On $q$-analogues of two-one formulas for multiple harmonic sums and multiple zeta star values

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Abstract Recently, the present authors jointly with Tauraso found a family of binomial identities for multiple harmonic sums (MHS) on strings $(\{2\}^a, c, \{2\}^b)$ that appeared to be useful for proving new congruences for MHS as well as new relations for multiple zeta values. Very recently, Zhao generalized this set of MHS identities to strings with repetitions of the above patterns and, as an application, proved the two-one formula for multiple zeta star values conjectured by Ohno and Zudilin. In this paper, we extend our approach to $q$-binomial identities and prove $q$-analogues of two-one formulas for multiple zeta star values.

Keywords Multiple harmonic sum · Multiple zeta value · $q$-Analogues of multiple zeta values · Two-one formulas · $q$-Binomial identity

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1 Introduction

Let $s_1, \ldots, s_m$ be positive integers, $s_1 \geq 2$. The multiple zeta star and zeta values are defined by the convergent series
\[ \zeta^*(s) = \zeta^*(s_1, \ldots, s_m) = \sum_{k_1 \geq \ldots \geq k_m \geq 1} \frac{1}{k_1^{s_1} \ldots k_m^{s_m}}, \] (1)

\[ \zeta(s) = \zeta(s_1, \ldots, s_m) = \sum_{k_1 > \ldots > k_m \geq 1} \frac{1}{k_1^{s_1} \ldots k_m^{s_m}}. \] (2)

The origin of these numbers goes back to the correspondence of Euler with Goldbach in 1742–1743 (see [16]) and Euler’s paper [11] that appeared in 1776. Euler studied double zeta values and established some important relation formulas for them. For example, he proved that

\[ 2\zeta^*(n, 1) = (n + 2)\zeta(n + 1) - \sum_{i=1}^{n-2} \zeta(n - i)\zeta(i + 1), \quad n \geq 2, \]

which, in particular, implies the simplest but nontrivial relation:

\[ \zeta^*(2, 1) = 2\zeta(3) \text{ or equivalently, } \zeta(2, 1) = \zeta(3). \] (3)

The systematic study of multiple zeta values began in the early 1990s with the works of Hoffman [14] and Zagier [31] and has continued with increasing attention in recent years (see survey articles [4, 15, 30, 36]). There is rather little known about individual evaluations of multiple zeta values. By contrast, they satisfy a wealth of interesting relations coming from different approaches. The so-called double shuffle relations arise as a result of product of multiple zeta values considered in terms of iterated integrals and nested sums, respectively. These are not enough to imply all relations among multiple zeta values and one of the basic problems in the theoretical study of multiple zeta values is to describe a complete set of such relations (see [17]).

There are also a lot of recent contributions on \(q\)-analogues of multiple zeta (star) values (see [5, 21, 32]). Actually, there are many possible ways to \(q\)-extend the multiple zeta (star) values and the main difficulty here is to find appropriate \(q\)-analogues.

Different problems may dictate different \(q\)-analogues of the same object but the main advantage of the \(q\)-theory is to provide new insights into classical non-\(q\)-settings. Among several \(q\)-analogues of the multiple zeta values that have been explored in the recent years, a \(q\)-analogue introduced by Bradley [5] and Zhao [32], independently, as

\[ \zeta_q(s_1, \ldots, s_m) = (1 - q)^{s_1 + \ldots + s_m} \sum_{k_1 > k_2 > \ldots > k_m > 0} \prod_{j=1}^{m} q^{(s_j-1)k_j} / (1 - q^{s_j}k_j) \] (4)

received a lot of attention. Many properties of this function including analytic continuation, \(q\)-double shuffle multiplication rules and a number of linear relations have been studied in [5, 7, 25, 32]. Another version of a \(q\)-analogue for classical multiple zeta values proposed by Ohno et al. [21], which reads as

\[ \zeta_q(s_1, \ldots, s_m) = \sum_{k_1 > k_2 > \ldots > k_m > 0} q^{k_1} / (1 - q^{k_1}s_1) \ldots (1 - q^{k_m}s_m), \]
was studied in detail in [8]. In connection with the irrationality problems for zeta values, Zudilin [35] considered the following $q$-analogue of (non-multiple) zeta values (see also [18,19,26])

$$Z_q(s + 1) = \sum_{n=1}^{\infty} \sigma_s(n) q^n, \quad s = 0, 1, 2, \ldots ,$$

(5)

where $\sigma_s(n) = \sum_{d|n} d^s$ stands for the sum of powers of the divisors. Very recently, Bachmann and Kühn [1] introduced a multiple $q$-extension of series (5), which is a generating function of the multiple divisor sums

$$\sigma_{s_1, \ldots, s_m}(n) = \sum_{\substack{u_1 v_1 + \cdots + u_m v_m = n \\ u_1 > \cdots > u_m > 0}} v_1^{s_1} \cdots v_m^{s_m}.$$ 

These $q$-analogues of multiple zeta values arise from the calculation of the Fourier expansion of multiple Eisenstein series and possess a good algebraic structure. One more specific $q$-analogue of the multiple zeta values and its connection to certain problems in enumerative geometry and related conjectures have been considered very recently by Okounkov [2,24]. Finally, we mention a very interesting $q$-deformation of the Riemann zeta function proposed by Cherednik [9] who conjectured a $q$-analogue of the Riemann hypothesis and studied the localization of zeros of the $q$-zeta functions.

Many different $q$-models [5,6,20,25,36] were introduced and studied in attempts to explain and extend the known relations for multiple zeta values. However, there are several examples of linear relations and one of them is the two-one formula (see [23], [21, Section 3]), for which $q$-analogues had not been known until recently.

