We propose an estimation procedure for discrete choice models of differentiated products with possibly high-dimensional product attributes. In our model, high-dimensional attributes can be determinants of both mean and variance of the indirect utility of a product. The key restriction in our model is that the high-dimensional attributes affect the variance of indirect utilities only through finitely many indices. In a framework of the random-coefficients logit model, we show a bound on the error rate of a $l_1$-regularized minimum distance estimator and prove the asymptotic linearity of the de-biased estimator.

1 Introduction

There are many occasions in which high-dimensional product attributes are available for estimating and predicting demands for differentiated products, especially in contexts where machine learning techniques such as pattern recognition and natural language processing can generate high-dimensional representation of product attributes. For example, we can consider a consumer choice over clothes. In addition to the typical characteristics of prices, countries of origin, and materials, there are numerous varieties of design patterns that can influence a consumer’s choice. The choice over books is dependent on their contents. With a natural language processing technique, we can potentially represent their contents with a high-dimensional semantic vectors. In this paper, we investigate a framework that enables us to integrate these potentially informative but not-yet-fully-used product attributes in estimation, inference, and prediction of demands for differentiated products.

We are not the first considering the estimation of high-dimensional discrete choice model. Gillen et al. (2019) studied an estimation of a model that extends Berry et al. (1995, henceforth BLP)’s random-coefficients discrete-choice model with high-dimensional attributes and applied to the study of political campaign for elections. In their model, the product attributes can grow exponentially relative to the sample size; however, random coefficient is only allowed for the price. Under this assumption, the high-dimensional attributes can only affect the mean indirect utility. Thus, they can apply a $l_1$-regularized least squares method to the inverted mean indirect utility to select relevant variables.

Recent methodological developments in statistics and econometrics allows us to apply a high-dimensional estimation procedure to a broader class of models. In particular, the recent work of Belloni et al. (2018) provides a set of useful results for $l_1$-regularized minimum distance estimation following the techniques developed by Frank and Friedman (1993) and Tibshirani (1996). We derive the property of our $l_1$-regularized minimum distance estimator and its debiased version based on their results. Because the objective function of the BLP model is non-linear in the parameters, we use the contraction inequality theorem of Ledoux and Talagrand (1991) to control the tail probability of the estimation error. This contraction inequality exploits the Lipschitz continuity of the objective function with respect to a single index. In the context of the BLP discrete choice model with non-linear but smooth moment conditions, we show that the analogue principle can be applied to the case with multiple indices. This allows us to have fixed number of random coefficients on the indices of potentially high-dimensional attributes.
The last section concludes. In the next section, we describe our model and introduce the regularized GMM (RGMM) problem for our model. In section 3, we show the probability bound for the estimation error from the regularized GMM problem. In section 4, we consider the de-biased procedure for the proper inference. The last section concludes.

2 Demand estimation as a regularized GMM

2.1 Model

Consider there are J products in each market $i \in \{1, \ldots, n\}$. Let us denote $[\cdot]$ for an integer indicates index set \{1, \ldots, \cdot\}. Each product $j$ shares a non-zero demand $S_{ij} \in (0, 1)$ in each market $i$. Each product $j$ in a market $i$ has observed $L$ attributes $\{x_{ijl}\}_{l \in [L]}$ including cost of attaining the product $j$, $-p_{ij}$, and an unobserved attribute $\xi_{ij}$.

In this paper, we consider the high-dimensionality in the product attributes $x_{ij}$. The key restriction is that we assume there are $G$ known finite partitions of the high-dimensional product characteristics $[L]$. In particular, we consider the following indirect utility for a product $j$ in a market $i$:

$$u_{ij} = \sum_{g \in [G]} \sum_{l \in L_g} x_{ijl}(\beta_l + \gamma_l \tilde{\beta}_g) + \xi_{ij} + \epsilon_{ij}$$

where $\epsilon_{ij}$ is an idiosyncratic error term and $L_g$ is mutually exclusive subset of $[L]$ for each $g \in [G]$ such that $\cup_{g \in [G]} L_g = [L]$. Note that $\sum_{g \in [G]} \sum_{l \in L_g} x_{ijl} \gamma_l = \sum_{g \in [G]} x_{ijg} \gamma_l \tilde{\beta}_g$ with $x_{ijl} = [x_{ijl}]_{l \in L_g}$ and $\gamma_g = [\gamma_l]_{l \in L_g}$ and $\tilde{\beta}_g \sim iid N(0, 1)$ capture the mean heterogeneity in the utility for the product $j$ in the market $i$. We assume $\beta_l$ is independent of $\tilde{\beta}_g$, and the group specific variance of the random coefficient $\beta_l$ is normalized to 1 for each corresponding vector $\gamma_l$. Therefore, the variance of the indirect utility of the product $j$ in the market $i$ is $\sum_{g \in [G]} (\sum_{l \in L_g} x_{ijg} \gamma_l \tilde{\beta}_g)^2$. This finite indices restriction allows us to apply the contraction inequality theorem, Ledoux and Talagrand (1991), which is in principle applied to a Lipschitz transformation of a single index. If the random coefficients are not restricted, then the number of indices grows to infinity as the number of attributes grows infinity, and we cannot apply the contraction inequality to bound the estimation error. For the reminder of the paper, let $g(l)$ represents the group $g \in [G]$ that the attribute $l$ belongs to.

The mean utility is the same as usual differentiated product demand model such as the BLP model. The heterogeneity term is different from the usual random coefficient model. This model can be seen as a special case of the usual random coefficient model that the product attributes $x_{ijl}$ and $x_{ijl'}$ share the same individual preference shock $\beta_l$ if $l, l' \in L_g$. In other words, consumers observe the set of characteristics $\{x_{ijl}\}_{l \in L_g}$ as a common index characteristics of $x_{ijg} \gamma_l$, but individuals may have different preference over the index $x_{ijg} \gamma_l$. In the example of the design pattern, we share the same objective descriptions of the design, but the subjective preference over the descriptions differ across people. An important feature is that we do not restrict $\beta_l = \gamma_l$. Therefore, the mean $\sum_{l \in L_g} x_{ijl} \beta_l$ and the variance $\sum_{l \in L_g} x_{ijl} \gamma_l \tilde{\beta}_g$ of each utility load from a group of characteristics $L_g$ may be unrelated each other.

2.2 Moment condition

Suppose that $\epsilon_{ij}$ follows iid Type-I extreme value distribution, then the parameters $\theta \equiv \{\beta', \gamma'\}'$ and observed characteristics $x_{ij}$ pin down the share of the product $j$ in the market $i$ as the function of mean utility vector $x_i \beta + \xi_i$ such that

$$s_j(x_i, x_i \beta + \xi_i; \theta) = \frac{\exp(x_i \beta + \xi_i + \sum_g x_{ijg} \gamma_l \tilde{\beta}_g)}{1 + \sum_{j' \in [J]} \exp(x_{ij'} \beta + \xi_{ij'} + \sum_g x_{ij'g} \gamma_l \tilde{\beta}_g)}$$

where $F \sim N_{|G|}(0, I_{|G|})$ and $S_{ij}$ denotes the population share of the product $j$ in the market $i$.

Here we assume that the unobserved product type $\xi_{ij}$ is mean independent of a vector of instruments $w_{ij}$,

$$E[\xi_{ij} | w_{ij}] = 0$$

where the expectation is taken over the markets $i$ for each $j \in [J]$. This conditional moment restriction leads to a set of unconditional moment conditions with transformed vector of $K$ instruments for each product $j$, $h_{jk}(w_{ij}), k \in [K]$.

\[1\] To simplify the argument, we ignore the measurement error issue of the observed market share from the population share for now.
such that
\[ E[\xi_i h_{jk}(w_{ij})] = 0 \]
for each \( j \in [J] \). Berry (1994) shows that there exists unique inverse functions of the share \( s_j(x, \cdot; \theta) \), such that
\[ s_j(x_i, s_{-1}(x_i, S_i; \theta); \theta) = S_{ij}. \tag{1} \]

Therefore, the moment condition is now
\[ E[(s_j^{-1}(x_i, S_i; \theta) - x_i' \beta) h_{jk}(w_{ij})] = 0, \forall j \in [J], k \in [K]. \]

Now let
\[ \xi_j(\tilde{x}; \theta) \equiv s_j^{-1}(x, S; \theta) - x_i' \beta \]
where \( \tilde{x} \equiv (x', S', w')' \).

### 2.3 Regularized GMM problem

Following Belloni et al. (2018), we consider a regularized GMM approach for the moment condition above. In particular, let \( f(\tilde{x}; \theta) \) be the score function vector of \((J, K)\) elements with
\[ f_{jk}(\tilde{X}; \theta) \equiv \xi_j(\tilde{X}; \theta) h_{jk}(W), \]
for each \( j \in [J] \) and \( k \in [K] \) where \( \tilde{X} \equiv (X', S', W)', \) and let
\[ f(\theta) \equiv A E[f(\tilde{X}; \theta)] \]
and
\[ \hat{f}(\theta) \equiv \hat{A} E_n[f(\tilde{X}; \theta)] \]
with some weight matrix \( A \) and its estimate \( \hat{A} \), where \( E_n[\cdot] \) represents sample mean of a random vector. For now, let \( A = \hat{A} = I \).

The regularized GMM estimator \( \hat{\theta} \) solves the following optimization problem
\[ \min_{\theta \in \Theta} \| \theta \|_1 : \| \hat{f}(\theta) \|_\infty \leq \lambda \]
for some regularization parameter \( \lambda \).

### 3 Bounds on the estimation error

Belloni et al. (2018) show the rate of convergence for the estimation error under two conditions in addition to the regularization condition which is the constraint of the optimization problem shown above. Below we cite their statement under three high-level conditions

**Proposition 1** (Proposition 3.1 of Belloni et al. (2018)). Assume the following three conditions:

1. (Regularization) The regularization parameter \( \lambda \) satisfies
   \[ \| \hat{f}(\theta_0) \|_\infty \leq \lambda \]
   with probability at least \( 1 - \alpha \)
2. (Identifiability) The population moment function satisfies the following:
   \[ \{ \| f(\theta) - f(\theta_0) \|_\infty \leq \epsilon, \theta \in R(\theta_0) \} \]
   implies
   \[ \| \theta - \theta_0 \| \leq r(\epsilon; \theta_0, l) \]
   for all \( \epsilon > 0 \) where \( R(\theta_0) \equiv \{ \theta \in \Theta : \| \theta \|_1 \leq \| \theta_0 \|_1 \} \), and \( r(\cdot; \theta_0, l) \) is a weakly increasing rate function depending on the semi-norm \( l \).
3. (Empirical moment restriction) The empirical moment function satisfies
   \[ \sup_{\theta \in R(\theta_0)} \| \hat{f}(\theta) - f(\theta) \|_\infty \leq \epsilon_n \]
   with probability at least \( 1 - \delta_n \).

