The Adapted Ordering Method in Representation Theory

Beatriz Gato-Rivera

Instituto de Matemáticas y Física Fundamental, CSIC,
Serrano 123, Madrid 28006, Spain

NIKHEF-H, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands

ABSTRACT

In 1998 the Adapted Ordering Method was developed for the representation theory of the superconformal algebras. This method, which proves to be very powerful, can be applied to most algebras and superalgebras, however. It allows: to determine maximal dimensions for a given type of singular vector space, to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams. In this article we present the Adapted Ordering Method for general algebras and superalgebras which admit a triangulation and review briefly the results obtained for the Virasoro algebra and for the $N = 2$ and Ramond $N = 1$ superconformal algebras.
1 Introduction

In 1998 the Adapted Ordering Method was developed, by M. Dörrapf and B. Gato-Rivera, for the study of the representation theory of the superconformal algebras. This method was applied successfully to the $N = 2$ superconformal algebras (topological, Neveu-Schwarz, Ramond and twisted) and to the Ramond $N = 1$ superconformal algebra, allowing to obtain rigorous proofs for several conjectured results, as well as many new results, especially for the case of the twisted $N = 2$ superconformal algebra and the case of the Ramond $N = 1$ superconformal algebra. The Adapted Ordering Method, which proves to be very powerful, can be applied to most algebras and superalgebras, however. It allows: to determine maximal dimensions for a given type of singular vector space, to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams. The purpose of this article is precisely to fill the existing gap, providing the description of the Adapted Ordering Method for a general algebra or superalgebra which admits a triangular decomposition into creation, annihilation and zero mode operators, like is the case for most Lie algebras and superalgebras.

For the given algebra or superalgebra one defines freely generated modules over a highest weight vector, denoted as Verma modules. A Verma module is in general not irreducible, but it contains submodules which are freely generated over, at least, one highest weight vector different from the highest weight vector of the Verma module. These vectors are usually referred to as singular vectors. The irreducible highest weight representations are then obtained as the quotients of the Verma modules divided by all their submodules. Surprisingly, the complete set of singular vectors do not generate all the submodules in the case of Verma modules which contain subsingular vectors. The reason is that subsingular vectors are singular vectors of the quotient space, but not of the Verma module itself. In this case one has to divide further by the submodules generated by the subsingular vectors, repeating this division procedure successively, if necessary.

On the Verma modules one introduces a hermitian contravariant form. The vanishing of the corresponding determinant indicates the existence of at least one singular vector. The determinant may not detect the whole set of singular vectors, however, neither does it give the dimension of the space of singular vectors with some given weights. There could be in fact more than one linearly independent singular vectors with the same weights. Therefore, the dimensions of the spaces of singular vectors have to be found by an independent procedure, like the Adapted Ordering Method which puts upper limits on the dimensions of these singular spaces. For most weight spaces of a Verma module these upper limits on the dimensions are trivial and, as a consequence, we obtain a rigorous proof that there cannot exist any singular vectors for these weights. For some weights, however, one may find necessary conditions that allow singular vector spaces to exist, either only one-dimensional, as is the case for the Virasoro algebra, or even higher dimensional spaces, as it happens for the $N = 2$ and Ramond $N = 1$ superconformal algebras.

The idea for developing the Adapted Ordering Method originated, in rudimentary form, from a procedure due to A. Kent for the study of the representations of the Virasoro algebra. For this purpose the author analytically continued the Virasoro Verma modules, yielding ‘generalised’ Verma modules, where he constructed generalised singular vectors expressions in terms of analytically continued Virasoro operators. This analytical continuation is not necessary, however, for the Adapted Ordering Method, nor is it necessary to construct singular vectors in order to apply it. The underlying idea is the concept of adapted orderings for all the possible terms of the ‘would be’ singular vectors. An adapted ordering is a criterion, satisfying certain requirements, to decide which of two given terms is the bigger one.
In what follows, in section 2 we will describe the Adapted Ordering Method for a general algebra or superalgebra which admits a triangulation and, as an example, we will apply this method to the Virasoro algebra. In section 3 we will review briefly the results obtained for the $N = 2$ and the Ramond $N = 1$ superconformal algebras, as an illustration of the power of this method. Section 4 is devoted to conclusions.

