Time-dependent Andreev bound states of a quantum dot coupled to two superconducting leads

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Sub-gap transport properties of a quantum dot (QD) coupled to two superconducting and one metallic leads are studied theoretically, solving the time-dependent equation of motion by the Laplace transform technique. We focus on time-dependent response of the system induced by a sudden switching on the QD-leads couplings, studying the influence of initial conditions on the transient currents and the differential conductance. We derive analytical expressions for measurable quantities and find that they oscillate in time with the frequency governed by the QD-superconducting lead coupling and acquire damping, due to relaxation driven by the normal lead. Period of these oscillations increases with the superconducting phase difference $\phi$. In particular, for $\phi = \pi$ the QD occupancy and the normal current evolve monotonically (without any oscillations) to their stationary values. In such case the induced electron pairing vanishes and the superconducting current is completely blocked. We also analyze time-dependent development of the Andreev bound states. We show, that the measurable conductance peaks do not appear immediately after sudden switching of the QD coupling to external leads but it takes some finite time-interval for the system needs create these Andreev states. Such time-delay is mainly controlled by the QD-normal lead coupling.

I. INTRODUCTION

Transient effects of the quantum dot (QD) systems have been intensively studied over last years, providing useful insight into the electron transport properties. These effects could be of special importance in experiments on nanoscopic devices, where different types of time-dependent pulses can effectively control the electron flow. Transient effects have been studied, both theoretically and experimentally for the QDs coupled to the metallic (conducting) electrodes\textsuperscript{1,67} and in the presence of superconducting reservoirs\textsuperscript{68–84}. Numerical theoretical approaches have been developed to deal with such time-dependent problems, e.g. the iterative influence-functional path integral\textsuperscript{25}, Keldysh formalism and time-dependent partition-free approach\textsuperscript{40} weak-coupling continuous-time Monte-Carlo methods\textsuperscript{69} and many other techniques\textsuperscript{33,61}.

The coherent oscillations and current beats have been found in a short time scale response of a system upon abrupt change of the bias voltage\textsuperscript{9,14}. From the periods of the current beats it is possible to estimate the values of the QDs energy levels or the hopping parameters between them\textsuperscript{38,61,65}. The transient current characteristics can be also used to determine the spin relaxation time in some QD systems\textsuperscript{2}. Such phenomena have been investigated for QDs coupled to the normal leads as a result of the bias voltage pulse\textsuperscript{5,22,29,31,36,41,53,60,75} driven by an arbitrary time-dependent bias\textsuperscript{26,27,40,53,61} by a sequence of rectangular pulses applied to the input lead\textsuperscript{17,35} or to the contact gradually switched on in time\textsuperscript{25}. The transient dynamics has been also studied for QD after a sudden symmetrical connection to the leads\textsuperscript{27,78,85} or asymmetrically coupled to electrodes following a sudden change of the QD energy levels\textsuperscript{11}. The transient heat generation driven by a step-like pulse bias with the Anderson-Holstein model or the time-dependent current through QD suddenly coupled to a vibrational mode has been studied in nanostructures with the normal\textsuperscript{19,39,47,56,63} or superconducting electrodes\textsuperscript{71}.

Technological progress in the real-time detection of single electrons has opened a possibility for studying electron transport from a perspective of the stochastic processes. Among theoretical tools for investigating the electron hopping statistics there are e.g. the full counting statistics (FCS) and the waiting time distribution (WTD)\textsuperscript{54,55,62,66,73,79}. These theoretical techniques have been successfully applied to investigations of the transient processes via QD coupled to the normal lead\textsuperscript{52} or in hybrid systems with superconductors\textsuperscript{66,73,79}. Time-dependent processes are often investigated numerically, however, in exceptional cases some analytical results can give the deeper insight into considered problem. For instance, WTD in the normal lead–QD–superconducting junction exhibit the coherent oscillations between the empty and doubly occupied QD\textsuperscript{73}. Similarly, some analytical calculations are possible for the energy transport in the polaronic regime described within the FCS method\textsuperscript{59}, for transient dynamics after a quench\textsuperscript{64}, for a phononic heat transport in the transient regime\textsuperscript{65} or for transient heat generation under a step-like bias pulse\textsuperscript{44}.

In this paper we analyze the sub-gap transport properties of a system comprising of a single QD which is tunnel coupled to: one metallic (normal) and two superconducting electrodes, focusing on transient effects driven by abrupt coupling of these constituents. It is natural that oscillations of the transient current would appear as a result of such quench, and the should depend on initial conditions of the system. Such hybrid nanostructures with QD between the normal and superconducting electrodes, reveal many interesting effects with potential applications in nanoelectronics, spintronics or quantum computing\textsuperscript{29,30,42,63,64}. The superconducting reservoir af-
fects the QD via proximity effect, and could be responsible for the Cooper-pair tunneling and Josephson currents, even in absence of any bias voltages. Additional normal electrode coupled to the system allows for good control of the electron transport$^{39,40}$ and could significantly affect the transient phenomena. Our goal is to investigate analytically the time-dependent QD occupation, the currents flowing from the normal and superconducting leads, the induced QD pairing, the conductance and the time evolution of the Andreev bound states (ABS)$^{39,40}$. The formation of ABS signifies that superconducting correlations are induced in the QD via the proximity effect. We investigate appearance in time of these states and study their spin-dependence. To perform analytical time-dependent calculations we assume that superconducting gap of both superconducting leads is the largest energy scale and we put it equal to infinity. Nevertheless, the realistic physics in the Andreev transport regime is still captured in this limit. Knowledge of the analytical formulas allows us to find the answers to such questions as: (i) how do the considered quantities and their characteristics depend on the QD energy levels or the individual coupling of the QD with a given lead, (ii) what is the time period and frequency of these time-dependent quantities, and many related issues. Our investigations allow us also to analyze time evolution of the Andreev bound states and their dependence on the phase difference between the superconducting reservoirs. In our calculations we apply the equation of motion method for the second quantization operators and obtain their analytical form using the Laplace transform technique. Numerical calculations could provide results only for a specific choice of parameters and would not give deep insight into specific dependence of here considered quantities of our system. In this context the analytical calculations are much more general and could have some advantage over numerical data.

The paper is organized as follows. In Sec. III we present our model and discuss the theoretical formalism. The time-dependent QD occupancy is analyzed in Sec. IV whereas Sec. V is devoted to the proximity-induced pairing effects. The normal and superconducting transient currents through the QD are analyzed in Sec. VI and in Sec. VII we discuss the subgap conductance. In the last Sec. VIII we draw the main conclusions of our study.

II. MODEL AND THEORETICAL DESCRIPTION

The system under consideration consists of a QD placed between two superconducting leads (S1 and S2) and one metallic electrode, N, see Fig. 1. The model Hamiltonian for this system can be written in the following form: $H = H_{Sj} + H_{S2} + H_N + H_{QD} + H_{int}$, where $H_{Sj}$ ($j = 1, 2$) describes electrons in the left or right superconducting lead $S_j$, $H_{S2}$ describes the QD, $H_N$ describes the normal lead, $H_{QD}$ describes the QD−QD transitions between external leads and the QD are established by the tunnel Hamiltonian:

$$H_{int} = \sum_{k, \sigma} V_{k\sigma} c_{k\sigma}^+ c_{\sigma} + \sum_{j=1,2} \sum_{\sigma} V_{q_j\sigma} c_{q_j\sigma}^+ c_{\sigma} + H.c.$$ (2)

We assume that the electron dispersion in all leads is spin-independent and impose the order parameters, $\Delta_j$, of the superconducting leads to be phase-dependent, $\Delta_j = |\Delta_j| \exp(i\varphi_j)$. In our notation $k$ ($q_j$) shall denote itinerant states of the normal (superconducting) lead. Correlations are neglected in our calculations.

We are going to study time-response of this system on abrupt switching of the coupling parameters. We shall thus calculate the time-dependent QD occupations, $n_\sigma(t)$ and the currents flowing from the leads, $j_N\sigma(t)$, $j_{Sj}\sigma(t)$. Additionally we will compute $\langle c_1(t)c_1(t) \rangle$, which characterized the electron pairing induced at QD via proximity effect. In what follows we assume that all couplings between the QD and the leads are suddenly switched on at $t = 0^+$ (for $t \leq 0$ the QD is decoupled from the leads). The time evolution of the considered quantities for $t > 0$ depends on the initial QD filling and the chemical potentials. As time goes to infinity, we reproduce the stationary limit results known from the corresponding system. In this paper we use the Laplace transform method and our strategy in the calculations is as follows: we construct the closed set of the equation of motion for creation and annihilation operators (in the Heisenberg representation) $c_{\sigma}(t)$, $\bar{c}_{\sigma}(t)$, $c_{q_j\sigma}(t)$, $\bar{c}_{q_j\sigma}(t)$, $c_{\sigma}^+(t)$, $\bar{c}_{\sigma}(t)$, using the Laplace transformations for these differential equations we obtain the set of coupled algebraic forms $e(s) = \int_0^\infty dt e^{-st} e(t)dt$ for all considered operators. For instance, the QD occupation $n_\sigma(t)$ can be found from the relation

$$n_\sigma(t) = \langle \mathcal{L}^{-1}\{c_\sigma^+(s)\}(t) \cdot \mathcal{L}^{-1}\{c_\sigma(s)\}(t) \rangle$$ (3)
where \( \mathcal{L}^{-1}\{a(s)\}(t) \) stands for the inverse Laplace transform of \( a(s) \) and \((\ldots)\) is the statistical averaging.

