REPRESENTABILITY OF RELATIVELY FREE AFFINE ALGEBRAS OVER A NOETHERIAN RING

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Abstract. Over the years questions have arisen about T-ideals of (noncommutative) polynomials. But when evaluating a noncentral polynomial in subalgebras of matrices, one often has little control in determining the specific evaluations of the polynomial. One way of overcoming this difficulty in characteristic 0, is to reduce to multilinear polynomials and utilizing the representation theory of the symmetric group. But this technique is unavailable in characteristic \( p > 0 \).

An alternative method, which succeeds, is the process of “hiking” a polynomial, in which one specializes its indeterminates in several stages, to obtain a polynomial that contains Capelli polynomials, in order to get control on its evaluations. This method was utilized on homogeneous polynomials in the proof of Specht’s conjecture for affine algebras over fields of positive characteristic.

In this paper we develop hiking further to nonhomogeneous polynomials, to apply to the representability question. Kemer proved in 1988 that every affine relatively free PI algebra over an infinite field, is representable. In 2010, the first author of this paper proved more generally that every affine relatively free PI algebra over any commutative Noetherian unital ring is representable. We present a different, complete, proof, based on hiking nonhomogeneous polynomials, over finite fields. We then obtain the full result over a Noetherian commutative ring, using Noetherian induction on T-ideals.

The bulk of the proof is for the case of a base field of positive characteristic. Here, whereas the usage of hiking is more direct than in proving Specht’s conjecture, one must consider nonhomogeneous polynomials when the base ring is finite, which entails certain difficulties to be overcome.

In the appendix we show how hiking can be adapted to prove the involutory versions, as well as various graded and nonassociative theorems.

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2010 Mathematics Subject Classification. Primary: 16R10, 16R40, 16W10; Secondary: 16G20, 17B50, 17C05.
Key words and phrases. Polynomial identity, relatively free, representable, T-ideal, hiking.
This research was supported by the Israel Science Foundation [grant number 1994/20].
In this paper, starting with the hiking technique from [16], we give a full exposition of Theorem 1.1 that relatively free affine PI-algebras over an arbitrary commutative Noetherian ring are representable. The crux of the matter is for algebras over a finite field. To prove that an affine algebra $A$ is representable, it is enough to embed it into a representation. This paper provides a detailed analysis of the conditions under which such embeddings exist.
an algebra that is a finite module over a commutative Noetherian base ring. (Anan’in [4] proved a far more general result, extended even further in [36].)

Our approach in this representability theorem follows the Shirshov program of [11, Section 2.4] If the algebra \( A \) is integral over its center, then Shirshov’s celebrated theorem [11, Theorem 2.2.2] shows that \( A \) is finite and then representable. There is a celebrated method, due to Razmyslov and Schelter, of adjoining characteristic values of the elements of a prime PI-algebra \( A \) in order to make it integral, and the challenge in the proof is to adjoin characteristic values to arbitrary PI-algebras, by means of a polynomial. (In the prime PI-case, this is the Capelli polynomial of [11, Definition 1.20].) This might seem circular, since one obtains characteristic values in matrices, and we want to prove that the algebra is representable; actually we take a maximal representable T-ideal \( I \) of \( A \), and compute in \( A/I \). Then we “hike” a given identity of \( A \) which is not an identity of \( A/I \) until it sufficiently resembles a Capelli polynomial that we can adjoin the characteristic values and obtain our finite module. The hiking process is summarized in the Canonization Theorem for Polynomials (Theorem 3.6):

Suppose \( f(x_1, \ldots, x_\ell) \) is a nonidentity of \( \widetilde{A}_0 \). Then the T-ideal of \( f \) contains a critical non-identity of \( \widetilde{A}_0 \) (defined in Definition 3.9, which involves Capelli-like polynomials).

This idea first appeared in [8], which provides the basis for our proof, but several aspects remained to be worked out. Here are the key ingredients in this paper:

- Description of “full quivers” of algebras, which involves collecting “Canonization theorems” that provide quivers with special properties, in §2.4. Some of the more esoteric sorts of quivers in [16] can be avoided, since, having already verified Specht’s conjecture (in the affine case) we have more control over the given T-ideal.
- The main tool in utilizing the combinatorics of polynomials is “hiking,” to provide special properties of a polynomial in a given T-ideal. The discussion here of hiking is the main contribution of this paper. The procedure here is more involved than in [16] since it involves nonhomogeneous polynomials, cf. Theorem 4.1, and is described below in several stages.
- Hiking is coordinated with Noetherian induction on T-ideals, in the sense that one takes a minimal non-representable counterexample, and then reduces it further to prove that it is representable after all. This is to be done by with the help of Shirshov’s theorem, by means of adjoining characteristic coefficients of matrices. But this further reduction is quite delicate, due to ambiguity in defining the characteristic coefficients.

On the one hand we want to use a famous trick of Amitsur (Lemma 6.5) to extract the characteristic coefficients from an alternating polynomial \( f \), but when our polynomial \( f \) is nonhomogeneous and involves alternating components of different multiplicities, we cannot apply Lemma 6.5 directly, and need first to hike \( f \), cf. §6. It is convenient to tie them by a different matrix method of calculating characteristic coefficients, given in Definition 6.15.

- In order to identify the matrix action with polynomials, we must utilize a Noetherian module rather than an algebra, as explained in §6.3.3 and modding out by this module enables us to apply Spechtian induction.
The reduction from the Noetherian case to the field case is a relatively straightforward application of classical Noetherian induction, included for completeness in §7.1.

Further applications should be possible in varied settings, including the proof of an involutory version as well as various graded theorems and nonassociative theorems, cf. the appendix (§8).

We work with algebras over a commutative Noetherian ring $C$, often a field $F$, with special emphasis on the possibility that $F$ is finite. $\bar{F}$ denotes the algebraic closure of $F$. A finitely generated algebra is called affine. A (noncommutative) polynomial is an element of the free associative algebra $C\{x\}$ on countably many generators.

A polynomial identity (PI) of an algebra $A$ over $C$ is a noncommutative polynomial which vanishes identically for any substitution in $A$. We use [32], [11] as a general reference for PIs. A $T$-ideal of $C\{x\}$ is an ideal $I$ of $C\{x\}$ closed under all algebra endomorphisms $C\{x\} \to C\{x\}$. We write $\text{id}(A)$ for the $T$-ideal of PIs of an algebra $A$.

Conversely, for any $T$-ideal $I$ of $C\{x\}$, each element of $I$ is a PI of the algebra $C\{x\}/I$, and $C\{x\}/I$ is relatively free, in the sense that for any PI-algebra $A$ with $\text{id}(A) \supseteq I$, and any $a_1, a_2, \ldots \in A$, there is a natural homomorphism $C\{x\}/I \to A$ sending $x_i \mapsto a_i$ for $i = 1, 2, \ldots$.

When $A = C\{x\}/I$ is relatively free, and $\overline{J} = J/I$ for a $T$-ideal $J \supseteq I$ of $F\{x\}$, we also call $\overline{J}$ a $T$-ideal of $A$. ($\overline{J}$ is invariant under all endomorphisms of $A$.)

### 1.1. Representability.

We mostly follow [11, §1.6] and [36]. An algebra $A$ over a field $F$ is called representable if it is embeddable as an $F$-subalgebra of $M_n(K)$ for a suitable field $K \supseteq F$.

An algebra $A$ over a commutative ring $C$ is called weakly representable if it is embeddable as a $C$-subalgebra of a finite dimensional algebra over a commutative Noetherian $C$-algebra $K$. This definition is weaker than [11, Definition 1.6.1] in order to avoid Bergman’s example [18] of a finite ring not embeddable into matrices over a commutative ring; any finite ring is weakly representable in our sense.

Obviously any representable algebra is weakly representable. On the other hand, by [36] any Noetherian algebra over a field which is finite over its center is representable, so “representable” and “weakly representable” coincide for algebras over a field.

Any weakly representable algebra is PI, but an easy counting argument of Lewin [30] leads to the existence of non-representable affine PI-algebras over any field.

Nevertheless, the representability question for relatively free affine algebras has considerable independent interest, and the purpose of this paper is to give a full proof of the following results:

**Theorem 1.1.** Every relatively free affine PI-algebra over an arbitrary field is representable;

and, more generally,

**Theorem 1.2.** Every relatively free affine PI-algebra over an arbitrary commutative Noetherian ring is weakly representable.

Kemer obtained Theorem 1.1 over infinite fields by means of the following amazing results:

**Theorem 1.3** ([11, Theorem 6.3.1], [28]).
(1) Every affine PI-algebra over an infinite field (of arbitrary characteristic) is PI-equivalent to a finite dimensional (f.d.) algebra.

(2) Every PI-algebra of characteristic 0 is PI-equivalent to the Grassmann envelope of a finite dimensional (f.d.) algebra.

An immediate consequence of Theorem 1.3(1) is that every relatively free affine PI-algebra over an infinite field is representable, since it can be constructed with generic elements obtained by adjoining commutative indeterminates to the f.d. algebra.

Remark 1.4. For graded associative algebras and various nonassociative affine algebras of characteristic 0, the finite basis of T-ideals has been established in the case when the operator algebra is PI (Iltyakov [24, 25] for alternative and Lie algebras, and Vais and Zelmanov [42] for Jordan algebras) but in many cases the representability question for relatively free affine algebras remains open for nonassociative algebras, so presumably is more difficult than the finite basis of T-ideals. The obstacle is getting started via some analog of Lewin’s theorem [30], which often is not yet available. Belov [9] obtained representability of relatively free alternative or Jordan algebras satisfying all identities of some finite dimensional algebra.

1.2. Overview of the proof of Theorem 1.1.

Kemer deduced from Theorem 1.3 the solution of Specht’s problem (the finite basis of T-ideals) for an affine algebra over an infinite field, in which he applied combinatorial techniques to representable algebras. The approach here for positive characteristic is the reverse, since Theorem 1.3 can fail over finite fields.

In [12]–[14], [16], summarized in [17], we have provided a complete proof for the affine case of Specht’s problem in arbitrary characteristic (The non-affine case has counterexamples, cf. [6, 7]):

Theorem 1.5 ([16]). Any affine PI-algebra over an arbitrary commutative Noetherian ring satisfies the ACC on T-ideals.

So we may start with the solution of Specht’s problem and apply Noetherian induction to prove representability of affine relatively free PI-algebras over finite fields. Together with Theorem 1.3 we then have Theorem 1.1 (These methods also work in characteristic 0, but then rely on Kemer’s solution of Specht’s problem in characteristic 0, which in turn requires his representability theorem in characteristic 0.)

Then we apply facts about torsion in rings, collected in §7, to obtain Theorem 1.2.

2. Preliminaries to the proof of Theorems 1.1 and 1.2

We fix the following notation: We start with the free associative affine algebra $C\{x\} = C\{x_1, \ldots, x_\ell\}$ in $\ell$ indeterminates, and a T-ideal $\mathcal{I}$. This gives us the relatively free algebra

$$A = C\{x\}/\mathcal{I}.$$  

We say that the T-ideal $\mathcal{I}$ is (weakly) representable if the affine algebra $A$ is (weakly) representable.

2.1. The underlying approach.

Note that the direct sum of two weakly representable algebras is weakly representable, and the direct sum of two representable algebras over a field is representable.
Remark 2.1. The proof of Theorems 1.1 and 1.2 goes along the following version of Noetherian induction:

We shall show that every $T$-ideal $\mathcal{I}$ is weakly representable, and is representable when $C$ is a field. Taking a $T$-ideal $\mathcal{I}$ maximal with respect to $A$ not weakly representable, we call $A$ a Specht minimal counterexample.

For $C$ a field, a key technique is to embed $A$ into a direct sum of two relatively free algebras, one of which is a homomorphic image of $A$ and thus representable by Specht induction, and the other of which is representable by some structural argument.

We call this argument Specht induction. It was utilized by Kemer in his proof of Specht’s conjecture, cf. [11, Proposition 6.6.31] and can be treated abstractly, applicable also to nonassociative algebras:

Lemma 2.2. Suppose that the relatively free $F$-algebra $A$ is a Specht minimal counterexample to representability, and $f$ is a polynomial generating a $T$-ideal $\langle f \rangle_T$ which annihilates some $T$-ideal of $A$, and $A/\mathcal{I}$ is representable for some $T$-ideal $\mathcal{I}$ for which $\mathcal{I} \cap \langle f \rangle_T = 0$. Then we have a contradiction to $A$ being a Specht minimal counterexample.

