Topological Interpretation of Interactive Computation

Emanuela Merelli\(^1\) and Anita Wasilewska\(^2\)

\(^1\) Department of Computer Science, University of Camerino, Italy
emanuela.merelli@unicam.it

\(^2\) Department of Computer Science, Stony Brook University, Stony Brook, NY, USA
anita@cs.stonybrook.edu

Abstract. It is a great pleasure to write this tribute in honor of Scott A. Smolka on his 65th birthday. We revisit Goldin, Smolka hypothesis that persistent Turing machine (PTM) can capture the intuitive notion of sequential interaction computation. We propose a topological setting to model the abstract concept of environment. We use it to define a notion of a topological Turing machine (TTM) as a universal model for interactive computation and possible model for concurrent computation.

Keywords: Persistent Turing machine; Topological environment; Topological Turing machine.

1 Introduction

In 2004, Scott A. Smolka worked with Dina Goldin\(^3\) and colleagues on a formal framework for interactive computing; the persistent Turing machine (PTM) was at the heart of their formalization \([1,2,3]\). A PTM is a Turing machine (TM) dealing with persistent sequential interactive computation a class of computations that are sequences (possibly infinite) of non-deterministic 3-tape TMs. A computation is called sequential interactive computation because it continuously interacts with its environment by alternately accepting an input string on the input-tape and computing on the work-tape a corresponding output string to be delivered on the output-tape. The computation is persistent, meaning that the content of the work-tape persists from one computation step to the next by ensuring a memory function.

The definition of PTM was based on Peter Wegner’s interaction theory developed to embody distributed network programming.

Interaction is more powerful than rule-based algorithms for computer problem solving, overturning the prevailing view that all computing is expressible as algorithms \([4,5]\).

\(^3\) The work was developed in connection of the celebration of Paris Kanellakis for his 50th birthday. They were his first and last Ph.D student.
Since in this framework interactions are more powerful than rules-based algorithms they are not expressible by an initial state described in a finite terms. Therefore, one of the four Robin Gandy’s principles (or constraints) for computability is violated, as stated in [6]. The need to relax such constraints allows one to think that interactive systems might have a richer behavior than algorithms, or that algorithms should be seen from a different perspective. Although PTM makes the first effort to build a TM that accepts infinite input, we strongly support the idea that the interaction model should also include the formal characterization of the notion of environment.

In this paper, we focus on Smolka et al. original point of view on persistent and interactive computation. We revisit and formalize a concept of computational environment for PTM following Avi Wigderson’s machine learning paradigm in [7].

Many new algorithms simply ’create themselves’ with relatively little intervention from humans, mainly through interaction with massive data.

We use the notion of computational environment to define class of abstract computable functions as sets of relations between inputs and outputs of PTM. The computational environment depends on time and space. It can evolve and so the effectiveness of these functions depends on a given moment and a given context.

Computational environment is defined in terms of ambient space. The ambient space is a generalization of a notion of ambient manifold introduced in [8] to describe the topological quantum computation model.

We do it in such a way that the infinite computation can be reduced to a set of relations, constrained within its ambient space by loops of non-linear interactions. The ambient space is not necessarily a vector space, hence there is a problem of linearity and non-linearity of computation. The non-linearity originated from the shape that can be associated to the ambient space, which can be obtained by the topological analysis of the set of data provided by the real environment. Figure [1] shows the synthesis of this concept. The ambient space and PTM can be thought as mathematical representation of complex systems, merely defined as systems composed of many non-identical elements, constituent agents living in an environment and entangled in loops of non-linear interactions.

We built a topological PTM to model both the behavior of an interactive machine and its computational environment. The main idea of the generalization is that output-tape is forced to be connected to the input-tape through a feedback loop. The latter can be modeled in a way that the input string can be affected by the last output strings, and by the current state of the computational environment. A state of a topological PTM becomes a set of input and output relations constrained to an environment whose geometric representation formally defines the context of the computation. If many topological PTMs share the same computational environment, the computation becomes a stream of interactions of concurrent processes, which at higher dimension can be seen as a collection of

4 https://www.ias.edu/ideas/mathematics-and-computation
streams, such as an $n$-string braid as examined in topological quantum information \cite{8}. In this scenario, the computational environment, envisaged as a discrete geometric space, may even evolve while computations take place.

The informal description given above depicts the environment. We define it as follows. Given a PTM, let $X$ be a set of its input and output strings. Since the computational environment depends on time and space. In this case the time is represented by collection of steps. For each step $i$ in time, we define an equivalence relation $\sim_i$ on $X$ such that $input_i$ in $X$ there exists an operator $f_i$ such that $f_i(input_i) = output_i$.

In classical Turing machine the set of operators $f_i$ is called rules or transformations. Our goal is to build an environment where this set of functions $f_i$ can be discovered. Each element of $X$ represents a transition from one state of the machine to a next guided by the operator $f_i$ (unknown for the model) constrained over the computational environment. The mathematical objects we are looking for should reflect the collective properties of the set $X$ in a natural way to support the discovery of the set of operators $f_i$. These operators allow us to represent $X$ as a union of quotient spaces of the set of equivalence classes $X/\sim_i$ of all the feasible relations hidden in $X$. The resulting functional matrix of $f_i$, also called interaction matrix, represents the computational model or what we called the learnt algorithm \cite{9}.

