STABLY FREE MODULES AND THE UNSTABLE CLASSIFICATION
OF 2-COMPLEXES

JOHN NICHOLSON

Abstract. For all $k \geq 2$, we show that there exists a group $G$ and a non-free stably free $ZG$-module of rank $k$. We use this to show that, for all $k \geq 2$, there exist homotopically distinct finite 2-complexes with fundamental group $G$ and with Euler characteristic exceeding the minimal value over $G$ by $k$. This resolves Problem D5 in the 1979 Problem List of C. T. C. Wall. We also explore a number of generalisations and present a potential application to the topology of closed smooth 4-manifolds.

1. Introduction

An important concept in algebra and topology is that of stabilisation. When faced with an intractable classification problem, it is often possible to classify up to a weaker notion of stable equivalence. For example, a projective $R$-module can be `stabilised' by taking a direct sum with the free module $R$, and we say that two projective $R$-modules $P$ and $Q$ are (reduced) stably equivalent if they are related by a sequence of stabilisations, i.e. if $P \oplus R^n \cong Q \oplus R^m$ for some $n, m \geq 0$. While it is typically difficult to classify the finitely generated projective $R$-modules up to $R$-isomorphism, the set of stable equivalence classes form a group $\tilde{K}_0(R)$ which is computable, at least in principle, using the methods of algebraic $K$-theory. Given such a stable classification, it then remains to classify the objects within a stable class up to isomorphism. This is the corresponding unstable classification.

The aim of this article will be to study the unstable classification of three closely related objects, where $G$ is a fixed group and $\rightsquigarrow$ denotes the stabilisation:

(I) Finitely generated (non-zero) projective $ZG$-modules up to $ZG$-isomorphism, with $P \rightsquigarrow P \oplus ZG$.

(II) Finite 2-complexes $X$ with $\pi_1(X) \cong G$ up to homotopy equivalence, with $X \rightsquigarrow X \vee S^2$.

(III) Closed smooth 4-manifolds $M$ with $\pi_1(M) \cong G$ up to homeomorphism, with $M \rightsquigarrow M \# (S^2 \times S^2)$.

In each case, the stable classification can be computed in principle. For (I), the stable classification is given by $\tilde{K}_0(ZG)$, as above. For (II), the stable classification is trivial, i.e. for all $X, Y$ we have $X \vee nS^2 \cong Y \vee mS^2$ for some $n, m \geq 0$ [72, Theorem 13]. For (III), the stable classification can be reduced to the computation of certain bordism groups using Kreck’s modified surgery [45].

Our main result is the construction of examples which illustrate qualitative features in the unstable classifications of projective $ZG$-modules (I) and finite 2-complexes (II). We will show:

Theorem A. For all $k \geq 2$, there exists a group $G$ and a finitely generated projective $ZG$-modules $P$ and $Q$ such that $P \oplus ZG \cong Q \oplus ZG$ and $Q \cong Q_0 \oplus ZG^k$ for some $ZG$-module $Q_0$, but $P \ncong Q$.

Theorem B. For all $k \geq 2$, there exists finite 2-complexes $X$ and $Y$ such that $X \vee S^2 \cong Y \vee S^2$ and $Y \cong Y_0 \vee kS^2$ for some finite 2-complex $Y_0$, but $X \ncong Y$.

In each case, for $k \geq 2$, we can take $G = \ast_{i=1}^k T$ to be the free product of $k$ copies of the trefoil group $T = \langle x, y \mid x^2 = y^3 \rangle$. The examples in Theorem A are in the simplest case where $Q = ZG^k$ and so $P$ is a non-free stably free $ZG$-module of rank $k$. The examples $X, Y$ in Theorem B are distinguished by showing that $\pi_2(X) \cong P$ and $\pi_2(Y) \cong ZG^k$ as $ZG$-modules.

We will now review unstable classification for (I)-(III) before discussing each case individually. We give generalisations of Theorems A and B and applications to smooth 4-manifolds (III).

2020 Mathematics Subject Classification. Primary 57K20; Secondary 20C12, 55P15, 57M05.
1.1. Background on unstable classification. For a set $S$, define a stabilisation to be a function $\Sigma : S \to S$ such that $\ell(a) := \sup\{k \geq 0 : \Sigma^n(a) = \Sigma^{n+k}(b)\}$ for some $b \in S$, $n \geq 0 < \infty$ where $n, k$ are integers. That is, for all $a \in S$, there does not exist an infinite sequence $b_k \in S$ for $k \geq 1$ such that, for each $k$, we have $\Sigma^n(a) = \Sigma^{n+k}(b_k)$ for some $n$. The stabilisations in (I)-(III) each satisfy this condition (see Section 2). The function $\Sigma$ is then fixed point free and we have a level function $\ell : S \to \mathbb{Z}_{\geq 0}$ which is surjective and satisfies $\ell(\Sigma(a)) = \ell(a) + 1$ for $a \in S$. For $n \geq 0$, we say that $a \in S$ has level $n$ if $\ell(a) = n$ and is at the minimal level if $\ell(a) = 0$.

We can view $(S, \Sigma)$ as a directed graph with vertex set $S$ and edges between $a$ and $\Sigma(a)$ for each $a \in S$. Let $=_{st}$ denote the corresponding (reduced) stable equivalence relation, i.e. $a =_{st} b$ if $\Sigma^n(a) = \Sigma^m(b)$ for some $n, m \geq 0$. For each stable equivalence class $c \in S / =_{st}$, the subgraph $S_c = \{a \in S : a =_{st} c\}$ is a directed tree which is graded by $\ell_c := \ell |_{S_c} : S_c \to \mathbb{Z}_{\geq 0}$. The goal of the unstable classification is then to determine the trees $S_c$ for each $c \in S / =_{st}$.

Whilst determining the trees $S_c$ explicitly will often be intractable, further progress can be made by asking for structural features of the trees. For $k \geq 0$, we say that $S_c$ has cancellation at level $k$ if $\Sigma(a) = \Sigma(b)$ implies $a = b$ for all $a, b \in S_c$ with $\ell(a) = \ell(b) \geq k$, or equivalently if $|\ell^{-1}(n)| = 1$ for all $n \geq k$ (see Fig. 1). We say that $S$ has cancellation at level $k$ if this holds for all $c \in S / =_{st}$. If such a $k$ exists, then we have a kind of ‘stable range’ where elements $a \in S$ with $\ell(a) \geq k$ are classified by $\ell(a) \in \mathbb{Z}_{\geq 0}$ and the stable class $c$.

We say that $S_c$ (resp. $S$) has cancellation if it has cancellation at level 0, or equivalently if $\ell_c$ (resp. $\ell$) is bijective (see Fig. 1b).

![Figure 1](image)

**Figure 1.** Branching phenomena for the graded trees $S_c$. The vertical height is $\ell(\cdot) \in \mathbb{Z}_{\geq 0}$.

For a group $G$, let $P(\mathbb{Z}G)$, $\text{HT}(G)$ and $\mathcal{M}(G)$ denote the sets with stabilisation described in (I), (II) and (III) respectively. Each of these sets have cancellation when $G = \{1\}$ is the trivial group. For $P(\mathbb{Z})$, this is trivial. For $\text{HT}(\{1\})$ this is an exercise in homotopy theory [54, Theorem 1]. For $\mathcal{M}(\{1\})$, this follows from deep results of Donaldson [18] and Freedman [26] (see [47, p14]).

When $G$ is a finite group, $P(\mathbb{Z}G)$, $\text{HT}(G)$ and $\mathcal{M}(G)$ each have cancellation at level one (see Fig. 1b). This was shown for $P(\mathbb{Z}G)$ by Swan [66], for $\text{HT}(G)$ by Browning [14] and for $\mathcal{M}(G)$ by Hambleton-Kreck [30]. The three proofs share a number of techniques; in fact, Hambleton-Kreck gave a new proof of case (II) before using these techniques as the basis for their proof of case (III) (see [31]). On the other hand, examples have been given to illustrate that cancellation (at level 0) fails for an arbitrary finite group $G$. This was shown for $P(\mathbb{Z}Q_{32})$ by Swan [67], where $Q_{32}$ is the quaternion group of order 32, by Swan [67], for $\text{HT}(\mathbb{Z}/5^3)$ by Metzler [57] and for $\mathcal{M}(\mathbb{Z}/5^3)$ by Kreck-Schafer [47] by applying the boundary of thickenings construction (see Section 1.4) to the examples of Metzler.

Much less is known about the case where $G$ is infinite. In particular, it has previously not been known whether there exists $G$ such that $P(\mathbb{Z}G)$ fails cancellation at any level $\geq 1$, $\text{HT}(G)$ fails cancellations at any level $\geq 2$, or $\mathcal{M}(G)$ fails cancellation at any level $\geq 1$. We will now discuss each case in turn.

1.2. Projective $\mathbb{Z}G$-modules. The rank of a finitely generated projective $\mathbb{Z}G$-module $P$, denoted by $\text{rank}_G(P)$, is the rank of the free abelian group $P_G = \mathbb{Z} \otimes \mathbb{Z}G$ $P$. We will focus on the case of stably free $\mathbb{Z}G$-modules $S$ where $\text{rank}_G(S) = m - n$ for any $n, m \geq 0$ such that $S \oplus \mathbb{Z}G^n \cong \mathbb{Z}G^m$. Let
SF(ZG) ⊆ P(ZG) denote the subset of the stably free ZG-modules. If ℓ = ℓ_{SF(ZG)} is the level function on P(ZG) restricted to SF(ZG), then ℓ(S) = rank_{SF(G)}(S) − 1 for S ∈ SF(ZG) (Proposition 2.1).

1.2.1. Main result. Examples of non-free stably free ZG-modules of rank one have been constructed over various torsion free groups \([8, 19, 32]\) and groups of the form \(G = F_n \times H\) for \(H\) finite \([41, 42, 61]\).

However, there has previously been no known examples of (a) a non-free stably free ZG-module of rank \(\geq 2\) over a group \(G\) (see \([41, \text{p.} 623]\), \([41, \text{p.} \text{xiii}]\)), or (b) a non-free projective ZG-module of rank \(\geq 2\) over a torsion free group \(G\) (see \([41, \text{p.} 2950]\)). For each \(k \geq 2\), Theorem \(A\) is the statement that there exists \(G\) for which \(P(ZG)\) fails cancellation at level \(k\). Since \(G = \ast_{i=1}^k T\) is torsion free, this gives examples for both (a) and (b) above.

We now state a more detailed and general version of Theorem \(A\) motivated by applications to 2-complexes in Theorem \(B\). Let cd\((G)\) denote the cohomological dimension of \(G\) and recall that ZG-modules \(M, N\) are Aut\((G)\)-isomorphic if \(M \cong N_\theta\) are ZG-isomorphic for some \(\theta \in \text{Aut}(G)\), where \(N_\theta\) is the abelian group \(N\) equipped with the \(G\)-action \(g : x := \theta(g) \cdot x\) for \(g \in G\) and \(x \in N\).

**Theorem A′.** For all \(k \geq 1\) and \(d \geq 2\), there exists a finitely presented group \(G\) with cd\((G)\) = \(d\) and infinitely many stably free ZG-modules \(S_i\) of rank \(k\) which are distinct up to Aut\((G)\)-isomorphism and such that, for all \(i\) \(S_i \not\cong S \oplus \text{ZG}\) for any ZG-module \(S\).

When \(d = 2\), we take \(G = \ast_{i=1}^k T\) where \(T\) is the trefoil group. Here the case \(k = 1\) was shown by Berridge-Dunwoody \([8]\). For \(d \geq 3\), the groups are constructed by applying the operation 

\[ G \rightsquigarrow (G \ast \langle q \mid - \rangle) \ast_{\langle q \ast \langle r \mid - \rangle \rangle} \]

to \(\ast_{i=1}^k T\) a total of \((d-2)\) times.

We will now explain the key idea behind our proof in the \(d = 2\) case, i.e. when \(G = \ast_{i=1}^k T\). Let \(\mathbb{F}\) be a field and let \(G = \ast_{i=1}^k M_i\) be a free product of \(G\). We say that an \(\mathbb{F}\)-module \(M\) is induced if there exists \(\mathbb{F}G_i\)-modules \(M_i\) and an \(\mathbb{F}G\)-module isomorphism

\[ M \cong \bigoplus_{\iota_i(M_i) = \mathbb{F}G \otimes_{\mathbb{F}G_i} M_i} \bigoplus_{\iota_i(M_i)} \mathbb{F}G = \mathbb{F}G \oplus M_i \]

where \(\iota_i(M_i) = \mathbb{F}G \otimes_{\mathbb{F}G_i} M_i\) is the inclusion of \(G\). It follows from Bergman’s theorem on modules over rings \([7]\) that, if \(M\) has no \(\mathbb{F}G\) summand, then \(M\) is uniquely determined by the \(M_i\) (Corollary 3.4).

For \(G = \ast_{i=1}^k T\), we will show that the ZG-modules \(S_i\) of Theorem \(A\) are distinct by showing that the corresponding \(\mathbb{F}_p[\ast_{i=1}^k T/T''']\)-modules \(\mathbb{F}_p[\ast_{i=1}^k T/T'''] \otimes_{\text{ZG}} S_i\) are distinct, where \(\mathbb{F}_p\) is the finite field of characteristic \(p\) and \(T'''\) is the second derived subgroup. This is achieved using Bergman’s theorem. To show that the modules have no \(\mathbb{F}_p[\ast_{i=1}^k T/T''']\) summand, we use that the group ring \(\mathbb{F}_p[T/T''']\) is stably finite (see Section 3.1) since \(T/T''\) is polycyclic and so is a sofic group \([29]\). This strategy was proposed by Evans in \([24]\), though an example was never given.

1.2.2. Cancellation bounds. If ZG is Noetherian of Krull dimension \(d_G\) and \(d = d_G + 1\), then stably free ZG-modules of rank \(\geq d\) are free \([6, \text{Chapter IV}]\). If \(G\) is polycyclic-by-finite, then ZG is Noetherian and it is conjectured that these are the only such groups (see \([18, \text{p.} \text{p328}]\)). If ZG is not Noetherian, then such a bound \(d\) can often still be found; for example, if \(G\) is a free group, then we can take \(d = 0\) \([5]\). This raises the question of whether, given a group \(G\), there always exist a bound \(d\) such that every stably free ZG-module of rank \(\geq d\) is free. We will show that there does not.

**Theorem C.** There exists a group \(G\) such that, for all \(k \geq 1\), there is a non-free stably free ZG-module of rank \(k\).

We construct examples over the non-finitely generated group \(G = \ast_{i=1}^\infty T\).

1.3. Finite 2-complexes. A finite 2-complex will be taken to mean a connected finite 2-dimensional CW-complex. Let \(G\) be finitely presented and let \(\ell = \ell_{HT(G)}\). Then \(\ell(X) = \chi(X) - \chi_{\text{min}}(G)\) for all \(X \in HT(G)\), where \(\chi_{\text{min}}(G) := \min\{\chi(X) : X \in HT(G)\}\) (Proposition 2.1).

Recall that a finite presentation \(P\) for a group \(G\) has an associated presentation complex \(X_P\) which is a finite 2-complex with \(\pi_1(X_P) \cong G\). Conversely, every finite 2-complex \(X\) with \(\pi_1(X) \cong G\) is homotopy equivalent to \(X_P\) for some finite presentation \(P\) for \(G\) (see, for example, \([35, \text{p61}]\)).
1.3.1. *Stably free modules and 2-complexes.* We will now discuss the link between stably free $\mathbb{Z}G$-modules and finite 2-complexes. This is the basis for our strategy for proving Theorem B (see below).

If $X$ is a finite 2-complex with $\pi_1(X) \cong G$, then $\pi_2(X)$ can be viewed as a $\mathbb{Z}G$-module via the isomorphism $\pi_2(X) \cong \pi_2(\tilde{X})$ and the monodromy action. This has the property that $\pi_2(X \vee S^2) \cong \pi_2(X) \oplus \mathbb{Z}G$ as $\mathbb{Z}G$-modules. If $\text{gd}(G) = 2$, then $\pi_2(X)$ is a stably free $\mathbb{Z}G$-module of rank $\ell(X) \in \mathbb{Z}_{\geq 0}$ (Proposition 7.3). In particular, there is a level preserving map

$$\pi_2 : \text{HT}(G) \to \text{SF}(\mathbb{Z}G) \cup \{0\}$$

and $\ell_{\text{SF}(\mathbb{Z}G)}(\pi_2(X)) = \ell_{\text{HT}(G)}(X) - 1$, where we take $\ell_{\text{SF}(\mathbb{Z}G)}(\{0\}) := -1$. The trefoil group $T$ has $\text{gd}(T) = 2$ and it was shown by Dunwoody [20] that $\text{HT}(T)$ fails cancellation at level one using a non-free stably free $\mathbb{Z}T$-module of rank one [19].

1.3.2. *Main result.* The question of whether, for each $k \geq 2$, there exists a group $G$ such that $\text{HT}(G)$ fails cancellation at level $k$ has been raised in a number of variants (see [22, Problem C] and [35, p124]). Most notably, the following version appeared in the 1979 Problems List of C. T. C. Wall [70]. If such an $X$ exists then, if $X_0$ is a finite 2-complex such that $\pi_1(X_0) \cong \pi_1(X)$ and $\ell(X_0) = 0$, then $X$ and $X_0 \vee kS^2$ are homotopy distinct finite 2-complexes at level $k$ in $\text{HT}(\pi_1(X))$.

**Problem 1.1** (Problem D5 from Wall’s list [70]). For each $k \geq 2$, does there exists a finite 2-complex $X$ such that $\ell(X) = k$ and $X \not\cong Y \vee S^2$ for any finite 2-complex $Y$?

**Remark 1.2.** Wall’s list contains eight problems concerning 2-complexes [70 List D]. Some are classical, some are due to Wall and others were suggested by participants at the 1977 Durham Symposium on Homological Group Theory. Each problem asks whether examples exist which illustrate certain phenomena. The only examples previously found were the finite 2-complexes of Metzler [58] (see also [52]) which are homotopy equivalent but not simple homotopy equivalent, resolving Problem D6. Bestvina-Brady [9] showed that there is a counterexample to the Eilenberg-Ganea conjecture (Problem D4) or the Whitehead conjecture (Problem D7).

Our main result is an affirmative answer to this question for all $k \geq 2$. We will also pursue the following natural generalisation to higher dimensions. For $n \geq 2$, a $(G, n)$-complex is an $n$-complex $X$ with $\pi_1(X) \cong G$ and such that the universal cover $\tilde{X}$ is $(n-1)$-connected. Equivalently, $X$ is the $n$-skeleton of a $K(G, 1)$. Let $\text{HT}(G, n)$ denote the set of finite $(G, n)$-complexes up to homotopy. Then $X \rightsquigarrow X \vee S^n$ is a stabilisation (Proposition 2.3).

