Atoms in infinite dimensional free sequence-set algebras

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Abstract. A. Tarski proved that the $m$-generated free algebra of $CA_\alpha$, the class of cylindric algebras of dimension $\alpha$, contains exactly $2^m$ zero-dimensional atoms, when $m \geq 1$ is a finite cardinal and $\alpha$ is an arbitrary ordinal. He conjectured that, when $\alpha$ is infinite, there are no more atoms other than the zero-dimensional atoms. This conjecture has not been confirmed or denied yet. In this article, we show that Tarski’s conjecture is true if $CA_\alpha$ is replaced by $D_\alpha$, $G_\alpha$, but the $m$-generated free $Crs_\alpha$ algebra is atomless.

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1. Introduction

Let $K$ be a class of similar algebras, and let $m$ be any cardinal. The $m$-generated free algebra over the class $K$ is denoted by $\mathfrak{F}r_m K$, and is defined to be the unique algebra (up to isomorphism) that satisfies the following conditions: (1) $\mathfrak{F}r_m K$ is generated by a set $X$ of cardinality $m$, and (2) The only term relations holding between the elements of the generating set $X$ are those that hold for all elements in all algebras in $K$. In other words, free algebras over the class $K$ are algebras that store the equations which are true in $K$. For a precise definition of free algebras and more about their properties, we refer to any book in universal algebra, e.g. [2] and [1, chapter 0].

Free algebras play an important role in universal algebra, and especially in the theory of Boolean algebras with operators (BAO’s), see, e.g., [4,6], [3, section 5.6] and [13]. One of the first things to investigate about these free algebras is whether they are atomic or not, i.e., whether their Boolean reducts
are atomic or not. An atomic Boolean algebra is an algebra in which below every non-zero element there is an atom, i.e., a minimal non-zero element.

Cylindric algebras are special BAO’s that were introduced by A. Tarski around 1947. These are Boolean algebras equipped with unary operations, called cylindrifications, and constant symbols, called diagonals. These algebras capture the intrinsic algebraic side of first order logic (FOL), see [1, section 4.3]. Let $\alpha$ be any ordinal.

**Definition 1.1.** A cylindric algebra of dimension $\alpha$ is an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, 0, 1, c_i, d_{ij} \rangle_{i,j \in \alpha},$$

where $A$ is a non-empty set, $+, \cdot$ are binary operations, $-, c_i$ are unary operations, $0, 1, d_{ij}$ are constant symbols, and $\mathfrak{A}$ satisfies the following postulates for every $x, y \in A$ and every $i, j, k \in \alpha$:

(CA 0) $\langle A, +, \cdot, -, 0, 1 \rangle$ is Boolean algebra,

(CA 1) $c_i 0 = 0$,

(CA 2) $x + c_i x = c_i x$,

(CA 3) $c_i (x \cdot c_i y) = c_i x \cdot c_i y$,

(CA 4) $c_i c_j x = c_j c_i x$,

(CA 5) $d_{ii} = 1$,

(CA 6) if $k \neq i, j$, then $d_{ij} = c_k (d_{ik} \cdot d_{kj})$,

(CA 7) if $i \neq j$, then $c_i (d_{ij} \cdot x) \cdot c_i (d_{ij} \cdot -x) = 0$.

The class of all cylindric algebras of dimension $\alpha$ is denoted by $\text{CA}_\alpha$. Atoms in the free cylindric algebras correspond to finitely axiomatizable complete and consistent theories of FOL. In the present paper, we are interested in the case when $\alpha$ is infinite, so $\alpha \geq \omega$ is assumed throughout the paper. The atomicity of the finite dimensional free algebras is discussed in [4, 17, 18, 19].

We will prove some results connected to a conjecture of A. Tarski [1, Remark 2.5.12 and Problem 2.6]. This conjecture is concerned with atoms and zero-dimensional elements in the finitely generated free cylindric algebras of dimension $\alpha$. An element $a$ of a cylindric algebra $\mathfrak{A}$ is said to be zero-dimensional if it is a fixed-point of all the cylindrifications, i.e., if $c_i a = a$ for each $i \in \alpha$.

In this section, we outline the background for this conjecture and we state our main theorems. Let $m$ be any cardinal. Let $K \subseteq \text{CA}_\alpha$ be a class of cylindric algebras containing at least one non-trivial algebra (i.e., having more than one-element). The following are true:

(Fact 1) If $m$ is infinite, then $\mathfrak{F}r_m K$ is atomless. This is due to D. Pigozzi [1, Theorem 2.5.13], and the proof can be easily generalized for any class of Boolean algebras with operators.

(Fact 2) Assume that $m$ is finite. In [1, Theorem 2.5.11], it is proved that $\mathfrak{F}r_m K$ contains exactly $2^m$ zero-dimensional atoms. (Tarski’s proof is only about $\text{CA}_\alpha$, but it works verbatim for the class $K$.)