Then with probability at least \( 1 - \alpha - \delta_n \),
\[ \| \hat{\theta} - \theta_0 \|_1 \leq r(\lambda + \epsilon_n; \theta_0, l). \]
3.1 Identifiability condition

For the second condition of the identifiability, Belloni et al. (2018) offers the following sufficient condition

Assumption 1 (Condition NLID for exactly sparse parameters). Suppose that there exist $T \subset [L]$ with cardinality $s$ such that $\theta_0 \neq 0$ only for $l \in T$.

For each $q \in \{1, 2\}$, suppose that there exists a sequence $\mu_n$ such that

$$k(\theta_0, l_q) = \inf_{\theta \in \mathcal{R}(\theta_0): \|\theta - \theta_0\|_q > 0} \left\| G(\theta - \theta_0) \right\|_\infty / \|\theta - \theta_0\|_q \geq s^{-1/q} \mu_n$$

where $G$ is the Jacobian matrix of $f(\theta)$.

Suppose further that

$$\{ \| f(\theta) - f(\theta_0) \|_\infty \leq \epsilon, \theta \in \mathcal{R}(\theta_0) \}$$

implies that

$$\| G(\theta - \theta_0) \|_\infty/2 \leq \epsilon$$

for all $\epsilon \leq \epsilon^*$ for some $\epsilon^*$.

The last condition in the above assumption is specific to the non-linear problem. Nevertheless, this assumption does not bind in our model because our target moment function is continuously differentiable everywhere.

The second condition in assumption regulates the modulus of continuity $k(\theta_0, l)$. Lemma 3.1 of Belloni et al. (2018) offers a sufficient condition for the second condition in exactly sparse model. For the linear IV regression model, for any sub-vector of covariates $X$, we need some sub-vector of instruments $W$ such that $E(W'X)$ is non-singular. In other words, there exists some instruments that are strong for any sub-vector of endogenous covariates. In our context of the BLP model, the Jacobian matrix is $JK \times 2L$ matrix with each $f$ entry for $jk$ element as

$$G_{jk,l}(\tilde{X}, \theta) = \begin{cases} E[h_{jk}(W)X_{jl}] & \text{for } 1 \leq l \leq L \\ E[h_{jk}(W)D_j(\theta) f(\tilde{\beta}; \tilde{X}, \theta) \sum_{j'=1}^{J} (1 - s_j(\tilde{\beta}; \tilde{X}, \theta)) X_{j'1}] & \text{for } L + 1 \leq l \leq 2L \end{cases}$$

where $s(\tilde{\beta}; \tilde{X}, \theta)$ is $J \times 1$ vector of $s_j(\tilde{\beta}; \tilde{X}, \theta) = \frac{\exp(X_j'\beta + \xi_j(X, \theta) + \sum_{g} X_{jg}'\gamma_g \beta_g)}{1 + \sum_{j' \in [J]} \exp(X_{j'}'\beta + \xi_{j'}(X, \theta) + \sum_{g} X_{j'g}'\gamma_g \beta_g)}$ and $D_j(\theta)$ is the $j$-th row of the inverse matrix of the Jacobian matrix of $s(\tilde{X}; \theta) \equiv s(X, X'\beta + \xi(X, \theta); \theta)$ vector with respect to the mean utility vector. Therefore, the modulus of continuity condition requires that the variables $h_{jk}(W)$ serve as the strong instruments for the attribute $l$ of the product $j$, $X_{jl}$, as well as the weighted sum of the attributes $l$ of the products $j'$ across the market $X_{j'1}$. This is not a strong restriction for the most of the attributes as we often assume the attributes are exogenous. For the endogenous attributes, we need to be cautious on the restriction as the instruments are not necessarily strong in particular when the asymptotic is considered for the size of markets $J$ rather than the number of markets $n$. See Armstrong (2014) for the relevant discussion.

Lemma 1 (Lemma 3.4 of Belloni et al., 2018), Under assumption for all $0 < \epsilon \leq \epsilon^*$

$$\{ \| f(\theta) - f(\theta_0) \|_\infty \leq \epsilon, \theta \in \mathcal{R}(\theta_0) \}$$

implies

$$\| \theta - \theta_0 \|_l \leq r(\epsilon; \theta_0, l) \leq 2\epsilon s^{1/q} \mu_n^{-1}.$$
and
\[ \xi_j(\nu; \bar{X}) \equiv \xi(\bar{X}; \theta) \]
such that \( \nu_{j', g} \equiv X_{j', g}\theta \) for every \( j' \in [J] \) and \( g \in \{0, 1, \ldots, G\} \).
Also, let
\[ \nu_{j'g0} \equiv X_{j'g0} \theta = \begin{cases} X_{j'0} \beta_0 & \text{if } g = 0 \\ X_{j'g0} \gamma_0 & \text{if } g > 0. \end{cases} \]

In lemma 4 in the Appendix, we show that the score functions \( f_{jk} \) are Lipschitz continuous in \( \nu_{j', g} \) uniformly for every \( \nu_{j', g} \) with the Lipschitz constant \( J \) times some universal constant. Using this property, we employ the Ledoux-Talagrand contraction inequality as follows:

**Theorem 1.** In addition to the assumptions for lemma 4 in the Appendix, suppose the following:

1. \( \sup_{\Delta \in \Theta} \vartheta \in [J], k \in [K] \mathbb{E}_{\theta} \text{Var}(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \leq B_{1n}^2, \text{ and} \)
2. \( \max_{j \in [J], l \in [L], k \in [K]} \mathbb{E}_n(X_{j,l}^2 h_{jk}^2(W)) \leq B_{2n}^2, \text{ and } \|n^{-1/2} \mathbb{E}_n(f(\tilde{X}, \theta_0))\|_{\infty} \leq n^{-1/2} l_n, \text{ with probability at least } 1 - \delta_n / 6 \)

then,
\[ \sup_{\theta \in \mathcal{R} \in [\theta_0]} \|\hat{f}(\theta) - f(\theta)\|_\infty \leq n^{-1/2}(\tilde{I}_n + l_n) \]
with probability at least \( 1 - \delta_n \), where
\[ \tilde{I}_n \equiv C(B_{1n} + (J^2 G)(2\sqrt{2} B_{2n} \sup_{\theta} \|\theta\|_1 \log^{1/2}(8J^2GKL / \delta_n))) \]
with a universal constant \( C \).

**Proof.** In the same argument of theorem 3.2 of [Belloni et al. (2018)](https://www.duke.edu/~vanligc/research/), \( \|n^{-1/2} \mathbb{E}_n(f(\tilde{X}, \theta_0))\|_{\infty} \leq n^{-1/2} l_n \) implies that we only need to bound the following empirical process
\[ \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \mathbb{E}_n(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)). \]

By taking \( t^2 > 16 B_{1n}^2 \), we may apply Chebyshev inequality and symmetrization lemma (Lemma 2.3.7 of [van der Vaart and Wellner (1996)](https://www.duke.edu/~vanligc/research/)) so that
\[
P \left( \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \left| \mathbb{G}_n(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \right| > t \right) \leq 4P \left( \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \left| \mathbb{G}_n(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \right| > t/4 \right)
\]
where \( \sigma \) is iid Rademacher variable taking \( -1 \) and \( 1 \) with equal probability independent of all the others.

Following the step 1 of lemma D.3 of [Belloni et al. (2018)](https://www.duke.edu/~vanligc/research/), by conditioning on \( \Omega_n \equiv \{\max_{j \in [J], l \in [L], k \in [K]} \mathbb{E}_n(X_{j,l}^2 h_{jk}^2(W)) \leq B_{2n}^2 \}, \) we consider bounding the tail probability conditional on the event \( \Omega_\text{and} \tilde{X} \),
\[
P \left( \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \left| \mathbb{G}_n(\sigma(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \right| > t/4 \bigg| \Omega_n, \tilde{X} \right).
\]

From now on, omit the conditioning for the notational simplicity. By Markov inequality, we have
\[
P \left( \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \left| \mathbb{G}_n(\sigma(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \right| > t/4 \right) \leq \mathbb{E}_{\sigma} \exp \left( \phi \max_{j \in [J], k \in [K]} \sup_{\Delta \in \Theta} \left| \mathbb{G}_n(\sigma(f_{jk}(\tilde{X}, \theta_0 + \Delta \theta) - f_{jk}(\bar{X}, \theta_0)) \right| \right) / \exp(t/4\phi) \]
where \( \phi \equiv t / (16J^2 GB_{2n}^2 \sup_{\theta} \|\theta\|_1^2) \).

5
By the mean value theorem, there exists a mean value vector \( \tilde{\nu} \) as a function of \( \Delta \nu \equiv \nu - \nu_0 \) given the fixed matrix of \( X \) such that

\[
f_{jk}(\tilde{X}; \theta_0 + \Delta \theta) - f_{jk}(\tilde{X}; \theta_0) = \sum_{g=0}^{G} \sum_{j' \in [j]} \frac{d \xi_j(\tilde{\nu}; \tilde{X})}{d \nu_{j'g}}(\nu_{j'g} - \nu_{j'g0}) h_{jk}(W).
\]

Let \( N \) be the support of \( \nu \) given the conditioning \( \tilde{X} \) and the parameter space \( \Theta \). Therefore, we have

\[
E \sigma \exp \left( \phi \max_{j \in [J], k \in [K]} \sup_{\Delta \nu \in \Theta} \left| \xi_n(\phi)(f_{jk}(\tilde{X}; \theta_0 + \Delta \theta) - f_{jk}(\tilde{X}; \theta_0)) \right| \right)
\leq E \sigma \exp \left( \phi \sum_{g=0}^{G} \sum_{j' \in [J]} \frac{d \xi_j(\tilde{\nu}; \tilde{X})}{d \nu_{j'g}}(\nu_{j'g} - \nu_{j'g0}) h_{jk}(W) \right)
\leq E \sigma \exp \left( JG \phi \max_{j \in [J], g \in [G], k \in [K]} \sup_{\Delta \nu \in \Theta} \left| \xi_n(\phi)(\nu_{j'g} - \nu_{j'g0}) h_{jk}(W) \right| \right)
\leq J^2GK \max_{j \in [J], g \in [G], k \in [K]} E \sigma \exp \left( JG \phi \sup_{\Delta \nu \in \Theta} \left| \xi_n(\phi)(\nu_{j'g} - \nu_{j'g0}) h_{jk}(W) \right| \right).
\]

The first inequality follows from the mean value theorem and the fact that the supremum over \( \Delta \theta \) is dominated by the supremum over \( \Delta \nu \) which are constrained conditional on each realization of the matrix \( X \). The second inequality follows from the triangular inequality. The third inequality follows from the union bound over \( j', g \) by taking the maximum over \( j', g \) indices. Finally, we take the union bound over \( j, j', g, k \) indices.

