Internal quark symmetries and colour
$SU(3)$ entangled with $Z_3$-graded Lorentz algebra

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Abstract

In the current version of QCD the quarks are described by ordinary
Dirac fields, organized in the following internal symmetry multiplets: the
$SU(3)$ colour, the $SU(2)$ flavour, and broken $SU(3)$ providing the family
triplets. In this paper we argue that internal and external (i.e. space-time)
symmetries are entangled at least in the colour sector in order to intro-
duce the spinorial quark fields in a way providing all the internal quark’s
degrees of freedom which do appear in the Standard Model. Because
the $SU(3)$ colour algebra is endowed with natural $Z_3$-graded discrete au-
tomorphisms, in order to introduce entanglement the $Z_3$-graded version
of Lorentz algebra with its vectorial and spinorial realizations are con-
sidered. The colour multiplets of quarks are described by 12-component
colour Dirac equations, with a $Z_3$-graded triplet of masses (one real and
a Lee-Wick complex conjugate pair). We show that all quarks in the
Standard Model can be described by the 72-component master quark sextet
of 12-component coloured Dirac fields, which is required in order to
implement the faithful spinorial representation of the $Z_3$-graded Lorentz
transformations.
1 Introduction

In the current version of Quantum Chromodynamics the massive quarks are treated as Dirac fermions endowed with additional internal degrees of freedom. In the minimal version, Standard Model displays the exact $SU(3)$ colour and the $SU(2)$ flavour symmetries, as well as strongly broken $SU(3)$ describing three quark families [1], [2], [3]).

If we introduce “master quark Dirac field” supposed to incorporate all internal quark symmetries, we should deal with $4 \times 3 \times 2 \times 3 = 72$-component fermionic master field (the first factor 4 corresponds to the four degrees of freedom of classical Dirac spinor, the next factor 3 stays for three colours, next factor 2 gives flavours, and the last factor 3 corresponds to the three families). Our aim here is to look for a framework introducing algebraic and group-theoretical structure which permits to incorporate all the internal quark symmetries enumerated above in some irreducible representations of $Z_3$-graded generalization of Lorentz algebra.

Because in the quark sector of Standard Model e.g. $u$ and $d$ quarks behave as fermions, the two-quark states $uu$ or $dd$ should be excluded, unless there are extra parameters distinguishing the states in a pair. This was the origin of colour degrees of freedom, and of an exact $SU(3)$ colour symmetry treating quarks as colour triplets which incorporate three distinct eigenstates, labeled as red, green, and blue. With such enlargement of the Hilbert space describing single quark states we arrive in Sect. 3 at new 12-component fermionic colour Dirac field, introduced in [4], [5], which is covariant under $Z_3$ symmetry group and contains colour Dirac $\Gamma^\mu$ matrices whose structure is described symbolically by the following tensor product:

$$M_3(C) \otimes H_2 \otimes H_2$$

$M_3(C)$ represents colour $3 \times 3$ matrices, $H_2$ are the $2 \times 2$ Hermitian ones, and in our approach the generalized $12 \times 12$-dimensional generalized Dirac $\Gamma^\mu$-matrices employ in $M_3$ sector the generators of the particular ternary Clifford algebra discussed in Sect. 2; similar constructions were recently considered in [6], [7].

The colour symmetry is somehow hidden in Nature, because the states with non-zero colour charges are not observed in experiments due to the quark confinement mechanism [8], so that in a quark model we deal only with composite hadronic states described asymptotically by the Dirac and Klein-Gordon (KG) equations. We argue that on the level of single quark states one should not postulate the description of colour quark triplets by standard Dirac fields, and we propose instead its 12-component colour generalization, incorporating the $Z_3$-grading and generating colour entanglement.

In this new approach (see Sect. 3) free quark field components satisfy a sixth-order generalization of Klein-Gordon’s equation, which factorizes into the triple product of the standard Klein-Gordon operator with real mass, and a pair of Klein-Gordon operators with complex-conjugated Lee-Wick masses (see e.g. [9], [10]). In such a way we obtain a set of three $Z_3$-graded mass parameters $m$, $jm$
and \( j^2 m, j = e^{\frac{2\pi i}{3}} \), covariant under the \( Z_3 \)-symmetry acting on complex energy plane. By a suitable choice of \( 12 \times 12 \) colour Dirac equations one can introduce the colour-entangled quark triplet with one real mass and a pair of complex masses \( jm \) and its complex conjugate \( j^2 m \) forming together a \( Z_3 \)-graded triplet.

Such construction permits to introduce in Sect. 4 the vectorial realization of \( Z_3 \)-graded Lorentz group acting on a triplet of replicas of one and the same four-momentum vector (see also [11], [12]). In Sect. 5, we present the full description of spinorial realizations of the \( Z_3 \)-graded Lorentz algebra which we introduced in [13] (see also [12], [14]).

In order to describe all quark symmetries, both the exact one (colour) and broken ones (flavour, generations), we consider in Sect. 6 in explicit way the action of \( Z_3 \)-graded Lorentz algebra \( \mathcal{L} = \mathcal{L}^{(0)} \oplus \mathcal{L}^{(1)} \oplus \mathcal{L}^{(2)} \) on the sextet of generalized \( 12 \times 12 \)-component colour Dirac \( \Gamma^\mu \) matrices which provides the irreducible representation of algebra \( \mathcal{L} \). It appears that such a sextet requires - as a module of which \( \mathcal{L} \) can act faithfully - six \( 12 \)-component colour Dirac fields which span 72 quark states and takes into account all known internal symmetries of the standard quark model, describing colour, flavour and generations. Further, in Sect. 7 we introduce chiral flavour doublets and we show how to define chiral colour spinors in the framework using the colour Dirac equations.

In a short outlook in final Sect. 8 we are pointing out the differences between our approach and other results dealing with possible modifications of internal symmetries sector in the quark models. Some problems which may occur in the procedure of supplementing our model with gauge interactions are also briefly addressed.

2 Ternary Clifford algebra, \( Z_3 \)-symmetry and the \( SU(3) \) algebra

In a recent series of papers ([4], [16], [17], [11], [15]) ternary algebraic structures have been introduced and discussed. Among others, a ternary generalization of Clifford algebra with two generators (see e.g. in [6], [7]) is of particular interest for high energy physics due to its close relation with the Lie algebra of the \( SU(3) \) group appearing as an exact colour symmetry and as a broken symmetry mixing the three quark families.

The standard \( 3 \times 3 \) matrix basis of ternary Clifford algebra (which was first considered in XIX-th century by Cayley [18] and Sylvester [19], who called its

\[ \text{3} \]

\[ \text{1} \text{Such triplet can be also realized by the } Z_3 \text{-graded set of generalized Wick rotations by the angles 0, 120 and 240 degrees.} \]

\[ \text{2In } [12] \text{ the } Z_3 \text{-graded Poincaré algebra is introduced in an alternative way, realized on a } Z_3 \text{-graded triplet of Minkowskian space replicas: one real and two complex-conjugate ones.} \]
elements “nonions” looks as follows:

\[
Q_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & j \\
j^2 & 0 & 0
\end{pmatrix},
Q_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & j^2 \\
j & 0 & 0
\end{pmatrix},
Q_3 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

\[
Q_1^\dagger = \begin{pmatrix}
0 & 0 & j \\
1 & 0 & 0 \\
0 & j^2 & 0
\end{pmatrix},
Q_2^\dagger = \begin{pmatrix}
0 & 0 & j^2 \\
1 & 0 & 0 \\
0 & j & 0
\end{pmatrix},
Q_3^\dagger = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

(2)

where \( j \) is the third primitive root of unity,

\[ j = e^{\frac{2\pi i}{3}}, \quad j^2 = e^{\frac{4\pi i}{3}}, \quad 1 + j + j^2 = 0. \]

(4)

and \( M^\dagger \) denotes the hermitian conjugate of matrix \( M \). We see that all the matrices (2, 3) are non-Hermitian. To complete the basis of 3 × 3 traceless matrices, we must add to (2) and (3) the following two linearly independent diagonal matrices:

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & j & 0 \\
0 & 0 & j^2
\end{pmatrix},
B^\dagger = \begin{pmatrix}
1 & 0 & 0 \\
0 & j^2 & 0 \\
0 & 0 & j
\end{pmatrix}.
\]

(5)

In what follows, we shall often use alternative notation \( I_A, \ A = 1, 2, \ldots, 8 \), with

\[
I_1 = Q_1, \quad I_2 = Q_2, \quad I_3 = Q_3, \quad I_4 = Q_1^\dagger, \quad I_5 = Q_2^\dagger, \quad I_6 = Q_3^\dagger, \quad I_7 = B, \quad I_8 = B^\dagger
\]

and can also add \( I_0 = 1 \). The Hermitian conjugation \( I_A^\dagger (A = 1, 2, \ldots, 8) \) provides the following permutation of indices \( A \to A^\dagger \):

\[
A = (1, 2, 3, 4, 5, 6, 7, 8) \to A^\dagger = (4, 5, 6, 1, 2, 3, 8, 7).
\]

(6)

(7)

(8)

We can introduce as well the standard complex conjugation \( M \to M^\bar{} \), which leads to the relations

\[
\bar{I}_A = (\bar{Q}_1 = Q_2, \bar{Q}_2 = Q_1, \bar{Q}_3 = Q_3, \bar{Q}_1^\dagger = Q_2^\dagger, \bar{Q}_2^\dagger = Q_3^\dagger, \bar{Q}_3^\dagger = Q_1^\dagger, \quad \bar{B} = B^\dagger) = I_{\bar{A}},
\]

(9)

which corresponds to another permutation of indices \( A \),

\[
A = (1, 2, 3, 4, 5, 6, 7, 8) \to \bar{A} = (2, 1, 3, 5, 4, 6, 8, 7).
\]

(10)

The 3 × 3 matrices \( Q_3 \) and \( Q_3^\dagger \) are real, while \( Q_2 = \bar{Q}_1 \) are mutually complex conjugated, as well as their Hermitean counterparts \( Q_2^\dagger = \bar{Q}_1^\dagger \).

The matrices (2) and (3) are endowed with natural \( \mathbb{Z}_3 \)-grading

\[
\text{grade}(Q_k) = 1, \quad \text{grade}(Q_k^\dagger) = 2,
\]

(11)
Out of three independent $\mathbb{Z}_3$-grade 0 ternary (i.e. three-linear) combinations, only one leads to a non-vanishing result. One can simply check that both $j$ and $j^2$ ternary skew commutators do vanish
\begin{align}
\{Q_1, Q_2, Q_3\}_j &= Q_1Q_2Q_3 + jQ_3Q_1Q_2 = 0, \\
\{Q_1, Q_2, Q_3\}_{j^2} &= Q_1Q_2Q_3 + j^2Q_3Q_1Q_2 = 0,
\end{align}
as well as the odd permutation, e.g. $Q_2Q_1Q_3 + jQ_1Q_3Q_2 + j^2Q_3Q_2Q_1 = 0$.

In contrast, the totally symmetric combination does not vanish but it is proportional to the $3 \times 3$ identity matrix $I_0 = 1_3$: \begin{equation} Q_aQ_bQ_c + Q_bQ_cQ_a + Q_cQ_aQ_b = 3 \eta_{abc} 1_3, \quad a, b, \ldots = 1, 2, 3. \tag{14} \end{equation}
with $\eta_{abc}$ given by the following non-zero components
\begin{equation} \eta_{111} = \eta_{222} = \eta_{333} = 1, \quad \eta_{123} = \eta_{231} = \eta_{312} = j^2, \quad \eta_{213} = \eta_{321} = \eta_{132} = j \tag{15} \end{equation}
and all other components vanishing. The above relation can be used as definition of ternary Clifford algebra (see e.g. [21], [4]).

Analogous set of relations is formed by Hermitian conjugates $Q^\dagger_a := \bar{Q}_a^T$ of matrices $Q_a$, which we shall endow with dotted indices $\dot{a}, \dot{b}, \ldots = 1, 2, 3$. They satisfy the relation
\begin{equation} Q^2_a = Q^\dagger_\dot{a} \tag{16} \end{equation}
as well as the identities conjugate to the ones in (14)
\begin{equation} Q^\dagger_a Q^\dagger_b Q^\dagger_c + Q^\dagger_b Q^\dagger_c Q^\dagger_a + Q^\dagger_c Q^\dagger_a Q^\dagger_b = 3 \eta_{\dot{a}\dot{b}\dot{c}} 1_3, \text{with } \eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{abc}. \tag{17} \end{equation}

It is obvious that any similarity transformation of the generators $Q_a$ keeps the ternary anti-commutator invariant. As a matter of fact, if we define $\bar{Q}_a := S^{-1}Q_aS$, with $S$ a non-singular $3 \times 3$ matrix, the new set of generators will satisfy the same ternary relations, because it follows that
\begin{equation} \bar{Q}_a\bar{Q}_b\bar{Q}_c = S^{-1}Q_aSS^{-1}Q_bSS^{-1}Q_cS = S^{-1}(Q_aQ_bQ_c)S, \tag{18} \end{equation}
and on the right-hand side we have the unit matrix which commutes with all other matrices, so that $S^{-1}1_3S = 1_3$.

Here is the full multiplication table of the associative algebra of eight basis matrices $I_A (A = 1, 2, \ldots 8)$. 

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It is also worthwhile to note that the six matrices $Q_a$ and $Q^\dagger_b$ together with two traceless diagonal matrices $B$ and $B^\dagger$ from (2, 3) form the basis for certain $\mathbb{Z}_3$-graded representation of the SU(3)-algebra, as it was shown by V. Kac in 1994 (see [22]).

All these matrices are cubic roots of the $3 \times 3$ unit matrix, i.e. their cubes are all equal to $\mathbb{1}_3$. One can observe that two traceless matrices $I_2 = Q_2$ and $I_7 = B$ generate, by consecutive multiplications, full 8-dimensional basis of the SU(3) algebra. The full basis of $3 \times 3$ traceless SU(3) matrices is generated by all possible powers and products of $B$ and $Q_2$, and is displayed in Table 1 below.

We endow the two diagonal matrices $B$ and $B^\dagger = B^2$ with $\mathbb{Z}_3$ grade 0, the matrices $Q_a$ with $\mathbb{Z}_3$ grade 1, and their three hermitian conjugates $\bar{Q}^\dagger_b$ with $\mathbb{Z}_3$ grade 2. Under matrix multiplication the grades are additive modulo 3.

The eight matrices $B, B^\dagger, Q_a, Q^\dagger_b$ can be mapped faithfully onto the canonical Gell-Mann basis of the SU(3) algebra. The Lie algebra of the commutators between the generators $I_A$ is given in Appendix I. The linear combinations of matrices $I_A$ producing the Gell-Mann matrices are given in Appendix II.

