DISCOUNT-SENSITIVE EQUILIBRIA IN ZERO-SUM
STOCHASTIC DIFFERENTIAL GAMES

Beatris A. Escobedo-Trujillo
Engineering Faculty
Universidad Veracruzana
Coatzacoalcos, Ver. 96538, México

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Abstract. We consider infinite-horizon zero-sum stochastic differential games with average payoff criteria, discount-sensitive criteria and, infinite-horizon undiscounted reward criteria which are sensitive to the growth rate of finite-horizon payoffs. These criteria include, average reward optimality, strong 0-discount optimality, strong -1-discount optimality, 0-discount optimality, bias optimality, F-strong average optimality and overtaking optimality. The main objective is to give conditions under which these criteria are interrelated.

1. Introduction. This paper extends to zero-sum stochastic differential games the concepts of \( m \)-discount optimality and strong \( m \)-discount optimality \((m = -1, 0)\) defined in [17] for a general class of Markov diffusion processes. As well as, Flynn’s F-strong average optimality concept given in [13] for discrete-time Markov control processes on Borel spaces. Moreover, under previous results that guarantee the existence of average reward, discounted, bias and overtaking equilibria (see [7]), we give conditions under which: F-strong average optimality and average reward optimality are equivalent; average reward optimality and strong -1-discount optimality are equivalent, as well as, strong 0-discount optimality implies bias optimality and bias optimality implies 0-discount optimality. Thus, we ensure that strong-1-discount, strong 0-discount and F-strong average optimal strategies exist.

As Jasso-Fuentes points out in [17], the study of \( m \)-discount optimality, for an integer \( m \geq -1 \), has been done implicitly when analyzing the coefficients of the Laurent series that appears in Blackwell optimality (see [26], for details). Explicit analysis for particular values of \( m \), say \( m = -1, 0 \), has been done in [14, 15, 30] for discrete-time Markov decision processes, and in [25] for continuous-time controlled Markov chains with a countable state spaces. See also [6, 31] for more details on these criteria.

On the other hand, as shown in [7, 24], bias and overtaking optimality criteria are refinements of the average reward optimality criterion; see e.g. [13, 15, 16]. These criteria choose an average optimal strategy with maximal expected reward growth as the time horizon goes to infinity.

Flynn in 1980 introduced the F-strong average criterion for problems with long finite horizons (see [11]). This criterion has been studied in [13] for discrete-time.
Markov control processes on Borel spaces, and recently in [27] for continuous-time Markov decision processes.

So, as far as we can tell, this article is the first one to study $m$-discount optimality and F-strong average optimality for zero-sum stochastic differential games.

Under our perspective, this work has the following novelties:

1. Generalizes, $m$-discount optimality and strong -$m$-discount optimality ($m = -1, 0$) concepts which have been studied in [17] for controlled diffusion processes, in [31] for Markov decision processes and [25] for continuous-time controlled Markov chains with a countable state spaces. Our approach in this paper is mainly based on [17].

2. Generalizes the concept of F-strong average optimality which has been studied in [13] in the context discrete-time Markov control processes on Borel spaces and in [27] for continuous-time Markov decision processes.

3. Analyzes the interrelationships between: average optimality, F-strong average optimality, $m$-discount optimality, strong $m$-discount optimality, bias optimality and overtaking optimality.

4. Studies stochastic differential games with additive structure (also known as separable; see e.g. [5, 18, 23]) to prove that bias optimality implies strong 0-discount optimality. Moreover, using the vanishing discount technique, shows that a sequence of $\alpha$-discount optimal strategies weakly converge to an average optimal strategy. This last fact is an additional assumption in previous works; see e.g. [17].

This paper is organized as follows. In Section 2 we introduce the dynamic system and our main assumptions. In section 3 we define the $\alpha$-discounted optimality criterion we are concerned with, and we summarize some known results on the existence of $\alpha$-discounted optimal strategies. In section 4 we introduce average payoff optimality criterion we are interested in, and we give conditions that ensure the existence of average optimal strategies. Section 5 generalizes the concept of F-strong average optimality, and it proves that this criterion and average reward optimality are equivalent. Section 6 is dedicated to prove that strong -1-discount optimality and average reward optimality are equivalent concepts. In Section 7, we prove that strong 0-discount optimality implies bias optimality and bias optimality implies 0-discount optimality. In Section 8 we prove that bias optimality implies strong 0-discount optimality. To this proof, we work with a class of stochastic differential games with additive structure and bounded coefficients. Section 9 deals with some technical complements that allow to prove that a sequence of $\alpha$-discount optimal strategies weakly converge to an average optimal strategy. This last is crucial to prove that bias optimality implies strong 0-discount optimality. Finally, we conclude in section 10 with some concluding remarks.

2. The game model and main assumptions.

2.1. The dynamic system. Let us consider an $n$-dimensional diffusion process $x(\cdot)$ controlled by two players and evolving according to the stochastic differential equation

$$dx(t) = b(x(t), u_1(t), u_2(t))dt + \sigma(x(t))dW(t), \ x(0) = x_0, \ t \geq 0, \ (1)$$

where $b : \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are given functions, and $W(\cdot)$ is a $d$-dimensional standard Brownian motion. The sets $U_1 \subset \mathbb{R}^{m_1}$ and $U_2 \subset \mathbb{R}^{m_2}$ are Borel sets called the action set for player 1 and player 2, respectively. Moreover,
for $i = 1, 2$, $u_i(\cdot)$ is a $U_i$-valued stochastic process representing the control actions of player $i$ at each time $t \geq 0$.

**Notation.** For vectors $x$ and matrices $A$ we use the usual Euclidean norms
\[ |x|^2 := \sum_k x_k^2 \quad \text{and} \quad |A|^2 := \text{Tr}(AA') = \sum_{k,p} A_{k,p}^2, \]
where $A'$ and Tr$(\cdot)$ denote the transpose and the trace of a square matrix, respectively.

**Assumption 1.**

(a) The action sets $U_1$ and $U_2$ are compact.

(b) The drift coefficient $b(x, u_1, u_2)$ is continuous on $\mathbb{R}^n \times U_1 \times U_2$, and $x \mapsto b(x, u_1, u_2)$ satisfies a Lipschitz condition uniformly in $(u_1, u_2) \in U_1 \times U_2$; that is, there exists a positive constant $K_1$ such that
\[ \sup_{(u_1, u_2) \in U_1 \times U_2} |b(x, u_1, u_2) - b(y, u_1, u_2)| \leq K_1|x - y| \quad \text{for all } x, y \in \mathbb{R}^n. \]

(c) There exists a positive constant $K_2$ such that for all $x, y \in \mathbb{R}^n$,
\[ |\sigma(x) - \sigma(y)| \leq K_2|x - y|. \]

(d) (Uniform ellipticity). The matrix $a(x) := \sigma(x)\sigma'(x)$ satisfies that, for some constant $K_3 > 0$,
\[ x' a(y) x \geq K_3 |x|^2 \quad \text{for all } x, y \in \mathbb{R}^n. \]

**Remark 1.** Lipschitz conditions on $b$ and $\sigma$ in Assumption 1(a)-(c), along with the compactness of $U_1$ and $U_2$ implies that there is a constant $\bar{K} \geq K_1 + K_2$ such that
\[ \sup_{(u_1, u_2) \in U_1 \times U_2} |b(x, u_1, u_2)| + |\sigma(x)| \leq \bar{K}(1 + |x|) \quad \text{for all } x \in \mathbb{R}^n. \]

Let $C^2(\mathbb{R}^n)$ be the space of real-valued functions $\nu(x)$ on $\mathbb{R}^n$ which are twice continuously differentiable. For $(u_1, u_2) \in U_1 \times U_2$, and $\nu$ in $C^2(\mathbb{R}^n)$, let
\[ L^{u_1, u_2} \nu(x) := \sum_{i=1}^n b^i(x, u_1, u_2) \partial_i \nu(x) + \frac{1}{2} \sum_{i,j=1}^n a^{ij}(x) \partial_{ij}^2 \nu(x), \quad (2) \]
where $b^i$ is the $i$-th component of $b$, and $a^{ij}$ is the $(i, j)$-component of the matrix $a(\cdot)$ defined in Assumption 1(d).

**Strategies of the players.** Let $\mathcal{P}(U_1)$ be the space of probability measures on $U_1$ endowed with the topology of weak convergence. The space $\mathcal{P}(U_2)$ is defined similarly.

For our purpose, in which we study non cooperative or Nash equilibria, we can restrict ourselves to consider randomized stationary strategies defined as follows.

**Definition 2.1.** A stationary randomized strategy for player 1 (resp. player 2) is a probability measure on $U_1$ (resp. $U_2$).

**Remark 2.** In general, the set of deterministic (ordinary) strategies for zero-sum stochastic differential games (SDGs) is such that, except for a quite restricted class of games (such as scalar linear SDGs, see [4, 20], and the references therein), one cannot assure the existence of a Nash equilibrium in the set of ordinary strategies for the players, see for instance [29]. By this reason, we enlarge the sets of ordinary strategies to include randomized strategies so that an equilibrium can be found in this new set.
Thecnically, our hypotheses ensure the existence of Nash equilibria for the α-discounted criteria (section 3), as well as, for average criterion (section 4) in the class of stationary strategies for all players (see, for instance [3, 7, 22] ) which is crucial in our study. Further, it is worth to mention that recurrence and ergodicity properties of the state system (1) can be easily verified through the use of stationary strategies, but for a more general class of strategies, this can be hard to handle.

The interpretation of a stationary randomized strategy is as follows. If player 1 observes the system (1) at time $t \geq 0$, then player 1 chooses his/her action in $U_1$ according to the probability measure $\phi \in \mathcal{P}(U_1)$. 

When the players use strategies $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ we write the drift coefficient $b$ in (1) as

$$
\begin{align*}
\Phi(x, \phi, \psi) := & \int_{U_1} \int_{U_1} b(x, u_1, u_2) \phi(du_1) \psi(du_2).
\end{align*}
$$

Moreover, recalling (2), for $h \in C^2(\mathbb{R}^n)$, let

$$
\begin{align*}
\mathcal{L}^{\phi, \psi}h(x) := & \int_{U_1} \int_{U_1} \mathcal{L}^{u_1, u_2}h(x) \phi(du_1) \psi(du_2).
\end{align*}
$$

**Remark 3.** Assumption 1 ensures that, for each pair of strategies $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ there exists an almost surely unique strong solution of (1) which is a Markov-Feller process. Furthermore, for each pair of strategies $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$, the operator $\mathcal{L}^{\phi, \psi}h$ in (4) becomes the infinitesimal generator of (1). (For more details, see the arguments of [2, Theorem 2.2.12].)