The natural extension of Problem 1.1 to $(G, n)$-complexes was considered by Dyer in [21,22] (see [21 p378]). However, there were still no examples found at level $k \geq 2$. We will show:

**Theorem B’.** For all $n \geq 2$ and $k \geq 0$, there exists a group $G$ and infinitely many homotopically distinct finite $(G, n)$-complexes $X_i$ at level $k$ such that $X_i \not\cong Y \vee S^n$ for any finite $(G, n)$-complex $Y$.

For $n = 2$ and $k \geq 1$, we take $G = \ast_{i=1}^k T$ and $X_i = \bigvee_{j=1}^k X_{P_i}$ for $i \geq 1$ where

$$P_i = \langle x, y, a, b \mid x^2 = y^3, a^2 = b^3, x^{2i+1} = a^{2i+1}, y^{3i+1} = b^{3i+1} \rangle$$

are the presentations of Harlander-Jensen [52]. We prove the $X_i$ are homotopically distinct by showing that the $\mathbb{Z}G$-modules $\pi_2(X_i)$, which are stably free, coincide with the examples behind Theorem 1.

For $k = 0$, we take $G = (T * \langle q \mid - \rangle) *_{\langle r \mid - \rangle} \langle r \mid - \rangle$ and $X_i = X_{Q_i}$ where

$$Q_i = \langle a, b, c \mid a^2 = b^3, [a^2, b^{2i+1}], [a^2, c^{3i+1}] \rangle$$

for the group constructed by Lustig [53]. For $n \geq 3$ and $k \geq 1$, the complexes are constructed from Theorem 1 using Wall’s theorem on the realisability of chain complexes in dimensions $\geq 3$ (see Proposition 7.1). The case $k = 0$ requires a generalisation of the examples of Lustig (see Section 8.2).

In Corollary 8.3, we will point out that Theorem B’ shows that, for each $n \geq 2$ and $k \geq 2$, there exists a finitely presented group $G$ such that the syzygies $\Omega_n^{G}(\mathbb{Z})$ have non-cancellation at level $k \geq 0$. This resolves a problem raised by Johnson in [41 p.xiii].
1.3.3. Homotopy classification over free products. The key ingredient in the proofs of Theorems A and B is Bergman’s theorem on $FG$-modules for a field $F$ and a group $G = \ast_{i=1}^k G_i$ (see Section 1.2). For a finite 2-complex $X$ with $\pi_1(X) \cong G$, this can be applied to determine the structure of $\pi_2(X) \otimes F$ as an $FG$-module (see Propositions 10.1 and 10.5). This raises the question of whether a general classification of $P(ZG)$ and $HT(G)$ can be obtained for $G = \ast_{i=1}^k G_i$. We will show that it cannot since, in general, we lose information by passing from $ZG$-modules to $FG$-modules.

**Theorem 1.3.** Let $k \geq 2$. Then:

(i) There exists a group $G = \ast_{i=1}^k G_i$ and a finite 2-complex $X$ with $\pi_1(X) \cong G$ such that $\pi_2(X)$ is not an induced $ZG$-module.

(ii) There exists a group $G = \ast_{i=1}^k G_i$ and a finite 2-complex $X$ with $\pi_1(X) \cong G$ such that $\pi_2(X)$ has two induced module structures

$$\pi_2(X) \cong t_1(M_1) \oplus \cdots \oplus t_k(M_k) \cong t_1(M'_1) \oplus \cdots \oplus t_k(M'_k)$$

such that, for each $i$, $M_i$ and $M'_i$ have no $ZG_i$ summands and are not $\text{Aut}(G_i)$-isomorphic.

For (i), we take $G = \ast_{i=1}^k (Z/p_i)^2$ for distinct primes $p_i$ and our results are a minor extension of those of Hog-Angeloni–Lustig–Metzler [34]. For (ii), we take $G = \ast_{i=1}^k (Z/p_i)^3$ for distinct primes $p_i$ such that $p_i \equiv 1 \pmod{4}$. These examples combine ideas from [34] with those of Metzler [57].

1.4. Smooth 4-manifolds. A 4-manifold will be assumed to be closed, smooth and connected. Alongside $\mathcal{M}(G)$, we can also consider the set $\mathcal{M}^{\text{Diff}}(G)$ of 4-manifolds $M$ with $\pi_1(M) \cong G$ up to diffeomorphism. Whilst $\mathcal{M}(\{1\})$ has cancellation, the existence of exotic smooth structures on simply connected 4-manifolds shows that $\mathcal{M}^{\text{Diff}}(\{1\})$ fails cancellation. There are only a few examples where cancellation is known to fail for $\mathcal{M}(G)$ (see [14] Sections 5.4, 7.10). On the other hand, for each $k \geq 1$, it is currently open whether there exists $G$ such that either $\mathcal{M}(G)$ or $\mathcal{M}^{\text{Diff}}(G)$ fail cancellation at level $k$. The question for $\mathcal{M}^{\text{Diff}}(G)$ was considered by Kreck [16, p.198] and Crowley [12, Problem 10B].

The following is a potential application of stably free $ZG$-modules to the unstable classification of 4-manifolds. A $ZG$-module $S$ is said to be geometrically realisable if there exists a finite 2-complex $X$ such that $\pi_1(X) \cong G$ and $\pi_2(X) \cong S$. Let $S^* = \text{Hom}_{ZG}(S, ZG)$ denote the dual (see Section 3.3).

**Theorem 1.4.** Let $G$ be a finitely presented group such that $\text{gd}(G) = 2$ and suppose there exists a stably free $ZG$-module $S$ which is geometrically realisable and such that $S \oplus S^*$ is not a free $ZG$-module. Then both $\mathcal{M}^{\text{Diff}}(G)$ and $\mathcal{M}(G)$ fail cancellation at level $k$.

**Remark 1.5.** If $G$ is a finitely presented group such that $\text{gd}(G) = 2$, then every stably free $ZG$-module is geometrically realisable if and only if $G$ has the D2 property (see Section 7). In particular, if a stably free $ZG$-module $S$ exists such that $S \oplus S^*$ is free, then either $G$ is a counterexample to Wall’s D2 problem or both $\mathcal{M}^{\text{Diff}}(G)$ and $\mathcal{M}(G)$ fail cancellation at level $k$.

We do not know whether or not a stably free $ZG$-module exists which satisfies these conditions, or even whether there exists any group $G$ and a stably free $ZG$-module $S$ such that $S \oplus S^*$ is non-free (see Problem 1.5). Such examples would also provide further examples for Theorem A since $S \oplus S^*$ would be a stably free $ZG$-module of even rank $\geq 2$. The natural candidates to investigate are the examples of Berridge-Dunwoody (Theorem 5.1) which are geometrically realisable by Harlander-Jensen (Theorem 8.2). The proof of Theorem 1.4 uses the boundary of thickenings construction which, given a finite 2-complex $X$, assigns a closed smooth 4-manifold $M(X)$ given by the boundary of a smooth regular neighbourhood of an embedding $i : X \hookrightarrow \mathbb{R}^5$ (see Section 11.1).

**Organisation of the paper.** The paper is organised as follows. In Section 2, we will fill in further details on unstable classification, building upon Section 1.3. Sections 3.4 will be devoted to the proof of Theorem A. Sections 6.5 will be devoted to the proof of Theorem B. In Section 9, we will prove Theorem 1.3. In Section 10, we will explore applications to 4-manifolds, culminating in a proof of Theorem 1.4. Finally, in Section 12, we will propose a number of directions in the study of $P(ZG)$, $HT(G)$ and $\mathcal{M}(G)$. In particular, we pose Problems 11.4 concerning projective
ZG-modules and Problems 31, 33 concerning finite 2-complexes. We hope that Theorems 31, 32 and 33 go some way towards convincing the reader that progress on these problems is possible.

Acknowledgements. I would like to thank Martin Dunwoody, Jens Harlander, F. E. A. Johnson and Mark Powell for useful correspondence and a number of helpful comments. This work was supported by EPSRC grant EP/N509577/1 and the Heilbronn Institute for Mathematical Research.

2. Preliminaries on \( P(ZG), \ HT(G, n) \) and \( \mathcal{M}(G) \)

The aim of this section will be to establish basic properties of the set \( P(ZG) \) of ZG-isomorphism classes of finitely generated non-zero projective ZG-modules, the set \( HT(G, n) \) of homotopy types of finite \((G, n)\)-complexes, and the set \( \mathcal{M}(G) \) of homeomorphism classes of closed smooth 4-manifolds \( M \) with \( \pi_1(M) \cong G \).

Recall from the introduction that, for a set \( S \), a stabilisation is a function \( \Sigma : S \rightarrow S \) such that \( \ell(a) := \sup\{k \geq 0 : \Sigma^{n+k}(b) \text{ for some } b \in S, n \geq 0\} < \infty \) where \( n, k \) are integers. We then have a level function \( \ell : S \rightarrow \mathbb{Z}_{\geq 0} \) which is surjective and satisfies \( \ell(\Sigma(a)) = \ell(a) + 1 \) for \( a \in S \). We will write \( \ell = \ell_S \) when we want to emphasise the set \( S \). Let \( =_{\text{st}} \) denote the corresponding (reduced) stable equivalence relation, i.e. \( a =_{\text{st}} b \) if \( \Sigma^n(a) = \Sigma^n(b) \) for some \( n, m \geq 0 \).

Recall that the rank of a finitely generated projective ZG-module \( P \), denoted by \( \text{rank}_{ZG}(P) \), is the rank of the free abelian group \( P_G = \mathbb{Z} \otimes_{ZG} P \).

**Proposition 2.1.** Let \( G \) be a group and let \( \ell = \ell_{P(ZG)} \). Then:

(i) \( \Sigma : P(ZG) \rightarrow P(ZG) \), \( P \mapsto P \oplus ZG \) is a stabilisation, i.e. \( \ell(P) < \infty \) for all \( P \in P(ZG) \).

(ii) For each \( P \in P(ZG) \), we have \( \ell(P) = \text{rank}_{ZG}(P) - \text{rank}_{\text{min}}([P]) \) and, for each \( c \in \bar{K}_0(ZG) \),

\[
\text{rank}_{\text{min}}(c) := \min\{\text{rank}_{ZG}(P_0) : P_0 \in P(ZG), [P_0] = c \in \bar{K}_0(ZG)\}
\]

is the minimal rank of a projective ZG-module in the class \( c \).

**Remark 2.2.** If \( c = 0 \), then \( \text{rank}_{\text{min}}(c) = 1 \) and so \( \ell(P) = \text{rank}_{ZG}(P) - 1 \) for \( P \) a stably free ZG-module.

It is not known whether \( \text{rank}_{\text{min}}(c) = 1 \) for all groups \( G \) and all \( c \in \bar{K}_0(ZG) \) (see Problem A2).

**Proof.** (i) Recall that, for \( P \in P(ZG) \), we defined:

\[
\ell(P) := \sup\{k \geq 0 : P \oplus ZG^r \cong Q \oplus ZG^{r+k} \text{ for some } Q \in P(ZG), r \geq 0\}.
\]

If \( P \oplus ZG^r \cong Q \oplus ZG^{r+k} \), then \( \text{rank}_{ZG}(P) = \text{rank}_{ZG}(Q) + k \) and so \( k \leq \text{rank}_{ZG}(P) \). In particular, we have \( \ell(P) \leq \text{rank}_{ZG}(P) < \infty \).

(ii) Let \( P \in P(ZG) \). Since \( \text{rank}_{ZG}(P \oplus ZG) = \text{rank}_{ZG}(P) + 1 \), we have that \( \ell(P) - \text{rank}_{ZG}(P) \) is invariant under the operation \( P \mapsto P \oplus ZG \) and so is a function of the class \( [P] \in \bar{K}_0(ZG) \). Let \( P_0 \in P(ZG) \) be such that \( [P_0] = c \) and \( \text{rank}_{ZG}(P_0) = \text{rank}_{\text{min}}(c) \), where the minimal value exists since it is bounded below by 0. Since \( \text{rank}_{ZG}(\cdot) \) is minimal at \( P_0 \), and \( \ell(\cdot) - \text{rank}_{ZG}(\cdot) \) is constant on \( c \), it follows that \( \ell(\cdot) \) is minimal at \( P_0 \), i.e. \( \ell(P_0) = 0 \). Hence \( \ell(P) - \text{rank}_{ZG}(P) = - \text{rank}_{ZG}(P_0) = - \text{rank}_{\text{min}}(c) \). □

For \( n \geq 2 \) and \( X \) a finite \((G, n)\)-complex, define the directed Euler characteristic \( \chi(X) = (-1)^n \chi(X) \).

**Proposition 2.3.** Let \( n \geq 2 \), let \( G \) be a group of type \( F_n \) and let \( \ell = \ell_{HT(G, n)} \). Then:

(i) \( \Sigma : HT(G, n) \rightarrow HT(G, n), X \mapsto X \vee S^n \) is a stabilisation, i.e. \( \ell(X) < \infty \) for all \( X \in HT(G, n) \).

(ii) For each \( X \in HT(G, n) \), we have \( \ell(X) = \chi(X) - \chi_{\text{min}}(G, n) \) where

\[
\chi_{\text{min}}(G, n) := \min\{\chi(X) : X \in HT(G, n)\}
\]

is the minimal directed Euler characteristic in \( HT(G, n) \), which always exists.

**Proof.** (i) Recall that, for \( X \in HT(G, n) \), we defined:

\[
\ell(X) := \sup\{k \geq 0 : X \vee rS^n \cong Y \vee (r+k)S^n \text{ for some } Y \in HT(G, n), r \geq 0\}.
\]

If \( X \vee rS^n \cong Y \vee (r+k)S^n \), then \( \text{rank}_{ZG}(H_n(X)) = \text{rank}_{ZG}(H_n(Y)) + k \) and so \( k \leq \text{rank}_{ZG}(H_n(X)) \).

(ii) This is analogous to the proof of Proposition 2.1(ii) and so will be omitted for brevity. □
Proposition 2.4. Let $G$ be a finitely presented group and let $\ell = \ell_M(G)$. Then:

(i) $\Sigma : \mathcal{M}(G) \to \mathcal{M}(G), M \mapsto M \#(S^2 \times S^2)$ is a stabilisation, i.e. $\ell(M) < \infty$ for all $M \in \mathcal{M}(G)$.

(ii) For each $M \in \mathcal{M}(G)$, we have $\ell(M) = \frac{1}{2}(\chi(M) - \chi_{\min}([M]))$ where, for $c \in \mathcal{M}(G)/\equiv_{st}$,

$$\chi_{\min}(c) := \min\{\chi(M) : M \in c\}$$

is the minimal Euler characteristic of a 4-manifold in $c$, which always exists.

Remark 2.5. The same holds with $\mathcal{M}(G)$ replaced by $\mathcal{M}^{\text{Diff}}(G)$.

Proof. (i) Recall that, for $M \in \mathcal{M}(G)$, we defined:

$$\ell(M) := \sup\{k \geq 0 : M \# r(S^2 \times S^2) \cong N \# (r + k)(S^2 \times S^2), \text{ for some } N \in \mathcal{M}(G), r \geq 0\}.$$ 

If $M \# r(S^2 \times S^2) \cong N \# (r + k)(S^2 \times S^2)$, then $\text{rank}_\mathbb{Z}(H_2(M)) = \text{rank}_\mathbb{Z}(H_2(N)) + 2k$. This implies that $k \leq \frac{1}{2}\text{rank}_\mathbb{Z}(H_n(M))$.

(ii) This is analogous to the proof of Proposition 2.1 (ii) and so will be omitted for brevity. \qed

3. Preliminaries on $RG$-modules

Let $G$ be a group, let $R$ be a ring and let $RG$ denote the group ring of $G$ with coefficients in $R$. We will now develop the necessary preliminaries on $RG$-modules.

3.1. Stably free $RG$-modules. For a ring $R$, a finitely generated (left) $R$-module $S$ is stably free if there exists $n, m$ such that $S \oplus R^n \cong R^m$. In order to have a well-defined notion of rank, certain conditions on $R$ must be imposed:

(I) For all $n, m, R^n \cong R^m$ implies $n = m$ (invariant basis number property)

(II) For all $n, m, S \oplus R^n \cong R^m$ implies $n \leq m$ (surjective rank property)

(III) For all $n, S \oplus R^n \cong R^n$ implies $S = 0$ (stable finiteness property)

Suppose $R$ satisfies (I). If $S$ is a stably free $R$-module, then we can define the rank of $S$ to be $\text{rank}(S) = m - n$ for any $n, m$ such that $S \oplus R^n \cong R^m$. If $R$ satisfies (II), then $\text{rank}(S) \geq 0$ for all $S$. If $R$ satisfies (III), then $S \neq 0$ implies that $\text{rank}(S) \geq 1$.

It is straightforward to see that (III) $\Rightarrow$ (II) $\Rightarrow$ (I). Conversely, examples were given by Cohn [17] to show that $(R \neq 0) \not\Rightarrow (I) \not\Rightarrow (II) \not\Rightarrow (III)$. Rings which satisfy (III) are also known as weakly finite and satisfy the equivalent condition that, for all $n$, one-sided inverses in $M_n(R)$ are two-sided, i.e. $uv = 1$ if and only if $vu = 1$.

We would now like to determine when conditions (I)-(III) hold for $RG$. The following is a consequence of [17] Proposition 2.4, Theorem 2.6.

Proposition 3.1. Let $R$ be a commutative ring and let $G$ be a group. Then $RG$ has the surjective rank property, and hence also the invariant basis number property.

It remains to determine when $RG$ is stably finite. It was shown by Kaplansky [43] that, if $F$ is a field of characteristic 0, then $FG$ is stably finite for all groups $G$. This implies that $ZG$ is stably finite since $ZG \subseteq QG$. Kaplansky conjectured that this holds for all fields $F$, but this remains open.

The best result for general fields $F$ is the following theorem of Elek-Szabó [23], which built upon earlier work of Ara, O’Meara and Perera [24] Theorem 3.4.

Theorem 3.2. Let $F$ be a field and let $G$ be a sofic group. Then $FG$ is stably finite.

For a definition of sofic, see [23] p430. For our purposes, it suffices to note that $G = 1$ is sofic and that sofic groups are closed under direct/free products, direct/inverse limits, subgroups, and that the extension of an amenable group (see [2] p227) by a sofic group is sofic. There is no known example of a non-sofic group.

All groups which will be considered in this article are sofic. We can therefore assume, when needed, that non-trivial stably free $FG$-modules have rank $\geq 1$.
3.2. **RG-modules over free products.** Fix groups $G_1, \cdots, G_n$, let $G = \ast_{k=1}^n G_k$ denote the free product and let $\iota_k : G_k \hookrightarrow G$ denote the inclusion map for each $k$.

