Generalised Split Octonions and their transformation in
SO(7) symmetry

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Abstract
Generators of SO(8) group have been described by using direct product of the Gamma matrices and the Pauli Sigma matrices. We have obtained these generators in terms of generalized split octonion also. These generators have been used to describe the rotational transformation of vectors for SO(7) symmetry group.

1 Introduction
Among all the division algebras, octonion forms the largest normed division algebra [1][2][3]. These are an algebraic structure defined on the 8-dimensional real vector space and they are non-associative and non-commutative extension of the algebra of quaternions. Günaydin and Gürsey discussed the Lie algebra of $G_2$ group and its embedding in SO(7) and SO(8) groups [4]. A dynamical scheme of quark and lepton family unification based on non-associative algebra has also been discussed [5]. Generators of SO(8) are constructed by using Octonion structure tensors [6] and the representations of these generators are also given as the products of Octonions. Lassig and Joshi [7] introduced the
bi-modular representation of octonions and formulated the $SO(8)$ gauge theory equivalent to the octonionic construction, also the peculiarities of the eight dimensional space has been described by Gamba \[8\]. Octonionic description has been used to describe the various applications in quantum chromo dynamics \[9\], in the study of representations of Clifford algebras \[10\], non associative quantum mechanics \[11\], in the study of symmetry breaking \[12\], in the study of flavor symmetry \[13\], in proposals of unified field theory models \[14\], unitary symmetry \[15\], octonionic gravity \[16\].

In the present study, we have described the generalized split octonionic description to represent the 8-dimensional algebra. We have obtained the generators of $SO(8)$ group with the help of generalized split octonions as well as direct product of Gamma and Pauli Sigma matrices. Since $SO(7) \subset SO(8)$, therefore the presented representation also gives the generators of $SO(7)$ symmetry group. These generators of $SO(7)$ group have been used to describe the rotational transformation in 7 dimensional space.

2 $SO(8)$ Symmetry group generators by using Direct Product of Gamma and Sigma Matrices

$SO(8)$ represents the special orthogonal group of eight-dimensional rotations having 28 generators. By using direct product of Pauli matrices and Gamma matrices, we have constructed the eight dimensional representation of $SO(8)$ Symmetry.

Dirac Gamma matrices $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are a set of conventional matrices with specific anti-commutation relations and they generate a matrix representation of the Clifford algebra $C\ell_{1,3}(R)$.

$$\gamma^0 = \begin{bmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{bmatrix} \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix} \quad \forall j = 1, 2, 3,$$  \hspace{1cm} (1)

where $\sigma^0$ is a $2 \times 2$ unit matrix and $\sigma^j$ are Pauli matrices defined as

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  \hspace{1cm} (2)

The product of Gamma matrices are given by

$$\gamma^4 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (3)
Here we intend to describe the generators of $SO(8)$ symmetry group by using the direct product of Gamma matrices with Pauli sigma matrices. For this, we have chosen the following representations,

\[
\begin{align*}
\beta_0 &= I_4 \otimes \sigma^0; \\
\beta_1 &= -i \gamma^4 \otimes \sigma^2 = [b_{16} - b_{25} + b_{38} - b_{47} + b_{52} - b_{61} + b_{74} - b_{83}]; \\
\beta_2 &= \sigma^1 \otimes \gamma^3 = [b_{17} - b_{28} - b_{35} + b_{46} + b_{53} - b_{64} - b_{71} + b_{72}]; \\
\beta_3 &= \sigma^1 \otimes \gamma^1 = [b_{18} + b_{27} - b_{36} - b_{45} + b_{54} + b_{63} - b_{72} - b_{81}]; \\
\beta_4 &= \gamma^4 \otimes \sigma^0 = [b_{11} + b_{22} + b_{33} + b_{44} - b_{55} - b_{66} - b_{77} - b_{88}]; \\
\beta_5 &= -i \gamma^3 \otimes \sigma^0 = [b_{16} - b_{25} - b_{37} + b_{46} + b_{52} - b_{61} - b_{74} + b_{83}]; \\
\beta_6 &= -i \gamma^2 \otimes \sigma^0 = [-b_{17} - b_{28} + b_{35} + b_{46} + b_{53} + b_{64} - b_{71} - b_{82}]; \\
\beta_7 &= -i \gamma^1 \otimes \sigma^0 = [b_{18} - b_{27} + b_{36} - b_{45} - b_{54} + b_{63} - b_{72} + b_{81}]; \\
\end{align*}
\] (4)

where $I_4$ is $4 \times 4$ identity matrix, $\beta_0$ is the $8 \times 8$ identity matrix and $\beta_a$, ($a=1,\ldots,7$) represents $8 \times 8$ matrices. $b_{ij}$ represents the $8 \times 8$ matrices in which $ij$th matrix element is unity and rest elements are zero. $\beta_a$ represents the 7 generators of $SO(8)$ symmetry group. It also satisfies the anti-commutation relation as

\[
\beta_a \beta_b + \beta_b \beta_a = 2\delta_{ab}. \quad (5)
\]