Proof. We can embed $A$ into $(A/\mathcal{I}) \oplus \langle f \rangle_T$. But $\langle f \rangle_T$ is representable by Specht induction, so $A$ is representable, contrary to assumption.

The bulk of this paper consists of the proof of Theorem 1.1. From now on, $A$ is assumed to be a Specht minimal counterexample. The proof relies on the ideas of the proof of Kemer’s representability theorem of relatively free affine algebras given in [10, 11]. Much of this paper is devoted to elaborating the theory of [13] and [14] in the field-theoretic case, as described in [16, 17], and there is a considerable overlap with [8].

Until §7, we work over a field $F$. Since the result is known in characteristic 0, we assume through Section 6.3.3 that $F$ has characteristic $p > 0$.

Remark 2.3. In view of Lewin’s theorem [30], any $T$-ideal $\mathcal{I}$ contains a representable $T$-ideal, which by Theorem 1.3 is contained in a maximal representable $T$-ideal $\mathcal{I}_0$ of $A$ contained in $\mathcal{I}$, which we aim to show is equal to $\mathcal{I}$. Thus $A_0 := F\{x\}/\mathcal{I}_0$ is representable (and affine over $F$). Assuming that $\mathcal{I}_0 \neq \mathcal{I}$, we have reduced to the case where $A_0$ is representable but every nonzero $T$-ideal of $A_0$ contained in $\mathcal{I}$ and properly containing $\mathcal{I}_0$ is not representable.

Our goal is to arrive at a contradiction by starting with a polynomial $f \in \mathcal{I} \setminus \mathcal{I}_0$ (in other words, a non-identity of $A_0$ in $\mathcal{I}$) and adjusting $f$ to a polynomial $\tilde{f}$ in $\mathcal{I} \setminus \mathcal{I}_0$, such that the $T$-ideal $\mathcal{I}_1 \subseteq \mathcal{I}$ generated by $\mathcal{I}_0$ and $\tilde{f}$ is a representable $T$-ideal. In this manner, we do not need to introduce parameters of induction, as opposed to Kemer’s approach described in [10, 11].

In our proof, we start with $A_0 = F\{x\}/\mathcal{I}_0$ of Remark 2.3.

Notation 2.4. Being representable, $A_0 \subset M_n(K)$ with $K$ algebraically closed, and we fix this particular representation. The “Zariski closure” $\tilde{A}_0$ of $A_0$ in $M_n(K)$ with respect to the Zariski topology [17 § 3.1] is PI-equivalent to $A_0$, and so we work throughout with $\tilde{A}_0$. We emphasize that $\mathcal{I}_0$ is the $T$-ideal of identities of $\tilde{A}_0$ as well as of $A_0$. (When $F$ is infinite then we may assume that $K = F$, cf. [17, Remark 3.1], but the
situation for $F$ finite is more delicate.} By the version of Wedderburn’s Principal Theorem \cite[Theorem 2.5.37]{33}, $\overline{A_0} = S \oplus J$ as vector spaces, where $J$ is the radical of $\overline{A_0}$ and $S \cong \overline{A_0}/J$ is a semisimple subalgebra of $\overline{A_0}$. Thus $S$ is a direct product of matrix algebras $R_1 \times \cdots \times R_k$, called \textbf{Wedderburn blocks}, which we want to view along the diagonal of $M_n(K)$, although possibly with some identification of coordinates, which are to be described graphically. By the Braun-Kemer-Razmyslov theorem, cf. \cite[Exercise 2.3.7]{19}, $J$ is nilpotent, so we take $t := t_{\overline{A_0}}$ maximal such that $J' \neq 0$. This description of $A$ as $(R_1 \times \cdots \times R_k) \oplus J$ is called \textbf{Wedderburn block form}.

2.1.1. Multilinearization versus quasi-linearization.

The well-known linearization process of a polynomial can be described in two stages: First, writing a polynomial $f(x_1, \ldots, x_n)$ as

$$f(0, x_2, \ldots, x_n) + (f(x_1, \ldots, x_n) - f(0, x_2, \ldots, x_n)),$$

one sees by iteration that any T-ideal is additively spanned by T-ideals of polynomials for which each indeterminate appearing nontrivially appears in each of its monomials, cf. \cite[Exercise 2.3.7]{34}. Then we could define the \textbf{linearization process} by introducing a new indeterminate $x'_i$ and passing to

$$f(x_1, \ldots, x_i + x'_i, \ldots, x_m) - f(x_1, \ldots, x_i, \ldots, x_m) - f(x_1, \ldots, x'_i, \ldots, x_m).$$

This process, applied repeatedly, yields a multilinear polynomial in the same T-ideal. In characteristic 0 the multilinearization process can be reversed by taking $x'_i = x_i$, implying that every T-ideal is generated by multilinear polynomials. But this fails in positive characteristic, and more generally when integers are not invertible, as exemplified by the Boolean identity $x^2 - x$, so we need an alternative. To handle characteristic $p > 0$, Kemer \cite{29} considered the following modification, which we review from \cite{17}.

\textbf{Definition 2.5.} A polynomial $f \in \mathcal{I}$ is \textbf{i-quasi-linear} on an algebra $A$ if

$$f(\ldots, a_i + a'_i, \ldots) = f(\ldots, a_i, \ldots) + f(\ldots, a'_i, \ldots)$$

for all $a_i, a'_i \in A$; $f$ is \textbf{A-quasi-linear} if $f$ is i-quasi-linear on $A$ for all $i$.

Suppose $f(x_1, x_2, \ldots) \in F\{x\}$ has degree $d_i$ in $x_i$. The \textbf{i-partial linearization step} of $f$ is

$$\Delta_i f := f(x_1, x_2, \ldots, x_{i,1} + \cdots + x_{i,d_i}, \ldots) - \sum_{j=1}^{d_i} f(x_1, x_2, \ldots, x_{i,j}, \ldots)$$  \hspace{1cm} (1)$$

where the substitutions were made in the $i$ component, and $x_{i,1}, \ldots, x_{i,d_i}$ are new variables.

When $\Delta_i f(A) = 0$, then $f$ is i-quasi-linear on $A$, so given a non-identity $f$ of $A$ we apply \cite[11] at most $\deg f$ times repeatedly, if necessary, to each $x_i$ in turn, to obtain a non-identity of $A$ in the T-ideal of $f$, that is quasi-linear.

\textbf{Proposition 2.6} (Special case of \cite[Theorem 1.4]{21}, also cf. \cite[Corollary 2.13]{15}). Assume $\text{char} F = p > 0$. For any non-identity $f$ of $A$, the T-ideal generated by $f$ contains a quasi-linear non-identity of $A$, for which the degree in each indeterminate is a $p$-power.
We apply all this to $\widetilde{A}_0$. We just say that an $\widetilde{A}_0$-quasi-linear polynomial $f$ is quasi-linear. In view of Proposition 2.6 we assume from now on that our polynomial $f(x_1, \ldots, x_n)$ is quasi-linear. When specializing $x_i$ to an element $\bar{x}_i$ of $\widetilde{A}_0$, we call the substitution $\bar{x}_i$ radical if $\bar{x}_i \in J$, and semisimple if $\bar{x}_i \in S$. The substitution $\bar{x}_i$ is pure if it is radical or semisimple. The substitution $f(\bar{x}_1, \ldots, \bar{x}_n)$ of $f(x_1, \ldots, x_n)$ is pure if each $\bar{x}_i$ is pure. Writing any substitution $\bar{x}_i$ as a sum of radical and semisimple substitutions, since $f(x_1, \ldots, x_m)$ is quasi-linear, we can reduce all substitutions in $J + S$ to pure substitutions (in $J \cup \bigcup R_k$). In particular, $f$ has a nonzero specialization where all substitutions $\bar{x}_i$ are pure.

Any semisimple substitution $\bar{x}_i$ is in $S$ and thus in a block (or in glued blocks) of some degree $n_i$, which we also call the degree of $\bar{x}_i$. A radical substitution $\bar{x}_i$ is somewhat more subtle. It is viewed as an edge connecting two vertices in blocks, say of degrees $n_{i_1}$ and $n_{i_2}$. If these blocks are not glued, then we call this substitution a bridge of degrees $n_{i_1}$ and $n_{i_2}$. A bridge is proper if $n_{i_1} \neq n_{i_2}$.

2.1.2. Folds in a polynomial.

As in Kemer’s proof of Specht’s conjecture and the proof given in [11], our first task is to estimate $d := [\widetilde{A}_0 : F]$ and $t$ in terms of polynomials.

**Definition 2.7.** A polynomial $f(x_1, \ldots, x_m)$ $m$-Alternates in $x_{i_1}, \ldots, x_{i_m}$ if

$$f(\ldots, x_{i_1}, \ldots, x_{i_2}, \ldots, x_{i_m}, \ldots) = \text{sgn}(\pi)f(\ldots, x_{\pi(i_1)}, \ldots, x_{\pi(i_2)}, \ldots, x_{\pi(i_m)}, \ldots)$$

for any $\pi \in S_m$.

For example, $c_m$ denotes the Capelli polynomial in $2m$ distinct indeterminates, which is alternating in the first $m$ indeterminates (i.e., switches sign when interchanging two of these indeterminates). Thus, for any field $F$, $c_{2m^2}$ is an identity of $M_{m-1}(F)$ but not an identity of $M_m(F)$.

We need several sets of alternating indeterminates.

**Definition 2.8.** A polynomial $f(x_1, \ldots, x_n)$ is $\mu$-fold $m$-Alternating if $f$ alternates in $\mu$ disjoint sets of indeterminates \{x_{j_1}, \ldots, x_{j_m}\}, 1 \leq r \leq \mu$.

**Remark 2.9.** Given a polynomial $f(x_1, \ldots, x_n)$, one way of increasing the number of $m$-alternating folds is by replacing an indeterminate $x_i$ occurring linearly in $f$ by $x_i c_m(x_{n+1}, \ldots, x_{n+m^2})$.

For example $\text{h}_{m,i}(y)$ denotes a multilinear central polynomial $\text{h}_{m,i}(y_1, \ldots, y_{m^2})$ for $M_m(F)$, alternating in specific indeterminates $y_1, \ldots, y_{m^2}$ which are all distinct. This is done by inserting a fold into a multilinear central polynomial, cf. [11] pp. 37,38].

2.2. Comparison with Kemer’s method.

The proof of Theorem 1.5 in [16] is somewhat different from Kemer’s proof. Kemer brought in some combinatoric definitions:

**Definition 2.10.** $\beta(A_0)$ is the largest $\beta$ such that, for any $\mu$, there is a $\mu$-fold $\beta$-alternating non-identity of $A_0$.

$\gamma(A_0)$ is the largest $\gamma$ such that, for arbitrarily large $\mu$ there is a $\mu$-fold, $\beta(\widetilde{A}_0)$-alternating and $(\gamma - 1)$-fold, $(\beta(\widetilde{A}_0) + 1)$-alternating non-identity of $A_0$. Such a polynomial is called a $\mu$-Kemer polynomial for $A_0$, [11] Definition 6.6.7].
The pair \( (\beta(A_0), \gamma(A_0)) \) is called the Kemer index of \( A_0 \), which we order lexicographically.

The Zariski closed algebra \( \overline{A_0} \) is full with respect to a monomial \( g \) if some nonzero substitution of \( g \) passes through all the blocks of the quiver.

A multilinear polynomial \( f \) has Property K on a f.d. algebra \( W \) if \( f \) vanishes under any specialization with fewer than \( t - 1 \) radical substitutions.

In this case, in characteristic 0, Kemer’s First Lemma ([11, Proposition 6.5.2]) says that, for \( F = \overline{F} \) algebraically closed, if \( \overline{A_0} \) is full then \( \beta(\overline{A_0}) = [\overline{A_0} : F] \); Kemer’s Second Lemma ([11, Proposition 6.6.31]) says that when \( \overline{A_0} \) is not PI-equivalent to a finite direct product of algebras of lower Kemer index, \( \gamma(\overline{A_0}) \) is the index of nilpotence of \( J \), and \( A \) has multilinear \( \mu \)-Kemer polynomials for arbitrarily large \( \mu \).

