Hyers-Ulam stability of quadratic forms in 2-normed spaces

Abstract: In this paper, we obtain Hyers-Ulam stability of the functional equations
\[ f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w), \]
\[ f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w) \]
and
\[ f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w) \]
in 2-Banach spaces. The quadratic forms \( ax^2 + bxy + cy^2 \), \( ax^2 + by^2 \) and \( axy \) are solutions of the above functional equations, respectively.

Keywords: linear 2-normed space, quadratic form, Hyers-Ulam stability

MSC: 39B52, 39B72

1 Introduction

We introduce some definitions on 2-Banach spaces [1], [2].

Definition 1.1. Let \( X \) be a real linear space with \( \dim X \geq 2 \) and \( \|\cdot, \cdot\| : X^2 \to \mathbb{R} \) be a function. Then \( (X, \|\cdot, \cdot\|) \) is called a linear 2-normed space if the following conditions hold:
(a) \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent,
(b) \( \|x, y\| = \|y, x\| \),
(c) \( \|ax, y\| = |a|\|x, y\| \),
(d) \( \|x, y + z\| \leq \|x, y\| + \|x, z\| \)
for all \( a \in \mathbb{R} \) and \( x, y, z \in X \). In this case, the function \( \|\cdot, \cdot\| \) is called a 2-norm on \( X \).

Definition 1.2. Let \( \{x_n\} \) be a sequence in a linear 2-normed space \( X \). The sequence \( \{x_n\} \) is said to be convergent in \( X \) if there exists an element \( x \in X \) such that
\[ \lim_{n \to \infty} \|x_n - x, y\| = 0 \]
for all \( y \in X \). In this case, we say that a sequence \( \{x_n\} \) converges to the limit \( x \), simply denoted by \( \lim_{n \to \infty} x_n = x \).

Definition 1.3. A sequence \( \{x_n\} \) in a linear 2-normed space \( X \) is called a Cauchy sequence if for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( m, n \geq N \), \( \|x_m - x_n, y\| < \varepsilon \) for all \( y \in X \). For convenience, we will write

Won-Gil Park: Department of Mathematics Education, College of Education, Mokwon University, Daejeon 35349, Republic of Korea; E-mail: wgpark@mokwon.ac.kr

*Corresponding Author: Jae-Hyeong Bae: Humanitas College, Kyung Hee University, Yongin 17104, Republic of Korea; E-mail: jhbae@khu.ac.kr
lim_{n \to \infty} \|x_n - x_m, y\| = 0 for a Cauchy sequence \{x_n\}. A 2-Banach space is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space which will be used to prove the stability results.

**Lemma 1.4.** [3] Let \((X, \| \cdot \|)\) be a linear 2-normed space and \(x \in X\). Then the following properties hold:

(a) If \(\|x, y\| = 0\) for all \(y \in X\), then \(x = 0\).
(b) \(\|x, z\| - \|y, z\| \leq \|x - y, z\|\) for all \(x, y, z \in X\).
(c) If a sequence \(\{x_n\}\) is convergent in \(X\), then \(\lim_{n \to \infty} \|x_n, y\| = \|\lim_{n \to \infty} x_n, y\|\) for all \(y \in X\).

In 1940, Ulam [4] suggested the stability problem of functional equations concerning the stability of group homomorphisms:

Let a group \(G\) and a metric group \(H\) with the metric \(\rho\) be given. For each \(\varepsilon > 0\), the question is whether or not there is a \(\delta > 0\) such that if \(f : G \to H\) satisfies \(\rho(f(xy), f(x)f(y)) < \delta\) for all \(x, y \in G\), then there exists a group homomorphism \(h : G \to H\) satisfying \(\rho(f(x), h(x)) < \varepsilon\) for all \(x \in G\).

The case of approximately additive mappings was solved by Hyers [5] under the assumption that \(G\) and \(H\) are Banach spaces. In 1978, Rassias [6] generalized the result of Hyers as follows: Let \(f : G \to H\) be a mapping between Banach spaces and let \(0 < p < 1\) be fixed. If \(f\) satisfies the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]

for some \(\theta \geq 0\) and for all \(x, y \in G\), then there exists a unique additive mapping \(A : G \to H\) such that

\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2 - \theta^p} \|x\|^p
\]

for all \(x \in G\).

