Period doubling bifurcation and chaos in oscillations of two interacting cavitation bubbles under ultrasonic

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Abstract. Two interacting bubbles oscillator under ultrasonic is a highly nonlinear system. The nature of the cavitation will be understood more deeply in the study of two interacting bubbles’ internal mechanism. In this paper, the model to describe the oscillation motion of the two interacting bubbles in the sound field is the Keller-Miksis equation. The dynamics of two interacting bubbles has been studied by adopting the methods of time domain wave, phase trajectory, Poincaré section, power spectrum and bifurcation diagram. We also studied the nonlinear oscillations of bubbles with different initial radii and characteristics of bifurcation structure of normalized radius as ultrasonic pressure amplitude changed. Results show that two bubbles oscillation traverses the similar mode of period doubling bifurcation to cascade chaos. Furthermore, the motion of two interacting bubbles shows discrepant oscillation characteristics under the action of same sound field. In addition, the periodic window and chaotic region of bubble’s oscillation occur alternately with the increase of pressure amplitude.

1. Introduction

The study on ultrasonic cavitation mechanism of microbubbles in a liquid is a hot topic in many fields, e.g. sonochemistry [1-2], medical diagnostics and therapy [3-4], food processing [5], ultrasonic-assisted soldering [6] and fluids engineering et al. [7]. When the liquid containing cavitation, bubbles is radiated by a strong sound field, the oscillation of bubble radius shows strong nonlinearity. The nonlinear characteristics of bubble oscillations in acoustic cavitation have been concerned by more and more scholars. The most significant discovery is the existence of period doubling bifurcation route to access to chaos in bubble nonlinear oscillations. The bifurcation structure of bubble radius oscillation has a similarity with other nonlinear oscillators, implying that they have universal characteristics. At the beginning of studying chaotic behaviors of bubble oscillations, Lauterborn W et al. [8] found the chaos in bubble oscillations under high intensity sound by using holographic camera at the speed of 69300 holograms per second. The discovery makes people have a great interest in it. Parlitz U et al. [9] found that the phenomena of period 1, period 2, period 3 and chaos state in a single bubble oscillation under the action of sound. However, at that time, the resonance frequency, subharmonics, harmonic dispersion and other issues were more focused. In recent years, people pay more attention to the cause, evaluation indicators of bifurcation and chaos. Behnia S et al. [10] studied a single cavitation bubbles oscillation by methods of bifurcation diagrams and Lyapunov exponent and
found rich nonlinear features of period doubling bifurcation. Sojahrood A J et al. [11] studied the bifurcation structure of ultrasound contrast agents (UCAs) under excitation. Selecting the appropriate ultrasonic frequency can increase the section of UCAs and improve the curative effect. Due to the model of single bubble, the interaction between bubbles has not yet considered, so it doesn’t simulate the real situation well.

However, in practical application, cavitation bubbles present in the form of cluster. We need to consider the interaction of bubbles. The model of most classic interacting bubbles is two bubbles oscillate under ultrasonic. Jiang L et al. [12] studied the frequency spectrum of the noise emitted by the interaction of bubbles. The interaction between bubbles exerts an additional nonlinear effect on the oscillation of the bubble. Sugita N et al. [13] studied a bifurcation structure of two spherical bubbles nonlinear oscillation. The energy localization is resulted from symmetry-breaking bifurcation of the state oscillation. At present, most of the problems of two bubble dynamics are concerned with the study of their cavitation effects, collapse, and the effect of dual-frequency [14]. There are few studies on two bubbles’ chaos characteristics.

In this paper, we investigate two bubbles nonlinear oscillation motion in sound field, and analyze the period doubling route to access to chaos. The time-domain wave, phase trajectory, Poincaré section, power spectrum and bifurcation diagram are used to study the dynamics of two spherical bubbles. We find that the oscillation of the two interacting bubbles in the sound field has complicated nonlinear characteristic which is similar to a single bubble’s oscillation. We compare the difference in the oscillation of two interacting bubbles with different original radii under the same excitation. Results show that the disparate nonlinear features of two bubbles with different initial radius under the action of sound. Meanwhile, smaller bubble is more intense than the larger one. In addition, the pressure threshold of smaller bubble access chaos is less than the large bubble.