Sometimes we write $x(\cdot)$ as $x^{\phi, \psi}(\cdot)$ to emphasize the dependence on $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$. Also, we shall denote by $\mathbb{P}^{\phi, \psi}(t, x, \cdot)$ the corresponding transition probability of the process $x^{\phi, \psi}(\cdot)$, i.e., $\mathbb{P}^{\phi, \psi}(t, x, B) := \mathbb{P}(x^{\phi, \psi}(t) \in B | x(0) = x)$ for every Borel set $B \subset \mathbb{R}^n$ and $t \geq 0$. The associated conditional expectation is written as $\mathbb{E}^{\phi, \psi}_x(\cdot)$.

Theorem 4.3 of [1] can be used to prove that, for each $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$, the probability measure $\mathbb{P}^{\phi, \psi}(t, x, \cdot)$ is absolutely continuous with respect to Lebesgue’s measure $\lambda$ for every $t \geq 0$ and $x \in \mathbb{R}^n$. Hence, there exists a transition density function $p^{\phi, \psi}(t, x, y) \geq 0$ such that

$$
\mathbb{P}^{\phi, \psi}(t, x, B) = \int_B p^{\phi, \psi}(t, x, y)dy
$$

for every Borel set $B \subset \mathbb{R}^n$.

2.2. **Recurrence and ergodicity.** Next assumption (a Lyapunov-like condition) guarantees the positive recurrence of the diffusion (1).

**Assumption 2.** There exists a function $w \in C^2(\mathbb{R}^n)$, with $w \geq 1$, and constants $d \geq c > 0$ such that

(i) $\lim_{|x| \to \infty} w(x) = +\infty$, and

(ii) $\mathcal{L}^{\phi, \psi}w(x) \leq -cw(x) + d$ for each $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ and $x \in \mathbb{R}^n$.

Under the Assumption 2, for each $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$, the Markov process $x^{\phi, \psi}(\cdot)$ is positive recurrent with a unique invariant probability measure $\mu_{\phi, \psi}$ (see [2] or [19]), which

$$
\mu_{\phi, \psi}(w) := \int_{\mathbb{R}^n} w(x) \mu_{\phi, \psi}(dx) < \infty.
$$
Moreover, for every \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2), x \in \mathbb{R}^n,\) and \(t \geq 0\) an application of Dynkin’s formula to \(e^{ct}w(x(t))\) and Assumption \(2(ii)\) yield
\[
\mu_{\phi,\psi}(w) \leq \frac{d}{c},
\]
and
\[
\mathbb{E}_x^{\phi,\psi}w(x(t)) \leq e^{-ct}w(x) + \frac{d}{c}(1 - e^{-ct}). \tag{7}
\]

**Remark 4.** By Assumption \(1,\) every \(t > 0\) and each compact set \(C \subset \mathbb{R}^n,\) there exists \(\rho = \rho(t,C) > 0\) such that \(p_{\phi,\psi}(t,x,y) \geq \rho\) for all \(x,y \in C\) and \((\phi,\psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2).\) Now, by the construction of fundamental solution in Section 6.4 of [10], for each \(t > 0,\) \(p^{w_1,w_2}(t,x,y)\) is continuous in \(x,y \in \mathbb{R}^n\) and in \((u_1, u_2) \in U_1 \times U_2.

**Definition 2.2.** Let \(w\) be the function in Assumption \(2.\) Let \(\mathcal{B}_w(\mathbb{R}^n)\) be the normed linear space of real-valued measurable functions \(\nu\) on \(\mathbb{R}^n\) with finite \(w\)-norm, which is defined as
\[
\|\nu\|_w := \sup_{x \in \mathbb{R}^n} \frac{|\nu(x)|}{w(x)}.
\]

**Assumption 3.** Suppose that for every \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2),\) the process \(x^{\phi,\psi}(\cdot)\) in \(1\) is uniformly \(w\)-exponentially ergodic, that is, there exist constants \(C > 0\) and \(\delta > 0\) such that
\[
\sup_{(\phi,\psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)} \|\mathbb{E}_x^{\phi,\psi}[\nu(x(t))] - \mu_{\phi,\psi}(\nu)\| \leq Ce^{-\delta t} \|\nu\|_w(x) \tag{8}
\]
where \(\mu_{\phi,\psi}(\nu) := \int_{\mathbb{R}^n} \nu(x)\mu_{\phi,\psi}(dx)\) for all \(x \in \mathbb{R}^n, t \geq 0,\) and \(\nu \in \mathcal{B}_w(\mathbb{R}^n).\)

Sufficient conditions for the exponential ergodicity of the process \(x^{\phi,\psi}(\cdot)\) can be seen in [16, Theorem 2.7].

2.3. **Reward rate function.** Let \(r : \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}\) be a measurable function, which we call the payoff (or reward/cost) rate function; that is, \(r\) is the reward rate function for player 1, and it is interpreted as the cost rate function for player 2. The payoff rate satisfies the following conditions:

**Assumption 4.** (a) The function \(r(x, u_1, u_2)\) is continuous on \(\mathbb{R}^n \times U_1 \times U_2\) and locally Lipschitz in \(x,\) uniformly with respect to \((u_1, u_2) \in U_1 \times U_2;\) that is, for each \(R > 0,\) there exists a constant \(K(R) > 0\) such that
\[
\sup_{(u_1, u_2) \in U_1 \times U_2} |r(x, u_1, u_2) - r(y, u_1, u_2)| \leq K(R)|x - y| \text{ for all } |x|, |y| \leq R.
\]

(b) \(r(\cdot, u_1, u_2)\) is in \(\mathcal{B}_w(\mathbb{R}^n)\) uniformly in \((u_1, u_2);\) that is, there exists \(M > 0\) such that for all \(x \in \mathbb{R}^n\)
\[
\sup_{(u_1, u_2) \in U_1 \times U_2} |r(x, u_1, u_2)| \leq Mw(x).
\]

(c) \(r(x, u_1, u_2)\) is upper semicontinuous (u.s.c) and concave in \(u_1 \in U_1\) for every \((x, u_2) \in \mathbb{R}^n \times U_2,\) and lower semicontinuous (l.s.c) and convex in \(u_2 \in U_2\) for every \((x, u_1) \in \mathbb{R}^n \times U_1.\)

When players use the pair of strategies \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) we write the payoff rate \(r\) as in \((3),\) that is,
\[
r(x, \phi, \psi) := \int_{U_2} \int_{U_1} r(x, u_1, u_2)\phi(du_1)\psi(du_2). \tag{9}
\]
The next lemma provides important facts.
Lemma 2.3. Under Assumptions 1 and 4, for each fixed $h \in C^2(\mathbb{R}^n) \cap B_w(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the functions $r(x, \phi, \psi)$ and $\mathcal{L}^{\phi, \psi}h(x)$ are upper semicontinuous (u.s.c) in $\phi \in \mathcal{P}(U_1)$ and lower semicontinuous (l.s.c) in $\psi \in \mathcal{P}(U_2)$.

Proof. Under assumptions 1(b) and 4(c) functions $r(x, u_1, u_2)$ and $b(x, u_1, u_2)$ are upper semicontinuous in $u_1 \in U_1$, lower semicontinuous in $u_2 \in U_2$, and bounded on $U_1 \times U_2$ for each $x \in \mathbb{R}^n$. Hence, by the definition of weak convergence of probability measures, the result follows. \hfill \Box

Now, let $w$ be the function in Assumption 2. In addition to the space $B_w(\mathbb{R}^n)$ in Definition 2.2 we consider the space $B_w(\mathbb{R}^n \times U_1 \times U_2)$, which consists of the real-valued measurable function $v$ on $\mathbb{R}^n \times U_1 \times U_2$ such that

$$\sup_{(u_1, u_2) \in U_1 \times U_2} |v(x, u_1, u_2)| \leq M_v(x) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

(10)

where $M_v$ is a positive constant depending of $v$. By Assumption 4(b), the payoff rate $r$ belongs to $B_w(\mathbb{R}^n \times U_1 \times U_2)$. Note that $B_w(\mathbb{R}^n)$ is contained in $B_w(\mathbb{R}^n \times U_1 \times U_2)$, because we can write any function $v \in B_w(\mathbb{R}^n)$ as $v(x) = v(x, u_1, u_2)$ for all $(u_1, u_2) \in U_1 \times U_2$. As in (9) for $v \in B_w(\mathbb{R}^n \times U_1 \times U_2)$ we write

$$v(x, \phi, \psi) := \int_{U_2} \int_{U_1} v(x, u_1, u_2) \phi(du_1) \psi(du_2).$$

3. Discounted optimality. In this section, we are interested in existence of Nash equilibria for $\alpha-$discounted criteria. This criteria is defined as follows.

Definition 3.1. Fix $v \in B_w(\mathbb{R}^n \times U_1 \times U_2)$ and $\alpha$ a positive constant. The expected $\alpha-$discounted-$v$-payoff when the players use $(\phi, \psi)$ in $\mathcal{P}(U_1) \times \mathcal{P}(U_2)$, given the initial state $x \in \mathbb{R}^n$, is

$$V_\alpha(x, \phi, \psi, v) := \mathbb{E}^\phi_x \left[ \int_0^\infty e^{-\alpha t} v(x(t), \phi, \psi)dt \right].$$

(11)

Moreover, when $v \equiv r$, we simply write $V_\alpha(x, \phi, \psi) := V_\alpha(x, \phi, \psi, r)$.

The following result shows a bound of the $\alpha-$discounted-$v$-payoff (11) in a certain sense. We will omit its proof, because it is a direct consequence of (7) and (8).

Proposition 1. Assumption 2, (7) and (10) imply that the expected $\alpha-$discounted-$v$ payoff $V_\alpha(\cdot, \phi, \psi, v)$ belongs to the space $B_w(\mathbb{R}^n \times U_1 \times U_2)$ for each $(\phi, \psi)$ in $\mathcal{P}(U_1) \times \mathcal{P}(U_2)$; in fact, for each $x$ in $\mathbb{R}^n$ we have

$$\sup_{(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)} |V_\alpha(x, \phi, \psi, v)| \leq \mathcal{M}_v(\alpha)w(x)$$

with $\mathcal{M}_v(\alpha) := M_v \frac{\alpha + d}{\alpha c}$. (12)

Here, $c$ and $d$ are as in Assumption 2, and $M_v$ is the constant in (10).

Definition 3.2. We say that a pair of stationary strategies $(\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ is $\alpha-$discount optimal if

$$V_\alpha(x, \phi, \psi^*) \leq V_\alpha(x, \phi^*, \psi^*) \leq V_\alpha(x, \phi^*, \psi) \quad \text{for every} \quad (\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2).$$

In the $\alpha$-discounted stochastic differential game, player 1 wishes to maximize $\phi \rightarrow V_\alpha(x, \phi, \psi)$, while player 2 wishes to minimize $\psi \rightarrow V_\alpha(x, \phi, \psi)$. 

