FOURIER COEFFICIENTS AND A FILTRATION ON $\text{Shv}(\text{Bun}_G)$

S. LYSENKO

Abstract. We define a filtration by DG-subcategories on the DG-category $\text{Shv}(\text{Bun}_G)$ of sheaves on the moduli of $G$-torsors on a curve, which is stable under the action of Hecke functors. We formulate a conjecture relating this filtration with another filtration on the spectral side of the categorical geometric Langlands conjecture. We also formulate a conjectural compatibility with the parabolic induction.

1. Introduction

1.0.1. The idea to associate with a nilpotent orbit in $\mathfrak{g}$ a set of Fourier coefficients of automorphic forms on $G$ has appeared first maybe in [17]. More general conjectures at the level of functions were proposed in [16]. Further works in this direction in the theory of automorphic forms include, in particular, [19, 20, 21, 22]. A conceptual formulation appears, in particular, as ([19], Conjectures 4.2 and 4.3).

Connections between nilpotent orbits in $\mathfrak{g}$ and representations of finite reductive groups also were used in preceding works of Lusztig on the classification of such representations [23].

1.0.2. We propose a way to fit the above ideas and, in particular, ([19], Conjectures 4.2 and 4.3) into the global nonramified geometric Langlands program as it is formulated in [2], [3]. Our proposal makes sense in several contexts.

In the context of $\mathcal{D}$-modules we propose some filtrations on the DG-categories $\mathcal{D}\text{-Mod}(\text{Bun}_G)$ and $\text{IndCoh}_{\text{Nilp}}(\text{LocSys}(\check{G}))$, which are expected to correspond to each other under the conjectural geometric Langlands equivalence ([2], Conjecture 1.1.6)

$$\mathcal{D}\text{-Mod}(\text{Bun}_G) \xrightarrow{\Rightarrow} \text{IndCoh}_{\text{Nilp}}(\text{LocSys}(\check{G}))$$

In the restricted constructible context we propose a similar filtrations on both sides and a similar refinement of ([3], Conjecture 21.2.7). More precisely, in this context we propose a filtration on $\text{Shv}(\text{Bun}_G)$ which is shown to be preserved by Hecke functors. The desired filtration on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ (also preserved by Hecke functors) is obtained by restriction from that on $\text{Shv}(\text{Bun}_G)$.

1.1. Conventions.

1.1.1. Work in the constructible context over an algebraically closed field $k$ of characteristic $p \geq 0$ in the sense of ([3], Section 1.1.1). We assume the characteristic of $p$ very good in the sense of ([3], Section D.1.1), the precise definition is found in [24]. Since $k$ is algebraically closed, we systematically ignore the Tate twists. We use the notations and conventions for the constructible sheaves theory from [3]. In particular, $e$ denotes the field of coefficients of our sheaf theory.
1.1.2. Let $G$ be reductive connected group over $k$, $X$ a smooth proper irreducible curve over $k$. In the case $p = 0$ the nilpotent orbits in $\mathfrak{g} = \text{Lie}G$ are classified in [8] (or [20] for a short summary). Our assumption on $p$ implies that there is a $G$-invariant isomorphism $B : \mathfrak{g} \cong \mathfrak{g}^*$, which we fix.

We use the Fourier transform, in the case $p > 0$ it is normalized ‘to preserve perversity as much as possible’. For this, we pick an injective character $\psi : F_p \to e^*$ and denote by $\mathcal{L}_\psi$ the corresponding Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1$. In the case $p = 0$ the Artin-Schreier sheaf does not exist and we always use the Kirillov model to get a similar definition of the Fourier transform as in ([13], Appendix A).

We let $\tilde{G}$ be the Langlands dual as a group over $e$. We also fix a $\tilde{G}$-invariant isomorphism $\tilde{k} : \tilde{\mathfrak{g}} \cong \tilde{\mathfrak{g}}^*$.

Write $\mathcal{X}(\mathfrak{g})$ (resp., $\mathcal{X}(\tilde{\mathfrak{g}})$) for the set of nilpotent orbits in $\mathfrak{g}$ (resp., in $\tilde{\mathfrak{g}}$). Let $N(\mathfrak{g}) \subset \mathfrak{g}$ be the variety of nilpotent elements, and similarly for $N(\tilde{\mathfrak{g}})$. If a maximal torus and a Borel subgroup $T \subset B \subset G$ are picked, we write $\Lambda$ for the coweights lattice of $T$, and $\Lambda^+$ for the dominant coweights of $G$.

1.1.3. Recall the categories $\text{DGCat}_{\text{cont}}$, $\text{DGCat}^{\text{non-cocmpl}}$ defined in ([15], ch. I.1, Section 10.3.1 and 10.3.3). For $C \in \text{DGCat}_{\text{cont}}$ equipped with a $t$-structure, $C^\triangledown$ denotes the heart of $C$. For a prestack $Y$ locally of finite type we write $\text{Shv}(Y)_{\text{constr}} \subset \text{Shv}(Y)$ for the $G$-subcategory of constructible objects defined in ([3], Section F.2).

Similarly, let $Y$ be an ind-algebraic stack $Y$ written as $Y \cong \text{colim}_{i \in I} Y_i$ with $I$ is a filtered $\infty$-category, $Y_i$ an algebraic stack locally of finite type such that for a map $i \to i'$ in $I$ the transition map $\alpha_{i,i'} : Y_i \to Y_{i'}$ is a closed immersion. By definition, $\text{Shv}(Y) \cong \text{lim}_{i \in I} \text{Shv}(Y_i)$ in $\text{DGCat}_{\text{cont}}$ with respect to the $!$-restrictions. Passing to the left adjoints, we can also rewrite it as $\text{Shv}(Y) \cong \text{colim}_{i \in I} \text{Shv}(Y_i)$ in $\text{DGCat}_{\text{cont}}$ with respect to the $!$-direct images. For $i \to i'$ as above $(\alpha_{i,i'}) : \text{Shv}(Y_i)_{\text{constr}} \to \text{Shv}(Y_{i'})_{\text{constr}}$, and we get a diagram

$$I \to \text{DGCat}^{\text{non-cocmpl}}, \ i \mapsto \text{Shv}(Y_i)_{\text{constr}}$$

Define $\text{Shv}(Y)_{\text{constr}}$ as $\text{colim}_{i \in I} \text{Shv}(Y_i)_{\text{constr}}$ taken in $\text{DGCat}^{\text{non-cocmpl}}$. Note that $\text{Shv}(Y)_{\text{constr}} \subset \text{Shv}(Y)$ is a full subcategory.

Let $1 - \text{Cat}$ is the $\infty$-category of $\infty$-categories introduced in ([15], ch. I.1, Section 1.1.1). Recall that the forgetful functor $\text{DGCat}^{\text{non-cocmpl}} \to 1 - \text{Cat}$ preserves filtered colimits. So, $\text{Shv}(Y)_{\text{constr}} \subset \text{Shv}(Y)$ is the full subcategory of those objects, which come as $!$-direct image under $Y_i \to Y$ of some object of $\text{Shv}(Y_i)_{\text{constr}}$.

For the convenience of the reader recall the definition of the perverse t-structure on $Y$. By definition, $\text{Shv}(Y)_{\leq 0}$ is the smallest full subcategory containing $\text{Shv}(Y_i)_{\leq 0}$ for $i \in I$, closed under extensions and small colimits. This t-structure is accessible and compatible with filtered colimits. The inclusion $\text{Shv}(Y)_{\text{constr}} \subset \text{Shv}(Y)$ is compatible with this t-structure, so $\text{Shv}(Y)_{\text{constr}}$ inherits a t-structure from $\text{Shv}(Y)$.

1.1.4. Acknowledgements. I am grateful to Sam Raskin for answering my questions and very useful discussions. I also thank Joakim Faergeman for his comments about a preliminary version of this paper and fruitful email correspondence.

\footnote{This follows from [27].}
2. Main results and conjectures

2.1. Fourier coefficients.

2.1.1. Let $x \in \mathfrak{g}$ be nilpotent and $\emptyset := \emptyset_x \subset \mathfrak{g}$ the nilpotent orbit through $x$. Let $\sigma : \mathfrak{sl}_2 \to \mathfrak{g}$ be the corresponding $\mathfrak{sl}_2$-triple, so $x$ is the nilpositive element of this triple (in the sense of [26], Definition 4.2.1). Write $(H, x, Y)$ for the corresponding standard $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.

Let $\mathfrak{g}_i \subset \mathfrak{g}$ be the subspace, on which $\sigma(\mathbb{G}_m)$ acts by $i$, so $\mathfrak{g} = \oplus_i \mathfrak{g}_i$. Let $P \subset G$ be the parabolic such that $\mathfrak{p} := \text{Lie } P = \oplus_{i \geq 0} \mathfrak{g}_i$ is the Jacobson-Morozov parabolic subalgebra of $x$ (26, Definition 4.3.2). Let $U \subset P$ be its unipotent radical, then $\mathfrak{u} := \text{Lie } U = \oplus_{i > 0} \mathfrak{g}_i$. We also get the subgroup $V \subset U$ such that $\mathfrak{v} := \text{Lie } V = \oplus_{i \geq 2} \mathfrak{g}_i$.

Let $M \subset P$ be the Levi such that $\mathfrak{m} = \text{Lie } M = \mathfrak{g}_0$. Set $\mathfrak{v}' = \oplus_{i \geq 3} \mathfrak{g}_i$. By the theory of $\mathfrak{sl}_2$-representations, $\mathfrak{v}' \subset [\mathfrak{u}, x]$. Let $V' \subset V$ be the subgroup such that $\text{Lie } V' = \mathfrak{v}'$. So, $\mathfrak{v}/\mathfrak{v}' \sim \mathfrak{g}_2$ as $M$-modules.

Recall that $x$ is called even iff $\mathfrak{g}_1 = 0$ (26, Definition 4.3.6). If $x$ is even then the above shows that $\mathfrak{g}_2 \supseteq \mathfrak{v}/[\mathfrak{v}, \mathfrak{v}]$.

By (8, Lemma 4.1.4), $Px = \emptyset \cap \mathfrak{v}$ is open and dense in $\mathfrak{v}$. More precisely, let

$$\mathcal{P} = \{ Z \in \mathfrak{g}_2 \mid \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \}$$

Then $\mathcal{P} \subset \mathfrak{g}_2$ is the open $M$-orbit on $\mathfrak{g}_2$, and $\emptyset \cap \mathfrak{v} = \mathcal{P} + \mathfrak{v}' = Px$.

Similarly, let $\mathcal{P}^- = \{ Z \in \mathfrak{g}_{-2} \mid \mathfrak{g}^Z \cap \mathfrak{g}_2 = 0 \}$. Choosing an opposite Borel in $\mathfrak{sl}_2$, from the above one gets the following. The subscheme $\mathcal{P}^- \subset \mathfrak{g}_{-2}$ is the open $M \times \mathbb{G}_m$-orbit on $\mathfrak{g}_{-2}$. Note that $x \in \mathcal{P}$ and $Y \in \mathcal{P}^-$. Let $\mathfrak{m}^x$ be the stabilizer of $x$ in $\mathfrak{m}$. Then $\mathfrak{m}^x$ is reductive (cf. [26], Lemma 4.3.4). Note that for any $i \geq 0$, the restriction of $B$ provides a nondegenerate pairing $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to k$.

We underline that if $x \neq 0$ then $\mathfrak{g}_2 \neq 0$.

2.1.2. Let $(V/V')_0^* \subset (V/V')^*$ be the open $M \times \mathbb{G}_m$-orbit. It is understood that $\mathbb{G}_m$ acts by scalar multiplications. If $p = 0$ then we identify $(V/V')^*$ with $\mathfrak{g}_{-2}$ as above, so this open orbit is $\mathcal{P}^-$. We underline that if $x \neq 0$ then $\mathfrak{g}_2 \neq 0$.

2.1.3. In the case of $x$ odd we have $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{p}_{>2} := \oplus_{i \geq 4} \mathfrak{g}_i$. Indeed, for $m \geq 2$ the map $\text{ad}_x : \mathfrak{g}_{m-2} \to \mathfrak{g}_m$ is surjective by the representation theory of $\mathfrak{sl}_2$. So, in this case

$$\mathfrak{v}/[\mathfrak{v}, \mathfrak{v}] \sim \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

2.1.4. Consider the diagram

$$\text{Bun}_{P/V'} \xrightarrow{p} \text{Bun}_P \xrightarrow{q} \text{Bun}_G$$

For $K \in Shv(\text{Bun}_G)$ we want to consider the Fourier transform of $p_! q^* K$ with respect to $V/V'$.

2.1.5. If $\mathcal{F}$ is a $P/V'$-torsor on $X$ write $\mathcal{F}_M$ for the induced $M$-torsor on $X$, and similarly for $\mathcal{F}_{P/V}$. Since $V/V'$ is a linear representation of $M$, $(V/V')_{\mathcal{F}_M}$ is a vector bundle on $X$. A datum of a torsor $\mathcal{F}'$ under $(V/V')_{\mathcal{F}_M}$ on $X$ is the same as an exact sequence of vector bundles on $X$

$$0 \to (V/V')_{\mathcal{F}_M} \to ? \to \emptyset_X \to 0$$
Namely, \( \mathcal{F}' \) is the torsor of sections of (2) over 1.

The stack \( \text{Bun}_{P/V'} \) classifies \( \mathcal{F}_M \in \text{Bun}_M \) and a torsor under \((U/V')_{\mathcal{F}_M}\) on \( X \).

2.1.6. Case of \( x \) even. If \( x \) is an even nilpotent element ([20], Definition 4.3.6) then \( V = U \). In this case \( \text{Bun}_{P/V'} \) classifies \( \mathcal{F}_M \in \text{Bun}_M \) and an exact sequence (2) on \( X \).

Let \( \mathcal{Y}_M \) be the stack classifying \( \mathcal{F}_M \in \text{Bun}_M \) and a torsor under \((U/V')_{\mathcal{F}_M}\) on \( X \).

(3)
\[
s : (V/V')_{\mathcal{F}_M} \to \Omega
\]

In this case the Fourier transform functor
\[
\text{Four} : \text{Shv}(\text{Bun}_{P/V'}) \to \text{Shv}(\mathcal{Y}_M)
\]
is defined as follows. We have a diagram of projections
\[
\begin{array}{c}
\mathbb{A}^1 \\
\uparrow \text{ev}
\end{array}
\]
\[
\mathcal{Y}_M \overset{a}{\leftarrow} \mathcal{Y}_M \times_{\text{Bun}_M} \text{Bun}_{P/V'} \overset{b}{\rightarrow} \text{Bun}_{P/V'},
\]
and the map \( \text{ev} \) sends a point of \( \mathcal{Y}_M \times_{\text{Bun}_M} \text{Bun}_{P/V'} \) to the pairing of (3) with (2).