To apply Ledoux-Talagrand contraction inequality (Theorem 4.12, [Ledoux and Talagrand, 1991]) in bounding the following term

\[
\exp \left( JG \phi \sup_{\Delta \nu \in \Theta} \left| \xi_n(\phi)(\nu_{j'g} - \nu_{j'g0}) h_{jk}(W) \right| \right),
\]

let

\[
\psi^{j'g}(\Delta \nu_{j'g}) \equiv \left[ \frac{d \xi_j(\tilde{\nu}; \tilde{X})}{d \nu_{j'g}}(\Delta \nu_{j'g}) \right] h_{jk}(W)
\]

then by lemma4 we have

\[
|\psi^{j'g}(\Delta \nu_{j'g})| \leq C_1 J|\Delta \nu_{j'g}|
\]

uniformly over \( \tilde{\nu} \).

Since \( \psi^{j'g}(0) = 0 \), Ledoux-Talagrand contraction inequality via corollary3 applies so that

\[
\frac{J^2GK \max_{j \in [J], k \in [K], g \in [G]} E \sigma \exp \left( JG \phi \sup_{\Delta \nu_{j'g} \in \Theta} \left| \xi_n(\phi)(\nu_{j'g} h_{jk}(W)) \right| \right)}{\exp(t/4\phi)} \\
\leq \frac{J^2GK \max_{j \in [J], k \in [K], g \in [G]} E \sigma \exp(C_1J^2G^2 \sup_{\Delta \nu_{j'g} \in \Theta} |\xi_n(\phi)(\nu_{j'g} h_{jk}(W))|)}{\exp(t/4\phi)} \\
\leq \frac{J^2GK \max_{j \in [J], k \in [K], g \in [G]} E \sigma \exp(C_1J^2G^2 \sup_{\Delta \theta \in \Theta} |\xi_n(\phi)(h_{jk}(W)X_{j'g} \Delta \theta_g)|)}{\exp(t/4\phi)}.
\]
By Holder inequality, we have
\[
J^2 G K \max_{j',j \in [J], k \in [K], l \in [L]} \frac{E_\sigma \exp(C_1 J^2 G \phi \sup_{\Delta \theta \in \Theta} |G_{n} \sigma h_{j,k}(W) X_{j',l} \Delta \theta|)}{\exp(t/4\phi)} 
\leq J^2 G K \max_{j',j \in [J], k \in [K], l \in [L]} \frac{E_\sigma \exp(C_1 J^2 G \phi |G_{n} \sigma h_{j,k}(W) X_{j',l} \sup_{\Delta \theta} \| \Delta \theta \|_1)}{\exp(t/4\phi)} 
\leq J^2 G K \max_{j',j \in [J], k \in [K], l \in [L]} \frac{2 \exp(2C^2_1 J^4 G^2 \phi^2 E_n (h_{j,k}(W)^2 X_{j',l}^2) \sup_{\Delta \theta \in \Theta} \| \Delta \theta \|_1^2)}{\exp(t/4\phi)} 
\leq J^2 G K \max_{j',j \in [J], k \in [K], l \in [L]} \frac{2 \exp(2C^2_1 J^4 G^2 \phi^2 B^2_{2n} \sup_{\theta \in \Theta} \| \theta \|_1^2)}{\exp(t/4\phi)}
\]
from the symmetry of distribution and sub-Gaussianity. Then the stated bound is achieved by following the analogue argument of Lemma D.3 of Belloni et al. (2018).

Then the following statement shows the error rate of RGMM BLP estimator:

**Theorem 2.** Under assumption[7] and assumptions for theorem[7] Assume further that
\[
\| \hat{f}(\theta_0) \|_\infty \leq \lambda
\]
with probability at least 1 − α, then we have for each q ∈ {1, 2}
\[
\| \theta - \theta_0 \|_q \leq 2n^{1/q} \mu_n^{-1} n^{-1/2} (\tilde{t}_n + t_n)
\]
with probability at least 1 − α − δ_n, where
\[
\tilde{t}_n \equiv C(B_{1n} + (J^2 G)(2\sqrt{2}B_{2n} \sup_{\theta} \| \theta \|_1 \log^{1/2}(8J^2 G K L/\delta_n)))
\]
with a universal constant C.

### 4 De-biased RGMM

Given the RGMM estimator \( \hat{\theta} \), it is recommended that we update the estimate in order to make a proper inference. De-biased Lasso, or De-biased RGMM procedure in Belloni et al. (2018) takes the following steps

1. Estimate the RGMM \( \hat{\theta} \)
2. Estimate the plug-in gradient
\[
\hat{G} = \partial_{\theta} \hat{f}(\hat{\theta})
\]
and the plug-in var-cov matrix
\[
\hat{\Omega} = E_n f(\hat{X}; \hat{\theta}) f(\hat{X}; \hat{\theta})'
\]
3. Solve the minimization problem of
\[
\min_{\gamma \in \mathbb{R}^{2L \times JK}} \sum_{t \in [2L]} \| \gamma_t \|_1
\]
subject to
\[
\| \gamma_t \hat{\Omega} - (\hat{G})_t \|_\infty \leq \lambda^2_t
\]
for some regularization parameters \( \lambda^2_t \)
4. Solve the minimization problem of
\[
\min_{\mu \in \mathbb{R}^{JK \times 2L}} \sum_{j \in [JK]} \| \mu_j \|_1
\]
subject to
\[
\| \mu_j \hat{G} - e_j' \|_\infty \leq \lambda^\mu_j
\]
for some regularization parameters \( \lambda^\mu_j \), where \( e_j \) is a coordinate vector with 1 in the \( j \)-th position and 0 elsewhere.
5. Update the RGMM estimator as $\hat{\theta} - \hat{\mu} \hat{f}(\hat{\theta})$.

First of all, we need to provide maximal inequalities for the auxiliary estimators $\hat{\gamma}$ and $\hat{\mu}$. The strategy follows the parallel argument of the maximal inequality for $\hat{\theta}$. Therefore, we need the following modulus of continuity conditions for $\gamma$ and $\mu$.

**Assumption 2.** Suppose that there exists a sequence $\mu_n$, such that

$$\inf_{\gamma \in \mathbb{R}; \|\gamma - \gamma_0\|_1 > 0} \|\gamma - \gamma_0\|_\infty / \|\gamma - \gamma_0\|_1 \geq s^{-1} \mu_n$$

and

$$\inf_{\mu \in \mathbb{R}; \|\mu - \mu_0\|_1 > 0} \|(\mu - \mu_0)G^n\|_\infty / \|\mu - \mu_0\|_1 \geq s^{-1} \mu_n.$$  

Note that the first condition requires that the variance matrices constructed from any elements of the score functions $f_{jk}(X, \theta_0)$ is non-singular, and the second condition follows if all eigenvalues of $G^n\Omega$ are bounded in absolute values from zero uniformly over $n$.

Then given a choice of penalty parameters $\lambda_\gamma^n$ and $\lambda_\mu^n$, we achieve the maximal inequality for $L^1$-norm of $\hat{\gamma}_l$ and $\hat{\mu}_j$.

**Lemma 2** (Lemma 3.7 of Belloni et al. (2018)). Let $t_{\alpha_n}^\Omega$ and $t_{\alpha_n}^G$ such that

$$n^{1/2}\|\hat{\Omega} - \Omega\|_\infty \leq \gamma_{\alpha_n}$$

and

$$n^{1/2}\|\hat{G} - G\|_\infty \leq \gamma_{\alpha_n}$$

with probability $1 - \delta_n$. Suppose $\max_{l \in [2L]} \|\gamma_0\|_1 \leq \bar{C}$, and $\max_{j \in [JK]} \|\mu_0\|_1 \leq \bar{C}$.

Let $\lambda_\gamma^n$ satisfy

$$n^{1/2} \lambda_\gamma^n \geq C t_{\alpha_n}^\Omega + t_{\alpha_n}^G,$$

and $\lambda_\mu^n$ satisfy

$$n^{1/2} \lambda_\mu^n \geq 2\bar{C}^2 t_{\alpha_n}^\Omega + \bar{C}^2 \max_{l \in [2L]} n^{1/2} \lambda_\gamma^n$$

$$\lambda_\mu^n \leq n^{-1/2} t_{\alpha_n}^\gamma,$$

for $l \in [2L]$ and $j \in [JK]$. Suppose assumption 2 holds. Then with probability $1 - 3\delta_n$, we have

$$\max_{l \in [2L]} \|\hat{\gamma}_l - \gamma_0\|_1 \leq \frac{s L_n(2 + \bar{C})}{\mu \sqrt{n}}$$

and with probability $1 - \delta_n$

$$\max_{j \in [JK]} \|\hat{\mu}_j - \mu_0\|_1 \leq \frac{s L_n(2 + \bar{C})}{\mu \sqrt{n}}.$$  

Next, we need maximal inequalities for the norms $\|\hat{G} - G\|_\infty, \|\hat{G} - G\|_\infty$, and $\|\hat{\Omega} - \Omega\|_\infty$ where $\hat{G} = \hat{G}(\hat{\theta})$ with $\hat{\theta}$ as the intermediate value of $\hat{\theta}$ and $\theta_0$. Unlike lemma 3.7 of Belloni et al. (2018) which assumes the tail probability bound for the process $\hat{G} - G$, we need certain modification of lemma as we do for theorem 1.