Remark 5. Isaac’s condition (see [9, Theorems 1 and 2] or [28, Theorem 4.2] for details). As a consequence of (12), function $V_\alpha$ is finite in $\mathcal{P}(U_1) \times \mathcal{P}(U_2)$ for each $x \in \mathbb{R}^n$. Moreover, by Assumption 4(c) the mapping $\phi \to V_\alpha(x, \phi, \psi)$ is u.s.c and concave on the compact set $\mathcal{P}(U_1)$, whereas $\psi \to V_\alpha(x, \phi, \psi)$ is l.s.c and convex on the compact set $\mathcal{P}(U_2)$, then we obtain for all $x \in \mathbb{R}^n$. 

$$\sup_{\phi \in \mathcal{P}(U_1)} \inf_{\psi \in \mathcal{P}(U_2)} V_\alpha(x, \phi, \psi) = \inf_{\psi \in \mathcal{P}(U_2)} \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi).$$  

(13)

Definition 3.3. The $\alpha-$discounted game value is defined as the common value (13), that is,

$$V_\alpha(x) := \sup_{\phi \in \mathcal{P}(U_1)} \inf_{\psi \in \mathcal{P}(U_2)} V_\alpha(x, \phi, \psi)$$

(14)

$$= \inf_{\psi \in \mathcal{P}(U_2)} \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi).$$

(15)

Definition 3.4. We say that a function $v_\alpha$ in $C^2(\mathbb{R}^n) \cap B_w(\mathbb{R}^n)$ and a pair of strategies $(\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ verify the $\alpha-$discount Bellman equations if

$$\alpha v_\alpha(x) = r(x, \phi^*, \psi^*) + \mathcal{L}^{\phi^*, \psi^*} v_\alpha(x)$$

(16)

for all $x \in \mathbb{R}^n$.

Our next theorem states the existence of a saddle point equilibria for the $\alpha-$discounted stochastic differential game. Moreover, it also ensures that the value of the game given in (14) verifies the so-called $\alpha-$discount Bellman equations (15)-(17). Its proof is given in [3, Theorem 2.1].

Theorem 3.5. Suppose that Assumptions 1, 2, 3 and 4 hold. Then:

(a) There exists a function $v_\alpha \in C^2(\mathbb{R}^n) \cup B_w(\mathbb{R}^n)$ and a pair of strategies $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$, such that verifies the $\alpha-$discount Bellman equations (15)-(17).

(b) Function $v_\alpha$ in (15)-(17) is the value of the discounted game, that is, $v_\alpha(x) = V_\alpha(x)$.

(c) There exists a pair $(\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ of $\alpha-$discount optimal strategies satisfying (15)-(17).

4. Average optimality. In this section we show existence of Nash equilibria for average criterion, this criteria is defined as follows:

Definition 4.1. For each $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$, $v \in B_w(\mathbb{R}^n \times U_1 \times U_2)$, and $T \geq 0$, let

$$J_T(x, \phi, \psi, v) := \mathbb{E}_x \psi \left[ \int_0^T v(x(t), \phi, \psi) dt \right]$$

(18)

be the $v-$ total expected payoff of $(\phi, \psi)$ over the time interval $[0, T]$, when the initial state is $x \in \mathbb{R}^n$. The $v-$payoff of $(\phi, \psi)$ given the initial state $x$ is

$$J(x, \phi, \psi, v) := \limsup_{T \to \infty} \frac{1}{T} J_T(x, \phi, \psi, v).$$

(19)

For $v \equiv r$, (19) is called long -run average payoff (also known as the ergodic payoff).
Definition 4.4. We say that a constant \( J \) is an average optimal (AO) strategy (also known as a saddle point for the average payoff Bellman equations) if, for every \( (\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2) \), the linearity of \( \nu \to \mathcal{L}^{\phi, \psi} \nu \), Assumptions 1(b), 4(c) and Lemma 2.3 yield

\[
\sup_{\phi \in \mathcal{P}(U_1)} \inf_{\psi \in \mathcal{P}(U_2)} \{ r(x, \phi, \psi) + \mathcal{L}^{\phi, \psi} \nu(x) \} = \inf_{\psi \in \mathcal{P}(U_2)} \sup_{\phi \in \mathcal{P}(U_1)} \{ r(x, \phi, \psi) + \mathcal{L}^{\phi, \psi} \nu(x) \}.
\]

Definition 4.3. The game value for average payoff is given by

\[
\mathcal{J} = \sup_{\phi \in \mathcal{P}(U_1)} \inf_{\psi \in \mathcal{P}(U_2)} \tau(\phi, \psi) = \inf_{\psi \in \mathcal{P}(U_2)} \sup_{\phi \in \mathcal{P}(U_1)} \tau(\phi, \psi) \quad (22)
\]

Definition 4.4. We say that a constant \( J \) is an average optimal strategy if, for every \( x \in \mathbb{R}^n \),

\[
J = r(x, \phi^*, \psi^*) + \mathcal{L}^{\phi^*, \psi^*} h(x) \quad (23)
\]

Next result, which can be found in [3, Theorem 2.2] and [7, Theorem 4.1], establishes the equivalence between an average optimal strategy and a canonical strategy. More precisely, [3] deals with the existence of a constant \( J \), a function \( h \) and a Nash equilibrium for the long-run average payoff such that (23)-(25) holds, and
proves the equivalence between an average optimal strategy and a canonical strategy.

**Theorem 4.5.** If Assumptions in Theorem 3.5 hold, then:

(i) There exists a solution \((J, h) \in \mathbb{R}^n \times C^2(\mathbb{R}^n) \cup B_w(\mathbb{R}^n)\) satisfying the average optimality equations \((23)-(25)\), with
\[
J := \lim_{\alpha \to 0} V_\alpha(0).
\]
Moreover, the constant \(J = \overline{J}\), the value of the average payoff game \((22)\), and the function \(h\) is unique up to additive constants; \(h\) is unique under the additional condition that \(h(0) = 0\).

(ii) There exists a pair of canonical strategies.

(iii) A pair of strategies is average optimal if, and only if, it is canonical.

Theorem 4.5 shows that diagram 1 below hold.

**Diagram 1.** We have
\[\text{CS} \iff \text{AO}.\]

5. **F-strong average optimality.** The main objective in this section is to prove that F-strong average optimality and average optimality are equivalent concepts.

Next definition generalizes the concept of F-strong average given in [13] for discrete-time Markov control processes on Borel spaces. For this end, let \(J_T(x, \phi, \psi, r)\) be as in \((18)\) with \(v \equiv r\).

**Definition 5.1.** A strategy \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is said to be F-strong average optimal (abbreviated F-strong AO) if
\[
\lim_{T \to \infty} \frac{1}{T}[J_T(x, \phi^*, \psi^*, r) - \sup_{\phi \in \mathcal{P}(U_1)} J_T(x, \phi, \psi^*, r)] = 0,
\]
and
\[
\lim_{T \to \infty} \frac{1}{T}[J_T(x, \phi^*, \psi^*, r) - \inf_{\psi \in \mathcal{P}(U_2)} J_T(x, \phi^*, \psi, r)] = 0.
\]

**Theorem 5.2.** Suppose that Assumptions 1, 2, 3 and 4 hold. Then a strategy \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is F-strong AO if and only if it is average optimal.

**Proof.** Suppose that \((\phi^*, \psi^*)\) is a F-strong AO strategy. Then, from \((19)\) and \((26)\) it follows that
\[
J(x, \phi^*, \psi^*, r) = \lim_{T \to \infty} \frac{1}{T} J_T(x, \phi^*, \psi^*, r) = \lim_{T \to \infty} \sup_{\phi \in \mathcal{P}(U_1)} \frac{1}{T} J_T(x, \phi, \psi^*, r).
\]
Now,
\[
\sup_{\phi \in \mathcal{P}(U_1)} \frac{1}{T} J_T(x, \phi, \psi^*, r) \geq \frac{1}{T} J_T(x, \phi, \psi^*, r) \quad \text{for all } \phi \in \mathcal{P}(U_1),
\]
therefore, letting \(T \to \infty\) in last inequality, from \((28)\) we deduce that
\[
J(x, \phi^*, \psi^*, r) = \lim_{T \to \infty} \sup_{\phi \in \mathcal{P}(U_1)} \frac{1}{T} J_T(x, \phi, \psi^*, r) \geq J(x, \phi, \psi^*, r) \quad \text{for all } \phi \in \mathcal{P}(U_1).
\]
Using \((27)\) we can obtain,
\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
\]
\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
\]
\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
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\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
\]
\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
\]
\[
J(x, \phi^*, \psi^*, r) \leq J(x, \phi^*, \psi^*, r),
\]
hence, by (29) and (30), \((\phi^*, \psi^*)\) is an average optimal.

Now, suppose that \((\phi^*, \psi^*)\) is average optimal, then by Definition 4.2
\[
\sup_{\phi \in \mathcal{P}(U_1)} J(x, \phi, \psi^*, r) \leq J(x, \phi^*, \psi^*, r).
\]
(31)

But, we know that
\[
J(x, \phi^*, \psi^*, r) \leq \sup_{\phi \in \mathcal{P}(U_1)} J(x, \phi, \psi^*, r).
\]
(32)

So, (26) follows from (31) and (32). By a similar argument, we can obtain the equation (27) implying that \((\phi^*, \psi^*)\) is \(F\)-strong average optimal.

DIAGRAM 2. As a consequence of Theorem 4.5 and 5.2, we have
\[F\text{-strong AO} \iff AO \iff CS.\]

We conclude this section with a generalization of the \(m\)-discount optimality concepts given in [17].

**Definition 5.3.** \((m\text{-discount optimality.})\) Let \(m \geq -1\) be an integer. A strategy \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is called
(i) \(m\)-discount optimal if
\[
\liminf_{\alpha \to 0} \alpha^{-m} [V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi, \psi^*)] \geq 0 \quad \text{for all } \phi \in \mathcal{P}(U_1),
\]
(33)
and
\[
\limsup_{\alpha \to 0} \alpha^{-m} [V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi^*, \psi)] \leq 0 \quad \text{for all } \psi \in \mathcal{P}(U_2),
\]
(34)
(ii) strong \(m\)-discount optimal if
\[
\lim_{\alpha \to 0} \alpha^{-m} [V_\alpha(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi^*)] = 0.
\]
(35)
\[
\lim_{\alpha \to 0} \alpha^{-m} [V_\alpha(x, \phi^*, \psi^*) - \inf_{\psi \in \mathcal{P}(U_2)} V_\alpha(x, \phi^*, \psi)] = 0.
\]
(36)

It is evident that strong \(m\)-discount optimality implies \(m\)-discount optimality.

In this article it concerned with the cases \(m = -1, 0\).

