we can limit ourselves to the subsystem $\alpha_4 = \alpha_5$, which coincides with (2.2). Note that this case was considered in [6], but there the corresponding argument was left out for reasons of space, so we present the proof in full here. It is also easy to see that (2.2) is invariant under permutations of the variables $\alpha_1, \alpha_2, \alpha_3$.

Suppose $\alpha_1 \neq \alpha_2$. Setting $\alpha_2 W_1 - \alpha_1 W_2$ equal to zero we obtain the relation

$$\alpha_1 + \alpha_2 = \alpha_3 + \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^2}.$$ 

Note that if all the quantities $\alpha_1, \alpha_2, \alpha_3$ are different, then we also obtain the relation

$$\alpha_2 + \alpha_3 = \alpha_1 + \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^2}$$

so that taking the difference of the last two equations we arrive at a contradiction: $\alpha_1 = \alpha_3$.

Thus in view of the symmetry of the system, we shall assume that $\alpha_1 \neq \alpha_2$ and $\alpha_2 = \alpha_3$. Then from the penultimate equation we obtain $\alpha_2^2 = \alpha_4^2$, and since $W_2 = 0$, it follows that $T = \alpha_1 / \alpha_2^2$. Substituting all these relations in the equation $W_4 = 0$ we see that $\alpha_1 + \alpha_2 = 0$. In combination with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 = 1$, up to the symmetries in Lemma 3, all these relations give us the stationary point

$$\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

The case $\alpha_1 = \alpha_2 = \alpha_3$ remains: here (2.2) degenerates into $\alpha_4 = \sqrt{5}\alpha_1$, which in combination with the equations $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 = 1$ and $\alpha_4 = \alpha_5$, up to the symmetries in Lemma 3 immediately gives us the stationary point

$$\left(\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{\sqrt{5}}{\sqrt{13}}, \frac{\sqrt{5}}{\sqrt{13}}\right).$$

Now we return to the general case. Getting an expression for $T$ from the equation $W_1(S) = 0$ we obtain

$$T = \frac{\alpha_2}{\alpha_1 \alpha_3} + \frac{\alpha_3}{\alpha_1 \alpha_2} - \frac{\alpha_1}{\alpha_2 \alpha_3} - \frac{2}{\alpha_1} + \frac{\alpha_1}{\alpha_4} + \frac{\alpha_1}{\alpha_5}. $$
Since \( 0 = W_4 \alpha_5 - W_5 \alpha_4 = V_4 \alpha_5 - V_5 \alpha_4 \), we arrive at the following equation

\[
\frac{(\alpha_4^2 - \alpha_5^2)}{\alpha_2 \alpha_3 \alpha_4 \alpha_5} [\alpha_1 \alpha_2 \alpha_3 + \alpha_4 \alpha_5 (\alpha_2 + \alpha_3)] = 0.
\]

As we explained earlier, we can assume that \( \alpha_4 \neq \pm \alpha_5 \). Hence substituting

\[
\alpha_1 = -\frac{\alpha_4 \alpha_5 (\alpha_2 + \alpha_3)}{\alpha_2 \alpha_3}
\]

into the equation \( V_2 \alpha_3 - V_3 \alpha_2 = 0 \) yields

\[
\frac{(\alpha_2 - \alpha_3)(4 \alpha_2 \alpha_3 + \alpha_4^2 + \alpha_5^2)}{\alpha_4 \alpha_5} = 0.
\]

We shall consider two cases: 1) \( \alpha_2 - \alpha_3 = 0 \) and 2) \( 4 \alpha_2 \alpha_3 + \alpha_4^2 + \alpha_5^2 = 0 \).

1) In this case the equation \( W_4 = 0 \) takes the following form:

\[
\alpha_5^2 \alpha_3 \alpha_4^2 (2 \alpha_2^2 - 3 \alpha_5^2) + \alpha_5^4 (2 \alpha_5^2 - 3 \alpha_2^2) = 0.
\]

For convenience we set \( \alpha_5 = 1 \). If we find a nontrivial solution, we can normalize it, but from this point on, the equations will no longer be homogeneous. Setting

\[
\alpha_4^2 = \frac{\alpha_2^2 (3 - 2 \alpha_2^2)}{2 - 3 \alpha_2^2}
\]

in \( W_2 = 0 \) we obtain the biquadratic equation \( 4 \alpha_4^2 - 6 \alpha_2^2 + 5 = 0 \), which has no real roots.

2) Setting

\[
\alpha_2 = -\frac{\alpha_4^2 + \alpha_5^2}{4 \alpha_3}
\]

in the equation \( W_2 = 0 \), we obtain an expression in the numerator which is biquadratic with respect to \( \alpha_3 \):

\[
2 \alpha_4^6 \alpha_5^2 + 4 \alpha_4^4 \alpha_5^4 + 2 \alpha_4^2 \alpha_5^6 + 32 \alpha_3^4 \alpha_4^2 \alpha_5^2 - 19 \alpha_3^2 \alpha_4^4 \alpha_5^2 - 19 \alpha_3^2 \alpha_4^2 \alpha_5^4 - \alpha_3^2 \alpha_5^6 - \alpha_3^2 \alpha_5^6,
\]

which must be equal to zero. Obtaining an expression for \( \alpha_3 \) from this equation and substituting the result in \( W_4 = 0 \), after some calculations we obtain an equation for \( \alpha_4 \) and \( \alpha_5 \):

\[
(3 \alpha_4^4 - 2 \alpha_2^2 \alpha_5^2 + 3 \alpha_5^4)(\alpha_4^2 + \alpha_5^2)^4 \times \left[ \alpha_4^4 + 18 \alpha_4^2 \alpha_5^2 + \alpha_5^4 + (\alpha_4^2 + \alpha_5^2) \sqrt{\alpha_4^4 + 34 \alpha_4^2 \alpha_5^2 + \alpha_5^4} \right] = 0,
\]

which has no real roots either. The proof of Lemma 4 is complete.

The next lemma demonstrates that, by comparison with a similar system for equation (2.2), system (4.1) has no essentially new conditionally stationary points either.
Lemma 5. All the conditionally stationary points of (4.1) are obtained from the point
\[(0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\]
by the action of \(G\).

Proof. Let \(S = (\alpha_1, \alpha_2, \alpha_3, a_4, \alpha_5) \in S^4\) be a conditionally stationary point of (4.1), so that there exists a curve
\[\gamma(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha(u), \alpha_5(u)), \quad u \in (-\varepsilon, \varepsilon), \quad S = \gamma(0),\]
with the properties detailed above. Note that here \(u\) is a smooth parameter, which does not necessarily coincide with the independent variable of system (4.1) with respect to which derivatives are taken.

First we note that we can assume that \(\alpha_4(0) \neq \alpha_5(0)\) (although formally system (2.1) can have new conditionally stationary points by comparison with (2.2), for \(\alpha_4 = \alpha_5\), it is easy to see that the corresponding argument in [6] rules this out).

Hence, taking account of the symmetry group \(G\), we can conclude that the field \(W\) can only have singularities for \(\alpha_1(0) = 0, \alpha_2(0) = 0, \alpha_3(0) = 0\) or \(\alpha_5(0) = 0\).