In order to characterize the set of operators $\{f_i\}$, we decided to analyze the set $X$ of environmental data by a persistent homology, a procedure used in topological data analysis (TDA). TDA is a subarea of the computational topology that filters the optimal space among simplicial complexes \cite{10}. A simplicial complex can be seen as a generalization of the notion of graph, where the relations are not binary between vertices, but n-ary among simplices. A simplex expresses any relation among points. For example, a 0-dimensional simplex is a unary relation of a single point, a 1-dimensional simplex is a binary relation of two points (a line), a 2-dimensional simplex is a three points relation (a full triangle), and so on. For the interested reader, Appendix 1 gives some useful definitions for algebraic and computational topology. Although a simplicial complex allows us to shape the environment as a discrete topological space, the new model of PTM also requires to express the feedback loop between the output at step $i$ with the input at the next step $i+1$ of the computation. To this end, we follows a recent approach proposed in the context of big data and complex systems for embedding a set of correlation functions (e.g. the encoding of a given data set) into a field theory of data which is relied on a topological space naturally identified with a simplicial complex \cite{11}. The resulting mathematical structure is a fiber bundle \cite{12}, whose components are summarized in Figure 1.

The framework consists of three topological spaces $B, H, G$, and one projection map $\pi$. The base space $B$, the set of input/output strings embedded in a simplicial complex; the fiber $H$, the set of all possible computations (the set of $f_i$) constrained by the ‘gauged’ transformations over the base point $P_B$ of the fiber; the total space $G$ of the fiber bundle obtained by the product of the other two spaces ($G = B \times H$), and the projection map $\pi : G \to G/H$ that allows
Fig. 1. Example of topological interpretation of computation. The base space $B$ is a two-dimensional handlebody of genus 3, such as a trifold. The small red circles around some points of the fiber space $H$ indicate the presence of states that make the computation inconsistent. The violet lines over the base space $B$ show the corresponding unfeasible paths to be avoided due to the topological constraints imposed by the base space. The non-linear transformations of the fibers states, induced by the projection map $\pi$ over the simplicial complex, guarantees the choice of the admissible paths with respect to the topology of the base space. The lines marked with a black cross correspond to inconsistent states of the system, which do not exist in the topological interpretation. The picture at the bottom right corner is an example of computation, it refers to the notion of contextuality [18], informally a family of data – a piece of information – which is locally consistent and globally inconsistent.

us to go from the total space $G$ to the base space $B$ obtained as a quotient space of the fiber $H$. In Figure 1 the $\pi$ projection map is represented by dashed lines and used to discover if the geometry of the base space can constrain the ongoing computation in order to predict and avoid unfeasible transformations, the red lines in the figure. In our model, the obstructions that characterize the ambient space and constraint the computation are represented by the presence of $n$-dimensional holes ($n > 1$) in the geometry of the topological space. In our framework the holes represent the lack of specific relations among input and output of topological Turing machine. It means that the topological space, in our
representation as simplicial complex, has a non trivial topology. As an example, in Figure 1 the base space \( B \) is two-dimensional handlebody of genus 3. The formal description of the proposed approach rests on three pillars: i) algebraic and computational topology for modeling the environment as a simplicial space \( B \); ii) field theory to represent the total space \( G \) of the machine as a system of global coordinates that changes according to the position \( P_B \) of the observer respect to the reference space \( H \), and iii) formal languages to enforce the semantic interpretation of the system behaviour into a logical space of geometric forms, in terms of operators \( f_i \) that here we call correlation functions in the space of the fiber \( H \).

Consequently, an effective PTM is nothing but a change of coordinates, consistently performed at each location according to the ‘field action’ representing the language recognized by the machine.

While the algorithmic aspect of a computation expresses the effectiveness of the computation, the topological field theory constraints the effectiveness of a computation to a specific environment where the computation might take place at a certain time in space.

It is right here to recall Landin’s metaphor of the ball and the plane, introduced to describe the existence of a double link between a program and machine [13]:

One can think of the ball as a program and the plane as the machine on which it runs. ... the situation is really quite symmetric; each constrains the other [14].

Alan Turing himself, in his address to the London Mathematical Society in 1947, said

... if a machine is expected to be infallible, it cannot also be intelligent [15].

It is becoming general thinking that intelligence benefits from interaction and evolves with something similar to adaptability checking [9]. Accordingly, the PTM, and its topological interpretation seem to be a good starting point for modeling concurrent processes as interactive TMs [19]. Also considering that the set of PTMs reveals to be isomorphic to a general class of effective transition systems as proved in Smolka et al. in [1]. This result allows to make the hypothesis that the PTM captures the intuitive notion of sequential interactive computation [2], in analogy to the Church-Turing hypothesis that relates Turing machines to algorithmic computation.

