ON SOBOLEV ROUGH PATHS

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Abstract. We introduce the space of rough paths with Sobolev regularity and the corresponding concept of controlled Sobolev paths. Based on these notions, we study rough path integration and rough differential equations. As main result, we prove that the solution map associated to differential equations driven by rough paths is a locally Lipschitz continuous map on the Sobolev rough path space for any arbitrary low regularity $\alpha$ and integrability $p$ provided $\alpha > 1/p$.

Keywords: Itô–Lyons map, Sobolev space, rough differential equation, rough path.
MSC 2020 Classification: 60L20.

1. Introduction

Loosely speaking, a rough path in the sense of T. Lyons [Lyo98] is a path $X$ from $[0, 1]$ taking values in a suitable algebraic structure, namely the step-$N$ free nilpotent group $G^N(R^d)$, and possessing sufficient regularity such as $\alpha$-Hölder continuity or $1/\alpha$-variation for $\alpha > 1/N$. While rough path theory found many successful applications over the past two decades, its original motivation is to study so-called rough differential equations (RDEs)

\begin{equation}
    dY_t = V(Y_t) dX_t, \quad Y_0 = y_0, \quad t \in [0, 1],
\end{equation}

where $y_0 \in \mathbb{R}^e$ is an initial value and $V$ is a smooth vector field on $\mathbb{R}^e$ mapping into the linear operators from $\mathbb{R}^d$ to $\mathbb{R}^e$.

As long as the driving signal $X = X$ is simply a path $X:[0,1] \to \mathbb{R}^d$, which is at least weakly differentiable with $p$-integrable derivative, that is, $X$ belongs to the Sobolev space $W^1_p$, the rough differential equation (1.1) is a classical object in analysis known as controlled ordinary differential equations, see, e.g., [Fil88]. Controlled differential equations appear in several different areas of analysis or geometry. The regularity of the driving signal plays an important role there, as well as the metric properties of the spaces of driving signals, e.g., its reflexivity or strict convexity: for instance in sub-riemannian geometry, see [Mon02], the geodesic problem leads to a minimal energy problem for horizontal paths, which can be solved due to the Hilbert space structure of the space of driving signals. Or in machine learning, where controlled differential equations can be regarded as a continuous depth version for deep feed forward neural networks, see [CLT20], the target problem in the spirit of the Chow-Rashevskii theorem leads to a minimal energy problem involving again strong metric properties of the space of driving signals.

The problem of choosing appropriate spaces of rough paths becomes considerably more involved when path regularity decreases, since non-linear effects appear. As soon as the driving signal $X$ is a sample path of a stochastic process like a Brownian motion, the RDE (1.1) (then also known as stochastic differential equation) cannot be treated anymore by classical methods from real analysis, cf. [Lyo91], but needs stochastic methods. An alternative way,
which fully clarifies the nature of the appearing non-linearity, is to assume that the driving signal $X$ is a rough path in the sense of T. Lyons. Then, the theory of rough paths establishes that (1.1) possesses a unique solution $Y$ and the solution map $X \mapsto Y$ is locally Lipschitz continuous with respect to suitable rough path metrics. In the context of rough path theory the map $X \mapsto Y$ is often called Itô–Lyons map. For more detailed introductions to rough path theory we refer to \cite{LC07, Lej09, FV10, FH14}.

It is well-known that the space of rough paths can be introduced in various ways by postulating different regularity properties in the definition of a rough path. Of course, all these rough path spaces share the fundamental feature that the Itô–Lyons map is locally Lipschitz continuous with respect to the corresponding distances on the underlying rough path spaces, see e.g. \cite{FV10} or \cite{FP18} and the references therein. The aim of the present paper is to introduce the fractional Sobolev regularity as defining regularity property of a rough path and show the local Lipschitz continuity of the Itô–Lyons map.

The metric structure of Sobolev spaces $W^{\alpha}_p$ for $1 < p < +\infty$ offers many favourable properties which are not provided by the frequently used distances on the rough path spaces such as Hölder or $p$-variation distances. Among others, let us mention for instance that the real-valued Sobolev spaces are known to be strictly convex, separable, reflexive, UMD Banach spaces of martingale type 2. Some of these properties are essential to solve optimization problems or to set up stochastic integration. Furthermore, Sobolev settings allow for better moment estimates in the context of stochastic partial differential equations. In the theory of regularity structure \cite{Hai14} and of paracontrolled distributions \cite{GIP15}, which are both closely related to rough path theory, the aforementioned favourable properties lead to a recent effort to introduce Sobolev distances or the even more general Besov distances in these theories, see e.g. \cite{HL17, LPT20a, HR20} for regularity structures and e.g. \cite{PT16, MP19, Hos20} for paracontrolled distributions.

The fractional Sobolev spaces appear naturally in the study of differential equations in classical analysis and of stochastic differential equations, for example, when working with Cameron–Martin spaces. These Sobolev spaces appear even in the context of rough differential equations, see \cite{CF10}. However, Sobolev distances on the space of rough paths are not used so far. The main reason for this steams from the fact that, in general, it was unclear so far whether the Itô–Lyons map is locally Lipschitz continuous with respect to the inhomogeneous Sobolev distance, without losing regularity\footnote{i.e. mapping $X$ with Sobolev regularity $\alpha$ to $Y$ with Sobolev regularity $\beta$ for $\beta < \alpha$}.

In Section 2 and 3 we introduce the space of Sobolev rough paths and the corresponding space of controlled paths of Sobolev type. As a first step, we demonstrate that these spaces lead to rough path integration with its known properties and to standard stability results as usually offered by rough path theory. Our approach is based on a novel discrete characterization of (non-linear) Sobolev spaces (see \cite{LPT20a}) in combination with classical estimates from rough path theory and Sobolev-variation embedding theorems (see also \cite{FV10}).

In Section 4 and 5 we manage to obtain the local Lipschitz continuity of the Itô–Lyons map acting on the space of Sobolev rough paths with arbitrary low regularity $\alpha > 0$ and integrability $p$ such that $\alpha > 1/p$. Although our proof again utilizes some of the sophisticated estimates from rough path theory, the Sobolev distances creates some new challenges mainly because of its missing direct link to a control function. Indeed, let us recall that numerous definitions of rough path spaces rely on metrics closely related the concept of so-called control functions $\omega$, which provide good estimates of increments of the type $|Y_t - Y_s| \leq \omega(s, t)$ such
as the $p$-variation norm with $\omega(s, t) := \|Y\|_{p_{\text{var}}[s, t]}^p$. While these type of estimates make it convenient to work $p$-variation or related semi-norms, the fractional Sobolev norm does not come with such convenient estimates of increments of rough paths.

The present work confirms that the Sobolev regularity offers a suitable topology on the space of rough paths and that the solution theory for rough differential equations naturally extends the classical solution theory of controlled ordinary differential equations based on Sobolev spaces. Additionally, this guarantees the access to the above mentioned favourable properties.

**Organization of the paper:** In Section 2 we introduce the space of Sobolev rough paths. Controlled paths of Sobolev type are discussed in Section 3 and rough differential equations driven by Sobolev rough paths are studied in Section 4. The local Lipschitz continuity of the Itô–Lyons map acting on the space of Sobolev rough paths with arbitrary low regularity is provided in Section 5.

**Acknowledgment:** C. Liu and J. Teichmann gratefully acknowledge support by the ETH foundation, D. Prömel and J. Teichmann gratefully acknowledge support by SNF Project 163014.

## 2. Sobolev rough path space

The definition of a rough path in the sense T. Lyons [Lyo08] basically consists of two components: an algebraic structure and an analytic regularity condition. While we work with the standard algebraic structure, we shall introduce a Sobolev regularity, which is in contrast to the common approaches in rough path theory, cf. [LCL07, Lej09, FV10, FH14].

We start by recalling some basic notation and definitions from rough path theory, as used e.g. in [FV10], and introduce the underlying algebra structure, which can be conveniently described by the free nilpotent Lie group $G^N(\mathbb{R}^d)$. Let $\mathbb{R}^d$ be the Euclidean space with norm $|\cdot|$ for $d \in \mathbb{N}$. The tensor algebra over $\mathbb{R}^d$ is defined by

$$T(\mathbb{R}^d) := \bigoplus_{n=0}^{\infty} (\mathbb{R}^d)^{\otimes n}$$

where $(\mathbb{R}^d)^{\otimes n}$ denotes the $n$-tensor space of $\mathbb{R}^d$ with the convention $(\mathbb{R}^d)^{\otimes 0} := \mathbb{R}$. We equip $T(\mathbb{R}^d)$ with the standard addition $+$, tensor multiplication $\otimes$ and scalar product.

