Analytical Modeling of Stress-Strain Behaviors under Static and Cyclic Conditions from a Microstructural Viewpoint*

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In order to understand nonlinear behaviors of stress-strain responses of metallic materials under static and cyclic conditions from the viewpoint of their microscopic structures, we theoretically investigate an analytical model by applying some mathematical techniques such as the soliton theory and the theory of nonlinear systems which develop greatly in recent years. The analytical model is based on the typical atomic chain model which includes topological defects and consists of the thermal effect, the interactions of atoms, the friction from the environment and the external force. In addition, the analytical model is developed from the typical atomic chain model by including the effects which can relate with work hardening and internal friction. In the static case, we show that inelastic behavior is displayed by the analytical model. In the cyclic case, by using the analytical model, we obtain the nonlinear behavior of the hysteresis loop which qualitatively corresponds to well-known experimental data better than that of the typical model.

Key Words: Dislocation, Micromechanics, Nonlinear Problem, Stress-Strain Behavior, Mathematical Techniques

1. Introduction

Functional Materials (namely materials which behave with functional properties) have been attracting research interests. The functional properties of such materials are significantly affected by the microstructure of the materials(3). Thus, the mechanical behaviors of metallic materials such as stress-strain responses from the microstructural viewpoint are of fundamental interest in the field of material science. In fact, these days, for example, the nano-scale metallic contact formed by the several atomic chains is created and examined mainly in terms of the mechanical behaviors to develop the nano-scale electric devices(2)–(4).

For analysis of the mechanical behaviors of such metallic materials, generally there are two approaches. That is, one is computational and the other is analytical. Molecular Dynamics (MD) simulation is one of the computational approaches. It is applied for various problems of the crystalline solids to show the mechanical properties of the atomic structure(2)–(6). For example, the large-scale (100 million atoms) MD simulations in three dimensions have been conducted to establish the local rules for mesoscopic simulations of interacting dislocations(6). Such a computational approach is powerful but it completely depends on the computer performance. Thus, it would have a limit to the analysis. On the other hand, the analytical approaches have also been conducted greatly. For example, the one dimensional atomic chain model which describes the atoms allocated regularly as an internal structure of the crystalline metallic materials has been investigated mathematically and applied to discuss the mechanical properties of the metallic materials including topological defects such as dislocations(7)–(9). Such an analytical approach is useful to understand the microstructural mechanisms explicitly but it has mathematical difficulties in analysis due to the complicated problems such as non-linearity and many degrees of freedom. Thus, it is difficult to treat the complicated cases, for instance, nonlinear behaviors such as plasticity and inelasticity in stress-strain curve. Recently, however, some mathematical techniques such as the soliton theory and the theory of nonlinear systems have developed greatly(10)–(12). Therefore, it is not only important for the interpretation of the mechanical properties in complicated systems, but also useful for the reduction of the computational cost in numerical approach, to apply such mathematical techniques to the analysis.

In the present paper, we propose and theoretically investigate an analytical model by applying some mathematical techniques in order to understand nonlinear behaviors...
of stress-strain responses of metallic materials under static and cyclic conditions from the viewpoint of their microscopic structures. The analytical model is based on the typical atomic chain model which includes topological defects and consists of the thermal effect, the interactions of atoms, the friction from the environment and the external force\(^{(9)}\). In addition, the analytical model is developed from the typical atomic chain model by including the effects which can relate with work hardening and internal friction. The analytical model is represented by a pair of equations, that is, one is a nonlinear equation and the other is a coupled equation. By analyzing these equations theoretically, we derive and discuss the stress-strain behaviors under the static and cyclic conditions.

2. Atomic Chain Model

In this section, we introduce the typical atomic chain model which describes the motion of the atom in the crystalline metallic materials, and derive its equation which represents the dynamics of the atom with the topological defect\(^{(9)}\). Then, we mention that the typical atomic chain model is not sufficient to describe the nonlinear behaviors of stress-strain responses of metallic materials.

