THE INTERRELATION OF THE SPECIAL DOUBLE CONFLUENT HEUN EQUATION AND THE EQUATION OF RSJ MODEL OF JOSEPHSON JUNCTION REVISITED

SERGEY I. TERTYCHNIY

Abstract. The explicit formulas for the maps interconnecting the sets of solutions of the special double confluent Heun equation and the equation of the RSJ model of overdamped Josephson junction in case of shifted sinusoidal bias are given. The approach these are based upon leans on the extensive application of eigenfunctions of certain linear operator acting on functions holomorphic on the universal cover of the punctured complex plane. The functional equation the eigenfunctions noted obey is derived, the matrix form of the monodromy transformation they infer is given.

1. Introduction

Let us consider a linear operator \( L_C \) which sends holomorphic function \( E = E(z) \) to the function \( L_C[E] \) which is computed as follows

\[
L_C : E(z) \mapsto L_C[E](z) = 2\omega z^{-\ell-1} \left( E'(z) - \mu E(z) \right)
\]

and which is also holomorphic everywhere except zero. In (1), the symbols \( \ell, \mu, \omega \) denote the constant parameters complying the restrictions \( \mu \neq 0 \neq \omega \) but otherwise arbitrary. It is natural to choose the domain \( \Omega \) of \( L_C \) to be some set (say, a linear space) of functions which is invariant with respect to its action. This assumption requires, obviously, of the domain \( \Omega \) of the members of \( \Omega \) to be invariant with respect to the map \( C : z \mapsto z^{-1} \), or, at least, to have a nonempty intersection with the image \( C\Omega \) produced by the latter map. The punctured complex plane \( \mathbb{C}^* = \mathbb{C} \setminus 0 \) is an example of such a domain and, for the sake of definiteness, we shall utilize it, provisionally, in the role of \( \Omega \). Then, after some work, we shall be able to specify a more appropriate \( \Omega \) realization.

The following statement holds true:

Lemma 1. The equality

\[
L_C \circ L_C[E] = E
\]

Supported in part by RFBR grant N 17-01-00192.
takes place if and only if the function $E = E(z)$ obeys the equation

$$
\begin{align*}
z^2 E'' + \left((\ell + 1)z + \mu(1 - z^2)\right)E' + (-\mu(\ell + 1)z + \lambda)E &= 0,
\end{align*}
$$

where $\lambda = (2\omega)^{-2} - \mu^2$.

The proof of the lemma immediately follows from the identity

$$
\mathcal{L}_C \circ \mathcal{L}_C [E](z) \equiv E(z) - (2\omega)^2 \cdot \text{lhs}(3)
$$

in which $\text{lhs}(3)$ stands for the left-hand side expression of the equation (3) and which is established by means of a straightforward computation.

The ordinary second order linear homogeneous differential equation (3) arising here belongs to the family of so called double confluent Heun equations (often referred to with acronym DCHE) which is discussed, in particular, in Ref.s [1, 2]; see also the online resource [3] for more recent bibliography. A generic DCHE is identified by four constant parameters while Eq. (3) involves only three ones. Accordingly, it was suggested to name Eq. (3) the special double confluent Heun equation (sDCHE, accordingly) which term is here adopted for definiteness as well.

It is important to emphasize that the equation (3) is segregated within the DCHE family due to its intimate relation to the non-linear first order ordinary differential equation

$$
\dot{\varphi} + \sin \varphi = B + A \cos \omega t,
$$

in which $\varphi = \varphi(t)$ is the unknown function, the symbols $A, B, \omega$ stand for some real constants, and $t$ is a free real variable, the dot denoting derivation with respect to $t$.

Eq. (6) and its generalizations arise in analysis of a number of models in physics, mechanics and geometry [4, 5]. Most widely it is known as the equation utilized in the so called RSJ model of Josephson junction [6, 7, 8, 9, 10], which applies when the effect of the electric capacitance of junction is negligible (so called overdamped Josephson junctions).

The basic points of relationship between the equations (3) and (6) (first noted in Ref. [13]), which imparts notable physical significance to investigation of properties of the former, are discussed below, the approach utilized here being new. It is based on the extensive use of the eigenfunctions of the operator (1). The principal point here is that any such eigenfunction is automatically a solution to Eq. (3) (for the appropriate value of the parameter $\lambda$). Indeed, the following statements holds true:

**Lemma 2.** An eigenfunction of the operator $\mathcal{L}_C$ with eigenvalue $\nu \neq 0$ obeys the equation (3) with $\lambda = \nu^2/(2\omega)^2 - \mu^2$. If the parameter constraint (4) is met then $\nu^2 = 1$.

\footnote{see the footnote in the page 12 therein}
Proof. The lemma assertion follows in obvious way from the same identity (3).

\[ \square \]

Remark 1. Omitting above the case \( \nu = 0 \), one does not forfeit any non-trivial relationships since there is, obviously, the only eigenfunction \( E(z) = \exp(\mu z) \) of \( L_C \) with the zero eigenvalue.

Motivated by the above two lemmas, we may adopt the set of solutions to Eq. (3), which constitutes a 2-dimensional linear space, as the functional space \( \Omega \) on which the action of the operator \( L_C \) has to be considered.

Remark 2. On the complex plane, the linear differential equation (3) suffers of the only singular point \( z = 0 \). Its solutions are thus holomorphic everywhere except at zero. To be more exact, solutions holomorphic thereat may, in principle, exist but only on the specific “tuned” constant parameter collection of lower dimension, see Refs. [14] [15] [16], and even on it only a single (up to constant factors) solution is regular at zero whereas all other ones are not. It means that, when considering common domain of functions constituting \( \Omega \), one must remove the center \( z = 0 \) from it. Then, starting from the complex plane, the punctured one, \( \mathbb{C}^* = \mathbb{C} \setminus 0 \), which is not simply connected, arises. A non-trivial solution \( E \) to Eq. (3), apart from sparse exceptions, can not live on \( \mathbb{C}^* \), however. The point is that the analytic continuation of a generic solution along non-homotopic curves evading zero will produce different values at the point where they meet, leading therefore to a multi-valued function. The non-uniqueness arises here since the correct domain for solutions to Eq. (3) is not a subset of \( \mathbb{C} \) (such as \( \mathbb{C}^* \)) but a Riemann surface reducing here to the universal cover of \( \mathbb{C}^* \).

