CONDITION FOR THE INTERSECTION OCCUPATION MEASURE TO BE ABSOLUTELY CONTINUOUS

X. Chen

UDC 519.21

Given i.i.d. \( \mathbb{R}^d \)-valued stochastic processes \( X_1(t), \ldots, X_p(t), \ p \geq 2 \), with stationary increments, a minimal condition is provided for the occupation measure

\[
\mu_t(B) = \int_{[0,t]^p} 1_B(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p))ds_1 \ldots ds_p, \quad B \subset \mathbb{R}^{d(p-1)},
\]

to be absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{d(p-1)} \). An isometry identity related to the resulting density (known as intersection local time) is also established.

1. Main Theorem

Let \( X(t) \) be a stochastic process taking values in \( \mathbb{R}^d \) with \( X(0) = 0 \) and let \( p_t(x) \ (x \in \mathbb{R}^d) \) be the density function of \( X(t) \). Assume that, for any \( 0 \leq s < t \),

\[
X(t) - X(s) \overset{d}{=} X(t - s).
\] (1.1)

Let \( X_1(t), \ldots, X_p(t) \) be independent copies of \( X(t) \). Given \( t_1, \ldots, t_p \geq 0 \) and \( x \in \mathbb{R}^{d(p-1)} \), the intersection local time \( \alpha(t_1, \ldots, t_p, x) \) of \( X_1(t), \ldots, X_p(t) \) formally written as

\[
\alpha(t_1, \ldots, t_p, x) = \int_{0}^{t_1} \ldots \int_{0}^{t_p} \delta_x(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p))ds_1 \ldots ds_p \quad (1.2)
\]
is defined as the Radon–Nikodym derivative of the occupation measure

\[
\mu_{t_1, \ldots, t_p}(B) = \int_{0}^{t_1} \ldots \int_{0}^{t_p} 1_B(X_1(s_1) - X_2(s_2), \ldots, X_{p-1}(s_{p-1}) - X_p(s_p))ds_1 \ldots ds_p \quad (1.3)
\]

with respect to the Lebesgue measure on \( \mathbb{R}^{d(p-1)} \). The most investigated setting is when \( X(t) \) is a Brownian motion. The criterion for the mutual intersection of independent Brownian motions was completed by Dvoretzky, Erdős, and Kakutani [3, 4] in the 1950s. Their work was followed by the extensive investigations, either of the trajectory properties of the Brownian intersection local times (see, e.g., [1, 7, 8]) or of the extensions to some other stochastic processes (see, e.g., [2, 5, 6]).

University of Tennessee, Knoxville, USA; e-mail: xchen@math.utk.edu.

Published in Ukrains’kyi Matematychnyi Zhurnal, Vol. 72, No. 9, pp. 1304–1312, September, 2020. Ukrainian DOI: 10.37863/umzh.v72i9.6278. Original article submitted August 20, 2020.

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A critical step in constructing the intersection local times is to establish the absolute continuity of \( \mu_{t_1, \ldots, t_p}(\cdot) \) with respect to the Lebesgue measure on \( \mathbb{R}^{d(p-1)} \). In the literature, this is mostly done either for Gaussian processes \([2, 6]\) or for Markov processes \([5]\) with Gaussian/Markovian property used as the main tool. In the present paper, we do this without using the Gaussian/Markovian property.

The main result of the paper is the following theorem:

**Theorem 1.1.** Under (1.1), assume that there exists \( \theta > 0 \) such that

\[
\int_{\mathbb{R}^d} \left[ \int_0^\infty e^{-\theta t} p_t(x) ds \right]^p dx < \infty. \tag{1.4}
\]

Then

\[
\mathbb{P} \left\{ \mu_{t_1, \ldots, t_p} \text{ is absolutely continuous for all } t_1, \ldots, t_p \geq 0 \right\} = 1. \tag{1.5}
\]

Further, the density \( \alpha(t_1, \ldots, t_p, x) \) given by (1.2) lives in the \( L^2 \)-space, i.e.,

\[
\mathbb{P} \left\{ \int_{\mathbb{R}^{d(p-1)}} [\alpha(t_1, \ldots, t_p, x)]^2 dx < \infty \text{ for all } t_1, \ldots, t_p \geq 0 \right\} = 1 \tag{1.6}
\]

and satisfies the isometry identity

\[
\mathbb{E} \int_{\mathbb{R}^{d(p-1)}} [\alpha(t_1, \ldots, t_p, x)]^2 dx = \int_{\mathbb{R}^d} \left[ \prod_{j=1}^p \int_0^{t_j} \int_{t_j}^{t_j+s} \left\{ p_s(x) + p_s(-x) \right\} ds \right] dx \tag{1.7}
\]

for any \( t_1, \ldots, t_p \geq 0 \).

