Newtonian and single layer potentials for the Stokes system with $L^\infty$ coefficients and the exterior Dirichlet problem

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Abstract. A mixed variational formulation of some problems in $L^2$-based Sobolev spaces is used to define the Newtonian and layer potentials for the Stokes system with $L^\infty$ coefficients on Lipschitz domains in $\mathbb{R}^3$. Then the solution of the exterior Dirichlet problem for the Stokes system with $L^\infty$ coefficients is presented in terms of these potentials and the inverse of the corresponding single layer operator.

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1. Introduction

Let $u$ be an unknown vector field, $\pi$ be an unknown scalar field, and $f$ be a given vector field defined on an exterior Lipschitz domain $\Omega_- \subset \mathbb{R}^3$. Let also $\mathbb{E}(u)$ be the symmetric part of the gradient of $u$, $\nabla u$. Then the equations

$$\mathcal{L}_\mu(u, \pi) := \text{div}(2\mu\mathbb{E}(u)) - \nabla \pi = f, \quad \text{div } u = 0 \text{ in } \Omega_-$$

(1.1)
determine the Stokes system with a known viscosity coefficient $\mu \in L^\infty(\Omega_-)$. This linear PDE system describes the flows of viscous incompressible fluids, when the inertia of such a fluid can be neglected. The coefficient $\mu$ is related to the physical properties of the fluid (for further details we refer the reader to the books [45] and [23] and the references therein).

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The methods of layer potential theory have a main role in the analysis of boundary value problems for elliptic partial differential equations (see, e.g., [13, 17, 30, 36, 39, 43, 48]). Fabes, Kenig and Verchota [21] obtained mapping properties of layer potentials for the constant coefficient Stokes system in $L^p$ spaces. Mitrea and Wright [43] have used various methods of layer potentials in the analysis of the main boundary problems for the Stokes system with constant coefficients in arbitrary Lipschitz domains in $\mathbb{R}^n$. The authors in [32] have obtained mapping properties of the constant coefficient Stokes layer potential operators in standard and weighted Sobolev spaces by exploiting results of singular integral operators. Gatica and Wendland [24] used the coupling of mixed finite element and boundary integral methods for solving a class of linear and nonlinear elliptic boundary value problems. The authors in [33] used the Stokes and Brinkman integral layer potentials and a fixed point theorem to show an existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems with data in $L^p$, Sobolev, and Besov spaces (see also [34]). All above results are devoted to elliptic boundary value problems with constant coefficients.

Potential theory plays also a main role in the study of elliptic boundary value problems with variable coefficients. Dindoš and Mitrea [19] have obtained well-posedness results in Sobolev spaces for Poisson problems for the Stokes and Navier-Stokes systems with Dirichlet condition on $C^1$ and Lipschitz domains in compact Riemannian manifolds by using mapping properties of Stokes layer potentials in Sobolev and Besov spaces. Chkadua, Mikhailov and Natroshvili [14] obtained direct segregated systems of boundary-domain integral equations for a mixed boundary value problem of Dirichlet-Neumann type for a scalar second-order divergent elliptic partial differential equation with a variable coefficient in an exterior domain of $\mathbb{R}^3$ (see also [13]). Hofmann, Mitrea and Morris [29] considered layer potentials in $L^p$ spaces for elliptic operators of the form $L=-\text{div}(A\nabla u)$ acting in the upper half-space $\mathbb{R}^{n+1}_+$, $n \geq 3$, or in more general Lipschitz graph domains, with an $L^\infty$ coefficient matrix $A$, which is $t$-independent, and solutions of the equation $Lu=0$ satisfy interior De Giorgi-Nash-Moser estimates. They obtained a Calderón-Zygmund type theory associated to the layer potentials, and well-posedness results of boundary problems for the operator $L$ in $L^p$ and endpoint spaces.

Our variational approach is inspired by that developed by Sayas and Selgas in [46] for the constant coefficient Stokes layer potentials on Lipschitz boundaries, and is based on the technique of Nédélec [44]. Girault and Sequeira [26] obtained a well-posed result in weighted Sobolev spaces for the Dirichlet problem for the standard Stokes system in exterior Lipschitz domains of $\mathbb{R}^n$, $n = 2, 3$. Băcuță, Hassell and Hsiao [8] developed a variational approach for the standard Brinkman single layer potential and used it in the analysis of the time dependent exterior Stokes problem with Dirichlet boundary condition in $\mathbb{R}^n$, $n = 2, 3$. Barton [7] constructed layer potentials for strongly elliptic differential operators in general settings by using the Lax-Milgram theorem, and generalized various properties of layer potentials for
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harmonic and second order elliptic equations. Brewster et al. in [9] have used a variational approach and a deep analysis to obtain well-posedness results for boundary problems of Dirichlet, Neumann and mixed type for higher order divergence-form elliptic equations with $L^\infty$ coefficients in locally $(\epsilon, \delta)$-domains and in Besov and Bessel potential spaces. Brewster et al. have studied the Dirichlet problem for the Stokes system with nonsmooth coefficients, and proved the unique solvability of the problem in Sobolev spaces on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ with a small Lipschitz constant when the coefficients have vanishing mean oscillations with respect to all variables. Choi and Lee [15] have studied the Dirichlet problem for the Stokes system with nonsmooth coefficients, and proved the unique solvability of the problem in Sobolev spaces on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ with a small Lipschitz constant when the coefficients have vanishing mean oscillations with respect to all variables. Choi and Yang [16] obtained the existence and pointwise bound of the fundamental solution for the Stokes system with measurable coefficients in $\mathbb{R}^n$, $n \geq 3$, whenever the weak solutions of the system are locally Hölder continuous. Alliot and Amrouche [3] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces. Also, Amrouche and Nguyen [5] proved existence and uniqueness results in weighted Sobolev spaces for the Poisson problem with Dirichlet boundary condition for the Navier-Stokes system in exterior Lipschitz domains in $\mathbb{R}^3$.

The purpose of this work is to show the well-posedness result of the Poisson problem of Dirichlet type for the Stokes system with $L^\infty$ coefficients in $L^2$-based Sobolev spaces on an exterior Lipschitz domain in $\mathbb{R}^3$. We use a variational approach that reduces this boundary value problem to a mixed variational formulation. A similar variational approach is used to define the Newtonian and layer potentials for the Stokes system with $L^\infty$ coefficients on Lipschitz surfaces in $\mathbb{R}^3$, by using the weak solutions of some transmission problems in $L^2$-based Sobolev spaces. Finally, the mapping properties of these layer potentials are used to construct explicitly the solution of the exterior Dirichlet problem for the Stokes system with $L^\infty$ coefficients. The analysis developed in this paper confines to the case $n = 3$, due to its practical interest, but the extension to the case $n \geq 3$ can be done with similar arguments.

2. Functional setting and useful results

Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, i.e., an open connected set whose boundary $\partial \Omega$ is locally the graph of a Lipschitz function. Assume that $\partial \Omega$ is connected. Let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$ denote the exterior Lipschitz domain. Let $\mathcal{E}_\pm$ denote the operators of extension by zero outside $\Omega_\pm$.

2.1. Standard $L^2$-based Sobolev spaces and related results

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse defined on the $L^1(\mathbb{R}^3)$ functions and generalized to the space of tempered distributions $\mathcal{S}^*(\mathbb{R}^3)$ (i.e., the topological dual of the space $\mathcal{S}(\mathbb{R}^3)$ of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^3$). The Lebesgue space of (equivalence classes of) measurable, square integrable functions on $\mathbb{R}^3$ is denoted by $L^2(\mathbb{R}^3)$, and by $L^\infty(\mathbb{R}^3)$ we denote the space of (equivalence classes of) essentially bounded measurable functions on $\mathbb{R}^3$. Let $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)^3$
denote the $L^2$-based Sobolev (Bessel potential) spaces

$$H^1(\mathbb{R}^3) := \{ f \in S^*(\mathbb{R}^3) : \| f \|_{H^1(\mathbb{R}^3)} = \| F^{-1}[(1+|\xi|^2)^{\frac{1}{2}}F] \|_{L^2(\mathbb{R}^3)} < \infty \},$$

$$H^1(\mathbb{R}^3)^3 := \{ f = (f_1, f_2, f_3) : f_j \in H^1(\mathbb{R}^3), \ j = 1, 2, 3 \}$$

(2.1) (2.2)

Now let $\Omega'$ be $\Omega_+, \Omega_-$ or $\mathbb{R}^3$. We denote by $\mathcal{D}(\Omega') := C_0^\infty(\Omega')$ the space of infinitely differentiable functions with compact support in $\Omega'$, equipped with the inductive limit topology. Let $\mathcal{D}^*(\Omega')$ denote the corresponding space of distributions on $\Omega'$, i.e., the dual space of $\mathcal{D}(\Omega')$. Let us consider the space

$$H^1(\Omega') := \{ f \in \mathcal{D}^*(\Omega') : \exists F \in H^1(\mathbb{R}^3) \text{ such that } F|_{\Omega'} = f \},$$