However, as a consequence of such a complication, the map \( C : z \mapsto z^{-1} \) involved in Eq. (1) losses the uniqueness of its “implementation”. Indeed, the map \( C \) may now have only a single fixed point; hence one has either \( C 1 = 1 \) and \( Cz \neq z \) for all the other points \( z \) of the \( E \) domain, or it holds \( C(-1) = -1 \) and \( Cz \neq z \) otherwise. There is therefore no point playing role of \(-1\) in the former case (the first \( C \) “implementation”) and similarly for \(+1\) in the latter case (the alternative, second, “implementation” of the map \( C \)). These subtleties go beyond the scope of the present notes, however. In order to focus on the principal points of the problem indicated in their title, we restrict consideration to a subset of the true domain of functions verifying Eq. (3). Namely, we consider it to be the open set obtained from \( \mathbb{C}^* \) by removal of the negative half of the real axis, \( \Omega^* := \mathbb{C}^* \setminus \mathbb{R}_{<0} \). The resulting (sub-)domain is simply connected and any holomorphic function is single-valued on it. Besides, the properties of the map \( C \) remains (locally) “standard”
and claims no precautions — at the price of dropping out of $-1 \notin \Omega^*$, as well as other negative real numbers, in particular.

We can now consider in more details the properties of the operator $\mathcal{L}_C$ and show how they enables one to exhibit the close relationship of the equations (3) and (6).

2. Basic properties of eigenfunctions of the operator $\mathcal{L}_C$

Let the equation (3) with fixed parameters $\ell, \lambda, \mu$ such that $\lambda + \mu^2 \neq 0$ be given. Then one can resolve Eq. (4) with respect to $\omega$ (the scaling parameter in (1)), i.e. select it obeying the equation

$$4 \omega^2 (\lambda + \mu^2) = 1. \tag{7}$$

With such $\omega$, we define the operator $\mathcal{L}_C$ by the formula (1) and assume that it acts on the space $\Omega$ of solutions to Eq. (3). In view of the lemma 5, an eigenfunction of the operator $\mathcal{L}_C$ belonging to $\Omega$ may correspond either to the eigenvalue $+1$ or to the eigenvalue $-1$. We denote such eigenfunctions (if they exist) by the symbols $E_{\{+\}}$ and $E_{\{-\}}$, respectively.

One has the following important statements.

**Lemma 3.**

- If a solution $E = E(z)$ to Eq. (3) is an eigenfunction of the operator $\mathcal{L}_C$ then it solves the Cauchy problem for this equation with initial data obeying one of the two constraints

$$E'(1) = (\pm (2\omega)^{-1} + \mu)E(1). \tag{8}$$

These correspond to the eigenvalues $\pm 1$, respectively.

- The eigenfunctions $E_{\{\pm\}}$, if exist, obey the constraint

$$E_{\{+\}}(z)E_{\{-\}}(1/z) + E_{\{-\}}(z)E_{\{+\}}(1/z) = 2e^{\mu(z+1/z-2)}E_{\{+\}}(1)E_{\{-\}}(1). \tag{9}$$

**Corollary 4.**

- $E_{\{\pm\}}(1) \neq 0$ for any eigenfunction of the operator $\mathcal{L}_C$.

- There may exist not more than two, up to constant factors, eigenfunctions of the operator $\mathcal{L}_C$ with eigenvalues $\pm 1$ (which, if exist, are linear independent, obviously).

Accordingly, the functions $E_{\{\pm\}}$ would provide the basis of the linear space $\Omega$.

**Lemma proof.** In accordance with lemma 2 each eigenfunction of the operator $\mathcal{L}_C$ verifies Eq. (3). Next, by definition, the property of being

---

2 and also at price of impossibility of explanation of the formula (49) below within such a framework.
an eigenfunction of $L_C$ with the eigenvalue either $+1$ or $-1$ is equivalent to the equalities

$$E_{\{\pm\}}'(z) = \pm (2\omega)^{-1} z^{-\ell-1} E_{\{\pm\}}(1/z) + \mu E_{\{\pm\}}(z).$$

Evaluating them at $z = 1$, one obtains Eqs. (8).

Further, considering Eq. (9), let us denote as $U = U(z)$ the difference of its left-hand side and right-hand side expressions. Computing its derivative and eliminating the derivatives $E_{\{\pm\}}'$ by means of the equation (10), the equation $U' = \mu \cdot (z - 1/z) \cdot U$ arises. Since $U(1) = 0$, obviously, this linear homogeneous first order differential equation forces $U$ to coincide with its trivial null solution implying $U(z) \equiv 0$. The lemma is thus proven. \[\square\]

**Remark 3.** Another constraint which the functions $E_{\{\pm\}}$ obey reads

$$E_{\{\pm\}}'(z) E_{\{\pm\}}(z) - E_{\{\pm\}}(z) E_{\{\pm\}}'(z) = \omega^{-1} z^{-\ell-1} e^{\mu(z+1/z-2)} E_{\{\pm\}}(1) E_{\{\pm\}}(1)$$

It follows from consideration of the Wronskian for Eq. (3) which applies since $E_{\{\pm\}}$ verify the latter.

