Quantum homotopy perturbation method for nonlinear dissipative ordinary differential equations

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While quantum computing provides an exponential advantage in solving linear differential equations, there are relatively few quantum algorithms for solving nonlinear differential equations. In our work, based on the homotopy perturbation method, we propose a quantum algorithm for solving n-dimensional nonlinear dissipative ordinary differential equations (ODEs). Our algorithm first converts the original nonlinear ODEs into the other nonlinear ODEs which can be embedded into finite-dimensional linear ODEs. Then we solve the embedded linear ODEs with quantum linear ODEs algorithm and obtain a state ε-close to the normalized exact solution of the original nonlinear ODEs with success probability Ω(1). The complexity of our algorithm is $O(g\eta T \text{poly}(\log(nT/\epsilon)))$, where $\eta, g$ measure the decay of the solution. Our algorithm provides exponential improvement over the best classical algorithms or previous quantum algorithms in $n$ or $\epsilon$.

Keywords: Quantum algorithm, nonlinear dissipative ordinary differential equations, homotopy perturbation method

I. INTRODUCTION

Nonlinear differential equations appear in many fields, such as fluid dynamics, biology, finance, etc. In general, the analytical solutions of nonlinear differential equations cannot be obtained effectively. Numerical methods are often used to solve nonlinear differential equations. However, when solving high-dimensional nonlinear differential equations, too many computing resources are required, which may exceed the computing power of classic computers. It is important to develop more efficient algorithms for solving nonlinear differential equations.

Quantum computing provides a promising way to speed up the solution of various equations. In recent years many quantum algorithms have been developed to solve various equations, such as system of linear equations [1–4], Poisson equation [5], Dirac equation [6], heat equation [7], linear ordinary differential equations (ODEs) [8–11], linear partial ODEs [12, 13] and so on [14–16].

However, because of the linearity of quantum mechanics, solving nonlinear equations with quantum computing is challenging, some related algorithms are proposed [17–25]. An early quantum algorithm for solving nonlinear ordinary differential equations (ODEs) is proposed in [17], but the complexity of the algorithm increases exponentially with the evolution time. In [19], the authors proposed a variational quantum algorithm for solving nonlinear differential equations and demonstrate the algorithm by solving 1-dimensional nonlinear Schrödinger equation. However, when the equations become complicated, whether the parameterized quantum circuit used in their work is

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We focus on an initial value problem described by the $n$-dimensional quadratic ODEs. The problem to be solved is defined as

$$\frac{du}{dt} = F_1 u + F_2 u \otimes u, \quad u(0) = u_{in},$$

(1)
where \( u, u_{in} \in \mathbb{R}^n \), \( F_1 \in \mathbb{R}^{n \times n} \), \( F_2 \in \mathbb{R}^{n \times n^2} \) are time independent sparse matrices. The sparsity of \( F_1, F_2 \) is \( s \), which means the number of non-zero elements in each row or column of \( F_1, F_2 \) does not exceed \( s \). We assume \( F_1 \) is a normal matrix and the eigenvalues \( \lambda_i \) of \( F_1 \) satisfy \( \text{Re}(\lambda_n) \leq \cdots \leq \text{Re}(\lambda_1) < 0 \). We also assume oracles \( O_{F1}, O_{F2} \) and \( O_u \) are given, \( O_{F1}, O_{F2} \) are used to extract the non-zero position and value of \( F_1, F_2 \) respectively, and \( |u_{in}\rangle \) is prepared with \( O_u \). In specific, \( O_{F1}, O_{F2} \) and \( O_u \) are defined as

\[
\begin{align*}
O_{F11}(j)|k\rangle &= |j\rangle |f_1(j,k)\rangle, \\
O_{F12}(j)|k\rangle |z\rangle &= |j\rangle |k\rangle |z \oplus F_{1j,k}\rangle, \\
O_{F13}(j)|0\rangle &= |j\rangle |g(j)\rangle, \\
O_{F21}(j)|k\rangle &= |j\rangle |f_2(j,k)\rangle, \\
O_{F22}(j)|k\rangle |z\rangle &= |j\rangle |k\rangle |z \oplus F_{2j,k}\rangle, \\
O_u|0\rangle &= |u_{in}/\|u_{in}\||.
\end{align*}
\]

(2)

where \( f_1(j,k) \) and \( f_2(j,k) \) represent the column number of the \( k \)-th non-zero element in \( j \)-th row of \( F_1, F_2 \) respectively, \( g(j) \) satisfies \( f_1(j,g(j)) = j \), here we treat the diagonal element of \( F_1 \) as a non-zero element. \( O_{F13} \) is used to construct an oracle of a matrix related to \( F_1 \), the details are shown in the proof of Lemma 5.

We define a parameter \( K \) which characterizes the nonlinearity of Eq.(1),

\[
K := \frac{4\|u_{in}\|\|F_2\|}{|\text{Re}(\lambda_1)|},
\]

(3)

We assume \( K \geq \|u_{in}\| \), if this is not satisfied, we rescale \( u \) to \( \zeta u \) with a suitable constant \( \zeta \) which keeps \( \frac{4\|u_{in}\|\|F_2\|}{|\text{Re}(\lambda_1)|} \) unchanged and makes \( K \geq \|u_{in}\| \). In this paper we use spectral norm, it means \( \| \cdot \| = \| \cdot \|_2 \).

## III. QUANTUM HOMOTOPY PERTURBATION METHOD

In this section, we give the whole process of quantum homotopy perturbation method for solving Eq.(1). It contains four steps:

1. Using homotopy perturbation method to transform Eq.(1) into Eq.(6), a series of nonlinear ODEs with variable \( \nu_t \).
2. Embedding Eq.(6) into Eq.(8), linear ODEs with variable \( \bar{y} \).
3. Solving Eq.(8) with quantum algorithm proposed in [9].
4. Measurement.

The following four subsections introduce the details of these four steps.

### A. Homotopy perturbation method

Firstly, we introduce the process of homotopy perturbation method [28–30] for solving Eq.(1). Using homotopy method, we construct homotopy \( \nu(t,p) : \mathbb{R}^+ \times [0,1] \rightarrow \mathbb{R}^n \), which satisfies

\[
H(\nu, p) = \frac{d\nu}{dt} - F_1 \nu - pF_2 \nu^{\otimes 2} = 0, \quad \nu(0,p) = u_{in}.
\]

(4)
Assuming $\nu$ is represented as
\[
\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots + p^c\nu_c. \tag{5}
\]
Substituting Eq.(5) into Eq.(4), then equating the terms with identical powers of $p$, we have the following equations:
\[
\frac{d\nu_0}{dt} - F_1\nu_0 = 0, \quad \nu_0(0) = u_{in},
\]
\[
\frac{d\nu_1}{dt} - F_1\nu_1 - F_2\nu_0 \otimes \nu_0 = 0, \quad \nu_1(0) = 0,
\]
\[
\frac{d\nu_2}{dt} - F_1\nu_2 - F_2(\nu_0 \otimes \nu_1 + \nu_1 \otimes \nu_0) = 0, \quad \nu_2(0) = 0,
\]
\[
\cdots
\]
\[
\frac{d\nu_c}{dt} - F_1\nu_c - F_2 \sum_{j=0}^{c-1} \nu_j \otimes \nu_{c-1-j} = 0, \quad \nu_c(0) = 0. \tag{6}
\]
When $p = 1$ in Eq.(4), we have
\[
\tilde{\nu} = \nu_0 + \nu_1 + \nu_2 + \cdots + \nu_c. \tag{7}
\]
The difference between $\tilde{\nu}$ and $\nu$ is analyzed in Sect. VI A.

B. Linear embedding

Secondly, Eq.(6) is embedded into the linear ODEs defined in Eq.(8):
\[
\frac{d\vec{y}}{dt} = A\vec{y}, \quad \vec{y}(0) = y_{in}, \tag{8}
\]
where $\vec{y} = [y_0, y_1, \ldots, y_c]$, $\vec{y}_i$ satisfies
\[
\vec{y}_i = \left\{ \begin{array}{ll}
[\nu_0 + \nu_1 + \cdots + \nu_c], & i = 0, \\
[y_{i,0}, y_{i,1}, \ldots, y_{i,\beta_i-1}], & 1 \leq i \leq c,
\end{array} \right.
\]
where $\beta_i$ represents the number of items in $\vec{y}_i$, $\vec{y}_{i,j}$ represents $j$-th item of $\vec{y}_i$, it is represented as
\[
\vec{y}_{i,j} = \otimes_{k=0}^{i} \nu_{a_{i,j,k}}, \quad a_{i,j,k} \text{ satisfies }
\]
\[
a_{i,j,k} \geq 0, \quad i + 1 \leq \sum_{k=0}^{i} (a_{i,j,k} + 1) \leq c + 1. \tag{10}
\]
By Eq.(10), $\beta_i$ satisfies
\[
\beta_i = \left\{ \begin{array}{ll}
1, & i = 0, \\
\sum_{k=1}^{c} \binom{c}{i}, & 1 \leq i \leq c.
\end{array} \right.
\]
We set $\vec{y}_{i,0} = \nu_0^{\otimes i+1}$, $i = 1, 2, \ldots, c$, then by Eq.(6), $y_{in}$ is written as
\[
y_{in} = [u_{in}], [u_{in}^{\otimes 2}, 0, \ldots, 0], [u_{in}^{\otimes 3}, 0, \ldots, 0] \ldots, [u_{in}^{\otimes c+1}]. \tag{12}
\]
We define \( \vec{a}_{i,j} = [a_{i,j,0}, a_{i,j,1}, \ldots, a_{i,j,i}] \), the mapping \( \vec{a}_{i,j} \mapsto j \) is one-to-one mapping, so we can construct the following two operations

\[
O_{a_1}|\vec{a}_{i,j}|0\rangle = |\vec{a}_{i,j}|j\rangle,
\]

\[
O_{a_2}|i\rangle|j\rangle|0\rangle = |i\rangle|j\rangle|\vec{a}_{i,j}\rangle
\]

with \( O(c) \)-qubit quantum arithmetic circuit, and the gate complexity is \( O(\text{poly}(c)) \). \( O(\text{poly}(c)) \) will not influence the complexity of our algorithm, so the complexity of \( O_{a_1} \) and \( O_{a_2} \) can be ignored in the following analysis. The dimension of \( \vec{y}_i \) is \( n^{i+1} \beta_i \), so the dimension of \( \vec{y} \) is

\[
N = \sum_{i=0}^{c} n^{i+1} \beta_i = (n + 1)^{c+1} - 1 - cn \approx (n + 1)^{c+1}.
\]  