Let us find the Laplace transforms of operators \( c_r(t) \) and \( c_{q_r}(t) \) which are required to calculate the QD occupancy \( \langle c_r(t)c_r(t) \rangle \equiv n_r(t) \), the QD induced pairing \( \langle c_\up(t)c_\down(t) \rangle \) and the currents flowing from the leads. We write the Laplace transformed equations of motions for the closed set of twelve operators (in the Heisenberg representation): \( c_\up, c_\down, c_{k\up}, c_{k\down}, c_{q_\up}, c_{q_\down}, c_{q_{1\up}}, c_{q_{1\down}}, c_{q_{2\up}}, c_{q_{2\down}}, \)

\[ (s + i\varepsilon) c_\up = -i \sum_{r=q_{1\up}, q_{2\up}} V_r c_r(s) + c_\up(0), \] \[ (s + i\varepsilon) c_{q_\up} = -i V_{q_\up} c_\up(s) - i\Delta_j c_{q_{2\down}}(s) + c_{q_{1\down}}(0), \] \[ (s - i\varepsilon) c_{q_{1\down}} = i V_{q_{1\down}} c_{q_\up}(s) - i\Delta_j c_{q_{2\down}}(s) + c_{q_{1\down}}(0), \] \[ (s + i\varepsilon) c_{q_{2\down}} = -i V_{q_{2\down}} c_\up(s) + c_{q_{2\down}}(0), \] \[ (s - i\varepsilon) c_{q_{2\down}} = i \sum_{r=q_{1\up}, q_{2\up}} V_r c_r(s) + c_{q_{2\down}}(0), \] \[ (s - i\varepsilon) c_{q_{1\down}} = i V_{q_{1\down}} c_{q_\up}(s) - i\Delta_j c_{q_{1\down}}(s) + c_{q_{1\down}}(0), \] \[ (s + i\varepsilon) c_{q_{1\down}} = -i V_{q_{1\down}} c_\up(s) - i\Delta_j c_{q_{2\down}}(s) + c_{q_{1\down}}(0), \] \[ (s - i\varepsilon) c_{q_{2\down}} = i V_{q_{1\down}} c_{q_\up}(s) + c_{q_{2\down}}(0). \]

From Eqs. 4a and Eqs. 5a we get

\[ c_\up(s)M_\up(s) \equiv A(s) - iK(s)c_\up(0), \]

\[ c_{q_\up}(s)M_{q_\up}(s) \equiv B(s) - iK(s)c_{q_\down}(s), \]

where

\[ K(s) = \sum_{j=1,2} \frac{V_{q_j} \Delta_j}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2}, \]

\[ A(s) = -\sum_{j=1,2} \frac{V_{q_j} \left( \Delta_j c_{q_{1\down}}(0) + i(s - i\varepsilon) c_{q_{1\down}}(0) \right)}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2}, \]

\[ -i \sum_k \frac{V_k c_k(0)}{s - \varepsilon_k} + c_\up(0), \]

\[ B(s) = \sum_{j=1,2} \frac{V_{q_j} \left( \Delta_j c_{q_{1\down}}(0) + i(s + i\varepsilon) c_{q_{1\down}}(0) \right)}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2}, \]

\[ + \frac{1}{s - \varepsilon_k} c_{q_{1\down}}(0) \]

\[ + \frac{1}{s - \varepsilon_k} c_{q_{2\down}}(0), \]

\[ M_\up^{(+/-)}(s) = s \pm i\varepsilon_\sigma + \sum_{j=1,2} \frac{V_{q_j}^2 (s \mp i\varepsilon_{q_j})}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2} + \sum_k \frac{V_k^2}{s \pm \varepsilon_k}. \]

Solving Eqs. 6a, 6b, we obtain for \( c_\up(s) \)

\[ c_\up(s) = \frac{M_\up^{(+)}(s)A(s) - iK(s)B(s)}{M_\up^{(+)}(s)M_\up^{(-)}(s) + K(s)K^*(s)}. \]

Repeating the same procedure to the set of operators: \( c_\down, c_{q_\down}, c_{q_{1\down}}, c_{q_{2\down}}, \) and \( c_{q_{2\down}} \) one can get

\[ c_{q_\down}(s) = \frac{M_\down^{(+)}(s)B^*(s) + iK(s)A^*(s)}{M_\down^{(+)}(s)M_\down^{(-)}(s) + K(s)K^*(s)}. \]

Laplace transforms of \( c_{q_\down}^{(+)} \) and \( c_{q_{2\down}}^{(+)} \) can be obtained, taking the hermitian conjugation of \( c_\up \) and \( c_\down \), respectively.

In the wide-band limit approximation and for \( |\Delta_j| = \infty \) the functions \( M_\up^{(+/-)}(s) \) and \( K(s) \) can be expressed in the following analytical forms: \( M_\up^{(+/-)}(s) = s \pm i\varepsilon_\sigma + \Gamma_N/2 \) and \( K(s) = \left( \Gamma_{S_1} e^{i\varphi_1} + \Gamma_{S_2} e^{i\varphi_2} \right)/2 \). Here we have assumed \( \Gamma_{S_1}/\Gamma_{S_2} = 2\pi \sum_{k/q} V_k^2/\varepsilon_\sigma \delta(e - e_{k/q}) \) and \( \varepsilon_\sigma = \varepsilon_k, \varepsilon_{q_\down} = \varepsilon_{q_{1\down}} - \varepsilon_\sigma = \varepsilon_{q_{2\down}}. \) As an example, let us present explicit form of the Laplace transform for \( c_\up(t) \)

\[ c_\up(s) = \frac{1}{(s - s_3)(s - s_4)} \left\{ \left( s - i\varepsilon_\down + \frac{\Gamma_N}{2} \right) \right. \]

\[ - \sum_{j=1,2} \frac{i V_{q_j} (s - i\varepsilon_j) c_{q_j}(0) + V_{q_j} \Delta_j c_{q_{2\down}}(0)}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2} \]

\[ - \frac{i}{2} \left( \Gamma_{S_1} e^{i\varphi_1} + \Gamma_{S_2} e^{i\varphi_2} \right) \left[ c_{q_\up}(0) + i \sum_k \frac{V_k c_{q_k}(0)}{s - \varepsilon_k} \right] \]

\[ + \sum_{j=1,2} \frac{i V_{q_j} (s + i\varepsilon_j) c_{q_\down}(0) + V_{q_j} \Delta_j c_{q_{1\down}}(0)}{s^2 + \varepsilon_{q_j}^2 + |\Delta_j|^2} \right\}, \]

where

\[ s_{3,4} = \frac{1}{2} \left[ -i(\varepsilon_\up - \varepsilon_\down) - \Gamma_N \pm i\delta \right] \]

\[ \delta = (\varepsilon_\up + \varepsilon_\down)^2 + \Gamma_1 \] and \( \Gamma_1 = \Gamma_{S_1}^2 + \Gamma_{S_2}^2 + 2\Gamma_{S_1} \Gamma_{S_2} \cos(\varphi_1 - \varphi_2). \)

Note, that in the formula 13 there appears the finite superconducting energy gap \( \Delta_j \). The limit \( |\Delta_j| = \infty \) will be imposed later on, when computing the expectation values of the product of two corresponding operators, e.g. \( \langle c_\up(t)c_\down(t) \rangle \) or \( \langle c_{q_\down}(t)c_{q_\down}(t) \rangle \). Additionally, expression for \( c_{q_\down}(s) \) needed for calculations of the currents flowing between the QD and the superconducting leads can be obtained from Eqs. 4b, 5b, 11, 12 and it reads

\[ c_{q_\down}(s) = \frac{1}{s^2 + \varepsilon_{q_\down}^2 + |\Delta_j|^2} \left\{ (s - i\varepsilon_{q_\down})(c_{q_\down}(0) - iV_{q_\down} c_\down(s)) \right. \]

\[ + \alpha V_{q_j} \Delta_j c_{q_{1\down}}(s) - i\alpha \Delta_j c_{q_{2\down}}(s) \right\}, \]
where \( \alpha = +(-) \) for \( \sigma = \uparrow(\downarrow) \). Using these formulas for \( c_{\sigma}(s) \) and \( c_{\bar{\sigma}}(s) \) we can analytically determine the QD occupancy, pairing parameter, subgap currents and its differential conductance.

In the following we set \( e = \hbar = k_B = 1 \) and make use of the wide-band limit approximation. All numerical calculations shall be performed for \( \Gamma_S = \Gamma_{S_2} = \Gamma_S \) and \( \mu_N = 0 \), unless stated otherwise. The energies, currents and time are expressed in units of \( \Gamma_S, \varepsilon \Gamma_S/\hbar \) and \( \hbar/\Gamma_S \), respectively. We assume the chemical potentials of superconducting leads \( \mu_{S_1} = \mu_{S_2} = 0 \) to be grounded. For experimentally available values of \( \Gamma_S, \Gamma_S \sim 200 \mu eV \) the typical time and current units would be \( \sim 3.3 p s e c \) and \( \sim 48 n A \), respectively.

