On a Functional Equation of Ruijsenaars

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Abstract

We obtain the general solution of the functional equation

\[ \sum_{I \subseteq \{1,2,\ldots,n\} \mid |I| = k} \left( \prod_{j \notin I} h(x_j - x_i)h(x_i - x_j - i\beta) - \prod_{j \notin I} h(x_i - x_j)h(x_j - x_i - i\beta) \right) = 0. \]

This equation, introduced by Ruijsenaars, guarantees the commutativity of \( n \) operators associated with the quantum Ruijsenaars-Schneider models.

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1 Introduction

The purpose of this paper is to investigate the functional equation

$$\sum_{I \subseteq \{1, 2, \ldots, n\} \atop |I| = k} \left( \prod_{i \in I} h(x_j - x_i) h(x_i - x_j - i\beta) - \prod_{i \in I} h(x_i - x_j) h(x_j - x_i - i\beta) \right) = 0. \quad (1)$$

Here \(\beta\) is an arbitrary positive number and the sum is over all subsets with \(k\) elements. As we will describe shortly, this equation underlies the quantum integrability of the Ruijsenaars-Schneider models. We will establish

**Theorem 1** The general solution of the functional equation (1) analytic in a neighbourhood of the real axis with either a simple pole at the origin or an array of such poles at \(np\) on the real axis \((n \in \mathbb{Z})\) is given by

$$h(x) = b \frac{\sigma(x + \nu)}{\sigma(x) \sigma(\nu)} e^{\alpha x}. \quad (2)$$

Before turning to the proof of this theorem let us place this work in context. Some years ago Ruijsenaars and Schneider [23] initiated the investigation of mechanical models obeying the Poincaré algebra

$$\{H, B\} = P, \quad \{P, B\} = H, \quad \{H, P\} = 0. \quad (3)$$

Here \(H\) will be the Hamiltonian of the system generating time-translations, \(P\) is a space-translation generator and \(B\) the generator of boosts. Their study was motivated in part by seeking mechanical models that described soliton interactions. The models they discovered were found to possess other nice features: they were in fact integrable and a quantum version of them naturally existed. Ruijsenaars and Schneider began with the ansatz

$$H = \sum_{j=1}^{n} \cosh p_j \prod_{k \neq j} f(x_j - x_k), \quad P = \sum_{j=1}^{n} \sinh p_j \prod_{k \neq j} f(x_j - x_k), \quad B = \sum_{j=1}^{n} x_j.$$

With this ansatz and the canonical Poisson bracket \(\{p_i, x_j\} = \delta_{ij}\) the first two Poisson brackets of (3) involving the boost operator \(B\) are automatically satisfied. The remaining Poisson bracket is then

$$\{H, P\} = -\sum_{j=1}^{n} \partial_j \prod_{k \neq j} f^2(x_j - x_k)$$

$$- \frac{1}{2} \sum_{j \neq k} \cosh(p_j - p_k) \prod_{l \neq j} f(x_j - x_l) \prod_{m \neq k} f(x_k - x_m) \left( \partial_j \ln f(x_k - x_j) + \partial_k \ln f(x_j - x_k) \right)$$

and for the independent terms proportional to \(\cosh(p_j - p_k)\) to vanish we require that \(f'(x)/f(x)\) be odd. This entails that \(f(x)\) is either even or odd\(^1\) and in either case \(F(x) = f^2(x)\) is even. Supposing that \(f(x)\) is so constrained, then the final Poisson bracket is equivalent to the functional equation

$$\{H, P\} = 0 \iff \sum_{j=1}^{n} \partial_j \prod_{k \neq j} f^2(x_j - x_k) = 0. \quad (4)$$

\(^1\)Ruijsenaars and Schneider assume \(f(x) = f(-x)\).
Observe that upon dividing (5) by \( \beta \) and letting \( \beta \to 0 \) this yields (6) with \( F(x) = h(x)h(-x) \) when \( k = 1 \).

For \( n = 3 \) equation (6) may be written in the form

\[
\begin{vmatrix}
 1 & 1 & 1 \\
 F(x) & F(y) & F(z) \\
 F'(x) & F'(y) & F'(z)
\end{vmatrix} = 0, \quad x + y + z = 0,
\]

where \( F(x) = f^2(x) \). Ruijsenaars and Schneider [23] showed that \( F(x) = \phi(x) + c \) satisfies (6) and further satisfies (6) for all \( n \). This same functional equation (without assumptions on the parity of the function \( F(x) \)) has arisen in several settings related to integrable systems. It arises when characterising quantum mechanical potentials whose ground state wavefunction (of a given form) is factorisable [13, 24, 19]. More recently it has been shown [4, 8, 21, 2, 12, 13, 14, 15, 16, 17, 18]. The analytic solutions to (3) were characterised by Buchstaber and Perelomov [4] while more recently a somewhat stronger result with considerably simpler proof was obtained by the authors [5]. One has

**Theorem 2** (23) Let \( F \) be a three-times differentiable function satisfying the functional equation (3). Then, up to the manifest invariance \( F(z) \to \alpha F(\delta z) + \beta \), the solutions of (3) are one of \( F(z) = \phi(z + d) \), \( F(z) = e^z \) or \( F(z) = z \). Here \( \phi \) is the Weierstrass \( \phi \)-function and \( 3d \) is a lattice point of the \( \phi \)-function.

