RICCI CURVATURE AND CONFORMALITY OF RIEMANNIAN MANIFOLDS TO SPHERES

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Abstract. In this paper we give bounds on the least eigenvalue of the conformal Laplacian and the Yamabe invariant of a compact Riemannian manifold in terms of the Ricci curvature and the diameter and deduce a sufficient condition for the manifold to be conformally equivalent to a sphere.

Résumé: Soit \((M, g)\) une variété riemannienne compacte sans bord de dimension \(n\). En utilisant des bornes inférieures sur la courbure de Ricci et le diamètre de \((M, g)\), on minore la plus petite valeur propre du laplacien conforme ainsi que l’invariant de Yamabe de cette variété. On en déduit certaines conditions pour que \((M, g)\) soit conformément difféomorphe à la sphère unité de même dimension.

Motivation

The purpose of the paper is to give conditions on some topological or geometrical invariants of a smooth compact Riemannian manifold \((M, g)\) without boundary, to be conformally diffeomorphic to the sphere of the same dimension equipped with its canonical metric. This problem has an interesting history. Indeed, the first approach was based on the use of the conformal automorphisms group of the manifold denoted by \(C(M, g)\). It was shown that the non-compactness of the connected component of the identity in \(C(M, g)\) implies that \((M, g)\) is conformally equivalent to the sphere \(S^n\) for \(n \geq 3\) (this is due to M.Obata, [19] and [20]). Unfortunately, there was a gap in [20] involving Obata’s use of a certain theorem and K.R.Gutschera in [14] gave some counterexamples and finally J.Lafontaine [17] completed the proof in 1988. The compactness of the whole group \(C(M, g)\) was shown by J. Ferrand [10]. An alternate approach to the problem based on the conformal scalar curvature theory was provided by R. Schoen [23].

We have to notice that many mathematicians obtained various conditions for a Riemannian manifold to be isometric to a sphere, ones used the infinitesimal conformal transformations (see M. Obata [21], C.C. Hsiung-L.W Stern [15], K. Yano-T. Nagano [28, …]) and others gave conditions on the sectional curvatures and

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the Ricci curvature or particular bounds for a certain eigenvalue of the Laplacian on \((M, g)\) and the Ricci curvature (see S.Deshmukh and A. Al.Eid [9], ...).

In this paper, we obtain a condition involving the Ricci and the scalar curvature for \((M, g)\) to be conformally equivalent to a sphere. This will be based on comparison results for the Yamabe invariant. The main ingredients are symmetrization process and isoperimetric comparison results due to P.Berard, G.Besson and S.Gallot [6], which have been extended by the second author in [11] and [12].

1. Introduction and Statement of Main Results

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\) without boundary. We denote by \(\text{Ric}_g\) its Ricci curvature, \(r_0\) the infimum of \(r(x) = \inf\{\text{Ric}_g(u, u), u \in T_x M, |u| = 1\}\), the least eigenvalue of \(\text{Ric}_g\) on the tangent space \(TM\), \(R\) its scalar curvature and \(d\) its diameter.

The isoperimetric profile of \((M, g)\) is defined by

\[
h(s) = \inf\{\frac{\text{vol}\partial\Omega}{\text{vol}(M, g)}, \Omega \subset M \ s.t \ \frac{\text{vol}\partial\Omega}{\text{vol}(M, g)} = s\}, \ s \in [0, 1]
\]

where \(\Omega \subset M\) are smooth domains with regular boundaries.

Let \(Is(s)\) be the isoperimetric profile of the model space: the unit sphere \(S^n\) of \(\mathbb{R}^{n+1}\) equipped with its canonical metric, that is

\[
Is(s) = \frac{\text{vol}(\partial B(s))}{\text{vol} S^n}
\]

where \(B(s)\) is a geodesic ball of \(S^n\) such that \(\frac{\text{vol}B(s)}{\text{vol} S^n} = s\).

