K3 SURFACES WITH A SYMPLECTIC AUTOMORPHISM
OF ORDER 11

IGOR V. DOLGACHEV AND JONGHAE KEUM

Abstract. We classify possible finite groups of symplectic automorphisms of K3 surfaces of order divisible by 11. The characteristic of the ground field must be equal to 11. The complete list of such groups consists of five groups: the cyclic group of order 11, 11 ⋊ 5, \( L_2(11) \) and the Mathieu groups \( M_{11}, M_{22} \). We also show that a surface \( X \) admitting an automorphism \( g \) of order 11 admits a \( g \)-invariant elliptic fibration with the Jacobian fibration isomorphic to one of explicitly given elliptic K3 surfaces.

1. Introduction

Let \( X \) be a K3 surface over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). An automorphism \( g \) of \( X \) is called symplectic if it preserves a regular 2-form of \( X \). In positive characteristic \( p \), an automorphism of order a power of \( p \) is called wild. A wild automorphism is symplectic. A subgroup \( G \) of the automorphism group \( \text{Aut}(X) \) is called symplectic if all elements of \( G \) are symplectic, and wild if it contains a wild automorphism.

It is a well-known result of V. Nikulin that the order of a symplectic automorphism of finite order of a complex K3 surface takes value in the set \{1, 2, 3, 4, 5, 6, 7, 8\}. This result is true over an algebraically closed field \( k \) of positive characteristic \( p \) if the order is coprime to \( p \). The latter condition is automatically satisfied if \( p > 11 \) \cite{DK2}. If \( p = 11 \), a K3 surface \( X_\varepsilon \) defined by the equation of degree 12 in \( \mathbb{P}(1, 1, 4, 6) \)

\[
y^2 + x^3 + \varepsilon x^2 t_0^4 + t_1^{11} t_0 - t_0^{11} t_1 = 0, \quad \varepsilon \in k,
\]

admits a symplectic automorphism of order 11

\[
g_\varepsilon : (t_0, t_1, x, y) \mapsto (t_0, t_0 + t_1, x, y).
\]

The main result of the paper is the following.

**Theorem 1.1.** Let \( G \) be a finite group of symplectic automorphisms of a K3 surface \( X \) over an algebraically closed field of characteristic \( p \geq 0 \). Assume that the order of \( G \) is divisible by 11. Then \( p = 11 \) and \( G \) is isomorphic to one of the following five groups

\[
C_{11}, 11 : 5 = 11 \times 5, L_2(11) = \text{PSL}_2(\mathbb{F}_{11}), M_{11}, M_{22}.
\]
Moreover, the following assertions are true.

(i) For any element \( g \in G \) of order 11, \( X \) admits a \((g)\)-invariant elliptic pencil \(|F|\) and \( X \) is \( C_{11} \)-equivariantly isomorphic to a torsor of one of the surfaces \( X_\varepsilon \) equipped with its standard elliptic fibration.

(ii) If \( X = X_\varepsilon \) and \( G \) contains an element of order 11 leaving invariant both the standard elliptic fibration and a section, then \( G \cong C_{11} \) if \( \varepsilon \neq 0 \) and \( G \) is isomorphic to a subgroup of \( L_2(11) \) if \( \varepsilon = 0 \).

The surface \( X_0 \) is a supersingular K3 surface with Artin invariant 1 isomorphic to the Fermat surface

\[
x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.
\]

In a recent paper of Kondo [Ko] it is proven that both \( M_{11} \) and \( M_{22} \) appear as symplectic automorphism groups of \( X_0 \). An element \( g \) of order \( p = 11 \) in these groups leaves invariant an elliptic pencil with no \( g \)-invariant section, and we do not know whether the \( g \)-invariant elliptic pencil has no sections or has a section but no \( g \)-invariant section. Thus the surface \( X_0 \) admits three maximal finite simple symplectic groups of automorphisms isomorphic to \( L_2(11) \), \( M_{11} \) and \( M_{22} \).

**Corollary 1.2.** A finite group \( G \) acts symplectically and wildly on a K3 surface over an algebraically closed field of characteristic 11 if and only if \( G \) is isomorphic to a subgroup of \( M_{23} \) of order divisible by 11 and having 3 or 4 orbits in its natural action on a set of 24 elements.

**Acknowledgment**

The authors are grateful to S. Kondô for many fruitful discussions.

**Notation**

For an automorphism group \( G \) or an automorphism \( g \) of \( X \), we denote by \( X^g \) the fixed locus with reduced structure, i.e. the set of fixed points of \( g \).

A subset \( T \) of \( X \) is \( G \)-invariant if \( g(T) = T \) for all \( g \in G \). In this case we say \( G \) leaves \( T \) invariant.

An elliptic pencil \(|E|\) on \( X \) is \( G \)-invariant if \( g(E) \in |E| \) for all \( g \in G \). In this case we say \( G \) leaves \(|E|\) invariant.

We also use the following notations for groups:

\( C_n \) the cyclic group of order \( n \), sometimes denoted by \( n \),
\( m : n = m \rtimes n \) the semi-direct product of cyclic groups \( C_m \) and \( C_n \),
\( M_n \) the Mathieu group of degree \( n \),
\#\( G \) the cardinality of \( G \),
\( V^g \) the subspace of \( g \)-invariant vectors of \( V \).
2. The surfaces $X_0$ and $X_1$

Let $p = 11$ and $X_\varepsilon$ be the K3-surface from (1.1). The surface $X_\varepsilon$ has an elliptic pencil defined by the projection to the $t_0, t_1$ coordinates

$$f_\varepsilon : X_\varepsilon \to \mathbb{P}^1.$$ 

We will refer to it as the standard elliptic fibration. Its zero section, the section at infinity, will be denoted by $S_\varepsilon$. It is immediately checked that the surface $X_\varepsilon$ is nonsingular. Computing the discriminant $\Delta_\varepsilon$ of the Weierstrass equation of the general fibre of the elliptic fibration on $X_\varepsilon$ we find that

$$\Delta_\varepsilon = -t_0^2(t_1^{11} - t_1t_0^{10})(5t_1^{11} - 5t_1t_0^{10} + 4\varepsilon^3t_0^{11}).$$

This shows that the set of singular fibres of the elliptic fibration on $X_0$ (resp. $X_\varepsilon, \varepsilon \neq 0$) consists of 12 irreducible cuspidal curves (resp. one cuspidal fibre and 22 nodal fibres). The automorphism $g_\varepsilon$ given by (1.2) is symplectic and of order 11. It fixes pointwisely the cuspidal fibres over the point $\infty = (0,1)$ and has 1 orbit (resp. 2 orbits) on the set of remaining singular fibres. It leaves invariant the zero section $S_\varepsilon$. The quotient surface $X_\varepsilon/(g_\varepsilon)$ is a rational elliptic surface with a double rational point of type $E_8$ equal to the image of the singular point of the fixed fibre. A minimal resolution of the surface has one reducible non-multiple fibre of type $\tilde{E}_8$ and one irreducible singular cuspidal fibre (resp. 2 nodal fibres).

Proposition 2.1. Let $X$ be a K3 surface over an algebraically closed field $k$ of characteristic 11. Assume that $X$ admits an automorphism $g$ of order 11. Assume also that $X$ admits a $(g)$-invariant elliptic fibration $f : X \to \mathbb{P}^1$ with a section $S$. Then there exists an isomorphism $\phi : X \to X_\varepsilon$ of elliptic surfaces such that $\phi g \phi^{-1} = \tau g_\varepsilon$ for some translation automorphism $\tau$ of $X_\varepsilon$. In particular, if $g(S) = S$ then $\phi g \phi^{-1} = g_\varepsilon$.

