INVARIANT SUBSPACES OF $RL^1$

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Abstract. In this note we extend D. Singh and A. A. W. Mehanna’s invariant subspace theorem for $RH^1$ (the real Banach space of analytic functions in $H^1$ with real Taylor coefficients) to the simply invariant subspaces of $RL^1$ (the real Banach space of functions in $L^1$ with real Fourier coefficients).

Let $T$ denote the unit circle in the complex plane, and let $L^p$ denote the Lebesgue spaces on $T$ with respect to Lebesgue measure normalized so that the Lebesgue measure of $T$ is 1. We use the standard notation $H^p$ to denote the subspace of $L^p$ consisting of those functions in $L^p$ whose negative Fourier coefficients vanish. Let

$$RH^p = \{ f \in H^p : \text{the Fourier (Taylor) coefficients of } f \text{ are real} \}.$$ 

An invariant subspace is a (closed) subspace invariant under multiplication by the coordinate function. D. Singh and A. A. W. Mehanna [3] gave a characterization of the invariant subspaces of $RH^1$. Specifically, they proved the following result.

Singh-Mehanna. Let $M$ be an invariant subspace of $RH^1$. Then there exists a unique (up to a constant multiple of modulus one) inner function, $I$, in $RH^1$ such that $M = IRH^1$.

Let $RL^p = \{ f \in L^p : \text{the Fourier coefficients of } f \text{ are real} \}$. A simply invariant subspace is an invariant subspace, $M$, whose image under multiplication by the coordinate function is strictly contained in $M$. (In $RH^1$ every invariant subspace is simply invariant.) In this note we extend Singh and Mehanna’s result to the simply invariant subspaces of $RL^1$.

Main Theorem. Let $M$ be a simply invariant subspace of $RL^1$. Then there exists a unique (up to a constant multiple of modulus one) unimodular function, $U$, in $RL^1$ such that $M = URH^1$.

To prove this theorem we follow the approach of Singh and Mehanna. We will first prove an analogous result in $RL^2$ and then use this result to prove the Main Theorem. Singh and Mehanna’s proof weighs heavily on the inner-outer factorization of functions in $H^1$. In general, $L^1$ functions do not have such a factorization. As we will soon see, however, the members of $M$ have a nice factorization which will prove useful in the proof of the Main Theorem.

Theorem 1. Let $M$ be a simply invariant subspace of $RL^2$. Then there exists a unique (up to a constant multiple of modulus one) unimodular function, $U$, in $RL^2$ such that $M = URH^2$.

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To prove this, we need to understand the form of the doubly invariant subspaces of \( RL^2 \). These are the invariant subspaces, \( \mathcal{M} \), of \( RL^2 \) for which multiplication by the coordinate function takes \( \mathcal{M} \) onto \( \mathcal{M} \).

**Theorem 2.** If \( \mathcal{M} \) is a doubly invariant subspace of \( RL^2 \) then \( \mathcal{M} = 1_E RL^2 \) where \( E \) is a measurable subset of \( T \).

In the above theorem, \( 1_E \) denotes the characteristic function of the set \( E \). The following proof is a slightly simplified version of the proof given in [1] for the doubly invariant subspaces of \( L^2 \). Although our proof is stated for \( RL^2 \), it works equally well in the \( L^2 \) setting.

**Proof.** If \( 1 \) is in \( \mathcal{M} \) then \( \mathcal{M} = RL^2 \). In this case \( E = T \). If \( 1 \) is not in \( \mathcal{M} \), let \( q \) be the orthogonal projection of \( 1 \) onto \( \mathcal{M} \). Then \( 1 - q = 1_{E^c} \), so we have that \( 1_E \in \mathcal{M} \) and \( 1_{E^c} \in \mathcal{M}^+ \). By the double invariance of \( \mathcal{M} \) we have that \( 1_E RL^2 \) is contained in \( \mathcal{M} \) and that \( 1_{E^c} RL^2 \) is contained in \( \mathcal{M}^+ \). We also have \( 1_E RL^2 + 1_{E^c} RL^2 = RL^2 \) and \( 1_E RL^2 \cap 1_{E^c} RL^2 = \{0\} \). Hence \( \mathcal{M} = 1_E RL^2 \) as desired. \( \square \)

We are now ready to prove Theorem 1. Our proof follows the proof given in [1] for the simply invariant subspaces of \( L^2 \). We include it for completeness.

