Planar Traveling Waves For Nonlocal Dispersion Equation With Monostable Nonlinearity

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Abstract

In this paper, we study a class of nonlocal dispersion equation with monostable nonlinearity in \(n\)-dimensional space

\[
\begin{aligned}
    u_t - J * u + u + d(u(t, x)) &= \int_{\mathbb{R}^n} f_\beta(y)b(u(t-\tau, x-y))dy, \\
    u(s, x) &= u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where the nonlinear functions \(d(u)\) and \(b(u)\) possess the monostable characters like Fisher-KPP type, \(f_\beta(x)\) is the heat kernel, and the kernel \(J(x)\) satisfies \(\hat{J}(\xi) = 1 - K|\xi|^\alpha + o(|\xi|^\alpha)\) for \(0 < \alpha \leq 2\). After establishing the existence for both the planar traveling waves \(\phi(x \cdot e + ct)\) for \(c \geq c_\star\) (\(c_\star\) is the critical wave speed) and the solution \(u(t, x)\) for the Cauchy problem, as well as the comparison principles, we prove that, all noncritical planar wavefronts \(\phi(x \cdot e + ct)\) are globally stable with the exponential convergence rate \(t^{-n/\alpha}e^{-\mu t}\) for \(\mu_\tau > 0\), and the critical wavefronts \(\phi(x \cdot e + c_\star t)\) are globally stable in the algebraic form \(t^{-n/\alpha}\). The adopted approach is Fourier transform and the weighted energy method with a suitably selected weight function. These rates are optimal and the stability results significantly develop the existing studies for nonlocal dispersion equations.

Keywords: Nonlocal dispersion equations, traveling waves, global stability, the Fisher-KPP equation, time-delays, weighted energy, Fourier transform.

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## 1 Introduction

For the gradient flow to an order parameter describing the state of a solid material, for example, a perfect crystal with two different orientations, it is usually described by a convolution model of phase transition in the form [2, 4, 8, 23, 24]

\[ u_t = J * u - u + F(u), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad (1.1) \]

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( J(x) \) is a non-negative and radial kernel with unit integral, and

\[ (J * u)(t, x) = \int_{\mathbb{R}^n} J(x-y)u(t, y)dy. \quad (1.2) \]

As showed in [4, 11], when the kernel \( J(x) \) has a second momentum, for example, \( J \) is compact-supported or Gaussian-like kernel \( J \sim e^{-x^2} \), its Fourier transform looks like

\[ \hat{J}(\xi) = 1 - K||\xi||^2 + o(||\xi||^2), \quad K > 0, \]

then the effect of the nonlocal dispersion \( J * u - u \) is almost the same to the linear diffusion \( K\Delta u \):

\[ J * u - u \approx K\Delta u, \]

which informs us to expect that the behaviors of the solutions to the nonlocal dispersion equation and the linear diffusion equation are almost identical [4, 8, 23, 24]

\[ u_t = J * u - u \iff u_t = K\Delta u. \]

Notice that, comparing with the heat equations, the solutions for the nonlocal dispersion equations usually loss the spatial regularity, but have much better regularity in time, see Remark 4.2 below for details.

In general, \( J(x) \) may not have a second momentum, let us say,

\[ \hat{J}(\xi) = 1 - K||\xi||^2 + o(||\xi||^2) \quad \text{as} \quad \xi \to 0 \quad \text{for} \quad \alpha \in (0, 2). \]
One example is the Cauchy law by taking $J(x) = \frac{1}{1+|x|^2}$ which implies its Fourier transform mentioned above with $\alpha = 1$. In this case, the behavior of the solutions to the nonlocal dispersion equation is almost identical to the fractional diffusion equation \cite{4, 8, 23, 24}

$$u_t = J \ast u - u \Leftrightarrow u_t = K \Delta^{\alpha/2} u.$$  

Equation (1.1) represents also the dynamical population model of single species in ecology \cite{13}, where $u(t, x)$ is the density of population at location $x$ and time $t$, and $J(x - y)$ is thought of as the probability distribution of jumping from location $y$ to location $x$, and $J \ast u = \int_{\mathbb{R}^n} J(x - y) u(t, y) dy$ is the rate at which individuals are arriving to position $x$ from all other places, while $-u(x, t) = -\int_{\mathbb{R}^n} J(x - y) u(t, x) dy$ stands the rate at which they are leaving the location $x$ to travel to all other places. In this case, under the consideration of the effects from birth rate and death rate, the equation (1.1) is usually written as follows

$$u_t = J \ast u - u + b(u(t - \tau, x)) - d(u(t, x)), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+,$$  

where $b(u(t - \tau, x))$ is the birth rate function, $d(u(t, x))$ is the death rate function, and $\tau > 0$ is the mature age of the single species, which is usually called the time-delay. Furthermore, if we consider the distribution of all matured population, the effect of birth rate is then involved in whole space $\mathbb{R}^n$ \cite{20, 39, 47}, and the equation is expressed as

$$\frac{\partial u}{\partial t} - J \ast u + u + d(u(t, x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+,$$  

where $f_\beta(y)$, with $\beta > 0$, is the heat kernel in the form of

$$f_\beta(y) = \frac{1}{(4\pi \beta)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4\beta}}, \quad \text{with} \quad \int_{\mathbb{R}^n} f_\beta(y) dy = 1.$$  

Notice that, by using the property of heat kernel

$$\lim_{\beta \to 0^+} \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy = b(u(t - \tau, x)),$$

we then derive the equation (1.3) as a limit of the equation (1.4) by taking $\beta \to 0^+$, and further derive the regular nonlocal dispersion equation (1.1) from the equation (1.3) by taking the time-delay $\tau = 0$ and $F(u) = b(u) - d(u)$. In particular, if we set $d(u) = u^2$ and $b(u) = u$, then, from (1.1) we get the classical Fisher-KPP equation with nonlocal dispersion

$$u_t = J \ast u - u + u(1 - u).$$  

So, the equations (1.1) and (1.3) and (1.6) all are the special cases of the equation (1.4).

In this paper, we will concentrate ourselves to the Cauchy problem for the more generalized equation (1.3) with non-locality of birth rate

$$\begin{cases}
\frac{\partial u}{\partial t} - J \ast u + u + d(u(t, x)) = \int_{\mathbb{R}^n} f_\beta(y) b(u(t - \tau, x - y)) dy, \\
u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n.
\end{cases}$$  

(1.7)
When $\tau = 0$ (no time-delay), then the above equation is reduced to

$$
\begin{cases}
\frac{\partial u}{\partial t} - J * u + u + d(u) = \int_{\mathbb{R}^n} f_\beta(y)b(u(t, x - y))dy, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}^n.
\end{cases}
$$ (1.8)

We will also discuss how the time-delay $\tau$ effects the property of the solutions.

For the equation (1.1) in 1D case, when $F(u)$ is bistable, namely, two constant equilibria $u_-$ and $u_+$ both are the stable nodes (the typical example is the Huxley equation with $F(u) = u(u-a)(1-u)$ for $0 < a < 1$), Bates et al [2] and Chen [6] proved that the traveling waves are globally stable as $t \to +\infty$.

In this paper, we consider another important type of equations with monostable nonlinearity. The typical example in this case is Fisher-KPP equation with $F(u) = u(1-u)$. Hence, throughout this paper, we assume that the death rate $d(u)$ and birth rate $b(u)$ capture the following characters of monostable nonlinearity:

(H1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;

(H2) $b'(0) > d'(0) \geq 0$ and $0 \leq b'(u_+) < d'(u_+)$;

(H3) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$.

These characters are summarized from the classical Fisher-KPP equation, see also the monostable reaction-diffusion equations in ecology, for example, the Nicholson’s blowflies equation [37, 38, 39, 47] with

$$d(u) = \delta u \quad \text{and} \quad b(u) = pue^{-au}, \quad p > 0, \delta > 0, a > 0$$

and $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{b}{\delta} > 0$ under the consideration of $1 < \frac{b}{\delta} \leq e$; and the age-structured population model [19, 20, 39, 42, 44] with

$$d(u) = \delta u^2 \quad \text{and} \quad b(u) = pe^{-\gamma u}, \quad \delta > 0, \quad p > 0, \quad \gamma > 0,$$

and $u_- = 0$ and $u_+ = \frac{b}{\delta} e^{-\gamma}$.

Clearly, under the hypothesis (H1)-(H3), both $u_- = 0$ and $u_+ > 0$ are constant equilibria of the equation (1.7), and $u_- = 0$ is unstable and $u_+$ is stable for the spatially homogeneous equation associated with (1.7), this is why we call the equation (1.7), including (1.1) and (1.3) and (1.8), as monostable.

On the other hand, we also assume the kernel $J(x)$ satisfying:

(J1) $J(x) = \prod_{i=1}^{n} J_i(x_i)$, where $J_i(x_i)$ is smooth, and $J_i(x_i) = J_i(|x_i|) \geq 0$ and $\int_{\mathbb{R}} J_i(x_i)dx_i = 1$ for $i = 1, 2, \ldots, n$, and $\int_{\mathbb{R}} |y_1|J_1(y_1)e^{-\lambda y_1}dy_1 < \infty$ for $\lambda_\ast > 0$ defined in (2.3) and (2.4);

(J2) Fourier transform of $J(x)$ satisfies $\hat{J}(\xi) = 1 - K|\xi|^\alpha + o(|\xi|^\alpha)$ as $\xi \to 0$ with $\alpha \in (0, 2]$ and $K > 0$.

A planar traveling wavefront to the equation (1.7) for $\tau \geq 0$ is a special solution in the form of $u(t, x) = \phi(x \cdot e + ct)$ with $\phi(\pm\infty) = u_{\pm}$, where $c$ is the wave speed, $e$ is a unit vector of the basis of $\mathbb{R}^n$. 

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Without loss of generality, we can always assume $e = e_1 = (1, 0, \cdots, 0)$ by rotating the coordinates. Thus, the planar traveling wavefront $\phi(x \cdot e_1 + ct) = \phi(x_1 + ct)$ satisfies, for $\tau \geq 0$,

\[
\begin{cases}
    c\phi' - J \ast \phi + \phi + d(\phi) = \int_{\mathbb{R}^n} f_\beta(y)b(\phi(\xi_1 - y_1 - ct))dy,
    \\
    \phi(\pm\infty) = u_\pm,
\end{cases}
\tag{1.9}
\]

where $' = \frac{d}{d\xi_1}$ and $\xi_1 = x_1 + ct$. Let

\[
f_{i\beta}(y_i) := \frac{1}{(4\pi\beta)^{1/2}}e^{-\frac{y_i^2}{4\beta}}. \tag{1.10}
\]

Then

\[
f_\beta(y) := \prod_{i=1}^n f_{i\beta}(y_i), \quad \text{and} \quad \int_{\mathbb{R}} f_{i\beta}(y_i)dy_i = 1, \quad i = 1, 2, \cdots, n, \tag{1.11}
\]

and (1.9) is reduced to, for $\tau \geq 0$,

\[
\begin{cases}
    c\phi' - J_1 \ast \phi + \phi + d(\phi) = \int_{\mathbb{R}} f_{1\beta}(y_1)b(\phi(\xi_1 - y_1 - ct))dy_1, \\
    \phi(\pm\infty) = u_\pm.
\end{cases} \tag{1.12}
\]

The main purpose of this paper is to study the global asymptotic stability of planar traveling wavefronts of the equations (1.7) and (1.8) with or without time-delay, respectively, in particular, in the case of the critical wave $\phi(x_1 + c\ast t)$. Here the number $c_\ast$ is called the critical speed (or the minimum speed) in the sense that a traveling wave $\phi(x_1 + ct)$ exists if $c \geq c_\ast$, while no traveling wave $\phi(x_1 + ct)$ exists if $c < c_\ast$.

