A ONE-RADIUS THEOREM ON A SPHERE WITH PRICKED POINT

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Abstract. We consider local properties of mean periodicity on the two-dimensional sphere $S^2$. According to the classical properties of periodic functions, each function continuous on the unit circle $S^1$ and possessing zero integrals over any interval of a fixed length $2r$ on $S^1$ is identically zero if and only if the number $r/\pi$ is irrational. In addition, there is no non-zero continuous function on $\mathbb{R}$ possessing zero integrals over all segments of fixed length and their boundaries. The aim of this paper is to study similar phenomena on a sphere in $\mathbb{R}^3$ with a pricked point. We study smooth functions on $S^2 \setminus (0,0,-1)$ with zero integrals over all admissible spherical caps and circles of a fixed radius. For such functions, we establish a one-radius theorem, which implies the injectivity of the corresponding integral transform. We also improve the well-known Ungar theorem on spherical means, which gives necessary and sufficient conditions for the spherical cap to belong to the class of Pompeiu sets on $S^2$. The proof of the main results is based on the description of solutions $f \in C^\infty (S^2 \setminus (0,0,-1))$ of the convolution equation $(f * \sigma_r)(\xi) = 0$, $\xi \in B_{\pi-r}$, where $B_{\pi-r}$ is the open geodesic ball of radius $\pi-r$ centered at the point $(0,0,1)$ on $S^2$ and $\sigma_r$ is the delta-function supported on $\partial B_r$. The key tool used for describing $f$ is the Fourier series in spherical harmonics on $S^1$. We show that the Fourier coefficients $f_\nu(\theta)$ of the function $f$ are representable by series in Legendre functions related with the zeroes of the function $P_\nu (\cos r)$. Our main results are consequence of the above representation of the function $f$ and the corresponding properties of the Legendre functions. The results obtained in the work can be used in solving problems associated with ball and spherical means.

Keywords: spherical means, Pompeiu transform, Legendre functions, convolution

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1. INTRODUCTION

Let $r$ be a fixed positive number. An obvious property of non-zero $2r$-periodic functions on the real axis is the absence of the anti-period equal to $2r$. In other words, if a function $f$ defined on $\mathbb{R}$ satisfies the relations

$$f(x+r)-f(x-r)=0, \quad f(x+r)+f(x-r)=0, \quad x \in \mathbb{R},$$

then $f \equiv 0$. In terms of integral means, this implies that each continuous function on $\mathbb{R}$ having zero integrals over all segments $K_r = [x-r,x+r]$ and over its boundaries $\partial K_r = \{x \pm r\}$ vanishes identically; as usually, the integral over $\partial K_r$ is introduced as the sum of the values of the functions at points in the set $\partial K_r$.

This fact admits non-trivial generalizations for various multi-dimensional spaces, see [1]-[2]. In particular, if the function $f \in C(\mathbb{R}^n)$, $n \geq 2$, has zero integrals over all balls and spheres of a fixed radius, then $f \equiv 0$, see [2]. The statements of such kind are called one radius theorems.

In the present work we study functions on a pricked two-dimensional sphere having zero integrals over all admissible spherical caps and circumferences of a fixed radius. For such...
functions we establish a new one radius theorem specifying one of the results in work [6]. We also observe that an intermediate result of the work is an improving of the known theorem by P. Ungar on spherical means [7], see Theorem 4.1 in Section 4.

2. Main result

Let \( S^2 = \{\xi \in \mathbb{R}^3 : |\xi| = 1\} \), \( \xi_1, \xi_2, \xi_3 \) be Cartesian coordinates of a point \( \xi \in S^2 \),

\[ S' = \{\xi \in S^2 : \xi_3 \neq -1\} \].

The distance \( d(\xi, \eta) \) between points \( \xi, \eta \in S^2 \) is calculated by the formula

\[ d(\xi, \eta) = \arccos(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3). \]

In particular,

\[ d(\xi, 0) = \arccos \xi_3, \quad \text{where} \quad 0 = (0, 0, 1). \]

The set

\[ B_r(\eta) = \{\xi \in S^2 : d(\xi, \eta) < r\}, \quad 0 < r < \pi, \]

is called an open geodesic ball (spherical cap) on \( S^2 \) of the radius \( r \) centered at the point \( \eta \). Its boundary

\[ \partial B_r(\eta) = \{\xi \in S^2 : d(\xi, \eta) = r\} \]

is a geodesic circumference of the radius \( r \) on \( S^2 \) centered at the point \( \eta \). In the same way, the set

\[ \overline{B_r(\eta)} = B_r(\eta) \cup \partial B_r(\eta) \]

is called a closed geodesic ball on \( S^2 \) of the radius \( r \) centered at the point \( \eta \).

