The supersingular locus of the Shimura variety of GU(2, n − 2)

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Abstract

We study the supersingular locus of a reduction at an inert prime of the Shimura variety attached to GU(2, n − 2). More concretely, we realize irreducible components of the supersingular locus as closed subschemes of flag schemes over Deligne–Lusztig varieties defined by explicit conditions. Moreover we study the intersections of the irreducible components. A stratification of Deligne–Lusztig varieties defined using a power of Frobenius action appears in the description of the intersections.

1 Introduction

The Shimura varieties play important roles in the study of number theory. One way to approach the arithmetic of Shimura varieties is to construct integral models and study their reductions. Among other things, the geometry of the supersingular loci of reductions of Shimura varieties is an important topic. One of the striking results in this direction is the study of the supersingular locus of a reduction of the Shimura variety of GU(1, n − 1) at an inert prime by Vollaard–Wedhorn in [VW11], where they give a description of the supersingular locus and their intersections in terms of Deligne–Lusztig varieties. This result is crucially used in [KR11].

A long standing problem since [VW11] is to extend such a result to unitary groups of other signatures. The only result in this line is a work [HP14] of Howard–Pappas on the GU(2, 2)-case using an exceptional isomorphism. A source of difficulty is that the Shimura variety of GU(2, n − 2) is not fully Hodge–Newton decomposable in the sense of [GHN19, Definition 3.1] if n ≥ 5. In such a case, we can not expect that the supersingular locus is a union of Deligne–Lusztig varieties by [GHN19, Theorem B].

On the other hand, the study of the supersingular locus is essentially reduced to a study of an affine Deligne–Lusztig variety via the Rapoport–Zink uniformization. Further, a construction of irreducible components of an affine Deligne–Lusztig variety under some unramified condition is given by Xiao–Zhu in [XZ17]. In their construction, we can rephrase the source of difficulty in the following way: Even though the affine Deligne–Lusztig variety related to the Shimura variety of GU(2, n − 2) is defined using a minuscule cocharacter, non-minuscule cocharacters appear in the construction of its irreducible components if n ≥ 5.

The objective of this paper is to get an explicit description of the irreducible components of the affine Deligne–Lusztig variety related to the Shimura variety of GU(2, n − 2) in terms of Deligne–Lusztig varieties.

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Let $F$ be a non-archimedean local field. Let $L$ be the completion of the maximal unramified extension of $F$. Let $G$ be the unramified general unitary group of degree $n$ over $F$. Let $\mu$ be a cocharacter of $G$ corresponding to $z \mapsto (\text{diag}(z, z, 1, \ldots, 1), 1)$ under an isomorphism $G_L \simeq \text{GL}_n \times \mathbb{G}_m$. Let $X_{\mu, r}(1)$ denote the affine Deligne–Lusztig variety for the dual $\mu^\vee$ of $\mu$ and $1 \in G(L)$. We put $r = \lfloor n/2 \rfloor$. Then $X_{\mu, r}(1)$ has $r$ isomorphism classes of irreducible components, whose representatives are given by $X_{\mu, r, x_0}^b(\tau_i^*)$ for $1 \leq i \leq r$ as explained in §5. If $i = 1$ or $i = n/2$, then $X_{\mu, r, x_0}^b(\tau_i^*)$ is isomorphic to the perfection of a Deligne–Lusztig variety as shown in Proposition 8.1 and Proposition 8.2. Assume that $2 \leq i \leq \lceil (n-1)/2 \rceil$. Then we construct a kind of Demazure resolution $X_i$ of $X_{\mu, r, x_0}^b(\tau_i^*)$. On the other hand, we construct a vector bundle $\mathcal{V}_i$ of rank $2i-1$ over a perfection $Y_i$ of a Deligne–Lusztig variety. Let $\text{Par}_i(\mathcal{G}_Y)$ denote the flag scheme parametrizing subvector bundles $\mathcal{W} \subset \mathcal{V}_i$ of rank $i - 1$.

**Theorem 1.1** (Theorem 8.3). The scheme $X_i$ is isomorphic to the closed subscheme of $\text{Par}_i(\mathcal{G}_Y)$ defined by an orthogonality condition on $\mathcal{W}$ (cf. (8.4)).

An open subscheme of $X_i$ is isomorphic to an open dense subscheme $\hat{X}_{\mu, r, x_0}^b(\tau_i^*)$ of $X_{\mu, r, x_0}^b(\tau_i^*)$. Therefore we can describe $\hat{X}_{\mu, r, x_0}^b(\tau_i^*)$ inside $\text{Par}_i(\mathcal{G}_Y)$ as we do in Proposition 8.3. However, it is important to describe the entire $X_i$ to study the intersection of irreducible components of $X_{\mu, r}(1)$. We give a description of the intersections of the irreducible components in most cases. As an interesting new phenomenon, we see that an intersection is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\text{pf}}$ defined by two equations

$$\sum_{i=1}^n x_i^{q+1} = 0, \quad \sum_{i=1}^n x_i^{q^3+1} = 0.$$ 

This is a stratification of a Deligne–Lusztig variety with respect to relative positions of parabolic subgroups and their twists by the third power of the Frobenius action. Such an intersection did not appear in the preceding researches in fully Hodge–Newton decomposable cases. Our study does not cover all the intersections in general because of some technical difficulty, but it does cover all the cases if $n \leq 6$.

We explain the contents of each section. In §2 we recall a terminology on relative positions in flag schemes. We also give some gluing constructions of reductive schemes. In §3 we recall Deligne–Lusztig variety and its Bruhat stratification. We give also a new stratification using twists by a power of Frobenius map. We study the irreducibility of the stratification in some unitary case. In §4 we recall affine Grassmannian and Satake cycles. In §5 we recall and generalize results on equidimensionality of Satake cycles in [Hai06]. In §6 we recall a construction of irreducible components of affine Deligne–Lusztig varieties in [XZ17]. In §7 we explain the setting of a unitary group and apply the result in §5 to the unitary case. In §8 we give an explicit description of irreducible components. In §9 we study the intersection of irreducible components. In §10 we explain the results in the $n = 6$ case as an example. In §11 we explain a relation between the affine Deligne–Lusztig varieties and the supersingular locus of reductions of Shimura varieties in our case.

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2 Flag scheme

2.1 Relative position

Let $G$ be a reductive group scheme over a scheme $S$. Let $\text{Par}(G)$ be the scheme of parabolic subgroups of $G$. Let $\text{Dyn}(G)$ be the scheme of Dynkin for $G$ constructed in [SGA3-3, XXIV, 3.3].

**Remark 2.1.** If $(\mathcal{F}, M, R)$ is a splitting of $G$ in the sense of [SGA3-3, XXII, Définition 1.13] and $\Delta$ is a set of simple roots, then we have a canonical isomorphism

$$\text{Dyn}(G) \simeq \Delta_{\mathcal{F}}.$$  \hfill (2.1)

This is stated in [SGA3-3, XXIV, 3.4 (iii)] choosing a pinning, but the isomorphism actually depends only on $(\mathcal{F}, M, R)$ and $\Delta$.

Let $\text{Oc}(\text{Dyn}(G))$ be the scheme of sets of open and closed subschemes of $\text{Dyn}(G)$ (cf. [SGA3-3, XXVI, 3.1]). We have a projective smooth morphism

$$\mathfrak{t}: \text{Par}(G) \to \text{Oc}(\text{Dyn}(G))$$

of schemes as [SGA3-3, XXVI, Théorème 3.3]. For $t, t' \in \text{Oc}(\text{Dyn}(G))(\mathcal{F})$, we put

$$\text{Par}_t(G) = t^{-1}(t), \quad \text{Par}_{t, t'}(G) = (t \times t)^{-1}(t, t').$$

We recall results from [SGA3-3, XXVI, 4.5.3, 4.5.4]. Let $\text{Stand}(G)$ be the scheme of pairs of parabolic subgroups of $G$ in mutually standard positions. Let $\text{TypeStand}(G)$ be the scheme of types of mutually standard positions in $G$. The natural morphism

$$\mathfrak{t}_2: \text{Stand}(G) \to \text{TypeStand}(G)$$

is smooth and a quotient of $\text{Stand}(G)$ by the conjugacy action of $G$. We have a commutative diagram

\[
\begin{array}{ccc}
\text{Stand}(G) & \xrightarrow{\mathfrak{t}_2} & \text{TypeStand}(G) \\
\downarrow & & \downarrow q_{\mathcal{F}} \\
\text{Par}(G) \times_{\mathcal{F}} \text{Par}(G) & \xrightarrow{t \times t} & \text{Oc}(\text{Dyn}(G)) \times_{\mathcal{F}} \text{Oc}(\text{Dyn}(G)).
\end{array}
\]

Let $\mathcal{P}$ be a parabolic subgroup scheme of $G$. Let $\text{Par}(G; \mathcal{P})$ be the scheme of parabolic subgroups of $G$ in standard positions relative to $\mathcal{P}$. Let $t \in \text{Oc}(\text{Dyn}(G))(\mathcal{F})$. We put

$$\text{Par}_t(G; \mathcal{P}) = \text{Par}(G; \mathcal{P}) \cap \text{Par}_t(G).$$

Then we have a morphism

$$\mathfrak{t}_{\mathcal{P}}: \text{Par}_t(G; \mathcal{P}) \to q_{\mathcal{F}}^{-1}(t(\mathcal{P}), t)$$
induced by \( t_2 \). For an \( \mathcal{I} \)-scheme \( \mathcal{I}' \) and \( r \in (q_\mathcal{g}^{-1}(t(\mathcal{P}), t))(\mathcal{I}') \), we define \( \text{Par}_r(\mathcal{I}; \mathcal{P}) \) by the fiber product

\[
\begin{array}{ccc}
\text{Par}_r(\mathcal{I}; \mathcal{P}) & \xrightarrow{r} & \mathcal{I}' \\
\downarrow & & \downarrow \\
\text{Par}_r(\mathcal{I}; \mathcal{P}) & \xrightarrow{t_\mathcal{g}} & q_\mathcal{g}^{-1}(t(\mathcal{P}), t).
\end{array}
\]

**Remark 2.2.** Let \( \mathcal{D} \) be a parabolic subgroup scheme of \( \mathcal{G} \). Let \( \mathcal{I}' \) be an \( \mathcal{I} \)-scheme. we write \( \mathcal{G}', \mathcal{P}', \mathcal{D}' \) for the base change of \( \mathcal{G}, \mathcal{P}, \mathcal{D} \) to \( \mathcal{I}' \). Assume that a maximal torus \( \mathcal{T}' \) of \( \mathcal{I}' \) is contained in \( \mathcal{P}' \cap \mathcal{D}' \). Then we have a natural isomorphism

\[
W_{\mathcal{P}'}(\mathcal{I}') \backslash W_{\mathcal{P}'}(\mathcal{I}') / W_{\mathcal{P}'}(\mathcal{I}') \cong q_\mathcal{g}^{-1}(t(\mathcal{P}), t(\mathcal{D})) \times \mathcal{I}' \tag{2.2}
\]

over \( \mathcal{I}' \) as in [SGA3-3, XXVI. 4.5.3].

**Notation 2.3.** Assume that \( \mathcal{G} \) is split and \( \mathcal{I} \) is connected. Let \( (\mathcal{F}, M, R) \) be a splitting of \( \mathcal{G} \) and \( \Delta \) be a set of simple roots. Let \( (W, S) \) be the Coxeter system of \( (M, R, \Delta) \). For \( I \subset S \), let \( W_I \) be the subgroup of \( W \) generated by \( I \), and let \( t(I) \) be the element of \( \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \) corresponding to \( I \) under (2.1). Conversely, let \( t(I) \) be the subset of \( S \) corresponding to \( t \) under (2.1) for \( t \in \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \). We simply write \( W_I \) for \( W(t(I)) \).

### 2.2 Inner gluing

**Definition 2.4.** Let \( \mathcal{G}_0 \) be a reductive group scheme over a scheme \( \mathcal{I}_0 \). Let \( \mathcal{I} \) be a scheme over \( \mathcal{I}_0 \). An inner gluing over \( \mathcal{I} \) of \( \mathcal{G}_0 \) as a pair \( (\mathcal{G}, \varphi) \), where \( \mathcal{G} \) is a reductive group scheme over \( \mathcal{I} \) and \( \varphi \) is a global section of the Zariski sheaf

\[
\text{Isom}_{\mathcal{I}}(\mathcal{G}_0 \times_{\mathcal{I}_0} \mathcal{I}, \mathcal{G}) / \text{Inn}_{\mathcal{I}}(\mathcal{G}_0 \times_{\mathcal{I}_0} \mathcal{I})
\]

on \( \mathcal{I} \).

**Remark 2.5.** Let \( \mathcal{V} \) be a vector bundle of rank \( n \) on \( \mathcal{I} \). We put \( \mathcal{G} = \text{Aut}_{\mathcal{I}}(\mathcal{V}) \). By taking Zariski local trivializations of \( \mathcal{V} \), we obtain an inner gluing \( (\mathcal{G}, \varphi_{\mathcal{V}}) \) over \( \mathcal{I} \) of \( \text{GL}_n \). This is independent of the choice of trivializations, because a difference of trivializations induces an inner automorphism of \( \text{GL}_n \).

**Lemma 2.6.** Let \( \pi: \mathcal{I} \rightarrow \mathcal{I}_0 \) be a morphism of schemes. Let \( \mathcal{G}_0 \) be a reductive group scheme over \( \mathcal{I}_0 \). Let \( (\mathcal{G}, \varphi) \) an inner gluing over \( \mathcal{I} \) of \( \mathcal{G}_0 \).