**Lemma 3.** Suppose the following

1. 

$$\max_{j \in [J], k \in [K], l \in [L]} E_n \left[ h_{jk}^2(W) \max_{j' \in [J], l' \in [L]} X_{j',l'}^2 \max_{j' \in [J], l' \in [L]} X_{j',l'}^2 \right] \leq C,$$

2. with probability $1 - \delta_n$, we have

$$\max_{j \in [J], k \in [K], l \in [L]} \mathbb{E}_n \left[ h_{jk}^2(W) \max_{j' \in [J], l' \in [L]} X_{j',l'}^2 \max_{j' \in [J], l' \in [L]} X_{j',l'}^2 \right] \leq B_n^2,$$

and

$$\max_{j, j' \in [J], k, k' \in [K], l \in [L]} \mathbb{E}_n \left[ h_{jk}^2(W) h_{j',k'}^2(W) X_{j,l}^2 X_{j',l}^2 \right] \leq B_n^2$$

with probability $1 - \delta_n$.  

8
3. with probability $1 - \delta_n$, we have
\[ \|\hat{\theta} - \theta_0\|_q \leq \Delta_{qn} \]
for $q \in \{1, 2\}$.

4. $\max_{j \in [J], k \in [K]} E[f_{jk}^4(\tilde{X}; \theta_0)] \leq C$, and $n^{-1/2}E\left[\max_{i \in [n]} \|f(\tilde{X}_i; \theta_0)\|_4^4\right] \leq \min\{\delta_n, \log^{-1/2}(JKL)\}$

5. $\max_{j \in [J], k \in [K], I \in [2L]} E\left[G_{jk, l}(\tilde{X}; \theta_0)^2\right] \leq C$

and
\[ n^{-1/2}E[\max_{i \in [n]} \|G(\tilde{X}_i; \theta_0)\|_\infty^2] \leq \min\{\delta_n, \log^{-1/2}(2JKL)\}. \]

Then with probability $1 - C^\prime \delta_n$ we have
\[ \|\hat{G} - G\|_\infty \leq C' \sqrt{n^{-1}\log(2JKL)} + C' J^2GB_n\Delta_{1n} \sqrt{n^{-1}\log(2J^2GKL/\delta_n)} + C' J^{3/2}\Delta_{2n}. \]
\[ \|\hat{G} - \hat{G}\|_\infty \leq C' \sqrt{n^{-1}\log(2JKL)} + C' J^2GB_n\Delta_{1n} \sqrt{n^{-1}\log(2J^2GKL/\delta_n)} + C' J^{3/2}\Delta_{2n} \]
and
\[ \|\hat{\Omega} - \Omega\|_\infty \leq C' \sqrt{n^{-1}\log(JK)} + C' J^2GB_n\Delta_{1n} \sqrt{n^{-1}\log(2J^2GKL/\delta_n)} + 2J^{3/2}C(J^{3/2}\Delta_{2n}^2 + \Delta_{2n}). \]

Proof. We follow the proof of lemma 3.9 of Belloni et al. (2018). First we bound $\|\hat{G} - G\|_\infty$. Note that $\sqrt{n}\|\hat{G} - G\|_\infty$ is bounded by the sum of the following three terms from triangular inequality:

(1.1) $\max_{j \in [J], k \in [K], I \in [2L]} |G_n(G_{jk, l}(\tilde{X}; \hat{\theta}) - G_{jk, l}(\tilde{X}; \theta_0))|

(1.2) $\max_{j \in [J], k \in [K], I \in [2L]} |G_nG_{jk, l}(\tilde{X}; \theta_0)|

(1.3) $\max_{j \in [J], k \in [K], I \in [2L]} n^{1/2}|E[G_{jk, l}(\tilde{X}; \hat{\theta}) - G_{jk, l}(\tilde{X}; \theta_0)]|

From lemma C.1 (4) of Belloni et al. (2018), the second term, (1.2), is bounded by
\[ \max_{j \in [J], k \in [K], I \in [2L]} |G_nG_{jk, l}(\tilde{X}; \theta_0)| \leq C \max_{j \in [J], k \in [K], I \in [2L]} \frac{1}{n} \left[ E\left[G_{jk, l}(\tilde{X}; \theta_0)\right]^2\right] \log^{1/2}(2JKL)
\[ + n^{-1/2}C_2 \left\{ E\left[\max_{i \leq n} \|G(\tilde{X}_i; \theta_0)\|_\infty^2\right]\delta_n^{-1} + \delta_n^{-1/2} + \log(2JKL)\right\} \]
\[ \leq C \log^{1/2}(2JKL) \]
with probability $1 - \delta_n$ by the condition 5.
For the last term, (1.3), we use the linear expansion of $G_{jk,t}(\tilde{X}; \theta)$ into $\sum_{j' \in [J], g \in [G]} h_{jk}(W)B_{j', g}(X_t)(\nu_{j'g} - \nu_{j'g0})$ from lemma 6 so that

$$\max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E[G_{jk,t}(\tilde{X}, \hat{\theta}) - G_{jk,t}(\tilde{X}, \theta_0)] \right|$$

$$\leq (1) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ \sum_{j' \in [J], g \in [G]} h_{jk}(W)B_{j', g}(X_t)(\nu_{j'g} - \nu_{j'g0}) \right] \right|$$

$$\leq (2) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ \sum_{j' \in [J], g \in [G]} h_{jk}(W)B_{j', g}(X_t)X_{j', g}(\theta_g - \theta_g0) \right] \right|^{1/2}$$

$$\leq (3) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ \sum_{j' \in [J], t' \in [L]} [h_{jk}(W)B_{j', g(t')}(X_t)X_{j', t'}(\theta_{t'0} - \theta_{t'00})^2] \right] \right|^{1/2}$$

$$\leq (4) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ \frac{h^2_{jk}(W)}{\max_{j' \in [J], t' \in [L]} B_{j', g(t')}(X_t)X_{j', t'}(\theta_{t'0} - \theta_{t'00})^2} \right] \right|^{1/2}$$

$$\leq (5) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ h^2_{jk}(W)\max_{j' \in [J], t' \in [L]} X_{j', t'}^2 \max_{j' \in [J], t' \in [L]} X_{j', t'}^2 \left( \theta_{t'0} - \theta_{t'00} \right)^2 \right] \right|^{1/2}$$

$$\leq (6) \max_{j \in [J], k \in [K], t \in [2L]} n^{1/2} \left| E \left[ h^2_{jk}(W)\max_{j' \in [J], t' \in [L]} X_{j', t'}^2 \max_{j' \in [J], t' \in [L]} X_{j', t'}^2 \left( \theta - \theta_0 \right)^2 \right] \right|^{1/2} \leq Cn^{1/2}J^{3/2}2n$$

with probability $1 - \delta_n$, where (1) and (5) follows from lemma 6 (2) follows from the definition of $\nu_{j'th}$, (3) follows from the monotonicity of the $L_p$ norm, (4) follows from the union bound, and (6) follows from the condition 1.

The first term, (1.1), is the empirical process in terms of the $G_{jk,t}$ instead of $f_{jk}$. Note that

$$E_n Var(G_{jk,l}(\tilde{X}; \hat{\theta}) - G_{jk,l}(\tilde{X}; \theta_0)) \leq E_n \left( \sum_{j' \in [J], t \in [L]} h_{jk}(W)B_{j', g(t)}(\theta_t - \theta_0) \right)^2$$

$$\leq E_n \left( h^2_{jk}(W)\max_{j' \in [J], t \in [L]} B_{j', g(t)}X_{j', t}^2 \right) \left( \theta - \theta_0 \right)^2$$

$$\leq C(J^2E_n \left( h^2_{jk}(W)\max_{j' \in [J], t \in [L]} X_{j', t}^2 \max_{j' \in [J], t \in [L]} X_{j', t}^2 \left( \theta - \theta_0 \right)^2 \right) \leq C(J^2B_{n}^2\Delta_t^2)$$

by Hölder inequality and lemma 6.

Then the conditions for the Corollary holds with $B_{1n} = JB_n\Delta_1n$, and $B_{2n} = B_n$ so that

$$\max_{j \in [J], k \in [K], l \in [2L]} |G_n(G_{jk,l}(\tilde{X}; \hat{\theta}) - G_{jk,l}(\tilde{X}; \theta_0))| \leq J^2GCB_n\Delta_1n \log^{1/2}(2J^2GKL/\delta_n)$$

with probability $1 - 7\delta_n$.

In the same argument of Belloni et al (2018), $\|\tilde{G} - \hat{G}\|_{\infty}$ has the same bound as $\|\tilde{G} - \hat{G}\|_{\infty}$. Finally, we consider $\|\hat{\Omega} - \Omega\|$. As we do for $\|\tilde{G} - \hat{G}\|_{\infty}, n^{-1/2}\|\Omega - \Omega\|$ is bounded by the sum of the following three terms

$$\max_{j,j' \in [J], k,k' \in [K]} \left| G_n(f_{jk}(\tilde{X}; \hat{\theta})f_{j'k'}(\tilde{X}; \hat{\theta}) - f_{jk}(\tilde{X}; \theta_0)f_{j'k'}(\tilde{X}; \theta_0)) \right|$$

$$\max_{j,j' \in [J], k,k' \in [K]} \left| G_n(f_{jk}(\tilde{X}; \theta_0)f_{j'k'}(\tilde{X}; \theta_0)) \right|$$

$$\max_{j,j' \in [J], k,k' \in [K]} n^{1/2} \left| E[f_{jk}(\tilde{X}; \hat{\theta})f_{j'k'}(\tilde{X}; \hat{\theta}) - f_{jk}(\tilde{X}; \theta_0)f_{j'k'}(\tilde{X}; \theta_0)) \right|.$$
First, we have for (2.1.1),
\[
\begin{align*}
\max_{j,j' \in [J],k,k' \in [K]} |\mathbb{G}_n(f_{jk}(\hat{X}; \hat{\theta}) - f_{jk}(\bar{X}; \bar{\theta})) (f_{j'k'}(\hat{X}; \hat{\theta}) - f_{j'k'}(\bar{X}; \bar{\theta}))| & \\
& \leq \max_{j,j' \in [J],k,k' \in [K]} n^{1/2} \mathbb{E}_n[(f_{jk}(\hat{X}; \hat{\theta}) - f_{jk}(\bar{X}; \bar{\theta}))^2]^{1/2} \mathbb{E}_n[(f_{j'k'}(\hat{X}; \hat{\theta}) - f_{j'k'}(\bar{X}; \bar{\theta}))^2]^{1/2} + n^{1/2} \mathbb{E}[(f_{jk}(\hat{X}; \hat{\theta}) - f_{jk}(\bar{X}; \bar{\theta}))^2]^{1/2} \mathbb{E}[(f_{j'k'}(\hat{X}; \hat{\theta}) - f_{j'k'}(\bar{X}; \bar{\theta}))^2]^{1/2} \leq n^{1/2} C^2 B_n^2 J^2 \Delta_{2n}^2
\end{align*}
\]
with probability $1 - C\delta_n$ and a universal constant $C$, where the second and the last inequalities are by Hölder inequality and Lemma 4.

