Noncritical Weighted Hardy’s Inequalities with compact perturbations

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Abstract

Let Ω be a bounded domain of \( \mathbb{R}^N \) \((N \geq 1)\) whose boundary \( \partial \Omega \) is a \( C^2 \) compact manifolds. In the present paper we shall study a variational problem relating the weighted Hardy inequalities with sharp missing terms established in [4]. As weights we adopted powers of the distance function \( \delta(x) \) to the boundary \( \partial \Omega \).

1 Introduction

Let Ω be a bounded domain of \( \mathbb{R}^N \) \((N \geq 1)\) whose boundary \( \partial \Omega \) is a \( C^2 \) compact manifolds. In [4] we have established \( N \)-dimensional weighted Hardy’s inequalities with weight function being powers of the distance function \( \delta(x) = \text{dist}(x, \partial \Omega) \) to the boundary \( \partial \Omega \). In this paper we shall study a variational problem relating to these new inequalities.

We prepare more notations to describe our results. Let \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). By \( L^p(\Omega, \delta^{\alpha p}) \) we denote the space of Lebesgue measurable functions with weight \( \delta^{\alpha p} \), for which

\[
||u||_{L^p(\Omega, \delta^{\alpha p})} = \left( \int_{\Omega} |u|^p \delta^{\alpha p} \, dx \right)^{1/p} < \infty.
\]  

(1.1)

\( W^{1,p}_{\alpha,0}(\Omega) \) is given by the completion of \( C_c^\infty(\Omega) \) with respect to the norm defined by

\[
||u||_{W^{1,p}_{\alpha,0}(\Omega)} = ||\nabla u||_{L^p(\Omega, \delta^{\alpha p})}.
\]  

(1.2)

Then \( W^{1,p}_{\alpha,0}(\Omega) \) becomes a Banach space with the norm \( || \cdot ||_{W^{1,p}_{\alpha,0}(\Omega)} \). Under these preparation we recall the noncritical weighted Hardy inequality in [4]. In particular, we have the simplest one:

\[
\int_{\Omega} |\nabla u|^p \delta^{\alpha p} \, dx \geq \mu \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \quad \text{for} \quad u \in W^{1,p}_{\alpha,0}(\Omega),
\]  

(1.3)

where \( \mu \) is a positive constant independent of \( u \). If \( \alpha = 0 \) and \( p = 2 \), then (1.3) is a well-known Hardy’s inequality and valid for a bounded domain \( \Omega \) of \( \mathbb{R}^N \) with Lipschitz boundary (c.f. [6], [8], [11], [13]). If \( \Omega \) is convex and \( \alpha = 0 \), then (1.3) with \( \mu = (1 - 1/p)^p \) holds for arbitrary \( 1 < p < \infty \) (see [13], [14]).

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The best possible $\mu$ in (1.3) is given by the quantity

$$\inf_{u \in W^{1,p}_{\alpha,0}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p \delta^p dx}{\int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx},$$

which depends on $p, \alpha$ and $\Omega$.

In this paper we consider the following variational problem

$$J_\alpha^\lambda = \inf_{u \in W^{1,p}_{\alpha,0}(\Omega), u \neq 0} \chi_\alpha^\lambda(u)$$

(1.5)

where $\lambda \in \mathbb{R}$ and

$$\chi_\alpha^\lambda(u) = \frac{\int_{\Omega} |\nabla u|^p \delta^p dx - \lambda \int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx}{\int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx}.$$ (1.6)

Note that $J_0^\lambda$ gives the best constant in (1.3). Clearly, the function $\lambda \mapsto J_\alpha^\lambda$ is non-increasing on $\mathbb{R}$ and $J_\alpha^\lambda \to -\infty$ as $\lambda \to \infty$.

