Two-grid algorithm for a system of singularly perturbed reaction-diffusion equations on Shishkin mesh

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Abstract. A system of two coupled singularly perturbed reaction-diffusion equations is considered. A cascadic two-grid algorithm of high accuracy based on a uniform with respect to small parameters convergent difference scheme for such problem has been developed. To increase the accuracy of the difference solution by an order, the Richardson extrapolation method was applied. The results of some numerical experiments are discussed.

1. Introduction

A system of two coupled singularly perturbed ordinary differential equations is considered. These systems of equations have applications in electroanalytical chemistry when investigating diffusion processes complicated by chemical reactions, in several models of some various physical phenomena, such as the turbulent interaction of waves and currents and in other applications, see [1–6] and the references therein.

It is well known that the use of the classical difference schemes for a singularly perturbed problem leads to large errors for small values of perturbation parameters. There are special (a uniform with respect to small parameters convergent) numerical methods for singularly perturbed differential equations and for systems of such equations [1–12]. The construction of numerical methods is much more complicated in the case of systems of singularly perturbed equations with several perturbation parameters, because in such problems, multiple (double in the case of two equations) boundary layers appear that have different characteristic scales [3–6]. Difference schemes of higher accuracy for singular perturbation boundary value problems are very important, see [6, 13–17] and the references therein. In [6] a hybrid finite difference scheme of HODIE type with the third order (except for a logarithmic factor) uniform convergence is constructed for a system of two coupled singularly perturbed reaction-diffusion problems on a piecewise uniform Shishkin mesh for the case in which the diffusion parameter associated with each equation of the system has a different order of magnitude. The computational cost of the HODIE method is higher than the central difference scheme with the second order (except for a logarithmic factor) uniform convergence, but the errors associated with central differences are considerably greater. Thus the HODIE method is more efficient in practice than the classical central difference scheme.
The two-grid method was investigated in [17–27] and in other works. According to the conception of the two-grid algorithm, if the difference scheme is resolved based on iterations then at first the problem is solved on a coarse mesh. Secondly, the new mesh solution is interpolated to nodes of the fine mesh and is used as the initial guess for following iterations. It leads to reduction of the number of iterations on the fine mesh and also reduces the number of arithmetical operations. To increase \( \varepsilon \)-uniform accuracy of the difference scheme on Shishkin mesh without additional calculations, we use Richardson extrapolation formula \([14, 28, 29]\) in two-grid method. Such an approach for coupled systems is original.

In [1, 4] a second order numerical method for a system of two coupled singularly perturbed ordinary differential equations based on the Shishkin mesh converges uniformly with respect to both singular perturbation parameters.

The aim of this work is to investigate a two-grid method using Richardson extrapolation for a system of two coupled singularly perturbed ordinary differential equations based on the difference scheme of a second order of accuracy on the Shishkin mesh.

**Notation:** Let \( \|f\| = \max_{x \in \Omega} |f(x)| \) be the norm of a continuous argument function and \( \|\mathbf{f}\| = \max_{k=1,2} \|f\| \). Analogously \( \|f^N\|_N = \max_{0 \leq i \leq N} |f^N_i| \) and \( \|\mathbf{f}^N\|_N = \max_{k=1,2} \|f^N_k\|_N \) be the discrete norm of the mesh function. Let \( \mathbf{f}|_\Omega \) be the projection of a function \( f(x) \) on a mesh \( \Omega \). For any function \( f \in C[0, 1] \) we set \( f_i = f(x_i) \) and if \( \mathbf{f} \in C[0, 1]^2 \) then \( \mathbf{f}|_\Omega = (f_1, f_2)^T \). Here \( C \), sometimes subscripted, denotes a generic positive constant that is independent of the perturbation parameters \( \varepsilon, \mu \) and the number of the mesh intervals.