Thus the even solutions of (3) are precisely those obtained by Ruijsenaars and Schneider. Until this year the general solution to (3) remained unknown when the authors established

**Theorem 3** (13) The general even solution of (3) amongst the class of meromorphic functions whose only singularities on the Real axis are either a double pole at the origin, or double poles at \( np \) (\( p \) real, \( n \in \mathbb{Z} \)) is:

a) for all odd \( n \) given by the solution of Ruijsenaars and Schneider while

b) for even \( n \geq 4 \) there are in addition to the Ruijsenaars-Schneider solutions the following:

\[
F_1(z) = \sqrt{(\phi(z) - e_2)(\phi(z) - e_3)} = \frac{\sigma_2(z)\sigma_3(z)}{\sigma^2(z)}
\]

\[
= \frac{\theta_3(v)\theta_4(v)}{\theta_1^2(v)} \frac{\theta_1^2(0)}{4\omega^2\theta_3(0)\theta_4(0)} = b\frac{dn(u)}{sn^2(u)}
\]

\[
F_2(z) = \sqrt{(\phi(z) - e_1)(\phi(z) - e_3)} = \frac{\sigma_1(z)\sigma_3(z)}{\sigma^2(z)}
\]

\[
= \frac{\theta_2(v)\theta_4(v)}{\theta_1^2(v)} \frac{\theta_1^2(0)}{4\omega^2\theta_2(0)\theta_4(0)} = b\frac{dn(u)}{sn^2(u)}
\]

\[
F_3(z) = \sqrt{(\phi(z) - e_1)(\phi(z) - e_2)} = \frac{\sigma_1(z)\sigma_2(z)}{\sigma^2(z)}
\]

\[
= \frac{\theta_2(v)\theta_3(v)}{\theta_1^2(v)} \frac{\theta_1^2(0)}{4\omega^2\theta_2(0)\theta_3(0)} = b\frac{cn(u)dn(u)}{sn^2(u)}
\]

\[\text{As an aside we remark that the delta function potential } \delta \delta(x) \text{ of many-body quantum mechanics on the line, which has a factorisable ground-state wavefunction, can be viewed as the } \alpha \to 0 \text{ limit of } -b/a \sinh^2(-x/\alpha + \pi i/3) \text{ with } \pi \alpha = 6b. \text{ Thus all of the known quantum mechanical problems with factorisable ground-state wavefunction are included in (3).}\]
Here
\[
\sigma_\alpha(z) = \frac{\sigma(z + \omega_\alpha)}{\sigma(\omega_\alpha)} e^{-z\zeta(\omega_\alpha)}, \quad u = \sqrt{e_1 - e_3}z, \quad v = \frac{z}{2\omega}, \quad b = e_1 - e_3
\]
with \(\omega_1 = \omega, \omega_2 = -\omega - \omega'\) and \(\omega_3 = \omega'\), and we have given representations in terms of the Weierstrass elliptic functions, theta functions and the Jacobi elliptic functions \[26\]. For appropriate ranges of \(z\) the solutions are real. Their degenerations yield all the even solutions with only a double pole at \(x = 0\) on the real axis. These degenerations may in fact coincide with the degenerations of the Ruijsenaars-Schneider solution. One can straightforwardly verify these new solutions do in fact satisfy (4) for even \(n\) \[3\] but new techniques had to be developed to show these exhaust the solutions.

The models discovered by Ruijsenaars and Schneider not only exhibit an action of the Poincaré algebra but were completely integrable as well. In particular Ruijsenaars and Schneider demonstrated the Poisson commutativity for their solutions of the light-cone quantities
\[
S_{\pm k} = \sum_{I \subseteq \{1,2,\ldots,n\} \mid |I| = k} \exp \left( \pm \sum_{i \in I} p_i \right) \prod_{i \in I, j \notin I} f(x_i - x_j).
\]
(6)
Then \(H = (S_1 + S_{-1})/2\) and \(P = (S_1 - S_{-1})/2\). (Note the even/oddness of the functions \(f(x)\) means that there really are only \(n\) functionally independent quantities.) It is an open problem whether the new solutions of theorem 3 yield integrable systems. We know that these new solutions do not always yield Poisson commuting quantities using the ansatz of Ruijsenaars and Schneider, but as yet we cannot rule out other Poisson commuting conserved quantities \[3\].