Next we analyze the structure of matrix \( A \). We have

\[
\frac{d\vec{y}_{i,j}}{dt} = (\sum_{j=0}^{i} I_n ^{\otimes j} \otimes F_1 \otimes I_n ^{\otimes i-j})\vec{y}_{i,j} + \sum_{k=0}^{i} (I_n ^{\otimes k} \otimes F_2 \otimes I_n ^{\otimes i-k})\nu_{a_{i,j,k}} \otimes \cdots \otimes \nu_{a_{i,j,k-1}} \otimes (\sum_{l=0}^{a_{i,j,k-1}} \nu_{a_{i,j,k-l}} \otimes \nu_{a_{i,j,k-l+1}} \otimes \cdots \otimes \nu_{a_{i,j,i}}),
\]

where \( \nu_{a_{i,j,0}} \otimes \cdots \otimes \nu_{a_{i,j,k-1}} \otimes \nu_{a_{i,j,k-1-l}} \otimes \nu_{a_{i,j,k-l+1}} \otimes \cdots \otimes \nu_{a_{i,j,i}} \in \mathcal{y}_{i+1} \), so Eq.(8) can be written as

\[
\frac{d}{dt} \begin{pmatrix} \vec{y}_0 \\ \vec{y}_1 \\ \vdots \\ \vec{y}_{c-1} \\ \vec{y}_c \end{pmatrix} = \begin{pmatrix} A_{0,0} & A_{0,1} & A_{1,2} \\ A_{1,1} & \ddots & \ddots \\ \vdots & \ddots & A_{c-1,c-1} & A_{c-1,c} \\ A_{c-1,c-1} & \ddots & A_{c,c} \end{pmatrix} \begin{pmatrix} \vec{y}_0 \\ \vec{y}_1 \\ \vdots \\ \vec{y}_{c-1} \\ \vec{y}_c \end{pmatrix},
\]

\( A_{i,i} \) is \( n^{i+1} \beta_i \) dimensional square matrix, which is represented as

\[
A_{i,i} = I_{\beta_i} \otimes (\sum_{j=0}^{i} I_n \otimes F_1 \otimes I_n ^{\otimes i-j}),
\]

\( A_{i,i+1} \) is \( n^{i+1} \beta_i \times n^{i+2} \beta_{i+1} \) dimensional matrix, its elements are determined by Eq.(16). \( |y(t)\rangle \) is defined to represent \( \vec{y}(t) \):

\[
|y(t)\rangle = \sum_{i=0}^{c} \sum_{j=0}^{\beta_i-1} |i, j\rangle |y_{i,j}(t)\rangle.
\]
C. Quantum linear ODEs algorithm

Thirdly, Eq.(8) is solved with the quantum algorithm proposed in [9]. \( \vec{y}(t) \) is written as

\[
\vec{y}(t) = e^{At} \vec{y}(0).
\]  

(20)

We define \( T_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!} \). When \( k \) is large enough and the evolution time \( h \) is relatively short (for example, \( h \leq \frac{1}{\|A\|} \)), we have \( \vec{y}(h) \approx T_k(Ah) \vec{y}(0) \). This approximate solution can be used as the initial condition for the next approximation, repeating this procedure \( m \) steps we have the approximation of \( \vec{y}(mh) \).

Next we introduce the details of the algorithm proposed in [9]. Let \( m, k, p \in \mathbb{Z}^+ \) and define

\[
C_{m,k,p}(A) := \sum_{j=0}^{d} |j\rangle \langle j| \otimes I - \sum_{i=0}^{m-1} \sum_{j=1}^{k} |i(k+1) + j\rangle \langle i(k+1) + j - 1| \otimes A/j
\]

\[
- \sum_{i=0}^{m-1} \sum_{j=0}^{k} |(i + 1)(k+1)\rangle \langle i(k+1) + j| \otimes I - \sum_{j=d-p+1}^{d} |j\rangle \langle j - 1| \otimes I,
\]  

(21)

where \( d := m(k+1) + p \), \( I \) is an \( N \)-dimensional unit matrix. We consider the linear system

\[
C_{m,k,p}(Ah)|x\rangle = |0\rangle |y_{in}\rangle,
\]  

(22)

where \( |y_{in}\rangle \in \mathbb{C}^N, h \in \mathbb{R}^+ \). After evolving \( m \) steps, the approximate solution of \( k \)-order Taylor series is obtained, and the solution remains unchanged at \( p \) steps. The solution of Eq.(22) is represented as \( |x\rangle = C_{m,k,p}(Ah)^{-1}|0\rangle |y_{in}\rangle \), it can also be written as

\[
|x\rangle = \sum_{i=0}^{m-1} \sum_{j=0}^{k} |i(k+1) + j\rangle |x_{i,j}\rangle + \sum_{j=0}^{p} m(k+1) + j \rangle |x_{m,j}\rangle.
\]

(23)

By Eq.(21), \( |x_{i,j}\rangle \) satisfies

\[
|x_{0,0}\rangle = |y_{in}\rangle,
\]

\[
|x_{i,0}\rangle = \sum_{j=0}^{k} |x_{i-1,j}\rangle, \quad 1 \leq i \leq m,
\]

\[
|x_{i,1}\rangle = Ah|x_{i,0}\rangle, \quad 0 \leq i < m,
\]

\[
|x_{i,j}\rangle = (Ah/j)|x_{i,j-1}\rangle, \quad 0 \leq i < m, 2 \leq j \leq k,
\]

\[
|x_{m,j}\rangle = |x_{m,j-1}\rangle, \quad 1 \leq j \leq p.
\]

(24)
Then we have
\[ |x_{0,0}⟩ = |y_{in}⟩, \]
\[ |x_{0,j}⟩ = ((Ah)^j/j!) |x_{0,0}⟩, \quad 1 \leq j \leq k, \]
\[ |x_{1,0}⟩ = T_k(Ah) |x_{0,0}⟩ \approx \exp(Ah) |y_{in}⟩, \]
\[ |x_{1,j}⟩ = ((Ah)^j/j!) |x_{1,0}⟩, \quad 1 \leq j \leq k, \]
\[ |x_{2,0}⟩ = T_k(Ah) |x_{1,0}⟩ \approx \exp(2Ah) |y_{in}⟩, \]
\[ \vdots \]
\[ |x_{m−1,0}⟩ = T_k(Ah) |x_{m−2,0}⟩ \approx \exp(Ah(m−1)) |y_{in}⟩, \]
\[ |x_{m−1,j}⟩ = ((Ah)^j/j!) |x_{m−1,0}⟩, \quad 1 \leq j \leq k, \]
\[ |x_{m,0}⟩ = T_k(Ah) |x_{m−1,0}⟩ \approx \exp(Ahm) |y_{in}⟩, \]
\[ |x_{m,j}⟩ = |x_{m,0}⟩ \approx \exp(Ahm) |y_{in}⟩, \quad 1 \leq j \leq p, \]

|xi,0⟩ is the approximate solution of the system at time \( ih \), \( i \in \{0, 1, 2, \ldots, m\} \). \( |xm,0⟩ = |xm,1⟩ = \cdots = |xm,p⟩ \) is the approximate solution of \( \bar{y}(t) = e^{At}y_{in} \) at \( t = mh \).

\section*{D. Measurement}

Finally, we measure some qubit registers of \( |x⟩ \) and get a state \( \epsilon \)-close to the normalized solution of Eq.(1). The measurement is divided into two steps: (1) Measure the first qubit register of \( |x⟩ \) which is defined in Eq.(23), if the result is \( |m(k+1)+j⟩, j = 0, 1, \ldots, p \), we have \( |y(t)⟩ \) in the second qubit register of \( |x⟩ \). (2) Measure the first qubit register of \( |y(t)⟩ \) which is defined in Eq.(19), if the result is \( |0, 0⟩ \), we get a state \( \epsilon \)-close to \( |u(t)/∥u(t)∥⟩ \) in the qubit second register of \( |y(t)⟩ \).

This measurement step is probabilistic, the success probability is analyzed in Sect.VII.

\section*{IV. STATE PREPARATION AND ORACLE CONSTRUCTION}

In this section, we give the preparation of \( |y_{in}⟩ \) and oracle construction of \( A \).

\subsection*{A. State preparation}

We first discuss the preparation of \( |y_{in}⟩ \), the result is shown in Lemma 1.

\textbf{Lemma 1} By Eq.(12), \( |y_{in}⟩ \) is defined as
\[ |y_{in}⟩ = \frac{1}{\sqrt{M}} ∑_{i=0}^{c} |i, 0⟩|u_{m}^{(i+1)}⟩, \quad (26) \]

where \( M = ∑_{j=i}^{c} ∥u_{m}∥^{2(i+1)} \). Given \( O_{u} \) defined in Eq.(2), \( |y_{in}⟩ \) can be prepared by querying \( O_{u} \) \( O(c) \) times.

\textbf{Proof} First we prepare
\[ |ψ⟩ = \frac{1}{\sqrt{M}} ∑_{i=0}^{c} ∥u_{m}∥^{2(i+1)} |i, 0⟩. \quad (27) \]
Then we execute controlled $O_u$ operation $C - O_u = \sum_{i=0}^{c} |i\rangle \langle i| \otimes O_u^{i+1}$ on $|\psi\rangle$,

$$|y_{in}\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{c} |i\rangle |u_{in}^{i+1}\rangle.$$ \hfill (28)

The query complexity of $O_u$ is $O(c)$.