If \( p > 0 \) then for \( K \in \text{Shv}(\text{Bun}_{P/V'}) \) set
\[
\text{Four}(K) = (a) b^* K \otimes \text{ev}^* L_\psi [\text{dim. rel.(act)}]
\]

Though \( \text{Four} \) depends on \( \psi \), we do not express this in our notation.

Let \( \mathcal{Y}'_M \subset \mathcal{Y}_M \) be the open substack given by the property that \( s \) at the generic point of \( X \) lies in \((V/V')^*_0\).

2.1.7. Case of \( x \) odd. Assume \( x \) odd. Let \( \mathcal{X}_P \) be the stack classifying \( \mathcal{F} \in \text{Bun}_{P/V'} \) and a \((V/V')_{\mathcal{F}_M}\)-torsor \( \bar{\mathcal{F}} \) on \( X \) given also by an exact sequence (2). Then \((V/V')_{\mathcal{F}_M}\) is a subsheaf of the sheaf of automorphisms of the \( P/V'\)-torsor \( \mathcal{F} \). We get the action map \( \text{act} : \mathcal{X}_P \to \text{Bun}_{P/V'} \) sending \((\mathcal{F}, \bar{\mathcal{F}})\) to the quotient of \( \mathcal{F} \times \bar{\mathcal{F}} \) by the diagonal action of \((V/V')_{\mathcal{F}_M}\).

In fact, \( \mathcal{X}_P \) identifies with the stack classifying \( \mathcal{F}, \mathcal{F}' \in \text{Bun}_{P/V'} \) and an isomorphism
\[
\mu_P : \mathcal{F} \times_{P/V'} P/V \cong \mathcal{F}' \times_{P/V'} P/V
\]
of \( P/V \)-torsors on \( X \). Namely, for \((\mathcal{F}, \mathcal{F}', \mu_P)\) as above let \( \bar{\mathcal{F}} \) be the sheaf of isomorphisms \( \mathcal{F} \cong \mathcal{F}' \) of \( P/V'\)-torsors on \( X \) compatible with \( \mu_P \). Then \((V/V')_{\mathcal{F}_M}\) acts on \( \bar{\mathcal{F}} \) via its action on \( \mathcal{F} \), and in this way \( \bar{\mathcal{F}} \) becomes a \((V/V')_{\mathcal{F}_M}\)-torsor on \( X \).

Let \( \mathcal{Y}_P \) be the stack classifying \( \mathcal{F} \in \text{Bun}_{P/V'} \) and a section (3). We define the Fourier transform
\[
\text{Four}_P : \text{Shv}(\text{Bun}_{P/V'}) \to \text{Shv}(\mathcal{Y}_P)
\]
as follows. Consider the diagram
\[
\begin{array}{c}
\mathbb{A}^1 \\
\uparrow \text{ev}_P
\end{array}
\]
\[
\mathcal{Y}_P \overset{a_P}{\leftarrow} \mathcal{Y}_P \times_{\text{Bun}_{P/V'}} \mathcal{X}_P \overset{b_P}{\rightarrow} \mathcal{X}_P \overset{\text{act}}{\rightarrow} \text{Bun}_{P/V'},
\]
where \( a_P, b_P \) are the projections. The map \( \text{ev}_P \) sends a point of \( \mathcal{Y}_P \times_{\text{Bun}_{P/V'}} \mathcal{X}_P \) to the pairing of (2) with (3). In the case \( p > 0 \) for \( K \in \text{Shv}(\text{Bun}_{P/V'}) \) set
\[
\text{Four}_P(K) = (a_P) b_P \text{act}^* K \otimes \text{ev}^*_P L_\psi [\text{dim. rel.(act \circ b_P)}]
\]
Let $\mathcal{Y}_P^0 \subset \mathcal{Y}_P$ be the open substack given by the property that $s$ at the generic point of $X$ lies in $(V/V')^0$.

**Definition 2.1.8.** We say that $K \in \text{Shv}(Bun_G)$ has no Fourier coefficients corresponding to $\mathcal{O}$ if

- $\text{Four}_F(p_!q^*K)$ vanishes over $\mathcal{Y}_P^0$ for $x$ is odd;
- $\text{Four}(p_!q^*K)$ vanishes over $\mathcal{Y}_M^0$ for $x$ even.

Let $\mathcal{F}_\mathcal{O} \subset \text{Shv}(Bun_G)$ be the full subcategory of those $K \in \text{Shv}(Bun_G)$ which have no Fourier coefficients corresponding to $\mathcal{O}$.

Note that $\mathcal{F}_\mathcal{O} \in \text{DGCat}_{\text{cont}}$, the embedding $\mathcal{F}_\mathcal{O} \hookrightarrow \text{Shv}(Bun_G)$ is continuous.

2.1.9. Write Nilp for the global nilpotent cone as defined in ([3], Appendix D). Recall that it can be seen as a mapping stack

$$\text{Maps}(X, N(g)/G \times G_m) \times \text{Maps}(X, B(G_m)) \{\Omega_X\}$$

Write $\text{Shv}_{\text{Nilp}}(Bun_G) \subset \text{Shv}(Bun_G)$ for the DG-subcategory of sheaves with singular support in Nilp as defined in ([3], Section F.8).

More generally, for a closed $G$-invariant subscheme $Y \subset N(g)$ write $\text{Shv}_{Y}(Bun_G)$ for the full DG-subcategory of those objects, whose singular support is contained in

$$\text{Maps}(X, Y/(G \times G_m)) \times \text{Maps}(X, B(G_m)) \{\Omega_X\}$$

**Definition 2.1.10.** Let $F_{\mathcal{O}} \subset \text{Shv}(Bun_G)$ be the full subcategory equal to $\bigcap_{\mathcal{O}' \in \mathcal{O}} \mathcal{F}_{\mathcal{O}'}$. More generally, for a closed $G$-invariant subscheme $Y \subset N(g)$ let $F_Y \subset \text{Shv}(Bun_G)$ be the full subcategory $\bigcap_{\mathcal{O}' \in \mathcal{O}} \mathcal{F}_{\mathcal{O}'}$. Let $F_{\text{Nilp},Y}$ be the intersection $F_Y \cap \text{Shv}_{\text{Nilp}}(Bun_G)$ inside $\text{Shv}(Bun_G)$.

Note that $F_Y \in \text{DGCat}_{\text{cont}}$, the embedding $F_Y \hookrightarrow \text{Shv}(Bun_G)$ is continuous. If $\mathcal{O} \subset \mathcal{O}'$ are nilpotent orbits then $F_{\mathcal{O}} \subset F_{\mathcal{O}'}$ is a full continuous embedding (and the same for Nilp-versions).

**Theorem 2.1.11.** For any nilpotent orbit $\mathcal{O}$ in $g$, $\mathcal{F}_\mathcal{O}$ is preserved by Hecke functors acting on $\text{Shv}(Bun_G)$.

**Remark 2.1.12.** Theorem 2.1.11 implies that for $\mathcal{O} \in \mathcal{X}(g)$, the full DG-subcategory $F_{\text{Nilp},\mathcal{O}}$ of $\text{Shv}_{\text{Nilp}}(Bun_G)$ is stable under the action of Hecke functors.

**Conjecture 2.1.13.** Let $Y \subset N(g)$ be a closed $G$-invariant subscheme. Then

$$F_{\text{Nilp},Y} = \text{Shv}_Y(Bun_G)$$

**Remark 2.1.14.** i) For $Y = N(g) - \mathcal{O}_{\text{reg}}$ and the sheaf theory being ind-holonomic $\mathcal{D}$-modules Conjecture 2.1.13 is proved in the striking paper ([10], Theorems B, C). They show that $\text{Shv}_Y(Bun_G)$ coincides with the category of the so-called anti-tempered objects of $\text{Shv}_{\text{Nilp}}(Bun_G)$. Their argument does not seem to work for $\ell$-adic sheaves for example.
ii) Assume that the sheaf theory is ind-holonomic D-modules, and $G$ is semi-simple and simply-connected. Then for $Y = \{0\}$ one has the inclusion $\text{Shv}(\text{Bun}_G) \subset F_{\text{Nilp}, Y}$. Indeed, by (9), Proposition 4.1.4.1), any irreducible local system on $\text{Bun}_G$ is trivial, the claim easily follows.

2.1.15. If we need to underline the dependence on $G$, we will write $\mathcal{F}_\mathcal{O}(G)$, $F_\mathcal{O}(G)$ and so on.

2.1.16. The Lusztig-Spaltenstein duality map $d : \mathcal{X}(\mathfrak{g}) \to \mathcal{X}(\hat{\mathfrak{g}})$ is constructed in ([29], [7]). This is the order reversing map denoted by $d_{BV}$ in [1]. The image of $d$ is called the set of special nilpotent orbits $\mathcal{X}(\hat{\mathfrak{g}})^{sp} \subset \mathcal{X}(\hat{\mathfrak{g}})$. We similarly have $d : \mathcal{X}(\hat{\mathfrak{g}}) \to \mathcal{X}(\mathfrak{g})$, and the restriction $\mathcal{X}(\mathfrak{g})^{sp} \to \mathcal{X}(\hat{\mathfrak{g}})^{sp}$ of $d$ is known to be bijective. One more important property of $d$ is that for any $\mathcal{O} \in \mathcal{X}(\mathfrak{g})$, $\mathcal{O} \subset D^2(\mathcal{O})$. By (7, Corollary A.3), for $\mathcal{O} \in \mathcal{X}(\mathfrak{g})$, $D^2(\mathcal{O})$ is the unique smallest special nilpotent orbit dominating $\mathcal{O}$.

In general, $d$ exchanges the zero orbit $\mathcal{O}_0$ and the regular nilpotent orbit $\mathcal{O}_{\text{reg}}$. If $\mathfrak{g}$ is simple it admits the subregular nilpotent orbit $\mathcal{O}_{\text{subreg}}$, the unique open orbit in $\text{Nilp} - \mathcal{O}_{\text{reg}}$. It is known that $d$ exchanges $\mathcal{O}_{\text{subreg}}$ and the so called minimal special nilpotent orbit. (In general, the minimal nonzero nilpotent orbit $\mathcal{O}_{\text{min}}$ is not special).

2.1.17. If $\mathcal{O} \in \mathcal{X}(\mathfrak{g})$, let $\mathcal{O}^\mathfrak{sp} \subset N(\mathfrak{g})$ be the union of $\mathcal{O}'$ for all $\mathcal{O}' \in \mathcal{X}(\mathfrak{g})^{sp}$, which are strictly less than $\mathcal{O}$. Definition 2.1.10 for $Y = \mathcal{O}^\mathfrak{sp}$ gives a full subcategory $F_{\mathcal{O}^\mathfrak{sp}} \subset F_\mathcal{O}$. For $\mathcal{O} \neq \mathcal{O}_0$ set

$$F_{\text{Nilp}, \mathcal{O}^\mathfrak{sp}} = F_{\mathcal{O}^\mathfrak{sp}} \cap \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$$

If $\mathcal{O} = \mathcal{O}_0$ we let $F_{\text{Nilp}, \mathcal{O}^\mathfrak{sp}}$ denote the DG-subcategory of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ containing only the zero object.

The first question, independent of the Langlands correspondence is to see which successive subquotients in the filtration $F_{\mathcal{O}}$ for $\mathcal{O} \in \mathcal{X}(\mathfrak{g})$ vanish. We expect that restricting the filtration to $\mathcal{X}(\mathfrak{g})^{sp}$, we do not lose any information. More precisely, we propose the following.

**Conjecture 2.1.18.** Let $\mathcal{O} \in \mathcal{X}(\mathfrak{g})$ be non special. Then $F_{\mathcal{O}} = F_{\mathcal{O}^\mathfrak{sp}}$.

2.1.19. **Example.** As a motivation for Conjecture 2.1.18, consider the first nontrivial example of $G = \text{Sp}_4$. The Hasse diagram of nilpotent orbits in $\mathfrak{g}$ is

$$\mathcal{O}_0 \subset \mathcal{O}_{\text{min}} \subset \mathcal{O}_{\text{subreg}} \subset \mathcal{O}_{\text{reg}}$$

in this case. In the notations of (8, Section 5.1), they are given by the following partitions $\mathcal{O}_0 = (1^4)$, $\mathcal{O}_{\text{min}} = (2, 1^2)$, $\mathcal{O}_{\text{subreg}} = (2^2)$, $\mathcal{O}_{\text{reg}} = (4)$. All of them are special except $\mathcal{O}_{\text{min}}$. So, in this case we expect $F_{\mathcal{O}_{\text{min}}} = F_{\mathcal{O}_0}$.

This equality is motivated by a result in the classical theory of automorphic forms (25, Theorem 3, see also [18]) saying that any infinite-dimensional representation of $\text{Sp}_4$ over a local non-archimedian field admits either a Whittaker or Bessel model. The analog of this in the global setting would be that an infinite-dimensional automorphic representation of $G(\mathbb{A})$ admits a nonzero Fourier coefficient either for $\mathcal{O}_{\text{reg}}$ or $\mathcal{O}_{\text{subreg}}$, here $\mathbb{A}$ is the ring of adeles of a curve over a finite field.
2.1.20. A much more general support for Conjecture 2.1.18 is provided by the main result of [22], which is concerned precisely with an analog of Conjecture 2.1.18 at the classical level of automorphic forms (they also study an analogous question for representations of reductive groups over local non-archimedean fields).

2.1.21. Question. Consider the case of \( X = \mathbb{P}^1 \). Then one has the Shatz stratification of \( \text{Bun}_G \) by locally closed substacks \( \text{Bun}_{G, \lambda} \) indexed by \( \lambda \in \Lambda^+ \). Write \( IC_\lambda \) for the IC-sheaf of the stratum \( \text{Bun}_{G, \lambda} \). Given \( \lambda \in \Lambda^+ \) find the smallest closed \( G \)-invariant subset \( Y \subset N(\mathfrak{g}) \) such that \( IC_\lambda \in F_Y \).

