Monogenic fields with odd class number Part I: odd degree

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Abstract

We bound the average number of 2-torsion elements in the class group of monogenised fields of odd degree (and compute it precisely conditional on a tail estimate) using an orbit parametrisation of Wood [31, 32]. Curiously, we find that the average number of non-trivial 2-torsion elements in the class group of monogenised fields of any given odd degree is twice the value predicted for the full family of fields of that degree by the Cohen–Lenstra–Martinet–Malle heuristic! We also find that monogenicity has an increasing effect on the average number of non-trivial 2-torsion elements in the narrow class group.

In addition, we obtain unconditional statements for monogenised rings of odd degree. For an order \( \mathcal{O} \), denote by \( \mathbb{Z}_2(\mathcal{O}) \) the group of 2-torsion ideals of \( \mathcal{O} \). We show that the average value of the difference \( |\text{Cl}_2(\mathcal{O})| - 2^{1-r_1-r_2} |\mathbb{Z}_2(\mathcal{O})| \) over all monogenised orders \( \mathcal{O} \) of fixed signature \((r_1, r_2)\) is \( 1 + 2^{1-r_1-r_2} \). For 3-torsion in quadratic orders, 2-torsion in cubic orders, and 2-torsion in orders arising from odd degree binary forms, work of Bhargava–Varma [9, 10] and Ho–Shankar–Varma [22] shows that the corresponding difference averaged over the full family of orders is equal to 1. This shows that monogenicity has an increasing effect not only on the class group of fields, but also on the class group of orders.

Our method gives a dual proof of a result of Bhargava–Hanke–Shankar [5] in the cubic case, reveals an interesting structure underpinning the deviation of these averages from those expected for the full families, and extends to the case of monogenised rings and fields of even degree at least 4.

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3 Reduction theory
1 Introduction

The Cohen–Lenstra–Martinet–Malle heuristics which were developed in a series of ground-breaking works [14, 16, 17, 15, 25], constitute our best conjectural description of the distribution of the $p^\infty$-part of the class group, $\mathrm{Cl}(K)[p^\infty]$, over families of number fields $K$ of fixed degree and signature ordered by discriminant for “good” primes $p$. We say that a prime $p$ is “good” if it is coprime to the degree of the field and “bad” otherwise.

So far, only two cases of these heuristics have been settled. In 1971, Davenport and Heilbronn [20] calculated the average number of 3-torsion elements in the class group of quadratic fields. In 2005, Bhargava calculated the average number of 2-torsion in the class group of cubic fields.

**Theorem 1** (Davenport–Heilbronn, [20]). Let $(r_1, r_2)$ denote the signature. The average number of 3-torsion elements in the class group of isomorphism classes of quadratic fields ordered by discriminant is:

| $(r_1, r_2)$ | $\text{Avg}(\#\text{Cl}_3(K))$ |
|-------------|---------------------------------|
| $(2, 0)$    | $3/4$                           |
| $(0, 1)$    | $2$                             |

**Theorem 2** (Bhargava, [2]). Let $(r_1, r_2)$ denote the signature. The average number of 2-torsion elements in the class group of isomorphism classes of cubic fields ordered by discriminant is:

| $(r_1, r_2)$ | $\text{Avg}(\#\text{Cl}_2(K))$ |
|-------------|---------------------------------|
| $(3, 0)$    | $5/4$                           |
| $(1, 1)$    | $3/2$                           |
The heuristics are expected to hold under any natural ordering on the family of fields and not just when ordering by discriminant. In [22], Ho–Shankar–Varma found evidence to support this expectation by showing that the average number of 2-torsion elements in the class group of fields associated to binary $n$-ic forms, ordered either by naive height or by Julia invariant, coincided with the value predicted from the Cohen–Lenstra–Martinet–Malle heuristics.

**Theorem 3** (Ho–Shankar–Varma [22]). Let $n \geq 3$ be an odd integers. Let $\mathcal{R}$ be the family of fields associated to binary $n$-ic forms ordered by naive height or by Julia invariant. The average number of 2-torsion elements in the class group of fields in $\mathcal{R}$ satisfies the bound:

$$\text{Avg}(\text{Cl}_2, \mathcal{R}) \leq 1 + \frac{1}{2^{r_1 + r_2 - 1}}$$

with equality conditional on a tail estimate.

Interestingly, the work of Bhargava–Varma [9, 10] and Ho–Shankar–Varma [22] showed that these averages remain the same when one imposes finitely many local conditions or even an acceptable family of local conditions. A set of local conditions is called acceptable if for large enough primes $p$ it includes all fields with discriminant indivisible by $p^2$.

It then becomes natural to ask about the effect of global conditions on averages of this kind. In [5], Bhargava–Hanke–Shankar showed that monogenicity had the effect of doubling the average number of non-trivial 2-torsion elements in the class group!

**Theorem 4** (Bhargava–Hanke–Shankar, [5]). Let $(r_1, r_2)$ denote the signature. The average number of 2-torsion elements in the class group of isomorphism classes of monogenised cubic fields ordered by “naive” height is:

| $(r_1, r_2)$ | Avg$(\#\text{Cl}_2(K): K \text{ is monogenic})$ |
|------------|-------------------------------------------|
| (3, 0)     | 3/2                                      |
| (1, 1)     | 2                                        |

In this paper, we show that this behaviour is not unique to cubic fields but indeed holds for any odd degree. More precisely, we prove that the average number of non-trivial 2-torsion elements in the class group of monogenised fields of odd degree ordered by naive height is twice that predicted by the Cohen–Lenstra–Martinet–Malle heuristic for the full family of fields.

1.1 Monogenised fields and rings

A number field $K$ of degree $n$ is said to be monogenic if $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$. The element $\alpha$ is called a monogeniser of the field $K$. A monogenised field is the data $(K, \alpha)$ of a monogenic field together with a choice of monogeniser.

It is expected that 100% of monogenic fields possess a unique monogeniser up to transformations of the form $\alpha \mapsto \pm \alpha + m$ for some $m \in \mathbb{Z}$, see [8]. This motivates the following definition.

**Definition 5.** Two monogenised fields $(K, \alpha)$ and $(K', \alpha')$ are said to be isomorphic if there exists a field isomorphism from $K$ to $K'$ taking $\alpha$ to $\pm \alpha' + m$ for some $m \in \mathbb{Z}$.

Thus, the expectation is that working with isomorphism classes of monogenised fields is statistically equivalent to working with isomorphism classes of monogenic fields. Nevertheless, if a statement holds for a “positive proportion” of monogenised fields, the same statement holds for “infinitely many” monogenic fields by using the arguments of [22] combined with the construction of strongly quasi-reduced elements from [8].
The height we choose for monogenised fields/rings has a convenient interpretation. Each isomorphism class of monogenised field contains a unique element \((K, \alpha_0)\) with the property that \(0 \leq \text{tr}(\alpha_0) < n\). If \(f(x) = x^n + a_1x^{n-1} + \ldots + a_n\) is the minimal polynomial of \(\alpha_0\), we define the naive height of the isomorphism class to be:

\[
H\left( [(K, \alpha_0)] \right) = \max_i \{|a_i|^{1/i}\}.
\]

Similarly, we can define monogenised rings and the naive height of their isomorphism classes. In Section 2 we will see that the set of monogenised rings is in natural bijection with the set of monic degree \(n\) polynomials. This equips monogenised rings with natural local measures, and we can speak of families of fields in \(\mathcal{R}^{r_1,r_2}\) associated with sets of local specifications \((\Sigma_p)_p\) on monic polynomials.

We denote by \(\mathcal{R}^{r_1,r_2}\) the collection of isomorphism classes of monogenised orders of signature \((r_1,r_2)\) ordered by naive height and by \(\mathcal{R}^{r_1,r_2}_{\text{max}}\) the subcollection consisting of maximal orders. For an monogenic ring \(\mathcal{O} \in \mathcal{R}^{r_1,r_2}\), denote by \(\text{Cl}_2(\mathcal{O})\) the 2-torsion subgroup of the ideal class group \(\text{Cl}(\mathcal{O})\) of \(\mathcal{O}\) and by \(\mathcal{I}_2(\mathcal{O})\) the group of 2-torsion ideals of \(\mathcal{O}\).

### 1.2 Outline of the results

In our main theorem for fields, we find the average number of 2-torsion elements in the class group and narrow class group of monogenised fields ordered by naive height, conditional on a tail estimate.

**Theorem 6 (Main theorem for monogenic fields).** Let \(\mathcal{R} \subset \mathcal{R}^{r_1,r_2}_{\text{max}}\) be a family of fields corresponding to an acceptable collection of local specifications \(\Sigma = (\Sigma_p)_p\).

The average number of 2-torsion elements in the class group of fields in \(\mathcal{R}\) satisfies the bound:

\[
\text{Avg}(\text{Cl}_2, \mathcal{R}) \leq 1 + \frac{2}{2^{r_1+r_2-1}}
\]

with equality conditional on a tail estimate.

The average number of 2-torsion elements in the narrow class group of fields in \(\mathcal{R}\) satisfies the bound:

\[
\text{Avg}(\text{Cl}^+_2, \mathcal{R}) \leq 1 + \frac{1}{2^{2r_1-1}} + \frac{1}{2^{r_2}}
\]

with equality conditional on a tail estimate.

These averages have several interesting consequences. Indeed, we obtain all of the corollaries of Ho–Shankar–Varma [22] with the added adjective “monogenic”.

**Corollary 7.** Let \(n \geq 3\) be an odd integer, \((r_1, r_2)\) a choice of signature, and \(\mathcal{R} \subset \mathcal{R}^{r_1,r_2}_{\text{max}}\) a family of monogenised fields corresponding to an acceptable family of local specifications \(\Sigma = (\Sigma_p)_p\).

1. The proportion of fields in \(\mathcal{R}\) which have odd class number is at least

\[
1 - \frac{2}{2^{r_1+r_2-1}}.
\]

2. The proportion of fields in \(\mathcal{R}\) which have odd narrow class number is at least

\[
1 - \frac{1}{2^{n-1}} - \frac{1}{2^{r_2}}.
\]
In particular, there are infinitely many degree $n$ monogenic $S_n$-fields with signature $(r_1, r_2)$ which have odd class number and infinitely many which have units of every signature.

We also deduce asymptotic lower bounds for the number of monogenic fields having odd class numbers when these fields are ordered by discriminant just as in [22].

**Corollary 8.** Let $n \geq 3$ be an odd integer, $(r_1, r_2)$ a choice of signature, and $\mathcal{R} \subset \mathcal{R}_{\max}^{r_1,r_2}$ a family of monogenised rings corresponding to an acceptable family of binary forms. Then the following asymptotic estimates hold.

1. $\# \{ R \in \mathcal{R} : |\text{Disc}(R)| < X \text{ and } 2 \nmid |\text{Cl}(R)| \} \gg X^{n(n+1)/2 - 1/(n-1)}$.
2. If $r_2 \neq 0$, then
   $\# \{ R \in \mathcal{R} : |\text{Disc}(R)| < X \text{ and } 2 \nmid |\text{Cl}^+(R)| \} \gg X^{n(n+1)/2 - 1/(n-1)}$.

We also obtain an unconditional statement for very large families of monogenised rings. We call a family of monogenised orders very large if for large enough $p$, there are no local conditions at $p$.

**Theorem 9** (Main theorem for monogenic rings). Let $n \geq 3$ be an odd integer and $(r_1, r_2)$ a choice of signature. Let $\mathcal{R} \subset \mathcal{R}^{r_1,r_2}$ be a very large family of monogenised orders with the property that local conditions at $2$ are given modulo $2$.

1. The average over $O \in \mathcal{R}$ of the quantity
   $$|\text{Cl}_2(O)| - \frac{1}{2^{r_1+r_2-1}} |\mathcal{I}_2(O)|$$
   is equal to $1 + \frac{1}{2^{r_1+r_2-1}}$.
2. The average over $O \in \mathcal{R}$ of the quantity
   $$|\text{Cl}_2^+(O)| - \frac{1}{2^{r_2}} |\mathcal{I}_2(O)|$$
   is equal to $1 + \frac{1}{2^{r_2}}$.

**Remark 10.** We remark that our arguments give the “dual” proof of the monogenic result of Bhargava–Hanke–Shankar [5] in the cubic case. Indeed, they work with pairs of half-integral symmetric matrices while we work with the dual lattice which consists of pairs of integral symmetric matrices. The advantage is that our “dual” proof generalises to all higher dimension since Wood’s parametrisation [32] continues to hold, while the parametrisation in terms pairs of half-integral symmetric matrices does not.

### 1.3 Strategy and organisation of the paper

In Section 2 we parametrise isomorphism classes of monogenised rings of degree $n$ in terms of certain monic polynomials of degree $n$ with integer coefficients (SPACE A). In Section 2 we use Wood’s parametrisation [32] to express the 2-torsion ideal classes of rings in $\mathcal{R}^{r_1,r_2}$ in terms of certain $\text{SL}_n(\mathbb{Z})$ orbits of pairs of $n \times n$ integral symmetric matrices $(A, B)$ subject to the constraint $\det(A) = (-1)^{n+1}$ (SPACE B). The main results are then proven by finding asymptotic formulae for
the number of elements of (SPACE A) and of (SPACE B) of height at most \(X\), and then evaluating the limit of the ratio as \(X\) tends to infinity. The asymptotic formula for (SPACE B) was computed by Bhargava–Shankar–Wang in [8] and we recall it at the end of Section 2.

The constraint \(\det(A) = (-1)^{n-1}/2\) complicates the direct application of techniques from the geometry of numbers because one needs to count orbits for the action of the group \(\text{SL}_n(\mathbb{Z})\) on the hypersurface defined by \(\det(A) = (-1)^{n-1}/2\). To circumvent this complication, we borrow an idea of [5] and “linearise” the problem by noting that the collection \(\mathcal{L}_\mathbb{Z}\) of \(\text{SL}_n(\mathbb{Z})\) equivalence classes of symmetric integral matrices of determinant \((-1)^{n-1}/2\) is finite. Counting \(\text{SL}_n(\mathbb{Z})\) orbits on the space of pairs \((A,B)\) with the constraint \(\det(A) = (-1)^{n-1}/2\) is thus reduced to counting \(\text{SO}_{A_0}(\mathbb{Z})\) orbits on the space of pairs \((A_0,B)\) over all \(A_0 \in \mathcal{L}_\mathbb{Z}\).

In Sections 3-6, the geometry of number techniques developed in [7, 6] are applied to count the relevant orbits in these slices. However, the arguments are complicated by the fact that we consider an infinite set of representations simultaneously and the fact that the \(A_0\) have maximal \(\mathbb{Q}\)-anisotropic subspaces of varying dimensions. The latter is a novel feature not present in [4] or [28] which both dealt with split orthogonal groups. In Section 5, we adapt the sieves of [22] to restrict the count to orbits associated to invertible ideal classes of orders and to maximal orders. For maximal orders, the lower bound obtained is conditional on a tail estimate just as in [22].

In Section 9, the counts on the individual slices are aggregated into the full count by summing over all the elements of \(\mathcal{L}_\mathbb{Z}\). Calculating this sum is delicate because it relies on evaluating non-trivial masses for each of the \(A_0\) slices at the 2-adic place and the Archimedean place. We find these masses in Section 7 and Section 8 by establishing equidistribution results.

### 1.4 Generalisations

The methods of this paper can be used to study averages of monogenised fields in even degree, as well as averages for rings and fields of odd degree associated to binary forms with leading coefficient \(N\) via Wood’s parametrisation [31].

In Part II of this two-part series [29], we apply the techniques developed in this paper and compute the average number of 2-torsion elements in the class group, narrow class group and oriented class group of monogenised fields of even degree at least 4. In that case, the parametrisations are much more subtle, and there is the presence of genus theory.

For rings and fields of odd degree associated to polynomials with leading coefficient \(N\), our methods can be applied directly to give asymptotic formulas. But now the equi-distribution results present much more difficulty and involve classification theorems for pairs of forms over \(\mathbb{Z}/p\mathbb{Z}\) as given by Dickson. In forthcoming work [30], Swaminathan shows that these averages in both the even and the odd cases can be reduced to the monogenic averages together with an additional term correcting for cuspidal over-count!

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2 The parametrisations

2.1 The parametrisation of monogenised \( n \)-ic rings

In order to count the number of monogenised \( n \)-ic rings having bounded height, we will use the following parametrisation in terms of binary \( n \)-ic forms:

**Definition 11.** Let \( U = \text{Sym}_n(2) \) denote the space of binary \( n \)-ic forms. We denote by \( U_1 \subset U \) the space of all monic binary \( n \)-ic forms \( f(x, y) = x^n + a_{n-1}x^{n-1} + \ldots + a_0y^n \). The group \( \text{GL}_2 \) acts on \( U \) via the twisted action \( \gamma \cdot f(x, y) := \det(\gamma)^{-1}f((x, y) \cdot \gamma) \) for \( \gamma \in \text{GL}_2 \) and \( f \in U \). Let \( F \subset \text{GL}_2 \) denote the group of lower triangular unipotent matrices. Then the action of \( F \) on \( U \) preserves \( U_1 \) and yields an action of \( F \) on \( U_1 \).

We say that a pair \((R, \alpha)\) is a monogenised \( n \)-ic ring if \( R \) is an \( n \)-ic ring and \( \alpha \) is an element of \( R \) such that \( R = \mathbb{Z}[\alpha] \). Two monogenised \( n \)-ic rings \((R, \alpha)\) and \((R, \alpha')\) are said to be isomorphic if \( R \) and \( R' \) are isomorphic via a ring isomorphism sending \( \alpha \) to \( \alpha' + m \) for some \( m \in \mathbb{Z} \). We then have the following explicit parametrisation of monogenised \( n \)-ic rings in terms of the orbit data introduced above:

**Theorem 12.** There is a natural bijection between isomorphism classes of monogenised \( n \)-ic rings and \( F(\mathbb{Z}) \)-orbits on \( U_1(\mathbb{Z}) \).

**Proof.** Consider the map sending a monic binary \( n \)-ic form \( f(x, y) \in U_1(\mathbb{Z}) \) to the monogenised \( n \)-ic ring \( R_f := \left( \frac{\mathbb{Z}[\theta]}{(f(\theta, 1))}, \theta \right) \). This map descends to a map from \( F(\mathbb{Z}) \backslash U_1(\mathbb{Z}) \) to isomorphism classes of monogenised \( n \)-ic rings which we denote by \( \Phi \). Indeed, if \( g = \gamma \cdot f \) for \( \gamma = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \in F(\mathbb{Z}) \), then \( g(\theta, 1) = f(\theta + m, 1) \) and the monogenised ring \( \left( \frac{\mathbb{Z}[\theta]}{(f(\theta, 1))}, \theta \right) \) is isomorphic to the monogenised ring \( \left( \frac{\mathbb{Z}[\theta]}{(f(\theta + m, 1))}, \theta \right) \) through \( \theta \mapsto \theta + m \). To verify that \( \Phi \) is surjective, note that it was already surjective as a map from monic binary \( n \)-ic forms to monogenised \( n \)-ic rings. To verify that \( \Phi \) is injective, suppose that \( \Phi(f) = \left( \frac{\mathbb{Z}[\theta]}{(f(\theta, 1))}, \theta \right) \) is isomorphic to \( \Phi(g) = \left( \frac{\mathbb{Z}[\omega]}{(g(\omega, 1))}, \omega \right) \). Then \( \theta \mapsto \omega + m \) for some \( m \in \mathbb{Z} \) under this isomorphism. Consequently, \( f(\theta, 1) = 0 \) in \( \Phi(f) \) means that \( f(\omega + m, 1) = 0 \) in \( \Phi(g) \). In other words, the polynomial \( g(\omega, 1) \) divides \( f(\omega + m, 1) \). But since both are monic, we must have \( g(\omega, 1) = f(\omega + m, 1) \). Thus, \( g = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} \cdot f \) and \( g = f \) in \( F(\mathbb{Z}) \backslash U_1(\mathbb{Z}) \).