Proof. Let

$$y^2 + x^3 + A(t_0, t_1)x + B(t_0, t_1) = 0$$

be the Weierstrass equation of the $g$-invariant elliptic pencil, where $A$ (resp. $B$) is a binary form of degree 8 (resp. 12). Since $f$ does not admit a non-trivial 11-torsion section ([DK2], Proposition 2.11), $g$ acts non-trivially on the base of the fibration. After a linear change of the coordinates $(t_0, t_1)$ we may assume that $g$ acts on the base by

$$g : (t_0, t_1) \mapsto (t_0, t_1 + t_0).$$

We know that a $g$-invariant elliptic fibration has one $g$-invariant irreducible cuspidal fibre $F_0$ and either 22 irreducible nodal fibres forming two orbits, or 11 irreducible cuspidal fibres forming one orbit ([DK1], p.124). Thus the discriminant polynomial $\Delta = -4A^3 - 27B^2$ must have one double root (corresponding to the fibre $F_0$) and either one orbit of double roots or two orbits of simple roots. We know that the zeros of $A$ correspond to either cuspidal fibres or nonsingular fibres with “complex multiplication” automorphism of order 6. Since this set is invariant with respect to our automorphism of
order 11 acting on the base, we see that the only possibility is $A = ct_0^8$ for some constant $c \in k$. We obtain $\Delta = -4ct_0^{12}t_1^{24} - 27B^2$. Again this uniquely determines $B$ and hence the surface. Since $B$ is of degree 12 and invariant under the action of $g$ on the base, it must be of the form

$$B = a(t_1^{11} - t_1t_0^{10})t_0 + bt_0^{12},$$

for some constants $a, b$. One can rewrite the above Weierstrass equation in the form

$$y^2 + x^3 + \varepsilon x^2t_0^4 + a(t_1^{11}t_0 - t_0^{11}t_1) + b't_0^{12} = 0.$$ 

A suitable linear change of variables $u_0 = t_0, u_1 = t_1 + dt_0$ makes $b' = 0$ without changing the action of $g$ on the base. Thus $X \cong X_\varepsilon$ as an elliptic surface. Let $\phi : X \to X_\varepsilon$ be the isomorphism. The composite

$$\phi g \phi^{-1} g^{-1} : X_\varepsilon \to X_\varepsilon,$$

acts trivially on the base, hence must be a translation automorphism. Since $\phi$ maps the zero section $S$ of $f : X \to \mathbb{P}^1$ to the zero section $S_\varepsilon$ of $f_\varepsilon : X_\varepsilon \to \mathbb{P}^1$ and $g_\varepsilon(S_\varepsilon) = (S_\varepsilon)$, the last assertion follows. $\square$

**Lemma 2.2.** Let $\varepsilon = 0$. For any translation automorphism $\tau$ of $X_0$, the composite automorphisms $\tau g_0$ and $g_0 \tau$ are of order 11.

**Proof.** Let $f : X \to B$ be any elliptic surface with a section $S$. Recall that its Mordell-Weil group $\text{MW}(f)$ is isomorphic to the quotient of the Neron-Severi group by the subgroup generated by the divisor classes of $S$ and the components of fibres. Thus any automorphism $g$ of $X$ which preserves the class of a fibre and the section $S$ acts linearly on the group $\text{MW}(f)$. Assume $\text{MW}(f)$ is torsion free. Suppose $g$ is of finite order $n$ with rank $\text{MW}(f)^n = 0$ and let $\tau$ be a translation automorphism identified with an element of $\text{MW}(f)$. Then, for any $s \in \text{MW}(f)$ we have

$$\tau g(s) = g(s) + \tau, \quad (\tau g)^n(s) = g^n(s) + g^{n-1}(\tau) + \ldots + g(\tau) + \tau = s.$$ 

The last equality follows from that the linear action of $g - 1_X$ on $\text{MW}(f)$ is invertible. This shows that $(\tau g)^n$ acts identically on $\text{MW}(f)$. It also acts identically on the class of a fibre. Thus $(\tau g)^n$ acts identically on the Neron-Severi lattice.

Apply this to our case $\varepsilon = 0$, when $g = g_0$ is a symplectic automorphism of order 11 of $X_0$. We will see in the proof of Proposition 2.9 that $\text{MW}(f_0)$ is torsion free. By Lemma 2.3(iii) below, rank $\text{MW}(f_0)^9 = 0$. Since the surface $X_0$ is supersingular (see Remark 2.7), by a theorem of Ogus \[Og\], an automorphism acting identically on the Picard group must be the identity. Thus $\tau g_0$ is a symplectic automorphism of order 11 for any section $\tau$. $\square$

An interesting question: Is there a $\tau$ such that the fixed locus $X_0^{\tau g_0}$ consists of an isolated point, the cusp of a cuspidal curve fixed pointwise by $g_0$? We do not know any example of a symplectic automorphism of order 11 with an isolated fixed point.
Lemma 2.3. Let $X$ be a K3 surface over an algebraically closed field $k$ of characteristic 11. Assume that $X$ admits an automorphism $g$ of order 11. Then the following assertions are true.

(i) $X$ admits a ($g$)-invariant elliptic pencil $|F|$;
(ii) rank Pic($X/(g)$) = 2;
(iii) for any $l \neq 11$, $\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^g = \text{rank } \text{Pic}(X)^g = 2$;
(iv) rank Pic($X$) = 2, 12 or 22.

Proof. To prove (i), assume first that $X$ does not admit a ($g$)-invariant elliptic pencil and $X^g$ is a point. This case could happen only if the sublattice $N$ of the Picard group of a minimal resolution of $X/(g)$ generated by irreducible components of exceptional curves is 11-elementary, and $N^\bot$ is an even lattice of rank 2. This is contained in the proof of Proposition 2.9 of [DK2]. The intersection matrix of $N^\bot$ is of the form

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$$

Since $N^\bot$ is indefinite and 11-elementary,

$$\det N^\bot = 4ab - c^2 = -1, \quad -11 \quad \text{or} \quad -121.$$  

In the first case, $N^\bot \cong U$, where $U$ is an even indefinite unimodular lattice. The second case cannot occur, since no square of an integer is congruent to 3 modulo 4. Assume the third case. Since $N^\bot$ is 11-elementary, all of the coefficients of the matrix must be divisible by 11, and hence $N^\bot \cong U(11)$. Therefore, in any case $N^\bot$ contains an isotropic vector. This is enough to deduce that $X$ admits a ($g$)-invariant elliptic pencil by the same proof as in Proposition 2.9 of [DK2].

Let $|F|$ be a ($g$)-invariant elliptic pencil. It follows from [DK1], p. 124, that the elliptic fibration has one cuspidal fibre and 22 nodal fibres, or 12 cuspidal fibres. The automorphism $g$ leaves one cuspidal fibre $F_0$ over a point $s_0 \in \mathbb{P}^1$ invariant.

Assertion (ii) follows from [DK1], where we proved that $X/(g)$ is a rational elliptic surface with no reducible fibres, and its minimal resolution is an extremal elliptic surface, i.e. the sublattice of the Picard group generated by irreducible components of fibres is of corank 1. It is proven in [HN], Proposition 3.2.1, that for any $l \neq p$ coprime with the order of $g$

$$\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^g = \dim H^2_{\text{et}}(X/(g), \mathbb{Q}_l).$$

In fact it is true for all $l \neq p$ because of the invariance of the characteristic polynomial of an endomorphism of a smooth algebraic variety. Now by (ii),

$$\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^g = \dim H^2_{\text{et}}(X/(g), \mathbb{Q}_l) = \text{rank } \text{Pic}(X/(g)) = 2.$$  

Since $g$ fixes the class of a fibre and an ample divisor, rank Pic($X$)$^g \geq 2$. This proves (iii).

