Estimating E-Bayesian of Parameters of Inverse Weibull Distribution Using a Unified Hybrid Censoring Scheme

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Abstract

The combination of generalization Type-I hybrid censoring and generalization Type-II hybrid censoring schemes, scheme creates a new censoring called a Unified hybrid censoring scheme. Therefore, in this study, the E-Bayesian estimation of parameters of the inverse Weibull (IW) distribution is obtained under the unified hybrid censoring scheme, and the efficiency of the proposed method was compared with the Bayesian estimator using Monte Carlo simulation as well as, we use a real data set for practical purposes. Finally, we showed that in all schemes the E-Bayesian estimation parameters are better than their Bayesian estimations.

Key Words: E-Bayesian Estimation; Unified Hybrid Censoring Scheme; Inverse Weibull Distribution.

Mathematical Subject Classification: 62F15, 62N02

1. Introduction

Consider a lifetime test with \( n \) units. Suppose that the units have independent and identically lifetime with the probability density function \( f(x; \theta) \) and the cumulative distribution function \( F(x; \theta) \), and \( Y_{1:n} < \cdots < Y_{n:n} \) are the lifetime of the units until their failure. For the first time, Epstein (1954) investigated a scheme in a survival experiment in which the experiment ended at time \( T^* = \min(Y_{r:n}, T) \) and the values of \( T \) and \( r \) were pre-determined. Childs et al. (2003) called this Type-I hybrid censoring. In this scheme, there may be very few failures up to time \( T \). Childs et al. (2003) investigated a scheme in which the experiment ended at time \( T^* = \max(Y_{r:n}, T) \). This scheme was called the Type-II hybrid censoring scheme. Obviously, this scheme does not have a problem with the previous scheme. Even before time \( T \), all units can fail, but the time to test is not predictable. Chandrasekar et al. (2004) introduced two Types of generalization hybrid censoring of Type I and II, so that the problem has somewhat improved the previous two schemes (not having the minimum failure in the Type-I hybrid censoring scheme and prolonging the test time in the Type-II hybrid censoring scheme).

In general Type-I hybrid censoring scheme, suppose \( T \in (0, \infty) \) and the values of \( k \) and \( r \) such that \( k < r \) are predetermined. If the \( k^{th} \) failure occurs before time \( T \), the experiment at \( \min(Y_{r:n}, T) \) and if, after time \( T \), the experiment ends at \( Y_{k:n} \). Therefore, this scheme guarantees at least \( k \) failures.

In general Type-II hybrid censoring scheme, assume that \( r \) and \( T_1, T_2 \in (0, \infty) \), so that \( T_1 < T_2 \), are constant and predetermined values. If the \( r^{th} \) failure occurs before time \( T_1 \), the experiment at time \( T_1 \), if between \( T_1 \) and \( T_2 \), occurs at time \( Y_{r:n} \), and if after \( T_2 \), the experiment ends at \( T_2 \). Therefore, this scheme guarantees that the experiment ends up at time \( T_2 \).
The combination of the above scheme creates a new censoring called a Unified hybrid censoring scheme. This scheme was first introduced by Balakrishnan et al. (2008). In this scheme, the values \( T_1, T_2, r, \) and \( k \), so that \( T_1 < T_2 \) and \( k < r \), are predetermined before the experiment begins. If the \( k \)th failure occurs before time \( T_1 \), the experiment at time \( \min \{Y_{r:n}, T_1, T_2 \} \), if between \( T_1 \) and \( T_2 \), occurs at time \( \min \{Y_{r:n}, T_2 \} \), and if after \( T_2 \), the experiment ends at \( Y_{r:n} \). In this censoring, one of the following six occurrences occurs. Suppose that for \( j = 1, 2 \), \( d_j \) the number of failures is up to \( T_j \). In this case, we have six types of observations.

1. If \( 0 < Y_{r:n} < Y_{r:n} < T_1 < T_2 \), the experiment ends at time \( T_1 \) with \( D \) failures.
2. If \( 0 < Y_{r:n} < T_1 < Y_{r:n} < T_2 \), the experiment ends with the failure of \( r \)th.
3. If \( 0 < Y_{r:n} < T_1 < T_2 < Y_{r:n} \), the experiment ends at time \( T_2 \) with \( d_2 \) failures.
4. If \( 0 < T_1 < Y_{r:n} < Y_{r:n} < T_2 \), the experiment ends at time \( Y_{r:n} \).
5. If \( 0 < T_1 < Y_{r:n} < T_2 < Y_{r:n} \), experiment ends at time \( T_2 \) with \( d_2 \) failures.
6. If \( 0 < T_1 < T_2 < Y_{r:n} < Y_{r:n} \), The experiment ends with the failure of \( k \)th.