3. Explicit representations of eigenfunctions of $L_C$

As it has been mentioned, the eigenfunctions of the operator $L_C$ can be utilized for description of the Eq. (3) solution space. However, one should show, firstly, that they do exist. Their definition is equivalent to the claim of fulfillment of one of the equations (10). However, the latter are not the classical ODEs since the unknown involved therein depends on two distinct arguments. Hence the corresponding well-known theorems of existence of solutions can not be directly applied. An independent proof of existence of eigenfunctions of $L_C$ has thus to be given. To that end, let us consider the following

**Lemma 5.** Let the holomorphic function $\Phi = \Phi(z)$ defined in a simply connected vicinity of the point $z = 1$ obey the Riccati equation

$$z \Phi' + (2\omega)^{-1} (\Phi^2 - 1) = (\ell + \mu(z + z^{-1})) \Phi,$$

and the holomorphic function $\Psi = \Psi(z)$ obey the (subsidiary) linear homogeneous first order ODE

$$2i \omega z \Psi' = (\Phi + \Phi^{-1}) \Psi.$$

Let also

$$|\Phi(1)| = 1 \text{ and } \Psi(1) = 1.$$

Then the expressions

$$E_{\{\pm\}}(z) := 2^{-1} e^{\mu(z+1/z-2)/2} z^{-\ell/2} \times$$

$$\left( \frac{1 \pm i}{\sqrt{2}} (\Psi(z) \Phi(z))^{1/2} + \frac{1 \mp i}{\sqrt{2}} (\Psi(1/z) / \Phi(1/z))^{1/2} \right)$$
determine the two eigenfunctions of the operator $L_C$ with eigenvalues $\pm 1$, respectively, provided neither of them is the identically zero function. In the latter case, another function (14) is still a proper (non-trivial) eigenfunction of $L_C$.

Proof outline. To verify the asserted property of a function $E_{\pm}(z)$, one has to compute its derivative and to examine the fulfillment of the corresponding equation among Eqs (10). In our case, utilizing the equations (11) and (12), the aforementioned derivative is expressed in terms of products of the same functions $\Phi$ and $\Psi$ with the same arguments $z$ and $1/z$ which are involved in the definition (14). Subsequent algebraic simplification establishes the identical vanishing of coefficients in front of all the remaining products of $\Psi$ and $\Phi$. □

The existence of the functions $\Phi$ and $\Psi$ in vicinity of the point $z = 1$ is ensured by the wellknown theorem of existence of local solution of Cauchy problem for ordinary differential equations. In case of Eqs (11) and (12), one can however say more.

Lemma 6. Let the parameters $\ell, \mu, \omega$ are real and $\omega > 0$. In case of initial conditions obeying the constraints (13), the solution $\Phi(z)$, $\Psi(z)$ of the Cauchy problem for the system of equations (11) and (12) exists in some vicinity of the “punctured unit circle”

$$\mathcal{S}^1 = \{ z \in \mathbb{C}, |z| = 1, z \neq -1 \},$$

both functions $\Phi(z)$, $\Psi(z)$ having no zeros therein.

Proof. Let us restrict Eq. (11) to the unit circle embedded into $\mathbb{C}$ and parameterized by means of the substitutions

$$z = e^{i\omega t}, \quad \Phi(z) = e^{i\varphi(t)}, \quad t \in \Xi := (-\pi\omega^{-1}, \pi\omega^{-1}) \subset \mathbb{R}.

Then we obtain exactly Eq. (6) with the parameters

$$A = 2\omega \mu, \quad B = \omega \ell.$$

Similar transformation of Eq. (12) leads to the equation

$$\dot{P}(t) = \cos \varphi(t),$$

where the function $P(t)$ is related to the original unknown $\Psi(z)$ through the equation

$$e^{P(t)} = \Psi(e^{i\omega t}).$$

For any real $A, B,$ and $\omega$, Eq. (6) is solvable on any segment of the real axis for any real initial data $\varphi(t_0) = \varphi_0$ set up at any prescribed real $t_0$. Moreover, the corresponding solution is a real-analytic function. Accordingly, let some real $\varphi_0$ be fixed and let the real-analytic function $\varphi(t)$ verify Eq. (6) on the segment $\Xi$, obeying the initial condition $\varphi(0) = \varphi_0$. Let us also introduce on the same domain $\Xi$ the real-analytic function $P(t) = \int_0^t \cos \varphi(\tilde{t}) \, d\tilde{t}$. 

The analytic continuation of the map \((\mathbb{C} \supset \mathbb{R} \supset) \Xi \ni t \mapsto e^{i\omega t} \in \mathbb{S}^1(\subset \mathbb{C}^*)\) establishes the holomorphic diffeomorphism of some vicinity of the segment \(\Xi\) to some vicinity of the punctured unit circle \(\mathbb{S}^1\) \((\subset \mathbb{C}^*)\), the former being in smooth bijection with the latter. The holomorphic functions \(\Phi\) and \(\Psi\) arising as the induced pullbacks of analytic continuations of the real analytic functions \(e^{i\varphi(t)}\) and \(e^{P(t)}\), respectively, verify Eq.s \((11), (12)\). By definition, they have no zeros on \(\mathbb{S}^1\); moreover, \(|\Phi| = 1\) whereas \(\Psi\) is real and strictly positive thereon. Hence there exist no their zeros in some vicinity of \(\mathbb{S}^1\) as well. Besides, in accordance with definitions and the posing of the Cauchy problem for the function \(\varphi\), it holds
\[
(18) \quad \Phi(1) = e^{i\varphi_0}, \quad \Psi(1) = 1
\]
(where \(\varphi_0\) can be chosen arbitrary real) and Eq.s \((13)\) are thus fulfilled. The lemma is proven. \(\square\)

**Remark 4.** The non-uniqueness of the square root function involved in Eq. \((14)\) is eliminated by means of the assignment to the functions \(\Phi^{1/2}, \Phi^{-1/2},\) and \(\Psi^{1/2}\) (the pullbacks of) the analytic continuations of the functions \(\exp \frac{1}{2} \varphi(t), \exp -\frac{1}{2} \varphi(t),\) and \(\exp \frac{1}{2} \int_0^t \cos \varphi(\tilde{t}) d\tilde{t}\), respectively.

**Remark 5.** The requirement of the above lemma for the constant parameters to be real is motivated by Eq.s \((17)\), in which the constants \(A, B, \omega\) are constrained by the meaning assigned to them in the physical or geometrical problems in which Eq. \((6)\) is utilized. Similarly, the variable \(t\) is there interpreted as a (rescaled) time or length. To keep the contact with applications, we assume below the above reality conditions to be fulfilled throughout. At the same time, it is worth noting that the existence results (and most formulas evading application of complex conjugation) remain valid for sufficiently small variations of the parameters shifting them from the real axis.