1) We start with the case when all the four relations hold: \(\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = 0, \alpha_4(0) = 1\) (the case \(\alpha_4(0) = -1\) reduces to this by a symmetry in the group \(G\)). Then we set
\[\alpha_i = c_i u^{k_i}(1 + o(1)), \quad u \to 0,\]
where \(c_i \neq 0, k_i \geq 1, i = 1, 2, 3, 5\). Consider the following function:
\[\alpha_5 W_4 - \alpha_4 W_5 = \frac{\alpha_4^2 - \alpha_5^2}{\alpha_4 \alpha_5} \alpha_1 + (\alpha_4^2 - \alpha_5^2) \left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) = \frac{c_1}{c_5} u^{k_1 - k_5}(1 + o(1)) + \frac{1}{c_2} u^{-k_2}(1 + o(1)) + \frac{1}{c_3} u^{-k_3}(1 + o(1)) = o(1).\]

It is obvious that if \(k_2 \neq k_3\), then the last summands contain terms of maximal growth having distinct growth orders, so the first summand cannot compensate for both of them, which leads to a contradiction. So we must have \(k_2 = k_3\). Hence we are either in case 1, a): \(k_1 \geq k_5\), when \(c_2 = -c_3\), or in case 1, b): all the three summands display the same maximal order of growth to infinity. In this case \(k_1 - k_5 = -k_2 = -k_3\), so \(k_5 = k_1 + k_2\), and since the three leading terms must compensate for one another we have
\[\frac{c_1}{c_5} + \frac{1}{c_2} + \frac{1}{c_3} = 0.\]  

\[\text{(4.3)}\]

\[\begin{align*}
W_2 \alpha_3 - W_3 \alpha_2 &= W_4 \alpha_5 - W_5 \alpha_4 = W_1 \alpha_2 - W_2 \alpha_1 = W_1 = o(1).
\end{align*}\]

Then we obtain relations for the coefficients that either lead to a contradiction or yield the required solutions. The authors thank N. Kruzhilin for pointing out these inconsistencies. — The authors’ note to the English edition.
In case 1, b) consider the function

\[ \alpha_2 W_1 - \alpha_1 W_2 = 2 \frac{\alpha_0^2}{\alpha_3} - 2 \frac{\alpha_1^2}{\alpha_3} - 2 \alpha_2 + \alpha_1^2 \alpha_2 \alpha_3 \alpha_5 \frac{\alpha_4^2 + \alpha_5^2}{\alpha_4 \alpha_5} + \alpha_1 \frac{\alpha_4^2 + \alpha_5^2 - 2 \alpha_2^2}{\alpha_4 \alpha_5} \]

\[ = - \frac{2 c_1}{c_3} u^{2k_1 - k_2} (1 + o(1)) + \frac{c_2 c_2}{c_5} u^{2k_1 + k_2 - 2k_5} (1 + o(1)) + \frac{c_1}{c_5} u^{k_1 - k_5} (1 + o(1)) \]

\[ = - \frac{2 c_2}{c_3} u^{2k_1 - k_2} (1 + o(1)) + \frac{c_2 c_2}{c_5} u^{-k_2} (1 + o(1)) + \frac{c_1}{c_5} u^{-k_2} (1 + o(1)) = o(1). \]

Since the order of the leading term in the first summand is definitely greater that those in the second and third summands, for equality we require that

\[ \frac{c_2^2 c_2}{c_5^2} + \frac{c_1}{c_5} = 0, \]

and in view of (4.3), this yields \(1/c_3 = 0\), which is a contradiction.

In case 1, a) (that is, when \(k_2 = k_3, k_1 \geq k_5\) and \(c_2 = -c_3\)) we consider \(W_1\) and \(W_2\) (we absorb in \(o(1)\) the part that we know does not contain terms with greatest growth order in this case):

\[ W_1 = \frac{c_1^2}{c_3^2} u^{2k_1 - 2k_2} (1 + o(1)) = o(1), \quad W_2 = - \frac{c_1}{c_2} u^{k_1 - k_2} \frac{1}{c_5} u^{-k_5} = o(1). \]

It follows from the first relation that \(k_1 \geq k_2 + 1\). Then in the second relation we have a contradiction.

We see that the four variables cannot vanish simultaneously for \(u = 0\).

2) Next we consider the cases when precisely three variables vanish. First assume that we have case 2, a): \(\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = 0, \alpha_4(0) = a_4 \neq 0\) and \(\alpha_5(0) = a_5 \neq 0\). Then

\[ \alpha_5 W_4 - \alpha_4 W_5 = (a_4^2 - a_5^2) \left( \frac{1}{c_2} u^{-k_2} (1 + o(1)) + \frac{1}{c_3} u^{-k_3} (1 + o(1)) \right) = o(1). \]

Hence necessarily \(k_2 = k_3\) and \(c_2 = -c_3\). Again, we consider the functions \(W_1\) and \(W_2\):

\[ W_1 = -4(1 + o(1)) + \frac{c_1^2}{c_2} u^{2k_1 - 2k_2} (1 + o(1)) = o(1), \]

\[ W_2 = - \frac{a_4^2 + a_5^2}{a_4 a_5} (1 + o(1)) - \frac{c_1}{c_2} u^{k_1 - k_2} (1 + o(1)) = o(1). \]

It follows from the first relation that \(c_1^2/c_2^2 = 4\), and then the second relation shows that \(a_4 = a_5\), but we have ruled this possibility out. Therefore, case a) is impossible.

We consider case 2, b): \(\alpha_1(0) = a_1 \neq 0, \alpha_2(0) = \alpha_3(0) = \alpha_5(0) = 0\) and \(\alpha_4(0) = a_4 \neq 0\). Then

\[ \alpha_5 W_4 - \alpha_4 W_5 \]

\[ = \frac{a_4 a_1}{c_5} u^{-k_5} (1 + o(1)) + \frac{a_4}{c_2} u^{-k_2} (1 + o(1)) + \frac{a_4}{c_3} u^{-k_3} (1 + o(1)) = o(1). \]
Thus \(k_2 = k_3 = k_5 = k\) and
\[
\frac{a_1}{c_5} + \frac{1}{c_2} + \frac{1}{c_3} = 0.
\]

Now we have
\[
W_4 = \frac{a_4}{c_5} \left( c_5 a_1^3 - \frac{a_1^4}{c_5} - \frac{a_4^3}{2c_3} - \frac{a_4^3}{2c_3} + \frac{a_4}{2c_2} + \frac{a_4}{2c_2} \right) u^{-2k}(1 + o(1)) + O(1) = o(1),
\]
\[
W_5 = \left( \frac{c_5 a_1^3 - \frac{a_1^4}{c_5} - \frac{a_4^3}{2c_3} - \frac{a_4^3}{2c_3} - \frac{a_4}{2c_2} - \frac{a_4}{2c_2} \right) u^{-k}(1 + o(1)) = o(1).
\]

Hence the corresponding coefficients of \(u^{-2k}\) and \(u^{-k}\) vanish. Taking their difference (after scaling the first coefficient) we obtain
\[
\frac{a_4}{c_3} + \frac{a_4}{c_2} + \frac{a_1}{c_5} = 0.
\]

In combination with the previous equality of a similar form this allows us to conclude that \(a_4 = 1\), so that \(a_1 = 0\), contradicting our assumptions.

Consider case 2, c): \(\alpha_1 = \alpha_2 = \alpha_5 = 0\), \(\alpha_3(0) = a_3 \neq 0\) and \(\alpha_4(0) = a_4 \neq 0\). Then
\[
\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4 c_1}{c_5} u^{k_1-k_5}(1 + o(1)) + \frac{a_4}{c_2} u^{-k_2}(1 + o(1)) = o(1),
\]
which shows that \(k_1 - k_5 = -k_2\) and \(c_1 c_2 + a_4 c_5 = 0\). Hence
\[
W_5 = \frac{a_4(1-a_4^2)}{2c_2} u^{-k_2}(1 + o(1)) + O(1) = o(1),
\]
that is, \(a_4 = 1\), contradicting the assumptions made in this case.

