Technical communiqué

On the lack of monotonicity of Newton–Hewer updates for Riccati equations

Mohammad Akbari *, Bahman Gharesifard, Tamas Linder

Department of Mathematics and Statistics, Queen’s University, Kingston, ON K7L 3N6, Canada

A R T I C L E   I N F O

Article history:
Received 26 October 2020
Received in revised form 6 May 2021
Accepted 19 May 2021
Available online 12 July 2021

Keywords:
Discrete algebraic Riccati equation
LQR optimal control

A B S T R A C T

We provide a set of counterexamples for the monotonicity of the Newton–Hewer method (Hewer, 1971) for solving the discrete-time algebraic Riccati equation in dynamic settings, drawing a contrast with the Riccati difference equation (Caines and Mayne, 1970).

1. Introduction

This note investigates the monotonicity properties of iterative methods for solving the discrete-time algebraic Riccati equation (DARE), which is given by

\[ P = A^T PA - A^T PB (B^T PB + R)^{-1} B^T PA + Q. \]  

(1)

As is well-known, given the discrete-time linear control system

\[ x(t + 1) = Ax(t) + Bu(t), \quad x(0) = x_0, \]

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) are the system’s state and controller at time \( t \geq 0 \), respectively, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \), and the cost function

\[ J(u) = \sum_{t=0}^{\infty} (x(t)^T Q x(t) + u(t)^T R u(t)), \]

where \( Q \in \mathbb{R}^{n \times n}, R \in \mathbb{R}^{m \times m} \) are positive-definite matrices, and assuming the controllability of the system \((A, B)\) and observability of \((A, Q^{1/2})\), the optimal controller which minimizes \( J \) is given by \( u^*(t) = - (B^T PB + R)^{-1} B^T PA x(t) \).

\* Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Maria Letizia Corradini under the direction of Editor André L. Tits.

Corresponding author.
E-mail addresses: 13mav1@queensu.ca (M. Akbari), bahman.gharesifard@queensu.ca (B. Gharesifard), tamas.linder@queensu.ca (T. Linder).

https://doi.org/10.1016/j.automatica.2021.109788
0005-1098/© 2021 Elsevier Ltd. All rights reserved.

where \( P \) satisfies (1), see, e.g., equation (3.3-21), p. 54 in Anderson and Moore (1990).

There are several classical iterative methods for solving the DARE in the literature, including the ones proposed in Caines and Mayne (1970), Hewer (1971), algebraic methods (Rodman & Lancaster, 1995), and semi-definite programming (Balakrishnan & Vandenberghe, 2003). In particular, these iterative methods generate a sequence of positive-definite matrices which converges to the solution of the DARE. Our main focus in this paper is on two commonly used methods, the so-called Riccati difference equation (Caines & Mayne, 1970), which provably converge to the fixed point solution of the DARE, and what we call the Newton–Hewer method which was introduced by Hewer in Hewer (1971), and uses a Newton-based update to generate a sequence of positive-definite matrices which monotonically converge to the solution of the DARE when initialized at a stable policy. Let us describe these in more detail:

The Riccati difference equation is given by

\[ P_{t+1} = A^T P_t A - A^T P_t B (B^T P_t B + R)^{-1} B^T P_t A + Q. \]

It has been shown that the right hand side of this dynamics is monotone as a function of \( P_t \), in the sense that if \( P_{t+1} \geq P_t \geq 0 \) implies \( P_{t+1} \geq P_{t+1} \geq 0 \), see De Souza (1989, Lemma 3.1). Furthermore, this dynamics is monotone as a function of \((A, B, Q, R)\) in the following sense: If \( P_t \geq P_t \geq 0 \) and

\[
\begin{pmatrix}
Q & A^T \\
A & -BR^{-1}B^T
\end{pmatrix} \geq
\begin{pmatrix}
\hat{Q} & A^T \\
A & -\hat{B}R^{-1}\hat{B}^T
\end{pmatrix},
\]

then \( P_{t+1} \geq \hat{P}_{t+1} \geq 0 \), see Freiling, Jank, and Abou-Kandil (1996), Wimmer (1992). Notably and important to the discussion we will have in the next section, as long as (2) is satisfied, this
monotonicity property holds even when the parameters $A, B, Q, R$ are time-varying.