We consider a microstructure of crystalline metallic materials shown in Fig. 1. From the microscopic viewpoint, the crystalline metallic material has an ordered structure, for instance, like a perfect fcc structure, because the atoms are allocated regularly. Namely, this means the microstructure consists of several atomic chain layers. We focus on one of the atomic chain layers in the metallic material at equal spacing \(a\) with the arrangement of the atoms spacing \(b\). We consider the dynamics of the atom in one atomic chain under the external force \(F\). In Fig. 2, we illustrate the dynamics of the atom in the atomic chain. In Fig. 2 (a), we assume one space exists as the topological defect which is made by missing the particle in one atomic chain layer. In the atomic chain, we consider the motion of the \(i\)th atom \((i = 1, 2, \cdots, N; N: \text{the total number of the atoms in one chain})\) which is located at the next place of the topological defect with the interactions of the other atoms. The interactions between the \(i\)th atom and the other atoms may be given by the two types of forces classified for convenience sake. One is the interaction between the nearest neighbor atoms in the same atomic chain which is illustrated by the spring force. The other is the interaction among the atoms except the atoms in the same chain, which is represented by the periodical potential force. This picture of the motion of the atoms is well-known as Frenkel-Kontorova model\(^{(7)}\). In addition to the above picture, for more practical analysis, we consider the thermal effect, the friction of the environment and the external force as the essential effects on the mechanical properties. Considering these effects, we illustrate the dynamical motion of the \(i\)th atom in the atomic chain as the following procedure in Fig. 2 (a) – (c). First, the \(i\)th atom exists with the interactions of the atoms and the thermal force (Fig. 2 (a)). Next, it changes the displacement \(u_i\) from the equilibrium point by the external force \(F\) against the friction of the environment (Fig. 2 (b)). Finally, the \(i\)th atom moves for the one space (Fig. 2 (c)). This is the motion of the atom described by the typical atomic chain model.

Considering the motion of the \(i\)th atom, we will find that the change of the displacement \(u_i\) of the \(i\)th atom can be represented by the propagation of the kink type wave. Actually, from the above observation, we can obtain the Langevin equation which has the kink type wave solution by considering the equation of the motion for the dynamics of the \(i\)th atom with the topological defect;

\[
0 = -\Gamma \frac{d u_i(t)}{d t} + \frac{K}{b^2} (u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) - \frac{\partial U_i}{\partial u_i} + F + \xi_i(t),
\]

where \(i = 1 \sim N\) (Let us recall that \(N\) is the total number of
the atoms in one atomic chain), \( \Gamma, K, U_i \) and \( \xi_i \) are the frictional coefficient, the spring coefficient, the potential energy of \( i \)th atom and the thermal fluctuation, respectively. The properties of the thermal effect are given by Gaussian white noise:

\[
\begin{align*}
\langle \xi(t) \rangle_{\text{av}} &= 0, \\
\langle \xi(t) \xi(t') \rangle_{\text{av}} &= 2 \Gamma k_B T \delta_{ij} \delta(t-t'),
\end{align*}
\]

where \( \langle \cdot \rangle_{\text{av}} \) means the ensemble average and the characters: \( k_B \) and \( T \) are Boltzmann constant and temperature. Here, we mention that, in the left hand side of Eq. (1), we suppose the inertia is sufficiently small and, therefore, negligible, because we consider the motion is very slow against time due to the large friction of the environment\(^{13}\).

