THE ADJOINT REIDEMEISTER TORSION
FOR THE CONNECTED SUM OF KNOTS

JOAN PORTI AND SEOKBEO M YOON

Abstract. Let $K$ be the connected sum of knots $K_1, \ldots, K_n$. It is known that the $\text{SL}_2(\mathbb{C})$-character variety of the knot exterior of $K$ has a component of dimension $\geq 2$ as the connected sum admits a so-called bending. We show that there is a natural way to define the adjoint Reidemeister torsion for such a high-dimensional component and prove that it is locally constant on a subset of the character variety where the trace of a meridian is constant. We also prove that the adjoint Reidemeister torsion of $K$ satisfies the vanishing identity if each $K_i$ does so.

1. Introduction

Let $M$ be a compact oriented 3-manifold with torus boundary and $\mathcal{X}(M)$ be the character variety of irreducible representations $\pi_1(M) \to \text{SL}_2(\mathbb{C})$. It happens very often that $\mathcal{X}(M)$ has a component of dimension 1. For instance, if the interior of $M$ admits a hyperbolic structure of finite volume, then the distinguished component is 1-dimensional [Thu79] and if $M$ contains no closed essential surface, then every component is 1-dimensional [CCG+94].

Once we fix a simple closed curve $\mu$ on the boundary torus $\partial M$, the adjoint Reidemeister torsion is defined as a meromorphic function on each 1-dimensional component of $\mathcal{X}(M)$ under a mild assumption [Por97, Dub03]. It enjoys fruitful interaction with quantum field theory and carries several conjectures consequently. See, for instance, [DG13, OT15, GKZ]. Recently, it is conjectured in [GKY] that the adjoint Reidemeister torsion satisfies a certain vanishing identity with respect to the trace function as follows.

Conjecture 1.1. Suppose that the character variety $\mathcal{X}(M)$ consists of 1-dimensional components and the interior of $M$ admits a hyperbolic structure of finite volume. Then for generic $c \in \mathbb{C}$ we have

$$\sum_{[\rho] \in \mathcal{X}_c^\mu(M)} \frac{1}{\tau_{\mu}(M; \rho)} = 0$$

where $\mathcal{X}_c^\mu(M)$ is the pre-image of $c \in \mathbb{C}$ under the trace function $\mathcal{X}(M) \to \mathbb{C}$ of $\mu \subset \partial M$ and $\tau_{\mu}(M; \rho)$ is the adjoint Reidemeister torsion associated to $\mu$ and a representation $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$.

As mentioned earlier, there are several 3-manifolds satisfying the conditions required in Conjecture 1.1. However, there are also several examples of 3-manifolds with torus boundary whose character varieties have high-dimensional components. The simplest one might be (the knot exterior of) the connected sum of knots. We refer to [CL96, PP13, Che21] for other examples. Two immediate problems when we consider Conjecture 1.1 for such 3-manifolds are that

(P1) the adjoint Reidemeister torsion is not defined for a component of dimension $\geq 2$;

(P2) the sum in the equation (1) does not make sense as the level set $\mathcal{X}_c^\mu(M)$ is no longer finite.

Related to these problems, we address the following question.

Question 1.2. Is the adjoint Reidemeister torsion defined and locally constant on $\mathcal{X}_c^\mu(M)$?

If the answer of Question 1.2 is positive, then the sum in the equation (1) makes sense for $M$ in an obvious way: by taking one representative on each connected component of $\mathcal{X}_c^\mu(M)$.

The main purpose of the paper is to investigate Question 1.2 and Conjecture 1.1 for the connected sum of knots. Let $K$ be the connected sum of knots $K_1, \ldots, K_n$ in $S^3$ and $\mu$ be a meridian. We denote by $M$ and $M_j$ the knot exteriors of $K$ and $K_j$, respectively. For technical reasons, we assume that for $1 \leq j \leq n$

(C) the level set $\mathcal{X}_c^\mu(M_j)$ consists of finitely many $\mu$-regular characters with the canonical longitude having trace other than $\pm 2$ for generic $c \in \mathbb{C}$.

For example, one may choose $K_j$ as a two-bridge knot or a torus knot. It is known that the character variety $\mathcal{X}(M)$ has a component of dimension $\geq 2$ as the connected sum admits a so-called bending. We refer to [JM87, PP13, KN20] for details on the bending construction.
Theorem 1.3. Let $K$ be the connected sum of knots $K_1, \ldots, K_n$ satisfying the above condition (C) and $\mu$ be a meridian. Then there is a natural way to define the adjoint Reidemeister torsion on $X^c_{\mu}(M)$ for generic $c \in \mathbb{C}$ which is locally constant.

Theorem 1.4. Let $K$ be the connected sum of knots $K_1, \ldots, K_n$ satisfying the above condition (C) and $\mu$ be a meridian. Then the knot exterior $M$ of $K$ satisfies the equation (1) if each $M_j$ does so.

It is proved in [Yoo20] that every hyperbolic two-bridge knot satisfies the equation (1) for a meridian. We thus obtain the following corollary.

Corollary 1.5. The knot exterior of the connected sum of hyperbolic two-bridge knots satisfies the equation (1) for a meridian.

Remark 1.6. Conjecture 1.1 was derived from the 3d-3d correspondence under the assumption that the interior of $M$ admits a hyperbolic structure. We refer to [GKY, Section 3] for details. It fails without the assumption since torus knot exteriors do not satisfy the equation (1). However, Theorem 1.4 and Corollary 1.5 suggest that one can relax the hyperbolicity condition, as the connected sum of knots is never hyperbolic.

The paper is organized as follows. In Section 2, we briefly recall basic definitions on the sign-refined Reidemeister torsion. We define the adjoint Reidemeister torsion for the connected sum of knots in Sections 3.1 and 3.2, and prove Theorems 1.3 and 1.4 in Section 3.3.

2. Review on the sign-refined Reidemeister torsion

2.1. The Reidemeister torsion of a chain complex. Let $C_\ast$ be a chain complex of vector spaces over a field $F$ 

$$C_\ast = (0 \to C_n \xrightarrow{\partial_n} \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0)$$

and $H_\ast(C_\ast)$ be the homology of $C_\ast$. For a basis $c_\ast$ of $C_\ast$ and a basis $h_\ast$ of $H_\ast(C_\ast)$ the Reidemeister torsion is defined as follows. Here and throughout the paper, every basis and tuple is assumed to be ordered. For $0 \leq i \leq n$ we choose a lift $\tilde{h}_i$ of $h_i$ to $C_i$ and a tuple $b_i$ of vectors in $C_i$ such that $\partial_i b_i$ is a basis of $\partial_i C_i$. Then the tuple $c'_i = (\partial_{i+1} b_{i+1}, \tilde{h}_i, b_i)$ is another basis of $C_i$. Letting $A_i$ be the basis transition matrix taking $c_i$ to $c'_i$, we have

$$\text{tor}(C_\ast, c_\ast, h_\ast) = \prod_{i=0}^n \det A_i^{(-1)^{i+1}} \in F^*.$$ 

Also, the sign-refined Reidemeister torsion is defined as

$$\text{Tor}(C_\ast, c_\ast, h_\ast) = (-1)^{|C_\ast|} \text{tor}(C_\ast, c_\ast, h_\ast) \in F^*, \quad |C_\ast| = \sum_{i=0}^n \alpha_i(C_\ast) \beta_i(C_\ast)$$

where $\alpha_i(C_\ast) = \sum_{j=0}^i \dim C_j$ and $\beta_i(C_\ast) = \sum_{j=0}^i \dim H_j(C_\ast)$.