### 2. Preliminaries

Consider the boundary value problem:

\[
L\mathbf{u}(x) = \begin{cases} 
-\varepsilon u''_1(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \\
-\mu u''_2(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x),
\end{cases}
\]

\( u_1(0) = A_1, \quad u_1(1) = B_1, \quad u_2(0) = A_2, \quad u_2(1) = B_2, \tag{1} \)

where \( x \in \Omega = (0, 1) \), the functions \( a_{ij} \) and \( f_i, i, j = 1, 2 \) are sufficiently smooth and

\[
0 < \varepsilon \leq 1, \quad 0 < \mu \leq 1, \quad \min_{x \in [0,1]} \{a_{11}(x) + a_{12}(x), \quad a_{11}(x) + a_{12}(x)\} > \alpha > 0, \tag{2}
\]

and a continuous matrix \( \mathbf{A} = \{a_{ij}\}_{i,j=1}^2 \) that is an \( L_0 \)-matrix, i.e., it has positive diagonal entries and non-positive off-diagonal entries.

The solution \( u(x) \) has two boundary layers near \( x = 0 \) and \( x = 1 \).

If conditions (2) are satisfied, the solution problem (4) is uniformly bounded with respect to \( \varepsilon \) and \( \mu \):

\[
\|\mathbf{u}\| \leq C_0 = \alpha^{-1}\|\mathbf{f}\| + \|\mathbf{u}(0)\| + \|\mathbf{u}(1)\|.
\]

According to [1, 3, 4, 6], for the decomposition of the solution (4) with the regular component \( v \) and the layer component \( w \)

\[
\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad L\mathbf{v} = \mathbf{f} \text{ on } \Omega, \quad \mathbf{v} = A^{-1}\mathbf{f} \text{ on } \partial\Omega, \\
L\mathbf{w} = \mathbf{0} \text{ on } \Omega, \quad \mathbf{w} = \mathbf{u} - \mathbf{v} \text{ on } \partial\Omega,
\]
we have the following estimates for $0 \leq x \leq 1$:

\[
\begin{align*}
|\overline{v}_i^{(k)}| & \leq C, \quad k = 0, 1, 2, \\
|\overline{v}_0^{(3)}| & \leq C \varepsilon^{-1/2}, \quad |\overline{v}_2^{(3)}| \leq C \mu^{-1/2}, \\
|\overline{w}_1(x)| & \leq C \beta(x), \quad |\overline{w}_2(x)| \leq C \beta(x), \\
|\overline{w}_1^{(k)}(x)| & \leq C \left( \varepsilon^{-k/2} B_\varepsilon(x) + \mu^{-k/2} B_\mu(x) \right), \quad k = 1, 2, \\
|\overline{w}_2^{(k)}(x)| & \leq C \mu^{-k/2} B_\mu(x), \quad k = 1, 2, \\
|\overline{w}_1^{(k)}(x)| & \leq C \left( \varepsilon^{-k/2} B_\varepsilon(x) + \mu^{-k/2} B_\mu(x) \right), \quad k = 3, 4, 5, 6, \\
|\overline{w}_2^{(k)}(x)| & \leq C \mu^{-1} \left( \varepsilon^{-(2-k)/2} B_\varepsilon(x) + \mu^{-(2-k)/2} B_\mu(x) \right), \quad k = 3, 4, 5, 6,
\end{align*}
\]

where $B_\varepsilon(x) = e^{-x/\sqrt{\alpha/\varepsilon}} + e^{-(1-x)/\sqrt{\alpha/\varepsilon}}$, $B_\mu(x) = e^{-x/\sqrt{\alpha/\mu}} + e^{-(1-x)/\sqrt{\alpha/\mu}}$.

According to [4, 6, 10], we specify a mesh:

\[
\Omega_N = \{ x_i : x_i = x_{i-1} + h_i, \quad x_0 = 0, \quad x_N = 1, \quad i = 1, 2, \ldots, N \},
\]

where

\[
\begin{align*}
&h_i = \frac{8\varepsilon}{N}, \quad 1 \leq i \leq \frac{7N}{8}, \quad \frac{7N}{8} \leq i \leq N; \\
&h_i = \frac{8(\sigma_{\mu} - \sigma_\varepsilon) N}{N}, \quad \frac{3N}{4} \leq i \leq \frac{7N}{8};
\end{align*}
\]

\[
\sigma_\mu = \min \left\{ \frac{1}{4}, 2\sqrt{\frac{\mu}{\alpha} \ln N} \right\}, \quad \sigma_\varepsilon = \min \left\{ \frac{1}{8}, 2\sqrt{\frac{\varepsilon}{\alpha} \ln N} \right\},
\]

note that without loss of generality we assume $\varepsilon < \mu$.

