QUASITORIC MANIFOLDS HOMEOMORPHIC TO HOMOGENEOUS SPACES

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Abstract. We present some classification results for quasitoric manifolds $M$ with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which admit an action of a compact connected Lie-group $G$ such that $\dim M/G \leq 1$. In contrast to Kuroki’s work [7, 6] we do not require that the action of $G$ extends the torus action on $M$.

1. Introduction

Quasitoric manifolds are certain $2n$-dimensional manifolds on which an $n$-dimensional torus acts such that the orbit space of this action may be identified with a simple convex polytope. They were first introduced by Davis and Januszkiewicz [2] in 1991.

In [7, 6] Kuroki studied quasitoric manifolds $M$ which admit an extension of the torus action to an action of some compact connected Lie-group $G$ such that $\dim M/G \leq 1$. Here we drop the condition that the $G$-action extends the torus action in the case where the first Pontrjagin-class of $M$ is equal to the negative of a sum of squares of elements of $H^2(M)$. In this note all cohomology groups are taken with coefficients in $\mathbb{Q}$. We have the following two results.

Theorem 1.1. Let $M$ be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which is homeomorphic (or diffeomorphic) to a homogeneous space $G/H$ with $G$ a compact connected Lie-group. Then $M$ is homeomorphic (diffeomorphic) to $\prod S^2$. In particular, all Pontrjagin-classes of $M$ vanish.

Theorem 1.2. Let $M$ be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Assume that the compact connected Lie-group $G$ acts smoothly and almost effectively on $M$ such that $\dim M/G = 1$. Then $G$ has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore $M$ is diffeomorphic to a $S^2$-bundle over a product of two-spheres.

The proofs of these theorems are based on Hauschild’s study [4] of spaces of $q$-type. A space of $q$-type is defined to be a topological space $X$ satisfying the following cohomological properties:

- The cohomology ring $H^*(X)$ is generated as a $\mathbb{Q}$-algebra by elements of degree two, i.e. $H^*(X) = \mathbb{Q}[x_1, \ldots, x_n]/I_0$ and $\deg x_i = 2$.
- The defining ideal $I_0$ contains a definite quadratic form $Q$.

The note is organised as follows. In section 2 we show that a quasitoric manifold $M$ with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ is of $q$-type. In section 3 we prove Theorem 1.1. In section 4 we recall some properties of cohomogeneity one manifolds. In section 5 we prove Theorem 1.2.

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2. Quasitoric manifolds with \( p_1(M) = -\sum a_i^2 \)

In this section we study quasitoric manifolds \( M \) with \( p_1(M) = -\sum a_i^2 \) for some \( a_i \in H^2(M) \). To do so we first introduce some notations from [4] and [5, Chapter VII]. For a topological space \( X \) we define the topological degree of symmetry of \( X \) as

\[
N_t(X) = \max\{\dim G; G \text{ compact Lie-group, } G \text{ acts effectively on } X\}
\]

Similarly one defines the semi-simple degree of symmetry of \( X \) as

\[
N_s^a(X) = \max\{\dim G; G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}
\]

and the torus-degree of symmetry as

\[
T_t(X) = \max\{\dim T; T \text{ torus, } T \text{ acts effectively on } X\}.
\]

In the above definitions we assume that all groups act continuously.

Another important invariant of a topological space \( X \) used in [4] is the so called embedding dimension of its rational cohomology ring. For a local \( \mathbb{Q} \)-algebra \( A \), we denote by \( \text{edim} A \) the embedding dimension of \( A \). By definition, we have \( \text{edim} A = \dim_\mathbb{Q} m_A/m_A^2 \), where \( m_A \) is the maximal ideal of \( A \). In case that \( A = \bigoplus_{i \geq 0} A^i \) is a positively graded local \( \mathbb{Q} \)-algebra, \( m_A \) is the augmentation ideal \( A_+ = \bigoplus_{i \geq 2} A^i \). If furthermore \( A \) is generated by its degree two part, then \( m_A^2 = \bigoplus_{i \geq 2} A^i \). Therefore for a quasitoric manifold \( M \) over the polytope \( P \) we have \( \text{edim} H^*(M) = \dim_\mathbb{Q} H^2(M) = m - n \) where \( m \) is the number of facets of \( P \) and \( n \) is its dimension.