In order to analyze the above equation theoretically, we simplify the equation by the mathematical procedure. Namely, in Eq. (1), we consider, as a simplest case, the periodical potential is given by \( U_i \equiv (A/p)(1 - \cos pu_i) \) where \( A \) and \( p \equiv 2\pi/b \) are constant. Supposing the lattice spacing \( b \) is sufficiently small, by the limiting procedure: \( u(ib,t) \rightarrow u(x,t) \) as \( b \rightarrow +0 \) with \( u(ib,t) \equiv u_i(t) \) and \( x \equiv ib \), we obtain the partial differential equation:

\[
0 = -\Gamma \frac{\partial^2 u(x,t)}{\partial t^2} + K \frac{\partial^2 u(x,t)}{\partial x^2} - A \sin pu(x,t) + F + \xi(x,t),
\]

where \( \langle \xi(x,t) \xi(x',t') \rangle_{\text{av}} = 2 \Gamma k_B T \delta(x-x') \delta(t-t') \).

In this study, we can regard the above procedure as valid, because the lattice spacing \( b \) in Eq. (1) is sufficiently small. Therefore, we can use the continuous equation (3) for the analysis of the mechanical behavior of the atomic chain.

Recently, the mechanical properties derived from the equation of the atomic chain model (3) have been investigated\(^{9)}\). As a result, it is known that the mechanical response of the atomic chain model is nearly equivalent with the result from the well-known typical linear viscoelastic model\(^{9)}\). This means that the atomic chain model itself will not induce the nonlinear behaviors of stress-strain responses of metallic materials. Therefore, in the next section, we extend the atomic chain model (Eq. (3)) phenomenologically by considering the effects which can relate with the work hardening and the internal friction as the actual phenomena in the inside of metallic materials.

3. Extension of the Atomic Chain Model and Analysis

3.1 Extension of the atomic chain model

In the previous section, we introduce the atomic chain model and mention that the atomic chain model is not sufficient to analyze the nonlinear behaviors of stress-strain responses. Therefore, we extend the equation of the atomic chain model Eq. (3) phenomenologically by considering the effects which can relate with the work hardening and the internal friction as the actual phenomena in the inside of metallic materials.
cussed by using the reaction-diffusion equations for self-organization of dislocation cell structures which are derived from the different context \(^{(16)}\). In addition, the similar equations to Eq. (5) are derived from the time dependent Ginzburg-Landau equation by Ohta \(^{(17)}\). He has applied the equations to the theoretical analysis for martensitic aging in shape memory alloys and induced the interesting results \(^{(17)}\). In the present study, we define the above extended atomic chain model as an analytical model and analyze the stress-strain properties by using Eq. (5).

3.2 Analysis of the extended model

In this section, we analyze the extended model given in the previous section by using some mathematical techniques \(^{(10)-(12)}\). (The details and the actual calculations are shown in Appendix A).

First of all, for the analysis of Eq. (5), we focus on the case that the atoms driven by the external force migrate one after another in the same direction by the existence of the topological defects such as dislocations in the atomic chain. In order to obtain this situation, we consider the traveling wave solution of the equation (As we have observed previously, in Fig. 2, we illustrate that the change of the displacement of the atom is connected to the propagation of the kink type wave). Hence, using the parameter \( \chi \), we consider the transformation:

\[
\chi = x + \int_0^t v(t') d t',
\]

where the parameter \( v \) designates the velocity of the traveling wave of migrating atoms. Next, in order to concentrate on the main motion of the atoms, we take the ensemble average for the thermal fluctuation \( \xi \) and extract the main term by the perturbation expansion for the variables such as the displacement \( u \). Setting the Landau’s symbol for the higher order terms \( O \) and the perturbation parameter \( \omega \), from the first equation of (5), we obtain, for the displacement of the atom \( u \) with respect to \( O(\omega^3) \):

\[
0 = K \frac{d^2 u_0(\chi)}{d \chi^2} - f(u_0(\chi)),
\]

where \( u_0 \) corresponds with the zeroth order term derived from the perturbation expansion of the displacement of the atom \( u \). Therefore, by solving the above equation and obtaining \( u_0 \) explicitly, we can get the main motion of the atoms. Here we encounter one serious problem, that is, it is difficult to solve Eq. (8), because Eq. (8) is the nonlinear equation. In fact, we can recall that the potential force \( f \) is given by the periodical (6). Hence, in order to settle this problem, we apply the theory of the soliton system, which has been greatly developed recently as one of the nonlinear theories \(^{(10)-(11)}\). By using the theory of the soliton system, we can solve Eq. (8) exactly, though the equation is nonlinear. Actually, we can get \(^{(9)}\) a solution:

\[
u_0(\chi) = \frac{4}{p} \arctan(e^\alpha \chi), \quad \alpha \equiv \sqrt{\frac{Ap}{K}}.
\]

