Near Ascending HNN-Extensions and a Combination Result for Semistability at Infinity

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Abstract

Semistability at $\infty$ is an asymptotic property of finitely presented groups that is needed in order to effectively define the fundamental group at $\infty$ for a 1-ended group. It is an open problem whether or not all finitely presented groups have semistable fundamental group at $\infty$. While many classes of groups are known to contain only semistable at $\infty$ groups, there are only a few combination results for such groups. Our main theorem is such a result.

Main Theorem. Suppose $G$ is the fundamental group of a connected reduced graph of groups, where each edge group is infinite and finitely generated, and each vertex group is finitely presented and either 1-ended and semistable at $\infty$ or has an edge group of finite index. Then $G$ is 1-ended and semistable at $\infty$.

An important part of the proof of this result is the semistability part of the following:

Theorem. Suppose $H_0$ is an infinite finitely presented group, $H_1$ is a subgroup of finite index in $H_0$, $\phi : H_1 \to H_0$ is a monomorphism and $G = H_0*_{\phi}$ is the resulting HNN extension. Then $G$ is 1-ended and semistable at $\infty$. If additionally, $H_0$ is 1-ended, then $G$ is simply connected at $\infty$. 

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1 Introduction

A connected locally finite complex $K$ has semistable fundamental group at $\infty$ if any two proper rays converging to the same end of $K$ are properly homotopic. If for any compact $C \subset K$ there is a compact $D \subset K$ such that loops in $K - D$ are homotopically trivial in $K - C$, then $K$ is simply connected at $\infty$. If $G$ is a finitely presented group then $G$ has semistable (respectively simply connected) fundamental group at $\infty$ if for some (equivalently any) finite complex $X$ with $\pi_1(X) = G$, the universal cover of $X$ has semistable (respectively simply connected) fundamental group at $\infty$. Both semistability and simple connectivity at $\infty$ are quasi-isometry invariants of finitely presented groups ([Geo08] and [Bri93]). If a finitely presented group $G$ has semistable fundamental group at $\infty$ (respectively is simply connected at $\infty$) then $H^2(G, \mathbb{Z}G)$ is free abelian (respectively trivial) (See Theorems 16.5.1 and 16.5.2, [Geo08]). A open question (apparently due to H. Hopf) ask if $H^2(G, \mathbb{Z}G)$ is free abelian for all finitely presented groups $G$. There are many classes of groups, all of whose (finitely presented) members are know to have semistable fundamental group at $\infty$. These include word hyperbolic groups (combining work of B. Bowditch [Bow99], G. Levitt [Lev98], G. Swarup [Swa96] and M. Bestvina and G. Mess [BM91]), CAT(0) cube groups (S. Shepherd [She]), Artin and Coxeter groups (Theorems 1.1 and 1.4, [Mih96]), many relatively hyperbolic groups ((Theorem 1.1, [MS21]) and (Theorem 1.5, [HM])) and groups containing a finitely generated infinite sub-commensurated subgroup of infinite index (Theorem 1.9, [Mih16]). There are only a few semistability combination results. Perhaps the most significant one states:

**Theorem 1.1 ([MT92]) (MTComb)** If $G$ is the fundamental group of a finite graph of groups where each vertex group is finitely presented with semistable fundamental group at $\infty$, and each edge group is finitely generated, then $G$ has semistable fundamental group at $\infty$.

Notice that finitely generated free groups trivially have semistable fundamental group at $\infty$. Exotic 1-ended groups can be written as $G_1*_{G_2}G_3$ for $G_i$ a finitely generated free group. These groups have semistable fundamental group at $\infty$ by Theorem 1.1.

**Example 1.2 (Examples3)** Let $F_i$ be the free group of rank $i > 0$. Three groups $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are constructed in [Rat07]. Each group $\Lambda_i$ can be
decomposed in two ways as amalgamated products $F_9 \ast_{F_{81}} F_9$, such that $F_{81}$ has index 10 in both $F_9$ factors. The group $\Lambda_1$ is a finitely presented, torsion free simple group. The groups $\Lambda_2$ and $\Lambda_3$ are not simple, but $\Lambda_2$ is virtually simple and $\Lambda_3$ has no non-trivial finite quotients.

Since the $F_{81}$ edge group has finite index in each vertex group, it is commensurated in each $\Lambda_i$. This gives a second way of seeing the $\Lambda_i$ have semistable fundamental group at $\infty$ (see Theorem 2.2). Our main theorem generalizes this phenomenon.

**Theorem 1.3 (SSDecomp)** Suppose $G$ is the fundamental group of a connected reduced graph of groups, where each edge group is infinite and finitely generated, and each vertex group is finitely presented and either 1-ended and semistable at $\infty$ or has an edge group of finite index. Then $G$ is 1-ended and semistable at $\infty$.

First a clarification. If an edge $e$ is a loop at a vertex $v$ in our graph, then the group $G_v$ has two subgroups $G_1$ and $G_2$, both of which are isomorphic to $G_e$. There is an isomorphism $\phi : G_1 \to G_2$ determining a subgroup of $G$ that is an HNN-extension of $G_v$. If either $G_1$ or $G_2$ has finite index in $G_v$, we say $G_e$ has finite index in $G_v$.

In the setting of this theorem, even when the graph is an edge path loop (with at least 2 vertices) and each edge group has finite index in its initial vertex, it seems that one cannot use commensurability ideas in order to prove semistability. Instead, for such a loop, the intersection of the edge groups will have finite index in the first vertex group producing an important near ascending HNN-extension subgroup as considered in the following result.

**Theorem 1.4 (MMFI)** Suppose $H_0$ is an infinite finitely presented group, $H_1$ is a subgroup of finite index in $H_0$, $\phi : H_1 \to H_0$ is a monomorphism and $G = H_0 \ast_\phi$ is the resulting HNN extension. Then $G$ is 1-ended and semistable at $\infty$. If additionally, $H_0$ is 1-ended, then $G$ is simply connected at $\infty$.

If $H_0 = H_1$ then $G$ is an ascending HNN-extension with base group $H_0$. This case of Theorem 1.4 was proved as (Theorem 3.1, [Mih85]), but is insufficient for our purposes. The proof of Theorem 1.4 is fundamentally more difficult than that of (Theorem 3.1, [Mih85]). This theorem is a critical piece of our proof of the Main Theorem.
Semistability preliminaries are covered in §2. We prove that a near ascending HNN-extension of an infinite finitely generated group is 1-ended in §3. The proof of Theorem 1.4 is presented in §4 and the proof of the Main Theorem is concluded in §5.

2 Semistability and End Preliminaries (SS)

While semistability makes sense for multiple ended spaces, we are only interested in 1-ended spaces in this article. Suppose $K$ is a locally finite connected CW complex. A ray in $K$ is a continuous map $r : [0, \infty) \to K$. A continuous map $f : X \to Y$ is proper if for each compact subset $C$ of $Y$, $f^{-1}(C)$ is compact in $X$. Proper rays $r, s : [0, \infty) \to K$ converge to the same end if for any compact set $C$ in $K$, there is an integer $k(C)$ such that $r([k, \infty))$ and $s([k, \infty))$ belong to the same component of $K - C$. The space $K$ has semistable fundamental group at $\infty$ if any two proper rays $r, s : [0, \infty) \to K$ that converge to the same end are properly homotopic (there is a proper map $H : [0,1] \times [0, \infty) \to X$ such that $H(0,t) = r(t)$ and $H(1,t) = s(t)$).

Note that when $K$ is 1-ended, this means that $K$ has semistable fundamental group at $\infty$ if any two proper rays in $K$ are properly homotopic. Suppose $C_0, C_1, \ldots$ is a collection of compact subsets of a locally finite 1-ended complex $K$ such that $C_i$ is a subset of the interior of $C_{i+1}$ and $\bigcup_{i=0}^{\infty} C_i = K$, and $r : [0, \infty) \to K$ is proper, then $\pi_1^\infty(K,r)$ is the inverse limit of the inverse system of groups:

$$\pi_1(K-C_0, r) \leftarrow \pi_1(K-C_1, r) \leftarrow \cdots$$

This inverse system is pro-isomorphic to an inverse system of groups with epimorphic bonding maps if and only if $K$ has semistable fundamental group at $\infty$ (see Theorem 2.1 of [Mih83] or Theorem 16.1.2 of [Geo08]).

(Theorem 2.1, [Mih83]) and (Lemma 9, [Mih86a]) promote several equivalent notions of semistability. We will use the second formulation in the next result to prove Theorem 1.4 and the third formulation in the proof of Theorem 1.3.

**Theorem 2.1 (ssequiv)** Suppose $K$ is a 1-ended, locally finite and connected CW-complex. Then the following are equivalent:

1. $K$ has semistable fundamental group at $\infty$. 

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2. For some (equivalently any) proper ray \( r : [0, \infty) \to K \) and compact set \( C \), there is a compact set \( D \) such that for any third compact set \( E \) and loop \( \alpha \) based on \( r \) and with image in \( K - D \), \( \alpha \) is homotopic rel\( \{r\} \) to a loop in \( K - E \), by a homotopy with image in \( K - C \).

3. For any compact set \( C \) there is a compact set \( D \) such that if \( r \) and \( s \) are rays based at \( v \) and with image in \( K - D \), then \( r \) and \( s \) are properly homotopic rel\( \{v\} \), by a proper homotopy in \( K - C \).

If in addition \( K \) is simply connected, then:

4. If \( r \) and \( s \) are rays based at \( v \) then \( r \) and \( s \) are properly homotopic rel\( \{v\} \).

In a connected locally finite CW-complex \( K \), every proper ray is properly homotopic to a proper edge path ray and every loop at a vertex of \( K \) is homotopic (by a small homotopy) to an edge path loop.

A subgroup \( H \) of a group \( G \) is commensurated in \( G \) if for each \( g \in G \), the subgroup \( H \cap (g^{-1}Hg) \) has finite index in both \( H \) and \( g^{-1}Hg \). The following two results will be used in our proof of Theorem 1.3. There is a notion of semistability at \( \infty \) for finitely generated groups that agrees with our definitions when the group is finitely presented. We mention this fact since it comes up in the following two results.

**Theorem 2.2** ([CM14]) (MainCM) If a finitely generated group \( G \) has an infinite, finitely generated, commensurated subgroup \( Q \), and \( Q \) has infinite index in \( G \), then \( G \) is 1-ended and semistable at \( \infty \).

**Theorem 2.3** (Theorem 3, [Mih86b]) (MComb) Suppose \( A \) and \( B \) are finitely generated 1 or 2-ended subgroups of the finitely generated group \( G \) and both \( A \) and \( B \) are semistable at \( \infty \). If the set \( A \cup B \) generates \( G \) and the group \( A \cap B \) contains a finitely generated infinite subgroup then \( G \) has semistable fundamental group at \( \infty \).