Note that in the first case, \( d_1 = d_2 = D, T_1 < Y_{(D+1):n} \) and \( r \leq D \), so that the experiment of \( D + 1 \)th does not occur before \( T_1 \), and in the third and fifth cases, \( T_2 < Y_{(d_2+1):n} \) and \( k \leq d_2 \) are such that That the \( d_1 + 1 \)th experiment does not occur before \( T_2 \). If \( c \) is the stopping point and \( d \) is the number of failures until time \( c \), then, the likelihood function of this hybrid censored sample is as follows:

\[
L(\theta|y) = \frac{n!}{(n-d)!} \prod_{i=1}^{d} f(y_{i:n}; \theta) [1 - F(c)]^{n-d}
\]

where \( y = (y_{1:n}, \ldots, y_{d:n}) \), \( \{D, d_1, d_2, k, r\} \), and \( \{T_1, T_2, Y_{r:n}, Y_{r:n}\} \).

If the random variable \( Y \) has a Weibull distribution with the pdf

\[
f(y; \alpha, \lambda) = \alpha \lambda y^{\alpha-1} e^{-\lambda y^\alpha}, \quad y > 0,
\]

then the random variable \( X = \frac{1}{y} \) has an IW distribution with the pdf

\[
f(x; \alpha, \lambda) = \alpha \lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}, \quad x > 0.
\]

The quantities \( \alpha > 0 \) and \( \lambda > 0 \) are the shape and scale parameters respectively. From now on it will be denoted by \( IW(\alpha, \lambda) \). If \( X \) follow \( IW(\alpha, \lambda) \), then the distribution function of \( X \) is given by

\[
F(x; \alpha, \lambda) = e^{-\lambda x^{-\alpha}}, \quad x > 0.
\]

The IW model has been derived as a suitable model for describing the degradation phenomena of mechanical components, such as the dynamic components of diesel engines, see for example Murthy et al. (2004). The physical failure process given by Erto and Rapone (1984) also leads to the IW model. Erto and Rapone (1984) showed that the IW model provides a good fit to survival data such as the times to breakdown of an insulating fluid subject to the action of constant tension, see also Nelson (1982). Interpretation of IW distribution in the context of load strength relationship for a component was provided by Calabria and Pulcini (1994). In reliability engineering research, IW distribution is often used in statistical analysis of life time and response time data. Khan et al. (2008) in their theoretical analysis of IW distribution mention that numerous failure characteristics such as wear out periods and infant mortality can be modeled through IW distribution. They mention about the wide range of areas in reliability analysis where IW distribution model can be used successfully. Shafiee et al. (2016) mention that IW distribution is an appropriate model for situations where hazard function is unimodal. They further mention the distribution as one of the popular distributions in complementary risk problems.

The hierarchical Bayesian prior distribution was primarily introduced by Lindley and Smith (1972). Then it was examined by Han (1997), and E-Bayesian and hierarchical Bayesian methods were introduced. Recently, E-Bayesian and hierarchical Bayesian methods have been used by Han (2009, 2011) to estimate exponential distribution parameter and estimation of the reliability of the binomial distribution, by Jaheen and Okasha (2011) to estimate of the reliability of the Type 12 distribution based on Type II progressive censoring samples, by Wang et al. (2012) and Yousefzadeh (2017) to estimate Pascal distribution parameters, by Yaghoobzadeh (2018) to estimate of scale parameter of gompertz distribution under type II censoring schemes based on fuzzy data. Also, Han (2017) gives the property of E-Bayesian estimation and hierarchical Bayesian estimation of the system reliability parameter. In this study, E-Bayesian of \( \alpha \) and \( \lambda \) parameters of IW distribution Based On an unified hybrid censored sample using square error loss function (
\(L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2\) are described in Section 2. A numerical example and a Monte Carlo simulation are presented in Section 3 for illustrative purposes. Section 4 is the conclusions.

2. Estimating the E-Bayesian of \(\alpha\) and \(\lambda\) Parameters

Suppose \(y_{1:n}, \ldots, y_{n:n}\) is a random sample based on unified hybrid censored scheme, and are identical to the probability density function (2). Also, assume that \(A\) and \(B\) are independent, and each has a prior gamma distribution as follows

\[\pi_1(\alpha|a_1, b_1) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \quad \alpha > 0,\]
\[\pi_2(\lambda|a_2, b_2) \propto \lambda^{a_2-1}e^{-b_2\lambda}, \quad \lambda > 0\]

Where \(a_1, b_1, a_2, b_2\) are positive and known values. The derivative of \(\pi(\alpha|a_1, b_1)\) with respect to \(\alpha\) is

\[
\frac{d\pi(\alpha|a_1, b_1)}{d\alpha} = \frac{a_1^{-1}e^{-b_1\alpha}}{\Gamma(a_1)}((a_1 - 1) - b_1\alpha)
\]