2.2. Spectral part and the Langlands correspondence.

2.2.1. Recall the algebraic stack \( \text{Arth}_G(X) \) over \( \text{Spec} e \) defined in ([3], 14.2.2). In the notations of loc.cit. it classifies right t-exact symmetric monoidal functors \( h : \text{Rep}(\check{G}) \rightarrow \text{QLisse}(X) \), which we think of as \( \check{G} \)-local systems \( \sigma \) on \( X \), and \( A \in H^0(X, h(\check{g}^*)) \). As in loc.cit., let \( \text{Nilp} \subset \text{Arth}_G(X) \) denote the closed conical subset of those \( (\sigma, A) \) that \( A \) is nilpotent. That is, for any local on \( X \) trivialization of \( \sigma \), one requires that \( A \) takes values in \( N(\mathfrak{g}) \subset \check{g}^* \). Here we used the isomorphism \( \check{\kappa} \) fixed in Section 1.1.2. Given \( O' \in X(\mathfrak{g}) \), one similarly gets the locus denoted \( \overline{O'} \subset \text{Arth}_G(X) \) by abuse of notations.

This gives a filtration on
\[
\text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\mathfrak{g}}^{\text{restr}}(X))
\]
indexed by nilpotent orbits in \( \mathfrak{g} \). Namely, for \( O' \in X(\mathfrak{g}) \) one has the full DG-subcategory
\[
S_{\overline{O'}} = \text{IndCoh}_{\overline{O'}}(\text{LocSys}_{\mathfrak{g}}^{\text{restr}}(X))
\]
If \( O \subset \overline{O'} \) then \( S_{\overline{O}} \subset S_{\overline{O'}} \).

The following is our main conjecture, it proposes a geometric counterpart of ([19], Conjecture 4.2 and 4.3).

**Conjecture 2.2.2.** For any \( O \in X(\mathfrak{g})^{\text{sp}} \) there is an equivalence
\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G)/F_{\text{Nilp}, < O^{\text{sp}}} \xrightarrow{\sim} \text{IndCoh}_{\overline{O}}(\text{LocSys}_{\mathfrak{g}}^{\text{restr}}(X))
\]
compatible with the Hecke actions. Here \( d : X(\mathfrak{g}) \rightarrow X(\check{\mathfrak{g}}) \) is the Lusztig-Spaltenstein duality map. For \( O = O_0 \) this recovers the conjectural geometric Langlands equivalence
\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\sim} \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\mathfrak{g}}^{\text{restr}}(X))
\]
of ([3], Conjecture 21.2.7).

**Remark 2.2.3.** Take \( O = O_{\text{reg}} \in X(\mathfrak{g})^{\text{sp}} \) and \( Y = N(\mathfrak{g}) - O_{\text{reg}} \). In view of Conjectures 2.1.18 and 2.1.19 we expect that
\[
\text{Shv}_{\text{Nilp}, < O^{\text{sp}}} = F_{\text{Nilp}, Y} = \text{Shv}_Y(\text{Bun}_G)
\]
coincides with the category \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{anti-temp}} \) of anti-tempered objects of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).
In this case (5) would become the tempered geometric Langlands conjecture as formulated in ([10, Section 1.6.2]):

\[ \text{Shv}_{Nilp}(Bun_G) / \text{Shv}_{Nilp}(Bun_G)^{anti-temp} \cong \text{QCoh}(\text{LocSys}_{G}^{estr}(X)) \]

**Remark 2.2.4.** Assume \( G \) simple and take \( \mathfrak{O} \) to be the minimal special nilpotent orbit \( \mathfrak{O}_{\text{min}}^{sp} \) in \( \text{N}(g) \), so \( d(\mathfrak{O}) = \mathfrak{O}_{\subreg} \). By definition, \( <\mathfrak{O}^{sp} = \{0\} \). In this case assuming Conjecture 2.1.13 we get \( F_{\text{Nilp},<\mathfrak{O}^{sp}} = \text{Shv}_0(Bun_G) \). So, Conjecture 2.2.2 predicts in this case an equivalence

\[ \text{Shv}_{Nilp}(Bun_G) / \text{Shv}_0(Bun_G) \cong \text{IndCoh}_{\subreg}(\text{LocSys}_{G}^{estr}(X)) \]

If our sheaf theory is that of ind-holonomic \( D \)-modules then the latter equivalence is established in [9].

### 2.3. Compatibility with the parabolic induction.

2.3.1. The idea of the compatibility of Conjecture 2.2.2 with the parabolic induction essentially appears already in David Ginzburg’s paper [16].

Recall the definition of induced orbits ([8, Section 7.1]). For a parabolic subalgebra \( p \subset g \) with Levi decomposition \( p = m \oplus u \) let \( \mathfrak{O} \in \mathfrak{X}(m) \). By ([8, Theorem 7.1.1]), there is a unique nilpotent orbit \( \text{Ind}_m^g(\mathfrak{O}) \in \mathfrak{X}(g) \) such that

\[ \text{Ind}_m^g(\mathfrak{O}) \cap (\mathfrak{O} + u) \subset \mathfrak{O} + u \]

is an open dense subset. It is the nilpotent orbit in \( g \) induced from \( \mathfrak{O} \). Recall that \( \dim \text{Ind}_m^g(\mathfrak{O}) = \dim \mathfrak{O} + 2 \dim u \), and \( \text{Ind}_m^g(\mathfrak{O}) \) depends only on \( m \) and not on the choice of a parabolic subalgebra containing \( m \) ([8, Theorem 7.1.3]). Therefore, we also write \( \text{Ind}_m^g(\mathfrak{O}) := \text{Ind}_m^g(\mathfrak{O}) \). The induction of the zero orbit \( \text{Ind}_m^g(\mathfrak{O}_0) \) is called a Richardson orbit.

The obtained map \( \text{Ind}_m^g : \mathfrak{X}(m) \to \mathfrak{X}(g) \) has the following remarkable property.

**Proposition 2.3.2** ([8, Theorem 8.3.1], [7, Proposition A.2]). Let \( \mathfrak{O}_m \in \mathfrak{X}(m) \) and \( \mathfrak{O} \in \mathfrak{X}(g) \) containing \( \mathfrak{O}_m \). Then one has

\[ d(\mathfrak{O}) = \text{Ind}_m^g(d(\mathfrak{O}_m)) \]

2.3.3. For a parabolic subgroup \( P \subset G \) with Levi quotient \( M \) consider the diagram of natural maps

\[ \begin{array}{ccc}
\text{Bun}_M & \xleftarrow{q^P} & \text{Bun}_P \\
\text{Bun}_G & \xrightarrow{p^P} & \text{Bun}_G 
\end{array} \]

The Eisenstein series functor \( \text{Eis}_! : \text{Shv}(\text{Bun}_M) \to \text{Shv}(\text{Bun}_G) \) is defined by

\[ \text{Eis}_!(\mathcal{K}) = p_!^P(q^P)^*\mathcal{K} \]

2.3.4. Let also \( \widetilde{\text{Bun}}_P \) be the Drinfeld compactification of \( \text{Bun}_P \) defined in [6]. More precisely, if \( [G,G] \) is simply-connected then \( \text{Bun}_P \) is defined as in loc.cit., and in general one corrects this definition as in ([28, Section 7.4]). Then \( \text{Bun}_P \subset \text{Bun}_P \) is an open substack, and the diagram (7) extends to the following one

\[ \begin{array}{ccc}
\text{Bun}_M & \xleftarrow{\tilde{q}^P} & \tilde{\text{Bun}}_P \\
\text{Bun}_G & \xrightarrow{\tilde{p}^P} & \text{Bun}_G 
\end{array} \]
The compactified Eisenstein series functor $\text{Eis}_* : \text{Shv}(\text{Bun}_M) \to \text{Shv}(\text{Bun}_G)$ is defined as in [6] by

$$\text{Eis}_*(\mathcal{K}) = \tilde{p}^P((\tilde{q}^P)^*\mathcal{K} \otimes \text{IC}_{\tilde{\text{Bun}}_P}),$$

where $\text{IC}_{\tilde{\text{Bun}}_P}$ is the IC-sheaf on $\tilde{\text{Bun}}_P$.

2.3.5. It is expected that both $\text{Eis}_!$ and $\text{Eis}_*$ restrict to the functors $\text{Shv}_{\text{Nilp}}(\text{Bun}_M) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, that is, preserve the nilpotence of the singular support.

Moreover, the so obtained conjectural functor

$$\text{Eis}_! : \text{Shv}_{\text{Nilp}}(\text{Bun}_M) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$$

is expected to be compatible with the conjectural equivalence in the same sense as in the context of $D$-modules. Recall that the analog of $\text{Eis}_!$ for the $D$-module context is compatible with the conjectural equivalence in the sense of ([2], Section 1.2.3).

Based on the existing works at the level of functions ([17]-[21]), we suggest the following.

**Conjecture 2.3.6.** Let $\mathcal{O} \in \mathcal{X}(\mathfrak{m})$, $\mathcal{O}' = \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\mathcal{O}) \in \mathcal{X}(\mathfrak{g})$ and $K \in F_\mathcal{O}(M)$. Then $\text{Eis}_!(K) \in F_{\mathcal{O}'}(G)$.

2.3.7. **Example.** The constant sheaf $e_{\text{Bun}_M}$ on $\text{Bun}_M$ lies in $F_{\mathcal{O}_0}(M)$. Let $\mathcal{O}' = \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\mathcal{O}_0)$ be the corresponding Richardson orbit. For $G = \text{GL}_n$ the analog at the level of functions of the property $\text{Eis}_!(e_{\text{Bun}_M}) \in F_{\mathcal{O}'}(G)$ appears as ([16], Conjecture 5.1).

2.3.8. **Question.** Should we expect the analog of Conjecture 2.3.6 with $\text{Eis}_!$ replaced by $\text{Eis}_*$?

### 3. Examples

In this section we prove Theorem 2.1.11 for a regular nilpotent $x$. We also discuss the example of the minimal nilpotent orbit.

#### 3.1. Regular nilpotent orbit.

3.1.1. Pick a maximal torus and a Borel subgroup $T \subset B \subset G$. Let $J$ denote the set of vertices of the Dynkin diagram. For $i \in J$ let $\alpha_i$ (resp., $\check{\alpha}_i$) denote the corresponding coroot (resp., root) of $(T,G)$. For $i \in J$ pick a nonzero vector $x_{\check{\alpha}_i}$ in the corresponding root subspace $X_{\check{\alpha}_i}$ of $\mathfrak{g}$. Take $x = \sum_{i \in J} x_{\check{\alpha}_i}$. Recall that $x$ is always a distinguished nilpotent ([29], Definition 4.3.1), and the corresponding Jacobson-Morozov parabolic subalgebra is $\mathfrak{b} = \text{Lie} B$. Recall that each distinguished nilpotent is even by ([8], Theorem 8.2.3), so $x$ is even. Let $\mathcal{O}$ denote the nilpotent orbit of $\mathfrak{g}$ containing $x$, we refer to it as the regular nilpotent orbit. By $w_0$ we denote the longest element of the Weyl group of $(T,G)$.

In the rest of Section 3.1 we prove Theorem 2.1.11 for $x$ regular.

3.1.2. Let $\hat{\Lambda}$ be the character lattice of $T$, $\Lambda$ be the cocharacter lattice. For a $T$-torsor on some base and $\lambda \in \hat{\Lambda}$ let $\mathcal{L}_x^\lambda_T$ denote the line bundle obtained from $\mathcal{F}_T$ via extension of scalars by $\lambda : T \to \mathbb{G}_m$. 

3.1.3. We have $V/V' \simeq \bigoplus_{i \in \mathcal{I}} X_{\alpha_i}$ as representations of $T$. The stack $\text{Bun}_{B/V'}$ classifies pairs $\mathcal{F}_T \in \text{Bun}_T$, and an exact sequence on $X$

$$0 \to \bigoplus_{i \in \mathcal{I}} \mathcal{L}^\alpha_i \mathcal{F}_T \to \mathcal{O}_X \to 0$$

The stack $\mathcal{Y}_T$ classifies $\mathcal{F}_T \in \text{Bun}_T$ and a section

$$s : \bigoplus_{i \in \mathcal{I}} \mathcal{L}^\alpha_i \mathcal{F}_T \to \Omega$$

on $X$. Write $s_{\alpha_i} : \mathcal{L}^\alpha_i \mathcal{F}_T \to \Omega$ for the corresponding component of $s$. The open substack $\mathcal{Y}_T^0 \subset \mathcal{Y}_T$ is given by the property that for any $i \in \mathcal{I}$, $s_{\alpha_i} \neq 0$.

3.1.4. Pick a closed point $y \in X$. The Hecke functors

$$H_y^-, H_y^- : \text{Rep}(\hat{G}) \times \text{Shv}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)$$

are defined as in ([5], 3.2.4). This definition is also equivalent to the one from ([4], 5.3.6). Let $\Lambda^+$ be the set of dominant coweights. Let $\text{Gr}_{G,y}$ denote the affine grassmanian of $G$ at $y$. For $\lambda \in \Lambda^+$ we denote by $\mathcal{A}^\lambda_{G}$ the IC-sheaf of $\mathcal{G}^\lambda$ as in ([5], 3.2.1).

Let $y \mathcal{H}_G$ be the Hecke stack classifying $\mathcal{F}_G, \mathcal{F}'_G \in \text{Bun}_G$ and $\beta : \mathcal{F}_G \to \mathcal{F}'_G |_{X-y}$. We have the diagram

$$\text{Bun}_G \xrightarrow{h^-} y \mathcal{H}_G \xrightarrow{h^-} \text{Bun}_G,$$

where $h^-$ (resp., $h^+$) sends the above point to $\mathcal{F}_G$ (resp., to $\mathcal{F}'_G$). Let $\mathcal{G}_y \to \text{Bun}_G$ be the $G(\mathcal{O}_y)$-torsor classifying $\mathcal{F}_G \in \text{Bun}_G$ and its trivialization over the formal disk $D_y$ around $y$. We have the isomorphisms

$$\text{id}^l, \text{id}^r : y \mathcal{H}_G \xrightarrow{\simeq} \mathcal{G}_y \times^{G(\mathcal{O}_y)} \text{Gr}_{G,y}$$

such that the projection to $\text{Bun}_G$ corresponds to $h^-, h^+ : y \mathcal{H}_G \to \text{Bun}_G$ respectively. For $\mathcal{F} \in \text{Shv}(\text{Bun}_G), \mathcal{S} \in \text{Perv}_{G(\mathcal{O}_y)}(\text{Gr}_{G,y})$ we have the corresponding twisted products $(\mathcal{F} \boxtimes \mathcal{S})^l, (\mathcal{F} \boxtimes \mathcal{S})^r$ normalized as in loc.cit. We then set

$$H_y^-(\mathcal{S}, \mathcal{F}) = \mathcal{S} \ast \mathcal{F} = h_1^-(\mathcal{F} \boxtimes (\ast \mathcal{S}))$$

and

$$H_y^+(\mathcal{S}, \mathcal{F}) = \mathcal{F} \ast \mathcal{S} = h_1^+(\mathcal{F} \boxtimes \mathcal{S})$$

Here we denoted by $\ast$ the covariant self-functor on $\text{Perv}_{G(\mathcal{O}_y)}(\text{Gr}_{G,y})$ coming from $G(\mathcal{O}_y) \to G(\mathcal{O}_y), g \mapsto g^{-1}$.

