On the linearization of the automorphism groups of algebraic domains.

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1 Linearization Theorem and applications

Let $D$ be a domain in $\mathbb{C}^n$ and $G$ a topological group which acts effectively on $D$ by holomorphic automorphisms. In this paper we are interested in projective linearizations of the action of $G$, i.e., a linear representation of $G$ in some $\mathbb{C}^{N+1}$ and an equivariant imbedding of $D$ into $\mathbb{P}^N$ with respect to this representation. Since $G$ acts effectively, the representation in $\mathbb{C}^{N+1}$ must be faithful. In our previous paper [11], however, we considered an example of a bounded domain $D \subset \mathbb{C}^2$ with an effective action of a finite covering $G$ of the group $SL_2(\mathbb{R})$. In this case the group $G$ doesn’t admit a faithful representation. The example shows that a linearization in the above sense doesn’t exist in general.

In the present paper we give a criterion for the existence of the projective linearization for birational automorphisms. The domains we discuss here are open connected sets defined by finitely many real polynomial inequalities or connected finite unions of such sets. These domains are called algebraic. For instance, in the above example the domain $D$ is algebraic.

Definition 1.1

1. A Nash map is a real analytic map

$$f = (f_1, \ldots, f_m): U \to \mathbb{R}^m$$

(where $U \subset \mathbb{R}^n$ is open) such that for each of the components $f_k$ there is a non-trivial polynomial $P_k$ with

$$P_k(x_1, \ldots, x_n, f_k(x_1, \ldots, x_n)) = 0$$

for all $(x_1, \ldots, x_n) \in U$.

2. A Nash manifold $M$ is a real analytic manifold with finitely many coordinate charts $\phi_i: U_i \to V_i$ such that $V_i \subset \mathbb{R}^n$ is Nash for all $i$ and the transition functions are Nash (a Nash atlas).
3. A Nash group is a Nash manifold with a group operation \((x, y) \mapsto xy^{-1}\) which is Nash with respect to all Nash coordinate charts.

In the above example the group \(SL_2(\mathbb{R})\) and its finite covering \(G\) are Nash groups (The universal covering of \(SL_2(\mathbb{R})\) is a so-called locally Nash group). Moreover, the action \(G \times D \to D\) is also Nash. Since the linearization doesn’t exist here, we need a stronger condition on the action of \(G\).

A topological group \(G\) is said to be a group of birational automorphisms of a domain \(D \subset \mathbb{C}^n\) if we are given an effective (continuous) action \(G \times D \to D\) such that every element \(g \in G\) defines an automorphism of \(D\) which extends to a birational automorphism of \(\mathbb{C}^n\). By the degree of a Nash map \(f\) we mean the minimal natural number \(d\) such that all polynomials \(P_k\) in Definition 1.1 can be chosen such that their degrees don’t exceed \(d\). Finally, under a biregular map between two algebraic varieties we understand an isomorphism in sense of algebraic geometry. Our main result is the following linearization criterion. It will be proved in section 3.

**Theorem 1.1** Let \(D \subset \mathbb{C}^n\) be a algebraic domain and \(G\) a group of birational automorphisms of \(D\). The following properties are equivalent:

1. \(G\) is a subgroup of a Lie group \(\hat{G}\) of birational automorphisms of \(D\) which extends the action of \(G\) and has finitely many connected components;
2. \(G\) is a subgroup of a Nash group \(\hat{G}\) of birational automorphisms of \(D\) which extends the action of \(G\) to a Nash action \(\hat{G} \times D \to D\);
3. \(G\) is a subgroup of a Nash group \(\hat{G}\) such that the action \(G \times D \to D\) extends to a Nash action \(\hat{G} \times D \to D\);
4. the degree of the automorphism \(\phi_g: D \to D\) defined by \(g \in G\) is bounded;
5. there exists a projective linearization, i.e. a linear representation of \(G\) in some \(\mathbb{C}^{N+1}\) and a biregular imbedding \(i: \mathbb{P}^n \hookrightarrow \mathbb{P}^N\) such that the restriction \(i|_D\) is \(G\)-equivariant.