We see that any solution to Eq. \((6)\) generates a pair of eigenfunctions of the operator \(L_C\) which are defined by Eq.s \((14)\) in terms of the functions \(\Phi(z)\) and \(\Psi(z)\) the above lemma operates with. However, one of them (not both, though) may prove to be identical zero. To get control of appearance of such a “pathology”, we need the next property of eigenfunctions of \(L_C\) derived below.

**Lemma 7.** Let us define the sequence of pairs of functions \(\{a_k(z), b_k(z)\}\), \(k = 1, 2, \ldots,\) holomorphic everywhere except zero by means of the following recurrent scheme:
\[
(19) \quad a_1 = \mu, \quad b_1 = \pm (2\omega)^{-1} z^{-\ell-1} ;
\]
\[
(20) \quad a_{k+1} = \mu a_k \mp (2\omega)^{-1} z^{-\ell-1} b_k + a_k', \quad b_{k+1} = \pm (2\omega)^{-1} z^{-\ell-1} a_k - \mu z^{-2} b_k + b_k'.
\]
Let also the function $E_{i\pm}$ obey the equation $L_C E_{i\pm} = \pm E_{i\pm}$. Then its derivatives admit the following representation:

$$
\frac{d^k}{dz^k} E_{i\pm}(z) = a_k(z) E_{i\pm}(z) + b_k(z) E_{i\pm}(1/z), \quad k = 1, 2, \ldots
$$

In particular, it holds

$$
\frac{d^k}{dz^k} E_{i\pm}(1) = (a_k(1) + b_k(1)) E_{i\pm}(1), \quad k = 1, 2, \ldots
$$

**Proof.** Let us apply the mathematical induction. The induction base, the case $k = 1$, reduces to the equality which, in view of (19), is equivalent just to the equation $L_C E_{i\pm} = \pm E_{i\pm}$ fulfilled by construction.

Next, let us compute the derivative of the both sides of Eq. (21) for some fixed $k$, eliminating afterwards $E'_{i\pm}$ on the right by means of Eq. (21) get with $k = 1$, and eliminating the derivatives $a'_k, b'_k$ with the help of Eq.s (20). As it can be shown by a straightforward computation, the result reduces to the same equation (21) with the index $k$ replaced with $k + 1$. The induction step has thus been carried out and the lemma proof is accomplished.

**Corollary 8.** The function $E_{i\pm}(z)$ defined by Eq. (14) is the identically zero function if and only if $E_{i\pm}(1) = 0$.

We apply the corollary to clarification of the conditions leading to identically zero function $E_{i\pm}$ defined by Eq. (14). Indeed, substituting therein $z = 1$ and taking into account Eq.s (18), one gets

$$
E_{i\pm}(1) = \mp \sin \frac{\phi_0}{2}(\phi_0 \mp \pi/2).
$$

Hence one of the functions $E_{i\pm}$ can, indeed, be identical zero and this takes place if and only if $\phi_0 = \pi/2 (\text{mod } \pi)$.

**Remark 6.**

- A variation of the initial data $\phi_0 = \varphi(0)$ for solution to Eq. (6) results in appearance of some additional constant factors and this is the only distinction of the functions $E_{i\pm}$, obtained by means of Eq.s (14), from the “fiducial” ones corresponding to, say, $\phi_0 = 0$. Besides, with respect to the case $\phi_0 = 0$, the absolute values of these factors do not exceed 1.
- In case of identical vanishing of one of the functions $E_{i\pm}$, the corresponding sum in brackets in Eq.s (14) vanishes. Then the same sum but with the opposite choice of the signs amounts to twice its first summand. Accordingly, the following factorized representation of the nontrivial eigenfunction $E_{i\pm}$ still produced by one of Eq.s (14) arises:

$$
E_{i\pm} \propto (e^{\mu(z+1/z-2)} z^{-\ell} \Psi(z) \Phi(z))^{1/2}.
$$

As we have mentioned, this situation occurs for $\phi_0 = \pi/2 (\text{mod } \pi)$.  


Resuming, we have our first key

**Theorem 9.** Let a solution $\varphi(t)$ to the equation (6) on the segment $\Xi = (-\pi \omega^{-1}, \pi \omega^{-1})$ be given. Then the analytic continuations of the functions $\exp(i \varphi(t))$ and $\exp(\int_0^t \cos \varphi(t) dt)$ from $\Xi$ to some its vicinity in $\mathbb{C}$, converted by means of the transformation (16) to the functions $\Phi(z)$ and $\Psi(z)$ holomorphic in some vicinity of the punctured circle (15), determine therein the two solutions $E_{\{\pm\}} = E_{\{\pm\}}(z)$ to Eq. (3) by means of the formulas (14). The functions $E_{\{\pm\}}$ are linear independent unless one of them is the identically zero function that takes place if and only if $\varphi(0) = \pi/2 \ (\text{mod} \ 2\pi)$ (leading to $E_{\{\pm\}}(z) \equiv 0$) or $\varphi(0) = -\pi/2 \ (\text{mod} \ 2\pi)$ (leading to $E_{\{\pm\}}(z) \equiv 0$, respectively). The linear independent functions $E_{\{\pm\}}$ constitute the basis of the space of solutions to Eq. (3), $\Omega$.

The functions $E_{\{\pm\}}$ are also the eigenfunctions of the linear operator $\mathcal{L}_C$ defined by Eq. (1) with eigenvalues $\pm 1$, respectively; $\mathcal{L}_C$ is, thus, represented in the basis $\{E_{\{+\}}, E_{\{-\}}\}$ by the diagonal matrix $\text{diag}(1, -1)$; accordingly, the linear space $\Omega$ is invariant with respect to the operator $\mathcal{L}_C$ which defines its involutive automorphism.

**Corollary 10.** The eigenfunctions of the operator $\mathcal{L}_C$ with eigenvalues $\pm 1$ are exactly the non-trivial solutions to Eq. (3) which obey the initial data constraint (8).

**Remark 7.** If a function $E$ verifies Eq. (3) then, obviously, either it is itself an eigenfunction of $\mathcal{L}_C$ or the functions $\text{const} \cdot (E \pm \mathcal{L}_C E)$ constitute just a pair of such eigenfunctions which are linear independent. Adjusting $\text{const}$, they can be made real (self-conjugated, see the next section).