2. The model and solution methods

2.1. Model of two interacting bubbles oscillation

Two interacting bubbles oscillate regularly under the action of a sound field, as shown in Figure 1. The subscripts $i$ and $j$ refer to the two bubbles with different original radius $R_{i0}$ and $R_{j0}$ respectively. $D$ is the distance between $O_i$ and $O_j$, which locate the centers of two bubbles. $P$ is any point in the sound field beyond the bubbles. In our simulation, we set $D \gg R_{i0}$ and $R_{i0} \gg R_{j0}$.

Figure 1. The oscillation model of two interacting bubbles in a sound field

When the bubbles are to collapse, the velocity of the bubble wall is much slower than that of the sound in the liquid medium. Therefore, the modified Keller-Miksis equation [15] is adopted to describe the bubbles dynamics. The equation of bubble $i$ oscillation is as follows:
\[
(1-\frac{\ddot{R}}{c})R\dddot{R} + 3(1-\frac{\ddot{R}}{2c})\dddot{R} = \frac{1}{\rho}(1+\frac{\ddot{R}}{c})[p_i - p_i(t)] + \frac{R}{\rho c \, dt}[p_i - p_i(t)] - F(R_i)
\]  

(1)

Here, \(R_i(t)\) is the instantaneous radius of the bubble \(i\). \(\ddot{R}(t)\) is the bubble wall velocity, \(\dddot{R}(t)\) is the bubble wall acceleration, \(\rho\) is the density of the liquid, \(c\) is the speed of sound in the medium. The term of \(F(R_i)\) represents the interaction of the bubble \(j\) to the bubble \(i\) and can be expressed as follows:

\[
F(R_j) = \frac{R_i^2 \dddot{R}_i + 2R_i \dddot{R}_j^2}{D}
\]  

(2)

\(p_{\infty}\) is the pressure far away from the bubbles and involves the static atmosphere pressure and the periodic acoustic excitation, which is recorded as:

\[
p_{\infty}(t) = P_{\infty} + P_{a} \sin \omega t
\]  

(3)

where \(P_{\infty}\) is the ambient pressure, \(P_{a}\) and \(\omega\) are the pressure amplitude and angular frequency of the excitation source, respectively. The pressure at the bubble wall is denoted as \(p_i\), which can be obtained according to the bubble wall pressure equilibrium conditions:

\[
p_i = \frac{2\sigma}{R_i} + \frac{4\mu \ddot{R}_i}{R_i} - p_g - p_v
\]  

(4)

where \(\sigma\) is surface tension, \(\mu\) is the viscosity coefficient of the liquid medium. The pressure inside the bubble including the vapor pressure \(p_v\) and the pressure of non-condensable gas \(p_g\). The pressure of the gas inside the bubble follows polytropic process regularity:

\[
p_g = p_{g0}(\frac{R_{g0}}{R_i})^{\kappa}
\]  

(5)

where \(\kappa\) is the polytropic exponent and is set to 4/3 when the gas is mainly diatomic molecules. \(p_{g0}\) is the initial pressure inside the bubble and its value is determined by the steady state of the bubble in the liquid medium when there is no sound excitation:

\[
0 = p_{g0}(\frac{R_{g0}}{R_i})^{\kappa} + p_v - P_{\infty} - \frac{2\sigma}{R_g}
\]  

(6)

where \(R_{g}\) is the equilibrium radius of the bubble, and the initial pressure in the bubble can be expressed as:

\[
p_{g0} = \frac{2\sigma}{R_g} - p_v + P_{\infty}
\]  

(7)

\(\omega_0\) is the linear resonant frequency of two interacting bubbles oscillator and can be represented as follows according to the literature [16]:
\[
\omega_i^2 = \frac{D_i^2}{D^2 R_{i0} - R_{i0} R_{j0}} \left\{ \frac{1}{\rho R_{i0}} \left[ 3\kappa (p_0 + 2\sigma) - \frac{2\sigma}{R_{i0}} \right] - \frac{1}{\rho D} \left[ 3\kappa (p_0 + 2\sigma) - \frac{2\sigma}{R_{j0}} \right] \right\}
\]

(8)

In this paper, we choose water as the medium. From Harr-Galaggher-Kell equation[17] at ambient temperature \(T=25^\circ C\) and ambient pressure \(P_{\infty}=1\) atm, the density of the liquid is \(\rho=997\) kg/m\(^3\), the speed of sound in liquid is \(c=1497\) m/s, the surface tension of the liquid is \(\sigma=0.072\) N/m, the liquid coefficient viscosity is \(\mu=0.00089\) kg/m-s, the vapor pressure is \(p_v=3166.8 P_{\infty}\), and the equilibrium radius of the bubble is \(R_E=10\) um.

