RESOLUTION OF SINGULAR FIBERS OF AN S¹-MANIFOLD

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Abstract. In this paper, we present a construction of resolution of discrete singular fibers of a closed 5-manifold that admits a locally free S¹-action, and prove its compatibility with the resolution of cyclic surface singularities in the quotient space by the S¹-action.

1. Introduction

S¹-actions on odd-dimensional manifolds naturally arise in the study of contact manifolds. In particular, any almost regular contact manifold admits a locally free S¹-action [14], [15]. Recent research progress in the field of Sasakian geometry further fosters the interest in odd-dimensional manifolds that admit an S¹-action. Investigation along this line has been carried out, for instance, in [7], [8], and resulted in greater insight into the topology of Sasakian manifolds. As the structure of Sasakian 3-manifolds is now well-understood and classified [1], attention has been drawn to the classification of Sasakian 5-manifolds. For a compact 5-manifold, it is known that M admits a Sasakian structure if and only if it admits a quasiregular Sasakian structure [13]. In the latter case, it admits a natural locally free S¹-action, and the quotient M/S¹ has a Kähler orbifold structure (see [2, Theorem 7.1.3]) with points in the singular locus corresponding to the fibers with non-trivial stabilizers in M.

Motivated by this, we consider a closed 5-manifolds M that admits a locally free, effective S¹-action α such that each orbit admits an S¹-invariant complex neighborhood (see paragraphs preceding Definition 2.2), called a complex S¹-manifold (M, α) hereafter; the quotient M/S¹ is a complex surface with singularities. The paper presents a construction of resolution of discrete singular fibers in (M, α), and shows that it is compatible with the resolution for surface cyclic quotient singularities in M/S¹ given in [12] (see also [10]).

Theorem 1.1. Given a complex S¹-manifold (M, α) with discrete singular fibers S_i, i = 1, ..., n, there exists an S¹-manifold (Mω, αω) and disjoint S¹-invariant 3-dimensional subspace P_i of Mω, i = 1, ..., n, such that

1. αω is free;
2. Mω − ∪_{i=1}^nP_i is S¹-equivariantly diffeomorphic to M − ∪_{i=1}^nS_i;
3. Mω/S¹ → M/S¹ is the resolution of M/S¹ as a singular complex surface;
4. P_i is the union Q_{ij} ∪ ... ∪ Q_{ij(c(i))} with Q_{ij} diffeomorphic to S^2 × S^1, 1 ≤ j < c(i), and Q_{c(i)} diffeomorphic to S^3, and Q_{ij} ∩ Q_{ij'} an S¹-invariant circle in Q_{ij}, Q_{ij'} when j' = j + 1 and empty otherwise, where c(i) is the length of the singular fiber S_i (see Definition 3.1).

We remark that the subspace P_i is a reminiscence to the chain of 2-spheres in the plumbing construction in 4-dimensions.

Notation and basic facts about S¹-manifolds are reviewed in Section 2, and the proof of Theorem 1.1 occupies Sections 3.1, 3.2 and 3.3.
2. Preliminaries

2.1. $S^1$-manifold. Throughout the paper, $S^1$ is the unit circle in the complex plane, and $(M, \alpha)$ denotes a closed smooth 5-manifold $M$ equipped with a locally free, effective $S^1$-action $\alpha : S^1 \times M \to M$. Since $M$ is compact, $\alpha$ is proper and the stabilizer of every point $p \in M$ is finite (i.e. almost free). In particular, by the slice theorem [11, Theorem 3.8] (see [9, Theorem 5.6], [3] and [5]), the orbit space $O := M/S^1$ is an orbifold [11, Theorem 1.5]. Denote by $\pi$ the quotient map $M \to O$, and call $(M, \alpha)$ an $S^1$-manifold.

Since $\alpha$ is effective, the principal orbit type $M_{\text{reg}}$ corresponds to points in $M$ with the trivial stabilizer. By the principal orbit type theorem [6, Theorem 5.14] (see also [9, Theorem 1.32]), it is open, dense and connected; in addition, $M_{\text{reg}} \to M_{\text{reg}}/S^1$ is a principle $S^1$-bundle.

**Definition 2.1.** The orbit $S$ of a point $p \in M$ is called a regular fiber if $p \in M_{\text{reg}}$; otherwise it is called a singular fiber.

In general, $M - M_{\text{reg}}$ is a union of smooth submanifolds of $M$. The present note concerns mainly the case where $M - M_{\text{reg}}$ is a discrete, and hence finite, set.

