Bogoliubov Theory of Dipolar Bose Gas in Weak Random Potential

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M. Ghabour and A. Pelster, PRA 90, 063636 (2014)
Outline

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2 Theoretical Description
   • Model
   • Bogoliubov Theory
   • Quantum, Thermal, Disorder Fluctuations

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   • Superfluid Zero-Temperature Depletion

5 Finite-Temperature Effects

6 Outlook
Introduction

- **Superfluid Helium in Porous Media:**
  - Experiment: Reppy et al., *PRL* **61**, 1950 (1988)

- **Theory:** K. Huang and H. F. Meng, *PRL* **69**, 644 (1992)

*FIG. 1.* The normalized superfluid-density data for helium contained in the three porous media are shown as a function of temperature. The solid curve gives the temperature dependence for the superfluid density of bulk helium.
Introduction

- Laser Speckles:
  - Experiment: Inguscio et al., *PRL* 95, 070401 (2005)

- Theory: J. W. Goodman. *Speckle Phenomena in Optics: Theory and Applications*. Roberts & Co Publ (2010)
Introduction

● Dipolar BEC:
  ● Experiment: Griesmaier et al., *PRL* 94, 160401 (2005)

  ● Theory: K. Góral et al., *PRA* 61, 051601(R) (2000)
Grand-Canonical Hamiltonian

- Grand-Canonical Hamiltonian of dipolar Bose gas in weak random potential

\[
\hat{K} = \int d^3x \hat{\psi}^\dagger(x) \left( \frac{-\hbar^2 \nabla^2}{2m} - \mu + U(x) \right) \hat{\psi}(x) + \frac{1}{2} \int d^3x \int d^3x' \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') V(x, x') \hat{\psi}(x') \hat{\psi}(x)
\]

- Properties
  - Disorder potential $U(x)$
  - Chemical potential $\mu$
  - Two-Body Interaction $V(x, x') = V_\delta(x - x') + V_{dd}(x - x')$
  - Bose quantized fields

\[
\left[ \hat{\psi}(x), \hat{\psi}^\dagger(x') \right] = \delta(x - x'), \quad \left[ \hat{\psi}(x), \hat{\psi}(x') \right] = \left[ \hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x') \right] = 0
\]
Random Potential

- Disorder Ensemble Average

\[ \langle \bullet \rangle_{\text{dis}} = \int DU P[U](\bullet), \quad \int DU P[U] = 1 \]

- Assumption

\[ \langle U(x) \rangle_{\text{dis}} = 0, \quad \langle U(x)U(x') \rangle_{\text{dis}} = R(x - x') \]

- Delta-Correlated Potential

\[ R(x - x') = R_0 \delta(x - x') \]

\( R_0 \) denotes disorder strength
**Two-Body Interaction**

- **Contact Interaction**
  \[ V_\delta(x - x') = g \delta(x - x') \]
  
  \[ g = 4\pi a\hbar^2/m \] with s-wave scattering length \( a \)

- **Dipolar Interaction**
  \[ V_{dd}(x) = \frac{C_{dd}}{4\pi} \frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} \]
  
  for dipoles aligned along \( z \)-axis direction

- **Dipolar Interaction Strength due to Magnetic or Electric dipole moments**
  \[ C_{dd} = \mu_0 d_m^2, \quad C_{dd} = d_e^2/\varepsilon_0 \]
  
  \( \mu_0 \) denotes vacuum magnetic permeability and \( \varepsilon_0 \) denotes vacuum dielectric constant
Momentum Space Representation

- Hamiltonian

\[ \hat{H} = \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \hat{a}_k^{\dagger} \hat{a}_k + \frac{1}{v} \sum_{p,k} U_{p-k} \hat{a}_p^{\dagger} \hat{a}_k \]
\[ + \frac{1}{2v} \sum_{p,k,q} V_q \hat{a}_{k+q}^{\dagger} \hat{a}_{p-q}^{\dagger} \hat{a}_p \hat{a}_k \]

- Creation/Annihilation Operators

\[ \left[ \hat{a}_k, \hat{a}_{k'}^{\dagger} \right] = \delta_{kk'}, \left[ \hat{a}_k, \hat{a}_{k'} \right] = 0 \]
\[ \left[ \hat{a}_{k'}^{\dagger}, \hat{a}_{k'}^{\dagger} \right] = 0 \]