**Remark 13.** The results above hold for all \( n \) (even or odd).

**Remark 14.** The expectation is that 100\% of monogenic fields have a unique monogenised representative, i.e., 100\% of all monogenic fields can be expressed as \( \mathbb{Z}[\pm \theta] \) in a unique way up to translation of \( \pm \theta \) by an integer. So, in what follows, we are counting the monogenic fields if we believe this expectation. Furthermore, Hilbert’s irreducibility theorem tells us that a proportion of 100\% of monogenic/monogenised fields have Galois group \( S_n \).

2.2 The parametrisation of ideal classes of order 2 in monogenised rings

We now present the parametrisation of order 2 ideal classes in monogenic rings by pairs of symmetric matrices. This parametrisation is due to Wood, in [31] and [32], and in the case \( n = 3 \) due to the work of Bhargava, [1]. The statements that we use are essentially the same as those of Ho–Shankar–Varma, in [22]. We first describe the parametrisation for integral domains. We then specialise to \( \mathbb{R}, \mathbb{Z}_p, \) and \( \mathbb{Z} \).

We define the space of pairs of symmetric matrices.
Definition 15. Let $T$ be a base ring. Let

$$V(T) = T^2 \otimes \text{Sym}^2(T^n)$$

be the space of pairs of symmetric $n \times n$ matrices with coefficients in $T$. The group $\text{SL}_n(T)$ acts on $V(T)$ by change of basis. In other words, if $\gamma \in \text{SL}_n(T)$ and $(A, B) \in V(T)$, we define

$$\gamma(A, B) = (\gamma^t A \gamma, \gamma^t B \gamma),$$

where $\gamma^t$ denotes the transpose of $\gamma$.

There is a natural map from this space of pairs of matrices, $V(T)$, to the space of polynomials, $U(T)$, called the resolvent map.

Definition 16 (The resolvent map $\pi$). Let $T$ be a base ring. We define the resolvent map $\pi: V(T) \to U(T)$ by

$$(A, B) \mapsto (-1)^{n-1} \det(Ax - B).$$

The resolvent map $\pi$ is $\text{SL}_n(T)$ invariant. We write $f_{(A, B)} := \pi(A, B)$ for the resolvent form of the $\text{SL}_n(T)$-equivalence class of the pair $(A, B)$. We say that a pair $(A, B) \in V(T)$ is non-degenerate if the associated binary form $f_{(A, B)}$ is non-degenerate (i.e. has non-zero discriminant).

Now, let $T$ be an integral domain and $f \in U_1(T)$. The following result of Wood parametrises ideal classes of the ring $R_f = \frac{T[[x]]}{(f(x))}$ in terms of $\text{SL}_n(T)$-orbits on $V(T)$.

Theorem 17 (Wood \cite{1}, \cite{2}). Take a non-degenerate binary $n$-ic form $f \in U_1(T)$ and let $R_f = \frac{T[[x]]}{(f(x))}$. We have a bijection between $\text{SL}_n(T)$-orbits of pairs $(A, B) \in V(T)$ with $f_{(A, B)} = f$ and equivalence classes of pairs $(I, \delta)$ where $I \subset R_f$ is an ideal of $R_f$ and $\delta \in R_f^\times$ such that $I^2 \subset \delta R_f^{n-3}$ as ideals and $N(I)^2 = N(\delta) N(R_f^{n-3})$. The classes $(I, \delta)$ and $(I', \delta')$ are equivalent if there exists a $\kappa \in K_f^\times$ with the property that $I = \kappa I'$ and $\delta = \kappa^2 \delta$.

They also describe the stabilisers.

Lemma 18 (Ho–Shankar–Varma, \cite{22}). The stabiliser of $(A, B)$ corresponding to $R_f$ and the pair $(I, \delta)$ in $\text{SL}_n(T)$ correspond to the norm 1 elements of the 2-torsion in $R_f$, $R_f^2/2|_{N=1}$.

2.3 The parametrisation over fields and $\mathbb{Z}_p$

Over a field or $\mathbb{Z}_p$, for some prime $p$, the parametrisation reduces to the following.

Lemma 19 (Ho–Shankar–Varma, \cite{22}). Let $f$ be a monic separable non-degenerate binary $n$-ic form with coefficients in $T$, for $T$ a field or $\mathbb{Z}_p$. The projective $\text{SL}_n(T)$-orbits of $V(T)$ with invariant binary $n$-ic form $f$ are in bijection with $\left(\frac{R_f^\times}{(R_f^\times)^2}\right)_{N=1}$.

2.4 The parametrisation over $\mathbb{Z}$

In this section, we relate the integral orbits to 2-torsion in the class group following Ho–Shankar–Varma.

Let $\mathcal{O}$ be an order in a degree $n$ number field whose galois group is $S_n$. We let $H(\mathcal{O})$ denote the set of pairs $(I, \delta)$ consisting of a fractional ideal $I \subset \mathcal{O}$ and an element $\delta \in K^\times$ such that $I^2 \subset (\delta)$,
\[ N(I)^2 = N(\delta) \] and such that the ideal \( I \) is projective (i.e. invertible as a fractional ideal). The set \( H(O) \) is equipped with a natural composition law defined by component wise multiplication.

There is a map from \( H(O) \) to the 2-torsion of the class group of the order \( O \) given by forgetting about the \( \delta \) component. The fibres of this map depend only on the rank of the unit group. It is also possible to relate \( H(O) \) to 2-torsion in the narrow class group of \( O \). The following is done in Ho–Shankar–Varma.

**Lemma 20** (Ho–Shankar–Varma, [22]). Let \( O \) be an order in an \( S_n \)-number field of degree \( n \) and signature \((r_1, r_2)\). Then

\[
|H(O)| = 2^{r_1 + r_2 - 1}|Cl_2(O)|.
\]

Furthermore, if \( H^+(O) \) denotes the subgroup of \( H(O) \) consisting of pairs \((I, \delta)\) such that \( \delta \) is positive under every real embedding of the fraction field of \( O \), then

\[
|H^+(O)| = 2^{r_2}|Cl_2^+(O)|.
\]

Finally, we record two theorems from Ho–Shankar–Varma. They characterise those elements of \( V(\mathbb{Z}) \) which correspond to the 2-torsion subgroup of the ideal group of \( O \), \( I_2(O) \). That is the group of fractional ideals of \( O \) with the property that \( I^2 = O \). If \( O \) is a maximal order, the only element in \( I_2(O) \) is the trivial element of the class group of \( O \).

**Definition 21.** A pair \((A, B) \in V(\mathbb{Q}) \) is said to be reducible if the quadrics corresponding to \( A \) and \( B \) in \( \mathbb{P}^{n-1}(\mathbb{Q}) \) have a dimension \((n - 1)/2 \) common rational isotropic subspace.

**Lemma 22** (Ho–Shankar–Varma, [22]). Let \((A, B) \) be a projective element \( V(\mathbb{Z}) \) with primitive, irreducible, and non-degenerate resultant form. Let \((I, \delta)\) be the corresponding pair. Then \( \delta \) is a square in \( (R_f \otimes \mathbb{Q})^\times \) if and only if \((A, B) \) is reducible.

The following result characterises the elements of \( V \) which correspond to elements of \( I_2(O) \) and is due to Ho–Shankar–Varma.

**Lemma 23** (Ho–Shankar–Varma, [22]). Let \( O_f \) be an order corresponding to the integral primitive irreducible and non-degenerate binary \( n \)-ic form \( f \). Then \( I_2(O_f) \) is in natural bijection with the set of projective reducible \( SL_n(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \cap \pi^{-1}(f) \).

Later, we will show that these elements make up most of the cusp and only a negligible fraction of the elements in the main body.

### 2.5 The counting problem

We count the average number of 2-torsion elements in the class group of monogenised rings and fields of odd degree. To make sense of this, we order the monogenic fields using the naive height on the minimal polynomial of a generator of the ring of integers whose trace is in \([0, n)s\). Take a monic integral polynomial \( f(x) = x^n + a_1 x^{n-1} + \ldots + a_n \in \mathbb{Z}[x] \). We define the naive height of \( f \) by:

\[
H(f) := \max\{|a_i|^{1/i}\} = \max\{|a_1|, |a_2|^{1/2}, \ldots, |a_n|^{1/n}\}.
\]

Note that \( H \) has the property that

\[
H(\lambda B) = \lambda H(f)
\]
so that \( H \) is homogeneous of degree 1. This will be needed when we apply arguments from the geometry of numbers. The goal of the paper is to determine the following averages:

\[
\lim_{X \to \infty} \frac{\sum_{O \in R} |\text{Cl}_2(O)| - |I_2(O)|}{\sum_{O \in R} 1}
\]

and

\[
\lim_{X \to \infty} \frac{\sum_{O \in R} |\text{Cl}_2^+(O)| - |I_2(O)|}{\sum_{O \in R} 1},
\]

where \( R \subset \mathfrak{R}^{r_1,r_2} \) is any acceptable family of monogenic rings (an acceptable family is one which includes all rings with squarefree discriminant).

The asymptotic formula for the denominator comes from the work of Bhargava–Shankar–Wang, \cite{Bhargava2013}.

**Theorem 24** (Bhargava–Shankar–Wang, \cite{Bhargava2013}). Let \( S = (S_p) \) be an acceptable collection of local specifications. If \( 0 \leq b < n \) is fixed and \( U_{1,b} \) denotes the set of monic polynomial whose \( x^{n-1} \) coefficient is \( b \), then we have

\[
\left| U_{1,b}^r(S)_{\leq X} \right| = \text{Vol}(U_{1,b}^r(\mathbb{R})_{<X}) \prod_p \text{Vol}(S_p) + o(X^{\frac{n(n+1)}{2} - 1}).
\]

As \( \text{Vol}(S_{\infty,H<X}) \) grows like \( X^{\frac{n(n+1)}{2} - 1} \), the main term dominates the error term. There is a power saving error term in their work, but we will not need it in what follows.

### 3 Reduction theory

Fix an element \( A \in \mathcal{L}_Z \) and \( \delta \in \mathcal{T}(r_2) \). We build a finite cover of the fundamental domain for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( V_{r_2,\delta}^A(\mathbb{R}) \).

**Definition 25.** The height of an element in \( B \in V_{r_2,\delta}^A \) is defined to be the height of the associated resolvent polynomial. That is, \( H(B) := H \left( (-1)^{n-1} \det(Ax - B) \right) \).

The construction of \cite{Bhargava2013} can be adapted to give a fundamental set \( R_{r_2,\delta}^A \) for the action of \( \text{SO}_A(\mathbb{R}) \) on \( V_{r_2,\delta}^A(\mathbb{R}) \) (which could be empty) with the following properties:

1. The set \( R_{r_2,\delta}^A \) is a semi-algebraic.

2. If \( R_{r_2,\delta}^A(X) \) denotes the set of elements of height at most \( X \), then the coefficients of elements \( B \in R_{r_2,\delta}^A(X) \) are bounded by \( O(X) \). The implied constant is independent of \( B \).

We define an indicator function that records whether \( V_{r_2,\delta}^A(\mathbb{R}) \) is empty.
Definition 26. We define the indicator function
\[
\chi_A(\delta) := \begin{cases} 
1 & \text{if } V_A^{r_2,\delta}(\mathbb{R}) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}.
\]

We can build now build a cover of a fundamental domain for the action of \(SO_A(\mathbb{Z})\) on \(V_A^{r_2,\delta}(\mathbb{R})\). To do so, we pick a fundamental domain \(F_A\) for the action of \(SO_A(\mathbb{Z})\) on \(SO_A(\mathbb{R})\) and act on \(V_A^{r_2,\delta}(\mathbb{R})\). This gives a \(\sigma(r_2)\) cover of a fundamental domain for the action of \(SO_A(\mathbb{Z})\) where \(\sigma(r_2) = 2^{r_1 + r_2 - 1}\) is the size of the stabiliser in \(SO_A(\mathbb{R})\) of an element \(v \in V_A^{r_2,\delta}(\mathbb{R})\).

Proposition 27. Let \(F_A\) be a fundamental domain for the action of \(SO_A(\mathbb{Z})\) on \(SO_A(\mathbb{R})\). Then
1. If \(\chi_A(\delta) = 1\), \(F_A \cdot R_A^{r_2,\delta}\) is an \(\sigma(r_2)\)-fold cover of a fundamental domain for the action of \(SO_A(\mathbb{Z})\) on \(V_A^{r_2,\delta}(\mathbb{R})\), where we regard \(F_A \cdot R_A^{r_2,\delta}\) as a multiset.
2. If \(\chi_A(\delta) = 0\), then \(\emptyset\) is a fundamental domain.

Proof. The stabiliser in \(SO_A(\mathbb{R})\) of an element \(B \in V_A^{r_2,\delta}(\mathbb{R})\) coincides with the stabiliser in \(SL_n(\mathbb{R})\) of \((A, B)\) which has size \(\sigma(r_2)\). \(\square\)

Remark 28. The characteristic functions will be used to define the Archimedean mass and will make the steps of the computation in Section 9 more transparent.

4 Averaging and cutting off the cusp

There are two distinct cases, the case where \(A\) is anisotropic over \(\mathbb{Q}\) and the case where \(A\) is isotropic over \(\mathbb{Q}\). In each case, we need to establish that: 1) the number of absolutely irreducible integral points in the cuspidal region is negligible, and 2) the number of reducible integral points in the main body is negligible.

We define absolutely irreducible points and reducible points and set the notation for the remainder of this section.

Definition 29. An element \(v \in V(\mathbb{Z})\) is said to be absolutely irreducible if \(v\) does not correspond to the identity element in the class group and the resolvent of \(v\) corresponds to an order in an \(S_n\)-field. An element which is not absolutely irreducible is said to be reducible.

We have the following theorem which gives conditions on reducibility. It is a restatement of the criterion which appears in Ho–Shankar–Varma \([22]\).

Theorem 30 (Reducibility criterion, \([22]\)). Let \((A, B) \in V(\mathbb{Z})\) be such that all the variables in one of the following sets vanish. Then \((A, B)\) is reducible.
1. The squares: \(\{a_{ij}, b_{ij} \mid 1 \leq i, j \leq \frac{n-1}{2}\}\).
   \(\) These pairs correspond to the identity element in the class group.
2. The rectangles: \(\{a_{ij}, b_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq n - k\}\) for some \(1 \leq k \leq n - 1\).
   \(\) These pairs correspond to the resolvent having repeated roots.

Definition 31. Let \(A\) be a fixed quadratic form in \(\mathbb{L}_\mathbb{Z}\) and fix \(0 \leq b < n\). We let \(V_A \subset V\) denote the space of pairs \((A, B)\), where \(B\) is arbitrary. Note that the resolvent map takes \(V_A\) to \(U\). Now, we let \(V_{A,b}\) denote the inverse image under the resolvent map of the set \(U_b\). It is easy to see that \(V_{A,b}\) is an affine subspace of \(V_A\) of dimension \(\frac{n(n+1)}{2} - 1\).
Definition 32. Let \( S \subset V_{A,b}^{r_2,\delta}(\mathbb{Z}) := V_{A,b}^{r_2,\delta}(\mathbb{R}) \cap V_{A,b}(\mathbb{Z}) \) be an \( \text{SO}_A(\mathbb{Z}) \) invariant set. Denote by \( N(S; X) \) the number of absolutely irreducible \( \text{SO}_A(\mathbb{Z}) \)-orbits on \( S \) that have height bounded by \( X \). For any \( L \subset V_A(\mathbb{Z}) \), let \( L^\text{irr} \) denote the set of absolutely irreducible elements. Note that any absolutely irreducible element has a resolvent form corresponding to an order \( \mathcal{O} \) in an \( S \) number field and so \( \mathcal{O}^\times [2] \cap H = 1 \) is trivial. As a result, the stabiliser in \( \text{SO}_A(\mathbb{Z}) \) of absolutely irreducible elements is trivial.

Therefore, we have

\[
N(S; X) = \frac{1}{\sigma(r_2)} \#\{ F_A \cdot R_{A,b}^{r_2,\delta}(X) \cap S^\text{irr} \}.
\]

The goal of this section is to obtain an asymptotic formula for \( N_H(S; X) \).

4.1 The case of \( A \) anisotropic over \( \mathbb{Q} \)

When \( A \) is anisotropic, we can pick a compact fundamental domain \( \mathcal{F}_A \) for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( \text{SO}_A(\mathbb{R}) \). It then follows that \( \mathcal{F}_A \cdot R_{A,b}^{r_2,\delta} \) is bounded. To estimate the number of absolutely irreducible integral points in the fundamental domain for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( V_{A,b}^{r_2,\delta} \), we can apply results from the geometry of numbers directly. The goal here is to use Davenport’s refinement of the Lipschitz method on \( \mathcal{F}_A \cdot R_{A,b}^{r_2,\delta}(X) \) to obtain the desired asymptotic formula.

We will need the following version of Davenport’s lemma.

Lemma 33 (Davenport’s Lemma). Let \( E \subset \mathbb{R}^n \) be a bounded semi-algebraic multiset with maximum multiplicity at most \( m \) which is defined by \( k \) algebraic inequalities of each having degree at most \( l \). Let \( E' \) be the image of \( E \) under any upper/lower triangular unipotent transformation. Then the number of integral points in \( E' \) counted with multiplicity is

\[
\text{Vol}(E) + O_{m,k,l} \left( \max\{ \text{Vol}(E), 1 \} \right)
\]

where \( \text{Vol}(E) \) denotes the greatest \( d \)-dimensional volume of a projection of \( E \) onto a \( d \)-dimensional coordinate hyperplane for \( 1 \leq d \leq n - 1 \).

Remark 34. We note that although Davenport’s lemma holds in the more general setting of \( o \)-minimal structures, the most common use is in the semi-algebraic setting.

Lemma 35. The number of integral points in \( \mathcal{F}_A \cdot R_{A,b}^{r_2,\delta} \) which are not absolutely irreducible is bounded by \( o \left( X^{\frac{n(n+1)}{2} - 1} \right) \).

Proof. We may write

\[
V(\mathbb{Z}) = \left( \cup V(\mathbb{Z})^{\neq k} \right) \bigcup V(\mathbb{Z})^{\text{red}},
\]

where \( V(\mathbb{Z})^{\text{red}} \) denotes elements which are reducible in the sense of Theorem 30.