## 2 The Adapted Ordering Method

Let $A$ denote an algebra or superalgebra which admits a triangular decomposition into creation, annihilation and zero mode operators: $A = A^- \oplus A^+ \oplus A^0$, and let $U(A)$ be the universal enveloping algebra of $A$, that also decomposes as $U(A) = U(A)^- \oplus U(A)^+ \oplus U(A)^0$. The Cartan subalgebra $H_A$ is contained in $A^0$ but does not need to be identical to $A^0$. In general, an eigenvector with respect to the Cartan subalgebra with weights given by the set $\{l_i\}$, in particular a singular vector $\Psi_{\{l_i\}}$, can be expressed as a sum of products of creation operators acting on a highest weight vector with weights $\{\Delta_i\}$:

$$
\Psi_{\{l_i\}} = \sum_{m_1,m_2,...} \sum_{a, b, c, ...} k^{-a} a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ... X_{\{l_i\}}^{a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ...} |\{\Delta_i\}\rangle ,
$$

where $a, a^{-}, ..., b, b^{-}, ...$ are the creation operators of the algebra (some zero mode operators should also be included if they act as creation operators), $X_{\{l_i\}}^{a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ...}$ are the products of the creation operators: $a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ...$, with total weights $\{l_i\}$, which will be denoted simply as terms, and $k^{-a} a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ... \in \mathbb{C}$ are coefficients which depend on the given term. A non-trivial term $Y$ then refers to a term with non-trivial coefficient $k_Y$.

Now let us define the set $C_{\{l_i\}}$ as the set of all the terms with weights $\{l_i\}$:

$$
C_{\{l_i\}} = \{X_{\{l_i\}}^{a^{-m_1} a^{-m_2} ... b^{-n_1} b^{-n_2} ...} : m_1, m_2, ... n_1, n_2, ... \in \mathbb{N}_0\} ,
$$

and let $O$ denote a total ordering on $C_{\{l_i\}}$ with global minimum, that is an ordering such that any two different terms in $C_{\{l_i\}}$ are ordered with respect to each other. Thus $\Psi_{\{l_i\}}$ in Eq. (1) needs to contain an $O$-smallest $X_0 \in C_{\{l_i\}}$ with $kX_0 \neq 0$ and $kY = 0$ for all $Y \in C_{\{l_i\}}$ with $Y <_O X_0$ and $Y \neq X_0$. We define an adapted ordering on $C_{\{l_i\}}$ as follows:

**Definition 2.1** A total ordering $O$ on $C_{\{l_i\}}$ with global minimum is called adapted to the subset $C^A_{\{l_i\}} \subset C_{\{l_i\}}$ in the Verma module $V_{\{\Delta_i\}}$ if for any element $X_0 \in C^A_{\{l_i\}}$ at least one annihilation operator $\Gamma$ exists for which $\Gamma X_0 |\{\Delta_i\}\rangle$ contains a non-trivial term $\tilde{X}$

$$
\Gamma X_0 |\{\Delta_i\}\rangle = (k_{\tilde{X}} \tilde{X} + ... ...) |\{\Delta_i\}\rangle
$$

which is absent, however, for all $\Gamma X |\{\Delta_i\}\rangle$, where $X$ is any term $X \in C_{\{l_i\}}$ such that $X_0 \not<_O X$. The complement of $C^A_{\{l_i\}}$, $C^K_{\{l_i\}} = C_{\{l_i\}} \setminus C^A_{\{l_i\}}$ is the kernel with respect to the ordering $O$ in the Verma module $V_{\{\Delta_i\}}$.

In this definition $\Gamma$ should also include zero mode operators if they act as annihilation operators. A crucial point now is that one needs to find suitable, clever orderings in order to obtain the smallest possible kernels. The reason is that the size of the kernel puts a limit on the dimension of the corresponding singular vector space, as stated in the following theorem:
The Adapted Ordering Method.

**Theorem 2.B** Let \( \mathcal{O} \) denote an adapted ordering on \( C_{\{l_i\}}^A \) at weights \( \{l_i\} \) with kernel \( C_{\{l_i\}}^K \) for a given Verma module \( \mathcal{V}_{\{\Delta_i\}} \). If the ordering kernel \( C_{\{l_i\}}^K \) has \( n \) elements, then there are at most \( n \) linearly independent singular vectors \( \Psi_{\{l_i\}} \) in \( \mathcal{V}_{\{\Delta_i\}} \) with weights \( \{l_i\} \).