**What is computation?** Turing, Church, and Kleene independently formalized the notion of computability with the notion of Turing machine, \( \lambda \)-calculus, partial recursive functions. Turing machine manipulates strings over a finite alphabet, \( \lambda \)-calculus manipulate \( \lambda \)-terms, and \( \mu \)-recursive functions manipulate natural numbers. The Church-Turing thesis states that

every effective computation can be carried out by a Turing machine or equivalently a certain informal concept (algorithm) corresponds to a certain mathematical object (Turing machine) [16].
The demonstration lies on the fact that the three notions of computability are formally equivalent. In particular, the Turing machine is a model of computation like a finite states control unit with an unbounded tape used to memorize strings of symbols. A deterministic sequence of computational steps transforms a finite input string in the output string. For each step of the computation, a Turing machine contains all the information for processing input in output, an algorithmic way to computing a function, those functions that are effectively computable. The Universal TM is the basic model of all effectively computable functions, formally defined by a mathematical description.

**Definition 1 (Turing machine)** A Turing machine (TM) is $\mathcal{M} = \langle Q, \Sigma, \mathcal{P} \rangle$,

- $Q$ is a finite set of states;
- $\Sigma$ is a finite alphabet containing the blank symbol $\#$; $L$ and $R$ are special symbols.
- $\mathcal{P} \subseteq Q \times \Sigma \times Q \times \{L, R\}$, the set of configurations of $\mathcal{M}$.

A computation is a chain of elements of $\mathcal{P}$ such that the last one cannot be linked to any possible configuration of $\mathcal{P}$.

The multi-tape Turing machine is a TM equipped with an arbitrary number $k$ of tapes and corresponding heads.

**Definition 2 (k-tape Turing machine)** A non-deterministic $k$-tape TM is a quadruple $\langle Q, \Sigma, \mathcal{P}, s_0 \rangle$, where

- $Q$ is a finite set of states; $s_0 \in Q$ is the initial state and $h \notin Q$ is the halting state.
- $\Sigma$ is a finite alphabet containing the blank symbol $\#$. $L$ and $R$ are special symbols.
- $\mathcal{P} \subseteq Q \times \Sigma^k \times (Q \cup \{h\}) \times (\Sigma \cup \{L, R\})^k$ is the set of configurations.

The machine makes a transition from its current configuration (state) to a new one (possibly the halt state $h$). For each of the $k$ tapes, either a new symbol is written at the current head position or the position of the head is shifted by one location to the left (L) or right (R).

The above definitions of TMs do not take into account the notion of environment; the input is implicitly represented in the configurations $\mathcal{P}$ of $\mathcal{M}$ machine modulo feasible relations. The objective of this contribution is to represent the environment explicitly in a way such that the admissible relations are naturally determined. Our view is supported by a recent, even though not formal, definition of computation.

*Computation is the evolution process of some environment via a sequence of simple and local steps.*

A Computational environment is the base space over which the process of transformation of an input string happens. For the TM, an environment is any configuration of $P$ of a machine $M$, from the initial one to the final one. It is a closed set – represented by the functional matrix, – whose feasible relations should be known a priori to assure the algorithmic aspect of the computation. Indeed, in TM the environment does not evolve, it remains unchanged during the computation.

If we consider the environment as an open set - the set of configurations may changes along the way due to computation - accordingly, the set of feasible relations may change. As Section 3 describes, one way to capture this variation is to associate a topology to the space of all possible configurations and use the global invariants of the space to classify the relations in categories whose elements are isomorphic to those of some model of computation, such as the TM. In this setting, the local steps (feasible relations) – the functional matrix – are affected by global topology. As a consequence, the evolution of an environment corresponds to a change of the topological invariants. Then the classical TM is equivalent to working with a space of states whose topology is invariant, which allows the process of transformation to run linearly.

While an interactive computation takes into account the non-linearity of the computation due to the structure of the transformations characterizing it. The non-linearity is implied by the topology of the base space $B$, and induced by the semi-direct product factorization of the transformation group, the simplicial analog of the mapping class group, denoted by $G_{MC}$.

In the viewpoint of computation as a process, the global context induces non-linear interactions among the processes affecting the semantic domain of the computation. The semantic object associated to TM, that is the function that TM computes, or the formal language that it accepts, becomes an interactive transition system for a PTM. In the topological setting it changed into the pair of $\langle$function, structure$\rangle$, entangled as a unique object. The function represents the behavior and the structure the context. Formally represented by the fiber subgroup in the semi-direct product form of the group of computations (connected to process algebra), denoted by $G_{AC}$, and $G_{MC}$ the group of self-mapping of the topological spaces (the environments self-transformations algebra, i.e. automorphisms which leave the topology invariant), quotient by the set of feasible relations. The new semantic object, a gauge group $G = G_{AC} \wedge G_{MC}$, provides another way to understand the meaning of contextuality [17], as a tool to distinguish effective computation from interactive computations. That is to identify configurations that are 'locally consistent, but globally inconsistent', as shown in Figure 1 and informally summarised in the following sentence.

"Contextuality arises where we have a family of data which is locally consistent but globally inconsistent."

Section 3 introduces the new interpretation. We leave the formal definition and full formalization of the theory corresponding to the group of computations for an evolving environment as future work.
2 Interactive Computation

In this section, we recall the definition of the persistent Turing Machine, \textit{PTM} as defined by Smolka et al. in [1] and the related notion of \textit{environment} introduced in their earlier work [2]. We introduce the definitions needed to support the construction of a new topological model that is a generalization of the \textit{PTM}. The new model allows one to re-interpret the classic scheme of computability, which envisages a unique and complete space of problems.