Consider the finite difference scheme [1, 3, 4] on the Shishkin mesh (3):

\[
L_{\varepsilon, \mu}^N \overline{U}_j^N = \begin{bmatrix} -\varepsilon \delta^2 & 0 \\ 0 & -\mu \delta^2 \end{bmatrix} \overline{U}_j^N + A(x_j) \overline{U}_j^N = \overline{f}(x_j),
\]

where $h_j = (h_j + h_{j+1})/2$ and

\[
\delta^2 u_j^N = \frac{1}{h_j} \left( \frac{u_{j+1}^N - u_j^N}{h_{j+1}} - \frac{u_j^N - u_{j-1}^N}{h_j} \right).
\]

The solution of the difference scheme (4) can be resolved based on iterations [1, 12]

\[
\begin{align*}
L_{1, j}^{N} U_{1, j}^{m+1,N} &= -\varepsilon \delta^2 U_{1, j}^{m+1,N} + a_{11, j} U_{1, j}^{m+1,N} = f_{1, j}^{N} - a_{12, j} U_{2, j}^{m,N}, \\
L_{2, j}^{N} U_{2, j}^{m+1,N} &= -\varepsilon \delta^2 U_{2, j}^{m+1,N} + a_{22, j} U_{2, j}^{m+1,N} = f_{2, j}^{N} - a_{21, j} U_{1, j}^{m+1,N}.
\end{align*}
\]

According to [4], the following theorem is valid.

**Theorem 1** Let $\overline{u}(x)$ be the solution to linear problem (4) with sufficiently smooth coefficients and right-hand side, and let $\overline{U}_N$ be the solution to the difference scheme (4). Then on the Shishkin mesh (3) for a constant $C$ we have:

\[
||\overline{u}\rvert_N - \overline{U}_N||_N \leq C \Delta N = C \ln^2 N/N^2,
\]

uniformly with respect to $\varepsilon$ and $\mu$. 

3. Two-grid method using Richardson extrapolation

To decrease the required number of arithmetical operations for resolving the difference scheme, a two-grid method is proposed. According to the idea of the two-grid algorithm [20], at first, the initial problem (4) is solved on a coarse mesh. Secondly, the found mesh solution is interpolated to the nodes of the fine mesh and is used as the initial guess for following iterations. It leads to reduction of the number of iterations on the fine mesh and also reduces the number of arithmetical operations. Note that the interpolation formula must be uniform with respect to the singular perturbation parameters \( \varepsilon \) and \( \mu \) else the accuracy of the found mesh solution may be lost [21–26, 30–32].

Thus, let \( \Omega_n \) be the Shishkin mesh corresponding to (3) and containing \( n \) mesh intervals, where we take \( n \ll N \). At first, problem (4) is preliminarily solved on the mesh \( \Omega_n \). The iterations on the mesh \( \Omega_n \) are performed until the inequality

\[
\| \tilde{U}_{m,n} - \tilde{U}^n \|_n \leq \Delta_n.
\]

is satisfied. Then the mesh solution \( \tilde{U}_{m,n} \) found on the mesh \( \Omega_n \) is interpolated to the nodes of the initial mesh \( \Omega_N \) using an appropriate interpolation which is uniform with respect to the parameters \( \varepsilon \) and \( \mu \) and for some constant \( C \)

\[
\| \text{Int}(\tilde{u}_{\Omega_n}, x) - \tilde{u}(x) \| \leq C \Delta_n.
\]