- Two-Body Interaction

\[ V_q = g \left[ 1 + \epsilon_{dd}(3 \cos^2 \theta - 1) \right] \]

\[ \epsilon_{dd} = C_{dd}/3g \] denotes relative dipolar interaction strength
Bogoliubov Prescription

- **Creation/annihilation operators**

  \[ \hat{a}_0 |N_0\rangle_0 = \sqrt{N_0} |N_0 - 1\rangle_0, \quad \hat{a}^\dagger_0 |N_0\rangle_0 = \sqrt{N_0 + 1} |N_0 + 1\rangle_0 \]

- **\(N_0 \gg 1\)**, we replace operators by c-number

  \[ \hat{a}_0 \approx \hat{a}^\dagger_0 \approx \sqrt{N_0} \]

- **Simplified Hamiltonian**

  \[
  \hat{\mathcal{K}}' = \left( -\mu + \frac{1}{v} U_0 \right) N_0 + \frac{1}{2v} V_0 N_0^2 \\
  + \frac{1}{2} \sum_k \left( \frac{\hbar^2 k^2}{2m} - \mu \right) (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-k}) + \frac{1}{\sqrt{N_0}} \sum_k U_{k,0} (\hat{a}_k^\dagger + \hat{a}_{-k}) \\
  + \frac{1}{2v} N_0 \sum_k (V_0 + V_k) (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-k}) + \frac{1}{2v} N_0 \sum_k V_k (\hat{a}_k^\dagger \hat{a}_k^\dagger + \hat{a}_k \hat{a}_{-k})
  \]

- This approximation is justified in weakly interacting systems
- for weak disorder, disorder fluctuations decouple in lowest order
Bogoliubov Transformation

- Diagonalizing the simplified Hamiltonian via inhomogeneous Bogoliubov transformation

\[
\hat{a}_k = u_k \hat{\alpha}_k - v_k \hat{\alpha}^\dagger_{-k} - z_k, \\
\hat{a}^\dagger_k = u_k \hat{\alpha}^\dagger_k - v_k \hat{\alpha}_{-k} - z_k^* 
\]

- New operators \( \hat{\alpha}_k, \hat{\alpha}^\dagger_k \) satisfy bosonic commutation relations

\[
\left[ \hat{\alpha}_k, \hat{\alpha}^\dagger_{k'} \right] = \delta_{kk'}, \left[ \hat{\alpha}_k, \hat{\alpha}_{k'} \right] = \left[ \hat{\alpha}^\dagger_k, \hat{\alpha}^\dagger_{k'} \right] = 0
\]

K. Huang and H. F. Meng, *PRL* 69, 644 (1992)
Bogoliubov Amplitudes And Translation

- Bogoliubov amplitudes $u_k$ and $v_k$ read ($n_0 = N_0/v$)

$$u_k^2 = \frac{1}{2} \left[ \frac{\hbar^2 k^2}{2m} - \mu + n_0(V_0 + V_k) \right] E_k + 1, \quad v_k^2 = \frac{1}{2} \left[ \frac{\hbar^2 k^2}{2m} - \mu + n_0(V_0 + V_k) \right] E_k - 1$$

- Translation $z_k$ reads

$$z_k = \frac{1}{\sqrt{N_0}} \frac{U_k}{E_k^2} \left( \frac{\hbar^2 k^2}{2m} - \mu + n_0 V_0 \right)$$

- Bogoliubov quasi-particle dispersion

$$E_k = \sqrt{\left[ \frac{\hbar^2 k^2}{2m} - \mu + n_0 (V_0 + V_k) \right]^2 - (n_0 V_k)^2}$$
*Hamiltonian after disorder ensemble average*

\[
\langle \hat{\mathcal{H}}' \rangle_{\text{dis}} = \nu \left( -\mu n_0 + \frac{1}{2} V_0 n_0^2 \right) \\
+ \frac{1}{2} \sum_k \left\{ E_k - \left[ \frac{\hbar^2 k^2}{2m} - \mu + n_0 (V_0 + V_k) \right] \right\} \\
+ \frac{1}{2} \sum_k E_k \left( \hat{\alpha}_k^\dagger \hat{\alpha}_k + \hat{\alpha}_{-k}^\dagger \hat{\alpha}_{-k} \right) \\
- \sum_k n_0 R_k \frac{E_k^2}{E_k^2} \left( \frac{\hbar^2 k^2}{2m} - \mu + n_0 V_0 \right)
\]
Grand-Canonical Potential

