Real equiangular lines in dimension 18 and the Jacobi identity for complementary subgraphs

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Abstract

We show that the maximum cardinality of an equiangular line system in \( \mathbb{R}^{18} \) is at most 59. Our proof includes a novel application of the Jacobi identity for complementary subgraphs. In particular, we show that there does not exist a graph whose adjacency matrix has characteristic polynomial \((x - 22)(x - 2)^{12}(x + 6)^{15}(x + 8)^2\).

1 Introduction

A real equiangular line system is a set of lines through the origin in Euclidean space where the angle between any pair of lines is the same. A classical problem in elliptical geometry is to determine the maximum number \( N(d) \) of equiangular lines in Euclidean space \( \mathbb{R}^d \) for a fixed dimension \( d \). In general, computing \( N(d) \) is a difficult problem dating back to a paper of Haantjes \[21\] in 1948. In the 1970s, Seidel et al. \[28, 30, 33\] made important progress on the study of real equiangular lines, including the discovery of Gerzon’s absolute bound: \( N(d) \leq d(d + 1)/2 \). Furthermore, in 1973 \[28\], the value of \( N(d) \) was settled for all \( d \leq 13 \), \( d = 15 \), and \( d = 21, 22, 23 \). In 2000, by giving an infinite construction of large equiangular line systems, De Caen \[8\] showed that \( N(d) = \Theta(d^2) \).

In order to study \( N(d) \), it is important to understand \( N_\alpha(d) \): the maximum number of equiangular lines in \( \mathbb{R}^d \) with a fixed common angle \( \arccos(\alpha) \) where \( \alpha \in (0, 1) \). Using Neumann’s observation \[28\], we know that if \( N_\alpha(d) \) is greater than \( 2d \) then \( \alpha \) must be equal to \( 1/a \) where \( a \) is an odd integer. Lemmens and Seidel \[28\] showed that \( N_{1/3}(d) = 28 \) for \( 7 \leq d \leq 15 \) and \( N_{1/3}(d) = 2(d - 1) \) for \( d \geq 15 \). Within the past decade, there has been great progress in the study of equiangular lines and its variants, involving a wide array of techniques from many areas of pure and applied mathematics. Cao, Koolen, Lin, and Yu \[9\] showed that \( N_{1/5}(d) = 276 \) for \( 23 \leq d \leq 185 \) and \( N_{1/5}(d) = \left\lfloor \frac{3(d - 1)}{2} \right\rfloor \) for \( d \geq 185 \). Asymptotically, this pattern continues. Indeed, Jiang, Tidor, Yao, Zhang, and Zhao \[24\], showed that when \( a \geq 3 \) is an

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odd integer, $\alpha = 1/a$, and $d$ is large enough, $N_\alpha(d) = \left\lfloor \frac{a+1}{a} (d - 1) \right\rfloor$, thereby proving a recent conjecture of Jiang and Polyanskii [23]. Other related progress includes improvements to the relative bound and various other useful constraints for $N_\alpha(d)$ [3, 13, 25, 26, 27, 31, 37]. In low dimensional Euclidean spaces, substantial improvements to upper and lower bounds for $N(d)$ have been achieved through new spectral methods and new constructions of equiangular lines [16, 17, 18, 19, 20, 29, 35]. We refer to the exposition of Kao and Yu [25] for a more in depth overview of the recent history of the problem.

Recently, the authors together with Yatsyna [18, 19] solved the problem of determining $N(14)$, $N(16)$, and $N(17)$. The smallest $d$ for which $N(d)$ is currently unknown is $d = 18$. In a series of papers from the last few years [19, 29, 35], the lower bound for $N(18)$ has increased from 48 to 57. Augmenting this recent progress, our main result of this paper is a new upper bound for $N(18)$:

**Theorem 1.1.** $N(18) \leq 59$.

We obtain Theorem 1.1 by studying the Seidel matrices that correspond to systems of equiangular lines as follows. Suppose $\{v_1, \ldots, v_n\}$ is a set of unit spanning vectors for an equiangular line system of cardinality $n$ in $\mathbb{R}^d$ with $n > d$. For all $i \neq j$, the inner product of $v_i$ and $v_j$ is equal to $\pm\alpha$ for some $\alpha \in (0, 1)$. The Seidel matrix $S$ corresponding to this line system is defined as $S = (G - I)/\alpha$ where $G$ is the Gram matrix for the set of vectors $\{v_1, \ldots, v_n\}$.

We begin by enumerating all possible polynomials that could be the characteristic polynomial $\text{Char}_S(x) := \det(xI - S)$ of a Seidel matrix that corresponds to a system of 60 equiangular lines in $\mathbb{R}^{18}$. We then apply the techniques developed in [18, 19] to rule out the existence of Seidel matrices whose characteristic polynomial is equal to any of the candidate characteristic polynomials enumerated above. However, these techniques are not quite strong enough to prove Theorem 1.1 - one last candidate characteristic polynomial $(x + 5)^{42}(x - 11)^{15}(x - 15)^3$ remained resilient to all known methods of establishing the nonexistence of a corresponding Seidel matrix. Thus, the novelty in this paper is to apply the Jacobi identity from [7] to develop a method to rule out this one last possibility.

As shown in [16], the existence of a Seidel matrix with characteristic polynomial equal to $(x+5)^{42}(x-11)^{15}(x-15)^3$ is equivalent to the existence of a (regular) graph with characteristic polynomial $(x - 22)(x - 2)^{42}(x + 6)^{15}(x + 8)^2$. In a spectral sense, such a graph is rather close to being strongly regular. It tends to be notoriously difficult to establish nonexistence results for strongly regular graphs. One can observe that the earlier improvements to the upper bounds for $N(17)$, $N(19)$, and $N(20)$ relied on elaborate nonexistence results for certain strongly regular graphs [1, 2, 6].

In Table 1, we give the latest update on the values of lower and upper bounds for $N(d)$ where $d \leq 43$, including the improvement from this paper. (See Sequence A002853 in The On-Line Encyclopedia of Integer Sequences [34].) For a more extensive history on the developments of the bounds, we again refer the reader to [25]. It is worth noting that for $d \leq 41$, the only remaining unknown values of $N(d)$ are $N(18)$, $N(19)$, and $N(20)$.

The outline of the paper is as follows. In Section 2 using the so-called polynomial enumeration algorithm of [18, 19], we enumerate all 44 possible candidate characteristic
polynomials for a Seidel matrix corresponding to a system of 60 equiangular lines in \( \mathbb{R}^{18} \). We employ methods from [18, 19] to show that 43 out of these 44 candidate characteristic polynomials cannot be the characteristic polynomial of any Seidel matrix. The computational parts of these methods ran in Magma [5] and Mathematica [36]. The total running time of all the computations used in this paper is less than 40 minutes running on a modern PC. The Magma [5] implementation of all the computations in this paper is available on GitHub [15]. Section 3 is devoted to ruling out the sole surviving candidate characteristic polynomial \((x+5)^{42}(x-11)^{15}(x-15)^{3}\). There, we use the Jacobi identity for complementary subgraphs and derive necessary algebraic conditions for a graph to be an induced subgraph of \( \Gamma \), a regular graph in the switching class of a putative Seidel matrix whose characteristic polynomial is \((x+5)^{42}(x-11)^{15}(x-15)^{3}\).

## 2 Candidate characteristic polynomials

### 2.1 Enumerating candidates

In this section, we find all the candidate characteristic polynomials of a Seidel matrix that corresponds to an equiangular line system of cardinality 60 in \( \mathbb{R}^{18} \). See Theorem 2.2.

By [18, Lemma 3.1], we derive the following lemma where \( \kappa = 11 \) and \( \theta = 13 \):

**Lemma 2.1.** Let \( S \) be a Seidel matrix corresponding to 60 equiangular lines in \( \mathbb{R}^{18} \). Then

\[
\text{Char}_S(x) = (x+5)^{42}(x-11)^{6}\phi(x),
\]

for some monic polynomial \( \phi \) of degree 12 in \( \mathbb{Z}[x] \) all of whose zeros are greater than \(-5\).

Let \( S \) be a Seidel matrix corresponding to an equiangular line system of cardinality 60 in \( \mathbb{R}^{18} \). The next step is to find feasible polynomials for \( \phi(x) = \sum_{i=0}^{12} b_i x^{12-i} \) where

\[
\text{Char}_S(x) = (x+5)^{42}(x-11)^{6}\phi(x).
\]

Clearly \( b_0 = 1 \). We can find \( b_1 \) and \( b_2 \), using \( \text{tr} \ S = 0 \) and \( \text{tr} \ S^2 = 60 \cdot 59 = 3540 \) together with Newton’s identities: \( b_1 = -144 \), and \( b_2 = 9486 \). Thus, we need to find all totally-real, integer polynomials \( \phi(x) \) such that \( b_0 = 1 \), \( b_1 = -144 \), and \( b_2 = 9486 \). We note that the top three coefficients of \( \phi(x-1) \) are: 1, -156, and 11136 respectively.

Let \( p(x) = \sum_{i=0}^{n} a_i x^{n-i} \) be a monic polynomial in \( \mathbb{Z}[x] \). Following [18], we say \( p(x) \) is **type 2** if \( 2^i \) divides \( a_i \) for all \( i \geq 0 \) and **weakly type 2** if \( 2^{i-1} \) divides \( a_i \) for all \( i \geq 1 \). By
Lemma 2.7 and Lemma 2.8], the polynomial $\phi(x - 1)$ must be type 2. Furthermore, just as in [18, 19], we employ the polynomial enumeration algorithm of [18, Section 2.3] to establish the following theorem. We refer to [18] for details.

**Theorem 2.2.** Let $S$ be a Seidel matrix corresponding to 60 equiangular lines in $\mathbb{R}^{18}$. Then

(i) $\text{Char}_S(x)$ is one of the 39 polynomials listed in Table 2.