Suppose that we have a short exact sequence of chain complexes

$$0 \to C'_\ast \to C_\ast \to C''_\ast \to 0 \tag{2}$$

with bases $c_\ast, c'_\ast$, and $c''_\ast$ of $C_\ast, C'_\ast$, and $C''_\ast$, respectively. It is proved in [Tur86, Lemma 3.4.2] that if $c_\ast, c'_\ast$, and $c''_\ast$ are compatible with respect to the sequence (2), i.e., $c_\ast = (c'_\ast, c''_\ast)$, then

$$\text{Tor}(C_\ast, c_\ast, h_\ast) = (-1)^{v+u} \text{Tor}(C'_\ast, c'_\ast, h'_\ast) \text{Tor}(C''_\ast, c''_\ast, h''_\ast) \text{tor}(H) \tag{3}$$

where $h_\ast, h'_\ast$, and $h''_\ast$ are bases of $H_\ast(C_\ast), H_\ast(C'_\ast)$, and $H_\ast(C''_\ast)$, respectively. Here

$$v = \sum_i \alpha_{i-1}(C'_\ast) \alpha_i(C''_\ast), \tag{4}$$

$$u = \sum_i ((\beta_i(C_\ast) + 1)(\beta_i(C'_\ast) + \beta_i(C''_\ast)) + \beta_{i-1}(C'_\ast) \beta_i(C''_\ast)), \tag{5}$$

and tor($H$) is the Reidemeister torsion of the long exact sequence induced from (2) with respect to $h_\ast, h'_\ast$, and $h''_\ast$. We refer to [Tur86, Tur02] for details.
2.2. The adjoint Reidemeister torsion of a CW-complex. Let \( \mathfrak{g} \) be the Lie algebra of \( SL_2(\mathbb{C}) \) and fix a basis of \( \mathfrak{g} \) as
\[
e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
Note that the Killing form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) is given by
\[
\langle \begin{pmatrix} b & a \\ c & -b \end{pmatrix}, \begin{pmatrix} b' & a' \\ c' & -b' \end{pmatrix} \rangle = 8bb' + 4ac' + 4ca' .
\]

Let \( X \) be a finite CW-complex and \( \rho : \pi_1(X) \to SL_2(\mathbb{C}) \) be a representation. We consider a cochain complex
\[
C^\ast(X; \mathfrak{g}_\rho) = \text{Hom}_{\mathbb{Z}[\pi_1 X]} \left( C_\ast(\tilde{X}; \mathbb{Z}), \mathfrak{g} \right)
\]
where \( \tilde{X} \) is the universal cover of \( X \). Here \( \mathfrak{g} \) is viewed as a \( \mathbb{Z}[\pi_1 X] \)-module through the adjoint action \( \text{Ad} \rho : \pi_1(X) \to \text{Aut}(\mathfrak{g}) \) associated to \( \rho \). We denote the cohomology of \( C^\ast(X; \mathfrak{g}_\rho) \) by \( H^\ast(X; \mathfrak{g}_\rho) \) and call it the twisted cohomology. Note that \( H^0(X; \mathfrak{g}_\rho) \) coincides with the set of invariant vectors in \( \mathfrak{g} \) under the \( \pi_1(X) \)-action.

Let \( \Sigma \) be a 2-torus with a usual CW-structure: one 0-cell \( p \), two 1-cells \( \mu \) and \( \lambda \), and one 2-cell \( \Sigma \) as in Figure 1 (left). We choose their lifts (to the universal cover of \( \Sigma \)) as in Figure 1 (right) and fix an order of the cells by \( c_i \leftrightarrow e_k \) for \( 1 \leq i \leq n \) and \( 1 \leq k \leq 3 \) by assigning \( \tilde{c}_i \leftrightarrow e_k \) and every cell of \( \tilde{X} \) that is not a lift of \( c_i \) to 0. Then the tuple
\[
c_X = \left( c_1^{(1)}, c_1^{(2)}, c_1^{(3)}, \ldots, c_n^{(1)}, c_n^{(2)}, c_n^{(3)} \right)
\]
is a basis of \( C^\ast(X; \mathfrak{g}_\rho) \). We refer to it as the geometric basis.

Let \( C_* (\Sigma; \mathbb{R}) \) be the ordinary chain complex of \( X \) with the real coefficient. Note that the tuple \( c_X \) is a basis of \( C_* (\Sigma; \mathbb{R}) \). For an orientation \( o_X \) of the \( \mathbb{R} \)-vector space \( H_* (\Sigma; \mathbb{R}) \) we define
\[
\epsilon(o_X) = \text{sgn}(\text{Tor}(C_* (\Sigma; \mathbb{R}), c_X, h_X)) \in \{ \pm 1 \}
\]
where \( h_X \) is any basis of \( H_* (\Sigma; \mathbb{R}) \) positively oriented with respect to \( o_X \) and \( \text{sgn}(x) \) is the sign of \( x \in \mathbb{R}^* \).

**Definition 2.1.** For a basis \( h_X \) of \( H^\ast(X; \mathfrak{g}_\rho) \) and an orientation \( o_X \) of \( H_* (\Sigma; \mathbb{R}) \) the adjoint Reidemeister torsion is defined as
\[
\tau(X; \rho, h_X, o_X) = \epsilon(o_X) \cdot \text{Tor}(C^\ast(X; \mathfrak{g}_\rho), c_X, h_X) \in \mathbb{C}^* .
\]
The above definition does not depend on the order, orientations, and lifts of \( c_i \)'s. Moreover, it does not depend on the choice of a basis of \( \mathfrak{g} \) if the Euler characteristic of \( X \) is zero

Note that every notion in this section associated to \( \rho \) is invariant under conjugating \( \rho \) up to an appropriate isomorphism. In particular, the adjoint Reidemeister torsion is invariant under the conjugation.

**Example 2.2.** Let \( \Sigma \) be a 2-torus with a usual CW-structure: one 0-cell \( p \), two 1-cells \( \mu \) and \( \lambda \), and one 2-cell \( \Sigma \) as in Figure 1 (left). We choose their lifts (to the universal cover of \( \Sigma \)) as in Figure 1 (right) and fix an order of the cells by \( c_{\Sigma} = (p, \mu, \lambda, \Sigma) \). Let \( o_\Sigma \) be the orientation of \( H_* (\Sigma; \mathbb{R}) \) induced from \( c_{\Sigma} \) so that \( \epsilon(o_\Sigma) = 1 \).