The nonlocal dispersion equation (1.1) has been extensively studied recently. Chasseigne et al [4] and Cortazar et al [8] showed that the linear nonlocal dispersion equation (1.1) (with $F = 0$) is almost equivalent to the linear diffusion equation, and the asymptotic behavior of the solutions to the linear equation of nonlocal dispersion is exactly the same to the corresponding linear diffusion equation. Ignat and Rossi [23, 24] further obtained the asymptotic behavior of the solutions to the nonlinear equation (1.1). Garcia-Melian and Quirós [17] investigated the blow up phenomenon of the solution to the equation (1.1) with $F(u) = u^p$, and gave the Fujita critical exponent. Regarding the structure of special solutions to (1.1) like traveling wave solutions, early in 1997 Bates et al [2] and Chen [6] established the existence of the traveling waves for (1.1) with bistable nonlinearity, and proved their global stability. For (1.1) with monostable nonlinearity, recently Coville and his collaborators [9, 10, 11, 12] studied the existence and uniqueness (up to a shift) of traveling waves. See also the existence/nonexistence of traveling waves by Yagisita [53] and the existence of almost periodic traveling waves by Chen [5]. However, the stability of traveling waves for the nonlocal equation (1.1) (including (1.1) and (1.3)) with monostable nonlinearity is almost not related, except a special case for the fast waves with large wave speed to the 1D age-structured population model by Pan et al [44]. As we know, such a problem is also very significant but challenging, because the equations of Fisher-KPP type possess an unstable node, different from the bistable case, this unstable node usually causes a serious difficulty in the stability proof, particularly, for the critical traveling waves. The main interest in this paper is to investigate the stability of traveling waves to (1.7) with $\tau > 0$ and (1.8) with $\tau = 0$. An easy to follow method will be introduced for the stability proof to the nonlocal dispersion equations.
In this paper, we will first investigate the linearized equation of (1.7), and derive the optimal decay rates of the solution to the linearized equation by means of Fourier transform. This is a crucial step for get the optimal convergence for the nonlocal stability of traveling waves. Then, we will technically establish the global existence and comparison principles of the solution to the n-D nonlinear equation with nonlocal dispersion (1.7). Inspired by [43] for the classical Fisher-KPP equations and the further developments by [39, 40], by ingeniously selecting a weight function which is dependent on the critical wave speed $c_*$, and using the weighted energy method and the Green function method with the comparison principles together, we will further prove that, all noncritical planar traveling waves $\phi(x \cdot e + ct)$ are exponentially stable in the form of $t^{-\frac{\mu}{\alpha}}e^{-\mu \tau}$ for some constant $\mu = \mu(\tau)$ such that $0 < \mu_{\tau} \leq \mu_0$ for $\tau \geq 0$; and all critical planar traveling waves $\phi(x \cdot e + c_* t)$ are algebraically stable in the form of $t^{-\frac{\mu_{\tau}}{\alpha}}$. These convergence rates are optimal and the stability results significantly develop the existing studies on the nonlocal dispersion equations. We will also show that the time-delay $\tau$ will slow down the convergence of the the solution $u(t, x)$ to the noncritical planar traveling waves $\phi(x \cdot e + ct)$ with $c > c_*$, and cause the higher requirement for the initial perturbation around the wavefronts.

For the stability of traveling waves to other modeling equations, we refer to the classical and significant contributions in [1, 3, 7, 14, 16, 21, 22, 25, 29, 30, 31, 32, 33, 37, 38, 39, 42, 43, 45, 46, 49, 50, 51, 52] for reaction-diffusion equations and [15, 18, 26, 27, 34, 35, 36, 48, 54] for fluid dynamical systems, and the references therein.

The paper is organized as follows. In section 2, we will state the existence of the traveling waves, and their stability. In section 3, we will give the solution formulas to the linearized dispersion equations of (1.7) and (1.8), and derive the optimal decay rates by Fourier transform with energy method together. In section 4, we will prove the global existence of the solution to (1.7) and establish the comparison principle. In section 5, based on the results obtained in sections 3 and 4, by using the weighted energy method, we will further prove the stability of planar traveling waves including the critical and noncritical waves. Finally, in section 6, we will give some particular applications of our stability theory to the classical Fisher-KPP equation with nonlocal dispersion and the Nicholson’s blowflies model, and make a concluding remark to a more general case.

Before ending this section, we make some notations. Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0$ and $c_i > 0$ $(i = 0, 1, 2, \cdots)$ represent specific constants. $j = (j_1, j_2, \cdots, j_n)$ denotes a multi-index with non-negative integers $j_i \geq 0$ $(i = 1, \cdots, n)$, and $|j| = j_1 + j_2 + \cdots + j_n$. The derivatives for multi-dimensional function are denoted as

$$\partial^j_x f(x) := \partial^{j_1}_{x_1} \cdots \partial^{j_n}_{x_n} f(x).$$

For a $n$-D function $f(x)$, its Fourier transform is defined as

$$\mathcal{F}[f](\eta) = \hat{f}(\eta) := \int_{\mathbb{R}^n} e^{-ix \cdot \eta} f(x) dx,$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \hat{f}(\eta) d\eta.$$

Let $I$ be an interval, typically $I = \mathbb{R}^n$. $L^p(I)$ $(p \geq 1)$ is the Lebesque space of the integrable functions defined on $I$, $W^{k,p}(I)$ $(k \geq 0, p \geq 1)$ is the Sobolev space of the $L^p$-functions $f(x)$ defined on the
interval $I$ whose derivatives $\partial^j_x f$ with $|j| = k$ also belong to $L^p(I)$, and in particular, we denote $W^{k,2}(I)$ as $H^k(I)$. Further, $L^p_w(I)$ denotes the weighted $L^p$-space for a weight function $w(x) > 0$ with the norm defined as

$$\|f\|_{L^p_w} = \left( \int_I w(x) |f(x)|^p \, dx \right)^{1/p},$$

$W^{k,p}_w(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{W^{k,p}_w} = \left( \sum_{|j|=0}^k \int_I w(x) |\partial^j_x f(x)|^p \, dx \right)^{1/p},$$

and $H^k_w(I)$ is defined with the norm

$$\|f\|_{H^k_w} = \left( \sum_{|j|=0}^k \int_I w(x) |\partial^j_x f(x)|^2 \, dx \right)^{1/2}. $$

Let $T > 0$ be a number and $B$ be a Banach space. We denote by $C^0([0, T], B)$ the space of the $B$-valued continuous functions on $[0, T]$, $L^2([0, T], B)$ as the space of the $B$-valued $L^2$-functions on $[0, T]$. The corresponding spaces of the $B$-valued functions on $[0, \infty)$ are defined similarly.

## 2 Traveling Waves and Their Stabilities

As we mentioned before, the existence and uniqueness (up to a shift) of traveling waves for the equation (1.1) were proved in [9, 10, 11, 12], particular, in a recent work by Yagisita [53] for the nonlinearity $F(u)$ monostable. Without any difficulty, these results can be extended to the nonlocal equation (1.7) with time-delay with the help of comparison principle established in Section 4, when $d(u)$ and $b(u)$ satisfy the monostable features (H1)-(H3). We state these results as follows without detailed proof.

**Theorem 2.1** Under the conditions (H1)-(H3) and (J1)-(J2), for the time-delay $\tau \geq 0$, there exist a minimum wave speed (also called the critical wave speed) $c_* > 0$ such that

- when $c \geq c_*$, there exits a monotone traveling wavefront $\phi(x_1 + ct)$ of (1.9) connecting $u_\pm$ exists;
- when $c < c_*$, no traveling wave $\phi(x_1 + ct)$ exists.

Here $(c_*, \lambda_*)$ with $c_* > 0$ and $\lambda_* > 0$ is given by

$$H_{c_}(\lambda_*) = G_{c_}(\lambda_*) = G_{c_*}'(\lambda_*), \quad (2.1)$$

where

$$H_{c}(\lambda) = b'(0)e^{3\lambda^2 - \lambda c}\tau, \quad G_{c}(\lambda) = c\lambda - E_c(\lambda) + d'(0), \quad E_c(\lambda) = \int_{\mathbb{R}} J_1(y_1)e^{-\lambda y_1}dy_1 - 1, \quad (2.2)$$

namely, $(c_*, \lambda_*)$ is the tangent point of $H_{c}(\lambda)$ and $G_{c}(\lambda)$ specified as

$$b'(0)e^{3\lambda^2 - \lambda c_\tau} = c_*\lambda_* - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda_\tau y_1}dy_1 + 1 + d'(0), \quad (2.3)$$

$$b'(0)(2\beta\lambda_* - c_\tau)e^{3\lambda^2 - \lambda c_\tau} = c_* + \int_{\mathbb{R}} y_1J_1(y_1)e^{-\lambda_\tau y_1}dy_1. \quad (2.4)$$

Furthermore, it can be verified:
In the case of \( c > c_* \), there exist two numbers depending on \( c \): \( \lambda_1 = \lambda_1(c) > 0 \) and \( \lambda_2 = \lambda_2(c) > 0 \) as the solutions to the equation \( H_c(\lambda_i) = G_c(\lambda_i) \), i.e.,

\[
b'(0)e^{\beta \lambda_i^2 - \lambda_i ct} = c\lambda_i - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda_i y_1}dy_1 + d'(0), \quad i = 1, 2, \tag{2.5}
\]

such that

\[
H_c(\lambda) < G_c(\lambda) \quad \text{for} \quad \lambda_1 < \lambda < \lambda_2, \tag{2.6}
\]

and particularly,

\[
H_c(\lambda_*) < G_c(\lambda_*) \quad \text{with} \quad \lambda_1 < \lambda_* < \lambda_2. \tag{2.7}
\]

In the case of \( c = c_* \), it holds

\[
H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \text{with} \quad \lambda_1 = \lambda_* = \lambda_2. \tag{2.8}
\]

When \( \xi_1 = x_1 + ct \to \pm \infty \), for all \( c \geq c_* \), the traveling wavefronts \( \phi(x_1 + ct) \) converge to \( u_\pm \) exponentially as follows

\[
|\phi(\xi_1) - u_\pm| = O(1)e^{-\lambda^\pm|\xi_1|}. \tag{2.9}
\]

Here \( \lambda^- = \lambda_1(c) > 0 \) is given in \((2.5)\), and \( \lambda^+ = \lambda^+(c) > 0 \) is the unique root determined by the following equation

\[
-c\lambda^+ - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda^+ y_1}dy_1 + d'(u_+) = b'(u_+)e^{\beta(\lambda^+)^2 - \lambda^+ ct}. \tag{2.10}
\]

For easily understanding all cases mentioned in the above, we show them in Figure 2.1.