### 4. Self-Conjugation Property of Eigenfunctions of the Operator $\mathcal{L}_C$

The explicit formulas for eigenfunctions of the operator $\mathcal{L}_C$ enables one an easy establishing of their invariance with respect to the complex conjugation. However, analogous relations for the functions $\Phi$ and $\Psi$ involved in $E_{\{\pm\}}$ definition (14) have to be derived up front. To that end, let us introduce the following auxiliary working definition.

**Definition.** Let $\Upsilon(z)$ be any function holomorphic in some connected open subset of $\mathbb{C}$ containing the point $z = 1$. We shall name the function

$$\tilde{\Upsilon}(z) := \overline{\Upsilon(1/\bar{z})}$$

(25)

**Remark 8.** The above definition obviously implies that
The function dual to holomorphic function is also holomorphic in some open set containing the point \( z = 1 \); the intersection of the domains of \( \Upsilon \) and \( \tilde{\Upsilon} \) is open, non-empty, and also contains that point.

“The duality map” \( \tilde{\cdot} : \Upsilon \mapsto \tilde{\Upsilon} \) is involutive; in particular, the function \( \Upsilon(z) \) is, in turn, dual to the function \( \tilde{\Upsilon}(z) \).

**Lemma 11.** Let the holomorphic function \( \Phi = \Phi(z) \) be a solution to Eq. (11) obeying the constraint \( |\Phi(1)| = 1 \) (cf. Eqs. (13)). Then

\[
\Phi(z)\tilde{\Phi}(z) = 1.
\]

To prove the lemma, we note first that the function \( \tilde{\Phi} = \tilde{\Phi}(z) \) dual to solution \( \Phi(z) \) to Eq. (11) obeys the equation

\[
z\tilde{\Phi}' + (i2\omega)^{-1}(\tilde{\Phi}^2 - 1) = -(\ell + \mu(z + z^{-1}))\tilde{\Phi}.
\]

Then a straightforward computation shows that, as a consequence of (11) and (27), it holds

\[
\frac{d}{dz}(\Phi(z)\tilde{\Phi}(z) - 1) = (-2i\omega z)^{-1}(\Phi(z) + \tilde{\Phi}(z))(\Phi(z)\tilde{\Phi}(z) - 1).
\]

Now let us introduce an auxiliary sequence of functions \( \delta_n \) (in fact, polynomials) of three arguments \( z, \Phi, \tilde{\Phi} \) which all are regarded here, for a time, as free complex variables. (It is worth noting that the functions \( \delta_n \) depends also on the parameters \( \ell, \mu, \omega \) but these their arguments will be suppressed for the sake of the symbolism simplicity.) The functions \( \delta_n \) are defined by means of the following recursive scheme:

\[
\begin{align*}
\delta_1 &= z(\Phi + \tilde{\Phi}), \\
\delta_{n+1} &= (\Phi + \tilde{\Phi} + 4i\omega n)\delta_n - 2i\omega z^2\frac{\partial \delta_n}{\partial z} \\
& \quad + (z(\Phi^2 - 1) - 2i\omega(\ell z + \mu(z^2 + 1))\Phi)\frac{\partial \delta_n}{\partial \Phi} \\
& \quad + (z(\tilde{\Phi}^2 - 1) + 2i\omega(\ell z + \mu(z^2 + 1))\tilde{\Phi})\frac{\partial \delta_n}{\partial \tilde{\Phi}}, \quad n = 1, 2, \ldots
\end{align*}
\]

We utilize them for introduction of the functions

\[
\Lambda_n(z, \Phi, \tilde{\Phi}) := (-2i\omega z)^{-n}\delta_n(z, \Phi, \tilde{\Phi})(\Phi\tilde{\Phi} - 1), \quad n = 1, 2, \ldots
\]

**Lemma 12.** Under the conditions of the lemma 11, it holds

\[
\frac{d}{dz}\Lambda_n(z, \Phi(z), \tilde{\Phi}(z)) = \Lambda_{n+1}(z, \Phi(z), \tilde{\Phi}(z)), \quad n = 1, 2, \ldots
\]

**Proof.** It is easy to show that, in view of Eq. (11) and Eq. (27), the above assertion is equivalent to Eq. (29) for \( n = 1 \) and to Eq. (30) for \( n > 1 \).

\( \square \)
Corollary 13. Under the conditions of the lemma [11], it holds
\[ \frac{d^n}{dz^n}(\Phi(z)\tilde{\Phi}(z) - 1) = \Lambda_n(z, \Phi(z), \tilde{\Phi}(z)), \quad n = 1, 2, \cdots. \]

Proof. In case \( n = 1 \) the above equation follows from Eqs. (28) and (29), and the definition (31). It is extended to higher derivative orders \( n = 2, 3, \cdots \) by means of the mathematical induction based on Eq. (32). \( \square \)

Corollary 14. Under conditions of the lemma [11], all the derivatives of the function \( \Phi(z)\tilde{\Phi}(z) - 1 \) vanish at the point \( z = 1 \).

Proof. In accordance with \( \Lambda_n \) definition (31) and Eq. (33), for any \( n = 1, 2, \cdots \) the derivative \( \frac{d^n}{dz^n}(\Phi(z)\tilde{\Phi}(z) - 1) \) factorizes into a function holomorphic in vicinity of the point \( z = 1 \) times the function \( \Phi(z)\tilde{\Phi}(z) - 1 \) itself. The latter is zero at the unity (since \( \Phi(1)\tilde{\Phi}(1) = |\Phi(1)|^2 = 1 \)); accordingly, its derivative is zero thereat as well. \( \square \)

Proof of the lemma [11]. Since the function \( \Phi(z)\tilde{\Phi}(z) - 1 \) is analytic at the point \( z = 1 \), the above corollary implies its identical vanishing and thus the validity of the assertion of the lemma [11]. \( \square \)

Similarly to above, let us consider how the function \( \tilde{\Psi} = \tilde{\Psi}(z) \) dual to solution \( \Psi = \Psi(z) \) to Eq. (12) is related to \( \Psi \). A straightforward computation establishes fulfillment of the equation
\[ 2i\omega z \tilde{\Psi}' = (\tilde{\Phi} + \tilde{\Phi}^{-1})\tilde{\Psi}. \]