1. The section \( \varphi \) induces isomorphisms

\[
\text{Oc}(\text{Dyn}(\mathcal{G}_0)) \times_{\mathcal{I}_0} \mathcal{I} \xrightarrow{\sim} \text{Oc}(\text{Dyn}(\mathcal{G})),
\]

\[
\text{TypeStand}(\mathcal{G}_0) \times_{\mathcal{I}_0} \mathcal{I} \xrightarrow{\sim} \text{TypeStand}(\mathcal{G})
\]

which are compatible with \( q_{\mathcal{G}_0} \) and \( q_{\mathcal{G}} \).

2. Assume that \( \mathcal{G}_0 \) is split and \( \mathcal{I}_0 \) is connected. Let \( (\mathcal{G}_0, M, R) \) be a splitting of \( \mathcal{G}_0 \) and \( \Delta \) be a set of simple roots. Let \( (W, S) \) be the Coxeter system of \( (M, R, \Delta) \). Let \( t_0, t'_0 \in \text{Oc}(\text{Dyn}(\mathcal{G}_0))(\mathcal{I}_0) \). Let \( t, t' \in \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \) denote the pullbacks to \( \mathcal{I} \) of \( t_0, t'_0 \). Then \( \varphi \) induces an isomorphism

\[
(W_{t_0} \backslash W_{t_0'})_{\mathcal{I}} \xrightarrow{\sim} q_{\mathcal{G}}^{-1}(t, t').
\]
Proof. There is a Zariski covering \( \{ U_\lambda \}_\lambda \in \Lambda \) of \( S \) and a family of isomorphisms \( \varphi_\lambda : G_0 \times_{S_0} U_\lambda \cong G \times_S U_\lambda \) such that \( \varphi_\lambda \) is compatible with \( \varphi|_{U_\lambda} \). Then the family of isomorphisms \( \varphi_\lambda \) induce isomorphisms

\[
Oc(Dyn(G_0)) \times_{S_0} U_\lambda \cong Oc(Dyn(G \times_S U_\lambda)).
\]

These isomorphisms glue together to give the first isomorphism in the claim (1) by [SGA3-3, XXIV, 3.4 (iv)].

The family of isomorphisms \( \varphi_\lambda \) induce also isomorphisms

\[
\text{Stand}(G_0) \times_{S_0} U_\lambda \cong \text{Stand}(G \times_S U_\lambda).
\]

By taking the quotients by the conjugacy actions of \( G_0 \times_{S_0} U_\lambda \cong G \times_S U_\lambda \), we obtain isomorphisms

\[
\text{TypeStand}(G_0) \times_{S_0} U_\lambda \cong \text{TypeStand}(G \times_S U_\lambda).
\]

These isomorphisms glue together to give the second isomorphism in the claim (1) because we take quotients by conjugacy actions. By the constructions, two isomorphisms in the claim (1) are compatible with \( q_{G_0} \) and \( q_G \).

By (1) we have an isomorphism

\[
q^{-1}_{G_0}(t_0, t_0') \times_{S_0} S \cong q^{-1}_G(t, t') \tag{2.3}
\]

induced by \( \varphi \). The claim (2) follows from [SGA3-3, XXII, Proposition 3.4] and (2.3). \( \square \)

3 Stratification of Deligne–Lusztig variety

3.1 Deligne–Lusztig variety

Let \( G_0 \) be a connected reductive group over \( \mathbb{F}_q \). We take a maximal torus and a Borel subgroup \( T_0 \subset B_0 \subset G_0 \) over \( \mathbb{F}_q \). We write \( G, B \) and \( T \) for the base changes to \( \mathbb{F}_q \) of \( G_0, B_0 \) and \( T_0 \). Let \((W, S)\) be the Coxeter system of \( G \) with respect to \( T \) and \( B \). For \( I, J \subset S \), we write \( \text{Par}_I(G) \) and \( \text{Par}_{I,J}(G) \) for \( \text{Par}_{t(I)}(G) \) and \( \text{Par}_{t(I),t(J)}(G) \).

For \( I, J \subset S \) and \( w \in W \), we put

\[
\text{Par}_{I,J}(G)[w] = t_2^{-1}(r_w);
\]

where \( r_w \in (q_G^{-1}(t(I), t(J)))(\mathbb{F}_q) \) corresponds to \([w] \in W_I \setminus W/W_J \) by Lemma 2.6 (2). Let \( \text{Par}_{I,J}(G)_{\leq[w]} \) be the closed reduced subscheme of \( \text{Par}_{I,J}(G) \) determined by

\[
\bigcup_{[w'] \leq[w]} \text{Par}_{I,J}(G)[w'].
\]

Let \( F \) be the \( q \)-th power Frobenius endomorphism of \( G \) obtained from \( G_0 \). Let \( I \subset S \) and \( w \in W \). For \( * \in \{ [w], \leq[w] \} \) with \([w] \in W_I \setminus W/W_{F(I)} \), let \( X_F^I(*) \) be the locally closed subscheme of \( \text{Par}_I(G) \) defined by the fiber product

\[
\begin{align*}
X_F^I(*) \quad & \xrightarrow{\text{Par}_{I,F(I)}(G)_*} \text{Par}_{I,F(I)}(G), \\
\text{Par}_I(G) \quad & \xrightarrow{(\text{id}_F)} \text{Par}_I(G) \times \text{Par}_{F(I)}(G).
\end{align*}
\]
Let $3.2$ Bruhat stratification
which sends a parabolic subgroup $P$ of type $G$ of type $I$ to a unique parabolic subgroup $P'$ of $G$ of type $J$ containing $P$.

$3.3$ Stratification relative to Frobenius twists
For $1 \leq i \leq m$, let $F_i$ be an Frobenius endomorphism of $G$ which descends it to an algebraic group over a finite filed. Let $w_1, \ldots, w_m \in W$. For $* \in \{[w'], \leq [w']\}$ with $[w'] \in W_1 \setminus W/W_{F_i(j)}$ and $1 \leq i \leq m$, let $X_{F_i}^{*_1, \ldots, *_m}$ be the locally closed subscheme of $X_{F_i}$ defined by the fiber product

\[ X_{F_i}^{*_1, \ldots, *_m} \rightarrow \prod_{1 \leq i \leq m} \text{Par}_{F_i(j)}(G), \]

\[ \text{Par}_j(G) \prod_{1 \leq i \leq m} (\text{Par}_j(G) \times \text{Par}_{F_i(j)}(G)). \]

Then $X_{F_i}^{*_1, \ldots, *_m}([w_1], \ldots, [w_m])$ for $[w_i] \in W_1 \setminus W/W_{F_i(j)}$ and $2 \leq i \leq m$ give a stratification of $X_{F_i}^{*_1}([w_1])$.

$3.4$ Unitary case
We put $V_0 = F_{q^d}^d$ equipped with the hermitian form

\[ F_{q^2}^d \times F_{q^2}^d \rightarrow F_{q^2}; (a_1)_{1 \leq i \leq d}, (a'_1)_{1 \leq i \leq d} \rightarrow \sum_{i=1}^d a_i^* a'_i. \]

We put $G_0 = GU(V_0)$. By taking the first factor of the decomposition

\[ F_{q^2} \otimes F_q F_{q^2} \simeq F_{q^2} \times F_{q^2}; \ a \otimes b \mapsto (ab, ab^q), \]

we have an isomorphism

\[ G \simeq \text{GL}_d \times G_m. \quad (3.1) \]

Let $T \subset B \subset G$ be the maximal torus and the Borel subgroup determined by the diagonal torus $T_d$ and the upper triangular subgroup $B_d$ of $\text{GL}_d$ under $(3.1)$. Let $(W_G, \{s_1, \ldots, s_{d-1}\})$
be the Coxeter system of $G$ with respect to $T$ and $B$, where $s_i$ corresponds to the simple root
\[ T_d \times \mathbb{G}_m \to \mathbb{G}_m; \quad (\text{diag}(x_1, \ldots, x_d), z) \mapsto x_iz^{-1} \]
of $GL_d \times \mathbb{G}_m$ under (3.1). For $1 \leq i_1 < \cdots < i_t \leq d - 1$, we put
\[ I_{d-t}^{i_1, \ldots, i_t} = \{ s_i \}_{i \in (1, \ldots, d-1) \setminus \{ i_1, \ldots, i_t \}}. \]

**Lemma 3.1.** Assume that $2 \leq i \leq d/2$. The schemes $X_{I_{d}^{i-1, i}}^{F,F^2,F^3}([1], \leq [s_{i-1}], [1])$ and $X_{I_{d}^{i-1, d-i}}^{F,F^2,F^3}([1], \leq [s_{d-i}])$ are irreducible.

**Proof.** The scheme $X_{I_{d}^{i-1, d-i}}([1])$ is irreducible by [BR06, Theorem 1]. Hence, it suffices to show the following claims:

1. The image of
\[ \pi_{I_{d}^{i-1, d-i}, I_{d}^{i-1}}: X_{I_{d}^{i-1, d-i}}([1]) \to X_{I_{d}^{i-1}}([1]) \]
on $\mathbb{F}_q$-valued points is equal to $X_{I_{d}^{i-1}}^{F,F^2,F^3}([1], \leq [s_{i-1}], [1])(\mathbb{F}_q)$.

2. The image of
\[ \pi_{I_{d}^{i-1, d-i}, I_{d}^{i-1}}: X_{I_{d}^{i-1, d-i}}([1]) \to X_{I_{d}^{i-1}}([1]) \]
on $\mathbb{F}_q$-valued points is equal to $X_{I_{d}^{i-1}}^{F,F^2,F^3}([1], \leq [s_{d-i}])((\mathbb{F}_q)$.

We show the claim (1). We equip $\mathbb{F}_q^d$ with the paring
\[ \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q; \quad ((x_i)_{1 \leq i \leq d}, (y_i)_{1 \leq i \leq d}) \mapsto \sum_{i=1}^d x_iy_i. \quad (3.2) \]
For an $\mathbb{F}_q$-vector subspace $V \subset \mathbb{F}_q^d$, let $V^\perp$ denote the orthogonal complement of $V$ with respect to the paring (3.2). The $q$-th power Frobenius element $F$ acts on $\mathbb{F}_q^d$. A point of $X_{I_{d}^{i-1, d-i}}([1])(\mathbb{F}_q)$ corresponds to a filtration $0 \subset V_1 \subset V_2 \subset \mathbb{F}_q^d$ such that $\dim V_1 = i - 1$, $\dim V_2 = d - i$ and
\[ V_1 \subset F(V_1^+) \subset V_2 \subset F(V_1^+), \quad (3.3) \]
The condition (3.3) implies
\[ V_1 + F^2(V_1) \subset F(V_2^+), \quad (3.4) \]
Therefore we have
\[ F^3(V_1) \subset F(V_1 + F^2(V_1)) \subset F^2(V_2^+) \subset F(V_2) \subset V_1^+ \cap F^2(V_1^+) \subset V_1^+. \quad (3.5) \]
The conditions (3.3), (3.4) and (3.5) imply that $V_1$ defines a point of
\[ X_{I_{d}^{i-1}}^{F,F^2,F^3}([1], \leq [s_{i-1}], [1])(\mathbb{F}_q), \]
because $\dim F(V_1^+) = i$. To show the claim (1), it suffices to show that the image of
\[ \pi_{I_{d}^{i-1, d-i}, I_{d}^{i-1}} \text{ on } \mathbb{F}_q\text{-valued points contains} \]
\[ X_{I_{d}^{i-1}}^{F,F^2,F^3}([1], [s_{i-1}], [1])(\mathbb{F}_q), \quad (3.6) \]
because $X_{i,1,d-i}([1])$ is proper. A point of (3.6) gives an $\mathbb{F}_q$-vector subspace $V_1 \subset \mathbb{F}_q^d$ of dimension $i - 1$ such that

$$V_1 \subset F(V_1^\perp), \quad \dim(V_1 + F^2(V_1)) = i, \quad F^3(V_1) \subset V_1^\perp. \quad (3.7)$$

The condition implies

$$F(V_1 + F^2(V_1)) \subset V_1^\perp \cap F^2(V_1^\perp)$$

and $\dim V_1^\perp \cap F^2(V_1^\perp) = d - i$. We take $V_2 \subset \mathbb{F}_q^d$ such that $F(V_2) = V_1^\perp \cap F^2(V_1^\perp)$. Then $(V_1, V_2)$ defines a point of $X_{i,1,d-i}([1])(\mathbb{F}_q)$ whose image under $\pi_{i,1,d-i,\mu}$ is the point of (3.6) corresponding to $V_1$. Therefore we obtain the claim (1).

The claim (2) is proved similarly. \qed

4 Affine Grassmannian

Let $F$ be a non-archimedean local field with residue field $k = \mathbb{F}_q$. Let $\mathcal{O}$ be the ring of integers of $F$. Let $\varpi$ be a uniformizer of $F$. For a perfect $k$-algebra $R$, we put

$$W_\mathcal{O}(R) = \lim_{\rightarrow n} W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n,$$

$$D_R = \text{Spec}(W_\mathcal{O}(R)) \quad \text{and} \quad D_R^* = \text{Spec}(W_\mathcal{O}(R)[1/\varpi]).$$

For an affine group scheme $H$ of finite type over $\mathcal{O}$, we define the jet group $L^+H$ and the loop group $LH$ by

$$L^+H(R) = H(W_\mathcal{O}(R)), \quad LH(R) = H(W_\mathcal{O}(R)[1/\varpi]).$$

We put $L = W_\mathcal{O}(\mathbb{F}_q)[1/\varpi]$. We note that $LH(\mathbb{F}_q) = H(L)$.

Let $G$ be a reductive group scheme over $\mathcal{O}$. Let $T$ be the abstract Cartan subgroup. Let $\Phi \subset X^\times(T)$ denote the set of roots of $G$. We fix a Borel subgroup $B \subset G$, which determines the semi-group of dominant coweights $X^\bullet(T)^+ \subset X^\bullet(T)$. Let $U$ be the unipotent radical of $B$. Let $\rho \in X_\bullet(T)_Q$ be the half sum of all positive roots.