Fix a prime \( p \). Let \( V(\mathbb{F}_p)^{=k} \) denote the set of elements whose resolvent factors into a product of an irreducible factor of degree \( k \) and \( n - k \) distinct linear factors. Let \( V(\mathbb{F}_p)^{\text{irr}} \) denote the set of elements of \( V(\mathbb{F}_p)^{\text{irr}} \) with the property that every lift to \( V(\mathbb{Z}) \) does not belong to \( V(\mathbb{Z})^{\text{red}} \). We obtain the same estimates as in Ho–Shankar–Varma, and this finishes the proposition.

Theorem 36. Let \( A \in \mathcal{L}_A \) be anisotropic over \( \mathbb{Q} \). We have

\[
N(V_{A,b}^{r_2,\delta}(\mathbb{Z}); X) = \frac{1}{\sigma(r_2)} \text{Vol} \left( \mathcal{F}_A \cdot R_{A,b}^{r_2,\delta}(X) \right) + o(X^{\frac{n(n+1)}{2} - 1}).
\]
4.2 The case of $A$ isotropic over $\mathbb{Q}$

Suppose now that $A$ is isotropic over $\mathbb{Q}$. Then there exists for some unique pair $p, q$ such that $n = p + q$ and $\frac{n-1}{2} \equiv q \mod 2$, there exists an element $g_A \in SL_n(\mathbb{Q})$ such that $g_A^t A g_A = A_{pq}$ where

$$A_{pq} := \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & \pm I_{[p-q]}
\end{pmatrix}.$$

The $\pm$ on the identity block is determined by the sign of $p - q$. We define $m$ to be the minimum of $p$ and $q$, $m = \min\{p, q\}$.

**Remark 37.** In general, ($n$-monogenic or even), we can take $A$ to a matrix of the form above over $\mathbb{Q}$ with the identity block replaced by an anisotropic quadratic form over $\mathbb{Q}$ having the same determinant as $A$ and in diagonal form.

Now for $K = \mathbb{R}$ or $\mathbb{Q}$, we consider the maps

$$\sigma_V: V^{r_2, \delta}_{A,b} \to V^{r_2, \delta}_{A_{pq}, b}$$

$$\sigma_A: SO_A(K) \to SO_{A_{pq}}(K)$$

defined by $\sigma_V(A, B) = (A_{pq}, g_A^t B g_A)$ and $\sigma_A(h) = g_A^t h (g_A)^{-1}$. We note that

$$H(A, B) = H(\sigma_V(A, B))$$

since $\pi \circ \sigma_V = \pi$. Furthermore, $\sigma_V(h \cdot v) = \sigma_A(h) \cdot \sigma_V(v)$.

Now, we denote by $\mathcal{L} \subset V^{r_2, \delta}_{A_{pq}, b}(\mathbb{R})$ the lattice $\sigma_V \left( V^{r_2, \delta}_{A_{pq}, b}(\mathbb{Z}) \right)$. We denote by $\Gamma \subset SO_{A_{pq}}(\mathbb{R})$ the subgroup $\sigma_A(SO_A(\mathbb{Z}))$. This subgroup is commensurable with $SO_{A_{pq}}(\mathbb{Z})$. Therefore, there exists a fundamental domain $\mathcal{F}$ for the action of $\Gamma$ on $SO_{A_{pq}}(\mathbb{R})$ which is contained in a finite union of $SO_{A_{pq}}(\mathbb{Q})$ translates of a Siegel domain, $\bigcup g_i S$ for $g_i \in SO_{A_{pq}}(\mathbb{Q})$. This is known from [12].

The choice of the standard $A_{pq}$ as above is convenient at this point. Indeed, we may now choose as our Siegel domain $\mathcal{S}$, the product $NTK$ where we choose $K$ to be compact, $N$ to be a subgroup of the group of lower triangular matrices with 1 on the diagonal and $T$ to be

$$T := \begin{pmatrix}
(t_1^{-1} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & (t_m^{-1} & \cdots \\
0 & \cdots & 0 & t_1
\end{pmatrix} : t_1/t_2 > c, \ldots, t_{m-1}/t_m > c, t_m > c$$

for some constant $c > 0$. This can be found in many sources, see for instance [11], [27], or [26].
Note that $s_i = t_i/t_{i+1}$, $0 \leq i \leq m - 1$ and $s_m = t_m$ forms a set of simple roots. Moreover, if we denote by $e^\rho$ the exponential of the half sum of the positive roots counted with multiplicities, we have

$$e^\rho(H) = \prod_{i=1}^{m} t_i^{\frac{p+q}{2} - i}$$
$$= \prod_{i=1}^{m} \left( \prod_{j=i}^{m} s_j \right)^{\frac{p+q}{2} - i}$$
$$= \prod_{i=1}^{m} s_i^{\left( \sum_{j=1}^{i} \frac{p+q}{2} - j \right)}$$
$$= \prod_{i=1}^{m} s_i^{i\left( \frac{p+q}{2} - 1 \right)}.$$

We now fix some notation for our choice of Haar measure on $G = \text{SO}_{Apq}$. We let $dg$ denote the Haar measure on $G$, $dn$ denote the Haar measure on the unipotent group $N$, and $dk$ denote the Haar measure on the compact group $K$. For every $1 \leq i \leq m$ we write $d^x t_i = \frac{dt_i}{t_i}$ and $d^x s_i = \frac{ds_i}{s_i}$. Furthermore, we write $dt = \prod_{i=1}^{m} dt_i$, $d^x t = \prod_{i=1}^{m} d^x t_i$ and $ds = \prod_{i=1}^{m} ds_i$, $d^x s = \prod_{i=1}^{m} d^x s_i$.

Changing variables between the $t$-coordinates and the $s$-coordinates gives us

$$dt = \left( \prod_{i=1}^{m} s_i^{i-1} \right) ds.$$

We thus find

$$d^x t = \frac{1}{t_1 \cdots t_m} dt$$
$$= \frac{1}{\prod_{i=1}^{m} s_i^{i-1}} \left( \prod_{i=1}^{m} s_i^{i-1} \right) ds$$
$$= \frac{1}{s_1 \cdots s_m} ds$$
$$= d^x s.$$

Therefore, the Haar measure is given in $NAK$-coordinates by

$$dg = e^{-2\rho(H)} du d^x t dk$$
$$= \prod_{i=1}^{m} t_i^{2i-p-q} du d^x t dk$$
$$= \prod_{i=1}^{m} s_i^{i(i+1-p-q)} du d^x s dk.$$ 

We define the main body and the cuspidal region of the multiset $F_A \cdot R^{r_2, \delta}_{A,b}$.

**Definition 38** (Main body and cuspidal region). The main body consists of all the elements of $F_A \cdot R^{r_2, \delta}_{A,b}$ for which $|b_{11}| \geq 1$. The cuspidal region consists of all the elements for which $|b_{11}| < 1$. 

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We are now ready to cut off the cuspidal region.

**Construction 39** (Partial order on the coordinates of $V_A$). We construct a partial order on the $n(n + 1)/2$ coefficients $\{b_{ij}\}$ for $i \leq j$. These define a set of coordinates on $B$ which we denote by $U$.

The partial order records the scaling of the different elements of $B$ under the action of the torus.

**Definition 40.** The weight $w(b_{ij})$ of an element $b_{ij} \in U$ is the factor by which $b_{ij}$ scales under the action of $(t_i^{-1}, \ldots, t_{m+1}^{-1}, 1, \ldots, 1, t_{m}, \ldots, t_1) \in T$.

We compute the weights in both the $t$-coordinates and the $s$-coordinates on $T$, recalling that $i \leq j$:

1. $w(b_{11}) = t_1^{-2} = s_1^{-2} \cdots s_m^{-2}$;
2. $w(b_{ij}) = t_i^{-1}t_j^{-1} = s_i^{-1} \cdots s_j^{-1}s_{j+1}^{-2} \cdots s_n^{-2}$ if $i \leq m$ and $j \leq m$;
3. $w(b_{ij}) = t_i^{-1} = s_i^{-1} \cdots s_n^{-1}$ if $i \leq m$ and $m + 1 \leq j \leq m + |p - q|$
4. $w(b_{ij}) = t_i^{-1}t_{n+1-j} = s_i^{-1} \cdots s_{n-j}^{-1}$ if $i \leq m$ and $m + |p - q| + 1 \leq j \leq n$;
5. $w(b_{ij}) = 1$ if $m + 1 \leq i \leq m + |p - q|$ and $m + 1 \leq j \leq m + |p - q|$
6. $w(b_{ij}) = t_{n+1-j} = s_{n-j+1} \cdots s_n$ if $m + 1 \leq i \leq m + |p - q|$ and $m + |p - q| + 1 \leq j \leq n$;
7. $w(b_{ij}) = t_{n-i+1}t_{n-j+1} = s_{n-i+1} \cdots s_{n-j}s_{n-j+1}^{2} \cdots s_n^{2}$ if $m + |p - q| + 1 \leq i \leq n$ and $m + |p - q|$.

We are now ready to define a partial order on $U$.

**Definition 41** (A partial order on subsets of $U$). Let $b$ and $b'$ be two elements of the set of coordinates $U$. We say that $b < b'$ if in the expression for $w(b)$ in the $s$-coordinates, the exponents of the variables $s_1, \cdots, s_m$ are smaller than or equal to the corresponding exponents appearing in the expression for $w(b')$ in the $s$-coordinates. The relation $<$ defines a partial order on $U$.

**Example 42.** We have $b_{11} < b_{m+1,m+1}$ because $w(b_{11}) = s_1^{-2} \cdots s_m^{-2}$ while $w(b_{m+1,m+1}) = 1 = s_1^0 \cdots s_m^0$. On the other hand, $b_{1,n-2}$ and $b_{2,n-3}$ cannot be compared in $<$ because $w(b_{1,n-2}) = s_1^{-1}s_2^{-1}$ while $w(b_{2,n-3}) = s_2^{-1}s_3^{-1}$. The important thing to note about the partial order $(U, <)$ is that if $i \leq i'$ and $j \leq j'$ then $b_{ij} < b_{i'j'}$.

We now take a closer look at the process of cutting off the cusp in a specific case before moving on to the general case.

**Example 43** (Base case of cusp cutting induction for $|p-q| > 1$). As the first non-trivial interesting example, let us consider the case $n = 5$, $m = 1$. Then $A_{14}$ is given by

$$A_{14} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix}.$$
The torus is

\[ T = \begin{cases} \left( \begin{array}{ccc} t^{-1} & 1 & 1 \\ 1 & 1 & t \end{array} \right) : t > c \end{cases}. \]

The s-coordinates are the same as the t-coordinates and the Haar measure takes the form

\[ dg = du \frac{1}{t^3} d^xt^kd. \]

We now record how elements of the torus act on elements of \( K \cdot R^{r_2, \delta}_{A_{14}, b}(X) \). Remember that the action of \( G = SO_{A_{14}} \) is given by conjugation \( g \cdot v = gvg^t \). Letting \( T \) act on \( K \cdot R^{r_2, \delta}_{A_{14}, b}(X) \) we find

\[ t \cdot v = \begin{pmatrix} t^{-2} & t^{-1} & t^{-1} & t^{-1} & 1 \\ t^{-1} & 1 & 1 & 1 & t \\ t^{-1} & 1 & 1 & 1 & t \\ t^{-1} & 1 & 1 & 1 & t \\ 1 & t & t & t & t^2 \end{pmatrix} O(X). \]

From this, it is easy to read off the weights.

Finally, the elements of the group \( N \) have the form:

\[ \begin{pmatrix} A_1 \\ B_2 \\ I_{|p-q|} \\ C \\ B_3 \\ A_3 \end{pmatrix} \]

where \( A_1 \) and \( A_2 \) are lower triangular with 1s on the diagonal and there are relations among \( A_1, A_3, B_2, B_3 \) and \( C \).

For any subset of \( U \) containing \( b_{11} \), we now want to estimate

\[ \tilde{I}(U_1, X) = X^{14} \int_{t \leq c} \prod_{b_{ij} \notin U_1} w(b_{ij}) \frac{d^xt^k}{t^3}. \]

For this example, we can do this very concretely. Remember that in \( T_X \) we have the bound \( t < CX \). We calculate some values to get a better feel for the situation.

\[ \tilde{I}(\emptyset, X) = X^{14} \int_{t = c}^{CX} \frac{d^xt^k}{t^3} = X^{14} \int_{t = c}^{CX} \frac{dt}{t^4} = O(X^{11}). \]

If \( U_1 \) is any subset of the middle block, we also obtain

\[ \tilde{I}(U_1, X) = X^{14} \int_{t = c}^{CX} \frac{d^xt^k}{t^3} = X^{14} \int_{t = c}^{CX} \frac{dt}{t^4} = O(X^{11} - \#U_1). \]

Now, let us examine strict subsets of the first ending at the off-anti-diagonal. By the monotonicity structure of \( \prec \), we only need to find three different integrals.

\[ \tilde{I}(\{b_{11}, b_{12}, b_{13}\}, X) = X^{11} \int_{t = c}^{CX} \frac{d^xt^k}{t^3} = X^{11} \int_{t = c}^{CX} dt = O(X^{12}). \]
\[
\tilde{I}(\{b_{11}, b_{12}\}, X) = X^{12} \int_{t = c}^{CX} t^3 d^x t t^3 = X^{12} \int_{t = c}^{CX} \frac{dt}{t} = O(X^{12} \ln X) = O_\epsilon(X^{12+\epsilon})
\]

\[
\tilde{I}(\{b_{11}\}, X) = X^{13} \int_{t = c}^{CX} t^2 d^x t t^2 = X^{13} \int_{t = c}^{CX} \frac{dt}{t^2} = O(X^{12})
\]

Therefore,
\[
N(V_A(\mathbb{Z})(U_1); X) = O_\epsilon(X^{12+\epsilon})
\]

for all \(U_1 \subset U\) such that \(b_{11} \in U_1\). The number of absolutely irreducible elements in the cusp which have height at most \(X\) is thus \(O_\epsilon(X^{14-1+\epsilon})\) and we have cut off the cusp!

**Example 44** (Base case of cusp cutting induction for \(|p - q| = 1\)). We now do the case \(n = 5\), \(m = 2\) before moving on to cutting off the cusp in the general case. We see that torus elements act as follows:

\[
 t \cdot v = \begin{pmatrix} t_1^{-2} & t_1^{-1}t_2^{-1} & t_1^{-1}t_2 & 1 \\ t_2^{-1}t_1^{-1} & t_2^{-1} & t_2 & t_1 \\ t_1^{-1} & 1 & t_2 & t_2t_1 \\ t_2^{-1} & 1 & t_1 & t_1t_2 \\ t_1^{-1} & 1 & t_2 & t_1t_2 & t_1 \end{pmatrix} O(X).
\]

From this, it is easy to read off the weights. The Haar measure takes the form

\[
dg = du \frac{1}{t_1t_2} d^x t dk = du \frac{1}{s_1s_2^3} d^x s dk.
\]

For any subset of \(U\) containing \(b_{11}\), we now want to estimate

\[
\tilde{I}(U_1, X) = X^{14 - \#U_1} \int_{t \in TX} \prod_{b_{ij} \not\in U_1} w(b_{ij}) \frac{d^x s}{s_1s_2^3}.
\]

We only need to look at proper subsets of \(U_0 = \{b_{11}, b_{12}, b_{13}, b_{22}\}\) which are left-closed and up-closed. In this case we can exclude the subset \(\{b_{11}, b_{12}, b_{22}\}\) by the **Squares** part of the criterion for irreducibility. and recall that we have the bound \(s_1, s_2 < CX\). Let’s compute:

\[
\tilde{I}(\{b_{11}\}, X) = X^{13} \int_{s_1, s_2 = c}^{CX} s_1^{\frac{3}{2}} s_2^{\frac{5}{2}} d^x s = X^{13} \int_{s_1, s_2 = c}^{CX} \frac{d^x s}{s_1s_2^3} = O(X^{10})
\]

\[
\tilde{I}(\{b_{11}, b_{12}\}, X) = X^{12} \int_{s_1, s_2 = c}^{CX} s_1^{\frac{3}{4}} s_2^{\frac{5}{4}} d^x s = X^{12} \int_{s_1, s_2 = c}^{CX} \frac{d^x s}{s_1s_2^3} = O(X^{12+\epsilon})
\]

\[
\tilde{I}(\{b_{11}, b_{12}, b_{13}\}, X) = X^{11} \int_{s_1, s_2 = c}^{CX} s_1^{\frac{3}{4}} s_2^{\frac{5}{4}} d^x s = X^{11} \int_{s_1, s_2 = c}^{CX} \frac{d^x s}{s_1s_2^3} = O(X^{12+\epsilon})
\]

\[
\tilde{I}(\{b_{11}, b_{12}, b_{13}\}, X) = X^{11} \int_{s_1, s_2 = c}^{CX} s_1^{\frac{3}{4}} s_2^{\frac{5}{4}} d^x s = X^{11} \int_{s_1, s_2 = c}^{CX} \frac{d^x s}{s_1s_2^3} = O(X^{12+\epsilon})
\]

\[
\tilde{I}(\{b_{11}, b_{12}, b_{22}\}, X) = X^{11} \int_{s_1, s_2 = c}^{CX} s_1^{\frac{3}{2}} s_2^{\frac{5}{2}} d^x s = X^{11} \int_{s_1, s_2 = c}^{CX} \frac{d^x s}{s_1s_2^3} = O(X^{13+\epsilon}).
\]

The number of absolutely irreducible elements in the cusp which have height at most \(X\) is thus \(O_\epsilon(X^{14-1+\epsilon})\) and we have cut off the cusp!
We recall that we had

\[ N_H(S; X) = \frac{1}{\sigma(r_2)} \# \{ \mathcal{F}_A \cdot R_{s, \delta}^r(X) \cap S^{\text{irr}} \}. \]

Now, let \( G_0 \) be a bounded open \( K \)-invariant ball in \( \text{SO}_{A,p}(\mathbb{R}) \). We can average the above expression by the usual trick to obtain

\[ N_H(S; X) = \frac{1}{\sigma(r_2) \text{Vol}(G_0)} \int_{h \in \mathcal{F}_A} \# \left\{ hG_0 R_{s, \delta}^r(X) \cap S^{\text{irr}} \right\} dh. \]

Now, again we may use classical arguments to see that the number of absolutely irreducible integral points in the cusp which have height at most \( X \) is

\[ O \left( \int_{t \in T} \# \left\{ tG_0 R_{s, \delta}^{r, \gamma}(X) \cap S^{\text{irr}} \right\} \prod_{i=1}^m s_i^{i(i+1-p-q)} d^X s \right). \]

**Definition 45.** Let \( U_1 \subset U \) be a subset of the set of coordinates. We define

\[ V_A(\mathbb{R})(U_1) = \{ B \in V_A(\mathbb{R}) : |b_{ij}(B)| < 1 \text{ if and only if } b_{ij} \in U_1 \} \]

and

\[ V_A(\mathbb{Z})(U_1) = V_A(\mathbb{Z}) \cap V_A(\mathbb{R})(U_1). \]

It thus suffices to show that

\[ N(V_A(\mathbb{Z})(U_1); X) = O_{\epsilon} \left( X^{\frac{n(n+1)}{2}-1+\epsilon} \right) \]

for all \( U_1 \subset U \) such that \( b_{11} \in U_1 \).