### III. Quantum Dot Occupancy

Let us consider the time-dependent QD occupancy after abrupt coupling (at \( t = 0^+ \)) to the normal and superconducting electrodes. We assume no bias voltage between electrodes and make use of the wide band limit approximation and impose \( \Delta_j = \infty \). Under these assumptions the QD occupation, \( n_{\sigma}(t) \), reads (cf. \ref{eq:quantumDotOccupancy}) for N-QD-S and \ref{eq:quantumDotOccupancy} for N-QD-N systems:

\[
n_{\sigma}(t) = \mathcal{L}^{-1} \left\{ \frac{s + i \varepsilon_{\sigma -} + \Gamma_N/2}{(s - s_1)(s - s_2)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{s - i \varepsilon_{\sigma -} + \Gamma_N/2}{(s - s_3)(s - s_4)} \right\}(t) n_{\sigma}(0) \\
+ \frac{\Gamma_{12}}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)} \right\} \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)} \right\}(t)(1 - n_{-\sigma}(0)) \\
+ \sum_{k_1,k_2} V_{k_1} V_{k_2} \mathcal{L}^{-1} \left\{ \frac{s + i \varepsilon_{\sigma -} + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i \varepsilon_{k_1})} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{s - i \varepsilon_{\sigma -} + \Gamma_N/2}{(s - s_3)(s - s_4)(s + i \varepsilon_{k_2})} \right\}(t) \langle c^+_{k_1 \sigma}(0) c_{k_2 \sigma}(0) \rangle \\
+ \frac{\Gamma_{12}}{4} \sum_{k_1,k_2} V_{k_1} V_{k_2} \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)(s + i \varepsilon_{k_1})} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i \varepsilon_{k_2})} \right\}(t) \langle c_{k_1 - \sigma}(0) c^{+}_{k_2 - \sigma}(0) \rangle ,
\]

where \( s_{1,2} = \frac{1}{2} \left[ \varepsilon_{\uparrow} - \varepsilon_{\downarrow} \right] - \Gamma_N \pm i \sqrt{\delta} \), and for \( \sigma = \downarrow \) one should replace \( s_{1,2} \leftrightarrow s_{3,4} \), respectively. The first two terms describe the transient QD charge oscillations which depend on the initial QD occupations. The last two terms (with the sums over \( k \)) are related to the normal lead and they give non-vanishing and non-oscillating contribution to \( n_{\sigma}(t) \), regardless of the initial conditions. Note that in Eq. \ref{eq:quantumDotOccupancy} the terms involving the expectation values of the product of electron annihilation and creation operators \( c_{\sigma}(s) \) and \( c^+_{\bar{\sigma}}(s) \) of the superconducting lead electrons do not appear. Such terms take e.g. the following integral form (cf. \ref{eq:quantumDotOccupancy}):

\[
\frac{\Gamma_S}{2\pi} \int_{-\infty}^{+\infty} d\varepsilon f_S(\varepsilon) \mathcal{L}^{-1} \left\{ \frac{\left( s + i \varepsilon_{\downarrow} + \Gamma_{\Delta} \right) \left( s + i \varepsilon \right)}{(s - s_1)(s - s_2)(s^2 + \varepsilon^2 + |\Delta_j|^2)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{\left( s - i \varepsilon_{\downarrow} + \Gamma_{\Delta} \right) \left( s - i \varepsilon \right)}{(s - s_3)(s - s_4)(s^2 + \varepsilon^2 + |\Delta_j|^2)} \right\}(t) , \tag{16}
\]

where \( f_S(\varepsilon) \) is the Fermi distribution function. It is easy to check numerically that the above integral over the energy is smaller and smaller with increasing \( |\Delta_j| \). Thus in our calculations for \( |\Delta_j| = \infty \) we can neglect all terms involving operators \( \hat{c}_{\mathbf{q} \sigma}(0) \). The formula \ref{eq:quantumDotOccupancy} can be further elaborated and after some algebra one rewrites the two first terms explicitly while the third and fourth terms can be expressed by integrals over the energy in the normal lead spectrum

\[
n_{\sigma}(t) = e^{-\Gamma_N t} \left[ n_{\sigma}(0) + (1 - n_{\sigma}(0) - n_{-\sigma}(0)) \sin^2 \left( \frac{\sqrt{\delta} t}{2} \right) \right] \mathcal{L}^{-1} \left\{ \frac{\Gamma_{12}}{\delta} \right\} \\
+ \frac{\Gamma_N}{2\pi} \int_{-\infty}^{+\infty} d\varepsilon f_N(\varepsilon) \mathcal{L}^{-1} \left\{ \frac{\left( s - i \varepsilon - \sigma + \Gamma_N/2 \right) \left( s + i \varepsilon \right)}{(s - s_1)(s - s_2)(s - i \varepsilon)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{\left( s + i \varepsilon - \sigma + \Gamma_N/2 \right) \left( s - i \varepsilon \right)}{(s - s_3)(s - s_4)(s + i \varepsilon)} \right\}(t) \\
+ \frac{\Gamma_N}{8\pi} \Gamma_{12} \int_{-\infty}^{+\infty} d\varepsilon (1 - f_N(\varepsilon)) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)(s + i \varepsilon)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i \varepsilon)} \right\}(t) .
\]

Here \( f_N(\varepsilon) \) is the electron Fermi distribution function for the normal lead and for \( \sigma = \downarrow \) the replacement \( (s_1,s_2) \leftrightarrow (s_3,s_4) \) should be done. The phase difference \( \phi \) enters Eq. \ref{eq:quantumDotOccupancy} only through the function \( \cos \phi \), therefore the QD occupancy satisfies the symmetry relation \( n_{\sigma}(\phi) = n_{\sigma}(\phi + 2\pi) \). Note that the part which depends on the initial QD filling oscillates with the period \( 2\pi/\sqrt{\delta} \). These oscillations depend on the QD electron energies, \( \varepsilon_{\uparrow} + \varepsilon_{\downarrow} \), both couplings \( \Gamma_{S_1}, \Gamma_{S_2} \) and the phase difference \( \phi \) of the superconducting order parameters, \( \phi = \varphi_1 - \varphi_2 \).
The oscillations are damped due to the exponential factor $e^{-\Gamma_N t}$ and in the asymptotic time limit the information about the initial QD occupation is entirely washed out. From Eq. 17 we infer that, when QD is coupled only to the superconducting leads and the initial conditions are $n_\sigma(0) = (1, 0)$ or $(0, 1)$, the time-dependent QD occupancy does not change at all (independently of $\phi$ and $\Gamma_{S_1/2}$). In this case the QD is occupied only by one electron which cannot be exchanged with the superconducting reservoirs due to the infinity large energy gaps. For the initial conditions $n_\sigma(0) = (1, 1)$ or $(0, 0)$ the oscillations of the QD occupancy oscillates with the time period $T = \frac{\pi}{\Delta}$ for $\phi \neq \pi$ independently of $\Gamma_{S_1/2}$ or for $\phi = \pi, \Gamma_{S_1} \neq \Gamma_{S_2}$. These oscillations, however, disappear for $\phi = \pi$ and $\Gamma_{S_1} = \Gamma_{S_2}$ as shown in Fig. 2.

The formula (17) for $\Gamma_N = 0$ resembles the Rabbi oscillations of a typical two-level quantum system described by the effective Hamiltonian $H_{eff} = \frac{1}{2} \left( \Gamma_{S_1} e^{i \phi_1} + \Gamma_{S_2} e^{i \phi_2} \right) c_\uparrow c_\downarrow + h.c. + \sum_\sigma \varepsilon_\sigma n_\sigma$. Assuming at $t = 0$ the QD is empty, $n_\sigma(0) = 0$, we can calculate the probability $P(t)$ of finding the QD in the doubly occupied configuration, $n_\uparrow = n_\downarrow = 1$. Within the standard treatment of a two-level system we have:

$$P(t) = \frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2} \sin^2 \left( \frac{\sqrt{\Gamma_{12} + (E_1 - E_2)^2} t}{2} \right),$$

where $E_1 = 0$ and $E_2 = \varepsilon_\uparrow + \varepsilon_\downarrow$ are energies of the empty and double occupied configurations, respectively. This formula can be rewritten as $P(t) = \frac{\Gamma_{12}}{\Gamma_{12} + (E_1 - E_2)^2} \sin^2 \left( \frac{\sqrt{\Gamma_{12} + (E_1 - E_2)^2} t}{2} \right)$ and becomes identical with our expression (17) obtained for $n_\sigma(0) = 0$, $\Gamma_N = 0$.

To illustrate such analytical results and to reveal influence of the phase difference of two superconducting leads on the QD occupation in Fig. 2 we show $n_\uparrow(t)$ and $n_\downarrow(t)$ with respect to time and $\phi$ for $\varepsilon_\sigma = 0$ (upper panel) and for the Zeeman splitting $\varepsilon_\sigma = -\varepsilon_\sigma = 0.5$ (bottom panel). We consider here the symmetric coupling $\Gamma_{S_1} = \Gamma_{S_1} = \Gamma_S$ and assume the initial conditions $n_\sigma(0) = (0, 0)$. Note that for $\varepsilon_\sigma = 0$ the QD occupancy becomes spin-independent, i.e. $n_\sigma(t) = n_{\sigma}(t)$ (see Eq. 17). For $t \to \infty$ it always tends to 0.5, regardless of the superconducting phase difference. In absence of any phase–difference we observe the oscillations of $n_\sigma(t)$ with the period $T = \pi/\Gamma_S$ which are damped according to the exponential function $e^{-\Gamma_N t}$. Notice, that period of these oscillations is twice shorter compared to the oscillations in the N-QD-S system. For $\phi \neq 0$ these oscillations are characterized by the phase-dependent period $T = \pi/\Gamma_S \cos(\phi/2)$]. For the special case $\phi = \pi$ ($\Gamma_{12} = 0$) the oscillations disappear and the QD charge develops in time exactly in the same way as for the QD coupled only to the normal lead (with $\varepsilon_\sigma = 0$), e.g.,

$$n_\sigma(t) = n_\sigma(0)e^{-\Gamma_N t}$$

$$+ \frac{\Gamma_N}{\pi} e^{-\Gamma_N t/2} \int_{-\infty}^{+\infty} d\varepsilon f_N(\varepsilon) \cos(\Gamma_N t/2) \cos(\varepsilon t) \left( \frac{\Gamma_N/2}{\varepsilon^2 + \Gamma_N^2} \right).$$

For $\mu = 0$ and the zero temperature case we obtain $n_\sigma(t) = \frac{1}{2} + e^{-\Gamma_N t} \left( n_\sigma(0) - \frac{1}{2} \right)$. It means that for $n_\sigma(0) = (0, 0)$ or $(1, 1)$ the QD occupation increases or decreases monotonically in time without any oscillations, changing from zero (one) to 0.5 (Fig. 2 upper panel).