The following result is due to Bérard, Besson and Gallot [6]:

If \((M, g)\) satisfies

\[
r_0d^2 \geq (n - 1)\varepsilon \alpha^2 \ (\varepsilon \in \{-1, 0, 1\} \ and \ \alpha \in \mathbb{R}_+),
\]

then \(\forall s \in [0, 1]\)

\[
dh(s) \geq a(n, \varepsilon, \alpha)Is(s).
\]

where \(a(n, \varepsilon, \alpha)\) is a constant depending on \(n, \varepsilon\) and \(\alpha\) as follows

\[
a(n, \varepsilon, \alpha) = \begin{cases} 
\alpha \sigma_n^{\frac{1}{n}} \left[2 \int_0^\pi (\cos t)^{n-1} dt \right]^{-\frac{\alpha}{n}}, & \text{if } \varepsilon = 1 \\
(1 + n \sigma_n)^{\frac{1}{n}} - 1, & \text{if } \varepsilon = 0 \\
ac(\alpha), & \text{if } \varepsilon = -1,
\end{cases}
\]

where \(\sigma_n = \int_0^\pi (\sin t)^{n-1} dt\) and \(c(\alpha)\) is the unique root of the equation \(\sigma_n (\cosh y)^n = \sinh y \int_y^{y+\alpha} (\cosh t)^{n-1} dt\). The Inequality [3] is sharper than the one given by M. Gromov [13].

Let \((M, g)\) be a compact Riemannian manifold without boundary satisfying relation [3]. In [12] (Theorem 5), the first author obtained a lower bound for the least eigenvalue of the operator \(\Delta + C\), where \(\Delta\) is the Laplacian of \((M, g)\) and \(C\) is a potential.

Following the proof of [12], we give in the first step a lower bound for the least eigenvalue of the conformal Laplacian of \((M, g)\), \(L = c_n \Delta + R\) with \(c_n = 4 \frac{n-2}{n+2}\).

We begin by providing some notations: let \(V\) denote the volume of \((M, g)\) and \(\omega_n\) the one of \((S^n, \text{can})\). For given positive reals \(r_1, r_2\) \((0 < r_1, r_2 < \pi)\), let \(B(S, r_1)\), \(B(N, r_2)\) be the geodesic ball of \(S^n\) of center the south pole \(S\) and radius \(r_1\) (resp. the geodesic ball of \(S^n\) of center the north pole \(N\) and radius \(r_2\)).
Proposition 1. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\) without boundary satisfying relation (3). Let \(\mu_1(M)\) (resp. \(\rho_1(S^n)\)) denotes the least eigenvalue of the conformal laplacian \(L = c_n \Delta + R\) on \(M\) (resp. of \(c_n \Delta + h_+ - h_-\) on the unit sphere \(S^n\)) acting on functions with

\[
h_+ = \begin{cases} (d/a)^2 \sup R_+ & \text{on } B(S, r_1) \\ 0 & \text{on the complementary in } S^n \end{cases}
\]
and

\[
h_- = \begin{cases} (d/a)^2 \sup R_- & \text{on } B(N, r_2) \\ 0 & \text{on the complementary in } S^n, \end{cases}
\]

where \(r_1, r_2\) satisfy:

\[
\omega_n^{-1} \text{vol} B(S, r_1) = V^{-1} \| R_+ \|_{L^1(M)} / \| R_+ \|_{L^{\infty}(M)}
\]
and

\[
\omega_n^{-1} \text{vol} B(N, r_2) = V^{-1} \| R_- \|_{L^1(M)} / \| R_- \|_{L^{\infty}(M)}.
\]

Then

\[
\mu_1(M) \geq (a/d)^2 \rho_1(S^n).
\]

In the second step, we give a lower bound for the first Yamabe invariant of \((M, g)\) which we denote by \(\lambda(M)\). Let \(\rho(S^n)\) be the least eigenvalue on \(S^n\) of

\[
c_n \Delta u + hu = \rho u^{\frac{n+2}{n-2}}
\]
where \(h = h_+ - h_-\) is given by (6) and (7), we have

Theorem 1. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\) without boundary satisfying relation (3). We have

\[
\lambda(M) \geq (a/d)^2 \beta \rho(S^n)
\]

Since the Yamabe invariant of \((M, g)\) is bounded from above by the one of the sphere, we derive the following rigidity result.