The Newton-Hewer method (Hewer, 1971) is given by
\begin{equation}
  P_{t+1} = A_t^* P_{t+1} A_t + K_t^* B K_t + Q, \\
  A_t = A - B K_t, \\
  K_t = (B^T P_t B + R)^{-1} B^T P_t A_t, \\
\end{equation}
and it has been shown that if the system is controllable, by initializing with a stable $K_0$, i.e., $\rho(A - B K_0) < 1$, where $\rho(\cdot)$ denotes the spectral radius, $P_t$ converges monotonically, i.e., $P_1 \geq P_2 \geq \ldots \geq P_t$, where $P_t$ is the solution of (1).

The monotonicity property of the Riccati difference equation has been used to derive a robust stability condition for finite-horizon robust LQR problem (Zou & Gupta, 2000). In addition, a boundedness result for the solution of the Riccati difference equation has been derived using this property (De Nicolao, 1992; Freiling et al., 1996). These applications motivate us to ask the natural question whether the Newton–Hewer dynamics has the monotonicity property that Riccati difference equation enjoys.

We will show in this note that this is not the case in general, i.e., $P_t \geq P_t^* \geq 0$ does not necessarily imply $P_{t+1} \geq P_{t+1}^* \geq 0$, by providing two counterexamples, each aimed to demonstrate a facet of this lack of monotonicity.

2. Counterexamples

The construction of our examples is done for the scalar case, and for this reason, we write the Newton–Hewer dynamics in this scenario. We assume that $Q$ and $R$ are positive real numbers, and do not change with time. By setting $n = m = 1$, the dynamics (3) can be written as:
\begin{equation}
  P_{t+1} = \frac{A_t^2 B^2 P_t^2 R + QB^2 Q_t^2 + 2 QB^2 P_t R + QR^2}{(P_t B^2 + R + AR)(P_t B^2 + R - AR)}. \\
\end{equation}
By taking derivative, it can be shown that $P_{t+1}$ as a function of $P_t$ is increasing for $P_t > P^*$ and decreasing for $P_t < P^*$, where $P^*$ is the solution to (1). For a stable policy $K_t$, $P_t$ will be larger than $P^*$ and monotonicity holds (Hewer, 1971). We have depicted $P_{t+1}$ as a function of $P_t$ in Fig. 1, and it can be observed that the Newton–Hewer dynamics is increasing for $P_t \geq P^*$, where $P^*$ is at the intersection of the line $P_t = P_{t+1}$ and Newton–Hewer dynamics. We now show that if the system has time-varying $Q$ and $R$, the stabilizability properties of the controller do not necessarily imply that the system is monotone, drawing a contrast with the Riccati difference equation.

To this end, note that this graph depends on $A, B, R$ and $Q$, and if one of these parameters changes, $P^*$ and the graph will change. To elaborate on this, we use Fig. 2 where we have depicted $P_{t+1}$ as a function of $P_t$ for two different Newton–Hewer dynamics with $(A, B, R, Q) = (1, 1, 1, 1)$ and $(A, B, R, Q) = (1, 1, 1, 2)$. In Fig. 1, $P^*_t$ and $P^*_2$ refer to the solution to the DARE (1) for the systems $(A, B, R, Q) = (1, 1, 1, 1)$ and $(A, B, R, Q) = (1, 1, 1, 2)$, respectively. If $Q_1 = 1, Q_{t+1} = 2$ and $K_t$ are such that $P^*_1 < P_{t+1}^* < P_{t+2}^*$, then the system for the next time step uses the orange graph to update $P_{t+1}$, and the reader can observe – we prove this with carefully chosen numerical values below – that this can lead to failure of monotonicity, i.e., $P_t \leq P_t^*$ does not necessarily imply $P_{t+1} \leq P_{t+1}$. Note that the system will remain monotone if $Q_{t+1} < Q_t$ for all $t$, in case the other system parameters $A, B, R$ remain fixed. Using this observation, we now explicitly construct the counterexample.