Two $S^1$-manifolds $(M, \alpha), (M', \alpha')$ are said to be $S^1$-equivalent if there exists a diffeomorphism between $M, M'$ that respects $\alpha, \alpha'$. By the slice theorem, each orbit $S$ of $p$ in an $S^1$-manifold $(M, \alpha)$ admits an $S^1$-invariant neighborhood $U_5$ which is $S^1$-equivalent to a model $S^1$-manifold $V_p := S^1 \times \mathbb{R}^4/\rho_p$, where $\rho_p$ is a faithful representation of the isotropy subgroup $\Gamma_p$ of $p$ on the real vector space $\mathbb{R}^4$, and the $S^1$-action on $V_p$ is induced by the $S^1$-action on $S^1$. Note that $\rho_p$ is faithful since $\alpha$ is effective.

The present paper concerns the case where $\mathbb{R}^4$ is regarded as the complex vector space $\mathbb{C}^2$ and $\rho_p$ is complex linear; we call such a $V_p := S^1 \times \mathbb{C}^2/\rho_p$ a model complex $S^1$-manifold.

**Definition 2.2.** A special complex $S^1$-manifold is an $S^1$-manifold with $M_{\text{sing}} := M - M_{\text{reg}}$ discrete, and for every orbit $S$, there exists an $S^1$-invariant neighborhood $U_5$ which is $S^1$-equivalent to a model $S^1$-manifold.

**Definition 2.3.** Given an orbit $S$ of $p \in M_{\text{sing}}$, we say it is a singular fiber of type $\frac{1}{r}(1, a)$, $r, a$ two positive coprime integers, if the stabilizer $\Gamma_p$ is isomorphic to $\mathbb{Z}_r = \langle g \rangle$ such that $\rho_p$ is conjugate to the representation

$Z_r \to \mathbb{C}^2$

$g \mapsto \begin{bmatrix} e^{\frac{2\pi i}{r}} & 0 \\ 0 & e^{2\pi i a} \end{bmatrix}$.

Up to conjugation, a faithful representation $\rho$ of $Z_r$ on $\mathbb{C}^2$ is conjugate to

$Z_r \to \mathbb{C}^2$

$g \mapsto \begin{bmatrix} e^{\frac{2\pi i}{r_1}} & 0 \\ 0 & e^{2\pi i a_i} \end{bmatrix}$

for some positive integers $r_1, r_2$ with their least common multiplier $r$, and $a_i, i = 1, 2$ integers coprime to $r_i, i = 1, 2$.

If $r_1, r_2 < r$, then $(M, \alpha)$ has non-discrete set of singular fibers, that is, $M_{\text{sing}}$ contains some 3-dimensional submanifolds of $M$. On the other hand, $M_{\text{sing}}$ is discrete if and only if $r_1 = r_2 = r$. Up to a change of generator, the representation
$\rho$ can be normalized as follows:

$$\mathbb{Z}_r \to \mathbb{C}^2$$

$$g \mapsto \begin{bmatrix} e^{2\pi i \frac{1}{r}} & 0 \\ 0 & e^{2\pi i \frac{a}{r}} \end{bmatrix}$$

for some $a > 0$ coprime to $r$. Therefore, we have the following.

**Lemma 2.1.** Every discrete singular fiber is of type $\frac{1}{r}(1, a)$, for some coprime positive integers $r, a$.

From now on, $(M, \alpha)$ is assumed to be a special complex $S^1$-manifold.

### 2.2. Neighborhood of a singularity.

In this subsection, we examine the $S^1$-structure of a neighborhood of a singular fiber of $\frac{1}{r}(1, a)$ type. Let $S$ be a singular fiber, and $U_5$ an $S^1$-invariant neighborhood of $S$ which is $S^1$-equivalent to a model complex $S^1$-manifold $S^1 \times \mathbb{C}^2 / \mathbb{Z}_r$, where $\mathbb{Z}_r$, generated by $g$, acts on $\mathbb{C}^2$ by the representation

$$\rho : \mathbb{Z}_r \to \mathbb{C}^2$$

$$g \mapsto \begin{bmatrix} e^{2\pi i \frac{1}{r}} & 0 \\ 0 & e^{2\pi i \frac{a}{r}} \end{bmatrix}$$

and the action of $\mathbb{Z}_r$ on $S^1$ is given by letting $g = e^{2\pi i \frac{1}{r}}$. The $S^1$-action on $S^1 \times \mathbb{C}^2 / \mathbb{Z}_r$ is induced by the $S^1$-action on the first factor of $S^1 \times \mathbb{C}^2$.

