Alpha-factor in multimode lasers

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Abstract. A theory of the spectral linewidth for a multi-mode laser is presented and used to explain experiment results. It turned out that linewidth is much greater than predicted by Schawlow-Townes theory. Henry attributed this enhancement to variation of real part of refractive index with carrier density. Spontaneous emissions cause changes to imaginary part of refractive index, which is coupled to changes in a real counterpart. This approach is expanded to cover multi-mode lasers and interactions between modes.

1. Introduction
The main forte of a laser is the ability to generate monochromatic light. However, in reality, spectrum’s linewidth has a minimum. Several factors contribute to it, and spontaneous emission is one of them. Schlow and Townes in [1] showed that it indeed causes broadening due to randomness of phase of an emitted photon. However, experiments showed that semiconductor laser has much broader linewidth than predicted (by ∼50 times).

Then Henry has built a theory around carrier density alteration caused by emissions in [2]. The main idea is that a single emission changes imaginary part of a refractive index profile. This change is coupled to a change in a real part via Kramers-Kronig relation. The ratio between them he called the alpha-factor. It turned out that linewidth broadening is proportional to the square of alpha-factor.

However Henry’s theory was built for single-mode lasers. At present numerous applications of multimode lasers (such as comb lasers) are being developed. For example, high-precision spectroscopy [3], wavelength-division multiplexing [4] and sub-pm distance measurements [5]. Yet excessive linewidth broadening can be the key factor in their success. Thus we aim for expanding Henry’s theory to lasers with multiple modes.

2. Theory
2.1. Phase deviation for a single emission
There are two ways through which spontaneous recombination can change phase: via immediate phase change due to radiated photon’s phase randomness (Δφ′k) and via alpha-factor (Δφ″k).

Consider βk as k-th mode’s complex amplitude, for which |βk|^2 = I_k equals amount of photons in k-th the mode. Then, if there was i-th emission in k-th mode we find from geometry

$$\Delta \phi'_k = I_k^{-1/2} \sin \theta_i ,$$

(1)
where $\theta_i$ is emitted photon’s phase.

To find $\Delta \phi''$ assume $\omega_k$ and $k_k$ - $k$-th mode’s frequency and wavenumber. For simplicity it’s aligned with $z$ axis. Then, the mode’s electric field is written as:

$$E_k \sim \beta_k \exp(\imath \omega_k t - \imath k_k z) + \beta_k^* \exp(-\imath \omega_k t + \imath k_k z) \quad (2)$$

Wave equation for it:

$$\frac{\partial^2 E_k}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 (\epsilon_k E_k)}{\partial t^2} \quad (3)$$

In order to take into account $\epsilon_k$ time dependency due to dispersion and time dependence of $\beta_k$, we write:

$$\epsilon_k E_k \sim \left( \epsilon_k \beta_k + \frac{\imath \epsilon_k}{\omega_k} \frac{\partial \beta_k}{\partial t} \right) \exp(\imath \omega_k t - \imath k_k z) + c.c. \quad (4)$$

Going back to (3) and assuming that high order derivatives are negligibly small:

$$\frac{2 \omega_k}{c^2} \left( \epsilon_k + \frac{\omega_k}{2} \frac{\partial \epsilon_k}{\partial \omega_k} \right) \frac{\partial \beta_k}{\partial t} = \left( \frac{\omega_k^2}{c^2} \epsilon_k - k_k^2 \right) \beta_k \quad (5)$$

A permittivity is related to a refractive index via $\epsilon_k = (n_k' - \imath n_k'')^2$. Then the left part of the equation can be rewritten with a group velocity $v_g$:

$$\epsilon_k + \frac{\omega_k}{2} \frac{\partial \epsilon_k}{\partial \omega_k} = n_k' + \omega_k \frac{\partial \epsilon_k}{\partial \omega_k} = \frac{c}{v_g} n_k' \quad (6)$$

The imaginary part of $n_k$ can be expressed with $g_k$ and $\alpha_k$ - $k$-th mode’s material gain and media’s attenuation respectively:

$$g_k - \alpha_k = - \frac{2 \omega_k}{c} n_k'' \quad (7)$$

If $k$-th mode radiates, $n_k'' = 0$ in equilibrium and so $k_k = n_k' \omega_k/c$.