The conditions of the following result are the most general we know in order to define ends of a space. In this article we are only interested in locally finite CW-complexes.

**Theorem 2.4** (ES) Suppose \( X \) is connected, locally compact, locally connected and Hausdorff and \( C \) is compact in \( X \), then \( C \ union all bounded components of (X - C) \) is compact, and \( X - C \) has only finitely many unbounded components. Here bounded means compact closure.
If $X$ satisfies hypotheses of Theorem 2.4, then the **number of ends** of $X$ is the largest number of unbounded components one can obtain in the compliment of a compact subset of $X$. If this number is unbounded, then $X$ has infinitely many ends. If $K$ is a locally finite, connected CW complex, then $K$ has 1-end if and only if any two proper rays in $K$ converge to the same end. Note that if two proper rays are properly homotopic, then they converge to the same end.

3 Near Ascending HNN-Extensions are 1-ended (NHH1E)

If $A$ is a group with finite generating set $\mathcal{A}$ then let $\Gamma(A, \mathcal{A})$ be the Cayley graph of $A$ with respect to $\mathcal{A}$. This means that the vertex set of $\Gamma(A, \mathcal{A})$ is $A$ and there is an edge (labeled $a$) between the vertex $v$ and $w$ if $va = w$ for some $a \in \mathcal{A}$. The following result is somewhat standard. We include it for completeness.

**Theorem 3.1 (MMFI1)** Suppose $H_0$ is an infinite finitely generated group, $H_1$ is a subgroup of finite index in $H_0$, $\phi : H_1 \to H_0$ is a monomorphism and $G = H_0 \ast_\phi$ is the resulting HNN extension. Then $G$ is 1-ended.

**Proof:** Choose a finite generating set $\mathcal{H}_0$ for $H_0$ where $\mathcal{H}_1 \subset \mathcal{H}_0$ generates $H_1$. Then $\mathcal{H}_0$ along with $t$ (the stable letter) generates $G$. Consider:

$$
\Gamma_1 = \Gamma(H_1, \mathcal{H}_1) \subset \Gamma_0 = \Gamma(H_0, \mathcal{H}_0) \subset \Gamma = \Gamma(G, \{t\} \cup \mathcal{H}_0).
$$

If $v$ (an element of $G$) is a vertex of $\Gamma$, then $v \Gamma_0$ and $v \Gamma_1$ are translates of $\Gamma_0$ and $\Gamma_1$ respectively. Note that if $w$ is a vertex of $v \Gamma_i$ ($i \in \{0, 1\}$) then $v \Gamma_i = w \Gamma_i$. Our metric on $\Gamma$ is the edge path metric. If $v$ is a vertex of $\Gamma$ and $M > 0$ an integer, then $B(v, M)$ is the ball of radius $M$ about $v$. Let $\ast$ be the identity vertex of $\Gamma$. There is a continuous map $P : \Gamma \to \mathbb{R}$ such that $P(\ast) = 0$, each edge labeled $t$ is mapped to an interval $[n, n + 1]$ for some $n \in \mathbb{Z}$ and each edge labeled by an element of $\mathcal{H}_0$ is mapped to an integer. This map extends the homomorphism $G \to \mathbb{Z}$ that kills the normal closure of $H_0$.

We show that for any integer $N$ there is an integer $M > N$ such that any two vertices of $\Gamma - B(\ast, M)$ can be joined in $\Gamma - B(\ast, N)$. There are only finitely many $v \Gamma_0$ that intersect $B(\ast, N)$ non-trivially. Choose $M$ such
that for each such \( v \Gamma_0 \), all bounded components of \( v \Gamma_0 - B(\ast, N) \) belong to \( B(\ast, M) \) (see Theorem 2.4). The next lemma follows easily from the fact that \( A_1 \) has finite index in \( A_0 \).

**Lemma 3.2 (FIClose)** Suppose \( A_1 \subset A_0 \) are finite generating sets for the groups \( A_1 \subset A_0 \) and that \( A_1 \) has finite index in \( A_0 \). Then there is an integer \( N_{3.2}(A_0, A_0, A_1, A_1) \) such that if \( v \) and \( w \) are vertices of \( \Gamma(A_0, A_0) \) then there is an edge path in \( \Gamma(A_0, A_0) \) of length \( \leq N_{3.2}(G_0, A_0, G_1, A_1) \) from \( v \) to a vertex of \( w \Gamma_1(A_1, A_1) \).

From this point on we let \( N_{3.2} = N_{3.2}(H_0, H_0, H_1, H_1) \). Suppose \( e = (v, w) \) is an edge labeled \( t \) and either \( v \) or \( w \) is in \( B(\ast, M) \). Let \( \alpha_v \) be an edge path at \( v \) such that each edge of \( \alpha \) is labeled by an element of \( H_1 \) and such that the end point \( a \) of \( \alpha_v \) is in \( \Gamma - B(\ast, M + 1) \). Let \( e_1 = (a, b) \) be the edge at \( a \) labeled \( t \) and note that \( e_1 \) has image in \( \Gamma - B(\ast, M) \). For each \( h \in H_1 \), we have \( t^{-1}ht = \phi(h) \in H_0 \) and so there is an edge path \( \beta_w \) in \( w \Gamma_0 \) from \( w \) to \( b \). Now suppose \( x \) and \( y \) are vertices of \( \Gamma - B(\ast, M) \). We want to find a path in \( \Gamma - B(\ast, N) \) connecting \( x \) and \( y \). Let \( \tau \) be any edge path from \( x \) to \( y \) in \( \Gamma \). If \( e = (v, w) \) is an edge of \( \tau \) labeled \( t \), such that either \( v \) or \( w \) belongs to \( B(\ast, M) \) then replace \( e \) by the path \( (\alpha_v, e_1, \beta_w^{-1}) \) (above). In this way we produce an edge path \( \tau_1 \) from \( x \) to \( y \) such that each \( t \) labeled edge of \( \tau_1 \) belongs to \( \Gamma - B(\ast, M) \). Each maximal subpath of \( \tau_1 \) (none of whose edges is labeled \( t \)) has both end points in \( \Gamma - B(\ast, M) \). Hence it suffices to show the end points of such paths can be joined by an edge path in \( \Gamma - B(\ast, N) \).

Recall that \( P: \Gamma \to \mathbb{R} \) and \( P(v \Gamma_0) = P(v) \). We have:

\[
P(B(\ast, N)) = [-N, N].
\]

Suppose \( \gamma \) is an edge path with end points in \( \Gamma - B(\ast, M) \), each edge of \( \gamma \) is labeled by an element of \( H_0 \) and \( \gamma \) intersects \( B(\ast, N) \), then \( -N \leq P(\gamma) \leq N \). The strategy is to “slide” \( \gamma \) upward to a path \( \gamma_k \) in level \( N + 1 \) (which automatically avoids \( B(\ast, N) \)). Assume the initial point of \( \gamma \) is \( v \) and the terminal point is \( w \) (see Figure 1). By the definition of \( M \), the vertex \( v \) is in an unbounded component of \( v \Gamma_0 - B(\ast, N) \). Hence there is an edge path \( \tau_1 \) in \( v \Gamma_0 - B(\ast, N) \) from \( v \) to \( v_1 \in v \Gamma_0 - B(\ast, M + N_{3.2} + 1) \). Similarly choose \( \tau_1' \) in \( v \Gamma_0 - B(\ast, N) \) from \( w \) to \( w_1 \in v \Gamma_0 - B(\ast, M + 1) \). Note that:

\[
v \Gamma_0 = v_1 \Gamma_0 = w \Gamma_0 = w_1 \Gamma_0.
\]

By the definition of \( N_{3.2} \) we choose an edge path \( \tau_2 \) of length \( \leq N_{3.2} \) in \( w_1 \Gamma_0 - B(\ast, M + 1) \) from \( v_1 \) to \( v_2 \in w_1 \Gamma_1 - B(\ast, M + 1) \). Let \( e = (v_2, v_3) \).
and $e' = (w_1, w_2)$ be edges labeled $t$. Let $\gamma'$ be an edge path in $w_1\Gamma_1$ from $v_2$ to $w_1$. The conjugation relations $t^{-1}ht = \phi(h)$ for $h \in \mathcal{H}_1$ (applied to each edge of $\gamma'$) give an edge path $\gamma_1$ in $v_3\Gamma_0$ from $v_3$ to $w_2$. Then:

$$\{v_3, w_2\} \subset v_3\Gamma_0 - B(*, M).$$

Note that the paths $(\tau_1, \tau_2, e)$ (from $v$ to $v_3$) and $(\tau'_1, e')$ (from $w$ to $w_2$) have image in $\Gamma - B(*, N)$. Hence we need only show there is an edge path in $\Gamma - B(*, N)$ from $v_3$ to $w_2$. Note that $P(\gamma_1) = P(\gamma) + 1 \geq -N + 1$. Repeating this process at most $2N + 1$ times produce a path from $v$ to $w$ in $\Gamma - B(*, N)$, and so $G$ is 1-ended.  

4 The Proof of Theorem 1.4 (PFMMFI)

This theorem generalizes both the Main Theorem and Theorem 3.1 of [Mih85] (where $H_0 = H_1$ and $G$ is an ascending HNN-extension of $H_0$). Our proof of Theorem 1.4 has similarities to the one in [Mih85], but requires several new ideas.

Proof: Theorem 3.1 implies that $G$ is 1-ended (even when $H_0$ is finitely generated and not finitely presented). Again we consider the following finite
presentation of $G$:

$$\mathcal{P} = \langle t, H_0 : R, t^{-1}ht = \hat{\phi}(h) \text{ for all } h \in H_1 \rangle$$

where $\mathcal{P}_0 = \langle H_0 : R \rangle$ is a finite presentation for $H_0$. We assume $H_1 \subset H_0$ and $\mathcal{P}_1 = \langle H_1 : R_1 \rangle$ (for some $R_1 \subset R_0$) is a finite presentation for $H_1$. Here $\hat{\phi} : F(H_1) \to F(H_0)$ is a homomorphism of free groups realizing $\phi$ and the relations $t^{-1}ht = \hat{\phi}(h)$ are called conjugation relations. Let

$$q : F(H_0 \cup \{t\}) \to G$$

be a quotient homomorphism with kernel equal to the normal closure of $R$ union the set of conjugation relations.