Consider the local diffeomorphism

$$\pi : S^1 \times \mathbb{C}^2 \to S^1 \times \mathbb{C}^2$$

$$(w, z_1, z_2) \mapsto (w^r, w^{-1}z_1, w^{-a}z_2)$$

which descends to a diffeomorphism from $S^1 \times \mathbb{C}^2 / \mathbb{Z}_r$ to $S^1 \times \mathbb{C}^2$, and therefore, the neighborhood $U_5$ is $S^1$-equivalent to the $S^1$-manifold $(S^1 \times \mathbb{C}^2, \alpha_0)$ with $\alpha_0$ given by

$$S^1 \times S^1 \times \mathbb{C}^2 \to S^1 \times \mathbb{C}^2$$

$$(t, u, v_1, v_2) \mapsto (t^r, t^{-1}u, t^{-a}v_2).$$

### 3. Resolution of singularities via charts

#### 3.1. Reduction on type $\frac{1}{a}(1, a)$ singular fibers.

Let $S$ be a singular fiber of $(M, \alpha)$ $S$. We measure its complexity by continued fractions as follows.

**Definition 3.1.** Suppose $S$ is of type $\frac{1}{a}(1, a)$, and $[b_1, \ldots, b_n]$ is the continued fraction of $\frac{1}{a}$, namely

$$\frac{r}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}}.$$ 

Then we define the length of the singular fiber to be $n$.

Note that the fraction can be calculated by the recursive formula:

$$r = ab_1 - a_1, 0 < a_1 < a$$

$$a = a_1b_2 - a_2, 0 < a_2 < a_1$$

$$a_i = a_{i+1}b_{i+2} - a_{i+2}, 0 < a_{i+2} < a_{i+1}$$

$\ldots$. (3.1)

Now we present a construction that reduces a type $\frac{1}{a}(1, a)$ singular fiber to a type $\frac{1}{a_1}(1, a_1)$ singular fiber, where $\frac{1}{a_1} = [b_2, \ldots, b_n]$ and $\frac{1}{a} = [b_1, b_2, \ldots, b_n]$. 

Consider the five manifold $\mathbb{C} \times S^3$ equipped with the $S^1$-action
\[
\delta_a : S^1 \times (\mathbb{C} \times S^3) \to \mathbb{C} \times S^3
\]
\[
(x, y_1, y_2) \mapsto (t^n x, t^{1-1} y_1, t^{-a} y_2),
\]
where $(x, y_1, y_2) \in \mathbb{C} \times S^3 \subset \mathbb{C}^3$.

**Lemma 3.1.** $(\mathbb{C} \times S^3, \delta_a)$ has only one singular fiber and it is of type $\frac{a}{a}$.

**Proof.** Observe first that the manifold $\mathbb{C} \times S^3$ can be obtained by gluing manifolds $X := \mathbb{C} \times S^1 \times \mathbb{C}$ and $X' := \mathbb{C} \times C \times S^1$ via the diffeomorphism
\[
X' \supset \mathbb{C} \times C^* \times S^1 \xrightarrow{f} \mathbb{C} \times S^1 \times C^* \subset X
\]
where $C^* = \mathbb{C} - \{0\}$. The manifold $\mathbb{C} \times S^3$ can be identified with $X \cup_f X'$ via the embeddings:
\[
i : X \to \mathbb{C} \times S^3
\]
\[
(p, q_1, q_2) \mapsto (p|q|, \frac{q_1}{|q|}, \frac{q_2}{|q|}),
\]
\[
i' : X' \to \mathbb{C} \times S^3
\]
\[
(p', q_1', q_2') \mapsto (p'|q'|, \frac{q_1'}{|q'|}, \frac{q_2'}{|q'|}),
\]
where $q = (q_1, q_2)$ and $q' = (q_1', q_2')$. It is not difficult to check the embeddings respect $f$, and hence induces a diffeomorphism between $X \cup_f X'$ and $\mathbb{C} \times S^3$.