This is the expression which describes the main motion of the atoms in the atomic chain. Namely, this solution shows the behavior of the atom which is affected by the physical factors discussed in the previous section and driven by the external force. From this solution, we will see that the higher the temperature or the magnitude of the external force \( \chi \), the faster the motion of the atoms is, because the traveling velocity of atoms \( v \) in the parameter \( \chi \) increases \(^{(9)}\). However, such behaviors of atoms are restricted by the second equation of (5). Actually, we will find that the displacement of the atom \( u_0 \) is constrained by the second equation of (5), that is, the motion of the atoms is restricted by the effects which can relate with the work hardening and the internal friction. By calculating the second equation of (5) with the above solution (9), we obtain the relation:

\[
d\varepsilon(t) = \frac{\pi^2 \Phi}{2 a A \Gamma} \int ds e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)} \left( \frac{a \varepsilon(t)-\varepsilon(s)}{b} \right) \left( \frac{a \varepsilon(t)-\varepsilon(0)}{b} \right) e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)} - \omega \pi^2 \Phi \int ds e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)} \left( \frac{a \varepsilon(t)-\varepsilon(s)}{b} \right) \left( \frac{a \varepsilon(t)-\varepsilon(0)}{b} \right) e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)},
\]

where \( \sigma \) and \( \varepsilon \) are the ensemble averaged stress and strain and \( \Phi \) is the distribution per length of the topological defects. This is the equation which designates the stress-strain relation induced by the dynamical motion of the atom. In the following section, using Eq. (10), we discuss the stress-strain properties under the static and cyclic conditions.

4. Discussion

4.1 Stress-strain property under static condition

In Eq. (10), as a static condition, we suppose that the static strain is given by \( \varepsilon(t) = C_e (t+t_0) \), where \( C_e \) and \( C_e \) are the initial strain and the constant strain velocity. Then, we obtain the stress response:

\[
\sigma(t) = C_e \frac{2 a A \Gamma}{\pi^2 b^2 \Phi} + \omega \frac{\Lambda}{b^2} \int_0^t ds e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)} \left( \frac{a C_e (t-s)}{b} \right) \left( \frac{a C_e (t-s)}{b} \right) e^{-\frac{\omega}{\tau_d} u_0(t-b/\Phi)}.
\]

This stress response to the strain given by the constant strain velocity is shown in Fig. 3. In this figure, for convenience, the stress and strain are normalized by a certain reference stress \( \sigma_N \) and reference strain \( \varepsilon_N \) respectively in order to get the stress-strain response which is independent of the unit system, because we are interested in the qualitative behavior of Eq. (11). Each value in Eq. (11) is given \(^{(3)}(14),(18)\)–(20) by \( \omega \Lambda/(b^2 \Phi_N) = 1 \text{ m}^{-1} \), \( \tau_d = 400 \text{ s} \), and \( C_e/(b \Phi) = 2.975 \text{ m/s} \) and \( (2 a A \Gamma C_e)/(\pi^2 b^2 \Phi \sigma_N) = 1.2104 \).

Figure 3 shows the stress-strain curve behaves nonlinear. This means that Eq. (11) can describe even the inelastic deformation, though the conventional atomic chain...
model represents only elastic deformation, that is, the linear stress-strain relation\(^9\). This is because the effects which can relate with the internal friction and the work hardening are taken into account. Those effects are represented by the second and third terms in the right hand side of Eq. (11), which are originally derived from the second and third terms in the right hand side of Eq. (10). These terms give rise to the time delay of the motion of the atoms, and therefore cause the inelastic behavior.