According to Han (1997), \(a_1\) and \(b_1\) should be chosen to guarantee that \(\pi(\alpha|a_1, b_1)\) is a decreasing function of \(\alpha\). Thus, \(b_1 > 0\) and \(0 < a_1 < 1\). Given \(a_1 = 1\), and the larger the value of \(b_1\), the thinner the tail of the density function is. Berger (1985) showed that the thinner tailed prior distribution often reduces the robustness of the Bayesian estimation. Consequently, the hyperparameter \(b_1\) should be chosen under the restriction \(0 < b_1 < c_1\), where \(c_1\) is a given upper bound (\(c_1\) is a positive constant). In this study, we only consider the case when \(a_1 = 1\). In this case, the density function \(\pi(\alpha|a_1, b_1)\) becomes

\[\pi(\alpha|b_1) = b_1 e^{-b_1\alpha}, \quad \alpha > 0,\]

Also, we consider the prior distribution \(b_1\) as \(\pi(b_1) = \frac{1}{c_1}, 0 < b_1 < c_1\).

As the same way, \(\pi(\lambda|a_2, b_2)\) becomes

\[\pi(\lambda|b_2) = b_2 e^{-b_2\lambda}, \quad \lambda > 0,\]

and \(\pi(b_2) = \frac{1}{c_2}, 0 < b_2 < c_2\). Therefore, the Bayesian estimation of each function of \(\alpha\) and \(\lambda\) as \(h(\alpha, \lambda)\), under square error loss function is as follows.

\[\hat{h} = E(\alpha, \lambda|y)(h(\alpha, \lambda)) = \left(\int_0^\infty \int_0^\infty h(\alpha, \lambda)\pi(\alpha, \lambda|y)d\alpha d\lambda\right)/\left(\int_0^\infty \int_0^\infty \pi(\alpha, \lambda|y)d\alpha d\lambda\right)\]

where

\[\pi(\alpha, \lambda|y) \propto \alpha^d \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha + 1)} e^{-b_1\alpha} \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} f_{\lambda|\alpha}(a_1^*, b_1^*, j)\]

where \(f_{\lambda|\alpha}(a_1^*, b_1^*, j)\) is the gamma distribution with the parameter \(a_1^* = d + 1\) of shape and with the scale parameter as follows.

\[b_1^* = b_2 + \sum_{i=1}^d y_{i:n} + jc^{-a}\]

By considering \(h(\alpha, \lambda) = \alpha\) and \(h(\alpha, \lambda) = \lambda\) in relation (6), the Bayesian estimations for the \(\alpha\) and \(\lambda\) representing the symbols \(\hat{\alpha}_B(b_1, b_2)\) and \(\hat{\lambda}_B(b_1, b_2)\), respectively, are as follows.

\[\hat{\alpha}_B(b_1, b_2) = \frac{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \left(\frac{\alpha}{b_1^*}\right)^{d+1} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha + 1)} e^{-b_1\alpha} d\alpha}{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n-d-j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+1}} \left(\prod_{i=1}^d y_{i:n}\right)^{-(\alpha + 1)} e^{-b_1\alpha} d\alpha\}
\]
\[ \hat{\lambda}_B(b_1, b_2) = \frac{(d + 1) \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha}{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha} \] (8)

The definition for E-Bayesian estimation was originally proposed by Han (2009), relate as follows.

**Definition 1:** With \( \hat{\theta}_B(b_1, b_2) \) being continuous,

\[ \hat{\theta}_{EB} = \int \hat{\theta}_B(b_1, b_2) \pi(b_1, b_2) \, db_1 \, db_2 \] (9)

is called the E-Bayesian estimation of \( \theta \) (briefly E-Bayesian estimation, the full name should be expected Bayesian estimation), which is assumed to be finite, where \( D \) is the domain of \( b_1 \) and \( b_2 \), \( \hat{\theta}_B(b_1, b_2) \) is the Bayesian estimation of \( \theta \) with hyper parameters \( b_1 \) and \( b_2 \), and \( \pi(b_1, b_2) \) is the density function of \( b_1 \) and \( b_2 \) over \( D \). Definition 1 indicates that the E-Bayesian estimation of \( \theta \) is just the expectation of the Bayesian estimation of \( \theta \) for all the hyperparameters.