![Figure 1](image-url)
Here $I_k$ is the identity matrix of size $k$. It follows that $\dim H^i(\Sigma; \mathbb{Z}) = 1$ for $i = 0, 2$, $\dim H^i(\Sigma; \mathbb{Z}) = 2$ for $i = 1$, and $\dim H^i(\Sigma; \mathbb{Z}) = 0$ otherwise. Let $P = \frac{1}{\pi} \in H^0(\Sigma; \mathbb{Z})$ and define maps

$$
\psi^0 : C^0(\Sigma; \mathbb{Z}) \to \mathbb{C}, \ \alpha \mapsto \langle \alpha(\overline{\mu}), P \rangle,
$$

$$
\psi^1 : C^1(\Sigma; \mathbb{Z}) \to \mathbb{C}^2, \ \alpha \mapsto \left( \langle \alpha(\overline{\mu}), P \rangle, \langle \alpha(\lambda), P \rangle \right),
$$

$$
\psi^2 : C^2(\Sigma; \mathbb{Z}) \to \mathbb{C}, \ \alpha \mapsto \langle \alpha(\overline{\Sigma}), P \rangle.
$$

One easily checks that each map $\psi^i$ induces an isomorphism $H^i(\Sigma; \mathbb{Z}) \to \mathbb{C}$ ($\mathbb{C}^2$ if $i = 1$). For simplicity we use the same notation $\psi^i$ for these isomorphisms. We choose a basis $h^i_{\Sigma}$ of $H^i(\Sigma; \mathbb{Z})$ by the pre-image of the standard basis of $\mathbb{C}$ ($\mathbb{C}^2$ if $i = 2$) under $\psi^i$. Explicitly, we have $h^0_{\Sigma} = (p(2), h^2_{\Sigma} = (\mu(2), \lambda(2))$, and $h^2_{\Sigma} = (\Sigma(2))$. Choosing a tuple $b^i$ of vectors in $C^i(\Sigma; \mathbb{Z})$ as $b^i = (p(1), p(3))$, $b^i = (\lambda(1), \lambda(3))$, and $b^2 = \emptyset$, we obtain

$$
\tau(\Sigma; \rho, h_{\Sigma}, o_{\Sigma}) = -1 \cdot (m^2 - 1)(m^2 - 1) \cdot (-m^2 - 1)(m^2 - 1)^{-1} = 1.
$$

Note that a different choice of $P \in H^0(\Sigma; \mathbb{Z})$ changes the basis $h_{\Sigma}$ but still we have $\tau(\Sigma; \rho, h_{\Sigma}, o_{\Sigma}) = 1$.

2.3. The adjoint Reidemeister torsion of a knot exterior. Let $M$ be the knot exterior of a knot $K \subset S^3$ with any given triangulation. It is well-known that $\dim H_i(M; \mathbb{R}) = 1$ for $i = 0, 1$ and $\dim H_i(M; \mathbb{R}) = 0$ otherwise. We choose the orientation $o_M$ of $H_i(M; \mathbb{R})$ induced from a basis $h_M = (\rho, \mu)$ of $H_1(M; \mathbb{R})$ where $\rho$ is a point in $M$ and $\mu$ is a meridian of $K$ oriented arbitrarily.

Let $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be a representation of the knot group. For the sake of simplicity, we assume that $m \neq \pm 1$ and $\Delta_K(m^2) \neq 0$ where $m$ is an eigenvalue of $\rho(\mu)$ and $\Delta_K$ is the Alexander polynomial of $K$. It follows that if $\rho$ is reducible, then it should be abelian (see e.g. [BZH14]). Therefore, $\rho$ is either irreducible (Section 2.3.1) or abelian (Section 2.3.2).

2.3.1. Irreducible representations. Suppose that $\rho$ is irreducible. In this case we further assume that $\mu$ is $\mu$-regular [Por97, Definition 3.21], i.e., $\dim H^1(M; \mathbb{Z}) = 1$ and the inclusion $\mu : M \to M$ induces an injective map $H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{Z})$. We choose an element $P \in H^0(\Sigma; \mathbb{Z})$, where $\Sigma = \partial M$, and define maps

$$
\psi^1 : C^1(\Sigma; \mathbb{Z}) \to \mathbb{C}, \ \alpha \mapsto \langle \alpha(\overline{\mu}), P \rangle,
$$

$$
\psi^2 : C^2(\Sigma; \mathbb{Z}) \to \mathbb{C}, \ \alpha \mapsto \langle \alpha(\overline{\Sigma}), P \rangle,
$$

where $\overline{\mu}$ and $\overline{\Sigma}$ are lifts of $\mu$ and $\Sigma$ (to the universal cover of $M$) respectively satisfying $\overline{\mu} \subset \overline{\Sigma}$. Here the boundary torus $\Sigma$ is oriented as in Stokes’ theorem. It is proved in [Por97] that the $\mu$-regularity implies that $\psi^i$ induces an isomorphism $H^i(M; \mathbb{Z}) \to \mathbb{C}$ for $i = 1, 2$. We define

$$
\tau_p(M; \rho) = \tau(M; \rho, h_M, o_M)
$$

where $h_M$ is a basis of $H^i(M; \mathbb{Z})$ given by the pre-image of the standard basis of $\mathbb{C}$ under $\psi^i$. Note that a different choice of $P \in H^0(\Sigma; \mathbb{Z})$ changes the basis $h_M$ but not the value of $\tau_p(M; \rho)$.

2.3.2. Abelian representations. Suppose that $\rho$ is abelian. This case might not be that interesting, as it essentially reduces to the case of Alexander polynomial. We however present explicit setups here for Section 3.

Lemma 2.3. We have $\dim H^i(M; \mathbb{Z}) = 1$ for $i = 0, 1$ and $\dim H^i(M; \mathbb{Z}) = 0$ otherwise.

Proof. We choose any Wirtinger presentation of the knot group $\pi_1(M) = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_{n-1} \rangle$.

Recall that the corresponding 2-dimensional cell complex $X$ consists of one 0-cell $p$, $n$ 1-cells $g_1, \ldots, g_n$, and $n - 1$ 2-cells $r_1, \ldots, r_{n-1}$. It is known that $X$ is simple homotopic equivalent to $M$ and thus we may use $X$ instead of $M$. We choose a lift of the base point $p$ arbitrarily and the lifts of other cells accordingly. Then with respect to the geometric basis, the boundary maps $\delta^0 : C^0(X; \mathbb{Z}) \to C^1(X; \mathbb{Z})$ and $\delta^1 : C^1(X; \mathbb{Z}) \to C^2(X; \mathbb{Z})$ are given as

$$
\delta^0 = \begin{pmatrix} \Phi(g_1 - 1) \\ \vdots \\ \Phi(g_n - 1) \end{pmatrix}, \quad \delta^1 = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial g_1}) & \cdots & \Phi(\frac{\partial r_1}{\partial g_n}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_{n-1}}{\partial g_1}) & \cdots & \Phi(\frac{\partial r_{n-1}}{\partial g_n}) \end{pmatrix}
$$

where $\Phi$ is the $\mathbb{Z}$-linear extension of $\text{Ad}\rho$ and $\partial r_j/\partial g_i$ denotes the Fox free differential. Recall that up to conjugation

$$
\rho(g_1) = \cdots = \rho(g_n) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad m \neq \pm 1.
$$
It is clear that $\text{Im} \delta^0 \cong \mathbb{C}^2$, $\text{Ker} \delta^0 \cong \mathbb{C}$ and $\dim H^0(X; \mathfrak{g}_\rho) = 1$. On the other hand, $\delta^1$ is surjective since $\Delta_K(1) \neq 0$ and $\Delta_K(m^2) \neq 0$. It follows that $\dim H^2(X; \mathfrak{g}_\rho) = 0$ and $\dim H^1(X; \mathfrak{g}_\rho) = 1$ since the Euler characteristic of $X$ is zero. Explicitly, the twisted cohomology of $X$ is generated by

$$(7) \quad C^0(X; \mathfrak{g}_\rho) \ni \alpha \text{ s.t. } \alpha(\bar{p}) = e_2, \quad C^1(X; \mathfrak{g}_\rho) \ni \alpha \text{ s.t. } \alpha(\bar{g}_i) = e_2 \ \forall i.$$ 

