Marciani Normal Form
of context-free grammars

Giacomo Marciani
gmarciani@acm.org
Department of Civil Engineering and Computer Science Engineering
University of Rome Tor Vergata, Italy

ABSTRACT
In this paper, we prove the semidecidability of the problem of saying whether or not a context-free grammar generates a regular language. We introduce the notion of context-free grammar in Marciani Normal Form. We prove that a context-free grammar in Marciani Normal Form always generates a regular language.

CCS Concepts
• Theory of computation → Grammars and context-free languages;

Keywords
automata theory, formal languages, context-free grammars, regular languages, grammar normal forms

1. INTRODUCTION
In Section 2, we give some preliminary definitions about language equations and the regularity problem. In particular, we introduce the class of bilateral-linear language equations, the pseudo-regular partition of productions and the looking forward property.
In Section 3, we state and prove Marciani’s Rule, which exposes a method to solve the bilateral-linear language equations.
In Section 4, we define the Marciani Normal Form, and we prove that a context-free grammar in such a form always generates a regular language.
In Section 5, we prove the semidecidability and the undecidability of the context-free regularity problem, by the application of the previous results.
In Section 6, we give some applicative examples of the Marcian’s Rule and the Marciani Normal Form.

2. DEFINITIONS
We recall definitions known in the literature, for convenience of the reader.

It is well known that a language equation can be classified, at first instance, according to the position of its unknown. In particular, we typically have the following definitions.

Definition 1. A left-linear (ll) language equation in the unknown is a language equation of the form

\[ r = ar + s \]  (1)

Definition 2. A right-linear (rl) language equation in the unknown is a language equation of the form

\[ r = ra + s \]  (2)

A grammar in which all productions for a given non-terminal identify ll or rl language equations, always generates a regular language.
The problem of saying whether or not a context-free grammar generates a regular language is very important. In fact, a context-free grammar has greater expressiveness than a regular one. On the other hand, a regular grammar is much lighter in terms of computational complexity. We call this problem Context-Free (CF) Regularity Problem.

Definition 3. The Context-Free (CF) Regularity Problem is the subset defined as follows:

\[ \mathbb{R} = \{ G \in [G_{CS}] \land L(G) \in [L_{REG}] \} \]  (3)

where \( G \) is the encoding of a context-free grammar \( G \) as a string in \( [0,1]^{*} \), \( [G_{CS}] \) is the class of context-free grammars, and \( [L_{REG}] \) is the class of regular languages.

We now introduce the new definitions, which will be essential in subsequent sections.

Definition 4. A bilateral-linear language equation in the unknown is a language equation of the form

\[ r = ar + rb + s \]  (4)

Definition 5. A grammar \( G = (V_{T}, V_{N}, S, P) \) enjoys the looking forward property if and only if the digraph \( D_{G} \), constructed as follows, does not have any cycle.

Let \( D_{G} \) be a digraph such that for all non-terminal symbol in \( V_{T} \) there exists a node in \( D_{G} \), and for all production \( A \rightarrow \alpha \) in \( P \) there exists an arc from \( A \) to every non-terminal symbol in \( \alpha \), with the exception of \( A \) itself.
3. MARCIANI’S RULE

It is well known that Arden’s Rule permits the resolution of left-linear and right-linear language equations [3]. In particular, by Arden’s Rule, given the language equation

$$r = ar + rb + s$$

in the unknown $$r$$, its least solution is $$a^*s$$. Likewise, given the language equation $$r = ra + s$$ in the unknown $$r$$, its least solution is $$sa^*$$.

Now, we state and prove a theorem that permits the resolution of both-linear language equations.

**Theorem 1.** Given the language equation $$r = ar + rb + s$$ in the unknown $$r$$, its least solution is $$a^*sb^*$$.

**Proof.** Let us divide the proof in the following three points.

a) We will first show that the language equation $$r = ar + rb + s$$ is equivalent to the language equation $$r = a^*(rb+s)$$.

b) Then, we will show that $$a^*sb^*$$ is a solution for $$r$$ of the language equation $$r = a^*(rb+s)$$. We prove by induction on $$i, j \geq 0$$.

$$z \subseteq z$$ holds because $$z = a^*(zb + s)$$.

(Step i) We have to show the following implication $$\bigcup_{i \geq 0} a^*s \subseteq z \rightarrow \bigcup_{i \geq 0} a^*s \subseteq z$$. This holds because $$\bigcup_{i \geq 0} a^*s \subseteq a \bigcup_{i \geq 0} a^*s \subseteq z$$.

(Step i, j \geq 0) We have to show the following implication $$\bigcup_{i,j \geq 0} a^*s \subseteq z \rightarrow \bigcup_{i,j \geq 0} a^*s \subseteq z$$. This holds because $$\bigcup_{i,j \geq 0} a^*s \subseteq z$$.