For $\lambda \in \Lambda^+$ let $y \mathcal{P}^\lambda_G$ be the closed substack of $y \mathcal{H}_G$ which identifies under $\text{id}^l$ with $\mathcal{G}_y \times^{G(\mathcal{O}_y)} \mathcal{G}^\lambda_{G}$. This notation agrees with that of ([5], Section 3.2.4). For a point $(\mathcal{F}_G, \mathcal{F}'_G, \beta) \in y \mathcal{P}^\lambda_G$ we say that $\mathcal{F}'_G$ is in the position $\leq \lambda$ with respect to $\mathcal{F}_G$. This is equivalent to $\mathcal{F}_G$ being in the position $\leq -w_0(\lambda)$ with respect to $\mathcal{F}'_G$.

3.1.5. Let $K \in \mathcal{F}_0$ and $\lambda \in \Lambda^+$. We will show that $\mathcal{A}^\lambda_G \ast K \in \mathcal{F}_0$.
3.1.6. Pick a \( k \)-point of \( Y^0 \) given by \((\mathcal{F}_T', s')\). We check that the \(*\)-fibre of 
\[
\text{Four}(pq^*(A^\lambda_G \ast K))
\]
at this point vanishes. Our argument will be compatible with the field extensions \( k \to k' \), so this is sufficient.

Let \( \Lambda_{ad}^+ \) be the set of dominant coweights for \( G_{ad} = G/Z(G) \). Let 
\[
\text{cond}(s') = \sum_{z \in X} \text{cond}(s')_z
\]
be the \( \Lambda_{ad}^+ \)-valued divisor on \( X \) given by the property: if \( i \in I \) then \( \langle \text{cond}(s')_z, \alpha_i \rangle \) is the order of zero of \( s'_i : \mathcal{L}^i_{\mathcal{F}_T'} \hookrightarrow \Omega \) at \( z \).

3.1.7. Let \( \text{Bun}_{\mathcal{F}_T'} \) be the stack classifying \( \mathcal{F}_T' \subseteq \text{Bun}_B \) and an isomorphism \( \gamma' : \mathcal{F}_T' \times_B T \cong \mathcal{F}_T' \). Denote by \( ev_s : \text{Bun}_{\mathcal{F}_T'} \to \mathcal{G}_a \) the composition
\[
\text{Bun}_{\mathcal{F}_T'} \to \prod_{i \in I} H^1(X, \mathcal{L}^i_{\mathcal{F}_T'}) \to \prod_{i \in I} H^1(X, \Omega) \to \prod_{i \in I} \mathbb{G}_a \to \mathcal{G}_a
\]
Let \( q' : \text{Bun}_{\mathcal{F}_T'} \to \text{Bun}_G \) be the natural map. It suffices to show that
\[
(9) \quad \text{R} \Gamma_c(\text{Bun}_{\mathcal{F}_T'}, ev_s^* \mathcal{L}_\psi \otimes q'^*(A^\lambda_G \ast K)) = 0
\]

3.1.8. We claim that \((9)\) identifies canonically with
\[
(10) \quad \text{R} \Gamma_c(\text{Bun}_G, K \otimes ((q'_* ev_s^* \mathcal{L}_\psi) \ast A^\lambda_G))
\]
Indeed, to see this we may assume \( K \) constructible. Then \((9)\) identifies with
\[
\text{R} \Gamma_c(\text{Bun}_G, (A^\lambda_G \ast K) \otimes q'_* ev_s^* \mathcal{L}_\psi) \to \text{D} \text{R} \text{Hom}(A^\lambda_G \ast K, q'_* ev_s^* \mathcal{L}_\psi[-1][2])
\]
By \((\Pi), 5.3.9\), the functor \( \text{Shv} (\text{Bun}_G) \to \text{Shv} (\text{Bun}_G), L \mapsto A^\lambda_G \ast L \) is left adjoint to the functor \( L \mapsto L \ast A^\lambda_G \). So, \((9)\) identifies with
\[
\text{D} \text{R} \text{Hom}(K, (q'_* ev_s^* \mathcal{L}_\psi[-1][2]) \ast A^\lambda_G),
\]
which in turn identifies with \((10)\) as desired.

3.1.9. Define the stack \( \text{Bun}_{\mathcal{F}_T'} \) as in \(\Pi\). More precisely, if \( [G,G] \) is simply-connected this is exactly the definition from \(\Pi\). In this case it classifies \( \mathcal{F}_T \in \text{Bun}_G \) and a collection of nonzero maps over \( X \)
\[
\kappa^{\bar{\lambda}} : \mathcal{L}^{\bar{\lambda}}_{\mathcal{F}_T'} \hookrightarrow \mathcal{V}^{\bar{\lambda}}_{\mathcal{F}_T'}, \bar{\lambda} \in \bar{\Lambda}^+
\]
subject to the Plücker relations. Here \( \bar{\Lambda}^+ \) is the set of dominant weights for \( G \), and \( \mathcal{V}^{\bar{\lambda}} \) denotes the Weyl module corresponding to \( \bar{\lambda} \) (cf. \(\Pi\), Section 1.4).

Let also \( y_{\infty} \text{Bun}_{\mathcal{F}_T'} \) be the version of \( \text{Bun}_{\mathcal{F}_T'} \), where the maps \( \kappa^{\bar{\lambda}} \) are allowed to have any poles at \( y \).

If \( [G,G] \) is not simply-connected, one changes the definition of \( \text{Bun}_{\mathcal{F}_T'} \) as is explained in Schieder’s paper \((28), \text{Section } 7\) and similarly for \( y_{\infty} \text{Bun}_{\mathcal{F}_T'} \).
3.1.10. As in (11, Section 5.3), one defines the Hecke action of \( \text{Perv}_{G(O_y)}(\text{Gr}_{G,y}) \) on \( \text{Shv}(y,_{\infty} \text{Bun}_U^T) \). So, for \( S \in \text{Perv}_{G(O_y)}(\text{Gr}_{G,y}) \), \( \mathcal{T} \in \text{Shv}(y,_{\infty} \text{Bun}_U^T) \) we get the objects \( \mathcal{T} \ast S \) and \( \partial \ast \mathcal{T} \) in \( \text{Shv}(y,_{\infty} \text{Bun}_U^T) \).

3.1.11. Let \( j : \text{Bun}_U^T \to y,_{\infty} \text{Bun}_U^T \) be the embedding. Let \( \tilde{q} : y,_{\infty} \text{Bun}_U^T \to \text{Bun}_G \) be the projection. Since

\[
\tilde{q} : \text{Shv}(y,_{\infty} \text{Bun}_U^T) \to \text{Shv}(\text{Bun}_G)
\]

commutes with the actions of the Hecke functors at \( y \), (10) identifies with

\[
(11) \quad \Gamma_c(\text{Bun}_G, K \otimes \tilde{q}((j!ev^*_x, \mathcal{L}_\psi) * \mathcal{A}_G)) \cong \Gamma_c(y,_{\infty} \text{Bun}_U^T, \tilde{q}^* K \otimes ((j!ev^*_x, \mathcal{L}_\psi) * \mathcal{A}_G))
\]

3.1.12. For \( \nu \in \Lambda \) let \( S^\nu \subset \text{Gr}_G \) denote the \( U(F) \)-orbit through \( t^\nu \in \text{Gr}_G \), here \( F \) is our local field (cf. (11), Section 7.1.1).

3.1.13. For \( \nu \in \Lambda \) let \( y,_{\nu} \tilde{\text{Bun}}_U^T \subset y,_{\infty} \text{Bun}_U^T \) be the locally closed substack given by the property (in the case of \([G,G]\) simply-connected) that for any \( \lambda \in \tilde{\Lambda} \), the map

\[
\mathcal{L}_\mathcal{T}_G^\lambda(\nu, \lambda) \to \mathcal{V}_G^\lambda
\]

are regular over \( X \) and have no zeros at \( y \). (We leave it to a reader to define an analog of this stack for \( G \) arbitrary reductive).

The stacks \( y,_{\nu} \tilde{\text{Bun}}_U^T \) for \( \nu \in \Lambda \) form a stratification of \( y,_{\infty} \text{Bun}_U^T \).

3.1.14. Fix \( \nu \in \Lambda \). It suffices to show that the contribution of the stratum \( y,_{\nu} \tilde{\text{Bun}}_U^T \) to integral (10) vanishes.

Set \( \mathcal{T}_G = \mathcal{T}_G(-\nu y) \). We have the open immersion \( \text{Bun}_U^T \to y,_{\nu} \tilde{\text{Bun}}_U^T \). By construction, the \(*\)-restriction of \((j!ev^*_x, \mathcal{L}_\psi) * \mathcal{A}_G^\lambda \) to \( y,_{\nu} \tilde{\text{Bun}}_U^T \) is the extension by zero from \( \text{Bun}_U^T \).

3.1.15. Let \( Z \) be the stack classifying \((\mathcal{T}_B, \gamma') \in \text{Bun}_U^T \), for which we set \( \mathcal{T}_G = \mathcal{T}_B \times_B G \), and \((\mathcal{T}_G, \mathcal{F}_G, \beta) \in \mathcal{Y}_G \) such that \( \mathcal{T}_G \) is in the position \( \leq \lambda \) with respect to \( \mathcal{T}_G \). So,

\[
Z = y,_{\mathcal{T}_G} \times_{\text{Bun}_G} \text{Bun}_U^T
\]

where we used the map \( h \to \) to form the fibred product.

Pick a trivialization \( \epsilon : \mathcal{T}_T \to \mathcal{T}_T \mid_{D_y} \), where \( \mathcal{T}_T^0 \) denotes the trivial \( T \)-torsor. This gives rise to a \( U(O_y) \)-torsor denoted \( \mathcal{U} \) over \( \text{Bun}_U^T \); it classifies trivializations \( \mathcal{T}_B \to \mathcal{T}_B \mid_{D_y} \). We have a projection \( h_Z : Z \to \text{Bun}_U^T \) sending the above point to \((\mathcal{T}_B, \gamma') \). It realizes \( Z \) as a fibration

\[
(12) \quad \mathcal{U} \times_{U(O_y)} \text{Gr}_G \to Z
\]
Let $Z' \subset Z$ be the substack that identifies via $h_Z^+$ with

$$U^c \times U^{(0)} (\mathcal{G}_G \cap S^v)$$

The stack $Z'$ classifies $(\mathcal{F}_B', \gamma', \mathcal{F}_G, \beta) \in Z$ such that $\mathcal{F}_B'$ induces a $B$-structure on $\mathcal{F}_G$, which we denote as $\mathcal{F}_B$, and such that the isomorphism $\gamma' : \mathcal{F}_B \times_B T \cong \mathcal{F}_T |_{X-y}$ extends to an isomorphism $\gamma : \mathcal{F}_B \times_B T \cong \mathcal{F}_T$ on $X$.

Let $h_Z' : Z' \to \text{Bun}_{\mathcal{F}_T}^c$ be the map that sends the above point to $(\mathcal{F}_B, \gamma)$.

### 3.1.16. The $*$-restriction of $(j! \text{ev}_s^*(\mathcal{L}_\psi)) \ast A_G^\lambda$ under $\text{Bun}_{\mathcal{F}_T}^c \hookrightarrow \text{Bun}_{\mathcal{F}_T}^c$ identifies with

$$(h_Z')!(A_G^\lambda \overset{i}{\to} (\mathcal{G}_G \cap S^v) \otimes \text{ev}_s^*(\mathcal{L}_\psi))^r,$$

here $A_G^\lambda \overset{i}{\to} (\mathcal{G}_G \cap S^v)$ denotes the $*$-restriction.

Consider the projection $\text{pr} : \Lambda = \Lambda_{G_{ad}}$. If for any $i \in \mathcal{I}$,

$$\langle \text{cond}(s')_y + \text{pr}(\nu), \tilde{\alpha}_i \rangle \geq 0$$

then we get a point $(\mathcal{F}_T, s') \in \mathcal{Y}_T^0$. Namely, for $i \in \mathcal{I}$ the datum of $s'$ gives rise to an inclusion $s_{\tilde{\alpha}_i} : \mathcal{L}_T^b_s \hookrightarrow \Omega$. We then have the morphism $\text{ev}_s : \text{Bun}_{\mathcal{F}_T}^c \to \mathcal{G}_a$ defined along the same lines as $\text{ev}_{s'}$.

**Lemma 3.1.17.** If for any $i \in \mathcal{I}$ the condition (14) holds then (13) is isomorphic to $\text{ev}_s^*\mathcal{L}_\psi \otimes \mathcal{M}$ for some constant complex $\mathcal{M} \in \text{Shv}(\text{Spec } k)$. Otherwise, (13) vanishes.

The proof of Lemma 3.1.17 is given in Section 3.1.19.

### 3.1.18. End of the proof of Theorem 2.1.11 for $x$ regular. Let $q : \text{Bun}_{\mathcal{F}_T}^c \to \text{Bun}_G$ be the natural map. By Lemma 3.1.17, the contribution of the substack $\text{Bun}_{\mathcal{F}_T}^c$ in (11) identifies with

$$\text{R}^\infty_{\mathcal{G}}(\text{Bun}_{\mathcal{F}_T}^c, q^*K \otimes \text{ev}_s^*\mathcal{L}_\psi)$$

tensored by some constant complex. The latter cohomology vanishes by definition of $\mathcal{F}_G$. We are done. □

### 3.1.19. Proof of Lemma 3.1.17.** A detailed argument for any $x$ even is given in Section 4 below. Here we only give a sketch.

As in (112), Section 4), one defines the Whittaker category $\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c)$ corresponding to $s'$. Namely, first one defines a full Serre abelian subcategory

$$\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright \subset \text{Shv}(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$$

It is singled out by an equivariance condition under some groupoid. More precisely, an object $K \in \text{Shv}(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$ by definition lies in $\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$ if for any constructible perverse subsheaf $K' \subset K$, $K'$ is equivariant under the corresponding groupoid. Our approach to defining $\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$ is justified by Remark 3.1.21 below.

Now $\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$ is presentable and closed under colimits in $\text{Shv}(y, \infty \text{Bun}_{\mathcal{F}_T}^c)^\triangleright$.

