ORBITAL STABILITY OF STANDING WAVES FOR SCHRÖDINGER TYPE EQUATIONS WITH SLOWLY DECAYING LINEAR POTENTIAL

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Abstract. In this paper, kinds of Schrödinger type equations with slowly decaying linear potential and power type or convolution type nonlinearities are considered.

By using the concentration compactness principle, the sharp Gagliardo-Nirenberg inequality and a refined estimate of the linear operator, the existence and orbital stability of standing waves in $L^2$-subcritical and -critical case are established in a systematic way.

2010 Mathematics Subject Classification: 35Q41, 35B35, 35J20.

Keywords: orbital stability, concentration compactness principle, standing waves, slowly decaying potential, Schrödinger type equations.

1. Introduction and main results

In this paper, we consider the following nonlinear Schrödinger type equations with slowly decaying linear potential

$$
\begin{align*}
&i\partial_t u + \Delta u + V(x)u + f(u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
&u(0, x) = u_0(x) \in H^1(\mathbb{R}^N), \quad x \in \mathbb{R}^N,
\end{align*}
$$

where $N \geq 3$, $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $V(x) = \gamma|x|^{-\alpha}$, $\gamma \in \mathbb{R}$, $0 < \alpha < 2$, and $f : \mathbb{C} \to \mathbb{C}$ is assumed to be one of the following five types:

- **Type 1** $f(u) = |u|^{p_1-1}u$ with $1 < p_1 < (N + 2)/(N - 2)$;
- **Type 2** $f(u) = (I_{\beta} \ast |u|^q) |u|^{q_1-2}u$ with $0 < \beta < N$, $1 + \beta/N < q < (N + \beta)/(N - 2)$ and $I_{\beta}$ being the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by
  $$I_{\beta}(x) = \frac{\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma(\frac{N}{2})\pi^{N/2}2^\beta |x|^{N-\beta}},$$
  with $\Gamma$ denoting the Gamma function (see [27], P.19);
- **Type 3** $f(u) = |u|^{p_1-1}u + |u|^{p_2-1}u$ with $1 < p_1 < p_2 < (N + 2)/(N - 2)$;
- **Type 4** $f(u) = (I_{\beta} \ast |u|^q_1) |u|^{q_1-2}u + (I_{\beta} \ast |u|^q_2) |u|^{q_2-2}u$ with $0 < \beta < N$ and $1 + \beta/N < q_1 < q_2 < (N + \beta)/(N - 2)$;
- **Type 5** $f(u) = (I_{\beta} \ast |u|^q) |u|^{q-2}u + |u|^{p-1}u$ with $0 < \beta < N$, $1 + \beta/N < q < (N + \beta)/(N - 2)$ and $1 < p < (N + 2)/(N - 2)$.

The Schrödinger operator $i\partial_t + \Delta$ arises in various physical contexts such as nonlinear optics and plasma physics, see [10, 30, 34]. The nonlinearity enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium. The potential $V(x)$ can be thought of as modeling inhomogeneities in the medium. In particular, the operator $-\Delta - \frac{\gamma}{|x|^\alpha}$ with Coulomb potential provides a quantum mechanical description of the Coulomb force between two charged particles and
corresponds to having an external attractive long-range potential due to the presence of a positively charged atomic nucleus, see [24, 28].

By a standing wave, we mean a solution of (1.1) in the form \( u(t, x) = e^{i\omega t} \varphi(x) \), where \( \omega \in \mathbb{R} \) is a constant and \( \varphi \in H^1(\mathbb{R}^N) \) satisfies the stationary equation

\[
-\Delta \varphi - \frac{\gamma}{|x|^s} \varphi + \omega \varphi = f(\varphi), \quad x \in \mathbb{R}^N.
\]

For \( f(u) \) is of Type 1, the existence of groundstate to (1.2) was studied in Fukaya and Ohta [13] and Fukuizumi and Ohta [15], and the uniqueness of the positive radial solution was studied in Benguria and Jeanneret [2]. For \( f(u) \) is of Type 2, the nonexistence, existence and uniqueness of positive solution to (1.2) were studied in [9, 20, 21, 22].

When \( \gamma = 0 \) and \( f(u) \) is of Type 1, (1.1) is invariant under the scaling transform

\[
u(t, x) \rightarrow \lambda^{\frac{2}{N-2}} u(\lambda^2 t, \lambda x)
\]

and if \( p = p_c := 1 + 4/N \), the transform keeps the mass invariant and the nonlinearity is called \( L^2 \)-critical. In this case, Cazenave [6] proved that the groundstate solution of (1.2) is orbitally stable for all \( \omega > 0 \) if \( p < p_c \), while is unstable for all \( \omega > 0 \) if \( p_c < p < (N + 2)/(N - 2) \). The instability of the boundstate solution with \( p = p_c \) was proved by Weinstein [33]. When \( f(u) \) is of Type 2, the transform

\[
u(t, x) \rightarrow \lambda^{\frac{2}{N-2}} u(\lambda^2 t, \lambda x)
\]

keeps (1.1) invariant and \( q = q_c := 1 + (2 + \beta)/N \) is the \( L^2 \)-critical exponent. In this case, Cazenave and Lions [7] showed the existence and orbital stability of standing waves for \( N = 3 \), \( q = 2 \) and \( \beta = 2 \). Recently, in the \( L^2 \)-subcritical case, that is, \( 1 + \beta/N < q < 1 + (2 + \beta)/N \), Wang et al. [32] studied the orbital stability of standing waves to (1.1). When \( f(u) \) is of Type 3, see [14, 29] for the existence, orbital stability and strong instability of standing waves and see [1, 8, 18, 23] for that of Type 5.

When \( \gamma \neq 0 \), it is easy to see that equation (1.1) does not enjoy scaling invariance even pace translation invariance. In this case, when \( f(u) \) is of Type 1, Fukaya and Ohta studied the existence and orbital stability of standing waves for \( 1 < p < 1 + 4/N \) in [15] and the strong instability of standing waves for \( \gamma > 0 \) and \( 1 + 4/N < p < (N + 2)/(N - 2) \) in [13]. When \( f(u) \) is of Type 2, \( N = 3 \) and \( \beta = 2 \), Cazenave and Lions [7] and Lions [22] established the existence and orbital stability of standing waves for \( q = 2 \), and Cheng and Yang [9] studied that for \( q > 2 \) and close to 2. More precisely, in (1.2), taking \( \omega \) as a fixed parameter, it is known that every solution \( v \in H^1(\mathbb{R}^N) \) to (1.2) is a critical point of the functional \( S_\omega \) defined by

\[
S_\omega(v) := \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 - \frac{\gamma}{|x|^s} |v|^2 + \omega |v|^2 \right) dx - \int_{\mathbb{R}^N} F(v) dx,
\]

where \( F(s) = \int_0^s f(\tau) d\tau \). Denote by \( \mathcal{M}_\omega^{\text{cr}} \) the set of all non-negative minimizers for

\[
\inf \{ S_\omega(v) : v \in H^1(\mathbb{R}^N) \setminus \{0\}, \ v \text{ a solution of (1.2)} \}.
\]