**Remark 1.1.** It is worthy to remark that (1.3) is never valid in the critical case that $\alpha \geq 1 - 1/p$. Nevertheless, we have established in this case a variant of weighted Hardy’s inequalities in [4] (cf. [10]). In a coming paper [3], we shall treat general weighted Hardy’s inequalities with compact perturbations and study relating variational problems including the critical case that $\alpha \geq 1 - 1/p$.

This paper is organized in the following way: The main result is described in Section 2. Section 3 is devoted to the proof of main result.

## 2 Main results

Our main result is the following.

**Theorem 2.1.** Assume that $\Omega$ is a bounded domain of class $C^2$ in $\mathbb{R}^N$. Assume that $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then there exists a constant $\lambda^* \in \mathbb{R}$ such that:

1. If $\lambda \leq \lambda^*$, then $J_\lambda^\alpha = \Lambda_{\alpha,p}$. If $\lambda > \lambda^*$, then $J_\lambda^\alpha < \Lambda_{\alpha,p}$.

Here

$$\Lambda_{\alpha,p} = \left(1 - \alpha - \frac{1}{p}\right)^p.$$ (2.1)

Moreover, it holds that:

2. If $\lambda < \lambda^*$, then the infimum $J_\lambda^\alpha$ in (1.3) is not attained.
3. If $\lambda > \lambda^*$, then the infimum $J_\lambda^\alpha$ in (1.3) is attained.

**Remark 2.1.**

1. In Theorem 2.1, it remains for $\lambda = \lambda^*$ of the open problem whether the infimum $J_\lambda^\alpha$ in (1.3) is attained or not.
2. For the case of $\alpha = 0$ and $p = 2$, it is shown that the infimum $J_\lambda^0$ in (1.3) is attained if and only if $\lambda > \lambda^*$. See [6].
3. For the case of $\alpha = 0$ and $\lambda = 0$, the value of the infimum $J_0^0$ in (1.3) and its attainability are studied in [13].
4. In the assertion 3 of Theorem 2.1, if \( \lambda > \lambda^* \) then the minimizer \( u \) for the variational problem (1.5) is a non-trivial weak solution of the following Euler-Lagrange equation:

\[-\text{div}(\delta^{\alpha p}|\nabla u|^{p-2}\nabla u) - \lambda \delta^{\alpha p}|u|^{p-2}u = J_{\lambda}^\alpha \delta^{(\alpha-1)p}|u|^{p-2}u \quad \text{in} \ D'(\Omega).\]

**Corollary 2.1.** Under the same assumptions as in Theorem 2.1 there exists a constant \( \lambda \in \mathbb{R} \) such that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_{\Omega} |\nabla u|^p \delta^{\alpha p} dx \geq \Lambda_{\alpha,p} \int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx + \lambda \int_{\Omega} |u|^p \delta^{\alpha p} dx. \tag{2.2}
\]

For each small \( \eta > 0 \), by \( \Omega_\eta \) we denote a tubular neighborhood of \( \partial \Omega \):

\[
\Omega_\eta = \{ x \in \Omega : \delta(x) = \text{dist}(x, \partial \Omega) < \eta \}. \tag{2.3}
\]

Then we have the following inequality of Hardy type which is crucial in the proof of Theorem 2.1.

**Theorem 2.2.** Assume that \( \Omega \) is a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Assume that \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Assume that \( \eta \) is a sufficiently small positive number. Then we have that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_{\Omega_\eta} |\nabla u|^p \delta^{\alpha p} dx \geq \Lambda_{\alpha,p} \int_{\Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx, \tag{2.4}
\]

where \( \Lambda_{\alpha,p} \) is defined by (2.1).

In [4] we have more precise estimate than (2.4).

**Corollary 2.2.** Under the same assumptions as in Theorem 2.2 there exists a positive constant \( \gamma \) such that for \( u \in W^{1,p}_{\alpha,0}(\Omega) \)

\[
\int_{\Omega} |\nabla u|^p \delta^{\alpha p} dx \geq \gamma \int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx. \tag{2.5}
\]

For any bounded domain \( \Omega \subset \mathbb{R}^N \) we can prove the following:

**Theorem 2.3.** Assume that \( \Omega \) is a bounded domain of \( \mathbb{R}^N \). Assume that \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Then the followings are equivalent with each other.