Considering the $\mathbb{Q}$-representation of the cyclic group (g) of order 11 on Pic($X$) $\otimes \mathbb{Q}$, we get (iv) from (iii). $\square$
Corollary 2.4. Let $X$ be a K3 surface over an algebraically closed field $k$ of characteristic 11. Assume that $X$ admits an automorphism $g$ of order 11. Then $X$ is isomorphic to a torsor of one of the elliptic surfaces $X_\varepsilon$. The order of this torsor in the Shafarevich-Tate group of torsors is equal to 1 or 11.

Proof. Let $f_J: J \to \mathbb{P}^1$ be the Jacobian fibration of the elliptic fibration $f: X \to \mathbb{P}^1$ defined by the $g$-invariant elliptic pencil. Let $J^o$ be the open subset of $J$ whose complement is the set of singular fibres of $f_J$. We know that the fibres of $f$ are irreducible. By a result of M. Raynaud, this allows us to identify $J^o$ with the component $\text{Pic}^0_{X/\mathbb{P}^1}$ of the relative Picard scheme of invertible sheaves of degree 0 (see [CD], Proposition 5.2.2). The automorphism $g$ acts naturally on the Picard functor and hence on $J^o$. Since $J$ is minimal, it acts biregularly on $J$. This action preserves the elliptic fibration on $J$ and defines an automorphism of order 11 on the base. This implies that there exists an $C_{11}$-equivariant isomorphism of elliptic surfaces $J$ and $X_\varepsilon$.

The assertion about the order of the torsor follows from the existence of a section or an 11-section of $f$. In fact, let $Y$ be a nonsingular relatively minimal model of the elliptic surface $X/(g)$ with the elliptic fibration induced by $f$. It is a rational elliptic surface. Let $F_0$ be the $g$-invariant fibre of $f$ over a point $s_0 \in \mathbb{P}^1$. The singular fibres of the elliptic fibration $f': Y \to \mathbb{P}^1$ over $\mathbb{P}^1 \setminus \{s_0\}$ are either two irreducible nodal fibres ($\varepsilon \neq 0$) or one cuspidal irreducible fibre ($\varepsilon = 0$). The standard argument in the theory of elliptic surfaces shows that the fibre of $f'$ over $s_0$ is either of type $\tilde{E}_8$ or $\tilde{D}_8$. This fibre is not multiple if and only if $f'$ has a section. The pre-image of this section is a section of $f$ making $X$ the trivial torsor. A singular fibre of additive type can be multiple only if the characteristic is positive, and the multiplicity $m$ must be equal to the characteristic (see [CD], Proposition 5.1.5). In this case an exceptional curve of the first kind on $Y$ is a $m$-section. The pre-image of this multi-section on $X$ is a $m$-section, in our case an 11-section.

□

Remark 2.5. Note that, even in the case $X = X_\varepsilon$, the $g$-invariant fibration may be different from the standard elliptic fibration. In other words a non-trivial torsor of an elliptic surface could be isomorphic to the same surface. This strange phenomenon could happen only in positive characteristic and only for torsors of order divisible by the characteristic. We do not know an example where this strange phenomenon really occurs. In Kondo’s example, the $g$-invariant elliptic fibration for an element $g$ of order 11 in $G = M_{11}$ or $M_{22}$ may have a section (but no $g$-invariant section!). If this happens, it is isomorphic to the standard elliptic fibration and hence $g$ is conjugate to $\tau g_\varepsilon$ as we have seen in Proposition 2.1.

Lemma 2.6. Suppose $p = 11$. Then there is a finite subgroup $K_\varepsilon$ of $\text{Aut}(X_\varepsilon)$ satisfying the following property:

...
(i) $K_\varepsilon$ leaves invariant both the standard elliptic fibration of $X_\varepsilon$ and the zero section $S_\varepsilon$ which is the section at infinity.

(ii) $K_0 \cong \text{GU}_2(11)/(\pm I) \cong L_2(11) : 12$ and $K_1 \cong 11 : 4$, where the first factor in the semi-direct product is a symplectic subgroup and the second factor a non-symplectic subgroup.

(iii) The image of $K_\varepsilon$ in $\text{Aut}(\mathbb{P}^1)$ is equal to the subgroup $\text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$ which leaves the set $V(\Delta_\varepsilon)$ invariant.

(iv) $\text{Aut}(\mathbb{P}^1, V(\Delta_0)) \cong \text{PGU}_2(11) \cong L_2(11).2$ and $\text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon)) \cong 11 : 2$ if $\varepsilon \neq 0$.

Proof. Assume $\varepsilon = 0$. After a linear change of variables

$$t_0 = \alpha^1 t'_0 + \alpha t'_1, \quad t_1 = t'_0 + t'_1,$$

where $\alpha \in \mathbb{F}_{11^2} \setminus \mathbb{F}_{11} \subset k^*$, we can transform the polynomial $t_0 t_1^1 - t_0^{11} t_1$ to the form $\lambda t_0^{12} + \mu t_1^{12}$. After scaling, it becomes of the form $f = t_0^{12} + t_1^{12}$.

Now notice that this equation represents a hermitian form over the field $\mathbb{F}_{11^2}$, hence the finite unitary group $\text{GU}_2(11)$ leaves the polynomial $f$ invariant. The group $\text{GU}_2(11)$ acts on the surface

$$(2.2) \quad X_0 \cong V(y^2 + x^3 + t_0^{12} + t_1^{12})$$

in an obvious way, by acting on the variables $t_0, t_1$ and identically on the variables $x, y$. Note that

$$(t_0, t_1, x, y) = (\lambda t_0, \lambda t_1, \lambda^4 x, \lambda^6 y)$$

in $\mathbb{P}(1, 1, 4, 6)$ for all $\lambda \in k^*$. In particular $(t_0, t_1, x, y) = (-t_0, -t_1, x, y)$, so $-I \in \text{GU}_2(11)$ acts trivially on $X_0$. Note also that $\text{SU}_2(11)$ and hence $\text{PSU}_2(11)$ acts symplectically on $X_0$. The action of $\text{PSU}_2(11)$ is faithful because it is a simple group. Take $K_0 = \text{GU}_2(11)/(\pm I)$ and consider the homomorphism

$$\text{det} : K_0 \to (\mathbb{F}_{11^2})^*.$$

It is known that

$$U_2(11) = \text{PSU}_2(11) \cong \text{PSL}_2(\mathbb{F}_{11}) = L_2(11).$$

If $A \in \text{GU}_2(11)$, then $(\text{det} A)^{12} = (\text{det} A)(\text{det} \bar{A}) = \text{det} A^2 \bar{A} = \text{det} I = 1$, so the image of $\text{det}$ is a cyclic group of order dividing 12. On the other hand, if $\zeta \in \mathbb{F}_{11^2}$ is a 12-th root of unity, the unitary matrix

$$
\begin{pmatrix}
1 & 0 \\
0 & \zeta
\end{pmatrix}
$$

generates an order 12 subgroup of $K_0$, which acts on $X_0$ non-symplectically. This proves (i) and (ii).