We now explain how the description of reducible elements gives us a priori bounds on the coordinates \( s_i \). Let \( C \) be an absolute constant such that \( CX \) bounds the absolute value of all the coordinates of elements \( B \in G_0 R_{s, \delta}^{r, \gamma}(X) \).

If \( (s_1^{-1}, \ldots, s_m^{-1}, 1, \ldots, 1, s_m, \ldots, s_1) \in T \) and \( CXw(b_{i_0,n-i_0}) < 1 \) for some \( i_0 \in \{1, \ldots, m\} \), then \( CXw(b_{ij}) < 1 \) for all \( i \leq i_0 \) and \( j \leq n - i_0 \). This comes from the **Rectangles** part of the criterion for reducibility. Therefore, we may assume that

\[ s_i < CX \]

for all \( i \in \{1, \ldots, m\} \).

Let us write \( T_X \) to denote the set of \( t = (s_1^{-1}, \ldots, s_m^{-1}, 1, \ldots, 1, s_m, \ldots, s_1) \in T \) which satisfy this condition.

Now Davenport’s lemma gives us

\[ N(V(\mathbb{Z})(U_1); X) = O \left( \int_{t \in T_X} \text{Vol}(tG_0 R_{s, \delta}^{r, \gamma}(X) \cap V(\mathbb{R})(U_1)) \prod_{i=1}^m s_i^{i(i+1-p-q)} d^X s \right) \]

\[ = O \left( X^{\frac{n(n+1)}{2}-1} \# U_1 \int_{t \in T_X} \prod_{b_{ij} \not\in U_1} w(b_{ij}) \prod_{i=1}^m s_i^{i(i+1-n)} d^X s \right). \]

So, we have reduced our problem to one of estimating the following integrals.
Definition 46. The active integral of $U_1 \subset U$ is defined by
\[
\tilde{I}(U_1, X) := X^{\frac{n(n+1)}{2} - \#U_1} \int_{t \in T_X} \prod_{b_{ij} \notin U_1} w(b_{ij}) \prod_{i=1}^m s_i^{i(i+1-n)} d^x s.
\]

Recall, that $b_{ij} \prec b_{i_0 j_0}$ when $i \leq i_0$ and $j \leq j_0$. Therefore, if $U_1 \subset U$ contains $b_{i_0 j_0}$ but not $b_{ij}$, then
\[
\tilde{I}(U_1 \setminus \{b_{i_0 j_0}\} \cup \{b_{ij}\}, X) \geq \tilde{I}(U_1, X).
\]
As a result, in order to obtain an upper bound for $\tilde{I}(U_1, X)$ we may assume that if $b_{i_0 j_0} \in U_1$, then $b_{ij} \notin U_1$ for all $i \leq i_0$ and $j \leq j_0$.

Furthermore, suppose $U_1$ contains any element on, or on the right of, the off anti-diagonal within the first $m$-rows. In that case, we are in the case of Rectangles in the criterion for reducibility and so $N(V(Z)(U_1); X) = 0$.

Definition 47. We define the subset $U_0 \subset U$ as the set of coordinates $b_{ij}$ such that $i \leq j$, $i \leq m$, and $i + j \leq n - 1$.

Now, if $|p - q| = 1$, every element in $V(Z)(U_0)$ is reducible and it suffices to consider $\tilde{I}(U_1, X)$ for all $U_1 \subset U_0$. On the other hand if $|p - q| > 1$ we need to consider all $U_1 \subset U$.

Since the product of the weight over all the coordinates is 1, we make the following definition.

Definition 48. We define for a subset $U_1 \subset U$
\[
I(U_1, X) = X^{\frac{(n+1)}{2} - \#U_1} \tilde{I}(U_1, X) = X^{-\#U_1} \int_{t \in T_X} \prod_{b_{ij} \in U_1} w(b_{ij})^{-1} \prod_{i=1}^m s_i^{i(i+1-n)} d^x s.
\]

We are now ready to states and prove the main cusp cutting lemma.

Lemma 49 (Main cusp cutting estimate). Let $U_1$ be a non-empty proper subset of $U_0$. Then we have the estimate
\[
I(U_1, X) = O(\epsilon(X^{1+\epsilon})).
\]
We also have $I(\emptyset) = O(\epsilon(X^{\frac{1}{2}m(m+1)(2m-3n+4)+\epsilon}) and I(U_0) = O(\epsilon(X^m(2m+1-n)+\epsilon)$.

Proof. We prove this lemma via a combinatorial argument using induction on $m$. Recall that $n = 2m + |p - q|$. The cases $|p - q| = 1$ and $|p - q| > 1$ turn out to be slightly different. We handle them separately.

To start, let us assume that $|p - q| > 1$. First, we compute $I(U_0, X)$
\[
I(U_0, X) = X^{-\#U_0} \int_{t \in T_X} \prod_{b_{ij} \in U_0} w(b_{ij})^{-1} \prod_{i=1}^m s_i^{i(i+1-n)} d^x s.
\]
\[
= X^{-m(n-(m+1))} \int_{t \in T_X} (t_2^{n-2+1} t_3^{n-4+2} t_4^{n-6+2} \ldots t_m^{n-2m+2}) \prod_{i=1}^m t_i^{2i-n} d^x t
\]
\[
= X^{-m(n-(m+1))} \int_{t \in T_X} t_1 t_2^{n} \ldots t_m d^x t
\]
\[
= X^{-m(n-(m+1))} \int_{T_X} s_1^{3} s_2^{5} \ldots s_m^{2m-1} d^x s
\]
where $U$ estimate in the induction step.

Consequently, we see that

$$I(\emptyset, X) = \int_{s_1, \ldots, s_n = c}^{CX} \prod_{i=1}^{m} s_i^{i(i+1-n)} d^X s = O \left( X^{\frac{1}{2}m(m+1)(2m-3n+4)} \right) = O \left( X^{\frac{1}{2}m(m+1)(-4m-3p-q+4)} \right).$$

Now, let $U'_1$ denote $U_0 \setminus U_1$. Define $I'_m(U'_1, X) := I(U_1, X)$. Then we have:

$$I'_m(U'_1, X) = X^{#U'_1 - m(n-(m+1))} \int_{b_{ij} \in U'_1}^{TX} \left( \prod_{b_{ij} \in U'_1} w(b_{ij}) \right) t_1 t_2 \cdots t_m d^X t.$$

We now work out the base case of the induction. When $m = 1$, we have

$$I_1(\emptyset, X) = O_\epsilon \left( X^{-|p-q|+\epsilon} \right)$$

$$I_1(\{b_{11}\}) = O_\epsilon \left( X^{1-|p-q|+\epsilon} \right)$$

$$I_1(\{b_{11}, \ldots, b_{1k}\}) = O_\epsilon \left( X^{1-|p-q|+\epsilon} \right)$$

$$I_1(U_0, X) = O_\epsilon \left( X^{1-|p-q|+\epsilon} \right).$$

In particular, we see that when $|p-q| > 1$, all these quantities are $O_\epsilon(X^{-1+\epsilon})$. We will use this estimate in the induction step.

Now, suppose that $m \geq 2$.

Now, for any decomposition $k = k_1 + k_2$ we have:

$$\int_{c}^{CX} s^k d^X s \ll_{\epsilon, C} \int_{c}^{CX} s^{k_1} d^X s \int_{c}^{CX} s^{k_2} d^X s.$$

Consequently, we see that $I'_m(U'_1, X)$ is bounded by the product

$$I'_m(U'_1, X) \leq J_m(U'_2, X) K_m(U'_3, X),$$

where $U'_2$ consist of all the elements of $U'_1$ in the first row, $U'_3$ consists of the rest of the elements of $U'_1$, and

$$J_m(U'_2, X) = \left( X^{#U'_2 - (n-2)} \int_{s_1, \ldots, s_n = c}^{CX} \prod_{b_{ij} \in U'_2} w(b_{ij}) \right) s_1 s_2 \cdots s_m d^X s.$$
\[ K_m(U_3', X) = \left( X^\#U_3' - \#U_0 + (n-2) \right) \int_{s_2, \ldots, s_n = c}^{CX} \left( \prod_{b_{ij} \in U_3'} w(b_{ij}) \right) s_2 s_3^2 \cdots s_m^2 d^x s. \]

Note that \( K_m(U_3', X) = I_{m-1}(U_3', X) \) and we can estimate it by induction. Now, \( U_1 \) is left-closed and non-empty and hence the subset \( U_2' \) is either empty or of the form \( \{b_1 k, \ldots, b_{n-2} k\} \) for \( k \geq 2 \).

Now, if \( U_2' = \emptyset \):

\[ J_m(U_2', X) = O_\epsilon \left( X^{2m-1-n+2} \right) = O_\epsilon \left( X^{1-|p-q|+\epsilon} \right) = O_\epsilon \left( X^{-1+\epsilon} \right). \]

Now, if \( k = 2 \), then:

\[ J_m(U_2', X) = X^{-1} \int_{s_1, \ldots, s_n = c}^{CX} t_1^{3-n} s_1 s_2 s_2 \cdots s_m d^x s \]
\[ = O_\epsilon \left( X^{-1+4m(3-n)-(m-1)+(2m-1)+\epsilon} \right) \]
\[ = O_\epsilon \left( X^{-1+3m-m(2m+|p-q|)+m+\epsilon} \right) \]
\[ = O_\epsilon \left( X^{-1+4m-2m^2-m|p-q|+\epsilon} \right) \]
\[ = O_\epsilon \left( X^{-1+\epsilon} \right). \]

Now, if \( k = 3 \), then:

\[ J_m(U_2', X) = X^{-2} \int_{s_1, \ldots, s_n = c}^{CX} t_1^{4-n} s_1 s_2 s_2 \cdots s_m d^x s \]
\[ = O_\epsilon \left( X^{-2+4m(4-n)+(2m-1)+\epsilon} \right) \]
\[ = O_\epsilon \left( X^{-3+6m-2m^2-m|p-q|+\epsilon} \right) \]
\[ = O_\epsilon \left( X^{-1+\epsilon} \right). \]

If \( 4 \leq k \leq m \), then:

\[ J_m(U_2', X) = X^{1-k} \int_{s_1, \ldots, s_n = c}^{CX} t_1^{-(n-2)-k+1} t_k^{-1} \cdots t_m^{-1} t_3 s_1 s_2^2 \cdots s_m d^x s \]
\[ = X^{1-k} \int_{s_1, \ldots, s_n = c}^{CX} t_1^{-(n-2)-k+1} t_3 \cdots t_k s_1 s_2^2 \cdots s_m d^x s \]
\[ = X^{1-k} \int_{s_1, \ldots, s_n = c}^{CX} t_1^{-(n-2)-k+1} s_3 s_4^2 \cdots s_k^{-3} s_1 s_2^2 \cdots s_m d^x s \]
\[ = O_\epsilon \left( X^{1-k + \frac{k-3}{2} (k-2) + 2m-1+m(k+1-2m-|p-q|)+\epsilon} \right) \]
The lemma now follows by induction on $m$. Squares is not a problem though! By the irreducible elements in the cusp at all in this case! It is thus safe to start the induction at $m=1$. Now, for $m=1$ of the induction, we have that $J_1(\emptyset, X)$ is $O(X^{-1})$, while the rest of the cases are $O(1)$. This is not a problem though! By the Squares part of the reducibility criterion there are in fact no irreducible elements in the cusp at all in this case! It is thus safe to start the induction at $m=2$. The calculations of example 3.16 now show that the same estimate of $O_\epsilon(X^{-1+\epsilon})$ hold for this base case. Now, for $m \geq 3$ the rest of the estimates obtained above remain valid. That is

$$J_m(U_2', X) = O_\epsilon(X^{-1+\epsilon}).$$

If $m + 1 \leq k \leq m + |p - q|$, then:

$$J_m(U_2', X) = X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} t_m \cdot \cdots \cdot t_m s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} t_3 \cdot \cdots \cdot t_m s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} s_3 s_4 \cdots s_{m-1} s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= O_\epsilon \left( X^{1-k+\frac{m-3(m-2)}{2}+2m-1+m(k+1-2m-|p-q|)+\epsilon} \right)$$

$$= O_\epsilon(X^{-1+\epsilon}).$$

If $m + |p - q| + 1 \leq k \leq n - 2$, then:

$$J_m(U_2', X) = X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} t_{n-2-k+1} \cdot \cdots \cdot t_m s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} t_3 \cdot \cdots \cdot t_{n-k-1} s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= X^{1-k} \int_{s_1, \ldots, s_n=\epsilon}^{CX} t_1^{-(n-2)-k+1} s_3 s_4 \cdots s_{n-k+1} s_1 s_2^2 \cdots s_m^2 d^k s$$

$$= O_\epsilon \left( X^{1-k+\frac{(n-k-1)(n-k)}{2}+2m-1+m(k+1-2m-|p-q|)+\epsilon} \right)$$

$$= O_\epsilon(X^{-1+\epsilon}).$$

Therefore, in all cases we find

$$J_m(U_2', X) = O_\epsilon(X^{-1+\epsilon}).$$

The lemma now follows by induction on $m$ used to bound $I_{m-1}'(U_3', X)$ by $O_\epsilon(X^{-1+\epsilon})$.

We now explain how to deal with the case $|p - q| = 1$. Recall that in this case we can use the Squares part of the reducibility criterion and assume that $U_1 \neq U_0$. Here, in the base case $m=1$ of the induction, we have that $J_1(\emptyset, X)$ is $O(X^{-1})$, while the rest of the cases are $O(1)$. This is not a problem though! By the Squares part of the reducibility criterion there are in fact no irreducible elements in the cusp at all in this case! It is thus safe to start the induction at $m=2$. The calculations of example 3.16 now show that the same estimate of $O_\epsilon(X^{-1+\epsilon})$ hold for this base case. Now, for $m \geq 3$ the rest of the estimates obtained above remain valid. That is

$$J_m(U_2', X) = O_\epsilon(X^{-1+\epsilon})$$
if \( U'_2 \neq \emptyset \), while \( J_m(\emptyset, X) = O(1) \).

So we can rewrite the induction step as follows. If \( U'_2 \) is non-empty, then the lemma follows by induction on \( m \) used to bound \( I'_{m-1}(U'_3, X) \) by \( O(\epsilon) \). If on the other hand \( U'_2 \) is empty, then \( U'_3 \) must be non-empty since \( U'_1 \) is non-empty (because \( U_1 \neq U_0 \)). If \( U'_3 \neq U_0\{b_1, \ldots, b_{1n-2}\} \), then by induction \( I'_{m-1}(U'_3, X) = O(\epsilon) \). So the last outstanding case is when \( U_1 = \{b_1, \ldots, b_{1n-2}\} \).

In this case, we compute directly:

\[
I(U_0, X) = X^{2-n} \int_{T_X} \prod_{b_{ij} \in U_1} w(b_{ij})^{-1} \prod_{i=1}^{m} s_i^{i(i+1-n)} d^\times s.
\]

\[
= X^{2-n} \int_{s_1, \ldots, s_n = c} \prod_{i=1}^{m} s_i^{i(i+1-n)} d^\times s
\]

\[
= X^{2-n} \int_{s_1, \ldots, s_n = c} \prod_{i=2}^{m} s_i^{i(i+1-n) + n - 2 + 1} d^\times s
\]

\[
= O_\epsilon \left(X^{-\frac{1}{6}m^2 + 3m - 13}\right).
\]

This is \( O_\epsilon(X^{-1+\epsilon}) \) when \( m \geq 2 \). This completes the proof of the main cusp cutting lemma.

\[\Box\]

**Remark 50.** The proof of the previous theorem was inspired by the induction argument of [28].

**Remark 51.** The proof shows that we get much better error terms for \( I(U_1, X) \) than in the statement of the theorem.

**Proposition 52.** The number of absolutely irreducible elements in the cusp which have height at most \( X \) is \( O_\epsilon \left( X^{\frac{n(n+1)}{2} - 1 + \epsilon} \right) \).

As in the anisotropic cases, we find that the number of reducible elements in the main body is negligible.

**Lemma 53.** The number of integral points in the main body of \( \mathcal{F}_A \cdot R_{A, b}^{r_2, \delta} \) which are not absolutely irreducible is bounded by \( o \left( X^{\frac{n(n+1)}{2} - 1} \right) \).

**Proof.** The proof is identical to the anisotropic case. \[\Box\]

**Theorem 54.** Let \( A \in \mathcal{L}_Z \) be isotropic over \( \mathbb{Q} \). We have

\[
N(V_{A,b}^{r_2, \delta}(\mathbb{Z}); X) = \frac{1}{\sigma(r_2)} \text{Vol} \left( \mathcal{F}_A \cdot R_{A, b}^{r_2, \delta}(X) \right) + o \left( X^{\frac{n(n+1)}{2} - 1} \right).
\]

**Remark 55.** By Witt’s decomposition theorem, the results of this section hold in the odd \( N \)-monogenic case verbatim.
5 Sieving to very large and acceptable collections

In this section, we determine the asymptotic formulas for certain families of rings and fields. The results of this section are adaptations of those found in [22]. We begin with the definition of a family of local specifications.

Fix $A \in \mathcal{L}_\mathbb{Z}$ and an integer $0 \leq b \leq n - 1$.

**Definition 56** (Collection of local specifications and the associated set). We say that a family $\Lambda_{A,b} = (\Lambda_{A,b,\nu})_{\nu}$ of subsets $\Lambda_{A,b,\nu} \subset V_{A,b}(O_\nu)$ indexed by the places $\nu$ of $\mathbb{Q}$ is a collection of local specifications if: 1) for each finite prime $p$ the set $\Lambda_{A,b,p} \subset V_{A,b}(\mathbb{Z}_p) \setminus \{\Delta = 0\}$ is an open subset which is non-empty and whose boundary has measure 0; and 2) at $\nu = \infty$, we have $\Lambda_{A,b,\infty} = V_{A,b}^{r_2,\delta}(\mathbb{R})$ for some integer $r_2$ with $0 \leq r_2 \leq \frac{n-1}{2}$ and $\delta \in T(r_2)$. We associate the set $V(\Lambda_{A,b}) := \{v \in V_{A,b}(\mathbb{Z}): \forall \nu (v \in \Lambda_{A,b,\nu})\}$ to the collection of local specifications $\Lambda_{A,b} = (\Lambda_{A,b,\nu})_{\nu}$.

5.1 Sieving to projective elements

**Definition 57.** For a prime $p$, we denote by $V_{A,b}(\mathbb{Z}_p)^{\text{proj}}$ the set of elements $v \in V_{A,b}(\mathbb{Z}_p)$ which correspond to a projective pair $(I, \delta)$ (i.e. with the property that $I^2 = (\delta)$) under the parametrisation.

We have

$$V_{A,b}^{r_2,\delta}(\mathbb{Z}) = V_{A,b}^{r_2}(\mathbb{Z}) \cap \left( \bigcap_p V_{A,b}^{\text{proj}}(\mathbb{Z}) \right).$$

**Definition 58.** We denote by $W_{A,b,p}$ the set of elements in $V_{A,b}(\mathbb{Z})$ that do not belong to $V_{A,b}^{\text{proj}}(\mathbb{Z}_p)$.

We need estimates for the number of elements in $W_{A,b,p}$ for large $p$. We have the following theorem whose proof is an almost verbatim adaptation of the one presented in [22] to the case at hand.