We denote by \( d\xi \) and \( dl(\xi) \) the differential of the area and the length on \( S^2 \), respectively.

The main result of the present work is the following spherical analogue of a one radius theorem.

**Theorem 2.1.** Let \( r \) be a fixed number in the interval \((0; \pi)\), \( f \in C^\infty(S') \) and the following conditions hold:

1) the function \( f \) has zero integrals with respect to the measure \( d\xi \) over each closed geodesic ball of radius \( r \) on \( S^2 \) lying in \( S' \);

2) the function \( f \) has non-zero integrals with respect to the measure \( dl(\xi) \) over each geodesic circumference of the radius \( r \) on \( S^2 \) located in \( S' \).

Then \( f \equiv 0 \).

It is interesting to compare this result with a one radius theorem obtained in work [6]. Theorem 2 in [6] shows that if \( 0 < r \leq \pi/2 \), \( f \in C(S') \) and

\[ \int_B f(\xi)d\xi = \int_{\partial B} f(\xi)dl(\xi) = 0 \quad (2.1) \]

for each closed geodesic ball \( B \) of radius \( r \) located in \( S' \), then \( f \equiv 0 \). Moreover, as \( \pi/2 < r < \pi \), there exist non-zero functions \( f \) satisfying conditions \( 2.1 \). If \( f \) is smooth on \( S' \) and satisfies conditions 1), 2) in Theorem 2.1 for some \( r \in (0; \pi) \), then \( f \equiv 0 \).

For other one radius theorem, we refer to [2]–[5] and the references therein.
3. Main notations

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{Z}+, \mathbb{C} \) be the sets of natural, integer, non-negative integer and complex numbers, respectively. We denote by \( P^\mu_\nu (\mu, \nu \in \mathbb{C}) \) the Legendre functions of first kind on \((-1, 1), \) that is,

\[
P^\mu_\nu (x) = \frac{1}{\Gamma(1-\mu)} \left( \frac{1+x}{1-x} \right)^{\frac{\mu}{2}} F \left( -\nu, \nu+1; 1-\mu; \frac{1-x}{2} \right), \quad \mu \notin \mathbb{N},
\]

\[
P^\mu_\nu (x) = (-1)^{\mu} (1-x^2)^{\frac{\mu}{2}} \left( \frac{d}{dx} \right)^\mu P_\nu (x), \quad \mu \in \mathbb{N},
\]

where \( F \) is the Gauss hypergeometric function, \( \Gamma \) is the Gamma function and \( P_\nu = P^0_\nu, \) see [8, Ch. 3, Sect. 3.4, Eq. (25)]. They satisfy Meier-Dirichlet integral representation

\[
P^\mu_\nu (\cos \theta) = \sqrt{\frac{2}{\pi \Gamma \left( \frac{1}{2} - \mu \right)}} \int_0^\theta (\cos t - \cos \theta)^{-\mu-\frac{1}{2}} \cos \left( \frac{\nu + 1}{2} t \right) dt \quad (3.1)
\]
as \( \theta \in (0; \pi], \) \( \Re \mu < \frac{1}{2}. \) The Legendre functions of second kind on \((-1, 1)\) are defined by the identity

\[
\frac{(1-x^2)^{\mu/2}Q^\mu_\nu (x)}{2^{\mu} \pi^{3/2}} = \cot \left( \frac{\pi}{2} (\nu + \mu) \right) \frac{x F \left( \frac{1-\nu-\mu}{2}, \frac{\nu-\mu}{2} + 1; \frac{3}{2}; x^2 \right)}{\Gamma \left( \frac{1+\nu-\mu}{2} \right) \Gamma \left( -\frac{\nu+\mu}{2} \right)} - \frac{1}{2} \tan \left( \frac{\pi}{2} (\nu + \mu) \right) \frac{F \left( -\frac{\nu+\mu}{2}, \frac{1+\nu-\mu}{2}; \frac{1}{2}; x^2 \right)}{\Gamma \left( \frac{1+\nu-\mu}{2} \right) \Gamma \left( 1 + \frac{\nu-\mu}{2} \right)}, \quad -\nu - \mu \notin \mathbb{N},
\]

\( Q_\nu = Q^0_\nu, \quad -\nu \notin \mathbb{N}. \)

They are related with \( P^\mu_\nu \) as follows:

\[
P^\mu_\nu (-x) = P^\mu_\nu (x) \cos \left( \pi (\nu + \mu) \right) - \frac{2}{\pi} Q^\mu_\nu (x) \sin \left( \pi (\nu + \mu) \right), \quad (3.2)
\]
see [8] Ch. 3, Sect. 3.4, Eqs. (14), (15), (20), (21)]. Moreover,

\[
(1-x^2) \left( \frac{d}{dx} P^\mu_\nu (x) \right) - Q^\mu_\nu (x) = 2^{\mu+1} \pi^{3/2} \frac{\Gamma \left( 1 + \frac{\nu+\mu}{2} \right)}{\Gamma \left( \frac{1+\nu-\mu}{2} \right) \Gamma \left( 1 + \frac{\nu-\mu}{2} \right)}, \quad (3.3)
\]
see [8] Ch. 3, Sect. 3.4, Formula (25)].