We finish this section with applications of Theorem 1.1. In the previous paper [11] we gave sufficient conditions on \(D\) and \(G\) such that \(G\) is a Nash group and the action \(G \times D \to D\) is Nash. The condition on \(D\) is to be bounded and to have a non-degenerate boundary in the following sense.

**Definition 1.2** A boundary of a domain \(D \subset \mathbb{C}^n\) is called non-degenerate if it contains a smooth point where the Levi-form is non-degenerate.

The group \(G\) is taken to be the group \(Aut_a(D)\) of all holomorphic Nash (algebraic) automorphisms of \(D\). We proved in [11] that, if \(D\) is a algebraic bounded domain with non-degenerate boundary, the group \(Aut_a(D)\) is closed.
in $\text{Aut}(D)$ and carries a unique structure of a Nash group such that the action $\text{Aut}_a(D) \times D \to D$ is Nash with respect to this structure.

Now let $G = \text{Aut}_b(D) \subset \text{Aut}_a(D)$ be the group of all birational automorphisms of $D$. Then $G$ satisfies the property 3 in Theorem 1.1 with $\hat{G} = \text{Aut}_a(D)$. By the property 2, $G$ is a subgroup of a Nash group of birational automorphisms of $D$. Since $G$ contains all the birational automorphisms of $D$, $G$ is itself a Nash group with the Nash action on $D$. We obtain the following corollary.

**Corollary 1.1** Let $D \subset \mathbb{C}^n$ be a bounded algebraic domain with non-degenerate boundary. Then the group $\text{Aut}_b(D)$ of all birational automorphisms of $D$ is Nash with the Nash action on $D$ which admits a projective linearization, i.e., there exist a representation of $\text{Aut}_b(D)$ in some $\mathbb{C}^{N+1}$ and a biregular imbedding $i: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ such that the restriction $i|_D$ is $\text{Aut}_b(D)$-equivariant.

Furthermore, S. Webster (see [10]) established the following sufficient conditions on $D$ which make its automorphisms birational. Let $D$ be a algebraic domain. The theory of semialgebraic sets (see Benedetti-Risler [2]) implies that the boundary $\partial D$ is contained in finitely many irreducible real hypersurfaces. Several of them, let say $M_1, \ldots, M_k$, have generically non-degenerate Levi forms. If $\partial D$ is non-degenerate in sense of Definition 1.2, such hypersurfaces exist. The complexifications $M_i$’s of $M_i$’s are defined to be their complex Zariski closures in $\mathbb{C}^n \times \overline{\mathbb{C}^n}$ where $M_i$’s are totally real imbedded via the diagonal map $z \mapsto (z, \bar{z})$. It follows that $M_i$’s are the irreducible complex hypersurfaces. Furthermore, the so-called Segre varieties $Q_{iw}$’s, $w \in \mathbb{C}^n$ are defined by

$$Q_{iw} := \{ z \in \mathbb{C}^n \mid (z, \bar{w}) \in M_i \}.$$  

The complexifications and Segre varieties are the important biholomorphic invariants of a domain $D$ and play a decisive role in the reflection principle.

Now we are ready to formulate the conditions of S. Webster.

**Definition 1.3** A algebraic domain is said to satisfy the condition $(W)$ if for all $i$ the Segre varieties $Q_{iw}$ uniquely determine $z \in \mathbb{C}^n$ and $Q_{iw}$ is an irreducible hypersurface in $\mathbb{C}^n$ for all $z$ off a proper subvariety $V_i \subset \mathbb{C}^n$.

The Theorem of S. Webster (see [10], Theorem 3.5) can be formulated in the following form:

**Theorem 1.2** Let $D \subset \mathbb{C}^n$ be a algebraic domain with non-degenerate boundary which satisfies the condition $(W)$. Further, let $f \in \text{Aut}(D)$ be an automorphism which is holomorphically extendible to a smooth boundary point with non-degenerate Levi-form. Then $f$ is birationally extendible to the whole $\mathbb{C}^n$.