The purpose of the present paper is to establish $q$-analogues of the two-one formulas for multiple zeta star values conjectured by Ohno and Zudilin [23] and proved very recently by Zhao [33].

Zhao’s proof is based on generalizations of binomial identities for finite multiple harmonic sums found in [12] to strings with repeating collections of twos and ones. In this paper, we extend this approach to $q$-analogues of multiple harmonic sums and, as a limit case, obtain corresponding results for $q$-zeta values.

We begin with some basic notation. Let $q$ be a real number with $0 < q < 1$. The $q$-analogue of a non-negative integer $n$ is defined as

$$[n]_q = \sum_{k=0}^{n-1} q^k = \frac{1 - q^n}{1 - q}.$$ 

For any real number $a$, put

$$(a)_0 := (a; q)_0 := 1, \quad (a)_n := (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1.$$ 

Let $n, m$ denote integers. Then the Gaussian $q$-binomial coefficient is defined by
\[ \left[ \frac{n}{m} \right] = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise.} \end{cases} \]

For non-negative integers \( n, m \) and \( s = (s_1, \ldots, s_m) \in \mathbb{Z}^m \), \( t = (t_1, \ldots, t_m) \in \mathbb{Z}^m \), define \( q \)-analogues of multiple harmonic sums

\[ H_n^*[s] = H_n^*[s_1, \ldots, s_m] = \sum_{n \geq k_1 \geq \ldots \geq k_m \geq 1} \frac{q^{k_1}}{[k_1]^q} \cdots \frac{q^{k_m}}{[k_m]^q}, \]

\[ \mathcal{H}_n[s; t] = \mathcal{H}_n[s_1, \ldots, s_m; t_1, \ldots, t_m] = \sum_{n \geq k_1 > \ldots > k_m \geq 1} \prod_{j=1}^m \frac{q(t_j-1)k_j(1+q^{k_j})}{[k_j]^q}, \]

with the convention that \( \mathcal{H}_n[s; t] = 0 \) if \( n < m \), and \( H_n^*[\emptyset] = \mathcal{H}_n[\emptyset; \emptyset] = 1 \) for all \( n \geq 0 \) and \( m = 0 \).

We consider \( q \)-analogues of multiple zeta star values (1) and multiple zeta values (2), which are defined by

\[ \zeta_q^*[s] = \zeta_q^*[s_1, \ldots, s_m] := \sum_{k_1 \geq \ldots \geq k_m \geq 1} \frac{q^{k_1}}{[k_1]^q} \cdots \frac{q^{k_m}}{[k_m]^q} \]  \hspace{1cm} (6)

and

\[ \hat{\zeta}_q[s; t] = \hat{\zeta}_q[s_1, \ldots, s_m; t_1, \ldots, t_m] \]

\[ = \sum_{k=1}^{\infty} \frac{q^{k^2+(t_1-1)k}(1+q^k)}{[k]^q} \mathcal{H}_{k-1}[s_2, \ldots, s_m; t_2, \ldots, t_m]. \]

If \( m = 0 \), we put \( \zeta_q^*[\emptyset] = \hat{\zeta}_q[\emptyset; \emptyset] = 1 \).

**Theorem 1.1** Let \( m \in \mathbb{N}, s_1, \ldots, s_m \in \mathbb{N}_0 \). Then

\[ \zeta_q^*[\{2\}^{s_1}, 1, \{2\}^{s_2}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{p=(2s_1+1)\circ(2s_2+1)\circ\ldots\circ(2s_m+1)} \hat{\zeta}_q[p; \bar{p}], \]  \hspace{1cm} (7)

where \( \circ \) is either comma or plus, and the string \( \bar{p} := (s_1+1) \circ (s_2+1) \circ \ldots \circ (s_m+1) \) is associated with the string \( p \). This means that the choice of commas and pluses in \( p = (2s_1+1) \circ (2s_2+1) \circ \ldots \circ (2s_m+1) \) and \( \bar{p} = (s_1+1) \circ (s_2+1) \circ \ldots \circ (s_m+1) \) is the same.

In the next theorem, we consider the two-one strings ending with 2.

Let

\[ \bar{\zeta}_q[s; t] = \bar{\zeta}_q[s_1, \ldots, s_m; t_1, \ldots, t_m] \]

\[ = \sum_{k_1 > \ldots > k_m \geq 1} (-1)^{k_m-1} q^{k_1^2-k_m(k_m-1)/2} \prod_{j=1}^m \frac{(1+q^{k_j}) q(t_j-1)k_j}{[k_j]^q}. \]
Theorem 1.2 Let $m, s_1, \ldots, s_m \in \mathbb{N}_0, s_{m+1} \in \mathbb{N}$. Then

$$\zeta^*[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1, \{2\}^{m+1}] = \sum_{p=(2s_1+1) \circ \cdots \circ (2s_m+1) \circ (2s_{m+1})} \bar{\zeta}_q[p, \bar{p}],$$

(8)

where $\circ$ is either comma or plus, and the string $\bar{p} := (s_1+1) \circ \cdots \circ (s_m+1) \circ (s_{m+1})$ is associated with the string $p$. This means that the choice of commas and pluses in $p = (2s_1+1) \circ \cdots \circ (2s_m+1) \circ (2s_{m+1})$ and $\bar{p} = (s_1+1) \circ \cdots \circ (s_m+1) \circ (s_{m+1})$ is the same.