Hereafter \( r \) is a fixed number in the interval \((0; \pi].\) It follows from (3.1) that the function

\[
h(\nu) = P_\nu (\cos r) = P^0_\nu (\cos r)
\]
is an entire function in the variable \( \nu \) of exponential type \( r. \) It possesses infinitely many zeroes, all zeroes are real, simple and are located symmetrically with respect to the point \(-\frac{1}{2}\) and lie outside the segment \([-1; 0], \) see [3] Part 2, Ch. 3. We denote the set of the zeroes of this function in the interval \((0; +\infty)\) by the symbol \( N(r), \) that is,

\[
N(r) = \{ \nu > 0 : P_\nu (\cos r) = 0 \}.
\]

We also let

\[
Z(r) = \{ l \in \mathbb{N} : P_l (\cos r) = 0 \}.
\]

We note that

\[
Z(\pi/2) = N(\pi/2) = \{ 2k + 1, k \in \mathbb{Z}+ \}.
\]

Moreover, the set \( \{ r \in (0, \pi) : Z(r) \neq \emptyset \} \) is countable and everywhere dense in the interval \((0, \pi), \) see [2].

We introduce spherical coordinates \( \varphi, \theta \) on \( \mathbb{S}^2 \) as follows:

\[
\xi_1 = \sin \theta \sin \varphi, \quad \xi_2 = \sin \theta \cos \varphi, \quad \xi_3 = \cos \theta, \quad \varphi \in (0, 2\pi), \quad \theta \in (0, \pi);
\]
as above $\xi_1$, $\xi_2$, $\xi_3$ are the Cartesian coordinates of a point $\xi \in S^2$. We let

$$p_{\nu,k}(\theta) = P_\nu^{-k}(\cos \theta), \quad \nu \in \mathbb{C}, \ k \in \mathbb{Z}.$$  \hfill (3.4)

The function $S_{\nu,k}$ is real analytic on $S'$. At that,

$$L(S_{\nu,k}) = -\nu(\nu + 1)S_{\nu,k}, \quad \nu \geq 1,$$

where $L$ is the Laplace operator on $S^2$, that is,

$$L = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2},$$

see the proof of Lemma 4.1 below.

With each function $f \in C(S')$, we associate the Fourier series

$$f \sim \sum_{k \in \mathbb{Z}} f^k,$$  \hfill (3.6)

whose terms are defined by the identities

$$f^k(\xi) = f_k(\theta) e^{ik\varphi}, \quad f_k(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\sin \theta \sin \alpha, \sin \theta \cos \alpha, \cos \theta) e^{-i\alpha \theta} d\alpha.$$

If $f \in C^\infty(S')$, then series (3.6) converges to $f$ in the standard topology of the space $C^\infty(S')$, see [4, Ch. 11, Sect. 11.1]. Relation (3.6) implies the formula

$$f^k(\xi) = \frac{1}{2\pi} \int_0^{2\pi} f(\tau_\alpha \xi) e^{ika} d\alpha, \quad \tau_\alpha = (\xi_1 \cos \alpha - \xi_2 \sin \alpha, \xi_1 \sin \alpha + \xi_2 \cos \alpha, \xi_3).$$  \hfill (3.7)

Let $O(3)$ be an orthogonal group in $\mathbb{R}^3$,

$$B_r = B_r(0) = \{ \xi \in S^2 : \xi_3 > \cos r \} = \{ (\varphi, \theta) : 0 \leq \theta < r \},$$

$$S_r = S_r(0) = \{ \xi \in S^2 : \xi_3 = \cos r \} = \{ (\varphi, \theta) : \theta = r \}.$$

We let

$$U_r(S') = \left\{ f \in C(S') : \int_{S_r} f(\tau \xi) d\xi = 0 \quad \forall \tau \in O(3) : rB_r \subset S' \right\}.$$  

The class $U_r(S')$ can be regarded as a set of the functions $f \in C(S')$ satisfying the convolution equation $f * \sigma_r = 0$ in the ball $B_{\pi - r}$, where $\sigma_r$ is the delta-function supported on $S_r$.