Let $\text{Gr}_G$ denote the affine Grassmannian over $k$ of $G$ defined by $\text{Gr}_G = LG/L^+G$. For a finite etale extension $\mathcal{O}'$ of $\mathcal{O}$ with residue field $k'$, we have a natural isomorphism

$$(\text{Gr}_G)_{k'} \cong \text{Gr}_{G_{\mathcal{O}'}} \quad (4.1)$$

by the construction. We simply write $\text{Gr}$ for $\text{Gr}_G$ if there is no confusion. Then $\text{Gr}$ is an ind-perfectly projective scheme by [BS17, Corollary 9.6]. Let $\mathcal{E}^0$ denote the trivial $G$-torsor over $\mathcal{O}$. For a perfect $k$-algebra $R$, we have

$$\text{Gr}(R) = \left\{ (\mathcal{E}, \beta) \mid \begin{array}{l} \mathcal{E} \text{ is a } G \text{-torsor on } D_R, \\ \beta: \mathcal{E}|_{D_R^*} \cong \mathcal{E}^0|_{D_R^*} \text{ is a trivialization.} \end{array} \right\} \quad (4.2)$$

(cf. [Zhu17, Lemma 1.3]). We sometimes write $\beta: \mathcal{E} \rightarrow \mathcal{E}^0$ for $\beta: \mathcal{E}|_{D_R^*} \cong \mathcal{E}^0|_{D_R^*}$ in (4.2), and call it a modification. Given a point $(\mathcal{E}, \beta)$, one can define a relative position invariant $\text{inv}(\beta) \in X_\bullet(T)^+$.

Let $\mu \in X_\bullet(T)^+$. The Schubert variety $\text{Gr}_\mu$ is the closed subscheme of $\text{Gr}_G$ parametrizing pairs $(\mathcal{E}, \beta)$ such that $\text{inv}(\beta) \leq \mu$. The Schubert cell $\text{Gr}_\mu$ is the open subscheme of $\text{Gr}_\mu$ parametrizing pairs $(\mathcal{E}, \beta)$ such that $\text{inv}(\beta) = \mu$. 8
For a sequence $\mu_* = (\mu_1, \ldots, \mu_n)$ of positive dominant coweights, let $\Gr_{\mu_*}$ be the scheme over $k$ parametrizing sequences of modifications $(\beta_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1})_{1 \leq i \leq n}$ with $\mathcal{E}_0 = \mathcal{E}$ such that $\text{inv}(\beta_i) \leq \mu_i$ for each $i$. The open subscheme $\Gr_{\mu_*} \subset \Gr_{\mu_*}$ is defined by the condition that $\text{inv}(\beta_i) = \mu_i$ for each $i$. The convolution map $m_{\mu_*} : \Gr_{\mu_*} \rightarrow \Gr_T$ sends a sequence of modifications to the composition $(\mathcal{E}_n, \beta_1 \circ \cdots \circ \beta_n)$.

Let $\lambda_* = (\lambda_1, \ldots, \lambda_l)$ and $\mu_* = (\mu_1, \ldots, \mu_n)$ be two sequences. We write

$$\Gr_{\lambda_*|\mu_*} = \Gr_{\lambda_*} \times_{\Gr_T} \Gr_{\mu_*}, \quad \Gr_{\lambda_*|\mu_*}^0 = \Gr_{\lambda_*} \times_{\Gr_T} \Gr_{\mu_*}^0,$$

where the products are over the convolution maps $m_{\lambda_*} : \Gr_{\lambda_*} \rightarrow \Gr_T$, $m_{\mu_*} : \Gr_{\mu_*} \rightarrow \Gr_T$ and their restrictions. We write

$$m_{\lambda_*|\mu_*} : \Gr_{\lambda_*|\mu_*}^0 \rightarrow \Gr_T$$

for the natural projection. We simply write $m$ for $m_{\lambda_*|\mu_*}$ if there is no confusion. For $1 \leq j \leq l$, we define

$$\text{pr}_j : \Gr_{\lambda_*|\mu_*}^0 \rightarrow \Gr_{(\lambda_1, \ldots, \lambda_j)}$$

by sending $((\alpha_i)_{1 \leq i \leq l}, (\beta_i)_{1 \leq i \leq n})$ to $(\alpha_i)_{1 \leq i \leq j}$.

An irreducible component of $\Gr_{\lambda_*|\mu_*}^0$ of dimension $\langle \rho, |\lambda_*| + |\mu_*| \rangle$ is called a Satake cycle. Let $S_{\lambda_*|\mu_*}$ be the set of Satake cycles in $\Gr_{\lambda_*|\mu_*}^0$. We sometimes write $\Gr_{\lambda_*|\mu_*}^{0,a}$ instead of $a \in S_{\lambda_*|\mu_*}$ for the Satake cycle. We put

$$\Gr_{\lambda_*|\mu_*}^{0,a} = \Gr_{\lambda_*|\mu_*}^{0} \cap \Gr_{\lambda_*|\mu_*}^{0,a}.$$

We fix an embedding $T \subset B$. Let $\mu \in X_*(T)$. Let $\mathcal{O}' = \mathcal{O}_L = W_{\mathcal{O}}(\overline{k})$ or a finite etale extension of $\mathcal{O}$ which splits $G$. For $\alpha \in \Phi$, let $U_{\alpha,\mathcal{O}'}$ denote the root subgroup of $G_{\mathcal{O}'}$ corresponding to $\alpha$. Let $P_{\mu,\mathcal{O}'}$ denote the parabolic subgroup of $G_{\mathcal{O}'}$ generated by $T_{\mathcal{O}'}$ and $U_{\alpha,\mathcal{O}'}$ for $\alpha \in \Phi$ such that $\langle \alpha, \mu \rangle$.

We write $\varpi^\mu$ for $\mu(\varpi) \in G(L) = LG(\overline{k})$. Let $[\varpi^\mu]$ denote the point of $\Gr_T$ determined by $\varpi^\mu$. For $\mu \in X_*(T)^+$, let $S_\mu$ be the Schubert cell $\Gr_\mu$ is the $L^+G$-orbit of $[\varpi^\mu]$.

For $\lambda \in X_*(T)$, let $S_\lambda$ be the $(LU)_{\overline{k}}$-orbit of $\varpi^\lambda$ in $\Gr_T$. For $\lambda \in X_*(T)$ and $\mu \in X_*(T)^+$, an irreducible component of $S_\lambda \cap \Gr_\mu$ is called a Mirković–Vilonen cycle after [MV07]. Let $MV_\mu(\lambda)$ be the set of the Mirković–Vilonen cycles in $S_\lambda \cap \Gr_\mu$. We sometimes write $(S_\lambda \cap \Gr_\mu)^b$ instead of $b \in MV_\mu(\lambda)$ for the Mirković–Vilonen cycle.

Let $(\hat{G}, \hat{B}, \hat{T})$ be the Langlands dual over $\overline{k}$ of $(G, B, T)$. For $\mu \in X_*(T)^+ = X^*(\hat{T})^+$, let $V_\mu$ denote the irreducible algebraic representation of $\hat{G}$ of highest weight $\mu$. For an algebraic representation $V$ of $\hat{G}$ and $\lambda \in X_*(T) = X^*(\hat{T})$, let $V(\lambda)$ denote the $\lambda$-weight space of $V$. Then we have

$$|MV_\mu(\lambda)| = \dim V_\mu(\lambda) \quad (4.3)$$

by [GHKR09] Proposition 5.4.2 and [Zhu17] Corollary 2.8.

For $\nu, \mu \in X_*(T)^+$ and $\lambda \in X_*(T)$ such that $\nu + \lambda \in X_*(T)^+$, there is an injective map

$$i^\text{MV}_{\nu} : S_{(\nu,\mu),\nu+\lambda} \rightarrow MV_\mu(\lambda)$$

constructed by [XZ17] Lemma 3.2.7.
5 Equidimensionality of Satake cycles

Lemma 5.1. The morphism \( m_{\mu*} : \text{Gr}_{\mu*} \to \text{Gr}_K \) is Zariski-locally trivial on \( \hat{\text{Gr}}_\lambda \).

Proof. Taking the base change to an unramified extension of \( \mathcal{O} \), we may assume that \( G \) is split by \([4.1]\). As in the proof of \([\text{Hai06, Lemma 2.1}]\), it suffices to show that
\[
L^+ G \to L^+ G/(L^+ G \cap \varpi^\lambda L^+ G \varpi^{-\lambda})
\]
has a section Zariski-locally. Since \( L^+ U/(L^+ U \cap \varpi^\lambda L^+ U \varpi^{-\lambda}) \) is an open subscheme of \( L^+ G/(L^+ G \cap \varpi^\lambda L^+ G \varpi^{-\lambda}) \), it suffice to show that
\[
L^+ U \to L^+ U/(L^+ U \cap \varpi^\lambda L^+ U \varpi^{-\lambda})
\]
has a section. We fix an identification \( G_{\alpha} \cong U_{\alpha,\mathcal{O}} \) for a positive root \( \alpha \). For a positive root \( \alpha \), let \( L^+_{<\alpha,\lambda} U_{\alpha,\mathcal{O}} \) be a closed subscheme of \( L^+ U_{\alpha,\mathcal{O}} \) defined by the condition \( x_i = 0 \) for \( i \geq \langle \alpha, \lambda \rangle \) for a point \( \sum_{i=0}^{\infty} \varpi^i[x_i] \) of \( L^+ U_{\alpha,\mathcal{O}} \). Then the composition
\[
\prod_{\alpha} L^+_{<\alpha,\lambda} U_{\alpha,\mathcal{O}} \to L^+ U \to L^+ U/(L^+ U \cap \varpi^\lambda L^+ U \varpi^{-\lambda})
\]
is an isomorphism. Hence we have a section. \( \square \)

Lemma 5.2. Assume that \( \mu \) is a dominant minuscule cocharacter and \( w \in W \). We have an isomorphism
\[
S_{w\mu} \cap \text{Gr}_{\mu} \cong L^+ U_T/((L^+ U)_T \cap \varpi^w(L^+ U)_T \varpi^{-w})
\]
In particular, \( S_{w\mu} \cap \text{Gr}_{\mu} \) is a perfection of an affine space of dimension \( \langle \rho, |\mu| - \lambda \rangle \).

Proof. The first claim follows from \([\text{XZ17, (3.2.3)}]\). The second claim follows from the first one as in the proof of \([\text{Hai06, Lemma 3.2}]\). \( \square \)

Theorem 5.3. Assume that \( \mu_i \) are minuscule. For a point \( y \) of \( \hat{\text{Gr}}_\lambda \), the fiber of \( m_{\mu*} : \text{Gr}_{\mu*} \to \text{Gr}_K \) at \( y \) is equidimensional of dimension \( \langle \rho, |\mu| - \lambda \rangle \).

Proof. This is proved in the same way as \([\text{Hai06, Theorem 3.1}]\) using Lemma 5.1 and Lemma 5.2 instead of \([\text{Hai06, Lemma 2.1 and 3.2}]\) respectively. \( \square \)

Proposition 5.4. Assume that each \( \mu_i \) is a sum of minuscule cocharacters. Then, for a point \( y \) of \( \text{Gr}_\lambda \), any irreducible component of the fiber \( m_{\mu*}^{-1}(y) \) whose generic point belongs to \( \text{Gr}_{\mu*} \) has dimension \( \langle \rho, |\mu| - \lambda \rangle \).

Proof. This follows from Theorem 5.3 in the same way as \([\text{Hai06, Proposition 4.1}]\). \( \square \)

6 Affine Deligne–Lusztig variety

Recall that \( L = W_\mathcal{O}(\mathbb{T})[1/\varpi] \). Let \( b \in G(L) \) and \( \mu \in X_\bullet(T) \). Let \( \sigma \) denote the \( q \)-th power Frobenius element. We define the affine Deligne–Lusztig variety \( X_{\mu}(b) \) by
\[
X_{\mu}(b) = \{ g(L^+ G)_T \in \text{Gr}_T \mid g^{-1}b\sigma(g) \in (L^+ G)_T \varpi^\mu(L^+ G)_T \}.
\]
Let $B(G)$ be the set of $\sigma$-conjugacy classes of $G(L)$. We define $B(G, \mu) \subset B(G)$ as [Kol97, 6.2]. Then $X_\mu(b)$ is non-empty if and only if $[b] \in B(G, \mu)$ by [Gas10 Theorem 5.1].

An element of $B(G)$ is called unramified if it is contained in the image of the natural map $B(T) \to B(G)$. Let $B(G)_{ur}$ denote the set of unramified elements of $B(G)$. For $\chi \in \mathbb{X}^\bullet(T)$, we put

$$\Gamma = \frac{1}{|\langle \sigma \rangle \chi|} \sum_{\chi' \in \mathbb{X}^\bullet(T)_Q} \chi'. $$

The natural paring $\mathbb{X}_\bullet(T) \times \mathbb{X}^\bullet(T) \to \mathbb{Z}$ induces a paring $\langle \ , \rangle : \mathbb{X}_\bullet(T)_\sigma \times \mathbb{X}^\bullet(T)_Q \to \mathbb{Q}$. We put

$$\mathbb{X}_\bullet(T)_\sigma^+ = \{[\lambda] \in \mathbb{X}_\bullet(T)_\sigma \mid \langle [\lambda], \overline{\alpha} \rangle \geq 0 \text{ for every } \alpha \in \Delta\}. $$

Then we have a bijection

$$\mathbb{X}_\bullet(T)_\sigma^+ \cong B(G)_{ur}; \ [\lambda] \mapsto [\varpi^\lambda] $$

as in [XZ17 Lemma 4.2.3].