Kemer’s First and Second Lemma are the keys to Kemer’s proof, with the pair \( (\beta(\overline{A_0}), \gamma(\overline{A_0})) \) forming the basis for induction. But one relies on characteristic 0, in order to stay within the T-ideal \( I \) when multilinearizing. In characteristic \( p \) one only has quasi-linearization, so we need some alternative form of inductio n. Moreover, there is no obvious way to pass to basic algebras since we are working with \( I_0 \), not \( I \). So we turn to a different method not relying on these parameters.

2.2.1. The alternative method of full quivers.

Let us review some of the main techniques we need for the proof. The reader can refer to [17] for further details. We rely on two languages: quivers \( \Gamma \) of the representations of \( A \) on one hand, versus the combinatorial language of polynomials on the other hand.

First we bring in the language of quivers. In [13] we considered the full quiver of a representation of an associative algebra over a field, and determined properties of full quivers by means of a close examination of the structure of the closure under the Zariski topology, studied in [12]. Then we modified \( f \) by means of a “hiking procedure” in order to force \( f \) to have certain combinatorial properties, and used this to carve out a T-ideal \( J \) from inside a given T-ideal; modding out \( J \) lowers the quiver in some sense, and then one obtains Specht’s conjecture by induction. Hiking turns out to be a powerful but intricate tool.

Our approach here is similar, but with some variation. Here we need not mod out by \( J \), but do need \( J \) to be representable. We start the same way, but one of the key steps in [16] fails since we must cope with non-multilinear polynomials, and we need a way of getting around it.

2.3. Review of full quivers.

One of the most useful tools in representation theory is the quiver of a f.d. algebra, for which we present a modification (for the Zariski closed algebra \( \overline{A_0} \)) more pertinent to PI-theory.

We need an explicit description, but which may distinguish among Morita equivalent algebras since matrix algebras of different size are not PI-equivalent. The full quiver of \( A_0 \), or of its Zariski closure \( \overline{A_0} \), is a directed graph \( \Gamma \), having neither double edges nor cycles, with the following information attached to the vertices and edges:

The vertices of the full quiver of \( \overline{A_0} \) correspond to the diagonal matrix blocks arising in the semisimple part \( S \), whereas the arrows come from the radical \( J \). Every vertex likewise corresponds to a central idempotent in a corresponding matrix block of \( M_n(K) \).
The vertices are ordered, say from 1 to k, and an edge always takes a vertex to a vertex of higher order. There are identifications of vertices, i.e., of matrix blocks, called diagonal gluing, and identification of edges, called off-diagonal gluing. Gluing of vertices in full quivers is identical, as in \( \{ (\alpha^* 0) : \alpha \in \tilde{F} \} \), or Frobenius, as in \( \{ (\alpha 0 \alpha^q) : \alpha \in \tilde{F} \} \) where \( |F| = q \).

Each vertex is labeled with a roman numeral (I, II etc.); glued vertices are labeled with the same roman numeral. A vertex can be either filled or empty. The first vertex listed in a glued matrix block is also given a pair of subscripts — the matrix degree \( n_i \) and the cardinality of the corresponding field extension of \( F \) (which, when finite, is denoted as a power \( q^{t_i} \) of \( q = |F| )\).

Superscripts indicate the Frobenius twist between glued vertices, induced by the Frobenius automorphism \( a \mapsto a^q \); this could identify \( a^{q_1} \) with \( a^{q_2} \) for powers \( q_1, q_2 \) of \( q \) (or equivalently \( a^{q_2/q_1} \) when \( q_1 < q_2 \)); we call this \( (q_1, q_2) \)-Frobenius gluing.

Off-diagonal gluing (i.e., gluing among the edges) includes Frobenius gluing (which only exists in nonzero characteristic) and proportional gluing obtained by multiplying by an accompanying scaling factor \( \nu \). Proportional Frobenius gluing is Frobenius gluing combined at the same time with proportional gluing.

Examples are given in [14].

2.4. Review of the three canonization theorems for quivers.

Since arbitrary gluing is difficult to describe, we need some “canonization” theorems to “improve” the gluing. The first theorem shows that we have already specified enough kinds of gluing.

**Theorem 2.11** (First Canonization Theorem, cf. [13, Theorem 6.12]). The Zariski closure \( \tilde{A}_0 \) of any representable affine PI-algebra \( A_0 \) has a representation for whose full quiver every gluing is proportional Frobenius.

For the Second Canonization Theorem we grade paths according to the following rule:

**Definition 2.12.** When \( |F| = q < \infty \), we write \( \mathcal{M}_\infty \) for the multiplicative monoid \( \{1, q, q^2, \ldots, \epsilon\} \), where \( ea = \epsilon \) for every \( a \in \mathcal{M}_\infty \). (In other words, \( \epsilon \) is the zero element adjoined to the multiplicative monoid \( \langle q \rangle \).) Let \( \overline{\mathcal{M}} \) be the semigroup \( \mathcal{M}_\infty / \sim \) where \( \sim \) is the equivalence relation obtained by matching the degrees of glued variables: When two vertices have a \( (q_1, q_2) \)-Frobenius twist, we identify 1 with \( \frac{q_1}{q_2} \) in the respective matrix blocks, and use \( \overline{\mathcal{M}} \) to grade the paths.

**Definition 2.13.** A full quiver is primitive (called basic in [14]) if it has a unique initial vertex \( r \) and unique terminal vertex \( s \), and all of its gluing above the diagonal is proportional Frobenius. A primitive full quiver \( \Gamma \) is canonical if any two paths from the vertex \( r \) to the vertex \( s \) have the same grade.

**Theorem 2.14** (Second Canonization Theorem, cf. [14, Theorem 3.7]). Any relatively free algebra is a subdirect product of algebras whose full quivers are primitive.
Any primitive full quiver $\Gamma$ of a representable relatively free algebra can be modified (via a change of base) to a canonical full quiver.

In view of this result, we may reduce to the case that the full quiver of our polynomial $f$ is primitive. The Third Canonization Theorem \cite[Theorem 3.12]{14} describes what happens when one mods out a “nice” $T$-ideal, so is not relevant, since all we need is to find a representable $T$-ideal, which we do later by another method.

3. The Canonization Theorem for Polynomials

We continue the proof of Theorem \cite[Theorem 1.1]{1} following the strategy outlined in Remark \cite[Remark 2.3]{23}.

We have two languages: quivers and their representations on one hand, versus the combinatorial language of identities on the other hand. We are given a polynomial

$$f(x_1, \ldots, x_m) = \sum g_j(x_1, \ldots, x_m) \in I \setminus I_0,$$

for monomials $g_j$.

3.1. The geometric aspect.

First we consider the geometrical aspect, using quivers. A branch $B$ of $f$ is a path that appears in a nonzero specialization of some monomial of $g_j$.

The length of the branch $B$ is its number of arrows, excluding loops, which equals its number of vertices (say $k$) minus 1. Thus, a typical branch has vertices of various matrix degrees $n_j$, $j = 1, 2, \ldots, k$. We call $(n_1, \ldots, n_k)$ the degree vector \cite[Definition 2.32]{10} of the branch $B$. The descending degree vector is obtained by ordering the entries of the degree vector to put them in descending order lexicographically (according to the largest $n_j$ which appears in the distinct glued matrix blocks, excluding repetitions, taking the multiplicity into account in the case of Frobenius gluing). We write the descending degree vector as $(\pi(n_1), \ldots, \pi(n_k))$. Thus, $\pi(n_1) = \max \{n_1, \ldots, n_k\}$.

We denote the largest $n_j$ appearing in a nonzero specialization of the quiver as $\tilde{n}$, and fix this substitution $\tilde{x}_1, \ldots, \tilde{x}_n$ for the time being. Any other substitution is denoted $\tilde{x}'$. A proper bridge connecting vertices of degree $n_i \neq n_j$ is an $\tilde{n}$-bridge if $n_i$ or $n_j$ is $\tilde{n}$. But there also is the possibility that a radical substitution connects two glued blocks both of the same degree $\tilde{n}$, in which case we call it $\tilde{n}$-internal.

**Definition 3.1.** A branch $B$ of $f$ is dominant if it has the maximal number of $\tilde{n}$-bridges, has maximal length $k$ with regard to this property, and has the maximal number of vertices of $\tilde{n}$-bridges among these in the lexicographic order, and then we continue down the line to $\tilde{n} - 1$, etc. The depth of a dominant branch $B$ is the number of times $\tilde{n}$ appears in its degree vector.

We work with a dominant branch $B$ of the quiver $\Gamma$ in $f$. A branch is pseudo-dominant if it has the same configuration of bridges (although perhaps with different multiplicity) as $B$. We define the pseudo-dominant components of $f$ to be those sums of monomials whose branches are pseudo-dominant with the same degrees.

(This extra complication of pseudo-dominant components only arises when the polynomial $f$ is nonhomogeneous.) Our goal is somehow to force every nonzero substitution
of $f$ into a pseudo-dominant branch by considering each degree in turn from $\tilde{n}$ down. After the first two stages of hiking, given in §4.2, §4.3, $f$ will contain some term
\[ h_m = h_{m,1}g_1h_{m,2}g_2 \cdots g_t h_{m,t+1}, \]
the product of $t + 1$ copies of distinct polynomials $h_{m,i}$ of the same degree $2m^2$; we call the $h_{m,i}$ the components of $h$. We focus first on semisimple substitutions having matrix degree $\tilde{n}$, and put $h = h_{\tilde{n}}$.

**Lemma 3.2.** Any nonzero specialization of $h$ has a component consisting solely of semisimple substitutions (all of the same degree).

**Proof.** Otherwise every component has a radical substitution, so we have a product of $t + 1$ radical elements, which is 0 by definition of $t$. \qed

Viewing a substitution of $x_i$ as corresponding to an edge in the quiver, we have two degrees, one for each vertex of the edge.

**Definition 3.3.** Suppose that $m$ is one of the two degrees of the substitution $\overline{x_i}$. A substitution $\overline{x_i}$ of $x_i$ is $m$-right if one of the two degrees of $\overline{x_i}$ is $m$; $\overline{x_i}$ is $m$-wrong if both degrees of $\overline{x_i}$ differ from $m$.

We write right (resp. wrong) for $\tilde{n}$-right (resp. $\tilde{n}$-wrong).

One delicate point: An internal radical bridge, say from one matrix block of degree $m$ to a different matrix block of degree $m$, is technically “$m$-right” according to this definition, but must be dealt with separately.

**Remark 3.4.**

1. In view of Lemma 3.2, a wrong substitution could lead to $h$ (and thus $f$) having a component with semisimple substitutions in a matrix block of the wrong degree.

2. Also, we must contend with the possibility that right substitutions of dominant branches could cancel, thereby not yielding nonzero evaluations of $f$.

**Remark 3.5.** Suppose $f(x_1, \ldots, x_\ell)$ is a full nonidentity of $\widetilde{A_0}$, via the dominant branch $\mathcal{B}$ say of degrees $m_1, \ldots, m_k$ having some number $k$ of bridges, and $k'$ internal radical substitutions. By Theorem 2.14, any wrong nonzero substitution may be assumed to have $k$ bridges since otherwise we may apply induction to the number of semisimple components in the full quiver. On the other hand, taking the nonzero substitution of $\mathcal{B}$ with $k'$ maximal, any wrong substitution has at most $k'$ internal radical substitutions.

Our objective is to modify $f$ to a non-identity of $\widetilde{A_0}$, containing a Capelli component which enables us to use combinatorial methods to calculate characteristic coefficients in a Shirshov extension, with multiplication by elements of $\widetilde{A_0}$. We prove the following main result, enabling us to correspond quivers with properties of polynomials, and which leads directly to the representability theorem.

**Theorem 3.6 (Canonization Theorem for Polynomials).** Suppose $f(x_1, \ldots, x_\ell)$ is a nonidentity of $\widetilde{A_0}$. Then the $T$-ideal of $f$ contains a critical non-identity of $\widetilde{A_0}$ (defined in Definition 3.9).
3.2. **Explicit description of the Canonization Theorem for Polynomials.**

The set-up of the Canonization Theorem for Polynomials, based on “hiking,” is done in several stages:

1. Eliminate unwanted semisimple substitutions.
2. Make sure that the remaining substitutions are in the “correct” semisimple components.
3. Locate an “atom” (see Definition 3.8) inside the polynomial where we can compute the action of characteristic coefficients.