We know that the group $\text{GU}_2(11)$ leaves the polynomial $f$ invariant. Thus its image $\text{PGU}_2(11)$ in $\text{Aut}(\mathbb{P}^1)$ must coincide with $\text{Aut}(\mathbb{P}^1, V(\Delta_0))$. It is known that $\text{PGU}_2(11)$ is a maximal subgroup in the permutation group $S_{12}$ and $\text{PGU}_2(11) \cong \text{PGL}_2(\mathbb{F}_{11}) \cong L_2(11).2$, a non-split extension. The quotient group is generated by the image of the automorphism : $(t_0, t_1) \to (t_0, \zeta t_1)$, where $\zeta \in \mathbb{F}_{11^2}$ is a 12-th root of unity. This proves (iii) and (iv).
Assume $\varepsilon \neq 0$. An element of $\text{PGL}_2(k)$ leaving $V(\Delta_\varepsilon)$ invariant must either leave all factors of $\Delta_\varepsilon$ from (2.1) invariant or interchange the second and the third factors. It can be seen by computation that the group $\text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$ is generated by the following 2 automorphisms
\[
e(t_0, t_1) = (t_0, t_1 + t_0), \quad i(t_0, t_1) = (t_0, -t_1 + bt_0)
\]
where $b$ is a root of $b^{11} - b + 3\varepsilon^3 = 0$. The order of $e$ (resp. $i$) is 11 (resp. 2) and $i$ normalizes $e$. We see that they lift to automorphisms of $\tilde{X}_\varepsilon$
\[
e(t_0, t_1, x, y) = (t_0, t_1 + t_0, x, y), \quad i(t_0, t_1, x, y) = (t_0, -t_1 + bt_0, -x + 3\varepsilon t_0^4, \sqrt{-1} y)
\]
and we take $K_\varepsilon = (\tilde{e}, \tilde{i})$. Clearly $\tilde{i}$ is non-symplectic of order 4 and normalizes $\tilde{e}$ which is symplectic of order 11, and both leave invariant the zero section $S_\varepsilon$.

Remark 2.7. The equation (2.2) makes $X_0$ a weighted Delsarte surface according to the definition in [Go]. It follows from loc.cit. that $X_0$ is a supersingular surface with Artin invariant $\sigma = 1$. It follows from the uniqueness of such surface that $X_0$ is also isomorphic to the Fermat quartic
\[x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,
\]
the Kummer surface associated to the product of supersingular elliptic curves, and the modular elliptic surface of level 4 (see [Sh]). We do not know whether the surface $X_\varepsilon, \varepsilon \neq 0$, is supersingular. By Lemma 2.6, we know that rank $\text{Pic}(X_\varepsilon) = 2, 12$ or 22.

Definition 2.8. The subgroup $K_\varepsilon \subset \text{Aut}(X_\varepsilon)$ from Lemma 2.6 contains a symplectic subgroup leaving invariant the standard elliptic fibration of $X_\varepsilon$, isomorphic to $L_2(11)$ if $\varepsilon = 0$ and to $C_{11}$ if $\varepsilon = 1$. Denote this subgroup by $H_\varepsilon$. It leaves invariant the zero section $S_\varepsilon$ of the elliptic fibration.

The group $H_\varepsilon$ acts on the base curve $\mathbb{P}^1$ and we have a homomorphism
\[\pi : H_\varepsilon \to \text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon)),
\]
which is an embedding. The image $\pi(H_\varepsilon)$ is equal to the unique index 2 subgroup of $\text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$.

Proposition 2.9. Let $G$ be a finite group of symplectic automorphisms of the surface $X_\varepsilon$ leaving invariant the standard elliptic fibration of $X_\varepsilon$. Let
\[\psi : G \to \text{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))
\]
be the natural homomorphism. Then $\psi$ is an embedding. If in addition $G$ is wild and leaves invariant the zero section $S_\varepsilon$, then $G$ is contained in $H_\varepsilon$.

Proof. Let $\alpha \in \text{Ker}(\psi)$. Then $\alpha$ acts trivially on the base curve. Since $p > 3$, $\alpha$ being symplectic must be a translation by a torsion section. It is known that there is no $p$-torsion in the Mordell-Weil group of an elliptic K3 surface if the characteristic $p > 7$ ([DK2]), and there are no other torsion sections because no symplectic automorphism of order coprime to $p$ can have more than 8 fixed points (Theorem 3.3 [DK2]), while the fibration has 12 or 23
singular fibres. Hence $\alpha$ must be the identity automorphism. This proves that $\psi$ is an embedding.

If $\psi$ is surjective, then $\#G = 2\cdot \#L_2(11)$ or $2.11$, which cannot be the order of a wild symplectic group in characteristic 11, by Proposition 3.3 and Lemma 4.1. Thus $\psi$ is not surjective. From this we see that if $G$ is wild, then $\psi(G)$ is contained in the unique index 2 subgroup $\pi(H_2)$ of Aut$(\mathbb{P}^1, V(\Delta_2))$. If an element $\alpha \in G$ and an element $h \in H_2$ have the same image in Aut$(\mathbb{P}^1, V(\Delta_2))$, then $\alpha h^{-1}$ is a translation by a section. If $\alpha$ leaves invariant the zero section $S^g_\epsilon$, so does $\alpha h^{-1}$, hence $\alpha h^{-1}$ is the identity automorphism. This proves the second assertion. \hfill \Box

3. A Mathieu representation

From now on $X$ is a K3 surface over an algebraically closed field of characteristic $p = 11$ and $G$ a group of symplectic automorphisms of $X$ of order divisible by 11.

Lemma 3.1. Let $S$ be a normal projective rational surface with an isolated singularity $s$. Then

$$e_c(S \setminus \{s\}) \geq 2,$$

where $e_c$ denotes the $l$-adic Euler-Poincaré characteristic with compact support.

Proof. Let $f : S' \to S$ be a minimal resolution of $S$. Let $E$ be the reduced exceptional divisor. Then $e_c(E) = 1 - b_1(E) + b_2(E) \leq 1 + b_2(E)$. Since the intersection matrix of irreducible components of $E$ is negative definite, we have $b_2(S') \geq 1 + b_2(E)$. This gives

$$e_c(S \setminus \{s\}) = e_c(S' \setminus E) = e_c(S') - e_c(E) \geq 2 + b_2(S') - (1 + b_2(E)) \geq 2. \hfill \Box

Lemma 3.2. Let $g$ be an automorphism of $X$ of order 11. Assume that $X^g$ is a point. Then the cyclic group $(g)$ is not contained in a larger symplectic cyclic subgroup of Aut$(X)$.

Proof. Let $H = (h)$ be a symplectic cyclic subgroup of Aut$(X)$ containing $(g)$. Write $\#H = 11r$ and $g = h^r$. Without loss of generality, we may assume that $r$ is a prime, and by Theorem 3.3 of [DK2], may further assume that $r = 2, 3, 5, 7$, or 11.

Assume $r \neq 11$. Let $f = h^{11}$. Then $f$ is symplectic of order $r = 2, 3, 5, \text{ or } 7$. By Theorem 3.3 of [DK2], $X^f$ is a finite set of points of cardinality < 11. The order 11 automorphism $g$ acts on $X^f$, hence acts trivially. Thus $X^f \subset X^g$, but $\#X^f \geq 3$, a contradiction. Thus $r = 11$.

Let $x \in X$ be the fixed point of $g$, and $y \in X/(g)$ be its image. Let $V = X/(g) \setminus \{y\}$. We claim that the quotient group $H = H/(g)$ acts freely on $V$. To see this, suppose that $h(z) = g^i(z)$ for some point $z \in X$, some $g^i \in (g)$. Then $g(z) = h^{11}(z) = g^{11i}(z) = 1_X(z) = z$, so $z = x$. This proves the claim.
By Lemma 2.3 for any \( l \neq 11 \), \( \dim H^2_{et}(X, \mathbb{Q}_l) = 2 \). This implies that
\[
\text{Tr}(g^*H^2_{et}(X, \mathbb{Q}_l)) = 0.
\]
By the Trace formula of S. Saito [S], \( l_x(g) = \text{Tr}(g^*|H^*_{et}(X, \mathbb{Q}_l)) = 2 \), where \( l_x(g) \) is the intersection index of the graph of \( g \) with the diagonal at the point \( (x,x) \). The formula of Saito ([S], Theorem 7.4, or [DK2], Lemma 2.8) gives \( e_c(V) = 3 \). Since the group \( H/g \) acts freely on \( V \), \( e_c(V/\bar{H}) = 3/\#\bar{H} \). Applying Lemma 3.1 to the surface \( S = X/H \), we obtain that \( \bar{H} \) is trivial. \( \square \)

Lemma 3.3. Let \( G \) be a finite group of symplectic automorphisms of a K3 surface \( X \) over an algebraically closed field of characteristic \( p = 11 \). Then
\[
\text{ord}(g) \in \{1, 2, 3, 4, 5, 6, 7, 8, 11\}
\]
for all \( g \in G \).