Let $C^{1, \text{var}}([0, 1]; \mathbb{R}^d)$ be the space of all continuous functions $Z : [0, 1] \to \mathbb{R}^d$ of finite variation. For $N \in \mathbb{N}$ and a path $Z \in C^{1, \text{var}}([0, 1]; \mathbb{R}^d)$, its step-$N$ signature is defined by

$$S_N(Z)_{s, t} := \left(1, \int_{s < u < t} dZ_{u}, \ldots, \int_{s < u_1 < \cdots < u_N < t} dZ_{u_1} \otimes \cdots \otimes dZ_{u_N}\right)$$

$$\in T^N(\mathbb{R}^d) := \bigoplus_{k=0}^{N} (\mathbb{R}^d)^{\otimes k} \subset T(\mathbb{R}^d),$$

cf. [FV10] Definition 7.2. The corresponding space of all these lifted paths is the step-$N$ free nilpotent group (w.r.t. $\otimes$)

$$G^N(\mathbb{R}^d) := \{S_N(Z)_{0, 1} : Z \in C^{1, \text{var}}([0, 1]; \mathbb{R}^d)\} \subset T^N(\mathbb{R}^d).$$
On $G^N(\mathbb{R}^d)$ one usually works with two types of complete metrics: the first metric is given by
\[
\rho(g, h) := \max_{i=1,\ldots,N} |\pi_i(g - h)| \quad \text{for} \quad g, h \in G^N(\mathbb{R}^d),
\]
where $\pi_i$ denotes the projection from $\bigoplus_{i=0}^N(\mathbb{R}^d)^{\otimes i}$ onto the $i$-th level. We set $|g| := \rho(g, 1)$ for $g \in G^N(\mathbb{R}^d)$. The second one is the Carnot–Caratheodory metric $d_{cc}$, which is given by
\[
d_{cc}(g, h) := \|g^{-1} \otimes h\|_{cc} \quad \text{for} \quad g, h \in G^N(\mathbb{R}^d),
\]
where $\|\cdot\|_{cc}$ is the Carnot–Caratheodory norm defined via [FV10, Theorem 7.32], cf. [FV10, Definition 7.41]. These two metrics are in general not equivalent (unless $d = 1$) in the sense that there exist constants $C_1, C_2$ such that $C_1 \rho(g, h) \leq d_{cc}(g, h) \leq C_2 \rho(g, h)$ for all $g, h \in G^N(\mathbb{R}^d)$. However, one has
\[
\rho(g, h) \leq Cd_{cc}(g, h) \quad \text{and} \quad d_{cc}(g, h) \leq C\rho(g, h)^{1/N},
\]
for some constant $C > 0$, uniformly on bounded sets (w.r.t. the Carnot–Caratheodory norm), see [FV10, Proposition 7.49]. In the following we equip the free nilpotent Lie group $G^N(\mathbb{R}^d)$ with the Carnot–Caratheodory metric $d_{cc}$, which turns $G^N(\mathbb{R}^d)$ into a complete geodesic metric space. For a path $X : [0, 1] \to G^N(\mathbb{R}^d)$, we set $X_{s:t} := X_s^{-1} \otimes X_t$ for any subinterval $[s, t] \subset [0, 1]$. We refer to [FV10, Chapter 7] for a more comprehensive introduction to $G^N(\mathbb{R}^d)$.

Next we introduce the analytic regularity conditions required on a rough path. A partition $\pi$ of an interval $[s, t]$ is a collection of finitely many essentially disjoint interval covering $[s, t]$, i.e., $\mathcal{P} := \{[t_{k-1}, t_k] : s = t_0 < t_1 < \cdots < t_n = t, n \in \mathbb{N}\}$. In this case we write $\mathcal{P} \subset [s, t]$ indicating that $\mathcal{P}$ is a partition of the interval $[s, t]$. Furthermore, for such a partition $\mathcal{P}$ and a function $\chi : \{(u, v) : s \leq u < v \leq t\} \to \mathbb{R}$ we use the abbreviation
\[
\sum_{[u, v] \in \mathcal{P}} \chi(u, v) := \sum_{i=0}^{n-1} \chi(t_i, t_{i+1}).
\]
In the following, if not otherwise specified, $(E, d)$ denotes a metric space and $C([0, 1]; E)$ stands for the set of all continuous functions $f : [0, 1] \to E$. We can obtain a metric thereon by $d_{cc}(f, g) := \sup_{0 \leq t \leq 1} d(f(t), g(t))$. If $E$ is normed vector space with norm $\|\cdot\|$, we define $\|f\|_{\infty} := \sup_{0 \leq t \leq 1}\|f(t)\|$. The $q$-variation of a function $f \in C([0, 1]; E)$ is defined by
\[
(2.1) \quad \|f\|_{q, \text{var}; [s, t]} := \left( \sup_{\mathcal{P} \subset [s, t]} \sum_{[u, v] \in \mathcal{P}} d(f_u, f_v)^q \right)^{1/q}, \quad q \in [1, +\infty),
\]
where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[s, t]$. The set of all functions $f \in C([0, 1]; E)$ with $\|f\|_{q, \text{var}} := \|f\|_{q, \text{var}; [0, 1]} < \infty$ is denoted by $C^{q, \text{var}}([0, 1]; E)$. For $r \in \mathbb{R}_+$ we set
\[
[r] := \sup\{n \in \mathbb{Z} : n \leq r\} \quad \text{and} \quad [r] := \sup\{n \in \mathbb{Z} : n < r\}.
\]
The space of all weakly geometric rough paths of finite $q$-variation is then given by
\[
\Omega^q := C^{q, \text{var}}([0, 1]; G^{|q|}(\mathbb{R}^n)) := \left\{ X \in C([0, 1]; G^{|q|}(\mathbb{R}^n)) : \|X\|_{q, \text{var}} < \infty \right\},
\]
where $\|\cdot\|_{q, \text{var}}$ is the $q$-variation with respect to the metric space $(G^{|q|}(\mathbb{R}^n), d_{cc})$ as defined in (2.1). Let us remark that $\|\cdot\|_{q, \text{var}}$ on $\Omega^q$ is often called the homogeneous rough path
norm because it is homogeneous with respect to the dilation map on $T^{[q]}(\mathbb{R}^n)$, cf. \cite{FP18} Definition 7.13]. The $q$-variation norm is frequently used in rough path theory but it is well-known that there exists a cascade of good metrics to measure the regularity of a rough path, see e.g. \cite{FP18} for a discussion about rough path metrics.

In contrast to the commonly used metrics in rough path theory, we shall consider fractional Sobolev metrics. For this purpose, let recall the definition of Sobolev regularity for functions mapping into a metric space $E$. For $\alpha \in (0,1)$, $p \in (1, +\infty)$ and a function $f \in C([0,T];E)$ we define the \textit{fractional Sobolev regularity} by

\begin{equation}
\|f\|_{W^\alpha_p;[s,t]} := \left( \int_{[s,t]^2} \frac{d(f(u),f(v))^p}{|v-u|^\alpha} \, du \, dv \right)^{1/p}
\end{equation}

and in the case of $p = +\infty$ we set

$$
\|f\|_{W^\alpha_p;[s,t]} := \sup_{u,v \in [s,t]} \frac{d(f(u),f(v))}{|v-u|^\alpha},
$$

where $[s,t] \subset [0,1]$. The latter case is also known as Hölder regularity. Furthermore, we set $\|f\|_{W^\alpha_p} := \|f\|_{W^\alpha_p;[0,1]}$. The space $W^\alpha_p([0,1];E)$ consists of all continuous functions $f: [0,1] \to E$ such that $\|f\|_{W^\alpha_p} < \infty$. Note that, for a continuous function $f: [0,1] \to E$, the fractional Sobolev (semi)-distance can be equivalently defined in a discrete way by

\begin{equation}
\|f\|_{W^\alpha_p;1} := \left( \sum_{j \geq 0} 2^{j(\alpha p-1)} \sum_{m=0}^{2^j-1} d_{cc}(f(\frac{m}{2^j}), f(\frac{m+1}{2^j}))^p \right)^{1/p},
\end{equation}

see \cite{LPT20a} Theorem 2.2], that is, there exist two constants $C_1, C_2 > 0$ such that

$$
C_1 \|f\|_{W^\alpha_p;1} \leq \|f\|_{W^\alpha_p} \leq C_2 \|f\|_{W^\alpha_p;1}, \quad f \in C([0,1];E).
$$

The Sobolev topology leads naturally to the notion of (fractional) Sobolev rough paths.

\textbf{Definition 2.1} (Sobolev rough path). Let $\alpha \in (0,1)$ and $p \in (1, +\infty)$ be such that $\alpha > 1/p$.

The space $W^\alpha_p([0,1];G^{1/\alpha}(\mathbb{R}^d))$ consists of all paths $X: [0,1] \to G^{1/\alpha}(\mathbb{R}^d)$ such that

$$
\|X\|_{W^\alpha_p} := \left( \int_{[0,1]^2} \frac{d_{cc}(X_s, X_t)^p}{|t-s|^{\alpha p+1}} \, ds \, dt \right)^{1/p} < +\infty.
$$

The space $W^\alpha_p([0,1];G^{1/\alpha}(\mathbb{R}^d))$ is called the \textit{weakly geometric Sobolev rough path space} and $X \in W^\alpha_p([0,1];G^{1/\alpha}(\mathbb{R}^d))$ is called a \textit{weakly geometric rough path of Sobolev regularity $(\alpha,p)$} or short \textit{Sobolev rough path}.

\textbf{Remark 2.2}. Assuming that $\alpha \in (0,1)$ and $p \in (1, +\infty)$ with $\alpha > 1/p$, every weakly geometric rough path of Sobolev regularity $(\alpha,p)$ is also Hölder continuous of order $\alpha - 1/p$, which can be seen with the help of the Garsia–Rodemich–Rumsey inequality, see e.g. \cite{FP18} Theorem A.1].

Furthermore, the metric structure provided by the Sobolev metric allows to conveniently approximate Sobolev rough path by geodesic interpolations along the dyadic numbers, see \cite{LPT20a} Section 3.3].

