JACOBI–TRUDY FORMULA FOR GENERALISED SCHUR POLYNOMIALS

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Abstract. Jacobi–Trudy formula for a generalisation of Schur polynomials related to any sequence of orthogonal polynomials in one variable is given. As a corollary we have Giambelli formula for generalised Schur polynomials.

To Grisha Olshanski on his 65-th birthday with admiration

1. Introduction

The classical Jacobi–Trudy formula expresses the Schur polynomials

\[
S_{\lambda}(x_1, \ldots, x_n) = \frac{\det \begin{bmatrix}
x_{\lambda_1+\lambda_2-1}^{x_{\lambda_1+\lambda_2}} & \cdots & x_{\lambda_1+\lambda_2} \\
x_{\lambda_1+\lambda_2-2}^{x_{\lambda_1+\lambda_2-1}} & \cdots & x_{\lambda_1+\lambda_2-2} \\
\vdots & \ddots & \vdots \\
x_{\lambda_1}^{x_{\lambda_2}} & \cdots & x_{\lambda_1}^{x_{\lambda_2}} \\
x_{\lambda_1}^{x_{\lambda_1}} & \cdots & x_{\lambda_1}^{x_{\lambda_1}}
\end{bmatrix}}{\Delta_n(x)}
\]

(1)

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a partition and \( \Delta_n(x) = \prod_{i<j}(x_i - x_j) \), as the determinant

\[
S_{\lambda}(x_1, \ldots, x_n) = \frac{\det \begin{bmatrix}
h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+l-1} \\
h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+l-2} \\
\vdots & \ddots & \vdots & \vdots \\
h_{\lambda_l-t+1} & h_{\lambda_l-t+2} & \cdots & h_{\lambda_l}
\end{bmatrix}}{\Delta_n(x)}
\]

(2)

where \( l = l(\lambda) \) and \( h_i = h_i(x_1, \ldots, x_n) \) are the complete symmetric polynomials (see [1]). Note that these polynomials \( h_i \) are particular case of Schur polynomials \( S_{\lambda} \), corresponding to the partition \( \lambda = (i) \) consisting of one part.

In this note we give a version of this formula, which is valid for the following generalisation of Schur polynomials related to any sequence of orthogonal polynomials in one variable.

More precisely, let \( \{\varphi_i(z)\} \), \( i = 0, 1, 2, \ldots \) be a sequence of polynomials in one variable, which satisfy a three-term recurrence relation

\[
z\varphi_i(z) = \varphi_{i+1}(z) + a(i)\varphi_i(z) + b(i)\varphi_{i-1}(z)
\]

(3)

with \( \varphi_0 \equiv 1 \), \( \varphi_{-1} \equiv 0 \) (for example, a sequence of the orthogonal polynomials [2]). The corresponding generalised Schur polynomials \( S(x_1, \ldots, x_n|a, b) \) are
defined for any partition \( \lambda \) and two infinite sequences \( a = \{a_i\}, b = \{b_i\} \) by the Weyl-type formula

\[
S_\lambda(x_1, \ldots, x_n|a, b) = \frac{\prod_{i=1}^{n} \varphi_{\lambda_i}(x_i)}{\Delta_n(u)}.
\]

For \( n = 1 \) we assume that \( \Delta_1 \equiv 1 \), so for \( \lambda = (i) \) we have \( S_\lambda(x_1|a, b) = \varphi_i(x_1) \). If the initial sequence \( \varphi_i(z) \) was orthogonal with measure \( d\mu(z) \) the polynomials \( S_\lambda(x_1, \ldots, x_n|a, b) \) are orthogonal with respect to the measure

\[
\Omega(z) = \Delta_\lambda^2(z) \prod_{i=1}^{n} d\mu(z_i).
\]

Alternatively, the generalised Schur polynomials can be defined in this case as the polynomials of the triangular form

\[
S_\lambda(x_1, \ldots, x_n|a, b) = \sum_{\mu \preceq \lambda} K_{\lambda, \mu}(a, b)m_{\mu}(x_1, \ldots, x_n),
\]

which are orthogonal with respect to the measure (5). Here \( m_{\mu}(x_1, \ldots, x_n) \) are monomial symmetric polynomials [1] and the notation \( \mu \preceq \lambda \) means that \( \mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k \) for all \( 1 \leq k \leq n \).