Define

$$\text{Shv}^W(y, \infty \text{Bun}_{\mathcal{F}_T}^c) \subset \text{Shv}(y, \infty \text{Bun}_{\mathcal{F}_T}^c)$$

(15)
as the full subcategory of those objects whose all perverse cohomology sheaves lie in $Shv^W_{y,\infty \mathbf{Bun}_{T}^{\mathbb{T}}}$). Then $Shv^W_{y,\infty \mathbf{Bun}_{T}^{\mathbb{T}}}$ in DGCat_{cont}, this category is equipped with a t-structure induced from that on $Shv(y,\infty \mathbf{Bun}_{T}^{\mathbb{T}})$, so that the embedding (15) is continuous and t-exact.

The Hecke action of $Perv_{G(\mathcal{O})}(\mathbf{Gr}_{G,y})$ on $Shv(y,\infty \mathbf{Bun}_{T}^{\mathbb{T}})$ preserves the full subcategory $Shv^W_{y,\infty \mathbf{Bun}_{T}^{\mathbb{T}}}$. For any locally closed substack $Z \subset y,\infty \mathbf{Bun}_{T}^{\mathbb{T}}$ stable under the corresponding groupoid, we may similarly define the version $Shv^W_{Z}$ of the Whittaker category for $s'$. In particular, this holds for $Z = \mathbf{Bun}_{T}^{\mathbb{T}}$, and we get $Shv^W_{\mathbf{Bun}_{T}^{\mathbb{T}}}$. By the above, (13) is an object of $Shv^W_{\mathbf{Bun}_{T}^{\mathbb{T}}}$). Our claim follows from Lemma 3.1.20 below.

**Lemma 3.1.20.** If there is $i \in I$ such that (14) does not hold then $Shv^W_{\mathbf{Bun}_{T}^{\mathbb{T}}}$ vanishes. Otherwise, any object of $Shv^W_{\mathbf{Bun}_{T}^{\mathbb{T}}}$ is isomorphic to $\text{ev}_s^* L_{\psi} \otimes M$ for some constant complex $M \in Shv(\text{Spec} \mathcal{O})$.

**Proof.** This is analogous to ([11], Lemma 6.2.8). □

**Remark 3.1.21.** Recall that for a classical algebraic stack locally of finite type, $Perv(\mathcal{Y})$ is defined as $\left(Shv(\mathcal{Y})^{\text{constr}}\right)^\vee$. If $\mathcal{Y}$ is moreover of finite type, then the natural functor $\text{Ind}(Perv(\mathcal{Y})) \to Shv(\mathcal{Y})^\vee$ is an equivalence. However, this is not true in general for $\mathcal{Y}$ locally of finite type. In the latter case an object of $Perv(\mathcal{Y})$ is not necessarily compact in $Shv(\mathcal{Y})^\vee$. This happens for example for $\mathcal{Y} = \sqcup_{j \in \mathbb{Z}} \text{Spec} \mathcal{O}$.

### 3.2. Minimal nilpotent orbit.

#### 3.2.1. Assume $\mathfrak{g}$ simple. Let $\bar{\alpha}$ be the highest root of $T$. Pick a nonzero element $x \in X_{\bar{\alpha}}$ in the corresponding root subspace. Let $\mathbb{O}_x$ be the nilpotent $G$-orbit through $x$. This is the minimal nilpotent orbit ([8], Theorem 4.3.3). Let $P \subset G$ be the parabolic such that $\text{Lie} P = \mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ is the Jacobson-Morozov parabolic subalgebra of $\mathfrak{g}$. It is known that $\mathfrak{g}_i = 0$ for $|i| > 2$. Recall that $U \subset P$ denotes the unipotent radical of $P$, and $V \subset U$ the subgroup with $\text{Lie} V = \mathfrak{g}_2$ in this case. The group $P$ is called the Heisenberg parabolic of $G$. It is known that $U$ is a Heisenberg group with the center $V$, and $\dim \mathfrak{g}_2 = 1$. Let $M$ be the Levi of $P$ with $\text{Lie} M = \mathfrak{g}_0$.

Since $V$ is a 1-dimensional linear representation of $M$, the open $M \times \mathbb{G}_m$-orbit on $V^*$ is $V^* - \{0\}$.

### 4. Case of even $x$

#### 4.1. In Section 4 we assume $x$ even and prove Theorem 2.1.11 in this case. Keep notations of Section 2.

#### 4.1.1. If $[G,G]$ is simply connected, one has the stack $\mathbf{Bun}_{P}$ defined in ([5], Section 1). If $[G,G]$ is not simply-connected one modifies the definition of $\mathbf{Bun}_{P}$ as is outlined in ([28], Section 7.4).
4.1.2. Pick a maximal torus and Borel subgroups \( T \subset B_M \subset M \) such that \( H \in \text{Lie} T \). This yields a Borel subgroup \( B \subset P \), which is the preimage of \( B_M \) under \( P \to M \). Let \( \mathcal{I}_M \subset \mathcal{J} \) be the subset corresponding to simple roots of \( M \). Let \( \Lambda_{G,P} \) denote the quotient of \( \Lambda \) be the subgroup generated by \( \alpha_i, i \in \mathcal{I}_M \). This is the algebraic fundamental group of \( M \). Let \( \Lambda_{G,P}^+ \) be the lattice dual to \( \Lambda_{G,P} \). Let \( \Lambda_{G,P}^+ \subset \Lambda_{G,P} \) be the subset of those weights, which are dominant for \( B \).

4.1.3. Pick a closed point \( y \in X \), our Hecke functors are applied at \( y \in X \). Let \( K \in \mathcal{F}_\Omega, \lambda \in \Lambda^+ \). We want to show that \( A^*_{G_k} \ast K \in \mathcal{F}_\Omega \).

4.1.4. Pick a \( k \)-point of \( \mathcal{Y}_M^0 \) given by \((\mathcal{F}^I, s')\), where \( s' : (V/V')_{\mathcal{F}^I_M} \to \Omega \). We check that the \(*\)-fibre of \( \text{Four}(p_!q^* (A^*_{G_k} \ast K)) \) at this point vanishes. Our argument will be compatible with field extensions \( k \to k' \) as in Section 3 so this is sufficient.

Recall that \( U = V \). Let \( \text{Bun}_{\mathcal{F}^I_M} \) be the stack classifying \( \mathcal{F}^I_M \in \text{Bun}_P \) and an isomorphism \( \gamma' : \mathcal{F}^I_P \times_P M \cong \mathcal{F}^I_M \). Let \( \text{Bun}_{\mathcal{F}^I_{V'/V}} \) be the stack classifying a \( P/V'\)-torsor \( \mathcal{F}^I_{P/V'} \) together with an isomorphism \( \mathcal{F}^I_{P/V'} \times_{P/V'} M \cong \mathcal{F}^I_M \). So, \( \text{Bun}_{\mathcal{F}^I_{V'/V}} \) classifies exact sequences
\[
0 \to (V/V')_{\mathcal{F}^I_M} \to U \to \mathcal{O}_X \to 0
\]
on \( X \).

Let \( ev_{s'} : \text{Bun}_{\mathcal{F}^I_M} \to \mathcal{G}_a \) be the composition \( \text{Bun}_{\mathcal{F}^I_M} \to \text{Bun}_{\mathcal{F}^I_{V'/V}} \to \mathcal{G}_a \), where the second map is the pairing of \((\ref{eq:pairing})\) with \( s' \).

Let \( \mathcal{Q}' : \text{Bun}_{\mathcal{F}^I_M} \to \text{Bun}_G \) be the natural map. It suffices to show that
\[
\text{R} \Gamma_c (\text{Bun}_{\mathcal{F}^I_M}, ev_{s'}^* \mathcal{L}_\psi \otimes q'^* (A^*_{G_k} \ast K)) = 0
\]

4.1.5. As in Section 3.1.8 the object \((\ref{eq:smash})\) identifies canonically with
\[
\text{R} \Gamma_c (\text{Bun}_G, K \otimes (q' ev_{s'}^* \mathcal{L}_\psi) \ast A^*_{G_k})
\]

4.1.6. Let \( \text{Bun}_{\mathcal{F}^I_M} \) denote the version of \( \text{Bun}_P \), where we fix the \( M \)-torsor to be \( \mathcal{F}^I_M \). If \([G,G]\) is simply-connected then, for a test scheme \( S \) of finite type, an \( S \)-point of \( \text{Bun}_{\mathcal{F}^I_M} \) is given by \( \mathcal{F}^I_G \) on \( X \times S \) and a collection of maps of coherent sheaves
\[
\kappa^V : (\mathcal{V}^U)_{\mathcal{F}^I_M} \to \mathcal{V}_{\mathcal{F}^I_G}
\]
on \( X \times S \) for every \( G \)-module \( \mathcal{V} \), such that for any \( k \)-point \( s \in S \), its restriction \( \kappa^V_s \) to \( X \times s \) is injective, and Plücker relations hold \((\ref{eq:plucker}), \text{Section } 1.1\). If \([G,G]\) is not simply-connected one corrects the above definition as in \((\ref{eq:correct}), \text{Section } 7.4\).

Denote by \( y_{\infty} \text{Bun}_{\mathcal{F}^I_M} \) the version of \( \text{Bun}_{\mathcal{F}^I_M} \), where the maps \( \kappa^V_s \) are allowed to have any poles at \( y \). If \([G,G]\) is not simply-connected, one defines \( y_{\infty} \text{Bun}_{\mathcal{F}^I_M} \) analogously.
4.1.7. As in (14, Section 4.1.4), one defines the Hecke action of $\text{Perv}_{G(\mathfrak{q}_y)}(\text{Gr}_{G,y})$ on $\text{Shv}_{(y,\infty)\tilde{\text{Bun}}_U^M}$. For $S \in \text{Perv}_{G(\mathfrak{q}_y)}(\text{Gr}_{G,y})$, $\mathcal{F} \in \text{Shv}_{(y,\infty)\tilde{\text{Bun}}_U^M}$ we get the corresponding objects $\mathcal{F} \ast S$ and $S \ast \mathcal{F}$ in $\text{Shv}_{(y,\infty)\tilde{\text{Bun}}_U^M}$. It is easy to see that the above Hecke action preserves the full subcategory

$$\text{Shv}_{(y,\infty)\tilde{\text{Bun}}_U^M}^{\text{constr}}$$

of constructible objects (defined as in Section 1.1.3).

4.1.8. Let $j : \text{Bun}_U^M \hookrightarrow (y,\infty)\tilde{\text{Bun}}_U^M$ be the open embedding, $\tilde{q} : (y,\infty)\tilde{\text{Bun}}_U^M \to \text{Bun}_G$ be the projection. Since $\tilde{q}^* : \text{Shv}_{(y,\infty)\tilde{\text{Bun}}_U^M} \to \text{Shv}(\text{Bun}_G)$ commutes with the actions of Hecke functors at $y$, (18) identifies with (19)

$$\mathcal{R}\Gamma_c(\text{Bun}_G, K \otimes \tilde{q}^*((j_1ev_y^*L_{\psi}) \ast A^\lambda_G)) \to \mathcal{R}\Gamma_c((y,\infty)\tilde{\text{Bun}}_U^M, (\tilde{q}^*)^*K \otimes ((j_1ev_y^*L_{\psi}) \ast A^\lambda_G))$$

4.1.9. For $\theta \in \Lambda_{G,P}$ let

$$y,\theta \tilde{\text{Bun}}_U^M \subset (y,\infty)\tilde{\text{Bun}}_U^M$$

be the locally closed substack defined as follows. For $[G,G]$ simply-connected it is given by the property that for any $\tilde{\lambda} \in \tilde{\Lambda}^+_G$ the map $\kappa_{\tilde{\lambda}}$ gives rise to a regular map

$$\mathcal{L}_{\mathcal{F}_M}^\lambda(-\langle \theta, \tilde{\lambda} \rangle y) \to \mathcal{V}_{\mathcal{F}_G}^\lambda$$

over the whole of $X$, which moreover has no zeros in a neighbourhood of $y$. Recall that here $\mathcal{V}_{\mathcal{F}_G}^\lambda$ denotes the corresponding Weyl module for $G$. One adapts the above definition to the case of any $G$ reductive using the correction from (28, Section 7).

The stacks $y,\theta \tilde{\text{Bun}}_U^M$ for $\theta \in \Lambda_{G,P}$ form a stratification of $(y,\infty)\tilde{\text{Bun}}_U^M$. We claim that the contribution of each stratum $y,\theta \tilde{\text{Bun}}_U^M$ to the integral (19) vanishes.

For $\theta \in \Lambda_{G,P}$ define the open substack $y,\theta \text{Bun}_U^M \subset y,\theta \tilde{\text{Bun}}_U^M$ (in the case of $[G,G]$ simply-connected) by the property that for any $\tilde{\lambda} \in \tilde{\Lambda}^+_{G,P}$ the maps (20) have no zeros on the whole of $X$. We leave it to a reader to adapt this definition for the general $G$ reductive.

By construction, the $*$-restriction of $(j_1ev_y^*L_{\psi}) \ast A^\lambda_G$ to $y,\theta \tilde{\text{Bun}}_U^M$ is the extension by zero from $y,\theta \text{Bun}_U^M$. Write

$$i_\theta : y,\theta \text{Bun}_U^M \hookrightarrow (y,\infty)\tilde{\text{Bun}}_U^M$$

for the natural inclusion.
4.1.10. For \( \theta \in \Lambda_{G,P} \) denote by \( \text{Gr}^\theta_M \) be the connected component of \( \text{Gr}_M \) containing the element \( t^\lambda M(\mathbb{O}) \) for any \( \lambda \in \Lambda \) over \( \theta \). Let \( \text{Gr}^\theta_P \) be the preimage of \( \text{Gr}^\theta_M \) under the natural map \( \text{Gr}_P \to \text{Gr}_M \).

We have a natural map \( \text{Gr}^\theta_P \to \text{Gr}_G \). At the level of reduced finite-dimensional subschemes of \( \text{Gr}_G \), for any closed subscheme of finite type \( S \subset \text{Gr}_G \) this gives a stratification of \( S \) by \( S \cap \text{Gr}^\theta_P, \theta \in \Lambda_{G,P} \).