**Lemma 2.1.** Let \( M \) be a quasitoric manifold with \( p_1(M) = -\sum a_i^2 \) for some \( a_i \in H^2(M) \). Then \( M \) is a manifold of q-type.

**Proof.** The discussion at the beginning of section 3 of [8] together with Corollary 6.8 of [2, p. 448] shows that there are a basis \( u_{n+1}, \ldots, u_m \) of \( H^2(M) \) and \( \lambda_{i,j} \in \mathbb{Z} \) such that

\[
p_1(M) = \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left( \sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2.
\]

Therefore

\[
0 = \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left( \sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2 + \sum_{i} a_i^2
\]

\[
= \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left( \sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2 + \sum_{j} \left( \sum_{i=n+1}^{m} \mu_{i,j} u_i \right)^2
\]

with some \( \mu_{i,j} \in \mathbb{Q} \) follows.

Because

\[
\sum_{i=n+1}^{m} X_i^2 + \sum_{j=1}^{n} \left( \sum_{i=n+1}^{m} \lambda_{i,j} X_i \right)^2 + \sum_{j} \left( \sum_{i=n+1}^{m} \mu_{i,j} X_i \right)^2
\]

is a positive definite bilinear form the statement follows. \( \square \)

**Proposition 2.2.** Let \( M \) be a quasitoric manifold of q-type over the \( n \)-dimensional polytope \( P \). Then we have for the number \( m \) of facets of \( P \):

\[
m \geq 2n
\]
Proof. By Theorem 3.2 of [4, p. 563], we have
\[ n \leq T_t(M) \leq \text{edim } H^*(M) = m - n. \]
Therefore we have \( 2n \leq m \). □

Remark 2.3. The inequality in the above proposition is sharp, because for \( M = S^2 \times \cdots \times S^2 \) we have \( m = 2n \) and \( p_1(M) = 0 \).

By Theorem 5.13 of [4, p. 573], we have for a manifold \( M \) of q-type that \( N^{ss}_t(M) \leq \text{dim } M + \text{edim } M \). Hence, for a quasitoric manifold \( M \), we get:

Proposition 2.4. Let \( M \) as in Proposition 2.3. Then we have
\[ N^{ss}_t(M) \leq 2n + m - n = n + m. \]

Remark 2.5. The inequality in the above proposition is sharp because for \( M = S^2 \times \cdots \times S^2 \) we have \( m = 2n \) and \( SU(2) \times \cdots \times SU(2) \) acts on \( M \) and has dimension \( 3n \).

3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 1.1. Recall from Lemma 2.1 that a quasitoric manifold \( M \) with first Pontrjagin-class equal to the negative of the sum of squares of elements of \( H^2(M) \) is a manifold of q-type.

Let \( M \) be a quasitoric manifold over the polytope \( P \) which is also a homogeneous space and is of q-type.

Let \( G \) be a compact connected Lie-group and \( H \subset G \) a closed subgroup such that \( M \) is homeomorphic or diffeomorphic to \( G/H \). Because \( \chi(M) > 0 \) and \( M \) is simply connected, we have \( \text{rank } G = \text{rank } H \) and \( H \) is connected. Therefore we may assume that \( G \) is semi-simple and simply connected.

Let \( T \) be a maximal torus of \( G \). Then \((G/H)^T\) is non-empty. By Theorem 5.9 of [4, p. 572], the isotropy group \( G_x \) of a point \( x \in (G/H)^T \) is a maximal torus of \( G \). Hence, \( H \) is a maximal torus of \( G \).

Now it follows from Theorem 3.3 of [4, p. 563] that
\[ T_t(G/H) = \text{rank } G. \]

Because \( M \) is quasitoric, we have \( n \leq T_t(G/H) \). Combining these inequations, we get
\[ \text{dim } G - \text{dim } H = \text{dim } M = 2n \leq 2 \text{rank } G. \]
This equation implies that \( \text{dim } G \leq 3 \text{rank } G \).