(ii) $\text{Char}_S(x) \in \left\{ (x + 5)^{42}(x - 9)^{3}(x - 11)^{6}(x - 13)^{9}, \right.$

\( (x + 5)^{42}(x - 11)^{14}(x - 13)^{3}(x - 17), \)

\( (x + 5)^{42}(x - 9)^{2}(x - 11)^{9}(x - 13)^{6}(x - 15), \)

\( (x + 5)^{42}(x - 11)^{10}(x - 13)^{6}(x^2 - 22x + 109) \),

(iii) or $\text{Char}_S(x) = (x + 5)^{42}(x - 11)^{15}(x - 15)^{3}$.

The rest of the paper is dedicated to ruling out the existence of Seidel matrices whose characteristic polynomial is equal to any of the 44 given in Theorem 2.2.

### 2.2 Certificates of infeasibility

First, we rule out 39 of the 44 candidate characteristic polynomials in Theorem 2.2, using certificates of infeasibility, which we define below.

Denote by $\mathcal{S}_n$ the set of all Seidel matrices of order $n$. Given a positive integer $e$, define the set $\mathcal{P}_{n,e} = \{ \text{Char}_S(x) \mod 2^e \mathbb{Z}[x] : S \in \mathcal{S}_n \}$. We will require the following upper bound on the cardinality of $\mathcal{P}_{n,e}$ for odd $n$.

**Theorem 2.3 ([20, Corollary 3.13]).** Let $n$ be an odd integer and $e$ be a positive integer. Then the cardinality of $\mathcal{P}_{n,e}$ is at most $2(\frac{e-2}{2})^+1$.

For $n = 59$ and $e = 7$, randomly generating Seidel matrices of order 59 is sufficient to obtain 2048 characteristic polynomials in distinct congruence classes modulo $2^e \mathbb{Z}[x]$. Since $2048 = 2^{(\frac{e-2}{2})^+1}$ is the maximum possible number of elements of $\mathcal{P}_{59,7}$, it follows that $|\mathcal{P}_{59,7}| = 2048$. Hence, we have explicitly constructed the set $\mathcal{P}_{59,7}$ and we will use it in Definition 2.5 below.

For $M$ a real symmetric matrix of order $n$ and $\mathcal{N}$ a subset of $\{1, 2, \ldots, n\}$, we denote by $M[\mathcal{N}]$ the principal submatrix of $M$ formed by the rows and columns that are indexed by the elements of $\mathcal{N}$. We use $\overline{\mathcal{N}}$ to denote the set complement of $\mathcal{N}$, i.e., $\overline{\mathcal{N}} := \{1, \ldots, n\} \setminus \mathcal{N}$. Next we introduce the notion of interlacing, motivated by Cauchy’s interlacing theorem:

**Theorem 2.4 ([10, 12, 22]).** Let $M$ be a real symmetric matrix of order $n$ having eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and suppose $M[\overline{i}]$, for some $i \in \{1, \ldots, n\}$, has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$. Then

\[ \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n. \]

Let $f(x) = \prod_{i=0}^{e}(x - \lambda_i)$ and $g(x) = \prod_{i=1}^{e}(x - \mu_i)$ such that $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_e$, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_e$. We say that $g$ **interlaces** $f$ if $\lambda_0 \leq \mu_1 \leq \lambda_1 \leq \cdots \leq \mu_e \leq \lambda_e$. 

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Definition 2.5. Let $p(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree $n$ and write

$$p(x) = \sum_{t=0}^{n} a_t x^{n-t},$$

where $a_0 = 1$, $a_1 = 0$, and $a_2 = -\binom{n}{2}$. An **interlacing characteristic polynomial** for $p(x)$ is defined to be a totally-real, integer polynomial $f(x) = \sum_{t=0}^{n-1} b_t x^{n-1-t}$ such that

(i) $b_0 = 1$, $b_1 = 0$, $b_2 = -\binom{n-1}{2}$,

(ii) $f(x)$ interlaces $p(x)$,

(iii) $f(x - 1)$ is weakly type 2 and is type 2 if $n - 1$ is even,

(iv) $f(x)$ is in a congruence class of $\mathcal{P}_{n-1,7}$, if $n - 1$ is odd.

Denote by $\text{Deck}(p)$ the set of all interlacing characteristic polynomials for $p(x)$. The next lemma follows from [19, Theorem 5.1].

Lemma 2.6. Suppose $p(x)$ is the characteristic polynomial of a Seidel matrix $S$. Then there exist nonnegative integers $n_f$ for each $f(x) \in \text{Deck}(p)$ such that

$$\sum_{f(x) \in \text{Deck}(p)} n_f \cdot f(x) = p'(x).$$

(1)

The (row) vector $n$ indexed by $\text{Deck}(p)$ whose $f(x)$-entry is $n_f$, for each $f(x) \in \text{Deck}(p)$ is called an **interlacing configuration** for $p(x)$. We can use Lemma 2.6 to show that there does not exist a Seidel matrix having $p(x)$ as its characteristic polynomial using information about interlacing configurations.

The **coefficient vector** of a polynomial $f(x) = \sum_{t=0}^{n-1} c_t x^{n-1-t}$ of degree $n - 1$ is defined to be the (row) vector $(c_0, c_1, \ldots, c_{n-1})$. Given a set $\mathfrak{P}$ of polynomials each of degree $n - 1$, the **coefficient matrix** $\text{Coeff}(\mathfrak{P})$ is defined as the $|\mathfrak{P}| \times n$ matrix whose rows are the coefficient vectors for each polynomial in $\mathfrak{P}$. If $\mathfrak{P}$ is a singleton, i.e., $\mathfrak{P} = \{f(x)\}$ then we merely write $\text{Coeff}(\mathfrak{P})$ as $\text{Coeff}(f)$ and if its columns require a particular order (e.g., in Lemma 2.16) then we write $\text{Coeff}(f_1, \ldots, f_k)$ for $\text{Coeff}(\mathfrak{P})$ where $\mathfrak{P} = \{f_1(x), \ldots, f_k(x)\}$. Note that (1) can be written as a vector equation as

$$n \cdot \text{Coeff}(\text{Deck}(p)) = \text{Coeff}(p').$$

(2)

We write $\Lambda_p$ for the set of distinct zeros of the polynomial $p(x)$ and define the polynomial $\text{Min}_p(x) := \prod_{\lambda \in \Lambda_p} (x - \lambda)$. By Definition 2.5, if $f(x) \in \text{Deck}(p)$ then $f(x)$ interlaces $p(x)$ and the top three coefficients of $f(x)$ are fixed (see (i) of Definition 2.5). Consequently, we have the following lemma, which is similar to Lemma 5.4 in [20].
Lemma 2.7. Let \( p(x) \) be a polynomial in \( \mathbb{Z}[x] \). Suppose \( \text{Min}_p(x) = \sum_{i=0}^{e} a_i x^{e-i} \). Then, for all \( f(x) \in \text{Deck}(p) \),

\[
f(x) = \frac{p(x)}{\text{Min}_p(x)} \sum_{i=0}^{e-1} b_i x^{e-1-i},
\]

where \( b_0 = 1, b_1 = a_1, b_2 = a_2 + n - 1, \) and \( b_i \in \mathbb{Z} \) for \( i \in \{3, \ldots, e-1\} \).

Lemma 2.7 above enables us to use the polynomial enumeration algorithm of [18, Section 2.3] to construct \( \text{Deck}(p) \), similar to what we did in [18, 19].

We write \( x \geq 0 \) to indicate that all entries of the vector \( x \) are nonnegative. In the following corollary, note that the polynomial \( p(x) \) divides \( \text{Min}_p(x)p'(x) \) and \( \text{Min}_p(x)f(x) \) for each \( f(x) \in \text{Deck}(p) \).

Corollary 2.8. Let \( p(x) \) be a polynomial where \( \text{Min}_p(x) \) has degree \( e \). Suppose there exists a vector \( c \in \mathbb{R}^e \) such that

\[
\text{Coeff} \left( \left\{ \frac{\text{Min}_p(x)f(x)}{p(x)} : f(x) \in \text{Deck}(p) \right\} \right) c \geq 0 \quad \text{and} \quad \text{Coeff} \left( \left\{ \frac{\text{Min}_p(x)p'(x)}{p(x)} \right\} \right) c < 0.
\]

Then \( p(x) \) is not the characteristic polynomial of any Seidel matrix.

Proof. By Lemma 2.6, Lemma 2.7, and Farkas’ Lemma (see [18, Theorem 4.1]), it follows that \( p(x) \) does not have an interlacing configuration. \( \square \)

We call the vector \( c \) from Corollary 2.8 a certificate of infeasibility for \( p(x) \). In this paper, certificates of infeasibility are written as tuples. Now we can rule out 39 of the 44 candidate characteristic polynomials of Theorem 2.2.

Lemma 2.9. There does not exist a Seidel matrix \( S \) whose characteristic polynomial is equal to any of the polynomials in Table 2.

Proof. Each polynomial in Table 2 is listed together with a certificate of infeasibility. \( \square \)

Now we continue by ruling out the candidate characteristic polynomials from part (ii) of Theorem 2.2 using techniques from [19].

2.3 Multiplicity of interlacing characteristic polynomials

The next lemma puts restrictions on the entries of an interlacing configuration.

Lemma 2.10 ([19, Lemma 6.2]). Let \( M \) be a real symmetric matrix of order \( n \) with eigenvalue \( \lambda \) of multiplicity \( m \). Suppose there exists a \( k \)-subset \( I \) of \( \{1, \ldots, n\} \) such that, for each \( i \in I \), the principal submatrix \( M[I] \) has an eigenvalue \( \lambda \) of multiplicity \( m \) or \( m + 1 \). Then \( M \) has a principal submatrix of order \( n - k \) with eigenvalue \( \lambda \) of multiplicity at least \( m \).