For $\tau \in \mathbb{X}_\bullet(T)$, we write $X_\mu(\tau)$ for $X_\mu(\varpi^\tau)$. We assume that $b = \varpi^\tau$ for $\tau \in \mathbb{X}_\bullet(T)$ such that $[\tau] \in \mathbb{X}_\bullet(T)_\sigma^+$. We can define the twisted centralizer $J_\tau$ over $\mathcal{O}$ for $\varpi^\tau$ as in [XZ17 4.2.13]. We note that $J_\tau = G$ if $[b] \in B(G)_{ur}$ is basic.

We assume that $G$ satisfies [XZ17 Hypothesis 4.4.1]. Further, we assume that $Z_G$ is connected.

Let $\lambda \in \mathbb{X}_\bullet(T)$ such that $[\lambda] = [\tau] \in \mathbb{X}_\bullet(T)_\sigma^+$. We take $\delta_\lambda \in \mathbb{X}_\bullet(T)$ such that $\lambda = \tau + \delta_\lambda - \sigma(\delta_\lambda)$. Let $b \in \mathbb{MV}_\mu(\lambda)$. We take $\nu_\lambda \in \mathbb{X}_\bullet(T)$ as in [XZ17 Lemma 4.4.3]. We put $\nu_b = \lambda + \nu_\lambda - \sigma(\nu_\lambda)$. Then we have an isomorphism

$$X_\mu(b) = X_\mu(\tau) \simeq X_\mu(\nu_b); \ gL^+G \mapsto \varpi^{\delta_\lambda+\nu_b}gL^+G. \quad (6.1) $$

Let $a \in S_{(\nu_b, \nu)}\lambda+\nu_b$ be the unique element such that $b = i_{\nu_b}^{\mathbb{MV}}(a)$. We define $X_{\mu, \nu_b}(\tau_b)$ by the fibre product

$$X_{\mu, \nu_b}(\tau_b) \longrightarrow \text{Gr}_{(\nu_b, \mu)}\tau_b + \sigma(\nu_b) $$

Further, we define $X_{\mu, \nu_b}^a(\tau_b)$ by the fibre product

$$X_{\mu, \nu_b}^a(\tau_b) \longrightarrow \text{Gr}_{(\nu_b, \mu)}\tau_b + \sigma(\nu_b) $$

Let $x_0$ denote $[1] \in J_\tau(F)/J_\tau(\mathcal{O})$. We put

$$\hat{X}_{\mu, \nu_b}^{b,x_0}(\tau_b) = X_{\mu, \nu_b}(\tau_b) \cap \text{Gr}_{\nu_b}. $$

Let $X_{\mu, \nu_b}^a(\tau_b)$ denote the closure of $\hat{X}_{\mu, \nu_b}^{b,x_0}(\tau_b)$ in $X_{\mu, \nu_b}^a(\tau_b)$. 

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By [XZ17, Theorem 4.4.14], there is a bijection between the set

$$\bigsqcup_{\lambda \in \mathfrak{X}^*(T), \, [\lambda] = [\tau] \in \mathfrak{X}_d(T)^+} \text{MV}_\mu(\lambda) \times J_r(F)/J_r(O)$$

and the set of irreducible components of $X_\mu(b)$ given by

$$(b, [g]) \mapsto X^{b,[g]}(\tau_b) := gX^{b,x}(\tau_b),$$

where we regard $X^{b,[g]}(\tau_b)$ as a subscheme of $X_\mu(b)$ by (6.1).

## 7 Unitary group

### 7.1 Setting

Let $F_2$ be the quadratic unramified extension of $F$. Let $O_2$ denote the ring of integers of $F_2$. Let $\varpi$ be a uniformizer of $F$. We put $\Lambda = O_2^n$ equipped with the hermitian form

$$O_2^n \times O_2^n \to O_2; \, ((a_i)_{1 \leq i \leq d}, (a'_i)_{1 \leq i \leq d}) \mapsto \sum_{i=1}^n \sigma(a_i)a'_i.$$ 

We put $G = \text{GU}(\Lambda)$. By taking the first factor of the decomposition

$$O_2 \otimes_O O_2 \simeq O_2 \times O_2; \, a \otimes b \mapsto (ab, a\sigma(b)),$$

we have an isomorphism

$$G_{O_2} \simeq \text{GL}_n \times \mathbb{G}_m. \quad (7.1)$$

We put $V = \Lambda \otimes_O F$. Let $\hat{G} = \text{GL}_n \times \mathbb{G}_m$ denote the dual group over $\mathbb{Q}_\ell$ with a maximal torus $\hat{T}$ and a Borel subgroup $\hat{B}$, which are the diagonal torus and the upper triangular subgroup on the $\text{GL}_n$-component. For an index $i \in \{1, \ldots, n\}$, we will use the notation $i' = n + 1 - i$. The group $\hat{X}^*(\hat{T})$ has a basis $\{\varepsilon_i\}_{i=0}^n$, where $\varepsilon_0$ is the projection to the $\mathbb{G}_m$-component and $\varepsilon_i$ is the character of $\hat{T}$ given by evaluating the $(i, i)$ entry for $i \geq 1$. In the following, all cocharacters of $T$ (equivalently, characters of $\hat{T}$) will be written according to this basis. We have $\sigma(\varepsilon_0) = \varepsilon_0$ and $\sigma(\varepsilon_i) = \varepsilon_{i'}$ for $1 \leq i \leq n$.

### 7.2 Satake cycle

Let $\mu = \varepsilon_1 + \varepsilon_2 \in \mathfrak{X}_d(T)$. We put $r = [n/2]$. We put

$$\nu_i = \varepsilon_1 + \cdots + \varepsilon_{i-1} - \varepsilon_{i'} - \cdots - \varepsilon_1, \quad \tau_i = 0$$

for $1 \leq i \leq [(n-1)/2]$, and

$$\nu_r = \varepsilon_1 + \cdots + \varepsilon_{r-1}, \quad \tau_r = \varepsilon_1 + \cdots + \varepsilon_n$$

if $n$ is even. We put $\lambda_i = -\varepsilon_i - \varepsilon_{i'}$ for $1 \leq i \leq r$.

**Lemma 7.1.** For $\lambda \in (1-\sigma)\mathfrak{X}_d(T)$, we have $\text{MV}_{\nu_i}(\lambda) \neq \emptyset$ if and only if $\lambda \in \{\lambda_1, \ldots, \lambda_r\}$. Further, $\text{MV}_{\nu_i}(\lambda_i)$ is a singleton for $1 \leq i \leq r$. 

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Proof. For $\lambda \in X_\bullet(T)$, we have $\dim V_{\mu^*}(\lambda) \leq 1$, and $V_{\mu^*}(\lambda)$ is nonzero if and only if $\lambda = -\epsilon_i - \epsilon_j$ for some $1 \leq i < j \leq n$. If $-\epsilon_i - \epsilon_j \in (1 - \sigma)X_\bullet(T)$ for some $1 \leq i < j \leq n$, we must have $j = i'$. Hence the claim follows from (4.3).

Let $1 \leq i \leq r$. Note that $\sigma(\nu_i^*) = \lambda_i + \nu_i^* - \tau_i^*$. Let $b_i$ be the unique element of $MV_{\mu^*}(\lambda_i)$. There is $a_i \in S(\nu_i^*, \mu^*)_{\lambda_i + \nu_i^*}$ such that $\iota_{\nu_i^*}^{MV}(a_i) = b_i$. Since $\iota_{\nu_i^*}^{MV}$ is injective by [XZ17, Lemma 3.2.7], the set $S(\nu_i^*, \mu^*)_{\lambda_i + \nu_i^*}$ is also a singleton.

We study the Satake cycle $\Gr_{\nu_i^*, \mu^*}(\sigma(\nu_i^*))$.

**Lemma 7.2.** We have

$$\Gr_{\nu_i^*, \mu^*}^{0, a_i} = \Gr_{\nu_i^*, \mu^*}^{0}.$$  

**Proof.** By the definition, $\Gr_{\nu_i^*, \mu^*}^{0}(\sigma(\nu_i^*))$ is equal to the inverse image of $\Gr_{\sigma(\nu_i^*)}$ under the convolution morphism

$$m_{\nu_i^*, \mu^*} : \Gr_{\nu_i^*, \mu^*} \to \Gr.$$  

By Lemma 5.1 and Proposition 5.2, $\Gr_{\nu_i^*, \mu^*}^{0}(\sigma(\nu_i^*))$ is irreducible, since $S(\nu_i^*, \mu^*)_{\lambda_i + \nu_i^*}$ is a singleton. Therefore, we obtain the claim.

We do not use the following lemma in the sequel, but it shows that a study of intersections of affine Deligne–Lusztig varieties is more subtle than intersections of Satake cycles.

**Lemma 7.3.** (1) The actions of $L^+G$ on $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_2}$ are transitive.

(2) The Satake cycle $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ contains $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_1}} |_{\nu_1}$.

**Proof.** We show (1). It suffices to show that the number of the orbits under the action of $L^+G$ on $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is 2. Let $(L^+G)_{\nu_1}$ be the stabilizer of $[\nu_1] \in \Gr_{\nu_1}$ in $L^+G$. Since the action of $L^+G$ on $\Gr_{\nu_1}$ is transitive, it suffices to show that the number of the orbits in $m_{\nu_i^*, \mu^*}^{-1}(\nu_1) \cdot (\nu_1)$ under the action of $(L^+G)_{\nu_1}$ is 2. These orbits are in a bijection with $(P_{\nu_1}, \nu_1) \cdot \Gr_{\nu_1} / (P_{\nu_1}, \nu_1) T$. Hence the number of the orbits is 2.

We show (2). Since the morphism $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2} \to \Gr_{\nu_2}$ is perfectly proper, the image of $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ under the morphism is equal to $\Gr_{\nu_2}$. Hence the intersection of $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is not empty.

If the intersection of $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is not empty, then $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ contains $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ because $L^+G$ acts transitively on $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_2}$ is stable under the action of $L^+G$. Then $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ contains $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_2}$, since $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is dense in $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$.

If the intersection of $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is empty, the intersection of $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is not empty. Then $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_2}$ contains $\Gr_{\nu_i^*, \mu^*}^{0, a_{\nu_2}} |_{\nu_1}$ because $L^+G$ acts transitively on $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ and $\Gr_{\nu_i^*, \mu^*}^{0} |_{\nu_1}$ is stable under the action of $L^+G$. 


8 Irreducible Components

We note that $[1] \in B(G, \mu^*)$ is the basic class. By results in [8] and Lemma 7.1, $X_{\mu^*}(1)$ has $r$ isomorphism classes of irreducible components. The $r$ isomorphism classes of irreducible components are given by $X_{\mu^*}^{b_i} \tau_i^*(\tau_i^*)$ for $1 \leq i \leq r$, where $X_{\mu^*}^{b_i} \tau_i^*(\tau_i^*)$ is the closures in $X_{\mu^*}(\tau_i^*)$ of $X_{\mu^*}^{b_i} \tau_i^*(\tau_i^*)$ fitting in the cartesian diagram

$$
\begin{aligned}
&\xymatrix{X_{\mu^*}^{b_i} \tau_i^*(\tau_i^*) \ar[r] & \text{Gr}_{\tau_i^*}^{0,\eta} \ar[d] \ar[r] & \text{Gr}_{\tau_i^*}^{0,\eta + \sigma(\nu_i^*)} \\
&\hat{\text{Gr}}_{\nu_i^*} \ar[u] \ar[r]^{1 \times p_\tau^\star} & \hat{\text{Gr}}_{\nu_i^*} \times \text{Gr}_{\tau_i^* + \sigma(\nu_i^*)}.}
\end{aligned}
$$

If $i = 1$, the above construction defines a Deligne–Lusztig variety. If $i = 2$ and $n \geq 5$, this defines a variety that is not a Deligne–Lusztig variety.

By (4.1) and (7.1), we have an isomorphism

$$
\text{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} \simeq \text{Gr}_{\mathbb{G}_m} \times \mathbb{G}_m.
$$

We put $\mathcal{E}^0 = \mathcal{O}_2^n$ and $\mathcal{L}^0 = \mathcal{O}_2$ and view them as trivial vector bundles on $D_{\mathbb{F}_{q^2}}$. For any perfect $\mathbb{F}_{q^2}$-algebra $R$,

$$
\text{Gr}_{\mathbb{G}_m} \times \mathbb{G}_m(R) = \left\{(\mathcal{E}, \mathcal{L}, \beta, \beta') \mid \begin{array}{l}
\mathcal{E} \text{ is a vector bundle on } D_R \text{ of rank } n, \\
\mathcal{L} \text{ is a line bundle on } D_R, \\
\beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ and } \\
\beta' : \mathcal{L}|_{D_R^*} \simeq \mathcal{L}^0|_{D_R^*} \text{ are trivializations.}
\end{array}\right\}
$$

(8.2)

by (4.2). Under the identification by (8.1), the Frobenius endomorphism of $\text{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ sends $(\mathcal{E}, \mathcal{L}, \beta, \beta')$ in (8.2) to

$$
(F(\mathcal{E}^\star) \otimes \mathcal{L}, F(\mathcal{E}^\star)^{-1} \otimes \beta', \beta')
$$

where

$$
F(\mathcal{E}^\star)^{-1} \otimes \beta' : (F(\mathcal{E}^\star) \otimes \mathcal{L})|_{D_R} \simeq (F(\mathcal{E}^0)^\star) \otimes \mathcal{L}^0)|_{D_R} = \mathcal{E}^0|_{D_R}.
$$

We regard $\text{Gr}_{\mathbb{G}_m}$ as an open and closed sub-ind-scheme of $\text{Gr}_{\mathbb{G}_m} \times \mathbb{G}_m$ by

$$
(\mathcal{E}, \beta) \mapsto (\mathcal{E}, \mathcal{L}^0, \beta, \text{id})
$$

If $\lambda \in \mathbb{X}_*(T)$ is trivial on $\mathbb{G}_m$-component under the identification (7.1), then we view $\text{Gr}_{G,\lambda}$ as a subscheme of $\text{Gr}_{\mathbb{G}_m} \subset \text{Gr}_{\mathbb{G}_m} \times \mathbb{G}_m$ under the identification (8.1). Under the identification by (8.1), the Frobenius endomorphism of $\text{Gr}_G \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ becomes

$$
(\mathcal{E}, \beta) \mapsto (F(\mathcal{E}^\star), F(\mathcal{E}^\star)^{-1})
$$

in $\text{Gr}_{\mathbb{G}_m}$.