When \( \varphi_i(z) \) is the sequence of classical Jacobi polynomials [2], the corresponding generalised Schur polynomials coincide with the multidimensional Jacobi polynomials with parameter \( \theta = 1 \) (see Lassalle [4] and Okounkov-Olshanski [5]).

Denote the polynomials \( S_\lambda(x_1, \ldots, x_n|a, b) \) with \( \lambda = (i, 0, \ldots, 0) \) as \( h_i(x) \) and extend this sequence for negative \( i \) by assuming that \( h_i(x) \equiv 0 \) for \( i < 0 \). Extend also the sequence of coefficients \( a(i) \) and \( b(i) \) to the negative \( i \) arbitrarily and define recursively the polynomials \( h_i^{(r)}(x_1, \ldots, x_n) \) by the relation

\[
h_i^{(r+1)} = h_i^{(r)} + a(i + n - 1)h_i^{(r)} + b(i + n - 1)h_{i-1}^{(r)}
\]

with initial data \( h_i^{(0)}(x) = h_i(x) \). One can check that \( h_i^{(r)}(x) \equiv 0 \) whenever \( i + r < 0 \) and that the definition of the polynomials \( h_i^{(r)}(x_1, \ldots, x_n) \) does not depend on the extension of the coefficients to the negative \( i \) provided

\[
r \leq i + 2n - 2.
\]

In particular, all the entries of the formula (9) below are well defined.

Our main result is the following
Theorem 1.1. The generalised Schur polynomials satisfy the following Jacobi-Trudy formula:

\[
S_\lambda(x_1, \ldots, x_n|a,b) = \begin{vmatrix}
  h_{\lambda_1} & h_{\lambda_1}^{(1)} & \ldots & h_{\lambda_1}^{(l-1)} \\
  h_{\lambda_2-1} & h_{\lambda_2}^{(1)} & \ldots & h_{\lambda_2}^{(l-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\lambda_{l+1}} & h_{\lambda_{l+1}}^{(1)} & \ldots & h_{\lambda_{l+1}}^{(l-1)}
\end{vmatrix}
\]

where \( l = l(\lambda) \).

This gives a universal proof of the Jacobi–Trudy and Giambelli formulas for usual Schur polynomials as well as for the characters of symplectic and orthogonal Lie algebras (see [3]) and for the factorial Schur polynomials for usual Schur polynomials as well as for the characters of symplectic and orthogonal Lie algebras [1, 6]. Another interesting case, which seems to be new, is the Jacobi–Trudy formula for the multidimensional Jacobi polynomials with parameter \( \theta = 1 \).

2. Proof

We start with the following lemma.

Lemma 2.1. The following equality holds

\[
h_i^{(r)}(x_1, \ldots, x_n) - x_1 h_i^{(r-1)}(x_1, \ldots, x_n) = h_{i+1}^{(r-1)}(x_2, \ldots, x_n)
\]

for all \( r,i \) satisfying the relation (8).

Proof. The proof is by induction in \( r \). When \( r = 1 \) we have from definition

\[
h_i^{(1)}(x_1, \ldots, x_n) = h_i + a(i+n-1)h_i(x_1, \ldots, x_n)
\]

\[
= \Delta_n(x)^{-1}
\begin{vmatrix}
  0 & (x_2 - x_1)\varphi_{i+n-1}(x_2) & \ldots & (x_n - x_1)\varphi_{i+n-1}(x_n) \\
  \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2) & \ldots & \varphi_{n-2}(x_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \ldots & 1
\end{vmatrix}
\]

Subtracting the first column from the others we get

\[
= \Delta_n(x)^{-1}
\begin{vmatrix}
  0 & (x_2 - x_1)\varphi_{i+n-1}(x_2) & \ldots & (x_n - x_1)\varphi_{i+n-1}(x_n) \\
  \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2) - \varphi_{n-2}(x_1) & \ldots & \varphi_{n-2}(x_n) - \varphi_{n-2}(x_1) \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & \ldots & 0
\end{vmatrix}
\]

\[
= \Delta_{n-1}(x)^{-1}
\begin{vmatrix}
  \varphi_{i+n-1}(x_2) & \ldots & \varphi_{i+n-1}(x_n) \\
  \varphi_{n-2}(x_2) - \varphi_{n-2}(x_1) & \ldots & \varphi_{n-2}(x_n) - \varphi_{n-2}(x_1) \\
  \vdots & \ddots & \vdots \\
  \varphi_{n-2}(x_2) - \varphi_{n-2}(x_1) & \ldots & \varphi_{n-2}(x_n) - \varphi_{n-2}(x_1) \\
  \varphi_{n-2}(x_2) - \varphi_{n-2}(x_1) & \ldots & \varphi_{n-2}(x_n) - \varphi_{n-2}(x_1) \\
\end{vmatrix}
= h_{i+1}(x_2, \ldots, x_n).
\]