Spontaneous emissions cause variations in density of charge carriers. Let’s consider a emission in $m$-th mode which caused change in $n_m''$. Due to homogeneous broadening a “bump” in $n''(\omega)$ around $\omega_m$ will appear:

$$n''(\omega) \rightarrow n''(\omega) + F(\omega - \omega_m) \Delta n_m'' \quad \text{Figure 1. $n''(\omega)$ change upon emission.}$$

We assume $F(\Delta \omega) = (1 + (\Delta \omega/\Gamma)^2)^{-1}$ where $\Gamma$ is a broadening parameter [6].

Due to Kramers-Kronig relation $k$-th mode’s permittivity will change (see Fig.1)

$$\epsilon_k = (n_k + \Delta n_k' - \imath \Delta n_k'')^2 \approx n_k^2 - 2 \imath \Delta n_k'' (F_{km} + \imath \alpha_{km}) \quad (8)$$

Here we substituted $F_{km} = \Delta n_k'' / \Delta n_m''$ which corresponds to homogeneous broadening, and $\alpha_{km} = \Delta n_k'/\Delta n_m''$ - inter-mode alpha-factor. With (6) and (8) $\partial \beta_k/\partial t$ can be factored out from (5):

$$\frac{\partial \beta_k}{\partial t} = - \frac{\omega_k}{c} v_g \Delta n_m'' (\Gamma_{km} + \imath \alpha_{km}) \beta_k \quad (9)$$
Because m-th mode radiates, \( n_m'' = 0 \), upon emission we can write \( \Delta n_m'' = -\frac{c}{2\omega_m} (g_m - \alpha_m) \).

Also it’s known that \( (g_m - \alpha_m) v_q = G_m - \gamma_m \), where \( G_m \) and \( \gamma_m \) are m-th mode’s modal gain and modal attenuation respectively:

\[
\frac{\partial \beta_k}{\partial t} = \frac{\omega_k}{\omega_m} \frac{G_m - \gamma_m}{2} (\Gamma_{km} + i\alpha_{km}) \beta_k \tag{10}
\]

For the clearer picture we will convert \( \beta_k \) to intensity and phase:

\[
I_k = \beta_k \beta_k^* \Rightarrow \dot{I}_k = \frac{\omega_k}{\omega_m} (G_m - \gamma_m) \Gamma_{km} I_k \tag{11}
\]

\[
\phi_k = \frac{1}{2i} \ln \left( \frac{\beta_k}{\beta_k^*} \right) \Rightarrow \dot{\phi}_k = \frac{1}{2i} \frac{\dot{\beta}_k \beta_k^* - \beta_k \dot{\beta}_k^*}{I} = \frac{\omega_k}{2\omega_m} (G_m - \gamma_m) \alpha_{km} \tag{12}
\]

Since we don’t know how \( G_m - \gamma_m \) changes over time, we can bind it to intensity, which behaviour is known:

\[
\dot{I}_m = (G_m - \gamma_m) I_m \Rightarrow G_m - \gamma_m = \frac{\dot{I}_m}{I_m} \tag{13}
\]

Assume that i-th emission into m-th mode happened at \( t = 0 \). Then the intensity will increase by 1: \( I_m \to I_m + 1 \). After certain period of time \( I_m \) will return to its equilibrium value. This way with (12) and (13) we can find total phase change:

\[
\Delta \phi_k'' = \int_0^\infty \dot{\phi}_k dt = \frac{1}{2i} \frac{\omega_k}{\omega_m} \alpha_{km} \int_0^\infty (G_m - \gamma_m) dt = \frac{1}{2i} \frac{\omega_k}{\omega_m} \alpha_{km} \ln \left( \frac{I_m}{I_m + 1} \right) \tag{14}
\]

For radiating modes \( I_m \gg 1 \), also for simplicity \( \omega_k/\omega_m \approx 1 \). This way (14) may be simplified:

\[
\Delta \phi_k'' \approx -\frac{\alpha_{km}}{2I_m} \tag{15}
\]

2.2. Linewidth broadening

Together with out-of-phase component \( \Delta \phi_k' \) (which non-zero only for \( k = m \)) we have found a total phase change for a single emission:

\[
\Delta \phi_k = -\frac{\alpha_{km}}{2I_m} + \frac{1}{I_m^{1/2}} \left[ \delta_{km} \sin \left( \theta_i \right) - \alpha_{km} \cos \left( \theta_i \right) \right] \tag{16}
\]