Furthermore, the rest 21 generators of $SO(8)$ symmetry group are obtained by the following relation

\[
\beta_{ab} = \frac{1}{2i}[\beta_a, \beta_b]. \quad (6)
\]

Thus the 7 generators $\beta_a$ and 21 generators $\beta_{ab}$ form the total 28 generators of the $SO(8)$ symmetry group. In this way, we have represented generators of $SO(8)$ symmetry group by using the direct product of Gamma matrices and Pauli matrices. In the succeeding sections, we show that these generators could also been obtained with the help of generalized split octonions.

### 3 Split Octonions

Split Octonion algebra $\mathfrak{H}$ with its split base units is defined as

\[
\begin{align*}
 u_0 &= \frac{1}{2} (e_0 + ie_7), \quad u_0^* = \frac{1}{2} (e_0 - ie_7); \\
 u_j &= \frac{1}{2} (e_j + ie_{j+3}) \quad u_m^* = \frac{1}{2} (e_j - ie_{j+3}). \\
\end{align*}
\] (7)
These basis elements satisfy the following algebra

\[ u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^*, \quad u_i^* u_j^* = -u_j^* u_i^* = \epsilon_{ijk} u_k; \]
\[ u_i u_j^* = -\delta_{ij} u_0, \quad u_i^* u_j = -\delta_{ij} u_0^*; \]
\[ u_0 u_i = u_i u_0^* = u_i, \quad u_0^* u_i^* = u_i^* u_0 = u_i^*; \]
\[ u_i u_0 = u_0 u_i^* = 0, \quad u_i^* u_0 = u_0^* u_i = 0; \]
\[ u_0 u_0^* = u_0^* u_0 = 0, \quad u_0^2 = u_0, \quad u_0^{*2} = u_0^*. \] (8)

These relations are invariant under \( G_2 \) transformation. Split octonion multiplication has been shown in Table 1.

| \( u \) | \( u_0^* \) | \( u_1^* \) | \( u_2^* \) | \( u_3^* \) | \( u_0 \) | \( u_1 \) | \( u_2 \) | \( u_3 \) |
|---|---|---|---|---|---|---|---|---|
| \( u_0^* \) | \( u_0^* \) | \( u_1^* \) | \( u_2^* \) | \( u_3^* \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( u_1^* \) | \( 0 \) | \( 0 \) | \( u_3 \) | \( -u_2 \) | \( u_1^* \) | \( -u_0^* \) | \( 0 \) | \( 0 \) |
| \( u_2^* \) | \( 0 \) | \( -u_3 \) | \( 0 \) | \( u_1 \) | \( u_2^* \) | \( 0 \) | \( -u_0^* \) | \( 0 \) |
| \( u_3^* \) | \( 0 \) | \( u_2 \) | \( -u_1 \) | \( 0 \) | \( u_3^* \) | \( 0 \) | \( 0 \) | \( -u_0^* \) |
| \( u_0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( u_0 \) | \( u_1 \) | \( u_2 \) | \( u_3 \) |
| \( u_1 \) | \( u_1 \) | \( -u_0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( u_3^* \) | \( -u_2^* \) |
| \( u_2 \) | \( u_2 \) | \( 0 \) | \( -u_0 \) | \( 0 \) | \( 0 \) | \( -u_3^* \) | \( 0 \) | \( u_1^* \) |
| \( u_3 \) | \( u_3 \) | \( 0 \) | \( 0 \) | \( -u_0 \) | \( 0 \) | \( u_2 \) | \( -u_1^* \) | \( 0 \) |

Table 1: Split Octonion multiplication table

## 4 Generalized Split Octonions

Let us consider a spinor \( U \) generated from split octonions \( u_k \) and conjugate of split octonions \( u_k^* \) as

\[ U = \begin{pmatrix} u_k \\ u_k^* \end{pmatrix} \] (9)

where \( k = 0, 1, 2, 3. \)