**B. Oracle construction of $A$**

Before introducing oracle construction of $A$, we analyze some features of $A$, including sparsity, upper bound of $\|A\|$ and eigenvalue of $A$. The results are shown in Lemma 2, Lemma 3 and Lemma 4.

*Lemma 2* The sparsity of the matrix $A$ is $O(sc^2)$.

*Proof* The sparsity $A_{i,i}$ is $(i + 1)sc$. The sparsity of $A_{0,1}$ is $sc(c + 1)/2$, when $i \geq 1$, the sparsity of $A_{i,i+1}$ is $\max\{s(c-i), s(i+1)\}$. Therefore, the sparsity of matrix $A$ is $O(sc^2)$.

*Lemma 3* $\|A\|$ satisfies $\|A\| \leq (c + 1)(\|F_1\| + \|F_2\|)$.

*Proof* By the definition of $A_{i,i}$, we have

$$\|A_{i,i}\| \leq (c + 1)\|F_1\|, \quad i \in [c + 1]_0,$$ \hfill (29)

where $[c + 1]_0 = [0, 1, \ldots, c]$ and in this paper, for any $i \in \mathbb{N}$, $[i + 1]_0 = [0, 1, \ldots, i]$. Then we analyze upper bound of $\|A_{i,i+1}\|$. We have $A_{0,1}A_{0,1}^T = \frac{c(c+1)}{2} F_2F_2^T$, then $\|A_{0,1}\|$ satisfies

$$\|A_{0,1}\| = \sqrt{\frac{c(c+1)}{2}} \|F_2\| < (c + 1)\|F_2\|.$$ \hfill (30)

When $i \geq 1$, $A_{i,i+1}$ can be regarded as $\beta_i \times \beta_{i+1}$ dimensional block matrix, each block unit is an $n_i+1 \times n_{i+2}$ dimensional matrix and has the form $I^{\otimes j} \otimes F_2 \otimes I^{\otimes i-j}$, $j \in [i+1]_0$. From the structure of $A_{i,i+1}$, the number of non-zero block unit in each row or column of $A_{i,i+1}$ is no more than $c$, so $A_{i,i+1}$ can be divided into at most $c$ matrices which have only one non-zero block unit in each row or column. Therefore,

$$\|A_{i,i+1}\| \leq c\|F_2\|, \quad i \in [1, 2, \ldots, c - 1].$$ \hfill (31)

Combining Eq.(29), Eq.(30) and Eq.(31), $\|A\|$ satisfies

$$\|A\| \leq \|\text{diag}(A_{0,0}, A_{1,1}, \ldots, A_{c,c})\| + \|\text{diag}(A_{0,1}, A_{1,2}, \ldots, A_{c-1,c})\|$$

$$= \max_{i=0}^{c}\{\|A_{i,i}\|\} + \max_{i=0}^{c-1}\{\|A_{i,i+1}\|\}$$

$$\leq (c + 1)(\|F_1\| + \|F_2\|).$$ \hfill (32)

*Lemma 4* The eigenvalue $\gamma_i$ of matrix $A$ satisfies $\text{Re}(\gamma_i) < 0, i \in [N]_0$.

*Proof* From the structure of $A$, the eigenvalues of $A$ are the sets of eigenvalues of $A_{i,i}$ for $i \in [c + 1]_0$. The eigenvalue of $A_{i,i}$ is the sum of $i + 1$ eigenvalues of $F_1$. For any $j \in [1, 2, \ldots, n]$, eigenvalue of $F_1 \lambda_j$ satisfies $\text{Re}(\lambda_j) < 0$, so the real part of all eigenvalues of $A_{i,i}$ is less than 0. Therefore, the eigenvalue $\gamma_i$ of $A$ satisfies $\text{Re}(\gamma_i) < 0, i \in [N]_0$.

Next, we introduce the way to construct oracle $O_A$ of $A$. $O_A$ gives the way to extract non-zero element position and value of $A$. We first give Lemma 5.
Lemma 5 Let matrix \( B(m) = \sum_{j=0}^{m} I_n^j \otimes F_1 \otimes I_n^{m-j} \). Oracles \( O_{m,1} \) and \( O_{m,2} \) are defined as
\[
O_{m,1}|\vec{j}\rangle|l\rangle = |\vec{j}\rangle |g_m(\vec{j}, l)\rangle,
\]
\[
O_{m,2}|\vec{j}\rangle|\vec{k}\rangle|z\rangle = |\vec{j}\rangle |\vec{k}\rangle |z \oplus B(m)_{\vec{j}, \vec{k}}\rangle,
\]
where \( \vec{j} = j_m j_{m-1} \ldots j_0 \), \( j_i \in [n]_0 \). \( g_m(\vec{j}, l) \) represents the column number of \( l \)-th non-zero element in \( j \)-th row of \( B(m) \), \( g_m(\vec{j}, l) \) is also written as \( g_m(\vec{j}, l) = k_m k_{m-1} \ldots k_0 \), \( k_i \in [n]_0 \). Then \( O_{m,1} \), \( O_{m,2} \) can be constructed by querying \( O_{F1} \) \( m + 1 \) times.

Proof When \( m = 0 \), \( B(0) = F_1 \), \( O_{0,1} \), \( O_{0,2} \) can be constructed by querying \( O_{F1} \) once. Assuming when \( m \geq 1 \), Oracles \( O_{m-1,1} \), \( O_{m-1,2} \) of \( B(m-1) \) are constructed by querying \( O_{F1} \) \( m \) times. \( B(m) \) is also written as
\[
B(m) = I \otimes B(m-1) + F_1 \otimes I_n^{m}.
\]
Generally, we treat the diagonal element of \( F_1 \) as a non-zero element, so the sparsity of \( B(m) \) is \((m + 1)c - m\). \( g_m(\vec{j}, k) \) can be represented as
\[
g_m(\vec{j}, k) = \begin{cases} 
  f_1(j_m, k) j_{m-1} \ldots j_0, & 0 \leq k < g(j_m), \\
  j_m g_m(\vec{j}, k) j_{m-1} \ldots j_0, & 0 \leq k - g(j_m) \leq m(c - 1) + 1, \\
  f_1(j_m, k - m(c - 1)) j_{m-1} \ldots j_0, & g(j_m) + 1 \leq k - m(c - 1) \leq c - 1,
\end{cases}
\]
where \( f_1(j, k) \) and \( g(j) \) are defined in Eq.(2). Then \( O_{m,1} \) can be constructed with Eq.(35), the construction process needs to query \( O_{m-1,1}, O_{F11}, O_{F13} \) once each.

On the other hand, the element of \( B(m) \) is written as
\[
B(m)_{\vec{j}, \vec{k}} = \delta_{j_m k_m} B(m-1)_{j_{m-1} \ldots j_0 \ldots k_{m-1} \ldots k_0} + \delta_{j_{m-1} \ldots j_0 \ldots k_{m-1} \ldots k_0} B_{j_{m-1} \ldots j_0 \ldots k_{m-1} \ldots k_0}.
\]
By Eq.(36), \( O_{m,2} \) can be constructed by querying \( O_{m-1,2} \) and \( O_{F12} \) once each. Therefore \( O_{m,1} \) and \( O_{m,2} \) can be constructed by querying \( O_{m-1,1}, O_{m-1,2}, O_{F11}, O_{F12} \) and \( O_{F13} \) once each.

From the above analysis, \( O_{m,1} \) and \( O_{m,2} \) can be constructed by querying \( O_{F1} \) \( m + 1 \) times.

Lemma 6 The oracle \( O_A \) of \( A \) can be constructed by querying \( O_{F1} \) \( O(c) \) times and querying \( O_{F2} \) \( O(1) \) times.

Proof To construct \( O_A \), we need to construct oracles of \( A_{i,i} \) and \( A_{i,i+1} \). We first consider \( A_{i,i} \), we define \( B(i) = \sum_{j=0}^{i} I_n^j \otimes F_1 \otimes I_n^{m-1-j} \), by Lemma 5, the oracle of \( B(i) \) can be constructed by querying \( O_{F1} \) \( i + 1 \) times. By Eq.(18), the oracle of \( A_{i,i} \) can be constructed by querying oracle of \( B(i) \) once.

Next we consider \( A_{i,i+1} \). When \( i = 0 \), \( A_{0,1} = [F_2, F_2, \ldots, F_2] \), the oracle of \( A_{0,1} \) is constructed by querying \( O_{F2} \) once. When \( i \geq 1 \), \( A_{i,i+1} \) is also regarded as \( \beta_i \times \beta_{i+1} \) dimensional block matrix \( D(i) \), the dimension of each block unit is \( n^i+1 \times n^{i+2} \). The number of non-zero block unit in \( j \)-th row of \( D(i) \) is \( \sum_{k=0}^{i} a_{i,j,k} \). Consider the \( l \)-th non-zero block unit in \( j \)-th row of \( D(i) \), \( l \) is represented as \( l = \sum_{k=0}^{i} a_{i,j,k} + i_2 \), where \( i_1 \in [i]_0, i_2 \in \mathbb{N} \), the \( l \)-th non-zero block unit is \( I_n^{i_1} \otimes F_2 \otimes I_n^{i_2} \), and the corresponding \( \vec{a}_{i+1,j} \) is represented as
\[
\vec{a}_{i+1,j} = [a_{i,j,0}, \ldots, a_{i,j,i_1}, i_2, a_{i,j,i_1+1} - 1 - i_2, a_{i,j,i_2+1}, \ldots, a_{i,j,i_2}],
\]
\( j' \) can be obtained from \( \vec{a}_{i+1,j} \) with \( O_{d1} \). The oracle of the non-zero block unit \( I_n^{i_1} \otimes F_2 \otimes I_n^{i_2} \) is constructed by querying \( O_{F2} \) once. By realizing the above process with quantum circuit, we construct
an oracle that extracts the non-zero position of $A_{i,i+1}$. The specific implementation process is
\[
\begin{align*}
|j, b_i, b_{i-1}, \ldots, b_0\rangle &\rangle |l, l_0\rangle |0, 0\rangle \\
O_a &\rightarrow |j, b_i, b_{i-1}, \ldots, b_0\rangle |l, l_0\rangle |i_1, i_2\rangle \\
O_b &\rightarrow |j, b_i, b_{i-1}, \ldots, b_0\rangle |j', b_i, b_{i-1}, \ldots, b_0\rangle |i_1, i_2\rangle \\
O_F &\rightarrow |j, b_i, b_{i-1}, \ldots, b_0\rangle |j', b_i, b_{i-1}, \ldots, f_2(b_i, b_{i-1}, \ldots, b_0)|i_1, i_2\rangle \\
\text{uncompute} &\rightarrow |j, b_i, b_{i-1}, \ldots, b_0\rangle |j', b_i, b_{i-1}, \ldots, f_2(b_i, b_{i-1}, \ldots, b_0)|0, 0\rangle.
\end{align*}
\]

There are some ancilla qubits to represent $|\tilde{a}_{i,j}\rangle$ and $|\tilde{a}_{i+1,j'}\rangle$ in the process shown in Eq. (38). For simplicity, we ignore the compute and uncompute process of $|\tilde{a}_{i,j}\rangle$ and $|\tilde{a}_{i+1,j'}\rangle$.