The situation changes in the presence of the Zeeman splitting (bottom panel). For symmetric splitting of around $\mu \Gamma_N = 0$, $\varepsilon_\uparrow = -\varepsilon_\downarrow$, the first term of Eq. (17) depends only on the phase difference $\phi$ and $\Gamma_N$. Its contribution to the final QD occupancy is the same for arbitrary values of $\varepsilon_\sigma$. On the other hand the two last terms in Eq. (17) depend separately on $\varepsilon_\sigma$. For $\phi = \pi$ the contribution from these terms is identical with the case of the QD coupled only to the normal lead. For $t = 40$ and $\varepsilon_\uparrow = -\varepsilon_\downarrow = 0.5$ (bottom panel in Fig. 2), the contribution from $\uparrow$ ($\downarrow$) is $\sim 0.03$ ($\sim 0.95$). For $\phi = 0$ such contributions become $\sim 0.49$ and $\sim 0.51$, respectively. One can thus control the QD occupancy by means of the phase difference parameter $\phi$.

Let us analyze more carefully variation of the QD occupancy against the phase difference $\phi$. In Fig. 3 (upper panel) we present the ABS energies of the proximitized QD, $E_{\alpha\beta} = E_{\alpha} - \varepsilon_\beta$, ($\alpha = \pm, \beta = \pm \uparrow / \downarrow$), where $E_{\alpha} = \frac{1}{2}(\varepsilon_\uparrow + \varepsilon_\downarrow) + \alpha \sqrt{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + 4 \Gamma_S^2 \cos^2 \frac{\phi}{2}}$ is the quasiparticle energy representing a superposition of the empty and double occupied states. In the lower panel...
we show the QD occupancies \( n_{\uparrow}(t), n_{\downarrow}(t) \) and the difference \( n_{\downarrow}(t) - n_{\uparrow}(t) \) for \( \Gamma_N = 0.02 \) obtained for particular times \( t \). QD occupancy rapidly changes for such values of \( \phi \) which satisfy the relation \( E_{s+} = E_{s-} \), i.e. for \( \phi = \pi \pm \arccos \frac{\varepsilon_{\uparrow}}{\varepsilon_{\downarrow}} \) (here \( \varepsilon_{\uparrow}, \varepsilon_{\downarrow} > 0 \)). Exactly for such values of \( \phi \) we observe the transition \( 0 - \pi \) in our system. This transition is clearly visible in the long time (steady) limit. In our case for \( \Gamma_N = 0.02 \) this time equals \( 200 \text{ u.t.} \) (approximately equal to \( \frac{1}{2\Gamma_N} \)). For greater \( \Gamma_N, \Gamma_N = 0.1 \), such transition is observed (although it is smeared around \( \phi = \pi \pm \pi/3 \)) already for \( t = 40 \text{ u.t.} \), see the lower panel in Fig. 2. At very early stage of the time evolution such \( 0 - \pi \) transition is only weakly manifested by the time-dependent magnetization \( n_{\downarrow}(t) - n_{\uparrow}(t) \). On the other hand, oscillations of the QD occupancies hardly detect existence of this transition. However, already for \( t \approx \frac{1}{\Gamma_N} = 50 \text{ u.t.} \) this transition is well marked on the occupancy curves as well as on \( n_{\downarrow}(t) - n_{\uparrow}(t) \). Notice the decreasing amplitude and increasing frequency of the QD occupancies versus time. These transient characteristics are described by the factor \( \sin^2(2\Gamma_N t \cos(\phi/2)/|\phi|) e^{-\Gamma_N t} \), see the first term of Eq. (17). Let us emphasize, that despite oscillatory character of \( n_{\sigma}(t) \), the resulting magnetization \( n_{\downarrow}(t) - n_{\uparrow}(t) \) is a smooth function of \( \phi \).

### IV. INDUCED ON-DOT PAIRING

We shall now calculate the pairing amplitude \( \chi(t) = \langle c_{\uparrow}(t)c_{\downarrow}(t) \rangle \) driven by the proximity effect, assuming absence of any bias voltage \( \mu_N = 0 \). Using the expressions for \( c_{\uparrow}(s) \) and \( c_{\downarrow}(s) \) obtained in Sec. II we find

\[
\chi(t) = -\frac{i}{2} (\Gamma_{S1} e^{i\phi_1} + \Gamma_{S2} e^{i\phi_2}) \times (20)
\]

\[
\frac{1}{(s-s_1)(s-s_2)} (t)
\]

\[
\frac{1}{(s-s_3)(s-s_4)} (t) + \frac{\Gamma_N}{2\pi} \Phi^* \]

where

\[
\Phi_{\sigma} = \int_{-\infty}^{\infty} d\varepsilon L^{-1} \left\{ \frac{1}{(s-s_1)(s-s_2)(s+i\varepsilon)} \right\} (t) \]

\[
L^{-1} \left\{ \frac{s+i\varepsilon+\Gamma_N/2}{(s-s_3)(s-s_4)(s-\i\varepsilon)} \right\} (t) (1-f_N(\varepsilon))
\]

\[-\int_{-\infty}^{\infty} d\varepsilon f_N(\varepsilon) L^{-1} \left\{ \frac{s+i\varepsilon+\Gamma_N/2}{s-s_1(s-s_2)(s-\i\varepsilon)} \right\} (t)
\]

\[
L^{-1} \left\{ \frac{1}{(s-s_3)(s-s_4)} (t) \right\}
\]

and the replacement \((s_1,s_2) \to (s_3,s_4)\) should be made for \( \sigma = \downarrow \). The terms proportional to \( n_{\downarrow}(0) \) and \((1-n_{\downarrow}(0))\) in the above relation can be expressed analytically, and

\[
\chi(t) = -\frac{i}{2} \left( \Gamma_{S1} e^{i\phi_1} + \Gamma_{S2} e^{i\phi_2} \right)
\]

\[
\frac{1}{2\pi} \frac{\Gamma_N}{\Phi_{\uparrow}^* + (n_{\downarrow}(0) + n_{\uparrow}(0) - 1)} e^{-\Gamma_N t}
\]

\[
\times \left[ \sqrt{\sin \left( \sqrt{\delta^2} t + i(\varepsilon_{\uparrow} + \varepsilon_{\downarrow}) \cos(\sqrt{\delta} t) \right)} \right] \frac{\sin \left( 2\Gamma_N \cos(\phi/2) t \right)}{2\Gamma_N t}
\]

Notice, that for \( \Gamma_{S1} = \Gamma_{S2} = \Gamma_S \) and \( \phi = \pi \) the factor \( \Gamma_{S1} e^{i\phi_1} + \Gamma_{S2} e^{i\phi_2} = 2\Gamma_S e^{i\phi_2} \) vanishes therefore the on-dot pairing \( \langle c_{\downarrow}(t)c_{\downarrow}(t) \rangle \) is absent (see upper and bottom panels in Fig. 2 for \( \phi = \pi \)). However, for \( \phi \neq \pi \) and \( \varepsilon_{\uparrow} + \varepsilon_{\downarrow} = 0 \) we have

\[
\chi(t) = -\frac{i}{2} \left( \frac{\Gamma_N}{\Gamma_S} \cos \left( \frac{\phi}{2} \Phi_{\uparrow}^* + i n_{\downarrow}(0) + n_{\uparrow}(0) - 1 \right) \times e^{-\Gamma_N t} \cos \left( \frac{\phi}{2} t \right) \right)
\]

where we have used the obvious property \( \Re \Phi_{\sigma} = 0 \), corresponding to \( \mu_N = 0 \). The imaginary part of \( \chi(t) \) oscillates with the same period as the QD occupancy, i.e. with \( T = \pi/|\Gamma_S| \cos(\phi/2)| \), but the real part changes monotonically from zero to some constant value without any oscillations (upper panel). We can notice, that
we find we infer that

\[ (25) \]

tion of time and the phase difference \( \phi \) of the corresponding electrode obtained from the evolution of the total number of electrons. Such electron currents can be induced transient currents. Such electron currents can be obtained from the evolution of the total number of electrons of the corresponding electrode. For the normal electrodes, respectively. These currents depend on time due to the abrupt coupling of all parts of the considered system. For \( t > 0 \), even at zero bias voltage, there are induced transient currents. Such electron currents can be obtained from the evolution of the total number of electrons of the corresponding electrode. For the normal

the imaginary part of \( \chi \) vanishes when the QD is filled by a single electron at the initial time \( t = 0 \). On the other hand, the real part of \( \chi \) is a non-vanishing function irrespective of the initial conditions. It is worth mentioning, that for \( \Gamma_N = 0 \) (i.e. Josephson junction setup) and for \( \varepsilon_1 - \varepsilon_2 = 0 \) the expression for \( \chi(t) \) becomes purely imaginary and is characterized by undamped oscillations inducing d.c. current (see the next section V). In general, from the analysis of Eq. \( 24 \) we infer that the QD induced pairing satisfies the symmetry relation \( \chi(t, \phi) = \chi(t, \phi + 4\pi) \). In particular, for \( t = \infty \), it becomes symmetric with respect to \( \phi = 2\pi \).