**Remark 1.1.** In a special case where \( X(t) \) is symmetric, i.e., \( X(-t) \overset{d}{=} X(t) \) (or \( p_s(x) = p_s(-x) \)), for every \( t \geq 0 \), the isometry identity (1.7) becomes

\[
\mathbb{E} \int_{\mathbb{R}^{d(p-1)}} [\alpha(t_1, \ldots, t_p, x)]^2 dx = 2^p \int_{\mathbb{R}^d} \left[ \prod_{j=1}^p \int_0^{t_j} \int_{t_j}^{t_j+s} p_s(x) ds \right] dx. \tag{1.8}
\]

**Proof of Theorem 1.1.** For any measure \( \mu \) on \( \mathbb{R}^{d(p-1)} \), its Fourier transform is defined as

\[
\hat{\mu}(\lambda_1, \ldots, \lambda_{p-1}) = \int_{\mathbb{R}^{d(p-1)}} \exp \left\{ i \sum_{j=1}^{p-1} \lambda_j x_j \right\} \mu(dx_1 \ldots dx_{p-1}).
\]

For any \( \theta > 0 \), we define a random measure as follows:

\[
\mu_\theta(B) = \int_{(\mathbb{R}^+)^p} \exp \left\{ -\theta \sum_{j=1}^p t_j \right\} \mu_{t_1, \ldots, t_p}(B) dt_1 \ldots dt_p.
\]
Note that
\[ \mu_{t_1, \ldots, t_p}(\cdot) \leq \exp \left\{ \theta \sum_{j=1}^{p} t_j \right\} \mu_{\theta}(\cdot), \quad t_1, \ldots, t_p \geq 0. \]

To prove (1.5) and (1.6), all that we need is to establish the almost sure absolute continuity of \( \mu_{\theta} \) (for some \( \theta > 0 \)) and the square integrability of the consequential density of the measure \( \mu_{\theta}(\cdot) \). According to the Plancherel–Parseval theorem (Theorem B.3 in [1, p. 302]), this is validated by the integrability
\[
\int_{(\mathbb{R}^d)^{p-1}} |\mathbb{E}[\hat{\mu}_{\theta}(\lambda_1, \ldots, \lambda_{p-1})]|^2 d\lambda_1 \ldots d\lambda_{p-1} < \infty. \tag{1.9}
\]

To establish (1.9) and isometry (1.7), we first prove that
\[
\int_{(\mathbb{R}^d)^{p-1}} \mathbb{E}[\hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1})]^2 d\lambda_1 \ldots d\lambda_{p-1}
= (2\pi)^{(p-1)} \int_{\mathbb{R}^d} \left[ \prod_{j=1}^{p} \int_{0}^{t_j} (t_j - s) \left\{ p_s(x) + p_s(-x) \right\} ds \right] \left[ \prod_{j=1}^{p} \int_{0}^{t_j} \exp \left\{ i \sum_{j=1}^{p-1} \lambda_j (X_j(s_j) - X_{j+1}(s_{j+1})) \right\} ds_1 \ldots ds_p \right] dx, \quad t_1, \ldots, t_p \geq 0. \tag{1.10}
\]

Note that
\[
\hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) = \int_{[0,t_1] \times \ldots \times [0,t_p]} \exp \left\{ i \sum_{j=1}^{p-1} \lambda_j (X_j(s_j) - X_{j+1}(s_{j+1})) \right\} ds_1 \ldots ds_p
= \prod_{j=1}^{p} \int_{0}^{t_j} \exp \left\{ i(\lambda_j - \lambda_{j-1})X_j(s) \right\} ds.
\]

Here and elsewhere, we follow the convention that \( \lambda_0 = \lambda_p = 0 \).