In order to obtain the Lipschitz continuity of the solution map associated to differential equations driven by Sobolev rough paths, we need to introduce an inhomogeneous Sobolev
distance \( \hat{\rho}_{W_p} \) on \( W^\alpha_p([0, 1]; G^{1/\alpha}(\mathbb{R}^d)) \) defined by

\[
\hat{\rho}_{W_p}(X^1, X^2) := \sum_{k=1}^{[d]} \rho^{(k)}_{W_p}(X^1, X^2), \quad \text{for} \quad X^1, X^2 \in W^\alpha_p([0, 1]; G^{1/\alpha}(\mathbb{R}^d)),
\]

where

\[
\rho^{(k)}_{W_p}(X^1, X^2) := \left( \sum_{j \geq 0} 2^{j(\alpha p-1)} \sum_{i=1}^{2^j} \pi_k(X^1_{(i-1)2^{-j}, i2^{-j}} - X^2_{(i-1)2^{-j}, i2^{-j}})^\frac{1}{p} \right)^\frac{1}{p}.
\]

Note that \( \hat{\rho}_{W_p} \) is the inhomogeneous counterpart of the discretely defined homogeneous Sobolev norm \([2,3]\), which is equivalent to the (classical) Sobolev metric as defined in \([2,2]\), see \([LPT20c]\) Theorem 2.2. Therefore, by using the equivalence of the homogeneous norms on the Carnot group \( G^{1/\alpha}(\mathbb{R}^d) \) (see \([FV10]\) Theorem 7.44), one can verify that \( \hat{\rho}_{W_p}(X^1, X^2) < +\infty \) for \( X^1, X^2 \) in \( W^\alpha_p([0, 1]; G^{1/\alpha}(\mathbb{R}^d)) \).

**Remark 2.3.** Note that there is also a canonical way to introduce the inhomogeneous Sobolev distance analogously to the integral definition of the homogeneous Sobolev norm \([2,2]\), which is expected to be equivalent to the discretely defined inhomogeneous Sobolev distance \([2,4]\).

Moreover, already in the case of homogeneous Sobolev norms, it was a challenging task to show the equivalence of the Sobolev norm via integrals \([2,2]\) and the discretely defined Sobolev norm \([2,3]\), see \([LPT20a]\).

In the following we frequently use the abbreviations: For two real functions \( a, b \) depending on variables \( x \) we write \( a \lesssim b \) or \( a \lesssim x \) \( b \) if there exists a constant \( C(z) > 0 \) such that \( a(x) \leq C(z) \cdot b(x) \) for all \( x \), and \( a \sim b \) if \( a \lesssim b \) and \( b \lesssim a \) hold simultaneously.

### 3. On controlled paths of Sobolev type

After having introduced the space of Sobolev rough paths, one wants to ensure that Sobolev rough paths lead to a fully fledged rough path integration and allow to set up a solution theory for rough differential equations. In order to demonstrate the difficulties arising by working with Sobolev rough paths, let us consider first the two level case \( W^\alpha_p([0, 1]; G^2(\mathbb{R}^d)) \) for \( \alpha \in (1/3, 1/2) \) and \( p \in [1, +\infty) \) with \( \alpha > 1/p \). We shall deal with the general case in Section 4 and 5. In this section we follow the approach using controlled paths as introduced
Proof. By [PP16, Theorem 4.9], one has

The corresponding space of all such controlled paths with respect to a given rough path $X$ is $C^{\frac{1}{\alpha}}_{\var}(\[0,1];G^2(\mathbb{R}^d))$ if

(i) $(Y,Y') \in C^{\frac{1}{\alpha}}_{\var}(\[0,1];\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e) \oplus \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d;\mathbb{R}^e))$ and

(ii) $R^Y: \Delta \rightarrow \mathbb{R}^e$, given by $R^Y_{s,t} := Y_{s,t} - Y'_{s,t} X_{s,t}$ for $(s,t) \in \Delta$, satisfies $\|R^Y\|_{\frac{1}{\alpha}} \var < +\infty$.

The corresponding space of all such controlled paths with respect to $X$ is $C^{\frac{1}{\alpha}}(\[0,1];G^2(\mathbb{R}^d))$ is denoted by $D^1_{\var X}([0,T];\mathbb{R}^e)$.

By the Sobolev-variation embedding theorem [FV06 Theorem 2] the Sobolev rough path space $W^\alpha_p([0,1];G^2(\mathbb{R}^d))$ can be embedded into the space $C^{\frac{1}{\alpha}}_{\var}(\[0,1];G^2(\mathbb{R}^d))$. In particular, this implies that for any controlled path $(Y,Y') \in D^1_{\var X}([0,T];\mathbb{R}^e)$ the standard rough path integral $\int Y \, dX$ exists, cf. [PP16 Theorem 4.9], [Gub04 Theorem 1] and [FH13 Theorem 4.10], and $\int Y \, dX$ possesses the same $1/\alpha$-variation as the rough path $X$. In the next lemma, we make a first observation how these statements transfer into the Sobolev setting.

**Lemma 3.1.** Let $X = (X,X)$ be a Sobolev rough path in $W^\alpha_p([0,1];G^2(\mathbb{R}^d))$ for $\alpha \in (1/3,1/2)$ and $p \in (1, +\infty)$ with $\alpha > 1/p$. Let $(Y,Y') \in D^1_{\var X}([0,T];\mathbb{R}^e)$ be an $\mathbb{R}^e$-valued controlled rough path. Then, the rough path integral $\int Y \, dX$ exists and belongs to the Sobolev space $W^\alpha_p([0,1];\mathbb{R}^e)$ for every $\alpha' < \alpha$.

Before proving Lemma 3.1 let us recall the notion of control functions: A function $\omega: \Delta \rightarrow [0, +\infty)$ is called control function if $\omega(s,s) = 0$ for $s \in [0,1]$ and $\omega$ is super-additive.

**Proof.** By [PP16 Theorem 4.9], one has

\[
\left| \int_s^t Y_r \, dX_r - Y_s X_{s,t} - Y'_s X_{s,t} \right| \lesssim \|R^Y\|_{\frac{1}{\alpha}} \|X\|_{\frac{1}{\alpha}} + \|Y'\|_{\frac{1}{\alpha}} + \|X\|_{\frac{1}{\alpha}} \var \|X\|_{\frac{1}{\alpha}} \var.
\]

Now we fix an $\alpha' < \alpha$. Thanks to the discrete characterization of Sobolev rough path [LPT20a Theorem 2.2], in order to show that $\int Y_r \, dX_r \in W^\alpha_p([0,1];\mathbb{R}^e)$ it suffices to prove that

\[
\left\| \int Y_r \, dX_r \right\|_{W^\alpha_p \var(1)}^p := \sum_{j=0}^\infty \sum_{i=1}^{2^j} 2^{i(\alpha'p-1)} \left| \int_{\frac{j}{2^j}}^{\frac{j+1}{2^j}} Y_r \, dX_r \right|^p < +\infty.
\]
Indeed, applying (5.1), we get

$$\left\| \int Y_t \, dX_t \right\|_{W^{\alpha}_p;[0,1]} \leq \sum_{j=0}^{\infty} \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \left\| \int_{\frac{j-1}{2^j}}^{\frac{j}{2^j}} Y_t \, dX_t - Y_{\frac{j-1}{2^j}} X_{\frac{j-1}{2^j}} + Y_{\frac{j}{2^j}} X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]} \left\| X_{\frac{j-1}{2^j}} - X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]}$$

$$+ \sum_{j=0}^{\infty} \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \left\| Y_t \, dX_t - Y_{\frac{j-1}{2^j}} X_{\frac{j-1}{2^j}} + Y_{\frac{j}{2^j}} X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]} \left\| X_{\frac{j-1}{2^j}} - X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]}$$

Now we estimate separately each of the terms of the above sum.

For the third term the same reasoning leads to

$$\sum_{j=0}^{\infty} \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \left\| Y_t \, dX_t - Y_{\frac{j-1}{2^j}} X_{\frac{j-1}{2^j}} + Y_{\frac{j}{2^j}} X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]} \left\| X_{\frac{j-1}{2^j}} - X_{\frac{j}{2^j}} \right\|_{W^{\alpha}_p;[0,1]}$$

For the second term, by [LPT20a] Proposition 4.3 we observe that

$$\left\| X \right\|_{W^{\alpha}_p;[0,1]} \leq \left\| X \right\|_{W^{\alpha}_p;[0,1]} \left\| t - s \right\|_{W^{\alpha}_p;[0,1]}$$

for all $s < t$, which implies that

$$\left\| X \right\|_{\frac{1}{2}}, var;[0,1] \leq \left\| X \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| X \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| Y \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| Y \right\|_{\frac{2}{2}}, var;[0,1]$$

and consequently that

$$\left\| Y \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| Y \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| Y \right\|_{\frac{2}{2}}, var;[0,1] \leq \left\| Y \right\|_{\frac{2}{2}}, var;[0,1]$$

Since $\left\| X \right\|_{\frac{2}{2}}, var;[0,1]$ is a control function, it follows that

$$\sum_{j=0}^{\infty} \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \left\| Y_t \, dX_t \right\|_{W^{\alpha}_p;[0,1]} \leq \sum_{j=0}^{\infty} 2^{-j(\alpha p - 1)} \left\| Y_t \, dX_t \right\|_{W^{\alpha}_p;[0,1]}$$

and since $\alpha p - 1 > 0$, the sum on the right hand side converges.

For the third term the same reasoning leads to

$$\sum_{j=0}^{\infty} \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \left\| Y_t \, dX_t \right\|_{W^{\alpha}_p;[0,1]} \leq \sum_{j=0}^{\infty} 2^{-j(\alpha p - 1)} \left\| Y_t \, dX_t \right\|_{W^{\alpha}_p;[0,1]}$$

(3.2)
Thanks to the assumption that $\alpha' < \alpha$, the sum on the right hand side of the above inequality converges. Hence, the proof is completed. □

From Lemma [3.1] we see that, without adapting the regularity of the controlled path $(Y, Y')$, one can only guarantee that $\int Y \, dX$ belongs to the Sobolev space $W^{\alpha'}_p([0, 1]; \mathbb{R}^e)$ for every $\alpha' < \alpha$. In words, the rough path integral has less regularity than the rough path $X$. This observation motivates us to introduce a Sobolev topology also on the space of controlled paths.