4. Auxiliary statements

We denote by $D_k$ the differential operator defined on the space $C^1(0, \pi)$ as follows:

$$(D_k u)(\theta) = (\sin \theta)^k \frac{d}{d\theta} \left( \frac{u(\theta)}{(\sin \theta)^k} \right), \quad u \in C^1(0, \pi).$$

Let $Id$ be the identity mapping.

**Lemma 4.1.** The identities hold:

$$D_k p_{\nu,k} = (k - \nu)(k + \nu + 1)p_{\nu,k+1},$$
$$D_{-k} p_{\nu,k} = p_{\nu,k-1},$$
$$L + \nu(\nu + 1)Id)(p_{\nu,k}(\theta)e^{ik\varphi}) = 0.$$  \hfill (4.1)  \hfill (4.2)
Proof. Employing the formula
\[
(1 - x^2) \frac{dP_\nu'(x)}{dx} = -\nu x P_\nu'(x) + (\nu + \mu) P_{\nu-1}'(x),
\]
see [8, Ch. 3, Sect. 3.8, Eq. (19)], we find
\[
p_{\nu,k}'(\theta) = \nu \csc \theta p_{\nu,k}(\theta) + \frac{(k - \nu)}{\sin \theta} p_{\nu-1,k}(\theta).
\]
This implies
\[
D_k p_{\nu,k}(\theta) = \frac{(k - \nu)}{\sin \theta}(p_{\nu-1,k}(\theta) - \cos \theta p_{\nu,k}(\theta)),
\]
\[
(4.3)
\]
\[
D_{-k} p_{\nu,k}(\theta) = \frac{1}{\sin \theta}((\nu + k) \cos \theta p_{\nu,k}(\theta) - (\nu - k)p_{\nu-1,k}(\theta)).
\]
\[
(4.4)
\]
Since
\[
P_{\nu-1}'(x) - x P_\nu'(x) = (\nu - \mu + 1)\sqrt{1 - x^2} P_{\nu-1}'(x),
\]
\[
(\nu - \mu)x P_\nu'(x) - (\nu + \mu) P_{\nu-1}'(x) = \sqrt{1 - x^2} P_{\nu+1}'(x),
\]
see [8, Ch. 3, Eqs. (15), (17)], by (4.3) and (4.4) we arrive at (4.1).

On a function \( u \) of the form \( u(\xi) = v(\theta)e^{ik\varphi} \), the operator \( L \) acts by the rule
\[
(Lu)(\xi) = (\ell_k v)(\theta)e^{ik\varphi},
\]
\[
(4.5)
\]
where
\[
\ell_k = \frac{d^2}{d\theta^2} + \csc \theta \frac{d}{d\theta} - \frac{k^2}{\sin^2 \theta}Id.
\]
The operator \( \ell_k \) can be represented as
\[
\ell_k = D_{-k-1}D_k - k(k + 1)Id = D_{k-1}D_{-k} - k(k - 1)Id.
\]
\[
(4.6)
\]
Now relation (4.2) follows (4.6) and (4.1). \( \square \)

Lemma 4.2. \( \square \)

(i) Let \( \varepsilon, \theta \in (0, \pi), k \in \mathbb{Z}_+ \). Then as \( \nu \to \infty \) and \( |\arg \nu| < \pi - \varepsilon \), the asymptotic identity
\[
p_{\nu,k}(\theta) = \sqrt{\frac{2}{\pi \sin \theta}} \cos((\nu + \frac{1}{2})\theta - \frac{\pi}{2}(2k + 1)) + O\left(\frac{e^{\theta|\text{Im}\nu|}}{|\nu|^{k+\frac{1}{2}}}\right)
\]
holds uniformly in \( \theta \) over each segment \([\alpha, \beta] \subset (0, \pi)\).

(ii) If \( \nu \in \mathbb{C}, \theta \in (0, \pi), k \in \mathbb{Z}_+ \), then
\[
|p_{\nu,k}(\theta)| \leq \frac{1}{k!} \left(\sin \frac{\theta}{2}\right)^k \left(\cos \frac{\theta}{2}\right)^{-k-1} e^{\theta|\text{Im}\nu|}.
\]
\[
(4.8)
\]
(iii) Let \( 0 < a < \pi, s, k \in \mathbb{Z}_+ \). Then
\[
\max_{\theta \in [0,a]} \left| \frac{d^s p_{\nu,k}(\theta)}{d\theta^s} \right| = O(\nu^{s-k}), \quad \nu \to +\infty.
\]
\[
(4.9)
\]
Proof. Taking into consideration (3.4), by formula (3.1) we have
\[
p_{\nu,k}(\theta) = \frac{(\sin \theta)^{-k}}{\sqrt{2\pi \Gamma(k + \frac{1}{2})}} \int_{-\theta}^{\theta} (\cos t - \cos \theta)^{k-\frac{1}{2}} e^{i(\nu + \frac{1}{2})t} dt.
\]
\[
(4.10)
\]
By (4.10) and asymptotic expansion of Fourier integrals, see [10, Ch. 2, Proof of Theorem 10.2], we obtain (4.7).
To prove (4.8), we again employ (4.10). Then

\[ |p_{\nu,k}(\theta)| \leq \frac{(\sin \theta)^{-k}}{2\pi \Gamma\left(k + \frac{1}{2}\right)} \int_{-\theta}^{\theta} (\cos t - \cos \theta)^{k - \frac{1}{2}} dt e^{\theta |\Im \nu|}. \]