(2.3)

where $\cdot|_{\Omega'}$ is the restriction operator to $\Omega'$. The space $\tilde{H}^1(\Omega')$ is the closure of $\mathcal{D}(\Omega')$ in $H^1(\mathbb{R}^3)$. This space can be also characterized as

$$\tilde{H}^1(\Omega') := \left\{ \tilde{f} \in H^1(\mathbb{R}^3) : \operatorname{supp} \tilde{f} \subseteq \Omega' \right\}.$$  

(2.4)

Similar to definition (2.2), $H^1(\Omega')^3$ and $\tilde{H}^1(\Omega')^3$ are the spaces of vector-valued functions whose components belong to the scalar spaces $H^1(\Omega')$ and $\tilde{H}^1(\Omega')$, respectively (see, e.g., [38]). The Sobolev space $\tilde{H}^1(\Omega')$ can be identified with the closure $\tilde{H}^1(\Omega')$ of $\mathcal{D}(\Omega')$ in the norm of $H^1(\Omega')$ (see, e.g., [42, (3.11), (3.13)], [38, Theorem 3.33]). The space $\mathcal{D}(\Omega')$ is dense in $H^1(\Omega')$, and the following spaces can be isomorphically identified (cf., e.g., [38, Theorem 3.14])

$$(H^1(\Omega'))^* = \tilde{H}^{-1}(\Omega'), \quad H^{-1}(\Omega') = (\tilde{H}^1(\Omega'))^*.$$  

(2.5)

For $s \in [0, 1]$, the Sobolev space $H^s(\partial \Omega)$ on the boundary $\partial \Omega$ can be defined by using the space $H^s(\mathbb{R}^2)$, a partition of unity and the pull-backs of the local parametrization of $\partial \Omega$, and $H^{-s}(\partial \Omega) = (H^s(\partial \Omega))^*$. All the above spaces are Hilbert spaces. For further properties of Sobolev spaces on bounded Lipschitz domains and Lipschitz boundaries, we refer to [1, 31, 38, 43, 47].

A useful result for the next arguments is given below (see, e.g., [17], [31, Proposition 3.3]).

**Lemma 2.1.** Assume that $\Omega := \Omega_+ \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected boundary $\partial \Omega$ and denote by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the corresponding exterior domain. Then there exist linear and bounded trace operators $\gamma_\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial \Omega)$ such that $\gamma_\pm f = f|_{\partial \Omega}$ for any $f \in C_0^\infty(\overline{\Omega}_\pm)$. These operators are surjective and have (non-unique) bounded linear right inverse operators $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega_\pm)$.

The jump of a function $u \in H^1(\mathbb{R}^3 \setminus \partial \Omega)$ across $\partial \Omega$ is denoted by $[\gamma(u)] := \gamma_+(u) - \gamma_-(u)$. For $u \in H^1_{\text{loc}}(\mathbb{R}^3)$, $[\gamma(u)] = 0$. The trace operator $\gamma : H^1(\mathbb{R}^3) \to H^{\frac{1}{2}}(\partial \Omega)$ can be also considered and is linear and bounded\(^1\).

If $X$ is either an open subset or a surface in $\mathbb{R}^3$, then we use the notation $\langle \cdot, \cdot \rangle_X$ for the duality pairing of two dual Sobolev spaces defined on $X$.

\(^1\)The trace operators defined on Sobolev spaces of vector fields on $\Omega_\pm$ or $\mathbb{R}^3$ are also denoted by $\gamma_\pm$ and $\gamma$, respectively.
2.2. Some weighted Sobolev spaces and related results

For a point \( \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \), its distance to the origin is denoted by \( |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \). Let \( \rho \) denote the weight function

\[
\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}.
\]

(2.6)

For \( \lambda \in \mathbb{R} \), we consider the weighted space \( L^2(\rho^\lambda; \mathbb{R}^3) \) given by

\[
f \in L^2(\rho^\lambda; \mathbb{R}^3) \iff \rho^\lambda f \in L^2(\mathbb{R}^3),
\]

(2.7)

which is a Hilbert space when it is endowed with the inner product and the associated norm,

\[
(f, g)_{L^2(\rho^\lambda; \mathbb{R}^3)} := \int_{\mathbb{R}^3} f g \rho^{2\lambda} d\mathbf{x}, \quad \|f\|_{L^2(\rho^\lambda; \mathbb{R}^3)} := (f, f)_{L^2(\rho^\lambda; \mathbb{R}^3)}.
\]

(2.8)

We also consider the weighted Sobolev space

\[
\mathcal{H}^1(\mathbb{R}^3) := \left\{ f \in D'(\mathbb{R}^3) : \rho^{-1} f \in L^2(\mathbb{R}^3), \nabla f \in L^2(\mathbb{R}^3)^3 \right\}
\]

(2.9)

which is a Hilbert space with respect to the inner product

\[
(f, g)_{\mathcal{H}^1(\mathbb{R}^3)} := (f, g)_{L^2(\rho^{-1}; \mathbb{R}^3)} + (\nabla f, \nabla g)_{L^2(\mathbb{R}^3)^3}
\]

(2.10)

and the associated norm

\[
\|f\|_{\mathcal{H}^1(\mathbb{R}^3)} := \left\|\rho^{-1} f\right\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)^3}
\]

(2.11)

(cf. [28]; see also [5]). The spaces \( L^2(\rho^\lambda; \Omega^-) \) and \( \mathcal{H}^1(\Omega^-) \) can be similarly defined, and \( D(\Omega^-) \) is dense in \( \mathcal{H}^1(\Omega^-) \) (see, e.g., [28, Theorem I.1], [27, Ch.1, Theorem 2.1]). The seminorm

\[
|f|_{\mathcal{H}^1(\Omega^-)} := \|\nabla f\|_{L^2(\Omega^-)^3}
\]

(2.12)

is equivalent to the norm of \( \mathcal{H}^1(\Omega^-) \) defined as in (2.11), with \( \Omega^- \) in place of \( \mathbb{R}^3 \) (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]). The weighted spaces \( L^2(\rho^{-1}; \Omega^+) \) and \( \mathcal{H}^1(\Omega^+) \) coincide with the standard spaces \( L^2(\Omega^+) \) and \( \mathcal{H}^1(\Omega^+) \), respectively (with equivalent norms).

Note that the result in Lemma 2.1 extends also to the weighted Sobolev space \( \mathcal{H}^1(\Omega^-) \). Therefore, there exists a linear bounded exterior trace operator

\[
\gamma_- : \mathcal{H}^1(\Omega^-) \to H^{\frac{3}{2}}(\partial\Omega),
\]

(2.13)

which is also surjective (see [46, p. 69]). Moreover, the embedding of the space \( \mathcal{H}^1(\Omega^-) \) into \( \mathcal{H}^1(\Omega^-) \) and Lemma 2.1 show the existence of a (non-unique) linear and bounded right inverse \( \gamma^{-1}_- : H^{\frac{3}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega^-) \) of operator (2.13) (see [32, Lemma 2.2], [40, Theorem 2.3, Lemma 2.6]).

Let \( \mathcal{H}^1(\Omega^-) \subset \mathcal{H}^1(\Omega^-) \) denote the closure of \( D(\Omega^-) \) in \( \mathcal{H}^1(\Omega^-) \). This space can be characterized as

\[
\mathcal{H}^1(\Omega^-) = \{ v \in \mathcal{H}^1(\Omega^-) : \gamma_- v = 0 \text{ on } \partial\Omega \}
\]

(2.14)
Then define the conormal derivative operator \( C \) for the Stokes system with \( L \). This formula suggests the following weak definition of the generalized conormal derivative operator \( C \) for the Stokes system with \( L \). Let \( \mu \) be a continuously differentiable viscosity coefficient in \( \mathbb{R}^3 \) and can be isomorphically identified with the space \( \mathcal{H}(\Omega) \) in \( \mathcal{H}(\mathbb{R}^3) \). In the sequel we assume that the viscosity coefficient \( \mu \) of the Stokes system (1.1) belongs to \( L(\Omega) \) and there exists a constant \( c_\mu > 0 \), such that

\[
c_{\mu}^{-1} \leq \mu \leq c_\mu \text{ a.e. in } \mathbb{R}^3.
\]  

(3.1)

Let \( \mathbb{E}(u) := \frac{1}{\mu}(\nabla u + (\nabla u)^T) \) be the strain rate tensor. If \((u, \pi) \in C^1(\Omega \times \mathbb{R}^3) \), we can define the classical interior and exterior conormal derivatives (i.e., the boundary traction fields) for the Stokes system (1.1) with continuously differentiable viscosity coefficient \( \mu \) by the well-known formula

\[
t_{\mu}^{\pm}(u, \pi) := \gamma \pm (-\pi I + 2\mu E(u)) \nu,
\]  

where \( \nu \) is the outward unit normal to \( \Omega \), defined a.e. on \( \partial \Omega \). Then for any function \( \varphi \in \mathcal{D}(\mathbb{R}^3) \) we obtain the first Green identity

\[
\pm \langle t_{\mu}^{\pm}(u, \pi), \varphi \rangle_{\partial \Omega} = 2\langle \mu \mathbb{E}(u), \mathbb{E}(\varphi) \rangle_{\Omega} + \langle \pi, \text{div} \varphi \rangle_{\Omega} + \langle \mathcal{L}_{\mu}(u, \pi), \varphi \rangle_{\Omega}.
\]

This formula suggests the following weak definition of the generalized conormal derivative for the Stokes system with \( \mu \) in the setting of \( L^2 \)-weighted Sobolev spaces (cf., e.g., [17, Lemma 3.2], [32, Lemma 2.9], [34, Lemma 2.2], [40, Definition 3.1, Theorem 3.2], [43, Theorem 14.4]).