Since every Nash automorphism $f \in \text{Aut}_a(D)$ extends holomorphically to generic boundary points, we obtain the following Corollary.
Corollary 1.2 Let $D \subset \mathbb{C}^n$ be a bounded algebraic domain which satisfies the condition (W). Then the whole group $\text{Aut}_a(D)$ is projective linearizable, i.e. there exist a representation of $\text{Aut}_a(D)$ in some $\mathbb{C}^{N+1}$ and a biregular imbedding $i: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ such that the restriction $i|_D$ is $\text{Aut}_a(D)$-equivariant.

To obtain the extendibility of the whole group $\text{Aut}(D)$ of holomorphic automorphisms, we consider the algebraic domains in sense of Diederich-Fornæss (see [4]).

Definition 1.4 A domain $D \subset \subset \mathbb{C}^n$ is called algebraic if there exists a real polynomial $r(z, \bar{z})$ such that $D$ is a connected component of the set
\[
\{z \in \mathbb{C}^n \mid r(z, \bar{z}) < 0\}
\]
and $dr(z) \neq 0$ for $z \in \partial D$.

The following fundamental result for such domains is due to K. Diederich and J. E. Fornæss (see [4]).

Theorem 1.3 Let $D \subset \subset \mathbb{C}^n$ be an algebraic domain. Then $\text{Aut}_a(D) = \text{Aut}(D)$.

Thus we obtain the linearization of the whole automorphism group $\text{Aut}(D)$.

Theorem 1.4 Let $D \subset \subset \mathbb{C}^n$ be an algebraic domain which satisfies the condition (W). Then the group $\text{Aut}(D)$ is projective linearizable, i.e. there exist a representation of $\text{Aut}(D)$ in some $\mathbb{C}^{N+1}$ and a biregular imbedding $i: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ such that the restriction $i|_D$ is $\text{Aut}(D)$-equivariant.

Further corollaries are devoted to the constructions of complexifications.

2 Complexifications

Using the linearization criterion we establish here existences of complexifications. To every real Lie group $G$ one can associate its complexification (see Hochschild [3]) defined as follows.

Definition 2.1 Let $G$ be a real Lie group. A complex Lie group $G^C$ together with a Lie homomorphism $\nu: G \to G^C$ is called a complexification of $G$ if for a given Lie homomorphism $\phi$ from $G$ into a complex Lie group $H$, there exists exactly one holomorphic Lie homomorphism $\phi^C: G^C \to H$ such that $\phi = \phi^C \circ \nu$. A real Lie group $G$ is called holomorphically extendible if the map $\nu: G \to G^C$ is injective.

A complexification always exists and is unique up to biholomorphisms (see Hochschild [3] and Heinzner [6] and [7]). Further, one defines the complexification of an action (see Heinzner [6]).
Definition 2.2 Let a real Lie group $G$ act on a complex space $X$ by holomorphic automorphisms. A complex space $X^\mathbb{C}$ together with a holomorphic action of $G^\mathbb{C}$ and a $G$-equivariant map $i : X \to X^\mathbb{C}$ is called a $G$-complexification of $X$ if to every holomorphic $G$-equivariant map $\phi : X \to Y$ into another complex space $Y$ with a holomorphic action of $G^\mathbb{C}$ there exists exactly one holomorphic $G^\mathbb{C}$-equivariant map $\phi^\mathbb{C}$ such that $\phi = \phi^\mathbb{C} \circ i$.

A $G$-complexification is unique up to biholomorphic $G^\mathbb{C}$-equivariant maps provided it exists. P. Heinzner proved in [6] the existence of a $G$-complexification of $X$ with properties that $i : X \to X^\mathbb{C}$ is an open imbedding and $X^\mathbb{C}$ is Stein in case $G$ is compact and $X$ is a Stein space.

Now the projective linearization in Theorem 1.1 implies the existence of complexifications in our situation.