Since split octonion represents the non associative and non division algebra. It cannot be represented in terms of matrices. However, there exists some real matrices \([2, 4]\) which are associated with split
octonions that can be obtained as

\[
\begin{bmatrix}
  u_0 \\
  u_1 \\
  u_2 \\
  u_3 \\
  u_0^* \\
  u_1^* \\
  u_2^* \\
  u_3^*
\end{bmatrix} =
\begin{bmatrix}
  u_0 \\
  u_1 \\
  u_2 \\
  u_3 \\
  u_0^* \\
  u_1^* \\
  u_2^* \\
  u_3^*
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  u_1 \\
  u_2 \\
  u_3 \\
  u_0^* \\
  u_1^* \\
  u_2^* \\
  u_3^*
\end{bmatrix},
\]

(10)

where 8×8 matrix term in the right side represents the left operator for split octonion basis element $u_0$ and we will denote it as $U_{L0}$ in our further calculations and equation (10) could be written as

\[
u_0[U] = U_{L0}[U].
\]

(11)

Similarly we have calculated four 8×8 matrices corresponding to split octonions and four 8×8 matrices corresponding to conjugate of split octonions with the help of the split octonion multiplication table 1.

Left multiplication of split octonion units by $U$ gives

\[
\begin{align*}
U_{L0} &= [a_{11} + a_{22} + a_{33} + a_{44}]; \\
U_{L1} &= [a_{38} - a_{47} + a_{52} - a_{61}]; \\
U_{L2} &= [-a_{28} + a_{46} + a_{53} - a_{71}]; \\
U_{L3} &= [a_{27} - a_{36} + a_{54} - a_{81}].
\end{align*}
\]

(12)

Left multiplication of conjugate of split octonions with $U$ are given as

\[
\begin{align*}
U_{L0}^* &= [a_{44} + a_{55} + a_{66} + a_{77}]; \\
U_{L1}^* &= [a_{16} - a_{25} + a_{64} - a_{74}]; \\
U_{L2}^* &= [a_{17} - a_{35} - a_{64} + a_{82}]; \\
U_{L3}^* &= [a_{18} - a_{45} + a_{63} - a_{72}],
\end{align*}
\]

(13)

where $U_{L0}$, $U_{L1}$, $U_{L2}$, $U_{L3}$, $U_{L0}^*$, $U_{L1}^*$, $U_{L2}^*$, $U_{L3}^*$ represent the 8×8 matrix representations named generalized split octonions and $a_{ij}$ represents the 8×8 matrix whose $i_j^{th}$ element is 1 and other terms are zero. By using the different combinations of equation (12) and (13), we construct the following 8×8 matrices
\begin{align*}
U_0 &= U_{L0} + U^*_{L0} = \begin{bmatrix}
\sigma_0 & 0 & 0 & 0 \\
0 & \sigma_0 & 0 & 0 \\
0 & 0 & \sigma_0 & 0 \\
0 & 0 & 0 & \sigma_0 \\
\end{bmatrix} ; \\
U_1 &= U_{L1} + U^*_{L1} = \begin{bmatrix}
0 & 0 & \sigma_2 & 0 \\
0 & 0 & 0 & \sigma_2 \\
\sigma_2 & 0 & 0 & 0 \\
0 & \sigma_2 & 0 & 0 \\
\end{bmatrix} ; \\
U_2 &= U_{L2} + U^*_{L2} = \begin{bmatrix}
0 & 0 & 0 & \sigma_3 \\
0 & 0 & -\sigma_3 & 0 \\
0 & \sigma_3 & 0 & 0 \\
-\sigma_3 & 0 & 0 & 0 \\
\end{bmatrix} ; \\
U_3 &= U_{L3} + U^*_{L3} = \begin{bmatrix}
0 & 0 & 0 & \sigma_1 \\
0 & 0 & -\sigma_1 & 0 \\
0 & \sigma_1 & 0 & 0 \\
-\sigma_1 & 0 & 0 & 0 \\
\end{bmatrix} ; \\
U_4 &= U_{L0} - U^*_{L0} = \begin{bmatrix}
\sigma_0 & 0 & 0 & 0 \\
0 & \sigma_0 & 0 & 0 \\
0 & 0 & -\sigma_0 & 0 \\
0 & 0 & 0 & -\sigma_0 \\
\end{bmatrix} ; \\
U_5 &= U_{L1} - U^*_{L1} = \begin{bmatrix}
0 & 0 & \sigma_2 & 0 \\
0 & 0 & 0 & -\sigma_2 \\
\sigma_2 & 0 & 0 & 0 \\
0 & -\sigma_2 & 0 & 0 \\
\end{bmatrix} ; \\
U_6 &= U_{L2} - U^*_{L2} = \begin{bmatrix}
0 & 0 & 0 & -\sigma_0 \\
0 & 0 & \sigma_0 & 0 \\
0 & \sigma_0 & 0 & 0 \\
-\sigma_0 & 0 & 0 & 0 \\
\end{bmatrix} ; \\
U_7 &= U_{L3} - U^*_{L3} = \begin{bmatrix}
0 & 0 & 0 & \sigma_2 \\
0 & 0 & \sigma_2 & 0 \\
0 & 0 & -\sigma_2 & 0 \\
-\sigma_2 & 0 & 0 & 0 \\
\end{bmatrix} .
\end{align*}