Let $R$ be a ring. If $M_k$ is an $RG_k$-module, then $\iota_k \# (M_k) = R \otimes_{RG_k} M_k$ is an $RG$-module. We say that an $RG$-module $M$ is induced if there exists $RG_k$-modules $M_k$ and an $RG$-module isomorphism

$$M \cong \iota_1 \# (M_1) \oplus \cdots \oplus \iota_n \# (M_n).$$

We now define two special types of map between induced $RG$-modules. Firstly, if $M = \bigoplus_{k=1}^n \iota_k \# (M_k)$ and $M' = \bigoplus_{k=1}^n \iota_k \# (M'_k)$ are induced $RG$-modules, then an $RG$-module homomorphism $f : M \to M'$ is called an induced isomorphism if there exists $RG_k$-module homomorphisms $f_k : M_k \to M'_k$ such that $f = \bigoplus_{k=1}^n f_k$.

Now, let $M = \bigoplus_{k=1}^n \iota_k \# (M_k)$ be an induced $RG$-module and suppose there exists $a$ for which $M_a \cong M'_a \oplus RG_a$ for some $RG_a$-module $M'_a$. Then, for any $b \neq a$, there is an isomorphism

$$f_{a,b} : a \# (M_a \oplus RG_a) \oplus b \# (M_b) \to a \# (M'_a) \oplus b \# (M_b \oplus RG_b)$$

induced by $\iota_{a,b} \# (RG_a) \cong RG \cong \iota_{a,b} \# (RG_b)$. We define a free transfer isomorphism on $a,b$ to be the isomorphism $F_{a,b} : M \to M'$ which extends $f_{a,b}$ by the identity map on the other components and where

$$M' = \iota_1 \# (M_1) \oplus \cdots \oplus \iota_n \# (M'_a) \oplus \cdots \oplus \iota_n \# (M'_n).$$

The following can be viewed as a special case of Bergman's theorem on modules over coproducts of rings [7]. We now restrict to the case where $R = \mathbb{F}$ is a field.

**Theorem 3.3** (Bergman). Let $M$ be a finitely generated induced $\mathbb{F}$-module. Then:

(i) If $M' \subseteq M$ is a submodule, then $M'$ is an induced $\mathbb{F}$-module.

(ii) If $M'$ is an induced $\mathbb{F}$-module, then $M \cong M'$ if and only if they are connected by a sequence of induced isomorphisms and free transfer isomorphisms.

For the convenience of the reader, we will briefly outline how this can be deduced from Bergman's results. Here will will use the terminology from [7, p1-4].

**Proof (outline).** Firstly, note that $FG$ is a the coproduct of the $\mathbb{F}$-rings $FG_k$ which are faithful since they come equipped with natural injections $\iota_k : FG_k \hookrightarrow FG$.

Part (i) follows immediately from [7, Theorem 2.2]. For part (ii), suppose $f : M \to M'$ is an isomorphism of $\mathbb{F}G$-modules. By [7, Theorem 2.3], and the remark on [7, p3], $f$ is the composition of induced isomorphisms, free transfer isomorphisms and transvections. Since transvections are module automorphisms, omitting them from the composition still leaves an isomorphism of $\mathbb{F}G$-modules.

**Corollary 3.4.** Let $M = \bigoplus_{k=1}^n \iota_k \# (M_k)$ be a finitely generated induced $\mathbb{F}G$-module and suppose each $M_k$ has no direct summand of the form $\mathbb{F}G_k$. Then:

(i) If $M' = \bigoplus_{k=1}^n \iota_k \# (M'_k)$ is an induced $\mathbb{F}G$-module, then $M \cong M'$ as $\mathbb{F}G$-modules if and only if $M_k \cong M'_k$ as $\mathbb{F}G_k$-modules for all $k$.

(ii) $M$ has no direct summand of the form $\mathbb{F}G$.

**Proof.** Part (i) follows from Theorem 3.3 (ii) since, if the $M_k$ have no direct summands of the form $\mathbb{F}G_k$, then there are no free transfer isomorphisms by definition.

To see part (ii) note that, if $M \cong M' \oplus \mathbb{F}G$, then $M' \subseteq M$ is a submodule and so is an induced $\mathbb{F}G$-module by Theorem 3.3 (i). If $M' = \bigoplus_{k=1}^n \iota_k \# (M'_k)$, then $M \cong \iota_1 \# (M'_1 \oplus \mathbb{F}G_1) \oplus \bigoplus_{k=2}^n \iota_k \# (M'_k)$ which contradicts the result from (i).

3.3. **RG-modules up to Aut(G)-isomorphism.** If $M$ is an $RG$-module and $\theta \in \text{Aut}(G)$, then we can define $M_\theta$ to be the $RG$-module with the same underlying $R$-module as $M$ but with $G$-action given by $g \cdot _{M_\theta} m = \theta(g) \cdot _M m$ for $g \in G$ and $m \in M$. We say that $RG$-modules $M$ and $M'$ are $\text{Aut}(G)$-isomorphic if $M \cong (M')_\theta$ are isomorphic as $RG$-modules for some $\theta \in \text{Aut}(G)$. This is equivalent to the existence of a $\theta$-isomorphism $f : M \to M'$ which is an $R$-module isomorphism for which $f(g \cdot m) = \theta(g) \cdot f(m)$ for $g \in G$ and $m \in M$.
This has a number of basic properties. In particular, if $M$ and $M'$ are $RG$-modules and $\theta \in \text{Aut}(G)$, then $(M \oplus M')_\theta \cong M_\theta \oplus (M')_\theta$, and $(RG)_\theta \cong RG$ for all $\theta \in \text{Aut}(G)$.

Recall that a subgroup $N \subseteq G$ is characteristic if $\theta(N) = N$ for all $\theta \in \text{Aut}(G)$. We also say that a surjective map $f : G \to H$ is characteristic if $\text{Ker}(f) \subseteq G$ is characteristic and, if so, there is an induced map $\tilde{f} : \text{Aut}(G) \to \text{Aut}(H)$.

The following is straightforward (see, for example, [59, Corollary 7.4]).

**Proposition 3.5.** Let $G$ be a group, let $f : G \to H$ be characteristic and let $\tilde{f} : \text{Aut}(G) \to \text{Aut}(H)$ be the map induced by $f$. If $M$ is an $RG$ module and $\theta \in \text{Aut}(G)$, then $f_\theta(M_\theta) \cong (f_\theta(M))_\theta$ are isomorphic as $RG$-modules. We will make use of the following in Section 11 on the unstable classification of 2-complexes. Recall that projective modules are reflexive $H$-modules.

**Theorem 3.7** (Kurosh subgroup theorem). Let $G = G_1 \ast \cdots \ast G_n$ where each $G_k$ is indecomposable and not infinite cyclic. For each $k$, let $f_k : G_k \to H_k$ be characteristic and such that, if $G_i \cong G_j$, then $H_i \cong H_j$ and $f_i, f_j$ differ by automorphisms of $G_i, H_i$.

If $f : G \to H_1 \ast \cdots \ast H_n$ is the map with $f|_{G_k} = f_k$, then $f$ is characteristic.

Our proof will be a routine application of the following version of the Kurosh subgroup theorem [60, Theorem 5.1].

Let $G = G_1 \ast \cdots \ast G_n$. If $H \subseteq G$ is a subgroup, then

$$H = F(X) \ast \left( \bigwedge_{k=1}^n g_k H_k g_k^{-1} \right)$$

where $F(X)$ is the free group on a set $X$, $H_k \subseteq G_k$ is a subgroup and $g_k \in G$.

**Proof of Proposition 3.5.** Let $\varphi \in \text{Aut}(G)$. Then $\varphi(G_k) \subseteq G$ is indecomposable and not infinite cyclic and so, by the Kurosh subgroup theorem, we have $\varphi(G_k) = g_{ik} H_{ik} g_{ik}^{-1}$ for some subgroup $H_{ik} \subseteq G_{ik}$. Since $\varphi$ is an automorphism, we have:

$$G = \ast_{k=1}^n (g_{ik} H_{ik} g_{ik}^{-1}) \leq \ast_{k=1}^n (g_{ik} G_{ik} g_{ik}^{-1}) \leq \ast_{k=1}^n (g_k G_k g_k^{-1}) = G$$

which implies that $H_{ik} = G_{ik}$ and that the $i_k$ are distinct.

Let $N_k = \text{Ker}(f_k) \subseteq G_k$ and note that $N = \text{Ker}(f)$ is generated by the subgroups $gN_k g^{-1}$ for $g \in G$. If $\varphi \in \text{Aut}(G)$, then the above implies that $\varphi|_{G_k} = c_{g_{ik}} \circ \varphi_{i,k}$ where $\varphi_{i,k} : G_i \to G_{ik}$ is an isomorphism and $c_{g_{ik}} : G_{ik} \to G$ is conjugation by $g_{ik}$. Since $f_i, f_k$ differ by automorphisms of $G_i, G_{ik}$, we have $\varphi_{i,k} (N_i) = N_{ik}$ for some $\varphi_{i,k} \in \text{Aut}(G_{ik})$ and so $\varphi_{i,k} (N_i) = N_{ik}$ since $N_{ik}$ is characteristic. Hence $\varphi(gN_k g^{-1}) = (g_{ik} N_{ik} g_{ik}^{-1}) \subseteq N$ and so $N$ is characteristic.

3.4. **Duals of $RG$-modules.** We will make use of the following in Section 11 on the unstable classification of 4-manifolds. Recall that $RG$ comes equipped with an involution

$$\tilde{\tau} : RG \to RG, \quad \sum_{i=1}^r n_i g_i \mapsto \sum_{i=1}^r n_i g_i^{-1}$$

where $n_i \in R$, $g_i \in G$. This is an anti-isomorphism of rings.

If $M$ is a (left) $RG$-module, then define $M^* = \text{Hom}_{RG}(M, RG)$ to be the (left) $RG$-module with $RG$-action given by letting

$$(\lambda \varphi) : m \mapsto \varphi(m)^{\lambda}$$

for $\lambda \in RG$ and $\varphi \in M^*$. This satisfies a number of basic properties such as that $(RG^n)^* \cong RG^n$ as $RG$-modules. The following facts about duals of projective $RG$-modules are standard. Part (ii) says that projective modules are reflexive.

**Proposition 3.8.** Let $P$ be a finitely generated projective $RG$-module. Then:

(i) $P^*$ is a finitely generated projective $RG$-module

(ii) The evaluation map $\text{ev}_P : P \to P^**, \ x \mapsto (\varphi \mapsto \varphi(x))$ is an isomorphism of $RG$-modules.
Proof. (i) If $P \oplus Q \cong RG^n$, then $P^* \oplus Q^* \cong (RG^n)^* \cong RG^n$.
(ii) If $P \oplus Q \cong RG^n$, then $ev_{RG^n} = ev_P \oplus ev_Q$ so $ev_P$ is an isomorphism since $ev_{RG^n}$ is. \qed

We will now prove that dualising commutes with the action of Aut($G$) defined in Section 3.3.

Proposition 3.9. Let $M$ be an $RG$-module and let $\theta \in$ Aut($G$). Then there is an isomorphism of $RG$-modules

$$(M_\theta)^* \cong (M^*)_\theta.$$ 

In particular, if $RG$-modules $M$ and $N$ are Aut($G$)-isomorphic, then $M^*$ and $N^*$ are Aut($G$)-isomorphic.

Proof. For each $\alpha \in$ Aut($G$), there is an isomorphism of $RG$-modules given by

$$\alpha_* : RG \to RG_\alpha, \quad \sum_{i=1}^r n_i g_i \mapsto \sum_{i=1}^r n_i \alpha(g_i)$$

where $n_i \in R$ and $g_i \in G$. Given this, define

$$f : M^* \to (M_\theta)^*, \quad \varphi \mapsto (\theta^{-1})_* \circ \varphi$$

where we are viewing $\varphi : M \to RG$ as an $RG$-homomorphism $\varphi : M_\theta \to RG_\theta$.

We claim that $f$ is a $\theta^{-1}$-isomorphism. This gives the desired result since it implies that $M^* \cong ((M_\theta)^*)_{\theta^{-1}}$ and so $(M^*)_\theta \cong (M^*)_\theta$ by applying $(\cdot)_\theta$ to both sides. Firstly, it is clear that this is an $R$-module homomorphism, and is an isomorphism with inverse given by post-composing with $\theta_*$. For $g \in G$, $\varphi \in M^*$ and $m \in M$, we have

$$f(g \cdot \varphi)(m) = (\theta^{-1})_*((\varphi(m)g^{-1}) = ((\theta^{-1})_* \circ \varphi)(m)(\theta^{-1}(g))^{-1}$$

and so $f(g \cdot \varphi) = \theta^{-1}(g) \cdot f(\varphi)$, as required. \qed

4. Groups of finite cohomological dimension

We will now recall some basic facts about groups with finite cohomological dimension which are due to Serre [62]. A standard reference is the notes of Bieri [10].

A group $G$ has cohomological dimension $n$, written $cd(G) = n$, if $n$ is the smallest integer for which there exists a projective resolution of $ZG$-modules of the form:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to Z \to 0.$$ 

This is equivalent to asking that $H^n(G; M) = 0$ for all $i > n$ and all $ZG$-modules $M$ [10] Proposition 5.1(a). If no such $n$ exists, then we take $cd(G) = \infty$.

A group $G$ is said to be of type FL if, for some $n \geq 0$, there exists a resolution of finitely generated free $ZG$-modules of the form:

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to Z \to 0$$

The following is [10] Propositions 1.5, 4.1(b)].

Proposition 4.1. Let $G$ be a group with $cd(G) = n$. If $G$ is of type FL, then there exists a resolution of finitely generated free $ZG$-modules of the form:

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to Z \to 0.$$ 

We now recall how these conditions are related under amalgamated free products and direct products. The following is [10] Proposition 2.13(a), Proposition 6.1].

Lemma 4.2. Let $G = G_1 *_H G_2$ for groups $G_1$, $G_2$ with a common subgroup $H$.

(i) If $G_1$, $G_2$ are of type FL and $H$ is of type FL, then $G$ is of type FL
(ii) If $n = \max\{cd(G_1), cd(G_2)\} < \infty$ and $cd(H) < n$, then $cd(G) = n$.

The following is a consequence of more general results on group extensions which can be found in [10] Proposition 2.7, Theorem 5.5].
Lemma 4.3. Let $G = G_1 \times G_2$ for groups $G_1, G_2$.

(i) If $G_1, G_2$ are of type FL, then $G$ is of type FL

(ii) If $\text{cd}(G_1), \text{cd}(G_2) < \infty$, $G_1$ is of type FL and $H^n(G_1; \mathbb{Z}G_1)$ is $\mathbb{Z}$-free for $n = \text{cd}(G_1)$, then $\text{cd}(G) = \text{cd}(G_1) + \text{cd}(G_2)$.

We will now give a construction of groups which will be the basis for our examples in Theorem A in the case $d \geq 3$. This is inspired by a construction of Lustig [33].

Let $G$ be a group and let $m \geq 2$ be an integer. Then define

$$G_+ = (G \times (q \mathbb{Z})) / [r \mathbb{Z}, G],$$

which is isomorphic to $(G \times (q \mathbb{Z})) *_{(q \mathbb{Z}, m)} (r \mathbb{Z})$. For integers $n_1, \ldots, n_{m-1} \geq 2$, we can define $G(n)$ inductively by letting $G_1 = G$ and $G_{i+1} = (G(i))_+$ for $i \geq 1$. We will label the new generator by $r_i$. The choice of $m_i \geq 2$ will not matter for the purposes of this article; it suffices to consider the case $m_i = 2$.

Let $\iota : G \to G(n)$ be the composition of the natural maps $G_{(i)} \to G_{(i+1)}$ and let $f : G(n) \to G$ be the map which sends $r_i \mapsto 1$ for each $i$. We have that $f \circ \iota = \text{id}_G$ and so $\iota$ is injective, $f$ is surjective and $G$ is a retract of $G(n)$.

Proposition 4.4. Let $n \geq 1$ and let $G$ be a finitely presented group of type FL with $\text{cd}(G) = d$. Then:

(i) $G(n)$ is a finitely presented group of type FL with $\text{cd}(G(n)) = n + d - 1$

(ii) The map $f : G(n) \to G$ is characteristic.


\begin{proof}
In order to prove this, we will first need the following lemma. The proof is identical to the one given in [33] p174.

Lemma 4.5. Let $G$ be a torsion free group and let $G_+ = (G \times (q \mathbb{Z})) / [r \mathbb{Z}, G]$ for some $m \geq 2$. Then the map $f : G_+ \to G$ which sends $r \mapsto 1$ is characteristic.

\end{proof}

\begin{proof}[Proof of Proposition 4.4]
It is clear that $G(n)$ is finitely presented. We now prove (i) by induction, noting that it is trivial in the case $n = 1$.

Suppose (i) holds for $n$ and note that $G(n+1) \cong (G(n) \times \mathbb{Z}) / \mathbb{Z}$. It is well known that $K(\mathbb{Z}, 1) \simeq S^1$ and so $\mathbb{Z}$ is of type FL, $\text{cd}(\mathbb{Z}) = 1$ and $H^1(\mathbb{Z}; \mathbb{Z}[\mathbb{Z}]) = 0$. By Lemma 4.3 $G(n) \times \mathbb{Z}$ is of type FL and $\text{cd}(G(n) \times \mathbb{Z}) = n + d$. By Lemma 4.2 this implies that $G(n+1)$ is of type FL and $\text{cd}(G(n+1)) = n + d$ as required.

Since $\text{cd}(G(n)) < \infty$, $G(n)$ is torsion free for all $n$ [10 Proposition 4.11]. By Lemma 4.5 this implies that the map $f_{i+1} : G_{(i+1)} \to G_{(i)}$, $r_{i+1} \mapsto 1$ is characteristic for all $i \geq 1$. Hence $f = f_n \circ f_{n-1} \circ \cdots \circ f_2$ is characteristic by composition.
\end{proof}

5. Proof of Theorem A

Recall that the trefoil group $T$ is defined as $\pi_1(S^3 \setminus N(K))$ where $N(K)$ is the knot exterior of the trefoil knot $K \subseteq S^3$. It has presentation $\mathcal{P} = \langle x, y \mid x^2 = y^3 \rangle$.

Let $T''$ denote the second derived subgroup of $T$, i.e. $T'' = (T')'$, and let $f : T \to T/T''$ be the quotient map. Note that $T/T''$ is polycyclic and so $T'' = \mathbb{Z}$.

The following was shown by P. H. Berridge and M. J. Dunwoody [8], building upon previous work of Dunwoody [20].

Theorem 5.1 (Berridge-Dunwoody). There exists infinitely many rank one stably free $\mathbb{Z}T$-modules $S_i$ for $i \geq 1$ such that:

(i) $S_i \oplus \mathbb{Z}T \cong \mathbb{Z}T^2$.

(ii) There exists distinct primes $p_i$ for which $\mathbb{F}_{p_i} \otimes f_{p_i}(S_j) \cong \mathbb{F}_{p_i}[T/T'']$ are isomorphic as $\mathbb{F}_{p_i}[T/T'']$-modules if and only if $i = j$.