Observe that the maximal possible dimension \( n \) does not imply that all the singular vectors of the corresponding type are \( n \)-dimensional. From this theorem one deduces that if \( C_{\{l_i\}}^K = \emptyset \) for a given Verma module, then there are no singular vectors with weights \( \{l_i\} \) in it. That is:

**Theorem 2.C** Let \( \mathcal{O} \) denote an adapted ordering on \( C_{\{l_i\}}^A \) at weights \( \{l_i\} \) with trivial kernel \( C_{\{l_i\}}^K = \emptyset \) for a given Verma module \( \mathcal{V}_{\{\Delta_i\}} \). A singular vector \( \Psi_{\{l_i\}} \) in \( \mathcal{V}_{\{\Delta_i\}} \) with weights \( \{l_i\} \) is therefore trivial.

In addition, the coefficients with respect to the ordering kernel \( C_{\{l_i\}}^K \) uniquely identify a singular vector. Since the size of the ordering kernels are small (one or two, rarely three, terms), it turns out that a few (one, two, \( \ldots \) ) coefficients completely determine a singular vector no matter its size, what allows to find easily product expressions for descendant singular vectors. This is summarized in the following theorem:

**Theorem 2.D** Let \( \mathcal{O} \) denote an adapted ordering on \( C_{\{l_i\}}^A \) at weights \( \{l_i\} \) with kernel \( C_{\{l_i\}}^K \) for a given Verma module \( \mathcal{V}_{\{\Delta_i\}} \). If two singular vectors \( \Psi_{\{l_i\}}^1 \) and \( \Psi_{\{l_i\}}^2 \) with the same weights \( \{l_i\} \) have \( k_X^1 = k_X^2 \) for all \( X \in C_{\{l_i\}}^K \), then

\[
\Psi_{\{l_i\}}^1 \equiv \Psi_{\{l_i\}}^2.
\]

Proof: Let us consider \( \Psi_{\{l_i\}} = \Psi_{\{l_i\}}^1 - \Psi_{\{l_i\}}^2 \), which does not contain any terms of the ordering kernel \( C_{\{l_i\}}^K \), simply because \( k_X^1 = k_X^2 \) for all \( X \in C_{\{l_i\}}^K \). As \( C_{\{l_i\}} \) is a totally ordered set with respect to \( \mathcal{O} \) which has a global minimum, the non-trivial terms of \( \Psi_{\{l_i\}} \), provided \( \Psi_{\{l_i\}} \) is non-trivial, need to have an \( \mathcal{O} \)-minimum \( X_0 \in C_{\{l_i\}}^A \). Thus the coefficient \( k_{X_0} \) of \( X_0 \) in \( \Psi_{\{l_i\}} \) must be non-trivial. As \( \mathcal{O} \) is adapted to \( C_{\{l_i\}}^A \) one can find an annihilation operator \( \Gamma \) such that \( \Gamma X_0 \{\{\Delta_i\}\} \) contains a non-trivial term (for a suitably chosen basis depending on \( X_0 \)) that cannot be created by \( \Gamma \) acting on any other term of \( \Psi_{\{l_i\}} \) which is \( \mathcal{O} \)-larger than \( X_0 \). But \( X_0 \) was chosen to be the \( \mathcal{O} \)-minimum of \( \Psi_{\{l_i\}} \). Therefore, \( \Gamma X_0 \{\{\Delta_i\}\} \) contains a non-trivial term that cannot be created from any other term of \( \Psi_{\{l_i\}} \). The coefficient of this term is obviously given by \( ck_{X_0} \) with \( c \) a non-trivial complex number. But \( \Psi_{\{l_i\}} \) is also a singular vector and must be annihilated by any annihilation operator, in particular by \( \Gamma \). It follows that \( k_{X_0} = 0 \), contrary to our original assumption. Thus, the set of non-trivial terms of \( \Psi_{\{l_i\}} \) is empty and therefore \( \Psi_{\{l_i\}} = 0 \). This results in \( \Psi_{\{l_i\}}^1 = \Psi_{\{l_i\}}^2 \). \( \square \)

Theorem 2.D states, therefore, that if two singular vectors with the same weights, in the same Verma module, agree on the coefficients of the ordering kernel, then they are identical. If the ordering kernel is trivial we consequently find \( 0 \) as the only vector that can satisfy the highest weight conditions. This observation provides a proof for Theorem 2.C:

Proof: The trivial vector \( 0 \) satisfies any annihilation conditions for any weights \( \{l_i\} \). As the ordering kernel is trivial the components of the vectors \( 0 \) and \( \Psi_{\{l_i\}} \) agree on the ordering kernel and using theorem 2.D we obtain \( \Psi_{\{l_i\}} = 0 \). \( \square \)
The Adapted Ordering Method.

