Spectral Renormalization Group and Local Decay in the Standard Model of the Non-relativistic Quantum Electrodynamics

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Abstract

We prove the limiting absorption principle for the standard model of the non-relativistic quantum electrodynamics (QED) and for the Nelson model describing interactions of electrons with phonons. To this end we use the spectral renormalization group technique on the continuous spectrum in conjunction with the Mourre theory.

I Introduction

The mathematical framework of the theory of non-relativistic matter interacting with the quantized electro-magnetic field (non-relativistic quantum electrodynamics) is well established. It is given in terms of the standard quantum Hamiltonian

\[ H^{SM}_g = \sum_{j=1}^{n} \frac{1}{2m_j} (i \nabla_{x_j} + gA(x_j))^2 + V(x) + H_f \]  

acting on the Hilbert space \( \mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f \), the tensor product of the state spaces of the particle system and the quantized electromagnetic field. Here \( SM \) stands for 'standard model'. The notation above and units we use are explained below. This model describes, in particular, the phenomena of emission and absorption of radiation by systems of matter, such as atoms and molecules, as well as other

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processes of interaction of quantum radiation with matter. It has been extensively studied in the last decade, see the books [50, 26] and reviews [2, 31, 32, 37, 39] and references therein for a partial list of contributions.

For reasonable potentials $V(x)$ the operator $H^{SM}_g$ is self-adjoint and its spectral and resonance structure - and therefore dynamics for long but finite time-intervals - is well understood (see e.g. [1, 16, 25, 27, 28, 29, 30, 48] and references therein for recent results). However, we still know little about its asymptotic dynamics. In particular, the full scattering theory for this operator does not, at present, exist (see, however, [17, 18, 19, 13, 14]).

A key notion connected to the asymptotic dynamics is that of the local decay. This notion also lies at the foundation of the construction of the modern quantum scattering theory. It states that the system under consideration is either in a bound state, or, as time goes to infinity, it breaks apart, i.e. the probability to occupy any bounded region of the physical space tends to zero and, consequently, average distance between the particles goes to infinity. In our case, this means that the photons leave the part of the space occupied by the particle system.

Until recently the local decay for the Hamiltonian $H^{SM}_g$ is proven only for the energies away from $O(g^2)$-neighborhoods of the ground state energy, $e_g$, and the ionization energy. However, starting from any energy, the system eventually winds up in a neighborhood of the ground state energy. Indeed, while the total energy is conserved, the photons carry away the energy from regions of the space where matter is concentrated. Hence understanding the dynamics in this energy interval is an important matter. Recently, the local decay was proven for states in the spectral interval $(\epsilon_g, \epsilon_g + \epsilon_{gap}/12)$ for the Hamiltonian $H^{SM}_g$ [20]. Here $\epsilon_{gap} := \epsilon_1^{(p)} - \epsilon_0^{(p)}$, where $\epsilon_0^{(p)}$ and $\epsilon_1^{(p)}$ are the ground state and the first excited state energies of the particle system. In this paper we give another prove of this fact.

However, the main goal of this paper is to develop a new approach to time-dependent problems in the non-relativistic QED which combines the spectral renormalization group (RG), developed in [8, 9, 5] (see also [21]), with more traditional spectral techniques such as Mourre estimate. The key here is the result that the stronger property of the limiting absorption principle (LAP) propagates along the RG flow.

Now, we explain the units and notation employed in (11). We use the units in which the Planck constant divided by $2\pi$, speed of light and the electron mass are equal to 1($\hbar = 1$, $c = 1$ and $m = 1$). In these units the electron charge is equal to $-\sqrt{\alpha}$, where $\alpha = \frac{e^2}{4\pi \hbar c} \approx \frac{1}{137}$ is the fine-structure constant, the distance, time and energy are measured in the units of $\hbar/mc = 3.86 \cdot 10^{-11}$ cm, $\hbar/mc^2 = 1.29 \cdot 10^{-21}$ sec and $mc^2 = 0.511 MeV$, respectively (natural units). We show below that one can set $g := \alpha^{3/2}$. 
Our particle system consists of \( n \) particles of masses \( m_j \) (the ratio of the mass of the \( j \)-th particle to the mass of an electron) and positions \( x_j \), where \( j = 1, \ldots, n \). We write \( x = (x_1, \ldots, x_n) \). The total potential of the particle system is denoted by \( V(x) \). The Hamiltonian operator of the particle system alone is given by

\[
H_p := -\sum_{j=1}^{n} \frac{1}{2m_j} \Delta x_j + V(x),
\]

where \( \Delta x_j \) is the Laplacian in the variable \( x_j \). This operator acts on a Hilbert space of the particle system, denoted by \( \mathcal{H}_p \), which is either \( L^2(\mathbb{R}^3n) \) or a subspace of this space determined by a symmetry group of the particle system.

The quantized electromagnetic field is described by the quantized (in the Coulomb gauge) vector potential

\[
A(y) = \int (e^{iky} a(k) + e^{-iky} a^*(k)) \frac{\chi(k) d^3k}{(2\pi)^3 \sqrt{2|k|}},
\]

where \( \chi \) is an ultraviolet cut-off: \( \chi(k) = 1 \) in a neighborhood of \( k = 0 \) and it vanishes sufficiently fast at infinity, and its dynamics, by the quantum Hamiltonian

\[
H_f = \int d^3k \; a^*(k) \omega(k) a(k),
\]

both acting on the Fock space \( \mathcal{H}_f \equiv F \). Above, \( \omega(k) = |k| \) is the dispersion law connecting the energy, \( \omega(k) \), of the field quantum with wave vector \( k \), \( a^*(k) \) and \( a(k) \) denote the creation and annihilation operators on \( F \) and the right side can be understood as a weak integral. The families \( a^*(k) \) and \( a(k) \) are operator valued generalized, transverse vector fields:

\[
a^\#(k) := \sum_{\lambda \in \{0,1\}} e_\lambda(k)a^\#_\lambda(k),
\]

where \( e_\lambda(k) \) are polarization vectors, i.e. orthonormal vectors in \( \mathbb{R}^3 \) satisfying \( k \cdot e_\lambda(k) = 0 \), and \( a^\#_\lambda(k) \) are scalar creation and annihilation operators satisfying standard commutation relations. See Supplement for a brief review of the definitions of the Fock space, the creation and annihilation operators acting on it and the definition of the operator \( H_f \).

In the natural units the Hamiltonian operator is of the form (1.1) but with \( g = e \) and with \( V(x) \) being the total Coulomb potential of the particle system. To obtain expression (1.1) we rescale this original Hamiltonian appropriately (see [10]). Then we relax the restriction on \( V(x) \) and consider standard generalized \( n \)-body potentials (see e.g. [44]), \( V(x) = \sum_i W_i(\pi_i x) \), where \( \pi_i \) are a linear maps.
from $\mathbb{R}^{3n}$ to $\mathbb{R}^{m_i}$, $m_i \leq 3n$ and $W_i$ are Kato-Rellich potentials (i.e. $W_i(\pi, x) \in L^{p_i}(\mathbb{R}^{m_i})+L^\infty(\mathbb{R}^{3n})$) with $p_i = 2$ for $m_i \leq 3$, $p_i > 2$ for $m_i = 4$ and $p_i \geq m_i/2$ for $m_i > 4$, see [47, 41]). In order not to deal with the problem of center-of-mass motion, which is not essential in the present context, we assume that either some of the particles (nuclei) are infinitely heavy or the system is placed in an external potential field.

One verifies that $H_f$ defines a positive, self-adjoint operator on $\mathcal{F}$ with purely absolutely continuous spectrum, except for a simple eigenvalue $0$ corresponding to the eigenvector $\Omega$ (the vacuum vector, see Supplement). Thus for $g = 0$ the low energy spectrum of the operator $H^{SM}_0$ consists of the branches $[\epsilon_i^{(p)}]$, and of the eigenvalues $\epsilon_i^{(p)}$ sitting at the top of the branch points (‘thresholds’) of the continuous spectrum. The absence of gaps between the eigenvalues and thresholds is a consequence of the fact that the photons (and the phonons) are massless. This leads to hard and subtle problems in the perturbation theory, known collectively as the infrared problem.

As was mentioned above, in this paper we prove the local decay property for the Hamiltonian $H^{SM}_g$. In fact, we prove a slightly stronger property - the limiting absorption principle - which states that the resolvent sandwiched by appropriate weights has Hölder continuous limit on the spectrum. To be specific, let $B$ denote the self-adjoint generator of dilatations on the Fock space $\mathcal{F}$. It can be expressed in terms of creation- and annihilation operators as

$$B = \frac{i}{2} \int d^3k \ a^*(k) \left\{ k \cdot \nabla_k + \nabla_k \cdot k \right\} a(k).$$

We further extend it to the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}$. Let $\langle B \rangle := (1 + B^2)^{1/2}$. Our goal is to prove the following

**Theorem I.1.** Let $g \ll \epsilon_g^{(p)}$ and let $\Delta \subset (\epsilon_g, \epsilon_g + \frac{1}{12} \epsilon_g^{(p)})$, where $\epsilon_g$ is the ground state energy of $H$. Then

$$\langle B \rangle^{-\theta} (H^{SM}_g - \lambda \pm i0)^{-1} \langle B \rangle^{-\theta} \in C^\nu(\Delta).$$

for $\theta > 1/2$ and $0 < \nu < \theta - \frac{1}{2}$.

The above theorem has the following consequence. (In what follows functions of self-adjoint operators are defined by functional calculus.)

**Corollary I.2.** For $\Delta$ as above and for any function $f(\lambda)$ with $\text{supp} f \subseteq \Delta$ and for $\nu < \theta - \frac{1}{2}$, we have

$$\|\langle B \rangle^{-\theta} e^{-iHt} f(H) \langle B \rangle^{-\theta}\| \leq Ct^{-\nu}.$$
The statement follows from (I.6) and the formula

$$\langle B \rangle^{-\theta} e^{-iHt} f(H) \langle B \rangle^{-\theta} = \int_{-\infty}^{\infty} d\lambda f(\lambda)e^{-i\lambda t}\text{Im}\langle B \rangle^{-\theta}(H - \lambda - i0)^{-1}\langle B \rangle^{-\theta}$$

(see e.g. [47] and a detailed discussion in [20]).

**Remark I.3.** Let $\Sigma_p := \inf \sigma(H_p)$. We expect that the method of this paper can be extended to the energy interval $\sigma(H) \setminus \sigma_{pp}(H)$ for the Nelson model and $(\sigma(H) \setminus \sigma_{pp}(H)) \cap (-\infty, \Sigma_p - \varepsilon]$ for some $\varepsilon > 0$, for QED.

Previously the limiting absorption principle and local decay estimates were proven in [9, 11] for the standard model of non-relativistic QED and for the Nelson model away from neighborhoods of the ground state energy and ionization threshold. In [22, 23] they were proven for the Nelson model near the ground state energy and for all values of the coupling constant, but under rather stringent assumptions, including that on the infra-red behavior of the coupling functions (see also [8, 10, 43, 49] for earlier works). Finally, as was mentioned above, it was proven in a neighbourhood of the ground state energy in [20].

Our approach consists of three steps. First, following [48], we use a generalized Pauli-Fierz transform to map the QED Hamiltonian Eq. (I.1) into a new Hamiltonian $H_{g}^{PF}$ whose interaction has a better, in some sense, infra-red behaviour. To this new Hamiltonian we apply sufficiently many iterations of the renormalization map obtaining at the end a rather simple Hamiltonian which we investigate further with the help of the Mourre estimate. This proves LAP for the latter Hamiltonian. Since, as we prove in this paper, the renormalization map preserves the LAP property we conclude from this that the Hamiltonian $H_{g}^{PF}$ enjoys the LAP property as well. The size of the interval and the number of iterations of the RG map depends on the distance of a spectral point of interest to the ground state energy.