Theorem 2. A compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\) without boundary satisfying condition (3) with \(\epsilon = 1\) and

\[
(a/d)^2 \beta \rho(S^n) \geq \lambda(S^n)
\]
is conformally diffeomorphic to the unit sphere \(S^n\)

Notice that in the particular case where the scalar curvature is constant, we have \(\lambda(M) = RV = (\frac{n}{2})^2 \beta \rho(S^n)\) and condition (13) is only satisfied when \(M\) is conformally diffeomorphic to the sphere.

As a consequence of Proposition 1 and Theorem 1 and in the case where \(Ric \geq n-1\), we obtain the following result:

Corollary 1. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\) without boundary satisfying \(Ric \geq n-1\). Then

\[
\mu_1(M) \geq n(n-1) = \mu_1(S^n)
\]
(15) \[ \lambda(M) \geq n(n-1)V^\frac{2}{n} = \left(\frac{V}{\omega_n}\right)^\frac{2}{n} \lambda(S^n). \]

Where \( \lambda(M) \) denote the Yamabe invariant of \((M, g)\) and \( \lambda(S^n) \) the one of \(S^n\).

We have to point out that (14) and (15) are optimal in the case where the metric \(g\) is Einstein.

The inequality (15) was proved by J. Petean and S. Ilias in [22] and [16] respectively by using analogous methods.

2. Yamabe Problem and conformal invariant \( \lambda(M) \)

**Yamabe question:** Given a compact Riemannian manifold \((M, g)\) without boundary of dimension \(n \geq 3\), is there a metric \(\tilde{g}\) conformal to \(g\) which has constant scalar curvature \(R_{\tilde{g}} = \lambda\)? We write \(\tilde{g} = u^{\frac{4}{n-2}}g\), \(u > 0\). By a simple computation we obtain:

(16) \[ R_{\tilde{g}} = u^{-\frac{4}{n-2}}(c_n \Delta u + Ru) \]

where \(R\) is the scalar curvature and \(\Delta\) the Laplacian of \((M, g)\). Hence the Yamabe problem is equivalent to solve:

(17) \[ c_n \Delta u + Ru = \lambda u^{\frac{n+2}{n-2}}, \quad u > 0 \]

We will use the following notations: \(p = \frac{2n}{n-2}\) and \(L = c_n \Delta + R\).

The operator \(L\) is called the conformal laplacian of \((M, g)\). Equation (17) can be rewritten as

(18) \[ Lu = \lambda u^{p-1}, \quad u > 0. \]

Yamabe noticed that (17) is the Euler-Lagrange equation of the functional:

(19) \[ Q_0(\tilde{g}) = \frac{\int_M R_{\tilde{g}} dv_{\tilde{g}}}{\left(\int_M dv_{\tilde{g}}\right)^\frac{2}{p}} \]

when restricted to a conformal class \([\tilde{g}] = \{hg / h \in C^\infty(M), h > 0\}\), where \(dv_{\tilde{g}}\) is the volume form of \((M, \tilde{g})\) and \(h = u^{\frac{4}{n-2}}, \quad u > 0\). In fact, on \([\tilde{g}]\) we can write \(Q_0(\tilde{g}) = Q_0(u^{p-2}g) = J(u)\), where

(20) \[ J(u) = \frac{\int_M uLudv_{\tilde{g}}}{\left(\int_M u^p dv_{\tilde{g}}\right)^\frac{2}{p}} = \frac{\int_M (c_n |\nabla u|^2 + Ru^2)dv_{\tilde{g}}}{\|u\|_p^2}. \]

We call \(J(u)\) the Yamabe quotient of \((M, g)\). Let \(u\) be a positive function in \(C^\infty(M)\) and a critical point of \(J\), then it is easy to see that \(u\) satisfies equation (17) with \(\lambda = J(u)\).