In particular, the $S_i$ are distinct up to $\mathbb{Z}G$-module isomorphism.

Remark 5.2. For $i \geq 0$, let $M_i = \text{Ker}(\bigoplus \mathbb{Z}) : \mathbb{Z}T^2 \to \mathbb{Z}T)$ be the relation module for the generating set $\{x^{2i+1}, y^{3i+1}\}$, which is a stably free $\mathbb{Z}T$-module of rank one. It was shown in [8] that...
Theorem 5.1. It is not known whether or not the 

\[ M_{\text{cd}}(\text{cd}(\text{stably free } Z_{12}) \rightleftharpoons \text{map induced by the characteristic by Proposition 3.6.}

For each 

\[ n \geq 1 \] and 

\[ 1 \leq m \leq k, \] there exists infinitely many stably free \( ZG(n) \)-modules 

\[ \hat{S}_i \] for 

\[ i \geq 1 \] such that:

(i) \( \hat{S}_i \oplus ZG(n) \cong ZG(n)^{m+1} \).

(ii) \( \hat{S}_i \) has no direct summand of the form \( ZG(n) \).

(iii) The \( \hat{S}_i \) for \( i \geq 1 \) are distinct up to \( \text{Aut}(G(n)) \)-isomorphism of \( ZG(n) \)-modules.

Note that the case \( m = k \) is sufficient to establish Theorem A. This result shows that the tree of stably free \( ZG(n) \)-modules has branching at all ranks \( 1 \leq m \leq k \). We do not know whether branching occurs at ranks \( \geq k+1 \), even in the case \( G = T \).

In order to prove Theorem 5.3 we will begin with the following lemma.

**Lemma 5.4.** Let \( f_j : T_j \rightarrow (T_j \mod{T_j}'' = \text{the quotient maps and let} \)

\[ f : G \rightarrow (T_1 / T_1'') \ast \cdots \ast (T_k / T_k'') \]

be the map induced by the \( f_j \). Then \( f \) is characteristic.

**Proof.** For any group \( G \), it is well known that \( G' \subset G \) is characteristic and so \( G'' \subset G \) is characteristic also. Hence \( f_j \) is characteristic for each \( j \). Since \( T \) is indecomposable and not infinite cyclic, \( f \) is characteristic by Proposition 5.6.

For simplicity, we will begin by proving Theorem 5.3 in the case \( n = 1 \), i.e. where \( G(n) = G \). From now on, fix \( 1 \leq m \leq k \). For integers \( i_1, \cdots, i_m \), define

\[ S_{i_1, \cdots, i_m} = f_{i_1}(S_{i_1}) \oplus \cdots \oplus f_{i_m}(S_{i_m}) \]

where \( f_j : T_j \rightarrow G \) is the inclusion map. We will now prove the following as a consequence of Bergman’s theorem, which we will apply by using Corollary 3.3.

**Proposition 5.5.** For integers \( i_1, \cdots, i_m \), we have:

(i) \( S_{i_1, \cdots, i_m} \oplus ZG \cong ZG^{m+1} \).

(ii) \( S_{i_1, \cdots, i_m} \) has no direct summand of the form \( ZG \).

(iii) \( \hat{S}_{i_1, \cdots, i_m} \) are \( \text{Aut}(G) \)-isomorphic as \( ZG \)-modules then, as sets, we have \( \{i_1, \cdots, i_m\} = \{i_1', \cdots, i_m'\} \).

**Proof.** Part (i) is a straightforward consequence of Theorem 5.1 (i).

Let \( G = *_{j=1}^n T_j / T_j'' \) and let \( \hat{i}_j : T_j / T_j'' \rightarrow G \) be inclusion. By Theorem 5.1 (ii), there exists \( p \) such that \( F_p \otimes f_{\hat{i}_j}(S_{i_j}) \neq F_p[T_j / T_j'' \] for all \( j \). Fix \( p \) and note that:

\[ F_p \otimes f_{\hat{i}_j}(S_{i_1, \cdots, i_m}) \cong \bigoplus_{j=1}^m F_p \otimes (f \circ i_j)_{\hat{g}}(S_{i_j}) \cong \bigoplus_{j=1}^m i_j_{\hat{g}}(F_p \otimes f_{\hat{i}_j}(S_{i_j})) \]

is an induced \( F_p G \) module. In order to show that Corollary 3.3 applies, it remains to show that \( F_p \otimes f_{\hat{i}_j}(S_{i_j}) \) has no direct summand of the form \( F_p[T_j / T_j'' \].

If \( F_p \otimes f_{\hat{i}_j}(S_{i_j}) \cong S \oplus F_p[T_j / T_j'' \], then \( S \oplus F_p[T_j / T_j'']^2 \cong F_p[T_j / T_j'']^2 \). Since \( T_j / T_j'' \) is polycyclic, it is amenable and so not includes. By Theorem 5.1, \( F_p[T_j / T_j''] \) is stably finite and so \( S = 0 \). Hence \( F_p \otimes f_{\hat{i}_j}(S_{i_j}) \cong F_p[T_j / T_j''] \), which is a contradiction.

To show (ii) note that, if \( S_{i_1, \cdots, i_m} \) has a direct summand \( ZG \), then \( F_p \otimes f_{\hat{i}_j}(S_{i_1, \cdots, i_m}) \) has a direct summand \( F_p G \). This contradicts Corollary 3.3 (ii).
To show (iii), suppose that \( \{i_1, \ldots, i_m\} \neq \{i'_1, \ldots, i'_m\} \) as sets. By symmetry, we can assume that there exists \( i'_r \notin \{i_1, \ldots, i_m\} \). Let \( p = p_{i'_r} \) in the notation of Theorem 5.1. By the argument above, \( \mathbb{F}_p \otimes f'_\#(S_{i_1, \ldots, i_m}) \) has no direct summand of the form \( \mathbb{F}_p \mathbb{G} \). On the other hand, \( \mathbb{F}_p \otimes f'_\#(S_{i'_1, \ldots, i'_m}) \cong \mathbb{F}_p[T_r/T'_r] \) which implies that

\[
\mathbb{F}_p \otimes f'_\#(S_{i'_1, \ldots, i'_m}) \cong \bigoplus_{j=1}^m \mathbb{F}_{p_{i'_r}} \otimes f'_\#(S_{i'_1, \ldots, i'_r}) \oplus \mathbb{F}_p \mathbb{G}
\]

If \( S_{i_1, \ldots, i_m} \cong S_{i'_1, \ldots, i'_m} \) are Aut(\( G \))-isomorphic, then \( S_{i_1, \ldots, i_m} \cong (S_{i'_1, \ldots, i'_m})_\theta \) for some \( \theta \in \text{Aut}(G) \). By Lemma 5.3, \( f \) is characteristic and so, by Proposition 3.5, \( f'_\#((S_{i'_1, \ldots, i'_m})_\theta) \cong (f'_\#(S_{i'_1, \ldots, i'_m}))_\theta \) for some \( \theta \in \text{Aut}(G) \). In particular, we have:

\[
\mathbb{F}_p \otimes f'_\#(S_{i_1, \ldots, i_m}) \cong (\mathbb{F}_p \otimes f'_\#(S_{i'_1, \ldots, i'_m}))_\theta \cong (\mathbb{F}_p \mathbb{G}^\theta)_\theta \cong \mathbb{F}_p \mathbb{G}^m
\]

which is a contradiction. \( \square \)

**Proof of Theorem 5.6.** Let \( \iota: G \hookrightarrow G(n) \) and \( f: G(n) \rightarrow G \) be as defined in Section 3. This satisfies \( f \circ \iota = \text{id}_G \) and, by Proposition 3.3, \( f \) is characteristic. Define \( \tilde{S}_i = \iota_\#(S_{i_1, \ldots, i_m}) \), where \( i_j = i \) for all \( j \). By Proposition 3.5, it is now straightforward to check that the \( \tilde{S}_i \) has the required properties. \( \square \)

We conclude this section with extended remarks on Theorem A' and Theorem 5.3.

### 5.0.1 Relation modules.###

By Remark 5.2, \( S_1 \) is the relation module for the generating set \( \{x^{2i+1}, y^{3i+1}\} \) of \( T \). It follows that \( S_{i_1, \ldots, i_m} \) is the relation module for the generating set \( \{x^{2i_1+1}, y^{3i_1+1}\}_{i=1}^k \) of \( G = T_1 \ast \cdots \ast T_k \) where \( T_i = \langle x_i, y_i \mid x_i^2 = y_i^3 \rangle \).

### 5.0.2 Change of field.###

In the proof of Proposition 5.5, the \( \mathbb{Z}G \)-modules were distinguished by passing to \( \mathbb{F}_p \mathbb{G} \) for various \( p \). An alternate approach is to instead pass to \( \mathbb{Q}G \) and use the results of Lewin [19]. If \( \mathbb{F} = \mathbb{F}_p \) or \( \mathbb{Q} \), then one can show:

**Theorem 5.6.** Let \( k \geq 1 \) and let \( G = T_1 \ast \cdots \ast T_k \). Then there exists a stably free \( \mathbb{Z}G \)-module \( S \) of rank \( k \) such that \( S \otimes \mathbb{F} \) is a non-free stably free \( \mathbb{F} \mathbb{G} \)-module.

It would be interesting to know if one could detect infinitely many distinct stably free \( \mathbb{Z}G \)-modules of rank \( k \) on \( \mathbb{F}G \) for some \( \mathbb{F} = \mathbb{F}_p \) or \( \mathbb{Q} \), even in the case \( k = 1 \).

### 5.0.3 Alternate constructions.###

There are more ways to deduce Theorem A in the case \( d \geq 3 \) from the case \( d = 2 \). By Proposition 5.5, the and the proof of Theorem 5.3, it suffices to find a finitely presented group \( G \) with \( \text{cd}(G) = d \) and a characteristic quotient \( f: G \rightarrow \ast_{i=1}^N T \) for some \( N \geq k \). Two such constructions are as follows.

1. Let \( G = *_{i=1}^r (\ast_{j=1}^n T_{i,j})_{(d-1)} \) where \( 1 \leq n_1 \leq \cdots \leq n_r \) and \( N = \sum_{i=1}^r n_i \). Then \( \text{cd}(G) = d \) and there is a characteristic quotient \( f: G \rightarrow \ast_{i=1}^N T \). For example, we can take \( G = (\ast_{i=1}^r T_{i,j})_{(d-1)} \) as above, or \( G = *_{i=1}^r T_{(d-1)} \) (see Theorem 5.1).

2. Let \( G = (\ast_{i=1}^N T) \times \Gamma \) where \( \Gamma \) is a finitely presented group with \( \text{cd}(\Gamma) = d-2 \), \( Z(\Gamma) = 1 \) and which does not contain \( \ast_{i=1}^N T \) as a direct factor. By Lemma 1.3, we have \( \text{cd}(G) = d \). If \( N \geq 2 \), then \( Z(\ast_{i=1}^N T) = 1 \) and it can be deduced from [28, Corollary 2.2] that \( f: G \rightarrow \ast_{i=1}^N T \) is characteristic.

For example: If \( d = 3 \), let \( \Gamma \) be a free group of rank \( 2 \). If \( d = 4 \), let \( \Gamma \) be a surface group of genus \( \geq 2 \). If \( d \geq 5 \), let \( \Gamma \subseteq L \) be a cocompact torsion free lattice in a non-compact simple Lie group \( L \) with dimension \( d-2 \) over its maximal compact subgroup. Note that there are infinitely many such \( \Gamma \) up to commensurability. I am indebted to F. E. A. Johnson for this observation.

### 6. Module invariants of CW-complexes###

Let \( X \) be a CW-complex and recall that its cellular chain complex \( C_*(\tilde{X}) \) is a chain complex of free \( \mathbb{Z}[\pi_1(X)] \)-modules under the monodromy action. The chain homotopy type of \( C_*(\tilde{X}) \) is a homotopy invariant for \( X \) and so, for all \( n \), the \( \mathbb{Z}[\pi_1(X)] \)-module \( H_n(C_*(\tilde{X})) \) is also a homotopy invariant.
If \( G \) is a group and \( \rho : \pi_1(X) \cong G \), then every \( \mathbb{Z}[\pi_1(X)] \)-module \( M \) can be converted to a \( \mathbb{Z}G \)-module with action \( g \cdot \mathbb{Z}G \cdot m := \rho^{-1}(g) \cdot \mathbb{Z}G \cdot m \) for \( g \in G \) and \( m \in M \). In this notation, \( H_n(C_*(\tilde{X}))_\rho \) is a \( \mathbb{Z}G \)-module. We will denote this by \( H_n(X; \mathbb{Z}G) \) when \( \rho \) is understood. If \( \rho' : \pi_1(X) \cong G \) and \( \theta = \rho \circ (\rho')^{-1} \in \text{Aut}(G) \), then \( H_n(C_*(\tilde{X}))_{\rho'} \cong (H_n(C_*(\tilde{X}))_\rho)_\theta \). In particular, the \( \text{Aut}(G) \)-isomorphism class of \( H_n(X; \mathbb{Z}G) \) is a homotopy invariant and is independent of the choice of \( \rho \).

The aim of this section will be to consider how \( H_n(X; \mathbb{Z}G) \) changes under wedge product. We will also give a mild variation of this invariant under group quotients.

6.1. Homology of a wedge product. The following is presumably well-known. However, we were not able to locate a suitable reference in the literature.

**Proposition 6.1.** Let \( X_1, X_2 \) be CW-complexes with a single 0-cell such that \( \pi_1(X_k) \cong G_k \). Let \( X = X_1 \vee X_2 \) which has \( \pi_1(X) \cong G \). Let \( G = G_1 \ast G_2 \). Then:

\[
C_*(\tilde{X}) = \begin{cases} 
\mathbb{Z}G, & \text{if } i \geq 1 \\
\mathbb{Z}G, & \text{if } i = 0 
\end{cases}
\]

where \( \partial_i = \iota_1\#(\partial_i^{X_1}) \oplus \iota_2\#(\partial_i^{X_2}) \) for \( i \geq 2 \), \( \partial_1 = (i_1\#(\partial_1^{X_1}), i_2\#(\partial_1^{X_2})) \) and \( \partial_0 = \varepsilon_G \).

**Proof.** It suffices to compute an explicit model for \( \tilde{X} \) in terms of \( \tilde{X}_1 \) and \( \tilde{X}_2 \). Such a model, which is often attributed to Scott-Wall \([62]\), is provided by taking the graph of spaces structure on \( X = X_1 \vee X_2 \) and lifting it to \( \tilde{X} \).

Define a graph \((V, E)\) with vertex set \( V = V(X_1) \cup V(X_2) \) where \( V(X_1) \) is the set of elements in \( G_1 \ast G_2 \) with final term in \( G_2 \), i.e. the identity \( e \) as well as the elements of the form \( g_n \cdots g_1 g_i \) for \( n \geq 1 \) where \( g_i \in G_2 \setminus \{1\} \) when \( i \) is odd and \( g_i \in G_1 \setminus \{1\} \) otherwise. Define \( V(X_2) \) similarly. Note that, whilst \( V(X_1) \cap V(X_2) = \{1\} \) as subsets of \( G_1 \ast G_2 \), the elements \( 1 \in V(X_i) \) are not identified in \( V \).

Define \( E = \bigsqcup_{v \in V(X_1)} (G_1 \setminus \{1\})_v \cup \bigsqcup_{v \in V(X_2)} (G_2 \setminus \{1\})_v \cup \{1, 1\} \) where, for each \( v \in V(X_1) \) and \( g \in G_1 \setminus \{1\} \), we have a directed edge \( e_{v, v} = (g)_v \) from \( v \) to \( v g \) which is labeled by \( g \in G \). Similarly for \( V(X_2) \) and \( G_2 \). The edge \( e_{1, 1} \) from \( 1 \in V(X_1) \) to \( 1 \in V(X_2) \) is labeled by \( 1 \in G \).

Let \( * \in X_1 \) denote the 0-cell and, for each \( g \in G_i \), let \( *_g \in \tilde{X}_i \) denote its corresponding lift. Our model is the CW-complex

\[
X_{(V, E)} = \left( \bigsqcup_{v \in V(X_1)} (\tilde{X}_1)_v \cup \bigsqcup_{v \in V(X_2)} (\tilde{X}_2)_v \right) / \sim
\]

where, if we have a directed edge \( e_{v_1, v_2} \in E \) with label \( g \in G \), then \( (*_g)_{v_1} \sim (*_1)_{v_2} \) where, if \( v_1 \in V(X_1) \), then \( (*_g)_{v_1} \in (\tilde{X}_1)_{v_1} \) and similarly for \( (*_1)_{v_2} \). By comparing with the construction in \([62]\), we have \( \tilde{X} \simeq X_{(V, E)} \).

We now determine the induced action of \( G = G_1 \ast G_2 \) on \( X_{(V, E)} \). Note that \( G_1 \) acts \( (\tilde{X}_1)_1 \) by monodromy and freely permutes the \( *_g \in (\tilde{X}_1)_1 \). This action extends to all of \( X_{(V, E)} \) inductively, and similarly for the action of \( G_2 \) on \( (\tilde{X}_2)_2 \). Since \( G = (G_1, G_2) \), this determines the full action of \( G \) on \( X_{(V, E)} \).