**Definition 3.1.** Let \( \mu \in L^\infty(\Omega) \) satisfy assumption (3.1). Let

\[
\mathcal{H}(\Omega, \mathcal{L}_{\mu}) := \{(u, \pi, \mathbf{f}) \in \mathcal{H}(\Omega)^3 \times L^2(\partial \Omega) \times \mathcal{H}^{-1}(\Omega)^3 : \mathcal{L}_{\mu}(u, \pi) = \mathbf{f} \text{ on } \partial \Omega, \text{ div } u = 0 \text{ in } \Omega \}.
\]

(3.3)

Then define the conormal derivative operator \( t_{\mu}^{\pm} : \mathcal{H}(\Omega, \mathcal{L}_{\mu}) \to H^{-\frac{1}{2}}(\partial \Omega)^3 \),

\[
\mathcal{H}(\Omega, \mathcal{L}_{\mu}) \ni (u, \pi, \mathbf{f}) \mapsto t_{\mu}^{\pm}(u, \pi, \mathbf{f}) \in H^{-\frac{1}{2}}(\partial \Omega)^3,
\]

(3.4)

\[
\pm \langle t_{\mu}^{\pm}(u, \pi, \mathbf{f}), \Phi \rangle_{\partial \Omega} := 2\langle \mu \mathbb{E}(u), \mathbb{E}(\gamma^{-1}_\mathbf{f}) \rangle_{\Omega} + \langle \pi, \text{div} \gamma^{-1}_\mathbf{f} \rangle_{\Omega}
\]

\[
\pm \langle \gamma^{-1}_\mathbf{f}, \Phi \rangle_{\Omega} = 0 \forall \Phi \in H^\frac{1}{2}(\partial \Omega)^3.
\]

(3.5)

where \( \gamma^{-1}_\mathbf{f} : H^\frac{1}{2}(\partial \Omega)^3 \to \mathcal{H}(\Omega)^3 \) is a (non-unique) bounded right inverse of the trace operator \( \gamma_\mathbf{f} : \mathcal{H}(\Omega)^3 \to H^\frac{1}{2}(\partial \Omega)^3 \).
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We use the simplified notation $t_{\mu}^{\pm}(u_{\pm}, \pi_{\pm})$ for $t_{\mu}^{\pm}(u_{\pm}, \pi_{\pm}; 0)$. The following assertion can be proved similar to [41, Theorem 5.3], [32, Lemma 2.9].

**Lemma 3.2.** Let $\mu \in L^\infty(\mathbb{R}^{3})$ satisfy assumption (3.1). Then for all $w_{\pm} \in H^{1}(\Omega^{\pm}),$ and $(u_{\pm}, \pi_{\pm}, f_{\pm}) \in \mathcal{H}^{1}(\Omega^{\pm}, \mathcal{L}_{\mu})$ the following identity holds

\[
\pm \langle t_{\mu}^{\pm}(u_{\pm}, \pi_{\pm}, f_{\pm}), \gamma_{\pm} w_{\pm} \rangle_{\partial \Omega} = 2\langle \mu E(u_{\pm}), E(w_{\pm}) \rangle_{\Omega^{\pm}} - \langle \pi_{\pm}, \text{div } w_{\pm} \rangle_{\Omega^{\pm}} + \langle f_{\pm}, w_{\pm} \rangle_{\partial \Omega^{\pm}}. \tag{3.6}
\]

Let $\gamma$ denote the trace operator from $H^{1}(\mathbb{R}^{3})$ to $H^{1/2}(\partial \Omega)^{3}$ (cf., e.g., [40, Theorem 2.3, Lemma 2.6], [8, (2.2)]). For $(u_{\pm}, \pi_{\pm}, f_{\pm}) \in \mathcal{H}^{1}(\Omega^{\pm}, \mathcal{L}_{\mu})$, let

\[
u := \hat{E}_{+} u_{+} + \hat{E}_{-} u_{-}, \quad \pi := \hat{E}_{+} \pi_{+} + \hat{E}_{-} \pi_{-}, \quad f := \tilde{f}_{+} + \tilde{f}_{-} \tag{3.7}
\]

Moreover, if $f = 0$, we define

\[
t_{\mu}(u, \pi) := t_{\mu}(u, \pi; 0) = t_{\mu}^{+}(u_{+}, \pi_{+}) - t_{\mu}^{-}(u_{-}, \pi_{-}). \tag{3.9}
\]

Then Lemma 3.2 leads to the following result.

**Lemma 3.3.** Let $\mu \in L^\infty(\mathbb{R}^{3})$ satisfy assumption (3.1). Also let $(u_{\pm}, \pi_{\pm}, f_{\pm}) \in \mathcal{H}^{1}(\Omega^{\pm}, \mathcal{L}_{\mu})$ and let $(u, \pi, f)$ be defined as in (3.7). Then for all $w \in H^{1}(\mathbb{R}^{3})$, the following formula holds

\[
\langle [t_{\mu}(u, \pi; f)], \gamma w \rangle_{\partial \Omega} = 2\langle \mu E(u_{+}), E(w) \rangle_{\Omega^{+}} + 2\langle \mu E(u_{-}), E(w) \rangle_{\Omega^{-}} - \langle \pi, \text{div } w \rangle_{\mathbb{R}^{3}} + \langle f, w \rangle_{\mathbb{R}^{3}}. \tag{3.10}
\]

We also need the following particular case of Lemma 3.3 when $f = 0$.

**Lemma 3.4.** Let $\mu \in L^\infty(\mathbb{R}^{3})$ satisfy assumption (3.1). Also let $(u_{\pm}, \pi_{\pm}, 0) \in \mathcal{H}^{1}(\Omega^{\pm}, \mathcal{L}_{\mu})$. Then for all $w \in H^{1}(\mathbb{R}^{3})$,

\[
\langle [t_{\mu}(u, \pi)], \gamma w \rangle_{\partial \Omega} = 2\langle \mu E(u_{+}), E(w) \rangle_{\Omega^{+}} + 2\langle \mu E(u_{-}), E(w) \rangle_{\Omega^{-}} - \langle \pi, \text{div } w \rangle_{\mathbb{R}^{3}}. \tag{3.11}
\]

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Recall that the function $\mu \in L^\infty(\mathbb{R}^{3})$ satisfies conditions (3.1). Next, we define the Newtonian and single layer potentials for the $L^{\infty}$ coefficient Stokes system (1.1).

4.1. Variational solution of the variable-coefficient Stokes system in $\mathbb{R}^{3}$

First we show the following useful well-posedness result.