- Grand-canonical potential \( \Omega_{\text{eff}} = -\beta^{-1} \ln Z_G \), where

\[
Z_G = \text{Tr} \, e^{-\beta \langle \hat{K}' \rangle_{\text{dis}}} \]

reduces to

\[
\Omega_{\text{eff}} = v \left( -\mu n_0 + \frac{1}{2} V_0 n_0^2 \right) + \frac{1}{2} \sum_k \left\{ E_k - \left[ \frac{\hbar^2 k^2}{2m} - \mu + n_0 (V_0 + V_k) \right] \right\} + \sum_k \frac{1}{\beta} \ln \left( 1 - e^{-\beta E_k} \right) - \sum_k \frac{n_0 R_k}{E_k^2} \left( \frac{\hbar^2 k^2}{2m} - \mu + n_0 V_0 \right)
\]
Grand-Canonical Free Energy

Extremizing with respect to $n_0$ for fixed chemical potential $\mu$, we find that the grand-canonical potential $\Omega_{\text{eff}}$ reduces up to first order in all fluctuations to the grand-canonical free energy

$$\mathcal{F} = - \frac{v \mu^2}{2 V_0}$$

$$+ \frac{1}{2} \sum_k \left[ E_k - \left( \frac{\hbar^2 k^2}{2m} + \mu \frac{V_k}{V_0} \right) \right]$$

$$+ \sum_k \frac{1}{\beta} \ln \left( 1 - e^{-\beta E_k} \right)$$

$$- \sum_k \frac{R_k}{E_k^2} \frac{\hbar^2 k^2}{2m} \frac{\mu}{V_0}$$
Condensate Depletion, $n - n_0 = n' + n_{th} + n_R$

- Due to quantum fluctuations
  \[ n' = \frac{1}{2v} \sum_k \left( \frac{\hbar^2 k^2}{2m} + nV_k \right) \frac{1}{E_k} - 1 \]

- Due to thermal fluctuations
  \[ n_{th} = \frac{1}{\sqrt{u}} \sum_k \frac{\hbar^2 k^2}{2m} + nV_k \frac{1}{E_k} \frac{1}{e^{\beta E_k} - 1} \]

- Due to external random potential
  \[ n_R = \frac{1}{\sqrt{u}} \sum_k \frac{nR_k}{E_k^4} \left( \frac{\hbar^2 k^2}{2m} \right)^2 \]

$E_k = \sqrt{\left( \frac{\hbar^2 k^2}{2m} \right)^2 + nV_k \frac{\hbar^2 k^2}{m}}$ denotes Bogoliubov dispersion relation
Condensate Depletion ($T = 0$)

- Due to quantum fluctuations
  
  $$n' = \frac{8}{3\sqrt{\pi}} \left( na \right)^{3/2} Q^{3/2}_{\frac{3}{2}}(\epsilon_{dd})$$

  A. R. P. Lima and A. Pelster, *PRA* 84, 041604(R) (2011)

- Due to external random potential
  
  $$n_R = \frac{m^2 R_0}{8\hbar^4 \pi^{3/2}} \sqrt{\frac{n}{a}} Q_{-\frac{1}{2}}(\epsilon_{dd})$$

  C. Krumnow et al., *PRA* 84, 021608(R) (2011); B. Nikolic et al., *PRA* 88, 013624 (2013)

- Functions describe dipolar effect expressed analytically
  
  $$Q_\alpha(\epsilon_{dd}) = (1 - \epsilon_{dd})^\alpha \, _2F_1 \left( -\alpha, \frac{1}{2}; \frac{3}{2}; \frac{-3\epsilon_{dd}}{1 - \epsilon_{dd}} \right)$$

  $$_2F_1$$ denotes hypergeometric function
**Figure:** (1) Dipolar enhancement function $Q_\alpha(\epsilon_{dd})$ versus relative dipolar interaction strength $\epsilon_{dd}$ for different values of $\alpha$: -5/2 (brown), -3/2 (pink), -1/2 (red), 1/2 (black), 3/2 (blue), 5/2 (green).
Model