Corollary 2.1 Let $D \subset \mathbb{C}^n$ be a algebraic domain and $G$ a Lie group of birational automorphisms of $D$ which satisfies one of the equivalent properties in Theorem 1.1. Then the group $G$ is holomorphically extendible and there exists a smooth $G$-complexification $D^\mathbb{C}$ of $D$ such the map $i : D \to D^\mathbb{C}$ is an open imbedding.

Proof. By property 5 in Theorem 1.1 $G$ is a subgroup of the complex Lie group $GL_N(\mathbb{C})$. By Definition 2.1, $G$ is holomorphically extendible. Let $i$ be the embedding of $\mathbb{C}^n \supset D$, given by Theorem 1.1. Since the stability group $H \subset GL_N(\mathbb{C})$ of the complex projective variety $X := i(\mathbb{P}^n)$ is a complex Lie group and $G \subset H$, the Definition 2.1 yields a holomorphic action of $G^\mathbb{C}$ on $X$. We claim that $D^\mathbb{C} := G^\mathbb{C} \cdot \overline{D} \subset X$ is the required $G$-complexification of $D$. Indeed, let $\phi : D \to Y$ be a $G$-equivariant holomorphic map into another complex space $Y$ with a holomorphic action of $G^\mathbb{C}$. To define the required in Definition 2.2 map $\phi^\mathbb{C}$ we take a point $z \in D^\mathbb{C}$ which is always of the form $z = Ax$ with $A \in G^\mathbb{C}$ and $x \in D$. Then we set $\phi^\mathbb{C}(z) := A\phi(x)$. Why is $\phi^\mathbb{C}(z)$ independent of the representation $z = Ax$? Because the holomorphic map $A \mapsto A\phi(x)$ is determined by values on the maximal totally real subgroup $G$: for $A \in Aut_b(D)$ one has $A\phi(x) = \phi(Ax)$. We obtain a well-defined $G^\mathbb{C}$-equivariant map $\phi^\mathbb{C} : D \to D^\mathbb{C}$ with the property $\phi = \phi^\mathbb{C} \circ i$ (because for $z \in D$ one can choose $A = 1$). The holomorphicity of $\phi^\mathbb{C}$ is obtained by fixing $A$ in the formula $\phi^\mathbb{C}(z) := A\phi(x)$.

For the algebraic domains we obtain the following Corollaries.

Corollary 2.2 Let $D \subset \subset \mathbb{C}^n$ be a algebraic domain with non-degenerate boundary. Then the group $Aut_b(D)$ is holomorphically extendible and there exists an $Aut_b(D)$-complexification of $D$.

Corollary 2.3 Let $D \subset \subset \mathbb{C}^n$ be a algebraic domain with non-degenerate boundary which satisfies the condition $(W)$. Then the group $Aut_a(D)$ is holomorphically extendible and there exists an $Aut_a(D)$-complexification of $D$.
Corollary 2.4 Let $D \subset \subset \mathbb{C}^n$ be an algebraic domain which satisfies the condition (W). Then the group $\text{Aut}(D)$ is holomorphically extendible and there exists an $\text{Aut}(D)$-complexification of $D$.

3 Proof of the main Theorem

Let $D$ be a algebraic domain and $G$ a group of birational automorphisms of $D$. We prove the equivalence of the properties in Theorem 1.1 in the direction of the following two chains: $2 \implies 3 \implies 4 \implies 5 \implies 2$ and $2 \implies 1 \implies 4$.

$2 \implies 3$. The proof is trivial. □

$3 \implies 4$. Let $G$ be a subgroup of a Nash group $\hat{G}$ such that the action $G \times D \to D$ extends to a Nash action $\hat{G} \times D \to D$. We prove the statement for arbitrary Nash manifold $\hat{G}$ and Nash map $\hat{G} \times D \to D$ by induction on $\dim G$. It is obvious for $\dim G = 0$. Let $U \subset \hat{G}$ be a Nash coordinate chart and $\phi_i(g): D \to \mathbb{R}$ be the $i$th coordinate of $\phi_g: D \to D$ for $g \in U$. Since the map $\phi_i: U \times D \to \mathbb{R}$ is Nash, it satisfies a polynomial equation $P(g, x, \phi_i(g, x)) \equiv 0$. This yields polynomial equations of the same degree for all $g \in U$ outside a proper Nash submanifold. This submanifold has lower dimension and the statement is true for it by induction. In summary, we obtain the boundness of the degree for the whole neighborhood $U$ and, since the Nash atlas is finite, for $G$. □