3.3. **The unmixed case.**

We quickly dispose first of the following easy case, following Kemer. We say a substitution is *unmixed* if it does not involve any bridges, i.e., all substitutions are in a single Wedderburn block. Here we need only multiply by a Capelli polynomial of the matrix degree, and then may proceed directly to the method of §6.

Although easy, this aspect is crucial to our proof, since substitutions alone are not sufficient to take care of examples such as the non-finitely generated T-space of \(\{[x_1, x_2]x_1^{p-1}x_2^{p-1} : k \in \mathbb{N}\}\) in the Grassmann algebra with two generators; also see [22, 23].

Furthermore, it provides the base for our induction on \(\tilde{n}\).

3.4. **The mixed case: Introducing the hiking procedure.**

To attain the proof of the Canonization Theorem for Polynomials, we must turn to the mixed case. In our combinatorics we need to cope with the danger that our substitutions are wrong, or the base field of the semisimple component is of the wrong size. To prevent this, we insert substitutions of multilinear polynomials for indeterminates inside \(f\), called *hiking*, which force the substitutions to become 0 in such situations. In other words, hiking replaces \(f\) by a more complicated polynomial in its T-ideal, which yields a zero value when we apply a wrong substitution to the original indeterminates of \(f\). The notion of hiking passes from branches of quivers to combinatorics of nonidentities, showing how to modify a non-identity of \(\tilde{A}_0\) to another non-identity whose algebraic operations leave us in the same quiver.

As stated in the introduction, we need to provide hiking for quasilinear polynomials. The hiking procedure requires three different stages.

Actually, \(f\) has three kinds of variables which play important roles:

- Core variables, used for exclusive absorption inside the radical (such as variables which appear in commutators with central polynomials of Wedderburn blocks),
- variables used for hiking,
- variables inside Capelli polynomials used for computing the actions of characteristic coefficients.

**Example 3.7.** An easy example of the underlying principle: If \(k = 2\) with \(n_1 > n_2\), then the quiver \(\Gamma\) consists of two blocks and an arrow connecting them, so we replace a variable \(y\) of \(f\) with a radical substitution, \(h_{n_1,1}|h_{n_1,2}, z|y|h_{n_2}\). The corresponding specialization remains in the radical. Here we are ready to utilize the techniques given below in §6 to compute characteristic coefficients, bypassing the complications of hiking.
Definition 3.8. Given a polynomial $f(x_1, \ldots, x_\ell)$ and another polynomial $g$, we write $f_{x_i \mapsto g}$ to denote that $g$ is substituted for $x_i$. We say that $f$ is hiked to $\tilde{f} := f_{x_i \mapsto g}$ (at $x_i$) if $g$ is linear in $x_i$.

We call the replacement $g$ of $x_i$ an atom of the hiked polynomial. A molecule is the product of atoms.

The motivation for hiking is that the hiked polynomial $\tilde{f}$ lies in the T-ideal of $f$ but combinatorially we have greater control over the nonzero substitutions of $\tilde{f}$.

Suppose we replace the polynomial $f$, with a radical substitution. We replace it and have a hiked polynomial. But to continue, we shall need a rather intricate analysis.

Definition 3.9. A polynomial is bonded (of length $d$) if it can be written in the form

$$\sum_u g_{u,1}h_{\tilde{n}}(y)g_{u,2}h_{\tilde{n}}(y) \cdot \ldots \cdot g_{u,d}h_{\tilde{n}}(y)g_{u,d+1}$$

for suitable polynomials $g_{u,i}$ (perhaps constant) in which the $y$ indeterminates do not occur. (In other words the $y$ indeterminates occur only in the $h_{\tilde{n}}(y)$.) The $h_{\tilde{n}}$ are called the bonds and are the alternating polynomials where we examine substitutions.

A bonded polynomial $f(x_1, \ldots, x_i; y, y', y''; z, z')$ is critical if any nonzero substitution of the $y_i$ is right.

Thus the bonds are attached to atoms. If $f$ is hiked to various polynomials $f_j$ we also say it can be hiked to $\sum f_j$.

Note that the situation is complicated by the fact that if the $x_i$ repeat then the atoms repeat, and thus the variables $y$ repeat.

(Likewise for other indeterminates that appear once the hiking is initiated.)

Remark 3.10. First suppose that the depth $u = k$, i.e., all $n_j = \tilde{n}$, and there are no nonzero external radical substitutions. In other words, the only nonzero substitutions involve specializing all the $x_i$ to semisimple elements in blocks of degree $\tilde{n}$. Then we simply replace $f$ by $hf$, which trivially is bonded, and Theorem 3.6 is proved. So in the continuation, we assume that $u < k$, which means there is some nonzero substitution $f(x_1, \ldots, x_k)$ in our dominant branch $B$, for which some $\bar{x}_j$ is an $\tilde{n}$-bridge. We pass to this $\bar{x}_1, \ldots, \bar{x}_k$ in what follows, and call it our working substitution.

4. THE HIKING THEOREM FOR POLYNOMIALS

In this section we prove the Canonization Theorem for Polynomials, by means of a more technical version to handle the mixed case.

Theorem 4.1 (Hiking Theorem for Polynomials). Suppose $f(x_1, \ldots, x_\ell)$ is a non-identity of $\tilde{A}_0$, possibly with a mixed or pure substitution. Then $f$ can be hiked to a critical nonidentity in which all of the substitutions of the $x_i$ in the dominant branch $B$ are right.

The proof of Theorem 4.1 is through a succession of hiking steps in order both to eliminate “wrong” substitutions and then bonding, i.e., insert $h_{\tilde{n}}$ into the polynomial. The latter is achieved by replacing $z_i$ by $h_{\tilde{n}}z_i$ and $z'_i$ by $z'_ih_{\tilde{n}}$; i.e., we pass to $f_{z_i \mapsto h_{\tilde{n}}z_i}, z'_i \mapsto z'_ih_{\tilde{n}}$.

The hiking procedure is performed in three different stages.
4.1. Preliminary hiking.

Our initial use of hiking is to resolve some technical issues. First, we want to eliminate the effect of \((q_1, q_2)\)-Frobenius gluing for \(q_1 \neq q_2\), since it can complicate bonding. Toward this end, we substitute \(z_{i'} c_n(y)^{n/q_2}\) for \(z_{i'}\), for each instance of Frobenius gluing. It makes the Frobenius gluing identical on \(f\).

We also need the base fields of the components all to be the same. When \(\mathcal{B}'\) is another branch with the same degree vector, and the corresponding base fields for the \(i\)-th vertex of \(\mathcal{B}\) and \(\mathcal{B}'\) are \(n_i\) and \(n_i'\) respectively, we take \(t_i = q^{n_i'}\) and replace \(x_i\) by \((c_{n_i}^t - c_n)x_i\). This cuts off specializations to matrices over finite fields of the wrong order.

4.2. First stage of hiking.

We have a quasi-linear nonidentity \(f\) of the Zariski closed algebra \(\widetilde{A_0}\) for which we have a working substitution in some branch \(\mathcal{B}\), where \(\overline{x_{i_1}}\) in \(\widetilde{A_0}\) is an \(n\)-bridge, corresponding to an edge in the full quiver whose initial vertex is labeled by \((n_{i_1}, t_{i_1})\) and whose terminal vertex is labeled by \((n_{i_1+1}, t_{i_1+1})\) where \(n = \max\{n_{i_1}, n_{i_1+1}\}\). We replace \(x_{i_1}\) by \(c_{n_{i_1}} z_{i_1}[x_{i_1}, h_{n-1} z_{i_1+1}c_{n_{i_1+1}}]\) (where as always the \(c_{n_i}\) involve new indeterminates in \(v\)), and \(z_{i_1}, z_{i_1+1}\) also are new indeterminates which we call “auxiliary indeterminates”; this yields a quasi-linear polynomial in which any substitution of \(x_{i_1}\) into a diagonal block of degree \(\ell < n_{i_1}\) or a bridge which is not an \(n\)-bridge is 0. For each semisimple substitution \(\overline{x_{i_1}}\) in a block of degree \(n_{i_1}\), taking \([x_{i_1}, h_{n_{i_1}}]\) yields 0. This removes all semisimple component substitutions in \(h\) of such \(x_i\) whose degree is too “small,” i.e., less than \(n_i\). For the time being, we could still have radical substitutions, but the first stage of hiking does prepare for their elimination in the second stage.

The number of extra \(n\)-bridges in a specialization of \(c_{n_{i_1}}(v) z_{i_1}[x_{i_1}, h_{n_{i_1}}(v)]z_{i_1+1} c_{n_{i_1+1}}(v)\) is called its (first stage) bridge contribution. (In other words, one takes the total number of bridges, and subtracts 1 if \(\overline{x_{i_1}}\) is an \(n\)-bridge.)

**Lemma 4.2.** Any nonzero specialization of \(h\) is either \(n\)-semisimple, or its bridge contribution is positive.

**Proof.** By definition, if the bridge contribution is 0 then every substitution has to be semisimple or a \((j, j)\)-bridge for some \(j\). If \(n\) does not appear then the graph would have such bridges. \(\square\)

**Lemma 4.3.** After the first stage of hiking, a wrong specialization of an \(n\)-semisimple element cannot be \(m\)-semisimple for \(m < n\) unless its bridge contribution is at least 2.

**Proof.** When evaluating \(h_n\) on semisimple elements of degree \(m\) we get 0 unless we pass away from the \(m\)-semisimple component, which requires two bridges. \(\square\)

**Lemma 4.4.** After the first stage of hiking, a wrong specialization of an \(n\)-semisimple element is either \(n\)-semisimple or its bridge contribution is at least 1.

**Proof.** When evaluating \(h_n\) on semisimple elements of degree \(m\) we get 0 unless we pass away from the \(m\)-semisimple component, which requires a bridge. \(\square\)

Thus, the first stage of hiking does not instantly zero out bridges for wrong specializations, \(\overline{x_{i_1}}\), but does prepare for their elimination in the second stage.

Appending the Capelli polynomials also sets the stage for eliminating other unwanted substitutions in the second stage.
After repeated applications of first stage hiking, we wind up with a new polynomial $f(x_1, \ldots, x_t; v; z)$ where we still have our original indeterminates $x_i$ but have adjoined new indeterminates.

4.3. Second stage of hiking.

**Example 4.5.** To introduce the underlying principle, here is a slightly more complicated example. Consider the quiver of three arrows, from degree 2 to degree 1, degree 1 to degree 1, and finally from degree 1 to degree 1.

First we multiply on the left by $c_4[h_{1,1}, z_1]z_2$. The second substitution could have an unwanted position inside the first matrix block of degree 2, since $c_4[h_{1,1}, z_1]z_2$ could be evaluated in the larger component. We take $f_{x_1 \to y'y} - f_{x_1 \to y'y}$, i.e., we multiply by a central polynomial $h_2$ on the left and subtract it from a parallel substitution of $h_2$ on the right. The unwanted substitution then cancels out with the other substitution and leaves 0.

In the second stage of hiking, in the blended case, we arrange for all previously unassigned nonzero substitutions to be pure radical.

Suppose $f(x_\ldots, x_t; y; z; z')$ is already hiked after the first stage, and in the branch $\mathcal{B}$ the indeterminate $z_i$ occurs of degree $d_i$ and the indeterminate $z_{u+1}$ occurs of degree $d'$, where $1 \leq j \leq u$.

**Proposition 4.6.** There are three cases to consider:

(i) There is a string $x_{i-1}x_i \ldots x_jx_{j+1}$ where $x_i, \ldots, x_j$ are all semisimple of the same degree $x_{n_1}$ whereas $x_{i-1}x_i, x_jx_{j+1}$ are both $\tilde{n}$-bridges.

We take the polynomial

$$f_{z_i \to h_0(y')^t z_i} - f_{z_{u+1} \to h_{n}(y')^t z_i},$$

where the branch $\mathcal{B}$ has depth $u$ and $t_i$ designates the maximal degree of $x_i$ in a monomial of $\mathcal{B}$, where $y'$ is a fresh set of indeterminates.