Ruijsenaars \[22\] also investigated the quantum version of the classical models he and Schneider introduced. From the outset he sought operator analogues of the light-cone quantities \(\hat{S}_k\). He showed that (for \(k = 1,\ldots,n\))
\[
\hat{S}_k = \sum_{I \subseteq \{1,2,\ldots,n\} \mid |I| = k} \prod_{i \in I, j \notin I} h(x_j - x_i) \exp \left( -\sqrt{-1} \beta \sum_{i \in I} \partial_i \right) \prod_{i \in I, j \notin I} h(x_i - x_j)
\]
pairwise commute if and only if \(\hat{S}_k\) held for all \(k\) and \(n \geq 1\). Further he was able to show \(\hat{S}_k\) led to a solution of \(\hat{S}_k\), the solution being related to the earlier Ruijsenaars-Schneider solution via
\[
\frac{\sigma(x + \nu)\sigma(x - \nu)}{\sigma^2(x)\sigma^2(\nu)} = \phi(\nu) - \phi(x).
\]
Ruijsenaars \[22\] suggested that this solution was “most likely unique” but was unable to prove this. A consequence of our classical analysis are the possible functions \(F(x) = h(x)h(-x)\). A natural question to ask is whether there is a solution to \(\hat{S}_k\) corresponding to our new solutions. If not, then the Ruijsenaars solution is indeed unique. Our theorem proves the uniqueness of the Ruijsenaars solutions.

We shall now turn to the proof of theorem 1 using the transform method developed in \[11\].

2 Proof of Theorem 1

In this section we solve \(\hat{S}_k\) by constructing the Fourier transform of this equation for \(k = 1\). In addition to the function \(F(z) = h(z)h(-z)\) let us introduce
\[
g(z, \beta) = h(-z)h(z - i\beta).
\]
The Fourier transform of \( \mathcal{F} \) will be in terms of the Fourier transform \( \hat{g}(k, \beta) \) of \( g(z, \beta) \). We will show how, after a judicious choice of parametrisation, the Fourier transform of \( \mathcal{F} \) when \( k = 1 \) leads to precisely the same equation encountered when studying the Fourier transform of \( \mathcal{F} \). Theorem 3 then gives the general solutions for \( g(z, \beta) \). We find that we can write

\[
g(z, \beta) = h(-i\beta) \left( F_1(z - i\beta) - F_1(z) - F_1(\nu - i\beta) + F_1(\nu) \right)
\]

where \( F_1(z) = \int F(\nu) d\nu \) (notice the \( \nu \) instead of \( \beta \)). The new solutions of theorem 3 do not have the factorisation property \( \hat{g}(x, \beta) \), whereby we establish that for \( k = 1 \) the only solutions to \( \mathcal{F} \) are \( \hat{g}(x, \beta) \) and \( \hat{g}(x, \beta) \) are given by \( \mathcal{F} \). There are however that these solutions satisfy \( \mathcal{F} \) for all \( k \), and so the theorem will be proved. Our strategy will be to take the Fourier transform for functions of increasing complexity, first considering those functions with only a pole at the origin and vanishing at infinity; next we consider similar functions decaying to a constant at infinity; finally we consider those functions with a periodic array of poles along the real axis including the origin.

Before we derive the equation for \( \hat{g} \) we look at the properties of \( g \) and the conditions that these properties demand of \( \hat{g} \). The original solutions of Ruijsenaars and Schneider for \( F(z) \) can be expressed as

\[
F(z) = A \varphi(z, g_2, g_3) + B,
\]

where \( A, B, g_2 \) and \( g_3 \) are constants. These are even functions of \( z \) with a double pole at the origin. The new solutions of \( \mathcal{F} \) have a similar structure but do not possess the addition of the arbitrary constant \( B \). However, in both cases we can use the constant \( A \) and the scaling properties of \( \varphi(z) \) to restrict our choice of \( F(z) \), without loss of generality, so that \( z^2 F(z) \to -1 \) as \( z \to 0 \) and that when the solution has finite real period we take this to be \( 2\pi \). Let \( \nu \) be a zero of \( F(z) \). Thus for the solutions \( \mathcal{F} \) we may express the constant \( B \) in terms of \( \nu \) as

\[
F(z) = \varphi(\nu) - \varphi(z).
\]

The condition \( F(\nu) = 0 \) then requires \( h(\nu) h(-\nu) = 0 \). We fix \( h(z) \) by demanding that \( h(z) = 0 \) at \( z = -\nu \). This entails \( g(\nu, \beta) = g(-\nu + i\beta, \beta) = 0 \). Also since \( F(z) \) has a double pole at \( z = 0 \), \( h(z) \) must have a simple pole at \( z = 0 \) and we choose \( h(z) \) so that \( zh(z) \to +1 \) as \( z \to 0 \). Thus \( g(z, \beta) \) must have simple poles at \( z = 0 \) and \( z = i\beta \) with \( zg(z) \to -h(-i\beta) \) as \( z \to 0 \) and \( (z - i\beta) g(z) \to h(-i\beta) \) as \( z \to i\beta \).