The induction step is straightforward check using the relation (7).
Now we are ready to prove the Jacobi–Trudy formula. The proof is by induction in $l = l(\lambda)$. If $l = 1$ then the formula follows from the definition of $h_i$. Suppose that $l > 1$. We will use the bracket $\{g(x_1, \ldots, x_n)\}$ to denote the result of the alternation:

$$\{g(x_1, \ldots, x_n)\} = \sum_{w \in S_n} \varepsilon(w) g(x_{w(1)}, \ldots, x_{w(n)}).$$

We claim that

$$\{h_i^{(r)}(x_1, \ldots, x_n) x_1^{n-1} x_2^{n-2} \ldots x_n^0\} = \{x_1^r \varphi_{i+n-1}(x_1) x_2^{n-2} \ldots x_n^0\}$$

for any $r \leq i + 2n - 2$. Indeed, for $r = 0$ this true by definition and the induction step follows easily from relations (3) and (7). From this we have

$$\begin{pmatrix}
    h_{\lambda_1} & h_{\lambda_1}^{(1)} & \ldots & h_{\lambda_1}^{(l-1)} \\
    h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\lambda_i-l+1} & h_{\lambda_i-l+1}^{(1)} & \ldots & h_{\lambda_i-l+1}^{(l-1)}
\end{pmatrix}
\begin{pmatrix}
    x_1^{n-1} x_2^{n-2} \ldots x_n^0
\end{pmatrix} = \begin{pmatrix}
    \varphi_{\lambda_1+n-1}(x_1) & x_1 \varphi_{\lambda_1+n-1}(x_1) & \ldots & x_1^{l-1} \varphi_{\lambda_1+n-1}(x_1) \\
    h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\lambda_i-l+1} & h_{\lambda_i-l+1}^{(1)} & \ldots & h_{\lambda_i-l+1}^{(l-1)}
\end{pmatrix}
\begin{pmatrix}
    x_2^{n-2} \ldots x_n^0
\end{pmatrix}.$$
3. GIAMBELLI FORMULA

As a corollary we have the following Giambelli formula for generalised Schur functions. Let us denote the generalised Schur polynomials corresponding to the hook Young diagrams as

\[ S_{(u|v)}(x) = S_{(u+1,1^v)}(x). \]

**Theorem 3.1.** The generalised Schur polynomials satisfy the following Giambelli formula

\[
S_\lambda(x_1, \ldots, x_n|a,b) = \left| \begin{array}{ccc}
S_{(\lambda_1-1|\lambda'_1-1)} & S_{(\lambda_1-1|\lambda'_2-2)} & \cdots & S_{(\lambda_1-1|\lambda'_r-r)} \\
S_{(\lambda_2-2|\lambda'_1-1)} & S_{(\lambda_2-2|\lambda'_2-2)} & \cdots & S_{(\lambda_2-2|\lambda'_r-r)} \\
\vdots & \vdots & \ddots & \vdots \\
S_{(\lambda_r-r|\lambda'_1-1)} & S_{(\lambda_r-r|\lambda'_2-2)} & \cdots & S_{(\lambda_r-r|\lambda'_r-r)} \end{array} \right|
\]

where \( r \) is the number of the diagonal boxes of \( \lambda \).

**Proof.** The proof follows the same line as Macdonald’s proof of the usual Giambelli formula (see [1], Ch.1, Section 3, Example 21), but we give the proof here for the reader’s convenience.