(14)

Here $U_A$ ($A=1,2,...,7$) gives the seven generators of $SO(8)$ symmetry group and the other 21 generators of $SO(8)$ group could be obtained by taking the commutation relation of these matrices i.e.

\[ U_{AB} = \frac{1}{2i} [U_A, U_B] \forall A, B = 1, 2, ..., 7. \]  

(15)

$U_A$ and $U_{AB}$ give the 28 generators of $SO(8)$ group. These 28 generators of $SO(8)$ group are the same as those obtained by the direct product of sigma and Gamma matrices. Now we would find the connection between generalized split octonions and direct product of sigma and Gamma matrices.
From equation (14) and equation (15), we have

\[ \beta_1 \implies -i \gamma^4 \otimes \sigma^2 = U_1 = U_{L1} + U_{L1}^*; \]
\[ \beta_2 \implies \sigma^1 \otimes \gamma^3 = U_2 = U_{L2} + U_{L2}^*; \]
\[ \beta_3 \implies \sigma^1 \otimes \gamma^1 = U_3 = U_{L3} + U_{L3}^*; \]
\[ \beta_4 \implies \gamma^4 \otimes \sigma^0 = U_4 = U_{L0} - U_{L0}^*; \]
\[ \beta_5 \implies -i \gamma^3 \otimes \sigma^2 = U_5 = U_{L1} - U_{L1}^*; \]
\[ \beta_6 \implies -i \gamma^2 \otimes \sigma^0 = U_6 = U_{L2} - U_{L2}^*; \]
\[ \beta_7 \implies -i \gamma^1 \otimes \sigma^2 = U_7 = U_{L3} - U_{L3}^*. \]  

These 21 generators \( U_{AB} \) represent the generators of \( SO(7) \) group since \( SO(7) \subset SO(8) \). Furthermore, with the help of these 21 generators of \( SO(7) \) group, we would like to describe the rotational transformation in seven dimensional space.

## 5 Rotational transformation for \( SO(7) \) symmetry

Let us consider a general spinor \( \psi \) in seven dimensional space. Under this symmetry, the spinor \( \psi \) transforms as

\[ \psi \mapsto \psi' = \exp \left( \sum_{A=1}^{7} f_A U_A \right) \psi \]
\[ = e^{X} \psi \]  

where vector \( X \) is,

\[ X = \sum_{A=1}^{7} f_A U_A. \]  


with $f_1, f_2, f_3, \ldots, f_7$ as the components of the vector. On expanding $X$,

$$
X = \begin{bmatrix}
  f_4 & 0 & 0 & 0 & 0 & if_1 - f_5 & if_2 - f_6 & if_3 - f_7 \\
  0 & f_4 & 0 & 0 & f_5 - if_1 & 0 & if_3 + f_7 & -if_2 - f_6 \\
  0 & 0 & f_4 & 0 & f_6 - if_2 & -if_3 - f_7 & 0 & if_1 + f_5 \\
  0 & 0 & 0 & f_4 & if_1 + f_5 & if_2 + f_6 & if_3 - f_7 & -f_4 \\
  -if_1 - f_5 & 0 & -if_3 - f_7 & f_6 - if_2 & 0 & -f_4 & 0 & 0 \\
  if_2 - f_6 & f_7 - if_3 & 0 & if_1 - f_5 & 0 & 0 & -f_4 & 0 \\
  if_3 - f_7 & if_2 - f_6 & f_5 - if_1 & 0 & 0 & 0 & -f_4 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & -f_4
\end{bmatrix}
$$