For \( f(u) \) is of Type 1, \( 1 < p < 1 + 4/N \) and \( \gamma \in \mathbb{R} \), [15] proved that there exist \( \omega_0 > 0 \) and \( \omega_* > \omega_0 \) such that \( \mathcal{M}_\omega^{\text{cr}} \) is not empty for any \( \omega \in (\omega_0, \infty) \) and \( e^{i\omega t} \varphi_\omega(x) \) is orbitally stable for any \( \varphi_\omega(x) \in \mathcal{M}_\omega^{\text{cr}} \) and \( \omega \in (\omega_*, \infty) \), by using a sufficient condition for orbital stability, that is, the positive definite of the operator \( S''_\omega(\varphi_\omega) \). Similar results were obtained by [9] for Type 2 nonlinearities, \( N = 3 \), \( \beta = 2 \), \( q > 2 \) close to 2. Note that \( \omega_0 \) and \( \omega_* \) are not given explicitly, and moreover, they did not consider the \( L^2 \)-critical case. Hence, in this paper, we further discuss the existence and orbital stability of standing waves of (1.1).
Note that we may take $\omega$ as unknown in (1.2). Indeed, for any $\rho > 0$, if we define

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{\gamma}{|x|^\alpha} |u|^2 \right) \, dx - \int_{\mathbb{R}^N} F(u) \, dx,$$

(1.5)\[A_\rho = \inf_{u \in M_\rho} E(u), \quad M_\rho = \{ u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = \rho \}\]

and

$$G_\rho = \{ u \in H^1(\mathbb{R}^N) : u \in M_\rho, \ E(u) = A_\rho \}.$$

Then the Lagrange multiplier theorem implies that for any $u \in G_\rho$, there exists $\lambda \in \mathbb{R}$ such that

$$-\Delta u - \frac{\gamma}{|x|^\alpha} u - f(u) = -\lambda u.$$

Hence, $e^{i\lambda t} u(x)$ is a standing wave to (1.1) with initial data $u_0(x) = u(x)$. One usually calls $e^{i\lambda t} u(x)$ the orbit of $u$. Moreover, if $u \in G_\rho$, then $e^{i\theta} u(x) \in G_\rho$ for any $\theta \in \mathbb{R}$.

In this paper, we consider the orbital stability of $G_\rho$. For this, we give the following definition of orbital stability which is similar to that in [31].

**Definition 1.1.** The set $G_\rho$ is said to be orbitally stable if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any initial data $u_0$ satisfying

$$\inf_{v \in G_\rho} \| u_0 - v \|_{H^1(\mathbb{R}^N)} < \delta,$$

the corresponding solution $u$ to (1.1) satisfies

$$\inf_{v \in G_\rho} \| u(t) - v \|_{H^1(\mathbb{R}^N)} < \epsilon$$

for all $t \geq 0$.

**Remark 1.2.** (1) Note that our definition of orbital stability is different from that in [15] for which we do not know whether or not $M_\rho^G$ and $G_\rho$ are single point sets.

(2) For the lack of scaling invariance, there is not direct connection between $M_\rho^G$ and $G_\rho$. However, for equation

$$\begin{cases}
i\partial_t u + \Delta u + (I_\beta \ast |u|^q)|u|^{q-2}u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\u(0, x) = u_0(x) \in H^1(\mathbb{R}^N), & x \in \mathbb{R}^N,
\end{cases}$$

there is some equivalence between $M_\rho^G$ and $G_\rho$, see [32].

To study the orbital stability of standing waves of (1.1), we first make the following assumption on $V(x)$ and $f(u)$.

**Assumption A.** Let $V(x) = \gamma |x|^{-\alpha}$, $\gamma \in \mathbb{R}$, $0 < \alpha < 2$ and $f(u)$ be one of Types 1-5 such that the following local well posedness holds for (1.1):

For any $u_0 \in H^1(\mathbb{R}^N)$, there exists $T = T(\|u_0\|_{H^1(\mathbb{R}^N)}) > 0$ such that (1.1) admits a unique solution $u(t) \in C([0, T], H^1(\mathbb{R}^N))$ with $u(0) = u_0$ satisfying the conservation of mass and energy:

$$E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0, T].$$

Moreover, let $[0, T^*)$ be the maximal interval on which the solution $u(t, x)$ is well defined: if $T^* < \infty$, then $\|u(t, x)\|_{H^1(\mathbb{R}^N)} \to \infty$ as $t \uparrow T^*$.
Remark 1.3. (1) When $f(u)$ is of Type 1, $N \geq 3$, $1 < p < 2N/(N-2) - 1$, $0 < \alpha < 2$ and $\gamma \in \mathbb{R}$, the local well posedness for (1.1) is established in Section 6.4 and Theorem 9.2.5 of Cazenave [5].

(2) When $f(u)$ is of Type 1, $N = 3$, $p = 3$, $1 < \alpha < 2$ and $\gamma < 0$, by using the Strichartz estimates established by Mizutani [26] for the linear operator of (1.1), Guo et al. [17] obtained the local well posedness for (1.1). Moreover, they also considered the global well posedness and scattering.

(3) When $f(u)$ is of Type 1, $N = 3$, $\alpha = 1$, $1 < p < 5$ and $\gamma \in \mathbb{R}$, the local well posedness is established by Miao et al. [25]. Moreover, they also studied the global existence, scattering and blow up criterion for (1.1).

(4) When $f(u)$ is of Type 2, $\beta = 2$, $q = 2$, $\alpha = 1$ and $\gamma > 0$, the local well posedness is established by Ginibre and Velo [16]. Moreover, $T^* = \infty$.

Under Assumption A, by using the concentration compactness principle, the sharp Gagliardo-Nirenberg inequality and a refined estimate of the linear operator $-\Delta - \gamma |x|^{-\alpha}$, we can obtain the following theorems.

Theorem 1.4. Let $N \geq 3$, $0 < \alpha < 2$, $f(u) = |u|^{p-1}u$, $V(x)$ and $f(u)$ satisfy Assumption A. Assume one of the following conditions hold:

(1) $\gamma < 0$, $1 < p < 1 + 2\alpha/N$, $\rho > 0$;
(2) $\gamma > 0$, $1 < p < 1 + 4/N$, $\rho > 0$;
(3) $\gamma > 0$, $p = 1 + 4/N$, $0 < \rho < \|Q_{p}\|_2^2$, where $Q_{p}$ is the unique positive radial solution of equation $-\Delta Q + Q = Q^p$;

then $G_{\rho}$ is not empty and orbitally stable.

Theorem 1.5. Let $N \geq 3$, $0 < \alpha < 2$, $\beta \in (0, N)$, $f(u) = (I_\beta * |u|^q)|u|^{q-2}u$, $V(x)$ and $f(u)$ satisfy Assumption A. Assume one of the following conditions hold:

(1) $\gamma < 0$, $1 + \beta/N < q < 1 + (\alpha + \beta)/N$, $\rho > 0$;
(2) $\gamma > 0$, $1 + \beta/N < q < 1 + (2 + \beta)/N$, $\rho > 0$;
(3) $\gamma > 0$, $q = 1 + (2 + \beta)/N$, $0 < \rho < \|W_q\|_2^2$, where $W_q$ is a radically ground state solution of the elliptic equation $-\Delta W + W = (I_\beta * |W|^q)|W|^{q-2}W$;

then $G_{\rho}$ is not empty and orbitally stable.

Theorem 1.6. Let $N \geq 3$, $0 < \alpha < 2$, $f(u) = |u|^{p_1-1}u + |u|^{p_2-1}u$, $V(x)$ and $f(u)$ satisfy Assumption A. Assume one of the following conditions hold:

(1) $\gamma > 0$, $1 < p_1 < p_2 < 1 + 4/N$, $\rho > 0$;
(2) $\gamma > 0$, $1 < p_1 < p_2 < 1 + 4/N$, $0 < \rho < \|Q_{p_2}\|_2^2$;
(3) $\gamma > 0$, $1 < p_1 < 1 + 2\alpha/N$, $p_1 < p_2 < 1 + 4/N$, $\rho > 0$;
(4) $\gamma > 0$, $1 < p_1 < 1 + 2\alpha/N$, $p_1 < p_2 < 1 + 4/N$, $0 < \rho < \|Q_{p_2}\|_2^2$;

then $G_{\rho}$ is not empty and orbitally stable.

