Numerical experiments of flux difference splitting methods with high resolution scheme for supersonic flows

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Abstract. In this work, we have carried out the assessment of a high resolution scheme for unsteady compressible flow. For high order spatial accuracy, we have used fifth order weighted essentially non oscillatory (WENO) scheme. This scheme is applied to four flux difference splitting (FDS) methods: Harten-Lax-van Leer (HLL), Roe solver, Harten-Lax-van Leer-Contact (HLLC), and Rusanov methods. We have compared results of these flux schemes with each other. WENO scheme is used for the reconstruction of left and right state variable across the cell interface for high resolution. The reconstruction procedure is performed in terms of primitive variables instead of conservative variable, in order to avoid spurious oscillation. We have considered two test cases: shock wave reflection and supersonic viscous flow over a flat plate, to access the performance of FDS schemes. An explicit third order TVD Runge-Kutta method is used for advancement of solution in time. The present results are compared with available numerical solutions. WENO-HLLC has good shock capturing capabilities as compare to WENO-Roe, WENO-HLL and WENO-Rusanov methods. It also provides best results inside and outside the boundary layer.

1. Introduction
High speed compressible flows exhibit different type of flow phenomena, for example reflection of shock wave, shock-shock and shock boundary layer interaction. Strong shock waves and density are the major parameters for the selection of appropriate numerical method. Among different kind of shock capturing method, upwind method have gained more popularity [1]. In order to indentify upwind direction, two approaches are well known i.e flux difference splitting (FDS) [2] and flux vector splitting (FVS) [3]. FDS schemes capture shocks with better accuracy than flux vector splitting [4–6]. Further FVS schemes result in poorer resolution of discontinuities as compared to FDS, particularly in stationary contact and shear waves.

Among different FDS schemes, we have considered Harten-Lax-van Leer (HLL) [7], Roe solver [8], Harten-Lax-van Leer-Contact (HLLC) [9] and Rusanov method [10] for our study. HLL is two wave configuration type solver, separating three constant states. Roe solver utilises average eigenvalues, average eigenvectors and wave strengths in order to calculate numerical flux. HLLC (C for contact discontinuity) is three wave configuration model, incorporating two additional intermediate star states. Rusanov method is the one wave configuration approximate Riemann solver.

FDS schemes are based on upwind approach, which generally has first order accuracy. For high order spacial accuracy, high resolution schemes [11–13] are the most appropriate choice in the study of compressible flows [2]. In our study we have used fifth order weighted essentially non oscillatory
(WENO) scheme [14–16] for solving hyperbolic equations. It utilises the idea of adaptive stencils for the reconstruction procedure of variable. We considered reconstruction of primitive variables instead of conservative variable, as it avoids spurious oscillation near the discontinuity and maintains uniform high order accuracy in smooth region. Reconstruction procedure produces two different values, called left and right state extrapolated values of conservative variable at each cell interface. Numerical flux is calculated by a monotone function of these two extrapolated values.

HLL, Roe, HLLC and Rusanov method with fifth order WENO scheme have been termed as WENO-HLL, WENO-Roe, WENO-HLLC and WENO Rusanov method. For the advancement of solution in time, we have used third order accurate TVD Runge-Kutta method. We have carried out simulation for two cases: shock wave reflection from a flat plate and supersonic viscous flow over a flat plate. For these problems, we have investigated the robustness of each method inside a boundary layer.

2. General Framework in Two Dimensional Space

Navier-Stokes equation in conservative form for unsteady compressible viscous flow is written as

\[ \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x_1} + \frac{\partial \mathbf{G}}{\partial x_2} = \left( \frac{\partial \tilde{\mathbf{F}}}{\partial x_1} + \frac{\partial \tilde{\mathbf{G}}}{\partial x_2} \right) \]

in which

\[ \mathbf{U} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ u_1 (E + p) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho u_2 \\ \rho u_2 u_1 \\ \rho u_2^2 + p \\ u_2 (E + p) \end{bmatrix} \]

\[ \tilde{\mathbf{F}} = \begin{bmatrix} 0 \\ \sigma_{11} \\ \sigma_{12} \\ u_1 \sigma_{11} + u_2 \sigma_{12} - q_1 \end{bmatrix}, \quad \tilde{\mathbf{G}} = \begin{bmatrix} 0 \\ \sigma_{12} \\ \sigma_{22} \\ u_1 \sigma_{12} + u_2 \sigma_{22} - q_2 \end{bmatrix} \]