(Fact 3) The 0-generated free algebra $\mathfrak{F}r_0 K$ contains exactly one atom, namely $-c_0 - d_{01}$, and this atom is zero dimensional, see [1, Theorem 2.5.11]. We show that it contains no other atom. Indeed, let
Thus, these classes are not subclasses of CA. To point out to the facts that CA cannot be replaced by Gs, we have that Gs ̸= 0 in the algebra FRm Cs. Again, similarly to our argument in (Fact 3),
\[ A \models c_i (a \cdot d_{ij}) = c_i (c_i a \cdot d_{ij}) = c_i a \cdot c_i d_{ij} = a \cdot 1 = a \neq 0, \]
\[ A \models c_i (a \cdot -d_{ij}) = c_i (c_i a \cdot -d_{ij}) = c_i a \cdot c_i -d_{ij} = a \cdot c_i -d_{ij} \neq 0. \]
Hence, \( a \cdot d_{ij} \neq 0 \) and \( a \cdot -d_{ij} \neq 0 \) are true in \( A \). Thus, the same is true in the free algebra \( \mathfrak{F} m Cs \), so \( a \) is not an atom, as desired. \( \square \)

In this article, we investigate whether the above theorem remains true if Cs is replaced by any of the relativized classes of cylindric algebras Crs, D, GS, GA. These classes will be defined in the next section. For now, it is enough to point out to the facts that Cs \( \subseteq \) CA, so we can use (Fact 2) and (Fact 3) mentioned above. On the other hand, some cylindric axioms fail to be true in the other classes Crs, D, and GS, e.g., the commutativity axiom (CA 4). Thus, these classes are not subclasses of CA, and hence (Fact 2) and (Fact 3) cannot apply.
The notion of a relativized algebra was introduced in algebraic logic by L. Henkin. Relativization was proved potent in obtaining positive results in logic, see, e.g., [14,9] and [7]. For some properties of these classes, see, e.g., [12,5] and [8]. For instance, contrary to $G_\alpha$, the equational theories of the classes $Crs_\alpha$ and $G_\alpha$ are decidable [4,7,10]. The decidability of the equational theory of the class $D_\alpha$ remains open. In the present paper, we will prove the following theorem.

**Theorem 1.3. (Main Result 1)** Let $\alpha \geq \omega$ be an infinite ordinal and let $m \geq 1$ be a finite cardinal. The following are true:

1. The free algebra $Fr_{m-1}Crs_\alpha$ is atomless.
2. The free algebra $Fr_mK$ is not atomic, but it contains exactly $2^m$-many atoms each of which is zero-dimensional, when $K$ is any of $D_\alpha$, $G_\alpha$ or $Gs_\alpha$.

The proof of Theorem 1.2 depends essentially on the fact that $Gs_\alpha$ is generated by locally finite dimensional algebras. The same is not true for the relativized classes of cylindric algebras, therefore the same argument cannot be used to prove Theorem 1.3. We will prove a stronger theorem, Theorem 2.4 in the next section, which will imply Theorem 1.3.

The proofs of Theorem 1.3 and Theorem 2.4 go by showing that there are no elements in the free algebras that are disjoint from all the diagonals $d_{ij}$. Theorem 1.4 below shows that the same is not true for $CA_\alpha$, therefore for settling the conjecture for $CA_\alpha$, one has to use other techniques, too.

**Theorem 1.4. (Main Result 2)** Let $\alpha \geq 2$ be any ordinal and let $m \geq 1$ be a finite cardinal. Then, there is $x \in Fr_mCA_\alpha$ such that $x \neq 0$ and $x \leq -d_{ij}$ for every $i, j \in \alpha \sim 2$ with $i \neq j$.

It is worth note that atomicity of these free algebras correspond to the failure of a version of Gödel’s incompleteness theorem for the corresponding logics. For more detail about this correspondence, see [4,15] and [17]. For results concerning the atomicity of free algebras of logics, one can see [1,3,4,6,11,13,15,16,17,18,19,20].

2. Algebras of sets of sequences

Throughout, fix an infinite ordinal $\alpha$. A function with domain $\alpha$ is called a sequence of length $\alpha$ (a sequence for short). For every $i \in \alpha$ and every two sequences $f, g$, we write $f \equiv_i g$ if and only if $g = f(i/u)$ for some $u$, where $f(i/u)$ is the sequence which agrees with $f$ everywhere except that it’s value at $i$ equals $u$. Let $V$ be a set of sequences of length $\alpha$. Such a set is called an $\alpha$-dimensional unit. The smallest set $U$ that satisfies $V \subseteq \alpha U$ is called the base of the unit $V$.

Let $i, j \in \alpha$. Define the $ij$-diagonal of the unit $V$ as follows:

$$D_{ij}^{[V]} = \{f \in V : f(i) = f(j)\}.$$
For each \( X \subseteq V \), define
\[
C_i^{[V]} X = \{ f \in V : (\exists g \in X) \ f \equiv_i g \}.
\]
This is called the \( V \)-cylindrification of \( X \) in the direction \( i \). When no confusion is likely, we omit the superscript \( [V] \) from the above defined objects.

**Definition 2.1.** The full cylindric-like algebra over the unit \( V \) is an algebra of the form
\[
\mathfrak{P}(V) \overset{\text{def}}{=} \langle \mathcal{P}(V), \cup, \cap, \sim, \emptyset, V, C_i, D_{ij} \rangle_{i,j \in \alpha},
\]
where \( \mathcal{P}(V) \) is the family of all subsets of \( V \), \( \cup, \cap, \sim \) are the Boolean set-theoretic operations, \( \emptyset \) is the empty set, and the \( C_i \)'s and the \( D_{ij} \)'s are as defined above.

Let \( K \) be a class of algebras of same signature. Then, \( IK, SK \) and \( HK \) are the classes that consist of the isomorphic copies, subalgebras and homomorphic images, respectively, of the members of \( K \).

**Definition 2.2.** We define the following classes of cylindric-like set algebras.

- The class of all relativized cylindric set algebras is given by
  \[
  \text{Crs}_\alpha = \text{IS}\{\mathfrak{P}(V) : V \text{ is an } \alpha\text{-dimensional unit}\}.
  \]

- The class of diagonalizable cylindric set algebras \( D_\alpha \) is given by
  \[
  \text{D}_\alpha = \text{IS}\{\mathfrak{P}(V) : V \text{ is a diagonalizable } \alpha\text{-dimensional unit}\},
  \]
  where an \( \alpha \)-dimensional unit \( V \) is diagonalizable if \( f(i/f(j)) \in V \), for each \( f \in V \) and each \( i, j \in \alpha \).