As a consequence, it holds
\[ \frac{d}{dz}(\Psi - \tilde{\Psi}) = (4i\omega z)^{-1}(\Phi + \hat{\Phi})(\Psi - \tilde{\Psi}), \]
provided the functions \( \Phi = \Phi(z) \) and \( \hat{\Phi} = \hat{\Phi}(z) \) (mutually dual) obey Eq. (26). Let us notice now that, as the functions \( \Phi \) and \( \hat{\Phi} \) are given, Eq. (35) can be regarded as a linear homogeneous first order ODE for the holomorphic function \( \delta = \delta(z) = \Psi(z) - \tilde{\Psi}(z) \) which is correctly defined in the intersection of the domains of the functions \( \Phi \) and \( \hat{\Phi} \) (with zero removed, if necessary). As a consequence, one may claim that the function \( \delta \) either has no zeros in its domain or is the identically zero function. But if the function \( \Psi(z) \) complies with the “initial condition” (18) then \( \tilde{\Psi}(1) = 1 \) as well implying \( \delta(1) = 0 \). Thus \( \delta(z) \equiv 0 \) at least in a connected vicinity of the point \( z = 1 \). We have therefore proven the following

Theorem 15. Let the functions \( \Phi(z) \) and \( \Psi(z) \) be holomorphic in some connected open subset of \( \mathbb{C}^* \) containing the point \( z = 1 \) and obey therein the system of equations (11), (12); let the constraints (13) be also fulfilled. Then the equation (26) and the equation
\[ \tilde{\Psi}(z) = \Psi(z) \]
hold true.
Corollary 16. Under the conditions of the theorem 15, it holds $|\Phi| = 1$ and $\Im \Psi = 0$ on the punctured unit circle (15).

Proof. Since $\overline{z} = z^{-1}$ on the unit circle in $\mathbb{C}$, the assertions to be proven follow from Eqs. (26) and (36). □

Remark 9. We have shown, in particular, that any holomorphic functions $\Phi, \Psi$ obeying conditions of the theorem 15 determine the smooth real valued functions $\varphi(t), P(t)$ verifying the equations (6) and (18), respectively.

Now a short straightforward computation leaning on Eqs. (26) and (36) proves the following

Theorem 17. Let the functions $\Phi(z)$ and $\Psi(z)$ obey the system of equations (11), (12) and the constraints (13). Then the functions $E_{(+)}(z)$ and $E_{(-)}(z)$ defined by Eqs. (14) are real (self-conjugated), i.e. obey the constraints

$$ E_{(\pm)}(\overline{z}) = E_{(\pm)}(z). $$

5. Representation of general solution to the equation of RSJ model in terms of solutions to special double confluent Heun equation

Having outlined the way of constructing of solutions to Eq. (3) from solutions to Eq. (6), we have to account for the inverse relationship. It can be expressed in the form of the following

Theorem 18. Let the holomorphic functions $E_{(+)}(z)$ and $E_{(-)}(z)$ be the real (self-conjugated, see Eq. (37)) eigenfunctions of the operator $\mathcal{L}_C$ defined by Eq. (1) with the corresponding eigenvalues $\pm 1$; let also $\alpha$ be an arbitrary real constant. We define the holomorphic functions $\Phi(z)$ and $\Theta(z)$ as follows:

$$ \Phi(z) := -iz \frac{\cos(\frac{1}{2} \alpha) E_{(+)}(z) + i \sin(\frac{1}{2} \alpha) E_{(-)}(z)}{\cos(\frac{1}{2} \alpha) E_{(+)}(1/z) - i \sin(\frac{1}{2} \alpha) E_{(-)}(1/z)}, $$

$$ \Theta(z) := -i \frac{\cos(\frac{1}{2} \alpha) E_{(+)}^2(1) E_{(-)}(z) + i \sin(\frac{1}{2} \alpha) E_{(-)}^2(1) E_{(+)}(z)}{E_{(+)}(1) E_{(-)}(1) \left( \cos(\frac{1}{2} \alpha) E_{(+)}(z) + i \sin(\frac{1}{2} \alpha) E_{(-)}(z) \right)}. $$

Then

- the continuous function $\varphi(t)$ of the real variable $t$ determined by the equation

$$ e^{i \varphi(t)} = \Phi(e^{i \omega t}) $$

is well defined, real valued, smooth and verifying Eq. (6):
the functions $P(t)$ and $Q(t)$ defined as follows

\begin{equation}
P(t) := -\log(-\Im \Theta(e^{i\omega t})), \quad Q(t) := \Re \Theta(e^{i\omega t})
\end{equation}

are well defined, real valued, smooth and are related to the function $\varphi(t)$ by subsequent quadratures as follows

\begin{equation}
P(t) = \int_0^t \cos \varphi(\tilde{t}) \, d\tilde{t}, \quad Q(t) = \int_0^t e^{-P(\tilde{t})} \sin \varphi(\tilde{t}) \, d\tilde{t}.
\end{equation}

**Remark 10.** In view of the lemma 5, the both functions $E_{\pm}(z)$ obey Eq. (3) and one learns from lemmas 5 and 6 that they always exist. Hence, the functions $\Phi(z)$ and $\Theta(z)$, as well as the functions $\varphi(t), P(t), Q(t)$ which they give rise to, are built (and always can be built) upon solutions of this equation.

**Theorem proof.** We proceed noting that since the functions $E_{\pm}(z)$ obey a linear homogeneous second order differential equation with coefficients holomorphic everywhere except at zero (Eq. (3) times $z^{-2}$), they are themselves holomorphic everywhere except, perhaps, at zero. Besides, in accord with the corollary 8, $E_{+}(1) \neq 0 \neq E_{-}(1)$ that disavows the source of an a priori conceivable fault of the definition (39).