4. MARCIANI NORMAL FORM

We introduce the notion of context-free grammars in Marciani Normal Form (MNF). We prove that every MNF grammar generates a regular language.

**Definition 7.** A context-free grammar $$G = (V, V_n, S, P)$$ is said to be in Marciani Normal Form (MNF) iff $$G$$ enjoys the looking forward property and there exists a pseudo-

regular partition $$\mathcal{T}_A$$ for all symbol $$A \in V_T$$.

**Theorem 2.** A context-free grammar $$G = (V, V_n, S, P)$$ in Marciani Normal Form always generates a regular language.

**Proof.** If $$G$$ is in Marciani Normal Form, then for each non-terminal $$A$$ there exists a pseudo-regular partition $$\mathcal{T}_A$$. Notice that to every pseudo-regular partition $$\mathcal{T}_A = \{\{p \in P_A, A \rightarrow \alpha A\}, \{p \in P_A, A \rightarrow \alpha B\}, \{p \in P_A, A \rightarrow \gamma\}\}$$ can be associated a both-linear language equation $$A = \alpha A + A\beta + \gamma$$. By the application of the Marciani’s Rule, we know that the least solution of the previous equation is $$\alpha^*\gamma^*$$, so $$L(A) = L(\alpha^*\gamma^*)$$.

As a consequence of the previous results and the looking forward property, there exists a regular expression $$e$$ such that $$L(S) = L(e)$$, then $$L(S)$$ is a regular language, that is $$G$$ is regular.

5. SEMIDECIDABILITY OF THE CF REGULARITY PROBLEM

We know that the context-free regularity problem is undecidable [3], due to its reduction to the undecidable Post Correspondence Problem [2].

We prove the semidecidability and the undecidability of the context-free regularity problem.

**Theorem 3.** Given a context-free grammar, the problem of saying whether or not there exists an equivalent grammar in Marciani Normal Form is semidecidable and undecidable.

**Proof.** Let us consider a context-free grammar in Chomsky Normal Form. Let us derive an equivalent grammar by unfolding every production, until getting the productions for the axiom only. Now, it’s easy to check if the derived grammar is in Marciani Normal Form. So, given any context-free grammar in Chomsky Normal Form, it is always possible to check if there exists an equivalent context-free grammar in Marciani Normal Form.

As this possibility holds for grammars in Marciani Normal Form, then it holds for every context-free grammar [3].

**Theorem 4.** The context-free regularity problem is semidecidable and undecidable.

**Proof.** Follows from the semidecidability and undecidability of the problem of saying whether or not, given a context-free grammar in Marciani Normal Form, there exists an equivalent grammar in Marciani Normal Form, and from the regularity of the language generated by a context-free grammar in Marciani Normal Form.

6. EXAMPLES

We now see three applicable examples of the Marciani Normal Form and Marciani’s Rule. Let us first consider a very simple application.
Example 1. Let us consider the context-free grammar $G$ with axiom $S$ and the following productions:

$$S \rightarrow abcS|S|ghi|\varepsilon$$

The grammar is in MNF, thus we know it generates a context-free language. The language generated by $G$ is denoted by the regular expression:

$$(abc)^*(ghi + \varepsilon)(def)^*$$

Let us now consider a slightly more complex application.

Example 2. Let us consider the context-free grammar $G$ with axiom $S$ and the following productions:

$$S \rightarrow aAS|SB|CD|z|\varepsilon$$

A grammar is in MNF, thus we know it generates a context-free language. The language generated by $G$ is denoted by the regular expression:

$$(au^*mu^*)^*(g^*ih^*prq^* + \varepsilon)(x^*ny^*def)^*$$

We now consider an application on a grammar in the notable Chomsky Normal Form (CNF) [1].

Example 3. Let us consider the context-free grammar $G$ with axiom $S$ and the following productions:

$$S \rightarrow AS|SB|CD|z|\varepsilon$$

The grammar is in MNF, thus we know it will generate a context-free language. The language generated by $G$ is denoted by the regular expression:

$$(u^*mu^*)^*(iu^*x^*rx^* + z + \varepsilon)(x^*nx^*)^*$$

7. CONCLUSIONS

We introduced the notion of context-free grammar in Marciani Normal Form (MNF). We proved that a context-free grammar in such a form always generates a regular language. Such a demonstration states the semidecidability of the CF Regularity Problem, which is the problem of saying whether or not a context-free grammar generates a regular language. We gave representative examples of applying Marciani’s Rule, showing the simplicity of determining the solution for the CF Regularity Problem.

8. REFERENCES

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