(19)

which is a traceless Hermitian matrix. In compact form $X$ can be written as,

$$
X = \begin{bmatrix}
  A & B^+ \\
  B & -A
\end{bmatrix}
$$

(20)

where $A$ and $B$ are given as,

$$
A = \begin{bmatrix}
  f_4 & 0 & 0 & 0 \\
  0 & f_4 & 0 & 0 \\
  0 & 0 & f_4 & 0 \\
  0 & 0 & 0 & f_4
\end{bmatrix};
$$

(21)

$$
B = \begin{bmatrix}
  0 & if_1 - f_5 & if_2 - f_6 & if_3 - f_7 \\
  f_5 - if_1 & 0 & if_3 + f_7 & -if_2 - f_6 \\
  f_6 - if_2 & -if_3 - f_7 & 0 & if_1 + f_5 \\
  f_7 - if_3 & if_2 + f_6 & -if_1 - f_5 & 0
\end{bmatrix}.
$$

(22)

In equation (20), the term $B$ corresponds to the split octonions ($u_1, u_2, u_3$) and $B^+$ corresponds to conjugate of split octonions ($u_1^*, u_2^*, u_3^*$). Matrix $A$ represents the unit split octonions $u_0$ and $-A$ corresponds to $u_0^*$. Matrices $A$ and $B$ are independent of each other and they correspond to $u_0$ and $u_j$ respectively.

Furthermore, the constructed $8 \times 8$ matrices given in equation (16) are being used to describe the rotation in $SO(7)$ Symmetry. As an infinitesimal rotation by an angle $\theta$ in the plane ($k, l$) is obtained by the following operator [1].

$$
R_{kl} = 1 + \theta U_k U_l
$$

(23)
which acts on a vector \( X \) to form a rotated vector \( X' \) as,

\[
X' = R_{kl} X R_{kl}^{-1}
\]  

(24)

By using equation (24), the rotation operator \( R_{12} \) becomes,

\[
R_{12} = \begin{bmatrix}
1 & 0 & 0 & i\theta & 0 & 0 & 0 & 0 \\
0 & 1 & i\theta & 0 & 0 & 0 & 0 & 0 \\
0 & -i\theta & 1 & 0 & 0 & 0 & 0 & 0 \\
-i\theta & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & i\theta \\
0 & 0 & 0 & 0 & 0 & 1 & i\theta & 0 \\
0 & 0 & 0 & 0 & -i\theta & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -i\theta & 0 & 0 & 1 \\
\end{bmatrix}.
\]  

(25)

The rotation \( R_{12} \) gives the rotated vector \( X' \) as follows,

\[
X' = X + 2\theta \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & if_2 & -if_1 & 0 \\
0 & 0 & 0 & 0 & -if_2 & 0 & 0 & if_1 \\
0 & 0 & 0 & 0 & if_1 & 0 & 0 & if_2 \\
0 & 0 & 0 & 0 & 0 & -if_1 & -if_2 & 0 \\
0 & if_2 & -if_1 & 0 & 0 & 0 & 0 & 0 \\
-if_2 & 0 & 0 & if_1 & 0 & 0 & 0 & 0 \\
if_1 & 0 & 0 & if_2 & 0 & 0 & 0 & 0 \\
0 & -if_1 & -if_2 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]  

(26)

While transforming these matrices in terms of rotations, we find that the upper half of transformation corresponds to \( u_j \) and lower half corresponds to \( u^*_j \). So here we will be calculating the rotational transformation of vectors by using split octonions \( u_j \) and Hermitian conjugate of split octonions \( u^*_j \) respectively. Since \( \theta \) is an infinitesimal rotation, therefore neglecting \( \theta^2 \) terms. Rotation \( R_{12} \) transforms the components of a vector \( X \) as follows:

\[
f_1 \mapsto f_1 + 2\theta [f_2 - if_5]; \\
f_2 \mapsto f_2 + 2\theta [-f_1 - if_6].
\]  

(27)

We have calculated all the possible rotations of \( R_{kl} \) and also calculated the corresponding transformation of all combinations \( kl \). These are shown in table 2.