According to [1, 4], on the Shishkin mesh the accuracy of the linear interpolation formula is uniform with respect to the parameter \( \varepsilon \) for some constant \( C \):

\[
\| \text{Int}(\tilde{u}_{\Omega_n}, x) - \tilde{u}(x) \| \leq C \frac{\ln^2 n}{n^2}.
\]

Therefore, we have for some constant \( C \)

\[
\| \tilde{u}_{\Omega_n} - \tilde{U}_{m,n} \|_n \leq C \Delta_n.
\]

Now we specify an initial guess for the iterations on the initial mesh \( \Omega_N \) and we have for some constant \( C \)

\[
\| \tilde{U}^{0,N} - \tilde{u} \|_N \leq C \Delta_n, \quad \tilde{U}^{0,N} = [\text{Int}(\tilde{U}_{m,n}^n, x)]_{\Omega_N}.
\]

Thus, using iterations on the coarse mesh and an appropriate interpolation, an initial guess \( \tilde{U}^{0,N} \) for iterations on the mesh \( \Omega_N \) with an accuracy of \( O(\Delta_n) \) is constructed. It remains to perform iterations on the mesh \( \Omega_N \) until an accuracy of \( O(\Delta_N) \) is achieved.

To increase the accuracy of the difference scheme (4) to the third order (except for a logarithmic factor), the Richardson extrapolation [14, 21–23, 28, 29] is applied in two-grid method. For this reason the mesh \( \Omega_n \) must have the same value of the parameters \( \sigma_1 \) and \( \sigma_2 \) as the mesh \( \Omega_N \). Thus these meshes are nested that is

\[
\Omega_n = \{ X_j \} \subset \Omega_N = \{ x_i \}.
\]

Let us \( N = kn \), where \( k \) is some integer. Then obviously that the initial mesh \( \Omega_N \) can be obtained from the coarse mesh \( \Omega_n \) by dividing each interval into \( k \) equal parts.

Let \( \tilde{U}^n \) be the solution of the difference scheme on the mesh \( \Omega_n \). According to Richardson method we introduce \( \tilde{U}^{Nn} \) on the mesh \( \Omega_N \). At first, let us define \( \tilde{U}^{Nn} \) on the mesh \( \Omega_n \) as

\[
\tilde{U}^{Nn}(X_j) = k_n \tilde{U}^n(X_j) + k_N \tilde{U}^N(X_j), \quad X_j \in \Omega_n.
\]

where

\[
k_n = -n^2/(N^2 - n^2) = -1/(k^2 - 1), \quad k_N = N^2/(N^2 - n^2) = k^2/(k^2 - 1).
\]

At the nodes of the initial mesh don’t coinciding with nodes of the coarse mesh, we define \( \tilde{U}^{Nn}(x_i) \) for any \( x_i \in \Omega_N \) by using the linear interpolation.

So, we constructed the mesh solution \( \tilde{U}^{Nn} \) on the mesh \( \Omega_N \) using Richardson extrapolation.

\[
\| \tilde{U}^{Nn} - \tilde{U}^{0,N} \|_{\Omega_N} \leq C \Delta_N.
\]
4. Results of numerical experiments

Consider the following boundary value problem:
\[
\begin{align*}
-\varepsilon u''_1(x) + u_1(x) - 0.5 u_2(x) &= f_1(x), \\
-\mu u''_2(x) - 2 u_1(x) + 4 u_2(x) &= f_2(x), \\
u_1(0) &= 3, \quad u_1(1) = 3, \quad u_2(0) = 0, \quad u_2(1) = 0,
\end{align*}
\]

where \(0 < x < 1\) and \(\bar{f}(x)\) corresponds to the exact solution:
\[
\begin{align*}
u_1(x) &= e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}} - x + x^2 + \cos^2(\pi x), \\
u_2(x) &= e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}} - 1/\sqrt{\varepsilon} - x + x^2 + \sin(\pi x).
\end{align*}
\]

Note that \(\|u_1\| \leq 3, \|u_2\| \leq 3\).