It now remains to read off the cell structure of \( X_{(V, E)} \) under this \( G \)-action. For \( i \geq 1 \), the \( i \)-cells lie in the interior of the copies of \( \tilde{X}_1 \), \( \tilde{X}_2 \) and so are unaffected by the relation \( \sim \). This implies that:

\[
C_i(X_{(V, E)}) = \bigoplus_{v \in V(X_1)} v \cdot C_i(\tilde{X}_1) \oplus \bigoplus_{v \in V(X_2)} v \cdot C_i(\tilde{X}_2)
\]

as an abelian group. Since \( G \) acts on the \( V(X_j) \) in the natural way, and the elements of \( V(X_j) \) are coset representatives for \( G/G_j \), we have that:

\[
\bigoplus_{v \in V(X_1)} v \cdot C_i(\tilde{X}_1) \cong \mathbb{Z}G \otimes_{\mathbb{Z}G_j} C_i(\tilde{X}_j) \cong \iota_j\#(C_i(\tilde{X}_j))
\]
as \(ZG\)-modules. We can determine \(C_0(X;E)\) and the \(\partial_i\) similarly.

\[\text{Corollary 6.2. Let } X_1 \text{ and } X_2 \text{ be CW-complexes with a single 0-cell such that } \pi_1(X_i) \cong G_i. \text{ Let } X = X_1 \vee X_2 \text{ which has } \pi_1(X) \cong G \text{ where } G = G_1 \ast G_2. \text{ Then:}
\begin{align*}
H_n(X;ZG) &\cong \iota_1#(H_n(X;ZG_1)) \oplus \iota_2#(H_n(X;ZG_2)).
\end{align*}\]

\[\text{Remark 6.3. This could be deduced from the Mayer-Vietoris sequence for homology with local coefficients \([71\text{ Theorem 2.4}]\), though the above argument is more direct.}\]

\[\text{6.2 Homology under group quotients. Let } X \text{ be a CW-complex with } \rho : \pi_1(X) \cong G \text{ and let } C_\ast(\bar{X})_\rho \text{ be the corresponding chain complex of } ZG\text{-modules. If } f : G \to H \text{ is a quotient of groups, then } f_\#(C_\ast(\bar{X})_\rho) \text{ is a chain complex of free } ZH\text{-modules with boundary maps } id_{ZH} \otimes \partial_i, \text{ and } H_n(f_\#(C_\ast(\bar{X})_\rho)) \text{ is a } ZH\text{-module. We will denote this by } H_n(X;ZH) \text{ when } f \text{ and } \rho \text{ are understood.}\]

\[\text{Subject to conditions on } f, \text{ this give an additional homotopy invariant for } X.\]

\[\text{Proposition 6.4. If } f \text{ is characteristic, then the } Aut(H)\text{-isomorphism class of } H_n(X;ZH) \text{ is a homotopy invariant and is independent of the choice of } \rho.\]

\[\text{Proof. If } C_\ast(\bar{X})_\rho \cong C_\ast(\bar{Y})_\rho' \text{ are chain homotopic as chain complexes of } ZG\text{-modules, then } f_\#(C_\ast(\bar{X})_\rho) \cong f_\#(C_\ast(\bar{Y})_\rho') \text{ are chain homotopic as chain complexes of } ZH\text{-modules. Let } \theta \in Aut(G). \text{ Since } f \text{ is characteristic, Proposition 5.5 implies that } f_\#((C_\ast(\bar{X})_\rho)_\theta) \cong (f_\#(C_\ast(\bar{X})_\rho))_\theta \text{ for some } \theta \in Aut(H). \text{ The result now follows.}\]

7. Algebraic classification of finite \((G,n)\)-complexes

A \((G,n)\)-complex is an \(n\)-dimensional CW-complex \(X\) such that \(\pi_1(X) \cong G\) and the universal cover \(\tilde{X}\) is \((n-1)\)-connected. By contracting a maximal spanning tree, \(X\) is homotopy equivalent to a \((G,n)\)-complex with a single 0-cell. For convenience, we will now assume that a \((G,n)\)-complex has a single 0-cell which is the basepoint.

If \(i \geq 2\), then \(\pi_i(X) \cong \pi_i(\tilde{X})\) as abelian group. In this way, we can view \(\pi_i(X)\) as a \(ZG\)-module under the monodromy action. If \(2 \leq i < n\), then \(\pi_i(X) = 0\) since \(\tilde{X}\) is \((n-1)\)-connected. If \(i = n\), then the Hurewicz theorem implies that:

\[\pi_n(X) \cong H_n(\tilde{X};Z) \cong H_n(X;ZG)\]
as \(ZG\)-modules. In particular, Corollary 6.2 applies to \(\pi_n(X)\).

7.1. Algebraic \(n\)-complexes and the D2 problem. Let \(G\) be a group. An algebraic \(n\)-complex over \(ZG\) is an exact chain complex:

\[E = (F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} Z \to 0)\]

where the \(F_i\) are finitely generated stably free \(ZG\)-modules.

Let \(\text{Alg}(G,n)\) denote the equivalence classes of algebraic \(n\)-complexes over \(ZG\) up to chain homotopy equivalences of the unaugmented complex \((F_i, \partial_i)_{i=1}^n\). The \(n\)th homotopy group of \(E\) is the \(ZG\) module \(\pi_n(E) = \ker(\partial_n)\) and is an invariant of the chain homotopy class of \(E\). If \(n \geq 2\), we can assume the \(F_i\) are free since every algebraic \(n\)-complex is chain homotopy equivalent to such a complex.

Let \(\text{PHT}(G,n)\) denote the polarised homotopy types of finite \((G,n)\)-complexes, i.e. the homotopy types of pairs \((X, \rho)\) where \(\rho : \pi_1(X) \cong G\). If \((X, \rho) \in \text{PHT}(G,n)\), then \(C_\ast(\tilde{X})_\rho\) is a chain complex of \(ZG\)-modules such that \(H_0(C_\ast(\tilde{X})_\rho) \cong Z\) and \(H_i(C_\ast(\tilde{X})_\rho) = 0\) for \(1 \leq i < n\). In particular, there is a map:

\[\Psi : \text{PHT}(G,n) \to \text{Alg}(G,n).\]

Recall that a finitely presented group \(G\) has the \(D2\) property if every finite CW-complex \(X\) such that \(\pi_1(X) \cong G\), \(H_i(\tilde{X};Z) = 0\) for \(i > 2\) and \(H^{i+1}(X;M) = 0\) for all finitely generated \(ZG\)-modules \(M\) is homotopy equivalent to a finite 2-complex. The following is a mild improvement of Wall’s results on finiteness conditions for CW-complexes due to Johnson [39] and Mannan [55]. This precise version follows from [11 Corollary 8.27] in the case \(n \geq 3\) and [60 Theorem 2.1] in the case \(n = 2\).
Proposition 7.1. Let $G$ be a finitely presented group. If $n \geq 3$, then $\Psi$ is bijective. If $n = 2$, then $\Psi$ is injective and is bijective if and only if $G$ has the D2 property.

Remark 7.2. The first part is often vacuous since there are finitely presented groups $G$ for which no algebraic $n$-complex over $\mathbb{Z}G$ exists for all $n \geq 3$. The first example was found by Stallings in [65] (see also [10, Proposition 2.14]) and was later generalised to a class of right-angled Artin groups by Bestvina-Brady [9, Main Theorem].

7.2. Realising $\mathbb{Z}G$-modules by algebraic $n$-complexes. The $n$th stable syzygy $\Omega_n^G(\mathbb{Z})$ is the set of $\mathbb{Z}G$-modules $M$ for which $M \oplus \mathbb{Z}G^i \cong \pi_{n-1}(E) \oplus \mathbb{Z}G^j$ for some $i, j \geq 0$ and some algebraic $(n-1)$-complex $E$ over $\mathbb{Z}G$. We will denote this by $\Omega_n(\mathbb{Z})$ when the choice of $G$ is clear from the context. This is well-defined and does not depend on the choice of $E$ [41, Theorem 8.9]. It also comes with a map:

$$\pi_n : \text{Alg}(G, n) \to \Omega_{n+1}(\mathbb{Z}).$$

The following can be found in [41, Proposition 8.18].

Proposition 7.3. Let $n \geq 2$ and let $G$ be an infinite finitely presented group of type FL such that $H^{n+1}(G; \mathbb{Z}) = 0$. Then $\pi_n$ is bijective.

The following is a straightforward consequence of Propositions 4.4 and 7.3.

Proposition 7.4. Let $G$ be a finitely presented group of type FL with $\text{cd}(G) = d$.

(i) If $n \geq d$, then $\Omega_n(\mathbb{Z})$ is the set of stably free $\mathbb{Z}G$-modules

(ii) If $n \geq d$, then $\pi_n : \text{Alg}(G, n) \to \Omega_{n+1}(\mathbb{Z})$ is bijective

(iii) If $n = d - 1$, then $0 \not\in \text{Im}(\pi_n : \text{Alg}(G, n) \to \Omega_{n+1}(\mathbb{Z}))$.

Remark 7.5. This implies that, for $n \geq 2$, $\pi_n : \text{Alg}(G, n) \to \Omega_{n+1}(\mathbb{Z})$ is not surjective whenever $\text{cd}(G) = n + 1$ (for example, $G = \mathbb{Z}^{n+1}$). This was noted in [41, p107].

It is possible to see that $\text{cd}(G) \leq n$ implies that $\pi_n : \text{Alg}(G, n) \to \Omega_{n+1}(\mathbb{Z})$ is surjective directly (see, for example, [33, Theorem 4]). The following is now clear.

Corollary 7.6. Let $n \geq 3$ and let $G$ be a finitely presented group of type FL with $\text{cd}(G) = n$. Then $\pi_n$ gives a one-to-one correspondence between homotopy types of finite $(G, n)$-complexes and $\text{Aut}(G)$-isomorphism classes of stably free $\mathbb{Z}G$-modules.

Finally, we note the following where $\text{rank}(P)$ denotes the stably free rank of $P$.

Proposition 7.7. Let $G$ be a finitely presented group of type FL with $\text{cd}(G) = d$ and let $n \geq d - 1$. Then $\chi(X) = k + \chi_{\min}(G, n)$ if and only if:

$$\text{rank}(\pi_n(X)) = k + \min\{\text{rank}(\pi_n(X_0)) : X_0 \text{ a finite } (G, n)\text{-complex}\}.$$ 

In particular, if $n \geq \max\{0, d\}$, then $k = \text{rank}(\pi_n(X))$.

8. Proof of Theorem B

We will now prove Theorem B separately in the two cases of non-minimal Euler characteristic ($k \geq 1$) and minimal Euler characteristic ($k = 0$). Throughout, $T_i \cong T$ will denote the trefoil group and $G_{(n)}$ will be as defined in Section 4.

8.1. Finite $(G, n)$-complexes with non-minimal Euler characteristic. The aim of this section will be to prove the following. Note that, in the case $n \geq 3$, we could also take $G$ to be one of the other groups listed at the end of Section 5.

Theorem 8.1. Let $n \geq 2$, let $k \geq 1$ and let $G = (T_1 \ast \cdots \ast T_k)_{(n-1)}$. Then, for all $1 \leq m \leq k$, there exists infinitely many finite $(G, n)$-complexes $\hat{X}_i$ such that:

(i) $\pi_n(\hat{X}_i) \cong \hat{S}_i$ as $\mathbb{Z}G$-modules (where $\hat{S}_i$ is as defined in Theorem 5.3)

(ii) $\chi(\hat{X}_i) = m + \chi_{\min}(G, n)$
(iii) $\hat{X}_i \not\cong Y_i \vee S^2$ for any finite $(G, n)$-complex $Y_i$.

Since the Aut($G$)-isomorphism class of $\pi_n(\hat{X}_i)$ is a homotopy invariant, it follows that the $\hat{X}_i$ are homotopically distinct by Theorem 8.3. By restricting to the case $m = k$, this implies Theorem 8.3 for $k \geq 1$.

We will begin with the case $n = 2$, where $G = T_1 \ast \cdots \ast T_k$. Let $S_i$ be the stably free $ZT$-modules from Theorem 5.1 and, for $1 \leq m \leq k$, recall that:

$$S_{i_1, \ldots, i_m} = i_1 \#(S_{i_1}) \oplus \cdots \oplus i_m \#(S_{i_m}).$$

The case of interest will be $\hat{S}_i = S_{i_1, \ldots, i_m}$ where $i_j = i$ for all $j$.

The main result which we will use is the following, which is [32] Theorem 4.5.

**Theorem 8.2** (Harlander-Jensen). The trefoil group $T$ has presentations

$$P_i = \langle x, y, a, b \mid x^2 = y^3, a^2 = b^3, x^{2i+1} = a^{2i+1}, y^{3i+1} = b^{3i+1} \rangle$$

for $i \geq 0$. For each $i$, there exists $\ell_i$ such that $S_i \cong \pi_2(X_{P_{\ell_i}})$ as $ZT$-modules.

**Remark 8.3.** Note that $P_0 \cong \langle x, y \mid x^2 = y^3, 1 \rangle$ and $P_1$ is homotopy equivalent to the presentation found by Dunwoody in [20].

Let $X_i = X_{P_{\ell_i}}$ for each $i \geq 1$. For integers $i_j \geq 1$, define:

$$X_{i_1, \ldots, i_n} = X_{i_1} \vee \cdots \vee X_{i_n}$$

which is a finite 2-complex with $\pi_1(X_{i_1, \ldots, i_n}) \cong T_1 \ast \cdots \ast T_k$. Let $\hat{X}_i = X_{i_1, \ldots, i_n}$ where $i_j = i$ for all $j$. By repeated application of Corollary 8.2 we have that $\pi_2(X_{i_1, \ldots, i_n}) \cong S_{i_1, \ldots, i_n}$ and so $\pi_2(\hat{X}_i) \cong \hat{S}_i$.

Since $\text{rank}(\hat{S}_i) = m$, we have that $\chi(\hat{X}_i) = m + \chi_{\text{min}}(G)$ by Proposition 7.7. Finally, if $\hat{X}_i \cong Y_i \vee S^2$, then:

$$\hat{S}_i \cong \pi_2(\hat{X}_i) \cong \pi_2(Y_i) \oplus (ZG \otimes Z \pi_2(S^2)) \cong \pi_2(Y_i) \oplus ZG$$

which is a contradiction since $\hat{S}_i$ has no summand of the form $ZG$ by Theorem 5.3. This completes the proof of Theorem 8.1 in the case $n = 2$.

We will now consider the case $n \geq 3$, where $G = (T_1 \ast \cdots \ast T_k)(n-1)$. By Theorem 5.3 there exists stably free $ZG$-modules $\hat{S}_i$ of rank $m$ and which have no summand of the form $ZG$. By Proposition 4.4 we have that $\text{cd}(G) = n$ and so, by Corollary 7.6, there exists finite $(G, n)$-complexes $\hat{X}_i$ such that $\pi_n(\hat{X}_i) \cong \hat{S}_i$. We can now argue similarly to the case $n = 2$. This completes the proof of Theorem 8.1.

**8.1.1. Application to Syzygies.** We now discuss consequences of Theorem 8.1 for syzygies. Recall that a $ZG$-module $M_0 \in \Omega_n(Z)$ is minimal if $M \in \Omega_n(Z)$ implies that $M \oplus ZG^i \cong M_0 \oplus ZG^j$ for some $i \leq j$. For $k \geq 0$, we say that $M \in \Omega_n(Z)$ has level $k$ if $M \oplus ZG^i \cong M_0 \oplus ZG^j$ where $j = i + 1$ and $M_0$ is minimal. If $X$ is a finite $(G, n)$-complex, then $\pi_n(X) \in \Omega_n(Z)$.

**Corollary 8.4.** For all $n \geq 3$ and $k \geq 1$, there exists a group $G$ and infinitely many $ZG$-modules $M_i \in \Omega_n(Z)$ at level $k$ which are distinct up to Aut($G$)-isomorphism.

8.1.2. $F$-homotopy. For a field $F$, an $F$-homotopy equivalence is a map $f : X \to Y$ such that $\pi_1(f)$ is a group isomorphism and $\pi_i(f) \otimes F : H_i(X) \otimes F \to H_i(Y) \otimes F$ is bijective for $i \geq 2$. If $F = \mathbb{F}_p$ or $\mathbb{Q}$ then, similarly to Theorem 5.6, we can show:

**Theorem 8.5.** If $n \geq 2$, $k \geq 1$ and $G = (T_1 \ast \cdots \ast T_k)(n-1)$, then there exists $F$-homotopically distinct finite $(G, n)$-complexes $X_1$, $X_2$ with $\chi(X_i) = k + \chi_{\text{min}}(G, n)$.

8.2. Finite $(G, n)$-complexes with minimal Euler characteristic. The following is the main result of [33].

**Theorem 8.6** (Lustig). Let $G = T_2$. Then there exists infinitely many homotopically distinct finite 2-complexes $X_i$ for $i \geq 1$ such that $\pi_1(X_i) \cong G$ and $\chi(X_i) = 1$. 


The aim of this section will be to give the following generalisation of this result:

**Theorem 8.7.** Let \( n \geq 2 \) and let \( G = T(n) \). Then there exists infinitely many finite \((G,n)\)-complexes \( \tilde{X}_i \) for \( i \geq 1 \) such that:

(i) \( H_n(\tilde{X}_i; ZT) \cong S_i \) as \( ZT \)-modules (where \( S_i \) is as defined in Theorem 8.1)

(ii) \( \chi(\tilde{X}_i) = \chi_{\min}(G,n) \)

**Remark 8.8.** This corrects a statement made in [33 Section 5] where it was suggested that \( \chi(X_i) = 1 + \chi_{\min}(T(2)) \). In fact, we have \( \chi(\tilde{X}_i) = \chi_{\min}(T(n),n) = 1 - n \).

By Proposition 6.1 the \( \text{Aut}(T) \)-isomorphism class of \( H_n(\tilde{X}_i; ZT) \) is a homotopy invariant and so the \( \tilde{X}_i \) are homotopically distinct by Theorem 5.1. Hence this implies Theorem 8.1 in the case \( k = 0 \).

We will begin with the following lemma, which can be verified directly.

**Lemma 8.9.** Let \( n \geq 2 \), let \( G \) be a group and let \( E = (ZG^d_i, \delta_j)_{i=1}^n \in \text{Alg}(G,n) \). If \( G_+ = (G \times \langle q \mid - \rangle) *_{(q=r^2)} \langle r \mid - \rangle \), then:

\[
E_+ = (ZG^d_+ \xrightarrow{\delta^+_1} \cdots \xrightarrow{\delta^+_n} ZG^d_{n+1} \xrightarrow{\delta^+_1} ZG_+ \xrightarrow{\epsilon_G} Z \to 0) \in \text{Alg}(G_+, n+1)
\]

where \( \delta^+_1 = (d_1, \cdots, d_n) \), \( \delta^+_2 = (d_2, \cdots, d_n) \) and \( \delta^+_i = (d_1, \cdots, d_{i-1} \cdots, d_{i+1}, \cdots, d_n) \) for \( i \geq 3 \). The \( \delta^+ \) are the induced maps and we take \( \delta^+_{n+1} = 0, \delta^+_{n+1} = 0 \).

**Remark 8.10.** This also works when \( G_+ = (G \times \langle q \mid - \rangle) *_{(q=r^m)} \langle r \mid - \rangle \) for \( m \geq 2 \).

Let \( \mathcal{P} = \langle x, y \mid x^2 = y^2 \rangle \) be the standard presentation of \( T \) and note that:

\[
C_\ast(\tilde{X}_P) \cong (ZT \xrightarrow{\partial_1} ZT^2 \xrightarrow{\partial_1} ZT \xrightarrow{\epsilon_T} Z \to 0) \in \text{Alg}(T, 2)
\]

where \( \partial_1 = (x, y-1) \) and \( \partial_2 = (x+1, -y^2+y+1) \). This has \( \pi_2(C_\ast(\tilde{X}_P)) = 0 \).

For each \( n \geq 1 \), define \( \tilde{E}_n \in \text{Alg}(T(n), n+1) \) by \( \tilde{E}_1 = C_\ast(\tilde{X}_P) \) and \( \tilde{E}_n = (\tilde{E}_{n-1})_+ \) for \( n \geq 2 \) using Lemma 8.9. Let \( E_n \in \text{Alg}(T(n), n) \) denote the restriction to the first \( n+1 \) terms in \( \tilde{E}_n \). Note that \( \pi_{n+1}(\tilde{E}_n) = 0 \) and so \( \pi_n(E_n) = \text{Im}(\partial_{E_{n+1}}) \cong ZT(n) \).