4.1.11. As in [6] write \( \text{Gr}^\theta_M \subset \text{Gr}_M \) for the positive part of the affine grassmanian for \( M \). It classifies an \( M \)-torsor \( \mathcal{F}_M \) on \( X \) with a trivialization \( \beta_M : \mathcal{F}_M \to \mathcal{F}_M^0 \mid_{X-y} \) such that for any finite-dimensional \( G \)-module \( V \) the map

\[
\beta_M : V_{\mathcal{F}_M}^U \to V_{\mathcal{F}_M^0}^U
\]

is regular over \( X \). For \( \theta \in \Lambda_{G,P} \) set

\[
\text{Gr}^\theta_{M,+} = \text{Gr}^\theta_M \cap \text{Gr}^\theta_M^+.
\]

Let \( \text{Gr}_M(\mathcal{F}_M^t) \) be the stack classifying \( \mathcal{F}_M \in \text{Bun}_M \) and an isomorphism \( \beta_M : \mathcal{F}_M \to \mathcal{F}_M^0 \mid_{X-y} \). A trivialization \( \epsilon_M : \mathcal{F}_M^t \to \mathcal{F}_M^0 \mid_{D_\eta} \) yields \( \text{Gr}_M(\mathcal{F}_M^t) \to \text{Gr}_M \). Let

\[
\text{Gr}^\theta_M(\mathcal{F}_M^t) \subset \text{Gr}_M(\mathcal{F}_M^t)
\]

be the substack corresponding to \( \text{Gr}^\theta_M \) under this identification. It is independent of a choice of \( \epsilon_M \). One defines \( \text{Gr}^\theta_{M,}\mathcal{F}_M^t, \text{Gr}^\theta_{M,+}(\mathcal{F}_M^t) \) similarly.

4.1.12. Let \( \text{Gr}^\theta_{M,-}(\mathcal{F}_M^t) \) be the stack classifying a \( M \)-torsor \( \mathcal{F}_M \) on \( X \), an isomorphism \( \beta_M : \mathcal{F}_M \to \mathcal{F}_M^0 \mid_{X-y} \) such that

\[
(\mathcal{F}_M, \beta_M) \in \text{Gr}^{-\theta,+}_M(\mathcal{F}_M^t)
\]

For \( \theta \in \Lambda_{G,P} \) the stack \( y_\theta \text{Bun}_U^{\mathcal{F}_M} \) classifies a \( P \)-torsor \( \mathcal{F}_P \) on \( X \) for which we set \( \mathcal{F}_M = \mathcal{F}_P \times_P M \), and an isomorphism \( \beta_M : \mathcal{F}_M \to \mathcal{F}_M^0 \mid_{X-y} \) such that \( (\mathcal{F}_M, \beta_M) \in \text{Gr}^{-\theta,-}_M(\mathcal{F}_M^t) \). Let

\[
q_\theta : y_\theta \text{Bun}_U^{\mathcal{F}_M} \to \text{Gr}^{-\theta,-}_M(\mathcal{F}_M^t)
\]

be the projection sending the above point to \( (\mathcal{F}_M, \beta_M) \).

4.1.13. Pick \( \theta \in \Lambda_{G,P} \). Pick a \( k \)-point \( \eta = (\mathcal{F}_M, \beta_M) \in \text{Gr}^{-\theta,-}_M(\mathcal{F}_M^t) \). The fibre of \( q_\theta \) over \( \eta \) identifies with \( \text{Bun}_U^{\mathcal{F}_M} \). Write

\[
l_\eta : \text{Bun}_U^{\mathcal{F}_M} \hookrightarrow y_\theta \text{Bun}_U^{\mathcal{F}_M}
\]

for the corresponding closed immersion.

Say that \( \eta \) has positive \( s' \)-conductor if the map

\[
(V/V')_{\mathcal{F}_M} \xrightarrow{\beta_M} (V/V')_{\mathcal{F}_M} \xrightarrow{s'} \Omega \mid_{X-y}
\]

initially defined over \( X - y \) extends to a regular map \( (V/V')_{\mathcal{F}_M} \xrightarrow{s} \Omega \) on \( X \). If \( \eta \) has positive \( s' \)-conductor then one gets a morphism \( ev_s : \text{Bun}_U^{\mathcal{F}_M} \to \mathbb{G}_a \) defined along the same lines as \( ev_{s'} \).
Lemma 4.1.14. If $\eta$ has positive $s'$-conductor then

$$i_{\eta}^*(\langle j_1 e v_s^* L_{\psi} \rangle \ast \mathcal{A}^1_G)$$

is isomorphic to $e v_s^* L_{\psi} \otimes M$ for some constant complex $M \in \text{Shv}(\text{Spec } k)$. Otherwise, it vanishes.

The proof of Lemma 4.1.14 is given in Section 4.2.12.

4.1.15. End of the proof of Theorem 2.1.11 for $x$ even. Recall that we picked $\theta \in \Lambda_{G,P}$ and a $k$-point $\eta = (\mathcal{F}_M, \beta_M) \in \text{Gr}^\theta_{M} (\mathcal{F}_M)$. It suffices to show that

$$\langle q_\theta \rangle ! \left( i_{\theta}(\bar{q})^* K \otimes i_{\theta}^*(\langle j_1 e v_s^* L_{\psi} \rangle \ast \mathcal{A}^1_G) \right)$$

vanishes. For this in turn it suffices to show that the $s'$-fibre of the latter complex at $\eta$ vanishes (our argument is compatible with field extensions). Let $q : \text{Bun}^{\mathcal{F}_M}_U \rightarrow \text{Bun}_G$ be the natural map. We must show that

$$\text{R} \Gamma_c (\text{Bun}^{\mathcal{F}_M}_U, q^* K \otimes i_{\theta}^*(\langle j_1 e v_s^* L_{\psi} \rangle \ast \mathcal{A}^1_G)) = 0$$

By Lemma 4.1.14 we may assume that $\eta$ has positive $s'$-conductor. Then our claim follows from Lemma 4.1.14 because

$$\text{R} \Gamma_c (\text{Bun}^{\mathcal{F}_M}_U, q^* K \otimes e v_s^* L_{\psi}) = 0$$

by definition of $\mathcal{F}_\theta$. Indeed, $s$ and $s'$ coincide at the generic point of $X$, so $(\mathcal{F}_M, s) \in \mathcal{V}_M^0$. Theorem 2.1.11 is proved for $x$ even. $\square$

4.2. Definition of the $W$-category.

4.2.1. As in (12, Section 4) or (13, Section 2), we define a full DG-subcategory $\text{Shv}^W_{(\mathcal{F}_M, s)} \subset \text{Shv}_{\mathcal{F}_M}$ attached to $s'$. Our strategy is first to define a full Serre abelian subcategory $\text{Shv}^W_{(\mathcal{F}_M, s)} \subset \text{Shv}_{\mathcal{F}_M}$ of the heart $\text{Shv}_{\mathcal{F}_M}$ of $\text{Shv}_{(\mathcal{F}_M, s)}$. Then we let $\text{Shv}^W_{(\mathcal{F}_M, s)} \subset \text{Shv}_{\mathcal{F}_M}$ be the full subcategory of those objects of $\text{Shv}_{\mathcal{F}_M}$ whose all perverse cohomology sheaves lie in $\text{Shv}^W_{(\mathcal{F}_M, s)}$.

We give the definition under the assumption that $[G, G]$ is simply-connected, the general case is done similarly adding the correction from (128, Section 7).

4.2.2. Let $\tilde{z} \in X$ be a closed point different from $y$. Let $(V/V')^\text{reg}_z$ (resp., $(V/V')^\text{mer}_z$) denote the group scheme (resp., group ind-scheme) of sections of $(V/V')_{\mathcal{F}_M}$ over $D_\tilde{z}$ (resp., over $D_\tilde{z}^*$). Here ‘reg’ stands for ‘regular’, and ‘mer’ stands for ‘meromorphic’.

More generally, let $\tilde{z} = \{z_1, \ldots, z_m\}$ be a finite collection of pairwise different closed points of $X - y$. Let $D_{\tilde{z}}$ be the formal neighbourhood of $\tilde{z} = U_{i=1}^m z_i$. Replacing $D_\tilde{z}$ by $D_{\tilde{z}} \ni \prod D_{z_i}$, one similarly defines $(V/V')^\text{reg}_{\tilde{z}}, (V/V')^\text{mer}_{\tilde{z}}$.

For $1 \leq i \leq m$ let $\chi_{z_i} : (V/V')^\text{mer}_{z_i} \rightarrow \mathbb{G}_a$ be the character given as the composition

$$(V/V')^\text{mer}_{z_i} \xrightarrow{\iota} \Omega(F_{z_i}) \xrightarrow{\text{Res}} \mathbb{G}_a$$

Here $F_{z_i}$ is the completion of the field of rational functions on $X$ at $z_i$. Define the character

$$\chi_{\tilde{z}} : (V/V')^\text{mer}_{\tilde{z}} \rightarrow \mathbb{G}_a$$
as \( \sum_{i=1}^{m} \chi_{zi} \). Note that

\[
(V/V')^{\text{mer}}_z / (V/V')^{\text{reg}}_z
\]
can be seen as the affine grassmanian \( \text{Gr}(V/V')_{\mathcal{F}_M,z} \) for the group scheme \((V/V')_{\mathcal{F}_M,z}\) at \(z\). The latter classifies \((V/V')_{\mathcal{F}_M,z}\)-torsors on \(D_z\) together with a trivialization over \(D_z^\circ\).

Since \( \chi_z \) is trivial on \((V/V')_{\mathcal{F}_M,z}\), \( \chi_z \) writes as the composition

\[
(V/V')^{\text{mer}}_z \to \text{Gr}(V/V')_{\mathcal{F}_M,z} \to \mathbb{G}_a
\]

4.2.3. Let \( \sim_{Y,M}^\infty \text{Bun}_{U,z} \), \( \beta \) be the open substack given by the property that all the maps \( \gamma \) with an isomorphism \( \beta \) acting trivially on \( F \sum \) can be seen as the affine grassmanian \( \text{Gr}(V/V')_{\mathcal{F}_M,z} \) for the group scheme \((V/V')_{\mathcal{F}_M,z}\) at \(z\). The latter classifies \((V/V')_{\mathcal{F}_M,z}\)-torsors on \(D_z\) together with a trivialization over \(D_z^\circ\).

4.2.4. Let \( \chi_z \) be a point of \( \sim_{Y,M}^\infty \text{Bun}_{U,z} \), \( \gamma \) we get a \( P \)-torsor \( \mathcal{F}'_P \) on \(D_z\) with an isomorphism \( \gamma' : \mathcal{F}'_P \times_P M \to \mathcal{F}'_M \) over \(D_z^\circ\).

4.2.5. For a point of \( \sim_{Y,M}^\infty \text{Bun}_{U,z} \), \( \beta \) and \( \beta_z \) give rise to an isomorphism \( \beta_{\gamma,\chi_n} : \mathcal{F}_P \to \mathcal{F}'_P \mid_{D_z^\circ} \) making the diagram commutative

\[
(V^U)_\mathcal{F}_M \xrightarrow{\chi_n} \mathcal{V}_{\mathcal{F}_G}(\infty y) \quad \quad \quad \mathcal{V}_{\mathcal{F}_G}(\infty y) \xrightarrow{\chi_n} (V^U)_\mathcal{F}_M
\]

for any finite-dimensional \( G \)-module \( V \).

For a point of \( \mathcal{G}_z \) we get two \( P \)-torsors \( \mathcal{F}_P \) and \( \mathcal{F}'_P \) on \(D_z\), and \( \beta_{\gamma,\chi_n} \) gives rise to an isomorphism \( \beta_{\gamma,\chi_n} : \mathcal{F}_P \to \mathcal{F}'_P \mid_{D_z^\circ} \) making the diagram commute

\[
\mathcal{F}_P \times_P M \xrightarrow{\gamma} \mathcal{F}'_M \quad \downarrow \beta_{\gamma,\chi_n} \\
\mathcal{F}'_P \times_P M
\]

Here \( \beta_{\gamma,\chi_n} \) is the extension of scalars of \( \beta_{\gamma,\chi_n} \) under \( P \to M \). So, \( \beta_{\gamma,\chi_n} \) extends to a regular map over \(D_z\).

4.2.5. For a point of \( \mathcal{G}_z \) as above let \( N \) be the sheaf of isomorphisms \( \mathcal{F}_P \to \mathcal{F}'_P \) of \( P \)-torsors over \(D_z\) compatible with \( \beta_{\gamma,\chi_n} \). The sheaf of automorphisms of \( \mathcal{F}_P \) on \(D_z^\circ\) acting trivially on \( \mathcal{F}_P \times_P M \) is \( V_{\mathcal{F}_P} \) over \(D_z\). This sheaf acts on \( N \) via its action on \( \mathcal{F}_P \). This way \( N \) becomes a torsor under \( V_{\mathcal{F}_P} \) over \(D_z^\circ\) together with a trivialization \( \beta_N \) over \(D_z^\circ\). Namely, \( \beta_{\gamma,\chi_n} \) gives the corresponding trivialization. The extension of scalars of \( \beta_N \) via \( V_{\mathcal{F}_P} \to (V/V')_{\mathcal{F}_M,z} \) gives a morphism

\[
\mathcal{G}_z \to \text{Gr}(V/V')_{\mathcal{F}_M,z}
\]

Define \( \chi_N : \mathcal{G}_z \to \mathbb{G}_a \) as the composition

\[
\mathcal{G}_z \to \text{Gr}(V/V')_{\mathcal{F}_M,z} \xrightarrow{\chi_N} \mathbb{G}_a
\]
4.2.6. We have a diagram of projections
\[
\begin{array}{ccc}
\mathcal{G}_\bar{z} & \xrightarrow{h_{\bar{z}}} & \mathcal{G}_\bar{z} \\
\xrightarrow{h_{\bar{z}}} & & \xrightarrow{h_{\bar{z}}} \\
\mathcal{G}_\bar{z} & \xrightarrow{h_{\bar{z}}} & \mathcal{G}_\bar{z} \\
\end{array}
\]
Here \( h_{\bar{z}} \) (resp., \( h_{\bar{z}} \)) sends the above point to \((\mathcal{F}_G, \kappa')\) (resp., to \((\mathcal{F}_G, \kappa')\)). This way \( \mathcal{G}_\bar{z} \) gets a structure of a groupoid over \( \mathcal{G}_\bar{z} \).

4.2.7. We first define \( \mathcal{Shv}_{W, \text{constr}, \heartsuit} \) good at \( \bar{z} \subset (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit} \) as the full subcategory of those objects which are \((G, \chi^*_\mathcal{L}_{\psi})\)-equivariant. The precise sense of this is as in [12], Section 4.7.

Let now \( \bar{z}' \) be another such collection of points and \( \bar{z}'' = \bar{z} \cup \bar{z}' \). Then we may consider the versions of the above objects for \( \bar{z}' \) and \( \bar{z}'' \). As in [13], Section 2.5), if \( \bar{z} \) is not empty then for an object \( K \in (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}'')^{\text{constr}})^{\heartsuit} \) the \((G, \chi^*_\mathcal{L}_{\psi})\)-equivariance and \((G, \chi^*_\mathcal{L}_{\psi})\)-equivariance are equivalent.