In Banach spaces, Bae and Park [7–9] investigated the stability problem of some functional equations:

\[
\begin{align*}
  f(x + y, z + w) + f(x - y, z - w) &= 2f(x, z) + 2f(y, w), \quad (1) \\
  f(x + y, z - w) + f(x - y, z + w) &= 2f(x, z) + 2f(y, w), \quad (2)
\end{align*}
\]

and

\[
  f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w). \quad (3)
\]

The quadratic forms \(f_1, f_2, f_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) given by \(f_1(x, y) := ax^2 + bxy + cy^2\), \(f_2(x, y) := ax^2 + by^2\) and \(f_3(x, y) := axy\) are solutions of (1), (2) and (3), respectively.

In 2011, Park [10] investigated approximate additive, Jensen and quadratic mappings in 2-Banach spaces. In this paper, we also investigate the stability of the functional equations (1), (2) and (3) in 2-Banach spaces with different assumptions from [10].

### 2 Results

Throughout this paper, let \(X\) be a normed space and \(Y\) a 2-Banach space.

**Theorem 2.1.** Let \(p \in (0, 2), \varepsilon > 0, \delta, \eta \geq 0\) and let \(f : X \times X \to Y\) be a surjective mapping such that

\[
\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)^p + \eta(\|u\| + \|v\|) \quad (4)
\]

for all \(x, y, z, w, u, v \in X\). Then there exists a unique mapping \(F : X \times X \to Y\) satisfying (1) such that

\[
\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{\varepsilon}{3} + \frac{2\delta}{4 - 2p}(\|x\|^p + \|y\|^p) + \frac{\eta}{3}(\|u\| + \|v\|) \quad (5)
\]

for all \(x, y, u, v \in X\).
Proof. Letting $y = x$ and $w = z$ in (4), we have
\[
\left\| f(x, z) + \frac{1}{4} [f(0, 0) - f(2x, 2z)], f(u, v) \right\| \leq \frac{\varepsilon}{4} + \frac{1}{2} \delta(\|x\|^p + \|z\|^p) + \frac{\eta}{4}(\|u\| + \|v\|)
\]
for all $x, z, u, v \in X$. Thus we obtain
\[
\left\| \frac{1}{4j} f(2^j x, 2^j z) - \frac{1}{4j+1} f(0, 0) + f(2^{j+1} x, 2^{j+1} z), f(u, v) \right\| \leq \frac{\varepsilon}{4j+1} + \frac{2j}{j+1} \delta(\|x\|^p + \|z\|^p) + 2^{-2(j+1)} \eta(\|u\| + \|v\|)
\]
for all $x, z, u, v \in X$ and all $j$. Replacing $z$ by $y$ in the above inequality, we see that
\[
\left\| \frac{1}{4j} f(2^j x, 2^j y) - \frac{1}{4j+1} f(0, 0) + f(2^{j+1} x, 2^{j+1} y), f(u, v) \right\| \leq \frac{\varepsilon}{4j+1} + 2^{j-1} \delta(\|x\|^p + \|y\|^p) + 2^{-2(j+1)} \eta(\|u\| + \|v\|)
\]
for all $x, y, u, v \in X$ and all $j$. For given integers $l, m(0 \leq l < m)$, we get
\[
\left\| \frac{1}{4j} f(2^j x, 2^j y) - \frac{1}{4m} f(0, 0) + f(2^m x, 2^m y), f(u, v) \right\| \leq \sum_{j=l}^{m-1} \left[ \frac{\varepsilon}{4j+1} + 2^{j-1} \delta(\|x\|^p + \|y\|^p) + 2^{-2(j+1)} \eta(\|u\| + \|v\|) \right]
\]
(6)
for all $x, y, u, v \in X$. By (6), the sequence $(\frac{1}{4j} f(2^j x, 2^j y))$ is a Cauchy sequence for all $x, y \in X$. Since $Y$ is complete, the sequence $(\frac{1}{4j} f(2^j x, 2^j y))$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x, y) := \lim_{j \to \infty} \frac{1}{2} f(2^j x, 2^j y)$ for all $x, y \in X$. By (4), we have
\[
\left\| \frac{1}{4j} f(2^j (x + y), 2^j (z + w)) + \frac{1}{4j} f(2^j (x - y), 2^j (z - w)) - \frac{2}{4j} f(2^j x, 2^j z) - \frac{2}{4j} f(2^j y, 2^j w), f(u, v) \right\|
\leq \frac{1}{4j} \left( \varepsilon + 2j \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \eta(\|u\| + \|v\|)) \right)
\]
for all $x, y, z, w, u, v \in X$ and all $j$. Letting $j \to \infty$, we see that $F$ satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (6), one can obtain the inequality (5). If $G : X \times X \to Y$ is another mapping satisfying (1) and (5), we obtain
\[
\|F(x, y) - G(x, y), f(u, v)\| = \frac{1}{4n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\|
\leq \frac{1}{4n} \|f(2^n x, 2^n y) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\|
\leq \frac{2}{4n} \left( \varepsilon^p + \frac{2n^{p+1} \delta(\|x\|^p + \|y\|^p) + \eta(\|u\| + \|v\|)}{4 - 2p} \right)
\to 0 \text{ as } n \to \infty
\]
for all $x, y, u, v \in X$. Hence the mapping $F$ is the unique mapping satisfying (1), as desired. \(\square\)