Therefore, with respect to (8) and (9) and definition (1), the E-Bayesian estimations \( \alpha (\hat{\alpha}_{EB}) \) and \( \lambda (\hat{\lambda}_{EB}) \) are as follows.

\[ \hat{\alpha}_{EB} = \frac{1}{c_1 c_2} \int_{c_1}^{c_2} \int_0^\infty \left( \frac{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha}{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha} \right) \, db_1 \, db_2 \]

\[ \hat{\lambda}_{EB} = \frac{1}{c_1 c_2} \int_{c_1}^{c_2} \int_0^\infty \left( \frac{(d + 1) \sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha}{\sum_{j=0}^{n-d} \frac{(-1)^j}{j!(n - d - j)!} \int_0^\infty \frac{\alpha^d}{b_1^{d+2}} \left( \prod_{i=1}^d y_{i:n} \right)^{-(a+1)} e^{-b_1 \alpha} d\alpha} \right) \, db_1 \, db_2 \]

### 3. Numerical Experiments

In this section, a numerical example and a Monte Carlo simulation are presented to illustrate all the estimation methods described in the section 2.

#### 3.1. Simulation Study

In this section, the simulation results are presented for comparing different unified hybrid censored schemes and the performance of estimation of Bayesian and E-Bayesian parameters are based on the mean square error (MSE) criterion. For this purpose, we generate a random sample of 50 of the IW distribution with \( \alpha = 2.5 \) and \( \lambda = 0.05 \). Then, the Bayesian, and E-Bayesian estimations of \( \alpha \) and \( \lambda \) were estimated. The performance of all estimates have been compared numerically of the MSE value. This process, have been irritated 1000 times, and the average all estimates and their MSEs were estimated and are listed in Tables 1 to 4. The simulation is conducted by R software.

Drawn upon the simulation results, we found out that:

1. According to Tables 1 and 2, in both the cases \((b_1, b_2)\), for fixed \( r, k, \) and \( T_2 \), when \( T_1 \) is increased, the performance of the E-Bayesian estimation of the parameters \( \alpha \) and \( \lambda \) is more than their Bayesian estimations. Also, the MSE of all estimators decreases with increasing \( T_1 \), and the numerical value of the estimators approaches the real values of the parameters by increasing \( T_1 \).

2. According to Tables 3 and 4, in both the cases \((b_1, b_2)\), for fixed \( r, k, \) and \( T_1 \), when \( T_2 \) is increased, the performance of the Bayesian estimation of the parameters \( \alpha \) and \( \lambda \) is more than their E-Bayesian estimations. Also, the MSE of all estimators decreases with increasing \( T_2 \), and the numerical value of the estimators approaches the real values of the parameters by increasing \( T_2 \).
Table 1. Estimate, the mean square error for $\alpha$ and $T_2 = 100$

| $(b_1, b_2)$ | $(k, r)$ | $T_1$ | $\hat{\alpha}_B$ | $\hat{\alpha}_{EB}$ | MSE($\hat{\alpha}_B$) | MSE($\hat{\alpha}_{EB}$) |
|--------------|----------|-------|-------------------|---------------------|----------------------|----------------------|
| (1.5, 2)     | (11, 20) | 80    | 1.0084356         | 1.3116907           | 0.9722919            | 0.9582604            |
|              |          | 85    | 1.6295644         | 1.8872927           | 0.9291750            | 0.8110088            |
|              |          | 95    | 2.2896661         | 2.5735351           | 0.8350252            | 0.7265117            |
| (15, 20)     |          | 80    | 0.9002296         | 1.4889474           | 1.2943128            | 1.0106295            |
|              |          | 85    | 2.0657321         | 2.4462040           | 1.0263026            | 0.9604438            |
|              |          | 95    | 2.3750732         | 2.6011219           | 0.9364447            | 0.8430386            |
| (18, 20)     |          | 80    | 0.9603170         | 1.2968166           | 0.7655433            | 0.7446662            |
|              |          | 85    | 1.0417214         | 2.1999589           | 0.6260000            | 0.5859107            |
|              |          | 95    | 1.8491109         | 2.5619906           | 0.5438144            | 0.4673512            |
| (12, 25)     |          | 80    | 3.968900          | 2.3963982           | 1.7479096            | 1.0443045            |
|              |          | 85    | 2.897733          | 2.4516401           | 0.9601863            | 0.8931725            |
|              |          | 95    | 2.753317          | 2.5567550           | 0.8093399            | 0.7713700            |
| (12, 35)     |          | 80    | 3.118313          | 2.2918441           | 1.0693892            | 0.9788446            |
|              |          | 85    | 2.965890          | 2.3658907           | 1.0433348            | 0.8797319            |
|              |          | 95    | 2.770518          | 2.5194235           | 0.8389121            | 0.6179893            |
| (2.5, 3)     | (11, 20) | 80    | 1.264413          | 1.6612946           | 1.2980096            | 1.2388655            |
|              |          | 85    | 1.536037          | 2.2058489           | 1.1631997            | 1.1217848            |
|              |          | 95    | 2.234662          | 2.5497267           | 0.9900068            | 0.9096081            |
| (18, 20)     |          | 80    | 1.860379          | 2.1271919           | 1.6419446            | 1.5753926            |
|              |          | 85    | 2.0782481         | 2.2608274           | 0.9715295            | 0.8473973            |
|              |          | 95    | 2.9904854         | 2.5826947           | 0.7014477            | 0.6473973            |
| (12, 25)     |          | 80    | 1.7583538         | 2.1329156           | 1.4209078            | 1.0292515            |
|              |          | 85    | 2.0836629         | 2.2943991           | 1.0031588            | 0.9421133            |
|              |          | 95    | 2.9015001         | 2.5290020           | 0.9064419            | 0.8547526            |
| (12, 35)     |          | 80    | 1.0192650         | 2.0194721           | 1.1086039            | 1.0041249            |
|              |          | 85    | 1.8166442         | 2.2617467           | 0.9388403            | 0.8259847            |
|              |          | 95    | 2.1086032         | 2.5030348           | 0.8166442            | 0.7741249            |
Table 2. Estimate, the mean square error for $\lambda$ and $T_2 = 100$