Note that the limiting $q \to 1$ case of (7) is Ohno–Zudilin’s two-one formula from [23]:

$$\zeta^*[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{p=(2s_1+1) \circ \cdots \circ (2s_m+1)} 2^l(p) \zeta(p),$$

(9)

where $l(p)$ is the length of the string $p$. It is clear that $l(p) = \sigma(p) + 1$, where $\sigma(p)$ is the number of commas in $p = (2s_1+1) \circ \cdots \circ (2s_m+1)$.

Similarly, taking the limit $q \to 1$ in (8) gives Zhao’s two-one formula from [33, Theorem 1.2].

Remark 1 Note that the series on the right-hand sides of (7) and (8) contain quadratic powers of the parameter $q$. This implies that the series converge rapidly and therefore, formulas (7), (8) can be used for fast calculation of the $q$-zeta star values (6) on strings of twos and ones.

Remark 2 The generalizations of Theorems 1.1 and 1.2 to arbitrary collections of arguments as well as their limiting cases ($q \to 1$) related to the corresponding results for ordinary multiple zeta values (1), (2) can be found in [13].

Remark 3 It is easy to see that for $m = 0$ and $s_{m+1} = 0$, Theorem 1.2 is also true. In this case, it corresponds to the trivial telescoping sum:

$$1 = \zeta^*[\emptyset] = \bar{\zeta}_q[0; 0] = \sum_{k=1}^{\infty} (-1)^{k-1} (1 + q^k)q^{k(k-1)/2}.$$

In the simplest cases $m = 0$ and $m = 1$, Theorems 1.2 and 1.1 give $q$-analogues of the following well-known formulas for ordinary zeta values (see [29,34], [22, p. 292]):

$$\zeta^*([2]^s, 1) = 2\zeta(2s+1),$$

(10)

$$\zeta^*([2]^s) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2s}} = 2(1 - 2^{1-2s})\zeta(2s).$$

Note that (10) is a particular case of (9).
Corollary 1.1 Let $s$ be a nonnegative integer. Then

\[ \zeta_q^\ast[2^s] = \frac{1 + q^k}{[k]^2_q} (-1)^k (1 - q^{k(1-2s+2+sk)}, \quad (11) \]

\[ \zeta_q^\ast[2^s, 1] = \frac{1 + q^k}{[k]_{q^{2s+1}}} q^{k^2 + sk}, \quad (12) \]

**Remark 4** Note that an alternative proof of formula (11) can be found in [10].

For $s = 1$, from formula (12) we get a new $q$-analogue of Euler’s formula (3) that becomes

\[ \zeta_q^\ast[2, 1] = \sum_{k=1}^{\infty} \frac{1 + q^k}{[k]_q} q^{k(k+1)}. \quad (13) \]

Another $q$-analogue of formula (3) in terms of series (4) was proved in [5, Cor. 7] and has the form $\zeta_q(2, 1) = \zeta_q(3)$. The detailed survey on various generalizations and proofs of formula (3) can be found in [3]. Note that no extension of Apéry’s proof leading to the irrationality of a $q$-analogue of $\zeta(3)$ is known. In this respect, formula (13) may be quite helpful: firstly, in view of the fast convergence of the series (13) and secondly, because of the fact that the irrationality proof of $\zeta(3)$ as a double series, $\zeta(2, 1)$, is known (see [27,28]).

**2 $q$-Binomial identities**

In this section, we prove some auxiliary $q$-binomial identities, which are $q$-analogues of those proved in [12, Lemma 2.1].

**Lemma 2.1** For integers $n \geq 1$, $l \geq 0$ we have

\[ \sum_{k=l+1}^{n} (1 + q^k) \frac{[n]_q}{[n+k]_q} (-1)^k q^{k(k-1)/2} = \frac{[l]_q - [n]_q}{[l]_q} \frac{[n]_q}{[n+l]_q} q^{l(l-1)/2}, \quad (14) \]

\[ \sum_{k=l+1}^{n} (1 + q^k) \frac{[k]_q [n]_q}{[n+k]_k} q^{k(k-1)} = ([n]_q - [l]_q) \frac{[n]_q}{[n+l]_q} q^{l^2}, \quad (15) \]

\[ \sum_{k=1}^{n} \frac{1 + q^k}{[k]_q} \frac{[n]_q}{[n+k]_k} q^{k^2} = \sum_{m=1}^{n} \frac{q^m}{[m]_q}, \quad (16) \]

Moreover, if $l \geq 1$ then

\[ \sum_{k=l}^{n} \frac{q^k}{[k]_q^2} \frac{[k]_l}{[k+l]_l} = \frac{q^l [n]_l}{[l]_q [n+l]_l}. \quad (17) \]

**Proof** It is easy to show that if we put
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\[ F(n, k) = (1 + q^k) \binom{n}{k} \binom{n+k}{k} (-1)^{k-1} q^{k(k-1)/2} \]

\[ G(n, k) = \frac{q^{n+k} - 1}{q^k + 1} F(n, k), \]

then

\[ (1 - q^n) F(n, k) = G(n, k + 1) - G(n, k) \quad (18) \]

for all positive integers \( n, k \). Summing \((18)\) over \( k \) from \( l + 1 \) to \( n \) we obtain

\[ (1 - q^n) \sum_{k=l+1}^{n} F(n, k) = \sum_{k=l+1}^{n} (G(n, k + 1) - G(n, k)) \]

\[ = G(n, n + 1) - G(n, l + 1) \]

\[ = -G(n, l + 1) = \frac{1 - q^{n+l+1}}{1 + q^{l+1}} F(n, l + 1) \]

\[ = (1 - q^{n-l}) \binom{n}{l} \binom{n+l}{l} (-1)^l q^{(l+1)/2}, \]

which implies \((14)\).