V. SUBGAP CURRENTS

Let us consider the currents \( j_{N\sigma}(t) \) and \( j_{S\sigma}(t) \) flowing between the QD and the normal or superconducting electrodes, respectively. These currents depend on time due to the abrupt coupling of all parts of the considered system. For \( t > 0 \), even at zero bias voltage, there are induced transient currents. Such electron currents can be obtained from the evolution of the total number of electrons of the corresponding electrode. For the normal

the imaginary part of \( \chi \) vanishes when the QD is filled by a single electron at the initial time \( t = 0 \). On the other hand, the real part of \( \chi \) is a non-vanishing function irrespective of the initial conditions. It is worth mentioning, that for \( \Gamma_N = 0 \) (i.e. Josephson junction setup) and for \( \varepsilon_1 - \varepsilon_2 = 0 \) the expression for \( \chi(t) \) becomes purely imaginary and is characterized by undamped oscillations inducing d.c. current (see the next section V). In general, from the analysis of Eq. \( 24 \) we infer that the QD induced pairing satisfies the symmetry relation \( \chi(t, \phi) = \chi(t, \phi + 4\pi) \). In particular, for \( t = \infty \), it becomes symmetric with respect to \( \phi = 2\pi \).

[FIG. 4. The real (upper panel) and imaginary (bottom panel) parts of the QD induced pairing \( \chi(t) = (c_1(t)c_\sigma(t)) \) as a function of time and the phase difference \( \phi_1 - \phi_2 \).]

\[
j_{N\sigma}(t) = 2\sqrt{2} \sum_k V_{k\sigma} e^{-ik_\sigma t}(c_\sigma^+(t)c_\sigma(0)) - \Gamma_N n_\sigma(t),
\]

where we have assumed the energy-independent normal lead spectrum. Using the formulas of Sec. III we find

\[
j_{N\sigma}(t) = \frac{\Gamma_N}{\pi} \Re \left( \int_{-\infty}^{+\infty} d\varepsilon f_N(\varepsilon) e^{-i\varepsilon t} \right) - \Gamma_N n_\sigma(t).
\]

Inserting the inverse Laplace transform and using the expression for \( n_\sigma(t) \) one can obtain analytical relation for \( j_{N\sigma}(t) \). However, this solution for arbitrary \( t \) cannot be written in relatively compact (or transparent) form,
so we restrict ourselves to the asymptotics \( t = \infty \)

\[
j_{N\sigma} = \frac{\Gamma_N}{\pi} \int d\varepsilon \left\{ f_N(\varepsilon) \left[ \mathcal{R} \left( \frac{i(\varepsilon + \varepsilon_{-\sigma}) + \Gamma_N}{(\Gamma_N/2 + i\varepsilon_{++})(\Gamma_N/2 + i\varepsilon_{+-})} \right) \right. \\
- \frac{\Gamma_N}{2} \left( (\varepsilon + \varepsilon_{-\sigma})^2 + \frac{\Gamma_N^2}{4} \right) \left( \frac{\Gamma_N}{4} + \varepsilon_{+-} \right) \right\} \frac{\Gamma_N \Gamma_{12}}{8 \left( \frac{\Gamma_N}{4} + \varepsilon_{+-}^2 \right) \left( \frac{\Gamma_N}{4} + \varepsilon_{++}^2 \right)} \left( 1 - f_N(\varepsilon) \right),
\]

(26)

where \( \varepsilon_{\alpha\beta} = \varepsilon + E_{\alpha\beta} \) and \( E_{\alpha\beta} \) are the quasiparticle energies of the proximitized QD.

Fig. 3 (upper panel) shows the time-dependent current flowing from the normal lead to the QD as function of the phase difference \( \phi = \varphi_1 - \varphi_2 \) obtained for the unbiased system \( \varepsilon_{\sigma} = 0 \). At a beginning the current starts to flow from the normal electrode to the empty QD. In a next stage, electrons tunnel in both directions with the characteristic oscillations. These damped oscillations are clearly visible and for \( t \to \infty \) the current vanishes for all \( \phi \). The period of these oscillations increases with \( \phi \), similarly to the behavior observed for the QD occupancy. Exceptionally, for \( \phi = \pi \), the current tends to its asymptotic value without any oscillations according to the formula (valid for the zero temperature, \( \varepsilon_{\sigma} = 0 \) and \( \Gamma_S = \Gamma_S^2 \)):

\[
j_{N\sigma}(t) = \Gamma_N e^{-\Gamma_N t} \left( \frac{1}{2} - n_{\sigma}(0) \right).
\]

(27)

We can notice, that right after the abrupt coupling (at \( t = 0^+ \)) the large value of transient current \( j_{N\sigma} \) is induced in the system \((\sim \Gamma_N^2)\) which is artifact of the WBL approximation. We have checked that by applying a more realistic (smooth) QD-Leads coupling profile the initial current would gradually increase, revealing the same period of oscillations and other overall features.

The situation looks a bit different for the currents, flowing between the QD and superconducting leads. To calculate these currents we start from the standard formula \( j_{S\sigma}(t) = 2\sum_{\alpha} \langle \delta (\varepsilon) V_{\alpha\beta}(c_{\alpha\uparrow}^\dagger(t)c_{\beta\downarrow}(t)) \rangle \) and use the Laplace transforms for \( c_{\alpha\uparrow}^\dagger(s) \) and \( c_{\beta\downarrow}(s) \), obtaining

\[
j_{S\sigma}(t) = \frac{\Gamma_S}{2} e^{-\Gamma_N t} \left[ 1 - n_{\sigma}(0) - n_{-\sigma}(0) \right] \\
\cdot \cos(\phi/2) \sin(2\Gamma_S|\cos(\phi/2)|t) \\
+ \frac{\Gamma_N \Gamma_S^2}{2\pi} \cos^2(\phi/2) \mathcal{R} \{ \Phi_{\sigma} \} - \frac{\Gamma_N \Gamma_S^2}{4\pi} \sin \phi \mathcal{R} \{ \Phi_{\sigma} \}.
\]

(28)

As usually, the replacement \( (s_1, s_2) \leftrightarrow (s_3, s_4) \) should be made for \( \sigma = \downarrow \). After straightforward algebra we can derive more explicit form for the superconducting current

\[
j_{S\sigma}(t) = \frac{\Gamma_{S\sigma}}{2} (1 - n_{\sigma}(0) - n_{-\sigma}(0)) e^{-\Gamma_N t} \\
\left[ (\Gamma_{S\uparrow\downarrow} \cos \phi + \Gamma_{S\downarrow\uparrow}) \frac{\sqrt{6}}{\sqrt{2}} \sin(\sqrt{6}t) \right] \\
+ \frac{\Gamma_N \Gamma_{S\sigma}}{4\pi} \mathcal{R} \{ \Gamma_{S\uparrow\downarrow} + \Gamma_{S\downarrow\uparrow} \mathcal{C} \}.
\]

(29)

Using the relation for the induced pairing, Eq. 20 the above current can be recast as \( j_{S\sigma}(t) = \mathcal{R} \{ \langle c_{\sigma}(t)c_{\sigma}(t) \rangle \} \sin(\phi - \varphi_d) \), where \( \varphi_d \) is the argument (phase) of \( \langle c_{\sigma}(t)c_{\sigma}(t) \rangle \), we obtain (e.g. 21):

\[
j_{S\sigma}(t) = \Gamma_S \langle c_{\sigma}(t)c_{\sigma}(t) \rangle \sin(\varphi_d - \varphi_d),
\]

(30)

where \( j = 1, 2 \). Inspecting \( \mathcal{R} \{ \langle c_{\sigma}(t)c_{\sigma}(t) \rangle \} \) we conclude that the currents flowing between the QD and a given superconducting lead does not depend on spin, \( j_{S\sigma}(t) = j_{S\downarrow\sigma}(t) \), irrespective of the spin dependent QD energy levels. This is a consequence of the fact that the QD can exchange charge with the superconducting leads only vi pairs of opposite spin electrons.

This formula simplifies for the case \( \Gamma_S = \Gamma_S = \Gamma_S \equiv \Gamma_S \) and \( \varepsilon_{\uparrow} = \varepsilon_{\downarrow} = 0 \), when we obtain

\[
j_{S\sigma}(t) = \frac{\Gamma_S}{2} e^{-\Gamma_N t} \left[ 1 - n_{\sigma}(0) - n_{-\sigma}(0) \right] \\
\cdot \cos(\phi/2) \sin(2\Gamma_S|\cos(\phi/2)|t) \\
+ \frac{\Gamma_N \Gamma_S^2}{2\pi} \cos^2(\phi/2) \mathcal{R} \{ \Phi_{\sigma} \} - \frac{\Gamma_N \Gamma_S^2}{4\pi} \sin \phi \mathcal{R} \{ \Phi_{\sigma} \}.
\]

(31)

For \( \mu_N = 0 \) the real part of \( \Phi_{\sigma} \), \( \mathcal{R} \{ \Phi_{\sigma} \} \), vanishes and in such case for \( \phi = \pi \) and identical couplings to both superconducting leads the currents \( j_{S\sigma}(t) \) vanish.