The \textit{PTM} provides a new way of interpreting \textit{TM} computation, based on dynamic stream semantics (comparable to behavior as a linear system). A \textit{PTM} is a non-deterministic 3-tape \textit{TM} (N3TM) that performs an infinite sequence of classical \textit{TM} computations. Each such computation starts when the \textit{PTM} reads input from its \textit{input-tape} and ends when the \textit{PTM} produces an output on its \textit{output-tape}. The additional \textit{work-tape} retains its content from one computational step to the next to carry out the persistence.

\textbf{Definition 3 (Smolka, Goldin Persistent Turing machine)} A persistent Turing machine (\textit{PTM}) is a N3TM having a read-only \textit{input-tape}, a read/write \textit{work-tape}, and a write-only \textit{output-tape}.

Let $w_i$ and $w_0$ denote the content of the input and output tapes, respectively, while $w$ and $w'$ the content of work-tape, and $\#$ empty content, then

- an \textit{interaction stream} is an infinite sequence of pairs of $(w_i, w_o)$ representing a computation that transforms $w_i$ in $w_o$;
- a \textit{macrostep} of \textit{PTM} is a computation step denoted by $w \xrightarrow{w_i/w_o} w'$, that starts with $w$ and ends with $w'$ on the work-tape and transforms $w_i$ in $w_o$;
- a \textit{PTM computation} is a sequence of \textit{macrosteps}.

$w \xrightarrow{w_i/\mu} s_{\text{div}}$ denotes a macrostep of a computation that diverges (that is a non-terminating computation); $s_{\text{div}}$ is a particular state where each divergent computations falls, and $\mu$ is special output symbol signifying divergence; $\mu \notin \Sigma$.

Moreover, the definition of the interactive transition system (\textit{ITS}) equipped with three notions of behavioral equivalence – \textit{ITS} isomorphism, interactive bisimulation, and interaction stream equivalence – allows them to determine the \textit{PTMs} equivalence.

\textbf{Definition 4 (Interactive transition system)} Given a finite alphabet $\Sigma$ not containing $\mu$, an \textit{ITS} over $\Sigma$ is a triple $(S, m, r)$ where

- $S \subseteq \Sigma^* \cup \{s_{\text{div}}\}$ is the set of states;
- $m \subseteq S \times \Sigma^* \times S \times (\Sigma^* \cup \{\mu\})$ is the transition relation;
- $r$ denotes the initial state.

It is assumed that all the states in $S$ are reachable from $r$. Intuitively, a transition $\langle s, w_i, s', w_o \rangle$ of an \textit{ITS} states that while the machine is in the state $s$ and having received the input string $w_i$ from the environment, the \textit{ITS} transits to state $s'$ and output $w_o$. 

Unfortunately, the sake of space economy forced to omit most of the results; we only recall Theorem 24, Theorem 32 and Thesis 50 (in the sequel renumbered Theorem 1, Theorem 2 and Thesis 1, respectively) and address the reader eager for more information to the original article [1].

**Theorem 1** The structures $\langle M, =_{ms} \rangle$ and $\langle T, =_{iso} \rangle$ are isomorphic.

Theorem 1 states that there exists a one-to-one correspondence between the class of PTMs, denoted by $M$ up to macrostep equivalence, denoted by $=_{ms}$, and the class of ITSs, denoted by $T$ up to isomorphism, denoted by $=_{iso}$.

**Theorem 2** If a PTM $M$ has unbounded nondeterminism, then $M$ diverges.

Theorem 2 states that a PTM $M$ diverges if there exists some $w \in \text{reach}(M)$, $w_i \in \Sigma^*$ such that there is an infinite number of $w_o \in \Sigma^* \cup \{\mu\}$, $w' \in \Sigma^* \cup \{s_{div}\}$, such that $w \xrightarrow{w_i/w_o} w'$.

**Thesis 1** Any sequential interactive computation can be performed by a PTM.

Like the Church-Turing Thesis, Thesis 1 cannot be proved. Informally, each step of a sequential interactive computation, corresponding to a single input/output-pair transition, is algorithmic. Therefore, by the Church-Turing Thesis, each step is computable by a TM. A sequential interactive computation may be history-dependent, so state information must be maintained between steps. A PTM is just a TM that maintains state information on its work-tape between two steps. Thus, any sequential interaction machine can be simulated by a PTM with possibly infinite input.

**The PTM environment.** In her earlier work [2], D. Goldin proposed a notion of environment to highlight that the class of behaviors captured by the TM, the class of algorithmic behaviors, is different from that represented by the PTM model, the sequential interactive behaviors. The conceptualization of the environment provides the observational characterization of PTM behaviors given by the input-output streams. In fact, given two different environments $O_1$ and $O_2$ and a PTM machine $M$, the behavior of $M$ observed by interacting with an environment $O_1$ can be different if observed by interacting with $O_2$. Also, given two machines $M_1$ and $M_2$ and one environment $O$, if the behaviors of the two machines are equal (one can be reduced to the other), they must be equivalent in $O$. This claim gives the go-ahead to Theorem 3. Any environment $O$ induces a partitioning of $M$ into equivalence classes whose members appear behaviorally equivalent in $O$; the set of equivalence classes is denoted by $\beta_O$. Indeed, the equivalences of the behaviors of two PTMs can be expressed by the language represented in the set of all interaction streams.