Theorem 1.7. Let $N \geq 3$, $0 < \alpha < 2$, $\beta \in (0, N)$, $f(u) = (I_\beta * |u|^p)|u|^{p-2}u + (I_\beta * |u|^q)|u|^{q-2}u$, $V(x)$ and $f(u)$ satisfy Assumption A. Assume one of the following conditions hold:

(1) $\gamma < 0$, $1 + \beta/N < q_1 < 1 + (\alpha + \beta)/N$, $q_1 < q_2 < 1 + (2 + \beta)/N$, $\rho > 0$;
(2) $\gamma < 0$, $1 + \beta/N < q_1 < 1 + (\alpha + \beta)/N$, $q_2 = 1 + (2 + \beta)/N$, $0 < \rho < \|W_{q_2}\|_2^2$;
(3) $\gamma > 0$, $1 + \beta/N < q_1 < q_2 < 1 + (2 + \beta)/N$, $\rho > 0$;
(4) $\gamma > 0$, $1 + \beta/N < q_1 < q_2 = 1 + (2 + \beta)/N$, $0 < \rho < \|W_{q_2}\|_2^2$;

then $G_{\rho}$ is not empty and orbitally stable.
Theorem 1.8. Let $N \geq 3$, $0 < \alpha < 2$, $\beta \in (0, N)$, $f(u) = (I_B * |u|^\eta)|u|^{\eta-2} u + |u|^{p-1}u$, $V(x)$ and $f(u)$ satisfy Assumption A. Assume one of the following conditions hold:

1. $\gamma > 0$, $1 < p < 1 + 4/N$, $1 + \beta/N < q < 1 + (2 + \beta)/N$, $\rho > 0$;
2. $\gamma < 0$, $1 < p < 1 + 4/N$, $q = 1 + (2 + \beta)/N$, $0 < \rho < \|W_q\|^2_2$;
3. $\gamma > 0$, $p = 1 + 4/N$, $1 + \beta/N < q < 1 + (2 + \beta)/N$, $0 < \rho < \|Q_p\|^2_2$;
4. $\gamma > 0$, $p = 1 + 4/N$, $q = 1 + (2 + \beta)/N$, $(\sqrt{\rho}/\|Q_p\|_2)^{4/N} + (\sqrt{\rho}/\|W_q\|_2)^{(2\beta+4)/N} < 1$;
5. $\gamma < 0$, $1 < p < 1 + 2\alpha/N$, $1 + \beta/N < q < 1 + (2 + \beta)/N$, $\rho > 0$;
6. $\gamma < 0$, $1 < p < 1 + 2\alpha/N$, $q = 1 + (2 + \beta)/N$, $0 < \rho < \|W_q\|^2_2$;
7. $\gamma < 0$, $1 < p < 1 + 4/N$, $1 + \beta/N < q < 1 + (\alpha + \beta)/N$, $\rho > 0$;
8. $\gamma < 0$, $p = 1 + 4/N$, $1 + \beta/N < q < 1 + (\alpha + \beta)/N$, $0 < \rho < \|Q_p\|^2_2$;

then $G_\rho$ is not empty and orbitally stable.

We should point out that, among the methods used in the study of orbital stability of standing waves, the profile decomposition method plays an important role in recent studies, see [4] and [12]. In [4], the authors considered a Schrödinger equation with inverse-square potential, i.e. (1.1) with $\alpha = 2$ and $\gamma < (N - 2)^2/4$. By using the equivalence of $\|u\|^2_2$ and $\int_{\mathbb{R}^N} (|\nabla u|^2 - \gamma |x|^{-\alpha} |u|^2) dx$, Bensouilah [3] obtained the profile decomposition of a bounded sequence in $H^1(\mathbb{R}^N)$ related to the problem, and based of which, [4] studied the orbital stability of standing waves. However, there is not an equivalence between $|\nabla u|^2_2$ and $\int_{\mathbb{R}^N} (|\nabla u|^2 - \gamma |x|^{-\alpha} |u|^2) dx$ for $\alpha \in (0, 2)$ and we cannot obtain the profile decomposition in this case. But in view of that $\int_{\mathbb{R}^N} (|\nabla u|^2 - \gamma |x|^{-\alpha} |u|^2) dx$ can be controlled by $\epsilon \|u\|^2_2$ and a function of $\|u\|^2_2$ (see Lemma 2.2), and by carefully examining the application of concentration compactness principle in the study of orbital stability of standing waves (see [7] and [22]), we can solve the problem by using the concentration compactness principle in a systematic way. In fact, the profile decomposition can be look as another equivalent description of the concentration compactness principle, see [12].

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorems 1.4-1.8.

Notation. Throughout this paper, we use the following notation. $C > 0$ stands for a constant that may be different from line to line when it does not cause any confusion. $L^r(\mathbb{R}^N)$ with $1 \leq r < \infty$ denotes the Lebesgue space with the norms $\|u\|_r = (\int_{\mathbb{R}^N} |u|^r dx)^{1/r}$. $H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm $\|u\|_{H^1(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx)^{1/2}$. $B_R(x)$ denotes the ball in $\mathbb{R}^N$ centered at $x$ with radius $R$. $B_R^c(x) = \mathbb{R}^N \setminus B_R(x)$. $\chi_B(x) = 1$ if $x \in B$, and $= 0$ if $x \notin B$.

2. Preliminaries

The following generalized Gagliardo-Nirenberg inequality can be found in [33].

Lemma 2.1. Let $N \geq 3$ and $0 < \eta < 4/(N - 2)$, then the following sharp Gagliardo-Nirenberg inequality

$$\|u\|_{L^{N+2}_{N+2}}^2 \leq C_{GN}(\eta) \|u\|_2^{2+\eta(2-N)/2} \|\nabla u\|_2^{2N/2}$$

holds for any $u \in H^1(\mathbb{R}^N)$. The sharp constant $C_{GN}(\eta)$ is

$$C_{GN}(\eta) = \frac{2(\eta + 2)}{4 - (N - 2)\eta} \left( \frac{4 - (N - 2)\eta}{N\eta} \right)^{N\eta/4} \frac{1}{\|Q_{\eta+1}\|_2^{\eta}},$$

where $Q_{\eta+1}$ is defined in Theorem 1.4.
Next, we give a refined estimate for the linear operator $-\Delta - \gamma |x|^{-\alpha}$.

**Lemma 2.2.** Let $N \geq 3$, $0 < \alpha < 2$ and $\gamma \in \mathbb{R}$. Then for any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon, \|u\|_2) > 0$ such that

$$\epsilon \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^\alpha} \, dx \geq -\delta(\epsilon, \|u\|_2)$$

for any $u \in H^1(\mathbb{R}^N)$.

