Interdependence of dynamical signals and topology: Detecting the influential nodes in networks

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By studying various dynamical processes, including coupled maps, cellular automata and coupled differential equations, on five different kinds of known networks, we found a positive relation between signal correlation and node’s degree. Thus a method of identifying influential nodes in dynamical systems is proposed, its validity is studied, and potential applications on real systems are discussed.

Dynamical processes on networks have been highly concerned, the network topological structure often plays a crucial role in determining the system's dynamical features. Various dynamical systems on complex networks have been widely investigated, such as transient activation in neuron networks, synchronization, virus spreading or information diffusion. Due to the complex connectivity and the dynamical nonlinearity, an exact mathematical analysis of general dynamical processes on networks is often highly intricate. In general dynamical systems on complex networks, the collective behavior of nodes remains unclear.

Recently, studying dynamical signals (the time series) presents a different way to understand the dynamical processes on complex networks. For example, Ref. \textsuperscript{3} shows effectiveness to simultaneously characterize the dynamics of several thousands nodes by scaling of fluctuations; Ref. \textsuperscript{4} gives an example to reconstruct the nonlinear dynamics in a network from observed dynamical signals. Beside theoretical interests, studying the dynamical signal may provide practical applications, e.g., for some dynamical systems, it would be difficult or even impossible to directly obtain the topological information of the background networks, such as financial systems \textsuperscript{5}, the neuronal functional networks \textsuperscript{6}, etc., thus a key problem arises that how to obtain the topological information from the dynamical signals.

In the paper, based on principal component analysis \textsuperscript{7}, we have investigated the cross-correlation matrix of the dynamical signals, which reflects the interdependence of the nodes on the network, and studied the relation between the cross-correlation matrix and the topological connectivity. We found that in most cases there is a positive relation between the nodes’ degrees and the components of the principal eigenvector, the eigenvector corresponded with the largest eigenvalue, of the cross-correlation matrix, i.e., the larger degree often corresponds to larger components. Thus, we suggest a method to obtain the topology from the dynamical signals. Three types of coupled dynamical models, namely, coupled maps, cellular automata and coupled differential equations, and five kinds of complex networks are studied. To be specific, the dynamical models are coupled logistic maps, avalanche processes \textsuperscript{8, 9}, and integrate-and-fire models \textsuperscript{10}. The complex network models include the Erdős-Rényi (ER) random graph \textsuperscript{11} (which has a Poisson degree distribution), the Barabási-Albert (BA) model \textsuperscript{12} (which has a power law degree distribution with exponent $\lambda = 3.0$), the generalized random graphs with scale-free (SF) degree distribution \textsuperscript{13}, the lattice embedded SF model \textsuperscript{14}, and the nearest neighbor growth model \textsuperscript{15} (which has an exponential degree distribution). In the simulation, the exponent of degree distribution for SF random graph and lattice embedded SF networks is fixed at $\lambda = 3.0$ to make the results comparable with that of BA model.

We start with a general expression of dynamics on a network. The connecting network is described by its adjacent matrix $A_{ij}$ equals to 1 if node $i$ and node $j$ have a common edge (then $j$ is said to be a neighbor of $i$ and vice versa) and 0 if not, and $k_i = \sum_{j=1}^{N} A_{ij}$ is the degree of node $i$, $x_i(t)$ is the output signal of node $i$. The cross-correlation matrix is determined as $C_{ij} = \sum_{\sigma_i, \sigma_j} \langle (x_i(x) - \langle x_i \rangle)(x_j(x) - \langle x_j \rangle) \rangle$, where $\sigma_i$ is the variance: $\sigma_i = \sqrt{\langle x_i^2 \rangle - \langle x_i \rangle^2}$, and the average is over a period of length $L_d$. Here we concern only about the strength of the correlation between the nodes, rather than the signs, thus we define the correlation strength matrix: $C_{ij} = |C_{ij}|$. Then its principal eigenvector could be determined, and the relation between the components of the principal eigenvector and the nodes’ degrees is investigated in detail.