For \( n \geq 2 \), let \( \Delta_n = \partial_{E_n}^n \) denote the final boundary map in \( E_n \), so that:

\[
\Delta_1 = \partial_1 \cdot (r_1+1), \quad \Delta_n = \partial_n \cdot \left( \begin{array}{cc}
\overbrace{v_n} & 0 \\
\overbrace{0} & -\Delta_{n-1}
\end{array} \right) : \text{ZT}^{n+1}(n) \to \text{ZT}^{n+1}(n)
\]

where \( v_n = (r_n-2, -1, -1, \cdots, -1, -1, -1) \). Here \( \Delta_1 \) is defined for the purposes of this definition and does not coincide with \( \partial_{E_1} = \partial_1 \).

Let \( \alpha_n, \beta_n \) denote the last two row vectors in \( \Delta_n \), which are defined by:

\[
\alpha_1 = (x-1)(r_1+1), \quad \beta_1 = (y-1)(r_1+1)
\]

\[
\alpha_n = (0, \cdots, 0, r_n-1, 0, -\alpha_{n-1}), \quad \beta_n = (0, \cdots, 0, r_n-1, 1, -\beta_{n-1})
\]

For each \( i \geq 0 \), let \( \alpha_n^{(i)} = \Sigma x \alpha_n, \beta_n^{(i)} = \Sigma y \beta_n \) where \( \Sigma x = \sum_{j=0}^{2i} x^j, \Sigma y = \sum_{j=0}^{3i} y^j \).

We will now show that, where we adopt the notation of Section 6.2.

**Proposition 8.11.** For \( n \geq 2 \), let \( \Delta_n^{(i)} \) be the matrix \( \Delta_n \) but with \( \alpha_n, \beta_n \) replaced by \( \alpha_n^{(i)}, \beta_n^{(i)} \), and let \( E_n^{(i)} \) be the resolution \( E_n \) but with \( \Delta_n \) replaced by \( \Delta_n^{(i)} \). Then:

(i) \( E_n^{(i)} \in \text{Alg}(T(n), n) \)

(ii) \( \text{if } f : T(n) \to T \text{, then } H_n(E_n^{(i)}; ZT) \cong \text{Ker}(\cdot \left( \begin{array}{cc}
\frac{x-1}{(r_n-1)\Sigma_x} & -\frac{y^2+y+1}{(r_n-1)\Sigma_y} \\
\frac{0}{(r_n-1)\Sigma_x} & \frac{(1-x^2+y^2)(r_n+1)}{(y-1)(r_n-1)}
\end{array} \right) \) as \( ZT \)-modules.

For the convenience of the reader, we will write this explicitly in the case \( n = 2 \):

\[
E_2^{(i)} = (ZT^3(2) \xrightarrow{(x+1, y-1)} \text{Ker}(\cdot \left( \begin{array}{cc}
\frac{x-1}{(r_n-1)\Sigma_x} & -\frac{y^2+y+1}{(r_n-1)\Sigma_y} \\
\frac{0}{(r_n-1)\Sigma_x} & \frac{(1-x^2+y^2)(r_n+1)}{(y-1)(r_n-1)}
\end{array} \right) \) as \( ZT \)-modules.
\]

\[
E_2^{(i)} = (ZT^3(2) \xrightarrow{(x+1, y-1)} \text{Ker}(\cdot \left( \begin{array}{cc}
\frac{x-1}{(r_n-1)\Sigma_x} & -\frac{y^2+y+1}{(r_n-1)\Sigma_y} \\
\frac{0}{(r_n-1)\Sigma_x} & \frac{(1-x^2+y^2)(r_n+1)}{(y-1)(r_n-1)}
\end{array} \right) \) as \( ZT \)-modules.
\]
In order to prove this, we will first need the following technical lemma.

**Lemma 8.12.** Let $G$ be a group with $T \subseteq G$. For $i = 1, 2, 3$, there exists $\lambda_i, \mu_i \in \mathbb{Z}T \subseteq \mathbb{Z}G$ such that, for all $r \in G$, we have:

\[
(r - 1, 0, 1 - x) = \lambda_1 \cdot (\Sigma_x(r - 1), 0, 1 - x^{2i+1}) + \lambda_2 \cdot (0, \Sigma_y(r - 1), 1 - y^{3i+1}) + \lambda_3 \cdot (\partial_2, 0) \cdot (r - 1) \\
(0, r - 1, 1 - y) = \mu_1 \cdot (\Sigma_x(r - 1), 0, 1 - x^{2i+1}) + \mu_2 \cdot (0, \Sigma_y(r - 1), 1 - y^{3i+1}) + \mu_3 \cdot (\partial_2, 0) \cdot (r - 1)
\]

**Proof of Proposition 8.7.** To prove (i), it suffices to show that $\text{Im}(\Delta^{(i)}_T) = \text{Im}(\Delta_n)$ for $i \geq 1$. We have $\text{Im}(\Delta^{(i)}_T) \subseteq \text{Im}(\Delta_n)$, so it remains to show $\alpha_n, \beta_n \in \text{Im}(\Delta^{(i)}_T)$.

By the proof of Lemma 8.12 we have $\mathbb{Z}T \cdot \{x - 1, y - 1\} = \mathbb{Z}T \cdot \{x^{2i+1} - 1, y^{3i+1} - 1\}$. It follows that $\mathbb{Z}T \cdot \{\alpha_1, \alpha_2\} = \mathbb{Z}T \cdot \{\alpha_1^{(i)}, \beta_1^{(i)}\}$ which implies that $\alpha_1, \beta_1 \in \text{Im}(\Delta^{(i)}_T)$. The case $n = 2$ is done in Lemma 8.12 which provides $\lambda_i$ such that:

\[
\alpha_2 = \lambda_1 \cdot \alpha_2^{(i)} + \lambda_2 \cdot \beta_2^{(i)} + \lambda_3 (r_2^2 - 1) \cdot (\partial_2, 0)
\]

and similarly for $\mu_i$ and $\beta_2$. Let $\gamma_1, \ldots, \gamma_n$ denote the first $n - 1$ rows of $\Delta_n$, the remaining two rows being $\alpha_n, \beta_n$. It is now straightforward to see that:

\[
\alpha_n = \lambda_1 \cdot \alpha_n^{(i)} + \lambda_2 \cdot \beta_n^{(i)} + \lambda_3 (-1)^n (r_n^2 - 1) \cdot \gamma_n + \sum_{i=2}^{n-1} (-1)^i (r_n^2 - 1) \cdot \gamma_i
\]

for $n \geq 2$, and similarly for $\beta_n$. Hence $\alpha_n, \beta_n \in \text{Im}(\Delta^{(i)}_T)$ for all $n \geq 2$.

To prove (ii), note that $H_n(E_n^{(i)}; \mathbb{Z}T) = \text{Ker}(f_\#(\Delta^{(i)}_T))$. For each $n \geq 2$, we have:

\[
f_\#(\Delta^{(i)}_T) = \begin{pmatrix} f_\#(v_n) & 0 \\ 0 & -f_\#(\Delta^{(i)}_{n-1}) \end{pmatrix}.
\]

Since $f_\#(v_n) = (0, \ldots, 0, (-1)^{n-2}\partial_2)$ is injective, this implies that $\text{Ker}(f_\#(\Delta^{(i)}_T)) = \text{Ker}(-f_\#(\Delta^{(i)}_{n-1}))$ and so, by induction:

\[
\text{Ker}(f_\#(\Delta^{(i)}_T)) \cong \text{Ker}(f_\#(\Delta^{(i)}_1)) = \text{Ker}(\left( \begin{pmatrix} 2(x^{2i+1} - 1) \\ 2(y^{3i+1} - 1) \end{pmatrix} \right)) = \text{Ker}(\left( \begin{pmatrix} x^{2i+1} - 1 \\ y^{3i+1} - 1 \end{pmatrix} \right)). \quad \square
\]

Let $G = T(n)$. For each $i \geq 1$, there exists $\ell_i$ such that $\text{Ker}(\left( \begin{pmatrix} x^{2\ell_i+1} - 1 \\ y^{3\ell_i+1} - 1 \end{pmatrix} \right)) \cong S_i$ where the $S_i$ are as defined in the discussion following Theorem 7.4.

If $n \geq 3$, then Proposition 7.1 implies that there exists finite $(G, n)$-complexes $\hat{X}_i$ such that $C_*(X) \cong E_n^{(\ell_i)}$ are chain homotopy equivalent where $X$ is the universal cover of $\hat{X}_i$. This is also true when $n = 2$ by taking $\hat{X}_i = X_i = \mathcal{P}_i$, where:

\[
\mathcal{P}_i = \langle a, b, c \mid a^2 = b^3, [a^2, b^{2i+1}], [a^2, c^{3i+1}] \rangle
\]

are the presentations given by Lustig in [53].

By Proposition 8.11 $H_n(X_i; \mathbb{Z}T) \cong S_i$ as $\mathbb{Z}T$-modules. It is straightforward to see that:

\[
\text{rank}(\pi_n(E_n^{(\ell_i)})) = \text{rank}(\pi_n(E_n)) = 1.
\]

By Proposition 3.5 $\text{cd}(G) = n + 1$ and so $0 \notin \text{Im}(\pi_n : \text{PHT}(G, n) \to \Omega_{n+1}(\mathbb{Z}))$ by Proposition 7.4. Hence, by Proposition 7.4 we have $\chi(X_i) = \chi_{\min}(G, n)$. This completes the proof of Theorem 8.7. By combining with Theorem 8.11 this completes the proof of Theorem [87].

9. **Proof of Theorem [C]**

The aim of this section will be to prove the following theorem which its Theorems [C] We will also give an application of this to the construction of non-finite $(G, n)$-complexes. The proofs are similar to that of Theorems [A] and [B] and so many of the details will be omitted.

We will let $T$ denote the trefoil group.

**Theorem 9.1.** Let $d \geq 2$ and let $G = \ast_{i=1}^\infty T_{i(d-1)}$. Then $\text{cd}(G) = d$ and, for all $k \geq 1$, there exists infinitely many stably free $\mathbb{Z}G$-modules of rank $k$ which are distinct up to $\text{Aut}(G)$-isomorphism.
Let $S_i$ denote the stably free $\Z T$-modules of Theorem 5.1 and let $\iota_j : T_j \hookrightarrow G$.

Proof. Let $k \geq 1$ and let $\hat{S}^{(k)}_i = \bigoplus_{j=1}^k \iota_{j,\#}(S_i)$ for $i \geq 1$. Since $\hat{S}^{(k)}_i \oplus \Z G \cong \Z G^{k+1}$, the $\hat{S}^{(k)}_i$ are stably free $\Z G$-modules of rank $k$. Let $f : G \twoheadrightarrow \ast_{j=1}^k T_j / T''_j$ be induced by the characteristic quotients $f_j : (T_j)/(d-1) \twoheadrightarrow T_j / T''_j$. This is characteristic by a mild generalisation of Proposition 5.6 which applies since $T_j$ is finitely generated.

For $p$ prime, we have that $F_p \otimes f_\#(\hat{S}^{(k)}_i) \cong \bigoplus_{j=1}^k \iota_{j,\#}(F_p \otimes f_\#(S_i))$ where $\iota_j : T_j / T''_j \hookrightarrow \ast_{j=1}^k T_j / T''_j$ is the inclusion map. Similarly to the proof of Theorem 5.3 there exists primes $p_i$ for $i \geq 1$ such that $F_p \otimes f_\#(S_i) \cong F_{p_i}[T_j / T''_j]$ if and only if $i = j$. Since Theorem 5.3 and Corollary 5.4 also holds for infinite free products (see [7]), we get the $F_p \otimes f_\#(\hat{S}^{(k)}_i)$ are distinct up to $\Aut(G)$-isomorphism. Since $f$ is characteristic, the $\hat{S}^{(k)}_i$ are distinct up to $\Aut(G)$-isomorphism also.

Theorem 9.2. Let $n \geq 2$ and let $G = \ast_{i=1}^\infty T_{n-1}$. Then there exists an aspherical $(G,n)$-complex $Y$ such that, for all $k \geq 1$, there are infinitely many homotopically distinct $(G,n)$-complexes $X_i$ with $X_i \vee S^n \cong Y \vee (k+1)S^n$.

Proof. By Lemma 8.9 there exists $\hat{E}_{n-1} \in \Alg(T_{n-1}, n)$ with $\pi_n(\hat{E}_{n-1}) = 0$. If $n \geq 3$, then Proposition 7.1 implies that there exists a finite $(G,n)$-complex $Y_0$ such that $C_\ast(Y_0) \cong \hat{E}_{n-1}$ are chain homotopy equivalent. This is also true when $n = 2$ by taking $Y_0 = X_\mathcal{P}$ where $\mathcal{P} = \langle x, y \mid x^2 = y^3 \rangle$ is the standard presentation for $T$. Hence, for all $n \geq 2$, $Y = \vee_{i=1}^\infty Y_0$ is an aspherical $(G,n)$-complex.

For all $i \geq 1$, let $X_i = \bigvee_{j=1}^k \tilde{X}_i \vee \bigvee_{j=k+1}^\infty Y_0$ where the $\tilde{X}_i$ are the finite $(T_{n-1}, n)$-complexes such that $\pi_n(\tilde{X}_i) \cong S_i$ which were constructed in Theorem 8.1. Then $X_i$ is a $(G,n)$-complex such that:

$$\pi_n(X_i) \cong \bigoplus_{j=1}^k \iota_{j,\#}(\pi_n(\tilde{X}_i)) \oplus \bigoplus_{j=k+1}^\infty \iota_{j,\#}(\pi_n(Y)) \cong \bigoplus_{j=1}^k \iota_{j,\#}(S_i) = \hat{S}^{(k)}_i.$$ 

Since the $\hat{S}^{(k)}_i$ are distinct up to $\Aut(G)$-isomorphism, this implies that the $X_i$ are homotopically distinct. By Theorem 5.1 and Proposition 7.1 we have that $\tilde{X}_i \vee S^n \cong Y_0 \vee 2S^n$. It follows that $X_i \vee S^n \cong Y \vee (k+1)S^n$, as required.

10. Some remarks on induced module decompositions

Recall that Theorems [A] and [B] concerned stably free $\Z G$-modules and finite 2-complexes $X$ with $\pi_1(X) \cong G$ where $G = \ast_{i=1}^k G_i$. In our example, $\pi_2(X) \otimes \Z_p$ was an induced $\Z_p G$-module whose component $\Z_p T$-modules $M_i$ were unique up to $\Z_p T$-isomorphism where $G_i = T$ is the trefoil group.

The aim of this section will be to investigate the extent to which this applies to all groups of the form $G = \ast_{i=1}^k G_i$ and to $\pi_2(X)$ rather than just $\pi_2(X) \otimes \Z$. For simplicity, we will restrict to the case of 2-complexes. However, all results have analogues for $(G,n)$-complexes for $n \geq 3$. The main result is Theorem 10.3 which was stated in the introduction. Part (i) will be proven as Theorem 10.2 and part (ii) will be proven as Theorem 10.4.

10.1. Existence of induced module decompositions. We will begin by considering the question of existence. From now on, we will take $\Z$ to be a field.

Proposition 10.1 (Existence over $\Z[G_1 \ast \cdots \ast G_k]$). Let $X$ be a finite 2-complex with $\pi_1(X) \cong G_1 \ast \cdots \ast G_k$. Then $\pi_2(X) \otimes \Z$ is an induced $\Z[G_1 \ast \cdots \ast G_k]$-module.

Proof. Let $X_i$ be a finite 2-complex with $\pi_1(X_i) \cong G_i$. Then $\pi_1(\vee_{i=1}^k X_i) \cong \ast_{i=1}^k G_i$ and so there exists $a, b \geq 0$ such that $X \vee aS^2 \cong \vee_{i=1}^k X_i \vee bS^2$. This implies that

$$(\pi_2(X) \otimes \Z) \oplus FG^a \cong \iota_1(\pi_2(X) \otimes \Z) \oplus \bigoplus_{j=1}^k \iota_{j,\#}(\pi_2(X_j) \otimes \Z)$$

and so $\pi_2(X) \otimes \Z$ is a submodule of an induced $\Z[G_1 \ast \cdots \ast G_k]$-module. Hence, by Theorem 5.3 $\pi_2(X) \otimes \Z$ is an induced $\Z[G_1 \ast \cdots \ast G_k]$-module.

□
Theorem 10.2 (Non-existence over \( \mathbb{Z}[G_1 \ast \cdots \ast G_k] \)). For all \( k \geq 2 \), there exists a finite 2-complex \( X \) with \( \pi_1(X) \cong G_1 \ast \cdots \ast G_k \) such that \( \pi_2(X) \) is not an induced \( \mathbb{Z}[G_1 \ast \cdots \ast G_k] \)-module.

In order to prove this, we will need the following method of proving that presentation complexes are homotopy equivalent. If \( \mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle \), then an elementary transformation on \( \mathcal{P} \) is an operation that replaces a relator \( r_i \) with:

1. \( \omega r_i \omega^{-1} \) for a word \( \omega \in F(x_1 \cdots, x_n) \) (conjugation)
2. \( r_i^{-1} \) (inversion)
3. \( r_ir_j \) or \( r_jr_i \) for some \( j \neq i \) (left or right multiplication).

We say that two group presentations \( \mathcal{P} \) and \( \mathcal{Q} \) are \( Q \)-equivalent if they are related by a sequence of elementary transformations. If \( \mathcal{P} \) and \( \mathcal{Q} \) are \( Q \)-equivalent, then \( X_\mathcal{P} \) and \( X_\mathcal{Q} \) are (simple) homotopy equivalent.

We begin by noting the following, which is a generalisation of [34] Theorem 3.

Proposition 10.3. Let \( k \geq 1 \) and let \( m_i, n_i \geq 1 \) for \( i = 1, \ldots, k \). Suppose there exists integers \( r_i, q_i \) such that \( (q_i, q_j) = 1 \) for all \( i \neq j \) and, for all \( i \), we have:

\[
r_i m_i - 1 = n_i q_i, \quad r_i \equiv 1 \mod{n_i}, \quad (m_i, n_i) \neq 1.
\]

Then \( G = \left< \frac{1}{k=1} (\mathbb{Z}/m_i \times \mathbb{Z}/n_i) \right> \) has a presentation

\[
\mathcal{P} = \langle a_1, b_1, \ldots, a_k, b_k | a_1 m_1, \ldots, a_k m_k, a_1 b_1 a_1^{-1} b_1^{-1}, \ldots, a_k b_k a_k^{-1} b_k^{-1}, b_1^{n_1}, \ldots, b_k^{n_k} \rangle
\]

of deficiency \(-1\). Furthermore, if \( \mathcal{P}_i = \langle a, b | a^n, b^m, [a, b] \rangle \) is the standard presentation for \( \mathbb{Z}/m_i \times \mathbb{Z}/n_i \), then \( X_\mathcal{P} \) and \( \cup_{i=1}^{k} X_{\mathcal{P}_i} \) are \( Q \)-equivalent. To see this, note that we can replace \( b_1^{n_1} \cdots b_k^{n_k} \) by left-multiplying by \( b_i^{-n_i} \) for \( 1 \leq i \leq k \). Since \( r_i \equiv 1 \mod{n_i} \), we can then replace \( a_i b_i a_i^{-1} b_i^{-1} \) by successively right-multiplying by \( b_i^{n_i} \).