**Lemma 4.1.** Let $a_{\mu}(\cdot, \cdot) : H^{1}(\mathbb{R}^{3})^{3} \times H^{1}(\mathbb{R}^{3})^{3} \to \mathbb{R}$ and $b(\cdot, \cdot) : H^{1}(\mathbb{R}^{3})^{3} \times L^{2}(\mathbb{R}^{3}) \to \mathbb{R}$ be the bilinear forms given by

\[
a_{\mu}(u, v) := 2\langle \mu E(u), E(v) \rangle_{\mathbb{R}^{3}}, \quad \forall u, v \in H^{1}(\mathbb{R}^{3})^{3}, \tag{4.1}
\]

\[
b(v, q) := -\langle \text{div } v, q \rangle_{\mathbb{R}^{3}}, \quad \forall v \in H^{1}(\mathbb{R}^{3})^{3}, \quad \forall q \in L^{2}(\mathbb{R}^{3}). \tag{4.2}
\]
Also let \( \mathcal{L} : H^1(\mathbb{R}^3)^3 \to \mathbb{R} \) be a linear and bounded map. Then the mixed variational formulation

\[
\begin{cases}
  a_\mu(u, v) + b(v, p) = \mathcal{L}(v), & \forall \; v \in H^1(\mathbb{R}^3)^3, \\
  b(u, q) = 0, & \forall \; q \in L^2(\mathbb{R}^3)
\end{cases}
\]

(4.3)
is well-posed. Hence, (4.3) has a unique solution \((u, p) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)\) and there exists a constant \(C = C(\mu) > 0\) such that

\[
\|u\|_{H^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \leq C\|\mathcal{L}\|_{H^{-1}(\mathbb{R}^3)^3}.
\]

(4.4)

**Proof.** By using conditions (3.1) and definition (2.11) of the norm of the weighted Sobolev space \(H^1(\mathbb{R}^3)\) we obtain that

\[
|a_\mu(u, v)| \leq 2c_\mu \|E(u)\|_{L^2(\mathbb{R}^3)^3} \|E(v)\|_{L^2(\mathbb{R}^3)^3} \\
\leq 2c_\mu \|u\|_{H^1(\mathbb{R}^3)} \|v\|_{H^1(\mathbb{R}^3)^3}, \quad \forall \; u, v \in H^1(\mathbb{R}^3)^3.
\]

(4.5)

Moreover, by using the Korn type inequality for functions in \(H^1(\mathbb{R}^3)^3\),

\[
\|\text{grad } v\|_{L^2(\mathbb{R}^3)^3} \leq 2\|E(v)\|_{L^2(\mathbb{R}^3)^3}
\]

(4.6)

cf., e.g., [46, (2.2))] and since the seminorm

\[
\|g\|_{H^1(\mathbb{R}^3)} : = \|\nabla g\|_{L^2(\mathbb{R}^3)^3}
\]

(4.7)
is a norm in \(H^1(\mathbb{R}^3)^3\) equivalent to the norm defined by (2.11) (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]), there exists a constant \(c_1\) such that

\[
a_\mu(u, u) \geq 2c_\mu^{-1} \|E(u)\|_{L^2(\mathbb{R}^3)^3}^2 \geq c_\mu^{-1} \|\nabla u\|_{L^2(\mathbb{R}^3)^3}^2 \\
\geq c_\mu^{-1}c_1 \|u\|_{H^1(\mathbb{R}^3)^3}^2, \quad \forall \; u \in H^1(\mathbb{R}^3)^3.
\]

(4.8)

Inequalities (4.5) and (4.8) show that \(a_\mu(\cdot, \cdot) : H^1(\mathbb{R}^3)^3 \times H^1(\mathbb{R}^3)^3 \to \mathbb{R}\) is a bounded and coercive bilinear form. Moreover, since the divergence operator

\[
\text{div} : H^1(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)
\]

(4.9)
is bounded, then the bilinear form \(b(\cdot, \cdot) : H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}\) is bounded as well. In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]) and also

\[
H^{1,\text{div}}(\mathbb{R}^3)^3 := \{ w \in H^1(\mathbb{R}^3)^3 : \text{div } w = 0 \}
\]

\[
= \{ w \in H^1(\mathbb{R}^3)^3 : b(w, q) = 0, \; \forall \; q \in L^2(\mathbb{R}^3) \}.
\]

In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]), and hence the operator

\[
-\text{div} : H^1(\mathbb{R}^3)^3 / H^{1,\text{div}}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)
\]

is an isomorphism. Then by Lemma A.2(ii) the bounded bilinear form \(b(\cdot, \cdot) : H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}\) satisfies the inf-sup condition (A.7). Hence, there exists \(\beta_0 \in (0, \infty)\) such that

\[
\inf_{q \in L^2(\mathbb{R}^3) \setminus \{0\}} \sup_{w \in H^1(\mathbb{R}^3)^3 \setminus \{0\}} \frac{b(w, q)}{\|w\|_{H^1(\mathbb{R}^3)^3} \|q\|_{L^2(\mathbb{R}^3)}} \geq \beta_0.
\]

(4.10)
By applying Theorem A.4, with $X = \mathcal{H}^1(\mathbb{R}^3)^3$, $M = L^2(\mathbb{R}^3)$, $V = \mathcal{H}^1_{\text{div}}(\mathbb{R}^3)^3$, we conclude that the mixed variational formulation (4.3) has a unique solution $(u, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(\mu) > 0$ such that $(u, p)$ satisfies inequality (4.4).

Next we use the result of Lemma 4.1 in order to show the well-posedness of the $L^\infty$ coefficient Stokes system in the space $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (see also [2, Theorem 3] for the constant-coefficient case).

**Theorem 4.2.** Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy conditions (3.1). Then the $L^\infty$ coefficient Stokes system

$$
\begin{aligned}
\text{div } (2\mu \mathbb{E}(u)) - \nabla \pi &= \ell, & \ell \in \mathcal{H}^{-1}(\mathbb{R}^3)^3, \\
\text{div } u &= 0, & \text{in } \mathbb{R}^3,
\end{aligned}
$$

has a unique solution $(u, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C_0 = C_0(\mu) > 0$ such that

$$
\|u\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \leq C_0\|\ell\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}.
$$

**Proof.** Note that the Stokes system (4.11) is equivalent to the variational problem (4.3) as follows from the density of $\mathcal{D}(\mathbb{R}^3)^3$ in the space $\mathcal{H}^1(\mathbb{R}^3)^3$ (cf., e.g., [28], [46, Proposition 2.1]). Then the well-posedness result of the Stokes system with $L^\infty$ coefficients (4.11) follows from Lemma 4.1. □

4.2. Newtonian potential for the Stokes system with $L^\infty$ coefficients

The well-posedness of problem (4.11) allows us to define the Newtonian potential for the Stokes system with $L^\infty$ coefficients as follows.

**Definition 4.3.** For any $\ell \in \mathcal{H}^{-1}(\mathbb{R}^3)^3$, we define the Newtonian velocity and pressure potentials for the Stokes system with $L^\infty$ coefficients as

$$
\mathcal{N}_{\mu;\mathbb{R}^3}\ell := u, \quad Q_{\mu;\mathbb{R}^3}\ell := \pi,
$$

where $(u, \pi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ is the unique solution of problem (4.11) with the given datum $\ell$.

Moreover, the well-posedness of problem (4.11) yields the continuity of the above operators as stated in the following assertion (cf. also [32, Lemma A.3] for $\mu = 1$).

**Lemma 4.4.** The Newtonian velocity and pressure potential operators

$$
\mathcal{N}_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \rightarrow \mathcal{H}^1(\mathbb{R}^3)^3, \quad Q_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)
$$

are linear and continuous.

4.3. Single layer potential for the Stokes system with $L^\infty$ coefficients

For a given $\phi \in H^{-\frac{3}{2}}(\partial \Omega)^3$, we now consider the following transmission problem for the Stokes system with $L^\infty$ coefficients

$$
\begin{aligned}
\text{div } (2\mu \mathbb{E}(u_\phi)) - \nabla \pi_\phi &= 0, & \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
\text{div } u_\phi &= 0, & \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
[t_{\mu}(u_\phi, \pi_\phi)] &= \phi, & \text{on } \partial \Omega,
\end{aligned}
$$

□
and show that this problem has a unique solution \((u_\varphi, \pi_\varphi) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)\) (cf. also [46, Proposition 5.1] for \(\mu = 1\)). Note that the membership of \(u_\varphi\) in \(H^1(\mathbb{R}^3)^3\) implies the transmission condition
\[
[\gamma(u_\varphi)] = 0 \quad \text{on } \partial \Omega, \tag{4.16}
\]
and the first equation in (4.15) implies also that the jump \([t_\mu(u_\varphi, \pi_\varphi)]\) is well defined.