Now we employ Lemma 2.10 to rule out the existence of a Seidel matrix corresponding to either of two of the candidate characteristic polynomials from part (ii) of Theorem 2.2.
Lemma 2.11. There does not exist a Seidel matrix $S$ with characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 9)^3(x - 11)^6(x - 13)^9.$$  

Proof. Suppose a Seidel matrix $S$ has characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 9)^3(x - 11)^6(x - 13)^9.$$  

Then $\text{Deck}(\text{Char}_S) = \{f_1(x), f_2(x)\}$, where

$$f_1(x) = (x + 5)^{41}(x - 9)(x - 11)^7(x - 13)^8,$$
$$f_2(x) = (x + 5)^{41}(x - 9)^3(x - 11)^5(x - 13)^8(x^2 - 19x + 82).$$

and we find that there is only one possible interlacing configuration $(n_1, n_2) = (28, 32)$. By Lemma 2.10 the Seidel matrix $S$ has a principal submatrix $T$ of order 32 with eigenvalue 11 of multiplicity at least 6. Furthermore, by Theorem 2.4 we also have that $\text{Char}_T(x)$ is divisible by $(x + 5)^{14}$. Therefore, $\text{Char}_T(x)$ is divisible by $(x + 5)^{14}(x - 11)^6$. This is a contradiction since $\text{tr}T^2 = 32 \cdot 31 = 992 = 1076 = 14 \cdot (-5)^2 + 6 \cdot 11^2$. 

Lemma 2.12. There does not exist a Seidel matrix $S$ with characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 11)^{14}(x - 13)^3(x - 17).$$  

Proof. Suppose a Seidel matrix $S$ has characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 11)^{14}(x - 13)^3(x - 17).$$  

Then $\text{Deck}(\text{Char}_S) = \{f_1(x), f_2(x), f_3(x)\}$, where

$$f_1(x) = (x + 5)^{41}(x - 11)^{13}(x - 13)^3(x^2 - 23x + 106),$$
$$f_2(x) = (x + 5)^{41}(x - 11)^{13}(x - 13)^2(x^2 - 36x^2 + 405x - 1382),$$
$$f_3(x) = (x + 5)^{41}(x - 11)^{13}(x - 13)^2(x - 17)(x^2 - 19x + 82).$$

The polynomial $f_2(x)$ has 105 interlacing characteristic polynomials but it has a corresponding certificate of infeasibility $(51115513410, 0, 6301221, 484710, 37285)$. Therefore, there does not exist a Seidel matrix with characteristic polynomial $f_2(x)$. Furthermore, there is only one possible interlacing configuration $(n_1, n_2, n_3) = (33, 0, 27)$ where $n_2 = 0$. By Lemma 2.10 the Seidel matrix $S$ has a principal submatrix $T$ of order 27 with eigenvalue 13 of multiplicity at least 3. Furthermore, by Theorem 2.4 we also have that $\text{Char}_T(x)$ is divisible by $(x + 5)^9$. Therefore, $\text{Char}_T(x)$ is divisible by $(x + 5)^9(x - 13)^3$. This is a contradiction since $\text{tr}T^2 = 27 \cdot 26 = 702 < 732 = 9 \cdot (-5)^2 + 3 \cdot 13^2$. 

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2.4 Warranted polynomials

In this section, we will use the notion of a \textit{warranted} polynomial. Let \( p(x) \) be a polynomial. Suppose that \( p(x) \) has at least one interlacing configuration and \( \text{Min}_p(x) \) has degree \( e \). We say that \( f(x) \in \text{Deck}(p) \) is \( p(x)\)-\textbf{warranted} if the \( f(x) \)-entry of every interlacing configuration for \( p(x) \) is positive.

Let \( f(x) \in \text{Deck}(p) \). By Farkas’ Lemma (see \[18\] Theorem 4.1]), if there exists \( c \in \mathbb{R}^e \) such that the \( h(x) \)-entry of

\[
\text{Coeff} \left( \left\{ \frac{\text{Min}_p(x)h(x)}{p(x)} : h(x) \in \text{Deck}(p) \right\} \right) c
\]

is negative for \( h(x) = f(x) \), nonnegative for \( h(x) \in \text{Deck}(p) \backslash \{ f(x) \} \), and

\[
\text{Coeff} \left( \left\{ \frac{\text{Min}_p(x)p'(x)}{p(x)} \right\} \right) c < 0,
\]

then \( f(x) \) is \( p(x) \)-warranted. The vector \( c \) is called the \textbf{certificate of warranty} for \( f(x) \). In this paper, certificates of warranty are written as tuples. The next lemma follows directly from the definition of warranted polynomials.

\begin{lemma}
Let \( S \) be a Seidel matrix of order \( n \). Suppose that \( f(x) \in \text{Deck}(\text{Char}_S) \) is \( \text{Char}_S(x) \)-\textbf{warranted}. Then there exists \( i \in \{1, \ldots, n\} \) such that \( \text{Char}_{S[i]}(x) = f(x) \).
\end{lemma}

Now we employ Lemma 2.13 to rule out the existence of a Seidel matrix corresponding to one of the candidate characteristic polynomials from part (ii) of Theorem 2.2

\begin{lemma}
There does not exist a Seidel matrix \( S \) with characteristic polynomial

\[ \text{Char}_S(x) = (x + 5)^{42}(x - 9)^2(x - 11)^9(x - 13)^6(x - 15). \]

\end{lemma}

\textbf{Proof.} Suppose a Seidel matrix \( S \) has characteristic polynomial

\[ \text{Char}_S(x) = (x + 5)^{42}(x - 9)^2(x - 11)^9(x - 13)^6(x - 15). \]

Then \( \text{Deck}(\text{Char}_S) = \{ f_1(x), \ldots, f_7(x) \} \), where

\[
\begin{align*}
f_1(x) &= (x + 5)^{41}(x - 9)(x - 11)^8(x - 13)^5(x^4 - 43x^3 + 673x^2 - 4529x + 11026), \\
f_2(x) &= (x + 5)^{41}(x - 9)(x - 11)^8(x - 13)^5(x^4 - 43x^3 + 673x^2 - 4525x + 10966), \\
f_3(x) &= (x + 5)^{41}(x - 9)^2(x - 11)^8(x - 13)^6(x^2 - 21x + 94), \\
f_4(x) &= (x + 5)^{41}(x - 9)(x - 11)^8(x - 13)^5(x^4 - 43x^3 + 673x^2 - 4513x + 10818), \\
f_5(x) &= (x + 5)^{41}(x - 9)(x - 11)^9(x - 13)^5(x^3 - 32x^2 + 321x - 978), \\
f_6(x) &= (x + 5)^{41}(x - 9)(x - 10)(x - 11)^9(x - 13)^6(x^2 - 20x + 83), \\
f_7(x) &= (x + 5)^{41}(x - 9)(x - 11)^9(x - 13)^6(x^2 - 19x + 74).
\end{align*}
\]
The polynomial $f_1(x)$ is $\text{Char}_S(x)$-warranted with certificate of warranty 

$$(45911387, 0, 0, 10146, 0)$$

and it has 208 interlacing characteristic polynomials. However, we arrive at a contradiction since $f_1(x)$ has a corresponding certificate of infeasibility:

$$(1132367732240930, 0, 0, 71075114863, 649903361, 64522780, 6545789).$$

2.5 Seidel-compatible polynomials

In this section, we introduce the notions of angles and Seidel-compatibility for polynomials. Let $f(x) = \frac{p(x)}{\text{Min}_p(x)}$, $f(x) \in \text{Deck}(p)$. For each $\lambda \in \Lambda_p$, define the angle $\alpha_\lambda(f)$ of $f(x)$ with respect to $\lambda$ as

$$\alpha_\lambda(f) := \sqrt{\frac{f(\lambda)}{\text{Min}_p(\lambda)}}.$$ 

Now we can introduce the notion of compatibility for polynomials with respect to $p(x)$. Let $\Sigma_p$ be the set of simple zeros of $p(x)$ and define $\text{Sim}_p(x)$ as

$$\text{Sim}_p(x) := \prod_{\xi \in \Sigma_p} (x - \xi).$$

Let $f(x)$ and $g(x)$ be distinct polynomials in $\text{Deck}(p)$. Define $\text{Mult}_p(x) := \text{Min}_p(x)/\text{Sim}_p(x)$. We say that $f(x)$ and $g(x)$ are $p(x)$-Seidel-compatible if there exists $\delta \in \{\pm 1\}^{\Sigma_p}$ such that

$$\sum_{\lambda \in \Sigma_p} \text{Mult}_p(\lambda)\delta(\lambda)\alpha_\lambda(f)\alpha_\lambda(g) \equiv R_p \pmod{2},$$

where

$$R_p := \begin{cases} 
\text{Mult}_p(1) + \text{Mult}_p(0), & \text{if deg } p \text{ is odd}; \\
(\text{Mult}_p(1) - \text{Mult}_p(-1))/2, & \text{if deg } p \text{ is even}.
\end{cases}$$

Any polynomial $p(x)$ is considered to be $p(x)$-Seidel-compatible with itself. Note that the definition of Seidel compatibility given here is a slight variation of the definition given in [19].

**Lemma 2.15.** Let $S$ be a Seidel matrix of order $n$. Suppose that $f(x) \in \text{Deck}(\text{Char}_S)$ is $\text{Char}_S(x)$-warranted. Then for all $j \in \{1, \ldots, n\}$, the polynomial $\text{Char}_{S[j]}(x)$ is $\text{Char}_S(x)$-Seidel-compatible with $f(x)$.