Let $\Gamma_1 \subset \Gamma_0 \subset \Gamma = \Gamma(\mathcal{P})$ be the Cayley 2-complexes for $\mathcal{P}_1$, $\mathcal{P}_0$ and $\mathcal{P}$ respectively. The 1-skeleton of $\Gamma$ is the Cayley graph of $G$ with respect to the generating set $H_0 \cup \{t\}$. At each vertex there is a 2-cell attached according to each relation of $\mathcal{P}$. If the edges of an edge path in $\Gamma$ are labeled by $S^{\pm 1}$ for $S \subset H_0$, then we call it an $S$ path. If $v \in G$ is a vertex of $\Gamma$ then $v\Gamma_1 \subset v\Gamma_0 \subset \Gamma$. Note that any edge path in $v\Gamma_0$ is an $H_0$ path. There is a continuous map $p : \Gamma \to \mathbb{R}$ such that each vertex is mapped to an integer, a (directed) edge $e$ labeled $t$ is mapped linearly by $p$ to an interval $[n, n+1]$ (so that $p(e(0)) = n$ and $p(e(1)) = n + 1$) and $p(v\Gamma_0) = \{p(v)\}$. A conjugation 2-cell is mapped to an interval $[n, n+1]$. We say a vertex, edge or 2-cell is in level $n$ if $p$ maps it to $n$. If $X$ is a 2-complex, let $X^{(1)}$ be the 1-skeleton of $X$, endowed with the edge path metric. Suppose $A$ is a subcomplex of $X$ and $A^{(1)}$ is the 1-skeleton of $A$. Let $B(A^{(1)}, n)$ be the ball of radius $n$ in $X^{(1)}$ around $A^{(1)}$. For $n \geq 1$, let $St^n(A)$ be $B(A^{(1)}, n)$ union all 2-cells whose boundaries belong to $B(A^{(1)}, n)$.

$$(*) \text{ Let } N_0 \text{ be such that } |\hat{\phi}(h)| + 3 \leq N_0 \text{ for all } h \in H_1. \text{ Then any (closed) 2-cell arising from a conjugation relation belongs to } St^{N_0}(v) \text{ for any vertex of that cell.}$$

All homotopies of paths will be relative to end points and we simply say the paths are homotopic. Note that $(*)$ implies:
Lemma 4.1 (oldslide) Suppose $\alpha$ is an $H_1$ path from $v$ to $w$ in $\Gamma$. Let $e$ and $d$ be the edges labeled $t$ at $v$ and $w$ respectively. Then the edge path $(e^{-1}, \alpha, d)$ is homotopic to an $H_0$ path $\beta$ by a homotopy $H$ with image in $St^{N_0}(im(\alpha))$. Furthermore $p(H) \subset [p(v), p(v + 1)]$.

We apply the next lemma in both this section and in the proof of Lemma 5.1.

Lemma 4.2 (SSL1) Suppose $A_0$ and $A_1$ are finitely presented groups, and $A_1$ has finite index in $A_0$. Let $Q = \langle A_0 : R \rangle$ be a finite presentation of $A_0$ such that $A_1 \subset A_0$ generates $A_1$ (so that the Cayley graph $\Gamma(A_1, A_1)$ is a subset of the Cayley 2-complex $\Gamma(Q)$). There is an integer $N_{4,2}(Q)$ such that if $\alpha$ is an edge path in $\Gamma(Q)$ with $\alpha(0) = v$ and end point in $v\Gamma(A_1, A)$ then $\alpha$ is homotopic to an edge path in $v\Gamma(A_1, A_1)$ by a homotopy $H$ in $\Gamma(Q)$ with image in $St^{N_{4,2}(Q)}(im(\alpha))$.

Proof: Say $\alpha = (e_1, \ldots, e_n)$ with vertices $v = v_0, \ldots, v_n$. If $i \in \{1, \ldots, n-1\}$ then there is an edge path $\phi_i$ of length $\leq N_{3,2} = N_{3,2}(A_0, A_0, A_1, A_1)$ from $v_i$ to $w_i \in v\Gamma(A_1, A_1)$. Define $\phi_0$ and $\phi_n$ to be trivial paths and define $w_0 = v_0$ and $w_n = v_n$. Choose an integer $M$ such that if two vertices of $\Gamma(A_1, A_1)$ are within $2N_{3,2} + 1$ in $\Gamma(Q)$ then there is an edge path in $\Gamma(A_1, A_1)$ of length $\leq M$ connecting these vertices. Choose $N_{4,2}(Q)$ so that any edge path loop in $\Gamma(Q)$ of length $\leq 2N_{3,2} + M + 1$ is homotopically trivial in $St^{N_{4,2}(Q)}(a) \subset \Gamma(Q)$ for any vertex $a$ of the loop. For $i \in \{1, \ldots, n\}$ let $\tau_i$ be an $H_1$ edge path (in $v\Gamma(A_1, A_1)$) of length $\leq M$ from $w_{i-1}$ to $w_i$. Each loop $(\phi_i, \tau_{i+1}, \phi_{i+1}^{-1}, e_{i+1}^{-1})$ is homotopically trivial in $St^{N_{4,2}(Q)}(v_i) \subset \Gamma(Q)$. This implies that $\alpha$ is homotopic to $(\tau_1, \ldots, \tau_n)$, by a homotopy in $St^{N_{4,2}(Q)}(im(\alpha)) \subset \Gamma(Q)$.  

Next we recall some basic group theory for HNN-extensions. Suppose $w \in F(H_0)$ and $q(w) = 1 \in G$. Then the number of letters of $w$ labeled $t$ is the same as the number of letters of $w$ labeled $t^{-1}$. If some letter of $w$ is labeled $t$, then (cyclically), there is a subword $t^{-1}w't$ of $w$ such that $q(w') \in H_1$. Suppose $\alpha$ is an edge path in $\Gamma$ with consecutive vertices $v_0, \ldots, v_n$. If the edge $(v_{i-1}, v_i)$ is labeled $a_i$ and is directed from $v_i$ to $v_{i+1}$ then let $b_i = a_i$. If this edge is directed $(v_i, v_{i-1})$ then let $b_i = a_i^{-1}$. We say the word $b_1 \cdots b_n$ ($\in F(H_0 \cup \{t\})$) corresponds to $\alpha$ and we write $\bar{\alpha} = b_1 \cdots b_n$.

An edge path has backtracking if there are adjacent edges $e$ and $e^{-1}$. The next lemma effectively allows us to move lowest level maximal subpaths of
certain loops upward. Lemmas 4.3 and 4.4 are used in the simply connected at ∞ part of our result. The no backtracking hypothesis is not an issue for simply connected at ∞ arguments, but when a base ray must be respected in the semistability part of our theorem, we cannot assume there is (cyclically) no backtracking in certain loops.

**Lemma 4.3 (smallmove)** Suppose γ is an edge path loop in Γ such that: at least one edge of γ is labeled t; p(γ) = [A, B]; and (cyclically) γ has no backtracking. Assume α is (cyclically) a maximal subpath of γ such that p(α) = A. Let e and d be the edges of γ that (cyclically) immediately precede and follow α, so that β = (e, α, d) is (cyclically) a subpath of γ. Then qα ∈ H1 and β = t−1αt and β is homotopic to an H0 path τ by a homotopy H with image in StN0+N4 (im(α)) and with p(H) ⊂ [A, A + 1].

**Proof:** The maximality of α implies β = t−1αt. We next argue that q(α) ∈ H1. By the remarks preceding this lemma, there is (cyclically) a subword of γ of the form t−1βt where q(β) ∈ H1. If β = α then indeed q(α) ∈ H1. Otherwise, α and β do not overlap. Furthermore (since γ has no backtracking) e and d are not adjacent to β (see Figure 2). If β = α then replace t−1βt in γ by an H0-word. This produces a word with two fewer t letters, but the t letters corresponding to d and e remain. Continuing, we eventually conclude that q(α) ∈ H1. Let v be the initial vertex of α, then the end point of α is in vΓ1.

By Lemma 4.2, α is homotopic to an edge path α1 in vΓ1 by a homotopy H (in vΓ0) with image in StN4 (im(α)). In particular, the image of pH is p(v) = A. Now we can slide α1 up one level (between t labeled edges) to a path τ, by a homotopy H which only uses 2-cells arising from conjugation relations on the letters of α1 (see Figure 2). Note that the image of this
homotopy lies between levels \( A \) and \( A + 1 \) and \( \hat{H} \) has image in \( \text{St}^{N_0}(im(\alpha_1)) \). Combine \( \tilde{H} \) and \( \hat{H} \) to form the homotopy \( H \) and finish the proof. \( \square \)

**Lemma 4.4 (slideupward)** Suppose \( C \) is a compact subcomplex of \( \Gamma \), \( p(C) \subset [A, B] \) and \( \gamma \) is an edge path loop (without backtracking) in \( \Gamma - C \). Then either \( \gamma \) is homotopically trivial in \( \Gamma - C \) or there is a homotopy in \( \Gamma - C \) of \( \gamma \) to an edge path loop \( \gamma_1 \) such that \( p(\gamma_1) \) has image in \( [A - 1, \infty) \), the homotopy fixes the edges of \( \gamma \) in levels \( A - 1 \) and above and the only edges of \( \gamma_1 \) above level \( A - 1 \) (including \( t \)-labeled edges connecting levels \( A - 1 \) and \( A \)) are edges of \( \gamma \).

**Proof:** Let \( \bar{\gamma} \in F(\mathcal{H}_0 \cup \{t\}) \) be the (reduced) word labeling \( \gamma \). If \( \gamma \) has image in a single level, then either \( \gamma \) lies in a level at or above level \( A - 1 \) (and we are finished) or \( \gamma \) lies in a level below level \( A - 1 \). In the later case, \( \gamma \) is homotopically trivial in that level (by a homotopy avoiding \( C \)). Instead, we assume some letter of \( \bar{\gamma} \) is \( t \).

Suppose \( \alpha \) is (cyclically) a maximal subpath of \( \gamma \) in the lowest level of \( \Gamma \) traversed by \( \gamma \). If \( p(\alpha) \geq A - 1 \) we are finished. If \( e \) and \( d \) are the edges preceding and following \( \alpha \), then Lemma 4.3 implies that \( (e, \alpha, d) \) is homotopic to an \( \mathcal{H}_0 \) path \( \tau \) (with \( p(\tau) = p(\alpha) + 1 \)) by a homotopy \( H \) with \( p(H) \subset (-\infty, A - 1] \). In particular, \( H \) avoids \( C \). Replace \( (e, \alpha, d) \) in \( \gamma \) by \( \tau \), eliminate backtracking and continue applying Lemma 4.3 to (cyclically) maximal subpaths in a lowest level until all edges of the resulting path are either in a single level below level \( A - 1 \) (in which case \( \gamma \) is homotopically trivial in \( \Gamma - C \)) or in levels \( A - 1 \) and above. \( \square \)

Now assume that \( H_0 \) is 1-ended. In this case, our goal is to show that \( G \) is simply connected at \( \infty \). Let \( C \) be a finite subcomplex of \( \Gamma \) and assume that \( p(C) \subset [A, B] \). Let \( D \) be a finite subcomplex of \( \Gamma \) containing \( \text{St}^{(B-A+2)(N_{4,2}+N_0)}(C) \). Also assume that if \( g \in G \) then \( g\Gamma_0 - D \) is a single unbounded component (since \( H_0 \) is one ended). Let \( \gamma \) be an edge path loop in \( \Gamma - D \). It suffices to show that \( \gamma \) is homotopically trivial in \( \Gamma - C \). The idea is to move \( \gamma \) to a loop in a single level above level \( B \) by a homotopy that avoids \( C \). Once there, it is homotopically trivial in that level and hence that homotopy avoids \( C \). Suppose \( p(\gamma) = [A_1, B_1] \).