Before stating our main stability theorems, let us technically choose a weight function:

\[
w(x_1) = \begin{cases} 
e^{-\lambda_*(x_1-x_*)}, & \text{for} \ x_1 \leq x_*, \\ 1, & \text{for} \ x_1 > x_*, \end{cases} \tag{2.11}
\]

where \( \lambda_* = \lambda_*(c_*) > 0 \) is given in \((2.3)\) and \((2.4)\), and \( x_* > 0 \) is a sufficiently large number such that,

\[
0 < d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_3(y)b'(\phi(x_* - y_1 - ct))dy < d'(u_+) - b'(u_+). \tag{2.12}
\]

The selection of \( x_* \) in \((2.12)\) is valid, because of \( d'(u_+) - b'(u_+) > 0 \) (see(H2)). In fact, we have

\[
\lim_{\xi_1 \to \infty} d'(\phi(\xi_1)) = d'(u_+)
\]

Figure 2.1: (a): the case of \( c > c_* \); (b): the case of \( c = c_* \); and (c): the case of \( c < c_* \).
> b'(u_+)
= \int_{\mathbb{R}^n} f_\beta(y) \left[ \lim_{\xi_1 \to \infty} b'(\phi(\xi_1 - y_1 - c\tau)) \right] dy
= \lim_{\xi_1 \to \infty} \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy,

which implies that, by (H3), there exists a unique \( x_* \gg 1 \) such that, for \( \xi_1 \in [x_*, \infty) \)

\[
d'(u_+) - b'(u_+)
> d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(\xi_1 - y_1 - c\tau)) dy
\geq d'(\phi(x_*)) - \int_{\mathbb{R}^n} f_\beta(y) b'(\phi(x_* - y_1 - c\tau)) dy
> 0.
\] (2.13)

**Theorem 2.2 (Stability of planar traveling waves with time-delay)** Under assumptions (H1)-(H3) and (J1)-(J2), for a given traveling wave \( \phi(x_1 + ct) \) of the equation (1.7) with \( c \geq c_* \) and \( \phi(\pm \infty) = u_\pm \), if the initial data \( u_0(s,x) \) is bounded in \([u_-, u_+]\) and \( u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n)) \) and \( \partial_s (u_0 - \phi) \in L^1([\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n)) \) with \( m > \frac{n}{2} \), then the solution of (1.7) uniquely exists and satisfies:

- When \( c > c_* \), the solution \( u(t, x) \) converges to the noncritical planar traveling wave \( \phi(x_1 + ct) \)
exponentially

\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{2}} e^{-\mu_* t}, \quad t > 0,
\] (2.14)

where

\[
0 < \mu_* < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\},
\] (2.15)

and \( \varepsilon_1 = \varepsilon_1(\tau) \) such that \( 0 < \varepsilon_1 < 1 \) for \( \tau > 0 \), and \( \varepsilon_1 = \varepsilon_1(\tau) \to 0^+ \) as \( \tau \to +\infty \);

- When \( c = c_* \), the solution \( u(t, x) \) converges to the critical planar traveling wave \( \phi(x_1 + c_* t) \)
algebraically

\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{n}{2}}, \quad t > 0.
\] (2.16)

However, when the time-delay \( \tau = 0 \), then we have the following stronger stability for the traveling waves but with a weaker condition on initial perturbation.

**Theorem 2.3 (Stability of planar traveling waves without time-delay)** Under assumptions (H1)-(H3) and (J1)-(J2), for a given traveling wave \( \phi(x_1 + ct) \) of the equation (1.8) with \( c \geq c_* \) and \( \phi(\pm \infty) = u_\pm \), if the initial data \( u_0(x) \) is bounded in \([u_-, u_+]\) and \( u_0 - \phi \in H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n) \) with \( m > \frac{n}{2} \), then the solution of (1.8) uniquely exists and satisfies:

- When \( c > c_* \), the solution \( u(t, x) \) converges to the noncritical planar traveling wave \( \phi(x_1 + ct) \)
exponentially

\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{2}} e^{-\mu_0 t}, \quad t > 0,
\] (2.17)

where

\[
0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\};
\] (2.18)
When $c = c^*$, the solution $u(t, x)$ converges to the critical planar traveling wave $\phi(x_1 + c^* t)$ algebraically

$$\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c^* t)| \leq C(1 + t)^{-\frac{n}{2}}, \quad t > 0.$$ (2.19)

**Remark 2.4**

1. Comparing Theorem 2.2 with time-delay and Theorem 2.3 without time-delay, we realize that, the sufficient condition on the initial perturbation around the wave in the case with time-delay is stronger than the case without time-delay, but the convergence rate to the noncritical waves $\phi(x_1 + ct)$ for $c > c^*$ in the case with time-delay is weaker than the case without time-delay, see (2.15) for $\mu \leq \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)] < G_c(\lambda_*) - H_c(\lambda_*)$, and (2.18) for $\mu_0 \leq G_c(\lambda_*) - H_c(\lambda_*)$, and $\varepsilon_1 = \varepsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$. This means, the time-delay $\tau > 0$ effects the stability of traveling waves a lot, not only the higher requirement for the initial perturbation, but also the slower convergence rate for the solution to the noncritical traveling waves.

2. The convergence rates showed both in Theorem 2.2 and Theorem 2.3 are explicit and optimal, particularly, the algebraic decay rates for the solution converging to the critical waves. Actually, all of them are derived from the linearized equations.

### 3 Linearized Nonlocal dispersion Equations

In this section, we will derive the solution formulas for the linearized nonlocal dispersion equations with or without time-delay, as well as their optimal decay rates, which will play a key role in the stability proof in section 5.

Now let us introduce the solution formula for linear delayed ODEs [28] and the asymptotic behaviors of the solutions [41].

**Lemma 3.1** ([28]) Let $z(t)$ be the solution to the following linear time-delayed ODE with time-delay $\tau > 0$

$$\begin{cases}
\frac{d}{dt} z(t) + k_1 z(t) = k_2 z(t - \tau) \\
z(s) = z_0(s), \quad s \in [-\tau, 0].
\end{cases}$$ (3.1)

Then

$$z(t) = e^{-k_1(t+\tau)} k_2 t z_0(-\tau) + \int_{-\tau}^{0} e^{-k_1(t-s)} k_2 e^{k_2(t-\tau-s)} z_0(s) + k_1 z_0(s) ds,$$ (3.2)

where

$$k_2 := k_2 e^{k_1 \tau},$$ (3.3)
and $e^{\hat{k}_2 t}$ is the so-called delayed exponential function in the form

$$e^{\hat{k}_2 t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + \frac{\hat{k}_2 t}{\tau}, & 0 \leq t < \tau, \\ 1 + \frac{\hat{k}_2 t}{\tau} + \frac{\hat{k}_2^2 (t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots, & \vdots \\ 1 + \frac{\hat{k}_2 t}{\tau} + \frac{\hat{k}_2^2 (t-\tau)^2}{2!} + \cdots + \frac{\hat{k}_2^m (t-(m-1)\tau)^m}{m!}, & (m-1)\tau \leq t < m\tau, \\ \vdots & \vdots \\ \end{cases}$$

(3.4)

and $\tilde{e}^{\hat{k}_2 t}$ is the fundamental solution to

$$\begin{align*}
\frac{d}{dt} z(t) &= \bar{k}_2 z(t-\tau) \\
z(s) &= 1, \quad s \in [-\tau, 0].
\end{align*}$$

(3.5)

Lemma 3.2 ([11]) Let $k_1 \geq 0$ and $k_2 \geq 0$. Then the solution $z(t)$ to (3.1) (or equivalently (3.2)) satisfies

$$|z(t)| \leq C_0 e^{-k_1 t} \tilde{e}^{\hat{k}_2 t},$$

(3.6)

where

$$C_0 := e^{-k_1 \tau} |z_0(-\tau)| + \int_{-\tau}^{0} e^{k_1 s} |z'_0(s) + k_1 z_0(s)| ds,$$

(3.7)

and the fundamental solution $\tilde{e}^{\hat{k}_2 t}$ with $\bar{k}_2 > 0$ to (3.3) satisfies

$$\tilde{e}^{\hat{k}_2 t} \leq C(1 + t)^{-\gamma} e^{\hat{k}_2 t}, \quad t > 0,$$

(3.8)

for arbitrary number $\gamma > 0$.

Furthermore, when $k_1 \geq k_2 \geq 0$, there exists a constant $\varepsilon_1 = \varepsilon_1(\tau)$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $\varepsilon_1 = 1$ for $\tau = 0$, and $\varepsilon_1 = \varepsilon_1(\tau) \to 0^+$ as $\tau \to +\infty$, such that

$$e^{-k_1 t} \tilde{e}^{\hat{k}_2 t} \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0,$$

(3.9)

and the solution $z(t)$ to (3.1) satisfies

$$|z(t)| \leq C e^{-\varepsilon_1 (k_1 - k_2) t}, \quad t > 0.$$  

(3.10)

Now, we consider the following linearized nonlocal time-delayed dispersion equation (which will be derived in section 5 for the proof of stability of traveling wavefronts)

$$\begin{align*}
\frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_\epsilon y_1} v(t, x - y) dy + c_1 v \\
= c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_\epsilon (y_1 + \epsilon \tau)} v(t - \tau, x - y) dy,
\end{align*}$$

(3.11)

for some given constant coefficients $c, c_1$ and $c_2$, where $c \geq c_s$ is the wave speed.
We are going to derive its solution formula as well as the asymptotic behavior of the solution. By taking Fourier transform to (3.17), we get

\[
\mathcal{F}\left[\int_{\mathbb{R}^n} J(y)e^{-\lambda_s y_1}v(t, x - y)dy\right](t, \eta) = \int_{\mathbb{R}^n} e^{-ix\cdot \eta}\left(\int_{\mathbb{R}^n} J(y)e^{-\lambda_s y_1}v(t, x - y)dy\right)dx \\
= \int_{\mathbb{R}^n} J(y)e^{-\lambda_s y_1}\left(\int_{\mathbb{R}^n} e^{-ix\cdot \eta}v(t, x - y)dy\right)dx \\
= \int_{\mathbb{R}^n} J(y)e^{-\lambda_s y_1}\left(\int_{\mathbb{R}^n} e^{-i(x+y)\cdot \eta}v(t, x)dx\right)dy \\
= \left(\int_{\mathbb{R}^n} e^{-iy\cdot \eta}J(y)e^{-\lambda_s y_1}dy\right)\hat{v}(t, \eta),
\]

(3.12)

and

\[
\mathcal{F}\left[c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}v(t - \tau, x - y)dy\right](t - \tau, \eta) \\
= c_2\int_{\mathbb{R}^n} e^{-ix\cdot \eta}\left(\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}v(t - \tau, x - y)dy\right)dx \\
= c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}\left(\int_{\mathbb{R}^n} e^{-ix\cdot \eta}v(t - \tau, x - y)dy\right)dx \\
= c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}\left(\int_{\mathbb{R}^n} e^{-i(x+y)\cdot \eta}v(t - \tau, x)dx\right)dy \\
= c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}e^{-iy\cdot \eta}\left(\int_{\mathbb{R}^n} e^{-ix\cdot \eta}v(t - \tau, x)dx\right)dy \\
= \left(c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}e^{-iy\cdot \eta}dy\right)\hat{v}(t - \tau, \eta),
\]

(3.13)

we have

\[
\frac{d\hat{v}}{dt} + A(\eta)\hat{v} = B(\eta)\hat{v}(t - \tau, \eta), \quad \text{with } \hat{v}(s, \eta) = \hat{v}_0(s, \eta), \quad s \in [-\tau, 0],
\]

(3.14)

where

\[
A(\eta) := c_1 - \int_{\mathbb{R}^n} J(y)e^{-\lambda_s y_1}e^{-iy\cdot \eta}dy
\]

(3.15)

and

\[
B(\eta) := c_2\int_{\mathbb{R}^n} f_\beta(y)e^{-\lambda_s (y_1+ct)}e^{-iy\cdot \eta}dy.
\]

(3.16)

By using the formula of the delayed ODE (3.2) in Lemma 3.1, we then solve (3.14) as follows

\[
\hat{v}(t, \eta) = e^{-A(\eta)(t+\tau)}e_{\tau}^{B(\eta)t}\hat{v}_0(-\tau, \eta) \\
+ \int_{-\tau}^{0} e^{-A(\eta)(t-s)}e_{\tau}^{B(\eta)(t-\tau-s)}\left[\partial_s \hat{v}_0(s, \eta) + A(\eta)\hat{v}_0(s, \eta)\right]ds,
\]

(3.17)

where

\[
B(\eta) := B(\eta)e^{A(\eta)\tau}.
\]

(3.18)

Then, by taking the inverse Fourier transform to (3.17), we get

\[
v(t, x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot \eta}e^{-A(\eta)(t+\tau)}e_{\tau}^{B(\eta)t}\hat{v}_0(-\tau, \eta)d\eta
\]

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and its derivatives
\[
\partial_{x_j}^k v(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i x \cdot \eta} (i \eta_j)^k e^{-A(\eta)(t+\tau)} e^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta) d\eta + \int_0^\tau \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i x \cdot \eta} (i \eta_j)^k e^{-A(\eta)(t-s)} e^{\mathcal{B}(\eta)(t-\tau-s)}
\]
\[
\times \left[ \partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right] ds \quad (3.19)
\]
for \( k = 0, 1, \ldots \) and \( j = 1, \ldots, n \).