- The class of locally square cylindric set algebras \( G_\alpha \) is given by
  \[
  \text{G}_\alpha = \text{IS}\{\mathfrak{P}(V) : V \text{ is a union of } \alpha\text{-dimensional squares}\},
  \]
  where a union of \( \alpha \)-dimensional squares is an \( \alpha \)-dimensional unit of the form \( V = \bigcup_{i \in I} \alpha U_i \) for some family of non-empty sets \( \{U_i : i \in I\} \).

- The class of generalized cylindric set algebras \( Gs_\alpha \) is given by
  \[
  \text{Gs}_\alpha = \text{IS}\{\mathfrak{P}(V) : V \text{ is a union of mutually disjoint } \alpha\text{-dimensional squares}\},
  \]
  where \( V \) is a union of mutually disjoint \( \alpha \)-dimensional squares if there is a family of mutually disjoint non-empty sets \( \{U_i : i \in I\} \) such that \( V = \bigcup_{i \in I} \alpha U_i \).

We note that \( \text{Crs}_\alpha, D_\alpha, Gs_\alpha \) and \( \text{HG}_\alpha \) are varieties, and it is still open whether \( G_\alpha = \text{HG}_\alpha \). Since we are dealing with many classes, it is more convenient to prove a general theorem which implies Theorem 1.3. We need to generalize our definitions, too.

**Definition 2.3.** Let \( \mathcal{U} \) be a class of \( \alpha \)-dimensional units.

- We say that \( \mathcal{U} \) supports diagonalization iff: for all \( V \in \mathcal{U}, f \in V, i, j \in \alpha \),
  \[
  V \cup \{f(i/f(j))\} \in \mathcal{U}.
  \]
We say that $\mathcal{U}$ requires diagonalization iff $\mathcal{U}$ contains a singleton, and, for all $V \in \mathcal{U}$, $f \in V$, $i, j \in \alpha$,

$$f(i/f(j)) \in V.$$ 

For any class $\mathcal{U}$ of $\alpha$-dimensional units, if $\mathcal{U}$ requires diagonalization then $\mathcal{U}$ must also support diagonalization (since in this case $V \cup \{f(i/f(j))\} = V$, for each $V \in \mathcal{U}$), but the converse is not necessarily true. Let $K$ be a class of similar algebras, then $V(K)$ stands for the smallest variety containing $K$. One can easily see that each of the classes $G_{\alpha}, D_\alpha$ and $H_{\alpha}$ can be viewed as $V(\{\mathfrak{P}(V) : V \in \mathcal{U}\})$ for some class of $\alpha$-dimensional units $\mathcal{U}$ that require diagonalization. The same is not true for the class $C_{\alpha}$-$\alpha$. However, $C_{\alpha}$-$\alpha$ can be introduced as the variety generated by the class of full algebras of all $\alpha$-dimensional units.

**Theorem 2.4.** Let $K$ be a class of cylindric-type algebras such that $V(K) = V(\{\mathfrak{P}(V) : V \in \mathcal{U}\})$ for some class $\mathcal{U}$ of $\alpha$-dimensional units. Let $m \geq 1$ be a finite cardinal. The following are true:

1. If $\mathcal{U}$ supports diagonalization, then the free algebra $\mathfrak{F}_mK$ is not atomic.
2. If $\mathcal{U}$ requires diagonalization, then the following are true:
   a. $\mathfrak{F}_mK$ contains exactly $2^m$-many atoms.
   b. All the atoms of $\mathfrak{F}_mK$ are zero-dimensional.
   c. There is a decomposition $\mathfrak{F}_mK \cong \mathfrak{A} \times \mathfrak{B}$ such that $|A| = 2^m$, $\mathfrak{A}$ is atomic and $\mathfrak{B}$ is atomless.
3. If $\mathcal{U}$ is the class of all $\alpha$-dimensional units, then $\mathfrak{F}_{m-1}K$ is atomless.

Theorem 2.4 implies Theorem 1.3 since the classes of $D_{\alpha}$-units, $G_{\alpha}$-units and $G_{\alpha}$-units require diagonalization, while the class of $C_{\alpha}$-$\alpha$-units supports diagonalization. We divide the proof of Theorem 2.4 into some lemmas and propositions. Throughout the remaining part of this paper, fix classes $K$ and $\mathcal{U}$, and a cardinal $m \geq 1$ satisfying the assumptions of the Theorem 2.4.

### 3. The atomic part in $\mathfrak{F}_mK$

Let $c_{\alpha}$ be the algebraic type of cylindric-like algebras, it consists of binary operations $+$, $\cdot$, unary operations $\alpha$, $c_i$ ($i \in \alpha$) and constant symbols $0, 1, d_{ij}$ ($i, j \in \alpha$). Let $Y$ be any set, the set of all terms $T_{\alpha}(Y)$ generated by $Y$ in the signature $c_{\alpha}$ is defined to be the smallest set satisfying:

- $Y \subseteq T_{\alpha}(Y)$ and $0, 1, d_{ij}$ for each $i, j \in \alpha$,
- For each $\tau, \sigma \in T_{\alpha}(Y)$, we have $\tau + \sigma, \tau, \tau \in T_{\alpha}(Y)$,
- For each $\tau \in T_{\alpha}(Y)$ and each $i \in \alpha$, we have $c_i\tau \in T_{\alpha}(Y)$.