6. Strong \(-1\)-discount optimality. Our main objective in this section is to prove that strong \(-1\)-discount optimality and average reward optimality are equivalent concepts (Theorem 6.5). To this end, (a) we define the discrepancy function (37); (b) the \(\alpha\)-discount payoff with \(v \equiv r\) is expressed in terms of the constants \((J, h)\) that satisfy the average payoff optimality equations (23)-(25) (Lemma 6.2); (c) some properties of \(V_\alpha(x, \phi, \psi, r)\) are given (Lemma 6.3); (d) Lemma 6.4 will provide some important facts. We prove each of these lemmas in the sequel.

**Definition 6.1.** Let \((J, h)\) be a pair satisfying the average payoff optimality equations (23)-(25). We define the discrepancy function \(\Delta\) as
\[
\Delta(x, \phi, \psi) = r(x, \phi, \psi) - J + \mathcal{L}^{\phi, \psi} h(x).
\]
(37)

**Remark 7.** Let \((\phi^*, \psi^*)\) be an average optimal strategies. By (24), \(r(x, \phi, \psi^*) - J + \mathcal{L}^{\phi, \psi^*} h(x) \leq 0\) for all \(\phi \in \mathcal{P}(U_1)\), implying that
\[
\Delta(x, \phi, \psi^*) \leq 0 \quad \forall \ \phi \in \mathcal{P}(U_1).
\]
(38)

Similarly, using the equation (25), we obtain
\[
\Delta(x, \phi^*, \psi) \geq 0 \quad \forall \ \psi \in \mathcal{P}(U_2).
\]
(39)
Moreover, by Theorem 4.5(iii) the average optimal strategies satisfies the average payoff optimality equations, in particular, equation (23), then

\[ \Delta(x, \phi^*, \psi^*) = 0. \]  

(40)

The following lemma relates the expected \( \alpha \)-discounted reward criteria for the different reward rates \( \Delta, h \) and \( r \).

**Lemma 6.2.** For \( (\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2) \), \( \alpha > 0 \) and \( x \in \mathbb{R}^n \), the \( \alpha \)-discounted payoff \( V_{\alpha}(x, \phi, \psi) \) can be written as

\[ V_{\alpha}(x, \phi, \psi) = \frac{1}{\alpha} J + h(x) - \alpha V_{\alpha}(x, \phi, \psi, h) + V_{\alpha}(x, \phi, \psi, \Delta). \]  

(41)

**Proof.** Taking \( v \equiv \Delta \) in equation (11) we have

\[ V_{\alpha}(x, \phi, \psi, \Delta) = \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} \Delta(x(t), \phi, \psi) dt \right] \]

\[ = \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} r(x(t), \phi, \psi) dt \right] - J \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} dt \right] \]

\[ + \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} \mathcal{L}^{\phi, \psi} h(x(t)) dt \right] \quad \text{by (37)} \]

\[ = V_{\alpha}(x, \phi, \psi) - \frac{1}{\alpha} J + \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} \mathcal{L}^{\phi, \psi} h(x(t)) dt \right]. \]  

(42)

On the other hand, an application of Dynkin’s formula to \( e^{-\alpha t} h(x(t)) \), \( t \geq 0 \), gives

\[ \mathbb{E}_x^{\phi, \psi} \left[ e^{-\alpha t} h(x(t)) \right] = h(x) + \mathbb{E}_x^{\phi, \psi} \left[ \int_0^t e^{-\alpha s} \left[ \mathcal{L}^{\phi, \psi} h(x(s)) - \alpha h(x(s)) \right] ds \right], \]

taking limit as \( t \to \infty \) in this expression yields

\[ \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} \mathcal{L}^{\phi, \psi} h(x(t)) dt \right] = -h(x) + \alpha \mathbb{E}_x^{\phi, \psi} \left[ \int_0^{\infty} e^{-\alpha t} h(x(t)) dt \right] \]

\[ = -h(x) + \alpha V_{\alpha}(x, \phi, \psi, h). \]  

(43)

Finally, the stated result follows from (42) and (43). \( \Box \)

**Lemma 6.3.** Let \( (\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2) \) and \( v \in \mathcal{B}_w(\mathbb{R}^n \times U_1 \times U_2) \). Under Assumptions 1, 2, 3 and 4, for all \( x \in \mathbb{R}^n \),

(a) \( \lim_{\alpha \downarrow 0} |\alpha V_{\alpha}(x, \phi, \psi, v) - \pi(\phi, \psi)| = 0, \)

(b) For each \( t \geq 0, \)

\[ |\mathbb{E}_x^{\phi, \psi} v(x(t), \phi, \psi) - \mathbb{E}_0^{\phi, \psi} v(x(t), \phi, \psi)| \leq \tilde{M}_v \delta e^{-\delta t} w(x), \]  

(44)

with \( \tilde{M}_v := \delta^{-1} C M_v[1 + w(0)] \) and \( C, \delta \) as in (8).

(c) For each \( \alpha > 0, \)

\[ |V_{\alpha}(x, \phi, \psi, v) - V_{\alpha}(0, \phi, \psi, v)| \leq \tilde{M}_v w(x), \]  

(45)

with \( \tilde{M}_v \) as in (44).
Proof. (a) Pick an arbitrary strategy \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) and note that
\[
|\alpha V_\alpha(x, \phi, \psi, v) - \psi(\phi, \psi)| \leq \alpha \int_0^\infty e^{-\alpha t}|\mathbb{E}_x^\phi v(x(t), \phi, \psi)|dt \\
\leq \frac{\alpha c M(v(x))}{\alpha + \delta},
\]
where the second inequality follows from (8). This implies the result.

(b) For every \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2), x \in \mathbb{R}^n,\) and \(t \geq 0,\) we have
\[
|\mathbb{E}_x^\phi v(x, \phi, \psi) - \mathbb{E}_0^\phi v(x, \phi, \psi)| \leq |\mathbb{E}_x^\phi v(x, \phi, \psi)| \\
+ |\mathbb{E}_0^\phi v(x, \phi, \psi)| \\
\leq C e^{-\delta t} M(v(x) + v(0)) \quad \text{(by (8) and (10))}.
\]

This yields (44) since \(w \geq 1.\)

(c) By (44)
\[
|V_\alpha(x, \phi, \psi, v) - V_\alpha(0, \phi, \psi, v)| \\
\leq \int_0^\infty e^{-\alpha t}|\mathbb{E}_x^\phi v(x(t), \phi, \psi)|dt \\
\leq \frac{\alpha c M(v(x))}{\alpha + \delta},
\]
and thus (45) follows. \(\square\)

**Lemma 6.4.** Let \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) average optimal. Under assumptions given in Lemma 6.3, the following statements hold for all \(x \in \mathbb{R}^n,\)

a) \(\limsup_{\alpha \downarrow 0} \sup_{\phi \in \mathcal{P}(U_1)} |\alpha V_\alpha(x, \phi, \psi^*) - J| \leq 0,\)

b) \(\liminf_{\alpha \downarrow 0} \inf_{\psi \in \mathcal{P}(U_2)} |\alpha V_\alpha(x, \phi^*, \psi) - J| \geq 0.\)

**Proof.** Let \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) be an average optimal strategy. By properties (38) and (39) of \(\Delta\) we obtain that \(\Delta(x, \phi, \psi^*) \leq 0\) for all \(\phi \in \mathcal{P}(U_1)\) and \(\Delta(x, \phi^*, \psi) \geq 0\) for all \(\psi \in \mathcal{P}(U_2).\) The latter implies, by taking \(v \equiv \Delta\) in (11), that
\[
\alpha V_\alpha(x, \phi, \psi^*, \Delta) \leq 0 \quad \forall \ \phi \in \mathcal{P}(U_1),
\]
and
\[
\alpha V_\alpha(x, \phi^*, \psi, \Delta) \geq 0 \quad \forall \ \psi \in \mathcal{P}(U_2).
\]

Now, combining Lemma 6.2, (47) and (48), it follows that
\[
\alpha V_\alpha(x, \phi, \psi^*) \leq J + \alpha h(x) - \alpha^2 V_\alpha(x, \phi, \psi^*, h) \quad \forall \ \phi \in \mathcal{P}(U_1),
\]
and
\[
\alpha V_\alpha(x, \phi^*, \psi) \geq J + \alpha h(x) - \alpha^2 V_\alpha(x, \phi^*, \psi, h) \quad \forall \ \psi \in \mathcal{P}(U_2).
\]

These last inequalities implies
\[
\sup_{\phi \in \mathcal{P}(U_1)} \alpha V_\alpha(x, \phi, \psi^*) \leq J + \alpha h(x) - \sup_{\phi \in \mathcal{P}(U_1)} \alpha^2 V_\alpha(x, \phi, \psi^*, h), \quad \text{(49)}
\]
and
\[
\inf_{\psi \in \mathcal{P}(U_2)} \alpha V_\alpha(x, \phi^*, \psi) \geq J + \alpha h(x) - \inf_{\psi \in \mathcal{P}(U_2)} \alpha^2 V_\alpha(x, \phi^*, \psi, h). \quad \text{(50)}
\]
The result follows taking the limit as \( \alpha \to 0 \) in both sides of inequalities (49) and (50), because \( h \) and \( V_\alpha(x, \phi, \psi^*, h) \) are in \( \mathcal{B}_w(\mathbb{R}^n) \) (see Proposition 1).

\[ \]

**Theorem 6.5.** Suppose that assumptions in Lemma 6.3 hold. Then a strategy \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is average optimal if and only if it is strong -1-discount optimal.

**Proof.** Let \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) be an average optimal strategy. Then, by Lemma 6.3(a) with \( v \equiv r \) and Lemma 6.4, we get

\[
0 = \pi(\phi^*, \psi^*) - J \leq \lim_{\alpha \to 0} \alpha[V_\alpha(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi^*)] \leq 0, \tag{51}
\]

and

\[
0 = \pi(\phi^*, \psi^*) - J \geq \lim_{\alpha \to 0} \alpha[V_\alpha(x, \phi^*, \psi^*) - \inf_{\psi \in \mathcal{P}(U_1)} V_\alpha(x, \phi^*, \psi)] \geq 0. \tag{52}
\]

So, by (51) and (52), \((\phi^*, \psi^*)\) is strong -1-discount optimal.