**Proof of Theorem 1.** Since \( \mathcal{M} \) is simply invariant there exists a \( U \) in \( \mathcal{M} \) of norm one. \( U \) is orthogonal to \( e^{in\theta} \mathcal{M} \) for all natural numbers \( n \geq 1 \). So by the symmetry of the inner product on \( RL^2 \) we get

\[
\int_{-\pi}^{\pi} e^{in\theta} |U|^2 (e^{i\theta}) d\theta = 0 \quad \text{for all integers } n \neq 0.
\]

Thus \( U \) has constant modulus one. The set \( \{e^{in\theta} U\}_{n=\infty}^{n=-\infty} \) spans a doubly invariant subspace in \( RL^2 \). Since \( U \) does not vanish on a set of positive measure, we have that this doubly invariant subspace is \( RL^2 \). The span of \( \{e^{in\theta} U\}_{n=0}^{n=\infty} \) is \( U RH^2 \) and is contained in \( \mathcal{M} \). If we show that the set \( \{e^{in\theta} U\}_{n<0} \) is contained in \( \mathcal{M}^+ \), then we can conclude that \( U RH^2 \) is all of \( \mathcal{M} \). Showing that the set \( \{e^{in\theta} U\}_{n<0} \) is contained in \( \mathcal{M}^+ \) is the same as showing that \( U \) is orthogonal to \( e^{in\theta} \mathcal{M} \) for all natural numbers \( n > 0 \). This is true by our choice of \( U \). Hence, \( \mathcal{M} = U RH^2 \) as desired. It remains to prove the uniqueness of \( U \). If \( I \) is another unimodular function such that \( \mathcal{M} = I RH^2 \), then we have \( U/I RH^2 = RH^2 \). Since the inverse of a unimodular function is its complex conjugate, we have that both \( U T \) and \( \overline{U T} \) are in \( RH^2 \). This implies that \( U/I \) is a constant of modulus one. \( \square \)

Let \( \mathcal{M} \) be a simply invariant subspace of \( RL^1 \). Then \( \mathcal{M} \) is a subset of \( L^1 \). The complexification of \( \mathcal{M} \), \( \overline{\mathcal{M}} \otimes C L^1 \), is then a simply invariant subspace of \( L^1 \). By a classical result [1], \( \overline{\mathcal{M}} \otimes C L^1 = \psi H^1 \), where \( \psi \) is a unimodular function in \( L^1 \). So \( \mathcal{M} \) is contained in \( \psi H^1 \). It follows that every element of \( \mathcal{M} \) has a unique unimodular-outer factorization.
Before we prove the Main Theorem we prove several technical lemmas. Let $f$ be an element of $L^1$. Define $f^*(e^{i\theta}) = \overline{f(e^{-i\theta})}$.

**Lemma 1.** For $f$ in $L^1$, $\hat{f}(n) = \hat{f}^*(n)$.

**Proof.** Let $f$ be an element of $L^1$. Then

$$
\hat{f}^*(n) = \int_{-\pi}^{\pi} f^*(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f(e^{-i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f(e^{-i\theta}) e^{in\theta} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = \hat{f}(n).
$$

\[\Box\]

**Corollary.** For $f$ in $L^1$, $f$ is in $RL^1$ if and only if $f = f^*$.

**Lemma 2.** If $O$ is outer then $O^*$ is outer.

**Proof.** Recall that a function $f$ in $H^p$ is outer if and only if

$$
\log |f(0)| = \int_{-\pi}^{\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}.
$$

Since $O$ is outer we have

$$
\log |O(0)| = \int_{-\pi}^{\pi} \log|O(e^{i\theta})| \frac{d\theta}{2\pi}.
$$

By Lemma 1 $|O(0)| = |O^*(0)|$ and the negative Fourier coefficients of $O^*$ are zero. A change of variable shows that

$$
\int_{-\pi}^{\pi} \log|O(e^{i\theta})| \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \log|O^*(e^{i\theta})| \frac{d\theta}{2\pi}.
$$

Thus

$$
\log |O^*(0)| = \int_{-\pi}^{\pi} \log|O^*(e^{i\theta})| \frac{d\theta}{2\pi}.
$$

so $O^*$ is outer. \[\Box\]

**Lemma 3.** Let $f$ be an element of $RL^1$ such that $f$ has a unique factorization $f = UO$, where $U$ is a unimodular function and $O$ is an outer function. Then $U$ and $O$ are in $RL^1$.