8.1 Component for $\nu_1$

An analogue of a component in [VW11].
Proposition 8.1. The irreducible component $X_{\nu_{1}^*,\mu}^{b_{1},x_{0}}(\tau_{1}^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^{0}$ bounded by $\nu_{1}^*$ such that $\varpi F(\mathcal{E}^\nu) \subset \mathcal{E}$. In particular, it is isomorphic to $X_{\nu_{1}^*,\mu}^{b_{1},x_{0}}([1])^{pf}$.

Proof. We have

$$\text{Gr}_{\nu_{1}^*,\mu}^{0,*}|_{\nu_{1}} = \text{Gr}_{\nu_{1}^*,\mu}^{0,*} = \text{Gr}_{\nu_{1}^*,\mu}^{0,*}$$

since $\nu_{1}$ is minuscule. Hence we have $X_{\nu_{1}^*,\mu}^{b_{1},x_{0}}(\tau_{1}^*) = X_{\mu,\nu_{1}^*}(\tau_{1}^*)$. By the definition, $X_{\mu,\nu_{1}^*}(\tau_{1}^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^{0}$ bounded by $\nu_{1}^*$ such that $F(\mathcal{E}^\nu) \rightarrow \mathcal{E}$ is bounded by $\mu^*$. The condition that $F(\mathcal{E}^\nu) \rightarrow \mathcal{E}$ is bounded by $\mu^*$ is equivalent to $\varpi F(\mathcal{E}^\nu) \subset \mathcal{E}$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}^\nu/\mathcal{E}^{0} \subset \frac{1}{\varpi} \mathcal{E}^{0}/\mathcal{E}^{0}$.

8.2 Components for $\nu_{r}$ when $n$ is even.

A generalization of a component in [HP14].

Proposition 8.2. The irreducible component $X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}(\tau_{r}^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^{0}$ bounded by $\nu_{r}^*$ such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\nu)$. In particular, it is isomorphic to $X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}([1])^{pf}$.

Proof. We have

$$\text{Gr}_{\nu_{r}^*,\mu}^{0,*}|_{\nu_{r}} = \text{Gr}_{\nu_{r}^*,\mu}^{0,*} = \text{Gr}_{\nu_{r}^*,\mu}^{0,*}$$

since $\nu_{r}^*$ and $\tau_{r}^* + \nu_{r}$ are minuscule. Hence we have $X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}(\tau_{r}^*) = X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}(\tau_{r}^*) = X_{\mu,\nu_{r}^*}(\tau_{r}^*)$. It is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^{0}$ bounded by $\nu_{r}^*$ such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\nu)$.

By the definition, $X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}(\tau_{r}^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^{0}$ bounded by $\nu_{r}^*$ such that $\frac{1}{\varpi} F(\mathcal{E}^\nu) \rightarrow \mathcal{E}^{0}$ is bounded by $\mu^*$. The condition that $\frac{1}{\varpi} F(\mathcal{E}^\nu) \rightarrow \mathcal{E}^{0}$ is bounded by $\mu^*$ is equivalent to $\varpi \mathcal{E} \subset F(\mathcal{E}^\nu)$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/\mathcal{E}^{0} \subset \frac{1}{\varpi} \mathcal{E}^{0}/\mathcal{E}^{0}$.

Example 8.3. Assume that $n = 4$. Then $X_{\nu_{r}^*,\mu}^{b_{r},x_{0}}(\tau_{r}^*)$ is isomorphic to the perfection of the Fermat hypersurface defined by

$$x_{1}^{q+1} + x_{2}^{q+1} + x_{3}^{q+1} + x_{4}^{q+1} = 0$$

in $\mathbb{P}^{3}$. This is a component which appears in [HP14].

8.3 Non-minuscule case

Let $2 \leq i \leq [(n - 1)/2]$. We put

$$\nu_{i,+} = \varepsilon_{1} + \cdots + \varepsilon_{i-1}, \quad \nu_{i,-} = -\varepsilon_{i} - \cdots - \varepsilon_{1,\nu}.$$ 

We put $\xi = \varepsilon_{1} + \cdots + \varepsilon_{n-2}$. Let $(\text{Gr}_{\nu_{i,+}}^{*} \times \text{Gr}_{\nu_{i,-}}^{*})_{\xi}$ be a subspace of $\text{Gr}_{\nu_{i,+}}^{*} \times \text{Gr}_{\nu_{i,-}}^{*}$ defined by the condition that

$$\mathcal{E}^{\beta_{+}} \rightarrow \mathcal{E}^{0} \rightarrow \mathcal{E}_{+}$$

is bounded by $\xi$ for for a point $(\mathcal{E}^{\beta_{+}} \rightarrow \mathcal{E}^{0}, \mathcal{E}^{\beta_{-}} \rightarrow \mathcal{E}^{0})$ of $\text{Gr}_{\nu_{i,+}}^{*} \times \text{Gr}_{\nu_{i,-}}^{*}$. Let

$$(\text{Gr}_{\nu_{i,+}}^{*} \times \text{Gr}_{\nu_{i,-}}^{*} \rightarrow \text{Gr}_{\nu_{i,+}}^{*} \times \text{Gr}_{\nu_{i,-}}^{*} \xi)$$
be a subspace of $\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}(\nu_{i-},\nu_{i+})$ defined by the condition that

$$E_{-} \xrightarrow{\beta_{-}} \frac{\beta_{-}}{E_{+}} \xrightarrow{\beta_{+}} \frac{\beta_{+}}{E_{0}} \xrightarrow{\beta_{0}} \frac{\beta_{0}}{E_{+}}$$

is bounded by $\xi$ for a point

$$(E \xrightarrow{\beta_{-}} E_{+} \xrightarrow{\beta_{+}} E_{0} \xrightarrow{\beta_{0}} E_{+})$$

of $\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}(\nu_{i-},\nu_{i+})$. We have a natural morphism

$$\pi_{1}: (\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}(\nu_{i-},\nu_{i+}))_{\xi} \rightarrow \text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}}.$$ 

Let $\mathcal{Y}_{i}$ be a vector bundle of rank $2i - 1$ over $(\text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}})_{\xi}$ defined by $E_{+}/E_{-}$, where $(E_{+}, E_{-})$ is a point of $(\text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}})_{\xi}$. We put $\mathcal{Y}_{i} = \text{ Aut}(\mathcal{Y}_{i})$. Let $t_{i,Z} \in \text{ Oc}(\text{Dyn}(\text{GL}_{2i-1,Z})(Z))$ be the image under

$$t(Z): \text{ Par}(\text{GL}_{2i-1,Z})(Z) \rightarrow \text{ Oc}(\text{Dyn}(\text{GL}_{2i-1,Z})(Z))$$

of the parabolic subgroup of $\text{GL}_{2i-1,Z}$ defined as the stabilizer of $Z^{i-1} \subset Z^{i-1} \oplus Z^{i} = Z^{2i-1}$. Let

$$t_{i} \in \text{ Oc}(\text{Dyn}(\mathcal{Y}_{i}))((\text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}}))_{\xi} \quad (8.3)$$

the element determined from $t_{i,Z}$ by Remark 2.3 and Lemma 2.6 [2]

We define a morphism

$$\Psi: (\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}_{\nu_{i}} \times \text{Gr}(\nu_{i-},\nu_{i+}))_{\xi} \rightarrow \text{ Par}_{t_{i}}(\mathcal{Y}_{i})$$

by sending

$$(E \xrightarrow{\beta_{-}} E_{+} \xrightarrow{\beta_{+}} E_{0} \xrightarrow{\beta_{0}} E_{+})$$

to the stabilizer of $E/E_{-} \subset E_{+}/E_{-}$. Then $\Psi$ is an isomorphism. Note that a natural morphism

$$\pi_{0}: (\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}_{\nu_{i}} \times \text{Gr}(\nu_{i-},\nu_{i+}))_{\xi} \rightarrow \text{ Gr}_{\nu_{i}}$$

is an isomorphism over $\text{ Gr}_{\nu_{i}}$.

Recall that $X_{\mu_{i}}^{\nu_{i}}(\tau_{i})$ and $X_{\mu_{i}}^{\nu_{i}}(\tau_{i})$ are closed subspaces of $\text{ Gr}_{\nu_{i}}$. The condition for the subspace

$$\pi_{0}^{-1}(X_{\mu_{i}}^{\nu_{i}}(\tau_{i})) \subset (\text{Gr}(\nu_{i+},\nu_{i-}) \times \text{Gr}_{\nu_{i}} \times \text{Gr}(\nu_{i-},\nu_{i+}))_{\xi}$$

is that $E \subset F(E_{\nu_{i}})^{\perp} \subset \frac{1}{\nu_{i}} E$.

For a point $(E_{+}, E_{-})$ of $(\text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}})_{\xi}$, we put $\mathcal{W} = E/E_{-} \subset E_{+}/E_{-}$, which is a subvector bundle of rank $i - 1$. Let $\mathcal{W}^{\perp} \subset E_{+}/E_{-}$ be the annihilator of $\mathcal{W}$. Then we have $\mathcal{W}^{\perp} = E_{\nu_{i}}^{\perp}/E_{\nu_{i}}^{\perp}$.

Let $Y_{i}$ be the closed subspace of $(\text{Gr}_{\nu_{i+}} \times \text{Gr}_{\nu_{i-}})_{\xi}$ defined by the conditions

1. $E_{+} \subset F(E_{\nu_{i}}^{\perp})$,
2. $E_{-} \subset F(E_{\nu_{i}}^{\perp})$,
3. $\pi_{0}^{*}E_{\nu_{i}}^{\perp} \subset F(E_{-})$. 

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Then we have $Y_i = X_{\pi^{-1}_{\mu}([1])^\mu}$. Assume that $(E_+, E_-)$ is a point of $Y_i$. The condition $E \subset F(E^\vee)$ is equivalent to that the image of $W$ under the natural morphism

$$
\phi_1 : E_+/E_- \to F(E^\vee)/F(E^\vee_+) = F(E_+^\vee/E^\vee_+)
$$

is contained in $F(W^\perp)$. The condition $F(E^\vee) \subset \frac{1}{\omega}E$ is equivalent to that the image of $F(W^\perp)$ under the natural morphism

$$
\phi_2 : E^0/\omega E^0 \to E_+/E_-
$$

is contained in $W$. We put $X_i = \pi_0^{-1}(X_{\mu^*, \nu^*}(\tau^*_i)) \cap \pi_1^{-1}(Y_i)$. Then $X_i$ is the subscheme of $\pi_1^{-1}(Y_i)$ cut out by the conditions

$$
\phi_1(W) \subset F(W^\perp), \quad \phi_2(F(\omega W^\perp)) \subset W. \tag{8.4}
$$

$$
\phi_2(F(\omega W^\perp)) \subset W. \tag{8.5}
$$

Let $\pi_0'$ and $\pi_1'$ be the restrictions of $\pi_0$ and $\pi_1$ to $X_i$ respectively. We have

$$
\xymatrix{
\pi_0^{-1}(X_{\mu^*, \nu^*}(\tau^*_i)) \ar[r] \ar[d]_{\pi_0} & X_i \ar[r] \ar[d]_{\pi_1} & \pi_1^{-1}(Y_i) \ar[d]_{\pi_1} \\
X_{\mu^*, \nu^*}(\tau^*_i) \ar[r]_{\pi_1'} & Y_i. 
}
$$

We note that $\pi_0$ and $\pi_0'$ are isomorphisms over $\hat{X}_{\mu^*, \nu^*}(\tau^*_i)$.

**Lemma 8.4.** The inverse image $\pi_0^{-1}(X_{\mu^*, \nu^*}(\tau^*_i))$ is contained in $X_i$.

**Proof.** Let $(E, E_+, E_-)$ be a point of $\pi_0^{-1}(X_{\mu^*, \nu^*}(\tau^*_i))$. Then we have $E_- = E \cap E_0$. By the condition $F(E^\vee) \subset \frac{1}{\omega}E$, we have $\omega F(E^\vee) \subset E \cap E_0 = E_-$. This implies $\omega F^{-1}(E^\vee) \subset E$ by taking the dual. Hence we have $\omega F^{-1}(E^\vee) \subset E \cap E_0 = E_-$. This means that $(E_+, E_-)$ is a point of $Y_i$. \qed

Let $\mathcal{G}_i$ denote the restriction of $\mathcal{G}_i$ to $Y_i$. We have an isomorphism

$$
\Psi_{Y_i} : \pi_1^{-1}(Y_i) \simeq \text{Par}_i(\mathcal{G}_i) \tag{8.6}
$$

induced by $\Psi$.

**Theorem 8.5.** The closed subscheme $X_i \subset \pi_1^{-1}(Y_i) \simeq \text{Par}_i(\mathcal{G}_i)$ is defined by the condition $\phi_1(W) \subset F(W^\perp)$.
Proof. It suffices to show that the condition (8.5) is automatic. The condition (8.3) is equivalent to $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$ under (8.6). Let $(\mathcal{E}_+ \cap \mathcal{E}_-)$ be a point of $\pi^{-1}_1(Y_i)$. Then we have

$$\varpi F^{-1}(\mathcal{E}^\vee) \subset \mathcal{E}_- \subset \mathcal{E}.$$ 

By taking the dual, we have $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}_-$. Hence the condition $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$ is satisfied. 

\begin{proposition}
Proposition 8.6. The scheme $\hat{X}_{\mu^*,x_0}^{\phi}(\tau_i^*)$ is isomorphic to the subscheme of $\pi^{-1}_1(Y_i)$ defined by the condition $\mathcal{E} \subset F(\mathcal{E}^\vee)$ and $\mathcal{E} \cap \mathcal{E}_0 = \mathcal{E}_-$. 