Taking account of the symmetries in the group \(G\) we see that we have fully investigated case 2): there are no conditionally stationary points in this case.

3) Assume that precisely two variable vanish for \(u = 0\). We shall consider all the possible cases (modulo the action of \(G\)).

Suppose we have case 3, a): \(\alpha_1(0) = \alpha_2(0) = 0\), \(\alpha_3(0) = a_3 \neq 0\), \(\alpha_4(0) = a_4 \neq 0\) and \(\alpha_5(0) = a_5 \neq 0\). Then
\[
W_4 = \frac{a_3 a_4}{c_1 c_2} u^{-k_1-k_2}(1 + o(1)) = o(1),
\]
which is a contradiction.

Consider case 3, b): \(\alpha_1(0) = \alpha_5(0) = 0\), \(\alpha_2(0) = a_2 \neq 0\), \(\alpha_3(0) = a_3 \neq 0\) and \(\alpha_4(0) = a_4 \neq 0\). Then
\[
\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4 c_1}{c_5} u^{k_1-k_2}(1 + o(1)) = o(1),
\]
so that \(k_1 \geq k_5\). Suppose \(k_1 > k_5\); then
\[
W_4 = \left( \frac{a_3 a_4}{a_3 c_1} + \frac{a_3 a_4}{a_2 c_1} - \frac{2 a_2 a_3 a_4}{c_1} \right) u^{-k_1}(1 + o(1)) = o(1),
\]
which yields $a_2^2 = a_3^2$. If $a_2 = a_3$, then

$$W_4 = \frac{a_4}{c_5a_2} \left( -\frac{2a_2^2}{a_4} + a_4 + 4a_4a_2^2 - 4\frac{a_4^2}{a_4} - a_4^3 \right) u^{-k_5}(1 + o(1)) = o(1),$$

$$W_3 = \frac{1}{c_5} \left( -a_4 + \frac{2a_2^2}{a_4} + 4a_2a_4 - 4\frac{a_4^2}{a_4} - a_4^3 \right) u^{-k_5}(1 + o(1)) = o(1).$$

The expressions in brackets must vanish, so subtracting one expression from the other we obtain $a_2^4 = 2a_2^3$. Without loss of generality we can set $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{2}$ and $a_4 = \frac{1}{\sqrt{2}}$. Then we arrive at a contradiction as follows:

$$W_5 = -2\sqrt{2} + o(1) = o(1).$$

On the other hand if $a_3 = -a_2$, then we again obtain a contradiction:

$$W_1 = -4 + o(1) = o(1).$$

Now let $k_1 = k_5$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = a_4 \left( \frac{a_4}{a_2} + \frac{a_4}{a_3} + \frac{c_1}{c_5} \right)(1 + o(1)) = o(1).$$

We express $c_1$ in terms of the other parameters and obtain

$$\alpha_2 \alpha_3 W_1 - \alpha_1 \alpha_3 W_2 = \left( \frac{a_2 a_4}{a_3} + 4a_2^2 + a_4^2 \right)(1 + o(1)) = o(1),$$

$$\alpha_2 \alpha_3 W_1 - \alpha_1 \alpha_2 W_3 = \left( \frac{a_3 a_4^2}{a_2} + 4a_3^2 + a_4^2 \right)(1 + o(1)) = o(1).$$

From these two equations it necessarily follows that

$$(a_2^2 - a_3^2) \left[ 4 + \frac{a_4^2}{a_2a_3} \right] = 0.$$

The first factor cannot vanish for otherwise

either $-\frac{a_2 a_4^2}{a_2} + 4a_2^2 + a_4^2 = 4a_4^2 = 0$ or $\frac{a_4}{a_2} + \frac{a_4}{a_3} + \frac{c_1}{c_5} = \frac{c_1}{c_5} = 0.$

The second factor cannot vanish since otherwise we have

$$-\frac{4a_3 a_2 a_3}{a_2} + 4a_3^2 - 4a_2 a_3 = 0.$$

Consider case 3, c): $\alpha_2(0) = \alpha_3(0) = 0$, $\alpha_1(0) = a_1 \neq 0$, $\alpha_4(0) = a_4 \neq 0$ and $\alpha_5(0) = a_5 \neq 0$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = (a_4^2 - a_3^2) \left( \frac{a_1}{a_4 a_5} (1 + o(1)) + \frac{1}{c_2 u k_2} (1 + o(1)) + \frac{1}{c_3 u k_3} (1 + o(1)) \right) = o(1).$$
The last relation gives us a contradiction because \( k_2, k_3 \geq 1 \), and the other parameters do not vanish.

Consider case 3, d): \( \alpha_2(0) = \alpha_5(0) = 0, \alpha_1(0) = a_1 \neq 0, \alpha_3(0) = a_3 \neq 0 \) and \( \alpha_4(0) = a_4 \neq 0 \). Then
\[
\alpha_2 W_3 - \alpha_3 W_2 = \frac{a_3 a_4}{c_5} u^{-k_5} (1 + o(1)) = o(1),
\]
which is a contradiction.

4) It remains to analyse the case when any one variable takes the value zero.

Assume that we have case 4, a): \( \alpha_1(0) = 0 \) and \( \alpha_i(0) = a_i \neq 0, i = 2, \ldots, 5 \). Then
\[
\alpha_5 W_4 - \alpha_4 W_5 = (a_4^2 - a_5^2) \frac{a_2 + a_3}{a_2 a_3} (1 + o(1)),
\]
which means that \( a_2 = -a_3 \). Then
\[
W_1 = -4 + o(1),
\]
which leads to a contradiction.

Consider case 4, b): \( \alpha_2(0) = 0 \), while the other \( \alpha_i(0) = a_i \neq 0 \). Then
\[
\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4^2 - a_5^2}{c_2} u^{-k_2} (1 + o(1)),
\]
and we obtain a contradiction.

Consider case 4, c): \( \alpha_5(0) = 0 \), while the other \( \alpha_i(0) = a_i \neq 0 \). Then we also arrive at contradiction:
\[
\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_1 a_4}{c_4} u^{-k_5} (1 + o(1)).
\]

The proof of Lemma 5 is complete.

4.3. The behaviour of trajectories in a neighbourhood of the initial point.

The following lemma was proved in [6].

**Lemma 6.** Stationary solutions of system (4.1) are associated with locally conical metrics on \( \overline{M} \), and trajectories of (4.1) asymptotically tending to (conditionally) stationary solutions correspond to asymptotically (locally) conical metrics on \( \overline{M} \).

We set
\[
J = \{ 0, -\alpha_2, \alpha_2, \alpha_4, \alpha_5 \} \in S^4 \mid \alpha_2 > 0, \alpha_4 \geq \alpha_5 > 0 \}.
\]

By Lemma 2, to construct a regular metric on \( \mathcal{M}_2/\mathbb{Z}_p \) we must find a trajectory of system (4.1) going out of a point of the form \((0, -\lambda, \lambda, \mu, \nu)\), where \( 2\lambda^2 + \mu^2 + \nu^2 = 1 \). The symmetries in Lemma 3 let us limit ourselves to the case when the initial point \( S_0 = (0, -\lambda, \lambda, \mu, \nu) \) lies in the region \( J \), which is a geodesic triangle in the 2-sphere \( \{ \alpha_1 = \alpha_2 + \alpha_3 = 0 \} \).