**Theorem 4.5.** Let \(\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3\) be given. Then the transmission problem (4.15) has the following equivalent mixed variational formulation: Find \((u_\varphi, \pi_\varphi) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)\) such that
\[
\begin{aligned}
2\langle \mu E(u_\varphi), E(v) \rangle_{\mathbb{R}^3} - \langle \pi_\varphi, \text{div } v \rangle_{\mathbb{R}^3} &= \langle \varphi, \gamma v \rangle_{\partial \Omega}, \forall v \in H^1(\mathbb{R}^3)^3, \\
\langle \text{div } u_\varphi, q \rangle_{\mathbb{R}^3} &= 0, \forall q \in L^2(\mathbb{R}^3).
\end{aligned} \tag{4.17}
\]
Moreover, problem (4.17) is well-posed. Hence (4.17) has a unique solution \((u_\varphi, \pi_\varphi) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)\), and there exists a constant \(C = C(\mu)\) such that
\[
\|u_\varphi\|_{H^1(\mathbb{R}^3)^3} + \|\pi_\varphi\|_{L^2(\mathbb{R}^3)} \leq C\|\varphi\|_{H^{-\frac{1}{2}}(\partial \Omega)^3}. \tag{4.18}
\]

**Proof.** The equivalence between the transmission problem (4.15) and the variational problem (4.17) follows from the density of the space \(D(\mathbb{R}^3)^3\) in \(H^1(\mathbb{R}^3)^3\) and formula (3.11), while the well-posedness of the variational problem (4.17) is an immediate consequence of Lemma 4.1 with the linear form \(\ell : H^1(\mathbb{R}^3)^3 \to \mathbb{R}\) given by
\[
\ell(v) := \langle \varphi, \gamma v \rangle_{\partial \Omega} = \langle \gamma^* \varphi, v \rangle_{\mathbb{R}^3}, \forall v \in H^1(\mathbb{R}^3)^3,
\]
and hence \(\ell = \gamma^*\varphi\), where \(\gamma^* : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{-1}(\mathbb{R}^3)^3\) is the adjoint of the trace operator \(\gamma : H^1(\mathbb{R}^3)^3 \to H^{\frac{1}{2}}(\partial \Omega)^3\).

Theorem 4.5 leads to the following definition (cf. [46, p. 75] for \(\mu = 1\)).

**Definition 4.6.** For any \(\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3\), we define the single layer velocity and pressure potentials for the Stokes system with \(L^\infty\) coefficients (1.1) as
\[
V_{\mu;\partial \Omega} \varphi := u_\varphi, \quad Q^{s}_{\mu;\partial \Omega} \varphi := \pi_\varphi, \tag{4.19}
\]
and the potential operators \(V_{\mu;\partial \Omega} : V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{\frac{1}{2}}(\partial \Omega)^3\) and \(K^{s}_{\mu;\partial \Omega} : V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{\frac{1}{2}}(\partial \Omega)^3\) as
\[
V_{\mu;\partial \Omega} \varphi := \gamma u_\varphi, \quad V^{s}_{\mu;\partial \Omega} \varphi := \frac{1}{2} \left( t^+_\mu(u_\varphi, \pi_\varphi) + t^-_\mu(u_\varphi, \pi_\varphi) \right), \tag{4.20}
\]
where \((u_\varphi, \pi_\varphi)\) is the unique solution of problem (4.15) in \(H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)\).

The next result shows the continuity of single layer velocity and pressure operators for the variable coefficient Stokes system (cf. [46, Proposition 5.2], [32, Lemma A.4, (A.10), (A.12)] and [43, Theorem 10.5.3] in the case \(\mu = 1\)).

**Lemma 4.7.** The following operators are linear and continuous
\[
V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{1}(\mathbb{R}^3)^3, \quad Q^{s}_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to L^2(\mathbb{R}^3), \tag{4.21}
\]
\[
V^{s}_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{\frac{1}{2}}(\partial \Omega)^3, \quad K^{s}_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \to H^{-\frac{1}{2}}(\partial \Omega)^3. \tag{4.22}
\]
Newtonian and single layer potentials for the Stokes system

Proof. The continuity of operators (4.21) and (4.22) follows from the well-posedness of the transmission problem (4.15) and Definition 4.6.

The next result yields the jump relations of the single layer potential and its conormal derivative across $\partial \Omega$ (see also [46, Proposition 5.3] for $\mu = 1$).

Lemma 4.8. Let $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3$. Then almost everywhere on $\partial \Omega$,

\[
[\gamma V_{\mu;\partial \Omega} \varphi] = 0, \tag{4.23}
\]

\[
[t^\mu_{\nu}(V_{\mu;\partial \Omega} \varphi, Q^s_{\mu;\partial \Omega} \varphi)] = \varphi, \quad t^\pm_{\nu}(V_{\mu;\partial \Omega} \varphi, Q^s_{\mu;\partial \Omega} \varphi) = \pm \frac{1}{2} \varphi + K^*_{\mu;\partial \Omega} \varphi. \tag{4.24}
\]

Proof. Formulas (4.23) and (4.24) follow from Definition 4.6 and the transmission condition in (4.16), as well as the transmission condition in the third line of (4.15).

Let $\mathbb{R} \nu = \{c \nu : c \in \mathbb{R}\}$. Let $\text{Ker} \{T : X \rightarrow Y\} := \{x \in X : T(x) = 0\}$ denote the null space of the map $T : X \rightarrow Y$. We next obtain the main properties of the single layer potential operator (cf., e.g., [43, Theorem 10.5.3], and [8, Proposition 3.3(c)]) and [46, Proposition 5.4] for $\mu = 1$ and $\alpha \in [0, \infty)$).

Lemma 4.9. The following properties hold

\[
V_{\mu;\partial \Omega} \nu = 0 \text{ in } \mathbb{R}^3, \quad Q^s_{\mu;\partial \Omega} \nu = -\chi_{\Omega+}, \tag{4.25}
\]

\[
\text{Ker} \{V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial \Omega)^3\} = \mathbb{R} \nu, \tag{4.26}
\]

\[
V_{\mu;\partial \Omega} \varphi \in H^{\frac{3}{2}}(\partial \Omega)^3, \quad \forall \varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3, \tag{4.27}
\]

where $\chi_{\Omega+} = 1$ in $\Omega_+$, $\chi_{\Omega+} = 0$ in $\Omega_-$, and

\[
H^{\frac{3}{2}}(\partial \Omega)^3 := \{\phi \in H^{\frac{3}{2}}(\partial \Omega)^3 : \langle \nu, \phi \rangle_{\partial \Omega} = 0\}. \tag{4.28}
\]

Proof. First, we consider the transmission problem (4.15) with the datum $\varphi = \nu \in H^{-\frac{1}{2}}(\partial \Omega)^3$. Then the solution of this problem is given by

\[
(u_{\nu}, \pi_\nu) = (0, -\chi_{\Omega+}) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3). \tag{4.29}
\]

Indeed, the pair $(u_{\nu}, \pi_\nu)$ satisfies the equations and the transmission condition in (4.15), as well as the transmission condition (4.16), and, in view of formula (3.11) and the divergence theorem,

\[
\langle [t_{\mu}(u_{\nu}, \pi_\nu)], \gamma \nu \rangle_{\partial \Omega} = -\langle \pi_\nu, \text{div } v \rangle_{\mathbb{R}^3} = \langle \nu, \gamma \nu \rangle_{\partial \Omega}, \quad \forall v \in D(\mathbb{R}^3)^3. \tag{4.30}
\]

Then by formula (2.3), Lemma 2.1, the dense embedding of the space $D(\mathbb{R}^3)^3$ in $H^1(\mathbb{R}^3)^3$, and the above equality, we obtain that $\langle [t_{\mu}(u_{\nu}, \pi_\nu)], \Phi \rangle_{\partial \Omega} = \langle \nu, \Phi \rangle_{\partial \Omega}$ for any $\Phi \in H^{\frac{1}{2}}(\partial \Omega)^3$. Hence, $[t_{\mu}(u_{\nu}, \pi_\nu)] = \nu$, as asserted. Then Definition 4.6 implies relations (4.25). Moreover, $V_{\mu;\partial \Omega} \nu = 0$, i.e., $\mathbb{R} \nu \subseteq \text{Ker} \{V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial \Omega)^3\}$.

Now let $\varphi_0 \in \text{Ker} \{V_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3 \rightarrow H^{\frac{1}{2}}(\partial \Omega)^3\}$. Let $(u_{\varphi_0}, \pi_{\varphi_0}) = (V_{\mu;\partial \Omega} \varphi_0, Q^s_{\mu;\partial \Omega} \varphi_0) \in H^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ be the unique solution of problem (4.15) with datum $\varphi_0$. Since $\gamma u_{\varphi_0} = 0$ on $\partial \Omega$, formula (3.11) yields that

\[
0 = \langle [t_{\mu}(u_{\varphi_0}, \pi_{\varphi_0})], \gamma u_{\varphi_0} \rangle_{\partial \Omega} = a_{\mu}(u_{\varphi_0}, u_{\varphi_0}). \tag{4.31}
\]
and hence $u_{\varphi_0} = 0$, $\pi_{\varphi_0} = c\chi_{\Omega_1}$ in $\mathbb{R}^3$, where $c \in \mathbb{R}$. In view of formula (3.11),

$$\langle [t_\mu(u_{\varphi_0}, \pi_{\varphi_0})], \gamma w \rangle_{\partial \Omega} = -\langle \pi_{\varphi_0}, \text{div} \ w \rangle_{\mathbb{R}^3} = -c\langle \nu, \gamma w \rangle_{\partial \Omega}, \forall w \in \mathcal{D}(\mathbb{R}^3)^3,$$

and, thus, $\varphi_0 = [t_\mu(u_{\varphi_0}, \pi_{\varphi_0})] = -c\nu$. Hence, formula (4.26) follows.