2.2. Nonlinear numerical tools

The oscillating state of bubbles under ultrasound plays a crucial role in cavitation. The aim of this paper is to find the nonlinear oscillation features between bubbles with different initial radius. In this paper, time domain wave, phase trajectory, Poincaré section, power spectrum and bifurcation diagram are used to study the dynamics of cavitation bubbles. The bifurcation diagrams were made with the pressure \(P_a\) as the control parameter and with different bubble initial radius. The ultrasonic pressure amplitude ranges from 0 to 5 atm. The different oscillation states of bubble \(i\) and \(j\) were observed under excitation pressure at \(P_a=0.15\), 1.5 and 4.5 atm. The time domain wave shows the oscillation regularity of the bubbles excited by the sound field. Through the relationship between the bubbles’ radius and the bubble wall velocity, the phase trajectory studies the change rules of the two bubbles’ motion state. By selecting the appropriate Poincaré section and then intercepting corresponding points on the plane phase trajectory left, the motion state of the bubbles can be judged concisely. Through the Fourier transform, any periodic signal can be expressed as the superposition of the fundamental wave and the harmonic wave. The power spectrum of any aperiodic signal is a continuous spectral structure. However, when studying a single cavitation bubble’s radiation in the sound field with frequency \(\omega\), Plesset[18] found it is the nonlinear response that results in harmonic dispersion. That is to say, apart from sound field generating integral multiple frequencies, there are also subharmonics. The emergence of waves means that bifurcation has arrived. Therefore, the power spectrum is an important method to analyze the bubble motion from the period oscillation to the chaos.

3. Results and discussion

Fig. 2 shows the bifurcation structure of the normalization radius \(R_i\) and \(R_j\) of the bubbles \(i\) and \(j\) with amplitude \(P_a\). The ultrasonic frequency \(f_s=2 f_i=658\) kHz. The distance between two bubbles \(D=300 R_E\). The initial radii of bubble \(i\) and \(j\) are 10 um and 9 um respectively. Via the period doubling bifurcation, both bubbles’ oscillation accessed the chaotic state. What’s more, it alternately appeared periodic and chaotic motion. In Fig. 2(b), when \(0<P_a<1.11\) atm, the oscillation of bubble \(j\) is in stable period 1. With the increasing of \(P_a\), the bubble \(j\) starts to enter a more organized region of period 2 motion state for \(1.11\) atm \(< P_a < 2.57\) atm. then, when \(2.57\) atm \(< P_a < 2.93\) atm, the bubble \(j\) state is in period 4. After all the period regions, the structure of diagram bifurcates into a chaotic region with \(2.93\) atm \(< P_a < 3.66\) atm for the first time. When \(3.66\) atm \(< P_a < 4.28\) atm, the chaotic region alternates with the periodic window; when \(P_a > 4.28\) atm, the bubble \(j\) enters the chaotic region for the second time. The bifurcation diagram of the large bubble \(i\) in Fig. 2(a) also has a similar topology structure, but the interval range \((0<P_a<0.2\) atm\) of the large bubble in period 1 is smaller than the small bubble \(j\) \((0<P_a<1.12)\) atm. The range of period 2 \((0.2\) atm \(< P_a < 2.57)\) atm) of large bubble \(i\) is significantly larger than the small bubble \(j\) \((1.12)\) atm \(< P_a < 2.56)\) atm.