Similarly, putting

\[ F(n, k) = \binom{n}{k} \binom{n+k}{k} (1 - q^{2k}) q^{k(k-1)}, \]

\[ G(n, k) = \frac{q^n - 1}{q^k + 1} F(n, k) \]

we conclude that

\[ F(n, k) = G(n, k) - G(n, k + 1) \quad (19) \]

for all positive integers \( n, k \). Summing both sides of \((19)\) over \( k \) from \( l + 1 \) to \( n \), we have

\[ \sum_{k=l+1}^{n} F(n, k) = G(n, l + 1) - G(n, n + 1) = G(n, l + 1) = \frac{1 - q^n}{1 - q^{n+l+1}} F(n, l + 1), \]

and the identity \((15)\) follows.

For proving \((16)\), we define for integers \( m \geq 0, k \geq 1 \) two functions:

\[ F(m, k) = \binom{m}{k} \binom{m+k}{k} 1 + q^k \]

\[ G(m, k) = \frac{q^{m+k+1}}{q^k} \frac{1 - q^k}{1 - q^{m-k+1}} F(m, k). \]

Then it is readily seen that

\[ F(m, k) - F(m + 1, k) = G(m, k + 1) - G(m, k). \quad (20) \]
Summing both sides of (20) over \( m \) from 0 to \( n - 1 \) we get
\[
\sum_{m=0}^{n-1} (G(m, k + 1) - G(m, k)) = \sum_{m=0}^{n-1} (F(m, k) - F(m + 1, k)) = F(0, k) - F(n, k) = -F(n, k).
\] (21)

Summing once again both sides of (21) over \( k \) from 1 to \( n \), we easily obtain
\[
\sum_{k=1}^{n} F(n, k) = \sum_{m=0}^{n-1} \sum_{k=1}^{n} (G(m, k) - G(m, k + 1)) = \sum_{m=0}^{n-1} (G(m, 1) - G(m, n + 1)) = \sum_{m=0}^{n-1} G(m, 1) = \sum_{m=0}^{n-1} \frac{q^{m+1}}{1 - q^{m+1}},
\]
which implies (16).

To prove (17), we put
\[
F(l, k) = \frac{q^{k-l} \left[ k \right]_q}{\left[ k + l \right]_q}, \quad G(l, k) = \frac{(1 - q^{k-l})(1 - q^{k+l})}{q^{k-l}} F(l, k).
\]

Then it is easy to see that
\[
(1 - q^l)^2 F(l, k) = G(l, k + 1) - G(l, k).
\] (22)

Summing (22) over \( k \) from \( l \) to \( n \) we get
\[
(1 - q^l)^2 \sum_{k=l}^{n} F(l, k) = \sum_{k=l}^{n} (G(l, k + 1) - G(l, k)) = G(l, n + 1) - G(l, l) = G(l, n + 1) = (1 - q^l)^2 \left[ \frac{n}{l} \right],
\]
and the identity (17) follows. \( \square \)

3 Identities for multiple harmonic sums

In this section, we prove \( q \)-binomial identities for multiple harmonic sums whose arguments are strings of twos and ones. We begin with some special cases and then extend them to arbitrary strings of twos and ones.
Theorem 3.1 Let $a, b$ be integers satisfying $a \geq 0$, $b \geq 1$. Then for any positive integer $n$,

$$H_n^*[[2]^a] = \sum_{k=1}^{n} \frac{1 + q^k}{[k]^2_q} \binom{n}{k} (-1)^{k-1} q^{k(k-1)/2+ak},$$  \(23\)

$$H_n^*[[2]^a, 1] = \sum_{k=1}^{n} \frac{1 + q^k}{[k]^{2a+1}_q} \binom{n}{n+k} q^{k^2+ak},$$  \(24\)

$$H_n^*[[2]^a, 1, \{2\}^b] = -\sum_{k=1}^{n} \frac{1 + q^k}{[k]^{2a+b+1}_q} \binom{n}{n+k} (-1)^k q^{k(k+1)/2+(a+b)k}$$

$$\quad - \sum_{k=1}^{n} \frac{1 + q^k}{[k]^{2a+1}_q} \binom{n}{n+k} q^{k^2+ak} \sum_{j=1}^{k-1} \frac{(-1)^{j}(1 + q^j)q^{bj-j(j+1)/2}}{[j]^{2b}_q}.$$  \(25\)

Proof We show that (23) is true by induction on $a$. For $a = 0$ the equality follows from (14). Suppose that the formula is true for $a > 0$. Then by the induction assumption and identity (17) we easily conclude that

$$H_n^*[[2]^{a+1}] = \sum_{k=1}^{n} \frac{q^k}{[k]^2_q} H_n^*[[2]^a] = \sum_{k=1}^{n} \frac{q^k}{[k]^2_q} \sum_{l=1}^{k} \frac{1 + q^l}{[l]^2_q} \binom{k}{l} (-1)^{k-1} q^{l(l-1)/2+al}$$

$$= \sum_{k=1}^{n} \frac{1 + q^l}{[l]^2_q} (-1)^{l-1} q^{l(l-1)/2+al} \sum_{k=1}^{n} \frac{q^k}{[k]^2_q} \binom{k}{k+l}$$

$$= \sum_{l=1}^{n} \frac{1 + q^l}{[l]^{2a+2}_q} \binom{n}{n+l} (-1)^{l-1} q^{l(l-1)/2+(a+1)l}$$

and the formula is proved.