Inserting Galilean transformations $x' = x + ut$, $t = t'$, and field operator in Heisenberg picture $\hat{\psi}'(x', t') = \hat{\psi}(x, t)e^{\frac{i}{\hbar}mv_s x}$, Hamiltonian reads

$$\hat{\mathcal{K}} = \frac{1}{2} \sum_k \left[ \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} \right] (\hat{a}_k^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-k})$$

$$+ \frac{1}{2} \sum_k \hbar k (u - v_s) (\hat{a}_k^\dagger \hat{a}_k - \hat{a}_{-k}^\dagger \hat{a}_{-k})$$

$$+ \frac{1}{2v} \sum_{p,k} U_{p-k} (\hat{a}_p^\dagger \hat{a}_k + \hat{a}_{-k}^\dagger \hat{a}_{-p})$$

$$+ \frac{1}{2v} \sum_{p,k,q} V_{q} \hat{a}_{k+q}^\dagger \hat{a}_p^\dagger \hat{a}_{p-q} \hat{a}_p \hat{a}_k$$

$$\mu_{\text{eff}} = \mu - \frac{1}{2} m v_s^2 + m u v_s$$ denotes effective chemical potential
Diagonalized Hamiltonian

- Hamiltonian after disorder ensemble average

\[
\langle \hat{\mathcal{C}}' \rangle_{\text{dis}} = v \left( -\mu_{\text{eff}} n_0 + \frac{1}{2} V_0 n_0^2 \right) \\
+ \frac{1}{2} \sum_k' \left\{ E_k - \left[ \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + n_0 (V_0 + V_k) \right] \right\} \\
+ \sum_k' \left[ E_k + \hbar k (u - v_s) \right] \hat{\alpha}_k^\dagger \hat{\alpha}_k \\
- \sum_k n_0 R_k \left( \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + n_0 V_0 \right) \frac{E_k^2 - [\hbar k (u - v_s)]^2}{E_k^2 - [\hbar k (u - v_s)]^2}
\]
Grand-Canonical Effective Potential

- Up to second order in $\hbar k (u - v_s)$

$$\Omega_{\text{eff}} = v \left( -\mu_{\text{eff}} n_0 + \frac{1}{2} V_0 n_0^2 \right)$$

$$+ \frac{1}{2} \sum_k \left\{ E_k - \left[ \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + n_0(V_0 + V_k) \right] \right\}$$

$$+ \sum_k \frac{1}{\beta} \ln \left( 1 - e^{-\beta E_k} \right) - \sum_k \frac{\beta e^{\beta E_k} [\hbar k (u - v_s)]^2}{2 \left( e^{\beta E_k} - 1 \right)^2}$$

$$- \sum_k \frac{n_0 R_k}{E_k^2} \left[ \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + n_0 V_0 \right]$$

$$- \sum_k \frac{n_0 R_k}{E_k^4} \left[ \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + n_0 V_0 \right] [\hbar k (u - v_s)]^2$$
System Momentum

- Extremizing yields in zeroth order $\mu_{\text{eff}} = n_0 V_0$
- From thermodynamic relation $P = \left(-\frac{\partial \Omega_{\text{eff}}}{\partial u}\right)_{v, T, \mu}$ we find
  $$P = m v \left(n v_s + n_n v_n\right)$$
- Normal fluid density decompose to $n_{nij} = n_{Rij} + n_{thij}$, $v_n = u - v_s$
- Contribution due to external random potential
  $$n_{Rij} = \frac{1}{\sqrt{\pi}} \sum'_{k} \frac{2nR_k \hbar^2 k_i k_j}{m \left(\frac{\hbar^2 k^2}{2m}\right) \left(\frac{\hbar^2 k^2}{2m} + 2nV_k\right)^2}$$
- Contribution due to thermal fluctuations
  $$n_{thij} = \frac{1}{\sqrt{\pi}} \sum'_{k} \frac{\beta}{m} \frac{\hbar^2 k_i k_j}{e^{\beta E_k}} \frac{e^{\beta E_k}}{(e^{\beta E_k} - 1)^2}$$
Superfluid Depletion \( (T = 0) \)

- In direction parallel to dipole polarization
  \[
  n_{R\parallel} = \frac{m^2 R_0}{2\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}} J_{-\frac{1}{2}}(\epsilon_{dd})
  \]