Now, suppose that \((\phi^*, \psi^*)\) is strong -1-discount optimal. Then, \((\phi^*, \psi^*)\) satisfies the equations (35) and (36) with \( m = -1 \), that is,

\[
\lim_{\alpha \to 0} \alpha V_\alpha(x, \phi^*, \psi^*) = \lim_{\alpha \to 0} \sup_{\phi \in \mathcal{P}(U_1)} \alpha V_\alpha(x, \phi, \psi^*), \tag{53}
\]

\[
\lim_{\alpha \to 0} \alpha V_\alpha(x, \phi^*, \psi^*) = \lim_{\alpha \to 0} \inf_{\psi \in \mathcal{P}(U_2)} \alpha V_\alpha(x, \phi^*, \psi). \tag{54}
\]

Lemma 6.3(a) with \( v \equiv r \) implies that

\[
\lim_{\alpha \to 0} \alpha V_\alpha(x, \phi^*, \psi^*) = \pi(\phi^*, \psi^*). \tag{55}
\]

Therefore, there exist the limits of the expressions of right-hand side in equations (53) and (54).

On the other hand,

\[
\sup_{\phi \in \mathcal{P}(U_1)} \alpha V_\alpha(x, \phi, \psi^*) \geq \inf_{\psi \in \mathcal{P}(U_2)} \sup_{\phi \in \mathcal{P}(U_1)} \alpha V_\alpha(x, \phi, \psi) = \alpha V_\alpha(x), \tag{56}
\]

where last equation is by Definition 3.3. Moreover,

\[
|\alpha V_\alpha(x) - J| \leq \alpha|V_\alpha(x) - V_\alpha(0)| + |\alpha V_\alpha(0) - J| = \alpha M w(x) + |\alpha V_\alpha(0) - J| \quad \text{(by Proposition 6.3(c) with } v \equiv r). \tag{57}
\]

Therefore, letting \( \alpha \to 0 \) in last inequality, we conclude from Theorem 4.5(i) that

\[
\liminf_{\alpha \to 0} \alpha V_\alpha(x) = J. \tag{57}
\]

So, combining expressions (53), (55), (56), and (57) we obtain

\[
\pi(\phi^*, \psi^*) = \lim_{\alpha \to 0} \sup_{\phi \in \mathcal{P}(U_1)} \alpha V_\alpha(x, \phi, \psi^*) \geq J, \tag{58}
\]

by similar arguments, it’s shown that

\[
\pi(\phi^*, \psi^*) = \lim_{\alpha \to 0} \inf_{\psi \in \mathcal{P}(U_2)} \alpha V_\alpha(x, \phi^*, \psi) \leq J. \tag{59}
\]

Then, by (53)-(59), we have \( \pi(\phi^*, \psi^*) = J \). Therefore, by Theorem 4.5, \((\phi^*, \psi^*)\) is average optimal. \qed
Combining Theorem 4.5, Theorem 5.2, and Theorem 6.5, it can be seen that under Assumptions 1, 2, 3 and 4, implications in diagram 3 below hold.

**DIAGRAM 3.**
F-strong AO \(\iff\) CS \(\iff\) AO \(\iff\) Strong-1-discount optimal.

7. **Strong 0-discount optimality.** The main objective in this section is to prove that under our Assumptions: (a) strong 0–discount optimality implies bias optimality, and (b) bias optimality implies 0–discount optimality. To this end, we introduce the bias optimality concept.

**Definition 7.1.** Let \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\). The **bias** of \((\phi, \psi)\) is the function \(h_{\phi, \psi} \in B_w(\mathbb{R}^n)\) given by

\[
h_{\phi, \psi}(x) := \int_0^\infty [\mathbb{E}_x^\phi \mathcal{R}(x(t), \phi, \psi) - \tau(\phi, \psi)]dt \quad \text{for all } x \in \mathbb{R}^n.
\]

**Definition 7.2.** (Bias optimality) We say that a pair of average optimal strategies \((\phi^*, \psi^*)\) is **bias optimal** if

\[
h_{\phi, \psi^*}(x) \leq h_{\phi^*, \psi^*}(x) \leq h_{\phi^*, \psi}(x)
\]

for every \(x \in \mathbb{R}^n\) and every pair of average optimal strategies \((\phi, \psi)\). The function \(h_{\phi^*, \psi^*}\) is called the **optimal bias function**.

**Remark 8.** Assumptions 1, 2, 3 and 4 guarantee the following (for proof see, for instance [7, Proposition 5.2 and Theorem 5.1])

(a) If \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is average optimal, then its bias \(h_{\phi, \psi}(x)\) and any function \(h\) of the average optimality equations (23)-(25) coincide up to an additive constant; in fact,

\[
h_{\phi, \psi}(x) = h(x) - \mu_{\phi, \psi}(h) \quad \text{for all } x \in \mathbb{R}^n.
\]

Moreover, \(h_{\phi, \psi}(x)\) is in \(C^2(\mathbb{R}^n) \cup B_w(\mathbb{R}^n)\).

(b) Bias optimal strategies set is non empty.

(c) For each \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\), the pair \((\tau(\phi, \psi), h_{\phi, \psi})\) is the unique solution of the Poisson equation

\[
\tau(\phi, \psi) = r(x, \phi, \psi) + \mathcal{L}^\phi \psi h_{\phi, \psi}.
\]

(d) The \(\mu_{\phi, \psi}\)-expectation of the bias function is zero, i.e., \(\mu_{\phi, \psi}(h_{\phi, \psi}) = 0\).

**Theorem 7.3.** Suppose that Assumptions 1, 2, 3 and 4 hold. Then a stationary strategy \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is bias optimal if it is strong 0-discount optimal.

**Proof.** Suppose that \((\phi^*, \psi^*)\) is strong 0-discount optimal but it is not bias optimal. Now, we observe that

\[
V_\alpha(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi^*) \leq V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi, \psi^*)
\]

for all \(\phi \in \mathcal{P}(U_1)\).

On the other hand, by Lemma 6.2

\[
V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi^*, \psi^*) = \alpha[V_\alpha(x, \phi, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)]
\]

\[
+ V_\alpha(x, \phi^*, \psi^*, \Delta) - V_\alpha(x, \phi, \psi^*, \Delta).
\]

Suppose that the pair \((\phi, \psi^*)\) is bias optimal. Then, by definition, \((\phi, \psi^*)\) is average optimal. So, \(\Delta(x, \phi, \psi^*) = 0\). Moreover, since \((\phi^*, \psi^*)\) is strong 0-discount
relationship with the bias optimality criterion. We define it because of its

Proof. Suppose that \( (\phi^*, \psi^*) \) is bias optimal and \( (\phi^*, \psi^*) \) is not, we obtain from (60) that the right-hand side of (65) is strictly negative. This fact together with (63) give that

\[
\lim_{\alpha \to 0} [V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi, \psi^*)] < 0.
\]

This contradicts the hypothesis that \( (\phi^*, \psi^*) \) is strong 0-discount optimal. Then, \( (\phi^*, \psi^*) \) must be bias optimal.

**Theorem 7.4.** Suppose that assumptions in Theorem 7.3 hold. If \( (\phi^*, \psi^*) \) is bias optimal, then, it is 0-discount optimal.

**Proof.** Suppose that \( (\phi^*, \psi^*) \) is bias optimal. Then, by definition is average optimal, which gives that \( \Delta(x, \phi^*, \psi^*) = 0 \). Thus, by definition of \( V_\alpha \) in (11), \( V_\alpha(x, \phi^*, \psi^*, \Delta) = 0 \). This fact and (41) give that,

\[
V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi, \psi^*) = \alpha [V_\alpha(x, \phi, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)] + V_\alpha(x, \phi^*, \psi^*, \Delta) - V_\alpha(x, \phi, \psi^*, \Delta) \\
\geq \alpha [V_\alpha(x, \phi^*, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)] = \alpha [V_\alpha(x, \phi^*, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)]
\]

because by (39), \( V_\alpha(x, \phi, \psi^*, \Delta) \leq 0 \) for every \( \phi \in \mathcal{P}(U_1) \) and \( \psi^* \) average optimal. Similarly,

\[
V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi^*, \psi) = \alpha [V_\alpha(x, \phi^*, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)] + V_\alpha(x, \phi^*, \psi^*, \Delta) - V_\alpha(x, \phi^*, \psi, \Delta) \\
\leq \alpha [V_\alpha(x, \phi^*, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)] = \alpha [V_\alpha(x, \phi^*, \psi^*, h) - V_\alpha(x, \phi^*, \psi^*, h)]
\]

because by (38), \( V_\alpha(x, \phi, \psi^*, \Delta) \geq 0 \) for every \( \psi \in \mathcal{P}(U_2) \) and \& \( \phi^* \) average optimal.

Therefore, taking \( \lim \inf \) in both sides of (66) and \( \lim \sup \) in inequality (67) as \( \alpha \to 0 \) from Lemma 6.3 we deduce that

\[
\lim_{\alpha \to 0} [V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi, \psi^*)] \geq \mu_{\phi, \psi^*}(h) - \mu_{\phi^*, \psi^*}(h) \geq 0,
\]

for all \( \phi \in \mathcal{P}(U_1) \), and

\[
\lim_{\alpha \to 0} [V_\alpha(x, \phi^*, \psi^*) - V_\alpha(x, \phi^*, \psi)] \leq \mu_{\phi^*, \psi}(h) - \mu_{\phi^*, \psi^*}(h) \leq 0,
\]

for all \( \psi \in \mathcal{P}(U_2) \), because \( (\phi^*, \psi^*) \) is bias optimal; see (60) and (61). Finally, result follows from (68) and (69).

By Theorems 7.3 and 7.4 the implications in diagram 4 below, hold.

**DIAGRAM 4.**

Strong 0-discount optimal \( \Rightarrow \) Bias optimal \( \Rightarrow \) 0-discount optimal.

A criterion which is “sensitive” to the growth rate of the finite-horizon reward/cost is the criterion of overtaking optimality. We define it because of its relationship with the bias optimality criterion.
Definition 7.5. A pair of strategies \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is said to be overtaking optimal if for each \((\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) and \(x \in \mathbb{R}^n\) we have
\[
\lim_{T \to \infty} \inf \{ J_T(x, \phi^*, \psi^*, r) - J_T(x, \phi, \psi^*, r) \} \geq 0,
\]
and
\[
\lim_{T \to \infty} \sup \{ J_T(x, \phi^*, \psi^*, r) - J_T(x, \phi^*, \psi, r) \} \leq 0.
\]

Remark 9. (a) Comparing inequalities given in the Definitions 4.2 and 7.5 together with (19), we can see that if \((\phi^*, \psi^*) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)\) is overtaking optimal in \( \mathcal{P}(U_1) \times \mathcal{P}(U_2) \), then it is average optimal.
(b) Let \((\phi^*, \psi^*)\) and \((\phi, \psi)\) be average optimal strategies. Then, a simple use of the Definition 4.2 gives
\[
\tau(\phi^*, \psi^*) = \tau(\phi, \psi) = \tau(\phi^*, \psi) = \tau(\phi^*, \psi).
\]

Remark 10. Authors in [7] identify the set of average optimal strategies and then, within this set, they look strategies that, for instance, maximize/minimize the bias. In our framework, the relationship between overtaking optimality and bias optimality is similar to the obtained by authors in [7], Theorem 6.1, but the converse holds only in the class of average optimal strategies.