Now let $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3$. By using the first formula in (4.20), we obtain for any $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3$ that

$$\langle \mathcal{V}_{\mu;\partial \Omega}(\varphi), \mu \rangle_{\partial \Omega} = \langle \gamma u_{\varphi}, \nu \rangle_{\partial \Omega} = \langle \text{div} \ u_{\varphi}, 1 \rangle_{\Omega} = 0,$$

where $u_{\varphi} = V_{\mu;\partial \Omega} \varphi$. Thus, we get relation (4.27). $\square$

Next we use the notation $[\cdot]$ for the equivalence classes of the space $H^{-\frac{1}{2}}(\partial \Omega, \Lambda^1 \mathcal{T}M)/\mathbb{R} \nu$. Thus, any $[\varphi] \in H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu$ can be written as $[\varphi] = \varphi + \mathbb{R} \nu$, where $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3$.

Exploiting properties (4.26) and (4.27), we now show the following invertibility result (cf. [43, Theorem 10.5.3], [8, Proposition 3.3(d)], [46, Proposition 5.5] for $\mu = 1$ and $\alpha \geq 0$ constant).

**Lemma 4.10.** The following operator is an isomorphism

$$\mathcal{V}_{\mu;\partial \Omega} : H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu \rightarrow H^{\frac{3}{2}}(\partial \Omega)^3. \quad (4.32)$$

**Proof.** We use arguments similar to those for Proposition 5.5 in [46]. First, Lemma 4.7 and the membership relation (4.27) imply that the linear operator in (4.32) is continuous. We show that this operator is also $H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu$-elliptic i.e., that there exists a constant $c = c(\partial \Omega)$ such that

$$\langle \mathcal{V}_{\mu;\partial \Omega} [\varphi], [\varphi] \rangle_{\partial \Omega} \geq c \| [\varphi] \|^2_{H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu}, \forall [\varphi] \in H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu. \quad (4.33)$$

Let $[\varphi] \in H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu$. Thus, $[\varphi] = \varphi + \mathbb{R} \nu$, where $\varphi \in H^{-\frac{1}{2}}(\partial \Omega)^3$. In view of formula (3.11), Definition 4.6, relations (4.26), (4.27), and inequality (4.8),

$$\langle \mathcal{V}_{\mu;\partial \Omega}([\varphi]), [\varphi] \rangle_{\partial \Omega} = \langle \mathcal{V}_{\mu;\partial \Omega}(\varphi), \varphi \rangle_{\partial \Omega} = \langle \gamma u_{\varphi}, [t_\mu(u_{\varphi}, \pi_{\varphi})] \rangle_{\partial \Omega}$$

$$= a_\mu(u_{\varphi}, u_{\varphi}) \geq c_\mu^{-1} \| u_{\varphi} \|^2_{H^1(\mathbb{R}^3)^3}, \quad (4.34)$$

where $u_{\varphi} = V_{\mu;\partial \Omega} \varphi$ and $\pi_{\varphi} = Q^s_{\mu;\partial \Omega} \varphi$. Now we use the property that the trace operator

$$\gamma : H^1_{\text{div}}(\mathbb{R}^3)^3 \rightarrow H^{\frac{3}{2}}(\partial \Omega)^3 \quad (4.35)$$

is surjective having a bounded right inverse $\gamma^{-1} : H^{\frac{3}{2}}(\partial \Omega)^3 \rightarrow H^1_{\text{div}}(\mathbb{R}^3)^3$ (cf., e.g., [46, Proposition 4.4]). Hence, for any $\Phi \in H^{\frac{3}{2}}(\partial \Omega)^3$, we have that $w \equiv \gamma^{-1} \Phi \in H^1_{\text{div}}(\mathbb{R}^3)^3$. Then there exists $c' \equiv c'(\partial \Omega) \in (0, \infty)$ such that

$$\| [\varphi], \Phi \|_{\partial \Omega} = ||\varphi, \Phi\|_{\partial \Omega} = ||(t_\mu(u_{\varphi}, \pi_{\varphi}), \gamma w)\|_{\partial \Omega} = |a_\mu(u_{\varphi}, w)| \quad (4.36)$$

$$\leq 2c_\mu \| u_{\varphi} \|_{H^1(\mathbb{R}^3)^3} \| \gamma^{-1} \Phi \|_{H^1(\mathbb{R}^3)^3} \leq 2c_\mu c' \| u_{\varphi} \|_{H^1(\mathbb{R}^3)^3} \| \Phi \|_{H^{\frac{3}{2}}(\partial \Omega)^3},$$

where the first equality in (4.36) follows from the relation $[\varphi] = \varphi + \mathbb{R} \nu$ and the membership of $\Phi$ in $H^{\frac{3}{2}}(\partial \Omega)^3$, the second equality follows from Definition 4.6, and the third equality is a consequence of formula (3.11). Since the space $H^{\frac{3}{2}}(\partial \Omega)^3$ is the dual of the space $H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu$, formula (4.36) yields that

$$\| [\varphi] \|_{H^{-\frac{1}{2}}(\partial \Omega)^3/\mathbb{R} \nu} \leq 2c_\mu c' \| u_{\varphi} \|_{H^1(\mathbb{R}^3)^3}. \quad (4.37)$$
Then by (4.34) and (4.37) we obtain inequality (4.33), and the Lax-Milgram lemma yields that operator (4.32) is an isomorphism. □

Remark 4.11. The fundamental solution of the constant-coefficient Stokes system in $\mathbb{R}^3$ is well known and leads to the construction of Newtonian and boundary layer potentials via the integral approach (see, e.g., [17, 36, 43, 48]). In view of Theorems 4.2 and 4.5, the Newtonian and single layer potentials provided by the variational approach (in the case $\mu = 1$) coincide with classical ones expressed in terms of the fundamental solution, since they satisfy the same boundary value problems (4.11) and (4.15), respectively (see also [46, Proposition 5.1] for $\mu = 1$). The assumption $\mu = 1$ is a particular case of a more general case of $L^\infty$ coefficients analyzed in this paper. We also note that an alternative approach, reducing various boundary value problems for variable-coefficient elliptic partial differential equations to boundary-domain integral equations, by employing the explicit parametrix-based integral potentials, was explored in, e.g., [12, 13, 14].

5. Exterior Dirichlet problem for the Stokes system with $L^\infty$ coefficients

In this section we analyze the exterior Dirichlet problem for the Stokes system with $L^\infty$ coefficients

\[
\begin{aligned}
\begin{cases}
\operatorname{div} (2\mu \mathcal{E}(u)) - \nabla \pi = f & \text{in } \Omega_-, \\
\operatorname{div} u = 0 & \text{in } \Omega_-, \\
\gamma_- u = \phi & \text{on } \partial \Omega, 
\end{cases}
\end{aligned}
\tag{5.1}
\]

with given data $(f, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{1/2}(\partial \Omega)^3$.

5.1. Variational approach

First, we use a variational approach and show that problem (5.1) has a unique solution $(u, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ (see also [26, Theorem 3.4] and [3, Theorem 3.16] for the constant-coefficient Stokes system).

**Theorem 5.1.** Assume that $\mu \in L^\infty(\Omega_-)$ satisfies conditions (3.1). Then for all given data $(f, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{1/2}(\partial \Omega)^3$ the exterior Dirichlet problem for the $L^\infty$ coefficient Stokes system (5.1) is well posed. Hence problem (5.1) has a unique solution $(u, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ and there exists a constant $C \equiv C(\partial \Omega; \mu) > 0$ such that

\[
\|u\|_{\mathcal{H}^1(\Omega_-)^3} + \|\pi\|_{L^2(\Omega_-)} \leq C \left(\|f\|_{\mathcal{H}^{-1}(\Omega_-)^3} + \|\phi\|_{H^{1/2}(\partial \Omega)^3}\right). \tag{5.2}
\]

**Proof.** First, we note that the density of the space $\mathcal{D}(\Omega_-)^3$ in $\mathcal{H}^1(\Omega_-)^3$ implies that the exterior Dirichlet problem (5.1) has the following equivalent variational formulation: Find $(u, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ such that

\[
\begin{aligned}
\begin{cases}
2\langle \mu \mathcal{E}(u), \mathcal{E}(\tilde{v}) \rangle_{\Omega_-} - \langle \pi, \operatorname{div} \tilde{v} \rangle_{\Omega_-} = -\langle f, \tilde{v} \rangle_{\Omega_-}, & \forall \tilde{v} \in \mathcal{H}^1(\Omega_-)^3, \\
\langle \operatorname{div} u, q \rangle_{\Omega_-} = 0, & \forall q \in L^2(\Omega_-), \\
\gamma_- u = \phi & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\tag{5.3}
\]
Next, we consider \( u_0 \in \mathcal{H}^1(\Omega_-)^3 \) such that
\[
\begin{aligned}
\text{div} \, u_0 &= 0 \quad \text{in } \Omega_-,
\gamma_- u_0 &= \phi \quad \text{on } \partial \Omega.
\end{aligned}
\tag{5.4}
\]