The integral in the right hand side is estimated as follows:

\[
\int_{0}^{\theta} (\cos t - \cos \theta)^{k - \frac{1}{2}} dt = \int_{\cos \theta}^{1} (x - \cos \theta)^{k - \frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} \\
\leq \frac{1}{\sqrt{1 + \cos \theta}} \int_{\cos \theta}^{1} (x - \cos \theta)^{k - \frac{1}{2}} (1 - x)^{-\frac{1}{2}} dx \\
= \frac{\sqrt{\pi} 2^{k-\frac{1}{2}} \Gamma\left(k + \frac{1}{2}\right)}{k!} \left(\sin \frac{\theta}{2}\right)^{2k} \left(\cos \frac{\theta}{2}\right)^{-1},
\]

and this proves estimate (4.8).

Finally, let us prove (4.9). As \(a < \pi/2\), estimate (4.9) is implied by the integral representation

\[ p_{\nu,-k}(\theta)e^{ik\psi} = \frac{i^k \Gamma(\nu + k + 1)}{2\pi \Gamma(\nu + 1)} \int_{-\pi}^{\pi} (\cos \theta + i \sin \theta \cos(\psi - \varphi))^{\nu} e^{ik\psi} d\psi, \quad \theta \in (0, \pi/2) \]

and the identity

\[ p_{\nu,-k}(\theta) = (-1)^k \frac{\Gamma(\nu + k + 1)}{\Gamma(\nu - k + 1)} p_{\nu,k}(\theta), \]

see [8] Ch. 3, Sect. 3.7, Eqs. (25), (26); Sect. 3.3.1, Eq. (7); Sect. 3.4, Eq. (5)]. On the other hand, asymptotic expansion (4.7) and the second relation in (4.1) show that

\[ \max_{0 < \alpha < \theta < \beta < \pi} \left| \frac{d^s p_{\nu,k}(\theta)}{d\theta^s} \right| = O(\nu^{s-k-1/2}), \quad \nu \to +\infty. \]

Employing these two cases, we obtain statement (iii).

\[ \square \]

**Lemma 4.3.** (i) The identity holds

\[ Z(r) = Z(\pi - r). \]

(ii) If \(p_{\nu,0}(r) = 0\), then \(p_{\nu,1}(r) \neq 0\).

(iii) If \(p_{\nu,0}(r) = 0\), then \(Q_{\nu}(\cos r) \neq 0\).

**Proof.** Statement (i) is implied by the definition of the set \(Z(r)\) and the relation

\[ P_{n}(-x) = (-1)^n P_{n}(x), \quad n \in \mathbb{Z}, \]

see [8] Ch. 3, Sect. 3.4, Formula (19)].

We assume that \(p_{\nu,0}(r) = p_{\nu,1}(r) = 0\) for some \(\nu \in \mathbb{C}\). Then

\[ p_{\nu,0}(r) = p'_{\nu,0}(r) = 0 \]

and

\[ \frac{d^2}{d\theta^2} p_{\nu,0}(\theta) + \operatorname{ctg} \theta \frac{d}{d\theta} p_{\nu,0}(\theta) + \nu(\nu + 1) p_{\nu,0}(\theta) = 0, \]

see (4.1), (4.2) and (4.5). Then by the uniqueness of the solution to the Cauchy problem for a second order ordinary differential equation we obtain \(p_{\nu,0} \equiv 0\) and this contradicts the definition of \(P_{\nu}\).

Finally, the formula

\[ (1 - x^2) \left( P_{\nu}(x) \frac{d}{dx} Q_{\nu}(x) - Q_{\nu}(x) \frac{d}{dx} P_{\nu}(x) \right) = 1, \]

see (3.3), shows that the identities \(P_{\nu}(\cos r) = 0\) and \(Q_{\nu}(\cos r) = 0\) can not hold simultaneously. This completes the proof. \[ \square \]
Lemma 4.4. Let
\[ \delta(\mu, \nu) = \int_0^r p_{\nu,0}(\theta)p_{\mu,0}(\theta) \sin \theta d\theta, \quad \mu, \nu \in N(r). \]
Then \( \delta(\mu, \nu) = 0 \) as \( \mu \neq \nu \) and
\[ \delta(\nu, \nu) > \frac{c}{\nu^2}, \] (4.11)
where constant \( c > 0 \) is independent of \( \nu \).