Using Theorem 2.D it is also easy to prove Theorem 2.B:

Proof: Suppose there were more than \( n \) linearly independent singular vectors \( \Psi_{\{\lambda\}} \) in \( \mathcal{V}_{\{\Delta\}} \) with weights \( \{\lambda\} \). We choose \( n+1 \) linearly independent singular vectors among them \( \Psi_1, \ldots, \Psi_{n+1} \). The ordering kernel \( C_{\{\lambda\}}^k \) has the \( n \) elements \( X_1, \ldots, X_n \). Let \( k_{jk} \) denote the coefficient of the term \( X_j \) in the vector \( \Psi_k \) in a suitable basis decomposition. The coefficients \( k_{jk} \) thus form a \( n \) by \( n+1 \) matrix \( M \). The homogeneous system of linear equations \( M\lambda = 0 \) thus has a non-trivial solution \( \lambda^0 = (\lambda^0_1, \ldots, \lambda^0_{n+1}) \) for the vector \( \lambda \). We then form the linear combination \( \Psi = \sum_{i=1}^{n+1} \lambda^0_i \Psi_i \). Obviously, the coefficient of \( X_j \) for the vector \( \Psi \) is just given by the \( j \)-th component of the vector \( M\lambda \) which is trivial for \( j = 1, \ldots, n \). Hence, the coefficients of \( \Psi \) are trivial on the ordering kernel. On the other hand, \( \Psi \) is a linear combination of singular vectors and therefore it is also a singular vector. Due to theorem 2.D one immediately finds that \( \Psi \equiv 0 \) and therefore \( \sum_{i=1}^{n+1} \lambda_i \Psi_i = 0 \). This, however, contradicts the assumption that \( \Psi_1, \ldots, \Psi_{n+1} \) are linearly independent. \( \square \)

As a simple example of the Adapted Ordering Method we will see now the application of this method to the Virasoro algebra \( \mathcal{V} \), which has been extensively studied in the literature\(^6\)\(^-\)\(^8\). This algebra is given by the commutation relations

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad [C, L_m] = 0, \quad m, n \in \mathbb{Z},
\]

where \( C \) commutes with all operators of \( \mathcal{V} \) and can hence be taken to be constant \( c \in \mathbb{C} \). \( \mathcal{V} \) can be written in its triangular decomposition: \( \mathcal{V} = \mathcal{V}^- \oplus \mathcal{V}^0 \oplus \mathcal{V}^+ \), where \( \mathcal{V}^+ = \text{span}\{L_m : m \in \mathbb{N}\} \) is the set of annihilation operators, \( \mathcal{V}^- = \text{span}\{L_{-m} : m \in \mathbb{N}\} \) is the set of creation operators, and the Cartan subalgebra is given by \( \mathcal{V}^0 = \text{span}\{L_0, C\} \). For elements of \( \mathcal{V} \) that are eigenvectors of \( L_0 \) with respect to the adjoint representation the \( L_0 \)-eigenvalue is called the level \( l \). For the universal enveloping algebra \( U(\mathcal{V}) \), elements of the form \( L_{-p_1} \cdots L_{-p_q}, \ p_q \in \mathbb{Z} \) for \( q = 1, \ldots, I, I \in \mathbb{N} \), are at level \( l = \sum_{q=1}^I p_q \). Note that annihilation operators \( L_m \in \mathcal{V}^+ \) have negative level \( l = -m \).

A representation with \( L_0 \)-eigenvalues bounded from below contains a vector with \( L_0 \)-eigenvalue \( \Delta \) which is annihilated by \( \mathcal{V}^+ \), a highest weight (h.w.) vector \( |\Delta\rangle \):

\[
\mathcal{V}^+ |\Delta\rangle = 0, \quad L_0 |\Delta\rangle = \Delta |\Delta\rangle.
\]

The Verma module \( \mathcal{V}_\Delta \) built on \( |\Delta\rangle \) is \( L_0 \)-graded in a natural way. The corresponding \( L_0 \)-eigenvalue is called the conformal weight and is written for convenience as \( \Delta + l \), where \( l \) is the level. Any proper submodule of \( \mathcal{V}_\Delta \) needs to contain a singular vector \( \Psi_l \) that is not proportional to the h.w. vector \( |\Delta\rangle \) but still satisfies the h.w. vector conditions with conformal weight \( \Delta + l \):

\[
\mathcal{V}^+ \Psi_l = 0, \quad L_0 \Psi_l = (\Delta + l)\Psi_l.
\]

Now we will see the total ordering on \( C_l \) defined by Kent\(^5\) for the Virasoro algebra. One has to take into account, however, that Kent used the following ordering to show that in his generalised Verma modules vectors at level 0 satisfying the h.w. conditions are actually proportional to the h.w. vector. Using the Adapted Ordering technology, though, one deduces that this ordering already implies that all Virasoro singular vectors are unique at their levels up to proportionality, simply because the ordering kernel for each \( l \in \mathbb{N} \) has just one element: \( L^l_{-1} \).