Particularly, we can choose \( u_0 \) as the solution of the Dirichlet problem for a constant-coefficient Brinkman system
\[
\begin{aligned}
(\Delta - \alpha \mathbb{I}) u_0 - \nabla \pi_0 &= 0, \quad \text{div} \, u_0 = 0 \quad \text{in } \Omega_-,
\gamma_- u_0 &= \phi \quad \text{on } \partial \Omega,
\end{aligned}
\tag{5.5}
\]

where \( \alpha > 0 \) is an arbitrary constant. The solution is given by the corresponding double layer potential
\[
\begin{aligned}
u_0 = W_{\alpha;\partial \Omega} \left( \frac{1}{2} \mathbb{I} + K_{\alpha;\partial \Omega} \right)^{-1} \phi,
\end{aligned}
\tag{5.6}
\]

where \( K_{\alpha;\partial \Omega} : H^{\frac{2}{3}}(\partial \Omega)^3 \to H^{\frac{2}{3}}(\partial \Omega)^3 \) is the corresponding Brinkman double-layer boundary potential operator. Note that

\[
\begin{aligned}
(W_{\alpha} h)_j(x) := \int_{\partial \Omega} S_{ij\ell}^\alpha(x, y) \nu_\ell(y) h_i(y) \, d\sigma_y.
\end{aligned}
\tag{5.7}
\]

The explicit form of the kernel \( S_{ij\ell}^\alpha(x, y) \) can be found in [48, (2.14)-(2.18)] and [36, Section 3.2.1].

In addition, the operator \( \frac{1}{2} \mathbb{I} + K_{\alpha;\partial \Omega} : H^{\frac{2}{3}}(\partial \Omega)^3 \to H^{\frac{2}{3}}(\partial \Omega)^3 \) is an isomorphism, and \( u_0 \) belongs to the space \( H^1(\Omega_-)^3 \) and satisfies (5.5), and hence (5.4). Moreover, the embedding \( H^1(\Omega_-)^3 \subset H^1(\Omega_-)^3 \) shows that \( u_0 \) belongs also to the space \( \mathcal{H}^1(\Omega_-)^3 \) (see also [26, Lemma 3.2, Remark 3.3]).

Then with the new variable \( \hat{u} := u - u_0 \in \mathcal{H}^1(\Omega_-)^3 \), the variational problem (5.3) reduces to the following mixed variational formulation (c.f. Problem (Q) in p. 324 of [26] for the constant-coefficient Stokes system): Find \( (\hat{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-) \) such that
\[
\begin{aligned}
\begin{cases}
 a_{\mu;\Omega_-}(\hat{u}, v) + b_{\Omega_-}(v, \pi) = \mathfrak{f}_{\mu;\Omega_0}(v), & \forall \, v \in \mathcal{H}^1(\Omega_-)^3, \\
b_{\Omega_-}(\hat{u}, q) &= 0, \quad \forall \, q \in L^2(\Omega_-),
\end{cases}
\end{aligned}
\tag{5.8}
\]

where \( a_{\mu;\Omega_-} : \mathcal{H}^1(\Omega_-)^3 \times \mathcal{H}^1(\Omega_-)^3 \to \mathbb{R} \) and \( b_{\Omega_-} : \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R} \) are the bilinear forms given by
\[
\begin{aligned}
a_{\mu;\Omega_-}(w, v) &= 2\langle \mu \mathbb{E}(w), \mathbb{E}(v) \rangle_{\Omega_-}, \quad \forall \, v, w \in \mathcal{H}^1(\Omega_-)^3, \\
b_{\Omega_-}(v, q) &= -\langle \text{div} \, v, q \rangle_{\Omega_-}, \quad \forall \, v \in \mathcal{H}^1(\Omega_-)^3, q \in L^2(\Omega_-),
\end{aligned}
\tag{5.9}
\]

and \( \mathfrak{f}_{\mu;\Omega_0} : \mathcal{H}^1(\Omega_-)^3 \to \mathbb{R} \) is the linear form given by
\[
\begin{aligned}
\mathfrak{f}_{\mu;\Omega_0}(v) := -\left( (f, \hat{E}v)_{\Omega_-} + 2\langle \mu \mathbb{E}(u_0), \mathbb{E}(v) \rangle_{\Omega_-} \right), \quad \forall \, v \in \mathcal{H}^1(\Omega_-)^3.
\end{aligned}
\tag{5.11}
\]

Here we took into account that the spaces \( \mathcal{H}^1(\Omega_-)^3 \) and \( \mathcal{H}^1(\Omega_-)^3 \) can be identified through the isomorphism \( \hat{E}_- : \mathcal{H}^1(\Omega_-)^3 \to \mathcal{H}^1(\Omega_-)^3 \). Note that
\[
\hat{E}_- \mathcal{H}^1(\Omega_-)^3 := \left\{ v \in \mathcal{H}^1(\Omega_-)^3 : \text{div} \, v = 0 \text{ in } \Omega_- \right\}
\]
Moreover, the formula \( (5.12) \), inequality \( (3.1) \) and the Hölder inequality yield that
\[
|a_{\mu;\Omega_-}(v_1, v_2)| \leq 2c_\mu \|E(v_1)\|_{L^2(\Omega_-)^{3\times3}} \|E(v_2)\|_{L^2(\Omega_-)^{3\times3}}
\[
\leq 2c_\mu \|v_1\|_{\mathcal{H}^1(\Omega_-)^3} \|v\|_{\mathcal{H}^1(\Omega_-)^3}, \quad \forall \ v_1, v_2 \in \mathcal{H}^1(\Omega_-)^3. \tag{5.13}
\]
Moreover, the formula
\[
2\|\mathbf{v}\|_{L^2(\Omega_-)^{3\times3}}^2 = \|\mathbf{v}\|_{L^2(\Omega_-)^{3\times3}}^2 + \|\mathbf{v}\|_{L^2(\Omega_-)^{3\times3}}^2, \quad \forall \ \mathbf{v} \in \mathcal{D}(\Omega_-)^3 \tag{5.14}
\]
(cf., e.g., the proof of Corollary 2.2 in [46]), and the density of the space \( \mathcal{D}(\Omega_-)^3 \) in \( \mathcal{H}^1(\Omega_-)^3 \) show that the same formula holds also for any function in \( \mathcal{H}^1(\Omega_-)^3 \). Therefore, we obtain the following Korn type inequality
\[
\|\text{grad} \ \mathbf{v}\|_{L^2(\Omega_-)^{3\times3}} \leq 2^{\frac{1}{2}} \|E(\mathbf{v})\|_{L^2(\Omega_-)^{3\times3}}, \quad \forall \ \mathbf{v} \in \mathcal{H}^1(\Omega_-)^3. \tag{5.15}
\]
Then by using inequality \( (5.15) \), the equivalence of seminorm \( (2.11) \) to the norm \( (2.11) \) in the space \( \mathcal{H}^1(\Omega_-)^3 \), and assumption \( (3.1) \) we deduce that there exists a constant \( C = C(\Omega_-) > 0 \) such that
\[
\|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3} \leq C \|\text{grad} \ \mathbf{u}\|^2_{L^2(\Omega_-)^{3\times3}} \leq C \|E(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times3}} \leq 2Cc_\mu \|E(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times3}} = 2Cc_\mu a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}), \quad \forall \ \mathbf{u} \in \mathcal{H}^1(\Omega_-)^3,
\]
and accordingly that
\[
a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}) \geq \frac{1}{2Cc_\mu} \|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3}, \quad \forall \ \mathbf{u} \in \mathcal{H}^1(\Omega_-)^3. \tag{5.16}
\]
In view of inequalities \( (5.13) \) and \( (5.16) \) it follows that the bilinear form \( a_{\mu;\Omega_-}(\cdot, \cdot) : \mathcal{H}^1(\Omega_-)^3 \times \mathcal{H}^1(\Omega_-)^3 \to \mathbb{R} \) is bounded and coercive. Moreover, arguments similar to those for inequality \( (5.13) \) imply that the bilinear form \( b_{\Omega_-}(\cdot, \cdot) : \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R} \) and the linear form \( \mathbf{f}_{\mu;\mathbf{u}_0} : \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R} \) given by \( (5.10) \) and \( (5.11) \), are also bounded. Since the operator
\[
\text{div} : \mathcal{H}^1(\Omega_-)^3 \to L^2(\Omega_-) \tag{5.17}
\]
is surjective (cf., e.g., [26, Theorem 3.2]), then by Lemma A.2, the bounded bilinear form \( b_{\Omega_-}(\cdot, \cdot) : \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R} \) satisfies the inf-sup condition
\[
\inf_{q \in L^2(\Omega_-)^3 \setminus \{0\}} \sup_{v \in \mathcal{H}^1(\Omega_-)^3 \setminus \{0\}} \frac{b_{\Omega_-}(v, q)}{\|v\|_{\mathcal{H}^1(\Omega_-)^3} \|q\|_{L^2(\Omega_-)}} \geq \beta_D \tag{5.18}
\]
with some constant \( \beta_D > 0 \) (cf. [26, Theorem 3.3]). Then Theorem A.4 (with \( X = \mathcal{H}^1(\Omega_-)^3 \) and \( M = L^2(\Omega_-) \)) implies that the variational problem \( (5.8) \) has a unique solution \((\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)\). Moreover, the pair \((\mathbf{u}, \pi) = (\mathbf{u} + \mathbf{u}_0, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)\), where \( \mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3 \) satisfies relations \( (5.4) \), is the unique solution of the mixed variational formulation \( (5.3) \) and depends continuously on the given data \( (\mathbf{f}, \phi) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^1(\partial\Omega)^3 \). The equivalence between the variational problem \( (5.3) \) and the exterior Dirichlet problem \( (5.1) \) shows that problem \( (5.1) \) is also well-posed, as asserted. \( \square \)
5.2. Potential approach