**Theorem 59.** We have

$$N(\cup_{p \geq M} W_{A,b,p}, X) = O \left( \frac{X^{n(n+1)/2 - 1}}{M^{1-\epsilon}} \right) + o \left( X^{n(n+1)/2} \right)$$

where the implied constant is independent of $X$ and $M$.

**Proof.** One shows just as in [22] that $W_{A,b,p} \subset V(\mathbb{Z}_p)$ is the pre-image of some subset of $V_{A,b}(\mathbb{F}_p)$ under the reduction modulo $p$ map by using Nakayama’s lemma. Making the necessary adjustments, the proof proceeds just as in [22], noting that the reduction modulo $p$ of $W_{A,b,p}$ has codimension greater than 2 in $V_{A,b}(\mathbb{F}_p)$ (being non-projective modulo $p$ and having discriminant divisible by $p$ give at least 2 conditions).

We now define the concept of very large collections of local specifications and state the asymptotic formula. Roughly, a collection of local specifications is very large if for large enough primes $p$, it includes all elements of $V$ which are projective at $p$.

**Definition 60** (Very large collection of local specifications). Let $\Lambda_{A,b} = (\Lambda_{A,b,\nu})_{\nu}$ be a collection of local specifications. We say that $\Lambda_{A,b}$ is very large if for all but finitely many primes, the sets $\Lambda_{A,b,p}$ contains all projective elements of $V_{A,b}(\mathbb{Z}_p)$. If $\Lambda_{A,b}$ is very large, we also say that the associated set $V(\Lambda_{A,b})$ is very large.
Theorem 61. Let \( r_2 \) be an integer such that \( 0 \leq r_2 \leq \frac{n-1}{2} \) and let \( \delta \in T(r_2) \). Then for a very large collection of local specifications \( \Lambda_{A,b} \) such that \( \Lambda_{A,b,\infty} = V_{A,b}^{r_2,\delta}(\mathbb{R}) \), we have
\[
N(\mathcal{V}(\Lambda_{A,b}^\delta), X) = \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_A \cdot R_{A,b}^{\delta}(X)) \prod_p \text{Vol}(\Lambda_{A,b,p}) + o\left( X^{\frac{n(n+1)}{2}-1} \right),
\]
where the volume of subsets of \( V_{A,b}^{\delta}(\mathbb{R}) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}) \) has covolume 1 and the volumes of subsets of \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) has measure 1.

5.2 Sieving to acceptable sets conditional on a tail estimate

We now define the concept of acceptable collections of local specifications and state the asymptotic formula. Roughly, a collection of local specifications is acceptable if for large enough primes \( p \), it includes all fields with discriminant indivisible by \( p^2 \).

Definition 62 (Acceptable collection of local specifications). Let \( \Lambda_{A,b} = (\Lambda_{A,b,\nu})_\nu \) be a collection of local specifications. We say that \( \Lambda_{A,b} \) is acceptable if for all but finitely many primes, the set \( \Lambda_{A,b,p} \) contains all elements of \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) whose discriminant is not divisible by \( p^2 \). If \( \Lambda_{A,b} \) is acceptable, we also say that the associated set \( \mathcal{V}(\Lambda_{A,b}) \) is acceptable.

We have the following unconditional asymptotic inequality.

Theorem 63. Let \( \Lambda_{A,b} = (\Lambda_{A,b,\nu})_\nu \) be an acceptable collection of local specifications.
\[
N(\mathcal{V}(\Lambda_{A,b}), X) \leq \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_A \cdot R_{A,b}^{\delta}(X)) \prod_p \text{Vol}(\Lambda_{A,b,p}) + o\left( X^{\frac{n(n+1)}{2}-1} \right),
\]
where the volume of subsets of \( V_{A,b}^{\delta}(\mathbb{R}) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}) \) has covolume 1 and the volumes of subsets of \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) has measure 1.

The following tail estimates are known for \( n = 3 \) and likely to be true for \( n \geq 5 \). Indeed they follow from a suitable version of the \( abc \) conjecture by work of Granville.

Definition 64. Let \( p \) be a prime. We denote by \( \mathcal{W}_{A,b,p} \) the set of elements \( v \in V_{A,b}(\mathbb{Z}) \) such that \( p^2 \mid \Delta(v) \).

Conjecture 65 (Conjectural tail estimates). We have
\[
N(\cup_{p \geq M} \mathcal{W}_{A,b,p}, X) = O\left( \frac{X^{\frac{n(n+1)}{2}-1}}{M^{1-\epsilon}} \right) + o\left( X^{\frac{n(n+1)}{2}-1} \right)
\]
where the implied constant is independent of \( X \) and \( M \).

We have the following asymptotic formula conditional on the preceding tail estimates.

Theorem 66. Suppose that the preceding tail estimates hold. Let \( r_2 \) be an integer such that \( 0 \leq r_2 \leq \frac{n-1}{2} \) and let \( \delta \in T(r_2) \). Then for an acceptable collection of local specifications \( \Lambda_{A,b} \) such that \( \Lambda_{A,b}(\infty) = V_{A,b}^{r_2,\delta}(\mathbb{R}) \) we have
\[
N(\mathcal{V}(\Lambda_{A,b}^\delta), X) = \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_A \cdot R_{A,b}^{\delta}(X)) \prod_p \text{Vol}(\Lambda_{A,b,p}) + o\left( X^{\frac{n(n+1)}{2}-1} \right),
\]
where the volume of subsets of \( V_{A,b}^{\delta}(\mathbb{R}) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}) \) has covolume 1 and the volumes of subsets of \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) are computed with respect to the Euclidean measure normalized so that \( V_{A,b}^{\delta}(\mathbb{Z}_p) \) has measure 1.
6 The product of local volumes and the local mass

6.1 The change of measure formula

To compute the volumes of sets and multi-sets in $V_{A,b}(\mathbb{R})$ and $V_{A,b}(\mathbb{Z}_p)$, we have the following version of the change of variable formula. Let $dv$ denote the Euclidean measure on $V_{A,b}$ and $U_{A,b}$ respectively normalized so that $V_{A,b}(\mathbb{Z})$ and $U_{A,b}(\mathbb{Z})$ have co-volume 1. Furthermore, let $\omega$ be an algebraic differential form generating the rank 1 module of top degree left-invariant differential forms on $SO_A$ over $\mathbb{Z}$.

**Proposition 67** (Change of measure formula). Let $K = \mathbb{Z}_p$, $\mathbb{R}$ or $\mathbb{C}$, let $| \cdot |$ denote the usual absolute value on $K$ and let $s: U_{1,b}(K) \to V_{A,b}(K)$ be a continuous map such that $\pi(f) = f$ for each $f \in U_{1,b}$. Then there exists a rational non-zero constant $J_A$, independent of $K$ and $s$, such that for any measurable function $\phi$ on $V_{A,b}(K)$, we have:

$$\int_{SO_A(K) \cdot s(U_{1,b}(K))} \phi(v) \ dv = |J_A| \int_{f \in U_{1,b}(K)} \int_{g \in SO_A(K)} \phi(g \cdot s(f)) \omega(g) \ df$$

$$\int_{V_{A,b}(K)} \phi(v) \ dv = |J_A| \int_{f \in U_{1,b}(K)} \left( \sum_{v \in V_{A,b}(K) \cap \pi^{-1}(f)} \frac{1}{\#Stab_{SO_A(\mathbb{Z}_p)}(v)} \int_{g \in SO_A(K)} \phi(g \cdot v) \omega(g) \right) \ df$$

where $\frac{V_{A,b}(K) \cap \pi^{-1}(f)}{SO_A(K)}$ denotes a set of representatives for the action of $SO_A(\mathbb{Z}_p)$ on $V_{A,b}(\mathbb{Z}_p) \cap \pi^{-1}(f)$.

We can simplify the second integral above by introducing a local mass.

**Definition 68** (Local mass formula). Let $p$ be a prime, $f \in U_{1,b}(\mathbb{Z}_p)$ and $A \in \mathcal{L}_1$. We define the local mass of $f$ at $p$ in $A$, $m_p(f, A)$ to be

$$m_p(f, A) := \sum_{v \in V_{A,b}(\mathbb{Z}_p) \cap \pi^{-1}(f)} \frac{1}{\#Stab_{SO_A(\mathbb{Z}_p)}(v)}$$

We now have the following formula for the local volumes appearing in the asymptotic formula.

**Proposition 69.** We have

$$\text{Vol} \left( \mathcal{F}_A \cdot R_{A,b}^{x,\delta}(X) \right) = \chi_A(\delta) |J_A| \text{Vol}(\mathcal{F}_A^{\delta}) \text{Vol}(U(\mathbb{R})^2_{H < X}) \text{Vol}(R^{x,\delta}(X)) \text{Vol}(U(\mathbb{R})^2_{H < X}) \text{Vol}(R^{x,\delta}(X)) \text{Vol}(U(\mathbb{R})^2_{H < X})$$

Let $S_p \subset U_{1,b}(\mathbb{Z}_p)$ be a non-empty open set whose boundary has measure 0. Consider the set $\Lambda_{A,b,p} = V_{A,b}(\mathbb{Z}_p) \cap \pi^{-1}(S_p)$. Then we have

$$\text{Vol}(\Lambda_{A,b,p}) = |J_A| \text{Vol}(SO_A(\mathbb{Z}_p)) \int_{f \in S_p} m_p(f, A) \ df$$

6.2 Computing the local masses

We now define the infinite mass and compute $m_p(f, A)$ for all $p \neq 2, \infty$. 

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Definition 70 (The infinite mass). Let $A \in \mathcal{L}_Z$ and $r_2$ be an integer such that $0 \leq r_2 \leq \frac{n-1}{2}$. The infinite mass of $A$ with respect to $r_2$ is defined to be

$$m_\infty(r_2, A) = \sum_{\delta \in T(r_2)} \chi_A(\delta).$$

The following lemma isolates the main properties of the local masses for all $p$, including the Archimedean place.

Lemma 71 (Main properties of the local masses). The local masses $m_p(f, A)$ and $m_\infty(A)$ have the following properties.

1. If $\gamma \in \text{SL}_n(\mathbb{Z}_p)$, we have
   $$m_p(f, \gamma^t A \gamma) = m_p(f, A).$$

2. If $\gamma \in \text{SL}_n(\mathbb{R})$, we have
   $$m_\infty(r_2, \gamma^t A \gamma) = m_\infty(r_2, A).$$

3. In particular, if $A_1$ and $A_2$ are unimodular integral matrices lying in the same genus, we have
   $$m_p(f, A_1) = m_p(f, A_2)$$
   for all primes $p$ and
   $$m_\infty(r_2, A_1) = m_\infty(r_2, A_2).$$

4. The sum of $m_2(f, A)$ over a set of representatives for the unimodular orbits of the action of $\text{SL}_n(\mathbb{Z}_2)$ on $\text{Sym}_n(\mathbb{Z}_2)$ is
   $$\sum_{\substack{A \in \text{Sym}_n(\mathbb{Z}_2) \\
\text{det}(A) = 1 \in \mathbb{Z}_2^*}} m_2(f, A) = 2^{n-1}.$$

5. The sum of $m_p(f, A)$ over a set of representatives for the unimodular orbits of the action of $\text{SL}_n(\mathbb{Z}_p)$ on $\text{Sym}_n(\mathbb{Z}_p)$ is
   $$\sum_{\substack{A \in \text{Sym}_n(\mathbb{Z}_p) \\
\text{det}(A) = 1 \in \mathbb{Z}_p^*}} m_p(f, A) = 1.$$

6. The sum of $m_\infty(r_2, A)$ over a set of representatives for unimodular the orbits of the action of $\text{SL}_n(\mathbb{R})$ on $\text{Sym}_n(\mathbb{R})$ is
   $$\sum_{\substack{A \in \text{Sym}_n(\mathbb{R}) \\
\text{det}(A) = 1 \in \mathbb{R}^*}} m_\infty(r_2, A) = 2^{r_1-1}.$$

We can now compute the local masses for all $p \neq 2, \infty$.

Corollary 72 (Local masses for $p \neq 2, \infty$). For $A \in \mathcal{L}_Z$ and $p \neq 2, \infty$ we have

$$m_p(f, A) = 1.$$
7 Point count and the 2-adic mass

Remark 73. In the proofs, we assume that the local conditions are modulo 2. Nevertheless, in the case of rings which are maximal at the prime 2, we can remove this assumption. Indeed, we can use the even, non-degenerate, \( \mathbb{F}_2 \) quadratic form on the \( \mathbb{F}_2 \) vector space \((R_f^x/(R_f^y)^2))_{N \equiv 1} \) introduced [4], whose kernel is the set of split forms, to calculate the local masses exactly. Doing so gives the same values as those obtained here at each maximal form \( f \), and so we can remove the assumption that the local conditions are modulo 2 in our theorems concerning fields.

In this section, we calculate the integral of the 2-mass, \( \int_{f \in S_2} m_2(f, A) \, df \), by comparing the 2-adic volumes of different indefinite special orthogonal groups.

By the classification of quadratic forms over \( \mathbb{Z}_2 \), there are only two determinant \((-1)^{\frac{n-1}{2}}\) classically integral quadratic forms over \( \mathbb{Z}_2 \) of odd dimension \( n \) up to \( \text{SL}_n(\mathbb{Z}_2) \) equivalence. See for instance [23] or [19]. They are

For \( n \equiv 1 \mod 4 \) we can take:

\[
\mathcal{M}_1 = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \\ \end{pmatrix}, \quad \mathcal{M}_{-1} = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \\ \end{pmatrix}.
\]

For \( n \equiv 3 \mod 4 \) we can take:

\[
\mathcal{M}_1 = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \\ \end{pmatrix}, \quad \mathcal{M}_{-1} = \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \\ \end{pmatrix}.
\]

Remark 74. We have chosen the subscripts to match the Hasse–Witt symbol of the bilinear form.

First, we state the basic constraint obtained in the last section.

Lemma 75. For any \( f \in U_{1,b}(\mathbb{Z}_2) \), we have

\[
m_2(f, \mathcal{M}_1) + m_2(f, \mathcal{M}_{-1}) = 2^{n-1}.
\]

We give names to the integral of the 2-adic masses over \( S_2 \).

Definition 76. We define

\[
c_2(n, \mathcal{M}_1) = \int_{f \in S_2} m_2(f, \mathcal{M}_1) \, df, \quad c_2(n, \mathcal{M}_{-1}) = \int_{f \in S_2} m_2(f, \mathcal{M}_{-1}) \, df.
\]
In this notation, the lemma above states
\[ c_2(M_1) + c_2(M_{-1}) = 2^{n-1} \text{Vol}(S_2). \]

Therefore, in order to determine the value of \( c_2(M_1) \) and \( c_2(M_{-1}) \), it is sufficient to find their ratio. We recall the following expressions which were a corollary of the change of measure formula:
\[
\text{Vol}(\Lambda_{\mathfrak{Mr}_1}(2)) = |J_{\mathfrak{Mr}_1}|_2 \text{Vol}(\text{SO}_{\mathfrak{Mr}_1}(Z_2)) \int_{f \in S_p} m_2(f, M_1) \, df
\]
\[= c_2(n, M_1) \left( |J_{\mathfrak{Mr}_1}|_2 \text{Vol}(\text{SO}_{\mathfrak{Mr}_1}(Z_2)) \right) \]
and
\[
\text{Vol}(\Lambda_{\mathfrak{Mr}_{-1}}(2)) = |J_{\mathfrak{Mr}_{-1}}|_2 \text{Vol}(\text{SO}_{\mathfrak{Mr}_{-1}}(Z_2)) \int_{f \in S_{-2}} m_p(f, M_{-1}) \, df
\]
\[= c_2(n, M_{-1}) \left( |J_{\mathfrak{Mr}_{-1}}|_2 \text{Vol}(\text{SO}_{\mathfrak{Mr}_{-1}}(Z_2)) \right). \]

We now come to the heart of the argument. It is composed of two parts. First, we show that \( \text{Vol}(\Lambda_{\mathfrak{Mr}_1}(2)) = \text{Vol}(\Lambda_{\mathfrak{Mr}_{-1}}(2)) \) and \( J_{\mathfrak{Mr}_1} = J_{\mathfrak{Mr}_{-1}} \). Second, we calculate ratio of the volumes \( \text{Vol}(\text{SO}_{\mathfrak{Mr}_1}(Z_2)) \) and \( \text{Vol}(\text{SO}_{\mathfrak{Mr}_{-1}}(Z_2)) \). Together, this will yield the value of the ratio of \( c_2(M_1) \) and \( c_2(M_{-1}) \), and hence their individual values.

We deduce the equality of volumes \( \text{Vol}(\Lambda_{\mathfrak{Mr}_1}(2)) = \text{Vol}(\Lambda_{\mathfrak{Mr}_{-1}}(2)) \) using an argument which uses the that \( M_1 \) and \( M_{-1} \) have the same reduction modulo 2.

**Lemma 77 (Point count).** Let \( S_2 \subset U_{1,b}(Z_2) \) be a local condition on the space of monic polynomials at the prime 2 defined modulo 2. Denote by \( \Lambda_{\mathfrak{Mr}_1}(2) \) and \( \Lambda_{\mathfrak{Mr}_{-1}}(2) \) the pre-images in \( V_{\mathfrak{Mr}_{1,b}}(Z_2) \) and \( V_{\mathfrak{Mr}_{-1,b}}(Z_2) \) respectively of \( S_2 \) under the resolvent map \( \pi \). Then the volumes of these two sets are equal
\[
\text{Vol}(\Lambda_{\mathfrak{Mr}_1}(2)) = \text{Vol}(\Lambda_{\mathfrak{Mr}_{-1}}(2)).
\]

**Proof.** The canonical representatives \( M_1 \) and \( M_{-1} \) are equal modulo 2. Since \( \Lambda_{\mathfrak{Mr}_1}(2) \) and \( \Lambda_{\mathfrak{Mr}_{-1}}(2) \) are defined by imposing congruence conditions modulo 2 on \( V_{\mathfrak{Mr}_{1,b}}(Z_2) \) and \( V_{\mathfrak{Mr}_{-1,b}}(Z_2) \), the result follows. \( \square \)

**Lemma 78.** The rational numbers giving the Jacobian change of variables for \( M_1 \) and \( M_{-1} \) are equal when the volume forms on the associated special orthogonal groups are those associated to point counting modulo increasing powers of \( p \):
\[
J_{\mathfrak{Mr}_1} = J_{\mathfrak{Mr}_{-1}}.
\]

In particular, their 2-adic valuations are the same
\[
|J_{\mathfrak{Mr}_1}|_2 = |J_{\mathfrak{Mr}_{-1}}|_2.
\]

**Proof.** We sketch the proof of the change of variables formula from Bhargava–Shankar in \( \square \) and explain how to deduce the lemma from their argument. The proof of the change of variables formula over all arithmetic fields proceeds by proving that over \( \mathbb{C} \) the identity
\[
\int_{\text{SO}_A(\mathbb{C}) \cdot s(U_{1,b}(\mathbb{C}))} \phi(v) \, dv = |J_A| \int_{f \in U_{1,b}(\mathbb{C})} \int_{\gamma \in \text{SO}_A(\mathbb{C})} \phi(\gamma \cdot s(f)) \omega(\gamma) \, df
\]

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holds for some non-zero rational number $\mathcal{J}_A$. The principle of permanence of identities then gives the result for $K = \mathbb{Z}_p$, $\mathbb{R}$, or $\mathbb{C}$ with the same $\mathcal{J}_A$.