In this paper we consider also the Nelson model. In that model the total system consisting of the particle system coupled to the quantized field is described by the Hamiltonian

$$H_{g}^{N} = H_{0}^{N} + I_{g}^{N}, \quad \text{(I.8)}$$

acting on the state space, $\mathcal{H} = \mathcal{H}_{p} \otimes \mathcal{F}$, where now $\mathcal{F}$ is the Fock space for phonons, i.e. spinless, massless Bosons. Here $g$ is a positive parameter - a coupling constant - which we assume to be small, and

$$H_{0}^{N} = H_{p}^{N} + H_{f}, \quad \text{(I.9)}$$

where $H_{p}^{N} = H_{p}$ and $H_{f}$ are given in (I.2) and (I.4), respectively, but, in the last case, with the scalar creation and annihilation operators, $a$ and $a^{*}$, and the
interaction operator is

\[ I_g = g \int \frac{d^3k}{|k|} \kappa(k) \left\{ e^{-ikx} a^*(k) + e^{ikx} a(k) \right\} \]  \hspace{1cm} (I.10)

(we can also treat terms quadratic in \( a \) and \( a^* \) but for the sake of exposition we leave such terms out). Here, \( \kappa = \kappa(k) \) is a real function with the property that

\[ \|\kappa\|_\mu := \left( \int \frac{d^3k}{|k|^{3+2\mu}} |\kappa(k)|^2 \right)^{1/2} < \infty, \]  \hspace{1cm} (I.11)

for some (arbitrarily small, but) strictly positive \( \mu > 0 \). In the following, we fix \( \kappa \) with \( \|\kappa\|_\mu = 1 \) and vary \( g \). It is easy to see that the operator \( I_g \) is symmetric and bounded relative to \( H_0^N \), in the sense of Kato \[47, 41\], with an arbitrarily small constant. Thus \( H_0^N \) is self-adjoint on the domain of \( H_0^N \) for arbitrary \( g \).

Of course, for the Nelson model we can take an arbitrary dimension \( d \geq 1 \) rather than the dimension 3. Our approach can handle the interactions quadratic in creation and annihilation operators, \( a \) and \( a^* \), as it is the case for the operator \( H_{SM}^g \). All the results mentioned above for the standard model Hamiltonian \( H_{SM}^g \) holds also for the Nelson model one, \( H_N^g \) with \( \mu > 0 \).

In order not complicate matters unnecessary we will think about the creation and annihilation operators used below as scalar operators rather than operator-valued transverse vector functions. We explain at the end of Appendix how to reinterpret the corresponding expression for the vector - photon - case.

\section{Generalized Pauli-Fierz Transform}

We describe the generalized Pauli-Fierz transform mentioned in the introduction (see \[48\]). We define the following Hamiltonian

\[ H_g^{PF} := e^{-igF(x)} H_{SM}^g e^{igF(x)}, \]  \hspace{1cm} (II.1)

which we call the generalized Pauli-Fierz Hamiltonian. In order to keep notation simple, we present this transformation in the one-particle, \( n = 1 \), case:

\[ F(x) = \sum_\lambda \int \left( \bar{f}_{x,\lambda}(k)a_\lambda(k) + f_{x,\lambda}(k)a_\lambda^*(k) \right) \frac{d^3k}{\sqrt{|k|}}, \]  \hspace{1cm} (II.2)

with the coupling function \( f_{x,\lambda}(k) \) chosen as

\[ f_{x,\lambda}(k) := \frac{e^{-ikx} \chi(k)}{(2\pi)^3 \sqrt{2|k|}} \varphi(|k|^{1/4} \varepsilon_\lambda(k) \cdot x). \]
The function $\varphi$ is assumed to be $C^2$, bounded, having a bounded second derivative and satisfying $\varphi'(0) = 1$. We compute

$$H_g^{PF} = \frac{1}{2}(p - gA_1(x))^2 + V_g(x) + H_f + gG(x)$$  \hspace{1cm} (II.3)

where $A_1(x) = A(x) - \nabla F(x)$, $V_g(x) := V(x) + 2g^2 \sum \omega |f_{x,\lambda}(k)|^2 d^3k$ and

$$G(x) := -i \sum \int \omega(\bar{f}_{x,\lambda}(k)a_{\lambda}(k) - f_{x,\lambda}(k)a_{\lambda}^*(k)) \frac{d^3k}{\sqrt{|k|}}$$  \hspace{1cm} (II.4)

(The terms $gG$ and $V_g - V$ come from the commutator expansion $e^{-igF(x)}H_fe^{igF(x)} = -ig[F, H_f] - g^2[F, [F, H_f]]$.) Observe that the operator family $A_1(x)$ is of the form

$$A_1(x) = \sum \int (e^{ikx}a_{\lambda}(k) + e^{-ikx}a_{\lambda}^*(k)) \frac{\chi_{\lambda,x}(k) d^3k}{(2\pi)^3 \sqrt{2|k|}}$$  \hspace{1cm} (II.5)

where the coupling function $\chi_{\lambda,x}(k)$ is defined as follows

$$\chi_{\lambda,x}(k) := e_{\lambda}(k)e^{-ikx}\chi(k) - \nabla_x f_{x,\lambda}(k).$$

It satisfies the estimates

$$|\chi_{\lambda,x}(k)| \leq \text{const min}(1, \sqrt{|k|} \langle x \rangle),$$  \hspace{1cm} (II.6)

with $\langle x \rangle := (1 + x^2)^{1/2}$, and

$$\int \frac{d^3k}{|k|} |\chi_{\lambda,x}(k)|^2 < \infty.$$  \hspace{1cm} (II.7)

Using the fact that the operators $A_1$ and $G$ have much better infra-red behavior than the original vector potential $A$ we can use our approach and prove the limiting absorption principle for $H_g^{PF}$ and $B$:

$$\langle B \rangle^{-\theta}(H_g^{PF} - z)^{-1}\langle B \rangle^{-\theta} \text{ Hölder continuous in } z.$$  \hspace{1cm} (II.8)

Now we show that estimate (II.8) and the additional restriction on the spectral interval imply the limiting absorption principle for $H_g^{SM}$. Let $B_1 := e^{-igF(x)}Be^{igF(x)}$. We compute

$$B_1 = B + gC$$  \hspace{1cm} (II.9)

where $C := -i[F(x), B]$. Note that the operator $C$ contains a term proportional to $x$. Now, let a function $f$ be supported in $(-\infty, \Sigma_p)$. Then, using that $(H_g^{SM} - z)^{-1} = e^{igF(x)}(H_g^{PF} - z)^{-1}e^{-igF(x)}$, we obtain
\begin{align*}
\langle B \rangle^{-\theta} f(\mathcal{H}_g^{SM})^2 (\mathcal{H}_g^{SM} - z)^{-1} \langle B \rangle^{-\theta} &= D E(z) D^* \\
\text{where } D := \langle B \rangle^{-\theta} f(\mathcal{H}_g^{SM}) (B_i)^{\theta} e^{igf(x)} \text{ and } E(z) := \langle B \rangle^{-\theta} (\mathcal{H}_g^{PF} - z)^{-1} \langle B \rangle^{-\theta}.
\end{align*}

The operator $D$ is bounded by standard operator calculus estimates and the fact that $e^{\delta(x)} f(\mathcal{H}_g^{SM})$ is bounded for $\delta > 0$ sufficiently small. Furthermore, the operator-family $E(z)$ is Hölder continuous by the assumed result. Now, for $z \in (-\infty, \Sigma_p - \varepsilon]$ for some $\varepsilon > 0$ the previous conclusion remains true even if remove the cut-off function $f(\mathcal{H}_g^{SM})$.

We mention for further references that the operator (I.13) can be written as

\begin{equation}
\mathcal{H}_g^{PF} = \mathcal{H}_0^{PF} + I_g^{PF},
\end{equation}

where $\mathcal{H}_0 = \mathcal{H}_0 + 2g^2 \sum_{\lambda} \int \omega |f_{x,\lambda}(k)|^2 d^3 k + g^2 \sum_{\lambda} \int \frac{|\chi_{\lambda}(k)|^2}{(2\pi)^6} d^3 k$, with $\mathcal{H}_0$ defined in (I.9), and $I_g^{PF}$ is defined by this relation. Note that the operator $I_g^{PF}$ contains linear and quadratic terms in the creation and annihilation operators, with the coupling functions (form-factors) in the linear terms satisfying estimate (II.6) and with the coupling functions in the quadratic terms satisfying a similar estimate. Moreover, the operator $\mathcal{H}_0^{PF}$ is of the form $\mathcal{H}_0^{PF} = \mathcal{H}_p^{PF} + \mathcal{H}_f$ where

\begin{equation}
\mathcal{H}_p^{PF} := \mathcal{H}_p + 2g^2 \sum_{\lambda} \int |k||f_{x,\lambda}(k)|^2 d^3 k + g^2 \sum_{\lambda} \int \frac{|\chi_{\lambda}(k)|^2}{|k|} d^3 k
\end{equation}

where $\mathcal{H}_p$ is given in (I.2).

\section{The Smooth Feshbach-Shur Map}

In this section, we review and extend, in a simple but important way, the method of isospectral decimations or Feshbach-Schur maps introduced in [8, 9] and refined in [5]. For further extensions see [24]. At the root of this method is the isospectral smooth Feshbach-Shur map acting on a set of closed operators and mapping a given operator to one acting on much smaller space which is easier to handle.

Let $\chi, \overline{\chi}$ be a partition of unity on a separable Hilbert space $\mathcal{H}$, i.e. $\chi$ and $\overline{\chi}$ are positive operators on $\mathcal{H}$ whose norms are bounded by one, $0 \leq \chi, \overline{\chi} \leq 1$, and

\footnote{In [8, 9, 5] this map is called the Feshbach map. As was pointed out to us by F. Klopp and B. Simon, the invertibility procedure at the heart of this map was introduced by I. Schur in 1917; it appeared implicitly in an independent work of H. Feshbach on the theory of nuclear reactions in 1958, where the problem of perturbation of operator eigenvalues was considered. See [24] for further discussion and historical remarks.}
\( \chi^2 + \overline{\chi}^2 = 1 \). We assume that \( \chi \) and \( \overline{\chi} \) are nonzero. Let \( \tau \) be a (linear) projection acting on closed operators on \( \mathcal{H} \) s.t. operators from its image commute with \( \chi \) and \( \overline{\chi} \). We also assume that \( \tau(1) = 1 \). Assume that \( \tau \) and \( \chi \) (and therefore also \( \overline{\chi} \)) leave \( \text{dom}(H) \) invariant \( \text{dom}(\tau(H)) = \text{dom}(H) \) and \( \chi \text{dom}(H) \subset \text{dom}(H) \). Let \( \overline{\tau} := 1 - \tau \) and define

\[
H_{\tau,\chi} := \tau(H) + \chi^\# \overline{\tau}(H) \chi^\#.
\]

where \( \chi^\# \) stands for either \( \chi \) or \( \overline{\chi} \).

Given \( \chi \) and \( \tau \) as above, we denote by \( D_{\tau,\chi} \) the space of closed operators, \( \mathcal{H} \), on \( \mathcal{H} \) which belong to the domain of \( \tau \) and satisfy the following conditions:

\( \tau(H) \chi \) is (bounded) invertible on \( \text{Ran} \overline{\chi} \),

\( \overline{\tau}(H) \chi \) and \( \chi \overline{\tau}(H) \) extend to bounded operators on \( \mathcal{H} \).

(For more general conditions see [5, 24].)

