Five-dimensional black holes in heterotic string theory

Maro Cvitan\textsuperscript{1,3}, Predrag Dominis Prester\textsuperscript{2,3}, Andrej Ficnar\textsuperscript{3}, Silvio Pallua\textsuperscript{3}, and Ivica Smolič\textsuperscript{3}

\textsuperscript{1} International School for Advanced Studies (SISSA/ISAS), Via Beirut 2–4, 34014 Trieste, Italy
\textsuperscript{2} Physics Department, Faculty of Arts and Sciences, University of Rijeka
Omladinska 14, HR-51000 Rijeka, Croatia
\textsuperscript{3} Theoretical Physics Department, Faculty of Science, University of Zagreb
p.p. 331, HR-10002 Zagreb, Croatia

We review recent results on near-horizon static black hole solutions and entropy in \( R^2 \)-corrected \( N = 2 \) SUGRA in \( D = 5 \), focusing on actions connected to heterotic string compactified on \( K3 \times S^1 \). Comparison with \( \alpha' \)-perturbative results, results obtained by using simple Gauss-Bonnet \( R^2 \)-correction, OSV conjecture and microscopic stringy description (for small black holes) shows that situation in \( D = 5 \) is, in a sense, even more interesting than in \( D = 4 \).

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1 5-dimensional black holes in higher derivative \( N = 2 \) SUGRA

Bosonic part of the Lagrangian for the \( N = 2 \) supergravity action in five dimensions is given by

\[
4\pi^2 L_0 = 2\partial^\alpha A^\alpha_i \partial_\alpha A^i_\alpha + A^2 \left( \frac{D}{4} - 3R - \frac{v^2}{2} \right) + N \left( \frac{D}{2} + R + 3v^2 \right) + 2\mathcal{N} v_{ab} F_{ab}^I + \mathcal{N} \left( F_{ab}^I F_{ab}^J + \frac{1}{2} \partial_a M^I \partial^a M^J \right) + \frac{e^{-1}}{24} c_{IJK} A^I_a F_{bc}^J F_{de}^K \epsilon^{abdec} \tag{1}
\]

where \( A^2 = A^i_{ab} A^i_{ab} \) and \( v^2 = v_{ab} v^{ab} \). Also,

\[
N = \frac{1}{6} c_{IJK} M^I M^J M^K, \quad N_I = \partial_I N = \frac{1}{2} c_{IJK} M^J M^K, \quad N_{IJ} = \partial_I \partial_J N = c_{IJK} M^K \tag{2}
\]

A bosonic field content of the theory is the following. We have Weyl multiplet which contains the fünfbein \( e^\mu_a \), the two-form auxiliary field \( v_{ab} \), and the scalar auxiliary field \( D \). There are \( n_V \) vector multiplets enumerated by \( I = 1, \ldots, n_V \), each containing the one-form gauge field \( A^I \) (with the two-form field strength \( F^I = dA^I \)), and the scalar \( M^I \). Scalar fields \( A^i_\alpha \), which are belonging to the hypermultiplet, can be gauge fixed and the convenient choice is given by \( A^2 = -2, \partial_\alpha A^i_\alpha = 0 \).

Lagrangian (1) can be obtained from 11-dimensional SUGRA by compactifying on six-dimensional Calabi-Yau spaces. Then \( M^I \) have interpretation as moduli (volumes of \( (1,1) \)-cycles), and \( c_{IJK} \) as intersection numbers. Condition \( \mathcal{N} = 1 \) is a condition of real special geometry.

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Action (1) is invariant under SUSY variations, which when acting on the purely bosonic configurations are given with

\[ \delta \psi^i = \mathcal{D}_\mu \xi^i + \frac{1}{2} \gamma^{ab} \epsilon_{\mu ab} \xi^i - \gamma^i \eta^i \]

\[ \delta \xi^i = \mathcal{D}_\mu \gamma^i - 2 \gamma^{a} \epsilon^{i} \mathcal{D}_\alpha \gamma \gamma^{b} \xi^i - 2 \gamma^{a} \epsilon^{i} \mathcal{D}_\alpha \xi^i + 4 \gamma^{a} \cdot \mathcal{D}_\alpha \xi^i \]