**Proof.** It obviously holds for $\gamma \leq 0$. Now we prove the lemma for $\gamma > 0$. By domain decomposition and the Hölder inequality, we know

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^\alpha} \, dx = \int_{B_1(0)} \frac{|u|^2}{|x|^\alpha} \, dx + \int_{B_1^*(0)} \frac{|u|^2}{|x|^\alpha} \, dx$$

\begin{equation}
\leq |||x|^{-\alpha} \chi_{B_1(0)}||_r ||u^2||_{r'} + |||x|^{-\alpha} \chi_{B_1^*(0)}||_s ||u^2||_s' = C_1 ||u||_{2r'}^2 + C_2 ||u||_{2s'}^2,
\end{equation}

where $1/r + 1/r' = 1$, $1/s + 1/s' = 1$, $r < N/\alpha$, $s > N/\alpha$, $N/\alpha - r$ and $s - N/\alpha$ are both sufficiently small. Hence, $r' - N/(N - \alpha) > 0$ and $N/(N - \alpha) - s > 0$ are small. By the Gagliardo-Nirenberg inequality (Lemma 2.1), we know

$$||u||_{2r'} \leq C ||u||_{2}^{N/r'-(N-2)} ||\nabla u||_{2}^{N-N/r'}.$$

Noting that $|N - N/r' - \alpha|$ is sufficiently small and $0 < \alpha < 2$, and by using the Young inequality

$$a^{1/q}b^{1/q'} \leq \frac{a}{q} + \frac{b}{q'}, \ a, b > 0, \ 1/q + 1/q' = 1,$$

we have for any $\epsilon_1 > 0$, there exists $\delta_1 = \delta_1(\epsilon_1, \|u\|_2)$ such that

$$||u||_{2}^{N/r'-(N-2)} ||\nabla u||_{2}^{N-N/r'} = ||u||_{2}^{N/r'-(N-2)} ||\nabla u||_{2}^{N-N/r'} \leq \epsilon_1 ||\nabla u||_{2}^2 + \delta_1(\epsilon_1, \|u\|_2).$$

The same estimates hold for $||u||_{2s'}^2$. In view of (2.1)-(2.3), we complete the proof. \qed

The following concentration compactness principle is cited from Lemma III.1 in [22].

**Lemma 2.3.** Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying:

$$\int_{\mathbb{R}^N} |u_n|^2 \, dx = \lambda,$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ satisfying one of the three possibilities:

(i) (compactness) there exists $\{y_{n_k}\}_{k=1}^\infty \subset \mathbb{R}^N$ such that $|u_{n_k}(-\cdot + y_{n_k})|^2$ is tight, i.e.,

$$\forall \epsilon > 0, \ \exists R < \infty, \ \int_{B_R(y_{n_k})} |u_{n_k}(x)|^2 \, dx \geq \lambda - \epsilon;$$

(ii) (vanishing) $\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_{n_k}(x)|^2 \, dx = 0$ for all $R < \infty$;

(iii) (dichotomy) there exists $\sigma \in (0, \lambda)$ such that for any $\epsilon > 0$, there exist $k_0 \geq 1$, $R_1 = R_1(\epsilon) > 0$, $\{y_{n_k}\}_{k=1}^\infty \subset \mathbb{R}^N$ and $u_n^{(1)}$, $u_n^{(2)}$ bounded in $H^1(\mathbb{R}^N)$ satisfying for
The following well-known Hardy-Littlewood-Sobolev inequality can be found in [19].

**Lemma 2.4.** Let \( N \geq 1 \), \( p, r > 1 \) and \( 0 < \beta < N \) with \( 1/p + (N - \beta)/N + 1/r = 2 \). Let \( u \in L^p(\mathbb{R}^N) \) and \( v \in L^r(\mathbb{R}^N) \). Then there exists a sharp constant \( C(N, \beta, p) \), independent of \( u \) and \( v \), such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^{N-\beta}} \leq C(N, \beta, p) \|u\|_p \|v\|_r.
\]

If \( p = r = \frac{2N}{N+\beta} \), then

\[
C(N, \beta, p) = C_\beta(N) = \pi^{\frac{N-\beta}{2}} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{N+\beta}{2})} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-\frac{\beta}{N}}.
\]

**Remark 2.5.** By the Hardy-Littlewood-Sobolev inequality above and the Sobolev embedding theorem, we obtain

\[
\left( I_\beta * |u|^p \right) |u|^p \leq C \left( \int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\beta}} \right)^{1+\beta/N} \leq C \|u\|_{\dot{H}^1(\mathbb{R}^N)}^{2p}
\]

for any \( p \in \left[ \frac{N+\beta}{N}, \frac{N+\beta}{N-2} \right] \), where \( C > 0 \) is a constant depending only on \( N \), \( \beta \) and \( p \).

The following generalized Gagliardo-Nirenberg inequality for the convolution problem can be found in [11].

**Lemma 2.6.** Let \( N \geq 3 \), \( 0 < \beta < N \), \( 1 + \beta/N < p < (N + \beta)/(N - 2) \), then

\[
\int_{\mathbb{R}^N} \left( I_\beta * |u|^p \right) |u|^p dx \leq C(\beta, p) \|\nabla u\|_2^{Np-N-\beta} \|u\|_2^{N+\beta-Np-2p}.
\]

The best constant \( C(\beta, p) \) is defined by

\[
C(\beta, p) = \frac{2p}{2p-Np+N+\beta} \left( \frac{2p-Np+N+\beta}{Np-N-\beta} \right)^{(Np-N-\beta)/2} \|W_p\|_2^{2-2p},
\]

where \( W_p \) is defined in Theorem 1.5. In particular, in the \( L^2 \)-critical case, i.e., \( p = 1 + \frac{2N}{N+\beta} \), \( C(\beta, p) = p \|W_p\|_2^{2-2p} \).

**Remark 2.7.** Note that \( W_p \) may be not unique, but the groundstate has the same \( L^2 \)-norm, see [11].
3. Proof of the main results

Proof of Theorem 1.4. We will prove this theorem in seven steps.

Step 1. We show that $A_{\rho} > -\infty$, that is, $A_{\rho}$ is well defined. For any $u \in M_{\rho}$, we have $\|u\|^2_{2} = \rho$, and by Lemmas 2.1 and 2.2, we have for any $\epsilon > 0$,

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^{N}} \left( |\nabla u|^2 - \frac{\gamma}{|x|^\alpha} |u|^2 \right) dx - \frac{1}{p+1} \int_{\mathbb{R}^{N}} |u|^{p+1} dx
\]

(3.1)

\[
\geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \int_{\mathbb{R}^{N}} |\nabla u|^2 dx - \delta(\epsilon, \|u\|_2)
- \frac{1}{p+1} C_{GN}(p-1) \|u\|_2^2 + (p-1)(2-N)/2 \|\nabla u\|_2^{(p-1)N/2}.
\]

For $1 < p < 1 + 4/N$, we have $0 < (p-1)N/2 < 2$. Thus, by the Young inequality, we obtain from (3.1) that

\[
E(u) \geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \int_{\mathbb{R}^{N}} |\nabla u|^2 dx - \delta(\epsilon, \|u\|_2) - \delta_1(\epsilon, \|u\|_2) - \frac{\epsilon}{2} \|\nabla u\|_2^2
\]

\[
= \left( \frac{1}{2} - \epsilon \right) \|\nabla u\|_2^2 - \delta_2(\epsilon, \|u\|_2) \geq -\delta_2(\epsilon, \|u\|_2) > -\infty
\]

by choosing $\epsilon = 1/4$.

For $p = 1 + 4/N$, we have

\[
\frac{(p-1)N}{2} = 2, 2 + \frac{(p-1)(2-N)}{2} = \frac{4}{N}, \quad \frac{1}{p+1} = \frac{N}{2N+4}
\]

and $C_{GN}(p-1) = (N+2)/N \|Q_{p}\|_2^{-4/N}$. Thus, we obtain from (3.1) that

\[
E(u) \geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \int_{\mathbb{R}^{N}} |\nabla u|^2 dx - \delta(\epsilon, \|u\|_2) - \frac{1}{2} \left( \frac{\|u\|_2}{\|Q_{p}\|_2} \right)^{4/N} \|\nabla u\|_2^2
\]

\[
= \frac{1}{2} \left( 1 - \left( \frac{\|u\|_2}{\|Q_{p}\|_2} \right)^{4/N} - \epsilon \right) \|\nabla u\|_2^2 - \delta(\epsilon, \|u\|_2) > -\infty
\]

by choosing $\epsilon > 0$ small enough since $\|u\|_2 > \|Q_{p}\|_2$.