Denote \( H_0 := \tau(H) \) and \( W := \overline{\tau}(H) \). Then \( H_0 \) and \( W \) are two closed operators on \( \mathcal{H} \) with coinciding domains, \( \text{dom}(H_0) = \text{dom}(W) = \text{dom}(H) \), and \( H = H_0 + W \). We remark that the domains of \( \chi W \chi, \overline{\chi} W \overline{\chi}, H_{\tau,\chi}, \) and \( H_{\overline{\tau},\chi} \) all contain \( \text{dom}(H) \).

The smooth Feshbach-Schur map (SFM) maps operators on \( \mathcal{H} \) to operators on \( \mathcal{H} \) by \( H \mapsto F_{\tau,\chi}(H) \), where

\[
F_{\tau,\chi}(H) := H_0 + \chi W \chi - \chi W \overline{\tau} H_{\tau,\chi}^{-1} \overline{\tau} W \chi.
\]

Clearly, it is defined on the domain \( D_{\tau,\chi} \).

Remarks

- The definition of the smooth Feshbach-Schur map given above is the same as in [21] and differs from the one given in [5]. In [5] the map \( F_{\tau,\chi}(H) \) is denoted by \( F_\chi(H, \tau(H)) \) and the pair of operators \( (H, T) \) are referred to as a Feshbach pair.

- The Feshbach-Schur map is obtained from the smooth Feshbach-Schur map by specifying \( \chi = \) projection and, usually, \( \tau = 0 \).

We furthermore define the maps entering some identities involving the Feshbach-Schur map:

\[
Q_{\tau,\chi}(H) := \chi - \overline{\chi} H_{\tau,\chi}^{-1} \overline{\tau} W \chi,
\]

\[
Q_{\tau,\chi}^\#(H) := \chi - \chi W \overline{\tau} H_{\tau,\chi}^{-1} \tau \chi.
\]

Note that \( Q_{\tau,\chi}(H) \in \mathcal{B}(\text{Ran} \chi, \mathcal{H}) \) and \( Q_{\tau,\chi}^\#(H) \in \mathcal{B}(\mathcal{H}, \text{Ran} \chi) \).

The smooth Feshbach map of \( H \) is isospectral to \( H \) in the sense of the following theorem.
Theorem III.1. Let $\chi$ and $\tau$ be as above. Then we have the following results.

(i) $0 \in \rho(H) \iff 0 \in \rho(F_{\tau,\chi}(H))$, i.e. $H$ is bounded invertible on $\mathcal{H}$ if and only if $F_{\tau,\chi}(H)$ is bounded invertible on $\text{Ran} \chi$.

(ii) If $\psi \in \mathcal{H} \setminus \{0\}$ solves $H\psi = 0$ then $\varphi := \chi\psi \in \text{Ran} \chi \setminus \{0\}$ solves $F_{\tau,\chi}(H)\varphi = 0$.

(iii) If $\varphi \in \text{Ran} \chi \setminus \{0\}$ solves $F_{\tau,\chi}(H)\varphi = 0$ then $\psi := Q_{\tau,\chi}(H)\varphi \in \mathcal{H} \setminus \{0\}$ solves $H\psi = 0$.

(iv) The multiplicity of the spectral value $\{0\}$ is conserved in the sense that $\dim \ker H = \dim \ker F_{\tau,\chi}(H)$.

(v) If one of the inverses, $H^{-1}$ or $F_{\tau,\chi}(H)^{-1}$, exists then so is the other and they are related as

$$H^{-1} = Q_{\tau,\chi}(H) F_{\tau,\chi}(H)^{-1} Q_{\tau,\chi}(H)^{\#} + \chi H_{\tau,\chi}^{-1} \chi, \quad (\text{III.6})$$

and

$$F_{\tau,\chi}(H)^{-1} = \chi H^{-1} \chi + \chi T^{-1} \chi.$$  

This theorem is proven in [5] (see [24] for a more general result). Now we establish a key result relating smoothness of the resolvent of an operator on its continuous spectrum with smoothness of the resolvent of its image under a smooth Feshbach-Schur map. Let $B_\theta := \langle \theta \rangle^{-\theta}$. In what follows $\Delta$ stands for an open interval in $\mathbb{R}$.

Theorem III.2. Assume a self-adjoint operator $B$ and a $C^\infty$ family $H(\lambda)$, $\lambda \in \Delta$, of closed operators satisfy the following conditions: $\forall \lambda \in \Delta, H_{\tau,\chi}(\lambda) \in D_{\tau,\chi}$ and

$$\text{ad}_B^j(A) \text{ is bounded } \forall j \leq 1, \quad (\text{III.7})$$

where $A$ stands for one of the operators $A = \chi, \bar{\chi}, \chi W, W\chi, \partial^k_\lambda(\chi H_{\tau,\chi}(\lambda)^{-1}\bar{\chi}) \forall k$.

If $H(\lambda) \in \text{dom}(F_{\tau,\chi})$, then for any $\nu \geq 0$ and $0 < \theta \leq 1$,

$$B_\theta(F_{\tau,\chi}(H(\lambda)) - i0)^{-1} B_\theta \in C^\nu(\Delta) \Rightarrow B_\theta(H(\lambda) - i0)^{-1} B_\theta \in C^\nu(\Delta). \quad (\text{III.8})$$

Proof. We use identity (III.6) with $H$ replaced by $H(\lambda) - i\varepsilon$. Since $\tau(1) = 1$ we have that $(H(\lambda) - i\varepsilon)_{\tau,\chi}^{\#} = H(\lambda)_{\tau,\chi}^{\#} - i\varepsilon$, where $\chi^{\#}$ is either $\chi$ or $\bar{\chi}$. Furthermore, on $\text{Ran} \chi$, the operator family $[(H(\lambda) - i\varepsilon)_{\tau,\chi}]^{-1}$ is differentiable in $\lambda$ and analytic in $\varepsilon$ and can be expanded as

$$[(H(\lambda) - i\varepsilon)_{\tau,\chi}]^{-1} = [H(\lambda)_{\tau,\chi}]^{-1} + i\varepsilon[H(\lambda)_{\tau,\chi}]^{-1}\chi^2[H(\lambda)_{\tau,\chi}]^{-1} + O(\varepsilon^2).$$
This implies the relation
\[
\lim_{\varepsilon \to 0} B_\theta[F_{r,\chi}(H(\lambda) - i\varepsilon)]^{-1} B_\theta = B_\theta[F_{r,\chi}(H(\lambda)) - i0]^{-1} B_\theta. \tag{III.9}
\]
Conditions (III.7) and the formula \(B_\theta = C_\theta \int_0^\infty \frac{d\omega}{\omega^{1/2}} (\omega + 1 + B^2)^{-1}\), where \(C_\theta := [\int_0^\infty \frac{d\omega}{\omega^{1/2}} (\omega + 1)^{-1}]^{-1}\), imply that the operators
\[
B_\theta \chi B_\theta^{-1}, B_\theta \chi B_\theta^{-1}, B_\theta[H(\lambda)_{r,\chi}]^{-1} B_\theta^{-1} \tag{III.10}
\]
and the transposed operators (i.e., \(B_\theta^{-1} \chi B_\theta^*, \) etc.) are bounded and \(C^\infty(\Delta)\) in \(\lambda\). This property shows that \(B_\theta^{-1} Q B_\theta^*\) and \(B_\theta^{-1} Q^* B_\theta^*\) are bounded and smooth in \(\lambda \in \Delta\). This together with (III.9), \(H(\lambda) \in \text{dom}(F_{r,\chi})\) and (III.6) implies the theorem.

IV A Banach Space of Hamiltonians

We construct a Banach space of Hamiltonians on which the renormalization transformation is defined. Let \(\chi_1(r) \equiv \chi_{r \leq 1}\) be a smooth cut-off function s.t. \(\chi_1 = 1\) for \(r \leq 1\), \(= 0\) for \(r \geq 11/10\) and \(0 \leq \chi_1(r) \leq 1\) and \(\sup |\partial^s_r \chi_1(r)| \leq 30 \forall r\) and for \(n = 1, 2\). We denote \(\chi_{r}(\rho) \equiv \chi_{r \leq \rho} := \chi_1(r/\rho) \equiv \chi_{r/\rho \leq 1}\) and \(\chi_{\rho} \equiv \chi_{H_\rho \leq \rho}\).

Let \(B_1^n\) denotes the unit ball in \(\mathbb{R}^{3d}, I := [0, 1]\) and \(m, n \geq 0\). Given functions \(w_{0,0} : [0, \infty) \to \mathbb{C}\) and \(w_{m,n} : I \times B_1^{m+n} \to \mathbb{C}, m + n > 0\), we consider monomials, \(W_{m,n} = W_{m,n}[w_{m,n})\), in the creation and annihilation operators of the form \(W_{0,0} := w_{0,0}[H_f]\) (defined by the operator calculus), for \(m = n = 0\), and
\[
W_{m,n}[w_{m,n}] := \int_{B_1^{m+n}} \frac{dk_{(m,n)}}{|k_{(m,n)}|^{1/2}} a^*(k_{(m)}) w_{m,n}[H_f; k_{(m)}, n] a(k_{(n)}), \tag{IV.1}
\]
for \(m + n > 0\). Here we used the notation
\[
k_{(m)} := (k_1, k_2, k_m) \in \mathbb{R}^{dm}, \quad a^*(k_{(m)}) := \prod_{i=1}^m a^*(k_i), \tag{IV.2}
\]
\[
k_{(m,n)} := (k_{(m)}, k_{(n)}), \quad dk_{(m,n)} := \prod_{i=1}^m d^d k_i \prod_{i=1}^n d^d k_{i}, \tag{IV.3}
\]
\[
|k_{(m,n)}| := |k_{(m)}| \cdot |k_{(n)}|, \quad |k_{(m)}| := |k_1| \cdots |k_m|. \tag{IV.4}
\]

We assume that for every \(m, n\) with \(m + n > 0\) the function \(w_{m,n}[r, k_{(m,n)}]\) is \(s\) times continuously differentiable in \(r \in I\), for almost every \(k_{(m,n)} \in B_1^{m+n}\), and weakly differentiable in \(k_{(m,n)} \in B_1^{m+n}\), for almost every \(r \in I\). As a function of \(k_{(m,n)}\), it is totally symmetric w. r. t. the variables \(k_{(m)} = (k_1, \ldots, k_m)\) and \(\tilde{k}_{(n)} = (\tilde{k}_1, \ldots, \tilde{k}_n)\) and obeys the norm bound
\[
\|w_{m,n}\|_{\mu,s} := \sum \|\partial^s_r (k_i k_{(n)}) a_{(m)}\|_{\mu} < \infty, \tag{IV.5}
\]
where \( q := (q_1, \ldots, q_{m+n}) \), \((k\partial_k)^q := \prod_{i=1}^{m+n} (k_j \cdot \nabla_{k_j})^{q_i}\), with \( k_{m+j} := \tilde{k}_j \), and where the sum is taken over the indices \( n \) and \( q \) satisfying \( 0 \leq n + |q| \leq s \) and where \( \mu \geq 0 \) and
\[
\|w_{m,n}\|_\mu := \max_j \sup_{r \in I, k_{(m,n)} \in B_{m+n}^\mu} \|k_j|^{-\mu} w_{m,n}[r; k_{(m,n)}]\]. \tag{IV.6}
\]

Here and in what follows \( k_j \) is the \( j \)-th 3-dimensional components of the \( k \)-vector \( k_{(m,n)} \) over we take the supremum. For \( m+n = 0 \) the variable \( r \) ranges in \([0, \infty)\) and we assume that the following norm is finite:
\[
\|w_{0,0}\|_{\mu,s} := |w_{0,0}(0)| + \sum_{1 \leq n \leq s} \sup_{r \in I} \|\partial_n^r w_{0,0}(r)\| \tag{IV.7}
\]
(for \( s = 0 \) we drop the sum on the r.h.s.). (This norm is independent of \( \mu \) but we keep this index for notational convinience.) The Banach space of these functions is denoted by \( \mathcal{W}_{\mu,s} \). Moreover, \( W_{m,n}[w_{m,n}] \) stresses the dependence of \( W_{m,n} \) on \( w_{m,n} \). In particular, \( W_{0,0}[w_{0,0}] := w_{0,0}[H_f] \).