Taking into account the error of the scheme (4) we stop iterations on the mesh \(\Omega_N\) when the following condition is satisfied
\[
\|L^N \bar{U}^{m,N} - \bar{f}^N\|_N \leq \alpha \Delta_N,
\]

where \(\alpha\) corresponds to (2). Then from the estimate
\[
\|U^{m,N} - \bar{U}^N\|_N \leq \alpha^{-1} \|L^N U^{m,N} - \bar{f}^N\|_N
\]
follows:
\[
\|U^{m,N} - \bar{U}^N\|_N \leq \Delta_N.
\]

We take \(U^0,N_2 = (0, 0, \ldots, 0)^T\).

Table 1 contains the number of iterations for a two-grid method for \(\mu = 1\) on the mesh \(\Omega_N\) in the case of \(n = N/2\) for various values of \(N\) and \(\varepsilon\) in upper line. The number of iterations on the mesh \(\Omega_n\) is given in brackets. The number of iterations for a one-grid method depending on \(N\) is given in the bottom line of the table.

**Table 1.** The number of iterations for one-grid and two-grid methods for \(\mu = 1\)

| \(\varepsilon\) | 64     | 128    | 256    | 512    | 1024   | 2048   | 4096   | 8192   |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1               | 1(2)   | 1(2)   | 1(3)   | 1(3)   | 1(3)   | 1(4)   | 1(4)   |        |
|                 | 2      | 3      | 3      | 3      | 4      | 4      | 4      |        |
| 10^{-2}         | 1(3)   | 1(3)   | 1(4)   | 1(4)   | 1(5)   | 1(5)   | 1(5)   | 1(6)   |
|                 | 3      | 4      | 5      | 5      | 6      | 6      |        |        |
| 10^{-3}         | 2(3)   | 2(3)   | 2(4)   | 2(4)   | 2(5)   | 2(5)   | 2(6)   | 2(6)   |
|                 | 3      | 4      | 5      | 5      | 6      | 6      |        |        |
| 10^{-5}         | 2(3)   | 3(3)   | 3(4)   | 3(4)   | 3(5)   | 3(5)   | 3(6)   | 3(6)   |
|                 | 3      | 4      | 4      | 5      | 6      | 6      | 6      |        |
| 10^{-7}         | 2(3)   | 3(3)   | 3(4)   | 3(4)   | 3(5)   | 3(5)   | 3(6)   | 3(6)   |
|                 | 3      | 4      | 4      | 5      | 6      | 6      | 6      |        |

Table 2 contains the number of iterations for a two-grid method for \(\mu = 10^{-3}\) on the mesh \(\Omega_N\) in the case of \(n = N/2\) for various values of \(N\) and \(\varepsilon\) in upper line. The number of iterations on the mesh \(\Omega_n\) is given in brackets. The number of iterations for a one-grid method depending on \(N\) is given in the bottom line of the table.
Table 2. The number of iterations for one-grid and two-grid methods for $\mu = 10^{-3}$

| $\varepsilon$ | $N$       | 64   | 128  | 256  | 512  | 1024 | 2048 | 4096 | 8192 |
|---------------|-----------|------|------|------|------|------|------|------|------|
| $10^{-1}$     | 2(2)      | 2(3) | 2(3) | 2(3) | 2(4) | 2(4) | 2(4) | 2(4) | 2(4) |
| $10^{-2}$     | 2(4)      | 2(4) | 2(5) | 2(6) | 2(7) | 2(8) | 2(8) | 2(8) | 2(9) |
| $10^{-3}$     | 2(5)      | 2(5) | 2(6) | 2(7) | 2(8) | 2(9) | 2(9) | 2(9) | 2(10)|

Table 3 contains the number of iterations for a two-grid method for $\mu = 10^{-5}$ on the mesh $\Omega_N$ in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ in upper line. The number of iterations on the mesh $\Omega_n$ is given in brackets. The number of iterations for a one-grid method depending on $N$ is given in the bottom line of the table.