We prove the second identity also by induction on $a$. For $a = 0$ its validity follows from (16). Assume the formula holds for $a > 0$. Then by the induction assumption and formula (17), we easily obtain

$$H_n^*[[2]^{a+1}, 1] = \sum_{k=1}^{n} \frac{q^k}{[k]^2_q} H_n^*[[2]^a, 1] = \sum_{k=1}^{n} \frac{q^k}{[k]^2_q} \sum_{l=1}^{k} \frac{1 + q^l}{[l]^{2a+1}_q} \binom{k}{k+l} q^{l^2+al}$$

$$= \sum_{l=1}^{n} \frac{1 + q^l}{[l]^{2a+3}_q} \sum_{k=l}^{n} \frac{q^k}{[k]^2_q} \binom{k}{k+l} = \sum_{l=1}^{n} \frac{1 + q^l}{[l]^{2a+3}_q} \binom{n}{n+l} q^{l^2+(a+1)l},$$

as required.

Finally, to prove (25), we rewrite it in the form
\[ H_n^*[\{2\}^a, 1, \{2\}^b] = -\sum_{k=1}^{n} \frac{(-1)^k A_{n,k} q^{(a+b+1)k}}{[k]_q^{2(a+b)+1}} - \sum_{k=1}^{n} \frac{A_{n,k} q^{k(k+1)/2+ak}}{[k]_q^{2a+1}} V_{k-1}(2b), \]

where

\[ A_{n,k} = (1 + q^k) \frac{\binom{n}{k}}{\binom{n+k}{k}} q^{k(k-1)/2}, \quad \text{and} \quad V_k(2s) = \frac{(-1)^j (1 + q^j) q^{sj-j(j+1)/2}}{[j]_q^s}. \]

and proceed by induction on \( n \). For \( n = 1 \) the formula is true, since \( H_1^*[\{2\}^a, 1, \{2\}^b] = q^{a+b+1} \). For \( n > 1 \) we use the equality

\[ H_n^*[\{2\}^a, 1, \{2\}^b] = \sum_{l=0}^{a} \frac{q^{n(a-l)}}{[n]_q^{2(a-l)}} H_{n-1}^*[\{2\}^l, 1, \{2\}^b] + q^{(a+1)n} \frac{H_n^*[\{2\}^b]}{[n]_q^{2a+1}} \]

and apply the induction assumption and formula (23) to get

\[ H_n^*[\{2\}^a, 1, \{2\}^b] = -\sum_{l=0}^{a} \frac{q^{n(a-l)}}{[n]_q^{2(a-l)}} \sum_{k=1}^{n-1} \frac{(-1)^k A_{n-1,k} q^{(b+l+1)k}}{[k]_q^{2(b+l)+1}} - \sum_{l=0}^{a} \frac{q^{n(a-l)}}{[n]_q^{2(a-l)}} \sum_{k=1}^{n-1} \frac{A_{n-1,k} q^{k(k+1)/2+lk} V_{k-1}(2b)}{[k]_q^{2l+1}} - \frac{q^{(a+1)n}}{[n]_q^{2a+1}} \sum_{k=1}^{n} \frac{(-1)^k A_{n,k} q^{bk}}{[k]_q^{2b}}. \]

Changing the order of summation, summing over \( l \), and noting that

\[ A_{n-1,k} \sum_{l=0}^{a} \frac{[n]_q^{2l}}{[k]_q^{2a}} q^{(k-n)l} = A_{n,k} \left( \frac{[n]_q^{2a}}{[k]_q^{2a}} q^{(k-n)a} - \frac{[k]_q^{2}}{[n]_q^{2}} q^{n-k} \right) \]

we obtain

\[ H_n^*[\{2\}^a, 1, \{2\}^b] = -\sum_{k=1}^{n} \frac{(-1)^k A_{n,k} q^{(a+b+1)k}}{[k]_q^{2(a+b)+1}} - \sum_{k=1}^{n} \frac{A_{n,k} q^{k(k+1)/2+ak}}{[k]_q^{2a+1}} V_{k-1}(2b) \]

\[ + \frac{q^{(a+1)n}}{[n]_q^{2a+2}} \sum_{k=1}^{n} A_{n,k} q^{k(k-1)/2} [k]_q V_{k-1}(2b) \]

\[ + \frac{q^{(a+1)n}}{[n]_q^{2a+2}} \sum_{k=1}^{n} \frac{(-1)^k A_{n,k} q^{bk}}{[k]_q^{2b-1}} - \frac{q^{(a+1)n}}{[n]_q^{2a+1}} \sum_{k=1}^{n} \frac{(-1)^k A_{n,k} q^{bk}}{[k]_q^{2b}}. \]
Noting that by (15),
\[(1 + q^j) \sum_{k=j+1}^{n} A_{n,k} q^{k(k-1)/2} [k]_q = A_{n,j} ([n]_q - [j]_q) q^{j(j+1)/2},\]
we can simplify the double sum on the right-hand side of (27) to get
\[
\sum_{k=1}^{n} A_{n,k} q^{k(k-1)/2} [k]_q V_{k-1}[2b] = \sum_{j=1}^{n} \frac{(-1)^j (1 + q^j) q^{b_j-j(j+1)/2}}{[j]_q^{2b}} \times \sum_{k=j+1}^{n} A_{n,k} q^{k(k-1)/2} [k]_q
\]
\[
= \sum_{j=1}^{n} \frac{(-1)^j A_{n,j} q^{b_j} ([n]_q - [j]_q)}{[j]_q^{2b}} + \sum_{j=1}^{n} \frac{(-1)^j A_{n,j} q^{b_j}}{[j]_q^{2b-1}}.
\]
(28)

Now from (27) and (28) we conclude the proof. □

The next two theorems generalize identities (23)–(25) to strings of arbitrary collections of twos and ones.