Therefore,
\[
|\hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1})|^2 = \hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \bar{\hat{\mu}}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1})
= \prod_{j=1}^{p} \int_{0}^{t_j} \int_{0}^{t_j} \exp \left\{ i(\lambda_j - \lambda_{j-1})(X_j(s) - X_j(r)) \right\} ds dr.
\]

We take expectations on both sides. By the independence of \( X_1, \ldots, X_p \) and by the increment stationarity given in (1.1), we get
\[
\mathbb{E}|\hat{\mu}(\lambda_1, \ldots, \lambda_{p-1})|^2 = \prod_{j=1}^{p} \int_{0}^{t_j} \int_{0}^{t_j} \mathbb{E} \exp \left\{ i(\lambda_j - \lambda_{j-1})(X_j(s) - X_j(r)) \right\} ds dr
\]
\[
= \prod_{j=1}^{p} \left\{ \int \int_{\{0 \leq r < s \leq t_j\}} \mathbb{E} \exp \left\{ i(\lambda_j - \lambda_{j-1})X(s - r) \right\} \right. \\
+ \int \int_{\{0 \leq s < r \leq t_j\}} \mathbb{E} \exp \left\{ -i(\lambda_j - \lambda_{j-1})X(r - s) \right\} dsdr \left. \right\} \\
= 2^p \prod_{j=1}^{p} \int_{0}^{t_j} (t_j - s) \varphi_s(\lambda_j - \lambda_{j-1})ds = 2^p \prod_{j=1}^{p} Q_{t_j}(\lambda_j - \lambda_{j-1}),
\]

where
\[
\varphi_t(\lambda) = \frac{1}{2} \left\{ \mathbb{E}e^{i\lambda X(t)} + \mathbb{E}e^{-i\lambda X(t)} \right\} = \mathbb{E}e^{iRX(t)}, \quad Q_t(\lambda) = \int_{0}^{t} (t - s) \varphi_s(\lambda)ds,
\]

and \( R \) is a Bernoulli random variable independent of \( X(t) \) with the distribution
\[
\mathbb{P}\{R = 1\} = \mathbb{P}\{R = -1\} = 1/2.
\]

Integrating both sides, we find
\[
\int_{(\mathbb{R}^d)^{p-1}} |\hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1})|^2 d\lambda_1 \ldots d\lambda_{p-1} = 2^p \int_{(\mathbb{R}^d)^{p-1}} \prod_{j=1}^{p} Q_{t_j}(\lambda_j - \lambda_{j-1})d\lambda_1 \ldots d\lambda_{p-1}
\]
\[
= 2^p \int_{(\mathbb{R}^d)^{p-1}} \prod_{j=1}^{p} Q_{t_j}(\lambda_j - \lambda_{j-1})d\lambda_1 \ldots d\lambda_{p-1}
\]
\[
= 2^p \int_{(\mathbb{R}^d)^{p-1}} Q_{t_p}(\gamma_1 - \ldots - \gamma_{p-1}) \prod_{j=1}^{p-1} Q_{t_j}(\gamma_j)d\gamma_1 \ldots d\gamma_{p-1},
\]

where the last step follows from the substitution \( \gamma_1 = \lambda_1, \gamma_2 = \lambda_2 - \lambda_1, \ldots, \gamma_{p-1} = \lambda_{p-1} - \lambda_{p-2} \) (we now recall the convention that \( \lambda_0 = \lambda_p = 0 \)).

We set
\[
G_t(x) = \int_{0}^{t} (t - s) \frac{p_s(x) + p_s(-x)}{2} ds.
\]

The following steps are required to justify the applicability of the Fubini theorem. Note that
\[
\int_{\mathbb{R}^d} G_t(x)dx = \int_{0}^{t} (t - s) \left[ \int_{\mathbb{R}^d} \frac{p_s(x) + p_s(-x)}{2} dx \right] ds = \int_{0}^{t} (t - s)ds = \frac{t^2}{2} < \infty.
\]
Since
\[ Q_t(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \cdot x} G_t(x) \, dx, \]
we have \(|Q_t(\lambda)| \leq t^2 / 2\). In particular, for any \(\varepsilon > 0\),
\[ \int_{(\mathbb{R}^d)^{p-1} \times \mathbb{R}^d} G_{t_p}(x) \prod_{j=1}^{p-1} |Q_{t_j}(\lambda_j)| \exp \left\{ -\frac{\varepsilon}{2} |\lambda_j|^2 \right\} \, d\lambda_1 \ldots d\lambda_{p-1} \, dx \]
\[ \leq \left( \prod_{j=1}^{p-1} \frac{t_j^2}{2} \right) \left( \int_{\mathbb{R}^d} \exp \left\{ -\frac{\varepsilon}{2} |\lambda_j|^2 \right\} \, d\lambda \right)^{p-1} < \infty. \]