| $(b_1, b_2)$ | $(k, r)$ | $T_1$ | $\hat{\lambda}_B$ | $\hat{\lambda}_{EB}$ | $\text{MSE}(\hat{\lambda}_B)$ | $\text{MSE}(\hat{\lambda}_{EB})$ |
|--------------|-----------|-------|-------------------|-------------------|--------------------------|--------------------------|
| (1.5, 2)     | (11, 20)  | 80    | 0.01002324        | 0.02018169        | 0.36583143               | 0.15168012               |
|              |           | 85    | 0.01388574        | 0.02082876        | 0.24540068               | 0.11009306               |
|              |           | 95    | 0.07551424        | 0.05698260        | 0.13283998               | 0.09900334               |
| (15, 20)     |           | 80    | 0.00854059        | 0.00881930        | 0.15417093               | 0.11921365               |
|              |           | 85    | 0.00985218        | 0.02058568        | 0.08650327               | 0.07451532               |
|              |           | 95    | 0.07288751        | 0.05165167        | 0.07900718               | 0.06650365               |
| (18, 20)     |           | 80    | 0.01298031        | 0.02154525        | 0.18194735               | 0.11264996               |
|              |           | 85    | 0.02697772        | 0.03872098        | 0.11207455               | 0.09469526               |
|              |           | 95    | 0.03939734        | 0.05098933        | 0.10112370               | 0.08978639               |
| (12, 25)     |           | 80    | 0.09020828        | 0.02195403        | 0.09984382               | 0.08134841               |
|              |           | 85    | 0.08257069        | 0.03250745        | 0.07433193               | 0.06636528               |
|              |           | 95    | 0.03631063        | 0.06295896        | 0.06405985               | 0.05471148               |
| (12, 35)     |           | 80    | 0.01250740        | 0.02827069        | 1.01471148               | 0.99984382               |
|              |           | 85    | 0.01295896        | 0.03920828        | 0.88134841               | 0.78433193               |
|              |           | 95    | 0.02195403        | 0.05631063        | 0.76635288               | 0.69405985               |
| (2.5, 3)     | (11, 20)  | 80    | 0.01217904        | 0.02987496        | 0.72154542               | 0.66436972               |
|              |           | 85    | 0.02047071        | 0.03594608        | 0.52135136               | 0.40379023               |
|              |           | 95    | 0.03962807        | 0.05881317        | 0.30398702               | 0.28200912               |
| (18, 20)     |           | 80    | 0.01714689        | 0.02243666        | 0.68180587               | 0.51885518               |
|              |           | 85    | 0.02714689        | 0.03948523        | 0.42243629               | 0.37489675               |
|              |           | 95    | 0.03172106        | 0.04950494        | 0.27156115               | 0.11788791               |
| (12, 25)     |           | 80    | 0.01152291        | 0.02391643        | 0.63932464               | 0.51376332               |
|              |           | 85    | 0.02841107        | 0.03120920        | 0.43404868               | 0.39150965               |
|              |           | 95    | 0.03992588        | 0.05114683        | 0.27755420               | 0.14960132               |
| (12, 35)     |           | 80    | 0.01637617        | 0.03174601        | 0.21078954               | 0.10921629               |
|              |           | 85    | 0.03275874        | 0.04146268        | 0.16296289               | 0.08314580               |
|              |           | 95    | 0.04064067        | 0.05030860        | 0.12041944               | 0.07720966               |
Table 3. Estimate, the mean square error for $\alpha$ and $T_1 = 45$