Now let us consider the identity

\begin{equation}
\text{i} e^{i\varphi(t)} (\dot{\varphi}(t) + \sin \varphi(t) - \omega(\ell + 2\mu \cos \omega t)) = \frac{d}{dt} (e^{i\varphi(t)} - \Phi(e^{i\omega t}))
\end{equation}

\begin{equation}
\hphantom{\text{i} e^{i\varphi(t)} (\dot{\varphi}(t) + \sin \varphi(t) - \omega(\ell + 2\mu \cos \omega t))} + \left(2^{-1}(e^{i\varphi(t)} + \Phi(e^{i\omega t})) - i\omega(\ell + 2\mu \cos \omega t) \right) (e^{i\varphi(t)} - \Phi(e^{i\omega t}))
\end{equation}

which takes place for arbitrary smooth function $\varphi(t)$ and which is proven by means of straightforward computation taking into account the $\Phi$ definition (38) and Eq. (10). It follow from (43), obviously, that if Eq. (40) is fulfilled then $\varphi(t)$ verifies Eq. (6) with $A = 2\omega \mu, B = \omega \ell$ (cf. Eq.s (17)).

Further, let us note that, since the functions $E_{\pm}(z)$ are real, in case of real $\alpha$, one has the following equalities:

\[ \Phi(z) = i z^{-\alpha} \frac{\cos(\frac{\pi}{2} \alpha) E_{\ell+1}(z) - i \sin(\frac{\pi}{2} \alpha) E_{\ell-1}(z)}{\cos(\frac{\pi}{2} \alpha) E_{\ell+1}(1/z) + i \sin(\frac{\pi}{2} \alpha) E_{\ell-1}(1/z)} \equiv \Phi(1/z)^{-1}. \]

For $z = e^{i\omega t}$ and real $t$, it holds $1/z = z$. Accordingly, one infers from above that $\Phi(e^{i\omega t}) = \Phi(e^{i\omega t})^{-1}$ and, consequently, $|\Phi(e^{i\omega t})| = 1$. Then Eq. (10) yields $|e^{i\varphi(t)}| = 1$, and the real-valued smooth function $\varphi(t)$ is determined in terms of the logarithm of the non-zero smooth function $\Phi(e^{i\omega t})$ in the standard way. The first assertion of the theorem is therefore proven.
Addressing now to the second assertion, let us introduce, in addition to the function \( \Theta(z) \), the function \( \tilde{\Theta}(z) \) as follows:

\[
\tilde{\Theta}(z) := i \cos(\frac{1}{2}\alpha) E_{\frac{1}{2}+1}(1) E_{\frac{1}{2}-1}(1/z) - E_{\frac{1}{2}+1}(1/z) - i \sin(\frac{1}{2}\alpha) E_{\frac{1}{2}+1}(1) - i \sin(\frac{1}{2}\alpha) E_{\frac{1}{2}-1}(1/z) .
\]

The functions \( \Theta = \Theta(z) \) and \( \tilde{\Theta} = \tilde{\Theta}(z) \) prove obeying the following system of the two linear homogeneous first order differential equations

\[
i \omega z \Theta' = - \Phi - 1 (\Theta - \tilde{\Theta}) , \quad i \omega z \tilde{\Theta}' = \Phi(\Theta - \tilde{\Theta}) .
\]

This is a direct consequence of definitions and Eq. (10).

A straightforward verification also based on definitions shows that for real eigenfunctions \( E_{\pm} \) (and for real constant \( \alpha \)) it holds

\[
\Theta(z) = \tilde{\Theta}(1/z) , \quad \text{i.e. the function } \Theta \text{ defined by means of a separate formula (44) is actually dual to the function } \Theta \text{ (see Eq. (25)). As a consequence, one gets } \tilde{\Theta}(e^{i \omega t}) = \Theta(e^{i \omega t}) . \quad \text{Then Eq.s (45) yield the equation}
\]

\[
\frac{d}{dt} \Theta(e^{i \omega t}) = - \Phi(e^{i \omega t})^{-1} (\Theta(e^{i \omega t}) - \Theta(e^{i \omega t})) .
\]

Separating its real and imaginary parts and taking into account Eq. (40), one gets

\[
\frac{d}{dt} \Re \Theta(e^{i \omega t}) = - \Im \Theta(e^{i \omega t}) \sin \varphi(t) ,
\]

\[
\frac{d}{dt} \Im \Theta(e^{i \omega t}) = - \Re \Theta(e^{i \omega t}) \cos \varphi(t) .
\]

In case of given real valued function \( \varphi(t) \), the latter equation determining \( \Im \Theta \) is integrated by means of a quadrature. Then the former one is integrated by means of another quadrature. The integration constants are fixed making use of the initial conditions \( \Re \Theta(e^{i \omega t})|_{t=0} = \Re \Theta(1) = 0 , \) \( \Im \Theta(e^{i \omega t})|_{t=0} = \Im \Theta(1) = -1 \) which directly follow from \( \Theta \) definition (39) evaluated at the point \( z = 1 \). The ultimate result of integration is just the formulas (42). The theorem proof has been accomplished.

Let us note that for \( t = 0 \) Eq.s (40) and (38) are equivalent to the equation

\[
E_{\frac{1}{2}-1}(1) \sin(\frac{1}{2}\varphi(0) - \frac{\pi}{4}) \sin(\frac{1}{2}\alpha) + E_{\frac{1}{2}+1}(1) \cos(\frac{1}{2}\varphi(0) - \frac{\pi}{4}) \cos(\frac{1}{2}\alpha) = 0 .
\]

Obviously, it is solvable with respect to the angular parameter \( \alpha \) for any given real \( \varphi(0) \) (recall that the values of the functions \( E_{\frac{1}{2} \pm} \) are real in case of real argument). Conversely, for any \( \alpha \in [0, 2\pi) \) some “initial data” \( \varphi_0 = \varphi(0) \in [0, 2\pi) \) obeying Eq. (43) can be found. We obtain, therefore, the following

**Corollary 19.** Eq.s (40), (38) enable one to obtain any solution to Eq. (4), representing it in terms of solutions to Eq. (3).
6. Conclusion

Resuming, we have shown that any solution to Eq. (6) can be converted to solutions to Eq. (3) by means of a quadrature and analytic continuation of two real analytic functions (theorem 9). Moreover, in a generic case, a basis of the solutions space of Eq. (3) is then produced which is constituted by the eigenfunctions of the operator $L_C$ (defined by Eq. (1)) and these are real (theorem 17).