For a simple simply connected Lie-group \( G' \) we have \( \text{dim } G' \geq 3 \text{rank } G' \) and \( \text{dim } G' = 3 \text{rank } G' \) if and only if \( G' = SU(2) \). Therefore we have \( G = \prod SU(2) \) and \( M = \prod SU(2)/T^1 = \prod S^2 \). This proves Theorem 1.1.

4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian \( G \)-manifolds \( M \) with orbit space a compact interval \([-1, 1]\). We follow [3, p. 39-44] in this discussion.

We fix a normal geodesic \( c : [-1, 1] \to M \) perpendicular to all orbits. We denote by \( H(c) \) the principal isotropy group \( G_{c(0)}' \), which is equal to the isotropy group \( G_{c(t)}' \) for \( t \in [-1, 1] \), and by \( K^\pm \) the isotropy groups of \( c(\pm 1) \).

Then \( M \) is the union of tabular neighbourhoods of the non-principal orbits \( Gc(\pm 1) \) glued along their boundary, i.e., by the slice theorem we have
\[ M = G \times K^- D_- \cup G \times K^+ D_+, \]
where \( D_\pm \) are discs. Furthermore \( K^\pm/H = \partial D_\pm = S_\pm \) are spheres.
Note that $M$ may be reconstructed from the following diagram of groups.

\[ \begin{array}{ccc}
G & \rightarrow & K^+ \\
\downarrow & & \downarrow \\
H & \rightarrow & H
\end{array} \]

The construction of such a group diagram from a cohologeneity one manifold may be reversed. Namely, if such a group diagram with $K^\pm/H = S^\pm$ spheres is given, then one may construct a cohologeneity one $G$-manifold from it. We also write these diagrams as $H \subset K^-, K^+ \subset G$.

Now we give a criterion for two group diagrams yielding up to $G$-equivariant diffeomorphism the same manifold $M$.

**Lemma 4.1** ([3, p. 44]). The group diagrams $H \subset K^-, K_1^+ \subset G$ and $H \subset K^-, K_2^+ \subset G$ yield the same cohologeneity one manifold up to equivariant diffeomorphism if there is an $a \in N_G(H)^0$ with $K_1^+ = aK_2^+a^{-1}$.

5. Quasitoric manifolds with cohologeneity one actions

In this section we study quasitoric manifolds $M$ which admit a smooth action of a compact connected Lie-group $G$ which has an orbit of codimension one. As before we do not assume that the $G$-action on $M$ extends the torus action. We have the following lemma:

**Lemma 5.1.** Let $M$ be a quasitoric manifold of dimension $2n$ which is of $q$-type. Assume that the compact connected Lie-group $G$ acts almost effectively and smoothly on $M$ such that $\dim M/G = 1$. Then we have:

1. The singular orbits are given by $G/T$ where $T$ is a maximal torus of $G$.
2. The Euler-characteristic of $M$ is $2\#W(G)$.
3. The principal orbit type is given by $G/S$, where $S \subset T$ is a subgroup of codimension one.
4. The center $Z$ of $G$ has dimension at most one.
5. $\dim G/T = 2n - 2$.

**Proof.** At first note that $M/G$ is an interval $[-1,1]$ and not a circle because $M$ is simply connected. We start with proving (1). Let $T$ be a maximal torus of $G$. By passing to a finite covering group of $G$ we may assume $G = G' \times Z'$ with $G'$ a compact connected semi-simple Lie-group and $Z'$ a torus. Let $x \in M$. Then the isotropy group $G_x$ has maximal rank in $G$. Therefore $G_x$ splits as $G'_x \times Z'$. By Theorem 5.9 of [4, p. 572], $G'_x$ is a maximal torus of $G'$. Therefore we have $G_x = T$.