Let $B(M)$ denote the operator that extracts the behavior of a given machine $M$, and $O(M)$ a mapping that associates any machine $M$ to the class of the behaviors feasible for the environment $O$. Therefore, each machine can be classified by analyzing its interaction streams with the two operators, $B$ and $O$. 
Definition 5 (Environment) Given a class $\mathcal{M}$ of PTMs and a set of suitable domains $\beta_{\mathcal{O}}$, that is the set of equivalence classes of feasible behaviours. An environment $\mathcal{O}$ is a mapping from machines to some domains $\mathcal{O} : \mathcal{M} \rightarrow \beta_{\mathcal{O}}$ and the following property holds:

$$\forall M_1, M_2 \in \mathcal{M}, \text{ if } B(M_1) = B(M_2) \text{ then } \mathcal{O}(M_1) = \mathcal{O}(M_2)$$

When $\mathcal{O}(M_1) \neq \mathcal{O}(M_2)$, we say that $M_1$ and $M_2$ are distinguishable in $\mathcal{O}$; otherwise, we say that $M_1$ and $M_2$ appear equivalent in $\mathcal{O}$.

Theorem 3 Let $\Theta$ denote the set of all possible environments. The environments in $\Theta$ induce an infinite expressiveness hierarchy of PTM behaviors, with TM behaviors at the bottom of the hierarchy.

So far, we have assumed that all the input streams are all feasible. However, this is not a reasonable assumption for the context in which interactive machines normally run. Typically an environment can be constrained and limited by some obstructions when generating the output streams. In our view, this is the case where the space of all possible configurations lies on a topological space with not trivial topology. In order to contribute to this theory, in the following we will tackle the issue of specifying these constraints, and relating them to the PTM model.

3 Topological Interpretation of Interactive Computation

Topological environment This section deals with the notion of topological environment as an integral part of the model of topological computation. In a classical TM the environment is not represented (Def 1), whereas in a PTM the environment is a mapping between the class of PTMs and their feasible domains. As described above the two functions $B$ and $O$ permit to identified the behavior of a PTM machine by observing its stream of interactions. In this case the environment $\mathcal{O}$ is a static mapping that associates machines with an equivalent behavior $B(M)$ to the same equivalence class. In this case the environment plays the role of an observer. In our approach the environment is part of the system that evolves together with the behavior of the machine over time step $i$. The environment constrains the behavior of a machine PTM so as the output generated by the machine affects the evolution of the environment.

To detect dynamic changes in the environment, we propose to define a dynamic analysis of the set of all the interactions streams available at any single PTM computation step $i$. Since interaction streams are infinite sequences of pairs of the form $(w_i,w_o)$ representing the input and output strings of PTMs computation step $i$, we use the set $\mathcal{P}$ of PTM configurations to represent them.

The resulting model of computation consists of two components entangled and coexisting during the interactive computation, a functional unit of computation and a self-organizing memory.
In our model, the infinite input of the PTM should be seen as a feedback loop of a dynamic system. Its functional behavior is represented by a class T of ITS constrained by the information contained in the self-organizing memory associated with the notion of topological environment. The data structure used to store information is the simplicial complex $S_P$, that is a topological space $S$ constructed over the set of PTM configuration $P$. The $S_P$ is equipped with a finite presentation in terms of homology groups whose relations are fully representable. In this view the PTM functional behavior can be determined by $S_P$ modulo ITS isomorphism. We operate in a discrete setting where full information about topological space is inherent in their simplicial representation. Appendix 1 provides some useful definitions for algebraic and computational topology.

**Definition 6 (Topological environment)** Given the set of PTM configurations $P_i$ available at a given time $i$, the topological environment is the simplicial complex $S_P$, constructed over $P_i$.

The topological environment $S_P$, as any topological space is equipped with a set of invariants that are important to understand the characteristics of the space. For the sake of simplicity we will refer to topological space as a continuous space. The $n$-dimensional holes, the language of paths, the homology and the genus are topological invariants. The $n$-dimensional holes are determined during the process of filtration, called persistent homology, that is used to construct a topological space starting from a set of points. The numbers of holes and their associated dimensions are determined by the homology structure fully represented by the homology groups associated with a topological space. Also the homology is a topological invariant of the space, it is always preserved by homeomorphisms of the space.

A path in a topological space $S$ is a continuous function $f : [0, 1] \rightarrow S$ from the unit interval to $S$. Paths are oriented, thus $f(0)$ is the starting point and $f(1)$ is the end-point, if we label the starting point $v$ and the end-point $v'$, we call $f$ a path from $v$ to $v'$ as shown in Figure 2-(a). Two paths $a$ and $b$, that is two continuous functions, from a topological space $S$ to a topological space $S'$ are homotopic if one can be continuously deformed into the other. Being homotopic is an equivalence relation on the set of all continuous functions from $S$ to $S'$. The homotopy relation is compatible with function composition.