Step 2. For any $\rho > 0$ there exists $C_1 > 0$ such that $A_{\rho} \leq -C_1 < 0$. Indeed, for any $\varphi \in M_\rho$ and $\lambda > 0$, we define $\varphi_\lambda(x) = \lambda^{N/2}\varphi(\lambda x)$. It is easy to check that $\|\varphi_\lambda\|_2^2 = \|\varphi\|_2^2 = \rho$ and

\[
E(\varphi_\lambda) = \frac{\lambda^2}{2} \int_{\mathbb{R}^{N}} |\nabla \varphi|^2 dx - \frac{\gamma\lambda^\alpha}{2} \int_{\mathbb{R}^{N}} |\varphi|^2 dx - \frac{\lambda^{(p-1)N/2}}{p+1} \int_{\mathbb{R}^{N}} |\varphi|^{p+1} dx.
\]

If $\gamma > 0$, by choosing $\lambda > 0$ sufficiently small, we have $E(\varphi_\lambda) < 0$ since $\alpha < 2$. If $\gamma < 0$ and $(p-1)N/2 < \alpha$, by choosing $\lambda > 0$ sufficiently small, we have $E(\varphi_\lambda) < 0$.

Let $\{u_n\}_{n=1}^\infty \subset M_\rho$ be a minimizing sequence of $E$, that is,

\[
\|u_n\|_2^2 = \rho \quad \text{and} \quad \lim_{n \to \infty} E(u_n) = A_\rho.
\]

Similarly to Step 1, one can show that $\{u_n\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ such that one of the three possibilities in Lemma 2.3 holds.
Step 3. The vanishing case in Lemma 2.3 does not occur. Suppose by contradiction that
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_{n_k}|^2 \, dx = 0.
\]
By Lion’s lemma, \( u_{n_k} \to 0 \) in \( L^s(\mathbb{R}^N) \) as \( k \to \infty \) for all \( s \in (2, 2N/(N - 2)) \). Hence,
\[
\int_{\mathbb{R}^N} |u_{n_k}|^{p+1} \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_{n_k}|^2 \, |x|^{-\alpha} \, dx \to 0
\]
by (2.1), and thus
\[
\lim_{k \to \infty} E(u_{n_k}) = \lim_{k \to \infty} \frac{1}{2} \|\nabla u_{n_k}\|^2 \geq 0,
\]
which contradicts \( A_\rho < 0 \).

Step 4. We show that the dichotomy case in Lemma 2.3 does not occur.

Firstly, according to Section I.2 in [22], we know
\[
(3.2) \quad A_\rho \leq A_\eta + A_{\rho-\eta}^\infty \text{ for any } \eta \in [0, \rho),
\]
where \( A_0 = 0, A_{\rho-\eta}^\infty = \inf_{M_{\rho-\eta}} E^\infty(u) \) and \( E^\infty(u) = \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \).

Next, we claim that
\[
(3.3) \quad A_{\theta \eta} \leq \theta A_\eta \text{ for any } \eta \in (0, \rho) \text{ and } \theta \in (1, \rho/\eta].
\]
Indeed, choose a sequence \( \{\varphi_n\}_{n=1}^\infty \subset M_\eta \) such that \( \lim_{n \to \infty} E(\varphi_n) = A_\eta \). Then \( \|\sqrt{\theta} \varphi_n\|^2 = \theta \eta \) and
\[
A_{\theta \eta} \leq \liminf_{n \to \infty} E(\sqrt{\theta} \varphi_n)
\]
\[
= \liminf_{n \to \infty} \theta \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi_n|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|\varphi_n|^2}{|x|^\alpha} \, dx \right) - \frac{\theta^{\frac{p+1}{p+1}}}{p+1} \int_{\mathbb{R}^N} |\varphi_n|^{p+1} \, dx
\]
\[
\leq \liminf_{n \to \infty} \theta \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi_n|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|\varphi_n|^2}{|x|^\alpha} \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |\varphi_n|^{p+1} \, dx \right)
\]
\[
= \theta A_\eta
\]
since \( \theta > 1 \) and \( p > 1 \). Hence the claim holds. It follows from (3.2), (3.3) and Lemma II.1 in [22], we have
\[
(3.4) \quad A_\rho < A_\eta + A_{\rho-\eta} \leq A_\eta + A_{\rho-\eta}^\infty \text{ for any } \eta \in (0, \rho).
\]

Finally, suppose by contradiction that (iii) in Lemma 2.3 holds. Denote
\[
\begin{align*}
\tilde{u}_{n_k} &= u_{n_k}^{(1)} + u_{n_k}^{(2)} + v_{n_k}, \\
\tilde{d}_{n_k} &= \operatorname{dist}(\operatorname{Supp} u_{n_k}^{(1)}, \operatorname{Supp} u_{n_k}^{(2)}), \\
\tilde{\sigma}_{n_k} &= \int_{\mathbb{R}^N} |u_{n_k}^{(1)}|^2 \, dx \quad \text{and} \quad \tilde{\eta}_{n_k} = \int_{\mathbb{R}^N} |u_{n_k}^{(2)}|^2 \, dx.
\end{align*}
\]
Then we may assume without loss of generality that
\[
\tilde{d}_{n_k} \to \infty, \quad \tilde{\sigma}_{n_k} \to \tilde{\sigma}, \quad \tilde{\eta}_{n_k} \to \tilde{\eta}
\]
as \( k \to \infty \) with \( \sigma, \tilde{\eta} \in (0, \rho) \), \( |\sigma - \sigma| \leq \epsilon \) and \( |\tilde{\eta} - (\rho - \sigma)| \leq \epsilon \). By direct calculation, we obtain that

\[
\int_{\mathbb{R}^N} \frac{|u_{nk}|^2}{|x|^{\alpha}} \, dx = \int_{\mathbb{R}^N} \frac{|u_{nk}^{(1)} + u_{nk}^{(2)} + v_{nk}|^2}{|x|^{\alpha}} \, dx
\]

(3.5)

\[
= \int_{\mathbb{R}^N} \frac{|u_{nk}^{(1)}|^2 + |u_{nk}^{(2)}|^2 + |v_{nk}|^2 + 2\text{Re}(u_{nk}^{(1)} \bar{v}_{nk}) + 2\text{Re}(u_{nk}^{(2)} \bar{v}_{nk})}{|x|^{\alpha}} \, dx
\]

\[
= \int_{\mathbb{R}^N} \frac{|u_{nk}^{(1)}|^2 + |u_{nk}^{(2)}|^2}{|x|^{\alpha}} \, dx + \delta(\epsilon),
\]

where \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). If \( \{y_{nk}\}_{k=1}^{\infty} \) is bounded, in view of (2.1), we have

\[
\int_{\mathbb{R}^N} \frac{|u_{nk}^{(2)}|^2}{|x|^{\alpha}} \, dx = \int_{|x-y_{nk}| \geq d_{nk}} \frac{|u_{nk}^{(2)}|^2}{|x|^{\alpha}} \, dx \leq \int_{|x-y_{nk}| \geq d_{nk}} \frac{|u_{nk}^{(2)}|^2}{|x|^{\alpha}} \, dx \to 0
\]

as \( k \to \infty \) since \( d_{nk} \to \infty \). Similarly, if \( \{y_{nk}\}_{k=1}^{\infty} \) is unbounded, then

\[
\int_{\mathbb{R}^N} \frac{|u_{nk}^{(1)}|^2}{|x|^{\alpha}} \, dx = \int_{|x-y_{nk}| \leq 2R_1} \frac{|u_{nk}^{(1)}|^2}{|x|^{\alpha}} \, dx \leq \int_{|x-y_{nk}| \leq 2R_1} \frac{|u_{nk}^{(1)}|^2}{|x|^{\alpha}} \, dx \to 0.
\]