We will now obtain an equation for \( \hat{g} \) by taking the Fourier transform of \( \mathcal{F} \). Set \( z_j = x_j + iy_j \) and denote by \( E(k, z_n) \) the \( k = 1 \) equation \( \mathcal{F} \), where \( z \) is the vector \( (z_1, z_2, ..., z_{n-1}) \). We define the \((n - 1)\)-dimensional Fourier transform by

\[
\hat{E}(k, z_n, \beta) = \int_{\mathbb{R}^{n-1}} E(z, z_n, \beta) e^{-ik \cdot z} dz.
\]

However, since \( g(z, \beta) \) has a pole at the origin, we replace \( z_j \) by \( z_j + ie_j \) and assume that \( \epsilon_1 > \epsilon_2 > ... > \epsilon_n > 0 \) and that \( \epsilon_1 \) is small. We then assume in the definition of \( \hat{E} \) in \( \mathcal{F} \) that we integrate along the Real axis in the complex \( x_j + iy_j \) plane.

2.1 Functions with infinite real period vanishing at infinity

In the first instance we consider the class of solutions \( g(z, \beta) \) which have infinite real period and tend to zero at infinity with no other singularities on the real axis other than the pole
at the origin. Hence, when \( z_j \) has been replaced by \( z_j + i \epsilon_j \), the integrand will have no other singularities in the domain of integration provided \( \epsilon_1 < \beta \) which we take to be real and positive. The reduction of \( \hat{E} = 0 \) as \( \epsilon_1 \to 0 \), to an equation involving the generalised Fourier transform \( \hat{g}(k, \beta) \) follows the lines of (11). The definition of \( \hat{g} \) is given by

\[
\hat{g}(k, \beta) = \frac{1}{2} \left( \hat{g}_U(k, \beta) + \hat{g}_L(k, \beta) \right),
\]

where

\[
\hat{g}_U(k, \beta) = \int_{-\infty}^{\infty} g(z, \beta) e^{-ikz} dz, \quad \hat{g}_L(k, \beta) = \int_{-\infty}^{\infty} g(z, \beta) e^{-ikz} dz,
\]

are defined respectively to go over and under the pole at \( z = 0 \). Then we have

\[
\hat{g}_L - \hat{g}_U = 2\pi i \times \text{Residue } g(z, \beta)|_{z=0} = -2\pi i h(-i\beta)
\]

and

\[
\hat{g}_U = \hat{g} + i\pi h(-i\beta), \quad \hat{g}_L = \hat{g} - i\pi h(-i\beta).
\]

However the above definition causes a problem as \( \beta \to 0 \) in that

\[
\lim_{\beta \to 0} \hat{g}(k, \beta) \neq \hat{g}(k, 0).
\]

This is because when \( \beta > 0 \) both paths of integration in (12) lie below the pole at \( z = i\beta \), while if we put \( \beta = 0 \) and then use the definitions (11) and (12) we find that the upper path of integration will go above the pole which becomes a double pole at \( z = 0 \) as \( \beta \to 0 \). To overcome this difficulty we define a modified \( \hat{g}_U, \hat{g}_U M \), and a modified generalised Fourier transform \( \hat{g}_M \), by indenting the upper contour in (12) so that it goes over the pole at \( z = i\beta \). Then

\[
\hat{g}_U M = \hat{g}_U - 2\pi i \text{ residue of } g(z, \beta) e^{-ikz}|_{z=i\beta} = \hat{g}_U - 2\pi i h(-i\beta) e^{i\beta k}.
\]

Hence

\[
\hat{g}_M = \hat{g} - \pi i h(-i\beta) e^{i\beta k}.
\]

With these definitions it is easy to show that when \( g = -1/(z(z-i\beta)), \) \( \hat{g}_M = \pi(e^{i\beta k} - 1) \text{ sign}(k)/\beta \) with \( \lim_{\beta \to 0} \hat{g}_M = \pi |k| \), which is the generalised Fourier transform of \(-1/z^2\).

When we take the Fourier transform of (11) for \( k = 1 \) each of the terms \( \prod_{j \neq i} h(x_j - x_i)h(x_i - x_j - i\beta) = \prod_{j \neq i} g(x_i - x_j) \) in the sum reduces to a product of one dimensional Fourier transforms. Using the above definitions, one of four possibilities arise depending on the position of the poles:

\[
\int_{-\infty}^{\infty} g(u + i\epsilon)e^{-iku} du = \hat{g}_M(k) + i\pi h(-i\beta) - i\pi h(-i\beta) e^{i\beta k},
\]
\[
\int_{-\infty}^{\infty} g(u - i\epsilon)e^{-iku} du = \hat{g}_M(k) - i\pi h(-i\beta) + i\pi h(-i\beta) e^{i\beta k},
\]
\[
\int_{-\infty}^{\infty} g(-u + i\epsilon)e^{-iku} du = \hat{g}_M(k) + i\pi h(-i\beta) - i\pi h(-i\beta) e^{-i\beta k},
\]
\[
\int_{-\infty}^{\infty} g(-u - i\epsilon)e^{-iku} du = \hat{g}_M(k) - i\pi h(-i\beta) + i\pi h(-i\beta) e^{-i\beta k}.
\]
Thus for example (relevant to the $n = 3$ case)

\[
g(z_1 - z_2)g(z_1 - z_3) = \int_{\mathbb{R}^2} g(z_1 - z_2 + i(\epsilon_1 - \epsilon_2))g(z_1 - z_3 + i(\epsilon_1 - \epsilon_3))e^{-ik_1z_1 - ik_2z_2}dz_1dz_2
\]

\[
= e^{-i(k_1 + k_2)z_3} \int_{-\infty}^{\infty} dv e^v e^{-i(k_1 + k_2)v} \int_{-\infty}^{\infty} du e^{-i(k_1 + k_2)u} \times [\hat{g}_M(k_1 + k_2) + i\pi h(-i\beta) - i\pi h(-i\beta)e^{\beta(k_1 + k_2)}]
\]