The oracle that extracts the non-zero value of $A_{i,i+1}$ can also be constructed in a similar process. For any $j \in [\beta_i]_0$, $k \in [\beta_{i+1}]_0$, we can judge whether $D(i)_{j,k}$ is a non-zero block unit and use $l_{j,k}$ to represent the judgement. If $D(i)_{j,k}$ is a non-zero block unit, we can find $i_1$ and represent it as $I^{\otimes i_1} \otimes F_2 \otimes I^{\otimes i_1}$, then we use $O_{F_2}$ to extract the elements of the non-zero block unit. The whole process is shown as
\[
\begin{align*}
|j, k, b_{i-1}, \ldots, b_0\rangle |i_1, i_0\rangle &\rangle |0, 0\rangle |z\rangle \\
O_{a1, a2} &\rightarrow |j, k, b_{i-1}, \ldots, b_0\rangle |l, l_0\rangle |i_1, i_0\rangle |z\rangle \\
O_{F_2} &\rightarrow |j, k, b_{i-1}, \ldots, b_0\rangle |l, l_0\rangle |i_1, i_0\rangle |z \oplus l_{j,k} \times F_2 b_{i-1}, b_i', \prod_{t \in [i], t \neq i_1} \delta_{b_t b_i'}\rangle \\
\text{uncompute} &\rightarrow |j, k, b_{i-1}, \ldots, b_0\rangle |i_1, i_0\rangle |z \oplus D(i)_{j,k}\rangle.
\end{align*}
\]
The query complexity of $O_{F_2}$ in the above process is $O(1)$. Therefore, oracle of $A_{i,i+1}$ can be constructed by querying $O_{F_2}O(1)$ times.

After constructing the oracles of $A_{i,i}$ and $A_{i,i+1}$, the oracle of $A$ can be directly constructed by querying oracles of $A_{i,i}$ and $A_{i,i+1}$ once. So the oracle $O_A$ can be constructed by querying $O_{F_1}O(c)$ times and querying $O_{F_2}O(1)$ times.

V. CONDITION NUMBER

In this section, we give an upper bound of the condition number of $C_{m,k,p}(Ah)$ defined in Sect. III C. We first analyze the upper bound of $\|e^{At}\|$, we have the following lemma.

Lemma 7 Consider the matrix $A$ defined in Sect. III B, when
\[
\frac{(c + 1)\|F_2\|}{|\text{Re}(\lambda_1)|} \leq 1,
\]
for $t \geq 0$, $\|e^{At}\|$ satisfies
\[
\|e^{At}\| \leq c + 1.
\]

Proof We consider $A$ as a $c + 1$-dimensional block matrix. $A$ is divided into
\[
A = B + C,
\]
$B$ contains $A_{i,i}, i = 0, 1, \ldots, c$ and $C$ contains $A_{i,i+1}, i = 0, 1, \ldots, c - 1$. We analyze the upper
bound of $\|e^{At}\|$ according to the method introduced in [36], $e^{At}$ is written as

$$e^{(B+C)t} = e^{Bt} + \int_0^t e^{B(t-t_0)} C e^{(B+C)t_0} dt_0.$$  \hspace{1cm} (43)

Using this formula to expand $e^{(B+C)t_0}$ we obtain

$$e^{(B+C)t} = e^{Bt} + \int_0^t e^{B(t-t_0)} C e^{B(t_0-t_1)} e^{(B+C)t_1} dt_1 dt_0.$$  \hspace{1cm} (44)

Clearly, a repetition of this process gives

$$e^{(B+C)t} = e^{Bt} + \sum_{k=0}^{c-1} A_k(t) + R_c(t),$$

where

$$A_k(t) = \int_0^t \int_0^{t_0} \cdots \int_0^{t_{k-1}} e^{B(t-t_0)} C e^{B(t_0-t_1)} C \cdots C e^{Bt_k} dt_k \cdots dt_0$$

and

$$R_c(t) = \int_0^t \int_0^{t_0} \cdots \int_0^{t_{c-1}} e^{B(t-t_0)} C \cdots C e^{B(t_{c-1}-t_c)} C e^{B+C} t_c dt_c \cdots dt_0.$$  \hspace{1cm} (47)

Noting that

$$\|A_k(t)\| \leq \|e^{Bt}\| \|C\|^{k+1} \frac{t^{k+1}}{(k+1)!} = \|e^{Bt}\| (\|Ct\|)^{k+1} (k+1)!,$$  \hspace{1cm} (48)

$$\|e^{Bt}\| = e^{\text{Re}(\lambda_1)t}, \quad \|C\| \leq (c+1)\|F_2\|,$$  \hspace{1cm} (49)

and the matrix $[e^{B(t-t_0)} C] \cdots [e^{B(t_{c-1}-t_c)} C]$ is zero because it is the product of $c+1$, $(c+1)\times(c+1)$ strictly upper triangular block matrices and thus, $R_c(t) = 0$. Hence,

$$\|e^{At}\| = \|e^{(B+C)t}\| \leq e^{\text{Re}(\lambda_1)t} \sum_{k=0}^{c} \frac{(c+1)\|F_2\|^k}{k!}.$$  \hspace{1cm} (50)

By Lemma 14, Eq.(40) and Eq.(50),

$$\|e^{At}\| \leq c + 1.$$  \hspace{1cm} (51)

Then the upper bound of the condition number $\kappa_C$ of $C_{m,k,p}(Ah)$ is analyzed in Lemma 8.

**Lemma 8** Consider the matrix $C_{m,k,p}(Ah)$ defined in Sect.III C. Let $h \in \mathbb{R}^+$ and satisfies $\|Ah\| \leq 1$, $c, m, k, p \in \mathbb{Z}^+$. When $\frac{(c+1)\|F_2\|}{\text{Re}(\lambda_1)} \leq 1$, $k \geq 5$ and $\frac{2}{(k+1)!} m(c+1)(c+2) \leq 1$, the condition number $\kappa_C$ of $C_{m,k,p}(Ah)$ satisfies

$$\kappa_C \leq 2e\sqrt{k(m(k+1)+p)(c+2)},$$  \hspace{1cm} (52)

where $e$ is mathematical constant.
Proof First we analyze the upper bound of \(\|C_{m,k,p}(Ah)^{-1}\|\), we have

\[
C_{m,k,p}(Ah)|x| = |B|
\]

where \(|B| = \sum_{l=0}^d b_0|l|\), \(|l|\) represents an \(N\)-dimensional state. We define \(|b'| = b_0|l|\),

\[
\sum_{l=0}^d \|\|b'||^2 = \|\|B\|^2.
\]

For any \(l \in [d]_0\), we define

\[
|x^l\rangle := C_{m,k,p}(Ah)^{-1}|b\rangle = \sum_{n=0}^d |x^l_n\rangle.
\]

We consider two cases: \(0 \leq l < m(k+1)\) and \(m(k+1) \leq l \leq d\).

When \(0 \leq l < m(k+1)\), assuming \(l = a(k+1) + b\), \(0 \leq a < m\), \(0 \leq b \leq k\). Then based on definition of \(x\), we have

\[
|x_{i,j}^l\rangle = 0, \quad 0 \leq i < a, 0 \leq j \leq k, \\
|x_{a,j}^l\rangle = 0, \quad 0 \leq j < b, \\
|x_{a,b}^l\rangle = b!(Ah)^{j-b}/j!|b\rangle, \quad b \leq j \leq k \\
|x_{a+1,0}^l\rangle = T_{b,k}(Ah)|b\rangle, \\
|x_{a+1,j}^l\rangle = (Ah)^j/j!|x_{a+1,0}\rangle, \\
|x_{a+2,0}^l\rangle = T_k(Ah)|x_{a+1,0}\rangle = T_k(Ah)T_{b,k}(Ah)|b\rangle, \\
\vdots \\
|x_{m,0}^l\rangle = T_{b,k}(Ah)|x_{m-1,0}\rangle = (T_k(Ah))^{m-a-1}T_{b,k}(Ah)|b\rangle, \\
|x_{m,j}^l\rangle = |x_{m,0}\rangle = (T_k(Ah))^{m-a-1}T_{b,k}(Ah)|b\rangle, \quad 1 \leq j \leq p,
\]

where \(|x_{i,j}^l\rangle = |x_{i(k+1)+j}\rangle\) and \(T_{b,k}(Ah) := \sum_{j=b}^k \frac{b!(Ah)^{j-b}}{j!}\). By Lemma 15, we have

\[
\|e^{Ah(m-a-1)} - T_k^{m-a-1}(Ah)\| \leq \frac{2}{(k+1)!} (m - a - 1)(c + 1)(c + 2),
\]

by Lemma 7, \(\|e^{Ah(m-a-1)}\| \leq c + 1\), then we have

\[
\|(T_k(Ah))^{m-a-1}\| \leq c + 2.
\]