Because $\dim G - \dim T$ is even, $x$ is contained in a singular orbit. In particular we have

\[ \chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-), \]

where $G/K^\pm$ are the singular orbits. Furthermore we may assume that $G/K^+$ contains a $T$-fixed point. This implies

\[ \chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G'). \]

Now assume that all $T$-fixed points are contained in the singular orbit $G/K^+$. Then we have $(G/K^-)^T = \emptyset$. This implies

\[ \chi(M) = \chi(G/K^+) = \#W(G'). \]
Now Theorem 5.11 of [4] p. 573 implies that $M$ is the homogeneous space $G'/G' \cap T = G/T$. This contradicts our assumption that $\dim M/G = 1$.

Therefore both singular orbits contain $T$-fixed points. This implies that they are of type $G/T$. This proves (1). (2) follows from (5.1) and (5.2).

Now we prove (3) and (5). Let $S \subset T$ be a minimal isotropy group. Then $T/S$ is a sphere of dimension $\text{codim}(G/T, M) - 1$. Therefore $S$ is a subgroup of codimension one in $T$ and $\text{codim}(G/T, M) = 2$.

If the center of $G$ has dimension greater than one, then $\dim Z' \cap S \geq 1$. That means that the action is not almost effective. Therefore (4) holds. □

By Lemma 5.1, we have with the notation of the previous section that $K^\pm$ are maximal tori of $G$ containing $H = S$. In the following we will write $G = G' \times Z'$ with $G'$ a compact connected semi-simple Lie-group and $Z'$ a torus.

Because $K^\pm$ are maximal tori of the identity component $Z_G(S)_0$ of the centraliser of $S$, there is some $a \in Z_G(S)^0$ such that $K^+ = aK^-a^{-1}$. By Lemma 4.1 we may assume that $K^+ = K^- = T$. Now from Theorem 4.1 of [9] p. 198 it follows that $M$ is a fiber bundle over $G/T$ with fiber the cohomogeneity one manifold with group diagram $S \subset T, T \subset T$. Therefore it is a $S^2$-bundle over $G/T$.

Lemma 5.2. Let $M$ and $G$ as in the previous lemma. Then we have

$$T_t(M) \leq \text{rank} G' + 1.$$ 

Proof. At first we recall the rational cohomology of $G/T$. By [1] p. 67, we have

$$H^*(G/T) \cong H^*(BT)/I$$

where $I$ is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of $G$. Therefore it follows that

$$\dim \mathbb{Q} H^{\text{odd}}(G/T) = 0 \quad \text{and} \quad \dim \mathbb{Q} H^2(G/T) = \text{rank} G'.$$

Therefore the Serre spectral sequence for the fibration $S^2 \to M \to G/T$ degenerates. Hence, we have

$$H^*(M) = H^*(G/T) \otimes H^*(S^2)$$

as $H^*(G/T)$-modules. In particular, we have

$$\dim \mathbb{Q} H^2(M) = \dim \mathbb{Q} H^2(G/T) + \dim \mathbb{Q} H^2(S^2) = \text{rank} G' + 1.$$ 

Therefore

$$T_t(M) \leq \text{edim} H^*(M) = \dim \mathbb{Q} H^2(M) = \text{rank} G' + 1$$

follows. □

Theorem 5.3. Let $M$ and $G$ as in the previous lemmas. Then $G$ has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore $M$ is diffeomorphic to a $S^2$-bundle over a product of two-spheres.

Proof. Because $M$ is quasitoric we have $n \leq T_t(M)$. By Lemma 5.1 we have

$$\dim G' = \text{rank} G' = \dim G/T = 2n - 2.$$ 

Now Lemma 5.2 implies

$$\dim G' = 2n - 2 + \text{rank} G' \leq 3 \text{rank} G'.$$

Therefore $\prod SU(2)$ is a finite covering group of $G'$. This implies the statement about the finite covering group of $G$.

It follows that $G/T = \prod S^2$. Therefore $M$ is a $S^2$-bundle over $\prod S^2$. □

Now Theorem 1.2 follows from Theorem 5.3 and Lemma 2.3.
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