1. There exists a positive number \( \gamma \) such that the inequality (2.5) is valid for every \( u \in W^{1,p}_{\alpha,0}(\Omega) \).

2. For a sufficiently small positive number \( \eta \), there exists a positive number \( \kappa \) such that the inequality (2.4) with \( \Lambda_{\alpha,p} \) replaced by \( \kappa \) is valid for every \( u \in W^{1,p}_{\alpha,0}(\Omega) \).

For the proofs of Theorem 2.2, Corollary 2.2 and Theorem 2.3 see in [4].

### 3 Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1.
3.1 Upper bound of $J^\alpha_\lambda$

First, we prove the assertion 1 of Theorem 2.1.

**Lemma 3.1.** Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. For any $\varepsilon > 0$ and any $\eta > 0$ there exists a function $h \in W^{1,p}_{\alpha,0}((0,\eta))$ such that

$$\int_0^\eta |h'(t)|^p t^{\alpha p} dt \leq (\Lambda_{\alpha,p} + \varepsilon) \int_0^\eta |h(t)|^{p(\alpha-1)} dt,$$

where $\Lambda_{\alpha,p}$ is defined by (2.1).

**Proof.** Since the inequality (3.1) is invariant with respect to scaling, we may assume that $\eta = 2$. Put

$$h(t) = \begin{cases} t^\beta & \text{if } t \in (0,1), \\ 2 - t & \text{if } t \in [1,2) \end{cases}$$

with $\beta > 1 - \alpha - 1/p$. Then we see that $h \in W^{1,p}_{\alpha,0}((0,2))$,

$$\int_0^2 |h'(t)|^p t^{\alpha p} dt = \frac{\beta^p}{p(\beta - 1 + \alpha + 1/p)} + C_{\alpha,p}$$

and

$$\int_0^2 |h(t)|^{p(\alpha-1)} dt = \frac{1}{p(\beta - 1 + \alpha + 1/p)} + D_{\alpha,p},$$

where

$$C_{\alpha,p} = \int_1^2 t^{\alpha p} dt \quad \text{and} \quad D_{\alpha,p} = \int_1^2 (2-t)^{p(\alpha-1)} dt$$

are constants independent of $\beta$. It follows from (3.2) and (3.3) that

$$\frac{\int_0^\eta |h'(t)|^p t^{\alpha p} dt}{\int_0^\eta |h(t)|^{p(\alpha-1)} dt} \to \Lambda_{\alpha,p} \quad \text{as } \beta \to 1 - \alpha - \frac{1}{p} + 0,$$

which implies (3.1) with $\eta = 2$. Therefore we obtain the desired conclusion. 

**Lemma 3.2.** Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then it holds that

$$J^\alpha_\lambda \leq \Lambda_{\alpha,p}$$

for all $\lambda \in \mathbb{R}$.

**Proof.** Since the boundary $\partial \Omega$ is of class $C^2$, there exists an $\eta_0 > 0$ such that for any $\eta \in (0,\eta_0)$ and every $x \in \Omega_\eta$ we have a unique point $\sigma(x) \in \partial \Omega$ satisfying $\delta(x) = |x - \sigma(x)|$. The mapping

$$\Omega_\eta \ni x \mapsto (\delta(x), \sigma(x)) = (t, \sigma) \in (0,\eta) \times \partial \Omega$$

is a $C^2$ diffeomorphism, and its inverse is given by

$$(0,\eta) \times \partial \Omega \ni (t,\sigma) \mapsto x(t,\sigma) = \sigma + t \cdot n(\sigma) \in \Omega_\eta,$$
where \( n(\sigma) \) is the inward unit normal to \( \partial \Omega \) at \( \sigma \in \partial \Omega \). For each \( t \in (0, \eta) \), the mapping