Once again, we choose an element $P \in H^0(\Sigma; \mathfrak{g}_\rho) = H^0(M; \mathfrak{g}_\rho)$ and define

$$\psi^0 : C^0(M; \mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\bar{p}), P \rangle,$$

$$\psi^1 : C^1(M; \mathfrak{g}_\rho) \to \mathbb{C}, \quad \alpha \mapsto \langle \alpha(\bar{\mu}), P \rangle,$$

where $\bar{p}$ and $\bar{\mu}$ are lifts of $p$ and $\mu$ (to the universal cover of $M$) respectively satisfying $\bar{p} \subset \bar{\mu}$. It is clear from the equation (7) that $\psi^i$ induces an isomorphism $H^i(M; \mathfrak{g}_\rho) \to \mathbb{C}$ for $i = 0, 1$. We define

$$\tau_\mu(M; \rho) = \tau(M; \rho, h_M, o_M)$$

where $h^1_M$ is a basis of $H^1(M; \mathfrak{g}_\rho)$ given by the pre-image of the standard basis of $\mathbb{C}$ under $\psi^i$. In fact, one can compute that

$$\tau_\mu(M; \rho) = \frac{\Delta_K(m^2) \Delta_K(m^{-2})}{(m - m^{-1})^2}$$

up to sign, but we would not use this fact in this paper.

3. The connected sum of knots

Let $K$ be the connected sum of knots $K_1, \ldots, K_n$ in $S^3$. We denote by $M$ and $M_j$ the knot exteriors of $K$ and $K_j$, respectively. It is known that the JSJ decomposition of $M$ consists of a composing space and $M_1, \ldots, M_n$.

3.1. A composing space. Let $D_1, \ldots, D_n$ be mutually disjoint discs in the interior of a disc $D^2$ and let $W = D^2 \setminus \text{int}(D_1 \cup \cdots \cup D_n)$ be a planar surface. Here $\text{int}(X)$ denotes the interior of $X$. A composing space $Y$ is a compact 3-manifold $W \times S^1$ having $n + 1$ boundary tori $\Sigma_j = \partial D_j \times S^1$ ($1 \leq j \leq n$) and $\Sigma = \partial D^2 \times S^1$. Letting $\mu = \{pt\} \times S^1$ and $\lambda_j = \partial D_j \times \{pt\}$, we have

$$\pi_1(Y) = \{\mu, \lambda_1, \ldots, \lambda_n | [\mu, \lambda_1] = \cdots = [\mu, \lambda_n] = 1\}.$$ 

One can check that $H_0(Y; \mathbb{R}) \cong \mathbb{R}$ is generated by a point $p \in Y$, $H_1(Y; \mathbb{R}) \cong \mathbb{R}^{n+1}$ is generated by $\mu, \lambda_1, \ldots, \lambda_n$, and $H_2(Y; \mathbb{R}) \cong \mathbb{R}^n$ is generated by $\Sigma_1, \ldots, \Sigma_n$. We choose the orientation $\gamma_Y$ of $H_*(Y; \mathbb{R})$ induced from a basis $h_Y = (p, \mu, \lambda_1, \ldots, \lambda_n, \Sigma_1, \ldots, \Sigma_n)$ of $H_*(Y; \mathbb{R})$. Here we orient $\mu, \lambda_j$, and $\Sigma_j$ as in Example 2.2 and Stokes’ theorem.

Let $\rho : \pi_1(Y) \to \text{SL}_2(\mathbb{C})$ be a representation with $\text{tr} \rho(\mu) \neq \pm 2$ and $\text{tr} \rho(\lambda_j) \neq \pm 2$ for some $1 \leq j \leq n$. Since $\mu$ commutes with all $\lambda_j$’s, we have up to conjugation

$$\rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\lambda_j) = \begin{pmatrix} l_j & 0 \\ 0 & l_j^{-1} \end{pmatrix}$$

for some $m \neq \pm 1$ and $l_j \in \mathbb{C}^*$. Note that there is no relation among $m, l_1, \ldots, l_n$.

**Proposition 3.1.** We have

$$(9) \quad \dim H^i(Y; \mathfrak{g}_\rho) = \begin{cases} 1 & i = 0, \\ n + 1 & i = 1, \\ n & i = 2, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** We first compute the twisted cohomology of $W$. Since $W$ retracts to the wedge sum $V$ of $n$ circles $\lambda_1, \ldots, \lambda_n$ (with the basepoint $p$), we may consider $V$ instead of $W$:

$$\mathfrak{g} \cong C^0(V; \mathfrak{g}_\rho) \xrightarrow{\delta^0} C^1(V; \mathfrak{g}_\rho) \cong \mathfrak{g}^\mathcal{G}, \quad \delta^0 = \begin{pmatrix} \text{Ad}_\rho(\lambda_1) - I_3 \\ \vdots \\ \text{Ad}_\rho(\lambda_n) - I_3 \end{pmatrix}.$$ 

From the equation (8) with the fact that $\text{tr} \rho(\lambda_j) \neq \pm 2$ for some $1 \leq j \leq n$, we have

$$(10) \quad \dim H^i(W; \mathfrak{g}_\rho) = \dim H^i(V; \mathfrak{g}_\rho) = \begin{cases} 1 & i = 0, \\ 3n - 2 & i = 1, \\ 0 & \text{otherwise}. \end{cases}$$
Without loss of generality, we assume that $l_1 \neq \pm 1$ and choose a basis $\mathbf{h}_W^i$ of $H^i(W; \mathfrak{g}_p)$ as
\[
\mathbf{h}_W^0 = p^{(2)}, \quad \mathbf{h}_W^1 = (\lambda_1^{(2)}, \ldots, \lambda_n^{(2)}; \lambda_2^{(1)}, \ldots, \lambda_n^{(1)}; \lambda_2^{(3)}, \ldots, \lambda_n^{(3)})).
\]
Here we choose a lift of $\rho$ arbitrarily and determines the lifts of other cells accordingly. Recall Section 2.2 that the notations $p^{(k)}$ and $\lambda_j^{(k)}$ make sense after we fix lifts of $\rho$ and $\lambda_j$.

We decompose $Y$ into two copies $Y_1$ and $Y_2$ of $W \times I$ where $I$ is an interval. It is clear that both $Y_1$ and $Y_2$ retract to $W$ and $Y_1 \cap Y_2 = W \sqcup W$. From the short exact sequence
\[
0 \to C^*(Y; \mathfrak{g}_p) \to C^*(Y_1; \mathfrak{g}_p) \oplus C^*(Y_2; \mathfrak{g}_p) \to C^*(W; \mathfrak{g}_p) \to 0,
\]
we obtain
\[
\mathcal{H} : 0 \to H^0(Y; \mathfrak{g}_p) \xrightarrow{\partial_0} H^0(W; \mathfrak{g}_p) \oplus H^0(W; \mathfrak{g}_p) \xrightarrow{\partial_0} H^0(W; \mathfrak{g}_p) + H^0(W; \mathfrak{g}_p)
\]
\[
\xrightarrow{d_0} H^1(Y; \mathfrak{g}_p) \xrightarrow{f_1} H^1(W; \mathfrak{g}_p) \oplus H^1(W; \mathfrak{g}_p) \xrightarrow{g_1} H^1(W; \mathfrak{g}_p) + H^1(W; \mathfrak{g}_p) \xrightarrow{d_1} H^2(W; \mathfrak{g}_p) \to 0.
\]
Fixing identifications $H^0(W; \mathfrak{g}_p) \cong \mathbb{C}$ and $H^1(W; \mathfrak{g}_p) \cong \mathbb{C}^{3n-2}$ with respect to $\mathbf{h}_W$, the matrix expressions of $g_0$ and $g_1$ in the sequence (12) are given by
\[
g_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} -I_n & 0 & 0 \\ 0 & -m^2 I_{n-1} & 0 \\ 0 & 0 & -m^{-2} I_{n-1} \end{pmatrix}
\]
\[
\end{pmatrix} = \dim \text{Im} g_0 = \dim \text{Im} f_0 = \dim \ker g_0 = 1,
\]
\[
\dim H^1(W; \mathfrak{g}_p) = 2 \dim H^1(W; \mathfrak{g}_p) - \dim \text{Im} g_1 = n.
\]
Also, we have $\dim H^1(Y; \mathfrak{g}_p) = n + 1$ since the Euler characteristic of $Y$ is zero. \[\square\]