Now, $\mathcal{J}_A$ can be realized as the Jacobian change of variables of the map

$$\psi_s^A : \text{SO}_A(\mathbb{C}) \times U_{1,b}(\mathbb{C}) \to V_b(\mathbb{C})$$

given by $\psi_s^A(\gamma, f) = \gamma \cdot s(f)$ for any analytic section $s : U_{1,b}(\mathbb{C}) \to V_{A,b}(\mathbb{C})$.

Thus, the lemma will follow from comparing the Jacobian change of variables of the maps $\psi_{s_{\mathcal{M}_1}}$ and $\psi_{s_{\mathcal{M}_{-1}}}$.

Now, fix a matrix $g \in \text{SL}_n(\mathbb{C})$ such that $\mathcal{M}_1 = g\mathcal{M}_{-1} g^t$.

Consider the map

$$\sigma_G : \text{SO}_{\mathcal{M}_1}(\mathbb{C}) \to \text{SO}_{\mathcal{M}_{-1}}(\mathbb{C})$$

defined by $\sigma_G(h) = ghg^{-1}$.

Now, fix an analytic section $s_1 : U_{1,b}(\mathbb{C}) \to V_{\mathcal{M}_1,b}(\mathbb{C})$ and define the analytic section $s_{-1} : U_{1,b}(\mathbb{C}) \to V_{\mathcal{M}_{-1},b}(\mathbb{C})$ to be $s_{-1} := g \cdot s_1$. The following diagram commutes

$$\begin{array}{ccc}
\text{SO}_{\mathcal{M}_1}(\mathbb{C}) \times U_{1,b}(\mathbb{C}) & \xrightarrow{\psi_{s_{\mathcal{M}_1}}} & V_b(\mathbb{C}) \\
\sigma_G \times \text{id} & & \downarrow g \cdot \\
\text{SO}_{\mathcal{M}_{-1}}(\mathbb{C}) \times U_{1,b}(\mathbb{C}) & \xrightarrow{\psi_{s_{\mathcal{M}_{-1}}}} & V_b(\mathbb{C})
\end{array}$$

The fact that $\det(g) = 1$ and that the Jacobian change of variable of $\psi_{s_{\mathcal{M}_1}}$ and $\psi_{s_{\mathcal{M}_{-1}}}$ are constants coupled with the diagram above implies that $\mathcal{J}_{\mathcal{M}_1} = \mathcal{J}_{\mathcal{M}_{-1}}$ as desired. In particular, the 2-adic valuations of those rational numbers are equal and we find $|\mathcal{J}_{\mathcal{M}_1}|_2 = |\mathcal{J}_{\mathcal{M}_{-1}}|_2$.

We can now use calculations related to the Smith–Minkowski–Siegel mass formula to find the ratio between the volumes of the 2-adic points of the special orthogonal groups $\text{SO}_{\mathcal{M}_1}$ and $\text{SO}_{\mathcal{M}_{-1}}$.

**Proposition 79.** We have the following ratio

$$\frac{c_2(n, \mathcal{M}_1)}{c_2(n, \mathcal{M}_{-1})} = \frac{\text{Vol}(\Lambda_{\mathcal{M}_1}(2))}{\text{Vol}(\Lambda_{\mathcal{M}_{-1}}(2))} \left( \frac{|\mathcal{J}_{\mathcal{M}_{-1}}|_2 \text{Vol}(\text{SO}_{\mathcal{M}_{-1}}(\mathbb{Z}_2))}{|\mathcal{J}_{\mathcal{M}_1}|_2 \text{Vol}(\text{SO}_{\mathcal{M}_1}(\mathbb{Z}_2))} \right) = \frac{\text{Vol}(\text{SO}_{\mathcal{M}_1}(\mathbb{Z}_2))}{\text{Vol}(\text{SO}_{\mathcal{M}_{-1}}(\mathbb{Z}_2))} = 2^{n-1} \pm_8 \frac{2^{n-1}}{2^{n-1} \mp_8 2^{n-2}},$$

where $\pm_8$ is $+$ if $n$ is congruent to 1 or 3 mod 8 and $-$ otherwise.

**Proof.** The first two equalities above follow directly from the preceding lemmata. The value of the ratio of the volumes of $\text{SO}_{\mathcal{M}_1}(\mathbb{Z}_2)$ and $\text{SO}_{\mathcal{M}_{-1}}(\mathbb{Z}_2)$ is inversely proportional to the ratio of the 2-adic densities of the lattices defined by $\mathcal{M}_1$ and $\mathcal{M}_{-1}$.

To find the ratio of the 2-adic densities of the lattices defined by $\mathcal{M}_1$ and $\mathcal{M}_{-1}$, it is sufficient to find the ratio of the diagonal factors $M_2(\mathcal{M}_1)$ and $M_2(\mathcal{M}_{-1})$ in the language of Sloane–Conway, [IS].

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There are general formulae for computing the value of the diagonal factors at every prime. These turn out to be rather subtle in the case of \( p = 2 \), depending not only on the form of each Jordan factor but on the full Jordan decomposition, in contrast to the case of odd primes.

Nevertheless, in our case we have that \( \mathcal{M}_1 \) and \( \mathcal{M}_{-1} \) are already in 2-adic Jordan form. Moreover, \( \mathcal{M}_1 \) is free, odd, and has octane value \( 1 \) (mod 8) if \( n \equiv 1, 3 \) (mod 8) and 5 (mod 8) if \( n \equiv 5, 7 \) (mod 8). On the other hand, \( \mathcal{M}_{-1} \) is free, odd, and has octane value \( 5 \) (mod 8) if \( n \equiv 1, 3 \) (mod 8) and 1 (mod 8) if \( n \equiv 5, 7 \) (mod 8).

We may now apply the formulae for the diagonal factor of Conway–Sloane, \cite{18}, to find that the diagonal factors have the form

\[
M_2(\mathcal{M}_1) = \frac{1}{2 \prod_{i=1}^{n-1} (1 - 2^{-2i})} \frac{1}{1 \mp 8 2^{-\frac{n-1}{2}}}
M_2(\mathcal{M}_{-1}) = \frac{1}{2 \prod_{i=1}^{n-1} (1 - 2^{-2i})} \frac{1}{1 \pm 8 2^{-\frac{n-1}{2}}}.\]

This completes the proof of the last equality of the lemma.

We have obtained the values for the 2-adic mass.

**Corollary 80.** The 2-adic masses satisfy the following identities:

\[
c_2(n, \mathcal{M}_1) + c_2(n, \mathcal{M}_{-1}) = 2^{n-1} \text{Vol}(S_2)
\]

\[
c_2(n, \mathcal{M}_0) - c_2(n, \mathcal{M}_{-1}) = \pm 8 2^{\frac{n-1}{2}} \text{Vol}(S_2),
\]

where \( \pm 8 \) is + if \( n \) is congruent to 1 or 3 mod 8 and – otherwise. In particular, we find:

\[
c_2(n, \mathcal{M}_1) = \frac{1}{2} \left( 2^{n-1} \pm 8 2^{\frac{n-1}{2}} \right) \text{Vol}(S_2)
\]

\[
c_2(n, \mathcal{M}_{-1}) = \frac{1}{2} \left( 2^{n-1} \mp 8 2^{\frac{n-1}{2}} \right) \text{Vol}(S_2).
\]

8 A spectral theorem for the indefinite special orthogonal groups

We now describe the distribution of the \( \delta \) among the \( V_A \) by using a version of the spectral theorem for \( SO_A \) combined with a modified parametrisation of the \( SO_A(\mathbb{R}) \) orbits on \( V_A(\mathbb{R}) \cap \pi^{-1}(f) \).

We begin by presenting an alternative parametrisation of the \( SO_A \) orbits on \( V_A \cap \pi^{-1}(f) \)

**Theorem 81.** Let \( A \in \mathcal{L}_\mathbb{Z} \). Consider the bilinear form \( W_A = \langle \cdot, \cdot \rangle_A \) whose matrix is \( A \). Then, given \( f \in U_1 \), there is a natural correspondence between \( SO_{W_A} \)-conjugacy classes of self-adjoint operators with characteristic polynomial \( f \) and \( SO_A \)-orbits on \( V_A \cap \pi^{-1}(f) \).

**Proof.** We note that for each \( A \in \mathcal{L}_\mathbb{Z} \) we can represent the orbit data

\[
\left( SO_A, V_A \cap \pi^{-1}(f) \right)
\]

where \( SO_A \) acts via \( g \cdot (A, B) = (A, g^tBg) \), in terms of the orbit data

\[
\left( SO_{W_A}, \left\{ M \in \text{Mat}_n \mid f(AM^tM - I_x)^{-1} = \det(Ix - M) \right\} \right)
\]

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where $\text{SO}_{W_A}$ acts on the set of $M$ via $\gamma \cdot M = \gamma M \gamma^{-1}$. Note that this last space is the space of self-adjoint operators with characteristic polynomial $\pm f$. Indeed, consider the maps

$$(A, B) \mapsto A^{-1}B$$

and

$$M \mapsto (A, AM).$$

These maps are immediately seen to descend to isomorphisms on the quotients.

**Remark 82.** The parametrisation above does not depend on the base.

We can now modify an argument of Bhargava–Gross [4] to describe the real $\text{SO}_A$ orbits on $V_A \cap \pi^{-1}(f)$. As a corollary, we find the distribution of the elements $\delta$ of $T(r_2)$ among the slices $V_A$ over $\mathbb{R}$.

We first explore the example of $f$, having no complex roots before presenting the general case.

**Example 83.** We first deal with the case of $f \in \mathbb{R}[x]$ having no complex roots. First, it is clear that the number of $\text{SO}_A(\mathbb{R})$ orbits on $V_A(\mathbb{R}) \cap \pi^{-1}(f)$ is equal to the number of $\delta$ which lie in $V_A$. By the above theorem, the number of $\delta$ which lie in $V_A(\mathbb{R}) \cap \pi^{-1}(f)$ is thus the number of $\text{SO}_{W_A}$ orbits on the $W_A$ self-adjoint operators with characteristic polynomial $f$. Indeed, such an operator is diagonalisable and has $n$ eigenspaces of dimension 1. The quadratic space defined by $W_A$ decomposes as an orthogonal direct sum of these spaces. For the signatures to match, we must have a subset of $q_A$ of these eigenspaces being negative definite for $W_A$. The subset of these eigenspaces which are negative definite determines the real orbit of $T$. We need some notation for the signature of the quadratic space $W_A$. Let’s suppose that $A$, and thus the quadratic space defined by $W_A$, has signature $(p_A, q_A)$. Therefore, the number of $\delta$ which land in $V_A(\mathbb{R}) \cap \pi^{-1}(f)$ is $(n)$. We now proceed to the general case. We want to find the distribution of the $\delta$ which land in $V_A(\mathbb{R}) \cap \pi^{-1}(f)$ when $f$ has complex roots. Towards this goal, let’s suppose that $f$ has $r_1$ real roots and $2r_2$ complex roots. Now consider a $W_A$ self-adjoint operator $T$ with characteristic polynomial $f$. This operator has $r_1$ eigenspaces of dimension 1 and $r_2$ eigenspaces of dimension 2. The quadratic space defined by $W_A$ decomposes as an orthogonal direct sum of these eigenspaces. To proceed with the argument, we need some notation for the signature of the quadratic space $W_A$. Let’s suppose that $A$, and thus the quadratic space defined by $W_A$, has signature $(p_A, q_A)$. Now, since each of the 2-dimensional eigenspaces has signature $(1, 1)$ we deduce that:

1. $p_A \geq r_2$;
2. $q_A \geq r_2$; and
3. the number of 1-dimensional eigenspaces which are negative definite for $W_A$ is

$$q_A - r_2.$$

The subset of the 1-dimensional eigenspaces which are negative definite determines the real orbit of $T$. Therefore, the number of $\delta$ which land in $V_A(\mathbb{R}) \cap \pi^{-1}(f)$ is indeed

$$\binom{r_1}{q_A - r_2}.$$
Remark 84. The argument presented above is an extension of an idea of Bhargava–Gross which dealt with $A$ totally split over $\mathbb{R}$.

We have computed the value of the infinite mass, $m_\infty(f, A)$, and we record the distribution obtained above as an equidistribution result.

**Theorem 85** (Equi-distribution of $T(r_2)$ in $\mathrm{SO}_n(\mathbb{R})/\mathrm{Sym}_q(\mathbb{R})$). Let $A \in \mathcal{L}_n$ and $0 \leq r_2 \leq \frac{n-1}{2}$. If $A$ has $q$ negative eigenvalues, the infinite mass $m_\infty(r_2, A)$, which is the number of $\delta$ in $T(r_2)$ such that the matrix $A_\delta$ associated to $\delta$ has $q$ negative eigenvalues, is given by

$$m_\infty(r_2, A) = \left( \frac{r_1}{q - r_2} \right).$$

In particular, if $q < r_2$, there are no $\delta$ which land in any $V_A$ for which $A$ has signature $(n - q, q)$.

For the final calculation, we are interested in the sum of the total masses over all genera which have a fixed Hasse–Witt symbol. We write $c_{\infty, 0}$ for the sum of the infinite masses across all genera which have Hasse–Witt symbol equal to 1 and $c_{\infty, 2}$ for the sum of the infinite masses across all genera which have Hasse–Witt symbol equal to $-1$.

**Lemma 86.** Let $l$ be an odd number. Then

$$\sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k} \right) = 2^{l-1}$$

and

$$\sum_{k=0}^{\frac{l-1}{2}} (-1)^k \left( \frac{l}{2k} \right) = \pm 2^{l-1}$$

where $\pm' \equiv +$ if $l$ is congruent to $-1$ or $1 \mod 8$ and $-$ otherwise. Furthermore, we have

$$\sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k+1} \right) = 2^{l-1}$$

and

$$\sum_{k=0}^{\frac{l-1}{2}} (-1)^k \left( \frac{l}{2k+1} \right) = \pm 2^{l-1}$$

where $\pm'' \equiv +$ if $l$ is congruent to $1$ or $3 \mod 8$ and $-$ otherwise.

**Proof.** These identities follow from the binomial formula. Indeed, for the first one, note that $2^l = (1+1)^l = \sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k} \right) + \sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k+1} \right)$ while $0 = (1-1)^l = \sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k} \right) - \sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k+1} \right)$. Thus $\sum_{k=0}^{\frac{l-1}{2}} \left( \frac{l}{2k} \right) = 2^{l-1}$. For the second one, examine $(1+i)^l$ in the complex plane. For notational clarity, let us write $\alpha = (1+i)^l$. On the one hand, $\alpha = \sum_{k=0}^{l} \left( \frac{l}{k} \right) i^k = \left( \sum_{k=0}^{\frac{l-1}{2}} (-1)^k \left( \frac{l}{2k} \right) \right) + i \left( \sum_{k=0}^{\frac{l-1}{2}} (-1)^k \left( \frac{l}{2k+1} \right) \right)$. On the other hand, $\alpha = 2^l e^{i\pi} \tau$ since $1+i = \sqrt{2} e^{\pi \tau}$. So we find $\Re(\alpha) = \pm \Im(\alpha)$, $2^l = \Re(\alpha)^2 + \Im(\alpha)^2$ and $\Re(\alpha) > 0$ if and only if $l$ is congruent to $-1, 0, 1 \mod 8$. Thus $\sum_{k=0}^{\frac{l-1}{2}} (-1)^k \left( \frac{l}{2k} \right) = \pm 2^{l-1}$. The proof of the second set of identities follows from examining $\Im(\alpha)$. \[\square\]
We have obtained the values for the total infinite mass.

**Corollary 87.** The total infinite masses satisfy the following identities:

\[
\begin{align*}
    c_{\infty,0} + c_{\infty,2} &= 2^{r_1 - 1} \\
    c_{\infty,0} - c_{\infty,2} &= \pm 8^{r_1 - \frac{1}{2}},
\end{align*}
\]

where \( \pm 8 \) is \(+\) if \( n \) is congruent to \( 1 \) or \( 3 \) mod \( 8 \) and \( -\) otherwise.

In particular, we find:

\[
\begin{align*}
    c_{\infty,0} &= \frac{1}{2} \left( 2^{r_1 - 1} \pm 8^{r_1 - \frac{1}{2}} \right) \\
    c_{\infty,2} &= \frac{1}{2} \left( 2^{r_1 - 1} \mp 8^{r_1 - \frac{1}{2}} \right).
\end{align*}
\]

9 Statistical consequences

We now calculate the average 2–torsion by assembling all the elements we have developed.