The blank spaces in table 2 show that these components are unchanged under the transformations. We have seen that rotational transformations of the vector \( f_4 \) corresponding to \( R_{14}, R_{21}, \)
Table 2: Rotational transformation of SO(7) symmetry group

|   | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|-------|-------|-------|-------|-------|-------|-------|
| $R_{12}$ | $f_2 - i f_5$ | $-f_1 - i f_6$ |       |       |       |       |       |
| $R_{13}$ | $f_3 - i f_5$ |       | $-f_1 - i f_7$ |       |       |       |       |
| $R_{14}$ | $f_4 - i f_5$ |       |       | $-f_1$ |       |       |       |
| $R_{15}$ | $-i f_1$ |       |       |       | $-i f_5$ |       |       |
| $R_{16}$ | $f_6 - i f_5$ |       |       |       |       | $-f_1 + i f_2$ |       |
| $R_{17}$ | $f_7 - i f_5$ |       |       |       |       |       | $-f_1 + i f_3$ |
| $R_{23}$ | $f_3 - i f_6$ | $-f_2 - i f_7$ |       |       |       |       |       |
| $R_{24}$ | $f_4 - i f_6$ |       |       | $-f_2$ |       |       |       |
| $R_{25}$ | $f_5 - i f_6$ |       |       |       | $-f_2 + i f_1$ |       |       |
| $R_{26}$ | $-i f_3$ |       |       |       | $-i f_6$ |       |       |
| $R_{27}$ | $f_7 - i f_6$ |       |       |       |       | $-f_2 + i f_3$ |       |
| $R_{34}$ | $f_4 - i f_7$ | $-f_3$ |       |       |       |       |       |
| $R_{35}$ | $f_5 - i f_7$ | $-f_3 + i f_1$ |       |       |       |       |       |
| $R_{36}$ | $f_6 - i f_7$ |       |       | $-f_3 + i f_2$ |       |       |       |
| $R_{37}$ | $-i f_3$ |       |       |       | $-i f_7$ |       |       |
| $R_{45}$ | $f_5$ | $-f_4 + i f_1$ |       |       |       |       |       |
| $R_{46}$ | $f_6$ |       |       | $-f_4 + i f_2$ |       |       |       |
| $R_{47}$ | $f_7$ |       |       |       | $-f_4 + i f_3$ |       |       |
| $R_{56}$ | $f_6 + i f_1$ | $-f_5 + i f_2$ |       |       |       |       |       |
| $R_{57}$ | $f_7 + i f_1$ |       |       | $-f_5 + i f_3$ |       |       |       |
| $R_{67}$ | $f_7 + i f_2$ | $-f_6 + i f_3$ |       |       |       |       |       |

$R_{34}$ rotations give $-f_1$, $-f_2$, $-f_3$ and the transformation of the same corresponding to $R_{45}$, $R_{46}$, $R_{47}$ gives $f_5$, $f_6$, $f_7$. This transformation represents rotations corresponding to SO(7) group corresponding to split octonions and conjugate of split octonions. These transformation are obtained by the combinations of $U_A (A=1,2,3,5,6,7)$ with $U_4 = U_{L0} - U_{0}^*$. Therefore, these transformations are corresponding to $u_0$ and $u_0^*$ respectively. Rest of the transformation have been obtained by different combinations of $U_A$ with $U_B$ (A, B = 1,2,3,5,6,7) except $U_4$. These transformations are obtained by using the split octonion $u_j$ and Hermitian conjugate of split octonion $u_j^*$ respectively.

## 6 Discussion

The generators of SO(8) and SO(7) groups have been generated by using the split octonion as a spinor and the same relation has been also described with the help of direct product of Gamma matrices and Pauli sigma matrices. By using these generators we have obtained the 21 rotational transformation in the $R_{kl}$ plane for SO(7) group. An infinitesimal rotation transformations of SO(7) group is defined by using generators of SO(8) symmetry. These representations define the generators of 7 and 8 dimensional orthogonal groups in terms of split octonionic descriptions.
of the Clifford groups. Rotational transformations in seven dimensional space by using direct product of gamma matrices with generators of $SU(2)$ group has been described. We have used these generators to describe the rotational transformation in seven dimensions for $SO(7)$ symmetry corresponding to split octonion $u_0, u_j$ and its Hermitian conjugate $u_0^*, u_j^*$ respectively. Dirac Gamma matrices $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ are matrices with satisfy specific anti commutation relations that ensure they generate a matrix representation of the Clifford algebra $C\ell_{1,3}(R)$. It is also possible to define higher-dimensional gamma matrices by using the direct product of Dirac Gamma matrices with $SU(N)$ group generator. This will make higher dimensional Clifford algebra. Clifford algebra can provide infinitesimal spatial rotations and Lorentz boosts. This generalized split octonion algebra can be further used to describe the higher dimensional algebra. Also one can calculate the rotational transformations and boosts in higher dimensional algebra by using direct product of Gamma matrices with generators of $SU(N)$ group.

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