Therefore, it is interesting to study the effect of the existence of holes (at any dimension) in a topological space $S$ (for simplicity the discussion is made thinking of $S$ as a 2D surface) built from the space of configurations $P$ where a sequential interactive computation takes place as a sequential composition of paths. Figures 2(b) and -(c) show the composition of two paths $a$ and $b$, and the proof that they are not homotopic, respectively. Given two-cycle paths, $a$ and $b$, with a point in common in $x$, if the composition of the two paths $ab$ or $ba$ is not commutative, the two composed paths are not equivalent. In this case, the two cycle paths, $a$ and $b$ can be considered the generators of a topological space with one 2-dimensional hole, as shown in Figure 3. Each generator represents a distinct class of paths, $[a]$ those going around the neck, and $[b]$ those around the belt of the torus, respectively.
Computable functions and topological space. We start taking into account those classes of problems whose computable functions are defined over a space $S$ endowed with a trivial topology, and it is a Vector Space. Figure 4 shows how an algorithmic computation $A$ associated with the function $f_A: S \rightarrow S$, evolves over $S$, representing the space of the states. Each state $v$ is defined by a vector that moves over $S$ driven by the configurations of the TM. In Figure 4 from left to right, the first two pictures represent a successful computation and a computation with an infinite loop, respectively. When the algorithm moves the vector towards a boundary, see the last picture, the computation is deadlocked. This happens because $S$ has not been defined globally. In fact, the boundary breaks the translational symmetry. If we allow the boundary to disappear by adding an extra-relation, global in nature, we obtain a global topology that is not trivial – the space is characterized by a not empty set of $n$-dimensional holes ($n \geq 2$). Figure 5 shows how the computation with a deadlock on the plane could have succeeded if the manifold of the space is a torus.

Figure 6 shows how we can transform a rectangle, 2-dimensional space $S$ homomorphic to 2-manifold with boundary, into a cylinder and then into a torus by adding two relations among the generators of the manifold $P$ that will be proved to be without boundary.

Hence, we proceed to analyze those classes of problems whose computable functions are defined over a space $S$ endowed with non-trivial topology. The class of functions $F_S$ effectively computable over a space $S$, and for each single function $f_A \in F_S$ and a couple of points $v, v' \in S$, we associate a computation
Fig. 3. From cycling paths to generators of a space $S$.

Fig. 4. a) successful computation, b) computation with an infinite loop, c) computation with a deadlock.

$f_A(v) = v'$, as a path that connects the two points $v$ and $v'$ in the space $S$. The path can be semantically interpreted as an interaction stream.

In Figure 4, the first two pictures from left to right, show that a close path $\pi$ in a surface that starts and ends to a fixed point $P_B$ is homotopic to 0; it means that any $\pi$ can be reduced to the point $P_B$. The class of behavioral equivalence to $\tau$ denoted by $[\pi]$ belongs to space or subspace space with trivial topology $g = 0$ ($g$ is the genus). The other pictures show irreducible paths belonging to space with a topological genus $g \neq 0$. E.g., if $g = 1$, i.e. is a torus there are three different classes of behaviors: i) the set of closed paths homotopic to 0. In this case, we are given a local interpretation and we are not aware that at the global level the genus can be different from 0; ii) the set of closed paths homotopic to the first generator $a$ of the homology group of the topological space $S$. The cycle fixed on the base point $P_B$ can be used to reduce any path going around the belt of the torus to $a$ by a continuous deformation; iii) similar to the previous set, but the paths are homotopic to the second generator $b$ of $S$. The cycle fixed
on the base point \( P_B \) goes around the belt of the torus. The last picture shows the composition of paths.

The interpretation of interaction streams over a \( S \) is indeed nothing but its identification with an element of the path algebra corresponding to a quiver representation of the transformation group \( \mathcal{G} \) of \( S \), say \( Q \) (or, more generally, a set of quivers, over some arbitrary ring). The different ways to reach any point \( p \in \mathcal{P} \) from \( P_B \) generate a path algebra \( A \) whose elements are describable words in a language \( \mathcal{L} \). Any point of \( \mathcal{P} \) can be related to any other point by a group element. By selecting a point \( p_0 \) of \( \mathcal{P} \) as a unique base point, every point of \( \mathcal{P} \) is in one-to-one correspondence with an element of such group \( \mathcal{G}_{MC} \approx \mathcal{MCG} \), the simplicial analog of the mapping class group. \( \mathcal{G}_{MC} \) is a group of transformations which do not change the information hidden in the data, such as the group of diffeomorphisms that do not change the topology of the base space. \( \mathcal{MCG} \) is an algebraic invariant of a topological space, that is a discrete group of symmetries. Since the algebras manipulate the data, the transformations applied to space are ‘processes’ carried on through the fiber, which is the representation space of the process algebra. Whenever \( Q \) can give the representation of the algebra, the algebra can be exponentiated to a group \( \mathcal{G}_{AP} \) and \( t \) a gauge group. We have now all the ingredients for defining a fiber bundle enriched with a group \( \mathcal{G} = \mathcal{G}_{AP} \wedge \mathcal{G}_{MC} \), called gauge group, (see Figure 1). Summarizing, fiber bundle is the mathematical structure that allows us to represent computation and its context (the environment) as a unique model. In terms of \( TM \), the context represents the transition function, also called the functional matrix.

While the algorithmic aspect of a computation expresses the effectiveness of the computation, the topology provides a global characterization of the environment.