It is geometrically natural to choose a basis $\mathbf{h}_Y^i$ of $H^i(Y; \mathfrak{g}_p)$ as
\[
\mathbf{h}_Y^0 = p^{(2)}, \quad \mathbf{h}_Y^1 = (\mu^{(2)}, \lambda_1^{(2)}, \ldots, \lambda_n^{(2)}), \quad \mathbf{h}_Y^2 = (\Sigma_1^{(2)}, \ldots, \Sigma_n^{(2)}).
\]
Alternatively, we may describe the basis $\mathbf{h}_Y$ as follows (as in Example 2.2). Let $P = \frac{1}{2} \xi e_2 \in H^0(Y; \mathfrak{g}_p)$ and consider isomorphisms
\[
\psi^0 : H^0(Y; \mathfrak{g}_p) \to \mathbb{C}, \quad \alpha \mapsto (\alpha(\tilde{P}), P),
\]
\[
\psi^1 : H^1(Y; \mathfrak{g}_p) \to \mathbb{C}^{n+1}, \quad \alpha \mapsto (\langle \alpha(\tilde{P}), P \rangle, \langle \alpha(\tilde{\lambda}_1), P \rangle, \ldots, \langle \alpha(\tilde{\lambda}_n), P \rangle),
\]
\[
\psi^2 : H^2(Y; \mathfrak{g}_p) \to \mathbb{C}^n, \quad \alpha \mapsto (\langle \alpha(\Sigma_1), P \rangle, \ldots, \langle \alpha(\Sigma_n), P \rangle).
\]
Then the basis $\mathbf{h}_Y^i$ maps to the standard basis of $\mathbb{C}$, $\mathbb{C}^{n+1}$, or $\mathbb{C}^n$ under $\psi^i$ accordingly.

**Proposition 3.2.** $\tau(Y; \rho, \mathbf{h}_Y, \alpha_Y) = (-1)^{n-1} (m - m^{-1})^{2n-2}$

**Proof.** Recall that $Y$ decomposes into two copies $Y_1$ and $Y_2$ of $W \times I$ with $Y_1 \cap Y_2 = W \sqcup W$ and that $W$ retracts to $V$, the wedge sum of $n$ circles $\lambda_1, \ldots, \lambda_n$ with the base point $p$.

We construct $V \times I$ from two copies of $V$ (regarding them as $V \times \partial I$) by adding cells $p \times I$, $\lambda_1 \times I$, $\ldots$, $\lambda_n \times I$. Choose cell orders of $V$, $V \times I$, and $V \times S^1$ as
\[
\bullet \ c_V = (p, \lambda_1, \ldots, \lambda_n),
\]
\[
\bullet \ c_{V \times I} = (c_V, c_V, c_V) \text{ where } c_V = (p \times I, \lambda_1 \times I, \ldots, \lambda_n \times I),
\]
\[
\bullet \ c_{V \times S^1} = (c_V, c_V, c_V).
\]
Then the basis transition between \((c_Y \times I, c_Y \times I)\) and \((c_Y, c_Y, c_Y \times S^1)\) is an even permutation. On the other hand, for \(h_{Y \times S^1} = (p, \mu, \lambda_1, \ldots, \lambda_n, \Sigma_1, \ldots, \Sigma_n) (= h_Y)\) a straightforward computation shows that

\[
\text{Tor}(C_*(V \times S^1; \mathbb{R}), c_Y \times S^1, h_{Y \times S^1})
\]

\[
= (-1)^{|C_*(V \times S^1; \mathbb{R})|} \det \begin{pmatrix} I_n & 0 & 0 \\ I_n & I_n & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \det \begin{pmatrix} -I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = 1.
\]

Note that \(|C_*(V \times S^1; \mathbb{R})|\) is obviously even.

We choose any triangulation of \(Y\) and cell orders \(c_Y, c_Y, c_Y\) according to \(c_Y \times S^1, c_Y \times I, \) and \(c_Y\), respectively. Applying the formula (3) to the short exact sequence (11), we obtain

\[
1 = (-1)^{s+u} \text{Tor}(C^*(Y; g_p), c_Y, h_Y) \text{Tor}(H)
\]

after canceling out the torsion terms for \(W \simeq Y\). Here \(\text{Tor}(H)\) is the Reidemeister torsion of the long exact sequence (12) with respect to \(h_Y\) and \(h_{W'}\). Note that the basis transition between \((c_{Y'}, c_{Y'}, c_Y)\) and \((c_Y, c_Y, c_Y)\) is an even permutation. One easily checks from the definitions (4) and (5) that \(v \equiv 0\) and \(u \equiv \sum_i \beta_i(C^*(Y; g_p)) \equiv n - 1\) in modulo 2. To simplify notations, we rewrite the sequence (12) as

\[
0 \to H^0 \to H^1 \to H^2 \to H^3 \to H^4 \to H^5 \to H^6 \to 0
\]

where the first and second rows are identified with respect to \(h_Y\) and \(h_{W'}\). We choose a tuple \(b^i\) of vectors in \(H^i\) as

\[
b^0 = e_1, b^1 = e_1, b^2 = e_1, b^3 = (e_2, e_3, \ldots, e_{n+1}),
\]

\[
b^4 = (e_{n+1}, e_{n+2}, \ldots, e_{2n-4}), b^5 = (e_1, e_2, \ldots, e_n), b^6 = \emptyset
\]

where \(e_k\) is a unit vector whose coordinates are all zero, except one at the \(k\)-th coordinate. Then the basis transition matrix \(A_i\) at \(H^i\) (see Section 2.2) is given by

\[
A_0 = I_1, A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A_3 = I_{n+1}
\]

\[
A_4 = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_{2n-2} & 0 \\ 0 & 0 & I_{3n-2} \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & -m^2 I_{n-1} & 0 & 0 \\ -m^2 I_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \end{pmatrix}, A_6 = I_n.
\]