Here we have set $u = z_2 - z_1$, $v = z_1 - z_3$, $\epsilon' = \epsilon_1 - \epsilon_2$ and $\epsilon'' = \epsilon_1 - \epsilon_2$. The common factor of $h(-i\beta)$ in these expressions suggests the rewriting $\hat{g}_M(k, \beta) = -iI(k, \beta)\beta h(-i\beta)$. Then in the limit $\beta \to 0$, $\hat{g}_M(k, 0) = I(k, 0)$. If we use this we find that the $n = 3$ equation (14) can be written as $\sum_1^3 J_j = 0$, where for example

\[
J_1 = \left( I(k_1 + k_2, \beta) \frac{\beta}{\pi} - 1 + e^{\beta(k_1 + k_2)} \right) \left( I(-k_2, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta k_2} \right)
\]

\[
+ \left( I(-k_1 - k_2, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta(k_1 + k_2)} \right) \left( I(k_2, \beta) \frac{\beta}{\pi} + 1 - e^{\beta k_2} \right).
\]

This corresponds to $g(z_1 - z_2)g(z_1 - z_3) - g(z_2 - z_1)g(z_3 - z_1)$ in the sum, with similar definitions for $J_2$ and $J_3$:

\[
J_2 = \left( I(-k_1 - k_2, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta(k_1 + k_2)} \right) \left( I(k_1, \beta) \frac{\beta}{\pi} - 1 + e^{\beta k_1} \right)
\]

\[
+ \left( I(k_1 + k_2, \beta) \frac{\beta}{\pi} - 1 + e^{\beta(k_1 + k_2)} \right) \left( I(-k_1, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta k_1} \right),
\]

\[
J_3 = \left( I(k_1, \beta) \frac{\beta}{\pi} - 1 + e^{\beta k_1} \right) \left( I(k_2, \beta) \frac{\beta}{\pi} - 1 + e^{\beta k_2} \right)
\]

\[
- \left( I(-k_1, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta k_1} \right) \left( I(-k_2, \beta) \frac{\beta}{\pi} + 1 - e^{-\beta k_2} \right).
\]

At this point we introduce what, with hindsight, will prove a judicious change of variable: set

\[
I(k, \beta) = \hat{G}(k, \beta) \frac{(e^{\beta k} - 1)}{(k\beta)}.
\]

This change of variable is suggested by a careful examination of the series for $I(k, \beta) = \sum_0^\infty I_j(k, \beta)$ that results from $\sum_1^3 J_j = 0$ with the expressions above. Whatever, when $\beta \to 0$, $I \to \hat{G}(k, 0)$ and we know (as $F(x)$ is even) that $\hat{G}(k, 0)$ is even. An examination of the equation for $\hat{G}(k, \beta)$ (say by a series expansion) shows further that $\hat{G}(k, \beta)$ itself is even. This fact leads to simplifications. When $\hat{G}(k, \beta)$ is even $J_1$ simplifies to

\[
J_1 = -\frac{(e^{\beta k_1} - 1)}{(k_1 + k_2)k_2} \left( \hat{G}(k_2, \beta) - \pi k_2 \right) \left( \hat{G}(k_1 + k_2, \beta) + \pi(k_1 + k_2) \right).
\]
and similarly for $J_2, J_3$. We then find the exponential factors involving $\beta$ are common to all $J_i$ and so (10) can finally be expressed, in the case $n = 3$ as

\[
\left( k_2 \mathcal{G} (k_1, \beta) + k_1 \mathcal{G} (k_2, \beta) \right) \mathcal{G}(k_1 + k_2, \beta) - (k_1 + k_2) \mathcal{G} (k_1, \beta) \mathcal{G}(k_2, \beta) = \pi^2 k_1 k_2 (k_1 + k_2) .
\]

(17)

Now this is precisely the equation for the Fourier transform of $F(z)$ obtained in [11][eqn. 5.5] when studying [4] for $n = 3$. The only solutions of the required from are $\mathcal{G}(k, \beta) \equiv \mathcal{G}(k) = F(k) = \pi k \coth(\pi k/a)$ and its limit as $a \to 0$ namely $\pi |k|$. Hence

\[
\hat{g}_M(k, \beta) = -i \beta h(-i\beta) \frac{e^{\beta k} - 1}{\beta k} F(k) ,
\]

(18)
or

\[
g(z, \beta) = h(-i\beta) \left( \int F(z - i\beta ) \, dz - \int F(z) \, dz \right) .
\]