$4 \implies 5$. Here is a sketch of the proof. The idea is to imbed the group $G$ into a complex algebraic variety so that the action on $D$ is given by a rational mapping. Using this mapping we construct a collection of homogeneous polynomials on $\mathbb{C}^{n+1}$ which generate a finite-dimensional linear subspace, invariant with respect to the action of $G$. These polynomials yield the required projective linearization.

The imbedding of $G$ is obtained by associating to every element $g \in G$ the Chow coordinates of the complex Zariski closure of the graph of the automorphism defined by $g$ (see Shafarevich, page 65). The main problem here is that the Chow scheme $C$ has infinitely many disjoint components parameterized by dimensions and degrees of subvarieties. In order to concern finitely many components of $C$ we have required the degree of automorphisms $\phi_g: D \to D$ to be bounded.

To every $g \in G$ one associates the $n$-dimensional (complex) Zariski closure $\tilde{\Gamma}_g \subset \mathbb{P}^n \times \mathbb{P}^n$ of the graph $\Gamma_g \subset D \times D$ of the automorphism defined by $g$. To regard $\tilde{\Gamma}_g$ as a subvariety of some $P^N$ let us consider the Segre imbedding:

$$v([z_0, ..., z_n], [w_0, ..., w_n]) = [z_i w_j]_{0 \leq i \leq n, 0 \leq j \leq n},$$

$$v: \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{n^2 + 2n}.$$
We set $N = n^2 + 2n$ and obtain a family of subvarieties $\rho(g) := v(\tilde{\Gamma}_g) \subset \mathbb{P}^N$ parameterized by $g \in G$. The family $V$ of all algebraic subvarieties of $\mathbb{P}^n$ is parameterized by the Chow scheme $C$ (see Shafarevich, page 65). Different automorphisms $g \in G$ define different subvarieties $v(\tilde{\Gamma}_g) \subset \mathbb{P}^N$ and one obtains an imbedding $\rho$ of $G$ in the Chow scheme $C$. The (complex) dimension of the subvarieties $v(\tilde{\Gamma}_g)$ is $n$. The degree of $v(\tilde{\Gamma}_g) \subset \mathbb{P}^N$ is the intersection number with $N - n$ generic linear hyperplanes $\{L_1 = 0\}, \ldots, \{L_{N-n} = 0\}$. It is equal to the intersection number of $\tilde{\Gamma}_g \subset \mathbb{P}^N \times \mathbb{P}^N$ with divisors $v^*L_1, \ldots, v^*L_{N-n}$. By the Bezout theorem this intersection number is bounded.

Thus, $G$ lies in fact in finitely many components of the Chow scheme $C$. Let $C_0$ denote the union of these components and $V_0$ the corresponding family of subvarieties of $\mathbb{P}^N$. We obtain an imbedding $\rho$ of $G$ in a complex projective variety $C_0$.

**Lemma 3.1** The imbedding $\rho: G \to C_0$ is continuous.

**Proof.** Assume the contrary. Then there exists a sequence $g_n \to g$ in $G$ such that no subsequence $\rho(g_{n(k)})$ converges to $\rho(g)$. On the other hand, since the degree of $\rho(g_n)$ is bounded, there exists a subsequence $\rho(g_{n(k)})$ which converges in $C_0$. This follows from the Theorem of Bishop (see e.g. F. Campana, page 4). Let $A \in C_0$ be the limes cycle of this subsequence. Our cycles lie in $v(\mathbb{P}^n \times \mathbb{P}^n)$ and we identify them with the preimages in $\mathbb{P}^n \times \mathbb{P}^n$. Since the action $G \times D \to D$ is continuous, the cycle $A$ contains the graph of $\phi_g$ and therefore its Zariski closure $\rho(g)$.