**Definition 2.E** On the set \( C_l \) of Virasoro operators we introduce the total ordering \( \mathcal{O}_l \) for \( l \in \mathbb{N} \). For two elements \( X_1, X_2 \in C_l, \ X_1 \neq X_2 \), with \( X_i = L_{-m_i^1} \cdots L_{-m_i^1} L_{-1}^l, n_i = l - m_i - \ldots - m_i^1 \), or
$X_i = L_{-1}^l$, $i = 1, 2$ we define

$$X_1 <_c v X_2 \quad \text{if} \quad n^1 > n^2.$$  

(8)

If, however, $n^1 = n^2$ we compute the index $j_0 = \min\{j : m_j^1 - m_j^2 \neq 0, j = 1, \ldots, \min(I_1, I_2)\}$. We then define

$$X_1 <_c v X_2 \quad \text{if} \quad m_{j_0}^1 < m_{j_0}^2.$$  

(9)

For $X_1 = X_2$ we set $X_1 <_c v X_2$ and $X_2 <_c v X_1$.

The index $j_0$ describes the first mode, read from the right to the left, for which the generators in $X_1$ and $X_2$ ($L_{-1}$ excluded) are different. For example, in $C_8$ one has $L_{-2}L_{-2}L_{-2}L_{-1}^2 <_c v L_4L_{-2}L_{-2}^2$ with index $j_0 = 2$. Observe that $L_{-1}^3 \in C_l$ is the global $\mathcal{O}_V$-minimum in $C_l$. Now using the Adapted Ordering Method one finds the following theorem$^1$.

**Theorem 2.F** The ordering $\mathcal{O}_V$ is adapted to $C_l^A = C_l \setminus \{L_{-1}^l\}$ for each level $l \in \mathbb{N}$ and for all Verma modules $\mathcal{V}_\Delta$. The ordering kernel is given by the single element set $C_l^K = \{L_{-1}^l\}$.

For example let us consider the set of terms at level 3, $C_3 = \{L_{-1}^3, L_{-2}L_{-1}, L_{-3}\}$. One finds the total ordering $L_{-1}^3 <_c v L_{-2}L_{-1} <_c v L_{-3}$, which is adapted to $C_3^A = \{L_{-2}L_{-1}, L_{-3}\}$ with the ordering kernel $C_3^K = \{L_{-1}^3\}$. To see this one has to compute the action of the annihilation operators $\Gamma \in \{L_1, L_2, L_3\}$ on the three terms. In fact, the action of $L_1$ already reveals the structure of $C_3^A$, as $L_1L_{-2}L_{-1}|\Delta\rangle$ contains the term $L_{-2}L_{-1}$ that is absent in $L_1L_{-3}|\Delta\rangle$. The action of the three annihilation operators on $L_{-1}^3|\Delta\rangle$, however, produce terms that are also created by the action of these operators on $L_{-2}L_{-1}|\Delta\rangle$ and/or $L_{-3}|\Delta\rangle$.

Finally, from the previous theorem one now deduces the known result about the uniqueness of the Virasoro singular vectors$^5$.

**Theorem 2.G** If the Virasoro Verma module $\mathcal{V}_\Delta$ contains a singular vector $\Psi_l$ at level $l$, $l \in \mathbb{N}$, then $\Psi_l$ is unique up to proportionality.

### 3 Results for the superconformal algebras

As an illustration of the power of the Adapted Ordering Method, in this section we will review briefly the results obtained for the $N = 2$ and Ramond $N = 1$ superconformal algebras. This method has been applied to the topological, to the Neveu-Schwarz and to the Ramond $N = 2$ algebras in Ref. 1, to the twisted $N = 2$ algebra in Ref. 2 and to the Ramond $N = 1$ algebra in Ref. 3. As the representation theory of these superconformal algebras has different types of Verma modules, one has to introduce different adapted orderings for each type and the corresponding kernels also allow different degrees of freedom.