(19)

It is then easy to check that for

\[
\hat{F} (k) = \pi k \coth(\pi k/a) , \quad F(z) = -\frac{1}{4} a^2 / \sinh^2 \left( \frac{1}{2} a z \right) ,
\]

(20)

we have

\[
g(z, \beta) = \frac{a}{2} h(-i\beta) \left( \coth \frac{1}{2} a (z - i\beta) - \coth \left( \frac{1}{2} a z \right) \right) \sinh \left( \frac{1}{2} i a \beta \right) \sinh \left( \frac{1}{2} a (z - i\beta) \right) \equiv h(-z) h(z - i\beta) ,
\]

(21)

with $h(z) = 1/2a / \sinh(1/2az)$ giving the appropriate factorisation $F(z) = h(-z) h(z)$.

Similarly (or let $a \to 0$) for $\hat{F} (k) = \pi |k|$ and $F(z) = -1/z^2$ we have

\[
g(z, \beta) = h(-i\beta) \left( \frac{1}{z - i\beta} - \frac{1}{z} \right) = h(-i\beta) \frac{-i\beta}{(z - i\beta)(z - i\beta)} \equiv h(-z) h(z - i\beta) ,
\]

(22)

with $h(z) = 1/z$ and $F(z) = h(-z) h(z)$. Both these factorisations are only defined up to the shift by the exponential $h(z) \to h(z)e^{\alpha z}$ given in [2].

For the cases $n \geq 4$ we also find that the same set of transformations of $\hat{g}_M$ reduce equation (10) to the original equation for the transform $F$ encountered in the study of [4]. For odd $n$ we only have the solutions [19] and [22] while for even $n$ we have in addition to these solutions the solution

\[
\hat{F} (k) = \pi k \tanh \left( \frac{\pi k}{2a} \right) , \quad F(z) = -\frac{a^2 \cosh az}{\sinh^2 az} .
\]

(23)

Now however we have

\[
g(z, \beta) = ah(-i\beta) \left( \frac{1}{\sinh a (z - i\beta)} - \frac{1}{\sinh az} \right) ,
\]

(24)

which cannot be written in the form $g(z, \beta) = h(-z) h(z - i\beta)$. Thus these do not yield solutions to [3].
2.2 Functions with infinite real period constant at infinity

The solutions (8) of Ruijsenaars and Schneider contain the addition of an arbitrary constant that, in the hyperbolic limit, corresponds to the function not vanishing at infinity. The new solutions of (11), of which (23) is an example, do not have this degree of freedom: the hyperbolic degenerations of these solutions all tend to zero at infinity. To deal with functions which do not tend to zero at infinity we must deal with distributional Fourier transforms. The addition of a constant to \( F(z) \) requires the addition of a similar constant to \( g(z, \beta) \), since for example if \( F \to a^2 \) as \( z \to \infty \) then \( g \to a^2 \) as \( z \to \infty \). For \( n = 3 \) we can easily verify that \( g(z, \beta) \to g(z, \beta) + \text{constant leaves (1) invariant} \). However, this is not automatically the case when \( n > 3 \). We find that for \( n > 3 \) the transformation \( g(z, \beta) \to g(z, \beta) + A \), requires extra conditions on \( g \) to leave the equation invariant. When \( n = 4 \), there is only one extra condition which is automatically satisfied for the solution \( g \) corresponding to the solution \( F \) given by Ruijsenaars and Schneider, but not for the solution \( F \) corresponding to the new solutions of (11). We believe that for \( n > 4 \), the Ruijsenaars and Schneider solution automatically satisfy all the extra conditions but that the new solutions given in theorem 3 do not. The addition of an arbitrary constant \( A \), to \( g \), requires the addition of \( A \delta(k) \) to \( \hat{g}(k, \beta) \). For the solution \( \hat{g} \) which are otherwise well behaved functions of \( k \) we can easily obtain the solution for \( g(z, \beta) \) and the corresponding solution for \( h(z) \), when they exist. For example, instead of (20) we add an arbitrary constant to \( F(z) \) and, demanding that \( F(\nu) = 0 \), we have

\[
F(z) = \frac{1}{4} a^2 \left\{ \frac{1}{\sinh^2 \left( \frac{1}{2} a \nu \right)} - \frac{1}{\sinh^2 \left( \frac{1}{2} a z \right)} \right\},
\]

so that we have

\[
g(z, \beta) = \frac{1}{2} ah(-i\beta) \frac{\sinh(-\frac{1}{2}a i \beta)}{\sinh(-\frac{1}{2}a z) \sinh \frac{1}{2} a (z - i \beta)} + A.
\]

Since by definition \( g(z, \beta) = h(-z) h(z - i \beta) \) and \( h(z) \) has a zero at \( z = -\nu \), we have \( g(\nu, \beta) = 0 \), as indicated earlier. Hence

\[
A \equiv A(\beta, \nu) = -\frac{1}{2} ah(-i\beta) \frac{\sinh(-\frac{1}{2}a i \beta)}{\sinh(-\frac{1}{2}a \nu) \sinh \frac{1}{2} a (\nu - i \beta)}.
\]