(ii) There is a string $x_1x_i \ldots x_jx_{j+1}$ where $x_1, \ldots, x_j$ are all semisimple of the same degree $x_{n_1}$ whereas $x_jx_{j+1}$ is an $\tilde{n}$-bridge. We take the polynomial

$$f_{z_1 \to h_0(y')^t z_1}.$$  

(iii) There is a string $x_{i-1}x_i \ldots x_kx_k$ where $x_i, \ldots, x_{k-1}$ are all semisimple of the same degree $x_{n_1}$ whereas $x_{k-1}x_k$ is an $\tilde{n}$-bridge. We take the polynomial

$$f_{z_i \to h_0(y')^t z_i}.$$  

This procedure was described so far without Frobenius twists. To eliminate superfluous Frobenius twists, we also perform the substitutions

$$h_\tilde{n}(y')^q x_j - x_j h_{\tilde{n}}^q(y'),$$

where $q_1 \neq q_2$ range over the various powers of $p$.

This hiking zeroes out semisimple substitutions of highest degree (namely $\tilde{n}$), but not a radical substitution at the $u$ block.

**Proof.** (Note that (i) is the usual case, but we also need (ii) and (iii) to handle terms lying at either end of the polynomial.) The expression (2) yields zero on a semisimple substitution, but not on a radical substitution, since exactly one of the two summands
of (2) would be 0. Likewise the other cases yield zero on a semisimple substitution, but not on a radical substitution.

Multiplying by all substitutions of (5) annihilates all non-identity Frobenius twists. □

Lemma 4.7. The second stage of hiking forces any nonzero specialization of an \( \tilde{n} \)-bridge also to be an \( \tilde{n} \)-bridge.

Proof. In order to provide a nonzero value, at least one of its vertices must be of degree \( \tilde{n} \). But if both were \( \tilde{n} \) the evaluation would be 0, by Lemmas 4.2–4.4 and Remark 4.6. Thus we get an \( \tilde{n} \)-bridge. □

Lemma 4.8. After the first and second stages of hiking, the positions of semisimple substitutions of degree \( \tilde{n} \) in nonzero evaluations are fixed; in other words, semisimple substitutions of degree \( \tilde{n} \) are \( \tilde{n} \)-right.

Proof. Lemma 4.7 “uses up” all the places for \( \tilde{n} \)-bridges, since more \( \tilde{n} \)-bridges would yield a substitution contradicting the maximality of the number of \( \tilde{n} \)-bridges in \( \mathcal{B} \). If \( \mathcal{B} \) has no semisimple substitutions of degree \( \tilde{n} \) then there is no room for any semisimple substitutions of degree \( \tilde{n} \) whatsoever, and we are done.

But if \( \mathcal{B} \) has a semisimple substitution of degree \( \tilde{n} \), that substitution must border an \( \tilde{n} \)-bridge, fixing the order of the pair of indices in the \( \tilde{n} \)-bridge, and thus fixing the positions of all the gaps of index \( \tilde{n} \) between \( \tilde{n} \)-bridges, so again we are done. □

Remark 4.9. Although this is already taken care of in the proof, we could have removed finite components simply by substituting \( x_i^m - x_i^\ell \) for \( x_i \), for suitable \( \ell, m \).

4.4. Conclusion of the proof of the Hiking and Canonization Theorems for Polynomials.

Proof of the Hiking Theorem for Polynomials (Theorem 4.1). Just iterate the hiking procedure down from \( \tilde{n} \). (It might well be that the right substitutions to degree \( \tilde{n} \) cancel, cf. Remark 3.4(2), and then we continue to \( \tilde{n} - 1 \); when we finally get to 1 then we are in the unmixed case, which was handled in §3.3. □

Proof of the Canonization Theorem for Polynomials (Theorem 3.6). After finishing the hiking, one obtains the bond (with a nonzero specialization) by replacing \( f \) by

\[
\tilde{f}_{z_u \mapsto c_{\tilde{n}}(y)^{1}\mapsto z_u, \ u \mapsto z'_{u+1} \mapsto c_{u}(y)^{1}.}
\]

□

Example 4.10. Let us run through the hiking procedure, taking

\[
\tilde{A}_0 = \begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
\end{pmatrix}.
\]

We have the full quiver

\[
I_3 \rightarrow \Pi_2,
\]

and take the nonidentity \( f = x_1[x_2, x_3]x_4 + x_4[x_2, x_3]x_1^2 \). We denote the second matrix component, as \( B = M_2(K) \).
We have nonzero specializations with $\overline{\mathbb{F}}_1$ in the first matrix component, $\overline{\mathbb{F}}_2$ an external radical specialization, and $\overline{\mathbb{F}}_3$ in $B$, but also we have a nonzero specialization of all variables into $B$. To avoid this situation, we replace $f$ by
\[
\tilde{f}(x; y; z; z') := f(c_3(y)c_3(y')^2zx_1z', x_2, x_3, x_4) - f(c_3(y)zx_1z'c_3(y'), x_2, x_3, x_4)
\]
\[
= (c_3(y)c_3(y')^2zx_1z'[x_2, x_3]x_4 + x_4[x_2, x_3]^2c_3(y)c_3(y')^2zx_1z'c_3(y)zx_1z')
\]
\[- (c_3(y)zx_1z'c_3(y')[x_2, x_3]x_4 + x_4[x_2, x_3]^2c_3(y)zx_1z'c_3(y)zx_1z')(c_3(y'))],
\]
where we see the specialization of highest degree in the first matrix component has been eliminated. We can eliminate the nonzero specializations of $c_3(y'')$ of degree 1 by taking $\tilde{f}(x; y; y'; c_3(y'')z; z') - \tilde{f}(x; y; y'; z; z'c_3(y''))$ which leaves us only with a radical specialization and a critical polynomial with a single bond $c_3(y'')$.

Note how quickly the polynomial becomes complicated even though we have hiked only one of the original indeterminates.

**Remark 4.11.** Other examples of hiking are given in [16]. The main difference between the hiking procedure of this paper and that of stage 3 hiking of [16] is in the treatment of the Frobenius automorphism. Stage 4 hiking from [16] is also analogous.

## 5. Removing ambiguity of matrix degree

Any polynomial $f$ of Theorem [5.5] can be written as a sum of homogeneous components $\sum f_j$, where $f_j$ has the same matrix degree for each monomial. We want to reduce to homogeneous components.

**Definition 5.1.** A hiked polynomial is uniform if there is some indeterminate $x_i$ for which, in each of its monomials, the atom obtained from hiking $x_i$ is semisimple of the same matrix degree.

Our objective in this section is to hike to a uniform polynomial. First we use [5.8] to dispose of the easy case where each hiked monomial has a semisimple atom (Definition [5.8]).

**Definition 5.2.** A radical element of a molecule is isolated if multiplication by any radical element on the left or right is zero.

**Remark 5.3.** The product of two isolated elements is 0, by definition.

**Proposition 5.4.** Any polynomial with a nonzero substitution can be hiked to a uniform polynomial, with a nonzero substitution.

**Proof.** Multiply $x_i$ by a new indeterminate $x'_i$ and hike that without making the substitution zero. We are done unless it provides a radical substitution. Since $J^{t+1} = 0$, we get an isolated element after at most $t$ hikes. An extra occurrence of $x_i$ which is hiked on must then be semisimple. \(\square\)

**Lemma 5.5.** We may hike further so that all matrix components of size $\hat{n}$ are defined over the same field.
Proof. In Proposition 5.4 we have just reduced to the case where all monomials have atoms of some $x_i$ of the same matrix degree $\tilde{n}$ (and the substitutions of the $x_i$ are all semisimple), but we next must contend with the possibility that the different matrix components may be defined over different fields. But these all have the same characteristic $p$ (the characteristic of $F$), so have sizes say $p^{t_{i1}}$ and $p^{t_{i2}}$, and we modify Proposition 4.6 by applying the appropriate Frobenius maps $x \mapsto p^{t_{ij}}$ at the various bonds. □

6. Characteristic coefficient-absorbing polynomials inside $T$-ideals

Having started with our $T$-ideal $I$ and a $\tilde{A}_0$-quasi-linear polynomial $f \in I$ with a nonzero evaluation (where we identify a representation $\tilde{A}_0$ with the full quiver of $f$), we have seen how to hike $f$ in various stages to get specific properties and still have a nonzero substitution. Utilizing all of these hiking procedures and replacing $I$ by the $T$-ideal generated by this hiked polynomial, we may make the following assumptions on $f$:

- All monomials of $f$ have the same matrix degree $\tilde{n}$, and over the same finite base field, although the multiplicities might vary because of gluing;
- $f$ is uniform, so a radical substitution into an auxiliary indeterminate $z$ inside $f$ yields 0; hence we have a substitution action of semisimple elements, preserving $I$ (the $T$-ideal generated by the substituted polynomial is obviously contained in the $T$-ideal $I$ generated by $f$).

To understand this substitution action, we want to utilize the well-understood properties of semisimple matrices (especially the coefficients of their characteristic polynomials, which we call characteristic coefficients). We follow the treatment of coefficient-absorbing polynomials from [16, Theorem 4.26] and [17, §6.3], although we can skip much of the discussion there because we already have obtained a bonded polynomial (see Definition 3.9).

Using Theorem 3.6 we work with quasi-linear polynomials and pinpoint semisimple substitutions of degree $\tilde{n}$, in order to utilize the well-understood properties of semisimple matrices (especially the characteristic coefficients of their simple components).

6.1. Characteristic coefficients.

Over a field $K$, any matrix $a \in M_n(K)$ can be viewed either as a linear transformation on the $n$-dimensional space $V = K^n$, and thus having Cayley-Hamilton polynomial $f_a$ of degree $n$, or (via left multiplication) as a linear transformation $\tilde{a}$ on the $n^2$-dimensional space $\tilde{V} = M_n(K)$ with Cayley-Hamilton polynomial $f_{\tilde{a}}$ of degree $n^2$. The matrix $\tilde{a}$ can be identified with the matrix

$$a \otimes I \in M_n(K) \otimes M_n(K) \cong M_{n^2}(K),$$

so its eigenvalues have the form $\beta \otimes 1 = \beta$ for each eigenvalue $\beta$ of $a$. But there are only finitely many components in the representation, so $\tilde{a}$ is algebraic in $M_n(K)$.

We recall a basic observation of Zariski and Samuels:

Lemma 6.1. Any characteristic coefficient of an element which is integral over a commutative ring $C$, is itself integral over $C$.

Proof. If $\alpha$ denotes the characteristic coefficient, then $\alpha$ is generated by powers of roots of the minimal polynomial of the given element. □
From this, we conclude:

**Proposition 6.2** ([13 Proposition 2.4]). Suppose \( a \in M_n(F) \). Then the characteristic coefficients of \( a \) are integral over the \( F \)-algebra \( C_a \) generated by the characteristic coefficients of \( a \).

**Proof.** The integral closure of \( C_a \) contains all the eigenvalues of \( a \), which are the eigenvalues of \( a \), so the characteristic coefficients of \( a \) also belong to the integral closure. \( \square \)

We are about to adjoin (finitely) many integral elements. Recall from Remark 2.3 that \( A_0 \) is a “minimal representable cover” of the algebra \( A \). We can define the characteristic coefficients via polynomials.

**Definition 6.3.** Given a quasi-linear polynomial \( f(x; y) \) in indeterminates labeled \( x_i, y_i \), we say \( f \) is characteristic coefficient-absorbing with respect to its full quiver \( \Gamma \) if the linear span of \( f(A_0) \) absorbs multiplication by any characteristic coefficient of any element in each bonded (diagonal) matrix block.

**Remark 6.4.** In view of Proposition 5.4 we may use characteristic coefficients for semisimple elements. We recall that we are working in characteristic \( p > 0 \). In order to guarantee that the semisimple substitutions are indeed semisimple as matrices, we take the Jordan decomposition of the matrix \( a = s + r \) where \( s \) is semisimple and \( r \) is nilpotent with \( sr = rs \), and then observe that if \( r^k = 0 \) and \( q \) is a \( p \)-power greater than \( k \), then
\[
a = (s + r)^q = s^q + r^q = s^q + 0 = s,
\]
which is semisimple. This leads us to take \( q \)-powers of matrices, and \( q \)-powers of characteristic coefficients.

**Lemma 6.5** (as in [15 Lemma 3.6]). Write an \( \tilde{n} \)-alternating polynomial \( f \) as a sum of homogeneous components \( \sum f_j \). Each \( f_j \) is characteristic coefficient absorbing in the blocks of degree \( \tilde{n} \).