Further we shall use the basis [6] for the description of the generators of colour algebra, which satisfies the Lie-algebraic relations with particular properties of complex structure constants (see Appendix I, relation 141).

|     | $Q_1$ | $Q_2$ | $Q_3$ | $Q_1^\dagger$ | $Q_2^\dagger$ | $Q_3^\dagger$ | $B$ | $B^\dagger$ |
|-----|-------|-------|-------|----------------|----------------|----------------|-----|-------------|
| $Q_1$ | $Q_1^\dagger$ | $j^2Q_1^3$ | $jQ_2^1$ | 1              | $B^\dagger$   | $B$            | $jQ_2$ | $j^2Q_3$   |
| $Q_2$ | $jQ_3^2$ | $Q_2^\dagger$ | $j^2Q_1^1$ | $B$            | 1              | $B^\dagger$   | $jQ_3$ | $j^2Q_1$   |
| $Q_3$ | $j^2Q_2^1$ | $jQ_1^1$ | $Q_3^\dagger$ | $B^\dagger$   | $B$            | 1              | $jQ_1$ | $j^2Q_2$   |
| $Q_1^\dagger$ | 1     | $j^2B$ | $jB^\dagger$ | $Q_1$          | $j^2Q_3$      | $jQ_2$         | $Q_3^\dagger$ | $Q_2^\dagger$ |
| $Q_2^\dagger$ | $jB^\dagger$ | 1     | $j^2B$ | $jQ_3$         | $Q_2$          | $j^2Q_1$      | $Q_1^\dagger$ | $Q_3^\dagger$ |
| $Q_3^\dagger$ | $j^2B$ | $jB^\dagger$ | 1     | $j^2Q_2$       | $jQ_1$         | $Q_3$          | $Q_2^\dagger$ | $Q_1^\dagger$ |
| $B$     | $Q_2$ | $Q_3$ | $Q_1$ | $jQ_3^1$      | $jQ_1^1$       | $jQ_2^1$       | $B^\dagger$   | 1           |
| $B^\dagger$ | $Q_3$ | $Q_1$ | $Q_2$ | $j^2Q_2^1$    | $j^2Q_3^1$     | $j^2Q_1^1$     | 1              | $B$         |

Table I: The multiplication table of nonion algebra
3 The $Z_3$-graded Dirac’s equation

We shall construct a generalized equation for quarks, incorporating not only their half-integer spin and particle-antiparticle content (due to charge conjugation, producing anti-quark states), but also the new discrete degree of freedom, the colour, taking three possible values.

Let us describe three different two-component fields (Pauli spinors), which will be distinguished by three colours, the “red” for $\varphi_+$, the “blue” for $\chi_+$, and the “green” for $\psi_+$; more explicitly

$$\varphi_+ = \begin{pmatrix} \varphi_1^1 \\ \varphi_2^1 \end{pmatrix}, \quad \chi_+ = \begin{pmatrix} \chi_1^2 \\ \chi_2^2 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}. \quad (19)$$

We follow the minimal scheme which takes into account the existence of spin by using Pauli spinors on which the 3-dimensional momentum operator acts through $2 \times 2$ matrix describing the scalar product $\sigma \cdot \mathbf{p}$.

To acknowledge the existence of anti-particles, we should also introduce three “anti-colours”, denoted by a “minus” underscript, corresponding to “cyan” for $\varphi_-$, “yellow” for $\chi_-$ and “magenta” for $\psi_-; here, too, we employ the two-component columns:

$$\varphi_- = \begin{pmatrix} \varphi_1^1 \\ \varphi_2^1 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} \chi_1^2 \\ \chi_2^2 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix}. \quad (20)$$

As a result, the six Pauli spinors (19) and (20) will form a twelve-component entity which we shall call “coloured Dirac spinor”. This construction reflects the overall $Z_3 \times Z_2 \times Z_2$ symmetry: one $Z_2$ group corresponds to the spin $\frac{1}{2}$ dichotomic degree of freedom, described by eigenstates; the second $Z_2$ is required in order to represent the particle-anti-particle symmetry, and the $Z_3$ group corresponding to colour symmetry.

The “coloured” Pauli spinors should satisfy first order equations conceived in such a way that they propagate all together as one geometric object, just like $E$ and $B$ components of Maxwell’s tensor in electrodynamics, or the pair of two-component Pauli spinors which are not propagating separately, but constitute one single entity, the four-component Dirac spinor.

This leaves not much space for the choice of the system of intertwined equations. Here we present the ternary generalization of Dirac’s equation, intertwining not only particles with antiparticles, but also the three “colours” in such a way that the entire system becomes invariant under the action of the $Z_3 \times Z_2$ group.

The set of linear equations for three Pauli spinors endowed with colours, and another three Pauli spinors corresponding to their anti-particles characterized by ”anti-colours” involves together twelve complex functions. The twelve components could describe three independent Dirac particles, but here they are intertwined in a particular $Z_3 \times Z_2$ graded manner, mixing together not only particle-antiparticle states, but the three colours as well.
Let us follow the logic that led from Pauli’s to \((Z_2\text{-graded})\) Dirac’s equation and extend it to the colours acted upon by the \(Z_3\) group. In the expression for the energy operator (Hamiltonian) the mass term is positive when it describes particles, and acquires negative sign when we pass to anti-particles, i.e. one gets the change of sign each time when particle-antiparticle components are interchanged.

We shall now assume that mass terms should acquire the factor \(j\) when we switch from the red component \(\varphi\) to the blue component \(\chi\), and another \(j\)-factor when we switch from blue component \(\chi\) to the green component \(\psi\). We remind that we use the notation introduced in \([4]\), \(j = e^{\frac{2\pi i}{3}}\), \(j^2 = e^{\frac{4\pi i}{3}}\), \(j^3 = 1\), and \(1 + j + j^2 = 0\).

The momentum operator will be non-diagonal, as in the Dirac equation, systematically intertwining not only particles with anti-particles, but also colours with anti-colours. The system that satisfies all these assumptions can be introduced in the following manner \([4, 11]\):

Let us first choose the basis in which particles with a given colour and the particles with corresponding anti-colour are grouped in pairs:

\[
(\varphi^+, \varphi^-, \chi^+, \chi^-, \psi^+, \psi^-)^T.
\]

where \(\varphi^\pm\), \(\chi^\pm\) and \(\psi^\pm\) are two-component Pauli spinors defined by eqs. \([19]\) and \([20]\).

In such a basis our “coloured Dirac equation” takes the following form in terms of six Pauli spinors:

\[
\begin{align*}
E \varphi^+ &= mc^2 \varphi^+ + c \sigma \cdot \mathbf{p} \chi^- , \\
E \varphi^- &= -mc^2 \varphi^- + c \sigma \cdot \mathbf{p} \chi^+ , \\
E \chi^+ &= j mc^2 \chi^+ + c \sigma \cdot \mathbf{p} \psi^- , \\
E \chi^- &= -j mc^2 \chi^- + c \sigma \cdot \mathbf{p} \psi^+ , \\
E \psi^+ &= j^2 mc^2 \psi^+ + c \sigma \cdot \mathbf{p} \varphi^- , \\
E \psi^- &= -j^2 mc^2 \psi^- + c \sigma \cdot \mathbf{p} \varphi^+ ,
\end{align*}
\]

Let us remark that while in the Schroedinger picture the energy \(E\) and the momentum \(\mathbf{p}\) are represented by differential operators

\[
E \to -i \hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \to -i \hbar \mathbf{\nabla},
\]

in \([22]\) we use their Fourier-transformed image, in which \(E\) and \(\mathbf{p}\) are interpreted as multiplication by the corresponding numerical eigenvalues.

The particle-antiparticle \(Z_2\)-symmetry is obtained if \(m \to -m\) and simultaneously \((\varphi^+, \chi^+, \psi^+) \to (\varphi^-, \chi^-, \psi^-)\) and vice versa; the \(Z_3\)-colour symmetry is realized by multiplication of mass \(m\) by \(j\) each time the colour changes, i.e. more explicitly, \(Z_3\) symmetry is realized by the following mappings:

\[
m \to jm, \quad \varphi^\pm \to \chi^\pm \to \psi^\pm \to \varphi^\pm ,
\]

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\[ m \rightarrow j^2 m, \quad \varphi_{\pm} \rightarrow \psi_{\pm} \rightarrow \chi_{\pm} \rightarrow \varphi_{\pm}. \]  

The system of equations (22) can be written using 12 \times 12 matrices acting on the 12-component colour spinor \( \Psi \) build up from six “coloured” Pauli spinors. In shortened form we can write

\[ \mathcal{E} \Psi = \begin{bmatrix} c^2 \mathcal{M} + c \mathcal{P} \end{bmatrix} \Psi, \]  

where \( \mathcal{E} = E \mathbb{I}_{12} \), with \( \mathbb{I}_{12} \) denoting the 12 \times 12 unit matrix, and the matrices \( \mathcal{M} \) and \( \mathcal{P} \) given explicitly below:

\[
\mathcal{M} = \begin{pmatrix}
m \mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -m \mathbb{I}_2 & 0 & 0 & 0 & 0 \\
0 & 0 & jm \mathbb{I}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -jm \mathbb{I}_2 & 0 & 0 \\
0 & 0 & 0 & 0 & jm \mathbb{I}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -j^2 m \mathbb{I}_2 \\
\end{pmatrix}
\]

\[
\mathcal{P} = \begin{pmatrix}
0 & 0 & 0 & \sigma \cdot p & 0 & 0 \\
0 & 0 & \sigma \cdot p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma \cdot p \\
0 & 0 & 0 & 0 & \sigma \cdot p & 0 \\
0 & \sigma \cdot p & 0 & 0 & 0 & 0 \\
\sigma \cdot p & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  

The two matrices \( \mathcal{M} \) and \( \mathcal{P} \) in (27) and (28) are 12 \times 12-dimensional: all the entries in \( \mathcal{M} \) are proportional to the 2 \times 2 unit matrix, and the entries in the second matrix \( \mathcal{P} \) contain 2 \times 2 Pauli’s sigma-matrices, so \( \mathcal{P} \) is as well a 12 \times 12 matrix. The energy matrix operator \( \mathcal{E} \) is proportional to the 12 \times 12 unit matrix.

One can easily see that the diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,

\[
E^6 \varphi_+ = m^6 c^{12} \varphi_+ + c^6 \| p \|^6 \varphi_+, \\
E^6 \varphi_- = m^6 c^{12} \varphi_- + c^6 \| p \|^6 \varphi_.
\]  

and similarly all other components. It is convenient to use the tensor product notation for the description of the matrices \( \mathcal{E}, \mathcal{M} \) and \( \mathcal{P} \).

Using two 3 \times 3 matrices \( B \) and \( Q_3 \) defined in (2),

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & j & 0 \\
0 & 0 & j^2 \\
\end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{pmatrix},
\]

the 12 \times 12 matrices \( M \) and \( P \) can be represented as the following tensor products:

\[
\mathcal{E} = E \mathbb{I}_3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 = E \mathbb{I}_{12}, \quad \mathcal{M} = m \otimes \sigma_3 \otimes \mathbb{I}_2, \quad \mathcal{P} = Q_3 \otimes \sigma_1 \otimes (\sigma \cdot p)
\]
where \( \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Let us rewrite the system (22) involving six coupled two-component spinors as one linear equation for the “colour Dirac spinor” \( \Psi \), conceived as column vector containing twelve components of three “colour” fields, in the basis (21) given by \( \Psi = [\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]^T \), with energy and momentum operators \( \mathcal{E} \) and \( \mathcal{P} \) on the left hand side and the mass operator \( \mathcal{M} \) on the right hand side:

\[
E \mathbb{I}_3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \Psi - Q_3 \otimes \sigma_1 \otimes c \sigma \cdot p \Psi = mc^2 B \otimes \sigma_3 \otimes \mathbb{I}_2 \Psi. \tag{32}
\]

Like in the case of the standard Dirac equation, let us transform this equation in a way that the mass operator becomes proportional to the unit matrix. For such a purpose, we multiply the equation (32) on the left by the matrix \( B^\dagger \otimes \sigma_3 \otimes \mathbb{I}_2 \).

Now we get the following equation which enables us to interpret the energy and the momentum as components of the Minkowskian four-vector \( c p^\mu = [E, c p] \):

\[
E B^\dagger \otimes \sigma_3 \otimes \mathbb{I}_2 \Psi - Q_2 \otimes (i \sigma_2) \otimes c \sigma \cdot p \Psi = mc^2 \mathbb{I}_3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \Psi, \tag{33}
\]

where we used the fact that \( (\sigma_3)^2 = \mathbb{I}_2 \), \( B^\dagger B = \mathbb{I}_3 \) and \( B^\dagger Q_3 = Q_2 \). The sixth power of this operator gives the same result as before,

\[
[E B^\dagger \otimes \sigma_3 \otimes \mathbb{I}_2 - Q_2 \otimes (i \sigma_2) \otimes c \sigma \cdot p]^6 = [E^6 - c^6 p^6] \mathbb{I}_{12} = m^6 c^{12} \mathbb{I}_{12} \tag{34}
\]

It is also worth to note that taking the determinant on both sides of the eq. (33) yields the twelfth-order equation:

\[
(E^6 - c^6 | p |^6)^2 = m^{12} c^{24}. \tag{35}
\]

There is still certain arbitrariness in the choice of \( 3 \times 3 \) matrix factors \( B^\dagger \) and \( Q_2 \) in the colour Dirac operator (33). This is due to the choice of \( j = e^{\frac{2\pi i}{3}} \) as the generator of the representation of the finite \( \mathbb{Z}_3 \)-symmetry group. If \( j^2 \) is chosen instead, in (33) the matrix \( B^\dagger \) will be replaced by \( B, Q_2 \) by \( Q_1 \), which is its complex conjugate; the remaining terms keep the same form.

The equation (33) can be written in a concise manner by introducing the \( 12 \times 12 \) matrix colour Dirac operator \( \Gamma^\mu p_\mu \) using Minkowskian space-time indices and metric \( \eta_{\mu\nu} = \text{diag}(+,-,-,-) \):

\[
\Gamma^\mu p_\mu \Psi = mc \mathbb{I}_{12} \Psi, \quad \text{with} \quad p^0 = \frac{E}{c}, \quad p^k = \begin{bmatrix} p^x, p^y, p^z \end{bmatrix}. \tag{36}
\]

with \( 12 \times 12 \) matrices \( \Gamma^\mu \) \( (\mu = 0, 1, 2, 3) \) defined as follows:

\[
\Gamma^0 = B^\dagger \otimes \sigma_3 \otimes \mathbb{I}_2, \quad \Gamma^k = Q_2 \otimes (i \sigma_2) \otimes \sigma^k \tag{37}
\]