### 4.2 Stress-strain property under cyclic condition

As a cyclic condition, we consider that the cyclic strain is given by

\[
\varepsilon(t) = 0.02 + 0.425 \sin(0.1 t). \tag{12}
\]

Then, applying this cyclic strain to Eq. (10), we obtain the stress-strain response to the cyclic strain. The behavior is shown in Fig. 4. In the figure, as an example, we mainly refer to the results of the cold worked materials such as copper\(^{14,18-20}\). Each constant in Eq. (10) and the number of the cycles should be discussed in detail and quantitatively adjusted each other. However, as mentioned in the previous section, in the present study, we are interested in the qualitative behavior of Eq. (10). Therefore, we adopt the following conditions for the figure. That is, each value in Eq. (10) is given\(^3,14,18-20\) by \(\omega \Lambda/(b^2 \sigma_N) = 0.22 \text{ m}^{-1}\), \(\tau_d = 400 \text{ s}\), \(a/(b \Phi) = 4.4 \text{ m}\) and \((2a \alpha \Gamma)/(\pi^2 b^2 \Phi \sigma_N) = 1 \text{ s}\). The stress and strain are normalized by a certain reference stress \(\sigma_N\) and reference strain \(\varepsilon_N\) respectively, and the appropriately scaled numbers of the cycle are denoted for reference.

In order to compare the result of the present extended model with that of the well-known conventional linear viscoelastic model, the stress-strain response of the conventional model under the same cyclic strain is shown in Fig. 5. In this figure, as the conventional linear viscoelastic model, we use Voigt model\(^{21}\) which is given by

\[
\eta \frac{d \varepsilon_V(t)}{dt} + G \varepsilon_V(t) = \sigma_V(t), \tag{13}
\]

where \(\sigma_V\) and \(\varepsilon_V\) are the stress and strain induced by the Voigt model. In the same way of Fig. 4, the stress and strain are normalized by a certain reference stress \(\sigma_N\) and reference strain \(\varepsilon_N\) respectively. For a comparison to Fig. 4, each constant is given by \(G/\sigma_N = 0.4\) and \(\eta/\sigma_N = 1 \text{ s}\).

Comparing Fig. 4 with Fig. 5, we can find that the hysteresis loop given by the present extended model has not the elliptic shape like the loop presented by the linear viscoelastic model, but the distorted shape derived from the nonlinearity of the model. This resembles the empirical property of the hysteresis loop which is given by the low cycle fatigue experiment\(^{14,18-20}\). Thus, it implies that, under the cyclic strain condition, the extended model qualitatively presents more realistic results than the conventional viscoelastic models. Moreover, we can understand that the distortion of the hysteresis loop is mainly caused by the second and third terms in the right hand side of Eq. (10), though the effects of the higher order terms for the perturbation expansion of Eq. (5) are not considered in the present analysis. Namely, it is analytically shown that the distortion of the hysteresis loop which is observed as
one of the characteristic properties in Fig. 4 is induced by the time delay of the atomic motion caused by the effects which can relate with the work hardening and the internal friction.

4.3 Relation to the conventional linear viscoelastic model

In the previous sections, by using the present analytical model, we show that the nonlinear behaviors are induced by the time delay of the atomic motion caused by the effects which can relate with the work hardening and the internal friction. For the proof of this result, we clarify the relation between the present model (Eq. (10)) and the conventional linear viscoelastic model (Eq. (13)). Namely, by neglecting the effect of the time delay of the atomic motion in the present model, we derive the conventional linear viscoelastic model from the present model.