Assuming Newtonian fluid, viscous stress, \( \sigma_{ij} \), can be expressed as

\[ \sigma_{ij} = 2\mu \left( S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right) \]

where \( S_{ij} \) is the strain rate tensor, \( \mathbf{U} \) is the conserved variables, \( \mathbf{F} \) and \( \mathbf{G} \) are the inviscid flux, \( \tilde{\mathbf{F}} \) and \( \tilde{\mathbf{G}} \) are the viscous flux in the \( x_1 \) and \( x_2 \) direction [17]. We have used Sutherland’s empirical equation for the calculation of dynamic viscosity, \( \mu \), given by

\[ \mu = (T/T_\infty)^{3/2} \left( \frac{T_\infty + C_o}{T + C_o} \right) \]

where \( T_\infty \) is free stream temperature and \( C_o = 110 \, K \).

We have used finite difference approach to discretise the equations on a uniform Cartesian grid. The rectangular domain is divided into grid points (using co-located grid approach) denoted by the subscript \( i, j \). Semi-discrete form of equation (1) is given as

\[ \frac{d\mathbf{U}_{ij}}{dt} = - \left( \frac{\Delta \mathbf{F}_{ij} - \Delta \tilde{\mathbf{F}}_{ij}}{\Delta x_1} + \frac{\Delta \mathbf{G}_{ij} - \Delta \tilde{\mathbf{G}}_{ij}}{\Delta x_2} \right) = \mathbf{L}_{ij}(\mathbf{U}) \]

where \( \Delta \mathbf{F}_{ij} = \mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}, \Delta \tilde{\mathbf{F}}_{ij} = \tilde{\mathbf{F}}_{i+1/2,j} - \tilde{\mathbf{F}}_{i-1/2,j}, \Delta \mathbf{G}_{ij} = \mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}, \Delta \tilde{\mathbf{G}}_{ij} = \tilde{\mathbf{G}}_{i,j+1/2} - \tilde{\mathbf{G}}_{i,j-1/2}. \mathbf{U}_{ij} \) is the conserved variable at the \( (i,j) \)th grid node, \( \Delta x_i \) is the grid spacing in \( x_i \).
direction, \( F_i \) are the inviscid flux vectors at the left and right cell boundaries, and \( G_i \) are the inviscid flux vectors at the bottom and top cell boundaries. We have used central-difference scheme to discretise the viscous flux vectors \((\tilde{F}_i, \tilde{G}_i)\) at the cell boundaries. In this study, we have adopted third order accurate TVD Runge-Kutta method [18] for integration of equation (5) which consists of following steps

\[
U^{n+1}_{ij} = U^n_{ij} - \Delta t \frac{\partial}{\partial x} (F_{i+1/2} - F_{i-1/2})
\]

for describing properties of different flux schemes.

3.1. HLL method

HLL Riemann solver defines the conservative variable \( U \) as

\[
U = \begin{cases} 
U^l & C^l > 0, \\
U_* & C^l \leq 0 \leq C^r, \\
U^r & C^r < 0,
\end{cases}
\]

where intermediate stage \( U_* \) is

\[
U_* = \frac{C^r U^r - C^l U^l - (F^r - F^l)}{C^r - C^l},
\]

and the corresponding flux is given by

\[
F_* = \frac{C^r F^l - C^l F^r - C^l C^r (U^r - U^l)}{C^r - C^l}.
\]

Intercell flux is as follows:

\[
F = \begin{cases} 
F^l & C^l > 0, \\
F_* & C^l \leq 0 \leq C^r, \\
F^r & C^r < 0,
\end{cases}
\]

In the preceding equations \( C^l \) and \( C^r \) are the left and right wave speeds calculated by adaptive Riemann solver [19].

3.2. Roe’s method

In this method, the inviscid flux is discretised as

\[
F = \frac{1}{2} (F^l - F^r) - \frac{1}{2} \sum_{i=1}^{4} \tilde{\alpha}_i |\tilde{\lambda}_i| \tilde{p}^{(i)},
\]
where $\alpha_i$ is wave strength, $\lambda_i$ is eigenvalue and $\vec{p}^{(i)}$ is the right eigenvector (see [20] for further definition of these parameters).