\[ \delta \Omega^{Ii} = - \frac{1}{4} \gamma^{a} \cdot F^I \cdot \epsilon^i - \frac{1}{2} \gamma^{a} \partial_\alpha M^I \cdot \epsilon^i - M^I \cdot \eta^i \]

\[ \delta \zeta^a = (3 \eta^a - \gamma \cdot \nu \epsilon^i) A^a_j \]

where \( \psi^i \) is gravitino, \( \xi^i \) auxiliary Majorana spinor (Weyl multiplet), \( \delta \Omega^{Ii} \) gaugino (vector multiplets), and \( \zeta^a \) is a fermion field from hypermultiplet.

In [1] a four-derivative part of the action was constructed by supersymmetric completion of the mixed gauge-gravitational Chern-Simons term \( \mathcal{A} \wedge \epsilon \epsilon \epsilon \mathcal{R} \wedge \mathcal{R} \). The bosonic part of the action is

\[ 4 \pi^2 \mathcal{L}_1 = \frac{e_I}{24} \left[ \frac{e^{-1}}{16} \epsilon_{abcd} A^I_{abcd} C^{abcd} C^{cd} + M^I \left[ \frac{1}{8} C^{abcd} C_{abcd} + \frac{1}{12} \partial^2 - 3 \epsilon_{abcd} v^b v^c d \right] 
+ 4 \epsilon_{abcd} v^b v^c d \right] + \frac{8}{3} \epsilon_{abcd} v^b v^c d \partial^2 \mathcal{D}_a \mathcal{D}_b \mathcal{D}_c \mathcal{D}_d 
- \frac{2}{3} \epsilon_{abcd} v^b v^c d \mathcal{D}_a \mathcal{D}_b \mathcal{D}_c \mathcal{D}_d 
+ \epsilon_{abcd} v^b v^c d \mathcal{D}_a \mathcal{D}_b \mathcal{D}_c \mathcal{D}_d 
\right] \]

where \( e_I \) are some constant coefficients, \( C^{abcd} \) is the Weyl tensor, and \( \mathcal{D}_a \) is the conformal covariant derivative.

We are interested in extremal black hole solutions of the action obtained by combining (1) and (3):

\[ \mathcal{A} = \int d^5 \sqrt{-g} \mathcal{L} = \int d^5 \sqrt{-g} (\mathcal{L}_0 + \mathcal{L}_1) \]

The action (5) is quartic in derivatives and generally probably too complicated for finding complete analytical black hole solutions even in the simplest spherically symmetric case. But, if one is more modest and interested just in a near-horizon behavior (which is enough to find the entropy) of extremal black holes, there is a smart way to do the job - Sen’s entropy function formalism [2].

For five-dimensional spherically symmetric extremal black holes near-horizon geometry is expected to be \( AdS_2 \times S^3 \), which has \( SO(2, 1) \times SO(4) \) symmetry. If the Lagrangian can be written in a manifestly diffeomorphism covariant and gauge invariant way, it is expected that near the horizon the complete background should respect this symmetry. In our case it means that near-horizon geometry should be given with

\[ ds^2 = v_1 \left( -x^2 dt^2 + \frac{dx^2}{x^2} \right) + v_2 d\Omega_3^2 \]

\[ F^I (x) = -e^I , \quad v_1 (x) = V , \quad M^I (x) = M^I , \quad D(x) = D \]

where \( v_1, e^I, M^I, V, \) and \( D \) are constants. All covariant derivatives are vanishing. If one defines

\[ f = \int S^3 \sqrt{-g} \mathcal{L} \]
where right hand side is evaluated on the background \( \mathcal{N} \), then equations of motion are equivalent to
\[
0 = \frac{\partial f}{\partial \hat{v}_1}, \quad 0 = \frac{\partial f}{\partial \hat{v}_2}, \quad 0 = \frac{\partial f}{\partial \bar{M}^I}, \quad 0 = \frac{\partial f}{\partial \bar{V}}, \quad 0 = \frac{\partial f}{\partial \bar{D}}.
\] (8)