Corollary 2.2. Let $f : X \times X \to Y$ be a surjective mapping such that
\[
\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon
\]
for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (1) such that
\[
\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{\varepsilon}{3}
\]
for all $x, y, u, v \in X$.

Proof. Taking $\delta = \eta = 0$ in Theorem 2.1, we have the desired result. \(\square\)

In the case $p > 2$ in Theorem 2.1, one can also obtain a similar result.
Theorem 2.3. Let \( p \in (0, 2) \), \( \varepsilon > 0 \) and \( \delta, \eta \geq 0 \), and let \( f : X \times X \rightarrow Y \) be a surjective mapping such that
\[
||f(x+y, z-w)+f(x-y, z+w)-2f(x, z)-2f(y, w), f(u, v)|| \leq \varepsilon + \delta(||x||^p + ||y||^p + ||z||^p + ||w||^p) + \eta(||u|| + ||v||)
\] (7)
for all \( x, y, z, w, u, v \in X \). Then there exists a unique mapping \( F : X \times X \rightarrow Y \) satisfying (2) such that
\[
||f(x, y) - F(x, y), f(u, v)|| \leq \frac{3}{2} \varepsilon + \frac{3}{4} \delta \left( \frac{17}{2} ||y||^p \right) + \frac{5}{3} \eta(||u|| + ||v||) + \frac{1}{3} ||f(0, 0)||
\] (8)
for all \( x, y, u, v \in X \).

Proof. Letting \( y = x \) and \( w = -z \) in (7), we obtain that
\[
||f(x, z) + f(x, -z) - \frac{1}{2} \left[ f(0, 0) + f(2x, 2z) \right], f(u, v)|| \leq \frac{1}{2} \varepsilon + \delta(||x||^p + ||z||^p) + \frac{\eta}{2}(||u|| + ||v||)
\] (9)
for all \( x, z, u, v \in X \). Putting \( x = 0 \) in (9), we get
\[
||f(0, z) + f(0, -z) - \frac{1}{2} \left[ f(0, 0) + f(0, 2z) \right], f(u, v)|| \leq \frac{1}{2} \varepsilon + \delta||z||^p + \frac{\eta}{2}(||u|| + ||v||)
\]
for all \( z, u, v \in X \). Replacing \( z \) by \(-z\) in the above inequality, we have
\[
||f(0, -z) + f(0, z) - \frac{1}{2} \left[ f(0, 0) + f(0, -2z) \right], f(u, v)|| \leq \frac{1}{2} \varepsilon + \delta||z||^p + \frac{\eta}{2}(||u|| + ||v||)
\]
for all \( z, u, v \in X \). By the above two inequalities, we see that
\[
||f(0, 2z) - f(0, -2z), f(u, v)|| \leq \varepsilon + \delta||z||^p + 2\eta(||u|| + ||v||)
\] (10)
for all \( z, u, v \in X \). Setting \( y = x \) and \( w = z \) in (7), we have
\[
||f(2x, 0) + f(0, 2z) - 4f(x, z), f(u, v)|| \leq \varepsilon + 2\delta(||x||^p + ||z||^p) + \eta(||u|| + ||v||)
\] (11)
for all \( x, z, u, v \in X \). Replacing \( z \) by \(-z\) in the above inequality, we see that
\[
||f(2x, 0) + f(0, -2z) - 4f(x, -z), f(u, v)|| \leq \varepsilon + 2\delta(||x||^p + ||z||^p) + \eta(||u|| + ||v||)
\] (12)
for all \( x, z, u, v \in X \). By (11) and (12), we know that
\[
||f(x, z) - f(x, -z) - \frac{1}{4} \left[ f(0, 2z) - f(0, -2z) \right], f(u, v)|| \leq \frac{1}{2} \varepsilon + 2\delta(||x||^p + ||z||^p) + \eta(||u|| + ||v||)
\] (13)
for all \( x, z, u, v \in X \). By (9) and (13), we get
\[
||f(x, z) - \frac{1}{8} \left[ f(0, 2z) - f(0, -2z) \right] - \frac{1}{4} \left[ f(0, 0) + f(2x, 2z) \right], f(u, v)|| \leq \varepsilon + 2\delta(||x||^p + ||z||^p) + \eta(||u|| + ||v||)
\]
for all \( x, z, u, v \in X \). By (10) and the above inequality, we have
\[
||f(x, z) - \frac{1}{4} \left[ f(0, 0) + f(2x, 2z) \right], f(u, v)|| \leq \frac{1}{8} \left[ 9\varepsilon + 16\delta||x||^p + 17||z||^p + 10\eta(||u|| + ||v||) \right]
\]
for all \( x, z, u, v \in X \). Thus we obtain that
\[
\left\| \frac{1}{4^l} f(2^l x, 2^l z) - \frac{1}{4^{l+1}} \left[ f(0, 0) + f(2^{l+1} x, 2^{l+1} z) \right], f(u, v) \right\| \leq \frac{1}{8} \left\| \frac{9}{4^l} \varepsilon + \frac{\delta}{2(2-p)} \left( 16||x||^p + 17||z||^p \right) + \frac{10\eta}{4^l}(||u|| + ||v||) \right\|
\]
for all \( x, z, u, v \in X \) and all \( l \). Hence, for given integers \( m \) and \( n \) with \( 0 \leq l < m \), we see that
\[
\left\| \frac{1}{4^l} f(2^l x, 2^l y) - \frac{1}{4^m} f(2^m x, 2^m y) - \frac{1}{3} \left( \frac{1}{4^i} - \frac{1}{4^m} \right) f(0, 0), f(u, v) \right\|
\]
for all $x, y, u, v \in X$. By assumption, the sequence $\{\frac{1}{2j}f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2j}f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x, y) := \lim_{j \to \infty} \frac{1}{2j}f(2^j x, 2^j y)$ for all $x, y \in X$. By (7), we have