Theorem 5.1 asserts the well-posedness of the exterior Dirichlet problem for the Stokes system with $L^\infty$ coefficients. However, if the given data $(f, \phi)$ belong to the space $H^{-1}(\Omega_-)^3 \times H^2_0(\partial \Omega)^3$, then the solution can be expressed in terms of the Newtonian and single layer potential and of the inverse of the single layer operator as follows (cf. [26, Theorem 3.4] for $\mu > 0$ constant, [22, Theorem 10.1] and [37, Theorem 5.1] for the Laplace operator).

Theorem 5.2. If $f \in H^{-1}(\Omega_-)^3$ and $\phi \in H^2_0(\partial \Omega)^3$ then the exterior Dirichlet problem (5.1) has a unique solution $(u, \pi) \in H^1(\Omega_-)^3 \times L^2(\Omega_-)$, given by

$$
\begin{align*}
\text{u} &= \mathcal{N}_{\mu;\mathbb{R}^3}(\tilde{f})|_{\Omega_-} + V_{\mu;\partial \Omega} \left( V_{\mu;\partial \Omega}^{-1} \left( \phi - \gamma_-(\mathcal{N}_{\mu;\mathbb{R}^3}(\tilde{f})) \right) \right), \\
\text{\pi} &= Q_{\mu;\mathbb{R}^3}(\tilde{f})|_{\Omega_-} + Q_{\mu;\partial \Omega} \left( V_{\mu;\partial \Omega}^{-1} \left( \phi - \gamma_-(\mathcal{N}_{\mu;\mathbb{R}^3}(\tilde{f})) \right) \right) \text{ in } \Omega_-,
\end{align*}
$$

where $\tilde{f}$ is an extension of $f$ to an element of $H^1(\mathbb{R}^3)^3$.

Proof. The result follows from Definition 4.3 and Lemmas 4.7, 4.8, 4.10. □

Appendix A. Mixed variational formulations and their well-posedness property

Here we make a brief review of well-posedness results due to Babuška [6] and Brezzi [10] for mixed variational formulations related to bounded bilinear forms in reflexive Banach spaces. We follow [20, Section 2.4], [11], [25, §4].

Let $X$ and $\mathcal{M}$ be reflexive Banach spaces, and let $X^*$ and $\mathcal{M}^*$ be their dual spaces. Let $a(\cdot, \cdot) : X \times X \to \mathbb{R}$, $b(\cdot, \cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Then we consider the following abstract mixed variational formulation.

For $f \in X^*$, $g \in \mathcal{M}^*$ given, find a pair $(u, p) \in X \times \mathcal{M}$ such that

$$
\begin{align*}
\begin{cases}
a(u, v) + b(v, p) &= f(v), \quad \forall \ v \in X, \\
b(u, q) &= g(q), \quad \forall \ q \in \mathcal{M}.
\end{cases}
\end{align*}
$$

(A.1)

Let $A : X \to X^*$ be the bounded linear operator defined by

$$
\langle Av, w \rangle = a(v, w), \quad \forall \ v, w \in X,
$$

(A.2)

where $\langle \cdot, \cdot \rangle := X^* \langle \cdot, \cdot \rangle_X$ is the duality pairing of the dual spaces $X^*$ and $X$. We also use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing $\mathcal{M}^* \langle \cdot, \cdot \rangle_{\mathcal{M}}$. Let $B : X \to \mathcal{M}^*$ and $B^* : \mathcal{M} \to X^*$ be the bounded linear and transpose operators given by

$$
\langle Bv, q \rangle = b(v, q), \quad \langle v, B^* q \rangle = \langle Bv, q \rangle, \quad \forall \ v \in X, \ \forall \ q \in \mathcal{M}.
$$

(A.3)

In addition, we consider the spaces

$$
V := \text{Ker } B = \{ v \in X : b(v, q) = 0, \ \forall \ q \in \mathcal{M} \},
$$

(A.4)

$$
V^\perp := \{ T \in X^* : \langle T, v \rangle = 0, \ \forall \ v \in V \}.
$$

(A.5)

Then the following well-posedness result holds (cf., e.g., [20, Theorem 2.34]).
Theorem A.1. Let $X$ and $M$ be reflexive Banach spaces, $f \in X^*$ and $g \in M^*$, and $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : X \times M \rightarrow \mathbb{R}$ be bounded bilinear forms. Let $V$ be the subspace of $X$ defined by (A.4). Then the variational problem (A.1) is well-posed if and only if $a(\cdot, \cdot)$ satisfies the conditions
\[
\exists \lambda > 0 \text{ such that } \inf_{u \in V \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(u, v)}{\|v\|_X \|v\|_X} \geq \lambda, \quad (A.6)
\]
and $b(\cdot, \cdot)$ satisfies the inf-sup (Ladyzhenskaya-Babuška-Brezzi) condition,
\[
\exists \beta > 0 \text{ such that } \inf_{q \in M \setminus \{0\}} \sup_{v \in X \setminus \{0\}} \frac{b(v, q)}{\|v\|_X \|q\|_M} \geq \beta. \quad (A.7)
\]
Moreover, there exists a constant $C$ depending on $\beta, \lambda$ and the norm of $a(\cdot, \cdot)$, such that the unique solution $(u, p) \in X \times M$ of (A.1) satisfies the inequality
\[
\|u\|_X + \|p\|_M \leq C (\|f\|_{X^*} + \|g\|_{M^*}). \quad (A.8)
\]
In addition, we have (see [20, Theorem A.56, Remark 2.7], [4, Theorem 2.7]).

Lemma A.2. Let $X, M$ be reflexive Banach spaces. Let $b(\cdot, \cdot) : X \times M \rightarrow \mathbb{R}$ be a bounded bilinear form. Let $B : X \rightarrow M^*$ and $B^* : M \rightarrow X^*$ be the operators defined by (A.3), and let $V = \text{Ker} B$. Then the following results are equivalent:

(i) There exists a constant $\beta > 0$ such that $b(\cdot, \cdot)$ satisfies condition (A.7).

(ii) $B : X/V \rightarrow M^*$ is an isomorphism and $\|Bw\|_{M^*} \geq \beta \|w\|_{X/V}$ for any $w \in X/V$.

(iii) $B^* : M \rightarrow V^\perp$ is an isomorphism and $\|B^*q\|_{X^*} \geq \beta \|q\|_M$ for any $q \in M$.

Remark A.3. Let $X$ be a reflexive Banach space and $V$ be a closed subspace of $X$. If a bounded bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is coercive on $V$, i.e., there exists a constant $c_a > 0$ such that
\[
a(w, w) \geq c_a \|w\|_X^2, \quad \forall w \in V, \quad (A.9)
\]
then the conditions (A.6) are satisfied as well (see, e.g., [20, Lemma 2.8]).

The next result known as the Babuška-Brezzi theorem is the version of Theorem A.1 for Hilbert spaces (see [6], [10, Theorems 0.1, 1.1, Corollary 1.2]).

Theorem A.4. Let $X$ and $M$ be two real Hilbert spaces. Let $a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ and $b(\cdot, \cdot) : X \times M \rightarrow \mathbb{R}$ be bounded bilinear forms. Let $f \in X^*$ and $g \in M^*$. Let $V$ be the subspace of $X$ defined by (A.4). Assume that $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is coercive and that $b(\cdot, \cdot) : X \times M \rightarrow \mathbb{R}$ satisfies the inf-sup condition (A.7). Then the variational problem (A.1) is well-posed.

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