The $S^1$-action $\delta_a$ on $\mathbb{C} \times S^3$ restricts to $S^1$-actions on $X, X'$ via (3.3), (3.4) as follows:
\[
\kappa_a : S^1 \times X \to X
\]
\[
(t, p, q_1, q_2) \mapsto (t^n p, t^{1-1} q_1, t^{-a} q_2)
\]
\[
\kappa_a' : S^1 \times X' \to X'
\]
\[
(t, p', q_1', q_2') \mapsto (t^n p', t^{1-1} q_1', t^{-a} q_2').
\]
This shows $S_0 = \{(0, 0)\} \times S^1$ is a singular fiber of $(X', \kappa_a')$; on the other hand, the $S^1$-action $\kappa$ on $X$ is free since $q_1$ is never zero.

To determine the type of the singular fiber $S_0$, consider the representation of $Z_a = \langle g \rangle$ on $\mathbb{C}^2$:
\[
\rho : Z_a \to \mathbb{C}^2
\]
\[
g \mapsto \begin{bmatrix} e^{2\pi i y} & 0 \\ 0 & e^{2\pi i x} \end{bmatrix}
\]
and the natural homomorphism
\[
i : Z_a \to S^1
\]
\[
g \mapsto e^{2\pi i x}.
\]
Equip $\tilde{X} = \mathbb{C}^2 \times S^1$ with the $S^1$-action acting on the second factor. Then the quotient $\mathbb{C}^2 \times S^1/Z_a$ given by $\rho, i$ is $S^1$-equivalent to $(X', \kappa_a')$ via the map
\[
\tilde{X}' = \mathbb{C}^2 \times S^1 \to X'
\]
\[
(\tilde{p}', \tilde{q}_1', \tilde{q}_2') \mapsto (\tilde{p}'(\tilde{q}_1')^{-1}(\tilde{q}_2')^{-1}, (\tilde{q}_1')^{-1}, (\tilde{q}_2')^{-1}).
\]
Note that the map sends the orbit of the $Z_a$-action to a point because $r = ab_1 - a_1$. Thus by Definition 2.3 the fiber is of type $\frac{a}{a}(1, a_1)$.
\[\square\]
Suppose $S$ is a singular fiber of $(M, \alpha)$ of type $\frac{1}{r}(1, a)$ with length $n$. Then by (3.2), we can identify an $S^1$-invariant neighborhood $U_5$ of $S$ with $S^1 \times C^2$ such that $S = S^1 \times \{(0, 0)\}$ and the $S^1$-action on $U_5$ is realized as follows:

$$S^1 \times U_5 \to U_5 \quad (t, u, v_1, v_2) \mapsto (t^r u, t^{-1} v_1, t^{-a} v_2). \quad (3.6)$$

Consider the $S^1$-equivariant embedding

$$e : (U_5 - S) \to \mathbb{C} \times S^3 \quad (u, v_1, v_2) \mapsto (u|v_1|, \frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}), \quad (3.7)$$

where $v = (v_1, v_2)$, and glue the manifold $(M - S, \alpha)$ and $(\mathbb{C} \times S^3, \delta_0)$ via $e$. The resulting new special complex $S^1$-manifold $(M', \alpha')$ has all singular fibers the same as $(M, \alpha)$ except that the singular fiber $S$ is now replaced with a (singular) fiber of smaller length. By induction, we eventually get a special complex $S^1$-manifold $(M_\alpha, \alpha_\omega)$ with $\alpha_\omega$ free, and this proves the first assertion of Theorem 1.1.

We remark that $e$ can be decomposed into the embeddings (3.3) and (3.4) in terms of $X, X'$ as follows:

$$U_5 \supset S^1 \times S^1 \times \mathbb{C} \to X \quad (u, v_1, v_2) \mapsto (u|v_1|, \frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}) \quad (3.8)$$

$$U_5 \supset S^1 \times S^1 \times \mathbb{C}' \to X' \quad (u, v_1, v_2) \mapsto (u|v_2|, \frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}). \quad (3.9)$$

(3.8), (3.9) come in handy when proving the rest assertions in Theorem 1.1.

3.2. The topology of the 3-dimensional subspace $P_i$. To understand the structure of $P_i$ in Theorem 1.1, we note that without loss of generality, it may be assumed that $(M, \alpha)$ has only one singular fiber $S$. In particular, if $a = 1$, $\alpha_\omega = \alpha'$, and in $M_\omega$, $U_5$ is replaced with $X, X'$ and the singular fiber $S$ replaced with the 3-sphere, $P := R \cup R'$, where $R = 0 \times S^1 \times \mathbb{C}, R' = 0 \times \mathbb{C} \times S^1$ and $f$ is given in (3.2). For the general case, we denote by $(X_i, \kappa_{a_i}), (X'_i, \kappa'_{a_i})$ the $S^1$-manifolds $\mathbb{C} \times S^1 \times \mathbb{C}, \mathbb{C} \times \mathbb{C} \times S^1$ equipped with the $S^1$-actions