Proof. If the order \( \text{ord}(g) \) of \( g \in G \) is coprime to the characteristic \( p = 11 \), then by Theorem 3.3 of [DK2]
\[
\text{ord}(g) \in \{1, \ldots, 8\}.
\]
It remains to show that \( G \) cannot contain any element of order \( 11r, r > 1 \). Assume the contrary, and let \( h \in G \) be an element of order \( 11r \). We may assume that \( r \) is a prime and hence \( r = 2, 3, 5, 7, \) or \( 11 \). Let \( g = h^r \) and \( f = h^{11} \). We see that \( g \) is of order 11. By Lemma 3.2 \( X^g \) cannot be a point, hence must be a cuspidal curve ([DK1]). Denote this curve by \( F \). It is easy to see that \( F \) is \( h \)-invariant, i.e. \( h(F) = F \).

Assume \( r = 11 \). Then \( h \) acts on the base curve \( \mathbb{P}^1 \) of the pencil \( |F| \) faithfully, however, using the Jordan canonical form we see that \( \mathbb{P}^1 \) does not admit an automorphism of order \( 11^2 \).

Next, assume that \( r = 2, 3, 5, 7 \). By Theorem 3.3 of [DK2],
\[
3 \leq \#X^f \leq 8.
\]
Since \( r \) is prime to 11,
\[
X^h = X^f \cap X^g.
\]
Clearly \( g \) acts on the finite set \( X^f \), and this action cannot be of order 11. This means that \( g \) acts trivially on \( X^f \), i.e. \( X^f \subset X^g = F \). Thus
\[
X^h = X^f.
\]
This means that \( h \) acts on \( F \) with \( \#X^f \) fixed points. But no nontrivial action on a rational curve can fix more than 2 points. A contradiction. \( \square \)

A Mathieu representation of a finite group \( G \) is a 24-dimensional representation on a vector space \( V \) over a field of characteristic zero with character \( \chi(g) = \epsilon(\text{ord}(g)) \),
where

\[ \epsilon(n) = 24(n \prod_{p|n} (1 + \frac{1}{p}))^{-1}, \quad \epsilon(1) = 24. \]

The number

\[ \mu(G) = \frac{1}{\#G} \sum_{g \in G} \epsilon(\text{ord}(g)) \]

is equal to the dimension of the subspace \( V^G \) of \( V \). Here \( V^G \) is the linear subspace of vectors fixed by \( G \). The natural action of a finite group \( G \) of symplectic automorphisms of a complex K3 surface on the singular cohomology

\[ H^*(X, \mathbb{Q}) = \bigoplus_{i=0}^4 H^i(X, \mathbb{Q}) \cong \mathbb{Q}^{24} \]

is a Mathieu representation with

\[ \mu(G) = \dim H^*(X, \mathbb{Q})^G \geq 5. \]

From this Mukai deduces that \( G \) is isomorphic to a subgroup of \( M_{23} \) with at least 5 orbits. In positive characteristic, if \( G \) is wild, then the formula for the number of fixed points is no longer true and the representation of \( G \) on the \( l \)-adic cohomology, \( l \neq p \),

\[ H^*_{et}(X, \mathbb{Q}_l) = \bigoplus_{i=0}^4 H^i_{et}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24} \]

is not Mathieu in general. However, in our case we have the following:

**Proposition 3.4.** Let \( G \) be a finite group acting symplectically on a K3 surface \( X \) over a field of characteristic 11. Then the representation of \( G \) on \( H^*_\text{et}(X, \mathbb{Q}_l), l \neq 11 \), is a Mathieu representation with \( \dim H^*_{et}(X, \mathbb{Q}_l)^G \geq 3. \)

**Proof.** Note that rank \( \text{Pic}(X)^G \geq 1 \), and the second assertion follows. It remains to prove that the representation is Mathieu. By Lemma 3.3 it is enough to show this for automorphisms of order 11. Let \( g \in G \) be an element of order 11. We need to show that the character \( \chi(g) \) of the representation on the \( l \)-adic cohomology \( H^*_{et}(X, \mathbb{Q}_l) \) is equal to \( \epsilon(11) = 2. \) Since

\[ \chi(g) = \text{Tr}(g^*|H^*_{et}(X, \mathbb{Q}_l)), \]

it suffices to show that \( \text{Tr}(g^*|H^2_{et}(X, \mathbb{Q}_l)) = 0 \), or \( \dim H^2_{et}(X, \mathbb{Q}_l)^g = 2. \) Now the result follows from Lemma 2.3. \( \square \)

### 4. Determination of Groups

In this section we determine all possible finite groups which may act symplectically and wildly on a K3 surface in characteristic 11. We use only purely group theoretic arguments.
Proposition 4.1. Let $G$ be a finite group having a Mathieu representation over $\mathbb{Q}$ or over $\mathbb{Q}_l$ for all prime $l \neq 11$. Then

$$\#G = 2^a 3^b 5^c 7^d 11^e 23^f$$

for $a \leq 7, b \leq 2, c \leq 1, d \leq 1, e \leq 1, f \leq 1$.

Proof. If the representation is over $\mathbb{Q}$, this is the theorem of Mukai (Mukai (Theorem (3.22)). In his proof, Mukai uses at several places the fact that the representation is over $\mathbb{Q}$. The only essential place where he uses that the representation is over $\mathbb{Q}$ is Proposition (3.21), where $G$ is assumed to be a 2-group containing a maximal normal abelian subgroup $A$ and the case of $A = (\mathbb{Z}/4)^2$ with $\#(G/A) \geq 2^4$ is excluded by using that a certain 2-dimensional representation of the quaternion group $Q_8$ cannot be defined over $\mathbb{Q}$. We use that $G$ also admits a Mathieu representation on 2-adic cohomology, and it is easy to see that the representation of $Q_8$ cannot be defined over $\mathbb{Q}_2$. $\square$

The following lemma is of purely group theoretic nature and its proof follows an argument employed by S. Mukai [Mu].

Lemma 4.2. Let $G$ be a finite group having a Mathieu representation over $\mathbb{Q}$ or over $\mathbb{Q}_l$ for all prime $l \neq 11$. Assume $\mu(G) \geq 3$. Assume that $G$ contains an element of order 11, but no elements of order $> 11$. Then the order of $G$ is equal to one of the following:

$$11, \; 5.11, \; 2^2 3.5.11, \; 2^4 3^2 5.11, \; 2^7 3^2 5.7.11.$$  

Proof. Since $G$ has no elements of order 23, by Proposition 4.1 we have

$$\#G = 2^a 3^b 5^c 7^d 11^e 23^f,$$

(a $\leq 7, b \leq 2, c \leq 1, d \leq 1$).

Let $S_q$ be a $q$-Sylow subgroup of $G$ for $q = 5, 7$ or 11. Then $S_q$ is cyclic and its centralizer coincides with $S_q$. Let $N_q$ be the normalizer of $S_q$. Since $G$ does not contain elements of order $5k, 7k, 11k$ with $k > 1$, the index $m_q := [N_q : S_q]$ is a divisor of $q - 1$. Since it is known that the dihedral groups $D_{14}$ and $D_{22}$ do not admit a Mathieu representation, we have $m_7 = 1$ or 3, and $m_{11} = 1$ or 5. Let $a_n$ be the number of elements of order $n$ in $G$. Then $a_q = \frac{\#G(q-1)}{qm_q}$. As in [Mu], we have

(4.1)  

$$\mu(G) = \frac{1}{\#G} \sum e(n)a_n = 8 + \frac{1}{\#G} (16 - 2a_3 - 4a_4 - 4a_5 - 6a_6 - 5a_7 - 6a_8 - 6a_{11}).$$

Case 1. The order of $G$ is divisible by 7.