Now we are going to derive the asymptotic behavior of \( v(t, x) \).

**Proposition 3.3 (Optimal decay rates for \( \tau > 0 \))** Suppose that \( v_0 \in C([-\tau, 0]; H^{m+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \) and \( \partial_s v_0 \in L^1([-\tau, 0]; H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \) for \( m \geq 0 \), and let
\[
\begin{align*}
\tilde{c}_1 & := c_1 - \int_{\mathbb{R}^n} J(y) e^{-\lambda_1 y} dy, \\
\tilde{c}_3 & := c_2 - \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_3 (y+\tau)} dy > 0.
\end{align*}
\quad (3.21)
\]
If \( \tilde{c}_1 \geq \tilde{c}_3 \), then there exists a constant \( \varepsilon_1 = \varepsilon_1(\tau) \) as showed in \( (3.11) \) satisfying \( 0 < \varepsilon_1 < 1 \) for \( \tau > 0 \), such that the solution of the linearized equation \( (3.11) \) satisfies
\[
\| \partial_{x_j}^k v(t) \|_{L^2(\mathbb{R}^n)} \leq C \varepsilon_1^k v_0 e^{-\varepsilon_1 \tilde{c}_1 - \varepsilon_3 t}, \quad t > 0,
\quad (3.22)
\]
for \( k = 0, 1, \ldots, [m] \) and \( j = 1, \ldots, n \), where
\[
\varepsilon_1^k v_0 := \left( \| v_0(-\tau) \|_{L^1(\mathbb{R}^n)} + \| v_0(-\tau) \|_{H^k(\mathbb{R}^n)} + \int_{-\tau}^0 \| (v'_{0s}, v_0)(s) \|_{L^1(\mathbb{R}^n)} + \| (v'_{0s}, v_0)(s) \|_{H^k(\mathbb{R}^n)} \right) ds.
\quad (3.23)
\]
Furthermore, if \( m > \frac{n}{2} \), then
\[
\| v(t) \|_{L^\infty(\mathbb{R}^n)} \leq C \varepsilon_1^m v_0^{-\frac{n}{2}} e^{-\varepsilon_1 \tilde{c}_1 - \varepsilon_3 t}, \quad t > 0.
\quad (3.24)
\]
Particularly, when \( \tilde{c}_1 = \tilde{c}_3 \), then
\[
\| v(t) \|_{L^\infty(\mathbb{R}^n)} \leq C \varepsilon_1^m v_0^{-\frac{n}{2}}, \quad t > 0.
\quad (3.25)
\]
**Proof.** Let
\[
I_1(t, \eta) := (i \eta_j)^k e^{-A(\eta)(t+\tau)} e^{\mathcal{B}(\eta)t} \hat{v}_0(-\tau, \eta),
\quad (3.26)
\]
\[
I_2(t-s, \eta) := (i \eta_j)^k e^{-A(\eta)(t-s)} e^{\mathcal{B}(\eta)(t-\tau-s)} \left[ \partial_s \hat{v}_0(s, \eta) + A(\eta) \hat{v}_0(s, \eta) \right].
\quad (3.27)
\]
Then, \( (3.20) \) is reduced to
\[
\partial_{x_j}^k v(t, x) = \mathcal{F}^{-1}[I_1](t, x) + \int_{-\tau}^0 \mathcal{F}^{-1}[I_2](t-s, x) ds.
\quad (3.28)
\]
So, by using Parseval’s equality, we have

\[
\|e^{\tilde{A}(\eta)t}\| = e^{-c_1t}\left|\exp\left(t \int_{\mathbb{R}^n} J(y)e^{\lambda_y}y_1 e^{-iy_\eta dy}\right)\right|
\]

\[= e^{-c_1t}\exp\left(t \int_{\mathbb{R}^n} J(y)e^{-\lambda_y}y_1 \cos(y \cdot \eta)dy\right)
\]

\[= e^{-c_1t}\exp\left(-t \int_{\mathbb{R}^n} J(y)e^{-\lambda_y}y_1 (1 - \cos(y \cdot \eta))dy\right)
\]

\[= e^{-k_1t}, \quad \text{with } k_1 := \tilde{c} + \int_{\mathbb{R}^n} J(y)e^{-\lambda_y}y_1 (1 - \cos(y \cdot \eta))dy, \quad (3.30)
\]

Note that, using (3.15), (3.16), and the facts \(e^{t^2} \geq 1\) for all \(x \in \mathbb{R}\), and \(\int_{\mathbb{R}^n} J(y) \sin(y \cdot \eta)dy = 0\) because \(J(y)\) is even and \(\sin(y \cdot \eta)\) is odd, and \(\int_{\mathbb{R}^n} J(y)dy = 1\), we have

\[
\exp\left(-t \int_{\mathbb{R}^n} J(y)e^{-\lambda_y}y_1 (1 - \cos(y \cdot \eta))dy\right)
\]

\[= \exp\left(-t \int_{\mathbb{R}^n} J(y)e^{-\lambda_y}y_1 + e^{\lambda_y}y_1 2 - \cos(y \cdot \eta))dy\right)
\]

\[= \exp\left(-t \int_{\mathbb{R}^n} J(y)(1 - \cos(y \cdot \eta))dy\right)
\]

\[= \exp\left(-t \int_{\mathbb{R}^n} J(y)[1 - \cos(y \cdot \eta)]]dy\right)
\]

\[= e^{(J(\eta)-1)t} \quad (3.31)
\]

and

\[|B(\eta)| \leq c_2 \int_{\mathbb{R}^n} f_y(y)e^{-\lambda_y(y_1+\tau \tau)}dy = c_3 =: k_2, \quad (3.32)
\]

and

\[|B(\eta)| = |B(\eta)e^{A(\eta)\tau}| \leq c_3 e^{k_1 \tau} \leq k_2 e^{k_1 \tau} =: \tilde{k}_2, \quad (3.33)
\]

and further

\[|e^{B(\eta)\tau}| \leq e^{\tilde{k}_2 \tau}. \quad (3.34)
\]

If \(\tilde{c}_1 \geq c_3\), from (J2), namely, \(1 - J(\eta) = k\|\eta\|^\alpha - o(\|\eta\|^\alpha) > 0\) as \(\eta \to 0\), then \(k_1 = \tilde{c}_1 + 1 - J(\eta) \geq c_3 = k_2\). Using (3.30), (3.31), (3.34) and (3.9) in Lemma 3.2, we obtain

\[
\|I_1(t)\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} e^{-A(\eta)(t+\tau)} e^{B(\eta)\tau} \tilde{v}_0(-\tau, \eta)\|\eta\|^{2k} d\eta
\]

\[\leq C \int_{\mathbb{R}^n} (e^{-k_1(t+\tau)} e^{\tilde{k}_2 \tau})^2 |\tilde{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta
\]

\[\leq C \int_{\mathbb{R}^n} (e^{-\epsilon_1(k_1 - k_2)\tau})^2 |\tilde{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta
\]

\[= C e^{-2\epsilon_1(\tilde{c}_1 - c_3)\tau} \int_{\mathbb{R}^n} e^{-2\epsilon_1(1-J(\eta))\tau} |\tilde{v}_0(-\tau, \eta)|^2 |\eta_j|^{2k} d\eta. \quad (3.35)
\]
Again from (J2), there exist some numbers $0 < K_1 < K$, $0 < \delta < 1$ and $\tilde{a} > 0$, such that

$$
\begin{align*}
K_1 |\eta|^\alpha &\leq 1 - \hat{J} (\eta) \leq K |\eta|^\alpha, & \text{as } |\eta| \leq \tilde{a}, \\
\delta := K_1 \tilde{a}^\alpha &\leq 1 - \hat{J} (\eta) \leq K |\eta|^\alpha, & \text{as } |\eta| \geq \tilde{a}.
\end{align*}
$$

Therefore, we have

$$
\int_{\mathbb{R}^n} e^{-2\varepsilon_1 (1 - \hat{J} (\eta)) t} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta
= \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1 (1 - \hat{J} (\eta)) t} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta + \int_{|\eta| \geq \tilde{a}} e^{-2\varepsilon_1 (1 - \hat{J} (\eta)) t} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta
\leq \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1 K_1 |\eta|^\alpha t} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta + \int_{|\eta| \geq \tilde{a}} e^{-2\varepsilon_1 \delta t} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta
\leq \|\hat{v}_0 (\tau, \eta)\|_{L^\infty (\mathbb{R}^n)}^2 t^{-\frac{n+2k}{\alpha}} \int_{|\eta| \leq \tilde{a}} e^{-2\varepsilon_1 K_1 |\eta|^\alpha t} |\eta_j|^{\frac{1}{2}} d\eta |\eta_j|^{\frac{1}{2}} d\eta
+ e^{-2\varepsilon_1 \delta t} \int_{|\eta| \geq \tilde{a}} |\hat{v}_0 (\tau, \eta)|^2 |\eta_j|^{2k} d\eta
\leq C \|\hat{v}_0 (\tau, \eta)\|_{L^2 (\mathbb{R}^n)}^2 + \|\hat{v}_0 (\tau, \eta)\|_{H^k (\mathbb{R}^n)}^2 t^{-\frac{n+2k}{\alpha}} e^{-\varepsilon_1 (\hat{c}_1 - c_3) t}.
\tag{3.37}
\end{align*}
$$

Substitute (3.37) into (3.35), we obtain

$$
\|I_1 (t)\|_{L^2 (\mathbb{R}^n)} \leq C \|\hat{v}_0 (\tau, \eta)\|_{L^1 (\mathbb{R}^n)} + \|\hat{v}_0 (\tau, \eta)\|_{H^k (\mathbb{R}^n)} t^{-\frac{n+2k}{\alpha}} e^{-\varepsilon_1 (\hat{c}_1 - c_3) t}.
\tag{3.38}
$$

Thus, in a similar way, we can also prove

$$
\|I_2 (t - s)\|_{L^2 (\mathbb{R}^n)}
= \left( \int_{\mathbb{R}^n} \left| e^{-A (\eta) (t - s)} e^{B (\eta) (t - s)} \right|^2 \left| \partial_s \hat{v}_0 (s, \eta) + A (\eta) \hat{v}_0 (s, \eta) \right|^2 |\eta_j|^{2k} d\eta \right)^{\frac{1}{2}}
\leq C e^{-\varepsilon_1 (\hat{c}_1 - c_3) t} \left( \int_{\mathbb{R}^n} e^{-2\varepsilon_1 (1 - \hat{J} (\eta)) t} \left( |\eta_j|^{2k} |\partial_s \hat{v}_0 (s, \eta) | + |\eta_j|^{2k} |\hat{v}_0 (s, \eta) |^2 \right) d\eta \right)^{\frac{1}{2}}
\leq C t^{-\frac{n+2k}{\alpha}} e^{-\varepsilon_1 (\hat{c}_1 - c_3) t} \left( \| (\partial_s v_0, v_0) (s) \|_{L^1 (\mathbb{R}^n)} + \| (\partial_s v_0, v_0) (s) \|_{H^k (\mathbb{R}^n)} \right).
\tag{3.39}
$$

Substituting (3.38) and (3.39) to (3.24), we immediately obtain (3.22).

Similarly, we can prove (3.24). We omit the details. Thus, we complete the proof of Proposition 3.34.

For $\tau = 0$, the equation (3.11) is reduced to

$$
\begin{align*}
\frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x_1} - \int_{\mathbb{R}^n} J (y) e^{-\lambda_0 y_1 v} (t, x - y) dy + c_1 v
= c_2 \int_{\mathbb{R}^n} f_\beta (y) e^{-\lambda_0 (y_1 + c r) v} (t, x - y - c r e_1) dy,
\end{align*}
$$

Taking Fourier transform to (3.40), as showed in (3.14), we have

$$
\frac{d\hat{v}}{dt} = [B (\eta) - A (\eta)] \hat{v}, \quad \text{with } \hat{v} (0, \eta) = \hat{v}_0 (\eta),
\tag{3.41}
$$
where $A(\eta)$ and $B(\eta)$ are given in (3.15) and (3.16) with $\tau = 0$, respectively. Integrating (3.41) yields

$$\dot{v}(t, \eta) = e^{-[A(\eta) - B(\eta)]t} \hat{v}_0(\eta).$$

Taking the inverse Fourier transform, we get the solution formula

$$v(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\eta} e^{-[A(\eta) - B(\eta)]t} \hat{v}_0(\eta) d\eta.$$

Then, a similar analysis as showed before can derive the optimal decay of the solution in the case without time-delay as follows. The detail of proof is omitted.