Here we used the equation (13) with the fact that \(f_0(e_1) = e_1 + e_2, f_1(e_1) = 0, f_1(e_{j+1}) = e_j + e_{3n-2+j}, d_0(e_1) = e_1, d_1(e_j) = e_j\) for \(1 \leq j \leq n\). It follows that

\[
\text{tor}(H) = -\det A_5^{-1}
\]

\[
= (-1)^{n-1} \det \begin{pmatrix} I_{2n-2} & 0 & -m^2 I_{n-1} & 0 \\ 0 & 0 & 0 & -m^2 I_{n-1} \\ -I_{2n-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{3n-2} \end{pmatrix}^{-1}
\]

\[
= (-1)^{n-1} \det \begin{pmatrix} (1 - m^2) I_{n-1} & 0 \\ 0 & (1 - m^2) I_{n-1} \end{pmatrix}^{-1}
\]

\[
= (m - m^{-1})^{2-2n}.
\]

Note that the third equation follows from the determinant formula for a block matrix. We conclude that

\[
\text{Tor}(C^*(Y; g_p), c_Y, h_Y) = (-1)^{n+1}(m - m^{-1})^{2n-2}.
\]

This completes the proof, since we have \(\epsilon(o_Y) = 1\) from the equation (15). \(\square\)

**Remark 3.3.** We have \(\tau(Y; \rho, h_Y, o_Y) = 1\) for \(n = 1\). It agrees with the computation given in Example 2.2 since \(Y\) retracts to a 2-torus when \(n = 1\).
3.2. The knot exterior of the connected sum. The composing space \( Y \) has \( n + 1 \) boundary tori \( \Sigma_1, \ldots, \Sigma_n \), and \( \Sigma \). For \( 1 \leq j \leq n \) we glue the knot exterior \( M_j \) of \( K_j \subset S^3 \) to \( Y \) by using a homeomorphism \( \partial M_j \to \Sigma_j \) that maps the meridian and canonical longitude of \( K_j \) to \( \mu \) and \( \lambda_j \), respectively. The resulting manifold \( M = \partial M = \Sigma \) is and is the knot exterior of the connected sum of \( K_1, \ldots, K_n \). We refer to [Jac80, IX.21–22] for details. We choose the orientation \( \sigma_M \) of \( H_*(M; \mathbb{R}) \) as in Section 2.3, i.e., the one induced from the basis \( h_M = (pt, \mu) \) of \( H_0(M; \mathbb{R}) \).

Let \( \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) be an irreducible representation. We denote by \( m \) and \( l_j \) eigenvalues of \( \rho(\mu) \) and \( \rho(\lambda_j) \) respectively as in the equation (8). For simplicity we assume that

\[
\Delta K_j = \sum_{j=1}^{n} \Delta K_j(n^2) \neq 0 \text{ for all } 1 \leq j \leq n
\]

where \( \Delta K_j \) is the Alexander polynomial of \( K_j \). It follows that each restriction \( \rho_j : \pi_1(M_j) \to \text{SL}_2(\mathbb{C}) \) of \( \rho \) is either reducible or abelian. We further assume that if \( \rho_j \) is irreducible, then

\[
l_j \neq \pm 1 \text{ and } \rho_j \text{ is } \mu \text{-regular.}
\]

Without loss of generality, we assume that \( \rho_1, \ldots, \rho_k \) are abelian and \( \rho_{k+1}, \ldots, \rho_n \) are irreducible where \( k \) should be less than \( n \), otherwise \( \rho \) becomes abelian. In particular, \( l_j \neq \pm 1 \) for some \( 1 \leq j \leq n \).

Proposition 3.4. We have

\[
\dim H^i(M; \mathfrak{g}_\rho) = \begin{cases} n - k & i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}
\]

Proof. From the short exact sequence

\[
0 \to C^*(M; \mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} C^*(M_j; \mathfrak{g}_\rho) \oplus C^*(Y; \mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} C^*(\Sigma_j; \mathfrak{g}_\rho) \to 0,
\]

we have

\[
0 \to C^0(M; \mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} H^0(M_j; \mathfrak{g}_\rho) \oplus H^0(Y; \mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} H^0(\Sigma_j; \mathfrak{g}_\rho) \to 0.
\]

(20)

With respect to the bases \( h_{\Sigma_j}, h_{M_j}, \) and \( h_Y \) given in Example 2.2 and Sections 2.3 and 3.1, the map \( G_0 \) in the sequence (20) agrees with

\[
G_0 : \mathbb{C}^{k+1} \to \mathbb{C}^n, \ (x_1, \ldots, x_k, y) \mapsto (x_1 - y, \ldots, x_k - y, -y, \ldots, -y).
\]

It follows that \( \dim \ker G_0 = \dim \ker D_0 = k + 1 \) and \( \dim \operatorname{im} D_0 = n - k - 1 \). Also, the matrix expression of \( G_1 : \mathbb{C}^{2n+1} \to \mathbb{C}^{2n} \) is given by

\[
G_1 = \begin{pmatrix}
1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\kappa_1 & 0 & 0 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & 0 & \cdots & 0 \\
\cdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\
0 & \cdots & \kappa_n & 0 & 0 & \cdots & -1
\end{pmatrix}
\]

where \( \kappa_j = (h_{M_j}(\lambda_j), P) \). Since we obtain an invertible matrix (of size \( 2n \)) from \( G_1 \) by deleting the \( (n + 1) \)-st column, we have \( \dim \operatorname{im} G_1 = 2n \) and \( \dim \ker G_1 = \dim \operatorname{im} F_1 = 1 \). It follows that \( \dim H^1(M; \mathfrak{g}_\rho) = \dim \operatorname{im} F_1 + \dim \operatorname{im} D_0 = n - k \). On the other hand, \( G_2 \) is surjective, since the restriction map \( H^2(Y; \mathfrak{g}_\rho) \to \bigoplus_{j=1}^{n} H^2(\Sigma_j; \mathfrak{g}_\rho) \) is an isomorphism (see the equation (14)). It follows that \( \dim H^3(M; \mathfrak{g}_\rho) = 0 \) and \( \dim H^2(M; \mathfrak{g}_\rho) = n - k \) since the Euler characteristic of \( M \) is zero. \( \square \)

We let \( P = \frac{1}{8} e_2 \in H^0(Y; \mathfrak{g}_\rho) \) and define maps

\[
\psi^1 : H^1(M; \mathfrak{g}_\rho) \to \mathbb{C}, \ \alpha \mapsto \langle \alpha(\mu), P \rangle,
\]

\[
\psi^2 : H^2(M; \mathfrak{g}_\rho) \to \mathbb{C}^{n-k}, \ \alpha \mapsto \langle \alpha(\Sigma_{k+1}), P \rangle, \ldots, \langle \alpha(\Sigma_n), P \rangle.
\]
Lemma 3.5. $\psi^1$ induces an isomorphism $H^1(M; g_\rho)/\text{Im } D_0 \to \mathbb{C}$ and $\psi^2$ is an isomorphism.

Proof. It is clear that $\psi^1$ is compatible with the isomorphism $H^1(M_j; g_\rho) \to \mathbb{C}$, $\alpha \mapsto \langle \alpha(\overline{\mu}), P \rangle$ for $1 \leq j \leq n$. In particular, $\psi^1$ is surjective. On the other hand, it follows from the sequence (20) that an element of $\text{Im } D_0$ maps to the trivial element in $H^1(M_j; g_\rho)$ under the restriction map $H^1(M_j; g_\rho) \to H^1(M_j; g_\rho)$. Therefore, $\psi^1$ induces a map $H^1(M; g_\rho)/\text{Im } D_0 \to \mathbb{C}$ which is an isomorphism since $\dim H^1(M; g_\rho) = n - k$ and $\dim \text{Im } D_0 = n - k - 1$. The second claim that $\psi^2$ is an isomorphism is obvious from the sequence (20).