\[
\partial \Omega \ni \sigma \mapsto \sigma_t(\sigma) = x(t, \sigma) \in \Sigma_t = \{ x \in \Omega : \delta(x) = t \}
\]

is a also a \( C^2 \) diffeomorphism of \( \partial \Omega \) onto \( \Sigma_t \), and its Jacobian satisfies

\[
|\text{Jac} \sigma_t(\sigma) - 1| \leq ct \quad \text{for any } \sigma \in \partial \Omega,
\]

where \( c \) is a positive constant depending only on \( \eta_0, \partial \Omega \) and the choice of local coordinates. Since \( n(\sigma) \) is orthogonal to \( \Sigma_t \) at \( \sigma_t(\sigma) = \sigma + t \cdot n(\sigma) \in \Sigma_t \), it follows that for every integrable function \( v \) in \( \Omega_\eta \),

\[
\int_{\Omega_\eta} v(x)dx = \int_0^\eta dt \int_{\Sigma_t} v(\sigma_t)ds_t
\]

\[
= \int_0^\eta dt \int_{\partial \Omega} v(x(t, \sigma))|\text{Jac} \sigma_t(\sigma)|ds,
\]

where \( ds \) and \( ds_t \) denote surface elements on \( \partial \Omega \) and \( \Sigma_t \), respectively. Hence \( (3.6) \) together with \( (3.5) \) implies that for every integrable function \( v \) in \( \Omega_\eta \),

\[
\int_0^\eta (1 - ct)dt \int_{\partial \Omega} |v(x(t, \sigma))|ds \leq \int_{\Omega_\eta} |v(x)|dx
\]

\[
\leq \int_0^\eta (1 + ct)dt \int_{\partial \Omega} |v(x(t, \sigma))|ds.
\]

Let \( \varepsilon > 0 \), and let \( \eta \in (0, \eta_0) \). Take \( h \in W^{1,p}_{\alpha,0}(0, \eta) \) be a function satisfying \( (3.8) \). Put

\[
u(x) = \begin{cases} h(\delta(x)) & \text{if } x \in \Omega_\eta, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases}
\]

Since \( |\nabla \nu(x)| = |h'(\delta(x))| \) for \( x \in \Omega_\eta \) by \( |\nabla \delta(x)| = 1 \), it follows from \( (3.8) \) that

\[
\int_{\Omega_\eta} |\nabla \nu|^p \delta^{\alpha p}dx \leq (1 + c\eta)|\partial \Omega| \int_0^\eta |h'(t)|^p t^{\alpha p}dt,
\]

which implies \( u \in W^{1,p}_{\alpha,0}(\Omega) \) by \( \text{supp } u \subset \Omega_\eta \). On the other hand, by \( (3.7) \) and \( (3.8) \) we have that

\[
\int_{\Omega_\eta} |u|^p \delta^{(\alpha - 1)p}dx \geq (1 - c\eta)|\partial \Omega| \int_0^\eta |h(t)|^p t^{\alpha - 1}dt.
\]

Since \( \text{supp } u \subset \Omega_\eta \), by combining \( (3.10) \), \( (3.11) \) and the estimate

\[
\int_{\Omega_\eta} |u|^p \delta^{\alpha p}dx \leq \eta^p \int_{\Omega_\eta} |u|^p \delta^{(\alpha - 1)p}dx,
\]

we obtain that

\[
\chi^\alpha(u) \leq \frac{1 + c\eta}{1 - c\eta} \int_0^\eta |h'(t)|^p t^{\alpha p}dt + |\lambda|\eta^p.
\]

This together with \( (3.1) \) implies that

\[
J^\alpha \leq \frac{1 + c\eta}{1 - c\eta}(\Lambda_{\alpha,p} + \varepsilon) + |\lambda|\eta^p.
\]

Letting \( \eta \to +0 \) in \( (3.12) \), \( (3.3) \) follows. Therefore it concludes the proof. \( \square \)
Lemma 3.3. Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then there exists a $\lambda \in \mathbb{R}$ such that $J_\lambda^\alpha = \Lambda_{\alpha,p}$.