Under non-equilibrium conditions \( \mu_N \neq 0 \), for symmetric couplings and for \( \varepsilon_{\sigma} = -\varepsilon_{\sigma} \), the asymptotic (for \( t \to \infty \)) value of the superconducting current can be ex-
pressed as
\[
    j_{S_1} = \frac{\Gamma_2 \Gamma_S}{4\pi} \left\{ \int \frac{(1 - f_N(\varepsilon))d\varepsilon}{(\Gamma_N^2/4 + \varepsilon^2_+)} \right. \\
    - \int \frac{f_N(\varepsilon)d\varepsilon}{(\Gamma_N^2/4 + \varepsilon^2_+)} \left. \right\} \cos^2 \left( \frac{\phi}{2} \right)
\]
\[
    - \frac{\Gamma_N \Gamma_2^2}{4\pi} \left\{ \int \frac{(1 - f_N(\varepsilon))(\varepsilon + \varepsilon_\uparrow)d\varepsilon}{(\Gamma_N^2/4 + \varepsilon^2_+)} \right. \\
    - \int \frac{f_N(\varepsilon)(\varepsilon - \varepsilon_\uparrow)d\varepsilon}{(\Gamma_N^2/4 + \varepsilon^2_+)} \left. \right\} \sin \phi
\]
where $\varepsilon_{\alpha\beta} = \varepsilon + E_{\alpha\beta}$ and $E_{\alpha\beta}$ denote quasiparticle energies of the proximitized QD. Notice, that the first term in the above formula vanishes for zero temperature and $\mu_N = 0$. In this case the superconducting current can be rewritten to the following (Josephson-like) formula
\[
    js_1 = \frac{\Gamma_S}{4\pi} \sin \frac{\phi}{2} \left| \cos \frac{\phi}{2} \right|
\]
\[
    \left[ \frac{\varepsilon^2_\uparrow - \Gamma_2^2 - \Gamma_S^2 \cos^2 \left( \frac{\phi}{2} \right)}{\Gamma_2 \Gamma_S N_1 \cos \left( \frac{\phi}{2} \right)} \right] - \frac{\pi}{2} \right].
\]
Let us remark that the formula for the current, Eq. 33, can be used to determine the coupling value $\Gamma_S$. As the time-oscillations are described by the first term of Eq. 31 then for a given $\phi$ the oscillating part of $js_1(t)$ is proportional to $\sin(2\Gamma_S \cos(\phi/2)/t)$. The period of these oscillations $T = \frac{\pi}{\Gamma S \cos(\phi/2)}$ for the system characterized by a sufficiently small $\Gamma_S$ and $\phi \approx \pi$ should be experimentally detectable.

Lower panel in Fig. 5 presents the current $j_{S_1}(t)$ as a function of $\phi$ for $n_\uparrow(0) = n_\downarrow(0) = 0$. The current oscillates with a damping amplitude and for large time it tends to a non-zero asymptotic value given in Eq. 33. The asymptotic value of the current does not depend on the initial QD occupancies, see Eq. 29. However, the transient currents are different for the QD initial occupancies, $n_\sigma(0) = (0, 0), (1, 1)$ and for $n_\sigma(0) = (0, 1), (1, 0)$. In the first case the current indicates rather rich time-dependent structure before it attains the asymptotic value. This is a consequence of the Rabi-like oscillations (damped via $e^{-\Gamma_N t}$ due to the coupling with normal lead) between the empty and double occupied QD configurations and is described by the first term of Eq. 29 which depends on the factor $(1 - n_\sigma(0) - n_{-\sigma}(0))$. This factor disappears for the initial occupancies $n_\sigma(0) = (0, 1)$ or $(1, 0)$ and all time-dependence of $j_{S_1}(t)$ is described by the last term of Eq. 29. This term, however, in contrast to the former case does not introduce any visible oscillations for small $\Gamma_N$ but enters the formulas for $j_{S_1}$, irrespective of the initial conditions. From Fig. 5 we can learn that at short time after the quench the current is symmetric with respect to $\phi = \pi$. This symmetry, however, is quickly lost and in the long time scale.

\[
    \text{FIG. 6. Time dependent currents flowing between the QD and the superconducting leads, } j_{S_1}(t), j_{S_2}(t) \text{ as a function of the phase difference } \varphi_1 - \varphi_2 \text{ in the presence of the Zeeman splitting } \varepsilon_\uparrow = -\varepsilon_\downarrow = 0.5 \text{ (upper panel). In the bottom panel the asymptotic spin up currents (solid curves) and corresponding QD occupancies (broken curves) obtained for } t = \infty \text{ are shown for the Zeeman splitting: } \varepsilon_\uparrow = -\varepsilon_\downarrow = 0, 0.25, 0.5, 0.75 \text{ and 1.0. The parameters are: } \Gamma_{S_1} = \Gamma_{S_2} = \Gamma_S, \Gamma_N = 0.2 \text{ and } n_\sigma(0) = 0.
\]

In Fig. 6 we present time dependent currents $j_{S_1 \uparrow}$ and $j_{S_2 \uparrow}$ vs. the phase difference $\phi$ for the finite Zeeman splitting of energy levels, $\varepsilon_\uparrow = -\varepsilon_\downarrow = 0.5\Gamma_S$. Both currents oscillate with the period, dependent on the phase difference $\phi$. As before, this period increases with $\phi$ and for $\phi = \pi$ the currents do not flow in the system. Comparing such $\phi$-dependence of the currents with those presented in Fig. 5 (lower panel) for $\varepsilon_\sigma = 0$ we observe a different behavior, especially at asymptotic large time. In presence of the Zeeman splitting the asymptotic currents almost vanish for some $\phi$ interval around $\phi = \pi$. To study this effect in more detail we show in Fig. 6 bottom panel, the superconducting currents for several values of the Zeeman splittings (solid lines for $\varepsilon_\uparrow = -\varepsilon_\downarrow = 0, 0.25, 0.5, 0.75$ and 1.0 (in units of $\Gamma_S$). As one can see, in absence of the Zeeman splitting the current does not flow only for $\phi = 0, \pi$. In presence of the Zeeman term the zero-value superconducting current interval of $\phi$ increases, but at the same time the maximal values of the currents diminish. For $\varepsilon_\uparrow = -\varepsilon_\downarrow \gg 1$ the superconducting currents do not flow. The corresponding asymptotic occupancies
of the QD, \( n_\uparrow(\phi, t = \infty) \) are shown in Fig. 6, bottom panel (broken lines). One can notice, that the occupancies decrease monotonically with \( \phi \) and remain very low for the zero-current interval of \( \phi \). The changes of the QD occupancies in presence of the Zeeman splitting reflect phasal-dependence of the superconducting currents. These changes are related to \( 0 - \pi \) transition and will be discussed in the next paragraph (compare \( \phi \)-dependence of \( n_\uparrow \) for \( \varepsilon_\uparrow = -\varepsilon_\downarrow = 0.5 \) and \( \Gamma_N = 0.1 \) shown in the lower panel of Fig. 3).

In Fig. 7 we analyze the time dependence of \( j_{S_\uparrow}(t) \) for some selected values of time \( t \), starting from the quench \( t = 0 \) until nearly the asymptotically large times. In the lower panel, \( \Gamma_N = 0.02 \), the \( \phi \)-dependence of the current demonstrates abrupt changing of the current value at points corresponding to \( E_{++} = E_{--} \) (see upper panel in Fig. 5). These jumps of the current are clearly visible for large times. However, for larger \( \Gamma_N \), e.g. for \( \Gamma_N = 0.1 \) (upper panel, Fig. 7) the \( \phi \)-dependence of the current even for asymptotically large times does not show such sharp changes. Notice, that the time at which the current achieves constant (in time) values is much shorter in comparison to the case of \( \Gamma_N = 0.02 \). In both regimes of \( \Gamma_N \) we can estimate this time as \( \frac{1}{\Gamma_N} \) (compare the results for \( \phi \)-dependence of \( n_\sigma(t) \)). For small time \( 0 - \pi \) transition is not visible but for larger time it becomes evident in spite of the oscillations. Such transition is very well visible in the asymptotics, where the oscillations vanish. For larger \( \Gamma_N \) the current tends to its asymptotic value (without time-oscillations) in much shorter time than for smaller \( \Gamma_N \), due to the damping factor \( e^{-\Gamma_N t} \) [see the first term in Eq. 34].