Therefore,

\[
\|(Ah)^{j-b}\| \leq \|(Ah)\|^{j-b} \leq 1, \quad b \leq j \leq k, \\
\|T_{b,k}(Ah)\| \leq T_{b,k}(\|Ah\|) \leq e^\|Ah\| \leq e, \\
\|T_k(Ah)\| \leq T_k(\|Ah\|) \leq e^\|Ah\| \leq e, \\
\|(T_k(Ah))^{m-a-1}T_{b,k}(Ah)\| \leq \|(T_k(Ah))^{m-a-1}\||T_{b,k}(Ah)\| \leq e(c + 2).
\]
By Eq.(56) and Eq.(59), \( \|x_{i,j}^l\| \) satisfies
\[
\|x_{a+1,0}^l\| \leq \|T_{b,k}(Ah)\| \|\|b^l\|\| \leq e\|\|b^l\|\|,
\]
\[
\|x_{a,j}^l\| \leq e(c+2)\|\|b^l\|\|,
\]
\[
\|x_{m,j}^l\| \leq e(c+2)\|\|b^l\|\|,
\]
therefore,
\[
\|C_{m,k,p}(Ah)^{-1}|b^l\|\|^2 \leq (m(k+1)+p)(e(c+2))^2\|\|b^l\|\|^2.
\] (61)

When \( m(k+1) \leq l \leq d \), assuming \( l = m(k+1)+b \), \( 0 \leq b \leq p \), \( x_{i,j} \) satisfies
\[
|x_{i,0}^l\| = 0, \quad 0 \leq i < m, 0 \leq j \leq k,
\]
\[
|x_{b,j}^l\| = 0, \quad 0 \leq j < b,
\]
\[
|x_{b,j}^l\| = |b^l\|, \quad b \leq j \leq p,
\]
then
\[
\|C_{m,k,p}(Ah)^{-1}|b^l\|\|^2 \leq p\|\|b^l\|\|^2.
\] (63)

From Eq.(61) and Eq.(63), for any \(|B\rangle\), \( \|C_{m,k,p}(Ah)^{-1}|B\rangle\| \) satisfies
\[
\|C_{m,k,p}(Ah)^{-1}|B\rangle\|\|^2 \leq (m(k+1)+p)\sum_{l=0}^{d} \|C_{m,k,p}(Ah)^{-1}|b^l\|\|^2 \leq (m(k+1)+p)^2(e(c+2))^2\|\|B\|\|^2,
\] (64)
then
\[
\|C_{m,k,p}(Ah)^{-1}\| \leq (m(k+1)+p)e(c+2).
\] (65)

From Lemma 4 in [9], we have
\[
\|C_{m,k,p}(Ah)\| \leq 2\sqrt{k}.
\] (66)

Therefore, by Eq.(66) and Eq.(65), we have \( \kappa_C \leq 2e\sqrt{k}(m(k+1)+p)(c+2) \).

VI. SOLUTION ERROR

In this section, we analyze the solution error of our algorithm. The error mainly comes from two aspects: (1) Homotopy perturbation method truncation error. The solution \( \tilde{u}(t) \) defined in Eq.(7) is an approximate solution of Eq.(1), the error bound is determined by the truncation order \( c \), in Sect.VIA, we analyze the convergence condition of Eq.(7) and give the error bound. (2) Linear ODEs solution error. We solve the linear ODEs defined in Eq.(8) with the quantum algorithm proposed in [9]. This algorithm also generates intermediate error, we analyze the error bound in Sect.VIB.
A. Homotopy perturbation method truncation error

We first analyze homotopy perturbation method truncation error, the result is shown in Lemma 9.

**Lemma 9** Consider the nonlinear ODEs defined in Eq. (1), the solution of Eq. (1) obtained by homotopy perturbation method is written as $\tilde{u}(t) = \sum_{i=0}^{c} \nu_i(t)$, $u(t)$ represents the exact solution of Eq. (1). When $K < 1$ and $c > \log_{1/K} \frac{1}{\epsilon(1-K)}$, $\tilde{u}(t)$ satisfies

$$\|u(t) - \tilde{u}(t)\| \leq \epsilon.$$  \hspace{1cm} (67)

**Proof** $u(t)$ can be represented as $u(t) = \sum_{i=0}^{\infty} \nu_i(t)$, Eq. (67) is transformed into

$$\| \sum_{j=c+1}^{\infty} \nu_j(t) \| < \epsilon.$$  \hspace{1cm} (68)

To prove Eq. (68), we analyze the upper bound of $\|\nu_i\|$ defined in Eq. (6). $\nu_0(t) = e^{Fi t} u_{in}$, we have

$$\|\nu_0(t)\| \leq \| e^{Fi t} \| \times \| u_{in} \| \leq \| u_{in} \|.$$ \hspace{1cm} (69)

We define $K_1 = \frac{\| u_{in} \| F_2}{|Re(\lambda_1)|}$ and assume when $j \leq i$, $\|\nu_j\|$ satisfies

$$\|\nu_j\| \leq \alpha_j K_1 \| u_{in} \|, \quad j = 0, 1, \ldots, i.$$ \hspace{1cm} (70)

Then $\|\nu_{i+1}(t)\|$ satisfies

$$\|\nu_{i+1}(t)\| = \left\| \int_0^t e^{Fi(t-\tau)} F_2 \sum_{j=0}^i \nu_j(\tau) \otimes \nu_{i-j}(\tau) d\tau \right\|$$

$$\leq \int_0^t \| e^{Fi(t-\tau)} F_2 \| \left( \sum_{j=0}^i \alpha_j \alpha_{i-j} \right) K_1 \| u_{in} \|^2 d\tau$$

$$\leq \left( \sum_{j=0}^i \alpha_j \alpha_{i-j} \right) K_1 \| u_{in} \|^2 \| F_2 \| \int_0^t \| e^{Re(\lambda_1)(t-\tau)} \| d\tau$$

$$\leq \left( \sum_{j=0}^i \alpha_j \alpha_{i-j} \right) K_1 \| u_{in} \|^2 \| F_2 \| \frac{1 - e^{Re(\lambda_1)t}}{|Re(\lambda_1)|}$$

$$\leq \left( \sum_{j=0}^i \alpha_j \alpha_{i-j} \right) K_1^{i+1} \| u_{in} \|.$$ \hspace{1cm} (71)

By Eq. (70), Eq. (71), $\alpha_i$ can be defined as

$$\alpha_{i+1} = \sum_{j=0}^i \alpha_j \alpha_{i-j}, \quad \alpha_0 = 1.$$ \hspace{1cm} (72)

**Eq. (72) is the catalan sequence [37]** and satisfies

$$\alpha_i = \frac{1}{i+1} \binom{2i}{i} \approx \frac{4^i}{i^{3/2} \sqrt{\pi}} < 4^i.$$ \hspace{1cm} (73)
Combining Eq.(3), Eq.(70), Eq.(71), Eq.(72) and Eq.(73), for any \( i \in \mathbb{N} \), \( \| \nu_i(t) \| \) has the upper bound

\[
\| \nu_i(t) \| < (4K_1)^i \| u_m \| \leq K^{i+1}.
\]

Substituting Eq.(74) into Eq.(68), we have

\[
\sum_{i=c+1}^{\infty} K^{i+1} < \epsilon.
\]

Therefore, when \( K < 1 \) and \( c > \log_{1/K} \frac{1}{\epsilon(1-K)} \), we have \( \| u(t) - \tilde{u}(t) \| \leq \epsilon \).

B. Linear ODEs solution error

Then we analyze the error of solving the linear ODEs defined in Eq.(8), we have a similar conclusion with Theorem 6 in [9], the difference comes from the upper bound of \( \| e^{A_j} \| \) or \( \| T_k^j(A) \| \) in our work is different from their work. Our result is shown in Lemma 10.

**Lemma 10** Consider the linear ODEs defined in Eq.(8) and the system of linear equations defined in Eq.(22). Let \( h \in \mathbb{R}^+ \) satisfies \( \| Ah \| \leq 1 \). When \( \frac{2}{(k+1)!} m(c+1)(c+2) \leq 1 \), we have

\[
\| y(jh) - |x_{j,0}\rangle \| \leq \frac{2j(c+1)(c+2)\| y_m \| }{(k+1)!}, \quad j = 0, 1, \ldots, m.
\]

**Proof** The exact solution of Eq.(8) at \( t = jh \) is \( |y(jh)\rangle \), it is written as

\[
|y(jh)\rangle = e^{A_jh} |y(0)\rangle.
\]

The solution of Eq.(22) \( |x_{j,0}\rangle \) satisfies

\[
|x_{j,0}\rangle = T_k^j(Ah)|x_{0,0}\rangle,
\]

where \( T_k(Ah) = \sum_{l=0}^{k} \frac{(Ah)^l}{l!} \), the initial value satisfies \( |y(0)\rangle = |x_{0,0}\rangle = |y_m\rangle \). By Lemma 7 and Lemma 15, \( \| y(jh) \| - |x_{j,0}\rangle \| \) satisfies

\[
\| y(jh) \| - |x_{j,0}\rangle \| \leq \| e^{A_jh} - T_k^j(Ah) \| \| y_m \| \leq \frac{2j(c+1)(c+2)\| y_m \| }{(k+1)!}, \quad j = 0, 1, \ldots, m.
\]

VII. SUCCESS PROBABILITY

As introduced in Sect.III D, there are two probabilistic steps in our method. This section gives a lower bound of the success probability of these two steps. The results are shown in Lemma 11 and Lemma 12.

Firstly, we analyze the success probability of getting \( |x_{m,i}\rangle, i = 0, 1, \ldots, p \) when measuring \( |x\rangle \). By setting appropriate conditions, Lemma 11 gives the same conclusion as Theorem 7 in [9].