Table 3. The number of iterations for one-grid and two-grid methods for $\mu = 10^{-5}$

| $\varepsilon$ | $N$       | 64   | 128  | 256  | 512  | 1024 | 2048 | 4096 | 8192 |
|---------------|-----------|------|------|------|------|------|------|------|------|
| $10^{-1}$     | 2(2)      | 2(3) | 2(3) | 2(3) | 2(4) | 2(4) | 2(4) | 2(4) | 2(4) |
| $10^{-2}$     | 2(5)      | 2(5) | 2(6) | 2(7) | 2(8) | 2(8) | 2(8) | 2(8) | 2(9) |
| $10^{-3}$     | 2(6)      | 2(7) | 2(8) | 2(9) | 2(9) | 2(9) | 2(9) | 2(9) | 2(10)|

Table 4 contains $T_{TGM}/T_{OGM}$, where $T_{TGM}$ is the implementation time for a two-grid method in the case of $n = N/2$ and $T_{OGM}$ is the implementation time for a one-grid method for $\mu = 10^{-3}$ (left table) and for $\mu = 10^{-5}$ (right table) on the mesh $\Omega_N$ for various values of $N$ and $\varepsilon$.

Table 4. The implementation time for $\mu = 10^{-3}$ (left) and for $\mu = 10^{-5}$ (right)

| $\varepsilon$ | $N$       | 256  | 1024 | 4096 | 8192 | 32768|
|---------------|-----------|------|------|------|------|------|
| $10^{-1}$     | 0.806     | 0.733| 0.708| 0.691| 0.667|
| $10^{-2}$     | 0.744     | 0.675| 0.687| 0.648| 0.613|
| $10^{-3}$     | 0.723     | 0.667| 0.674| 0.703| 0.622|
| $10^{-5}$     | 0.755     | 0.828| 0.746| 0.726| 0.705|
| $10^{-7}$     | 0.701     | 0.676| 0.645| 0.628| 0.623|

| $\varepsilon$ | $N$       | 256  | 1024 | 4096 | 8192 | 32768|
|---------------|-----------|------|------|------|------|------|
| $10^{-1}$     | 0.804     | 0.758| 0.707| 0.728| 1.275|
| $10^{-2}$     | 0.743     | 0.668| 0.690| 0.654| 0.628|
| $10^{-3}$     | 0.762     | 0.735| 0.652| 0.704| 0.619|
| $10^{-5}$     | 0.809     | 0.694| 0.652| 0.627| 0.663|
| $10^{-7}$     | 0.815     | 0.782| 0.757| 0.728| 0.752|
So, the main part of iterations in a two-grid method performed on the coarse mesh. It leads to essential decreasing of the implementation time.

Table 5 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ for $\mu = 1$.

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $2.41e-4$ | $3.91e-5$ | $2.47e-6$ | $1.39e-7$ |
| $10^{-1}$     |     | $6.97e-4$ | $2.17e-5$ | $3.59e-6$ | $1.27e-7$ |
| $10^{-2}$     |     | $7.80e-3$ | $5.11e-4$ | $3.34e-5$ | $2.18e-6$ |
| $10^{-4}$     |     | $1.52e-1$ | $1.18e-2$ | $7.46e-4$ | $4.67e-5$ |
| $10^{-6}$     |     | $1.60e-1$ | $1.96e-2$ | $1.93e-3$ | $1.74e-4$ |
| $10^{-7}$     |     | $1.59e-1$ | $1.97e-2$ | $1.95e-3$ | $1.75e-4$ |

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $2.92e-4$ | $4.10e-6$ | $1.84e-6$ | $2.06e-8$ |
| $10^{-1}$     |     | $2.63e-4$ | $2.90e-4$ | $1.05e-5$ | $3.77e-7$ |
| $10^{-2}$     |     | $1.74e-3$ | $1.07e-4$ | $6.54e-6$ | $6.81e-6$ |
| $10^{-4}$     |     | $4.95e-2$ | $1.54e-3$ | $8.98e-5$ | $5.44e-6$ |
| $10^{-6}$     |     | $5.18e-2$ | $9.07e-4$ | $9.06e-6$ | $5.89e-7$ |
| $10^{-7}$     |     | $5.10e-2$ | $9.07e-4$ | $9.08e-6$ | $5.89e-7$ |

Table 6 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ for $\mu = 10^{-3}$.