$$\kappa_{a_i} : S^1 \times X_i \to X_i \quad (t, p, q_1, q_2) \mapsto (t^{a_i-1} p', t^{-1} q'_1, t^{-a_i})$$

$$\kappa'_{a_i} : S^1 \times X'_i \to X'_i \quad (t, p', q'_1, q'_2) \mapsto (t^{a_i-1} p', t^{-1} q'_1, t^{-a_i} q'_2),$$

respectively, $i = 0, \ldots, n - 1$, with $n$ being the length of the singular fiber $S$ and $a_{i-1} = r, a_0 = a$ and $a_i > 0$ given by (3.1). Then we have the following lemma.

**Lemma 3.2.** Let $R_i, R_i'$ be the $S^1$-invariant subspaces $0 \times S^1 \times \mathbb{C} \times \mathbb{C} \times S^1 \times \mathbb{C} \times S^1 \times \mathbb{C} \times S^1 \times S^1 \times \mathbb{C}$, respectively, $i = 0, \ldots, n - 1$, and $R'_{n-1}$ the $S^1$-invariant subspace $0 \times \mathbb{C} \times S^1 \times \mathbb{C} \times S^1 \times \mathbb{C} \times S^1 \times \mathbb{C}$, such that the resolution $(M_\alpha, \alpha_\omega)$ is obtained by gluing $U_5 - S$ to $X_\omega := X_0 \cup g_1 X_1 \cup g_2 X_2 \cup \cdots \cup g_{n-1} X_{n-1} \cup g X'_n \cup X'_n$ via an $S^1$-equivariant embedding $\epsilon_\omega$ that restricts to

$$U_5 - (S^1 \times 0 \times \mathbb{C}) = S^1 \times \mathbb{C} \times \mathbb{C} \to X_0.$$
given in (3.8), and \( g \) is the diffeomorphism in (3.20). Furthermore, \( U_S - S \) is \( S^3 \)-equivalent to \( X_\omega - R_\omega \), with

\[
R_\omega := R_0 \cup q_1 (C_1^+ \cup R_j) \cup q_2 \cdots \cup q_{n-1} (C_{n-1}^+ \cup R_{n-1}) \cup g R_{n-1}.
\]

Lemma 3.2 implies the second and forth assertions of Theorem 1.6 where \( P \) corresponds to \( R_\omega \) here, and \( Q_j := R_{j-1} \cup q_j C_j^+ \), \( j = 1, \ldots, n - 1 \), and \( Q_n := R_{n-1} \cup g R_{n-1} \). It is then not difficult to check \( Q_n \) is diffeomorphic to a 3-sphere, \( Q_j \) diffeomorphic to \( S^3 \times S^1 \), and \( Q_j \cap Q_{j+1} \) the circle \( R_j \cap C_j^+ = 0 \times S^1 \times 0 \), \( j = 1, \ldots, n - 1 \).

**Proof.** We prove by induction. The case \( n = 1 \) is clear, so we may assume \( n > 1 \), and we apply the resolution construction to the singular fiber \( S_0 \) in \( (M', \alpha') \). To do so, we observe first that \( X' \) is an \( S^1 \)-invariant neighborhood of \( S_0 \), but the \( S^1 \)-action is not quite the same as \( U_S \) in form, so we “normalize” \( X' \) by the \( S^1 \)-equivariant diffeomorphism

\[
X' \rightarrow X'' = S^1 \times \mathbb{C}^2
\]

(3.10)

The induced \( S^1 \)-action on \( X'' \) is then given by

\[
S^1 \times X'' \rightarrow X''
\]

(3.11)

\[
(p', q_1, q_2) \mapsto (q_2^{-1} q_1, p'(q_2)^{b_2}).
\]

With the new coordinates, we can now apply the resolution construction to \( X'' \). Take a copy of \( \mathbb{C} \times S^1 \) and identify it with \( X_1 \cup \hat{X}' \) as in (3.3), (3.4), where \( X_1, \hat{X}' \) are copies of \( X, X' \), respectively, and \( f_1 = \hat{f} \). Think of \( X'' \) as the union of \( S^1 \times \mathbb{C}^* \times \mathbb{C} \cup S^1 \times \mathbb{C} \). Then as in (3.8) and (3.9), we have the gluing maps:

\[
X'' \supset S^1 \times \mathbb{C}^* \times \mathbb{C} \rightarrow X_1
\]

\[
(u, v_1, v_2) \mapsto (u|v_1|, u|v_1|, v_2)
\]