From the Theorem 1.1 we see that

\[
S_{(u|v)}(x) = \left| \begin{array}{cccc}
1 & h_{0} & h_{0}^{(1)} & \cdots & h_{0}^{(v)} \\
0 & h_{1} & h_{1}^{(1)} & \cdots & h_{1}^{(v)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & h_{u-v} & h_{u-v}^{(1)} & \cdots & h_{u-v}^{(v)} \end{array} \right|
\]

In this formula \( u \geq 0, v \geq 0 \), but we can define the functions \( S_{(u|v)}(x) \) by the same formula for all integers \( u \) and nonnegative integers \( v \). It is easy to check that this defines them correctly and that for \( u \) negative \( S_{(u|v)}(x) = 0 \) except when \( u + v = -1 \), in which case \( S_{(u|v)}(x) = (-1)^v \).

Now consider the following matrix of the size \( j \times (j + 1) \)

\[
H^{(j)} = \begin{pmatrix}
h_{0} & h_{0}^{(1)} & \cdots & h_{0}^{(j)} \\
h_{-1}^{(1)} & h_{-1}^{(1)} & \cdots & h_{-1}^{(j)} \\
\vdots & \vdots & \ddots & \vdots \\
h_{-j} & h_{-j}^{(1)} & \cdots & h_{-j}^{(j)}
\end{pmatrix}
\]

and denote by \( \Delta_i^{(j)} \), \( 1 \leq i \leq j + 1 \) the determinant of its sub-matrix without the \( i \)-th column multiplied by \(( -1 )^{i-1}\). If \( i > j + 1 \) we set by definition \( \Delta_i^{(j)} = 0 \). One can check also that \( \Delta_i^{(k-1)} = (-1)^{k-1} \).

1We are very grateful to G. Olshanski, who pointed out this to us.
For any partition \( \lambda \) consider the matrices
\[
A = \begin{pmatrix}
h_{\lambda_1} & h_{\lambda_1}^{(1)} & \ldots & h_{\lambda_1}^{(l-1)} \\
h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\
\vdots & \vdots & \ddots & \vdots \\
h_{\lambda_t-t+1} & h_{\lambda_t-t+1}^{(1)} & \ldots & h_{\lambda_t-t+1}^{(l-1)}
\end{pmatrix}, \quad B = \begin{pmatrix}
\Delta_1^{(l-1)} & \Delta_2^{(l-2)} & \ldots & \Delta_1^{(0)} \\
\Delta_2^{(l-1)} & \Delta_2^{(l-2)} & \ldots & \Delta_2^{(0)} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_t^{(l-1)} & \Delta_t^{(l-2)} & \ldots & \Delta_t^{(0)}
\end{pmatrix}.
\]
Note that \( B \) is upper-triangular with respect to the anti-diagonal with the anti-diagonal elements \((-1)^{k-1}\), so the determinant of \( B \) is identically equal to 1, while the determinant of \( A \) by Theorem 1.1 coincides with \( S_{\lambda}(x_1, \ldots, x_n|a, b) \).

From linear algebra and definition of \( S_{(a|v)} \) we have
\[
AB = \begin{pmatrix}
S_{(\lambda_1-1|l-1)} & S_{(\lambda_1-1|l-2)} & \ldots & S_{(\lambda_1-1|0)} \\
S_{(\lambda_2-2|l-1)} & S_{(\lambda_2-2|l-2)} & \ldots & S_{(\lambda_2-2|0)} \\
\vdots & \vdots & \ddots & \vdots \\
S_{(\lambda_t-t|l-1)} & S_{(\lambda_t-t|l-2)} & \ldots & S_{(\lambda_t-t|0)}
\end{pmatrix}.
\]
Taking the determinants of both sides we see that \( S_{\lambda}(x_1, \ldots, x_n|a, b) \) equals to the determinant of the last matrix. In this matrix there are many zeros since for \( k > r \) we have \( \lambda_k - k < 0 \) and therefore \( S_{(\lambda_k-k|l-j)} = (-1)^{l-j} \) if \( k = \lambda_k - k + l - j = -1 \) and 0 otherwise. This means that in the \( k \)-th row with \( k > r \) there is only one non-zero element \((-1)^{l-j}\) with \( l - j = k - \lambda_k - 1 \). This reduces the calculation of the determinant to the \( r \times r \) matrix with the remaining columns having the numbers \( \lambda_j' - j, \quad j = 1, \ldots, r \). Indeed, for any \( \lambda \) of length \( l \) with \( r \) boxes on the diagonal the union of two sets \( \{k-\lambda_k-1\}, \quad k = r+1, \ldots, l \) and \( \{\lambda_j' - j\}, \quad j = 1, \ldots, r \) is the set \( \{0, 1, 2, \ldots, l-1\} \) as it follows, for example, from the identity
\[
\sum_{i=1}^{l} t^i (1 - t^{-\lambda_i}) = \sum_{j=1}^{r} (t^\lambda_j - j + 1) - t^{\lambda_j} \]
(see [1], Ch.1, Section 1, Example 4). The check of the sign completes the proof.