**Definition 4.2.8.** Let
\[
\mathcal{Shv}_{W, \text{constr}, \heartsuit} \subset (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit}
\]
be the full subcategory of \( \mathcal{K} \) such that for any nonempty finite collection \( \bar{z} \) as above, the restriction of \( \mathcal{K} \) to \((\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit} \) lies in \( \mathcal{Shv}_{W, \text{constr}, \heartsuit} \).

One checks that \( \mathcal{Shv}_{W, \text{constr}, \heartsuit} \) is an abelian Serre subcategory of \( (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit} \).

**Definition 4.2.9.** Let
\[
\mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \subset \mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\heartsuit}
\]
be the full subcategory of \( \mathcal{K} \) such that for any nonempty finite collection \( \bar{z} \) as above, the restriction of \( \mathcal{K} \) to \( (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit} \) lies in \( \mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \).

One checks that \( \mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \) is a Serre abelian subcategory of \( (\mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\text{constr}})^{\heartsuit} \) closed under small colimits.

**Definition 4.2.10.** Let
\[
\mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \subset \mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z})^{\heartsuit}
\]
be the full subcategory of those objects \( \mathcal{K} \) whose all perverse cohomology sheaves lie in \( \mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \).

We conclude that \( \mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \in \text{DGCat}_{\text{cont}} \), the inclusion \( (\text{22}) \) is continuous, and \( \mathcal{Shv}_{W}^{\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}} \) is compatible with the perverse t-structure on \( \mathcal{Shv}(\mathcal{G}, \mathcal{F}_M U, \text{good at } \bar{z}) \), so it inherits a t-structure (compatible with filtered colimits).
Let also
\[ Shv^W(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}_{\text{constr}}) = Shv(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}_{\text{constr}}) \cap Shv^W(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}) \]
Then \( Shv^W(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}_{\text{constr}}) \in \text{DGCat}^{\text{non-cocmpl}} \), and this DG-category inherits a t-structure from \( Shv(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}_{\text{constr}}) \).

Since in our definitions \( z \) are chosen distinct from \( y \), one gets the following.

**Proposition 4.2.11.** The Hecke action of \( \text{Perv}_{G(\mathbb{O})}(\text{Gr}_{G,y}) \) on \( Shv(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}) \) preserves the full subcategories \( Shv^W(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}) \) and \( Shv^W(y,\infty \widetilde{\text{Bun}}_U^{\mathcal{T}_M}_{\text{constr}}) \).

**4.2.12. Proof of Lemma 4.1.14.** For \( \theta \in \Lambda_{G,P} \) one similarly defines the versions
\[ Shv^W(y,\theta \text{Bun}_U^{\mathcal{T}_M}), \quad Shv^W(y,\theta \text{Bun}_U^{\mathcal{T}_M}_{\text{constr}}) \]
of \( W \)-categories. As soon as a closed point \( \eta = (\mathcal{T}_M, \beta_M) \in \text{Gr}_{G,M}(\mathcal{T}_M) \) is picked, the closed immersion (21) is stable under the action of the corresponding groupoids. So, we similarly get the categories \( Shv^W(\text{Bun}_U^{\mathcal{T}_M}), Shv^W(\text{Bun}_U^{\mathcal{T}_M}_{\text{constr}}) \). By functoriality, \( \iota^*\iota^* \) sends \( Shv^W(y,\text{Bun}_U^{\mathcal{T}_M}) \) to \( Shv^W(\text{Bun}_U^{\mathcal{T}_M}) \) and preserves constructibility. Our claim is reduced to Lemma 4.2.13 below.

**Lemma 4.2.13.** If \( \eta \) has positive \( s' \)-conductor then any object of \( Shv^W(\text{Bun}_U^{\mathcal{T}_M}_{\text{constr}}) \) is isomorphic to \( ev^*_{s}L_{s} \otimes M \) for some constant complex \( M \in Shv(\text{Spec} k) \). Otherwise, \( Shv^W(\text{Bun}_U^{\mathcal{T}_M}_{\text{constr}}) \) vanishes.

*Proof.* This is analogous to (11), Lemma 6.2.8.

**Theorem 2.1.11** is proved for \( x \) even.

---

### 5. Case of \( x \) odd

5.1. In Section 5 we assume \( x \) odd and prove Theorem 2.1.11 in this case. Keep notations of Section 2

5.1.1. Recall that we picked a closed point \( y \in X \), and our Hecke functors are applied at \( y \in X \). Let \( K \in \mathcal{F}_\Omega, \lambda \in \Lambda^+ \). We want to show that \( A_G^\lambda \ast K \in \mathcal{F}_\Omega \).

5.1.2. Pick a \( k \)-point of \( y^0_P \) given by \( \mathcal{F}^1 \in \text{Bun}_{P/V'} \) and \( s' : (V/V')_{\mathcal{T}_M} \rightarrow \Omega \), where \( \mathcal{T}_M^s = \mathcal{T}_M^1 \times_{P/V'} M \). We will show that the \( s' \)-fibre of Four\(_P(p_!q^*(A_G^\lambda \ast K)) \) at this point vanishes (our argument will be compatible with field extensions as in the previous sections).

Let \( \mathcal{X}_{\mathcal{T}_M} \) be the stack classifying exact sequences on \( X \)
\[
0 \rightarrow (V/V')_{\mathcal{T}_M} \rightarrow ? \rightarrow \mathcal{O}_X \rightarrow 0
\]
(23)
Let \( ev_{s'} : \mathcal{X}_{\mathcal{T}_M} \rightarrow \mathbb{A}^1 \) be the map sending (23) to its pairing with \( s' \).

Denote by \( \text{Bun}_U^{\mathcal{T}_M} \) the stack classifying a \( P \)-torsor \( \mathcal{F}_P \) on \( X \) and an isomorphism of \( M \)-torsors \( s' : \mathcal{F}_P \times_P M \cong \mathcal{F}_M^s \) on \( X \).
Recall the map act : \( X_P \to \text{Bun}_{P/V'} \) defined in Section 2.1.7. Let \( X_1 \) be the stack classifying an exact sequence (23), for which we get the the \( P/V' \)-torsor \( \mathcal{F}'_{P/V'} = \text{act}(\mathcal{F}^1, (23)) \) on \( X \), and a lifting of \( \mathcal{F}'_{P/V'} \) to a \( P \)-torsor \( \mathcal{F}'_P \). Note that \( \mathcal{F}'_P \) is equipped with \( \gamma' : \mathcal{F}'_P \times_P M \to \mathcal{F}'_M \).

Write \( \text{act}_1 : X_1 \to \text{Bun}_{\mathcal{F}^1} \) for the map sending the above point to \( (\mathcal{F}'_P, \gamma') \). Let \( \text{ev}_1 : X_1 \to G_a \) be the map sending the above point to the pairing of (23) with \( s' \).

In fact, \( X_1 \) classifies a \( P \)-torsor \( \mathcal{F}'_P \) with an isomorphism \( \mathcal{F}'_P \times_P (P/V) \to \mathcal{F}^1 \times_{P/V'} (P/V) \) of \( P/V \)-torsors.

Write \( \text{Bun}_{\mathcal{F}^1} \) for the stack classifying a \( P/V \)-torsor \( \mathcal{F}^1 \) together with an isomorphism \( \mathcal{F}^1 \times_{P/V} (P/V) \to \mathcal{F}^1 \times_{P/V'} (P/V) \). So, \( \text{Bun}_{\mathcal{F}^1} \) classifies exact sequences

\[
0 \to (U/V) \mathcal{F}^1_{M} \to ? \to \mathcal{O}_X \to 0
\]
on \( X \). We get a cartesian square

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\text{act}_1} & \text{Bun}_{\mathcal{F}^1} \\
\downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{\mathcal{F}^1_{P/V}} & \text{Bun}_{\mathcal{F}^1_{U/V}}
\end{array}
\]

where \( \mathcal{F}^1_{P/V} = \mathcal{F}^1 \times_{P/V'} (P/V) \).

5.1.3. Set

\[
\mathcal{E} = (\text{act}_1)^* \mathcal{L}_\psi
\]

Let \( q' : \text{Bun}_{\mathcal{F}^1} \to \text{Bun}_G \) be the natural map. For \( p > 0 \) it suffices to show that

\[
R \Gamma_c(\text{Bun}_{\mathcal{F}^1}, \mathcal{E} \otimes (q')^*(A_G^\lambda \ast K)) = 0
\]

5.1.4. The complex (24) identifies canonically with

\[
R \Gamma_c(\text{Bun}_G, (A_G^\lambda \ast K) \otimes q'_!(\mathcal{E}))
\]

Now as in Section 3.1.8 one identifies (24) canonically with

\[
R \Gamma_c(\text{Bun}_G, K \otimes ((q'_!(\mathcal{E}) \ast A_G^\lambda)))
\]

5.1.5. Let the stacks \( \text{Bun}_{\mathcal{F}^1} \), \( y, \infty \text{Bun}_{\mathcal{F}^1} \) be as in Section 4.1.6. As in Section 4.1.8 we have the maps

\[
j : \text{Bun}_{\mathcal{F}^1} \to y, \infty \text{Bun}_{\mathcal{F}^1}, \quad q' : y, \infty \text{Bun}_{\mathcal{F}^1} \to \text{Bun}_G,
\]

and (25) identifies canonically with

\[
R \Gamma_c(\text{Bun}_G, K \otimes q'_!(j_!(\mathcal{E}) \ast A_G^\lambda)) \to R \Gamma_c(y, \infty \text{Bun}_{\mathcal{F}^1}, (q')^*K \otimes (j_!(\mathcal{E}) \ast A_G^\lambda))
\]

(26)
5.1.6. Recall the stratification of $\widetilde{\text{Shv}}_{G}$ by locally closed substacks $y,\theta \widetilde{\text{Bun}}_{U}$ for $\theta \in \Lambda_{G,P}$ from Section 4.1.9. We claim that the contribution of each stratum $y,\theta \widetilde{\text{Bun}}_{U}$ to (26) vanishes.

By construction, the $*$-restriction of $(j_{!}\mathcal{E})*A_{G}^{\lambda}$ to $y,\theta \widetilde{\text{Bun}}_{U}$ is the extension by zero from $y,\theta \widetilde{\text{Bun}}_{U}$. So, we must show that for each $\theta \in \Lambda_{G,P}$,

$$\text{R} \Gamma_{c}(y,\theta \widetilde{\text{Bun}}_{U},i_{\theta}^{*}(q)^{*}K \otimes i_{\theta}^{*}((j_{!}\mathcal{E})*A_{G}^{\lambda})) = 0$$

Recall the map $q_{\theta} : y,\theta \widetilde{\text{Bun}}_{U} \to \text{Gr}^{\theta,-}(\mathcal{F}_{M})$ from Section 4.1.1.12. It suffices to show that

$$(q_{\theta})_{!}i_{\theta}^{*}((q)^{*}K \otimes ((j_{!}\mathcal{E})*A_{G}^{\lambda})) = 0$$

This will be done in Section 5.3 after introducing a suitable version of the $W$-category.

5.2. Definition of the $W$-category.

5.2.1. Define the full DG-subcategory $\text{Shv}^{W}(y,\infty \widetilde{\text{Bun}}_{U}) \subset \text{Shv}(y,\infty \widetilde{\text{Bun}}_{U})$ attached to $s'$ as in Section 4.2 with some changes. Here are the details.

As in Section 4.2.2 for a collection of pairwise distinct closed points $z = \{z_{1},\ldots,z_{m}\} \subset X - y$ we have the same objects

$$(V/V')_{z}^{\text{reg}}, \quad (V/V')_{z}^{\text{mer}}, \quad \chi_{z} : (V/V')_{z}^{\text{mer}} \to \mathbb{G}_{a}$$

and

$$\tilde{\chi}_{z} : \text{Gr}(V/V')_{z}^{\text{mer}} \to \mathbb{G}_{a}$$

The open substack

$$y,\infty \widetilde{\text{Bun}}_{U}; \text{good at } z \subset y,\infty \widetilde{\text{Bun}}_{U}$$

is defined as in Section 4.2.2.

5.2.2. The definition of the groupoid $\mathcal{G}_{z}$ is changed compared to Section 4.2.3 as follows.

Let $\mathcal{G}_{z}$ be the ind-stack classifying two points of $y,\infty \widetilde{\text{Bun}}_{U}; \text{good at } z$, namely $(\mathcal{F}_{G},\kappa^{V})$ and $(\mathcal{F}_{G}',\kappa'^{V})$, and an isomorphism $\beta_{z} : \mathcal{F}_{G} \to \mathcal{F}_{G}'|_{X - \cup_{i}z_{i}}$ subject to the following conditions.

First, for any finite-dimensional $G$-module $\mathcal{V}$ the diagram commutes

$$(\mathcal{V})|_{\mathcal{F}_{G}} \xrightarrow{\kappa^{V}} \mathcal{V}_{\mathcal{F}_{G}}(\infty y) \xrightarrow{\kappa'^{V}} \mathcal{V}_{\mathcal{F}_{G}}(\infty y)$$

So, for a point of $\mathcal{G}_{z}$ we get two $P$-torsors $\mathcal{F}_{P}$ and $\mathcal{F}_{P}'$ on $D_{z}$, and $\beta_{z}$ gives rise to an isomorphism $\beta_{P,z} : \mathcal{F}_{P} \to \mathcal{F}_{P}'|_{D_{z}}$ making the diagram commute

$$\mathcal{F}_{P} \times_{P} M \xrightarrow{\gamma} \mathcal{F}_{M}' \xrightarrow{\gamma'} \mathcal{F}_{P}' \times_{P} M$$
Here $\tilde{\beta}_{P,\bar{z}}$ is the extension of scalars of $\beta_{P,\bar{z}}$ under $P \to M$. So, $\tilde{\beta}_{P,\bar{z}}$ extends to a regular map over $D_{\bar{z}}$.