We fix three numbers \( \mu, 0 < \xi < 1 \) and \( s \geq 0 \) and define Banach space
\[
\mathcal{W}_{\mu,s}^{\mu,s} \equiv \mathcal{W}_{\mu,s}^{\mu,s} := \bigoplus_{m+n \geq 0} \mathcal{W}_{\mu,s}^{\mu,s}, \tag{IV.8}
\]
with the norm
\[
\|w\|_{\mu,s,\xi} := \sum_{m+n \geq 0} \xi^{-(m+n)} \|w_{m,n}\|_{\mu,s} < \infty. \tag{IV.9}
\]

Clearly, \( \mathcal{W}_{\mu,s}^{\mu,s'} \subset \mathcal{W}_{\mu,s}^{\mu,s} \) if \( \mu' \geq \mu, s' \geq s \) and \( \xi' \leq \xi \).

**Remark IV.1.** Though we use the same notation, the Banach spaces, \( \mathcal{W}_{\xi,\mu,s} \), etc., introduced above differ from the ones used in [28 ?]. The latter are obtained from the former by setting \( q = 0 \) in (IV.5). To extend estimates of [28 ?] to the present setting one has to estimate the effect of the derivatives \((k\partial_k)^q\) which is straightforward.

The following basic bound, proven in [2], links the norm defined in (IV.6) to the operator norm on \( \mathcal{B}[\mathcal{F}] \).

**Theorem IV.2.** Fix \( m, n \in \mathbb{N}_0 \) such that \( m + n \geq 1 \). Suppose that \( w_{m,n} \in \mathcal{W}_{\mu,0}^{\mu,0} \), and let \( W_{m,n} \equiv W_{m,n}[w_{m,n}] \) be as defined in (IV.7). Then \( \forall \rho > 0 \)
\[
\|(H_f + \rho)^{-m/2} W_{m,n} (H_f + \rho)^{-n/2}\| \leq \|w_{m,n}\|_0, \tag{IV.10}
\]
and therefore
\[
\|\chi_{\rho} W_{m,n} \chi_{\rho}\| \leq \frac{\rho^{(m+n)(1+\mu)}}{\sqrt{m!n!}} \|w_{m,n}\|_\mu, \tag{IV.11}
\]
where \( \| \cdot \| \) denotes the operator norm on \( \mathcal{B}^{\mathcal{F}} \).
Theorem [IV.2] says that the finiteness of $\|w_{m,n}\|_\mu$ insures that $W_{m,n}$ defines a bounded operator on $B[\mathcal{F}]$.

Now with a sequence $w := (w_{m,n})_{m+n \geq 0}$ in $\mathcal{W}^{\mu,s}$ we associate an operator by setting

$$H(w) := W_{0,0}(w) + \sum_{m+n \geq 1} \chi_1 W_{m,n}(w) \chi_1,$$  \hspace{1cm} (IV.12)

where we write $W_{m,n}(w) := W_{m,n}(w_{m,n})$. This form of operators on the Fock space will be called the generalized normal (or Wick) form. Theorem [IV.2] shows that the series in (IV.12) converges in the operator norm and obeys the estimate

$$\|H(w) - W_{0,0}(w)\| \leq \xi \|w_1\|_{\mu,0,\xi},$$  \hspace{1cm} (IV.13)

for any $w = (w_{m,n})_{m+n \geq 0} \in \mathcal{W}^{\mu,0}$. Here $w_1 = (w_{m,n})_{m+n \geq 1}$. Hence we have the linear map

$$H : w \rightarrow H(w)$$  \hspace{1cm} (IV.14)

from $\mathcal{W}^{\mu,0}$ into the set of closed operators on the Fock space $\mathcal{F}$. Furthermore the following result was proven in [2].

**Theorem IV.3.** For any $\mu \geq 0$ and $0 < \xi < 1$, the map $H : w \rightarrow H(w)$, given in (IV.12), is one-to-one.

Define the spaces $\mathcal{W}^{\mu,s}_{op} := H(\mathcal{W}^{\mu,s})$ and $\mathcal{W}^{\mu,s}_{mn,op} := H(\mathcal{W}^{\mu,s}_{mn})$. Sometimes we display the parameter $\xi$ as in $\mathcal{W}^{\mu,s}_{op,\xi} := H(\mathcal{W}^{\mu,s}_{\xi})$. Theorem [IV.3] implies that $H(\mathcal{W}^{\mu,s})$ is a Banach space under the norm $\|H(w)\|_{\mu,s,\xi} := \|w\|_{\mu,s,\xi}$. Similarly, the other spaces defined above are Banach spaces in the corresponding norms.

Recall that $B$ denotes the dilation generator on the Fock space $\mathcal{F}$ (see (I.5)). Let

$$\chi_{\rho} \equiv \chi_{H_f \leq \rho} \quad \text{and} \quad \overline{\chi}_{\rho} \equiv \chi_{H_f \geq \rho}$$  \hspace{1cm} (IV.15)

be a smooth partition of unity, $\chi_{\rho}^2 + \overline{\chi}_{\rho}^2 = 1$. Let $F_{\rho} := F_{\tau \chi_{\rho}}$. We have

**Lemma IV.4.** Let $\chi^\#_{\rho}$ be either $\chi_{\rho}$ or $\overline{\chi}_{\rho}$. If $H \in \mathcal{W}^{\mu,1}_{op}$, then the operators

$$\text{ad}_B^j(\chi^\#_{\rho}), \quad H_f^{-1}\text{ad}_B^j(W_{00}) \quad \text{and} \quad \text{ad}_B^j(H - W_{00})$$

are bounded \hspace{1cm} (IV.16)

for $j \leq 1$. In particular, condition (III.7) with $\tau(H) := W_{00}$, and therefore property (III.8), with $\chi = \chi_{\rho}$, hold for $H(\lambda) \in C^\infty(\Delta, \mathcal{W}^{\mu,1}_{op}) \cap \text{dom}(F_{\rho})$.

**Proof.** The result follows from the following relations

$$[B, a^\#(k)] = \pm i(k \cdot \nabla_k + \frac{d}{2})a^\#(k),$$  \hspace{1cm} (IV.17)

$$i[B, H_f] = H_f, \quad i[B, f(H_f)] = H_f f'(H_f).$$  \hspace{1cm} (IV.18)
Using these relations we show, in particular, that if \( H \in \mathcal{W}_\mu^{\xi,1} \), then for any \( j \leq 1 \), \( \text{ad}_B^j(W) \in \mathcal{W}_\mu^{1-j,\xi,\xi'} \) \( \forall \xi' < \xi \) and

\[
\|\text{ad}_B(H_{mn})\|_{\mathcal{W}_\mu^{0,0}} \leq c(m + n + 1)\|H_{mn}\|_{\mathcal{W}_\mu^{1,1}}. \tag{IV.19}
\]

Eqn (III.25) implies that \( \text{ad}_B^j(\chi^\#) \) and \( H_f^{-1}\text{ad}_B^j(T) \) are bounded and Eqn (IV.19) implies that \( \text{ad}_B^j(W) \) are bounded, for \( j \leq 1 \).

## V The Renormalization Transformation \( \mathcal{R}_\rho \)

In this section we present an operator-theoretic renormalization transformation based on the smooth Feshbach-Schur map related closely to the one defined in [5] and [8, 9]. We fix the index \( \mu \) in our Banach spaces at some positive value \( \mu > 0 \).

The renormalization transformation is homothetic to an isospectral map defined on a subset of a suitable Banach space of Hamiltonians. It has a certain contraction property which insures that (upon an appropriate tuning of the spectral parameter) its iteration converges to a fixed-point (limiting) Hamiltonian, whose spectral analysis is particularly simple. Thanks to the isospectrality of the renormalization map, certain properties of the spectrum of the initial Hamiltonian can be studied by analyzing the limiting Hamiltonian.

The renormalization map is defined below as a composition of a decimation map, \( F_\rho \), and two rescaling maps, \( S_\rho \) and \( A_\rho \). Here \( \rho \) is a positive parameter - the photon energy scale - which will be chosen later.

The \textit{decimation of degrees of freedom} is done by the smooth Feshbach map, \( F_\tau,\chi \). Except for the first step, the decimation map will act on the Banach space \( \mathcal{W}_{\mu,s}^s \). The operators \( \tau \) and \( \chi \) will be chosen as

\[
\tau(H) = W_{00} := w_{00}(H_f) \quad \text{and} \quad \chi = \chi_\rho \equiv \chi_{\rho^{-1}H_f \leq 1}, \tag{V.1}
\]

where \( H = H(w) \) is given in Eqn (IV.12). With \( \tau \) and \( \chi \) identified in this way we will use the notation

\[
F_\rho \equiv F_{\tau,\chi_\rho}. \tag{V.2}
\]

The following lemma shows that the domain of this map contains the following polydisc in \( \mathcal{W}_{\mu,s}^{\mu,s} \),

\[
\mathcal{D}^{\mu,s}(\alpha, \beta, \gamma) := \left\{ H(w) \in \mathcal{W}_{\mu,s}^{\mu,s} \mid |w_{0,0}[0]| \leq \alpha, \right. \\
\left. \text{sup}_{r \in [0, \infty)} |w_{0,0}[r] - 1| \leq \beta, \quad \|w_1\|_{\mu,s,\xi} \leq \gamma \right\}, \tag{V.3}
\]

for appropriate \( \alpha, \beta, \gamma > 0 \). Here \( w_1 := (w_{m,n})_{m+n \geq 1} \).
Lemma V.1. Fix $0 < \rho < 1$, $\mu > 0$, and $0 < \xi < 1$. Then it follows that the polidisc $D^{\mu,1}(\rho/8, 1/8, \rho/8)$ is in the domain of the Feshbach map $F_\rho$.

Proof. Let $H(w) \in D^{\mu,1}(\rho/8, 1/8, \rho/8)$. We remark that $W := H[w] - W_{0,0}[w]$ defines a bounded operator on $\mathcal{F}$, and we only need to check the invertibility of $H(w)\chi_\rho$ on $\text{Ran} \chi_\rho$. Now the operator $T + E = W_{0,0}[w]$ is invertible on $\text{Ran} \chi_\rho$ since for all $r \in [3\rho/4, \infty)$

$$\text{Re}w_{0,0}[r] \geq r - |w_{0,0}[r] - r| \geq r(1 - \sup_r |w'_{0,0}[r] - 1|) - |w_{0,0}[0]| \geq \frac{3\rho}{4}(1 - 1/8) - \frac{\rho}{8} \geq \frac{\rho}{2}. \quad (V.4)$$

On the other hand, by (IV.11), $|W| \leq \xi \rho/8 \leq \rho/8$. Hence $\text{Re}(W_{0,0}[w] + W) \geq \frac{\rho}{3}$ on $\text{Ran} \chi_\rho$, i.e. $H(w)\chi_\rho$ is invertible on $\text{Ran} \chi_\rho$. \hfill \Box

We introduce the scaling transformation $S_\rho : \mathcal{B}[\mathcal{F}] \rightarrow \mathcal{B}[\mathcal{F}]$, by

$$S_\rho(1) := 1, \quad S_\rho(a^\#(k)) := \rho^{-d/2} a^\#(\rho^{-1}k), \quad (V.5)$$

where $a^\#(k)$ is either $a(k)$ or $a^*(k)$, and $k \in \mathbb{R}^d$. On the domain of the decimation map $F_\rho$ we define the renormalization map $\mathcal{R}_\rho$ as

$$\mathcal{R}_\rho := \rho^{-1} S_\rho \circ F_\rho. \quad (V.6)$$

Remark V.2. The renormalization map above is different from the one defined in [5]. The map in [5] contains an additional change of the spectral parameter $\lambda := -\langle H \rangle_\Omega$.