(3.12)

and hence, the composition

\[
X_s \xrightarrow{f^{-1}} X'_s \xrightarrow{(3.10)} X'' \xrightarrow{(3.12)} X_1,
\]

(3.13)

is given by the assignment

\[
(p, q_1, q_2) \mapsto (q_2^{-1} q_1, p|q_2|^{b_2} (\frac{q_2}{|q_2|})^{b_1}),
\]

where \( X_s := \mathbb{C} \times S^1 \times \mathbb{C}^* \subset X \), \( X'_s := \mathbb{C} \times \mathbb{C}^* \times S^1 \subset X' \), and \( X'' := S^1 \times \mathbb{C}^* \times \mathbb{C} \subset X'' \). The composition (3.13) restricts to the following assignment

\[
(0, q_1, q_2) \mapsto (q_2^{-1} q_1, 0)
\]

(3.14)

on \( 0 \times S^1 \times \mathbb{C}^* \).

By induction, \( g_i, i = 2, \ldots, n - 1 \) satisfying conditions required in Lemma 3.2 and an embedding \( e'' \) from \( X'' - S_0 \) to

\[
X'' := X_1 \cup g_2 X_2 \cup g_3 \cdots \cup g_{n-1} X_{n-1} \cup g X'_{n-1},
\]

such that the resolution of \( S_0 \) in \( (M', \alpha') \) is obtained by gluing \( X'' \) to \( X'' - S_0 \) via \( e'' \). Let \( X_0 := X \), and \( g_1 \) be the composition (3.13). Then \( e'' \) and (3.8) induce a gluing embedding from \( U_S - S \) to

\[
X_0 := X_0 \cup g_1 X_1 \cup g_2 \cdots \cup g_{n-1} X_{n-1} \cup g X'_{n-1}.
\]
Furthermore, by (3.14), we have
\[
X_\omega \supset R_0 \cup R_1 \cup g_2 \cdots \cup g_{n-1} (R_{n+1}^1 \cup R_{n+1}) \cup R_n^2 \cup R_{n+1}^2
\]
\[
= R_0 \cup g_1 (R_1^1 \cup R_1) \cup g_2 \cdots \cup g_{n-1} (R_{n+1}^1 \cup R_{n+1}) \cup R_n^2 \cup R_{n+1}^2,
\]
and there the claim. \(\square\)

Note that if \(M\) is simply connected, then \(M_\omega\) is diffeomorphic to \(M \# n(S^1 \times S^2)\).

3.3. Compatibility with resolution in 4-dimensions. Given a complex 2-dimensional orbifold \(O\) and let \(p\) be an isolated singularity of type \(\frac{1}{2}(1, a)\). Then based on Reid’s model \([12]\), one can resolve the singularity by replacing a neighborhood \(\hat{U}_p\) of \(p\) with a 4-manifold given by gluing \(n + 1\) copies of \(\mathbb{C}^2\), denoted by \(Y_0, \ldots, Y_n\). The gluing maps \(f_i\) between \(Y_i, i = 0, \ldots, n\), is given by
\[
f_i : Y_i - (\mathbb{C} \times 0) \to Y_{i+1}
\]
\[
(\xi_i, \eta_i) \mapsto (\eta_i^{-1}, \xi_i(\eta_i)^{b_i}),
\]
and the embedding from \(\hat{U}_p - p \simeq \mathbb{C}^2 - \{(0, 0)\}\) to \(Y_1 \cup f_1 \cup \cdots \cup f_{n+1} \cup Y_n\) given by
\[
\hat{U}_p - V_0 \to Y_0
\]
\[
(v_1, v_2) \mapsto (v_1^{e^{2\pi i/a}}, v_2^{e^{2\pi i/a}})
\]
\[
\hat{U}_p - V_1 \to Y_1
\]
\[
(v_1, v_2) \mapsto (v_1^{e^{2\pi i/a}}, v_1^{e^{2\pi i/a}}v_2^{e^{2\pi i/a}}b_i)
\]
\[
\vdots\]
\[
\vdots\]
(3.15)
allows us to glue \(\hat{U}_p - \{p\}\) with \(Y_0 \cup f_0 Y_1 \cup f_1 \cup \cdots \cup f_{n+1} \cup Y_n\), where \(V_i\) is \(\mathbb{C} \times 0, 0 \times \mathbb{C}\) or their union depending on the assignments in (3.15), and \(\frac{a}{a} = [b_1, \ldots, b_n]\) is the fraction expression of \(\frac{a}{a}\).