Looking again at the third term [3.2] in the proof of Lemma 3.1, we notice that to ensure $\int Y \, dX$ belongs to the Sobolev space $W^\alpha_\beta([0, 1]; \mathbb{R}^e)$ separately, one has to find conditions on $R^Y$ such that the series

$$
\sum_{j=0}^\infty \sum_{i=1}^{2^j} 2^{j(\alpha p - 1)} \| R^Y \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} \| X \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]}^p
$$

converges. Applying the estimates

$$
\| X \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} \lesssim \| X \|_{W^\alpha_\beta([0, 1]; \mathbb{R}^e)} 2^{-j(\alpha + \beta)}
$$

to the above series, we essentially need the following condition:

$$
\sum_{j=0}^\infty \sum_{i=1}^{2^j} \| R^Y \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} \| X \|_{W^\alpha_\beta([0, 1]; \mathbb{R}^e)}^p < +\infty.
$$

More explicitly, we need that $\| R^Y \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]}$ can be compared to $2^{-j \beta}$ for some $\beta > 0$ uniformly over all $i = 1, \ldots, 2^j$ and $j \geq 1$. This consideration naturally leads us to invoke the so-called mixed Hölder-variation space introduced in [FP18]: we shall require that $R^Y$ satisfies that

$$
\sup_{[u, v] \in \mathcal{P}} \sum_{i=1}^{2^j} \left\| R^Y \right\|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} < +\infty.
$$

Once this is the case, then it follows immediately that

$$
\| R^Y \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} \leq C 2^{-2j(\alpha + \beta)}
$$

for all $i$ and $j$ with $C$ denoting the supremum in (3.3), and then as $\alpha p - 1 > 0$ it holds that

$$
\sum_{j=0}^\infty \sum_{i=1}^{2^j} \| R^Y \|_{2^{j \var}[\frac{i-1}{2^j}, \frac{i}{2^j}]} \| X \|_{W^\alpha_\beta([0, 1]; \mathbb{R}^e)}^p \leq \sum_{j=0}^\infty 2^{-2j(\alpha p - 1)} \sum_{i=1}^{2^j} \| X \|_{W^\alpha_\beta([0, 1]; \mathbb{R}^e)}^p \leq \left( \sum_{j=0}^\infty 2^{-2j(\alpha p - 1)} \right) \| X \|_{W^\alpha_\beta([0, 1]; \mathbb{R}^e)}^p < +\infty,
$$

as wished.

Inspired by the above observations, we introduce the following function space: Let $(B, \| \cdot \|)$ be a Banach space. For $\beta \in (0, 1)$ and $q \geq 1$ we use $V^\beta_q(\Delta; B)$ to denote the space of all
continuous functions \( f \in C(\Delta; B) \) such that
\[
\sup_{P} \sum_{[u,v] \in P} \frac{||f||_{\text{var},[u,v]}^q}{|u - v|^{3q - 1}} < +\infty.
\]
Moreover, for \([s, t] \subset [0, 1]\) we define
\[
||f||_{\tilde{V}^q_{\beta};[s, t]} := \left( \sup_P \sum_{[u,v] \in P} \frac{||f||_{1/\beta \text{-var},[u,v]}^q}{|u - v|^{3q - 1}} \right)^{\frac{1}{q}}
\]
and \( ||f||_{\tilde{V}^q_{\beta};[0, 1]} := ||f||_{\tilde{V}^q_{\beta};[0, 1]} \).

Let us remark that, if the remainder term \( R^Y \) attached to a controlled rough path \((Y, Y')\) satisfies additionally that \( R^Y \in \tilde{V}^{2\alpha}_{\frac{q}{2}}(\Delta; E) \), then the rough integral \( \int Y dX \) is an element in \( W^{\alpha}_{p}[0, 1]; \mathbb{R}^e \), by the previous discussion.

Furthermore, if we want to apply the Banach fixed point theorem to obtain existence and uniqueness results for rough differential equations driven by Sobolev signals \( X \) within the Sobolev framework, the Sobolev regularity of controlled paths is necessary, i.e., \((Y, Y')\) should be an element in \( W^{\alpha}_{p}([0, 1]; \mathcal{L}([\mathbb{R}^d, \mathbb{R}^e]) \times W^{\alpha}_{p}([0, 1]; \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^e)) \). In particular, since \( Y_{s,t} = Y_s'X_s,t + R^Y_{s,t} \), from the discrete characterization of Sobolev norms (2.3) we see that in this case \( R^Y \) satisfies
\[
\sum_{j=0}^{\infty} \sum_{i=1}^{2^j \alpha p - 1} \frac{||R^Y||_{1/2;[\frac{j}{2^j}; \frac{j+1}{2^j}}^p}{2^{j \alpha p - 1}} < +\infty.
\]
Hence, let us denote by \( \hat{W}^{\beta}_{q}(\Delta; \mathbb{R}^n) \) the space of all such \( f \in C(\Delta; \mathbb{R}^n) \) such that
\[
||f||_{\hat{W}^{\beta}_{q}} := \left( \sum_{j=0}^{\infty} \sum_{i=1}^{2^j \alpha p - 1} \frac{||f||_{C(\frac{j}{2^j}; \frac{j+1}{2^j}}^p}{2^{j \alpha p - 1}} \right)^{\frac{1}{p}} < +\infty.
\]
Hence, in the Sobolev setting the natural definition of controlled paths goes as follows.

**Definition 3.2.** Let \( X \) be an element in \( W^{\alpha}_{p}([0, 1]; \mathbb{R}^d) \). A pair \((Y, Y')\) is called a controlled path of Sobolev type \((\alpha, p)\) if \( Y \in W^{\alpha}_{p}([0, 1]; \mathbb{R}^n) \), \( Y' \in W^{\alpha}_{p}([0, 1]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^n)) \) and \( R^Y_{s,t} := Y_{s,t} - Y'_{s,t}X_{s,t} \) satisfies that \( R^Y \in \tilde{V}^{2\alpha}_{\frac{q}{2}}(\Delta; \mathbb{R}^n) \cap \hat{W}^{2\alpha}_{\frac{q}{2}}(\Delta; \mathbb{R}^n) \). The space of all such controlled rough paths is denoted by \( \mathcal{D}^{\alpha,p}_{X}([0, 1]; \mathbb{R}^n) \), which is equipped with the norm
\[
||(Y, Y')||_{\mathcal{D}^{\alpha,p}_{X}} := ||Y'||_{W^{\alpha}_{p}} + ||R^Y||_{\tilde{V}^{2\alpha}_{\frac{q}{2}}} + ||R^Y||_{\hat{W}^{2\alpha}_{\frac{q}{2}}} + ||Y_0|| + ||Y'_0||.
\]

**Remark 3.3.** From the definition of \( \tilde{V}^{2\alpha}_{\frac{q}{2}}(\Delta; \mathbb{R}^n) \) we can immediately see that if \( R^Y \in \tilde{V}^{2\alpha}_{\frac{q}{2}}(\Delta; \mathbb{R}^n) \), then it also has finite 1/\(2\alpha\)-variation. Hence, applying Sobolev-variation embedding results (see [FY09], Theorem 2) to \((Y, Y')\), it follows that every controlled path of Sobolev type \((\alpha, p)\) is a controlled path with finite 1/\(\alpha\)-variation. Moreover, using the discrete characterization of Sobolev norms, we can also see that \( ||Y||_{W^{\alpha}_{p}} \) can be estimated by \( ||R^Y||_{\tilde{V}^{2\alpha}_{\frac{q}{2}}} + ||Y'||_{W^{\alpha}_{p}} + ||Y_0|| + ||X||_{W^{\alpha}_{p}} \). Finally, we remark that \( \mathcal{D}^{\alpha,p}_{X}([0, 1]; \mathbb{R}^n) \), \( ||\cdot||_{\mathcal{D}^{\alpha,p}_{X}} \) is a Banach space.
With the notion of Sobolev rough paths and controlled paths of Sobolev type, one can recover many stability properties known for controlled paths with finite $q$-variations (e.g. under rough path integration, compositions of smooth functions, ...) also for controlled paths of Sobolev type. Let us just mention some of them here.

**Lemma 3.4.** Let $X$ be a Sobolev rough path in $W^\alpha_p([0,1];G^2(\mathbb{R}^d))$, $(Y,Y') \in \mathcal{D}_X^{\alpha,p}([0,1];\mathbb{R}^n)$ be an controlled path of Sobolev type. Let $I_X(Y) := \int Y \, dX$ be the rough path integral obtained as in Lemma 3.3. Then, one has:

(i) $(I_X(Y),Y)$ belongs to $\mathcal{D}_X^{\alpha,p}([0,1];\mathbb{R}^n)$.

(ii) If $\tilde{X}$ is another rough path in $W^\alpha_p([0,1];G^2(\mathbb{R}^d))$ and $(\tilde{Y},\tilde{Y}') \in \mathcal{D}_X^{\alpha,p}([0,1];\mathbb{R}^n)$, then

$$
\| R^I(Y) - R^I(\tilde{Y}) \|_{\tilde{Y}^{2\alpha}} + \| R^I(Y) - R^I(\tilde{Y}) \|_{\tilde{Y}'^{2\alpha}} + \| Y - \tilde{Y}' \|_{W^p} + \rho_\alpha(Y) + \rho_\alpha(\tilde{Y},\tilde{X}) + \rho_\alpha(X,\tilde{X})
$$

where $R^I(Y)$ and $R^I(\tilde{Y})$ are the remainder terms of $(I_X(Y),Y)$ and $(I_\tilde{X}(\tilde{Y}),\tilde{Y})$, respectively.