Note that the equational theory of $K$ coincides with the equational theory of $\{\mathfrak{P}(V) : V \in \mathcal{U}\}$. So, for example, whenever we say that $K \not\models \tau = 0$, for some term $\tau \in T_{\alpha}(Y)$, we will assume that there is a unit $V \in \mathcal{U}$, $f \in V$ and an evaluation $\iota : Y \rightarrow P(V)$ such that $(V, f, \iota) \models \tau$. The latter means that $f$ is a member of the interpretation of $\tau$ in the algebra $\mathfrak{P}(V)$ under the evaluation $\iota$. Examples of equations that are true in the class $K$ (cf., [11, Theorem 9.4] and [1, Theorem 1.2.6(ii) and Theorem 1.2.11]): For $i, j \in \alpha$ with $i \neq j$, 

Suppose that \( \text{Lemma 3.1.} \)

Consequently, \( X^q = 0 \).

Let \( X = \{x_0, \ldots, x_{m-1}\} \) be the generating set of the free algebra \( S_r m K \).

Let \( q \in X \{−1, 1\} \), we call such function a choice function for \( X \). For each \( x_k \in X \), let \( x_k \) if \( q(x_k) = 1 \) otherwise let \( x_k = −x_k \). Finally, we define \( X^q = x_0^q \cdots x_{m-1}^q \).

**Lemma 3.1.** Suppose that \( U \) requires diagonalization. Let \( i, j \in \alpha \) be such that \( i \neq j \), and let \( q \in X \{−1, 1\} \) be a choice function. Then,

\[
X^q \cdot c_0 - d_{01} = X^q \cdot c_i - d_{ij}.
\]

Consequently, \( X^q \cdot c_0 - d_{01} \) is a zero-dimensional element in \( S_r m K \).

**Proof.** Suppose that \( U, i, j \) and \( q \) are as required above. Let \( V \in U, f \in V \) and \( \iota : X \rightarrow \mathcal{P}(V) \) be such that \( (V, f, \iota) \models X^q \cdot c_i - d_{ij} \). Then \( f(i) = f(j) \).

Suppose that \( f(0) \neq f(i) \). Since \( U \) requires diagonalization then

\[
f(i/f(0)) \in V, \quad (V, f(i/f(0)), \iota) \models -d_{ij}.
\]

This implies that \( (V, f, \iota) \models c_i - d_{ij} \) which contradicts the assumptions. Hence, \( f(0) = f(i) \), and similarly one can show that \( f(1) = f(i) \). Now, we show that \( (V, f, \iota) \models -c_0 - d_{01} \). Suppose towards a contradiction that \( (V, f, \iota) \models c_0 - d_{01} \).

Then there is \( u \) in the base of \( V \) and \( g = f(0/u) \) such that \( (V, g, \iota) \models -d_{01} \). Hence, \( u \neq f(0) \). By assumptions, \( g_1 = g(i/g(0)) \in V \) and \( g_2 = g_1(0/g(j)) \in V \). Then \( g_2 = f(i/u) \), which implies \( (V, f, \iota) \models c_i - d_{ij} \). This contradicts the assumptions. Thus, \( (V, f, \iota) \models -c_0 - d_{01} \). We have shown that \( K \models X^q \cdot c_i - d_{ij} \leq X^q \cdot c_0 - d_{01} \). The desired follows by the symmetry of indices.

Let \( \tau = \text{def} X^q \cdot c_i - d_{ij} \). To show that \( \tau \) is zero-dimensional, we need to prove that \( c_i \tau = \tau \). By the first part, we have

\[
c_i \tau = c_i(X^q \cdot c_i - d_{ij}) = c_iX^q \cdot c_i - d_{ij} \leq -c_i - d_{ij} \leq d_{ij}.
\]

Thus,

\[
\tau = d_{ij} \cdot \tau = d_{ij} \cdot c_i(d_{ij} \cdot \tau) \quad \text{by (Eq 7)}
\]

\[
= d_{ij} \cdot c_i \tau = c_i \tau.
\]

Hence, \( \tau = X^q \cdot c_0 - d_{01} \) is zero-dimensional and we are done. \( \square \)

Now, we will show that each of the zero-dimensional elements, given in the above lemma, is an atom in the free algebra \( S_r m K \).
Lemma 3.2. Suppose that \( \mathcal{U} \) requires diagonalization. Let \( i, j \in \alpha \) be such that \( i \neq j \), and let \( q \in X \{ -1, 1 \} \) be a choice function. Then \( X^q \cdot c_0 - d_{01} \) is an atom in the free algebra \( \mathfrak{T}_m K \).

Proof. Suppose that \( \mathcal{U} \) requires diagonalization. Let \( i, j \in \alpha \) be such that \( i \neq j \), and let \( q \in X \{ -1, 1 \} \) be a choice function. Let \( \tau := X^q \cdot c_0 - d_{01} \). Let \( \{ w \} \in \mathcal{U} \), such unit is guaranteed to exist by the assumption that \( \mathcal{U} \) requires diagonalization. Define \( \nu : X \rightarrow \{ 0, \{ w \} \} \) as follows: For each \( x_k \in X \), let \( \nu(x_k) = \{ w \} \) if \( q(x_k) = 1 \) and \( \nu(x_k) = \emptyset \) otherwise. Clearly, \( \{ w \}, w, \nu \models \tau \), i.e., \( \tau \) is not zero in \( \mathfrak{T}_m K \). To prove that \( \tau \) is an atom in \( \mathfrak{T}_m K \), it is enough to prove the following: For any term \( \sigma \in T_\alpha (X) \),

\[
\text{either } K \models \tau \cdot \sigma = 0 \quad \text{or} \quad K \models \tau \cdot -\sigma = 0.
\]

(3.1)

We prove (3.1) by induction on the complexity of the term \( \sigma \). Obviously, (3.1) holds if \( \sigma = x_k \) for some \( x_k \in X \). Also, Lemma 3.1 guarantees that (3.1) is true if \( \sigma = d_{ij} \) for some \( i, j \in \alpha \). It is not hard to see that the induction step holds for the Boolean connectives. We will show that the induction step also holds for the cylindrification operations. To do that, we will use the fact that cylindrifications are additive and complemented operators. Let \( \sigma = c_k \sigma' \) for some \( \sigma' \in T_\alpha (X) \) and \( k \in \alpha \). Remember that \( \tau \) is zero-dimensional, so \( K \models c_k \tau = \tau \) and \( K \models c_k - \tau = -\tau \). By the induction hypothesis, we have one of the following cases.