The multiplication rules for \( B, B^\dagger, Q_A \) and \( Q_B^\dagger \) \( (A, B, ... = 1, 2, 3) \) are given in the Table 1.
The 12-component colour Dirac equation (36) is invariant under an arbitrary similarity transformation, i.e. if we set
\[ \Psi' = R \Psi, \quad (\Gamma^{\mu})' = R \Gamma^{\mu} R^{-1} \] then
\[ (\Gamma^{\mu})' p^{\mu} \Psi' = mc \Psi', \quad (38) \]
we get obviously
\[ [(\Gamma^{\mu})' p^{\mu}]^6 = (p_0^6 - |p|^6) \mathbb{1}_{12} \quad (39) \]
Following the formulae (37) for the colour Dirac \( \Gamma^{\mu} \)-matrices we see that they are neither real \( \bar{\Gamma}^{\mu} \neq \Gamma^{\mu} \) nor Hermitian \( (\Gamma^{\mu})' \neq \Gamma^{\mu} \). From the colour Dirac equation (33) one gets the following equations for complex-conjugated \( \bar{\Psi} \) and Hermitean-conjugated \( \Psi' \):
\[ \bar{\Gamma}^{\mu} p^{\mu} \bar{\Psi} = mc \bar{\Psi}, \quad p^{\mu} \Psi' (\Gamma^{\mu})' = mc \Psi', \quad (40) \]
where \( \bar{\Psi} \) is a column, \( \Psi' \) is a row, \( \bar{\sigma}_k = -\sigma_2 \sigma_k \sigma_2, \ \sigma_k = \sigma^k, \ \sigma_0 = \sigma^0 = \mathbb{1}_2, \) and
\[ \bar{\Gamma}^0 = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \bar{\Gamma}^k = Q_1 \otimes (i \sigma_2) \otimes \bar{\sigma}^k, \quad (\Gamma^{\mu})' = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \bar{\Gamma}^k = Q_1 \otimes \sigma_3 \otimes \sigma^k, \quad (41) \]
Further, the second equation of (40) can be written in terms of the matrices (37) if we introduce the Hermitian-adjoint colour Dirac spinor \( \Psi^H = \Psi' C \), where the \( 12 \times 12 \)-matrix \( C \) satisfies the relation
\[ (\Gamma^{\mu})' C = C \Gamma^{\mu}. \quad (42) \]
It can be also shown that neither \( \bar{\Gamma}^{\mu} \) nor \( (\Gamma^{\mu})' \) can be obtained via similarity transformation (38).
To obtain a general solution of the colour Dirac equation one should use its Fourier transformed version (see (36)). In the momentum space it becomes:
\[ (\Gamma^{\mu} p^{\mu} - m \mathbb{1}_{12}) \hat{\Psi}(p) = 0. \quad (43) \]
The sixth power of the matrix \( \Gamma^{\mu} p^{\mu} \) is diagonal and proportional to \( m^6 \), so that we have
\[ (\Gamma^{\mu} p^{\mu})^6 - m^6 \mathbb{1}_{12} = (p_0^6 - |p|^6 - m^6) \mathbb{1}_{12} = 0. \quad (44) \]
Now we should find the inverse of the matrix \( (\Gamma^{\mu} p^{\mu} - m \mathbb{1}_{12}) \). Let us note that the sixth-order expression on the left-hand side in (44) can be factorized as follows:
\[ (\Gamma^{\mu} p^{\mu})^6 - m^6 = \left( (\Gamma^{\mu} p^{\mu})^2 - m^2 \right) \left( (\Gamma^{\mu} p^{\mu})^2 - j m^2 \right) \left( (\Gamma^{\mu} p^{\mu})^2 - j^2 m^2 \right). \quad (45) \]
The first factor can be expressed as the product of two linear operators, one of which defines the colour Dirac equation (36) (see also (43):
\[ (\Gamma^{\mu} p^{\mu})^2 - m^2 = (\Gamma^{\mu} p^{\mu} - m) (\Gamma^{\mu} p^{\mu} + m) \quad (46) \]
Therefore the inverse of the Fourier transform of the linear operator defining the colour Dirac equation (43) is given by the following matrix:

$$[\Gamma^\mu p_\mu - m]^{-1} = \frac{(\Gamma^\mu p_\mu + m) \left( (\Gamma^\mu p_\mu)^2 - j \, m^2 \right) \left( (\Gamma^\mu p_\mu)^2 - j^2 \, m^2 \right)}{(p_0^6 - | \mathbf{p} |^6 - m^6)}. \quad (47)$$

The inverse of the six-order polynomial can be decomposed into a sum of three expressions with second-order denominators, multiplied by the common factor of the fourth order. Let us denote by $\Omega$ the sixth root of $(| \mathbf{p} |^6 + m^6)$,

$$\Omega = \sqrt[6]{| \mathbf{p} |^6 + m^6}, \quad (48)$$

along with five other root values obtained via multiplication by consecutive powers of the sixth root of unity, $q = e^{\frac{2\pi i}{6}}$. Recalling relation (4) and that $q^2 = j$, we have the identity

$$\left( p_0^6 - \Omega^6 \right) = (p_0^2 - \Omega^2) ((p_0^2 - j\Omega^2) (p_0^2 - j^2\Omega^2)$$

which leads to the decomposition formula

$$\frac{1}{(p_0^6 - | \mathbf{p} |^6 - m^6)} = \frac{1}{3 \Omega^4} \left[ \frac{1}{p_0^2 - \Omega^2} + \frac{j}{p_0^2 - j\Omega^2} + \frac{j^2}{p_0^2 - j^2\Omega^2} \right]$$

or equivalently,

$$\frac{1}{(p_0^6 - | \mathbf{p} |^6 - m^6)} = \frac{1}{3 \Omega^4} \left[ \frac{1}{p_0^2 - \Omega^2} + \frac{1}{j^2 p_0^2 - \Omega^2} + \frac{1}{j p_0^2 - \Omega^2} \right]$$

As long as there is a non-zero mass term, we do not encounter the infrared divergence problem at $| \mathbf{p} | \to 0$. Each of the three inverses of a second-order polynomial can be in turn expressed as a sum of simple first-order poles, e.g.

$$\frac{1}{j^2 - \Omega^2} = \frac{j}{2 \Omega} \left[ \frac{1}{p_0 - j^2\Omega} - \frac{1}{p_0 + j^2\Omega} \right] = \frac{j^2}{2 \Omega} \left[ \frac{1}{j p_0 - \Omega} - \frac{1}{j p_0 + \Omega} \right], \quad (52)$$

and similarly for other terms in (50). After such a substitution in (47), six $Z_6$-graded simple poles do appear, Figure (1) illustrating the location of these six poles in the complex energy plane. In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane.

The first term in the decomposition (50) of the coulour Dirac propagator (50) presents two simple poles on the real line, while the second and the third terms display two simple poles each, located on complex straight lines $\text{Im} p_0 = j \text{Re} p_0$ and $\text{Im} p_0 = j^2 \text{Re} p_0$.

One can add that in the propagators given by formula (50), the non-standard residua $\pm j$ and $\pm j^2$ should be justified by suitable form of the $Z_3$-graded commutators describing quantum oscillator algebra of colour quark field excitations.
It should be stressed that the colour Dirac equation (40) breaks the Lorentz symmetry \( O(1,3) \) reducing it to \( O_3 \), because the \( 3 \times 3 \)-matrices describing “colour” are different for the \( \Gamma^0 \) and \( \Gamma^k \) components. However we shall show in the following Section 4 that one can introduce a \( \mathbb{Z}_3 \)-graded generalization of the Lorentz transformations, acting in covariant way on three “replicas” of the energy-momentum four-vector introduced above. Analogous extensions of space-time were discussed in [29], [28].

4 \( \mathbb{Z}_3 \)-graded set of three complex four-momenta and \( \mathbb{Z}_3 \)-graded Lorentz transformations

The mass shell condition (35) for coloured Dirac equation can be decomposed into the usual relativistic Klein-Gordon invariant multiplied by a strictly positive factor which can be interpreted as generating the form-factor for quark propagator with given mass \( m \).

\[
C_6 = p_6^2 - \Omega^6 = (p_6^2 - |p|^2)(p_6^2 + p_6^2 |p|^2 + |p|^4) = m^6 c^6, \quad (53)
\]

The sixth-order polynomial \( C_6 \) can be further decomposed into the product of the following three second-order polynomials,

\[
C_6 = \left( C_2 \right)^0 \left( C_2 \right)^1 \left( C_2 \right)^2, \quad (54)
\]

with

\[
\left( C_2 \right)^0 = p_6^2 - p^2, \quad \left( C_2 \right)^1 = j p_6^2 - p^2, \quad \left( C_2 \right)^2 = j^2 p_6^2 - p^2. \quad (55)
\]
Let us denote by superscripts $(0)$, $(1)$ and $(2)$ the four-momenta with quadratic invariants given by $C_2$, $C_2$ and $C_2$.

We get explicitly

\[(p_0)^2 - (0) = C_2, \quad (p_0)^2 - (1) = C_2, \quad (p_0)^2 - (2) = C_2,\]

From any real four-vector $p_{\mu}$ one can define its two “replicas” $(1) p_{\mu}$ and $(2) p_{\mu}$ with $p_{\mu}$ in the complex plane, obtained by the generalized Wick rotations by $j$ and by $j^2$ in the following way. Let us introduce three $4 \times 4$ matrices acting on Minkowskian four-vectors:

\[
(0) A = \text{diag} (1, 1, 1, 1) = i_1, \quad (1) A = \text{diag} (j^2, 1, 1, 1), \quad (2) A = \text{diag} (j, 1, 1, 1),
\]

providing a (reducible) matrix representation of the cyclic $Z_3$ group,

\[(r+s) A A = A.\]

where the superscripts $(r+s)$ are added modulo 3, e.g. $1+2 \rightarrow 0$, $2+2 \rightarrow 1$, etc.

Acting on a given four-vector $p_{\mu} = (p_0, p)$ by one of the matrices $A$ we produce its three $Z_3$-graded “replicas” belonging correspondingly to sectors $(r) C_2$:

\[
(0) \rightarrow (r) = (r) A p_{\mu} \rightarrow (0), \quad (1) \rightarrow (r) = (r) A p_{\mu} \rightarrow (1), \quad (2) \rightarrow (r) = (r) A p_{\mu} \rightarrow (2).
\]

In what follows, we shall use for both the Lorentz boosts and the Wick rotations a short-hand notation:

\[
p_{\mu} = L_{\nu}^{\mu} p_{\nu} \rightarrow p_{\mu} = p, \quad p_{\mu} = A_{\nu}^{\mu} p_{\nu} \rightarrow p_{\mu} = A p.
\]

It should be stressed here that the spacetime remains Minkowskian, with one real time and three real spatial coordinates; however, the components of $(1) p_{\mu}$ and $(2) p_{\mu}$ can take on particular $Z_3$-graded complex values. Three “replicas” (59) are the images of the same four-vector which can be obtained by $Z_3$-valued Wick rotations in complex energy plane.

In particular, let us denote by $L_{00}$ the classical Lorentz transformations which map the real Minkowskian momenta $p_{\mu}$ into $p_{\mu}$

\[
(0) L_{00} = L_{00}^{\mu} p_{\nu} = p_{\mu} = (0) p = (0),
\]

where lower indices $(00)$ mean that we transform $C_2$ into itself, and the superscript $(0)$ says that we deal with the classical Lorentz transformations. The
zero-grade Lorentz transformations can be extended to the mappings of four-vectors \( \mathbf{p} \) belonging to sector \( C_2 \) onto four-vectors \( \mathbf{p}' \) belonging to sector \( C_2 \), with \( r, s = 0, 1, 2 \). Let us apply a Lorentz boost transforming a four-vector from the sector \( s \), \( \mathbf{p} \) into a vector from another sector \( r \), \( \mathbf{p}' \). Using the shorthand notations (60) and (61), we have:

\[
(\mathbf{r})_{\mathbf{p}}' = (\mathbf{r})_{\mathbf{A}}(\mathbf{0})_{\mathbf{p}} = (\mathbf{r})_{\mathbf{A}^{-1}}L_{00}(\mathbf{0})_{\mathbf{p}} = (\mathbf{r})_{\mathbf{A}^{-1}}L_{rs}(\mathbf{s})_{\mathbf{A}}^{-1}(\mathbf{0})_{\mathbf{p}} = (\mathbf{r} - \mathbf{s})_{L_{rs}}(\mathbf{s})_{\mathbf{p}} \quad (62)
\]

where

\[
L_{rs} = (\mathbf{r})_{\mathbf{A}}(\mathbf{0})_{\mathbf{L}_{00}}(\mathbf{s})_{\mathbf{A}}^{-1},
\]

describes the Lorentz transformation from sector \( s \) onto sector \( r \), and where the superscript \( (r-s) \) accordingly to the \( Z_3 \)-grading is taken modulo 3.

In order to provide the formulae for \( Z_3 \)-graded boosts in explicit form we will choose the four-vector \( \mathbf{p}_u = (p_0, \mathbf{p}) \) as restricted to the plane \((0, 1)\), with the three-vector \( \mathbf{p} \) aligned along the first spatial axis. In such a frame the Lorentz rotations reduce only to the boost in \((0, 1)\) plane, which is given by the following transformation:

\[
\begin{pmatrix}
  p_0' \\
  p_1'
\end{pmatrix}
= \begin{pmatrix}
  \cosh & \sinh \\
  \sinh & \cosh
\end{pmatrix}
\begin{pmatrix}
  p_0 \\
  p_1
\end{pmatrix},
\]

(64)

Subsequently, we get the following triplet of homogeneous transformations:

\[
L_{00}, \quad L_{11}, \quad L_{22}:
\]

\[
L_{00}(u) = \begin{pmatrix}
  \cosh & \sinh \\
  \sinh & \cosh
\end{pmatrix}, \quad L_{11}(u) = \begin{pmatrix}
  \cosh & j^2 \sinh \\
  j^2 \sinh & \cosh
\end{pmatrix}, \quad L_{22}(u) = \begin{pmatrix}
  \cosh & j \sinh \\
  j \sinh & \cosh
\end{pmatrix}
\]

(65)

preserving respectively the bilinear forms \( (r) \), \( C_2 \). (see (55)).

The matrices (65) are self-adjoint:

\[
L_{00} = L_{00}^\dagger, \quad L_{11} = L_{11}^\dagger, \quad L_{22} = L_{22}^\dagger
\]

(66)

The generalized Lorentz boosts (65) conserve the group property: the product of two Lorentz boosts acting in the \( r \)-th sector is a boost of the same type. Indeed, we see from (65) that the product of two boosts acting in the \( r \)-th sector \((r = 0, 1, 2)\) looks as follows (no summation over \( r \)):

\[
L_{rr}(u) \cdot L_{rr}(v) = L_{rr}(u + v).
\]

(67)

If we look at three fourdimensional Lorentz boost transformations on planes \((0, i), \ i = 1, 2, 3\), the respective set of three independent “classical” Lorentz
boosts belonging to \( L_{00} \) requires the introduction of three \( 4 \times 4 \) matrices with three independent parameters \( u, v, w \):

\[
\begin{pmatrix}
  
  \begin{pmatrix}
    \text{chu} & \text{shu} & 0 & 0 \\
    \text{shu} & \text{chu} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
  \end{pmatrix} & \begin{pmatrix}
    \text{chu} & 0 & \text{shu} & 0 \\
    0 & 1 & 0 & 0 \\
    \text{shu} & 0 & \text{chu} & 0 \\
    0 & 0 & 1 & 0
  \end{pmatrix} & \begin{pmatrix}
    \text{chu} & 0 & 0 & \text{shu} \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    \text{shu} & 0 & 0 & \text{chu}
  \end{pmatrix}
\end{pmatrix}
\]  

(68)

Next, let us consider the general set of matrices (see (63)) transforming the \( s \)-th sector into the \( r \)-th one,

\[
(p^r_\mu)^{(r-s)\nu} = (L_{rs})^\nu_\mu, \quad r, s = 0, 1, 2, \quad r \neq s.
\]  

(69)

There are two types of such matrices: raising and lowering the \( Z_3 \) by 1. For the sake of simplicity, let us firstly consider the two-dimensional case (i.e. \( \mu, \nu = 0, 1 \) in (69). The matrices \( 2 \times 2 \) raising the \( Z_3 \) index \((r) \) of the generalized four-momenta \( p_\mu \) to \( (r+1) \) are:

\[
\begin{align*}
L_{10}^{(1)} &= \begin{pmatrix}
  j^2 \text{chu} & j^2 \text{shu} \\
  j \text{shu} & \text{chu}
\end{pmatrix},
L_{21}^{(1)} &= \begin{pmatrix}
  j^2 \text{chu} & j \text{shu} \\
  j \text{shu} & \text{chu}
\end{pmatrix},
L_{02}^{(1)} &= \begin{pmatrix}
  j^2 \text{chu} & \text{shu} \\
  j \text{shu} & \text{chu}
\end{pmatrix}
\end{align*}
\]  

(70)

The determinants of the matrices (70) are equal to \( j^2 \).