Let \( \mathcal{L}_\mathbb{Z} \) denote a set of representatives of unimodular integral matrices under the action of \( \text{SL}_n(\mathbb{Z}) \). Denote by \( \mathcal{G}_\mathbb{Z} \) the set of genera of quadratic \( n \)-ary forms containing a unimodular integral element. Notice that \( \mathcal{G}_\mathbb{Z} \) partitions \( \mathcal{L}_\mathbb{Z} \). We have to estimate the following sum:

\[
\sum_{\begin{subarray}{c}
\mathcal{O} \in \mathcal{L}_\mathbb{R}, \\
H(\mathcal{O}) < X
\end{subarray}} 2^{r_1 + r_2 - 1} |\text{Cl}_2(\mathcal{O})| - |\mathcal{I}_2(\mathcal{O})| \prod_p \text{Vol}(S_p) + o(1).
\]

Now, by the preceding sections, we know that this sum is equal to:

\[
\sum_{0 \leq b < n} \sum_{\delta \in \mathcal{T}(r_2)} \sum_{A \in \mathcal{L}_\mathbb{Z}} N_H(\mathcal{V}(A_{\Lambda^\delta_{A,b}}), X) \prod_p \text{Vol}(S_p).
\]

Expanding, we find:

\[
\sum_{\delta \in \mathcal{T}(r_2)} \sum_{A \in \mathcal{L}_\mathbb{Z}} \frac{1}{\sigma(r_2)} \text{Vol}(F_A) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \prod_{p \neq 2} m_p(A) \frac{f_{f \in \mathcal{S}_2} m_2(f, A) df}{\text{Vol}(\mathcal{S}_2)}.
\]

The indicator functions come into play at this point.

\[
\sum_{\delta \in \mathcal{T}(r_2)} \sum_{A \in \mathcal{L}_\mathbb{Z}} \frac{1}{\sigma(r_2)} \chi_A(\delta) \text{Vol}(F_A) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \frac{f_{f \in \mathcal{S}_2} m_2(f, A) df}{\text{Vol}(\mathcal{S}_2)}.
\]
We now break up the collection \( \mathcal{L}_2 \) into genera and sum over the forms in each genus separately before summing over the distinct genera. Since, both the characteristic function and the \( p \)-adic masses are constant over the forms in a single genus, they factor out of the inner sum.

\[
= \sum_{\delta \in T(r_2)} \sum_{\vartheta \in \mathcal{G}_2} \sum_{A \in \mathcal{G} \cap \mathcal{L}_2} \frac{1}{\sigma(r_2)} \chi_A(\delta) \text{Vol}(\mathcal{A}) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \frac{\int_{f \in \mathcal{L}_2} m_2(f, A) df}{\text{Vol}(\mathcal{S}_2)}
\]

\[
= \sum_{\delta \in T(r_2)} \sum_{\vartheta \in \mathcal{G}_2} \frac{1}{\sigma(r_2)} \chi_{\mathcal{G}}(\delta) \frac{\int_{f \in \mathcal{L}_2} m_2(f, \mathcal{G}) df}{\text{Vol}(\mathcal{S}_2)} \left( \sum_{A \in \mathcal{G} \cap \mathcal{L}_2} \text{Vol}(\mathcal{A}) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \right)
\]

Now, the inner sum gives the Tamagawa number of the special orthogonal group of an integral form in a genus. It is known to always be equal 2, see for instance [24] and [19]. We denote it by \( \tau(\text{SO}) \).

\[
= \tau(\text{SO}) \sum_{\delta \in T(r_2)} \sum_{\vartheta \in \mathcal{G}_2} \frac{1}{\sigma(r_2)} \chi_{\mathcal{G}}(\delta) \frac{\int_{f \in \mathcal{L}_2} m_2(f, \mathcal{G}) df}{\text{Vol}(\mathcal{S}_2)}
\]

At this point, we simplify the sum using the fact that \( \sigma(r_2) = \frac{1}{2^{r_1 + r_2 - 1}} \), the value of the 2-adic mass, the value of the infinite mass, and the classification of genera of unimodular integral quadratic forms as it appears in [13] or [19].

\[
= \frac{\tau(\text{SO})}{2^{r_1 + r_2 - 1}} \sum_{\delta \in T(r_2)} \sum_{\vartheta \in \mathcal{G}_2} \chi_{\mathcal{G}}(\delta) \frac{\int_{f \in \mathcal{L}_2} m_2(f, \mathcal{G}) df}{\text{Vol}(\mathcal{S}_2)}
\]

\[
= \frac{\tau(\text{SO})}{2^{r_1 + r_2 - 1}} \sum_{\vartheta \in \mathcal{G}_2} \sum_{\delta \in T(r_2)} \chi_{\mathcal{G}}(\delta) \left( \frac{\int_{f \in \mathcal{L}_2} m_2(f, \mathcal{G}) df}{\text{Vol}(\mathcal{S}_2)} \right) \left( \sum_{\delta \in T(r_2)} \chi_{\mathcal{G}}(\delta) \right)
\]

\[
= \frac{\tau(\text{SO})}{2^{r_1 + r_2 - 1}} \left( c_2(\mathcal{M}_1) \sum_{k \geq 0} \left( \frac{n}{4k} \right) + c_2(\mathcal{M}_1) \sum_{k \geq 0} \left( \frac{n}{4k + 2} \right) \right)
\]

\[
= \frac{\tau(\text{SO})}{2^{r_1 + r_2 - 1}} \left( c_2(\mathcal{M}_1)c_{\infty,0} + c_2(\mathcal{M}_1)c_{\infty,2} \right)
\]

Now, these values being known from previous sections, we substitute them and simplify.

\[
= \frac{1}{2^{r_1 + r_2 - 1}} \cdot 2 \cdot \frac{1}{4} \left( \left( 2^{n-1} \pm 8 \cdot 2^{\frac{n-1}{2}} \right) \left( 2^{r_1} \pm 8 \cdot 2^{\frac{r_1-1}{2}} \right) \right)
\]

\[
= \frac{1}{2^{r_1 + r_2 - 1}} \cdot 2 \cdot \frac{1}{4} \left( \left( 2^{n-1} - 2^{\frac{n-1}{2}} \right) \left( 2^{r_1} - 2^{\frac{r_1-1}{2}} \right) \right)
\]

We now present the corresponding computation for the narrow class group. The setup is the same as above, with the exception that we now sum over \( \delta_{\neq 0} = (11 \cdots 1) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \) instead of
the entire collection \( T(r_2) \). Apart from this, the computation proceeds in the same way as above and we leave it in its raw form.

\[
\sum_{O \in \mathcal{R}} 2^{r_2} |\text{Cl}_2(O)| - |I_2(O)| \left( \sum_{0 \leq b < n} \text{Vol}(U_{1,b}^T(\mathbb{R}) < X) \right) \prod_p \text{Vol}(S_p) + o(1)
\]

\[
= \left( \sum_{0 \leq b < n} \text{Vol}(U_{1,b}^T(\mathbb{R}) < X) \right) \prod_p \text{Vol}(S_p)
\]

\[
= \sum_{A \in \mathcal{Z}_2} \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_A) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \prod_{p \neq 2} m_p(A) \frac{\int_{f \in S_2^2} m_2(f, A) df}{\text{Vol}(S_2^2)}
\]

\[
= \sum_{A \in \mathcal{Z}_2} \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_A) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \frac{\int_{f \in S_2^2} m_2(f, A) df}{\text{Vol}(S_2^2)}
\]

\[
= \sum_{G \in \mathcal{G}_{\mathcal{Z}_2}} \frac{1}{\sigma(r_2)} \chi_A(\delta_{\geq 0}) \text{Vol}(\mathcal{F}_A) \prod_p \text{Vol}(\text{SO}_A(\mathbb{Z}_p)) \frac{\int_{f \in S_2^2} m_2(f, A) df}{\text{Vol}(S_2^2)}
\]

\[
= \tau(\text{SO}) \sum_{G \in \mathcal{G}_{\mathcal{Z}_2}} \frac{1}{\sigma(r_2)} \chi_G(\delta_{\geq 0}) \frac{\int_{f \in S_2^2} m_2(f, G) df}{\text{Vol}(S_2^2)}
\]

\[
= \frac{\tau(\text{SO})}{2^{r_1 + r_2 - 1}} \sum_{G \in \mathcal{G}_{\mathcal{Z}_2}} \chi_G(\delta_{\geq 0}) \frac{\int_{f \in S_2^2} m_2(f, G) df}{\text{Vol}(S_2^2)}
\]

\[
= \frac{1}{2^{r_1 + r_2 - 1}} \cdot 2 \cdot \frac{1}{2} \left( 2^{n-1} + 2^\frac{n-1}{2} \right)
\]

\[
= 2^{r_2} + \frac{2^{r_2}}{2^{n-1}}.
\]

Thus we obtain the following averages for the class group and the narrow class group of monogenic fields.

\[
\text{Avg}_{\text{monogenic}}(\text{Cl}_2) = \lim_{X \to \infty} \frac{1}{\sum_{H(O) < X} \sum_{O \in \mathcal{R}} 1} \sum_{H(O) < X} \sum_{O \in \mathcal{R}} |\text{Cl}_2(O)|
\]
\[
\lim_{X \to \infty} \frac{1}{2^{r_1 + r_2 - 1}} \left( \frac{\sum_{0 \leq b < n} \sum_{\delta \in T(r_2)} \sum_{A \in \mathcal{L}} N_H(\mathcal{V}(\Lambda_{A,b}^\delta), X)}{\sum_{\mathcal{O} \in \mathfrak{O}_L} \frac{1}{H(\mathcal{O}) < X}} + 1 \right)
= \frac{1}{2^{r_1 + r_2 - 1}} \left( \left( 2^{r_1 + r_2 - 1} + 1 \right) + 1 \right)
= 1 + \frac{2}{2^{r_1 + r_2 - 1}}
= 1 + \frac{1}{2^{r_1 + r_2 - 2}}.
\]

\[
\text{Avg}_{\text{monogenic}} \left( \text{Cl}_2 \right) = \lim_{X \to \infty} \frac{\sum_{\mathcal{O} \in \mathfrak{O}_L} \left| \text{Cl}_2^+(\mathcal{O}) \right|}{\sum_{\mathcal{O} \in \mathfrak{O}_L} \frac{1}{H(\mathcal{O}) < X}}
= \lim_{X \to \infty} \frac{1}{2^{r_2}} \left( \frac{\sum_{0 \leq b < n} \sum_{A \in \mathcal{L}} N_H(\mathcal{V}(\Lambda_{A,b}^\delta), X)}{\sum_{\mathcal{O} \in \mathfrak{O}_L} \frac{1}{H(\mathcal{O}) < X}} + 1 \right)
= \frac{1}{2^{r_2}} \left( \left( 2^{r_2} + \frac{2^{r_2}}{2^{r_2 - 1}} \right) + 1 \right)
= 1 + \frac{1}{2^{r_2 - 1}} + \frac{1}{2^{r_2}}.
\]

**Remark 88.** The calculations in this section go through in the even case.

**Example 89.** In the case \((n, r_1, 2r_2) = (3, 3, 0)\) we find

\[
\text{Avg}_{\text{monogenic}} \left( \text{Cl}_2 \right) = 1 + \frac{2}{2^2} = \frac{3}{2},
\]

\[
\text{Avg}_{\text{monogenic}} \left( \text{Cl}_2^+ \right) = 1 + \frac{1}{2} + 1 = \frac{5}{2}.
\]

In the case \((n, r_1, 2r_2) = (3, 1, 2)\) we find

\[
\text{Avg}_{\text{monogenic}} \left( \text{Cl}_2 \right) = 1 + \frac{2}{2^1} = 2
\]

\[
\text{Avg}_{\text{monogenic}} \left( \text{Cl}_2^+ \right) = 1 + \frac{1}{2} + \frac{1}{2} = 2.
\]

Therefore, we recover the result of Bhargava–Hanke–Shankar in the monogenic cubic case \[5\].

We can deduce lower density estimates.

**Corollary 90.** Let \(n \geq 3\) be an odd integer, \((r_1, r_2)\) a choice of signature, and \(\mathfrak{M} \subset \mathfrak{M}^\text{max}_{r_1, r_2}\) a family of monogenised rings corresponding to an acceptable family of binary forms. Then the following positive proportion estimates hold.
1. The proportion of maximal orders in $\mathcal{R}$ which have odd class number is at least
\[
1 - \frac{1}{2^{r_1 + r_2 - 2}}.
\]

2. The proportion of maximal orders in $\mathcal{R}$ which have odd narrow class number is at least
\[
1 - \frac{1}{2^{n-1}} - \frac{1}{2^{r_2}}.
\]

Consequently, a proportion of at least
\[
1 - \frac{1}{2^{n-1}} - \frac{1}{2^{r_2}}
\]
maximal orders in $\mathcal{R}$ have a narrow class number equal to their class number. In particular, there are infinitely many monogenic number fields with units of every signature.

We can also deduce asymptotic lower bounds for the number of monogenic fields having odd class numbers when these fields are ordered by discriminant.

**Corollary 91.** Let $n \geq 3$ be an odd integer, $(r_1, r_2)$ a choice of signature, and $\mathcal{R} \subset \mathcal{R}_{\text{max}}^{r_1, r_2}$ a family of monogenised rings corresponding to an acceptable family of binary forms. Then the following asymptotic estimates hold.

1. \[
\# \{ \mathcal{O} \in \mathcal{R} : |\text{Disc}(\mathcal{O})| < X \text{ and } 2 \nmid |\text{Cl}(\mathcal{O})| \} \gg X \frac{n(n+1)/2-1}{n(n-1)}.
\]

2. If $r_2 \neq 0$, then
\[
\# \{ \mathcal{O} \in \mathcal{R} : |\text{Disc}(\mathcal{O})| < X \text{ and } 2 \nmid |\text{Cl}^+(\mathcal{O})| \} \gg X \frac{n(n+1)/2-1}{n(n-1)}.
\]

Finally, we can state the unconditional result for the concerning average 2-torsion in the class group and narrow class group of monogenic rings.

**Theorem 92.** Let $n \geq 3$ be an odd integer and $(r_1, r_2)$ a choice of signature.

1. The average over $\mathcal{O} \in \mathcal{R}^{r_1, r_2}$ of
\[
|\text{Cl}_2(\mathcal{O})| - \frac{1}{2^{r_1 + r_2 - 1}} |I_2(\mathcal{O})|
\]
is equal to \(1 + \frac{1}{2^{r_1 + r_2 - 1}}\).

2. The average over $\mathcal{O} \in \mathcal{R}^{r_1, r_2}$ of
\[
|\text{Cl}_2^+(\mathcal{O})| - \frac{1}{2^{r_2}} |I_2(\mathcal{O})|
\]
is equal to \(1 + \frac{1}{2^{r_2}}\).

Furthermore, these numbers do not change when we average over any very large family in $\mathcal{R}^{r_1, r_2}$. 

38
A  Supplement on cutting off the cusp

We justify the claim made in the proof of the cusp cutting lemma that the exponents for \(X\), obtained when estimating \(J_m(U'_2, X)\) in the different cases for \(k\), are at most \(-1\).

- *(If \(k = 2.\)) We need to estimate
  \[-1 + 4m - 2m^2 - m|p - q|.
  
  Denote this quantity by \(F(m, |p - q|)\). Note that \(\frac{\partial F}{\partial m} = -4m - |p - q| + 4\). Thus for \((m \geq 3, |p - q| = 1)\) and \((m \geq 2, |p - q| > 1)\), \(F\) is decreasing in \(m\) and \(|p - q|\).
  The initial values are \(F(3, 1) = -10\) and \(F(2, 3) = -7\).

- *(If \(k = 3.\)) We need to estimate
  \[-3 + 6m - 2m^2 - m|p - q|.
  
  Denote this quantity by \(F(m, |p - q|)\). Note that \(\frac{\partial F}{\partial m} = -4m - |p - q| + 6\). Thus for \((m \geq 3, |p - q| = 1)\) and \((m \geq 2, |p - q| > 1)\), \(F\) is decreasing in \(m\) and \(|p - q|\).
  The initial values are \(F(3, 1) = -4\) and \(F(2, 3) = -3\).

- *(If \(4 \leq k \leq m.\)) We need to estimate
  \[1 - k + \frac{(k - 3)(k - 2)}{2} + 2m - 1 + m(k + 1 - 2m - |p - q|).
  
  Denote this quantity by \(F(k, m, |p - q|)\). Note that \(\frac{\partial F}{\partial k} = k + m - \frac{7}{2}\). Thus, on our range of \(k\), we find that \(F\) is increasing. Consequently,
  \[
  \left(\max_{4 \leq k \leq m} F(k, m, |p - q|)\right) = F(m, m, |p - q|) \\
  = \frac{1}{2}(-2m|p - q| - m^2 - m + 6).
  
  Thus for \((m \geq 4, |p - q| = 1)\) and \((m \geq 4, |p - q| > 1)\) the maximum of \(F\) as \(k\) ranges from 4 to \(m\) is decreasing in \(m\) and in \(|p - q|\).
  The initial values are \(F(4, 4, 1) = -11\) and \(F(4, 4, 3) = -19\).

- *(If \(m + 1 \leq k \leq m + |p - q|\).) We need to estimate
  \[1 - k + \frac{(m - 3)(m - 2)}{2} + 2m - 1 + m(k + 1 - 2m - |p - q|).
  
  Denote this quantity by \(F(k, m, |p - q|)\). Note that \(\frac{\partial F}{\partial k} = m - 1\). Thus, on our range of \(k\), we find that \(F\) is increasing. Consequently,
  \[
  \left(\max_{m + 1 \leq k \leq m + |p - q|} F(k, m, |p - q|)\right) = F(m + |p - q|, m, |p - q|)
  
  39
Thus for \((m \geq 3, |p - q| = 1)\) and \((m \geq 2, |p - q| > 1)\) the maximum of \(F\) as \(k\) ranges from \(m + 1\) to \(m + |p - q|\) is decreasing in \(m\) and in \(|p - q|\).

The initial values are \(F(4, 3, 1) = -4\) and \(F(5, 2, 3) = -3\).

\[\begin{align*}
&= \frac{1}{2}(-2|p - q| - m^2 - m + 6).
\end{align*}\]

\(\bullet\) (If \(m + |p - q| + 1 \leq k \leq n - 2\).) We need to estimate

\[1 - k + \frac{(n - k - 1)(n - k)}{2} + 2m - 1 + m(k + 1 - 2m - |p - q|).\]

Denote this quantity by \(F(k, m, |p - q|)\). Note that \(\frac{\partial F}{\partial k} = -|p - q| + k - m - \frac{1}{2}\). Thus, on our range of \(k\), we find that \(F\) is increasing. Consequently,

\[
\left( \max_{m + |p - q| + 1 \leq k \leq n - 2} F(k, m, |p - q|) \right) = F(n - 2, m, |p - q|) = -|p - q| - m + 3.
\]

Thus for \((m \geq 3, |p - q| = 1)\) and \((m \geq 2, |p - q| > 1)\) the maximum of \(F\) as \(k\) ranges from \(m + |p - q| + 1\) to \(n - 2\) is decreasing in \(m\) and in \(|p - q|\).

The initial values are \(F(5, 3, 1) = -1\) and \(F(5, 2, 3) = -2\).

Therefore, we conclude that in the cusp cutting lemma \(J_m(U_2', X) = O_\epsilon(X^{-1+\epsilon})\) for all \(m \geq 2, |p - q| > 1\) and for all \(m \geq 3, |p - q| = 1\).

**B  A wall crossing phenomenon**

In this section, we solve the problem of determining the indicator functions

\[\chi_A(\delta) := \begin{cases} 1 & \text{if } \cap V^r_{\delta}(\mathbb{R}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}\]

exactly. The results of this section will not be needed in the final calculation, and as a result, we don’t attempt to provide complete proofs. Nevertheless, we include them because they uncover a beautiful phenomenon at play: wall crossing.

First, we show that we can associate a wall structure to the system of polynomials \(U_1(\mathbb{R}^{r_2})\) and \(\delta \in T(r_2)\) based on the ordering of the real roots of the polynomials in \(U_1(\mathbb{R}^{r_2})\). We then show that the signature of the first element of the pair \((A_\delta, B_\delta)\) associated to \(\delta\) via \(s_\delta\) obeys a nice set of “wall crossing” rules. These rules allow us to determine \(\chi_A(\delta)\).

**B.1 The wall crossing structure**

We represent elements of \(U_1(\mathbb{R}^{r_2})\) by their roots. Of course, to do this we must choose an ordering of the roots. Each ordering allows us to see \(U_1(\mathbb{R}^{r_2})\) as a subset of \(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\) whose last \(r_2\) entries are not real. We will construct our wall crossing structure in this larger space, \(\mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2}\).
We represent $T(r_2)$ simply as the set of elements of $\{\pm1\}^{r_1} \times \{1\}^{r_2}$ with an even number of $-1$ entries.

We now define walls, chambers and levels for the space 
\[
\left(\mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2}, T(r_2)\right).
\]

**Definition 93.** A **wall** is a subset of $\mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2}$ of the form \(\{x_i = x_j\}\) for $i \neq j$ and $(x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}) \in \mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2}$.

**Definition 94.** A **chamber** is a subset of $\mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2}$ of the form:
\[
\left\{(x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}) \in \mathbb{R}^{r_1} \times (\mathbb{C} \setminus \mathbb{R})^{r_2} : x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(r_1)}\right\}
\]
for some permutation $\sigma$ of the set $\{1, \ldots, r_1\}$. We denote the set of chambers by $\mathcal{C}$. It has size $(r_1!)$. The **principal chamber** is the one corresponding to the identity permutation.