Let us consider the simple case of the QD coupled solely to superconducting leads, assuming \( \Gamma_{S_\uparrow} = \Gamma_{S_\downarrow} = \Gamma_S \), \( \varepsilon_\uparrow = -\varepsilon_\downarrow \) and \( n_\uparrow(0) = n_\downarrow(0) = 0 \). In this case

\[
n_\sigma(t) = \sin^2(\Gamma_S \cos(\phi/2)t), \tag{34}
\]

\[
\begin{align*}
\Gamma_{S_\uparrow, S_\downarrow}(t) &= \frac{\Gamma_S}{2} \cos(\phi/2) \sin(2\Gamma_S |\cos(\phi/2)|t). \tag{35}
\end{align*}
\]

The QD occupancy and the current do not depend on spin and, in addition, both superconducting currents, \( j_{S_\uparrow, S_\downarrow}(t) \), are exactly identical. Note, however, that for \( \varepsilon_\uparrow + \varepsilon_\downarrow \neq 0 \) these currents differ one from another, see Eq. 29 and their difference equals \( \Gamma_S^2 \sin \phi \sin(\sqrt{\delta t})(\varepsilon_\uparrow + \varepsilon_\downarrow)/\delta \). The current \( j_{S_\uparrow, S_\downarrow} \) vanishes for \( \phi = \pi \) and \( \Gamma_{S_\uparrow} = \Gamma_{S_\downarrow} \). For different couplings, \( \Gamma_{S_\uparrow} \neq \Gamma_{S_\downarrow} \), the current does not vanish, even for \( \phi = \pi \). For instance \( j_{S_\uparrow, S_\downarrow} \) in this case (for \( \varepsilon_\sigma = 0 \)) is found to be \( j_{S_\uparrow, S_\downarrow}(t) = \frac{\Gamma_S}{2} \sin[(\Gamma_{S_\uparrow} - \Gamma_{S_\downarrow})t] \).

It would be interesting to consider the transition from the permanently oscillating superconducting currents in the system of the QD placed only between two superconducting leads (\( \Gamma_N = 0 \)) to finite constant asymptotic values of such currents in the presence of the third normal lead (\( \Gamma_N \neq 0 \)), see e.g. bottom panel in Fig. 6. From Eq. 34 we see, that for \( \Gamma_N \neq 0 \) the current consists of two parts. The first one corresponds to the transient oscillations damped by the factor \( e^{-\Gamma_N t} \), whereas the second one is described by the imaginary part of \( \Phi_\sigma \). This part of the current slowly evolves in time to some non-zero asymptotic value given in Eq. 35. Asymptotic value of this current vanishes with decreasing \( \Gamma_N \), therefore the oscillating part is damped less and less effectively, and simultaneously the imaginary part of \( \Phi_\sigma \) vanishes thereby the current oscillates with constant amplitude \( \frac{\Gamma_S}{2} \cos(\phi/2) \), Eq. 35.

VI. DIFFERENTIAL SUBGAP CONDUCTANCE

The last part of our studies is devoted to the subgap time-dependent Andreev conductance \( G_\sigma(\mu, t) = \frac{\partial j_{N\sigma}(t)}{\partial \mu}, \) expressing it in units of \( \frac{e^2}{h} \). We investigate this quantity as a function of the bias voltage (\( \mu = \mu_N \)) applied to the normal lead. Using the expressions for the current and QD charge, Eqs. 17 and 20 we obtain at zero
we display position of the quasiparticle
of the QD-leads couplings (we denote such time-scale
the limit of large time the conductance is characterized
mal electrode, $\Gamma_{\phi}$ (bottom panel), in the presence of weakly coupled nor-
$\phi$ $\pi/2$ and $3\pi/4$, respectively (upper panel). The
bottom panel shows the result for $\Gamma_N = 0.1$ where different
time scales, $\tau_1$, $\tau_2$ and $\tau_3$ are indicated. For negative values
of $\mu$ the results are symmetrical. The QD energy levels are:
$\varepsilon_0 = 0$ and $\Gamma_{S_1} = \Gamma_{S_2} = \Gamma_S$.

\[ G_\sigma(\mu, t) = \Re \left[ \Gamma_N e^{-i\mu t} L^{-1} \left\{ \frac{s + i\varepsilon - \sigma + \Gamma_N/2}{(s-s_1)(s-s_2)(s-i\mu)} \right\} (t) \right. \]
\[ + \left. \frac{\Gamma_N^2 \Gamma_1}{8} L^{-1} \left\{ \frac{1}{(s-s_3)(s-s_4)(s+i\mu)} \right\} (t) \right. \]
\[ - \frac{\Gamma_N^2}{2} L^{-1} \left\{ \frac{s + i\varepsilon - \sigma + \Gamma_N/2}{(s-s_1)(s-s_2)(s-i\mu)} \right\} (t) \]
\[ - \frac{\Gamma_N^2}{2} L^{-1} \left\{ \frac{s - i\varepsilon - \sigma + \Gamma_N/2}{(s-s_3)(s-s_4)(s+i\mu)} \right\} (t) \]

where for $\sigma = \downarrow$ the replacement $(s_1, s_2) \rightarrow (s_3, s_4)$ should
be made. Notice, that for $\varepsilon_1 = \varepsilon_\downarrow$ the conductance is spin
independent ($G_\uparrow = G_\downarrow = G$). In Fig. 8 we plot the
time-dependent conductance $G_\sigma(\mu, t) = G$ as a function of $\mu$
for different phase difference between the superconducting
leads, i.e. for $\phi = 0$ (upper panel) and for $\phi = 0.85\pi$
(bottom panel), in the presence of weakly coupled normal
electrode, $\Gamma_N = 0.1\Gamma_S$ ($\Gamma_{S_1} = \Gamma_{S_2} = \Gamma_S = 1$) and
$\varepsilon_\sigma = 0$. The process of forming the Andreev subgap
states is clearly visible. We observe that for $\phi = 0$ in
the limit of large time the conductance is characterized
by two well pronounced maxima appearing at $\mu \approx \pm \Gamma_S$
whose half-widths gradually shrink in time. These maxima
appear after some time-interval after abrupt switching
of the QD-leads couplings (we denote such time-scale
by $\tau_1$ see Fig. 9). This characteristic time is needed to
build up two distinct maxima of $G$ is longer depends on
the phase difference $\phi$ – compare the upper and bottom
panels in Fig. 8. Time-evolution of such quasiparticle
peaks allows us to estimate how fast the Andreev quasi-
particles appear in the system and thus it is desirable to
study this process in more detail.

By inspecting $G_\sigma(\mu, t)$ in Fig. 8 we observe, that up to
some specific time, $\tau_1$, a broad one-peaked structure of $G$
is present. Then, the conductance rapidly transforms in
time into two-peak structure. The position of each
quasiparticle peak evolves in time to its steady limit value
(that time is called $\tau_2$) and finally the width and height of
peaks are established after the time $\tau_3$ (see Fig. 2 bottom
panel). In Fig. 9 we display position of the quasiparticle
peaks maxima vs. time and $\mu$ for different values of $\Gamma_N$
and $\phi$ indicated in the legend (upper panel). As one can
see, the moment of appearance of two-peak structure, $\tau_1$,
depends on both $\phi$ and $\Gamma_N$. However, for $\phi = 0$ this
time only slightly depends on $\Gamma_N$. With increasing $\phi$
it increases with remarkable dependence on $\Gamma_N$ (for a given
$\phi$ it increases with $\Gamma_N$). The time scale for appearance
of the two-peak structure is very small and for $\phi = 0$
it equals approximately $2.5$ u.t., for $\phi = \pi/2$ it changes
from $\sim 3$ u.t. for $\Gamma_N = 0.1$ up to $\sim 4$ u.t. for $\Gamma_N = 0.5$,
and for $\phi = 3\pi/4$ it changes from $\sim 6$ u.t. for $\Gamma_N = 0.1$
up to $\sim 8.5$ u.t. for $\Gamma_N = 0.5$, respectively (see upper
panel, Fig. 9). Positions of the maxima versus µ evolve in time during approximately τ₂ ≈ 12 n.t. (the bold parts of lines in the bottom panel) and attain their steady-state values. Note, that the asymptotic quasiparticle peaks heights and widths are achieved with the envelope function 1 − exp(−t/τ₁), where τ₁ = Γ₁(37) can be deduced from the explicit expression for Gₖ(µ, t) in which the long living terms proportional to exp(−Γ₁ₕt) are present.

Let us consider a few special cases, for which the simpler analytical formulas can be given. In the limit φ = π, ε₀ = 0 and Γₘ = Γ, the conductance takes the form (here G₂ = Gₐ = G):

\[
G(µ, t) = \frac{Γₙ}{Γₙ^2 + µ^2} \left[ Γₙ^2 - e^{-Γₙt/2} \right],
\]

(37)

In this case the zero bias conductance reads \( G(µ = 0, t = 2 \ln 2/Γₙ) \) and for \( t = 2 \ln 2/Γₙ \) it reaches the optimal value equal to 0.5 and it vanishes for \( t \rightarrow ∞ \).

As differential conductance depends on the couplings Γ₁, Γ₂ and φ only through Γ₁₂ so different choices of these parameters can lead to the same values of \( Gₙ \). Note, that vanishing conductance \( Gₙ(µ, t = ∞) \) for \( φ = π \) is obtained, assuming the symmetric couplings Γ₁ = Γ₂. For Γ₁ ≠ Γ₂ even for \( φ = π \) the conductance looks quite different. Assuming, e.g. \( Γ₁ = k \) we obtain Γ₁₂ = \( \sqrt{Γ₁^2(1 + k^2 + 2k \cos φ)} \), than for \( k ≠ 1 \) and \( φ = π \) one has Γ₁₂ = \( Γ₁^2(1 - k)^2 \). The same value of Γ₁₂ one can obtain for \( k = 1 \) and \( φ = \arccos(\frac{1 - k^2}{2}) \).