**Lemma 11** Consider the system of linear equations defined in Eq.(22). Let \( h \in \mathbb{R}^+ \) satisfies \( \| Ah \| \leq 1 \). \( g = \max_{t \in [0, mh]} \| y(t) \| / \| y(mh) \| \), \( m, k, p \in \mathbb{Z}^+ \). When \( (k+1)! \geq 50m(c+1)(c+2)g \),
we have

\[ \frac{\|x_m,0\|^2}{\|x\|^2} \geq \frac{1}{p + 77mg^2}. \]  

(80)

**Proof** As introduced before, \( |x_{m,j}⟩ = |x_{m,0}\rangle \) for \( j \in [p]_0 \), we define

\[ |x_{\text{good}}⟩ := \sum_{j=0}^{p} |m(k+1)+j⟩|x_{m,j}⟩ \]  

(81)

and

\[ |x_{\text{bad}}⟩ := \sum_{i=0}^{m-1} \sum_{j=0}^{k} |i(k+1)+j⟩|x_{i,j}⟩ \]  

(82)

We see \( |x⟩ = |x_{\text{good}}⟩ + |x_{\text{bad}}⟩ \) and \( ⟨x_{\text{good}}|x_{\text{bad}}⟩ = 0 \), then

\[ \|x\|^2 = \|x_{\text{good}}\|^2 + \|x_{\text{bad}}\|^2 = (p+1)\|x_{m,0}\|^2 + \|x_{\text{bad}}\|^2. \]  

(83)

Next we give a lower bound of \( \|x_{m,0}\| \) and an upper bound of \( \|x_{\text{bad}}\| \). We define \( q = \|y(mh)⟩\| \), by Lemma 10 and \( (k+1)! \geq 50m(c+1)(c+2)g \), we have

\[ \|x_{i,0}⟩ - |y(ih))⟩\| \leq 0.04q, \ 0 \leq l \leq m. \]  

(84)

By the definition of \( g \), \( \|y(ih))⟩\| \leq gq \) for any \( i \in [m-1]_0 \), then we have

\[ \|x_{i,0}\| \leq (g + 0.04)q \leq 1.04gq, \ 0 \leq i \leq m-1. \]  

(85)

and

\[ 0.96q \leq \|x_{m,0}\| \leq 1.04q. \]  

(86)

For any \( i \in [m]_0 \), \( |x_{i,j}⟩ = (Ah/j)|x_{i,j-1}\⟩ \), we have

\[ |x_{i,j}⟩ = \frac{(Ah)^{j-1}}{j!}|x_{i,1}\⟩, \ 2 \leq j \leq k. \]  

(87)

We have \( \|Ah\| \leq 1 \), therefore,

\[ \|x_{i,j}\| \leq \frac{\|x_{i,1}\|}{j!}, \ 2 \leq j \leq k. \]  

(88)
Next, based on \( |x_{i+1,0}⟩ = |x_{i,0}⟩ + \sum_{j=1}^{k} |x_{i,j}⟩ \), we have
\[
2.08gq \geq |||x_{i+1,0}⟩|| + |||x_{i,0}⟩|| \\
\geq |||x_{i+1,0}⟩ - |x_{i,0}⟩|| \\
\geq |||x_{i,1}⟩|| - \sum_{j=2}^{k} |||x_{i,j}⟩|| \\
\geq \left( 1 - \sum_{j=2}^{k} \frac{1}{j!} \right) |||x_{i,1}⟩|| \\
\geq (3 - e)|||x_{i,1}⟩||,
\]
then
\[
|||x_{i,1}⟩|| \leq \frac{2.08gq}{3 - e}, \quad 0 \leq i \leq m - 1.
\]
and
\[
|||x_{i,j}⟩|| \leq \frac{2.08gq}{j!(3 - e)}, \quad 0 \leq i \leq m - 1, \quad 1 \leq j \leq k.
\]
\[
|||x_{bad}⟩|| \text{ satisfies}
\]
\[
|||x_{bad}⟩||^2 = \sum_{i=0}^{m-1} |||x_{i,0}⟩||^2 + \sum_{i=0}^{m-1} \sum_{j=1}^{k} |||x_{i,j}⟩||^2 \\
\leq 1.04^2mg^2q^2 + m \sum_{j=1}^{k} \frac{(2.08gq)^2}{(j!)^2(3 - e)^2} \\
\leq 70.9mg^2q^2.
\]
The last step of Eq.(92) is derived from the inequality \( \sum_{j=1}^{k} \frac{1}{j!^2} \leq I_0(2) - 1 < 1.28 \), where \( I_0(2) < 2.28 \) a modified Bessel function of the first kind[9]. Combining Eq.(86) and Eq.(92), we have
\[
\frac{|||x_{m,0}⟩||^2}{|||x⟩||^2} \geq \frac{(0.96q)^2}{p(0.96q)^2 + 70.9mg^2q^2} \\
\geq \frac{1}{p + 77mg^2}.
\]
Secondly, we analyze the success probability of the second probabilistic step. After the first measurement, the desired state is the state \( |y(T)⟩ \) defined in Eq.(19). Then we measure the first qubit register of \( |y(T)⟩ \), if the result is \( |0,0⟩ \), we have a state \( \epsilon \)-close to \( |u(T)⟩/||u(T)⟩|| \) in the second qubit register of \( |y(T)⟩ \). The lower bound of the success probability in this measurement is analyzed in Lemma 12.

Lemma 12 Let \( T \in \mathbb{R}^+ \), \( \eta' = K/||\tilde{u}(T)⟩|| \). When \( K < \sqrt{2}/2 \), we have
\[
\frac{|||y_0(T)⟩||^2}{|||y(T)⟩||^2} \geq \frac{1 - 2K^2}{1 - 2K^2 + 2(\eta')^2}.
\]
Proof We rearrange the components of \( \vec{y} \),

\[
\vec{y} = [\vec{y}_0, \vec{y}_1, \ldots, \vec{y}_c],
\]  

(95)

where \( \vec{y}_0 = \sum_{j=0}^{c} \vec{y}_j \), when \( i \geq 1 \), the component of \( \vec{y}_i \) is represented as \( \nu_{a_i,0} \otimes \nu_{a_i,1} \otimes \cdots \otimes \nu_{a_i,k} \), \( a_i,j \) satisfies

\[
k \geq 1, a_i,j \geq 0, \sum_{j=0}^{k} (a_i,j + 1) = i + 1.
\]  

(96)

By Eq.(96), the number of elements in \( \vec{y}_i \) is \( 2^i - 1 \). By Eq.(74) and Eq.(96), for any \( j \in [2^i - 1]_0 \), we have \( \| \vec{y}_{i,j} \| \leq K^{i+1} \). Therefore,

\[
\| \vec{y}_i \|^2 \leq (2^i - 1)K^{2(i+1)} < (2K^2)^i.
\]  

(97)

By Eq.(97) and \( \| \vec{y}_0 \| = \| \vec{y}_0 \| = \| \vec{u}(T) \| = K/\eta' \), we have

\[
\| \vec{y}_0 \|^2 \leq (K/\eta')^2 \leq (K/\eta')^2 + \sum_{i=1}^\infty (2K^2)^i.
\]  

(98)

\[
\geq \frac{(K/\eta')^2}{(K/\eta')^2 + \frac{2K^2}{1 - 2K^2}}
\]  

(99)

\[
= \frac{1 - 2K^2}{1 - 2K^2 + 2(\eta')^2}.
\]  

(100)

VIII. MAIN RESULT

In this section, we give the main result of our work.

Theorem 1 Given \( n \)-dimensional nonlinear dissipative ODEs \( \frac{du}{dt} = F_1 u + F_2 u^2 \) defined in Eq.(1) and construct the linear ODEs \( \frac{d\vec{u}}{dt} = A\vec{u} \) defined in Eq.(8). Let \( T > 0 \), \( g = \max_{t \in [0,T]} \| y(t) \| / \| y(T) \| \), \( \eta = \| u_{in} \| / \| u(T) \| \), \( c = \left\lceil \log_{1/K} \left( \frac{4\| u_{in} \|}{(1-K)(1-\eta)} \right) \right\rceil \). When \( K < \sqrt{2}/2 \) and \( \frac{(c+1)\| F_2 \|}{4T(1-KT)} \leq 1 \), there exists a quantum algorithm to produce \( | u_{out}(T) \rangle \) which satisfies \( \| u_{out}(T) - u(T) \| / \| u(T) \| \| \leq \epsilon \) with \( \Omega(1) \) success probability. The query complexity of the algorithm for \( O_{F_1}, O_{F_2}, O_u \) is

\[
O \left( \frac{gnsT(\| F_1 \| + \| F_2 \|)}{\sqrt{1 - 2K^2}\| u_{in} \|} \right) \left( \log \left( \frac{gnsT(\| F_1 \| + \| F_2 \|)}{\epsilon(1 - 2K^2)\| u_{in} \|} \right) \right). \]  

(101)

The gate complexity of this algorithm is larger than its query complexity by a factor of

\[
O \left( \log \left( \frac{n gnsT(\| F_1 \| + \| F_2 \|)}{\epsilon(1 - 2K^2)\| u_{in} \|} \right) \right). \]  

(102)

Proof Whole process. First we show the whole process of our algorithm. We define \( \eta' = \eta K/\| u_{in} \| \) and set

\[
\epsilon \leq 0.1 \sqrt{1 - 2K^2/\eta'}, \epsilon_1 = \frac{\epsilon K}{4\eta}, \delta = \epsilon \sqrt{1 - 2K^2} \]  

(103)
and construct the N-dimensional linear ODEs defined in Eq. (8)

\[ \frac{d\vec{y}}{dt} = A\vec{y}, \ \vec{y}(0) = y_{in}. \]  

(104)

We also set \( h = T/\|T\|A\| \), \( m = p = T/h = [T\|A\|] \), \( k = \left\lfloor \frac{\log(\Omega)}{\log(\log(\Omega))} \right\rfloor \), where \( \Omega = 50m(c + 1)(c + 2)g/\delta \). Then we construct the linear system defined in Eq. (22) and solve the linear system with the algorithm proposed in [2]. The normalized solution of Eq. (22) is represented as

\[ |x\rangle = \sum_{l=0}^{d} |l\rangle|x_l\rangle, \]  

(105)

where \( d = m(k + 1) + p \). \( |x\rangle \) can also be represented as

\[ |\bar{x}\rangle = \sum_{l=0}^{d} \alpha_l |l\rangle|\bar{x}_l\rangle, \]  

(106)

where |\bar{x}_l\rangle \) is a normalized state. Assuming the output state of quantum linear system algorithm[2] is

\[ |\bar{x}'\rangle = \sum_{l=0}^{d} \alpha'_l |l\rangle|\bar{x}'_l\rangle. \]  

(107)

By Theorem 5 in [2], we make |\bar{x}'\rangle \) satisfies

\[ \||\bar{x}\rangle - |\bar{x}'\rangle\| \leq \delta. \]  

(108)

Then we execute measurement step discussed in Sect. III D and get a state \( \epsilon \)-close to \( |u(T)/\|u(T)|\|\rangle \) with success probability \( O((1 - 2K^2)/(g\eta)^2) \). We can amplify the success probability to \( \Omega(1) \) with quantum amplitude amplification algorithm[38] by running quantum linear system algorithm \( O(g\eta'/\sqrt{1 - 2K^2}) \) times.