Let us start with the topological $N = 2$ superconformal algebra $T$. It contains the Virasoro generators $L_m$ with trivial central extension, a Heisenberg algebra $H_m$ corresponding to the U(1) current, and the fermionic generators $G_m$ and $Q_m$ corresponding to two anticommuting fields with conformal weights 2 and 1 respectively. $T$ satisfies the (anti-)commutation relations$^{15}$.
The set of annihilation operators $T^+$ is spanned by the generators with positive index, the set of creation operators $T^-$ is spanned by the generators with negative index, and the zero modes are given by $T^0 = \text{span}\{L_0, H_0, C, G_0, Q_0\}$. The Cartan subalgebra is generated by $H_T = \text{span}\{L_0, H_0, C\}$, where $C$ can be taken to be constant $c \in \mathbb{C}$, and the fermionic generators $\{G_0, Q_0\}$ classify the different choices of Verma modules.

A h.w. vector $|\Delta, h\rangle^N$ is an eigenvector of $H_T$ with eigenvalue $\Delta$, $H_0$ eigenvalue $h$, and vanishing $T^+$ action. Additional vanishing conditions $N$ are possible with respect to the operators $G_0$ and $Q_0$, resulting as follows. One can distinguish four different types of h.w. vectors $|\Delta, h\rangle^N$ labeled by a superscript $N \in \{G, Q, GQ\}$, or no superscript at all: h.w. vectors $|\Delta, h\rangle^G$ annihilated by $G_0$ but not by $Q_0$ ($G_0$-closed), h.w. vectors $|\Delta, h\rangle^Q$ annihilated by $Q_0$ but not by $G_0$ ($Q_0$-closed), h.w. vectors $|0, h\rangle^{GQ}$ annihilated by both $G_0$ and $Q_0$ (chiral), with zero conformal weight necessarily, and finally undecomposable h.w. vectors $|0, h\rangle$ that are neither annihilated by $G_0$ nor by $Q_0$ (no-label), also with zero conformal weight. Hence we have four different types of Verma modules: $V_{\Delta, h}^{G}$, $V_{\Delta, h}^{Q}$, $V_{0, h}^{GQ}$ and $V_{0, h}$, built on the four different types of h.w. vectors.

For elements $X$ of $T$ which are eigenvectors of $H_T$ with respect to the adjoint representation one defines the level $l$ as $[L_0, X] = lX$ and the charge $q$ as $[H_0, X] = qX$. In particular, elements of the form $X = \mathcal{L}_l \ldots \mathcal{L}_1 \mathcal{H}_h \ldots \mathcal{H}_1 \mathcal{Q}_q \ldots \mathcal{Q}_1 \mathcal{G}_g \ldots \mathcal{G}_1$, and any reorderings of it, have level $l = \sum_{j=1}^{l} j + \sum_{j=1}^{h} h_j + \sum_{j=1}^{g} q_j + \sum_{j=1}^{g} g_j$ and charge $q = g - q$. The Verma modules are naturally $\mathbb{N}_0 \times \mathbb{Z}$ graded with respect to the $H_T$ eigenvalues relative to the eigenvalues $(\Delta, h)$ of the h.w. vector. For a vector $\Psi_{l,q}$ in $V_{\Delta, h}^N$, the $L_0$-eigenvalue is $\Delta + l$ and the $H_0$-eigenvalue is $h + q$ with the level $l \in \mathbb{N}_0$ and the relative charge $q \in \mathbb{Z}$.

The singular vectors are annihilated by $T^+$ and may also satisfy additional vanishing conditions with respect to the operators $G_0$ and $Q_0$. Therefore one also distinguishes singular vectors of the types $\Psi_{l,q}^{G}$, $\Psi_{l,q}^{Q}$, $\Psi_{l,q}^{GQ}$ and $\Psi_{l,q}$. As there are 4 types of Verma modules and 4 types of singular vectors one might think of 16 different combinations of singular vectors in Verma modules. However, no-label and chiral singular vectors do not exist neither in chiral Verma modules $V_{0, h}^{GQ}$ nor in no-label Verma modules $V_{0, h}$ (with one exception: chiral singular vectors exist at level 0 in no-label Verma modules). Using the Adapted Ordering Method one has to introduce adapted orderings for the remaining 12 combinations, whose kernels give upper limits for the dimensions of the corresponding singular vector spaces. One finds that for most charges $q$ singular vectors do not exist. For the case of the Verma modules $V_{\Delta, h}^G$ built on $G_0$-closed h.w. vectors $|\Delta, h\rangle^G$, for $c \neq 3$, the maximal dimensions for the singular vector spaces $\Psi_{l,q}^{G}$, $\Psi_{l,q}^{Q}$, $\Psi_{l,q}^{GQ}$ and $\Psi_{l,q}$ are given as follows:

| $\Psi_{l,q}^{G}$ | $q = -2$ | $q = -1$ | $q = 0$ | $q = 1$ | $q = 2$ |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| $\Psi_{l,q}^{Q}$ | $|\Delta, h\rangle^G$ | 0         | 1         | 2         | 1         | 0         |
| $\Psi_{l,q}^{GQ}$ | $|\Delta, h\rangle^G$ | 1         | 2         | 1         | 0         | 0         |
| $\Psi_{l,q}^{GQ}$ | $|\Delta, h\rangle^G$ | 0         | 1         | 1         | 0         | 0         |
| $\Psi_{l,q}$ | $|\Delta, h\rangle^G$ | 0         | 1         | 1         | 0         | 0         |
and the spectral flows one constructs a one-to-one mapping between the Ramond singular vectors in Neveu-Schwarz Verma modules only exist for
vectors of only the type $\Psi^G_0$ whereas the ones built in chiral or antichiral Verma modules correspond to topological singular vectors. This implies 12 that all of them exist already at level 1. The four types of two-dimensional singular vector spaces of Tab. a also exist starting at level 2, and four examples at level 3 were constructed 12 as well. For the case of the three-dimensional singular vector spaces in no-label Verma modules in Tab. b, the corresponding types of singular vectors have been constructed at level 1 generating one-dimensional as well as two-dimensional 1 spaces, but no further search has been done for the three-dimensional spaces.

Tables Tab. a and Tab. b prove the conjecture made in Ref. 12, using the algebraic mechanism denoted the cascade effect, about the possible existing types of topological singular vectors. In addition, low level examples were constructed 12 for all these types, what proves that all of them exist already at level 1. The four types of two-dimensional singular vector spaces of Tab. a also exist starting at level 2, and four examples at level 3 were constructed 12 as well. For the case of the three-dimensional singular vector spaces in no-label Verma modules in Tab. b, the corresponding types of singular vectors have been constructed at level 1 generating one-dimensional as well as two-dimensional 1 spaces, but no further search has been done for the three-dimensional spaces.

Transferring the dimensions we have found in tables Tab. a and Tab. b to the Neveu-Schwarz $N = 2$ algebra 17–21 is straightforward as this algebra is related to the topological $N = 2$ algebra through the topological twists $T_{W}^{\pm}$: $L_m = L_m \pm 1/2 H_m$, $H_m = \pm H_m$, $G_m = G_m^{\pm}$ and $Q_m = G_m^{\mp}$, where $G_m^{\pm}$ are the half-integer moded fermionic generators. As a result, the standard Neveu-Schwarz h.w. vectors correspond to $G_0$-closed topological h.w. vectors, whereas the chiral (antichiral) Neveu-Schwarz h.w. vectors, annihilated by $G_{-1/2}^+$ ($G_{-1/2}^-$), correspond to chiral topological h.w. vectors. This implies 10, 12 that the standard, chiral and antichiral Neveu-Schwarz singular vectors correspond to topological singular vectors of the types $\Psi^{G}_{l,q,|\Delta| G}$ and $\Psi^{QQ}_{l,q,|\Delta| G}$, whereas the ones built in chiral or antichiral Neveu-Schwarz modules correspond to topological singular vectors of only the type $\Psi^{G}_{l,q,|\Delta| G}$. As a consequence, by untwisting the first row of table Tab. a one recovers the results 13, 14 that in Verma modules of the Neveu-Schwarz $N = 2$ algebra singular vectors only exist for charges $q = 0, \pm 1$ and two-dimensional spaces only exist for uncharged singular vectors. By untwisting the third row of table Tab. a one gets a proof for the conjecture 12 that chiral singular vectors in Neveu-Schwarz Verma modules only exist for $q = 0, 1$ whereas antichiral singular vectors only exist for $q = 0, -1$. The untwisting of the first row of table Tab. b, finally, proves the conjecture 10, 12 that in chiral Neveu-Schwarz Verma modules $\mathcal{V}_{NS,ch}^{h,2h}$ singular vectors only exist for $q = 0, -1$, whereas in antichiral Verma modules $\mathcal{V}_{NS,ah}^{h,2h}$ singular vectors only exist for $q = 0, 1$.