(a) Either, \( K \models \tau \cdot \sigma' = 0 \). In this case, we have

\[
K \models \tau \cdot \sigma = \tau \cdot c_k \sigma' = c_k \tau \cdot c_k \sigma' = c_k (\tau \cdot \sigma') = 0.
\]

(b) Or, \( K \models \tau \cdot -\sigma' = 0 \). Here,

\[
K \models \tau \cdot -\sigma = \tau \cdot -\sigma' = -(\tau + \sigma') = -c_k (\tau + c_k \sigma') = -c_k (\tau + \sigma') \leq -\tau + \sigma' = 0.
\]

We have proved (3.1). Therefore, \( \tau = X^q \cdot c_0 - d_{01} \) is an atom in \( \mathfrak{T}_m K \). \( \square \)

We showed that \( \mathfrak{T}_m K \) contains at least \( 2^m \) zero-dimensional atoms if \( \mathcal{U} \) requires diagonalization. Note that, in this case, the sum of all these atoms in \( \mathfrak{T}_m K \) is equal to \( -c_0 - d_{01} \) which can be shown to be zero-dimensional element by the argument used in Lemma 3.1. In the following section, we will prove that \( \mathfrak{T}_m K \) does not contain any extra atom.

4. The non-atomic part in \( \mathfrak{T}_m K \)

For each term \( \sigma \in T_\alpha (X) \), we let \( \text{index}(\sigma) \) be the set of all indices \( i \in \alpha \) that appear in \( \sigma \). Let \( \Gamma \subseteq \alpha \) and let \( f, g \) be two sequences of length \( \alpha \). We write \( f \equiv_\Gamma g \) if and only if \( f(k) = g(k) \) for each \( k \in \alpha \sim \Gamma \). We start with the following proposition.

Proposition 4.1. Suppose that \( \mathcal{U} \) supports diagonalization. Then there is no atom below \( c_0 - d_{01} \) in the free algebra \( \mathfrak{T}_m K \). 

Proof. Suppose that \( \mathcal{U} \) supports diagonalization. Let \( \tau \in T_\alpha(X) \) be a cylindric term that satisfies \( K \not\models \tau \cdot c_0 - d_{01} = 0 \). We prove that \( \tau \cdot c_0 - d_{01} \) is not an atom in \( \mathfrak{F}_m K \). Let \( V \in \mathcal{U} \), \( f \in V \) and \( \iota \) be an evaluation such that \( (V, f, \iota) \models \tau \cdot c_0 - d_{01} \). We can find \( g \in V \) such that \( g = f(0/u) \), for some \( u \neq f(1) \), and \( (V, g, \iota) \models -d_{01} \). Let \( \Gamma = \text{index}(\tau) \cup \{0, 1\} \), since every term is built up from finitely many symbols in the signature \( ct_\alpha \) then \( \Gamma \) must be finite. Let \( i, j \in \alpha \sim \Gamma \) be such that \( i \neq j \).

Case 1: Suppose that \( g(i) = g(j) \). Recall that \( g(0) \neq g(1) \). So, without loss of generality, we may assume that \( g(0) \neq g(j) \). Let

\[
W = \{ h \in V : h(i) = h(j) \} \quad \text{and} \quad V' = V \cup \{g(i/g(0))\}.
\]

Note that \( V' \in \mathcal{U} \) because \( \mathcal{U} \) supports diagonalization. Define the evaluations \( \iota_1, \iota_2 : X \to \mathcal{P}(V') \) as follows: For each \( x_k \in X \),

\[
\iota_1(x_k) = \iota(x_k) \cap W \quad \text{and} \quad \iota_2(x_k) = \iota(x_k) \cup \{g(i/g(0))\}.
\]

For each \( \sigma \in T_\alpha(X) \) and each \( h \in V \), if \( \text{index}(\sigma) \subseteq \Gamma \) and \( h \equiv_\Gamma g \), then

\[
(V', h, \iota_1) \models \sigma \iff (V, h, \iota) \models \sigma \iff (V', h, \iota_2) \models \sigma. \tag{4.1}
\]

This can be shown by a simple induction argument on the complexity of the term \( \sigma \) as follows. Obviously, (4.1) is true for the case when \( \sigma = x_k \in X \) and when \( \sigma = d_{k\lambda} \), \( k, \lambda \in \Gamma \). Also, it is not hard to see that the induction step holds for the Boolean connectives. Now, suppose that \( \sigma = c_k \sigma' \) and \( \text{index}(\sigma) \subseteq \Gamma \).