From (8), Remark 8(d), Remark 9, (71) and (72), we can obtain the following implications.

i) If \((\phi^*, \psi^*)\) is overtaking optimal, then it is bias optimal.
ii) If \((\phi^*, \psi^*)\) is bias optimal, then it is overtaking optimal in the class of average optimal strategies.

By Theorems 7.3, 7.4, and Remark 10 the implications in diagram 5, hold.

DIAGRAM 5.

Overtaking optimal
\[ \Downarrow \Uparrow \] in AO

Strong 0-discount optimal \(\implies\) Bias optimal \(\implies\) 0-discount optimal
8. **Stochastic differential games with additive structure.** To obtain that bias optimality implies strong 0–discount optimality, in this section we work with a class of stochastic differential games with additive structure and bounded coefficients. This kind of games has been recently studied in [8, 18, 23]. The following additional assumption is required.

**Assumption 5.** (a) There exist measurable functions $b_1 : \mathbb{R}^n \times U_1 \to \mathbb{R}$ and $b_2 : \mathbb{R}^n \times U_2 \to \mathbb{R}$ with the same properties from $b$ given in Assumption 1 (continuity, linear growth and Lipschitz), such that the diffusion coefficient $b$ is given by

$$b(x, u_1, u_2) := b_1(x, u_1) + b_2(x, u_2).$$

(b) There exist bounded measurable functions $v_1 : \mathbb{R}^n \times U_1 \to \mathbb{R}$ and $v_2 : \mathbb{R}^n \times U_2 \to \mathbb{R}$ with the same properties from $r$ given in Assumption 4 such that the function $v$ in (10) can be expressed as

$$v(x, u_1, u_2) := v_1(x, u_1) + v_2(x, u_2).$$

(c) There exist density functions $p^{\phi}(t, x, y)$ and $p^{\psi}(t, x, y)$ with the same properties from $p^{\phi, \psi}(t, x, y)$ such that

$$p^{\phi, \psi}(t, x, y) := p^{\phi}(t, x, y) + p^{\psi}(t, x, y).$$

Throughout this section, by Assumption 5 the expected $\alpha$–discounted-$v$-payoff (Definition 11) and the $v$–payoff (19) are expressed as follows.

$$V_\alpha(x, \phi, \psi, v) := \mathbb{E}_x^{\phi, \psi} \left[ \int_0^\infty e^{-\alpha t} v_1(x(t), \phi) dt \right] + \mathbb{E}_x^{\phi, \psi} \left[ \int_0^\infty e^{-\alpha t} v_2(x(t), \psi) dt \right].$$

$$J(x, \phi, \psi, v) := \int_{\mathbb{R}^n} v_1(x(t), \phi) \mu_{\phi, \psi}(dx) + \int_{\mathbb{R}^n} v_2(x(t), \psi) \mu_{\phi, \psi}(dx).$$

**Remark 11.** All the results in previous sections are valid without the Assumption 5.

8.1. **Bias optimality implies strong 0–discount optimality.** In this subsection, we prove that if the game has additive structure and bounded coefficients, then, bias optimality implies strong 0-discount optimality.

**Remark 12.** If one of the players, say player 2, fixed a strategy $\psi \in \mathcal{P}(U_2)$, then we have a control problem for player 1. We say that $\phi^* \in \mathcal{P}(U_1)$ is an optimal response to $\psi \in \mathcal{P}(U_2)$ if

$$V_\alpha(x, \phi^*, \psi) = \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi).$$

Similarly, we say that $\psi^* \in \mathcal{P}(U_2)$ is an optimal response to $\phi \in \mathcal{P}(U_1)$ if

$$V_\alpha(x, \phi, \psi^*) = \inf_{\psi \in \mathcal{P}(U_2)} V_\alpha(x, \phi, \psi).$$

It is easy verify that if $\phi^*$ is an optimal response to $\psi^*$ and, conversely, then $(\phi^*, \psi^*)$ is $\alpha$-discount optimal.

The following result plays an important role in our analysis. Its proof is given in the Appendix.

**Lemma 8.1.** Suppose that assumptions in Theorem 7.3 and Assumption 5 hold true. Let $\{\alpha_m : m = 1, 2, \ldots\}$ be a sequence of positive numbers converging to zero and let $(\phi_{\alpha_m}, \psi_{\alpha_m})$ be a sequence of $\alpha_m$-discount optimal strategies. If $\phi_{\alpha_m} \to \phi^*$ in $\mathcal{P}(U_1)$, and $\psi_{\alpha_m} \to \psi^*$ in $\mathcal{P}(U_2)$, then, $(\phi^*, \psi^*)$ is average optimal.
Lemma 8.2. Suppose that Assumptions in Lemma 8.1 hold. For each $\alpha > 0$, let $(\phi_\alpha, \psi_\alpha) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ be an $\alpha$-discount optimal strategy such that $\phi_\alpha \to \phi$ and $\psi_\alpha \to \psi$, weakly. Then,

(a) If $v \in \mathcal{B}_w(\mathbb{R}^n \times U_1 \times U_2)$ is continuous on $U_1 \times U_2$, then for all $x \in \mathbb{R}^n$

$$\lim_{\alpha \to 0} [\alpha V_\alpha(x, \phi_\alpha, \psi_\alpha, v)] = \overline{v}(\phi, \psi),$$

with $\overline{v}(\phi, \psi) = \int_{\mathbb{R}^n} v_1(x, \phi) \mu_{\phi, \psi}(dx) + \int_{\mathbb{R}^n} v_2(x, \psi) \mu_{\phi, \psi}(dx)$.

(b) For all pair $(\phi^*, \psi^*)$ of average optimal strategies

$$\liminf_{\alpha \to 0} [V_\alpha(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_\alpha(x, \phi, \psi^*)] \geq \mu_{\phi^* - h} - \mu_{\phi, \psi^*}(-h).$$

$$\limsup_{\alpha \to 0} [V_\alpha(x, \phi^*, \psi^*) - \inf_{\psi \in \mathcal{P}(U_2)} V_\alpha(x, \phi^*, \psi)] \leq \mu_{\phi^*, \psi^*}(-h) - \mu_{\phi^*, \psi}(-h).$$

Proof. (a) Note that we can write the $\alpha-$discounted-$v$-payoff (11) as

$$|\alpha V_\alpha(x, \phi_\alpha, \psi_\alpha, v)| = \alpha |V_\alpha(x, \phi, \psi, v)| - V_\alpha(x, \phi, \psi, v) + |\alpha V_\alpha(x, \phi, \psi, v)|.$$

Moreover, observe that

$$|V_\alpha(x, \phi_\alpha, \psi_\alpha, v) - V_\alpha(x, \phi, \psi, v)| \leq \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} \left[ \int_0^\infty e^{-\alpha t} v_1(x(t), \phi_\alpha) dt \right]$$

$$+ \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} \left[ \int_0^\infty e^{-\alpha t} v_2(x(t), \psi_\alpha) dt \right]$$

$$- \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} \left[ \int_0^\infty e^{-\alpha t} v_2(x(t), \psi) dt \right]$$

$$\leq \int_0^\infty e^{-\alpha t} |\mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi_\alpha) - \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi)| dt$$

$$+ \int_0^\infty e^{-\alpha t} |\mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_2(x(t), \psi_\alpha) - \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_2(x(t), \psi)| dt,$$

and by Assumption 5 the first integral in right-hand side of (78) is finite, because

$$|\mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi_\alpha) - \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi)| \leq M |\mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} w(x) + \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} w(x)|$$

$$\leq 2M[e^{-ct} w(x) + \frac{d}{c}(1 - e^{-ct})].$$

Now,

$$|\mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi_\alpha) - \mathbb{E}_{x}^{\phi_\alpha, \psi_\alpha} v_1(x(t), \phi)| \leq \int_{\mathbb{R}^n} |p^{\phi_\alpha, \psi_\alpha}(t, x, y) v_1(y, \phi_\alpha) - p^{\phi_\alpha, \psi_\alpha}(t, x, y) v_1(y, \phi)| dy$$

$$\leq \int_{\mathbb{R}^n} |p^{\phi_\alpha, \psi_\alpha}(t, x, y) - p^{\phi}(t, x, y)| v_1(y, \phi_\alpha) - v_1(y, \phi)| dy$$

$$+ \int_{\mathbb{R}^n} |p^{\phi_\alpha}(t, x, y) - p^{\phi}(t, x, y)| v_1(y, \phi)| dy$$
by Assumption 5 and the weak convergence criterion we obtain that the first integral in the last inequality from right-hand side of (80) converge to zero. Whereas, by the continuity of $u_1 \rightarrow p^{u_1}(t, x, y)$ and $u_2 \rightarrow p^{u_2}(t, x, y)$ (recall Remark 4) we deduce that $p^{\phi_1}(t, x, y) \rightarrow p^{\phi}(t, x, y)$ and $p^{\phi_2}(t, x, y) \rightarrow p^{\psi}(t, x, y)$ in $L^1(\mathbb{R}^n)$ for all $v_1 \in C_b(\mathbb{R}^n)$ (bounded functions space); therefore, the second and third integral in (80) converge to zero.

By similar arguments it can be deduced that

$$
|E^{\phi_1, \psi_1}v_2(t, \psi_1) - E^{\phi_2, \psi_2}v_2(t, \psi)| \leq 2M_1[e^{-\epsilon t}w(x) + \frac{d}{c}(1 - e^{-\delta t})],
$$

and

$$
|E^{\phi_1, \psi_1}v_2(t, \psi_1) - E^{\phi_2, \psi_2}v_2(t, \psi_1)| \rightarrow 0 \text{ as } \alpha \rightarrow 0.
$$

Hence, by (79)-(82), the right-hand side of (78) will converge to zero by the Lebesgue’s dominated convergence theorem. So, as $\alpha \rightarrow 0$

$$
|V_0(x, \phi_\alpha, \psi_\alpha, v) - V_0(x, \phi, \psi, v)| \rightarrow 0, \text{ for each } x \in \mathbb{R}^n,
$$

whereas, by Lemma 6.3(a)

$$
\alpha V_0(x, \phi, \psi, v) \rightarrow \mu(\phi, \psi) \text{ as } \alpha \rightarrow 0.
$$

Using (77), (83) and (84) we get (74).

(b) We only prove inequality (75), since inequality (76) is proved of similar manner.