By using the inequality

\[
\left| \sum_{j=1}^{l} a_j x_j^{r} - \sum_{j=1}^{l} a_j x_j^{r} \right| \leq C \sum_{j \neq k} |a_j| |a_k|^{r-1}
\]

with \( r > 1 \) and \( l \geq 2 \), and in view of \( u_{nk}^{(1)} u_{nk}^{(2)} = 0 \), we obtain that

\[
\left| \int_{\mathbb{R}^N} \big( |u_{nk}^{(1)} + u_{nk}^{(2)} + v_{nk}|^{p+1} - |u_{nk}^{(1)}|^{p+1} - |u_{nk}^{(2)}|^{p+1} \big) \, dx \right|
\]

\[
\leq \int_{\mathbb{R}^N} \big( |v_{nk}|^{p+1} + |u_{nk}^{(1)}||v_{nk}|^p + |u_{nk}^{(2)}||v_{nk}|^p + |u_{nk}^{(1)}|^{p+1} |v_{nk}| + |u_{nk}^{(2)}|^{p+1} |v_{nk}| \big) \, dx
\]

\[
= \delta(\epsilon),
\]

which implies that

\[
\int_{\mathbb{R}^N} |u_{nk}|^{p+1} \, dx = \int_{\mathbb{R}^N} |u_{nk}^{(1)}|^{p+1} \, dx + \int_{\mathbb{R}^N} |u_{nk}^{(2)}|^{p+1} \, dx + \delta(\epsilon)
\]

(3.9)

for any \( k \geq k_0 \), where \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). By (3.5)-(3.9), we obtain

\[
A_{\rho} = \lim_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{nk}|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_{nk}|^2}{|x|^{\alpha}} \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u_{nk}|^{p+1} \, dx \right)
\]

\[
\geq \liminf_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{nk}^{(1)}|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_{nk}^{(1)}|^2}{|x|^{\alpha}} \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u_{nk}^{(1)}|^{p+1} \, dx \right) + \delta(\epsilon)
\]

\[
+ \liminf_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{nk}^{(2)}|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_{nk}^{(2)}|^2}{|x|^{\alpha}} \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u_{nk}^{(2)}|^{p+1} \, dx \right)
\]

\[
\geq A_{\rho}^{\infty} + A_{\tilde{\eta}}^{\infty} + \delta(\epsilon) \quad \text{or} \quad A_{\rho}^{\infty} + A_{\tilde{\eta}} + \delta(\epsilon).
\]

Letting \( \epsilon \to 0 \), we obtain that \( A_{\rho} \geq A_{\sigma} + A_{\rho-\sigma}^{\infty} \) or \( A_{\rho} \geq A_{\sigma}^{\infty} + A_{\rho-\sigma} \), which contradicts (3.4). Hence, (iii) in Lemma 2.3 does not occur.
Step 5. From Steps 3 and 4 and Lemma 2.3, there exists a subsequence \(\{u_{n_k}\}_{k=1}^{\infty}\) and \(u \in H^1(\mathbb{R}^N)\) such that
\[ u_{n_k} \to u \text{ in } L^2(\mathbb{R}^N), \quad u_{n_k} \to u \text{ in } H^1(\mathbb{R}^N) \]
as \(k \to \infty\). Consequently, \(u_{n_k} \to u\) in \(L^s(\mathbb{R}^N)\) for \(s \in [2, 2^*)\) by the Hölder inequality and the Sobolev inequality. Hence,
\[
A_p = \lim_{k \to \infty} E(u_{n_k}) = \lim_{k \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} |u_{n_k}|^2 dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} |u_{n_k}|^{p+1} dx
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} dx.
\]
By the definition of \(A_p\), we see that \(u\) is a minimizer of \(A_p\), \(\lim_{k \to \infty} \|\nabla u_{n_k}\|^2_2 = \|\nabla u\|^2_2\), and hence \(u_{n_k} \to u\) in \(H^1(\mathbb{R}^N)\).

Step 6. We show that under Assumption A, if \(1 < p < 1 + 4/N\) and \(u_0 \in H^1(\mathbb{R}^N)\) or \(p = 1 + 4/N\) and \(\|u_0\|_2 < \|Q_p\|_2\), then the Cauchy problem (1.1) admits a global solution \(u(t) \in C([0, \infty), H^1(\mathbb{R}^N))\) with \(u(0) = u_0\).

Indeed, by Assumption A, it suffices to bound \(\|\nabla u(t)\|_2\) in the existence time. By the conversation law, Lemmas 2.1 and 2.2, we know
\[
\|\nabla u(t)\|_2^2 = 2E(u(t)) + \frac{2}{p + 1} \|u(t)\|_{p+1} + \int_{\mathbb{R}^N} \gamma \|u(t)\|^2 dx
\]
\[
\leq 2E(u(0)) + \epsilon \|\nabla u(t)\|_2^2 + \delta(\epsilon, \|u(t)\|_2)
\]
\[
+ \frac{2}{p + 1} C_{NG}(p - 1)\|u(t)\|^{2+(p-1)(2-N)/2}_2 \|\nabla u(t)\|_2^{(p-1)N/2}.
\]
Similarly to Step 1, for \(p = 1 + 4/N\), we have
\[
\|\nabla u(t)\|_2^2 \leq 2E(u(0)) + \epsilon \|\nabla u(t)\|_2^2 + \delta(\epsilon, \|u(t)\|_2) + \left(\frac{\|u(t)\|_2}{\|Q_p\|_2}\right)^{4/N} \|\nabla u(t)\|_2^2,
\]
Since \(\|u(t)\|_2 = \|u_0\|_2 < \|Q_p\|_2\), by choosing \(\epsilon > 0\) sufficiently small, the above estimate implies the boundedness of \(\|\nabla u(t)\|_2\). For \(1 < p < 1 + 4/N\), we have
\[
\|\nabla u(t)\|_2^2 \leq 2E(u(0)) + \epsilon \|\nabla u(t)\|_2^2 + \delta(\epsilon, \|u(t)\|_2) + \epsilon \|\nabla u(t)\|_2^2 + \delta_1(\epsilon, \|u(t)\|_2)
\]
and we arrive at the conclusion.

Step 7. We prove that \(G_p\) is orbitally stable. Suppose by contradiction that there exist sequences \(\{u_{0,n}\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)\) and \(\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}^+\) and a constant \(\epsilon_0 > 0\) such that for all \(n \geq 1\),
\[
\inf_{v \in G_p} \|u_{0,n} - v\|_{H^1(\mathbb{R}^N)} < \frac{1}{n}
\]
and
\[
\inf_{v \in G_p} \|u_n(t_n) - v\|_{H^1(\mathbb{R}^N)} \geq \epsilon_0,
\]
where \(u_n(t)\) is the solution to (1.1) with initial data \(u_{0,n}\).