We claim that $A = \rho(g)$. This yields a contradiction with the choice of $\rho(g_{n(k)})$. Indeed, otherwise there exists a horizontal of vertical $n$-dimensional projective subspace $H \subset \mathbb{P}^n \times \mathbb{P}^n$ ($H = \{z\} \times \mathbb{P}^n$ or $H = \mathbb{P}^n \times \{w\}$) such that the intersection number of $A$ and $H$ is more than one. Then the intersection number of $\rho(g_{n(k)})$ and $H$ is also more than one which contradicts to the birationality of $\phi_{g_{n(k)}}$.

Further let $\tilde{G}$ be the complex Zariski closure of $G$ in $C_0$.

**Lemma 3.2** The action $\phi: G \times D \to D$ extends to a rational map $\tilde{\phi}: \tilde{G} \times \mathbb{P}^n \to \mathbb{P}^n$.

**Proof.** We begin with the construction of the graph $\Gamma_\tilde{\phi} \subset \tilde{G} \times \mathbb{P}^n \to \mathbb{P}^n$ of $\tilde{\phi}$. For this we regard $\mathbb{P}^n \times \mathbb{P}^n$ as a subset of $\mathbb{P}^N$ (via the Segre imbedding $v$ in (1)). We then define $\Phi = \Gamma_\tilde{\phi}$ to be the intersection of the Chow family $V_0$ with $\tilde{G} \times \mathbb{P}^n \to \mathbb{P}^n$. This is a complex algebraic variety. Moreover, for $g \in G$ and $x \in D$ the fibre $\Phi_{(g,x)} \subset \mathbb{P}^n$ consists of the single point $g(x)$. Since the set $G \times D$ is Zariski dense in $\tilde{G} \times \mathbb{P}^n$, this is true for every generic fibre of $\Phi$. This means that $\Phi$ is the graph of a rational map $\tilde{\phi}: \tilde{G} \times \mathbb{P}^n \to \mathbb{P}^n$.

The projective variety $\tilde{G}$ is imbedded in a projective space $\mathbb{P}^m$. The map $\tilde{\phi}$ can be extended to a rational map from $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^n$. Such map is given by
\( n+1 \) polynomials \( P_1(x,y), \ldots, P_{n+1}(x,y) \), homogeneous separately in \( x \in \mathbb{C}^{m+1} \) and \( y \in \mathbb{C}^{n+1} \). Let \( h \) be a fixed homogeneous polynomial on \( \mathbb{C}^{n+1} \). Then the function

\[
(x,y) \mapsto h(P_1(x,y), \ldots, P_{n+1}(x,y))
\]

is a separately homogeneous polynomial on \( \mathbb{C}^{m+1} \times \mathbb{C}^{n+1} \). The algebra \( \mathbb{C}_h[x,y] \) of such polynomials is equal to the tensor product \( \mathbb{C}_h[x] \otimes \mathbb{C}_h[y] \). Therefore there exist polynomials \( \varphi_i \in \mathbb{C}_h[x], \psi_i \in \mathbb{C}_h[y], i = 1, \ldots, l \) such that

\[
h(P_1(x,y), \ldots, P_{n+1}(x,y)) = \sum_{i=1}^{l} \varphi_i(x)\psi_i(y).
\]

For \( x = g \in G \) fixed we obtain

\[
\alpha_*(f^{-1})h = \sum_{i=1}^{l} c_i \psi_i(y),
\]

where \( \alpha_* \) denotes the associated action of \( G \) on homogeneous polynomials.

In other words, the orbit of \( h \) via the action of \( G \) is contained in the finite-dimensional subspace \(< \psi_1, \ldots, \psi_l > \subset \mathbb{C}_h[y] \). The linear hull of this orbit is a finite-dimensional \( G \)-invariant subspace containing \( h \).