Inductively we show that for \( 0 \leq k \leq B - A + 2 \), the loop \( \gamma \) is either homotopically trivial in \( \Gamma - C \) (as desired) or homotopic to an edge path loop \( \mu_k \) by a homotopy in \( \Gamma - C \) such that \( \mu_k \) has image in \( \Gamma - \text{St}^{(B-A+2-k)(N_{4,2}+N_0)}(C) \).
Figure 3: Sliding $\gamma$ Up

and $p(\mu_k) \subset [A - 1 + k, \infty)$. Note that for $k = B - A + 2$, we have $p(\mu_k) \subset [B + 1, \infty)$ (so $\mu_k$ is above $C$). If $A_1 \geq A - 1$, then let $\mu_0 = \gamma$. If $A_1 < A - 1$ then Lemma 4.4 implies that either $\gamma$ is homotopically trivial in $\Gamma - C$ (and we are finished) or there is a homotopy in $\Gamma - C$ of $\gamma$ to an edge path loop $\gamma'_1$ such that $p(\gamma'_1) \subset [A - 1, \infty)$, the homotopy fixes the edges of $\gamma$ in levels $A - 1$ and above, and the only edges of $\gamma'_1$ above level $A - 1$ (including $t$-labeled edges connecting levels $A - 1$ and $A$) are edges of $\gamma$. If $\gamma'_1$ has image in level $A - 1$ then it is homotopically trivial in that level (hence missing $C$) and again we are finished. Let $\gamma_1$ be obtained from $\gamma'_1$ by eliminating backtracking (which can only occur in level $A - 1$). If $\alpha_1$ is a maximal subpath of $\gamma_1$ in level $A - 1$ and $e$ and $d$ are the edges of $\gamma_1$ that (cyclically) immediately precede and follow $\alpha_1$, then $e$ and $d$ are edges of $\gamma$ labeled $t$. In particular if $v$ and $w$ are the initial and end points of $\alpha_1$ then $v$ and $w$ are in the single unbounded component of $v\Gamma_0 - D$. Hence there is an $H_0$ path $\alpha_2$ (without backtracking) in $v\Gamma_0 - D$ from $v$ to $w$. Now $\alpha_1$ and $\alpha_2$ are homotopic by a homotopy in $v\Gamma_0$ (in level $A - 1$) and hence in $\Gamma - C$ (see Figure 3). Let $\mu_0$ be obtained from $\gamma_1$ by replacing each maximal $H_0$ subpath $\alpha_1$ of $\gamma_1$ with a corresponding $\alpha_2$. Note that there is no backtracking in $\mu_0$. We have that $\gamma$ is homotopic to $\mu_0$ by a homotopy in $\Gamma - C$, the image of $\mu_0$ is in $\Gamma - D$ and $p(\mu_0) \subset [A - 1, \infty)$.
If $A_1 > A - 1$, let $\mu_1 = \gamma$. Otherwise let $\alpha_2$ be a maximal subpath of $\mu_0$ in level $A - 1$. Let $e$ and $d$ be the edges of $\mu_0$ that (cyclically) immediately precede and follow $\alpha_2$ (see Figure 3). The path $\beta = (e, \alpha_2, d)$ is such that $\beta = t^{-1} \bar{\alpha}_2 t$. Lemma 4.3 implies that $\beta$ is homotopic to an $\mathcal{H}_0$ path $\tau$ by a homotopy $H$ with image in $\text{St}_{N_0 + N_1 + 2}(\text{im}(\alpha_2))$ and with $p(H) \subset [A - 1, A]$. In particular $H$ (and $\tau$) avoids $\text{St}_{(B-A+1)(N_2+N_0)}(C)$. For all such maximal subpaths $\alpha_2$ of $\mu_0$ in level $A - 1$ replace $(e, \alpha_2, d)$ by the corresponding $\tau$. Eliminate backtracking (all in level $A$) and call the resulting path $\mu_1$. The result is that $\mu_0$ (and $\gamma$) is homotopic to an edge path loop $\mu_1$ in level $A - 1$ replace $(e, \alpha_2, d)$ by a homotopy in $\Gamma - \text{St}_{N_0 + N_1}(\text{im}(\mu_1))$. This completes the proof that if $\mathcal{H}_0$ is 1-ended then $G$ is simply connected at $\infty$.

Now we turn to the semistability part of our theorem. The next lemma allows us to replace a path with labeling $(t^{-1}, \bar{\alpha}, t)$ (where $\bar{\alpha}$ is an $\mathcal{H}_0$ word) with a path comprised of subpaths either in the level above the level of $\alpha$ or in the complement of an arbitrary compact set $E$. In Figure 4 the path $(e^{-1}, \alpha, d)$ is homotopic to $(\gamma_1, c^{-1}, \tau_2, b, \gamma_2^{-1})$ where $p(\gamma_1) = p(\alpha) + 1$ and the path $(c^{-1}, \tau_2, b)$ has image in $\Gamma - E$. 

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Lemma 4.5 (UpOut) Suppose:

A. $D$ is a compact subcomplex of $\Gamma$ such that for each vertex $v \in \Gamma$ the space $v\Gamma_0 - D$ is a union of unbounded path components, and

B. $\beta = (e^{-1}, \alpha, d)$ is an edge path of $\Gamma$ such that $\beta = t^{-1}\alpha t$, $e$ and $d$ are edges of $\Gamma - D$, $\alpha$ is an $H_0$ path and the end points of $\alpha$ can be joined by a path $\alpha'$ in $\Gamma - D$ such that $p(\alpha'(t)) \leq p(\alpha)$ for all $t$.

Then, for any compact set $E \subset \Gamma$ such that $D \subset E$, there is a homotopy $M$ of $\beta$ to a path $(\gamma_1', c^{-1}, \tau_2, b, \gamma_2')$ such that:

1. $\tau_2$, $\gamma_1'$ and $\gamma_2'$ are $H_0$ paths,
2. $b$ and $c$ are $t$ edges,
3. $(c^{-1}, \tau_2, b)$ has image in $\Gamma - E$,
4. for $i \in \{1, 2\}$, the end points of $\gamma_i'$ can be connected by an edge path $\gamma_i''$ with image in $\Gamma - D$ such that $p(\gamma_i''(t)) \leq p(\gamma_i')$ for all $t$, and
5. $p(M)$ has image in $[p(\alpha), p(\alpha) + 1]$.

Proof: Suppose $\rho_1$ and $\rho_2$ are paths with the same end points. We use the notation $\rho_1 \sim^K \rho_2$ when $K$ is a homotopy of $\rho_1$ to $\rho_2$ relative to end points. Say the initial (terminal) vertex of $\alpha$ is $v$ ($w$). Since $v\Gamma_1 \subset v\Gamma_0$ and $w \in v\Gamma_0 - D$, hypothesis (A.) implies that there is an $H_0$ path $\tau_1$ at $w$ with image in $v\Gamma_0 - D$ such that the end point $z$ of $\tau_1$ belongs to $v\Gamma_1 - St^{N_{3,2} + 1}(E)$. (See Figure 4.) Let $\gamma_1$ be an $H_1$ path from $v$ to $z$ and $K_1$ a homotopy in $v\Gamma_0$ realizing:

$$(\alpha, \tau_1) \sim^{K_1} \gamma_1.$$ 

Lemma 3.2 implies there is an $H_0$ edge path $\tau_2$ of length $\leq N_{3,2}$ from $z$ to $y \in w\Gamma_1$. Let $\gamma_2$ be an $H_1$ path from $w$ to $y$ and $K_2$ a homotopy in $w\Gamma_0$ realizing:

$$(\tau_1, \tau_2) \sim^{K_2} \gamma_2.$$ 

Let $c$ and $b$ be the $t$ edges at $z$ and $y$ respectively. Since $z \in \Gamma - St^{N_{3,2} + 1}(E)$ we have that $(c^{-1}, \tau_2, b)$ has image in $\Gamma - E$. At this point conclusions (2.) and (3.) are satisfied. By Lemma 4.1, there are $H_0$ paths $\gamma_1'$ and $\gamma_2'$ such that:

$$(e^{-1}, \gamma_1, c) \sim^{L_1} \gamma_1'$$ and $$(d^{-1}, \gamma_2, b) \sim^{L_2} \gamma_2'.$$
with \( p(L_i) \subset [p(\alpha), p(\alpha) + 1] \). Conclusion (I.) is satisfied. Hypothesis (B.) and the definitions of \( \tau_1 \) and \( \tau_2 \) imply that \( \gamma_1'' = (e^{-1}, \alpha', \tau_1, c) \) is a path in \( \Gamma - D \) connecting the end points of \( \gamma_1' \), and \( \gamma_2'' = (d^{-1}, \tau_1, \tau_2, b) \) is a path in \( \Gamma - D \) connecting the end points of \( \gamma_2' \). Conclusion (4.) is satisfied. Combining \( K_i \) and \( L_i \) to form the homotopy \( M_i \) we have:

\[(e^{-1}, \alpha, \tau_1, c) \sim_{M_1} \gamma_1' \text{ and } (d^{-1}, \tau_1, \tau_2, b) \sim_{M_2} \gamma_2'\]
with \( p(M_i) = [p(\alpha), p(\alpha) + 1] \). The homotopy \( M_2 \) implies that:

\[\tau_1 \sim_{M_2} (d, \gamma_2', b^{-1}, \tau_2^{-1})\]
Substituting for \( \tau_1 \) in \( M_1 \) we have:

\[(e^{-1}, \alpha, d, \gamma_2', b^{-1}, \tau_2^{-1}, c) \sim \gamma_1'\]
and so:

\[(e^{-1}, \alpha, d) \sim (\gamma_1', c^{-1}, \tau_2, b, \gamma_2'^{-1})\]
where \( p(M) = [p(\alpha), p(\alpha) + 1] \), so that (5.) is satisfied. □

Use of the next lemma is one of the primary difference between the proof of (Theorem 3.1, [Mih85]) and the current argument. It is one of the main ingredients in the proof of Theorem 1.4. If one thinks of \( t \)-labeled edges of \( \Gamma \) as vertical and \( H_0 \) labeled edges as horizontal, then Lemma 4.6 explains how to push a vertical edge \( e \) that is “far enough” from a compact set \( C \subset \Gamma \), to a vertical edge \( e_1 \) that is “arbitrarily far” from \( C \), by a homotopy that misses \( C \).