**Lemma 7.** A unique trajectory of system (4.1) goes out of the above-mentioned point \( S_0 = (0, -\lambda, \lambda, \mu, \nu) \) into the domain \( \alpha_1 < 0 \).
Proof. Consider an open ball $U \subset \mathbb{R}^2$ of small radius $\varepsilon < 1 - \mu^2 - \nu^2$ in the system of coordinates $x = \alpha_1, y = \alpha_2 + \alpha_3$. Then in the neighbourhood $J \times U$ of the domain $J$ the variables $x, y, z = \alpha_4 > 0, w = \alpha_5 > 0$ form a local system of coordinates on $S^4$,

$$S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \left( x, \frac{1}{2} \left( y - \sqrt{2} \sqrt{1 - z^2 - x^2 - w^2 - \frac{y^2}{2}} \right), \frac{1}{2} \left( y + \sqrt{2} \sqrt{1 - z^2 - x^2 - w^2 - \frac{y^2}{2}} \right), z, w \right).$$

In these new coordinates the vector field $V(S)$ looks as follows:

$$V_1(S) = -4 + \frac{x^2}{z^2} + \frac{x^2}{w^2} + 2 \frac{x^2}{2x^2 + y^2 + z^2 + w^2 - 1},$$

$$V_2(S) = -\sqrt{2} - 2x^2 - 2z^2 - 2w^2 - y^2 (-2zw + 2x^2 + 2xw^2 - x^3)$$

$$+ \frac{xzw(y + \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})}{2zw + y^3 - y + x^2y},$$

$$V_3(S) = \sqrt{2 - 2x^2 - 2z^2 - 2w^2 - y^2} \frac{-2zw + 2x^2 + 2xw^2 - x^3}{xzw(y - \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})}$$

$$+ \frac{xzw(y - \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})}{2zw + y^3 - y + x^2y},$$

$$V_4(S) = -\frac{x}{z} - \frac{y}{w} - \frac{y(w^2 - z^2)}{w(x^2 + y^2 + z^2 + w^2 - 1)},$$

$$V_5(S) = -\frac{x}{w} - \frac{y}{z} - \frac{y(z^2 - w^2)}{z(x^2 + y^2 + z^2 + w^2 - 1)}.$$

In this system of coordinates the field $W$, which is tangent to $S^4$, has the components

$$W_x = W_1, \quad W_y = W_2 + W_3, \quad W_z = W_4, \quad W_w = W_5,$$

with the $W_i$ defined by (4.1).

In $J \times U$ we consider the system

$$\frac{d}{dv} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} xW_x \\ xW_y \\ xW_z \\ xW_w \end{pmatrix},$$

(4.4)

where $du = xdv$. Obviously, (4.4) has the same trajectories as (4.1). Since $z, w > 0$, $xW$ is a smooth field and its stationary points in $J \times U$ are described by $x = y = 0$, so that all of them lie on $J$. Now consider the linearization of system (4.4) in a neighbourhood of the point $S_0$:

$$\frac{dx}{dv} = -4x, \quad \frac{dy}{dv} = \frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu \nu} x + 4y, \quad \frac{dz}{dv} = 0, \quad \frac{dw}{dv} = 0.$$
The linearized system is degenerate, with eigenvalues $-4, 4, 0, 0$ and eigenvectors
\[ e_1 = \left(8, \frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu}, 0, 0\right), \]
\[ e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1), \]
respectively.

The tangent space to $J$ is spanned by $e_3$ and $e_4$. Calculations show that
\[ \langle (0, 0, a, b), xW \rangle |_{xW} \rightarrow 0 \text{ as } (x, y, z, w) \rightarrow S_0. \]
This means that trajectories of (4.1) meet $J$ at right angles. This allows us to consider the behaviour of the system in the $(x, y)$- and $(z, w)$-planes separately. In the $(x, y)$-plane we consider the parabolas
\[ F_1(x, y) = -\frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu} x + 8y - \alpha x^2 = 0, \]
\[ F_2(x, y) = -\frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu} x + 8y + \alpha x^2 = 0 \]
and the line $x = -\delta$, where $\alpha, \delta > 0$. They form a bounded region $\Gamma \subset U$. At points of the first parabola we have
\[ \langle \nabla F_1, (xW_x, xW_y) \rangle = \frac{d}{dv} \left( -\frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu} x + 8y - \alpha x^2 \right) = 12\alpha x^2 + O(x^2 + y^2). \]
The resulting expression vanishes only on $J$. Obviously, for sufficiently large values of the parameter $\alpha$ the angle between the outward normal $\nabla F_1$ and $(xW_x, xW_y)$ is acute, that is, the projections of trajectories onto $(x, y)$ cross the first parabola going out of $\Gamma$. Hence these trajectories leave the region $\Gamma \times J$. In a similar way, for large $\alpha$ the angle between the inward normal $\nabla F_2$ and $(xW_x, xW_y)$ is obtuse, so again the trajectories intersecting the second cylindrical surface $\{ F_2 = 0 \} \times J$ go outwards. The parabolas meet at $(0, 0)$ and the projections of trajectories intersect the parabolas, therefore there exists a trajectory whose projection comes into the point $(0, 0)$ for sufficiently small $\delta$. Since trajectories reach $J$ at a right angle, for sufficiently large $\alpha$ and sufficiently small $\delta$ there exists a trajectory of the system coming in $S_0$. Obviously, the projection of its tangent vector at $(x, y) = (0, 0)$ coincides with the tangent vector $\left(8, \frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu}\right)$ to the parabolas and the tangent vector itself is equal to $e_1$.

The $x$-coordinate converges to zero as $e^{-4v}$. Hence with respect to the parameter $u$ the trajectory of (4.1) comes into $S_0$ in finite time $u_0$, because in a neighbourhood of $S_0$ we have $-4u = c_1 e^{-4v} + c_2$ asymptotically. Note that in the domain $x < 0$ the parameters $u$ and $v$ are ‘inversely proportional’ since $c_1 < 0$. We see that for each $S_0 \in J$ there exists a unique trajectory leaving $S_0$ in finite time and going in the domain $\alpha_1 = x < 0$ with tangent vector at $S_0$ equal to $-e_1$. The proof of Lemma 7 is complete.

We have shown that for each point $S_0$ system (4.1) has a trajectory going out of this point. Hence by Lemma 2 there exists a metric on $\mathcal{M}_2/\mathbb{Z}_p$ which is regular in a neighbourhood of the orbifold $\mathcal{O}$.
4.4. Now we describe the behaviour of trajectories of the system at infinity. Recall that we are looking for asymptotically locally conical metrics.