Both the computation and the environment can be represented as groups (algebras), and their interaction is captured as the set of accessible transformations of the semi-direct product of the two groups, carrying constrained by the restric-
The pictures (A–D) summarize the main steps to transform a space $S$ of PTM into a topological space $S_P$. The construction is obtained by gluing together – put in relation – the two boundaries of the space $S$, $a$ and $b$ respectively, which become the generators $a$ and $b$ of the new space $S_P$. The topological space $S_P$, finite but not limited, naturally supports the notion of the environment of PTM.

Definition 7 (Topological Turing machine) A Topological Turing machine (TTM) is a group $G$ consisting of all interaction streams generated by the group of PTMs entangled with the group of all transformations of the topological space $S_P$ preserving the topology. Formally $G = G_{AP} \wedge G_{MC}$, where $G_{AP}$ is the group of PTMs and $G_{MC}$ the simplicial analog of the mapping class group.

Proposition 1 If $G$ is automatic, the associated language $L$ is regular. Since the representations of $G$ can then be constructed in terms of quivers $Q$ with relations induced by the corresponding path algebra induced by PTMs, the syntax of $L$ is fully contained in $T$ and its semantics in $M$.

Definition 8 (Constrained interactive computation) An interactive computation is constrained if it is defined over a topological space $S_P$ and it is an element of the language of paths of $S_P$.

Theorem 4 Any constrained interactive computation is an effective computation for a TTM.

Thesis 2 Any concurrent computation can be performed by a TTM.

4 Final remarks

In 2013, Terry Tao in his blog [20] posted this question: if there is any computable group $G$ which is "Turing complete" in the sense that the halting problem for
any Turing machine can be converted into a question of the above form. In other words, there would be an algorithm which, when given a Turing machine $T$, would return (in a finite time) a pair $x_T, y_T$ of elements of $G$ with the property that $x_T, y_T$ generate a free group in $G$ if and only if $T$ does not halt in finite time. Or more informally: can a 'group' be a universal Turing machine?

Acknowledgements

E. M. thanks Mario Rasetti for bringing her to conceive a new way of thinking about computer science and for numerous and lively discussions on topics related to this article; and Samson Abramsky with his group for insightful conversations on the topological interpretation of contextuality and contextual semantics. E. M. and A.W. thank the anonymous referees for suggesting many significant improvements.

Funding statements. We acknowledge the financial support of the Future and Emerging Technologies (FET) programme within the Seventh Framework Programme (FP7) for Research of the European Commission, under the FP7 FET-Proactive Call 8 - DyMCS, Grant Agreement TOPDRIM, number FP7-ICT-318121.

References

1. D.Q. Goldin, S.A. Smolka, P.C. Attie, E.L. Sondereggera. Turing machines, transition systems, and interaction. Information and Computation 194, 2004. - ENTCS
2. D. Goldin. Persistent Turing Machines as a Model of Interactive Computation. LNCS, Vol.1762, Springer, 2000.

3. D.Q. Goldin, S.A. Smolka, P. Wegner. Interacting Computation: The new paradigm. Springer, 2006.

4. P. Wegner. Why Inter. is More P Than Algorit. CACM, Vol. 40, No.5, ACM, 1997.

5. P. Wegner. Interactive foundations of computing. TCS, Vol.192, Elsevier, 1998.

6. R.O Gandy. Churchs Thesis and Principles for Mechanisms. J. Barwise, H. J. Keisler and K. Kunen, eds, The Kleene Symposium, North-Holland Publishing Company, 1980.

7. A. Wigderson. Mathematics and Computation. IAS, Draft: March 2018.

8. S. Garrone, A. Marzuoli, M. Rasetti. Spin networks, quantum automata and link invariants. Journal of Physics: Conference Series 33, 2006.

9. E. Merelli, M. Pettini, M. Rasetti. Topology driven modeling: the IS metaphor. Natural Computing Vol.14, No.3, 2015.

10. P. Wegner. Interactive foundations of computing. TCS, Vol.192, Elsevier, 1998.

11. M. Rasetti, E. Merelli. Topological Field Theory of Data: mining data beyond complex networks. Ed. P. Contucci, Laganà, In Advances in disordered systems, random processes and some applications. Cambridge University Press, 2016.

12. N. Steenrod. The topology of Fiber Bundles. Princeton Mathematical Series. Princeton University Press, 1951.

13. P.J. Landin. A Program Machine Symmetric Automata Theory. Machine Intelligence Vol. 5, ed. Meltzer and Michie, Edinburgh University Press.

14. S. Abramsky. An algebraic characterisation of concurrent composition. ArXiv 1406.1965v1, 2014.

15. A. M. Turing. Lecture to the London Mathematical Society, 20 February 1947. Quoted in B. E. Carpenter and R. W. Doran (eds.), A. M. Turing’s Ace Report of 1946.

16. H. Lewis, C.H. Papadimitriou. Elements of the Theory of Computation. 2nd Ed. Prentice Hall, 1998.

17. S. Abramsky. Contextuality: At the Borders of Paradox. Categories for the Working Philosophers, Ed. by Elaine Landry. 2017.

18. S. Abramsky. Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity. arXiv:1406.7380v1, 2014.

19. S. Abramsky. What are the Fundamental Structures of Concurrency? We still dont know! Electronic Notes in Theoretical Computer Science Vo.162, 2006.