Figure 3(a) describes the radius time-domain variation of oscillation normalization of two bubbles excited by acoustic wave while the pressure amplitude is \(P_1=0.15\) atm. The simulation time we have chosen is six times of the excitation period. From the figure, the bubble presents 6 periodic sinusoidal oscillation and is in a period 1 state. The steady state of oscillation curve repeats again and again with the periodic excitation. Moreover, the smaller bubble oscillates more intensely than the larger one if the excitation energy is constant. Fig. 3(b) is a phase diagram which represents the relationship...
between the fluctuation velocity of bubbles’ wall and their normalized radius. The phase diagram of two bubbles’ motion trajectory is a stable closed curve, and the Poincaré section map displays the only one point, as shown in Figure 3(c). The power spectrums of the bubble \( i \) and \( j \) focus on the fundamental frequency, as shown in Fig. 3(d). Only a small fraction of energy appears at their double frequency.

**Figure 2.** Bifurcation diagrams of dimensionless bubble radius - pressure amplitude with different initial radius (a) \( R_{i0}=10\text{um} \); (b) \( R_{j0}=9\text{um} \)

**Figure 3.** Bubble oscillation with period 1 (a) The time-domain diagram; (b) phase diagram; (c) Poincaré section; (d) power spectrum
Figure 4. bubble oscillation with period 2 (a) The time-domain diagram; (b) phase diagram; (c) Poincaré section; (d) power spectrum

Fig. 4(a) is the time-domain variation diagram of the normalized radius of two interacting bubble oscillations while pressure $P_2=1.5$atm. The simulation time we selected is also six times of the excitation period. But from the figure, there are only 3 period oscillations, implying the two interacting bubbles are in period 2. In other words, the motion rules of the bubbles will repeat every two excitation cycles. Meanwhile each bubble oscillation has its own characteristics. The trajectories in the phase diagram of Fig. 4(b) are cycled twice, leaving two points on the Poincaré section respectively, as shown in Fig. 4(c). As shown in Fig. 4(d), the power spectrum of bubbles oscillation includes not only the fundamental frequency $f_s$ and its multiplication component $nf_s$, but also the subharmonic wave $f_s/2,3f_s/2,5f_s/2$ and so on. These phenomena indicate that bubbles oscillation begins to access to chaos state. The oscillation of bubbles is more intense, and cavitation is more likely to occur.
Figure 5. bubble oscillation with chaos (a) The time-domain diagram; (b) phase diagram; (c) Poincaré section; (d) power spectrum

The oscillations of bubbles enter stable chaos state as excitation pressure increases gradually. We studied the motion of the bubbles under the excitation pressure $P_3=4.5$ atm, as shown in Fig. 5(a). The simulation time we selected is forty times of the excitation period. Both of the two bubbles lose their periodical motion characteristics and show no regularity in radius variation. At this time, the trajectory in the phase diagram of Fig. 5(b) is no longer the form of a limit cycle, but a curve that will never be closed. The characteristic of reciprocating movement reflects that the phase trajectories converge within bounded regions and thus form strange attractors. The Poincaré section map in Fig. 5(c) is not a finite point set or a closed curve, but a scattered point set with a fractal structure. According to the Poincaré section mapping theory, at this moment the bubbles have entered the chaotic oscillation state. Fig. 5(d) shows that the power spectrum of two bubbles’ oscillation presents a continuous spectrum structure. It is shown that the bubbles will access the stable chaos region when the excitation pressure is large enough by all of the nonlinear methods. From the perspective of cavitation effects, severe cavitation has taken place under the sound field, which is beneficial to improving the effect of ultrasonic cavitation treatment.

4. Conclusion
In this paper, the nonlinear characteristics of two interacting bubbles under the action of strong sound field were studied. We studied the effects of the ultrasonic pressure amplitude on bubbles’ motion under the conditions of bubbles with different initial radii by nonlinear methods of the phase trajectory, Poincaré section, power spectrum, and bifurcation diagram. Results show that two bubbles oscillation traverses the similar mode of period doubling bifurcation to cascade stable chaos region. Meanwhile, the periodic windows interspersed with chaotic regions, indicating the existence of transient chaos. The continuous structure on power spectrum and strange attractors will appear when acoustic wave amplitude exceeds the pressure threshold. It means that the oscillation of two interacting bubbles will be intense chaotic motion. In addition, we found that the oscillation of the smaller bubble is more drastic.

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