4. Particular cases

As a corollary we have the following well-known cases of the Jacobi–Trudy formula.

1. When \( a(i) = b(i) = 0 \) for all \( i \geq 0 \) we have \( \varphi_i(z) = z^i \) and (9) clearly coincides with the usual Jacobi–Trudy formula for Schur polynomials.

2. The characters of the orthogonal Lie algebra \( so(2n + 1) \) correspond to the case when \( a(i) = 0, b(i) = 1 \) for \( i > 0 \) and \( a(0) = -1, b(0) = 0 \) and the polynomials \( \varphi_i(z) = x^i + x^{i-1} + \cdots + x^{-1}, \quad z = x + x^{-1} \). Using the recurrence relation (7), having in this case the form
\[
h_i^{(r+1)} = h_{i+1}^{(r)} + h_{i-1}^{(r)}
\]
we can rewrite the general Jacobi-Trudy formula (9) in the form known in representation theory (see Prop. 24.33 in Fulton-Harris [3]): the character $\chi_\lambda$ is the determinant of the $l \times l$ matrix whose $i$-th row is

$$(h_{\lambda_{i-1}+1} - h_{\lambda_i-2} + h_{\lambda_i-1} + \cdots - h_{\lambda_i-1} + h_{\lambda_i+1}).$$

The same is true for the characters of the even orthogonal Lie algebra $\mathfrak{so}(2n)$, where $a(i) = 0$ for all $i \geq 0$ and $b(i) = 2$ and $\varphi_i(z) = x^i + x^{-i}$, $z = x + x^{-1}$ (see Prop. 24.44 in [3]) and for the symplectic Lie algebra $\mathfrak{sp}(2n)$, when $a(i) = 0$, $b(i) = 1$ for all $i \geq 0$ and $\varphi_i(z) = x^i + x^{-i} + \cdots + x^{-i}$, $z = x + x^{-1}$ (Prop. 24.22 in [3]). Note that the change of $a(0)$ and $b(1)$ does not affect the definition of the relevant $h_i^{(r)}$ for $r > 0$.

3. The factorial Schur polynomials [6] correspond to the special case when $b_i = 0$, so that

$$\varphi_i(z) = (z - a(0))(z - a(1)) \cdots (z - a(i - 1)), \quad i > 0.$$ 

The Jacobi–Trudy formula for them can be found in [1], Ch.1, Section 3, Example 20.

4. For $a(i)$, $b(i)$ given by

$$a(x) = -\frac{2p(p + 2q + 1)}{(2x - p - 2q - 1)(2x - p - 2q + 1)},$$

$$b(x) = \frac{2x(2x - 2q - 1)(2x - 2p - 2q - 1)(2x - 2p - 4q - 2)}{(2x - p - 2q)(2x - p - 2q - 1)^2(2x - p - 2q - 2)}$$

we have the Jacobi-Trudy formula for the multidimensional Jacobi polynomials with $k = -1$, which seems to be new.

5. INFINITE-DIMENSIONAL AND SUPER VERSIONS

Let us assume now that the coefficients $a(i)$ and $b(i)$ of the recurrence relation are rational functions of $i$. In that case we can define the generalised Schur functions (which are the infinite-dimensional version of $S_\lambda(x|a,b)$) in the following way (cf. Okounkov-Olshanski [5]).