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $7.50e-2$ | $4.92e-3$ | $3.08e-4$ | $1.95e-5$ |
| $10^{-1}$     |     | $6.57e-2$ | $4.33e-3$ | $2.73e-4$ | $1.72e-5$ |
| $10^{-2}$     |     | $5.06e-2$ | $3.32e-3$ | $2.07e-4$ | $1.37e-5$ |
| $10^{-4}$     |     | $4.86e-2$ | $4.06e-3$ | $2.55e-4$ | $1.58e-5$ |
| $10^{-6}$     |     | $1.46e-1$ | $1.67e-2$ | $1.60e-3$ | $1.42e-4$ |
| $10^{-7}$     |     | $1.79e-1$ | $2.00e-2$ | $1.86e-3$ | $1.65e-4$ |

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $1.27e-2$ | $5.91e-5$ | $6.83e-7$ | $2.59e-7$ |
| $10^{-1}$     |     | $1.12e-2$ | $9.21e-5$ | $2.88e-5$ | $4.50e-6$ |
| $10^{-2}$     |     | $8.49e-3$ | $7.55e-4$ | $3.88e-5$ | $9.74e-6$ |
| $10^{-4}$     |     | $1.17e-2$ | $1.24e-3$ | $7.60e-5$ | $4.66e-6$ |
| $10^{-6}$     |     | $7.29e-2$ | $1.60e-3$ | $9.71e-5$ | $5.95e-6$ |
| $10^{-7}$     |     | $7.48e-2$ | $1.38e-3$ | $1.06e-4$ | $1.29e-5$ |

Table 7 contains the error norm for a one-grid method (left table) and for a two-grid method with Richardson extrapolation (right table) in the case of $n = N/2$ for various values of $N$ and $\varepsilon$ for $\mu = 10^{-5}$.

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $1.62e-1$ | $1.97e-2$ | $1.94e-3$ | $1.75e-4$ |
| $10^{-1}$     |     | $1.58e-1$ | $1.93e-2$ | $1.91e-3$ | $1.72e-4$ |
| $10^{-2}$     |     | $1.51e-1$ | $1.84e-2$ | $1.81e-3$ | $1.64e-4$ |
| $10^{-4}$     |     | $1.01e-1$ | $1.27e-2$ | $1.25e-3$ | $1.12e-4$ |
| $10^{-6}$     |     | $4.87e-2$ | $6.21e-3$ | $6.09e-4$ | $5.44e-5$ |
| $10^{-7}$     |     | $9.48e-2$ | $1.18e-2$ | $1.16e-3$ | $1.04e-4$ |

| $\varepsilon$ | $N$ | 64 | 256 | 1024 | 4096 |
|---------------|-----|----|-----|------|------|
| $1$           |     | $5.25e-2$ | $9.15e-4$ | $1.28e-5$ | $1.95e-7$ |
| $10^{-1}$     |     | $5.15e-2$ | $9.41e-4$ | $4.82e-5$ | $2.49e-6$ |
| $10^{-2}$     |     | $4.89e-2$ | $8.83e-4$ | $4.51e-5$ | $7.13e-6$ |
| $10^{-4}$     |     | $3.29e-2$ | $1.31e-3$ | $8.09e-5$ | $5.01e-6$ |
| $10^{-6}$     |     | $1.15e-2$ | $1.11e-3$ | $8.94e-5$ | $7.66e-6$ |
| $10^{-7}$     |     | $3.48e-2$ | $2.07e-3$ | $1.12e-4$ | $6.98e-6$ |

It follows from Tables 5-7 that the application Richardson extrapolation in a two-grid method increases the accuracy of the difference scheme to $O(\ln^3 N/N^3)$ uniformly with respect to both singular perturbation parameters.
Finally, we note that the constructed two-grid method with Richardson extrapolation for a system of two coupled singularly perturbed ordinary differential equations on the Shishkin mesh has the computational cost lower than the central difference scheme, and the accuracy is considerably greater. Thus such algorithm is more efficient in practice thanks to the simple structure.