Proof. Let $\eta > 0$ be a sufficiently small number as in Theorem 2.2. For any $u \in W_{\alpha,0}^{1,p}(\Omega) \setminus \{0\}$, by using Hardy’s inequality (2.4) and the estimate

$$\int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx \leq \eta^{-p} \int_{\Omega} |u|^p \delta^{\alpha p} dx,$$

we have that

$$\Lambda_{\alpha,p} \int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx = \Lambda_{\alpha,p} \int_{\Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx + \Lambda_{\alpha,p} \int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} dx \leq \int_{\Omega} |\nabla u|^p \delta^{\alpha p} dx + \Lambda_{\alpha,p} \eta^{-p} \int_{\Omega} |u|^p \delta^{\alpha p} dx,$$

which implies that

$$\chi_\lambda^\alpha(u) \geq \Lambda_{\alpha,p}$$

for $\lambda \leq -\Lambda_{\alpha,p} \eta^{-p}$. Consequently, it holds that $J_\lambda^\alpha \geq \Lambda_{\alpha,p} \eta^{-p}$. This together with (3.4) implies the desired conclusion. \hfill \Box

Lemma 3.4. Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Let $1 < p < \infty$ and $\alpha < 1 - 1/p$. Then the function $\lambda \mapsto J_\lambda^\alpha$ is Lipschitz continuous on $\mathbb{R}$.

Proof. Let $\lambda, \bar{\lambda} \in \mathbb{R}$. Then it holds that for any $u \in W_{\alpha,0}^{1,p}(\Omega) \setminus \{0\}$

$$|\chi_\lambda^\alpha(u) - \chi_{\bar{\lambda}}^\alpha(u)| = |\lambda - \bar{\lambda}| \frac{\int_{\Omega} |u|^p \delta^{\alpha p} dx}{\int_{\Omega} |u|^p \delta^{(\alpha-1)p} dx} \leq M |\lambda - \bar{\lambda}|,$$

where $M = \sup_{x \in \Omega} \delta(x)$ is a positive constant depending only on $\Omega$. Hence we see that

$$|J_\lambda^\alpha - J_{\bar{\lambda}}^\alpha| \leq M |\lambda - \bar{\lambda}|$$

for $\lambda, \bar{\lambda} \in \mathbb{R}$. It completes the proof. \hfill \Box

Proof of the assertion 1 of Theorem 2.1

By Lemma 3.3 and $\lim_{\lambda \to \infty} J_\lambda^\alpha = -\infty$, the set $\{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}$ is non-empty and upper bounded. Hence the sup $\{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}$ exists finitely. Put

$$\lambda^* = \sup \{ \lambda \in \mathbb{R} : J_\lambda^\alpha = \Lambda_{\alpha,p} \}. \quad (3.13)$$

Since the function $\lambda \mapsto J_\lambda^\alpha$ is non-increasing on $\mathbb{R}$, it follows from Lemma 3.2 and Lemma 3.3 that $J_\lambda^\alpha = \Lambda_{\alpha,p}$ for $\lambda < \lambda^*$ and $J_\lambda^\alpha < \Lambda_{\alpha,p}$ for $\lambda > \lambda^*$. Further, by Lemma 3.4 we have the equality $J_{\lambda^*}^\alpha = \Lambda_{\alpha,p}$. Therefore the assertion 1 of Theorem 2.1 is valid. \hfill \Box