**Proof.** (i) We have already shown that with $(Y,Y') \in \mathcal{D}_X^{\alpha,p}([0,1];\mathbb{R}^n)$, the rough path integral $I_X(Y)$ is well-defined and belongs to $W^\alpha_p([0,1];\mathbb{R}^n)$. Hence, to show the item (i), it only remains to check that the remainder term $R^I(Y) := \int_s^t Y \, dX - Y_sX_{s,t}$ belongs to $\tilde{Y}^{2\alpha}(\Delta;\mathbb{R}^n)$.

By [PP16, Theorem 4.9], we note again that

$$
\| R^I(Y) \|_{\tilde{Y}^{2\alpha}} + \| R^I(Y) \|_{\tilde{Y}'^{2\alpha}} + \| Y - \tilde{Y}' \|_{W^p} + \rho_\alpha(Y) + \rho_\alpha(\tilde{Y},\tilde{X}) + \rho_\alpha(X,\tilde{X})
$$

where $R^I(Y)$ and $R^I(\tilde{Y})$ are the remainder terms of $(I_X(Y),Y)$ and $(I_\tilde{X}(\tilde{Y}),\tilde{Y})$, respectively.

(ii) First, since $Y_{s,t} = Y'_sX_{s,t} + R^Y_{s,t}$ and $\tilde{Y}_{s,t} = \tilde{Y}'_s\tilde{X}_{s,t} + R^\tilde{Y}_{s,t}$, we have

$$
\| Y_{s,t} - \tilde{Y}_{s,t} \| \lesssim \| Y'_s \| X_{s,t} - \tilde{X}_{s,t} \| + \| Y'_s - \tilde{Y}'_s \| X_{s,t} \| + \| R^Y_{s,t} - R^\tilde{Y}_{s,t} \|.
$$

By taking $s = \frac{i-1}{t}$, $t = \frac{i}{2}$ for $j \geq 0$ and $i = 1, \ldots, 2^j$ and using the discrete characterization of Sobolev norms as before, we can check that

$$
\| Y - \tilde{Y} \|_{W^p} \lesssim \| X - \tilde{X} \|_{W^p} + \| Y - \tilde{Y} \|_{W^p} + \| R^Y - R^\tilde{Y} \|_{\tilde{Y}^{\alpha}}.
$$

Next we bound the term $\| R^I(Y) - R^I(\tilde{Y}) \|_{\tilde{Y}^{2\alpha}}$. In the first step above we have seen that $R^I(Y) = Y'_sX_{s,t} + h^Y_{s,t}$ with the residue function $h^Y_{s,t}$ having finite $1/3\alpha$ variation. Similarly
$R_{s,t}^{h}(\tilde{Y}) = \tilde{Y}_{s,t}^{h} + h_{s,t}^{\tilde{Y}}$ for some $h_{s,t}^{\tilde{Y}}$ of finite 1/3$\alpha$ variation. Moreover, from the classical sewing lemma (cf. [FHL13]) we also know that
d$h_{s,u,t} := h_{s,t}^{\tilde{Y}} - h_{s,u}^{\tilde{Y}} - h_{u,t}^{\tilde{Y}} = -R_{s,u}^{Y}X_{u,t} - Y_{s,u}^{\var}X_{u,t},$
and the similar relation holds for $\delta h_{s,u,t}^{\tilde{Y}}$ for $s < u < t$. Then, since $3\alpha > 1$, the sewing lemma
applied to the difference $\delta h_{s,u,t}^{\tilde{Y}} - \delta h_{s,u,t}^{\tilde{Y}}$ leads to the bound

$$|h_{s,t}^{\tilde{Y}} - h_{s,t}^{\tilde{Y}}| \lesssim \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\frac{1}{2\alpha}{\text{var}},[s,t]} \|X\|_{\frac{1}{2\alpha}{\text{var}},[s,t]} + \|R_{s}^{\tilde{Y}}\|_{\frac{1}{2\alpha}{\text{var}},[s,t]} |X_{s,t} - \tilde{X}_{s,t}|$$

$$+ \|\tilde{Y}'\|_{\frac{1}{\alpha}{\text{var}},[s,t]} \|X - \tilde{X}\|_{\frac{1}{\alpha}{\text{var}},[s,t]} + \|\tilde{Y}'\|_{\frac{1}{\alpha}{\text{var}},[s,t]} \|X\|_{\frac{1}{2\alpha}{\text{var}},[s,t]}.$$ 

Now, inserting $s = \frac{i-1}{2^j}$ and $t = \frac{i}{2^j}$, we can follow the same lines as in the proof of Lemma 3.1
to deduce that

$$\sum_{j=0}^{\infty} 2^j(\alpha - 1) \sum_{i=1}^{2^j} \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\frac{1}{2\alpha}{\text{var}},[\frac{i}{2^j},\frac{i+1}{2^j}]} \|X\|_{\frac{1}{2\alpha}{\text{var}},[\frac{i}{2^j},\frac{i+1}{2^j}]}$$

$$\lesssim \sum_{j=0}^{\infty} 2^j(\alpha - 1) \sum_{i=1}^{2^j} \|X\|_{\frac{1}{2\alpha}{\text{var}},[0,1]} \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(1)},[\frac{i}{2^j},\frac{i+1}{2^j}]} 2^{-j(\alpha - 1)}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-j(\frac{2\alpha}{2} - 1)} \|X\|_{\tilde{W}_{2\alpha}^{(2)},[0,1]} \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(1)},[\frac{i}{2^j},\frac{i+1}{2^j}]} 2^{-j(\alpha - 1)}$$

Thus, for $F_{s,t}^{1} := \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\frac{1}{2\alpha}{\text{var}},[s,t]} \|X\|_{\frac{1}{2\alpha}{\text{var}},[s,t]}$, we obtain that

$$\|F_{s,t}^{1}\|_{\tilde{W}_{\frac{1}{2\alpha}}^{(1)}} \lesssim \|X\|_{\tilde{W}_{2\alpha}^{(2)},[0,1]} \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(1)},[0,1]}.$$ 

Applying the same reasoning to $F_{s,t}^{2} := \|R_{s}^{\tilde{Y}}\|_{\frac{1}{2\alpha}{\text{var}},[s,t]} \|X_{s,t} - \tilde{X}_{s,t}\|$, $F_{s,t}^{3} := \|\tilde{Y}'\|_{\frac{1}{\alpha}{\text{var}},[s,t]} \|X - \tilde{X}\|_{\frac{1}{\alpha}{\text{var}},[s,t]}$ and $F_{s,t}^{4} := \|Y' - \tilde{Y}'\|_{\frac{1}{\alpha}{\text{var}},[s,t]} \|X\|_{\frac{1}{\alpha}{\text{var}},[s,t]}$ and noting that $|h_{s,t}^{\tilde{Y}} - h_{s,t}^{\tilde{Y}}| \lesssim \sum_{i=1}^{4} F_{s,t}^{i}$, we can conclude that

$$\|h_{s,t}^{Y} - h_{s,t}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(2)}} \lesssim \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(1)}} + \tilde{\rho}W_{p}^{(1)}(X, \tilde{X}),$$

which in turn implies that $\|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(2)}} \lesssim \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(1)}} + \|Y' - \tilde{Y}'\|_{W_{p}^{(1)}} + \tilde{\rho}W_{p}^{(1)}(X, \tilde{X}).$

A similar calculation also provides a similar bound for $\|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(2)}}$, which completes the proof of (ii).

\[\square\]

**Remark 3.5.** The proof of the Lemma 3.4 illustrates the reason why we choose the discrete Sobolev norm $\|\cdot\|_{\tilde{W}_{2\alpha}^{(2)}}$ instead of $\|\cdot\|_{\tilde{W}_{p}^{(1)}}$ in Definition 3.2 because in general one only has

$$\|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{p}^{(1)}} \lesssim \|R_{s}^{Y} - R_{s}^{\tilde{Y}}\|_{\tilde{W}_{2\alpha}^{(2)}} + \|Y' - \tilde{Y}'\|_{W_{p}^{(1)}} + \tilde{\rho}W_{p}^{(1)}(X, \tilde{X})^{\frac{1}{2}},$$

so that we do not have a (local) Lipschitz estimates.
The same regularity condition for the second order term \( X \) appears in the framework of paracontrolled distributions when working with Sobolev spaces, see [PT16, Definition 5.1].

**Remark 3.6.** Recall that the rough path integration coincides with the classical Young integration if \( \alpha > 1/2 \). For the Young integral is well-known that the integration operator is continuous with to the Sobolev distance, see, e.g., [Kam94] and [Zah98, Zah01]. This in line with Lemma 3.4. In the case \( \alpha > 1/2 \) the second order term \( X \) does not appear, therefore, the Sobolev distance \( \hat{\rho}_{W^p} \) can be equivalently defined in its integral form, which dominates the distance \( \rho_{\hat{\nu}} \), see [PP18, Corollary 2.12]. However, for the rough path distances we (currently) cannot avoid the use of \( \rho_{\hat{\nu}} \), see also Remark 3.7 below.

Controlled paths of Sobolev type are also stable under compositions of smooth functions. For \( n \in \mathbb{N} \) let \( C^n_0(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \) be the space of \( n \)-times continuously differentiable functions \( f: \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e) \) such that \( f \) and its derivatives of up to order \( n \) are bounded.

**Lemma 3.7.** Let \( F \in C^q_0(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \) and \( (Y, Y') \in \mathcal{D}^{\alpha,p}_X([0,1]; \mathbb{R}^e) \). Then, one has:

(i) \( (F(Y), F(Y')) := (F(Y), DF(Y)Y') \in \mathcal{D}^{\alpha,p}_X([0,1]; \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)) \).