We mention here some properties of the scaling transformation. It is easy to check that $S_\rho(H_f) = \rho H_f$, and hence

$$S_\rho(\chi_\rho) = \chi_1 \quad \text{and} \quad \rho^{-1} S_\rho(H_f) = H_f, \quad (V.7)$$

which means that the operator $H_f$ is a fixed point of $\rho^{-1}S_\rho$. Further note that $E \cdot 1$ is expanded under the scaling map, $\rho^{-1}S_\rho(E \cdot 1) = \rho^{-1}E \cdot 1$, at a rate $\rho^{-1}$. (To control this expansion it is necessary to suitably restrict the spectral parameter.)

Now we show that the interaction $W$ contracts under the scaling transformation. To this end we remark that the scaling map $S_\rho$ restricted to $\mathcal{W}^{\mu,s}_{op}$ induces a scaling map $s_\rho$ on $\mathcal{W}^{\mu,s}$ by

$$\rho^{-1}S_\rho(H(w)) =: H(s_\rho(w)), \quad (V.8)$$

where $s_\rho(w) := (s_\rho(w_{m,n}))_{m+n \geq 0}$, and it is easy to verify that, for all $(m, n) \in \mathbb{N}_0^2$,

$$s_\rho(w_{m,n})[r, k_{m,n}] = \rho^{m+n-1} w_{m,n} [\rho r, \rho k_{m,n}], \quad (V.9)$$
We note that by Theorem IV.2, the operator norm of \( W_{m,n} [s_\rho(w_{m,n})] \) is controlled by the norm
\[
\| s_\rho(w_{m,n}) \|_\mu = \max_j \sup_{r \in I, k \in B_{m+n}} \rho^{m+n-1} \left| \frac{w_{m,n}[\rho r, \rho k_{m,n}]}{|k_j|^\mu} \right| \rho^{m+n-1} \| w_{m,n} \|_\mu.
\]
Hence, for \( m + n \geq 1 \), we have
\[
\| s_\rho(w_{m,n}) \|_\mu \leq \rho^\mu \| w_{m,n} \|_\mu. \tag{V.10}
\]
Since \( \mu > 0 \), this estimate shows that \( S_\rho \) contracts \( \| w_{m,n} \|_\mu \) by at least a factor of \( \rho^\mu < 1 \). The next result shows that this contraction is actually a dominating property of the renormalization map \( R_\rho \) along the ‘stable’ directions. Below, recall, \( \chi_1 \) is the cut-off function introduced at the beginning of Section III. Define the constant
\[
C_\chi := \frac{4}{3} \left( \sum_{n=0}^{2} \sup_{r} |\partial_r^n \chi_1| + \sup_{r} |\partial_r \chi_1|^2 \right) \leq 200. \tag{V.11}
\]

**Theorem V.3.** Let \( \epsilon_0 : H \to \langle H \rangle_\Omega \) and \( \mu > 0 \) (see (V.4)). Then for the absolute constant \( C_\chi \) given in (V.11) and for any \( s \geq 1 \), \( 0 < \rho < 1/2 \), \( \alpha, \beta \leq \frac{\rho}{8} \) and \( \gamma \leq \frac{\rho}{8c_\chi} \) we have
\[
R_\rho - \rho^{-1} \epsilon_0 : D^{\mu,s}(\alpha, \beta, \gamma) \to D^{\mu,s}(\alpha', \beta', \gamma'), \tag{V.12}
\]
continuously, with \( \xi := \frac{\sqrt{2}}{4C_\chi} \) (in the definition of the corresponding norms) and
\[
\alpha' = 3C_\chi \left( \frac{\gamma^2}{2\rho} \right), \beta' = \beta + 3C_\chi \left( \frac{\gamma^2}{2\rho} \right), \gamma' = 128C_\chi^2 \rho^s \gamma. \tag{V.13}
\]
With some modifications, this theorem follows from [5], Theorem 3.8 and its proof, especially Equations (3.104), (3.107) and (3.109). For the norms (IV.5) with \( q = 0 \) it is presented in [48], Appendix I. A generalization to the \( q > 0 \) case is straightforward.

**Remark V.4.** Subtracting the term \( \rho^{-1} \epsilon_0 \) from \( R_\rho \) allows us to control the expanding direction during the iteration of the map \( R_\rho \). In [5] such a control was achieved by using the change of the spectral parameter \( \lambda \) which controls \( \langle H \rangle_\Omega \) (see remark in Appendix I).
Proposition V.5. Let $\Delta$ be an open interval in $\mathbb{R}$, $\mu > 0$ and let $\rho$ and $\xi$ be as in Theorem V.3. Then for $H(\lambda) \in C^\infty(\Delta, \mathcal{D}^{\mu, 1}(\alpha, \beta, \gamma))$, with $\alpha, \gamma < \frac{\rho}{8}$, $\beta \leq \frac{1}{8}$, the following is true for $1 \geq \theta > 0$ and $\nu \geq 0$

$$B_\theta(R_\rho(H(\lambda)) - i0)^{-1}B_\theta \in C^\nu(\Delta) \Rightarrow B_\theta(H(\lambda) - i0)^{-1}B_\theta \in C^\nu(\Delta).$$  

(V.14)

Proof. By Theorem V.3, $\forall \lambda \in \Delta, H(\lambda) \in \text{dom}(R_\rho)$. Then Lemma IV.4 and invariance of the operator $B_\theta$ under the rescaling $S_\rho$ imply the result.  

VI Renormalization Group

In this section we describe some dynamical properties of the renormalization group $R^n_\rho \forall n \geq 1$ generated by the renormalization map $R_\rho$. A closely related iteration scheme is used in [5]. First, we observe that $\forall w \in \mathbb{C}, R_\rho(wH_f) = wH_f$ and $R_\rho(w1) = \frac{1}{\rho}w1$. Hence we define $M_{fp} := \mathbb{C}H_f$ and $M_u := \mathbb{C}1$ as candidates for a manifold of the fixed points of $R_\rho$ and an unstable manifold for $M_{fp} := \mathbb{C}H_f$. The next theorem identifies the stable manifold of $M_{fp}$ which turns out to be of the (complex) codimension 1 and is foliated by the (complex) co-dimension 2 stable manifolds for each fixed point in $M_{fp}$. This implies in particular that in a vicinity of $M_{fp}$ there are no other fixed points and that $M_u$ is the entire unstable manifold of $M_{fp}$.

We introduce some definitions. As an initial set of operators we take $D := \mathcal{D}^{\mu, 2}(\alpha_0, \beta_0, \gamma_0)$ with $\alpha_0, \beta_0, \gamma_0 \ll 1$. (The choice $s = 2$ of the smoothness index in the definition of the polidiscs is dictated by the needs of the Mourre theory applied in the next section.) We also let $D_s := \mathcal{D}^{\mu, 2}(0, \beta_0, \gamma_0)$ (the subindex $s$ stands for 'stable', not to be confused with the smoothness index $s$ which in this section is taken to be 2). We fix the scale $\rho$ so that

$$\alpha_0, \beta_0, \gamma_0 \ll \rho \leq \min\left(\frac{1}{2}, C^2 \chi\right)$$  

(VI.1)

where, recall, the constant $C^2 \chi$ is appears in Theorem V.3 and is defined in (VI.11). Below we will use the $n-$th iteration of the numbers $\alpha_0, \beta_0$ and $\gamma_0$ under the map (VI.13):

$$\alpha_n := c \left(\rho^{-1}(c\rho^\mu)^{n-1}\gamma_0\right)^2,$$

$$\beta_n = \beta_0 + \sum_{j=1}^{n-1} c \left(\rho^{-1}(c\rho^\mu)^{j}\gamma_0\rho\right)^2,$$

$$\gamma_n = (c\rho^\mu)^n \gamma_0.$$
For \( H \in D \) we denote \( H_u := \langle H \rangle \Omega \) and \( H_s := H - \langle H \rangle \Omega 1 \) (the unstable- and stable-central-space components of \( H \), respectively). Note that \( H_s \in D_s \).

Recall that a complex function \( f \) on an open set \( D \) in a complex Banach space \( W \) is said to be analytic if \( \forall \xi \in W, f(H + \tau \xi) \) is analytic in the complex variable \( \tau \) for \( |\tau| \) sufficiently small (see [12]). Our analysis uses the following result from [21]:

**Theorem VI.1.** Let \( \delta_n := \nu_n \rho^n \) with \( 4\alpha_n \leq \nu_n \leq \frac{1}{18} \). There is an analytic map \( e : D_s \rightarrow \mathbb{C} \) s.t. \( e(H) \in \mathbb{R} \) for \( H = H^* \) and

\[
U_{\delta_n} \subset \text{dom}(\mathcal{R}_\rho^n) \text{ and } \mathcal{R}_\rho^n(U_{\delta_n}) \subset D^{n,2}(\rho/8, \beta_n, \gamma_n)
\]

where \( U_{\delta} := \{ H \in D | |e(H_s) + H_u| \leq \delta \} \). Moreover, \( \forall H \in U_{\delta_n} \) and \( \forall n \geq 1 \), there are \( E_n \in \mathbb{C} \) and \( w_n(r) \in \mathbb{C} \) s.t. \( |E_n| \leq 2\nu_n, |w_n(r) - 1| \leq \beta_n, w_n \) is \( C^2 \),

\[
\mathcal{R}_\rho^n(H) = E_n + w_n(H_f)H_f + O_{W_{op}}(\gamma_n),
\]

\( E_n \) and \( w_n(r) \) are real if \( H \) is self-adjoint and, as \( n \to \infty \).

Moreover, one can show that \( w_n(r) \) converge in \( L^\infty \) to some number (constant function) \( w \in \mathbb{C} \) (21).

This theorem implies that \( M_{fp} := \mathbb{C}H_f \) is (locally) a manifold of the fixed points of \( \mathcal{R}_\rho \) and \( M_u := \mathbb{C}1 \) is an unstable manifold and the set

\[
M_s := \bigcap_n U_{\delta_n} = \{ H \in D | e(H_s) = -H_u \}
\]

is a local stable manifold for the fixed point manifold \( M_{fp} \) in the sense that \( \forall H \in M_s \exists w \in \mathbb{C} \) s.t.

\[
\mathcal{R}_\rho^n(H) \rightarrow wH_f \text{ in the sense of } W_{op}^s
\]
as \( n \to \infty \). Moreover, \( M_s \) is an invariant manifold for \( \mathcal{R}_\rho : M_s \subset \text{dom}(\mathcal{R}_\rho) \) and \( \mathcal{R}_\rho(M_s) \subset M_s \), though we do not need this property here and therefore we do not show it. The next result reveals the spectral significance of the map \( e \):

**Theorem VI.2.** Let \( H \in D \). Then the number \( E := e(H_s) + H_u \) is an eigenvalue of the operator \( H \). Moreover, if \( H \) is self-adjoint, then it is the ground state energy of \( H \).