The matrices lowering the \( Z_3 \) index by one (or increasing it by 2, what is equivalent from the point of view of the \( Z_3 \)-grading) are:

\[
\begin{align*}
L_{01}^{(2)} &= \begin{pmatrix}
  j \text{chu} & \text{shu} \\
  j \text{shu} & \text{chu}
\end{pmatrix},
L_{12}^{(2)} &= \begin{pmatrix}
  j \text{chu} & j^2 \text{shu} \\
  j^2 \text{shu} & \text{chu}
\end{pmatrix},
L_{20}^{(2)} &= \begin{pmatrix}
  j \text{chu} & \text{shu} \\
  j \text{shu} & \text{chu}
\end{pmatrix}
\end{align*}
\]  

(71)

The determinants of the matrices (71) are equal to \( j \).

The above two sets of three matrices each are mutually Hermitian-adjoint:

\[
L_{01} = L_{10}, \quad L_{12} = L_{21}, \quad L_{20} = L_{02}
\]  

(72)

We recall that the superscript over each matrix \( L_{rs} \) is equal to the difference of its lower indices, i.e. \((t) = (r - s)\).

The matrices \( L_{rs}^{(r)} \) and \( L_{rs}^{(s)} \) \((r, s = 0, 1, 2)\) raising or lowering respectively the \( Z_3 \)-grade of the four-momentum vectors \( p_\mu \) do not form a Lie group. However, together with matrices \( L_{rs}^{(0)} \) they can be used as building blocks in bigger \( 12 \times 12 \) matrices forming a \( Z_3 \)-graded generalization of the Lorentz group. This construction is possible due to the chain rule obeyed by these matrices, which due to the definition (63) display the group property. We have:

\[
L_{rs}^{(r-s)} \left( p_0, p_1; u \right) L_{st}^{(s-t)} \left( p_0, p_1; v \right) = L_{rt}^{(r-t)} \left( p_0, p_1; (u + v) \right).
\]  

(73)

In order to pass to arbitrary four-momentum vectors \( p_\mu \), \( \mu = 0, 1, 2, 3 \) one should embed the \( 2 \times 2 \) matrices (70 - 71) into \( 4 \times 4 \) matrices in a way analogous
to passing from the $2 \times 2$ boost matrices $L_{00}$ (see (65)) to the triplet of boosts in planes $(0, i), i = 1, 2, 3$ described by the $4 \times 4$ matrices (68).

If we write a $Z_3$-extended four-momentum vector $(p^\mu, p^\mu, p^\mu)^T$ as a column with 12 entries, we can introduce three boost sectors $\Lambda, (r = 0, 1, 2)$ of the generalized $Z_3$-graded Lorentz group as $12 \times 12$ matrices as follows:

$$
\begin{align*}
(0) \Lambda &= \begin{pmatrix} (0) L_{00} & 0 & 0 \\
0 & (0) L_{11} & 0 \\
0 & 0 & (0) L_{22} \end{pmatrix} \\
(1) \Lambda &= \begin{pmatrix} 0 & 0 & (1) L_{02} \\
(1) L_{10} & 0 & 0 \\
0 & (1) L_{21} & 0 \end{pmatrix} \\
(2) \Lambda &= \begin{pmatrix} 0 & (2) L_{01} & 0 \\
0 & 0 & (2) L_{12} \\
(2) L_{20} & 0 & 0 \end{pmatrix}
\end{align*}
(74)
$$

It should be stressed that in each of the $12 \times 12$ matrices $(r) \Lambda, r = 0, 1, 2$ the triplets of $4 \times 4$ matrices $(r-s) L_{rs}$ are obtained from the standard classical Lorentz boost by applying the definition (63), i.e. each $\Lambda$-matrix depends only on three parameters defining three independent Lorentz boosts.

One can show that the matrices (74) display the following $Z_3$-graded multiplication rules:

$$
\begin{align*}
\Lambda \cdot (0) \subset \Lambda, & \quad \Lambda \cdot (1) \subset \Lambda, & \quad \Lambda \cdot (2) \subset \Lambda, & \quad \Lambda \cdot \Lambda \subset \Lambda, \\
\Lambda \cdot (r) \subset (1), & \quad \Lambda \cdot (2) \subset (1), & \quad \Lambda \cdot \Lambda \subset \Lambda.
\end{align*}
(75)
$$

$$
\begin{align*}
(0) \Lambda \cdot \Lambda \subset \Lambda, & \quad (1) \Lambda \cdot \Lambda \subset \Lambda, & \quad (2) \Lambda \cdot \Lambda \subset \Lambda.
\end{align*}
(76)
$$

where $\Lambda$ $(r = 0, 1, 2)$ denote the $Z_3$-graded sectors of the full set of $12 \times 12$ matrix Lorentz group which includes also the $Z_3$-graded $O(3)$ spatial rotations.

The multiplication table (75-76) with the $Z_3$-graded structure can be described in a compact way using the bold-face symbols $\Lambda$ as follows:

$$(r) \Lambda \cdot (s) \subset (r+s)|_3 \Lambda, \text{ with } r, s, .. = 0, 1, 2, \text{ (r + s) taken modulo 3.} \quad (77)
$$

The construction of $Z_3$-graded $O(3)$ rotations completing the $Z_3$-graded boosts $\Lambda$ is as follows. Let us denote by $R_i$ the usual space rotation around the $i$-th axis, represented as a $3 \times 3$ matrix. When incorporated into the four-vector representation of the Lorentz group, it becomes a sub-matrix of a $4 \times 4$ Lorentzian matrix according to the formula $R_i = \begin{pmatrix} 1 & 0 \\
0 & R_{ii} \end{pmatrix}$. The $Z_3$-graded space rotations supplementing the $Z_3$-graded boosts (74) are constructed as the following $12 \times 12$ matrices:

$$
\begin{align*}
(0) R_i &= I_3 \otimes R_i, & \quad (1) R_i &= Q_3^1 \otimes R_i, & \quad (2) R_i &= Q_3 \otimes R_i,
\end{align*}
(78)
$$
where the choice of the colour generators $Q_3^\dagger$ and $Q_3$ is consistent with the colour Dirac equations \((32,33)\).

The $\mathbb{Z}_3$-graded infinitesimal generators of the Lorentz boosts can be obtained by considering the matrices \((r)\Lambda\) with infinitesimal boost parameters (i.e. taking the differential) what amounts to the replacements of the entries $sh$ by $1$, and of all other entries, $chu$ and $1$ alike, by $0$. The resulting $12 \times 12$ matrices are the Lie algebra generators of the generalized Lorentz boosts, which we shall denote as $K_i$, $r = 0, 1, 2$. By taking their commutators we obtain the $\mathbb{Z}_3$-graded generators of the space rotations \((r + s)\mod 3)\):

\[
[K_i, K_j] = -\epsilon_{ijk} J_k
\]  

In such a way we obtain the full set of generators of the $\mathbb{Z}_3$-graded Lorentz algebra which satisfy the following commutation relations:

\[
\begin{align*}
[J_i^{(r)}, J_k^{(s)}] &= \epsilon_{i j k l} J_l^{(r+s)}, \\
[J_i^{(r)}, K_k^{(s)}] &= \epsilon_{i j k l} K_l^{(r+s)}, \\
[K_i^{(r)}, K_k^{(s)}] &= -\epsilon_{i j k l} J_l^{(r+s)}. 
\end{align*}
\]  

which were firstly introduced and studied in \((13)\).

Let us consider \((r)\Lambda\) as $3 \times 3$ matrices, with their matrix elements represented by $4 \times 4$ blocks. \((0)L_{rs}\) (see \((69)\))

The matrices \((0)\Lambda\) are Hermitian by virtue of formula \((66)\), while \((1)\Lambda\dagger = (2)\Lambda\)
or equivalently, \((\Lambda)\dagger = \Lambda\) as a result of formula \((72)\).

Any group structure of $12 \times 12$ matrices is preserved under the similarity transformations,

\[
\Lambda \rightarrow \tilde{\Lambda} = U \Lambda U^{-1},
\]

but the above Hermitian properties of $\Lambda$-matrices are conserved only if the transformation matrices are unitary. The proof is immediate: let us denote by $U = U \otimes 1_4$ a $12 \times 12$ matrix obtained as a tensor product of a $3 \times 3$ complex valued $U$ matrix by the unit $4 \times 4$ matrix $1_4$. (Obviously, $U\dagger = U\dagger \otimes 1_4$). Let us firstly define \((0)\Lambda \rightarrow U(0)\Lambda U^{-1}\) and impose the Hermiticity conditions on the transformed matrices $U(0)\Lambda U^{-1}$. Because the matrix $(0)\Lambda$ is Hermitian, we have

\[
\left(\frac{(0)\Lambda U U^{-1}}{U\dagger} \right)^\dagger = (U^{-1})\dagger \frac{(0)\Lambda}{U\dagger} = U(0)\Lambda U^{-1},
\]

the matrix $U(0)\Lambda U^{-1}$ is Hermitian, too, if the similarity matrices are unitary, if i.e. $U\dagger = U^{-1}$, according to the formula $U = U \otimes 1_4$ it follows that $U\dagger = U^{-1}$. Hermitian conjugation relations between the matrices $(0)\Lambda$ and $(1)\Lambda$ are also
preserved after similarity transformation if the similarity matrices obey the same unitarity condition $U^\dagger = U^{-1}$.

In this way we introduced the symmetry $SU(3)$ acting on the vector representation of the $Z_3$-graded Lorentz group. The $3 \times 3$ matrices $U$ appearing in the $12 \times 12$ matrices $U$ during the unitary similarity transformations leave the $4 \times 4$ Lorentzian blocks unaffected, in agreement with the well known “no-go theorems” by Coleman and Mandula and O’Raifeartaigh ([23], [24]).

We point out that in order to obtain the entire $Z_3$-graded Lorentz group we should add as well the $Z_3$-graded extension of space rotations, also represented as $12 \times 12$ matrices, with building blocks made of $4 \times 4$ matrices, just like the $Z_3$-graded boosts. As in the case of Lorentz boosts, besides the rotations that leave the transformed 3-momentum in the same sector, one gets also $12 \times 12$ matrices with non diagonal $4 \times 4$ entries, which map one of the $Z_3$-graded sectors onto another one.

We conclude that the full set of $Z_3$-graded $O(3)$ subgroup elements can be represented by $12 \times 12$ matrices and incorporated in the $Z_3$-graded Lorentz group.

In this Section we were considering the vectorial realizations of the $Z_3$-graded Lorentz group which can be also extended to the realizations of $Z_3$-graded Poincaré algebra (see also ([12])) In the next two sections we will present our main result: how the sextet of the colour Dirac matrices $\Gamma^\mu$ appears in the construction of faithful spinorial $72 \times 72$ matrix representation of the $Z_3$-graded Lorentz algebra. In such a way we will be able to incorporate all internal symmetries of quark sector appearing in the Standard Model into a group-theoretical framework.

5 $Z_3$-graded generalized Lorentz algebra and its spinorial matrix realization

The $12 \times 12$ matrices $\Gamma^\mu$ (see ([37])) appearing in the coloured Dirac equation (36), (see also ([37])) are linked with the $Z_3$-graded generalization of classical Lorentz symmetries. In particular, the $Z_3 \otimes Z_2 \otimes Z_2$ structure of $\Gamma^\mu$-matrices implies that due to the identities $(Q_a)^3 = (Q_a^\dagger)^3 = \mathbb{1}_3$, $B^3 = (B^\dagger)^3 = \mathbb{1}_3$ and $(\sigma_i)^2 = \mathbb{1}_2$ their sixth powers are proportional to the unit matrix $\mathbb{1}_{12}$ (see also ([35])).

Let us first derive the $Z_3$-graded Lorentz algebra, which follows from the covariance properties of the colour Dirac equation (36).

The two standard commutators of $\Gamma^\mu$ matrices, namely

$$J_i = \frac{i}{2} \epsilon_{ijk} [\Gamma^j, \Gamma^k], \quad K_1 = \frac{1}{2} [\Gamma_1, \Gamma_0]$$

(82)

provide only the first step towards the construction of the generators of a $Z_3$-graded Lorentz algebra. Surprisingly, one can check that the generators
\( (J_i^{(0)}, K_i^{(0)}) \) satisfying the standard \( D = 4 \) Lorentz algebra relations

\[
\begin{align*}
[ J_i^{(0)}, K_k^{(0)} ] &= -\epsilon_{ikl} J_l^{(0)}, \\
[ J_i^{(0)}, K_k^{(0)} ] &= \epsilon_{ikl} K_l^{(0)}, \\
[ J_i^{(0)}, J_k^{(0)} ] &= \epsilon_{ikl} J_l^{(0)}.
\end{align*}
\] (83)

\( \gamma \)

Using the definition of standard colour \( \Gamma \)-matrices \( (37) \) and substituting it in \( (82) \) and \( (85) \), we get

\[
\begin{align*}
J_i &= -\frac{i}{2} Q_1 \otimes I_2 \otimes \sigma_i, \\
K_i &= -\frac{1}{2} Q_1 \otimes Q_1 \otimes \sigma_i, \\
J_i^{(0)} &= -\frac{i}{2} I_3 \otimes I_2 \otimes \sigma_i, \\
K_i^{(0)} &= -\frac{1}{2} Q_3 \otimes Q_1 \otimes \sigma_i.
\end{align*}
\] (86)

In order to introduce the \( Z_3 \)-graded Lorentz algebra

\[ \mathcal{L} = L^{(0)} \oplus L^{(1)} \oplus L^{(2)} \] (87)

where \( L^{(0)} = (J_i^{(0)}, K_i^{(0)}), \quad L^{(1)} = (J_i^{(1)}, K_i^{(1)}), \quad L^{(2)} = (J_i^{(2)}, K_i^{(2)}) \), one should supplement the relations \( (85) \) by the pairs of other possible double commutators:

\[
\begin{align*}
[ J_i, [J_j, K_k]] &= (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) K_l^{(2)}, \\
[ K_i, [K_j, J_k]] &= (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) J_l^{(1)}.
\end{align*}
\] (88)

In particular, besides the representation \( (86) \) we get the following realizations:

\[
\begin{align*}
J_i^{(1)} &= -\frac{i}{2} Q_3 \otimes I_2 \otimes \sigma_i, \\
K_i^{(2)} &= -\frac{1}{2} Q_3 \otimes Q_1 \otimes \sigma_i.
\end{align*}
\] (89)

The three-linear double commutators in \( (85) \) and \( (88) \) are related with \( Z_3 \)-grading; when taken into account, the full set of \( Z_3 \)-graded relations defining the \( Z_3 \)-graded Lorentz algebra introduced in \( (13) \) (where \( r, s, = 0, 1, 2, r + s \) are taken modulo 3), results in the following set of commutation relations \( (13) \):

\[
\begin{align*}
\left[ J_i^{(r)}, J_k^{(s)} \right] &= \epsilon_{ikl} J_l^{(r+s)}, \\
\left[ J_i^{(r)}, K_k^{(s)} \right] &= \epsilon_{ikl} K_l^{(r+s)}, \\
\left[ K_i^{(r)}, K_k^{(s)} \right] &= -\epsilon_{ikl} J_l^{(r+s)}.
\end{align*}
\] (90)

From the commutators \( \left[ K^{(1)}, K^{(1)} \right] \cong J^{(2)} \) and \( \left[ J^{(1)}, J^{(1)} \right] \cong J^{(2)} \) one gets

the realization of remaining generators of \( \mathcal{L} \),

\[
\begin{align*}
J_i^{(2)} &= -\frac{i}{2} Q_3 \otimes I_2 \otimes \sigma_i, \\
K_m^{(1)} &= -\frac{1}{2} Q_3 \otimes \sigma_1 \otimes \sigma_m.
\end{align*}
\] (91)
The formulae \[ (86), \ (89) \] and \[ (91) \] describe the spinorial realization of the Lie algebra \( \mathcal{L} \) which is implied by the choice \[ (37) \] of matrices \( \Gamma^\mu \). Let us introduce a unified notation englobing all possible choices of \( \Gamma^\mu \)-matrices \( (A \neq B) \)

\[
\Gamma^0_{(A,\alpha)} = I_A \otimes \sigma_\alpha \otimes \sigma^0, \quad \Gamma^i_{(B,\beta)} = I_B \otimes (i\sigma_\beta) \otimes \sigma^i, \tag{92}
\]

where \( I_0 = \mathbb{1}_3 \), \( I_A \) with \( A = 1, 2, \ldots, 8 \) are given in \[ (96) \], and \( \alpha, \beta = 2, 3 \) but \( \{\sigma_\alpha, \sigma_\beta\} = 0 \) i.e. we always have either \( \alpha = 2, \beta = 3 \) or \( \alpha = 3, \beta = 2 \).