Proof. The natural morphism $\pi^{-1}_0(\hat{X}_{\mu^*,x_0}^{\phi}(\tau_i^*)) \to \hat{X}_{\mu^*,x_0}^{\phi}(\tau_i^*)$ is an isomorphism. Hence the claim follows from Lemma 8.4 and Theorem 8.5. 
\end{proposition}

9 Intersections

Let $x, x' \in J_\tau(F)/J_\tau(\mathcal{O})$. Let $\Lambda_x$ and $\Lambda_{x'}$ be the lattices of $V$ determined by $x$ and $x'$. We put

$$l_{x,x'} = \text{length}_{\mathcal{O}}(\Lambda_x/(\Lambda_x \cap \Lambda_{x'})).$$

Let $\mathcal{E}_x$ and $\mathcal{E}_{x'}$ be the modifications of $\mathcal{E}^0$ corresponding to $\Lambda_x$ and $\Lambda_{x'}$. Let $P_{x,x'}$ be the parabolic subgroup of $G$ that is the stabilizer of the filtration

$$\varpi \Lambda_x \subset \varpi^2 \Lambda_{x'} + \varpi \Lambda_x \subset (\Lambda_x \cap \varpi \Lambda_{x'}) + \varpi \Lambda_x \subset (\Lambda_x \cap \Lambda_{x'}) + \varpi \Lambda_x \subset \Lambda_x.$$ 

We note that $\varpi \Lambda_x \subset \Lambda_{x'}$ if and only if $\varpi \Lambda_{x'} \subset \Lambda_x$ by taking dual with respect to the hermitian paring. We put

$$d_1 = \dim((\varpi^2 \Lambda_{x'} + \varpi \Lambda_x)/\varpi \Lambda_x),$$

$$d_2 = \dim((\Lambda_x \cap \varpi \Lambda_{x'}) + \varpi \Lambda_x)/\varpi \Lambda_x).$$

We note that $d_1 + d_2 = l_{x,x'}$.

9.1 Intersection of components for $\nu_i$ and $\nu_{i'}$, where $i, i' \neq r$ if $n$ is even.

9.1.1 Different hyperspecial subgroups

We assume that $x \neq x'$. For a subscheme $X$ of $X_{\mu^*,x}^{\phi}(\tau_i^*)$, let $X_{P_{x,x'},[w]}^\phi$ be the inverse image of $X_{\mu^*,x}^{\phi}(\tau_i^*)$ under $X \hookrightarrow X_{\mu^*,x}^{\phi}(\tau_i^*) \to X_{\mu^*,x}^{\phi}(\tau_i^*)$.

We recall that

$$\hat{X}_{\mu^*,x}^{\phi}(\tau_i^*) = X_{\mu^*,x}^{\phi}(\tau_i^*) \setminus X_{\mu^*,x}^{\phi}(\tau_i^*)^\phi.$$ 

Assume that $i \leq i'$. For $j_1, j_2 \in \mathbb{N}$ such that $i - 1 - d_2 \leq j_1 \leq i - 1$ and $d_2 - i \leq j_2 \leq n - i - d_2 - j_1$, we define $w_{j_1,j_2} \in S_n$ by

$$w_{j_1,j_2}(j) = \begin{cases} j + j_1 & \text{if } i - j_1 \leq j \leq d_2, \\
+j_1 - j_2 - 1 & \text{if } d_2 + 1 \leq j \leq d_2 + j_1, \\
+j_2 & \text{if } n - j_2 - i + 1 \leq j \leq n - d_2, \\
+j_2 - i - j_2 & \text{if } n - d_2 + 1 \leq j \leq n - d_2 + j_2, \\
+j & \text{otherwise.} \end{cases}$$

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We put $\mathcal{E}_{x,x'} = (\mathcal{E}_x + \mathcal{E}_x') \cap \frac{1}{\omega} \mathcal{E}_x$ and $\mathcal{E}_{x,x'}^- = (\mathcal{E}_x \cap \mathcal{E}_x') + \omega \mathcal{E}_x$. Let $\mathcal{E}_+$ and $\mathcal{E}_-$ be the universal vector bundles on $X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]}$. We note that
\[
\text{length}((\mathcal{E}_+ + \mathcal{E}_{x,x'})/\mathcal{E}_+^+) = j_1, \quad \text{length}((\mathcal{E}_- + \mathcal{E}_{x,x'})/\mathcal{E}_-^-) = j_2.
\]
We put $\mathcal{E}_{+, -} = \mathcal{E}_+ \cap (\mathcal{E}_- + \mathcal{E}_x')$ and $d_{j1,j2} = j_2 - j_1 + 2i - 1 - d_2$. We note that
\[
\mathcal{E}_{+, -} = \mathcal{E}_- + \mathcal{E}_+ \cap \mathcal{E}_x' \subset F(\mathcal{E}_-^\vee) \cap (F(\mathcal{E}_-^\vee) + \mathcal{E}_x') = F(\mathcal{E}_{+, -}^\vee)
\]
using $\mathcal{E}_+ \subset F(\mathcal{E}_+^\vee)$ and $\mathcal{E}_- \subset F(\mathcal{E}_-^\vee)$.

**Lemma 9.1.** We have $\text{length}(\mathcal{E}_{+, -}/\mathcal{E}_-) = d_{j1,j2}$. Further $\mathcal{E}_{+, -}/\mathcal{E}_-$ is a vector bundle on $X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]}$.

**Proof.** We have
\[
\text{length}(\mathcal{E}_+ / \mathcal{E}_{+, -}) = \text{length}((\mathcal{E}_+ + \mathcal{E}_x')/(\mathcal{E}_- + \mathcal{E}_x'))
\]
\[
= \text{length}((\mathcal{E}_+ + \mathcal{E}_x')/(\omega \mathcal{E}_x + \mathcal{E}_x')) - \text{length}((\mathcal{E}_- + \mathcal{E}_x')/(\omega \mathcal{E}_x + \mathcal{E}_x'))
\]
\[
= \text{length}((\mathcal{E}_+ + \mathcal{E}_x')/(\mathcal{E}_x + \mathcal{E}_x')) + \text{length}((\mathcal{E}_x + \mathcal{E}_x')/(\omega \mathcal{E}_x + \mathcal{E}_x')) - j_2
\]
\[
= j_1 + \text{length}((\mathcal{E}_x \cap \omega \mathcal{E}_x')/(\omega \mathcal{E}_x \cap \omega \mathcal{E}_x')) - j_2 = j_1 + d_2 - j_2.
\]
Hence we obtain the first claim. By the above equalities, $\text{length}(\mathcal{E}_+ / \mathcal{E}_{+, -})$ is constant on $X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]}$. Hence $\mathcal{E}_+ / \mathcal{E}_{+, -}$ is a vector bundle on $X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]}$ by [BS17] Lemma 7.3. Therefore $\mathcal{E}_{+, -}/\mathcal{E}_-$ is also a vector bundle. □

Let $\mathcal{G}_{j1,j2}$ be the restriction of $\mathcal{G}$ to $X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]}$. Let
\[
t_{j1,j2} \in \text{Oc}(\text{Dyn}(\mathcal{G}_{j1,j2}))(X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]})
\]
denote the restriction of $t_i$ in (8.3). Let $\mathcal{P}_{j1,j2}$ be the parabolic subgroup of $\mathcal{G}_{j1,j2}$ determined by
\[
\mathcal{E}_- \subset \mathcal{E}_+ \cap (\mathcal{E}_- + \mathcal{E}_x') \subset \mathcal{E}_+.
\]
We put $l_{j1,j2} = i' - 1 - j_2 - d_1$. We define $s_{j1,j2} \in S_{2i-1}$ by
\[
s_{j1,j2}(j) = \begin{cases} j + l_{j1,j2} & \text{if } i - l_{j1,j2} \leq j \leq d_{j1,j2}, \\ j + i - 1 - d_{j1,j2} - l_{j1,j2} & \text{if } d_{j1,j2} + 1 \leq j \leq d_{j1,j2} + l_{j1,j2}, \\ j & \text{otherwise.} \end{cases}
\]
Let $r_{j1,j2}$ be the element of
\[
(q_{j1,j2}^{-1}(t(\mathcal{P}_{j1,j2}),t_{j1,j2}))(X_{n-1,n-1}^i([1])^{\text{pf}}_{p_{x,x'},[w_{j1},j2]})
\]
corresponding to $[s_{j1,j2}]$ by Lemma 2.6(2).

**Proposition 9.2.** Assume that $X_{\mu,x'}^b(\tau_i^* \cap X_{\mu,x'}^b(\tau_i^*)$ is not empty. Then we have $1 \leq l_{x,x'} \leq i + i' - 1$.

The subscheme $X_{\mu,x'}^b(\tau_i^* \cap X_{\mu,x'}^b(\tau_i^*) \subset X_{\mu,x'}^b(\tau_i^*)$ is the locus defined by the condition that $\mathcal{E}_+ \cap \omega \mathcal{E}_x' \subset \mathcal{E}_+ \subset \frac{1}{\omega} \mathcal{E}_{x,x'}$, $\omega \mathcal{E}_x' \subset \mathcal{E}_- \subset \mathcal{E}_x \cap \frac{1}{\omega} \mathcal{E}_x'$,
\[
\text{length}((\mathcal{E}_+ + \mathcal{E}_{x,x'})/\mathcal{E}_{x,x'}) \leq \lceil (i' - i + d_2 - d_1)/2 \rceil
\]
and
\[ \text{length}(\mathcal{E} + \mathcal{E}_{+,-})/\mathcal{E}_{+,-}) + \text{length}(\mathcal{E}_{-} + \mathcal{E}_{x,x'})/\mathcal{E}_{x,x'} = i' - 1 - d_1. \]

In particular, \( \hat{X}_{\mu}^{\text{b},x}(\tau^*_i) \cap \hat{X}_{\mu}^{\text{b},x'}(\tau^*_j) \) is the union of \( \left( \hat{X}_{\mu}^{\text{b},x}(\tau^*_i) \cap \hat{X}_{\mu}^{\text{b},x'}(\tau^*_j) \right)_{P,|w,j|2} \) for \( j_1, j_2 \in \mathbb{N} \) such that \( j_1 + d_2 - i \leq j_2 \leq j_1 + d_2 - i + 1 \),

\[ i - 1 - d_2 \leq j_1 \leq i - 1 - d_1, \]

\[ i' - i - d_1 \leq j_2 \leq \min\{[(i' - i + d_2 - d_1)/2], n - i - d_2 - j_1\}. \]

Further we have
\[ \left( \hat{X}_{\mu}^{\text{b},x}(\tau^*_i) \cap \hat{X}_{\mu}^{\text{b},x'}(\tau^*_j) \right)_{P,|w,j|2} = \hat{X}_{\mu}^{\text{b},x}(\tau^*_i)_{P,|w,j|2} \cap \text{Par}_{j_1,j_2}((\mathcal{F}_{j_1,j_2}; \mathcal{P}_{j_1,j_2})^{\mu}). \]

**Proof.** The intersection \( \hat{X}_{\mu}^{\text{b},x}(\tau^*_i) \cap \hat{X}_{\mu}^{\text{b},x'}(\tau^*_j) \) is parametrized by \( \mathcal{E} \leadsto \mathcal{E}_x \) which is equal to \( \nu^*_i \) such that \( \mathcal{E} \subset \mathcal{F}(\mathcal{E}_x) \subset \mathcal{F}_{\mathcal{E}} \) and \( \mathcal{E} \leadsto \mathcal{E}_x' \) is equal to \( \nu^*_i \). Let \( \mathcal{E} \) be a point of \( \hat{X}_{\mu}^{\text{b},x}(\tau^*_i) \cap \hat{X}_{\mu}^{\text{b},x'}(\tau^*_j) \). We put
\[ \mathcal{E}_+ = \mathcal{E} + \mathcal{E}_x, \quad \mathcal{E}_- = \mathcal{E} \cap \mathcal{E}_x, \quad \mathcal{E}_+ = \mathcal{E} + \mathcal{E}_x', \quad \mathcal{E}_- = \mathcal{E} \cap \mathcal{E}_x'. \]

Then we have
\[ \text{length}(\mathcal{E}_x/\mathcal{E}_-) = i, \quad \text{length}(\mathcal{E}/\mathcal{E}_-) = i - 1, \]
\[ \text{length}(\mathcal{E}_x'/\mathcal{E}_-) = i', \quad \text{length}(\mathcal{E}/\mathcal{E}_-) = i' - 1. \]

Hence we have \( \text{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}_-')) \leq i' - 1 \) and \( \text{length}(\mathcal{E}_x/(\mathcal{E}_- \cap \mathcal{E}_-')) \leq i + i' - 1 \). Therefore the inclusion \( \mathcal{E}_- \cap \mathcal{E}_-' \subset \mathcal{E}_x \cap \mathcal{E}_x' \) implies that
\[ 1 \leq l_{x,x'} \leq i + i' - 1. \]

We have \( \mathcal{E}_x + \mathcal{E}_x' \subset \mathcal{E}_x \) and \( \mathcal{E}_x + \mathcal{E}_x' \subset \mathcal{E}_x \), since \( \mathcal{E}_x \subset \mathcal{E} \). We have \( \mathcal{E}_+ \subset \mathcal{E}_x \cap \mathcal{E}_x' = \mathcal{E}_x \cap \mathcal{E}_x' \), and \( \mathcal{E}_- \subset \mathcal{E}_x \cap \mathcal{E}_x' \), since \( \mathcal{E}_+ \subset \mathcal{E}_x \) and \( \mathcal{E} \subset \mathcal{E}_x \).