(ii) If \( X \) is another rough path in \( W^p_\alpha([0,1]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \) and \( (\tilde{Y}, \tilde{Y}') \in \mathcal{D}^{\alpha,p}_X([0,T]; \mathbb{R}^e) \), then

\[
\| R^F(Y) - R^F(\tilde{Y}) \|_{\mathcal{D}^\alpha_\alpha} + \| R^F(Y') - R^F(\tilde{Y}') \|_{\mathcal{D}^\alpha_\alpha} \\
\lesssim \| R^Y - R^{\tilde{Y}} \|_{\mathcal{D}^\alpha_\alpha} + \| Y' - \tilde{Y}' \|_{W^p_\alpha} + \rho_{\hat{\nu}}(X, \tilde{X}) + \hat{\rho}_{W^p}(X, \tilde{X})
\]

where \( R^F(Y) \) and \( R^F(\tilde{Y}) \) are the remainder terms of \( (F(Y), F(Y')) \) and \( (F(\tilde{Y}), F(\tilde{Y}')) \), respectively.

**Proof.** The proof follows by very similar arguments as in the proof of Lemma 3.4, which can be adapted to the present setting without further difficulties. \( \square \)

The stability results (Lemma 3.4 and 3.7) allow to apply a Banach fixed point argument to show that differential equations driven by Sobolev rough paths along smooth enough vector fields admit a unique solution of the same Sobolev regularity as the driven signals. Moreover, the solution depends continuously on the driven signals in a locally Lipschitz manner. We summarize these facts in the next theorem:

**Theorem 3.8.** Suppose \( X \) is a rough path in \( W^p_\alpha([0,1]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \) and \( V \in C_0^q(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \). Then, the rough differential equation

\[
Y_t = y_0 + \int_0^t V(Y_s) \, dX_s, \quad t \in [0,1],
\]

admits a unique solution \( Y \in W^\alpha_\alpha([0,1]; \mathbb{R}^e) \). Furthermore, If \( \tilde{X} \) is another rough path in \( W^\alpha_\alpha([0,1]; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)) \) and \( \tilde{Y} \) is the solution to the differential equation driven by \( \tilde{X} \) along \( V \) with initial value \( y_0 \), then it holds that

\[
\| Y - \tilde{Y} \|_{W^\alpha_\alpha} \lesssim \rho_{\hat{\nu}}(X, \tilde{X}) + \hat{\rho}_{W^p}(X, \tilde{X}),
\]

where the proportional constant only depends on \( p, \alpha, X, \tilde{X}, \) and \( V \).

As Theorem 3.8 can also be derived as a special case of Theorem 5.1 we only outline here the main steps of the proof. However, in the present level-2 setting it is more transparent.
to see why $\rho_{\nu_{\rho}} + \rho_{W_{\rho}}$ appear in our stability estimates, in particular, in the local Lipschitz continuity of the map associated to differential equations driven by Sobolev rough paths.

Proof. Let $\Phi^V$ be solution mapping defined on $D^{\varphi}_{\rho}(\mathbb{R}^e)$ into itself, which is given by

$$\Phi^V((Y, Y')) := \left(y_0 + \int V(Y)\,dX, \, V(Y)\right).$$

By Lemma 3.3 and 3.7 it is straightforward to check that $\Phi^V$ is a local contraction, and therefore the rough differential equation admits a unique local solution. Then a routine argument in theory of differential equations allows us to paste local solutions together to get a unique global solution. The estimates of $\|Y - \tilde{Y}\|_{W_{\rho}}$ follows then from the corresponding estimates of the remainder terms in Lemma 3.3 and 3.7. We note that every estimates contains the term $\rho_{\nu_{\rho}}(X, X) + \rho_{W_{\rho}}(X, X)$. For more details we refer the reader to [FH11] Chapter 8.

Although the setup therein is the H"older case, one can copy all proofs verbatim to the current Sobolev setting based on the notion of paracontrolled distributions but not on classical rough path spaces. The paracontrolled distribution approach avoids the use of the sewing lemma but does not directly extend to less regular driving signals. \hfill \Box

Remark 3.9. In the case $\alpha \in (1/3, 1/2)$ the continuity of the Itô–Lyons map was established in [LTI16] also in a Sobolev setting based on the notion of paracontrolled distributions but not on classical rough path spaces. The paracontrolled distribution approach avoids the use of the sewing lemma but does not directly extend to less regular driving signals.

4. Rough differential equations driven by Sobolev rough paths

We consider the controlled differential equation

$$dY_t = V(Y_t)\,dX_t, \quad Y_0 = y_0, \quad t \in [0, 1],$$

for a driven signal $X \in C^{r,\varphi}([0, 1]; \mathbb{R}^d)$, an initial value $y_0 \in \mathbb{R}^e$ and a vector field $V = (V_1, \ldots, V_d) : \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e)$. Let $\text{Lip}^\alpha := \text{Lip}^\alpha(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e))$ be the space of all $\alpha$-Lipschitz continuous functions $V : \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ in the sense of E. Stein for $\alpha > 0$, equipped with the usual norm $\|V\|_{\text{Lip}^\alpha}$, see [FV10] Definition 10.2).

As discussed in the Introduction, if $r > 2$, it is not sufficient to take “only” a $\mathbb{R}^d$-valued path $X$ as input to the system (4.1) in order to develop a pathwise solution theory. Therefore, we require in the following the driven signal to be a rough path $X$. For a given weakly geometric rough path $X \in C^{r,\varphi}([0, 1]; G^{[\gamma]}(\mathbb{R}^d))$, $Y \in C([0, 1]; \mathbb{R}^e)$ is said to be a solution to the rough differential equation

$$dY_t = V(Y_t)\,dX_t, \quad Y_0 = y_0, \quad t \in [0, 1],$$

if there exist a sequence $(X^n) \subset C^{1,\varphi}([0, 1]; \mathbb{R}^d)$ such that

$$\lim_{n \to \infty} \sup_{0 \leq s \leq t \leq T} d_{\varphi}(S_r(X^n)_{s,t}, X_{s,t}) = 0, \quad \sup_n \|S_r(X^n)\|_{1,\varphi} < +\infty,$$

and the corresponding solutions $Y^n$ to equation (4.1) converge uniformly on $[0, T]$ to $Y$ as $n \to \infty$, cf. [FV10] Definition 10.17. By [FV10] Theorem 10.14 and Corollary 10.15, given a rough path $X \in C^{r,\varphi}([0, 1]; G^{[\gamma]}(\mathbb{R}^d))$ and a vector field $V \in \text{Lip}^{\gamma - 1}$ with $\gamma > r \geq 1$, there exists a solution $Y$ to the equation (4.2) such that for any $[s, t] \subset [0, T]$,

$$|Y_t - Y_s - \mathcal{E}_V(Y_s, X_{s,t})| \lesssim (\|V\|_{\text{Lip}^{\gamma - 1}} \|X\|_{1,\varphi} [s, t])^{\gamma},$$

Consequently, $Y$ is a solution to the rough differential equation (4.1).
where $\mathcal{E}_V(Y_s, X_{s,t})$ denotes the step-$[r]$ Euler scheme (cf. [FV10] Definition 10.1)), namely,

$$
(4.4) \quad \mathcal{E}_V(Y_s, X_{s,t}) := \sum_{k=1}^{[r]} \sum_{i_1, \ldots, i_k \in \{1, \ldots, d\}} V_{i_1} \ldots V_{i_k} I(Y_s) \pi_k(X_{s,t})^{i_1, \ldots, i_k},
$$

where $I$ is the identity map on $\mathbb{R}^e$ and $\pi_k(X_{s,t})^{i_1, \ldots, i_k}$ denotes the $(i_1, \ldots, i_k)$-component of $\pi_k(X_{s,t}) \in (\mathbb{R}^d)^{\otimes k}$.

Instead of using the classical notation of weakly geometric rough paths of finite $r$-variation, we shall consider the driven signal $X$ of the controlled differential equation (1.2) to be a Sobolev rough path in $W^\alpha_p([0, 1]; G^{\frac{1}{2p}}(\mathbb{R}^d))$ with $\alpha \in (0, 1)$ and $p \in [1, +\infty)$ such that $\alpha > \frac{1}{p}$, cf. Definition 2.1. From Sobolev embedding theorems, see e.g. [FV06, Theorem 2], we know that $X$ still belongs to $C^{r,\var}$, with $r := \frac{1}{\alpha}$, for some $r \geq 1$. Hence, if the vector field $V$ in (1.2) belongs to Lip,$^{-1}$ then by classical results from rough path theory, as stated above, there exists a solution $Y \in C^{r,\var}$ to the rough differential equation (1.2). The following proposition shows that in this case we even obtain the solution $Y$ to be of Sobolev regularity. Namely, $Y$ has exactly the same Sobolev regularity as the driving signal $X$.