3.2 $J_\lambda^\alpha$ is not attained when $\lambda < \lambda^*$

Next, we prove the assertion 2 of Theorem 2.1

Proof of the assertion 2 of Theorem 2.1

Suppose that for some $\lambda < \lambda^*$ the infimum $J_\lambda^\alpha$ in (1.5) is attained at an element $u \in W_{\alpha,0}^{1,p}(\Omega) \setminus \{0\}$. Then, by the assertion 1 of Theorem 2.1 we have that

$$\chi_\lambda^\alpha(u) = J_\lambda^\alpha = \Lambda_{\alpha,p} \quad (3.14)$$
and for \( \lambda < \bar{\lambda} < \lambda^* \)
\[
\chi_{\bar{\lambda}}^\alpha(u) \geq J_{\bar{\lambda}}^\alpha = \Lambda_{\alpha,p}.
\]  
(3.15)

From (3.14) and (3.15) it follows that
\[
(\bar{\lambda} - \lambda) \int_{\Omega} |u|^p \delta^{\alpha p} dx \leq 0.
\]
Since \( \bar{\lambda} - \lambda > 0 \), we conclude that
\[
\int_{\Omega} |u|^p \delta^{\alpha p} dx = 0,
\]
which contradicts \( u \neq 0 \) in \( W^{1,p}_{\alpha,0}(\Omega) \). Therefore it completes the proof.

\[ \square \]

3.3 Attainability of \( J_{\lambda}^\alpha \) when \( \lambda > \lambda^* \)

At last, we prove the assertion 3 of Theorem 2.1.

Let \( \{u_k\} \) be a minimizing sequence for the variational problem (1.5) normalized so that
\[
\int_{\Omega} |u_k|^p \delta^{(\alpha-1)p} dx = 1 \quad \text{for all } k.
\]  
(3.16)

Since \( \{u_k\} \) is bounded in \( W^{1,p}_{\alpha,0}(\Omega) \), by taking a suitable subsequence, we may assume that there exists a \( u \in W^{1,p}_{\alpha,0}(\Omega) \) such that
\[
\nabla u_k \overset{\text{weak}}{\rightharpoonup} \nabla u \quad \text{in } (L^p(\Omega, \delta^{\alpha p}))^N,
\]  
(3.17)
\[
u_k \overset{\text{weak}}{\rightharpoonup} u \quad \text{in } L^p(\Omega, \delta^{(\alpha-1)p})
\]  
(3.18)
and
\[
u_k \rightharpoonup u \quad \text{in } L^p(\Omega, \delta^{\alpha p})
\]  
(3.19)
by Hardy’s inequality (2.5) and the compact embedding \( W^{1,p}_{\alpha,0}(\Omega) \hookrightarrow L^p(\Omega, \delta^{\alpha p}) \).

Under these preparation we establish the properties of concentration and compactness for the minimizing sequence, respectively.

**Proposition 3.1.** Let \( \Omega \) be a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Let \( 1 < p < \infty \) and \( \alpha < 1 - 1/p \). Let \( \lambda \in \mathbb{R} \). Let \( \{u_k\} \) be a minimizing sequence for (1.5) satisfying (3.10), (3.17),  (3.18) and (3.19) with \( u = 0 \). Then it holds that
\[
\nabla u_k \rightharpoonup 0 \quad \text{in } (L^p_{\text{loc}}(\Omega))^N
\]  
(3.20)
and
\[
J_{\lambda}^\alpha = \Lambda_{\alpha,p}.
\]  
(3.21)

**Proof.** Let \( \eta > 0 \) be a sufficiently small number as in Theorem 2.2. By Hardy’s inequality (2.4) and (3.10) we have that
\[
\int_{\Omega_n} |\nabla u_k|^p \delta^{\alpha p} dx \geq \Lambda_{\alpha,p} \int_{\Omega_n} |u_k|^p \delta^{(\alpha-1)p} dx
\]
\[
= \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_n} |u_k|^p \delta^{(\alpha-1)p} dx \right),
\]
and so
\[
\chi_\alpha(\mu_k) \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_n} |\mu_k|^{p\delta^{(\alpha-1)p}} \, dx \right)
+ \int_{\Omega \setminus \Omega_n} |\nabla \mu_k|^{p\delta^{\alpha p}} \, dx - \lambda \int_{\Omega} |\mu_k|^{p\delta^{\alpha p}} \, dx. \tag{3.22}
\]