By using a Hölder inequality, we derive that the functional \(J\) is bounded from below. The infimum

(21) \[ \lambda(M) = \inf \{Q_0(\tilde{g}) / \tilde{g} \in [\tilde{g}]\} = \inf \{J(u) / u \in C^\infty(M), u > 0\} \]

is a conformal invariant, which means that it is determined by the conformal class and is independent of the choice of the initial metric \(g\) in the conformal class. It is called the Yamabe invariant of \((M, g)\).

We have the following results

**Theorem A:** ([27], [26], [1], [2]): The Yamabe problem can be solved on any compact manifold \(M\) with \(\lambda(M) < \lambda(S^n)\).
Theorem B: ([27], [1], [2]): For any compact Riemannian manifold \((M, g)\) without boundary, we always have \(\lambda(M) \leq \lambda(S^n) = n(n-1)\omega_n^2\).

Theorem A reduces the resolution of Yamabe problem to the estimate of the invariant \(\lambda(M)\). In fact, if we can find a function \(u \in L^2(M)\) such that \(J(u) < \lambda(S^n)\), then \(\lambda(M) < \lambda(S^n)\), hence the Yamabe problem has a solution. In this way T.Aubin [2] proved the conjecture in the two following cases:

1) \((M, g)\) is not a conformally flat compact Riemannian manifold of dimension \(n \geq 6\).
2) \((M, g)\) is a locally conformally flat compact Riemannian manifold of dimension \(n \geq 3\) and finite Poincaré group, not conformal to \((S^n, can)\).

R. Schoen [24] solved all the remaining cases of the Yamabe problem, using the positive mass theorem.

We remark that for the case where \((M, g)\) is conformal to \(S^n\), the Yamabe problem clearly has a solution. If \(\Phi : M \rightarrow S^n\) is a conformal diffeomorphism then \(\Phi^*(g_0) = fg\), where \(g_0\) is the standard metric of \(S^n\) and \(f\) a positive function in \(C^\infty(M)\), clearly \(fg\) has constant scalar curvature.

Besides the proof of T.Aubin and R.Schoen of the Yamabe problem, another proof by A.Bahri [3], A.Bahri-H.Brézis [5] of the same conjecture is available using the theory of critical points at infinity.

3. Symmetrization method and applications

In the following we give lower bounds for the first eigenvalue of the conformal Laplacian of the manifold \((M, g)\), that we denote by \(\mu_1(M)\) and its Yamabe invariant \(\lambda(M)\). The method we use here is inspired by the one used in [11] and [12]. We begin by the case of the least eigenvalue of the Laplacian and the proof obtain Proposition 1.

Proof of Proposition 1: Let \((M, g)\) be a compact Riemannian manifold which satisfies the isoperimetric inequality [3]. One can apply the symmetrization process described in [7], [8] and [24] or [11] and [12] to symmetrize a smooth function \(f\) into a radial function \(f^*\) on the model space \((S^n, can)\). The function \(f^*\) is in \(H^1(S^n)\), radial (w.r.t the north pole) and satisfies

\[
\begin{align*}
\omega_n \int_M f^q dg & = \ V \int_{S^n} f^* \ q dv, \text{ for all real } q \geq 1 \\
\omega_n \int_M |\nabla f|^2 dg & \geq V(\frac{\theta}{2})^2 \int_{S^n} |\nabla f^*|^2 \ dv
\end{align*}
\]

The first identity of (22) derives from the coarea formula (see [4]) and the second can be proved through coarea formula, isoperimetric inequality of [6] and the Cauchy-Schwarz inequality.