Derivatives over electric field strengths \( e^I \) are giving (properly normalized) electric charges:
\[
q_I = \frac{\partial f}{\partial e^I}.
\] (9)

The entropy (equal to the Wald formula [3]) is given with
\[
S_{bh} = 2\pi (q_I e^I - f)
\] (10)

It is immediately obvious that though the system (8), (9) is algebraic, it is in generic case too complicated to be solved in direct manner, and that one should try to find some additional information. Such additional information can be obtained from supersymmetry. It is known that there should be 1/2 BPS black hole solutions, for which it was shown in [4] that near the horizon supersymmetry is enhanced fully. This means that in this case we can put all variations in (3) to zero, which one can use to express all unknowns in terms of one. To fix remaining unknown we just need one equation from (8), where the simplest is the one for \( \bar{D} \).

Typically one is interested in expressing the results in terms of charges, not field strengths, and this is achieved by using (9). One gets [5–7]
\[
8 c_{IJK} \bar{M}^J \bar{M}^K = q_I + \frac{c_I}{8}, \quad \bar{M}^I \equiv \sqrt{\hat{v}_1} M^I,
\] (11)

where we introduced scaled moduli \( \bar{M}^I \). The entropy becomes
\[
S_{(BPS)} = \frac{8\pi}{3} c_{IJK} \bar{M}^I \bar{M}^J \bar{M}^K
\] (12)

A virtue of this presentation is that if one is interested only in entropy, then it is enough to consider just (11) and (12), in which the sole effect of the higher derivative terms are just constant shifts of charges \( q_I \to q_I + c_I/8 \). It was shown in [8] that (12) agrees with the OSV conjecture [9], after proper treatment of uplift from \( D = 4 \) to \( D = 5 \) is made.

## 2 Heterotic black holes – non-BPS solutions

We shall be especially interested in the case when prepotential is of the form
\[
\mathcal{N} = \frac{1}{2} M^1 c_{ij} M^i M^j, \quad i, j > 1
\] (13)

where \( c_{ij} \) is a regular matrix with an inverse \( e^{ij} \). In this case, which corresponds to \( K3 \times T^2 \) 11-dimensional compactifications, it is easy to show that the entropy of BPS black holes is given with
\[
S_{(BPS)} = 2\pi \sqrt{\frac{1}{2} \hat{q}_i e^{ij} \hat{q}_j}, \quad \hat{q}_I = q_I + \frac{c_I}{8}
\] (14)

When additionally \( c_1 = 24, c_i = 0 \), our action is equivalent to the (consistently truncated) tree-level effective action of heterotic string compactified on \( K3 \times S^1 \).

One especially interesting (and simple) case is given with \( STU \)-prepotential \( \mathcal{N} = M^1 M^2 M^3 \), which corresponds to heterotic string on \( T^4 \times S^1 \). Black hole solutions are characterised by three integer charges
usually denoted as $m = q_1$, $n = q_2$ and $w = q_3$. Constructed BPS black hole solutions are physically acceptable for $m \geq 0$ and $n, w > 0$. The entropy is now

$$S_{bh}^{(BPS)} = 2\pi \sqrt{nw(m + 3)}. \quad (15)$$

It is remarkable that here non-BPS solutions (for almost all values of charges) were also analytically constructed [7]. For example, for $m \geq 1$, $n < 0$, $w > 0$ the entropy is

$$S_{bh}^{(n-BPS)} = 2\pi \sqrt{|n|w(m - 1/3)}. \quad (16)$$

Properties of these non-BPS solutions suggest possibility that they are descending from BPS states either in $D > 5$ and/or $N > 2$ [7].

Motivated by results from $D = 4$, we have studied solutions when $R^2$-correction is given purely by Gauss-Bonnet density. We obtained that the entropy is different from the one following from $R^2$ SUSY action (14). This mismatch was not present in $D = 4$.