That means \( k \in \Gamma \) and \( \text{index}(\sigma') \subseteq \Gamma \), too. Let \( h \in V \) be such that \( h \equiv_\Gamma g \). Note that for any \( h' \in V' \), if \( h \equiv_k h' \) then \( h' \equiv_\Gamma g \) and so \( h' \in V \). Now by the induction hypothesis, we have

\[
(V', h, \iota_1) \models \sigma \iff (\exists h' \in V') [h \equiv_k h' \text{ and } (V', h', \iota_1) \models \sigma']
\]

\[
\iff (\exists h' \in V) [h \equiv_k h' \text{ and } (V', h', \iota_1) \models \sigma']
\]

\[
\iff (\exists h' \in V) [h \equiv_k h' \text{ and } (V', h', \iota) \models \sigma']
\]

\[
\iff (V, h, \iota) \models \sigma.
\]

Similarly, \( (V', h, \iota_2) \models \sigma \iff (V, h, \iota) \models \sigma \). We have shown that (4.1) is true. Thus, in particular, we have

\[
(V', f, \iota_1) \models \tau \cdot c_0 - d_{01} \quad \text{and} \quad (V', f, \iota_2) \models \tau \cdot c_0 - d_{01}. \tag{4.2}
\]

By the choice of \( \iota_1 \), we have \( (V', h, \iota_1) \models -x_0 \) for each \( h \notin W \). Hence,

\[
(V', f, \iota_1) \models -c_0(-d_{01} \cdot c_i(x_0 \cdot -d_{ij})). \tag{4.3}
\]

On the other hand, \( (V', g(i/g(0)), \iota_2) \models x_0 \cdot -d_{ij} \) and \( (V', g, \iota_2) \models -d_{01} \cdot c_i(x_0 \cdot -d_{ij}) \). Whence, it follows that

\[
(V', f, \iota_2) \models c_0(-d_{01} \cdot c_i(x_0 \cdot -d_{ij})). \tag{4.4}
\]

Therefore, by (4.2), (4.3) and (4.4), \( \tau \cdot c_0 - d_{01} \) is not an atom in \( \mathfrak{F}_m K \).

Case 2: Suppose that \( g(i) \neq g(j) \). Let

\[
W = \{ h \in V : h(i) \neq h(j) \} \quad \text{and} \quad V' = V \cup \{g(i/g(j))\}.
\]
Again, the assumption on $\mathcal{U}$ guarantees that $V' \in \mathcal{U}$. Define the evaluations $\nu_1, \nu_2 : X \to \mathcal{P}(V')$ as follows: For each $x_k \in X$,
\[
\nu_1(x_k) = \nu(x_k) \cap W \quad \text{and} \quad \nu_2(x_k) = \nu_1(x_k) \cup \{g(i/g(j))\}.
\]
Similarly to the above case, cf. (4.1), one can easily show that
\[
(V', f, \nu_1) \models \tau \cdot c_0 - d_{01} \quad \text{and} \quad (V', f, \nu_2) \models \tau \cdot c_0 - d_{01}.
\]
Moreover, the choice of $\nu_1$ and $\nu_2$ implies
\[
(V', f, \nu_1) \models -c_0(-d_{01} \cdot c_i(x_0 \cdot d_{ij})) \quad \text{and} \quad (V', f, \nu_2) \models c_0(-d_{01} \cdot c_i(x_0 \cdot d_{ij})).
\]
Therefore, again by (4.5) and (4.6), $\tau \cdot c_0 - d_{01}$ is not an atom in $\mathfrak{F}_m K$. \qed

Now, we are ready to prove the main result of this paper.

\textbf{Proof of Theorem 2.4.} Let $\mathcal{U}$ be a class of $\alpha$-dimensional units and let $m \geq 1$ be a finite cardinal.

(1) If $\mathcal{U}$ supports diagonalization, then Proposition 4.1 implies that $\mathfrak{F}_m K$ is not atomic.

(2) Suppose that $\mathcal{U}$ requires diagonalization. By Lemma 3.1, Lemma 3.2 and Proposition 4.1, we have shown that $\mathfrak{F}_m K$ contains exactly $2^m$-many atoms, each of which is zero-dimensional. Let $\mathfrak{A} \subseteq \mathfrak{F}_m K$ be the subalgebra generated by $\{a \cdot c_0 - d_{01} : a \in \mathfrak{F}_m K\}$, and let $\mathfrak{B} \subseteq \mathfrak{F}_m K$ be the subalgebra generated by $\{a \cdot c_0 - d_{01} : a \in \mathfrak{F}_m K\}$. It is not hard to see that $a \mapsto (a \cdot c_0 - d_{01}, a \cdot c_0 - d_{01})$ is an isomorphism from $\mathfrak{F}_m K$ onto $\mathfrak{A} \times \mathfrak{B}$. Clearly, $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the desired of item (c).

(3) Suppose that $\mathcal{U}$ is the class of all $\alpha$-dimensional units. Let $Y$ be the generating set of the free algebra $\mathfrak{F}_{m-1} K$. Let $\tau \in T_\alpha(Y)$ be any term such that $K \not\models \tau = 0$. We will show that $\tau$ is not an atom in $\mathfrak{F}_{m-1} K$.

Let $V \in \mathcal{U}$, $f \in V$ and $\iota : Y \to \mathcal{P}(V)$ be an evaluation such that $(V, f, \iota) \models \tau$. Let $\Gamma = \text{index}(\tau)$ and let $i, j \in \alpha \sim \Gamma$ be such that $i \neq j$. Pick brand new elements $a, b$ that are not in the base of $V$ such that $a = b \iff f(i) \neq f(j)$. For every $h \in V$ with $h \equiv_\Gamma f$, let $h^*$ be the sequence given as follows: $h^*(i) = a$, $h^*(j) = b$ and $h^*(k) = h(k)$, for every $k \in \alpha \sim \{i, j\}$. Set $V^* = \{h^* : h \in V \text{ and } h \equiv_\Gamma f\}$. Define the evaluation $\iota^* : Y \to \mathcal{P}(V^*)$ as follows. For each $y \in Y$, let
\[
\iota^*(y) = \{h^* \in V^* : h \in \iota(y)\}.
\]

We are going to show that for every $\sigma \in T_\alpha(Y)$ and every $h \in V$, if $\text{index}(\sigma) \subseteq \Gamma$ and $h \equiv_\Gamma f$ then
\[
(V, h, \iota) \models \sigma \iff (V^*, h^*, \iota^*) \models \sigma.
\]

This can be shown by an induction on the complexity of the term $\sigma$ as follows. Obviously, (4.7) is true for the case when $\sigma = x_k \in X$ and when $\sigma = d_{k\lambda}$, $k, \lambda \in \Gamma$. Again, it is not hard to see that the induction step holds for the Boolean connectives. Now, suppose that $\sigma = c_k \sigma'$ and $\text{index}(\sigma) \subseteq \Gamma$. 