**Proof.** The proof can be formulated in the language of [10 Theorem J, Equation 1.19, page 27] (with the same proof), as follows, writing \( T_{a,j} \) for the transformation given by left multiplication by \( a \):
\[
\alpha_k f_j(a_1, \ldots, a_t, r_1, \ldots, r_m) = \sum f_j(T_{a}^{k_1}a_1, \ldots, T_{a}^{k_t}a_t, r_1, \ldots, r_m),
\]
summed over all vectors \((k_1, \ldots, k_t)\) with each \(k_i \in \{0, 1\}\) and \(k_1 + \cdots + k_t = k\), where \(\alpha_k\) is the \(k\)-th characteristic coefficient of a linear transformation \( T_{a}: V \rightarrow V \). \( \square \)

Since the sole purpose of the hypotheses of Lemma 6.5 was to obtain the conclusion \( \square \), we merely assume \( \square \).

**Lemma 6.6.** For any polynomial \( f(x_1, x_2, \ldots) \) quasi-linear in \( x_1 \) with respect to a matrix algebra \( M_n(F) \), satisfying \( \square \), there is a homogeneous component \( \tilde{f} \) in the \( T \)-ideal generated by \( f \) which is characteristic coefficient absorbing.

**Proof.** Take the polynomial of Lemma 6.5 after we zero out the substitutions of all but one of the components. \( \square \)
Remark 6.7. Notation as in \([2]\), where \(f = f_j\), the Cayley-Hamilton identity for \(n \times n\) matrices is

\[
0 = \sum_{k=0}^{n} (-1)^k \alpha_k f(a_1, \ldots, a_t, r_1, \ldots, r_m) \lambda^{n-k}
\]

\[
= \sum_{k=0}^{n} (-1)^k \sum_{k_1 + \cdots + k_t = k} f(T_a^{k_1} a_1, \ldots, T_a^{k_t} a_t, r_1, \ldots, r_m) \lambda^{n-k},
\]

which is thus an identity in the \(T\)-ideal generated by \(f\).

Definition 6.8. We call this identity

\[
\sum_{k=0}^{n} (-1)^k \sum_{k_1 + \cdots + k_t = k} f(T_a^{k_1} a_1, \ldots, T_a^{k_t} a_t, r_1, \ldots, r_m) \lambda^{n-k}
\]

the Cayley-Hamilton identity induced by \(f\).

6.2. Controlling the action of characteristic coefficients.

Definition 6.9. Fixing \(0 \leq k < n\), we denote the \(k\)-th characteristic coefficient of \(a\), as \(\alpha_{\text{pol}}(a)\).

Now we use hiking to force the polynomial-defined characteristic coefficients of the matrices to commute with each other.

Proposition 6.10. One can hike \(f\) such that the characteristic coefficients \(\alpha_{\text{pol}}(a)\) of any matrix evaluation commute with each other.

Proof. Take homogeneous \(\hat{f}\) of Lemma 6.6 and one more indeterminate \(y''\). There is a Capelli polynomial \(\hat{c}_{n^2}(y'') := \hat{c}_{n^2}(y'', \ldots)\) and \(p\)-power \(\tilde{q}\) such that

\[
\hat{c}_{n^2}(\alpha_{k} y'') x_i c_{n^2}(y'') = \alpha_{\tilde{q}}(y_1) c_{n^2}(y'') x_i c_{n^2}(y'')
\]

(7)

on any diagonal block. Since characteristic coefficients commute on any diagonal block, we see from this that

\[
\hat{c}_{n^2}(y'') x_i c_{n^2}(y'') \hat{c}_{n^2}(z) x_i c_{n^2}(z) - \hat{c}_{n^2}(z) x_i c_{n^2}(z) \hat{c}_{n^2}(y'') x_i c_{n^2}(y'')
\]

(8)

vanishes identically on any diagonal block, where \(z = \alpha_{k} y''\). One concludes from this that substituting \(\hat{f}\) for \(x_i\) would hike \(\hat{f}\) one step further. But there are only finitely many ways of performing this particular hiking procedure. Thus, after a finite number of hikes, we arrive at a polynomial in which we have complete control of the substitutions, and the characteristic coefficients defined via polynomials commute. \(\square\)

Notation 6.11. Let \(S\) denote the finite set of products (of length up to the bound of Shirshov’s Theorem \([11\), Chapter 2\]) of components according to the Peirce decomposition (sub-Peirce components when considering rings without 1) of the generic generators of \(A_0\).

Let \(C\) be the algebra obtained by adjoining to \(F\) the characteristic coefficients of the elements of \(S\), and \(\tilde{A}_0 = \tilde{A}_0 \otimes_F C\), the algebra obtained by adjoining these characteristic coefficients to \(\tilde{A}_0\).

We introduce a commuting indeterminate \(\lambda_i\) for each of these finitely many characteristic coefficients \(\alpha_i\), \(i \in I\), define \(C'\) to be \(C[\lambda_i : i \in I]\), and \(A'\) to be \(\tilde{A}_0[\lambda_i : i \in I]\).
In this way, after hiking, the substitution action now is well-defined on any single monomial of our hiked polynomial \( f \), and is integral over the action involving a single component. Our total degree of integrality could be huge (although it is bounded, since the degree for each element is bounded).

6.3. Closed submodules.

**Definition 6.12.** We take \( C \) and \( \tilde{A}_0 \) from Notation [6.11] noting that \( C \) is central. An ideal \( U \) of \( \tilde{A}_0 \) is **closed** if \( CU = U \), i.e. if \( U \) absorbs multiplication by elements of \( C \).

We just saw in principle how to obtain closed \( T \)-ideals using a dominant branch, but we have to contend also with pseudo-dominant branches from \( f \). This is tricky, since we must contend with different degrees of \( \tilde{n} \). Even in the homogeneous case, different monomials might define different actions, so we do not have a single action that we can mod out. But, even worse, for nonhomogeneous polynomials \( f \), different components of the same matrix degree \( \tilde{n} \) may occur with different multiplicities in the monomials of the relatively free algebra \( A \), so the characteristic coefficient arguments of Proposition [6.10] may work differently for different monomials of \( f \). In order to identify these actions it is conceptually clearer to bring in another viewpoint of the coefficients of the characteristic polynomial, and unify it with these other actions.

The \( T \)-ideal \( I \) generated by the hiked polynomial \( f \) contains a nonzero \( T \)-ideal which is also an ideal of the algebra \( \tilde{A}_0 \). In principle, we shall use Shirshov’s Theorem [11, Chapter 2] to produce closed ideals, in order to extend our representation of \( \tilde{A}_0 \). In view of Shirshov’s theorem we only need to adjoin a finite number of elements to obtain \( C \).

6.3.1. Symmetrized characteristic coefficients.

Our discussion in adjoining characteristic coefficients involves ambiguities arising from different branches. We could bypass these by making identifications in Definition 6.16 below, but it seems clearer to identify everything with the following notion, in view of Lemma 6.14 below.

**Definition 6.13.** Given matrices \( a_1, \ldots, a_t \), the **symmetrized** \((k; j)\) characteristic coefficient is the \( j \)-elementary symmetric function applied to the \( k \)-characteristic coefficients of \( a_1, \ldots, a_t \).

For example, taking \( k = 1 \), the symmetrized \((1, j)\)-characteristic coefficients \( \alpha_t \) are

\[
\sum_{j=1}^{t} \text{tr}(a_j), \sum_{j_1 < j_2} \text{tr}(a_{j_1}) \text{tr}(a_{j_2}), \ldots, \prod_{j=1}^{t} \text{tr}(a_j).
\]

**Lemma 6.14.** Any characteristic coefficient \( \alpha_k \) is integral over the ring with all the symmetrized characteristic coefficients adjoined.

**Proof.** If \( \alpha_{k;j} \) denotes the \((n; j)\)-characteristic coefficient, then \( \alpha_k \) satisfies the usual polynomial \( \lambda^n + (-1)^j \sum_{j=1}^{n} \alpha_{k;j} \lambda^{n-j} \).

6.3.2. Computing the action of characteristic coefficients.

If the vertex corresponding to \( r \) has matrix degree \( n_i \), taking an \( n_i \times n_i \) matrix \( w \), we define \( \alpha_{\text{pol}}^q \) \((w) \) as in the action of Definition 6.9 and then the left action

\[
a_{u,v} \mapsto \alpha_{\text{pol}}^q (w)a_{u,v}.
\]
Likewise, for an \( n \times n \) matrix \( w \) we define the right action

\[
a_{u,v} \mapsto \alpha_{\text{pol}} \overline{q}(w) a_{u,v}.
\]  

(We only need the action when the vertex is non-empty; we forego the action for empty vertices.)

This action repeats according to the multiplicity \( m \) of the vertex, and so for each given \( u,v \) we take the left and right multiplication operators

\[
\phi^l : a_{u,v} \mapsto \alpha_{\text{pol}} \overline{q}(w)^m a_{u,v}, \quad \phi^r : a_{u,v} \mapsto \alpha_{\text{pol}} \overline{q}(w)^m a_{u,v}
\]

inside the endomorphism algebra of the module containing all the substitutions in dominant branches. This gives us an action on the right substitutions of branches.

We need to consider the endomorphism algebra and its invariants, in order to cope with possible cancellation in symmetric expressions in quasi-linearizations.

### 6.3.3. Resolving ambiguities.

The difficulty is that the action \( a \mapsto \phi(a) \) is not multiplicative in general, i.e. \( \phi(ab) \) need not be \( \phi(a)\phi(b) \).

We need to find a Noetherian module in whose endomorphisms \( A \) can be represented via the Cayley-Hamilton theorem. We need to coordinate two differing actions. Towards this end we introduce an auxiliary ring.

**Definition 6.15.** In the matrix ring \( M_n(K) \), we define

\[
\alpha_{\text{mat},k}(a) := \sum_{j=1}^{n} \sum_{i_1}^{i_k} e_{j,i_1} a e_{i_2,i_2} \cdots a e_{i_k,i_k} a e_{i_1,j},
\]

the inner sum taken over all index vectors of length \( k \).

The operator algebra generated by multiplication by elements \( \{\alpha_{\text{mat},k} : k \leq n\} \) over \( F \) (cf. Notation 6.11) is denoted \( \mathcal{T} \).

Thus \( \alpha_{\text{mat},k}(a) \) gives us the matrix evaluation of a characteristic coefficient, and \( \mathcal{T} \) provides the characteristic coefficients.

We have two versions of characteristic coefficients, one given in Definition 6.15 and the other in Remark 6.7, but the matrix version is not necessarily compatible with polynomial evaluations.

Since we may be in nonzero characteristic, in the main situation our quasi-linear hiked polynomials need not be homogeneous, and we also turn to the pseudo-dominant components.

**Definition 6.16.** Take generators \( \psi_1, \ldots, \psi_l \) of \( \mathcal{T} \), and formally define relations

\[
\overline{\psi}^k_j = \lambda_{j,0} + \lambda_{j,1} \overline{\psi}_j + \cdots + \lambda_{j,k-1} \overline{\psi}^{k-1}_j
\]

for commuting indeterminates \( \lambda_{j,u} \).

Taking this big ring \( A' \) of Notation 6.11. Let \( M = A'/\mathcal{J} \) where \( \mathcal{J} \) is the ideal generated by the elements \( (\lambda_{j,j'} - \gamma_{j,j'})d \) for \( d \) isolated (see Definition 5.7).

**Lemma 6.17.** \( \mathcal{J} \cap \mathcal{I} = 0 \). Hence \( \mathcal{I} \) embeds naturally into \( M \).

**Proof.** If \( f \in \mathcal{J} \cap \mathcal{I} \), then its isolated substitutions must be 0, but by hypothesis \( f \) has nonzero isolated substitutions. \( \square \)

Let \( W \) be the annihilator of \( \mathcal{J} \) in \( A' \). Then \( W \mathcal{J} = 0 \), implying:
Lemma 6.18. The action on $M$ is the same over $T$ and $C$.

Proof. The only possible discrepancy is isolated, so the difference in the action comes from the $\lambda_{j,j'} - \gamma_{j,j'}$, which are in $J$ by definition. □

Lemma 6.19. The algebra $M$ is a finite module over $C$, and in particular is Noetherian and representable.

Proof. Indeed, $M$ is a finite module over $C'$ in view of Shirshov’s Theorem. But $C'$ is finite over $C$, in view of Lemma 6.1, implying $M$ is finite over $C$. Thus $M$ is Noetherian. □

Lemma 6.20. For any polynomial $f$, each of its pseudo-dominant branches provide finite (Noetherian) $C$-submodules of $T$. Consequently, $T$ is integral and finite over $C$.

