An ISS characterization for discontinuous discrete-time systems

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Input-to-state stability (ISS) is an important concept to study robustness properties of nonlinear control systems. The ISS property can be, in particular, characterized by ISS Lyapunov functions. For discrete-time nonlinear control systems two different forms of ISS Lyapunov functions (implication-form and dissipation-form) are known to be equivalent if the dynamics are continuous. However, for discontinuous dynamics the equivalence is no longer satisfied. In this work, we discuss this phenomenon and, eventually, give a complete characterization of ISS in terms of ISS Lyapunov functions.

We consider nonlinear discrete-time systems of the form

$$x^+ = G(x, u)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and $G: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfying $G(0, 0) = 0$. Solutions at time $k \in \mathbb{N}$ with initial state $x(0) = \xi$ and input $u(\cdot)$ are denoted by $x(k, \xi, u(\cdot))$. For discrete time systems of the form (1) with unconstrained state and input spaces, solutions are uniquely defined. Thus, no additional properties on $G$ have to be required. In the following, $\mathcal{P}$ denotes the class of positive definite function $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ and $K, K_\infty, K\mathcal{L}$ are the standard comparison functions, see eg. [1].

**Definition 1** We call system (1) input-to-state stable (ISS) if there exist functions $\beta \in K\mathcal{L}$ and $\gamma \in K_\infty$ such that for all initial values $\xi \in \mathbb{R}^n$, all input functions $u(\cdot) \in \mathcal{C}_\infty$ and all times $k \in \mathbb{N}$ we have

$$\|x(k, \xi, u(\cdot))\| \leq \beta(\|\xi\|, k) + \gamma(\|u(\cdot)\|_\infty).$$

There are two standard forms of ISS Lyapunov functions to characterize the ISS property of system (1).

**Definition 2** Let $V: \mathbb{R}^n \to \mathbb{R}_+$ satisfy $\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|)$ for some $\alpha_1, \alpha_2 \in K_\infty$. Then $V$ is said to be a

- **dissipation-form ISS Lyapunov function** if there exist functions $\alpha \in K_\infty$ and $\sigma \in K$ such that for all $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ we have

$$V(G(\xi, \mu)) - V(\xi) \leq -\alpha(\|\xi\|) + \sigma(\|\mu\|).$$

- **implication-form ISS Lyapunov function** if there exist functions $\chi \in K$ and $\theta \in \mathcal{P}$ such that for all $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ we have

$$\|\xi\| \geq \chi(\|\mu\|) \Rightarrow V(G(\xi, \mu)) - V(\xi) \leq -\theta(V(\xi)).$$

Note that if $G$ in (1) is continuous then also the ISS LF can be assumed to be continuous. This follows from [2, Theorem 1], where the authors prove the equivalence between ISS of system (1) and the existence of a continuous (even smooth) dissipation-form ISS Lyapunov function. Moreover, by [2, Remark 3.3] both dissipation-form and implication-form ISS Lyapunov functions are equivalent for continuous dynamics $G$.

For discontinuous systems, it is shown in [3, Example 3.3] that the concept of an implication-form ISS Lyapunov functions is not strong enough to conclude ISS. Here, we give a slightly modified version of this example to show that not only continuity in the origin is necessary but also boundedness on bounded sets.

**Example 3** Let $\epsilon \in [0, 1/2]$ be given. The system $x^+ = G(x, w) = \kappa(x)\nu(w)$ with

$$\kappa(x) = \begin{cases} \frac{3}{2}, & |x| \in [0, \epsilon] \\ \frac{1}{|x|^2 - \epsilon}, & |x| \in (\epsilon, 1) \end{cases}, \quad \nu(w) = \begin{cases} 0, & |w| \in [0, \epsilon] \\ \frac{1}{2}(|w| - \epsilon^2), & |w| \in (\epsilon, 1) \end{cases}.$$ 

is not ISS. For instance, if $w \equiv 1$ and $\xi \in (\epsilon, 1)$ it holds $x(2k + 1, \xi, w) = \frac{2k}{\xi - \epsilon}$, which is a diverging trajectory. However, $V : \mathbb{R} \to \mathbb{R}_+, \xi \mapsto |\xi|$ is an implication-form ISS LF satisfying

$$|x| \geq |w| \Rightarrow V(G(x, w)) - V(x) \leq -\frac{1}{2}|x|,$$
which can be proved as follows:

For $|w| \in [0, \epsilon]$, $G(x, w) = 0$. Let $|w| \in (\epsilon, 1)$ and consider $|x| \in (\epsilon, 1)$, we get $|x| - \epsilon \leq |w| - \epsilon$, and thus $|G(x, w)| \leq \frac{1}{2}(|x| - \epsilon) \leq \frac{|w|}{2}$. Now if $|x| \geq 1$, we get $|G(x, w)| = \frac{1}{2} \|w\| + 2 \epsilon \frac{1}{2}(|w| - \epsilon)^2 \leq (\frac{1}{2} + \epsilon) \frac{1}{2}(|x| - \epsilon)(|w| - \epsilon) \leq \frac{1}{2}(|x| - \epsilon) \leq \frac{|w|}{2}$, since $\frac{1}{2} \leq 1$, $\frac{1}{2} + \epsilon < 1$ and $(|w| - \epsilon) < 1$. Finally, for $|x| \geq |w| \geq 1$, we get $|G(x, w)| \leq \frac{1}{2} \leq \frac{|w|}{2}$.

In [3], a motivating example was given to illustrate that not only continuity in the origin is necessary for ISS of a system. To overcome this issue, the authors in [3] provide the definition of a strong implication-form ISS Lyapunov function, i.e., an implication-form ISS Lyapunov function that satisfies, for some $\varphi \in \mathcal{K}$, the implication

$$
\|\xi\| < \varphi(\|\mu\|) \Rightarrow V(G(\xi, \mu)) \leq \varphi(\|\mu\|).
$$

Moreover, it is shown that for discontinuous discrete-time systems ISS is equivalent to the existence of a strong implication-form ISS Lyapunov function, see [3, Theorem 4.4].

Motivated by Example 3, we see that for ISS of a system not only continuity in the origin is necessary but also boundedness on bounded sets. Therefore, we follow a different procedure than that in [3]. In particular, we develop a new Lyapunov characterization of ISS, which does not require an additional property on the implication-form ISS Lyapunov function, but on the system dynamics itself. The essential condition is the following, which was originally introduced in [4].

**Definition 4** The function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in (1) is globally $\mathcal{K}$-bounded if there exist functions $\eta_1, \eta_2 \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$, we have

$$
\|G(\xi, \mu)\| \leq \eta_1(\|\xi\|) + \eta_2(\|\mu\|).
$$

Note that ISS of (1) implies global $\mathcal{K}$-boundedness, see [5, Remark 3.3]. Therefore, the global $\mathcal{K}$-boundedness is a necessary condition for ISS. Moreover, from [6, Lemma 5] we know that $G$ is $\mathcal{K}$-bounded if $G$ satisfies $G(0, 0) = 0$, is continuous in $(0, 0)$ and it is bounded on bounded sets. This exactly reflects the behaviour of the discontinuous system given in Example 3. In particular, any continuous $G$ satisfying $G(0, 0) = 0$ is globally $\mathcal{K}$-bounded.

Eventually, we are able to prove that the existence of an implication-form ISS Lyapunov function indeed implies ISS of the discontinuous system (1), provided that $G$ is globally $\mathcal{K}$-bounded (which is necessary for ISS). This main conclusion is summarized in the following result.

**Theorem 5** Consider system (1). Then the following are equivalent.

1. System (1) is ISS.
2. There exists a dissipation-form ISS Lyapunov function.
3. There exists an implication-form ISS Lyapunov function and $G$ is globally $\mathcal{K}$-bounded.

The proof of the theorem can be found in [6, Theorem 12], where we particularly present an extension of this observation for a more general ISS property with respect to closed sets. In addition to the dissipation and implication forms of ISS Lyapunov functions, an ISS Lyapunov function can be given in a so-called max-form, which is particularly useful in the context of large-scale systems, see [7] for more details.

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