**Lemma 4.6 (push)**  Given a compact subcomplex \( C \) of \( \Gamma \) there is a compact subcomplex \( D_{4.6}(C) \) of \( \Gamma \) containing \( St^{N_0 + N_{4.2}}(C) \) such that for any third
compact set \( E \) and any edge \( e \) labeled \( t \) in \( \Gamma - D_{1,6}(C) \), there is an edge path \((\alpha, e_1, \beta)\) from \( e(0) \) to \( e(1) \) such that \( \alpha \) and \( \beta \) are \( H_0 \) paths, the edge \( e_1 \) has image in \( \Gamma - E \) and label \( t \), and \( e \) is homotopic to \((\alpha, e_1, \beta)\) by a homotopy with image in \( \Gamma - C \).

**Proof:** Note that if \( w \) and \( v \) are vertices of \( \Gamma \) and \( w \in v \Gamma_0 \) then \( w \Gamma_0 = v \Gamma_0 \).

For \( K \) a subcomplex of \( \Gamma \), we take \( St(K) \) in \( \Gamma \). (See \((*)\) for the definition of \( N_0 \).) There are only finitely many \( v \Gamma_0 \) that intersect \( St^{N^4_2 + N_0}(C) \).

Choose \( D \) a finite subcomplex of \( \Gamma \) such that if \( v \Gamma_0 \cap St^{N_0 + N^4_2}(C) \neq \emptyset \), then \( St^{N_0 + N^4_2}(C) \) union all bounded components of \( v \Gamma_0 - St^{N_0 + N^4_2}(C) \) belong to \( D \) (see Theorem 2.4). Let \( v = e(0) \). If \( v \in \Gamma - D \) then by the definition of \( D \), there are edge paths in \( v \Gamma_0 - St^{N_0 + N^4_2}(C) \) from \( v \) to vertices arbitrarily far from \( C \). In particular there is an edge path \( \alpha_1 \) in \( v \Gamma_0 - St^{N_0 + N^4_2}(C) \) from \( v \) to \( w_1 \in v \Gamma_0 - St^{N^4_2 + 1}(E \cup D) \). By Lemma 3.2 there is an edge path \( \alpha_2 \) (of length \( \leq N_{3,2} \)) from \( w_1 \) to \( w \in v \Gamma_1 - St(E \cup D) \). In particular, \( \alpha_2 \) avoids \( D \). Then \( \alpha = (\alpha_1, \alpha_2) \) has image in \( v \Gamma_0 - St^{N_0 + N^4_2}(C) \). By Lemma 4.2, \( \alpha \) is homotopic to an edge path \( \alpha' \) (with image in \( v \Gamma_1 \)) by a homotopy in \( v \Gamma_0 - St^{N_0}(C) \). Let \( e_1 \) be the edge labeled \( t \) at \( w \). Since \( w \in v \Gamma_0 - St(E) \), \( e_1 \) avoids \( E \). Combining conjugation 2-cells (one for each edge of \( \alpha' \)), the path \( \alpha' \) is homotopic to an edge path \((e, \beta, e_1^{-1})\) by a homotopy in \( St^{N_0}(im(\alpha')) \subset \Gamma - C \). Let \( \beta = \beta^{-1} \).

Lemma 4.6 allows us to focus on edge paths in \( v \Gamma_0 \) \( (v \in G) \) with end points outside of a compact set \( E \). Theorem 2.4 implies that for any compact set \( E \subset \Gamma \) and \( v \in G \), \( E \) union all bounded components of \( v \Gamma_0 - E \) is compact. Since \( E \) only intersects finitely many \( v \Gamma_0 \) non-trivially, \( E \) is contained in a compact set \( E' \) such that for any \( v \in G \) the set \( v \Gamma_0 - E \) is a union of unbounded components.

The point of the next lemma is to show the path \( \gamma \) is homotopic to a path comprised of several \( H_0 \) subpaths in \( \Gamma - E \) and a path \( \gamma_1 \) “one level higher” than the path \( \gamma \). It will be inductively applied in Lemma 4.9.

**Remark 4.7 (Rslideup)** Lemma 4.8 is valid when \( F = \emptyset \). In this case, \( St^k(F) = \emptyset \) for all \( k \geq 0 \). We will applied this special case to paths \( \gamma \) in a level above the highest level of a certain compact set \( C \subset \Gamma \). In this case the homotopy \( H \) of Lemma 4.8 avoids \( C \) since \( p(H) \) has image above the highest level of \( C \).
Lemma 4.8 (slideup) Suppose $E$ and $F$ are finite subcomplexes of $\Gamma$, such that $E$ contains $St^{N_{4,2}+N_0}(F)$, and $g\Gamma_0 - E$ is a union of unbounded components for any $g \in G$. If $\gamma$ is an edge path in $v\Gamma_0 - St^{N_{4,2}+N_0}(F)$ with end points in $\Gamma - E$, then there is a homotopy $H$ with image in $\Gamma - F$, of $\gamma$ to an edge path $(\tau, e, \gamma, e'^{-1}, \tau'^{-1})$ where $e$ and $e'$ are edges labeled $t$; $\tau$ and $\tau'$ are $H_0$ paths in $\Gamma - E$; and $\gamma_1$ is an $H_0$ path in $\Gamma - F$ (so that $p(\gamma) + 1 = p(\gamma_1)$) with end points in $\Gamma - E$. Furthermore $p(H)$ has image in $[p(v), p(v) + 1]$.

Proof: In Figure 5, the path $(\tau_1, \tau_2)$ will be our path $\tau$ and combining the two homotopies $K_1$ and $K_2$ will give the homotopy $H$. Let $v$ be the initial and $w$ the terminal vertex of $\gamma$. Since $w\Gamma_0 - E$ is a union of unbounded components, there is an $H_0$ path $\tau'$ in $w\Gamma_0 - E$ from $w$ to a vertex $w_1 \in w\Gamma_0 - St(E)$. See Figure 5. There is an $H_0$ path $\tau_1$ in $v\Gamma_0 - E$ from $v$ to $v_1 \in v\Gamma_0 - St^{N_{4,2}+1}(E)$.

Note that:

$$v\Gamma_0 = w\Gamma_0 = v_1\Gamma_0 = w_1\Gamma_0.$$ 

By Lemma 3.2 there is an $H_0$ path $\tau_2$ of length $\leq N_{3,2}$ from $v_1$ to a vertex $v_2$ of $w_1\Gamma_1$. Note that $im(\tau_2) \subset v_1\Gamma - St(E)$ (and so $v_2 \in w_1\Gamma_1 - St(E)$). By Lemma 4.2 the path $\nu = (\tau_2^{-1}, \tau_1^{-1}, \gamma, \tau_1')$ is homotopic to an $H_1$ path $\gamma'$, by
a homotopy $K_1$ with image in $St^{N_4}_0(im(v))$. Since $E$ contains $St^{N_4+N_0}_0(F)$ and $\gamma$ avoids $St^{N_4+N_0}_0(F)$, the path $\nu$ avoids $St^{N_4+N_0}_0(F)$. Hence $K_1$ avoids $St^{N_0}_0(F)$. In particular, $\gamma'$ avoids $St^{N_0}_0(F)$.

Let $e$ be the edge labeled $t$ at $v_2$ (with end point $v_3$) and let $e'$ be the edge labeled $t$ at $w_1$ (with end point $w_2$). By Lemma 4.1, the path $\gamma'$ is homotopic to $(e,\gamma_1,e'^{-1})$ (where $\gamma_1$ is an $H_0$ path) by a homotopy $K_2$ with image in $St^{N_0}_0(im(\gamma'))$. Since $\gamma'$ avoids $St^{N_0}_0(F)$, $K_2$ (and $\gamma_1$) avoids $F$. Since $v_2,w_1 \in \Gamma-St(E)$, we have $v_3,w_2 \in \Gamma-E$. Now combine $K_1$ and $K_2$ to produce $H$ and let $\tau = (\tau_1,\tau_2)$ to finish the proof. □

**Lemma 4.9 (slideC)** Suppose $C$ and $E$ are finite subcomplexes of $\Gamma$ such that $p(C) \subset [A,B]$ and $E$ contains $St^{(B-A+2)(N_4+N_0)}_0(C)$. Also assume that if $g \in G$ then $g\Gamma_0-E$ is a union of unbounded components. If $\gamma$ is an edge path in $v\Gamma_0-St^{(B-A+2)(N_4+N_0)}_0(C)$, with end points in $\Gamma-E$ and $p(v) \in [A-1,B]$ then there is a homotopy of $\gamma$ (with image in $\Gamma-C$) to a path in $\Gamma-E$.

**Proof:** Apply Lemma 4.8 to $\gamma$ with $F = St^{(B-A+1)(N_4+N_0)}_0(C)$. This gives a homotopy $H'_1$ with image in $\Gamma-F$ of $\gamma$ to $(\beta_1,\gamma_1,\beta'_1)$ (here $\beta_1 = (\tau,e)$ and $\beta'_1 = (e'^{-1},\tau'^{-1})$) where $\beta_1$ and $\beta'_1$ have image in $\Gamma-E$ and $p(\gamma) + 1 = p(\gamma_1)$. Next, apply Lemma 4.8 to $\gamma_1$ with $F = St^{(B-A)(N_4+N_0)}_0(C)$ to produce $(\beta_2,\gamma_2,\beta'_2)$. So after applying Lemma 4.8 at most $B-A+2$ times, we have a homotopy $H'$ with image in $\Gamma-C$ of $\gamma$ to $(\beta,\tilde{\gamma},\beta')$ (the paths $\beta$ and $\beta'$ are the concatenations of the $\beta_i$ and $\beta'_i$ respectively) where $\beta$ and $\beta'$ have image in $\Gamma-E$ and $p(\tilde{\gamma}) = B+1$. Of course the image of $\tilde{\gamma}$ may intersect $E$, but $\tilde{\gamma}$ has image in level $B+1$ (above $C$) and we may apply Lemma 4.8 to $\tilde{\gamma}$ with $F = \emptyset$ (see Remark 4.7), to obtain a homotopy $\tilde{H}$ of $\tilde{\gamma}$ to $(\alpha_1,\tilde{\gamma}_1,\alpha'_1)$ where $\alpha_1$ and $\alpha'_1$ are $H_0$ paths in $\Gamma-E$, $p(\tilde{\gamma}_1) = B+2$ and $\tilde{H}$ has image in $[B+1,B+2]$ (and hence $\tilde{H}$ avoids $C$). Continue applying Lemma 4.8 (with $F = \emptyset$) to obtain $\alpha_2,\ldots,\alpha_n$ and $\alpha'_2,\ldots,\alpha'_n$ all with image in $\Gamma-E$ and path $\tilde{\gamma}_n$ with $p(\tilde{\gamma}_n) = B+n+1$ larger than the largest integer in $p(E)$. Let $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\alpha' = (\alpha'_1,\ldots,\alpha'_n)$. We have $\gamma$ is homotopic by a homotopy with image in $\Gamma-C$ to $(\beta,\alpha,\tilde{\gamma}_n,\alpha',\beta')$, a path with image in $\Gamma-E$. □