Since
\[
\int_{\Omega \setminus \Omega_n} |\mu_k|^{p\delta^{(\alpha-1)p}} \, dx \leq \eta^{-p} \int_{\Omega} |\mu_k|^{p\delta^{\alpha p}} \, dx,
\]

it follows from (3.19) with \( u = 0 \) that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\mu_k|^{p\delta^{(\alpha-1)p}} \, dx = 0. \tag{3.23}
\]

Hence, by (3.22), (3.23) and (3.19) with \( u = 0 \), we obtain that
\[
\lim sup_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla \mu_k|^{p\delta^{\alpha p}} \, dx \leq J_\alpha - \Lambda_{\alpha,p}.
\]

Since \( J_\alpha - \Lambda_{\alpha,p} \leq 0 \) by Lemma 3.2, we conclude that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla \mu_k|^{p\delta^{\alpha p}} \, dx = 0,
\]

and so
\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_n} |\nabla \mu_k|^p \, dx = 0.
\]

This shows (3.20). Moreover, letting \( k \to \infty \) in (3.22), it follows from (3.23), (3.24) and (3.19) with \( u = 0 \) that
\[
J_\alpha^p \geq \Lambda_{\alpha,p}.
\]

This together with Lemma 3.2 implies (3.21). Consequently it completes the proof.

**Proposition 3.2.** Let \( \Omega \) be a bounded domain of class \( C^2 \) in \( \mathbb{R}^N \). Let \( 1 < p < \infty \) and \( \alpha < 1 - \frac{1}{p} \). Let \( \lambda \in \mathbb{R} \). Let \( \{ \mu_k \} \) be a minimizing sequence for (1.5) satisfying (3.16), (3.17), (3.18) and (3.19) with \( u \neq 0 \). Then it holds that
\[
J_\alpha^p = \min(\Lambda_{\alpha,p}, \chi_\alpha^{\alpha}(u)). \tag{3.25}
\]

In addition, if \( J_\alpha^p < \Lambda_{\alpha,p} \), then it holds that
\[
J_\alpha^p = \chi_\alpha^{\alpha}(u), \tag{3.26}
\]

namely \( u \) is a minimizer for (1.5), and
\[
\mu_k \rightharpoonup u \ \text{in} \ W_{1,p}^\alpha(\Omega). \tag{3.27}
\]

**Proof.** Let \( \eta > 0 \) be a sufficiently small number as in Theorem 2.2. Then we have (3.22) by the same arguments as in the proof of Proposition 3.1. By the estimate
\[
\int_{\Omega \setminus \Omega_n} |\mu_k - u|^p \delta^{(\alpha-1)p} \, dx \leq \eta^{-p} \int_{\Omega} |\mu_k - u|^p \delta^{\alpha p} \, dx,
\]


\[ \lim_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} |u_k|^p \delta^{(\alpha-1)p} \, dx = \int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} \, dx. \] (3.28)

Since it follows from (3.17) that \( \nabla u_k \rightharpoonup \nabla u \) weakly in \( (L^p(\Omega \setminus \Omega_\eta, \delta^{ap}))^N \), by weakly lower semi-continuity of the \( L^p \)-norm, we see that
\[
\liminf_{k \to \infty} \int_{\Omega \setminus \Omega_\eta} |\nabla u_k|^p \delta^{ap} \, dx \geq \left( \liminf_{k \to \infty} \|\nabla u_k\|_{L^p(\Omega \setminus \Omega_\eta, \delta^{ap})} \right)^p \\
\geq \|\nabla u\|_{L^p(\Omega \setminus \Omega_\eta, \delta^{ap})} \\
= \int_{\Omega \setminus \Omega_\eta} |\nabla u|^p \delta^{ap} \, dx. \] (3.29)