Proof. As \( \mu \neq \nu \), the statement is implied by the identity
\[ (\mu - \nu)(\mu + \nu + 1) \int_0^r p_{\nu,0}(\theta)p_{\mu,0}(\theta) \sin \theta d\theta = \sin r(p_{\mu,0}(r)p_{\nu,0}'(r) - p_{\nu,0}(r)p_{\mu,0}'(r)), \]
see [8, Ch. 3, Sect. 3.12, Formula (3)]. It is sufficient to prove inequality (4.11) for sufficiently large \( \nu \in N(r) \). Suppose that \( \nu > \frac{\pi}{4} - \frac{1}{2} \). We let
\[ g(\theta, t) = (\cos t - \cos \theta)^{-\frac{1}{2}}, \quad 0 \leq t \leq \theta \leq \pi. \] (4.12)
Then by (3.1) we get
\[ \delta(\nu, \nu) = \int_0^r (p_{\nu,0}(\theta))^2 \sin \theta d\theta - \frac{2}{\pi^2} \int_0^r \sin \theta \left( \int_0^\theta g(\theta, t) \cos \left( \nu + \frac{1}{2} \right) t dt \right)^2 d\theta \]
\[ \geq \frac{2}{\pi^2} \int_0^\frac{\pi}{4(\nu+1/2)} \sin \theta \left( \int_0^\theta g(\theta, t) \cos \left( \nu + \frac{1}{2} \right) t dt \right)^2 d\theta \]
\[ \geq \frac{1}{\pi^2} \int_0^\frac{\pi}{4(\nu+1/2)} \sin \theta \left( \int_{\frac{\theta}{2}}^\theta g(\theta, t) dt \right)^2 d\theta. \] (4.13)
An internal integral in (4.13) is estimated as follows:
\[ \int_{\frac{\theta}{2}}^\theta g(\theta, t) dt = \int_{\frac{\theta}{2}}^{\cos \frac{\theta}{2}} (x - \cos \theta)^{-\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} \]
\[ \geq \frac{1}{\sin \theta} \int_{\cos \frac{\theta}{2}}^{\cos \frac{\theta}{2}} (x - \cos \theta)^{-\frac{1}{2}} dx = 2 \left( \cos \frac{\theta}{2} - \cos \theta \right) \frac{1}{\sin \theta}. \] (4.14)
Taking into consideration that
\[ \frac{\cos \frac{\theta}{2} - \cos \theta}{\sin \theta} = \frac{\sin \frac{3\theta}{4}}{2 \cos \frac{\theta}{2} \cos \frac{\theta}{4}} \geq \frac{1}{2} \frac{3\theta}{4} \geq \frac{3\theta}{4\pi} \]
as \( 0 < \theta < \frac{\pi}{4(\nu+1/2)} \), by (4.13) and (4.14) we obtain
\[ \delta(\nu, \nu) \geq \frac{4}{\pi^2} \int_0^\frac{\pi}{4(\nu+1/2)} \frac{3\theta}{4\pi} d\theta \]
and this implies (4.11). \( \Box \)

Lemma 4.5. Let \( r \in (0, \pi), \nu \in C, k \in Z \). Then for each \( \tau \in O(3) \) such that \( \tau B_r \subset B_\pi \) the identities hold:
\[ \int_{S_r} S_{\nu,k}(\tau \xi) d\xi = 2\pi \sin r p_{\nu,0}(r)S_{\nu,k}(\tau 0), \] (4.15)
\[ \int_{B_r} S_{\nu,k}(\tau \xi) d\xi = 2\pi \sin r p_{\nu,1}(r)S_{\nu,k}(\tau 0). \] (4.16)
Proof. By Pizzetti formula, see [11, Formula (20)] and (3.5), we have

\[
\int_{S_r} S_{\nu,k}(\tau \xi) dl(\xi) = 2\pi r \sin \left( S_{\nu,k}(\tau_0) + \sum_{m=1}^{\infty} \frac{L(L+2) \ldots (L+(m-1)m) S_{\nu,k}(\tau_0)}{(m!)^2} \left( \sin \frac{r}{2} \right)^{2m} \right)
\]

\[
= 2\pi r S_{\nu,k}(\tau_0) \cdot \left( 1 + \sum_{m=1}^{\infty} (-\nu(\nu+1))(2-\nu(\nu+1)) \ldots (m(m-1)-\nu(\nu+1)) \left( \sin \frac{r}{2} \right)^{2m} \right)
\]

\[
= 2\pi r S_{\nu,k}(\tau_0) \frac{\Gamma(m-\nu)\Gamma(m+\nu+1)}{\Gamma(-\nu)\Gamma(\nu+1)(m!)^2} \left( \sin \frac{r}{2} \right)^{2m}
\]

\[
= 2\pi r S_{\nu,k}(\tau_0) F\left(-\nu, \nu+1; 1; \left( \sin \frac{r}{2} \right)^2 \right).
\]