| $(b_1, b_2)$ | $(k, r)$ | $T_2$ | $\hat{\alpha}_B$ | $\hat{\alpha}_{EB}$ | MSE($\hat{\alpha}_B$) | MSE($\hat{\alpha}_{EB}$) |
|--------------|----------|-------|----------------|-----------------|-----------------|----------------|
| (1.5, 2)     | (11, 20) | 90    | 0.9957537      | 0.6339161       | 1.062178        | 1.214407       |
|              |          | 110   | 2.3118858      | 1.1531517       | 0.830641        | 0.719226       |
|              |          | 150   | 2.6339161      | 1.7957537       | 0.762178        | 0.614407       |
| (15, 20)     |          | 90    | 1.7065898      | 0.8937124       | 1.037677        | 1.954498       |
|              |          | 110   | 2.0185215      | 1.0552049       | 0.908985        | 1.490925       |
|              |          | 150   | 2.4611235      | 1.4480276       | 0.836010        | 0.959404       |
| (18, 20)     |          | 90    | 1.1308251      | 0.8127624       | 0.831108        | 0.927974       |
|              |          | 110   | 1.9309987      | 1.0086287       | 0.774999        | 0.864279       |
|              |          | 150   | 2.5122400      | 1.8806394       | 0.658258        | 0.749992       |
| (12, 25)     |          | 90    | 1.1822798      | 0.9285195       | 0.858445        | 0.9079119      |
|              |          | 110   | 1.9889285      | 1.008465        | 0.7596543       | 0.8111743      |
|              |          | 150   | 2.4374748      | 1.8500879       | 0.609414        | 0.7384733      |
| (12, 35)     |          | 90    | 1.5718727      | 1.1097288       | 1.389528        | 1.4073459      |
|              |          | 110   | 1.9807946      | 1.640983        | 1.006945        | 1.250638       |
|              |          | 150   | 2.6149597      | 2.0560069       | 0.8416344       | 0.9255014      |
| (2.5, 3)     | (11, 20) | 90    | 1.8285346      | 1.4242101       | 1.3840789       | 1.5504698      |
|              |          | 110   | 2.1850752      | 1.8895607       | 1.1109162       | 1.4645827      |
|              |          | 150   | 2.7077178      | 2.1627746       | 0.9614387       | 1.0019672      |
| (18, 20)     |          | 90    | 1.1078642      | 0.8345830       | 0.7908158       | 1.0080633      |
|              |          | 110   | 2.1139134      | 1.1009293       | 0.7324507       | 0.8379379      |
|              |          | 150   | 2.4554278      | 2.0875900       | 0.6246905       | 0.7161964      |
| (12, 25)     |          | 90    | 0.9926499      | 0.8910045       | 1.0783606       | 1.84205811     |
|              |          | 110   | 1.1296560      | 0.7888355       | 0.79102405      | 0.82499218     |
|              |          | 150   | 2.5287248      | 1.7491727       | 0.6991450       | 0.70316875     |
| (12, 35)     |          | 90    | 1.9232352      | 1.2534225       | 1.7885198       | 1.86673045     |
|              |          | 110   | 2.2694444      | 1.7918619       | 1.06024561      | 1.73262390     |
|              |          | 150   | 2.5128730      | 2.1681149       | 0.94087910      | 1.16673042     |
3.2 Application with real data set

In this subsection, a real data set is used to analyze $\alpha$ and $\lambda$ estimation methods. The data set represent repair times (in h) for an airborne communication transceiver. They were first analyzed by Von Alven (1964). The data is presented in Table 5. Before analyzing the data, we fit the IW model to this data set. We used the Kolmogorov-Smirnov (K-S) distance between the fitted the empirical distribution functions, and corresponding p-values. It is observed that for this data, the K-S and corresponding p-value are 0.0815 and 0.9197, respectively. We observe the IW model fit quite well to this data set.