Let \(m, n\) be nonnegative integers and \(s = (s_1, \ldots, s_m) \in \mathbb{Z}^m\), \(t = (t_1, \ldots, t_m) \in \mathbb{Z}^m\). Define the multiple nested sum
\[
\hat{\mathcal{H}}_n[s; t] = \hat{\mathcal{H}}_n[s_1, \ldots, s_m; t_1, \ldots, t_m] = \sum_{k=1}^{n} \binom{n}{k} q^{k^2 + (t_1-1)k} (1 + q^k) \mathcal{H}_{k-1}[s_2, \ldots, s_m; t_2, \ldots, t_m],
\]
and put \(\hat{\mathcal{H}}_n[\emptyset; t] = 0\) if \(n < m\), and \(\hat{\mathcal{H}}_n[\emptyset] = 1\) for all \(n \geq 0\) and \(m = 0\).

**Theorem 3.2** Let \(m \in \mathbb{N}\), \(s_1, \ldots, s_m \in \mathbb{N}_0\). Then for any positive integer \(n\),
\[
H_n^*[\{2\}^{s_1}, 1, \{2\}^{s_2}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{p=(2s_1+1) \circ (2s_2+1) \circ \cdots \circ (2s_m+1)} \hat{\mathcal{H}}_n[p; \overline{p}], \tag{29}
\]
where \(\circ\) is either comma or plus, and the string \(\overline{p} := (s_1 + 1) \circ (s_2 + 1) \circ \cdots \circ (s_m + 1)\) is associated with the string \(p\). This means that the choice of commas and pluses in \(p = (2s_1 + 1) \circ (2s_2 + 1) \circ \cdots \circ (2s_m + 1)\) and \(\overline{p} = (s_1 + 1) \circ (s_2 + 1) \circ \cdots \circ (s_m + 1)\) is the same.

**Proof** Note that for \(m = 1\), the theorem is true by (24). We proceed by induction on \(n + m\). If \(n = 1\), then \(H_1^*[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] = q^{s_1 + \cdots + s_m + m}\). On the other hand,
\( \hat{\mathcal{H}}_1[p_1, \ldots, p_r; \tilde{p}_1, \ldots, \tilde{p}_r] = 0 \) if \( r > 1 \) and \( \hat{\mathcal{H}}_1[p_1; \tilde{p}_1] = \hat{\mathcal{H}}_1[2s_1 + \cdots + 2s_m + m; s_1 + \cdots + s_m + m] = q^{s_1+\cdots+s_m+m} \), and therefore the equality in (29) holds trivially for \( n = 1 \) and any \( m \geq 1 \).

Now assume that \( n > 1 \), \( m \geq 2 \) and apply the expansion

\[
H^*_n[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{l=0}^{s_1} \frac{q^{n(s_1-l)}}{[n]_q^{2(s_1-l)}} H^*_{n-1}[\{2\}, 1, \{2\}^{s_2}, 1, \ldots, \{2\}^{s_m}, 1]
+ \frac{q^{(s_1+1)n}}{[n]_q^{s_1+1}} H^*_{n}[\{2\}^{s_2}, 1, \ldots, \{2\}^{s_m}, 1],
\]

then by induction, we have

\[
H^*_n[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{l=0}^{s_1} \frac{q^{n(s_1-l)}}{[n]_q^{2(s_1-l)}} \sum_{p=(2l+1)\circ(2s_2+1)\circ\ldots\circ(2s_m+1)} \hat{\mathcal{H}}_{n-1}[p; \tilde{p}]
+ \frac{q^{(s_1+1)n}}{[n]_q^{s_1+1}} \sum_{p=(2s_2+1)\circ\ldots\circ(2s_m+1)} \hat{\mathcal{H}}_{n}[p; \tilde{p}].
\]

Next, note that the set of all strings \((2l+1)\circ(2s_2+1)\circ\ldots\circ(2s_m+1)\) falls naturally into two nonintersecting classes \(K_1\) and \(K_2\) corresponding to the fixed choice of the first \(\circ\) as comma or as plus, respectively. Then any string from \(K_1\) has the form \(p = (2l+1, p_2, \ldots, p_r)\), where \((p_2, \ldots, p_r) = (2s_2+1)\circ\ldots\circ(2s_m+1)\), and any string from the second class \(K_2\) is given by \(p = (2l+2s_2+2+t_1, t_2, \ldots, t_u)\), where \((t_1, t_2, \ldots, t_u) = 0 \circ(2s_3+1)\circ\ldots\circ(2s_m+1)\).

Now considering the double sum from (30) and splitting the inner sum into two parts in accordance with the above subdivision of strings, we have

\[
\sum_{p=(2l+1)\circ(2s_2+1)\circ\ldots\circ(2s_m+1)} \hat{\mathcal{H}}_{n-1}[p; \tilde{p}] = \sum_{p=(2s_2+1)\circ\ldots\circ(2s_m+1)} \hat{\mathcal{H}}_{n-1}[2l+1, p; l+1, \tilde{p}]
+ \sum_{t=0\circ(2s_3+1)\circ\ldots\circ(2s_m+1)} \hat{\mathcal{H}}_{n-1}[2l+2s_2+2+t_1, t_2, \ldots, t_u; l+s_2, 2+t_1, t_2, \ldots, t_u],
\]

where \(\tilde{0} = 0\). Substituting (31) in (30) and summing over \(l\) by the formula

\[
\frac{[n-1]_q}{[n-1+k]_q} \sum_{l=0}^{s_1} \frac{[n]_q^{2l}}{[k]_q^{2l}} q^{(k-n)l} = \frac{[\ell]_q}{[n+k]_q} \left( \frac{[n]_q^{2s_1}}{[k]_q^{2s_1}} q^{(k-n)s_1} - \frac{[k]_q^{2s_1}}{[n]_q^{2s_1}} q^{n-k} \right),
\]