In order to neglect the effect of the time delay of the atomic motion in the present model, we suppose that \( \tau_d \) is sufficiently large in Eq. (10). Then, we obtain the equation:

\[
\frac{d \varepsilon(t)}{dt} = \frac{\pi^2 \Phi}{2a} \beta^2 \sigma(t) - \omega \frac{\pi^2 \Phi}{2a} \Lambda \left( \frac{a \varepsilon(t)-\varepsilon(0)}{b} \right). 
\]

By linearizing this equation with the solution (9) and neglecting the initial stress and strain, we can obtain the equation:

\[
\frac{2a \alpha \Gamma}{\pi^2 b^2 \Phi} \frac{d \varepsilon(t)}{dt} = \omega \Lambda \frac{a \alpha}{\pi b^2 \Phi} \varepsilon(t) = \sigma(t). 
\]

This linear ordinary differential equation is the similar equation of the Voigt model (Eq. (13)). This indicates that the Voigt model is implicated in the present model and therefore the consistency is ensured.

5. Concluding Remarks

In the present study, we proposed the analytical model which was developed from the typical atomic chain model by considering the effects which were able to relate with the work hardening and the internal friction. Then, we investigated the analytical model by applying some mathematical techniques and derived the stress-strain properties of metallic materials.

In the static case, the analytical model can describe the inelastic deformation which is not induced by the typical atomic chain model. Thus, we have analytically shown that the inelastic deformation is caused by the time delay induced by the effects which can relate with the work hardening and the internal friction.

In the cyclic case, the analytical model can describe the distortion of the hysteresis loop which appears as a result of the nonlinearity. Such a nonlinear behavior of the hysteresis loop qualitatively corresponds to well-known experimental data better than that of the conventional model. We have analytically shown that the distortion of the hysteresis loop is induced by the time delay caused by the effects which can relate with the work hardening and the internal friction.

The above results would imply that the analytical model is useful to describe the stress-strain properties, compared with the typical atomic chain model and the conventional linear viscoelastic model. We believe that the results in the present paper not only directly contribute to the interpretation and application of the mechanical properties of metallic materials from the microstructural viewpoint, but also lead one to the technical achievement of the numerical simulations for the large system and to the acquirement of the fundamental knowledge to attack more complex systems, in terms of controlling the mechanical properties from the microstructural viewpoint for the functional materials.

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Appendix

A. Derivation of the macroscopic stress-strain relation

In order to get the traveling wave solution of the first equation of Eq. (5), we change the variables \( x \) and \( t \) into the independent variables \( \chi \) and \( \tau \) by the transformation:

\[
\begin{align*}
\chi &= x + \int_0^t \nu(t') \, dt' \\
\tau &= t.
\end{align*}
\]

Considering the ensemble average of \( \xi \), we expand the variables with respect to the perturbation parameter \( \omega \) as:

\[
\begin{align*}
\langle u(x,t) \rangle_{\omega} &= u_0(\chi) + \omega v_1(\chi,\tau) + O(\omega^2), \\
\langle v(t) \rangle_{\omega} &= \omega v_1(\tau) + O(\omega^2), \\
\langle X(t) \rangle_{\omega} &= X_0 + \omega X_1(\tau) + O(\omega^2), \\
\langle \rho(x,t) \rangle_{\omega} &= \omega \rho_1(\chi,\tau) + O(\omega^2), \\
\langle f(u(x,t)) \rangle_{\omega} &= f(u_0(\chi)) + \omega f_1(\chi) + \omega \rho_1(\chi,\tau) + O(\omega^2), \\
\langle F(x,t) \rangle_{\omega} &= \omega F_1(\chi,\tau) + O(\omega^2),
\end{align*}
\]

where we suppose \( \nu_0 \equiv 0 \) and \( \rho_0 \equiv 0 \) for the balance of the external force \( F \) among the variables. Then, putting the above expanded variables into Eq. (5), we obtain the following equations, in \( O(\omega^0) \) with respect to \( u \);

\[
0 = K \frac{d^2 u_0(\chi)}{d\chi^2} - f(u_0(\chi)), \tag{17}
\]

in \( O(\omega^1) \) with respect to \( u \) and \( \rho \);
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial u_1(\chi, \tau)}{\partial \tau} = -Lu_1(\chi, \tau) - \frac{d X_1(\tau)}{d \tau} \frac{d u_0(\chi)}{d \chi} \\
+ F_1(\chi, \tau) - \rho_1(\chi, \tau),
\end{array} \right.
\tag{18}
\end{aligned}
\]