**Proof.** Set $p(x) = \text{Char}_S(x)$. Since $f(x) \in \text{Deck}(p)$ is $p(x)$-warranted, then, by Lemma 2.13 there exists $i \in \{1, \ldots, n\}$ such that $\text{Char}_{S[i]}(x) = f(x)$. Let $j \in \{1, \ldots, n\}$ and let $g(x)$ =
Char$_{S_{ij}}(x)$. If $f(x) = g(x)$ then we are done. Otherwise, we have $i \neq j$ and for each $\lambda \in \Sigma_p$, denote by $u_\lambda$ a unit eigenvector for $\lambda$. By the Spectral Decomposition Theorem, we have

$$Mult_p(S)_{i,j} = \sum_{\lambda \in \Sigma_p} Mult_p(\lambda) u_\lambda(i) u_\lambda(j) = \sum_{\lambda \in \Sigma_p} Mult_p(\lambda) \delta(\lambda) \alpha_\lambda(f) \alpha_\lambda(g)$$

where $\delta \in \{\pm 1\}^{\Sigma_p}$ and $u_\lambda(i)^2 = \alpha_\lambda(f)^2$, $u_\lambda(j)^2 = \alpha_\lambda(g)^2$ by [19] Lemmas 4.4 and 4.5. Clearly, $Mult_p(S)_{i,j}$ is an off-diagonal entries of $Mult_p(S)$, which is an integer matrix. Furthermore, the parities of the off-diagonal entries of $Mult_p(S)$ can be determined by using [16] Lemma 2.1. It follows that $Mult_p(S)_{i,j}$ also satisfies (3) and hence, $g(x)$ is $p(x)$-Seidel-compatible with $f(x)$. 

Now we employ Lemma 2.15 to rule out the last remaining candidate characteristic polynomial from part (ii) of Theorem 2.2.

**Lemma 2.16.** There does not exist a Seidel matrix $S$ with characteristic polynomial

$$Char_S(x) = (x + 5)^{42} (x - 11)^{10} (x - 13)^{6}(x^2 - 22x + 109).$$

**Proof.** Suppose a Seidel matrix $S$ has characteristic polynomial

$$Char_S(x) = (x + 5)^{42} (x - 11)^{10} (x - 13)^{6}(x^2 - 22x + 109).$$

Then $Deck(Char_S) = \{f_1(x), \ldots, f_{11}(x)\}$, where

$$f_1(x) = (x + 5)^{41}(x - 11)^9(x - 13)^5(x^4 - 41x^3 + 609x^2 - 3871x + 8886),$$

$$f_2(x) = (x + 5)^{41}(x - 11)^9(x - 13)^5(x^4 - 41x^3 + 609x^2 - 3867x + 8834),$$

$$f_3(x) = (x + 5)^{41}(x - 11)^9(x - 13)^6(x^3 - 28x^2 + 245x - 682),$$

$$f_4(x) = (x + 5)^{41}(x - 11)^9(x - 13)^5(x^4 - 41x^3 + 609x^2 - 3855x + 8678),$$

$$f_5(x) = (x + 5)^{41}(x - 11)^9(x - 13)^6(x^3 - 28x^2 + 245x - 670),$$

$$f_6(x) = (x + 5)^{41}(x - 11)^9(x - 13)^5(x^4 - 41x^3 + 609x^2 - 3851x + 8626),$$

$$f_7(x) = (x + 5)^{41}(x - 9)(x - 11)^9(x - 13)^6(x^2 - 19x + 74),$$

$$f_8(x) = (x + 5)^{41}(x - 5)(x - 11)^{11}(x - 13)^5(x - 14),$$

$$f_9(x) = (x + 5)^{41}(x - 11)^9(x - 13)^6(x^3 - 28x^2 + 245x - 654),$$

$$f_{10}(x) = (x + 5)^{41}(x - 5)(x - 10)(x - 11)^9(x - 13)^7,$$

$$f_{11}(x) = (x + 5)^{41}(x - 11)^{10}(x - 13)^6(x^2 - 17x + 58).$$

The polynomial $f_1(x)$ is $Char_S(x)$-warranted with certificate of warranty

$$(16485427, 0, 0, 0, -1859).$$

Only three polynomials, $f_1(x)$, $f_3(x)$, and $f_8(x)$, in $Deck(Char_S)$ are $Char_S(x)$-Seidel-compatible with $f_1(x)$. The vector $n = (207/4, 6, 9/4)$ is the unique solution to the equation $n \cdot Coeff(f_1, f_3, f_8) = Coeff(\text{Char}_S)$. However, the entries of $n$ are not all integers, which contradicts Lemma 2.6.

To complete the proof of Theorem 1.1 it remains to rule out one last candidate characteristic polynomial from Theorem 2.2.
3 The Jacobi identity for complementary subgraphs

In this section, we rule out the last remaining candidate characteristic polynomial from Theorem 2.2. That is, we prove the following theorem.

**Theorem 3.1.** There does not exist a Seidel matrix $S$ with characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 11)^{15}(x - 15)^3.$$ 

Let $\Gamma$ be a graph, $A$ be the adjacency matrix of $\Gamma$, and $\mathcal{T}, \mathcal{U}$ be subsets of $V(\Gamma)$. In the remainder of the paper, the rows and columns of the adjacency matrix $A$ are indexed by the vertices of $\Gamma$. Accordingly, we define the following notation.

- $\Gamma[\mathcal{T}]$ denotes the subgraph of $\Gamma$ induced on the vertices of $\mathcal{T}$.
- $A[\mathcal{T}, \mathcal{U}]$ denotes the submatrix of $A$ with rows from $\mathcal{T}$ and columns from $\mathcal{U}$.
- $A[\mathcal{T}]$ denotes the principal submatrix $A[\mathcal{T}, \mathcal{T}]$.

Let $v$ be a vertex of $\Gamma$ and let $v$ be a vector such that $v = A[\mathcal{T}, \{v\}]$. For $\mathcal{S} \subseteq \mathcal{T}$, denote by $v[\mathcal{S}]$ the vector $A[\mathcal{S}, \{v\}]$. Furthermore, in this section we occasionally consider the complement of a subset of $V(\Gamma)$, where similarly to before, we define $\overline{\mathcal{T}} := V(\Gamma) \setminus \mathcal{T}$. In order to prove Theorem 3.1 inspired by a paper of De Caen [7], we heavily use the following identity due to Jacobi:

**Lemma 3.2** (Jacobi). Let $\mathcal{T} \subseteq V(\Gamma)$ and suppose that $A$ is the adjacency matrix of $\Gamma$, then

$$\det(xI - A[\overline{\mathcal{T}}]) = \det(xI - A) \det((xI - A)^{-1}[\mathcal{T}]).$$

We use Lemma 3.2 to derive a necessary algebraic condition for induced subgraphs of $\Gamma$ (see Corollary 3.8 and Lemma 3.11).

Denote by $J$ the all-ones matrix and by $O$ the zero matrix. We use subscripts $J_{n,m}$ to indicate the all-ones matrix with $n$ rows and $m$ columns, or merely write $J_n$ for $J_{n,n}$. The all-ones vector (with $n$ entries) is denoted by $1_n$. The subscript is omitted if the order of the matrix (or vector) can be determined from context.

First we consider the orthogonal projection matrix of the dimension-2 eigenspace of a regular graph in the switching class of the putative Seidel matrix of Theorem 3.1. For the sake of contradiction, suppose that there exists a Seidel matrix $S$ with characteristic polynomial

$$\text{Char}_S(x) = (x + 5)^{42}(x - 11)^{15}(x - 15)^3.$$ 

By [16, Remark 5.12] there exists a (regular) graph $\Gamma$ with adjacency matrix $A$ of spectrum:

$$\text{Spec } A = \{[22]^4, [2]^1, [-6]^2, [-8]^2\}.$$ 

For the rest of this section, $\Gamma$ will remain fixed and will be repeatedly referred to without being specified again.
3.1 An equitable partition for $\Gamma$

Following [11, Section 3.1], we set $M = 3A^2 + 12A - 36I - 28J$, where $A$ is the adjacency matrix of $\Gamma$. Then $M$ is positive semidefinite with diagonal entries 2, so $|M_{ij}| \leq 2$ for all $i, j$. For $i \neq j$, the entry $(A^2 + 4A)_{ij} = (M_{ij} + 28)/3$ is an integer. Hence $M_{ij} \in \{-1, 2\}$ and $(A^2 + 4A)_{ij} \in \{9, 10\}$.

Now we claim that the relation where $i \sim j$ when $M_{ij} = 2$ is an equivalence relation. Indeed, if $i \sim j$ and $j \sim k$ for distinct $i, j, k$, then

$$2 = \text{rank } M \geq \text{rank } \begin{bmatrix} M_{ii} & M_{ij} & M_{ik} \\ M_{ij} & M_{jj} & M_{jk} \\ M_{ik} & M_{jk} & M_{kk} \end{bmatrix} = \text{rank } \begin{bmatrix} 2 & 2 & M_{ik} \\ 2 & 2 & 2 \\ M_{ik} & 2 & 2 \end{bmatrix}.$$ 

This forces $M_{ik} = 2$, or equivalently, $i \sim k$. Now, since $MJ = 0$ and the entries of $M$ are 2 and $-1$, each equivalence class has size 20.

Let $\Omega$ be the partition of $V(\Gamma)$ whose three parts are the three equivalence classes of the above relation. Just as for $\Gamma$, we also fix $\Omega$ for the remainder of this section. If we assume that $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ where $\{A_{ii} : i \in \{1, 2, 3\}\} = \{A[T] : T \in \Omega\}$ then

$$M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \otimes J_{20},$$

where $\otimes$ denotes the Kronecker product for matrices.

We summarise the above as a lemma:

**Lemma 3.3.** Suppose $A$ is the adjacency matrix of $\Gamma$ such that $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ where $\{A_{ii} : i \in \{1, 2, 3\}\} = \{A[T] : T \in \Omega\}$. Then

$$A^2 + 4A - 12I_{60} = 9J_{60} + I_3 \otimes J_{20}.$$ 

Now we show that $\Omega$ is an equitable partition of $\Gamma$.

**Lemma 3.4.** The partition $\Omega$ of $V(\Gamma)$ is equitable with quotient matrix

$$\begin{bmatrix} 2 & 10 & 10 \\ 10 & 2 & 10 \\ 10 & 10 & 2 \end{bmatrix}.$$ 

**Proof.** Suppose that $\Omega = \{T_1, T_2, T_3\}$. Assume that $A$ is the adjacency matrix of $\Gamma$ such that $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ where $A_{ii} = A[T_i]$ for each $i \in \{1, 2, 3\}$.