We choose now sufficiently many polynomials \( h_j, j = 1, \ldots, s \) which separate the points of \( \mathbb{C}^{n+1} \) and such that neither \( h_j \) nor the differentials \( dh_j \) nowhere vanish simultaneously. They lie in a finite-dimensional \( G \)-invariant subspace \( L \subset \mathbb{C}_h[y] \). Let \( (p_1, \ldots, p_{N+1}) \) be a collection of homogeneous polynomials which yields a basis of \( L \). The required representation of \( G \) is the action on \( L \) and the polynomial map \((p_1, \ldots, p_{N+1}): \mathbb{C}^{n+1} \to \mathbb{P}^N \) defines the required projective linearization. \( \square \)

5 \implies 2. Assume we are given a projective linearization of the action of \( G \) on \( D \). It follows that the given representation of \( G \) is faithful and we identify \( G \) with its image in \( GL_{N+1}(\mathbb{C}) \). We define now the group \( \hat{G} \supset G \) to be the subgroup of all \( g \in GL_{N+1} \) such that \( g(i(D)) = i(D) \). It follows that \( G \subset \hat{G} \).

We wish to prove that \( G \) is a Nash subgroup of \( GL_{N+1}(\mathbb{C}) \). For the proof we use the technique of semialgebraic sets and maps which are closely related to the Nash manifolds and maps. The semialgebraic subsets of \( \mathbb{R}^n \) are the sets of the form \( \{ P_1 = \cdots = P_k, Q_1 < 0, \ldots, Q < s \} \) and finite unions of them where \( P_1, \ldots, P_k \) and \( Q_1, \ldots, Q_s \) are real polynomials on \( \mathbb{R}^n \). More generally, the semialgebraic subsets of a Nash manifold \( M \) are the subsets which have semialgebraic intersections with every Nash coordinate chart. The semialgebraic maps between semialgebraic sets are any maps with semialgebraic graphs. The Nash submanifolds of \( \mathbb{R}^n \) are exactly semialgebraic real analytic submanifolds and the Nash maps are semialgebraic real analytic maps.

Now the graph \( \Gamma \subset GL_{N+1} \times i(D) \to \mathbb{P}^N \) of the restriction to \( i(D) \) of the linear action of \( GL_{N+1} \) on \( \mathbb{P}^N \) is a semialgebraic subset. The condition \( g(i(D)) = i(D) \)
on \( g \) defines a semialgebraic subset \( \hat{G} \subset GL_{N+1} \). We proved this in the previous paper (see [11], Lemma 6.2). Since \( \hat{G} \) is a subgroup, it is Nash. Thus, \( G \) in a subgroup of the Nash group \( \hat{G} \) of birational automorphisms of \( i(D) \cong D \) with required properties. \( \square \)

\[ 2 \implies 1. \] A Nash open subset of \( \mathbb{R}^n \) is semialgebraic and has therefore finitely many connected components (see Benedetti-Risler, [3], Theorem 2.2.1). The Nash group \( \hat{G} \) admits a finite Nash atlas and has also finitely many components. \( \square \)

\[ 1 \implies 4. \] Assume \( G \) is a subgroup of a Lie group \( \hat{G} \) of birational automorphisms of \( D \) with finitely many connected components. Consider the complex coordinates \( \phi_i: \hat{G} \times D \to \mathbb{C} \) of the action of \( \hat{G} \). For fixed \( g \in \hat{G} \) the map \( \phi_i(g): D \to \mathbb{C} \) extends to a rational map \( \hat{\phi}_i(g): \mathbb{C}^n \to \mathbb{C} \). These extensions define a map \( \hat{\phi}_i: \hat{G} \times \mathbb{C}^n \to \mathbb{C} \). A priori we don’t know whether this new map is real analytic or even continuous. To prove this we use the following result of Kazaryan [5]:

**Proposition 3.1** Let \( D' \) be a domain in \( \mathbb{C}^n \) and let \( E \subset D' \) be a nonpluripolar subset. Let \( D'' \) be an open set in a complex manifold \( X \). If \( f \) is a meromorphic function on \( D' \times D'' \) such that \( f(g, \cdot) \) extends to a meromorphic function on \( X \) for all \( g \in E \), then \( f \) extends to a meromorphic function in a neighborhood of \( E \times X \subset D' \times X \).