As to the representations of the Ramond $N = 2$ algebra 18–21, they are exactly isomorphic to the representations of the topological $N = 2$ algebra. Namely, combining the topological twists $T_{W}^{\pm}$ and the spectral flows one constructs a one-to-one mapping between the Ramond singular vectors...
and the topological singular vectors, at the same levels and with the same charges\textsuperscript{16}. Therefore the results of tables Tab.\textit{a} and Tab.\textit{b} can be transferred to the Ramond singular vectors simply by exchanging the labels $G \rightarrow (+)$, $Q \rightarrow (–)$, where the helicity $(\pm)$ denotes the vectors annihilated by the fermionic zero modes $G_0^\pm$, and by taking into account that the chiral and indecomposable \textit{no-helicity} Ramond vectors\textsuperscript{12,11,16}, require conformal weight $\Delta + l = c/24$.

The twisted $N = 2$ superconformal algebra\textsuperscript{18–21} is not related to the other three $N = 2$ algebras. It has mixed modes, integer and half-integer, for the fermionic generators, and the eigenvectors have no charge, as the U(1) current generators are half-moded, but they have \textit{fermionic parity}. The Adapted Ordering Method was worked out for the twisted $N = 2$ algebra in Ref. 2. The maximal dimension for the singular vector spaces in standard Verma modules was found to be two and these two-dimensional singular spaces were shown to exist by explicit computation, starting at level $3/2$. In Verma modules built on $G_0$-closed h.w. vectors, however, the singular vectors were found to be only one-dimensional. This method also allowed to propose a reliable conjecture for the coefficients of the relevant terms of all singular vectors, i.e. for the coefficients with respect to the ordering kernels, what made possible to identify all the cases of two-dimensional singular vector spaces for all levels, as well as to identify all $G_0$-closed singular vectors. The resulting expressions, in turn, led to the discovery of subsingular vectors for this algebra, and several explicit examples were also computed. Finally, the multiplication rules for singular vectors operators were derived using the ordering kernel coefficients, what set the basis for the analysis of the twisted $N = 2$ embedding diagrams.

Finally let us consider the $N = 1$ superconformal algebras\textsuperscript{6,21–23}. The structure of the h.w. representations of the Neveu-Schwarz $N = 1$ algebra has been completely understood in Ref. 24. The corresponding Verma modules do not contain two-dimensional singular vector spaces neither subsingular vectors. In the case of the Ramond $N = 1$ algebra, however, the application of the Adapted Ordering Method in Ref. 3 has shown that its representations have a much richer structure than previously suggested in the literature. In particular, it was found that standard Verma modules may contain two-dimensional singular vector spaces and also subsingular vectors. Moreover, the two-dimensional ordering kernels allowed to derive multiplication rules for singular vector operators and led to expressions for two-dimensional singular spaces. Using these multiplication rules descendant singular vectors were studied and embedding diagrams were derived for the rational models. In addition, this allowed to conjecture the ordering kernel coefficients of all singular vectors and therefore identify these vectors uniquely.

### 4 Conclusions and Final Remarks

We have presented the Adapted Ordering Method for general algebras and superalgebras provided they can be triangulated, like is the case for most Lie algebras and superalgebras. This method is based on the concept of adapted ordering, which implies that any singular vector needs to contain at least one non-trivial term included in the ordering kernel. The size of the ordering kernel therefore limits the dimension of the corresponding singular vector space. As a result the adapted orderings must be chosen such that the ordering kernels are as small as possible. Weights for which the ordering kernels are trivial do not allow any singular vectors in the corresponding weight spaces. On the other hand, non-trivial ordering kernels give us the maximal dimension of a possible singular vector space and uniquely define all singular vectors through the coefficients with respect to them.

The Adapted Ordering Method has been applied so far to the $N = 2$ and Ramond $N = 1$
superconformal algebras, allowing to prove several conjectured results as well as to obtain many new results, as we have reviewed. For example, this method allowed to discover subsingular vectors and two-dimensional spaces of singular vectors for the twisted $N=2$ and Ramond $N=1$ algebras\cite{2,3}. (For the other three isomorphic $N=2$ algebras two-dimensional singular spaces had been discovered\cite{12-14}, as well as subsingular vectors\cite{9-12}, before the Adapted Ordering Method was applied to them). We are convinced therefore that this method should be of very much help for the study of the representation theory of many other algebras, in particular the $N > 2$ superconformal algebras.

Acknowledgements

I am grateful to Christoph Schweigert for providing some information about triangulated algebras and to Bert Schellekens for reading carefully the manuscript. The work of the author is partially supported by funding of the spanish Ministerio de Educació n y Ciencia, Research Project BFM2002-03610.

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