This justifies the application of the Fubini theorem in the following way:
\[ \int_{(\mathbb{R}^d)^{p-1}} Q_{t_p}(\gamma_1 - \ldots - \gamma_{p-1}) \prod_{j=1}^{p-1} Q_{t_j}(\gamma_j) \exp \left\{ -\frac{\varepsilon}{2} |\gamma_j|^2 \right\} \, d\gamma_1 \ldots d\gamma_{p-1} \]
\[ = \int_{(\mathbb{R}^d)^{p-1}} \left[ \int_{\mathbb{R}^d} e^{-i(\gamma_1 + \ldots + \gamma_{p-1}) \cdot x} \prod_{j=1}^{p-1} Q_{t_j}(\gamma_j) \exp \left\{ -\frac{\varepsilon}{2} |\gamma_j|^2 \right\} \, d\gamma_j \right] \, dx \]
\[ = \int_{\mathbb{R}^d} G_{t_p}(x) \prod_{j=1}^{p-1} \int_{\mathbb{R}^d} e^{-i\lambda \cdot x} Q_{t_j}(\lambda) \exp \left\{ -\frac{\varepsilon}{2} |\lambda|^2 \right\} \, d\lambda \, dx \]
\[ = (2\pi)^{d(p-1)} \int_{\mathbb{R}^d} G_{t_p}(x) \prod_{j=1}^{p-1} (G_{t_j} * \phi_\varepsilon)(x) \, dx, \]
where \(\phi_\varepsilon(x)\) is the density of the \(d\)-dimensional normal distribution \(N(0, \varepsilon I_{d \times d})\) \((I_{d \times d}\) is the \((d \times d)\) identity matrix\) and the last step follows from the inverse Fourier transform.

We now let \(\varepsilon \to 0^+\) on both sides. First, by using (1.11), we can show that
\[ Q_{t_p}(\gamma_1 - \ldots - \gamma_{p-1}) \prod_{j=1}^{p-1} Q_{t_j}(\gamma_j) \geq 0 \quad \forall (\gamma_1, \ldots, \gamma_{p-1}) \in \mathbb{R}^{d(p-1)} \]
with proper substitution of the variables. By virtue of the monotonic convergence, the left-hand side increases to
\[ \int_{(\mathbb{R}^d)^{p-1}} Q_{t_p}(\gamma_1 - \ldots - \gamma_{p-1}) \prod_{j=1}^{p-1} Q_{t_j}(\gamma_j) \, d\gamma_1 \ldots d\gamma_{p-1}, \]
regardless of finiteness or infiniteness of the limit.
In view of (1.12), to prove (1.10), it suffices to show that
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} G_t^p(x) \prod_{j=1}^{p-1} (G_{t_j} \ast \phi_\varepsilon)(x) \, dx = \int_{\mathbb{R}^d} \left( \prod_{j=1}^{p} G_{t_j}(x) \right) \, dx.
\] (1.13)

Indeed, for any \( \theta > 0 \),
\[
\int_{0}^{\infty} e^{-\theta t} G_t(x) \, dt = \left( \int_{0}^{\infty} t e^{-\theta t} \, dt \right) \left( \int_{0}^{\infty} e^{-\theta t} p_t(x) + p_s(-x) \, dt \right)
= \theta^{-2} \int_{0}^{\infty} e^{-\theta t} p_t(x) + p_s(-x) \, dt.
\] (1.14)

Therefore, by virtue of condition (1.4), we obtain
\[
\int_{\mathbb{R}^d} G_t^p(x) \, dx < \infty \quad \forall t \geq 0.
\]

By Lemma 2.2.2 in [1, p. 28],
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \left| (G_{t_j} \ast \phi_\varepsilon)(x) - G_{t_j}(x) \right|^p \, dx = 0, \quad j = 1, \ldots, p - 1.
\]

Clearly, this leads to (1.13).

It remains to prove (1.9) and (1.7). For (1.9), we simply note that
\[
\int_{(\mathbb{R}^d)^{p-1}} \mathbb{E} |\bar{\mu}_\theta(\lambda_1, \ldots, \lambda_{p-1})|^2 d\lambda_1 \ldots d\lambda_{p-1}
= \int_{(\mathbb{R}^d)^{p-1}} \mathbb{E} \left| \int_{(\mathbb{R}^d)^p} dt_1 \ldots dt_p \, \exp \left\{ -\theta \sum_{j=1}^{p} t_j \right\} \bar{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 \, d\lambda_1 \ldots d\lambda_{p-1}
\leq \theta^{-p} \int_{(\mathbb{R}^d)^p} dt_1 \ldots dt_p \exp \left\{ -\theta \sum_{j=1}^{p} t_j \right\} \int_{(\mathbb{R}^d)^{p-1}} \mathbb{E} \left| \bar{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 \, d\lambda_1 \ldots d\lambda_{p-1}
= (2\pi)^{d(p-1)} \theta^{-p} \int_{(\mathbb{R}^d)^p} dt_1 \ldots dt_p \exp \left\{ -\theta \sum_{j=1}^{p} t_j \right\} \int_{\mathbb{R}^d} \left[ \prod_{j=1}^{p} \int_{0}^{t_j} (t_j - s) \{ p_s(x) + p_s(-x) \} \, ds \right] \, dx
condition for the intersection occupation measure to be absolutely continuous