| $(b_1, b_2)$ | $(k, r)$ | $T_2$ | $\hat{\lambda}_B$ | $\hat{\lambda}_{EB}$ | MSE($\hat{\lambda}_B$) | MSE($\hat{\lambda}_{EB}$) |
|--------------|----------|-------|----------------|------------------|-------------------|-------------------|
| (1.5, 2)     | (11, 20) | 90    | 0.03067451     | 0.01754875       | 0.089908104       | 0.356086732       |
|              |          | 110   | 0.04394579     | 0.02913630       | 0.070822873       | 0.236063086       |
|              |          | 150   | 0.05279717     | 0.03434380       | 0.051552638       | 0.164494967       |
| (15, 20)     | (11, 20) | 90    | 0.02808769     | 0.02030007       | 0.161504991       | 0.209289078       |
|              |          | 110   | 0.03180027     | 0.02436261       | 0.127928345       | 0.178411826       |
|              |          | 150   | 0.04985280     | 0.03367812       | 0.102193247       | 0.154849690       |
| (18, 20)     | (11, 20) | 90    | 0.02733432     | 0.01501363       | 0.089908104       | 0.137946388       |
|              |          | 110   | 0.04519281     | 0.02207054       | 0.082897327       | 0.106536726       |
|              |          | 150   | 0.05386042     | 0.03622065       | 0.068481334       | 0.08562029        |
| (12, 25)     | (11, 20) | 90    | 0.0302440      | 0.02761198       | 0.087186751       | 0.105403070       |
|              |          | 110   | 0.04360056     | 0.03335961       | 0.080150728       | 0.091038057       |
|              |          | 150   | 0.05222345     | 0.04173082       | 0.072282442       | 0.081038058       |
| (12, 35)     | (11, 20) | 90    | 0.02557485     | 0.01339570       | 0.139337354       | 0.197229546       |
|              |          | 110   | 0.03144091     | 0.02524225       | 0.107611978       | 0.172822455       |
|              |          | 150   | 0.04888983     | 0.03733944       | 0.081363791       | 0.151957044       |
| (2.5, 3)     | (11, 20) | 90    | 0.02766431     | 0.01985449       | 0.142921258       | 0.184132366       |
|              |          | 110   | 0.03932568     | 0.02065841       | 0.128337187       | 0.152294538       |
|              |          | 150   | 0.05599842     | 0.04142739       | 0.088214903       | 0.124841527       |
| (15, 20)     | (11, 20) | 90    | 0.01655514     | 0.01043330       | 0.221247390       | 0.269735181       |
|              |          | 110   | 0.03960635     | 0.03177747       | 0.136787245       | 0.182609868       |
|              |          | 150   | 0.04705481     | 0.03644419       | 0.091725247       | 0.099171033       |
| (18, 20)     | (11, 20) | 90    | 0.02150275     | 0.01451502       | 0.097394016       | 0.124558867       |
|              |          | 110   | 0.03133300     | 0.02014775       | 0.085431039       | 0.121805690       |
|              |          | 150   | 0.04753325     | 0.03622597       | 0.062548587       | 0.107863095       |
| (12, 25)     | (11, 20) | 90    | 0.01674199     | 0.01173110       | 0.090087877       | 0.153396965       |
|              |          | 110   | 0.03900441     | 0.02676016       | 0.088175630       | 0.134683630       |
|              |          | 150   | 0.51420139     | 0.03914584       | 0.060002891       | 0.115577656       |
| (12, 35)     | (11, 20) | 90    | 0.01086794     | 0.01283105       | 0.111859032       | 0.245538310       |
|              |          | 110   | 0.03178846     | 0.02982290       | 0.075802935       | 0.090254322       |
|              |          | 150   | 0.04825255     | 0.03503170       | 0.061635373       | 0.078183760       |
Table 5. Repair times (in h) for an airborne communication transceiver

| 0.2 | 0.3 | 0.5 | 0.5 | 0.5 | 0.5 | 0.6 | 0.6 | 0.7 | 0.7 | 0.7 | 0.8 | 1.0 | 1.0 | 1.0 | 1.1 | 1.3 | 1.5 | 1.5 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1.5 | 1.5 | 2.0 | 2.0 | 2.2 | 2.5 | 2.7 | 3.0 | 3.0 | 3.3 | 3.3 | 4.0 | 4.0 | 4.5 | 4.7 | 5.0 | 5.4 | 5.4 | 7.0 |
| 8.8 | 9.0 | 10.3 | 7.5 | 8.8 | 9.0 | 10.3 | 10.3 |

To compute the Bayesian and E-Bayesian estimations, since we do not have any prior information, we assumed that $b_1 = b_2 = 0.01$. Therefore, for $c_1 = c_2 = 1$, for these data, six unified hybrid censored schemes are considered under the following conditions.

- **Scheme 1**: $K = 14, r = 30, T_1 = 3.5, T_2 = 4.5$
- **Scheme 2**: $K = 19, r = 25, T_1 = 1.5, T_2 = 3$
- **Scheme 3**: $K = 19, r = 30, T_1 = 1.5, T_2 = 2$
- **Scheme 4**: $K = 30, r = 32, T_1 = 2, T_2 = 4$
- **Scheme 5**: $K = 30, r = 32, T_1 = 1, T_2 = 3$
- **Scheme 6**: $K = 18, r = 20, T_1 = 1, T_2 = 3$

In all schemes, the Bayesian and E-Bayesian estimates of the parameters have been obtained. These results are presented in Table 6. Also for this data, in a complete uncensored sample, the maximum likelihood estimation for parameters $\alpha$ and $\lambda$ are 1.011941 and 1.125229, respectively.