$$\left\| \frac{1}{2^j}f(2^j (x + y), 2^j (z - w)) + \frac{1}{2^j}f(2^j (x - y), 2^j (z + w)) - \frac{2}{4^j}f(2^j x, 2^j z) - \frac{2}{4^j}f(2^j y, 2^j w), f(u, v) \right\|$$

$$\leq \frac{1}{4^j} \delta \left( \|x\| + \|y\| + \|z\| + \|w\| + \eta(\|u\| + \|v\|) \right)$$

for all $x, y, z, w, u, v \in X$ and all $j$. Letting $j \to \infty$, we see that $F$ satisfies (2). Setting $l = 0$ and taking $m \to \infty$ in (14), one can obtain the inequality (8). If $G : X \times X \to Y$ is another mapping satisfying (2) and (8), by Theorem 3 in [8], we obtain that

$$\|F(x, y) - G(x, y), f(u, v)\| \leq \frac{1}{4^n} \|F(2^nx, 2^ny) - G(2^nx, 2^ny), f(u, v)\|$$

$$\leq \frac{1}{4^n} \|F(2^nx, 2^ny) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\|$$

$$\leq \frac{2}{4^n} \left\{ \frac{3}{2} \epsilon + \frac{2^n \delta}{4 - 2^p} \left( \|x\| + \frac{17}{2^p} \|y\| \right) \right\} + \frac{5}{3} \eta(\|u\| + \|v\|) + \frac{1}{3} \|f(0, 0)\|$$

$$\to 0 \text{ as } n \to \infty$$

for all $x, y, u, v \in X$ and all $j$. Hence the mapping $F$ is the unique mapping satisfying (2), as desired.

**Corollary 2.4.** Let $f : X \times X \to Y$ be a surjective mapping such that

$$\|f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \epsilon$$

for all $x, y, z, w, u, v \in X$ and all $z, w \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (2) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{3}{2} \epsilon + \frac{1}{3} \|f(0, 0)\|$$

for all $x, y, u, v \in X$.

**Proof.** Taking $\delta = \eta = 0$ in Theorem 2.3, we have the desired result.

In the case $p > 2$ in Theorem 2.3, one can also obtain a similar result.

**Theorem 2.5.** Let $p \in (0, 2]$, $\epsilon, \delta, \eta \geq 0$ and let $f : X \times X \to Y$ be a surjective mapping such that

$$\|f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \epsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|)$$

(15)

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying (3) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{3}{2} \epsilon + \frac{1}{4 - 2^p} \delta(\|x\|^p + \|y\|^p) + \frac{2}{3} \eta(\|u\| + \|v\|)$$

(16)

for all $x, y, u, v \in X$.