**Proof of correctness.** Then we analyze the error bound and the impact of error on success probability.

Assuming \( u(T) \) represents the exact solution. We define |\tilde{u}(T)\rangle \) as

\[ |\tilde{u}(T)\rangle = \sum_{i=0}^{c} \nu_i(T)\rangle = |y_{0,0}(T)\rangle. \]  

(109)

By Lemma 9 and our choice of parameter \( c \), we have

\[ \|\|u(T)\rangle - |\tilde{u}(T)\rangle\| \leq \epsilon_1. \]  

(110)

Then by Lemma 16 and \( \|\|u(T)\rangle\| = K/\eta' \), we have

\[ \|\|\tilde{u}(T)\rangle - |\bar{y}_{0,0}(T)\rangle\| \leq \frac{2\eta \epsilon_1}{K} = \epsilon/2. \]  

(111)

Let \( S := \{m(k + 1), m(k + 1) + 1, \ldots, m(k + 1) + p\} \). By Lemma 10 and our choice of parameter
For any $l \in S$, we have
$$
||x_l - y(T)|| \leq \delta ||y(T)||. \quad (112)
$$

By Eq. (112) and Lemma 16, we have
$$
||\bar{x}_l - \bar{y}(T)|| \leq 2\delta. \quad (113)
$$

By Lemma 11, for any $l \in S$, we have
$$
\alpha_l = \frac{||x_l||}{||x||} \geq \frac{1}{\sqrt{78mg}} \geq \delta. \quad (114)
$$

By Eq. (108), Eq. (114) and Lemma 17, we have
$$
||\bar{x}_l - \bar{x}'_l|| \leq \frac{2\delta}{\alpha_l - \delta}. \quad (115)
$$

Then, combine Eq. (113) and Eq. (115), we have
$$
||\bar{x}'_l - \bar{y}(T)|| \leq ||\bar{x}'_l - \bar{x}_l|| + ||\bar{x}_l - \bar{y}(T)|| \leq 2\delta \left(1 + \frac{1}{\alpha_l - \delta}\right). \quad (116)
$$

We default $\alpha_l$ is small enough, such as $\alpha_l < 0.5$, then by Eq. (114) and $\delta = \sqrt{\frac{1 - 2K^2}{30 \sqrt{78mg\eta'}}}$, we have
$$
2\delta(1 + \frac{1}{\alpha_l - \delta}) \leq 2\delta \sqrt{3} \frac{1}{\alpha_l} \leq \frac{\epsilon \sqrt{1 - 2K^2}}{5 \sqrt{3\eta'}}. \quad (117)
$$

$|y(T)\rangle$ and $|\bar{x}'_l\rangle$ are also written as
$$
|\bar{y}(T)\rangle = \sum_{w=0}^{c} \chi_w |\bar{y}_w(T)\rangle \quad (118)
$$
and
$$
|\bar{x}'_l\rangle = \sum_{w=0}^{c} \chi'_w |\bar{x}'_{l,w}\rangle. \quad (119)
$$

By Lemma 12, we have
$$
\chi_0 = \frac{||y_0(T)||}{||y(T)||} \geq \sqrt{\frac{1 - 2K^2}{(1 - 2K^2) + 2(\eta')^2}} \geq \frac{\sqrt{1 - 2K^2}}{\sqrt{3\eta'}}. \quad (120)
$$

By Eq. (116), Eq. (117), Eq. (120) and Lemma 17, we have
$$
|||\bar{x}'_{l,0} - \bar{y}_{0,0}(T)|| \leq \frac{2(|||\bar{x}'_l - \bar{y}(T)|||)}{\chi_0 - (|||\bar{x}'_l - \bar{y}(T)|||)} \leq \frac{2\epsilon}{5 - \epsilon} \leq \epsilon/2. \quad (121)
$$

We notice $|\bar{x}'_{l,0}\rangle$ is the output state of our algorithm, we have $|u_{out}(T)\rangle = |\bar{x}'_{l,0}\rangle$. Combining Eq. (111)
and Eq.\((121)\), we have
\[
\|\|\tilde{u}(T)\| - |u_{\text{out}}(T)\| \leq \|\|\tilde{u}(T)\| - |\tilde{y}_{0,0}(T)\|\| + \|\|\tilde{x}'_{1,0}\| - |\tilde{y}_{0,0}(T)\|\| \leq \varepsilon. \tag{122}
\]

On the other hand, caused by solution error, the success probability also changes. By Lemma\ 18, Eq.\((120)\) and \(\varepsilon \leq 0.1\sqrt{1 - 2K^2/\eta}\), we have
\[
\chi_0' \geq \chi_0 - \frac{1}{\sqrt{3\eta}} \sqrt{1 - 2K^2} \times \frac{1}{5} \varepsilon \geq \chi_0 - \varepsilon/2 \geq \frac{1 - 2K^2}{2\eta} \tag{123}
\]
and
\[
\alpha'_l \geq \alpha_l - \delta \geq \left(\frac{1}{\sqrt{78}} - \frac{1}{30\sqrt{78}}\right) \frac{1}{\sqrt{mg}} \geq \frac{1}{10\sqrt{mg}} \tag{124}
\]
then
\[
\sum_{l \in S} |\alpha'_l|^2 \geq \frac{p}{100mg^2} = \frac{1}{100g^2}, \tag{125}
\]
Therefore, the success probability of our algorithm is
\[
p = (\chi_0')^2 \times \left(\sum_{l \in S} |\alpha'_l|^2\right) \geq \frac{1 - 2K^2}{400(g\eta)^2}. \tag{126}
\]

**Complexity analysis.** Finally, we analyze the complexity of our algorithm.

By Lemma 2 and Lemma 3, the sparsity of \(A\) is \(c^2 s\) and \(\|A\| \leq (c + 1)(\|F_1\| + \|F_2\|)\). The sparsity \(s_C\) of \(C_{m,k,p}(Ah)\) satisfies
\[
s_C < k + c^2 s. \tag{127}
\]

By Lemma 8, the condition number \(\kappa_C\) of \(C_{m,k,p}(Ah)\) satisfies
\[
\kappa_C \leq 2e\sqrt{k}((m(k + 1) + p)(c + 2). \tag{128}
\]

By Theorem 5 in [2] and Eq.\((108)\), the query complexity of quantum linear system algorithm to oracle of \(C_{m,k,p}(Ah)\) and \(|0\rangle\langle y_{\text{in}}|\) is
\[
O(s_C\kappa_C\log(s_C\kappa_C/\delta)). \tag{129}
\]

By the definition of \(C_{m,k,p}(Ah)\), the oracle of \(C_{m,k,p}(Ah)\) can be constructed by querying oracle \(O_A\) once, by Lemma 6, \(O_A\) is constructed by querying \(O_{F_1}\) \(O(c)\) times and \(O_{F_2}\) \(O(1)\) times. By Lemma 1, \(|0\rangle\langle y_{\text{in}}|\) can be prepared by querying \(O_u\) \(O(c)\) times.

Substituting Eq.\((127)\), Eq.\((128)\) into Eq.\((129)\) and considering the choice of all parameters, the query complexity of solving Eq.\((22)\) for \(O_{F_1}\), \(O_{F_2}\) and \(O_u\) is
\[
O \left( sT(\|F_1\| + \|F_2\|)\log \left(\frac{gnsT\|F_1\|\|F_2\|}{\varepsilon(1 - 2K^2)\|y_{\text{in}}\|}\right) \right). \tag{130}
\]

Using amplitude amplification algorithm[38], we repeat the above process \(O(1/\sqrt{\eta})\) times and get \(|u_{\text{out}}(T)\) = \(\tilde{x}'_{1,0}(T)\) which satisfies \(\|\|u_{\text{out}}(T)\| - u(T)\|\| \leq \varepsilon\) with \(\Omega(1)\) success probability.
The query complexity of the whole process for $O_{F_1}$, $O_{F_2}$ and $O_u$ is

$$O\left(\frac{g\eta sT(\|F_1\| + \|F_2\|)}{\sqrt{1 - 2K^2\|u_{in}\|}}\right) \log\left(\frac{g\eta sT\|F_1\|\|F_2\|}{\epsilon(1 - 2K^2)\|u_{in}\|}\right)).$$

(131)

The gate complexity of this algorithm is larger than its query complexity by a factor of

$$O\left(\text{poly}\left(\log\left(\frac{ng\eta sT\|F_1\|\|F_2\|}{\epsilon(1 - 2K^2)\|u_{in}\|}\right)\right)\right).$$

(132)

IX. CONCLUSION AND DISCUSSION

In this paper, we presented a quantum homotopy perturbation method for solving nonlinear dissipative ODEs. The gate complexity of our algorithm is $O(g\eta T\text{poly}(\log(nT/\epsilon)))$. The complexity of the optimal classical algorithm for solving Eq.(1) is at least linear with $n$, the complexity of the algorithm proposed in [20] is linear with $1/\epsilon$, so our algorithm provides exponential improvement over the best classical algorithms or previous quantum algorithms in $n$ or $\epsilon$. $\eta$ and $g$ also affect the complexity of our algorithm, $\eta$ measures the decay of $u(T)$ and increases exponentially as $T$ increases, $g$ measures the decay of $\bar{y}(T)$ defined in Eq.(8) and also increases exponentially as $T$ increases. Our algorithm is effective when $T$ is relatively small which makes $\eta$ and $g$ small enough. $\eta$ and $g$ are also affected by $F_2$, when $T$ is relatively small, the trend of $\eta$ and $g$ increasing exponentially with $T$ may not be obvious due to the influence of $F_2$, this case makes our algorithm perform better. Our algorithm has the potential to accelerate the solution of various nonlinear equations, and can be applied to nonlinear problems in various fields, such as fluid dynamics, biology, finance, etc, thereby accelerating the research progress of nonlinear science.