**Lemma 8.** If $S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is a solution of (4.1), then the following relations hold:

1) $\frac{d}{du} \ln \left( \frac{\alpha_2}{\alpha_1} \right) = 2 \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1 \alpha_2 \alpha_3} + 2 \left( \alpha_2 - \frac{\alpha_4^2 + \alpha_5^2}{2 \alpha_4 \alpha_5} \alpha_1 \right) \left( \frac{1}{\alpha_1 \alpha_2} + \frac{1}{\alpha_4 \alpha_5} \right)$;

2) $\frac{d}{du} \ln \left( \frac{\alpha_2}{\alpha_3} \right) = 2(\alpha_2 - \alpha_3) \left( \frac{1}{\alpha_4 \alpha_5} + \frac{1}{\alpha_1 \alpha_2 \alpha_3} \left( \frac{\alpha_4^2 + \alpha_5^2}{2 \alpha_4 \alpha_5} \alpha_1 - \alpha_2 - \alpha_3 \right) \right)$;

3) $\frac{d}{du} \ln \left( \frac{\alpha_4}{\alpha_5} \right) = \left( \frac{\alpha_4 - \alpha_5}{\alpha_4 \alpha_5} \right)^2 \left( \frac{\alpha_1}{\alpha_4 \alpha_5} + \frac{\alpha_2 + \alpha_3}{\alpha_2 \alpha_3} \right)$;

4) $\frac{d}{du} \ln(\alpha_5^2) = -2\alpha_1(1 + \alpha_1^2)$ for $\alpha_5 = 0$;

5) $\frac{d}{du} \alpha_1 = \frac{(\alpha_2 - \alpha_3)^2}{\alpha_2 \alpha_3} \left( 1 - (\alpha_2 + \alpha_3)^2 \right)$ for $\alpha_1 = 0$;

6) $\frac{d}{du} (\alpha_1 - \alpha_2) = \left( \frac{\alpha_4 - \alpha_5}{\alpha_4 \alpha_5} \right) \left( \frac{\alpha_1^2}{\alpha_4 \alpha_5} + 1 \right)$ for $\alpha_1 = \alpha_2$;

7) $\frac{d}{du} (\alpha_2 + \alpha_3) = \frac{2}{\alpha_4 \alpha_5} (2\alpha_2^2 - \alpha_4^2 - \alpha_5^2)$ for $\alpha_2 + \alpha_3 = 0$;

8) $\frac{d}{du} (2\alpha_2^2 - \alpha_4^2 - \alpha_5^2) = \left( \alpha_2 + \alpha_3 \right) \left( \frac{4}{\alpha_1 \alpha_3} \left( \alpha_2 \alpha_3 + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \right) \right.$

$\left. - \frac{2\alpha_2}{\alpha_3} \left( \frac{\alpha_4^2 - \alpha_5^2}{\alpha_4 + \alpha_5} \right)^2 + 2 \frac{\alpha_4^2 + \alpha_5^2}{\alpha_4 \alpha_5} \right)$ for $2\alpha_2^2 = \alpha_4^2 + \alpha_5^2$;

9) $\frac{d}{du} \ln(\alpha_4 - \alpha_5) \sim \frac{2}{\alpha_3}$ for $\alpha_1 = \alpha_2$, $\alpha_4 - \alpha_5 \to 0$, $\alpha_3 \to 0$;

10) $\frac{d}{du} \ln \left( \frac{\alpha_2 - \frac{\alpha_4^2 + \alpha_5^2}{2 \alpha_4 \alpha_5} \alpha_1}{\alpha_1} \right) = \left( \frac{\alpha_4^2 - \alpha_5^2}{\alpha_4^2 + \alpha_5^2} \right)^2 \frac{1}{\alpha_3 \alpha_4 \alpha_5} \left( (\alpha_4^2 + \alpha_5^2)(\alpha_2 + \alpha_3) \right.$

$\left. + \alpha_2(\alpha_4^2 + \alpha_5^2 + 2\alpha_2 \alpha_3) \right)$ for $\alpha_2 = \frac{\alpha_4^2 + \alpha_5^2}{2 \alpha_4 \alpha_5} \alpha_1$.

Lemma 8 is proved by direct calculation. However, certain observations simplify the calculations significantly. Let

$$F(S) = F(\alpha_1, \ldots, \alpha_5) = \frac{P_1(S)}{P_2(S)}$$

be a homogeneous rational system of degree zero. Obviously,

$$\frac{d}{dt} \ln F(R) = \frac{1}{F(R)} \left( \frac{\partial F(R)}{\partial R} \right) \frac{dR}{dt} = \frac{1}{F(R)} \left( \frac{\partial F(R)}{\partial R} \right) \cdot V(R)$$

is a homogeneous rational function of $R$ of degree $-1$, therefore

$$\frac{d}{du} \ln F(S) = f \frac{d}{dt} \ln F(fS) = f \frac{d}{dt} \ln F(R) = \frac{d}{dt} \ln F(S)$$
(here and below we use the notation from (4.1) and (4.2)). Thus, to find \( \frac{d}{du} \ln F(S) \) it is sufficient to calculate \( \frac{d}{dt} \ln F(S) \), that is, to do the calculations in parts 1)–3) of Lemma 8 we can use system (2.1), which is pretty manageable, instead of the very cumbersome (4.1).

Now let \( F(S) \) be a rational function of degree \( k > 0 \). We will calculate its derivative at a point where \( F(S) = 0 \) (this is what we have in parts 4)–10) of Lemma 8). Obviously,

\[
\frac{d}{dt} F(R) = \left\langle \frac{\partial F(R)}{\partial R}, \frac{dR}{dt} \right\rangle = \left\langle \frac{\partial F(R)}{\partial R}, V(R) \right\rangle
\]

is a homogeneous rational function of degree \( k - 1 \) of \( R \). Then

\[
\frac{d}{du} F(S) = f \frac{d}{dt} (F(fS)f^{-k}) = f^{1-k} \frac{d}{dt} F(R) - kF(R)f^{-k-1} \frac{df}{du}
\]

\[
= \frac{d}{dt} F(S) - kF(R)f^{-k} \langle V(S), S \rangle = \frac{d}{dt} F(S) - kF(S) \langle V(S), S \rangle = \frac{d}{dt} F(S).
\]

So as before, in calculating the derivative of \( F(S) \) we use (2.1) in place of the intractable relations (4.1).

4.5. The behaviour of trajectories at infinity. The next assertion is the basis of the proof of Theorem 2.

**Proposition 1.** The trajectory of (4.1), which is specified by an initial point \( S_0 = (0, -\lambda, \lambda, \mu, \nu) \), \( \lambda > 0 \), \( \mu \geq \nu > 0 \), \( 2\lambda^2 + \mu^2 + \nu^2 = 1 \), displays one of the following patterns of asymptotic behaviour, depending on the parameter \( \mu \):

1) if \( \lambda = \frac{1}{2} \), then \( S(u) \) converges to the stationary point

\[
S_{\infty} = \left( -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)
\]

as \( u \to \infty \); the trajectory \( S(u) \) corresponds to the metric \( g_\alpha \) with \( \alpha = \sqrt{\mu^2 - \nu^2} \) in the family (3.1);

2) if \( \lambda < \frac{1}{2} \) and \( \mu = \nu \), then \( S(u) \) converges to the conditionally stationary points

\[
S'_{\infty} = \left( -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\]

as \( u \to \infty \);

3) if \( \lambda < \frac{1}{2} \) and \( \mu > \nu \), then \( S(u) \) converges to the point \( S_1 = (0, 0, 0, 1, 0) \) as \( u \to u_1 < \infty \);

4) if \( \lambda > \frac{1}{2} \), then \( S(u) \) converges to the point \( S_2 = (0, 0, 1, 0, 0) \) as \( u \to u_2 < \infty \).