**Proposition 3.4 (Optimal decay rates for $\tau = 0$)** Suppose that $v_0 \in H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $m \geq 0$, then the solution of the linearized equation (3.40) satisfies

$$\|\partial_x^k v(t)\|_{L^2(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{m+2k}{2}} e^{-(\tilde{c}_1 - c_3)t}, \ t > 0,$$

for $k = 0, 1, \cdots, [m]$ and $j = 1, \cdots, n$, where the positive constants $\tilde{c}_1$ and $c_3$ are defined in (3.21) for $\tau = 0$.

Furthermore, if $m > \frac{2}{5}$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{m}{\tilde{c}_1 - c_3}}, \ t > 0.$$  \hspace{1cm} (3.43)

Particularly, when $\tilde{c}_1 = c_3$, then

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\|v_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{H^k(\mathbb{R}^n)}) t^{-\frac{m}{\tilde{c}_1 - c_3}}, \ t > 0.$$  \hspace{1cm} (3.44)

### 4 Global Existence and Comparison principle

In this section, we prove the global existence and uniqueness of the solution for the Cauchy problem to the nonlinear equation with nonlocal dispersion (1.7), and then establish the comparison principle in n-D case by different proof approach to the previous work [5, 12].

**Proposition 4.1 (Existence and Uniqueness)** Let $u_0(s, x) \in C([-\tau, 0]; C(\mathbb{R}^n))$ with $0 = u_- \leq u_0(s, x) \leq u_+ \text{ for } (s, x) \in [-\tau, 0] \times \mathbb{R}^n$, then the solution to (1.7) uniquely and globally exists, and satisfies that $u \in C^1([0, \infty); C(\mathbb{R}^n))$, and $u_- \leq u(t, x) \leq u_+ \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

**Proof.** Multiplying (1.7) by $e^{\eta_0 t}$ and integrating it over $[0, t]$ with respect to $t$, where $\eta_0 > 0$ will be technically selected in (4.4) below, we then express (1.7) in the integral form

$$u(t, x) = e^{-\eta_0 t} u(0, x) + \int_0^t e^{-\eta_0 (t-s)} \left[ \int_{\mathbb{R}^n} J(x-y) u(s, y) dy + (\eta_0 - 1) u(s, x) \right] - d(u(s, x)) + \int_{\mathbb{R}^n} f_\beta(y) b(u(s - \tau, x - y)) dy ds.$$  \hspace{1cm} (4.1)

Let us define the solution space as, for any $T \in [0, \infty]$,

$$\mathfrak{B} = \left\{ u(t, x) | u(t, x) \in C([0, T] \times \mathbb{R}^n) \text{ with } u_- \leq u \leq u_+, \right. \left. u(s, x) = u_0(s, x), (s, x) \in [-\tau, 0] \times \mathbb{R}^n \right\},$$

\hspace{1cm} (4.2)
with the norm
\[\|u\|_B = \sup_{t \in [0,T]} e^{-\eta_0 t} \|u(t)\|_{L^\infty(\mathbb{R}^n)},\]

where
\[\eta_0 := 1 + \eta_1 + \eta_2, \quad \eta_1 := \max_{u \in [u_-, u_+]} |d'(u)|, \quad \eta_2 := \max_{u \in [u_-, u_+]} |b'(u)|.\]

Clearly, \(B\) is a Banach space.

Define an operator \(P\) on \(B\) by
\[P(u)(t, x) := e^{-\eta_0 t} u_0(0, x) + \int_0^t e^{-\eta_0 (t-s)} \left[ \int_{\mathbb{R}^n} J(x-y) u(s, y) dy + (\eta_0 - 1) u(s, x) \
- d(u(s, x)) + \int_{\mathbb{R}^n} f_\beta(y) b(u(s - \tau, x - y) dy \right] ds, \quad 0 \leq t \leq T,\]
and
\[P(u)(s, x) := u_0(s, x), \quad \text{for } s \in [-\tau, 0].\]

Now we are going to prove that \(P\) is a contracting operator from \(B\) to \(B\).

Firstly, we prove that \(P : B \rightarrow B\). In fact, if \(u \in B\), from (H1)-(H3), namely, \(0 = d(0) \leq d(u) \leq d(u_+), \quad 0 = b(0) \leq b(u) \leq b(u_+), \quad \text{and} \quad d(u_+) = b(u_+)\), and using the facts \(\int_{\mathbb{R}^n} J(x-y) dy = 1\), \(\int_{\mathbb{R}^n} f_\beta(y) dy = 1\), and
\[g(u) := (\eta_0 - 1) u - d(u) \quad \text{is increasing,}\]
which implies \(g(u_+) \geq g(u) \geq g(0) = 0 \quad \text{for } u \in [u_-, u_+], \text{ then we have}\)
\[0 = u_- \leq P(u) \leq e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0 (t-s)} \left[ \int_{\mathbb{R}^n} J(x-y) u_+ dy \
+ (\eta_0 - 1) u_+ - d(u_+) + \int_{\mathbb{R}^n} f_\beta(y) b(u_+) dy \right] ds\]
\[= e^{-\eta_0 t} u_+ + \int_0^t e^{-\eta_0 (t-s)} [\eta_0 u_+ - d(u_+) + b(u_+)] ds\]
\[= u_+.\]

This plus the continuity of \(P(u)\) based on the continuity of \(u\) proves \(P(u) \in B\), namely, \(P\) maps from \(B\) to \(B\).

Secondly, we prove that \(P\) is contracting. In fact, let \(u_1, u_2 \in B\), and \(v = u_1 - u_2\), then we have
\[P(u_1) - P(u_2) = \int_0^t e^{-\eta_0 (t-s)} \left[ \int_{\mathbb{R}^n} J(x-y) v(s, y) dy + (\eta_0 - 1) v(s, x) - [d(u_1(s, x)) - d(u_2(s, x))] \
+ \int_{\mathbb{R}^n} f_\beta(y) [b(u_2(s - \tau, x - y)) - b(u_2(s - \tau, x - y))] dy \right] ds.\]

So, we have
\[|P(u_1) - P(u_2)| e^{-\eta_0 t} \leq \int_0^t e^{-2\eta_0 (t-s)} \left( \eta_0 + \max_{u \in [u_-, u_+]} |d'(u)| \right) \|v\|_B ds\]
\[+ \max_{u \in [u_-, u_+]} |b'(u)| \left( \int_0^t e^{-2\eta_0 (t-s)} \|v\|_B ds \right), \quad \text{for } t \geq \tau \]
\[+ \max_{u \in [u_-, u_+]} |b'(u)| \left( \int_0^t e^{-2\eta_0 (t-s)} \|v\|_B ds \right), \quad \text{for } 0 \leq t \leq \tau\]
use the property of contracting operator instead of the differential equation (1.7), we will work on the integral equation (4.1), and sufficiently on the initial data. The proof is also new and easy to follow. Different from the previous works [5, 12], were proved in [5, 12]. Here we give a comparison principle in regularity in space.

\[ (\eta_0 + \eta_1)(1 - e^{-\eta_0 t}) + \eta_2(e^{-2\eta_0 t} - e^{-2\eta_0 t}) \|v\|_B \]

\[ \leq \frac{1}{2\eta_0} \frac{\eta_0 + 2\eta_1 + \eta_2}{2\eta_0} \|v\|_B \]

\[ = \frac{2\eta_0 - 1}{2\eta_0} \|v\|_B \]

\[ = : \rho \|v\|_B \] (4.11)

for \( 0 < \rho := \frac{2\eta_0 - 1}{2\eta_0} < 1 \), namely, we prove that the mapping \( P \) is contracting:

\[ \|P(u_1) - P(u_2)\|_B \leq \rho \|u_1 - u_2\|_B \] < \|u_1 - u_2\|_B. \] (4.12)

Hence, by the Banach fixed-point theorem, \( P \) has a unique fixed point \( u \) in \( B \), i.e., the integral equation (4.1) has a unique classical solution on \([0, T]\) for any given \( T > 0 \). Differentiating (4.1) with respect to \( t \), we get back to the original equation (1.7), i.e.,

\[ u_t = J * u - u + d(u(t, x)) + \int_{\mathbb{R}^n} f_\beta(y)b(u(t - \tau, x - y))dy, \] (4.13)

then we can easily confirm from the right-hand-side of (4.13) that \( u_t \in C([0, T] \times \mathbb{R}^n) \). This completes our proof. \( \square \)

**Remark 4.2** From the proof of Proposition (4.1), we realize that, when \( u_0(s, x) \in C^k([-\tau, 0] \times \mathbb{R}^n) \), then the solution of the time-delayed equation (1.7) holds \( u(t, x) \in C^{k+1}([0, \infty); C(\mathbb{R}^n)) \); while for the non-delayed equation (1.8) (i.e., \( \tau = 0 \)), if \( u_0(x) \in C(\mathbb{R}^n) \), then the solution of the non-delayed equation (1.8) holds \( u(t, x) \in C^\infty([0, \infty); C(\mathbb{R}^n)) \). This means that the solution to the nonlocal dispersion equation (1.7) possesses a really good regularity in time. However, the solutions for (1.7) lack the regularity in space.

Now we establish two comparison principle for (1.7). Although the comparison principle in 1D case were proved in [5] [12]. Here we give a comparison principle in n-D case with much weaker restriction on the initial data. The proof is also new and easy to follow. Different from the previous works [5] [12], instead of the differential equation (1.7), we will work on the integral equation (4.1), and sufficiently use the property of contracting operator \( P \).

Let \( \tilde{u}(t, x) \) be an upper solution to (1.7), namely

\[
\begin{cases}
\frac{\partial \tilde{u}}{\partial t} - J * \tilde{u} + \tilde{u} + d(\tilde{u}(t, x)) \geq \int_{\mathbb{R}^n} f_\beta(y)b(\tilde{u}(t - \tau, x - y))dy, \\
\tilde{u}(s, x) \geq u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n,
\end{cases}
\] (4.14)

where its integral form can be written as

\[
\tilde{u}(t, x) \geq e^{-\eta_0 t}\tilde{u}(0, x) + \int_0^t e^{-\eta_0(t-s)} \left[ \int_{\mathbb{R}^n} J(x-y)\tilde{u}(s, y)dy + (\eta_0 - 1)\tilde{u}(s, x) - d(\tilde{u}(s, x)) + \int_{\mathbb{R}^n} f_\beta(y)b(\tilde{u}(s - \tau, x - y))dy \right] ds, \quad t > 0
\] (4.15)

and let \( \bar{u}(t, x) \) be an lower solution to (1.7) satisfying (4.14) or (4.15) conversely. Then we have the following comparison result. 

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Proposition 4.3 (Comparison Principle) Let $\bar{u}(t, x)$ and $\bar{u}(t, x)$ be the classical lower and upper solutions to (1.7), with $u_\leq \leq \bar{u}(t, x)$, $\bar{u}(t, x) \leq u_\geq$, respectively, and satisfy $0 \leq \bar{u}(t, x) \leq u_\geq$ and $0 \leq \bar{u}(t, x) \leq u_\geq$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then $\bar{u}(t, x) \leq \bar{u}(t, x)$ for $(t, x) \in [0, \infty) \times \mathbb{R}^n$.

Proof. We need to prove $\bar{u}(t, x) - \bar{u}(t, x) \geq 0$ for $(t, x) \in [0, \infty) \times \mathbb{R}^n$, namely, $r(t) := \inf_{x \in \mathbb{R}^n} v(t, x) \geq 0$, where $v(t, x) := \bar{u}(t, x) - \bar{u}(t, x)$.