Coupled logistic maps. The dynamic of an individual node $i$ coupled with its neighbors is described by:

$$x_{i}^{t+1} = (1 - \varepsilon)f(x_i^t) + \frac{\varepsilon}{k_i} \sum_{j=1}^{N} A_{ij} f(x_j^t), \quad (1)$$

where $f(x)$ is the logistic map $f(x) = 1 - ax^2$, $\varepsilon \in [0, 1]$ is the coupling strength. For a given network, the parameter space $(\varepsilon, a)$ has two extreme cases: the synchronization regime and complete non-synchronization regime. In
the synchronization region, parts of or all the nodes have the consistent motion. The cross-correlation matrix may reflect the synchronization motion. Here, we investigate the system running in the state which is far from synchronization. Figure 1 is a typical show of the connection matrix and the correlation strength matrix, since the states have no particular orderliness, the latter is highly noised. The same holds for other network models and for other dynamical processes.

**FIG. 1:** (Color online) A direct show of network connection and correlation strength of coupled maps on a ER random graph, with $N = 100$, $\langle k \rangle = 6$. Left: Network connection, dot at $(i, j)$ indicates the connection of node $i$ and $j$. Right: Correlation strength, the brighter, the stronger. $a = 1.9$, $\varepsilon = 0.9$.

The relationship of the components of the principal eigenvector of the correlation strength matrix of the cooccurrence between nodes in an avalanche could unveil the linkages between them. The Bak-Tang-Wiesenfeld (BTW) sandpile model [14] on SF networks has been investigated by K. S. Goh et al. [15], which set the threshold of being toppled of each node $i$ be its degree $k_i$, unlike that of the uniform threshold height in lattice cases; and proposed a losing probability $f$ when the grains are transferring from a toppled node to its neighbors, as the sinks or open boundaries in lattice cases. The rule can be adopted on arbitrary networks. The correlation in this case is between the toppling events $s^n_i$ in avalanches, $s^n_i = 1$ if node $i$ toppled in the $n$th avalanche, and 0 if not. It is expected that the cooccurrence between nodes in an avalanche could unveil the linkages between them.
We suggest a method to locate the important or influential nodes (namely, the nodes that have large degrees) in dynamical systems on complex networks, according to the positive relation between the components of the principal eigenvector of the correlation strength matrix and the nodes’ degrees. First, one can measure the cross-correlation matrix from the output signal of each node, and the component values of the principal eigenvector of the correlation strength matrix could be calculated. The degree of a node is represented by its corresponding component value. Thus, the important or influential nodes are located as the nodes with larger component values. In Fig. 7 a typical result of the method is shown for coupled chaotic maps on a BA network. For other dynamical and network models, similar results can be obtained.

Further more, we define the efficiency of the locating method as $E(f) = n_m(f)/(N \cdot f)$, where $N$ is the total number of the nodes, and $f$ is the fraction of selected nodes with largest degrees, and the $n_m(f)$ is number of nodes that correctly matched. If the nodes were randomly ordered, the fraction that matches with other ordering, say, by degree, will be equal to the fraction that compared, thus $E(f) = f$. A typical result by our method for IFNs is shown in Fig. 4. The efficiency of the method in IFNs model indicates direct application potentials, i.e., identifying the most influencing neurons in the experiments in neuronal networks [12], (not necessarily through current correlation).

In conclusion, we studied coupled chaotic maps, avalanche processes, and integrate and fire neuron models, covering three types of coupled dynamical systems, on various network topologies, whose degree distribution varies from Poissonian to scale-free and to exponential, and of different local properties. All our results support the proposal that the components of the principal eigenvector of the correlation strength matrix have a positive relation with the nodes’ degrees, thus a representation
of nodes’ degrees, the ability to influence others and the importance to the system, could be realized by the components, which can be obtained through the dynamical processes. Further applications such as immunization of internet or contagious disease, identifying pivotal neurons, etc., can be investigated.

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FIG. 6: (Color online) A direct show of the efficiency of our method. Squares: the first 8 nodes with largest degrees; filled squares: nodes that are correctly located out by our method; empty square: the missed node, which is just the 8’th node among the first 8 nodes; small circles: ordinary nodes.

FIG. 7: (Color online) Efficiency of the locating method by ordering of the component values of the principal eigenvector. The parameters are the same as that of Fig. 4.

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