We remark that given an element of $T(r_2)$ and an element in $\mathcal{C}$ in a chamber we may use the algebraic expressions appearing in Wood’s parametrization to calculate $s_\delta(f) = (A_\delta(f), B_\delta(f))$ explicitly. Since the signature of a non-singular symmetric matrix is a continuous function in the entries and since chambers are path-connected, it follows that the signature of $A_\delta(\cdot)$ is constant on chambers.

**Definition 95.** The **level** of an element $\delta$ in $T(r_2)$ is its number of $-1$ entries. It is an even number between 1 and $r_1 - 1$.

These definitions out of the way, we now calculate $A_\delta$ explicitly to get a better handle on its behaviour across walls.

### B.2 Explicit formulas for $A_\delta$

We compute $A_\delta$ given by Melanie Wood’s parametrisation explicitly by choosing the basis
\[
\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}
\]
for $R_f$.

For the Chinese remainder theorem computations we assume:
\[
f(x) = (x - x_1) \cdots (x - x_{r_1})(x^2 + Y_1) \cdots (x^2 + Y_{r_2})
\]
with $Y_i > 0$. We now have 3 types of relations:

1. $\frac{1}{(x_j - x_i)}(x - x_i) + \frac{1}{x_i - x_j}(x - x_j) = 1$;
2. $\frac{1}{x_j + x_i}(x - x_i) + \frac{1}{x_i + Y_j}(x^2 + Y_j) = 1$;
3. $\frac{1}{x_i - Y_j}(x^2 + Y_i) + \frac{1}{Y_j - x_i}(x^2 + Y_j) = 1$.

The Chinese remainder theorem polynomials for the real roots are of the right degree (i.e. $\deg(f(x)) - 1$). Thus we find:
\[
\mathfrak{A}_{x_i0} = \frac{1}{\prod_{j \neq i_0} (x_{i_0} - x_j) \prod_k (x_{i_0}^2 + Y_k)} \left[(x_{i_0})^{i_0 + j}\right].
\]
The situation is slightly more complicated for complex roots because we now need to find the reduced form from the Chinese remainder theorem.

The polynomial \( p(x) \) for which we need to know the reduced \([x^{n-1}] (p(x))\) and \([x^{n-2}] (xp(x))\) coefficients is the following:

\[
p_{A_{i_0}}(x) = -\frac{\prod_j (x^2 - x_j) \prod_{k \neq i_0} (x^2 + Y_k)}{\prod_j (Y_{i_0} + x_j^2) \prod_{k \neq i_0} (Y_k - Y_{i_0})}.
\]

On the other hand:

\[
xp_{A_{i_0}}(x) = -x \frac{\prod_j (x^2 - x_j) \prod_{k \neq i_0} (x^2 + Y_k)}{\prod_j (Y_{i_0} + x_j^2) \prod_{k \neq i_0} (Y_k - Y_{i_0})}.
\]

We now need to find the leading coefficients of the remainder of these polynomials after division by \( f(x) \). The general equations are the following:

\[
p_{Y_{i_0}}(x) = Q_{p_{Y_{i_0}}}(x)f(x) + r_{p_{Y_{i_0}}}(x)
\]

\[
 xp_{Y_{i_0}}(x) = Q_{xp_{Y_{i_0}}}(x)f(x) + r_{xp_{Y_{i_0}}}(x)
\]

where \( \text{deg}(r_{p_{Y_{i_0}}}(x)), \text{deg}(r_{xp_{Y_{i_0}}}) < \text{deg}(f(x)) \).

We denote the leading coefficient of \( p_{Y_{i_0}}(x) \) by \( \Xi_{i_0} \) and the leading coefficient of \( xp_{Y_{i_0}}(x) \) by \( \Xi_{i_0,x} \).

A calculation shows

\[
\Xi_{i_0} = -\frac{1}{\prod_j (Y_{i_0} + x_j^2) \prod_{k \neq i_0} (Y_k - Y_{i_0})} \cdot \frac{1}{2i\sqrt{Y_{i_0}}} \cdot \left( \prod_j \left( i\sqrt{Y_{i_0}} + x_j \right) - \prod_j \left( -i\sqrt{Y_{i_0}} + x_j \right) \right).
\]

A similar calculation shows

\[
\Xi_{i_0,x} = -\frac{1}{\prod_j (Y_{i_0} + x_j^2) \prod_{k \neq i_0} (Y_k - Y_{i_0})} \cdot \frac{1}{2} \cdot \left( \prod_j \left( i\sqrt{Y_{i_0}} + x_j \right) + \prod_j \left( -i\sqrt{Y_{i_0}} + x_j \right) \right).
\]

We thus find

\[
\mathfrak{A}_{Y_{i_0}} = \begin{bmatrix}
\Xi_{i_0} & \Xi_{i_0,x} & -A_{i_0}\Xi_{i_0} & -A_{i_0}\Xi_{i_0,x} & A_{i_0}^2\Xi_{i_0} & \cdots \\
\Xi_{i_0,x} & -A_{i_0}\Xi_{i_0} & -A_{i_0}\Xi_{i_0,x} & A_{i_0}^2\Xi_{i_0} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}.
\]

We can take the sum of the \( \mathfrak{A} \) weighed by \( \delta \) to get an explicit formula for \( A_{\delta} \).

**Lemma 96** (Explicit formula for \( A_{\delta} \) in a chamber.). Consider \( f(x) = (x - x_1) \cdots (x - x_{r_1})(x^2 + Y_1) \cdots (x^2 + Y_{r_2}) \) with \( Y_j > 0 \) and that choice of ordering on its roots fixed (or equivalently a choice of chamber for \( f \)). Let \( \delta \in T(r_2) \). Then \( A_{\delta}(f) \) is given by

\[
A_{\delta}(f) = \mathfrak{A}_{Y_1} + \ldots + \mathfrak{A}_{Y_{r_2}} + \sum_{i=1}^{r_1} \delta_i \mathfrak{A}_{x_i}.
\]
As we can see from the above expressions, crossing from one chamber to another is equivalent to switching adjacent elements of $\delta$ while staying in the same chamber. As a result, we can assume from now that everything happens within the same chamber. As we exchange two adjacent real roots, we cross a wall and land inside a different chamber, but the resulting matrix is the same as if we had exchanged the corresponding entries of $\delta$. This phenomenon is illustrated in Figures 1 and 2.

It turns out that during the process of exchanging two roots, the matrix $A_\delta$ develops a singularity if the elements of $\delta$ corresponding to those roots have opposite sign. By studying the features of this singularity, we can find how the signature of the matrix $A_\delta$ changes.

### B.3 Singularities and wall crossing rules

The signature of $A_\delta(\cdot)$ is constant on chambers. We now examine how this signature varies when crossing a wall.

We begin with an example which shows that singularities arise at the level of the one-parameter family of matrices $A_\delta(f_t)$ for $f_t$ a continuous path which crosses a wall, i.e. when two real roots of $f$ collide. We also show how to leverage the appearance of these singularities into a wall-crossing formula.

**Example 97** (Singularities, wall crossing, and level jumping in the cubic case). Let $n = 3$ and $r_2 = 0$. Suppose that $f_t = (x - t)x(x + 10)$ for $t \in [-1, 1]$. Then $f_t$ is a continuous path between two chambers which crosses a wall in a “tame” way, that is such that a maximum of two roots are equal at any time. Let $\delta = (1 - 1 - 1)$. Then, from the formulas above, we can write $A_\delta(f_t)$ in the
Figure 2: Crossing a wall and swapping two adjacent entries of $\delta$ leads to the same result.

Following way:

$$\frac{1}{t(t-10)} \begin{pmatrix} 1 & t & t^2 \\ t & t^2 & t^3 \\ t^2 & t^3 & t^4 \end{pmatrix} - \frac{1}{10t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{10(10-t)} \begin{pmatrix} 1 & 10 & 10^2 \\ 10 & 10^2 & 10^3 \\ 10^2 & 10^3 & 10^4 \end{pmatrix}.$$

So we see that as $t$ approaches 0 from the left, the only entry which has a singularity is the top-left entry and that it has a singularity of the form $-\frac{1}{5t}$.

We now examine how the signature changes between $A_\delta(f_{-1})$ and $A_\delta(f_1)$.

We can start with the above form of the matrix and start to diagonalise it orthogonally. After the first step, we see the following:

$$\begin{pmatrix} -\frac{5}{t} + p(t) & 0 \\ 0 & M \end{pmatrix}$$

where $p(t)$ is a polynomial in $t$ and the entries of $M$ are polynomials in $t$. It follows that the eigenvalues of the matrix $M$ will be rational functions of $t$ and that there is a single one with a zero. That is we have the following situation.

$$\begin{pmatrix} -\frac{5}{t} + p(t) & 0 \\ 0 & t(a_0 + b_0t + \ldots) \end{pmatrix}.$$ 

Now, we claim that the constant $a_0$ has the same sign as $-5$. Indeed, we proceed by contradiction. Suppose that $a_0$ were positive. Then the following theorem of Wood [32] tells us the form of $B_\delta$ in terms of the form of $A_\delta$ in the basis $\{1, \theta, \ldots, \theta^{n-1}\}$ for $R_f$. 

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Theorem 98 (Wood, [32]). We have the identity
\[ \tilde{c}_{n-1}(r\theta) = \tilde{c}_{n-2}(r) \]
for any \( r \) in \( R_f \). In particular, \( B_\delta(f_1) \) does not have a singularity since \( A_\delta(f_1) \)'s only singularity occurs in the top left corner.

In particular, in the language of [3], this means that if \( a_0 \) is positive, then going from \( A_\delta(f_1) \) to \( A_\delta(f_{-1}) \) yields the same orbit which is a contradiction since this operation corresponds to swapping two distinct adjacent signs in \( \delta \) within the same chamber. Thus the number of negative eigenvalues when going from \( A_{-1} \) to \( A_1 \) increases by 2!

Now, we may ask in the same situation to compare the signature of \( A_{(111)} \) and of \( A_{(-1-11)} \) in the principal chamber. To do so, we consider the path:

\[
A_{(\frac{1}{t},t,1)}(f_t)
= \frac{1}{t-10} \begin{pmatrix}
1 & t & t^2 \\
t & t^2 & t^3 \\
t^2 & t^3 & t^4
\end{pmatrix} + \frac{1}{10t^2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{1}{10(10- t)} \begin{pmatrix}
1 & 10 & 10^2 \\
10 & 10^2 & 10^3 \\
10^2 & 10^3 & 10^4
\end{pmatrix}.
\]

Again, we see a single singularity appearing in the top left corner. It is a pole of order 2 of the form \( \frac{1}{t^2}(10 + b_0 t + \ldots) \). Similar arguments to above show that there is a single eigenvalue of \( \frac{1}{t^2}(c_0 + d_0 t + \ldots) \) with \( c_0 \) having the same sign as 10. Thus the number of negative eigenvalues going from \( A_{(111)} \) and of \( A_{(-1-11)} \) stays the same!

The arguments of the example above generalise to any degree and any number of real roots. The only thing left is to compute the signature of a single element. We do this by noting that the element \( (11 \cdots 1) \) corresponds to the distinguished orbit in the language of Bhargava–Gross, [4], and thus \( A \) is totally split. Overall, this gives us the following statement.

Definition 99. Let \( n = r_1 + 2r_2 \). Consider \( \mathcal{T}(r_2) \) whose elements consists of sequences of +1 and −1 of length \( r_1 \) containing an even number of −1. For \( \delta \in \mathcal{T}(r_2) \), we denote by \( \Omega_{-}(\delta) \) the number of negative eigenvalues of the matrix \( A_\delta \). We define the main checker to be the following sequence of length \( r_1 \)
\[
(-1)^{\frac{n-1}{2}} (+ - + - \ldots + +),
\]
seen as an element of \( \mathcal{T}(r_2) \) by padding the rest with 1’s.

Proposition 100 (Wall crossing and level jumping). The wall crossing and level jumping phenomena, now give us the following rules for computing \( \Omega_{-}(\delta) \).

1. \( \Omega_{-}(11\ldots11) = \frac{n-1}{2} \) if \( n \equiv 1 \) or \(-3 \) mod 8, while \( \Omega_{-}(11\ldots11) = \frac{n+1}{2} \) if \( n \equiv -1 \) or 3 mod 8.

2. (Level Jumping) We have \( \Omega_{-}(11\ldots11) \) is equal to \( \Omega_{-}(\delta) \) for all \( \delta \in \mathcal{T}(r_2) \) consisting of a block of −1 followed by a block of 1.

3. (Wall Crossing) If \( \delta' \) is obtained from \( \delta \) by switching two adjacent entries of \( \delta \) which have different signs, then \( \Omega_{-}(\delta') = \Omega_{-}(\delta) \pm 2 \) where the sign of ±2 is determined by the sign of the entry in the main checker where the negative entry of \( \delta \) lands after the switch.
B.4 The checker invariant

The main proposition of the last section implies equidistribution results and also leads us to the values of the infinite mass. We first introduce a quantity called the checker invariant.

**Definition 101.** (The checker invariant $c_{r_2}$) Fix a triple $(n, r_1, r_2)$ consisting of a choice of degree and a signature. The main checker is defined to be the element $\omega_{r_2} \in \mathcal{T}(r_2)$ given by $(+1, -1, +1, -1, \ldots, +1, +1, +1)$. The checker invariant is the map $c_{r_2}: \mathcal{T}(r_2) \to \mathbb{Z}$ given by:

$$c_{r_2}(\delta) = \frac{\langle \delta, \omega_{r_2} \rangle - 1}{4} + \frac{r_1 - 1}{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

The main theorem about the checker invariant is that its image lies in the even natural numbers, $c_{r_2}: \mathcal{T}(r_2) \to 2\mathbb{N}$, and that it is equal to $\Omega_-$.

**Theorem 102.** The checker invariant has the following properties.

1. $c_{r_2}(1, 1, \ldots, 1) = \Omega_-(1, 1, \ldots, 1)$
2. $c_{r_2}(\delta_{\min})$ for all $\delta_{\min} \in \mathcal{T}_{\min}(r_2)$
3. $c_{r_2}(\delta) = c_{r_2}(\delta_{1\leftrightarrow 1})$
4. $c_{r_2}(\delta) = c_{r_2}(\delta_{1\leftrightarrow -1}) \pm 2$

In particular, $c_{r_2}$ takes the same value as $\Omega_-$ on the element $(1, 1, \ldots, 1)$ of $\mathcal{T}(r_2)$ and satisfies the same wall crossing and level jumping rules. It follows that $c_{r_2}: \mathcal{T}(r_2) \to 2\mathbb{N}$ and $c_{r_2} = \Omega_-$.  

Using the checker invariant, we can now recover the archimedean equidistribution result and calculate the value of the infinite mass.

References

[1] Manjul Bhargava. Higher composition laws. II. On cubic analogues of Gauss composition. *Ann. of Math. (2)*, 159(2):865–886, 2004.

[2] Manjul Bhargava. The density of discriminants of quartic rings and fields. *Ann. of Math. (2)*, 162(2):1031–1063, 2005.

[3] Manjul Bhargava. Most hyperelliptic curves over $\mathbb{Q}$ have no rational points, 2013.

[4] Manjul Bhargava and Benedict H. Gross. The average size of the 2-Selmer group of Jacobians of hyperelliptic curves having a rational Weierstrass point. In *Automorphic representations and L-functions*, volume 22 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 23–91. Tata Inst. Fund. Res., Mumbai, 2013.

[5] Manjul Bhargava, Jonathan Hanke, and Arul Shankar. The mean number of 2-torsion elements in the class groups of $n$-monogenized cubic fields, 2020.

[6] Manjul Bhargava and Arul Shankar. Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves. *Ann. of Math. (2)*, 181(1):191–242, 2015.
[7] Manjul Bhargava and Arul Shankar. Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0. *Ann. of Math. (2)*, 181(2):587–621, 2015.

[8] Manjul Bhargava, Arul Shankar, and Xiaoheng Wang. Squarefree values of polynomial discriminants i, 2016.

[9] Manjul Bhargava and Ila Varma. On the mean number of 2-torsion elements in the class groups, narrow class groups, and ideal groups of cubic orders and fields. *Duke Math. J.*, 164(10):1911–1933, 2015.

[10] Manjul Bhargava and Ila Varma. The mean number of 3-torsion elements in the class groups and ideal groups of quadratic orders. *Proc. Lond. Math. Soc. (3)*, 112(2):235–266, 2016.

[11] Armand Borel. Ensembles fondamentaux pour les groupes arithmétiques. In *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962)*, pages 23–40. Librairie Universitaire, Louvain; Gauthier-Villars, Paris, 1962.

[12] Armand Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. of Math. (2)*, 75:485–535, 1962.

[13] J. W. S. Cassels. *Rational quadratic forms*, volume 13 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[14] H. Cohen and H. W. Lenstra, Jr. Heuristics on class groups of number fields. In *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, volume 1068 of *Lecture Notes in Math.*, pages 33–62. Springer, Berlin, 1984.

[15] H. Cohen and J. Martinet. Class groups of number fields: numerical heuristics. *Math. Comp.*, 48(177):123–137, 1987.

[16] Henri Cohen and Jacques Martinet. Étude heuristique des groupes de classes des corps de nombres. *J. Reine Angew. Math.*, 404:39–76, 1990.

[17] Henri Cohen and Jacques Martinet. Heuristics on class groups: some good primes are not too good. *Math. Comp.*, 63(207):329–334, 1994.

[18] J. H. Conway and N. J. A. Sloane. Low-dimensional lattices. IV. The mass formula. *Proc. Roy. Soc. London Ser. A*, 419(1857):259–286, 1988.

[19] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1993. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[20] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Roy. Soc. London Ser. A*, 322(1551):405–420, 1971.

[21] Alex Eskin, Zeév Rudnick, and Peter Sarnak. A proof of Siegel’s weight formula. *Internat. Math. Res. Notices*, (5):65–69, 1991.
[22] Wei Ho, Arul Shankar, and Ila Varma. Odd degree number fields with odd class number. *Duke Math. J.*, 167(5):995–1047, 2018.

[23] Burton W. Jones. A canonical quadratic form for the ring of 2-adic integers. *Duke Math. J.*, 11:715–727, 1944.

[24] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.

[25] Gunter Malle. On the distribution of class groups of number fields. *Experiment. Math.*, 19(4):465–474, 2010.

[26] P. Sawyer. Spherical functions on \( \text{SO}_0(p,q)/\text{SO}(p) \times \text{SO}(q) \). *Canad. Math. Bull.*, 42(4):486–498, 1999.

[27] P. Sawyer. Computing the Iwasawa decomposition of the classical Lie groups of noncompact type using the QR decomposition. *Linear Algebra Appl.*, 493:573–579, 2016.

[28] Arul Shankar and Xiaoheng Wang. Average size of the 2-selmer group of jacobians of monic even hyperelliptic curves. 2013.

[29] Artane Siad. Monogenic fields with odd class number part ii: even degree, in preparation.

[30] Ashvin Swaminathan. Average 2-torsion in class groups of rings associated to binary \( n \)-ic forms, 2020. in preparation.

[31] Melanie Matchett Wood. Rings and ideals parameterized by binary \( n \)-ic forms. *J. Lond. Math. Soc. (2)*, 83(1):208–231, 2011.

[32] Melanie Matchett Wood. Parametrization of ideal classes in rings associated to binary forms. *J. Reine Angew. Math.*, 689:169–199, 2014.