Recall that the basis $h^0_{\Sigma_j}$ of $H^0(\Sigma_j; g_\rho)$ gives us an isomorphism $\oplus_{j=1}^n H^0(\Sigma_j; g_\rho) \cong \mathbb{C}^n$. Denoting by $(\epsilon_1, \ldots, \epsilon_n)$ the standard basis of $\mathbb{C}^n$, we choose a basis of $\text{Im } D_0$ as

$$(D_0(\epsilon_{k+1}), \ldots, D_0(\epsilon_{n-1})).$$

Note that the equation (21) implies that the above tuple is indeed a basis of $\text{Im } D_0$. We then extend it to a basis $h^1_M$ of $H^1(M; g_\rho)$ by adding an element $\xi$ at the first position which maps to the standard basis of $\mathbb{C}$ under $\psi^1$:

$$h^1_M = (\xi, D_0(\epsilon_{k+1}), \ldots, D_0(\epsilon_{n-1})).$$

We also choose a basis $h^2_M$ of $H^2(M; g_\rho)$ by the pre-image of the standard basis of $\mathbb{C}^{n-k}$ under $\psi^2$ and define the adjoint Reidemeister torsion by

$$\tau_\mu(M; \rho) = \tau(M; \rho, h_M, o_M).$$

Note that the above definition reduces to the definition (6) of a knot exterior when $n = 1$.

**Lemma 3.6.** The definition (23) does not depend on the choice of $P \in H^0(Y; g_\rho)$ and the order of indices of $\Sigma_{k+1}, \ldots, \Sigma_n$.

**Proof.** If we replace $P$ by $cP$ for some $c \in \mathbb{C}^*$, then the basis transition matrices for $H^1(M; g_\rho)$ is $\frac{1}{i} I_{n-k}$ for both $i = 1, 2$ and thus $\tau_\mu(M; \rho)$ does not change. If we exchange two indices other than $n$, then the basis transition is clearly an odd permutation for both $H^1(M; g_\rho)$ and $H^2(M; g_\rho)$. Therefore $\tau_\mu(M; \rho)$ does not change. If we exchange the index $n$ with another one, then the basis transition for $H^2(M; g_\rho)$ is an odd permutation. On the other hand, since $\epsilon_{k+1} + \cdots + \epsilon_n \in \text{Im } G_0 = \text{Ker } D_0$ (see the equation (21)), we have $D_0(\epsilon_n) = -D_0(\epsilon_{k+1}) - \cdots - D_0(\epsilon_{n-1})$. It follows that the basis transition matrix for $H^1(M; g_\rho)$ has determinant $-1$ and thus $\tau_\mu(M; \rho)$ does not change.

**Theorem 3.7.** $\tau_\mu(M; \rho) = (m - m^{-1})^{2n-2} \tau_\mu(M_1; \rho_1) \cdots \tau_\mu(M_n; \rho_n)$.

**Proof.** Choose any triangulation of $M$ with any cell order $c_M$. We denote by $c_Y$ (resp., $c_M$, and $c_{\Sigma_j}$) the cell order restricted to $Y$ (resp., $M_j$ and $\Sigma_j$). Note that the Euler characteristics of $M, M_j, Y, \Sigma_j$ are even. It follows that we may assume that the number of $i$-dimensional cells in each $M_j, Y, \Sigma_j$ is even by applying the barycentric subdivision once. Let $e = 1$ (resp., $-1$) if the basis transition between $(c_{\Sigma_1}, \ldots, c_{\Sigma_n}, c_M)$ and $(c_{M_1}, \ldots, c_{M_n}, c_Y)$ is an even (resp., odd) permutation.

Applying the formula (3) to the short exact sequence (19), we obtain

$$e \cdot \prod_{j=1}^n \text{Tor}(C^*(M_j; g_\rho), c_{M_j}, h_{M_j}) \cdot \text{Tor}(C^*(Y; g_\rho), c_Y, h_Y)$$

$$= (-1)^{e+n} \text{Tor}(C^*(M; g_\rho), c_M, h_M) \cdot \prod_{j=1}^n \text{Tor}(C^*(\Sigma_j; g_\rho), c_{\Sigma_j}, h_{\Sigma_j}) \cdot \text{tor}(\mathcal{G}).$$

where $\text{tor}(\mathcal{G})$ is the Reidemeister torsion of the long exact sequence (20) with respect to $h_M, h_Y, h_{\Sigma_j},$ and $h_M$. It is clear from the definition (4) that $\nu$ is even since the number of $i$-dimensional cells in each $M, M_j, Y$ and $\Sigma_j$ is even for all $i$. Also, a direct computation from the definition (5) gives that $\nu \equiv n$ in modulo 2. Recall that there are two trivial terms $H^0(M; g_\rho)$ and $H^2(M; g_\rho)$ in $\mathcal{G}$. Ignoring these trivial terms, we rewrite $\mathcal{G}$ as

$$0 \to \mathcal{G}^0 \xrightarrow{G_0} \mathcal{G}^1 \xrightarrow{D_0} \mathcal{G}^2 \xrightarrow{F_1} \mathcal{G}^3 \xrightarrow{G_1} \mathcal{G}^4 \xrightarrow{D_1} \mathcal{G}^5 \xrightarrow{F_2} \mathcal{G}^6 \xrightarrow{G_2} \mathcal{G}^7 \to 0$$

where

- $\cong$ indicates that $\mathcal{G}^i$ is isomorphic to $\mathcal{G}^i$.
- $\mathcal{G}^i_{\Sigma_{k+1}}$ indicates that $\mathcal{G}^i$ is restricted to $\Sigma_{k+1}$.
- $\mathcal{G}^i_{\Sigma_n}$ indicates that $\mathcal{G}^i$ is restricted to $\Sigma_n$.
Then from the equation (26) we have

\[ h_{10} \] the conditions (16) and (17) in Section 3.2 are satisfied for generic
torsion is well-defined on the level set

\[ \mathcal{X} \] notations

\[ (25) \] 0

\[ (3) \] to the short exact sequence

\[ \pi \]

Proofs of Theorems 1.3 and 1.4.

Combining the equations (24) and (27) with Example 2.2 and Proposition 3.2, we obtain the desired formula.

\[ A_0 = I_{k+1}, \quad A_1 = \begin{pmatrix} I_k & -1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & -1 & I_{n-k-1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ I_{n-k-1} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \kappa_n & 0 & \cdots & -1 \end{pmatrix}. \]

\[ A_5 = I_{n-k}, \quad A_6 = \begin{pmatrix} I_{n-k} & 0 \\ 0 & I_n \end{pmatrix}, \quad A_7 = -I_n. \]

It follows that \( \text{tor}(G) = (-1)^{n-k} (-1)^{n-k-1} (-1)^n (-1)^{\frac{n(n+1)}{2}} (-1)^n = (-1)^{\frac{n(n+1)}{2}} + 1. \) Therefore, we conclude that

\[ (24) \quad e \cdot \prod_{j=1}^{n} \text{Tor}(C^*(M_j; g_p), c_{M_j}, h_{M_j}) \cdot \text{Tor}(C^*(Y; g_p), c_Y, h_Y) 
\]

\[ = (-1)^{\frac{n(n+1)}{2}} + 1 \text{ Tor}(C^*(M_j; g_p), c_{M_j}, h_{M_j}) \prod_{j=1}^{n} \text{Tor}(C^*(\Sigma_j; g_p), c_{\Sigma_j}, h_{\Sigma_j}). \]