Theorems V.2 and V.3 were proven in [48] for somewhat simper Banach spaces which do not contain the derivatives \((k \partial_k)^n\). However, an extension to the Banach spaces which are used in this paper is straightforward and is omitted here.
VII  Mourre Estimate

In this section we prove the Mourre estimate for the operator-family $H^{(n)}(\lambda) := R^n_\rho (H)$ with $\lambda := - H_u$. This gives the limiting absorption principle for $H^{(n)}(\lambda)$. The latter is then transferred with the help of Theorem III.2 to the limiting absorption principle for the operator $H$. In Section ?? this limiting absorption principle will be connected to the limiting absorption principle for the family $H_g - \lambda$, where $H_g$ is either $H_g^{PF}$ or $H_g^N$.

**Theorem VII.1.** Let $H(\lambda) = H(\lambda)^* \in C^\infty (\Delta, D^{\mu,2}(\alpha, \beta, \gamma))$, where $\Delta$ is an open interval in $\mathbb{R}$, and $\Delta^\delta := [\delta, \infty)$. If $\delta > \gamma$ and $\beta \leq \frac{1}{3}$, then

$$B_\theta (H(\lambda) - i 0)^{-1} B_\theta \in C^\nu (\Delta \cap E^{-1}(\Delta^\delta)), \quad (VII.1)$$

where $E : \lambda \rightarrow E(\lambda)$ with $E(\lambda) := \langle H(\lambda) \rangle_{\Omega}$, for any and $1/2 < \theta \leq 1$ and $\nu < \theta - \frac{1}{2}$.

**Proof:** In what follows we omit the argument $\lambda$. Let $E := w_{0,0} [0], T := w_{0,0} [H_f] - w_{0,0} [0]$ and $W := \sum_{m+n \geq 1} \chi_1 W_{m,n} [\omega] \chi_1$, so that $H = E 1 + T + W$. Let $H_1 := H - E = T + W$. We write $i [H_1, B] = \tilde{T} + \tilde{W}$, where $\tilde{T} := i [T, B] = T' (H_f) H_f$ and $\tilde{W} := i [W, B]$. By relation (III.28) we have for $s = 2$

$$||\tilde{W}||_{\mathcal{W}_{op}^{\mu,s-1}} \leq c \gamma,$$

where the $\xi$-parameter in the norm on the l.h.s. should be taken slightly smaller than the $\xi$-parameter in the Banach space $\mathcal{W}_{op}^{\mu,s}$ for $W$. The shift in the smoothness index from $s$ to $s - 1$ is due to the fact that the coupling functions for the operator $i [W, B]$ are $(k \cdot \nabla_k + \frac{3(m+n)}{2}) w_{m,n} (r, k)$, where $k := k^{(m,n)}$, and therefore loose one derivative compared to the coupling functions, $w_{m,n}(r, k)$, of $W$.

We write

$$i [H_1, B] = H_1 + \tilde{T} - \frac{1}{2} T + \tilde{W} - \frac{1}{2} W.$$

Remembering that the operator norm is dominated by the $\mathcal{W}_{op}^{\mu,0}$ - norm we see that the last two terms are bounded as

$$||\tilde{W} - \frac{1}{2} W|| \leq C \gamma. \quad (VII.2)$$

Furthermore using the estimate $|T'(r) - 1| < \beta$ and the definition of $\tilde{T}$ we find

$$\tilde{T}(r) - \frac{1}{2} T(r) \geq (1 - \beta) r - \frac{1}{2} (1 + \beta) r = \frac{1}{2} (1 - 3 \beta) r$$

and therefore

$$\tilde{T} - \frac{1}{2} T \geq \inf_{0 \leq r \leq \infty} \left( \tilde{T}(r) - \frac{1}{2} T(r) \right)$$
This gives \([H_1, B] \geq \frac{1}{2} H_1 - c\gamma\) and therefore \(\Delta' := (\frac{1}{2} \delta, \infty), \delta \gg \gamma,\)

\[
E_{\Delta'}(H_1) i [H_1, B] E_{\Delta'}(H_1) \geq \frac{1}{4} \delta E_{\Delta'}(H_1)^2.
\]

(VII.3)

This proves the Mourre estimate for the operator \(H_1 \equiv H_1(\lambda)\).

Moreover, since \(H(\lambda) \in C^\infty(\Delta, D^{\mu,2}(\alpha, \beta, \gamma)),\) we have that the commutators \([H_1, B] \) and \([[H_1, B], B]\) are bounded relative to the operator \(H_1\) (this is guaranteed by taking the index \(s = 2\) for the polidisc \(D^{\mu,s}(\alpha, \beta, \gamma)\)). Hence the standard Mourre theory is applicable and gives Hölder continuity in the spectral parameter \(\sigma\) as well as in the "operator \(H_1(\lambda)\)”, i.e. in \(\lambda\) (see [44]):

\[
B_\theta R_1(\lambda, \sigma) E_{\Delta'}(H_1(\lambda)) B_\theta \in C^\nu(\Delta \times \mathbb{R}),
\]

(VII.4)

where \(\nu < \theta - 1/2\), where we restored the argument \(\lambda\) in our notation and where \(R_1(\lambda, \sigma) := (H_1(\lambda) - \sigma)^{-1}\). Since

\[
B_\theta R_1(\lambda, \sigma) B_\theta = B_\theta R_1(\lambda, \sigma) E_{\Delta'}(H_1(\lambda)) B_\theta + B_\theta R_1(\lambda, \sigma)(1 - E_{\Delta'}(H_1(\lambda))) B_\theta
\]

(VII.5)

and since the last term on the right hand side is \(C^\nu(\Delta)\) in \(\lambda\) and \(C^\infty(\Delta^\delta)\) in \(\sigma\) we conclude from (VII.4) that

\[
B_\theta R_1(\lambda, \sigma) B_\theta \in C^\nu(\Delta \times \Delta^\delta).
\]

(VII.6)

Now take \(\sigma = E(\lambda) + i0\). Since by the condition of the theorem \(E(\lambda) := \langle H(\lambda) \rangle_{\Omega} \in C^\infty(\Delta)\) we conclude that (VII.1) holds.

\(\square\)

In the previous section the parameter \(\delta_n\) was allowed to change in a certain range (see Theorem V.2). In this section we make a particular choice of \(\delta_n\), namely \(\delta_n := \frac{1}{18} \rho^n\). Recall the definition of the set \(U_\delta\) in Theorem VI.1.

**Theorem VII.2.** Assume (VI.1). Let \(n \geq 1, \delta_n := \frac{1}{18} \rho^n\) and let \(H = H^* \in U_{\delta_n}\) and \(\Delta_{\delta_n} := [e(H_s) + \frac{2}{3} \delta_n, e(H_s) + \delta_n]\). Then

\[
B_\theta (H_s - \lambda - i0)^{-1} B_\theta \in C^\nu(\Delta_{\delta_n})
\]

(VII.7)

for any and \(1/2 < \theta \leq 1\) and \(\nu < \theta - \frac{1}{2}\).

**Proof.** Let \(D_n\) be the disc of the radius \(\delta_n\) centered at \(e(H_s)\). Since, by (VI.2), \(U_{\delta_n} \subset D(R^n_p)\), the operator \(H^{(n)}(\lambda) := R^n_p(H)\), with \(\lambda := -H_u\), is well defined. By (VI.2), \(D_n \ni \lambda \to H^{(n)}(\lambda) \in D^{\mu,2}(\frac{1}{8} \rho, \beta_{n-1}, \gamma_{n-1})\) is \(C^\infty\). Moreover,
\[ H^{(n)}(\lambda) = H^{(n)}(\lambda)^*, \forall \lambda \in D_n \cap \mathbb{R}. \]

Hence, since \( \rho \gg \gamma_{n-1}, \beta_{n-1} \leq \frac{1}{3} \), by (VI.1), and \( \Delta_{\delta_n} \subset D_n \), we have by Theorem VII.1 that
\[
B_\theta(H^{(n)}(\lambda) - i0)^{-1}B_\theta \in C^\alpha(D_n \cap E_{n-1}(\Delta^{1/3})),
\]
where \( 0 \leq \nu < \theta - 1/2 \) and, as before, \( E_n(\lambda) \equiv E_n(\lambda, H_s) := (H^{(n)}(\lambda))_u \), which, by the above conclusion, is \( C^\infty \). We need the following proposition to describe the set \( E_{n-1}(\Delta^{1/3}) \).

**Proposition VII.3.** Let \( n \geq 0, \delta_n := \frac{1}{18} \rho^2 \) and \( A_{\delta_n} := \{ \frac{2}{3} \delta_n \leq |\lambda - e(H_s)| \leq \delta_n \} \). For \( H \in U_{\delta_n} \) we denote \( E_n(\lambda, H_s) := (R^\alpha_{\nu}(H))_u \equiv (R^\alpha_{\nu}(H))_{\Omega}, \lambda = - H_u \). Then
\[
|E_n(\lambda, H_s)| \geq \frac{1}{50} \rho \text{ for } \lambda \in A_{\delta_n}.
\]

**Proof.** In this proof we do not display the argument \( H_s \). Let \( \lambda \in A_{\delta_n} \) with \( \delta_n \) given in the proposition. Define \( E_{0n}(\lambda) \) by the equation
\[
E_n(\lambda) = \rho^{-n}(E_{0n}(\lambda) - \lambda).
\]

The following estimate is shown in [48] (see Eqn (V.27) of the latter paper):
\[
|E_{0n}(\lambda) - e| \leq \frac{1}{5} |\lambda - e| + (1 - \rho)^{-1} \rho^{n+1} \alpha_{n+1}.
\]

This inequality and the definition of \( \alpha_n \) imply
\[
|E_{0n}(\lambda) - \lambda| \geq |\lambda - e| - |E_{0n}(\lambda) - e|
\]
\[
\geq \frac{4}{5} |\lambda - e| - 2\gamma_0^2 c (c^2 \rho^{2\mu+1})^{\frac{1}{n}} \rho^{-1}.
\]

Due to \( 2\gamma_0^2 c \rho^{-1} (c^2 \rho^{2\mu+1})^{\frac{1}{n}} \ll \rho^n \), (VII.11) gives
\[
|E_{0n}(\lambda) - \lambda| \geq \frac{1}{50} \rho^{n+1}.
\]

Due to (VII.9) this implies the statement of the proposition. \( \square \)

Proposition VII.3 says that
\[
E_n : \Delta_{\delta_n} \ni \lambda \mapsto E_n(\lambda) \in \Delta^{1/3}.
\]

Hence \( E_{n-1}(\Delta^{1/3}) \supset \Delta_{\delta_n} \). Since \( \Delta_{\delta_n} \subset D_n \), we have that
\[
B_\theta(H^{(n)}(\lambda) - i0)^{-1}B_\theta \in C^\alpha(\Delta_{\delta_n})
\]
which, due to Proposition V.5, gives (VII.7). \( \square \)
VIII  Initial Conditions for the Renormalization Group

Now we turn to the operator families $H_g - \lambda$, we are interested in. Here the operator $H_g = H_0 + gI$ is given either by (I.8) or by (II.11). These operators do not belong to the Banach spaces defined above. We define an additional renormalization transformation which acts on such operators and maps them into the disc $D^{\mu,s}(\alpha_0, \beta_0, \gamma_0)$ for some appropriate $\alpha_0, \beta_0, \gamma_0$.