We put \( j_1 = \text{length}(\mathcal{E}_x/\mathcal{E}_x) \) and \( j_2 = \text{length}(\mathcal{E}_x/\mathcal{E}_x') \). We have
\[ \text{length}(\mathcal{E}_x/(\mathcal{E}_- \cap \mathcal{E}_-')) = j_1 \quad \text{and} \quad \text{length}(\mathcal{E}_x/(\mathcal{E}_x \cap \mathcal{E}_x')) = j_2 + \text{length}(\mathcal{E}_x/\mathcal{E}_x'). \]

We have
\[ j_2 + i - d_2 = \text{length}(\mathcal{E}_- \cap \mathcal{E}_x)/(\mathcal{E}_-)/(\mathcal{E}_x)/(\mathcal{E}_x') \leq 1 + \text{length}(\mathcal{E}_x/\mathcal{E}_x') = 1. \]

Further we have
\[ \text{length}(\mathcal{E}_x/(\mathcal{E}_x')/(\mathcal{E}_x') = 1. \]

Therefore we obtain \( j_2 \leq [(i' - i + d_2 - d_1)/2] \).

Further, \( j_1 + d_2 - i \leq j_2 \leq j_1 + d_2 - i + 1 \) follows from \( \text{length}(\mathcal{E}_x/\mathcal{E}_x') = 1 \). This implies \( i - 1 - d_2 \leq j_1 \) and \( d_2 - i \leq j_2 \). We have \( j_1 \leq i - 1 - d_1 \) and \( j_2 \leq n - i - d_2 - j_1 \) by the inclusions \( \mathcal{E}_x + \mathcal{E}_x' \subset \mathcal{E}_x \cap \mathcal{E}_x' \) and \( \mathcal{E}_+ \cap \mathcal{E}_x' \subset \mathcal{E}_- \cap \mathcal{E}_x' \). The equality
\[ \text{length}(\mathcal{E}_x/(\mathcal{E}_x')/(\mathcal{E}_x') = 1. \]
and $\text{length}(\mathcal{E}/\mathcal{E}_{+,+}) \leq i - 1$ imply $j_2 \geq j' - i - d_1$.

We have

$$\text{length}(\mathcal{E}/\mathcal{E}_{+,+}) = \text{length}(\mathcal{E}/(\mathcal{E} \cap \mathcal{E}_{+,+})) = \text{length}(\mathcal{E}/(\mathcal{E} \cap \mathcal{E}_+))$$

$$= \text{length}(\mathcal{E}/(\mathcal{E}_+ + \mathcal{E}_+))$$

Hence, $\text{length}(\mathcal{E}/\mathcal{E}_{+,+}) = i' - 1 - j_2 - d_1$ if and only if $\text{length}(\mathcal{E}_+ + \mathcal{E}_+)/\mathcal{E}_+) = i' - 1$.

This implies the last claim. \hfill \square

### 9.1.2 Same hyperspecial subgroup

Assume that $x = x'$. It suffices to consider the case where $x = x' = x_0$, since all the hyperspecial subgroups are conjugate.

Let $2 \leq i \leq [(n - 1)/2]$. Let $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ be a point of $X_i$. Let $s$ be the rank of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$. We put $V_1 = \mathcal{E}/\mathcal{E}_-$ and take $V_2 \subset \mathcal{E}^0/\mathcal{E}_-$ and $V_3 \subset \mathcal{E}_+/\mathcal{E}_-$ such that projections induce isomorphisms $\mathcal{V}_2 \simeq (\mathcal{E} + \mathcal{E}^0)/\mathcal{E}$ and $\mathcal{V}_3 \simeq \mathcal{E}_+/\mathcal{E}^0$. An open neighbourhood of $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ in $\text{Gr}(i - 1, \mathcal{Y}_{i})$ under (8.6) is given by $\text{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3)$ sending $f \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3)$ to the inverse image $\mathcal{E}_f$ of

$$\{v + f(v) \mid v \in V_1\} \subset \mathcal{E}_+ / \mathcal{E}_-$$

in $\mathcal{E}_+$. By Theorem 8.5, the condition that $\mathcal{E}_f$ belongs to $X_i$ is equivalent to

$$\langle v + f(v), F(v' + f(v')) \rangle = 0 \quad (9.1)$$

in $\varphi^{-1}W_{O(R)}/W_{O(R)}$ for $v, v' \in V_1$. We write $f$ as $f_2 + f_3$ for $f_2 \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ and $f_3 \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_3)$. For $v, v' \in (\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$, the condition (9.1) is equivalent to

$$\langle v + f_2(v), F(f_3(v')) \rangle + \langle f_3(v), F(v' + f_2(v')) + f_3(v') \rangle = 0 \quad (9.2)$$

in $\varphi^{-1}W_{O(R)}/W_{O(R)}$.

Take a basis $v_1, \ldots, v_{s-1}$ of $V_1$ such that $v_1, \ldots, v_s$ form a basis of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$. Take a basis $v_1, \ldots, v_{2i-s-1}$ of $V_2$ and a basis $v_{2i-s}, \ldots, v_{2i-1}$ of $V_3$. Write $f(v_l)$ as $x_{l,j}v_1 + \cdots + x_{l,2i-1}v_{2i-1}$. Then the condition (9.2) is equivalent to

$$\langle v_l + \sum_{j=1}^{2i-s-1} x_{l,j}v_j, F\left( \sum_{k=2i-s}^{2i-1} x_{m,k}v_k \right) \rangle + \left( \sum_{k=2i-s}^{2i-1} x_{l,k}v_k, F\left( v_m + \sum_{j=1}^{2i-1} x_{m,j}v_j \right) \right) = 0$$

for $1 \leq l, m \leq s$. We can write this as

$$\langle v_l + \sum_{j=1}^{2i-s-1} x_{l,j}v_j, F(v_k) \rangle \rangle_{l,k}(x_{m,k})_{k,m} = -(x_{l,k})_{l,k}\langle v_k, F(v_m) + \sum_{j=1}^{2i-1} x_{m,j}v_j \rangle \rangle_{k,m}.$$ 

Taking the determinant, we obtain

$$\det(x_{l,k})_{l,k}\left( \det((v_l + \sum_{j=1}^{2i-s-1} x_{l,j}v_j, F(v_k)))_{l,k}(\det(x_{l,k})_{l,k})^{q-1} - (-1)^s \det((v_k, F(v_m) + \sum_{j=1}^{2i-1} x_{m,j}v_j))_{k,m} \right) = 0.$$ 

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The condition $\mathcal{E}_f \cap \mathcal{E}^0 = \mathcal{E}_-$ is equivalent to $\det(x_{l,k})_{l,k} \neq 0$. Hence, if $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ belongs to the closure of $\pi_0^{-1}(X_{\mu^*}^{b_{1,x_0}}(\tau_1^*))$, then we have $\det((v_k, F(v_m)))_{k,m} = 0$. This means $F^{-1}(\mathcal{E}_+^\nu) \subset \mathcal{E}$. Hence we have obtained the following proposition:

**Proposition 9.3.** The intersection

$$\pi_0^{-1}(X_{\mu^*}^{b_{1,x_0}}(\tau_1^*)) \cap \pi_0^{-1}(X_{\mu^*}^{b_{2,x_0}}(\tau_2^*))$$

is contained in the locus defined by the condition $F^{-1}(\mathcal{E}_+^\nu) \subset \mathcal{E}$.

Conversely, we assume that $F^{-1}(\mathcal{E}_+^\nu) \subset \mathcal{E}$. Then we may assume that $v_1$ is a basis of $F^{-1}(\mathcal{E}_+^\nu)/\mathcal{E}_-$, $v_i$ is an element of $(F(\mathcal{E}) \cap \mathcal{E}^0)/\mathcal{E}_-$ lifting a basis of $(F(\mathcal{E}) \cap \mathcal{E}^0)/\mathcal{E} \cap \mathcal{E}^0)$ such that $v_i \notin F^{-1}(\mathcal{E}_+^\nu)/\mathcal{E}_-$ and $v_{2i-s}$ is an element of $F(\mathcal{E})/\mathcal{E}_-$ lifting a basis of $(F(\mathcal{E}) + \mathcal{E}^0)/\mathcal{E}$. Further, we may assume that $v_1, \ldots, v_{i-1}$ and $v_{2i-1}, \ldots, v_{2i-s}, v_{2i-s-1}, \ldots, v_{i+1}$ form dual base with respect to the paring

$$\mathcal{E}/(F^{-1}(\mathcal{E}_+^\nu)) \times \mathcal{E}_+/F(\mathcal{E})_+; (v, v') \mapsto \langle F(v), v' \rangle,
$$

and that $\langle v_j, F(v_k) \rangle = 0$ for $i + 1 \leq j \leq 2i - 1$ and $i \leq k \leq 2i - 1$. Then the condition

\[ (9.1) \]

is equivalent to

\[
\langle x_{l,j} v_j, F \left( \sum_{k=i}^{2i-1} x_{m,k} v_k' \right) \rangle + \left\{ \begin{array}{ll}
\{ v_l + x_{l,i} v_i, F \left( \sum_{k=2i-s}^{2i-1} x_{m,k} v_k' \right) \} & \text{if } 1 \leq l \leq r, \\
\{ v_l + x_{l,i} v_i, F \left( \sum_{k=i}^{2i-1} x_{m,k} v_k' \right) \} & \text{if } s + 1 \leq l \leq i - 1,
\end{array} \right.
\]

(9.3)

for $1 \leq l, m \leq i - 1$.

We put

$$y = \det(x_{l,j})_{1 \leq l \leq s, 2i-1 \leq j \leq 2i-1}.$$

We want to show that the quotient of $k[[x_{l,j}]]_{1 \leq l \leq s, 1 \leq j \leq 2i-1}$ by the relation (9.3) is nonzero after inverting $y$.

**Proposition 9.4.** (1) The intersection

$$\pi_0^{-1}(X_{\mu^*}^{b_{1,x_0}}(\tau_1^*)) \cap \pi_0^{-1}(X_{\mu^*}^{b_{2,x_0}}(\tau_2^*))$$

is equal to the locus defined by the condition $F^{-1}(\mathcal{E}_+^\nu) = \mathcal{E}$.

(2) We have an isomorphism $X_{\mu^*}^{b_{1,x_0}}(\tau_1^*) \cap X_{\mu^*}^{b_{2,x_0}}(\tau_2^*) \simeq X_{l_1}^{F, F^3}[1, [1]]$ given by $\mathcal{E} \mapsto \mathcal{E}/(\mathcal{I}_- \mathcal{E}^0)$.

**Proof.** In this case, (9.3) becomes

$$\langle x_{l,3} v_3, F(x_{1,2} v_2 + x_{1,3} v_3) \rangle + \langle v_1 + x_{1,2} v_2, F(x_{1,3} v_3) \rangle = 0.$$

If the quotient of $k[[x_{1,2}, x_{1,3}]]$ by this relation is zero after inverting $x_{1,3}$, there is a positive integer $N$ such that $x_{1,3}^N$ is divisible by

$$\langle x_{1,3} v_3, F(x_{1,2} v_2 + x_{1,3} v_3) \rangle + \langle v_1 + x_{1,2} v_2, F(x_{1,3} v_3) \rangle$$

in $k[[x_{1,2}, x_{1,3}]]$. This does not happen because $\langle v_3, F(v_2) \rangle \neq 0$, which follows from $v_2 \notin F^{-1}(\mathcal{E}_+^\nu)/\mathcal{E}_-$. Hence we have (1) The claim (2) follows from (1).
By Proposition 9.4, $X_{\mu^*}^{b_1,x_0}(\tau_i^*) \cap X_{\mu^*}^{b_2,x_0}(\tau_2^*)$ is isomorphic to the perfect closed sub-scheme of $(\mathbb{P}^{n-1})^{pf}$ defined by two equations

$$\sum_{i=1}^{n} x_i^{p+1} = 0, \quad \sum_{i=1}^{n} x_i^{p+1} = 0.$$

9.2 Intersection of components for $\nu_i$ and $\nu_r$ when $n$ is even.

**Proposition 9.5.** If $i \neq r$, then $X^{b_i,x}(\tau_i^*) \cap X^{b_r,x}(\tau_r^*)$ is empty.

**Proof.** Assume $\mathcal{E}$ is a point of $X^{b_i,x}(\tau_i^*) \cap X^{b_r,x}(\tau_r^*)$. Then $\mathcal{E} \rightarrow F(\mathcal{E}^\vee)$ and $\varpi \mathcal{E} \rightarrow F(\mathcal{E}^\vee)$ are bounded by $\mu$. This is a contradiction. \qed

Assume that $x \neq x'$

**Proposition 9.6.** Assume that $X^{b_i,x}(\tau_i^*) \cap X^{b_r,x}(\tau_r^*)$ is not empty. Then we have $1 \leq l_{x,x'} \leq r - 1$ and $\varpi \Lambda_x \subset \Lambda_{x'}$. The intersection $X^{b_i,x}(\tau_i^*) \cap X^{b_r,x}(\tau_r^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}_x$ bounded by $\nu_r^*$ such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$ and $\mathcal{E} \rightarrow \mathcal{E}_{x'}$ is also bounded by $\nu_r^*$. In particular, it is isomorphic to

$$\{ H \in \text{Gr}^{pf}(r-1-l_{x,x'}, \varpi^{-1}(\Lambda_x \cap \Lambda_{x'})/((\Lambda_x + \Lambda_{x'})) \mid H \subset \text{Frob}(H^\perp) \}.$$

**Proof.** Assume that $\mathcal{E}$ is a point of $X^{b_i,x}(\tau_i^*) \cap X^{b_r,x}(\tau_r^*)$. Since $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$ and both $\mathcal{E} \rightarrow \mathcal{E}_x$ and $\mathcal{E} \rightarrow \mathcal{E}_{x'}$ are bounded by $\nu_r^*$, we have the following chain conditions:

$$\mathcal{E}_x \subset \mathcal{E} \subset \varpi^{-1} F(\mathcal{E}^\vee) \subset \varpi^{-1} \mathcal{E}_x,$$

$$\mathcal{E}_{x'} \subset \mathcal{E} \subset \varpi^{-1} F(\mathcal{E}^\vee) \subset \varpi^{-1} \mathcal{E}_{x'}.$$}

The inclusion follows from $\mathcal{E}_x \subset \mathcal{E} \subset \varpi^{-1} \mathcal{E}_{x'}$. Note that length$(\varpi^{-1} F(\mathcal{E}^\vee)/\mathcal{E}) = 2$, while both length$(\mathcal{E}/\mathcal{E}_x)$ and length$(\mathcal{E}/\mathcal{E}_{x'})$ are $r - 1$. Then $\mathcal{E}_x \cap \mathcal{E}_{x'}$ and $\mathcal{E}$ are related by

$$\mathcal{E}_x + \mathcal{E}_{x'} \subset \mathcal{E} \subset \varpi^{-1} F(\mathcal{E}) \subset \varpi^{-1} (\mathcal{E}_x \cap \mathcal{E}_{x'}).$$

Since $l_{x,x'} = \text{length}(\mathcal{E}_x + \mathcal{E}_{x'})/\mathcal{E}_x$, we have

$$l_{x,x'} = r - 1 - \text{length}(\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'})).$$

Hence we have $1 \leq l_{x,x'} \leq r - 1$.

The isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'}) \subset \varpi^{-1}(\mathcal{E}_x \cap \mathcal{E}_{x'})/(\mathcal{E}_x + \mathcal{E}_{x'})$. \qed

10 Example

In this section, we study in details the case where $n = 6$. We identify a moduli parametrizing modification $\mathcal{E} \subset \mathcal{E}_x$ bounded by $\nu_1$ with $(\mathbb{P}^5)^{pf}$ by taking a basis of $\Lambda_x$ such that the Hermitian paring is the standard one. Let $\mathbb{P}_{x,x',+}$ be the projective subspace of $(\mathbb{P}^5)^{pf}$ defined by the condition $\varpi \mathcal{E}_{x,x'}^+ \subset \mathcal{E}$. Let $\mathbb{P}_{x,x',-}$ be the projective subspace of $(\mathbb{P}^5)^{pf}$ defined by the condition $\varpi \mathcal{E}_{x,x'}^- \subset \mathcal{E}$. We note that $\mathbb{P}_{x,x',+}$ and $\mathbb{P}_{x,x',-}$ are isomorphic to $(\mathbb{P}^{5-d_2})^{pf}$ and $(\mathbb{P}^{d_2-1})^{pf}$ respectively.
10.1 Intersection of components for $\nu_1$

We may assume that $x \neq x'$. The intersection is not empty only if $l_{x,x'} = 1$. In this case, $d_1 = 0$, $d_2 = 1$, $j_1 = j_2 = 0$. The intersection is $\mathbb{P}_{x,x',-}$, which is a point given by $\mathcal{E}_x \cap \mathcal{E}_{x'}$.

10.2 Intersection of components for $\nu_1$ and $\nu_2$

If $x = x'$, then the intersection is isomorphic to the perfect closed subscheme of $(\mathbb{P}^5)^{pf}$ defined by two equations

$$\sum_{i=1}^{6} x_i^{q+1} = 0, \quad \sum_{i=1}^{6} x_i^{q^3+1} = 0.$$ 

We assume that $x \neq x'$.

10.2.1 $d_1 = 0$, $d_2 = 1$

In this case, $j_1 = 0$ and $j_2 = 1$. The intersection is equal to the perfect closed subscheme of $\mathbb{P}_{x,x',+}$ defined by equation

$$\sum_{i=1}^{6} x_i^{q+1} = 0.$$

10.2.2 $d_1 = 0$, $d_2 = 2$

In this case, $j_1 = 0$ and $j_2 = 1$. The intersection is $\mathbb{P}_{x,x',-}$, which is isomorphic to $(\mathbb{P}^1)^{pf}$.

Remark 10.1. If $d_1 = d_2 = 1$, then there is no $j_2 \in \mathbb{N}$ satisfying the condition in Proposition 10.2.

10.3 Intersection of components for $\nu_2$

Let $(\mathcal{E}_+, \mathcal{E}_-)$ be a point of $X_{t_1,4}^{4}([1])^{pf}$. The hermitian paring on $V$ induces a paring on $\mathcal{E}_+/\mathcal{E}_-$ since we have $\mathcal{E}_+ \subset F(\mathcal{E}_+)$ and $\mathcal{E}_- \subset F(\mathcal{E}_-)$. We take a basis $v_1, v_2, v_3$ of $\mathcal{E}_+/\mathcal{E}_-$ such that $v_1 \in F^{-1}(\mathcal{E}_+)/\mathcal{E}_-, v_2 \in \mathcal{E}_x/\mathcal{E}_-$. Let $\mathcal{E}$ be a point of $\hat{X}_{\mu^*,}^{b_2,x}(\tau_2^*)$ in the fiber of $(\mathcal{E}_+, \mathcal{E}_-)$ under

$$\pi: \hat{X}_{\mu^*,}^{b_2,x}(\tau_2^*) \to X_{t_1,4}^{4}([1])^{pf}.$$ 

We can take a generator $v = x_1v_1 + x_2v_2 + v_3$ of $\mathcal{E}/\mathcal{E}_-$ for $x_1, x_2 \in k$, since $\mathcal{E} \not\subset \mathcal{E}_x$. Then we have

$$\langle v, F(v) \rangle = x_1 \langle v_1, F(v_3) \rangle + x_2 \langle v_2, F(v_3) \rangle + x_3^3 \langle v_3, F(v_2) \rangle + \langle v_3, F(v_3) \rangle$$

because $\langle w, F(w) \rangle = 0$ for $w, w' \in \mathcal{E}_x/\mathcal{E}_-$ and $\langle v_3, F(v_1) \rangle = 0$. Hence the fiber of $(\mathcal{E}_+, \mathcal{E}_-)$ under $\pi$ is defined by

$$x_1 \langle v_1, F(v_3) \rangle + x_2 \langle v_2, F(v_3) \rangle + x_3 \langle v_3, F(v_2) \rangle + \langle v_3, F(v_3) \rangle = 0$$

in $(\mathbb{A}^2)^{pf}$. We note that $(\langle v_1, F(v_3) \rangle, \langle v_2, F(v_3) \rangle) \neq (0, 0)$ because $v_3 \not\in \mathcal{E}_x/\mathcal{E}_-$.

We describe the fiber of

$$\pi_{j_1,j_2}: \hat{X}_{\mu^*,}^{b_2,x}(\tau_2^*)_{\mathbb{P}_{x,x'}[w_{j_1,j_2}]} \cap \text{Par}_{j_1,j_2} (\mathcal{G}_{j_1,j_2}; \mathcal{P}_{j_1,j_2})^{pf} \to X_{t_1,4}^{4}([1])^{pf}_{\mathbb{P}_{x,x'}[w_{j_1,j_2}]}$$

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when
\[
\left( X_{\mu}^{b_2,x}(\tau_2) \cap X_{\mu}^{b_2,x'}(\tau_2') \right)_{p_{x,x'}[w_{j_1,j_2}]}
\]
is not empty.

10.3.1  \( d_1 = 0 \), \( d_2 = 1 \)
In this case, \( 0 \leq j_1 \leq 1 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 2 - j_1 \). The fiber of \( \pi_{j_1,0} \) is given by the condition \( E \not\subset E_{+,-} \).

10.3.2  \( d_1 = 0 \), \( d_2 = 2 \)
In this case, \( 0 \leq j_1 \leq 1 \) and \( 0 \leq j_2 \leq 1 \). We have \( d_{j_1,j_2} = 1 - j_1 + j_2 \). The fiber of \( \pi_{j_1,0} \) is given by the condition \( E \not\subset E_{+,-} \). The fiber of \( \pi_{j_1,1} \) is given by the condition \( E \subset E_{+,-} \).

10.3.3  \( d_1 = 1 \), \( d_2 = 1 \)
In this case, \( j_1 = 0 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 2 \). The fiber of \( \pi_{0,0} \) is given by the condition \( E \subset E_{+,-} \).

10.3.4  \( d_1 = 0 \), \( d_2 = 3 \)
In this case, \( j_1 = 0 \) and \( j_2 = 1 \). We have \( d_{j_1,j_2} = 1 \). The fiber of \( \pi_{0,1} \) is given by the condition \( E = E_{+,-} \).

10.3.5  \( d_1 = 1 \), \( d_2 = 2 \)
In this case, \( j_1 = 0 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 1 \). The fiber of \( \pi_{0,0} \) is given by the condition \( E = E_{+,-} \).

10.4 Intersection of components for \( \nu_3 \)
10.4.1  \( l_{x,x'} = 1 \)
The intersection is isomorphic to the perfection of the Fermat hypersurface defined by
\[
x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1} = 0
\]
in \( \mathbb{P}^3 \).

10.4.2  \( l_{x,x'} = 2 \)
The intersection is a point given by \( E_x + E_{x'} \).

11 Shimura variety
Let \( E \) be a quadratic imaginary field, and let \( V \) be an \( n \)-dimensional Hermitian space over \( E \) with signature \((2, n - 2)\) at infinity. Fix a prime \( p \neq 2 \) inert in \( E \). Further assume that \( V \otimes_E \mathbb{Q}_p \) contains a self-dual \( \mathbb{Z}_p^2 \) lattice \( \Lambda \). Let \( G = GU(V) \) be the general associated unitary group. We put \( G = GU(\Lambda) \) as before.
We take a basis of $V_C = V \otimes_E \mathbb{C}$ over $\mathbb{C}$ such that the Hermitian form is given by the matrix $\text{diag}(1, -1, -1)$. Let $h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathbb{G}_R$ be the morphism of algebraic groups over $\mathbb{R}$ such that $h(z)$ corresponds to $\text{diag}(z \cdot 1, \bar{z} \cdot 1, -1)$ for $z \in \mathbb{C}^\times$ under

$$G(\mathbb{R}) \subset \text{Aut}_E(V_C) \cong \text{GL}_n(\mathbb{C}),$$

where the last isomorphism is given by the basis taken above. Let $X$ be the $G(\mathbb{R})$-conjugacy class of $h$. Then $(G, X)$ is a Shimura datum.

We have an isomorphism

$$(\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m) \cong \mathbb{G}_m \times \mathbb{G}_m$$

of algebraic groups over $\mathbb{C}$ induced by the isomorphism $\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$; $a \otimes b \mapsto (ab, \bar{ab})$. We define $\mu_h$ by the composition

$$\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m \cong (\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m) \xrightarrow{h_\mathbb{C}} \mathbb{G}_C,$$

where the first morphism is the inclusion into the first factor. Let $\mu : \mathbb{G}_m \to \mathbb{G}_E$ be the morphism of algebra over $E$ such that $\mu(z)$ corresponds to $(\text{diag}(z \cdot 1, 1, -1), z)$ for $z \in E^\times$ under the isomorphism

$$\mathbb{G}_E \cong \text{GL}_n(E) \times \mathbb{G}_E$$

given by taking a basis of $V$ over $E$. Then $\mu_h$ and $\mu_\mathbb{C}$ are in the same $G(\mathbb{C})$-conjugacy class. We note that the reflex field $E(G, X)$ of $(G, X)$ is $E$ if $n \neq 4$ and $\mathbb{Q}$ if $n = 2$.

Let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup. Let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup. We put $K = K^p K_p \subset G(\mathbb{A}_f)$. Let $\text{Sh}_K(G, X)$ be the canonical model over $E(G, X)$ of the Shimura variety attached to $(G, X)$ and $K$. Let $\mathcal{M}(G, X)$ be the canonical integral model of $\text{Sh}_K(G, X)$ over $\mathcal{O}_{E(G, X), (p)}$ constructed in [Kis10].

Let $S_K(G, X)$ be the perfection of $\mathcal{M}(G, X) \otimes \mathbb{F}_p$. We have the Newton map

$$N : S_K(G, X)(\mathbb{F}_p) \to B(G, \mu^*)$$

as in [XZ17] 7.2.7. Let $[b] \in B(G, \mu^*)$ be the basic element. We write $S_K(G, X)|_{[b]}$ for the closed perfect subscheme of $S_K(G, X)$ defined by $N^{-1}([b])$. We call $S_K(G, X)|_{[b]}$ the supersingular locus of $S_K(G, X)$.

**Remark 11.1.** In [Kot92], a moduli space of abelian schemes with additional structures is constructed. It is isomorphic to a finite union of integral models of Shimura varieties. Under the isomorphism, a point of $S_K(G, X)|_{[b]}$ corresponds to a supersingular abelian variety.

We take a point $x \in S_K(G, X)|_{[b]}(\bar{\mathbb{F}}_p)$. We put $L = \text{W}(\mathbb{F}_p)[\frac{1}{p}]$. Then we have a basic element $b_x \in G(L)$ and an algebraic group $I_x$ over $\mathbb{Q}$ as in [XZ17] 7.2.9. We have an embeddings $I_x(\mathbb{Q}) \subset G(\mathbb{A}_f^p)$ and $I_x(\mathbb{Q}) \subset J_x(\mathbb{Q}_p)$ as in [XZ17] 7.2.13. Then we have the isomorphism

$$I_x(\mathbb{Q}) \backslash X_\mu^* (b_x) \times G(\mathbb{A}_f^p)/K^p \cong S_K(G, X)|_{[b]}$$

(11.1)

by [XZ17] Corollary 7.2.16. By the isomorphism (11.1) and results in the previous sections, we obtain a description of irreducible components of $S_K(G, X)|_{[b]}$ and their intersections if $K^p$ is sufficiently small.
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