Table 6. Bayesian and E-Bayesian estimations of parameters $\alpha$ and $\lambda$

| Scheme | $\hat{\alpha}_B$ | $\hat{\alpha}_{EB}$ | $\hat{\lambda}_B$ | $\hat{\lambda}_{EB}$ |
|--------|------------------|----------------------|-------------------|---------------------|
| 1      | 0.075438204      | 0.094656141          | 0.827177291       | 1.144378477         |
| 2      | 0.940139616      | 1.009714642          | 1.022434427       | 1.108393285         |
| 3      | 0.983522234      | 1.013935819          | 1.772656778       | 1.031561386         |
| 4      | 1.263046540      | 1.030439965          | 1.795299730       | 1.176216987         |
| 5      | 0.925065487      | 1.013107987          | 2.348933240       | 1.025538408         |
| 6      | 2.006398994      | 1.094094911          | 2.575507341       | 1.500216398         |

Table 6, shows that in all schemes, the E-Bayesian estimation of the parameters are closer to their estimated value in the complete sample. Therefore, estimating E-Bayesian parameters is better than their Bayesian estimations.

4. Conclusion

In this study, the Bayesian and E-Bayesian estimations of the inverse Weibull distribution parameters were obtained under the unified hybrid censored scheme with squared error loss function. In this study, six unified hybrid censored schemes are considered, and, using a real data set, we showed that in all schemes the E-Bayesian estimation parameters are better than their Bayesian estimations. Also, using Monte Carlo simulation, the conditions of superiority of the estimator were obtained with respect to each other.

Acknowledgments

The authors would like to thank the Editor and the anonymous referees for their constrictive comments and suggestions that appreciably improved the quality of presentation of this manuscript.
References

1. Berger, J. O. (1985), *Statistical Decision Theory and Bayesian Analysis*, second ed., Springer-Verlag, New York.
2. Calabria, R. and G. Pulcini. (1994), Bayes 2-sample prediction for the inverse Weibull distribution, *Communications in Statistics-Theory and Methods*, 23 (6), 1811–1824.
3. Childs, A., Chandrasekhar, B., Balakrishnan, N. and Kundu, D. (2003), Exact Likelihood Inference Based on Type-I and Type-II Hybrid Censored Samples from the Exponential distribution, *Annals of the Institute of Statistical Mathematics*, 55, 319-330.
4. Chandrasekhar, B., Childs, A. and Balakrishnan, N. (2004), Exact Likelihood Inference for the Exponential distribution under Generalized Type-I and Type-II Hybrid Censoring, Naval Research Logistics, 51, 994-1004.
5. Epstein, B. (1954), Truncated Life-Tests in the Exponential Case, *Annals of Mathematical Statistics*, 25, 555-564.
6. Erto, P. and Rapone, M. (1984), Non-informative and practical Bayesian confidence bounds for reliable life in the Weibull model, *Reliability Engineering*, 7, 181-191.
7. Han, M. (1997), The structure of hierarchical prior distribution and its applications, *Chinese Operations Research and Management Science*, 63, 31–40.
8. Han, M. (2009), E-Bayesian estimation and hierarchical Bayesian estimation of failure rate, *Applied Mathematical Modelling*, 33(4), 1915-1922.
9. Han, M. (2011), E-Bayesian estimation of the reliability derived from Binomial distribution, *Applied Mathematical Modelling*, 35, 2419-2424.
10. Han, M. (2017), The E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter, *Communications in Statistics-Theory and Methods*, 46(4), 1606-1620.
11. Jaheen, Z. F. and Okasha, H. M. (2011), E-Bayesian estimation for the Burr type XII model based on type-2 censoring, *Applied Mathematical Modelling*, 35, 4730-4737.
12. Khan, M. S., Pasha, G. R. and Pasha, A. H. (2008), Theoretical analysis of inverse Weibull distribution, WSEAS Transactions on Mathematics, 7(2), 30–38.
13. Lindley, D. V. and Smith, A. F. (1972), Bayes estimation for the linear model, *Journal of the Royal Statistical Society Series B*, 34, 1–41.
14. Murthy, D.N.P. Xie, M. and Jiang, R. (2004), *Weibull Models*, Wiley, New York.
15. Nelson, W. (1982), *Applied Lifetime Data Analysis*, Wiley, New York.
16. Shafiei, S., Darijani, S. and Saboori, H. (2016), Inverse Weibull power series distributions: properties and applications, *Journal of Statistical Computation and Simulation*, 86 (6), 1069–1094.
17. Von Alven, W.H. (1964), *Reliability engineering by ARINC*, Prentice-Hall, Englewood Cliffs.
18. Wang, J., Li, D. and Chen, D. (2012), E-Bayesian Estimation and Hierarchical Bayesian Estimation of the System Reliability Parameter, *Systems Engineering Procedia*, 3, 282-289.
19. Yousefzadeh, F. (2017), E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter based on asymmetric loss function, *Communications in Statistics-Theory and Methods*, 46(1), 1-8.
20. Yaghoobzadeh, S. S. (2018), Estimating E-bayesian and hierarchical bayesian of scalar parameter of Gompertz distribution under type II censoring schemes based on fuzzy data, *Communications in Statistics - Theory and Methods*, doi. org/10.1080/036109262017.1417438.