Let $H_{pg}$ denote either $H_{P,F}^p$ or $H_{N,F}^p$ and let $e_0^{(p)} < e_1^{(p)} < ...$ be the eigenvalues of $H_{pg}$, so that $e_0^{(p)}$ is its ground state energy. Let $P_p$ be the orthogonal projection onto the eigenspace corresponding to $e_0^{(p)}$. On Hamiltonians acting on $H_p \otimes H_f$ which were described above, we define the map

$$R^{(0)}_{\rho_0} = \rho_0^{-1} S_{\rho_0} \circ F_{\tau_0 \pi_0},$$

(VIII.1)

where $\rho_0 \in (0, e_{gap}^{(p)})$ is an initial photon energy scale (recall that $e_{gap}^{(p)} := e_1^{(p)} - e_0^{(p)}$ and $e_j^{(p)}$ are the eigenvalues of $H_p$) and where

$$\tau_0(H_g - \lambda) = H_0g - \lambda \text{ and } \pi_0 \equiv \pi_0[H_f] := P_p \otimes \chi_{H_f \leq \rho_0}.$$

(VIII.2)

for any $\lambda \in \mathbb{C}$. Recall the convention $\bar{\pi}_0 := 1 - \pi_0$. Define the set

$$I_0 := \{ z \in \mathbb{C} | \Re z \leq e_0^{(p)} + \frac{1}{2} \rho_0 \}.$$  

(VIII.3)

We assume $\rho_0 \gg g^2$. To simplify the notation we assume that the ground state energy, $e_0^{(p)}$, of the operator $H_p$ is simple (otherwise we would have to deal with matrix-valued operators on $H_f$). We have

**Theorem VIII.1.** Let $H_g$ be the Hamiltonian given either by (II.11) or by (I.8) and let $\rho_0 \gg g^2$, $\mu > -1/2$ and $\lambda \in I_0$. Then

$$H_g - \lambda \in \text{dom}(R^{(0)}_{\rho_0}).$$

(VIII.4)

Furthermore, define the family of operators $H^{(0)}_\lambda := R^{(0)}_{\rho_0}(H_g - \lambda) \mid \text{Ran}P_p \otimes I$. Then $H^{(0)}_\lambda = H^{(0)*}_\lambda$, for $\lambda \in I_0 \cap \mathbb{R}$, and

$$H^{(0)}_\lambda = \rho_0^{-1}(e_0^{(p)} - \lambda) \in D^{\mu,2}(\alpha_0, \beta_0, \gamma_0),$$

(VIII.5)

where, with $\mu$ as in Eqn (II.11), $\alpha_0 = O(g^2 \rho_0^{-1})$, $\beta_0 = O(g^2)$, and $\gamma_0 = O(g \rho_0^4)$, for $\lambda \in I_0$. Moreover, $R^{(0)}_{\rho_0}(H - \lambda)$ is analytic in $\lambda \in I_0$. In particular, these results apply to the Pauli-Fierz and Nelson Hamiltonians by taking $\mu = 1/2$ and $\mu > 0$, respectively.
Note that if $\psi(p)$ is a ground state of $H_{pg}$ with the energy $e_0(p)$ and $\psi_0 = \psi(p) \otimes \Omega$, then we have

$$e_0(p) - \lambda = \langle H - \lambda \rangle_{\psi_0}. \quad (VIII.6)$$

Theorem VIII.1 is proven in [48], Appendix II, for somewhat simper Banach spaces which do not contain the derivatives $(k \partial_k)^q$. However, an extension to the Banach spaces which are used in this paper is straightforward and is omitted here.

Note that $K := R_{\rho_0}^{(0)}(H_g - \lambda) \mid_{\text{Ran}(P_{pj} \otimes 1)} = \langle H_{0g} - \lambda \rangle \mid_{\text{Ran}(P_{pj} \otimes 1)}$ and therefore $\forall \lambda \in I_0 \cap \mathbb{R}, \sigma(K) = \sigma(H_{pg})/\{\lambda_j\} + [0, \infty) - \lambda$. Hence

$$\forall \lambda \in I_0 \cap \mathbb{R}, K \geq e_1(p) - e_0(p) - \frac{1}{8} \rho_0 \geq \frac{7}{8} (e_1(p) - e_0(p)).$$

Therefore $0 \notin \sigma(K)$. This, the relation $\sigma(R_{\rho_0}^{(0)}(H_g - \lambda)) = \sigma(H_\lambda^{(0)}) \cup \sigma(K)$ and Theorem III.1 imply that $H_\lambda^{(0)}$ is isospectral to $H_g - \lambda$ in the sense of Theorem III.1. Moreover, similarly to Proposition IV.3, and using the relation

$$R_{\rho_0}^{(0)}(H_g - \lambda)^{-1} = H_\lambda^{(0)}^{-1}(P_{pj} \otimes 1) + (H_{0g} - \lambda)^{-1}(\bar{P}_{pj} \otimes 1). \quad (VIII.7)$$

one shows the following result

**Proposition VIII.2.** Let $\mu > 0$, $\rho_0 \gg g^2$ and $\Delta_0 \subseteq I_0 \cap \mathbb{R}$. If $H_g$ is given in either (II.11) or (I.8), then

$$B_s(H_\lambda^{(0)} - i0)^{-1} B_s \in C^\nu(\Delta_0) \Rightarrow B_s(H_g - \lambda - i0)^{-1} B_s \in C^\nu(\Delta_0). \quad (VIII.8)$$

### IX Proof of Theorem I.1

Let $H_g$ be a Hamiltonian given in either (II.11) or (I.8). Recall the definition

$$H_\mu^{(0)} := R_{\rho_0}^{(0)}(H_g - \mu) \mid_{\text{Ran}P_{pj} \otimes 1}, \mu \in I_0. \quad (IX.1)$$

The r.h.s. is well defined according to Theorem VIII.1. By Equation (VIII.5), if $\mu \in I_0$, then

$$H_\mu^{(0)} - \rho_0^{-1}(e_0(p) - \mu) \in D^{n2}(\alpha_0, \beta_0, \gamma_0), \quad (IX.2)$$

where $\alpha_0, \beta_0$ and $\gamma_0$ are given in Theorem VIII.1. The condition (VI.1) is satisfied if

$$g^2 \rho_0^{-1}, g \rho_0^p \ll \rho \leq \frac{1}{2}, \quad (IX.3)$$

which can be arranged since by our assumption $g \ll 1$ and $\rho_0$ can be fixed anywhere in the interval $(0, e_{gap})$. 

Let $H_{\mu s} := (H^{(0)}_\mu)_s = H^{(0)}_\mu - \langle H^{(0)}_\mu \rangle_\Omega$ and $H_{\mu u} := (H^{(0)}_\mu)_u = \langle H^{(0)}_\mu \rangle_\Omega$, the stable-central and unstable components of the operator $H^{(0)}_\mu$, respectively (see Section VII), and let $e : D_s \to \mathbb{C}$ be the map introduced in Theorem VI.1. We introduce the subsets:

$$D_\delta := \{ \mu \in I_0 \mid |e(H_{\mu s}) + H_{\mu u}| \leq \delta \} \quad (IX.4)$$

and

$$E^\mu_\delta := \{ \lambda \in \mathbb{R} \mid \frac{\rho}{8} \delta \leq |\lambda - e(H_{\mu s})| \leq \delta \}. \quad (IX.5)$$

Recall, $\delta_n = \frac{1}{18} \rho^n$ for $n \geq 0$. Let $0 = e(H_{\epsilon_g s}) + H_{\epsilon_g u}$ be the solution to the equation $e(H_{\epsilon_g}) = -H_{\epsilon_g u}$ for $\mu$. By Theorem V.3, $0 = e(H_{\epsilon_g s}) + H_{\epsilon_g u}$ is the ground state energy of the operator $H^{(0)}_{\epsilon_g}$ and therefore, by Theorem II.1, $\epsilon_g$ is the ground state energy of the operator $H_{\epsilon_g}$. In the lemma below we show that for $g$ sufficiently small

$$D_{\delta_n} \setminus D_{\frac{\rho}{8} \delta_n} \ni \mu \mapsto -H_{\epsilon_{g u}} \in E^\mu_{\delta_n},$$

the latter equation yields, in turn, that

$$B_\theta(H^{(0)}_\mu - i0)B_\theta \in C^\nu(D_{\delta_n} \setminus D_{\frac{\rho}{8} \delta_n}),$$

which, due to Proposition VII.4, yields

$$B_\theta(H_{\epsilon_g} - \mu - i0)^{-1}B_\theta \in C^\nu(D_{\delta_n} \setminus D_{\frac{\rho}{8} \delta_n}). \quad (IX.6)$$

Let $\epsilon_g$ be the solution to the equation $e(H_{\mu s}) = -H_{\mu u}$ for $\mu$. By Theorem V.3, $0 = e(H_{\epsilon_g s}) + H_{\epsilon_g u}$ is the ground state energy of the operator $H^{(0)}_{\epsilon_g}$ and therefore, by Theorem II.1, $\epsilon_g$ is the ground state energy of the operator $H_{\epsilon_g}$. In the lemma below we show that for $g$ sufficiently small

$$D_{\delta_n} \setminus D_{\frac{\rho}{8} \delta_n}, \forall n \geq 0, \text{ cover } (\epsilon_g, \epsilon_g + \frac{1}{18} \rho_0), \quad (IX.7)$$

This together with (IX.6) implies the statement of Theorem I.1. \hfill \Box

**Lemma IX.1.** For $g$ sufficiently small, (IX.7) holds.

**Proof.** We claim that for $g$ sufficiently small and for $n \geq 0$

$$D(\epsilon_g, \frac{\rho_0}{4} \delta_n) \subset D_{\delta_n}. \quad (IX.8)$$

We prove this claim by induction in $n$. We assume it is true for $n \leq j - 1$ and prove it for $n = j$. For $j = 0$, the induction assumption is absent and so our proof of the induction step yields also the first step.

We introduce the notation $e(\mu) := e(H_{\mu s})$. First we use the relation $e(\epsilon_g) = -H_{\epsilon_g u}$ to obtain

$$|e(\mu) + H_{\mu u}| \leq |e(\mu) - e(\epsilon_g)| + |H_{\epsilon_g u} - H_{\mu u}|. \quad (IX.9)$$
Next, let $\Delta_0 E(\mu)$ be defined by the relation $H_{\mu u} =: \rho_0^{-1}(\mu - \epsilon_0) - \Delta_0 E(\mu)$. Then by (IX.2) and analyticity of $\Delta_0 E(\mu)$ in $I_0$, $|\partial_\mu \Delta_0 E(\mu)| \leq \alpha_0/\rho_0$. The last two relations imply

$$|H_{\epsilon g u} - H_{\mu u}| = |\rho_0^{-1}(\epsilon_g - \mu) + \Delta_0 E(\epsilon_g) - \Delta_0 E(\mu)| \leq \alpha_0(1 + \alpha_0)|\epsilon_g - \mu|. \quad (IX.10)$$

Recall the definition $E_n(\lambda, H_{\mu s}) := \langle R_\rho^n(H_{\mu s}) \rangle _{\Omega}$, $\lambda = -H_{\mu u}$. Now we estimate the first term on the r. h. s. of (IX.9). Define $\Delta_n E(\lambda, H_{\mu s}) := E_n(\lambda, H_{\mu s}) - \rho^{-1}E_{n-1}(\lambda, H_{\mu s})$. \quad (IX.11)

It is shown in [48], Eqns (V.24)-(V.25) that $e(H_s)$ satisfies the equation

$$e(H_s) = \sum_{i=1}^{\infty} \rho^i \Delta_i E(e(H_s), H_s), \quad (IX.12)$$

where the series on the right hand side converges absolutely by the estimate

$$|\partial_\lambda^m \Delta_n E(\lambda)| \leq \alpha_n(\frac{1}{12}\rho^{n+1})^{-m} \text{ for } n \leq j \text{ and } m = 0, 1, \quad (IX.13)$$

shown in [21]. The relation (IX.12) together with the definitions $e(\mu) := e(H_{\mu s})$ and $e(\epsilon_g) := e(H_{\epsilon g s}) = -H_{\epsilon g u}$ implies

$$e(\mu) = \sum_{i=1}^{\infty} \rho^i \Delta_i E(e(\mu), H_{\mu s}) \quad (IX.14)$$

and

$$e(\epsilon_g) = \sum_{i=1}^{\infty} \rho^i \Delta_i E(e(\epsilon_g), H_{\epsilon g s}). \quad (IX.15)$$