For the second term, (2.1.2), note that
\[
\begin{align*}
\mathbb{V}ar(\mathbb{G}_n((f_{j,k}(\hat{X}; \hat{\theta}) - f_{j,k}(\bar{X}; \bar{\theta})) f_{j',k'}(X, \theta))) & \leq \mathbb{V}ar(\mathbb{G}_n(h_{jk}(W) h_{j',k'}(W) X_j(\theta - \theta_0))) \\
& \leq C J^2 B_n^2 \Delta_{2n}^2
\end{align*}
\]
so that the conditions for corollary holds with $B_{1n} = JB_n \Delta_{1n}$ and $B_{2n} = B_n$, therefore,
\[
\max_{j,j' \in [J],k,k' \in [K]} |\mathbb{G}_n((f_{jk}(\hat{X}; \hat{\theta}) - f_{jk}(\bar{X}; \bar{\theta})) f_{j'k'}(\hat{X}; \hat{\theta}))| \leq \tilde{C} J^2 G B_n \Delta_{1n} \log^{1/2}(2 J^2 G K / \delta_n).
\]

For the remaining two terms, (2.2),
\[
\max_{j,j' \in [J],k,k' \in [K]} |\mathbb{G}_n f_{jk}(\hat{X}; \theta_0) f_{j'k'}(\hat{X}; \theta_0)|
\]
is bounded by
\[
C \max_{j,k} E[f_{jk}^4(\hat{X}; \theta_0)]^{1/2} \sqrt{\log(J K)} + Cn^{-1/2} E[\max_i |f(\hat{X}_i; \theta_0)|^4] \{n^{-1} + \log(J K)\} \leq C' \sqrt{\log(J K)}
\]
from Lemma C.1(4) of Belloni et al. (2018) under condition 4.
Finally, for (2.3),
\[
\max_{j,j' \in [J], k,k' \in [K]} n^{1/2} E[f_{j,k} (\hat{X}; \hat{\theta}) f_{j,k'} (\hat{X}; \hat{\theta}) - f_{j,k} (\tilde{X}; \theta_0)f_{j,k'} (\tilde{X}; \theta_0)] \\
\leq \max_{j,j' \in [J], k,k' \in [K]} n^{1/2} |E[f_{j,k} (\hat{X}; \hat{\theta}) (f_{j,k'} (\hat{X}; \hat{\theta}) - f_{j,k'} (\tilde{X}; \theta_0))] + |E[(f_{j,k} (\hat{X}; \hat{\theta}) - f_{j,k} (\tilde{X}; \theta_0)) f_{j,k'} (\tilde{X}; \theta_0)]| \\
+ 2n^{1/2} \max_{j,j' \in [J], k,k' \in [K]} |E[(f_{j,k} (\hat{X}; \hat{\theta}) - f_{j,k} (\tilde{X}; \theta_0)) f_{j,k'} (\tilde{X}; \theta_0)]|
\]
\[
\leq J^3 C \Delta_2^2 + n^{1/2} \max_{j,j' \in [J], k,k' \in [K]} E \left[ C^2 J^2 h^2_{j,k}(W) h^2_{j,k'}(W) \xi_j^2(\tilde{X}; \theta_0) \left( \sum_{j' \in [J], g \in [G]} X_{j',g}(\theta_0 - \theta_g) \right) \right]^{27/2} \\
\leq n^{1/2} J^3 C \Delta_2^2 + n^{1/2} \max_{j,j' \in [J], k,k' \in [K]} E \left[ C^2 J^2 h^2_{j,k}(W) h^2_{j,k'}(W) \xi_j^2(\tilde{X}; \theta_0) \max_{j,j' \in [J], k,k' \in [K]} \sum_{j' \in [J], k' \in [K]} (\theta_{j'k'} - \theta_{jk})^2 \right]^{1/2} \\
\leq n^{1/2} J^3 C \Delta_2^2 + n^{1/2} 2^{3/2} C \Delta_2 n
\]
by lemma 4 and Hölder inequality.

Combining these results, we attain the asymptotic linearity

**Theorem 3.** Suppose that
\[
\max_{j \in [J]} E[f_j^2(\tilde{X}; \theta_0)] \leq C
\]
and
\[
n^{-1/2} E \left[ \max_{i \in [n]} \|f(\tilde{X}_i; \theta_0)\|_\infty \right] \leq \min\{\delta_n, \log^{-1/2}(JKL)\}.
\]
Suppose that \(\|\hat{f}(\theta_0)\|_\infty \leq \lambda\) with probability at least \(1 - \alpha\), and assumptions for lemmas 1, 2 and 3, and theorem 7 with
\[
B_n + \bar{C} + \mu_n^{-1} \leq C.
\]
For \(\bar{\alpha} \geq 0\) and \(C' \geq 1\), let
\[
\tilde{\lambda} = C' J^{3/2} \max\{J^{3/2} \tilde{\lambda}, \bar{\lambda}\}
\]
with \(\bar{\lambda} \equiv n^{-1/2 + \bar{\alpha}} J^2 G \Phi^{-1}\left(1 - (2J^2 GKLn)^{-1}\right)\).
Then, setting \(\lambda^\gamma = \frac{1}{2} \lambda^\mu = \tilde{\lambda}\), we have with probability \(1 - \alpha - C\delta_n\),
\[
\sqrt{n}(\hat{\theta} - \theta_0) = -\mu_0 \gamma_0 \hat{f}(\theta_0) + r
\]
with \(\|r\|_\infty \leq C\nu_n\) where \(\hat{\theta}\) is the updated RGMM estimator provided that
\[
n^{-1/2 + 2\delta} s J^{3/2} \max\left\{n^{-1/2} J^4 G^2 \log^2(2J^2 GKLn), J^2 G \log(2J^2 GKLn)\right\} \leq u_n
\]
and
\[
\tilde{\lambda} \geq C J^{3/2} \max\{J^{3/2} D_{2n}^2, D_{2n}\}
\]
for some large enough \(C > 0\) where \(D_{2n} \equiv n^{-1/2} s^{1/2}(\log^{1/2}(2JKL) + J^2 G \log^{1/2}(2J^2 GKL/\delta_n))\).

**Proof.** First note that the rate terms of theorem 8 satisfies the following
\[
l_n \leq C' \log^{1/2}(2JKL)
\]
and
\[
l_n' \leq C' J^2 G \log^{1/2}(2J^2 GKL)
\]
with probability \(1 - \delta_n\) under the assumptions.
Therefore,
\[
\Delta_{qn} \leq C'n^{-1/2}s^{1/q}(l_n + l_n') \leq C''n^{-1/2}s^{1/q}(\log^{1/2}(2JKL) + J^2G \log^{1/2}(2J^2KGL)) \equiv D_{qn}.
\]
with probability at least \(1 - \alpha - \delta_n\).

To apply lemma 2, note that for \(s \geq 1\), we have
\[
n^{-1/2} \log^{1/2}(2JKL) \leq n^{-1/2}s^{1/q}\log(2JKL) \leq D_{qn}.
\]

Note also that
\[
D_{1n}n^{-1/2}J^2G \log^{1/2}(2J^2GKL/\delta_n) \leq D_{2n}^2.
\]

Therefore, by lemma 3, we have
\[
\max\{\|\hat{g} - g\|_\infty, \|\hat{g} - \hat{g}\|_\infty, \quad\|\hat{\Omega} - \Omega\|_\infty\} \leq CJ^{3/2}(J^{3/2}D_{2n}^2 + D_{2n}).
\]

with probability \(1 - C\delta_n\).

Now, let
\[
l_n^\Omega = l_n^G = n^{1/2}CJ^{3/2} \max\{J^{3/2}D_{2n}^2, D_{2n}\}
\]
and
\[
\lambda_j^\Omega = \frac{1}{2}\lambda_j^G = \bar{\lambda}
\]
so that
\[
n^{1/2}\bar{\lambda} \geq (\bar{C} + 1)n^{1/2}CJ^{3/2}(J^{3/2}D_{2n}^2 + D_{2n})
\]
and
\[
\bar{\lambda} \leq n^{-1/2}l_n = \hat{\lambda}.
\]

Then lemma 2 applies to get
\[
\max_{l \in [p]} \|\hat{g}_l - g_0\|_1 \leq C_1s\hat{\lambda}
\]
and similar definition for \(l_n'\) gives
\[
\max_{j \in [JK]} \|\hat{\mu}_j - \mu_{0j}\|_1 \leq C_2s\hat{\lambda}.
\]

By lemma 3.6 of Belloni et al. (2018), the decomposed error rates defined in the lemma, \(\bar{r}_1, \bar{r}_2\) and \(\bar{r}_3\) are bounded by
\[
\bar{r}_1 \leq \sqrt{n}\lambda\Delta_{1n}
\]
\[
\bar{r}_2 \leq Cn^{1/2}(\Delta_{2n}^2 + J^2\Delta_{2n})\Delta_{1n}
\]
\[
\bar{r}_3 \leq Cn^{-1/2}\hat{\lambda}\Delta_{1n}.
\]

Thus,
\[
\|r\| \leq Cu_n
\]
for \(u_n\) such that \(n^{-1/2+\bar{a}}J^{3/2} \max\{n^{-1/2}J^{3/2}J^4G^2 \log^{3/2}(2J^2GJKLn), J^2G \log(2J^2GJKLn)\} \leq u_n\) with probability \(1 - \alpha - C\delta_n\).

5 Conclusion

In this paper, we propose a \(l_1\)-penalized estimation for random coefficient logit model of differentiated product demands. Unlike the existing approach, our procedure allows for random coefficients on possibly high-dimensional attributes. Therefore, both of the mean and variance of the indirect utilities may be determined by high-dimensional but sparse set of attributes.