Conversely, let two real eigenfunctions of the operator $L_C$ with eigenvalues $+1$ and $-1$ be given. Then all the solutions to Eq. (6) can be obtained from Eqs. (38), (40) (theorem 18, corollary 19), as similar formulas, Eq. (39), determining the integrals (42) which are involved in the criterion of the so-called phase-lock [12, 13], the property manifested by solutions to Eq. (6) [8, 9, 10].

In total, the relationships indicated above establish the explicit 1-to-1 correspondence between solutions spaces of Eq. (3) and Eq. (6), essentially.

In conclusion, it is worth noting that the eigenfunctions of the operator $L_C$ (as well as this operator on its own, of course) are the important tools proving to be efficient in investigation of various problems related to sDCH(r) [3]. In particular, the following explicit matrix representation $M$ of the monodromy transformation\(^3\) of its space of solutions with respect to the basis \(\{E_{(+)}, E_{(-)}\}\) can be obtained\(^4\):

\[
M = e^{\mu} (2E_{(+)}(1)E_{(-)}(1))^{-1} \times \\
\begin{pmatrix}
E_{(+)}(-\hat{1})E_{(-)}(-\hat{1}) + E_{(+)}(-\hat{1})E_{(-)}(-\hat{1}) & E_{(+)}(-\hat{1})^2 - E_{(+)}(-\hat{1})^2 \\
E_{(-)}(-\hat{1})^2 - E_{(-)}(-\hat{1})^2 & E_{(-)}(-\hat{1})E_{(-)}(-\hat{1}) + E_{(-)}(-\hat{1})E_{(-)}(-\hat{1})
\end{pmatrix}.
\]

Here the symbols $\hat{-1}$ and $\hat{1}$ denote the preimages of $-1 \in \mathbb{C}^*$ in the Riemann surface\(^5\) — the domain of solutions to Eq. (3), namely, the preimages closest to the preimage of 1, of which $\hat{-1}$ is connected to the latter by an arc passed in counterclockwise direction while for $\hat{1}$ similar arc is passed clockwise. The above formula shows, in particular, that the diagonal elements of $M$ are real and coincide while off-diagonal ones are pure imaginary. It also follows from the equation (9) that $\det M = 1$. The two eigenvalues of the matrix (49) coincide if and only

---

\(^3\) The monodromy transformation sends a solution to Eq. (3) to another its solution which is obtained by means of the point-wise analytic continuation of the former along counterclockwise-oriented full circle arcs (or other curves homotopic to such arcs) encircling the singular center $z = 0$. On the set of solutions to Eq. (3), the monodromy transformation of solutions to Eq. (3) converts to the map $\varphi(t) \mapsto M\varphi(t) = \varphi(t + 2\pi/\omega)$ which explicitly represents the latter via the former.

\(^4\) The formula (49) has been derived in case of positive integer orders \(\ell\) and is, most likely, correct for all integer orders as well. The cases of other orders require additional examination.

\(^5\) See the remark 2.
if one of its off-diagonal elements vanishes and this condition can be utilized in yet another criterion of the phase-lock behavior for solutions to Eq. (6), the one utilizing the properties of eigenfunctions of the operator $L_C$.

References

[1] D. Schmidt, G. Wolf. Double confluent Heun equation, in: Heun’s differential equations, Rouv偕 (Ed.) Oxford Univ. Press, Oxford, N.Y., (1995), Part C.
[2] S. Yu. Slavyanov, W. Lay. Special Function: A Unified Theory Based on Singularities I Foreword by A. Seeger. Oxford; New York: Oxford University Press, 2000. — ISBN 0-19-850573-6
[3] Heun functions, their generalizations and applications, http://theheunproject.org/bibliography.html
[4] R. L. Foote. Geometry of the Prytz planimeter. Reports Math. Physics 42 (1998), 249–271.
[5] R. L. Foote, M. Levi, S. Tabachnikov. Tractrices, Bicycle Tire Tracks, Hachet Planimeters, and a 100-year-old Conjecture arXiv:1207.0834v1 (2012)
[6] W. C. Stewart. Current-voltage characteristics of Josephson junctions. Appl. Phys. Lett., 12, 277–280(1968).
[7] D. E. McCumber. Effect of ac impedance on dc voltage-current characteristics of superconductor weak-link junctions. J. Appl. Phys., 39, 3113-3118 (1968).
[8] A. Barone, G. Paterno Physics and applications of the Josephson effect John Wiley and Sons Inc. 1982
[9] P. Mangin, R. Kahn. Superconductivity An introduction, Springer, 2017
[10] В. В. Шмидт. Введение в физику сверхпроводников. Изд 2-е, М.: МСНМО, 2000 — В. V. Schmidt. Introduction to physics of superconductors, 2000 (in Russian).
[11] J. Guckenheimer, Yu. S. Ilyashenko. The duck and the devil: canards on the staircase. Mosc. Math. J., 2001, vol 1, 1 pp. 27–47
[12] S. I. Tertychniy. On asymptotic properties of solutions to equation $\dot{\phi} + \sin \phi = f$ for periodical $f$, Rus. Math. Survey, Vol. 55, N 1, p. 186-187 (2000).
[13] S. I. Tertychniy. Long-term behavior of solutions to the equation $\dot{\phi} + \sin \phi = f$ with periodic $f$ and the modeling of dynamics of overdamped Josephson junctions Preprint Arxiv:math-ph/0512058 (2005).
[14] S. I. Tertychniy. The modelling of a Josephson junction and Heun polynomials. arXiv:math-ph/0601064 (2006)
[15] V. M. Buchstaber, S. I. Tertychnyi. Explicit solution family for the equation of the resistively shunted Josephson junction model”. Theoret. and Math. Phys., 176:2 (2013), 965–986
[16] V. M. Buchstaber, S. I. Tertychnyi. Holomorphic solutions of the double confluent Heun equation associated with the RSJ model of the Josephson junction. Theoret. and Math. Phys., 182:3 (2015), 329–355