Let $C_1,C_2,\ldots$ be a collection of compact subcomplexes of $\Gamma$ such that $C_i$ is a subset of the interior of $C_{i+1}$ and $\bigcup_{i=1}^{\infty}C_i = \Gamma$. If $a$ and $b$ are vertices of $v\Gamma_0$ and $p(v) = n$, then define the degree of $(a,b)$ (written $deg(a,b)$) to be the largest integer $m$ such that there is an edge path $\alpha$ in $\Gamma-C_m$ from $a$ to $b$ such that $p(\alpha) \subset (-\infty,n]$. 19
The next result is in analogy with Lemma 4.4 of [Mih85]. The proof is elementary, but it is one of the main ideas of the overall argument. The other lemmas used in the proof of Theorem 1.4 explain how to carefully slide $H_0$ paths upward and $t$-edges horizontally. Lemma 4.10 will allow us to slide certain $H_0$ paths downward. It is in the proof of Lemma 4.10 where we need $H_0$ to be finitely presented (as opposed to only finitely generated).

**Lemma 4.10 (Like4.4)** Suppose $D$ is a compact subcomplex of $\Gamma$ and $W$ and $L$ are unbounded path components of $v\Gamma_0 - D$, (we allow $W = L$ as a possibility) then one of the following two statements holds:

(i) The set $W \times L$ contains a collection of pairs of vertices $(w_1, l_1), (w_2, l_2), \ldots$ such that all $w_j$ are distinct, all $l_j$ are distinct and $\deg(w_j, l_j) \geq j$, or

(ii) There are finite sets of vertices $S \subset W$ and $T \subset L$ such that all pairs of vertices of $(W - S) \times (L - T)$ have degree less than some fixed positive integer $N_1(W, L)$.

**Proof:** Begin selecting pairs of vertices $(w_j, l_j)$ satisfying the hypothesis of (i). If for some $j \geq 1$, $(w_j, l_j)$ cannot be selected to satisfy hypothesis (i), then all pairs of vertices of $(W - \{w_1, w_2, \ldots, w_{j-1}\}) \times (L - \{l_1, l_2, \ldots, l_{j-1}\})$ have degree less than $j$ (so that (ii) is satisfied). \qed

We are ready to combine our lemmas to prove the semistability part of Theorem 1.4. Let $C$ be a finite subcomplex of $\Gamma$ and $r$ be the proper ray at $\ast$, each of whose edges is labeled $t$. Choose integers $A$ and $B$ such that

$$p(C) \subset [A, B].$$

Let $D_1$ be a finite subcomplex of $\Gamma$ containing $St^{(B-A+2)(N_4+2N_0)}(C)$. Furthermore, if $v\Gamma_0$ intersects $D_1$ non-trivially and $p(v) = A - 1$ (there are only finitely many of these), then we require that all components of $v\Gamma_0 - D_1$ are unbounded. Next we define a finite subcomplex $D_2$ of $\Gamma$. Our first requirement is that $D_2$ contains $D_{1.6}(D_1)$ (see Figure 6). Suppose $v\Gamma_0$ intersects $D_1$ non-trivially, $p(v) = A - 1$ and that $W$ and $L$ are unbounded components of $v\Gamma_0 - D_1$ satisfying Lemma 4.10 (ii) (where $D_1$ plays the role of $D$). Then we require that $D_2$ contains the finite sets $S \subset W$ and $T \subset L$ and the set $C_{N_1(W, L)}$ given by Lemma 4.10 (recall that the $C_i$ are a cofinal sequence of finite subcomplexes of $\Gamma$). We also require that for each vertex $v \in \Gamma$ the
space $v\Gamma_0 - D_2$ is a union of unbounded components. An important fact follows immediately from Lemma 4.10:

(*) Suppose $v\Gamma_0$ intersects $D_1$ non-trivially, $p(v) = A - 1$, and $\alpha$ is an edge path in $\Gamma - D_2$ that begins and ends in $v\Gamma_0$ such that $p(\alpha) \subset (-\infty, A - 1]$. If the initial vertex of $\alpha$ belongs to the (unbounded) component $W$ of $v\Gamma_0 - D_1$ and the terminal vertex of $\alpha$ belongs to the (unbounded) component $L$ of $v\Gamma_0 - D_1$, then $W \times L$ satisfies condition (i) of Lemma 4.10.

Assume

$$p(D_2) \subset [A', B']$$

and $r([0, B']) \subset D_2$.

Let $E$ be an arbitrary finite subcomplex of $\Gamma$ (containing $D_2$). Enlarge $E$ so that for any vertex $v \in \Gamma$, $v\Gamma_0 - E$ is a union of unbounded path components. If $\gamma$ is an edge path loop in $\Gamma - D_2$ and based on $r$, our goal is to show that $\gamma$ is homotopic rel$\{r\}$ to an edge path loop in $\Gamma - E$ by a homotopy in $\Gamma - C$ (see Theorem 2.1(2)). Say $\gamma(0) = r(K)$ for some $K > B'$. If $\tau$ is an edge path ending at $r(K)$, each of whose edges is labeled $t^{-1}$, then $\gamma$ is homotopic rel$\{r\}$ by a homotopy in $\Gamma - D_2$ to $(\tau, \gamma, \tau^{-1})$ (see Figure 6). In particular, we may assume:

$$p(E) \subset [A'', B'']$$

and $\gamma(0) = r(K)$ for $K > B''$. 

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We cannot assume that \( \gamma \) is cyclically without backtracking. It may be that \( \gamma \) has the form \((\tau, \gamma', \tau')\) where cyclically \( \tau \) and \( \tau' \) backtrack. If we eliminating \( \tau \) and \( \tau' \) our path \( \gamma' \) may not be based on \( \tau \). Unfortunately, we cannot apply Lemmas 4.3 or 4.4. Instead, we will apply Lemma 4.5 to \( \gamma \), \( D_2 \) and \( E \). Repeated applications of this lemma allow us to move the maximal \( H_0 \) subpaths of \( \gamma \) that lie below level \( A-1 \) to level \( A-1 \) (modulo some paths in \( \Gamma - E \)). Notice that if \( \alpha \) is a maximal \( H_0 \) subpath of \( \gamma \) in a lowest level \( L < A-1 \) of \( \gamma \), then \( \alpha \) has image in \( \Gamma - D_2 \) so that hypotheses (A.) and (B.) of Lemma 4.5 are satisfied. Apply Lemma 4.5 to all maximal \( H_0 \) subpaths of \( \gamma \) in level 1. If \( M \) is one of these homotopies, then \( p(M) \subset [L, L+1] \) and so avoids \( C \). Say \( \hat{\gamma} \) is the resulting path. Then \( \gamma \) is homotopic to \( \hat{\gamma} \) by a homotopy that avoids \( C \). The only subpaths of \( \hat{\gamma} \) in level \( L \) have image in \( \Gamma - St(E) \). If \( \alpha \) is a maximal \( H_0 \) subpath of \( \hat{\gamma} \) in level \( L+1 \) then \( \alpha \) is a concatenation of \( H_0 \) subpaths of \( \gamma \) (which have image in \( \Gamma - D_2 \)) and paths of the form \( \gamma_i' \) arising from Lemma 4.5. As the end points of these \( \gamma_i' \) can be connected by edge paths in \( \Gamma - D_2 \) in levels \( \leq L+1 \), we conclude that if \( \hat{\alpha} \) is a maximal \( H_0 \) subpath of \( \hat{\gamma} \) in level \( L+1 \), then the end points of \( \hat{\alpha} \) can be joined by a path \( \beta \) in \( \Gamma - D_2 \) such that \( p(\beta(t)) \leq L+1 \) for all \( t \). If \( e \) and \( d \) are the \( t \) edges at the initial and terminal vertex of \( \hat{\alpha} \) then \( e \) (\( d \)) is either an edge of \( \gamma \) or is in \( \Gamma - E \). In either case, \( e \) and \( d \) avoid \( D_2 \). Hence Lemma 4.5 can be applied to \((e^{-1}, \hat{\alpha}, d)\). Repeated applications of Lemma 4.5 shows:

**Fact 1.** There is a homotopy \( \gamma M \gamma_1 \) with \( p(M) \subset \Gamma - C \) such that for any vertex \( v \) of \( \gamma_1 \) either \( p(v) \in [A-1, \infty) \) or \( v \in \Gamma - E \), the homotopy \( M \) fixes the edges of \( \gamma \) in levels \( A-1 \) and above, and the only edges of \( \gamma_1 \) in levels \( A \) and above are edges of \( \gamma \) (and hence these edges avoid \( D_2 \)). Any \( t \)-edge of \( \gamma_1 \) not in \( \Gamma - E \) is also a \( t \)-edge of \( \gamma \) and so is in \( \Gamma - D_2 \). Furthermore:

1. Suppose \( \tau \) is a maximal \( H_0 \) subpath of \( \gamma_1 \) with \( p(\tau) = A-1 \). Let \( v_0 \) and \( v_1 \) be the end points \( \tau \). Then Lemma 4.5 implies there is an edge path \( \beta \) with \( im(\beta) \subset \Gamma - D_2 \) connecting \( v_0 \) to \( v_1 \) with \( p(\beta) \subset (-\infty, A-1] \). If \( W (L) \) is the component of \( v\Gamma_0 - D_1 \) containing \( v_0 \) (\( v_1 \)) then \((*) \) implies \( W \times L \) satisfies condition (i) of Lemma 4.10.