Recall that $M = 3A^2 + 12A - 36I - 28J$. Using $(A + 8I)(A + 6I)(A - 2I) = 280J$ and Lemma 3.3, we deduce that

$$AM = \begin{bmatrix} -16 & 8 & 8 \\ 8 & -16 & 8 \\ 8 & 8 & -16 \end{bmatrix} \otimes J_{20}. \quad (4)$$
Let \( u_i \in T_i \) for \( i = 1, 2, 3 \) and denote by \( n_{ij} \) the number of neighbours of \( u_i \) in \( T_j \). Take the \( 3 \times 3 \) submatrix of (4) induced on \( \{u_1, u_2, u_3\} \) to obtain

\[
\begin{bmatrix}
n_{11} & n_{12} & n_{13} \\
n_{21} & n_{22} & n_{23} \\
n_{31} & n_{32} & n_{33}
\end{bmatrix}
\begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}
= \begin{bmatrix}
-16 & 8 & 8 \\
8 & -16 & 8 \\
8 & 8 & -16
\end{bmatrix}.
\]

Since \( \Gamma \) is regular with valency 22, we obtain

\[
\begin{bmatrix}
n_{11} & n_{12} & n_{13} \\
n_{21} & n_{22} & n_{23} \\
n_{31} & n_{32} & n_{33}
\end{bmatrix}
= \begin{bmatrix}
2 & 10 & 10 \\
10 & 2 & 10 \\
10 & 10 & 2
\end{bmatrix}.
\]

The statement of the lemma follows, since the choice of \( u_1, u_2, \) and \( u_3 \) was immaterial. \( \square \)

The following corollary is immediate.

**Corollary 3.5.** Let \( T \in \Omega \) and \( u \in T \). Then \( \Gamma[T] \) is a disjoint union of cycles and \( u \) is adjacent to precisely 10 vertices in \( T \).

### 3.2 Expressions for matrix inverses

Now we pursue expressions for the inverse of matrices that we may encounter when applying Lemma 3.2. Let \( D : \mathbb{R}[x] \to \mathbb{R}[x] \) be the linear operator defined by \( Dx^i := x^{i-1} \) for \( i > 0 \) and \( Dx^0 := 0 \).

**Lemma 3.6.** Suppose \( A \) is a real square matrix and \( m(x) \) denotes its minimal polynomial. Let \( f \) be a rational function over the real numbers. If \( m(f(x)) \) is nonzero, then we have

\[
(f(x)I - A)^{-1} = \frac{1}{m(f(x))} \sum_{k=1}^{\deg m} D^k m(f(x)) A^{k-1}.
\]

**Proof.** Observe that

\[
m(y) = \sum_{k=0}^{\deg m} (D^k m(t) - t D^{k+1} m(t)) y^k
\]

\[
m(y) - m(t) = (y-t) \sum_{k=1}^{\deg m} D^k m(t) \cdot y^{k-1}.
\]

(5)

Suppose we set \( y = A \) and \( t = f(x) \). By the Cayley-Hamilton theorem, \( m(A) = 0 \) and hence,

\[
-m(f(x))I = (A - f(x)I) \sum_{k=1}^{\deg m} D^k m(f(x)) A^{k-1}.
\]

The desired result then follows immediately. \( \square \)
Next we derive an expression for \((xI - A)^{-1}\), where \(A\) is the adjacency matrix of \(\Gamma\).

**Proposition 3.7.** Suppose \(A\) is the adjacency matrix of \(\Gamma\) such that
\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\]
where \(\{A_{ii} : i \in \{1, 2, 3\}\} = \{A[T] : \ T \in \Omega\}\). Then
\[
(xI - A)^{-1} = \frac{(x - 22)(x + 8)(A + (x + 4)I_{60}) + (9x + 82)J_{60} + (x - 22)I_3 \otimes J_{20}}{(x - 22)(x - 2)(x + 6)(x + 8)}.
\]

**Proof.** Let \(m(x) = (x - 22)(x - 2)(x + 6)(x + 8) = x^4 - 10x^3 - 244x^2 - 536x + 2112\) be the minimal polynomial of \(A\). We obtain
\[
D^1m(x) = x^3 - 10x^2 - 244x - 536,
\]
\[
D^2m(x) = x^2 - 10x - 244,
\]
\[
D^3m(x) = x - 10,
\]
\[
D^4m(x) = 1.
\]
By Lemma 3.6 we obtain
\[
m(x)(xI - A)^{-1} = A^3 + (x - 10)A^2 + (x^2 - 10x - 244)A + (x^3 - 10x^2 - 244x - 536)I.
\]

By Lemma 3.3
\[
A^3 = -4A^2 + 12A + 208J_{60} - 8I_3 \otimes J_{20} = 28A - 48I_{60} + 172J_{60} - 12I_3 \otimes J_{20}.
\]
Furthermore, \(A(I_3 \otimes J_{20}) = (I_3 \otimes J_{20})A = 10J_{60} - 8I_3 \otimes J_{20}\). The lemma then follows by simplifying (6).

Let \(\oplus\) denote the direct sum of matrices. Applying Proposition 3.7 and Lemma 3.2 together with the fact that the characteristic polynomial of a graph belongs to \(\mathbb{Z}[x]\), we obtain the following corollary:

**Corollary 3.8 (Subgraph condition I).** Let \(T, U \in \Omega\) such that \(T \neq U\). Suppose that \(X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\) is an adjacency matrix for \(\Gamma[T \cup S]\) where \(S \subseteq U\), \(X_{11} = X[T]\) and \(X_{22} = X[S]\). Then the determinant of
\[
(x + 4)I_{20 + |S|} + X + \frac{(9x + 82)J_{20 + |S|}}{(x - 22)(x + 8)} + \frac{J_{20} \oplus J_{|S|}}{x + 8}
\]
is in \(\frac{(x + 6)^5 + |S|}{(x - 22)(x - 2)^{22 - |S|}(x + 8)^2}\mathbb{Z}[x]\).

**Proof.** Suppose \(A\) is the adjacency matrix of \(\Gamma\) such that \(A[T \cup S] = X\). By Proposition 3.7
\[
(x - 2)(x + 6)(xI - A)^{-1}[T \cup S] = (x + 4)I_{20 + |S|} + X + \frac{(9x + 82)J_{20 + |S|}}{(x - 22)(x + 8)} + \frac{J_{20} \oplus J_{|S|}}{x + 8}.
\]
Since \(\det(xI - A[T \cup S]) \in \mathbb{Z}[x]\), the conclusion then follows from Lemma 3.2.
3.3 Five 4-cycles

In this subsection, we show that for each $T \in \Omega$, the local graph $\Gamma[T]$ is the disjoint union of five 4-cycles. Denote by $C_n$, the cycle graph on $n$ vertices.

Lemma 3.9. Let $T \in \Omega$. Then $\Gamma[T]$ is the disjoint union of at least 5 cycles.

Proof. Let $A$ be the adjacency matrix of $\Gamma$. By Corollary 3.8, we have that

$$\det \left( (x + 4)I + A[T] + \frac{10(x + 6)J}{(x - 22)(x + 8)} \right)$$

must be in

$$\frac{(x + 6)^5}{(x - 22)(x - 2)^2(x + 8)^2} \mathbb{Z}[x].$$

By Corollary 3.5, we know that $\Gamma[T]$ is a disjoint union of cycles. Let $\Psi$ be the multiset of eigenvalues (with multiplicity) of $A[T]$ whose eigenspaces are orthogonal to the all-ones vector and let $c$ be the number of connected components of $\Gamma[T]$. By the matrix determinant lemma, we have

$$\det \left( (x + 4)I + A[T] + \frac{10(x + 6)J}{(x - 22)(x + 8)} \right) = \frac{(x + 6)^c(x - 12)(x - 2)}{(x - 22)(x + 8)} \prod_{\lambda \in \Psi} (x + 4 + \lambda).$$

The lemma follows since, by the Perron-Frobenius theorem [14, Chapter 8], we have $|\lambda| < 2$ for each $\lambda \in \Psi$. 

Now we establish the main result of this subsection.

Lemma 3.10. Let $T \in \Omega$. Then $\Gamma[T]$ is the disjoint union of five 4-cycles.

Proof. Suppose that $\Gamma[T]$ is a graph of order 20 equals to a disjoint union of cycles. We will show that each cycle of $\Gamma[T]$ must be an even cycle. Let $u \in \overline{T}$ and let $A$ be the adjacency matrix of $\Gamma$. Applying Proposition 3.7, we write

$$(x - 2)(x + 6)(xI - A)^{-1}[T \cup \{u\}] = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} Q^T & R \end{pmatrix},$$