Our strategy bases on the contraction inequality by Ledoux and Talagrand (1991) as is used in Belloni et al. (2018) for the GMM procedure with a single index. We show that the \(l_1\)-regularized GMM estimation and its de-biased procedure are valid for the BLP model with fixed number of indices generated out of high-dimensional but sparse set of attributes.

Unfortunately, the contraction inequality principle does not apply to a fully flexible random coefficient BLP model as the number of indices grows in the same rate as the number of the attributes. Also, our current result does not accommodate the models with the number of products growing exponentially. These challenges are left for the future work.
Then for any pair of index values $\nu_c$ we consider the bound for the linear expansion of $s$ when every product has non-zero share in every market. Assume that $\xi$ with respect to the index $x'_{jg}\gamma_g$ when the share of each product for any $\tilde{\beta}$ fall in the shrinking range of $c_1/J$ and $c_2/J$. This is the same assumption employed in Berry, Linton, and Pakes (2004). All these arguments should apply to the special case that the number of product $J$ is a fixed constant when every product has non-zero share in every market.

**Lemma 4.** Let

$$s_j(\tilde{\beta}; \tilde{x}, \theta) = \frac{\exp(x'_{j} \beta + \xi_j(\tilde{x}; \theta) + \sum_{g \in \{G\}} x'_{jg} \gamma_g \tilde{\beta}_g) \cdot \sum_{j' \in [J]} \exp(x'_{j'} \beta + \xi_{j'}(\tilde{x}; \theta) + \sum_{g \in \{G\}} x'_{j'g} \gamma_g \tilde{\beta}_g)}{1 + \sum_{j' \in [J]} \exp(x'_{j'} \beta + \xi_{j'}(\tilde{x}; \theta) + \sum_{g \in \{G\}} x'_{j'g} \gamma_g \tilde{\beta}_g)}.$$ 

Assume that

$$\frac{c_1}{J} < s_j(\tilde{\beta}; \tilde{x}, \theta) < \frac{c_2}{J}$$

for almost every $\tilde{\beta} \in \mathbb{R}^G$ and $\tilde{x}$, and for every $j \in \{0, 1, \ldots, J\}$. Then for any pair of index values $\nu_{j'g}$ and $\nu_{j'g0}$ for each $g \in \{G\}$, $j' \in [J]$, we have

$$\left| \frac{d\xi_j(\tilde{\nu}; \tilde{x})}{d\nu_{j'g}} (\nu_{j'g} - \nu_{j'g0}) \right| \leq C_1 J |\nu_{j'g} - \nu_{j'g0}|$$

with a universal constant $C_1$ uniformly over $\tilde{\nu}$ and $\tilde{x}$.

**Proof.** First note that

$$\frac{d\xi(\tilde{x}; \theta)}{dx'_{jg}\gamma_g} = \left[ \frac{\partial s_j(\tilde{x}; \theta)}{\partial \xi} \right]^{-1} \left[ \frac{\partial s_j(\tilde{x}; \theta)}{\partial x'_{jg}\gamma_g} \right] = \left[ \frac{\partial s_j(\tilde{x}; \theta)}{\partial \xi} \right]^{-1} \int \tilde{e}_g(s(\tilde{\beta}; \tilde{x}, \theta) (1 - s_j(\tilde{\beta}; \tilde{x}, \theta))dF_{\tilde{\beta}},$$

where $s_j(\tilde{\beta}; \tilde{x}, \theta)$ is a vector of $s_j(\tilde{\beta}; \tilde{x}, \theta)$.

---

**References**

**ARMSTRONG, T. B.** (2016): “Large Market Asymptotics for Differentiated Product Demand Estimators with Economic Models of Supply,” *Econometrica*, 84, 1961–1980.

**BELLONI, A., V. CHERNOZHUKOV, D. CHETVERIKOV, C. HANSEN, AND K. KATO** (2018): “High-Dimensional Econometrics and Regularized GMM,” *arXiv preprint arXiv:1806.01888*.

**BERRY, S., J. LEVINSOHN, AND A. PAKES** (1995): “Automobile Prices in Market Equilibrium,” *Econometrica*, 841–890.

**BERRY, S., O. B. LINTON, AND A. PAKES** (2004): “Limit Theorems for Estimating the Parameters of Differentiated Product Demand Systems,” *The Review of Economic Studies*, 71, 613–654.

**BERRY, S. T.** (1994): “Estimating Discrete-Choice Models of Product Differentiation,” *Rand Journal of Economics*, 242–262.

**FOSTER, D. J. AND A. RAKHLIN** (2019): “Sl_\text{\textbar\textbar} Vector Contraction for Rademacher Complexity,” *arXiv preprint arXiv:1911.06468*.

**FRANK, I. AND J. FRIEDMAN** (1993): “A Statistical View of Some Chemometrics Regression Tools,” *Technometrics*, 35, 109–135.

**GILLEN, B. J., S. MONTERO, H. R. MOON, AND M. SHUM** (2019): “BLP-2LASSO for Aggregate Discrete Choice Models with Rich Covariates,” *The Econometrics Journal*, 22, 262–281.

**LEDoux, M. AND M. TalagRAND** (1991): *Probability in Banach Spaces: Isoperimetry and Processes*, Springer.

**TIBSHIRANI, R.** (1996): “Regression Shrinkage and Selection via the Lasso,” *Journal of the Royal Statistical Society, Series B*, 58, 267–288.

**Van der Vaart, A. AND J. WellNER** (1996): *Weak Convergence and Empirical Processes: With Applications to Statistics*, Springer.

---

**A Supporting Lemmas**

We consider the bound for the linear expansion of $\xi_j$ with respect to the index $x'_{jg}\gamma_g$ when the share of each product for any $\tilde{\beta}$ fall in the shrinking range of $c_1/J$ and $c_2/J$. This is the same assumption employed in Berry, Linton, and Pakes (2004). All these arguments should apply to the special case that the number of product $J$ is a fixed constant when every product has non-zero share in every market.
Then,

\[ \text{RHS} \leq \left| \sum_{j'' \in [J]} d_{jj''}(\theta) \int \tilde{\beta}_g s_{j''}(\tilde{\beta}; \tilde{x}, \theta)(1 - s_{j'}(\tilde{\beta}; \tilde{x}, \theta))dF_{\tilde{\beta}}(\nu_{j'g} - \nu_{j'g0}) \right| \]

where \( d_{jj''}(\theta) \) is \((j, j'')\) element of \( \left[ \frac{\partial \psi(x, \theta)}{\partial x} \right]^{-1} = D(\theta) \) matrix.

For the inverse matrix elements \( d_{jj''}(\theta) \), [Berry et al. (2004), p.657] show that the upper bound of the \( D \) matrix in the positive definite sense, i.e.,

\[ x' \left( \text{diag}(s_1, \ldots, s_J)^{-1} + \begin{pmatrix} 0 \\ \frac{1}{\delta_0} \end{pmatrix} - D(\theta) \right) x > 0 \]

for any non-zero vector \( x \) where \( s_j \) are the lower bounds of the shares satisfying the rate condition in the assumption. Thus, each element of \( |d_{jj''}(\theta)| \) is bounded above by \( \frac{1}{2\delta} + \frac{1}{2\delta_0} \leq 2J/c_1 \).

Now,

\[ \left| \sum_{j'' \in [J]} d_{jj''}(\theta) \int \tilde{\beta}_g s_{j''}(\tilde{\beta}; \tilde{x}, \theta)(1 - s_{j'}(\tilde{\beta}; \tilde{x}, \theta))dF_{\tilde{\beta}}(\nu_{j'g} - \nu_{j'g0}) \right| \]

\[ \leq \sum_{j'' \in [J]} |d_{jj''}(\theta)| \int |\tilde{\beta}_g s_{j''}(\tilde{\beta}; \tilde{x}, \theta)(1 - s_{j'}(\tilde{\beta}; \tilde{x}, \theta))dF_{\tilde{\beta}}| |\nu_{j'g} - \nu_{j'g0}| \]

\[ \leq \frac{2J}{c_1} \sum_{j'' \in [J]} \int |\tilde{\beta}_g s_{j''}(\tilde{\beta}; \tilde{x}, \theta)dF_{\tilde{\beta}}| |\nu_{j'g} - \nu_{j'g0}| \]

\[ \leq \frac{2J}{c_1} \int |\tilde{\beta}_g s_{j''}(\tilde{\beta}; \tilde{x}, \theta)dF_{\tilde{\beta}}| |\nu_{j'g} - \nu_{j'g0}| \]

\[ \leq \frac{2J}{c_1} \int |\tilde{\beta}_g| (1 - s_0(\tilde{\beta}; \tilde{x}, \theta)dF_{\tilde{\beta}}| |\nu_{j'g} - \nu_{j'g0}| \]

\[ \leq \frac{2J}{c_1} \int |\tilde{\beta}_g| dF_{\tilde{\beta}}| |\nu_{j'g} - \nu_{j'g0}| \leq C_1J|\nu_{j'g} - \nu_{j'g0}|. \]

Remark 1. One may achieve \( L_\infty \)-Lipschitz result for the vector of indices \( \{\nu_{j0}\}_{j \in [J]} \). The \( L_\infty \)-Lipschitz constant can be invariant to the number of products \( J \), from the assumption that \( \sum_j (1 - s_j(\tilde{\beta})) = s_0(\tilde{\beta}) \leq c_2/J \). From this property, the variance term of [Berry et al. (2004)] achieves \( J \) rate, instead of \( J^2 \). Therefore, the rate of convergence may be improved with respect to the number of products \( J \) relative to the one in this paper. Nevertheless, Ladeau-Talagrand contraction inequality does not apply with the \( L_\infty \)-Lipschitz case. While there is a recent study by [Foster and Rakhlin (2019)] showing a tail probability bound for the Rademacher average with \( L_\infty \)-Lipschitz mapping, it is not trivial to apply to our case.

Next we show the sufficient conditions for the empirical process at the true parameter value \( \theta_0 \) is bounded by the log(JK) rate.