The next step is to move all \( t \) edges of \( \gamma_1 \) outside of \( E \). Apply Lemma 4.6 (with \( C = D_1 \) and \( D = D_2 \)) to each \( t \)-edge of \( \gamma_1 \) not already in \( \Gamma - E \), to obtain a homotopy \( \gamma M_2 \gamma_2 \) in \( \Gamma - D_1 \) where each \( t \)-edge of \( \gamma_2 \) has image in \( \Gamma - E \). It is important to observe that if \( e \) is a \( t \) edge of \( \gamma_1 \) connecting levels \( A-1 \) and \( A \), then the homotopy \( M_2 \) uses Lemma 4.6 to replace \( e \) by a path \((\alpha, e_1, \beta) \) in \( \Gamma - D_1 \) connecting the end points of \( e \) such that \( p(\alpha) = A-1 \),
\[ p(\beta) = A \text{ and } e_1 \text{ (in } \Gamma - E) \text{ has label } t. \text{ Since } \alpha \text{ avoids } D_1, \text{ there is an analogue of (**) for } \gamma_2:\]

(**) If \( \tau \) is a maximal subpath of \( \gamma_2 \) in level \( A - 1 \), then \( \tau = (\alpha_0^{-1}, \tau', \alpha_1) \) where \( \tau' \) is a maximal subpath of \( \gamma_1 \) in level \( A - 1 \) and each of the paths \( \alpha_0 \) and \( \alpha_1 \) is either trivial (if the corresponding end point of \( \tau' \) is already in \( \Gamma - E \)) or comes from our replacement of edges labeled \( t \) at the end points of \( \tau' \) (see Figure 7). So \( \alpha_0 \) and \( \alpha_1 \) are in level \( A - 1 \), avoid \( D_1 \) and both end in \( \Gamma - E \). By (**) the end points of \( \tau' \) are in unbounded components \( W \) and \( L \) of \( v\Gamma_0 - D_1 \) and \( W \times L \) satisfies condition (i) of Lemma 4.10. As \( \alpha_0 \) and \( \alpha_1 \) avoid \( D_1 \), the end points of \( \tau \) are in \( W \) and \( L \) as well.

Again let \( \tau \) (with end points \( v_0 \) and \( v_1 \)) be a maximal subpath of \( \gamma_2 \) in level \( A - 1 \). Then \( v_0, v_1 \in v_0\Gamma_0 - E \). If \( W_i \) is the component of \( v_0\Gamma_0 - D_1 \) containing \( v_i \) (for \( i \in \{0, 1\} \)) then (**) implies that \( W_0 \times W_1 \) satisfies condition (i) of Lemma 4.10. Choose \( N > 1 \) such that \( E \subset C_N \) and vertex pairs \((w_0, w_1) \in W_0 \times W_1 \) such that \( \deg(w_0, w_1) \geq N \). For \( i \in \{0, 1\} \) let \( \psi_i \) be
an edge path in $W_i$ from $v_i$ to $w_i$. Let $\mu$ be an edge path from $w_0$ to $w_1$ in $\Gamma - C_M (\subset \Gamma - E)$ such that $p(\mu) \subset (-\infty, A - 1]$ (see Figure 8). Consider the edge path loop $\rho = (\tau, \psi_1, \mu^{-1}, \psi_0^{-1})$ (in levels $\leq A - 1$). The loop $\rho$ is homotopic (in the image of $\rho$) to a loop $\rho'$ without backtracking. Lemma 4.3 can then be applied to $\rho'$. After a sequence of backtracking elimination and applications of Lemma 4.3 the resulting loop has image in $v_0\Gamma_0$ and so is homotopically trivial in level $A - 1$. In particular $\tau$ is homotopic to $(\psi_0, \mu, \psi_1^{-1})$ by a homotopy in levels $\leq A - 1$ and hence in $\Gamma - C$. Since $\mu$ has image in $\Gamma - E$ and (for $i \in \{1, 2\}$) $\psi_i$ is an edge path in level $A - 1$ with end points in $\Gamma - E$ and image in $\Gamma - D_1$, we have shown:

(†) The path $\gamma$ is homotopic rel $\{r\}$ to an edge path loop $\gamma_3$ in $\Gamma - D_1$ by a homotopy in $\Gamma - C$ such that each $t$-edge of $\gamma_3$ has image in $\Gamma - E$ and each edge of $\gamma_3$ in any level $< A - 1$ is in $\Gamma - E$.

Next suppose $\alpha$ is a maximal subpath of $\gamma_3$ in level $L \in [A - 1, B]$. The end points of $\alpha$ are in $\Gamma - E$ and $\alpha$ has image in $\Gamma - D_1$. By Lemma 4.9, the path $\alpha$ is homotopic rel $\{0, 1\}$ to an edge path in $\Gamma - E$ by a homotopy in $\Gamma - C$. Apply Lemma 4.9 to all such $\alpha$. The resulting path, call it $\gamma_4$, is homotopic rel $\{0, 1\}$ to $\gamma_3$ by a homotopy in $\Gamma - C$. Notice that each edge of $\gamma_4$ in a level $\leq B$ is in $\Gamma - E$. At this point we need only be concerned about maximal subpaths of $\gamma_4$ which are in a level above level $B$ (and hence above the compact set $C'$). Repeated applications of Lemma 4.8 to such paths with $F = \emptyset$ (see Remark 4.7) move such paths out of $E$ by homotopies in $\Gamma - C$. This finishes the proof that $G$ has semistable fundamental group at $\infty$. \hfill \Box

## 5 The Proof of the Main Theorem

Before proving Theorem 1.3 we prove Lemma 5.1 (by an argument that parallels the proof of Theorem 3 of [Mih86b]).

**Lemma 5.1 (SS2)** Suppose $G = A *_C B$ or $G = A *_C$, where $A$ and $B$ are finitely presented, $C$ is infinite, finitely generated and in the first case, of finite index in $B$. If $A$ is 1-ended and semistable at $\infty$, then $G$ is 1-ended and semistable at $\infty$.

**Proof:** We consider the case $G = A *_C B$ first. If $B = C$ or $A = C$, then $A$ has finite index in $G$ and so $G$ is 1-ended and semistable at $\infty$ (since semistability and the number of ends of a group are quasi-isometry invariants). We assume
A \neq C \neq B. Let \( \mathcal{A} \cup \mathcal{B} \) be a finite generating set for \( G \), where \( \mathcal{A} \) generates \( A \), \( \mathcal{B} \) generates \( B \) and \( \mathcal{A} \cap \mathcal{B} = \mathcal{C} \) generates \( C \). Let \( \mathcal{P} \) be a finite presentation for \( G \) with generators \( \mathcal{A} \cup \mathcal{B} \). Let \( \Gamma \) be the Cayley 2-complex for \( \mathcal{P} \). We show \( \Gamma \) has semistable fundamental group at \( \infty \). Let \( \Gamma_H \subset \Gamma \) be the corresponding Cayley graph of \( H = A \cdot B \) or \( C \) with respect to \( \mathcal{A} \), \( \mathcal{B} \), and \( \mathcal{C} \) respectively. Choose a geodesic edge path line \( \ell = (\ldots, c_{-1}, c_0, c_1, \ldots) \) in \( \Gamma_C \) (so that each edge \( c_i \) has label \( c_i \in \mathcal{C}^{\pm 1} \)). Assume the initial vertex of \( c_0 \) is \( * \), the identity vertex of \( \Gamma_C \). Let \( r^+ = (c_0, c_1, \ldots) \) and \( r^- = (c_1^{-1}, c_2^{-1}, \ldots) \) be the two (opposite direction) geodesic rays (beginning at \( * \)) of \( \ell \). Recall that the vertices of \( \Gamma \) are the elements of \( G \). Given any compact set \( D \subset \Gamma \) there are only finitely many vertices \( v \in \Gamma \) such that both \( vr^+ \) and \( vr^- \) intersect \( D \).

Let \( s = (s_0, s_1, \ldots) \) be a proper edge path ray in \( \Gamma \) with initial point \( * \). Then the label of \( s_i \) is \( s_i \in \mathcal{A}^{\pm 1} \cup \mathcal{B}^{\pm 1} \). Let \( * = w_0, w_1, \ldots \) be the consecutive vertices of \( s \). For any vertex \( v \in \Gamma \) let \( B(v, N) \) be the ball of radius \( N \) about \( v \) in the 1-skeleton of \( \Gamma \). Select \( r_0 = r^+ \), and for \( i > 0 \), set \( r_i \) equal to \( w_i r^+ \) or \( w_i r^- \) so that \( r_i \) has finite index in \( \mathcal{C} \). Given any compact subset \( \Gamma \) of \( \Gamma \), only finitely many \( r_i \) intersect \( B(*, N) \). (See Figure 9.)

If \( s_i \in \mathcal{A}^{\pm 1} \) then the rays \( r_i \) and \((s_i, r_{i+1})\) are both \( \mathcal{A} \)-rays and so are properly homotopic \( rel\{w_i\} \) by a homotopy \( K_i \) in \( w_i \Gamma_A \) (see Theorem 2.1(4)). There may be infinitely many \( j \) such that \( w_j \Gamma_A = w_i \Gamma_A \). Let \( J_i = \{ j \geq 0 : w_j \Gamma_A = w_i \Gamma_A \} \). Theorem 2.1(3) implies that the set of all \( K_j \) such that \( j \in J_i \), can be selected so that only finitely many have image which intersects any given compact subset of \( \Gamma \). Since only finitely many translates of \( \Gamma_A \) intersect a given compact set, we may assume that: If \( s_i \in \mathcal{A}^{\pm 1} \) then \( K_i \) is defined and for any compact set \( D \), only finitely many such \( K_i \) have image that intersects \( D \).

If \( s_k \in \mathcal{B}^{\pm 1} \) then the ray \((s_k^{-1}, r_k)\) is a \( \mathcal{B} \) ray. Since \( C \) has finite index in \( B \), there is a proper \( \mathcal{C} \) ray \( q_{k+1} \) at \( w_{k+1} \) that is properly homotopic \( rel\{w_{k+1}\} \)
to \((s_k^{-1}, r_k)\) by a homotopy \(K_k'\) whose image is within \(N_{4,2} + N_{3,2}\) of the image of \(r_k\). (Let \(\beta_0\) be the trivial path at \(w_{k+1}\). If \(e_i\) is the \(i^{th}\) edge of \((s_k^{-1}, r_k)\) then let \(\beta_i\) be an edge path of length \(\leq N_{3,2}\) from the terminal point of \(e_i\) to a vertex of \(w_{k+1} \Gamma_C\). Apply Lemma 4.2 to the paths \((\beta_i^{-1}, e_i, \beta_i)\) and combine the resulting homotopies.)

Now \(q_{k+1}\) and \(r_{k+1}\) are both \(\mathcal{A}\) (in fact, \(\mathcal{C}\)) rays at \(w_{k+1}\) and so are properly homotopic by a homotopy \(K''_k\) in \(w_{k+1} \Gamma_A\). Again, Theorem 2.1(3) allows us to choose the \(K''_k\) such that for any compact set \(D\), only finitely many such \(K''_k\) have image that intersect \(D\).