On the other hand, applying the formula (3) to the short exact sequence

\[ (25) \quad 0 \to \bigoplus_{j=1}^{n} C_*(\Sigma_j; \mathbb{R}) \to \bigoplus_{j=1}^{n} C_*(M_j; \mathbb{R}) \oplus C_*(Y; \mathbb{R}) \to C_*(M; \mathbb{R}) \to 0, \]

we have

\[ (26) \quad e \cdot \prod_{j=1}^{n} \text{Tor}(C_*(M_j; \mathbb{R}), c_{M_j}, h_{M_j}) \cdot \text{Tor}(C_*(Y; \mathbb{R}), c_Y, h_Y) 
\]

\[ = (-1)^{u' + \nu} \prod_{j=1}^{n} \text{Tor}(C_*(\Sigma_j; \mathbb{R}), c_{\Sigma_j}, h_{\Sigma_j}) \cdot \text{Tor}(C_*(M; \mathbb{R}), c_M, h_M) \cdot \text{tor}(G'). \]

where \( \text{tor}(G') \) is the Reidemeister torsion of the long exact sequence induced from (25) with respect to the bases \( h_{\Sigma_j}, h_{M_j}, h_Y, \) and \( h_M. \) Repeating similar computations, we have \( u' = \nu = 0 \) in modulo 2 and \( \text{tor}(G') = (-1)^{\frac{n(n+1)}{2}}. \) Then from the equation (26) we have

\[ (27) \quad e \cdot \prod_{j=1}^{n} \epsilon(o_{M_j}) \cdot \epsilon(o_Y) = (-1)^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} \epsilon(o_{\Sigma_j}) \cdot \epsilon(o_M). \]

Combining the equations (24) and (27) with Example 2.2 and Proposition 3.2, we obtain the desired formula. \( \square \)

3.3. Proofs of Theorems 1.3 and 1.4. Recall that \( \mathcal{X}(M) \) is the character variety of irreducible representations \( \pi_1(M) \to \text{SL}_2(\mathbb{C}) \) and \( \mathcal{X}^\mu(M) \) is the pre-image of \( c \in \mathbb{C} \) under the trace function \( \mathcal{X}(M) \to \mathbb{C} \) of \( \mu. \) We use the notations \( \mathcal{X}(M_j) \) and \( \mathcal{X}^\mu(M_j) \) similarly for \( 1 \leq j \leq n. \) Since we assumed that

(C) the level set \( \mathcal{X}^\mu(M_j) \) consists of finitely many \( \mu \)-regular characters with the canonical longitude having trace

other than \( \pm 2 \) for generic \( c \in \mathbb{C}, \)

the conditions (16) and (17) in Section 3.2 are satisfied for generic \( c \in \mathbb{C}. \) It follows that the adjoint Reidemeister torsion is well-defined on the level set \( \mathcal{X}^\mu(M) \) for generic \( c \in \mathbb{C}. \)
Lemma 3.8. The connected components of $X_{\mu}^c(M)$ are the pre-images of the restriction map (i.e. induced by the inclusions $M_j \to M$):

$$\Phi : X_{\mu}^c(M) \to \prod_{j=1}^n X_{\mu}^c(M_j)$$

where $\alpha_j : \pi_1(M_j) \to \text{SL}_2(\mathbb{C})$ is the abelian representation with $\text{tr}(\alpha_j(\mu)) = c$

Proof. We first prove that $\Phi$ is surjective. Let $\rho_j$ be a representation $\pi_1(M_j) \to \text{SL}_2(\mathbb{C})$ satisfying $\text{tr}(\rho_j(\mu)) = c$ for $1 \leq j \leq n$. Since we assume that $c \neq \pm 2$, we can conjugate each $\rho_j$ so that $\rho_1(\mu) = \rho_2(\mu) = \cdots = \rho_n(\mu)$. This is sufficient to extend these representations to $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ which is irreducible since at least one of $\rho_j$’s is irreducible.

A point, say $p$, in the image of $\Phi$ is $([\alpha_1], \ldots, [\alpha_k], [\rho_{k+1}], \ldots, [\rho_n])$ up to reordering where $\alpha_1, \ldots, \alpha_k$ are abelian and $\rho_{k+1}, \ldots, \rho_n$ are irreducible. To analyze the pre-image $\Phi^{-1}(p)$, consider two characters in $\Phi^{-1}(p)$, those are conjugacy classes of irreducible representations $\rho$ and $\rho'$ of $\pi_1(M)$.

As the pre-images of $\Phi$ are connected and the image is discrete, those pre-images are the connected components.

Specifically, for $j = 1, \ldots, k, \rho_j = \rho_j'$. It is because of that the genericity assumption (16) implies that $\rho_j$ and $\rho_j'$ are abelian, and an abelian representation of a knot exterior is determined by the trace of $\mu$.

For $j = k+1, \ldots, n$, $\rho_j'$ and $\rho_j$ are conjugate by some matrix of $D$, because an irreducible representation is determined by its character.

Namely, $\rho$ and $\rho'$ differ by bending along some of the tori $\Sigma_{k+1}, \ldots, \Sigma_n$. Note that bending along all tori $\Sigma_{k+1}, \ldots, \Sigma_n$ simultaneously by the same matrix in $D$ does not change the conjugacy class. It follows that the pre-image $\Phi^{-1}(p)$ is homeomorphic to

$$\left(\frac{D \times \cdots \times D}{n-k}\right) \cong \frac{(\mathbb{C}^* \times \cdots \times \mathbb{C}^*)}{\mathbb{C}^*} \cong (\mathbb{C}^*)^{n-k-1}.$$

As the pre-images of $\Phi$ are connected and the image is discrete, those pre-images are the connected components. □

From Theorem 3.7 and Lemma 3.8, we obtain Theorem 1.3: the adjoint Reidemeister torsion is locally constant on $X_{\mu}^c(M)$. Note that the term $(m - m^{-1})^{2n-2}$ in Theorem 3.7 is the constant $(c^2 - 4)^{n-1}$ on $X_{\mu}^c(M)$.

On the other hand, we have

$$\frac{1}{(c^2 - 4)^{n-1}} \sum_{[\rho] \in X_{\mu}^c(M)} \frac{1}{\tau_{\mu}(M; \rho)} = \prod_{j=1}^n \left( \sum_{[\rho] \in X_{\mu}^c(M_j)} \frac{1}{\tau_{\mu}(M_j; \rho)} + \frac{1}{\tau_{\mu}(M_j; \alpha_j)} \right) - \prod_{j=1}^n \frac{1}{\tau_{\mu}(M_j; \alpha_j)}$$

Here $J$ runs on all subsets of $\{1, \ldots, n\}$ different from the whole set ($J$ is the subset of indexes $j$ such that the restriction to $\pi_1(M_j)$ is abelian, hence $J$ may be empty but not the whole set). The notation $[\rho] \in X_{\mu}^c(M)$ means that we take one representative on each connected component of $X_{\mu}^c(M)$, which agrees with the ordinary sum for $M_j$. This completes the proof of Theorem 1.4, because we have assumed that for each $j = 1, \ldots, n$:

$$\sum_{[\rho] \in X_{\mu}^c(M_j)} \frac{1}{\tau_{\mu}(M_j; \rho)} = 0.$$
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