That means \( k \in \Gamma \) and \( \text{index}(\sigma') \subseteq \Gamma \), too. Let \( h \in V \) be such that \( h \equiv_f f \).

By the induction hypothesis, we have
\[
(V^*, h^*, \iota^*) \models \sigma \iff (\exists g^* \in V^*) [g^*_k \equiv h^* \text{ and } (V^*, g^*, \iota^*) \models \sigma']
\[
\iff (\exists g \in V) [g \equiv_k h \text{ and } (V, g, \iota) \models \sigma']
\[
\iff (V, h, \iota) \models \sigma.
\]

We have shown that (4.7) is true. Thus, in particular, we have shown that \((V^*, f^*, \iota^*) \models \tau\). But, by the choice of \( a \) and \( b \), we have
\[
(V, f, \iota) \models d_{ij} \iff (V^*, f^*, \iota^*) \models -d_{ij}.
\]

Therefore, both \( \tau \cdot d_{ij} \) and \( \tau \cdot -d_{ij} \) are non-zero in the free algebra \( \mathfrak{F}_{m-1} K \), i.e., \( \tau \) is not an atom in \( \mathfrak{F}_{m-1} K \), as desired. \( \square \)

By the argument we used in (Fact 3), see page 2 herein, we know that each of the free algebras \( \mathfrak{F}_{0} \alpha \) and \( \mathfrak{F}_{0} \gamma \) contains exactly \( 2^0 = 1 \) atom which happens to be zero-dimensional. This argument cannot be used to obtain similar results for the 0-generated free algebras of the classes \( \alpha \) and \( \gamma \) because none of these is locally finite dimensional. Moreover, our method here to obtain the results concerning these classes depends essentially on the assumption \( m \geq 1 \), see Proposition 4.1.

**Problem.** Are there any non-zero-dimensional atoms in \( \mathfrak{F}_{0} \alpha \) or in \( \mathfrak{F}_{0} \gamma \)? Is any of \( \mathfrak{F}_{0} \alpha \) and \( \mathfrak{F}_{0} \gamma \) atomic?

## 5. On Tarski’s conjecture

Now, we prove Theorem 1.4. This theorem shows a difference between \( \mathfrak{F}_{m} \gamma_{\alpha} \) and \( \mathfrak{F}_{m} \alpha \) and points to the direction that Tarski’s conjecture [1, Remark 2.5.12] might fail. We will use several notions from the theory of cylindric algebras, e.g., generalized cylindrifications [1, Definition 1.7.1], substitutions [1, Definition 1.5.1], reducts of \( \alpha \), and neat reducts of \( \alpha \)'s [1, 2.6.28]. For instance, for each \( x, y \in \mathfrak{F}_{m} \alpha \) and each \( i, j \in \alpha \) with \( i \neq j \), we let
\[
x \oplus y \defeq (x \cdot -y) + (-x \cdot y), \quad c_{(2)} x \defeq c_0 c_1 x \quad \text{and} \quad s_j^i x \defeq c_i (x \cdot d_{ij}).
\]

**Proof of Theorem 1.4.** Remember that \( m \geq 1 \). Let \( x \) be one of the free generators of \( \mathfrak{F}_{m} \alpha \). We will define a \( \alpha \)-term \( \tau(x) \) with the desired property as follows: \( \tau(x) \defeq x \cdot \chi(x) \), where
\[
\chi(x) \defeq -c_{(2)}(c_0 x \oplus c_0 y) - c_{(2)}(c_1 x \oplus c_1 y)
- c_{(2)}[c_1 (d_{01} \cdot c_0 x) \cdot c_0 x - d_{01}]
- c_{(2)}[c_0 (d_{01} \cdot c_1 x) \cdot c_1 x - d_{01}],
\]
and \( y \defeq c_0 x \cdot c_1 x - x \). Clearly, \( \tau(x) \in \mathfrak{F}_{m} \alpha \). Now we prove that \( \tau(x) \neq 0 \) in \( \mathfrak{F}_{m} \alpha \). (We note that \( \tau(x) = 0 \) in \( \gamma_{\alpha} \) by Theorem 1.3.) \( \square \)

**Claim 1.** \( \tau(x) \neq 0 \) in \( \mathfrak{F}_{m} \alpha \).
Proof of Claim 1. It suffices to construct an \( \mathfrak{A} \in \text{CA}_\alpha \) such that \( \tau^{\mathfrak{A}}(a) \neq 0 \) for some \( a \in A \). Let \( p \in ^\alpha \alpha \) be the identity sequence defined by: \( p(i) = i \) for each \( i \in \alpha \). Let \( B = ^\alpha \alpha \cup \{p'\} \) for some \( p' \not\in ^\alpha \alpha \). Let \( h : B \to ^\alpha \alpha \) be defined by \( h(q) = q \) if \( q \in ^\alpha \alpha \) and \( h(p') = p \). Let \( i, j \in \alpha \) be such that \( i \neq j \) and let \( X \subseteq B \). Define, 
\[
\begin{align*}
d_{ij} &= B, \\
d_{ij} &= \{q \in ^\alpha \alpha : q(i) = q(j)\}, \\
c_iX &= \{q \in X : (\exists t \in X) h(q) \equiv i t(h(t))\}.
\end{align*}
\]
We construct \( \mathfrak{A} \) as follows: \( \mathfrak{A} = \langle \mathcal{P}(B), \cup, \cap, \sim, \emptyset, B, c_i, d_{ij} \rangle_{i,j \in \alpha} \). It is easy to check that \( \mathfrak{A} \) satisfies the postulates (CA 0)-(CA 7) of Definition 1.1. Therefore, \( \mathfrak{A} \in \text{CA}_\alpha \).