Then identity (4.15) is implied by (3.4) and the definition of the Legendre function. Employing (4.15) and (4.1), we obtain

\[
\int_{B_r} S_{\nu,k}(\tau \xi) d\xi = \int_{0}^{r} \int_{S_{\nu,k}} S_{\nu,k}(\tau \xi) dl(\xi) d\rho = 2\pi S_{\nu,k}(\tau_0) \int_{0}^{r} \sin \rho \ p_{\nu,0}(\rho) d\rho
\]

\[
= 2\pi S_{\nu,k}(\tau_0) \int_{0}^{r} \sin \rho \ (D_{-1} p_{\nu,1})(\rho) d\rho = 2\pi S_{\nu,k}(\tau_0) \int_{0}^{r} \frac{d}{d\rho} (p_{\nu,1}(\rho) \sin \rho) d\rho
\]

\[
= 2\pi r \ p_{\nu,1}(r) S_{\nu,k}(\tau_0).
\]

This completes the proof. \(\square\)

Lemma 4.6. Let \(f \in C^\infty(S')\). Then \(f \in U_r(S')\) if and only if for each \(k \in \mathbb{Z}\) the expansion holds:

\[
f^k(\xi) = \sum_{\nu \in N(r)} \alpha_{\nu,k} S_{\nu,k}(\xi), \quad \xi \in S',
\]

where \(\alpha_{\nu,k} \in \mathbb{C}\) and

\[
\alpha_{\nu,k} = O(\nu^{-a}) \quad \text{as} \quad \nu \to +\infty \quad \text{for each} \quad a > 0.
\]  

(4.17)

Lemma 4.6 is a particular case of the result established earlier by Vit.V. Volchkov [11, Thm. 16.6(ii)].

According Ungar theorem on spherical means [11], if a function \(f \in C(S^2)\) has zero integrals over all geodesic circumferences of the radius \(r\) and \(P_l(\cos r) \neq 0\) for each \(l \in \mathbb{N}\), then \(f \equiv 0\).

The next result specifies this fact.

Theorem 4.1. Let \(f \in C^\infty(S')\). Then the function \(f\) has zero integrals over all geodesic circumferences of the radius \(r\) on \(S^2\) lying in \(S'\) if and only if for each \(k \in \mathbb{Z}\) the expansion holds true:

\[
f^k(\xi) = \sum_{\nu \in Z(r)} \alpha_{\nu,k} S_{\nu,k}(\xi), \quad \xi \in S',
\]

where the coefficients \(\alpha_{\nu,k}\) satisfy condition (4.17).

Proof. First we assume that the integrals of \(f\) over all geodesic circumferences of the radius \(r\) on \(S^2\) located in \(S'\) vanish. By Lemma 4.6 we have

\[
f^k(\xi) = \sum_{\nu \in N(r)} \alpha_{\nu,k} S_{\nu,k}(\xi), \quad \xi \in S',
\]

where the coefficients \(\alpha_{\nu,k}\) satisfy condition (4.17). By formula (3.7), the integrals of \(f^k\) over all geodesic circumferences of the radius \(r\) on \(S^2\) lying in \(S'\) are also zero. In particular, since
\[ S_{\pi-r} = S_r(0, 0, -1), \]

we have

\[
\int_{S_{\pi-r}} f^k(a_\xi)d\xi = 0 \quad \text{as} \quad |t| < r,
\]

where

\[ a_\xi = (\xi_1, \xi_2 \cos t + \xi_3 \sin t, -\xi_2 \sin t + \xi_3 \cos t). \]

Writing this relation for the right hand side in (4.19) and employing Lemmata 4.2, 4.3 we find

\[
\sum_{\nu \in N(r)} \alpha_{\nu,k} \psi^\nu(-\cos r)\psi^\nu_k(t) = 0, \quad |t| < r.
\] (4.20)

We apply the differential operator \(D_{-|k|+1}D_{-|k|}\) to both sides of the above identity and taking into consideration (4.9) and (4.1), we obtain

\[
\sum_{\nu \in N(r)} \alpha_{\nu,k} \psi^\nu(-\cos r)\psi^\nu_0(t) = 0, \quad |t| < r.
\]

By (4.9) and Lemma 4.4 we then conclude that

\[
\alpha_{\nu,k} \psi^\nu(-\cos r) = 0, \quad \nu \in N(r).
\] (4.21)

In view of formula (3.2), identity (4.21) can be rewritten as

\[
\alpha_{\nu,k} \sin(\pi \nu)Q^\nu_r(\cos r) = 0, \quad \nu \in N(r).
\]

Then, in view of Statement (iii) of Lemma 4.3

\[
\alpha_{\nu,k} \sin(\pi \nu) = 0, \quad \nu \in N(r),
\]

and hence, \(\alpha_{\nu,k} = 0\) as \(\nu \in N(r), \nu \notin N\). In view of (4.19) this proves the necessary condition in Theorem 4.1.