The inequality of Hardy-Littlewood-Polya ([25] formulas (60) and (13)) implies

\[
\int_M Rf^2 dg \geq \beta \int_0^{\omega_n} \left[ R^+_{\ast}(V - \beta u) - R^-_{\ast}(\beta u) \right] f^2(u) du
\]

where \(\beta\) denotes the ratio \(\frac{V}{\omega_n}\), \(R^+_{\ast}(V - \beta u)\) is the increasing symmetric re-arrangement of \(R_\ast\) and \(R^-_{\ast}(\beta u)\) (respectively \(f^*(u)\)) the decreasing symmetric re-arrangement of \(R_-\) (respectively of \(f\)). Then we apply at the right handside of (23) the following Steffensen inequality (one can see D.S Mitrović [18]):
Theorem \([\text{(18)}]\) Let \(\varphi\) and \(\psi\) be two given integrable functions defined on the interval \((a, b)\) such that \(\varphi\) is decreasing and \(0 \leq \psi \leq 1\) on \((a, b)\), then:

\[
\int_{b-\gamma}^{b} \varphi(t) dt \leq \int_{a}^{b} \varphi(t) \psi(t) dt \leq \int_{a}^{a+\gamma} \varphi(t) dt,
\]

where \(\gamma = \int_{a}^{b} \psi(t) dt\).

We obtain

\[
\int_{M} R f^2 dv_g \geq \beta \left[ \sup R^*_+ \int_{\gamma^+}^{\omega_n} f^{*2}(u) du - \sup R^*_- \int_{0}^{\gamma^-} f^{*2}(u) du \right]
\]

with

\[
\gamma^+ = \omega_n - (\beta \sup R^*_+)^{-1} \int_{0}^{V} R^*_+(V - u) du
\]

and

\[
\gamma^- = (\beta \sup R^*_-)^{-1} \int_{0}^{V} R^*_-(u) du
\]

We identify \(R^*_+\) (respectively \(R^*_-\)) with a function \(R^*_+(\pi - r)\) (respectively \(R^*_-(r)\)) of the distance to the north pole. Let \(\tilde{R}_+\) and \(\tilde{R}_-\) be the radial functions defined on \(S^n\) as follows

\[
\tilde{R}_+(r) = \begin{cases} 
    (d/a)^2 \sup R^*_+ & \text{on } [\pi - r_1, \pi] \\
    0 & \text{on } [0, \pi - r_1]
\end{cases}
\]

and

\[
\tilde{R}_-(r) = \begin{cases} 
    (d/a)^2 \sup R^*_- & \text{on } [0, r_2] \\
    0 & \text{on } [r_2, \pi]
\end{cases}
\]

We have

\[
(d/a)^2 \sup R^*_+ \int_{\gamma^+}^{\omega_n} f^{*2}(u) du = \omega_n-1 \int_{\pi - r_1}^{\pi} \tilde{R}_+(r) f^{*2}(r)(\sin r)^{n-1} dr
\]

and

\[
(d/a)^2 \sup R^*_- \int_{0}^{\gamma^-} f^{*2}(u) du = \omega_n-1 \int_{0}^{r_2} \tilde{R}_-(r) f^{*2}(r)(\sin r)^{n-1} dr.
\]

Therefore, we obtain

\[
\int_{M} R f^2 dv_g \geq \beta \left( \frac{a}{d} \right)^2 \int_{S^n} (h_+ - h_-) f^{*2}(v) dv
\]

and finally using \([\text{(22)}]\),

\[
\frac{\int_{M} L f dv_g}{\int_{M} f^2 dv_g} \geq \left( \frac{a}{d} \right)^2 \frac{\int_{S^n} [c_n |df^*|^2 + (h_+ - h_-) f^{*2}] dv}{\int_{S^n} f^{*2} dv},
\]