Combine \(K_k'\) and \(K''_k\) to form \(K_k\), a proper homotopy of \(r_k\) to \((s_k, r_{k+1})\). Combining the \(K_i\) for \(i \geq 0\), provides a proper homotopy from \(r^+\) to \(s\) (see Figure 9). Hence every proper edge path ray at \(*\) is properly homotopic to \(r^+\) and \(G\) is semistable at \(\infty\) (see Theorem 2.1(4)). As any two proper rays in \(\Gamma\) are properly homotopic, the space \(\Gamma\) (and hence the group \(G\)) is 1-ended.

In the second case semistability follows directly from Theorem 1.1. It remains to show that \(A \ast_C \mathcal{C}\) is 1-ended. Let \(\mathcal{C} \subset \mathcal{A}\) be finite generating sets for \(\mathcal{C}\) and \(\mathcal{A}\) respectively. Let \(t\) be the stable letter and for each \(c \in \mathcal{C}\), let \(w_c\) be an \(\mathcal{A}\) word such that \(t^{-1}ct = w_c\) is a conjugation relation for \(G\). Let \(\Gamma\) be the Cayley graph of \(G\) with respect to \(\mathcal{A} \cup \{t\}\) and let \(\Gamma_C \subset \Gamma\) be the Cayley graph of \(\mathcal{C}\) with respect to \(\mathcal{C}\). Again choose a geodesic edge path line \(\ell = (\ldots, c_{-1}, c_0, c_1, \ldots)\) in \(\Gamma_C\), so that each edge \(c_i\) has label \(\bar{c}_i \in \mathcal{C}^\pm\). Assume the initial vertex of \(c_0\) is \(*\), the identity vertex of \(\Gamma_C\). Let \(r^+ = (c_0, c_1, \ldots)\) and \(r^- = (c_{-1}, c_{-2}, \ldots)\) be the two (opposite direction) geodesic rays (beginning at \(*\)) of \(\ell\). Given any compact set \(D \subset \Gamma\) there are only finitely many vertices \(v \in \Gamma\) such that both \(vr^+\) and \(vr^-\) intersect \(D\). Choose \(E(D)\) a finite subcomplex of \(\Gamma\) such that if \(v\) is a vertex of \(\Gamma - E\) then either \(vr^+\) or \(vr^-\) avoids \(D\). It is enough to show that any two vertices \(v\) and \(w\) in \(\Gamma - E\) can be joined by an edge path in \(\Gamma - D\). Let \(r_v\) be \(vr^+\) or \(vr^-\) avoiding \(D\). Similarly select \(r_w\) avoiding \(D\). It is enough to show that \(r_v\) and \(r_w\) converge to the same end of \(\Gamma\). Let \(\alpha = (a_0, a_1, \ldots, a_n)\) be an edge path from \(v\) to \(w\). Assume \(a_i\) has label \(\bar{a}_i \in \mathcal{A}^\pm \cup \{t^{\pm 1}\}\). For \(i \in \{1, \ldots, n\}\) let \(v_i\) be the initial vertex of \(a_i\) and let \(r_i\) be \(v_i r^+\). Let \(v_0 = v, v_{n+1} = w, r_0 = r_v\) and \(r_{n+1} = r_w\). It is enough to show that \(r_i\) and \(r_{i+1}\) converges to the same end of \(\Gamma\) for \(i \in \{0, \ldots, n\}\). Choose \(i \in \{0, 1, \ldots, n\}\). If \(\bar{a}_i \in \mathcal{A}^\pm\) then the proper rays \(r_i\) and \((a_i, r_{i+1})\) form a proper line in the 1-ended space \(v_i \Gamma_{A}\). Hence \(r_i\) and \((a_i, r_{i+1})\) converge to the same end of \(v_i \Gamma_{A}\). In particular, \(r_i\) and \(r_{i+1}\) converge to the same end of \(\Gamma\). If \(\bar{a}_i = t\) then write \(r_i = (e_1, e_2, \ldots)\). Let \(t_k\) be the edge labeled \(t\) at the initial vertex of \(e_k\). There is an \(\mathcal{A}\) path \(z_j\)
(labeled by one of the conjugation words $w_k^±$), connecting the end points of $(t_k^{-1}, e_k, t_{k+1})$. As $(z_1, z_2, \ldots)$ tracks $r_i$, it is proper and converge to the same end of $\Gamma$ as does $r_i$. Also $(z_1, z_2, \ldots)$ and $r_{i+1}$ are in the 1-ended space $v_{i+1}\Gamma_A$, so they converge to the same end of $\Gamma$. Hence $r_i$ and $r_{i+1}$ converge to the same end of $\Gamma$. If $\bar{a}_i = t^{-1}$ then consider the $t$ edges at the vertices of $r_{i+1}$ and proceed as before. □

**Proof:** (of Theorem 1.3) Suppose $\mathcal{G}$ is our graph of groups decomposition of $G$. First a reduction. Suppose there is a vertex $v$ of $\mathcal{G}$ such that $G_v$ is 1-ended and semistable at $\infty$. If there is an edge (loop) $e(v, v)$ of $\mathcal{G}$, then the HNN-extension of $G_v$ determined by $e(v, v)$ is 1-ended and semistable at $\infty$ by Lemma 5.1. In this situation, replace $v$ by a single vertex $v'$ with vertex group the HNN-extension of $G_v$. The resulting graph of groups is still a decomposition of $G$ and satisfies the hypothesis of our theorem. We may assume:

(0) If $v$ is a vertex of $\mathcal{G}$ and $G_v$ is 1-ended and semistable at $\infty$, then no edge at $v$ is a loop.

Next, suppose there is a vertex $v$ of $\mathcal{G}$ such that $G_v$ is not both 1-ended and semistable at $\infty$ and an edge (loop) $e(v, v)$ (giving an HNN-extension). Then there are subgroups $G_1$ and $G_2$ of $G_v$ (both isomorphic to $G_e$) and an isomorphism $\phi : G_1 \to G_2$ determining a subgroup of $G$ that is an HNN-extension of $G_v$. If either $G_1$ or $G_2$ has finite index in $G_v$, then Theorem 1.4 implies that the resulting HNN-extension (call it $G_v'$) of $G_v$ is 1-ended and semistable at $\infty$. In this case $v$ and the loop $e(v, v)$ are replaced by $v'$. If $\mathcal{G}$ has only 1-vertex $v$, then our hypotheses and reductions imply that $\mathcal{G}$ cannot have an edge so that $G = G_v$. Our hypotheses imply that $G$ is 1-ended and semistable at $\infty$. Assume the result is true for $\mathcal{G}$ with $N \geq 1$ vertices. Now assume $\mathcal{G}$ has $N + 1$ vertices. Suppose $e(v, w)$ is an edge of $\Gamma$, and the vertex group $G_v$ (of $\mathcal{G}$) is 1-ended and semistable at $\infty$. Our reduction (0) implies that $v \neq w$. If $G_e$ has finite index in $G_w$, then Lemma 5.1 implies that the subgroup of $G$ generated by $G_v$ and $G_w$ is 1-ended and semistable at $\infty$. If $G_w$ is 1-ended and semistable at $\infty$ then Theorem 2.3 implies that the subgroup of $G$ generated by $G_v$ and $G_w$ is 1-ended and semistable at $\infty$. In either case, reducing $\mathcal{G}$ by combining $e$ into a single vertex (with group generated by $G_v \cup G_w$) gives a graph of groups decomposition of $G$ with fewer vertices and satisfying the hypothesis of our theorem. Inductively, $G$ is 1-ended and semistable at $\infty$. So we may assume:
(1) If $v$ is a vertex of $G$ and $G_v$ is 1-ended and semistable at $\infty$, then each edge $e(v, w)$ of $G$ is such that $v \neq w$, $G_e$ has infinite index in $G_w$, and $G_w$ is not 1-ended and semistable at $\infty$.

Similarly, Theorems 1.4 and 2.2 imply that we may assume:

(2) No edge (loop) $e(v, v)$ of $G$ is such that $G_e$ has finite index in $G_v$ and no edge $e(v, w)$ with $v \neq w$ is such that $G_e$ has finite index in both $G_v$ and $G_w$.

By (1), there is a vertex $v_0 \in G$ such that $G_{v_0}$ is not 1-ended and semistable at $\infty$. By our hypothesis there is an edge $e_0 = e(v_0, v_1)$ of $G$ such that $G_{e_0}$ has finite index in $G_{v_0}$. By (2), $v_0 \neq v_1$ and $G_{e_0}$ has infinite index in $G_{v_1}$. By (1), $G_{v_1}$ is not 1-ended and semistable at $\infty$. By our hypothesis there is an edge $e_1 = e(v_1, v_2)$ of $G$ such that $G_{e_1}$ has finite index in $G_{v_1}$ (so $e_1 \neq e_0$) and by (2), $v_1 \neq v_2$ and $G_{e_1}$ has infinite index in $G_{v_2}$. By (1), $G_{v_2}$ is not 1-ended and semistable at $\infty$. Continuing, we arrive at a simple (non-crossing) edge path loop $L$ in $G$ (relabeling if necessary) $e_0, e_1, \ldots, e_n$ where $e_i = e(v_i, v_{i+1})$, $v_i \neq v_{i+1}$, $v_{n+1} = v_0$, no $G_{e_i}$ is 1-ended and semistable at $\infty$, and $G_{e_i}$ has finite index in $G_{v_i}$ and infinite index in $G_{v_{i+1}}$. It is enough to show the fundamental group $L$ of this loop of groups is 1-ended and semistable at $\infty$ (since then we could collapse the loop to a single vertex and the resulting graph of groups decomposition of $G$ would have fewer vertices and still satisfy the hypothesis of our theorem).

Let $A$ be the fundamental group of the graph of groups obtained by removing $e_n$ from $L$. We consider $L$ to be an HNN-extension of $A$, with stable letter $t$ and $t^{-1}ct \in G_{v_0}$ for $c \in G_{e_n} < G_{v_0}$. The group $C = G_{e_0} \cap G_{e_1} \cap \cdots \cap G_{e_n}$ has finite index in $G_{v_0}$. The subgroup of $L$ generated by $G_{v_0}$ and $t$ is an HNN extension with base $G_0$ and associated subgroup $C$ (of finite index in $G_0$). Theorem 1.4 implies this subgroup is finitely presented, 1-ended and semistable at $\infty$. Similarly each of the vertex groups of $L$ belong to a subgroup of $L$ that is finitely presented, 1-ended and semistable at $\infty$. Repeated applications of Theorem 2.3 imply that $L$ is 1-ended and semistable at $\infty$. $\square$

References

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