where

$$P = (x + 4)I + \frac{10(x + 6)J}{(x - 22)(x + 8)} + A[T],$$

$$Q = \mathbf{u} + \frac{9x + 82}{(x - 22)(x + 8)} \mathbf{1},$$

$$R = (x + 4) + \frac{10(x + 6)}{(x - 22)(x + 8)} = \frac{x^3 - 10x^2 - 222x - 644}{(x - 22)(x + 8)}.$$
and \( u = A[T, \{u\}] \). Using the properties of Schur complement [32, Theorem 3.1.1], we can write
\[
\det \left( (xI - A)^{-1}[T \cup \{u\}] \right) = \frac{\det P \cdot \det(R - Q^TP^{-1}Q)}{(x - 2)^{21}(x + 6)^{21}}.
\]
From Lemma 3.2, we obtain
\[
\det(xI - A[T \cup \{u\}]) = \frac{(x - 22)(x - 2)^{21}(x + 8)^2}{(x + 6)^6} \cdot \det P \cdot \det(R - Q^TP^{-1}Q).
\]
As in the proof of Lemma 3.9, note that the multiplicity of \((x + 6)\) as an eigenvalue of \( P \) is equal to the number of disjoint cycles of \( \Gamma[T] \) and this number is at most 6. Therefore,
\[
\frac{(x - 22)(x - 2)^{21}(x + 8)^2}{(x + 6)^6} \cdot \det P
\]
in its simplest form cannot have \((x + 6)\) as a factor of the numerator.
Applying the Sherman-Morrison formula [1], we have
\[
P^{-1} = ((x + 4)I + A[T])^{-1} - \frac{10J}{(x - 12)(x - 2)(x + 6)}.
\]
Note that
\[
\det(R - Q^TP^{-1}Q) = \frac{x^4 + 6x^3 - 102x^2 - 1016x - 2604}{(x - 12)(x + 6)(x + 8)} - u^T((x + 4)I + A[T])^{-1}u.
\]
Now suppose that \( T = \bigcup_{i=1}^t C_i \) where \( \Gamma[C_i] = C_{|C_i|} \) for each \( i \in \{1, \ldots, t\} \). Fix \( i \in \{1, \ldots, t\} \) and by Lemma 3.6 we obtain
\[
(x + 4)(x + 6)((x + 4)I + A[C_i])^{-1} = -\frac{1}{m_i(-x - 4)} \sum_{k=1}^{\deg m_i} D^k m_i(-x - 4)A[C_i]^{k-1}
\]
where \( m_i(x) = (x - 2)q_i(x) \) is the minimal polynomial of \( C_{|C_i|} \). Hence,
\[
(x + 6)((x + 4)I + A[C_i])^{-1} = \frac{1}{q_i(-x - 4)} \sum_{k=1}^{\deg m_i} D^k m_i(-x - 4)A[C_i]^{k-1}
\]
Define \( \omega(x) := (x + 6) \det(R - Q^TP^{-1}Q) \) and \( \tau_i(x) := (x + 6)((x + 4)I + A[C_i])^{-1} \). Then
\[
\omega(x) = \frac{x^4 + 6x^3 - 102x^2 - 1016x - 2604}{(x - 12)(x + 8)} - \sum_{i=1}^{t} u[C_i]^T \tau_i(x)u[C_i].
\]
Note that \( \det(xI - A[T \cup \{u\}]) \in \mathbb{Z}[x] \). Thus, the denominator of \( \det(R - Q^TP^{-1}Q) \) in its simplest form is not divisible by \((x + 6)\), since the other factor of \( \det(xI - A[T \cup \{u\}]) \) does
not have \((x + 6)\) as a factor in its numerator, as argued above. Hence, we have \(\omega(-6) = 0\). Next, we will show that \(\omega(-6) = 0\) implies that \(|C_i|\) is even for all \(i\). By (7), we have

\[
\tau_i(-6) = \frac{1}{q_i(2)} \sum_{k=1}^{\deg m_i} D^k m_i(2) A[C_i]^{k-1}.
\]

Using (5), we obtain

\[
(A[C_i] - 2I) \sum_{k=1}^{\deg m_i} D^k m_i(2) A[C_i]^{k-1} = m_i(A[C_i]) - m_i(2) I = 0,
\]

which implies that

\[
\sum_{k=1}^{\deg m_i} D^k m_i(2) A[C_i]^{k-1} = \varepsilon J_{|C_i|}
\]

for some constant \(\varepsilon\). Multiplying both sides from the right by the all-ones vector yields

\[
\varepsilon = \frac{1}{|C_i|} \sum_{k=1}^{\deg m_i} D^k m_i(2) \cdot 2^{k-1} = \frac{m_i'(2)}{|C_i|} = \frac{q_i(2)}{|C_i|}.
\]

It follows that

\[
\tau_i(-6) = \frac{1}{q_i(2)} \cdot \frac{q_i(2)}{|C_i|} J_{|C_i|} = \frac{J_{|C_i|}}{|C_i|}
\]

and therefore,

\[
\omega(-6) = 5 - \sum_{i=1}^{t} \frac{(1^T u[C_i])^2}{|C_i|}.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum_{i=1}^{t} \frac{(1^T u[C_i])^2}{|C_i|} \geq \frac{(1^T u)^2}{\sum_{i=1}^{t} |C_i|} = \frac{100}{20} = 5
\]

with equality if and only if there exists \(\delta\) such that \(1^T u[C_i] = \delta |C_i|\) for all \(i\). Since \(\omega(-6) = 0\), there exists \(\delta\) such that \(1^T u[C_i] = \delta |C_i|\) for all \(i\). Moreover, we have \(\delta = 1/2\) since \(20\delta = 1^T u = 10\). Since \(1^T u[C_i] = |C_i|/2\) is an integer, each cycle of \(\Gamma[T]\) must be even. Combined with Lemma 3.9, we conclude that \(\Gamma[T]\) is the disjoint union of five 4-cycles. \(\square\)

### 3.4 Pairwise compatible neighbourhoods

Let \(T \in \Omega\). In this subsection, we consider graphs resulting from joining an independent set of vertices from some \(U \in \Omega\) where \(U \neq T\) to the vertex set of \(\Gamma[T] = 5C_4\). The next lemma provides a necessary algebraic condition for such graph to be an induced subgraph of \(\Gamma\).
Lemma 3.11 (Subgraph condition II). Let $\mathcal{T}, \mathcal{U} \in \Omega$ such that $\mathcal{T} \neq \mathcal{U}$. Suppose $S \subseteq \mathcal{U}$ such that $|S| = s$ and the vertices in $S$ are pairwise nonadjacent. Let $A$ be the adjacency matrix of $\Gamma$. Suppose $A[\mathcal{T}, \mathcal{T} \cup S] = [X \ B]$, where $X = A[\mathcal{T}]$. Then the determinant of

$$(x + 4)I + \frac{10(x^2 + 4x - 30)J}{(x - 12)(x + 6)(x + 8)} - \frac{B^\top B}{x + 4} - \frac{B^\top (X^2 - (x + 4)X)B}{(x + 2)(x + 4)(x + 6)}$$

is in $\frac{(x + 6)^s}{(x - 12)(x - 2)^{23-s}(x + 2)^5(x + 4)^{10}(x + 8)} \mathbb{Z}[x]$.

Proof. Using properties of the Schur complement, we can write

$$\det \left( (x + 4)I + \begin{bmatrix} X & B \\ B^\top & O \end{bmatrix} \right) + \frac{(9x + 82)J}{x - 22} + \frac{J_{20} \oplus J_s}{x + 8} = \det P \cdot \det(R - Q^\top P^{-1}Q), \quad (8)$$

where

$$P = (x + 4)I_{20} + \frac{10(x + 6)}{(x - 22)(x + 8)}J_{20} + X;$$

$$Q = B + \frac{9x + 82}{x - 22}(x + 8)J_{20,s};$$

$$R = (x + 4)I_s + \frac{10(x + 6)}{(x - 22)(x + 8)}J_s.$$

By Lemma 3.10, the induced subgraph $\Gamma[\mathcal{T}] = 5C_4$. Thus, we can compute $\det P$ and $P^{-1}$ as follows:

$$\det P = \frac{(x - 12)(x - 2)(x + 2)^5(x + 4)^{10}(x + 6)^5}{(x - 22)(x + 8)}, \quad (9)$$

$$P^{-1} = \frac{I_{20}}{x + 4} - \frac{10J_{20}}{(x - 12)(x - 2)(x + 6)} + \frac{X^2 - (x + 4)X}{(x + 2)(x + 4)(x + 6)}.$$ 

By Lemma 3.14, we know that $B^\top 1_{20} = 101_s$. Hence

$$R - Q^\top P^{-1}Q = (x + 4)I_s + \frac{10(x^2 + 4x - 30)J_s}{(x - 12)(x + 6)(x + 8)} - \frac{B^\top B}{x + 4} - \frac{B^\top (X^2 - (x + 4)X)B}{(x + 2)(x + 4)(x + 6)}. \quad (10)$$

The lemma then follows from Corollary 3.8 with (8), (9), and (10).

Let $u \in \mathcal{T}$. First we consider graphs resulting from joining $u$ to $\mathcal{T}$. Based on Lemma 3.11 above, the following lemma yields conditions on the adjacency of $u$ with the vertices in $\mathcal{T}$.

Lemma 3.12. Let $\mathcal{T} \in \Omega$ and $u \in \mathcal{T}$. Let $A$ be the adjacency matrix of $\Gamma$. Suppose that $u = A[\mathcal{T}, \{u\}]$. Then

$$u^\top u = 10, \quad u^\top A[\mathcal{T}]u = 6, \quad \text{and} \quad u^\top A[\mathcal{T}]^2u = 28.$$
Proof. By Lemma 3.4, the vertex $u$ is adjacent to 10 vertices in $\Gamma[\mathcal{T}]$. Thus, $1^\top u = 10$. By Lemma 3.11,

\[
(x + 4) + \frac{10(x^2 + 4x - 30)}{(x - 12)(x + 6)(x + 8)} - \frac{10}{x + 4} - \frac{u^\top A[\mathcal{T}]^2 u}{(x + 2)(x + 4)(x + 6)} + \frac{u^\top A[\mathcal{T}] u}{(x + 2)(x + 6)}
\]

belongs to $\frac{(x + 6)}{(x - 12)(x - 2)^2(2)(x + 2)(x + 4)^{10}(x + 8)}\mathbb{Z}[x]$. Set $\alpha = u^\top A[\mathcal{T}]^2 u$ and $\beta = u^\top A[\mathcal{T}] u$. It follows that

\[
(x + 2)(x^5 + 10x^4 - 88x^3 - 1444x^2 - 5468x - 4656) - (x - 12)(x + 8)(\alpha - (x + 4)\beta)
\]

is in $(x + 6)^2 \mathbb{Z}[x]$. Reducing modulo $(x + 6)^2$ yields the simultaneous equations $16\alpha - 4\beta = 424$ and $132\alpha + 48\beta = 3984$. Thus $\alpha = 28$ and $\beta = 6$.