**Proposition 4.1.** Let $\alpha \in (0, 1)$ and $p \in [1, +\infty]$ be such that $\alpha > 1/p$. Suppose that $X \in W^\alpha_p([0, 1]; G^{\frac{1}{2p}}(\mathbb{R}^d))$ and $V \in \text{Lip}_{\var}$ for some $\gamma > 1/\alpha$. Then, for any initial condition $y_0 \in \mathbb{R}^e$ there exists a solution $Y$ to the rough differential equation (1.2) with $Y_0 = y_0$. Moreover, there exists a continuous increasing function $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $X \in W^\alpha_p([0, 1]; G^{\frac{1}{2p}}(\mathbb{R}^d))$ with $\sup_{t \in [0, 1]} \|X_t\|_{cc} \leq M$, one has

$$
\|Y\|_{W^\alpha_p} \lesssim f(M) \left( \|V\|_{\text{Lip}_{\var}} \|X\|_{W^\alpha_p} + (\|V\|_{\text{Lip}_{\var}} \|X\|_{W^\alpha_p})^\gamma \right).
$$

**Proof of Proposition 4.1.** Since we have that $X \in C^{r,\var}$, see [FV06, Theorem 2], there exists a solution $Y$ to the rough differential equation (1.2) with $Y_0 = y_0$ and (4.3) holds. As a consequence, for every $j \in \mathbb{N}$, one has

$$
\sum_{k=1}^{2^j} |Y_{(k-1)2^{-j}} - Y_{(k-1)2^{-j}}|^p \lesssim \sum_{k=1}^{2^j} |\mathcal{E}_V(Y_{(k-1)2^{-j}}, X_{(k-1)2^{-j}, k2^{-j}})|^p + \sum_{k=1}^{2^j} \|X\|^p_{\text{Lip}_{\var}((k-1)2^{-j}, k2^{-j})}.
$$

From the expression (4.3) we can deduce that

$$
|\mathcal{E}_V(Y_{(k-1)2^{-j}}, X_{(k-1)2^{-j}, k2^{-j}})| \lesssim \|V\|_{\text{Lip}_{\var}} |X_{(k-1)2^{-j}, k2^{-j}}|.
$$

Furthermore, by [FV10] (7.22) we have

$$
\|X_{(k-1)2^{-j}, k2^{-j}}\| \lesssim \max \left( 1, \sup_{t \in [0, 1]} \|X_t\|_{\text{cc}}^{(\frac{1}{\alpha})} \right) \rho(X_{k2^{-j}}, X_{(k-1)2^{-j}}).
$$

Hence, by assumptions we obtain that

$$
(4.5) \quad |\mathcal{E}_V(Y_{(k-1)2^{-j}}, X_{(k-1)2^{-j}, k2^{-j}})| \lesssim \rho(X_{k2^{-j}}, X_{(k-1)2^{-j}}).
$$

On the other hand, by [FPT18] Corollary 2.12 we get

$$
(4.6) \quad \|X\|^p_{\text{Lip}_{\var}((k-1)2^{-j}, k2^{-j})} \lesssim \|X\|^p_{W^\alpha_p((k-1)2^{-j}, k2^{-j})} 2^{-j(\alpha p - 1)}.
$$
Inserting (4.5) and (4.6) into the above estimate, we arrive at
\[
\sum_{j \geq 0} 2^{j/2} |Y_{k-2j} - Y_{(k-1)2-j}|^p \leq \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p + \sum_{k=1}^{2^{j/2}} \left( \|X\|_{W^\gamma_p; [(k-1)2-j, k2-j]}^p 2^{-j(\alpha-1)} \right)^\gamma.
\]
It follows that
\[
\sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p \leq \sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p
\]
\[
+ \sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \left( \|X\|_{W^\gamma_p; [(k-1)2-j, k2-j]}^p 2^{-j(\alpha-1)} \right)^\gamma.
\]
(4.7)

Applying [LPT20a, Theorem 2.2], for the Euclidean metric \(\rho\), to the first term in the right-hand side of inequality (4.7), we conclude that
\[
\sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p \lesssim \int_0^1 \rho(X_{u}, X_v)^p |v-u|^{\alpha+1} \, du \, dv.
\]
Invoking that \(\rho(g, h) \lesssim d_{cc}(g, h)\) locally uniformly on \(G[1, \alpha](\mathbb{R}^d)\), we can further deduce that
\[
\sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p \lesssim \int_0^1 \rho(X_{u}, X_v)^p |v-u|^{\alpha+1} \, du \, dv = \|X\|_{W^\alpha_p}^p
\]
and thus
\[
(4.8) \quad \sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \rho(X_{k2-j}, X_{(k-1)2-j})^p \lesssim \|X\|_{W^\alpha_p}^p.
\]
Let us now turn to the second term in the right-hand side of (4.7). Since \(\gamma > \frac{1}{\alpha} > 1\), the elementary inequality \(\sum |a_i|^\gamma \leq (\sum |a_i|)^\gamma\) implies that
\[
\sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \left( \|X\|_{W^\gamma_p; [(k-1)2-j, k2-j]}^p 2^{-j(\alpha-1)} \right)^\gamma \lesssim \left( \sum_{j \geq 0} 2^{-j(\alpha-1)(1-\frac{\gamma}{\alpha})} \|X\|_{W^\gamma_p; [(k-1)2-j, k2-j]}^p \right)^\gamma.
\]
Since \(1 - \frac{1}{\gamma} > 0\) and \(\alpha - 1 > 0\), using the super-additivity of the control function \(\omega(s, t) := \|X\|_{W^\gamma_p; [s, t]}^p\), we can immediately deduce that
\[
(4.9) \quad \sum_{j \geq 0} 2^{j(\alpha-1)} \sum_{k=1}^{2^{j/2}} \left( \|X\|_{W^\gamma_p; [(k-1)2-j, k2-j]}^p 2^{-j(\alpha-1)} \right)^\gamma \lesssim \|X\|_{W^\alpha_p}^p.
\]
Inserting the bounds (4.8) and (4.9) into inequality (4.7) and noting that the left-hand side of (4.7) is equivalent to the \(p\)-th power of the \(W^\alpha_p\)-norm of \(Y\) due to [LPT20a, Theorem 2.2], we finally obtain that
\[
\|Y\|_{W^\alpha_p}^p \lesssim \|X\|_{W^\gamma_p}^p + \|X\|_{W^\gamma_p}^p.
\]
where the proportionality constant depends continuously on $M$ and is increasing in $M$ (in fact, we may choose $f(M) := \max(1, M^{\frac{1}{2p}})$). This completes the proof. □

5. CONTINUITY OF THE ITÔ–LYONS MAP ON SOBOLEV SPACES

If the vector field $V$ belongs even to $\text{Lip}^\gamma$ rather than $\text{Lip}^{\gamma-1}$ for $\gamma > 1/\alpha$, then classical results from rough path theory (see, e.g., [FV10, Theorem 10.26]) imply the uniqueness of the solution $Y$ to the rough differential equation (4.2). Recalling that the solution $Y$ is an element of $W_p^\alpha([0, T]; \mathbb{R}^e)$ by Proposition 11, the Itô–Lyons map $\Phi$ given by

(5.1) $\Phi: \mathbb{R}^e \times \text{Lip}^\gamma \times W_p^\alpha([0, 1]; G^{\frac{1}{2p}}(\mathbb{R}^d)) \to W_p^\alpha([0, 1]; \mathbb{R}^e)$ via $\Phi(y_0, V, X) := Y,$

where $Y$ denotes the unique solution to rough differential equation (4.2) given the input $(y_0, V, X)$, is well-defined.

One of the central results of rough path theory is the local Lipschitz continuity of the Itô–Lyons map, which, of course, crucially depends on the chosen topology. We now establish the local Lipschitz continuity of the Itô–Lyons map acting on the space of Sobolev rough paths, as defined in (5.1).

Theorem 5.1. Let $\alpha \in (0, 1)$, $\gamma > 1$ and $p \in (1, +\infty)$ be such that $\alpha > 1/p$ and $\gamma > 1/\alpha$. Then, the Itô–Lyons map $\Phi$ as defined in (5.1) is locally Lipschitz continuous with respect to the initial value, vector field and the driving signal, that is, for $y_0^i \in \mathbb{R}^e$, $V^i \in \text{Lip}^\gamma$ and $X^i \in W_p^\alpha([0, 1]; G^{\frac{1}{2p}}(\mathbb{R}^d))$ satisfying

$$\|X^i\|_{W_p^\alpha} \leq b \quad \text{and} \quad |V^i|_{\text{Lip}^\gamma} \leq l, \quad i = 1, 2,$$

for some $b, l > 0$, with corresponding solution $Y^i = \Phi(y_0^i, V^i, X^i)$, there exists a constant $C = C(b, l, \gamma, \alpha, p, T) \geq 1$ such that

$$\|Y^1 - Y^2\|_{W_p^\alpha} \leq C \left( |V^1 - V^2|_{\text{Lip}^\gamma} + |y_0^1 - y_0^2| + \rho W_p^\alpha(X^1, X^2) + \rho \gamma^\alpha W_p^\alpha(X^1, X^2) \right).$$

Proof. A careful inspection of the proof of [FV10, Theorem 10.26] reveals that if $\omega$ is a control function on $\Delta$ and $\omega'$ is a non-negative function on $\Delta$ such that

$$\|X^i\|_\omega := \sup_{0 \leq s \leq t \leq 1} \frac{\|X^i_s\|_{\text{var}([s, t])}}{\omega(s, t)^\alpha} \leq 1 \quad \text{and} \quad \|X^i\|_{\omega'} := \sup_{0 \leq s \leq t \leq 1} \frac{\|X^i_s\|_{\text{var}([s, t])}}{\omega'(s, t)^\alpha} \leq 1,$$

for $i = 1, 2$, then for any $s < t$ in $[0, 1],$

(5.2) $|Y^1_{s, t} - Y^2_{s, t} | \lesssim (|y_0^1 - y_0^2| + |V^1 - V^2|_{\text{Lip}^\gamma} + l \rho_{\omega'}(X^1, X^2)) \omega'(s, t)^\alpha \exp(CT \omega(s, t) + CT \omega(0, 1))$ + $(|y_0^1 - y_0^2| + |V^1 - V^2|_{\text{Lip}^\gamma} + l \rho_{\omega'}(X^1, X^2)) \gamma^{-1} \omega(s, t)^\gamma \exp(CT \omega(0, 1)),

where $\rho_{\omega'}(X^1, X^2) := \sum_{k=1}^{\frac{1}{\alpha}} \sup_{0 \leq s \leq t \leq 1} | \frac{\sigma_0 (X^1_s - X^2_s)}{\omega(s, t)^\alpha} |$ and the same expression holds for $\rho_{\omega'}(X^1, X^2)$. Let us define