Proof. The first assertion is by Lemma 6.19. The second assertion follows since the elements of $T$ are integral, and we only need finitely many to generate $T$. □

Also recall that $C$ is finite over $F$.

6.4. Conclusion of the proof of Theorem 1.1

As mentioned earlier, we assume that $\text{char}(F) > 0$, since the result is known in characteristic 0.

We have reduced to the case that $f$ is $A$-quasi-linear and suitably hiked, picking one homogeneous component and zeroing out the other ones.

Now $I$ contains a nonzero $T$-ideal $I_1$ of $\widehat{A}_0$ generated by $\bar{q}$-characteristic coefficient-absorbing polynomials of $I$ in $C'\widehat{A}_0$. The ideal $I_1$ is representable by Lemma 6.19, implying $\langle f \rangle_T \cap I_1$ is representable, a contradiction by Lemma 2.2.

This concludes the proof of Theorem 1.1.

7. Proof of Theorem 1.2, over an arbitrary Noetherian ring $C$

We introduce new notation for the remainder of the paper. Let $A$ be a given relatively free affine PI-algebra over an arbitrary commutative Noetherian ring $C$. From now on, let $J$ denote the nilpotent radical of the Noetherian ring $C$. We take $t$ maximal such that $J^t \neq 0$, i.e., $J^{t+1} = 0$. Write $J$ as a finite intersection $P_1 \cap \cdots \cap P_j$ of prime ideals, with $j$ minimal possible. We call $j$ the irredundancy index of $J$. The proof is based on a triple induction in the following order: Specht induction on $A$, Noetherian induction on $C$, and usual induction on the irredundancy index.

7.1. Various aspects of torsion.

The difference for algebras over a Noetherian ring $C$ from the field-theoretic case is that modules over $C$ may have torsion.

Define $F := C/J$. If $C$ is local then $F$ is a field, and we shall see how to reduce to $F$-algebras. Thus, reduction to $C$ local is a crucial part of the proof.

7.1.1. Reduction to all torsion of $C$ contained in $J$.

We define $Q$ to be the set of elements of $A$ which have annihilators in $C \setminus J$. We claim that we can reduce to the case that $Q = 0$. Assume otherwise that $Q \neq 0$.

Given $0 \neq a \in Q$, define $S_a = \{c \in C \setminus J : c^k a = 0 \text{ for some } k\}$. We want to reduce to the case that $S_a = \emptyset$. For $c \in C$, define $I_{k,c} = \{a : c^k a = 0\}$.

Lemma 7.1. Given $c \in C$, there is some $n$ such that if $c \in S_a$ then $c^n a = 0$. 

\[\text{Lemma 6.18. The action on } M \text{ is the same over } T \text{ and } C.\]

\[\text{Proof. The only possible discrepancy is isolated, so the difference in the action comes from the } \lambda_{j,j'} - \gamma_{j,j'}, \text{ which are in } J \text{ by definition.} \]

\[\text{Lemma 6.19. The algebra } M \text{ is a finite module over } C, \text{ and in particular is Noetherian and representable.} \]

\[\text{Proof. Indeed, } M \text{ is a finite module over } C' \text{ in view of Shirshov’s Theorem. But } C' \text{ is finite over } C, \text{ in view of Lemma 6.1, implying } M \text{ is finite over } C. \text{ Thus } M \text{ is Noetherian.} \]

\[\text{Lemma 6.20. For any polynomial } f, \text{ each of its pseudo-dominant branches provide finite (Noetherian) } C\text{-submodules of } T. \text{ Consequently, } T \text{ is integral and finite over } C. \]

\[\text{Proof. The first assertion is by Lemma 6.19. The second assertion follows since the elements of } T \text{ are integral, and we only need finitely many to generate } T. \]

\[\text{Also recall that } C \text{ is finite over } F. \]

\[\text{6.4. Conclusion of the proof of Theorem 1.1} \]

\[\text{As mentioned earlier, we assume that } \text{char}(F) > 0, \text{ since the result is known in characteristic 0.} \]

\[\text{We have reduced to the case that } f \text{ is } A\text{-quasi-linear and suitably hiked, picking one homogeneous component and zeroing out the other ones.} \]

\[\text{Now } I \text{ contains a nonzero } T\text{-ideal } I_1 \text{ of } \widehat{A}_0 \text{ generated by } \bar{q}\text{-characteristic coefficient-absorbing polynomials of } I \text{ in } C'\widehat{A}_0. \text{ The ideal } I_1 \text{ is representable by Lemma 6.19, implying } \langle f \rangle_T \cap I_1 \text{ is representable, a contradiction by Lemma 2.2.} \]

\[\text{This concludes the proof of Theorem 1.1.} \]

\[\text{7. Proof of Theorem 1.2, over an arbitrary Noetherian ring } C \]

\[\text{We introduce new notation for the remainder of the paper. Let } A \text{ be a given relatively free affine PI-algebra over an arbitrary commutative Noetherian ring } C. \text{ From now on, let } J \text{ denote the nilpotent radical of the Noetherian ring } C. \text{ We take } t \text{ maximal such that } J^t \neq 0, \text{ i.e., } J^{t+1} = 0. \text{ Write } J \text{ as a finite intersection } P_1 \cap \cdots \cap P_j \text{ of prime ideals, with } j \text{ minimal possible. We call } j \text{ the irredundancy index of } J. \text{ The proof is based on a triple induction in the following order: Specht induction on } A, \text{ Noetherian induction on } C, \text{ and usual induction on the irredundancy index.} \]

\[\text{7.1. Various aspects of torsion.} \]

\[\text{The difference for algebras over a Noetherian ring } C \text{ from the field-theoretic case is that modules over } C \text{ may have torsion.} \]

\[\text{Define } F := C/J. \text{ If } C \text{ is local then } F \text{ is a field, and we shall see how to reduce to } F\text{-algebras. Thus, reduction to } C \text{ local is a crucial part of the proof.} \]

\[\text{7.1.1. Reduction to all torsion of } C \text{ contained in } J. \]

\[\text{We define } Q \text{ to be the set of elements of } A \text{ which have annihilators in } C \setminus J. \text{ We claim that we can reduce to the case that } Q = 0. \text{ Assume otherwise that } Q \neq 0. \]

\[\text{Given } 0 \neq a \in Q, \text{ define } S_a = \{c \in C \setminus J : c^k a = 0 \text{ for some } k\}. \text{ We want to reduce to the case that } S_a = \emptyset. \text{ For } c \in C, \text{ define } I_{k,c} = \{a : c^k a = 0\}. \]

\[\text{Lemma 7.1. Given } c \in C, \text{ there is some } n \text{ such that if } c \in S_a \text{ then } c^n a = 0. \]

Proof. $\mathcal{I}_{k,c}$ is clearly an ideal of $A$, and is a T-ideal, since for any endomorphism $\varphi$ of $A$, $\varphi^{k}\varphi(a) = (\varphi^{k}a) = \varphi(0) = 0$. Hence, we have an ascending chain of T-ideals $\mathcal{I}_{1,c} \subseteq \mathcal{I}_{2,c} \subseteq \ldots$, which must stabilize at some $\mathcal{I}_{n,c}$. This means that any element annihilating a power of $c$ must annihilate $c^n$.

**Lemma 7.2.** For any $0 \neq a \in Q$, if there is a counterexample $A$ to Theorem 1.1, then there is a counterexample $\bar{A}$ which is a homomorphic image of $A$ with $S_{\bar{a}} = \emptyset$, where $\bar{a}$ is the image of $a$ in $\bar{A}$.

**Proof.** Take $A$ a Specht minimal counterexample. Pick $c \in S_{a}$, and $\mathcal{I}_{n,c} \neq 0$, in view of Lemma 7.1. We have an injection $A \to A/\mathcal{I}_{n,c} \oplus (A \otimes_C (C/c^nC))$. But $A/\mathcal{I}_{n,c}$ is representable, by Specht induction, and $A \otimes_C (C/c^nC)$ is representable by Noetherian induction on $C$. Hence $A$ is representable, contrary to assumption. Since this holds for every $c \in S_{a}$, we conclude that $S_{a} = \emptyset$.

**Proposition 7.3.** If there is a counterexample $A$ to Theorem 1.1, then there is a counterexample $\bar{A}$ for which all elements of $C$ making elements of $A$ torsion lie in $J$.

**Proof.** Take $A$ a Specht minimal counterexample. We claim that $Q = 0$. Indeed otherwise we can take $a \neq 0$ in $A$ and contradict Lemma 7.2.

7.1.2. Reduction to irredundancy index of $J$ equaling 1.

**Corollary 7.4.** If there is a counterexample $A$ to Theorem 1.1, satisfying the conclusion of Proposition 7.3, then there is a counterexample in which the irredundancy index of $J$ is 1.

**Proof.** The intersection $P_1 \cap \cdots \cap P_j$ of prime ideals clearly is irredundant. We claim that $j = 1$. Indeed if $j > 1$ we can take $s \in P_1 \setminus J$. Localizing $A$ at $s$, we have $A$ embedded into $A[s^{-1}] \oplus (A \otimes_C (C/s'C))$. $A \otimes_C (C/s'C)$ is representable by Noetherian induction. Thus, it suffices to show that $A[s^{-1}]$ is representable. On the other hand, the kernel of the natural map $A \to A[s^{-1}]$ is $\text{Ann}(s^l)$, implying $C[s^{-1}]/P_i[s^{-1}] \cong (C/P_i)[s^{-1}]$ for each $i \geq 2$ (for if $s^k(c + P_i) = 0$ then $s^kc \in P_i$ and thus $c \in P_i$). Likewise $P_i[s^{-1}]$ is a prime ideal of $C[s^{-1}]$, since if $c'c'' \in P_i[s^{-1}]$ then some $s^k c'c'' \in P_i$, implying $c'c'' \in P_i$, so $c' \in P_i$ or $c'' \in P_i$.

By Lemma 7.1 if $c \in P_i$ for $2 \leq i \leq j$ then $s^nc \in J$, so $c \in J[s^{-1}]$. On the other hand, if $s^{-k}c$ is nilpotent then $s^nc = 0$, implying $c \in P_i$ for $2 \leq i \leq j$, and we conclude that $J[s^{-1}]$ is the nilradical of $C[s^{-1}]$, which has lower irredundancy index. So we are done by induction on $j$ once we manage to prove the case $j = 1$.

7.1.3. Reduction to $C$ local.

**Proposition 7.5.** If there is a counterexample $A$ to Theorem 1.1, then there is a counterexample $\bar{A}$ for which $C$ is local and all elements of $J$ making elements of $A$ torsion lie in $J$.

**Proof.** Take $A$ a Specht minimal counterexample. By Corollary 7.4 we may assume that the irredundancy index of $J$ equals 1. Localizing by all elements of $C \setminus J$ we may assume that $J$ is a maximal ideal of $C$, i.e., $C$ is local.
7.2. **Further reduction for $C$ Noetherian.**

Next, we consider the general case that $C$ is Noetherian; in view of the previous discussion we may assume that $C$ is a local Noetherian domain. If $J = 0$ then by Proposition 7.5 we may assume $A$ is torsion free over $C$. Localizing, we embed $A$ into a relatively free algebra over the field of fractions of $C$, so we could assume that $C$ is a field, which is discussed further in §7.3. Hence we assume that $J \neq 0$, i.e., $t \geq 1$.

There exists $s \in J$ for which $sA \neq 0$, since otherwise $J^tA = 0$, implying $A$ is a $C/J^t$-algebra, and we are done by Noetherian induction. Take $s \in J$ for which $sA \neq 0$. $A/\text{Ann } s$ is an algebra over $F$, since $sJ = 0$, implying $A$ is a $C/J$-algebra, and we are done by Noetherian induction. Take $s \in J$ for which $sA \neq 0$.

7.3. **Conclusion of the proof via the field case.**

We conclude the proof by one last application of hiking. Take some polynomial $f \in sA \setminus \{0\}$. The idea is to find a hiked polynomial in the $T$-ideal of $f$, with which we can then apply Shirshov’s theorem to utilize results from integrality.