If this is not true, then there exist some constants $\varepsilon > 0$ and $T > 0$ such that $r(t) > -\varepsilon e^{3\eta_0 T}$ for $t \in [0, T)$ and $r(T) = -\varepsilon e^{3\eta_0 T}$, where $\eta_0$ given in (4.3).

Since $\bar{u}(t, x)$ and $\bar{u}(t, x)$ are the lower and upper solutions to (1.7) and $\bar{u}(s, x) - \bar{u}(s, x) \geq 0$, for $s \in [-\tau, 0]$, and using (4.4) and (4.7), and noting $\bar{u}(t, x) - \bar{u}(t, x) \geq -\varepsilon e^{3\eta_0 T}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$, then we have, for $0 \leq t \leq T$,

$$
\begin{align*}
&\bar{u}(t, x) - \bar{u}(t, x) \\
\geq& e^{-\eta_0 t}[\bar{u}(0, x) - \bar{u}(0, x)] \\
&+ \int_0^t e^{-\eta_0 (t-s)} \left( \int_{\mathbb{R}^n} J(x) \bar{u}(s, y) - \bar{u}(s, y) \right) dy \\
&+ g(\bar{u}(s, x)) - g(\bar{u}(s, x)) \\
&+ \int_{\mathbb{R}^n} f_\beta(y)[b(\bar{u}(s-\tau, x-y)) - b(\bar{u}(s-\tau, x-y))] dy ds \\
\geq& \int_0^t e^{-\eta_0 (t-s)} \left( -\varepsilon e^{3\eta_0 s} - \max_{\zeta \in [u_\leq, u_\geq]} \zeta' e^{3\eta_0 s} \right) ds \\
&- \max_{u \in [u_\leq, u_\geq]} |b'(u)| \left\{ \begin{array}{ll}
-\varepsilon e^{3\eta_0 t} e^{3\eta_0 (s-\tau)} ds, & \text{for } t \geq \tau \\
0, & \text{for } 0 \leq t \leq \tau
\end{array} \right.
\end{align*}
$$

Thus, from the assumption we know

$$
-\varepsilon e^{3\eta_0 T} = \inf_{x \in \mathbb{R}^n} (\bar{u}(T, x) - \bar{u}(T, x)) \geq -\frac{2\eta_0 + 1}{4\eta_0} e^{3\eta_0 T},
$$

which is a contradiction for $\eta_0 > \frac{1}{2}$. Here, our $\eta_0$ defined in (4.3) satisfies $\eta_0 > 1$. Thus the proof is complete. $\square$

5 Global Stability of Planar Traveling Waves

The main purpose in this section is to prove Theorems 2.2 for all traveling waves including the critical traveling waves.

For given traveling wave $\phi(x_1 + ct)$ with the speed $c \geq c_\ast$ and the given initial data $u_\leq \leq u_0(s, x) \leq u_\geq$ for $(s, x) \in [-\tau, 0] \times \mathbb{R}^n$, let us define $U^+_0(s, x)$ and $U^-_0(s, x)$ as

$$
\begin{align*}
U^-_0(s, x) :=& \min\{\phi(x_1 + cs), u_0(s, x)\} \\
U^+_0(s, x) :=& \max\{\phi(x_1 + cs), u_0(s, x)\}
\end{align*}
$$

(5.1)
We see from (1.7) that \( V \) in the right-side of line above (5.6).
\[
\phi = \phi(x_1 + cs) \text{ with } \phi(0) \in H^m_0(\mathbb{R}^n) \in H^m(\mathbb{R}^n), \text{ we have } u_0 - \phi \in C([-\tau, 0]; C(\mathbb{R}^n)). \]
On the other hand, the traveling wave \( \phi(x_1 + cs) \) is smooth, then we can guarantee \( U_0^+(s, x) \in C([-\tau, 0]; C(\mathbb{R}^n)) \).
Thus, applying Proposition 4.1, we know that the solutions of (1.7) with the initial data

\[
Sobolev's \text{ embedding theorem } H^m(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \text{ we have } u_0 - \phi \in C([-\tau, 0]; C(\mathbb{R}^n)). \]

Thus, applying Proposition 4.1 we know that the solutions of (1.7) with the initial data \( U_0^+ (s, x) \) and \( U_0^- (s, x) \) globally exist, and denote them by \( U^+(t, x) \) and \( U^-(t, x) \), respectively, that is,

\[
\begin{aligned}
\frac{\partial U^\pm}{\partial t} - J * U^\pm + U^\pm + d(U^\pm) &= \int_{\mathbb{R}^n} f_\beta(y) b(U^\pm(t, x-y))dy, \\
U^\pm(s, x) &= U_0^\pm(s, x), \quad x \in \mathbb{R}^n, s \in [-\tau, 0].
\end{aligned}

(5.2)
\]

Then the comparison principle (Proposition 4.3) further implies

\[
\begin{aligned}
u_- \leq U^-(t, x) \leq u(t, x) &\leq U^+(t, x) \leq u_+ \\
u_- \leq U^-(t, x) \leq \phi(x_1 + ct) &\leq U^+(t, x) \leq u_+
\end{aligned}

(5.3)
\]

In what follows, we are going to complete the proof for the stability in three steps.

**Step 1. The convergence of \( U^+(t, x) \) to \( \phi(x_1 + ct) \)**

Let

\[
V(t, x) := U^+(t, x) - \phi(x_1 + ct), \quad V_0(s, x) := U_0^+(s, x) - \phi(x_1 + cs).
\]

(5.4)

It follows from (5.3) that

\[
V(t, x) \geq 0, \quad V_0(s, x) \geq 0.
\]

(5.5)

We see from (1.7) that \( V(t, x) \) satisfies (by linearizing it around 0)

\[
\frac{\partial V}{\partial t} - \int_{\mathbb{R}^n} J(y)V(t, x-y)dy + V + d'(0)V \\
= -Q_1(t, x) + \int_{\mathbb{R}^n} f_\beta(y)Q_2(t, x-y)dy + [d'(0) - d'(\phi(x_1 + ct))]V \\
+ \int_{\mathbb{R}^n} f_\beta(y)[b'(\phi(x_1 - y_1 + c(t - \tau)) - b'(0)]V(t, x-y)dy \\
=: I_1(t, x) + I_2(t, -x) + I_3(t, x) + I_4(t, x),
\]

(5.6)

with the initial data

\[
V(s, x) = V_0(s, x), \quad s \in [-\tau, 0],
\]

(5.7)

where

\[
Q_1(t, x) = d(\phi + V) - d(\phi) - d'(\phi)V
\]

(5.8)

with \( \phi = \phi(x_1 + ct) \) and \( V = V(t, x) \), and

\[
Q_2(t, x-y) = b(\phi + V) - b(\phi) - b'(\phi)V
\]

(5.9)

with \( \phi = \phi(x_1 - y_1 + c(t - \tau)) \) and \( V = V(t, x-y) \). Here \( I_i, i = 1, 2, 3, 4, \) denotes the \( i \)-th term in the right-side of line above (5.6).
From (H3), i.e., $d''(u) \geq 0$ and $b''(u) \leq 0$, applying Taylor formula to (5.8) and (5.9), we immediately have

$$Q_1(t, x) \geq 0 \quad \text{and} \quad Q_2(t - \tau, x - y) \leq 0,$$

which implies

$$I_1(t, x) \leq 0 \quad \text{and} \quad I_2(t - \tau, x) \leq 0. \quad (5.10)$$

From (H3) again, since $d'(\phi)$ is increasing and $b'(\phi)$ is decreasing, then $d'(0) - d'(\phi(x_1 + ct)) \leq 0$ and $b'(\phi(x_1 - y_1 + c(t - \tau))) - b'(0) \leq 0$, which imply, with $V \geq 0$,

$$I_3(t, x) \leq 0 \quad \text{and} \quad I_4(t - \tau, x) \leq 0. \quad (5.11)$$

Thus, applying (5.10) and (5.11) to (5.6), we obtain

$$\tau > 0,$$

From Proposition 4.1, we know that $\bar{\phi}$ satisfies

$$\begin{aligned}
\frac{\partial \bar{\phi}}{\partial t} - \nabla \cdot (V_\alpha \bar{\phi}) &= \beta \bar{\phi}, \\
0 &\leq \bar{\phi}(t, x) \leq \bar{\phi}(0, x),
\end{aligned} \quad (5.12)$$

Let $\bar{V}(t, x)$ be the solution of the following equation with the same initial data $V_0(s, x)$:

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial t} - J * \bar{V} + V_\alpha \bar{V} &= 0, \\
\bar{V}(s, x) &= V_0(s, x), \quad s \in [-\tau, 0], x \in \mathbb{R}^n.
\end{aligned} \quad (5.13)$$

From Proposition 4.1, we know that $\bar{V}(t, x)$ globally exists. Furthermore, (5.13) is actually a linear equation, and its solution is as smooth as its initial data. By the comparison principle (Proposition 4.3), we have

$$0 \leq V(t, x) \leq \bar{V}(t, x), \quad \text{for} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \quad (5.14)$$

Let

$$v(t, x) := e^{-\lambda_\alpha(x_1 + ct - x_\alpha)} \bar{V}(t, x). \quad (5.15)$$

From (5.13), $v(t, x)$ satisfies

$$\begin{aligned}
\frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_\alpha y_1} v(t, x - y) dy + c_1 v &= c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_\alpha(y_1 + ct)} v(t - \tau, x - y) dy, \\
\end{aligned} \quad (5.16)$$

where

$$c_1 := c_\alpha + 1 + d'(0) > 0, \quad \text{and} \quad c_2 := b'(0). \quad (5.17)$$

When $\tau = 0$, then (5.16) is reduced to

$$\frac{\partial v}{\partial t} - \int_{\mathbb{R}^n} J(y) e^{-\lambda_\alpha y_1} v(t, x - y) dy + c_1 v = c_2 \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_\alpha y_1} v(t, x - y) dy. \quad (5.18)$$

Applying Proposition 3.3 to (5.10) for $\tau > 0$ and Proposition 3.4 to (5.18) for $\tau = 0$, we obtain the following decay rates:

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}} e^{-c_1(t\tau - c_\alpha)t}, \quad \text{for} \ \tau > 0, \quad (5.19)$$

$$\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}} e^{-(c_1 - c_\alpha)t}, \quad \text{for} \ \tau = 0, \quad (5.20)$$
where \(0 < \varepsilon_1 = \varepsilon_1(\tau) < 1\), and \(c_3\) is defined in (5.21), which can be directly calculated as, by using the property (1.11),

\[
c_3 = b'(0) \int_{\mathbb{R}^n} f_\beta(y) e^{-\lambda_c(y_1 + ct)} dy = b'(0) \int_{\mathbb{R}} f_1\beta(y_1) e^{-\lambda_c(y_1 + ct)} dy_1 = b'(0)e^{\beta\lambda_c^2 - \lambda_c ct} > 0. \tag{5.21}
\]

and

\[
\tilde{c}_1 = c\lambda_c + 1 + d'(0) - \int J(y_1)e^{-\lambda_c y_1} dy_1 = c\lambda_c + d'(0) - E_c(\lambda_c). \tag{5.22}
\]

When \(c > c_*\), namely, the wave \(\phi(x_1 + ct)\) is non-critical, from (2.7) in Theorem 2.1 we realize

\[
\tilde{c}_1 := c\lambda_c + d'(0) - E_c(\lambda_c) = G_c(\lambda_c) > H_c(\lambda_c) = b'(0)e^{\beta\lambda_c^2 - \lambda_c ct} =: c_3. \tag{5.23}
\]

Thus, (5.19) and (5.20) immediately imply the following exponential decay for \(c > c_*\)

\[
\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}}e^{-\varepsilon_1\tilde{\mu}t}, \quad \text{for } \tau > 0, \tag{5.24}
\]

\[
\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}}e^{-\tilde{\mu}t}, \quad \text{for } \tau = 0, \tag{5.25}
\]

where

\[
\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_c) - H_c(\lambda_c) > 0. \tag{5.26}
\]

When \(c = c_*\), namely, the wave \(\phi(x_1 + ct)\) is critical, from (2.8) in Proposition 2.1 we realize

\[
\tilde{c}_1 := c\lambda_c + d'(0) - E_c(\lambda_c) = G_c(\lambda_c) = H_c(\lambda_c) = b'(0)e^{\beta\lambda_c^2 - \lambda_c ct} := c_3. \tag{5.27}
\]

Then, from (5.19) and (5.20), we immediately obtain the following algebraic decay for \(c = c_*\)

\[
\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}}, \quad \text{for all } \tau \geq 0. \tag{5.28}
\]

Since \(V(t, x) \leq \tilde{V}(t, x) = e^{\lambda_c(x_1 + ct - x_*)}v(t, \xi)\), and \(0 < e^{\lambda_c(x_1 + ct - x_*)} \leq 1\) for \(x_1 \in (-\infty, x_* - ct]\), we immediately obtain the following decay for \(V\).