Then we may express \( g \) as

\[
g(z, \beta) = \frac{1}{2} ah(-i\beta) \frac{\sinh \frac{1}{2} a (z + \nu)}{\sinh(\frac{1}{2}a \nu) \sinh(-\frac{1}{2}a z)} \frac{\sinh \frac{1}{2} a (z + \nu - i \beta)}{\sinh \frac{1}{2} a (z - i \beta)} \frac{\sinh(-\frac{1}{2}a i \beta)}{\sinh \frac{1}{2} a (\nu - i \beta)}
\]

\[
\equiv h(-z) h(z - i \beta),
\]

where

\[
h(z) = \frac{1}{2} a \frac{\sinh \frac{1}{2} a (z + \nu)}{\sinh(\frac{1}{2}a \nu) \sinh(\frac{1}{2}a z)},
\]

with

\[
h(-z) h(z) = a^2 \frac{\sinh \frac{1}{2} a (z - \nu) \sinh \frac{1}{2} a (z + \nu)}{\sinh^2 \left( \frac{1}{2} a \nu \right) \sinh^2 \left( \frac{1}{2} a z \right)} = \frac{a^2}{4} \left\{ \frac{1}{\sinh^2 \left( \frac{1}{2} a \nu \right)} - \frac{1}{\sinh^2 \left( \frac{1}{2} a z \right)} \right\},
\]

8
as required. Also observe that as \( \nu \to \infty \) we have \( h(z) \) tending to the solutions of the previous section times the exponential factor \( e^{az/2} \), and our factorisation is only unambiguous up to such terms.

A similar but easier calculation for \( F(z) = 1/\nu^2 - 1/z^2 \) gives

\[
g(z, \beta) = h(-i\beta) \frac{\nu (-i\beta) (-z + \nu) z + \nu - i\beta}{\nu - i\beta} (z - i\beta)^{-1/2},
\]

so that \( h(z) = (z + \nu) / (z\nu) \) with \( h(z) h(-z) = 1/\nu^2 - 1/z^2 \).

### 2.3 Periodic functions

For \( 2\pi \) periodic functions which have a periodic array of double poles at \( z = 2\pi p \) we have shown in [1] that the appropriate form of the transform \( \hat{g}(k, \beta) \) is \( \sum_{p=-\infty}^{\infty} a_p \delta(k - p) \) corresponding to a Fourier series \( \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} a_p e^{ipz} \) for \( g(z, \beta) \). The equation satisfied by \( \hat{g} \) is the same as that for the non-periodic case, but it is solved only at integer values of the \( \{k_p\} \). Thus, if \( \hat{g}(k, \beta) \) is the solution of the continuous case we have the solution \( a_p (\beta) = \hat{g}(p, \beta) \). The corresponding solution, \( g_{2\pi}(z, \beta) \), is the periodic extension of the continuous case expressed in the form

\[
g_{2\pi}(z, \beta) = \sum_{p=-\infty}^{\infty} g(z - 2\pi p)
\]

where \( g(z) \) is the nonperiodic solution determined by [19] with the particular forms [21], [22], [24], [28] and [31].

For the function \( F(z) = 1/\sinh^2 z \) this corresponds to the \( \varphi \) function. Since without loss of generality we have taken \( F(z) \) to satisfy \( z^2 F(z) = -1 \) as \( z \to 0 \), we can use the scaling property

\[
\varphi(z, g_2, g_3) = a^2 \varphi(az, g_2/a^2, g_3/a^6)
\]

to take \( F(z) = -\varphi(z) \). The addition of a constant to \( F \) gives rise to the addition of a constant to \( g \), thus

\[
F \to F + B \Rightarrow \int F \to \int F + Bz + B_1 \Rightarrow \int g \to \int g - iB\beta h(-i\beta).
\]

Hence a periodic solution corresponding to [21] is

\[
g(z, \beta) = -h(-i\beta) \left( \zeta(z) - \zeta(z - i\beta) + C(\beta) \right)
\]

where again \( C(\beta) \) is an arbitrary function of \( \beta \), which also depends on the parameter \( \nu \), and is determined by the condition \( g(\nu, \beta) = 0 \). Thus

\[
g(z, \beta) = -h(-i\beta) \left\{ \zeta(z) - \zeta(z - i\beta) - \zeta(\nu - i\beta) + \zeta(\nu - i\beta) \right\}
\]

\[
= h(-i\beta) \frac{\sigma(\nu) \sigma(-i\beta) \sigma(z + \nu) \sigma(z + \nu - i\beta)}{\sigma(\nu - i\beta) \sigma(-i\beta) \sigma(z - i\beta) \sigma(\nu)} = h(-z) h(z - i\beta)
\]

with \( h(z) = \sigma(z) = \sigma(z + \nu) / (\sigma(z) \sigma(\nu)) \).