Reid’s model can be viewed as a reduction where we first reduce the singularity to a singularity of type \(\frac{1}{2}(1, a)\) by gluing \(Y_p - p\) with \(Y \cup f(Y'/\mathbb{Z}_a)\), where \(Y, Y'\) are two copies of \(\mathbb{C}^2\) with \(\mathbb{Z}_a\) acting on \(Y'\) by
\[
(\zeta'_1, \zeta'_2) \mapsto (\epsilon^{e^{2\pi i/a}} \zeta'_1, \epsilon^{e^{2\pi i/a}} \zeta'_2), \quad \epsilon = e^{2\pi i/a}.
\]

To find the gluing map \(f\), we apply the embedding in (3.15) to \(Y' \simeq Y'\) with \(Y' \xrightarrow{i} Y''\)
\[
(\zeta'_1, \zeta'_2) \mapsto (\zeta''_1, \zeta''_1),
\]
and identify \(Y\) with \(Y_0\)\footnote{The use of \(Y''\) comes in handy when later we compare it with the resolution of the singular fiber of \((M, a)\).}. In particular, this implies the equality:
\[
((\zeta'_2)^a, (\zeta'_1)^{-a}_1) = (v_2^{-1}v_1^a, v_1^a(v_2v_1^{-a})b_n).
\]
(3.16)
The gluing map \(f\) can then be expressed in terms of coordinates as follows:
\[
f : (Y' \setminus \{0, 0\}) \to Y
\]
\[
(\zeta'_1, \zeta'_2) \mapsto ((\zeta''_2)^a \zeta''_1, (\zeta''_2)^{-a}).
\]
Similarly, we can write down the embedding
\[
(U_p - \{0, 0\}) \to Y \cup f(Y'/\mathbb{Z}_a)\]
as follows:

\[
(U_p - 0 \times \mathbb{C}) \to Y,
\]

\[
(v_1, v_2) \mapsto (v_1^r, v_2 v_1^{-a}),
\]

\[
(U_p - \mathbb{C} \times 0) \to Y',
\]

\[
(v_1, v_2) \mapsto (v_2^r, v_1 v_2^{-a}).
\]

(3.17)

(3.18)

Note the last map involves fraction, so it is well-defined only after quotient the \(\mathbb{Z}_a\)-action.

Observe that the resolution \(O'\) of \(O\) of the singular point \(p\) is topologically equivalent to the manifold

\[
((O - \hat{D}_U)/\mathbb{Z}_a) \cup_h (D_Y \cup_f (D'_Y/\mathbb{Z}_a)),
\]

(3.19)

where \(D_U, D_Y, D'_Y\) are the product of two unit disks in \(U_p, Y, Y'\), respectively, with the gluing map \(h\) induced by the map

\[
\partial D_U = S^1 \times D^2 \cup D^2 \times S^1 \to S^1 \times D^2 \cup_f (S^1 \times D^2/\mathbb{Z}_a)
\]

given by the assignments

\[
S^1 \times D^2 \to S^1 \times D^2 \subset D_Y
\]

\[
(v_1, v_2) \mapsto (v_1^r, v_2 v_1^{-a}),
\]

(3.20)

\[
D^2 \times S^1 \to S^1 \times D^2 \subset D'_Y
\]

\[
(v_1, v_2) \mapsto (v_2^r, v_1 v_2^{-a}),
\]

(3.21)

and \(f\) induced by the map

\[
D'_Y \supset D^2 \times S^1 \to D^2 \times S^1 \subset D_Y
\]

\[
((\zeta_1', \zeta_2') \mapsto (\zeta_1' (\zeta_2')^{-a}), (\zeta_2')^{-a}).
\]

(3.22)

To see the resolution given in Section 3.1 descends to the resolution for complex orbifolds topologically. We recall the map (3.5) from \(\hat{X}'\) to \(X'\):

\[
(\tilde{p}', \tilde{q}_1', \tilde{q}_2') \mapsto ((\tilde{q}_1')^{-1}, \tilde{q}_2' \tilde{q}_1'^{-a}),
\]

