A Geometrical Interpretation of Hyperscaling Breaking in the Ising Model

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In random percolation one finds that the mean field regime above the upper critical dimension can simply be explained through the coexistence of infinite percolating clusters at the critical point. Because of the mapping between percolation and critical behaviour in the Ising model, one might check whether the breakdown of hyperscaling in the Ising model can also be interpreted as due to an infinite multiplicity of percolating Fortuin-Kasteleyn clusters at the critical temperature $T_c$. Preliminary results suggest that the scenario is much more involved than expected due to the fact that the percolation variables behave differently on the two sides of $T_c$.

1. INTRODUCTION

The critical behaviour of statistical mechanical systems reduces to mean field theory above the upper critical dimension $d_u$. In this case, the scaling relations between the critical exponents of the phase transition remain valid except the so-called hyperscaling relations, i.e., the equalities in which the number $d$ of space dimensions of the system explicitly appears, like

$$2 - \alpha = \nu d. \quad (1)$$

For this reason one says that, above $d_u$, hyperscaling breaks. Some time ago, Coniglio\textsuperscript{1} proposed an interpretation of hyperscaling breaking for the pure random percolation problem: above the upper critical dimension the number of percolating clusters at the critical threshold, which is finite for $d \leq d_u$, becomes infinite. The presence of infinitely many interpenetrating clusters damps the fluctuations of the order parameter establishing the mean field regime.

The magnetization transition of the Ising model can be equivalently described as a percolation transition of suitably defined site-bond clusters\textsuperscript{2}. In particular, the magnetization coincides with the percolation order parameter. That suggests an analogous interpretation of hyperscaling breaking as for random percolation: we tried to test this conjecture. We present here preliminary results, based on Monte Carlo simulations, concerning the multiplicity of percolating clusters at criticality both in random percolation and in the Ising model above the upper critical dimension $d_u$.

2. BREAKDOWN OF HYPERSCALING IN RANDOM PERCOLATION

We start from an hypercubic lattice with a fraction $p$ of occupied sites. Near the critical density $p_c$, the singular part of the cluster number $K(p)_{\text{sing}}$ behaves as

$$K(p)_{\text{sing}} = \sum_s n(s, p)_{\text{sing}} \sim |p - p_c|^{2-\alpha}, \quad (2)$$

$n(s, p)$ is the number of clusters of size $s$)

The connectedness length $\xi(p)$, i.e. the typical radius of the largest finite cluster, diverges as

$$\xi(p) \sim |p - p_c|^{-\nu}. \quad (3)$$

From Eqs. (2) and (3) one obtains
\[ K(p)_{\text{sing}} \sim \xi^{(\alpha-2)/\nu}. \]

Let us now assume that the singular behaviour comes only from the critical clusters, i.e. the clusters whose radius is about \( \xi \). Say \( N_\xi \) the number of such clusters in a volume of the order \( \xi^d \). The singular part of the cluster number is given by

\[ \frac{N_\xi}{\xi^d} \sim K(p)_{\text{sing}} \sim \xi^{(\alpha-2)/\nu}. \] (5)

If \( N_\xi \) is of the order of unity, from Eq. (5) we obtain the hyperscaling relation (1); if instead \( N_\xi \) grows with \( \xi \), so that, at the critical point, \( N_\xi \to \infty \), then hyperscaling breaks down.

When there are infinitely many spanning clusters the percolation order parameter fluctuates over a distance \( \xi_1 \ll \xi \); this would simply explain why above \( d_u \) the mean field solution is valid. Besides, one finds that, above the upper critical dimension (for random percolation \( d_u = 6 \)),

\[ N_\xi \sim \xi^{d-d_u} = \xi^{d-6}. \] (6)

### 3. BREAKDOWN OF HYPERSCALING IN THE ISING MODEL

If one builds clusters by joining nearest-neighbouring spins of the same sign with the bond probability \( p_B = 1 - \exp(-2J/kT) \) (\( J \) is the Ising coupling), the magnetization transition of the Ising model becomes equivalent to the percolation transition of such clusters (called Fortuin-Kasteleyn or FK clusters): the percolation temperature coincides with the thermal critical point \( T_c \) and the critical percolation exponents are just the Ising exponents. Because of that one can interpret the Ising transition as a simple geometrical phenomenon. Therefore, the argument used above for the pure percolation problem can be extended to the Ising case. One expects then that the number of FK clusters \( N_\xi \) in a region of linear dimension \( \xi \) is of the order of unity below \( d_u = 4 \), while above \( d_u \)

\[ N_\xi \sim \xi^{d-4}. \] (7)

This implies that, above \( d_u \), there should be an infinite number of percolating FK clusters at the critical temperature.