Lemma 5. Assume there is some \( \sigma > 0 \) such that

\[ \max_{j \in [J], l \in [2L], k \in [K]} E_n(X_{jk}(h_{jk}(W))^2 \leq \sigma^2 \]

and

\[ \log(JKn) \left( E[\|h_{jk}(W)\|^4_{\infty} + 4\xi(\tilde{\xi}; \theta_0)^4 \right)^{1/2} \leq \sigma^2 \]

Then

\[ \|n^{-1/2} G_n(g(\tilde{X}, \theta_0))\|_{\infty} \leq n^{-1/2} C\sigma \sqrt{\log(JK)} \]

with probability at least \( 1 - \delta / \log^2(n) \).

Proof. The analogue argument in example 7 of Belloni et al (2018) in the application of lemma A.2 and A.3 shows the result. □
Lemma 6. Assume the assumptions for lemma 2. Let \( \nu_{j',g} \) and \( \nu_{j',g_0} \) as \( JG \) vectors of indices as previously defined. Then there exists a sequence \( B_{j',g}(x_l) \) which depends on the intermediate value of \( \theta \) and \( \theta_0 \) such that

\[
\frac{d\xi_j(x;\theta)}{d\theta} - \frac{d\xi_j(x;\theta_0)}{d\theta} = \sum_{j' \in [J] \in [G]} B_{j',g}(x_l)(\nu_{j',g} - \nu_{j',g_0}),
\]

and

\[
|B_{j',g}(x_l)| \leq \tilde{C} J \max_{j' \in [J]} |x_{j'}|
\]

with a universal constant \( \tilde{C} \) for any value of \( \theta \) and \( x_l \).

**Proof.** Observe that

\[
\frac{d\xi_j(x;\theta)}{d\theta} = D_j(\theta) \int \tilde{B}_{g_0}(x;\tilde{\beta};\tilde{x};\theta) \sum_{j' \in [J]} \left( 1 - \nu_{j'}(\tilde{\beta};\tilde{x};\theta) \right) x_{j'} dF_{\tilde{\beta}}.
\]

Below, we omit \( \tilde{x} \) as the arguments of \( s_j(x;\tilde{x};\theta) \) and \( s_j(\tilde{\beta};\tilde{x};\theta) \) for notational simplicity.

Now we have

\[
\left| \frac{d\xi_j(x;\theta)}{d\theta} - \frac{d\xi_j(x;\theta_0)}{d\theta} \right| = \left| \sum_{j',j'' \in [J]} \int \tilde{B}_{g_0}(x;\tilde{\beta};\tilde{x};\theta) \sum_{j' \in [J]} \left( 1 - \nu_{j'}(\tilde{\beta};\tilde{x};\theta) \right) x_{j'} dF_{\tilde{\beta}} \right|.
\]

Thus,

\[
\sum_{j',j'' \in [J]} \int \tilde{B}_{g_0}(x;\tilde{\beta};\tilde{x};\theta) \sum_{j' \in [J]} \left( 1 - \nu_{j'}(\tilde{\beta};\tilde{x};\theta) \right) x_{j'} dF_{\tilde{\beta}} = \sum_{j',j'' \in [J]} \int \tilde{B}_{g_0}(x;\tilde{\beta};\tilde{x};\theta) \sum_{j' \in [J]} \left( 1 - \nu_{j'}(\tilde{\beta};\tilde{x};\theta) \right) x_{j'} dF_{\tilde{\beta}}
\]

First consider expanding \( s_{j''}(\tilde{\beta};\tilde{x};\theta)(1 - s_{j'}(\tilde{\beta};\tilde{x};\theta)) - s_{j''}(\theta)(1 - s_{j'}(\theta)) \) with respect to \( \nu_{jg} - \nu_{jg_0} \). We have

\[
\frac{ds_{j''}(\theta)(1 - s_{j'}(\theta))}{d\nu_{jg}} = \int \tilde{B}_{g_0}s_{j''}(\tilde{\beta};\theta)s_{j'}(\tilde{\beta};\theta)(1 - 2s_j(\tilde{\beta};\theta))dF_{\tilde{\beta}}.
\]

Therefore, by the mean value theorem, there exists an intermediate value vector \( \tilde{\beta} \),

\[
s_{j''}(\theta)(1 - s_{j'}(\theta)) - s_{j''}(\theta_0)(1 - s_{j'}(\theta_0)) = \sum_{j=1}^{J} \sum_{g=0}^{G} \int \tilde{B}_{g_0}s_{j''}(\tilde{\beta};\theta)s_{j'}(\tilde{\beta};\theta)(1 - 2s_j(\tilde{\beta};\theta))dF_{\tilde{\beta}}(\nu_{jg} - \nu_{jg_0}).
\]

Next consider expanding \( d_{j',j''}(\theta) - d_{j',j''}(\theta_0) \) with respect to \( \nu_{jg} - \nu_{jg_0} \). Observe that

\[
\frac{DD^{-1}(\theta)}{d\nu_{jg}} = -D^{-1}(\theta) \left[ \frac{ds_{j''}(\theta)}{d\nu_{jg}} \frac{d\xi_j(\theta)}{d\nu_{jg}} \right]_{j_1,j_2} D^{-1}(\theta)
\]

\[
= -D^{-1}(\theta) \left[ \frac{ds_{j''}(\theta)}{d\nu_{jg}} s_{j_1}(\theta)(1 - s_{j_2}(\theta)) \right]_{j_1,j_2} D^{-1}(\theta)
\]

\[
= -D^{-1}(\theta) \left[ \int \tilde{B}_{g_0}s_{j_1}(\tilde{\beta};\theta)s_{j_2}(\tilde{\beta};\theta)(1 - 2s_j(\tilde{\beta};\theta))dF_{\tilde{\beta}} \right]_{j_1,j_2} D^{-1}(\theta)
\]

\[
= -\left[ \sum_{j_1 \in [J]} \sum_{j_2 \in [J]} d_{j_1,j_2}(\theta) s_{j_1}(\theta)(1 - 2s_j(\tilde{\beta};\theta))dF_{\tilde{\beta}} \right].
\]

Therefore, by the mean value theorem

\[
d_{j',j''}(\theta) - d_{j',j''}(\theta_0) = \sum_{j_1,j_2 \in [J]} \sum_{g=0}^{G} d_{j_1,j_2}(\tilde{\beta}) s_{j_1}(\tilde{\beta};\theta)(1 - 2s_j(\tilde{\beta};\theta))dF_{\tilde{\beta}}(\nu_{jg} - \nu_{jg_0}).
\]
Combining two results, we have
\[ \frac{d\xi_j(x; \theta)}{d\theta_t} - \frac{d\xi_j(x; \theta_0)}{d\theta_t} = \sum_{j=1}^J \sum_{g=0}^G B_{jg}(x_t)(\nu_{jg} - \nu_{jg0}) \]
where
\[ B_{jg}(x_t) = \sum_{j', j'' \in [J]} \int \hat{z}_{g(t)} d_{j'j''} s_{j'}(\tilde{\beta}; \tilde{\theta}) s_{j''}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta}))dF_{\tilde{\beta}}x_{j't} \]
\[ + \sum_{j', j'' \in [J]} \int \hat{z}_{g(t)} d_{j'j''} d_{j', j''} s_{j'}(\tilde{\beta}; \tilde{\theta}) s_{j''}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta})) dF_{\tilde{\beta}}x_{j't}. \]

For the second claim, observe that the absolute value of the first term of \(B_{j'g}(x_t)\) is bounded above by
\[ \frac{J}{C_1} \max_{j' \in [J]} |x_{j't}| \int \left| \hat{z}_{g(t)} \hat{\beta}_j \right| \sum_{j', j'' \in [J]} s_{j'}(\tilde{\beta}; \tilde{\theta}) s_{j''}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta})) dF_{\tilde{\beta}} \leq C_2 J \max_{j' \in [J]} |x_{j't}|. \]
because the crude bound of \(0 < s_j(\tilde{\beta}; \tilde{\theta}) < 1\) says that
\[ \left| \sum_{j', j'' \in [J]} s_{j'}(\tilde{\beta}; \tilde{\theta}) s_{j''}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta})) \right| \leq |(1 - s_0(\tilde{\beta}; \tilde{\theta}) - s_j(\tilde{\beta}; \tilde{\theta}))(1 - 2s_j(\tilde{\beta}; \tilde{\theta}))(1 - s_0(\tilde{\beta}; \tilde{\theta}) - s_j(\tilde{\beta}; \tilde{\theta}))(1 - 2s_j(\tilde{\beta}; \tilde{\theta}))| \leq |1 - 2s_j(\tilde{\beta}; \tilde{\theta})||1 + s_j(\tilde{\beta}; \tilde{\theta}) + s_j(\tilde{\beta}; \tilde{\theta})^2| \leq 3. \]

Similarly, the absolute value of the second term is bounded above by
\[ \frac{J^2}{C_3} \max_{j' \in [J]} |x_{j't}| \int \left| \hat{z}_{g(t)} \hat{\beta}_j \right| s_{j_1}(\tilde{\beta}; \tilde{\theta}) s_{j_2}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta})) s_{j'}(\tilde{\beta}; \tilde{\theta})(1 - 2s_j(\tilde{\beta}; \tilde{\theta})) dF_{\tilde{\beta}} \leq C_4 J \max_{j' \in [J]} |x_{j't}|. \]
since \( |d_{j_1j_2}(\theta) d_{j_2j_4}(\theta)| \leq |d_{j_1j_2}(\theta)||d_{j_2j_4}(\theta)|. \) Therefore, the statement claimed follows for a constant \( \tilde{C} \geq \max\{C_2, C_4\}. \)

\[ \square \]

**Corollary 1.** In addition to the assumptions for lemma\(^2\), suppose that

1. \( \sup_{\Delta \theta \in \Theta, j', j'' \in [J], k, k' \in [K]} \mathbb{E}_n \text{Var}(f_{jk}(\tilde{X}; \theta_0 + \Delta \theta) - f_{jk}(\tilde{X}; \theta_0)) f_{j'k'}(\tilde{X}; \theta_0) \leq B_1^2 \), and
2. \( \max_{j', j'' \in [J], t \in [L], k, k' \in [K]} \mathbb{E}_n (X_t h_{j,k}(W_t) h_{j', k'}(W_t) \xi_j(\tilde{X}; \theta_0))^2 \leq B_2^2 \) with probability at least \( 1 - \delta_n / 6 \),

then,
\[ \sup_{\theta \in \mathcal{R}(\theta_0), j', j'' \in [J], k, k' \in [K]} |\mathbb{G}_n(f_{j,k}(\tilde{X}; \theta) - f_{j,k}(\tilde{X}; \theta_0)) f_{j'k'}(\tilde{X}; \theta_0)| \leq \alpha^{-1/2} C(B_1 n + (J^2 G)(2\sqrt{2} B_2 n ||\theta - \theta_0||_1 \log 1/2 + 4 J^2 G K L/\delta_n) \]
with probability at least \( 1 - \delta_n \) with a universal constant \( C \).