We claim that there exists \(v \in G_p\) such that
\[
\lim_{n \to \infty} \|u_{0,n} - v\|_{H^1(\mathbb{R}^N)} = 0.
\]
Indeed, by (3.11), there exists \( \{v_n\}_{n=1}^{\infty} \subset G_\rho \) such that
\[
(3.13) \quad \|u_{0,n} - v_n\|_{H^1(\mathbb{R}^N)} < \frac{2}{n}.
\]
That \( \{v_n\}_{n=1}^{\infty} \subset G_\rho \) implies that \( \{v_n\}_{n=1}^{\infty} \subset M_\rho \) is a minimizing sequence of \( E \). So by Steps 3-5, there exists \( v \in G_\rho \) such that
\[
(3.14) \quad \lim_{n \to \infty} \|v_n - v\|_{H^1(\mathbb{R}^N)} = 0.
\]
Then the claim follows from (3.13) and (3.14) immediately. Hence,
\[
\lim_{n \to \infty} \|u_{0,n}\|_2^2 = \|v\|_2^2 = \rho, \quad \lim_{n \to \infty} E(u_{0,n}) = E(v) = A_\rho.
\]
By the conservation of mass and energy, we have
\[
\lim_{n \to \infty} \|u_n(t_n)\|_2^2 = \rho, \quad \lim_{n \to \infty} E(u_n(t_n)) = E(v) = A_\rho.
\]
Similarly to Step 1, \( \{u_n(t_n)\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N) \). Set
\[
\tilde{u}_n = \frac{\sqrt{\rho}u_n(t_n)}{\|u_n(t_n)\|_2},
\]
Then \( \|\tilde{u}_n\|_2^2 = \rho \) and
\[
E(\tilde{u}_n) = \left( \frac{\sqrt{\rho}}{\|u_n(t_n)\|_2} \right)^2 \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n(t_n)|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_n(t_n)|^2}{|x|^\alpha} dx \right)
- \left( \frac{\sqrt{\rho}}{\|u_n(t_n)\|_2} \right)^{p+1} \frac{1}{p+1} \int_{\mathbb{R}^N} |u_n(t_n)|^{p+1} dx
= \frac{\rho}{\|u_n(t_n)\|_2^2} E(u_n(t_n))
+ \left( \frac{\sqrt{\rho}}{\|u_n(t_n)\|_2} \right)^2 \left( \frac{\sqrt{\rho}}{\|u_n(t_n)\|_2} \right)^{p+1} \frac{1}{p+1} \int_{\mathbb{R}^N} |u_n(t_n)|^{p+1} dx,
\]
which implies that
\[
\lim_{n \to \infty} E(\tilde{u}_n) = \lim_{n \to \infty} E(u_n(t_n)) = A_\rho.
\]
Hence, \( \{\tilde{u}_n\}_{n=1}^{\infty} \subset M_\rho \) is a minimizing sequence of \( E \), and by Steps 3-5, there exists \( \tilde{v} \in G_\rho \) such that
\[
(3.15) \quad \lim_{n \to \infty} \|\tilde{u}_n - \tilde{v}\|_{H^1(\mathbb{R}^N)} = 0.
\]
By the definition of \( \tilde{u}_n \), we know
\[
(3.16) \quad \lim_{n \to \infty} \|\tilde{u}_n - u_n(t_n)\|_{H^1(\mathbb{R}^N)} = \lim_{n \to \infty} \left( 1 - \frac{\sqrt{\rho}}{\|u_n(t_n)\|_2} \right) \|u_n(t_n)\|_{H^1(\mathbb{R}^N)} = 0.
\]
(3.15) and (3.16) imply that
\[
\lim_{n \to \infty} \|u_n(t_n) - \tilde{v}\|_{H^1(\mathbb{R}^N)} = 0,
\]
which contradicts (3.12). The proof is complete.

**Proof of Theorem 1.5.** The proof follows from that of Theorem 1.4 by line to line, so we only point out the differences.
Step 1. We show that \( A_\rho > -\infty \). For any \( u \in M_\rho \), we have \( \|u\|_2^2 = \rho \), and by Lemmas 2.2 and 2.6, we have for any \( \epsilon > 0 \),
\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \frac{\gamma}{|x|^\alpha} |u|^2 \, dx - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\beta * |u|^q) |u|^q \, dx
\]
\[
\geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \delta(\epsilon, \|u\|_2) - \frac{\epsilon}{2} \|u\|_2^2 - \frac{1}{2q} C(\beta, q) \|u\|_2^{N+\beta-N+2q} \|\nabla u\|_2^{N+\beta-N-\beta}.
\]
(3.17)

For \( 1 + \beta/N < q < 1 + (\beta + 2)/N \), we have \( 0 < Nq - N - \beta < 2 \). Thus, by the Young inequality, we obtain from (3.17) that
\[
E(u) \geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \|\nabla u\|_2^2 - \delta_2(\epsilon, \|u\|_2) \geq -\delta_2(\epsilon, \|u\|_2) > -\infty
\]
by choosing \( \epsilon = 1/4 \).

For \( q = 1 + (\beta + 2)/N \), we have \( Nq - N - \beta = 2 \), \( N + \beta - Nq + 2q = (2\beta + 4)/N \), and \( \|C(\beta, q)\|_2^{(2\beta + 4)/N} \). Thus, we obtain from (3.17) that
\[
E(u) \geq \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \|\nabla u\|_2^2 - \delta(\epsilon, \|u\|_2) - \frac{\epsilon}{2} \left( \frac{\|u\|_2}{\|W_q\|_2} \right)^{2(\beta + 4)/N} \|\nabla u\|_2^2
\]
\[
= \frac{1}{2} \left( 1 - \left( \frac{\|u\|_2}{\|W_q\|_2} \right)^{2(\beta + 4)/N} - \epsilon \right) \|\nabla u\|_2^2 - \delta(\epsilon, \|u\|_2) > -\infty
\]
by choosing \( \epsilon > 0 \) small enough since \( \|u\|_2 < \|W_q\|_2 \).

Step 2. For any \( \rho > 0 \) there exists \( C_1 > 0 \) such that \( A_\rho \leq -C_1 < 0 \). Indeed, for any \( \varphi \in M_\rho \) and \( \lambda > 0 \), we define \( \varphi_\lambda(x) = \lambda^{N/2} \varphi(\lambda x) \). Then \( \|\varphi_\lambda\|_2^2 = \|\varphi\|_2^2 = \rho \) and
\[
E(\varphi_\lambda) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla \varphi_\lambda|^2 \, dx - \lambda^{N-\beta} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^\alpha} \, dx - \frac{\lambda^{Nq-N-\beta}}{2q} \int_{\mathbb{R}^N} (I_\beta * |\varphi|^q) |\varphi|^q \, dx.
\]
If \( \gamma > 0 \) or \( \gamma < 0 \) and \( Nq - N - \beta < \alpha \), by choosing \( \lambda > 0 \) sufficiently small, we obtain \( E(\varphi_\lambda) < 0 \).

Let \( \{u_n\}_{n=1}^\infty \subset M_\rho \) be a minimizing sequence of \( E \), that is,
\[
\|u_n\|_2^2 = \rho \quad \text{and} \quad \lim_{n \to \infty} E(u_n) = A_\rho.
\]
Then \( \{u_n\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^N) \) and there exists a subsequence \( \{u_{n_k}\}_{k=1}^\infty \) such that one of the three possibilities in Lemma 2.3 holds.