20. Mathoverflow. https://mathoverflow.net/questions/88368/can-a-group-be-a-universal-turing-machine
Appendix 1: Definitions of Algebraic and Computational Topology

Definition 9 Topology
A topology on a set $X$ is a family $T \subseteq 2^X$ such that
- If $S_1, S_2 \subseteq T$, then $S_1 \cap S_2 \in T$ (equivalent to: If $S_1, S_2, \ldots, S_n \in T$ then $\cap_{i=1}^n S_i \in T$).
- If $\{S_j| j \in J\} \subseteq T$, then $\cup_{j\in J} S_j \in T$.
- $\emptyset, X \in T$.

Definition 10 Topological spaces
The pair $(X,T)$ of a set $X$ and a topology $T$ is a topological space. We will often use the notation $X$ for a topological space $X$, with $T$ being understood.

Definition 11 Simplices
Let $u_0, u_1, \ldots, u_k$ be points in $\mathbb{R}^d$. A point $x = \sum_{i=0}^k \lambda_i u_i$ is an affine combination of the $u_i$, if the $\lambda_i$ sum to 1. The affine hull is the set of affine combinations. It is a k-plane if the $k+1$ points are affinely independent by which we mean that any two affine combinations, $x = \sum_{i=0}^k \lambda_i u_i$ and $y = \sum_{i=0}^k \mu_i u_i$, are the same iff $\lambda_i = \mu_i$ for all $i$. The $k+1$ points are affinely independent iff the $k$ vectors $u_i \ldots u_0$, for $1 \leq i \leq k$, are linearly independent. In $\mathbb{R}^d$ we can have at most $d$ linearly independent vectors and therefore at most $d+1$ affinely independent points.

A $k$-simplex is the convex hull of $k+1$ affinely independent points, $\sigma = \{u_0, u_1, u_2, \ldots, u_k\}$. Its dimension is $\dim \sigma = k$. Any subset of affinely independent points is again independent and therefore also defines a simplex of lower dimension.

Definition 12 Face
A face of $\sigma$ is the convex hull of a non-empty subset of the $u_i$ and it is proper if the subset is not the entire set. We sometimes write $\tau \leq \sigma$ if $\tau$ is a face and $\tau < \sigma$ if it is a proper face of $\sigma$. Since a set of $k+1$ has $2^{k+1} - 1$ subsets, including empty set, $\sigma$ has $2^{k+1} - 1$ faces, all of which are proper except for $\sigma$ itself. The boundary of $\sigma$, denoted as $\partial \sigma$, is the union of all proper faces, and the interior is everything else.

Definition 13 Simplicial complexes
A simplicial complex is a finite collection of simplices $K$ such that $\sigma \in K$ and $\tau \in K$, and $\sigma, \tau \in K$ implies $\sigma \cap \tau \in K$ is either empty or a face of both.

Definition 14 Filtration
A filtration of a complex $K$ is a nested sequence of subcomplex, $\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \ldots \subseteq K^m = K$. We call a complex $K$ with a filtration a filtered complex.

Definition 15 Chain group
The $k$-th chain group of a simplicial complex $K$ is $(C_k(K),+)$, let $\mathbb{F}$ be a field. The $\mathbb{F}$-linear space on the oriented $k$-simplices, where $[\sigma] = -[\tau]$ if $\sigma = \tau$ and $\sigma$ and $\tau$ have different orientations. An element of $C_k(K)$ is a $k$-chain, $\sum_q n_q[\sigma_q], n_q \in \mathbb{Z}, \sigma_q \in K$.

Definition 16 Boundary homomorphism
Let $K$ be a simplicial complex and $\sigma \in K, \sigma = [v_0, v_1, \ldots, v_k]$ The boundary homomorphism $\partial_k : C_k(K) \to C_{k-1}(K)$ is $\partial_k \sigma = \sum_i(-1)^i[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k]$ where $\hat{v}_i$ indicates that $v_i$ is deleted from the sequence.
Definition 17 Cycle and boundary
The k-th cycle group is $Z_k = \ker \partial_k$. A chain that is an element of $Z_k$ is a k-cycle. The k-th boundary group is $B_k = \text{im} \partial_{k+1}$. A chain that is an element of $B_k$ is a k-boundary. We also call boundaries bounding cycles and cycles not in $B_k$ nonbounding cycles.

Definition 18 Homology group
The k-th homology group is $H_k = Z_k/B_k = \ker \partial_k/\text{im} \partial_{k+1}$. If $z_1 = z_2 + B_k$, $z_1, z_2 \in Z_k$, we say $z_1$ and $z_2$ are homologous and denote it with $z_1 \sim z_2.$

Definition 19 k-th Betti number
The k-th Betti number $B_k$ of a simplicial complex $K$ is the dimension of the k-th homology group of $K$. Informally, $\beta_0$ is the number of connected components, $\beta_1$ is the number of two-dimensional holes or "handles" and $\beta_2$ is the number of three-dimensional holes or "voids" etc. . . .

Definition 20 Invariant
A topological invariant is a property of a topological space which is invariant under homeomorphisms. Betti numbers are topological invariants.

Definition 21 Genus
The genus is a topological invariant of a close (oriented) surface. The connected sum of $g$ tori is called a surface with genus $g$. genus refers to how many 'holes' the donut surface has.
As an example, a torus is homeomorphic to a sphere with a handle. Both of them have just one hole (handle). The sphere has $g = 0$ and the torus has $g = 1.$