The choice \( \alpha = 1 \) is not present in the formula \[ (92) \] because it is reserved for the description of symmetry generators \( \mathcal{L} \) (see \[ (89), (91) \]). Further, eight colour \( 3 \times 3 \) matrices \( I_A \ (A = 1, 2, \ldots, 8) \) with the multiplication rules given in Table 1 span the ternary basis of the \( SU(3) \) algebra (22, Sect. 8).

The characteristic feature of “colour” \( \Gamma \)-matrices is that the \( 3 \times 3 \) \( \{\sigma_\alpha, \sigma_\beta\} = 0 \) matrices \( \Gamma^\mu \) of classical spinorial Lorentz group (see \[ (83, 84) \], where \( A^i, B^k \), \( (i, k = 1, 2, 3) \) are the six real \( SL(2, \mathbb{C}) \) Lie group parameters

\[
S^{(0)} = \exp \left( a^i K_i^{(0)} + b^k J_k^{(0)} \right) \tag{94}
\]

we should introduce the following pairs of \( \Gamma^\mu \)-matrices

\[
\Gamma^\mu = (\Gamma^i_{(A,\alpha)}, \Gamma^0_{(A,\alpha)}) \quad \text{and} \quad \tilde{\Gamma}^\mu = (\Gamma^i_{(B,\beta)}, \Gamma^0_{(B,\beta)}) , \tag{95}
\]

where we have chosen \[ (93) \] \( \alpha = 3 \) and \( \beta = 2 \). Although for any choice of the first factor \( I_A \) in \( \Gamma^\mu \)'s (see \[ (92) \]) we have

\[
\left[ J_i^{(0)}, \Gamma^j_{(A,\alpha)} \right] = \epsilon_{ijl} \Gamma_{(A,\alpha)}^l, \quad \left[ J_i^{(0)}, \Gamma^0_{(A,\alpha)} \right] = 0 \tag{96}
\]

the boosts \( K_i^{(0)} \) (see \[ (86) \]) act covariantly only on doublets \( (\Gamma^\mu, \tilde{\Gamma}^\mu) \), with \( (A \neq B) \), because only for such a choice we can get the closure of commutation relations:

\[
\left[ K_i^{(0)}, \Gamma^j_{(A,\alpha)} \right] = \delta_i^j \Gamma^0_{(A,\alpha)}, \quad \left[ K_i^{(0)}, \Gamma^0_{(B,\beta)} \right] = \Gamma^i_{(B,\beta)},
\]

\[
\left[ K_i^{(0)}, \Gamma^j_{(B,\beta)} \right] = \delta_i^j \Gamma^0_{(B,\beta)}, \quad \left[ K_i^{(0)}, \Gamma^0_{(A,\alpha)} \right] = \Gamma^i_{(A,\alpha)}. \tag{97}
\]

It follows from \[ (96), (97) \] that the standard Lorentz covariance requires the pair of coloured Dirac equations described by the doublet \( (\Gamma^\mu, \tilde{\Gamma}^\mu) \) of coloured Dirac
matrices (see 95), which we shall call “Lorentz doublets”. In particular, the $\Gamma^\mu$ matrices \([37]\) from Sect. 3 should be supplemented by the following Lorentz doublet partner:

\[
\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^i_{(8;2)} = B^\dagger \otimes (i\sigma_2) \otimes \sigma^i. \quad (98)
\]

Further we will show that the Lorentz doublets of $\Gamma^\mu$-matrices required by the standard Lorentz covariance can be useful for the description of weak isospin (flavour) doublets of the $SU(2) \times U(1)$ electroweak symmetry. In such a way one can show that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of Standard Model are linked with the presence of standard Lorentz covariance which generates three 24-component Lorentz doublets of colour Dirac spinors.

Next, we will show that in order to obtain the closure of the faithful action of generators $(J^{(s)}_k, K^{(s)}_k)$ $(s = 0, 1, 2)$ which describe the $Z_3$-graded spinorial transformations of matrices $\Gamma^\mu$, we need two Hermitean-conjugate sextets $(\Gamma^\mu_{(a)}, \Gamma^\mu_{(a)} = (\Gamma^\mu_{(a)})^\dagger)$ $(a = 1, 2, \ldots, 6)$ of coloured $12 \times 12$ Dirac matrices. We will also show that the sextet $\Gamma^\mu_{(a)}$ defines three Lorentz doublets needed for the implementation of classical Lorentz covariance.

## 6 Irreducible spinorial representation of $Z_3$-graded Lorentz algebra and colour $\Gamma^\mu$ matrices as its module

### 6.1 Sextet of $\Gamma^\mu$-matrices following from the $Z_3$-graded Lorentz covariance

Let us choose $(J^{(s)}_k, K^{(s)}_k)$ as given by Eqs. (89), (91), and assume that $\Gamma^\mu_{(1)}$ describes the $\Gamma^\mu$-matrix (37) and respectively, $\tilde{\Gamma}^\mu_{(a)}$, its doublet partner (98). By calculating the multicommutators of $(J^{(s)}_k, K^{(s)}_k) \in L^{(1)}$ with the set $\Gamma^\mu_{(a)}$, $(a = 1, 2, \ldots, 6)$, we will show that the following sextet of $\Gamma$-matrices which break the Lorentz covariance is closed under the action of $L^{(1)}$:

\[
\begin{align*}
\Gamma^\mu_{(1)} &= \left( \Gamma^0_{(8;3)}, \Gamma^i_{(2;2)} \right) ; \quad \Gamma^\mu_{(4)} = \left( \Gamma^0_{(8;2)}, \Gamma^i_{(2;3)} \right) ; \\
\Gamma^\mu_{(2)} &= \left( \Gamma^0_{(2;2)}, \Gamma^i_{(4;3)} \right) ; \quad \Gamma^\mu_{(5)} = \left( \Gamma^0_{(2;3)}, \Gamma^i_{(4;2)} \right) ; \\
\Gamma^\mu_{(3)} &= \left( \Gamma^0_{(4;3)}, \Gamma^i_{(8;2)} \right) ; \quad \Gamma^\mu_{(6)} = \left( \Gamma^0_{(4;2)}, \Gamma^i_{(8;3)} \right). 
\end{align*}
\quad (99)
\]

It is easy to see that from the six components of the sextet (99) one can construct as well the set of six $\Gamma^\mu$-matrices $\Gamma^\mu_{(A;\alpha)}$, $A = 2, 4, 8$ and $\alpha = 2, 3$, which can be described as well as three Lorentz doublets (95), with $(A, B) = (2, 8), (2, 4)$ and $(4, 8)$. More explicitly,

\[
\begin{align*}
\Gamma^0_{(A;\alpha)} &= I_A \otimes \sigma_\alpha \otimes \mathbb{1}_2, \quad \Gamma^i_{(B;\beta)} = I_B \otimes (i\sigma_\beta) \otimes \sigma^i. 
\end{align*}
\quad (100)
\]
where \((I_A, I_B = Q_2, Q^I_1, B^l)\) and \(\alpha = 2, 3\). The construction of the \(Z_3\)-graded Lorentz generators \(J^{(r)}_k\), \(K^{(s)}_m\) employed as first tensorial factor the \(3 \times 3\) matrices \(1\) for \(r, s = 0\), \(Q_3\) for \(r, s = 1\) and \(Q^I_3\) for \(r, s = 2\). From the remaining six generators of the \(SU(3)\) Lie algebra in the Kac basis, only three do appear in the sextet \([100]\). In order to implement the full \(SU(3)\) colour symmetry, the remaining matrices \(Q_1, Q^I_2\) and \(B\) should be included in the module on which acts via commutation the spinorial representation of the \(Z_3\)-graded Lorentz algebra. This means that the following sextet should be also taken into consideration, obtained by replacing the matrices \(I_A\) by their complex conjugates, and keeping the remaining tensorial factors unchanged:

\[
\Gamma^{0}_{(A,\alpha)} = \hat{I}_A \otimes \sigma_{\alpha} \otimes \sigma^0, \quad \text{where} \quad \hat{I}_A = (Q_1, Q^I_2, B) \tag{101}
\]

The realization of \(L^{(2)}\) sector acting on colour \(\Gamma^\mu\) matrices is obtained by introducing the Hermitian-conjugate sextet \(\Gamma^{\mu}_{(a)} = (\Gamma^{\mu}_{(a)})^\dagger\)

\[
\Gamma^{\mu}_{(1)} = (\Gamma^{\mu}_{(1)})^\dagger = \left(\Gamma^{0}_{(7;3)}, \Gamma^{i}_{(5;2)}\right); \quad \Gamma^{\mu}_{(4)} = (\Gamma^{\mu}_{(4)})^\dagger = \left(\Gamma^{0}_{(7;2)}, \Gamma^{i}_{(5;3)}\right);
\]

\[
\Gamma^{\mu}_{(2)} = (\Gamma^{\mu}_{(2)})^\dagger = \left(\Gamma^{0}_{(5;2)}, \Gamma^{i}_{(1;3)}\right); \quad \Gamma^{\mu}_{(5)} = (\Gamma^{\mu}_{(5)})^\dagger = \left(\Gamma^{0}_{(5;3)}, \Gamma^{i}_{(1;2)}\right); \tag{102}
\]

\[
\Gamma^{\mu}_{(3)} = (\Gamma^{\mu}_{(3)})^\dagger = \left(\Gamma^{0}_{(1;3)}, \Gamma^{i}_{(7;2)}\right); \quad \Gamma^{\mu}_{(6)} = (\Gamma^{\mu}_{(6)})^\dagger = \left(\Gamma^{0}_{(1;2)}, \Gamma^{i}_{(7;3)}\right),
\]

linearly related with the tilded \(\Gamma\)-matrices \(\hat{\Gamma}^{\mu}_{(a)} = (\hat{\Gamma}^{\mu}_{(a)})^\dagger\) which are required by standard Lorentz covariance described by the grade 0 sector \(L^{(0)}\) (see \([83]\), \([84]\) and \([87]\), \([88]\)).

### 6.2 Lorentz doublets and classical Lorentz symmetry - sector \(L^{(0)}\)

The action of zero-grade rotation generators \(J^{(0)}_i\) on coloured matrices \(\Gamma^\mu\) is described by the eq. \((106)\). In particular, the space rotations leave the temporal component \(\Gamma^{0}_{(A,\alpha)}\) invariant, and transform the space components as the coordinates of a \(D = 3\) three-vector, while the commutators of boosts \(K^{(0)}_i\) with \((\Gamma^{0}_{(A,\alpha)}, \Gamma^{(B,3)}_{(i)})\) generate new \(\Gamma^\mu\)-matrices which permit to introduce the “Lorentz partners”.

Let us start with the first “standard” choice of colour \(\Gamma^\mu\)-matrices (see \([93]\))

\[
\Gamma^{\mu}_{(1)} = \left(\Gamma^{0}_{(8;3)}, \Gamma^{i}_{(2;2)}\right) = \left(B^l \otimes \sigma_3 \otimes 1_2, Q_2 \otimes (i\sigma_2 \otimes \alpha^i)\right). \tag{103}
\]

When iterated, the commutators of boosts \(K^{(0)}_i\) with \(\Gamma^{\mu}_{(1)}\)-matrices yields the following result:

\[
[K^{(0)}_i, \Gamma^{0}_{(8;3)}] = \Gamma^{i}_{(8;2)} = \Gamma^{i}_{(4)}, \quad [K^{(0)}_i, \Gamma^{j}_{(2;2)}] = \delta^j_i \Gamma^{0}_{(2;3)} = \delta^j_i \Gamma^{0}_{(4)}, \tag{104}
\]

\[
[K^{(0)}_i, \Gamma^{0}_{(2;3)}] = \Gamma^{i}_{(2;2)} = \Gamma^{i}_{(1)}, \quad [K^{(0)}_i, \Gamma^{0}_{(8;2)}] = \delta^0_i \Gamma^{0}_{(8;3)} = \delta^0_i \Gamma^{0}_{(1)}. \tag{105}
\]
Apparently, we obtain a classical Lorentz doublet \( \left( \Gamma^\mu_{(1)}, \tilde{\Gamma}^\mu_{(1)} \right) \), where \( \tilde{\Gamma}^\mu_{(1)} = \left( \Gamma^0_{(5)}, \Gamma^3_{(3)} \right) \) (see 99). It appears that in an analogous manner one can introduce classical Lorentz doublets for each colour \( \Gamma^\mu \)-matrix listed in (99) by adding to \( \Gamma^\mu_{(a)} \) the multiplet \( \tilde{\Gamma}^\mu_{(a)} = \left( \Gamma^0_{(b)}, \Gamma^i_{(c)} \right) \), where \( b = a + 4 \mod 6 \) and \( c = a + 2 \mod 6 \).

After the calculation of commutators of the boosts \( K_{i}^{(0)} \) with all \( \Gamma^\mu \)-matrices which appear in (99), the following sextet of Lorentz doublets \( (\Gamma^\mu_{(a)}, \tilde{\Gamma}^\mu_{(a)}) \), \( a = 1, 2, ..., 6 \) is obtained:

\[
\begin{align*}
\Gamma^\mu_{(1)} &= (\Gamma^0_{(8;3)}, \Gamma^i_{(2;2)}) = (\tilde{\Gamma}^0_{(3)}, \tilde{\Gamma}^i_{(5)}), & \tilde{\Gamma}^\mu_{(1)} &= (\Gamma^0_{(2;3)}, \Gamma^i_{(8;2)}) = (\Gamma^0_{(5)}, \Gamma^i_{(3)}); \\
\Gamma^\mu_{(2)} &= (\Gamma^0_{(2;2)}, \Gamma^i_{(4;3)}) = (\tilde{\Gamma}^0_{(4)}, \tilde{\Gamma}^i_{(6)}), & \tilde{\Gamma}^\mu_{(2)} &= (\Gamma^0_{(4;2)}, \Gamma^i_{(2;3)}) = (\Gamma^0_{(6)}, \Gamma^i_{(4)}); \\
\Gamma^\mu_{(3)} &= (\Gamma^0_{(4;3)}, \Gamma^i_{(8;2)}) = (\tilde{\Gamma}^0_{(5)}, \tilde{\Gamma}^i_{(1)}), & \tilde{\Gamma}^\mu_{(3)} &= (\Gamma^0_{(8;3)}, \Gamma^i_{(4;2)}) = (\Gamma^0_{(1)}, \Gamma^i_{(5)}); \\
\Gamma^\mu_{(4)} &= (\Gamma^0_{(8;2)}, \Gamma^i_{(2;3)}) = (\tilde{\Gamma}^0_{(6)}, \tilde{\Gamma}^i_{(2)}), & \tilde{\Gamma}^\mu_{(4)} &= (\Gamma^0_{(2;2)}, \Gamma^i_{(8;3)}) = (\Gamma^0_{(2)}, \Gamma^i_{(6)}); \\
\Gamma^\mu_{(5)} &= (\Gamma^0_{(2;3)}, \Gamma^i_{(4;2)}) = (\tilde{\Gamma}^0_{(1)}, \tilde{\Gamma}^i_{(3)}), & \tilde{\Gamma}^\mu_{(5)} &= (\Gamma^0_{(4;3)}, \Gamma^i_{(2;2)}) = (\Gamma^0_{(3)}, \Gamma^i_{(1)}); \\
\Gamma^\mu_{(6)} &= (\Gamma^0_{(4;2)}, \Gamma^i_{(8;3)}) = (\tilde{\Gamma}^0_{(2)}, \tilde{\Gamma}^i_{(4)}), & \tilde{\Gamma}^\mu_{(6)} &= (\Gamma^0_{(8;2)}, \Gamma^i_{(4;3)}) = (\Gamma^0_{(4)}, \Gamma^i_{(2)}),
\end{align*}
\]

where, we added, for the sake of completeness, the inverse formulae \( \tilde{\Gamma}^\mu_{(a)} \rightarrow \Gamma^\mu_{(a)} \).