The formula (4.1) gives

(4.2)  

$$\mu(G) \leq 8 + \frac{16}{\#G} - \frac{30}{7m_7} - \frac{60}{11m_{11}}.$$  

Since $\mu(G) \geq 3$, the numbers $m_{11}$ and $m_7$ must be greater than 1.
The number of 11-Sylow subgroups is equal to 2

Assume \( m_{11} = 5, m_7 = 3 \). Then \( \#G \) is divisible by 5, and the formula (4.1) gives

\[
\mu(G) \leq 8 + \frac{16}{\#G} - \frac{16}{5m_5} - \frac{10}{7} - \frac{12}{11}.
\]

If \( m_5 = 1 \), then this inequality gives \( \mu(G) < 3 \). If \( m_5 = 2 \), then the number of \( q \)-Sylow subgroups is equal to \( 2^{a-1} \cdot 3^b \cdot 7 \cdot 11, 2^a \cdot 3^{b-1} \cdot 5 \cdot 11, 2^a \cdot 3^b \cdot 5 \cdot 7 \) for \( q = 5, 7, 11 \) respectively. Taking \( q = 5 \) and applying Sylow’s theorem, we get \( a - b \equiv 0 \mod 4 \). Since \( 1 \leq a \leq 7, 1 \leq b \leq 2 \), the only solutions are \((a, b) = (5, 1), (6, 2)\). However, neither \( 2^5 \cdot 5 \cdot 11 \) nor \( 2^6 \cdot 3 \cdot 5 \cdot 11 \) is congruent to 1 modulo 7.

If \( m_5 = 4 \), then the number of \( q \)-Sylow subgroups is equal to \( 2^{a-2} \cdot 3^b \cdot 7 \cdot 11, 2^a \cdot 3^{b-1} \cdot 5 \cdot 11, 2^a \cdot 3^b \cdot 5 \cdot 7 \) for \( q = 5, 7, 11 \) respectively. A similar argument, shows that \( a - b \equiv 1 \mod 4 \) and the possible order is \( 2^7 \cdot 3^2 \cdot 5 \cdot 11 \).

Case 2. The order of \( G \) is divisible by 5, but not by 7.

The formula (4.1) gives

\[
\mu(G) \leq 8 + \frac{16}{\#G} - \frac{16}{5m_5} - \frac{60}{11m_{11}}.
\]

Assume that \( m_{11} = 1 \). Then this inequality gives \( \mu(G) < 3 \).

Assume \( m_{11} = 5 \). If \( m_5 = 1 \), then the number of \( q \)-Sylow subgroups is equal to \( 2^a \cdot 3^b \cdot 11, 2^a \cdot 3^b \) for \( q = 5, 11 \) respectively. By Sylow’s theorem, \( a - b \equiv 0 \mod 4 \), \( a + 8b \equiv 0 \mod 10 \). This system of congruences has only one solution \( a = b = 0 \) in the range \( a \leq 7, b \leq 2 \). This gives the possible order 511.

If \( m_5 = 2 \), then the number of \( q \)-Sylow subgroups is equal to \( 2^{a-1} \cdot 3^b \cdot 11, 2^a \cdot 3^b \) for \( q = 5, 11 \) respectively. By Sylow’s theorem, \( a - b \equiv 1 \mod 4 \), \( a + 8b \equiv 0 \mod 10 \). This system has only one solution \( a = 2, b = 1 \) in the range \( 1 \leq a \leq 7, b \leq 2 \). This gives the possible order 2^2 \cdot 3 \cdot 5 \cdot 11.

If \( m_5 = 4 \), then the number of \( q \)-Sylow subgroups is equal to \( 2^{a-2} \cdot 3^b \cdot 11, 2^a \cdot 3^b \) for \( q = 5, 11 \) respectively. By Sylow’s theorem, \( a - b \equiv 2 \mod 4 \), \( a + 8b \equiv 0 \mod 10 \). This system has only one solution \( a = 4, b = 2 \) in the range \( 2 \leq a \leq 7, b \leq 2 \). This gives the possible order 2^4 \cdot 3 \cdot 5 \cdot 11.

Case 3. The order of \( G \) is divisible by neither 5 nor 7.

In this case \( m_{11} \neq 5 \), and hence \( m_{11} = 1 \). Thus the formula (4.1) gives

\[
\mu(G) \leq 8 + \frac{16}{\#G} - \frac{60}{11}.
\]

The number of 11-Sylow subgroups is equal to \( 2^a \cdot 3^b \). By Sylow’s theorem, \( a + 8b \equiv 0 \mod 10 \). This congruence has 3 solutions \((a, b) = (0, 0), (2, 1), (4, 2)\) in the range \( a \leq 7, b \leq 2 \). The first gives the possible order 11. In the second and the third case, the inequality (4.1) gives \( \mu(G) < 3 \).
Proposition 4.3. In the situation of the previous lemma, $G$ is isomorphic to one of the following groups:

$$C_{11}, \ 11 : 5, \ L_2(11), \ M_{11}, \ M_{22}.$$ 

Proof. By Lemma 4.2, there are five possible orders for $G$

$$11, \ 5.11, \ 2^2.3.5.11, \ 2^4.3^2.5.11, \ 2^7.3^2.5.7.11.$$ 

In the first two cases, the assertion is obvious. The remaining possible 3 orders are exactly the orders of the 3 simple groups given in the assertion. The theory of finite simple groups shows that there is only one simple group of the order in each of these cases.

Assume the last 3 cases. It suffices to show that $G$ is simple.

Let $K$ be a proper normal subgroup of $G$ such that $G/K$ is simple. If $\# K$ is not divisible by 11, then an order 11 element of $G$ acts on the set $\text{Syl}_q(K)$ of $q$-Sylow subgroups of $K$. Since $\# \text{Syl}_q(K)$ is not divisible by 11 for any prime $q$ dividing $\# K$, the order 11 element $g$ must normalize a $q$-Sylow subgroup of $K$. If one of the numbers $q = 3, 5, 7$ divides $\# K$, then $g$ centralizes an element of one of these orders. This contradicts the assumption that $G$ does not contain an element of order $> 11$. If $q = 2$ divides $\# K$, then a 2-Sylow subgroup of $K$ is of order $\leq 2^7$, and hence $g$ centralizes an element of order 2, again a contradiction. So, we may assume that $11 | \# K$. If $\# K = 11$, then an order 2 element of $G$ normalizes $K$. Neither a cyclic group of order 22 nor a dihedral group of order 22 has a Mathieu representation, so $\# K > 11$.

If $K \cong 11 : 5$, then an order 2 element of $G$ normalizes the unique 11-Sylow subgroup of $K$, again a contradiction. If $\# K$ is one of the remaining three possibilities, then the group $G/K$ is of order $2^5.3.7$ or $2^3.7$ or $2^2.3$. In the first case an order 7 element of $G$ normalizes, hence centralizes a Sylow 11-subgroup of $K$, again a contradiction. Obviously in the other two cases $G/K$ cannot be simple. This proves that $G$ is simple. \hfill $\square$

Corollary 4.4. Let $G$ be a finite group acting symplectically and wildly on a $K3$ surface $X$ over a field of characteristic 11. Let $g$ be an element of order 11 in $G$. Then the normalizer of $(g)$ in $G$ must be isomorphic to $11 : 5$ if $\# G > 11$.