\[ = (2\pi)^{d(p-1)} \gamma^{-3p} \int_{\mathbb{R}^d} \prod_{j=1}^{p} \int_{0}^{\infty} e^{-\theta t} \left\{ p_t(x) + p_t(-x) \right\} dt \, dx \]

\[ \leq 2^p (2\pi)^{d(p-1)} \gamma^{-3p} \int_{\mathbb{R}^d} \left[ \int_{0}^{\infty} e^{-\theta t} p_t(x) dt \right]^p dx, \]

where the second, third, and fourth steps follow from the Jensen inequality, (1.10), and (1.14), respectively. Therefore, (1.9) follows from (1.4).

Thus, we have established (1.5) and (1.6). Further, by the Parseval identity, we get

\[ \int_{\mathbb{R}^{d(p-1)}} \left[ \alpha(t_1, \ldots, t_p, x) \right]^2 dx = (2\pi)^{-d(p-1)} \int_{(\mathbb{R}^d)^{p-1}} \left| \hat{\mu}_{t_1, \ldots, t_p}(\lambda_1, \ldots, \lambda_{p-1}) \right|^2 d\lambda_1 \ldots d\lambda_{p-1}. \]

Together with (1.10), this proves identity (1.7).

Theorem 1.1 is proved.

We complete this section by the following comment: The density \(\alpha(t_1, \ldots, t_p, x)\) addressed in Theorem 1.1 exists only in the form of equivalent class; this is a fact that brings some inconvenience, when it comes to application. Thus, it becomes ambiguous to talk about \(\alpha(t_1, \ldots, t_p, 0)\) for given \(t_1, \ldots, t_p\) because \(\alpha(t_1, \ldots, t_p, 0)\) represents a class of random variables such that any two members of this class are equal to each other with probability 1. The problem is to find a continuous modification of \(\alpha(t_1, \ldots, t_p, x)\). A standard procedure according to the Kolmogorov extension theory requires the local Hölder type of moment continuity

\[ \mathbb{E} \left| \alpha(t_1, \ldots, t_p, x) - \alpha(s_1, \ldots, s_p, y) \right|^m \leq C \left\{ \|t-s\| + \|x-y\| \right\}^\beta \quad (1.15) \]

for some \(m > 0\) and \(\beta > d + 1\). This cannot be achieved without additional assumptions. In the case where \(X(t)\) is Gaussian, e.g., (1.15) can be installed under certain nonlocal determinism conditions (Theorem 2.8 in [6]).

2. Applications to Gaussian Processes

Let \(X(t)\) be an \(\mathbb{R}^d\)-valued stochastic process satisfying our pointwise increment-stationarity given by (1.1). In addition, assume that \(X(t)\) is pointwise Gaussian:

\[ X(t) \sim \mathcal{N}(0, \Sigma(t)), \quad t \geq 0, \quad (2.1) \]

where \(\Sigma(t)\) is a nonnegative definite \((d \times d)\)-matrix.

**Theorem 2.1.** Assume that (1.1) and (2.1) are true. Condition (1.4) is satisfied if

\[ \int_{0}^{\infty} e^{-\theta t} \det(\Sigma(t))^{-\frac{p-1}{2p}} dt < \infty \quad \text{for some} \quad \theta > 0. \quad (2.2) \]

Consequently, all statements in Theorem 1.1 hold under (2.2).
**Proof.** Note that
\[
\int_{\mathbb{R}^d} \left[ \int_0^\infty e^{-\theta t} p_t(x) dt \right]^p dx = \int_0^{\infty} \left( \int_0^\infty dt_1 \ldots dt_p e^{-(t_1+\ldots+t_p)} \int_{\mathbb{R}^d} \prod_{j=1}^p p_t(x) dx \right)^{1/p} dt_p \leq \int_0^{\infty} \left( \int_{\mathbb{R}^d} \prod_{j=1}^p \left( p_t(x) \right)^p dx \right)^{1/p} dt_p = \left\{ \int_0^{\infty} \left[ \int_{\mathbb{R}^d} \left( p_t(x) \right)^p dx \right]^{1/p} dt \right\}^p.
\]