First note that the generalised Schur polynomials (4) are the linear combination of the usual Schur polynomials

$$S_\lambda(x_1, \ldots, x_n|a,b) = \sum_{\mu \subseteq \lambda} c_{\lambda,\mu}(n|a,b)S_\mu(x_1, \ldots, x_n),$$

where $c_{\lambda,\mu}(n|a,b)$ are some rational functions of $n$. The generalised Schur functions depend on the additional parameter $d$ and defined by

$$S_\lambda(x|a,b;d) = \sum_{\mu \subseteq \lambda} c_{\lambda,\mu}(d|a,b)S_\mu(x),$$ (14)
where $S_\mu(x)$ are the usual Schur functions [1]. They satisfy the following infinite dimensional version of the Jacobi–Trudy formula:

$$S_\lambda(x|a,b;d) = \left| \begin{array}{cccc} h_{\lambda_1} & h^{(1)}_{\lambda_1} & \cdots & h^{(l-1)}_{\lambda_1} \\ h_{\lambda_2-1} & h^{(1)}_{\lambda_2-1} & \cdots & h^{(l-1)}_{\lambda_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h^{(1)}_{\lambda_l-l+1} & \cdots & h^{(l-1)}_{\lambda_l-l+1} \end{array} \right|. \quad (15)$$

Here $l = l(\lambda)$, $h_i = S_\lambda(x|a,b;d)$ with $\lambda = (i)$ if $i \geq 0$ and $h_i \equiv 0$ if $i < 0$, $h^{(r)}_i = h^{(r)}_i(x|a,b;d)$ are defined for generic $d$ by the recurrence relation

$$h^{(r+1)}_i = h^{(r)}_{i+1} + a(i + d - 1)h^{(r)}_i + b(i + d - 1)h^{(r)}_{i-1} \quad (16)$$

with initial data $h^{(0)}_i = h_i$.

The generalised super Schur polynomials $S_\mu(x_1,\ldots,x_n;y_1,\ldots,y_m|a,b)$ can be defined by the same formula (14), where the Schur functions should be replaced by the super Schur polynomials $S_\mu(x_1,\ldots,x_n;y_1,\ldots,y_m)$ (see e.g. [1]) and $d$ must be specialised as the superdimension $d = n - m$ (provided the coefficients have no poles at $d = n - m$). Alternatively, $S_\mu(x_1,\ldots,x_n;y_1,\ldots,y_m|a,b)$ is the image of the corresponding generalised Schur function $S_\mu(x|a,b,d)$ under the homomorphism $\phi$ sending the power sums $p_k \in \Lambda$ to the super power sums $x_1^k + \cdots + x_n^k - y_1^k - \cdots - y_m^k$ with $d = n - m$. In the case of factorial Schur polynomials their super version had been introduced in a different way by Molev [8].

An important example corresponds to the sequences (12),(13). In this case the generalised Schur functions coincide with the Jacobi symmetric functions with parameter $k = -1$ (see [9]). These functions and their super versions play an important role in representation theory of the orthosymplectic Lie superalgebras [10].

Finally we would like to mention that the Jacobi-Trudy formula (9) can be rewritten in a dual form in terms of the conjugate partition in the spirit of Macdonald [1] (Ch.1, Section 3, Example 21) and Okounkov-Olshanski [7], Section 13.

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References

[1] I. Macdonald Symmetric functions and Hall polynomials. 2nd edition, Oxford Univ. Press, 1995.
[2] G. Szego Orthogonal polynomials. Fourth edition. AMS Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.
[3] W. Fulton, J. Harris, Representation theory. A first course. Graduate Texts in Mathematics 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
[4] M. Lassalle, *Polynomes de Jacobi généralisés*, C.R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 6, 425–428.

[5] A. Okounkov, G. Olshanski, *Limits of BC-type orthogonal polynomials as the number of variables goes to infinity*, Jack, Hall-Littlewood and Macdonald polynomials, 281–318, Contemp. Math., 417, Amer. Math. Soc., Providence, RI, 2006.

[6] L. C. Biedenharn and J. D. Louck, *A new class of symmetric polynomials defined in terms of tableaux*, Advances in Appl. Math. 10 (1989), 396–438.

[7] A. Okounkov, G. Olshanski, *Shifted Schur functions*. St. Petersburg Math. J. 9 (1998), no. 2, 239–300.

[8] A. Molev, *Factorial supersymmetric Schur functions and super Capelli identities*. AMS Translations (2), Vol. 181, AMS, Providence, R.I., 1997, 109–137.

[9] A.N. Sergeev, A.P. Veselov, *BC∞ Calogero-Moser operator and super Jacobi polynomials*. arXive: 0807.3858.

[10] A.N. Sergeev, A.P. Veselov, *Euler characters and super Jacobi polynomials*. In preparation.

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