Example 2.1 (Consider the Dynamics (3)). Let the system be scalar, i.e., $n = m = 1$, and let $A = 1, B = 1, R = 1$ be fixed and $Q_t$ be time-varying. Let $P_t$ be the sequence generated by (3) at each time step. Given that $A, B, R$ are fixed, $P_t$ is a function of $[Q_1, Q_2, \ldots, Q_t]$ and $K_0$, where $K_0$ is a stable policy at time 0. Let $P_1$ be the sequence generated by (3) with $A = 1, B = 1, R = 1$ and $Q_1$ and $K_0$. We claim that $P_1 \geq P_t^*$ does not necessarily imply that $P_{t+1} \geq P_{t+1}$. To prove this claim, we need to chose $K_0$ properly. Let $K_0 = \sqrt{3} - 1$,
which is a stabilizing policy. Hence, by (3) we have
\begin{equation}
  P_1 = \frac{K_0^2 R_1 + Q_1}{1 - (A - B K_0)^2} = \frac{4 - 2 \sqrt{3} + Q_1}{4 \sqrt{3} - 6}.
\end{equation}
Given this
\begin{equation}
  K_1 = \frac{B P_1 A}{B^2 P_1 + R} = \frac{4 - 2 \sqrt{3} + Q_1}{2 \sqrt{3} - 2 + Q_1}, \\
  P_2 = \frac{K_1^2 R_2 + Q_2}{1 - (A - B K_1)^2} = \frac{(8 - 4 \sqrt{3})Q_1 + (16 - 8 \sqrt{3})Q_2 + (Q_2 + 1)Q_1^2}{4(\sqrt{3} - 1)Q_1 + Q_1^2 - 68 + 40 \sqrt{3}}.
\end{equation}
If we choose 

Now let 

\[ Q_1 = 1 \text{ and } Q_2 = 2 \]

then 

\[ P_1 = 1.6547, \text{ and } P_2 = 2.7835. \]

If we choose 

\[ \hat{Q}_1 = 1 \text{ and } \hat{Q}_2 = 2, \]

then 

\[ \hat{P}_1 = 2.7321. \]

This demonstrates that given \( \hat{P}_1 \geq P_1 \), it does not follow that \( P_{t+1} \geq \hat{P}_{t+1} \). Fig. 3 shows the sequence \( P_t \) (dotted line) and \( \hat{P}_t \) (dashed line) where \( Q_1 = 1 \) and \( Q_2 = 2 \) for \( t \geq 2 \) and \( \hat{Q}_2 = 2 \). Furthermore, the sequence \( \hat{P}_t \), which is generated by the Riccati difference equation with initialization \( \hat{P}_1 = P_1 \) and the same parameters \( \hat{A} = A, \hat{B} = B, \hat{Q}_2 = Q_2, \hat{R} = R \) is shown (solid line) for six time steps. •

We conclude with providing an example which demonstrates another aspect of lack of monotonicity of Newton-Hewer dynamics.

**Example 2.2.** We consider two dynamics with the same \( Q \) and \( R \), albeit time-varying, but with different initial conditions \( K_0 \). Similar to the previous example, we assume \( n = m = 1 \), and \( A = 1, B = 1, R = 1 \) are fixed and \( Q \) is time-varying. We assume \( Q_2 \) is 1 for odd time steps and 1.1 for even time steps. If we choose \( K_0 = 0.7321 \) for the first system and \( \hat{K}_0 = 0.6180 \) for the second system, we will have \( P_1 = 1.6180 \) and \( \hat{P}_1 = 1.6547 \), and for the next time, we have \( P_2 = 1.7351 \) and \( \hat{P}_2 = 1.7347 \), which shows that the monotonicity does not hold. Fig. 4 illustrates the behaviour of two dynamics at the next time steps. •

**References**

Anderson, B. D., & Moore, J. B. (1990). *Optimal control: Linear quadratic methods*. Prentice-Hall.

Balakrishnan, V., & Vandenberge, L. (2003). Semidefinite programming duality and linear time-invariant systems. *IEEE Transactions on Automatic Control*, 48(1), 30–41.

Caines, P. E., & Mayne, D. Q. (1970). On the discrete time matrix Riccati equation of optimal control. *International Journal of Control*, 12(5), 785–794.

De Nicolao, G. (1992). On the time-varying Riccati difference equation of optimal filtering. *SIAM Journal on Control and Optimization*, 30(6), 1251–1269.

De Souza, C. E. (1989). On stabilizing properties of solutions of the Riccati difference equation. *IEEE Transactions on Automatic Control*, 34(12), 1313–1316.

Freiling, G., Jank, G., & Abou-Kandil, H. (1996). Generalized Riccati difference and differential equations. *Linear Algebra and its Applications*, 241, 291–303.

Hewer, G. (1971). An iterative technique for the computation of the steady state gains for the discrete optimal regulator. *IEEE Transactions on Automatic Control*, 16(4), 382–384.

Rodman, L., & Lancaster, P. (1995). *Oxford mathematical monographs, Algebraic riccati equations*. Wimper, H. (1992). Monotonicity and maximality of solutions of discrete-time algebraic Riccati equations. *Mathematical Systems, Estimation, and Control*, 2, 219–235.

Zou, J., & Gupta, Y. P. (2000). Robust stabilizing solution of the Riccati difference equation. *European Journal of Control*, 6(4), 384–391.