Hence we end the proof by using the fact that the least eigenvalue is the infimum of the Rayleigh quotient. \(\square\)
In the sequel we deal with the Yamabe invariant $\lambda(M)$ of $(M, g)$ introduced in \cite{21}. Since this invariant can be expressed in terms of Rayleigh quotient as

\begin{equation}
\lambda(M) = \inf_u \frac{\int_M (c_n |\nabla u|^2 + R u^2) dv_g}{(\int_M u^p dv_g)^{2/p}}
\end{equation}

where the infimum is taken over all smooth real-valued positive functions $u$ on $M$, we can use the same techniques introduced above in the aim to give bounds for $\lambda(M)$. Let $\rho(S^n)$ be the least eigenvalue on $S^n$ of

\begin{equation}
c_n \Delta u + hu = \rho u^{\frac{n+2}{n}}
\end{equation}

where $h = h_+ - h_-$ is given by \cite{10} and \cite{7}.

**Proof of Theorem 1.** We begin the proof by providing a lower bound for the Yamabe invariant $\lambda(M)$ with the use of the symmetrization method given in Proposition \cite{11}. For a positive function $f$ in $C^\infty(M)$, we consider its decreasing symmetric rearrangement $f^*$. Let $R$ be the scalar curvature of $(M, g)$ and $h$ the function defined on the unit sphere by \cite{10} and \cite{7}. Following the same steps as in the proof of Proposition \cite{11} and applying \cite{22} for $q = p$, we obtain

\begin{equation}
\frac{\int_M (c_n |\nabla f|^2 + R f^2) dv_g}{(\int_M f^p dv_g)^{2/p}} \geq \frac{\beta}{\beta^*} \frac{a}{d} \frac{\int_{S^n} (c_n |\nabla f^*|^2 + h f^* f^2) dv}{(\int_{S^n} f^p dv)^{2/p}},
\end{equation}

and hence

\begin{equation}
\lambda(M) \geq (a/d)^2 \beta^* \rho(S^n).
\end{equation}

\hfill \Box

**Proof of Theorem 2.** From Theorem \cite{11} we have

\[ \lambda(M) \geq (a/d)^2 \beta^* \rho(S^n). \]

On the other hand $\lambda(M)$ is upper bounded by $\lambda(S^n)$, and since in this case $R_\omega = 0$, we derive that $(a/d)^2 \beta^* \rho(S^n) \leq \lambda(S^n)$. Hence, if $(a/d)^2 \beta^* \rho(S^n) \geq \lambda(S^n)$, then the equality $\lambda(M) = \lambda(S^n)$ holds and $(M, g)$ is conformally diffeomorphic to the unit sphere $S^n$, which completes the proof. \hfill \Box

As a consequence of Proposition \cite{11} and Theorem \cite{11} in the case where $\text{Ric} \geq n - 1$ (take $\alpha = d$ and $\varepsilon = 1$ in \cite{23}), we obtain the corollary given in the introduction.

**Proof of corollary 4.** Since the scalar curvature $R \geq n(n-1)$ we derive the following inequalities

\[ \inf_f \frac{\int (c_n |\nabla f|^2 + R f^2) dv_g}{\int f^2 dv_g} \geq \inf_f \frac{\int (c_n |\nabla f|^2 + n(n-1) f^2) dv}{\int f^2 dv} \]

and

\[ \inf_f \frac{(\int c_n |\nabla f|^2 + R f^2) dv_g}{(\int f^p dv_g)^{2/p}} \geq \inf_f \frac{(\int c_n |\nabla f|^2 + n(n-1) f^2) dv}{(\int f^p dv)^{2/p}}, \]

where the infimum is taken over all smooth real-valued positive functions $f$ on $M$. The same arguments as in the proof of Proposition \cite{11} and Theorem \cite{11} enable us to lowerbound the right hand sides of these inequalities by $\mu_1(S^n)$ and $\beta^* \lambda(S^n)$ respectively. \hfill \Box
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