### 6.3 Sextet of colour \( \Gamma^\mu \)-matrices and representations of \( Z_3 \)-graded Lorentz algebra - sector \( L^{(1)} = J_j^{(1)} \oplus K^j_{j} \)

Calculating the commutators of matrices \( \Gamma^\mu_{(a)} \), with the generators \( (J_i^{(0)}, K_m^{(0)}) \in L^{(0)} \) was rather easy, because the only non-commuting tensorial factors were the 3 × 3 “colour” matrices, while in the remaining two \( Z_2 \times Z_2 \) factors matrices \( \sigma_i \) commuted with the 2 × 2 unit matrices. However, when we consider the commutators of the operators \( (J_i^{(r)}, K_m^{(s)}) \), \( r, s = 1, 2 \) with two colour Dirac matrices \( \Gamma^\mu_{(1)}, \Gamma^\mu_{(2)} \) defined above, we generate subsequently new commutators we need to calculate.

Let us observe how the new set (99) of \( \Gamma^\mu \)-matrices is produced. Calculating the commutators with the grade 1 generators we use the multiplication rule for tensor products of matrices:

\[
(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d),
\]

with \( a \cdot c \) and \( b \cdot d \) denoting ordinary matrix multiplication. The following formula will be helpful in our calculations:

\[
[a \otimes b, c \otimes d] = a \cdot c \otimes \{b, d\} - [a, c] \otimes \{d, b\}, \quad \text{where} \quad \{b, d\} = b \cdot d + d \cdot b, \quad [a, c] = a \cdot b - b \cdot a.
\]
We recall also the well known identities involving Pauli’s $\sigma$-matrices:
\[ \sigma^i \sigma^j = \delta^{ij} \mathbb{I}_2 + i \epsilon^{ijk} \sigma^k, \quad \\{ \sigma^i, \sigma^k \} = 2 \delta^{ik} \mathbb{I}_2 \] (109)

6.3.1 Grade 1 space rotations: sector $J_i^{(1)}$

Let us start with grade 1 rotations acting on $\Gamma^{\mu}_{(1)} = (\Gamma^0_{(8;3)}, \Gamma^i_{(2;2)})$, forming the $12 \times 12$ matrix valued four-vector \[\{37\}\] appearing in the colour Dirac equation \[\{36\}\]. With the use of the rules of matrix multiplication of tensor products, we arrive at the following sequences of commutators:
\[
\begin{align*}
[J_i^{(1)}, \Gamma^0_{(8;3)}] &= -\frac{\beta}{2} \Gamma^i_{(2;3)} = -\frac{\beta}{2} \Gamma^i_{(4)}; \\
[J_i^{(1)}, \Gamma^k_{(2;3)}] &= -\frac{1}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(4;3)} + \frac{\alpha}{2} \delta^k_i \Gamma^0_{(4;3)} = -\frac{1}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(2)} + \frac{\alpha}{2} \delta^k_i \Gamma^0_{(3)}; \\
[J_i^{(1)}, \Gamma^0_{(4;3)}] &= -\frac{\gamma}{2} \Gamma^i_{(8;3)} = -\frac{\gamma}{2} \Gamma^i_{(6)}; \\
[J_i^{(1)}, \Gamma^k_{(4;3)}] &= -\frac{j^2}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(8;3)} + \frac{j}{2} \delta^k_i \Gamma^0_{(8;3)} = -\frac{j^2}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(6)} + \frac{j}{2} \delta^k_i \Gamma^0_{(5)}; \\
[J_i^{(1)}, \Gamma^0_{(8;3)}] &= -\frac{\alpha}{2} \Gamma^i_{(2;3)} = -\frac{\alpha}{2} \Gamma^i_{(2)},
\end{align*}
\]
where we use the following shortened notation for the coefficients appearing on the right-hand side:
\[\alpha = j - j^2, \quad \beta = j^2 - 1, \quad \gamma = 1 - j, \quad (j = e^{\frac{2\pi i}{3}}).\] (111)

We see that with the relations \[\{110\}\], the six commutators with $J_i^{(1)}$ close on the following 36-component multiplet of obtained from six colour $\Gamma^0_{(a)}$ matrices:
\[\mathcal{C}^+ = \left( \Gamma^0_{(1)}, \Gamma^i_{(2)}, \Gamma^0_{(3)}, \Gamma^i_{(4)}, \Gamma^0_{(5)}, \Gamma^i_{(6)} \right)\] (112)
which describes the following triplet of $\Gamma^\mu$-matrices given by formula \[\{109\}\] with $\alpha = 3$:
\[\mathcal{C}^+ = \left( \Gamma^\mu_{(2;3)}, \Gamma^\mu_{(4;3)}, \Gamma^\mu_{(8;3)} \right)\] (113)

In a short-hand notation the relations \[\{110\}\] look as follows:
\[[J_i^{(1)}, \Gamma_{(2;3)}] \simeq \Gamma_{(4;3)}, \quad [J_i^{(1)}, \Gamma_{(4;3)}] \simeq \Gamma_{(8;3)}, \quad [J_i^{(1)}, \Gamma_{(8;3)}] \simeq \Gamma_{(2;3)},\] (114)

Now let us generate a new sequence of commutators starting from $\Gamma^k_{(2;2)}$:
\[
\begin{align*}
[J_i^{(1)}, \Gamma^k_{(2;2)}] &= -\frac{1}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(4;2)} + \frac{\alpha}{2} \delta^k_i \Gamma^0_{(4;2)} = -\frac{1}{2} \epsilon^i_{\alpha} k^m \Gamma^m_{(5)} + \frac{\alpha}{2} \delta^k_i \Gamma^0_{(6)};
\end{align*}
\]

25
\[
\left[ J_i^{(1)} , \Gamma_0^{(4,2)} \right] = -\frac{\gamma_i}{2} \Gamma_0^{(8,2)} = -\frac{\gamma_i}{2} \Gamma_0^{(3)};
\]
\[
\left[ J_i^{(1)} , \Gamma_k^{(4,2)} \right] = -\frac{j_i^2}{2} \epsilon_i^k m \Gamma_0^{m(8,2)} + \frac{\gamma_i}{2} \delta_i^k \Gamma_0^{(8,2)} = -\frac{1}{2} \epsilon_i^k m \Gamma_0^{m(3)} + \frac{\gamma_i}{2} \delta_i^k \Gamma_0^{(4)};
\]
\[
\left[ J_i^{(1)} , \Gamma_0^{(8,2)} \right] = -\frac{\gamma_i}{2} \Gamma_0^{(2,2)} = -\frac{\gamma_i}{2} \Gamma_0^{(1)};
\]
\[
\left[ J_i^{(1)} , \Gamma_0^{(8,2)} \right] = -\frac{j_i^2}{2} \epsilon_i^k m \Gamma_0^{m(2,2)} + \frac{\beta_i}{2} \delta_i^k \Gamma_0^{0(2,2)} = -\frac{j_i^2}{2} \epsilon_i^k m \Gamma_0^{m(1)} + \frac{\beta_i}{2} \delta_i^k \Gamma_0^{0(2,2)} ;
\]
\[
\left( J_i^{(1)} , \Gamma_0^{(2,2)} \right) = -\frac{\alpha_i^0}{2} \Gamma_0^{(4,2)} = -\frac{\alpha_i^0}{2} \Gamma_0^{(5)} ;
\]

We get the 36-component multiplet \( C^- \) of 12 x 12 dimensional matrices \( \Gamma_{\alpha}^{(4)} \):

\[
C^- = \left( \Gamma_0^{(2)} , \Gamma_1^{(1)} , \Gamma_4^{(4)} , \Gamma_3^{(6)} , \Gamma_5^{(1)} \right) = \left( \Gamma_{\alpha}^{(2,2)} , \Gamma_{\alpha}^{(4,2)} , \Gamma_{\alpha}^{(8,2)} \right) ,
\]

and the counterpart of the relations (114) for the triplet \( C^- \) is obtained by replacing in the corresponding \( \Gamma \)-matrices the indices \( (A:1) \) by \( (A:2) \). The union \( \mathcal{C}^- = (\mathcal{C}_+ \oplus \mathcal{C}^-) \) describes the 72-component sextet \( \Gamma_{\alpha}^{(4)} \) (see 106), and we obtain (116) from (112) by replacing in formula (100) \( \sigma_\alpha = \sigma_3 \) by \( \sigma_\alpha = \sigma_2 \).

6.3.2 Grade 1 boosts generated by \( K_i^{(1)} \)

By analogy with calculation of covariance under the generators \( J_i^{(1)} \) we consider now the closure of actions of the generators \( K_i^{(1)} \) on the multiplets of colour \( \Gamma_{\alpha}^{(4)} \) matrices. If we start with the boost transformations acting on the first standard colour matrices \( \Gamma_{(1)}^{(4)} = (\Gamma_0^{(8,3)} , \Gamma_0^{(2,2)}) \) of the sextet (103), we obtain the closure after the calculation of the following set of 12 commutators:

\[
\left[ K_i^{(1)} , \Gamma_0^{(8,3)} \right] = -\frac{j_i^2}{2} \Gamma_0^{(2,2)} = -\frac{j_i^2}{2} \Gamma_0^{(1)}
\]
\[
\left[ K_i^{(1)} , \Gamma_0^{(4,2)} \right] = \frac{\alpha_i}{2} \epsilon_i^k m \Gamma_0^{m(4,3)} - \frac{1}{2} \delta_i^k \Gamma_0^{0(4,3)} = \frac{\alpha_i}{2} \epsilon_i^k m \Gamma_0^{m(2)} - \frac{1}{2} \delta_i^k \Gamma_0^{0(2)} ;
\]
\[
\left[ K_i^{(1)} , \Gamma_0^{(8,2)} \right] = -\frac{j_i^2}{2} \Gamma_0^{(8,2)} = -\frac{j_i^2}{2} \Gamma_0^{(3)} ;
\]
\[
\left[ K_i^{(1)} , \Gamma_0^{(4,3)} \right] = -\frac{\gamma_i}{2} \epsilon_i^k m \Gamma_0^{m(4,3)} + \frac{j_i^2}{2} \delta_i^k \Gamma_0^{0(4,3)} = -\frac{\gamma_i}{2} \epsilon_i^k m \Gamma_0^{m(3)} + \frac{j_i^2}{2} \delta_i^k \Gamma_0^{0(4)} ;
\]
\[
\left[ K_i^{(1)} , \Gamma_0^{(2,2)} \right] = \frac{\beta_i}{2} \epsilon_i^k m \Gamma_0^{m(2,2)} - \frac{j_i^2}{2} \delta_i^k \Gamma_0^{0(2,2)} = \frac{\beta_i}{2} \epsilon_i^k m \Gamma_0^{m(4)} - \frac{j_i^2}{2} \delta_i^k \Gamma_0^{0(5)} ;
\]
\[
\left[ K_i^{(1)} , \Gamma_0^{(8,2)} \right] = -\frac{1}{2} \Gamma_0^{(4,2)} = -\frac{1}{2} \Gamma_0^{(5)} ;
\]

(117)
\[
\begin{align*}
&\left[K^{(1)}_i, \Gamma_k^{(2;3)}\right] = -\frac{\alpha}{2} \epsilon_i \kappa \Gamma_{m(4;2)} + \frac{1}{2} \delta^k \Gamma^{(4;2)} = -\frac{\alpha}{2} \epsilon_i \kappa \Gamma_{m(5)} + \frac{1}{2} \delta^k \Gamma^0(6) = \\
&\left[K^{(1)}_i, \Gamma_{0(4;2)}\right] = \frac{j^2}{2} \Gamma^{(8;3)} = \frac{j^2}{2} \Gamma_{(6)}
\end{align*}
\]

The relations (117) can be also expressed with a short-hand notation as follows

\[ [K^{(1)}, C^+] \simeq C^-, \quad [K^{(1)}, C^-] \simeq C^+. \quad (118) \]

We see that all 72 components of the sextet (106) are needed in order to obtain the irreducible representation closed under the action of the boost generators \( K^{(1)}_i \).

The pattern of the coefficients appearing on the right-hand side of these 12 commutators bears the imprint of the underlying \( Z_3 \times Z_2 \) symmetry. The six commutators of \( K^{(1)}_i \) with time-like components of \( \Gamma \)-matrices produce only the space-like components, multiplied by halves of all sixth-order roots of unity, i.e. \( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{3}{4} \), while the commutators with spatial components \( \Gamma_{\mu(4;2)} \) contain again the spatial components, multiplied by the coefficients \( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \) and time-like components \( \Gamma_{\mu(8;3)} \) with the coefficients \( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \). The full multiplication table of this Lie algebra over complex roots together with the diagram showing the structure constants on the complex plane are given in the Appendix I.

It is worth to observe that in the definitions (99) of the basic sextet \( \Gamma_{\mu} \) in the colour sector enters only the following triplet of colour generators \((I_2, I_4, I_8) = (Q_2, Q_1, B^1)\) which satisfies the following relations (see also the Table of commutators in Appendix I):

\[ [Q_2, Q_1] = \alpha Q_3, \quad [Q_3, Q_1] = \beta B^1, \quad [Q_3, B^1] = \gamma Q_2. \quad (119) \]

The closure of the action of \( Q_3 \) on the multiplet \((Q_2, Q_1, B^1)\) leads to the covariance of the 72-dimensional multiplet (102) under the action of the generators \((J_{(1)}^{(1)}, K_{(1)}^{1m}) \in L^{(1)}\), which do contain the matrix \( Q_3 \) as their first colour factor (see (111)).