From (2.2), we conclude that $\Sigma(t)$ is positive-definite almost everywhere in $t$. Therefore,
\[
\int_{\mathbb{R}^d} \left( p_t(x) \right)^p dx = (2\pi)^{-dp/2} \det(\Sigma(t))^{-p/2} \int_{\mathbb{R}^d} \exp \left\{ -\frac{p}{2} (x, \Sigma(t)^{-1} x) \right\} dx = p^{-d/2} (2\pi)^{-\frac{d(p-1)}{2}} \det(\Sigma(t))^{-\frac{p-1}{2}} \text{ a.e.}
\]

Hence, by condition (1.14), we get
\[
\int_{\mathbb{R}^d} \left[ \int_0^\infty e^{-\theta t} p_t(x) dt \right]^p dx \leq p^{-d/2} (2\pi)^{-\frac{d(p-1)}{2}} \left\{ \int_0^\infty e^{-\theta t} \det(\Sigma(t))^{-\frac{p-1}{2p}} dt \right\}^p < \infty.
\]

In the remaining part of this section, we consider two examples.

**Example 2.1.** Let $X(t)$ be a $d$-dimensional fractional Brownian motion with the Hurst parameter $(H_1, \ldots, H_d)$, $(H_1, \ldots, H_d) \in (0, 1)$. This means that the components $B^{H_1}_t(t), \ldots, B^{H_d}_d(t)$ of $X(t)$ are independent mean-zero Gaussian process with the covariance functions given by
\[
\text{Cov} \left( B^{H_j}_j(t), B^{H_j}_j(s) \right) = \frac{1}{2} (|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j}), \quad j = 1, \ldots, p.
\]

In particular, $\Sigma(t)$ is diagonal with diagonal elements $|t|^{2H_j}$, $j = 1, \ldots, p$. Hence,
\[
\det(\Sigma(t)) = \prod_{j=1}^p |t|^{2H_j} = t^{2(H_1+\ldots+H_d)}, \quad t \geq 0.
\]

Thus, condition (2.2) is equivalent to
\[
H_1 + \ldots + H_d < \frac{p}{p-1}.
\]
To compare it with the known result, we consider a special case where $H_1 = \ldots = H_d = H$. In this case, the above inequality becomes

$$dH < \frac{p}{p-1}. $$

This is the condition given by (5.7) in [2] for the existence of intersection local times of the fractional Brownian motions with identically distributed components.

**Example 2.2.** A 1-dimensional Ornstein–Uhlenbeck process $U_1(t)$ is a mean-zero stationary Gaussian process with covariance function

$$\text{Cov} \left( U_1(0), U_1(t) \right) = e^{-t/2}, \quad t \geq 0. \quad (2.3)$$

A $d$-dimensional Ornstein–Uhlenbeck process

$$U(t) = (U_1(t), \ldots, U_d(t))$$

contains i.i.d. 1-dimensional Ornstein–Uhlenbeck processes $U_1(t), \ldots, U_d(t)$ as components. In our subsequent discussion, $U(t)$ is a $d$-dimensional Ornstein–Uhlenbeck process. We set

$$X(t) = \int_0^t U(s)ds, \quad t \geq 0. $$

Then $X(t)$ satisfies (1.1) and (2.1). To compute $\det (\Sigma(t))$, we note that

$$\Sigma(t) = \int_0^t \int_0^t \text{Cov} \left( U(s), U(r) \right) ds dr = \int_0^t \int_0^t \exp \left\{ -\frac{1}{2} |s-r| \right\} I_{d\times d} ds dr = 4 \left[ t - 2(1 - e^{-t/2}) \right] I_{d\times d},$$

where $I_{d\times d}$ is the $(d \times d)$ identity matrix and the second inequality follows from (2.3). Hence,

$$\det (\Sigma(t)) = 4^d \left[ t - 2(1 - e^{-t/2}) \right]^d.$$ 

In particular,

$$\det (\Sigma(t)) \sim t^{2d} \quad (t \to 0^+).$$

Thus, condition (2.2) is equivalent to

$$d < \frac{p}{p-1}. $$

In other words, (2.2) holds if and only if $d = 1$.

The present paper was partially supported by the Simons Foundation (No. 585506).
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