**Lemma 5.1** Let \(V = V(t, x)\). Then

1. when \(c > c_*\), then

\[
\|V(t)\|_{L^\infty((-\infty, x_* - ct) \times \mathbb{R}^{n-1})} \leq C(1 + t)^{-\frac{n}{2}} e^{-\varepsilon_1\tilde{\mu}t}, \quad \text{for } \tau > 0; \tag{5.29}
\]

\[
\|V(t)\|_{L^\infty((-\infty, x_* - ct) \times \mathbb{R}^{n-1})} \leq C(1 + t)^{-\frac{n}{2}} e^{-\tilde{\mu}t}, \quad \text{for } \tau = 0; \tag{5.30}
\]

Here \(\tilde{\mu} := \tilde{c}_1 - c_3 = G_c(\lambda_c) - H_c(\lambda_c) > 0\) for \(c > c_*\).

2. when \(c = c_*\), then

\[
\|V(t)\|_{L^\infty((-\infty, x_* - ct) \times \mathbb{R}^{n-1})} \leq C(1 + t)^{-\frac{n}{2}}, \quad \text{for } \tau \geq 0. \tag{5.31}
\]

Next we prove \(V(t, x)\) exponentially decay for \(x \in [x_* - ct, \infty) \times \mathbb{R}^{n-1}\).
Lemma 5.2 For $\tau > 0$, it holds that

\[ \|V(t)\|_{L^\infty((x_0,-ct)\times\mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}} e^{-\mu_\tau t}, \quad \text{for } c > c_*, \] (5.32)
\[ \|V(t)\|_{L^\infty((x_0,-ct)\times\mathbb{R}^{n-1})} \leq Ct^{-\frac{n}{\alpha}}, \quad \text{for } c = c_*, \] (5.33)

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 \hat{\mu}\}$ for $c > c_*$. 

**Proof.** From (5.2) and (1.9), as set in (5.4) $V(t,x) := U^+(t,x) - \phi(x_0 + ct)$, we have

\[ \frac{\partial V}{\partial t} - J * V + V + d(\phi + V) - d(\phi) = \int_{\mathbb{R}^n} f_\beta(y)[b(\phi + V) - b(\phi)]dy. \] (5.34)

Applying Taylor expansion formula and noting (H_3) for $d'(u) \geq 0$ and $b''(u) \leq 0$, we have

\[ d(\phi + V) - d(\phi) = d'(\phi)V + d''(\phi)V^2 \geq d'(\phi)V, \] (5.35)
\[ b(\phi + V) - b(\phi) = b'(\phi)V + b''(\phi)V^2 \leq b'(\phi)V, \] (5.36)

where $\tilde{\phi}_i$ ($i = 1, 2$) are some functions between $\phi$ and $\phi + V$. Substituting (5.35) and (5.36) into (5.34), and noticing Lemma 5.1, we have

\[ \begin{cases} \frac{\partial V}{\partial t} - J * V + V + d'(\phi)V \leq \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(x_1 - y_1 + c(t - \tau)))V(t - \tau, x - y)dy, \\
V|_{x_1 \leq x_0 - ct} \leq C_2(1 + t) - \frac{n}{\alpha} e^{-\varepsilon_1 \hat{\mu} t}, \quad \text{for } t > 0, (x_2, \cdots, x_n) \in \mathbb{R}^{n-1} \\
V|_{x_1 = s} = V_0(s, x), \quad \text{for } s \in [-\tau, 0], x \in \mathbb{R}^n \end{cases} \] (5.37)

for some positive constant $C_2$.

Let

\[ \tilde{V}(t) = C_3(1 + t) - \frac{n}{\alpha} e^{-\mu_\tau t} \] (5.38)

for $C_3 \geq C_2 \geq \max_{(s,x) \in [-\tau,0] \times \mathbb{R}^n} |V_0(s, x)|$. As in (2.12), for given $0 < \varepsilon_0 < 1$, we can select a sufficiently large number $x_*$ such that, for $\xi_1 \geq x_* \gg 1$,

\[ d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(\xi_1 - y_1 - ct))dy \geq 0. \] (5.39)

Thus, we have

\[ \begin{align*}
\frac{\partial \tilde{V}}{\partial t} - J * \tilde{V} + \tilde{V} + d'(\phi)\tilde{V} - \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(\xi_1 - y_1 - ct))\tilde{V}(t - \tau)dy \\
= -\frac{n}{\alpha} C_3(1 + t + \tau) - \frac{n}{\alpha} e^{-\mu_\tau t} - \mu_\tau C_3(1 + t + \tau) - \frac{n}{\alpha} e^{-\mu_\tau t} \\
+ C_3(1 + t + \tau) - \frac{n}{\alpha} e^{-\mu_\tau t}d'(\phi(\xi_1)) \\
- C_3(1 + t) - \frac{n}{\alpha} e^{-\mu_\tau (t - \tau)} \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(\xi_1 - y_1 - ct))dy \\
= C_3(1 + t + \tau) - \frac{n}{\alpha} e^{-\mu_\tau t} \left\{ d'(\phi(\xi_1)) - \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(\xi_1 - y_1 - ct))dy \right\} - \mu_\tau \\
- \frac{n}{\alpha} (1 + t + \tau)^{-1} - \left( e^{\mu_\tau t} \left( \frac{1 + t}{1 + t + \tau} \right) - 1 \right) \int_{\mathbb{R}^n} f_\beta(y)b'(\phi(\xi_1 - y_1 - ct))dy \\
\geq C_3(1 + t + \tau) - \frac{n}{\alpha} e^{-\mu_\tau t} \left\{ \varepsilon_0 d'(u_+) - b'(u_+) \right\} - \mu_\tau - \frac{n}{\alpha} (1 + t + \tau)^{-1}
\end{align*} \]
by selecting a sufficiently small number

$$0 < \mu_\tau < d'(u_+) - b'(u_+) \quad \text{for} \quad c > c_*, \quad \mu_\tau = 0 \quad \text{for} \quad c = c_*,$$

and taking $t \geq l_0 \tau$ for a sufficiently large integer $l_0 \gg 1$. Hence, we proved that

$$\frac{\partial \tilde{V}}{\partial t} - J * \tilde{V} + \tilde{V} + d'(\phi)\tilde{V} \geq \int_{\mathbb{R}^n} f_\beta(y)\beta'(\phi(\xi_1 - y_1 - ct))\tilde{V}(t - \tau)dy,$$

for $t > l_0 \tau, \xi \in [x_*, +\infty) \times \mathbb{R}^{n-1}$

$$\tilde{V}|_{\xi_1 = x_*} = C_3(1 + \tau + t)^{-\frac{n}{\alpha}}e^{-\mu t} > C_2(1 + t)^{-\frac{n}{\alpha}}e^{-\varepsilon_1 \mu t}, \quad \text{for} \quad t > 0, (\xi_2, \cdots, \xi_n) \in \mathbb{R}^{n-1}$$

$$\tilde{V}(t) = C_3(1 + \tau + t)^{-\frac{n}{\alpha}}e^{-\mu t} > V_0(t, \xi), \quad \text{for} \quad t \in [-\tau, l_0 \tau], \xi \in \mathbb{R}^n.$$

Denote $\Omega := \{(x, t)|x_1 \geq x_* - ct, t \geq l_0 \tau, (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}\}$. Noticing the construction of \([5.37]\) and \([5.43]\), then similar to the proof of Proposition \([4.3]\), we know that

$$\tilde{V}(t) - V(t, x) \geq 0, \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times [-\tau, \infty) \setminus \Omega.$$

Thus the proof is complete. \(\square\)

For $\tau = 0$, it is easy to prove the corresponding results as follows.

**Lemma 5.3** For $\tau = 0$, it holds that

$$\|V(t)\|_{L^\infty([x_* - ct, \infty) \times \mathbb{R}^{n-1})} \leq C t^{-\frac{n}{\alpha}}e^{-\mu t}, \quad \text{for} \quad c > c_*, \quad \|V(t)\|_{L^\infty([x_* - ct, \infty) \times \mathbb{R}^{n-1})} \leq C t^{-\frac{n}{\alpha}}, \quad \text{for} \quad c = c_*,$$

with some constant $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 \mu_1\}$ for $c > c_*$. Combing Lemma \([5.1]\) Lemma \([5.3]\) we obtain the decay rates for $V(t, x)$ in $L^\infty(\mathbb{R}^n)$.

**Lemma 5.4** It holds that:

1. when $c > c_*$, then

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{\alpha}}e^{-\mu t}, \quad \text{for} \quad \tau > 0,$$

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{\alpha}}e^{-\mu_0 t}, \quad \text{for} \quad \tau = 0,$$

where $0 < \mu_\tau < \min\{d'(u_+) - b'(u_+), \varepsilon_1 [G_c(\lambda_*) - H_c(\lambda_*)]\}$ with $0 < \varepsilon_1 < 1$ for $\tau > 0$, and $0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$ for $\tau = 0$;

2. when $c = c_*$,

$$\|V(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{\alpha}}, \quad \text{for} \quad \text{all} \quad \tau \geq 0.$$

Since $V(t, x) = U^+(t, x) - \phi(x_1 + ct)$, Lemma \([5.4]\) give directly the following convergence for the solution in the cases with time-delay.
Lemma 5.5 It holds that:

1. when \( c > c_* \), then

\[
\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_* t}, \quad \text{for } \tau > 0, \tag{5.50}
\]

\[
\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \tag{5.51}
\]

where \( 0 < \mu_* < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\} \) with \( 0 < \varepsilon_1 < 1 \) for \( \tau > 0 \), and \( 0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\} \) for \( \tau = 0 \);

2. when \( c = c_* \), then

\[
\sup_{x \in \mathbb{R}^n} |U^+(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_* t}, \quad \text{for all } \tau \geq 0. \tag{5.52}
\]

Step 2. The convergence of \( U^-(t, x) \) to \( \phi(x_1 + ct) \)

For the traveling wave \( \phi(x_1 + ct) \) with \( c \geq c_* \), let

\[
V(t, x) = \phi(x_1 + ct) - U^-(t, x), \quad V_0(s, x) = \phi(x_1 + cs) - U^-_0(s, x). \tag{5.53}
\]

As in Step 1, we can similarly prove that \( U^-(t, x) \) converges to \( \phi(x_1 + ct) \) as follows.