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References
[1] Matthews S, O’Riordan E and Shishkin G 2002 Journal of Computational and Applied Mathematics 145 151–166
[2] Linss T and Madden N 2003 Computational Methods in Applied Mathematics 3 417–423
[3] Madden N and Stynes M 2003 IMA Journal of Numerical Analysis 23 627–644
[4] Linss T and Madden N 2004 Computing 73 121–133
[5] Shishkin G 2007 Computational Mathematics and Mathematical Physics 47 797–828
[6] Clavero C, Gracia J and Lisbona F 2010 Journal of Computational and Applied Mathematics 234 2501–2515
[7] Ilyin A 1969 Mat. Zametki 6 237–248
[8] Emel’yanov K 1973 The boundary value problems for the equations of mathematical physics (UB RAS, Yekaterinburg) pp 30–42
[9] Bakhalov N 1969 USSR Computational Mathematics and Mathematical Physics 9 139–166
[10] Shishkin G 1992 Grid Approximations of Singular Perturbation Elliptic and Parabolic Equations (UB RAS, Yekaterinburg)
[11] Miller J, O’Riordan E and Shishkin G 2012 Fitted numerical methods for singular perturbation problems (World Scientific)
[12] O’Riordan E, Stynes J and Stynes M 2009 Lecture Notes in Computer Science vol 5434 (Springer) pp 104–115
[13] Roos H G, Stynes M and Tobiska L 2008 Robust Numerical Methods for Singularly Perturbed Differential Equations (Springer Series in Computational Mathematics vol 24) (Springer, Berlin)
[14] Shishkin G and Shishkina L 2009 Difference methods for singular perturbation problems (Chapman and HALL/CRC monographs and surveys in pure and applied mathematics vol 140) (Chapman and HALL/CRC, Boca Raton)
[15] Shishkin G 2006 Zh. Vych. Mat. 9 81–108
[16] Kopteva N and Linss T 2001 Journal of Computational and Applied Mathematics 137 257–267
[17] Angelova I and Vulkov L 2008 Applications of Mathematics in Engineering and Economics vol 1067 (American Institute of Physics) pp 305–312
[18] Axelsson O and Layton W 1996 SIAM J. Numer. Anal. 33 2359–2374
[19] Xu J 1994 SIAM J. Sci. Comput. 15 231–237
[20] Vulkov L and Zadorin A 2010 International Journal of Numerical Analysis and Modeling 7 580–592
[21] Zadorin A and Tikhovskaya S 2013 Numerical Analysis and Applications 6 9–23
[22] Zadorin A and Tikhovskaya S 2013 Sib. Zh. Ind. Mat. 16 42–55
[23] Tikhovskaya S and Zadorin A 2015 Application of Mathematics in Technical and Natural Sciences vol 1684 (AIP Conference Proceedings) pp 090007–1–090007–8
[24] Tikhovskaya S 2014 Lobachevskii Journal of Mathematics 35 409–415
[25] Tikhovskaya S 2015 Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki. 157 60–74
[26] Zadorin A, Tikhovskaya S and Zadorin N 2015 Applied Numerical Mathematics 93 270–278
[27] Tikhovskaya S and Korbut M 2019 Journal of Physics: Conference Series vol 1210 (IOP Publishing) pp 012142–1–012142–8
[28] Natividad M and Stynes M 2003 Applied Numerical Mathematics 45 315–329
[29] Shishkin G and Shishkina L 2005 Differential Equations 41 1030–1039
[30] Zadorin A 2008 Computational Mathematics and Mathematical Physics 48 1634–1645
[31] Tikhovskaya S and Zadorin A 2016 Application of Mathematics in Technical and Natural Sciences vol 1773 (AIP Conference Proceedings) pp 100008–1–100008–9
[32] Zadorin A and Tikhovskaya S 2019 International Journal of Numerical Analysis and Modeling 16 590–608