- In direction perpendicular to dipole polarization
  \[
  n_{R\perp} = \frac{m^2 R_0}{4\hbar^4 \pi^{\frac{3}{2}}} \sqrt{\frac{n}{a}} \left[ Q_{-\frac{1}{2}}(\epsilon_{dd}) - J_{-\frac{1}{2}}(\epsilon_{dd}) \right]
  \]

C. Krumnow et al., *PRA* 84, 021608(R) (2011); B. Nikolic et al., *PRA* 88, 013624 (2013)

- Functions describe dipolar effect expressed analytically
  \[
  J_\alpha(\epsilon_{dd}) = \frac{1}{3} (1 - \epsilon_{dd})^\alpha \ _2F_1 \left( -\alpha, \frac{3}{2}; \frac{5}{2}; \frac{-3\epsilon_{dd}}{1 - \epsilon_{dd}} \right)
  \]

\(_2F_1\) denotes hypergeometric function
Ratios

Figure: (2) Ratios of superfluid depletions $n_{R\parallel}$ and $n_{R\perp}$ and condensate depletion $n_R$ versus relative dipolar interaction strength $\epsilon_{dd}$.

--- finite localization time

R. Graham and A. Pelster, *Int. J. Bif. Chaos* **19**, 2745 (2009)
**Figure:** (3) Superfluid depletions ratio $n_{R\parallel}/n_{R\perp}$ versus relative dipolar interaction strength $\epsilon_{dd}$. 

**Equation:** 

$$\frac{n_{R\parallel}}{n_{R\perp}}$$
Condensate Depletion \((T > 0)\)

- Condensate depletion due to thermal excitations

\[
\frac{n_{\text{th}}}{n} = \frac{\gamma^{-\frac{1}{6}} t^2}{2\pi^{\frac{1}{2}} \left( \zeta\left(\frac{3}{2}\right) \right)^{\frac{4}{3}}} I(\gamma, \epsilon_{\text{dd}}, t)
\]

Gas parameter \(\gamma = na^3\), and relative temperature \(t = T / T_c^0\),
\(T_c^0 = 2\pi \hbar^2 n^\frac{2}{3} / (\zeta(\frac{3}{2})^\frac{2}{3}) \text{mk}_\text{B}\) non-interacting Bose gas critical temperature

- The integral \(I(\gamma, \epsilon_{\text{dd}}, t)\) reads

\[
I(\gamma, \epsilon_{\text{dd}}, t) = \int_0^\infty dx \int_0^\pi d\theta \frac{x \sin \theta \left(1 + \frac{\alpha x^2}{8 \Theta^2}\right)}{\sqrt{\Theta + \frac{\alpha x^2}{16 \Theta}} \left(e^{\sqrt{x^2 + \frac{\alpha x^4}{16 \Theta^2}}} - 1\right)}
\]

with abbreviations \(\alpha = \left[t / \gamma^{\frac{1}{3}} \left(\zeta\left(\frac{3}{2}\right)\right)^\frac{2}{3}\right]^2\) and \(\Theta = 1 + \epsilon_{\text{dd}} \left(3 \cos^2 \theta - 1\right)\)
Depletion Plot

Figure: (4) Thermal fractional depletion $n_{\text{th}}/n$ versus relative temperature $t$ for different values of relative dipolar interaction strength $\epsilon_{dd} = 0$ (solid), $\epsilon_{dd} = 0.8$ (dotted), and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).
Below the critical temperature we approximate the condensate depletion analytically

\[
\frac{n_{\text{th}}}{n} = \frac{\pi^{3/2} \gamma^{-1/6} t^2}{6 \left( \zeta \left( \frac{3}{2} \right) \right)^{4/3}} Q^{-1/2} (\epsilon_{dd}) - \frac{\pi^{7/2} \gamma^{-5/6} t^4}{480 \left( \zeta \left( \frac{3}{2} \right) \right)^{8/3}} Q^{-5/2} (\epsilon_{dd}) + \ldots
\]

This reproduces the isotropic contact interaction case for vanishing dipolar interaction due to \( Q_\alpha(0) = 1 \)
For fractional condensate depletion values limited to \( \frac{n-n_0}{n} \leq \frac{1}{2} \)