Let \( a \defeq \{p\} \). Then \( b \defeq c_0a \cdot c_1a - a = \{p'\} \). It can be checked that \( c_0a = c_0b \), \( c_1a = c_1b \), \( c_1(d_{01} \cdot c_0a) \cdot c_0a \leq d_{01} \) and \( c_0(d_{01} \cdot c_1a) \cdot c_1a \leq d_{01} \), hence \( \chi^{\mathfrak{A}}(a) = 1 \), and \( \tau^{\mathfrak{A}}(a) = a \neq 0 \).

Claim 2. Let \( i, j \in \alpha \sim 2 \) be such that \( i \neq j \). Then \( \tau(x) \leq -d_{ij} \) in \( \mathfrak{F}_m \text{CA}_\alpha \).

Proof of Claim 2. Consider the following system \( E(X, Y) \) of equations:
\[
\begin{align*}
X \cdot Y &= 0, \quad X \neq 0, \\
c_iX &= c_iY \quad \text{for } i \in 2, \\
c_i(d_{01} \cdot c_kX) \cdot c_kX &\leq d_{01} \quad \text{for } \{i, k\} = 2.
\end{align*}
\]
Let \( \eta \defeq \eta(x) \defeq y \cdot \chi(x), \quad \chi \defeq \chi(x) \) and \( \tau \defeq \tau(x) \). First we show that \( E(\tau, \eta) \) holds in \( \mathfrak{F}_m \text{CA}_\alpha \).

(A) \( \tau \cdot \eta = 0 \) since \( \tau \leq x \) and \( \eta \leq -x \).

(B) \( \tau \neq 0 \) by Claim 1.

(C) Let \( i \in 2 \). By \( c_{(2)}\chi = \chi \), we have \( c_i\tau = c_i(x \cdot c_{(2)}\chi) = c_i x \cdot \chi \) and similarly \( c_i\eta = c_i y \cdot \chi \), hence \( c_i\tau \oplus c_i\eta = (c_i x \oplus c_i y) \cdot \chi = 0 \) since \( \chi \leq -c_{(2)}(c_i x \oplus c_i y) \).

This implies \( c_i \tau = c_i \eta \).

(D) Let \( i, k \in \alpha \) be such that \( \{i, k\} = 2 \). By \( c_{(2)}\chi = \chi \), we again have
\[
c_i(d_{01} \cdot c_k\tau) \cdot c_k \tau = [c_i(d_{01} \cdot c_kx) \cdot c_kx] \cdot \chi \leq d_{01}
\]
by \( \chi \leq -c_{(2)}(c_i(d_{01} \cdot c_kx) \cdot c_kx - d_{01}) \).

Now, let \( i, j \in \alpha \sim 2 \) be such that \( i \neq j \). Let \( s_{ij}^\tau \defeq c_i(\tau \cdot d_{ij}) \) and \( s_{ij}^\eta \defeq c_i(\eta \cdot d_{ij}) \). Assume that \( \tau \cdot d_{ij} \neq 0 \). Then \( s_{ij}^\tau \neq 0 \). By \( \{i, j\} \cap 2 = \emptyset \) and by [1, Section 1.5], we know that \( E(s_{ij}^\tau, s_{ij}^\eta) \) also holds in \( \mathfrak{F}_m \text{CA}_\alpha \). Let \( \mathcal{R} \defeq \mathfrak{R}_{2 \cup \{i\}} \mathfrak{F}_m \text{CA}_\alpha \) be the reduct of \( \mathfrak{F}_m \text{CA}_\alpha \) resulting by ignoring the operations that contain indices in \( \alpha \sim (2 \cup \{i\}) \). Consider the neat reduct \( \mathfrak{C} = \mathfrak{R}_c \mathcal{R} \) of the reduct \( \mathcal{R} \). Then, \( \mathfrak{C} \in G_2 \) by [3, Theorem 3.2.65]. Let \( \tau' \defeq s_{ij}^\tau \) and \( \eta' \defeq s_{ij}^\eta \). Then \( \tau', \eta' \in C \) and \( E(\tau', \eta') \) holds in \( \mathfrak{C} \). This is a contradiction since \( \mathfrak{C} \in G_2 \) and it is not difficult to verify that \( E(X, Y) \) fails in \( \mathfrak{C} \in G_2 \) for every \( X, Y \in C \). That means our assumption \( \tau \cdot d_{ij} \neq 0 \) cannot hold, i.e., \( \tau \leq -d_{ij} \). \( \square \)
Therefore, there is $x \in \mathfrak{F}_{m} CA_{\alpha}$ such that $x \neq 0$ and $x \leq -d_{ij}$ for every $i, j \in \alpha \sim 2$ with $i \neq j$. We note that this proof works to prove Theorem 1.4 if $\alpha$ is any arbitrary ordinal, but the theorem is interesting only for the case $\alpha \geq \omega$. □

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