3 Perturbative calculations in $\alpha'$

In view of above results, it is interesting to perturbatively calculate entropy of large 5-dimensional 3-charge extremal black holes up to $\alpha'$-order using low energy effective action of heterotic string (which is fully known only up to $\alpha'^2$-order). The main virtue is that this is a straightforward calculation giving unambiguous results. Using entropy function formalism, and taking special care for non manifestly covariant gravitational Chern-Simons term (using technique from [10]), we obtained for the entropy of BPS black holes [11]

$$S_{bh}^{(BPS)} = 2\pi \sqrt{nw(m + 3)} \left(1 + \frac{3}{2m} - \frac{9}{8m^2} + O(m^{-3})\right), \quad n, w, m > 0, \quad (17)$$

which is in agreement with the supersymmetric result, i.e., with (15) after being expanded in $1/m$.

For non-BPS black holes we obtained the entropy

$$S_{bh}^{(n-BPS)} = 2\pi \sqrt{|n|w(m - 1/3)} \left(1 + \frac{1}{2m} - \frac{1}{8m^2} + O(m^{-3})\right), \quad n < 0, w, m > 0, \quad (18)$$

which disagrees with both SUSY (15) and Gauss-Bonnet results already at $\alpha'$-order. Instead, our result (18) suggests the following formula

$$S_{bh}^{(n-BPS)} = 2\pi \sqrt{|n|w(m + 1)}. \quad (19)$$

Furthermore, if we take BPS formula (15) for granted, then we have been able to show that $\alpha'^3$ term in the non-BPS entropy formula (18) must be $1/16m^3$, which is again in agreement with the conjectured expression (19). Now, using AdS/CFT arguments, from (17) and (18) one infers that central charges satisfy $c_L - c_R = 12w$, which is indeed what is expected [12].

4 Small black holes

Extremely interesting is what happens when one takes $q_1 = 0$ in (14). For the $K3 \times S^1$ heterotic compactifications the entropy becomes

$$S_{bh}^{(BPS)} = 2\pi \sqrt{\frac{3}{2} c^3 q_i q_j}. \quad (20)$$

4 In heterotic string language $n$ and $w$ are momentum and winding number on $S^1$, and $m$ is the magnetic charge of antisymmetric $B_{\mu\nu}$ field.
On the other hand, for the action with Gauss-Bonnet $R^2$ term we obtain

$$S_{bh}^{(BPS)} = 4\pi \sqrt{c_3 q_i q_j} .$$  \hspace{1cm} (21)

These black holes are **small**, meaning that their horizon is generated (regularized) by higher-derivative terms in the action.

Contrary to the large black holes discussed before, for these small black holes microscopic stringy description is known (in the special 2-charge case of $T^4 \times S^1$ heterotic compactification microstates are well-known perturbative Dabholkar-Harvey states) for which statistical entropy was calculated [13, 14]. For BPS states microscopic entropy (for $nw \gg 1$) is exactly equal to the black hole entropy obtained from the action supplemented with just Gauss-Bonnet $R^2$-correction (21), and disagrees with the entropy obtained by using supersymmetric $R^2$-correction [20].

5 Conclusion

We have analysed near-horizon solutions for (both BPS and non-BPS) static spherically symmetric black holes of $N = 2$ SUGRA actions with $R^2$-corrections, which are effective actions of tree-level heterotic string compactified on $K3 \times S^1$ and $T^4 \times S^1$. In addition, we have also made calculations by taking for $R^2$-correction just the Gauss-Bonnet density. In $D = 5$, contrary to a situation in $D = 4$ (see [16] for a review), for these two types of higher-derivative corrections formulae for the entropy are not matching.

For large black holes, where full stringy microscopic description is not known, obtained entropies of BPS black holes are equal to the one obtained from OSV conjecture and topological string, and are consistent with perturbative results up to $\alpha'2$-order.

For small black holes microscopic description is known, with (asymptotic) statistical entropy exactly matching the result obtained by using Gauss-Bonnet $R^2$-correction.

The exact matchings obtained by using just the $R^2$-corrections in effective actions (which has infinite expansion) is surprising. For SUSY correction it has partial explanation through AdS$_3$/CFT$_2$ correspondence [17], but for small black holes where simple Gauss-Bonnet correction does the job (and SUSY correction fails) it is still a complete mystery. We believe that these issues deserve further investigation.

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5 For 2-charge black holes in $D > 5$ one needs in the action also higher Gauss-Bonnet densities [15].