The module $sA \neq 0$ is a module over $F = A/J$ since $sJ = 0$. Thus, we can use the theory of hiking on $sA$. Take a nonzero polynomial $f \in sA$, and let $M$ be the set of hiked polynomials from $f$, obtained via Theorem 4.1. $M$ is a module since hiking involves a series of four stages of substitutions, and multiplying a hiked polynomial by another polynomial yields a hiked polynomial. We take some hiked polynomial $0 \neq g \in M$. As in Lemma 6.19 we can use Shirshov’s theorem to adjoin finitely many elements to $C$ to obtain a commutative ring $C$ for which $C \otimes gA$ is finite over $C$ and thus over $C$; hence it is representable by Anan’in’s Theorem [4].

But $gA$ contains a critical nonidentity, which we can then hike to a critical nonidentity $h$. Viewing $hA \subset M_m(C')$ we define $\xi_i(a)$ to be the $i$-characteristic coefficient, i.e.,

$$a^m = \sum_{i=0}^{m-1} \xi_i(a)a^{m-i}.$$  

We use the module action of §6.3.3 in the general, nonhomogeneous case. Let $\psi_i$ be the substitution operator for hiked polynomials, and impose the relations

$$(\psi_i(a) - \xi_i(a))h = 0$$

to get a canonical homomorphism $\varphi : A \to \hat{A}$. Its kernel intersects $hA$ trivially. This induces a map canonical map $A \to \hat{A} \oplus (A/hA)$, which is an injection, by Lemma 2.2, proving that $A$ is representable.

8. **Appendix: Further applications of hiking, for other categories**

Specht’s Conjecture and representability of a $T$-ideal $I$ may be handled in some other categories of algebras, mutatis mutandis, since hiking is a formal process. In this brief appendix we show how to modify the proof for algebras with involution, and indicate how it could also work for other categories. We start by noting that the reduction to algebras over arbitrary Noetherian goes through as in Section 7, which is module-theoretic over the associative commutative base ring. So the issue is for algebras over a field. A $C$-algebra in some category is called **representable** if there is a 1:1 morphism to finite dimensional $K$-algebra in the category, for a suitable field $K$.
8.1. Algebras with Involution.

An involution of an algebra is an anti-automorphism (*)& order \( \leq 2 \). Involutions occur throughout algebra, in the theory of group algebras and Lie algebras, and more generally, Hopf algebras; matrix algebras with involution play a crucial role in defining the classical Lie algebras. One develops the theory in terms of (*)& in the category, as in \[33\] §2.13 and \[32\]. A (*)&-ideal is an ideal \( A \) such that \( A^* = A \). An algebra is (*)&-simple if it has no proper nonzero (*)&-ideals. \((C\{x\},*)\) denotes the free associative algebra with involution in the indeterminates \( x_0,x_1,x_1^*,\ldots \), where (*)& acts in the obvious way:

\[
(x_i^*)^* = x_i; \quad (x_{i_1}\cdots x_{i_t})^* = x_{i_t}^*\cdots x_{i_1}^*.
\]

Its elements \( f(x_1,x_1^*,x_2,x_2^*,\ldots,x_m,x_m^*) \) are called (*)&-polynomials, and are written here as \( f(x_1,x_2,\ldots,a_m) \). \( f(a_1,\ldots,a_m) \) is the specialization of \( f \), under substituting \( x_i \mapsto a_i \) and \( x_i^* \mapsto a_i^* \). For an \( F \)-algebra with involution \((A,*)\), \( f(A,*) \) denotes \( \{f(a_1,\ldots,a_m) : a_i \in A \} \). We say that \( f \) is a (*)&-identity of \((A,*)\) if \( f(A,*) = 0 \), and \((A,*)\) is a (*)&-PI-algebra if \((A,*)\) has a nonzero (*)&-identity. A crucial theorem of Amitsur \[3\] is that every (*)&-PI-algebra is a PI-algebra.

There are three standard involutions related to matrix algebras over a field \( K \) of characteristic \( \neq 2 \).

1. (Exchange type) \((A,*) = (M_n(K) \oplus M_n(K)^{op},\circ)\), where \((\circ)\) is the exchange involution \((a_1,a_2)^{\circ} = (a_2,a_1)\). \( d^+ = d^- = n^2 \).
2. (Orthogonal type) \((A,*) = (M_n(K),t)\), where \((t)\) is the transpose; \( d^+ = \frac{n(n+1)}{2} \) and \( d^- = \frac{n(n-1)}{2} \).
3. (Symplectic type) \((A,*) = (M_{2n}(K),s)\) where \((s)\) is \( e_{i,j}\): \n
\[
e_{i,j} = e_{j+n,i+n}, \quad e_{i,j+n}^* = -e_{j,i+n}, \quad e_{i+n,j}^* = -e_{j+n,i}, \quad \forall 1 \leq i,j \leq n.
\]

\[d^+ = \frac{n(n+1)}{2} \text{ and } d^- = \frac{n(n-1)}{2}.\]

Lemma 8.1. Any (*)&-simple algebra over an algebraically closed field can be put into one of these three forms.

A \( C \)-algebra with involution \((A,*)\) is called (*)&-representable if it is embeddable as a \( C \)-subalgebra with involution of a finite dimensional \( K \)-algebra with involution \((W,*)\), for a suitable field \( K \). In this case, by means of the regular representation, one can embed \( A \) into some \((M_m(K),*)\) and, tensoring by the algebraic closure of the fixed subfield of \( K \), assume that \( W = (M_n(K) \oplus M_n(K)^{\circ},\circ) \) or \( W = (M_n(K),*) \) where (*)& is a standard involution.

A (*)&-T-ideal is a T-ideal which is also invariant under (*)&.

8.1.1. ACC on (*)&-T-ideals in arbitrary characteristic.

Sviridova \[10\] proved the ACC on (*)&-T-ideals (Specht’s problem) over a field of characteristic 0; for affine algebras, the key step for affine (*)&-PI-algebras is the analog of \[28\] proved in \[39\] that any affine (*)&-PI-algebra satisfies precisely the same (*)&-identities as some finite dimensional (*)&-algebra, which is essentially the same as the (*)&-representability of relatively free affine (*)&-algebras.

For a field \( F \) of characteristic \( p > 0 \), we want to use hiking to obtain these theorems, much as in \[17\] and in the main text of this paper. Although the argument has not been published, the solution to Specht’s problem was described in \[35\], which we follow here in developing hiking of (*)&-polynomials.
Since the strategy outlined in [17, Remark 2.3] relied on full quivers on the Zariski closure in a representation of \((A, \ast)\), we need the analog for algebras with involution. A \((\ast)\)-T-ideal \(\mathcal{I}\) is **representable** if \((C\{x\}/CI, \ast)\) is a \((\ast)\)-representable algebra.

**Remark 8.2.** The program to prove Specht’s conjecture and representability of a \((\ast)\)-T-ideal \(\mathcal{I}\).

1. In view of Amitsur’s theorem, \(\mathcal{I}\) contains a \((\ast)\)-T-ideal \(\mathcal{I}_0\) that is representable, so we can work in the \((\ast)\)-representable algebra \((C\{x\}/CI_0, \ast)\), which we embed into matrices with over an algebraically closed field \(K\), with standard involution of transpose or symplectic type. (Exchange type is easily reduced to the non-involutory case by projecting to each component.) In other words, we may assume that \((A, \ast) \subseteq (M_n(K), \ast)\).

2. The Zariski closure is closed with respect to the involution \((\ast)\) of \(W\). (Proof: Applying \((\ast)\) to each polynomial relation for \(a\) yields a polynomial relation for \(a^*\).) Replacing \((A, \ast)\) by its Zariski closure, we may assume that \((A, \ast)\) is Zariski closed.

3. The Jacobson radical of the Zariski closure is a nilpotent \((\ast)\)-ideal.

4. Any Zariski closed \((\ast)\)-algebra \((A, \ast)\) can be decomposed into \((S, \ast) \oplus J\), where \((S, \ast) \cong (A/J, \ast)\) is a direct product of \((\ast)\)-simple algebras \((S_i, \ast)\), each of the form of Lemma 8.1. (Indeed, \(S\) is the direct product of matrix algebras over fields, since its center is Zariski closed and one mimics the classical proof of Wedderburn’s theorem in [33, Theorem 2.5.37], as first done by E. Taft, noting that symmetric idempotents (resp. antisymmetric) idempotents of \(A/J\) lift to symmetric idempotents (resp. antisymmetric) idempotents of \(A\).

5. There is a natural \((\ast)\)-version of the Wedderburn block form, and thus a quiver.

6. If a matrix \(a\) is semisimple, then so is \(a^*\), so quasi-linearization enables us to reduce to semisimple and radical substitutions.

7. The issues of \((\ast)\)-hiking are the same as noninvolutory hiking, since one can insert Capelli polynomials without \((\ast)\).

8. The \((\ast)\) version of Theorem 7.4 is proved in exactly the same way.

9. One can define characteristic coefficients using either matrices or polynomials. This is a bit tricky since, for \((\ast)\) of symplectic type, the pfaffian [32, Theorem 2.5.10] takes the place of the characteristic polynomial for symmetric elements, so the degree is \(\frac{n^2}{2}\).

10. Adjoin the characteristic coefficients using polynomials to get an integral extension, noting that the arguments in the text only used properties of modules over commutative rings.

11. Apply Shirshov’s theorem to get a \((\ast)\)-algebra finite over a \((\ast)\)-fixed commutative Noetherian algebra, which is \((\ast)\)-representable by the \((\ast)\)-analog of [36].

**Theorem 8.3.**

*Proof.* In the program of [17], go through the steps of Remark 2.3, where all the work was done, since hiking behaves in the same way. Hence the ACC for \((\ast)\)-T-ideals holds over an arbitrary field of characteristic \(p > 0\), and in light of Sviridova’s theorem, over any field, and thus over any commutative Noetherian ring.

8.1.2. \((\ast)\)-representability of relatively free affine \((\ast)\)-algebras over a field.
Theorem 8.4. Any relatively free affine \((\ast )\)-PI algebra over a commutative Noetherian ring is \((\ast )\)-representable.

Proof. Using Specht’s \((\ast )\)-Conjecture, we can take a maximal non-\((\ast )\)-representable T-ideal. Then we repeat the proof of Theorem 8.1 using Remark 8.2 where appropriate.

\(\Box\)

8.2. Other categories.

Some other categories of algebras are amenable to the program in Remark 8.2. We need a category for which Specht’s problem was solved in characteristic 0, and for which the simple objects over an algebraically closed field are easily characterized in terms of their identities, and have matrix-like descriptions in which one can take the Zariski closure and define hiking.

8.2.1. Alternative algebras.

Alternative algebras have a similar structure theory to associative algebras, in part because 2-generated alternative algebras are associative, by Artin’s theorem. (Of course one takes T-ideals in the free alternative algebra.) Thus any alternative PI-algebra satisfies a 2-generated identity, and Remark 6.4 is applicable. Ilyakov solved Specht’s problem for affine alternative algebras of characteristic 0. Shafer proved the Wedderburn principle theorem, and the only split simple alternative algebras are the split algebra of octonians (which are algebraic of degree 2 and satisfy the same 2-generated identities of \(M_2(F)\)) and the usual associative matrix algebras. Thus one can define a Zariski closure, and the Wedderburn block form, thereby providing hiking.

One proves the ACC on T-ideals over an arbitrary field by the induction procedure given in [17, Definition 7.1 and Lemma 7.2], and then the \((\ast )\)-representability of relatively free affine alternative \((\ast )\)-algebras over an arbitrary field, as in the main text.

8.2.2. Group-graded algebras.

Specht’s problem was solved for affine algebras of characteristic 0 graded by a finite group, by Aljadeff and Belov in [2]. The gradings in matrix algebras are described explicitly in [5]. In characteristic 0, the Jacobson radical often is graded [20], but the situation is messier in nonzero characteristic. Karasik developed the necessary structure theory in terms of \(G\)-simple PI-algebras, but their structure seems to be quite complicated, so some details need to be worked out in the Wedderburn block form.

8.2.3. Jordan algebras.

The Wedderburn decomposition \(S \oplus J\) into split simple Jordan algebras and the radical was discovered by Albert [1] for special Jordan algebras, and by Penico in general over a field of characteristic \(\neq 2\). One also knows the split simple Jordan algebras, characterized in terms of their polynomial identities, so we have the Zariski closure and Wedderburn block form, and thus can perform hiking. Vais and Zelmanov proved Kemer’s conjecture in characteristic 0, but representability remains open.

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