**Proof.** Since $f$ is in $RL^1$, by the above corollary we have that $f = f^*$. Thus $UO = U^*O^*$. Since $U^*$ is unimodular and $O^*$ is outer, the uniqueness of our factorization implies $U^* = U$ and $O^* = O$. By the above corollary we get $U$ and $O$ are in $RL^1$. \[\Box\]

We are now ready to prove the Main Theorem.
Proof of Main Theorem. Let $\mathcal{M}$ be a simply invariant subspace of $RL^1$. Then $\mathcal{M} \cap RL^2$ is an invariant (closed) subspace of $RL^2$. We will show that $\mathcal{M} \cap RL^2$ is dense in $\mathcal{M}$. Then $\mathcal{M} \cap RL^2$ is actually simply invariant, and by Theorem 4 $\mathcal{M} \cap RL^2 = U RH^2$. Hence $\mathcal{M}$ is of the form $U RH^1$, as desired.

We first show that $\mathcal{M} \cap RL^2$ is nonempty. Let $f$ be any nonzero element of $\mathcal{M}$. By Lemma 3 we have that $f = UO$, where $U$ is unimodular and $O$ is outer, with both $U$ and $O$ in $RL^1$. Since $O$ is outer it is actually a member of $RH^1$, so by Lemma 3.4 of [3] we may assume without loss of generality that $\sqrt{O}$ is in $RH^2$. Therefore $g := U\sqrt{O}$ is in $RL^2$. We now show that $g$ is also in $\mathcal{M}$. By Corollary 3.4 of [3] there exists a sequence of polynomials, $\{p_n\}$, in $RH^2$ such that 

$$\|\sqrt{O}p_n - 1\|_2 \to 0 \text{ as } n \to \infty.$$ 

Thus,

$$\|fp_n - g\|_1 = \|g\sqrt{O}p_n - g\|_1 \\ \leq \|g\|_2 \|\sqrt{O}p_n - 1\|_2 \quad \text{(by Cauchy-Schwarz)} \\ \to 0 \text{ as } n \to \infty.$$ 

Since $fp_n$ is in $\mathcal{M}$ for all $n$ by the invariance of $\mathcal{M}$, and since $\mathcal{M}$ is closed, we see that $g$ is in $\mathcal{M}$, as desired.

It remains to show that $\mathcal{M} \cap RL^2$ is dense in $\mathcal{M}$. Let $f$ be any nonzero element of $\mathcal{M}$. Then $f = UO$, where $U$ is unimodular and $O$ is outer, with both $U$ and $O$ in $RL^1$. As mentioned above, we assume without loss of generality that $\sqrt{O}$ is in $RH^2$. Let $\sqrt{O}_n = \sum_{k=0}^{n} a_k e^{ik\theta}$ be the partial sums of the Fourier series for $\sqrt{O}$. We know that $\sqrt{O}_n$ converges to $\sqrt{O}$ in $RL^2$. By the work above we know that $U\sqrt{O}_n$ is in $\mathcal{M}$ for all $n > 0$. Thus

$$\|U\sqrt{O}_n - f\|_1 = \|U\sqrt{O}(\sqrt{O}_n - \sqrt{O})\|_1 \\ \leq \|U\sqrt{O}\|_2 \|\sqrt{O}_n - \sqrt{O}\|_2 \quad \text{(by Cauchy-Schwarz)} \\ \to 0 \text{ as } n \to \infty.$$ 

We conclude that $\mathcal{M} \cap RL^2$ is dense in $\mathcal{M}$, as desired. 

References

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