**Lemma 3.3** The map \( \hat{\phi}_i: \hat{G} \times \mathbb{C}^n \to \mathbb{C} \) is real analytic.

**Proof.** The question of real analyticity of \( \hat{\phi}_i \) is local with respect to \( \hat{G} \) so we can take a real analytic coordinate neighborhood \( E \) in \( \hat{G} \), regarded as an open subset of \( \mathbb{R} \). The map \( \phi_i \) is real analytic in \( E \times D \) and extends therefore to a holomorphic function in a neighborhood \( D' \times D'' \) of \( E \times D'' \) in the complex manifold \( \mathbb{C} \times X \). Here we must replace \( D \) by a bit smaller neighborhood \( D'' \subset D \). The set \( E \), being an open subset of \( \mathbb{R} \), is nonpluripolar. By Proposition [5], \( \phi_i \) extends to a meromorphic functions in a neighborhood of \( E \times X \). The restriction \( \hat{\phi}_i \) is therefore real analytic. \( \square \)

According to the construction of Chow scheme (see Shafarevich, [8], p.65) every graph \( \Gamma_{\hat{\phi}_i} \subset \mathbb{C}^{2n} \subset \mathbb{P}^{2n} \) has its Chow coordinate in the Chow scheme \( C \). The Chow coordinates yield a continuous mapping \( f: \hat{G} \to C \). This is the universal property of the Chow scheme. It follows from the Theorems of D. Barlet on universality of the Barlet space and on the equivalence of the latter to the Chow scheme in case of projective space (see Barlet, [1]). Since \( G \) has finitely many components, the image in \( C \) is has also this property. But the

\[ ^1 \text{A subset } E \subset D' \text{ is called nonpluripolar if there are no plurisubharmonic functions } f: D' \to \mathbb{R} \cup \{-\infty\} \text{ such that } f|_E \equiv -\infty. \]
degree of variety is constant on the components of $C$. This implies that the degree of the variety in $\mathbb{P}^{2n}$ associated to $g \in \hat{G}$ is bounded. This implies that the degrees of defining polynomials are bounded and the statement is proven. □

References

[1] D. Barlet. Espace cycles analytique complexes de dimension finie. Number 482 in Seminaire F.Norguet, Lecture Notes in Math., pages 1–158. Springer, 1975.

[2] R. Benedetti and J.-J. Risler. Real algebraic and semi-algebraic sets. Actualites Mathematiques. Hermann Editeurs des Sciences et des Arts, 1990.

[3] F. Campana. Application de l’espace des cycles à la classification biméromorphe des espaces analytiques Kähleriens compacts. Equipe associée d’Analyse Globale. Institut Elie Cartan, 1980.

[4] K. Diederich and J. E. Fornæss. Applications holomorphes propre entre domaines à bord analytique réel. C. R. Acad. Sci. Paris, 307(I):321–324, 1988.

[5] P. Heinzner. Equivariant holomorphic extensions of real analytic manifolds. Technical Report 40, Mathematica Gottingensis, Schriftenreihe des Sonderforschungsbereichs Geometrie und Analysis, 1991.

[6] P. Heinzner. Geometric invariant theory on stein spaces. Math. Ann., 289:631–662, 1991.

[7] G. Hochschild. The structure of Lie groups. Holden-Day. San Francisco London Amsterdam, 1965.

[8] Kazaryan. Meromorphic continuation with respect to groups of variables. Math. USSR-Sb., 53:385–398, 1986.

[9] I. R. Shafarevich. Basic algebraic geometry. Springer, 1974.

[10] S. Webster. On the mapping problem for algebraic real hypersurfaces. Inventiones math., 43:53–68, 1977.

[11] D. Zaitsev. On the automorphism groups of algebraic bounded domains. to appear in Math. Ann., 1994.