Step 3. The vanishing case in Lemma 2.3 does not occur. Suppose by contradiction that
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_{n_k}|^2 \, dx = 0.
\]
Lion’s lemma implies that \( u_{n_k} \to 0 \) in \( L^s(\mathbb{R}^N) \) for any \( s \in (2, 2N/(N-2)) \). Hence,
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_{n_k}|^q) |u_{n_k}|^q \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|u_{n_k}|^2}{|x|^\alpha} \, dx \to 0
\]
according to (2.1) and (2.4), and then
\[
\lim_{k \to \infty} E(u_{n_k}) = \lim_{k \to \infty} \frac{1}{2} \|u_{n_k}\|_2^2 \geq 0,
\]
which contradicts $A_p < 0$.

**Step 4.** The dichotomy case in Lemma 2.3 does not occur. Similarly, we have

$$A_p \leq A_\eta + A_{\rho - \eta}^\infty$$

for any $\eta \in [0, \rho)$

and

$$A_{\theta \eta} \leq \theta A_\eta$$

for any $\eta \in (0, \rho)$ and $\theta \in (1, \rho/\eta]$, 

where $A_0 = 0$, $A_{\rho - \eta}^\infty = \inf_{M_{\rho - \eta}} E^\infty(u)$ and $E^\infty(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2q} \int_{R^N} (I_\beta * |u|^q)|u|^q dx$.

Consequently,

$$A_p < A_\eta + A_{\rho - \eta} \leq A_\eta + A_{\rho - \eta}^\infty$$

for any $\eta \in (0, \rho)$.

Suppose by contradiction that (iii) in Lemma 2.3 holds. By using (3.8) and $u_{n_k}^{(1)} u_{n_k}^{(2)} = 0$, we obtain that

$$\int_{R^N} (I_\beta * |u_{n_k}^{(1)}|) |u_{n_k}^{(1)}| q dx$$

$$= \int_{R^N} (I_\beta * |u_{n_k}^{(1)} + u_{n_k}^{(2)} + v_{n_k})^q|u_{n_k}^{(1)} + u_{n_k}^{(2)} + v_{n_k})^q dx$$

$$= \int_{R^N} (I_\beta * (|u_{n_k}^{(1)} + u_{n_k}^{(2)}|^q + |v_{n_k})^q + C(|u_{n_k}^{(1)} + u_{n_k}^{(2)})^q-1|v_{n_k}| + |u_{n_k}^{(1)} + u_{n_k}^{(2)}||v_{n_k}|^q-1)) dx$$

$$\times (|u_{n_k}^{(1)} + u_{n_k}^{(2)}|^q + |v_{n_k})^q + C(|u_{n_k}^{(1)} + u_{n_k}^{(2)})^q-1|v_{n_k}| + |u_{n_k}^{(1)} + u_{n_k}^{(2)}||v_{n_k}|^q-1)) dx$$

$$= \int_{R^N} (I_\beta * |u_{n_k}^{(1)} + u_{n_k}^{(2)}|^q) |u_{n_k}^{(1)} + u_{n_k}^{(2)}|^q dx + \delta(\epsilon)$$

$$= \int_{R^N} (I_\beta * |u_{n_k}^{(1)}|^q) |u_{n_k}^{(1)}|^q dx + \int_{R^N} (I_\beta * |u_{n_k}^{(2)}|^q) |u_{n_k}^{(2)}|^q dx$$

$$+ 2 \int_{R^N} (I_\beta * |u_{n_k}^{(1)}|^q) |u_{n_k}^{(2)}|^q dx + \delta(\epsilon),$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Since $0 < \beta < N$ and $1 + \beta/N < q < (N + \beta)/(N - 2)$, we can choose a constant $\delta > 0$ sufficiently small such that

$$0 < \beta + \delta < N \quad \text{and} \quad 1 + \frac{\beta + \delta}{N} \leq q \leq \frac{N + \beta + \delta}{N - 2},$$

which combines with $d_{n_k} = \text{dist}\{\text{Supp } u_{n_k}^{(1)}, \text{Supp } u_{n_k}^{(2)}\} \to \infty$ and (2.4) gives that

$$\int_{R^N} \int_{R^N} \frac{|u_{n_k}^{(1)}(x)|^q |u_{n_k}^{(2)}(y)|^q}{|x - y|^{N-\beta}} dxdy$$

$$\leq \frac{1}{d_{n_k}^\delta} \int_{R^N} \int_{R^N} \frac{|u_{n_k}^{(1)}(x)|^q |u_{n_k}^{(2)}|}{|x - y|^{N-\beta-\delta}} dxdy$$

$$\leq C \frac{1}{d_{n_k}^\delta} \left( \int_{R^N} |u_{n_k}|^{2q-\delta} dx \right)^{\frac{1+\delta}{2q-\delta}}$$

$$\leq C \frac{1}{d_{n_k}^\delta} \|u_{n_k}\|_{H^1(R^N)}^{2q} \to 0$$
as $k \to \infty$. Hence,

$$A_\rho = \lim_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{nk}|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_{nk}|^2}{|x|^\alpha} dx - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\beta * |u_{nk}|^q)|u_{nk}|^q dx \right)$$

$$\geq \liminf_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n(1)|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_n(1)|^2}{|x|^\alpha} dx - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\beta * |u_n(1)|^q)|u_n(1)|^q dx \right)$$

$$+ \liminf_{k \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^\alpha} dx - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\beta * |u_n|^q)|u_n|^q dx \right)$$

$$+ \delta(\epsilon)$$

$$\geq A_\sigma + A_\beta^\infty + \delta(\epsilon) \quad \text{or} \quad A_\sigma^\infty + A_\beta + \delta(\epsilon).$$

Letting $\epsilon \to 0$, we obtain that $A_\rho \geq A_\sigma + A_\rho^\infty - \sigma$ or $A_\rho \geq A_\sigma^\infty + A_\rho - \sigma$, which contradicts (3.18).

**Step 5.** From Steps 3 and 4 and Lemma 2.3, we know $u_{nk} \to u$ in $L^2(\mathbb{R}^N)$ and $u_{nk} \to u$ in $H^1(\mathbb{R}^N)$. Consequently, by the H"older inequality, the Sobolev inequality and the Hardy-Littlewood-Sobolev inequality, we know $u_{nk} \to u$ in $L^s(\mathbb{R}^N)$ for $s \in [2, 2^\ast]$ and

$$\int_{\mathbb{R}^N} (I_\beta * |u_{nk}|^q)|u_{nk}|^q dx \to \int_{\mathbb{R}^N} (I_\beta * |u|^q)|u|^q dx.$$ 

Hence, $u$ is a minimizer of $A_\rho$ and $u_{nk} \to u$ in $H^1(\mathbb{R}^N)$.

**Step 6.** Under Assumption A, if $1 + \beta/N < q < 1 + (2 + \beta)/N$ and $u_0 \in H^1(\mathbb{R}^N)$ or $q = 1 + (2 + \beta)/N$ and $\|u_0\|_2 < \|W_0\|_2$, then the Cauchy problem (1.1) admits a global solution $u(t) \in C([0, \infty), H^1(\mathbb{R}^N))$ with $u(0) = u_0$.

**Step 7.** Similarly, $G_\rho$ is orbitally stable. The proof is complete.

**Proof of Theorem 1.6.** It can be done by small modifications of the proof of Theorem 1.4, and we omit it.

**Proof of Theorem 1.7.** It can be done by small modifications of the proof of Theorem 1.5, and we omit it.

**Proof of Theorem 1.8.** It can be done by combining the proof of Theorems 1.4 and 1.5, and we omit it.

**Acknowledgements.** This work was supported by Tianjin Municipal Education Commission with the Grant no. 2017KJ173 “Qualitative studies of solutions for two kinds of nonlocal elliptic equations”.

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