5. Proof of the Main Theorem

In this section we complete the proof of Theorem I announced in Introduction. It remains to prove the assertion (ii).

Lemma 5.1. Assume $\varepsilon \neq 0$. Let $G \subset \text{Aut}(X_\varepsilon)$ be a finite wild symplectic subgroup. If an element $g \in G$ of order 11 leaves invariant the standard elliptic fibration with a $g$-invariant section, then $G = (g) \cong C_{11}$ and $G$ is conjugate to $H_\varepsilon = (g_\varepsilon)$. In particular, $H_\varepsilon$ is a maximal finite wild symplectic subgroup of $\text{Aut}(X_\varepsilon)$. 

Proof. Since $g$ leaves a section invariant, it must be a conjugate to $g_c$. So up to conjugation, we may assume that $g$ leaves the zero section $S_c$ invariant. Thus $g = g_c$ by Proposition 2.9.

Suppose $G > (g)$. Let $N$ be the normalizer of $(g)$ in $G$. Then $N \cong 11 : 5$ by Corollary 4.4.

Claim that $N$ leaves invariant the standard elliptic pencil $|F|$. It is enough to show that $h(F_0) = F_0$ for any $h \in N$, where $F_0 = X^g$ is a cuspidal curve in $|F|$. In fact, for any $x \in F_0$, we have $h(x) = hg(x) = g^i h(x)$ for some $i$, so $h(x) \in X(g) = F_0$, which proves the claim.

Next, claim that $N$ leaves invariant the zero section $S_c$. In fact, $h(S_c) = hg(S_c) = g^i h(S_c)$, so $(g)$ leaves invariant $h(S_c)$, and hence $h(S_c) = S_c$ as $g$ cannot leave invariant two distinct sections by Lemma 2.3 (iii).

Now Proposition 2.9 gives a contradiction. Hence, $G = (g)$. □

Lemma 5.2. Let $G \subset \text{Aut}(X_0)$ be a finite wild symplectic subgroup, isomorphic to $L_2(11)$. If an element $g \in G$ of order 11 leaves invariant both the standard elliptic fibration and a section, then $G$ is conjugate to $H_0$. In particular, if $G$ contains $g_0$ then $G = H_0$.

Proof. Replacing $G$ by a conjugate subgroup in $\text{Aut}(X_0)$, we may assume that $g$ leaves invariant both the standard elliptic fibration and the zero section $S_0$, i.e. $g = g_0$. We need to prove that $G = H_0$.

Let $|F|$ be the standard elliptic fibration. Then $g(S_0) = S_0$ and $X^g = F_0$, a cuspidal curve in $|F|$.

Let $N$ be the normalizer of $(g)$ in $G$. Then $N \cong 11 : 5$. The same argument as in the proof of Lemma 5.1 shows that $N$ leaves invariant both the standard elliptic pencil $|F|$ and the zero section $S_0$. By Proposition 2.9 $N \subset H_0$.

We have $N \subset G \cap H_0$. Suppose $G \cap H_0 = N$. Consider the $G$-orbit of the divisor class $[F] \in \text{Pic}(X_0)$,

$$G([F]) = \{ h([F]) \in \text{Pic}(X_0) | h \in G \}.$$ 

Clearly $N$ acts on it. Note

$$\# G([F]) = [G : N] = 12.$$ 

Thus $G([F])$ is the set of 12 different elliptic fibrations with a section. The automorphism $g$ cannot leave invariant an elliptic fibration other than $|F|$, hence fixes $[F]$ and has 1 orbit on the remaining 11 elliptic fibrations, which we denote by $[F_1], \ldots, [F_{11}]$.

Recall that $H_0$ leaves invariant the zero section $S_0$. The three divisor classes

$$[F], \sum_{j=1}^{11} [F_j], \quad [S_0]$$ 

are linearly independent over $\mathbb{C}$.
are $N$-invariant, and their intersection matrix is given as follows:

$$
\begin{pmatrix}
0 & 11m & 1 \\
11m & 110m & 11b \\
1 & 11b & -2
\end{pmatrix}
$$

where $m = F \cdot F$, $b = S_0 \cdot F$, $i \geq 1$. Its determinant is equal to

$$
242(m^2 + bm) - 110m,
$$

which cannot be 0 for any positive integers $m$ and $b$. This implies that

$$
\mu(N) = 2 + \text{rank } \text{Pic}(X_0)^N \geq 5,
$$

a contradiction to the equality $\mu(N) = 4$. This proves that $N$ is a proper subgroup of $G \cap H_0$. Since $N$ is a maximal subgroup of $G$, we have $G = H_0$. 

Note that $\mu(M_{11}) = \mu(M_{22}) = 3$ and $\mu(L_2(11)) = 4$. Note also that $L_2(11)$ is isomorphic to a maximal subgroup of both $M_{11}$ and $M_{22}$.

The following proposition completes the proof of Theorem 1.1 (ii).

**Proposition 5.3.** Let $G \subset \text{Aut}(X_0)$ be a finite wild symplectic subgroup. Assume that $G \cong M_{11}$ or $M_{22}$. Then no conjugate of $G$ in $\text{Aut}(X_0)$ contains the automorphism $g_0$ given by (1.2). In other words, no element of $G$ of order 11 can leave invariant both the standard elliptic fibration and a section. In particular, $H_0$ is a maximal finite wild symplectic subgroup of $\text{Aut}(X_0)$.

**Proof.** Suppose that a conjugate of $G$ contains $g_0$. Replacing $G$ by the conjugate, we may assume that $g_0 \in G$.

Let $K$ be a subgroup of $G$ such that $g_0 \in K \subset G$ and $K \cong L_2(11)$. Then by Lemma 5.2, $K = H_0$. Thus

$$
g_0 \in H_0 \subset G.
$$

Since $H_0 \cong L_2(11)$ is a maximal subgroup of $G$, its normalizer subgroup $N_G(H_0)$ coincides with $H_0$.

Let $|F|$ be the standard elliptic fibration on $X_0$, and $S_0$ the zero section. Then $g(S_0) = S_0$ and $X^g = F_0$, a cuspidal curve in $|F|$. Furthermore, both the section $S_0$ and the elliptic pencil $|F|$ are $H_0$-invariant (see Definition 2.8).

Consider the $G$-orbit of the divisor class $[F]$,

$$
G([F]) = \{h([F]) \in \text{Pic}(X_0) | h \in G\}.
$$

Consider the action of $H_0$ on it. By Proposition 2.8, the stabilizer subgroup $G_{[F]}$ of $[F]$ coincides with $H_0$. The automorphism $g_0$ cannot leave invariant two different elliptic fibrations, hence fixes $[F]$ and has orbits on the set $G([F]) \setminus \{[F]\}$ of cardinality divisible by 11. This implies that $H_0$ fixes $[F]$ and has orbits on the set $G([F]) \setminus \{[F]\}$ of cardinality divisible by 11. Write

$$
G([F]) = \{[F], [F_1], [F_2], ..., [F_{r-1}]\}
$$

$$
G([F]) = \{[F], [F_1], [F_2], ..., [F_{r-1}]\}.
$$
where \( r = \#G([F]) = [G : H_0] \). Let
\[
O_1 \cup O_2 \cup ... \cup O_s
\]
be the orbit decomposition of the index set \( \{1, 2, ..., r-1\} \) by the action of \( H_0 \). Since \( H_0 \) fixes \([F]\) and acts transitively on each \( O_i \), the intersection number \( F \cdot F_t \) is constant on the orbit \( O_i \) containing \( t \), i.e. \( F \cdot F_t = m_i \) for all \( t \in O_i \). Note that the divisor
\[
\mathcal{F} = \sum_{j=0}^{r-1} F_j
\]
is \( G \)-invariant, and
\[
(5.1) \quad \mathcal{F}^2 = \left( \sum_{j=0}^{r-1} F_j \right)^2 = rF_0 \cdot \sum_{j=0}^{r-1} F_j = r \sum_{i=1}^{s} m_i \#O_i.
\]

Next recall that \( H_0 \) leaves invariant the zero section \( S_0 \). Similarly we consider the \( G \)-orbit of the divisor class \([S_0]\)
\[
G([S_0]) = \{ h([S_0]) \in \text{Pic}(X_0) | h \in G \}.
\]
Let \( G_0 \) be the stabilizer subgroup of \([S_0]\). Since it contains \( H_0 \) and \( H_0 \) is maximal in \( G \), we obtain that \( G_0 = H_0 \) or \( G_0 = G \).