Second, we require that the extension of scalars
\[ F_p \times_p P/V \to F_p \times_p P/V \]
of $\beta_{P,\bar{z}}$ initially defined on $D_{\bar{z}}$ extends to an isomorphism of $P/V$-torsors over $D_{\bar{z}}$ denoted
\[ \beta_{P/V,\bar{z}}: F_p \times_p P/V \to F_p \times_p P/V|_{D_{\bar{z}}} \]

5.2.3. For a point of $G_{\bar{z}}$ let $\mathcal{N}$ be the sheaf of isomorphisms $\mathcal{F}_p \to \mathcal{F}_p'$ of $P$-torsors over $D_{\bar{z}}$ compatible with $\beta_{P/V,\bar{z}}$. The sheaf of automorphisms of $\mathcal{F}_p$ on $D_{\bar{z}}$ acting trivially on $\mathcal{F}_p \times_p P/V$ is $\mathcal{V}_{\mathcal{F}_p}$ over $D_{\bar{z}}$. This sheaf acts on $\mathcal{N}$ via its action on $\mathcal{F}_p$. This way $\mathcal{N}$ becomes a torsor under $\mathcal{V}_{\mathcal{F}_p}$ over $D_{\bar{z}}$ together with a trivialization $\beta_{\mathcal{N}}$ over $D_{\bar{z}}$. Namely, $\beta_{P,\bar{z}}$ gives the corresponding trivialization. The extension of scalars of $(\mathcal{N}, \beta_{\mathcal{N}})$ via $V \to V/V'$ gives a morphism
\[ G_{\bar{z}} \to \text{Gr}(V/V')_{\mathcal{F}'_p,\bar{z}} \]
Define $\chi_G: G_{\bar{z}} \to G_a$ as the composition
\[ G_{\bar{z}} \to \text{Gr}(V/V')_{\mathcal{F}'_p,\bar{z}} \xrightarrow{\chi_G} G_a \]

5.2.4. As in Section 4.2.6, we have a diagram of projections
\[ \text{Bun}_{U,\text{good at } \bar{z}} \xrightarrow{h_{\mathcal{G}}} \mathcal{G}_{\bar{z}} \xrightarrow{h_{G}} \text{Bun}_{U,\text{good at } \bar{z}} \]
Here $h_{\mathcal{G}}$ (resp., $h_G$) sends the above point to $(\mathcal{F}'_p, K^\nu)$ (resp., to $(\mathcal{F}_p, K^\nu)$). This way $G_{\bar{z}}$ gets a structure of a groupoid over $\text{Bun}_{U,\text{good at } \bar{z}}$.

5.2.5. The rest of the definition of $W$-categories goes through without changes. One first defines
\[ \text{Shv}^W_{\text{good at } \bar{z}} \subset (\text{Shv}(\text{Bun}_{U,\text{good at } \bar{z}})_{\text{constr}})^\mathcal{G} \]
as in Section 4.2.7 Then Definitions 4.2.8 4.2.9 4.2.10 apply in our situation of $x$ odd giving rise to the categories
\[ \text{Shv}^W(\text{Bun}_{U}^{\mathcal{F}'_p}), \, \text{Shv}^W(\text{Bun}_{U}^{\mathcal{F}'_p})_{\text{constr}} \]
with the same formal properties.

The analog of Proposition 4.2.11 holds for $x$ odd with the same proof.

5.2.6. Let $\eta = (\mathcal{F}_M, \beta_M) \in \text{Gr}_M(\mathcal{F}'_M)$, here $\mathcal{F}_M$ denotes a $M$-torsor on $X$, and $\beta_M: \mathcal{F}_M \to \mathcal{F}'_M |_{X-x}$ is an isomorphism.

The notion of positive $s'$-conductor for $\eta$ is defined as in Section 4.1.13 If $\eta$ has a positive $s'$-conductor then one gets a morphism $s: (V/V')_{\mathcal{F}_M} \to \Omega$ on $X$.

Recall that $\text{Bun}_{U}^{\mathcal{F}_M}$ classifies a $P$-torsor $\mathcal{F}_P$ together with an isomorphism $\gamma: \mathcal{F}_P \to \mathcal{F}_M$.

We get a natural map $i_{\mathcal{F}_M}: \text{Bun}_{U}^{\mathcal{F}_M} \to \text{Bun}_{U}^{\mathcal{F}'_M}$, which is equivariant under the
above groupoids. So, along the same lines one defines the versions $\text{Shv}^W(Bun^\mathcal{F}_M)$, $\text{Shv}^W(Bun^\mathcal{F}_M)^{\text{constr}}$. Set

$$\text{Shv}^W(Bun^\mathcal{F}_M)^{\text{constr}} = \text{Shv}^W(Bun^\mathcal{F}_M) \cap \text{Shv}^W(Bun^\mathcal{F}_M)^{\text{constr}}$$

The functor $i^*_\mathcal{F}_M$ restricts to

$$i^*_\mathcal{F}_M : \text{Shv}^W(y, \infty \rightsquigarrow \text{Bun}^\mathcal{F}_M) \to \text{Shv}^W(Bun^\mathcal{F}_M)$$

and preserves the constructibility.

5.2.7. Let $\text{Bun}^\mathcal{F}_M$ be the stack classifying a $P/V$-torsor $\mathcal{F}$ on $X$ together with a trivialization $\mathcal{F}_{P/V} \times_{P/V} M \to \mathcal{F}$ on $X$.

Let $X_{\mathcal{F}_M}$ be the stack classifying exact sequences

$$(28) \quad 0 \to (V/V')_{\mathcal{F}_M} \to ? \to \mathcal{O}_X \to 0$$

on $X$. If $\eta$ has a positive $s'$-conductor then one gets a morphism $ev_s : X_{\mathcal{F}_M} \to G_a$ given by the pairing of (28) with $s$.

5.2.8. The natural map $\nu : \text{Bun}^\mathcal{F}_M \to \text{Bun}^\mathcal{F}_{M/V}$ is surjective on the level of $k$-points.

Let $W^\mathcal{F}_M$ be the stack classifying a point $(\mathcal{F}, \gamma) \in \text{Bun}^\mathcal{F}_M$, and a torsor $\bar{\mathcal{F}}$ under $V_{\mathcal{F}_P}$ on $X$. We have a cartesian square

$$\begin{array}{ccc}
W^\mathcal{F}_M & \xrightarrow{\text{act}_{\mathcal{F}_M}} & \text{Bun}^\mathcal{F}_M \\
\downarrow \text{pr}_1 & & \downarrow \nu \\
\text{Bun}^\mathcal{F}_{M/U} & \xrightarrow{\nu} & \text{Bun}^\mathcal{F}_{M/U/V}
\end{array}$$

Here the action map $\text{act}_{\mathcal{F}_M}$ is defined similarly to the map $\text{act} : \mathcal{X}_P \to \text{Bun}_P$ from Section 2.1.7 and $\text{pr}_1$ sends the collection $(\mathcal{F}, \gamma, \bar{\mathcal{F}})$ to $(\mathcal{F}, \gamma)$.

Let $\tau_W : W^\mathcal{F}_M \to X_{\mathcal{F}_M}$ be the map sending a collection $(\mathcal{F}, \gamma, \bar{\mathcal{F}})$ as above to the $(V/V')_{\mathcal{F}_M}$-torsor $\bar{\mathcal{F}} \times_{V_{\mathcal{F}_P}} (V/V')_{\mathcal{F}_M}$, which we view as the exact sequence (28).

**Lemma 5.2.9.** If $\eta$ has a positive $s'$-conductor then for any $\mathcal{K} \in \text{Shv}^W(\text{Bun}^\mathcal{F}_M)^{\text{constr}}$ we have an isomorphism

$$\text{act}_{\mathcal{F}_M}^{**} \mathcal{K} \simeq \text{pr}_1^{**} \mathcal{K} \otimes r_W^{**} ev_s^* L_\psi$$

Otherwise, $\text{Shv}^W(\text{Bun}^\mathcal{F}_M)^{\text{constr}}$ vanishes.

**Proof.** This is analogous to ([11], Lemma 6.2.8). \hfill $\square$

5.3. **End of the proof of Theorem 2.1.11** for $x$ odd.

5.3.1. Recall that we fixed $\theta \in \Lambda_{G,P}$. Pick a $k$-point $\eta = (\mathcal{F}_M, \beta_M) \in \text{Gr}^\theta_{\mathcal{F}_M}(\mathcal{F}_M')$. The fibre of $\eta_\theta$ over $\eta$ identifies with $\text{Bun}_U^\mathcal{F}_M$. As in Section 4.1.13 we get the closed immersion (21). It suffices to show that the fibre of (27) at $\eta$ vanishes.
5.3.2. Note that $E \in Shv^{W}(\text{Bun}_{U}^{F})$. Set

$$\mathcal{K} = j^{*} \mathfrak{g}(\mathfrak{j} \mathcal{E}) \ast A_{G}^{\mathfrak{g}}.$$

Let $q : \text{Bun}_{U}^{F} \rightarrow \text{Bun}_{G}$ be the natural map. It remains to show that

$$R\Gamma_{c}(\text{Bun}_{U}^{F}, q^{*} K \otimes \mathcal{K}) = 0$$

By functoriality, $\mathcal{K} \in Shv^{W}(\text{Bun}_{U}^{F})$. By Lemma 5.2.9, we may and do assume that $\eta$ has a positive $s'$-conductor. It suffices to show that

$$\nu_{!}(q^{*} K \otimes \mathcal{K}) = 0,$$

To do so, we will show that $\nu^{*} \nu_{!}(q^{*} K \otimes \mathcal{K}) = 0$.

5.3.3. Pick a $k$-point $(\mathcal{F}_{P}, \gamma)$ of $\text{Bun}_{U}^{F}$. We check that the $*$-fibre of $\nu^{*} \eta(q^{*} K \otimes \mathcal{K})$ at $(\mathcal{F}_{P}, \gamma)$ vanishes. Let $\mathcal{F}_{P/V'} = \mathcal{F}_{P} \times_{P} P/V'$. Since at the generic point of $X$ we have $s = s'$, we get $(\mathcal{F}_{P/V'}, s) \in \mathcal{Y}_{P}$. Our claim follows from Lemma 5.2.9 and the fact that $K \in \mathcal{F}_{0}$. Namely, the fibre at $(\mathcal{F}_{P}, \gamma)$ of

$$(\text{pr}_{1})_{!}(r_{w}^{*} \text{ev}_{s}^{*} \mathcal{L}_{\psi} \otimes \text{act}_{w}^{*} q^{*} K)$$

vanishes by definition of $\mathcal{F}_{0}$. $\square$

REFERENCES

[1] Pr. N. Achar, An order-reversing duality map for conjugacy classes in Lusztig’s canonical quotient, Transformation Groups, vol. 8, 107 - 145 (2003)

[2] D. Arinkin, D. Gaitsgory, Singular support of coherent sheaves, and the geometric Langlands conjecture, Selecta Mathematica, vol. 21, 1 - 199 (2015)

[3] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, The stack of local systems with restricted variation and geometric Langlands theory with nilpotent singular support, arxiv version 2

[4] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, Y. Varshavsky, Duality for automorphic sheaves with nilpotent singular support, arxiv version 2

[5] A. Braverman, M. Finkelberg, D. Gaitsgory, I. Mirkovic, Intersection cohomology of Drinfeld’s compactifications, Sel. Math. New ser. vol. 8, 381 - 418 (2002), erratum in: Sel. Math. New ser. 10 (2004), 429 - 430

[6] A. Braverman, D. Gaitsgory, Geometric Eisenstein series, Invent. Math. 150, No. 2, 287-384 (2002)

[7] D. Barbasch, D. Vogan. Unipotent representations of complex semisimple groups. Ann. of Math. (2) 121.1 (1985), 41 - 110

[8] D. H. Collingwood, W. M. McGovern, Nilpotent Orbits In Semisimple Lie Algebras, 1993 by Chapman and Hall/CRC

[9] J. Faergeman, Quasi-Tempered Automorphic D-modules, arXiv:2210.09193

[10] J. Faergeman, S. Raskin, Non-vanishing of geometric Whittaker coefficients for reductive groups, arXiv:2207.02955

[11] E. Frenkel, D. Gaitsgory, K. Vilonen, Whittaker Patterns in the Geometry of Moduli Spaces of Bundles on Curves, Ann. of Math., Vol. 153, Issue 3, 699 - 748 (2001)

[12] D. Gaitsgory, On a vanishing conjecture appearing in the geometric Langlands correspondence, Vol. 160 (2004), Issue 2, 617 - 682

[13] D. Gaitsgory, Twisted Whittaker model and factorizable sheaves, Sel. math., New ser. 13, 617 (2008)

[14] D. Gaitsgory, S. Lysenko, Metaplectic Whittaker category and quantum groups: the "small" FLE, version April 21, 2020
[15] D. Gaitsgory, N. Rozenblyum, A study in derived algebraic geometry, corrected version available at https://people.math.harvard.edu/~gaitsgde/GL/
[16] D. Ginzburg, Certain conjectures relating unipotent orbits to automorphic representations, Israel J. of Math., vol. 151, 323 - 355 (2006)
[17] D. Ginzburg, S. Rallis, D. Soudry, On Fourier coefficients of automorphic forms of symplectic groups, Manuscripta Mathematica, vol. 111, 1-16 (2003)
[18] R. Howe, On a notion of rank for unitary representations of the classical groups, C.I.M.E. Summer School on Harmonic Analysis and Group Representations, Cortona, 1980, 223 - 331
[19] D. Jiang, Automorphic integral transforms for classical groups I: endoscopy correspondences, arXiv:1212.6525
[20] D. Jiang, B. Liu, Fourier coefficients for automorphic forms on quasisplit classical groups, arXiv:1412.7453
[21] D. Jiang, B. Liu, Arthur Parameters and Fourier coefficients for Automorphic Forms on Symplectic Groups, Annales de l’Institut Fourier, 66 (2016), no. 2, 477 - 519.
[22] D. Jiang, B. Liu, G. Savin, Raising nilpotent orbits in wave-front sets, Repr. Theory, vol. 20, 419 - 450 (2016)
[23] G. Lusztig, Characters of reductive groups over a finite field, Princeton University Press, Princeton, NJ, 1984
[24] George J. McNinch, Nilpotent orbits over ground fields of good characteristic, Math. Annalen vol. 329, 49 - 85 (2004)
[25] M. E. Novodvorsky, Automorphic L-funcitons for symplectic group GSp(4), in: Proceedings of Simposia in Pure Math., vol. 33 (1979), part 2, 87 - 95
[26] A. T. Rakotoarisoa, The Bala-Carter Classification of Nilpotent Orbits of Semisimple Lie Algebras, Ph.D. thesis, University of Ottawa (2017)
[27] N. Rozenblyum, Filtered colimits of \(\infty\)-categories, available at http://www.math.harvard.edu/~gaitsgde/GL/colimits.pdf
[28] S. Schieder, The Harder-Narasimhan stratification of the moduli stack of \(G\)-bundles via Drinfeld’s compactifications, Selecta Mathematica, vol. 21, 763 - 831 (2015)
[29] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Math. 946, Springer-Verlag (1982)

Institut Elie Cartan Lorraine, Université de Lorraine, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France

Email address: Sergey.Lysenko@univ-lorraine.fr