It can be recalled (see (102)) that in order to construct the Lorentz doublets \((\Gamma^{(a)}, \Gamma^{(d)}_a) (a = 1, 2, ..., 6)\) it is sufficient to use the components of the sextet of matrices \( \Gamma_{(a)}^\mu \) suffice (see (106), and again the relations (119) imply the closure of \( \Gamma_{(a)}^\mu \) under the actions of generators belonging to \( L^{(1)} \).
6.4 Representations of $Z_3$-graded Lorentz algebra - sector $L^{(2)} = J^{(2)}_i \oplus K^{(2)}_m$

The matrices of the sextet multiplet $\Gamma^\mu$ are complex and non-Hermitian. Due to the relations

\[
\left( J^{(1)}_i \right)^\dagger = - J^{(2)}_i, \quad \left( K^{(1)}_i \right)^\dagger = K^{(2)}_i, \quad \tag{120}
\]

in order to obtain the closed action of grade 2 generators one should introduce the Hermitean-conjugate sextet $\Gamma^\dagger$ of $\Gamma^\mu$-matrices \[(196)\] as

\[
(\Gamma^\mu)^\dagger = \left( \left( \Gamma^{(1)}_\mu \right)^\dagger, \ldots, \left( \Gamma^{(6)}_\mu \right)^\dagger \right) \quad \tag{121}
\]

One can deduce from the relations \[(110), \quad (117)\] and \[(120)\] the covariant actions of generators from the sector $L^{(2)}$ by using the formulae

\[
\left( \left[ J^{(1)}_i, \Gamma^\mu_{(A,\alpha)} \right] \right)^\dagger = - \left[ \left( J^{(1)}_i \right)^\dagger, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right] = \left[ J^{(2)}_i, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right], \quad \tag{122}
\]

\[
\left( \left[ J^{(2)}_i, \Gamma^\mu_{(A,\alpha)} \right] \right)^\dagger = - \left[ \left( J^{(2)}_i \right)^\dagger, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right] = \left[ J^{(1)}_i, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right], \quad \tag{123}
\]

\[
\left( \left[ K^{(1)}_i, \Gamma^\mu_{(A,\alpha)} \right] \right)^\dagger = - \left[ \left( K^{(1)}_i \right)^\dagger, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right] = \left[ K^{(2)}_i, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right], \quad \tag{124}
\]

\[
\left( \left[ K^{(2)}_i, \Gamma^\mu_{(A,\alpha)} \right] \right)^\dagger = - \left[ \left( K^{(2)}_i \right)^\dagger, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right] = \left[ K^{(1)}_i, \left( \Gamma^\mu_{(A,\alpha)} \right)^\dagger \right] \quad \tag{125}
\]

One obtains in such a way the irreducible actions of sector $L^{(2)}$ on the sextet \[(121)\] of Hermitean-conjugated $\Gamma^\mu$-matrices which describe the grade 2 counterparts of the relations \[(110), \quad (115)\] and \[(117)\]. In the sector $L^{(2)}$ one can introduce as well the Hermitean-conjugate Lorentz doublets \[(196)\] of $\Gamma^\mu$-matrices \[(196)\] with grade 2 generators.

The commutators of the sextet $\Gamma$ (see \[(102)\]) with grade 2 generators $J^{(2)}_i$ of spatial rotations form the same two closed sets of relations for the multiplets \[(112)\] and \[(116)\] (see also formula \[(100)\]), which are analogous to the realizations of the generators $J^{(1)}_i$, with grade 1. More explicitly,

\[
\left[ J^{(2)}_i, \Gamma^{(0)}_{(8;3)} \right] = - \frac{\beta}{2} \tilde{\Gamma}^{(3)}_i = - \frac{\beta}{2} \tilde{\Gamma}^{(6)}_i; \quad \tag{126}
\]

\[
\left[ J^{(2)}_i, \gamma \mid k \right]_{\Gamma^{(2)}} = - \frac{1}{2} \epsilon_{i k} m \Gamma^{(2)}_{(3;3)} - \frac{\alpha}{2} \delta^k_i \Gamma^{(0)}_{(2;3)} = - \frac{1}{2} \epsilon_{i k} m \tilde{\Gamma}^{(2)}_{(2)} - \frac{\alpha}{2} \delta^k_i \tilde{\Gamma}^{(0)}_{(1)}; \quad \tag{126}
\]

\[
\left[ J^{(2)}_i, \Gamma^{(0)}_{(2;3)} \right] = \frac{\gamma}{2} \tilde{\Gamma}^{(3)}_j = \frac{\gamma}{2} \tilde{\Gamma}^{(4)}_j; \quad \tag{126}
\]

\[
\left[ J^{(2)}_i, \gamma \mid k \right]_{\Gamma^{(0)}_{(2;3)}} = - \frac{j}{2} \epsilon_{i k} m \Gamma^{(0)}_{(3;3)} - \frac{\beta}{2} \delta^k_i \Gamma^{(0)}_{(3;3)} = - \frac{j}{2} \epsilon_{i k} m \tilde{\Gamma}^{(0)}_{(4)} - \frac{\beta}{2} \delta^k_i \tilde{\Gamma}^{(0)}_{(3)}; \quad \tag{126}
\]

\[
\left[ J^{(2)}_i, \Gamma^{(0)}_{(4;3)} \right] = - \frac{\alpha}{2} \tilde{\Gamma}^{(2)}_j = - \frac{\alpha}{2} \tilde{\Gamma}^{(2)}_j. \quad \tag{126}
\]
\[ J_{i}^{(2)} \Gamma^{(8;3)} = -\frac{j}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(4;3)} + \frac{\beta}{2} \delta_{i}^{k} \Gamma^{0}_{(4;3)} = -\frac{j}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(6)} + \frac{\beta}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(5)} \]

The next chain of commutators with \( J^{(2)} \) we begin with \( \Gamma^{k}_{(2;2)} \):

\[ J_{i}^{(2)} \Gamma^{k}_{(2;2)} = -\frac{j^{2}}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(8;2)} - \gamma \frac{\beta}{2} \delta_{i}^{k} \Gamma^{0}_{(8;2)} = -\frac{j^{2}}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(1)} - \frac{\gamma}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(1)} \]

\[ J^{(2)} \Gamma^{0}_{(8;2)} = \beta \frac{j}{2} \Gamma^{0}_{(4;2)} = \beta \tilde{\Gamma}^{0}_{(3)} \]

\[ J_{i}^{(2)} \Gamma^{k}_{(8;2)} = -\frac{j}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(4;2)} - \frac{\beta}{2} \delta_{i}^{k} \Gamma^{0}_{(4;2)} = -\frac{j}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(3)} - \frac{\beta}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(3)} \]

\[ J_{i}^{(2)} \Gamma^{0}_{(4;2)} = -\frac{\alpha}{2} \Gamma^{0}_{(2;2)} = -\frac{\alpha}{2} \Gamma^{0}_{(5)} \]  \hspace{1cm} (127)

\[ J_{i}^{(2)} \Gamma^{k}_{(4;2)} = -\frac{1}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(2;2)} + \frac{\alpha}{2} \delta_{i}^{k} \Gamma^{0}_{(2;2)} = -\frac{1}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(5)} + \frac{\alpha}{2} \delta_{i}^{k} \Gamma^{0}_{(4)} \]

\[ J_{i}^{(2)} \Gamma^{0}_{(2;2)} = \gamma \frac{j}{2} \Gamma^{0}_{(8;2)} = \gamma \tilde{\Gamma}^{0}_{(1)} \]

Similar series of commutators, starting with the “basic” colour Dirac matrix \( \Gamma^{0}_{(1)} \) and then continuing to closed commutators structure is produced by the following actions of boosts from the grade 2 Lorentz sector \( L^{(2)} \):

\[ K_{i}^{(2)} \Gamma^{0}_{(8;3)} = -\frac{j}{2} \Gamma^{0}_{(4;2)} = \frac{j}{2} \tilde{\Gamma}^{0}_{(3)} \]

\[ K_{i}^{(2)} \Gamma^{k}_{(2;2)} = -\gamma \frac{\beta}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(8;3)} - \frac{\beta}{2} \delta_{i}^{k} \Gamma^{0}_{(8;3)} = -\gamma \frac{\beta}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(4)} - \frac{\beta}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(3)} \]

\[ K_{i}^{(2)} \Gamma^{0}_{(2;2)} = \frac{j^{2}}{2} \Gamma^{0}_{(8;3)} = \frac{j^{2}}{2} \Gamma^{0}_{(4)} \]

\[ K_{i}^{(2)} \Gamma^{0}_{(4;2)} = -\frac{j}{2} \Gamma^{0}_{(8;2)} = -\frac{j}{2} \Gamma^{0}_{(1)} \]

\[ K_{i}^{(2)} \Gamma^{k}_{(2;3)} = -\frac{\alpha}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(2;3)} - \frac{1}{2} \delta_{i}^{k} \Gamma^{0}_{(2;3)} = -\frac{\alpha}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(2)} - \frac{1}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(1)} \]  \hspace{1cm} (128)

\[ K_{i}^{(2)} \Gamma^{0}_{(4;2)} = \frac{1}{2} \Gamma^{0}_{(2;3)} = \frac{1}{2} \Gamma^{0}_{(2)} \]

\[ K_{i}^{(2)} \Gamma^{k}_{(2;3)} = \gamma \frac{j}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(8;2)} + \frac{j}{2} \delta_{i}^{k} \Gamma^{0}_{(8;2)} = \gamma \frac{j}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(1)} + \frac{j}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(6)} \]

\[ K_{i}^{(2)} \Gamma^{0}_{(8;3)} = \frac{j}{2} \Gamma^{0}_{(8;3)} = \frac{j}{2} \Gamma^{0}_{(8)} \]

\[ K_{i}^{(2)} \Gamma^{0}_{(8;2)} = -\beta \frac{j}{2} \epsilon_{i}^{k} m \Gamma^{n}_{(4;3)} - \frac{j}{2} \delta_{i}^{k} \Gamma^{0}_{(4;3)} = -\beta \frac{j}{2} \epsilon_{i}^{k} m \tilde{\Gamma}^{n}_{(6)} - \frac{j}{2} \delta_{i}^{k} \tilde{\Gamma}^{0}_{(5)} \]
\[
\left[ K_i^{(2)}, \Gamma^{(4;3)}_0 \right] = -\frac{1}{2} \Gamma^{(2;2)} = -\frac{1}{2} \Gamma^{(5)}_i
\]

\[
\left[ K_i^{(2)}, \Gamma^{k(m)}_0 \right] = \frac{\alpha}{2} \epsilon^k \epsilon^{lm} \Gamma^{lm}_0 + \frac{1}{2} \delta^k_i \delta^{lm} \Gamma^{lm}_0 + \frac{1}{2} \delta^l_i \Gamma^{lm}_0
\]

Because from (119) follows the closure of the triplet

\[(I_1, I_5, I_7) \equiv (I_4^\dagger, I_2^\dagger, I_8^\dagger) = (Q_1, Q_2^\dagger, B),\]

under the action of \(Q_3^\dagger\) (compare with (6)) one can as well reproduce the covariant action of \(L^{(2)}\) on the Hermitian-conjugate doublets of \(\Gamma^\mu\)-matrices.

The general pattern of commutators in (126)-(128) better explains why the irreducible representation is described by the sextet of colour \(\Gamma^\mu\) matrices. The generators \(J_i^{(1)}, J_i^{(2)}, K^m, K^{(2)}_m\) contain as their \(3 \times 3\) matrix factors the elements \(Q_3\) and \(Q_3^\dagger\), which therefore cannot appear in the coloured \(\Gamma^\mu\)-matrices; the boosts contain as their second factor the matrix \(\sigma_1\), which as well cannot appear in the sextet (99). Starting from the first “standard” colour Dirac operator whose \(\Gamma\)-matrices contain \(B^\dagger\) and \(Q_2\), commutators with \(Q_3\) and \(Q_3^\dagger\) can generate in the colour sector only the third colour matrix \(Q_1^\dagger\), besides \(B^\dagger\) and \(Q_2\). This reduces the number of \(\Gamma^\mu\) matrices spanning the spinorial realization of the \(\mathbb{Z}_3\)-graded Lorentz algebra to six, characterized by three colour matrices \(I_A\) \((A = 2, 4, 8)\) and two Pauli matrices \(\sigma_\alpha\) \((\alpha = 2, 3)\).

If we start with complex conjugate Dirac operator (36), (37) with \(\Gamma^0 = B \otimes \sigma_3 \otimes I_2\) and \(\Gamma^i = Q_1 \otimes (i\sigma_2) \otimes \sigma_i\) (note that \(B\) is the complex conjugate of \(B^\dagger\) and \(Q_1\) is the complex conjugate of \(Q_2\)), we get the alternative sextet describing coloured Dirac equations for the complex-conjugated fields \(\bar{\Psi}\) (see (40)), which contains as its colour factors the matrices \(Q_2^\dagger, Q_1\) and \(B\).

7 Irreducible realizations of the \(\mathbb{Z}_3\)-graded Lorentz algebra and the full set of quark symmetries

7.1 Chiral colour doublets and flavour states

The flavour quark eigenstates in the Standard Model are represented by chirally projected Dirac spinors (see e.g. [25], [26]). If we introduce the \(D = 3+1\) Clifford algebra defined by the relations

\[
\left\{ \gamma^\mu, \gamma^\nu \right\} = 2 \eta^{\mu\nu} I_4, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1),
\]

and define

\[
\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (\gamma^5)^2 = -I_4
\]

then the standard chiral Dirac spinors are defined as

\[
\psi_\pm = \pm i \gamma^5 \psi_\pm = \frac{1}{2} (1 \pm i \gamma^5) \psi.
\]
The ψ± denote the four-component Dirac spinors satisfying the chirality conditions \( P_\pm \psi_\pm = \psi_\pm \), \( P_\mp \psi_\pm = 0 \), where \( P_\pm = \frac{1}{2} (1 \pm i \gamma^5) \) are the chiral projection operators.

For more clarity we shall use the following realization of Clifford algebra of Dirac matrices in terms of tensor products of 2 × 2 matrices:

\[
\gamma^0 = \sigma_3 \otimes 1_2, \quad \gamma^i = (i \sigma_2) \otimes \sigma^i, \quad \gamma^5 = -i \sigma_1 \otimes 1_2.
\]  

(132)

This enables us to express the colour Dirac 12 × 12-matrices in a more concise manner:

\[
\Gamma^0_{(1)} = \Gamma^0_{(8,3)} = B^\dagger \otimes \gamma^0, \quad \Gamma^i_{(1)} = \Gamma^0_{(2,2)} = Q_2 \otimes \gamma^i, \\
\tilde{\Gamma}^0_{(1)} = \Gamma^0_{(2,3)} = Q_2 \otimes \gamma^0, \quad \tilde{\Gamma}^i_{(1)} = \Gamma^0_{(8,2)} = B^\dagger \otimes \gamma^i.
\]  

(133)

The chiral projection operator acting on the matrices \( \Gamma_\mu^{(1)} \) can be now defined as follows:

\[
P_\pm = 1_3 \otimes \frac{1}{2} (1_2 \pm \sigma_1) \otimes 1_2,
\]

(134)

so that the chirally projected matrices \( \Gamma_\mu^{(1)\pm} = P_\pm \Gamma_\mu^{(1)} \) look as follows:

\[
P_\pm \Gamma^0_{(1)} = \Gamma^0_{(1)\pm} = \left( B^\dagger \otimes \frac{1}{2} (\sigma_3 \mp i \sigma_2) \otimes 1_2, \quad Q_2 \otimes \frac{1}{2} (i \sigma_2 \mp \sigma_3) \otimes \sigma^i \right) = \left( \frac{1}{2} (\Gamma^0_{(1)} \mp i \Gamma^0_{(4)}), \quad \frac{1}{2} (\Gamma^i_{(1)} \pm \Gamma^i_{(4)}) \right)
\]

(135)

By adding the relations

\[
P_\pm \Gamma^0_{(2)} = \Gamma^0_{(2)\pm} = \left( Q_2 \otimes \frac{1}{2} (\sigma_2 \pm \sigma_3) \otimes 1_2, \quad Q^\dagger_1 \otimes \frac{1}{2} (\sigma_3 \mp i \sigma_2) \otimes \sigma^i \right) = \left( \frac{1}{2} (\Gamma^0_{(2)} \pm \Gamma^0_{(5)}), \quad \frac{1}{2} (\Gamma^i_{(2)} \pm \Gamma^i_{(5)}) \right)
\]

(136)

\[
P_\pm \Gamma^0_{(3)} = \Gamma^0_{(3)\pm} = \left( Q^\dagger_1 \otimes \frac{1}{2} (\sigma_3 \mp \sigma_2) \otimes 1_2, \quad B^\dagger \otimes \frac{1}{2} (i \sigma_2 \mp \sigma_3) \otimes \sigma^i \right) = \left( \frac{1}{2} (\Gamma^0_{(3)} \pm \Gamma^0_{(6)}), \quad \frac{1}{2} (\Gamma^i_{(1)} \pm \Gamma^i_{(4)}) \right)
\]

(137)

one introduces chiral/anti-chiral triplets \( \Gamma_\mu^{(r)} \) (\( r = 1, 2, 3 \)) of colour \( \Gamma \)-matrices defined in terms of three pairs \( (\Gamma_\mu^{(1)}, \Gamma_\mu^{(4)}), (\Gamma_\mu^{(2)}, \Gamma_\mu^{(5)}), (\Gamma_\mu^{(3)}, \Gamma_\mu^{(6)}) \), or six matrices \( \Gamma_\mu^{(A,\alpha)} \) defined by formula (100). It is interesting to note that one can construct similar projection operators related with \( Z_3 \)-graded colour sectors. Let us define three projection operators:

\[
\Pi^{(0)} = \frac{1}{3} (1_4 + B + B^\dagger) \otimes 1_4,
\]

(138)
\begin{equation}
(1) \quad \Pi = \frac{1}{3}(\mathbb{1}_3 + j^2 B + j B^\dagger) \otimes \mathbb{1}_4,
\quad (2) \quad \Pi = \frac{1}{3}(\mathbb{1}_3 + j B + j^2 B^\dagger) \otimes \mathbb{1}_4.
\end{equation}

One checks easily that the three projectors (138, 139) satisfy the expected relations
\begin{equation}
[\Pi]_r^2 = \Pi, \quad r = 0, 1, 2, \quad \Pi \Pi = 0, \quad \text{for } r \neq s, \quad (0) \Pi + (1) \Pi + (2) \Pi = \mathbb{1}_3,
\end{equation}
i.e. we obtain a \(Z_3\)-graded generalization of the \(Z_2\)-graded standard chiral projectors \(P_{\pm}\), where \(P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+ P_- = P_- P_+ = 0\) and \(P_+ + P_- = \mathbb{1}\).