3.3. HLLC method
This method [9] is an advancement of the HLL method. In this method, the conservative variable is separated by two intermediate stages, $U^*_l$ and $U^*_r$, by the contact wave speed, $C_m$, as follows:

$$U = \begin{cases} U^l & C^l > 0, \\ U^*_l & C^l \leq 0 < C^m, \\ U^*_r & C^m \leq 0 \leq C^r, \\ U^r & C^r < 0, \end{cases}$$

(13)

and the intercell flux is written as

$$F = \begin{cases} F^l & C^l > 0, \\ F^*_l & C^l \leq 0 < C^m, \\ F^*_r & C^m \leq 0 \leq C^r, \\ F^r & C^r < 0, \end{cases}$$

(14)

The right ($k = r$) and left ($k = l$) states of the conservative variable, $U^k$, wave speed, $C^k$, and the corresponding interface fluxes, $F^k$, are defined by

$$U^k = \rho^k \left[ \begin{array}{c} C^k - u^k_1 \\ \frac{p^k}{\rho^k + f(C^*)} \end{array} \right]$$

(15)

$$F^*_k = F^k + C^k(u^k - u^l),$$

(16)

where $f(C^*)$ is given by

$$f(C^*) = (C^* - u^k) \left[ C^* + \frac{p^k}{\rho^k(C^* - u^k)} \right].$$

3.4. Rusanov’s method
This scheme is central-based one wave equation Godunov type method.

$$F = \frac{1}{2} (F^l + F^r) - \frac{C^+}{2} (U^r - U^l).$$

(17)

As described in [21], this method is obtained by choosing $C^l = -C^r$ in HLL method (equation (10)). Where wave speed $C^+ = \text{max}(C^l, C^r)$.

3.5. WENO method
The numerical flux $F_{i+1/2,j}$ can be expressed as monotone function of left ($U^l_{i+1/2,j}$) and right ($U^r_{i+1/2,j}$) state extrapolated values at the cell interface as

$$F_{i+1/2,j} = F_{i+1/2,j}(U^l_{i+1/2,j}, U^l_{i+1/2,j})$$

(18)

These left and right state extrapolated values are obtained by high order polynomial reconstruction. For scalar function $Q(x)$, fifth order accurate left state ($Q^l_{i+1/2,j}$) extrapolated value is given as
\[ Q_{i+1/2,j}^l = W_0 V_0 + W_1 V_1 + W_2 V_2 \]  

where \( V_k (k = 0, 1, 2, 3) \) is the interpolated value of \( k_{th} \) stencil given by

\[
V_0 = \left( \frac{1}{3} Q_{i-2} - \frac{7}{6} Q_{i-1} + \frac{11}{6} Q_i \right) \\
V_1 = \left( -\frac{1}{6} Q_{i-1} + \frac{5}{6} Q_i + \frac{1}{3} Q_{i+1} \right) \\
V_2 = \left( \frac{1}{3} Q_i + \frac{5}{6} Q_{i+1} - \frac{1}{6} Q_{i+2} \right) 
\]

and \( W_k (k = 0, 1, 2) \) are nonlinear weights given by

\[
W_k = \frac{\alpha_k}{\sum \alpha_i} \quad \text{and} \quad \alpha_i = \frac{d_i}{(\varepsilon + \beta_i)^2}. 
\]

Smoothness indicators \( \beta_i \) in above equation are defined as

\[
\beta_0 = (13/12)(Q_{i-2} - Q_{i-1} + Q_i)^2 + (1/4)(Q_{i-2} - Q_{i-1} + Q_i)^2 \\
\beta_1 = (13/12)(Q_{i-1} + Q_i + Q_{i+1})^2 + (1/4)(Q_{i-1} + Q_i + Q_{i+1})^2 \\
\beta_2 = (13/12)(Q_{i-1} + Q_i + Q_{i+1})^2 + (1/4)(Q_{i-1} + Q_i + Q_{i+1})^2 
\]

Optimal value of weight coefficients \( d_i \) in equation (21) for a given stencil are \( d_0 = 1/10, d_1 = 3/5 \) and \( d_2 = 3/10 \). Larger the value of smoothness indicator, less smooth is the value of \( Q(x) \) function in the stencil. Therefore, if there is some discontinuity inside any one of the stencil, its weight tends to zero and other stencil will be activated to reconstruct the numerical flux. Further detail regarding smoothness indicator are given in Jiang and Shu [22]. To avoid zero denominator, \( \varepsilon \) value is set as \( 10^{-6} \).