Assume \( m \neq k \):

\[
\Delta \phi_k = -\frac{\alpha_{km}}{2I_m} - \frac{\alpha_{km}}{I_m^{1/2}} \cos \left( \theta_i \right) \tag{17}
\]

First summand is negligibly small for \( I_m \gg 1 \). Then if \( R_m \) is a spontaneous emission rate into m-th mode, after time \( t \) average square of k-th mode deviation due to m-th mode will be:

\[
\left\langle \Delta \phi_k^2 \right\rangle = \frac{\alpha_{km}^2}{2I_m} R_m t \tag{18}
\]

Since emission is a random process, mode’s phase will do Brownian motion [2] and thus mode’s power spectrum will be Lorentzian with FWHM \( \Delta f_k \)

\[
\Delta f_k = \frac{\alpha_{km}^2}{4\pi I_m} R_m \tag{19}
\]
For $k = m$ from [2]:

$$\Delta f_k = \frac{1 + \alpha_{kk}^2}{4\pi I_k} R_k$$

Assuming that contribution from each emission is independent. Then in order to get full FWHM for mode, all modes’ contributions should be considered:

$$\Delta f_k = \frac{1}{4\pi I_k} R_k + \sum_m \frac{\alpha_{km}^2}{4\pi I_m} R_m$$  \hspace{1cm} (20)

2.3. Alpha-factor

Here $P\int \ldots$ stands for principal value integral. In our $n(\omega)$ notation Kramers-Kronig relation states that

$$n'(\omega) = 1 - \frac{2}{\pi} P\int_0^\infty \frac{\omega' n''(\omega')}{\omega'^2 - \omega^2} d\omega'$$  \hspace{1cm} (21)

Using the property of refractive index $n''(-x) = -n''(x)$, we change integration domain:

$$n'(\omega) = 1 - \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\omega' n''(\omega')}{\omega'^2 - \omega^2} d\omega'$$  \hspace{1cm} (22)

As shown earlier, spontaneous emission into $m$-th mode will cause change $n''(\omega) \rightarrow n''(\omega) \pm F(\omega \mp \omega_m) \Delta n''_m$. Particularly, for $k$-th mode $n'(\omega_k) \rightarrow n'(\omega_k) + \Delta n'(\omega_k)$.

$$\Delta n'_k = \Delta n'_{\omega_k} = -\frac{\Delta n''_m}{\pi} P\int_{-\infty}^{\infty} \frac{\omega' [F(\omega' - \omega_m) - F(\omega' + \omega_m)]}{\omega'^2 - \omega_k^2} d\omega'$$  \hspace{1cm} (23)

This integral can be computed with methods of complex analysis:

$$\alpha_{km} = \frac{\Delta n'_{\omega_k}}{\Delta n''_m} = -\frac{\xi}{1 + \xi^2}, \text{ where } \xi = \frac{\omega_m - \omega_k}{\Gamma}$$  \hspace{1cm} (24)

**Figure 2.** Alpha factor dependency on frequency difference between modes
3. Examples

In this section we estimate linewidth broadening due to alpha-factor for a QD laser with maximal modal gain $273 \text{ cm}^{-1}$, $\Gamma = 4 \text{ meV}$ and 0.2 eV frequency distance between modes. For clarity we approximate comb laser spectra with a stack of modes of equal intensity which is usually the case.

As seen on Fig.3, as number of active modes increases, overall linewidth broadening reduces. This can be explained by the fact that the strongest sources of noise are on sides of a spectra as the intensity here is smaller. And further away these sources are the less noise there will be.

On Fig.4 as $\Gamma/\Delta \omega$ increases, broadening increases as well due to expanding of $n''(\omega)$ “bump” (see Fig.1). However, as it reaches hundreds, opposite effect takes place - $\xi$ becomes too small for neighbouring modes (see Fig.2) and alpha-factors never reach significant values. However, such $\Gamma$ can’t be reached in practice.

4. Results

The method of calculating linewidth broadening of semiconductor lasers caused by spontaneous emissions was derived. This method requires usage of inter-mode alpha factor, which represents impact of spontaneous emission into one mode on refractive index of neighbouring modes. It turns out that inter-mode alpha-factor is zero for mode interaction with itself and reaches maximum for modes, whose frequencies difference equals to homogeneous broadening parameter.

We have shown that in order to reduce linewidth broadening one should increase a number of radiating modes, reduce homogeneous linewidth broadening and a reach higher intensity for each mode.

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