Lemma 5.6 It holds that:

1. when \( c > c_* \), then

\[
\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_* t}, \quad \text{for } \tau > 0, \tag{5.54}
\]

\[
\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_0 t}, \quad \text{for } \tau = 0, \tag{5.55}
\]

where \( 0 < \mu_* < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\} \) with \( 0 < \varepsilon_1 < 1 \) for \( \tau > 0 \), and \( 0 < \mu_0 < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\} \) for \( \tau = 0 \);

2. when \( c = c_* \), then

\[
\sup_{x \in \mathbb{R}^n} |U^-(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{a}{2}} e^{-\mu_* t}, \quad \text{for all } \tau \geq 0. \tag{5.56}
\]

Step 3. The convergence of \( u(t, x) \) to \( \phi(x_1 + ct) \)

Finally, we prove that \( u(t, x) \) converges to \( \phi(x_1 + ct) \). Since the initial data satisfy \( U^-_0(s, x) \leq u_0(s, x) \leq U^+_0(s, x) \) for \( (s, x) \in [-\tau, 0] \times \mathbb{R}^n \), then the comparison principle implies that

\[
U^-(t, x) \leq u(t, x) \leq U^+(t, x), \quad (t, x) \in R_+ \times \mathbb{R}^n.
\]

Thanks to Lemmas 5.5 and 5.6 by the squeeze argument, we have the following convergence results.

Lemma 5.7 It holds that:
1. when \( c > c_* \), then
\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{\mu}{\sqrt{\pi c}}} e^{-\mu \tau}, \quad \text{for } \tau > 0, \quad (5.57)
\]
\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{\mu}{\sqrt{\pi c}}} e^{-\mu \tau}, \quad \text{for } \tau = 0, \quad (5.58)
\]
where \( 0 < \mu < \min\lbrace d'(u_+) - b'(u_+), \varepsilon_1 \rbrace \) with \( 0 < \varepsilon_1 < 1 \) for \( \tau > 0 \), and
\( 0 < \mu_0 < \min\lbrace d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*) \rbrace \) for \( \tau = 0 \);

2. when \( c = c_* \), then
\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{\mu}{\sqrt{\pi c}}}, \quad \text{for all } \tau \geq 0. \quad (5.59)
\]

6 Applications and Concluding Remark

In this section, we first give the direct applications of Theorem 2.1-2.2 to the Nicholson’s blowflies type equation with nonlocal dispersion, and the classical Fisher-KPP equation with nonlocal dispersion. Then we point out that, the developed stability theory above can be also applied to the more general case.

6.1 Nicholson’s blowflies equation with nonlocal dispersion

For the equation (1.7), by taking \( d(u) = \delta u \) and \( b(u) = pu e^{-au} \) with \( \delta > 0, p > 0 \) and \( a > 0 \), we get the so-called Nicholson’s blowflies equation with nonlocal dispersion
\[
\begin{aligned}
& \left \{ \frac{\partial u}{\partial t} - J * u + u + \delta u(t, x) \right \} = p \int_{\mathbb{R}^n} f_\beta(y) u(t - \tau, x - y) e^{-au(t - \tau, x - y)} dy, \\
& u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}^n. 
\end{aligned} \quad (6.1)
\]
Clearly, there exist two constant equilibria \( u_- = 0 \) and \( u_+ = \frac{1}{a} \ln \frac{\mu}{\delta} \), and the selected \( d(u) \) and \( b(u) \) satisfy the hypothesis \((H_1)-(H_3)\) automatically under the consideration of \( 1 < \frac{\mu}{\delta} \leq \epsilon \). Let \( J(x) \) satisfy the hypothesis \((J_1)\) and \((J_2)\), from Theorem 2.1 and Theorem 2.2, we have the following existence of monostable traveling waves and their stabilities.

Theorem 6.1 (Traveling waves) Let \( J(x) \) satisfy \((J_1)\) and \((J_2)\). For \( \lambda_* \), there exists the minimal speed \( c_* > 0 \), such that

- when \( c \geq c_* \), the planar traveling waves \( \phi(x \cdot e_1 + ct) \) exist uniquely (up to a shift);
- when \( c < c_* \), the planar traveling waves \( \phi(x \cdot e_1 + ct) \) do not exist;

Here \( c_* > 0 \) and \( \lambda_* > 0 \) are determined by
\[
H_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \text{and} \quad H'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),
\]
where
\[
H_c(\lambda) = pe^{\beta \lambda^2 - \lambda c} \quad \text{and} \quad G_c(\lambda) = c\lambda - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda y_1} dy_1 + 1 + \delta.
\]
Particularly, when \( c > c_* \), then \( H_c(\lambda_*) < G_c(\lambda_*) \).
Theorem 6.2 (Stability of traveling waves) Let $J(x)$ satisfy $(J_1)$ and $(J_2)$, and the initial data be $u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ and $\partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H^{m+1}_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \leq u_0 \leq u_+$ for $(s, x) \in [-\tau, 0] \times \mathbb{R}^n$. Then the solution of (6.1) uniquely exists and satisfies:

- when $c > c_*$, then
  \[ \sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{\alpha}} e^{-\mu t}, \quad t > 0, \]  

for $0 < \mu < \min\{d'(u_+) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$, and $\varepsilon_1 = \varepsilon_1(\tau)$ such that $0 < \varepsilon_1 < 1$ for $\tau > 0$ and $\varepsilon_1 = 1$ for $\tau = 0$.

- when $c = c_*$, then
  \[ \sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{n}{\alpha}}, \quad t > 0. \]  

6.2 Fisher-KPP equation with nonlocal dispersion

For the equation (1.7), let $d(u) = u^2$, $b(u) = u$ and the delay $\tau = 0$, and take the limit of (1.7) as $\beta \to 0^+$, we get the classical Fisher-KPP equation with nonlocal dispersion without time-delay

\[
\begin{cases}
\frac{\partial u}{\partial t} - J * u + u = u(1 - u) \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}^n.
\end{cases}
\]  

(6.4)

Then we have the existence of the monostable traveling waves and their stabilities from Theorem 2.1 and Theorem 2.2.

Theorem 6.3 (Traveling waves) Let $J(x)$ satisfy $(J_1)$ and $(J_2)$. For (6.4), there exists the minimal speed $c_* > 0$, such that

- when $c \geq c_*$, the planar traveling waves $\phi(x \cdot e_1 + ct)$ exist uniquely (up to a shift);
- when $c < c_*$, the planar traveling waves $\phi(x \cdot e_1 + ct)$ do not exist.

Here $c_* := \lambda_* \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1$, and $\lambda_* > 0$ is determined by $\int_{\mathbb{R}} (1 + \varepsilon_1^2) J_1(y_1) e^{-\lambda_* y_1} dy_1 = 0$.

When $c > c_*$, then $H_c(\lambda_*) < G_c(\lambda_*)$, where $H_c(\lambda_*) = 1$ and $G_c(\lambda_*) = c\lambda_* - \int_{\mathbb{R}} J_1(y_1) e^{-\lambda_* y_1} dy_1 + 1$.

Theorem 6.4 (Stability of traveling waves) Let $J(x)$ satisfy $(J_1)$ and $(J_2)$, and the initial data be $u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n))$ with $m > \frac{n}{2}$, and $u_- \leq u_0 \leq u_+$ for $x \in \mathbb{R}^n$. Then the solution of (6.4) uniquely exists and satisfies:

- when $c > c_*$, then
  \[ \sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{\alpha}} e^{-\mu t}, \quad t > 0, \]  

for $0 < \mu < \min\{d'(u_+) - b'(u_+), G_c(\lambda_*) - H_c(\lambda_*)\}$;

- when $c = c_*$, then
  \[ \sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{n}{\alpha}}, \quad t > 0. \]  

(6.6)
6.3 Concluding Remark

Here we give a remark on the wave stability to the generalized equations with nonlocal dispersion. Let us consider a more general monostable equation with nonlocal dispersion

\[
\begin{aligned}
\frac{\partial u}{\partial t} = & J * u + u + d(u(t, x)) = F \left( \int_{\mathbb{R}^n} \kappa(y)b(u(t - \tau, x - y))dy \right),
\end{aligned}
\]

(6.7)

where \( J(x) \) satisfies (J1) and (J2) as mentioned before, and \( F(\cdot), d(u), b(u) \) and \( g(x) \) satisfy

(\( \mathcal{H}_1 \)) There exist \( u_- = 0 \) and \( u_+ > 0 \) such that \( d(0) = b(0) = F(0) = 0, d(u_+) = F(b(u_+)) \), \( d \in C^2[0, u_+] \), \( b \in C^2[0, u_+] \) and \( F \in C^2[0, b(u_+]) \);

(\( \mathcal{H}_2 \)) \( F'(0)b'(0) > d'(0) \geq 0 \) and \( 0 < F'(b(u_+))b'(u_+) < d'(u_+) \);

(\( \mathcal{H}_3 \)) \( d'(u) \geq 0, b'(u) \geq 0, d''(u) \geq 0 \) and \( b''(u) \leq 0 \) for \( u \in [0, u_+] \);

(\( \mathcal{H}_4 \)) \( F'(u) \geq 0 \) and \( F''(u) \leq 0 \) for \( u \in [0, b(u_+)] \);

(\( \mathcal{H}_5 \)) \( \kappa(x) \) is a smooth, positive and radial kernel with \( \int_{\mathbb{R}^n} \kappa(x)dx = 1 \) and \( \int_{\mathbb{R}^n} \kappa(x)e^{-\lambda x_1}dx < +\infty \) for all \( \lambda > 0 \).

Then, by a similar calculation, we can prove the existence of the traveling waves \( \phi(x_1 + ct) \) for \( c \geq c_* \), where \( c_* > 0 \) is a specified minimal wave speed, and that the noncritical traveling waves with \( c > c_* \) are exponentially stable and the critical waves with \( c = c_* \) are algebraically stable.

**Theorem 6.5 (Traveling waves)** Assume that (J1)-(J2) and (\( \mathcal{H}_1 \))-(\( \mathcal{H}_5 \)) hold. For (6.7), there exists the minimal speed \( c_* > 0 \), such that

- when \( c \geq c_* \), the planar traveling waves \( \phi(x \cdot e_1 + ct) \) exist uniquely (up to a shift);
- when \( c < c_* \), the planar traveling waves \( \phi(x \cdot e_1 + ct) \) do not exist;

Here \( c_* > 0 \) and \( \lambda_* = \lambda_*(c_*) > 0 \) are determined by

\[
\mathcal{H}_c(\lambda_*) = \mathcal{G}_c(\lambda_*) \quad \text{and} \quad \mathcal{H}'_c(\lambda_*) = \mathcal{G}'_c(\lambda_*) ,
\]

where

\[
\mathcal{H}_c(\lambda) = F'(0)b'(0) \int_{\mathbb{R}^n} e^{-\lambda y_1} \kappa(y)dy, \quad \mathcal{G}_c(\lambda) = c\lambda - \int_{\mathbb{R}} J_1(y_1)e^{-\lambda y_1}dy_1 + 1 + d'(0).
\]

When \( c > c_* \), then

\[
\mathcal{H}_c(\lambda_*) < \mathcal{G}_c(\lambda_*) .
\]

**Theorem 6.6 (Stability of traveling waves)** Assume that (J1)-(J2) and (\( \mathcal{H}_1 \))-(\( \mathcal{H}_5 \)) hold. Let the initial data be \( u_0 - \phi \in C([-\tau, 0]; H^m_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n)) \) and \( \partial_s(u_0 - \phi) \in L^1([-\tau, 0]; H^{m+1}_w(\mathbb{R}^n) \cap L^1_w(\mathbb{R}^n)) \) with \( m > \frac{n}{2} \), and \( u_- \leq u_0 \leq u_+ \) for \( x \in \mathbb{R}^n \). Then the solution of (6.7) uniquely exists and satisfies:
when $c > c_*$, then
\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + ct)| \leq C(1 + t)^{-\frac{n}{\mu_0}}e^{-\mu_0 t}, \quad t > 0,
\]
for $0 < \mu_0 < \min\{d'(u_-) - b'(u_+), \varepsilon_1[G_c(\lambda_*) - H_c(\lambda_*)]\}$, and $0 < \varepsilon_1 < 1$ for $\tau > 0$ and $\varepsilon_1 = 1$ for $\tau = 0$;

when $c = c_*$, then
\[
\sup_{x \in \mathbb{R}^n} |u(t, x) - \phi(x_1 + c_* t)| \leq C(1 + t)^{-\frac{n}{\mu}}, \quad t > 0.
\]

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