**Proof.** Consider the regions \( \Pi \) and \( \Gamma \) in \( S^4 \) defined by

\[
\Pi = \{ \alpha_1 \leq 0, \alpha_2 \leq \alpha_1, \alpha_3 \geq 0, \alpha_4 \geq \alpha_5 \geq 0, \alpha_2 + \alpha_3 \leq 0, 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2 \},
\]

\[
\Gamma = \{ \alpha_1 \leq 0, \alpha_2 \leq \alpha_1, \alpha_4 \geq \alpha_5 \geq 0, \alpha_2 + \alpha_3 \geq 0, 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2 \}.
\]
The boundaries of $\Pi$ and $\Gamma$ are formed by the following subsets of $S^4$ (which we call ‘walls’ in what follows):

\[
\begin{align*}
\Pi_1 &= \{\alpha_1 = 0, \; \alpha_2 \leq \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_2 &= \{\alpha_1 \leq 0, \; \alpha_2 = \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_3 &= \{\alpha_1 \leq 0, \; \alpha_2 \leq \alpha_1, \; \alpha_3 = 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_4 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 = \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_5 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 = \alpha_5 = 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_6 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 = 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\}, \\
\Pi_7 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 = \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_1 &= \{\alpha_1 = 0, \; \alpha_2 \leq \alpha_1, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \leq 0, \; 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_2 &= \{\alpha_1 \leq 0, \; \alpha_2 = \alpha_1, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \geq 0, \; 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_3 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_4 = \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \geq 0, \; 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_4 &= \{\alpha_1 \leq 0, \; \alpha_2 < \alpha_1, \; \alpha_4 = \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \geq 0, \; 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_5 &= \{\alpha_1 < 0, \; \alpha_2 < \alpha_1, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 = 0, \; 2\alpha_2^2 \geq \alpha_4^2 + \alpha_5^2\}, \\
\Gamma_6 &= \{\alpha_1 < 0, \; \alpha_2 < \alpha_1, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \geq 0, \; 2\alpha_2^2 = \alpha_4^2 + \alpha_5^2\}.
\end{align*}
\]

In addition, we partition $\Pi$ by the wall

\[\Pi_8 = \left\{\alpha_1 \leq 0, \; \alpha_2 = \frac{\alpha_4^2 + \alpha_5^2}{2\alpha_4\alpha_5}\alpha_1, \; \alpha_3 \geq 0, \; \alpha_4 \geq \alpha_5 \geq 0, \; \alpha_2 + \alpha_3 \geq 0, \; 2\alpha_2^2 \leq \alpha_4^2 + \alpha_5^2\right\}\]

into the subdomains

\[\Pi' = \Pi \cap \left\{\alpha_2 \leq \frac{\alpha_4^2 + \alpha_5^2}{2\alpha_4\alpha_5}\alpha_1\right\} \quad \text{and} \quad \Pi'' = \Pi \cap \left\{\frac{\alpha_4^2 + \alpha_5^2}{2\alpha_4\alpha_5}\alpha_1 \leq \alpha_2 \leq \alpha_1\right\}.
\]

In accordance with Lemma 7, the trajectory $S(u)$ goes out of the point

\[S_0 = (0, -\lambda, \lambda, \mu, \nu), \quad \lambda > 0, \quad \mu \geq \nu > 0, \quad 2\lambda^2 + \mu^2 + \nu^2 = 1,
\]

with tangent vector $e_1 = (-8, -\frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu}, 0, 0)$ (expressed in the system of coordinates $(\alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_5)$). If $\mu = \nu$, then (2.1) reduces to system (2.2), which was investigated in [6], and in the case $\mu = \nu$ assertions 2) and 4) follow from the results of [6]. For this reason we shall assume that $\mu > \nu$. If $\lambda < \frac{1}{2}$, then $\mu^2 + \nu^2 > \frac{1}{2}$. Now, $\alpha_2 + \alpha_3 < 0$ for the coordinates of $e_1$, therefore at the initial instant the trajectory $S(u)$ enters the domain $\Pi' \subset \Pi$. In a similar way, if $\lambda > \frac{1}{2}$, then $\mu^2 + \nu^2 < \frac{1}{2}$ and for $u$ close to $u_0$ this curve enters the domain $\Gamma$. Finally, if $\lambda = \frac{1}{2}$, then $\mu^2 + \nu^2 = \frac{1}{2}$, and the family of solutions explicitly described in (3.1) satisfies this condition; the trajectories of the solutions in (3.1) fill the intersection of $\Pi$ and $\Gamma$. We shall analyse the behaviour of trajectories in the regions $\Pi$ and $\Gamma$ thoroughly; in each of them we must consider two significantly different cases: when the trajectory attains the boundary of the region in finite time and when it remains in the interior of the region for all values of $u$.

We split the rest of the proof of Proposition 1 into Lemmas 9–12.
Lemma 9. Assume that \( \lambda < \frac{1}{2} \), so that the trajectory \( S(u) \) enters the domain \( \Pi' \) at the initial instant. Then \( S(u) \) intersects the boundary of \( \Pi \) for some \( u = u_1 < \infty \).

Proof. Assume the contrary: \( S(u) \) remains in \( \Pi \) for all values of \( u \). The region \( \Pi \) contains one stationary point

\[
S_2 = \left( -\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)
\]

and one conditionally stationary point

\[
S_3 = \left( -\frac{\sqrt{2}}{2\sqrt{3}}, -\frac{\sqrt{2}}{2\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right).
\]

Both of them lie in \( \Pi_2 \cap \Pi_4 \subset \partial \Pi \). Let \( S(u) \to S_2 \) as \( u \to \infty \). In a neighbourhood of \( S_2 \) we introduce the system of coordinates \( x = \alpha_1 + \frac{1}{\sqrt{5}}, y = \alpha_2 + \frac{1}{\sqrt{5}}, z = \alpha_3 - \frac{1}{\sqrt{5}}, w = \alpha_4 - \alpha_5 \) and consider the linearization of (4.1) in this neighbourhood:

\[
\begin{align*}
\frac{dx}{du} &= \frac{2}{\sqrt{5}}(-21x - y + 3z), \\
\frac{dy}{du} &= \frac{2}{\sqrt{5}}(-x - 21y + 3z), \\
\frac{dz}{du} &= \frac{2}{\sqrt{5}}(-9x - 9y + 7z), \\
\frac{dw}{du} &= -\frac{10}{\sqrt{5}} w.
\end{align*}
\]  

(4.5)

We can immediately verify that the system (4.5) has eigenvalues \(-8\sqrt{5}, -8\sqrt{5}, -2\sqrt{5}, 2\sqrt{5}\). The eigenvectors corresponding to the negative eigenvalues span the hyperplane \( x + y - 3z = 0 \). Hence there exists a hypersurface tangent to the hyperplane \( x + y - 3z = 0 \) which is formed by trajectories of (4.1). These enter \( S_2 \) exponentially, and no other trajectory of (4.1) comes into \( S_2 \). Clearly, this hypersurface contains the intersection \( \Pi \cap \Gamma \), which consists of the trajectories of solutions in the family (3.1). A direct calculation shows that in a neighbourhood of \( S_2 \) this hypersurface is transversal to the other walls of the regions \( \Pi \) and \( \Gamma \), but is disjoint from the regions proper. Thus no trajectory of system (4.1) can approach \( S_2 \) save the trajectories corresponding to (3.1).

Now let \( S(u) \to S_3 \) as \( u \to \infty \), so that in particular, \( \alpha_3 \to 0 \) and \( \alpha_4 - \alpha_5 \to 0 \). It follows from part 9) of Lemma 8 that \( \ln(\alpha_4 - \alpha_5) \) increases for \( u \to \infty \), which is a contradiction. Thus we see that \( S_2 \) and \( S_3 \) are not limit points of \( S(u) \).

Note that as follows from part 10) of Lemma 8, either a trajectory lies entirely in \( \Pi' \) or after traversing a wall of \( \Pi' \) it goes over to the domain \( \Pi'' \) and cannot then return. We consider these two cases separately.