Assume \( G_0 = H_0 \). Then all stabilizers are conjugate to \( H_0 \). Similarly as above we claim that \( g_0 \in H_0 \) fixes no elements of \( G([S_0]) \) other than \([S_0]\). If \( g_0h(S_0) = h(S_0) \) for some \( h \in G \), then \( g_0 \in H_0 h^{-1} \) and since all cyclic subgroups of order 11 in \( H_0 \) are conjugate inside \( H_0 \) we can write \( (g_0) = hh'(g_0)h^{-1}h^{-1} \) for some \( h' \in H_0 \). This implies \( hh' \in N_G((g_0)) \). Since \( \#N_G((g_0)) = \#N_{H_0}((g_0)) = 55 \) (see the proof of Lemma 4.2), we obtain that \( N_G((g_0)) = N_{H_0}((g_0)) \subset H_0 \), hence \( h \in H_0 \) and \( h(S_0) = S_0 \). This proves the claim and shows that \( H_0 \) has orbits on the set \( G(S_0) \setminus \{ S_0 \} \) of cardinality divisible by 11. Write
\[
G([S_0]) = \{ [S_0], [S_1], [S_2], ..., [S_{r-1}] \}.
\]
It is clear that the divisor
\[
\mathcal{S} = \sum_{j=0}^{r-1} S_j
\]
is \( G \)-invariant. Let \( S_0 \cdot F_t = b_i \) for \( t \in O_i \). Then we have
\[
(5.2) \quad \mathcal{F} \cdot \mathcal{S} = \left( \sum_{j=0}^{r-1} F_j \right) \cdot \left( \sum_{j=0}^{r-1} S_j \right) = rS_0 \cdot \sum_{j=0}^{r-1} F_j = r(1 + \sum_{i=1}^{s} b_i \#O_i).
\]
In either case \( G \cong M_{11} \) or \( M_{22} \), we know \( \mu(G) = 3 \) and hence the two divisors \( \mathcal{F} \) and \( \mathcal{S} \) are linearly dependent in \( \text{Pic}(X_0) \). This implies
\[
\mathcal{F}^2 \mathcal{S}^2 = (\mathcal{F} \cdot \mathcal{S})^2.
\]
Substituting from (5.1), (5.2), we get
\[ r\left(\sum_{i=1}^{s} m_i \# \mathcal{O}_i\right)S^2 = r^2(1 + \sum_{i=1}^{s} b_i \# \mathcal{O}_i)^2. \]

Since \( \# \mathcal{O}_i \equiv 0 \mod 11 \) for all \( i \) and \( r \equiv 1 \mod 11 \), the left hand side \( \equiv 0 \mod 11 \), but the right hand side \( \equiv 1 \mod 11 \), a contradiction.

Assume \( G_0 = G \). Then the divisor \( S = S_0 \) is \( G \)-invariant, and we have a simpler equality
\[ r\left(\sum_{i=1}^{s} m_i \# \mathcal{O}_i\right)S^2 = (1 + \sum_{i=1}^{s} b_i \# \mathcal{O}_i)^2, \]
again a contradiction. \( \square \)

**Remark 5.4.** In \([Ko]\) Kondo proves that the unique supersingular K3 surface \( X \) with Artin invariant 1 admits symplectic automorphism groups \( G \cong M_{11} \) or \( G \cong M_{22} \). It follows from the previous results that any element \( g \in G \) of order 11 leaves invariant an elliptic pencil without a \( g \)-invariant section. In fact, according to his construction of \( G \) on \( X \), one can show that \( \text{Pic}(X)^g \cong U(11) \), hence a \((g)\)-invariant elliptic pencil has only an 11-section.

It is known that the Brauer group of a supersingular K3 surface is isomorphic to the additive group of the field \( k \). It is well-known that the group of torsors of an elliptic fibration with a section is isomorphic to the Brauer group. We do not know which torsors admit a non-trivial automorphism of order \( p \) (maybe all?). We also do not know whether they define elliptic fibrations on the same surface \( X_0 \). Note that the latter could happen only for torsors of order divisible by \( p = \text{char}(k) \). It would be very interesting to see how the three groups \( L_2(11), M_{11} \) and \( M_{22} \) sit inside the infinite group \( \text{Aut}(X_0) \).

**Remark 5.5.** It follows from Lemma 2.6 that our surface \( X_0 \) admits a non-symplectic automorphism of order 12. By Remark 2.7, \( X_0 \) is supersingular with Artin invariant \( \sigma = 1 \). It follows from \([Ny]\) that the maximal order of a non-symplectic isomorphism of a supersingular surface with Artin invariant \( \sigma \) divides \( 1 + p^\sigma \). Thus 12 is the maximum possible order. What is the maximum possible non-symplectic extension of \( M_{11} \) or \( M_{22} \)?

**Remark 5.6.** A K3 surface may admit a non-symplectic automorphism of order 11 over any field of characteristic 0 or \( p \neq 2, 3, 11 \). The well-known example is the surface \( V(x^2 + y^3 + z^{11} + w^{66}) \) in \( \mathbb{P}(1, 6, 22, 33) \). It is interesting to know whether there exists a supersingular K3 surface \( X \) which admits a non-symplectic automorphism of order 11. It follows from \([Ny]\) that, if \( p \neq 2 \), then 11 must divide \( 1 + p^\sigma \), where \( \sigma \) is the Artin invariant of \( X \).

**References**

[Ar] M. Artin, *Supersingular K3 surfaces*, Ann. Ec. Norm. Sup., 4-e Serie, 7 (1974), 543–567.
[CD] F. Cossec., I. Dolgachev, *Enriques surfaces I*, Birkhäuser 1989.

[DK1] I. Dolgachev, J. Keum, *Wild p-cyclic actions on K3 surfaces*, J. Algebraic Geometry, 10 (2001), 101-131.

[DK2] I. Dolgachev, J. Keum, *Finite groups of symplectic automorphisms of K3 surfaces in positive characteristic*, math. AG/0403478, to appear in Ann. Math.

[Go] Y. Goto, *The Artin invariant of supersingular weighted Delsarte surfaces*, J. Math. Kyoto Univ., 36 (1996), 359–363.

[HN] G. Harder, M.S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. 212 (1975), 215–248.

[Ko] S. Kondō, *Maximal subgroups of the Mathieu group M_{23} and symplectic automorphisms of supersingular K3 surfaces*, math.AG/0511286.

[Mu] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. 94 (1988), 183-221.

[Ny] N. Nygaard, *Higher DeRham-Witt complexes on supersingular K3 surfaces*, Comp. Math. 42 (1980/81), 245–271.

[Og] A. Ogus, *Supersingular K3 crystals*, in “Journées de Géometrie Algébrique de Rennes”, Asterisque, vol. 64 (1979), pp. 3–86.

[S] S. Saito, *General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings*, Amer. J. Math. 109 (1987), 1009-1042.

[Sh] T. Shioda, *Supersingular K3 surfaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., 732, Springer, Berlin, 1979, pp. 564–591.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: idolga@umich.edu

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea

E-mail address: jhkeum@kias.re.kr