Hence, by letting \( k \to \infty \) in (3.22), from (3.19), (3.28) and (3.29) it follows that
\[
J_\alpha^\Lambda \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega \setminus \Omega_\eta} |u|^p \delta^{(\alpha-1)p} \, dx \right) + \int_{\Omega \setminus \Omega_\eta} |\nabla u|^p \delta^{ap} \, dx - \lambda \int_{\Omega} |u|^p \delta^{ap} \, dx. \] (3.30)

Letting \( \eta \to +0 \) in (3.30), we obtain that
\[
J_\alpha^\Lambda \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \right) + \int_{\Omega} |\nabla u|^p \delta^{ap} \, dx - \lambda \int_{\Omega} |u|^p \delta^{ap} \, dx. \] (3.31)

Since it holds that
\[
0 < \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \leq \liminf_{k \to \infty} \int_{\Omega} |u_k|^p \delta^{(\alpha-1)p} \, dx = 1 \] (3.32)
by \( u \neq 0 \), (3.16) and (3.18) and weakly lower semi-continuity of the \( L^p \)-norm, we have from (3.31) and (3.32) that
\[
J_\alpha^\Lambda \geq \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \right) + \chi_{\alpha}(u) \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \\
\geq \min(\Lambda_{\alpha,p}, \chi_{\alpha}(u)). \] (3.33)

This together with Lemma 3.2 implies (3.25). Moreover, by (3.25) and (3.33), we conclude that
\[
J_\alpha^\Lambda = \Lambda_{\alpha,p} \left( 1 - \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx \right) + \chi_{\alpha}(u) \int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx. \] (3.34)

In addition, if \( J_\alpha^\Lambda < \Lambda_{\alpha,p} \), then \( J_\alpha^\Lambda = \chi_{\alpha}(u) \) by (3.25), and so, it follows from (3.31) and (3.16) that
\[
\int_{\Omega} |u|^p \delta^{(\alpha-1)p} \, dx = 1 = \lim_{k \to \infty} \int_{\Omega} |u_k|^p \delta^{(\alpha-1)p} \, dx. \] (3.35)

(3.18) and (3.35) imply that
\[
\lim_{k \to \infty} u_k \to u \quad \text{in} \quad L^p(\Omega, \delta^{(\alpha-1)p}). \] (3.36)
Further, by (3.16), (3.19), (3.26) and (3.35), we obtain that
\[
\int_{\Omega} \left| \nabla u_k \right|^p \delta^{\alpha p} dx = \chi_{\lambda}^\alpha(u_k) + \lambda \int_{\Omega} |u_k|^p \delta^{\alpha p} dx \\
\rightarrow \chi_{\lambda}^\alpha(u) + \lambda \int_{\Omega} |u|^p \delta^{\alpha p} dx = \int_{\Omega} \left| \nabla u \right|^p \delta^{\alpha p} dx.
\]
This together with (3.17) implies that
\[
\nabla u_k \rightarrow \nabla u \quad \text{in} \quad (L^p(\Omega, \delta^{\alpha p}))^N, \tag{3.37}
\]
which shows (3.27). Consequently it completes the proof. ☐

Proof of the assertion 3 of Theorem 2.1 Let \( \lambda > \lambda^* \). Then \( J_\lambda^\alpha < \Lambda_{\alpha,p} \) by the assertion 1 of Theorem 2.1. Let \( \{u_k\} \) be a minimizing sequence for (1.5) satisfying (3.16), (3.17), (3.18) and (3.19). Then we see that \( u \neq 0 \) by Proposition 3.1. Therefore, by applying Proposition 3.2 we conclude that \( \chi_{\lambda}^\alpha(u) = J_\lambda^\alpha \), namely \( u \) is a minimizer for (1.5). It finishes the proof. ☐

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