we obtain

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\[ H^*_{n}[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] = \sum_{p=(2s_2+1)\cdots o(2s_m+1)} \hat{H}_n[2s_1 + 1, p; s_1 + 1, \bar{p}] \]

\[ = q^{n(s_1+1)} \left[ \begin{array}{c} n \\ 2s_1+1 \end{array} \right] \sum_{p=(2s_2+1)\cdots o(2s_m+1)} \hat{H}_n[-1, p; 0, \bar{p}] \]

\[ + q^{n(s_1+1)} \left[ \begin{array}{c} n \\ 2s_1+2 \end{array} \right] \sum_{p=(2s_2+1)\cdots o(2s_m+1)} \hat{H}_n[p; \bar{p}] \]

\[ + \sum_{t=0o(2s_3+1)\cdots o(2s_m+1)} \hat{H}_n[2s_2 + t_1, t_2, \ldots, t_u; s_2 + 1 + \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_u] \]

Noticing that the first and fourth sums on the right-hand side of (32) add up to

\[ \sum_{p=(2s_1+1)\cdots o(2s_m+1)} \hat{H}_n[p; \bar{p}], \]

we have

\[ H^*_{n}[\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1] - \sum_{p=(2s_1+1)\cdots o(2s_m+1)} \hat{H}_n[p; \bar{p}] \]

\[ = q^{n(s_1+1)} \left[ \begin{array}{c} n \\ 2s_1+1 \end{array} \right] \sum_{p=(2s_2+1)\cdots o(2s_m+1)} \hat{H}_n[-1, p; 0, \bar{p}] \]

\[ - q^{n(s_1+1)} \left[ \begin{array}{c} n \\ 2s_1+2 \end{array} \right] \sum_{p=(2s_2+1)\cdots o(2s_m+1)} \hat{H}_n[-1, p; 0, \bar{p}] \]

\[ - q^{n(s_1+1)} \left[ \begin{array}{c} n \\ 2s_1+2 \end{array} \right] \sum_{t=0o(2s_3+1)\cdots o(2s_m+1)} \hat{H}_n[2s_2 + t_1, t_2, \ldots, t_u; s_2 + 1 + \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_u]. \]

(33)

Finally, expanding \( \hat{H}_n[-1, p; 0, \bar{p}] \), rearranging the order of summation, and applying (15), we obtain

\[ \hat{H}_n[-1, p; 0, \bar{p}] = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \left[ \begin{array}{c} n+k \\ k \end{array} \right] \left( 1 + q^k \right) \]

\[ \times \sum_{l=1}^{k-1} \frac{q^{(p_1-1)l}(1 + q^l)}{[l]_{q}^{p_1}} \mathcal{H}_{l-1}[p_2, \ldots, p_r; \bar{p}_2, \ldots, \bar{p}_r] \]
\[
\sum_{l=1}^{n} \frac{q^{(p_{l-1})l}(1 + q^l)}{[l]_q^{p_l}} \mathcal{H}_{l-1}[p_2, \ldots, p_r; \bar{p}_2, \ldots, \bar{p}_r]
\times \sum_{k=1}^{n} \frac{[n]_q}{[n+k+k]_q} [k]_q q^{k^2-k} (1 + q^k)
= [n]_q \mathcal{H}_n[p; \bar{p}] - \mathcal{H}_n[p_1 - 1, p_2, \ldots, p_r; \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_r].
\]

Substituting the last expression in (33) and simplifying, we have
\[
H_n^*([2]^{s_1}, 1, \ldots, [2]^{s_m}, 1) - \sum_{p=(2s_1+1) \circ \cdots \circ (2s_m+1)} \mathcal{H}_n[p; \bar{p}]
= \frac{q^{n(s_1+1)}}{[n]_q^{2s_1+2}} \sum_{p=(2s_2+1) \circ \cdots \circ (2s_m+1)} \mathcal{H}_n[p_1 - 1, p_2, \ldots, p_r; \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_r]
- \frac{q^{n(s_1+1)}}{[n]_q^{2s_1+2}} \sum_{t=0}^{m} \mathcal{H}_n[2s_2 + t_1, t_2, \ldots, t_u; s_2 + 1 + t_1, t_2, \ldots, t_u].
\]

(34)

It is easy to see that the last two sums in (34) are equal and we obtain the required identity.

In the next theorem, we consider two-one strings ending with 2.

Let \(m\) be a nonnegative integer and \(s = (s_1, \ldots, s_m) \in \mathbb{Z}^m, t = (t_1, \ldots, t_m) \in \mathbb{Z}^m\).

We define the nested sum
\[
\mathcal{H}_n[s, t] = \sum_{n \geq k_1 > \cdots > k_m \geq 1} (-1)^k \frac{[n]}{[n+k_1]} \frac{q^{k^2-k}}{[k_1]} \prod_{j=1}^{m} \frac{(1 + q^{k_j}) q^{(t_j-1)k_j}}{[k_j]_q^{s_j}}
\]
and put \(\mathcal{H}_n[s, t] = 0\) if \(n < m\), and \(\mathcal{H}_n[\emptyset] = 1\) for all \(n \geq 0\) and \(m = 0\).