![Figure 1: The 21-vertex graph $\mathcal{S}$.](image)

By Corollary 3.5, we know that $u$ has 10 neighbours in $\mathcal{T}$. At first sight, there appears to be $\binom{20}{10}$ possible ways to assign vertices in $\mathcal{T}$ as neighbours of $u$. However, using Lemma 3.12 we will see that only 2560 of these 10-sets of $\mathcal{T}$ satisfy Corollary 3.13 below. Furthermore, all 2560 such graphs are isomorphic to the graph $\mathcal{S}$ in Figure 1.

For each $u \in V(\Gamma)$, we denote by $N(u)$ the subset of $V(\Gamma)$ consisting of vertices adjacent to $u$.

**Corollary 3.13.** Let $\mathcal{T} \in \Omega$ and $u \in \overline{\mathcal{T}}$. Then $\Gamma[N(u) \cap \mathcal{T}] = 3K_2 + 4K_1$.

**Proof.** Let $u = A[\mathcal{T}, \{u\}]$ and set $\alpha = u^\top A[\mathcal{T}]^2 u$ and $\beta = u^\top A[\mathcal{T}] u$. By Lemma 3.12, we have $\alpha = 28$, $\beta = 6$, and $1^\top u = 10$. By Lemma 3.10, we can write $\mathcal{T} = \bigcup_{i=1}^5 \mathcal{C}_i$ where
Thus, the determinant of Lemma 3.12, \(f\), where \(G\) to obtain that there are 2560 = \(5\cdot 2^8\cdot 3\) vertices in \(N\). The next lemma provides conditions on the adjacency of the independent set with the vertices in \(T\). Subsequently, we consider graphs resulting from joining an independent set of order two to \(T\). The next lemma provides conditions on the adjacency of the independent set with the vertices in \(T\).

\[\Gamma[C_i] = C_4\] for each \(i \in \{1, \ldots, 5\}\). Observe that

\[
\sum_{i=1}^{5}(1^T u[C_i])^2 = \sum_{i=1}^{5} u[C_i]^T J u[C_i] = \sum_{i=1}^{5} u[C_i]^T \left(\frac{A[C_i]^2}{2} + A[C_i]\right) u[C_i]
= \sum_{i=1}^{5} u[C_i]^T \frac{A[C_i]^2}{2} u[C_i] + \sum_{i=1}^{5} u[C_i]^T A[C_i] u[C_i]
= \frac{\alpha}{2} + \beta = 20.
\]

Combining with \(1^T u = 10\), we must have \(1^T u[C_i] = 2\) for each \(i \in \{1, \ldots, 5\}\). Next, observe that, for each \(i \in \{1, \ldots, 5\}\), we have \(u[C_i]^T A[C_i] u[C_i] \in \{0, 2\}\). Therefore, \(u^T A[T] u = \sum_{i=1}^{5} u[C_i]^T A[C_i] u[C_i] = 6\) implies that \(|\{i \in \{1, \ldots, 5\} : u[C_i]^T A[C_i] u[C_i] = 2\}| = 3\) and \(|\{i \in \{1, \ldots, 5\} : u[C_i]^T A[C_i] u[C_i] = 0\}| = 2\).

By Corollary 3.13 for \(T \in \Omega\) and \(u \in \overline{T}\), the induced subgraph \(\Gamma[T \cup \{u\}]\) is isomorphic to \(\Phi\) in Figure 1. Partition \(T\) as \(T = \bigcup_{i=1}^{2} C_i \cup \bigcup_{i=1}^{3} C_i^*\) such that \(\Gamma[C_i] = C_4\), \(\Gamma[C_i^*] = C_4\), \(\Gamma[N(u) \cap C_i] = K_1\) for \(i \in \{1, 2\}\), and \(\Gamma[N(u) \cap C_i^*] = K_2\) for \(i \in \{1, 2, 3\}\). This way, we see that there are \(560 = \binom{5}{2} \cdot 2^2 \cdot 4^3\) subsets in \(\binom{T}{10}\) that can be used as the neighbourhood of \(u\) to obtain \(\Phi = \Gamma[T \cup \{u\}]\).

Subsequently, we consider graphs resulting from joining an independent set of order two to \(T\). The next lemma provides conditions on the adjacency of the independent set with the vertices in \(T\).

**Lemma 3.14.** Let \(T, U \in \Omega\) and \(u, v \in U\) where \(T \neq U\), \(u \neq v\), and \(u \not\sim v\). Suppose that \(A\) is the adjacency matrix of \(\Gamma\) and \(A[T, \{u, v\}] = [u, v]\). Then

\[\langle u^T v, u^T A[T] v \rangle \in \{(4, 14), (5, 10), (6, 6)\}.
\]

**Proof.** Let \(B = A[T, \{u, v\}] = [u, v]\). Define \(\alpha = u^T v\) and \(\beta = u^T A[T] v\). Then, by Lemma 3.12

\[
B^T B = \begin{bmatrix} 10 & \alpha \\ \alpha & 10 \end{bmatrix} \quad \text{and} \quad B^T A[T] B = \begin{bmatrix} 6 & \beta \\ \beta & 6 \end{bmatrix}.
\]

Thus, the determinant of

\[
(x + 4)I + \frac{10(x^2 + 4x - 30)J}{(x - 12)(x + 6)(x + 8)} = \frac{B^T B}{x + 4} - \frac{40J}{(x + 2)(x + 4)(x + 6)} + \frac{B^T A[T] B}{(x + 2)(x + 4)}
\]

is equal to

\[
\frac{f(x)g(x)}{(x - 12)(x + 2)^2(x + 4)^2(x + 8)},
\]

where

\[
f(x) = x^5 + 6x^4 - (\alpha + 94)x^3 + (2\alpha + \beta - 950)x^2 + (104\alpha - 4\beta - 2704)x + 192\alpha - 96\beta - 1248,
\]

\[
g(x) = x^3 + 10x^2 + (\alpha + 22)x + 2\alpha - \beta + 18.
\]
By Lemma 3.11, we must have $f(x)g(x) \in (x + 6)^2 \mathbb{Z}[x]$. Reducing modulo $(x + 6)^2$ yields the simultaneous equations
\[
384\alpha^2 + 132\beta^2 + 624\alpha\beta - 12000\alpha - 6240\beta + 68400 = 0,
32\alpha^2 - 16\beta^2 - 56\alpha\beta + 560\alpha + 680\beta - 6000 = 0.
\]
It follows that $\beta + 4\alpha = 30$.

Let $w = A[\mathcal{T}]u$. By Lemma 3.10, we can partition the set $\mathcal{T}$ as $\mathcal{T} = \bigcup_{i=1}^5 C_i$ such that $\Gamma[C_i] = C_4$ for each $i \in \{1, \ldots, 5\}$. By Corollary 3.13, there are two indices, $a$ and $b$, such that $\Gamma[N(u) \cap C_i] = 2K_1$ for each $i \in \{a, b\}$. It follows that
\[
\mathbf{w}(w) = \begin{cases} 
0 & \text{if } w \in N(u) \cap (C_a \cup C_b); \\
1 & \text{if } w \in \mathcal{T} \setminus (C_a \cup C_b); \\
2 & \text{otherwise.}
\end{cases}
\]
Since $|N(v) \cap C_i| = 2$ for each $i \in \{1, \ldots, 5\}$, it follows that $\mathbf{w}^T \mathbf{v} \in \{6, 10, 14\}$. Furthermore, since $\beta = \mathbf{w}^T \mathbf{v}$ and $\beta + 4\alpha = 30$, we must have $(\alpha, \beta) \in \{(4, 14), (5, 10), (6, 6)\}$. 

Fix $\mathcal{T}, \mathcal{U} \subset \Omega$ with $\mathcal{T} \neq \mathcal{U}$ and $u \in \mathcal{U}$. By Corollary 3.13, the induced subgraph $\Gamma[\mathcal{T} \cup \{u\}]$ is isomorphic to $\mathfrak{G}$ in Figure 1. For each subset $\mathcal{K} \subset \binom{\mathcal{T}}{4}$, define the corresponding (characteristic) vector $\mathbf{v}_{\mathcal{K}} \in \{0, 1\}^T$ such that $\mathbf{v}_{\mathcal{K}}(w) = 1$ if $w \in \mathcal{K}$ and $\mathbf{v}_{\mathcal{K}}(w) = 0$ otherwise. Let $\mathbf{u}$ denote $\mathbf{v}_{N(u) \cap \mathcal{T}}$. Define the set $\mathfrak{B}_u \subset \{0, 1\}^T$ to consist of vectors $\mathbf{v}$ such that
\begin{itemize}
  \item $\mathbf{v}^T \mathbf{v} = 10$.
  \item $\mathbf{v}^T A[\mathcal{T}] \mathbf{v} = 6$.
  \item $\mathbf{v}^T A[\mathcal{T}]^2 \mathbf{v} = 28$.
  \item $(\mathbf{u}^T \mathbf{v}, \mathbf{u}^T A[\mathcal{T}] \mathbf{v}) \in \{(4, 14), (5, 10), (6, 6)\}$.
\end{itemize}

By Lemma 3.12 and Lemma 3.14, for any $v \in \mathcal{U}$ with $v \not\sim u$, we must have $\mathbf{v}_{N(v) \cap \mathcal{T}} \in \mathfrak{B}_u$. Form a graph $G_u$ with vertex set $\mathfrak{B}_u$ where $\mathbf{v} \in \mathfrak{B}_u$ and $w \in \mathfrak{B}_u$ are adjacent if and only if
\[
(\mathbf{v}^T \mathbf{w}, \mathbf{v}^T A[\mathcal{T}] \mathbf{w}) \in \{(4, 14), (5, 10), (6, 6)\}.
\]

Note that the graph $G_u$ does not depend on the choice of $u \in \mathcal{U}$. For each $u^* \in \mathcal{U}$, since $\Gamma[\mathcal{T} \cup \{u^*\}] = \Gamma[\mathcal{T} \cup \{u\}] = \mathfrak{G}$, there exists a permutation matrix $P$ such that $\mathbf{v}_{N(u^*) \cap \mathcal{T}} = P \mathbf{u}$ and $P^T A[\mathcal{T}] P = A[\mathcal{T}]$. Thus, $\mathbf{v} \in \mathfrak{B}_u$ if and only if $P \mathbf{v} \in \mathfrak{B}_{u^*}$ and furthermore, $G_u$ is isomorphic to $G_{u^*}$.