Again the solution for \( g \) for the new solutions of Byatt-Smith and Braden [1] can be written in a form similar to [14] but again cannot be factorised into the product \( h(-z) h(z - i\beta) \). For example consider \( F(z) = -c_n(z) \text{dn}(z)/\text{sn}(z)^2 \), whence \( \int F(z) dz = 1/\text{sn}(z) \). Thus from [14]

\[
g(z, \beta) = h(-i\beta) (1/\text{sn}(z - i\beta) - 1/\text{sn}(z))
\]

and a series approach shows that this cannot be written in factorised form.
2.4 Another Functional Equation

At this stage we have proven the theorem. We have shown that the Fourier transform of \( \mathcal{F} \) with \( k = 1 \) leads to studying

\[
g(z, \beta) \equiv h(-i\beta) (F_1(z - i\beta) - F_1(z) + C(\beta)) = h(-z) h(-i\beta). \tag{37}
\]

Here \( F_1(z) = \int F(z) \, dz \) and \( C(\beta) = -F_1(\nu - i\beta) + F_1(\nu) \), since we stipulated \( g(\nu, \beta) = 0 \). We found that \( F(z) \) had to be a solution given by theorem 3, and that amongst these the only such solutions allowing the desired factorisation were given by \( \mathcal{F} \). We shall conclude by showing that an analysis of \( (37) \) directly yields this result.

We view \( (37) \) as a functional equation for \( C, h \) and \( F_1 \), with \( g \) being consequently determined. We solve this subject to appropriate conditions that we inherit from the original problem:

\[
zh(z) \to a, \quad zF_1(z) \to a, \quad \text{and} \quad g(\nu, \beta) = 0. \tag{38}
\]

The last condition means that \( C(\beta) = -F_1(\nu - i\beta) + F_1(\nu) \), and so \( C(0) = 0 \). We observe that \( (37) \) is invariant under \( F_1(z) \to F_1(z) + Bz + B_1, \quad C(\beta) \to C(\beta) + i\beta B \).

We may fix this freedom by further requiring that

\[
C(0) = C'(0) = 0. \tag{39}
\]

To solve \( (37) \) we expand the equation as a power series in \( \beta \). The first non-trivial terms give sufficient equations to eliminate \( h(z) \) and \( h(-z) \) and derive a third order equation for \( F_1(z) \) in terms of the coefficients \( \{C_0, C_1, b_0, b_1, b_2\} \) in the expansions

\[
C(\beta) = \sum_{j=0}^{\infty} C_j \beta^{j+2} \quad \text{and} \quad h(-i\beta) = \sum_{j=0}^{\infty} b_j (-i\beta)^{j-1}. \tag{40}
\]

The resulting third order equation for \( F_1 \),

\[
(2F_1''(x)F_1'(x) - 3F_1'(x)^2)b_0^2 + 12C_0b_0^2 + 12F_1'(x)^2(b_1^2 - 2b_2b_0) + 24iC_1b_0^2F_1'(x) = 0, \tag{41}
\]

can be integrated to give

\[
F_1(z) = b_0 \zeta(z, g_2, g_3) + z b_0 \varphi(\nu, g_2, g_3), \tag{42}
\]

with \( C \) determined as

\[
C(\beta) = b_0 \zeta(\nu) - b_0 \zeta(\nu - i\beta) + i\beta b_0 \varphi(\nu). \tag{43}
\]

(The remaining constants in this are defined in terms of \( C_0, C_1, b_1 \) and \( b_2 \) below.)

The function \( h(z) \) is then determined by the equation

\[
\frac{h'}{h} = \frac{1}{2} F_1'' + \frac{1}{2} b_0 \varphi'(\nu) + \frac{b_1}{b_0}, \tag{44}
\]

with solution

\[
h(z) = b_0 \frac{\sigma(z + \nu)}{\sigma(z) \sigma(\nu)} e^{az}. \tag{45}
\]
The four constants \( \{C_0, C_1, b_1, b_2\} \) define the four constants appearing in (42-43) namely \( \wp(\nu), \xi(\nu), g_2 \) and \( \wp'(\nu) \), with \( g_3 \) given by \( \{\wp(\nu), \wp'(\nu)\} \) and \( g_2 \). Of course, to satisfy (38) we also require \( b_0 = \alpha \). The relations between the two sets of constants are given by

\[
C_0 = -\frac{1}{2}b_0\wp'(\nu), \quad C_1 = ib_0\left(\wp(\nu) - \frac{1}{2}g_3\right),
\]

\[
b_1 = b_0(\zeta(\nu) + \alpha), \quad b_2 = \frac{1}{2}b_0\left((\zeta(\nu) + \alpha)^2 - \wp(\nu)\right).
\]

The constant \( \alpha \) is arbitrary and does not affect the solution since it is clear that the quotient \( h(\nu) h(z - i\beta) / h(-i\beta) \) is independent of \( \alpha \). This is the ambiguity in the factorisation noted earlier.

We conclude that only the solutions of Ruijsenaars and Schneider, given by (2), yield solutions of (37) and consequently solutions of (1). Thus Theorem 1 is again proved.

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