**Figure:** (5) Validity range of Bogoliubov theory in the \( t - \gamma \) plane for (a) clean case, i.e. \( R_0 = 0 \), and (b) dirty case with \( R_0 = \frac{2\hbar^4 \pi^{\frac{3}{2}} n^{\frac{3}{2}}}{5m^2} \) for different values of relative dipolar interaction strength \( \epsilon_{dd} = 0 \) (red), \( \epsilon_{dd} = 0.5 \) (blue), \( \epsilon_{dd} = 0.8 \) (green).
The total fractional condensate depletion $\frac{\Delta n}{n}$ versus $\gamma$

**Figure:** (6) Fractional depletion $\Delta n/n$ versus gas parameter $\gamma$ for different values of relative dipolar interaction strength $\epsilon_{dd} = 0$ (red), $\epsilon_{dd} = 0.8$ (blue) and relative temperature $t = 0$ (solid), $t = 0.5$ (dotted) with the disorder strength $R_0 = \frac{2\hbar^4 \pi^{\frac{3}{2}} n^{\frac{1}{3}}}{5m^2}$. 
Superfluid Depletion \( (T > 0) \)

- Parallel to the dipoles

\[
\frac{n_{\text{th}\parallel}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}} \left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{8}{3}}} \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin \theta \cos^2 \theta e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}}}{\Theta^{\frac{5}{2}} \left(e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}} - 1\right)^2}
\]

- Perpendicular to the dipoles

\[
\frac{n_{\text{th}\perp}}{n} = \frac{\gamma^{-\frac{5}{6}} t^4}{8\pi^{\frac{1}{2}} \left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{8}{3}}} \int_0^\infty dx \int_0^\pi d\theta \frac{x^4 \sin^3 \theta e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}}}{2\Theta^{\frac{5}{2}} \left(e^{\sqrt{x^2 + \frac{\alpha x^4}{16\Theta^2}}} - 1\right)^2}
\]

with abbreviations \( \alpha = \left[t/\gamma^{\frac{1}{3}} \left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}\right]^2 \) and \( \Theta = 1 + \epsilon_{dd} (3 \cos^2 \theta - 1) \).
**Figure:** (7) Superfluid thermal fractional depletions $n_{\text{th}||}/n$ and $n_{\text{th}\perp}/n$ versus relative temperature $t$ for different values of relative dipolar interaction strength $\epsilon_{dd} = 0$ (solid), $\epsilon_{dd} = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).
Figure: (8) Superfluid thermal fractional depletions $n_{th\parallel}/n$ and $n_{th\perp}/n$ versus relative dipolar interaction strength $\epsilon_{dd}$ for different values of relative temperature $t = 0.2$ (solid), $t = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).
**Figure:** (9) Ratios of thermal superfluid depletions $n_{th\parallel}/n_{th\perp}$ versus relative temperature $t$ for different values of relative dipolar strength $\epsilon_{dd} = 0$ (solid), $\epsilon_{dd} = 0.6$ (dotted) and gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue).
Zero-Temperature Vicinity \((T \approx 0)\)

- Below the critical temperature we approximate the superfluid depletion analytically

- Parallel to the dipoles

\[
\frac{n_{th\parallel}}{n} = \frac{\pi^\frac{7}{2} \gamma^{-\frac{5}{6}} t^4}{15 \left(\zeta\left(\frac{3}{2}\right)\right)^\frac{8}{3}} J_{-\frac{5}{2}}(\epsilon_{dd}) + \ldots
\]

- Perpendicular to the dipoles

\[
\frac{n_{th\perp}}{n} = \frac{\pi^\frac{7}{2} \gamma^{-\frac{5}{6}} t^4}{30 \left(\zeta\left(\frac{3}{2}\right)\right)^\frac{8}{3}} \left[Q_{-\frac{5}{2}}(\epsilon_{dd}) - J_{-\frac{5}{2}}(\epsilon_{dd})\right] + \ldots
\]
Figure: (10) Ratios of thermal superfluid depletions $n_{th\parallel}/n_{th\perp}$ versus relative dipolar interaction strength $\epsilon_{dd}$ for different values of the gas parameter $\gamma = 0.01$ (red), $\gamma = 0.20$ (blue) and relative temperature $t = 0$ (solid-gray), $t = 0.5$ (dotted-dashed).
Further investigations

- Sound velocities in the anisotropic two-fluid model
- Anisotropic disorder potential
- Local density approximation
- Anisotropic trap potential
Thank You For Your Attention