To $\psi^* \in \mathcal{P}(U_2)$ fixed, by Assumption 5(b) with $v \equiv r$, the compactness of $\mathcal{P}(U_1)$ and Remark 12, there exist an $\alpha$-discount policy, $\phi_\alpha$, for the control problem associated to $\psi^*$, such that (73) hold. Hence, using Lemma 6.2 we have

$$
V_0(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_0(x, \phi, \psi^*) = V_0(x, \phi^*, \psi^*) - V_0(x, \phi_\alpha, \psi^*)
= \alpha[V_0(x, \phi_\alpha, \psi^*, h) - V_0(x, \phi^*, \psi^*, h)] + V_0(x, \phi^*, \psi^*, \Delta) - V_0(x, \phi_\alpha, \psi^*, \Delta),
\geq \alpha[V_0(x, \phi_\alpha, \psi^*, h) - V_0(x, \phi^*, \psi^*, h)],
$$

because $(\phi^*, \psi^*)$ is average optimal then, $\Delta(x, \phi^*, \psi^*) = 0$. Moreover, by properties of the discrepancy function, $\Delta$, we obtain that $\Delta(x, \phi_\alpha, \psi^*) \leq 0$ for all $\phi_\alpha \in \mathcal{P}(U_1)$ and $\psi^*$ average optimal. This last given $V_0(x, \phi^*, \psi^*, \Delta) = 0$ and $V_0(x, \phi_\alpha, \psi^*, \Delta) \leq 0$.

Now, taking lim inf in inequality (85) as $\alpha \rightarrow 0$, Lemma 6.3 and Lemma 8.2(a) gives

$$
\lim_{\alpha \rightarrow 0} [V_0(x, \phi^*, \psi^*) - \sup_{\phi \in \mathcal{P}(U_1)} V_0(x, \phi, \psi^*)] \geq \mu_{\phi^*, \psi^*}(h) - \mu_{\phi^*, \psi^*}(h)
= \mu_{\phi^*, \psi^*}(-h) - \mu_{\phi^*, \psi^*}(-h),
$$

and result follows.

\begin{theorem}
Suppose that assumptions in Theorem 7.3 hold. If $(\phi^*, \psi^*)$ is bias optimal, then, it is strong 0-discount optimal.
\end{theorem}
Definition 9.1. denote the closure of this set by \( \overline{\text{functions}} \) theorems indeed hold. To this end, we introduce the following notation.

Appendix: Proof Lemma 8.1. An example to illustrate our results can be build combining the

9. Appendix: Proof Lemma 8.1. Throughout this section, we work with a zero-sum stochastic differential game with additive structure. The proof of Lemma 8.1 is based on a result in [18], which is an extension to zero-sum stochastic differential games of Lemma A.16 in Arapostathis and Borkar [1]. We next write in our present this theorems in [18], and then we verify that the hypotheses in these theorems indeed hold. To this end, we introduce the following notation.

Let \( \mathcal{O} \) be a bounded domain in \( \mathbb{R}^n \), i.e., an open and connected subset of \( \mathbb{R}^n \) and denote the closure of this set by \( \overline{\mathcal{O}} \).

Definition 9.1. The set \( C^{2,\beta}(\mathcal{O}) \) is the normed subspace of \( C^2(\mathcal{O}) \) consisting of those functions \( f \) for which \( f, \partial_x f, \) and \( \partial^2_{x_i x_j} f \) satisfy a Hölder condition with exponent \( \beta \in ]0,1[ \) on \( \mathbb{R} \), for \( i, j = 1, 2, \ldots, n \).

Definition 9.2. Fix \( p \geq 1 \). The normed space \( L^p(\mathcal{O}) \) consists of all measurable functions \( f \) on \( \mathcal{O} \) for which \( \|f\|_{L^p(\mathcal{O})} < \infty \), where

\[
\|f\|_{L^p(\mathcal{O})} := \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}.
\]

Definition 9.3. The set \( W^{2,p}(\mathcal{O}) \) is the space of measurable functions \( f \) in \( L^p(\mathcal{O}) \) such that \( f \), and its first and second weak derivatives, \( \partial_x f, \partial^2_{x_i x_j} f \), are in \( L^p(\mathcal{O}) \) for all \( i, j = 1, \ldots, n \).

For every \( x \in \mathbb{R}^n \), \( (u_1, u_2) \) in \( U_1 \times U_2 \), \( \alpha > 0 \), and \( h \) in \( W^{2,p}(\mathcal{O}) \), let

\[
\hat{\Psi}(x, u_1, u_2, h, \alpha) := r_1(x, u_1) + r_2(x, u_2) + \sum_{i=1}^{n} b^i_1(x, u_1) \partial_i h(x) + \sum_{i=1}^{n} b^i_2(x, u_2) \partial_i h(x)
\]

\[- \alpha h(x),
\]

and

\[
\hat{\Psi}^{u_1, u_2}(h, x) := \hat{\Psi}(x, u_1, u_2, h, \alpha) + \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) \partial^2_{ij} h(x),
\]
with \(a\) as in Assumption 1(d) and \(b\) and \(v \equiv r\) as in Assumption 5.

We denote
\[
\Psi(x, \phi, \psi, h, \alpha) := \int_{U_2} \int_{U_2} \Psi(x, u_1, u_2, h, \alpha) \phi(du_1) \psi(du_2).
\]
\[
\tilde{\xi}_{\alpha}^{\phi, \psi} h(x) := \Psi(x, \phi, \psi, h, \alpha) + \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(x) \partial_{ij}^2 h(x). \tag{1}
\]
\[
\Phi_1(x, \psi, h, \alpha) := \sup_{\phi \in \mathcal{P}(U_1)} \hat{\Psi}(x, \phi, \psi, h, \alpha)
\]
\[
\Phi_2(x, \phi, h, \alpha) := \inf_{\psi \in \mathcal{P}(U_2)} \hat{\Phi}(x, \phi, \psi, h, \alpha).
\]
\[
\tilde{\xi}_{\alpha}^{\phi} h(x) := \Phi_1(x, \psi, h, \alpha) + \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(x) \partial_{ij}^2 h(x). \tag{2}
\]
\[
\tilde{\xi}_{\alpha}^{\psi} h(x) := \Phi_2(x, \phi, h, \alpha) + \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(x) \partial_{ij}^2 h(x). \tag{3}
\]

**Definition 9.4.** (Weak topology on \(\mathcal{P}(U_1)\)). Let \(C_0(U_1)\) be the space of continuous bounded functions on \(U_1\). A sequence \(\{\phi_m\}\) in \(\mathcal{P}(U_1)\) is said to converge weakly to \(\phi \in \mathcal{P}(U_1)\), and we denote such convergence as \(\phi_m \rightharpoonup \phi\), if
\[
\int_{U_1} h(u_1) \phi_m(du_1) \rightarrow \int_{U_1} h(u_1) \phi(du_1),
\]
for all \(h \in C_0(U_1)\). Similarly, we define the convergence in \(\mathcal{P}(U_2)\).

Using the convergence criterion given in [1] or [2], section 5 in [18] proves that the convergences \(\phi_m \rightarrow \phi\), \(\psi_m \rightarrow \psi\), \(\alpha_m \rightarrow \alpha\), and \(h_m \rightarrow h\), yield the following properties
\[
\lim_{m \rightarrow \infty} \left\{ r(\cdot, \phi_m, \psi_m) + \mathcal{L}^{\phi_m, \psi_m} h_m - \alpha_m h_m \right\} = r(\cdot, \phi, \psi) + \mathcal{L}^{\phi, \psi} h - \alpha h, \tag{4}
\]
\[
\lim_{m \rightarrow \infty} \sup_{\phi \in \mathcal{P}(U_1)} \left\{ r(\cdot, \phi, \psi_m) + \mathcal{L}^{\phi, \psi_m} h_m - \alpha_m h_m \right\} = \sup_{\phi \in \mathcal{P}(U_1)} \left\{ r(\cdot, \phi, \psi) + \mathcal{L}^{\phi, \psi} h - \alpha h \right\},
\]
\[
\lim_{m \rightarrow \infty} \inf_{\psi \in \mathcal{P}(U_2)} \left\{ r(\cdot, \phi_m, \psi) + \mathcal{L}^{\phi_m, \psi} h_m - \alpha_m h_m \right\} = \inf_{\psi \in \mathcal{P}(U_2)} \left\{ r(\cdot, \phi, \psi) + \mathcal{L}^{\phi, \psi} h - \alpha h \right\},
\]
with \(r(x, \phi, \psi) = r_1(x, \phi) + r_2(x, \psi)\). However, such results also hold for the weak convergence criterion (Definition 9.4).

The following result establishes the limit result referred to in (4).

**Theorem 9.5.** Let \(\mathcal{O}\) be a \(C^2\) domain and suppose that Assumptions 1, 3, 2 and 4 hold. In addition, assume that there exist sequences \(\{h_m\} \subset \mathcal{W}^{2,p}(\mathcal{O})\), \(\{\xi_m\} \subset L^p(\mathcal{O})\), with \(p > n\) (\(n\) is the dimension of (1)), \(\{(\phi_n, \psi_n)\} \subset \mathcal{P}(U_1) \times \mathcal{P}(U_2)\), and \(\{\alpha_m\} \subset (0, 1)\), satisfying the following:

(a) \(\tilde{\xi}_{\alpha_m}^{\phi_m, \psi_m} h_m = \xi_m\) in \(\mathcal{O}\) for \(m = 1, 2, \ldots\)

(b) There exists a constant \(M\) such that \(\|h_m\|_{\mathcal{W}^{2,p}(\mathcal{O})} \leq M\) for \(m = 1, 2, \ldots\)

(c) \(\xi_m\) converges in \(L^p(\mathcal{O})\) to some function \(\xi\).

(d) \(\alpha_m\) converges to some \(\alpha\)

(e) \(\phi_m \rightarrow \phi\) in \(\mathcal{P}(U_1)\) and \(\psi_n \rightarrow \psi\) in \(\mathcal{P}(U_2)\).
Then there exist a function $h \in W^{2,p}(\mathcal{O})$ and a subsequence $\{m_k\} \subset \{1, 2, \ldots\}$ such that $h_{m_k} \to h$ in the norm of $C^{1,\beta}(\mathcal{O})$ for $\beta < 1 - \frac{2}{p}$ as $k \to \infty$. Moreover,

$$\hat{L}_\alpha^\phi \psi h = \xi \text{ in } \mathcal{O}. \quad (7)$$

The following result deals with the convergence presented in (5).

**Theorem 9.6.** Assume that conditions of Theorem 9.5 are satisfied except that now we replace condition (a) of that theorem by the following:

(a'): $\hat{L}^{1,\psi}_m h_m = \xi_m$ in $\mathcal{O}$ for $m = 1, 2, \ldots$

Then, there exist a function $h \in W^{2,p}(\mathcal{O})$ and a subsequence $\{m_k\} \subset \{1, 2, \ldots\}$ such that $h_{m_k} \to h$ in the norm of $C^{1,\beta}(\mathcal{O})$ for $\beta < 1 - \frac{2}{p}$ as $k \to \infty$. Moreover,

$$\hat{L}^{1,\psi}_\alpha h = \xi \text{ in } \mathcal{O}. \quad (8)$$

The following result is analogous to Theorem 9.6.