We say that two \( \mathbb{Z}G \)-modules \( M \) and \( M' \) are \textit{stably isomorphic}, written \( M \cong_s M' \), if there exists \( a, b \geq 0 \) such that \( M \oplus \mathbb{Z}G^a \cong M' \oplus \mathbb{Z}G^b \).

Lemma 10.4. For \( 1 \leq i \leq k \), let \( M_i, M'_i \) be finitely generated \( \mathbb{Z}G_i \)-lattices such that

\[
t_{i1}(M_i) \oplus \cdots \oplus t_{ik}(M_k) \cong t_{i1}(M'_i) \oplus \cdots \oplus t_{ik}(M'_k)
\]

as \( \mathbb{Z}[G_1 \ast \cdots \ast G_k] \)-modules. Then \( M_i \cong_s M'_i \) for all \( 1 \leq i \leq k \).

Proof. For \( 1 \leq i \leq k \), let \( q_i : G_1 \ast \cdots \ast G_k \to G_i \) be the projection map. By applying \((q_i)_\#\) to the given isomorphism of \( \mathbb{Z}[G_1 \ast G_2] \)-modules, we get that

\[
M_i \bigoplus_{j=2}^{k} (q_i \circ t_{ij})_\#(M_j) \cong M'_i \bigoplus_{j=2}^{k} (q_i \circ t_{ij})_\#(M'_j)
\]

as \( \mathbb{Z}G \)-modules. If \( j \neq i \), then \( q_i \circ t_{ij} : G_j \to G_i, g \mapsto 1 \). If \( M \) is a finitely generated \( \mathbb{Z}G_j \)-module, then \((q_i \circ t_{ij})_\#(M) \cong ZG_1 \otimes \mathbb{Z}(Z \otimes ZG_j, M) \). If \( \mathbb{Z} \otimes \mathbb{Z}G_j, M \cong \mathbb{Z}^{rM} \oplus F_M \) for \( F_M \) a finite abelian group and \( r_M \geq 0 \), then \((q_i \circ t_{ij})_\#(M) \cong ZG_1^{rM} \oplus F_M G_1 \).

In particular, for some finite abelian groups \( F, F' \) and some \( r, r' \geq 0 \), we have \( M_1 \oplus ZG'_1 \oplus FG_1 \cong M'_1 \oplus ZG_1^{r'} \oplus F'G_1 \). Since \( M_1, M'_1 \) are \( \mathbb{Z}G_1 \)-lattices, this \( ZG_1 \)-isomorphism must induce isomorphisms \( FG_1 \cong F'G_2 \) and \( M_1 \oplus ZG'_1 \cong M'_1 \oplus ZG'_1 \). Hence \( M_i \cong_s M'_i \) and, by symmetry, we have that \( M_i \cong_s M'_i \) for all \( 1 \leq i \leq k \).

\]
Proof of Theorem 10.2. Let $p_1, \ldots, p_k$ be distinct primes and let $G = \prod_{i=1}^{k} \mathbb{Z}/p_i \mathbb{Z}$. By Proposition 10.3, $G$ has a presentation $\mathcal{P}$ of deficiency $-1$. We claim that $\pi_2(X_\mathcal{P})$ is not an induced $\mathbb{Z}[G_1, \ldots, G_k]$-module, where $G_i = \mathbb{Z}/p_i \mathbb{Z}$ for all $i$.

Suppose that $\pi_2(X_\mathcal{P}) = \pi_1(M_1) \oplus \cdots \oplus \pi_k(M_k)$ for $\mathbb{Z}G_i$-modules $M_i$. Again by Proposition 10.3, we have that $X_\mathcal{P} \vee (k-1)S^2 \simeq X_{P_1} \vee \cdots \vee X_{P_k}$ where the $P_i = \langle a, b \mid a^{p_i}, b^{p_i}, [a, b] \rangle$ are the standard presentations for $G_i$. Hence, we have:

$$t_1\#(M_1 \oplus \mathbb{Z}G_i^{k-1}) \oplus \bigoplus_{j=2}^{k} t_j\#(M_j) \cong \bigoplus_{j=1}^{k} t_j\#(\pi_2(X_\mathcal{P})).$$

By Lemma 10.4, this implies that $M_i \cong \pi_2(X_{P_i})$ for all $i$ and so $M_i \cong \mathbb{Z}G_i^r$ for some integers $r_i \leq s_i$. This gives that:

$$\pi_2(X_\mathcal{P}) \oplus \mathbb{Z}G_i^{s_i} \oplus \cdots \oplus \mathbb{Z}G_i^{k-1} \cong \pi_2(X_\mathcal{P}) \oplus \mathbb{Z}G_i^{r_i} \oplus \cdots \oplus \mathbb{Z}G_i^{s_i}.$$

By Proposition 2.1, $\sum s_i + k - 1 = \sum r_i \leq \sum s_i$ which is a contradiction. \qed

10.2. Uniqueness of induced module decompositions. We will now turn to the question of uniqueness. The following is an immediate consequence of Corollary 3.4.

Proposition 10.5 (Uniqueness over $\mathbb{F}[G_1, \ldots, G_k]$). Let $X$ be a finite $2$-complex with $\pi_1(X) = G_1 \ast \cdots \ast G_k$. If $\pi_2(X) \otimes \mathbb{F} \cong \pi_2(Y_1) \otimes \mathbb{F} \oplus \cdots \oplus \mathbb{F} \oplus \pi_2(Y_k)$ for $\mathbb{F}G_i$-modules $M_i$ such that $\mathbb{F}G_i \cong M_i$, then the $M_i$ are unique up to $\mathbb{F}G_i$-module isomorphism.

Theorem 10.6 (Non-uniqueness over $\mathbb{Z}[G_1, \ldots, G_k]$). For all $k \geq 2$, there exists infinite $n$-complexes $X_i$, $Y_i$ with $\pi_1(X_i) = \pi_1(Y_i) \cong G_i$ for $1 \leq i \leq k$ such that $\pi_2(X_1 \vee \cdots \vee X_k) \ncong \pi_2(Y_1 \vee \cdots \vee Y_k)$ but, for all $i$, $\mathbb{Z}G_i \ncong \pi_2(X_i)$, $\pi_2(Y_i)$ and $\pi_2(X_i) \not\cong \pi_2(Y_i)$ are not $\text{Aut}(G_i)$-isomorphic.

Remark 10.7. Note that this implies Theorem 3.3 (ii) since it implies that:

$$\pi_2(X_1 \cup \cdots \cup X_k) \cong t_1\#(\pi_2(X_1)) \oplus \cdots \oplus t_k\#(\pi_2(X_k)) \cong t_1\#(\pi_2(Y_1)) \oplus \cdots \oplus t_k\#(\pi_2(Y_k)).$$

In order to prove this, we will begin by proving the following. We note that this holds for a larger class of abelian groups than elementary abelian $p$-groups.

Proposition 10.8. Let $k \geq 2$ and let $p_i$ be distinct primes and $n_i \geq 1$ for $i = 1, \ldots, k$. If $\mathcal{P}_i$, $\mathcal{P}_i'$ are two presentations for $G_i = \mathbb{Z}/p_i^{n_i}$ with the same deficiency, then $X_{\mathcal{P}_1} \vee \cdots \vee X_{\mathcal{P}_k} \cong X_{\mathcal{P}_1'} \vee \cdots \vee X_{\mathcal{P}_k'}$.

Proof. For ease of notation, we will let $k = 2$. The general case is analogous. Let:

$$\mathcal{P}_r^{(i)} = \langle a_1, \ldots, a_{n_1}, a_1^{p_1}, \ldots, a_1^{p_1}, a_2^{p_2}, [a_1, a_2], \{[a_i, a_j] : i < j, (i, j) \neq (1, 2)\}\rangle$$

for $r \in \mathbb{Z}$ with $(r, p_1) = 1$. This is a presentation for $G_i$, and since the homotopy type of $\mathcal{P}_r^{(i)}$ can be shown to depend only on $r \mod p_1$, we can take $r \in (\mathbb{Z}/p_1)^\times$.

It was shown by Browning 15 (see also 27) of Proposition 9.2) that, if $\mathcal{P}$ is a presentation for $(\mathbb{Z}/p_1)^\times$, then $X_\mathcal{P} \simeq X_{\mathcal{P}'_1} \vee \ell S^2$ for some $r \in (\mathbb{Z}/p_1)^\times$, $\ell \geq 0$. It suffices to show that $X_{\mathcal{P}'_1} \vee X_{\mathcal{P}'_2} \simeq X_{\mathcal{P}_1} \vee X_{\mathcal{P}_2}$ for all $r \in (\mathbb{Z}/p_1)^\times$, $s \in (\mathbb{Z}/p_2)^\times$.

As in Proposition 10.3 there exists integers $r_i$, $q_i$ such that $(q_i, q_j) = 1$ for all $i \neq j$ and such that $r_i^{p_i} - 1 = p_i q_i$ and $r_i \equiv 1 \mod p_i$ for all $i$. Let $r, s$ be integers such that $(r, p_1) = 1$ and $(s, p_2) = 1$. If $(r q_1, s q_2) = 1$ then, by the same argument as given in Proposition 10.3, $G = G_1 * G_2$ has a presentation:

$$\mathcal{P}_{r,s} = \langle a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}, a_1^{p_1}, \ldots, a_1^{p_1}, b_1^{p_1}, \ldots, b_1^{p_1}, a_2^{p_2}, \ldots, b_2^{p_2}, a_2^{p_2}, a_2^{-1}, b_2^{p_2}, b_2^{-1}, [a_i, a_j], [b_i, b_j] : i < j, (i, j) \neq (1, 2)\rangle.$$

This form is general for all $r, s$, since, by Dirichlet’s theorem on arithmetic progressions, there exists $r', s'$ such that $r' \equiv r \mod p_1$, $s' \equiv s \mod p_2$ and $(r' q_1, s' q_2) = 1$. 

Let \((P_{r,s})_+\) denote the presentation \(P_{r,s}\) with the additional relation \(a_1^{p_1} \cdot b_1^{p_2} \sim b_1^{p_2}\) by left multiplying with \(a_1^{p_1}\), then replace \(a_2(a_1^{p_1})a_2^{-1}(a_1^{p_1})^{-r_1} \sim [a_2, a_1^{p_1}]\) by right multiplying with \(a_1^{r_1-1}\) (which works since \(r_1 \equiv 1 \mod p_1\)), and similarly \(b_2(b_1^{p_2})b_2^{-1}(b_1^{p_2})^{-r_2} \sim [b_2, b_1^{p_2}]\). This implies that \((P_{r,s})_+\) and \(P_{r,s}(1) \ast P_{r,s}(2)\) are \(Q\)-equivalent and so we have:

\[
X_{P_{r,s}} \vee S^2 \simeq X_{P_{r,s}+} \simeq X_{P_{r,s}(1)} \vee X_{P_{r,s}(2)}.
\]

Note that \(P_{r,s}\) differs from \(P_{r,s}(1)\) by changing \(a_2a_1a_2^{-1}a_1^{-r_1} \sim a_2(a_1^{p_1})a_2^{-1}(a_1^{p_1})^{-r_1}\). Since both relations hold in \(G\), we can add \(a_2(a_1^{p_1})a_2^{-1}(a_1^{p_1})^{-r_1}\) to \(P_{r,s}\) and add \(a_2a_1a_2^{-1}a_1^{-r_1}\) to \(P_{r,s}(1)\) to get that \(X_{P_{r,s}} \vee S^2 \simeq X_{P_{r,s}} \vee S^2\). By symmetry, we also have that \(X_{P_{r,s}} \vee S^2 \simeq X_{P_{r,s}} \vee S^2\) and so \(X_{P_{r,s}} \vee X_{P_{r,s}(2)} \simeq X_{P_{r,s}(1)} \vee X_{P_{r,s}(2)}\).

The following can be found in [60, Theorem 1.2 (3)(iv)]. This can also be deduced by combining the earlier work [43, Proposition 9] with [13, Theorem 1.7].

**Lemma 10.9.** Let \(G = (\mathbb{Z}/p)^n\) for \(p\) prime and \(n \geq 1\). Let \(\delta(G)\) denote the number of \(Aut(G)\)-isomorphism classes of modules \(\pi_2(X_P)\) for \(P\) a presentation with \(def(P) = def(G)\). If \(p = 2\), then \(\delta(G) = 1\) and, if \(p\) is odd, then:

\[
\delta(G) = \begin{cases} \left(\frac{p-1}{2}, n-1\right), & \text{if } n \text{ is even} \\ \left(\frac{p-1}{2}, \frac{n}{p}\right), & \text{if } n \text{ is odd}. \end{cases}
\]

**Proof of Theorem 11.0.** Let \(k \geq 2\) and, for \(i = 1, \ldots, k\), let \(p_i\) be distinct primes with \(p_i \equiv 1 \mod 4\) and let \(G_i = (\mathbb{Z}/p_i)^3\). By Lemma 10.9, we have that \(\delta(G_i) = 2\) and so there exists presentations \(P_i\), \(Q_i\) for \(G_i\) such that \(def(P_i) = def(Q_i) = def(G)\) and \(\pi_2(X_{P_i}) \not\cong \pi_2(X_{Q_i})\) are not \(Aut(G_i)\)-isomorphic.

Similarly to the proof of Theorem 10.2, \(\pi_2(X_{P_i})\), \(\pi_2(X_{Q_i})\) \(\in \Omega_2^{G_i}(\mathbb{Z})\) are minimal by 68 Proposition 2.1. This implies that \(ZG_i \not\cong \pi_2(X_{P_i})\), \(\pi_2(X_{Q_i})\) for all \(i\). By Proposition 10.8, we have that

\[
X_{P_1} \vee \cdots \vee X_{P_k} \simeq X_{Q_1} \vee \cdots \vee X_{Q_k}
\]

and so \(\pi_2(X_{P_1} \vee \cdots \vee X_{P_k}) \cong \pi_2(X_{Q_1} \vee \cdots \vee X_{Q_k})\), as required.

**11. The unstable classification of smooth 4-manifolds**

The aim of this section will be to discuss applications of stably free \(ZG\)-modules to the unstable classification of smooth 4-manifolds. Whilst we will restrict our attention to 4-manifolds, it is also possible to use the examples in Theorem 10.3 for \(n \geq 3\) to study the unstable classification of 2n-manifolds. For brevity, we will not discuss this here.

All manifolds will be assumed to be smooth and connected but not necessarily closed. We will let \(\cong\) denote homeomorphism and let \(\cong_{\text{Diff}}\) denote diffeomorphism.

**11.1. Boundary of thickenings construction.** Let \(X\) be a finite 2-complex. By [16] Statement 7.2, \(X\) is simple homotopy equivalent to a finite simplicial 2-complex \(X'\). By a general position argument, there exists an embedding \(i : X' \to \mathbb{R}^3\) and a smooth regular neighbourhood \(N(i)\) of this embedding which is unique up to diffeomorphism (see, for example, [47] Section II). We define \(M(X) := \partial N(i)\), which is a closed smooth 4-manifold. This depends a priori on the choice of \(X'\) and embedding \(i\), but we will omit these choices from the notation. For convenience, we will refer to \(M(X)\) as a *model* when we have fixed some choices of \(X'\) and \(i\) to obtain a well-defined manifold.

Recall that two closed smooth 4-manifolds \(M_1, M_2\) are *smoothly s-cobordant*, written \(\cong_{\text{Sob}}\), if there exists a smooth 5-manifold with boundary \(W\) such that \(\partial W \cong_{\text{Diff}} M_1 \sqcup M_2\) and, for \(i = 1, 2\), the induced inclusion maps

\[
i_i : M_i \hookrightarrow \partial W \hookrightarrow W
\]

are simple homotopy equivalences. If two closed smooth 4-manifolds are smoothly s-cobordant, then they are simple homeotopy equivalent and also stably diffeomorphic (see [44] Theorem 3.4)].
The following is implicit in [69] (see also [17] p15 and [11] Proposition 5 & 6). This gives a sense in which the construction $X \mapsto M(X)$ is well-defined.

**Proposition 11.1.** Let $X$ and $Y$ be finite 2-complexes such that $X \simeq_s Y$. Then, for any models $M(X)$ and $M(Y)$, we have $M(X) \cong_{sCob} M(Y)$. In particular, they are stably diffeomorphic.

The following special case will be useful later on.

**Lemma 11.2.** Let $X$ be a finite 2-complex. Then there exists models $M(X)$ and $M(X \vee S^2)$ such that $M(X \vee S^2) \cong_{Diff} M(X)$ #$(S^2 \times S^2)$.

**Proof.** Let $X'$ be a finite simplicial 2-complex such that $X \simeq_s X'$, let $i : X \hookrightarrow \mathbb{R}^5$, let $N(i)$ be a smooth regular neighbourhood of $i$ and let $M(X) = \partial N(i)$.

We have $X \vee S^2 \simeq X' \vee \Delta$ where $\Delta$ is a triangle and $X'$ and $\Delta$ are wedged at a 0-simplex so that $X' \vee \Delta$ is a finite simplicial 2-complex. By embedding $\Delta$ in a sufficiently small neighbourhood of the wedge point, we can extend $i$ to an embedding $i_+ : X' \vee \Delta \hookrightarrow \mathbb{R}^5$ so that $N(i_+) = N(i) \# (S^2 \times D^3)$ is a smooth regular neighbourhood of $i_+$. Furthermore, since $\partial N(i_+)$ and so, by Proposition 11.1, we have that $M(X \vee S^2) \cong_{Diff} \partial N(i) \# \partial (S^2 \times D^3) \cong_{Diff} M(X)$ #$(S^2 \times S^2)$. 

11.2. **Proof of Theorem 1.4.** The aim of this section will be to prove the following theorem from the introduction, which we restate here for convenience.

**Theorem 1.4.** Let $G$ be a finitely presented group such that $gd(G) = 2$ and suppose there exists a stably free $\mathbb{Z}G$-module $S$ which is geometrically realisable and such that $S \oplus S^*$ is not a free $\mathbb{Z}G$-module. Then both $\mathcal{M}^{Diff}(G)$ and $\mathcal{M}(G)$ fail cancellation at level $k$.