(3.23)

where the \(S^1\)-action acting on the second factor of \(\hat{X}\). Note \(\pi\) is an \(S^1\)-equivariant diffeomorphism, whereas \(\pi'\) is not a diffeomorphism. These two maps together with the gluing map \(f\) in (3.22) give us the composition \(\pi^{-1} \circ f \circ \pi'\) which in terms of the coordinates of \(X'\) can be written down as follows:

\[
(\tilde{p}', \tilde{q}_1', \tilde{q}_2') \mapsto (\tilde{p}'|\tilde{q}_1'| (\tilde{q}_1')^{-1} \tilde{q}_2', (\tilde{q}_1')^{-a} |\tilde{q}_1'|^{-a-1}).
\]

(3.24)

Quotient out the \(S^1\)-action, we obtain that \(\hat{X} = \hat{X}/S^1\), \(\hat{X}' = \hat{X}'/S^1\), and that the gluing map (3.24) descends to the following map

\[
\hat{X}' \supset \mathbb{C} \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^* \subset \hat{X}
\]

\[
(\tilde{p}', \tilde{q}_1') \mapsto (\tilde{p}'|\tilde{q}_1'| (\tilde{q}_1')^{-1} \tilde{q}_2', (\tilde{q}_1')^{-a} |\tilde{q}_1'|^{-a-1}).
\]

(3.25)

In particular, when restricted to \(D^2 \times S^1 \subset \hat{X}'\), it yields precisely the assignment in (3.22) since now \(|\tilde{q}_1'| = 1\).
Now, we want to show the gluing map $\epsilon$ in (3.7) descends to the gluing maps (3.17) and (3.18) in 4-dimensions. As with the case of $X, X'$, we consider a model complex $S^1$-manifold $\hat{U}_5 \simeq S^1 \times \mathbb{C}^2$ for $U_5$, as in (2.1):

$$\hat{U}_5 \to U_5$$

$$(\hat{u}, \hat{v}_1, \hat{v}_2) \mapsto (\hat{u}', \hat{v}_1^{-1}, \hat{v}_1\hat{v}_2^{-1} \hat{v}_1).$$

(3.25)

Compose the maps (3.25), (3.8), and (3.9), and combine them with the maps $\pi : \hat{X} \to X, \pi' : \hat{X}' \to X'$, we obtain the assignments:

$$\hat{U}_5 \supset S^1 \times \mathbb{C} \to \hat{X}$$

$$(\hat{u}, \hat{v}_1, \hat{v}_2) \mapsto (|\hat{v}_1|^{\frac{1}{\alpha}}|\hat{v}_1\hat{v}_2^{-1} \hat{v}_1|^{\alpha-1})$$

$$\hat{U}_5 \supset S^1 \times \mathbb{C} \to \hat{X}'$$

$$(\hat{u}, \hat{v}_1, \hat{v}_2) \mapsto (|\hat{v}_2|^{\frac{1}{\alpha}}|\hat{v}_2^{1-\frac{1}{\alpha}} \hat{v}_1|^{\frac{1}{\alpha}}).$$

(3.26)

(3.27)

which descend to the following assignments after quotienting out the $S^1$-action:

$$\mathbb{C} \times \mathbb{C} \to \hat{X}$$

$$(\hat{v}_1, \hat{v}_2) \mapsto (|\hat{v}_1|^{\frac{1}{\alpha}}|\hat{v}_1\hat{v}_2^{-1} \hat{v}_1|^{\alpha-1})$$

$$\mathbb{C} \times \mathbb{C} \to \hat{X}'$$

$$(\hat{v}_1, \hat{v}_2) \mapsto (|\hat{v}_2|^{\frac{1}{\alpha}}|\hat{v}_2^{1-\frac{1}{\alpha}} \hat{v}_1|^{\frac{1}{\alpha}}).$$

The assignment (3.26) restricts to the following on $S^1 \times D^2$:

$$S^1 \times D^2 \to S^1 \times D^2 \subset \hat{X}$$

$$(\hat{v}_1, \hat{v}_2) \mapsto (\hat{v}_1, \hat{v}_2^{1-\alpha}),$$

while the assignment (3.27) the following on $D^2 \times S^1$:

$$D^2 \times S^1 \to S^1 \times D^2 \subset \hat{X}'$$

$$(\hat{v}_1, \hat{v}_2) \mapsto (\hat{v}_2, \hat{v}_1^{1-\alpha}).$$

They are those in (3.20) and (3.21), respectively, so $M'/S^1$ is homeomorphic to $O'$, and hence, by induction, the third assertion of Theorem 1.1; this completes the proof.

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