**Theorem 9.7.** Assume the hypotheses of Theorem 9.5, except for condition (a), which we replace with:

(a''): $\hat{L}^{2,\psi}_m h_m = \xi_m$ in $\mathcal{O}$ for $m = 1, 2, \ldots$

Then, there exist a function $h \in W^{2,p}(\mathcal{O})$, and a subsequence $\{m_k\} \subset \{1, 2, \ldots\}$ satisfying $h_{m_k} \to h$ in the norm of $C^{1,\beta}(\mathcal{O})$ for $\beta < 1 - \frac{2}{p}$ as $k \to \infty$. Moreover,

$$\hat{L}^{2,\psi}_\alpha h = \xi \text{ in } \mathcal{O}. \quad (9)$$

9.1. **Proof Lemma 8.1. The vanishing discount technique.** We will prove existence of a Nash equilibrium for the long-run average payoff (19) with $v = r$ ($r = r_1 + r_2$) using the so-called vanishing discount approach. The idea is to impose conditions on an associated $\alpha$-discounted payoff model in such a way that, when $\alpha \downarrow 0$, we obtain average payoff optimality equations (23)-(25). To this end, when $(\phi_{\alpha_m}, \psi_{\alpha_m})$ becomes a Nash equilibrium for the $\alpha_m$-discounted payoff, we define:

$$v_{\alpha_m}(x) := V_\alpha(x, \phi_{\alpha_m}, \psi_{\alpha_m}),$$

and

$$h_{\alpha_m}(x) := v_{\alpha_m}(x) - v_{\alpha_m}(0).$$

The proof of the following proposition is the same as the proof of Proposition 4.4 in [18], therefore is omitted.

**Proposition 2.** Fix $R > 0$ and $\alpha > 0$. Suppose that Assumptions 1, 2, 3, 4, and 5 hold true. Let $\{\alpha_m\}$ be a sequence of positive numbers converging to zero and a sequence $(\phi_{\alpha_m}, \psi_{\alpha_m})$ of Nash equilibria associated to the $\alpha_m$-discounted payoff. Consider also a pair $(\phi, \psi) \in \mathcal{P}(U_1) \times \mathcal{P}(U_2)$ such that $\phi_{\alpha_m} \to \phi$ and $\psi_{\alpha_m} \to \psi$. Then, the following statements are true:

(a) There exists $g \in \mathbb{R}$ and a subsequence $\alpha_{m_k} (=: \alpha_m)$ of $\alpha_m$, such that

$$\alpha_m v_{\alpha_m}(0) \to g \text{ as } \alpha_m \to 0.$$

(b) Function $h_{\alpha_m}$ satisfies

$$\|h_{\alpha_m}\|_{W^{2,p}(B_R)} \leq \tilde{K}, \quad (8)$$

for some positive constant $\tilde{K}$ and as a consequence,

$$\|\alpha_m h_{\alpha_m}\|_{W^{2,p}(B_R)} \to 0, \text{ as } \alpha_m \to 0.$$
Proof Lemma 8.1. Let \((\phi_{\alpha_m}, \psi_{\alpha_m})\) be a Nash equilibrium for the \(\alpha_m\)-discounted payoff. Then, the pair \((\phi_{\alpha_m}, \psi_{\alpha_m})\) satisfies equations (15)-(17) in \(\mathbb{R}^n\) for each \(m = 1, 2, \ldots\). Replacing \(h_{\alpha_m}\) given in (8) to the equations (15)-(17), we get

\[
\alpha_m v_{\alpha_m}(0) = r(x, \phi_{\alpha_m}, \psi_{\alpha_m}) + L^{\phi_{\alpha_m}, \psi_{\alpha_m}} h_{\alpha_m}(x) - \alpha_m h_{\alpha_m}(x) \quad \text{for } x \in B_R.
\]

Finally, since previous convergence was uniform, we get that for each \(\epsilon > 0\), the third equality in (12), shows that

\[
\lim_{m \to \infty} \alpha_m v_{\alpha_m}(0) = \lim_{m \to \infty} r(x, \phi_{\alpha_m}, \psi_{\alpha_m}) + L^{\phi_{\alpha_m}, \psi_{\alpha_m}} h_{\alpha_m}(x) - \alpha_m h_{\alpha_m}(x) = g,
\]

In terms of the operators in (1), (2), and (3), expression (9) become

\[
\tilde{\mathcal{L}}_{\alpha_m} h_{\alpha_m}(x) = \alpha_m v_{\alpha_m}(0),
\]

and

\[
\tilde{\mathcal{L}}^{1, \psi_m}_{\alpha_m} h_{\alpha_m}(x) = \alpha_m v_{\alpha_m}(0),
\]

and

\[
\tilde{\mathcal{L}}^{2, \phi_m}_{\alpha_m} h_{\alpha_m}(x) = \alpha_m v_{\alpha_m}(0).
\]

Then, by Proposition 2, there exits a subsequence of \(\alpha_m\) such that, for \(R > 0\) fixed,

\[
\alpha_m v_{\alpha_m}(0) \to g,
\]

as well as

\[
\alpha h_{\alpha_m}(x) \to 0, \quad \text{in } W^{2,p}(B_R)
\]

It is easy to verify that results in Proposition 2 together with equation (9) yield that the hypotheses established in Theorems 9.5, 9.6 and 9.7 hold. Then, by invoking this theorem, and noting (10) and (11), we can claim the existence of a function \(h \in W^{2,p}(B_R)\) such that the following is satisfied \(h_{\alpha_m} \to h\) as \(\alpha_m \to 0\) uniformly on \(B_R\). Further,

\[
g = r(x, \phi^*, \psi^*) + L^{\phi^*, \psi^*} h(x)
\]

Last treatment can be extended for all \(x \in \mathbb{R}^n\) because \(R > 0\) was arbitrary. Finally, since previous convergence was uniform, we get that for each \(\epsilon > 0\), there exists a natural number \(N_\epsilon\) such that, for all \(m \geq N_\epsilon\),

\[
|h(x)| - |h_{\alpha_m}(x)| \leq |h(x) - h_{\alpha_m}(x)| < \epsilon.
\]

This yields

\[
|h(x)| \leq \epsilon + |h_{\alpha_m}(x)| 
\]

implying that \(h\) is in \(B_{\alpha_m}(\mathbb{R}^n)\).

It only remains to prove that \((\phi^*, \psi^*)\) is indeed a Nash equilibrium for the long run average payoff criterion with additive structure. To do that, first notice that the third equality in (12), shows that

\[
g \leq r(x, \phi^*, \psi) + L^{\phi^*, \psi} h(x); \quad x \in \mathbb{R}^n, \quad \psi \in \mathcal{P}(U_1).
\]
Next we apply Itô’s formula to function \( h \) and then take expectations, we get
\[
\mathbb{E}^\phi_0 \left[ h(x(T)) \right] = h(x) + \mathbb{E}^\phi_0 \left[ \int_0^T \mathcal{L}^\phi \psi h(x(t)) dt \right].
\] (14)

Replacing (14) in (13) and multiplying by \( 1/T \), we obtain
\[
g \leq \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r(x(t), \phi^*, \psi) dt \right] + \frac{1}{T} \mathbb{E}^\phi_0 \left[ |h(x(T)| - \frac{1}{T} h(x) \right].
\] (15)

Given that \( h \in \mathcal{B}_w(\mathbb{R}^n) \), we have that \( |h(x)| \leq ||h||_w w(x) \). This fact, and (7) lead to
\[
|\mathbb{E}^\phi_0 w(x(T))| \leq ||h||_w \mathbb{E}^\phi_0 w(x(T)) \leq ||h||_w [e^{-cT} w(x) + \frac{d}{c} (1 - e^{-cT})].
\]

Hence, letting \( T \to \infty \) in (15) and using this last inequality, we get by the additive structure of \( r \) (see Assumption 5)
\[
g \leq \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_1(x(t), \phi^*) dt \right] + \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_2(x(t), \psi) dt \right] =: \hat{J}(x, \phi^*, \psi).
\] (16)

Applying the same arguments but replacing second equality in (12) with the first one, we can also verify that
\[
g \geq \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_1(x(t), \phi^*) dt \right] + \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_2(x(t), \psi) dt \right] =: \hat{J}(x, \phi^*, \psi^*)
\] (17)

By mimicking the last procedure applied to the equations in (12), we get
\[
g \geq \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_1(x(t), \phi^*) dt \right] + \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^\phi_0 \left[ \int_0^T r_2(x(t), \psi) dt \right] =: \hat{J}(x, \phi^*, \psi).
\] (18)

Hence, by (16), (17) and (18) we have
\[
\hat{J}(x, \phi^*, \psi) \leq g = \hat{J}(x, \phi^*, \psi^*) \leq \hat{J}(x, \phi^*, \psi),
\]
so, \( (\phi^*, \psi^*) \) is a Nash equilibrium for the zero-sum stochastic differential game with additive structure.

10. **Concluding remarks.** In this work, we give conditions under which: (a) \( F \)-strong average optimality and average reward optimality are equivalent (Theorem 5.2); (b) average reward optimality and strong \(-1\)-discount optimality are equivalent (Theorem 6.5); (c) strong \(0\)-discount optimality implies bias optimality (Theorem 7.3); and (d) bias optimality implies \(0\)-discount optimality (Theorem 7.4) for a class of zero-sum stochastic differential games without additive structure and unbounded coefficients. Under previous results that guarantee the existence of average reward, discounted, bias and overtaking equilibria, we ensure diagram 6 hold. Moreover, we show that if the zero-sum stochastic differential game have additive structure and bounded coefficients, then, implications in diagram 7 hold. Differences between
Diagram 6 and 7 is that in this last bias optimality implies strong 0-discount optimality (Theorem 8.3).

**Diagram 6.** Games without additive structure and unbounded coefficients.

Overtaking optimal

\[
\downarrow\uparrow \quad \text{in } AO
\]

Strong 0-discount optimal \(\implies\) Bias optimal \(\implies\) 0-discount optimal

\[
\bigcap
\]

F-strong AO \(\iff\) CS \(\iff\) AO \(\iff\) strong-1-discount optimal

**Diagram 7.** Games with additive structure and bounded coefficients.

Overtaking optimal

\[
\downarrow\uparrow \quad \text{in } AO
\]

Strong 0-discount optimal \(\iff\) Bias optimal \(\implies\) 0-discount optimal

\[
\bigcap
\]

F-strong AO \(\iff\) CS \(\iff\) AO \(\iff\) strong-1-discount optimal

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E-mail address: bescobedo@uv.mx