**Proof.** Letting $y = x$ and $w = -z$ in (15), we gain

$$\|f(2x, 2z) + f(0, 0) - 2f(x, z) + 2f(x, -z), f(u, v)\| \leq \epsilon + 2 \delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|)$$

(17)

for all $x, z, u, v \in X$. Putting $x = z = \epsilon = \eta = 0$ in (17), we get $f(0, 0) = 0$. Putting $x = z = 0$ in (15), we get

$$\|f(y, -w) + f(-y, w) + 2f(y, w), f(u, v)\| \leq \epsilon + \delta(\|y\|^p + \|w\|^p) + \eta(\|u\| + \|v\|)$$
for all \( y, w, u, v \in X \). Replacing \( y \) by \( x \) and \( w \) by \( z \) in the above inequality, we have
\[
\|f(x, -z) + f(-x, z) + 2f(x, z), f(u, v)\| \leq \varepsilon + \delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \tag{18}
\]
for all \( x, z, u, v \in X \). Setting \( y = -x \) and \( w = z \) in (15), we obtain
\[
\|f(2x, 2z) - 2f(x, z) + 2f(-x, z), f(u, v)\| \leq 2\varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \tag{19}
\]
for all \( x, z, u, v \in X \). By (17) and (18), we gain
\[
\|f(2x, 2z) - 4f(x, z) + f(x, -z) - f(-x, z), f(u, v)\| \leq 2\varepsilon + 3\delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|)
\]
for all \( x, z, u, v \in X \). By (17) and (19), we get
\[
\|f(x, -z) - f(-x, z), f(u, v)\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|)
\]
for all \( x, z, u, v \in X \). By (17), (18) and (19), we have
\[
\|f(2x, 2z) - 4f(x, z), f(u, v)\| \leq 2\varepsilon + 3\delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|)
\]
for all \( x, z, u, v \in X \). Replacing \( x \) by \( 2^j x \) and \( z \) by \( 2^j z \) and dividing \( 4^{j+1} \) in the above inequality, we obtain that
\[
\left\| \frac{1}{4^j}f(2^j x, 2^j z) - \frac{1}{4^j}f(2^{j+1} x, 2^{j+1} z), f(u, v) \right\| \leq \frac{1}{4^j+1} \left[ 2\varepsilon + 3 \cdot 2^j \delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|) \right] \tag{20}
\]
for all \( x, z, u, v \in X \) and all \( j = 0, 1, 2, \cdots \). For given integers \( l, m \) \((0 \leq l < m)\), we obtain that
\[
\left\| \frac{1}{4^l}f(2^l x + y, 2^l (z - w)) + \frac{1}{4^l}f(2^l (x - y), 2^l (z + w)) - \frac{2}{4^l}f(2^l x, 2^l z) + \frac{2}{4^l}f(2^l y, 2^l w), f(u, v) \right\| \leq \frac{2}{4^l} \left[ 2\varepsilon + 3 \cdot 2^l \delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|) \right]
\]
for all \( x, y, z, w, u, v \in X \) and all \( j = 0, 1, 2, \cdots \). Letting \( j \to \infty \) in the above inequality, we see that \( F \) satisfies (3). Setting \( l = 0 \) and taking \( m \to \infty \) in (20), one can obtain the inequality (16). If \( G : X \times X \to Y \) is another mapping satisfying (3) and (16), by Theorem 3.1 in [9], we obtain that
\[
\|F(x, y) - G(x, y), f(u, v)\| = \frac{1}{4^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\|
\]
\[
\leq \frac{1}{4^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\|
\]
\[
\leq \frac{1}{4^n} \left[ \frac{4}{3} \varepsilon + \frac{6}{4 - 2^p} \delta(\|x\|^p + \|y\|^p) + \frac{4}{3} \eta(\|u\| + \|v\|) \right] \to 0 \text{ as } n \to \infty
\]
for all \( x, y, u, v \in X \). Hence the mapping \( F \) is the unique mapping satisfying (3), as desired. \( \square \)

In the case \( p > 2 \) in Theorem 2.5, one can also obtain a similar result.

**Corollary 2.6.** Let \( f : X \times X \to Y \) be a surjective mapping such that
\[
\|f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w), f(u, v)\| \leq \varepsilon
\]
for all \( x, y, z, w, u, v \in X \). Then there exists a unique mapping \( F : X \times X \to Y \) satisfying (2) such that

\[
\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{2}{3}\varepsilon
\]

for all \( x, y, u, v \in X \).

**Proof.** Taking \( \delta = \eta = 0 \) in Theorem 2.5, we have the desired result. \( \square \)

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