Equation (18) corresponds to Eq. (8) in the main text. In Eq. (18), the operator \( L \) is given by
\[
L \equiv -K \frac{\partial^2}{\partial \chi^2} + f'(u_0).
\tag{19}
\]

This operator is self-adjoint. The eigenvalue and the eigenfunction are represented by \( \lambda \) and \( \phi_i \), respectively.

Considering the solvability condition for the first equation of Eq. (18), we get the equation:
\[
0 = \int_{-\infty}^{\infty} d\chi \left( \frac{d u_0(\chi)}{d \chi} \right)^2,
\tag{20}
\]
and supposing the constant \( \alpha \) is sufficiently large, we can consider
\[
\frac{\alpha}{\pi} \int_{-\infty}^{\infty} dx f(x) \sech(\alpha x) \approx f(x).
\tag{28}
\]

Then, the integral part in Eq. (27) approximately becomes
\[
\int_{-\infty}^{\infty} d\chi \frac{d u_0(\chi)}{d \chi} u_0(x + X(s)) = b u_0(x + X(s)).
\tag{29}
\]

Thus, we obtain the equation:
\[
\Sigma \frac{d X_1(t)}{d t} = b F_1(t) - b \frac{\Lambda}{\tau_d} u_0(x + X(0)) e^{-\frac{t}{\tau_d}}.
\tag{30}
\]

We mention the relation between the strain and the ensemble averaged velocity \( \langle \dot{u} \rangle_{en} \). The strain per one topological defect with the displacement variable \( u \); \( \varepsilon_0 \) is given by
\[
\varepsilon(t) = \frac{u(x(t)) - \langle u(x(t)) \rangle_{en}}{a}.
\tag{31}
\]

Since it is considered that many topological defects exist from the macroscopic viewpoint, the observed strain \( \varepsilon \) is given by
\[
\varepsilon(t) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \Phi(x, t) \varepsilon_0(x, t) dx,
\tag{32}
\]
where \( \Phi = \Phi(x, t) \) and \( L \) are the distribution part length of the topological defects and the length of the atomic chain. Supposing that \( \Phi \) is independent of space and time and considering the derivation of \( t \), we obtain
\[
\frac{d \varepsilon(t)}{d t} = \frac{b}{a} \Phi(t) \varepsilon_0(x(t))_{en}.
\tag{32}
\]
This designates the relation between the strain and the ensemble averaged velocity. Calculating this equation, we have
\[ \varepsilon(t) = \varepsilon(0) + \frac{b}{a} \Phi(X(t) - X(0)) \]
\[ = \varepsilon(0) + \omega \frac{b}{a} \Phi(X(t) - X(0)) + O(\omega^2). \] (33)

Supposing that the stress \( \sigma \) is given by \( \sigma(t) = F(t)/b^2 \), we put the above equation into Eq. (30). Then, assuming that we can neglect the term \( O(\omega^2) \), we obtain the stress-strain relation:
\[
\frac{d \varepsilon(t)}{dt} = \frac{\pi^2 \Phi}{2a} b^2 \sigma(t)
- \frac{\omega}{2a} \int_0^t \int d s e^{-\frac{s}{\tau_d}} \left( \frac{a}{b} \Phi \right) \left( \frac{a \varepsilon(t) - \varepsilon(s)}{b} \Phi \right) e^{-\frac{s}{\tau_d}}.
\] (34)

This corresponds to Eq. (10) in the main text.

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