(5.3) $\omega(s, t) := \|X^1\|_{\text{var}([s, t])} + \|X^2\|_{\text{var}([s, t])} + \sum_{k=1}^{\frac{1}{\alpha}} \omega^{(k)} X^1, X^2 (s, t),$
where \( \omega_{X^1,X^2}(s,t) := \left( \frac{\|\pi_k(X^1_s - X^1_t)\|_{\varphi_p}}{\rho_{\varphi_p}(X^1_s,X^2_t)} \right)^{\frac{1}{\pi}} \) and \( \rho_{\varphi_p} \) is the inhomogeneous variation metric defined in [FV10, Definition 8.6]. Furthermore, we set

\[
\omega'(s,t) := \|X^1_{s,t}\|_{\varphi_p} + \|X^2_{s,t}\|_{\varphi_p} + \sum_{k=1}^{\frac{1}{\pi}} \omega_{X^1,X^2}(s,t)
\]

with \( \omega_{X^1,X^2}(s,t) := \left( \frac{\|\pi_k(X^1_s - X^1_t)\|_{\varphi_p}}{\rho_{\varphi_p}(X^1_s,X^2_t)} \right)^{\frac{1}{\pi}} \). By definition, we see that for such \( \omega \) and \( \omega' \) it holds that \( \|X^1\|_{\frac{1}{\pi}\omega} \leq 1 \) and \( \|X^1\|_{\frac{1}{\pi}\omega'} \leq 1 \) for \( i = 1, 2 \). Moreover, since

\[
|\pi_k(X^1_{s,t} - X^2_{s,t})| \leq \frac{\rho_{\varphi_p}(X^1_s,X^2_t)}{\rho_{\varphi_p}(X^1_s,X^2_t)} \omega_{\varphi_p}(X^1_s,X^2_t) \leq \omega(s,t) \omega_{\varphi_p}(X^1_s,X^2_t),
\]

we indeed have \( \rho_{\varphi_p}(X^1_s,X^2_t) \leq \rho_{\varphi_p}(X^1_s,X^2_t) \). By the same reasoning we can also deduce that \( \rho_{\varphi_p}(X^1_s,X^2_t) \leq \rho_{\varphi_p}(X^1_s,X^2_t) \). Although \( \omega' \) is not a control function, it holds that \( \omega'(s,t) \leq \omega(s,t) \) for all \( s < t \) in \([0,1]\). Hence, we can bound the \( \omega'(s,t) \) appeared in the exponential function in (5.2) by \( \omega(0,1) \). All above observations allow us to reduce estimate (5.2) to

\[
|Y^1_{s,t} - Y^2_{s,t}| \leq \left( |y^1_1 - y^2_1| + |V^1 - V^2|_{\text{Lip}(\gamma^{-1})} + \rho_{\varphi_p}(X^1_s,X^2_t) + \rho_{\varphi_p}(X^1_s,X^2_t) \right) \exp(Cl^{\frac{1}{\pi}} \omega(0,1)) 
\]

× \( \omega'(s,t)^{\alpha} + \omega(s,t)^{\gamma_{\alpha}} \).

For simplicity we denote

\[
F := \left( |y^1_1 - y^2_1| + |V^1 - V^2|_{\text{Lip}(\gamma^{-1})} + \rho_{\varphi_p}(X^1_s,X^2_t) + \rho_{\varphi_p}(X^1_s,X^2_t) \right) \exp(Cl^{\frac{1}{\pi}} \omega(0,1)),
\]

which is a constant independent of \((s,t)\). Then we obtain that

\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} |Y^1_{s,t} - Y^2_{s,t}| \leq F^p \left( \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega'((i - 1)2^{-j}, i2^{-j})^{\alpha p} \right) 
\]

+ \( \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega((i - 1)2^{-j}, i2^{-j})^{\gamma_{\alpha} p} \).

By definition, we have

\[
\omega'((i - 1)2^{-j}, i2^{-j})^{\alpha p} \leq d_{cc}(X^1_{(i-1)2^{-j},i2^{-j}}, X^1_{(i-1)2^{-j},i2^{-j}})^p + d_{cc}(X^2_{(i-1)2^{-j},i2^{-j}}, X^2_{(i-1)2^{-j},i2^{-j}})^p
\]

+ \( \sum_{k=1}^{\frac{1}{\pi}} |\pi_k(X^1_{(i-1)2^{-j},i2^{-j}} - X^2_{(i-1)2^{-j},i2^{-j}})|^\frac{1}{\pi} \rho_{\varphi_p}(X^1_s,X^2_t) \frac{1}{\pi} \).

In view of the definition of \( \hat{\rho}_{\varphi_p} \) we observe that

\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} |\pi_k(X^1_{(i-1)2^{-j},i2^{-j}} - X^2_{(i-1)2^{-j},i2^{-j}})|^\frac{1}{\pi} \hat{\rho}_{\varphi_p}(X^1_s,X^2_t) \frac{1}{\pi} = \hat{\rho}_{\varphi_p}(X^1_s,X^2_t) \frac{1}{\pi} \]
and thus
\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega((i - 1)2^{-j}, i2^{-j})^{\alpha p}
= \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} d_{cc}(X_{(i-1)2^{-j}}, X_{i2^{-j}})^p + \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} d_{cc}(X_{(i-1)2^{-j}}, X_{i2^{-j}})^p + \left\lfloor \frac{1}{\alpha} \right\rfloor.
\]

By \cite[Theorem 2.2]{LP20a} the right-hand side of the above inequality is bounded by the term \(C(\|X^1\|_{W^\alpha} + \|X^2\|_{W^\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor)\) for some constant \(C\) only depending on \(\alpha\) and \(p\), therefore the term
\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega((i - 1)2^{-j}, i2^{-j})^{\alpha p}
\]
is bounded by \(C(b^p + 1)\) due to our hypothesis.

On the other hand, in view of the definition of \(\omega\) (cf. (5.3)), one has
\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega((i - 1)2^{-j}, i2^{-j})^{\gamma \alpha p}
\leq \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \|X^1\|^{\gamma p}_{\text{var};[(i-1)2^{-j}, i2^{-j}]} + \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \|X^2\|^{\gamma p}_{\text{var};[(i-1)2^{-j}, i2^{-j}]}
+ \sum_{k=1}^{[\frac{1}{\alpha}]} \sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma} \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma}.
\]

From the proof of Proposition 4.1 we see that the right-hand side of the above inequality is bounded by \(C(\|X^1\|_{W^\alpha} + \|X^2\|_{W^\alpha})^\gamma\) for some constant \(C\) only depending on \(\alpha, p\) and \(\gamma\).

For the last term, note that in the proof of \cite[Theorem 3.3]{LP18} one has
\[
\rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma} \leq \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\gamma (\alpha - \frac{1}{p}) k}.
\]

Since \(\gamma > 1/\delta > 1\) and \(\rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma}\) is super-additive as a function on \(\Delta\), we can deduce that
\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma}
\leq \left( \sum_{j \geq 0} 2^{-j(\alpha p - 1)(1 - \frac{1}{\delta})} \sum_{i=1}^{2^j} \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma} \right)^{\gamma} \leq \rho_{\text{var};[(i-1)2^{-j}, i2^{-j}]}^{(k)}(X^1, X^2)^{\frac{\gamma}{\alpha} p \gamma},
\]
for every \(k = 1, \ldots, \left\lfloor \frac{1}{\alpha} \right\rfloor\). Hence, we obtain that
\[
\sum_{j \geq 0} 2^{j(\alpha p - 1)} \sum_{i=1}^{2^j} \omega((i - 1)2^{-j}, i2^{-j})^{\gamma \alpha p} \leq b^p \gamma + 1.
\]
Note that from the above estimate we also deduce that \( \omega(0, 1) \leq C(b^\frac{1}{\alpha} + 1) \) for some constant \( C \) only depending on \( \alpha, p \) and \( \gamma \). Now inserting all above estimates into (5.4) and using [LPT 20a, Theorem 2.2], we find that

\[
\| Y^1 - Y^2 \|_{W^\alpha p} \lesssim C \left( |V^1 - V^2|_{\text{Lip}^{-1}} + |y_0^1 - y_0^2| + \hat{\rho}_{V^\alpha p}(X^1, X^2) + \rho_{\hat{V}^\alpha p}(X^1, X^2) \right).
\]

\[\square\]

**Remark 5.2.** One would expect that the inhomogeneous mixed Hölder-variation distance \( \rho_{\hat{V}^\alpha p} \) is not needed for the continuity statement of Theorem 5.1 and that \( \rho_{\hat{V}^\alpha p} \) is dominated by the inhomogeneous Sobolev distance \( \hat{\rho}_{V^\alpha p} \) as one can observe for the homogeneous Sobolev norms. However, at least if one wants to follow a similar approach as developed in the present work, this would require an extensive study of the inhomogeneous distances: First, it seems to require to generalize the Sobolev-variation embedding theorem for functions \( f: [0, T] \to E \) provided in [FV06] to functions \( f: \Delta \to E \). Second, similar generalizations seem to be needed for the characterization of non-linear Sobolev spaces [LPT20a] as well as for the embedding results in [FP18]. These generalizations are outside the scope of the present article.

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