First assume that \( S(u) \in \Pi' \) for all \( u \). The relation in part 1) of Lemma 8 demonstrates that \( F_1 = \ln \frac{\alpha_2}{\alpha_1} \) is a decreasing function along the trajectory \( S(u) \) in \( \Pi' \). Hence as \( u \to \infty \), \( S(u) \) approaches the minimum level of the function \( F_1 \) on \( \Pi' \), which is \( \Pi_2 \cap \Pi' = \Pi_2 \cap \Pi_4 \cap \Pi' \). The relation in part 2) of Lemma 8 demonstrates that \( F_2 = \ln \frac{\alpha_3}{\alpha_5} \) is an increasing function along the trajectories in a neighbourhood of \( \Pi_2 \cap \Pi_4 \). Hence \( S(u) \) approaches the maximum level of \( F_2 \) in \( \Pi_2 \cap \Pi_4 \), so that \( \alpha_3 \to 0 \). Then however, it follows from part 9) of Lemma 8 that \( \alpha_4 - \alpha_5 \) is increasing, which contradicts the trajectory approaching the wall \( \Pi_4 \). This contradiction shows that the trajectory must go over to the domain \( \Pi'' \).
Suppose $\frac{d}{dx}$ demonstrates that the function $F(u)$ approaches the maximum level of the function $F_2$, that is, $\alpha_3 \to 0$. Next, in a neighbourhood of $\{\alpha_3 = 0\}$ $F_1$ is decreasing on trajectories, so $\alpha_1 - \alpha_2 \to 0$. An argument similar to the proof of the previous lemma (the part which deals with a neighbourhood of the wall $\Pi_3$) demonstrates that the trajectory either converges to the conditionally stationary point $S_2$ (which is impossible as we have just shown) or to the point $(0, 0, 0, 1, 0)$.

Thus we have shown that the trajectory $S(u)$ ‘attains’ the point $(0, 0, 0, 1, 0)$ in infinite time. However, it is easy to see that $\alpha_2^3$ is a smooth function in a neighbourhood of $(0, 0, 0, 1, 0)$, so we can take $\alpha_2^3$ for a new smooth parameter on $S(u)$. Now, we can attain the point $(0, 0, 0, 1, 0)$ only for some finite value of $\alpha_2^3$, and therefore only for a finite value of $u$. The proof of Lemma 9 is complete.

**Lemma 10.** Suppose $\lambda < \frac{1}{2}$ and assume that a trajectory $S(u)$ enters $\Pi'$ at the initial instant and intersects the boundary of the domain $\Pi$ at a point $S_1 = S(u_1)$, $u_1 < \infty$, for the first time. Then $S_1 = (0, 0, 0, 1, 0)$.

**Proof.** In fact, since $\Pi_4 \setminus (\Pi_1 \cup \Pi_3 \cup \Pi_5)$ lies in an invariant subspace of the system (4.1), the wall $\Pi_4$ can be attained in finite time only at points in which $\Pi_4$ intersects the walls $\Pi_1$, $\Pi_3$ and $\Pi_5$.

Let $S_1 = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Pi_1$. If $\alpha_3 > 0$, then $\alpha_2 < 0$ and $\alpha_2 \neq \alpha_3$. Hence $(\alpha_2 + \alpha_3)^2 < \alpha_2^2 + \alpha_3^2 \leq 1$ and the relation in part 5) of Lemma 8 demonstrates that the $\alpha_1$-coordinate is decreasing in a neighbourhood of the wall $\Pi_1$, so $\Pi_1$ cannot be attained in finite time from inside $\Pi_1$. Let $\alpha_3 = 0$ and $\alpha_2 \neq 0$. Then again $\alpha_1$ is decreasing, except perhaps for the case $\alpha_2 = -1$, when $S_1 = (0, -1, 0, 0, 0)$. However, in this case too the relation in part 5) of Lemma 8 demonstrates that in a neighbourhood of $S_1$, for $u < u_1$ the derivative of the variable $\alpha_1$ is equal to $-2\frac{1 + \alpha_2}{\alpha_3} < 0$, so $\alpha_1$ is decreasing, which is a contradiction. Hence the only possibility is when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $S_1 = (0, 0, 0, \alpha_4, \alpha_5)$. In this case assume that $\alpha_4 \geq \alpha_5 > 0$. Let $X = (x_1, x_2, x_3, x_4, x_5)$ be the tangent vector to $S(u)$ at $S_1$. Then we set
\[
\lim_{u \to u_1} \frac{\alpha_2(u)}{\alpha_1(u)} = h \quad \text{and} \quad \lim_{u \to u_1} \frac{\alpha_3(u)}{\alpha_1(u)} = f.
\]
Calculating the limits of $W(S(u))$ as $u \to u_1$ we obtain
\[
x_1 = \frac{h}{f} + \frac{f}{h} - 2 - \frac{1}{fh} \left( \frac{\alpha_2^2 - \alpha_5^2}{2\alpha_4\alpha_5} \right) \left( \frac{f}{h} + \frac{h}{f} \right),
\]
\[
x_2 = \frac{1}{f} - \frac{h^2}{f} + f \frac{\alpha_2^2 + \alpha_5^2}{\alpha_4\alpha_5} - \frac{\alpha_2^2 - \alpha_5^2}{2\alpha_4\alpha_5} \left( \frac{1}{f} - \frac{h}{f} \right),
\]
\[
x_3 = \frac{1}{h} - \frac{f^2}{h} + h \frac{\alpha_2^2 + \alpha_5^2}{\alpha_4\alpha_5} - \frac{\alpha_2^2 - \alpha_5^2}{2\alpha_4\alpha_5} \left( \frac{1}{h} - \frac{f}{h} \right).
\]
Now taking account of the equalities $x_2/x_1 = h$ and $x_3/x_1 = f$ we obtain
\[
h = f = 2 \frac{\alpha_4\alpha_5}{\alpha_4^2 + \alpha_5^2}.
\]
However, this means that the $x_1$-, $x_2$- and $x_3$-components of $X$ have the same sign, so that the trajectory cannot attain $S_1$ from the domain $\Pi$, which is a contradiction. Hence the wall $\Pi_1$ is attainable in finite time only if $\alpha_5 = 0$, that is, $S_1 = (0, 0, 0, 1, 0)$.

Now let $S_1 \in \Pi_5$. The relation in part 4) of Lemma 8 implies that $\alpha_5^2$ is increasing in a neighbourhood of $\Pi_5 \setminus \Pi_1$. Hence $S_1 \in \Pi_5 \cap \Pi_1$, so bearing in mind the above description of the trajectory in a neighbourhood of $\Pi_1$ we see that $S_1 = (0, 0, 0, 1, 0)$.

Let $S_1 \in \Pi_3$. Based of the above arguments we can assume that $\alpha_5 \neq 0$ and $\alpha_1 \neq 0$, so the component $W_3$ is smooth in a neighbourhood of $\Pi_3$. We take $\alpha_3$ for a smooth parameter in a neighbourhood of $u = u_1$. If $\alpha_1 \neq \alpha_2$ or $\alpha_4 \neq \alpha_5$, then the component of $W$ tangential to $\Pi_3$ in a neighbourhood of $S_1$ has order $1/\alpha_3$, so the trajectory cannot attain $\Pi_3$ in finite time. We see that $\alpha_1 = \alpha_2$, $\alpha_4 = \alpha_5$ and $S_1 = (\alpha, \alpha, 0, \sqrt{\frac{1-2\alpha^2}{2}}, \sqrt{\frac{1-2\alpha^2}{2}})$. Let $X = (x_1, x_2, x_3, x_4, x_5)$ be the tangent vector to $S(u)$ at $S_1$. Obviously,

$$\lim_{\varepsilon \to 0} W(S_1 + \varepsilon X) = \lim_{u \to u_1} W(S(u)) = X.$$ 

It is clear that $\lim_{u \to u_1} W_3(S(u)) = 0$, so $x_3 = 0$. Hence $X = 0$ and therefore $S_1$ is a conditionally stationary point, which cannot be attained in finite time.