We warn that some care needs to be taken. The percolation transition of the FK clusters in the Ising model is asymmetric on the two sides of \( T_c \): for \( T > T_c \), the percolation variables coincide with the thermal variables in any dimension, whereas for \( d > d_u = 4 \) and \( T < T_c \), the thermal fluctuations are different from the cluster fluctuations. There are indeed arguments suggesting that the critical percolation exponents below \( T_c \), except \( \beta \), are different from the Ising ones for \( d > 4 \) and that the upper critical dimension of the FK clusters is \( d_F K_u(T < T_c) = 6 \), like in random percolation.

### 4. RESULTS FOR RANDOM PERCOLATION

The first numerical studies on the multiplicity of infinite clusters at and above the upper critical dimension were carried on by de Arcangelis [3], but only small lattices could be used (up to \( 6^7 \)).

We performed Monte Carlo simulations for pure random site percolation in several dimensions on hypercubic lattices. In any space dimension we calculated the number of distinct spanning clusters for different lattices at the critical density \( p_c \). We noticed that, for \( d \leq 6 \), there is almost always just a single percolating cluster (or none), independently of the lattice size, whereas for \( d = 7 \) we observed an "explosion" of the multiplicity of percolating clusters, which increases with the lattice side \( L \). Our aim is to prove Eq. (6) which, at the critical point, assumes the following finite-size scaling form:

\[ N_\infty \propto L^{d-6}. \] (8)

In Fig. 1 we plot the multiplicity of the percolating clusters as a function of the linear dimension of the lattice. The linear fit of the data is excellent. We conclude that, at least for the particular case \( d = 7 \), Eq. (6) is correct.
5. RESULTS FOR THE ISING MODEL

If there is a mismatch between the percolation and the thermal variables for $T < T_c$, the percolation transition of the FK clusters would show quite an unusual behaviour, since the critical exponents would be different above and below $T_c$.

We investigated the Ising model in 5, 6 and 7 dimensions. We found that the behaviour below $T_c$ dominates, i.e. the finite-size scaling fits at $T_c$ return the values of the conjectured exponents for $T < T_c$. Fig. 3 shows the variation of the multiplicity $N_\infty$ of the percolating clusters with the lattice side $L$ for the 5-dimensional Ising model at $T_c$. We observe a clear decrease of $N_\infty$ with $L$, which is in contrast with Eq. (7)! On the other hand, since the scaling behaviour at $T < T_c$ dominates, this result would confirm that the upper critical dimension $d^{FK}_u (T < T_c) > 4$. We decided then to check whether $d^{FK}_u (T < T_c) = 6$, as conjectured. For this purpose we investigated the 7-dimensional Ising model and we determined for each lattice the peak of the percolating cluster multiplicity. In Fig. 4 we plot the values of the maxima versus the lattice side $L$: the points lie on a straight line, like in random percolation.

6. CONCLUSIONS

Our results clearly show that for random percolation hyperscaling breaks due to the presence of infinite percolating clusters at the critical point. For the Ising model the situation is more complicated because of the different behaviour of the percolation variables below and above $T_c$. We found that the upper critical dimension of the FK clusters below $T_c$ is six, like in random percolation. What really matters for our geometrical interpretation of hyperscaling breaking is a relation between $N_\xi$ and $\xi$ such that $N_\xi$ increases with $\xi$. It is then crucial to extrapolate numerically such a relation for $T \neq T_c$. Moreover, it is well known that the finite-size scaling behaviour above the upper critical dimension is anomalous because the relevant scale is no longer the correlation length $\xi$ but another length $\ell$. It would be interesting to check whether $\ell$ coincides with the above-mentioned geometrical fluctuation scale $\xi_1$.

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