## 7.2 The flavour and generations in quark models and chiral colour Dirac multiplets

Three generations of quarks (called also "three families") are known, each formed by a flavour ("weak isospin") doublet:
- \((u, d)\), or the "up - down" doublet ("First generation");
- \((s, c)\), or the "strange - charm" doublet ("Second generation");
- \((t, b)\), or the "top - bottom" doublet ("Third generation");

The flavour \(SU(2)\) symmetry is visible only if we consider chiral (left-handed) quarks, described by the doublet \((u, d)\). In the case of \(2^{\text{nd}}\) and \(3^{\text{rd}}\) generation, possible flavour symmetries exchanging \(c\) with \(s\) or \(t\) with \(b\) are strongly violated and the internal symmetry is used mostly for the classification purposes.

Historically, for the classification purposes, the flavours were firstly ordered into the \(SU(3)\) \((u, d, s)\) multiplet however at present we know that dynamically the coset \(SU(3)/U(2)\) is badly broken.

Having introduced chiral and anti-chiral colour Dirac matrices, we can define respective 12-component chiral colour Dirac spinors and corresponding chiral colour Dirac equations (see (22)). The chiral and anti-chiral states can be formed by pairs of quarks \((s, c)\) and \((t, b)\) of other generations, i.e. one can use the same scheme for the doublets in each of three generations.

The chiral structure of the flavour sector of the Standard Model becomes important when we consider together leptons and quarks, with leptons interacting weakly as a kind of fourth colour (see e.g. (27)). In our model one can introduce leptons as colourless quarks just by replacing the \(3 \times 2\) matriceses \(B\)
and $Q_a$ appearing in the tensor products defining the generalized $12 \times 12$ Dirac matrices by the $3 \times 3$ unit matrix.

In the Standard Model the fact that leptons and coloured quarks are coupled weakly in analogous way leads to an important feature of the chiral anomaly cancellation. We hope that in the next stage of development of our model with interaction vertices introduced, such a cancellation mechanism can be also naturally achieved.

8 Outlook

The Standard Model (SM) of elementary particles is without doubt very successful experimentally tested part of theoretical physics; however, its group-theoretical structure still requires further investigations. The internal symmetries are the product of three unitary groups $SU(3) \times SU(2) \times U(1)$, with chiral $SU(2)$ sector describing weak interactions and basic role of colour $SU(3)$ group describing strongly interacting gluons and quarks which are not observable as free asymptotic states. The full spectrum of quarks requires still another $SU(3)$ symmetry due to appearance of quark generations, which should be interpreted with the help of an additional geometric structure, describing from group-theoretical point of view the full set of all quarks as given by irreducible 72-dimensional representations of a new group which intertwines Lorentz and colour symmetries.

Unifying efforts in the literature ((see e.g. [25], [28], [30], [31], [32]) went along various paths, with two basic ways of unification: the first preserves the tensor product structure of space-time and internal symmetries, while the other one is more radical, intertwining the relativistic and internal colour symmetries. A well-known example of second type of unification scheme is provided by the known passage from bosonic symmetries to supersymmetries, which describe the supermultiplet containing commuting bosons and anti-commuting fermions both incorporated in one common $Z_2$-graded algebraic structure.

In our paper we deal exclusively with fundamental quark degrees of freedom, described by the collection of anti-commuting fields, with triplets of quark states with three different colours, which permit the introduction of $Z_3$-graded algebraic structure. The $Z_3$-grading does not change the fermionic statistics of quarks, but leads to particular link between relativistic (Lorentz) symmetry and internal (colour) symmetries, which in colour Dirac equations cease to be described by a tensor product group structure. The colour generators are naturally expressed in a $Z_3$-graded ternary basis (see Sect. 2) and the $SU(3)$ colour symmetries provide all possible $3 \times 3$ matrix choices of ternary basis (see the end of Sect. 4). It is the $Z_3$-covariance which in a field-theoretic description of quarks provides the passage from the three copies of 4-component standard Dirac fields to the 12-dimensional colour Dirac field (see Sect. 3).

The physical observation that the quarks are described by six colour triplets did lead us to the idea that one should look for an algebraic scheme which would provide a unifying 72-dimensional module incorporating all six colour triplets...
of fermions inside a single irreducible representation. We demonstrate in this paper how this goal can be achieved by introduction of the $Z_3$-graded Lorentz symmetries, which can be extended to $Z_3$-graded Poincaré group. We consider the vectorial representation of the $Z_3$-graded Poincaré algebra in Sect. 4, and the spinorial representation of the $Z_3$-graded Lorentz algebra in Sect. 5 and 6.

Usually the efforts to incorporate all existing quarks in a unique irreducible multiplet are restricted only to the discussion of internal degrees of freedom. Our approach, which leads to a $Z_3$-graded extension of standard relativistic symmetries, implies as well the modification of quark dynamics, what can be seen already from the wave equations for free quarks which imply dispersion relations of the sixth order, satisfied by all the components of the sextet of free $Z_3$-graded quark fields. The dispersion relations can be described as a triple product of mass shells, one with real mass and a pair of mutually conjugate complex ones, which lead to the appearance of complex wave vectors (see Sect. 3 and 4) and provide damped exponential solutions along with freely propagating waves.

Our model is formulated only on the preliminary kinematic level, without defining neither the Lagrangean, nor the interaction vertices. Still on the kinematic level it is important to describe besides quarks, also the leptons and gauge fields. Before we proceed further we observe that:

i) One should complete the quantum field-theoretic description of free quantum $Z_3$-graded fermionic quark fields, with algebra of field oscillators and Green function. In this paper we provided only the formulae for the quark propagators (see (47), (50), (51)). These formulae follow from the decomposition of basic quantum quark field satisfying sixth order equation into three Klein-Gordon like fields with $Z_3$-graded set of complex masses $m_s = j^s m$, ($s = 0, 1, 2$). Further one should consider the $Z_3$-covariant set of quantized free Klein-Gordon fields with respective oscillator algebras describing the field quanta that lead to three residua $(1, j, j^2)$ of three propagators in eq. (50). These residua describe three respective metrics in the Hilbert-Fock spaces associated with our three Klein-Gordon type free quantum fields.

ii) One should provide the $Z_3$-covariant interaction vertices, in particular the prescriptions for gauge field couplings. In order to obtain such results we should study the role of $Z_3$-graded Lorentz transformations in space-time (Sect. 3 and 4 were restricted only to the four-momentum space).

In order to be able to construct the action density and covariantize the space-time derivatives by introducing gauge fields we should find out how the $Z_3$-graded Lorentz and Poincaré generators act on the space-time coordinates. This is going to be the subject of our future research.
Appendix I

Here we give the multiplication table of the Lie algebra spanned by 8 generators: the six off-diagonal Q-matrices \((Q_a, Q_b^\dagger)\), \((a, b = 1, 2, 3)\) and the pair of diagonal matrices \((B, B^\dagger)\). The entries correspond to the ordinary commutators \([A, B] = AB - BA\).

The overall pattern becomes clearly visible if we express all complex coefficients appearing in the table of commutators in terms of three Greek letters. Let us introduce the following notation:

\[
\begin{align*}
    j - j^2 &= \sqrt{3} e^{i\frac{\pi}{2}} = \alpha, \\
    j^2 - 1 &= \sqrt{3} e^{i\frac{2\pi}{6}} = \beta, \\
    1 - j &= \sqrt{3} e^{-i\frac{\pi}{6}} = \gamma.
\end{align*}
\]

(141)

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
    & Q_1 & Q_2 & Q_3 & Q_1^\dagger & Q_2^\dagger & Q_3^\dagger & B & B^\dagger \\
\hline
Q_1 & 0 & -\alpha Q_3^\dagger & \alpha Q_2^\dagger & 0 & \beta B^\dagger & -\gamma B & -\beta Q_2 & \gamma Q_3 \\
\hline
Q_2 & \alpha Q_3^\dagger & 0 & -\alpha Q_1^\dagger & -\gamma B & 0 & \beta B^\dagger & -\beta Q_3 & \gamma Q_1 \\
\hline
Q_3 & -\alpha Q_2^\dagger & \alpha Q_1^\dagger & 0 & \beta B^\dagger & -\gamma B & 0 & -\beta Q_1 & \gamma Q_2 \\
\hline
Q_1^\dagger & 0 & \gamma B & -\beta B^\dagger & 0 & -\alpha Q_3 & \alpha Q_2 & \beta Q_3^\dagger & -\gamma Q_2^\dagger \\
\hline
Q_2^\dagger & -\beta B^\dagger & 0 & \gamma B & \alpha Q_3 & 0 & -\alpha Q_1 & \beta Q_1^\dagger & -\gamma Q_1^\dagger \\
\hline
Q_3^\dagger & \gamma B & -\beta B^\dagger & 0 & -\alpha Q_2 & \alpha Q_1 & 0 & \beta Q_2^\dagger & -\gamma Q_1^\dagger \\
\hline
B & \beta Q_2 & \beta Q_3 & \beta Q_1 & -\beta Q_3^\dagger & -\beta Q_1^\dagger & -\beta Q_2^\dagger & 0 & 0 \\
\hline
B^\dagger & -\gamma Q_3 & -\gamma Q_1 & -\gamma Q_2 & \gamma Q_2^\dagger & \gamma Q_1^\dagger & \gamma Q_3^\dagger & 0 & 0 \\
\hline
\end{array}
\]

Table 2. The commutators between eight \(3 \times 3\) generators of \(SU(3)\) algebra in Kac’s basis, with the coefficients \(\alpha, \beta, \gamma\) given by (141).

The multiplication table is obviously anti-symmetric, and all complex coefficients have the same absolute value \(\sqrt{3}\). If we renormalize the generators dividing every one by \(\sqrt{3}\), the new generators \(\tilde{Q}_a = Q_a/\sqrt{3}\) would satisfy the same commutator algebra with complex renormalized structure constants \(\tilde{\alpha} = \alpha/\sqrt{3}\), etc., with their moduli equal to 1.

The Lie algebra defined by this table is semi-simple, what can be seen from the property that each row and each column contains all six different coefficients, but each of them appearing only once.

Two \(3 \times 3\) blocks containing brackets between the generators \(Q_a\) or \(Q_a^\dagger\) display the same set of coefficients, equal to \(\pm \alpha\), while the Cartan subalgebra generators \(B\) and \(B^\dagger\) commute (see in Table 2 the last \(2 \times 2\) matrix block filled with zeros). The commutators of \(B\) and \(B^\dagger\) with \(Q_a\)’s or \(Q_a^\dagger\)’s result in raising the index \(a\) (\(1 \to 2, 2 \to 3, 3 \to 1\)) or lowering the index \(b\) (\(3 \to 2, 2 \to 1, 1 \to 3\)).
The structure of the $SU(3)$ Lie algebra is now clearly visible. It is illustrated by the root diagram, displaying the third roots of unity ($1, j, j^2$) and third roots of $-1$ ($-1, -j, -j^2$), as well as the roots ($\pm \alpha, \pm \beta, \pm \gamma$).

Figure 2: Left: The diagram of complex coefficients $\pm \alpha, \pm \beta, \pm \gamma; |\alpha| = |\beta| = |\gamma| = \sqrt{3}$; Right: the sixth roots of unity

Appendix II

Six traceless off-diagonal Gell-Mann matrices:

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (142)
$$

$$
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (143)
$$

supplemented by two traceless diagonal matrices spanning the Cartan subalgebra of $SU(3)$ Lie algebra

$$
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (144)
$$

The mapping between the Cartan subalgebras, $B$ and $B^\dagger$ on one side and $\lambda_3$ and $\lambda_8$ on the other side, is given by the following linear combinations:

$$
\frac{1}{j-1} B + \frac{1}{j^2-1} B^\dagger = \lambda_3, \quad -\frac{j}{\sqrt{3}} B - \frac{j^2}{\sqrt{3}} B^\dagger = \lambda_8, \quad (145)
$$

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or more explicitly,
\[
\frac{1}{j - 1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} + \frac{1}{j^2 - 1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda_3, \quad (146)
\]

\[
-\frac{j}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} - \frac{j^2}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \lambda_8. \quad (147)
\]

The six Gell-Mann matrices [142], [143] can be expressed as linear combinations of ternary Clifford algebra generators \(Q_a, Q_b^\dagger\) as follows:

\[
\lambda_1 = \frac{1}{3} \left( Q_1 + Q_2 + Q_3 + Q_1^\dagger + Q_2^\dagger + Q_3^\dagger \right),
\]
\[
\lambda_2 = \frac{i}{3} \left( Q_1 + Q_2 + Q_3 - Q_1^\dagger - Q_2^\dagger - Q_3^\dagger \right),
\]
\[
\lambda_4 = \frac{1}{3} \left( jQ_1 + j^2Q_2 + Q_3 + j^2Q_1^\dagger + jQ_2^\dagger + Q_3^\dagger \right),
\]
\[
\lambda_5 = \frac{i}{3} \left( jQ_1 + j^2Q_2 + Q_3 - j^2Q_1^\dagger - jQ_2^\dagger - Q_3^\dagger \right),
\]
\[
\lambda_6 = \frac{1}{3} \left( j^2Q_1 + jQ_2 + Q_3 + jQ_1^\dagger + j^2Q_2^\dagger + Q_3^\dagger \right),
\]
\[
\lambda_7 = \frac{i}{3} \left( j^2Q_1 + jQ_2 + Q_3 - j^2Q_1^\dagger - jQ_2^\dagger - Q_3^\dagger \right),
\]

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