We remark that the graph $G_u$ has 454 vertices. This can be proved without the use of a computer using Lemma 3.14, but we omit it since our proof is a bit tedious and the graph $G_u$ can be readily generated using a computer.

By Lemma 3.10, the induced subgraph $\Gamma[\mathcal{U}]$ is the disjoint union of five 4-cycles, $5C_4$. This implies that there exists a subset of 10 pairwise nonadjacent vertices in $\mathcal{U}$ including the vertex $u$. This subset corresponds to a clique of order 9 in the graph $G_u$. However, using Magma (and independently, Mathematica), we find that the largest clique in $G_u$ has 7 vertices. This contradicts our assumption that the graph $\Gamma$ exists and completes the proof of Theorem 3.1.
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In Table A we list all but five of the candidate characteristic polynomials from Theorem 2.2 together with their certificates of infeasibility.

|   | Polynomial                                                                 | Certificate                                                                 |
|---|---------------------------------------------------------------------------|----------------------------------------------------------------------------|
| 1 | $(x + 5)^{42}(x - 11)^{12}(x - 13)^3(x^3 - 39x^2 + 495x - 2049)$         | $(0, 0, 0, 34294, 10143, 1812)$                                           |
| 2 | $(x + 5)^{42}(x - 11)^{12}(x - 13)^2(x^3 - 41x^2 + 551x - 2423)$         | $(-305901333, 0, 0, -29631, -1854, 35)$                                    |
| 3 | $(x + 5)^{42}(x - 11)^{12}(x - 13)(x - 15)(x^2 - 28x + 191)$              | $(-1458935482, 0, 0, -81026, 9, 0)$                                       |
| 4 | $(x + 5)^{42}(x - 11)^{12}(x - 13)^2(x^3 - 26x + 157)$                   | $(0, 0, 0, -843, -338)$                                                   |
| 5 | $(x + 5)^{42}(x - 11)^{12}(x - 13)^2(x^2 - 24x + 139)(x^2 - 28x + 191)$  | $(-769246694472, 0, 0, -458256882, -36586502, -2928267, -235607)$          |
| 6 | $(x + 5)^{42}(x - 11)^{12}(x - 13)^2(x - 15)(x^2 - 26x + 161)$            | $(0, 0, 0, 0, 370, 171)$                                                  |
| 7 | $(x + 5)^{42}(x - 9)(x - 11)^{11}(x - 13)^4(x^3 - 28x + 191)$            | $(19949599256, 0, 0, 2370351, 198001, 16971)$                               |
|   | (x + 5)^12(x - 11)^14(x - 13)^2(x - 13)^2(x-15) (x^4 - 49x^2 + 395x - 23)  |
|---|--------------------------------------------------------------------------------------------------|
|   | (0, 0, 0, 0, 0, -12398, 0, 0, -31763185, -2280433, -162889)                            |
| 9 | (x + 5)^12(x - 11)^11(x - 13)^4(x^2 - 24x + 139) (x^2 - 26x + 161)                |
|   | (0, 0, 0, 0, 0, -1389, -790)                                                               |
| 10| (x + 5)^12(x - 11)^13(x - 13)^2(x - 15) (x^4 - 37x^2 + 447x - 1763)                   |
|   | (0, 0, 0, 0, -122674, -43601, -8823)                                                       |
| 11| (x + 5)^12(x - 11)^11(x - 13)^2(x - 15)^2 (x^2 - 24x + 139)                            |
|   | (0, 0, 0, 0, 420, 199)                                                                   |
| 12| (x + 5)^12(x - 11)^10(x - 13)^4(x^3 - 37x^2 + 443x - 1711)                             |
|   | (0, 0, 0, 0, 0, 234708, 62958, 8869)                                                      |
| 13| (x + 5)^12(x - 9)(x - 11)^10(x - 13)^3(x^2 - 26x + 161)                                |
|   | (0, 0, 0, 0, 0, 0, 125208, 41635, 8550)                                                   |
| 14| (x + 5)^12(x - 11)^11(x - 13)^4(x^4 - 50x^4 + 924x^4 - 7470x^2 + 22259)               |
|   | (0, 0, 0, -24742927, -5579245, -747551, -85438)                                          |
| 15| (x + 5)^12(x - 11)^10(x - 13)^4(x^4 - 24x + 139) (x^4 - 37x^2 + 447x - 1763)          |
|   | (0, 0, 0, 0, 0, 0, 0, 0, 2249, 1485)                                                     |
| 16| (x + 5)^12(x - 9)(x - 11)^12(x - 13)^2 (x^2 - 15)                                       |
|   | (196837694, 0, 0, 0, 55815, 5074)                                                         |
| 17| (x + 5)^12(x - 11)^11(x - 13)^2(x - 15) (x^2 - 24x + 139)^2                            |
|   | (0, 0, 0, -172391, -37294, -2664)                                                        |
| 18| (x + 5)^12(x - 11)^11(x - 13)^2 (x^2 - 24x + 131)                                       |
|   | (0, 0, 0, 0, 0, 0, -2265, 946)                                                            |
| 19| (x + 5)^12(x - 9)(x - 11)^10(x - 13)^6 (x^3 - 37x^2 + 447x - 1763)                     |
|   | (0, 0, 0, -34871227, -8914066, -1403713, -190406)                                       |
| 20| (x + 5)^12(x - 11)^10(x - 13)^4 (x^4 - 48x^3 + 850x^2 - 6576x + 18749)                |
|   | (0, 0, 0, 0, 0, -176, -103)                                                              |
| 21| (x + 5)^12(x - 11)^11(x - 13)^4 (x - 15) (x^3 - 35x^2 + 399x - 1477)                   |
|   | (0, 0, 0, 0, 0, 94177, -32024, -5780)                                                    |
| 22| (x + 5)^12(x - 9)(x - 11)^10(x - 13)^4 (x - 15) (x^2 - 24x + 139)                     |
|   | (0, 0, 0, 0, 0, 557458, -207778, -43657)                                                 |
| 23| (x + 5)^12(x - 11)^10(x - 13)^4 (x^2 - 24x + 139)^4                                    |
|   | (0, 0, 0, -415, -169)                                                                   |
| 24| (x + 5)^12(x - 11)^10(x - 13)^3 (x^3 - 35x^2 + 395x - 1433)                             |
|   | (27648369302, 0, 0, 3643351, 310262, 20593)                                              |
| 25| (x + 5)^12(x - 11)^10(x - 13)^3 (x^4 - 46x^2 + 780x^2 - 5778x + 15779)                |
|   | (0, 0, 0, 68304472, 16503986, 2380728, 292487)                                           |
| 26| (x + 5)^12(x - 11)^11(x - 13)^4 (x - 15) (x^2 - 22x + 113)                              |
|   | (0, 0, 0, 0, 108251, 29388, 4254)                                                        |
| 27| (x + 5)^12(x - 11)^10(x - 13)^4 (x^2 - 24x + 139) (x^3 - 35x^2 + 399x - 1477)         |
|   | (0, 0, 0, -1845487303, -379343200, -48329261, -5293439, -539442)                       |
|   | Candidate characteristic polynomials of Theorem 2.2 and their certificates of infeasibility. |
|---|-----------------------------------------------------------------------------------------------|
| 28. | $(x + 5)^42(x - 9)(x - 11)^8(x - 13)^9(x^2 - 24x + 139)^2$  
     | $(34043665264, 0, 0, 5251934, 519994, 51999)$                                               |
| 29. | $(x + 5)^42(x - 11)^{10}(x - 13)^9(x - 15)(x^2 - 20x + 95)$  
     | $(64088824162, 0, 0, 8048870, 611903, 47069)$                                             |
| 30. | $(x + 5)^42(x - 11)^9(x - 13)^9(x^3 - 33x^2 + 351x - 1207)$  
     | $(0, 0, 0, 0, 1139, 599)$                                                                 |
| 31. | $(x + 5)^42(x - 9)(x - 11)^8(x - 13)^6(x^3 - 35x^2 + 399x - 1477)$  
     | $(0, 0, 0, 0, -814967, -316527, -68325)$                                                   |
| 32. | $(x + 5)^42(x - 11)^9(x - 13)^8(x^2 - 22x + 113)(x^2 - 24x + 139)$  
     | $(0, 0, 0, 0, -1163, -700)$                                                               |
| 33. | $(x + 5)^42(x - 9)^2(x - 11)^7(x - 13)^7(x^2 - 24x + 139)$  
     | $(0, 0, 0, 0, 248, 129)$                                                                  |
| 34. | $(x + 5)^42(x - 11)^8(x - 13)^9(x^2 - 20x + 95)(x^2 - 24x + 139)$  
     | $(531003714336, 0, 0, 41982177, 3283852, 255629, 19664)$                                  |
| 35. | $(x + 5)^42(x - 9)(x - 11)^8(x - 13)^7(x^2 - 22x + 113)$  
     | $(-24681212876, 0, 0, -4267959, -431106, -43111)$                                        |
| 36. | $(x + 5)^42(x - 11)^9(x - 13)^6(x^3 - 33x^2 + 351x - 1191)$  
     | $(-6730490844, 0, 0, -798929, -33284, 0)$                                                 |
| 37. | $(x + 5)^42(x - 11)^8(x - 13)^7(x^3 - 31x^2 + 311x - 1009)$  
     | $(0, 0, 0, 0, 1329, 722)$                                                                |
| 38. | $(x + 5)^42(x - 9)(x - 11)^7(x - 13)^8(x^2 - 20x + 95)$  
     | $(-816690681, 0, 0, -119057, -4880, 14)$                                                    |
| 39. | $(x + 5)^42(x - 7)(x - 11)^9(x - 13)^8$  
     | $(143620, 0, 0, 253)$                                                                     |