Let $S_1 \in \Pi_2$. In view of the above, we can assume that $S_1 \notin \Pi_1 \cup \Pi_3 \cup \Pi_5$ and therefore $S_1 \notin \Pi_4$ and $\alpha_4 > \alpha_5 > 0$. Then relation 6) in Lemma 8 demonstrates that the function $\alpha_1 - \alpha_2$ increases in a neighbourhood of $\Pi_2$, so $\Pi_2$ cannot be attained in finite time.

Finally, relations 7) and 8) in Lemma 8 demonstrate that at points in the walls $\Pi_6 \setminus \Pi_7$ and $\Pi_7 \setminus \Pi_6$ the vector $W$ points inwards the domain, so for $\Pi_6$ and $\Pi_7$, only points in $\Pi_6 \cap \Pi_7$ can be attained in finite time. On the other hand the intersection $\Pi_6 \cap \Pi_7$ consists of the trajectories corresponding to the family of solutions (3.1), so this part also cannot be attained by trajectories $S(u)$ for $u = u_1$.

From the above we conclude that only the case $S_1 = (0, 0, 0, 1, 0)$ is possible under the assumptions of Lemma 10, which completes the proof.

**Lemma 11.** Assume that $\lambda > \frac{1}{2}$, so that the trajectory going out of the initial point lies in the domain $\Gamma$. Then $S(u)$ intersects the boundary of the domain $\Gamma$ for $u = u_1 < \infty$.

**Proof.** Assume the converse: $S(u)$ remains in $\Gamma$ for all $u$. First, by Lemma 4 there are no stationary points in the interior of $\Gamma$. Second, as already noted, the function $F_3$ is decreasing on trajectories of (4.1) in $\Gamma$, hence as $u \to \infty$, $S(u)$ approaches the set of points at which $F_3$ takes the minimum value in $\Gamma$, namely $\Gamma_3$. Now we observe that for $\alpha_4 = \alpha_5$ system (4.1) reduces to a system investigated in [6] for the same boundary values. Here the whole of our domain $\Gamma_3$ corresponds to the domain $\Gamma$ considered in the proof of Lemma 13 in [6]. The functions increasing or decreasing on trajectories that were used in the proof in [6] are well defined in the whole of $\Gamma$ and are increasing or decreasing, respectively, in some neighbourhood of $\Gamma_3$. Hence our argument in [6] shows that the trajectory $S(u)$ converges to the stationary point $S_2$ or to $(0, 0, 1, 0, 0)$, and in the second case the convergence takes finite time. Since the linearization (4.5) of system (4.1) in a neighbourhood of $S_2$, which we obtained in the first part of the proof of Lemma 9, demonstrates that $S(u)$
cannot tend to $S_2$, we have arrived at a contradiction and the proof of Lemma 11 is complete.

**Lemma 12.** Suppose $\lambda > \frac{1}{2}$, and assume that the trajectory from the initial point lies in the domain $\Gamma$ and intersects the boundary of $\Gamma$ for the first time in $S_1 = S(u_1)$, $u_1 < \infty$. Then $S_1 = (0, 0, 1, 0, 0)$.

**Proof.** Since $\Gamma_3 \setminus (\Gamma_1 \cup \Gamma_4)$ lies in a subspace invariant under the system (4.1), the wall $\Gamma_3$ can be attained in finite time only where it intersects the walls $\Gamma_1$ and $\Gamma_4$.

Consider the function $F_3 = \ln \frac{a_4}{a_5}$ on $S^4$. Formula 3) in Lemma 9 shows that $F_3$ is decreasing on the trajectories of system (4.1) in the domain $\Gamma$.

Let $S_1 = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Gamma_1$. If $\alpha_2 < 0$, then $\alpha_3 > 0$ and $\alpha_2 \neq \alpha_3$. Hence the $\alpha_1$-coordinate decreases in a neighbourhood of $\Gamma_1$, which is a contradiction. If $\alpha_2 = 0$ and $\alpha_3 \neq 0$, then $\alpha_1$ is not decreasing only when $\alpha_3 = 1$, that is, $S_1 = (0, 0, 1, 0, 0)$. Finally assume that $\alpha_2 = \alpha_3 = 0$, that is, $S_1 = (0, 0, 0, \alpha_4, \alpha_5)$. Repeating the argument used in the proof of Lemma 10 under the assumption that $\alpha_5 > 0$ word for word we arrive at a contradiction. Hence the remaining case is $\alpha_5 = 0$, so that $S_1 = (0, 0, 0, 1, 0)$. However, this means that in the interior of the region $\Gamma$ the function $F_3$ increases without limit on the trajectory, which contradicts it being decreasing.

Let $S_1 \in \Gamma_4$. Relation 4) in Lemma 8 means that $\alpha_2^2$ is increasing along trajectories in a neighbourhood of $\Gamma_4 \setminus \Gamma_1$, which leads to a contradiction (we have already shown that $\Gamma_1$ cannot be attained).

Finally, repeating the proof of Lemma 10 word for word we can show that the cases $S_1 \in \Gamma_2$, $S_1 \in \Gamma_5$ and $S_1 \in \Gamma_6$ are also impossible.

Assume that $S(u)$ remains in $\Gamma$ for all $u$. First, it follows from Lemma 4 that there are no stationary points in the interior of $\Gamma$. Second, as we pointed out above, $F_3$ is a decreasing function on trajectories of (4.1) in the domain $\Gamma$. Hence as $u \to \infty$, the trajectory $S(u)$ converges to the set of points at which $F_3$ takes the minimum value in $\Gamma$, that is, to $\Gamma_3$. Now we observe that for $\alpha_4 = \alpha_5$ system (4.1) reduces to a system investigated in [6] for the same boundary data, and the whole of our $\Gamma_3$ corresponds to the domain $\Gamma$ considered in the proof of Lemma 13 in [6].

The functions increasing or decreasing on trajectories that were used in the proof in [6], are well defined in the whole of our domain $\Gamma$ and are increasing or decreasing, respectively, in some neighbourhood of $\Gamma_3$. Hence our argument in [6] shows that the trajectory $S(u)$ tends to the stationary point $S_2$ or to $(0, 0, 1, 0, 0)$, and in the second case the convergence takes finite time. Since the linearization of system (4.1) in a neighbourhood of $S_2$ which we performed in the first part of the proof of the lemma demonstrates that $S(u)$ cannot tend to $S_2$, we have thus proved that in the case under consideration the trajectory $S(u)$ attains the point $(0, 0, 1, 0, 0)$ in finite time. The proof of Lemma 12 is complete.

Lemmas 9–12 complete the proof of Proposition 1.

Obviously, Proposition 1 yields Theorem 2. Indeed, trajectories in part 1) of Proposition 1 correspond to metrics in family 1) from Theorem 2, trajectories in part 2) correspond to metrics in family 2) from Theorem 2, and trajectories in parts 3) and 4) of Proposition 1 converge in finite time to singular points, so they correspond to incomplete metrics on $\mathcal{M}_2$. The fact that the metrics from family 2)
in Theorem 2 have the holonomy group $\text{Spin}(7)$ was proved in [6]. The proof of Theorem 2 is now complete.

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