Our algorithm only discusses time-independent homogeneous quadratic nonlinear ODEs. When solving time-dependent nonlinear ODEs, the algorithm proposed in [9] is not suitable, an alternative way is to use the algorithm proposed in [8] to solve the linear ODEs, then the dependence of complexity on error $\epsilon$ becomes $O(\text{poly}(1/\epsilon))$. Is it possible to optimize the complexity of time-dependent quadratic nonlinear ODEs to $O(\text{poly}(\log(1/\epsilon)))$ is an open question.

On the other hand, homotopy analysis method[39] and its derivatives[40–42] are similar to homotopy perturbation method. Whether we can use quantum computing to accelerate the execution process of homotopy analysis method and thus construct a quantum homotopy analysis method is also a question to be investigated further.

Furthermore, how to induce nonlinearity in quantum computing is a basic problem when solving nonlinear equations with quantum algorithm. A common method is producing multiple copies of the original system, some nonlinear quantum algorithms contain copy process[17, 19, 21]. In [43], a linearization technique of nonlinear classical dynamics based on Koopman-von Neumann method is proposed. [44] summarizes three classical linear embedding techniques, including Carleman embedding(Carleman linearization is also called Carleman embedding)[26, 27], coherent states embedding[27, 45] and position-space embedding[46], and then puts forward the prospects of these linear embedding techniques to construct effective quantum algorithms. An open question is whether there are other ways to induce nonlinearity in quantum computing.

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Appendix

In this appendix, we give some lemmas used in proving some conclusions of our work. Lemma 16, Lemma 17 and Lemma 18 are given in [9], we just list them again.

**Lemma 13** Let $\gamma \geq 1$, $m \in \mathbb{N}^+$, when $t \geq 0$, we have

$$
\sum_{j=0}^{m-1} \frac{t^j}{j!} \leq m. \tag{133}
$$

**Proof** We consider two cases: (1) $0 < t \leq m - 1$; (2) $t \geq m$.

When $0 < t \leq m - 1$, we can find $i \in [m-1]_0$ which satisfies $t \in (i, i+1]$, then we have

$$
\sum_{j=0}^{m-1} \frac{t^j}{j!} e^{-\gamma t} < \frac{t^i}{i!} e^{-\gamma t}. \tag{134}
$$

We define

$$
g(t) = \frac{t^i}{i!} e^{-\gamma t}, \quad 0 < t \leq m - 1. \tag{135}
$$

It is obvious that $g(t) \leq g\left(\frac{t}{\gamma}\right)$ for $0 < t \leq m - 1$. Using Stirling’s formula $i! \approx \sqrt{2\pi\left(\frac{i}{e}\right)^i}$, we have

$$
g(t) \leq m \frac{\left(\frac{t}{\gamma}\right)^i}{\sqrt{2\pi\left(\frac{i}{e}\right)^i}} e^{-i} \leq m \left(1 + \frac{1}{\gamma}\right)^i \leq m. \tag{136}
$$

Therefore,

$$
\sum_{j=0}^{m-1} \frac{t^j}{j!} e^{-\gamma t} \leq m. \tag{137}
$$

When $t \geq m$, we have

$$
\sum_{j=0}^{m-1} \frac{t^j}{j!} e^{-\gamma t} < \frac{t^m}{(m-1)!} e^{-\gamma t}. \tag{138}
$$

Similar with the case $0 < t \leq m - 1$, we also have

$$
m \frac{t^m}{(m-1)!} e^{-\gamma t} \leq m \frac{\left(\frac{m-1}{\gamma}\right)^m}{\sqrt{2\pi(m-1)}} \frac{1^m}{(m-1)^m} e^{-(m-1)} \leq m \left(1 + \frac{1}{\gamma}\right)^m \leq m. \tag{139}
$$

Therefore, for any $t \geq 0$, we have $\sum_{j=0}^{m-1} \frac{t^j}{j!} e^{-\gamma t} \leq m$. 
Lemma 14 Let $\gamma, \beta \in \mathbb{R}^+, m \in N^+, \text{ when } t \geq 0 \text{ and } \gamma/\beta \geq 1$, we have
\[
\sum_{j=0}^{m-1} \frac{(\beta t)^j}{j!} e^{-\gamma t} \leq m. \tag{140}
\]

Proof We define $t' = \beta t, \gamma' = \gamma/\beta$, then Eq.(140) is written as
\[
\sum_{j=0}^{m-1} \frac{(t')^j}{j!} e^{-\gamma' t'} \leq m. \tag{141}
\]
By Lemma 13, we directly obtain Eq.(141).

Lemma 15 Given a matrix $M$ satisfies $\|M\| \leq 1$ and $\|e^{Mt}\| \leq \Delta$ for any $t \geq 0$. Let $k, m \in N^+$ and satisfy $\frac{2}{(k+1)!} m \Delta (\Delta + 1) \leq 1$, $T_k(M) = \sum_{l=0}^{k} \frac{(Ah)^l}{l!}$. Then for any $l \in [m], we have
\[
\|e^{Ml} - T_k^l(M)\| \leq \frac{2l \Delta (\Delta + 1)}{(k+1)!}. \tag{142}
\]
Proof When $l = 0, \|e^{Ml} - T_k^l(M)\| = 0$. When $l = 1,$
\[
\|e^M - T_k^l(M)\| = \| \sum_{j=k+1}^{\infty} \frac{M^j}{j!} \| \leq \sum_{j=k+1}^{\infty} \| \frac{M^j}{j!} \| \leq \sum_{j=k+1}^{\infty} \frac{1}{j!} \leq \frac{2}{(k+1)!} \leq \frac{2}{(k+1)!} \Delta (\Delta + 1). \tag{143}
\]
Assuming when $l \leq l'$, we have
\[
\|e^{Ml} - T_k^l(M)\| \leq \frac{2l \Delta (\Delta + 1)}{(k+1)!}, \tag{144}
\]
then by $\|e^{Ml}\| \leq \Delta$, we have
\[
\|T_k^l(M)\| \leq \Delta + \frac{2}{(k+1)!} l \Delta (\Delta + 1) \leq \Delta + 1. \tag{145}
\]
When $l = l' + 1,$
\[
\|e^{M(l'+1)} - T_k^{l'+1}(M)\| = \| (e^M - T_k(M)) (\sum_{j=0}^{l'} T_k^{l'-j}(M)) \|
\leq \frac{2 \Delta}{(k+1)!} \times (\sum_{j=0}^{l'} \|T_k^{l'-j}(M)\|)
\leq \frac{2}{(k+1)!} (l' + 1) \Delta (\Delta + 1). \tag{146}
\]
Therefore, for any $l \in [m], we have $\|e^{Ml} - T_k^l(M)\| \leq \frac{2}{(k+1)!} l \Delta (\Delta + 1)$.

Lemma 16 (9). Let $|\psi\rangle$ and $|\varphi\rangle$ be two vectors such that $\||\psi\rangle\| \geq \alpha > 0$ and $|||\psi\rangle - |\varphi\rangle|| \leq \beta$. Then
\[
\left\| \frac{|\psi\rangle}{|||\psi\rangle||} - \frac{|\varphi\rangle}{|||\varphi\rangle||} \right\| \leq \frac{2 \beta}{\alpha}. \tag{147}
\]
Lemma 17 ([9]). Let $|\psi\rangle = \alpha|0\rangle|\psi_0\rangle + \sqrt{1-\alpha^2}|1\rangle|\psi_1\rangle$ and $|\varphi\rangle = \beta|0\rangle|\varphi_0\rangle + \sqrt{1-\beta^2}|1\rangle|\varphi_1\rangle$, where $|\psi_0\rangle$, $|\psi_1\rangle$, $|\varphi_0\rangle$, $|\varphi_1\rangle$ are unit vectors, and $\alpha, \beta \in [0, 1]$. Suppose $\|\psi\rangle - |\varphi\rangle\| \leq \delta < \alpha$. Then $\|\psi_0\rangle - |\varphi_0\rangle\| \leq \frac{\delta}{\alpha-\delta}$.

Lemma 18 ([9]). Let $|\psi\rangle = \alpha|0\rangle|\psi_0\rangle + \sqrt{1-\alpha^2}|1\rangle|\psi_1\rangle$ and $|\varphi\rangle = \beta|0\rangle|\varphi_0\rangle + \sqrt{1-\beta^2}|1\rangle|\varphi_1\rangle$, where $|\psi_0\rangle$, $|\psi_1\rangle$, $|\varphi_0\rangle$, $|\varphi_1\rangle$ are unit vectors, and $\alpha, \beta \in [0, 1]$. Suppose $\|\psi\rangle - |\varphi\rangle\| \leq \delta < \alpha$. Then $\beta \geq \alpha - \delta$.
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