The conductance \( Gₙ(µ, t) \) shown in Fig. 8 for \( φ = 0.85π \) and Γ₁ = Γ₂ = 1 is identical with that one calculated for \( Γ₁ = 0.5 \) and \( φ = \arccos(-\frac{2}{3}) \). It means that asymmetry in the couplings to superconducting leads Γ₁, Γ₂ can be effectively captured by the phase difference parameter, \( φ \). This conclusion refers also to the QD occupancy and the current flowing between the QD and the normal lead. Since explicit expression for \( Gₙ(µ, t) \) is rather lengthy, we skip it here and present only its asymptotic form \( t \rightarrow ∞ \) for Γ₁ = Γ₂ = Γ, ε₊ = −ε₋ \( (G₂ = Gₐ = G) \)

\[
G(µ) = \frac{Γ₁^2 Γ₂^2}{2 + µ^2} \cos^2 \left( \frac{φ}{2} \right) \left\{ \frac{1}{\left( Γ₁^2 + µ^2 \right)} \right\} \left\{ \frac{1}{\left( Γ₂^2 + µ^2 \right)} \right\}.
\]

(38)

where \( µ_{αβ} = µ + E_{αβ} \). For Γ₁ ≪ Γ₂ the asymptotic conductance has four maxima placed at \( µ \approx ±ε₊ ± Γ₂ | \cos \left( \frac{φ}{2} \right) | \) or equivalently at \( µ = E_{++, E_{+-}, E_{-+}} \) and \( E_{-} = 0 \).

\[
G(µ) = \frac{Γ₁^2 Γ₂^2}{2 + µ^2} \cos^2 \left( \frac{φ}{2} \right) \left\{ \frac{1}{\left( Γ₁^2 + µ^2 \right)} \right\} \left\{ \frac{1}{\left( Γ₂^2 + µ^2 \right)} \right\}.
\]

(39)

Fig. 10 presents the asymptotic conductance, \( G(µ, t = ∞) \), as a function of the bias voltage \( µ \) and the phase difference \( φ \). As one can see for \( φ = 0 \) two distinct maxima of \( G \) are visible (cf. Fig. 8 for \( t = 100 \)). For nonzero \( φ \), which satisfies the condition \( \cos(φ) > \frac{Γ₂^2 - Γ₁^2 + Γ₂^2}{2Γ₁ Γ₂} \), these two maxima appear at points \( µ = ±\sqrt{Γ₁^2 - Γ₂^2} \).

In the opposite case, there is only one maximum at \( µ = 0 \) whose height is reduced to zero value with \( φ → π \). In consequence, for \( φ = π \) and \( t = ∞ \) the conductance vanishes for all \( µ \). Note that, for the QD coupled only to one superconducting and one normal electrode, the zero-bias conductance is invariant under the replacement \( Γ₁ → Γ₂ \).

However, in our system with two superconducting leads this conclusion is no longer valid, even for the symmetric couplings case, \( Γ₁ = Γ₂ \). Such property is achieved only for \( φ = π \).

In the last part of our studies we discuss the time-evolution of the ABS for nonzero splitting of the QD energy levels. In the first case we consider the symmetric splitting around the zero energy (Fig. 11) \( ε₊ = −ε₋ = 0.5 \) for \( φ = 0.85π \) and in the second case the splitting is symmetric but around the nonzero energy value equal 0.5 (Fig. 12).
time after the quench). In Fig. 11 we analyze the approach to equilibrium of $G_T(\mu, t)$ for two values of $\Gamma_N$, $\Gamma_N = 0.1(0.02)$ upper (bottom) panel. We show only $G_T(\mu, t)$ as $G_\downarrow(\mu, t)$ is symmetric (with respect to $\mu = 0$) in relation to $G_T$. The maxima of $G_T$ for large time correspond to $E_{-\downarrow}$, $E_{++}$, $E_{--}$ and $E_{+-}$ ABS states (on the negative side of $\mu$-axis). It is interesting that the time evolution of $E_{-\downarrow}$, $E_{++}$ ABS is different from the evolution of $E_{--}$ and $E_{+-}$, respectively. The stationary values of the conductance peaks corresponding to $G_T$ and $G_\downarrow$ are all the same (according to Eq. 38) but the ABS $E_{-\downarrow}$ and $E_{++}$ begin to appear later than $E_{--}$ and $E_{+-}$. For $\Gamma_N = 0.1(0.02)$ this delay time can be approximately estimated as 30 (60) u.t. For stronger coupling $\Gamma_N$ (upper panel) the ABS peaks are wider in comparison to the case of weakly coupled normal electrode (bottom panel) and appear earlier than for smaller $\Gamma_N$. In Fig. 12 we show the phasal-dependence of $G_T$ and $G_\downarrow$ calculated for small time, $t = 10$ u.t. (upper panels), for $t = 30$ u.t. (middle panels) and for long time, $t = 100$ u.t. (bottom panels) at which the conductance attains the stationary values. In addition, in the upper panels the curves representing the localization of the ABS states on the $(\mu, \phi)$ plane are depicted. We observe essential difference with strong asymmetry between $G_T$ and $G_\downarrow$ at short period of time after the quench. The time evolution of $G_T$ ($G_\downarrow$) is limited to the appearance of $E_{-\downarrow}$ ($E_{--}$) ABS. Next, for larger time other Andreev states appear but the most visible are still the curves corresponding to $E_{-\downarrow}$ and $E_{--}$, respectively. Notice, that for $\phi = \pi$ and large time the conductance vanishes for both spins (cf. Eq. 38) as shown in the bottom panels. However, for smaller time after the quench all ABS states also vanish for $\phi = \pi$ except $E_{++}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12}
\caption{Phase-dependent Andreev conductance $G_T$ (left panels) and $G_\downarrow$ (right panels) (in $4e^2/h$ units) as a function of the bias voltage $\mu$ for $t = 10, 30$ and $100$ u.t. (upper, middle and bottom panels, respectively) after the quench. Other parameters: $\varepsilon_\uparrow = 1, \varepsilon_\downarrow = 0, \Gamma_N = 0.1, \Gamma_{S1} = \Gamma_{S2} = \Gamma_S$. The lower curves in all panels correspond to $\phi = 0$ and each next upper curve is shifted up by $\Gamma_S/2$, so the upper curves correspond to $\phi = 2\pi$. $E_{++}, E_{-\downarrow}, E_{--}, E_{+-}$ are represented by corresponding solid lines in the upper panels: they show the localization of the ABS (for $\Gamma_N = 0$) on the $(\mu, \phi)$ plane.

\textit{VII. CONCLUSIONS}

We have analyzed theoretically the transient sub-gap quasiparticles and transport properties of the QD coupled to one metallic and two superconducting electrodes (with large energy gaps), using the equation of motion for the second quantization operators and determining their Laplace transforms. Response of the system to the sudden coupling of its constituent parts and influence of the initial QD occupations on the induced electron pairing, the transient currents, and the differential conductance have been studied. The analytical formulas for these quantities have been derived and for some specific situations and their mutual relations have been analyzed. We have addressed the equilibrium case (with identical chemical potentials of all leads) and investigated the conductivity on non-equilibrium (biased) system.

We have shown that formulas for the QD occupation, $n_\sigma(t)$, on-dot pairing amplitude $\langle \epsilon_\downarrow(t) \epsilon_\uparrow(t) \rangle$ and charge current flowing between the QD and superconducting...}
leads, \( j_{S,\sigma}(t) \), consist of two parts. The first one depends on the initial QD occupations, but does not depend on the chemical potentials of external reservoirs. This term describes the oscillating transient behavior of the considered quantities which is damped via \( \exp(-\Gamma_N t) \) due to the QD-normal lead coupling. It is proportional to the factor \((1 - n_\uparrow(0) - n_\downarrow(0))\) and vanishes for some specific initial QD filling. The second part of the considered formulas depends mainly on the QD-normal leads coupling and results in monotonic time-dependence of the corresponding quantities. In contrast to the first part, it appears in all formulas regardless of the initial conditions.

Having analytical expressions for the physical observables we have shown, how the amplitude and time period of the transient oscillations depend on the model parameters \( \varepsilon_\sigma \), \( \Gamma_{S,\sigma} \), \( \Gamma_N \) and \( \varphi_1, \varphi_2 \). We have also presented a reminiscence of the Rabi-type oscillations between the empty and doubly-occupied configurations of the proximitized QD. We have found that for \( \varepsilon_\sigma = 0 \) the asymptotic QD occupancy does not depend on the superconducting leads phase difference \( \phi \) and it tends to half-filling. In presence of the Zeeman splitting \( \varepsilon_\sigma(t) \) relevantly depends on the phase difference \( \phi \), indicating signatures of \( 0 - \pi \) transition. Such transition is at smaller time (right after the quench) rather not much evident, but it becomes more and more clear at larger times, \( t \geq \frac{1}{\Gamma} \).

Finally, we have analyzed the time-dependent differential conductance as a function of the bias voltage between the normal and superconducting leads and we have inspected its phasic dependence. It has been found, that two-peak structure of the conductance (for \( \varepsilon_\sigma = 0 \)) known from the stationary transport properties, does emerge after some characteristic time-interval. This time scale increases with the phase differences \( \phi \). We have analyzed the spin-dependent conductance considering the Zeeman splitting of the QD levels, and found different temporal evolution of the corresponding Andreev bound states. Ultimately, for asymptotically large times, these Andreev peaks become symmetric and spin-independent. Our theoretical predictions could be verified by the present-day experimental methods and they could shed light on dynamics of the sub-gap quasiparticle states.

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