The right state \( (Q_{i+1/2,j}^r) \) can be easily calculated by symmetry. Spurious oscillations are produced if we use conservative variables for polynomial reconstruction. Therefore, left and right state extrapolated values are obtained in terms of primitive variable instead of conservative variables. Equation (19) is used to reconstruct polynomial for each primitive variable.

4. Numerical Results

This section includes performance assessment and comparison of WENO-HLL, WENO-Roe, WENO-HLLC and WENO-Rusanov method for shock wave reflection and supersonic laminar flow over a flat plate.

4.1. Shock wave reflection

We have carried out simulation for supersonic inviscid flow over a flat plate. A shock wave at an angle of \( 29^\circ \) to the horizontal with a Mach number 2.9 is incident over a reflecting wall. Length of the domain is 4 units and height is 1 unit. We have used rectangular domain of grid size of \( 150 \times 50 \) and CFL number 0.6. Dirichlet boundary conditions on left and top boundary are as follows

\[
\begin{align*}
(\rho, u, v, p)^{left} &= (1.0, 2.91, 0, 0.251), \\
(\rho, u, v, p)^{top} &= (1.697, 2.61934, -0.50633, 1.52819). 
\end{align*}
\]

Reflecting boundary condition has been used at bottom boundary, supersonic outflow boundary condition has been taken at right boundary. In figure 1 and 2, we have shown the pressure contours over the entire domain and pressure distribution along the centre line in vertical direction respectively. WENO-HLLC results in good resolution as compared to numerical results of Colella [23], and other
flux schemes produces small numerical oscillation. Similar behavior of HLLC was found in [4] for shock wave reflection over wedge.

![Fig. 1. Pressure contours](image)

Fig. 1. Pressure contours (a) WENO-HLL; (b) WENO-Roe; (c) WENO-HLLC; (d) WENO-Rusanov

![Fig. 2. Pressure distribution](image)

Fig. 2. Pressure distribution along centreline in y direction

4.2. Supersonic laminar flow over flat plate
We performed simulation for supersonic viscous flow with Mach number $M = 4.0$ over an insulated flat plate. Free stream Reynolds number, $Re_\infty = 1000$ and temperature, $T_\infty = 293$ K. Simulations were carried on a rectangular domain of $70 \times 70$ grid size. We considered Dirichlet free stream boundary condition at the inlet and upper boundary, supersonic outflow at right boundary and adiabatic wall for bottom boundary. The assessment of different flux schemes is measured by the velocity profile as shown in figure 3. For vertical direction we have used normalized y-distance as $Y_{\text{norm}} = (y/x)\sqrt{Re_x}$, where $Re_x$ is the local distance Reynolds number along x-axis. WENO-HLLC and WENO-HLL scheme show better accuracy as compared to the numerical solution of Anderson [24]. For small wall distance WENO-Roe scheme shows small oscillation, and smoothen as we move further. WENO-Rusanov scheme gives the least accuracy among all the above schemes discussed.
Mach contours for WENO-Roe and WENO-HLLC schemes as shown in figure 4b and 4c are smooth over entire domain, as compare to WENO-HLL (figure 4a) and WENO-Rusunov schemes (figure 4d).

Fig. 3. Velocity profile inside and outside the boundary layer at the trailing edge.

Fig. 4. Mach contours for flow past over a flat plate, Re=1000, Ma=4: (a) WENO-HLL; (b) WENO-Roe; (c) WENO-HLLC; (d) WENO-Rusanov
5. Conclusions

We have investigated the performance of fifth order accurate WENO scheme which is applied to four flux difference splitting scheme (HLL, Roe, HLLC, and Rusanov) for the reconstruction of high order numerical flux. Reconstruction is based on primitive variables rather than conservative variable, considering the fact that latter produces spurious oscillation. For accuracy in time, we have adopted third order accurate TVD Runge-Kutta method. Comparison of WENO-HLL, WENO-Roe, WENO-HLLC and WENO-Rusanov methods has been done for two different problems: shock wave reflection and supersonic viscous flow over a flat plate. The results show that WENO-HLLC scheme captures the flow with good accuracy both inside and outside the boundary layer. Further, it is capable of capturing strong shocks and discontinuities.

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