We will begin by recalling the basic algebraic topology of $M(X)$ for $X$ a finite 2-complex.

**Lemma 11.4.** Let $X$ be a finite 2-complex with $\pi_1(X) \cong G$. Then $M(X)$ is a closed smooth 4-manifold such that $\pi_1(M(X)) \cong G$ and there is an isomorphism of $\mathbb{Z}G$-modules:

$$\pi_2(M(X)) \cong H^2(X; \mathbb{Z}G) \oplus H_2(X; \mathbb{Z}G).$$

**Proof.** This follows from the argument given in [17] Section II in the case where $G$ is finite. The general case was proven in [28] Theorem 4.2 (see also [29] Lemma 5.7). 

**Lemma 11.5.** Let $G$ be a finitely presented group such that $\text{cd}(G) = 2$ and let $X$ be a finite 2-complex with $\pi_1(X) \cong G$. Then there are isomorphisms of $\mathbb{Z}G$-modules:

$$H_2(X; \mathbb{Z}G) \cong \pi_2(X), \quad H^2(X; \mathbb{Z}G) \cong \pi_2(X)^* \oplus H^2(G; \mathbb{Z}G).$$

Furthermore, $\pi_2(X)$ and $\pi_2(X)^*$ are stably free $\mathbb{Z}G$-modules.

**Proof.** The first part follows from the fact that $H_2(X; \mathbb{Z}G) \cong H_2(\tilde{X}) \cong \pi_2(X)$ (see, for example, [35] p81), and holds for any $G$ finitely presented. Since $\text{cd}(G) = 2$, this is stably free by [29] Lemma 5.4. By the universal coefficient spectral sequence (see [29] p11) applied to $X$, there is an exact sequence:

$$0 \to H^2(G; \mathbb{Z}G) \to H^2(X; \mathbb{Z}G) \to \pi_2(X)^* \to H^3(G; \mathbb{Z}G) \to 0.$$ 

Since $\text{cd}(G) = 2$, we have $H^3(G; \mathbb{Z}G) = 0$. The dual of a stably free module is stably free (the proof coincides with that of Proposition 5.8 (i)) and so $\pi_2(X)^*$ is stably free. This implies it is projective and so the exact sequence splits and so we obtain the required isomorphism of $\mathbb{Z}G$-modules:

$$H^2(X; \mathbb{Z}G) \cong \pi_2(X)^* \oplus H^2(G; \mathbb{Z}G).$$

**Proof of Theorem 1.4.** By assumption, there exists a finite 2-complex $X$ such that $\pi_1(X) \cong G$ and $\pi_2(X) \cong S$. Since $G$ is finitely presented with $\text{gd}(G) = 2$, there exists a finite aspherical 2-complex $Y_0$. Let $k$ be the rank of $S$ as a stably free $\mathbb{Z}G$-module and let $Y = Y_0 \cup kS^2$. By Lemma 11.2, there exists models $M(Y)$ and $M(Y_0)$ such that $M(Y) \cong_{Diff} M(Y_0) \# k(S^2 \times S^2)$. Fix a model $M(X)$. By [22] Theorem 13, there exists $r \geq 0$ such that $X \vee rS^2 \simeq_s Y \vee rS^2$ and so, by Proposition 11.4, we have that $M(X) \# t(S^2 \times S^2) \cong_{Diff} M(Y) \# t(S^2 \times S^2)$ for some $t \geq r$. 


We will now prove that $M(X) \not\cong M(Y)$. First note that, by combining Lemmas 11.4 and 11.5, we obtain isomorphisms of $\mathbb{Z}G$-modules:

$$\pi_2(M(X)) \cong S \oplus S^* \oplus H^2(G; \mathbb{Z}G), \quad \pi_2(M(Y)) \cong \mathbb{Z}G^2k \oplus H^2(G; \mathbb{Z}G).$$

Since $S$ is stably free and so projective, we have that $S^* \cong S$ by Proposition 3.3 (ii). Since $\text{cd}(G) = 2$, we have $H^2(G; \mathbb{Z}G)^* = 0$ [29 Proposition 4.6]. This gives:

$$\pi_2(M(X))^* \cong S^* \oplus S^* \oplus H^2(G; \mathbb{Z}G)^* \cong S \oplus S^*, \quad \pi_2(M(Y))^* \cong (\mathbb{Z}G^2k)^* \oplus H^2(G; \mathbb{Z}G)^* \cong \mathbb{Z}G^2k.$$

If $M(X) \cong M(Y)$ then, by the discussion at the start of Section 6, $\pi_2(M(X))$ and $\pi_2(M(Y))$ are $\text{Aut}(G)$-isomorphic. By Proposition 3.9 this implies that $\pi_2(M(X))^*$ and $\pi_2(M(Y))^*$ are $\text{Aut}(G)$-isomorphic. In particular, for some $\theta \in \text{Aut}(G)$, there are isomorphisms of $\mathbb{Z}G$-modules:

$$S \oplus S^* \cong \pi_2(M(X))^* \cong (\pi_2(M(Y))^*)_\theta \cong (\mathbb{Z}G^2k)_\theta \cong \mathbb{Z}G^2k.$$

This contradicts the hypothesis that $S \oplus S^*$ is non-free. Hence $M(X) \not\cong M(Y)$, as claimed.

Now let $\equiv$ denote either homeomorphism or diffeomorphism. Let $\ell \geq 0$ be minimal for which $M(X)\#\ell(S^2 \times S^2) \equiv M(Y)\#\ell(S^2 \times S^2)$. Since $M(X) \not\cong M(Y)$ implies $M(X) \not\cong M(Y)$, we have $\ell \geq 1$.

Let $M := M(X)\#(\ell-1)(S^2 \times S^2)$, $N := M(Y)\#(\ell-1)(S^2 \times S^2)$ and $N_0 := (M(Y)\#(\ell-1)(S^2 \times S^2)$.

By minimality of $\ell$, we have $M \not\cong N$ and $M\#(S^2 \times S^2) \equiv N\#(S^2 \times S^2)$. Since $M(Y) \equiv M(Y)\#(S^2 \times S^2)$, we have that $N \equiv N_0\#(S^2 \times S^2)$ by taking the connected sum with $(\ell-1)(S^2 \times S^2)$. This shows that both $\mathcal{M}^{\text{Diff}}(G)$ and $\mathcal{M}(G)$ fail cancellation at level $k$, as required.

12. Further directions

We will now collect together a list of open problems on unstable classification. The problems are arranged into lists (A) Finitely generated projective $\mathbb{Z}G$-modules, and (B) Finite 2-complexes. The spirit of these problems is to search for examples which illustrate structural features of each respective classification, much like Wall’s list of problems concerning finite 2-complexes [70 List D].

**Finitely generated projective $\mathbb{Z}G$-modules.** Recall that, for a projective $\mathbb{Z}G$-module $P$, the rank is defined as

$$\text{rank}_{\mathbb{Z}G}(P) = \text{rank}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P)$$

and the level is $\ell(P) = \max\{m - n : P \oplus \mathbb{Z}G^m \cong Q \oplus \mathbb{Z}G^n, Q \in \mathbb{Z}(\mathbb{Z}G)\}$.

**Problem A1.** Do non-zero finitely generated projective $\mathbb{Z}G$-modules have non-zero rank?

**Remark.** This is true if $G$ is finite by Swan [66] and, more generally, provided $G$ satisfies Bass’ strong conjecture on Hattori-Stallings rank. In particular, it holds if $(\mathbb{Q}, +) \not\cong G$ [29 Lemma 2.8]. This is true for stably free $\mathbb{Z}G$-modules over all groups $G$ since $\mathbb{Z}G$ is stably finite (see Section 8.1). In particular, if $G$ is torsion free and satisfies the Farrell-Jones conjecture, then $\tilde{K}_0(\mathbb{Z}G) = 0$ and so the statement holds (see Problem A6). It was pointed out by F. E. A. Johnson [37] that, by the examples of Akasaka [1], Problem A1 has a negative answer in the case of infinitely generated projective modules. More specifically, for every non-solvable finite group $G$ there is an infinitely generated projective $\mathbb{Z}G$-module $P$ which is not free and for which $\mathbb{Z} \otimes_{\mathbb{Z}G} P = 0$.

**Problem A2.** Does every stably class in $\tilde{K}_0(\mathbb{Z}G)$ contain a projective $\mathbb{Z}G$-module of rank one? That is, do we have $\ell(P) = \text{rank}_{\mathbb{Z}G}(P) - 1$ for all finitely generated projective $\mathbb{Z}G$-modules $P$?

**Remark.** This is true for $G$ finite by Swan [66]. Since the zero class in $\tilde{K}_0(\mathbb{Z}G)$ contains $\mathbb{Z}G$, this is also true for any group $G$ such that $\tilde{K}_0(\mathbb{Z}G) = 0$. For example, as above, this is true provided $G$ is torsion free and satisfies the Farrell-Jones conjecture.

**Problem A3.** For which $k \geq 2$ does there exist a group $G$ and finitely generated projective $\mathbb{Z}G$-modules $P$ and $Q$ such that $P \oplus \mathbb{Z}G^k \cong Q \oplus \mathbb{Z}G^k$ but $P \oplus \mathbb{Z}G^{k-1} \not\cong Q \oplus \mathbb{Z}G^{k-1}$? (see Fig. 12)

**Remark.** This is open for all $k \geq 2$. Examples here would give further examples of the type considered in Theorem A. Presumably constructing examples which are stably free $\mathbb{Z}G$-modules would be most straightforward.
Problem A4. Does every finitely presented group $G$ have a cancellation bound for projective $ZG$-modules? That is, does there exist a constant $d$ for which $P \oplus ZG \cong Q \oplus ZG$ implies $P \cong Q$ for finitely generated projective $ZG$-modules $P$ and $Q$ of rank $\geq d$? (see Fig. 2)

Remark. As explained in the introduction, examples do not exist when $ZG$ is Noetherian (such as if $G$ is polycyclic-by-finite). Examples were constructed in Theorem C for $G = \ast_{i=1}^{\infty} T$, which is not finitely presented. It would be interesting to know whether or not this example can be modified to give an example over a finitely presented group. For example, $G = \ast_{i=1}^{\infty} T \cong \ast_{i \in Z} T$, where $Z$ freely permutes the copies of $T$, which is isomorphic to $T \times C_{\infty}$. However, the non-free stably free modules of Theorem C all become free upon passage to $Z[T \times C_{\infty}]$ via extension of scalars. We are indebted to Sam Hughes for discussions on this point.

Problem A5. Does there exist a group $G$ and finitely generated projective $ZG$-modules $P$ and $Q$ such that $P \oplus ZG \cong Q \oplus ZG$ but $P \oplus P^* \not\cong Q \oplus Q^*$?

Remark. By Swan [66], there are no examples when $G$ is finite. This is motivated by Theorem 1.4. Given these applications, the main case of interest is therefore the case where $G$ is finitely presented with $\text{cd}(G) < \infty$ and type FL, and where $P$ is stably free.

Problem A6. Does there exist a torsion free group $G$ and a finitely generated projective $ZG$-module which is not stably free?

Remark. This is a well known problem and appeared, for example, in Wall’s problem list [70, Problem A1]. Note that the Farrell-Jones conjecture is a broad generalisation of this question. Many torsion free groups $G$ are known to satisfy the Farrell-Jones conjecture (see [51]) and so the question above has a negative answer in these cases. Whilst the examples obtained in this paper are all stably free, they do at least demonstrate that more elaborate projective modules exist in the case of torsion free groups.

Finite 2-complexes. Recall that a CW-complex $X$ is irreducible if $X \simeq Y \lor Z$ for CW-complexes $Y$, $Z$ implies that $Y$ or $Z$ is contractible.

Problem B1. Let $X_i, Y_i$ be irreducible non-simply connected finite 2-complexes. When does

$$X_1 \lor \cdots \lor X_k \simeq Y_1 \lor \cdots \lor Y_k$$

imply that $X_i \simeq Y_{\sigma(i)}$ for some $\sigma \in S_k$?

Remark. This is motivated by the results in Section 10. Here irreducibility is necessary since it rules out the following two situations:

(a) Exchange of subfactors: If $X \not\simeq Z$, $Y \not\simeq \ast$, then $(X \lor Y) \lor Z \simeq X \lor (Y \lor Z)$.

(b) Non-cancellation: If $X \lor S^2 \simeq Y \lor S^2$, $X \not\simeq Y$, then $X \lor (Z \lor S^2) \simeq Y \lor (Z \lor S^2)$. 

---

Figure 2. Further branching phenomena
The finite 2-complexes given in the proof of Theorem 10.6 are irreducible and so show that some further conditions must be imposed. This was shown to be true by Jajodia Corollary 4 in the case where the $X_i$, $X_i$, have a single 2-cell.

**Problem B2.** For which $k \geq 1$ do there exist finite 2-complexes $X_1$, $X_2$ with $\pi_1(X_1) \cong \pi_1(X_2)$ such that $X_1 \vee kS^2 \cong X_2 \vee kS^2$ and $X_1 \vee (k-1)S^2 \not\cong X_2 \vee (k-1)S^2$? (see Fig. 2a)

**Remark.** This is open for all $k \geq 1$. The question was asked in the case $k = 2$ can be found in Problem C] and later appeared in [22, p124]. Following the same method of proof of Theorem 12.1 one imagines that examples in Problem A3 which are stably free could lead to examples here.

**Problem B3.** Does there exist a finitely presented group $G$ such that, for infinitely many $k \geq 0$, there are homotopically distinct finite 2-complexes $X_1$, $X_2$ with $\pi_1(X_i) \cong G$ and $\chi(X_i) = k + \chi_{\min}(G)$?

**Remark.** This is the analogue of Problem A4 for finite 2-complexes. As with Problem B2, examples there which are stably free could lead to examples here. Note that this is equivalent to asking that, for infinitely many $k \geq 0$, there are homotopically distinct finite 2-complexes $X_1$, $X_2$ with $\pi_1(X_i) \cong G$ and $\chi(X_i) = k + \chi_{\min}(G)$.

**References**

[1] T. Akasaka, *A note on nonfinitely generated projective $\mathbb{Z}_2$-modules*, Proc. Amer. Math. Soc. 86 (1982), no. 3, 391.

[2] P. Ara, K. C. O’Meara, and F. Perera, *Stable finiteness of group rings in arbitrary characteristic*, Adv. Math. 170 (2002), no. 2, 224–238.

[3] V. A. Artamonov, *Projective nonfree modules over group rings of solvable groups*, Mat. Sb. (N.S.) 116 (158) (1981), no. 2, 232–244.

[4] V. A. Artamonov and A. A. Bovdi, *Integral group rings: groups of invertible elements and classical $K$-theory*, Algebra, Topology, Geometry, Vol. 27 (Russian), Itogi Nauk i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, Translated in J. Soviet Math. 57 (1991), no. 2, 2931–2958, pp. 3–43, 232.

[5] H. Bass, *Projective modules over free groups are free*, J. Algebra 1 (1964), 367–373.

[6] ———, *Algebraic $K$-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[7] G. M. Bergman, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. 200 (1974), 1–32.

[8] P. H. Berridge and M. J. Dunwoody, *Nonfree projective modules for torsion-free groups*, J. London Math. Soc. (2) 19 (1979), no. 3, 433–436.

[9] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997), no. 3, 445–470.

[10] R. Bieri, *Homological dimension of discrete groups*, second ed., Queen Mary College Mathematics Notes, Queen Mary College, Department of Pure Mathematics, London, 1981.

[11] I. Bokor, D. Crowley, S. Friedl, F. Hebestreit, D. Kasprzak, M. Land, and J. Nicholson, *Connected sum decompositions of high-dimensional manifolds*, 2019–20 MATRIX annals, MATRIX Book Ser., vol. 4, Springer, Cham, 2021, pp. 5–30.

[12] J. Bowden, D. Crowley, J. Davis, S. Friedl, C. Rovi, and S. Tillmann, *Open problems in the topology of manifolds*, 2019–20 MATRIX annals, MATRIX Book Ser., vol. 4, Springer, Cham, 2021, pp. 647–659.

[13] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982.

[14] W. J. Browning, *Homotopy types of certain finite CW-complexes with finite fundamental group*, Ph.D. thesis, Cornell University, 1978.

[15] ———, *Finite CW-complexes of cohomological dimension 2 with finite abelian $\pi_1$*, ETH preprint (unpublished) (1979).

[16] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, Vol. 10, Springer-Verlag, New York-Berlin, 1973.

[17] P. M. Cohn, *Some remarks on the invariant basis property*, Topology 5 (1966), 215–228.

[18] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. 18 (1983), no. 2, 279–315.

[19] M. J. Dunwoody, *Relation modules*, Bull. London Math. Soc. 4 (1972), 151–155.

[20] ———, *The homotopy type of a two-dimensional complex*, Bull. London Math. Soc. 8 (1976), no. 3, 282–285.

[21] M. N. Dyer, *Non-minimal roots in homotopy trees*, Pacific J. Math. 80 (1979), no. 2, 371–380.

[22] ———, *Trees of homotopy types of $(\pi, m)$-complexes*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 251–254.

[23] G. Elek and E. Szabó, *Sofic groups and direct finiteness*, J. Algebra 280 (2004), no. 2, 426–434.

[24] M. J. Evans, *Epi- morphisms between the free groups in a variety of groups*, J. Algebra 220 (1999), no. 2, 492–511.

[25] ———, *Relation modules of infinite groups*, Bull. London Math. Soc. 31 (1999), no. 2, 154–162.
[62] G. P. Scott and C. T. C. Wall, Topological methods in group theory, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 137–203.

[63] J. P. Serre, Cohomologie des groupes discrets, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), 1971, pp. 77–169. Ann. of Math. Studies, No. 70.

[64] A. J. Sieradski and M. N. Dyer, Distinguishing arithmetic for certain stably isomorphic modules, J. Pure Appl. Algebra 15 (1979), no. 2, 199–217.

[65] J. Stallings, A finitely presented group whose 3-dimensional integral homology is not finitely generated, Amer. J. Math. 85 (1963), 541–543.

[66] R. G. Swan, Induced representations and projective modules, Ann. of Math. (2) 71 (1960), 552–578.

[67] _______., Projective modules over group rings and maximal orders, Ann. of Math. (2) 76 (1962), 55–61.

[68] _______., Minimal resolutions for finite groups, Topology 4 (1965), 193–208.

[69] C. T. C. Wall, Classification problems in differential topology. IV. Thickennings, Topology 5 (1966), 73–94.

[70] _______., List of problems, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 369–394.

[71] G. W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978.

[72] J. H. C. Whitehead, Simplicial Spaces, Nuclei and m-Groups, Proc. London Math. Soc. (2) 45 (1939), no. 4, 243–327.

Department of Mathematics, Imperial College London, London, SW7 2AZ, U.K.

Email address: john.nicholson@imperial.ac.uk