We estimate the difference between these series. It follows from the analyticity of $E_n(\lambda, H_s)$ in $H_s$, see [21], Proposition V.3, that $\Delta_i E(\lambda, H_{\mu s})$ are analytic in $\mu \in D_\delta_i, i \leq j - 1$. Now, by the induction assumption $D_\delta_i \supset D(\epsilon_g, \rho_0^i/4 \delta_i)$ for $i \leq j - 1$. Hence using the Cauchy formula we conclude from (IX.13) that for $i \leq j - 1$

$$|\partial_\mu \Delta_i E(\lambda, H_{\mu s})| \leq \frac{4\alpha_i}{(1 - \rho)\rho_0 \delta_i} \text{ on } D(\epsilon_g, \rho_0^i/4 \delta_i).$$

The latter estimate together with (IX.13) gives

$$\sum_{i=1}^{\infty} \rho^i |\Delta_i E(e(\mu), H_{\mu s}) - \Delta_i E(e(\epsilon_g), H_{\epsilon g s})|$$
\[
\leq \sum_{i=1}^{j-1} \rho^i \left( \frac{\alpha_1}{\delta_i} |e(\mu) - e(\epsilon_g)| + \frac{4\alpha_i}{(1 - \rho)\rho_0 \delta_i} |\mu - \epsilon_g| \right) + 2 \sum_{i=j}^{\infty} \rho^i \alpha_i \\
\leq 20 \alpha_1 |e(\mu) - e(\epsilon_g)| + \frac{80\alpha_1}{(1 - \rho)\rho_0} |\mu - \epsilon_g| + 4\alpha_j \rho^j
\]
on $D(\epsilon_g, \frac{\rho_0}{4} \delta_j)$, where we used that $\delta_j = \frac{1}{18} \rho^j$. This estimate together with the relations (IX.14) and (IX.15) gives

\[
|e(\mu) - e(\epsilon_g)| \leq 40 \alpha_1 \frac{1}{1 - \rho} \delta_j + 160 \alpha_j \delta_j
\]
in $D(\epsilon_g, \frac{\rho_0}{4} \delta_j)$, provided $\alpha_1 \leq \frac{1}{40}$. This estimate together with (IX.9) and (IX.10) and the definition of $D_{\delta_n}$ implies (IX.8) with $n = j$, provided

\[
\frac{1 + \alpha_0}{4} + \frac{40\alpha_1}{1 - \rho} + 160 \alpha_0 \leq 1.
\]
(IX.17)

Remembering the definition of $\alpha_j$, we see that the latter conditions can be easily arranged by taking $g$ sufficiently small. This proves (IX.8).

Next we show that for $g$ sufficiently small

\[D_{\tau \delta_n} \subset D(\epsilon_g, 1.5\rho_0 \tau \delta_n), \text{ where } \tau = O(1).\]
(IX.18)
The proof of this embedding proceeds by induction in $n$ along the same lines as the proof of (IX.8) given above. We have

\[|e(\mu) + H_{\mu u}| \geq |H_{\epsilon_g u} - H_{\mu u}| - |e(\mu) - e(\epsilon_g)|.\]
Again using the equality in (IX.10) and the estimates $|\partial_\mu \Delta_0 E(\mu)| \leq \rho_0^{-1} \alpha_0$ and (IX.16), we find

\[|e(\mu) + H_{\mu u}| \geq \rho_0^{-1} (1 - \alpha_0 - \frac{160\alpha_1}{\rho(1 - \rho)}) |\mu - \epsilon_g| - 80 \rho^{-1} \alpha_j \delta_j
\]
in $D_{\delta_i}$, provided $\alpha_1 \leq \frac{1}{40}$. Let $\mu \in D_{\tau \delta_j}$. Then $|e(\mu) + H_{\mu u}| \leq \tau \delta_j$, which together with the previous estimate gives

\[|\mu - \epsilon_g| \leq \rho_0 (1 - \alpha_0 - \frac{160\delta_1}{\rho(1 - \rho)})^{-1} (\tau + \frac{80\alpha_j}{\rho}) \delta_j.
\]
This yields (IX.18), provided $g$ is sufficiently small.

Embeddings (IX.8) and (IX.18) with $\tau = \frac{g}{8}$ imply that

\[D_{\delta_n} \setminus D_{\frac{\rho_0}{16} \delta_n} \supset D(\epsilon_g, \frac{\rho_0}{4} \delta_n) \setminus D(\epsilon_g, \frac{3\rho_0}{16} \delta_n).
\]
Since $\forall n$, $\frac{\rho_0}{4} \delta_n > \frac{3\rho_0}{16} \delta_{n-1}$, the sets on the r. h. s. cover the interval $(\epsilon_g, \epsilon_g + \frac{1}{18} \rho_0)$ and therefore so do the sets on the l.h.s. Hence the lemma follows. \qed
Supplement: Background on the Fock space, etc

Let $\mathfrak{h}$ be either $L^2(\mathbb{R}^3, \mathbb{C}, d^3k)$ or $L^2(\mathbb{R}^3, \mathbb{C}^2, d^3k)$. In the first case we consider $\mathfrak{h}$ as the Hilbert space of one-particle states of a scalar Boson or a phonon, and in the second case, of a photon. The variable $k \in \mathbb{R}^3$ is the wave vector or momentum of the particle. (Recall that throughout this paper, the velocity of light, $c$, and Planck’s constant, $\hbar$, are set equal to 1.) The Bosonic Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{S}_n \otimes \mathfrak{h}^{\otimes n}, \quad (X.1)$$

where $\mathcal{S}_n$ is the orthogonal projection onto the subspace of totally symmetric $n$-particle wave functions contained in the $n$-fold tensor product $\mathfrak{h}^{\otimes n}$ of $\mathfrak{h}$; and $\mathcal{S}_0 \otimes 0 := \mathbb{C}$. The vector $\Omega := \bigoplus_{n=1}^{\infty} 0$ is called the vacuum vector in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $\left( \psi_n \right)_{n=0}^{\infty}$ of $n$-particle wave functions, which are totally symmetric in their $n$ arguments, and $\psi_0 \in \mathbb{C}$. In the first case these functions are of the form, $\psi_n(k_1, \ldots, k_n)$, while in the second case, of the form $\psi_n(k_1, \lambda_1, \ldots, k_n, \lambda_n)$, where $\lambda_j \in \{-1, 1\}$ are the polarization variables.

In what follows we present some key definitions in the first case only limiting ourselves to remarks at the end of this appendix on how these definitions have to be modified for the second case. The scalar product of two vectors $\Psi$ and $\Phi$ is given by

$$\langle \Psi, \Phi \rangle := \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} d^3k_j \overline{\psi_n(k_1, \ldots, k_n)} \varphi_n(k_1, \ldots, k_n). \quad (X.2)$$

Given a one particle dispersion relation $\omega(k)$, the energy of a configuration of $n$ non-interacting field particles with wave vectors $k_1, \ldots, k_n$ is given by $\sum_{j=1}^{n} \omega(k_j)$. We define the free-field Hamiltonian, $H_f$, giving the field dynamics, by

$$\left( H_f \Psi \right)_n(k_1, \ldots, k_n) = \left( \sum_{j=1}^{n} \omega(k_j) \right) \psi_n(k_1, \ldots, k_n), \quad (X.3)$$

for $n \geq 1$ and $\left( H_f \Psi \right)_n = 0$ for $n = 0$. Here $\Psi = (\psi_n)_{n=0}^{\infty}$ (to be sure that the r.h.s. makes sense we can assume that $\psi_n = 0$, except for finitely many $n$, for which $\psi_n(k_1, \ldots, k_n)$ decrease rapidly at infinity). Clearly that the operator $H_f$ has the single eigenvalue 0 with the eigenvector $\Omega$ and the rest of the spectrum absolutely continuous.

With each function $\varphi \in \mathfrak{h}$ one associates an annihilation operator $a(\varphi)$ defined as follows. For $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_n = 0$, for all but finitely many $n$, the vector $a(\varphi) \Psi$ is defined by

$$\left( a(\varphi) \Psi \right)_n(k_1, \ldots, k_n) := \sqrt{n+1} \int d^3k \overline{\varphi(k)} \psi_{n+1}(k, k_1, \ldots, k_n). \quad (X.4)$$
These equations define a closable operator $a(\varphi)$ whose closure is also denoted by $a(\varphi)$. Eqn (X.4) implies the relation

$$a(\varphi)\Omega = 0.$$  \hspace{1cm} (X.5)

The creation operator $a^*(\varphi)$ is defined to be the adjoint of $a(\varphi)$ with respect to the scalar product defined in Eq. (X.2). Since $a(\varphi)$ is anti-linear, and $a^*(\varphi)$ is linear in $\varphi$, we write formally

$$a(\varphi) = \int d^3k \, \overline{\varphi(k)} a(k), \quad a^*(\varphi) = \int d^3k \, \varphi(k) a^*(k),$$  \hspace{1cm} (X.6)

where $a(k)$ and $a^*(k)$ are unbounded, operator-valued distributions. The latter are well-known to obey the canonical commutation relations (CCR):

$$[a^\#(k), a^\#(k')] = 0, \quad [a(k), a^*(k')] = \delta^3(k-k'),$$  \hspace{1cm} (X.7)

where $a^\# = a$ or $a^*$.

Now, using this one can rewrite the quantum Hamiltonian $H_f$ in terms of the creation and annihilation operators, $a$ and $a^*$, as

$$H_f = \int d^3k \, a^*(k) \, \omega(k) \, a(k),$$  \hspace{1cm} (X.8)

acting on the Fock space $\mathcal{F}$.

More generally, for any operator, $t$, on the one-particle space $\mathcal{H}$ we define the operator $T$ on the Fock space $\mathcal{F}$ by the following formal expression

$$T := \int a^*(k)ta(k)dk,$$

where the operator $t$ acts on the $k$-variable ($T$ is the second quantization of $t$). The precise meaning of the latter expression can obtained by using a basis $\{\phi_j\}$ in the space $\mathcal{H}$ to rewrite it as

$$T := \sum_j \int a^*(\phi_j)a(t^*\phi_j)dk.$$

To modify the above definitions to the case of photons, one replaces the variable $k$ by the pair $(k,\lambda)$ and adds to the integrals in $k$ also the sums over $\lambda$. In particular, the creation and annihilation operators have now two variables: $a^\#_\lambda(k) \equiv a^\#(k,\lambda)$; they satisfy the commutation relations

$$[a^\#_\lambda(k), a^\#_{\lambda'}(k')] = 0, \quad [a_\lambda(k), a^*_\lambda(k')] = \delta_{\lambda,\lambda'}\delta^3(k-k').$$  \hspace{1cm} (X.9)

One can also introduce the operator-valued transverse vector fields by

$$a^\#(k) := \sum_{\lambda \in \{-1,1\}} e_\lambda(k)a^\#_\lambda(k),$$

where $e_\lambda(k) \equiv e(k,\lambda)$ are polarization vectors, i.e. orthonormal vectors in $\mathbb{R}^3$ satisfying $k \cdot e_\lambda(k) = 0$. Then in order to reinterpret the expressions in this paper
for the vector (photon) - case one either adds the variable $\lambda$ as was mentioned above or replaces, in appropriate places, the usual product of scalar functions or scalar functions and scalar operators by the dot product of vector-functions or vector-functions and operator valued vector-functions.

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