\[ \square \]

**Proof.** All the arguments in the proof of theorem 1 applies by replacing \( h_{j,k}(W_t) \) terms with \( h_{j,k}(W_t) h_{j', k'}(W_t) \xi_j(\tilde{X}; \theta_0) \).

\[ \square \]

**Corollary 2.** In addition to the assumptions for lemma\(^2\), suppose that
1. \( \sup_{\Delta \theta \in \Theta, j, l \in [J], k \in [K], t \in [2L]} E_n \text{Var}(G_{j,k,l}(\hat{X}; \hat{\theta}_0 + \Delta \theta) - G_{j,k,l}(X_{j,l})) \leq B_{2n}^2 \), and

2. \( \max_{j,l \in [J], k \in [K]} E_n (h_{j,k}(W_j) X_{j,l} \max_{j' \in [J]} |X_{j',l}|)^2 \leq B_{2n}^2 \) with probability at least \( 1 - \delta_n/6 \), then,

\[
\sup_{j, j' \in [J], k, k' \in [K], t \in [2L]} \| G_n (G_{j,k,l}(\hat{X}; \hat{\theta}) - G_{j,k,l}(X_{j,l}; \hat{\theta}_0)) \| \leq n^{-1/2} C (B_{1n}^2 + (J^2 G)(2 \sqrt{2} B_{2n}^2 \| \hat{\theta} - \hat{\theta}_0 \|_1 \log 1/2 (8 J^2 G KL/\delta_n)))
\]

with probability at least \( 1 - \delta_n \) with a universal constant \( C \).

**Proof.** By lemma 6 the gradient functions \( G_{jk,l}(\hat{X}; \hat{\theta}) \) can be linearly expanded with respect to \( \nu_{jg} - \nu_{jg0} \) indices and their coefficients depend on \( l \) only through the corresponding sub-vector of covariates \( X_l \). Therefore, all the arguments in the proof of theorem 1 applies by replacing \( f_{jk}(\hat{X}; \theta) \) terms with \( G_{jk,l}(\hat{X}; \theta) \) and \( h_{jk}(W_i) \) terms with \( h_{jk}(W_i) \max_{j' \in [J]} |X_{j',l}| \).

**Corollary 3.** (Based on Ledoux and Talagrand (1991), theorem 4.12) Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be convex and increasing. Let \( \mathcal{N} \) be a subset of \( \mathbb{R}^{n_j} \) and \( \mathcal{N}_i, \mathcal{N}_j \) for each \( i \in [n] \) and \( j \in [J] \) be \( i, j, (ij) \)-th coordinates of \( \mathcal{N} \). Let \( \sigma = \{ \sigma_i \}_{i=1}^n \) be independent Rademacher random variables taking \( \{-1, 1\} \) with equal probability. Let \( \phi_1 : \mathcal{N}_i \rightarrow \mathbb{R} \) be functions such that \( |\phi_1(i)| \leq 1 \) and \( |\phi_1(i_1)\nu_{ij} - \phi_1(i_2)\nu_{ij'}| \leq |\nu_{ij} - \nu_{ij'}| \) uniformly over \( i \in \mathcal{N}_i, \nu_{ij}, \nu_{ij'} \in \mathcal{N}_{ij} \) for every \( i \in [n] \) and \( j \in [J] \). Then

\[
E \left[ F \left( \frac{1}{2} \sup_{\nu \in \mathcal{N}} \sum_{i=1}^n \sigma_i \phi_1(i)\nu_{ij} \right) \right] \leq E \left[ F \left( \sup_{\nu \in \mathcal{N}_j} \sum_{i=1}^n \sigma_i \nu_{ij} \right) \right].
\]

**Proof.** The result follows from the proof of the Ledoux and Talagrand (1991), Theorem 4.12. Below, we state a modified sketch of the original proof. First, we want to show that

\[
E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sum_{i=1}^n \sigma_i \phi_1(i)\nu_{ij} \right) \right] \leq E \left[ G \left( \sup_{\nu \in \mathcal{N}_j} \sum_{i=1}^n \sigma_i \nu_{ij} \right) \right]
\]

for convex and increasing \( G : \mathbb{R} \rightarrow \mathbb{R} \). Once the above inequality holds, we would achieve the stated inequality by the symmetry of the distribution of the random variables multiplied with Rademacher variables.

We show the above inequality by conditioning and iteration. Let \( \sigma_{i > j} = \{ \sigma_j, \ldots, \sigma_n \} \). Now, order the \( 2^{n-j} \) support values of \( \sigma_{i > j} \). Let \( \sigma_{i > j}^r \) be a \( r \)th value in the ordered support values of \( \sigma_{i > j} \). As the Rademacher variables are independent,

\[
E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sum_{i=1}^n \sigma_i \phi_1(i)\nu_{ij} \right) \right] = \sum_{r=1}^{2^{n-1}} E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sum_{i=1}^n \sigma_i \phi_1(i)\nu_{ij} + \sum_{i=1}^n \sigma_i^r \phi_1(i)\nu_{ij} \right) \right] \left( \frac{1}{2} \right)^{n-1}
\]

If

\[
E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sigma_1 \phi_1(i)\nu_{ij} + t \right) \right] \leq E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sigma_1 \nu_{ij} + t \right) \right]
\]

then we have

\[
\sum_{r=1}^{2^{n-1}} E \left[ G \left( \sup_{\nu \in \mathcal{N}} \sum_{i=1}^n \sigma_i \phi_1(i)\nu_{ij} + \sum_{i=1}^n \sigma_i^r \phi_1(i)\nu_{ij} \right) \right] \left( \frac{1}{2} \right)^{n-1}
\]

\[
\leq \sum_{r=1}^{2^{n-1}} E \left[ G \left( \sup_{\nu_{ij} \in \mathcal{N}_{ij}, \nu_{ij'} \in \mathcal{N}_{ij'}} \sigma_1 \nu_{ij} + \sum_{i=1}^n \sigma_i^r \phi_1(i)\nu_{ij} \right) \right] \left( \frac{1}{2} \right)^{n-1},
\]

therefore, we achieve the target inequality by iterating over \( r > 1 \).

Now we show for all \( t_1, s_1 \in \mathcal{N}_1 \) and \( t_2, s_2 \in \mathcal{N}_2 \),

\[
\frac{1}{2} G (s_{1j} - \phi(s_2)s_{2j}) + \frac{1}{2} G (t_{1j} + \phi(t_2)t_{2j}) \leq \frac{1}{2} G (s_{1j} - s_{2j}) + \frac{1}{2} G (t_{1j} + t_{2j}).
\]
The remaining argument follows essentially the same argument of the proof of Ledoux and Talagrand (1991) but the fact that \( \phi(s) \) takes a vector argument. Nevertheless, a similar argument applies because it is uniformly bounded by constant. First, we may assume that
\[
t_{1j} + \phi(t_2)t_{2j} \geq s_{1j} + \phi(s_2)s_{2j}
\]
and
\[
s_{1j} - \phi(s_2)s_{2j} \geq t_{1j} - \phi(t_2)t_{2j}
\]
otherwise the two separate supremum under \( \sigma = 1 \) and \( \sigma = -1 \) is solved as a single supremum under common variables either \( (t_1, t_2) \) or \( (s_1, s_2) \) only. We distinguish between the following cases. When \( t_{2j} \geq s_{2j} \geq 0 \), we have
\[
t_{1j} + \phi(t_2)t_{2j} - s_{1j} + s_{2j} \geq s_{1j} + \phi(s_2)s_{2j} - s_{1j} + s_{2j}
\]
\[
\geq s_{2j} - |\phi(s_2)|s_{2j}
\]
and
\[
s_{2j} - \phi(s_2)s_{2j} \leq t_{2j} - \phi(t_2)t_{2j}
\]
from \( |\phi(t_2)t_{2j} - \phi(s_2)s_{2j}| \leq |t_{2j} - s_{2j}| \) and \( t_{2j} \geq s_{2j} \). Therefore, we have
\[
G(s_{1j} - \phi(s_2)s_{2j}) - G(s_{1j} - s_{2j}) \leq G(s_{1j} - s_{2j} + (1 - \phi(s_2))s_{2j}) - G(s_{1j} - s_{2j})
\]
\[
\leq G(t_{1j} + \phi(t_2)t_{2j} + (1 - \phi(s_2))s_{2j}) - G(t_{1j} + \phi(t_2)t_{2j})
\]
\[
\leq G(t_{1j} + s_{2j} - \phi(s_2)s_{2j} + \phi(t_2)t_{2j}) - G(t_{1j} + \phi(t_2)t_{2j})
\]
\[
\leq G(t_{1j} + t_{2j} - \phi(t_2)t_{2j} + \phi(t_2)t_{2j}) - G(t_{1j} + \phi(t_2)t_{2j})
\]
\[
\leq G(t_{1j} + t_{2j}) - G(t_{1j} + \phi(t_2)t_{2j})
\]
as \( G(\cdot + x) - G(\cdot) \) is increasing for any \( x \geq 0 \). Thus, the desired inequality is achieved.

The same argument applies with \( t \) replaced with \( s \) and \( \phi \) into \( -\phi \). The parallel argument holds when \( t_{2j} \leq s_{2j} \leq 0 \).

When \( t_{2j} \geq 0 \) and \( s_{2j} \leq 0 \),
\[
G(t_{1j} + \phi(t_2)t_{2j}) - G(t_{1j} + t_{2j}) \leq G(t_{1j} + |\phi(t_2)|t_{2j}) - G(t_{1j} + t_{2j}) \leq 0
\]
and
\[
G(s_{1j} - s_{2j}s_{2j}) - G(s_{1j} - s_{2j}) \leq G(s_{1j} - |\phi(s_2)|s_{2j}) - G(s_{1j} - s_{2j}) \leq 0.
\]
The parallel argument applies when \( t_{2j} \leq 0 \) and \( s_{2j} \geq 0 \).