We proceed to the sufficient condition. Assume that for each \(k \in \mathbb{Z}\) expansion (4.18) holds true. Then by (4.15) and Statement (i) of Lemma 4.3 we conclude that each Fourier coefficient \(f^k\) has zero integrals over all geodesic circumferences of the radius \(r\) on \(S^3\) lying in \(S\'). Therefore, the function \(f\) possesses the stated property.

5. Proof of Theorem 2.1

Suppose that a function \(f \in C^\infty(S')\) satisfies the assumptions of Theorem 2.1. Then it follows from the first condition of Theorem 2.1 and Theorem 4.1 that for each \(k \in \mathbb{Z}\) representation (4.18) holds true and the coefficients obey estimate (4.17). In view of the second condition of Theorem 2.1 and formula (3.7) we obtain

\[
\int_{B_r} f^k(a_\xi)d\xi = 0, \quad |t| < \pi - r.
\] (5.1)

Employing (5.1), (4.18), (4.16) and Lemma 4.2 we find

\[
\sum_{\nu \in \mathbb{Z}(r)} \alpha_{\nu,k} \psi^\nu_1(r)\psi^\nu_1_k(t) = 0, \quad |t| < \pi - r.
\]

In view of the arguing in the proof of Theorem 4.1 this yields

\[
\sum_{\nu \in \mathbb{Z}(r)} \alpha_{\nu,k} \psi^\nu_1(r)\psi^\nu_0(t) = 0, \quad |t| < \pi - r,
\]

which is equivalent to the identity

\[
\sum_{\nu \in \mathbb{Z}(\pi-r)} \alpha_{\nu,k} \psi^\nu_1(r)\psi^\nu_0(t) = 0, \quad |t| < \pi - r.
\]
see Statement (i) of Lemma 4.3. Now Lemma 4.4 shows that
\[ \alpha_{\nu,k} p_{\nu,1}(r) = 0, \quad \nu \in \mathcal{Z}(r). \]
But by Statement (ii) in Lemma 4.3, the identities \( p_{\nu,0}(r) = 0 \) and \( p_{\nu,1}(r) = 0 \) can not hold simultaneously. This is why \( \alpha_{\nu,k} = 0 \) as \( \nu \in \mathcal{Z}(r) \). This means that \( f^k = 0 \) and hence, \( f = 0 \). This completes the proof of Theorem 2.1.

\section*{BIBLIOGRAPHY}

1. C.A. Berenstein, R. Gay. A local version of the two-circles theorem // Israel J. Math. 55:3, 267–288 (1986).
2. V.V. Volchkov. Solution of the support problem for several function classes // Matem. Sborn. 188:9, 13–30 (1997). [Sb. Math. 188:9, 1279–1294 (1997).]
3. V.V. Volchkov. Integral geometry and convolution equations. Kluwer Academic Publishers, Dordrecht (2003).
4. V.V. Volchkov, Vit.V. Volchkov. Harmonic analysis of mean periodic functions on symmetric spaces and the Heisenberg group. Springer-Verlag, London (2009).
5. V.V. Volchkov, Vit.V. Volchkov. Offbeat integral geometry on symmetric spaces. Birkhäuser, Basel (2013).
6. V.V. Volchkov. On the injectivity of the local Pompeiu transform on the sphere // Matem. Zamet. 81:1, 59–69 (2007). [Math. Notes. 81:1, 51–60 (2007).]
7. P. Ungar. Freak theorem about functions on a sphere // J. London Math. Soc. 29:2, 100–103 (1954).
8. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. Higher transcendental functions. Vols. I-II. Robert E. Krieger Publishing Company, Malabar, Florida (1981).
9. E. Badertscher. The Pompeiu problem on locally symmetric spaces // J. Analyse Math. 57:1, 250–281 (1991).
10. E. Riekstins. Asymptotic expansions of integrals. Zinatne, Riga (1974). (in Russian).
11. C.A. Berenstein, L. Zalcman. Pompeiu’s problem on spaces of constant curvature // J. d’Analyse Math. 30:1, 113–130 (1976).

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