**Theorem 3.3** Let \(m, s_1, \ldots, s_m \in \mathbb{N}_0, s_{m+1} \in \mathbb{N}\). Then for any positive integer \(n\),
\[
H_n^*([2]^{s_1}, 1, \ldots, [2]^{s_m}, 1, [2]^{s_{m+1}}] = - \sum_{p=(2s_1+1) \circ \cdots \circ (2s_m+1) \circ (2s_{m+1})} \mathcal{H}_n[p; \bar{p}],
\]
(35)

where \(\circ\) is either comma or plus, and the string \(\bar{p} := (s_1 + 1) \circ \cdots \circ (s_m + 1) \circ (s_{m+1})\) is associated with the string \(p\). This means that the choice of commas and pluses in \(p = (2s_1+1) \circ \cdots \circ (2s_m+1) \circ (2s_{m+1})\) and \(\bar{p} = (s_1 + 1) \circ \cdots \circ (s_m + 1) \circ (s_{m+1})\) is the same.

Note that for \(m = 0\), identity (35) coincides with (23). For \(m = 1\), the theorem becomes (25). For \(m \geq 2\), the proof of Theorem 3.3 follows exactly by the same reasoning as the proof of Theorem 3.2 and is left to the reader.
Remark 1 Letting $q \to 1$ in (29), we get Zhao’s identity [33, Theorem 2.2]:

$$H'_n(\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1) = \sum_{p=(2s_1+1)\circ\cdots\circ(2s_m+1)} 2^{|p|} \hat{H}_n(p),$$

where

$$H'_n(p) = H'_n(p_1, p_2, \ldots, p_r) = \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_r \geq 1} \frac{1}{k_1^{p_1} k_2^{p_2} \cdots k_r^{p_r}},$$

$$\hat{H}_n(p) = \hat{H}_n(p_1, p_2, \ldots, p_r) = \sum_{n \geq k_1 > k_2 > \cdots > k_r \geq 1} \frac{(n)}{(n+k_1)} \frac{1}{k_1^{p_1} k_2^{p_2} \cdots k_r^{p_r}}$$

are ordinary multiple harmonic sums, and $l(p)$ is the length of the string $p$.

Similarly, letting $q \to 1$ in (35), we get a multiple harmonic sum identity from [33, Theorem 2.4] for two-one strings ending with 2:

$$H'_n(\{2\}^{s_1}, 1, \ldots, \{2\}^{s_m}, 1, \{2\}^{s_{m+1}}) = \sum_{p=(2s_1+1)\circ\cdots\circ(2s_{m+1})} 2^{|p|} \overline{H}_n(p),$$

where

$$\overline{H}_n(p) = \overline{H}_n(p_1, p_2, \ldots, p_r) = \sum_{n \geq k_1 > k_2 > \cdots > k_r \geq 1} \frac{(n)}{(n+k_1)} \frac{(-1)^{k_r-1}}{k_1^{p_1} k_2^{p_2} \cdots k_r^{p_r}},$$

and $l(p) = \sigma(p) + 1$ is the length of the string $p$, and $\sigma(p)$ is the number of commas in $p = (2s_1+1) \circ \cdots \circ (2s_m+1) \circ (2s_{m+1})$.

4 Applications to $q$-zeta values

In this section, we prove Theorems 1.1 and 1.2. For this purpose, we need a lemma that justifies limit transition from finite $q$-binomial identities for multiple harmonic sums to corresponding relations for multiple $q$-zeta values.

Lemma 4.1 Let $0 < q < 1$, $c, c_1, c_2 \in \mathbb{R}$, $c > 0$, and let $R_k$ be a sequence of real numbers satisfying $|R_k| < k^{c_1} q^{c_2 k}$ for all $k = 1, 2, \ldots$. Then

$$\lim_{n \to \infty} \sum_{k=1}^{n} q^{c_2 k} \left(1 - \frac{\left[n\right]}{\left[n+k\right]}ight) R_k = 0.$$

Proof First, we notice that for $1 \leq k \leq n/2$ and $n$ sufficiently large,
\[
1 - \frac{[n]}{[n+k]} = 1 - \frac{(q; q)_n^2}{(q; q)_{n-k}(q; q)_{n+k}}
\]
\[
= 1 - \frac{(1 - q^{n-k+1})(1 - q^{n-k+2}) \cdots (1 - q^n)}{(1 - q^{n+1})(1 - q^{n+2}) \cdots (1 - q^{n+k})} = O(q^{c_3 n}),
\]

where \(c_3\) is some positive constant independent of \(n\). Therefore, we have

\[
\left| n^{\frac{n}{2}} \sum_{k=1}^{n/2} q^{ck^2} \left( 1 - \frac{[n]}{[n+k]} \right) R_k \right| < O(q^{c_3 n}) \sum_{k=1}^{n/2} q^{c k^2} |R_k| < O(q^{c_3 n}) \sum_{k=1}^{n/2} k^{c_2} q^{ck^2+c_1 k}.
\]

On the other side, for \(n/2 \leq k \leq n\), we can apply the trivial inequality

\[
0 < 1 - \frac{[n]}{[n+k]} = 1 - \frac{(1 - q^{n-k+1})(1 - q^{n-k+2}) \cdots (1 - q^n)}{(1 - q^{n+1})(1 - q^{n+2}) \cdots (1 - q^{n+k})} < 1
\]

to obtain

\[
\left| \sum_{k=n/2}^{n} q^{ck^2} \left( 1 - \frac{[n]}{[n+k]} \right) R_k \right| < q^{cn^2/4} \sum_{k=n/2}^{n} |R_k| < n^{c_4} q^{cn^2/4 + c_5 n}, \tag{39}
\]

where \(c_4, c_5\) are some real constants independent of \(n\). Now letting \(n\) tend to infinity, by (38) and (39), we conclude the proof.

Now Theorems 1.1 and 1.2 easily follow from Theorems 3.2 and 3.3 by Lemma 4.1.

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